Difference equations for the higher rank XXZ model with a boundary

Takeo Kojima* and Yas-Hiro Quano†

*Department of Mathematics, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan
†Department of Medical Electronics, Suzuka University of Medical Science Kishioka-cho, Suzuka 510-0293, Japan

Abstract

The higher rank analogue of the XXZ model with a boundary is considered on the basis of the vertex operator approach. We derive difference equations of the quantum Knizhnik-Zamolodchikov type for 2N-point correlations of the model. We present infinite product formulae of two point functions with free boundary condition by solving those difference equations with N = 1.

1 Introduction

Representation theory of the affine quantum group plays an important role in the description of solvable lattice models and massive integrable quantum field theories in two dimensions [1, 2, 3]. For models with the affine quantum group symmetry the difference analogue of the Knizhnik-Zamolodchikov equations (quantum Knizhnik-Zamolodchikov equations) are satisfied by both correlation functions and form factors [4, 5].

Integrable models with boundary reflection have been also studied in lattice models and massive quantum theories. The boundary interaction is specified by the reflection matrix $K$ for lattice models [6], and by the boundary S-matrix for massive quantum theories [7]. It is shown in [8] that the space of states of the boundary XXZ model can be described in terms of vertex operators associated with the bulk XXZ model [9]. The explicit bosonic formulae of the boundary vacuum of the boundary XXZ model were obtained by using the bosonization of the vertex operators [10]. This approach is also relevant for other various models [11, 12, 13, 14, 15].

It is shown in [16] that correlation functions and form factors in semi-infinite XXZ/XYZ spin chains with integrable boundary conditions satisfy the boundary analogue of the quantum Knizhnik-Zamolodchikov equation. In this paper we establish the similar results for the $U_q(\widehat{sl}_n)$-analogue of XXZ
spin chain with a boundary magnetic field $h$:

$$
\mathcal{H}_B = \sum_{k=1}^{\infty} \left\{ \sum_{a,b=0}^{n-1} q^{n-1} e_{aa}^{(k+1)} e_{bb}^{(k)} + q^{-1} \sum_{a,b=0}^{n-1} e_{aa}^{(k+1)} e_{bb}^{(k)} - \sum_{a,b=0}^{n-1} e_{ab}^{(k+1)} e_{ba}^{(k)} \right\}
$$

where $-1 < q < 0$ and $0 \leq L \leq M \leq n - 1$. On the basis of the boundary vacuum states constructed in [11], we derive the boundary analogue of the quantum Knizhnik-Zamolodchikov equations for the correlation functions in the higher rank XXZ model with a boundary. We also obtain the two point functions by solving the simplest difference equations for free boundary condition.

The rest of this paper is organized as follows. In section 2 we review the vertex operator approach for the higher rank XXZ model with a boundary. In section 3 we derive the boundary quantum Knizhnik-Zamolodchikov equations for the $2N$-point correlation functions. In section 4 we obtain the two point functions by solving the difference equation with $N = 1$ for free boundary condition. In Appendix A we summarize the results of the bosonization of the vertex operators in $U_q(sl_n)$ [13]. In Appendix B we summarize the bosonic formulae of the boundary vacuum states [11].

## 2 Formulation

The higher rank XXZ model with boundary reflection was formulated in [11] in terms of the vertex operators of the quantum affine group $U_q(sl_n)$. For readers’ convenience let us briefly review the results in [11].

Throughout this paper we fix $n \in \mathbb{N}_{\geq 2}$, and also fix $q$ such that $-1 < q < 0$. The model is labeled by the three parameters $i, L, M$ such that $0 \leq L \leq M \leq n - 1$ and $i \in \{L, M\}$. In this paper we consider the following three cases:

(C1) $0 \leq L = M = i \leq n - 1$,

(C2) $0 \leq L = i < M \leq n - 1$,

(C3) $0 \leq L < M = i \leq n - 1$.

In what follows we denote the $q$-integer $(q^k - q^{-k})/(q - q^{-1})$ by $[k]$, and we use the following symbols:

$$
b(z) = \frac{q - q^{-1}z}{1 - z}, \quad c(z) = \frac{q - q^{-1}}{1 - z}. \tag{2.1}
$$

The nonzero entries of the R-matrix $R^{(i)VV}(z)$ are given by

$$
R^{(i)VV}(z)^{j_1,j_2}_{j_1,j_2} = r^{(i)VV}(z) \times \begin{cases} 
1, & j_1 = j_2 = k_1 = k_2 \\
b(q^2z), & j_1 = k_1 \neq j_2 = k_2, \\
-qc(q^2z), & j_1 = k_2 < j_2 = k_1, \\
-qzc(q^2z), & j_1 = k_2 > j_2 = k_1.
\end{cases} \tag{2.2}
$$

Here the scalar functions are

$$
r^{(i)VV}(z) = z^{-\delta_{i,0}}(q^{2n-1}z; q^{2n})_{\infty}(q^{2n}z; q^{2n})_{\infty}\left(q^{2n}z; q^{2n}z^{-1}; q^{2n} \right)_{\infty}, \tag{2.3}
$$

$$
(\nu z; z^2)_{\infty} = (1 - z^2)^{\nu/2}(1 - z^4)^{\nu/2} \cdots \left(1 - z^{2\nu}\right)^{\nu/2}, \quad \nu \in \mathbb{Z}.
$$
where
\[
(z; p_1, \cdots, p_m)_\infty = \prod_{k_1, \cdots, k_m=0}^\infty (1 - z p_1^{k_1} \cdots p_m^{k_m}).
\]

The boundary K-matrix \( K^{(i)}(z) \) is a diagonal matrix, whose diagonal elements are given by
\[
K^{(i)}(z)_j = \frac{\varphi^{(i)}(z)}{\varphi^{(i)}(1/z)} \times \begin{cases} 
z^2, & 0 \leq j \leq L - 1, \\
\frac{1 - rz}{1 - r/z}, & L \leq j \leq M - 1, \\
1, & M \leq j \leq n - 1,
\end{cases}
\]  
(2.4)

where we have set
\[
\varphi^{(i)}(z) = z^{A_{i,0} - 1} \left( \frac{q^{2n+2} z^2}{q^{4n} z^2; q^{4n}_\infty} \right) \times \begin{cases} 
1, & \text{for (C1)}, \\
\frac{(r q^{2n} z; q^{2n}_\infty)}{(r q^{2n-2M+2L} z; q^{2n}_\infty)}, & \text{for (C2)}, \\
\frac{(r^{-1} z; q^{2n}_\infty)}{(n^{-1} q^{2n+2M-2L} z; q^{2n}_\infty)}, & \text{for (C3)}.
\end{cases}
\]

They satisfy the boundary Yang-Baxter equation:
\[
K_2^{(i)}(z_2) R_2^{(i)}(z_1 z_2) K_1^{(i)}(z_1) R_1^{(i)}(z_1/z_2) = R_2^{(i)}(z_1/z_2) K_1^{(i)}(z_1) R_1^{(i)}(z_1 z_2) K_2^{(i)}(z_2).
\]
(2.6)

Let \( V = \mathbb{C}n \oplus \cdots \oplus \mathbb{C}n_{n-1} \) be the basic representation of \( U_q(sl_n) \), and let \( V_z \) be the evaluation representation of \( U_q(sl_n) \) in the homogeneous picture. Let \( V(\Lambda_i) \) be the irreducible highest weight module with the level 1 highest weight \( \Lambda_i \) (\( i = 0, \cdots, n - 1 \)). The type-I vertex operator \( \Phi^{(i,i+1)}(z) \) is an intertwining operator of \( U_q(sl_n) \) defined by
\[
\Phi^{(i,i+1)}(z) : V(\Lambda_{i+1}) \rightarrow V(\Lambda_i) \otimes V_z,
\]
(2.7)

where the superscripts \( i, i + 1 \) should be interpreted as elements in \( \mathbb{Z}_n \). Let us define the component of the vertex operators \( \Phi^{(i,i+1)}_j(z) \) as follows.
\[
\Phi^{(i,i+1)}_j(z) |u\rangle = \sum_{j=0}^{n-1} \Phi^{(i,i+1)}_j(z) |u\rangle \otimes v_j, \quad \text{for } |u\rangle \in V(\Lambda_{i+1}).
\]
(2.8)

The dual type-I vertex operator \( \Phi^{*(i+1,i)}(z) \) is an intertwining operator of \( U_q(sl_n) \) defined by
\[
\Phi^{*(i+1,i)}(z) : V(\Lambda_i) \otimes V_z \rightarrow \hat{V}(\Lambda_{i+1}).
\]
(2.9)

Let us define the components of the dual vertex operators \( \Phi^{*(i+1,i)}_j(z) \) as follows.
\[
\Phi^{*(i+1,i)}_j(z) |u\rangle \otimes v_j = \Phi^{*(i+1,i)}_j(z) |u\rangle, \quad \text{for } |u\rangle \in V(\Lambda_i).
\]
(2.10)

Let us summarize here the properties of the vertex operators:

**Commutation relations** The vertex operators satisfy the following commutation relation:
\[
\Phi^{(i-2,i-1)}_{j_2}(z_2) \Phi^{(i-1,i)}_{j_1}(z_1) = \sum_{j_1', j_2' = 0}^{n-1} R^{(i)}_{j_1'j_2'}(z_1/z_2) \Phi^{(i-2,i-1)}_{j_2'}(z_1) \Phi^{(i-1,i)}_{j_1'}(z_2).
\]
(2.11)
Concerning other commutation relations (3.34), (3.36) and (3.42), see section 3.

**Normalizations** We adopt the following normalizations:

\[
\Phi^{(i,i+1)}_i(z)|\Lambda_{i+1}\rangle = |\Lambda_i\rangle \otimes v_i + \cdots, \quad \Phi^{*(i+1,i)}_i(z)|\Lambda_i\rangle \otimes v_i = |\Lambda_{i+1}\rangle + \cdots, \tag{2.12}
\]

where $|\Lambda_i\rangle$ is the highest weight vector of $V(\Lambda_i)$.

**Invertibility** They satisfy the following inversion relation:

\[
g_n \Phi^{(i-1,i)}_j(z) \Phi^{*(i-1)}_j(z) = \text{id}, \tag{2.13}
\]

where

\[
g_n = \frac{(q^2; q^{2n})_\infty}{(q^{2n}; q^2)_\infty}.
\]

We define the normalized transfer matrix by

\[
T^{(i)}_B(z) = g_n \sum_{j=0}^{n-1} \Phi^{*(i,i-1)}_j(z) K^{(i)}(z) \Phi^{(i-1)}_j(z), \tag{2.14}
\]

Let the space $H^{(i)}$ be the span of vectors $|p\rangle = \otimes_{k=1}^{\infty} v_{p(k)}$, where $p: \mathbb{N} \to \mathbb{Z}/n\mathbb{Z}$ satisfies the asymptotic condition

\[
p(k) = k + i \in \mathbb{Z}/n\mathbb{Z}, \quad \text{for} \quad k \gg 1. \tag{2.15}
\]

As usual, the transfer matrix (2.14) and the Hamiltonian (1.1) are related by

\[
\frac{d}{dz} T^{(i)}_B(z) \bigg|_{z=1} = \frac{2q}{1 - q^2} \mathcal{H}_B + \text{const}, \quad \text{for} \quad h = \frac{r + 1}{r - 1} \times \frac{1 - q^2}{2q}. \tag{2.16}
\]

Note that the left hand side act on the space $V(\Lambda_i)$ while the right hand side acts on the space $H^{(i)}$.

Thus we can make the following identification:

\[
V(\Lambda_i) \simeq H^{(i)}. \tag{2.17}
\]

The boundary ground state and the dual boundary ground state are characterized by

\[
T^{(i)}_B(z)|i\rangle_B = |i\rangle_B, \quad (i = 0, \cdots, n-1), \tag{2.18}
\]

and

\[
B\langle i|T^{(i)}_B(z) = B\langle i|, \quad (i = 0, \cdots, n-1). \tag{2.19}
\]

Using the inversion relation, the eigenvalue problems (2.18) and (2.19) are reduced to

\[
K^{(i)}(z) \Phi^{(i-1,i)}_j(z)|i\rangle_B = \Phi^{(i-1,i)}_j(z^{-1})|i\rangle_B, \tag{2.20}
\]

and

\[
K^{(i)}(z) \Phi^{*(i,i-1)}_j(z^{-1}) = B\langle i| \Phi^{*(i,i-1)}_j(z). \tag{2.21}
\]

The bosonizations of vertex operators are given in [15]. The bosonic formulae of the boundary vacuum are given in [11]. For readers’ convenience we summarize the bosonizations of vertex operators in Appendix A and the bosonic formula of the boundary vacuum in Appendix B.
3 Boundary quantum Knizhnik-Zamolodchikov equations

The purpose of this section is to derive the q-difference equations for the correlation function of the higher rank XXZ spin chain with a boundary magnetic field. For $U_q(sl_2)$ case [14], the said difference equations are based on the duality relation of vertex operators

$$\Phi^*_i(\zeta) = \Phi_j(-q^{-1}\zeta),$$

in addition to (2.11), (2.20) and (2.21). For $n > 2$ case, however, the dual vertex operator $\Phi^*_j(z)$ is written in terms of $(n-1)$-st determinant of $\Phi_j(z)$'s. Thus it is not convenient to use the duality relation for the present case.

For $n > 2$, we use the explicit formulae of the boundary states to derive the boundary quantum Knizhnik-Zamolodchikov equations. In this section we establish the following simple relations:

$$\Phi^{(i+1,j)}_j(q^n z)|i\rangle_B = K^{(i,j)}(z)\Phi^{(i+1,j)}_j(q^n z)|i\rangle_B, \quad (j = 0, \cdots, n-1),$$

$$B\langle i|\Phi^{(i+1,j)}_j(1/(q^n z)) = K^{(i,j)}(z) B\langle i|\Phi^{(i+1,j)}_j(z/q^n), \quad (j = 0, \cdots, n-1),$$

where the functions $K^{(i,j)}(z)$ are given by (3.3), (3.11) and (3.18). The relations (3.1) in addition to the commutation relations (2.11), (3.34), (3.36) and (3.42) imply the q-difference equations of the present model.

3.1 Boundary state

In this subsection we use the symbols $P^*(z), Q^*(z), R^-(z), S^-(z)$, which are bosons defined in Appendix A. See Appendix A as for the definitions.

Let us first consider consider "(C1) $0 \leq L = M = i \leq n-1"$-case. Let us show the following relation:

$$\Phi^{(i+1,j)}_j(q^n z)|i\rangle_B = K^{(i,j)}(z)\Phi^{(i+1,j)}_j(q^n z)|i\rangle_B, \quad (j = 0, \cdots, n-1),$$

(3.2)

where

$$K^{(i,j)}(z) = \frac{2^{i+1} \phi^{(i,j)}(z)}{\phi^{(i,j)}(1/z)} \times \begin{cases} 
z^2, & (0 \leq j \leq L - 1 = i - 1), \\
1, & (i = L \leq j \leq n - 1),
\end{cases} \quad \text{for} \quad (C1),$$

(3.3)

and

$$\phi^{(i,j)}(z) = \delta_{i,0} \frac{(q^{4n} z^2, q^{4n})_\infty}{(q^{2n+2} z^2, q^{4n})_\infty}. \quad \text{(3.4)}$$

Multiply the both sides of (3.2) by $k^{(i)}_j(z)\phi^{(i,j)}(1/z)$, where

$$k^{(i)}_j(z) = \begin{cases} 
z, & 0 \leq j \leq i - 1, \\
1, & i \leq j \leq n - 1. \quad \text{(3.5)}
\end{cases}$$

Then the RHS of (3.2) is obtained from the LHS by changing $z \to 1/z$.

Bosonization formulae of $P^*(z), Q^*(z)$ and $|i\rangle_B$ imply the identity

$$e^{Q^*(q^n z)|i\rangle_B} = \frac{(q^{2n+2} z^{-2}; q^{4n})_\infty}{(q^{4n} z^2; q^{4n})_\infty} e^{P^*(q^n z)|i\rangle_B}. \quad \text{(3.6)}$$
By using this identity we have

\[ k_j^i(z)\varphi^{*i}(z)(1/z)\Phi_0^{*+1,i}(q^n z)|i\rangle_B = c_0^i e^{P^*(q^n z)} e^{\Lambda_1} |i\rangle_B, \tag{3.7} \]

where \( c_0^i \) is some constant. The relation \((3.2)\) with \( j = 0 \) follows form the fact that RHS of \((3.7)\) is symmetric under \( z \to 1/z \).

Invoking the bosonization of the dual vertex operators, we also have for \( j > 0 \) as follows:

\[
k_j^i(z)\varphi^{*i}(1/z)\Phi_j^{*+1,i}(q^n z)|i\rangle_B \\
= e_j^i \int \frac{dw_1}{w_1} \cdots \int \frac{dw_n}{w_n} k_j^{(i)}(z) \text{Int}(z, w_1, w_2, \cdots, w_j) e^{P^*(q^n z)} e^{\Lambda_1} |i\rangle_B, \tag{3.8} \]

where \( e_j^i \)'s are some constants. Here we set the integrand:

\[
\text{Int}(w_0, w_1, \cdots, w_j) = \frac{w_j \prod_{k=1}^j \left\{(1 - w_k^2) w_k^{-\delta k} (1 - q w_{k-1} w_k)\right\}}{\prod_{k=1}^j D(w_{k-1}, w_k)}, \tag{3.9} \]

where

\[ D(w_1, w_2) = (1 - q w_1 w_2)(1 - q w_1 / w_2)(1 - q w_2 / w_1)(1 - q/(w_1 w_2)). \]

Thus the relation \((3.2)\) with \( j > 0 \) follows from the identities

\[
\sum_{\epsilon_1,=\pm,\cdots,\epsilon_j,=\pm} \left\{ k_j^{(i)}(z^{-1}) \text{Int}(z, w_1^{\epsilon_1}, \cdots, w_j^{\epsilon_j}) - k_j^{(i)}(z) \text{Int}(z^{-1}, w_1^{\epsilon_1}, \cdots, w_j^{\epsilon_j}) \right\} = 0. \tag{3.10} \]

Let us consider “\((C2)\) \(0 \leq L = i < M \leq n - 1^n\)-case. From the same arguments as for \((C1)\), we have

\[
K^{*i}(z)_j^i = \frac{\varphi^{*i}(z)}{\varphi^{*i}(1/z)} \times \begin{cases} 
\frac{z^2}{1 - q^{2M+2L} z}, & (0 \leq j \leq L - 1), \\
\frac{1}{1 - q^{2M+2L} z^{-1}}, & (L \leq j \leq M - 1), \\
1, & (M \leq j \leq n - 1),
\end{cases} \tag{3.11} \]

where we have set

\[
\varphi^{*i}(z) = z^{\delta_{i,0}} \frac{(q^{4n} z^2; q^{4n})_\infty (rq^n z q^{2n})_\infty}{(q^{2n+2} z^2; q^{4n})_\infty (rq^{n+2L-2M} z q^{2n})_\infty} . \tag{3.12} \]

In this case the following relations are useful:

\[
e^{Q_i}(q^n z)|0\rangle_B = \frac{(rq^n z^{-1}; q^{2n})_\infty (q^n z^{-2}; q^{4n})_\infty}{(rq^{n-2M} z; q^{2n})_\infty} e^{P^*(q^n z)}|0\rangle_B, \tag{3.13} \]

\[
e^{Q_i}(q^n z)|i\rangle_B = \frac{(rq^{n+2L-2M} z^{-1}; q^{2n})_\infty (q^{n+2L-2M} z^{-2}; q^{4n})_\infty}{(rq^n z^{-1}; q^{2n})_\infty (q^{n+2L-2M} z q^{2n})_\infty} e^{P^*(q^n z)}|i\rangle_B, \tag{3.14} \]

and

\[
e^{S_i}(w)|i\rangle_B = g_j^{(i)}(w) e^{R_j^R(q^{n+1} w)}|i\rangle_B, \tag{3.15} \]

where

\[
g_j^{(i)}(q^{n+1} w) = \begin{cases} 
(1 - 1/w^2)(1 - q^{-n+2M-L}/(rw)), & j = L, \\
(1 - 1/w^2)(1 - q^{-n-M_L}/w), & j = M, \\
1 - 1/w^2), & j \neq L, M, \tag{3.16} \end{cases} \]
and

\[ g_j^{(i)}(q^{n+1}w) = \begin{cases} \frac{(1 - 1/w^2)}{(1 - q^n - 2M + L/w)}, & j = L, \\ \frac{(1 - 1/w^2)(1 - q^n - Mr/w)}{1 - 1/w^2}, & j = M, \quad (i \geq 1), \\ \frac{(1 - 1/w^2)}{1 - 1/w^2}, & j \neq L, M, \end{cases} \]  

(3.17)

Let us consider “(C3) 0 \leq L < M = i \leq n - 1”-case. Repeating the same procedure as in (C1), we have

\[ K^{(i)}(z) = \frac{\varphi^{(i)}(z)}{\varphi^{(i)}(1/z)} \times \begin{cases} \frac{z^2}{1 - q^n z - 1}, & (0 \leq j \leq L - 1), \\ \frac{1 - q^n + 2M - 2L - 1}{1 - q^n + 2M - 2L - 1}, & (L \leq j \leq M - 1), \quad \text{for (C3)}, \\ 1, & (M \leq j \leq n - 1), \end{cases} \]  

(3.18)

where we set

\[ \varphi^{(i)}(z) = \frac{(q^{2n+1}; q^{2n})_\infty (r^{-1}q^n; q^n)_\infty}{(q^{2n+2}; q^{2n})_\infty (r^{-1}q^n - 2; q^{2n})_\infty}. \]  

(3.19)

In this case the following relations are useful:

\[ e^{Q^*(q^n \mid i)}(z) \mid B = \frac{(r^{-1}q^n z - 1, q^{2n})_\infty (q^{2n+2}; q^{2n})_\infty}{(r^{-1}q^n z - 1, q^{2n})_\infty (q^{2n+2}; q^{2n})_\infty} e^P^{(q^n \mid i)}(z) \mid B, \quad (L = 0), \]  

(3.20)

\[ e^{Q^*(q^n \mid i)}(z) \mid B = \frac{(r^{-1}q^n z - 1, q^{2n})_\infty (q^{2n+2}; q^{2n})_\infty}{(r^{-1}q^n z - 1, q^{2n})_\infty (q^{2n+2}; q^{2n})_\infty} e^P^{(q^n \mid i)}(z) \mid B, \quad (L \geq 1), \]  

(3.21)

and

\[ e^{S^*_i(w)}(z) \mid B = g_j^{(i)}(w) e^{R^*_j(q)}(q^{(n+1)/w}) \mid i \rangle \mid B, \]  

(3.22)

where

\[ g_j^{(i)}(q^{n+1}w) = \begin{cases} \frac{(1 - 1/w^2)(1 - q^{n+2M-L}/(rw))}{1 - 1/w^2}, & j = L, \\ \frac{(1 - 1/w^2)}{1 - 1/w^2}, & j = M, \\ \frac{(1 - 1/w^2)}{1 - 1/w^2}, & j \neq L, M, \end{cases} \]  

(3.23)

### 3.2 Dual boundary state

From the same arguments as for the boundary state case, we can show the following relation:

\[ B\langle i | Q_j^{(i+1)}(1/(q^n z)) = K^{(i)}(z) \rangle_j B\langle i | Q_j^{(i+1)}(z/q^n)\rangle B\langle i | P^{(i)}(q^n) \rangle \mid i \rangle \mid B. \]  

(3.24)

For each case the following relations are useful:

(C1)-case: 0 \leq L = M = i \leq n - 1

\[ B\langle i | e^{P^*(q^n \mid i)} = \frac{(q^{2n+2}; q^{2n})_\infty}{(q^{2n+2}; q^{2n})_\infty} B\langle i | e^{Q^*(q^n \mid i)} \rangle, \]  

(3.25)

\[ B\langle i | e^{S^*_i(w)} = g_j^{(i)}(w) B\langle i | e^{R^*_j(q^2/w)} \rangle, \]  

(3.26)

where

\[ g_j^{(i)}(q^2w) = (1 - w^2). \]  

(3.26)
(C2)-case: $0 \leq L = i < M \leq n - 1$

$$
B(\langle i|e^{P(z/q^n)} = \frac{\langle q^{2n+2z^2}; q^n \rangle_{\infty}(q^{n+2L-2M}z; q^{2n})_{\infty}}{(q^{4n^2}; q^n)_{\infty}(q^{n+2L-2M}; q^{2n})_{\infty}} B\langle i|e^{Q(1/(q^n z))},
$$

(3.27)

where

$$
g_j^{(0)}(qw) = \begin{cases} 
\frac{(1 - w^2)}{(1 - q^{-L}w/r)}, & j = L, \\
\frac{(1 - w^2)}{(1 - q^{-L-1}w/r)}, & j = M, \\
(1 - w^2), & j \neq L, M,
\end{cases}
$$

and

$$
g_j^{(i)}(qw) = \begin{cases} 
\frac{(1 - w^2)(1 - q^{-L}w)}{(1 - q^{-L-1}w/r)}, & j = L, \\
\frac{(1 - w^2)}{(1 - q^{-L-1}w/r)}, & j = M, (i \geq 1), \\
(1 - w^2), & j \neq L, M,
\end{cases}
$$

(3.29)

(C3)-case: $0 \leq L < M = i \leq n - 1$

$$
B(\langle i|e^{P(z/q^n)} = \frac{\langle q^{2n+2z^2}; q^n \rangle_{\infty}(q^{n+2M-2L}z; q^{2n})_{\infty}}{(q^{4n^2}; q^n)_{\infty}(q^{n+2M-2L}; q^{2n})_{\infty}} B\langle i|e^{Q(1/(q^n z))},
$$

(3.30)

where

$$
g_j^{(i)}(qw) = \begin{cases} 
\frac{(1 - w^2)}{(1 - q^{-L}w/r)}, & j = L, \\
(1 - w^2)(1 - q^{M-2L}w/r), & j = M, \\
(1 - w^2), & j \neq L, M,
\end{cases}
$$

(3.31)

3.3 Correlation functions and difference equations

Let us consider the 2N-point correlation function:

$$
G^{(i)}(z_1, \cdots, z_N|z_{N+1}, \cdots, z_{2N})
$$

$$
= \sum_{j_1=0}^{n-1} \cdots \sum_{j_N=0}^{n-1} \sum_{j_{N+1}=0}^{n-1} \cdots \sum_{j_{2N}=0}^{n-1} v^*_{j_1} \otimes \cdots \otimes v^*_{j_N} \otimes v_{j_{N+1}} \otimes \cdots \otimes v_{j_{2N}}
$$

(3.32)

$$
\times G^{(i)}(z_1, \cdots, z_N|z_{N+1}, \cdots, z_{2N})^{j_1 \cdots j_N}_{j_{N+1} \cdots j_{2N}}
$$

where

$$
G^{(i)}(z_1, \cdots, z_N|z_{N+1}, \cdots, z_{2N})^{j_1 \cdots j_N}_{j_{N+1} \cdots j_{2N}} = B\langle i|\Phi_j^{(i-1)}(z_1) \cdots \Phi_{j_N}^{(i-N+1, \cdots, -N)}(z_N) \Phi_{j_{N+1}}^{(i-N+1)}(z_{N+1}) \cdots \Phi_{j_{2N}}^{(i-1, i)}(z_{2N})|i\rangle B. \quad (3.33)
$$

In order to derive $q$-difference equations, we use the commutation relations of vertex operators and the action formulae of vertex operators to the boundary state. In what follows we assume that $K^{(i)}(z)$ is a diagonal matrix whose diagonal elements are given by (3.3), (3.11) and (3.18).
The commutation relations between vertex operators of different types are given as follows [16]:

\[
\Phi_j^{(i,i+1)}(z_2)\Phi_j^{(i+1,i)}(z_1) = \sum_{k=0}^{n-1} R^{(i)V*V}(z_1/z_2)^{k,k}_{j,j} \Phi_k^{(i,i-1)}(z_1)\Phi_k^{(i-1,i)}(z_2),
\]

(3.34)

and

\[
\Phi_k^{(i,i+1)}(z_2)\Phi_j^{(i+1,i)}(z_1) = r^{(i)V*V}(z_1/z_2)\Phi_k^{(i,i-1)}(z_1)\Phi_k^{(i-1,i)}(z_2),
\]

(3.35)

Here the nonzero components are

\[
R^{(i)V*V}(z)^{k,k}_{j,j} = r^{(i)V*V}(z) \times \begin{cases} 
  b(z), & j = k, \\
  c(z), & j > k, \\
  zc(z), & j < k,
\end{cases}
\]

(3.38)

and

\[
R^{(i)V*V^*}(z)^{k,k}_{j,j} = r^{(i)V*V^*}(z) \times \begin{cases} 
  b(q^{2n}z), & j = k, \\
  q^{2n}zc(q^{2n}z)q^{2(k-j)}, & j > k, \\
  c(q^{2n}z)q^{2(k-j)}, & j < k,
\end{cases}
\]

(3.39)

where

\[
r^{(i)V*V}(z) = -qz^{-\delta_{i,k}} \frac{(q^{2n+z}; q^{2n})_{\infty}(q^{2n+2z}; q^{2n})_{\infty}}{(q^{2}; q^{2n})_{\infty}(q^{2n}; q^{2})_{\infty}},
\]

(3.40)

\[
r^{(i)V*V^*}(z) = -q^{-1}z^{-\delta_{i,k}} \frac{(q^{2n+z}; q^{2n})_{\infty}(q^{2n+2z}; q^{2n})_{\infty}}{(q^{2n+2}; q^{2n})_{\infty}(q^{2n}; q^{2})_{\infty}}.
\]

(3.41)

The commutation relations between the dual vertex operators are given as

\[
\Phi_{j_2}^{*(i+2,i+1)}(z_2)\Phi_{j_1}^{*(i+1,i)}(z_1) = \sum_{k_1,k_2=0}^{n-1} R^{(i)V*V^*}(z_1/z_2)^{k_1,k_2}_{j_1,j_2} \Phi_{k_1}^{*(i+2,i+1)}(z_1)\Phi_{k_2}^{*(i+1,i)}(z_2).
\]

(3.42)

Here the nonzero components are

\[
R^{(i)V*V^*}(z_1/z_2)^{k_1,k_2}_{j_1,j_2} = r^{(i)V*V^*}(z) \times \begin{cases}
  1, & j_1 = j_2 = k_1 = k_2, \\
  b(q^2 z), & j_1 = k_1 \neq j_2 = k_2, \\
  -qz c(q^2), & j_1 = k_2 < j_2 = k_1, \\
  -qc(q^2), & j_1 = k_2 > j_2 = k_1,
\end{cases}
\]

(3.43)

where

\[
r^{(i)V*V^*}(z) = r^{(i)V V}(z).
\]

(3.44)

Now we are in a position to derive boundary quantum Knizhnik-Zamolodchikov equations, which is a version of Cherednik’s equation [17]. From the commutation relations (2.11), (3.34), (3.36), (3.42)
and the boundary state identities (3.3) we obtain the following $q$-difference equations:

$$G^{(i)}(z_1 \cdots q^{-2n} z_j \cdots z_N | z_{N+1} \cdots z_{2N})$$

$$= R^V_{j-1} (z_j / (q^{2n} z_{j+1})) \cdots R^V_{j+1}(z_{j+1} / (q^{2n} z_{j+2})) K_j^{(i)}(z_j / q^{2n})$$

$$\times R^V_{j+1}(z_{j+1} / (q^{2n} z_{j+2})) \cdots R^V_{j+2}(z_{j+2} / (q^{2n} z_{j+3})) K_j^{(i)}(z_{j+2} / q^{2n}) (3.45)$$

$$\times R^V_{j+2}(z_{j+2} / (q^{2n} z_{j+3})) \cdots R^V_{j+3}(z_{j+3} / (q^{2n} z_{j+4})) K_j^{(i)}(z_{j+3} / q^{2n}) (3.46)$$

$$\times R^V_{j+3}(z_{j+3} / (q^{2n} z_{j+4})) \cdots R^V_{j+4}(z_{j+4} / (q^{2n} z_{j+5})) K_j^{(i)}(z_{j+4} / q^{2n}) (3.47)$$

$$\times R^V_{j+4}(z_{j+4} / (q^{2n} z_{j+5})) \cdots R^V_{j+5}(z_{j+5} / (q^{2n} z_{j+6})) K_j^{(i)}(z_{j+5} / q^{2n}) (3.48)$$

Here the coefficient matrices are given by (3.3), (3.4), (3.11), (3.18), (3.38), (3.39) and (3.43).

For $N = 1$, the equations (3.45) and (3.46) are as follows:

$$G^{(i)}(q^{-2n} z_1 | z_2) = K^{(i)}(z_1 / q^{2n}) R^{V}_{21}(z_2 z_1 / q^{2n}) K^{(i)}(q^n / z_2) R^{V}_{12}(z_1 / z_2) G^{(i)}(z_1 | z_2),$$

$$G^{(i)}(z_1 | q^{-2n} z_2) = R^{V}_2(z_2 / (q^{2n} z_1)) K^{(i)}(q^n / z_2) R^{V}_1(z_1 / z_2) K^{(i)}(z_2 | z_1).$$

4 Two point functions

The purpose of this section is to perform explicit calculations of two point functions for free boundary condition. In what follows we consider the case $i = L = M = 0$ and $N = 1$. In this case the boundary K-matrices $K^{(0)}(z)$ and $K^{*(0)}(z)$ become scalar matrices, i.e.

$$K^{(0)}(z) = \frac{\varphi^{(0)}(z)}{\varphi^{(0)}(z^{-1})} \times \text{id}, \quad K^{*(0)}(z) = \frac{\varphi^{*(0)}(z)}{\varphi^{*(0)}(z^{-1})} \times \text{id}.$$

The boundary quantum Knizhnik-Zamolodchikov equations thus reduces to:

$$G^{(0)}(q^{-2n} z_1 | z_2) = \frac{\varphi^{(0)}(z_1 / q^{2n}) \varphi^{*(0)}(q^n / z_1)}{\varphi^{(0)}(q^{2n} / z_1) \varphi^{*(0)}(z_1 / q^n)} R^{V}_{21}(z_2 z_1 / q^{2n}) R^{V}_{12}(z_1 / z_2) G^{(0)}(z_1 | z_2),$$

$$G^{(0)}(z_1 | q^{-2n} z_2) = \frac{\varphi^{(0)}(z_2) \varphi^{*(0)}(q^n / z_2)}{\varphi^{(0)}(1/z_2) \varphi^{*(0)}(z_2 / q^n)} R^{V}_{21}(z_2 / (q^{2n} z_1)) R^{V}_{12}(z_1 / z_2) G^{(0)}(z_1 | z_2).$$

Let us now introduce the scalar function $r(z_1 | z_2)$ by

$$r(z_1 | z_2) = A(z_1) A(q^n z_2) B(z_1 z_2) B(z_1 / z_2),$$
where

\[
A(z) = \frac{(q^{2n+2}; q_{z}^{2n}, q_{z}^{4n})_{\infty} (q^{4n+2}/z; q_{z}^{2n}, q_{z}^{4n})_{\infty}}{(q^{4n+2}; q_{z}^{2n}, q_{z}^{4n})_{\infty} (q^{6n}/z^2; q_{z}^{2n}, q_{z}^{4n})_{\infty}},
\]

(4.4)

\[
B(z) = \frac{(q^{2n}; q_{z}^{2n}, q_{z}^{4n})_{\infty} (q^{2n}/z; q_{z}^{2n}, q_{z}^{4n})_{\infty}}{(q^{2n+2}; q_{z}^{2n}, q_{z}^{4n})_{\infty} (q^{2n+2}/z; q_{z}^{2n}, q_{z}^{4n})_{\infty}}.
\]

(4.5)

Note that the function \( r(z_1|z_2) \) satisfies

\[
r(q^{-2n}z_1|z_2) = q^{-2n}z_1^{2n}V^* (z_1 z_2/q^{2n})_r V (z_1 z_2) \frac{\varphi^{(0)}(z_1/q^{2n}) \varphi^{(*)}(q^n/z_1)}{\varphi^{(0)}(q^{2n}/z_1) \varphi^{(*)}(z_1/q^n)} \times r(z_1|z_2),
\]

(4.6)

\[
r(z_1|q^{-2n}z_2) = q^{-2n}z_2^{2n}V^* (z_2/(q^{2n}z_1))_r V (z_1 z_2) \frac{\varphi^{(0)}(z_2) \varphi^{(*)}(q^n/z_2)}{\varphi^{(0)}(1/z_2) \varphi^{(*)}(z_2/q^n)} \times r(z_1|z_2).
\]

(4.7)

Let \( \tilde{G}(z_1|z_2)_j \) be the auxiliary function defined by

\[
\tilde{G}(z_1|z_2)_j = r(z_1|z_2)^{-1} G^{(0)}(z_1|z_2)_j.
\]

(4.8)

Then we have

\[
\sum_{j=0}^{n-1} \tilde{G}(q^{-2n}z_1|z_2)_j = \frac{1 - q^{2n}/(z_1 z_2)}{1 - z_1 z_2} \frac{1 - q^{2n} z_2/z_1}{1 - z_1/z_2} \sum_{j=0}^{n-1} \tilde{G}(z_1|z_2)_j,
\]

(4.9)

\[
\sum_{j=0}^{n-1} \tilde{G}(z_1|q^{-2n}z_2)_j = \frac{1 - q^{2n}/(z_1 z_2)}{1 - z_1 z_2} \frac{1 - q^{2n} z_2/z_1}{1 - z_2/z_1} \sum_{j=0}^{n-1} \tilde{G}(z_1|z_2)_j.
\]

(4.10)

From these obtain

\[
\sum_{j=0}^{n-1} B(0|\Phi_j^{(0,1)}(z_1)\Phi_j^{(1,0)}(z_2)|0)_B = c_0 \times \left( \left( q^{2n} z_1/z_2; q^{2n} \right)_{\infty} (q^{2n} z_2/z_1; q^{2n})_{\infty} (q^{2n} z_1 z_2; q^{2n})_{\infty} \left( q^{2n}/z_1 z_2; q^{2n} \right)_{\infty} \right)^{-1},
\]

(4.11)

where \( c_0 \) is a constant independent of spectral parameters \( z_1, z_2 \). By specializing the spectral parameters \( z_1 = z_2 \), we have

\[
c_0 = g_n^{-1} \times B(0|0)_B \times \left( \left( q^{2n+2}; q^{2n}, q^{2n} \right)_{\infty} \right)^2,
\]

(4.12)

where the norm \( B(0|0)_B \) is given as follows \[1]\n
\[
B(0|0)_B = \frac{1}{\sqrt{(q^{4n}; q^{4n})_{\infty}}} \prod_{j=1}^{n-1} \left( \sqrt{(q^{4n+2-2j}; q^{4n})_{\infty} (q^{4n+2-2j}; q^{4n})_{\infty}} \right)^{j(n-j)}.
\]

Let \( \omega \) satisfy \( \omega^n = 1 \) and \( \omega \neq 1 \). Then we have

\[
\sum_{j=0}^{n-1} (q^2 \omega)^j \tilde{G}(q^{-2n}z_1|z_2)_j = q^{2n} z_1^{2n} \sum_{j=0}^{n-1} (q^2 \omega)^j \tilde{G}(z_1|z_2)_j,
\]

(4.13)

\[
\sum_{j=0}^{n-1} (q^2 \omega)^j \tilde{G}(z_1|q^{-2n}z_2)_j = q^{2n} z_2^{2n} \sum_{j=0}^{n-1} (q^2 \omega)^j \tilde{G}(z_1|z_2)_j.
\]

(4.14)
From these we obtain
\[
\begin{align*}
\sum_{j=0}^{n-1} (q^2 \omega^k)^j B_0 \{ \Phi_j^{(0,1)}(z_1) \Phi_j^{(1,0)}(z_2) \} B_j
&= c_k r(z_1, z_2) \times \left\{ (q^{2n} z_2^2; q^{4n})_\infty (q^{2n} z_1^2; q^{4n})_\infty (q^{2n} z_2^2; q^{4n})_\infty \right\}^{-1}. \quad (4.15)
\end{align*}
\]

Here \(c_k\) are constants independent of spectral parameters \(z_1, z_2\).

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A Bosonization of vertex operators in \(U_q(\hat{sl}_n)\)

For readers’ convenience, we summarize the results of bosonizations of the vertex operators [15].

Let \(\mathbb{C}[[\hat{P}]]\) be the \(\mathbb{C}\)-algebra generated by the symbols \(\{e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}, e^{\lambda_{n-1}}\}\) which satisfy the following defining relations:

\[
e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i | \alpha_j)} e^{\alpha_i} e^{\alpha_j}, \quad (2 \leq i, j \leq n-1),
\]

\[
e^{\alpha_i} e^{\lambda_{n-1}} = (-1)^{\delta_i, n-1} e^{\lambda_{n-1}} e^{\alpha_i}, \quad (2 \leq i \leq n-1).
\]

For \(\alpha = m_2 \alpha_2 + \cdots + m_n \alpha_n + m_{n+1} \lambda_{n-1}\), we denote \(e^{m_2 \alpha_2} \cdots e^{m_n \alpha_n} e^{m_{n+1} \lambda_{n-1}}\) by \(e^\alpha\). Let \(((\alpha_s | \alpha_t))\) stand for the A-type Catran matrix whose matrix element \((\alpha_s | \alpha_t)\) is an integer. Let \(\mathbb{C}[\hat{Q}]\) be the \(\mathbb{C}\)-subalgebra of \(\mathbb{C}[[\hat{P}]]\) generated by the symbols \(\{e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}\}\) which satisfy the following defining relations:

\[
e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i | \alpha_j)} e^{\alpha_j} e^{\alpha_i}, \quad (1 \leq i, j \leq n-1).
\]

For \(\alpha_1 = -\sum_{r=2}^{n-1} r \alpha_r + n \lambda_{n-1}, \quad \lambda_i = -\sum_{r=i+1}^{n-1} (r - i) \alpha_r + (n - i) \lambda_{n-1}\).

Let us consider the \(\mathbb{C}\)-algebra generated by the bosons \(a_s(k)\) \((s \in \{1, \ldots, n-1\}, k \in \mathbb{Z})\) which satisfy the following defining relations:

\[
[a_s(k), a_t(l)] = \delta_{k+l,0} [(-1)^{(\alpha_s | \alpha_t)}] k^{-1}.
\]

The highest weight module \(V(\lambda_i)\) is realized as

\[
V(\lambda_i) = \mathbb{C}[a_s(-k), (s \in \{1, \ldots, n-1\}, k \in \mathbb{Z} \geq 0)] \otimes \mathbb{C}[\hat{Q}] e^{\lambda_i}.
\]

We consider \(\mathbb{C}[\hat{Q}] e^{\lambda_i}\) as a subspace of \(\mathbb{C}[[\hat{P}]]\). Here the actions of the operators \(a_s(k), \partial_\alpha, e^\alpha\) on \(V(\lambda_i)\) are defined as follows:

\[
a_s(k) f \otimes e^\beta = \begin{cases} a_s(k) f \otimes e^\beta, & (k < 0), \\ [a_s(k), f] \otimes e^\beta, & (k > 0), \end{cases}
\]
\[
\partial_{\alpha} f \otimes e^\beta = (\alpha|\beta)f \otimes e^\beta, \\
e^{\alpha} f \otimes e^\beta = f \otimes e^\alpha e^\beta.
\]

The inner product is explicitly given as follows:

\[
(\alpha_i|\bar{\alpha}_j) = \delta_{i,j}, \quad (\bar{\alpha}_i|\alpha_j) = \frac{i(n-j)}{n}, \quad (1 \leq i \leq j \leq n-1).
\]

\[
\Phi^{(i+1)}_{n-1}(z) = e^{P(z)} e^{Q(z)} e^{\Lambda_{n-1}} (q^{|n+1}z)^{\alpha_{n-1}+\frac{i(n+1)}{n}} (-1)^{\alpha_1-\frac{n-1}{n}}(n-1)^{-\frac{1}{2}(n-i)(n-i-1)}, \\
\Phi^*(i+1)(z) = e^{P^*(z)} e^{Q^*(z)} e^{\Lambda_1} ((-1)^{n-1}qz)^{\alpha_1+\frac{i}{2}} q^{(1)}_{n+1+i+1}, \\
\Phi^{((i+1)}_{j}(z) = c_j \int \cdots \int C_j \frac{dw_{j+1}}{2\pi i w_{j+1}} \cdots \frac{dw_{n-1}}{2\pi i w_{n-1}} \frac{1}{z} \frac{1}{(1-qw_{n-1}/z)(1-qz/w_{n-1})} \\
\times (1-qw_{n-2}/w_{n-1}) \cdots (1-qw_{j+2}/w_{j+1})(1-qw_{j+1}/w_{j+2}) \\
\times : \Phi^{((i+1)}_{n-1}(z) X_{n-1}(q^{|n+1}w_{n-1}) \cdots X_{j+1}(q^{|n+1}w_{j+1}) :.
\]

(A.1)

\[
\Phi^{*(i+1)}_j(z) = c^*_j \int \cdots \int C^*_j \frac{dw_1}{2\pi i w_1} \cdots \frac{dw_j}{2\pi i w_j} z \frac{1}{(1-qz/w_1)(1-qw_1/z)} \\
\times (1-qw_2/w_1) \cdots (1-qw_{j-1}/w_j)(1-qw_{j-1}/w_{j+1}) \\
\times : \Phi^{*(i+1)}_0(qw_1) \cdots X_j(qw_j) :.
\]

(A.2)

where \(c_j, c^*_j\) are appropriate constants. The contours \(C_j, C^*_j\) encircle \(w_l = 0\) anti-clockwise in such a way that

\[
C_j : \quad |q| < |w_{n-1}/z| < |q^{-1}|, \quad |q| < |w_l/w_{l+1}| < |q^{-1}|, \quad (l = j+1, \cdots, n-2),
\]

\[
C^*_j : \quad |q| < |w_1/z| < |q^{-1}|, \quad |q| < |w_{l+1}/w_l| < |q^{-1}|, \quad (l = 1, \cdots, j-1).
\]

Here we have used

\[
X^{-}_j(w) = e^{R^{-}_j(w)} e^{S^{-}_j(w)} e^{-\alpha_j w - \partial_{\alpha_j}},
\]

\[
P(z) = \sum_{k=1}^{\infty} a^*_n(-k) q^{\frac{2n+1}{2k} z^k}, \quad Q(z) = \sum_{k=1}^{\infty} a^*_n(k) q^{-\frac{2n+1}{2k} z^{-k}},
\]

\[
P^*(z) = \sum_{k=1}^{\infty} a^*_1(-k) q^{\frac{2}{2k} z^k}, \quad Q^*(z) = \sum_{k=1}^{\infty} a^*_1(k) q^{-\frac{2}{2k} z^{-k}},
\]

\[
R^{-}_j(w) = -\sum_{k=1}^{\infty} \frac{a^*_j(-k)}{[k]} q^{\frac{k}{2} w^k}, \quad S^{-}_j(w) = \sum_{k=1}^{\infty} \frac{a^*_j(k)}{[k]} q^{\frac{k}{2} w^{-k}},
\]

\[
a^*_n(k) = \sum_{l=1}^{n-1} \frac{(-[k])}{[k][n][l]} a^*_l(k), \quad a^*_n(-k) = \sum_{l=1}^{n-1} \frac{(-[n-l])}{[k][n][l]} a_l(k).
\]

\[
[a_j(k), a^*_n(-k)] = \delta_{j,n-1} \frac{[k]}{k}, \quad [a^*_j(k), a^*_n(-k)] = \delta_{j,1} \frac{[k]}{k}.
\]
B Bosonization of the boundary vacuum states

For readers’ convenience we summarize the bosonic formulae of the boundary vacuum states. Let us set the symmetric matrix as

\[
\hat{I}_{s,t}(k) = \begin{cases} 
0, & \text{if } st = 0, \\
\frac{[sk][tk]}{|k|^2[nk]}, & 1 \leq s \leq t \leq n - 1,
\end{cases}
\]

(B.1)

Let us consider the \(\mathbb{C}\)-algebra generated by the bosons \(a_s(k)\) \((s \in \{1, \ldots, n-1\}, k \in \mathbb{Z})\) which satisfy the following defining relations:

\[
[a_s(k), a_t(l)] = \delta_{k+l,0} \frac{[(\alpha_s|\alpha_t)k][k]}{k},
\]

where \(I(\alpha_s|\alpha_t)\) is an element of A-type Cartan matrix.

The boundary state has the form

\[
|i\rangle_B = e^F_i |i\rangle, \quad F_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \alpha_{s,t}(k) a_s(-k) a_t(-k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \beta_s^{(i)}(k) a_s(-k).
\]

Here the coefficients of the quadratic part are given by

\[
\alpha_{s,t}(k) = -q^{2(n+1)k/2} \hat{I}_{s,t}(k),
\]

and those of the linear part are given by

\[
\beta_s^{(i)}(k) = (q^{(n+3/2)k} - q^{(n+1/2)k}) \theta_k \sum_{s=1}^{n-1} \hat{I}_{s,t}(k)
\]

(B.2)

(B.3)

(B.4)

where

\[
\theta_k = \begin{cases} 
0, & k \text{ is odd}, \\
1, & k \text{ is even}.
\end{cases}
\]

The dual boundary state has the form

\[
B\langle i \rangle = \langle i | e^{G_i}, \quad G_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \gamma_{s,t}(k) a_s(k) a_t(k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \delta_s^{(i)}(k) a_s(k).
\]

Here the coefficients of the quadratic part are given by

\[
\gamma_{s,t}(k) = -q^{2k} \frac{2|k|}{[k]} \times \hat{I}_{s,t}(k),
\]

(B.5)

(B.6)
and those of the linear part are given by

\[ \delta^{(i)}_{\alpha}(k) = -(q^{k/2} - q^{-3k/2}) \theta_{\alpha} \sum_{s=1}^{n-1} \hat{I}_{\alpha,s}(k) \]  

(B.7)

\[ + \begin{cases} 
0, & (C1), \\
q^{(L-3/2)k} \hat{I}_{\alpha,L}(k) - q^{(2L-M-3/2)k} \hat{I}_{\alpha,M}(k), & (C2), \\
-q^{(-L-3/2)k} \hat{I}_{\alpha,L}(k) + q^{(M-2L-3/2)k} \hat{I}_{\alpha,M}(k), & (C3).
\end{cases} \]

(B.8)

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