A de Broglie-Bohm Like Model for Klein-Gordon Equation
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Abstract
A de Broglie-Bohm like model of Klein-Gordon equation, that leads to the correct Schrödinger equation in the low-speed limit, is presented. Under this theoretical framework, that affords an interpretation of the quantum potential, the main assumption of the de Broglie-Bohm interpretation—that the local momentum of particles is given by the gradient of the phase of the wave function—is not but approximately correct. Also, the number of particles is not locally conserved. Furthermore, the representation of physical systems through wave functions won’t be complete.

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1 Introduction
It is generally accepted that the Klein-Gordon equation has not a consistent de Broglie-Bohm like interpretation[3, p. 498-503][5]. However, in this paper we show that through the introduction of a hidden variable Φ such that:

\[ p_\mu = -\partial_\mu S - \partial_\mu \Phi, \]

where \( S \) is the phase of the wave function, a solution to this problem can be easily found, in such way that it leads to the correct Schrödinger equation in the low-speed limit, and affords a sound interpretation of the quantum potential.

Within the theoretical framework of this work, the main assumption of the de Broglie-Bohm interpretation of quantum mechanics is not valid and a picture of particles moving under the action of electromagnetic field alone is recovered. However, the number of particles is not locally conserved. Also, given that a hidden variable has been introduced—on the grounds of some general electrodynamical considerations, presented in the first section—the representation of physical systems through wave functions won’t be complete and, therefore,
as foreseen by Einstein, Podolsky, and Rosen, quantum mechanics won’t be a complete theory of motion.

2 On the Motion of Particles Under the Action of an Electromagnetic Field

Consider a non-stochastic ensemble of particles whose motion is described by means of a function

\[ \vec{r} = \vec{r} (\vec{x}, t), \]

such that \( \vec{r}(\vec{x}, t) \) represents the position, at time \( t \), of the particle that passes through the point \( \vec{x} \) at time zero—the \( \vec{x} \)-particle. This representation of motion coincides with the lagrangian representation used in hydrodynamics [1].

As to the function \( \vec{r}(\vec{x}, t) \), we suppose that it is invertible for any value of \( t \). In other words, that there is a function \( \vec{x} = \vec{x}(\vec{r}, t) \), that gives us the coordinates, at time zero, of the particle that passes through the point \( \vec{r} \) at time \( t \). Also, we suppose that \( \vec{r}(\vec{x}, t) \) is a continuous function, altogether with its derivatives of as higher order as needed to secure the validity of our conclusions.

According to the definition of \( \vec{r}(\vec{x}, t) \), the velocity of the \( \vec{x} \)-particle at time \( t \) is

\[ \vec{u}(\vec{x}, t) = \left( \frac{\partial \vec{r}}{\partial t} \right) \vec{x}. \]

Writing \( \vec{x} \) as a function of \( \vec{r} \) and \( t \), we get the velocity field at time \( t \):

\[ \vec{v}(\vec{r}, t) = \vec{u}(\vec{x}(\vec{r}, t), t). \]

As it is known from fluid kinematics, the corresponding field of acceleration is:

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}. \]

From this, considering that

\[ (\vec{v} \cdot \nabla) \vec{v} = \nabla v^2 / 2 - \vec{v} \times (\nabla \times \vec{v}) \]

we can write the corresponding field of force:

\[ \vec{f} = m \left( \vec{e} + \frac{1}{c^2} \vec{v} \times \vec{h} \right) \]

where

\[ \vec{e} = -\frac{1}{c^2} \frac{\partial \vec{a}}{\partial t} - \nabla a_0, \quad (1) \]

\[ \vec{h} = \vec{\nabla} \times \vec{a}, \quad (2) \]

\[ a_0 = -c^2 - v^2 / 2, \quad (3) \]

\[ \vec{a} = -c \vec{v}, \quad (4) \]
and \( c \) is a constant, with dimensions of speed, that we shall take equal to the velocity of light.

Equations (1) and (2) are formally analogous to the definition of the electric and magnetic fields from the electrodynamic potentials. From them we can prove that
\[
\nabla \cdot \vec{h} = 0,
\]
and
\[
\nabla \times \vec{e} = -\frac{1}{c} \frac{\partial \vec{h}}{\partial t},
\]
analogous to the first pair of Maxwell equations. Furthermore, the right members of equations (3) and (4), but for a constant factor, correspond to the low-speed approximation of the components of the relativistic four-velocity, which suggest us to investigate the properties of this field, defined as:
\[
v^{\mu} = \left( \frac{c}{\sqrt{1 - v^2/c^2}}, \frac{\vec{v}}{\sqrt{1 - v^2/c^2}} \right)
\]
Using four-dimensional tensorial notation, the derivative of \( v^{\mu} \) with respect to the proper time, along the world-line of the corresponding particle is:
\[
\frac{dv^{\mu}}{ds} = v^{\nu} \partial_{\nu} v^{\mu}
\]
From the identity
\[
v^{\nu} v^{\nu} = c^2
\]
we prove that
\[
v^{\nu} \partial_{\mu} v^{\nu} = 0,
\]
that allows us to write
\[
\frac{dv^{\mu}}{ds} = (\partial_{\nu} v^{\mu} - \partial_{\mu} v^{\nu}) v^{\nu}.
\]
If the four-force acting on the ensemble of particles has an electromagnetic origin, it is given by the expression
\[
f^{\mu} = \frac{q}{c} F^{\mu \nu} v^{\nu},
\]
where
\[
F^{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}
\]
is Faraday’s tensor, and \( A_{\mu} \) is the electrodynamic four-potential.

From the law of motion
\[
m \frac{dv^{\mu}}{ds} = f^{\mu},
\]
and equations (6) to (8) we prove that
\[
(\partial_{\mu} p^{\nu} - \partial_{\nu} p^{\mu}) v^{\nu} = 0,
\]
where
\[ p_\mu = m v_\mu + \frac{q}{c} A_\mu. \] (10)

There is a class of solutions of equations (9) where \( p_\mu \) is the four-gradient of a function of space-time coordinates
\[ p_\mu = -\partial_\mu \phi. \] (11)

Therefore,
\[ \partial_\mu p_\nu - \partial_\nu p_\mu = 0, \]
and (9) is obviously satisfied.

Equation (10) can be written in the form
\[ m v_\mu = -\partial_\mu \phi - \frac{q}{c} A_\mu \] (12)

If the four-potential meets the Lorentz condition,
\[ \partial^\mu A_\mu = 0, \] (13)
equation (12) is analogous to the decomposition of a classical, three-dimensional field, into an irrotational and a solenoidal part. The electromagnetic field appears thus as determining the four-dimensional vorticity of the field of kinetic momentum \( m v_\mu \).

Equation (5) tells us that functions \( \phi \) are not arbitrary, but are subject to the condition:
\[ (\partial^\mu \phi + \frac{q}{c} A^\mu)(\partial_\mu \phi + \frac{q}{c} A_\mu) = m^2 c^2, \] (14)
which is the relativistic Hamilton-Jacobi equation.\[ 2, \text{Ch. VIII}\]

The components of the kinetic momentum are:
\[ \left( \frac{K}{c}, \vec{p} \right) = \left( -\frac{1}{c} \frac{\partial \phi}{\partial t} - \frac{q}{c} \nabla \phi - \frac{q}{c} \vec{A}, \vec{V} \right), \] (15)
where \( V \) and \( \vec{A} \) are the components of the electrodynamic four-potential.

In the low-speed limit, we have
\[ K \approx mc^2 + \frac{p^2}{2m}. \]

From this and (15) we get:
\[ \frac{\partial \phi}{\partial t} + \frac{(\nabla \phi - \frac{q}{c} \vec{A})^2}{2m} + qV + mc^2 = 0, \] (16)
that, but for the constant term \( mc^2 \), is the non-relativistic Hamilton-Jacobi equation.

Notice that
\[ \int m v_\mu dx^\mu = -\frac{q}{c} \int A_\mu dx^\mu, \]
for any closed path in space time. In particular
\[ \oint \vec{p} \cdot d^3 \vec{x} = -\frac{q}{c} \oint \vec{A} \cdot d^3 \vec{x}, \]
for any closed path of simultaneous events in three-dimensional space.

3 A de Broglie-Bohm Like Interpretation of Klein-Gordon Equation

Consider the Klein-Gordon equation describing an ensemble of spin-less particles with mass \( m \) and charge \( q \):\[
(i\hbar \partial^\mu - \frac{q}{c} A^\mu)(i\hbar \partial_\mu - \frac{q}{c} A_\mu)\Psi = 0,
\]
where \( A^\mu \) satisfies the Lorentz condition (13). In other words:
\[
-\hbar^2 \partial^\mu \partial_\mu \Psi - \frac{2i\hbar}{c} A^\mu \partial_\mu \Psi + \frac{q^2}{c^2} A^\mu A_\mu \Psi = m^2 c^2 \Psi. \tag{17}
\]
We can easily show that
\[
\partial^\mu \left( \frac{i\hbar}{2} (\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*) - \frac{q}{c} A_\mu \Psi^* \Psi \right) = 0. \tag{18}
\]
By the Madelung substitution
\[
\Psi = \sqrt{\rho} e^{i S}, \tag{19}
\]
equation (20) is transformed into
\[
\partial^\mu (\rho (-\partial_\mu S - \frac{e}{c} A_\mu)) = 0, \tag{20}
\]
that, in view of (12), tells us that
\[
\rho = \Psi^* \Psi, \tag{21}
\]
and
\[
w_\mu = -\frac{\partial_\mu S}{m} - \frac{q}{mc} A_\mu, \tag{22}
\]
might be interpreted as the density of particles in the (local) system of reference, where the particles are at rest, and the field of four-velocity, respectively. (Hence \( p_\mu = -\partial_\mu S \).) This interpretation is not sound, however, because \( w_\mu \) is not an implicitly time-like four-vector. Actually, from (17) and (19), it can be shown that
\[
(-\partial^\mu S - \frac{q}{c} A^\mu)(-\partial_\mu S - \frac{q}{c} A_\mu) = m^2 c^2 + \hbar^2 \frac{\partial^\mu \partial_\mu \sqrt{\rho}}{\sqrt{\rho}}, \tag{23}
\]
in plain disagreement with (14).

However, considering that \( w_\mu \) has already the required four-vorticity, we’ll suppose that there is a function \( \Phi \) such that

\[
m v_\mu = -\partial_\mu S - \partial_\mu \Phi - \frac{q}{c} A_\mu.
\]

(24)

From (14) and (23) we can show that \( \Phi \) has to satisfy the condition

\[
2 m v_\mu \partial_\mu \Phi + \partial_\mu \Phi \partial_\mu \Phi = \frac{\hbar^2}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial t^2}.
\]

(25)

We’ll show that this is a sound hypothesis by proving that it leads to the correct equations in the low-speed limit, where we suppose that

\[
\frac{1}{c^2} \left( \frac{\partial \Phi}{\partial t} \right)^2 = 0, \quad \frac{\hbar^2}{c^2 \sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial t^2} = 0,
\]

in such way that (25) can be rewritten, using classical, three-dimensional, notation, as

\[
2 m \frac{\partial \Phi}{\partial t} + 2 m \vec{v} \cdot \vec{\nabla} \Phi - (\vec{\nabla} \Phi)^2 = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}.
\]

(26)

or,

\[
\frac{\partial \Phi}{\partial t} + (\vec{v} - \vec{\nabla} \Phi) \cdot \vec{\nabla} \Phi + \frac{(\vec{\nabla} \Phi)^2}{2m} = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}.
\]

(27)

In other words:

\[
\frac{(\nabla S - \frac{2}{c} \vec{A})}{m} \cdot \nabla \Phi + \frac{(\vec{\nabla} \Phi)^2}{2m} = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - \frac{\partial \Phi}{\partial t}.
\]

(28)

From this we can build a low-speed limit approximation of the kinetic energy:

\[
K \approx mc^2 + \frac{(\nabla S + \vec{\nabla} \Phi - \frac{2}{c} \vec{A})^2}{2m} = mc^2 + \frac{(\nabla S - \frac{2}{c} \vec{A})^2}{2m} - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - \frac{\partial \Phi}{\partial t},
\]

(29)

Finally, given that in this case,

\[
K = -\frac{\partial (S + \Phi)}{\partial t} - qV.
\]

(30)

we conclude that, in the low-speed limit:

\[
\frac{\partial S}{\partial t} + mc^2 + \frac{(\nabla S - \frac{2}{c} \vec{A})^2}{2m} - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + qV = 0,
\]

(31)

that, but for the constant term \( mc^2 \), is the classical Hamilton-Jacobi equation with quantum potential. From this and (20) the corresponding Schrödinger equation is easily recovered.
In the general case, we get a picture of classical particles moving under the action of the electromagnetic field alone: there is not quantum potential. The field of four-velocity is given by (24), where \( S \) is the phase of the wave function and \( \Phi \) is a hidden variable. \( \rho, S \) and \( \Phi \) are solutions of a system of non-linear differential equations: (20), (23), and

\[
(-\partial \mu S - \partial \nu \Phi - \frac{q}{c} A^\mu)(-\partial \mu S - \partial \mu \Phi - \frac{q}{c} A_\mu) = m^2 c^2,
\]

that, however, does not determine them completely. But for the restrictions imposed in the low-speed limit, the function \( \Phi \) could be equal to \( \phi - S \) for any solution of equation (14).

The number of particles is not locally conserved. From (20) and (24) we get:

\[
\partial \mu (\rho u^\mu) = -\frac{\partial \mu (\rho \partial \nu \Phi)}{m}.
\]

The \( \Phi \) function appears thus related to local creation/annihilation processes, which leads to the appearance of a quantum force in the corresponding hydrodynamic model.

Let

\[
T_{\mu\nu} = m \rho u_\mu u_\nu
\]

Then

\[
\partial \nu T_{\mu\nu} = m \rho u_\nu \partial \nu u_\mu + m u_\mu \partial \nu (\rho u_\nu) = \frac{q}{c} \rho \partial \nu F_{\mu\nu} - u_\mu \partial \nu (\rho \partial \nu \Phi)
\]

Equations (34) and (35) describe a pressure-less fluid of charged particles under the action of the electromagnetic fluid and a force:

\[
K_\mu = -v_\mu \partial \nu (\rho \partial \nu \Phi)
\]

associated—through (33)—to local creation/annihilation processes.

4 The Low-Speed Limit Revisited

In the low speed limit, the equations of quantum mechanics for spin-less particles can be proved to be equivalent to:

\[
m \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right) = q \vec{E} + \frac{q}{c} \vec{u} \times \vec{H} - \vec{\nabla} Q
\]

and

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0,
\]

where

\[
\rho = \Psi^* \Psi,
\]

\[
m \vec{u} = \nabla S - \frac{q}{c} \vec{A},
\]

\[
7
\]
and

\[ Q = -\frac{\hbar^2 \Delta\sqrt{\rho}}{2m \sqrt{\rho}}, \]  

is the quantum potential.

From (26)

\[ -\vec{\nabla} Q = -\frac{\partial \vec{\nabla} \Phi}{\partial t} - (\vec{v} \cdot \vec{\nabla}) \vec{\nabla} \phi - (\vec{\nabla} \phi \cdot \vec{v}) \vec{v} - \vec{\nabla} \phi \times (\vec{\nabla} \times \vec{v}) + \frac{\vec{\nabla} (\vec{\nabla} \Phi)^2}{2m} \]

On this basis, considering that in this limit the field of velocity is given by:

\[ m\vec{v} = \vec{\nabla} S + \vec{\nabla} \Phi - \frac{q}{c} \vec{A}, \]  

and that, therefore,

\[ m\vec{v} = m\vec{u} + \vec{\nabla} \Phi, \]

and

\[ m\vec{\nabla} \times \vec{v} = -\frac{q}{c} \vec{H} \]

equation (37) can be rewritten as:

\[ m \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = q\vec{E} + \frac{q}{c} \vec{v} \times \vec{H}. \]  

The continuity equation is transformed into:

\[ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = -\frac{\vec{\nabla} \cdot (\rho \vec{\nabla} \Phi)}{m}, \]  

Therefore, the number of particles is not locally conserved.

Equations (43) and (44) describe a flux of classical particles, under the action of the electromagnetic field.

5 Conclusions and Remarks

If the ideas exposed in this paper are proven to be valid:

1. The main assumption of the de Broglie-Bohm theory—that the local impulse of quantum particles is given by the gradient of the phase of the wave function—won’t be accurate. [See eq. (24).]

2. However, there will be still room for a classical interpretation of quantum phenomena, in terms of particles moving along well defined trajectories, under the action of the electromagnetic field.

3. The number of particles won’t be locally conserved.

4. Given that a hidden variable has been introduced—after some considerations on electrodynamics—the representation of physical systems through wave functions won’t be complete and, therefore, as foreseen by Einstein, Podolsky, and Rosen, quantum mechanics won’t be a complete theory of motion. Actually, but for the restrictions imposed in the low-speed limit, the function \( \Phi \) could be equal to \( \phi - S \) for any solution of equation (14).
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