A PERTURBATION OF SPACETIME LAPLACIAN EQUATION

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Abstract. We study a perturbation
\[ \Delta u + P|\nabla u| = h|\nabla u|, \]
of spacetime Laplacian equation in an initial data set \((M, g, p)\) where \(P\) is the
trace of the symmetric 2-tensor \(p\) and \(h\) is a smooth function.

Stern [Ste19] introduced a level set method of harmonic maps and he used it to simplify
proofs of some important results in scalar curvature geometry of three manifolds. See [BS19]
for a Neumann boundary version. Later developments include simplified proofs of positive
mass theorem [BKKS19] [HKK20], Gromov’s dihedral rigidity conjecture for three dimension
cubes and mapping torus of hyperbolic 3-manifolds [CK21], and hyperbolic positive mass
theorem [BHK+21].

The first approach to positive mass theorems used stable minimal surface [SY79],
Jang equation [SY81] and stable marginally outer trapped surfaces [EHLS16]. The
hyperbolic positive mass theorem could be established via a study of constant mean
curvature 2 surface in asymptotically hyperbolic manifolds under extra assumption
on the mass aspect function (see [ACG08]). See also a Jang equation proof [Sak21].

Gromov [Gro18, Gro21] further generalized minimal surface approach to stable
\(\mu\)-bubbles. The merit of a stable \(\mu\)-bubble is possibility of weaker scalar curvature
condition. We take a simplest example: if a three manifold \((M, g)\) admits a function
\(h \in C^\infty(M)\) such that the scalar curvature \(R_g\) satisfies
\[ R_g + h^2 - 2|\nabla^g h| \geq 0, \]
then the topology of a stable \(\mu\)-bubble of prescribed mean curvature \(h\) in \((M^3, g)\)
can be still classified using the stability condition and [FSS0].

We generalize the (spacetime) harmonic function approach of [BKKS19] [HKK20]
to deal the weaker condition similar to (0.1). An initial data set \((M, g, p)\) is a
Riemannian manifold \((M, g)\) equipped with an extra symmetric 2-form \(p\). We write
\(P = \text{tr } p\), the spacetime Hessian [HKK20] is defined to be
\[ \bar{\nabla}^2 u = \nabla^2 u + p|\nabla u|, u \in C^2(M). \]

Given a smooth function \(h\) on \(M\), we study the solution to the equation
\[ \Delta u + P|\nabla u| = h|\nabla u|. \]
The equation is a perturbation to the spacetime harmonic function, and the solution
\(u\) is spacetime harmonic if \(h = 0\). We follow [HKK20, Proposition 3.2] in detail and
establish the analogous proposition.

**Proposition 0.1.** Let \((U, g, p)\) be a 3-dimensional oriented compact initial data
set with smooth boundary \(\partial U\), having outward unit normal \(\eta\). Let \(u : U \to \mathbb{R}\) be a
spacetime harmonic function, and denote the open subset of \(\partial U\) on which \(|\nabla u| \neq 0\)
by $\partial_{x_0} U$. If $\bar{u}$ and $\underline{u}$ denote the maximum and minimum values of $u$ and $\Sigma_t$ are $t$-level sets, then
\[
\int_{\partial_{x_0} U} (\partial_t |\nabla u| + p(\nabla u, \eta))dA
\]
\[(0.4) \geq \int_{\underline{u}} \int_{\Sigma_t} \left( \frac{1}{2} \frac{\nabla^2 u}{|\nabla u|^2} + (\mu + J(\nu) + h^2 - 2hP + 2\langle \nu, \nabla h \rangle) - K \right) dAdt,
\]
where $\nu = \frac{\nabla u}{|\nabla u|}$, $K$ is Gauss curvature of the level set and $dA$ is the area element.

**Proof.** Recall the Bochner formula
\[
\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u \rangle.
\]
For $\delta > 0$ set $\phi_\delta = (|\nabla u|^2 + \delta)^{\frac{1}{2}}$, and use the Bochner formula to find
\[
\Delta \phi_\delta = \frac{\Delta |\nabla u|^2}{2\phi_\delta} - \frac{|\nabla |\nabla u|^2|^2}{4\phi_\delta^2}
\]
\[(0.5) \geq \frac{1}{\phi_\delta} (|\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 + Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u \rangle).
\]
On a regular level set $\Sigma$, the unit normal is $\nu = \frac{\nabla u}{|\nabla u|}$ and the second fundamental form is given by $A_{ij} = \frac{\nabla \nabla u}{|\nabla u|}$, where $\partial_i$ and $\partial_j$ are tangent to $\Sigma$. We then have
\[
|A|^2 = |\nabla u|^{-2}(|\nabla^2 u|^2 - 2|\nabla |\nabla u|^2 + |\nabla^2 u(\nu, \nu)|^2),
\]
and the mean curvature satisfies
\[(0.6) |\nabla u| H = \Delta u - \nabla^2 u.
\]
Furthermore by taking two traces of the Gauss equations
\[
2Ric(\nu, \nu) = R_g - R_{\Sigma} - |A|^2 + H^2,
\]
where $R_g$ is the scalar curvature of $U$ and $R_{\Sigma}$ is the scalar curvature of the level set $\Sigma$. Combining these formulas with (0.5) produces
\[
\Delta \phi_\delta \geq \frac{1}{\phi_\delta} \left( |\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 + \langle \nabla u, \nabla \Delta u \rangle + \frac{|\nabla u|^2}{2} (R_g - R_{\Sigma} + H^2 - |A|^2) \right)
\]
\[= \frac{1}{2\phi_\delta} \left( |\nabla^2 u|^2 + (R_g - R_{\Sigma})|\nabla u|^2 + 2\langle \nabla u, \nabla (-P|\nabla u| + h|\nabla u|) \rangle + (\Delta u)^2 - 2(\Delta u)|\nabla^2 u|^2 \right).\]
Let us now replace the Hessian with the spacetime Hessian via the relation (0.2), and utilize (0.3) to find
\[
\Delta \phi_\delta \geq \frac{1}{2\phi_\delta} \left( |\nabla^2 u - p|\nabla u|^2 + (R_g - R_{\Sigma})|\nabla u|^2 + 2\langle \nabla u, \nabla (-P|\nabla u| + h|\nabla u|) \rangle + (\Delta u)^2 - 2(\Delta u)|\nabla^2 u|^2 \right)
\]
\[+ (-P|\nabla u| + h|\nabla u|)^2 - 2(-P|\nabla u| + h|\nabla u|)|\nabla^2 u|^2 \right).\]
Noting that
\[
\langle \nabla u, \nabla |\nabla u| \rangle = \frac{1}{2} \langle \nu, \nabla |\nabla u|^2 \rangle = u^i \nabla_{i\nu} u = |\nabla u| \nabla^2 u,
\]
we have
\[
\Delta \phi_\delta \geq \frac{1}{2\phi_\delta} \left( |\nabla^2 u|^2 - p|\nabla u|^2 + (R_g - R_{\Sigma})|\nabla u|^2 + 2\langle \nabla u, \nabla (-P|\nabla u| + h|\nabla u|) \rangle + (\Delta u)^2 - 2(\Delta u)|\nabla^2 u|^2 \right)
\]
\[+ (-P|\nabla u| + h|\nabla u|)^2 - 2(-P|\nabla u| + h|\nabla u|)|\nabla^2 u|^2 \right).\]
and expanding $|\nabla^2 u - p |\nabla u|^2$, we have that
\[
\Delta \phi_\delta \geq \frac{1}{2\phi_\delta} \left( |\nabla^2 u|^2 - 2\langle p, \nabla^2 u \rangle |\nabla u| - |p|_g^2 |\nabla u|^2 + (R_g - R_\Sigma)|\nabla u|^2 + 2|\nabla u|\langle \nabla u, -\nabla P + \nabla h \rangle + (-P|\nabla u| + h|\nabla u|)^2 \right) .
\]
Regrouping using the energy density $2\mu = R_g + P^2 - |p|_g^2$,
\[
\Delta \phi_\delta \geq \frac{1}{2\phi_\delta} \left( |\nabla^2 u|^2 - 2\langle p, \nabla^2 u \rangle |\nabla u| + (2\mu - R_\Sigma)|\nabla u|^2 - 2|\nabla u|\langle \nabla u, \nabla P \rangle + |\nabla u|^2(h^2 - 2hP + 2\langle \nu, \nabla h \rangle) \right) .
\]
(0.7)

Consider an open set $\mathcal{A} \subset [\bar{u}, \bar{u}]$ containing the critical values of $u$, and let $\mathcal{B} \subset [\bar{u}, \bar{u}]$ denote the complementary closed set. Then integrate by parts to obtain
\[
\int_{\partial U} \langle \nabla \phi_\delta, v \rangle = \int_U \Delta \phi_\delta = \int_{u^{-1}(\mathcal{A})} \Delta \phi_\delta + \int_{u^{-1}(\mathcal{B})} \Delta \phi_\delta .
\]
According to Kato type inequality [HKK20 Lemma 3.1] and (0.5) there is a positive constant $C_0$, depending only on $\text{Ric}_g$, $P - h$ and its first derivatives such that
\[
\Delta \phi_\delta \geq -C_0|\nabla u| .
\]

An application of the coarea formula to $u : u^{-1}(\mathcal{A}) \to \mathcal{A}$ then produces
\[
(0.8) \quad -\int_{u^{-1}(\mathcal{A})} \Delta \phi_\xi \leq C_0 \int_{u^{-1}(\mathcal{A})} |\nabla u| = C_0 \int_{t \in \mathcal{A}} |\Sigma_t|dt ,
\]
where $|\Sigma_t|$ is the 2-dimensional Hausdorff measure of the $t$-level set $\Sigma_t$. Next, apply the coarea formula to $u : u^{-1}(\mathcal{B}) \to \mathcal{B}$ together with (0.7) to obtain
\[
\int_{u^{-1}(\mathcal{B})} \Delta \phi_\delta \\
\geq \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{|\nabla u|}{|\nabla u|^2} \left[ |\nabla^2 u|^2 + 2\mu - R_\Sigma - \frac{2}{|\nabla u|} \langle p, \nabla^2 u \rangle + \frac{2}{|\nabla u|} \langle \nabla u, \nabla P \rangle \right] dA dt \\
+ \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{1}{\phi_\delta} (h^2 - 2hP + 2\langle \nu, \nabla h \rangle) dA dt .
\]
Combining with (0.8) and
\[
\int_{u^{-1}(\mathcal{B})} \Delta \phi_\delta = \int_U \Delta \phi_\delta - \int_{u^{-1}(\mathcal{A})} \Delta \phi_\delta = \int_{\partial U} \partial_\gamma \phi_\delta - \int_{u^{-1}(\mathcal{A})} \Delta \phi_\delta ,
\]
we obtain
\[
\int_{\partial U} \partial_\gamma \phi_\delta + C_0 \int_{t \in \mathcal{A}} |\Sigma_t|dt \\
\geq \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{|\nabla u|}{|\nabla u|^2} \left[ |\nabla^2 u|^2 + 2\mu - R_\Sigma - \frac{2}{|\nabla u|} \langle p, \nabla^2 u \rangle + \frac{2}{|\nabla u|} \langle \nabla u, \nabla P \rangle \right] dA dt \\
+ \frac{1}{2} \int_{t \in \mathcal{B}} \int_{\Sigma_t} \frac{1}{\phi_\delta} (h^2 - 2hP + 2\langle \nu, \nabla h \rangle) dA dt .
\]
(0.9)
On the set $u^{-1}(B)$, we have that $|\nabla u|$ is uniformly bounded from below. In addition, on $\partial_{\neq 0} U$ it holds that

$$\partial_\eta \phi_\delta = \frac{|\nabla u|}{\phi_\delta} \partial_\eta |\nabla u| \to \partial_\eta |\nabla u| \text{ as } \delta \to 0.$$ 

Therefore, the limit $\delta \to 0$ may be taken in (0.9), resulting in the same bulk expression except that $\phi_\epsilon$ is replaced by $|\nabla u|$, and with the boundary integral taken over the restricted set. Furthermore, by Sard’s theorem (see [HKK20, Remark 3.3]) the measure $|A|$ of $A$ may be taken to be arbitrarily small. Since the map $t \mapsto |\Sigma_t|$ is integrable over $[u, \bar{u}]$ in light of the coarea formula, by then taking $|A| \to 0$ we obtain

Lastly integration by parts gives

$$\int_{U} \langle p, \nabla^2 u \rangle = \int_{U} p^{ij} \nabla_{ij} u = \int_{\partial U} p(\nabla u, \eta) - \int_{U} u^i \nabla^2 p_{ij},$$

and recalling that $J = \text{div}_g (p - Pg)$ and $R_{\Sigma_t} = 2K_{\Sigma_t}$, yields the desired result. □

Now we discuss two special boundary conditions of the equation (0.3) namely Dirichlet and Neumann boundary conditions and its geometric implications on the boundary contribution

$$\int_{\partial_{\neq 0} U} (\partial_\eta |\nabla u| + p(\nabla u, \nu))dA$$

in (0.3). We assume that $u \in C^2(U \cup S)$ where $S$ is a relatively open portion of $\partial U$. This is a valid assumption due to an existence theorem of [HKK20, Section 4].

We have the following:

**Lemma 0.2.** Assume the solution $u$ to (0.3) takes constant values on $S$, then

$$\partial_\eta |\nabla u| + p(\nabla u, \eta) = (-H_S - \text{tr}_S p + h)|\nabla u|.$$ 

**Proof.** Since $u$ is constant on $S$, then by using the decomposition of the Laplacian $\Delta$,

$$-P|\nabla u| + h|\nabla u| = \Delta u = H_S (\nabla u, \eta) + \nabla^2 u(\eta, \eta).$$

The decomposition is already used in (0.9). Since $u$ is constant on $S$, $\nabla u$ either point outward or inward of $U$. We calculate only the case when $\nabla u$ points outward, that is $\eta = \nabla u/|\nabla u|$, $\partial_\eta |\nabla u| + p(\nabla u, \eta)$

$$= \frac{|\nabla u|}{\nabla u} (\nabla^2 u)(\nabla u, \eta) + p(\nabla u, \eta)$$

$$= (\nabla^2 u)(\eta, \eta) + p(\eta, \eta)|\nabla u|$$

$$= -H_S |\nabla u| - P|\nabla u| + p(\eta, \eta)|\nabla u| + h|\nabla u|$$

$$= (-H - \text{tr}_S p + h)|\nabla u|.$$ □

**Lemma 0.3.** Suppose that $\partial_\nu = 0$ on $S$, then

$$\partial_\eta |\nabla u| = \frac{1}{|\nabla u|} B(\nabla u, \nabla u),$$

where $B$ is the second fundamental form of $S$ in $U$. 

Proof. First,
\[
\frac{\partial \eta}{\eta} |\nabla u| = \frac{1}{|\nabla u|} \eta_j \nabla_i u \nabla^j u = \frac{1}{|\nabla u|} [\nabla_i u \nabla^j (\eta_j \nabla^i u) - \nabla^j u \nabla^i u \eta_j] = -\frac{1}{|\nabla u|} B(\nabla u, \nabla u),
\]
we have used the boundary condition \( \partial u / \partial \eta = 0 \). \( \square \)

Note that we have not used \( u \) is a solution to (0.3). On a regular level set \( \Sigma_t \), let \( e_1 \) be a unit tangent vector of \( \partial \Sigma_t \), then \( \{ \eta, \nu = \frac{\nabla u}{|\nabla u|}, e_1 \} \) forms an orthonormal basis. We see that the geodesic curvature of \( \partial \Sigma_t \) in \( \Sigma_t \) is given by
\[
\kappa_{\partial \Sigma_t} = \langle \nabla e_1 \eta, e_1 \rangle
\]
since \( \frac{\partial u}{\partial \eta} = 0 \) implies that \( \eta \) is orthogonal to \( \partial \Sigma_t \) in \( \Sigma_t \) and points outward of \( \Sigma_t \). So
\[
B(\nu, \nu) = \langle \nabla_\nu \eta, \nu \rangle = (\langle \nabla_\nu \eta, \nu \rangle + \langle \nabla e_1 \eta, e_1 \rangle) - \langle \nabla e_1 \eta, e_1 \rangle = H_S - \kappa_{\partial \Sigma_t}.
\]
Therefore, we conclude the following.

**Lemma 0.4.** Assume the solution \( u \) to (0.3) takes constant values on \( S \), at a point in \( S \cap \partial \Sigma_t \) with \( \Sigma_t \) being a regular level set, then
\[
\frac{\partial \eta}{\eta} |\nabla u| + p(\nabla u, \eta) = (-H_S + p(\frac{\nabla u}{|\nabla u|}, \eta) + \kappa_{\partial \Sigma_t}) |\nabla u|.
\]

So to study rigidity questions on initial data sets with boundary, it is natural to assume
\[
\mu + J(\nu) + h^2 - 2hP + 2|\nabla h| \geq 0,
\]
and the convexity condition on the boundary
\[
H_S \geq |\text{tr}_S p - h|,
\]
or
\[
H_S \geq p(\eta, e_0)
\]
where \( e_0 \) is any unit vector on \( S \). With \( h = 0 \), the conditions on the boundary are termed boundary dominant energy conditions by [AdLM19] in their study of initial data sets with a noncompact boundary.

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