Data-Driven Participation Factors for Nonlinear Systems Based on Koopman Mode Decomposition

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Abstract—This paper develops a novel data-driven technique to compute the participation factors for nonlinear systems based on the Koopman mode decomposition. Provided that certain conditions are satisfied, it is shown that the proposed technique generalizes the original definition of the linear mode-in-state participation factors. Two numerical examples are provided to demonstrate the performance of our approach: one relying on a canonical nonlinear dynamical system, and the other based on the two-area four-machine power system. The Koopman mode decomposition is capable of coping with a large class of nonlinearity, thereby making our technique able to deal with oscillations arising in practice due to nonlinearities while being fast to compute and compatible with real-time applications.

Index Terms—Koopman mode decomposition, modal analysis, participation factors, nonlinear systems, stability.

I. INTRODUCTION

The participation factors are an important component of the so-called selective modal analysis proposed by Pérez-Arriaga et al. [1]. They are widely used in the power industry as they provide a measure of the relative contribution of modes to system states and vice versa. Applications include stability analysis [2], dynamic model reduction [3], and placement of power system stabilizers [4]. Alternative [5] and complementary [6] perspectives to the original definition of linear participation factors have also been proposed in the literature. For instance, Hashlamoun et al. [5] advocate that the initial states are uncertain, thereby casting the problem of computing the participation factors as a stochastic one, as opposed to a deterministic one; then, by relying on the definition of the mathematical expectation and by assuming that the initial states follow a uniform probability distribution, a dichotomy between mode-in-state and state-in-mode participation factors is suggested. Despite the existence of different views, the participation factors are a well-accepted metric of the dynamic performance of linear systems; therefore, its extension to nonlinear models is of practical interest since it is known that the analysis of power systems through model linearization does not provide an accurate picture of the modal characteristics when the system is operating under stressed conditions [7].

An attempt to go beyond the linear paradigm was made by Lesieutre et al. [8] by applying a transformation from state variables to harmonic variables to gain insight on the state-in-mode participation factors at the Hopf bifurcation point. Although interesting, their approach ultimately computed the participation factors of a transformed linear model associated with the stable limit cycle. A different approach was taken by Vital et al. [7], with the intent of studying inter-area modes of oscillation in stressed power systems following large disturbances, when nonlinearities play an important role. The idea in [7] is to compute the participation factors by considering up to the second-order terms in the Taylor series expansion of the nonlinear model and then applying the method of normal forms. The inclusion of third-order terms has also been exploited in [9]. Due to the importance of this line of research, the shortcomings of the method of normal forms have been investigated in [10]. Firstly, this method suffers from a heavy computational burden that will make it inapplicable to large-scale systems even if second-order terms are considered only. Secondly, it involves a highly nonlinear numerical problem that needs to be solved to retrieve the initial conditions. In an attempt to overcome these weaknesses, Pariz et al. [11] proposed the modal series method, which has the advantages of being valid under resonance conditions and not requiring nonlinear transformations. However, their approach is also restricted to polynomial nonlinearities, as is the case with the method of normal forms. Furthermore, the aforementioned approaches do not consider the state-in-mode participation factors, which are accounted for in this paper.

In face of the exposed challenges, it has been suggested in [12] and in [13] that the computation of the participation factors from measurements could either provide a solution to the aforementioned issues directly, or be complementary to model-based techniques such as the ones that rely on the Taylor series expansion of the power systems nonlinear model. The problem thus becomes one of estimating linear and nonlinear participation factors from the measurements. In this paper, this problem is addressed via the Koopman operator-theoretic framework [13]. Recently, following the work of Mezić et al. [14], [15] and of Rowley et al. [16], this framework based on the point spectrum of the Koopman operator, and henceforth referred to as the Koopman mode decomposition (KMD), has gained momentum as a powerful data-driven tool to analyze nonlinear dynamical systems. Our solution of data-driven participation factor is also based on the spectral properties of the Koopman operator, precisely the point spectrum and associated eigenfunctions, and the so-called Koopman modes [16]. To approximate the spectral objects, we resort to the extended dynamic mode decomposition (EDMD) [17], [18]. Then, we demonstrate how to compute linear and nonlinear participation factors from the measurements. To the authors’ best of knowledge, this is the first comprehensive application of the EDMD algorithm in power systems. Sako et al. [19], and Netto and Mili [20], [21] adopt the EDMD but did not consider nonlinear observables, which conversely are accounted for in the present work, thereby exploring the full potential of the KMD. The computation of the participation factors with the proposed technique is not restricted by the form of the underlying dynamical system. Furthermore, provided that certain conditions are satisfied, it is shown that our approach

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where \( x \in \mathbb{R}^n \) is the system state vector, and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a vector-valued nonlinear function. By performing a Taylor series expansion of (1) around a stable equilibrium point (SEP), and considering only the first-order term, we have
\[
\dot{x} = Ax, \tag{2}
\]
where \( A \in \mathbb{R}^{n \times n} \) is a Jacobian matrix, i.e., \( A := \nabla f(x) = \frac{\partial f}{\partial x} \), and \( x^\top \) denotes the transpose of \( x \). By assuming that all the eigenvalues \( \lambda_i, i = 1, \ldots, n \), of \( A \) are distinct, the eigendecomposition of \( A \) is given by
\[
A = [u_1 \ldots u_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} [v_1^\top \ldots v_n^\top] = UAV, \tag{3}
\]
where \( U \) and \( V \) are matrices containing, respectively, the right and the left eigenvectors of \( A \), and \( V = U^{-1} \). By applying a similarity transformation expressed as
\[
z = Vx, \tag{4}
\]
and using (3), the solution to (2) is given by
\[
x(t) = \sum_{j=1}^n (v_j^\top x_0) u_j e^{\lambda_j t}, \tag{5}
\]
where \( x_0 \) is the initial state. The time evolution of the \( i \)-th state in (5) is written as
\[
x_i(t) = \sum_{j=1}^n (v_j^\top x_0) u_{ij} e^{\lambda_j t} = \sum_{j=1}^n \sigma_{ij} e^{\lambda_j t}. \tag{6}
\]

Here we introduce the so-called contribution factors \([22]\).

**Definition 1:** The contribution factors of the linear system (2) are defined as
\[
\sigma_{ij} := (v_j^\top x_0) u_{ij}. \tag{7}
\]
They measure the contribution of mode \( j \) to the oscillations of state \( i \) for the initial state \( x_0 \).

Notice that besides the characteristics of the linear system (2) given by the eigendecomposition of \( A \), the contribution factors are also dependent on the initial state \( x_0 \). For power systems, this implies that the contribution factors are dependent on the network topology, the SEP around which (2) is linearized, as well as the location and duration of a given disturbance \([22]\). The nonlinear counterpart of (7) based on the KMD was pinpointed by Suzuki and Mezić in \([23]\), although they do not make reference to the term contribution factor. Whereas the contribution factors provide valuable information about the contribution of a mode to the dynamics of a state, a measure of the system performance that depends only on the characteristics of (2), and not on the system initial condition, \( x_0 \), is advantageous for capturing inherent system characteristics.

### A. Participation factors as originally proposed in \([1]\)

Now we introduce the original notion of participation factors based on (1).

**Definition 2:** The mode-in-state participation factors of the linear system (2) are defined as
\[
p_{ij} := v_{ij} u_{ij}. \tag{8}
\]
They provide a relative measure of the magnitude of the modal oscillations in a state when only that state is perturbed initially.

To derive (8), one makes use of (7) and selects \( x_0 = e_i \), where \( e_i \) is the unit vector along the \( i \)-th coordinate axis.

**Definition 3:** The state-in-mode participation factors of the linear system (2) are defined as
\[
p_{ij} := v_{ij} u_{ij}. \tag{9}
\]
They measure the relative participation of the \( j \)-th mode in the \( i \)-th state.

To derive (9), one substitutes (7) into (2) and finds \( \dot{z} = Ax \), yielding to \( z_i(t) = z_{i0} e^{\lambda_j t} = p_{ij} x_{0} e^{\lambda_j t} \). Then, by selecting \( x_0 = u_i, z_i(t) = (\sum_{j=1}^n v_{ij} u_{ij}) e^{\lambda_j t} \). Notice that as proposed in \([1]\), (8) and (9) have the same expression.

### B. Alternative definition of the participation factors \([3]\)

As shown above, Pérez-Arriaga et al. \([1]\) put \( x_0 = e_i \) in (7) to define the mode-in-state participation factors. Instead, Hashlamoun et al. \([5]\) start by considering that the initial state \( x_0 \) in (7) is uncertain and proceed from there.

**Definition 4:** In the set-theoretic formulation, the mode-in-state participation factors of the linear system (2) measuring relative influence of a given mode on a given set can be defined as
\[
p_{ij} := \operatorname{avg}_{x_0 \in S} \left( \frac{v_{ij}^\top x_0 u_{ij}}{x_{i0}} \right), \tag{10}
\]
whenever (10) exists, \( x_{i0} = \sum_{j=1}^n (v_j^\top x_0) u_{ij} \) is the value of \( x_i(t) \) at \( t = 0 \), and \( \operatorname{avg}_{x_0 \in S} \) is an operator that computes the average of a function over a set \( S \subset \mathbb{R}^n \). The average in (10) is an estimator of the mean of a random variable, which tends to the true mean of that random variable when it exists, which is represented by the expectation operator. That is, we have
\[
\pi_{ij} := E \left[ \frac{v_{ij}^\top x_0 u_{ij}}{x_{i0}} \right]. \tag{11}
\]

By assuming that the components of the initial state vector, \( x_0 \), are independent with zero mean, (11) reduces to (9). We refer to (11) as the probabilistic mode-in-state participation factors of the linear system (2).

**Definition 5:** The probabilistic state-in-mode participation factors of the linear system (2) are defined as
\[
\pi_{ij} := \begin{cases} E \left[ \frac{v_{ij} x_0}{z_{i0}} \right], & \text{if } \lambda_j \text{ is real,} \\ E \left[ \frac{(v_{ij} + v_{ij}^*) x_0}{z_{i0} + z_{i0}^*} \right], & \text{if } \lambda_j \text{ is complex,} \end{cases} \tag{12}
\]

where (12) generalizes the one proposed by Pérez-Arriaga et al. \([1]\) to nonlinear dynamical systems, in the case of mode-in-state participation factors.

The paper proceeds as follows. Section II briefly revisits the formulation of the linear participation factors. Section III introduces the proposed data-driven technique to compute linear and nonlinear participation factors based on the KMD. Section IV discusses some numerical results. Conclusions and ongoing work are provided in Section V.
whenever the expectation exists, \( z_0 := z(t = 0) = v_j^\top x_0 \), and \( \ast \) denotes complex conjugation. By assuming that the units of the state variables are scaled to ensure that the probability density function is such that the components of \( x_0 \) are jointly uniformly distributed over the unit sphere in \( \mathbb{R}^n \) centered at the origin, yields

\[
\pi_{ij} = \frac{(\mathcal{R}\{v_{ij}\})^2}{(\mathcal{R}\{v_j\})^\top \mathcal{R}\{v_j\}},
\]

(13)

where \( \mathcal{R}\{v_{ij}\} \) stands for the real part of \( v_{ij} \).

Notice that (13) differs from (9). The interested reader is referred to [5] for more details on the derivation of (13). Hereafter, we consider the definition of participation factors given by Hashlamoun et al. [5], namely Definitions 4 and 5.

### III. PARTICIPATION FACTORS FOR NONLINEAR SYSTEMS BASED ON KOOPMAN MODE DECOMPOSITION

We have considered a continuous-time formulation in the previous section. If instead of (2), a discrete-time autonomous system of the form \( x_k = Ax_{k-1} \) is assumed, we can show that

\[
x_k = \sum_{j=1}^{n} (v_j^\top x_0) u_j \mu_j^k,
\]

(14)

implying that (5) and (14) are equivalent with \( \mu_j^k = e^{\lambda_j s} \) for fixed \( s \). We consider a discrete-time formulation to introduce the KMD motivated by the fact that our approach is data-driven. Hence, let us consider a discrete-time autonomous nonlinear system given by \( x_k = F(x_{k-1}) \), where \( x \in \mathbb{R}^n \), \( M \) is the state space, and \( F : M \rightarrow M \). The Koopman operator is a linear operator \( K \) that acts on functions defined on \( M \) in the following manner:

\[
Kg(x_k) = g(F(x_k)),
\]

(15)

where \( g : M \rightarrow \mathbb{R} \). The eigenvalues, \( \mu_j \), and eigenfunctions, \( \varphi_j \), of \( K \) are defined as

\[
K\varphi_j(x_k) = \mu_j \varphi_j(x_k), \quad j = 1, 2, ... \tag{16}
\]

The set containing all \( \mu_j \) is called the point spectrum of the Koopman operator. Now, consider a vector-valued observable \( g : M \rightarrow \mathbb{C}^q \). As in (16), if all the elements of \( g \) lie within the span of the eigenfunctions, \( \varphi_j \), we have

\[
g(x_k) = \sum_{j=1}^{\infty} \varphi_j(x_k) \phi_j = \sum_{j=1}^{\infty} \varphi_j(x_0) \phi_j \mu_j^k, \tag{17}
\]

where \( \phi_j \in \mathbb{C}^q \) are the Koopman modes [16], and \( (\mu_j, \varphi_j, \phi_j) \) are referred to as the Koopman tuples. As stated by Susuki and Mezić [23], “the real part of \( \varphi_j(x_0) \phi_j \) determines the initial amplitude of modal dynamics”, and in fact define a nonlinear generalization of the linear contribution factors based on the KMD.

#### A. Extended Dynamic Mode Decomposition (EDMD)

Following Klus et al. [13], consider a set of snapshots pairs of the system states \( x_k, k = 0, ..., m \). Also, consider

\[
X = [x_0 \ldots x_{m-1}], \quad X' = [x_1 \ldots x_m], \tag{18}
\]

\( X, X' \in \mathbb{R}^{n \times m} \). In addition, consider a vector of observable functions, i.e. lifted states, defined as

\[
\gamma(x_k) = [\gamma_1(x_k) \ldots \gamma_q(x_k)]^\top,
\]

(19)

where \( \gamma : \mathbb{R}^n \rightarrow \mathbb{R}^q \), and define \( \Gamma_k = [\gamma(x_0) \ldots \gamma(x_{m-1})] \), \( \Gamma_{X'} = [\gamma(x_1) \ldots \gamma(x_m)] \). A finite-dimensional approximation to the Koopman operator \( K \) is estimated as

\[
K = \Gamma_{X'} \Gamma_k^\dagger,
\]

(20)

where \( \dagger \) denotes the Moore-Penrose pseudoinverse. A finite set of Koopman eigenvalues is approximated by the eigenvalues of \( K \), whereas the eigenfunctions \( \varphi_j \) are given by

\[
\varphi(x_k) = \Xi \gamma(x_k),
\]

(21)

where \( \Xi = [\xi_1^\top; \ldots; \xi_q^\top] \) contains the left eigenvectors of \( K \), and \( \varphi(x_k) = [\varphi_1(x_k) \ldots \varphi_q(x_k)]^\top \). Finally, in order to obtain the Koopman modes for the full-state observable, \( x_k \), let \( B \in \mathbb{R}^{n \times q} \) be a matrix defined as follows:

\[
x_k = B \gamma(x_k).
\]

(22)

From (21), we have that \( \gamma(x_k) = \Xi^{-1} \varphi(x_k) \) and

\[
x_k = B \gamma(x_k) = B \Xi^{-1} \varphi(x_k),
\]

(23)

where \( \Xi^{-1} \) contains the right eigenvectors of \( K \). Therefore, the Koopman modes are the column vectors \( \phi_j \), \( j = 1, ..., q \), of \( \Phi = B \Xi^{-1}, \Xi \in \mathbb{C}^{n \times q} \), and

\[
x_k = \sum_{j=1}^{q} \varphi_j(x_k) \phi_j = B \Xi^{-1} \varphi(x_k).
\]

(24)

**Remark 1:** By adopting the KMD, we trade a finite-dimensional nonlinear system by an infinite-dimensional linear one. From the numerical standpoint, (24) is a finite-dimensional approximation of (17).

Now we are in the position to state the main result of this paper.

#### B. Data-Driven Participation Factors for Nonlinear Systems

**Definition 6:** The data-driven mode-in-state participation factors for nonlinear systems based on the EDMD are defined as

\[
p_{ij} := \xi_{ij} \phi_{ij} + \sum_{r=1, r \neq i}^{q} \xi_{ij} \phi_{ij} \Xi \frac{\gamma_{r0}}{\gamma_{00}},
\]

(25)

where \( i = 1, ..., n \) and \( j = 1, ..., q \). Notice that as opposed to the linear case, \( q \geq n \) and the matrix of the mode-in-state participation factors is in general not square.

Now, we derive (25) and show that it is equivalent to (8) under certain conditions. To do that, we start from the definition (11) and apply the KMD (24) instead of the eigen-decomposition given by (14). Let us define \( p_{ij} \) as

\[
p_{ij} := E \left[ \frac{\varphi_j(x_0) \phi_{ij}}{\gamma_{00}} \right] = E \left[ \frac{\xi_{ij} \gamma_{00}}{\gamma_{00}} \right],
\]

(26)
whenever the expectation exists, \( \gamma_0 = [\gamma_1(x_0) \ldots \gamma_q(x_0)]^T \), and \( \gamma_{i0} \) is the value of \( \gamma_i(x(t)) \) at \( t = 0 \). Then,

\[
p_{ij} = \mathbb{E} \left[ \sum_{r=1}^q (\xi_{rj} \gamma_{r0}) \phi_{ij} \right] = \mathbb{E} \left[ (\xi_{ij} \gamma_{i0}) \phi_{ij} \right] + \mathbb{E} \left[ \sum_{r=1, r \neq i}^q (\xi_{rj} \gamma_{r0}) \phi_{ij} \right] = \xi_{ij} \phi_{ij} + \sum_{r=1, r \neq i}^q \xi_{rj} \phi_{ij} \mathbb{E} \left[ \frac{\gamma_{r0}}{\gamma_{i0}} \right], \tag{27}
\]

Case 1: Suppose that the observables are the identity map, i.e., \( \gamma(x_k) = x_k \). By assuming that the components of the initial state vector, \( x_0 \), follow a uniform probability density function and are statistically independent with zero mean, the second term on the right-hand side in (27) vanishes and the resultant expression

\[
p_{ij} = \xi_{ij} \phi_{ij} + \sum_{r=1, r \neq i}^q \xi_{rj} \phi_{ij} \mathbb{E} \left[ \frac{\gamma_{r0}}{\gamma_{i0}} \right] = \xi_{ij} \phi_{ij}. \tag{28}
\]

implies that (25) leads to (8). Notice that we rely on Lemma 1 stated in [24] to derive (28). In the nonlinear setting, the finite-dimensional approximation of the Koopman operator provides a data-driven approach for the computation of the mode-in-state participation factors.

Case 2: Suppose that the quotient between \( \gamma_{r0} \) and \( \gamma_{i0} \), \( r \neq i \), is an odd function. By assuming that the components of the initial state vector, \( x_0 \), are independent with zero mean, and by virtue of the law of the unconscious statistician [25], the second term in (27) vanishes and the following expression same as above holds:

\[
p_{ij} = \xi_{ij} \phi_{ij} + \sum_{r=1, r \neq i}^q \xi_{rj} \phi_{ij} \mathbb{E} \left[ \frac{\gamma_{r0}}{\gamma_{i0}} \right] = \xi_{ij} \phi_{ij}. \tag{29}
\]

Definition 7: The data-driven state-in-mode participation factors for nonlinear systems based on the EDMD are defined as

\[
\pi_{ij} := \frac{(\mathbb{R} \{ \xi_{ij} \})^2}{(\mathbb{R} \{ \xi_j \})^T \mathbb{R} \{ \xi_j \}}. \tag{30}
\]

For this, we have assumed that the observables \( \gamma_1(x_0), \ldots, \gamma_q(x_0) \) are jointly uniformly distributed over the unit sphere in \( \mathbb{R}^q \) centered at the origin. Notice that, in power systems, one can often normalize the acquired measurements or estimates, and their corresponding functions, to comply with this assumption.

We now sketch the derivation of (30). Suppose that \( \gamma(x_{k-1}) = \tilde{x}_{k-1} \), and \( \gamma(x_k) = \tilde{x}_k \). By making use of (20),

\[
\tilde{x}_k = K \tilde{x}_{k-1}. \tag{31}
\]

Now, by defining a similarity transformation \( z_k := \Xi \tilde{x}_k \) and substituting it into (31), we have

\[
z_k = \Xi K \Xi^{-1} z_{k-1} = \Omega z_{k-1}, \tag{32}
\]

where \( \Omega = \text{diag}(\mu_1, \ldots, \mu_q) \). We reach the final result (30) under the aforementioned assumption that follows [5]; here due to space limitation, the detailed derivation is omitted.

C. Example: A Canonical Nonlinear Dynamical System

Consider the autonomous dynamical system expressed as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\lambda_1(x_1 - x_2^2) \\
\lambda_2 x_2
\end{bmatrix}, \tag{33}
\]

with \( \lambda_1 = -1 \), \( \lambda_2 = -0.05 \), and \( x_0 = [-1; 2]^T \), and assume that \( x_0 \) follows a uniform probability density function. This system (33) has been studied by Brunton et al. in [26]. By selecting \( w_1 = x_1, w_2 = x_2 \), and \( w_3 = x_2^3 \), we have

\[
\begin{align*}
\dot{w}_1 &= \dot{x}_1 = \lambda_1(x_1 - x_2^2) = \lambda_1 w_1 - \lambda_1 w_3, \\
\dot{w}_2 &= \dot{x}_2 = \lambda_2 x_2 = \lambda_2 w_2, \\
\dot{w}_3 &= 3x_2 \dot{x}_2 = 2\lambda_2 x_2^2 = 2\lambda_2 w_3,
\end{align*}
\]

where \( [w_1; w_2; w_3]^T \) is the vector of the observable functions. Notice that this particular choice allows us to transform the two-dimensional nonlinear system (33) into a three-dimensional linear system (34) without any linearization. Although such a finite-dimensional transformation only exists for certain classes of nonlinear dynamical systems [26], this example elucidates the key idea of the Koopman operator-theoretic approach. In what follows, we first integrate (33) numerically with a time step of 0.01 seconds and use the results to build (18). Then, we select \( \gamma = [x_1; x_2; x_2^2]^T \), with \( \gamma_0 = [-1; 2; 4]^T \). By applying the EDMD, we obtain \( \mu = [-1; -0.05; -0.11]^T \),

\[
\begin{bmatrix}
1 & 0 & -1.1111 \\
0 & 1 & 0 \\
0 & 0 & 1.4948
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0.7433 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Finally, by making use of (29) and (30), we find

\[
P = \begin{bmatrix}
1 & 0 & 0.8259 \\
0 & 1 & 0 \\
0 & 0 & 0.5525 \\
0 & 1 & 0
\end{bmatrix}, \quad \Pi = \begin{bmatrix}
0.4475 & 0 & 0 \\
0 & 1 & 0 \\
0.5525 & 0 & 1
\end{bmatrix}.
\]

Notice that the sum of all the entries of a single row or column of \( P \) is not necessarily equal to 1. However, if the first two columns of \( P \) related to the linear modes 1 and 2 are taken into consideration, the proposed matrix of linear and nonlinear participation factors reduces to the one proposed by Pérez-Arriaga et al. [1]. It is important to mention that the choice of the vector of observables given by (19) plays a key role. Furthermore, the nonlinear participation factors are not necessarily restricted to the unit interval; in fact, if for instance we choose \( \lambda_1 = -1 \) and \( \lambda_2 = -0.4 \), we get \( p_{13} = 4.9029 \), i.e., the participation factor associated with the nonlinear mode 3 is equal to 4.9029 > 1. Conversely, the state-in-mode participation factors are always restricted to the interval (0, 1); see (30). We compute the time evolution of each modal variable \( w_j(t) \) for the following set of initial conditions:

\[
\gamma_0 = [0.1; 0; 0]^T, \quad \gamma_0 = [0; 0.1; 0]^T, \quad \text{and} \quad \gamma_0 = [0; 0; 0.1]^T,
\]

which allows us to distinguish the influence of each observable function on each mode. The results depicted in Fig. [1] are in good agreement with \( \Pi \).

IV. NUMERICAL RESULTS IN POWER SYSTEMS

We carry out simulations on the two-area four-machine power system [10]. All the generators are represented by the sub-transient model equipped with an automatic voltage regulator and a fast-response exciter; the loads are modeled as constant admittances. The system operating condition is a highly stressed one, close to the point of voltage collapse,
characterized by a power flow of 457 MW from Area 1 to Area 2. A three-phase short-circuit is applied at Bus 5 and cleared after 10 milliseconds with no line switching. We rely on time-domain simulations only to emulate synchrophasor measurements, with a reporting rate of 120 frames/second. From the small-signal stability analysis, the system has one inter-area and two local electromechanical linear modes of oscillation, as presented in Table I.

**TABLE I**

| Mode       | Eigenvalue | Freq (Hz) | Damp. (%) |
|------------|------------|-----------|-----------|
| Inter-area | -0.0014 ± j 0.0242 | 0.2625 | 3.74 |
| Local (Area 1) | -1.2452 ± j 0.0146 | 1.2756 | 15.35 |
| Local (Area 2) | -1.7396 ± j 0.0730 | 1.2212 | 22.10 |

In order to estimate the Koopman tuples via EDMD, we select \( \gamma = [\delta^\top; \omega^\top; E_{fd}^\top; (\sin(\delta))^\top; (\cos(\delta))^\top; P_{gen}^\top]^\top \), where \( \delta, \omega, E_{fd}, \) and \( P_{gen} \) are vectors containing the generators’ rotor angle, rotor speed deviation, field voltage, and real power injection, respectively. Although this particular set of observables led to good results in all of the extensive tests that we have performed, we note that the choice of the observable functions for power systems, and in general \( [26] \), remains as an open problem and is out of the scope of the present work. We also remark that the generators’ rotor angle is not directly measured in practice, and should be estimated via a dynamic state estimator \([20],[21],[27],[28]\]. Likewise, brushless excitation systems are commonly found in practice and they do not allow us to directly measure the field voltage, which in this case shall be estimated as well. We assume that \( x_0 \) follows a uniform probability density function. To assess the estimation error \( \varepsilon \), we apply the Frobenius norm on the matrices containing the snapshots of the system state vector, \( x \), obtained from the time-domain simulation and from the EDMD, \( \hat{x} \). We find an adequate value \( \varepsilon = \| \hat{x} - x \|_F / \| x \|_F = 0.04\% \). The EDMD eigenvalues are shown in Table II. We notice that the pair of eigenvalues \( \mu_{4,5} \) is similar to the local (linear) mode of Area 2. Likewise, \( \mu_{6,7} \) refer to the local mode of Area 1, and \( \mu_{19,20} \) to the inter-area mode. Notice that, since the system has 4 generators and 6 observable functions are being selected, this results in 24 modes. Furthermore, because we adopt the center of angle reference frame, one eigenvalue is equal to zero, \( \mu_{16} = 0 \), which helps us to validate the estimation results. Table III displays the mode-in-state participation factors computed using \([23]\). Notice that the results are not in the unit interval as is the case for the model-based participation factors. In the model-based approach, if the eigenvalues are non-degenerate, each left eigenvector is orthogonal to all right eigenvectors except its corresponding one, and vice versa. This property does not hold for \( \Xi \), the matrix containing the left eigenvectors of \( K \), and \( \Phi \), the matrix containing the Koopman modes. We recommend to normalize the matrix containing the mode-in-state participation factors by row. From Table I we can see that \( \mu_{1,2} \) is a control mode with frequency equal to 5.92 Hz. From Table III we observe that \( \mu_{1,2} \) has the highest participation on the state \( E_{fd,t2} \), which is expected since generator 2 is electrically the closest to Bus 5 where a three-phase short-circuit has been applied to. Although the participation factors are supposedly independent of the disturbance duration and location, the proposed technique is data-driven and relies on the most excited modes in the dataset. The modes \( \mu_{8,9}, \mu_{14,15}, \) and \( \mu_{17,18} \) respectively have the first, second and third highest participation on the states \( \{ \delta_1, \delta_2, \delta_3, \delta_4 \} \). From Table III we observe that \( f_{8,9} = 0.83 \) Hz, \( f_{14,15} = 0.32 \) Hz, and \( f_{17,18} = 0.31 \) Hz, i.e., these are inter-area modes. Their frequency, however, differ from the linear inter-area modes. In this sense, the frequency of \( \mu_{12,13} \) is approximately twice the frequency of the linear inter-area mode.

![Evolution of the modal variables](image_url)
V. CONCLUSIONS

A novel data-driven technique that reveals both linear and nonlinear participation factors based on the Koopman mode decomposition has been proposed. Numerical simulations carried out on a canonical nonlinear dynamical system, and on the two-area four-machine power system, demonstrated the performance of our technique. Since the Koopman mode decomposition is capable of coping with a large class of nonlinearity, the proposed technique is applicable to complex oscillatory responses arising in practice due to nonlinearities, as is the case in power systems. To demonstrate the broadness of our technique, its performance under particular phenomena such as bifurcations will be evaluated and reported in future publications.

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