Converse bounds for classical communication over quantum networks

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Abstract

We explore the classical communication over quantum channels with one sender and two receivers, or with two senders and one receiver. First, for the quantum broadcast channel (QBC) and the quantum multi-access channel (QMAC), we study the classical communication assisted by non-signalling and positive-partial-transpose-preserving codes, and obtain efficiently computable one-shot bounds to assess the performance of classical communication. Second, we consider the asymptotic communication capability of communication over the QBC and QMAC. We derive an efficiently computable strong converse bound for the capacity region, which behaves better than the previous semidefinite programming strong converse bound for point-to-point channels. Third, we obtain a converse bound on the one-shot capacity region based on the hypothesis testing divergence between the given channel and a certain class of subchannels. As applications, we analyze the communication performance for some basic network channels, including the classical broadcast channels and a specific class of quantum broadcast channels.

1 Introduction

A fundamental goal of quantum information theory is to find the ultimate limits imposed on information processing and transmission by the laws of quantum mechanics. The classical capacity of a noisy point-to-point quantum channel is the maximal rate at which it can transmit classical information faithfully over asymptotically many uses of the channel. The Holevo-Schumacher-Westmoreland theorem [1, 2, 3] gives a characterization of the classical capacity of a general quantum channel.

In many situations of communication, there are usually more than one sender and receiver. It is also important to understand how well we could send information via network channels (e.g., broadcast channel, multiple-access channel, interference channel). The capacity region of a classical degraded broadcast channel can be written as a single-letter formula [4, 5], while no such characterization is known for general classical broadcast channels. Quantum broadcast channel is introduced in [6], and many information-theoretic results are now known for quantum broadcast channels. Refs. [6, 7] give a quantum generalization of the superposition coding method for classical communication over quantum broadcast channel. Ref. [8] gives the one-shot Marton inner bound for classical-quantum broadcast channels. A single-letter capacity region of the Hadamard quantum broadcast channel, as a quantum generalization of the degraded broadcast channel, is obtained in [9]. Refs. [10, 11, 12, 13] offer many other results on communication-related tasks for quantum broadcast channels.

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The single-letter characterization of the capacity region of classical multi-access channels is given in [14, 15, 16]. Ref. [17] shows the capacity region for classical-quantum multiple access channels admits a single letter characterization. Ref. [18] gives a regularized formula for the entanglement-assisted classical-classical capacity region for quantum multiple access channels, and Ref. [19] studies the classical-quantum and quantum-quantum capacity region.

The capacity of a channel provides a fundamental characterization of the asymptotic information transmission capabilities of the channel, since it is assumed that the senders are allowed to use the channel many times and the channel has no memory after each use. In practice, however, the sender may be forced to use the channel only once and the channel may not be memoryless, and one might be concerned with the tradeoff between the number of channel uses (code blocklength), communication rate and error probability. This is one of the driving forces behind the emerging field of one-shot information theory. In recent years, the quantum broadcast communication protocols were studied in [20] using the powerful convex split technique proposed in [21], and also studied in [22] using decoupling approach.

Another challenge for further study of the quantum channel capacity region is that it is rather difficult to calculate the regularized expression of the capacity region as well as a reasonable estimation, and that little is now known about the strong converse property of quantum network channel. In order to deal with these issues, one can consider the encoding scheme and the decoding scheme as a whole multi-bipartite operation, and impose suitable constraints on this operation, such as non-signalling (NS) operation, positive-partial-transpose-preserving (PPT) operation, and product operation [23, 24].

We consider two basic kinds of quantum network channels: the quantum broadcast channel with one sender and two receivers, and the quantum multi-access channel with two senders and one receiver. The motivation for this paper is two-fold. One is to offer an efficiently computable converse bound for classical communication over quantum network channels and provide insights into the study on the strong converse property for general network channels. The other is to study the performance of non-signalling- and PPT- codes in the network communication. In addition, our results on quantum channels may also shed lights on the study of communication over classical network channels.

The paper is structured as follows. In Section 2, we review some relevant preliminaries and set the notations. In Section 3, after deriving the one-shot communication fidelity of quantum broadcast channel assisted by non-signalling and PPT codes and its classical version, we give an semidefinite-programming strong converse bound on the asymptotic rate region which can be efficiently computed. We also give a hypothesis testing converse bound on the finite-blocklength communication rate. We investigate the non-signalling and PPT-assisted communication fidelity and strong converse bound for the quantum multi-access channel in Section 4, and draw a conclusion in Section 5.

2 Preliminaries

A quantum register $A$ is associated to a Hilbert space $\mathcal{H}_A$ equipped with a standard orthonormal basis $\{|j\}_A$. In this work, we only deal with finite-dimensional spaces, and the dimensions of systems $A, B, C$ are denoted by $d_A, d_B, d_C$ respectively. The linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ are always written with subscripts indicating the systems involved, for example, $X_{A\rightarrow B}$ and $Y_A$. The subscripts would be omitted when it is clear from context.

A quantum operation (or channel) $\mathcal{E}_{A\rightarrow B}$ with input system $A$ and output system $B$ is a completely positive (CP), trace preserving (TP) linear map from the linear operators on $\mathcal{H}_A$ to the linear operators on $B$. Since the subscript of an operator or operation indicates its input and output systems, we can write a product of operators or operations without the tensor symbol, and omit the identity operator $\mathbb{I}$ or identity operation $\text{id}$, which would make no confusion. For example, $X_AY_B \equiv Y_BX_A \equiv X_A \otimes Y_B$, $X_{AB}Y_{BC} \equiv (X_{AB} \otimes \mathbb{I}_C)(\mathbb{J}_A \otimes Y_{BC})$ and $\mathcal{E}_{B\rightarrow C}(X_{AB}) \equiv (\text{id}_A \otimes \mathcal{E}_{B\rightarrow C})X_{AB}$. We also write the partial trace of a multipartite operator by omitting the subscript the partial trace takes on.
for example, $X_B := \text{Tr}_A(X_{AB})$. The Choi-Jamiołkowski matrix (or Choi matrix for short) [25, 26] of a quantum operation $\mathcal{E}_{A \rightarrow B}$ is $J_\mathcal{E} = \sum_{i,j=1}^d |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$ where $\{|i\rangle\}$ is the standard basis of the input space $\mathcal{H}_A$. The Choi matrix can be equivalently written as $J_\mathcal{E} = (\text{id}_{\tilde{A}\rightarrow A} \otimes \mathcal{E}_{A \rightarrow B})\tilde{\psi}/\tilde{A}A$ where $\tilde{\psi}/\tilde{A}A = \sum_{i,j=1}^d |i\rangle\langle j|_{\tilde{A}} \otimes |i\rangle\langle j|_A$ is the unnormalised isotropic maximally entangled state and $\tilde{A}, A$ are isomorphic systems. The output of the channel $\mathcal{E}_{A \rightarrow B}$ with input $\rho_A$ can be recovered from $J_\mathcal{E}$ as $\mathcal{E}_{A \rightarrow B}(\rho_A) = \text{Tr}_A(J_\mathcal{E}^T \rho_A)$, where $T_A$ denotes the partial transpose on $A$. Given an operator $X_{A \rightarrow B} = \sum_{ij} x_{ij} |i\rangle\langle j|_A |i\rangle\langle j|_B$, define $\text{vec}(X) = \sum_{ij} x_{ij} |j\rangle_1 \otimes |i\rangle_2 \in \mathcal{H}_{AB}$. Throughout this paper, $\text{log}$ denotes the binary logarithm.

A positive semidefinite operator $P_{AB}$ is said to be a positive partial transpose (PPT) operator if $P_{AB}^{T_A} \geq 0$. A bipartite operation $Z_{AB \rightarrow A'B'}$ is PPT-preserving [27, 28] if it takes any PPT density operator to another PPT state. A bipartite operation $Z_{AB \rightarrow A'B'}$ is non-signalling from $A$ to $B$ if $\text{Tr}_B(Z_{AB \rightarrow A'B'}) = Z_{B \rightarrow B'} \text{Tr}_A$ for some operation $Z_{B \rightarrow B'}$, and $Z_{AB \rightarrow A'B'}$ is non-signalling from $B$ to $A$ if $\text{Tr}_A(Z_{AB \rightarrow A'B'}) = Z_{A \rightarrow A'} \text{Tr}_B$ for some operation $Z_{A \rightarrow A'}$. One-way non-signalling operations are also referred to as ‘semi-causal’ [29, 30].

A code in our network communication protocol is defined as some tripartite operation $\mathcal{X}$. We say a tripartite operation $\mathcal{X}_{ABC \rightarrow A'B'C'}$ is non-signalling (NS) and positive-partial-transpose-preserving (PPT) if and only if it is NS and PPT with respect to any bipartite cut. The condition is that its Choi matrix $X_{ABC'A'B'C'}$ satisfies [23, 31]

\begin{align*}
\text{CP} & \quad X_{ABCA'B'C'} \geq 0, \\
\text{TP} & \quad X_{ABC} = I_{ABC}, \\
\text{PPT} & \quad X_{TAA'} \geq 0, X_{TBB'} \geq 0, X_{TCC'} \geq 0, \\
A \nleftrightarrow BC & \quad X_{ABCB'C'} = \frac{1_A}{d_A} \otimes X_{BCB'C'}, X_{ABCA'} = \frac{1_B}{d_B} \otimes X_{ABA'}, \\
B \nleftrightarrow AC & \quad X_{ABCA'C'} = \frac{1_B}{d_B} \otimes X_{ACAC'}, X_{ABCB'} = \frac{1_C}{d_C} \otimes X_{BB'}, \\
C \nleftrightarrow AB & \quad X_{ABCA'B'} = \frac{1_C}{d_C} \otimes X_{ABAB'}, X_{ABC'C'} = \frac{1_A}{d_A} \otimes X_{CC'}. \tag{1}
\end{align*}

The unassisted code corresponds to some product tripartite operation $\mathcal{X} = \mathcal{E}_{A \rightarrow A'} D_{1,B \rightarrow B'} D_{2,C \rightarrow C'}$. A code class $\Omega$ is a set of codes satisfying certain properties. The set of NS and PPT tripartite operations, the set of NS tripartite operations, and the set of product operations are written as $\Omega = \text{NSPPT}$, $\Omega = \text{NS}$ and $\Omega = \text{ua}$, respectively.

Semidefinite programming (SDP), as a generalization of linear programming, has been proven to be a very useful tool in the theory of quantum information and computation (see, e.g., [23, 24, 32, 33, 34, 35] for a partial list.) SDP can be solved via interior point methods efficiently in theory as well as in practice. In this paper, we use the CVX software [36] and QETLAB [37] to solve SDPs.

## 3 Classical communication over quantum broadcast channel

Suppose Alice wants to send classical message labeled by $\{1, \ldots, m_1\}$ to Bob, and simultaneously send message labeled by $\{1, \ldots, m_2\}$ to Charlie, using the composite channel $\mathcal{M} = \mathcal{N} \circ \mathcal{X}$, where $\mathcal{X}$ is a tripartite operation as a coding scheme; see Fig. 1. The code $\mathcal{X}$ is chosen within some coding class $\Omega$ in order to make the overall channel $\mathcal{M}$ as close to a classical noiseless channel as possible. Thus the registers $A, B', C'$ can be assumed to be classical [24]. The classical register $A$ indeed consists of two subregisters $A_1$ and $A_2$, storing messages to be sent to Bob and Charlie, respectively.
Figure 1: Classical communication over quantum broadcast channel \( \mathcal{N} \) assisted by a code \( \mathcal{X} \). The tripartite code \( \mathcal{X} \) is designed in order for the whole operation \( \mathcal{M} \) to emulate a noiseless classical channel.

### 3.1 One-shot \( \varepsilon \)-error capacity

We first define several quantities to characterize the capability for communication over quantum broadcast channels.

**Definition 1.** The success probability of \( \mathcal{N}_{A'BC} \) to transmit messages of size \((m_1, m_2)\) assisted by code \( \mathcal{X}_{ABC \rightarrow A'B'C'} \) is defined as

\[
    p_s(\mathcal{N}, \mathcal{X}, m_1, m_2) = \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1, m_2} \text{Tr}(\mathcal{M}(|ij\rangle \langle ij|_A) |ij\rangle \langle ij|_{B'C'}),
\]

where \( \mathcal{M}_{A \rightarrow B'C'} = \mathcal{N} \circ \mathcal{X} \).

Moreover, the \( \Omega \)-assisted optimal success probability of \( \mathcal{N} \) to transmit messages of size \((m_1, m_2)\) is defined as

\[
    f_{\Omega}(\mathcal{N}, m_1, m_2) = \max_{\mathcal{X} \in \Omega} p_s(\mathcal{N}, \mathcal{X}, m_1, m_2).
\]

**Definition 2.** The \( \Omega \)-assisted one-shot \( \varepsilon \)-error classical capacity region of \( \mathcal{N} \) is defined as

\[
    \{(R_1, R_2) : f_{\Omega}(\mathcal{N}, 2^{R_1}, 2^{R_2}) \geq 1 - \varepsilon \}.
\]

The \( \Omega \)-assisted classical capacity region of \( \mathcal{N} \) is defined as

\[
    \{(R_1, R_2) : \lim_{n \to \infty} f_{\Omega}(\mathcal{N}^\otimes n, 2^{nR_1}, 2^{nR_2}) = 1 \}.
\]

We call \((R_1, R_2)\) a strong converse rate pair for \( \mathcal{N} \), if

\[
    \lim_{n \to \infty} f_{\text{us}}(\mathcal{N}^\otimes n, 2^{nR_1}, 2^{nR_2}) = 0.
\]

We show that the NSPPT-assisted optimal success probability of a broadcast channel can be formulated as the following SDP.

**Theorem 3.** The optimal success probability of a channel \( \mathcal{N}_{A' \rightarrow BC} \) to transmit messages of size
\( (m_1, m_2) \) assisted by NSPPT codes is given by

\[
f_{\text{NSPPT}}(N, m_1, m_2) = \max \text{ Tr } \left( J_N^T E_{1,A'BC} \right)
\]

\[
\text{s.t. } E_i, E_i^{T_B}, E_i^{T_C}, E_i^{T_{A'}} \geq 0 \text{ for } i = 1, 2, 3, 4, \text{ (CP, PPT)}
\]

\[
K_{A'BC} = K_{A'} \frac{1}{d_{BC}}, \text{ (BC } \not\leftrightarrow \text{ A)}
\]

\[
\text{Tr } K_{A'} = d_{BC}, \text{ (TP)}
\]

\[
E_i, \pi := \frac{1}{m_1 m_2} \text{ for } i = 1, 2, 3, \text{ (A } \not\leftrightarrow \text{ BC)}
\]

\[
E_{1,A'BC} + (m_2 - 1)E_{2,A'BC} = (E_{1,AB} + (m_2 - 1)E_{2,AB}) \otimes 1_{C}/d_{C};
\]

\[
E_{1,A'BC} + (m_1 - 1)E_{3,A'BC} = (E_{1,AC} + (m_1 - 1)E_{3,AC}) \otimes 1_{B}/d_{B},
\]

where \( K_{A'BC} := E_{1,A'BC} + (m_2 - 1)E_{2,A'BC} + (m_1 - 1)E_{3,A'BC} + (m_1 - 1)(m_2 - 1)E_{4,A'BC}. \)

**Proof.** Denoting \( M_{A \rightarrow BC'} = N_{A \rightarrow BC} \circ X_{ABC' \rightarrow A'BC'}, \)

\( D_{1,AB} = \sum_{i=1}^{m_1} |ii⟩⟨ii| \) and \( D_{2,A'BC} = \sum_{j=1}^{m_2} |jj⟩⟨jj|, \)

the success probability of \( N \) to transmit messages of size \( (m_1, m_2) \) assisted by code \( X \) is written as

\[
p_s(N, X, m_1, m_2) := \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1, m_2} \text{ Tr } (M(|ii⟩⟨ii|A)|ij⟩⟨ij|B'C')
\]

\[
= \frac{1}{m_1 m_2} \text{ Tr } (J_M, AB'C'(D_{1,A'B'} \otimes D_{2,A'BC'}))
\]

Since \( J_M = \text{ Tr } A'B'(J_N^{T} X_{ABC} A'B'C') \), we have

\[
f_{\text{NSPPT}}(N, m_1, m_2) = \frac{1}{m_1 m_2} \max_X \text{ Tr } \left( J_N^{T} X_{ABC} X (D_{1,A'B'} \otimes D_{2,A'BC'}) \right)
\]

\[
= \frac{1}{m_1 m_2} \max_X \text{ Tr } \left( J_N^{T} X_{ABC} A'B' (D_{1,A'B'} \otimes D_{2,A'BC'}) \right),
\]

where \( A' := \frac{1}{m_1 m_2} \sum_{\pi_1 \in S_{m_1}, \pi_2 \in S_{m_2}} (W_{\pi_1, A'B'} \otimes W_{\pi_2, A'BC'}) X (W_{\pi_1, A'B'} \otimes W_{\pi_2, A'BC'})^\dagger. \) Here \( S_{m_1}, S_{m_2} \)

are symmetric groups, and \( W_{\pi_1} \) and \( W_{\pi_2} \) are the operators permuting the basis with respect to \( \pi_1 \) and \( \pi_2 \) respectively. If \( X \) is a feasible solution to the optimization problem, so is \( A' \). Since \( X \geq 0 \) and the \( A, B', C' \) are classical, \( \tilde{X} \) can be written as

\[
\tilde{X} = D_{1,A'B'} \otimes D_{2,A'BC} \otimes E_{1,A'BC} + D_{1,A'B'} \otimes (1 - D_{2,A'BC}) \otimes E_{2,A'BC}
\]

\[
+ (1 - D_{1,A'B'}) \otimes D_{2,A'BC} \otimes E_{3,A'BC} + (1 - D_{1,A'B'}) \otimes (1 - D_{2,A'BC}) \otimes E_{4,A'BC}
\]

for some positive semidefinite operators \( E_i. \) The CP and PPT constraints are equivalent to

\[
E_i, E_i^{T_B}, E_i^{T_C}, E_i^{T_{A'}} \geq 0,
\]

for each \( i. \) Using Eq. (11) leads to a simplification

\[
f_{\text{NSPPT}}(N, m_1, m_2) = \max_{E_1} \text{ Tr } \left( J_N^T E_{1,A'BC} \right).
\]

Denoting

\[
K_{A'BC} := E_{1,A'BC} + (m_2 - 1)E_{2,A'BC} + (m_1 - 1)E_{3,A'BC} + (m_1 - 1)(m_2 - 1)E_{4,A'BC},
\]
the TP constraint $\hat{X}_{ABC} = 1_{ABC}$ is equivalent to

$$K_{BC} = 1_{BC}. \quad (15)$$

The NS constraint $BC \not\rightarrow A$ is equivalent to

$$K_{A'BC} = K_{A'} \frac{1_{BC}}{d_{BC}}. \quad (16)$$

Using this constraint, Eq. (15) becomes

$$\text{Tr} K_{A'} = d_{BC}. \quad (17)$$

The NS constraint $A \not\rightarrow BC$ is equivalent to $D_1 D_2 E_{1,BC} + D_1 (1 - D_2) E_{2,BC} + (1 - D_1) D_2 E_{3,BC} + (1 - D_1)(1 - D_2) E_{4,BC} = \frac{1}{d^4} 1_{ABCB} = \frac{1}{m_1 m_2} 1_{ABCB}. \quad (18)$$

Since $K_{BC} = 1_{BC}$, that $E_{4,BC} = \frac{1_{BC}}{m_1 m_2}$ follows from that $E_{i,BC} = \frac{1_{BC}}{m_1 m_2}$, $i = 1, 2, 3$. Since $\hat{X}_{ABC} = \frac{1}{m_1^2} 1_{ABC}$ and $\hat{X}_{ABCB} = \frac{1}{m_1} 1_{ABCB}$, the NS constraint $AB \not\rightarrow C$ and $AC \not\rightarrow B$ are satisfied.

The constraint $C \not\rightarrow AB$ is

$$D_1 E_{1,A'BC} + (m_2 - 1) D_1 E_{2,A'BC} + (1 - D_1) E_{3,A'BC} + (m_2 - 1)(1 - D_1) E_{4,A'BC}$$

$$= (D_1 E_{1,A'B} + (m_2 - 1) D_1 E_{2,A'B} + (1 - D_1) E_{3,A'B} + (m_2 - 1)(1 - D_1) E_{4,A'B}) \otimes \frac{1_C}{d_C}, \quad (19)$$

which is equivalent to

$$E_{1,A'BC} + (m_2 - 1) E_{2,A'BC} = (E_{1,A'B} + (m_2 - 1) E_{2,A'B}) \otimes \frac{1_C}{d_C}, \quad (20)$$

$$E_{3,A'BC} + (m_2 - 1) E_{4,A'BC} = (E_{3,A'B} + (m_2 - 1) E_{4,A'B}) \otimes \frac{1_C}{d_C}. \quad (21)$$

Similarly, the constraint $B \not\rightarrow AC$ is equivalent to

$$E_{1,A'BC} + (m_2 - 1) E_{3,A'BC} = (E_{1,A'C} + (m_2 - 1) E_{3,A'C}) \otimes \frac{1_B}{d_B}. \quad (22)$$

$$E_{2,A'BC} + (m_2 - 1) E_{4,A'BC} = (E_{2,A'C} + (m_2 - 1) E_{4,A'C}) \otimes \frac{1_B}{d_B}. \quad (23)$$

Notice that Eq. (21) is also implied by Eqs. (16) and (20), and that Eq. (23) is also implied by Eqs. (16) and (22).

Putting together the above constraints, we obtain the desired SDP characterization. \hfill \Box

### 3.2 Reduction to classical case

For a c-qq channel $N_{A' \rightarrow BC}$ with Choi matrix $J_N = \sum_{ij} |ij\rangle\langle ij|_{A'} \otimes \rho_{i,B} \otimes \sigma_{j,C}$, its optimal success probability can be simplified as follows.

Let $E_k = \sum_{ij} |ij\rangle\langle ij|_{A'} \otimes F_{k,ij,BC}$, then $K_{A'BC} = \sum_{ij} |ij\rangle\langle ij| \otimes (F_{1,ij} + (m_2 - 1) F_{2,ij} + (m_1 - 1) F_{3,ij} + (m_1 - 1)(m_2 - 1) F_{4,ij}) = \sum_{ij} |ij\rangle\langle ij| \otimes T_{ij}$. Thus $K_{A'} = \sum_{ij} |ij\rangle\langle ij| \otimes \text{Tr}(T_{ij})$. The constraint $BC \not\rightarrow A$ becomes $T_{ij} = \text{Tr}(T_{ij}) \frac{1_{BC}}{d_{BC}}$ for each $i, j$. The TP constraint becomes $\sum_{ij} \text{Tr}(T_{ij}) = d_{BC}$.

The constraint $A \not\rightarrow BC$ becomes $\sum_{ij} F_{k,ij} = \frac{1_{BC}}{m_1 m_2}$ for each $k$. The constraint $C \not\rightarrow AB$ becomes $F_{1,ij,BC} + (m_2 - 1) F_{2,ij,BC} = (F_{1,ij,B} + (m_2 - 1) F_{2,ij,B}) \otimes \frac{1_C}{d_C}$ for each $i, j$. The constraint $B \not\rightarrow AC$ becomes $F_{1,ij,BC} + (m_2 - 1) F_{3,ij,BC} = (F_{1,ij,C} + (m_1 - 1) F_{3,ij,C}) \otimes \frac{1_B}{d_B}$ for each $i, j$. 

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**Proposition 4.** Given a $c$-$q$ channel $\mathcal{N}_{A'\rightarrow BC}$ which outputs $\rho_{i,B}$ and $\sigma_{j,C}$ upon input $i,j$. The optimal success probability of the channel $\mathcal{N}_{A'\rightarrow BC}$ to transmit messages of size $(m_1,m_2)$ assisted by NSPPT codes is given by

$$f_{\text{NSPPT}}(\mathcal{N}, m_1, m_2) = \max \sum_{ij} \text{Tr}(F_{ij}(\rho_i^T \otimes \sigma_j^T))$$

s.t. $F_{k,ij}, F_{k,ij}^T \geq 0 \forall k, i,j$, (CP, PPT)

$$T_{ij} = \text{Tr}(T_{ij}) \mathbb{1}_{BC}/d_{BC}, \quad (BC \not\rightarrow A)$$

$$\sum_{ij} \text{Tr}(T_{ij}) = d_{BC}, \quad (TP)$$

$$\sum_{ij} F_{k,ij} = \mathbb{1}_{BC}/m_1m_2 \forall k, \quad (A \not\leftrightarrow BC)$$

$$F_{1,ij,BC} + (m_2 - 1)F_{2,ij,BC} = (F_{1,ij,B} + (m_2 - 1)F_{2,ij,B}) \otimes \mathbb{1}_C/d_C \forall i, j, \quad (C \not\leftrightarrow AB)$$

$$F_{1,ij,BC} + (m_1 - 1)F_{3,ij,BC} = (F_{1,ij,C} + (m_1 - 1)F_{3,ij,C}) \otimes \mathbb{1}_B/d_B \forall i, j, \quad (B \not\leftrightarrow AC)$$

where $T_{ij} = F_{1,ij} + (m_2 - 1)F_{2,ij} + (m_1 - 1)F_{3,ij} + (m_1 - 1)(m_2 - 1)F_{4,ij}$.

Furthermore, we can reduce to the case of classical channel. When $\rho_i, \sigma_j$ are all classical states, if $\{F_{k,ij}\}$ is feasible solution to the above SDP, so is $\{\Delta(F_{k,ij})\}$, where $\Delta$ is completely dephasing channel. Thus it suffices to consider the case that $F_{k,ij}$ are classical for a classical channel $\mathcal{N}$. Suppose a classical channel output $y = i', z = j'$ upon input $x = (i,j)$ with probability $p(i'j'|ij)$, and its Choi matrix is $\sum_{ij} p(i'j'|ij) \otimes p(i'j'|ij)\otimes |ij\rangle \langle ij|$. Let $E_k = \sum_{ij} p(i'j'|ij)\otimes \sigma_k(i'j'|ij)\otimes |ij\rangle \langle ij|$. In this case, the SDP reduces to a linear programming and the PPT constraints are implicitly satisfied.

**Theorem 5.** The optimal success probability of a classical channel $p(i'j'|ij)$ to transmit messages of size $(m_1,m_2)$ assisted by NS codes is given by

$$f_{\text{NS}}(\mathcal{N}, m_1, m_2) = \max \sum_{ij} p(i'j'|ij)\eta_1(i'j'ij)$$

s.t. $\eta_k(i'j'ij) \geq 0, \forall k,i', j', i, j,$

$$\mu(i'j'ij) = \eta_1(i'j'ij) + (m_2 - 1)\eta_2(i'j'ij) + (m_1 - 1)(m_2 - 1)\eta_4(i'j'ij),$$

$$\mu(ij) = d_{BC}\mu(i'j'ij) \forall i', j', (BC \not\leftrightarrow A)$$

$$\sum_{ij} \mu(ij) = d_{BC},$$

$$m_1m_2\mu_k(i'j') = 1 \forall k, (A \not\leftrightarrow BC)$$

$$d_C(\eta_1(i'j'ij) + (m_2 - 1)\eta_2(i'j'ij)) = \eta_1(i'j'ij) + (m_2 - 1)\eta_2(i'j'ij), \forall i', j', i, j, \quad (C \not\leftrightarrow AB)$$

$$d_B(\eta_3(i'j'ij) + (m_1 - 1)\eta_3(i'j'ij)) = \eta_3(i'j'ij) + (m_1 - 1)\eta_3(i'j'ij), \forall i', j', i, j, \quad (B \not\leftrightarrow AC)$$

where $\mu(ij) := \sum_{i'j'} \mu(i'j'ij)$, $\eta_k(i'j'ij) := \sum_{ij} \eta_k(i'j'ij)$.

Theorem 5 gives a linear-programming converse bound on the optimal success probability of unassisted communication over classical channel, and can be viewed as an extension of the works of [38, 39]. For the case of point-to-point classical communication, Polyanskiy, Poor and Verdú prove a general
converse bound on the one-shot $\varepsilon$-error capacity [38], and then Matthews gives a linear programming characterization for this converse bound via non-signalling code [39]. Here we present a linear programming converse bound for classical broadcast communication. It remains for future work to compare this bound with some existing results, e.g., [40].

3.3 Converse bound on classical capacity region based on NSPPT codes

Before introducing the strong converse for the broadcast capacity, we first present an SDP upper bound on the single-shot optimal success probability of NSPPT codes.

**Proposition 6.** For any quantum broadcast channel $\mathcal{N}_{A' \rightarrow BC}$ and given $m_1, m_2$,

$$f_{\text{NSPPT}}(N, m_1, m_2) \leq g(N, m_1, m_2),$$

where

$$g(N, m_1, m_2) := \min \text{ Tr } Q_{BC}$$

s.t. $m_1 m_2 \mathbb{1}_{A'} Q_{BC} \geq V_{A'BC} \geq -m_1 m_2 \mathbb{1}_{A'} Q_{BC},$

$$V_{A'BC} \geq Y_{A'BC} \geq -V_{A'BC},$$

$$Y_{A'BC} \geq Z_{A'BC} \geq -Y_{A'BC},$$

$$Z_{A'BC} \geq J_{A'BC} \geq -Z_{A'BC}.$$  

Furthermore, $g$ is submultiplicative in the sense that

$$g(N \otimes N', m_1 m'_1, m_2 m'_2) \leq g(N, m_1, m_2) g(N', m'_1, m'_2)$$

Consequently, $f_{\text{NSPPT}}(N^\otimes n, m_1^n, m_2^n) \leq g(N, m_1, m_2)^n$.

**Proof.** It follows from Eq. (7) that

$$f_{\text{NSPPT}}(N, m_1, m_2) \leq \max \text{ Tr } \left( J_N^T E_{1,A'BC} \right)$$

s.t. $E_1, E_1^{T A'}, E_1^{T B}, E_1^{T C} \geq 0,$

$$E_{1,BC} = \frac{\mathbb{1}_{BC}}{m_1 m_2}.$$  

The dual of SDP (29) is

$$\min \text{ Tr } Q_{BC}$$

s.t. $J_N^T + P_1^{T A'} + P_2^{T B} + P_3^{T C} - m_1 m_2 \mathbb{1}_{A'} Q_{BC} \leq 0$

$$P_i \geq 0.$$  

Here the Hermicity of $Q_{BC}$ is implicitly implied. Noticing that $\mathbb{1}_{A'BC}/d_{A'BC} > 0$ is a feasible solution to SDP (29), the optimal solutions to SDPs (29) and (30) coincide due to the Slater’s theorem.

Denoting $V := J_N^T + P_1^{T A'} + P_2^{T B} + P_3^{T C}$, $Y := J_N^T + P_1^{T A'} + P_2^{T B}$ and $Z := J_N^T + P_1^{T A'}$, it follows that the first constraint in (30) is equivalent to $m_1 m_2 \mathbb{1}_{A'} Q_{BC} \geq V$, that $P_1 \geq 0$ iff $V_{T C} \geq Y_{T C}$, that $P_2 \geq 0$ iff $Y_{T B} \geq Z_{T B}$, and that $P_1 \geq 0$ iff $Z_{T A'} \geq J_{N,BC}$. SDP (30) can be rewritten as

$$\min \text{ Tr } Q_{BC}$$

s.t. $m_1 m_2 \mathbb{1}_{A'} Q_{BC} \geq V_{A'BC},$

$$V_{A'BC} \geq Y_{A'BC},$$

$$Y_{A'BC} \geq Z_{A'BC},$$

$$Z_{A'BC} \geq J_{N,BC}.$$
By adding new constraints, the above SDP is no larger than $g(N; m_1, m_2)$.

It can be verified that $g$ is sub-multiplicative, since if $(Q, V, Y, Z)$ and $(Q', V', Y', Z')$ are feasible solutions to $g(N; m_1, m_2)$ and $g(N', m'_1, m'_2)$ respectively, then $(Q \otimes Q', V \otimes V', Y \otimes Y', Z \otimes Z')$ is feasible to $g(N \otimes N', m_1 m'_1, m_2 m'_2)$.

\begin{proof}
\end{proof}

\begin{theorem}
For a quantum broadcast channel $\mathcal{N}_{A' \rightarrow BC}$, if $R_1 + R_2 > C_g(N)$ then $(R_1, R_2)$ is a strong converse rate pair. Here

\[
C_g(N) := \log \min \ TrQ_{BC}
\]

\[\text{s.t. } \mathbb{1}_{A'}Q_{BC} \geq V'_{AB} \geq -\mathbb{1}_{A'}Q_{BC}
\]

\[V'_{AB} \geq Y'_{AB} \geq -V'_{AB}
\]

\[Y'_{AB} \geq Z'_{AB} \geq -Y'_{AB}
\]

\[Z'_{AB} \geq J'_{BC} \geq -Z'_{AB}.
\]

\end{theorem}

\begin{proof}
Actually $C_g(N)$ is given by $C_g(N) = \log \min \{m_1 m_2 : g(N; m_1, m_2) \leq 1\}$. Notice that $g$ is a strictly decreasing function of $m_1 m_2$ in the sense that $g(N; m_1, m_2) > g(N; m'_1, m'_2)$ for $m_1 m_2 < m'_1 m'_2$. Suppose $\log m_1 + \log m_2 > C_g(N)$, then $g(N; m_1, m_2) < 1$, and $g(N; m_1, m_2)^n \rightarrow 0$ as $n \rightarrow \infty$. It follows that $f_{\text{ua}}(N^\otimes n, m_1^n, m_2^n) \leq f_{\text{NSPPT}}(N^\otimes n, m_1^n, m_2^n) \leq g(N^\otimes n, m_1^n, m_2^n) \leq g(N, m_1, m_2)^n \rightarrow 0$. By definition, $(\log m_1, \log m_2)$ is a strong converse rate pair.

The authors of Ref. [24] give a strong converse bound for the point-to-point classical communication, that is, $C(N) \leq C_\beta(N)$, where $C(N)$ is asymptotic unassisted classical capacity of $\mathcal{N}$. When $\mathcal{N}$ is viewed as a point-to-point channel from $A'$ to $BC$, $C_\beta(N)$ is given by

\[
C_\beta(N) = \log \min \ Tr(S_{BC})
\]

\[\text{s.t. } \mathbb{1}_{A'}S_{BC} \geq R'_{AB} \geq -\mathbb{1}_{A'}S_{BC}
\]

\[R_{AB} \geq J'_{BC} \geq -R_{AB}.
\]

\end{proof}

\begin{proposition}
For any quantum broadcast channel $\mathcal{N}_{A' \rightarrow BC}$,

\[
C_g(N) \leq C_\beta(N).
\]

\end{proposition}

\begin{proof}
Suppose $(S^*, R^*)$ is an optimal solution to $C_\beta(N)$, it suffices to check that $Tr S^*$ is achievable by (32). Choose $V = Y = Z = (R^*)^T$ and $Q = (S^*)^T$. Since $\mathbb{1}_{A'}S^*_{BC} \geq (R^*)^T_{BC} \geq -\mathbb{1}_{A'}S^*_{BC}$, one has $\mathbb{1}_{A'}(S^*_{BC})^T \geq (R^*)^T \geq -\mathbb{1}_{A'}(S^*_{BC})^T$, hence $\mathbb{1}_{A'}Q_{BC} \geq V_{AB} \geq -\mathbb{1}_{A'}Q_{BC}$. The second and third constraints in (32) become trivial. It follows from $R^*_{AB} \geq J'_{BC} \geq -R^*_{AB}$ that $Z^T_{AB} \geq J_{BC} \geq -Z^T_{AB}$.

In some cases, our bound $C_g$ is strictly smaller than the bound $C_\beta$. For example, consider $\mathcal{N}_r(p) = E_0 p E_0^\dagger + E_1 p E_1^\dagger$ with $E_0 = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + \sqrt{\tau}|3\rangle\langle 3|$, and $E_1 = \sqrt{1-\tau}|0\rangle\langle 3|$. $0 \leq r \leq 1$. Fig. 2 shows that $C_g(\mathcal{N}_r) < C_\beta(\mathcal{N}_r)$ for this class of channels.

We now randomly choose quantum broadcast channel $\mathcal{N}_{A' \rightarrow BC}$ with $d_{A'} = 4, d_B = d_C = 2$, and compare our converse bound $C_g$ with the point-to-point bound in $C_\beta$, in Fig. 3. The numerical results suggest that $C_g < C_\beta$ holds for generic channel $\mathcal{N}$.
3.4 Converse bound on one-shot communication capacity based on hypothesis testing

In this subsection, following the work [41], we derive a converse bound on one-shot communication rate based on quantum hypothesis testing, and then obtain a converse bound on the asymptotic unassisted classical capacity.

Define

\[ Q_{BC} := \{ F \in \text{CP}(A' \rightarrow BC) : \exists \sigma \in \mathcal{S}(BC) \text{ s.t. } F(\rho_{A'}) \leq \sigma, \forall \rho \in \mathcal{S}(A') \}, \]
\[ Q_B := \{ F \in \text{CP}(A' \rightarrow BC) : \exists \sigma \in \mathcal{S}(B) \text{ s.t. } \text{Tr}_C(F(\rho_{A'})) \leq \sigma, \forall \rho \in \mathcal{S}(A') \}, \]
\[ Q_C := \{ F \in \text{CP}(A' \rightarrow BC) : \exists \sigma \in \mathcal{S}(C) \text{ s.t. } \text{Tr}_B(F(\rho_{A'})) \leq \sigma, \forall \rho \in \mathcal{S}(A') \}, \]

where \( \text{CP}(A' \rightarrow BC) \) denotes the set of completely positive linear maps from \( A' \) to \( BC \), and \( \mathcal{S} \) denotes the set of density operators.

The quantum hypothesis testing divergence [42] is defined as

\[ D_1^e(\rho||\sigma) := - \log \min \{ \text{Tr}(\sigma T) : \text{Tr}(\rho T) \geq 1 - \varepsilon, 0 \leq T \leq 1 \}. \]

Figure 2: The broadcast capacity bound \( C_g \) is strictly smaller than the point-to-point capacity bound \( C_\beta \) for channels \( \mathcal{N}_r \).

Figure 3: Plot of left-hand side of (34) vs. right-hand side for 1000 randomly generated channels \( \mathcal{N} \). We see the inequality (34) is strict for almost all points.
By hypothesis we have \( \phi \), where the maximum is over all pure state \( \epsilon \).

Take the tester \( T \) \{ \( F \) \} such that \( \text{Tr}(\rho_{\epsilon,\epsilon}) \geq 1 \). We have explored the classical communication over the quantum broadcast channel. Using similar approaches we can study the quantum multi-access channel; see Fig. 4.

\begin{equation}
R_1 \leq \min_{\mathcal{F} \in Q_B} \max_{\phi} D^\epsilon_h(\mathcal{N}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \| \mathcal{F}_{A'\rightarrow BC}(\hat{\phi}_{\hat{A}A'}) \),
\end{equation}

\begin{equation}
R_2 \leq \min_{\mathcal{F} \in Q_C} \max_{\phi} D^\epsilon_h(\mathcal{N}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \| \mathcal{F}_{A'\rightarrow BC}(\hat{\phi}_{\hat{A}A'}) \),
\end{equation}

\begin{equation}
R_1 + R_2 \leq \min_{\mathcal{F} \in Q_{BC}} \max_{\phi} D^\epsilon_h(\mathcal{N}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \| \mathcal{F}_{A'\rightarrow BC}(\hat{\phi}_{\hat{A}A'}) \),
\end{equation}

where the maximum is over all pure state \( \phi_{\hat{A}A'} \), and \( \hat{A} \) and \( A' \) are isomorphic systems.

**Proof.** By hypothesis we have \( f_{\text{ua}}(\mathcal{N}, m_1, m_2) \geq 1 - \varepsilon \) for \( m_1 = 2^{R_1} \) and \( m_2 = 2^{R_2} \).

In order to show Eq. (39), we need to show that for all \( \mathcal{F} \in Q_B \), there exists \( \phi_{\hat{A}A'} \) and 0 \( \leq T_{\hat{A}BC} \leq 1 \), such that \( \text{Tr}(\mathcal{N}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \geq T_{\hat{A}BC} \geq 1 - \varepsilon \) and \( \text{Tr}(\mathcal{F}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \leq \frac{1}{m_1} \).

Since \( f_{\text{ua}}(\mathcal{N}, m_1, m_2) \geq 1 - \varepsilon \), there is a choice of coding scheme such that \( \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1 m_2} \text{Tr}(\mathcal{N}(\rho_{ij,A'})(E_i,B \otimes F_{j,C})) \geq 1 - \varepsilon \). Here \{\( \rho_{ij,A'} \)\}_{i,j=1}^{m_1 m_2} is input state set of \( \mathcal{N} \) as encoding scheme, and two POVMs \{\( E_i,B \)\}_{i=1}^{m_1} \{\( F_{j,C} \)\}_{j=1}^{m_2} serve as decoding scheme. Notice that

\[ 1 - \varepsilon \leq \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1 m_2} \text{Tr}(\mathcal{N}_{A'\rightarrow BC}(\rho_{ij,A'})(E_i,B \otimes F_{j,C})) \]

\[ = \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1 m_2} \text{Tr}(J_{N_A'BC}(\rho_{ij,A'} \otimes E_i,B \otimes F_{j,C})) \]

\[ = \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1 m_2} \text{Tr}(N_{A'\rightarrow BC}(\phi_{\hat{A}A'})(\hat{\rho}_{\hat{A}}^{-1/2}(\rho_{ij,A'} \otimes E_i,B \otimes F_{j,C})(\hat{\rho}_{\hat{A}}^{-1/2})), \]

where \( \hat{\rho}_{\hat{A}} = \frac{1}{m_1 m_2} \sum_{i,j} \rho_{ij,A} \) with \( \rho_{ij,A} = \text{id}_{A'\rightarrow \hat{A}}(\rho_{ij,A'}) \), and \( \phi_{\hat{A}A'} = |\phi\rangle\langle \phi | \) with \( |\phi\rangle = \text{vec}(\text{id}_{\hat{A}A'}\hat{\rho}_{\hat{A}}^{-1/2}) \). Take the tester \( T = (\hat{\rho}_{\hat{A}}^{-1/2}) \sum_{i,j=1}^{m_1 m_2} (\rho_{ij,A'} \otimes E_i,B \otimes F_{j,C})(\hat{\rho}_{\hat{A}}^{-1/2} \), and it thus holds that \( 0 \leq T \leq 1 \) and \( \text{Tr}(\mathcal{N}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \geq 1 - \varepsilon \). Now the Eq. (39) follows.

For any \( \mathcal{F} \in Q_B \),

\[ \text{Tr}(\mathcal{F}_{A'\rightarrow BC}(\phi_{\hat{A}A'})) \geq 1 - \varepsilon \]

Similarly, for \( \mathcal{F} \in Q_C \), \( \text{Tr}(\mathcal{F}(\phi)T) \leq \frac{1}{m_2} \), and for \( \mathcal{F} \in Q_{BC} \), \( \text{Tr}(\mathcal{F}(\phi)T) \leq \frac{1}{m_1 m_2} \). Therefore Eqs. (39), (40) and (41) hold for any \( m_1, m_2 \) satisfying \( f_{\text{ua}}(\mathcal{N}, m_1, m_2) \geq 1 - \varepsilon \).

---

### 4 Classical communication over quantum multi-access channel

We have explored the classical communication over the quantum broadcast channel. Using similar approaches we can study the quantum multi-access channel; see Fig. 4. We now derive a strong converse rate region of quantum multi-access channel \( \mathcal{N}_{A'B'\rightarrow C} \).

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Similar to Def. 1, the success probability of $\mathcal{N}_{A'B'\to C}$ to transmit messages of size $(m_1, m_2)$ assisted by code $\mathcal{X}_{ABC\to A'B'C'}$ is defined as

$$p_s(\mathcal{N}, \mathcal{X}, m_1, m_2) = \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1, m_2} \text{Tr}(\mathcal{M}(|ij\rangle\langle ij|_AB)|ij\rangle\langle ij|_{C'})$$

where $\mathcal{M}_{AB\to C'} = \mathcal{N} \circ \mathcal{X}$. The $\Omega$-assisted optimal success probability is similarly defined as $f_\Omega(\mathcal{N}, m_1, m_2) = \max_{\mathcal{X} \in \Omega} p_s(\mathcal{N}, \mathcal{X}, m_1, m_2)$. $(R_1, R_2)$ is called a strong converse rate pair for $\mathcal{N}$ if

$$\lim_{n \to \infty} f_{\text{NSPPT}}(\mathcal{N}^n, 2^{nR_1}, 2^{nR_2}) = 0.$$ (45)

![Figure 4: Classical communication over quantum multi-access channel $\mathcal{N}$ assisted by general code $\mathcal{X}$.

We give the following characterization with proof put in Appendix A.

**Theorem 10.** The optimal success probability of quantum multi-access channel $\mathcal{N}_{A'B'\to C}$ to transmit messages of size $(m_1, m_2)$ assisted by non-signalling and PPT codes is given by

$$f_{\text{NSPPT}}(\mathcal{N}, m_1, m_2) = \max \text{ Tr}(J_N^{T_{A'B'C'}}E_{1,A'B'C})$$

s.t. $E_i, E_i^{T_{A'B'C}}, E_i^{T_{B'C}}, E_i^{T_C} \geq 0$,

$$\text{Tr} L_{A'B'} = d_C, \quad (\text{TP})$$

$$L_{A'B'C} = \frac{1}{d_C} L_{A'B'}, \quad (C \nrightarrow AB)$$

$$E_{i,C} = \frac{1}{m_1 m_2}, \quad (AB \nrightarrow C)$$

$$E_{1,A'C} = E_{2,A'C}, E_{3,A'C} = E_{4,A'C}, \quad (B \nrightarrow AC)$$

$$E_{1,B'C} = E_{3,B'C}, E_{2,B'C} = E_{4,B'C}, \quad (A \nrightarrow BC)$$

where $L_{A'B'C} := E_{1,A'B'C} + (m_2 - 1) E_{2,A'B'C} + (m_1 - 1) E_{3,A'B'C} + (m_1 - 1) (m_2 - 1) E_{4,A'B'C}$.

Similar to Prop. 6, we have a sub-multiplicative converse bound for quantum multi-access channels as follows. See Appendix A for the proof.

**Proposition 11.** For any quantum multi-access channel $\mathcal{N}_{A'B'\to C}$ and given $m_1, m_2$,

$$f_{\text{NSPPT}}(\mathcal{N}, m_1, m_2) \leq h(\mathcal{N}, m_1, m_2),$$

where

$$h(\mathcal{N}, m_1, m_2) := \min \text{ Tr} Q_C$$

s.t. $m_1 m_2 I_{A'B'C} \geq V_{A'B'C} \geq -m_1 m_2 I_{A'B'C}$,

$$V_{A'B'C} \geq Y_{A'B'C} \geq -V_{A'B'C},$$

$$Y_{A'B'C} \geq Z_{A'B'C} \geq -Y_{A'B'C},$$

$$Z_{A'B'C} \geq J_{N_{A'B'}} \geq -Z_{A'B'C}.$$ (48)
Furthermore, \( h \) is submultiplicative in the sense that \( h(N \otimes N', m_1 m_1', m_2 m_2') \leq h(N,m_1,m_2)h(N',m_1',m_2'). \) Consequently, \( f_{\text{NSPPT}}(N^{\otimes n}, m_1^n, m_2^n) \leq h(N,m_1,m_2)^n. \)

Similar to Theorem 7, we have

**Theorem 12.** For a quantum multi-access channel \( N_{A'B'\to C} \), if \( R_1 + R_2 > C_h(N) \), then \( (R_1,R_2) \) is a strong converse rate pair. Here

\[
C_h(N) := \log \min \text{ Tr } Q_C
\]

s.t. \( \mathbb{1}_{A'B'}Q_C \geq V_{A'B'C} \geq -\mathbb{1}_{A'B'}Q_C, \)
\[
Y_{A'B'C}^T \geq Z_{A'B'C}^T \geq -Y_{A'B'C}^T, \]
\[
Z_{A'B'C}^T \geq J_{N}^T \geq -Z_{A'B'C}^T. \quad (49)
\]

**Proof.** Indeed \( C_h(N) \) is given by \( C_h(N) = \log \min \{ m_1 m_2 : h(N,m_1,m_2) \leq 1 \} \). Notice that \( h \) is a strictly decreasing function of \( m_1 m_2 \). Suppose \( \log(m_1 + m_2) > C_h(N) \), then \( h(N,m_1,m_2) < 1 \), and \( h(N,m_1,m_2)^n \to 0 \) as \( n \to \infty \). It follows that

\[
f_{\text{ua}}(N^{\otimes n}, m_1^n, m_2^n) \leq f_{\text{NSPPT}}(N^{\otimes n}, m_1^n, m_2^n) \leq h(N^{\otimes n}, m_1^n, m_2^n) \leq h(N,m_1,m_2)^n \to 0. \quad (50)
\]

Then \( \log(m_1, m_2) \), by definition, is a strong converse rate pair of multi-access channel \( N \). \( \Box \)

We find that \( C_h \) is a better bound than \( C_\beta \) for quantum multi-access channels. In this case \( C_\beta \) is written as

\[
C_\beta(N) = \log \min \text{ Tr } (S_C)
\]

s.t. \( \mathbb{1}_{A'B'}S_C \geq R_{A'B'C}^T \geq -\mathbb{1}_{A'B'}S_C \)
\[
R_{A'B'C} \geq J_{N}^T \geq -R_{A'B'C}. \quad (51)
\]

**Proposition 13.** For any quantum multi-access channel \( N_{AB\to C} \),

\[
C_h(N) \leq C_\beta(N). \quad (52)
\]

In particular, this inequality can be strict for some channels.

**Proof.** Similar to the proof of Proposition 8, we take \( V = Y = Z = R_{A'B'C}^T \) and \( Q = S_T \), provided \( (S_C^*, R_{A'B'C}^*) \) is an optimal solution to SDP (51). So the optimal value \( \text{ Tr } (S_C^*) \) can be achieved by the SDP (49) of \( C_h \). Thus the inequality holds.

Considering the channel \( N_t \) defined in Prop. 8, Fig. 5 shows that \( C_h(N_t) < C_\beta(N_t) \) holds for this class of channels. The figure looks quite similar to Fig 2, but the values are indeed different. In fact numerical results suggest that this inequality is also strict for generic channels. \( \Box \)

5 **Discussions**

In summary, we have characterized the one-shot optimal average success probability of NS and PPT codes in classical communication over a given quantum network channel, in the form of SDP. By invoking the property of the SDPs, we have established strong converse rates for general quantum
broadcast and multi-access channels. We also have obtained converse bounds for the one-shot classical communication over network channels.

Although we did not deal with the channels with more than two senders or receivers, it can be expected to be a simple extension with more technical involvement. The approach used in our work can apply to the study of the quantum capacity of quantum channels and multi-partite entanglement distillation. It would be interesting to study further the property of the hypothesis testing divergence between a channel and a certain class of channels, which may help to explore more properties of the broadcast channel capacity region.

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A Proofs of Theorem 10 and and Proposition 11

Proof of Theorem 10. The success probability of code \( \mathcal{X}_{ABC \rightarrow A'B'C'} \) to transmit messages of size \((m_1, m_2)\) over channel \( \mathcal{N}_{A'B' \rightarrow C} \) is

\[
p_s(\mathcal{N}, \mathcal{X}, m_1, m_2) = \frac{1}{m_1 m_2} \sum_{i,j=1}^{m_1, m_2} \text{Tr}(|ij\rangle\langle ij| (\mathcal{N} \circ \mathcal{X}) (|ij\rangle\langle ij|))
\]

\[
= \frac{1}{m_1 m_2} \text{Tr}(J_{N,ABC'}^T X_{ABC'} A'B'C' (D_{1,AC'} \otimes D_{2,BC}'))
\]

\[
= \frac{1}{m_1 m_2} \text{Tr}(J_{N,ABC'}^T \tilde{X}_{ABC'} A'B'C' (D_{1,AC'} \otimes D_{2,BC}'))
\]

where \( D_{1,AC'} = \sum_{k=1}^{m_1} |kk\rangle\langle kk| \) and \( D_{2,BC} = \sum_{k=1}^{m_2} |kk\rangle\langle kk| \) and \( \tilde{X} = t_{AC'} t_{BC}(X) \).

The optimal success probability of \( \mathcal{N} \) to transmit messages of size \((m_1, m_2)\) assisted by \( \Omega \)-class codes is defined as

\[
f_\Omega(\mathcal{N}, m_1, m_2) = \max_{\mathcal{X} \in \Omega} p_s(\mathcal{N}, \mathcal{X}, m_1, m_2).
\]

Since the registers \( A, B, C' \) are classical, we have

\[
\tilde{X} = D_{1,AC'} D_{2,BC} E_{A'B'C} + D_{1,AC'} (1 - D_{2,BC} E_{A'B'C})
\]

\[
+ (1 - D_{1,AC'}) D_{2,BC} E_{A'B'C} + (1 - D_{1,AC'}) (1 - D_{2,BC}) E_{A'B'C}.
\]

Now let us consider the NSPPT codes. \( \tilde{X} \) is CP and PPT iff \( E_{i} \geq 0, E_{i}^{T_A} \geq 0, E_{i}^{T_B} \geq 0, E_{i}^{T_C} \geq 0 \) for \( i = 1, 2, 3, 4 \). Denoting \( L_{A'B'C} = E_{1,AB'C} + (m_2 - 1)E_{2,AB'C} + (m_1 - 1)(m_2 - 1)E_{3,AB'C} \), the CP constraint is \( L_{C} = 1_{C} \).

The NS constraint \( C \not\rightarrow AB \) is \( \tilde{X}_{ABC} = \frac{\text{tr}_C}{\text{tr}_{AB}} \tilde{X}_{ABC} \), equivalent to \( L_{A'B'C} = \frac{\text{tr}_C}{\text{tr}_{AB}} L_{A'B'} \). Thus the TP condition becomes \( \text{Tr} L_{A'B'} = d_{C} \). The NS constraint \( AB \not\rightarrow C \) is equivalent to \( E_{1,C} = \frac{1_C}{m_1 m_2} \).

The constraints \( AC \not\rightarrow B \) and \( BC \not\rightarrow A \) are implicitly implied. The constraint \( B \not\rightarrow AC \) is \( \tilde{X}_{ABC} = \frac{\text{tr}_A}{\text{tr}_{BC}} \tilde{X}_{AC} \), which is equivalent to \( E_{1,AC} = E_{2,AC} \) and \( E_{3,AC} = E_{4,AC} \). Similarly the NS constraint \( A \not\rightarrow BC \) is \( E_{1,B'C} = E_{3,B'C} \) and \( E_{2,B'C} = E_{4,B'C} \).

Putting together the above constraints, we obtain the SDP in Theorem 10.

Proof of Proposition 11. It is easy to see

\[
f_{\text{NSPPT}}(\mathcal{N}, m_1, m_2) \leq \max \text{Tr}(J_{N,AB'C}^T E_{1,AB'C})
\]

\[
\text{s.t. } E_{1}, E_{1}^{T_A}, E_{1}^{T_B}, E_{1}^{T_C} \geq 0,
\]

\[
E_{1,C} = \frac{1_C}{m_1 m_2}.
\]

The dual of the right-hand-side SDP is

\[
\min \text{Tr} Q_{C}
\]

\[
\text{s.t. } J_{N}^T + P_{1}^{T_A} + P_{2}^{T_B} + P_{3}^{T_C} \leq m_1 m_2 \mathbb{1}_{AB} Q_{C}.
\]

\[
P_{i} \geq 0.
\]
Introducing $V := J_N^T + P_{1A}^T + P_{2B}^T + P_{3C}^T$, $Y := J_N^T + P_{2B}^T + P_{3C}^T$ and $Z := J_N^T + P_{3C}^T$, we have the SDP (59) is equivalent to

\[
\begin{align*}
\min & \quad \text{Tr} \, Q_C \\
\text{s.t.} & \quad m_1 m_2 \mathbb{1}_{AB} Q_C \geq V, \\
& \quad V^T_A \geq Y^T_A, \\
& \quad Y^T_B \geq Z^T_B, \\
& \quad Z^T_C \geq J_N^T_{AB}.
\end{align*}
\]

By adding new constraints, the above SDP is no larger than $h(N, m_1, m_2)$. It can be readily verified that $h$ is sub-multiplicative. \qed