**Integrability of planar polynomial differential systems through linear differential equations.**

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Abstract

In this work, we consider rational ordinary differential equations \( dy/dx = Q(x, y)/P(x, y) \), with \( Q(x, y) \) and \( P(x, y) \) coprime polynomials with real coefficients. We give a method to construct equations of this type for which a first integral can be expressed from two independent solutions of a second–order homogeneous linear differential equation. This first integral is, in general, given by a non Liouvillian function.

We show that all the known families of quadratic systems with an irreducible invariant algebraic curve of arbitrarily high degree and without a rational first integral can be constructed by using this method. We also present a new example of this kind of families.

We give an analogous method for constructing rational equations but by means of a linear differential equation of first order.

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1 Introduction

This paper deals with rational ordinary differential equations such as
\[ \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \]  
(1)
where \( Q(x, y) \) and \( P(x, y) \) are coprime polynomials with real coefficients. We associate to this rational equation a planar polynomial differential system by introducing an independent variable \( t \) usually called time. Denoting by \( \dot{} = d/dt \), we have
\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]  
(2)
where \((x, y) \in \mathbb{R}^2\). This system defines the vector field \( \mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \) over \( \mathbb{R}^2 \) and, equivalently, the 1-form \( \omega = Q(x, y)dx - P(x, y)dy \). We indistinctively talk about equation (1) and system (2). Let \( d \) be the maximum degree of \( P \) and \( Q \). We say that system (2) is of degree \( d \). When \( d = 2 \), we say that (2) is a quadratic system.

In order to simplify notation, we define \( \mathbb{R}[x, y] \) as the ring of polynomials in two variables with real coefficients and \( \mathbb{R}(x, y) \) as the field of rational functions in two variables with real coefficients, that is, the quotient field of the previous ring. Analogous definitions stand for \( \mathbb{R}[x] \) and \( \mathbb{R}(x) \).

We have an equation (1) defined in a certain class of functions, in this case, the rational functions with real coefficients \( \mathbb{R}(x, y) \) and we consider the problem whether there is a first integral in another, possibly larger, class. For instance, as we will discuss later on, H. Poincaré stated the problem of determining when a system (2) has a rational first integral. A \( C^k \) function \( H : U \to \mathbb{R} \) such that it is constant on each trajectory of (2) and it is not locally constant is called a first integral of system (2) on \( W \).

To find an integrating factor or an inverse integrating factor for system (2) is closely related to finding a first integral for it. When considering the integrability problem we are also addressed to study whether an (inverse) integrating factor belongs to a certain given class of functions.

Definition 1 Let \( W \) be an open set of \( \mathbb{R}^2 \). A function \( \mu : W \to \mathbb{R} \) of class \( C^k(W) \), \( k > 1 \), that satisfies the linear partial differential equation
\[ \omega \wedge d\mu = \mu d\omega, \]  
(3)
is called an integrating factor of system (2) on \( W \).
It has been shown than an easier function to find which also gives additional properties for a differential system \(2\) is the inverse of an integrating factor, that is, \(V = 1/\mu\), which is called inverse integrating factor.

We note that \(\{V = 0\}\) is formed by orbits of system \(2\). The function \(\mu = 1/V\) defines on \(W \setminus \{V = 0\}\) an integrating factor of system \(2\), which allows the computation of a first integral of the system on \(W \setminus \{V = 0\}\). The first integral \(H\) associated to the inverse integrating factor \(V\) can be computed through the integral \(H(x, y) = \int \omega/V\), and the condition \(3\) for \(\mu = 1/V\) ensures that this line integral is well defined.

The inverse integrating factors play an important role in two of the most difficult open problems of qualitative theory of planar polynomial vector fields, which are the center problem and 16\(^{th}\) Hilbert problem. In [6], it has been noticed that for many polynomial differential systems with a center at the origin there is always an inverse integrating factor \(V\) globally defined in all \(\mathbb{R}^2\), which is usually a polynomial. However, the first integral for a polynomial differential system with a center at the origin can be very complicated.

We say that a function \(f(x, y)\) is an invariant for a system \(2\) if \(\omega \wedge df = kf\) with \(k(x, y)\) a polynomial of degree lower or equal than \(d - 1\), where \(d\) is the degree of the system. This polynomial \(k(x, y)\) is called the cofactor of \(f(x, y)\). In the previous equality and all along this paper we use the convention of identifying the space of functions over \(\mathbb{R}^2\) and the space of 2–forms over \(\mathbb{R}^2\). In case \(f(x, y) = 0\) defines a curve in the real plane, this definition implies that \(\omega \wedge df\) equals zero on the points such that \(f(x, y) = 0\). In case \(f(x, y)\) is a polynomial we say that \(f(x, y) = 0\) is an invariant algebraic curve for system \(2\).

Let us consider \(f(x, y) = 0\) an invariant algebraic curve for system \(2\), we will always assume that \(f(x, y)\) is an irreducible polynomial in \(\mathbb{R}[x, y]\). Otherwise, it can be shown that each of its factors is an invariant algebraic curve for system \(2\). We will denote by \(n\) the degree of the polynomial \(f(x, y)\).

In [12], G. Darboux gives a method for finding an explicit first integral for a system \(2\) in case that \(d(d+1)/2+1\) different irreducible invariant algebraic curves are known, where \(d\) is the degree of the system. In this case, a first integral of the form \(H = f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_r^{\lambda_r}\), where \(f_i(x, y) = 0\) is an invariant algebraic curve for system \(2\) and \(\lambda_i \in \mathbb{C}\) not all of them null, for \(i = 1, 2, \ldots, r\), \(r \in \mathbb{N}\), can be defined in the open set \(\mathbb{R}^2 \setminus \Sigma\), where \(\Sigma = \{(x, y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdot \ldots \cdot f_r)(x, y) = 0\}\). The functions of this type are called Darboux functions. We remark that, particularly, if \(\lambda_i \in \mathbb{Z}\), \(\forall i = 1, 2, \ldots, r\), \(H\) is a rational first integral for system \(2\). In this sense J. P. Jouanoulou [15], showed that if at least \(d(d+1)+2\) different irreducible invariant algebraic curves are known, then there exists a rational first integral.

The main fact used to prove Darboux’s theorem (and Jouanoulou’s improvement) is that the cofactor corresponding to each invariant algebraic curve is a
polynomial of degree \( \leq d - 1 \). Invariant functions can also be used in order to find a first integral for the system. This observation permits a generalization of Darboux’s theory which is given in [14], where, for instance, non-algebraic invariant curves with an algebraic cofactor for a polynomial system of degree 4 are presented. In our work, we give other families of systems with such invariant curves.

C. Christopher, in [9], studies the multiplicity of an invariant algebraic curve and gives the definition for exponential factor, which is a particular case of invariant for system (2).

**Definition 2** Let \( h, g \) be two coprime polynomials. The function \( e^{h/g} \) is called an exponential factor for system (2) if for some polynomial \( k \) of degree at most \( d - 1 \), where \( d \) is the degree of the system, the following relation is fulfilled: \( \omega \wedge d(e^{h/g}) = ke^{h/g} \). As before, we say that \( k(x, y) \) is the cofactor of the exponential factor \( e^{h/g} \).

We note that an exponential factor \( e^{h/g} \) does not define an invariant curve, but the next proposition, proved in [9], gives the relationship between both notions.

**Proposition 3** [9] If \( F = e^{h/g} \) is an exponential factor and \( g \) is not a constant, then \( g = 0 \) is an invariant algebraic curve, and \( h \) satisfies the equation \( \omega \wedge dh = (hk_g + gk_F) \) where \( k_g \) and \( k_F \) are the cofactors of \( g \) and \( F \), respectively.

All these previous results are closely related to a result of M. F. Singer [25] which represents an important progress in the resolution of the integrability problem when considering first integrals for a system (2) in the class of Liouville functions. Roughly speaking, we can define a Liouville function or a function which can be expressed by means of quadratures, as a function constructed from rational functions using composition, exponentiation, integration, and algebraic functions. A precise definition of this class of functions is given in [25].

**Theorem 4** [25] Let us consider the polynomial 1–form \( \omega = Qdx - Pdy \) related to system (2). System (2) has a Liouville first integral if, and only if, \( \omega \) has an inverse integrating factor of the form \( V = \exp \int \xi \), where \( \xi \) is a closed rational 1–form.

We notice that the conditions on the function \( V \) given in this Theorem can be restated as \( d\omega = \xi \wedge \omega \) and \( d\xi = 0 \).

Taking into account Theorem 4, C. Christopher [10] gives the following result, which precises the form of the inverse integrating factor.

**Theorem 5** [10] If system (2) has an inverse integrating factor of the form \( \exp \int \xi \) with \( d\xi = 0 \) and \( \xi = \xi_1dx + \xi_2dy \) where \( \xi_i, i = 1, 2, \) are rational functions in \( x \) and \( y \), then there exists an inverse integrating factor of system (2) of the form

\[
V = \exp\{D/E\} \prod C_i^{l_i},
\]

where \( D, E \) and the \( C_i \) are polynomials in \( x \) and \( y \), and \( l_i \in \mathbb{C} \).
Theorem 5 states that the search for Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors.

In [21], H. Poincaré stated the following problem concerning the integration of an equation (1): Give conditions on the polynomials $P$ and $Q$ to recognize when there exists a rational first integral. As the same H. Poincaré noticed, a sufficient condition to solve this problem consists on finding an upper bound for the degree of the invariant algebraic curves for a given system (2). From Darboux’s result, it is known that for every polynomial vector field, there exists an upper bound for the possible degrees of irreducible invariant algebraic curves. However, it is a hard problem to explicitly determine such an upper bound. Some bounds have been given under certain conditions on the invariant curves, see D. Cerveau and A. Lins Neto work [5], or on the local behavior of critical points, see M. Carnicer’s work [4].

In this sense, A. Lins Neto conjectured [16] that a polynomial system (2) of degree $d$ with an invariant algebraic curve of degree high enough (where this bound only depends on $d$) would have a rational first integral. This conjecture has been shown to be false by several counterexamples. J. Moulin-Ollagnier [19] gives a family of quadratic Lotka-Volterra systems, each with an invariant algebraic curve of degree $2\ell$, where $\ell$ is the parameter of the family, without rational first integral. A simpler example is given by C. Christopher and J. Llibre in [11]. In [8] a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree without a Darboux first integral nor a Darboux inverse integrating factor is given. All these counterexamples exhibit a Liouvillian first integral.

The natural conjecture at this step, also given by A. Lins Neto, see [17], after the counterexample of J. Moulin-Ollagnier appeared, is that a polynomial system (2) of degree $d$ with an invariant algebraic curve of degree high enough (where this bound only depends on $d$) has a Liouvillian first integral.

In this work we show a relationship between solutions of a class of systems (2) and linear homogeneous ordinary differential equations of order 2 of the form

$$A_2(x)w''(x) + A_1(x)w'(x) + A_0(x)w(x) = 0,$$

(4)

where $x \in \mathbb{R}$, $w'(x) = dw(x)/dx$ and $w''(x) = dw'(x)/dx$. We only consider equations (4) where $A_i(x) \in \mathbb{R}[x]$ for $0 \leq i \leq 2$ and $A_2(x) \neq 0$.

By means of a change of variable we rewrite an equation (4) as a polynomial differential system such that it has an invariant related to $w(x)$. In case $w(x)$ is a polynomial we get an invariant algebraic curve.

Moreover, we give an explicit first integral for all the systems built up by this method by means of two independent solutions of equation (4).

We give analogous results for a linear homogeneous ordinary differential equation
of order 1 such as
\[ w'(x) + A(x)w(x) = 0, \] (5)
where \( x \in \mathbb{R}, \ w'(x) = dw(x)/dx \) and \( A(x) \in \mathbb{R}(x) \). All these results are given in Section 2.

In Section 3 we consider all the families of quadratic systems with an algebraic curve of arbitrarily high degree known until the moment of composition of this paper and we show that they all belong to the construction explained in Section 2. The families of quadratic systems with an algebraic curve of arbitrarily high degree studied in this paper are the ones appearing in \([7, 8, 11, 19]\) and one example more first appearing in this work. This new example consists on a biparametrical family of quadratic systems, which we give an explicit expression of a first integral for, such that when one of the parameters is a natural number, say \( n \), the system exhibits an irreducible invariant algebraic curve of degree \( n \).

We give the explicit expression for the first integral of a certain system \( \mathbb{P} \) by means of invariant functions for it, and applying the Generalized Darboux’s Theory as explained in \([14]\), where a new kind of first integrals, not only the Liouvillian ones as in classical theories, is described. We exemplify this result with the families of systems depending on parameters described in Section 3. We remark that the first integrals that we give in Section 2 are not, in general, of Liouvillian type. However, these first integrals are Liouvillian at the values of parameters which correspond to the systems with algebraic solutions. In the Subsection 3.4, we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed by the method appearing in Section 2 from an equation (5).

A question suggested by these examples is whether there are polynomial systems which are not reversible nor Liouvillian integrable which have a center and can be integrated by means of Theorem 6, see Section 2. The work \([2]\) is related to this question as it gives an example of an analytic system, not polynomial, with a center which is not reversible nor Liouvillian integrable. All the known families of polynomial vector fields with a center at the origin are either Liouvillian integrable or reversible, see \([27, 28]\) for the definition of reversibility. In \([27, 28]\), Żołądek classifies all the reversible cubic systems with a center. The reversible systems may have a first integral not given by Liouville functions or no explicit form of a first integral may be known. For instance the reversible system \( \dot{x} = -y + x^4, \ \dot{y} = x \) has a first integral composed by Airy functions, see \([14]\), and no Liouvillian first integral exists. The system \( \dot{x} = -y^3 + x^2y^2/2, \ \dot{y} = x^3 \) is an example given by Moussu, see \([20]\), which has a center at the origin since it is a monodromic and reversible singular point and no explicit first integral is known for this system.

Since some examples of polynomial systems, which can be integrated by the method described in Section 2, appear after a birrational transformation, another suggested open question is if all the polynomial systems with a center are birra-
Sectionally equivalent to one derived from Theorem 6 or from Theorem 11.

2 Homogeneous linear differential equations of order \( \leq 2 \) and planar polynomial systems.

Let us consider a homogeneous linear differential equation of order 2:

\[
A_2(x) w''(x) + A_1(x) w'(x) + A_0(x) w(x) = 0, 
\]

where \( w'(x) = dw(x)/dx, w''(x) = dw'(x)/dx, A_i(x) \in \mathbb{R}[x], i = 0, 1, 2 \), and \( A_2(x) \neq 0 \).

**Theorem 6** Given \( g(x, y) = g_0(x, y)/g_1(x, y) \) with \( g_i(x, y) \in \mathbb{R}[x, y], g_1(x, y) \neq 0 \) and \( \partial g/\partial y \neq 0 \), each nonzero solution \( w(x) \) of equation (6) is related to a finite number of solutions \( y = y(x) \) of the rational equation

\[
\frac{dy}{dx} = A_2(x) \left( \frac{g_0 \partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial x} \right),
\]

by the functional change \( dw/dx = g(x, y)w(x) \), which implicitly defines \( y \) as a function of \( x \).

**Proof.** Let us consider equation (6) and the functional change \( dw/dx = g(x, y)w(x) \) where \( y = y(x) \), that is, \( y \) is implicitly defined as a function of \( x \). This change may also be written as \( w(x) = \exp \left( \int_{x_0}^{x} g(s, y(s))ds \right) \), where \( x_0 \) is any constant, and it is injective. We see that it is not necessarily bijective unless the maximum degree of \( g_1(x, y) \) and \( g_0(x, y) \) in the variable \( y \) equals 1. But it defines a finite number of functions \( y(x) \).

By this functional change, equation (6) becomes

\[
w(x) \left( A_0(x) + g A_1(x) + g^2 A_2(x) + A_2(x) \frac{dy}{dx} \frac{\partial g}{\partial y} + A_2(x) \frac{\partial g}{\partial x} \right) = 0.
\]

We have that \( w(x) \) is a nonzero solution of (6) so this equation is equivalent to the ordinary differential equation of first order (7). Therefore, each non-zero solution \( w(x) \) of equation (6) corresponds to a finite number of solutions \( y = y(x) \) of the planar polynomial system (7).

**Theorem 7** We consider the 1-form related to equation (7)

\[
\omega = \left( A_0(x) g_1^2 + A_1(x) g_1 g_0 + A_2(x) g_0^2 + A_2(x) \left( g_1 \frac{\partial g_0}{\partial x} - g_0 \frac{\partial g_1}{\partial x} \right) \right) dx
\]

\[
- A_2(x) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) dy.
\]
Let $w(x)$ be any nonzero solution of equation \((7)\). Then the curve defined by $f(x, y) = 0$, with $f(x, y) := g_1(x, y)w'(x) - g_0(x, y)w(x)$ is invariant for system \((7)\) and has the polynomial cofactor

$$k(x, y) = \left( A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y} \right) g_1 + A_2(x) g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left( \frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right).$$

**Proof.** Let us consider $f(x, y)$ as defined above and let us compute $\omega \wedge df$:

$$\omega \wedge df = \left( \frac{\partial g_1}{\partial x} w'(x) + g_1 w''(x) - \frac{\partial g_0}{\partial x} w(x) - g_0 w'(x) \right) A_2(x) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) + A_1(x) g_1 g_0 \left( \frac{\partial g_1}{\partial y} w'(x) - \frac{\partial g_0}{\partial y} w(x) \right)$$

$$+ A_2(x) g_0^2 + A_2(x) \left( g_1 \frac{\partial g_0}{\partial x} - g_0 \frac{\partial g_1}{\partial x} \right) \right]$$

$$= g_1 \left( \frac{\partial g_1}{\partial y} - \frac{\partial g_0}{\partial y} \right) A_2(x) w''(x) +$$

$$g_1 (A_1(x) g_0 + A_0(x) g_1) \left( \frac{\partial g_1}{\partial y} w'(x) - \frac{\partial g_0}{\partial y} w(x) \right) +$$

$$\left( A_2(x) \left( \frac{\partial g_0}{\partial x} \frac{\partial g_1}{\partial y} - \frac{\partial g_1}{\partial x} \frac{\partial g_0}{\partial y} \right) + A_2(x) g_0 \frac{\partial g_0}{\partial y} \right) (g_1 w'(x) - g_0 w(x))$$

Since $w(x)$ is a solution of \((6)\), we can substitute $A_2(x) w''(x)$ by $-A_1(x) w'(x) - A_0(x) w(x)$. Therefore,

$$\omega \wedge df = \left[ \left( A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y} \right) g_1 + A_2(x) g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left( \frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right) \right] f(x, y).$$

Then, we have that the function $f(x, y)$ is an invariant for system \((7)\) and has the written polynomial cofactor.

**Theorem 8** Let $\{w_1(x), w_2(x)\}$ be a set of fundamental solutions of equation \((6)\). We define $f_i(x, y) := g_1(x, y)w_i'(x) - g_0(x, y)w_i(x)$, $i = 1, 2$. Then, system \((7)\) has a first integral $H(x, y)$ defined by

$$H(x, y) := \frac{f_1(x, y)}{f_2(x, y)} = \frac{g_1(x, y) w_1'(x) - g_0(x, y) w_1(x)}{g_1(x, y) w_2'(x) - g_0(x, y) w_2(x)}.$$
Proof. By Theorem 7 we have that $f_i(x, y) := g_1(x, y)w_i'(x) - g_0(x, y)w_i(x)$, $i = 1, 2$, are invariants for system (7), both with the polynomial cofactor

$$k(x, y) = \left( A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y} \right) g_1 + A_2(x) g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left( \frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right).$$

We remark that $f_1/f_2$ cannot be constant since the two solutions $w_i(x)$, $i = 1, 2$, are independent. Therefore,

$$\omega \wedge dH = \frac{f_2(\omega \wedge df_1) - f_1(\omega \wedge df_2)}{f_2^2} = \frac{f_2 k_1 f_1 - f_1 k_1 f_2}{f_2^2} \equiv 0.$$

So, $H(x, y)$ is a first integral of system (7).

Lemma 9 The function defined by

$$q(x) := A_2(x) \exp \left( \int_{x_0}^{x} \frac{A_1(s)}{A_2(s)} ds \right)$$

is an invariant for system (7), with cofactor $(A_1(x) + A_2'(x)) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right)$.

We notice that $q(x)$ is a product of invariant algebraic curves and exponential factors for system (6), with complex exponents.

Proof. We compute $\omega \wedge dq$ and we have

$$\omega \wedge dq = \omega \wedge \left( \frac{A_1(x) + A_2'(x)}{A_2(x)} \right) q dx = (A_1(x) + A_2'(x)) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) q.$$

We notice that this algebraic cofactor has degree $\leq d - 1$ provided that system (7) has degree $d$.

Proposition 10 We use the same notation as in Theorem 4. Let $w(x)$ be a nonzero solution of (7) and we define $f(x, y) := w'(x) - g(x, y)w(x)$ and $q(x)$ as in Lemma 9. The function $V(x, y) = q(x)f(x, y)^2$ is an inverse integrating factor of system (7).

Proof. We only need to verify that $\omega \wedge dV + V d\omega = 0$. We have that

$$d\omega = - \left[ 2 A_0(x) g_1 \frac{\partial g_1}{\partial y} + A_1(x) \left( g_0 \frac{\partial g_1}{\partial y} + g_1 \frac{\partial g_0}{\partial y} \right) + 2 A_2(x) g_0 \frac{\partial g_0}{\partial y} \
+ 2 A_2(x) \left( \frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right) + A_2'(x) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) \right].$$
Then,
\[
\omega \wedge dV = \omega \wedge (2q f \, df + f^2 \, dq) = 2q f (\omega \wedge df) + f^2 (\omega \wedge dq)
\]

\[
= \left[ 2A_0(x) \frac{\partial g_1}{\partial y} + 2A_1(x) \frac{\partial g_0}{\partial y} + 2A_2(x) g_0 \frac{\partial g_0}{\partial y} + 2A_2(x) \left( \frac{\partial g_1}{\partial y} - \frac{\partial g_0}{\partial y} \right) \right] V
\]

\[
+ A'_2(x) \left( g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) V
\]

\[
= -V d\omega.
\]

We remark that Theorem 8 gives, in general, a non Liouvillian first integral for the planar polynomial systems (7). In Section 3 we analyze some polynomial systems constructed from Theorem 8 that have no Liouvillian first integral.

We consider now a linear homogeneous ordinary differential equation of order 1 such as

\[
w'(x) + A(x) w(x) = 0,
\]

where \( x \in \mathbb{R} \), \( w'(x) = dw(x)/dx \) and \( A(x) = A_0(x)/A_1(x) \) with \( A_i(x) \in \mathbb{R}[x] \) and \( A_1(x) \neq 0 \). We give analogous results for this case whose proofs are not given to avoid non useful repetitions.

**Theorem 11** Given \( g(x, y) = g_0(x, y)/g_1(x, y) \) with \( g_i(x, y) \in \mathbb{R}[x, y], g_1(x, y) \neq 0 \) and \( \partial g/\partial y \neq 0 \) and \( h(x) = h_0(x)/h_1(x) \) with \( h_i(x) \in \mathbb{R}[x] \) and \( h_1(x) \neq 0 \), each nonzero solution \( w(x) \) of equation (8) is related to a finite number of solutions \( y = y(x) \) of the rational equation

\[
\frac{dy}{dx} = \frac{A_1(x) h_0(x) g_1^2 - A_0(x) h_1(x) g_0 g_1 - A_1(x) h_1(x) \left( g_1 \frac{\partial g_0}{\partial x} - g_0 \frac{\partial g_1}{\partial x} \right)}{\quad A_1(x) h_1(x) \left( g_1 \frac{\partial g_0}{\partial y} - g_0 \frac{\partial g_1}{\partial y} \right) },
\]

by the functional change

\[
w(x) = g(x, y) - \exp \left( - \int_0^x A(s) \, ds \right) \left[ \int_0^x \exp \left( \int_0^s A(r) \, dr \right) h(s) \, ds \right].
\]
Theorem 12 We consider the 1-form related to equation (3)

\[ \omega = \left[ A_1(x) \, h_0(x) \, g_1^2 - A_0(x) \, h_1(x) \, g_0 \, g_1 - A_1(x) \, h_1(x) \left( g_1 \, \frac{\partial g_0}{\partial x} - g_0 \, \frac{\partial g_1}{\partial x} \right) \right] \, dx \\
- A_1(x) \, h_1(x) \left( g_1 \, \frac{\partial g_0}{\partial y} - g_0 \, \frac{\partial g_1}{\partial y} \right) \, dy. \]

Let \( w(x) \) be any nonzero solution of equation (8), that is, for \( C \in \mathbb{R} - \{0\} \) we have \( w(x) = C \exp \left( - \int_0^x A(s) \, ds \right) \).

Then, the function

\[ f(x, y) := g_1 \, w(x) - g_0 + g_1 \exp \left( - \int_0^x A(s) \, ds \right) \left[ \int_0^x \exp \left( \int_0^s A(r) \, dr \right) \, h(s) \, ds \right] \]

is invariant for the polynomial system (3), with the polynomial cofactor

\[ k(x, y) = -A_0(x) \, h_1(x) \, g_1 \, \frac{\partial g_0}{\partial y} + A_1(x) \, h_0(x) \, g_1 \, \frac{\partial g_1}{\partial y} \]

\[ + A_1(x) \, h_1(x) \left( \frac{\partial g_0}{\partial y} \, \frac{\partial g_1}{\partial x} - \frac{\partial g_1}{\partial y} \, \frac{\partial g_0}{\partial x} \right). \]

Lemma 13 The function \( q(x, y) = g_1(x, y) \exp \left( \int_0^x -A(s) \, ds \right) \) is an invariant for system (3) with the same polynomial cofactor as \( f(x, y) \).

Theorem 14 We use the same notation as in Theorem 12 and Lemma 13. The function \( H(x, y) \) defined by \( H(x, y) := f(x, y) / q(x, y) \) is a first integral for system (3) and the function \( V(x, y) := A_1(x) \, h_1(x) \, g_1(x, y) \, q(x, y) \) is an inverse integrating factor.

We remark that \( H(x, y) \) is a Liouvillian function and, therefore, a system (3) has always a Liouville first integral.

In Section 3 we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed following Theorem 14.

3 Examples of families of quadratic systems.

3.1 Quadratic systems with invariant algebraic curves of arbitrarily high degree linear in one variable.

We first consider the examples of families of quadratic systems with algebraic solutions of arbitrarily high degree appearing in [7]. In that work all the invariant
algebraic curves linear in the variable $y$, that is, defined by $f(x, y) = p_1(x)y + p_2(x)$, where $p_1$ and $p_2$ are polynomials, are determined.

The example appearing in [8] is a further study of an example appearing in [7] and the example given in [11] is also described in [7]. We show that all these quadratic systems, with an invariant algebraic curve of arbitrary degree can be constructed by the method explained in the previous section. Moreover, we give the explicit expression of a first integral for any value of the parameter $n$, even in the case when $n$ is not a natural number. If $n$ is a natural number, we obtain the invariant algebraic curves of arbitrary degree and a Liouvillian first integral. However, when $n \not\in \mathbb{N}$ we obtain polynomial systems with a non Liouvillian first integral.

As it is shown in [7], all these families of systems can be written, after an affine change of variables if necessary, in the form

$$\dot{x} = \Omega_1(x), \quad \dot{y} = (2n + 1)L'(x)\Omega_1(x) - \frac{n(n+1)}{2}\Omega_1(x)\Omega_1'(x) - L(x)^2 + y^2,$$  

(10)

where $\Omega_1(x)$ is any quadratic polynomial, $L(x)$ is any linear polynomial and $' = d/dx$. We have that system (10) has an invariant curve $f(x, y) = 0$, where $f(x, y) := p_1(x)y + \Omega_1(x)p_1'(x) - L(x)p_1(x)$, with a cofactor $y + L(x)$, where $p_1(x)$ is a solution of the second order linear differential equation

$$\Omega_1(x)w''(x) + (\Omega_1'(x) - 2L(x))w'(x) + \frac{n}{2}(4L'(x) - (n+1)\Omega_1''(x))w(x) = 0.$$  

(11)

In [7] it is shown that, in case $n \in \mathbb{N}$, an irreducible polynomial of degree $n$ belonging to a family of orthogonal polynomials is a solution of equation (11). For instance, when $\Omega_1(x) = 1$, we get the Hermite polynomials, when $\Omega_1(x) = x$, we get the Generalized Laguerre polynomials and when $\Omega_1(x) = 1 - x^2$, we get the Jacobi polynomials.

We consider again the general case in which $n \in \mathbb{R}$ and we define $A_2(x) := \Omega_1(x)$, $A_1(x) := \Omega_1'(x) - 2L(x)$ and $A_0(x) := \frac{n}{2}(4L'(x) - (n+1)\Omega_1''(x))$. We have the linear differential equation (11) in the same notation as in Theorem 6 and we consider $g(x, y) := \frac{L(x) - y}{A_2(x)}$.

The system obtained by the method explained in Section 2 exactly coincides with system (10). We consider a set of fundamental solutions of equation (11) $\{w_1(x), w_2(x)\}$ and applying Theorem 8 we have a first integral for system (10) for any value of the parameter $n \in \mathbb{R}$.

In case $n \in \mathbb{N}$ we have that $w_1(x)$ degenerates to a polynomial and $w_1'(x) - g(x, y)w_1(x) = 0$ coincides with the algebraic curve given in the work [7].
We explicitly give the first integral for each of the families described in [7] and for $n \in \mathbb{R}$. We have that $A_2(x)$ is a non-null quadratic polynomial in this case, and depending on its number of roots, we can transform it by a real affine change of variable to one of the following forms: $A_2(x) = 1, x, x^2, 1 - x^2, 1 + x^2$.

If $A_2(x) = 1$, we can choose $L(x) = x$ by an affine change of coordinates. A set of fundamental solutions $\{w_1(x), w_2(x)\}$ for (11) with $n \in \mathbb{R}$ is

$$w_1(x) = 2^n \sqrt{\pi} \left( \frac{1}{\Gamma\left(\frac{1-n}{2}\right)} \right) _1 F_1 \left( \left[ \frac{-n}{2}, \frac{1}{2}, x^2 \right] \right) - \frac{2x}{\Gamma\left(-\frac{n}{2}\right)} _1 F_1 \left( \left[ \frac{1-n}{2}, \frac{3}{2}, x^2 \right] \right),$$

$$w_2(x) = 2^n \sqrt{\pi} \left( \frac{1}{\Gamma\left(\frac{1-n}{2}\right)} \right) _1 F_1 \left( \left[ \frac{-n}{2}, \frac{1}{2}, x^2 \right] \right) + \frac{2x}{\Gamma\left(-\frac{n}{2}\right)} _1 F_1 \left( \left[ \frac{1-n}{2}, \frac{3}{2}, x^2 \right] \right),$$

where $\Gamma(x)$ is the Euler’s–Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ and $_1 F_1(a; b; x)$ is the confluent hypergeometric function defined by the series

$$ _1 F_1(a; b; x) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

with $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, the Pochhammer symbol. See [1] for further information about these functions. So, a first integral for this system is the expression given in Theorem 8: $H(x, y) := f_1(x, y) / f_2(x, y)$, where

$$f_{1,2}(x, y) = \pm \Gamma\left(\frac{1-n}{2}\right) \left[ 6 (xy - x^2 + 1) _1 F_1 \left( \left[ \frac{1-n}{2}, \frac{3}{2}, x^2 \right] \right) \right. \left. - 4 (n - 1) x^2 _1 F_1 \left( \left[ \frac{3-n}{2}, \frac{5}{2}, x^2 \right] \right) \right] + 3 \Gamma\left(-\frac{n}{2}\right) \left[ 2nx _1 F_1 \left( \left[ 1 - \frac{n}{2}, \frac{3}{2}, x^2 \right] \right) + (x - y) _1 F_1 \left( \left[ -\frac{n}{2}, \frac{1}{2}; x^2 \right] \right) \right].$$

When $n \in \mathbb{N}$, we have that (11) corresponds to the equation for Hermite polynomials and $w_1(x)$ coincides with the Hermite polynomial of degree $n$. The invariant algebraic curve given in [7] corresponds to $f_1(x, y) = 0$.

If $A_2(x) = x$, we choose $L(x) = \frac{1}{2}(x - \alpha)$, where $\alpha$ is an arbitrary real constant, and a set of fundamental solutions for (11) is:

$$w_1(x) = \frac{(\alpha + 1)_n}{\Gamma(n + 1)} _1 F_1 \left( \left[ -n, \alpha + 1; x \right] \right), \quad w_2(x) = x^{-\alpha} _1 F_1 \left( \left[ -\alpha - n, 1 - \alpha; x \right] \right).$$
The first integral for this system is
\[ H(x, y) = x^\alpha h_1(x, y)/h_2(x, y) \]
with
\[ h_1(x, y) = (2y - x + \alpha) (\alpha + 1) \frac{x}{1 + x} F_1(-n; \alpha + 1; x) - 2n x \frac{x}{1 + x} F_1(1 - n; \alpha + 2; x), \]
\[ h_2(x, y) = (2y - x + \alpha) (\alpha - 1) \frac{x}{1 - x} F_1(-\alpha - 1; 1 - \alpha; x) - 2(\alpha + n) x \frac{x}{1 - x} F_1(1 - \alpha - n; 2 - \alpha; x). \]
The first integral as given in Theorem 8 is \( f_1(x, y)/f_2(x, y) \) and we notice that
\[ H(x, y) = cf_1(x, y)/f_2(x, y) \]
where \( c \in \mathbb{R} \setminus \{0\} \). We do not write \( c \) in terms of the parameters of the system to simplify notation.

When \( n \in \mathbb{N} \), we have that \( L \) correponds to the equation of Generalized Laguerre polynomials and \( w_1(x) \) coincides with the Generalized Laguerre polynomial \( L_n^{(a)} \). The invariant algebraic curve given in [7] corresponds to \( f_1(x, y) = 0 \), where
\[ f_1(x, y) := w_1'(x) - g(x, y)w_1(x). \]

If \( A_2(x) = x^2 \), the birational transformation yet described in [7], \( x = 1/X \) and \( y = (1/X)(1/2 - Y) \), makes this case equivalent to the previous one.

If \( A_2(x) = 1 - x^2 \), we choose \( L(x) = \frac{1}{2}((\alpha + \beta)x + (\alpha - \beta)) \), where \( \alpha, \beta \) are two arbitrary real constants, and a set of fundamental solutions for \( L \) is:
\[ w_1(x) = \frac{(\alpha + 1)n}{\Gamma(n + 1)} \frac{x}{1 + x} F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1 - x}{2}\right), \]
\[ w_2(x) = (1 - x)^{-\alpha} \frac{x}{1 - x} F_1\left(-\alpha - n, 1 + \beta + n; 1 - \alpha; \frac{1 - x}{2}\right), \]
where \( F_1(a_1, a_2; b; x) \) is the hypergeometric function defined by
\[ F_1(a_1, a_2; b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k x^k}{(b)_k k!}. \]

The first integral given in Theorem 8 is
\[ H(x, y) = (1 - x)^{\alpha}h_1(x, y)/h_2(x, y), \]
where
\[ h_1 = n (1 + \alpha + \beta + n) (x^2 - 1) \frac{x}{1 + x} F_1\left(1 - n, 2 + \alpha + \beta + n; 2 + \alpha; \frac{1 - x}{2}\right) + \]
\[ (\alpha + 1)((\alpha + \beta)x + (\alpha - \beta) - 2y) \frac{x}{1 - x} F_1\left(-n, 1 + \alpha + \beta + n; 1 + \alpha; \frac{1 - x}{2}\right), \]
\[ h_2 = (\alpha - 1)((\alpha - \beta)x + (\alpha + b) + 2y) \frac{x}{1 - x} F_1\left(-\alpha - n, 1 + \beta + n; 1 - \alpha; \frac{1 - x}{2}\right) + \]
\[ (\alpha + n)(1 + \beta + n) (x^2 - 1) \frac{x}{1 - x} F_1\left(1 - \alpha - n, 2 + \beta + n; 2 - \alpha; \frac{1 - x}{2}\right). \]
The first integral as given in Theorem 8 is \( f_1(x, y)/f_2(x, y) \) and we notice that 
\( H(x, y) = cf_1(x, y)/f_2(x, y) \) where \( c \in \mathbb{R} - \{0\} \). As before, we do not write \( c \) in terms of the parameters of the system to simplify notation.

When \( n \in \mathbb{N} \), we have that (11) corresponds to the equation of Jacobi polynomials and \( w_1(x) \) coincides with the Jacobi polynomial \( P^{(\alpha,\beta)}_n(x) \) and the invariant algebraic curve given in [7] corresponds to \( f_1(x, y) = 0 \), where \( f_1(x, y) := w'_1(x) - g(x, y)w_1(x) \).

If \( A_2(x) = 1 + x^2 \) the complex affine change of variable \( x = iX \) makes this case equivalent to the previous one, as it is shown in [7].

We have re-encountered by this method all the examples appearing in [7] from a unified point of view. In addition, in this work we have given an explicit expression of a first integral for each case and for any value of the parameter \( n \in \mathbb{R} \). To this end, we have found invariants for the system and we have applied the generalization of Darboux’s method as explained in [14] to be able to construct a first integral which is, in general, of non Liouvillian type.

### 3.2 A Lotka-Volterra system

As it has been explained in the introduction, the first counterexample to Lins Neto conjecture was given by J. Moulin-Ollagnier in [19]. His example is a quadratic system with two invariant straight lines and an irreducible invariant algebraic curve \( f(x, y) = 0 \) of degree \( 2\ell \) when \( \ell \in \mathbb{N} \). This gives a family of systems depending on the parameter \( \ell \) which have a Darboux inverse integrating factor when \( \ell \in \mathbb{N} \) but no rational first integral. The method used in [19] only shows the existence of such invariant algebraic curve but no closed formula to compute it is given. We give an explicit expression for an invariant by means of Bessel functions for any value of \( \ell \in \mathbb{R} - \{\frac{1}{2}\} \) which, in the particular case \( \ell \in \mathbb{N} \) degenerates to the algebraic curve encountered in [19].

We show that after a birrational transformation, this example coincides with a system constructed with the method explained in Section 2. A birrational transformation is a rational change of variables whose inverse is also rational. This kind of transformations bring polynomial systems such (2) to polynomial systems and do not change the Liouvillian or non Liouvillian character of the first integral.

Let us consider the system appearing in [19] but assuming that \( \ell \in \mathbb{R} - \{\frac{1}{2}\} \)

\[
\dot{x} = x \left( 1 - \frac{x}{2} + y \right), \quad \dot{y} = y \left( -\frac{2\ell + 1}{2\ell - 1} + \frac{x}{2} - y \right).
\]  

(12)

We make the birrational transformation

\[
x = \frac{4uv}{1 - 2\ell}, \quad y = \frac{1 - 2\ell}{4v},
\]
whose inverse is

\[ u = xy, \quad v = \frac{1 - 2\ell}{4y}. \]

By this transformation, system (12) becomes

\[
\dot{u} = \frac{2u}{1 - 2\ell}, \quad \dot{v} = \frac{1 - 2\ell}{4} + \frac{2\ell + 1}{2\ell - 1} v + \frac{2u}{2\ell - 1} v^2.
\]

(13)

We notice that the equation for the orbits satisfied by the variable \( v \) as a function of \( u \) is a Ricatti equation.

Let us consider \( g(u, v) := v \) and the linear differential equation of order 2 given by

\[
u w''(u) + \frac{1}{2}(1 + 2\ell) w'(u) - \frac{1}{8}(1 - 2\ell)^2 w(u) = 0.
\]

(14)

Applying the method given in the previous section, this linear differential equation gives system (13) modulus a change of time.

A set of two fundamental solutions for equation (14) is given by

\[
w_1(u) = u^{(1-2\ell)/4} I_{\frac{1}{2} - \ell} \left( (1 - 2\ell) \sqrt{\frac{u}{2}} \right), \quad w_2(u) = u^{(1-2\ell)/4} I_{\frac{1}{2} + \ell} \left( (1 - 2\ell) \sqrt{\frac{u}{2}} \right),
\]

(15)

provided that \( \ell \) is not of the form \( \frac{1}{2}(1 - 2r) \), with \( r \) an integer number, because in this case \( w_1 \) and \( w_2 \) are linearly dependent. The function \( I_\nu(u) \) is the Modified Bessel function defined by the solution of the second order differential equation

\[
u u^2 w''(u) + u w'(u) - (u^2 + \nu^2) w(u) = 0,
\]

(16)

and being bounded when \( u \to 0 \) in any bounded range of \( \arg(u) \) with \( \Re(e(u)) \geq 0 \). See [1] for further information about this function.

Hence, by Theorem 8 we have that \( H(u, v) = f_1(u, v)/f_2(u, v) \), where \( f_i(u, v) := w_i'(u) - vw_i(u) \) for \( i = 1, 2 \), is a first integral for system (13). For a sake of simplicity we consider the following renaming of the independent variable \( u = 2z^2/(1 - 2\ell)^2 \). This is not a birational transformation and that’s why we only use it to simplify notation. The function \( H \) writes as:

\[
H = \frac{(1 - 2\ell)^2 I_{\frac{1+2\ell}{2}}(z) - 4\nu z I_{\frac{2\ell-1}{2}}(z)}{(1 - 2\ell)^2 I_{-\frac{1+2\ell}{2}}(z) - 4\nu z I_{-\frac{2\ell-1}{2}}(z)}.
\]

(17)

By Theorem 7 we have that \( f_i(u, v) \), \( i = 1, 2 \) are invariants with the same polynomial cofactor \( k \) for system (13), so the curve \( f(u, v) = 0 \) given by \( f(u, v) = \pi z^{2\ell+1}(f_2^2(u, v) - f_1^2(u, v)) \) is also an invariant. We multiply by \( \pi \) only for esthetic reasons.
Now we assume that \( \ell \in \mathbb{N} \) and we show that \( f = 0 \) defines an invariant algebraic curve. To this end we use the following formulas appearing in \([11, 23]\). When \( \nu - \frac{1}{2} = n \in \mathbb{Z} \), we define \( c(n) = -n\pi\sqrt{-1/2} \) and the following relation is satisfied:

\[
I_\nu(z) = -\frac{1}{\sqrt{z}} e^{c(n)} \sqrt{\frac{2}{\pi}} \left\{ \sinh (c(n) - z) \sum_{k=0}^{\lfloor \nu \rfloor - 1} \frac{(|\nu| + 2k - \frac{1}{2})!}{(2k)!(|\nu| - 2k - \frac{1}{2})!(2z)^{2k}} 
+ \cosh (c(n) - z) \sum_{k=0}^{\lfloor \nu \rfloor - 1} \frac{(|\nu| + 2k + \frac{1}{2})!}{(2k + 1)!(|\nu| - 2k - \frac{1}{2})!(2z)^{2k+1}} \right\},
\]

(18)

where \( \lfloor x \rfloor \) stands for the greatest integer \( k \) such that \( k \leq x \) and \( |\nu| \) stands for the absolute value.

From the former equation we obtain the following two equalities, with \( \nu - \frac{1}{2} = n \in \mathbb{Z} \) and \( \ell \in \mathbb{N} \),

\[
I_{\nu}^2(z) - I_{-\nu}^2(z) = \frac{2}{\pi z} \sum_{k=0}^{n} (-1)^{k+1} \frac{(2n - k)! (2n - 2k)!}{k!((n - k)!)^2} \left( \frac{1}{2z} \right)^{2(n-k)},
\]

(19)

\[
I_{\nu+\frac{1}{2}}(z)I_{-\nu+\frac{1}{2}}(z) - I_{-(\nu+\frac{1}{2})}(z)I_{-(\nu-\frac{1}{2})}(z) = (-1)^\ell \frac{2}{\pi z} \left[ \left( \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{(\ell + 2i)!}{(2i)!((\ell - 2i)! \left( \frac{1}{2z} \right)^{2i} \right) \left( \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{(\ell + 2j)!}{(2j + 1)!(\ell - 2j - 2)!} \left( \frac{1}{2z} \right)^{2j} \right) - \left( \sum_{i=0}^{\lfloor \frac{\ell - 1}{2} \rfloor} \frac{(\ell + 2i + 1)!}{(2i + 1)!(\ell - 2i - 1)!} \left( \frac{1}{2z} \right)^{2i+1} \right) \left( \sum_{j=0}^{\lfloor \frac{\ell - 1}{2} \rfloor} \frac{(\ell + 2j - 1)!}{(2j)!(\ell - 2j - 1)!} \left( \frac{1}{2z} \right)^{2j} \right) \right. \right].
\]

(20)

Then, we have that \( f_1(z, v) = (1 - 2\ell)^2 I_{\nu+\frac{1}{2}}(z) - 4vz I_{-\nu+\frac{1}{2}}(z) \) and \( f_2(z, v) = (1 - 2\ell)^2 I_{-(\nu+\frac{1}{2})}(z) - 4vz I_{-(\nu-\frac{1}{2})}(z) \), and we write \( f \) arranged in powers of \( v \):

\[
f(z, v) = \pi z^{2\ell+1} \left( (1 - 2\ell)^4 (I_{\nu+\frac{1}{2}}^2(z) - I_{-\nu+\frac{1}{2}}^2(z)) 
- 8vz(1 - 2\ell)^2 (I_{\nu+\frac{1}{2}}(z)I_{-\nu+\frac{1}{2}}(z) - I_{-(\nu+\frac{1}{2})}(z)I_{-(\nu-\frac{1}{2})}(z)) 
+ 16v^2z^2 (I_{\nu+\frac{1}{2}}^2(z) - I_{-\nu+\frac{1}{2}}^2(z)) \right).
\]

Let us consider each coefficient of \( v \) in \( f(z, v) \) separately and we will show that it is an even polynomial in the variable \( z \). The coefficient in \( f(z, v) \) of \( v^0 \) is:

\[
\pi z^{2\ell+1} (1 - 2\ell)^4 (I_{\nu+\frac{1}{2}}^2(z) - I_{-\nu+\frac{1}{2}}^2(z))
\]
which by equation (19) is an even polynomial in the variable $z$ of degree $2\ell$. The coefficient in $f(z,v)$ of $v^2$ is:

$$16\pi z^{2\ell+3}(I_{\ell-\frac{1}{2}}^2(z) - I_{-(\ell+\frac{1}{2})}^2(z)),$$

which also by equation (19) is an even polynomial in the variable $z$ of degree $2\ell + 2$.

Finally, the coefficient in $f(z,v)$ of $v^1$ is:

$$8\pi (1 - 2\ell)^2 z^{2\ell+2}(I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z) - I_{-(\ell+\frac{1}{2})}(z)I_{-(\ell-\frac{1}{2})}(z)),$$  

which by equation (20) is an even polynomial in the variable $z$ of degree $2\ell$.

Hence, we have that $f(z,v)$ is an even polynomial in the variable $z$ of total degree $2\ell + 4$. When rewriting $z = (1 - 2\ell)^2 u^2 / \sqrt{2}$ we have that $f(u,v)$ is a polynomial of total degree $\ell + 2$ which is irreducible. The fact of being irreducible is easily seen because it is a polynomial of degree two in $v$ and it cannot be decomposed in linear factors (the discriminant is not a polynomial raised to an even power) and the coefficients of $v^0$ and $v^2$ do not have any root in common.

Undoing the birational change of variables we get that $f(x,y)$ is an irreducible polynomial of degree $2\ell$ given by:

$$f(x,y) = x^{\ell+\frac{1}{2}} y^{\ell-\frac{1}{2}} \left[ 2y \left( I_{\ell+\frac{1}{2}}^2(z) - I_{-(\ell+\frac{1}{2})}^2(z) \right) + 2\sqrt{2} \sqrt{xy} \left( I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z) - I_{-(\ell+\frac{1}{2})}(z)I_{-(\ell-\frac{1}{2})}(z) \right) + x \left( I_{\ell+\frac{1}{2}}^2(z) - I_{-(\ell-\frac{1}{2})}^2(z) \right) \right],$$

where $z$ is the same variable as before, that is, $z = (1 - 2\ell)^2 u^2 / \sqrt{2}$.

By equation (17) we can write the first integral for system (12) for any value of $\ell \in \mathbb{R} - \{\frac{1}{2}(1 - 2r) \mid r \in \mathbb{N}\}$:

$$H(x,y) = \frac{\sqrt{2y} I_{\ell+\frac{1}{2}}(z) - \sqrt{x} I_{\ell-\frac{1}{2}}(z)}{\sqrt{2y} I_{-(\ell+\frac{1}{2})}(z) - \sqrt{x} I_{-(\ell-\frac{1}{2})}(z)}.$$

We have studied system (12) for any value of the parameter $\ell \in \mathbb{R} - \{\frac{1}{2}(1 - 2r) \mid r \in \mathbb{N}\}$ giving an explicit expression for a first integral using Theorem 6 and the Generalized Darboux’s theory as explained in [14]. This first integral is not of Liouvillian type. Moreover, we give one of its invariants with a polynomial cofactor. In the particular case $\ell \in \mathbb{N}$, this invariant is the invariant algebraic curve whose existence was proved in [19].
3.3 A new example of a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree

We give another example of a family of quadratic systems with an irreducible invariant algebraic curve of degree $2\ell$ when $\ell \in \mathbb{N}$, where $\ell$ is a parameter of the family. This family also depends on the parameter $a \in \mathbb{R}$.

Let us consider the quadratic system

\[
\begin{align*}
\dot{x} &= (2a-1)\ell x - a(2\ell - 1)y + 2a(a-\ell)(2\ell - 1)x^2 - 2a^2(2\ell - 1)^2 xy, \\
\dot{y} &= y(2(2a-1)\ell + 2a(2a - 2\ell - 1)(2\ell - 1)x - 4a^2(2\ell - 1)^2 y),
\end{align*}
\]

where $a, \ell \in \mathbb{R}$ which satisfy $a \neq 0$, $\ell \neq \frac{1}{2}$ and $(2\ell - 1)a^2 - 2\ell \neq 0$. An straightforward computation shows that system (22) has $y = 0$ and $y - x^2 = 0$ as invariant algebraic curves.

Let us consider the following birational transformation $x = Y, y = XY^2$ whose inverse is $X = y/x^2$ and $Y = x$. In these new variables system (22) becomes

\[
\begin{align*}
\dot{X} &= 2a(2\ell - 1)(X - 1)XY, \\
\dot{Y} &= ((2a - 1)\ell + a(2\ell - 1)(2a - 2\ell - X)Y - 2a^2(2\ell - 1)^2 XY^2) Y.
\end{align*}
\]

By a change of the time variable we can divide this system by $Y$ and the resulting system coincides with the one described in Theorem 6 taking $A_2(X) := 2X(X - 1)^2$, $A_1(X) := (2\ell - 2a + 3X)(X - 1)$, $A_0(X) := \ell (1 - 2a)$ and $g(X, Y) := a(2\ell - 1)Y/(X - 1)$. The equation $A_2(X)w''(X) + A_1(X)w'(X) + A_0(X)w(X) = 0$ has the following set of fundamental solutions in this case:

\[
\begin{align*}
w_1(X) &= (X - 1)^{-\ell} 2F_1 \left( \frac{1}{2} - \ell, -\ell; a - \ell; X \right), \\
w_2(X) &= (X - 1)^{-\ell} X^{1-a+\ell} 2F_1 \left( 1 - a, \frac{3}{2} - \ell; 2 - a + \ell; X \right).
\end{align*}
\]

By Theorem 7 $f_i(X, Y) = w_i'(X) - g(X, Y)w_i(X), i = 1, 2$, define invariants with a polynomial cofactor for system (23). Moreover, by Theorem 8 we have a non Liouvillian first integral given by $H(X, Y) = f_1(X, Y)/f_2(X, Y)$.

In the particular case $\ell \in \mathbb{N}$, we notice that $f_1(X, Y) = 0$ is a rational function. It is an easy computation to show that this rational function is a polynomial when rewritten in coordinates $x$ and $y$. This polynomial gives place to an invariant algebraic curve of degree $2\ell$ for system (22). That is, by undoing the birrational transformation, we deduce that $f_1(x, y)$ is an irreducible invariant algebraic curve for system (22), given by:

\[
f_1(x, y) = 2(a - \ell)(\ell + (2\ell - 1)a x)x^{2\ell - 1} 2F_1 \left( \frac{1}{2} - \ell, -\ell; a - \ell; \frac{y}{x^2} \right)
\]
$$+\ell(2\ell-1)x^{2\ell-3}(x^2-y)\;_2F_1\left(\frac{3}{2}-\ell, 1-\ell; 1+a-\ell; \frac{y}{x^2}\right).$$

It is easy to see that the polynomial $f_1(x, y)$ has degree $2\ell$ and the cofactor associated to the invariant algebraic curve $f_1(x, y) = 0$ is $\ell(2\ell-1)((2a-1)+4a(a-\ell)x-4(2\ell-1)a^2y)$.

The first integral for (22) is given by $H(x, y) = y^{a-\ell}f_1(x, y)/h(x, y)$, where

$$h(x, y) = 2(a-\ell-2)\left[(a-\ell-1)x^2 + (1-a-a(2\ell-1)x)y\right]x^{7-2a}\;_2F_1\left(1-a, \frac{3}{2}-a; 2-a-\ell; \frac{y}{x^2}\right) + (a-1)(2a-3)x^{5-2a}(x^2-y)y\;_2F_1\left(2-a, \frac{5}{2}-a; 3-a+\ell; \frac{y}{x^2}\right).$$

We notice that when both $a$ and $\ell$ belong to the set of natural numbers, we have that $h(x, y) = 0$ is an invariant algebraic curve different from $f_1(x, y) = 0$. Then we have a quadratic system with a rational first integral $H(x, y)$ with arbitrary degree.

### 3.4 A complete family of quadratic systems with a center at the origin

In this subsection we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed using Theorem 11. The family encountered corresponds to the reversible case, see [23].

The family of quadratic systems depends on twelve parameters, but up to affine transformations and positive time rescaling, we get a family of five essential parameters. We have taken a system (9) and we have chosen $g(x, y) := y^2$, $h(x) := 2x(dx-1)/(1+ax)$ and $A(x) := 2b/(1+ax)$, where $a, b, d$ are real parameters. Using Theorem 11 we have encountered the 3-parameter family of quadratic systems next described. We remark that in spite of the simplicity of the chosen polynomials $g(x, y)$, $A(x)$ and $h(x)$, we amazingly obtain the complete family of quadratic systems with a reversible center at the origin. We notice that other choices of the functions $g(x, y)$, $A(x)$ and $h(x)$ would give place to other families of polynomial systems.

Let us recall that a center is an isolated singular point of an equation with a neighborhood foliated of periodic orbits. When the linear approximation of an equation near a singular point has non-null purely imaginary eigenvalues, the point can be a center or a focus. To distinguish between these two possibilities is the so-called center problem. H. Poincaré gave a method to solve it by defining a numerable set of values, called Liapunov-Poincaré constants, which are all zero when the singular point is a center and at least one of them is not null when it is
a focus. When these constants are computed from a family of systems, they are polynomials on the coefficients of the family. Hilbert’s Nullstellensatz ensures that there always exists a finite number of independent polynomials which generates the whole ideal made up with all these Liapunov-Poincaré polynomials. The zero-set of these independent polynomials gives place to the center subfamilies. The reader is referred to [23, 24] for a survey on this subject.

The computation of these center cases for the family of quadratic systems was done by Dulac [13] for the case of complex systems and a proof for real systems is given in [18]. We also refer Bautin [3] who showed the existence of only three independent constants. The computation of the zero set of these three independent values gives place to four complete families of quadratic systems with a center at the origin which are described in [24].

Let us now consider an equation (9) such as \((1 + ax)w'(x) + 2bw(x) = 0\) and \(g(x, y)\) and \(h(x)\) as formerly defined. The rational equation as constructed in Theorem 11 is

\[
\frac{dy}{dx} = \frac{-x + dx^2 - by^2}{y + axy},
\]

which gives the corresponding quadratic planar system

\[
\dot{x} = y + axy, \quad \dot{y} = -x + dx^2 - by^2.
\]  

(24)

We suppose that \(ab(a + b)(a + 2b)(a + b + d) \neq 0\). In case this value is zero, the origin of system (24) is still a center but with a Darboux integrating factor instead of a Darboux first integral. This particular case can also be studied by our method, but we do not write it to avoid giving examples without essential differences.

By Theorem 12 we have that \(f(x, y)\) is an invariant of system (24) with cofactor \(-2by\), where \(f(x, y)\) is given by

\[
f(x, y) = b(a + b)(a + 2b)w(x) - (a + b + d)(1 + ax)^{-\frac{2b}{a}} - b(a + b)(a + 2b)y^2 + b(a + 2b)d x^2 - 2b(a + b + d)x + a + b + d,
\]

(25)

with \(w(x)\) any non-zero solution of \((1 + ax)w'(x) + 2bw(x) = 0\), that is, \(w(x) = C(1 + ax)^{-\frac{2b}{a}}\).

Choosing \(C = (a + b + d)(b(a + b)(a + 2b))^{-1}\) we get an invariant conic. System (24) has two invariant algebraic curves, the former conic with cofactor \(-2by\) and an invariant straight line given by \(1 + ax = 0\) with cofactor \(y\). The Darboux first integral

\[
H(x, y) = (1 + ax)^{\frac{2b}{a}}f(x, y)
\]

21
coincides with the first integral described in Theorem 14.

The origin of this system is a center since it is a monodromic singular point with a continuous first integral defined in a neighborhood of it. This example addresses to the thought that other families of polynomial systems of higher degree with a center at the origin can be easily obtained by this method, avoiding the cumbersome computation of Poincaré-Liapunov constants.

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