VARIATION OF MIXED HODGE STRUCTURES AND PRIMITIVE FORMS

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Abstract. Let \( \mathcal{V} \) be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. In [1] we showed that the opposite filtration \( \Psi = F_\infty^* W \) extends to a filtration \( \Psi \) of \( \mathcal{V} \) by flat sub-bundles, which pairs with the limit Hodge filtration \( \mathcal{F} \) of \( \mathcal{V} \), to define a 'polarized' \( \mathcal{C}'\)-variation of mixed Hodge structure, on a neighborhood of the origin. According to [MS] there is a 1-1 correspondence between these opposite filtrations and choice of primitive elements constituting a basis for Brieskorn lattice or Gauss-Manin vector bundle. Primitive forms provide a simple explanation of polarization in the limit and the complex structure \( \bar{\partial} \) on vanishing cohomology, in case of isolated hyper-surface singularities [1].

Introduction

One of the important subject of study in Hodge theory and also D-modules is the asymptotic behaviour of the variation of (mixed) Hodge structures or the possibility of extending D-module over compactifications of the ambient space. The concepts of Deligne extension, limit MHS, or extending the D-modules over the normal crossing divisor at infinity are examples of this question. The question of asymptotic behaviour of Riemann-hodge bilinear relations and Polarizations is still not answered clearly in Hodge theory.

For a holomorphic germ \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) with isolated singularity at \( 0 \in \mathbb{C}^{n+1} \), Our description of asymptotic fiber as the stack on 0 of the extended Gauss-Manin module identifies the new fiber with module of relative differentials \( \Omega_f = \Omega_{n+1}^+ X / df \wedge \Omega^n X \). A. Varchenko shows that residue pairing defines a self duality on \( \Omega_f \).

The Hodge filtration of the limit fiber is different from what is called limit Hodge filtration introduced by W. Schmid [SCH] in pure case, and G. Pearlstein in mixed case, or their polarizations!. Usually the new Hodge filtration is opposite to the limit Hodge filtration [P2]. Although there would exists an isomorphism exchanging them. One may ask if there could be a pairing polarizing both of them. This fact is true for VHS; However it is false for VMHS [P1], [P2] in general. This is for the period

Key words and phrases. Mixed Hodge structure, Isolated hypersurface singularity, Primitive forms.

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map of VMHS’s or their Gauss-Manin connection may have irregular singularities. In this case the limit Hodge filtration may not exist.

1. LIMIT FIBER

Consider a holomorphic germ \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) with isolated singularity at \( 0 \in \mathbb{C}^{n+1} \). The description of asymptotic fiber as the stack on 0 of the extended Gauss-Manin module identifies the new fiber with module of relative differentials \( \Omega_f = \Omega_{X,0}^{n+1}/df \wedge \Omega_{X,0}^n \).

The Hodge filtration of the limit fiber is different from the concept of limit Hodge filtration introduced by W. Schmid [SCH] and also its polarization. Usually the new Hodge filtration is opposite to the limit Hodge filtration. Although there exists an isomorphism exchanging them. One may ask if there could be a pairing polarizing both of them. This fact is true for pure VHS; However it is false for VMHS [P2] in general. This is for the period map of VMHS’s or their Gauss-Manin connection may have irregular singularities. In this case the limit Hodge filtration may not exist.

I have proved the existence of a real structure on \( \Omega_f \), by using a theorem of A. Kaplan [P1]. The point is the mixed Hodge metric theorem guarantees the existence of polarization in terms of \( I_{p,q} \) decomposition. We use this fact to show that Deligne-Hodge decomposition is split over \( \mathbb{R} \).

2. REVIEW OF NEARBY CYCLE MAP

The description of cohomology in terms of holomorphic differential forms by the de Rham isomorphism leads to the definition of Brieskorn Lattice

\[
H'' = f_* \frac{\Omega_{X,0}^{n+1}}{df \wedge d\Omega_{X,0}^n}
\]

The Brieskorn lattice is the stack at 0 of a locally free \( \mathcal{O}_T \)-module \( H'' \) of rank \( \mu \) with \( \mathcal{H}_T'' \cong \mathcal{H}_r \) and hence \( H'' \subset \mathcal{(i^1)H}_0 \). The regularity of the Gauss-Manin connection proved by Brieskorn and Malgrange implies that \( H'' \subset \mathcal{G}_0 \).

**Theorem 2.1. (Malgrange)**

\[ H'' \subset V^{-1} \]

The Leray residue formula can be used to express the action of \( \partial_t \) in terms of differential forms by \( \partial_t[df \wedge \omega] = [dw] \). In particular, \( sH'' \subset H'' \), and

\[
\frac{H''}{s.H''} \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge d\Omega_{X,0}^n} \cong \frac{\mathbb{C}\{z\}}{(\partial(f))}
\]

Since \( V^{-1} \) is a \( \mathbb{C}\{s\} \)-module the theorem implies that \( H'' \) is a free \( \mathbb{C}\{s\} \)-module of rank \( \mu \) and the action of \( s \) can be expressed in terms of differential forms by \( s[\omega] = [df \wedge \omega] \). The \( \mathbb{C} \)-isomorphism
(2) \[ \psi = \bigoplus_{-1 < \alpha \leq 0} \psi_\alpha : H_C = \bigoplus_{-1 < \alpha \leq 0} H_C^{-2\pi i \alpha} \to \bigoplus_{-1 < \alpha \leq 0} C^\alpha \cong V^{-1}/sV^{-1} \]

Then the monodromy \( M \) on \( H_C \) corresponds to \( \exp(-2\pi i.t \partial_t) \) on \( \bigoplus_{-1 < \alpha \leq 0} C^\alpha \).

The Hodge filtration \( F = (F_k)_{k \in \mathbb{Z}} \) on \( V^{>1} \) defined by J. Scherk and J. Steenbrink is the increasing filtration by the free \( \mathbb{C}\{s\}\)-modules

(3) \[ F_k = F^{n-k} = (s^{-k}H^n) \cap V^{>1} \]
of rank \( \mu \). Let \[ \Omega_f = \Omega^n_{C^{n+1}}/df \wedge \Omega^n_{C^{n+1}} \]
a space which has dimension \( \mu \), the Milnor number of \( f \), and which we identify with

(4) \[ \frac{F^k \mathcal{H}_{X,0}}{\partial_t^{-1}F^k \mathcal{H}_{X,0}}, \quad 0 \leq k < n \]
by the correspondence \([\omega] \to [\omega/(f - t)]\). As an \( \mathcal{O}_{T,0}\)-subquotient module of \( \mathcal{G}_0 \), the latter module inherits the filtration \( V \). It is given explicitly by:

(5) \[ V^\alpha(\frac{F^n \mathcal{H}_{X,0}}{\partial_t^{-1}F^n \mathcal{H}_{X,0}}) = [V^\alpha \cap F^n \mathcal{H}_{X,0} + \partial_t^{-1}F^n \mathcal{H}_{X,0}] \]

3. DESCRIPTION OF ASYMPTOTIC FIBER

Our goal would be to explain the polarization for the MHS on the extended fiber of Deligne extension (above 0) for isolated hypersurface singularities. We will explain later that this MHS is coming from an opposite Hodge filtration (opposite to limit mixed Hodge structure). The point is after compactification and obtaining the Brieskorn lattice, the variable in use to get the stack at 0 is not \( t \). The quotient \( \mathcal{G}/t.\mathcal{G} \) gives \( H^n(X_{\infty}) \) which is not new.

If one defines a map;

(6) \[ s : \Omega^{n+1}_{X,0} \to V^{>1} \]
by sending any form to the image of \( \omega/df \) in \( V^{>1} \), then the sub-module \( df \wedge d\Omega_{X,0}^{n-1} \) goes to 0. In this way we get a copy of the Brieskorn lattice \( H^n \) inside \( V^{>1} \).

The Hodge filtration on \( H^n(X_{\infty}, \mathbb{C}) \) is defined by

(7) \[ F^p H(X_{\infty})_\lambda = \psi_\alpha^{-1} \partial_t^{-p} \text{Gr}_V^{\alpha+n-p} \mathcal{H}(0) \]
Set $\beta = \alpha + n - p$. Because $\mathcal{H}^{(0)} \subset V^{-1}$, $Gr^\beta_V = 0$ for $\beta \leq -1$. Thus $F_p = 0, p \geq n$.

If we sum over the $\beta$ and $q_0$ to the quotient by $\mathcal{H}^{(-1)} = \partial_t^{-1}\mathcal{H}^{(0)}$, we obtain;

$$Gr^\text{•}_V \Omega_f = Gr^\text{•}_F H^n(X_\infty, \mathbb{C})$$

noting that

$$Gr^\beta_V \Omega_f \xrightarrow{t} Gr^\beta_{\beta+1} \Omega_f$$

The Brieskorn lattice $\mathcal{H}_0$ is by definition the cohomology module in degree $n + 1$ of the twisted de Rham complex. It is a free $\mathbb{C}[\partial_t^{-1}]$-module, which is identified with $\Omega_{X,0}/df \wedge d\Omega^n_{X,0}$, and its rank is $\mu = \dim \mathbb{C}\mathcal{H}^{0}/\partial_t^{-1}\mathcal{H}^{0}$, with

$$\mathcal{H}^{0}/\partial_t^{-1}\mathcal{H}^{0} = \Omega_{X,0}/df \wedge \Omega^n_{X,0} = \Theta^{C_{n+1},0}_{\partial f/\partial x_0, \ldots, \partial f/\partial x_n}$$

What was said, shows in order to get the stack at 0 of the Brieskorn or Gauss-manin module, one needs to choose the correct variable giving parametrization around 0. The Gauss-manin connection of isolated hyper-surface singularities is always invertible. This is characteristic property in this case. The identity $\Omega_f = \mathcal{H}^{(0)}/\partial_t^{-1}\mathcal{H}^{(0)}$ is what we interpret as the fiber over 0. Specifically we use $z = \partial_t^{-1}$ as the parametrizing variable at 0.

The deep insight of this fact needs some terminologies from $D$-module theory which is out of the task for this introductory text. However as a sketch; the Kashiwara-Malgrange $V$-filtration allows to give a precise explanation of nearby and vanishing cycle map for perverse sheaves or their corresponding $D$-module. The very definition of nearby map, express the nearby fibers in these sheaves denoted $\psi_t$, and the very definition of vanishing cycle map expresses the stack at 0, denoted $\phi_t$. Specifically $\psi_t \mathcal{H} = Gr^V_{\alpha} G, \alpha \neq 0$ is supported at $t \neq 0$, or the $t$ chart of $\mathbb{P}^1$. The map we are looking for is the other one mean $\phi_t \mathcal{H} = Gr^V_{0} G$. The difference between $\psi_t \mathcal{H}, \phi_t \mathcal{H}$ is some shift in the complexes that they appear as their cohomologies.

**Remark 3.1.** In analogy of the notation, the variable $z^{-1} = \partial_t$ may be considered as one around infinity. By infinity we mean where $z \neq 0$.

**Remark 3.2.** The module $\Omega_f$ is canonically isomorphic to the cohomology of the de Rham complex with compact support. In fact the following diagram is commutative up to a complex constant.

$$H^\text{•}(\Omega^\text{•}_X \cap df) \xrightarrow{\cong} \Omega_f = \Omega^n_{X,0}/df \wedge \Omega^n_X$$

\[\xrightarrow{\partial T_r} \mathbb{C}\]
In the chart around 0 the Gauss-Manin system is defined by
\begin{align}
G := \oplus_{p \in \mathbb{Z}} F^p z^{-p} \subset \mathbb{C}[z, z^{-1}] \otimes \mathcal{H}
\end{align}
it is a locally free $\mathbb{C}[z]$-module equipped with the connection $\nabla = \partial_z$,
\begin{align}
\partial_z (\oplus_p h_p z^{-p}) = - \oplus_p p h_p z^{-p-1}
\end{align}
such that Griffiths transversality holds. $\nabla$ is meromorphic at 0 and is integrable trivially. The restriction $G/z.G$, is naturally identified with graded space $\oplus_p F^p / F^{p+1}$.

The Gauss-Manin connection induces an endomorphism of degree $-1$ of the graded space $\oplus_p F^p / F^{p+1}$;
\begin{align}
\Phi : F^p / F^{p+1} \rightarrow F^{p-1} / F^p \otimes \Omega_X^1
\end{align}
which is the cup product with the Kodaira-Spencer class of the variation, $[SA3]$.

4. Hodge theory and Residue form

For a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated critical point; the local residue $Res_{f,0}(\omega, \eta)$ where, $\omega, \eta$ are differential forms; defines a symmetric bilinear pairing (Grothendieck Pairing= residue form) which is non-degenerate (Proved by Grothendieck). In fact after division by $df$ each of the forms $\omega$ and $\eta$ define a middle dimensional cohomology class of every local level hyper-surface of the function $f$. In this way, the forms $\omega$ and $\eta$ define two sections of the vanishing cohomology bundle. The asymptotic of the polarization form on vanishing cohomology gives a meromorphic function on a neighbourhood of the critical value $0 \in \mathbb{C}$. The residue of this function at $0 \in \mathbb{C}$ is equal to $Res_{0,f}(\omega, \eta)$ $[V]$, $[G]$, $[CIR]$.

**Theorem 4.1.** $[1]$ Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, be a holomorphic germ with isolated singularity at 0. There exists an isomorphism of polarized mixed Hodge structures $\Phi$ compatible with Leray residue $\omega \rightarrow \omega/df$ such that the following diagram is commutative up to complex a complex constant $C$;
\begin{align}
S : H^n(X_\infty) \times H^n(X_\infty) \longrightarrow \mathbb{C} \\
\downarrow (\Phi, \Phi) \\
(-1)^N Res_{f,0} : \Omega(f) \times \Omega(f) \longrightarrow \mathbb{C}
\end{align}
$N$ is a number operator. In other words;
\begin{align}
S(\omega/df, \eta/df) = C. res_{f,0}(\omega, (-1)^N.\eta)
\end{align}

**Theorem 4.2.** $[1]$ Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic isolated singularity germ. Using the Polarization form of the MHS on vanishing cohomology of $f$, the corresponding polarization on $\Omega_f$ is given by a set of forms $\{Res_k\}$ on $g^W_k \Omega_f$ which are induced from Grothendieck residue.
**Theorem 4.3.** Let $\mathcal{V}$ be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set

$$\Psi = F_{\infty}^{\mathcal{V}} \ast W$$

Then $\Psi$ extends to a filtration $\Psi$ of $\mathcal{V}$ by flat sub-bundles, which pairs with the Hodge filtration $F$ of $\mathcal{V}$, to define a polarized $\mathbb{C}$-variation of mixed Hodge structure, on a neighborhood of the origin.

5. **Primitive elements**

To the germ of isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$ one can associate a Frobenius structure, that is a product structure on the tangent space of a manifold given by residue pairing. One step in solving the Poincare-Birkhoff problem for the Gauss-Manin system of $f$ is to glue the different lattices in the Gauss-Manin vector space, to obtain trivial vector bundles over $\mathbb{C}P(1)$.

There is a 1-1 correspondence between such lattices and decreasing filtrations $\bigoplus_{\alpha \in [0,1]} H^\bullet_{\alpha}$ of $H_C$, that are stable by $N$ and are opposite to Steenbrink (limit) mixed hodge structure. Normally the opposite filtration is indexed by rational numbers, in contrast to the usual one that is indexed by non-negative integers.

Then the orthogonality relations for the opposite filtration would be of the form

$$(H^\alpha)^\perp = H^{n-p}_{1-\alpha} \text{ if } \alpha \neq 0$$

$$(H^0)^\perp = H^{n+1-p}_0$$

This implies that the new mixed Hodge structure on $H_C$ is Hodge-Tate and

$$H^\bullet = \sum_q F^q \cap W_{n+q-\bullet}$$

Thus we need to find some explanation of conjugation. Let $\phi_1, ..., \phi_\mu$ be a basis of $H_C$ and $(\alpha, s_\alpha)$ are the spectral pairs of $f$. One may choose the basis in a way that we get the following recursive relation;

$$\phi_{s(\alpha)}(t) = \partial_t^{-k} \prod_{j=0}^{k-1} (t \partial_t - s_\alpha) \phi_{s(1)}$$

In this way we reach to a set of forms $\phi_{s_{\alpha}(1)}$ indexed by spectral numbers which produce other basis elements by applying the operator $t \partial_t - \alpha$ successively. They also describe $Gr^F Gr^V H_C$ concretely. These forms are called primitive elements relative to the nilpotent operator induced by $t \partial_t - \alpha$ on $H_{\alpha}$. They provide information about the Jordan blocks structure in $H_C$. If we call the Jordan Block as $B_k := \langle N^j \omega_{k_0} \mid j = 0, ..., \nu_k \rangle,$

then it holds that;
\[ B_{\alpha,l} = B_{1-\alpha,\nu_k-1}, \quad \alpha \in ]0,1]\]
\[ B_{0,l} = B_{0,\nu_k-1}, \quad \alpha = 0 \]

according to [SA2], page 18.

**Remark 5.1.** The conjugate filtration \( \bar{F}^\bullet \) is not opposite to \( F^\bullet \) in general. It is opposite on the graded pieces of the weight filtration. The method we defined opposite filtration in the previous section gives a canonical way to define opposite filtration to \( F^\bullet \).

**Proposition 5.2.** [SA4] There is a 1-1 correspondence between opposite filtrations on \( H_C \) and free, rank \( \mu \), \( \mathbb{C}[t] \)-submodules \( G' \) on which the connection is logarithmic and \( G, G' \) define a trivial vector bundle on \( \mathbb{P}^1 \).

\( G' \) in theorem is given by;

\[ G' = \mathbb{C}[t]\langle [\omega_0], ..., t^{s_0}[\omega_0], ..., t^{s_r}[\omega_r] \rangle \]

The primitive elements provide good basis of the Brieskorn module that is they also prove the existence of a solution to Poincare-Birkhoff problem. In such a basis the matrix of the operator \( t \) gets the form;

\[ t = A_0 + A_1 \partial_t^{-1} \]

where \( A_0, A_1 \) are square matrices of size \( \mu \) and \( A_1 \) is a diagonal matrix. It holds [MS] that in such a basis the K. Saito Higher residue form [S1] takes the form

\[ K_f = \delta_{\kappa(i)_j} \partial_t^{-n-1}, \]

where \( \delta \) is the Kronecker delta and \( \kappa \) is an involution of the set \( \{1, ..., \mu\} \). According to [HI], a choice of basis for the \( I_{p,q}^0 \) pieces will provide the above special form for residue pairing.

"Thus the primitive elements explain a way to write the polarization in theorem 2.1, as far as the conjugation operator on \( \Omega_f = A_f \). This motivates the relation of primitive elements to a real structure on Jacobi ring of a holomorphic isolated singularity germ \( f \)."

Primitive elements sometimes called primitive forms allow us to define a germ of Frobenius manifold at origin on the moduli of universal un-foldings of \( f \), which is completely determined by the asymptotic behaviour of the associated VMHS, [P2].

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