Existence and Uniqueness of Positive Solutions to Nonlinear Systems of Equations With $\mathcal{M}$-Type Functions

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ABSTRACT
This paper considers the existence and uniqueness of positive solutions to a class of nonlinear systems of equations, which has wide applications in fluid mechanics and electrical circuit theory. In this paper, we define a new class of functions, called $\mathcal{M}$-type function, and then, we show some good properties of $\mathcal{M}$-type function as well as the relationship between this kind of functions and other kinds of functions. In particular, by fully exploiting the properties of $\mathcal{M}$-type function, we present that under a reasonable condition, the nonlinear system of equations with an $\mathcal{M}$-type function, which can be seen as a generalization of the multilinear system with a nonsingular $\mathcal{M}$-tensor, and a positive right-hand side vector has a unique positive solution. Meanwhile, for this class of problems, we give a preliminary and simple algorithm framework. What is more, under an additional assumption, we illustrate the feasibility of using Newton step when improving the original iteration algorithm to obtain better convergence. Numerical experiments show the stability and efficiency of the proposed algorithms.

INDEX TERMS
Iteration algorithm, $\mathcal{M}$-type function, Newton step, nonlinear system of equations, off-diagonally antitone, unique positive solution.

1. INTRODUCTION
It is well-known that the linear system of equations has been extensively studied in the literature. By taking advantages of special properties of structured matrices, many good results about the linear system of equations in both theory and algorithm have been obtained (see [25] for example). Furthermore, the nonlinear system of equations, which is given by

$$f(x) = b,$$

has also been widely studied since it was proposed. A series of papers research on the existence of solutions as well as the effective algorithms about this problem (see [2], [4], [18], [19], [23], [24] for more details).

Nowadays, multilinear system with an $\mathcal{M}$-tensor, which can be seen as a subclass of the nonlinear systems of equations, has received wide attention due to its diversiform applications, such as numerical partial differential equations and data mining (see [6], [13], [26], [27] for more details). More and more papers consider about the solvability of such a system and obtain many important conclusions. For example, for the multilinear system with a nonsingular $\mathcal{M}$-tensor, Ding and Wei [6] show that a positive right-hand side vector leads to the truth that this system has a unique positive solution; Gowda et al. [8] prove that if the right-hand side vector is relaxed to nonnegative, then the corresponding system has a nonnegative solution, which may not be unique. As for algorithms, a diverse range of literatures have proposed efficient methods to solve this kind of problems (see [9], [10], [12], [15], [26]).

Inspired by the works above, in this paper, we mainly consider a class of nonlinear systems of equations, which has a close connection with nonlinear network problems [20]. We define a new class of functions, called $\mathcal{M}$-type functions, which contains positive homogeneous polynomials generated by nonsingular $\mathcal{M}$-tensors as its subclass. Then, we show good properties of this kind of functions and present that $\mathcal{M}$-type function and $\mathcal{M}$-function are two different definitions by giving two examples. In particular, we take into account the nonlinear system of equations with an $\mathcal{M}$-type function and a positive right-hand side vector; and give a conclusion about the unique solvability of the positive solution to this
system. What is more, we find that, in the proof of the existence of positive solutions, an idea of algorithm for solving this problem arises spontaneously. Further, by studying the potential relationship between $\mathcal{M}$-type function and nonsingular $M$-matrix, we also discover that it is possible to use Newton step to improve this algorithm to accelerate convergence.

The rest of this paper is divided into the following five parts. In Section II, we briefly review some basic but significant definitions and conclusions, and more than that, we give the definition of $\mathcal{M}$-type function. In Section III, we present the properties of $\mathcal{M}$-type function and the relationship between this class of functions and other classes of functions. In Section IV, we investigate the existence and uniqueness of the positive solution to the nonlinear system of equations with an $\mathcal{M}$-type function and a positive right-hand side vector. And we present the resulting preliminary algorithm. Besides, we show the feasibility of adding a Newton step in the algorithm designed for this system with a reasonable assumption. In Section V, we present the numerical experiments to support the reliability and efficiency of the proposed algorithms. Conclusions are given in Section VI.

II. PRELIMINARIES

In this section, we are going to summarize some definitions and conclusions, and these results will play important roles in the subsequent analysis.

In this paper, let lowercase $x, y, z, \ldots$ denote vectors, capital letters $A, B, C, \ldots$ denote matrices and calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ denote tensors. Let the index set $[n] := \{1, 2, \ldots, n\}$, we denote $\mathbb{R}^n := \{x = (x_1, x_2, \ldots, x_n)^T \mid x_i \in \mathbb{R}, \forall i \in [n]\}$, $\mathbb{R}^+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\mathbb{R}^{+n} := \{x \in \mathbb{R}^n \mid x > 0\}$, where $x \geq 0$ and $x > 0$ represent $x_i \geq 0$ and $x_i > 0$ for all $i \in [n]$, respectively. For a set $S$, let $|S|$ denote the cardinality of $S$. Besides, for a continuously differentiable mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we use $F_i(x)$ and $F^i(x)$ for the $i$-th element of $F(x)$ and the Jacobian of $F$ at $x \in \mathbb{R}^n$, respectively.

First, we recall two definitions about mappings, which are shown below.

**Definition 1 ([5]):** The mapping $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is (1) isonote (antitone) on $D$, if for any $x, y \in D$,

$$x \leq y \Rightarrow F(x) \leq F(y) \quad (F(x) \geq F(y));$$

(2) inverse isonote on $D$, if for any $x, y \in D$,

$$F(x) \leq F(y) \Rightarrow x \leq y.$$

In the following, the relationship between an inverse isonote mapping and an isonote mapping is given.

**Lemma 1 ([23]):** Let the mapping $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then, $F$ is inverse isonote if and only if $F$ is injective and $F^{-1} : F(D) \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ is isonote.

With the above concepts, we can further recall the following definitions.

**Definition 2 ([17]):** The mapping $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is off-diagonally antitone if for any $x \in \mathbb{R}^n$ and any $i, j \in [n]$ with $i \neq j$, the functions $ψ_{ij} : \{t \in \mathbb{R} \mid x + te \in D\} \rightarrow \mathbb{R}$, which is given by

$$ψ_{ij}(t) = F(x + te),$$

are antitone, where $e \in \mathbb{R}^n$ is the unit basis vectors with the $j$-th component one and all others zero.

**Definition 3 ([17]):** If the mapping $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is off-diagonally antitone and inverse isonote, then, $F$ is called an $\mathcal{M}$-function.

Nowadays, tensor has become a popular topic. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ be an m-th order n-dimensional tensor and $x \in \mathbb{R}^n$ be a vector. Then, we know that $\mathcal{A}x^{m-1} \in \mathbb{R}^n$ and the $i$-th element of $\mathcal{A}x^{m-1}$ is given by

$$(\mathcal{A}x^{m-1})_i := \sum_{i_2, \ldots, i_m=1}^n a_{i_2\cdots i_m}x_{i_2}\cdots x_{i_m}, \forall i \in [n].$$

Recall that $\lambda \in \mathbb{R}$ is called an eigenvalue of $\mathcal{A}$ and $x \in \mathbb{R}^n\setminus\{0\}$ is called the corresponding eigenvector of $\mathcal{A}$ if $\mathcal{A}x^{m-1} = \lambda x^{m-1}$, where $x^{m-1} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. What is more, $\rho(A)$ is called the spectral radius of $\mathcal{A}$ if it is the maximum modulus of eigenvalues of $\mathcal{A}$ (see [14], [21] for more details).

Then, with the above definitions we review that a tensor $\mathcal{A} \in T_{m,n}$ is called an $\mathcal{M}$-tensor if $\mathcal{A} := s\mathcal{I} - B$, where $\mathcal{I}$ is the identity tensor, $B \geq 0$ is a nonnegative tensor and $s \geq \rho(B)$; and $\mathcal{A}$ is called a nonsingular $\mathcal{M}$-tensor if it is an $\mathcal{M}$-tensor with $s > \rho(B)$ (interested readers can refer to [22]). About the nonsingular $\mathcal{M}$-tensor, we have the following properties.

**Theorem 1 ([22]):** Let $\mathcal{A} \in T_{m,n}$ be a nonsingular $\mathcal{M}$-tensor. Then, the following statements hold:

1. (1) all of its diagonal entries and off-diagonal entries are positive and non-positive, respectively;
2. (2) there is an $x > 0$ such that $\mathcal{A}x^{m-1} > 0$.

Let tensor $\mathcal{A} \in T_{m,n}$ and vector $b \in \mathbb{R}^n$. Then, the multilinear system, which is also known as tensor equations, is to find an $x \in \mathbb{R}^n$ satisfying $\mathcal{A}x^{m-1} = b$. When $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and $b$ is a positive vector, Ding and Wei [6] give the following result:

**Lemma 2 ([6]):** Let $\mathcal{A} \in T_{m,n}$ be a nonsingular $\mathcal{M}$-tensor. Then, for any $b \in \mathbb{R}^n_{++}$, $\mathcal{A}x^{m-1} = b$ has a unique positive solution. Additionally, if $\mathcal{A}x^{m-1} = b \geq \hat{x} > 0$, then, we have $\hat{x} \geq \hat{x} > 0$.

Thus, we can draw a conclusion below.

**Proposition 1:** Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ be a nonsingular $\mathcal{M}$-tensor. Then, $f(x) = \mathcal{A}x^{m-1}$ is an $\mathcal{M}$-function on the set $D := \{x \in \mathbb{R}^n \mid x > 0, f(x) > 0\}$.

**Proof:** First, we show that the set $D$ is not empty. From item (2) of Theorem 1 we know that, there exists an $x \in \mathbb{R}_{++}^n$ such that $\mathcal{A}x^{m-1} \in \mathbb{R}_{++}^n$. Thus, $D \neq \emptyset$.

Next, on the one hand, since $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, from item (1) of Theorem 1 we know that, all diagonal entries and off-diagonal entries of $\mathcal{A}$ are positive and non-positive, respectively. That is, $a_{i,j} > 0$ for any $i \in [n]$ and the rest
elements of $\mathcal{A}$ are all non-positive. Thus, for any $i \in [n]$, 
\[
(A^m x)^i = \sum_{i_2, \ldots, i_m = 1}^n a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = a_{ii} x_i^{m-1} + \sum_{(i_2, \ldots, i_m) \neq (i, \ldots, i)} a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}.
\]

It is clear that $f$ is off-diagonally antitone on $\mathbb{R}_n^{++}$. On the other hand, from Lemma 2 it is easy to see that $f(x) = Ax^{m-1}$ is inverse isotone on $D$.

According to the definition of $M$-function and by combining the above two aspects together we obtain that, $f(x) = Ax^{m-1}$ is an $M$-function on $D$.

Let the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by
\[
f(x) = \begin{pmatrix}
a_{11} x_1^{\gamma_1} - P_1(x) \\
a_{22} x_2^{\gamma_2} - P_2(x) \\
\vdots \\
a_{nn} x_n^{\gamma_n} - P_n(x)
\end{pmatrix},
\]
where for any $i \in [n]$, $a_i > 0$ and
- $P_i(x) := 0$ and $\gamma_i \geq 1$ may not be an integer; or
- $P_i : \mathbb{R}^n \to \mathbb{R}$ is a polynomial with degree $d_i \geq 1$, and $\gamma_i \geq d_i$ may not be an integer. Besides, the coefficients of all monomials in $P_i$ are positive.

What is more, $\lim_{x_i \to \infty} P_i(x) = 0$, which implies that when $\gamma_i = d_i$, $P_i$ does not include the term $\xi x_i^{d_i}$, where $\xi \in \mathbb{R}_n^{++}$.

**Definition 4:** The mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an $M$-function if it is defined by (1).

Here, we give an example about the $M$-type function.

**Example 1:** Let mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
\[
f(x) = \begin{pmatrix}
2x_2^2 - x_1 x_3 - 4x_3^2 \\
9x_1^2 - x_2^2 \\
2x_1^2 - x_1 x_2 x_3 - 4x_3^2
\end{pmatrix}.
\]

Then, $f$ is an $M$-type function.

By fully exploiting the good properties of $M$-type functions, in this paper, we mainly consider the existence and uniqueness of the positive solution to the nonlinear system of equations given by
\[
f(x) = b,
\]
where the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is an $M$-type function defined by (1) and $b \in \mathbb{R}_n^{++}$ is a positive vector.

### III. $M$-TYPE FUNCTION

In this section, we will discuss the properties of $M$-type function, including the relationship between $M$-type function and other kinds of functions.

First, from the structure of $M$-type function, we have:

**Proposition 2:** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an $M$-type function defined by (1). Then, $f$ is off-diagonally antitone.

**Proposition 3:** Let $A \in T_{m,n}$ be a nonsingular $M$-tensor. Then, $g(x) = Ax^{m-1}$ is an $M$-type function.

Actually, mapping $g$ in Proposition 3 is a positive homogeneous polynomial, and such a class of mappings, which is generated by a nonsingular $M$-tensor, is a proper subclass of $M$-type functions.

From Proposition 2 we know that $M$-type function is off-diagonally antitone. $M$-function is also off-diagonally antitone. Then, what is the relationship between these two classes of functions? In the following, we use two examples to illustrate that these two definitions are different.

**Example 2:** Let mapping $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
\[
f_1(x) = \begin{pmatrix}
2x_1^2 - 9x_1 x_2 - x_3^2 \\
\frac{9x_1^2}{x_2^2} + 2
\end{pmatrix}.
\]

Then, $f_1$ is an $M$-function.

However, $f_1$ given by Example 2 is not an $M$-function. According to the definition of $M$-function, we know that $M$-function is inverse isotone. From Lemma 1, an inverse isotone mapping must be injective. Let $\tilde{x} = (1, 0)^T$ and $\tilde{x} = (-1, 0)^T$, then, $f_1(\tilde{x}) = f_1(\tilde{x}) = (2, 0)^T$. Thus, $f_1$ is not injective, and then, $f_1$ is not an $M$-function.

**Example 3:** Let mapping $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
\[
f_2(x) = \begin{pmatrix}
e^{x_1} + 2 \\
e^{x_2} + 1
\end{pmatrix}.
\]

Then, $f_2$ is an $M$-function.

Obviously, $f_2$ given by Example 3 is off-diagonally antitone and inverse isotone, hence, $f_2$ is an $M$-function. However, from the definition (1) we know that, an $M$-type function cannot contain constants as well as the terms like $e^{x_i}$. Therefore, $f_2$ is not an $M$-type function.

Combining Examples 2 and 3 together, then, $M$-type function and $M$-function are two different concepts.

It is well-known that $M$-function and $M$-matrix (refer to [7], [11], [16] for more information about $M$-matrix) has a close relationship: a matrix $A$ is an $M$-matrix if and only if the induced linear mapping is an $M$-function (see [23] for more details). Besides, from Proposition 3, it is easy to see that for a nonsingular $M$-matrix $B$, the function $g(x) = Bx$ is an $M$-type function. Now, we recall some properties of nonsingular $M$-matrix.

**Proposition 4 ([3]):** Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix. Then, the following statements are equivalent:
- (1) $A$ is a nonsingular $M$-matrix;
- (2) there is an $x > 0$ such that $Ax > 0$;
- (3) $A^{-1}$ exists, which is a nonnegative matrix;
- (4) all of the principal minors of $A$ are positive.

In the end of this section, we present a vital property of $M$-type function.

**Proposition 5:** Suppose that the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is an $M$-type function defined by (1) and $D := \{x \in \mathbb{R}^n \mid x > 0, f(x) > 0\} \neq \emptyset$. Then, there exists a vector $z \in \mathbb{R}_n^{++}$ such that $f(z) \geq b$ for any $b \in \mathbb{R}_n^{++}$.

**Proof:** Let $\hat{x} \in D$, then, we have $\hat{x} > 0$ and $f(\hat{x}) > 0$. If $f(\hat{x}) \geq b$, then $z := \hat{x}$ is the vector satisfying the condition.
Otherwise, there exists at least one index $i \in [n]$ such that $0 < f_i(x) < b_i$. Then, let $\theta := \max_{i \in [n]} \frac{b_i}{f_i(x)} > 1$, and we define

$$
\xi = \arg\max_{i \in [n]} \{ y_i \}, \quad \sigma = \arg\min_{i \in [n]} \{ y_i \}, \quad \text{and} \quad \tau = \frac{\gamma}{\theta}. 
$$

Thus, we have $\gamma \xi \geq y_i \geq d_i \geq 0$ for any $i \in [n]$, $\frac{\gamma}{\theta} \leq 1$ and $\tau > 1$. Let $z := \tau f(\hat{x}) > 0$, then we have

$$
f(z) \geq \left( \frac{\gamma}{\theta} f_1(\hat{x}), \frac{\gamma}{\theta} f_2(\hat{x}), \ldots, \frac{\gamma}{\theta} f_n(\hat{x}) \right)^T 
\geq \tau \frac{\gamma}{\theta} f(\hat{x}) = \theta f(\hat{x}) \geq b. 
$$

Hence, for any $b \in \mathbb{R}_{++}^n$, there exists a vector $z \in \mathbb{R}_{++}^n$ satisfying $f(z) \geq b$. \hfill \Box

\section{IV. THE MAIN RESULT}

\subsection{A. EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS}

Let $b \in \mathbb{R}_{++}^n$ be an arbitrary positive vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ be an $M$-type function defined by (1). In this section, we are going to present the existence and uniqueness of positive solutions to system (2). For notational convenience, we define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$
F(x) = f(x) - b. \quad (3)
$$

Then, finding solutions of (2) is equivalent to finding solutions to $F(x) = 0$.

For an $M$-type function $f : \mathbb{R}^n \to \mathbb{R}^n$ we make the following assumption:

\textit{Assumption 1:} The set $D := \{ x \in \mathbb{R}^n \mid x > 0, f(x) > 0 \}$ is not empty.

Actually, Assumption 1 is reasonable. Let tensor $A \in \mathbb{T}_{m,n}$ be a nonsingular $M$-tensor. We already know that $g(x) = x^m A x^{m-1}$ is an $M$-type function. Besides, item (2) of Theorem 1 shows that there exists an $x \in \mathbb{R}_{++}^n$ satisfying $x^m A x^{m-1} \in \mathbb{R}_{++}^n$, that is, the set $D$ is not empty. In the following, we will always assume that Assumption 1 holds.

First, we give some notations below. Let $x^k \in \mathbb{R}_{++}^n$ satisfying $F(x^k) = f(x^k) - b \in \mathbb{R}_{++}^n \setminus \{0\}$. Suppose we cannot find such a vector. From Proposition 5, we know that there always exists a $z \in \mathbb{R}_{++}^n$ such that $f(z) \geq b$. Then, we have $f(z) = b$, which implies that system $f(x) = b$ has a positive solution.

Then, for $x^k$ we have $x^k \in D$. We define $j \in [n]$ satisfying

$$
F_j(x^k) = \max_{i \in [n]} \{ F_i(x^k) \}, \quad (4)
$$

and $\alpha^k \in \mathbb{R}$ satisfying

$$
\alpha^k = -x_j^k. \quad (5)
$$

Besides, we also define a set of notations as follows:

$$
x_j^k(\lambda) := x_j^k + \lambda \alpha^k, \quad (6)
$$

and

$$
h(\lambda) := F_j(x_1^k, \ldots, x_{j-1}^k, x_j^k + \lambda \alpha^k, x_{j+1}^k, \ldots, x_n^k), \quad (7)
$$

where $j$ is defined by (4).

Then, for $x_j^k(\lambda)$ and $h(\lambda)$ defined by (6) and (7), we have the following conclusion.

\textit{Lemma 3:} For any $x^k \in \mathbb{R}_{++}^n$ satisfying $F(x^k) \in \mathbb{R}_{++}^n \setminus \{0\}$, the following two statements hold:

1. For any $\lambda \in (0, 1)$, we have $x_j^k(\lambda) > 0$;  
2. There is a scalar $\hat{\lambda} \in (0, 1)$ satisfying $h(\lambda) \geq 0$ for any $\lambda \in (0, \hat{\lambda}]$.

\textit{Proof:} (1) Since $x^k \in \mathbb{R}_{++}^n$, we have $x_j^k > 0$. Clearly, from (5) we know that

$$
x_j^k(\lambda) = x_j^k + \lambda \alpha^k = (1 - \lambda)x_j^k + \lambda \alpha^k > 0
$$

holds for any $\lambda \in (0, 1)$. Besides, we have $x_j^k(\lambda) < x_j^k$.

(2) Obviously, $h(0) = F_j(x^k) > 0$ based on the definition of $j$, and then, according to the continuity of the mapping $b$, we know that there exists a scalar $\hat{\lambda} \in (0, 1)$ such that for any $\lambda \in (0, \hat{\lambda})$, we have $h(\lambda) \geq 0$. \hfill \Box

From Lemma 3, we can see that for any $\hat{\epsilon} \in (0, 1)$, there always exists a positive integer $p$ such that $x_j^k + \hat{\epsilon} \alpha^k > 0$ and

$$
F_j(x_1^k, \ldots, x_{j-1}^k, x_j^k + \hat{\epsilon} \alpha^k, x_{j+1}^k, \ldots, x_n^k) \geq 0. 
$$

Then, we can easily obtain the following result:

\textit{Lemma 4:} Let $x^k \in \mathbb{R}_{++}^n$ satisfying $F(x^k) \in \mathbb{R}_{++}^n \setminus \{0\}$. Then, we have

$$
x_j^{k+1} := x_j^k + \hat{\epsilon} \alpha^k > 0 \quad \text{and} \quad x_{[n]\setminus[j]}^{k+1} := x_{[n]\setminus[j]}^k > 0, \quad (8)
$$

where $\hat{\epsilon} \in (0, 1)$ is a constant, and $p^k$ is the smallest positive integer satisfying

$$
F_j(x_1^k, \ldots, x_{j-1}^k, x_j^{k+1} + \hat{\epsilon} \alpha^k, x_{j+1}^k, \ldots, x_n^k) \geq 0. 
$$

Besides, we also have $F(x^{k+1}) \geq 0$.

\textit{Proof:} From Lemma 3, the fact that $\hat{\epsilon} \in (0, 1)$ and $p^k$ is a positive integer, we have $x_j^{k+1} = x_j^k + \hat{\epsilon} \alpha^k > 0$. From (8) and $\alpha^k < 0$ we know that

$$
0 < x^{k+1} \leq x^k. 
$$

Besides, since the $M$-type function $f$ is off-diagonally antitone, then, we have

$$
F_j(x_1^{k+1}, \ldots, x_{j-1}^{k+1}, x_j^k, x_{j+1}^{k+1}, \ldots, x_n^{k+1}) \geq 0, 
$$

and

$$
F_{[n]\setminus[j]}(x^{k+1}) \geq F_{[n]\setminus[j]}(x^k) \geq 0. 
$$

Thus, we have $F(x^{k+1}) \geq 0$. \hfill \Box

\textit{Remark 1:} Let $x^0 \in \mathbb{R}_{++}^n$ satisfying $F(x^0) \in \mathbb{R}_{++}^n \setminus \{0\}$. With the iterative approach showing in (8), we can obtain a nonincreasing positive sequence $\{x^k\}$, that is, $0 < x^{k+1} \leq x^k$ for any $k$. Besides, this sequence also satisfies $F(x^k) \geq 0$.\hfill \Box

\textit{Lemma 5:} Let $\{x^k\}$ be the sequence generated in Remark 1. Then, there exists a vector $x^* \in \mathbb{R}_{++}^n$ satisfying $x^k \to x^*$ as $k \to \infty$, and $F(x^*) \in \mathbb{R}_{++}^n$. 

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Proof: Since the nonincreasing positive sequence \(\{x^k\}\) is lower bounded, then, there exists a vector \(x^* \geq 0\) such that, as \(k \to \infty, x^k \to x^*\) and \(F(x^*) \geq 0\).

Now we will show that all the elements of \(x^*\) are positive, that is \(x^* > 0\). Suppose on the contrary there exists an index \(i \in [n]\) such that \(x^*_i = 0\). Then, according to the structure of the \(\mathcal{M}\)-type function \(f\), we have that \(f_i(x^*) = a_i(x^*_i)^{\gamma_i} - P_i(x^*) = -P_i(x^*) \leq 0\). Besides, the fact that \(b \in \mathbb{R}^n_+\) implies that \(b_j > 0\). Hence, we obtain that

\[ F_i(x^*) = f_i(x^*) - b_i = -P_i(x^*) - b_i < 0, \]

which is a contradiction. Thus, we have \(x^* \in \mathbb{R}^n_+\). \(\square\)

Theorem 2: Let \(x^* \in \mathbb{R}^n_+\) be the limit point of the sequence \(\{x^k\}\) generated in Remark 1. Then, \(x^*\) satisfies the system \(F(x^*) = 0\).

Proof: On the contrary, suppose that the limit point \(x^*\) of the sequence \(\{x^k\}\) does not satisfy \(F(x^*) = 0\). We define the index \(w\) to satisfy \(F_w(x^*) = \max_{i\in[n]}[F_i(x^*)]\geq0\). Let \(\alpha^w = -x^w\) and

\[ h_w(\gamma) = F_w(x_1^w, \ldots, x_{w-1}^w, x_w^w + \lambda \alpha^w, x_{w+1}^w, \ldots, x_n^w). \]

Obviously, \(h_w(0) = F_w(x^*) > 0\). Hence, according to the continuity of \(h\), there is a positive scalar \(\lambda \in (0, 1)\) such that \(h_w(\lambda) > 0\) for all \(\lambda \in (0,\bar{\lambda}]\). Then, it immediately follows that there is a positive integer \(\bar{\gamma}\) satisfying

\[ F_w(x_1^\bar{\gamma}, \ldots, x_{w-1}^\bar{\gamma}, (1 - \bar{\gamma}^\bar{\gamma})x_w^\bar{\gamma}, x_{w+1}^\bar{\gamma}, \ldots, x_n^\bar{\gamma}) > 0, \]

where \(\bar{\gamma} \in (0, 1)\) is the constant given in Lemma 4. Considering the continuity of \(F\), there is a neighbourhood \(N(x^*, \eta)\) of \(x^*\), where \(\eta > 0\), such that for any \(x \in N(x^*, \eta),\)

\[ F_w(x_1, \ldots, x_{w-1}, (1 - \bar{\gamma}^\bar{\gamma})x_w, x_{w+1}, \ldots, x_n) > 0. \]

In view of the fact that \(x^k \to x^*\) as \(k \to \infty\), for all sufficiently large \(k\), we have

\[ F_w(x_1^k, \ldots, x_{w-1}^k, (1 - \bar{\gamma}^\bar{\gamma})x_w^k, x_{w+1}^k, \ldots, x_n^k) > 0. \]

Then, for all sufficiently large \(k\), it follows that

\[ x_{w+1}^{k+1} = (1 - \bar{\gamma}^\bar{\gamma})x_w^k \leq (1 - \bar{\gamma}^\bar{\gamma})x_w^k, \]

where the inequality holds because of the definition of \(p^k\). Hence, by taking the limit we obtain that \(x_w^* \leq (1 - \bar{\gamma}^\bar{\gamma})x_w^k\). Since \(x_w^k > 0\), this is a contradiction. Thus, the limit point \(x^*\) satisfies \(F(x^*) = 0\). \(\square\)

Then, we immediately have the following conclusion:

Theorem 3: Let \(f : \mathbb{R}^n \to \mathbb{R}^n\) be an \(\mathcal{M}\)-type function defined by (1). If the set \(D = \{x \in \mathbb{R}^n \mid x > 0, f(x) > 0\}\) is not empty, then, \(f(x) = b\) has a unique positive solution for any \(b \in \mathbb{R}^n_+\).

Proof: From Theorem 2, it is obvious that the system \(f(x) = b\) has a positive solution. Now, we will show the uniqueness of the positive solution.

Suppose on the contrary that there exists two points \(x, y \in \mathbb{R}^n_+\) such that \(f(x) = b\) and \(f(y) = b\). Let \(\nu := \min_{i\in[n]} \frac{\nu_i}{y_i}\), then, we have \(x \geq \nu y\) and there exists an index \(j \in [n]\) such that \(x_j = \nu y_j\). If \(\nu < 1\), then

\[ f(\nu y) = (\nu^{p_1}a_1y_1^{\gamma_1} - P_1(\nu y), \ldots, \nu^{p_n}a_ny_n^{\gamma_n} - P_n(\nu y))^T \]

\[ \leq \nu^{\gamma_{\min}} b \]

\[ < b, \]

where \(\gamma_{\min} = \min_{i\in[n]}\{\gamma_i\}\). This implies that

\[ \left(\frac{b_i + P_i(\nu y))}{a_i}\right)^{\frac{1}{\gamma_i}} > \nu y_i, \quad \forall i \in [n]. \]

However, it is not difficult to see that for the index \(j\) we have

\[ \left(\frac{b_j + P_j(\nu y))}{a_j}\right)^{\frac{1}{\gamma_j}} \leq \left(\frac{b_j + P_j(x))}{a_j}\right)^{\frac{1}{\gamma_j}} = x_j = \nu y_j, \]

which is a contradiction. Hence, we have \(\nu \geq 1\), and then, \(x \geq y\). Repeat similar steps and we will obtain that \(y \geq x\). Combining these two parts together, we have \(x = y\). That is, the system \(f(x) = b\) has a unique positive solution.

From item (2) of Theorem 1 we know that for \(f(x) = \mathcal{A}x^{m-1}\), where \(\mathcal{A} \in \mathbb{T}_{m,n}\) is a nonsingular \(\mathcal{M}\)-tensor, \(D \neq \emptyset\). Then, we can directly get the result below, which has already shown in Section III.

Corollary 1 (f(6)): Let \(\mathcal{A} \in \mathbb{T}_{m,n}\) be a nonsingular \(\mathcal{M}\)-tensor. Then, \(\mathcal{A}x^{m-1} = b\) has a unique positive solution for any \(b \in \mathbb{R}^n_+\).

Actually, the above analysis also provide a simple but convergent method to find the unique positive solution of system (2), which can be summarized in Algorithm 1.

**Algorithm 1 (A Convergent Algorithm for System (2))**

1: Choose an initial point \(x^0 \in \mathbb{R}^n_+\) satisfying \(F(x^0) \geq 0\) and a constant \(\epsilon \in (0, 1)\).
2: while \(\|F(x^k)\| \neq 0\) do
3: Let \(f \in [n]\) satisfy \(f_j(x^k) = \max_{i\in[n]}[F_i(x^k)]\).
4: Let \(x^{k+1} = x^k + \epsilon \alpha^{k^k}, x_{k+1}^k, \ldots, x_n^k\).
5: for \(p = 1, 2, \ldots\) do
6: Compute \(h(p) = F_j(x^k, \ldots, x_{k-1}^k, x_k^k, x_{k+1}^k, \ldots, x_n^k)\).
7: If \(h(p) \geq 0\), let \(x^{k+1} = x^k + \epsilon \alpha^{k^k}\) and stop.
8: end for
9: end while

From the above analysis we know that, with an initial point \(x^0 > 0\) satisfying \(F(x^0) \geq 0\), Algorithm 1 can generate a non-increasing positive sequence \(\{x^k\}\) converging to the unique positive solution of system (2). However, at the \(k\)-th iteration of Algorithm 1, only one element of \(x^k\) decreases, which may result in more iterations and longer computing time. Therefore, although Algorithm 1 is simple and convenient, it may not be the best choice to solve this problem. Hence, we need to improve this algorithm.

The improvement plan varies from case to case. For example, since \(\mathcal{M}\)-type functions is a generalization of the
positive homogeneous polynomial generated by a nonsingular $M$-tensor, by studying the properties of $M$-type function, nonsingular $M$-tensor and nonsingular $M$-matrix, in the next subsection we show that, with Assumption 2 holding, which is stricter than Assumption 1, we may develop a technique to help us find the unique positive solution faster.

**B. AN IMPROVED METHOD OF ALGORITHM I**

In this subsection, inspired by [1], we mainly provide an idea for improving Algorithm 1 to obtain better convergence.

**Assumption 2:** The set $D := \{x \in \mathbb{R}^n \mid f(x) > 0\}$ is not empty and $f'(x)$ is a nonsingular $M$-matrix for any $x \in D$, where $f'(x)$ is the Jacobian of $f$ at $x \in \mathbb{R}^n$.

This assumption is also reasonable. Since for $g(x) = Ax^{m-1}$ where $A \in T_{m,n}$ is a nonsingular $M$-tensor, we know that $D \neq \emptyset$. Besides, Lemma 2.2 of [1] shows that the Jacobian $g'(x)$, where $x \in D$, is a nonsingular $M$-matrix. Thus, function $g(x) = Ax^{m-1}$ is an $M$-type function satisfying Assumption 2. Next, we will always assume that Assumption 2 holds.

In the following, we will give the framework of the improved algorithm, which is shown in Algorithm 2.

**Algorithm 2 (An Improved Algorithm for System (2))**

1. Choose an initial point $x^0 \in \mathbb{R}^n_{+}$ satisfying $F(x^0) \geq 0$ and two constants $\varepsilon_1, \varepsilon_2 \in (0, 1)$.
2. while $\|F(x^k)\| \neq 0$ do
3. Let $j \in [n]$ satisfy $F_j(x^k) = \max_{i \in [n]} \{F_i(x^k)\}, I_k = \{s \mid F_s(x^k) > 0\} \setminus \{j\}$ and $I_{\bar{k}} = \{t \mid F_t(x^k) = 0\}$.
4. Compute $h(p) = F_j(x^{k+1}_1, \cdots, x^k_{j-1}, x^k_j + \varepsilon_p, x^k_{j+1}, \cdots, x^k_n)$.
5. if $h(p) \geq 0$, let $x^{k+1}_j := x^k_j + \varepsilon_p$ and stop.
6. for $q = 0, 1, \cdots$ do
7. Compute $g(q) = F_j(x^{k+1}_1, \cdots, x^{k+1}_{j-1}, x^k_j, x^{k+1}_{j+1}, \cdots, x^k_n)$.
8. if $g(q) \geq 0$, let $x^{k+1}_j := x^k_j + \varepsilon_2 q^k$ and stop.
9. end for
10. for $q = 0, 1, \cdots$ do
11. Compute $g(q) = F_j(x^{k+1}_1, \cdots, x^{k+1}_{j-1}, x^k_j, x^{k+1}_{j+1}, \cdots, x^k_n)$.
12. if $g(q) \geq 0$, let $x^{k+1}_j := x^k_j + \varepsilon_2 q^k$ and stop.
13. end for
14. end while

**Remark 2:** Indeed, step 5 to step 8 is directly from Algorithm 1, which guarantees the convergence (see Theorem 4). And in step 9 to step 13, we employ an acceleration part, that is, a Newton step, to gain better speed of convergence. Therefore, Algorithm 2, which is an improved version of Algorithm 1, can also be seen as a Newton-type method.

We first show the existence of the vector $\beta^k$ given in step 9, that is, $[F'(x^k)]_{I_k \bar{k}}$ is always a nonsingular matrix.

**Proposition 6:** For any $x^k \in D$, $[F'(x^k)]_{I_k \bar{k}}$ is a nonsingular $M$-matrix.

**Proof:** Under Assumption 2, for any $x^k \in D$, the Jacobian matrix $f'(x^k)$ is a nonsingular $M$-matrix, then, so is the principal sub-matrix $[F'(x^k)]_{I_k \bar{k}}$. Thus, we obtain that $[F'(x^k)]_{I_k \bar{k}} = [F'(x^k)]_{I_k \bar{k}}$ is a nonsingular $M$-matrix.

Then, we briefly illustrate that this algorithm is well-defined and convergent. We define a set of notations as follows:

$$x^k(\mu) := x^k + \mu \beta^k$$

Similar to Lemma 3, for $x^k(\mu)$ and $g(\mu)$, which are defined by (9), we have the result below.

**Lemma 6:** At the $k$-th iteration of Algorithm 2, the following two statements hold:

1. for any $\mu \in (0, 1]$, we have $x^k(\mu) > 0$.
2. there is a scalar $\bar{\mu} \in (0, 1]$ satisfying $g(\mu) \geq 0$ for any $\mu \in (0, \bar{\mu}]$.

**Proof:** (1) First, from Proposition 6 we know that $[F'(x^k)]_{I_k \bar{k}}$ is always a nonsingular $M$-matrix. Combining with item (3) of Proposition 4, we have that $[F'(x^k)]_{I_k \bar{k}}$ is a nonnegative matrix. Hence, we can obtain that $\beta^k \geq 0$, which implies that $x^k(\mu) < x^k$ for any $\mu > 0$. Combining with item (1) of Proposition 4, we get that $x^k(\mu) > 0$.

(2) Without loss of generality, we may assume that $I_k$ consists of the first $|I_k|$ elements in $[n]$. For any $i \in I_k$,

$$g_i(\mu) = F_i(x^k(\mu)) = a_i(x^k(\mu))^\gamma_i - P_i(x^k(\mu)) - b_i$$

and

$$g_i(0) = F_i(x^k) = a_i(x^k)^\gamma_i - P_i(x^k) - b_i > 0.$$
Furthermore, from the analysis of item (1) we know that $x_i^k(\lambda) < x_i^k$, where $i \in I_k$. Then, it is easy to see that
\[
a_i \left(x_i^k(\mu)\right) < a_i \left(x_i^k\right)^{\gamma_i}, \quad P_i \left(x_i^k(\mu)\right) \leq P_i(x_i^k).
\] (10)

Hence, according to (10) and the continuity of the function $g$, by adjusting the value of $\mu \in (0, 1)$, we can ensure that $g_i(\mu)$ is nonnegative for any $i \in I_k$. That is, there exists a scalar $\tilde{\mu} \in (0, 1)$ such that $g_i(\mu) \geq 0$ for any $\mu \in (0, \tilde{\mu})$.

According to Lemma 6, we can see that for $k$-th iteration, for any $\epsilon \in (0, 1)$, there exists a nonnegative integer $q$ such that $x_i^k + \epsilon \beta_i^k > 0$ and $F_i(x_i^k + \epsilon \beta_i^k) \geq 0$. Then, from Lemmas 3 and 6, we can easily obtain that:

**Lemma 7:** At the $k$-th iteration of Algorithm 2, we have $x_i^{k+1} = x_i^k + \epsilon \beta_i^k > 0$ and $F_i(x_i^k + \epsilon \beta_i^k) \geq 0$. Hence, according to (10) and the continuity of the function $g$, by adjusting the value of $\mu \in (0, 1)$, we can ensure that $g_i(\mu)$ is nonnegative for any $i \in I_k$. That is, there exists a scalar $\tilde{\mu} \in (0, 1)$ such that $g_i(\mu) \geq 0$ for any $\mu \in (0, \tilde{\mu})$.

In this section, we will show the behaviors of Algorithms 1 and 2 on the nonlinear system of equations involving an $M$-type function and a positive right-hand side vector by implementing these two algorithms in Matlab R2016a. The reliability of Algorithms 1 and 2 will be presented by concrete numerical experiments. All the numerical experiments are conducted on a laptop with Inter(R) Core(TM) CPU i5-8250U @ 1.60GHz and 8 GB memory running Microsoft Window 10.

In the experiments, with Assumption 2 holding, we know that the set $D = \{x \mid f(x) > 0, x > 0\} \neq \emptyset$, and we first need to find a point $x$ satisfying $x > 0$ and $f(x) > 0$. We may employ the command ‘fmincon’ in Matlab to find such a point. And then, with the determination of vector $b$, by using the technique in Proposition 5, we can obtain a proper initial point $x_0$ for both Algorithm 1 and Algorithm 2, satisfying the condition that $x_0 > 0$ and $F(x_0) = f(x_0) - b \geq 0$. As for the stopping criterion for Algorithms 1 and 2, it is defined as
\[
||f(x) - b|| \leq 10^{-10}.
\]

In order to show the performance of the proposed algorithms, we use the following items to describe the effect of the algorithms: we use ‘iter’ for iterations, ‘time’ for computing time (in second) and ‘residue’ for the residue $||f(x^*) - b||$, where $x^*$ is an approximate solution. Besides, we notice that in these two algorithms, there exist some parameters. In Algorithm 1 we choose $\epsilon = 0.2$, and in Algorithm 2 we choose $\epsilon_1 = 0.2$ and $\epsilon_2 = 0.5$.

Now, we see a simple example which is shown below.

**Example 4:** We consider the function
\[
f(x) = \left(\frac{x_1^3 - 2x_1x_2}{3x_2^2 - x_1x_2}\right).
\]
defined on $\mathbb{R}^2$. First we show that this $M$-type function satisfies Assumption 2. Let $\bar{x} = (2, 1)^T$, then, we have $f(\bar{x}) = (4, 1)^T > 0$, which indicates that $D = \{x \mid x > 0, f(x) > 0\} \neq \emptyset$. And for any $x = (x_1, x_2)^T \in D$, we have $x_1 > 2x_2, 3x_2 > x_1, x_1 > \frac{x_2}{2}$ and $x_2 > \frac{x_2}{2}$. Besides, the Jacobian of $f(x)$ is given by
\[
f'(x) = \begin{pmatrix} 3x_1^2 - 2x_2 & -2x_1 \\ -x_2^2 & 9x_2^2 - 2x_1x_2 \end{pmatrix}.
\]

Let $y = (a, \frac{2a}{9})^T$, where $a > 0$. Then, we have $f'(x)y > 0$ for any $x \in D$ and it follows from item (2) of Proposition 4 that $f'(x)$ is a nonsingular $M$-matrix on $D$.

Next, five different $b$’s will be used to test Algorithms 1 and 2, and the results are summarized in Tables 1 and 2.

We can see from these two tables that both Algorithm 1 and Algorithm 2 can solve these five nonlinear systems of equations with different $b$’s efficiently. By comparing the data in Tables 1 and 2, we can see that Algorithm 1 behaves better than Algorithm 2 in terms of computing time, while Algorithm 2 has better performance than Algorithm 1 in terms of the number of iteration.
and 2, when directly employing FS to solve these nonlinear systems of equations, the iterations are fewer, the residues are smaller, but the computing time are longer. What is more, FS can not find the positive solution of the problem involving \( b = (89, 7)^T \) with the current initial point. And during the experiment on FS, we find that when changing the initial points, the other four cases may not find the correct solution, either. Besides, when using FS to solve system (2), which may have non-positive solutions, FS does not necessarily return a positive solution. For example, when employing FS to solve the system \( g(x) = b \), where \( g(x) = (x_1^2, x_2^2)^T \) is an \( \mathcal{M}\)-type function satisfying Assumption 2 defined on \( \mathbb{R}^2 \) and \( b = (1, 4)^T \), we may obtain the negative solution \( x^\ast = (-1, -2)^T \) other than the positive solution \( x^\dagger = (1, 2)^T \). Hence, FS has a certain dependence on the choice of the initial point. Algorithms 1 and 2, in contrast, are relatively stable algorithms and will always return positive solutions.

Now, we consider some higher dimensional problems, which are shown below.

**Example 5:** We consider the function

\[
f(x) = \begin{pmatrix} x_1^2 - x_2 \\ x_2^2 - x_3 \\ \vdots \\ x_n^2 - x_{n-1} \\ x_{n-1}^2 - x_n \\ x_{n-1}^2 - x_n \end{pmatrix}
\]

defined on \( \mathbb{R}^n \), where \( f_i(x) = x_i^2 - x_{i+1} \) for \( i \in [n-1] \). Obviously, \( f \) is an \( \mathcal{M}\)-type function. Now, we will show that \( f \) satisfies Assumption 2. First, it is easy to check that the set \( D = \{ x \mid x > 0, f(x) > 0 \} \neq \emptyset \). Then, from the fact that the Jacobian of \( f(x) \), where \( x \in D \), is a \( Z\)-matrix, we know that it is further a nonsingular \( M\)-matrix according to item (4) of Proposition 4. Thus, we can obtain that \( f \) is an \( \mathcal{M}\)-type function satisfying Assumption 2.

Since Algorithm 2 is an improved version of Algorithm 1, which is more efficient, here, we only show how Algorithm 2 behaves when dealing with Example 5 (in fact, we have also carried out this experiment on Algorithm 1 and the result shows that Algorithm 2 is indeed more efficient than Algorithm 1). We will test Algorithm 2 on function \( f \)'s with various sizes by choosing different \( n \). We randomly generate several \( b \)'s to carry on the experiments. The corresponding results are summarized in Table 4.

We can see from Table 4 that Algorithm 2 can efficiently find positive solutions to these problems within a short time. Besides, we also plot convergence curves of Algorithm 2 for all the cases in Table 4, which similarly show that Algorithm 2 may be linearly convergent.

From the examples given in this section we find that both Algorithm 1 and Algorithm 2 can effectively find the positive

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**TABLE 1.** Numerical results of Algorithm 1 solving Example 4.

| \( b \)     | \( \text{iter} \) | \( \text{time} \) | \( \text{residue} \) | \( \text{solution} \) |
|------------|-------------------|-------------------|--------------------|--------------------|
| \((1, 1)^T\) | 95                | 0.0032            | \( 9.34 \times 10^{-11} \) | \((1.573, 0.919)^T\) |
| \((3, 1)^T\) | 102               | 0.0053            | \( 8.80 \times 10^{-11} \) | \((1.883, 0.977)^T\) |
| \((1, 3)^T\) | 99                | 0.0028            | \( 5.95 \times 10^{-11} \) | \((1.745, 1.236)^T\) |
| \((15, 71)^T\) | 105              | 0.0025            | \( 7.09 \times 10^{-11} \) | \((3.330, 3.293)^T\) |
| \((89, 7)^T\) | 85                | 0.0025            | \( 9.64 \times 10^{-11} \) | \((4.780, 2.115)^T\) |

**TABLE 2.** Numerical results of Algorithm 2 solving Example 4.

| \( b \)     | \( \text{iter} \) | \( \text{time} \) | \( \text{residue} \) | \( \text{solution} \) |
|------------|-------------------|-------------------|--------------------|--------------------|
| \((1, 1)^T\) | 41                | 0.0043            | \( 8.54 \times 10^{-11} \) | \((1.573, 0.919)^T\) |
| \((3, 1)^T\) | 37                | 0.0087            | \( 9.53 \times 10^{-11} \) | \((1.883, 0.977)^T\) |
| \((1, 3)^T\) | 38                | 0.0058            | \( 8.26 \times 10^{-11} \) | \((1.745, 1.236)^T\) |
| \((15, 71)^T\) | 46               | 0.0054            | \( 4.90 \times 10^{-11} \) | \((3.330, 3.293)^T\) |
| \((89, 7)^T\) | 37                | 0.0044            | \( 2.72 \times 10^{-11} \) | \((4.780, 2.115)^T\) |

**TABLE 3.** Numerical results of FS solving Example 4.

| \( b \)     | \( \text{iter} \) | \( \text{time} \) | \( \text{residue} \) | \( \text{solution} \) |
|------------|-------------------|-------------------|--------------------|--------------------|
| \((1, 1)^T\) | 5                 | 0.0103            | \( 4.44 \times 10^{-16} \) | \((1.573, 0.919)^T\) |
| \((3, 1)^T\) | 4                 | 0.0111            | \( 1.33 \times 10^{-15} \) | \((1.883, 0.977)^T\) |
| \((1, 3)^T\) | 5                 | 0.0122            | \( 1.20 \times 10^{-14} \) | \((1.745, 1.236)^T\) |
| \((15, 71)^T\) | 8               | 0.0087            | \( 3.55 \times 10^{-15} \) | \((3.330, 3.293)^T\) |
| \((89, 7)^T\) | -                 | -                 | -                  | -                  |
solution of the nonlinear system of equations involving an $M$-type function and a positive right-hand side vector. They have certain advantages in terms of computing time. What is more, when dealing with large scale problems, the advantage of Algorithm 2 will be greater than Algorithm 1. Therefore, we can draw the conclusion that Algorithm 2 is a reliable and efficient algorithm.

VI. CONCLUSION
In this paper, we defined a new class of functions, called $M$-type function. We presented that this class of functions is different from the class of functions called $M$-function by constructing two examples. Then, we also showed some significant properties of $M$-type functions. What is more, we investigated the nonlinear system of equations with an $M$-type function and a positive right-hand side vector. We proved that with a rational assumption, the above system has a positive solution, and this positive solution is unique. In this process, we also got an idea of algorithm for this problem. Besides, we illustrated that when improving the above primary algorithm, under an additional condition, we can employ a Newton step to help us find the solution faster. Through several numerical experiments, we can see that both proposed algorithms are stable, reliable and efficient.

There are still many questions about $M$-type function to be considered in the future. For example, what better properties does $M$-type function have? What can we say about the nonlinear system of equations with an $M$-type function and a non-positive right-hand side vector? And, it is well-known that the complementarity problem has some connection with the nonlinear system of equations. Then, what can we say about the complementarity problem with an $M$-type function? Besides, according to the certain structural similarity between $M$-type function and $M$-function, can we extend the definition of $M$-type function so that it contains $M$-function as its subclass, and obtain some good results? All of these problems are worthy of our in-depth study.

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TABLE 4. Numerical results of Algorithm 2 solving Example 5.

| dimension $n$ | iter | time | residue |
|---------------|------|------|---------|
| $n = 10$      | 42   | 0.0070 | $5.48 \times 10^{-11}$ |
| $n = 15$      | 48   | 0.0075 | $6.19 \times 10^{-11}$ |
| $n = 20$      | 47   | 0.0111 | $6.28 \times 10^{-11}$ |
| $n = 25$      | 43   | 0.0081 | $3.38 \times 10^{-11}$ |
| $n = 30$      | 59   | 0.0074 | $8.49 \times 10^{-11}$ |
| $n = 35$      | 44   | 0.0087 | $6.98 \times 10^{-11}$ |
| $n = 40$      | 40   | 0.0091 | $6.71 \times 10^{-11}$ |
| $n = 45$      | 44   | 0.0088 | $7.50 \times 10^{-11}$ |
| $n = 50$      | 55   | 0.0146 | $8.00 \times 10^{-11}$ |
| $n = 100$     | 44   | 0.0177 | $7.23 \times 10^{-11}$ |
| $n = 200$     | 63   | 0.0431 | $9.43 \times 10^{-11}$ |
| $n = 300$     | 47   | 0.0642 | $6.71 \times 10^{-11}$ |
| $n = 400$     | 61   | 0.2883 | $9.40 \times 10^{-11}$ |