EQUIVARIANT $A_\infty$ ALGEBRAS FOR NONORIENTABLE LAGRANGIANS

AMITAI ZERNIK

Abstract. We set up an algebraic framework for the study of pseudoholomorphic discs bounding nonorientable Lagrangians, as well as equivariant extensions of such structures arising from a torus action.

First, we define unital cyclic twisted $A_\infty$ algebras and prove some basic results about them, including a homological perturbation lemma which allows one to construct minimal models of such algebras. We then construct an equivariant extension of $A_\infty$ algebras which are invariant under a torus action on the underlying complex. Finally, we construct a homotopy retraction of the Cartan-Weil complex to equivariant cohomology, which allows us to construct minimal models for equivariant cyclic twisted $A_\infty$ algebras.

In a forthcoming paper we will use these results to define and obtain fixed point expressions for the open Gromov-Witten theory of $\mathbb{RP}^{2n'} \to \mathbb{CP}^{2n'}$, as well as its equivariant extension.

Contents

1. Introduction 2
1.1. Some conventions 4
2. Twisted $A_\infty$ algebras 4
2.1. Twisted $A_\infty$ algebras as collections of maps 4
2.2. Twisted $A_\infty$ algebras as differentials; morphisms and homotopies 7
2.3. Pairing cocycles, cyclic and unital morphisms 11
3. Homological algebra of twisted $A_\infty$ algebras 12
3.1. Cyclic and unital homological algebra 14
3.2. Constructing retractions in simple cases 18
4. Equivariant cohomology 18
4.1. Basic definitions 18
4.2. Equivariant cohomology in the separated case 20
4.3. Equivariant angular and Euler forms 21
4.4. General equivariant pushforward and Poincare duality 24
5. Equivariant twisted $A_\infty$ algebras 27
5.1. The equivariant cyclic DGA, statement of results 27
5.2. Equivariant extension of invariant deformations 29
5.3. Cyclic unital equivariant retraction 31
6. Appendix. Orientation conventions 36
6.1. Terminology and pullback orientation. 36
6.2. Short exact sequences and ordered direct sums 36
6.3. Boundary. 37
6.4. Pushforward. 37
References 39
1. INTRODUCTION

This paper lays the algebraic foundations for our definition of the equivariant open Gromov-Witten invariants for $\mathbb{R}P^{2n'}$, and their computation using $A_{\infty}$ fixed-point localization. We briefly describe the geometric motivation, before outlining the main results.

Let $(M, \omega)$ be a compact symplectic manifold and $L \hookrightarrow M$ be an embedded Lagrangian submanifold. Such an embedding is called relatively spin if $L$ is oriented and $w_2(TL) \in \text{Im } (i^*) \subset H^2(L; \mathbb{Z}/2\mathbb{Z})$ (cf. Definition 3.1.1 in [4]). Fukaya [3] associates a $G$-gapped cyclic filtered $A_{\infty}$ algebra to every such relatively spin Lagrangian embedding. This is a cyclic deformation of the differential graded algebra (DGA) of differential forms on $L$, obtained by taking into account quantum corrections, namely, the effects of positive-energy pseudoholomorphic discs in $M$ bounded by $L$.

The Lagrangian embedding $\mathbb{R}P^{2n'} \hookrightarrow \mathbb{C}P^{2n'}$ is not relatively spin. Indeed, $\mathbb{R}P^{2n'}$ is not even orientable. On the other hand, Solomon [11] showed that for $n' = 1$ the moduli spaces associated with pseudoholomorphic discs bounding $\mathbb{R}P^2$ can be used to define invariants which are equivalent to Welschinger’s [12] signed counts of real rational planar curves. One upshot of extending the $A_{\infty}$ formalism to accommodate non-orientable $L$ such as $\mathbb{R}P^{2n'}$ is that it allows to generalize the definition of the Solomon-Welschinger invariants to all $n'$. Roughly speaking, this is because it allows to “keep tabs” of boundary corrections, which for $n' = 1$ happen to vanish but in higher dimensions become significant. This approach for defining invariants is based on [11].

Another feature of the embedding $\mathbb{R}P^{2n'} \hookrightarrow \mathbb{C}P^{2n'}$ is that it is equivariant with respect to an action of the rank $n'$ torus group $T$. This motivates one to look for $H^* (BT)$-valued invariants generalizing the classical invariants discussed in the previous paragraph, and compute them using fixed-point localization. Here the boundary contributions become significant already for $n' = 1$.

With this in mind, our goal in this paper is to set up an $A_{\infty}$ formalism that will (i) capture the quantum deformations associated with non-orientable Lagrangians, and (ii) handle equivariant extensions of $T$-invariant $A_{\infty}$ algebras. Let us now explain how this is carried out.

A non-orientable Lagrangian embedding $L \hookrightarrow M$ will be called relatively $Pin$ if $w_2(TL) + w_1(TL)^2 \in \text{Im } (i^*)$ (see [11]). In this case, the Maslov index of holomorphic discs may be odd, and one must allow for forms with values in the orientation local system, which leads to some subtle signs in the computations. In section 2 we introduce a generalization of the notion of a $G$-gapped filtered $A_{\infty}$ algebra which we call a twisted $A_{\infty}$ algebra (see Definition 2), that captures this situation. More precisely, in a forthcoming paper we will construct a twisted $A_{\infty}$ algebra for the $Pin$ Lagrangian embedding $\mathbb{R}P^{2n} \hookrightarrow \mathbb{C}P^{2n}$. It should be possible to construct such an algebra for any relatively $Pin^{-}$ Lagrangian embedding. As usual the easy-to-check Definition 2 is followed by an equivalent easy-to-use definition of twisted $A_{\infty}$ algebras, as tame differentials on a certain bar coalgebra, see Proposition 12. Cyclic and unital versions are also discussed.

In section 3 we prove the homological perturbation lemma for twisted $A_{\infty}$ algebras and discuss cyclic and unital versions too, see Theorem 23 and Proposition 27. This is an important computational tool, which enables one to construct minimal
models. To apply these results a certain retraction is needed, see Definitions 23 and 26.

The remainder of the paper is devoted to discussing the equivariant situation. When a manifold \( L \) is equipped with an action of the rank \( n' \) torus group \( \mathbb{T} \), one can construct the Cartan-Weil DGA which is a certain extension of the De Rham \( \mathbb{C} \)-DGA of \( L \) defined over \( H^* (B\mathbb{T}) = \mathbb{C} [\alpha_1, \ldots, \alpha_{n'}] \). In Section 4 we give a self-contained and fairly thorough account of this, including a discussion of the equivariant angular form (see Definition 34 and Proposition 35) and Poincare duality, Corollary 44.

In Section 5 we apply the Cartan-Weil theory to twisted \( A_{\infty} \) algebras. We define what it means for a twisted \( A_{\infty} \) algebra to be invariant under an action of a torus group (Definition 49) and show that in this case the twisted \( A_{\infty} \) algebra admits an equivariant extension, see Proposition 51. Since DGA’s are special cases of twisted \( A_{\infty} \) algebras (see Example 6), we can summarize the situation with the following commutative square of differential coalgebras.

\[
\begin{array}{ccc}
(B_{\mathbb{C}}, d + \wedge) & \xrightarrow{\bar{d} = 0} & (B_{\mathbb{C}}^{\mathbb{C}^\infty}, D + \wedge) \\
\tau = 0 & \Downarrow & \tau = 0 \\
(B_{\Lambda_0^G (\mathbb{C})}, m) & \xrightarrow{\bar{d} = 0} & (B_{\Lambda_0^G (\mathbb{C} [\mathbb{C}]), m}^{\mathbb{C}^\infty}, m^{\mathbb{C}^\infty})
\end{array}
\]

\((B_{\mathbb{C}}, d + \wedge)\) is the differential bar coalgebra over \( \mathbb{C} \), corresponding to the De Rham DGA of differential forms on some manifold \( L \) equipped with a torus action. \((B_{\mathbb{C}}^{\mathbb{C}^\infty}, D + \wedge)\) is the Cartan-Weil equivariant extension of this algebra over \( \mathbb{C} [\mathbb{C}] \).

\((B_{\Lambda_0^G (\mathbb{C})}, m)\) is some twisted \( A_{\infty} \) algebra which is a deformation of \((B_{\mathbb{C}}, d + \wedge)\), over the Novikov ring \( \Lambda_0^G (\mathbb{C}) \). We assume that this deformation is \( \mathbb{T} \)-invariant; for example, \((B_{\Lambda_0^G (\mathbb{C})}, m)\) might be the quantum deformation associated with the Lagrangian embedding \( \mathbb{R} \mathbb{P}^n \hookrightarrow \mathbb{C} \mathbb{P}^n \), in which case the algebra is \( \mathbb{T} \)-invariant because the \( \mathbb{T} \)-action on \( \mathbb{R} \mathbb{P}^n \) extends to \( \mathbb{C} \mathbb{P}^n \) and the associated moduli spaces of discs. Anyway, if \((B_{\Lambda_0^G (\mathbb{C})}, m)\) is invariant we can construct \((B_{\Lambda_0^G (\mathbb{C} [\mathbb{C}]), m}^{\mathbb{C}^\infty}, m^{\mathbb{C}^\infty})\) which extends both \((B_{\mathbb{C}}^{\mathbb{C}^\infty}, D + \wedge)\) and \((B_{\Lambda_0^G (\mathbb{C})}, m)\) over \( \Lambda_0^G (\mathbb{C} [\mathbb{C}]) \). See 33 for more details about this square of twisted \( A_{\infty} \) algebras.

When \( L \) has even cohomology the equivariant cohomology admits a perfect pairing (see Corollary 18), and we show that the equivariant extension of cyclic twisted \( A_{\infty} \) algebras is also cyclic in this case.

Theorem 53 states that when \( L \) has even cohomology, there exists a cyclic unital retraction of the Cartan-Weil complex to its cohomology. Such a retraction is needed in order to construct minimal models for equivariant \( A_{\infty} \) algebras. The proof involves the construction of a certain homotopy kernel (see Definition 55 and Proposition 56). It is a kind of equivariant Hodge-De Rham decomposition for even cohomology manifolds. This construction is central to the definition of the equivariant invariants of \( \mathbb{R} \mathbb{P}^n' \hookrightarrow \mathbb{C} \mathbb{P}^n' \) and their computation using fixed point localization, which will be the subject of a forthcoming paper.

Acknowledgments. I would like to thank my adviser Jake Solomon. He suggested the problem of open fixed-point localization and that studying the \( A_{\infty} \) formalism
in [3] could be a way to solve it. I also learned a lot from countless conversations we’ve had. I would like to thank Mohammed Abouzaid and Rahul Pandharipande for interesting discussions and ideas. Finally, I’d like to thank Pavel Giterman for his careful reading of the paper and many useful suggestions.

1.1. Some conventions. When we say an object $C$ is graded we mean it is equipped with a $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-grading. We call the $\mathbb{Z}$-component of the grading the codimension degree and the $\mathbb{Z}/2$ component of the grading the local system degree. We denote by $\text{cd} x$ (resp. Is $x$ the codimension degree (resp. the local system degree) of a homogenous element $x$. We will sometimes use the notation $x^{a,b}$ to indicate that $x^{a,b} \in C^{a,b}$ is homogeneous of degree $\text{deg} x = (a, b) \in (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$.

Maps will be grading-preserving, unless stated otherwise. We will denote by $C[p]$ the codimension degree shift of $C$ and by $C[p,q]$ the bidegree shift of $C$, so

$$(C[p])^{a,b} := C^{a+p,b}, \quad (C[p,q])^{a,b} := C^{a+p,b+q}. $$

Tensor products of graded objects are graded in the usual way.

We let $\text{FGVect}$ denote the category of filtered graded real vector spaces. The filtration on an object $X$ is denoted $\{F^E X\}$; it is indexed by $E \in \mathbb{R}_{\geq 0}$ and decreasing, if $E_1 \leq E_2$ we have $F^{E_2} X \subseteq F^{E_1} X$. The grading is by $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as above. Morphisms in $\text{FGVect}$ are required to preserve the filtration and the grading.

An object $V$ is called discrete if it is equipped with the discrete filtration, which satisfies $F^{\geq 0} V := \bigcup_{E \geq 0} F^E V = 0$.

Let $V_1, V_2$ be objects of $\text{FGVect}$. The usual graded tensor product $V_1 \otimes V_2$ is equipped with a filtration

$$(1) \quad F^E (V_1 \otimes V_2) := \bigcup_{E_1 + E_2 \geq E} F^{E_1} V_1 \otimes F^{E_2} V_2. $$

This turns $\text{FGVect}$ into a monoidal category. Let $\hat{\text{FGVect}}$ denote the full subcategory of $\text{FGVect}$ of objects which are complete with respect to the filtration; this category is monoidal with respect to the completed tensor product $\hat{\otimes}$. If $\Lambda$ is a monoid in $\hat{\text{FGVect}}$ we denote by $\text{BiMod}_\Lambda$ the category of $\Lambda$-bimodules in $\text{FGVect}$. It is a monoidal category with respect to the monoidal product $- \hat{\otimes} \Lambda -$, where the filtration on $V_1 \otimes_\Lambda V_2$ is given by the same formula (1) except the tensor products on both sides are taken over $\Lambda$.

If $A$ and $B$ are objects of a category $C$ we’ll denote by $C(A, B)$ the internal hom object (assuming it exists).

A local system on a topological space $X$ is a sheaf $\mathcal{L}$ which is locally isomorphic to the trivial local system, which is the constant sheaf $\underline{\mathbb{C}}$. If $E \xrightarrow{\pi} X$ is a rank $r$ vector bundle, the orientation local system of $E$, denoted $\text{Or} (E)$, is the sheafification of $U \mapsto H^r_{cv}(E|_U; \mathbb{C})$ where $H^r_{cv}$ denotes the compact vertical cohomology (see Bott and Tu [2] pg. 61) it is a local system on $X$. If $X$ is a manifold and $TX$ is the associated tangent bundle, $\text{Or} (TX)$ is called simply the orientation local system of $X$. An orientation for $X$ is an isomorphism $\text{Or} (TX) \simeq \underline{\mathbb{C}}$, and given an orientation we say $X$ is oriented. If no such orientation exists we say $X$ is non-orientable.

2. Twisted $A_\infty$ algebras

2.1. Twisted $A_\infty$ algebras as collections of maps. In this section we define twisted $A_\infty$ algebras. We follow the notation and conventions of [3] and introduce
modifications where they are needed. The differences are summarized in Remark 3.

**Definition 1.** (a) We say a subset $G \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}$ is a *submonoid* if it contains 0 = (0, 0) and is closed under addition. If $G$ is a submonoid we denote by $E : G \to \mathbb{R}_{\geq 0}$ and $\mu : G \to \mathbb{Z}$ the projections to each of the components.

(b) We say a submonoid $G \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}$ is a *discrete submonoid* if the following hold.

1. The image $E(G) \subset \mathbb{R}_{\geq 0}$ is discrete.
2. For each $\lambda \in \mathbb{R}_{\geq 0}$ the inverse image $E^{-1}(\lambda)$ is a finite set.

Let $G$ be a discrete submonoid, $R$ a $2\mathbb{Z} \oplus 0$-graded unital commutative algebra over $\mathbb{R}$. By that we mean that we think of $R$ as $(\mathbb{Z} \oplus 2\mathbb{Z})$-graded, but $R^{a,b} = 0$ unless $(a, b) \in 2\mathbb{Z} \oplus 0 \subset \mathbb{Z} \oplus 2\mathbb{Z}$. Let $C$ be a $(\mathbb{Z} \oplus 2\mathbb{Z})$-graded $R$-module; in particular, the structure maps are required to respect the grading.

**Definition 2.** A $G$-gapped twisted $A_\infty$ algebra structure on $C$ over $R$ is a collection $\{m_{k,\beta}\}$ of maps

$$m_{k,\beta} : C \otimes_R k \to C [2 - k - \mu(\beta), \mu(\beta) \mod 2]$$

for each $k \in \mathbb{Z}_{\geq 0}$ and $\beta \in G$, such that

1. $m_{0,0} = 0$,

and for every $\beta \in G$ and $k \geq 0$ we have

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} (-1)^* m_{k_1,\beta_1}(x_1, \ldots, x_{i-1}, m_{k_2,\beta_2}(x_i, \ldots, x_{i+k_2-1}), \ldots, x_k) = 0$$

for

$$* = \sum_{j=1}^{i-1} (cd x_j - 1) + \mu(\beta_2) \sum_{j=1}^{i-1} (ls x_j + cd x_j - 1) + \mu(\beta_2).$$

Depending on the context we may simply say that $(C, \{m_{k,\beta}\})$ is a *twisted $A_\infty$ algebra*, or even just an *algebra*.

A pairing $C \otimes_R C \xrightarrow{\langle,\rangle} R[-p,q]$ is called *antisymmetric* if

$$\langle u \otimes v \rangle = (-1)^{1+(cd u-1)(cd v-1)} \langle v \otimes u \rangle$$

and *non-degenerate* if for any $u \neq 0$ there exists some $v$ such that $\langle u, v \rangle \neq 0$.

**Definition 3.** We say $(C, \{m_{k,\beta}\}, \langle,\rangle)$ is a cyclic twisted $A_\infty$ algebra if $(C, \{m_{k,\beta}\})$ is a twisted $A_\infty$ algebra and

1. $\langle,\rangle : C \otimes_R C \to R[-p,q]$ is an antisymmetric, non-degenerate pairing,

2. for every $k \geq 0$ and $\beta \in G$ we have

$$\langle m_{k,\beta}(x_1, \ldots, x_k), x_0 \rangle = (-1)^\bigstar \langle m_{k,\beta}(x_0, \ldots, x_{k-1}), x_k \rangle$$

for $\bigstar = (cd x_0 - 1) \left( \sum_{j=1}^{k} (cd x_j - 1) \right) + \mu(\beta) ls x_0$, and

3. the induced pairing on $HC = H(C, m_{1,0})$ is perfect; that is the induced map $HC \to \text{Mod}_R[H(C, R[-p,q])]$ is an isomorphism. Here $\text{Mod}_R$ is the category of graded $R$-modules (cf. §1.3).
Note that $m_{1,0}^2 = 0$, so $HC$ is well-defined, and $\langle m_{1,0} x, y \rangle + \langle m_{1,0} y, x \rangle = 0$, so there is indeed an induced pairing on $HC$.

**Remark 4.** Our definition \ref{definition:discrete-submonoid} of a discrete submonoid differs from Definition 6.2 in \cite{notes} in that $\mu$ can immediately apparent that the sign in Eq \ref{eq:sign} reduces to the sign in the filtered $A_\infty$ relation (61) in \cite{notes}. Similarly the sign in Eq \ref{eq:filtered-sign} reduces to the sign in the cyclic symmetry condition (62) in \cite{notes}. In fact, for $\mu (G) < 2\mathbb{Z}$ we find that $G$-gapped cyclic $A_\infty$ algebra over $R = \mathbb{R}$ are in bijection with $G$-gapped cyclic filtered $A_\infty$ algebras in the sense of Definition 6.4 of \cite{notes}.

**Definition 5.** An element $e \in C^{0,0}$ is a strict unit of the twisted $A_\infty$ algebra $(C, \{ m_{k,\beta} \})$ if

\begin{equation}
m_{2,0} (e, x) = (-1)^{cd_1} m_{2,0} (x, e) = x
\end{equation}

and $m_{k,\beta} (\cdots, e, \cdots) = 0$ for all $(k, \beta) \neq (2, 0)$.

**Example 6.** Suppose $C$ is a $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-graded unital associative algebra over $R$ with product $\wedge$, and $d : C \to C [1] = C [1, 0]$ (cf. §11) is an $R$-linear differential satisfying the graded Leibniz rule

\[d (x \wedge y) = dx \wedge y + (-1)^{cd_1} x \wedge dy.\]

We will say that $(C, d, \wedge)$ is an $R$-DGA, or a DGA over $R$, in this case. In this case we can take the trivial discrete submonoid $G = \{ 0 \}$ and construct a unital twisted $A_\infty$ algebra $(C, \{ m_{k,\beta} \})$ over $G$ by setting

\begin{equation}
m_{1,0} (x) = (-1)^{cd_1} dx
\end{equation}

\begin{equation}
m_{2,0} (x, y) = (-1)^{cd_1} x \wedge y
\end{equation}

and $m_{k,0} = 0$ for $k \neq 1, 2$.

If $\tilde{f} : C \to R [-p, q \text{ mod } 2]$ is any $R$-linear map (for some $p \in \mathbb{Z}$ and $q \in \{ 0, 1 \}$) and we set

\[\langle x \otimes y \rangle := (-1)^{cd_0} x \wedge y \int x \wedge y,
\]

then $(C, \{ m_{k,\beta} \})$ satisfies Eq \ref{eq:filtered-sign}. It follows that $(C, \{ m_{k,\beta} \}, \langle \cdot \rangle)$ is cyclic if $\langle \cdot \rangle$ is non-degenerate and the induced pairing on $H (C, d)$ is perfect (making $H (C, d)$ a Frobenius algebra). In this case we say $(C, d, \wedge, \tilde{f})$ is a cyclic DGA (over $R$).

**Example 7.** Here is a special case of Example \ref{example:6} that will be important. Set $R = \mathbb{C}$. Let $L$ be a closed, non-orientable manifold. Take $C (L)$ to be the $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{C}$ vector space with

$$C^{a,b} (L) = \Omega^a \left( L; Or (TL)^{\otimes c_2} \right),$$

the smooth differential $a$-forms forms on $L$ with values in the local system $Or (TL)$ (cf. §11). The exterior derivative $d$ and wedge product $\wedge$ make $C (L)$ a DGA over $R = \mathbb{C}$. Integration $\int : C \to C \left[ {-n, 1 \text{ mod } 2} \right]$ turns it into a cyclic DGA, where we set $\int \omega = 0$ unless $\omega \in C^{n-1} (L)$.

**Definition 8.** If $(C, d, \wedge)$ is an $R$-DGA, a $G$-gapped twisted $A_\infty$ algebra structure $\{ m_{k,\beta} \}$ on $C$ over $R$ will be called a deformation of $(C, d, \wedge)$ if $m_{1,0}$ and $m_{2,0}$ are given by Eqs \ref{eq:deformation} \& \ref{eq:deformation} and $m_{k,0} = 0$ for $k \geq 3$. Similarly, a cyclic deformation of a cyclic DGA $(C, d, \wedge, \tilde{f})$ is a cyclic twisted $A_\infty$ algebra $(C, \{ m_{k,\beta} \}, \langle \cdot \rangle)$ where $\langle \cdot \rangle$ is
defined by Eq \((\ref{eq:1})\), such that \((C, \{m_{k,\beta}\})\) is a deformation of \((C, d, \wedge)\). A (cyclic) deformation is called *unital* if the unit of the DGA is also a unit for the twisted \(A_\infty\) algebra.

For a justification for this terminology, see Remark \([\ref{rem}13]\).

### 2.2. Twisted \(A_\infty\) algebras as differentials; morphisms and homotopies.

To better see what’s going on, and give cleaner and more general (or at least easy to generalize) definitions and proofs, we reinterpret twisted \(A_\infty\) algebras as differentials on a certain bar coalgebra, see Proposition \([\ref{prop}12]\) below (this is the easy-to-use equivalent definition we referred to in the introduction).

Fix a discrete submonoid \(G\) and a \((2\mathbb{Z} \oplus 0)\)-graded commutative unital real algebra \(R\). Consider the *Novikov ring* of formal, possibly infinite, sums

\[
\Lambda = \Lambda_0^G (R) := \left\{ \sum_{\beta \in G} a_\beta \mu(\beta) T^{E(\beta)} | a_\beta \in R \right\}
\]

(as a set, this is just the set of maps \(R^G\). The product is defined using the addition in \(G\)). It is complete with respect to the filtration of ideals \(\{F^E \Lambda\}_{E \in \mathbb{R}_\geq 0}\) given by

\[
F^E \Lambda := \left\{ \sum_{\{\beta \in G | E(\beta) \geq E\}} a_\beta \mu(\beta) T^{E(\beta)} | a_\beta \in R \right\}
\]

We give \(\Lambda\) a \(\mathbb{Z} \oplus 0 \subset (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\) grading by setting \(\deg \epsilon = (1, 0)\) and \(\deg T = (0, 0)\). We denote by \(1_\Lambda \in \Lambda\) the unit of \(\Lambda\). Thus, it becomes a monoid in \(\text{FGVect}\), and we denote by \(\text{BiMod}_\Lambda\) the category of \(\Lambda\)-bimodules - see \([\ref{def}1.1]\) for precise explanation of what this means.

Now let \(C\) be a \((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-graded \(R\)-module, taken with the discrete filtration (see \([\ref{def}1.1]\)). We construct an object \(C^G \in \text{BiMod}_\Lambda\) by setting \(C^G := \Lambda_0^G (R) \hat{\otimes} RC\) with the usual grading and filtration on the tensor product. The bimodule structure is defined in the following, non-symmetric, way:

\[
\epsilon r_1 T^{E_1} \otimes \left( c r^T \right) \otimes \epsilon r_2 T^{E_2} \mapsto (-1)^{(cd \epsilon + bs \epsilon + 1)r_1} c \epsilon r_1 + r_2 T^{E_1 + E_2}
\]

**Remark 9.** If \(R = \mathbb{R}\) and \(\mu (G) \subset 2\mathbb{Z}\) we can identify \(\Lambda_0^G (R)\) with \(\Lambda_0^G\) in Definition \(6.3\) of \([\ref{def}3]\) by setting \(\epsilon = \epsilon^2\). In this case \(C^G\) becomes a symmetric bimodule (i.e. a \(\Lambda\)-module). See also Remark \([\ref{rem}4]\).

We now define the *bar coalgebra* \(\B (C^G)\) associated with the bimodule \(C^G\). As an object of \(\text{BiMod}_\Lambda\) we have

\[
\B (C^G) := \bigoplus_{k \geq 0} (C^G \{1\})^{\hat{\otimes} \Lambda_k}
\]

where \(\hat{\otimes}\) is the coproduct in \(\text{BiMod}_\Lambda\), so

\[
\B (C^G) = \{ (x_0, ..., x_k, ...) \mid \exists E_i \rightarrow \infty \text{ s.t. } x_i \in F^{E_i} (C^G \{1\})^{\hat{\otimes} \Lambda_k} \}
\]

(cf. Definition 3.2.16 in \([\ref{def}4]\)). We denote by \(i_j : (C^G \{1\})^{\hat{\otimes} \Lambda_j} \rightarrow \B (C^G \{1\})\) and \(\pi_j : \B (C^G \{1\}) \rightarrow (C^G \{1\})^{\hat{\otimes} \Lambda_j}\) the structure maps associated with the coproduct \(\hat{\otimes}\). The *comultiplication* \(\Delta : \B (C^G) \rightarrow \B (C^G) \hat{\otimes}_\Lambda \B (C^G)\) is defined by
\[ \Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k) , \]
and the counit \( \eta \) is the projection \( \pi_0 : \mathcal{B}(C^G) \to \Lambda \). \( \mathcal{B}(C^G), \Delta, \eta \) is a coalgebra in the sense that it is a comonoid in the category \( \text{BiMod}_A \).

Morphisms between coalgebras are defined in the usual way, and are always assumed to be counital (dualizing the notion of unital algebra morphism).

**Definition 10.** If \( f_1, f_2 : (Q, \Delta, \eta) \to (Q', \Delta', \eta') \) are two coalgebra morphisms, a map \( h : Q[-d] \to Q' \) will be called an \((f_1, f_2)\)-coderivation of degree \( d \) if

\[ \eta' h = 0 \]

(so for us, coderivations are always counital) and

\[ \Delta \circ h = [(h \otimes f_2) \pm (f_1 \otimes h)] \circ \Delta \]

where

\[ ((h \otimes f_2) \pm (f_1 \otimes h))(x \otimes y) := hx \otimes f_2 y + (-1)^{d \cdot cd} \cdot f_1 x \otimes hy. \]

A differential on a coalgebra \( (Q, \Delta, \eta) \) is an \((\text{id}_Q, \text{id}_Q)\)-coderivation \( m : Q[-1] \to Q \) of degree 1 such that \( m^2 = 0 \).

Let \( C \) be an \( R \)-module and let \( \mathcal{B} = \mathcal{B}(C^G) \) be the corresponding bar coalgebra. Let \( \text{Coder}^1(\mathcal{B}) \) denote the set of \((\text{id}_B, \text{id}_B)\)-coderivations of degree 1. We have the following bijections of sets:

\[ \text{Coder}^1(\mathcal{B}(C^G)) \cong \frac{1}{\text{BiMod}_A(\mathcal{B}, C^G[2])} \prod_{k \geq 0} \text{BiMod}_A(C^G[1] \hat{\otimes}_A k, C^G[2]) \]

\[ \cong \prod_{k \geq 0} \text{Mod}_R(C \otimes \mu^k; C^G[2-k]) \cong \prod_{k \geq 0} \text{Mod}_R(C \otimes \mu^k; C[2-k-\mu(\beta)]) . \]

The bijection (1) is given by the maps \( m \mapsto \pi_1 \circ m \) and \( \Xi^2 \circ (\text{id}_B \otimes \rho \otimes \text{id}_B) \circ \Delta^2 \leftrightarrow \rho \) where \( \Delta^2 : \mathcal{B} \to \mathcal{B} \hat{\otimes}_A \mathcal{B} \hat{\otimes}_A \mathcal{B} \) is the reiteration of \( \Delta \) and \( \Xi^d : \mathcal{B} \hat{\otimes}_A C^G[1+d] \hat{\otimes}_A \mathcal{B} \to \mathcal{B}[d] \) is the unique continuous additive map which satisfies

\[ \Xi^d((x_1 \otimes \cdots \otimes x_a) \otimes y \otimes (z_1 \otimes \cdots \otimes z_b)) = (-1)^d \sum_{i=1}^{a} (cd x_i) \otimes x_a \otimes y \otimes z_1 \otimes \cdots . \]

(2) comes from the direct sum decomposition \( B = \hat{\otimes}_{k \geq 0} (C^G[1]) \hat{\otimes}_A k \). (3) is the extension/restriction of scalars adjunction for \( R \to \Lambda \), with a \( k \)-degree shift, and (4) comes from the isomorphism of \( R \)-modules \( C^G = \prod_{\beta \in G} e^{\mu(\beta)} \otimes E(\beta) \).

**Definition 11.** A coderivation \( m : \mathcal{B}(C^G) \to \mathcal{B}(C^G) \) will be called tame if \( (\pi_1 \circ m \circ i_0)(1_A) \in \mathcal{F}^{>0}(C^G[1]) \).

If \( C' \) is another \( R \)-module with \( \mathcal{B}' = \mathcal{B}(C'^G) \) the corresponding bar coalgebra, a morphism \( f : \mathcal{B} \to \mathcal{B}' \) will be called tame if \( (\pi'_1 \circ f \circ i_0)(1_A) \in \mathcal{F}^{>0}(C'^G[1]) \), where \( \pi'_1 : \mathcal{B}' \to C'^G[1] \) is the projection.

\[ ^1 \text{Indeed it is not hard to show \( \text{Im} \Delta \) is not contained in the incomplete tensor product \( \mathcal{B}(C^G) \otimes_A \mathcal{B}(C^G), \) generally speaking. So this is not the same as a coalgebra in the usual sense which "happens to be" complete. With this caveat pointed out, we will none-the-less find it convenient to refer to it simply as a coalgebra.} \]
The following simple proposition is important, in that it allows us to redefine twisted $A_\infty$ algebras as tame differentials on the bar coalgebra. We will work from this vantage point in the remainder of the paper.

**Proposition 12.** Let $G$ be a discrete submonoid, $R$ a commutative unital real $(2\mathbb{Z} \oplus 0)$-graded algebra, and $C$ a $(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-graded $R$-module. Under the bijection (7), tame differentials $m$ on $B = B(C^G)$ correspond to twisted $A_\infty$ algebra structures $\{m_{k,\beta}\}$ on $C$.

**Proof.** Let $m \in \text{Coder}^1(B)$. For $x_1, \ldots, x_k \in C$ we write

$$(\pi_1 \circ m \circ m)(x_1 \otimes \cdots \otimes x_k) = \sum_{\beta \in G} \lambda(\beta) C(x_1, \ldots, x_k; \beta)$$

for $C(x_1, \ldots, x_k; \beta) \in C$ and $\lambda(\beta) := e^{\mu(\beta)} T^E(\beta) \in \Lambda^{\mu(\beta), 0}$. Clearly $m^2 = 0$ iff $C(x_1, \ldots, x_k; \beta) = 0$ for all $x_1, \ldots, x_k \in C$ and $\beta \in G$. We compute $C(x_1, \ldots, x_k; \beta)$ as follows.

$$\lambda(\beta) C(x_1, \ldots, x_k; \beta) =$$

$$= \sum (-1)^{\Pi_1} \lambda(\beta_1) m_{k_1, \beta_1} (x_1 \otimes \cdots \otimes x_{i-1} \otimes$$

$$\otimes \lambda(\beta_2) m_{k_2, \beta_2} (x_{i} \otimes \cdots \otimes x_{i+k_2-1} \otimes x_{i+k_2} \otimes \cdots \otimes x_k)$$

$$= \lambda(\beta_1) \sum (-1)^{\Pi_1+\Pi_2} m_{k_1, \beta_1} (x_1 \otimes \cdots \otimes m_{k_2, \beta_2} (x_i \otimes \cdots \otimes x_{i+k_2-1}) \otimes \cdots) =$$

$$= \lambda(\beta) \sum (-1)^{\Pi_1+\Pi_2+\Pi_3} m_{k_1, \beta_1} (x_1 \otimes \cdots \otimes m_{k_2, \beta_2} (x_i \otimes \cdots \otimes x_{i+k_2-1}) \otimes \cdots)$$

where all the sums range over

$$\{(k_1, k_2, \beta_1, \beta_2, i) | k_1 + k_2 = k + 1, \beta_1 + \beta_2 = \beta, 1 \leq i \leq k - k_2 + 1\},$$

and the signs are

$$\Pi_1 = \sum_{j=1}^{i-1} (\text{cd} x_j - 1)$$

$$\Pi_2 = \sum_{i=1}^{j-1} (\text{cd} x_j + \text{ls} x_j + 1) \mu(\beta_2)$$

$$\Pi_3 = \mu(\beta_2) - 1$$

Clearly, the vanishing of $C(x_1, \ldots, x_k; \beta)$ is equivalent to Eq (3). Requiring $m$ to be tame is tantamount to Eq (2). \qed

We will often refer to a pair $(B, m)$ as a twisted $A_\infty$ algebra; by that we mean that $B = B(C^G)$ for some $C$ and $m$ is a tame differential on $B$.

**Remark 13.** Let $G, G'$ be two discrete submonoids, and let $R, R'$ be two unital commutative $(2\mathbb{Z} \oplus 0)$-graded real algebras. Denote $\Lambda := \Lambda_0^R(G)$ and $\Lambda' := \Lambda_0^{R'}(G')$, and suppose we have a map $\Lambda \rightarrow \Lambda'$. If $C$ is any $R$-module, we have a natural isomorphism

$$B_\Lambda (C \otimes_R \Lambda) \otimes \Lambda \rightarrow B_{\Lambda'} (C \otimes_R \Lambda')$$

which commute with $\Delta, \eta$ in the obvious way. Hereafter, we use a subscript as in $B_\Lambda (C \otimes_R \Lambda)$ to denote the underlying $\mathbf{FGVect}$ monoid over which the bar complex
is constructed, unless $\Lambda$ is clear from the context. It follows that any differential $m$ on $B_{\Lambda}(C \hat{\otimes}_{R} A)$ defines a differential $m := m \otimes 1_{N}$ on $B_{N}(C \hat{\otimes}_{R} A)$.

A special case of the above is when we take $G' = \{0\}$, $R' = R$, so $\Lambda' = R$. We then have a quotient map $\Lambda \to R' = R$ obtained by setting $T = 0, \epsilon = 0$. If $(C, \{m_{k,\beta}\})$ is a $G$-gapped twisted $A_{\infty}$ algebra over $R$, which is a deformation of some DGA $(C, d, \wedge)$ (cf. Definition $8$), then $m' = m \otimes 1_{R}$ is the twisted $A_{\infty}$ algebra which corresponds to $(C, d, \wedge)$ as in Example $5$.

**Definition 14.** (a) Let $(B, m), (B', m')$ be twisted $A_{\infty}$ algebras. A twisted $A_{\infty}$ morphism $f : (B, m) \to (B', m')$ is a tame coalgebra morphism $f : B \to B'$ which is a chain map: $f \circ m = m' \circ f$.

(b) Given two twisted $A_{\infty}$ morphisms $f_{1}, f_{2} : (B, m) \to (B', m')$ a homotopy $h : f_{1} \Rightarrow f_{2}$ is an $(f_{1}, f_{2})$-coderivation $h : B \rightarrow B'$ of degree $(-1)$ with $m'h + hm = f_{2} - f_{1}$.

**Remark 15.** The bijection $11$ generalizes easily to $(f_{1}, f_{2})$-coderivations of any degree $d$ if we set $\Xi^{d} = (f_{1} \otimes \rho \otimes f_{2}) \circ \Delta^{d} \leftrightarrow \rho$.

If $B = B(C^{G})$ and $B' = B(C'^{G})$ are two bar coalgebras and we let $Mor(B, B')$ denote the set of tame coalgebra morphisms $f : B \to B'$, then we have a bijection $f \mapsto \{f_{k, \beta} | f_{0,0} = 0\}$

$$ Mor(B, B') \simeq \prod_{(k, \beta) \neq (0,0)} \text{Mod}_{R}(C^{\otimes_{R} k}, C'[1 - k - \mu(\beta)]) $$

The maps $f_{k, \beta} : C^{\otimes_{R} k} \to C'[1 - k - \mu(\beta)]$ are uniquely determined by the following equation:

$$ \pi'_{1} \circ f \circ i_{k} = \sum_{\beta} \epsilon^{\mu(\beta)} T^{E(\beta)} f_{k, \beta} $$

where $i_{k} : (C^{G})^{\hat{\otimes}_{R} k} \to B(C^{G})$ and $\pi'_{1} : B' \to C'^{G}$ are the coproduct structure maps.

This is proved by writing down bijections, similar to Eq $11$. The bijection (1) in Eq $11$ is replaced by the bijection of sets

$$ Mor(B(C^{G}), B(C'^{G})) \simeq \{ \rho : B(C^{G}) \to C'^{G}[1] : (\rho \circ i_{1})(1_{A}) \in \mathcal{F}^{>0}(C'^{G}) \} $$

where $\rho$ maps to the morphism

$$ \hat{\rho} := \sum_{m \geq 0} i'_{m+1} \circ \rho^{\otimes(m+1)} \circ \Delta^{m} $$

where $i'_{m+1} : (C'^{G}[1])^{\hat{\otimes}_{A}(m+1)} \to B(C'^{G}[1])$ is the coproduct structure map. We need to require $(\rho \circ i_{1})(1_{A}) \in \mathcal{F}^{>0}(C'^{G})$ for the sum in $13$ to converge; see also the discussion following Eq (3.2.28) in $4$ (the requirement that differentials be tame will be used only later, in the proof of the homological perturbation lemma, Theorem $24$). The other modifications to Eq $11$ are straightforward.

At any rate, using these bijections one can spell out everything components: the twisted $A_{\infty}$ morphism relation $f \circ m = m' \circ f$, the formula for the composition of two morphisms $f_{1} \circ f_{2}$, and the homotopy equation $m'h + hm = f_{2} - f_{1}$, to name a few relations. Since we will avoid working in components, we only illustrate this principle with the following proposition.
Proposition 16. Let \( (C, \{ m_{k, \beta} \}) \) and \( \left( C', \left\{ m'_{k, \beta} \right\} \right) \) be twisted \( A_{\infty} \) algebras, with corresponding differential coalgebras \( (B, m) \) and \( (B', m') \). The bijection \( \Xi \) induces a bijection from the set of twisted \( A_{\infty} \) morphisms \( f : (B, m) \to (B', m') \) and sets of \( R \)-module maps

\[
\{ f_{k, \beta} : C^{\otimes n} k \to C' [1 - k - \mu(\beta)] \vert f_{0,0} = 0 \}_{k \geq 0, \beta \in G}
\]

such that for any \( k \geq 0, \beta \in G \) and \( x_1, ..., x_k \in C \) we have

\[
\sum (-1)^{kL} m_{r, \beta_0} (f_{k_1, \beta_1} (x_1, ..., x_{k_1}), ..., f_{k_r, \beta_r} (x_{k-k_r+1}, ..., x_k)) = \sum (-1)^{kR} x f_{k_1, \beta_1} (x_1, ..., m_{k_2, \beta_2} (x_i, ..., x_{i+k_2-1}), ..., x_k).
\]

The sum on the left hand side ranges over all \( r \)-tuples of pairs \( ((k_1, \beta_1), ..., (k_r, \beta_r)) \) and \( \beta_0 \in G \) with \( \sum_{j=1}^r k_j = k \) and \( \beta_0 + \sum_{j=1}^r \beta_j = \beta \). The sum on the right hand side ranges over all \( k_1, k_2 \geq 0 \) and \( \beta_1, \beta_2 \in G \) such that \( k_1 + k_2 = k + 1 \) and \( \beta_1 + \beta_2 = \beta \). The signs are as follows.

\[
\Xi_L = \sum_{i=1}^r \mu(\beta_i) + \sum_{i=1}^r \mu(\beta_b) \cdot (b - 1) + \sum_{j \leq \sum_{i=1}^{i-1} k_i} (l s x_j + c d x_j - 1),
\]

\[
\Xi_R = \sum_{j=1}^{i-1} (c d x_j - 1) + \mu(\beta_2) \cdot \sum_{j=1}^{i-1} (1 + ls x_j + cd x_j).
\]

Proof. Straightforward.

We will find it convenient to denote twisted \( A_{\infty} \) morphisms also as \( f : (C, \{ m_{k, \beta} \}) \to \left( C', \left\{ m'_{k, \beta} \right\} \right) \). This always means a map of the corresponding bar coalgebras or, equivalently, a set of \( R \)-module maps \( \{ f_{k, \beta} \} \) as in Proposition 16.

2.3. Pairing cocycles, cyclic and unital morphisms. Let \( G \) be a discrete submonoid, \( R \) a commutative unital \((2\mathbb{Z} \oplus 0)\)-graded real algebra. Let \( (B, m) \) be a \( G \)-gapped twisted \( A_{\infty} \) algebra over \( R \). Recall this means \( B = B(C^G) \) for some \((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-graded \( R \)-module \( C \).

We define \( CC(B) := BiMod_{\Lambda}(B, \Lambda) \) to be the complex of \( \Lambda \)-bimodule maps equipped with the differential \( m^* : CC(B) \to CC(B) [1] \) defined by \( m^* \phi = \phi \circ m \).

Note every pairing \( C \otimes_R C \xrightarrow{\langle \cdot , \cdot \rangle} R \) \([\text{mod } 2]\) defines an element \( \diamond = \langle \cdot , \cdot \rangle \otimes id_\Lambda \in CC^{-p,q}(B) \) by setting

\[
\diamond (x_1 \otimes x_2) = \langle x_1, x_2 \rangle
\]

for \( x_i \in C \subset C^G \) and

\[
\diamond (x) = 0
\]

for \( x \in (C^G [1])^{\otimes_k} \) for \( k \neq 2 \). Given \( \diamond \in CC^{-p,q}(B) \) we can “read off” the unique pairing \( \langle \cdot , \cdot \rangle \) which produced it.

Definition 17. An element \( \diamond \in CC^{-p,q}(B) \) will be called a pairing cocycle for \( (B, m) \) of degree \((p, q)\) if \( \diamond = \langle \cdot , \cdot \rangle \otimes id_\Lambda \) where \( \langle \cdot , \cdot \rangle : C \otimes_R C \to R \) \([\text{mod } 2]\) is an antisymmetric, non-degenerate pairing which induces a perfect pairing on \( H(C, m_{1,0}) \), and we have

\[
m^* \diamond = 0.
\]
We then have

**Proposition 18.** Pairing cocycles are in bijection with pairings \( \langle \cdot \rangle : C \otimes_R C \to R \) such that \((C, \{m_{k,\beta}\}, \langle \cdot \rangle)\) is a cyclic twisted \(A_\infty\) algebra (cf. Definition 19).

**Proof.** A straightforward computation shows equation (5) is equivalent to \(m^* \hat{\cdot} = 0\).

Thus we may refer to \((B, \mu, \hat{\cdot})\) as a cyclic twisted \(A_\infty\) algebra. Let \((B, \mu, \hat{\cdot}), (B', \mu', \hat{\cdot}')\) be cyclic twisted \(A_\infty\) algebras.

**Definition 19.** (a) A cyclic morphism \(f : (B, \mu, \hat{\cdot}) \to (B', \mu', \hat{\cdot}')\) is a twisted \(A_\infty\) morphism \((B, \mu) \to (B', \mu')\) such that \(f^* \hat{\cdot}' = \hat{\cdot}\).

(b) If \(e \in C^{0,0} \subset C^G \subset B\) (respectively, \(e' \in C^{0,0} \subset B'\)) is a unit for \((B, \mu)\) (resp. \((B', \mu')\)), a morphism \(f : (B, \mu) \to (B', \mu')\) will be called unital if \(fe = e'\).

**Remark 20.** If \(\mu(G) \subset 2\mathbb{Z}\) it is not hard to see that a morphism \(\{f_{k,\beta}\}\) is cyclic (resp. unital) iff Eqs. (73,74) (resp. (72)) in [3] hold.

**Remark 21.** For the purposes of this paper, pairing cocycles will serve just as a convenient book-keeping device. In future work we will like to use them to give a more meaningful definition of the homotopy theory of cyclic twisted \(A_\infty\) algebras along the lines of Kontsevich and Soibelman’s work [7].

### 3. Homological algebra of twisted \(A_\infty\) algebras

In this section we will show that under certain assumptions, one can construct minimal models for (possibly cyclic or unital) twisted \(A_\infty\) algebras. This is a central tool for analyzing the homotopy theory of twisted \(A_\infty\) algebras.

Let \((C, d)\) be a dg \(R\)-module. Namely, \(C\) is a \((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-graded \(R\)-module and \(d : C \to C[1]\) is an \(R\)-linear map which squares to zero. We denote \(HC = H(C, d)\) and define \(d'x := (-1)^{\text{deg} x}d x\).

**Definition 22.** A twisted \(A_\infty\) algebra \((C, \{m_{k,\beta}\})\) is called a perturbation of \((C, d)\) if \(m_{1,0} = d'\).

We will construct a minimal model for a perturbation of \((C, d)\) by transferring the coalgebra differential to \(HC := H(C, d)\). In order to carry this out we need some auxiliary data.

**Definition 23.** A retraction of \((C, d)\) to \(HC\) is a 3-tuple \((\Pi_{1,0}, I_{1,0}, h_{1,0})\) (see Remark 20) where

\[
\Pi_{1,0} : C \to HC, \quad I_{1,0} : HC \to C
\]

are chain maps:

\[
\Pi_{1,0}d' = 0, d'I_{1,0} = 0,
\]

which satisfy

\[
\Pi_{1,0}I_{1,0} = \text{id}_{HC},
\]

and \(h_{1,0} : C \to C[-1]\) is an \(R\)-module map such that

\[
d'h_{1,0} + h_{1,0}d'x = I_{1,0}\Pi_{1,0} - \text{id}_C.
\]

In addition, we require that the following *side conditions* hold:

\[
(15) \quad h_{1,0}^2 = 0 \quad \Pi_{1,0}h_{1,0} = 0 \quad h_{1,0}I_{1,0} = 0
\]
**Theorem 24.** Let \((B(C^G), m)\) be a perturbation of \((C, d)\) and \((\Pi_{1,0}, I_{1,0}, h_{1,0})\) a retraction of \((C, d)\) to \(HC\). Then there exists

1. a perturbation \(B((HC)^G), m^{can}\) of \((HC, 0)\),
2. twisted \(A_\infty\) morphisms
   \[
   \Pi : (B(C^G), m) \rightarrow (B((HC)^G), m^{can})
   \]
   and
   \[
   I : (B((HC)^G), m^{can}) \rightarrow (B(C^G), m)
   \]
   with \(\Pi \circ I = id_{B((HC)^G)}\), and
3. a homotopy \(h : id_{B(C^G)} \Rightarrow I \circ \Pi\).

**Remark 25.** Our somewhat odd notation for the retraction is justified by the fact that the \((1, 0)\) component (see Remark 15) of the coalgebra morphisms \(\Pi, I\) and homotopy \(h\) are the retraction’s \(\Pi_{1,0}, I_{1,0}\) and \(h_{1,0}\), respectively.

**Proof.** Let \(\{m_{k,\beta} : C^\otimes_n \rightarrow C[2 \cdot k - \mu (\beta)]\}\) denote the components of \(m\) and let \(\hat{m}_{k,\beta} : (C^G[1])^\otimes_n \rightarrow C^G[1]\) denote the corresponding coderivations under the bijection \([11]\), so \(m = \sum_k \hat{m}_{k,\beta}\). Let \(\partial' = m - \hat{m}_{1,0}\).

Set \(\mathcal{P} = I_{1,0} \circ \Pi_{1,0}\). We define morphisms \(\hat{\Pi}_{1,0} : B(C^G) \rightarrow B((HC)^G)\) and \(\hat{I}_{1,0} : B((HC)^G) \rightarrow B(C^G)\), and an \((\mathcal{P}, id_C)\)-coderivation \(\hat{h}_{1,0} : B(C^G)[+1] \rightarrow B(C^G)\), by setting

\[
\hat{\Pi}_{1,0} (x_1, \ldots, x_k) = \Pi_{1,0} (x_1) \otimes \cdots \otimes \Pi_{1,0} (x_k),
\]
\[
\hat{I}_{1,0} (y_1, \ldots, y_k) = I_{1,0} (y_1) \otimes \cdots \otimes I_{1,0} (y_k), \text{ and}
\]
\[
\hat{h}_{1,0} (x_1, \ldots, x_k) = \sum_{i=1}^k (-1)^{i-1} \sum_{j=1}^{i-1} \sum_{d} \partial' x_i \otimes \cdots \otimes x_{i-1} \otimes h_{1,0} x_i \otimes \mathcal{P} x_{i+1} \otimes \cdots \otimes \mathcal{P} x_k,
\]

for \(x_1, \ldots, x_k \in C^G, y_1, \ldots, y_k \in (HC)^G\), cf. Eqs \([113]\). We claim the following expressions satisfy the conditions set out in the theorem.

\[
m^{can} = \sum_{a=0}^{\infty} \hat{\Pi}_{1,0} \partial' \left(\hat{h}_{1,0} \partial'\right)^a \hat{I}_{1,0}
\]
\[
\Pi = \sum_{a=0}^{\infty} \hat{\Pi}_{1,0} \left(\partial' \hat{h}_{1,0}\right)^a
\]
\[
I = \sum_{a=0}^{\infty} \left(\hat{h}_{1,0} \partial'\right)^a \hat{I}_{1,0}
\]
\[
h = \sum_{a=0}^{\infty} \hat{h}_{1,0} \left(\partial' \hat{h}_{1,0}\right)^a.
\]
Let us explain why the infinite sums converge point-wise. Suffice it show that for every \( x \in \mathcal{B} = \mathcal{B} (C^G) \) and \( E \in \mathbb{R}_{\geq 0} \) there exists some \( N \) with \( \left( \partial' \hat{h}_{1,0} \right)^a x \in \mathcal{F}^E \mathcal{B} \) for all \( a \geq N \). Write \( x = x_0 + x_1 + \cdots + x_k + \cdots \) where \( x_k \in (C^G)^\otimes k \). We say \( x_k \) has length \( k \).

There exists some \( N' \) such that

(a) \( x_k \in \mathcal{F}^E \mathcal{B} (C^G) \) for all \( k > N' \) and

(b) \( \sum_{i=1}^{N'} E (\beta_i) \geq E \) for any sequence of \( N' \) elements \( \beta_i \in G, \beta_i \neq 0 \) (see Def. [1]).

Now we take \( N = 3N' \), and check that \( \left( \partial' \hat{h}_{1,0} \right)^a x \in \mathcal{F}^E \mathcal{B} \) for all \( a \geq N \). Write \( \partial' = \hat{m}_{\geq 2,0} + \hat{m}_{\neq 0} \) where \( \hat{m}_{\geq 2,0} = \sum_{k \geq 2} \hat{m}_{\geq 2,k} \) and \( \hat{m}_{\neq 0} = \sum_{\beta \neq 0} \hat{m}_{\beta} \). Since all the maps preserve the filtration, we find that \( \left( \partial' \hat{h}_{1,0} \right)^a x = \left( \hat{m}_{\geq 2,0} \hat{h}_{1,0} + \hat{m}_{\neq 0} \hat{h}_{1,0} \right)^a x \equiv N' \mod \mathcal{F}^E \mathcal{B} \), where \( x \equiv x_0 + x_1 + \cdots + x_{N'} \). Expand \( \left( \hat{m}_{\geq 2,0} \hat{h}_{1,0} + \hat{m}_{\neq 0} \hat{h}_{1,0} \right)^a \) into a sum of \( 2^a \) products. Suppose one of these products has \( a_+ \) factors of \( \hat{m}_{\neq 0} \hat{h}_{1,0} \), which may increase the length, and \( a_- \) factors of \( \hat{m}_{\geq 2,0} \hat{h}_{1,0} \) which decrease the length, so \( a_- \leq a_+ + N' \) or else the product vanishes on \( x \equiv N' \). Since we also have \( a_+ + a_- = a \) we find that for \( a \geq N = 3N' \) we have \( a_+ \geq N' \) which, by condition (b), implies that \( \left( \partial' \hat{h}_{1,0} \right)^a x \equiv N \in \mathcal{F}^E \mathcal{B} \). This completes the proof that (17) is well-defined.

The verification that (17) satisfy all the conditions of the theorem is a standard result, which we omit. See the coalgebra perturbation lemma (2.1.) of [20] for a very similar claim, with essentially the same proof.

We will explain the relation between this formalism and the more familiar formalism of labeled ribbon trees, which is also used by Fukaya in [3], in Remark 29 below.

3.1. Cyclic and unital homological algebra. Next we want to discuss unital and cyclic refinements of Theorem 24.

**Definition 26.** Given a pairing \( C \otimes_R C \to R [-p, q] \), a retraction \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) will be called cyclic if

\[
\langle h_{1,0} x, y \rangle + (-1)^{cd} x, h_{1,0} y \rangle = 0
\]

for any \( x, y \in C \). An element \( e \in C^{0,0} \) is a unit for the retraction \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) if \( h_{1,0} e = 0 \).

**Proposition 27.** (a) Let \( (\mathcal{B} (C^G), m, \langle \cdot \rangle) \) be a cyclic twisted \( A_{\infty} \) algebra, such that \( (\mathcal{B} (C^G), m) \) is a perturbation of \( (C, d) \). Let \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) be a cyclic unital retraction of \( (C, d) \) to \( HC \). Let \( I \) and \( \Pi \) be given by Eq (17). Then \( m^{can} \) is cyclic with respect to the induced pairing on \( HC \) and the morphism \( I \) is cyclic.

(b) Let \( (\mathcal{B} (C^G), m) \) be a twisted \( A_{\infty} \) algebra with unit \( e \). If \( e \) is a unit for the retraction \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) then \( \Pi_{1,0} e \) is a unit for \( m^{can} \) and the morphisms \( I, \Pi \) given by Eq (17) are unital.

**Remark 28.** Let \( f : (B, m, \langle \cdot \rangle) \to (B', m', \langle \cdot \rangle) \) be a cyclic morphism. \( f^* \langle \cdot \rangle = \langle \cdot \rangle \) means, in particular, that \( \langle f_{1,0} x, f_{1,0} y \rangle = \langle x, y \rangle \). Since \( \langle \cdot \rangle \) is non-degenerate, this implies that \( f_{1,0} \) must be injective. It follows that \( \Pi \) will not be cyclic, unless \( m_{1,0} = 0 \).
Even so, we should think of $\Pi_{1,0}$ as inducing a cyclic equivalence when a cyclic retraction is used. Cf. Remark 8.4 in [3].

Proof of Proposition 27. We prove (a). Let $\hat{\diamond}^H$ denote the pairing cocycle for $B(H(C^G, m_{1,0}))$. First we show $I$ is a cyclic morphism, i.e. that $I^*\hat{\diamond} = \hat{\diamond}^H$.

We choose some $x_1, \ldots , x_k \in C$, and compute:

\[
(I^*\hat{\diamond}) (x_1, \ldots , x_k) = 
\]

\[
= \sum_{a_1, a_2 \geq 0} \sum_{k_1 + k_2 = k} \hat{\diamond} \left( \pi_1 \left( \hat{h}_{1,0} \vartheta \right)^{a_1} \hat{I}_{1,0} (x_{k_1}) \otimes \pi_1 \left( \hat{h}_{1,0} \vartheta \right)^{a_2} \hat{I}_{1,0} (x_{k_1+1}) \right).
\]

We now show that the summands with $a_1 > 0$ vanish. Indeed, for some $\sigma \in C^G$ we can write

\[
\hat{\diamond} \left( \hat{h}_{1,0} \sigma \otimes \left( \hat{h}_{1,0} \vartheta \right)^{a_2} \hat{I}_{1,0} (x_{k_1+1} \otimes \cdots \otimes x_k) \right) = 
\]

\[
= \pm \hat{\diamond} \left( \sigma \otimes \hat{h}_{1,0} \pi_1 \left( \hat{h}_{1,0} \vartheta \right)^{a_2} \hat{I}_{1,0} (x_{k_1+1} \otimes \cdots \otimes x_k) \right) = 0
\]

where the last equality is by the side conditions (16). Terms with $a_2 > 0$ vanish for the same reason, which shows that $I$ is cyclic.

Now we show that $m_{\text{can}}^*$ is cyclic:

\[
0 = I^* m^* \hat{\diamond} = (m_{\text{can}}^*)^* I^* \hat{\diamond} = (m_{\text{can}}^*)^* \hat{\diamond}^H.
\]

We prove (b). Note that, since $m_{1,0} \mathbf{e} = 0$ and $h_{1,0} \mathbf{e} = 0$, we have $I_{1,0} \Pi_{1,0} \mathbf{e} = \mathbf{e}$. Let us check $\Pi$ is unital. Expand

\[
\pi_1 \Pi = \pi_1 \Pi_{1,0} \sum \prod_{j=1}^n \left( m_{k_j, \beta_j}^{(s_j)} \circ h_{1,0}^{(t_j)} \right)
\]

where for $\mathbf{y} = y_1 \otimes \cdots \otimes y_l$, $y_i \in C$, the operators $m_{k_j, \beta_j}^{(s_j)}, h_{1,0}^{(t_j)} : B(C^G) \to B(C^G)$ (degree shift not shown) are given by

\[
m_{k_j, \beta_j}^{(s_j)} \mathbf{y} = \begin{cases} 
(-1)^{\sum_{j=1}^s (cd y_j - 1)} \cdots y_s \otimes \lambda (\beta) m_{k_j, \beta} (y_{s+1}, \ldots , y_{s+k}) \otimes \cdots 
& \text{if } l \geq s + k \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
h_{1,0}^{(t_j)} \mathbf{y} = \begin{cases} 
(-1)^{\sum_{j=1}^s (cd y_j - 1)} y_1 \otimes \cdots \otimes y_l \otimes h_{1,0} (y_{t+1}) \otimes P y_{t+2} \otimes \cdots \otimes P y_l 
& \text{if } l \geq s + 1 \\
0 & \text{otherwise}
\end{cases}
\]

see (16). The sum in (19) ranges over all $a \geq 0$ and for each $a$ over $a$-tuples of quadruples $(k_j, \beta_j, s_j, t_j)$ with $k_j, s_j, t_j$ non-negative integers and $\beta_j \in G$. We use $\Pi_{1,0}^a$ to denote the composition of operators where the factor corresponding to $j = 1$ is applied first.
We focus on the contribution of one summand of (19) and consider

\[
(20) \quad \left( \pi_1 \Pi_{1,0} \prod_{j=1}^{a} (m_{k_j, \beta_j}^{(s_j)} \circ \hat{h}_1^{(t_j)}) \right)(x)
\]

for

\[
(21) \quad x = x_1 \otimes \cdots \otimes (x_r = e) \otimes \cdots \otimes x_k.
\]

Clearly, if \( a = 0 \) we have a nonzero contribution only for \( k = 0 \), in which case we get \( \Pi_{1,0}e \). We now show all the other contributions either cancel in pairs or vanish. Clearly, if \( a \geq 1 \) and \((k_j, \beta_j) \neq (2, 0)\) for all \( 1 \leq j \leq a \) then (20) vanishes since \( \Pi_{1,0} \prod_{j=1}^{a} (m_{k_j, \beta_j}^{(s_j)} \circ \hat{h}_1^{(t_j)}) \) has length \( \geq 2 \). Suppose, then, that we have some \( b \leq a \) with \( r_b \in \{s_j + 1, \ldots, s_j + k_j\} \) for all \( 1 \leq j \leq a \) and \( s_b = r_b - 1 \) or \( s_b = r_b - 2 \). Let us clean up the indexing, replacing \( x \) with \( \prod_{j=1}^{b-1} (m_{k_j, \beta_j}^{(s_j)} \circ \hat{h}_1^{(t_j)})x \) and \( r \) with \( r_b \), so \( x \) is still of the form (21). We update \( l \) accordingly and replace \( \pi_1 \Pi_{1,0} \prod_{j=1}^{a-b} (m_{k_j, \beta_j}^{(s_j)} \circ \hat{h}_1^{(t_j)}) \) with \( \pi_1 \Pi_{1,0} \prod_{j=1}^{a-b} (m_{k_j + (b-1), \beta_j - (b-1)}^{(s_j)} \circ \hat{h}_1^{(t_j + (b-1))}) \), so now \((k_1, \beta_1) = (2, 0)\) and \( s_1 = r - 1 \) or \( s_1 = r - 2 \). We call this process *bringing e to the front*.

Note that if we had \( r \neq 1, k \), then the same is true after we bring \( e \) to the front. In this case we have

\[
m_{1,0}^{(r-2)}x + m_{1,0}^{(r-1)}x =

- \frac{1}{2} \sum_{j \leq r-2} (cd x_j)^{-1} x_1 \otimes \cdots \otimes m_{2,0} (x_{r-1} \otimes e) \otimes x_{r+1} \otimes \cdots \otimes x_l

+ \frac{1}{2} \sum_{j \leq r-1} (cd x_j)^{-1} x_1 \otimes \cdots \otimes x_{r-1} \otimes m_{2,0} (e \otimes x_{r+1}) \otimes \cdots \otimes x_l = 0,
\]

which implies that contributions with \( i \neq 1, l \) can be canceled in pairs. We complete the proof by showing contributions with \( i = 1 \) or \( i = l \) vanish. This involves checking a few cases as follows. We again assume that we’ve brought \( e \) to the front, to simplify the indices.

**Case 1**, \( i = 1, s_1 = 0, a \geq 2 \): We show \( h_1^{(t_1)} m_{2,0} h_1^{(t_1)} x = 0 \). If \( t_1 = t_2 - 1 \) this is true because \( h_1^{(t_1)} = 0 \). If \( t_1 = 0 \) we use \( h_1^{(t_1)} e = 0 \). If \( 1 \leq t_1 \leq t_2 - 2 \) we use \( h_1^{(t_1)} e = 0 \). If \( t_1 \geq t_2 \) we use \( h_1^{(t_1)} e = 0 \).

**Case 2**, \( i = 1, s_1 = 0, a = 1 \): In this case we must have \( k = 2 \); \( \pi_1 \Pi_{1,0} m_{2,0} h_1^{(t_1)} x = 0 \) for \( t_1 = 0 \) because \( h_1^{(t_1)} e = 0 \) and for \( t_1 = 1 \) because \( \Pi_{1,0} h_1^{(t_1)} = 0 \).

**Case 3**, \( i = l, s_1 = l - 1, a \geq 2 \), and **Case 4**, \( i = l, s_1 = l - 1, a = 1 \): These are similar to the previous two cases.

The proof that \( \Pi_{1,0} e \) is a unit for \( m^{can} \) and that \( I \) is a unital morphism follow the same arguments and are omitted.
Remark 29. By Proposition 12 the differential
\[ m^{\text{can}} = \sum_{a=0}^{\infty} \tilde{\Pi}_{1,0} \partial'(h_{1,0} \partial')^a \tilde{l}_{1,0} \]
corresponds to some set of maps \( \{ m_{k,\beta}^{\text{can}} \} \). We have
\[ m_{k,\beta}^{\text{can}} = \sum_{\Gamma \in \text{Gr}(k,\beta)} m_{\Gamma} \]
where \( \text{Gr}(k,\beta) \) is as in Definition 9.1 of \([3]\): it is a set of isomorphism types of \( G \)-labeled rooted ribbon trees, which are represented by triples \((T, v_0, \beta(\cdot))\) satisfying conditions (1)-(4) at the beginning of Section 9 ibid. and Definition 9.1 (the only change is that for our discrete submonoid \( G \), which appears in condition (3), we do not assume \( \mu(G) \subset 2\mathbb{Z} \)). Note this equation is formally identical to Eq (117) of \([3]\). Indeed for \( \mu(G) \subset 2\mathbb{Z} \) the contribution \( m_{\Gamma} \) is also precisely as described there. Let us briefly explain how to see this.

Expand
\[ \pi_1 m^{\text{can}} = \sum \pi_1 \tilde{\Pi}_{1,0} \left( m_{k_{a+1},\beta_{a+1}}^{(s_{a+1})} \prod_{j=1}^{a} h_{1,0}^{(t_j)} m_{k_j,\beta_j}^{(s_j)} \right) \tilde{l}_{1,0} \]
(see the proof of Proposition 27 for an explanation of this notation) and apply each summand to \( y = y_1 \otimes \cdots \otimes y_k \).

It is not hard to see that the side conditions \([15]\) imply that \( t_j = s_j \), or else the contribution vanishes (so we must apply each \( h_{1,0} \) to the output of previous \( m_{k,\beta} \)). Moreover, since \( \mathcal{P} h_{1,0} = 0 \) we find that the sequence \( l_j - k_j - t_j \) is non-increasing (so each operation is applied either after, or to the right of, the previous operations). Here \( l_j \) is the length of \( \prod_{i=1}^{a} h_{1,0}^{(t_i)} m_{k_i,\beta_i} y \), with \( l_0 = l \). This defines an obvious map from the set of (generically) nonzero summands in Eq (22) to \( G \)-labeled ribbon rooted trees.

There is a map in the other direction: embed \( T \) in \( \mathbb{R}^2 \) so that \( v_0 \) has maximal height w.r.t. some linear projection \( \mathbb{R}^2 \to \mathbb{R} \). Order the vertices of \( T \) so that the vertices most distant from \( v_0 \) (or “deepest”) appear first, and vertices with the same distance are ordered from left to right (using the planar embedding). This determines a summand of Eq (22) in an obvious way, where the order of the operations is given by the order of the vertices of \( T \). It is easy to check that these maps define a bijection between isomorphism types of \( G \)-labeled rooted ribbon trees \( \Gamma \) and (generically) non-vanishing contributions to Eq (22). If \( \mu(G) \subset 2\mathbb{Z} \) this contribution is precisely \( m_{\Gamma} \) of Eq (117) in \([3]\).

By Proposition 16 the morphism \( I \) can also be decomposed as some \( \{ I_{k,\beta} \} \), and we have
\[ I_{k,\beta} = \sum_{\Gamma \in \text{Gr}(k,\beta)} f_{\Gamma} \]
for some \( f_{\Gamma} \) which, in case \( \mu(G) \subset 2\mathbb{Z} \) are identical to \( f_{\Gamma} \) of Eq (117) in \([3]\). The reasoning is essentially the same.

\(^{2}\)There is a typo there, and \( \text{Gr}(k,\beta) \) is written \( \text{Gr}(\beta,k) \). Compare with Eq (117) ibid.
3.2. Constructing retractions in simple cases. If \((C, d)\) is a dg \(R\)-module where \(R\) is a field, constructing a retraction of \((C, d)\) to \(HC\) is a simple matter; by choosing a splitting for the short exact sequences

\[
0 \to \ker d \to C \to \text{Im} d \to 0
\]

and

\[
0 \to \text{Im} d \to \ker d \to HC \to 0
\]

we obtain an internal direct sum decomposition

\[
C = H \oplus d(C) \oplus O
\]

where \(H \simeq HC\) and \(d(C) \simeq O\) so that in these components the differential \(d\) is given by

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \phi \\
0 & 0 & 0
\end{pmatrix},
\]

and then we can take \(I_{1,0} : H \to C\) and \(\Pi_{1,0} : C \to H\) to be the structure maps associated with the decomposition (23), and

\[
h_{1,0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \phi^{-1} & 0
\end{pmatrix}.
\]

Given any \(e \in C\) with \(de = 0\), it is easy to modify the above construction so that \(e \in H \subset C\) in (23), and then the retraction is unital.

Given an antisymmetric, non-degenerate pairing \(\langle \cdot, \cdot \rangle : C \otimes_R C \to R[-p, q]\), if the decomposition (23) is an orthogonal direct sum decomposition, then the associated retraction is cyclic.

In the situation of Example 7 we can use the Hodge-De Rham decomposition, see the discussion in [3] of the orientable case, the non-orientable case is the same.

4. Equivariant cohomology

In this section we introduce the Cartan-Weil complex associated with an equivariant manifold, and explain how to extend various constructions to this equivariant setting. Throughout this section, we will not use the \((\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-grading. Some objects will be \(\mathbb{Z}\)-graded, and local systems will be explicitly indicated.

4.1. Basic definitions. Fix a positive integer \(n'\) and let \(T = \mathbb{R}^{n'}/\mathbb{Z}^{n'}\) denote the torus group.

Suppose now \(X\) is a smooth \(T\)-manifold, possibly with boundary. Let \(\{\xi_j\}_{j=1}^{n'}\) denote the \(n'\) commuting vector fields on \(X\) with 1-periodic flows, which correspond to the standard basis of \(\mathbb{R}^{n'} = \text{Lie}(T)\) under the \(T\)-action. Suppose \(\mathcal{L}\) is a \(T\)-equivariant local system on \(X\). We now introduce the Cartan-Weil complex \(\mathcal{C}^{CW}(X; \mathcal{L})\) which is a graded deformation of the usual De Rham complex of \(\mathcal{L}\)-valued differential forms on \(X\). We will adopt the viewpoint that the cohomology of the Cartan-Weil complex is the equivariant cohomology of \(X\), though it should really be seen as a way to compute the cohomology of the homotopy quotient of \(X\) by the \(T\) action. We refer the reader to [1] and [5] for more comprehensive discussions of this rich subject. Our treatment will be focused on the applications we have in mind.
We denote by $L[\alpha_1, \ldots, \alpha_n] := L \otimes_{\mathbb{C}} \mathbb{C} [\alpha_1, \ldots, \alpha_n]$ the locally constant sheaf of $\mathbb{Z}$-graded abelian groups where $\deg \alpha_i = 2$. We will often denote $\mathbb{C} [\bar{\alpha}] = \mathbb{C} [\alpha_1, \ldots, \alpha_n]$, similarly for $\mathbb{C} [\bar{\alpha}]$, etc. As an $R$-module, the Cartan-Weil complex is

$$C^{CW}(X; L) = \Omega(X; L[\bar{\alpha}])^T,$$

the abelian group of $T$-invariant differential forms on $X$ with values in $L[\bar{\alpha}]$. It is $\mathbb{Z}$-graded by the sum of the De Rham and algebraic grading of $L[\bar{\alpha}]$. We equip it with the degree one $\mathbb{C} [\bar{\alpha}]$-linear differential $D : C^{CW}(C; L) \to C^{CW}(C; L)[1]$ given by

$$D = d - \sum_{j=1}^{n'} \alpha_j \iota_{\xi_j},$$

where $\iota_{\xi_j}$ denotes contraction with the vector field $\xi_j$ on $X$. It is easy to see that $D^2 = 0$ on $T$-invariant forms. If $L_1, L_2$ are two equivariant local systems on $X$ then the usual wedge product of forms preserves $T$-invariance and so defines a map

$$\wedge : C^{CW}(X; L_1) \otimes_R C^{CW}(X; L_2) \to C^{CW}(X; L_1 \otimes_{\mathbb{C}} L_2)$$

which satisfies the graded Leibniz rule and is associative in the obvious sense. It corresponds to the cup-product in equivariant cohomology. If $f : Y \to X$ is a $T$-equivariant map the usual pullback map of forms respects the invariant forms so we obtain a map

$$f^* : C^{CW}(Y; L) \to C^{CW}(X; f^* L).$$

If $(f : X \to Y, \kappa : Or(TX) \otimes f^* Or(TY)^\vee \to K^\vee \otimes f^* L)$ is an oriented proper submersion (see Definition 60) where $K$ and $L$ are $T$-equivariant local systems on $X$ and $Y$, respectively, then have a pushforward map

$$f_*^\kappa : C^{CW}(X; K) \to C^{CW}(Y; L)[\dim Y - \dim X]$$

defined by integration on the fiber. See the appendix 6 for our conventions regarding orientations and the pushforward. An important special case is pushforward to a point, which is defined when $X$ is compact:

$$\int : C^{CW}(X; Or(TX)^\vee) \to R[\dim X].$$

Remark 30. If $X$ is a topological space with a $T$-action, and $L$ is a local system on $X$, then being $T$-equivariant is a property and does not involve additional choices (unlike lifting the action to a vector bundle over $X$, for example). To see this, consider the maps $s, t : T \times X \to X$ where $s$ is the projection and $t$ is the action map. By definition, if $L$ is $T$-equivariant there exists an isomorphism $\phi : s^* L \simeq t^* L$, which we can think of as a nonzero global section $\hat{\phi}$ of $\text{Local}_{T \times X}(s^* L, t^* L)$, the internal hom object in the category of local systems on $T \times X$. $\hat{\phi}$ is required to satisfy the identity axiom, which specifies the value of $\hat{\phi}$ at $e \times X$. Since $e \times X$ intersects every connected component of $T \times X$ we find that this specifies the value of $\hat{\phi}$ everywhere. It follows that a $T$-action, if it exists, is unique.

This also explains why we did not need to assume that the local system isomorphism $\kappa$ is $T$-equivariant: any isomorphism between $T$-equivariant local systems $L_1, L_2$ on $X$ is automatically $T$-equivariant.
4.2. Equivariant cohomology in the separated case.

**Definition 31.** Let $\mathcal{L}$ be a local system on a manifold $X$. $X$ will be called $\mathcal{L}$-separated if $H^*(X; \mathcal{L})$ is concentrated either only in even or only in odd degrees.

**Proposition 32.** If $X$ is a $\mathbb{T}$-manifold and $\mathcal{L}$ is an equivariant local system on $X$ such that $X$ is $\mathcal{L}$-separated then there’s an isomorphism of $\mathbb{C}[\alpha]$-modules $\Phi : \mathbb{C}[\alpha] \otimes H(X; \mathcal{L}) \simeq H^{CW}(X; \mathcal{L})$

**Proof.** Let $m \triangleleft \mathbb{C}[\alpha]$ denote the maximal ideal generated by $\alpha_1, \ldots, \alpha_{n'}$. Consider the filtration of $\mathbb{C}^{CW}(X; \mathcal{L})$ by the dg $\mathbb{C}[\alpha]$-submodules

$$
\mathbb{C}^{CW}(L) = m^0 \mathbb{C}^{CW}(X; \mathcal{L}) \supset m^1 \mathbb{C}^{CW}(X; \mathcal{L}) \supset \cdots
$$

Let $\left(E_i = \bigoplus_{i \geq 0} m^i \mathbb{C}^{CW}(X; \mathcal{L})/m^{i+1} \mathbb{C}^{CW}(X; \mathcal{L}), d_1 \right), (E_2, d_2), \ldots$ denote the corresponding spectral sequence. All the terms are $\mathbb{Z}$-graded $\mathbb{C}[\alpha]$-modules, where the $\mathbb{C}[\alpha]$-linear differential $d_i$ has degree 1. We have $E_1 \simeq \mathbb{C}[\alpha] \otimes \mathbb{C}(L; \mathcal{L})^T$ and $d_1 = \text{id} \otimes d$, so $E_2 \simeq \mathbb{C}[\alpha] \otimes H(X; \mathcal{L})$, which is concentrated either in even or odd (total) degree. It follows that $d_2, d_3, \ldots$ must all vanish and the spectral sequence degenerates at the second page. This means that there’s an isomorphism of $\mathbb{C}[\alpha]$-modules

$$
(25) \quad \mathbb{C}[\alpha] \otimes H(X; \mathcal{L}) \simeq \bigoplus_{i \geq 0} m^i H^{CW}(X; \mathcal{L})/m^{i+1} H^{CW}(X; \mathcal{L}).
$$

We now show that there’s an isomorphism of $\mathbb{C}[\alpha]$-modules $\mathbb{C}[\alpha] \otimes H(X; \mathcal{L}) \simeq H^{CW}(X; \mathcal{L})$. It follows from Eq (25) that the map $q : H^{CW}(X; \mathcal{L}) \to H(X; \mathcal{L})$ obtained by setting $\alpha_j = 0$ for $1 \leq j \leq n'$, is surjective. By choosing a basis for $H(X; \mathcal{L})$, choosing $q$-preimages in $H^{CW}(X; \mathcal{L})$, and then choosing equivariant $D$-closed forms representing the corresponding cohomology classes, we define a $\mathbb{C}$-linear map of complexes $\sigma : (H(X; \mathcal{L}), 0) \to (H^{CW}(X; \mathcal{L}), D)$. Let $\text{id}_{\mathbb{C}[\alpha]} \otimes \sigma : (\mathbb{C}[\alpha] \otimes H(X; \mathcal{L}), 0) \to (H^{CW}(X; \mathcal{L}), D)$ denote the extension of scalars. We claim $\Phi := H(\text{id}_{\mathbb{C}[\alpha]} \otimes \sigma) : \mathbb{C}[\alpha] \otimes H(X; \mathcal{L}) \to H^{CW}(X; \mathcal{L})$ is an isomorphism. Since it is a $\mathbb{C}[\alpha]$-module map it respects the $m$-adic filtrations, and so (since $\bigcap_{i \geq 0} m^i = 0$ and the filtration on $H^{CW}(X; \mathcal{L})$ is complete at every degree) it suffices to check that the induced map of the associated graded modules

$$
\bigoplus_{i \geq 0} (m^i \otimes H(X; \mathcal{L}) / (m^{i+1} \otimes H(X; \mathcal{L})) \to \bigoplus_{i \geq 0} m^i H^{CW}(X; \mathcal{L})/m^{i+1} H^{CW}(X; \mathcal{L})
$$

is an isomorphism. For this it suffices to check that the induced map on the $E_2$ page is an isomorphism which is obvious by construction. \hfill \Box

**Corollary 33.** (equivariant Künneth formula) Suppose $X_1, X_2$ are $\mathbb{T}$-manifolds and for $i = 1, 2$ we have an equivariant local system $\mathcal{L}_i$ on $X_i$ such that $X_i$ is $\mathcal{L}_i$-separated. Then we have the following isomorphisms of $\mathbb{C}[\alpha]$-modules:

$$
H^{CW}(X_1 \times X_2; \mathcal{L}_1 \boxtimes \mathcal{L}_2) \simeq \mathbb{C}[\alpha] \otimes H(X_1; \mathcal{L}_1) \otimes H(X_2; \mathcal{L}_2) \simeq H^{CW}(X_1; \mathcal{L}_1) \otimes_{\mathbb{C}[\alpha]} H^{CW}(X_2; \mathcal{L}_2)
$$
We use the notation $\mathcal{L}_1 \boxtimes \mathcal{L}_2 := pr_i^\ast \mathcal{L}_1 \otimes pr_2^\ast \mathcal{L}_2$ for $pr_i : X_1 \times X_2 \to X_i$ the projection.

**Proof.** We find that $X_1 \times X_2$ is $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ separated, by the usual Künneth formula. So by Proposition 32 and the Künneth formula again we have isomorphisms of $\mathbb{C}[\bar{\alpha}]$-modules:

$$H^{CW}(X_1 \times X_2; \mathcal{L}_1 \boxtimes \mathcal{L}_2) \simeq \mathbb{C}[\bar{\alpha}] \otimes H(X_1 \times X_1; \mathcal{L}_1 \boxtimes \mathcal{L}_2) \simeq \mathbb{C}[\bar{\alpha}] \otimes H(X_1; \mathcal{L}_1) \otimes H(X_2; \mathcal{L}_2) \simeq (\mathbb{C}[\bar{\alpha}] \otimes H(X_1; \mathcal{L}_1)) \otimes_{\mathbb{C}[\bar{\alpha}]} (\mathbb{C}[\bar{\alpha}] \otimes H(X_2; \mathcal{L}_2)) \simeq H^{CW}(X_1; \mathcal{L}_1) \otimes_{\mathbb{C}[\bar{\alpha}]} H^{CW}(X_2; \mathcal{L}_2)$$

$\Box$

4.3. **Equivariant angular and Euler forms.** The remainder of this section is devoted to extending some ideas from Bott and Tu's book [2] to the equivariant setting.

Let $E \xrightarrow{\pi} B$ be a smooth $T$-equivariant rank $n$ vector bundle. Recall $Or(E)$ denotes the corresponding equivariant orientation local system on $B$ (see §1.1). Fix an auxiliary $T$-invariant bundle metric, and denote by $iS : S(E) \to E$ the $T$-submanifold consisting of unit vectors in $E$. The map $\pi_S = \pi \circ iS : S(E) \to B$ makes this a $T$-equivariant sphere bundle over $B$. Whenever we discuss the sphere bundle associated with a vector bundle, we'll assume that an invariant metric is fixed on $E$. The choice of metric is immaterial: if $\phi : E \simeq E'$ is an equivariant isomorphism of vector-bundles and we fix some invariants metrics on $E$ and $E'$, then there exists an equivariant isometry $\tilde{\phi} : E \to E'$ (“normalize” $\phi$).

We define an equivariant local system isomorphism

$$\kappa : Or(TS(E)) \otimes \pi_S^\ast Or(TB)^\vee \simeq (\pi_S^\ast Or(E)).$$

Let $\mathbb{R} \to iS^\ast TE$ be the bundle map corresponding to the outward normal vector, where $\mathbb{R}$ denotes the trivial line bundle on $S(E)$. We obtain equivariant short exact sequences of vector bundles:

$$0 \to iS^\ast E \to iS^\ast TE \to iS^\ast TB \to 0 \tag{26}$$

$$0 \to \mathbb{R} \to iS^\ast TE \to TS(E) \to 0$$

where in the second line we map $\mathbb{R}$ to an outward pointing radial vector. These sequences define $\kappa$, see [6].

**Definition 34.** An equivariant angular form for $E$ is a form $\phi \in \Omega(S(E); (\pi_S^\ast Or(E)^\vee)[\bar{\alpha}])^\mathbb{T}$ of degree $(n - 1)$ such that

(a) $\pi_S^\ast (\phi) = 1$ and

(b) $D\phi = -\pi^\ast e$ for some form $e \in \Omega(B; Or(E)^\vee[\bar{\alpha}])^\mathbb{T}$.

The degree $n$ form $e$ is called the Euler form associated with $\phi$; since $\pi$ is submersive it is uniquely determined by condition (b). See also pg.113 of [2].

**Proposition 35.** Let $E \xrightarrow{\pi} B$ be a $T$-equivariant vector bundle. An equivariant angular form for $\pi$ exists.

**Proof.** Consider the ansatz
\[ \phi = \sum_{J \in \mathbb{Z}_{\geq 0}^n} \phi_J \alpha^J \quad \text{and} \quad e = \sum_{J \in \mathbb{Z}_{\geq 0}^n} e_J \alpha^J \]

where

\[ \phi_J \in \Omega^{n-2|J|}(S(E); \pi^* \Omega^*(E)^\vee)^{\mathbb{T}} \quad \text{and} \quad e_J \in \Omega^{n-2|J|}(B; \Omega^*(E)^\vee)^{\mathbb{T}}. \]

By degree considerations, we see that condition (b) in Definition 34 is equivalent to requiring

\[ (b') \quad \pi^*_S(\phi_0) = 1. \]

We define a relation \((a_j)\) on \((27)\) for every \(J = (j_1, ..., j_{n'}) \in \mathbb{Z}_{\geq 0}^n\), by the following equation.

\[ (a_j) \quad d\phi_J - \sum_{\{1 \leq a \leq n'/\lambda_a \geq 1\}} \iota_{\xi_a} \phi_J - e_a = -\pi^*_S e_J \]

where \(e_a \in \mathbb{Z}_{\geq 0}^n\) denotes the \(a\)-th standard unit vector. Clearly, condition (a) in Definition 34 is equivalent to the logical conjunction \(\bigwedge (a_j)\), taken over all \(J \in \mathbb{Z}_{\geq 0}^n\).

Bott and Tu prove that there exist \(\phi_0 \in \Omega^{n-1}(S(E); \Omega^*(E)), e_0 \in \Omega^n(B; \mathbb{C})\) such that conditions \((a_0)\) and \((b')\) hold (see [2], pg. 121-122, for the orientable case; the non-orientable case is also discussed in that book). We may assume without loss of generality that \(\phi_0\) and \(e_0\) are \(\mathbb{T}\)-invariant by averaging. That is, we can replace \(\phi_0 \leftarrow \pi^* \phi_0\) and \(e_0 \leftarrow \pi^* e_0\), where \(\pi^*: C(L) \to C(L)^{\mathbb{T}}\) is the projection operator defined by averaging out the \(\mathbb{T}\)-action using the Haar measure on \(\mathbb{T}\). It is easy to check that \(\pi^*\) commutes with \(d\) as well as with equivariant pushforward and pullback, and induces a homotopy equivalence \((C(L), d) \simeq \left(C(L)^{\mathbb{T}}, d|_{C(L)^{\mathbb{T}}}\right)\).

We will continue to use such averaging arguments to establish invariance of all the forms we need, without explicit mention.

We now proceed by induction on \(t \geq 1\). It will be convenient to extend \(\phi_J\) and \(e_J\) by zeros to all of \(\mathbb{Z}_{\geq 0}^n\). We will also define \(t_a := \iota_{\xi_a}\). With this we can rewrite the relation \((a_j)\) as

\[ (a_j) \quad d\phi_J - \sum_{J \in \mathbb{Z}_{\geq 0}^n} t_a \phi_J - e_a = -\pi^*_S e_J. \]

Now, assume that \(\phi_J, e_J\) are given as in \((27)\) for all \(|J| \leq t\), such that \((a_j)\) holds for all \(|J| \leq t\). We show that we can define \(\phi_J, e_J\) for \(\{J : |J| = t+1\}\) so that \((a_j)\) holds for such \(J\) too.

We will need the following part of the Gysin sequence for \(S(E)\).

\[ H^{n-2t-2}(B; \Omega^*(E)^\vee) \xrightarrow{\pi^*_S} H^{n-2t-2}(S(E); \Omega^*(E)^\vee) \]

Focus on some \(J = (j_1, ..., j_{n'})\) with \(|J| = t + 1\). We have

\[ H^{-2t-1}(B) = 0 \xrightarrow{\pi^*_S} H^{-2t-1}(S(E); \pi^*_S \Omega^*(E)^\vee) \xrightarrow{\pi^*_S} H^{-2t-1}(S(E); \pi^*_S \Omega^*(E)^\vee) \xrightarrow{\pi^*_S} H^{-2t-1}(S(E); \pi^*_S \Omega^*(E)^\vee) \]
In the last equality we used \( d^2 = 0 \) and \( \sum_{1 \leq a, b \leq n'} t_a t_b = 0 \). Since \( \pi_S \) is a submersion, \( \pi_S^* \) is injective and we conclude that \( \sum_{a=1}^{n'} t_a e_{J-e_a} \) is \( d \)-closed and in the kernel of

\[
\pi_S^*: H^{n-2t+1}(B; \text{Or}(E)^\vee) \to H^{n-2t+1}(S(E); \pi_S^* \text{Or}(E)^\vee).
\]

The second row of the Gysin sequence above shows that \( \pi_S^* \) is injective on cohomology classes of degree \( n-2t-1 \) so we conclude that \( \sum_{a=1}^{n'} t_a e_{J-e_a} = \delta e'_J \) for some \( e'_J \in \Omega^{n-2(t+1)}(B; \text{Or}(E)^\vee)^T \).

Next we compute

\[
d \left( - \sum_{a=1}^{n'} t_a \phi_{J-e_a} + \pi_S^* e'_J \right) = \sum_{a=1}^{n'} t_a d\phi_{J-e_a} + \pi_S^* \delta e'_J =
\]

\[
= \sum_{a=1}^{n'} t_a \left( \sum_{b=1}^{n'} t_b \phi_{J-e_a - e_b} - \pi_S^* \delta e_{J-e_a} \right) + \pi_S^* \delta e'_J = \pi_S^* \left( \delta e'_J - \sum_{a=1}^{n'} t_a e_{J-e_a} \right) = 0.
\]

Using the top row of the Gysin sequence above we conclude that

\[
\left[ - \sum_{a=1}^{n'} t_a \phi_{J-e_a} + \pi_S^* e'_J \right] \in \text{Im} \left( \pi_S^* \right).
\]

In other words, there exists some \( x \in \Omega^{n-2t}(B; \text{Or}(E)^\vee)^T \) and some \( y \in \Omega^{n-1-2t}(S(E); \pi_S^* \text{Or}(E)^\vee)^T \) such that

\[
- \sum_{a=1}^{n'} t_a \phi_{J-e_a} + \pi_S^* e'_J = \pi_S^* x + dy.
\]

It follows that

\[
\phi_J = -y
\]

\[
e_J = e'_J - x.
\]

satisfy condition \((a_J)\). In this way we define \( \phi_J, e_J \) for all \( \{ J : |J| = t + 1 \} \), establishing the inductive step. This concludes the proof of the proposition. \( \square \)

We will not use the following proposition in this paper, but state it for completeness and future reference.
Proposition 36. Let $E \xrightarrow{\pi} B$ be an equivariant vector bundle. Choose an equivariant angular form $\phi$ for $\pi$, with associated equivariant Euler form $e$.

(a) $De = 0$. Moreover, if $\tilde{\phi}$ is another equivariant angular form for $\pi$ and $\tilde{e}$ is the associated equivariant Euler form then there exists some form $\gamma \in \Omega (B; Or (E)^\vee (a)]^\mathbb{T}$ such that

$$\tilde{e} - e = D\gamma.$$ 

(b) Let $f : B' \to B$ be a smooth equivariant map and $E' \xrightarrow{\pi'} B'$ the pullback bundle. If we use the pulled-back bundle metric on $E'$, we obtain a Cartesian square

$$\begin{array}{ccc}
S(E') & \xrightarrow{j} & S(E) \\
\downarrow & & \downarrow \\
B' & \xrightarrow{f} & B
\end{array}$$

and $\tilde{f}^* \phi$ is an angular form for $E'$; the associated Euler form is $f^* e$.

Proof. Part (a) is proved along the same lines as the previous proposition. Part (b) is immediate. $\square$

4.4. General equivariant pushforward and Poincaré duality. We now want to define pushforward along an equivariant embedding. Using the graph construction, this can be used to define pushforward along any equivariant map. As a special case, we obtain a Poincaré dual to an equivariant submanifold.

4.4.1. Blowing up. Before we can define the equivariant pushforward, we need to discuss a certain blow up construction that will allow us to construct the Thom form for an equivariant bundle.

Definition 37. (a) Let $E \xrightarrow{\pi} B$ be an equivariant vector bundle. The blow up $\tilde{E}$ of the zero section of $\pi$ is the $\mathbb{T}$-manifold with boundary $\tilde{E} := S(E) \times [0, \infty)$, where the action on $[0, \infty)$ is trivial. It comes equipped with an equivariant smooth map $\beta : \tilde{E} \to E$ so that $(v, r) \in S(E) \times [0, \infty)$ maps to $r \cdot i_{S(N)} (v)$ (cf. the discussion at the beginning of §4.3). $\tilde{\pi} := \pi \circ \beta : \tilde{E} \to B$ turns this into an equivariant fiber-bundle.

(b) Let $X \xrightarrow{i} Y$ be an equivariant embedding of closed $\mathbb{T}$-manifolds. Let $N = N_{X/Y}$ denote the associated equivariant normal bundle, and fix some equivariant tubular neighbourhood $\tilde{X} \subset N \subset Y$ of $X$ in $Y$. We define the blow up of $Y$ along $X$ by

$$\text{Bl}_X Y := \tilde{N} \cup \bigcup_{\beta \in \mathbb{T}} (Y \setminus X)$$

where we use the open embeddings $S(N) \times (0, \infty) \subset S(N) \times [0, \infty)$ and $S(N) \times (0, \infty) \times [0, \infty) \to N \setminus \text{zero section} \subset Y \setminus X$ to glue. The maps $\beta : S(N) \times [0, \infty) \to N \subset Y$ and $Y \setminus X \to Y$ glue to form a map $\beta_{X/Y} : \text{Bl}_X Y \to Y$.

We check $\text{Bl}_X Y$ is Hausdorff: if distinct points $x, y \in \text{Bl}_X Y$ satisfy $\beta_{X/Y} (x) \neq \beta_{X/Y} (y)$ then they can be separated in $Y$, otherwise they can be separated in $S(N_{\beta_{X/Y} (x)})$. It follows that $\text{Bl}_X Y$ is a manifold with boundary.
Remark 38. Our terminology is non-standard. What is usually referred to as the blow up of $Y$ along $X$ is obtained from the above construction by identifying antipodal points in $S(N) \subset Y$ (if we’re doing real algebraic geometry) or $S^1$ orbits associated with a $U(1)$ action (if we’re working over $\mathbb{C}$, so in particular, $N$ can be taken to be a Hermitian vector bundle). Either way, the usual construction produces a manifold without boundary. On the other hand, $\text{Bl}_X Y$ is relatively orientable, see Lemma 39.

The map $\beta_{X/Y}$ satisfies a natural universal property, which in particular means the blow up is essentially unique, independent of the choices (tubular neighbourhood, bundle metric) and explains our use of the definite noun (the blow up). We will not go into details since for the purposes of this paper, we do not care about the uniqueness of the construction.

Let $\tilde{E} = S(E) \times [0, \infty)$ be the blow up of $E$. We denote by $pr_s : \tilde{E} \to S(E)$ and $pr_r : \tilde{E} \to [0, \infty)$ the projections.

Lemma 39. (a) There is a natural equivariant local system isomorphism

$$\text{Or}(T \tilde{E} \beta^*) \simeq \beta^* \text{Or}(TE).$$

On the open set $U = \tilde{E} \setminus (S(E) \times \{0\})$ this isomorphism is induced by the isomorphism of vector bundles

$$d\beta : T\tilde{E} \to TE.$$

(b) If $X \hookrightarrow Y$ is an equivariant embedding of closed $\mathbb{T}$-manifolds, there’s a canonical isomorphism of local systems

$$\text{Or}(T\text{Bl}_X Y) \simeq \beta_{X/Y}^* \text{Or}(TY)$$

which agrees with (29) on $\tilde{N} \subset \text{Bl}_X Y$.

(c) If $\text{Im}(i) \neq Y$ then

$$\int_Y \omega = \int_{\text{Bl}_X Y} \beta_{X/Y}^* \omega$$

for any $\omega \in \Omega(Y; \text{Or}(TY)[\tilde{\alpha}]^\vee)^T$, where we use the isomorphism of part (b) to identify $\Omega(Y; \beta^* \text{Or}(TY)[\tilde{\alpha}]^\vee)^T = \Omega(Y; \text{Or}(T\text{Bl}_X Y)[\tilde{\alpha}]^\vee)^T$.

Proof. To construct (29) use the ordered direct sum decomposition

$$T\tilde{E} = pr_S^* TS(E) \oplus pr_r^* T[0, \infty)$$

and the short exact sequence

$$0 \to TS(E) \to \pi^*_{\tilde{E}} TE \xrightarrow{dr} \mathbb{R} \to 0$$

where $r$ is the distance from the zero section of $E$. $pr_r^* T[0, \infty)$ and $\mathbb{R}$ are both canonically oriented by the positive direction, and so the decompositions define the desired isomorphism by (6.2). It is easy to see that on $U$ this isomorphism agrees with $d\beta$.

To prove (b), write $\text{Bl}_X Y = U_1 \cup U_2$ where $U_1 = \tilde{N}$ and $U_2 = Y \setminus X$. For $i = 1, 2$ we have isomorphisms $\text{Or}(T\text{Bl}_X Y)|_{U_i} \simeq \beta_{X/Y}^* \text{Or}(TY)|_{U_i}$ (for $i = 1$ by part (a), for $i = 2$ by the identity). These isomorphisms agree on the intersection $U_1 \cap U_2 = U$. The result follows.
For part (c), note that if we restrict to the complement of an arbitrarily small neighbourhood of $X \hookrightarrow Y$, the integrands can be identified. The result follows. □

4.4.2. The equivariant Thom form. Fix once and for all some smooth function $\sigma : [0, \infty) \rightarrow [-1, 0]$ which satisfies $\sigma (r) = -1$ for $r \leq \frac{1}{2}$ and $\sigma (r) = 0$ for $r \geq 1$.

Let $E \xrightarrow{\pi} B$ be a rank $n$ equivariant vector bundle, let $\tilde{E} = S(E) \times [0, \infty)$ be the blow up of $E$. Choose an equivariant angular form $\phi \in \Omega (S(E); \text{Or} (E)^\vee [\bar{a}])^T$ for $E$. There is a unique form $\tau_E \in \Omega^n (E; \pi^* \text{Or} (E)^\vee [\bar{a}])^T$ which satisfies

$$\beta^* \tau_E = D (\text{pr}_E^* \phi \wedge \text{pr}_E^* \sigma).$$

Definition 40. $\tau_E$ is called an equivariant Thom form for $E$.

Note that $\tau_E$ is $D$-closed since it pulls back to a $D$-exact form on the blow up.

4.4.3. Pushforward along an equivariant embedding.

Definition 41. (cf. Definition 60) An oriented equivariant embedding is a pair $(i, \kappa)$ where $i : X \hookrightarrow Y$ is a $\mathbb{T}$-equivariant embedding of closed $\mathbb{T}$-manifolds and

$$\kappa : \text{Or} (TX) \otimes i^* \text{Or} (TY)^\vee \otimes K \simeq i^* \mathcal{L}$$

is an (equivariant, see Remark 30) isomorphism of local systems, where $K, \mathcal{L}$ are equivariant local systems on $X$ and on $Y$, respectively.

Let $(i, \kappa)$ be an oriented equivariant embedding, set $n = \dim Y - \dim X$, and let $N \xrightarrow{\pi} X$ denote the rank $n$ normal bundle associated with $i$:

$$0 \rightarrow TX \rightarrow i^* TY \rightarrow N \rightarrow 0.$$ (31)

Let $\tau_N \in \Omega (N; \text{Or} (TX)^\vee \otimes i^* \text{Or} (TY) [\bar{a}])^T$ be an equivariant Thom form for $N$, where we’ve used (31) to identify $\text{Or} (N)^\vee \simeq \text{Or} (TX)^\vee \otimes i^* \text{Or} (TY)$ (cf. 402).

Definition 42. Given an oriented equivariant embedding $(i, \kappa)$ we define an equivariant pushforward

$$i^*_\kappa : \Omega (X; \mathcal{K} [\bar{a}])^T \rightarrow \Omega (Y; \mathcal{L} [\bar{a}])^T [n]$$

by

$$i^*_\kappa (\omega) := \tau_N \wedge \pi^* \omega,$$

where we interpret the right hand side to mean the zero extension (to $Y$) of the compactly supported form $\tau_N \wedge \pi^* \omega$ which is defined on $N$.

Proposition 43. (Equivariant Poincare duality.) For any pair of $D$-closed forms $\omega_X \in \Omega (X; \mathcal{K} [\bar{a}])^T$, $\omega_Y \in \Omega (Y; (\mathcal{L} \otimes \text{Or} (TY))^\vee [\bar{a}])^T$, we have

$$\int_Y i^*_\kappa \omega_X \wedge \omega_Y = \int_X \omega_X \wedge i^* \omega_Y$$

where on the right we think of $\omega_X \wedge i^* \omega_Y$ as taking values in $\text{Or} (TX)^\vee$ using the identification $\mathcal{K} \otimes i^* (\mathcal{L} \otimes \text{Or} (TY))^\vee \simeq \text{Or} (TX)^\vee$ induced by $\kappa$. 
Proof. Denote by \( i_\partial : S(N) = S(N) \times \{0\} \to \text{Bl}_X Y \) the inclusion of the boundary. We compute

\[
\int_Y i_\partial^* \omega_X \wedge \omega_Y \overset{(1)}{=} \int_\tilde{N} D (pr_N^* \phi \wedge pr_N^* \sigma) \wedge \tilde{\pi}^* \omega_X \wedge \beta^* \omega_Y |_{\tilde{N}} = \int_\tilde{N} D (pr_N^* \phi \wedge pr_N^* \sigma \wedge \tilde{\pi}^* \omega_X \wedge \beta^* \omega_Y |_{\tilde{N}} ) \overset{(2)}{=} - \int_{S(N)} i_\partial^* (pr_N^* \phi \wedge pr_N^* \sigma \wedge \tilde{\pi}^* \omega_X \wedge \beta^* \omega_Y |_{\tilde{N}} ) = - \int_{S(N)} \phi \wedge \sigma (0) \wedge \tau_Y^* (\omega_X \wedge i_\partial^* \omega_Y ) = \int_X \omega_X \wedge i_\partial^* \omega_Y .
\]

In (1) we’ve used part (c) of Lemma 39 and the fact the integrand is supported in \( \tilde{N} \subset \text{Bl}_X Y \). In (2) we picked up a minus sign, since we think of \( S(N) \) as oriented as in 26, using the outward pointing radial vector. This is the same as thinking of \( S(N) \) as the boundary of the associated disc bundle \( D(N) \), see \( 63 \), which is opposite its orientation as the boundary of \( \text{Bl}_X Y \), which is the orientation used to apply Stokes’ Theorem \( 62 \). \( \square \)

Corollary 44. Given an equivariant isomorphism of local systems

\[
(32) \quad \kappa : \text{Or} (TX \otimes i^* \text{Or} (TY))^Y \simeq i^* \mathcal{L},
\]

we call \( \tau_{X/Y} := i_\partial^* 1 \) the \( (\mathcal{L}\text{-valued}) \) equivariant Poincare dual to \( X \) in \( Y \). We have

\[
\int_Y \tau_{X/Y} \wedge \omega = \int_X \omega |_X
\]

for any \( D \)-closed form \( \omega \in \Omega(Y; (\mathcal{L} \otimes \text{Or} (TY))^Y) \).

5. Equivariant twisted \( A_\infty \) algebras

5.1. The equivariant cyclic DGA, statement of results. Let \( L \) be an \( n \)-dimensional closed manifold equipped with a smooth \( \mathbb{T} \)-action. Since in our forthcoming papers we’ll need to consider non-orientable \( L \), we will assume that \( L \) is non-orientable. All of the results hold, with the obvious changes, for \( L \) oriented.

We reintroduce the \( (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \)-grading. We work over the unital commutative real algebra \( R = \mathbb{C}[\alpha] = \mathbb{C}[\alpha_1, ..., \alpha_n] \), where \( \deg \alpha_i = (2, 0) \). Denote by \( C^{\text{CW}} (L) \) the Cartan–Weil \( R \)-DGA:

\[
C^{\text{CW}} (L) := C^{\text{CW}} (L; \mathbb{C} \oplus C^{\text{CW}} (L; \text{Or} (TL)) \]

it is equipped with the differential \( D \) and wedge product \( \wedge \), see 41.1. It is \( (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \)-graded. The codimension \( \mathbb{Z} \)-component of the grading is as in 41.1 and the \( \mathbb{Z}/2 \)-component specifies the local system degree, as in Example 7.

We can think of \( C^{\text{CW}} (L) \) as a deformation over \( \mathbb{C}[\alpha] \) of the \( \mathbb{C} \)-DGA \( C (L) \), in the sense of Remark 13 as follows: take \( G = G' = \{0\} \), the trivial gapping monoids, \( R = \mathbb{C}[\alpha] \), and \( R' = \mathbb{C} \). We have \( \Lambda = \Lambda_\alpha^G (R) = \mathbb{C}[\alpha] \) and \( \Lambda' = \Lambda_\alpha^{G'} (R') = \mathbb{C} \), and there’s an obvious quotient map \( \Lambda \stackrel{\alpha=0}{\longrightarrow} \Lambda' \). Let \( (B^{\text{CW}} , D + \wedge) \) and \( (B, d + \wedge) \) denote the differential coalgebras corresponding to the DGA’s \( C^{\text{CW}} (L) \) and \( C (L) \), as in Example 6 and Proposition 12 (the sloppy-but-visual suggestive notation \( d + \wedge, D + \wedge \) for the differentials will not be used outside this introduction). We then have \( (d + \wedge) = (D + \wedge) \otimes_R \text{id}_{R'} \).
In the next subsection we’ll present a simple condition for when a deformation $(\mathcal{B}, m)$ of $C(L)$ can be extended to a deformation $(\mathcal{B}_{C^W}, m_{C^W})$ of $C_{C^W}(L)$. In this case we obtain a commutative square of extensions

$$
\begin{aligned}
\begin{array}{c}
\mathcal{B}_C, d + \Lambda \\
\mathcal{B}^{CW}_{C_{[\alpha]}}, d + \Lambda
\end{array}
\end{aligned}
\begin{array}{c}
\mathcal{B}^{CW}_{C_{[\alpha]}}, D + \Lambda \\
\mathcal{B}^{CW}_{C_{[\alpha]}}, m_{C^W}
\end{array}
\begin{array}{c}
\mathcal{B}^{CW}_{C_{[\alpha]}}, D + \Lambda \\
\mathcal{B}^{CW}_{C_{[\alpha]}}, m_{C^W}
\end{array}
$$

Things become more interesting when we consider cyclic symmetry. We use integration $\int : C^W(L) \to R[-n, 1]$, to define a pairing (cf. Eq (33)) $\langle \cdot \rangle^W : C^W(L) \otimes_R C^W(L) \to C^W(L)[-n, 1]$.

**Lemma 45.** $\langle \cdot \rangle^W$ is non-degenerate.

**Proof.** Suppose $0 \neq \omega \in C^W(L)$. Write $\omega = \sum_{\ell \in \mathbb{Z}^k} \omega^\ell \alpha^\ell$ with $\omega^\ell \in C(L)^T$ and $\alpha^\ell = \alpha^{(i_1, \ldots, i_n)} := \prod_{r=1}^{n} i_r$. Choose some $I_0$ so $\omega_{I_0} \neq 0$. Since $\langle \cdot \rangle$ is non-degenerate, there’s some $\omega' \in C(L)$ with $\langle \omega_{I_0}, \omega' \rangle \neq 0$. We have $0 \neq \langle \omega_{I_0}, \omega' \rangle = \langle \pi^T \omega_{I_0}, \pi^T \omega' \rangle = \langle \omega_{I_0}, \pi^T \omega' \rangle$ (see the proof of Proposition 45 for the definition and some properties of $\pi^T$) so we may assume without loss of generality that $\eta \in C(L)^T \subset C^W(L)$. Since the coefficient of $\alpha_{I_0}$ in $\langle \omega, \eta \rangle^W$ is $\langle \omega_{I_0}, \eta \rangle \neq 0$ we find that $\langle \cdot \rangle^W$ is non-degenerate. □

For general $L$, the induced pairing on cohomology need not be perfect. Therefore, we will need to introduce an additional assumption on $L$.

**Definition 46.** We say a closed manifold $L$ has even cohomology if

$$
H^{2i+1,j}(C(L), d) = 0
$$

for all $i, j$.

In particular we find that $L$ is both $\mathbb{C}$-separated and Or $(TL)$-separated, see Definition 51 so by Proposition 52 we have an isomorphism of $\mathbb{C}[\alpha]$-modules

$$
\Phi : \mathbb{C}[\alpha] \otimes \mathbb{C} H(L) \to H^W(L)
$$

(recall $H(L) = H^{0,0}(L) \oplus H^{1,1}(L)$ combines cohomology with values in $\mathbb{C}$ and in Or $(TL)$).

**Lemma 47.** We can choose the isomorphism $\Phi$ so that

$$
\langle \Phi(f_1 \otimes \psi_1), \Phi(f_2 \otimes \psi_2) \rangle^W = f_1 \cdot f_2 \cdot \langle \psi_1, \psi_2 \rangle
$$

for $\psi_1 \in H(L)$ and $f_1 \in \mathbb{C}[\alpha]$.

Hereafter, when there’s no risk of confusion, we may use $\langle \cdot \rangle$ and $\langle \cdot \rangle^W$ to denote the induced pairings on $H(L)$ and $H^W(L)$, respectively.

**Proof.** Fix some basis $\left\{ \gamma_i \right\}_{i=0}^{2N-1}$ for $H(L)$ so that $\left\{ \gamma_i \right\}_{i=0}^{N-1}$ is a basis for $H^{0,0}(L)$ and $\left\{ \gamma_i \right\}_{i=N}^{2N-1}$ is the dual basis for $H^{1,1}(L)$, i.e. we have $\langle \gamma_i, \gamma_{N+j} \rangle = \delta_{i,j}$ for $0 \leq i, j \leq N - 1$. It follows from $\mathbb{C}[\alpha]$-linearity of $\langle \cdot \rangle^W$ that

$$
C_{ij} := \langle \Phi(1 \otimes \gamma_i), \Phi(1 \otimes \gamma_{N+j}) \rangle = \delta_{ij} \mod m,
$$
so that \((C_{ij})\) is an invertible \(N \times N\) matrix with coefficients in \(\mathbb{C}[\alpha]\). Let \((C^\prime_{ij})\) denote its inverse, and let \(c : \mathbb{C}[\alpha] \otimes H(L) \to \mathbb{C}[\alpha] \otimes H(L)\) denote the \(\mathbb{C}[\alpha]\)-linear automorphism defined in the basis \(\gamma_i\) by the block-diagonal matrix \(\begin{pmatrix} \text{id} & 0 \\ 0 & C^\prime_{ij} \end{pmatrix}\). The isomorphism \(\Phi \circ c\) satisfies Eq (34).

\[\square\]

**Corollary 48.** If \(L\) has even cohomology, then the equivariant pairing \((\cdot)_{\text{CW}}\) induces a perfect pairing on \(H_{\text{CW}}(L)\).

We will see that the equivariant deformation of a cyclic invariant algebra is cyclic. In \[\S 5.3\] we will construct a unital cyclic retraction of \(C_{\text{CW}}(L)\) to \(H_{\text{CW}}(L)\). This means that we can apply Theorem 24 to construct minimal model s for equivariant extensions of invariant twisted \(A_\infty\) algebras.

### 5.2. Equivariant extension of invariant deformations.

Let \(B\) be a non-orientable closed \(\mathbb{T}\)-manifold, let \(C = C(L)\) be the associated DGA over \(R = \mathbb{C}\) (cf. Example \[\S 7\]). Let \(B = B_{\mathbb{C}[\alpha]}(C)\) be the associated bar coalgebra, defined over the Novikov ring \(A^G_0(\mathbb{C})\). For \(1 \leq a \leq n'\) define \(\iota_a' \circ x = (-1)^{cd_x} \iota_a x\). Let \(\iota_{\xi_i}\) denote the \((\text{id}_B, \text{id}_B)\)-coderivation of degree \(-1\) of \(B\), corresponding to \(\iota_{\xi_i}'\). Explicitly, we have

\[
\iota_{\xi_i}(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^k (-1)^{\sum_{j=1}^{i-1} (cd_{x_i} - 1)} x_1 \otimes \cdots \otimes x'_i x_i \otimes \cdots \otimes x_k.
\]

**Definition 49.** \((C(L), \{m_{k,\beta}\})\) will be called \(\mathbb{T}\)-invariant if

\[
(35) \quad \iota_{\xi_i}' m_{k,\beta} (x_1, \ldots, x_k) = \sum_{i=1}^k (-1)^{1 + \mu(\beta) + \sum_{j=1}^{i-1} (x_j - 1)} m_{k,\beta} (x_1, \ldots, \iota_{\xi_i}' x_i, \ldots, x_k)
\]

for all \((k, \beta) \neq (1, 0), x_1, \ldots, x_k \in C(L)\) and \(1 \leq a \leq n'\).

For \(j = 1, 2\) let \(h_j : B[-d_j] \to B\) be maps in \(\text{FGVect}_\mathbb{C}\), whose degree is \(d_j = cd h_j\). We define the graded commutator \([h_1, h_2] := h_1 \circ h_2 - (-1)^{d_1 \cdot d_2} h_2 \circ h_1\).

**Lemma 50.** If \(h_1, h_2\) are \((\text{id}_B, \text{id}_B)\)-coderivations then \([h_1, h_2]\) is a degree \((cd h_1 + cd h_2)\) \((\text{id}_B, \text{id}_B)\)-coderivation. In terms of the bijection \[\[17\]\], we have

\[
[h_1, h_2]_{k,\beta} (x_1 \otimes \cdots \otimes x_k) = \\
\sum (-1)^{cd h_2 \sum_{j<i} (cd x_j - 1)} h^1_{k_1,\beta_1} (\cdots \otimes h^2_{k_2,\beta_2} (x_i \otimes \cdots) \otimes \cdots) - \\
- \sum (-1)^{cd h_1 \sum_{j<i} (cd x_j - 1)} h^2_{k_2,\beta_2} (\cdots \otimes h^1_{k_1,\beta_1} (x_i \otimes \cdots) \otimes \cdots)
\]

**Proof.** The first statement holds for any coalgebra, and is proved by dualizing the proof of the corresponding statement for algebras. The second claim is straightforward.

The graded Jacobi identity

\[
(36) \quad (-1)^{cd h_1 h_3} [h_1, [h_2, h_3]] + (-1)^{cd h_1 cd h_2} [h_2, [h_3, h_1]] + (-1)^{cd h_2 cd h_3} [h_3, [h_1, h_2]] = 0
\]

holds, making the \((\text{id}_B, \text{id}_B)\)-coderivations into a graded Lie algebra.
If we set $\partial = m - \hat{m}_{1,0}$ then we can write Eq (35) as

$$[\xi, \partial] := i_{\xi} \partial + \partial i_{\xi} = 0 \text{ for } 1 \leq a \leq n',$$

which implies that

$$[i_{\xi}, m] = [i_{\xi}, \hat{m}_{1,0}] = (i'_{\xi} m_{1,0} + m_{1,0} i'_{\xi})^\wedge = -\hat{L}_{\xi}$$

where $\hat{L}_{\xi} (x_1 \otimes \cdots \otimes x_k) = \sum_{j=1}^{\infty} (x_1 \otimes \cdots \otimes L_{\xi} x_j \otimes \cdots \otimes x_k)$. Note that we have $[i_{\xi}, i_{\xi}] = 0$, $[i_{\xi}, \hat{L}_{\xi}] = 0$ and $[\hat{L}_{\xi}, \hat{L}_{\xi}] = 0$ for all $1 \leq a, b \leq n'$, by Lemma 50 and the commutativity of the vector fields $\{\xi_a\}$.

**Proposition 51.** Let $L$ be a non-orientable closed $\mathbb{T}$-manifold, and let $(C(L), d, \wedge)$ denote the associated $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-graded De Rham DGA over $\mathbb{C}$. Let $(C(L), \{m_{k,\beta}\})$ be a $G$-gapped $\mathbb{T}$-invariant deformation of $(C(L), d, \wedge)$. We define operations

$$m_{k,\beta}^{CW} : C^{CW}(L)^{\otimes k} \to C^{CW}(L)[2 - k - \mu(\beta), \mu(\beta) \mod 2]$$

as follows. For $(k, \beta) \neq (1, 0)$ we take $m_{k,\beta}^{CW}$ to be the $\mathbb{C}[\hat{a}]$-linear extension of the restriction of $m_{k,\beta}$ to the $\mathbb{T}$-invariant forms. For $(k, \beta) = (1, 0)$ we set $m_{1,0}^{CW} x = (-1)^{cd} D x$.

Then

(a) $(C^{CW}(L), \{m_{k,\beta}^{CW}\})$ is a $G$-gapped deformation of $(C^{CW}(L), D, \wedge)$ over $R = \mathbb{C}[\hat{a}]$.

(b) If $(C(L), \{m_{k,\beta}\})$ is unital, then $(C^{CW}(L), \{m_{k,\beta}^{CW}\})$ is a unital deformation of $(C^{CW}(L), D, \wedge)$.

Assuming, in addition, that $L$ has even cohomology, we have

(c) If $(C(L), \{m_{k,\beta}\}, \{\cdot\})$ is cyclic then $(C^{CW}(L), \{m_{k,\beta}^{CW}\}, \{\cdot\}^{CW})$ is a cyclic deformation of $(C^{CW}(L), D, \wedge, \{\cdot\})$.

**Proof.** To prove (a), we first show that the operations $\{m_{k,\beta}^{CW}\}$ are well-defined (that is, take invariant forms to invariant forms) and satisfy the twisted $A_\infty$ relations (3). One can check this directly, but the signs get somewhat cumbersome. It is convenient to switch to the bar coalgebra, using the setup introduced just before the proposition. Consider $\mathcal{B}' := \mathcal{B} \left( (C(L) \otimes \mathbb{C}[\hat{a}])^G \right) = \mathcal{B} \otimes \mathbb{C}[\hat{a}]$ (as always, completion is with respect to the energy filtration, with $\mathbb{C}[\hat{a}]$ discrete). We’ll use $i_{\xi}, \hat{L}_{\xi}$ and $m$ to denote, by the usual abuse of notation, the extension of scalars of these coderivations from $\mathcal{B}$ to $\mathcal{B}'$. Consider the $(\text{id}_{\mathcal{B}'}, \text{id}_{\mathcal{B}'})$-coderivation

$$m' := m - \sum_a \alpha_a i_{\xi_a}.$$ 

We have

$$[m', m'] = -2 \sum_a [\alpha_a i_{\xi_a}, m] = -2 \sum_a \alpha_a \hat{L}_{\xi_a}. \quad (37)$$
By the graded Jacobi identity,
\[ 0 = [i_{\xi_a}, [m', m']] + 2 [m', [i_{\xi_a}, m']] = 
\left[ i_{\xi_a} - 2 \sum_{b} \alpha_b \hat{L}_{\xi_b} \right] + 2 \left[ m', \left[ i_{\xi_a}, m - \sum_{b} \alpha_b \xi_b \right] \right] = -2 \left[ m', \hat{L}_{\xi_a} \right], \]
which, by Lemma 50 implies
\[ \text{Eq (38)} \quad \mathcal{L}_{\xi_a} m_{k,\beta} (x_1 \otimes \cdots \otimes x_k) = \sum_{i} m_{k,\beta} (\cdots \otimes \mathcal{L}_{\xi_a} x_i \otimes \cdots), \; (k, \beta) \neq (1, 0). \]

Eq (38) shows that \( m' \) preserves the sub-coalgebra
\[ \mathcal{B}^{\text{CW}} := \mathcal{B}_{\Lambda_{G}^{G}([\alpha])} \left( (C^{\text{CW}} (L))^{G} \right) \subset \mathcal{B}'. \]

By Eq (37) \( m_{\text{CW}}^{0} = m'|_{\mathcal{B}^{\text{CW}}} \) is a differential, whose components \( \left\{ m_{k,\beta}^{\text{CW}} \right\} \) are as prescribed in the statement of the proposition. Clearly \( m_{0,0}^{\text{CW}} = 0 \) so \( \left\{ m_{k,\beta}^{\text{CW}} \right\} \) define a twisted \( A_{\infty} \) algebra, which is easily seen to be a deformation of \( (C^{\text{CW}} (L), D, \wedge) \).

To establish (b) we note that if \( m_{k,\beta} \) is unital the unit \( e \) must be the constant 0-form 1. Since \( \mathcal{L}_{\xi_a} e = 0 \) and \( i_{\xi_a} e = 0 \) for all \( 1 \leq r \leq n' \), we find that \( e \) is also a unit for \( \left\{ m_{k,\beta}^{\text{CW}} \right\} \).

Finally we prove (c). First, we check that Eq (5) holds for \( \left\{ m_{k,\beta}^{\text{CW}} \right\} \). For \( (k, \beta) \neq (1, 0) \) this is immediate, and for \( (k, \beta) = (1, 0) \) it is a special case of Example 6. Then (c) follows from Lemma 15 and Corollary 18.

Remark 52. A \( \mathbb{T}^{*} \) module is an algebraic structure which captures the notion of a smooth action of a Lie group on a differential complex, see 5. This structure suffices to define an equivariant extension of the complex, in essentially the same way as for the De Rham complex of a manifold \( L \). We would like to point out that Definition 49 and the proof of Proposition 51 can be formulated in this more general language. In other words, instead of starting off from \( C = C (L) \) and working with deformations of the De Rham DGA, we can assume that \( (C, d) \) admits a \( \mathbb{T}^{*} \)-module structure, define when perturbations of \( d \) are \( \mathbb{T} \)-invariant, and for such perturbations obtain equivariant extensions as above.

5.3. Cyclic unital equivariant retraction. The main goal of this section is to prove the following.

Theorem 53. Let \( L \) be a non-orientable closed \( \mathbb{T} \)-manifold which has even cohomology. Then there exists a cyclic unital retraction of \( (C^{\text{CW}} (L), D) \) to \( H (C^{\text{CW}}) \).

Remark 54. All the results in this section continue to hold for \( L \) oriented, mutatis mutandis.

In proving this theorem we will construct an equivariant homotopy kernel, which represents the homotopy operator of the retraction, and plays a prominent role in defining equivariant open Gromov-Witten invariants. This result can be seen as a kind of equivariant extension of the Hodge-De Rham decomposition.
Let \( L \) be a non-orientable \( n \)-dimensional closed \( \mathbb{T} \)-manifold. For \( 0 \leq i \leq 2N - 1 \) choose \( \omega_i \in C^{CW}(L) \) such that (i) \( D\omega_i = 0 \), (ii) the cohomology classes \( \{[\omega_i]\}_{i=0}^{2N-1} \) form a basis for \( H^{CW}(L) \), (iii) \( \omega_i \in (C^{CW}(L))^{\bullet,0} \) for \( 0 \leq i \leq N - 1 \) and \( \omega_i \in (C^{CW}(L))^{\bullet,1} \) for \( N \leq i \leq 2N - 1 \), and (iv) \( \langle \omega_i, \omega_{j+N} \rangle = \delta_{i,j} \). A set of forms \( \{\omega_i\}_{i=0}^{2N-1} \) satisfying (i) - (iv) will be called an equivariant basis of forms for \( L \). An equivariant basis of forms exists; indeed, one can take \( \omega_i \) to be any form which represents \( \Phi(1 \otimes \gamma_i) \), see the proof of Lemma 47 and the discussion directly above it for explanation of the notation.

Consider the diagonal \( L = \Delta \overset{i}{\rightarrow} L \times L \), and let \( i_\Delta(L) \subset N_\Delta \subset L \times L \) be an equivariant tubular neighbourhood. Construct the blow up \( \beta_\Delta : L \times L \rightarrow \widetilde{L} \times L \), where \( \widetilde{L} \times L = Bl_\Delta(L \times L) = \widetilde{N}_\Delta \cup (L \times L \setminus \Delta) \), see 4.3.1. The blow up \( L \times L \) is a compact manifold with boundary \( i_\beta : S(N_\Delta) \rightarrow \widetilde{L} \times L \). For \( j = 1, 2 \) we denote by \( pr_j : L \times L \rightarrow L \) the projection and set \( \overline{\nu}_j = pr_j \circ \beta_\Delta \).

Note that
\[
0 \rightarrow TL \rightarrow i_\Delta^*(pr_1^*TL \oplus pr_2^*TL) \rightarrow N_\Delta \rightarrow 0.
\]
Since we have \( pr_j \circ i_\Delta = id_\Delta \) and \( i_\Delta \pi_S = \pi i_\partial \) we find that \( Or(N_\Delta) \simeq i_\Delta^* Or(pr_1^*TL) \) and
\[
\sigma_\Delta(N_\Delta) \vee \simeq i_\Delta^* \overline{\nu}_2^* Or(TL) \vee
\]

**Definition 55.** Let \( L \) be a non-orientable closed \( \mathbb{T} \)-manifold with even cohomology. An equivariant homotopy kernel \( \Lambda' \) for \( L \) is a form \( \Lambda' \in \Omega \left( \Delta \times L; \overline{\nu}_2^* Or(TL) \right) \) with the following properties.

(a) \( i_\Delta^* \Lambda' = -\phi + i_\Delta^* \beta_\Delta^* \Upsilon \) where \( \phi \in \Omega(S(N); i_\Delta^* \overline{\nu}_2^* Or(TL) \vee [\overline{\alpha}] ) \) \( \Upsilon \) is an equivariant angular form for \( S(N_\Delta) \) (see \ref{ang}) and \( \Upsilon \in \Omega(L \times L; pr_2^* Or(TL) \vee [\overline{\alpha}] ) \).

(b) \( D\Lambda' = \sum_{j=0}^{N-1} \overline{\nu}_1^* \omega_j \wedge \overline{\nu}_2^* \omega_{j+N} \) where \( \{\omega_i\}_{i=0}^{2N-1} \) is an equivariant basis of forms for \( L \).

**Proposition 56.** If \( L \) is a non-orientable closed \( \mathbb{T} \)-manifold which has even cohomology, an equivariant homotopy kernel for \( L \) exists.

**Proof.** Denote \( \mathcal{L} = pr_2^* Or(TL) \vee \). Use \ref{hom} to define the isomorphism of local systems

\[
\kappa : Or(TL) \otimes i_\Delta^* Or(TL) \simeq Or(N_\Delta) \simeq i_\Delta^* \mathcal{L}
\]

needed in order to construct an equivariant Poincare dual to the diagonal \( \tau_\Delta \) with values in \( pr_2^* Or(TL) \vee \) (see \ref{poincare}). Let \( \Lambda \in \Omega^{n-1}(\widetilde{L} \times L; \overline{\nu}_2^* Or(TL) \vee [\overline{\alpha}] ) \) be the unique form supported in \( N_\Delta \subset Bl_\Delta(L \times L) \) such that
\[
\Lambda|_{\bar{N}_\Delta} = pr_2^* \phi \wedge pr_1^* \sigma,
\]
so that \( D\Lambda = \beta_\Delta^* \tau_\Delta \) (cf. Eq \ref{poincare}).

Let \( \{\omega_i\}_{i=0}^{2N-1} \) be an equivariant basis for \( L \times L \). By Corollary \ref{equivariant} we have some elements \( f_{ij} \in \mathbb{C}[\overline{\alpha}] \) with
\[
|\tau| = \sum_{0 \leq i,j \leq 2N-1} f_{ij} \cdot (pr_1^* [\omega_i] \wedge pr_2^* [\omega_j])
\]
We claim
\[ f_{ij} = \delta_{i,j+N} := \begin{cases} 1 & i + j = N \mod 2N \\ 0 & \text{otherwise} \end{cases}. \]

To see this, we compute \( \int_{L \times L} \tau \wedge pr_1^* \omega_i \wedge pr_2^* \omega_j \) for \( 0 \leq i', j' \leq 2N - 1 \), in two ways. First we use Corollary 44:
\[ \int_{L \times L} \tau \wedge pr_1^* \omega_i \wedge pr_2^* \omega_j = \int_L \omega_i \wedge \omega_j = \delta_{i', j' + N}. \]

On the other hand, by Eq (40) we have:
\[ \int_{L \times L} \tau \wedge pr_1^* \omega_i \wedge pr_2^* \omega_j = \sum_{i,j} f_{ij} \int_{L \times L} pr_1^* (\omega_i \wedge \omega_j) \wedge pr_2^* (\omega_j \wedge \omega_j) = \sum_{i,j} f_{ij} \delta_{i,i'+N} \delta_{j, j'+N} = f_{i'+N,j'+N}. \]

Combining Eqs (41,42) we conclude \( f_{ij} = \delta_{i,j+N} \).

Now define \( \Lambda' = \Lambda + \beta_\Delta^* \mathcal{Y} \) where \( D \mathcal{Y} = \sum_{i=0}^{N-1} \overline{pr}_1^* \omega_i \wedge \overline{pr}_2^* \omega_{i+N} - \tau \) so
\[ D \Lambda' = \sum_{i=0}^{N-1} \overline{pr}_1^* \omega_i \wedge \overline{pr}_2^* \omega_{i+N} \]
and we have \( i_S^* \Lambda' = -\phi + i_S^* \beta_\Delta^* \mathcal{Y} \) by construction. \( \square \)

Given an equivariant homotopy kernel \( \Lambda' \) for \( L \) we define \( \mathbb{C}[\hat{\alpha}] \)-module maps \( \Pi_{1,0} : C_{CW}^\infty (L) \to H_{CW}^\infty (L) \), \( I_{1,0} : H_{CW}^\infty (L) \to C_{CW}^\infty (L) \), and \( h_{1,0} : C_{CW}^\infty (L) [+1] \to C_{CW}^\infty (L) \), by setting
\[ h_{1,0}^* x = \begin{cases} \overline{pr}_1! (\overline{pr}_2^* x \Lambda') & \text{ls } x = 0 \\ \overline{pr}_2! (\overline{pr}_1^* x \Lambda') & \text{ls } x = 1 \end{cases} \]
\[ I_{1,0} [\omega_i] = \omega_i, \quad 0 \leq i \leq 2N - 1 \]
\[ h_{1,0}^* x = \begin{cases} \overline{pr}_1! (\overline{pr}_2^* x \Lambda') & \text{ls } x = 0 \\ \overline{pr}_2! (\overline{pr}_1^* x \Lambda') & \text{ls } x = 1 \end{cases} \]

Let us explain how the pushforward is defined. We deal with the case \( \text{ls } x = 0 \), the other case is analogous. It is easy to see that \( \overline{pr}_1 \) is an equivariant proper submersion, so by Definition 43, we just need to supply an equivariant local system isomorphism
\[ Or \left( T \left( L \times L \right) \right) \otimes \overline{pr}_1^* Or (T L)^\vee \to \mathcal{K}^\vee \otimes \overline{pr}_1^* L, \]
where we have \( \mathcal{K} = \overline{pr}_2^* Or (T L)^\vee \) and require \( L = \mathbb{C} \) in this case (recall we need \( h_{1,0}^* \) to preserve the local system degree). By part (b) of Lemma 39 we have
\[ Or \left( T \left( L \times L \right) \right) \otimes \overline{pr}_1^* Or (T L)^\vee \simeq \beta_\Delta^* \left( Or (T L \times L) \otimes pr_1^* Or (T L)^\vee \right), \]
and we compose this with $\beta^* \kappa'$ where $\kappa'$ is the composition of the obvious local system isomorphisms

$$\text{Or}(T(L \times L)) \otimes \text{pr}_1^* \text{Or}(TL)^\vee \simeq \text{pr}_2^* \text{Or}(TL)^\vee \to K' \otimes \text{pr}_1^* L$$

for $K' = \text{pr}_2^* \text{Or}(TL)^\vee$.

**Lemma 57.** $(\Pi_{1,0}, I_{1,0}, h'_{1,0})$ satisfies all the conditions of Definition 23 for being a retraction of $(C^{CW}(L), D)$ to $H^{CW}(L)$, except possibly for the side conditions (15). Moreover, it satisfies Eq (18) and $h'_{1,0} 1_{0,0} = 0$ where $1_{0,0} \in (C^{CW}(L))_{0,0}$ denotes the constantly 1 zero-form.

**Proof.** $\Pi_{1,0} D = 0$ by Stokes' theorem. $D I_{1,0} = 0$ because $\omega_i$ are closed. $\Pi_{1,0} I_{1,0} = \text{id}_{H^{CW}(L)}$ since this is so for the basis elements by the assumption on the pairing.

Let us check that $D' h_{1,0} + h_{1,0} D' = I_{1,0} \Pi_{1,0} - \text{id}_{C^{CW}(L)}$. We assume $s x = 0$, the case $s x = 1$ is analogous.

\begin{equation}
(D' h_{1,0} + h_{1,0} D') x = (-1)^{cd} x [ -D \bar{p}_1 (\bar{p}_2^* x \Lambda') + \bar{p}_1 (\bar{p}_2^* D x \Lambda') ] = \tag{44}
\end{equation}

\begin{equation}
= (-1)^{cd} x [ -D \bar{p}_1 (\bar{p}_2^* x \Lambda') + \bar{p}_1 (D(\bar{p}_2^* x \wedge \Lambda')) + (-1)^{cd} x \bar{p}_1 (\bar{p}_2^* x \wedge D(\Lambda')) ]
\end{equation}

\begin{equation}
= (-1)^{cd} x [ -D \bar{p}_1 (\bar{p}_2^* x \Lambda') + \bar{p}_1 (D(\bar{p}_2^* x \Lambda')) ] + \bar{p}_1 (\bar{p}_2^* x \wedge \sum_{i=0}^{N-1} \bar{p}_1^* \omega_i \wedge \bar{p}_2^* \omega_{i+N} ) .
\end{equation}

We show

\begin{equation}
(-1)^{cd} x [ -D \bar{p}_1 (\bar{p}_2^* x \Lambda') + \bar{p}_1 (D(\bar{p}_2^* x \Lambda')) ] = -\text{id} x. \tag{45}
\end{equation}

Indeed,

\begin{equation}
(-1)^{cd} x [ -D \bar{p}_1 (\bar{p}_2^* x \Lambda') + \bar{p}_1 (D(\bar{p}_2^* x \Lambda')) ] = \tag{46}
\end{equation}

\begin{equation}
= (-1)^{cd} x (-1)^{cd + (n-1) + n} (-1) \pi_S! (i_S^* \bar{p}_2^* x (-\phi) + \pi_S^* i_{\Delta}^* \epsilon)
\end{equation}

The factor $(-1)^{cd + (n-1) + n}$ comes from Stokes' theorem [2]. The $(-1)$ immediately following it appears because we change from the orientation of $S(N_\Delta)$ as a boundary of $L \times L$ to the orientation as the boundary of the unit $D(N_\Delta)$. We have $i_S^* \bar{p}_2^* x = \pi_S^* x$ and $\pi_S! \phi = 1$, so by the projection formula [53]

\begin{equation}
(-1)^{cd} x (-1)^{cd + (n-1) + n} (-1) \pi_S! (i_S^* \bar{p}_2^* x (-\phi) + \pi_S^* i_{\Delta}^* \epsilon) = (-\text{id}) x,
\end{equation}

as claimed.

Next we show

\begin{equation}
\bar{p}_1! \left( \bar{p}_2^* x \wedge \sum_{i=0}^{N-1} \bar{p}_1^* \omega_i \wedge \bar{p}_2^* \omega_{i+N} ) \right) = I_{1,0} \Pi_{1,0} x. \tag{46}
\end{equation}

First, we claim $\bar{p}_2^* \beta^* = \text{pr}_1^* h'_{1,0}$. To see this, remove an arbitrarily small neighbourhood of $S(N_\Delta) \times \{0\} \subset L \times L$, so that $\beta$ can be taken to be the identity, and then
\( \kappa = \kappa' \) (see the second statement in Lemma 39 (b)). With this established, we compute:

\[
\tilde{pr}_{1!} \left( \pr_{2}^{*} x \wedge \sum_{i=0}^{N-1} \pr_{1}^{*} \omega_{i} \wedge \pr_{2}^{*} \omega_{i+1}^{N} \right) = pr_{1!} \left( \pr_{2}^{*} x \wedge \sum_{i=0}^{N-1} \pr_{1}^{*} \omega_{i} \wedge \pr_{2}^{*} \omega_{i+1}^{N} \right) = \\
= \sum_{i=0}^{N-1} pr_{1!} \left( \pr_{2}^{*} (x \wedge \omega_{i+1}^{N}) \wedge \pr_{1}^{*} \omega_{i} \right) = \sum_{i=0}^{N-1} \int_{L} (x \wedge \omega_{i+1}^{N}) \omega_{i} = I_{1,0} \Pi_{1,0} x.
\]

plugging in Eqs (15,16,10) on the last line of Eq (14) shows that \( Dh_{1,0} + h_{1,0} D = I_{1,0} \Pi_{1,0} \) - \text{id}_{C^{CW}(L)}.

Next we check that Eq (18) holds. We assume without loss of generality that \( ls x = 0 \) and \( ls y = 1 \); the other cases either vanish since the integrand takes values in the trivial local system, or can be derived from this case by the antisymmetry of the pairing \( \langle \cdot, \cdot \rangle \). We compute, using (9), Lemma 58 and the fact \( \deg p \tilde{r}_{2!} \) is even and \( \deg \Lambda' \) is odd:

\[
(-1)^{(cd x-1)(cd y)+(cd x-1)} \langle h_{1,0} x, y \rangle_{CW} = \int_{L} p \tilde{r}_{1!} \left( p \tilde{r}_{2}^{*} x \Lambda' \right) \wedge y = \\
= \int_{L} p \tilde{r}_{2!} x \wedge p \tilde{r}_{1} y = \int_{L} x \tilde{p} \tilde{r}_{2!} (\Lambda' p \tilde{r}_{1} y) = \\
= (-1)^{cd y} \int_{L} x \tilde{p} \tilde{r}_{2!} (p \tilde{r}_{1}^{*} \Lambda') = (-1)^{cd y} (-1)^{cd x (cd y-1) + cd x} \langle x, h_{1,0} y \rangle_{CW}
\]

which is equivalent to \( \langle h_{1,0} x, y \rangle + (-1)^{cd x} \langle x, h_{1,0} y \rangle = 0 \), as we wanted to show.

Finally, \( h_{1,0} 1^{0,0} = 0 \) since the degree of \( \Lambda' \) is \( n-1 \) and the dimension of the fiber of \( p \tilde{r}_{1} \) is \( n \).

The following is a simple variation of a result found in [9], where it is attributed to Lambe and Stasheff [8].

**Lemma 58.** Let \( (C, d) \) be a dg \( R \)-module. Let \( \Pi_{1,0} : C \to HC, I_{1,0} : HC \to C, \) and \( h_{1,0}^{t} : C \to C [-1] \) be \( R \)-module maps such that the 3-tuple \( (\Pi_{1,0}, I_{1,0}, h_{1,0}^{t}) \) satisfies all the conditions of Definition 52, except perhaps for the side conditions 13.

Set

\[
h_{1,0} = h_{1,0}^{t} \circ d \circ h_{1,0}^{t}.
\]

Then \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) is a retraction of \( (C, d) \) onto \( H(C, d) \). If Eq (18) holds for \( h_{1,0}^{t} \) then the retraction \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) is cyclic. If \( h_{1,0}^{t} e = 0 \) then \( e \) is a unit for \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \).

**Proof.** Straightforward.

**Proof of Theorem 53.** By Proposition 59, an equivariant homotopy kernel \( \Lambda' \) exists. We define \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) by (13), and then apply Lemma 57 and Lemma 58 to produce the desired cyclic unital retraction \( (\Pi_{1,0}, I_{1,0}, h_{1,0}) \) of \( C^{CW}(L) \) to \( H^{CW}(L) \).

\( \square \)
Remark 59. It is not hard to show that $h_{1,0}$ is also represented by a kernel. That is, there exists some $\Lambda \in \Omega \left( \widetilde{L} \times L; \widetilde{\text{pr}}_2^* \left( \text{Or} \left( TL \right)^{\nu} \right) [\bar{a}] \right)$ such that

$$h_{1,0} x = \begin{cases} \widetilde{\text{pr}}_1^* (\widetilde{\text{pr}}_2^* x \Lambda) & \text{ls } x = 0, \\ \widetilde{\text{pr}}_2^* (\widetilde{\text{pr}}_1^* x \Lambda) & \text{ls } x = 1. \end{cases}$$

6. Appendix. Orientation conventions

In this appendix we state some conventions regarding orientations and the definition of the pushforward of forms; we then specify the signs appearing in formulas involving the pushforward operation. We work in the non-equivariant context, but the conventions and results hold, \textit{mutatis mutandis}, for the equivariant setting.

6.1. Terminology and pullback orientation. It will be convenient to speak in terms of relative orientations. A bundle $E \rightarrow B$ is called \textit{relatively oriented}, or \textit{r-oriented} for short, if there is some specific isomorphism of local systems between $\text{Or} (E)$ and a local system on $B$ which is treated as known. The collection of known local systems is usually specified by presupposing some bundles to be r-oriented.

When we say a manifold with boundary $X$ is r-oriented we mean $TX$ is r-oriented.

As an example we state the following convention.

\textbf{Convention.} If $E \rightarrow B$ is r-oriented and $f : B' \rightarrow B$ is any smooth map, then the pullback bundle $f^* E \rightarrow B'$ is r-oriented using the canonical isomorphism of local systems $\text{Or} (f^* E) \simeq f^* \text{Or} (E)$.

6.2. Short exact sequences and ordered direct sums. In the smooth category, a short exact sequence of bundles over a base space $B$

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

can always be split

$$E \simeq E_1 \oplus E_2. \quad (47)$$

For the purposes of orientation, it is important to remember the order of the summands; to emphasize this we will sometimes call \textit{(47)} an \textit{ordered} direct sum decomposition. A short exact sequence always fixes the order of the summands as above.

If the rank of $E_i$ is $n_i$, and the rank of $E$ is $n = n_1 + n_2$, the wedge product $\wedge : \Omega^{n_1}_c (E_1) \otimes \Omega^{n_2}_c (E_2) \rightarrow \Omega^n_c (E_1 \oplus E_2)$ induces an isomorphism $H^{n_1}_c (E_1 | U) \otimes H^{n_2}_c (E_2 | U) \simeq H^n_c (E | U)$ for sufficiently small $U \subset B$, and thus an isomorphism

$$\text{Or} (E) \simeq \text{Or} (E_1) \otimes \text{Or} (E_2). \quad (48)$$

\textbf{Convention.} If any two of the three vector bundles in a short exact sequence, or an ordered direct sum decomposition, are r-oriented then the third is r-oriented by Eq \textit{(48)}. 

\textit{EQUIVARIANT} $A_{\infty}$ \textit{ALGEBRAS FOR NONORIENTABLE LAGRANGIANS} 36
6.3. **Boundary.** If $X$ is a manifold with boundary we have an ordered direct sum decomposition

\[ TX|_{\partial X} = \mathbb{R} \oplus T\partial X \]

which is the reverse of the ordered direct sum decomposition for the short exact sequence

\[ 0 \to T\partial X \to TX|_{\partial X} \xrightarrow{dc} \mathbb{R} \to 0 \]

where $c : X \to [0,1)$ is any collar coordinate. Henceforth we use $\mathbb{R}$ to denote the trivial real line bundle, if the base (in this case, $\partial X$) is clear from the context. $\mathbb{R}$ is canonically oriented, hence $r$-oriented, and we have the following.

**Convention.** An $r$-orientation on $X$ induces an $r$-orientation on $\partial X$ by Eq (49). In other words, we orient $\partial X$ so that an oriented basis for $TX$ is obtained from an oriented basis for $T\partial X$ by appending an outward normal vector at the beginning.

6.4. **Pushforward.**

**Definition 60.** A pair $(f, \kappa)$ will be called an **oriented proper submersion** if

1. $f : X \to Y$ is a surjective, proper map between manifolds without boundary, such that $df|_x$ is surjective for all $x \in X$.
2. $\kappa : Or(TX) \otimes f^* Or(TY)^{\vee} \simeq K^\vee \otimes f^* L$ is an isomorphism of local systems for $K$ (resp. $L$) some local system on $X$ (resp. $Y$).

**Definition 61.** We say $(f, \kappa)$ is an **oriented proper submersion with boundary** if the pair $(f, \kappa)$ is as in the previous definition, except $X$ now has (possibly empty) boundary (and $Y$ has no boundary) and we require in addition that $f|_{\partial X}$ be surjective and $d(f|_{\partial X})|_x$ be surjective for every $x \in \partial X$.

**Convention.** Let $(f, \kappa)$ be an oriented proper submersion with boundary. Our pushforward is defined so that

\[ f^\kappa_! (f^* \alpha \wedge \beta) = \alpha \wedge f^\kappa_! \beta \]

for all $\alpha \in \Omega(Y; \mathbb{C})$ and $\beta \in \Omega(X; \mathcal{K})$.

Roughly speaking, this means we write the differentials corresponding to coordinates of the fiber after the differentials pulled back from the base before integrating them out. This is in agreement with the definition on page 61 of [2].

With this convention we have the following properties. We omit the proofs, since they are obtained by nothing more than careful book-keeping of conventions from well-known proofs.

**Lemma 62.** (Stokes’ Theorem.) Let $(f, \kappa)$ be an oriented proper submersion with boundary. Use $\kappa : Or(TX) \otimes f^* Or(TY)^{\vee} \simeq K^\vee \otimes f^* L$ and [6.3] to define an isomorphism $\partial \kappa : Or(T\partial X) \otimes f|_{\partial X}^* Or(TY)^{\vee} \simeq (K|_{\partial X})^\vee \otimes f|_{\partial X}^* L$. We then have

\[ (f^\kappa_! d - df^\kappa_!) \omega = (-1)^{\deg \omega + \deg f_!} (f|_{\partial X})_!^{\partial \kappa} \]

Henceforth \( \deg f_! = \dim Y - \dim X \) denotes the degree shift of the map $f_!$. 


Lemma 63. (Projection Formula.) Let
\[ (f : X \to Y, \kappa : \text{Or}(TX) \otimes f^*\text{Or}(TY)^\vee \simeq K^\vee \otimes f^*L) \]
be an oriented proper submersion with boundary. Let \( L_1, L_2 \) be two additional local systems on \( Y \). Set \( K^+ := f^*L_1 \otimes K \otimes f^*L_2, \ L^+= L_1 \otimes L \otimes L_2 \). Denote by \( \kappa^+ : \text{Or}(TX) \otimes f^*\text{Or}(TY)^\vee \simeq K^\vee \otimes f^*L^+ \) the obvious isomorphism of local systems induced from \( \kappa \). Then for \( \alpha_1 \in \Omega(Y; L_1), \beta \in \Omega(X; K), \) and \( \alpha_2 \in \Omega(Y; L_2) \) we have
\[ f_1^+ ((f^*\alpha_1) \wedge \beta \wedge (f^*\alpha_2)) = (-1)^{\deg \alpha_2 \cdot \deg f^*} \alpha_1 \wedge (f^+ \beta) \wedge \alpha_2 \]

The following two lemmas will not be used, we state them for completeness and future reference.

Lemma 64. (Composition.) Let
\[ (f : X \to Y, \kappa : \text{Or}(TX) \otimes f^*\text{Or}(TY)^\vee \to K^\vee \otimes f^*K_Y) \]
be an oriented proper submersion with boundary and
\[ (g : Y \to Z, \kappa' : \text{Or}(TY) \otimes f^*\text{Or}(TZ)^\vee \to K^\vee \otimes g^*K_Z) \]
be an oriented proper submersion.
Denote by
\[ f^* \kappa' \circ \kappa : \text{Or}(TX) \otimes (gf)^* \text{Or}(TZ) \to K^\vee_X \otimes (gf)^*K_Z \]
the local system obtained from combining \( \kappa \) and \( f^* \kappa' \) in the obvious way. We then have
\[ (gf)^{f^* \kappa' \circ \kappa} = (-1)^{\deg f \cdot \deg g} g_k' \circ f_k^* \]

Lemma 65. (Push-pull property.) Let
\[ X' \xrightarrow{u'} X \xrightarrow{u} X' \hspace{1cm} \text{with} \hspace{1cm} (f, \kappa) \text{an oriented proper submersion with boundary.} \]
It follows that \( \partial X' = Y' \times_Y \partial X \), and \( f' \) is also an oriented proper submersion with boundary; indeed, since the fibers of \( f' \) are canonically isomorphic to the fibers of \( f \) we find that there’s a canonical isomorphism of local systems \( \text{Or}(TX') \otimes f'^*\text{Or}(TY')^\vee \simeq u'^* \{ \text{Or}(TX) \otimes f^*\text{Or}(TY)^\vee \} \), and we denote by
\[ \kappa^u : \text{Or}(TX') \otimes f'^*\text{Or}(TY')^\vee \to u'^* K^\vee \otimes f'^* (u^* L) \]
the local system isomorphism obtained from combining \( \text{[50]} \) and \( \kappa \) in the obvious way.
\[ u^* \circ f_k^u = (f')^{\kappa^u} \circ (u')^* \]
EQUIVARIANT $A_{\infty}$ ALGEBRAS FOR NONORIENTABLE LAGRANGIANS

REFERENCES

[1] M. F. Atiyah and R. Bott,  The moment map and equivariant cohomology, Topology 23 (1984), no. 1, 1–28.
[2] R. Bott and L. W. Tu,  Equivariant characteristic classes in the Cartan model, Geometry, analysis and applications (Varanasi, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 3–20.
[3] K. Fukaya,  Cyclic symmetry and adic convergence in Lagrangian Floer theory, Kyoto J. Math. 50 (2010), no. 3, 521–590.
[4] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono,  Lagrangian intersection Floer theory: anomaly and obstruction. Part I, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
[5] V. W. Guillemin and S. Sternberg,  Supersymmetry and equivariant de Rham theory, Mathematics Past and Present, Springer-Verlag, Berlin, 1999, With an appendix containing two reprints by Henri Cartan [ MR0042426 (13,107e); MR0042427 (13,107f)], doi:10.1007/978-3-662-03992-2.
[6] J. Huebschmann and T. Kadeishvili,  Small models for chain algebras, Math. Z. 207 (1991), no. 2, 245–280, doi:10.1007/BF01202738.
[7] M. Kontsevich and Y. Soibelman,  Notes on $A_{\infty}$-algebras, $A_{\infty}$-categories and non-commutative geometry, Homological mirror symmetry, Lecture Notes in Phys., vol. 757, Springer, Berlin, 2009, pp. 153–219.
[8] L. Lambe and J. Stasheff,  Applications of perturbation theory to iterated fibrations, Manuscripta Math. 58 (1987), no. 3, 363–376.
[9] L. A. o. Lambe,  Homological perturbation theory, Encyclopedia of Mathematics.
[10] J. Solomon and S. Tukachinsky,  Point like bounding chains in open gromov-witten theory, preprint.
[11] J. Solomon,  Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions, ProQuest LLC, Ann Arbor, MI, 2006, Thesis (Ph.D.)–Massachusetts Institute of Technology, also available at http://arxiv.org/abs/math/0606429
[12] J.-Y. Welschinger,  Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry, Invent. Math. 162 (2005), no. 1, 195–234.