Shor–Movassagh chain leads to unusual integrable model

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Abstract

The ground state of the Shor–Movassagh chain can be analytically described by the Motzkin paths. There is no analytical description of the excited states. The model is not solvable. We prove the integrability of the model without interacting part in this paper (free Shor–Movassagh). The Lax pair for the free Shor–Movassagh open chain is explicitly constructed. We further obtain the boundary K-matrices compatible with the integrability of the model on the open interval. Our construction provides a direct demonstration for the quantum integrability of the model, described by the Yang–Baxter algebra. Because the partial transpose of the R matrix is not invertible, the model does not have crossing unitarity and the integrable open chain cannot be constructed by the reflection equation (boundary Yang–Baxter equation).

Keywords: random walk model, integrable spin chain, boundary Yang–Baxter equation, crossing unitarity

1. Introduction

A spin-1 model called Shor–Movassagh chain [1] was shown to have unique ground state.³ The ground state can be expressed as a sum with respect to Motzkin paths. Among other interesting aspects, its entanglement entropy strongly exceeds the one exhibited by other previously known local models. Inspired by the discovery of the Affleck–Kennedy–Lieb–Tasaki (AKLT) model, studying such Hamiltonians with highly entangled spins is currently one of the most challenging and intriguing fields in quantum physics. Since this model is not exactly solvable,

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³ Its half-integer version is called the Fredkin spin chain [2, 3].
Figure 1. The local Hilbert space and its mapping to the steps in the ‘x–y’ plane.

it is still hard to describe thermodynamics. The model can be seen as a generalized Temperley–Lieb system. This is understood from the local Hamiltonian, since it is a representation of the generator of Temperley–Lieb algebra.

The full Hamiltonian of the uncolored\(^4\) Shor–Movassagh chain was described by Bravyi et al\(^4\) \cite{4}. It is a spin chain with a three-dimensional local Hilbert space given by a basis \(|u\rangle\), \(|f\rangle\), and \(|d\rangle\). Here \(u\), \(f\), and \(d\) are used to abbreviate ‘up’, ‘flat’, and ‘down’ of the Motzkin walks respectively. This identification comes from that the states in this system can be mapped to paths (Motzkin walks) in the ‘x–y’ plane as done in \cite{1}. So the state \(|u\rangle\) maps to the (1, 1) direction, \(|f\rangle\) maps to the (1, 0) direction, and \(|d\rangle\) maps to the (1, −1) direction in the ‘x–y’ plane, as shown in figure 1.

The Hamiltonian is given by (the coupling constant \(g\) was equal to 1 in the original paper):

\[
H_{SM} = H_{\text{boundary}} + \sum_{j=1}^{N-1} H_{\text{free}, j} + g H_{\text{int}, j} \tag{1.1}
\]

In the original paper the Hamiltonian densities \(H_{\text{free}}\) and \(H_{\text{int}}\) are given in terms of two-site projectors, as

\[
H_{\text{free}, j} = (|u_j f_{j+1}\rangle - |f_j u_{j+1}\rangle)(\langle u_j f_{j+1}| - \langle f_j u_{j+1}|) + (|d_j f_{j+1}\rangle - |f_j d_{j+1}\rangle)(\langle d_j f_{j+1}| - \langle f_j d_{j+1}|) \tag{1.2}
\]

\(^4\)The color was introduced by Shor & Movassagh \cite{1}.
and

\[ H_{\text{int},j} = \langle u_j d_{j+1} | - | f_j f_{j+1} \rangle (\langle u_j d_{j+1} | - | f_j f_{j+1} \rangle). \] (1.3)

The boundary operators for the open chain in the original paper are given by

\[ H_{\text{boundary}} = H_1 + H_N = |d_1 \rangle \langle d_1| + |u_N \rangle \langle u_N|. \] (1.4)

In this paper we will analyze the case where the coupling contact \( g = 0 \). We observe that the resulting system is integrable. By turning on \( g \), the system loses its symmetries and starts to ‘interact’. The corresponding term can be viewed as a ‘kinetic part’ of the Shor–Movassagh model. Thus we will call the case with \( g = 0 \) as the non-interacting Shor–Movassagh chain or the free Shor–Movassagh spin chain

\[ H_{\text{FSM,open}} = H_{\text{SM}|g=0} = \sum_{j=1}^{N-1} H_{\text{free},j} + H_{\text{boundary}}. \] (1.5)

We will denote the corresponding Hamiltonian \( H_{\text{FSM,open}} \) in the following.

To readers familiar with other similar models but not with the Bethe ansatz, the central building block is the \( R \)-matrix (4.1). It essentially describes the scattering between two quasiparticles in the spin chain. This \( R \)-matrix satisfies the Yang–Baxter equation (4.8), which states that many-body scattering can be reduced to two-body scattering. This fact ensures the integrability of the periodic free Shor–Movassagh chain.

The bulk of the Hamiltonian \( H_{\text{FSM,open}} \) (3.2) can be derived from the periodic transfer matrix (constructed by \( R \)-matrix) through a logarithmic trace identity. However, the open boundary case Hamiltonian \( H_{\text{FSM,open}} \) can not be constructed by the double row transfer matrix based on Sklyanin’s reflection equations [5]. This is because the partial transpose of this \( R \)-matrix is degenerate (4.12), it is not invertible [6]. And also the model does not have crossing unitarity relation or its substitute. One can not construct the open chain [8] in the ordinary way. Then whether such an integrable open chain exists is a problem.

Some remarks are in order. In our case, the main obstacle which prevents Sklyanin’s method from being proceeded is the non-invertibility of the partial transpose \( R_{12}^0(\lambda) \). To implement Sklyanin’s way to prove the existence of the commutative family, one has to insert the identity \( R_{12}^0(\lambda)R_{12}^0(\lambda)^{-1} \) inside the trace and then use the tricks of partial trace, partial transpose, and cyclic properties of trace. For details, see the proof of Theorem 1 in [5]. Also see the proof of theorem 2.4 in [7], and equation (2.2.11) in [8]. Basically, the proofs in [7, 8] are still under Sklyanin’s framework. By crossing unitarity, we mean equation (6) in [5]:

\[ R_{12}^0(\lambda)R_{12}^0(-\lambda - 2\eta) = \tilde{\rho}(\lambda), \] (1.6)

here \( \tilde{\rho}(\lambda) \) is a nonzero scalar function. Sometimes it has alternative forms, such as equations (2.6) and (2.9) in [7], and equation (1.2.13) in [8], also see the references therein. Generally speaking, it is a relation about the partial transpose of the \( R \)-matrix.

On the other hand, the traditional basis for applying the quantum inverse scattering method to a completely integrable system is to represent the equations of motion of the system into the Lax form [9, 10]. For quantum systems with periodic boundary conditions, the existence of the \( R \)-matrix allows one to express the original equations of motion in the quantum Lax form

5 In [7], the \( R \)-matrix is partially transposed, inverted and partially transposed again, see equation (2.6) in there. In [8], the relation is modified by a similarity transformation.
[11–13]. Meanwhile the explicit forms of Lax pairs for some physically important models have been given by many authors [14–16]. A natural problem is that there must exist one kind of revised form of the ordinary Lax formulation for models with open boundaries. It can be used to describe completely quantum integrable lattice spin open chains [17], in which the authors provided an alternative way to construct the quantum integrability of Heisenberg XXZ open chain. The method was further applied to the small-polaron model [18] and the Hubbard model [19]. In all these cases, the authors matched their results with the ones through solving the reflection equation (for these models the partial transposes of their $R$-matrices are invertible, and they have crossing unitarity).

This paper aims to present an explicit construction of the Lax pair and find the integrable conditions for the free Shor–Movassagh open chain. We should point out that such kind of model (the partial transpose of the $R$-matrix is non-invertible) is very rare. The Lax formulation method for open chain by Zhou & Guan [17, 19] does not require the model has these properties (although the previous models they handled do have such properties). That is why we plan to apply the method to Shor–Movassagh open chain. As a further result, we obtain its boundary $K$-matrices compatible with the integrability of the model. The calculation of $K$-matrices is not straightforward. Since for a certain model, the Lax pair operators, especially the $M$ operator are not unique. In most cases, Lax pair operators will not automatically lead to a solution of $K$-matrices. So our main task is to find a proper Lax pair that leads to correct $K$-matrices. This concept is implied in the works [17–19]. Only then the integrable open chain can really be constructed. The construction provides a demonstration for the quantum integrability of the model.

In this paper we will employ Lax formulation to approach the problem of the free Shor–Movassagh chain for open boundary case. The paper is organized as follows. In section 2, the Lax formulation for open quantum spin chain is introduced. In section 3, the Hamiltonian of free Shor-Movassagh chain with open boundary conditions is given in more detailed form. The Lax pair for the open free Shor–Movassagh chain is explicitly constructed in section 4. The corresponding boundary $K$-matrices are obtained in section 5. Section 6 is devoted to the conclusion.

2. Lax pair

Because the partial transpose of the free Shor–Movassagh chain $R$-matrix is not invertible, Sklyanin’s method [5] does not apply to this case. We try to construct integrability by using quantum Lax formulation.

We first recall the original work of Lax formulation [11–13] for quantum integrable models with general open boundary conditions in one dimension [18–20]. We consider an operator version of the auxiliary linear problem

$$
\Phi_{j+1} = L_j(\lambda)\Phi_j, \quad j = 1, 2, \ldots, N,
$$

with boundary equations

$$
\frac{d}{dt}\Phi_j = M_j(\lambda)\Phi_j, \quad j = 2, 3, \ldots, N, 
$$

$$
\frac{d}{dt}\Phi_N+1 = M_+(\lambda)\Phi_{N+1},
$$

$$
\frac{d}{dt}\Phi_1 = M_-(\lambda)\Phi_1. 
$$

(2.1)
Here \( L_j(\lambda), M_j(\lambda), \) and \( M_\pm(\lambda) \) are matrices\(^6\) depending on the spectral parameter \( \lambda \) and the dynamical variables. The spectral parameter \( \lambda \) does not depend on the time \( t \), and dynamical variables. Evidently, the consistency conditions for equations (2.1) and (2.2) yield the following Lax equations

\[
\frac{d}{dt} L_j(\lambda) = M_{j+1}(\lambda)L_j(\lambda) - L_j(\lambda)M_j(\lambda), \quad j = 2, 3, \ldots, N - 1, \tag{2.3}
\]

and boundary terms

\[
\frac{d}{dt} L_N(\lambda) = M_+(\lambda)L_N(\lambda) - L_N(\lambda)M_N(\lambda),
\]

\[
\frac{d}{dt} L_1(\lambda) = M_2(\lambda)L_1(\lambda) - L_1(\lambda)M_-(\lambda). \tag{2.4}
\]

If the equations of motion for the system can be expressed in the form of equations (2.3) and (2.4), with the boundary \( K_- \)-matrices exist as the solutions of equations (2.6) and (2.7) below, then we insist that the spin chain with open boundary conditions is completely integrable. Let \( T(\lambda) = L_N(\lambda) \ldots L_2(\lambda)L_1(\lambda) \) be the usual monodromy matrix [12] of the system. \( K_- (\lambda) \) and \( K_+ (\lambda) \) are the matrices for the left and the right boundary, respectively. The double row transfer matrix \( \tau(\lambda) \) of open chain is defined as the trace on the auxiliary space \( V_0 \) as

\[
\tau(\lambda) = \text{tr}_0 \left[ K_+(\lambda)T(\lambda)K_-(\lambda)T^{-1}(\lambda) \right]. \tag{2.5}
\]

From the Lax equations (2.3) and (2.4), it follows that the transfer matrix \( \tau(\lambda) \) does not depend on time \( t \). It is required that the boundary \( K \)-matrices satisfy the constraint conditions

\[
K_-(\lambda)M_-(\lambda) = M_-(\lambda)K_-(\lambda), \tag{2.6}
\]

\[
\text{tr}_0 \left[ K_+(\lambda)M_+(\lambda)\mathbb{U}_N(\lambda) \right] = \text{tr}_0 \left[ K_+(\lambda)\mathbb{U}_N(\lambda)M_+(\lambda) \right], \tag{2.7}
\]

where \( \mathbb{U}_N(\lambda) = T(\lambda)K_-(\lambda)T^{-1}(\lambda) \). These two equations can be regarded as reflection equations for \( K \)-matrices [19–21]. But one cannot get commutation relations from them as Sklyanin’s scheme does. This implies that the double row transfer matrices with different spectral parameters commute with each other,

\[
[\tau(\lambda), \tau(\mu)] = 0. \tag{2.8}
\]

Thus the system possesses an infinite number of independent conserved quantities, and it is completely integrable.

3. The free Shor–Movassagh chain with open boundary conditions

We denote the basis states by \( \{|u\}, |f\rangle, |d\rangle \} \), where \( u, f \) and \( d \) are used to abbreviate ‘up’, ‘flat’, and ‘down’ of Motzkin walks, respectively,

\[
|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |f\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.1}
\]

\(^6\)We remark that the matrix \( L_j \) acts on auxiliary space 0 and quantum space \( j, M_j \) acts on spaces 0, \( j - 1 \), and \( j, M_- \) (\( M_+ \)) acts on spaces 0 and 1 (\( N \)), respectively. We will give further details in later sections.
Then the Hamiltonian for the open free Shor–Movassagh chain can be expressed as

\[ H_{\text{FSM,open}} = H_1 + \sum_{j=1}^{N-1} \left[ \hat{U}_{j,j+1} + \hat{D}_{j,j+1} \right] + H_N. \]  

(3.2)

The operators \( \hat{U}_{j,j+1} \) and \( \hat{D}_{j,j+1} \) are projectors to the states

\[ |u_j, f_{j+1} \rangle - |f_j, u_{j+1} \rangle, \]  

(3.3)

and

\[ |d_j, f_{j+1} \rangle - |f_j, d_{j+1} \rangle \]  

(3.4)

respectively.

Then the operator \( \hat{U}_{j,j+1} \) can be expressed in terms of the states and standard basis,

\[ |u_j, f_{j+1} \rangle \langle u_j, f_{j+1} | - |u_j, f_{j+1} \rangle \langle f_j, u_{j+1} | + |f_j, u_{j+1} \rangle \langle f_j, f_{j+1} | - |f_j, u_{j+1} \rangle \langle u_j, f_{j+1} | = E_{11}^{j} E_{22}^{j+1} - E_{12}^{j} E_{21}^{j+1} + E_{21}^{j} E_{12}^{j+1}, \]  

(3.5)

and for the operator \( \hat{D}_{j,j+1} \),

\[ |d_j, f_{j+1} \rangle \langle d_j, f_{j+1} | - |d_j, f_{j+1} \rangle \langle f_j, d_{j+1} | + |f_j, d_{j+1} \rangle \langle f_j, f_{j+1} | = E_{11}^{j} E_{22}^{j+1} - E_{12}^{j} E_{21}^{j+1} + E_{21}^{j} E_{12}^{j+1}, \]  

(3.6)

Let us consider the general boundary terms

\[ H_1 = a_1 |u_1 \rangle \langle u_1 | + b_1 |f_1 \rangle \langle f_1 | + c_1 |d_1 \rangle \langle d_1 | = a_1 E_{11}^{1} + b_1 E_{22}^{1} + c_1 E_{33}^{1}, \]  

(3.7)

\[ H_N = a_N |u_N \rangle \langle u_N | + b_N |f_N \rangle \langle f_N | + c_N |d_N \rangle \langle d_N | = a_N E_{11}^{N} + b_N E_{22}^{N} + c_N E_{33}^{N}. \]  

(3.8)

We remark that this Hamiltonian with open boundaries cannot be derived from Sklyanin’s reflection equations [5]. In the following sections, we will construct the Hamiltonian (3.2) based on Lax formulation, and its integrability will be discussed.

4. Constructing the Lax pair operators for bulk and boundaries

Unless specified otherwise, here and below we adopt the standard notation used in the Algebraic Bethe Ansatz: for any matrix \( A \in \text{End}(V) \), \( A_j \) is an embedding operator in the tensor space \( V \otimes V \otimes \cdots \), which acts as \( A \) on the \( j \)th space and as an identity on the other factor spaces.
The $R$-matrix $R_{12}(\lambda) = P_{12}((\lambda + \eta)I - \lambda \hat{e}_{12})$ for the free Shor–Movassagh spin chain can be written as

$$R_{12}(\lambda) = \begin{pmatrix}
\lambda + \eta & \lambda & \lambda + \eta \\
\lambda & \eta & \lambda + \eta \\
\lambda + \eta & \eta & \lambda + \eta
\end{pmatrix}. \tag{4.1}
$$

Here $\lambda$ is the spectral parameter, $\eta$ is the crossing parameter, $P_{12}$ is the permutation operator, $\hat{e}_{12}$ is the corresponding generator \cite{22} of the Temperley–Lieb algebra, and $\frac{1}{2} \hat{e}_{12}$ is a projection operator.

$$\hat{e}_{12} = \hat{U}_{12} + \hat{D}_{12}. \tag{4.2}
$$

Here we give some discussions about this generator $\hat{e}_{12}$:

$$\hat{e}_{j,j+1} \hat{e}_{j+1,j+2} \hat{e}_{j,j+1} = \hat{e}_{j,j+1}, \tag{4.3}$$

$$\hat{e}_{j,j+1} \hat{e}_{j-1,j} \hat{e}_{j,j+1} = \hat{e}_{j,j+1}, \tag{4.4}$$

$$\hat{e}_{j,j+1}^2 = 2 \hat{e}_{j,j+1}, \tag{4.5}$$

$$\hat{e}_{j,j+1} \hat{e}_{k,k+1} = \hat{e}_{k+1,k} \hat{e}_{j,j+1}, \quad |j-k| > 1. \tag{4.6}$$

These are the relations for the generators of the Temperley–Lieb algebra. And it is known that the XXX spin chain can be realized using these generators \cite{23}. Comparing these relations to the standard definition of the Temperley–Lieb algebra given in \cite{23}

$$\hat{e}_{i,i+1}^2 = (q + q^{-1}) \hat{e}_{i,i+1}. \tag{4.7}
$$

With the other relations being the same as in equations (4.3), (4.4), and (4.6), we find that $q = 1$ in our case.\footnote{This confirms that the free Shor–Movassagh spin chain with periodic boundary conditions is integrable from the point view of both the Yang–Baxter algebra and Temperley–Lieb algebra. We further remark that the discussions for open boundary cases in references \cite{23–25} cannot be applied to the free Shor–Movassagh case. This is because the Temperley–Lieb algebra related $R$-matrix in there satisfies the crossing unitarity. And in their papers the two boundary Temperley–Lieb algebra is related to Sklyanin’s reflection equations.}

$R_{ij}(\lambda)$ is an embedding operator of $R$-matrix in the tensor space, which acts as an identity on the factor spaces except for the $i$th and $j$th ones. The $R$-matrix satisfies the quantum Yang–Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda + \nu)R_{23}(\nu) = R_{23}(\nu)R_{13}(\lambda + \nu)R_{12}(\lambda), \tag{4.8}
$$

$\lambda$ is the spectral parameter. $\eta$ is the crossing parameter. $P_{12}$ is the permutation operator, $\hat{e}_{12}$ is the corresponding generator \cite{22} of the Temperley–Lieb algebra, and $\frac{1}{2} \hat{e}_{12}$ is a projection operator.

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$$\hat{e}_{j,j+1}^2 = 2 \hat{e}_{j,j+1}, \tag{4.5}$$

$$\hat{e}_{j,j+1} \hat{e}_{k,k+1} = \hat{e}_{k+1,k} \hat{e}_{j,j+1}, \quad |j-k| > 1. \tag{4.6}$$

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$$R_{12}(\lambda)R_{13}(\lambda + \nu)R_{23}(\nu) = R_{23}(\nu)R_{13}(\lambda + \nu)R_{12}(\lambda), \tag{4.8}$$
and possesses the following properties:

\[
\text{Initial condition: } R_{12}(0) = \eta P_{12}, \quad (4.9)
\]

\[
\text{Unitarity: } R_{12}(\lambda)R_{21}(-\lambda) = (\eta + \lambda)(\eta - \lambda) \text{id}, \quad (4.10)
\]

\[
\text{Projection: } R_{12}(-\eta) = \eta P_{12}\bar{e}_{12} = -\eta \bar{e}_{12}. \quad (4.11)
\]

The partial transpose of this \( R \)-matrix is degenerate

\[
\det(R_{12}^{\dagger}(\lambda)) = 0. \quad (4.12)
\]

The elements of the third row, the third column, the seventh row, and the seventh column of the partial transpose \( R_{12}^{\dagger}(\lambda) \) are all zero. It is not invertible. Therefore the \( R \)-matrix does not satisfy crossing unitarity.

In the case of free Shor–Movassagh chain, the \( R \)-matrix (4.1) can be chosen as the \( L \)-matrix,

\[
L_{\eta}(\lambda) = R_{0}(\lambda). \quad (4.13)
\]

The matrix \( L_{\eta}(\lambda) \) acts on auxiliary space 0 and quantum space \( j \). Following the method in the references [14, 15, 26], we can calculate the corresponding \( M \)-matrix operator. Let us call a given \( M_{\beta}(\lambda) \) together with the corresponding \( M_{\eta}(\lambda) \): one set of \( M \)-matrix operators. As we mentioned before, a set of \( M \)-matrices: \( M_{\beta}(\lambda) \) in the bulk and \( M_{\eta}(\lambda) \) for the boundaries, it is not unique. Finding the proper set of \( M \)-matrix operator that can lead to integrable open boundaries needs comparison and tedious calculations. The difference between different sets of \( M \)-matrix operators can be some operator, say \( M_{\beta}(\lambda) - M_{\eta}(\lambda) = \tilde{O} \) (here we use \( \tilde{M}_{\eta}(\lambda) \) representing \( M_{\beta}(\lambda) \) in another set). The simplest case is that \( \tilde{O} \) is an identity operator multiplied by some scalar function. This fact is in accordance with the equations (2.3) and (2.4).

In the following, we directly present a set of \( M \)-matrix operator that can give rise to integrable boundaries. We can calculate the \( M \)-matrix\(^8\) \( M_{\beta}(\lambda) \) in the bulk,

\[
M_{\beta}(\lambda) = \begin{pmatrix}
M_{11}^{11} & M_{12}^{11} & M_{13}^{11} \\
M_{21}^{11} & M_{22}^{11} & M_{23}^{11} \\
M_{31}^{11} & M_{32}^{11} & M_{33}^{11}
\end{pmatrix}, \quad j = 2, \ldots, N, \quad (4.14)
\]

in which

\[
M_{11}^{11} = \frac{i\eta^2}{\eta^2 - \lambda^2} \left( \begin{array}{c}
2E_{j-1}^{11}E_{j}^{12} \\
-2E_{j-1}^{11}E_{j}^{12} \\
2E_{j-1}^{11}E_{j}^{12}
\end{array} \right),
\]

\[
M_{22}^{11} = \frac{i\eta^2}{\eta^2 - \lambda^2} \left( \begin{array}{c}
2(1 - 2E_{j-1}^{11}E_{j}^{12})(E_{j}^{11} + E_{j}^{12}) \\
-2E_{j-1}^{11}E_{j}^{12} \\
2E_{j-1}^{11}E_{j}^{12}
\end{array} \right),
\]

\[
M_{33}^{11} = \frac{i\eta^2}{\eta^2 - \lambda^2} \left( \begin{array}{c}
2E_{j-1}^{11}E_{j}^{12} \\
-2E_{j-1}^{11}E_{j}^{12} \\
2E_{j-1}^{11}E_{j}^{12}
\end{array} \right).\]

\(^8\) We point out that the matrix \( M_{\beta}(\lambda) \) acts non-trivially on spaces \( 0, j - 1, \) and \( j \). Here we only use \( \beta \) as subscript of \( M \) to follow the conventions in the previous papers mentioned above. \( M_{\beta}(\lambda) \) is the operator for the bulk (of the open boundary case) and also for the periodic case.
From the equations (2.3) and (2.4), it follows that the \( M_- (\lambda) \) matrix has the following form

\[
M_- (\lambda) = \begin{pmatrix}
D_1^{(1)} & D_1^{(2)} & A_- \\
D_1^{(2)} & D_1^{(3)} & A_+ \\
A_+ & D_1^{(4)} & D_1^{(5)} \\
D_1^{(4)} & D_1^{(5)} & C_+ \\
A_- & D_1^{(6)} & C_- \\
D_1^{(6)} & C_- & D_1^{(9)}
\end{pmatrix},
\]

(4.16)

with

\[
D_1^{(1)} = D_1^{(5)} = D_1^{(9)} = \frac{\eta^2}{\eta^2 - \lambda^2},
\]

\[
D_1^{(2)} = \frac{\hat{i}(b_1 - a_1)\eta^2}{\eta^2 - \lambda^2}, \quad D_1^{(4)} = \frac{i(a_1 - b_1)\eta^2}{\eta^2 - \lambda^2}, \quad D_1^{(6)} = \frac{\hat{i}(c_1 - b_1)\eta^2}{\eta^2 - \lambda^2}, \quad D_1^{(8)} = \frac{i(b_1 - c_1)\eta^2}{\eta^2 - \lambda^2},
\]

\[
D_1^{(3)} = \frac{\hat{i}(c_1 - a_1 + 1)\eta^2 + i(a_1 - c_1)\lambda^2}{\eta^2 - \lambda^2}, \quad D_1^{(7)} = \frac{i(a_1 - c_1 + 1)\eta^2 + i(c_1 - a_1)\lambda^2}{\eta^2 - \lambda^2},
\]

\[
A_- = \frac{i\eta(\eta + (b_1 - a_1)\lambda)}{\eta^2 - \lambda^2}, \quad A_+ = \frac{i\eta(\eta + (a_1 - b_1)\lambda)}{\eta^2 - \lambda^2},
\]

\[
C_- = \frac{i\eta(\eta + (b_1 - c_1)\lambda)}{\eta^2 - \lambda^2}, \quad C_+ = \frac{i\eta(\eta + (c_1 - b_1)\lambda)}{\eta^2 - \lambda^2}.
\]

(4.17)

The matrix \( M_- (\lambda) \) acts non-trivially on spaces 0, and 1.
The $M_+(\lambda)$ matrix\(^{10}\) has the following form

$$
M_+(\lambda) = \begin{pmatrix}
D_N^{(1)} & D_N^{(2)} & A'_+ \\
D_N^{(3)} & D_N^{(4)} & D_N^{(5)} \\
A'_- & D_N^{(6)} & D_N^{(7)} \\
& C'_- & D_N^{(8)} \\
& & & D_N^{(9)}
\end{pmatrix},
$$

(4.18)

with

$$
D_N^{(1)} = D_N^{(5)} = D_N^{(9)} = \frac{i\eta^2}{\eta^2 - \lambda^2}, \quad D_N^{(2)} = \frac{i(b_N - a_N)\eta^2}{\eta^2 - \lambda^2},
$$

$$
D_N^{(4)} = \frac{i(a_N - b_N)\eta^2}{\eta^2 - \lambda^2}, \quad D_N^{(6)} = \frac{i(c_N - b_N)\eta^2}{\eta^2 - \lambda^2}, \quad D_N^{(8)} = \frac{i(b_N - c_N)\eta^2}{\eta^2 - \lambda^2},
$$

$$
D_N^{(3)} = \frac{i(c_N - a_N + 1)\eta^2 + i(a_N - c_N)\lambda^2}{\eta^2 - \lambda^2},
$$

$$
D_N^{(7)} = \frac{i(a_N - c_N + 1)\eta^2 + i(c_N - a_N)\lambda^2}{\eta^2 - \lambda^2},
$$

$$
A'_- = \frac{i\eta(a_N + b_N - a_N)\lambda}{\eta^2 - \lambda^2}, \quad A'_+ = \frac{i\eta(a_N + b_N - a_N)\lambda}{\eta^2 - \lambda^2},
$$

$$
C'_- = \frac{i\eta(a_N + b_N - c_N)\lambda}{\eta^2 - \lambda^2}, \quad C'_+ = \frac{i\eta(a_N + c_N - b_N)\lambda}{\eta^2 - \lambda^2}.
$$

(4.19)

Thus we have obtained all the Lax pair operators for the free Shor–Movassagh spin chain with open boundaries. This also gives a straightforward proof of the integrability of the model.

\(^{10}\) The matrix $M_+(\lambda)$ acts non-trivially on spaces 0, and $N$.  

10
5. Boundary reflection matrices

Based on the proper set of $M$-matrix operators (4.14), (4.16), and (4.18) in the last section, we are now ready to compute the boundary $K$-matrices.\(^{11}\) We proceed to study the constraint conditions (2.6) and (2.7). Let us set

$$K_-(\lambda) = \begin{pmatrix} K^{11}_- & K^{22}_- \\ K^{22}_- & K^{33}_- \end{pmatrix}. \quad (5.1)$$

Substituting (4.16) into (2.6), after tedious calculations, we have the following constraints for $K_-$:

$$\begin{align*}
K^{11}_- &= -(a_1 - b_1)\lambda + \eta, \\
K^{22}_- &= -(c_1 - b_1)\lambda + \eta, \\
K^{33}_- &= -(b_1 - a_1)\lambda + \eta.
\end{align*} \quad (5.2)$$

Similarly, for $K_+(\lambda)$

$$K_+(\lambda) = \begin{pmatrix} K^{11}_+ & K^{22}_+ \\ K^{22}_+ & K^{33}_+ \end{pmatrix}. \quad (5.3)$$

In order to calculate $K_+(\lambda)$, one should note that for

$$U_N(\lambda) = L_N(\lambda)U_{N-1}(\lambda)L_N^{-1}(-\lambda), \quad (5.4)$$

the entries of $U_{N-1}(\lambda)$ commute with the elements of $L_N(\lambda)$ (since they act on different quantum spaces). This fact simplifies the calculations of the partial trace in equation (2.7). It also implies that the constraints of $K_+$ do not rely on $K_-$. Substituting (4.18) into (2.7), we have the constraints for $K_+$,

$$\begin{align*}
K^{11}_+[\eta + (a_N - b_N)\lambda] + K^{22}_+[\eta + (a_N - b_N)\lambda] \\
+ K^{33}_+[\eta + (c_N - b_N)\lambda] = 0, \\
K^{11}_+[\eta + (c_N - b_N)\lambda] + K^{22}_+[\eta + (c_N - b_N)\lambda] \\
+ K^{33}_+[\eta + (c_N - b_N)\lambda] = 0.
\end{align*} \quad (5.5)$$

The above constraint equations give the ratios of the elements in $K$-matrices. In principle, one can write down many $K$-matrices which make the model integrable. We remark that one can construct several kinds of open boundaries. To retrieve the Hamiltonian (3.2), an obvious solution can be related to two-boundary Temperley–Lieb algebra [24].\(^{12}\) Taking into account the scaling of the $K$-matrices, and setting $a_1 = c_1, a_N = c_N$, we have

$$K_-(\lambda) = \begin{pmatrix} (b_1 - a_1)\lambda + \eta & (a_1 - b_1)\lambda + \eta \\ (a_1 - b_1)\lambda + \eta & (b_1 - a_1)\lambda + \eta \end{pmatrix}, \quad (5.6)$$

\(^{11}\) We remark that in most cases, the $M$-matrix operators will not give rise to nontrivial $K$-matrices. Since ordinarily, the number of the constraint equations is equal to the number of unknown entries of $K$-matrices. All the equations are linear homogeneous equations, meaning that all the constant terms are zero. They will only lead to trivial zero solutions.

\(^{12}\) By setting $a_1 = c_1, a_N = c_N$, one can easily define the Temperley–Lieb generators $\hat{e}_0$ and $\hat{e}_N$ on the boundaries.
and
\[
K_{+}(\lambda) = \frac{2}{3}\left( (b_N - a_N)(\lambda + \eta) + \eta \right) \left( (b_N - a_N)(\lambda + \eta) + \eta \right) + \eta \left( (b_N - a_N)(\lambda + \eta) + \eta \right).
\]

(5.7)

Here \(a_1, b_1, a_N, \) and \(b_N\) are arbitrary free parameters describing the boundary effect. Unlike the case of the one dimensional Hubbard open chain [18, 19], in our case the \(K\)-matrices can only be derived from the Lax formulation. The reflection equations (solutions) [5] do not exist due to the non-invertibility of the partial transpose of the \(R\)-matrix.

Then the free Shor–Movassagh Hamiltonian with open boundaries (3.2) can be derived from the double row transfer matrix (2.5) in the following way:
\[
H_{\text{FSM,open}} = -\frac{\eta}{2} \frac{\partial}{\partial \lambda} \ln \tau(\lambda) \bigg|_{\lambda=0} + \text{const}.
\]

(5.8)

This is similar to the situation of the one dimensional Heisenberg spin chain.

6. Conclusion

We have presented the Lax pair for the free Shor–Movassagh open spin chain. Based on the proper Lax pair, we calculated the boundary \(K\)-matrices. The calculation is tedious but the results are simple. The double row monodromy matrix and transfer matrix of the spin chain have also been constructed. Note that unlike the case of Hubbard model [18–20], the \(K\)-matrices in our case cannot be obtained from the reflection equations and Temperley–Lieb algebra. For the current model, the non-invertibility of the partial transpose \(R_{12}^a(\lambda)\) blocks the proof of the transfer matrix commutativity in Sklyanin’s scheme. There is also no crossing unitarity (equation (6) in [5]) or its substitute in the current model. We further remark that the \(K_{+}\) and \(K_{-}\) here are not isomorphic. They cannot be related (mapped) to each other by reparametrization. See proposition 1 in [5] for the isomorphic case. The crossing unitarity builds up the isomorphism between \(K_{+}\) and \(K_{-}\). More precisely, if crossing symmetry is satisfied then there is a natural bijection between the sets of solutions (\(K\)-matrices) of the two reflection equations. More detailed discussion about the relations between the \(K\)-matrices is in reference [7], section III B.1. So the reflection equation and the dual reflection equation become equivalent in some sense. This property simplifies the process of algebraic Bethe ansatz for the models with open boundaries. The non-diagonal \(K\)-matrices can be derived by assuming non-diagonal boundary terms in the Hamiltonian (3.2). The integrability for the open free Shor–Movassagh chain does not follow from Sklyanin’s reflection equations; the construction of algebraic Bethe ansatz for the open chain will be hard. The algebraic Bethe ansatz scheme is based on the commutation relations among the elements of the monodromy matrix. We cannot derive that information from the Lax pair construction (2.6) and (2.7). In view of the behavior of the \(L\) (or \(R\)) matrix acting on local quantum space, one can assume spin up \(|u\rangle\) in (3.1)
as vacuum state, but it is hard to construct the Bethe state without the help of commutation relations. Since such kind of integrable model is very rare, there may exist some new physical phenomena caused by the unusual boundaries effects. It will become clear if we can tackle its spectrum problem.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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