A penalized exponential risk bound in parametric estimation

Spokoiny, Vladimir

Weierstrass-Institute,
Mohrenstr. 39, 10117 Berlin, Germany
spokoiny@wias-berlin.de

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Abstract

The paper offers a novel unified approach to studying the accuracy of parameter estimation by the quasi likelihood method. Important features of the approach are: (1) The underlying model is not assumed to be parametric. (2) No conditions on parameter identifiability are required. The parameter set can be unbounded. (3) The model assumptions are quite general and there is no specific structural assumptions like independence or weak dependence of observations. The imposed conditions on the model are very mild and can be easily checked in specific applications. (4) The established risk bounds are nonasymptotic and valid for large, moderate and small samples. (5) The main result is the concentration property of the quasi MLE giving an nonasymptotic exponential bound for the probability that the considered estimate deviates out of a small neighborhood of the “true” point.

In standard situations under mild regularity conditions, the usual consistency and rate results can be easily obtained as corollaries from the established risk bounds. The approach and the results are illustrated on the example of generalized linear and single-index models.

JEL codes: C13,C22.

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1 Introduction

One of the most popular approaches in statistics is based on the parametric assumption that the distribution \( P \) of the observed data \( Y \) belongs to a given parametric family
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$(P_\theta, \theta \in \Theta)$, where $\Theta$ is a subset in a finite dimensional space $\mathbb{R}^p$. In this situation, the statistical inference about $P$ is reduced to recovering $\theta$. The standard likelihood principle suggests to estimate $\theta$ by maximizing the corresponding log-likelihood function $L(Y, \theta)$. The classical parametric statistical theory focuses mostly on asymptotic properties of the difference between $\tilde{\theta}$ and the true value $\theta_0$ as the sample size $n$ tends to infinity. There is a vast literature on this issue. We only mention the book by Ibragimov and Khas’minskij (1981), which provides a comprehensive study of asymptotic properties of maximum likelihood and Bayesian estimators. The related analysis is effectively based on the Taylor expansion of the likelihood function near the true point under the assumption that the considered estimate well concentrates in a small (root-n) neighborhood of this point. In the contrary, there is only few results which establish this desired concentration property. Ibragimov and Khas’minskij (1981), Section 1.5, presents some exponential concentration bounds in the i.i.d. parametric case. Large deviation results about minimum contrast estimators can be found in Jensen and Wood (1998) and Sieders and Dzhaparidze (1987), while subtle small sample size properties of these estimators are presented in Field (1982) and Field and Ronchetti (1990). This paper aims at studying the concentration properties of a general parametric estimate. The main result describes some concentration sets for the considered estimate and establishes an exponential bound for deviating of the estimate out of such sets.

In the modern statistical literature there is a number of papers considering maximum likelihood or more generally minimum contrast estimators in a general i.i.d. situation, when the parameter set $\Theta$ is a subset of some functional space. We mention the papers Van de Geer (1993), Birgé and Massart (1993), Birgé and Massart (1998), Birgé (2006) and references therein. The studies mostly focused on the concentration properties of the maximum over $\theta \in \Theta$ of the log-likelihood $L(Y, \theta)$ rather on the properties of the estimator $\tilde{\theta}$ which is the point of maximum of $L(Y, \theta)$. The established results are based on deep probabilistic facts from the empirical process theory (see e.g. Talagrand (1996), van der Vaart and Wellner (1996), Boucheron et al. (2003)). Our approach is similar in the sense that the analysis also focuses on the properties of the maximum of $L(Y, \theta)$ over $\theta \in \Theta$. However, we do not assume any specific structure of the model. In particular, we do not assume independent observations and thus, cannot apply the methods from the empirical process theory.

The aim of this paper is to offer a general and unified approach to statistical estimation problem which delivers meaningful and informative results in a general framework under mild regularity assumptions. An important issue of the proposed approach is that it allows to go beyond the parametric case, that is, the most of results and conclusions continue to apply even if the parametric assumption is not precisely fulfilled. Then the
target of estimation can be viewed as the best parametric fit. Some other important features of the proposed approach are that the established risk bounds are nonasymptotic and equally apply to large, moderate and small samples and that the results describe nonasymptotic confidence and concentration sets in terms of quasi log-likelihood rather than the accuracy of point estimation. In the most of examples, the usual consistency and rate results can be easily obtained as corollaries from the established risk bounds. The results are obtained under very mild conditions which are easy to verify in particular applications. There is no specific assumptions on the structure of the data like independence or weak dependence of observations, the parameter set can be unbounded. Another interesting feature of the proposed approach that it does not require any identifiability conditions. Informally, one can say that whatever the quasi likelihood or contrast is, the corresponding estimate belongs with a dominating probability to the corresponding concentration set. Examples show that the resulting concentration sets are of right magnitude, in typical situations this is a root-n vicinity of the true point.

Now we specify the considered set-up. Let \( Y \) stand for the observed data. For notational simplicity we assume that \( Y \) is a vector in \( \mathbb{R}^n \). By \( P \) we denote the measure describing the distribution of the whole sample \( Y \). The parametric approach discussed below allows to reduce the whole description of the model to a few parameters which have to be estimated from the data. Let \((P_\theta, \theta \in \Theta)\) be a given parametric family of measures on \( \mathbb{R}^n \). The parametric assumption means simply that \( P = P_{\theta_0} \) for some \( \theta_0 \in \Theta \). The parameter vector \( \theta_0 \) can be estimated using the maximum likelihood (MLE) approach. Let \( L(Y, \theta) \) be the log-likelihood for the considered parametric model:

\[
L(Y, \theta) = \log \frac{dP_\theta}{dP_0}(Y),
\]

where \( P_0 \) is any dominating measure for the family \((P_\theta)\). The MLE estimate \( \tilde{\theta} \) of the parameter \( \theta_0 \) is given by maximizing the log-likelihood \( L(\theta) \):

\[
\tilde{\theta} = \arg\max_{\theta \in \Theta} L(Y, \theta).
\]  

(1.1)

Note that the value of the estimate will not be changed if the process \( L(Y, \theta) \) is multiplied by any positive constant \( \mu \).

The quasi maximum likelihood approach admits that the underlying distribution \( P \) does not belong to the family \((P_\theta)\). The estimate \( \tilde{\theta} \) from (1.1) is still meaningful and it becomes the quasi MLE. Later we show that \( \tilde{\theta} \) estimate the value \( \theta_0 \) defined by maximizing the expected value of \( L(Y, \theta) \):

\[
\theta_0 \overset{\text{def}}{=} \arg\max_{\theta \in \Theta} E L(Y, \theta)
\]

which is the true value in the parametric situation and can be viewed as the parameter of the best parametric fit in the general case.
Note that the presented set-up is quite general and the most of statistical estimation procedures can be represented as quasi maximum likelihood for a properly selected parametric family. In particular, popular least squares, least absolute deviations, M-estimates can be represented as quasi MLE.

The set-up of this paper is even more general. Namely, we consider a general estimate $\tilde{\theta}$ defined by maximizing a random field $L(\theta)$. The basic example we have in mind is the scaled quasi log-likelihood $L(\theta) = \mu L(Y, \theta)$ for some $\mu > 0$. In some cases, especially if the parameter set is unbounded, the scaling factor $\mu$ can also be taken depending on $\theta$, that is, $L(\theta) = \mu(\theta)L(Y, \theta)$. We focus on the properties of the process $L(\theta)$ as a function of the parameter $\theta$. Therefore, we suppress the argument $Y$ there. One has to keep in mind that $L(\theta)$ is random and depends on the observed data $Y$. The study focuses on the concentration properties of the estimate $\tilde{\theta}$ which is defined by maximization of the random process $L(\theta)$. Let

$$\theta_0 = \arg\max_{\theta} E L(\theta).$$

We also define $L(\theta, \theta_0) = L(\theta) - L(\theta_0)$. The aim of our study is to bound the value of the quasi maximum likelihood $L(\tilde{\theta}, \theta_0) = \max_{\theta} L(\theta, \theta_0)$. The basic assumption imposed on the process $L(\theta)$ is that the difference $L(\theta, \theta_0) = L(\theta) - L(\theta_0)$ has bounded exponential moments for every $\theta$. Our primary goal is to bound the supremum of such differences, or more precisely, to establish an exponential bound for the value $L(\tilde{\theta}, \theta_0)$. The standard approach of empirical process theory is to consider separately the mean and the centered stochastic deviations of the process $L(\theta)$. Here a slightly different standardization of the process $L(\theta)$ is used. Assume that the exponential moment for $L(\theta, \theta_0)$ is finite for all $\theta$. This enables us to define for each $\theta$ the rate function $M(\theta, \theta_0)$ which ensures the identity

$$E \exp\{L(\theta, \theta_0) + M(\theta, \theta_0)\} = 1.$$ 

This means that the process $L(\theta, \theta_0) + M(\theta, \theta_0)$ is pointwise stochastically bounded in a rather strict sense. We aim at establishing a similar bound for the maximum of $L(\theta, \theta_0) + M(\theta, \theta_0)$. It turns out that some payment for taking the maximum is necessary. Namely, we present a penalty function $\text{pen}(\theta)$ which ensures that the maximum of $L(\theta, \theta_0) + M(\theta, \theta_0) - \text{pen}(\theta)$ is bounded with exponential moments. Then we show that this fundamental fact yields a number of straightforward corollaries about the quality of estimation.

The paper is organized as follows. The next section presents the main result which describes an exponential upper bound for the (quasi) maximum likelihood. Section 2.2 discusses some implications of this exponential bound for statistical inference. In particular, we present a general likelihood-based construction of confidence sets and establish
an exponential bound for the coverage probability. We also show that the considered estimate well concentrates on the level set of the rate function $\mathfrak{M}(\theta, \theta_0)$. Under some standard conditions we show that such concentration sets become usual root-n neighborhoods of the target $\theta_0$. Sections 3 and 4 illustrate the obtained general results for two quite popular statistical models: generalized linear and single index. These models are very well studied, the existing results claim asymptotic normality and efficiency of the maximum likelihood estimate as the sample size grows to infinity. On the contrary, our study focuses on nonasymptotic deviation bounds and concentration properties of this estimate. The main result giving an exponential bound for the maximum likelihood is based on general results for the maximum of a random field described in Section 5.

2 Exponential bound for the maximum likelihood

This section presents a general exponential bound on the (quasi) maximum likelihood value in a quite general set-up. The main result concerns the value of maximum $\mathcal{L}(\tilde{\theta}) = \max_{\theta \in \Theta} \mathcal{L}(\theta)$ rather than the point of maximum $\tilde{\theta}$. Namely, we aim at establishing some exponential bounds on the supremum in $\theta$ of the random field

$$\mathcal{L}(\theta, \theta_0) \overset{\text{def}}{=} \mathcal{L}(\theta) - \mathcal{L}(\theta_0).$$

In this paper we do not specify the structure of the process $\mathcal{L}(\theta)$. The basic assumption we impose on the considered model is that $\mathcal{L}(\theta)$ is absolutely continuous in $\theta$ and that $\mathcal{L}(\theta)$ and its gradient w.r.t. $\theta$ have bounded exponential moments.

(E) The rate function $\mathfrak{M}(\theta, \theta_0)$ is finite for all $\theta \in \Theta$:

$$\mathfrak{M}(\theta, \theta_0) \overset{\text{def}}{=} -\log E \exp \{ \mathcal{L}(\theta, \theta_0) \}. $$

Note that this condition is automatically fulfilled if $P = P_{\theta_0}$ and $\mathcal{L}(\theta) = \mu \log (dP_{\theta}/dP_{\theta_0})$ with $\mu \leq 1$ provided that all $P_{\theta}$ are absolutely continuous w.r.t. $P_{\theta_0}$. With $\mu = 1$ and $\mathcal{L}(\theta) = \log (dP_{\theta}/dP_{\theta_0})$, it holds $\mathfrak{M}(\theta, \theta_0) = -\log E_{\theta_0}(dP_{\theta}/dP_{\theta_0}) = 0$. For $\mu < 1$, $\mathfrak{M}(\theta, \theta_0) = -\log E_{\theta_0}(dP_{\theta}/dP_{\theta_0})^\mu \geq 0$ for $\mu < 1$ by the Jensen inequality.

The main observation behind the condition (E) is that

$$E \exp \{ \mathcal{L}(\theta, \theta_0) + \mathfrak{M}(\theta, \theta_0) \} = 1.$$

Our main goal is to get a similar bound for the maximum of the random field $\mathcal{L}(\theta, \theta_0) + \mathfrak{M}(\theta, \theta_0)$ over $\theta \in \Theta$. Below in Section 2.2 we show that such a bound implies an exponential bound for the coverage probability for a confidence set $\mathcal{E}(\delta) = \{ \theta : \mathcal{L}(\theta, \theta) \leq \}$
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and that the estimate $\tilde{\theta}$ well concentrates on a set $A(r, \theta_0) = \{ \theta : \mathcal{M}(\theta, \theta_0) \leq r \}$ in the sense that the probability of the event $\{ \tilde{\theta} \notin A(r, \theta_0) \}$ is exponentially small in $r$.

Unfortunately, in some situations, the exponential moment of the maximum of $\mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0)$ can be unbounded. We present a simple example of this sort.

**Example 2.1.** Consider a Gaussian shift with only one observation $Y \sim N(\theta, 1)$ and suppose that the true parameter is $\theta_0 = 0$. Then the log-likelihood ratio $\mathcal{L}(\theta, \theta_0)$ reads as $\mathcal{L}(\theta, \theta_0) = Y \theta - \theta^2/2$, and it holds $\mathcal{M}(\theta, \theta_0) = 0$, $\sup_{\theta} \mathcal{L}(\theta) = Y^2/2$ and $E\theta_0 \exp\{\sup_{\theta} \mathcal{L}(\theta, \theta_0)\} = \infty$.

We therefore consider the penalized expression $\mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0) - \text{pen}(\theta)$, where the penalty function $\text{pen}(\theta)$ should provide some bounded exponential moments for $\sup_{\theta \in \Theta} [\mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0) - \text{pen}(\theta)]$.

To bound local fluctuations of the process $\mathcal{L}(\theta)$, we introduce an exponential moment condition on the stochastic component $\zeta(\theta)$:

$$\zeta(\theta) \overset{\text{def}}{=} \mathcal{L}(\theta) - E\mathcal{L}(\theta).$$

Suppose also that the random function $\zeta(\theta)$ is differentiable in $\theta$ and its gradient $\nabla \zeta(\theta) = \partial \zeta(\theta)/\partial \theta \in \mathbb{R}^p$ fulfills the following condition:

**(ED)** There exist some continuous symmetric matrix function $V(\theta)$ for $\theta \in \Theta$ and constant $\lambda^* > 0$ such that for all $|\lambda| \leq \lambda^*$

$$\sup_{\gamma \in S^p} \sup_{\theta \in \Theta} \log E \exp \left\{ 2\lambda \frac{\gamma^T \nabla \zeta(\theta)}{\sqrt{\gamma^T V(\theta) \gamma}} \right\} \leq 2\lambda^2. \quad (2.1)$$

Define for every $\theta, \theta' \in \Theta$, $d = \|\theta - \theta'\|$ and $\gamma = (\theta' - \theta)/d$

$$\mathcal{S}^2(\theta, \theta') \overset{\text{def}}{=} d^2 \int_0^1 \gamma^T V(\theta + td\gamma) \gamma dt.$$ 

Next, introduce for every $\theta^0 \in \Theta$ the local vicinity $\mathcal{B}(\epsilon, \theta^0)$ such that $\mathcal{S}(\theta, \theta^0) \leq \epsilon$ for all $\theta \in \mathcal{B}(\epsilon, \theta^0)$.

Let also the function $V(\cdot)$ from (ED) satisfy the following regularity condition:

**(V)** There exist constants $\epsilon > 0$ and $\nu_1 \geq 1$ such that

$$\sup_{\theta, \theta' \in \Theta : \mathcal{S}(\theta, \theta') \leq \epsilon} \sup_{\gamma \in S^p} \frac{\gamma^T V(\theta) \gamma}{\gamma^T V(\theta^0) \gamma} \leq \nu_1.$$
Now we are prepared to state the main result which gives some sufficient condition on the penalty function $\text{pen}(\theta)$ ensuring the desired penalized exponential bound. It is a specification of a more general result from Theorem 5.5 in Section 5.

Here and in what follows $\omega_p$ (resp. $Q_p$) denotes the volume (resp. the entropy number) of the unit ball in $\mathbb{R}^p$.

**Theorem 2.1.** Suppose that the conditions $(E)$ is fulfilled and $(ED)$ holds with some $\lambda^*$ and a matrix function $V(\theta)$ which fulfills $(V)$ for $\epsilon > 0$ and $\nu_1 \geq 1$. If for some $\varrho \in (0, 1)$ with $\varrho \epsilon/(1 - \varrho) \leq \lambda^*$, the penalty function $\text{pen}(\theta)$ fulfills

$$H_\epsilon(\varrho) \stackrel{\text{def}}{=} \log \left\{ \omega_p^{-1} \epsilon^{-p} \int_{\Theta} \sqrt{\det(V(\theta))} \exp\{-\varrho \text{pen}(\theta)\} d\theta \right\} < \infty$$

with $\text{pen}(\theta) = \inf_{\theta \in B(\epsilon, \theta_0)} \text{pen}(\theta)$, then

$$E \exp\left\{ \sup_{\theta \in \Theta} [\mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0) - \text{pen}(\theta)] \right\} \leq \Omega(\varrho)$$

where

$$\log \Omega(\varrho) = \frac{2 \epsilon^2 \varrho^2}{1 - \varrho} + (1 - \varrho)Q_p + H_\epsilon(\varrho) + p \log(\nu_1).$$

2.1 **Penalty via the norm $\|\sqrt{V^*(\theta - \theta_0)}\|$**

The choice of the penalty function $\text{pen}(\theta)$ can be made more precise if $V(\theta) \leq V^*$ for a fixed matrix $V^*$ and all $\theta$. This section describes how the penalty function can be defined in terms of the norm $\|\sqrt{V^*(\theta - \theta_0)}\|$.

**Theorem 2.2.** Let the conditions $(E)$ and $(ED)$ be fulfilled and in addition $V(\theta) \leq V^*$ for some matrix $V^*$ for all $\theta \in \Theta$. Let $\varrho \in (0, 1)$ and $\epsilon > 0$ be fixed to ensure $\varrho \epsilon/(1 - \varrho) \leq \lambda^*$. Suppose that $\kappa(t)$ is a monotonously decreasing positive function on $[0, +\infty)$ satisfying

$$\mathcal{P}^* \stackrel{\text{def}}{=} \omega_p^{-1} \int_{\mathbb{R}^p} \kappa(\|z\|) dz = \int_0^\infty \kappa(t)t^{p-1} dt < \infty.$$

Define

$$\text{pen}(\theta) = -\varrho^{-1} \log \kappa(\epsilon^{-1}\|\sqrt{V^*(\theta - \theta_0)}\| + 1).$$

Then the assertion (2.3) holds with

$$\log \Omega(\varrho) = \frac{2 \epsilon^2 \varrho^2}{1 - \varrho} + (1 - \varrho)Q_p + \log(\mathcal{P}^*).$$

**Proof.** This result is a straightforward corollary of Theorem 2.1 applied with $V(\theta) \equiv V^*$ and thus, condition $(V)$ is fulfilled with $\nu_1 = 1$. \qed
Here two natural ways of defining the penalty function \( \text{pen}(\theta) \): quadratic or logarithmic in \( \| \sqrt{V^*}(\theta - \theta_0) \| \). The functions \( \kappa(\cdot) \) and the corresponding \( P^* \)-values are:

\[
\kappa_1(t) = e^{-\delta_1(t-1)^2}, \quad P^*_1 = 1 + \omega_p^{-1}(\pi/\delta_1)^{p/2}, \\
\kappa_2(t) = (t + 1)^{-p-\delta_2}, \quad P^*_2 = \frac{p}{\delta_2},
\]

where \( \delta_1, \delta_2 > 0 \) are some constant and \([a]_+ = \max\{a, 0\}\). The corresponding penalties read as:

\[
\text{pen}_1(\theta) = \rho^{-1}\delta_1 \epsilon^{-2}\| \sqrt{V^*}(\theta - \theta_0) \|^2.
\]

\[
\text{pen}_2(\theta) = -\rho^{-1}(p + \delta_2) \log (\epsilon^{-1}\| \sqrt{V^*}(\theta - \theta_0) \| + 2).
\]

### 2.2 Some corollaries

Theorem 2.1 claims that the value \( L(\theta, \theta_0) + M(\theta, \theta_0) - \text{pen}(\theta) \) is uniformly in \( \theta \in \Theta \) stochastically bounded. In particular, one can plug the estimate \( \tilde{\theta} \) in place of \( \theta \):

\[
E \exp \left\{ \rho \left[ L(\tilde{\theta}, \theta_0) + M(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta}) \right] \right\} \leq \Omega(\rho). \tag{2.8}
\]

Below we present some corollaries of this result.

#### 2.2.1 Concentration properties of the estimator \( \tilde{\theta} \)

Define for every subset \( A \) of the parameter set \( \Theta \) the value

\[
\zeta(A) \overset{\text{def}}{=} \inf_{\theta \notin A} \{ M(\theta, \theta_0) - \text{pen}(\theta) \}. \tag{2.9}
\]

The next result shows that the estimator \( \tilde{\theta} \) deviates out of the set \( A \) with an exponentially small probability of order \( \exp\{-\zeta(\bar{A})\} \).

**Corollary 2.3.** Suppose (2.8). Then for any set \( A \subset \Theta \)

\[
P(\tilde{\theta} \notin A) \leq \Omega(\rho)e^{-\zeta(\bar{A})}.
\]

**Proof.** If \( \tilde{\theta} \notin A \), then \( M(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta}) \geq \zeta(A) \). As \( L(\tilde{\theta}, \theta_0) \geq 0 \), it follows

\[
\Omega(\rho) \geq E \exp \left\{ \rho \left[ L(\tilde{\theta}, \theta_0) + M(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta}) \right] \right\} \\
\geq E \exp \left\{ \rho \left[ M(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta}) \right] \right\} \geq e^{\zeta(A)} P(\tilde{\theta} \notin A)
\]

as required. \( \square \)

Two particular choices of the set \( A \) can be mentioned:

\[
A = A(r, \theta_0) = \{ \theta : M(\theta, \theta_0) \leq r \},
\]

\[
A = A'(r, \theta_0) = \{ \theta : M(\theta, \theta_0) - \text{pen}(\theta) \leq r \},
\]
For the set $A'(r, \theta_0)$, Corollary 2.3 yields

$$P(\tilde{\theta} \not\in A'(r, \theta_0)) = P(\mathcal{M}(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta}) \geq r) \leq \Omega(g)e^{-\varrho r}.$$ 

For the set $A(r, \theta_0)$, define additionally $b(r)$ as the minimal value for which

$$\mathcal{M}(\theta, \theta_0) - \text{pen}(\theta) \geq r - b(r), \quad \theta \not\in A(r, \theta_0),$$

or, equivalently,

$$b(r) = \sup_{\theta \not\in A(r, \theta_0)} \{r + \text{pen}(\theta) - \mathcal{M}(\theta, \theta_0)\}. \tag{2.10}$$

**Corollary 2.4.** Suppose (2.8). Then for any $r > 0$

$$P(\tilde{\theta} \not\in A(r, \theta_0)) = P(\mathcal{M}(\tilde{\theta}, \theta_0) \geq r) \leq \Omega(g)e^{-\varrho[r-b(r)]}.$$ 

In typical situations the value $\mathcal{M}(\theta, \theta_0)$ is nearly proportional to the sample size $n$ and is nearly quadratic in $\theta - \theta_0$ so that for a fixed $r$ the set $A(r, \theta_0)$ corresponds to a root-$n$ neighborhood of the point $\theta_0$. See below in Section 2.4 for a precise formulation.

### 2.2.2 Confidence sets based on $L(\tilde{\theta}, \theta)$

Next we discuss how the exponential bound can be used for establishing some risk bounds and for constructing the confidence sets for the target $\theta_0$ based on the maximized value $L(\tilde{\theta}, \theta)$. The inequality (2.8) claims that $L(\tilde{\theta}, \theta_0)$ is stochastically bounded with finite exponential moments. This implies boundness of the polynomial moments.

Define

$$b \overset{\text{def}}{=} b(0) = \sup_{\theta} \{\text{pen}(\theta) - \mathcal{M}(\theta, \theta_0)\}.$$ \tag{2.11}

**Corollary 2.5.** Suppose (2.8) and let $b$ from (2.11) be finite. Then

$$E \exp\{gL(\tilde{\theta}, \theta_0)\} \leq e^{gb}\Omega(g).$$

**Proof.** Observe that

$$E \exp\{g\mathcal{L}(\tilde{\theta}, \theta_0)\} \leq e^{gb}E \exp\{g[\mathcal{L}(\tilde{\theta}, \theta_0) + \mathcal{M}(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta})]\} \leq e^{gb}\Omega(g)$$

as required.

By the same reasons, one can construct confidence sets based on the (quasi) likelihood process. Define

$$\mathcal{E}(\bar{\zeta}) = \{\theta \in \Theta : L(\tilde{\theta}, \theta) \leq \bar{\zeta}\}.$$

The bound for $L(\tilde{\theta}, \theta_0)$ ensures that the target $\theta_0$ belongs to this set with a high probability provided that $\bar{\zeta}$ is large enough. The next result claims that $\mathcal{E}(\bar{\zeta})$ does not cover the true value $\theta_0$ with a probability which decreases exponentially in $\bar{\zeta}$.
Corollary 2.6. Suppose (2.8). For any $\zeta > 0$

$$P(\theta_0 \notin \mathcal{E}(\zeta)) \leq \Omega(\phi) \exp\{-g_3 + \phi b\}.$$  

Proof. The bound (2.8) implies for the event $\{\theta_0 \notin \mathcal{E}(\zeta)\} = \{\mathcal{L}(\tilde{\theta}, \theta_0) > \zeta\}$

$$P\{\theta_0 \notin \mathcal{E}(\zeta)\} \leq P\{\phi[\mathcal{L}(\tilde{\theta}, \theta_0) + \mathcal{M}(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta})] > g_3 - \phi b\}$$

$$\leq \exp\{-g_3 + \phi b\} \mathcal{E}\exp\{\phi[\mathcal{L}(\tilde{\theta}, \theta_0) + \mathcal{M}(\tilde{\theta}, \theta_0) - \text{pen}(\tilde{\theta})]\} \leq \Omega(\phi) \exp\{-g_3 + \phi b\}$$

as required. \hfill \square

2.3 Identifiability condition

Until this point no any identifiability condition on the model has been used, that is, the presented results apply even for a very poor parametrization. Actually, a particular parametrization of the parameter set plays no role as long as the value of maximum is considered. If we want to derive any quantitative result on the point of maximum $\tilde{\theta}$, then the parametrization matters and an identifiability condition is really necessary. Here we follow the usual path by applying the quadratic lower bound for the rate function $\mathcal{M}(\theta, \theta_0)$ in a vicinity of the point $\theta_0$. Suppose that the rate function $\mathcal{M}(\theta, \theta_0) = -\log \mathcal{E}\exp\{\mathcal{L}(\theta, \theta_0)\}$ is two times continuously differentiable in $\theta$. Obviously $\mathcal{M}(\theta_0, \theta_0) = 0$ and simple algebra yields for the gradient $\nabla \mathcal{M}(\theta, \theta_0) = d\mathcal{M}(\theta, \theta_0)/d\theta$:

$$\nabla \mathcal{M}(\theta, \theta_0)|_{\theta = \theta_0} = -\mathcal{E}\nabla \mathcal{L}(\theta)|_{\theta = \theta_0} = -\mathcal{E}\mathcal{L}(\theta_0) = 0$$

because $\theta_0$ is the point of maximum of $\mathcal{E}\mathcal{L}(\theta)$. The Taylor expansion of the second order in a vicinity of $\theta_0$ yields for all $\theta$ close to $\theta_0$ the following approximation:

$$\mathcal{M}(\theta, \theta_0) \approx (\theta - \theta_0)^\top I(\theta_0)(\theta - \theta_0)/2$$

with the matrix $I(\theta_0) = \mathcal{E}\nabla^2 \mathcal{M}(\theta, \theta_0)|_{\theta = \theta_0}$. So, one can expect that the rate function $\mathcal{M}(\theta, \theta_0)$ is nearly quadratic in $\theta - \theta_0$ in a neighborhood of the point $\theta_0$.

Corollary 2.7. Let (2.8) hold. Suppose that for some positive symmetric matrix $D$ and some $\tau > 0$, the function $\mathcal{M}(\theta, \theta_0)$ fulfills

$$\mathcal{M}(\theta, \theta_0) \geq (\theta - \theta_0)^\top D^2(\theta - \theta_0), \quad \theta \in \mathcal{A}(\tau, \theta_0),$$

(2.12)

Then for any $\zeta \leq \tau$

$$P(\|D(\tilde{\theta} - \theta_0)\|^2 > \zeta) \leq \Omega(\phi)e^{-\phi[3 - b(\phi)]}.$$
Proof. It is obvious that

\[ \{ \|D(\tilde{\theta} - \theta_0)\|^2 > \delta \} \subseteq \{ \|D(\tilde{\theta} - \theta_0)\|^2 > \delta, \tilde{\theta} \in A(r, \theta_0) \} \cup \{ \tilde{\theta} \notin A(r, \theta_0) \} \]
\[ \subseteq \{ \mathcal{M}(\tilde{\theta}, \theta_0) > \delta, \tilde{\theta} \in A(r, \theta_0) \} \cup \{ \mathcal{M}(\tilde{\theta}, \theta_0) > \delta \} \]
\[ = \{ \mathcal{M}(\tilde{\theta}, \theta_0) > \delta \} \]

and the result follows from Corollary 2.4.

In the next theorem we assume the lower bound (2.12) to be fulfilled on the whole parameter set \( \Theta \). The general case can be reduced to this one by using once again the concentration property of Corollary 2.4.

**Theorem 2.8.** Suppose \((E), (ED)\) with \( V(\theta) \leq V^* \) for a matrix \( V^* \). Let also for some \( a > 0 \)

\[ \mathcal{M}(\theta, \theta_0) \geq a^2(\theta - \theta_0)^\top V^*(\theta - \theta_0), \quad \theta \in \Theta. \tag{2.13} \]

Fix some \( a_1 \leq a \) and define \( \text{pen}(\theta) \) by

\[ \text{pen}(\theta) = a_1^2(\theta - \theta_0)^\top V^*(\theta - \theta_0). \tag{2.14} \]

Then with \( s = 1 - a_1^2/a^2 \) it holds

\[ \Omega(\varrho, s) \overset{\text{def}}{=} \log E \exp \left\{ \varrho \sup_{\theta} [\mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0) - \text{pen}(\theta)] \right\} \]
\[ \leq 2\varrho + (1 - \varrho)Q_p + \log \left( 1 + \frac{\omega_p^{-1}p^{p/2}}{(1 - \varrho)^{p/2}a_1^p} \right) \]
\[ \leq pC(\varrho) + p\log \left( |a^2(1 - s)(1 - \varrho)|^{-1/2} \right) \tag{2.15} \]

for some fixed constant \( C(\varrho) \). In addition, \( b(\tau) \) from (2.10) fulfills \( b(\tau) = 0 \) for all \( \tau \geq 0 \) yielding for any \( \zeta > 0 \) the concentration property and confidence bound:

\[ P(\tilde{\theta} \notin \mathcal{A}(\delta, \theta_0)) \leq \Omega(\varrho, s)e^{-\varrho\delta}, \quad \mathcal{A}(\delta, \theta_0) = \{ \theta : \mathcal{M}(\theta, \theta_0) \leq \delta \}, \]
\[ P(\theta_0 \notin \mathcal{E}(\delta)) \leq \Omega(\varrho, 0)e^{-\varrho\delta}, \quad \mathcal{E}(\delta) = \{ \theta : \mathcal{L}(\theta, \theta_0) \leq \delta \}. \]

Proof. We apply Theorem 2.2 with

\[ \varphi(t) = \exp \left\{ -(1 - \varrho)a_1^2(t - 1)^2 \right\} \]

leading for \( c^2 = (1 - \varrho)/\varrho \) and \( t = \epsilon^{-1}\|\sqrt{V^*}(\theta - \theta_0)\| \) to the formula (2.14) for \( \text{pen}(\theta) \).

By simple algebra

\[ \mathfrak{P}^* = \omega_p^{-1} \int_{\mathbb{R}^p} \varphi(\|\theta\|)d\theta = 1 + \omega_p^{-1} \frac{\pi^{p/2}}{(1 - \varrho)^{p/2}a_1^p}. \]
A penalized exponential risk bound in parametric estimation

cf. the bound (2.7) for $P^*$ with $\delta_1 = (1 - \varrho)a_1^2$. This implies the bound (2.15) for the
$\Omega(\varrho)$ because $p^{-1}Q_p$ and $p^{-1}\log \omega_p$ are bounded by some fixed constants.

The inequality (2.13) ensures for $r = M(\theta, \theta_0)$ that $\text{pen}(\theta) \leq a_1^2/a^2r$, i.e. $b(r) \leq a_1^2/a^2r$ and $b = b(0) = 0$. Finally, the concentration and coverage bounds follow from
Corollaries 2.4 and 2.6.

Remark 2.1. If the quadratic lower bound (2.13) is only fulfilled for $\theta$ from an elliptic
neighborhood $B(r, \theta_0) = \{\theta : \|\sqrt{V^*}(\theta - \theta_0)\| \leq r\}$ of the point $\theta_0$ with a sufficiently
large $r$, then it is reasonable to redefine the penalty function using the hybrid proposal:

$$\text{pen}(\theta) = \begin{cases} 
    a_1^2\|\sqrt{V^*}(\theta - \theta_0)\|^2, & \theta \in B(r, \theta_0), \\
    \varrho^{-1}(p + 1)(\|\sqrt{V^*}(\theta - \theta_0)\| + 2), & \theta \notin B(r, \theta_0).
\end{cases}$$

Then the bound (2.15) still applies with the obvious correction of the value $\Omega(\varrho,s)$. However, the values $b$ and $b(r)$ from (2.10) entering in our risk bounds have to be corrected depending on the behavior of the rate function $M(\theta, \theta_0)$ for $\theta \notin B'(r, \theta_0)$.

2.4 Discussion

This section collects some comments about the presented exponential bound.

Bounds for polynomial loss

Our concentration result is stated in terms if the rate function $M(\theta, \theta_0)$. Note that the
bounds (2.15) and (2.13) imply the usual result about the quadratic loss $\|\sqrt{V^*}(\tilde{\theta} - \theta_0)\|^2$:

$$P(\|\alpha\sqrt{V^*}(\tilde{\theta} - \theta_0)\|^2 > 3) \leq \Omega(\varrho,s)e^{-\Theta_3}.$$  

Note however, that the result (2.15) in terms of the rate function $M(\theta, \theta_0)$ is more
accurate because the lower bound (2.13) can be very rough. The bound (2.13) as well
as the bound $V(\theta) \leq V^*$ are only used to evaluate the constants in the exponential
risk bound. Moreover, if $\varrho$ or $s$ approaches one, the leading term in the risk bound is
$p\log((1-s)(1-\varrho)^{-1/2})$ which does not depend on $a$ or $V^*$.

Coverage probability and risk bounds

The result of Corollary 2.5 justifies the use of confidence set $E(3) = \{\theta : L(\tilde{\theta}, \theta) \leq 3\}$. However, the bound for the coverage probability given by this result is quite rough and
cannot be used for practical purposes. One has to apply one or another resampling
scheme to fix a proper value $3(\alpha)$ providing the prescribed coverage probability $1 - \alpha$.

The same remark applies to the result of Corollary 2.7. All these bounds are deduced
from rather rough exponential inequalities and constants shown there are not optimal.
However, the concentration property enables us to apply the classical one-step improvement technique to build a new estimate which achieves the asymptotic efficiency bound.

**Root-n consistency**

Suppose that there exists a constant $n$ (usually this constant means the sample size) such that the functions

$$v(\theta) \overset{\text{def}}{=} n^{-1}V(\theta), \quad m(\theta, \theta_0) \overset{\text{def}}{=} n^{-1}\mathfrak{M}(\theta, \theta_0)$$

are continuous and bounded on every compact set by constants which only depend on this set. In addition we assume similarly to (2.12) that for some fixed symmetric positive matrix $D_1$ and some $r > 0$, it holds in the vicinity $A(r, \theta_0)$ of the point $\theta_0$:

$$m(\theta, \theta_0) \geq (\theta - \theta_0)^\top D_1^2 (\theta - \theta_0), \quad v(\theta) \leq a^{-2}D_1^2.$$  \hspace{1cm} (2.16)

Then $\mathfrak{M}(\theta, \theta_0) \geq n(\theta - \theta_0)^\top D_1^2 (\theta - \theta_0)$ and the elliptic set $A^*(r, \theta_0) \overset{\text{def}}{=} \{\theta : (\theta - \theta_0)^\top D_1^2 (\theta - \theta_0) \leq r/n\}$ is a root-n neighborhood of the point $\theta_0$. By Theorem 2.8 the estimate $\tilde{\theta}$ deviates from this neighborhood with probability which decreases exponentially with $r$:

$$P\left(\|D_1(\tilde{\theta} - \theta_0)\|^2 > r/n\right) \leq \Omega(q, s)e^{-\rho r}.$$

**Local approximation**

The standard asymptotic theory of parameter estimation heavily uses the idea of local approximation: the considered (quasi) log likelihood is approximated by the log-likelihood of another simpler model in the vicinity of the true point yielding the local asymptotic equivalence of the original and the approximating model. The local asymptotic normality (LAN) condition is the most popular example of this approach; see Ibragimov and Khas’minskij (1981), Ch. 2, for more details. A combination of this idea with the concentration property of Corollary 2.4 can be used to derive sharp asymptotic risk bounds for the estimate $\tilde{\theta}$; see again Ibragimov and Khas’minskij (1981), Ch. 3. Similarly one can derive non asymptotic risk in the framework of this paper. However, a precise formulation of the related results is to be given elsewhere.

**Large and moderate deviation**

The obtained results can be used to derive large and moderate deviations for the estimate $\tilde{\theta}$; cf. Jensen and Wood (1998), Sieders and Dzhaparidze (1987). Particularly, the deviation result from Corollary 2.4 can be used to study the efficiency of the estimate $\tilde{\theta}$ in the Bahadur sense; see e.g. Arcones (2006) and reference therein.
3 Estimation in a generalized linear model

In this section we illustrate the general results of Sections 2 and 2.2 on the problem of estimating the parameter vector in the so-called generalized linear model. Let \( \mathcal{P} \) be an exponential family with the canonical parametrization (EFC) which means that the corresponding log-likelihood function can be written in the form

\[
\ell(y, v) = yv - d(v) + \ell(y)
\]

where \( d(\cdot) \) is a given convex function; see Green and Silverman (1994). The term \( \ell(y) \) is unimportant and it cancels in the log-likelihood ratio.

Let \( Y = (Y_1, \ldots, Y_n) \) be an observed sample. A generalized linear assumption means that the \( Y_i \)'s are independent, the distribution of every \( Y_i \) belongs to \( \mathcal{P} \) and the corresponding parameter linearly depends on given feature vectors \( \Psi_i \):

\[
Y_i \sim P_{\Psi_i^\top \theta}
\]

(3.1)

To be more specific we consider a deterministic explanatory variables \( \Psi_1, \ldots, \Psi_n \). The case of a random design can be considered in the same way.

The parametric assumption (3.1) leads to the log-likelihood \( L(\theta) = \sum_i \ell(Y_i, \Psi_i^\top \theta) \):

\[
L(\theta) = \sum_i \ell(Y_i, \Psi_i^\top \theta) = \sum_i \{ Y_i \Psi_i^\top \theta - d(\Psi_i^\top \theta) + \ell(Y_i) \}.
\]

(3.2)

Asymptotic properties of the MLE \( \tilde{\theta} = \arg\max_{\theta} L(\theta) \) are well studied. We refer to Fahrmeir and Kaufmann (1985), Lang (1996), Chen et al. (1999) and the book McCullagh and Nelder (1989) for further references. The results claim asymptotic consistency, normality and efficiency of the estimate \( \tilde{\theta} \).

Our approach is a bit different because we do not assume that the underlying model follow (3.1). The observations \( Y_i \) are independent, otherwise any particular structure is allowed. In particular, the distribution of every \( Y_i \) does not necessarily belong to \( \mathcal{P} \). The considered problem is the problem of the best parametric approximation of the data distribution \( P \) by the GLM’s of the form \( \prod_i P_{\Psi_i^\top \theta} \).

Example 3.1. [Mean regression] The least squares estimate \( \tilde{\theta} \) in the classical mean regression minimizes the sum of squared residuals:

\[
\tilde{\theta} = \arg\min_{\theta} \sum_i (Y_i - \Psi_i^\top \theta)^2.
\]

This estimate can be viewed as the quasi MLE for to the Gaussian homogeneous errors. However, many of its properties continue to hold even if the errors are not i.i.d. Gaussian. What we only need is the existence of exponential moments of the errors.
Example 3.2. [Poisson regression] Let $Y_i$ be some nonnegative integers, observed at “locations” $X_i$. Such data often appear in digital imaging, positron emission tomography, queueing and traffic theory, and many others. A natural way of modeling such data is to assume that every $Y_i$ is Poissonian with the parameter which depends on the locations $X_i$ through the regression vector $\Psi_i$. A generalized linear model assumes that the canonical parameter of the underlying Poisson distribution of $Y_i$ linearly depends on the vector $\Psi_i$ leading to the (quasi) MLE

$$\tilde{\theta} = \operatorname*{argmax}_\theta \sum_{i=1}^n \{ Y_i \Psi_i^\top \theta - \exp(\Psi_i^\top \theta) \}.$$\

For our further analysis we only require that every $Y_i$ has a bounded exponential moment, see below the condition (3.5) for a precise formulation.

In the general situation, for some $\mu > 0$ which will be fixed later, define

$$L(\theta) \overset{\text{def}}{=} \mu \sum_i \{ Y_i \Psi_i^\top \theta - \dot{d}(\Psi_i^\top \theta) \},$$\

where $b_i = EY_i$. This yields

$$\nabla L(\theta) \bigg|_{\theta = \theta_0} = \mu \sum_i \{ b_i - \dot{d}(\Psi_i^\top \theta_0) \} \Psi_i = 0,$$\

(3.3)

Next, for every $\theta$

$$\zeta(\theta) \overset{\text{def}}{=} L(\theta) - E\zeta(\theta) = \mu \sum_i (Y_i - b_i) \Psi_i^\top \theta,$$

$$\nabla \zeta(\theta) = \nabla \zeta = \mu \sum_i (Y_i - b_i) \Psi_i,$$

Let there exist a positive value $\lambda_1^+$, and for every $i \leq n$ the value $n_i > 0$ such that

$$\log E \exp \left\{ 2 \lambda \frac{Y_i - b_i}{n_i} \right\} \leq 2 \lambda^2, \quad |\lambda| \leq \lambda_1^+.$$\

(3.5)

In the case of Gaussian errors, one can take $n_i = s_i \overset{\text{def}}{=} E^{1/2}(Y_i - b_i)^2$. Define

$$V_1 \overset{\text{def}}{=} \sum_i n_i^2 \Psi_i \Psi_i^\top, \quad V = \mu^2 V_1.$$\

(3.6)

The matrix $V$ is a symmetric and non-negative. Denote $c_i(\gamma) \overset{\text{def}}{=} \gamma^\top \Psi_i n_i (\gamma^\top V_1 \gamma)^{-1/2}$ for any $\gamma \in S^p$. By definition, $|c_i(\gamma)| \leq 1$, however, usually $|c_i(\gamma)|$ is much smaller, of order $1/\sqrt{n_i}$. 
Lemma 3.1. Suppose (3.5). If for some $\mu > 0$, it holds $\mu|\hat{\psi}_i^T(\theta - \theta_0)|n_i \leq \lambda_i^*$ for all $i$ and all $\theta \in \Theta$, then (E) is satisfied. Moreover, if $\lambda$ is such that $|\lambda c_i(\gamma)| \leq \lambda_i^*$ for all $i$ and any $\gamma \in S^d$, then (ED) holds with this $\lambda$ and $V(\cdot) \equiv V$ from (3.6).

Proof. Let $|\lambda c_i(\gamma)| \leq \lambda_i^*$ for all $i$. Independence of the $Y_i$'s and (3.5) imply in view of $\sum_i c_i^2(\gamma) = 1$ that

$$\log E \exp \left\{ 2\lambda \frac{\gamma^T \nabla \zeta}{(\gamma^T V \gamma)^{1/2}} - 2\lambda^2 \right\} = \sum_i \log E \exp \left\{ 2\lambda c_i(\gamma) \frac{Y_i - b_i}{n_i} \right\} - 2\lambda^2 \leq 0.$$ 

This implies (ED) with $V(\theta) \equiv V$.$\Box$

An important feature of the GLM is that the gradient $\nabla \zeta(\theta)$ and hence, the corresponding matrix $V(\theta)$ do not depend on $\theta$. This automatically yields the condition $(V)$ with $\nu_1 = 1$ and any $\epsilon > 0$.

Now we consider the rate function $\mathcal{M}(\theta, \theta_0) \overset{\text{def}}{=} -\log E \exp\{\mathcal{L}(\theta, \theta_0)\}$. It holds

$$\mathcal{M}(\theta, \theta_0) = \mu \sum_i \left\{ d(\hat{\psi}_i^T \theta) - d(\hat{\psi}_i^T \theta_0) - \hat{\psi}_i^T (\theta - \theta_0)b_i \right\} - \sum_i \log E \exp \{ \mu \hat{\psi}_i^T (\theta - \theta_0) (Y_i - b_i) \}. \quad (3.7)$$

This function is smooth in $\theta$ and by (3.4)

$$\nabla \mathcal{M}(\theta, \theta_0)|_{\theta = \theta_0} = \mu \sum_i \{ d(\hat{\psi}_i^T \theta_0) - b_i \} \hat{\psi}_i = 0.$$ 

Moreover,

$$\nabla^2 \mathcal{M}(\theta, \theta_0) = \mu \sum_i \hat{d}(\hat{\psi}_i^T \theta) \hat{\psi}_i \hat{\psi}_i^T - \mu^2 \sum_i s_i^2(\theta) \hat{\psi}_i \hat{\psi}_i^T$$

where with $\xi_i = Y_i - b_i$ and $u = \theta - \theta_0$

$$s_i(\theta) \overset{\text{def}}{=} \left( E e^{\mu \hat{\psi}_i^T u \xi_i} \right)^{-2} \left\{ E e^{\mu \hat{\psi}_i^T u \xi_i} E e^{\mu \hat{\psi}_i^T u \xi_i} - \left( E e^{\mu \hat{\psi}_i^T u \xi_i} \right)^2 \right\}$$

Particularly, $s_i^2(\theta_0) = s_i^2 = \text{Var} Y_i$. If $u = \theta - \theta_0$ is small then $s_i(\theta)$ is close to $s_i$. The “identifiability” condition which would provide the concentration property from Theorem 2.7 means that for some fixed positive constants $\mu$ and $a$

$$\mathcal{M}(\theta, \theta_0) \geq a^2 (\theta - \theta_0)^T V(\theta - \theta_0) \quad (3.8)$$

at least for all $\theta$ from a vicinity of $\theta_0$. The next lemma presents some simple sufficient conditions.
Lemma 3.2. Let for a subset $\Theta_0 \subseteq \Theta$ hold:

$$\frac{1}{2} \sum_{i} \bar{d}(\psi_i^\top \theta) \psi_i \psi_i^\top \geq 2a_1 V_1, \quad \frac{1}{2} \sum_{i} s_i^2(\theta) \psi_i \psi_i^\top \leq a^2 V_1, \quad \theta \in \Theta_0,$$

for some positive constants $a_1 = a_1(\Theta_0), a = a(\Theta_0)$. Then (3.8) is fulfilled with any $\mu$ satisfying $\mu \leq a_1 / a^2$.

Proof. With $\mu \leq a_1 / a^2$ for any $\theta \in \Theta_0$

$$\frac{1}{2} \nabla^2 M(\theta, \theta_0) \geq 2\mu a_1 V_1 - a^2 \mu^2 V_1 = (2\mu^{-1} a_1 - a^2)V \geq a^2 V.$$

Now the result follows by the second order Taylor expansion of $M(\theta, \theta_0)$ at $\theta = \theta_0$. $lacksquare$

Now we are ready to state the main result for the GLM estimation problem which is a specification of Theorems 2.2 and 2.8.

Theorem 3.3. Suppose that the $Y_i$'s are independent and the point $\theta_0$ is defined by (3.3). Let there exist $\lambda^* > 0$ and the values $n_i$ such that (3.5) is fulfilled. Let, additionally, for some $\mu > 0$ and all $\theta \in \Theta$

$$\mu n_i |\psi_i^\top (\theta - \theta_0)| \leq \lambda^*_1, \quad i \leq n,$$

and with the matrices $V, V_1$ from (3.6) and some $\lambda^* > 0$:

$$\sup_{\gamma \in S \lambda^*} \lambda^* n_i |\gamma^\top \psi_i| \leq \lambda^*_1, \quad i \leq n.$$

Fix any $\rho < 1$ and $\epsilon > 0$ with $\rho \epsilon / (1 - \rho) \leq \lambda^*$. Define $\Psi^*$ and $\text{pen}(\theta)$ by (2.5) and (2.6). Then

$$\log E \exp \left\{ \sup_{\theta \in \Theta} \mathcal{L}(\theta, \theta_0) + M(\theta, \theta_0) - \text{pen}(\theta) \right\} \leq 2\epsilon^2 \rho^2 / (1 - \rho) + (1 - \rho) Q_p + \log(\Psi^*).$$

Let also there exist $a > 0$ such that the function $M(\theta, \theta_0)$ from (3.7) fulfills

$$M(\theta, \theta_0) \geq a^2 (\theta - \theta_0)^\top V(\theta - \theta_0).$$

Then for $s = 1 - a^2 / a^2$

$$P(M(\tilde{\theta}, \theta_0) > z) \leq \Omega(\rho, s)e^{-\epsilon s^3},$$

$$\log \Omega(\rho, s) \leq pC(\rho) + p \log(|a^2(1 - s)(1 - \rho)|^{-1/2})$$

and for the confidence set $\mathcal{E}(z) = \{ \theta : \mathcal{L}(\tilde{\theta}, \theta) \leq z \}$ holds with $\Omega(\rho) = \Omega(\rho, 0)$

$$P(\theta_0 \notin \mathcal{E}(z)) \leq \Omega(\rho)e^{-\epsilon s}.$$
4 Single-index regression

In this section we illustrate the general results of Sections 2 and 2.2 by the problem of estimating the index vector $\theta$ in the so called single-index regression model. Such models are frequently used in statistical modeling to overcome the “curse of dimensionality” problem, see Stone (1986).

Let $Y = (Y_1, \ldots, Y_n)$ be an observed sample. We assume that the $Y_i$’s are independent and the distribution of every $Y_i$ belongs to an exponential family $\mathcal{P}$ with canonical parametrization:

$$Y_i \sim P_{f_i},$$

where the underlying parameter $f_i$ can be different for each $i$. Regression analysis aims at explaining this parameter $f_i$ as a function of the explanatory vector $X_i \in \mathbb{R}^d$: $f_i = f(X_i)$ for some regression function $f(\cdot)$.

We again consider a deterministic design $X_1, \ldots, X_n$. The assumption $f_i = f(X_i)$ reduces the original problem to recovering the regression function $f(\cdot)$ from the observed data. However, in the case of a large $d$ this problem is too complex because of the design sparsity. This “curse of dimensionality” problem can be avoided by some dimensionality reduction assumption. Below we consider one possible assumption of this sort:

$$f(X_i) = g(x_i^\top \theta)$$

where $g(\cdot)$ is a univariate link function, while $\theta \in \mathbb{R}^d$ is an index vector. This assumption effectively means that the explanatory vector $X_i$ can be projected on the index $\theta$ and this projection can be used instead of the original vector without any information loss. Therefore, the primary goal in estimation of a single index model is in recovering the index vector $\theta$. There is a number of results in the literature about the quality of estimation of the index vector $\theta$. We mention Li (1991), Ichimura (1993), Hardle et al. (1993), Hristache et al. (2001), Xia et al. (2002), Climov et al. (2002), Delecroix et al. (2003), Yin et al. (2008) among many others.

Below we assume that the link function $g$ is given and it is sufficiently smooth. Note however, that the underlying model just follows (4.1). The considered problem is the problem of the best parametric approximation of the function $f(x)$ by single index function $g(x^\top \theta)$ with a fixed link function $g(\cdot)$. Such problem often occurs as an important building block in popular statistical procedures like logit regression, projection pursuit of neuronal networks. Note that for a linear link function $g(\cdot)$ we come back to generalized linear estimation.

The parametric assumption $f(X_i) = g(X_i^\top \theta)$ leads to the log-likelihood $L(\theta) =$
\[
\sum_i \ell(Y_i, g(X_i^\top \theta)) \quad \text{where} \quad \ell(y, v) = yv - d(v) + \ell(y)
\]
is the log-likelihood function for \( P \):

\[
L(\theta) = \sum_i \ell(Y_i, g(X_i^\top \theta)) = \sum_i \{ Y_i g(X_i^\top \theta) - d(g(X_i^\top \theta)) + \ell(Y_i) \}.
\] (4.3)

For some \( \mu > 0 \) whose value will be specified later, define

\[
\mathcal{L}(\theta) \overset{\text{def}}{=} \mu \sum_i \{ Y_i g(X_i^\top \theta) - d(g(X_i^\top \theta)) \}.
\]

We use the well known properties of the canonical exponential families: \( E_v Y = \dot{d}(v) \)
which implies

\[
E \mathcal{L}(\theta) = \mu \sum_i \{ \dot{d}(f_i) g(X_i^\top \theta) - d(g(X_i^\top \theta)) \},
\]

\[
\nabla E \mathcal{L}(\theta) = \mu \sum_i \{ \dot{d}(f_i) - \dot{d}(g(X_i^\top \theta)) \} \dot{g}(X_i^\top \theta) X_i.
\]

The target \( \theta_0 \) maximizes \( E \mathcal{L}(\theta) \):

\[
\theta_0 = \arg\max_{\theta} E \mathcal{L}(\theta) = \arg\max_{\theta} \sum_i \{ \dot{d}(f_i) g(X_i^\top \theta) - d(g(X_i^\top \theta)) \}.
\] (4.4)

This particularly yields

\[
\nabla E \mathcal{L}(\theta_0) = 0.
\]

Next, for every \( \theta \)

\[
\zeta(\theta) \overset{\text{def}}{=} L(\theta) - E \mathcal{L}(\theta) = \mu \sum_i \{ Y_i - \dot{d}(f_i) \} g(X_i^\top \theta),
\]

\[
\nabla \zeta(\theta) = \mu \sum_i \{ Y_i - \dot{d}(f_i) \} \dot{g}(X_i^\top \theta) X_i.
\]

It is easy to see that condition \( (E) \) is fulfilled if \( f_i + \mu \{ g(X_i^\top \theta) - g(X_i^\top \theta_0) \} \in \mathcal{U} \) for all \( i \) and all \( \theta \in \Theta \). Let \( n(v) \) be a function of \( v \) which ensures for some fixed \( \lambda^*_1 > 0 \) that

\[
\log E_v \exp \left\{ 2\lambda \frac{Y - \dot{d}(v)}{n(v)} \right\} \leq 2\lambda^2, \quad \lambda \leq \lambda^*_1.
\] (4.5)

Define

\[
V_1(\theta) \overset{\text{def}}{=} \sum_i n^2(f_i) |\dot{g}(X_i^\top \theta)|^2 X_i X_i^\top, \quad V(\theta) \overset{\text{def}}{=} \mu^2 V_1(\theta).
\] (4.6)

Then for any \( \gamma \in S^d \)

\[
E \exp \left\{ 2\lambda \frac{\gamma^\top \nabla \zeta(\theta)}{(\gamma^\top V(\theta) \gamma)^{1/2}} - 2\lambda^2 \right\} \leq 1
\]

provided that \( n(f_i)|\gamma^\top X_i| \leq \lambda^*_1 \) for all \( i \), which implies \((ED)\).
Now we consider the rate function $\mathcal{M}(\theta, \theta_0)$. As $\log E \exp \{\mu Y\} = d(v + \mu) - d(v)$, it holds

$$
\mathcal{M}(\theta, \theta_0) \overset{\text{def}}{=} -\log E \exp \{\mathcal{L}(\theta, \theta_0)\} = \sum_i \left\{ \tilde{d}(f_i) - d(f_i + \mu\delta_i(\theta)) + \mu d(g(X_i^\top \theta)) - \mu d(g(X_i^\top \theta_0)) \right\}
$$

with $\delta_i(\theta) \overset{\text{def}}{=} g(X_i^\top \theta) - g(X_i^\top \theta_0)$. For the gradient $\nabla \mathcal{M}(\theta, \theta_0)$ holds:

$$
\nabla \mathcal{M}(\theta, \theta_0) = \sum_i \left\{\tilde{d}(g(X_i^\top \theta)) - \tilde{d}(f_i + \mu\delta_i(\theta))\right\} \mu \tilde{g}(X_i^\top \theta) X_i.
$$

The equation $\nabla E \mathcal{L}(\theta_0) = 0$ implies $\nabla \mathcal{M}(\theta, \theta_0) |_{\theta = \theta_0} = 0$. Moreover,

$$
\nabla^2 \mathcal{M}(\theta, \theta_0) = \frac{\partial^2 \mathcal{M}(\theta, \theta_0)}{\partial \theta^2} = \sum_i \mu \left\{ \tilde{d}(g(X_i^\top \theta)) |\tilde{g}(X_i^\top \theta)|^2 + \mu \tilde{d}(g(X_i^\top \theta)) - \tilde{d}(f_i - \mu\delta_i(\theta)) \right\} \tilde{g}(X_i^\top \theta) X_i X_i^\top
$$

$$
- \mu^2 \sum_i \left\{ \tilde{d}(f_i + \mu\delta_i(\theta)) |\tilde{g}(X_i^\top \theta)|^2 \right\} X_i X_i^\top.
$$

The “identifiability” condition which would provide the concentration property of $\tilde{\theta}$ in an elliptic neighborhood of the point $\theta_0$ means that

$$
n^{-1} \sum_i \left\{ \tilde{d}(g(X_i^\top \theta)) |\tilde{g}(X_i^\top \theta)|^2 + \left[ \tilde{d}(g(X_i^\top \theta)) - \tilde{d}(f_i) \right] |\tilde{g}(X_i^\top \theta)| \right\} X_i X_i^\top \geq D_1^2
$$

for a positive matrix $D_1^2$. This condition ensures that with a proper choice of $\mu$, the value $\mathcal{M}(\theta, \theta_0)$ satisfies $\mathcal{M}(\theta, \theta_0) \geq C(\theta - \theta_0)^\top D_1^2 (\theta - \theta_0)$ for some $C = C(D_1) > 0$.

In the case of Gaussian regression $Y_i \sim N(f_i, \sigma^2)$, it holds $d(v) = v^2/(2\sigma^2)$, so that $\tilde{d}(v) = v/\sigma^2$ and $\tilde{d}(v) = \sigma^{-2}$. The identifiability condition reads now as

$$(n\sigma^2)^{-1} \sum_i \left\{ |\tilde{g}(X_i^\top \theta)|^2 + |g(X_i^\top \theta) - f_i| \tilde{g}(X_i^\top \theta) \right\} X_i X_i^\top \geq D_1^2.$$

**Theorem 4.1.** Suppose that $Y_i \sim P_{f_i} \in \mathcal{P}$ for some EFC $\mathcal{P}$. Let the point $\theta_0$ be defined by \((4.4)\) and $\bar{\theta} = \arg\max_{\theta} \mathcal{L}(\theta)$ be its estimate. Let also there exist $\lambda_1^* > 0$ and the function $n(v)$ such that \((4.5)\) is fulfilled. Let also for some $\mu^* > 0$

$$
f_i + \mu^* \{g(X_i^\top \theta) - g(X_i^\top \theta_0)\} \in \mathcal{U}, \quad i \leq n, \ \theta \in \Theta,
$$

and for some $\lambda^* > 0$ and the matrix $V_1(\theta)$ from \((4.6)\)

$$
\sup_{\gamma \in S^p} \frac{n(f_i)}{|\gamma^\top V_1(\theta)\gamma|^{1/2}} \lambda^* \leq \lambda_1^* \quad i \leq n, \ \theta \in \Theta.
$$
Then for any $\mu$ with $0 < \mu \leq \mu^*$, the conditions (E) and (ED) are fulfilled with $V(\theta)$ from (4.6). For any $\varrho < 1$ and $\epsilon > 0$ with $\varrho \epsilon / (1 - \varrho) \leq \lambda^*$, it holds

$$\log E \exp \left\{ \sup_{\theta \in \Theta} \left[ \mathcal{L}(\theta, \theta_0) + \mathcal{M}(\theta, \theta_0) - \text{pen}(\theta) \right] \right\} \leq 2 \epsilon^2 \varrho^2 / (1 - \varrho) + (1 - \varrho) Q_p + \log (\mathcal{P}^*),$$

where $\mathcal{P}^*$ and $\text{pen}(\theta)$ are defined by (2.5) and (2.6).

Let further there exist $a > 0$, and a matrix $V^*$ such that

$$\mathcal{M}(\theta, \theta_0) \geq a^2 (\theta - \theta_0)^\top D^2 (\theta - \theta_0), \quad V(\theta) \leq V^*, \quad \theta \in \Theta.$$  

Then for $s = 1 - a^2 / a^2$, it holds

$$P (\mathcal{M}(\tilde{\theta}, \theta_0) > \delta) \leq \Omega(\varrho, s) e^{-\delta s}$$

and for the confidence set $\mathcal{E}(\delta) = \{ \theta : \mathcal{L}(\tilde{\theta}, \theta) \leq \delta \}$ holds with $\mathcal{Q}(\varrho) = \mathcal{Q}(\varrho, 0)$

$$P (\theta_0 \not\in \mathcal{E}(\delta)) \leq \mathcal{Q}(\varrho) e^{-\delta s}.$$

5 A penalized exponential bound for a random field

Let $(\mathcal{Y}(\upsilon), \upsilon \in \Upsilon)$ be a random field on a probability space $(\Omega, \mathcal{F}, P)$, where $\Upsilon$ is a separable locally compact space. For any $\upsilon \in \Upsilon$ we assume the following exponential moment condition to be fulfilled:

$$(E) \quad \text{For every } \upsilon \in \Upsilon \quad E \exp \left\{ \mathcal{Y}(\upsilon) \right\} = 1.$$  

The aim of this section is to establish a similar exponential bound for a supremum of $\mathcal{Y}(\upsilon)$ over $\upsilon \in \Upsilon$. A trivial corollary of the condition (E) is that if the set $\Upsilon$ is finite with $N = \# \Upsilon$, then

$$E \exp \left\{ \sup_{\upsilon \in \Upsilon} \mathcal{Y}(\upsilon) \right\} \leq N.$$  

Unfortunately, in the general case the supremum of $\mathcal{Y}(\upsilon)$ over $\upsilon$ does not necessarily fulfill the condition of bounded exponential moments. We therefore, consider a penalized version of the process $\mathcal{Y}(\upsilon)$, that is, we try to bound the exponential moment of $\mathcal{Y}(\upsilon) - \text{pen}(\upsilon)$ for some penalty function $\text{pen}(\upsilon)$. The goal is to find a possibly minimal such function $\text{pen}(\upsilon)$ which provides

$$E \exp \left\{ \sup_{\upsilon \in \Upsilon} \left[ \mathcal{Y}(\upsilon) - \text{pen}(\upsilon) \right] \right\} \leq 1.$$
In the case of a finite set $\mathcal{Y}$, a natural candidate is $\text{pen}(\upsilon) = \log(\#\mathcal{Y})$. Below we show how this simple choice can be extended to the case of a general set $\mathcal{Y}$. There exists a number of results about a supremum of a centered random field which are heavily based on the theory of empirical processes. See e.g. the monographs van der Vaart and Wellner (1996), Van de Geer (2000), Massart (2007), and references therein. Our approach is a bit different. First the process $\bar{y}(\upsilon)$ does not need to be centered, instead we use the normalization $\mathbb{E} \exp\{\bar{y}(\upsilon)\} = 1$. Secondly we do not assume any particular structure of this process like independence of observations, so the methods of the empirical processes do not apply here. Finally, our analysis is focuses on the penalty function $\text{pen}(\cdot)$ rather then on the deviation probability of $\max_\upsilon \bar{y}(\upsilon)$.

5.1 A local bound

Define $\mathcal{M}(\upsilon) = \mathbb{E} \bar{y}(\upsilon)$, $\zeta(\upsilon) = \bar{y}(\upsilon) - \mathbb{E} \bar{y}(\upsilon)$, and denote $\zeta(\upsilon, \upsilon') = \zeta(\upsilon) - \zeta(\upsilon')$ for $\upsilon, \upsilon' \in \mathcal{Y}$. We assume a nonnegative symmetric function $\mathcal{D}(\upsilon, \upsilon')$ is given such that the following condition is fulfilled:

\[(\mathcal{E}) \quad \text{There exist numbers } \epsilon > 0 \text{ and } \lambda^* > 0, \text{ such that for any } \lambda \leq \lambda^* \]

\[
\sup_{\upsilon, \upsilon' \in \mathcal{Y} : \mathcal{D}(\upsilon, \upsilon') \leq \epsilon} \log \mathbb{E} \exp \left\{ 2\lambda \frac{\zeta(\upsilon, \upsilon')}{\mathcal{D}(\upsilon, \upsilon')} \right\} \leq 2\lambda^2.
\]

Let $\epsilon > 0$ be shown in condition $(\mathcal{E})$. Define for any point $\upsilon^0 \in \mathcal{Y}$ the “ball”

\[
\mathcal{B}(\epsilon, \upsilon^0) = \{ \upsilon : \mathcal{D}(\upsilon, \upsilon^0) \leq \epsilon \}.
\]

To state the result, we have to introduce the notion of local entropy. We say that a discrete set $\mathcal{D}(\epsilon, \mathcal{C})$ is an $\epsilon$-net in $\mathcal{C} \subseteq \mathcal{Y}$, if

\[
\mathcal{C} \subseteq \bigcup_{\upsilon^0 \in \mathcal{D}(\epsilon, \mathcal{C})} \mathcal{B}(\epsilon, \upsilon^0).
\]

(5.1)

By $N(\epsilon_0, \epsilon, \upsilon^0)$ for $\epsilon_0 \leq \epsilon$ we denote the local covering number defined as the minimal number of sets $\mathcal{B}(\epsilon_0, \cdot)$ required to cover $\mathcal{B}(\epsilon, \upsilon^0)$. With this covering number we associate the local entropy

\[
\mathcal{Q}(\epsilon, \upsilon^0) \overset{\text{def}}{=} \sum_{k=1}^{\infty} 2^{-k} \log N(2^{-k}\epsilon, \epsilon, \upsilon^0).
\]

Assume that $\upsilon^0 \in \mathcal{Y}$ is fixed. The following result controls the supremum in $\upsilon$ of the penalized process $\bar{y}(\upsilon) - \text{pen}(\upsilon)$ over the ball $\mathcal{B}(\epsilon, \upsilon^0)$.
Theorem 5.1. Assume (E) and (Ec). For any \( \rho \in (0, 1) \) with \( \rho/(1 - \rho) \leq \lambda^* \), any \( v^* \in Y \)

\[
\log E \exp \left\{ \sup_{v \in B(\epsilon, v^*)} \rho \left[ y(v) - \text{pen}(v) \right] \right\} \leq \frac{2\epsilon^2 \rho^2}{1 - \rho} + (1 - \rho) Q(\epsilon, v^*) - \rho \text{pen}(v^*)
\]

with

\[
\text{pen}(v^*) = \inf_{v \in B(\epsilon, v^*)} \text{pen}(v).
\]

Proof. We begin with some result which bounds the stochastic component of the process \( y(v) \) within the local ball \( B(\epsilon, v^*) \).

Lemma 5.2. Assume that \( \zeta(v) \) is a separable process satisfying condition (Ec). Then for any given \( v^* \in Y \), \( v^* \in B(\epsilon, v^*) \), and \( \lambda \leq \lambda^* \)

\[
\log E \exp \left\{ \frac{\lambda}{\epsilon} \sup_{v \in B(\epsilon, v^*)} \zeta(v, v^*) \right\} \leq Q(\epsilon, v^*) + 2\lambda^2.
\]

Proof. The proof is based on the standard chaining argument; see e.g. van der Vaart and Wellner (1996). Without loss of generality, we assume that \( Q(\epsilon, v^*) < \infty \). Then for any integer \( k \geq 0 \), there exists a \( 2^{-k}\epsilon \)-net \( D_k(\epsilon, v^*) \) in the local ball \( B(\epsilon, v^*) \) having the cardinality \( N(2^{-k}\epsilon, \epsilon, v^*) \). Using the nets \( D_k(\epsilon, v^*) \) with \( k = 1, \ldots, K - 1 \), one can construct a chain connecting an arbitrary point \( v \) in \( D_k(\epsilon, v^*) \) and \( v^* \). It means that one can find points \( v_k \in D_k(\epsilon, v^*) \), \( k = 1, \ldots, K - 1 \), such that \( D(v_k, v_{k-1}) \leq 2^{-k+1}\epsilon \) for \( k = 1, \ldots, K \). Here \( v_{K-1} \) means \( v \), and \( v_{0} \) means \( v^* \). Notice that \( v_k \) can be constructed recurrently: \( v_{k-1} = \tau_{k-1}(v_k) \), \( k = K, \ldots, 1 \), where

\[
\tau_{k-1}(v) = \arg \min_{v' \in D_{k-1}(\epsilon, v^*)} D(v', v').
\]

It obviously holds

\[
\zeta(v, v^*) = \sum_{k=1}^{K} \zeta(v_k, v_{k-1}).
\]

It holds for \( \zeta(v_k, v_{k-1}) = \zeta(v_k, v_{k-1})/D(v_k, v_{k-1}) \) that

\[
\zeta(v_k, v_{k-1}) = D(v_k, v_{k-1}) \zeta(v_k, v_{k-1}) = 2\epsilon c_k \zeta(v_k, v_{k-1})
\]

with \( c_k = D(v_k, v_{k-1})/(2 \epsilon) \leq 2^{-k} \). By condition (Ec) \( \log E \exp \left\{ 2\lambda \zeta(v_k, v_{k-1}) \right\} \leq 2\lambda^2 \).

Next,

\[
\sup_{v \in D_k(\epsilon, v^*)} \zeta(v, v^*) \leq \sum_{k=1}^{K} \sup_{v' \in D_k(\epsilon, v^*)} \zeta(v', \tau_{k-1}(v'))
\]

\[
\leq 2\epsilon \sum_{k=1}^{K} \sup_{v' \in D_k(\epsilon, v^*)} c_k \zeta(v', \tau_{k-1}(v')).
\]

(5.2)
Since $c_k \leq 2^{-k}$, the Hölder inequality and condition ($\mathcal{E})$ imply

$$\log \mathbb{E} \exp \left\{ \frac{\lambda}{\epsilon} \sup_{v \in \mathcal{D}_k(\epsilon, v^{\circ})} \zeta(v, v^*) \right\} \leq \log \mathbb{E} \exp \left\{ 2\lambda \sum_{k=1}^{K} \sup_{v' \in \mathcal{D}_k(\epsilon, v^{\circ})} c_k \xi(v', \tau_{k-1}(v')) \right\}$$

$$\leq \sum_{k=1}^{K} 2^{-k} \log \left[ \mathbb{E} \exp \left\{ \sup_{v' \in \mathcal{D}_k(\epsilon, v^{\circ})} 2^k c_k \times 2\lambda \xi(v', \tau_{k-1}(v')) \right\} \right]$$

$$\leq \sum_{k=1}^{K} 2^{-k} \log \left[ \sum_{v' \in \mathcal{D}_k(\epsilon, v^{\circ})} \mathbb{E} \exp \left\{ 2^k c_k \times 2\lambda \xi(v', \tau_{k-1}(v')) \right\} \right]$$

$$\leq \sum_{k=1}^{K} 2^{-k} \{ \log N(2^{-k} \epsilon, \epsilon, v^{\circ}) + 2\lambda^2 \}.$$ 

These inequalities and the separability of $\zeta(v, v^*)$ yield

$$\log \mathbb{E} \exp \left\{ \frac{\lambda}{\epsilon} \sup_{v \in \mathcal{B}(\epsilon, v^{\circ})} \zeta(v, v^*) \right\} = \lim_{K \to \infty} \log \mathbb{E} \exp \left\{ \frac{\lambda}{\epsilon} \sup_{v \in \mathcal{D}_k(\epsilon, v^{\circ})} \zeta(v, v^*) \right\}$$

$$\leq \sum_{k=1}^{\infty} 2^{-k} \{ 2\lambda^2 + \log N(2^{-k} \epsilon, \epsilon, v^{\circ}) \} \leq 2\lambda^2 + Q(\epsilon, v^{\circ})$$

which completes the proof of the lemma. \qed

Now define for a fixed a point $v^{\circ}$

$$v^* = \arg\min_{v \in \mathcal{B}(\epsilon, v^{\circ})} \{ M(v) + \text{pen}(v) \},$$

where $M(v) = -\mathbb{E}[y(v)]$. If there are many such points, then take any of them as $v^*$. Obviously

$$\sup_{v \in \mathcal{B}(\epsilon, v^{\circ})} \{ y(v) - \text{pen}(v) \} \leq y(v^*) - \text{pen}(v^*) + \sup_{v \in \mathcal{B}(\epsilon, v^{\circ})} \zeta(v, v^*).$$

Therefore, by the Hölder inequality and Lemma 5.2 with $\lambda = \epsilon \varrho/(1 - \varrho)$

$$\log \mathbb{E} \exp \left\{ \sup_{v \in \mathcal{B}(\epsilon, v^{\circ})} \varrho [y(v) - \text{pen}(v)] \right\}$$

$$\leq \varrho \log \mathbb{E} \exp \left\{ y(v^*) - \text{pen}(v^*) \right\} + (1 - \varrho) \log \mathbb{E} \exp \left\{ \frac{\varrho}{1 - \varrho} \sup_{v \in \mathcal{B}(\epsilon, v^{\circ})} \zeta(v, v^*) \right\}$$

$$\leq 2\epsilon^2 \varrho^2/(1 - \varrho) + (1 - \varrho) Q(\epsilon, v^{\circ}) - \varrho \text{pen}(v^*)$$

$$\leq 2\epsilon^2 \varrho^2/(1 - \varrho) + (1 - \varrho) Q(\epsilon, v^{\circ}) - \varrho \text{pen}_\epsilon(v^{\circ}).$$

which is the assertion of the theorem. \qed
5.2 A global exponential bound for the penalized process

This section presents some sufficient conditions on the penalty function $\text{pen}(\nu)$ which ensure the general exponential bound for the penalized process $Y(\nu) - \text{pen}(\nu)$. For simplicity we assume that the local entropy numbers $Q(\epsilon, \nu)$ are uniformly bounded by a constant $Q^*(Y)$. Let also $\pi$ be a $\sigma$-finite measure on the space $Y$ and $\pi(A)$ stand for the $\pi$-measure of a set $A \subset Y$. The standard proposal for $\pi$ is the usual Lebesgue measure.

**Theorem 5.3.** Assume $(E)$ and $(\Xi)$ with some fixed $\epsilon$ and $\lambda^*$. Let $\varrho < 1$ be such that $\varrho \epsilon/(1 - \varrho) \leq \lambda^*$. Let also $Q(\epsilon, \nu) \leq Q^*(Y)$ for all $\nu \in Y$. Let a $\sigma$-finite measure $\pi$ on $Y$ be such that for some $\nu \geq 1$

$$\sup_{\nu, \nu' : D(\nu, \nu') \leq \epsilon} \frac{\pi(B(\nu, \nu'))}{\pi(B(\nu, \nu'))} \leq \nu.$$  \hspace{1cm} (5.3)

Finally, let a function $\text{pen}(\nu)$ satisfy

$$\mathcal{H}_\epsilon(\varrho) \overset{\text{def}}{=} \log \int_Y \frac{1}{\pi(B(\nu, \nu'))} \exp\{-\varrho \text{pen}_\epsilon(\nu)\} d\pi(\nu) < \infty$$

with $\text{pen}_\epsilon(\nu) = \inf_{\nu' \in B(\nu, \nu')} \text{pen}(\nu)$. Then

$$E \exp\left\{\sup_{\nu \in Y} \varrho \left[ Y(\nu) - \text{pen}(\nu) \right] \right\} \leq \Omega(\varrho, \epsilon),$$  \hspace{1cm} (5.4)

where

$$\log \Omega(\varrho, \epsilon) = 2\epsilon^2 \varrho^2 \frac{1}{1 - \varrho} + (1 - \varrho)Q^*(Y) + \log \nu + \mathcal{H}_\epsilon(\varrho).$$  \hspace{1cm} (5.5)

**Proof.** We begin with a simple technical result which bounds the maximum of a given function via the weighted integral of the local maxima.

**Lemma 5.4.** Let $f(\nu)$ be a nonnegative function on $Y \subset \mathbb{R}^p$ and let for every point $\nu \in Y$ a vicinity $A(\nu)$ be fixed such that $\nu' \in A(\nu)$ implies $\nu \in A(\nu')$. Let also the measure $\pi(A(\nu))$ of the set $A(\nu)$ fulfill for every $\nu \in Y$

$$\sup_{\nu \in A(\nu)} \frac{\pi(A(\nu))}{\pi(A(\nu'))} \leq \nu.$$  \hspace{1cm} (5.6)

Then

$$\sup_{\nu \in Y} f(\nu) \leq \nu \int_Y f^*(\nu) \frac{1}{\pi(\nu)} d\pi(\nu)$$

with

$$f^*(\nu) \overset{\text{def}}{=} \sup_{\nu' \in A(\nu)} f(\nu').$$
Suppose the following condition is fulfilled:

because \( \nu \in A(\nu^0) \) implies \( \nu^0 \in A(\nu) \) and hence, \( f(\nu^0) \leq f^*(\nu) \). Now by (5.6)

\[
\int_{\mathcal{Y}} f^*(\nu) \frac{1}{\pi(A(\nu))} d\pi(\nu) \geq \int_{A(\nu^0)} f^*(\nu) \frac{1}{\pi(A(\nu))} d\pi(\nu) \geq f(\nu^0) \int_{A(\nu^0)} \frac{1}{\pi(A(\nu))} d\pi(\nu) = f(\nu^0)/\nu
\]

as required.

This result applied to \( f(\nu) = \exp \{ \varrho [\mathcal{Y}(\nu) - \text{pen}(\nu)] \} \) and \( A(\nu) = \mathcal{B}(\epsilon, \nu^0) \) implies

\[
\sup_{\nu \in \mathcal{Y}} \exp \left\{ \varrho [\mathcal{Y}(\nu) - \text{pen}(\nu)] \right\} \leq \nu \int_{\mathcal{Y}} \sup_{\nu \in \mathcal{B}(\epsilon, \nu^0)} \exp \left\{ \varrho [\mathcal{Y}(\nu) - \text{pen}(\nu)] \right\} \frac{d\pi(\nu)}{\pi(\mathcal{B}(\epsilon, \nu^0))}.
\]

This implies by Theorem 5.1

\[
\log \mathbb{E} \sup_{\nu \in \mathcal{Y}} \exp \left\{ \varrho [\mathcal{Y}(\nu) - \text{pen}(\nu)] \right\} \leq \frac{2\epsilon^2 \varrho^2}{1 - \varrho} + (1 - \varrho) \mathcal{Q}^*(\mathcal{Y}) + \log \left\{ \nu \int_{\mathcal{Y}} \exp \left\{ -\varrho \text{pen}(\nu^0) \right\} \frac{d\pi(\nu^0)}{\pi(\mathcal{B}(\epsilon, \nu^0))} \right\}
\]

and the assertion follows.

5.3 Smooth case

Here we discuss the special case when \( \mathcal{Y} \subset \mathbb{R}^p \), the process \( \mathcal{Y}(\nu) \) and its stochastic component \( \zeta(\nu) \) are absolutely continuous and the gradient \( \nabla \zeta(\nu) \) has bounded exponential moments. We also assume that \( \pi \) is the Lebesgue measure on \( \mathcal{Y} \).

Suppose the following condition is fulfilled:

\[
(\mathcal{E}D) \quad \text{There exist } \lambda^* > 0 \text{ and for each } \nu \in \mathcal{Y}, \text{ a symmetric non-negative matrix } H(\nu) \text{ such that for any } \lambda \leq \lambda^*
\]

\[
\sup_{\nu \in \mathcal{Y}} \sup_{\gamma \in \mathbb{S}^p} \log \mathbb{E} \exp \left\{ 2\lambda \frac{\gamma^\top \nabla \zeta(\nu)}{\|H(\nu)\gamma\|} \right\} \leq 2\lambda^2.
\]

The matrix function \( H(\nu) \) can be used for defining a natural topology in \( \mathcal{Y} \). Namely, for any \( \nu, \nu' \in \mathcal{Y} \) define \( \delta = \|\nu - \nu'\| \), \( \gamma = (\nu - \nu')/\delta \) and

\[
\mathcal{D}^2(\nu, \nu') \overset{\text{def}}{=} \|\nu - \nu'\|^2 \int_0^1 \gamma^\top H^2(\nu + t\delta \gamma) \gamma dt.
\]
Next, introduce for each \( \nu^0 \in \mathcal{T} \) and \( \epsilon > 0 \) the set

\[
\mathcal{B}(\epsilon, \nu^0) \overset{\text{def}}{=} \{ \nu : D(\nu, \nu^0) \leq \epsilon \}
\]

To state the result, we need one more condition on the uniform continuity of the matrix \( H(\nu) \) in \( \nu \).

\((H)\) There exist constants \( \epsilon > 0 \) and \( \nu_1 \geq 1 \) such that

\[
\sup_{\nu, \nu' : D(\nu, \nu') \leq \epsilon} \sup_{\gamma \in \mathbb{S}^p} \gamma^T H^2(\nu) \gamma \leq \nu_1.
\]

**Theorem 5.5.** Let \((E)\) be satisfied. Suppose that \((ED)\) holds with some \( \lambda^* \) and a matrix function \( H(\nu) \) which fulfills \((H)\). If for some \( \varrho \in (0, 1) \) and \( \epsilon > 0 \) with \( \varrho \epsilon / (1 - \varrho) \leq \lambda^* \), the penalty function \( \text{pen}(\nu) \) fulfills

\[
\mathcal{P}_\varrho(\epsilon) \overset{\text{def}}{=} \log \left\{ \omega_p^{-1} \epsilon^{-p} \int_{\mathcal{Y}} \det(H(\nu^0)) \exp \left\{ - \varrho \text{pen}(\nu^0) \right\} d\nu^0 \right\} < \infty
\]

with \( \text{pen}(\nu^0) = \inf_{\nu \in \mathcal{B}(\epsilon, \nu^0)} \text{pen}(\nu) \), then

\[
E \exp \left\{ \sup_{\nu \in \mathcal{T}} \varrho (y(\nu) - \text{pen}(\nu)) \right\} \leq \mathcal{Q}(\varrho, \epsilon) \tag{5.7}
\]

where

\[
\log \mathcal{Q}(\varrho, \epsilon) = \frac{2 \epsilon^2 \varrho^2}{1 - \varrho} + (1 - \varrho) \mathbb{Q}_p + \mathcal{P}_\varrho(\epsilon) + p \log(\nu_1)
\]

with \( \mathbb{Q}_p \) being the usual entropy number for the Euclidean ball in \( \mathbb{R}^p \).

**Proof.** First we show that the differentiability condition \((ED)\) implies the local moment condition \((E)\).

**Lemma 5.6.** Assume that \((ED)\) holds with some \( \lambda^* \). Then for any \( \nu^0 \in \mathcal{T} \) and any \( \lambda \) with \( |\lambda| \leq \lambda^*/\nu_1^{1/2} \), it holds

\[
\sup_{\nu \in \mathcal{B}(\epsilon, \nu^0)} \log E \exp \left\{ 2 \lambda \zeta(\nu, \nu^0) \right\} \leq 2 \lambda^2. \tag{5.8}
\]

**Proof.** For \( \nu \in \mathcal{B}(\epsilon, \nu^0) \), denote \( \delta = \|\nu - \nu^0\|, \gamma = (\nu - \nu^0) / \delta \). With this notation

\[
\zeta(\nu, \nu^0) = \delta \gamma^T \int_0^1 \nabla \zeta(\nu^0 + t \delta \gamma) dt.
\]

The condition \((H)\) implies for every \( t \in [0, 1] \) that

\[
\frac{\lambda \varrho}{{D}(\nu, \nu^0)} \leq \lambda \nu_1^{1/2} \leq \lambda^*.
\]
Now the Hölder inequality and \((ED)\) yield
\[
\log E \exp \left\{ 2\lambda \frac{\zeta(v,v^\circ)}{D(v,v^\circ)} - \lambda^2 \right\} \\
= \log E \exp \left\{ \int_0^1 \gamma^T \left[ \frac{2\lambda \partial}{D(v,v^\circ)} \nabla \zeta(v^\circ + t\partial \gamma) - \frac{2\lambda^2 \partial^2}{D^2(v,v^\circ)} H^2(v^\circ + t\partial \gamma) \gamma \right] dt \right\} \\
\leq \int_0^1 \log E \exp \left\{ \gamma^T \left[ \frac{2\lambda \partial}{D(v,v^\circ)} \nabla \zeta(v^\circ + t\partial \gamma) - \frac{2\lambda^2 \partial^2}{D^2(v,v^\circ)} H^2(v^\circ + t\partial \gamma) \gamma \right] \right\} dt \\
\leq 0
\]
as required.

Next we show that condition \((H)\) implies (5.3). Consider for every \(v^\circ \in T\) an elliptic neighborhood \(B'(\epsilon,v^\circ) = \{ v : \|H(v^\circ)(v - v^\circ)\| \leq \epsilon \} \).

**Lemma 5.7.** Assume \((H)\). Then

1. for any \(\epsilon > 0\) and any \(v \in T\)
   \[
   B'(\nu_1^{-1/2} \epsilon, v) \subset B(\epsilon, v) \subset B'(\nu_1^{1/2} \epsilon, v), \\
   B(\nu_1^{-1/2} \epsilon, v) \subset B'(\epsilon, v) \subset B(\nu_1^{1/2} \epsilon, v). 
   \]

2. For every \(v \in T\),
   \[
   \nu_1^{-p/2} \leq \epsilon^{-p} \pi(\epsilon) \det(H(v))/\omega_p \leq \nu_1^{p/2}, 
   \]
   where \(\omega_p\) is the Lebesgue measure of the unit ball in \(\mathbb{R}^p\).

3. condition (5.3) holds with \(\nu = \nu_1^p\).

**Proof.** Condition \((H)\) implies that for any \(v^\circ \in T\) and \(v \in B(\epsilon,v^\circ)\) that
\[
\nu_1^{-1} \gamma^T H^2(v^\circ) \gamma \leq \int_0^1 \gamma^T H^2(v^\circ + t\partial \gamma) \gamma dt \leq \nu_1 \gamma^T H^2(v^\circ) \gamma
\]
with \(\partial = \|v - v^\circ\|\) and \(\gamma = (v - v^\circ)/\partial\), which yields the first assertion of the lemma.

The Lebesgue measure of the ellipsoid \(B'(\epsilon,v)\) is equal to \(\omega_p \epsilon^p/\det(H(v))\). This and (5.9) imply the second assertion. This, in turn, implies (5.3) in view of \((H)\).

The next result claims that in the smooth case the local entropy number \(Q(\epsilon,v^\circ)\) is similar to the usual Euclidean situation.

**Lemma 5.8.** Assume \((H)\). Then \(\sup_{v \in \Theta} Q(\epsilon,v) \leq Q_p + p \log(\nu_1)\).
Proof. Fix any $\mathbf{v}^0 \in \mathcal{T}$. Linear transformation with the matrix $H^{-1}(\mathbf{v}^0)$ reduces the situation to the case when $H(\mathbf{v}^0) \equiv I$ and $\mathcal{B}'(\epsilon_0, \mathbf{v}^0)$ is a usual Euclidean ball for any $\epsilon_0 \leq \epsilon$. Moreover, by $(H)$, each elliptic set $\mathcal{B}'(\epsilon_0, \mathbf{v})$ for $\mathbf{v} \in \mathcal{B}(\epsilon, \mathbf{v}^0)$ is nearly an Euclidean ball in the sense that the ratio of its largest and smallest axes (which is the ratio of the largest and smallest eigenvalues of $H^{-1}(\mathbf{v}^0)H^2(\mathbf{v})H^{-1}(\mathbf{v}^0)$) is bounded by $\nu_1$. Therefore, for any $\epsilon_0 \leq \epsilon$, a Euclidean net $\mathcal{D}(\epsilon_0/\nu_1)$ with the step $\epsilon_0/\nu_1$ ensures a covering of $\mathcal{B}(\epsilon, \mathbf{v}^0)$ by the sets $\mathcal{B}(\epsilon_0, \mathbf{v}^0)$, $\mathbf{v}^0 \in \mathcal{D}(\epsilon_0/\nu_1)$. Therefore, the corresponding covering number is bounded by $(\nu_1\epsilon/\epsilon_0)^{p}$ yielding the claimed bound for the local entropy.

Now the result of theorem 5.5 is reduced to the statement of Theorem 5.3.

Computing of the penalty simplifies a lot when the matrix $H(\mathbf{v})$ is uniformly bounded by a matrix $H^*$, or, equivalently, condition $(H)$ is fulfilled for $H(\mathbf{v}) \equiv H^*$. Then one can define $\text{pen}(\mathbf{v})$ as a function of the norm $\|H^*(\mathbf{v} - \mathbf{v}_0)\|$ for a fixed $\mathbf{v}_0$.

**Theorem 5.9.** Assume additionally to the conditions of Theorem 5.5 that $H(\mathbf{v}) \leq H^*$ for a symmetric matrix $H^*$. Suppose that $\kappa(t)$ is a monotonously decreasing positive function on $[0, +\infty)$ satisfying

$$
P^* \overset{\text{def}}{=} \omega_p^{-1} \int_{\mathbb{R}^p} \kappa(\|\mathbf{u}\|)d\mathbf{u} = p \int_0^{\infty} \kappa(t)t^{p-1}dt < \infty. \quad (5.11)$$

Define

$$\text{pen}(\mathbf{v}) = -\varrho^{-1} \log \kappa(\epsilon^{-1}\|H^*(\mathbf{v} - \mathbf{v}_0)\| + 1)$$

Then

$$E \exp\left\{ \sup_{\mathbf{v} \in \mathcal{T}} \mathbb{E}[Y(\mathbf{v}) - \text{pen}(\mathbf{v})] \right\} \leq \Omega(\varrho, \epsilon) \quad (5.12)$$

with

$$\log \Omega(\varrho, \epsilon) = \frac{2\epsilon^2 \varrho^2}{1 - \varrho} + (1 - \varrho)Q_p + \log(P^*),$$

where $\omega_p$ is the volume of the unit ball in $\mathbb{R}^p$.

**Proof.** Let us fix $\mathbf{v}^0 \in \mathcal{T}$. Definition of the semi-metric $\mathcal{D}$ and condition $(H)$ imply for every $\mathbf{v} \in \mathcal{B}(\epsilon, \mathbf{v}^0)$ that

$$\|H^*(\mathbf{v}^0 - \mathbf{v})\| \leq \epsilon.$$

The triangle inequality and $(H)$ now imply for this $\mathbf{v}$ that

$$\epsilon^{-1}\|H^*(\mathbf{v} - \mathbf{v}_0)\| + 1 \geq \epsilon^{-1}\|H^*(\mathbf{v}^0 - \mathbf{v}_0)\|$$
and \( \text{pen}_\epsilon(v^*) \geq -\epsilon^{-1} \log \kappa(\epsilon^{-1}\|H^*(v^* - v_0)\|) \). Therefore, it follows by change of variables \( u = \epsilon H^*(v - v_0) \) that
\[
\omega_p^{-1} \epsilon^{-p} \int_{\mathbb{R}^p} \det(H^*) \exp\{-\epsilon \text{pen}_\epsilon(v)\} dv \leq \omega_p^{-1} \int_{\mathbb{R}^p} \kappa(\|u\|) du \\
\leq p \int_0^\infty \kappa(t)t^{p-1} dt = \mathbb{P}^*,
\]
and the result follows from Theorem 5.5.

Natural candidates for the function \( \kappa(\cdot) \) and the corresponding \( \mathbb{P}^* \)-values are:
\[
\kappa_1(t) = e^{-\delta_1(t-1)^2}, \quad \mathbb{P}^*_1 = 1 + \omega_p^{-1}(\pi/\delta_1)^{p/2},
\]
\[
\kappa_2(t) = \|1 + t\|^{-p-\delta_2}, \quad \mathbb{P}^*_2 = p/\delta_2,
\]
where \( \delta_1, \delta_2 > 0 \) are some constants. The result of Theorem 5.9 yields

**Corollary 5.10.** Under conditions of Theorem 5.9, the bound (5.12) holds with
\[
\text{pen}_1(v) = \epsilon^{-1}\delta_1 \epsilon^{-2}\|H^*(v - v_0)\|^2,
\]
\[
\log \Omega_2(\rho, \epsilon) = \frac{2\epsilon^2 \rho^2}{1-\rho} + (1-\rho)\Omega_{\rho} + \log(1 + \omega_p^{-1}|\pi/\delta_1|^{p/2}).
\]
\[
\text{pen}_1(v) = -\epsilon^{-1}(p + \delta_2) \log(\epsilon^{-1}\|H^*(v - v_0)\| + 2),
\]
\[
\log \Omega_1(\rho, \epsilon) = \frac{2\epsilon^2 \rho^2}{1-\rho} + (1-\rho)\Omega_{\rho} + \log(p/\delta_2).
\]

Sometimes it is useful to combine the functions \( \kappa_1(\cdot) \) and \( \kappa_2(\cdot) \) in the form
\[
\kappa(t) = \kappa_1(t)1(t \geq r) + \kappa_2(t)1(t \leq r)
\]
for a properly selected \( r \) which still ensures (5.11) with
\[
\mathbb{P}^* \leq \omega_p^{-1}|\pi/\delta_1|^{p/2} + pr^{-\delta_2}/\delta_2.
\]

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