DIMENSIONS OF COMPONENTS OF TENSOR PRODUCTS OF THE LINEAR GROUPS REPRESENTATIONS WITH APPLICATIONS TO BEURLING-FOURIER ALGEBRAS

BENOÎT COLLINS, HUN HEE LEE, AND PIOTR ŚNIADY

ABSTRACT. We give universal upper bounds on the relative dimensions of isotypic components of a tensor product of the linear group $\text{GL}(n)$ representations and universal upper bounds on the relative dimensions of irreducible components of a tensor product of the special linear group $\text{SL}(n)$ representations. This problem is motivated by harmonic analysis problems, and we give some applications of this result in the theory of Beurling-Fourier algebras.

1. INTRODUCTION

1.1. The main problem for linear groups $\text{GL}(n)$. In this paper we are interested in the following question: let $\lambda, \mu$ be two irreducible representations of the linear group $\text{GL}(n)$ and consider the decomposition of their tensor product $\lambda \otimes \mu$ into isotypic components. How big the dimension of such an isotypic component can be?

For irreducible representations $\lambda, \mu, \nu$ we denote by $c_{\lambda,\mu}^\nu$ the Littlewood-Richardson coefficient, i.e. the multiplicity of the irreducible representation $\nu$ in the Kronecker tensor product $\lambda \otimes \mu$. For an irreducible representation $\rho$ we denote by $d_\rho$ its dimension. With these notations, the dimension of the isotypic component $[\nu]$ of $\lambda \otimes \mu$ is equal to $c_{\lambda,\mu}^\nu d_\nu$. Our goal will be to give an upper bound for the fraction

\[ P_{\lambda,\mu}(\nu) := \frac{c_{\lambda,\mu}^\nu d_\nu}{d_\lambda d_\mu} \]

which can be interpreted as the relative dimension of the isotypic component $[\nu]$ in $\lambda \otimes \mu$.

2010 Mathematics Subject Classification. 05E10 (Primary) 22E46, 43A30, 47L30, 51F25 (Secondary).

Key words and phrases. representations of unitary groups, Kronecker tensor product of representations, Littlewood-Richardson coefficients, Young tableaux, Beurling-Fourier algebras.
Equation (1) defines a probability distribution $P_{\lambda, \mu}$ (called the Littlewood-Richardson measure) on irreducible representations. This probability measure can be interpreted as a distribution of a random irreducible component of the Kronecker tensor product $\lambda \otimes \nu$, where each irreducible component is sampled with a probability proportional to its dimension. Our problem can therefore equivalently formulated as finding an upper bound for the atoms of Littlewood-Richardson measure.

1.2. The main result for linear groups $GL(n)$. The main result of this paper is the following partial answer to the above problem.

**Theorem 1.1.** Let $n \geq 1$ be a fixed integer. There exists a constant $C_n$ such that for any irreducible representations $\lambda, \mu, \nu$ of $GL(n)$ the atom of the Littlewood-Richardson measure is bounded from above as follows:

$$P_{\lambda, \mu}(\nu) := \frac{c_{\lambda, \mu}^{\nu}}{d_{\lambda} d_{\mu}} \leq C_n \left( \frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right).$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$ are the components of the highest weight of $\lambda$ and $\mu$, respectively. The notations used in the above inequality will be recalled in Section 2. We postpone its proof to Section 5.

We will see that this result is optimal in a sense which will be clarified at the end of the paper.

1.3. The main result for special linear groups $SL(n)$. In this paper we are also interested in the following modification of the above problem: let $\lambda, \mu$ be two irreducible representations of the special linear group $SL(n)$ and consider the decomposition of their tensor product $\lambda \otimes \mu$ into irreducible components. How big the dimension of such an irreducible component can be?

A partial answer for this problem is given by the following result, which is a corollary to our main theorem:

**Corollary 1.2.** Let $n \geq 1$ be a fixed integer. There exists a constant $C_n$ such that for any irreducible representations $\lambda, \mu, \nu$ of $SL(n)$, if $\nu$ contributes (with multiplicity at least 1) to the decomposition of the Kronecker tensor product $\lambda \otimes \mu$ into irreducible components, then its relative dimension is bounded from above as follows:

$$\frac{d_{\nu}}{d_{\lambda} d_{\mu}} \leq C_n \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right).$$

The notations used in the above inequality will be recalled in Section 2.3, where we will also present its proof.
1.4. **The case of unitary groups and special unitary groups.** The representation theory of the unitary group $U(n)$ is exactly the same as that of the linear group $GL(n)$, namely the restriction map gives a one to one map; its inverse is given by the analytic continuation. In particular, the correspondence between irreducible representations and highest weights holds also for $U(n)$. For this reason in the formulation of Theorem 1.1 one can replace the representations of the linear groups $GL(n)$ by the representations of the unitary groups $U(n)$ and the result holds true without any modifications.

Analogous relationship holds between the representation theory of the special unitary group $SU(n)$ and the special linear group $SL(n)$, for this reason in the formulation of Corollary 1.2 one can replace representations of $SL(n)$ by representations of $SU(n)$.

1.5. **Applications to Beurling-Fourier algebras.** Our paper is motivated by the work of Mahya Ghandehari, Hun Hee Lee, Ebrahim Samei and Nico Spronk [GLSS12] and gives a proof of their conjecture (Conjecture 1, p. 19). Our main theorem implies that the conjecture is true for any integer $n \geq 2$, whilst it was proved for $2 \leq n \leq 5$ in an elementary way in [GLSS12].

In this subsection we briefly describe what are Beurling-Fourier algebras and implications of our main results on them. See [LS12, LST12] for the details on Beurling-Fourier algebras.

Let $G$ be a compact group and $\hat{G}$ be the set of equivalence classes of unitary irreducible representations of $G$. The Fourier algebra $A(G)$ of $G$ is defined as

$$A(G) := \left\{ f \in C(G) : \|f\|_{A(G)} := \sum_{\pi \in \hat{G}} d_{\pi} \left\| \hat{f}(\pi) \right\|_1 < \infty \right\}.$$  

Here, $\hat{f}(\pi)$ denotes the Fourier transform given by

$$\hat{f}(\pi) := \int_G f(x) \overline{\pi(x)} \, dx \in M_{d_{\pi}}(\mathbb{C})$$

where $dx$ denotes the normalized Haar measure on $G$; $\overline{\pi}$ denotes the conjugate representation of $\pi$; and $\|\cdot\|_1$ is the trace norm. It is well known that the Fourier algebra is actually a Banach algebra under the pointwise multiplication.

The Fourier algebra can be defined for any locally compact groups (see [Eym64]) and is regarded as one of the most fundamental examples of commutative Banach algebras associated to groups. When the (compact) group $G$ is abelian, $A(G)$ is nothing but the group algebra $L^1(\hat{G})$ of the Pontryagin dual $\hat{G}$, so that Fourier algebras are usually called the “dual” object of
group algebras. In general, Fourier algebras are quite far away from operator algebras (i.e. norm-closed subalgebras of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$) including $C^*$-algebras. However, by putting some weights on $A(G)$ for a compact group $G$ we can make weighted versions of $A(G)$ much closer to operator algebras.

We call a function $\omega : \widehat{G} \to [1, \infty)$ a \textit{weight} if

$$\omega(\sigma) \leq \omega(\pi) \omega(\pi')$$

for any $\pi, \pi' \in \widehat{G}$ and $\sigma \in \widehat{G}$ appearing as a component of the irreducible decomposition of $\pi \otimes \pi'$.

We define the \textit{Beurling-Fourier algebra} $A(G, \omega)$ by

$$A(G, \omega) := \left\{ f \in C(G) : \|f\|_{A(G, \omega)} = \sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \|f(\pi)\|_1 < \infty \right\}.$$  

There is a natural isometry between $A(G)$ and $A(G, \omega)$ (see [LS12] for the details), so that we can endow an operator space structure on $A(G, \omega)$ coming from $A(G)$ (as the predual of the group von Neumann algebra $\text{VN}(G)$) through this isometry. Then from the condition (4) one can show that $A(G, \omega)$ is a completely contractive Banach algebra under the pointwise multiplication ([LS12]).

Fundamental examples of weights on $\widehat{G}$ are given by the following polynomial dependence on dimensions of the representations. For $\alpha \geq 0$, we define $\omega_\alpha : \widehat{G} \to [1, \infty)$ by

$$\omega_\alpha(\pi) = d_\pi^\alpha \quad (\pi \in \widehat{G}).$$

Clearly $\omega_\alpha$ satisfies the condition (4), and so, it defines a weight on $\widehat{G}$; it is called the \textit{dimension weight of order $\alpha$}.

In [GLSS12, Theorem 4.9] it has been shown that $A(\text{SU}(n), \omega_\alpha)$ is completely isomorphic to an operator algebra under assumption that the estimate (3) for $\text{SU}(n)$ holds true (this assumption was referred to as [GLSS12, Conjecture 1]). Since our main result says that the conjecture is indeed true for all $n \geq 2$, this implies the following.

**Theorem 1.3.** Let $\omega_\alpha$ be the dimension weight of order $\alpha > \frac{d(\text{SU}(n))}{2} = \frac{n^2-1}{2}$ on $\text{SU}(n)$, $n \geq 2$. Then $A(\text{SU}(n), \omega_\alpha)$ is completely isomorphic to an operator algebra.

Note that the above result is not true for $U(n)$, $n \geq 2$ (in general, for any compact connected non-simple Lie groups, see [GLSS12, Theorem 4.7]) even though the representations of $U(n)$, $n \geq 2$ satisfy the estimate (2).
1.6. Viewpoint of representation theory and random matrix theory. 

The main result of this paper is also of intrinsic interest in representation theory and also random matrix theory. According to it, the ‘widths’ of representations tell something about the relative dimensions of the Littlewood-Richardson components, namely any irreducible representation appearing in the tensor product cannot have a too large relative dimension if the width of both tensored irreducible representations is large enough. This result was known for ‘typical’ irreducible representations (see e.g. [CS09]) but here we show that it holds true uniformly, at the expense of a worse, but asymptotically optimal estimate. Thus the difficulty of our main result lies in its uniformity.

Our estimate relies on a combinatorial lemma proved in Section 4 and it turns out that this lemma admits a direct counterpart in random matrix that has its own interest. We state this as Lemma 3.4.

1.7. Organization of the paper. In Section 2 we recall some notations and facts from representation theory. In Section 3 we give an auxiliary result: a convenient description of the probability distribution of the first coordinate $\mu_1$ of a random representation $\mu = (\mu_1, \mu_2, \ldots)$ distributed according to the Littlewood-Richardson measure $P_{\lambda, \mu}$. Following this description, Section 4 gathers the properties of this probability distribution which are necessary in order to prove our main theorem. Section 5 contains the proof of the main theorem, and in Section 6 we explain the sense in which our result is optimal.

2. Representation theory of classical groups

2.1. Representations of linear groups $GL(n)$ and weights. In this article $n \geq 1$ is a fixed integer. We say that $\lambda$ is a weight if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is such that $\lambda_1 \geq \cdots \geq \lambda_n$. We denote by $\hat{GL}(n)$ the collection of irreducible representations of the linear group $GL(n)$, up to equivalence.

There is a canonical bijective correspondence between the set $\hat{GL}(n)$ of (equivalence classes of) irreducible representations and the set of weights which to a representation associates its highest weight. In order to simplify the notation we will identify an irreducible representation of $GL(n)$ with the corresponding weight. We refer to [Ful97] for an extensive treatment of the subject. Throughout the whole paper, we work with the field of complex numbers $\mathbb{C}$. In particular, $GL(n)$ means the linear group $GL(n, \mathbb{C})$ and $SL(n)$ means the special linear group $SL(n, \mathbb{C})$.

2.2. Kronecker tensor product. If $\rho_1 : GL(n) \to \text{End} V_1$ and $\rho_2 : GL(n) \to \text{End} V_2$ are representations of the same group $GL(n)$, we denote by $\rho_1 \otimes \rho_2$ :
GL(n) → End(V_1 ⊗ V_2) their Kronecker tensor product given by the diagonal action on simple tensors:

\[(\rho_1 \otimes \rho_2)(g)(v \otimes w) := \rho_1(g)(v) \otimes \rho_2(g)(w)\]

for \(g \in GL(n), v \in V_1, w \in V_2\).

2.3. Representations of SL(n). Here we describe briefly the irreducible representations of the special linear group SL(n) of matrices of determinant one, and their relation with the irreducible representations of GL(n). It is known, cf [FH91, Section 15.5], that any irreducible representation of GL(n), when restricted to SL(n), yields again an irreducible representation. Besides, this map is surjective and its quotient can be precisely described as follows: two representations \(\lambda, \mu\) of GL(n) yield the same representation when restricted to SL(n) if and only if there exists an integer \(k\) such that \(\mu + k1 = \lambda\).

Unsurprisingly, the one-dimensional representation given by the determinant is trivial on SL(n) but non-trivial on GL(n). Its highest weight is equal to \(1 = (1, \ldots, 1)\). The highest weight of the trivial representation is equal to \((0, \ldots, 0)\). As we have seen, they restrict to the same representation of SL(n).

Put differently, it is possible to parametrize the irreducible representations of SL(n) as those weights \(\lambda = (\lambda_1, \ldots, \lambda_n)\) for which the last component is equal to zero: \(\lambda_n = 0\).

We are now ready to show Corollary 1.2, assuming that Theorem 1.1 holds true.

Proof that Theorem 1.1 implies Corollary 1.2. Let \(\lambda, \mu\) be (as in Corollary 1.2) representations of SL(n). We view them as weights such that their last components are equal to zero: \(\lambda_n = 0, \mu_n = 0\). These weights give rise to representations of GL(n) which will be denoted by \(\lambda, \mu\).

The tensor product \(\lambda \otimes \mu\) of representations of SL(n) is nothing else but a restriction of the tensor product \(\bar{\lambda} \otimes \bar{\mu}\) of representations of GL(n). Furthermore, the decomposition of \(\bar{\lambda} \otimes \bar{\mu}\) into irreducible components gives rise (by restriction) to a decomposition of \(\lambda \otimes \mu\) into irreducible components. It follows that the initial assumption that \([\nu]\) appears in the decomposition of the tensor product \(\lambda \otimes \mu\) implies that there exists some weight \(\nu'\) such that:

- \(\nu'\) contributes to the decomposition of the tensor product \(\bar{\lambda} \otimes \bar{\mu}\) of representations of GL(n); in other words \(c_{\lambda,\mu}^{\nu'} \geq 1\);
- \(\nu'\) is an irreducible representation of GL(n) which restricted to SL(n) coincides with representation \(\nu\).
We apply Theorem 1.1 for $\tilde{\lambda}, \tilde{\mu}, \nu'$; Equation (2) takes the form
\[
\frac{c_{\lambda, \mu}^{\nu'}}{d_\lambda d_\mu} = \frac{c_{\tilde{\lambda}, \tilde{\mu}}^{\nu'}}{d_{\tilde{\lambda}} d_{\tilde{\mu}}} \leq C_n \left( \frac{1}{\lambda_1} + \frac{1}{\mu_1} \right).
\]
Taking into account $c_{\lambda, \mu}^{\nu'} \geq 1$ this implies Equation (3) and finishes the proof of Corollary 1.2. \[\square\]

3. Littlewood-Richardson measure and Gelfand-Tsetlin patterns

In Section 3.1 we recall the definition of Gelfand-Tsetlin patterns. As we shall see, patterns provide a concrete model for Littlewood-Richardson measure (Lemma 3.3 (a)). For the purposes of the current paper we do not need this kind of result in full generality; for this reason in Section 3.2 we will state Lemma 3.1 which concerns the simplified setup: the first coordinate of a random weight distributed according to Littlewood-Richardson measure. This lemma is the key element of the proof of Lemma 4.1 in Section 4, which will be used in the proof of Theorem 1.1 (the main theorem). The remaining part of this section is devoted to the proof of Lemma 3.1.

3.1. Gelfand-Tsetlin patterns. Let $\lambda$ be a weight. We say that
\[
A = (a_l(i))_{l \in \{i, \ldots, n-1\}, \ i \in \{1, \ldots, n-1\}} \in \mathbb{Z}^{n(n-1)/2}
\]
is a Gelfand-Tsetlin pattern of shape $\lambda$ (or, shortly, pattern) if the following system of inequalities is fulfilled:
\[
a_1(1) \leq a_2(1) \leq \cdots \leq a_{n-2}(1) \leq a_{n-1}(1) \leq \lambda_1 \\
\forall i, \forall i, \forall i \\
a_2(2) \leq a_3(2) \leq \cdots \leq a_{n-1}(2) \leq \lambda_2 \\
\forall i, \forall i \\
\vdots \\
\forall i, \forall i \\
a_{n-1}(n-1) \leq \lambda_{n-1} \\
\forall i, \forall i \\
\lambda_n.
\]
(5)

This system of inequalities can be represented by an oriented graph $G$ from Figure[1]

The first row of (5) will deserve special attention, for this reason we will use simplified notation
\[
a_l := a_l(1) \quad \text{for} \ l \in \{1, \ldots, n-1\}.
\]
It will be also convenient to define

\[ a_n := \lambda_1. \]

Analogously, if \( B = (b_l(i)) \) is a pattern of shape \( \mu \) we denote

\[ b_l := b_l(1) \quad \text{for } l \in \{1, \ldots, n-1\} \]

and

\[ b_n := \mu_1. \]

3.2. Concrete realization of Littlewood-Richardson measure. The following proposition is the key component in the proof of Proposition 4.1. It gives a concrete realization of the first coordinate of a random weight distributed according to Littlewood-Richardson measure.

**Proposition 3.1.** Let \( \lambda, \mu \) be weights. Let \( A = (a_l(i)) \) be a random pattern of shape \( \lambda \) (sampled with the uniform distribution) and let \( B = (b_l(i)) \) be
a random pattern of shape $\mu$ (also sampled with the uniform distribution),
we assume that $A$ and $B$ are independent.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be a random weight distributed according to the
Littlewood-Richardson measure $P_{\lambda, \mu}$; then

$$\nu_1 \overset{\text{dist}}{=} \max_{k,l \geq 1, k+l=n+1} a_k + b_l,$$

where $\overset{\text{dist}}{=}$ denotes the equality of distributions of random variables.

We postpone its proof until Section 3.6. The remaining part of the current
section is devoted to preparation to this proof.

3.3. Polynomial representations. Polynomial irreducible representations
of $\text{GL}(n)$ play a special role. Such a polynomial representation corresponds
to a weight $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ are non-negative integers. A weight with this property is called a Young diagram
and can be represented graphically as shown on Figure 2 (we use the Eng-
lish notation for drawing Young diagrams). Polynomial representations are
associated to very rich combinatorial structures related to Young diagrams
and Young tableaux which we will explore in Section 3.4.

Many problems concerning irreducible representations can be reduced to
the special case of irreducible polynomial representations. This is also the
case for Lemma 3.1 the following lemma gives the details of this reduction.

**Lemma 3.2.** Assume that Lemma 3.1 is true under the additional assumption that weights $\lambda, \mu$ are Young diagrams. Then Lemma 3.1 is true in general, without such an assumption.

**Proof.** For $p \in \mathbb{Z}$ we denote by $\text{Det}^p : \text{GL}(n) \to \text{End}(\mathbb{C})$ the one-dimensional representation given by an appropriate power of the determinant:

$$\text{Det}^p(g) := (\det(g))^p \quad \text{for } g \in \text{GL}(n),$$
where the right-hand side should be interpreted as a $1 \times 1$ matrix, thus as an endomorphism of the one-dimensional vector space $\mathbb{C} = \mathbb{C}^1$. Representation $\text{Det}^p$ is irreducible and corresponds to the highest weight $p \mathbf{1} := (p, \ldots, p) \in \mathbb{Z}^n$.

Kronecker tensor product $\lambda \otimes \text{Det}^p$ of an irreducible representation $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\text{Det}^p$ is again an irreducible representation which corresponds to the shifted weight $\lambda + p \mathbf{1} := (\lambda_1 + p, \ldots, \lambda_n + p)$.

The dimensions of irreducible representations, Littlewood-Richardson coefficients and the Littlewood-Richardson measure are invariant under such shifts:

\[
\begin{align*}
d_{\lambda + p \mathbf{1}} &= d_{\lambda}, \\
c_{\lambda + p \mathbf{1}, \mu + q \mathbf{1}}^{\nu + (p + q) \mathbf{1}} &= c_{\lambda, \mu}^{\nu}, \\
P_{\lambda + p \mathbf{1}, \mu + q \mathbf{1}}(\nu + (p + q) \mathbf{1}) &= P_{\lambda, \mu}(\nu),
\end{align*}
\]

for arbitrary $p, q \in \mathbb{Z}$ and irreducible representations $\lambda, \mu, \nu$ of $\text{GL}(n)$.

We use notations of Lemma 3.1. We denote $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ and set $p := -\lambda_n$ and $q := -\mu_n$ so that weights $\lambda' := \lambda + p \mathbf{1}$ and $\mu' := \mu + q \mathbf{1}$ are Young diagrams. We also set $\nu' = (\nu'_1, \ldots, \nu'_n) := (p + q) \mathbf{1} + \nu$. Clearly, since $\nu$ is distributed according to Littlewood-Richardson measure $P_{\lambda, \mu}$ it follows that $\nu'$ is distributed according to Littlewood-Richardson measure $P_{\lambda', \mu'}$.

We define shifted patterns $A' = (a_l(i) + p)$ and $B' = (b_l(i) + q)$. Clearly $A'$ and $B'$ are random patterns of shape $\lambda'$ and $\mu'$ respectively.

We apply Lemma 3.1 to Young diagrams $\lambda'$, $\mu'$, random weight $\nu'$ and random patterns $A'$, $B'$. It follows that

\[
\nu_1 + (p + q) = \nu'_1 \overset{\text{dist}}{=} \max_{k, l \geq 1, \ k + l = n + 1} a'_k + b'_l = \max_{k, l \geq 1, \ k + l = n + 1} (a_k + p) + (b_l + q)
\]

which shows that Lemma 3.1 holds true for weights $\lambda$ and $\mu$ as desired. □

3.4. Young tableaux, Robinson-Schensted-Knuth correspondence and the plactic monoid. We recall some basic notations related to Young tableaux, Robinson-Schensted-Knuth correspondence and the plactic monoid. A good treatment of these topics is given in Part I of the book [Ful97].

3.4.1. Tableaux. A semi-standard tableau (or, shortly, tableau) $T$ is a filling of the boxes a given Young diagram $\lambda$ with letters from the alphabet $\{1, \ldots, n\}$ with the property that the filling should be weakly increasing along each row, and strictly increasing down a column, see Figure 3. The
1 1 1 1 1 2 2 2 3
2 2 2 2 3 3 3
3 3 3

**Figure 3.** Example of a tableau $T$ in the alphabet $\{1, 2, 3\}$ filling the Young diagram $(9, 7, 3)$ from Figure $2$. The boxes were colored in order to improve visibility. The corresponding word is given by $w(T) = (3, 3, 3, 2, 2, 2, 2, 3, 3, 3, 1, 1, 1, 1, 2, 2, 3)$.

The value of $n$ will be fixed so we do not have to specify it for each tableau separately. We also say that Young diagram $\lambda$ is the shape of tableau $T$.

For a given tableau $T$ we set $a_l(i)$ to be the number of boxes in the $i$th row of $T$ filled with numbers $\leq l$. It is easy to check that so defined $A = (a_l(i))$ is a pattern; furthermore for any Young diagram $\lambda$ this gives a bijective correspondence between tableaux of shape $\lambda$ and patterns of shape $\lambda$. In the following we will identify a tableau with the corresponding pattern.

3.4.2. **Words.** A word $w = (w_1, \ldots, w_\ell)$ is a sequence of the elements of the alphabet $\{1, \ldots, n\}$. We recall that the insertion tableau $P(w)$ of $w$ is defined as the semi-standard tableau obtained by Schensted row insertion algorithm applied iteratively to the letters $w_1, \ldots, w_\ell$. For a given tableau $T$ we denote by $w(T)$ the word obtained by reading the entries of $T$ along the lines, from left to right and from the bottom line to the top one, see Figure $3$. This word has a property that $T = P(w(T))$.

For a word $w = (w_1, \ldots, w_\ell)$ we denote by $\text{LI}(w)$ the length of the longest (weakly) increasing subsequence of $w$, i.e. the length of the longest sequence $i_1 < \cdots < i_k \in \{1, \ldots, \ell\}$ such that

$$w_{i_1} \leq \cdots \leq w_{i_k}.$$  

It is well-known that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the shape of the insertion tableau $P(w)$ then $\text{LI}(w) = \lambda_1$ is equal to the length of the first row of $\lambda$.

3.4.3. **Multiplication of tableaux, plactic monoid and plactic Littlewood-Richardson rule.** We consider the free monoid in alphabet $\{1, \ldots, n\}$, which is just the set of words equipped with a multiplication $\cdot$ given by concatenation of words. Let us identify two words $w$ and $w'$ (we denote it $w \equiv w'$) if and only if the corresponding insertion tableaux are equal: $P(w) = P(w')$. One can show that $w \equiv w'$ and $v \equiv v'$ implies that $w \cdot v \equiv w' \cdot v'$ thus multiplication $\cdot$ is well defined on the equivalence classes of $\equiv$. The set
of such equivalence classes of $\equiv$ equipped with multiplication $\cdot$ is called plactic monoid.

Map $P$ gives a bijection between the elements of the plactic monoid and tableaux; thus the multiplication in the plactic monoid can be used to define multiplication of tableaux which will be denoted by the same symbol $\cdot$. Alternatively, the product $S \cdot T := P(w(S) \cdot w(T))$ of tableaux $S$ and $T$ is defined as the insertion tableau corresponding to the concatenation of the words corresponding to the original tableaux.

Recall that the plactic Schur polynomial is defined as a formal sum

$$S_\lambda := \sum_T T$$

of all tableaux with shape $\lambda$. Plactic Littlewood-Richardson rule says that

$$(7) \hspace{0.5cm} S_\lambda \cdot S_\mu = \sum_\nu c^\nu_{\lambda,\mu} S_\nu,$$

where $c^\nu_{\lambda,\mu}$ are the usual Littlewood-Richardson coefficients.

3.4.4. Involution on tableaux. Let us consider an antiautomorphism $\alpha$ of the free monoid defined on the generators by $\alpha(i) := n+1-i$. Alternatively, $\alpha$ is an involution on words defined by reading the word backwards and by reversing the order in the alphabet. Plactic monoid can be equivalently described as the free monoid divided by plactic relations (Knuth relations) which are fulfilled by generators $x, y, z \in \{1, \ldots, n\}$:

$$y \cdot z \cdot x = y \cdot x \cdot z \hspace{0.5cm} \text{if } x < y \leq z,$$

$$x \cdot z \cdot y = z \cdot x \cdot y \hspace{0.5cm} \text{if } x \leq y < z.$$

Since $\alpha$ preserves these plactic relations, $\alpha$ gives rise to an antiautomorphism of the plactic monoid.

If we identify the elements of the plactic monoid with tableaux, the antiautomorphism $\alpha$ becomes an involution on the set of tableaux. It can be described explicitly as follows: for a given tableau $T$ we replace each entry $i$ by $\alpha(i) = n+1-i$ and we rotate the tableau by angle $\pi$, thus obtaining a skew tableau, see Figure 4. After rectifying it (by an application of Schützerberger’s jeu de taquin), we obtain $\alpha(T)$. Alternatively, $\alpha(T) = P(\alpha(w(T)))$. Greene’s theorem shows that involution $\alpha$ maps the set of tableaux of a given shape into itself.

3.5. Concrete model for Littlewood-Richardson measure. The following lemma is a simple reformulation of well-known combinatorics of the representation theory in the language of probability theory.

The following is the key ingredient for the proof of Proposition 3.1.
Lemma 3.3. Let $\lambda$, $\mu$ be Young diagrams. Let $S$ be a random Young tableau of shape $\lambda$ and let $T$ be a random Young tableau of shape $\mu$. We assume that $S$ and $T$ are sampled according to the uniform distribution given by their respective shape constraints, and that they are independent. Then,

(a) the distribution of the shape of the product $S \cdot T$ coincides with the Littlewood-Richardson measure $P_{\lambda,\mu}$;

(b) let $\nu = (\nu_1, \ldots, \nu_n)$ be a random Young diagram distributed according to the Littlewood-Richardson measure $P_{\lambda,\mu}$, then

$$\nu_1 \overset{\text{dist}}{=} \max_{k,l \geq 1, k+l = n+1} a_k(S) + a_l(T),$$

where $\overset{\text{dist}}{=}$ denotes the equality of distributions of random variables.

Proof. We will identify a probability measure on the set of tableaux with the appropriate formal linear combination of tableaux with coefficients given by appropriate probabilities. The dimension $d_\lambda$ is equal to the number of tableaux of the shape $\lambda$, therefore the normalized plactic Schur polynomial

$$\frac{1}{d_\lambda} S_\lambda$$

can be identified to the uniform probability measure on the set of tableaux of shape $\lambda$.

The plactic Littlewood-Richardson rule (7) can be equivalently written in the form

$$\left( \frac{1}{d_\lambda} S_\lambda \right) \cdot \left( \frac{1}{d_\mu} S_\mu \right) = \sum_\nu \left( \frac{d_\nu c_{\lambda,\mu}}{d_\lambda d_\mu} \right) \left( \frac{1}{d_\nu} S_\nu \right).$$

The left-hand side corresponds to the distribution of the random tableau $S \cdot T$. The right-hand side corresponds to the distribution of the random
tableau filling a random Young diagram with the distribution $P_{\lambda,\mu}$. By comparing the distribution of the shape of the Young tableaux contributing to both sides of the equality we finish the proof of the first part of the lemma.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be the shape of the tableau $S \cdot T$; from the first part of this lemma it follows that the distribution of $\nu$ is given by the Littlewood-Richardson measure $P_{\lambda,\mu}$. Clearly, the length of the first row of $\nu$ fulfills

$$\nu_1 = \text{LI} \left( w(S) \cdot w(T) \right) ;$$

it follows that

$$\nu_1 = \max_k \left[ \text{LI} \left( w(S) \mid_{\{1,\ldots,k\}} \right) + \text{LI} \left( w(T) \mid_{\{k,\ldots,n\}} \right) \right] ,$$

where $w \mid_A$ denotes the word $w$ with all letters which do not belong to $A$ omitted. In the following we will analyze the two summands contributing to the right-hand side of (8). We start with the first one.

We consider the tableau $S \mid_{\{1,\ldots,k\}}$ obtained by removing from $S$ all boxes with entries bigger than $k$. Clearly,

$$w(S) \mid_{\{1,\ldots,k\}} = w \left( S \mid_{\{1,\ldots,k\}} \right) .$$

In particular,

$$\text{LI} \left( w(S) \mid_{\{1,\ldots,k\}} \right) = a_k(S)$$

is the length of the first row of tableau $S \mid_{\{1,\ldots,k\}}$.

We turn now to the second summand on the right-hand side of (8). Clearly, for any word $w$

$$w \mid_{\{k,\ldots,n\}} = \alpha \left( \alpha(w) \mid_{\{1,\ldots,n+1-k\}} \right)$$

and

$$\text{LI} w = \text{LI} \alpha(w)$$

thus

$$\text{LI} \left( w \mid_{\{k,\ldots,n\}} \right) = \text{LI} \left( \alpha(w) \mid_{\{1,\ldots,n+1-k\}} \right) .$$

We define $T' = \alpha(T)$; thus random tableaux $T'$ and $T$ have the same distribution. We have

$$\text{LI} \left( w(T) \mid_{\{k,\ldots,n\}} \right) = \text{LI} \left( w(T') \mid_{\{1,\ldots,n+1-k\}} \right) = a_{n+1-k}(T') .$$

Equations (8), (9), (10) finish the proof. □

3.6. Proof of Proposition 3.1.

Proof of Proposition 3.1. In Lemma 3.2 we showed that it is enough to prove the result under additional assumption that $\lambda$ and $\mu$ are Young diagrams. We use part (b) of Lemma 3.3 and use the fact that there is a bijective correspondence between tableaux and patterns. □
3.7. **An application to random matrix theory.** In what follows, we state an interesting corollary of Proposition 3.1. This corollary is of purely random matrix nature, but to the best of our knowledge it seems to be new.

**Corollary 3.4.** Let $A$, $B$ be independent Hermitian random matrices of the same size $n \times n$. Assume that both the distribution of $A$ and the distribution of $B$ is invariant under unitary conjugation. Then the largest eigenvalue of $A + B$ is a random variable which has the same distribution as

$$\max_{k,l \geq 1, k+l=n+1} a_k + b_l,$$

where $a_k$ (resp. $b_k$) is the random variable obtained by taking the largest eigenvalue of the $k \times k$ upper left corner of $A$ (resp. $B$).

We will just sketch the main ideas of the proof and leave the details to the reader.

**Sketch of the proof.** Without loss of generality we can assume that the eigenvalues of $A, B$ are prescribed. Indeed, if they are random, the proof can be completed by conditioning over prescribed eigenvalues and a decomposition of measure type argument.

And if the eigenvalues of $A, B$ are prescribed, the result follows from Proposition 3.1 and successive applications of [CS09]. Indeed, in [CS09], it is proved that if $A$ is a unitarily invariant selfadjoint random matrix and $\lambda^N = (\lambda_1^N \geq \cdots \geq \lambda_n^N)$, is a tuple of sequences of integers such that $\lambda^N_i / N$ converges to the $i$th largest eigenvalue of $A$, then the law of $(a_1, \ldots, a_n)$ is the limit of the laws of $(a_1^N / N, \ldots, a_n^N / N)$ as appearing in Proposition 3.1 and corresponding to weight $\lambda^N$. A similar statement holds for a random matrix $B$ and $\mu^N = (\mu_1^N \geq \cdots \geq \mu_n^N)$. It has been also shown in [CS09] that the law of the largest eigenvalue of $A + B$ is the limit of the laws of $\nu^N_1 / N$, where $\nu^N$ is distributed according to the Littlewood-Richardson measure $P_{\lambda^N, \mu^N}$. We apply Proposition 3.1 to $\lambda^N$, $\mu^N$ and $\nu^N$ and pass to the limit. \[\square\]

4. **The first row of a random pattern**

The main result of this section is the following lemma giving an upper bound on the atoms of the distribution of the first row of a random pattern with a given shape. This proposition is the key in the proof of Theorem 1.1.

**Proposition 4.1.** There exists some constant $D_n$ with the following property. Let $\lambda$ be a weight and let $A = (a_i(i))$ be a random pattern with shape $\lambda$. Then for any $x \in \mathbb{Z}$ and $1 \leq k \leq n - 1$:

$$P(a_k = x) \leq D_n \frac{1}{\lambda_1 - \lambda_{n+1-k}}.$$
We postpone the proof of Proposition 4.1 until Section 4.3; the remaining part of the current section is a preparation for this proof.

4.1. Taking degeneracy into account. Let the weight $\lambda$ be fixed. The inequalities (5) define a convex polyhedron in the space $\mathbb{R}^{\frac{n(n-1)}{2}}$. For some choices of the weight $\lambda$ it might happen that this polyhedron is of dimension smaller then the maximal dimension $\frac{n(n-1)}{2}$. This creates some difficulties; in the following, we explain how to avoid them.

Restricting the system of inequalities (5) to one row and one column implies that

$$a_l(i) \leq \cdots \leq \lambda_i,$$

in other words if $\lambda_{n+i-l} = \lambda_i$ then automatically $a_l(i) = \lambda_i$. Such variables are trivial from our viewpoint, thus it is enough to restrict our attention to the index set

$$\mathcal{I} = \{(l, i) : l \in \{i, \ldots, n-1\}, i \in \{1, \ldots, n-1\}, \lambda_{n+i-l} < \lambda_i\}$$

and to study only variables $(a_l(i) : (l, i) \in \mathcal{I})$. We define $d = |\mathcal{I}|$. The set of solutions to the above system of inequalities (5) in integer numbers (respectively, real numbers) will be denoted by $\mathcal{D} \subset \mathbb{Z}^d$ (respectively, by $\mathcal{C} \subset \mathbb{R}^d$). Thus there is a natural bijective correspondence between patterns of shape $\lambda$ and the elements of $\mathcal{D}$.

We denote by $\tilde{\mathcal{G}}$ the oriented graph $\mathcal{G}$ in which:

- every vertex $A_l(i)$ with $(l, i) \notin \mathcal{I}$ is glued to the vertex $\Lambda_i$,
- all pairs of vertices $\Lambda_i$ and $\Lambda_j$ are glued together if $\lambda_i = \lambda_j$.

The graph $\tilde{\mathcal{G}}$ encodes all inequalities fulfilled by the variables $(a_l(i) : (l, i) \in \mathcal{I})$. The following lemma is elementary.

**Lemma 4.2.** The graph $\tilde{\mathcal{G}}$ is acyclic.

4.2. Continuous versus discrete. Our goal is to understand the uniform measure on $\mathcal{D}$. There is also a simpler object: the uniform measure on $\mathcal{C}$. In the following we investigate how these two measure are related to each other. The following Lemma addresses the question of how intersections of $(b + I)$ with $\mathcal{D}$ and $\mathcal{C}$ are related to each other, where the unit cube $I$ is defined as

$$I = \left\{(a_l(i)) : |a_l(i)| < \frac{1}{2}\right\} \subset \mathbb{R}^d.$$
Lemma 4.3. There is some constant $C > 0$ (which depends only on $n$) with the property that for any weight $\lambda$ and any lattice point $b \in \mathbb{Z}^d$

$$b \in \mathcal{D} \iff (b + I) \cap \mathcal{D} \neq \emptyset \iff \text{vol}[(b + I) \cap \mathcal{C}] \geq C \iff (b + I) \cap \mathcal{C} \neq \emptyset.$$  

Proof. Since the lattice point $b$ is the only element of $(b + I) \cap \mathbb{Z}^d$, if $(b + I) \cap \mathcal{D}$ is non-empty then it is equal to $\{b\}$. This explains why the first two conditions are equivalent.

Now we suppose that $b \in \mathcal{D}$. For $m \in \mathbb{Z}$ we denote

$$\mathcal{I}_m = \{(l, i) \in \mathcal{I} : b_l(i) = m\}$$

and we denote by $\mathcal{C}_m \subset \mathbb{R}^{[\mathcal{I}_m]}$ the set of solutions of the system of inequalities (5) over variables $a_l(i)$ such that $(l, i) \in \mathcal{I}_m$, subject to the additional requirement that

$$|a_l(i) - m| < \frac{1}{2}.$$ 

Since $(b + I) \cap \mathcal{C} = \prod_m \mathcal{C}_m$ (where, in the right hand side of this equality, with the obvious identification of the coordinates, the multiplication denotes the Cartesian product), it is enough to show that if $\mathcal{I}_m \neq \emptyset$, then vol $\mathcal{C}_m$ is bigger than some universal positive constant.

We denote by $\widehat{\mathcal{G}}_m$ the graph $\widehat{\mathcal{G}}$ restricted to the following vertices:

- vertices $A_l(i)$ with $(l, i) \in \mathcal{I}_m$,
- vertices $\Lambda_i$ with $\lambda_i = m$ (in fact, all such vertices from $\mathcal{G}$ are glued together so they correspond to a single vertex in $\widehat{\mathcal{G}}$).

The graph $\widehat{\mathcal{G}}_m$ encodes all inequalities fulfilled by the collection of variables $(a_l(i))$ over $(l, i)$ such that $|a_l(i) - m| < \frac{1}{2}$.

Since $\widehat{\mathcal{G}}_m$ is acyclic, it is possible to extend it to a linearly ordered set. Let us choose any such a linear extension. There are the following two cases:

- the graph $\widehat{\mathcal{G}}_m$ does not contain any vertex $\Lambda_\ell$; then the set of solutions which is compatible with the selected linear order is a simplex with the volume

  $$\frac{1}{|\mathcal{I}_m|!}.$$ 

- the graph $\widehat{\mathcal{G}}_m$ contains a vertex $\Lambda_\ell$; let us say that there are $p$ (respectively, $q$) vertices $A_l(i)$ which are smaller (respectively, bigger) than $\Lambda_\ell$ with $p + q = |\mathcal{I}_m|$; then the set of solutions which is compatible with the selected linear order is a product of two simplexes with the volume

  $$\frac{1}{2^{p+q}p!q!}.$$
Note that the simplex obtained by choosing a linear order has a smaller volume than $C_m$, so that the above cases give us a lower bound. Now this finishes the proof that the first condition implies the third one.

The third condition trivially implies the fourth condition.

Assume that $(b + I) \cap C \neq \emptyset$. Let $a$ be any element of this set. The system of inequalities (5) has a particularly nice form: if $a$ is a solution then also $\text{round}(a)$ is a solution, where $\text{round}$ denotes the (coordinate-wise) rounding of a real number to the closest integer. On the other hand $\text{round}(a) = b$ therefore $b \in D$ which finishes the proof that the fourth condition implies the first condition. 

\[ \square \]

4.3. Proof of Proposition 4.1.

Proof of Proposition 4.1. For $x \in \mathbb{Z}$ (respectively, $x \in \mathbb{R}$) we denote by $D^x \subset \mathbb{Z}^{d-1}$ (respectively, by $C^x \subset \mathbb{R}^{d-1}$) the set of integer (respectively, real) solutions of the system of inequalities (5) over variables $a_l(i)$, $(l, i) \in I$, $(l, i) \neq (k, 1)$, subject to the additional requirement that $a_k(1) = x$.

With respect to the subsets of $\mathbb{R}^{d-1}$ we denote by $\text{vol}_{d-1}$ the usual Lebesgue volume while with respect to the subsets of $\mathbb{Z}^{d-1}$ we denote by $\text{vol}$ the counting measure.

Now we fix $x \in \mathbb{Z}$. Lemma 4.3 implies that

\[ \int_{|x-y|<\frac{1}{2}} \text{vol}_{d-1} C^y \, dy = \sum_{b \in D^x} \text{vol}_{d} [(b + I) \cap C] \geq C \cdot \text{vol} D^x. \]

It follows that there exists some $y$ such that

\[ \text{vol}_{d-1} C^y \geq C \cdot \text{vol} D^x. \]

It is a simple exercise to check that for $x_0 \in \{\lambda_1, \lambda_{n+1-k}\}$ the set $C^{x_0}$ is nonempty. Let us select the value of $x_0$ for which

\[ |x_0 - y| \geq \frac{\lambda_1 - \lambda_{n+1-k}}{2} \]

and let us fix some $a \in C^{x_0}$.

Under the obvious identifications $a \in C^{x_0} \subset C \subset \mathbb{R}^d$ and $C^y \subset C \subset \mathbb{R}^d$ we can consider the convex cone having $a$ as the vertex and $C^y$ as the base. Clearly, $C$ as a convex set contains this cone. It follows that for $t = (1 - \alpha)x_0 + \alpha y$, with $0 < \alpha < 1$ we have

\[ \text{vol}_{d-1} C^t \geq \alpha^{d-1} \text{vol}_{d-1} C^y \]

hence

\[ \text{vol}_d C = \int_z \text{vol}_{d-1} C^z \, dz \geq \frac{\lambda_1 - \lambda_{n+1-k}}{2d} \text{vol}_{d-1} C^y. \]
Lemma 4.3 shows that

\[(13) \quad \vol D \geq \vol_d C.\]

Inequalities (11), (12), (13) imply that

\[
P(a_k(S) = x) = \frac{\vol D^x}{\vol D} \leq \frac{\text{Const}}{\lambda_1 - \lambda_{n+1-k}}.
\]

\[\Box\]

5. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. For a weight \(\lambda = (\lambda_1, \ldots, \lambda_n)\) we denote by \(\overline{\lambda} = (-\lambda_n, \ldots, -\lambda_1)\) the weight corresponding to the contragredient representation. Since \(d_\lambda = d_{\overline{\lambda}}\) and \(c^{\nu}_{\lambda,\mu} = c^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}\) therefore the inequality (2) holds for \(\lambda, \mu, \nu\) if and only if it holds for \(\overline{\lambda}, \overline{\mu}, \overline{\nu}\).

Let \(\lambda, \mu\) be fixed. By the pigeon hole principle, there exist \(i, j \in \{1, \ldots, n-1\}\) such that

\[
\lambda_i - \lambda_{i+1} \geq \frac{\lambda_1 - \lambda_n}{n-1},
\]

\[
\mu_j - \mu_{j+1} \geq \frac{\mu_1 - \mu_n}{n-1}.
\]

For \(i' = n - i\) and \(j' = n - j\) we have analogous inequalities

\[
\overline{\lambda}_{i'} - \overline{\lambda}_{i'+1} \geq \frac{\overline{\lambda}_1 - \overline{\lambda}_n}{n-1},
\]

\[
\overline{\mu}_{j'} - \overline{\mu}_{j'+1} \geq \frac{\overline{\mu}_1 - \overline{\mu}_n}{n-1}.
\]

Since \((i+j)+(i'+j') = 2n\), at least one of the following is true: \(i+j \leq n\) or \(i'+j' \leq n\). Therefore, without loss of generality we will assume that \(i+j \leq n\); if this is not the case, simply replace \(\lambda, \mu, \nu\) by \(\overline{\lambda}, \overline{\mu}, \overline{\nu}\).

Let \(A\) and \(B\) be as in Lemma 3.1. Equation (6) implies that

\[
P(\nu_1 = x) \leq \sum_{k,l \geq 1, k+l=n+1} P(a_k + b_l = x)
\]

thus it is enough to find appropriate bounds for the distribution of the sum \(a_k + b_l\) for each choice of \(k\) and \(l\) separately. The latter distribution is a convolution of two probability measures, thus

\[
P(a_k + b_l = x) \leq \min \left( \max_{z} P(a_k = z), \max_{z} P(b_l = z) \right)
\]

and it is enough to show that there is such a bound for \(a_k\) or for \(b_l\). Clearly,

\[
n + 1 - k \geq i + 1 \quad \vee \quad n + 1 - l \geq j + 1
\]
We will investigate these two cases separately.

In the first case, \( \lambda_1 - \lambda_{n+1-k} \geq \lambda_i - \lambda_{i+1} \geq \frac{\lambda_1 - \lambda_n}{n-1} \).

We apply Lemma 4.1 in this way
\[
P(a_k = z) \leq D_n \frac{1}{\lambda_1 - \lambda_{n+1-k}} \leq D_n \frac{n-1}{\lambda_1 - \lambda_n}.
\]

In the second case, \( \mu_1 - \mu_{n+1-l} \geq \mu_j - \mu_{j+1} \geq \frac{\mu_1 - \mu_n}{n-1} \).

We apply Lemma 4.1 for diagram \( \lambda' := \mu \) and \( k' = l \), in this way
\[
P(b_l = z) \leq D_n \frac{1}{\mu_1 - \mu_{n+1-l}} \leq D_n \frac{n-1}{\mu_1 - \mu_n}.
\]

This completes the proof. \( \Box \)

6. SATURATION OF THE BOUND

Here we show that our bound is saturated in some natural sense.

**Proposition 6.1.** For each \( n \), there exist two sequences \( (\lambda_N), (\mu_N) \) of irreducible representations of \( \text{GL}(n) \) (respectively, \( \text{SL}(n) \)) which tend to infinity with the property that the inequality (2) of Theorem 1.1 (respectively, inequality (3) of Corollary 1.2) is saturated up to a multiplicative constant that depends only on \( n \) and not on \( N \).

**Proof.** Take \( \lambda = (N, 0, \ldots, 0) \) and \( \mu = (M, 0, \ldots, 0) \). Then it is clear from Littlewood-Richardson rule that all the \( \nu \) for which there is a non-zero probability \( P_{\lambda,\mu} \) are of the form
\[
(A, B, 0, \ldots, 0)
\]
with the constraints that \( A \geq B \geq 0, A + B = N + M, A \geq \max(N, M) \). There are \( \min(N, M) \) choices. By pigeon hole principle, at least one of these weights has a probability at least
\[
\frac{1}{\min(N, M)}
\]
which is comparable to the bound obtained in our Corollary 1.2 and therefore also saturates the bound for the main Theorem 1.1. Note that it follows from the proof that the Littlewood-Richardson coefficients appearing in this proof can not be large. As a matter of fact, one can prove that they are all equal to 1 in this case (but we do not need it in order to complete the proof). \( \Box \)
The above proposition shows that, for example, if we wanted, for a given $N$, the following inequality

$$P_{\lambda, \mu}(\nu) \leq C_n \left( \frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right)^\alpha$$

to be true for all $\mu, \nu$, then necessarily, $\alpha \leq 1$, and actually $\alpha = 1$ is the best possible constant.

Note that if the quantifier of Theorem 1.1 is not on all choices of $\mu, \nu$ but just on some nice (possibly infinite) sets of pairs, then it is possible to obtain much better estimates.

As a first example, if in $\text{GL}(3)$, one takes the collection $\mu_N = \nu_N = (2N, N, 0)$, it is easy to see that the largest dimension of a Littlewood-Richardson factor that can occur in $\mu_n \otimes \nu_n$ is at most of order $N^3$, which is less than $N^6$. However if one in addition allows Littlewood-Richardson coefficients, then one obtains $N^5$. Here we still saturate Theorem 1.1 but not Corollary 1.2 any more.

As a second example, if one takes in $\text{GL}(4)$ the sequence $\mu_N = \nu_N = (3N, 2N, N, 0)$, one can see that the largest dimension of a Littlewood-Richardson summand that can occur in $\mu_n \otimes \nu_n$ is at most of order $N^6$, which is less than $N^{12}$. And if one in addition allows Littlewood-Richardson coefficients, then one obtains $N^9$. Here, we are away from saturation both for Theorem 1.1 and for Corollary 1.2.

**Acknowledgments**

B.C.’s research was supported by an NSERC Discovery grant and an ERA at the University of Ottawa. He wishes to thank the organizers of the EPSRC Symposium Workshop “Interacting particle systems, growth models and random matrices”, as well as Chungbuk National University and RIMS for their hospitality and the opportunity to meet with coworkers and make critical progress on the project. He also thanks Ebrahim Samei for enlightening discussions.

H.H.L.’s research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2005963).

In the initial phase of research, P.Š. was a holder of a fellowship of Alexander von Humboldt-Stiftung. P.Š.’s research has been supported by a grant of Deutsche Forschungsgemeinschaft (SN 101/1-1).

**References**

[CŠ09] Benoît Collins and Piotr Śniady. Representations of Lie groups and random matrices. *Trans. Amer. Math. Soc.*, 361(6):3269–3287, 2009.
[Eym64] Pierre Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[FH91] William Fulton and Joe Harris. *Representation theory*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[GLSS12] Mahya Ghandehari, Hun Hee Lee, Ebrahim Samei, and Nico Spronk. Some Beurling-Fourier algebras are operator algebras. Preprint arxiv:1208.4835v1, 2012.

[LS12] Hun Hee Lee and Ebrahim Samei. Beurling-Fourier algebras, operator amenability and Arens regularity. *J. Funct. Anal.*, 262:167–209, 2012.

[LST12] Jean Ludwig, Nico Spronk, and Lyudmila Turowska. Beurling-Fourier algebras of compact groups. *J. Funct. Anal.*, 262:463–499, 2012.

Département de Mathématique et Statistique, Université d’Ottawa, 585 King Edward, Ottawa, ON, K1N6N5 Canada
CNRS, Institut Camille Jordan Université Lyon 1, 43 Bd du 11 Novembre 1918, 69622 Villeurbanne, France

E-mail address: collins@uottawa.ca

Department of Mathematics, Chungbuk National University, 410 Sungbong-Ro, Heungduk-Gu, Cheongju 361-763, Korea

E-mail address: hhlee@chungbuk.ac.kr

Zentrum Mathematik, M5, Technische Universität München, Boltzmannstrasse 3, 85748 Garching, Germany

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

Institute of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland

E-mail address: piotr.sniady@tum.de, piotr.sniady@math.uni.wroc.pl