On the fate of the phantom dark energy universe in semiclassical gravity II: Scalar phantom fields

Jaume de Haro\textsuperscript{1,}*, Jaume Amoros\textsuperscript{1,}† and Emilio Elizalde\textsuperscript{2,}‡

\textsuperscript{1}Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
\textsuperscript{2}Instituto de Ciencias del Espacio (CSIC) & Institut d’Estudis Espacials de Catalunya (IEEC/CSIC)
Campus UAB, Facultat de Ciències, Torre C5-Parell-2a planta, 08193 Bellaterra (Barcelona) Spain

Quantum corrections coming from massless fields conformally coupled with gravity are studied, in order to see if they can lead to avoidance of the annoying Big Rip singularity which shows up in a flat Friedmann-Robertson-Walker universe filled with dark energy and modeled by a scalar phantom field. The dynamics of the model are discussed for all values of the two parameters, named $\alpha > 0$ and $\beta < 0$, corresponding to the regularization process. The new results are compared with the ones obtained in [1] previously, where dark energy was modeled by means of a phantom fluid with equation of state $P = \omega \rho$, with $\omega < -1$.

PACS numbers: 98.80.Qc, 04.62.+v, 04.20.Dw

Keywords: Dark energy, future singularities, semiclassical gravity

I. INTRODUCTION

Recent observations of distant type Ia supernovae, baryonic acoustic oscillations (BAO), anisotropies of the cosmic microwave background radiation (CMB), and some other, confirm that our universe expands in an accelerated way [2,3]. In fact, it seems that it is at present in a dark energy phase [4]. A proposal to explain this situation is to assume that the energy density of our universe is dominated by a phantom scalar field, that is, a model where the energy density and pressure are $\rho = -\frac{1}{2} \dot{\phi}^2 + V(\phi)$ and $P = -\frac{1}{2} \dot{\phi}^2 - V(\phi)$, being $\phi$ the scalar phantom field. In this case, future singularities are bound to appear in a finite time [5]. These singularities are undoubtedly there in the classical situation when no quantum effects are taken into account, but it seems feasible that, near the singularities, where the curvature has very high values, quantum effects could have the power to drastically modify the behavior of the universe, yielding a milder singularity or maybe even a non-singular model.

In this paper we extend the study carried out in [1] to a dark energy universe modeled by a scalar phantom field. In fact we will consider exponential potentials which give rise to a Big Rip singularity, and introduce quantum corrections in order to avoid these late time singularities. Specifically, we shall consider the quantum effects due to massless, conformally coupled fields. This is a special, workable case where the quantum vacuum stress tensor—which depends on two regularization parameters, here called $\alpha > 0$ and $\beta < 0$—and the semiclassical Friedmann equation, can be both calculated explicitly.

We will show, analytically and numerically, that quantum effects drastically modify the Big Rip singularity, rendering it of type III or turning it into a singularity in the contracting phase. In the first case (a type III singularity) the Hubble parameter does not diverge, but the energy density does tend towards infinity. In the other case (a singularity in the contracting phase) the Hubble parameter becomes finite and negative, and the energy density diverges towards minus infinity. What is important to note is that, in both cases, the Hubble parameter remains finite.

The paper is organized as follows. In the next Section, using the mathematical theory of dynamical systems, we study some phantom fields driving the universe to a Big Rip singularity. In Sect. III we introduce the quantum corrections due to a massless conformally coupled field, and we perform an analytic study of the semiclassical Friedmann equations. In Sect. IV a numerical analysis is carried out to check to good approximation the analytic results obtained in the previous Section. In Sect. V we analyze the problem in the context of loop quantum cosmology, where it has been stated that quantum corrections do completely avoid the Big Rip singularity. We will see that the way to obtain these conclusion is

* E-mail: jaime.haro@upc.edu
† E-mail: jaume.amoros@upc.edu
‡ E-mail: elizalde@ieec.uab.es, elizalde@math.mit.edu
in doubt, because they have been got in some places from an incorrectly modified Friedmann equation. In last Section we compare the results obtained for a phantom field with those that were derived for a phantom fluid model. The units to be used in the paper are: \( c = \hbar = M_p = 1 \), where \( M_p \) is the reduced Planck mass.

II. DARK ENERGY MODELED BY A PHANTOM FIELD

A phantom field \( \phi \) is modeled by an energy density of the kind \( \rho = -\frac{1}{2} \dot{\phi}^2 + V(\phi) \) and a pressure \( P = -\frac{1}{2} \dot{\phi}^2 - V(\phi) \). For this field the Friedmann and conservation equations are:

\[
\begin{align*}
H^2 &= \frac{1}{3} \left( -\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \\
0 &= -\dot{\phi} - 3H\dot{\phi} + \frac{dV}{d\phi}
\end{align*}
\]

From this system one deduces that \( \dot{\rho} = 3H\dot{\phi}^2 \), and thus \( H = \frac{1}{2} \dot{\phi}^2 > 0 \), which means that \( \frac{2}{a} = H + H^2 > 0 \), that is, in this model the universe is expanding in an accelerating way.

To analyze the dynamics of the system we start considering a power-law potential, i.e., \( V(\phi) = \lambda \phi^{2n} \) with \( \lambda > 0 \). The field equation can be written as follows

\[
d\left( \frac{1}{2} \dot{\phi}^2 + \bar{V}(\phi) \right) = -3H\dot{\phi}^2,
\]

where \( \bar{V}(\phi) = -V(\phi) \). This is a dissipative system, and the slow-roll conditions \( (V'/V)^2 \ll 1 \) and \( |V''/V| \ll 1 \) are satisfied when \( |\phi| \gg n \). Then, due to the attractor nature of the slow-roll regime, at late time, the solutions have the same behavior as the slow-roll solution, which satisfies the system

\[
\begin{align*}
H^2 &= \frac{1}{3} V(\phi) \\
0 &= -3H\dot{\phi} + \frac{dV}{d\phi}
\end{align*}
\]

Since the dynamics of the system decouples for \( \phi > 0 \) and \( \phi < 0 \), we only consider the domain \( \phi > 0 \), where the field obeys the equation \( \dot{\phi}_s = 2n \sqrt{\frac{2}{3}} \phi_{sr}^{n-1} \), which solution is

\[
\begin{align*}
\phi_{sr}(t) &= \left[ 2n(n-2) \sqrt{\frac{1}{3}} (t_s - t) \right]^{\frac{1}{n-2}} \quad \text{for } n > 2 \\
\phi_{sr}(t) &= \phi_{sr}(t_0) e^{4\sqrt{\frac{1}{3}} (t-t_0)} \quad \text{for } n = 2 \\
\phi_{sr}(t) &= \phi_{sr}(t_0) + 2 \sqrt{\frac{1}{3}} (t-t_0) \quad \text{for } n = 1.
\end{align*}
\]

Evaluating \( H_{sr}(t) = \sqrt{\frac{2}{3}} \phi_{sr}^{n}(t) \), one can see that the Big Rip singularity appears for \( n > 2 \).

The following remark is in order. To prove, in a rigorous way, the attractor nature of the slow roll solution, we may use the variables \( x = \frac{\phi}{\sqrt{n}} \) and \( y = \frac{\phi}{\sqrt{6}H_X} \). Then, the dynamical equation is

\[
\frac{dy}{dx} = -3(1 + y^2) \left[ 1 - \frac{V'}{\sqrt{6}gY} \right] = -3(1 + y^2) \left[ 1 - \frac{2n}{6y^2} \right].
\]

The slow roll solution is the curve \( \frac{dy}{dx} = 0 \), i.e., \( y = \frac{2n}{6x} \), and it is easy to verify that, for large values of \( x \), this is the leading term of the solution. Then, since \( \frac{dy}{dx} < 0 \) above this curve, and \( \frac{dy}{dx} > 0 \) below it, this definitely proves that the slow roll solution is an attractor at late times.

As a different specific example, we choose \( V(\phi) = V_0 e^{-2\phi/\phi_0} \) (being \( V_0 \) and \( \phi_0 \) two constant parameters). Then, with the change of function \( \phi = \phi_0 \ln(\psi) \) (now \( \psi \) belongs in the domain \((0, \infty)\)), the system becomes

\[
\begin{align*}
H^2 &= \frac{1}{3\psi^2} \left( -\frac{2\phi_0^2}{2} \dot{\psi}^2 + V_0 \right), \\
0 &= -\dot{\phi}_0 (\ddot{\psi} - \psi^2) - 3H \phi_0 \dot{\psi} - 2 \frac{\dot{\psi}}{\phi_0}.
\end{align*}
\]
In the expanding phase ($H > 0$), this system can be written as follows:

$$\phi_0 \ddot{\psi} + \frac{2}{\phi_0} \left( -\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0 \right) + \sqrt{3} \phi_0 \dot{\psi} \sqrt{\frac{\phi_0^2}{2}} \dot{\psi}^2 + V_0 = 0, \quad (7)$$

and dividing this equation by $\dot{\phi}$—i.e., using the variable $\phi$ as a time—one obtains the equation

$$\frac{d\dot{\psi}}{d\phi} = -\frac{2}{\phi_0^3} \left( -\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0 \right) - \frac{\sqrt{3}}{\phi_0} \sqrt{\frac{\phi_0^2}{2}} \dot{\psi}^2 + V_0 \equiv F(\dot{\psi}). \quad (8)$$

This is an autonomous first order differential equation, therefore, it can be completely studied just through the sign of the function $F$. From the Friedmann equation, one can see that the domain of $F$ is the interval $[-\frac{\sqrt{2V_0}}{\phi_0}, \frac{\sqrt{2V_0}}{\phi_0}]$. The zeros of $F$ are the points $\pm \frac{\sqrt{2V_0}}{\phi_0}$ and $-\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}}$. $F$ has a vertical asymptotic at zero. Finally, $F$ is positive in the interval $(-\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}}, 0)$ and negative in $(-\frac{\sqrt{2V_0}}{\phi_0}, -\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}} \cup (0, \frac{\sqrt{2V_0}}{\phi_0})$.

This all means that, using $\phi$ as time, the critical points $\frac{\sqrt{2V_0}}{\phi_0}$ and $-\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}}$ are repellers and the critical points 0 and $-\frac{\sqrt{2V_0}}{\phi_0}$ are attractors (see Pict. 1). However, in the domain $\psi < 0$, when the time $\phi$ increases, the cosmic time $t$ decreases, and vice versa, what means that, in terms of the cosmic time, the critical point $-\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}}$ is a global attractor while the other two critical points $\pm \frac{\sqrt{2V_0}}{\phi_0}$ are repellers (see Pict. 2).

In terms of the field $\phi$, the critical points obtained above are:

$$\psi = \frac{-\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}} \iff \phi(t) = \phi_0 \ln \left( \frac{t_s - t}{\phi_0 \sqrt{1 + \frac{2V_0}{\phi_0}}} \right); \quad H(t) = \frac{\phi_0^2}{2} \frac{t_s - t}{t_s - t}; \quad (9)$$

$$\psi = \pm \frac{\sqrt{2V_0}}{\phi_0} \iff \phi(t) = \phi_0 \ln \left( \frac{t_s - t}{\phi_0 \sqrt{1 + \frac{2V_0}{\phi_0}}} \right); \quad H(t) = 0,$$

where $t_s$ is an arbitrary constant. Then, since $-\frac{\sqrt{2V_0}}{\phi_0} \frac{1}{\sqrt{1 + \frac{2V_0}{\phi_0}}}$ is a global attractor, it follows that, except for the solutions $\psi = \pm \frac{\sqrt{2V_0}}{\phi_0}$, all the other have a Big Rip singularity.
III. QUANTUM CORRECTIONS

In this section we will study in detail what is the change in the dynamics in the model $V(\phi) = V_0 e^{-2\phi/\phi_0}$ when one takes into account quantum effects. It is well-known that for a massless, conformally coupled field, the anomalous trace is given by [1, 6]

$$T_{\text{vac}} = \alpha \Box R - \frac{\beta}{2} G,$$

where $R = 6 \left( \dot{H} + 2H^2 \right)$ the scalar curvature and $G = 24H^2 \left( \dot{H} + H^2 \right)$ the Gauss-Bonnet curvature invariant.

The coefficients, $\alpha$ and $\beta$, coming from dimensional regularization are [8]

$$\alpha = \frac{1}{2880\pi^2} \left( N_0 + 6N_{1/2} + 12N_1 \right) > 0,$$

$$\beta = \frac{-1}{2880\pi^2} \left( N_0 + 11N_{1/2} + 62N_1 \right) < 0,$$

with $N_0$ the number of scalar fields, $N_{1/2}$ the number of four-component neutrinos, and $N_1$ the number of electromagnetic fields, respectively.

In terms of the Hubble parameter, Eq. (10) is [9]

$$T_{\text{vac}} = 6\alpha \left( \ddot{H} + 12\dot{H}H + 7H\dot{H} + 4\dot{H}^2 \right) - 12\beta (H^4 + 2H^2 \dot{H}).$$

With the trace anomaly being $T_{\text{vac}} = \rho_{\text{vac}} - 3P_{\text{vac}}$ and, inserting (12) into the conservation equation, $\dot{\rho}_{\text{vac}} + 3H (\rho_{\text{vac}} + P_{\text{vac}}) = 0$, the modified energy density reads

$$\rho_{\text{vac}} = 6\alpha \left( 3H^2 \ddot{H} + H\dot{H} - \frac{1}{2} \dot{H}^2 \right) - 3\beta H^4,$$

and the semiclassical Friedmann equation becomes

$$H^2 = \frac{\rho + \rho_{\text{vac}}}{3},$$

with $\rho = -\frac{1}{2} \dot{\phi}^2 + V_0 e^{-2\phi/\phi_0}$.
Using the dimensionless variables \( \bar{t} = H_+ t, \bar{H} = H/H_+, \bar{Y} = \dot{H}/H_+^2, \bar{\psi} = \psi \) and \( \bar{\rho} = \frac{\rho}{H_+^2} \), with \( H_+ = \sqrt{-1/\beta} \), the semiclassical Friedmann equation and the conservation equation can be written as an autonomous system:

\[
\begin{align*}
\ddot{Y}' &= \bar{Y}, \\
\bar{Y}' &= \frac{1}{2\alpha T} \left( -\beta \bar{H}^2 + \beta \bar{\rho} - 6\alpha \bar{H}^2 \bar{Y} + \alpha \bar{Y}^2 + \beta \bar{H}^4 \right), \\
\bar{\varphi}' &= \frac{2}{\bar{\psi}} - 3\bar{H} \bar{\varphi} - 2 \frac{\bar{V}_0}{\bar{\phi}_0 \bar{\psi}}.
\end{align*}
\]

(15)

where \( \dot{} \) denotes derivative with respect to the time \( \bar{t} \), and we have defined the new parameters \( \bar{V}_0 = V_0/H_+^2 \) and \( \bar{\phi}_0 = \phi_0 \).

What we see at first sight from this system is that it does not have any critical point. It is also easy to show that the energy density \( \bar{\rho} \) evolves in accordance with the equation \( \bar{\rho}' = \bar{H} \bar{\phi}_0^2 \bar{\varphi}^2 \), which means that the energy density increases in the expanding phase, and decreases in the contracting one.

Now, we look for singular solutions of the system with the following behavior near to the singularity [10] [11]

\[ \bar{\psi}(\bar{t}) = A(\bar{t}_s - \bar{t}) + B(\bar{t}_s - \bar{t})^2 + \mathcal{O}((\bar{t}_s - \bar{t})^3); \quad \bar{H}(\bar{t}) = \bar{H}_0 + \delta \bar{H}(\bar{t}), \]

(16)

where \( A, B \) and \( \bar{H}_0 \) are some constants. Inserting these functions in the conservation equation, one obtains

\[ A = \sqrt{2V_0/\bar{\phi}_0}; \quad B = -\frac{3}{2} \bar{H}_0 A, \]

(17)

and inserting them in the semiclassical Friedmann equation and retaining the leading terms, one gets

\[ \delta \bar{H}''(\bar{t}) = \frac{\beta}{2\alpha} \frac{\bar{\varphi}_0^2}{\bar{t} - \bar{t}_s} \Rightarrow \delta \bar{H}(\bar{t}) = \frac{\beta}{2\alpha} \bar{\varphi}_0 \left. \frac{\bar{\varphi}}{\bar{\psi}} \right|_{\bar{t}_s} \ln \left( \frac{\bar{t}_s - \bar{t}}{T} \right), \]

(18)

where \( T \) is an integration constant.

What we observe here is that, when we introduce quantum corrections, the Big Rip singularity, for \( \bar{H}_0 > 0 \), is transformed into a type III singularity, because as \( \bar{t} \to \bar{t}_s \) one has \( \bar{H} \to \bar{H}_0, \bar{\rho} \to \infty \) and \( |p| \to \infty \). And, when \( \bar{H}_0 < 0 \), one gets \( \bar{H} \to \bar{H}_0 \) (contracting phase), \( \bar{\rho} \to -\infty \) and \( |P| \to \infty \).

In order to qualitatively study the system it is quite convenient, as in [12] [13], to perform the variable change \( \bar{\rho} \equiv \sqrt{\bar{H}} \). After what, the semiclassical Friedmann and conservation equations become

\[ \bar{\rho}'' = -\partial_{\bar{\rho}} W(\bar{\rho}, \bar{\rho}') - 3\epsilon \bar{\rho}^2 (\bar{\rho}')^2, \quad \bar{\rho}' = \bar{H} \bar{\varphi}_0^2 \frac{\bar{\varphi}^2}{\bar{\psi}^2}. \]

(19)

where \( W(\bar{\rho}, \bar{\rho}') = \frac{\bar{\beta}}{8\alpha} \left( \bar{\rho}^2 (1 - \frac{1}{3} \bar{\rho}^4) + \frac{\bar{\varphi}}{\bar{\rho}} \right), \) and \( \epsilon \equiv \text{sign}(\bar{H}) \).

For positive values of \( \bar{\rho} \), the potential \( W \) (Fig. 3 of Ref. [13]), has a unique zero, at \( \bar{\rho}_0 = (3/2)^{1/4} \left( 1 + \sqrt{1 + \frac{4}{3} \bar{\rho}} \right)^{1/4} \), and two critical points, at \( \bar{\rho}_\pm = \left( \frac{1 + \sqrt{1 - 1/3}}{2} \right)^{1/4} \) (\( \bar{\rho}_- < \bar{\rho}_+ \)). Thus, for \( \bar{\rho} > 1/4 \) there are no critical points, being the potential strictly increasing, from \( -\infty \) to \( \infty \). For \( \bar{\rho} < 1/4 \), the potential satisfies \( W(0) = -\infty \) and \( W(\infty) = \infty \), and exhibits a relative maximum, at \( \bar{\rho}_- \), and a relative minimum, at \( \bar{\rho}_+ \) (a hollow one). For very small values of \( \bar{\rho} \), at \( \bar{\rho}_- \) one has \( \bar{H}^2 \equiv \bar{\rho} \), that is, the system is close to the Friedmann phase and, at \( \bar{\rho}_+ \), one has \( |\bar{H}| \equiv 1 \), that is, the system is close to the de Sitter phase. On the other hand, for negative values of \( \bar{\rho} \), the potential only has a critical point at \( \bar{\rho}_+ \), and satisfies \( W(0) = W(\infty) = \infty \).

Now, assume that, initially, the system has an energy density which is positive, and that it is in the expanding phase (what does happen nowadays). Then, since in the expanding phase the energy density increases, this means that the slope of the potential is more steep and thus the system can evolve to the contracting phase. When it enters that phase, the energy density decreases and even it could be negative; if so, the system is confined in the decreasing phase because the potential satisfies \( W(0) = W(\infty) = \infty \).

What is important to stress here is that the system cannot remain all the time in the expanding phase due to the form of the potential, and also the fact that the energy density is increasing, in this phase. Three different situations may occur:

1. The system may develop a singularity in a finite time (type III singularity). This comes from Eq. (18).
2. The system may enter in the decreasing phase and the energy density becomes negative, and then the system cannot abandon this phase. In this situation the energy density could by $-\infty$ in a finite time.

3. The system may bounce infinitely many times (an oscillating universe).

This is what one can say by analytically studying the system. What we will do in next section is to perform a corresponding numerical study, which will to show that only the first two situations are actually possible.

IV. NUMERICAL ANALYSIS

In this section we numerically integrate the system (15), assuming that initially the system is in the Friedmann phase, that is, at time $t = 0$ the variables $(\bar{H}(0), \bar{Y}(0), \bar{\psi}(0), \bar{\phi}(0))$ satisfy the constraints

$$\bar{H}^2(0) = \frac{1}{3} \left( -\bar{Y}(0) + \frac{\bar{V}_0}{\bar{\psi}^2(0)} \right); \quad \bar{Y}(0) = \frac{\bar{\phi}_0^2 \bar{\phi}^2(0)}{2 \bar{\psi}^2(0)}.$$  \hspace{1cm} (20)

This means that the initial conditions depend on two variables. Next, to perform our calculations we choose as variables $(\bar{\psi}(0), \bar{\phi}(0))$, and also the following values for the parameters: $\bar{V}_0 = 24$ and $\bar{\phi}_0 = 4/\sqrt{3}$. Note that, from Eq. (20), with our choose of parameters the variable $\bar{\phi}(0)$ belongs to the interval $[-3, 3]$ and $\bar{\psi}(0)$ belongs to $(0, \infty)$.

In Fig. 2 we plot three simulations for different values of $\beta/\alpha$ (the system (15) depends on this quotient), the first being for $\beta/\alpha = -0.5$, the second for $\beta/\alpha = -1$, and the last one for $\beta/\alpha = -10$. The blue color means initial conditions which drive to a singularity of the form given by Eqs. (17) and (18) in the decreasing phase, that is, the Hubble parameter is negative and the energy density diverges to minus infinity. On the other hand, the red color means initial conditions which drive to a type III singularity.

In Fig. 3 we have integrated the system (15) for $\beta/\alpha = -10$, and we show the evolution of the Hubble parameter and of the energy density. In the first two plots the initial conditions are taken in the blue region of Fig. 1, given a universe evolving, at late times, in the decreasing phase with an energy density which diverges at late times. The last two plots correspond to initial conditions taken in the red region of Fig. 1, and they show a type III singularity.

![Figure 2: Three different simulations, for the values $\beta/\alpha = -0.5$, $\beta/\alpha = -1$, and $\beta/\alpha = -10$, respectively. Red points mean initial conditions driving to a type III singularity. Blue points, initial conditions driving to a singularity in the decreasing phase.](image-url)
V. PHANTOM FIELDS IN LOOP QUANTUM COSMOLOGY

For the flat FRW spacetime, Einstein’s theory is obtained from the Lagrangian $L = \frac{1}{2} R a^3 + L_{\text{matter}}$, where $a$ denotes the scalar factor and $L_{\text{matter}} = a^3 \rho = a^3 \left( -\frac{1}{2} \dot{\phi}^2 - V(\phi) \right)$. This Lagrangian can be written as follows

$$L = 3 \left( \frac{d(\dot{a}a^2)}{dt} - \dot{a}^2 a \right) + a^3 \rho,$$

what means that the same theory is obtained avoiding the total derivative, which gives the Lagrangian $L_E = -3\dot{a}^2 a + a^3 \rho$. The conjugate momentum of the scale factor is then given by $p = \frac{\partial L_E}{\partial \dot{a}} = -6\dot{a}a$, and thus the Hamiltonian is

$$H_E = \dot{a}p + a^3 \frac{\partial P}{\partial \dot{\phi}} - L_E = -\frac{p^2}{12a} + a^3 \rho = -3H^2 a^3 + a^3 \rho.$$  \hspace{1cm} (21)

On the other hand, in loop cosmology the following effective Hamiltonian, which captures the underlying loop quantum dynamics, is considered $[14,16]$

$$H_{LQC} = -3V \frac{\sin^2(\lambda \beta)}{\gamma^2 \lambda^2} + V \rho,$$  \hspace{1cm} (22)

where $\gamma$ is the Barbero-Immirzi parameter and $\lambda$ is a parameter with dimensions of length, which is determined by invoking the quantum nature of the geometry, that is, through identification of its square with the minimum eigenvalue of the area operator in LQG, which gives as a result $\lambda \equiv \sqrt{\frac{\sqrt{3}}{4}}$ (see [16]). Here $V$ is the physical volume $V = a^3$ and $\beta$ is canonically conjugated to $V$, and satisfies $\{\beta, V\} = \frac{1}{2} \frac{\partial}{\partial \beta}$, where $\{,\}$ is the Poisson bracket.

The Hamiltonian constraint is then given by $\frac{\sin^2(\lambda \beta)}{\gamma^2 \lambda^2} = \frac{4}{9}$, and the Hamiltonian equation yields the identity:

$$\dot{\beta} = \{V, H_{LQC}\} = -\frac{\gamma}{2} \frac{\partial H_{LQC}}{\partial \beta} \iff H = \frac{\sin(2\lambda \beta)}{2\gamma \lambda} \iff \beta = \frac{1}{2\lambda} \arcsin(2\lambda \gamma H).$$  \hspace{1cm} (23)
Writing this last equation as $H^2 = \frac{\sin^2(\lambda \beta)}{2\gamma^2 \lambda^2} (1 - \sin^2(\lambda \beta))$, and using the Hamiltonian constraint $H_{LQC} = 0 \iff \frac{\sin^2(\lambda \beta)}{2\gamma^2 \lambda^2} = \frac{\rho}{3}$, one gets the following modified Friedmann equation in loop quantum cosmology

$$H^2 = \frac{\rho}{3} \left(1 - \frac{\rho}{\rho_c}\right) \iff \frac{H^2}{\rho_c^2/12} + \frac{(\rho - \rho_c)^2}{\rho_c^2/4} = 1,$$

(24)

being $\rho_c = \frac{3}{2\gamma^2 \lambda^2}$. This equation, together with the conservation equation $-\dot{\phi}/3H\dot{\phi} + \frac{dV}{d\phi} = 0$, determine the dynamics of the universe in loop cosmology.

From the equation of the ellipse, one can easily check that the Hubble parameter belongs to the interval $[-\rho_c/12, \rho_c/12]$, and the energy density, $\rho$, to $[0, \rho_c]$, what means that there is not Big Rip. In fact, an exhaustive study of the potential $V = V_0 e^{-2\beta/\omega \phi}$ was performed in [17,18].

But here a problem appears. It is well-known that the current cosmological theories are built from two invariants, the scalar curvature $R$ and the energy density, one gets, in terms of the standard variables, the following Lagrangian Gauss-Bonnet gravity [19] the Lagrangian and then it cannot leave the decreasing phase, becoming singular at finite time, as we have shown, again numerically and $\rho$ escape towards infinity in finite time (this was proven in [1]). The same does not happen for a phantom field where the plane $\rho > 0$ is not invariant, and eventually, the system crosses this manifold, i.e., it can have negative energy density, and then it cannot leave the decreasing phase, becoming singular at finite time, as we have shown, again numerically and analytically.

VI. DISCUSSION AND COMPARISON WITH THE PHANTOM FLUID MODEL

In [11] we have studied in detail the case of a phantom fluid modeled by the EoS $P = \omega \rho$ with $\omega < -1$. There, we have shown that, in the case $-1 \leq \frac{\beta}{\omega} < 0$, there exists a one parameter family of solutions which evolves into the contracting Friedmann phase at late times and only a particular solution asymptotically converging towards the de contracting de Sitter universe. All the other solutions enter into the contracting phase and become singular at finite time, satisfying $\lim_{t \to t_s} H(t) = -\infty$ and $\lim_{t \to t_s^+} \rho(t) = 0$. On the other hand, we have shown in [11], that for $-1 > \frac{\beta}{\omega}$ almost all solutions describe a universe bouncing infinitely many times (an oscillating universe).

In the present paper, by studying a phantom field we have shown, both analytically and numerically, that all solutions are singular. Some of them display Type III singularities and the other ones are singular in the contracting phase, satisfying $H(t) \to H(t_s) < 0, \rho(t) \to -\infty$ and $|P(t)| \to \infty$, when $t \to t_s$.

The difference comes from the fact that, for a phantom fluid, when one considers the dynamics in $\mathbb{R}^3$ using the coordinates $(H, H, \rho)$, the manifold $\rho = 0$ is invariant. More precisely, the half plane $\rho = 0$ with $H > 0$ is a repeller, whereas when $H > 0$ it is an attractor. This means that, at late time, all the solutions go towards this half plane. Moreover, in the contracting phase there is a critical point $(-\sqrt{-1/\beta}, 0, 0)$ (the contracting de Sitter universe) which restricted to the plane $\rho = 0$ is a repeller. This means that only a solution tends asymptotically towards this point, while all the other escape towards infinity in finite time (this was proven in [11]). The same does not happen for a phantom field where the manifold $\rho = 0$ is not invariant, and eventually, the system crosses this manifold, i.e., it can have negative energy density, and then it cannot leave the decreasing phase, becoming singular at finite time, as we have shown, again numerically and analytically.
Acknowledgments. This investigation has been supported in part by MICINN (Spain), projects MTM2011-27739-C04-01, MTM2009-14163-C02-02, and FIS2010-15640, by the CPAN Consolider Ingenio Project, and by AGAUR (Generalitat de Catalunya), contracts 2009SGR 345, 994 and 1284. EE was also supported by MICINN (Spain), contract PR2011-0128, and his research was partly carried out while on leave at the Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755, USA.

[1] J. Haro, J. Amoros and E. Elizalde, Phys. Rev. D83, 123528 (2011).
[2] S. Perlmutter et al., Astrophys. J. 517, 565 (1999).
[3] A.G. Riess et al., Astron. J. 116, 1009 (1999).
[4] E. Komatsu et al., Astrophys. J. Suppl. Ser. 192, 18 (2011).
[5] S.K. Srivastava, Gen. Relativ. Gravit. 39, 241 (2007).
[6] S. Nojiri, S. Odintsov and S. Tsujikawa, Phys. Rev. D71, 063005 (2005).
[7] V. Mukhanov, Physical fundations of cosmology (Cambridge University Press, 2005).
[8] M.V. Fischetti, J.B. Hartle and B.L. Hu, Phys. Rev. D20, 1757 (1979).
[9] P.C.W. Davies, Phys. Lett. B68, 402 (1977).
[10] S. Nojiri and S. Odintsov, Phys. Rev. D70, 103522 (2004).
[11] E. Elizalde, S. Nojiri and S. Odintsov, Phys. Rev. D70, 043539 (2004).
[12] T. Azuma and S. Wada, Prog. Theor. Phys. 75, 845 (1986).
[13] S. Wada, Phys. Rev. D31, 2470 (1985).
[14] A. Ashtekar and P. Singh, Class. Quantum Grav. 29, 213001 (2011).
[15] P. Singh, Class. Quantum Grav. 26, 125005 (2009).
[16] P. Singh, J. Phys. Conf. Ser. 140, 012005 (2009).
[17] T. Naskar and J. Ward, Phys. Rev. D76, 063514 (2007).
[18] D. Smart and B. Gumjudpai, Phys. Rev. D76, 043514 (2007).
[19] G. Cognola, E. Elizalde, S. Nojiri, S. Odintsov and S. Zerbini, Phys. Rev. D73, 084007 (2006).
[20] J. de Haro, Future singularity avoidance in phantom dark energy models (to appear in JCAP), gr-qc:1204.5604 (2012).