BATALIN-VILKOVISKY STRUCTURE ON HOCHSCHILD COHOMOLOGY WITH COEFFICIENTS IN THE DUAL ALGEBRA

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ABSTRACT. We prove that Hochschild cohomology with coefficients in
$A^* = \text{Hom}_k(A, k)$ together with an $A$-structural map $\psi : A^* \otimes_A A^* \to A^*$ is a Batalin-Vilkovisky algebra. This applies to symmetric, Frobenius and monomial path algebras.

1. Introduction

Let $A$ be an associative unital algebra projective over a commutative
ring $k$. The Hochschild cohomology $k$-modules of $A$ with coefficients in an
$A$-bimodule $M$,

$$H^\ast(A, M) = \bigoplus_{n \geq 0} H^n(A, M)$$

have been introduced by Hochschild [6] and extensively studied since then.
Operations on cohomology have been defined, such as the cup product and
the Gerstenhaber bracket, making it into a Gerstenhaber algebra [4]. Tradler
showed [10] that for symmetric algebras this Gerstenhaber algebra structure
on cohomology comes from a Batalin-Vilkovisky operator (BV-operator) and
Menichi extended the result [8]. As Tradler mentions, it is important to
determine other families of algebras where this property holds. Lambre-
Zhou-Zimmermann proved that this is the case for Frobenius algebras with
semisimple Nakayama automorphism [7]. Independently, Volkov proved with
other methods that this holds for Frobenius algebras in which the Nakayama
automorphism has finite order and the characteristic of the field $k$ does not
divide it [11]. It has also been shown that Calabi-Yau algebras admit the
existence of a BV-operator [5], and that this BV-structure on its cohomology
is isomorphic to the one of the cohomology of the Koszul dual, for a Koszul
Calabi-Yau algebra [2]. More generally, for algebras with duality, see [7], a
BV-structure is equivalent to a Tamarkin-Tsygan calculus or a differential
calculus [7]. The proofs of [5], [7] and [10] have in common the use of Connes’
additional [3] on homology to define the BV-operator on cohomology.

We start by giving an interpretation of Connes’ differential in Hochschild
cohomology with coefficients in the $A$-bimodule $A^* = \text{Hom}_k(A, k)$. The use
of $A^*$ as bimodule of coefficients replaces the inner product which is in force

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for Frobenius algebras \([7], [10]\) as it is shown in Lemma 2.1 and Corollary 4.1. For symmetric algebras this induced BV-structure is isomorphic to the one given by Tradler in [10]. In the case of monomial path algebras we give a description of the \(A\)-bimodule structure of \(A^*\) that allows us to construct an \(A\)-structural map on \(A^*\).

To the knowledge of the author there is no other BV-operator entirely independent of Connes’ differential.

2. Connes’ differential

Connes’ differential is the map \(B : HH_n(A) \to HH_{n+1}(A)\) that makes the Hochschild theory of an algebra into a differential calculus [9]. It is given by

\[
B([a_0 \otimes \cdots \otimes a_n]) = \left[ \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \right].
\]

For an \(A\)-bimodule \(M\) the dual \(A\)-bimodule is denoted \(M^* = Hom_k(M,k)\). We consider the canonical \(A\)-bimodule structure on \(M^*\), that is \((afb)(x) = f(bxa)\) for all \(a, b \in A\), all \(f \in M^*\) and all \(x \in M\). Let

\[
\bar{B} : H^{n+1}(A, A^*) \to H^n(A, A^*)
\]
given by

\[
\bar{B}([f])([a_1 \otimes \cdots \otimes a_n])(a_0) := \sum_{i=0}^{n} (-1)^{ni} f(a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1})(1).
\]

It is straightforward to verify that it is well-defined. Let

\[
\mathcal{E} : H^n(A, M^*) \to H_n(A, M)^*
\]

be the morphism

\[
\mathcal{E}([f])([x \otimes a_1 \otimes \cdots \otimes a_n]) = f(a_1 \otimes \cdots \otimes a_n)(x),
\]

for all \(a_i \in A\), for \(i = 1, \cdots, n\), all \(x \in M\) and all \([f] \in H^{n+1}(A, M^*)\), see [11]. The evaluation map \(ev : H_n(A, M) \to H_n(A, M)^{**}\) can be composed with the \(k\)-dual of \(\mathcal{E}\) to get a morphism

\[
\varphi : H_n(A, M) \to H^n(A, M^*)^*
\]

which is given by

\[
\varphi([x \otimes a_1 \otimes \cdots \otimes a_n])([f]) = f(a_1 \otimes \cdots \otimes a_n)(x).
\]

For \(M = A\) we obtain a morphism \(\varphi : HH_n(A) \to H^n(A, A^*)^*\). The proof of the following lemma is straightforward.
Lemma 2.1. Let $k$ be a commutative ring and let $A$ be an associative and unital $k$-algebra. The following diagram is commutative

$$HH_n(A) \xrightarrow{B} HH_{n+1}(A)$$

$$\downarrow \varphi \quad \quad \downarrow \varphi$$

$$H^n(A, A^*) \xrightarrow{B^*} H^{n+1}(A, A^*).$$

If $k$ is a field then $\varphi$ is a monomorphism. If $k$ is a field and $HH_n(A)$ is finite dimensional then $\varphi : HH_n(A) \to H^n(A, A^*)$ is an isomorphism.

3. Batalin-Vilkovisky structure

A Gerstenhaber algebra is a triple $(\mathcal{H}^*, \cup, [, ,])$ such that $\mathcal{H}^*$ is a graded $k$-module, $\cup : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m}$ is a graded commutative associative product and $[, ,] : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m-1}$ is a graded Lie bracket such that it is antisymmetric $[f, g] = (-1)^{|f||g|}[g, f]$, it satisfies the Jacobi identity $[f, [g, h]] = [[f, g], h] + (-1)^{|f||g|}[g, [f, h]]$ as well as the Poisson identity $[f, g \cup h] = [f, g] \cup h + (-1)^{|f||g|}[g \cup [f, h]],$ for all homogeneous elements $f, g, h$ of $\mathcal{H}^*$. We denote by $|f|$ the degree of an homogeneous element $f \in \mathcal{H}^*$. A Batalin-Vilkovisky algebra (BV-algebra) is a Gerstenhaber algebra $(\mathcal{H}^*, \cup, [, ,])$ together with a morphism

$$\Delta : \mathcal{H}^{n+1} \to \mathcal{H}^n$$

such that $\Delta^2 = 0$ and

$$[f, g] = (-1)^{|f|+1}(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|}f \cup \Delta(g)).$$

Recall that $H^0(A, M) = MA = \{m \in M | ma = am \text{ for all } a \in A\}$ for an $A$-bimodule $M$.

Definition 3.1. Let $M$ be an $A$-bimodule. A morphism $\psi : M \otimes_A M \to M$ of $A$-bimodules is called an $A$-structural map if it is associative, that is

$$\psi(m_1 \otimes \psi(m_2 \otimes m_3)) = \psi(\psi(m_1 \otimes m_2) \otimes m_3)$$

for all $m_1, m_2, m_3 \in M$, and $\psi$ is unital in the sense that there is $1_M \in H^0(A, M)$ such that $\psi(1_M \otimes m) = \psi(m \otimes 1_M) = m$ for all $m \in M$.

Remark 3.1. Let $\psi : M \otimes_A M \to M$ be an $A$-structural map. Then the $\cup$-product

$$\cup : H^n(A, M) \otimes H^m(A, M) \to H^{n+m}(A, M \otimes_A M)$$

can be composed with $\psi$ to obtain

$$\cup_\psi : H^n(A, M) \otimes H^m(A, M) \to H^{n+m}(A, M),$$
that is
\[(f \cup_\psi g)(a_1 \otimes \cdots \otimes a_{n+m}) := \psi(f(a_1 \otimes \cdots \otimes a_n) \otimes g(a_{n+1} \otimes \cdots \otimes a_{n+m})).\]

Our assumptions on \(\psi\) imply that \(H^\bullet(A, M)\) is an associative and unital \(k\)-algebra.

We will denote \(H^\bullet_\psi(A, M)\) the \(k\)-algebra \(H^\bullet(A, M)\) endowed with the \(\cup_\psi\)-product. In case \(M = A^*\), we have the following.

**Lemma 3.1.** Let \(A\) be an associative unital \(k\)-algebra and let \(\psi : A^* \otimes_A A^* \to A^*\) be an \(A\)-structural map. Then \(H^\bullet_\psi(A, A^*)\) is a Gerstenhaber algebra.

**Proof.** Let \(d^*\) be the differential on the complex that calculates \(H^\bullet(A, A^*)\) and let \(f, g \in H^\bullet(A, A^*)\) be homogeneous elements. The following relation is well known, see [4],

\[
d^*(g \circ f) = d^*(g) \circ f + (-1)^{|f|} d^*(g \circ f),
\]

where \(g \circ f(a_1 \otimes \cdots \otimes a_{|f|+|g|-1})\) is by definition

\[
\sum_{i=0}^{|g|} (-1)^j g(a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+|f|-1}) \otimes a_{i+|f|} \otimes \cdots \otimes a_{|f|+|g|-1}),
\]

for \(j = (i-1)(|f| - 1)\). If \(f\) and \(g\) are cocycles, we get that the cup product is graded commutative and since \(\psi\) is \(k\)-linear we get that the \(\cup_\psi\)-product is graded commutative. Define the bracket in terms of \(\bar{B}\) and the \(\cup_\psi\)-product as

\[
[f, g]_\psi := (-1)^{|f|-1}|g|(\bar{B}(f \cup_\psi g) - \bar{B}(f) \cup_\psi g - (-1)^{|f|} f \cup_\psi \bar{B}(g)).
\]

Hence the graded \(k\)-module \(H^\bullet_\psi(A, A^*)\) with the \(\cup_\psi\)-product and the bracket \([, ]_\psi\) is a Gerstenhaber algebra. \(\square\)

**Theorem 3.1.** Let \(A\) be an associative unital \(k\)-algebra and let \(\psi : A^* \otimes_A A^* \to A^*\) be an \(A\)-structural map. Then the data \((H^\bullet_\psi(A, A^*), \cup_\psi, [ , ]_\psi, \bar{B})\) is a BV-algebra.

**Proof.** Since the following diagram is commutative

\[
\begin{array}{ccc}
HH_n(A) & \xrightarrow{B} & HH_{n+1}(A) \\
\varphi \downarrow & & \varphi \downarrow \\
H^n(A, A^*) & \xrightarrow{\bar{B}} & H^{n+1}(A, A^*)
\end{array}
\]

we have that \(\bar{B}^2 = 0\). Then \(H^\bullet_\psi(A, A^*)\) is a BV-algebra with the bracket defined as in the last lemma. \(\square\)
4. Frobenius and Symmetric algebras

Assume that $A$ is a symmetric algebra, i.e. a finite dimensional algebra with a symmetric, associative and non-degenerate bilinear form $<,> : A \otimes A \rightarrow k$, where associative means

$$<ab, c> = <a, bc>$$

for all $a, b, c \in A$. The bilinear form defines an isomorphism of $A$-bimodules $Z : A \rightarrow A^*$ given by $Z(a) = <a, - >$. It is shown in [10] that this defines a $BV$-operator on Hochschild cohomology, where $\Delta f$ is defined such that for $f \in HH^n(A)$ we have

$$< \Delta f(a_1 \otimes \cdots \otimes a_{n-1}), a_n> = \sum_{i=1}^{n}(-1)^{i(n-1)} <f(a_i \otimes \cdots \otimes a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{i-1}), 1>.$$

**Corollary 4.1.** If $A$ is a symmetric algebra, then there is an $A$-structural map $\psi : A^* \otimes_A A^* \rightarrow A^*$ such that the $BV$-algebras $HH^•(A)$ and $H^•_\psi(A, A^*)$ are isomorphic.

**Proof.** Let $Z : A \rightarrow A^*$ be the isomorphism of $A$-bimodules given by the bilinear form of $A$. We will denote $Z_s : HH^•(A) \rightarrow H^•_\psi(A, A^*)$ the isomorphism induced by composition with $Z$. Then the following diagram is commutative

$$\begin{array}{ccc}
HH^n(A) & \xrightarrow{\Delta} & HH^{n-1}(A) \\
Z_s \downarrow & & \downarrow Z_s \\
H^n(A, A^*) & \xrightarrow{\bar{B}} & H^{n-1}(A, A^*).
\end{array}$$

Indeed,

$$(\bar{B} \circ Z_s)([f])(a_1 \otimes \cdots \otimes a_{n-1})(a_0)$$

$$= B(Z \circ f)(a_1 \otimes \cdots \otimes a_{n-1})(a_0)$$

$$= \sum_{i=0}^{n-1}(-1)^{(n-1)i} (Z \circ f)(a_i \otimes \cdots \otimes a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{i-1})(1)$$

$$= Z \circ \left( \sum_{i=0}^{n-1}(-1)^{(n-1)i} f(a_i \otimes \cdots \otimes a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{i-1})(1) \right)$$

$$= (Z_s \circ \Delta)([f])(a_1 \otimes \cdots \otimes a_{n-1})(a_0).$$

Using the isomorphism given by the product $A \otimes_A A \cong A$ the transport of the algebra structure of $A$ to $A^*$ via $Z$ gives the $A$-structural map

$$\psi = Z \circ (Z \otimes Z)^{-1} : A^* \otimes_A A^* \rightarrow A^*.$$  

This isomorphism satisfies the associativity and unity conditions of remark 3.1, since the product of $A$ is associative and has a unit. Even more, there
are commutative diagrams where the vertical maps are isomorphisms

\[
\begin{array}{ccc}
HH^n(A) \otimes HH^m(A) & \xrightarrow{\cup} & HH^{n+m}(A) \\
Z_* \otimes Z_* & \xrightarrow{\cup_{\psi}} & Z_* \\
H^n(A, A^*) \otimes H^m(A, A^*) & \xrightarrow{\cup_{\psi}} & H^{n+m}(A, A^*)
\end{array}
\]

\[
\begin{array}{ccc}
HH^n(A) \otimes HH^m(A) & \xrightarrow{[\cdot, \cdot]} & HH^{n+m-1}(A) \\
Z_* \otimes Z_* & \xrightarrow{[\cdot, \cdot]_{\psi}} & Z_* \\
H^n(A, A^*) \otimes H^m(A, A^*) & \xrightarrow{[\cdot, \cdot]_{\psi}} & H^{n+m-1}(A, A^*)
\end{array}
\]

Indeed,

\[
Z_*(f) \cup_{\psi} Z_*(g) = \psi \circ (Z \otimes Z)(f \cup g) = Z \circ (Z \otimes Z)^{-1} \circ (Z \otimes Z)(f \cup g) = Z \circ (f \cup g) = Z_*(f \cup g),
\]

and

\[
[Z_* f, Z_* g]_{\psi} = (-1)^{|f|-1}|g| (\bar{b}(Z_* f \cup_{\psi} Z_* g) - \bar{b}(Z_* f) \cup_{\psi} Z_* g - (-1)^{|f|} Z_* f \cup_{\psi} \bar{b}(Z_* g))
\]

\[
= (-1)^{|f|-1}|g| (\bar{b}(Z_* (f \cup_{\psi} g)) - Z_*(\Delta f) \cup_{\psi} Z_* g - (-1)^{|f|} Z_* f \cup_{\psi} Z_*(\Delta g))
\]

\[
= (-1)^{|f|-1}|g| (Z_* \Delta (f \cup g) - Z_*(\Delta f \cup g) - (-1)^{|f|} Z_*(f \cup \Delta g))
\]

\[
= Z_* [f, g].
\]

Commutativity of these diagrams implies that the BV-algebras \( HH^\bullet(A) \) and \( H_\bullet^\bullet(A, A^*) \) are isomorphic. \( \square \)

**Remark 4.1.** Observe that choosing \( \Delta := (Z_*)^{-1} \bar{b} Z_* \) gives \( HH^\bullet(A) \) the structure of a BV-algebra.

Assume now that \( A \) is a Frobenius algebra, i.e. a finite dimensional algebra with a non-degenerate associative bilinear form \( < -, - > : A \times A \to k \). For every \( a \in A \) there exist a unique \( \mathcal{R}(a) \in A \) such that \( < a, - > = \mathcal{R}(a) >. \) The map \( \mathcal{R} : A \to A \) turns out to be an algebra isomorphism and is called the **Nakayama** automorphism of the Frobenius algebra \( A \). Following [7] we consider the \( A \)-bimodule \( A_{\mathcal{R}} \) whose underlying \( k \)-module is \( A \) and the corresponding actions are

\[
a x b = a x \mathcal{R}(b).
\]

Hence the morphism \( Z : A_{\mathcal{R}} \to A^\ast \) given by \( Z(a) = < a, - > \) is an isomorphism of \( A \)-bimodules [7]. The morphism

\[
\mu : A_{\mathcal{R}} \otimes_A A_{\mathcal{R}} \to A_{\mathcal{R}}
\]
given by \( \mu(a \otimes b) = a \mathcal{N}(b) \) is a morphism of \( A \)-bimodules since 
\[
\mu(ab \otimes_A cd) = ab \mathcal{N}(cd) = ab \mathcal{N}(c) \mathcal{N}(d) = ab \mathcal{N}(c)d = a \mu(b \otimes_A c)d
\]
and it is well-defined since 
\[
\mu(ac \otimes b) = \mu(a \mathcal{N}(c) \otimes b) = a \mathcal{N}(c) \mathcal{N}(b) = a \mathcal{N}(cb) = \mu(a \otimes cb)
\]
for all \( a, b, c, d \in A_\mathcal{R} \). It is also unital and associative since \( \mathcal{N}(1) = 1 \), and
\[
\mu(\mu(a \otimes b) \otimes c) = \mu(\mathcal{N}(a(b) \otimes c)) = \mu(a \otimes bc) = \mu(a \otimes b \mathcal{N}(c)) = \mu(a \otimes \mu(b \otimes c)).
\]
Then \( \psi = Z \circ \mu \circ (Z \otimes_A Z)^{-1} : A^* \otimes_A A^* \to A^* \) is an \( A \)-structural map.

**Corollary 4.2.** Let \( A \) be a Frobenius algebra with diagonalizable Nakayama automorphism, then the BV-algebras \( HH^\bullet_\mathcal{N}(A, A^*) \) and \( HH^\bullet(A, A_\mathcal{R}) \) are isomorphic.

**Proof.** Hochschild cohomology of \( A \) with coefficients in \( A_\mathcal{R} \) is isomorphic, see [7], to Hochschild cohomology of \( A \) with coefficients in \( A_\mathcal{R} \) corresponding to the eigenvalue 1 \( \in k \) of the linear transformation \( \mathcal{N} \),
\[
HH^\bullet(A, A_\mathcal{R}) \cong HH^\bullet_1(A, A_\mathcal{R}).
\]
The BV-operator of \( HH^\bullet(A, A_\mathcal{R}) \) is the transpose of Connes’ differential
\[
B_\mathcal{N}([a_0 \otimes \cdots \otimes a_n]) = \left[ \sum_{i=0}^{n} (-1)^{i}a_i \otimes \cdots \otimes a_n \otimes \mathcal{N}(a_0) \otimes \cdots \otimes \mathcal{N}(a_{i-1}) \right],
\]
with respect to the duality given in [7]. By finite dimensionality arguments, this morphism turns out to be the \( k \)-dual of \( \varphi \), namely
\[
\partial : HH_\bullet(A, A^*)^* \to HH^\bullet(A).
\]
The compatibility conditions for the \( \cup \)-product and the Gerstenhaber bracket are proved similarly. \( \square \)

5. **Monomial path algebras**

Let \( Q \) be a finite quiver with \( n \) vertices and consider a monomial path algebra \( A = kQ/\langle T \rangle \), that is, \( T \) is a subset of paths in \( Q \) of length greater or equal than 2. We do not require the algebra \( A \) to be finite dimensional. We write \( s(\omega) \) and \( t(\omega) \) for the source and the target of \( \omega \). A basis \( P \) of \( A \) is given the set of paths of \( Q \) which do not contain paths of \( T \). Let \( P^\vee \) be the dual basis of \( P \), and for \( \omega \in P \) we denote \( \omega^\vee \) its dual. Let \( \alpha \in P \) and define \( \omega/\alpha \) as the subpath of \( \omega \) that starts in \( s(\omega) \) and ends in \( s(\alpha) \) if \( \alpha \) is a subpath of \( \omega \) such that \( t(\alpha) = t(\omega) \), and zero otherwise. Let \( \beta \in P \) and define \( \beta \backslash \omega \) as the subpath of \( \omega \) that starts at \( t(\beta) \) and ends in \( t(\omega) \) if \( \beta \) is a subpath of \( \omega \) such that \( s(\beta) = s(\omega) \), and zero otherwise. The canonical \( A \)-bimodule
structure of $A^*$ is isomorphic to the one given by linearly extending the following action

\[ \alpha \cdot \omega \cdot \beta = (\beta \omega / \alpha) \cdot \beta. \]

Now we construct an $A$-structural map for $A^*$. For $\omega, \gamma \in P$ we define

\[ \omega \cdot \gamma = \begin{cases} (\gamma \omega) \cdot \beta & \text{if } t(\omega) = s(\gamma) \\ 0 & \text{otherwise} \end{cases} \]

and extend by linearity. Observe that $\gamma \beta \omega = \gamma \beta \omega$, then

\[ (\omega \cdot \beta) \cdot \gamma = (\beta \omega) \cdot \beta \gamma = (\beta \beta) \omega = \omega \cdot (\beta \cdot \gamma). \]

Therefore, by linearly extending $\psi(\omega \otimes \gamma) = \omega \cdot \gamma$ we get a morphism of $k$-modules

\[ \psi : A^* \otimes A \to A^*. \]

It is a morphism of $A$-bimodules since

\[ \alpha \cdot (\omega \cdot \gamma) \cdot \beta = (\beta \omega / \alpha) \cdot \beta = (\beta \beta) \omega = \omega \cdot (\beta \cdot \gamma). \]

The morphism $\psi$ is associative since the product of $A$ is associative. Let $e_1, \ldots, e_n$ be the idempotents of $A$ given by the vertices of $Q$. Define $1^* = e_1^* + \cdots + e_n^*$ and observe that if $\alpha$ is a basic element of $A$ of length greater or equal than one then

\[ 1^* \cdot \alpha = e_1^* \cdot \alpha + \cdots + e_n^* \cdot \alpha = 0 = \alpha \cdot e_1^* + \cdots + \alpha \cdot e_n^* = \alpha \cdot 1^* \]

for every $i = 1, \ldots, n$. Moreover,

\[ 1^* \cdot e_i = e_1^* \cdot e_i + \cdots + e_n^* \cdot e_i = e_i = e_i^* \cdot e_i + \cdots + e_i \cdot e_i = e_i \cdot 1^* \]

so we get that $1^* \in H^0(A, A^*)$. Finally,

\[ 1^* \cdot \omega = e_1^* \cdot \omega + \cdots + e_n^* \cdot \omega = e_i^* / (\omega) \cdot \omega = \omega \]

and analogously $\omega \cdot 1^* = \omega$. Therefore $\psi$ is an $A$-structural map.

**Corollary 5.1.** Let $A$ be a monomial path algebra. Then $H^*_\psi(A, A^*)$ is a BV-algebra.

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