A Note on Möbius Function and Möbius Inversion Formula of Fibonacci Cobweb Poset

Ewa Krot

Institute of Computer Science, Białystok University
PL-15-887 Białystok, ul.Sosnowa 64, POLAND
e-mail: ewakrot@wp.pl, ewakrot@ii.uwb.edu.pl

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Abstract

The explicit formula for Möbius function of Fibonacci cobweb poset \( P \) is given here for the first time by the use of Kwaśniowski’s definition of \( P \) in plane grid coordinate system [1].

1 Fibonacci cobweb poset

The Fibonacci cobweb poset \( P \) has been introduced by A.K.Kwaśniewski in [3, 4] for the purpose of finding combinatorial interpretation of fibonomial coefficients and their recurrence relation. At first the partially ordered set \( P \) (Fibonacci cobweb poset) was defined via its Hasse diagram as follows: It looks like famous rabbits grown tree but it has the specific cobweb in addition, i.e. it consists of levels labeled by Fibonacci numbers (the \( n \)-th level consist of \( F_n \) elements). Every element of \( n \)-th level \( (n \geq 1, n \in \mathbb{N}) \) is in partial order relation with every element of the \((n + 1)\)-th level but it’s not with any element from the level in which he lies (\( n \)-th level) except from it.

2 The Incidence Algebra \( I(P) \)

One can define the incidence algebra of \( P \) (locally finite partially ordered set) as follows (see [5, 6]):

\[
I(P) = \{ f : P \times P \rightarrow \mathbb{R}; \quad f(x,y) = 0 \quad unless \quad x \leq y \}.
\]
The sum of two such functions \( f \) and \( g \) and multiplication by scalar are defined as usual. The product \( H = f \ast g \) is defined as follows:

\[
h(x, y) = (f \ast g)(x, y) = \sum_{z \in P: x \leq z \leq y} f(x, z) \cdot g(z, y).
\]

It is immediately verified that this is an associative algebra over the real field (associative ring).

The incidence algebra has an identity element \( \delta(x, y) \), the Kronecker delta. Also the zeta function of \( P \) defined for any poset by:

\[
\zeta(x, y) = \begin{cases} 1 & \text{for } x \leq y \\ 0 & \text{otherwise} \end{cases}
\]

is an element of \( I(P) \). The one for Fibonacci cobweb poset was expressed by \( \delta \) in [1, 4] from where quote the result:

\[
\zeta = \zeta_1 - \zeta_0
\]

where for \( x, y \in \mathbb{N} \):

\[
\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y)
\]

\[
\zeta_0(x, y) = \sum_{k \geq 0} \sum_{s \geq 0} \delta(x, F_s + k) \sum_{r=1}^{F_s - k - 1} \delta(k + F_s + r, y).
\]

The knowledge of \( \zeta \) enables us to construct other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of \( P \). The one of them is Möbius function indispensable in numerous inversion type formulas of countless applications. It is known that the \( \zeta \) function of a locally finite partially ordered set is invertible in incidence algebra and its inversion is the famous Möbius function \( \mu \) i.e.:

\[
\zeta \ast \mu = \mu \ast \zeta = \delta.
\]

The Möbius function \( \mu \) of Fibonacci cobweb poset \( P \) was presented for the first time by the present author in [2]. It was recovered by the use of the recurrence formula for Möbius function of locally finite partially ordered set \( I(P) \) (see [5]):

\[
\begin{cases}
\mu(x, x) = 1 & \text{for all } x \in P \\
\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)
\end{cases}
\]
The Möbius function of Fibonacci cobweb poset was given by the following formula:

\[
\mu(x, y) = \begin{cases} 
0 & x > y \\
1 & x = y \\
0 & F_{k+1} \leq x, y \leq F_{k+2} - 1; \ x \neq y; \ k \geq 3 \\
-1 & F_{k+1} \leq x \leq F_{k+2} - 1 < F_{k+2} \leq y \leq F_{k+3} - 1 \\
(-1)^{n-k} \prod_{l=k+1}^{n-1} (F_l - 1) & F_{k+1} \leq x \leq F_{k+2} - 1, \ F_{n+1} \leq y \leq F_{n+2} - 1; \ n > k + 1,
\end{cases}
\]

(5)

where:

• the condition \(F_{k+1} \leq x, y \leq F_{k+2} - 1; \ x \neq y; \ k \geq 3\) means that \(x, y\) are different elements of \(k\)-th level;

• the condition \(F_{k+1} \leq x \leq F_{k+2} - 1 < F_{k+2} \leq y \leq F_{k+3} - 1\) means that \(x\) is an element of \(k\)-th level and \(y\) is an element of \((k + 1)\)-th level;

• the condition \(F_{k+1} \leq x \leq F_{k+2} - 1, \ F_{n+1} \leq y \leq F_{n+2} - 1; \ n > k + 1\) means that \(x\) is an element of \(k\)-th level and \(y\) is an element of \(n\)-th level.

The above formula allows us to find out the \(\mu\) function matrix (see [2]) but it is not good enough to be applied in compact form via Möbius inversion formula for cobweb poset. For this purpose more convenient, explicit formula is needed.

3 Plane grid coordinate system of \(P\)

In [1] A. K. Kwaśniewski defined cobweb poset \(P\) as infinite labeled graph oriented upwards as follows: Let us label vertices of \(P\) by pairs of coordinates: \(\langle i, j \rangle \in \mathbb{N} \times \mathbb{N}, \) where the second coordinate is the number of level in which the element of \(P\) lies (here it is the \(j\)-th level) and the first one is the number of this element in his level (from left to the right), here \(i\). We shall refer, (following [1]) \(\Phi_s\) as to the set of vertices (elements) of the \(s\)-th level, i.e.:

\[
\Phi_s = \{\langle j, s \rangle, \ 1 \leq j \leq F_s\}, \ s \in \mathbb{N}.
\]

For example \(\Phi_1 = \{\langle 1, 1 \rangle\}, \ \Phi_2 = \{\langle 1, 2 \rangle\}, \ \Phi_3 = \{\langle 1, 3 \rangle, \langle 2, 3 \rangle\}, \ \Phi_4 = \{\langle 1, 4 \rangle, \langle 2, 4 \rangle, \langle 3, 4 \rangle\}, \ \Phi_5 = \{\langle 1, 5 \rangle, \langle 2, 5 \rangle, \langle 3, 5 \rangle, \langle 4, 5 \rangle, \langle 5, 5 \rangle\} \ldots
Then $P$ is a labeled graph $P = (V, E)$ where

$$V = \bigcup_{p \geq 1} \Phi_p, \quad E = \{(\langle j, p \rangle, \langle q, p + 1 \rangle)\}, \quad 1 \leq j \leq F_p, \quad 1 \leq q \leq F_{p+1}. $$

Now we can define the partial order relation on $P$ as follows: let $x = \langle s, t \rangle, y = \langle u, v \rangle$ be elements of cobweb poset $P$. Then

$$(x \leq y) \iff [(t < v) \lor (t = v \land s = u)].$$

4 The Möbius function and Möbius inversion formula on $P$

The above definition of $P$ allows us to derive an explicit formula for Möbius function of cobweb poset $P$. To do this we can use the formula (5). Then for $x = \langle s, t \rangle, y = \langle u, v \rangle, 1 \leq s \leq F_t, 1 \leq u \leq F_v, t, v \in \mathbb{N}$ we have

$$\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) =$$

$$= \delta(t, v)\delta(s, u) - \delta(t + 1, v) + \sum_{k=2}^{\infty} \delta(t + k, v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1)$$

(6)

where $\delta$ is the Kronecker delta defined by

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}. $$

We can also derive more convenient then (11) formula for $\zeta$ function of $P$ (the characteristic function of partial order relation in $P$):

$$\zeta(x, y) = \zeta(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t + k, v).$$

(7)

The formula (6) enables us to formulate following theorem (see [5]):

**Theorem 4.1. (Möbius Inversion Formula of cobweb $P$)**

Let $f(x) = f(\langle s, t \rangle)$ be a real valued function, defined for $x = \langle s, t \rangle$ ranging in cobweb poset $P$. Let an element $p = \langle p_1, p_2 \rangle$ exist with the property that $f(x) = 0$ unless $x \geq p$. 

Suppose that
\[ g(x) = \sum_{\{y \in P : y \leq x\}} f(y). \]
Then
\[ f(x) = \sum_{\{y \in P : y \leq x\}} g(y)\mu(y, x). \]
But using coordinates of \( x, y \) in \( P \) i.e. \( x = \langle s, t \rangle, \ y = \langle u, v \rangle \) if
\[ g(\langle s, t \rangle) = \sum_{v=1}^{t-1} \sum_{u=1}^{F_v} (f(\langle u, v \rangle) + f(\langle s, t \rangle)) \]
then we have
\[ f(\langle s, t \rangle) = \sum_{v \geq 1} \sum_{u=1}^{F_v} g(\langle u, v \rangle)\mu(\langle s, t \rangle, \langle u, v \rangle) = \sum_{v \geq 1} \sum_{u=1}^{F_v} g(\langle u, v \rangle) \left[ \delta(v, t)\delta(u, s) - \delta(v + 1, t) + \sum_{k=2}^{\infty} \delta(v + k, t)(-1)^k \prod_{i=v+1}^{t-1} (F_i - 1) \right]. \]

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