THE DE BRUIJN–ERDŐS THEOREM FROM A HAUSDORFF MEASURE POINT OF VIEW

M. DOLEŽAL\textsuperscript{1,†}, T. MITSIS\textsuperscript{2} and CH. PELEKIS\textsuperscript{1,*†}

\textsuperscript{1}Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67, Praha 1, Czech Republic  
e-mails: dolezal@math.cas.cz, peleki chr@gmail.com

\textsuperscript{2}Department of Mathematics and Applied Mathematics, University of Crete, 70013 Heraklion, Greece  
e-mail: themis.mitsis@gmail.com

\textit{(Received October 14, 2018; revised June 28, 2019; accepted July 9, 2019)}

Abstract. Motivated by a well-known result in extremal set theory, due to Nicolaas Govert de Bruijn and Paul Erdős, we consider curves in the unit \(n\)-cube \([0,1]^n\) of the form

\[
A = \left\{ (x, f_1(x), \ldots, f_{n-2}(x), \alpha) : x \in [0,1] \right\},
\]

where \(\alpha\) is a fixed real number in \([0,1]\) and \(f_1, \ldots, f_{n-2}\) are injective measurable functions from \([0,1]\) to \([0,1]\). We refer to such a curve \(A\) as an \(n\)-de Bruijn–Erdős-set. Under the additional assumption that all functions \(f_i, i = 1, \ldots, n-2\), are piecewise monotone, we show that the Hausdorff dimension of \(A\) is at most 1 as well as that its 1-dimensional Hausdorff measure is at most \(n-1\). Moreover, via a walk along devil’s staircases, we construct a piecewise monotone \(n\)-de Bruijn–Erdős-set whose 1-dimensional Hausdorff measure equals \(n-1\).

1. Prologue, related work and main results

Here and later, \([n]\) denotes the set of positive integers \(\{1, \ldots, n\}\). The collection of all subsets of \([n]\) is denoted \(2^{[n]}\). Given \(i \in [n]\), we denote by \(\pi_i : [0,1]^n \to [0,1]^n\) the function that maps the point \((x_1, \ldots, x_n)\) to the point \((y_1, \ldots, y_n)\), where \(y_i = x_i\), and \(y_j = 0\) for \(j \neq i\). In other words, \(\pi_i\) is the projection onto the \(i\)-th coordinate. Given a finite set \(F\), we denote by \(|F|\) its cardinality. Finally, \(\lambda(\cdot)\) denotes the Lebesgue measure on the real line.

\textsuperscript{*}Corresponding author.
\textsuperscript{†}Research of Doležal was supported by the GAČR project 17-27844S and RVO: 67985840.
\textsuperscript{‡}Research of Peleki s was supported by the GAČR project 18-01472Y and RVO: 67985840.

\textbf{Key words and phrases:} de Bruijn–Erdős theorem, Hausdorff measure, devil’s staircase, piecewise monotone function.

\textbf{Mathematics Subject Classification:} 05D05, 28A78, 26A30.
We shall be interested in an extremal problem which is motivated by a particular result from extremal set theory. Extremal set theory (see [1,11]) is concerned with the problem of obtaining sharp estimates on the cardinality of a family $\mathcal{F} \subset 2^{[n]}$ under constraints that are described in terms of union, intersection or inclusion. This is a rapidly evolving area in combinatorics which interacts with various branches of mathematics and theoretical computer science including geometry, probability theory, analysis and complexity theory. Part of this interaction is based upon the idea that several results from extremal set theory have continuous counterparts. This is an idea that dates back to the 70's and, since its conception, several results have been reported in a measurable setting (see [10,19,20,22]) as well as in a vector space setting (see [3,17,18,23]). In this article we look at results of extremal set theory from a Hausdorff dimension point of view. In [13], Konrad Engel, Christian Reiher and the last two authors reported an analogue of Sperner’s theorem in this setting. An analogue of the Erdős–Ko–Rado theorem can be found in [12]. In this article, we investigate an extremal problem which is motivated by a well-known result from extremal set theory, due to de Bruijn and Erdős (see [7]), and reads as follows.

**Theorem 1.1** (de Bruijn–Erdős). Let $\mathcal{F} \subset 2^{[n]}$. Assume that for any two sets $A, B \in \mathcal{F}$ there exists a unique $i \in [n]$ such that $i \in A \cap B$. Then $|\mathcal{F}| \leq n$.

See [7], or [5, Theorem 7.3.1] for a proof of this result. Let us remark that the bound is sharp and is attained by the family $\mathcal{F} = \{ [n] \setminus \{1\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, n\} \}$, which is referred to in the literature as a near-pencil.

We look at the de Bruijn–Erdős theorem from a Hausdorff measure-theoretic perspective. Before being more precise, let us briefly mention that one can identify a set $A \subset [n]$ with a binary vector of length $n$: put 1 in the $i$-th coordinate if $i \in A$, and 0 otherwise. Notice that this correspondence is bijective and one may choose not to distinguish between subsets and binary vectors. With this remark in mind, the de Bruijn–Erdős theorem can be expressed by saying that if $A \subset \{0, 1\}^n$ is such that for every two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $A$ there exists a unique $i \in [n]$ with $x_i = y_i > 0$ then $|A| \leq n$. Inspired by the last observation, we introduce the following.

**Definition 1** ($n$-de Bruijn–Erdős-sets). Fix a positive integer $n \geq 2$. An $n$-de Bruijn–Erdős-set (or $n$-dBE-set for short) is a measurable set $A \subset [0, 1]^n$ such that for any two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $A$ there exists a unique $i \in [n]$ with $x_i = y_i$.

Given $i \in [n]$ and $\alpha \in [0, 1]$, let $H_{i, \alpha}^n$ denote the hyperplane $\{ (x_1, \ldots, x_n) \in [0, 1]^n : x_i = \alpha \}$. It is easy to see that the $n$-dimensional Lebesgue measure of an $n$-dBE-set equals zero: fix $a = (a_1, \ldots, a_n) \in A$ and notice that $A \subset$
Given this fact it is natural to look at the Hausdorff dimension of an \( n \)-dimensional \( \psi \)-set. Here and later, the term injective, measurable function. Then \( f \) is a strictly increasing function from \([0,1]\) into measurable disjoint \( S_j \)'s such that for every \( j \) the restriction of \( f \) to \( S_j \) is a strictly monotone function. An \( n \)-dimensional \( \psi \)-set, \( A \), is called piecewise monotone if it is of the form

\[
A = \{ (x, f_1(x), \ldots, f_{n-2}(x), \alpha) : x \in [0,1] \},
\]

where \( f_i \) are piecewise monotone functions, and \( \alpha \) is a continuous function on \( [0,1] \).
where $\alpha \in [0,1]$ is fixed and each function $f_i, i \in [n-2]$ is piecewise monotone.

It should be noted that without imposing this seemingly artificial restriction, little can be said about the Hausdorff dimension and the corresponding Hausdorff measure of an $n$-dBE-set. For example, the following, rather unexpected, result due to James Foran readily implies that the 1-dimensional Hausdorff measure of a 3-dBE-set can be arbitrarily large.

**Theorem 1.2** (Foran [16]). For every positive real $K$, there exists a measurable, injective function from $[0,1]$ to $[0,1]$ such that the 1-dimensional Hausdorff measure of its graph is larger than $K$.

Moreover, the next theorem states that the Hausdorff dimension of a measurable 3-dBE-set can also be large. This theorem is an immediate consequence of the main result from [6]. See also [9] for a related result.

**Theorem 1.3** (Coen et al. [6]). There exists a compact set $\Gamma \subset [0,1]^2$ whose projections onto the coordinate axes are injective and whose Hausdorff dimension is equal to 2.

Bearing the above in mind, our main result on the Hausdorff dimension and the corresponding Hausdorff measure of piecewise monotone $n$-dBE-sets reads as follows.

**Theorem 1.4.** Let $A \subset [0,1]^n$ be a measurable, piecewise monotone $n$-dBE-set. Then $\dim_H(A) \leq 1$ as well as

$$\mathcal{H}^1(A) \leq n - 1.$$

We prove Theorem 1.4 in Section 2. In Section 3, we show that the upper bound on the 1-dimensional Hausdorff measure of an $n$-dBE-set, provided by Theorem 1.4, is sharp. More precisely, we have the following.

**Theorem 1.5.** There exists a measurable, piecewise monotone $n$-dBE-set $A$ such that

$$\mathcal{H}^1(A) = n - 1.$$

The set constructed in the proof of Theorem 1.5 is a piecewise monotone $n$-dBE-set $A$ of the form

$$A := \{(x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)), \alpha) : x \in [0,1]\},$$

where $h$ is a singular function (i.e., continuous, strictly increasing, having derivative zero almost everywhere) and each $f_i, i = 1, \ldots, n-3$, is a strictly increasing function from $[0,1]$ to $[0,1]$ that maps a particular set of measure zero to a set of measure one. The existence of the latter functions is guaranteed by the following result, which may be of independent interest.
Theorem 1.6. For every \( S \subset [0, 1] \) with \( \lambda(S) = 1 \) there exists a strictly increasing function \( f: [0, 1] \to [0, 1] \) and \( N \subset S \) with \( \lambda(N) = 0 \) such that \( \lambda(f(N)) = 1 \).

2. Proof of Theorem 1.4

We begin with the case of 2-dBE-sets, which is rather trivial.

Theorem 2.1. Let \( A \) be a 2-dBE-set. Then \( \mathcal{H}^1(A) \leq 1 \).

Proof. Let \( x = (x_1, x_2), y = (y_1, y_2) \in A \) be distinct and notice that the hypothesis implies that either \( x_1 = y_1 \) or \( x_2 = y_2 \), but not both. Without loss of generality, assume that \( x_1 = y_1 \). Hence both \( x, y \) belong to the line \( \ell := \{(x_1, \alpha) : \alpha \in [0, 1]\} \). Now notice that every other point from \( A \) must belong to the line \( \ell \) as well and thus \( A \subset \ell \). The result follows. \( \square \)

So, from now on, we may assume that \( n \geq 3 \). The proof of Theorem 1.4 requires some definitions. We say that a family \( \mathcal{F} \) of pairwise disjoint subsets of the real line is left-right ordered if for every two distinct nonempty sets \( F_1, F_2 \in \mathcal{F} \) we have either \( \sup F_1 \leq \inf F_2 \) or \( \inf F_1 \geq \sup F_2 \). We say that \( \mathcal{F} \) is a left-right ordered partition of a subset \( A \) of the real line if it is a left-right ordered family as well as a partition of \( A \).

It is well known (see [2, p. 4]) that for every subset \( A \) of the real line and for every \( \delta, \varepsilon > 0 \) there is a cover \( \{U_i\}_{i=1}^\infty \) of \( A \) consisting of convex sets (i.e. intervals) such that the diameter of each \( U_i \) is at most \( \delta \) and such that \( \sum_i \text{diam}(U_i) \leq \mathcal{H}_\delta^1(A) + \varepsilon \). If for every \( i \) we define \( V_i := (U_i \setminus \bigcup_{j<i} U_j) \cap A \) then it is clear that \( \{V_i\} \) is a partition of \( A \) such that the diameter of each \( V_i \) is at most \( \delta \). It follows that for every subset \( A \) of the real line and for every \( \delta, \varepsilon > 0 \) we have

\[
\mathcal{H}_\delta^1(A) = \inf \left\{ \sum_i \text{diam}(V_i) \right\}
\]

where the infimum is taken over all at most countable left-right ordered partitions \( \{V_i\} \) of \( A \) such that the diameter of each \( V_i \) is at most \( \delta \).

Recall that a function \( f: F \subset \mathbb{R}^n \to \mathbb{R}^m \) is Lipschitz with constant \( c \) if

\[
|f(x) - f(y)| \leq c \cdot |x - y| \quad \text{for all } x, y \in F.
\]

The following result is well known (see [15, p. 24]).

Lemma 2.2. Fix positive integers \( n, m \) and let \( F \subset \mathbb{R}^n \). If \( f: F \to \mathbb{R}^m \) is a Lipschitz function with constant \( c \) then \( \mathcal{H}^s(f(F)) \leq c^s \mathcal{H}^s(F) \).

The proof of Theorem 1.4 is based upon the following lemmata.
Lemma 2.3. Let $A$ be a bounded subset of the real line and suppose that $\{V_i\}_i, \{W_j\}_j$ are at most countable left-right ordered partitions of $A$. Then $\{V_i \cap W_j\}_{i,j}$ is an at most countable left-right ordered partition of $A$ and
\[
\sum_{i,j} \text{diam}(V_i \cap W_j) \leq \min \left\{ \sum_i \text{diam}(V_i), \sum_j \text{diam}(W_j) \right\}.
\]

Proof. The fact that $\{V_i \cap W_j\}_{i,j}$ is an at most countable left-right ordered partition of $A$ is trivial. We will show that for every $i$ it holds
\[
\sum_j \text{diam}(V_i \cap W_j) \leq \text{diam}(V_i),
\]
then the assertion easily follows. To this end, fix an index $i$ as well as $\varepsilon > 0$ and, since $A$ is bounded, find a finite set $F = \{j_1, \ldots, j_n\}$ of indices $j$ such that
\[
\sum_j \text{diam}(V_i \cap W_j) \leq \sum_{k=1}^n \text{diam}(V_i \cap W_{j_k}) + \varepsilon.
\]

For every $k = 1, \ldots, n$, find $x_k, y_k \in V_i \cap W_{j_k}$ such that
\[
\text{diam}(V_i \cap W_{j_k}) \leq |x_k - y_k| + \frac{\varepsilon}{n}.
\]

Without loss of generality suppose that $x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_{n-1} \leq x_n \leq y_n$. Then
\[
\sum_j \text{diam}(V_i \cap W_j) \leq \sum_{k=1}^n \text{diam}(V_i \cap W_{j_k}) + \varepsilon \leq \sum_{k=1}^n (y_k - x_k + \frac{\varepsilon}{n}) + \varepsilon
\]
\[
= \sum_{k=1}^n (y_k - x_k) + 2\varepsilon \leq y_n - x_1 + 2\varepsilon \leq \text{diam}(V_i) + 2\varepsilon.
\]

As this holds for every $\varepsilon > 0$, the proof is finished. \(\Box\)

Lemma 2.4. Suppose that $f_0, f_1, \ldots, f_m : [0, 1] \to [0, 1]$ are strictly increasing real functions, all of them defined on a given subset $D$ of $[0, 1]$. Then
\[
\mathcal{H}^1 \left( \left\{ \sum_{k=0}^m f_k(x) : x \in D \right\} \right) \leq \sum_{k=0}^m \mathcal{H}^1(f_k(D)).
\]

Proof. We may assume that $m = 1$ (the general case follows by induction on $m$). Then it clearly suffices to show that for every $\delta > 0$ it holds
\[
\mathcal{H}^1_{2\delta}((f_1 + f_2)(D)) \leq \mathcal{H}^1_{\delta}(f_1(D)) + \mathcal{H}^1_{\delta}(f_2(D)).
\]

Acta Mathematica Hungarica 159, 2019
So fix $\delta > 0$ and $\varepsilon > 0$ and find at most countable left-right ordered partitions $\{V^1_i\}_i$ of $f_1(D)$ and $\{V^2_j\}_j$ of $f_2(D)$ such that the diameter of each $V^1_i$ and of each $V^2_j$ is at most $\delta$, and such that we have

$$\sum_i \text{diam}(V^1_i) \leq \mathcal{H}^1_\delta(f_1(D)) + \varepsilon \quad \text{and} \quad \sum_j \text{diam}(V^2_j) \leq \mathcal{H}^1_\delta(f_2(D)) + \varepsilon.$$ 

We define $\mathcal{F} := \{f^{-1}_1(V^1_i) \cap f^{-1}_2(V^2_j)\}_{i,j}$. The fact that both functions $f_1$ and $f_2$ are increasing clearly gives us that $\mathcal{F}_1 := \{f_1(F): F \in \mathcal{F}\}$ is an at most countable left-right ordered partition of $f_1(D)$. Moreover, Lemma 2.3 applied to the partitions $\{V^1_i\}_i$ and $\mathcal{F}_1$ gives us that the partition $\mathcal{F}_1$ satisfies

$$\sum_{F \in \mathcal{F}_1} \text{diam}(F) \leq \sum_i \text{diam}(V^1_i) \leq \mathcal{H}^1_\delta(f_1(D)) + \varepsilon.$$ 

Similarly $\mathcal{F}_2 := \{f_2(F): F \in \mathcal{F}\}$ is an at most countable left-right ordered partition of $f_2(D)$ and

$$\sum_{F \in \mathcal{F}_2} \text{diam}(F) \leq \mathcal{H}^1_\delta(f_2(D)) + \varepsilon.$$ 

It remains to observe that the family $\{ (f_1 + f_2)(F): F \in \mathcal{F} \}$ is a cover of $(f_1 + f_2)(D)$ (in fact, it is a left-right ordered partition) consisting of sets of diameter at most $2\delta$. Therefore

$$\mathcal{H}^1_{2\delta}((f_1 + f_2)(D)) \leq \sum_{F \in \mathcal{F}} \text{diam}((f_1 + f_2)(F))$$

$$= \sum_{F \in \mathcal{F}} \text{diam}(f_1(F)) + \sum_{F \in \mathcal{F}} \text{diam}(f_2(F)) \leq \mathcal{H}^1_\delta(f_1(D)) + \mathcal{H}^1_\delta(f_2(D)) + 2\varepsilon.$$ 

As this holds for every $\varepsilon > 0$, the proof is completed. \qed

**Lemma 2.5.** Suppose that $f_0, f_1, \ldots, f_m: [0,1] \rightarrow [0,1]$ are strictly monotone real functions, all of them defined on a given subset $D$ of $[0,1]$. Denote $G := \{(f_0(x), f_1(x), \ldots, f_m(x)): x \in D\}$. Then

$$\mathcal{H}^1(G) \leq \sum_{k=0}^m \mathcal{H}^1(f_k(D)).$$

**Proof.** Without loss of generality, we may assume that all the functions $f_0, f_1, \ldots, f_m$ are strictly increasing. Let us denote

$$S := \left\{ \sum_{k=0}^m f_k(x): x \in D \right\}.$$
Then the mapping $F: S \to G$ defined by $F(\sum_{k=0}^{m} f_k(x)) := (f_0(x), f_1(x), \ldots, f_m(x))$ is Lipschitz with constant 1, and it is surjective. Therefore $H^1(G) \leq H^1(S)$, by Lemma 2.2. The rest follows by Lemma 2.4. □

We now have all the necessary ingredients for the proof of the main result of this section.

**Proof of Theorem 1.4.** Notice that the first statement is a consequence of the second and so it is enough to show the second statement. Let $A$ be of the form

$$A = \{ (x, f_1(x), \ldots, f_{n-2}(x), \alpha): x \in [0,1]\},$$

where $\alpha \in [0,1]$ is fixed and each function $f_i, i \in [n-2]$ is piecewise monotone. Define the set

$$A' := \{ (x, f_1(x), \ldots, f_{n-2}(x)) : x \in [0,1]\}$$

and notice that the fact $H^1(A) = H^1(A')$ implies that it is enough to show $H^1(A') \leq n - 1$.

Since $f_i$, $i \in [n-2]$, is piecewise monotone, there exists an at most countable partition $\{S_{i,j}\}_j$ of $[0,1]$ into measurable sets such that the function $f_i$ restricted to each $S_{i,j}$ is strictly monotone. For every $(n-2)$-tuple $j = (j_1, \ldots, j_{n-2})$ let $S_j = S_{1,j_1} \cap \cdots \cap S_{n-2,j_{n-2}}$ and denote by $f_{i,j}$ the restriction of $f_i$ to the set $S_j$. Notice that the sets $\{S_j\}_j$ are pairwise disjoint and that each function $f_{i,j}$ is strictly monotone. For every $j$ denote

$$G_j := \{ (x, f_1(x), \ldots, f_{n-2}(x)) : x \in S_j \}.$$

Then, by Lemma 2.5 applied to the strictly monotone functions $f_{0,j} := id|_{S_j}$ and $f_{1,j}, \ldots, f_{n-2,j}$ (for every $j$), we have

$$H^1(A') = \sum_j H^1(G_j) \leq \sum_j \left( H^1(S_j) + \sum_{k=1}^{n-2} H^1(f_k(S_j)) \right)$$

$$= H^1([0,1]) + \sum_{k=1}^{n-2} H^1(f_k([0,1])) \leq n - 1,$$

as desired. □

### 3. Proofs of Theorem 1.5 and Theorem 1.6

We begin with the case $n = 3$ of Theorem 1.5. This proof is sketched in [16, p. 810], but we include here the details for the sake of completeness.
and for the purpose of illustrating the idea of our proof in higher dimensions.
The proof of the case $n = 3$ employs the following result from measure theory
(see [4, Proposition 5.5.4]).

**Lemma 3.1.** Let $f : [0, 1] \to [0, 1]$ be a function and let $E$ be a measurable
set such that at every point of $E$ the function $f(\cdot)$ is differentiable. Then

$$
\lambda(f(E)) \leq \int_E |f'(x)| \, dx.
$$

We can now proceed with the proof for the existence of a 3-dBE-set
whose 1-dimensional Hausdorff measure equals 2.

**Proof of Theorem 1.5 (case $n = 3$).** Let $h : [0, 1] \to [0, 1]$ be a continuous,
strictly increasing function having zero derivative almost every-
where. An example of such a function can be found in [24]. Now fix
a function and let $\lambda$ be such that

$$
\lambda(N_{\omega}) = 2.
$$

The proof of Theorem 1.5, for $n \geq 4$, is based upon Theorem 1.6. Hence
we now proceed with the proof of Theorem 1.6, which requires the following
lemma. In the proof, $\omega$ denotes the set of all nonnegative integers, $2^\omega$ the
set of all infinite dyadic sequences, $2^{<\omega}$ the set of all finite dyadic sequences,
$2^{\leq n}$ the set of all dyadic sequences of length at most $n$, and $2^n$ the set of all
dyadic sequences of length equal to $n$. Finally, given $s = (s_1, \ldots, s_n) \in 2^n$
and $i \in \{0, 1\}$, we denote by $s^{\uparrow\{i\}}$ the dyadic sequence $(s_1, \ldots, s_n, i)$.

**Lemma 3.2.** Let $I = [a, b] \subset [0, 1]$ be a closed interval and let $S_I \subset [0, 1]$
be such that $\lambda(I \setminus S_I) = 0$. Then there exists a set $N_I \subset S_I \cap I$
and a continuous non-decreasing function $f_I : I \to [0, 1]$ such that $f_I(a) = 0$, $f_I(b) = 1$,
$\lambda(N_I) = 0$ and $\lambda(f_I(N_I)) = 1$. 

*Acta Mathematica Hungarica 159, 2019*
Proof. By assumption we have \( \lambda(S_I \cap I) = \lambda(I) > 0 \) and the inner regularity of the Lebesgue measure implies that the set \( S_I \cap I \) contains a closed subset \( F \) with \( \lambda(F) > 0 \). In particular, \( F \) is uncountable. As every closed set has the perfect set property (see [21]), there is a nonempty perfect (i.e. closed and with no isolated points) subset \( \tilde{F} \) of \( F \). By induction on the length of \( s \), we will construct closed intervals \( J_s \), \( s \in 2^{\omega} \), such that for every \( n \in \omega \), the following conditions hold:

(i) \( J_{s^{\setminus\{0\}}} \cup J_{s^{\setminus\{1\}}} \subset J_s \) for every \( s \in 2^n \),

(ii) the upper endpoint of \( J_s^{\setminus\{0\}} \) is strictly below the lower endpoint of \( J_s^{\setminus\{1\}} \) for every \( s \in 2^n \),

(iii) \( \lambda(J_s) \leq \frac{1}{(n+1)2^n} \) for every \( s \in 2^n \),

(iv) the interior of \( J_s \) intersects \( \tilde{F} \) for every \( s \in 2^n \).

Simply begin the construction by setting \( J_0 = I \); then both conditions (iii)\(_0\) and (iv)\(_0\) are satisfied. Now suppose that we have already defined all intervals \( J_s \), \( s \in 2^{\leq n} \), for some \( n \in \omega \) such that conditions (i)\(_k\), (ii)\(_k\) hold for every \( k < n \) and such that conditions (iii)\(_k\), (iv)\(_k\) hold for every \( k \leq n \). By (iv)\(_n\) we know that the interior of \( J_s \) intersects \( \tilde{F} \) for every \( s \in 2^n \). As \( \tilde{F} \) is a perfect set, the intersection \( J_s \cap \tilde{F} \) is an infinite (even uncountable) set for every \( s \in 2^n \). So for every \( s \in 2^n \), we can find two points \( r_{s^{\setminus\{0\}}} < r_{s^{\setminus\{1\}}} \) in \( J_s \cap \tilde{F} \). Then for every \( s \in 2^n \), we can find closed intervals \( J_{s^{\setminus\{0\}}} \) and \( J_{s^{\setminus\{1\}}} \) containing \( r_{s^{\setminus\{0\}}} \) and \( r_{s^{\setminus\{1\}}} \), respectively, in such a way that conditions (i)\(_n\), (ii)\(_n\) and (iii)\(_{n+1}\) are satisfied. Then condition (iv)\(_{n+1}\) clearly holds as well. This finishes the construction.

Let \( N_I \) be the subset of \( I \) given by \( N_I = \bigcap_{n \in \omega} \bigcup_{s \in 2^n} J_s \). Then for every \( m \in \omega \), we have by condition (iii)\(_m\) that

\[
\lambda(N_I) \leq \lambda\left( \bigcup_{s \in 2^n} J_s \right) \leq \sum_{s \in 2^n} \lambda(J_s) \leq \frac{1}{m+1},
\]

which implies that \( \lambda(N_I) = 0 \). Also, for every \( x \in N_I \) and every \( n \in \omega \) there is \( s \in 2^n \) such that \( x \in J_s \), and by (iii)\(_n\) and (iv)\(_n\) there is \( y \in \tilde{F} \) such that \( |x - y| \leq \frac{1}{(n+1)2^n} \). It follows that every \( x \in N_I \) is in the closure of the closed set \( \tilde{F} \), and so \( N_I \subset \tilde{F} \subset F \subset S_I \cap I \).

For every \( x \in N_I \) there is (by (i)\(_n\) and (ii)\(_n\)) a unique \( \alpha_x \in 2^{\omega} \) such that \( x \in \bigcap_{n \in \omega} J_{\alpha_x} \). Moreover, if \( x, y \) are two distinct points from \( N_I \) then \( \alpha_x \neq \alpha_y \) by (iii)\(_n\). It clearly follows that the mapping \( \phi: N_I \to [0, 1] \) which maps each \( x \in N_I \) to the unique point from \( \bigcap_{n=1}^{\infty} [2\alpha_x(n)/3^n, 2\alpha_x(n+1)/3^n] \) is an order preserving homeomorphism of the set \( N_I \) onto the standard ternary Cantor set \( C \subset [0, 1] \). Let \( \Phi: I \to [0, 1] \) be the continuous non-decreasing extension of \( \phi \) which is linear on each open subinterval of \( I \setminus N_I \). Let \( c: [0, 1] \to [0, 1] \) be the Devil’s staircase (see [8]), i.e. a continuous non-decreasing function with

\[ Acta Mathematica Hungarica 159, 2019 \]
There exists a strictly increasing function with \( f_I(a) = c(0) = 0, f_I(b) = c(1) = 1 \) and \( \lambda(f_I(N_I)) = \lambda(c(C)) = 1 \). \( \square \)

**Proof of Theorem 1.6.** Let \( S \subseteq [0,1] \) be such that \( \lambda(S) = 1 \). Let \( I_n = [a_n, b_n] \), \( n \in \omega \), be an enumeration of all closed subintervals of \([0,1]\) with rational endpoints. By Lemma 3.2, we can find (by induction on \( n \)) sets \( N_{I_n} \subseteq S \setminus \bigcup_{m<n} N_{I_m} \), \( n \in \omega \), and continuous non-decreasing functions \( f_{I_n} : I_n \rightarrow [0,1] \), \( n \in \omega \), such that \( f_{I_n}(a_n) = 0, f_{I_n}(b_n) = 1, \lambda(N_{I_n}) = 0 \) and \( \lambda(f_{I_n}(N_{I_n})) = 1 \) for every \( n \in \omega \).

For every \( n \in \omega \), let \( g_n : [0,1] \rightarrow [0,1] \) be the unique continuous non-decreasing extension of \( f_{I_n} \) from \( I_n \) to the whole interval \([0,1]\). Finally, we define \( N = \bigcup_{n \in \omega} N_{I_n} \) and

\[
f(x) = \sum_{n \in \omega} \frac{1}{2^{n+1}} g_n(x), \quad \text{for } x \in [0,1].
\]

Clearly, \( N \subseteq S \) and \( \lambda(N) = 0 \). Since the sum in the definition of \( f \) converges uniformly and all summands are continuous functions, it follows that \( f \) is continuous as well. Also, \( f \) is the sum of non-decreasing functions and so it is non-decreasing as well. Moreover, if \( x < y \) are two points from \([0,1]\) then there exists \( m \in \omega \) such that \( x < a_m < b_m < y \). Now notice that

\[
f(y) - f(x) = \sum_{n \in \omega} \frac{1}{2^{n+1}} (g_n(y) - g_n(x)) \geq \frac{1}{2^{m+1}} (g_m(y) - g_m(x)) = \frac{1}{2^{m+1}} > 0,
\]

which implies that \( f \) is strictly increasing. It is obvious that \( f(0) = 0 \) and \( f(1) = 1 \), so it only remains to show that \( \lambda(f(N)) = 1 \).

To this end, let us fix \( m \in \omega \) for a while. If \( N_{I_n} \) and \( f_{I_n} \) were constructed in the same way as in the proof of Lemma 3.2, then \( N_{I_m} \) is closed and \( g_m \) is constant on every open subinterval of \([0,1] \setminus N_{I_m} \). So if \( f^{(m)} : [0,1] \rightarrow [0,1] \) is the (strictly increasing and continuous) function given by

\[
f^{(m)}(x) = \sum_{\substack{n \in \omega \\cap m \neq m}} \frac{1}{2^{n+1}} g_n(x), \quad \text{for } x \in [0,1],
\]

then for every open subinterval \((c,d)\) of \([0,1]\) which does not intersect \( N_{I_m} \) we have

\[
\lambda(f(c,d)) = f(d) - f(c) = f^{(m)}(d) - f^{(m)}(c) = \lambda(f^{(m)}((c,d))).
\]
Since the set \([0, 1] \setminus N_m\) is the union of countably many pairwise disjoint such subintervals \((c_i, d_i), i \in \omega\), and of a subset of \(\{0, 1\}\) it follows that
\[
(1) \quad \lambda(f([0, 1] \setminus N_m)) = \sum_{i \in \omega} \lambda(f((c_i, d_i))) = \sum_{i \in \omega} \lambda(f(m)((c_i, d_i)))
\]
\[
= \lambda(f(m)([0, 1] \setminus N_m)) \leq \lambda(f(m)([0, 1])) = f(m)(1) - f(m)(0) = 1 - \frac{1}{2^{m+1}}.
\]
On the other hand,
\[
(2) \quad \lambda(f([0, 1])) = f(1) - f(0) = 1.
\]
Comparing equations (1) and (2) yields \(\lambda(f(N_m)) \geq \frac{1}{2^{m+1}}\). Since this is true for every \(m \in \omega\) and all the sets \(N_m, m \in \omega\), are pairwise disjoint, the monotonicity of \(f\) implies
\[
\lambda(f(N)) = \sum_{m \in \omega} \lambda(f(N_m)) \geq \sum_{m \in \omega} \frac{1}{2^{m+1}} = 1,
\]
as required. \(\square\)

We now have all ingredients for the proof of Theorem 1.5, when \(n \geq 4\).

**Proof of Theorem 1.5 (Case \(n \geq 4\)).** Let \(h: [0, 1] \to [0, 1]\) be a singular function and let \(S_1 := h(\{x : h'(x) \neq 0\})\). Notice that Lemma 3.1 implies that \(\lambda(S_1) = 1\). By Theorem 1.6 there exists a strictly increasing function \(f_1: [0, 1] \to [0, 1]\) and a set \(N_1 \subset S_1\) with \(\lambda(N_1) = 0\) such that \(\lambda(f_1(N_1)) = 1\). Now for \(j = 2, \ldots, n-3\) define recursively \(S_j = S_{j-1} \setminus N_{j-1}\) and apply Theorem 1.6 to choose a strictly increasing function \(f_j: [0, 1] \to [0, 1]\) and a set \(N_j \subset S_j\) with \(\lambda(N_j) = 0\) such that \(\lambda(f_j(N_j)) = 1\). Let \(\alpha \in [0, 1]\) be fixed, and consider the set
\[
A := \{(x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)), \alpha) : x \in [0, 1]\}.
\]
Since the functions \(h, f_1, \ldots, f_{n-3}\) are strictly increasing it follows that \(A\) is an \(n\)-dBE-set. Moreover, it is enough to show that
\[
\mathcal{H}^1(\{(x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)) : x \in [0, 1]\}) = n - 1.
\]
To this end, we proceed as in the proof of the case \(n = 3\) and show that the set
\[
D = \{(x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)) : x \in [0, 1]\}
\]
can be written as the disjoint union of \(n - 1\) sets that project onto a particular coordinate to sets of measure one. Consider the sets
\[
D_1 := \{(x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x))) : h'(x) = 0\} \quad \text{and} \quad D_2 = D \setminus D_1.
\]
As in the proof of the case \( n = 3 \) it follows that \( \mathcal{H}^1(D_1) \geq 1 \). We now look at the set \( D_2 \). Let \( W_j = h^{-1}(N_j) \), for \( j = 1, \ldots, n - 3 \), and notice that \( \lambda(f_j(N_j)) = \lambda(f_j(h(W_j))) = 1 \). Since \( h(\cdot) \) is injective, it follows that we can write

\[
S_{n-3} = h(\{ x : h'(x) \neq 0 \} \setminus \{ W_1 \cup \cdots \cup W_{n-4} \}).
\]

Moreover, \( \lambda(S_{n-3}) = 1 \). Thus, denoting \( Q_1 := \{ x : h'(x) \neq 0 \} \setminus (W_1 \cup \cdots \cup W_{n-3}) \), it follows that the set \( D_2 \) can be written as the disjoint union of the sets \( D_{2,1}, D_{2,2}, \ldots, D_{2,n-2} \), where

\[
D_{2,j} = \{ (x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)) : x \in W_j \}, \quad \text{for } j = 1, \ldots, n - 3
\]

and

\[
D_{2,n-2} = \{ (x, h(x), f_1(h(x)), \ldots, f_{n-3}(h(x)) : x \in Q_1 \}.
\]

Since \( \lambda(h(Q_1)) = 1 \) it follows that the set \( D_{2,n-2} \) projects onto the second coordinate to a set of measure one; hence \( \mathcal{H}^1(D_{2,n-2}) \geq 1 \). Similarly for \( j = 1, \ldots, n - 3 \), since \( \lambda(f_j(h(W_j))) = 1 \) it follows that the set \( D_{2,j}, j = 1, \ldots, n - 3 \), projects onto the \( (j + 2) \)-th coordinate to a set of measure one; hence \( \mathcal{H}^1(D_{2,j}) \geq 1 \) as well. Thus

\[
\mathcal{H}^1(D) = \mathcal{H}^1(D_1) + \sum_{j=1}^{n-2} \mathcal{H}^1(D_{2,j}) \geq n - 1.
\]

Theorem 1.4 finishes the proof. \( \square \)

**Acknowledgement.** We are grateful to the anonymous referee for bringing to our attention references \([6]\) and \([9]\).

**References**

[1] I. Anderson, *Combinatorics of Finite Sets*, Corrected reprint of the 1989 edition, Dover Publications, Inc. (Mineola, NY, 2002).

[2] C.J. Bishop and Y. Peres, *Fractals in Probability and Analysis*, Cambridge Studies in Advanced Mathematics, **162**. Cambridge University Press (Cambridge, 2017).

[3] A. Blokhuis, A.E. Brouwer, A. Chowdhury, P. Frankl, T. Mussche, B. Patkós and T. Szőnyi, A Hilton–Milner theorem for vector spaces, *Electron. J. Combin.*, **17** (2010), Research Paper 71, 12 pp.

[4] V. I. Bogachev, *Measure Theory*, Vol. 1, Springer (2007).

[5] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press (1994).

[6] F. Coen, N. Gillman, T. Keleti, D. King and J. Zhu, Large sets with small injective projections, arXiv:1906.06288 (2019).

[7] N. G. de Bruijn and Erdős, On a combinatorial problem, *Indag. Math.*, **10** (1948) 421–423.
[8] O. Dovgoshey, O. Martio, V. Ryazanov and M. Vuorinen, The Cantor function, *Expo. Math.*, 24 (2006), 1–37.
[9] V. Eiderman and M. Larsen, A rare plane set with Hausdorff dimension 2, arXiv:1904.09034 (2019).
[10] K. Engel, A continuous version of a Sperner-type theorem, *Elektron. Informationsverarb. Kybernet.*, 22 (1986), 45–50.
[11] K. Engel, *Sperner Theory*, Encyclopedia of Mathematics and its Applications, 65, Cambridge University Press (Cambridge, 1997).
[12] K. Engel, T. Mitsis and C. Pelekis, A fractal perspective on optimal antichains and intersecting subsets of the unit n-cube, arXiv:1707.04856 (2017).
[13] K. Engel, T. Mitsis, C. Pelekis and C. Reiher, Projection inequalities for antichains, *Israel J. Math.* (to appear), arXiv:1812.06496.
[14] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press (1992).
[15] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Ltd. (Chichester, 1990).
[16] J. Foran, The length of the graph of a one to one function from [0, 1] to [0, 1], *Real Anal. Exchange*, 25 (1999/00), 809–816.
[17] P. Frankl and N. Tokushige, The Katona theorem for vector spaces, *J. Comb. Theory, Ser. A*, 120 (2013), 1578–1589.
[18] P. Frankl and R. M. Wilson, The Erdős–Ko–Rado theorem for vector spaces, *J. Combin. Theory, Ser. A*, 43 (1986), 228–236.
[19] G. O. H. Katona, Continuous versions of some extremal hypergraph problems, *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976)*, Vol. II, pp. 653–678, *Colloq. Math. Soc. János Bolyai*, 18, North-Holland, Amsterdam-New York, 1978.
[20] G. O. H. Katona, Continuous versions of some extremal hypergraph problems. II, *Acta Math. Acad. Sci. Hungar.*, 35 (1980), 67–77.
[21] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, 156, Springer-Verlag (New York, 1995).
[22] D. A. Klain and G. C. Rota, A continuous analogue of Sperner’s theorem, *Comm. Pure Appl. Math.*, 50 (1997), 205–223.
[23] M. Katchalski and R. Meshulam, An extremal problem for families of pairs of subspaces, *European J. Combin.*, 15 (1994), 253–257.
[24] A. C. Zaanen and W. A. J. Luxemburg, A real function with unusual properties, [Solution to Problem 5029], *Amer. Math. Monthly*, 70, (1963), 674–675.