Central limit theorem for statistics of subcritical configuration models.

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Abstract

We consider subcritical configuration models and show that the central limit theorem for any ad-
ditive statistic holds when the statistics satisfies a fourth moment assump-
tion, a variance lower bound and the degree sequence of the graph satisfies a growth condition. If the degree sequence is bounded, for well known statistics like component counts, log-partition function, and maximum cut-size which are Lipschitz under addition of an edge or switchings then the assumptions reduce to a linear growth condition for the variance of the statistic. Our proof is based on an application of the central limit theorem for martingale-difference arrays due to McLeish [18] to a suitable exploration process.

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1 Introduction

In this short note, we prove a central limit theorem for additive statistics of random graphs that come from subcritical configuration models. In a configuration model, we are given a degree sequence \( \{d^m_{n,i}\}_{i=1}^{n}, n \geq 1 \) such that \( m = \sum_{i=1}^{n} d_{n,i} \) is even. One attaches \( d_{n,i} \) half-edges or stubs to each vertex \( i, 1 \leq i \leq n \). The random multi-graph formed by pairing uniformly at random these half-edges or stubs is what is known as the configuration model \( G(G(n, \{d^m_{n,i}\}_{i=1}^{n})), [25, Chapter 7] \). Under the assumption of subcriticality (i.e. no giant component) and a growth condition on the degree sequence we prove a central limit theorem for any additive statistic of the graph having an appropriate variance lower bound. See Section 2 for a precise definition of the model and assumptions, along with the statement of the main result (see Theorem 2.1). Our results can possibly be extended to a larger class of random graph models which have similar constructions to the configuration model (see Remark 4.1).

Graphs on \( n \) vertices can be broadly divided into three classes: Dense graphs, those with number of edges being of order \( n^2 \); Sparse graphs with bounded (average) degree and consequently having order \( n \) edges; and in between are those whose average degree grows in the number of vertices, but only at sub-linear speed. Each class has a separate limiting theory. In this article we shall focus on a particular model that falls in the sparse graph regime. One feature in this class of random graphs is the following phase transition. If the expected degree (to be precise expectation of the size-biased degree distribution) of a vertex is larger than one, namely the super-critical phase, then there is a largest connected component referred to as the “giant” component which contains a positive proportion of all vertices. On the other hand, if the expected degree is smaller than one, namely sub-critical phase, then there is no “giant” component and all components are small.

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The study of random graphs has a rich history beginning with the pioneering work of Erdős-Rényi in 1960's (see [6, 16]). In recent years, the theory of random graphs has been significantly expanded by addition of newer models of random graphs such as the preferential attachment model, configuration model, inhomogeneous random graphs et al. (see [25, 26] for a thorough review of the subject). In [12, 26] the weak and strong law has been established for the giant component of a configuration model. Recently, a more general strong law result for additive Lipschitz statistics of configuration model has been shown in [24] using the interpolation method. For many of these models, Strong law or Weak law of large numbers for wide-range of statistics can often be proven now using local-weak convergence ([11]) or the interpolation method ([24]). Additionally, strong law for susceptibility of the configuration model has been shown via branching process approximation in [15].

However, central limit theorems for statistics of sparse random graphs especially non Erdős-Rényi models are harder to find. Central limit theorem’s for the size of the giant component for Erdős-Rényi graphs in the super-critical phase have been studied in [22, 5, 19, 7]. Similar limit theorems for k-core in the super-critical phase and susceptibility in the sub-critical phase have been studied in [13] and [14] respectively. Asymptotic normality for subtree counts in the sparse regime (and subgraph counts in the other regimes) has been shown in [23] and extensions of the same to functional limit theorems has been shown in [11]. Also, alternate proofs of the same via cumulant method and discrete Malliavin-Stein method can be found in [10] and [17] respectively.

As for the configuration model: a large deviation result for the empirical neighbourhood distribution was shown in [8] using the framework of local weak convergence; a central limit theorem for self-loops and multiple edges in configuration model was shown in [2] using the Chen-Stein method for the Poisson approximation; in [3], asymptotic variance of the giant component of the configuration model was determined under appropriate conditions on the degree sequence; and more recently in [4], this was extended to a central limit theorem as a consequence of a more general normal approximation result derived for local statistics of the configuration model using the Stein’s method.

As mentioned earlier, strong law of large numbers for additive Lipschitz functions with some additional assumptions has been shown in [24]. In this article we consider additive statistics in the sub-critical configuration model and formulate broad assumptions that need to be verified for a central limit theorem to hold. The assumptions, apart from subcriticality include a decay rate for the size of the largest component along with moment bounds and variance lower bounds for the statistic. We first provide a construction via edge exploration of the configuration model $G_n := G(n, \{d^n_i\}_{i=1}^n)$ for a specified degree sequence and prove a central limit theorem in Theorem 2.1. The exploration can be used to generate a martingale array sequence from the statistic for which we verify McLeish’s martingale-difference array central limit theorem, [18].

The rest of the paper is organized as follows: In Section 2, we state our model precisely as well as state our central limit theorem. In Section 3, we discuss some examples and applications of our result. We conclude with the proof of our main result in Section 4.

2 Model and Main Result

We begin by defining the statistic of interest. Let

$\mathcal{G} = \{G : G$ is a finite multi-graph\}

denote the collection of all finite multi-graphs. Our statistic is defined as a function $F : \mathcal{G} \to \mathbb{R}$ such that $F$ is invariant under graph isomorphism (i.e., $F(G_1) = F(G_2)$ if $G_1 \cong G_2$) and additive (i.e., $F(G) = F(G_1) + F(G_2)$ if $G$ is a vertex-disjoint union of $G_1$ and $G_2$).

We shall construct a random (multi-)graph $G_n := G(n, \{d^n_i\}_{i=1}^n)$ on the vertex set $V_n = [n] = \{1, \ldots, n\}$ with a specified degree sequence $d^n_1, \ldots, d^n_n$ as follows. Assume that $\sum_{i=1}^n d^n_i$ is even and
we shall denote \( 2m_n = \sum_{i=1}^{n} d_i^n \). We shall denote \( d_{\max}^n := \max_{1 \leq i \leq n} d_i^n \). We shall follow the standard construction via half-edges but with a breadth-first exploration and hence we shall describe the same in detail. Also, for convenience of reading we will refer to \( d_i^n \) as \( d_i \) without the superscript \( n \).

Let \( W(i) = \{(i,1), \ldots, (i,d_i)\} \) be the set of ordered half-edges incident on \( i \in [n] \) and \( \mathcal{H}_n := \cup_{i=1}^{n} W(i) \) be the total collection of half-edges on all the vertices in \([n]\). The half-edges are ordered as per the lexicographic order i.e., \((i,k) \leq (j,l)\) if \( i \leq j \) or \( i = j \) and \( k \leq l \). For all \( t \in [m_n] \), \( \mathcal{H}_n \) is partitioned into three sets \( A_t, C_t, U_t \) respectively the set of active, connected and unexplored half-edges with the initial configuration being \( A_1 = W(1), C_1 = \emptyset, U_1 = \mathcal{H}_n \setminus W(1), X_{1,n} = \emptyset \). Also our exploration shall ensure that both of the following events cannot happen for any \( k \in [n] \) : \( W(k) \cap A_t \neq \emptyset, U_t \cap W(k) \neq \emptyset \).

**Edge Exploration:** The exploration algorithm can be described as follows: At step \( t \in [m_n] \), choose the smallest edge \((v_t, i_t)\) (w.r.t. lexicographic order) in \( A_t \) and pair it uniformly at random with one of the other half-edges in \( A_t \cup U_t \), say \((j_t, l_t)\). Then if:

\[
(j_t, l_t) \in A_t, \quad \text{set} \quad A_{t+1} = A_t \setminus \{(v_t, i_t), (j_t, l_t)\} \quad \text{and} \quad U_{t+1} = U_t;
\]

\[
(j_t, l_t) \notin A_t, \quad \text{set} \quad A_{t+1} = A_t \cup W(j_t) \setminus \{(v_t, i_t), (j_t, l_t)\} \quad \text{and} \quad U_{t+1} = U_t \setminus W(j_t - 1).
\]

Set of connected half-edges at time \( t + 1 \) to be \( C_{t+1} = C_t \cup \{(v_t, i_t), (j_t, l_t)\} \) and the newly formed edge is denoted by \( X_{t,n} := \{(v_t, i_t), (j_t, l_t)\} \). Now, repeat the algorithm until \( A_{t+1} = U_{t+1} = \emptyset \) Since we are pairing two half-edges at every time-step, the algorithm will stop at \( t = m_n \).

Denoting the set of matchings on \( \mathcal{H}_n \) by \( \mathcal{M}_n \), we have generated a sequence of random elements of \( 2^{\mathcal{M}_n} - X_{1,n}, \ldots, X_{m_n,n} \). Given the sequence \( X_{1,n}, \ldots, X_{m_n,n} \), we construct the (multi-)graph \( G_n \) by placing an edge between \( i, j \in [n] \) for every pairing of half-edges \(((i, h), (j, l)) \in X_n := (\cup_{k=1}^{m_n} X_{k,n}) \). Thus \( G_n \in \sigma(X_{1,n}, X_{2,n}, \ldots, X_{m_n,n}) \). Further, we define \( \mathcal{F}_{k,n} = \sigma(X_{1,n}, X_{2,n}, \ldots, X_{k,n}) \).

Let \( F_n \equiv F(G_n) \) and for \( 1 \leq k \leq m_n \), let \( \Delta_{k,n} = \mathbb{E}(F_n \mid \mathcal{F}_{k,n}) - \mathbb{E}(F_n \mid \mathcal{F}_{k-1,n}) \) with

\[
C_n := \sup_{1 \leq k \leq m_n} \mathbb{E}(\Delta_{k,n}^2).
\]

We shall make the following assumptions: Let \( C_{\max}^n \) denote the largest connected component in \( G_n \). For some \( \kappa \geq 0 \) and a sequence \( \alpha_n \geq 1 \)

\[
C_n(d_{\max}^n)^2\alpha_n = o(n^{2\kappa}) \quad \text{and} \quad C_n(d_{\max}^n)^2 \mathbb{P}(C_{\max}^n > \alpha_n) = o(n^{2\kappa - 1}). \tag{G1}
\]

\[
\mathbb{V}(F_n) = \Omega(n^{\frac{3}{2}+\kappa}). \tag{F1}
\]

Now, we are ready to state our main result

**Theorem 2.1.** Assume \((G1)\) and \((F1)\). Then

\[
\frac{F_n - \mathbb{E}(F_n)}{\sqrt{\mathbb{V}(F_n)}} \xrightarrow{d} Z, \tag{2.1}
\]

with \( Z \) being a standard Normal random variable.

We note that in view of the assumptions in Theorem 2.1, we can get the following bounds on variance using \((4.5)\) and definition of \( C_n \) as well as \((G1)\): There exists a constant \( M \) such that for any \( \epsilon > 0 \) and large enough \( n \),

\[
\sqrt{nC_n}\alpha_n d_{\max}^n \leq \epsilon n^{1/2+\kappa} \leq \epsilon M \mathbb{V}(F_n) \leq \epsilon M m_n \sqrt{C_n} \leq \epsilon n d_{\max}^n \sqrt{C_n}. \tag{2.2}
\]
Thus, we can conclude that for our bounds to hold $\alpha_n = o(n)$ i.e., the configuration model has to be necessarily sub-critical.

The key tool in the proof of Theorem 2.1 is McLeish’s martingale-difference array central limit theorem, [18]. Our inspiration for this central limit theorem arose from powerful central limit theorems proven for geometric functionals of Poisson and Bernoulli point processes proved in [20, 21]. The advantage with the martingale-difference array central limit theorem is that it reduces the proof of central limit theorem to moment bounds and convergence in probability of the squared martingale-differences. In [20, 21], the latter is achieved by applying ergodic theorem to appropriate ergodic random fields constructed from the functional and the Poisson point process. However, in our case, the model does not have any underlying ergodicity or stationarity and so, to achieve the required $L_1$ convergence of squared martingale-differences, we use the ‘sub-criticality’ of the configuration model and additivity of the functional. We shall comment later in Section 3 about verifying the assumptions on the degree sequence and the function $F$. We also remark about possible extensions of our main theorems in Remark 4.1.

Our proof techniques require us to restrict to sub-critical configuration models (i.e., no giant component). This is in contrast to the results in [4] which apply to the super-critical regime as well. However, our mild assumptions on the locality of the statistics as well as that of degree sequence are less restrictive in some applications.

We emphasize that the variance lower bound condition shall usually be the most non-trivial condition in this article to verify and we do not have a method of showing the same in any of our examples. For example, the cardinality of a maximum independent set is a Lipschitz functional and satisfies strong law (see [24]) but it is much tightly concentrated on a random $d$-regular graph (see [9]). Also, the number of multiple edges as well as self-loops are Lipschitz statistics and with variance growing polynomially in $n$ but the variance growth isn’t sufficient to verify the assumptions of our theorems. However, a central limit theorem for the same has been shown in [2, Section 1.3].

Our general central limit theorem can be considered as reducing the task of proving a central limit theorem for many statistics of random graphs to that of proving variance lower bounds. However, we would like to mention that similar variance lower bound conditions appear in most general central limit theorems such as those in [10, 4]. Usually these variance lower bound conditions are verified in a case-specific manner like in [3].

### 3 Applications

We will prepare for our applications by recalling a couple of lemmas from the literature.

**Lemma 3.1. ([12, Theorem 1.1])** Let the $D_n$ be the degree of a randomly chosen vertex in $G(n, \{d^n_i\}_{i=1}^n)$. We assume that

$$D_n \xrightarrow{d} D, \mathbb{E}[D_n] \to \mathbb{E}[D] \in (0, \infty), \frac{\mathbb{E}[D_n(D_n - 1)]}{\mathbb{E}[D_n]} \to \frac{\mathbb{E}[D(D - 1)]}{\mathbb{E}[D]} \in [0, 1).$$

Also, assume that for some $\gamma > 3$, uniformly in $n, k$

$$P(D_n \geq k) = O(k^{1-\gamma}).$$

Then, there exists a constant $A$ so that

$$\mathbb{P}(|C_{\text{max}}| \geq An^{1/(\gamma-1)}) \to 0 \text{ as } n \to \infty.$$

Recall that $\mathcal{M}_n$ is the set of matching on $H_n$. Consider two matchings $m, m' \in \mathcal{M}_n$. We say $m, m'$ differ by a *switching* and denote it by $m \sim m'$ if there exists $i_1, i_2, i_3, i_4 \in H_n$ such that $(i_1, i_2), (i_3, i_4) \in m$
and \( m' = m - \{(i_1, i_2), (i_3, i_4)\} + \{(i_1, i_3), (i_2, i_4)\} \). Since \((i_1, i_2)\) corresponds to a pairing of half-edges, a switching is really a switching of a two pairs of half-edges. Given a matching \( m \in \mathcal{M}_n \), we denote the (multi)-graph obtained by pairing of half-edges matched in \( m \) as \( G(m) \) and we abbreviate \( F(G(m)) \) by \( F(m) \).

**Lemma 3.2.** If \( F \) is \( M \)-Lipschitz under switchings (i.e., \(|F(m) - F(m')| \leq M \) for \( m \sim m' \)) for some \( M < \infty \) or if \( F \) is \( M/4 \)-Lipschitz under edge-addition (i.e., \(|F(G) - F(G + (i, j))| \leq M/4 \) for any graph \( G \) on \([n]\) and \( 1 \leq i \neq j \leq n \)) for some \( M < \infty \), then we have that \( \sup_{n \geq 1} c_n \leq M^4 \).

**Proof.** Let \( m \sim m' \) be as above and let \( G = G(m), G' = G(m') \). Further, set \( G_1 = G - \{(i_1, i_2), (i_3, i_4)\} \). Then \( G' = G_1 + \{(i_1, i_3), (i_2, i_4)\} \). Now note that if \( F \) is \( M/4 \)-Lipschitz under edge-addition then we have that,

\[
|F(m) - F(m')| = |F(G) - F(G')| \leq |F(G) - F(G_1)| + |F(G_1) - F(G')| \leq M,
\]

i.e., \( F \) is \( M \)-Lipschitz under switchings. Thus it is enough to prove the first part of the Lemma i.e., under the assumption of \( M \)-Lipschitz under switchings. This is proven in [27, Proof of Theorem 2.19] or see [8, Section 7.1.2]. \( \Box \)

**Examples of statistics that are Lipschitz under edge-addition:**

We provide a few examples of statistics that are Lipschitz under edge-addition. Most of them can be found in [24, Section 1]. We present them here for completeness sake.

1. **Number of connected components:** \( F(G) := \beta_0(G) \) is the number of connected components. It is easy to see that this satisfies our assumptions of additivity and is \( 1 \)-Lipschitz under edge-addition.

2. **Components of fixed size:** Let \( G \) be a graph with components \( \Gamma_1, \ldots, \Gamma_L \) and \( |\Gamma| \) denotes the size (measured in number of vertices or edges) of the components. For \( K, p \in \mathbb{N} \), define

\[
F_{p,K}(G) = \sum_{i=1}^{L} |\Gamma_i|^p 1[|\Gamma_i| \leq K].
\]

For example, \( F_{0,K} \) is the number of components of size at most \( K \), \( F_{1}(K) \) is the total size of components of size at most \( K \). We note that \( F_{p,K} \) is \((2K)^p\)-Lipschitz. We shall consider an untruncated version (i.e., \( K = \infty \)) and hence non-Lipschitz later in Corollary 3.3.

3. **Maximum cut-size:** A cut \((S, T)\) is a partition of the vertex set into two disjoint sets (i.e., vertex set is the disjoint union of \( S \) and \( T \)). The size \( c(S, T) \) of a cut \((S, T)\) is the number of edges between \( S \) and \( T \). Define \( F(G) = \max\{c(S, T) : (S, T) \text{ is a cut of } G\} \). It can be verified that \( F \) is \( 1 \)-Lipschitz under edge-addition.

4. **Log-partition function:** Let \( S \) be a finite set with one map \( h : S \to (0, \infty) \) and a second symmetric map \( J : S \times S \to (0, \infty) \). Define the statistic

\[
F(G) := \log\left( \sum_{\sigma \in S^V} w(\sigma) \right)
\]

where

\[
w(\sigma) := \prod_{v \in V} h(\sigma_v) \prod_{(v, w) \in E} J(\sigma_v, \sigma_w)
\]

where we have used the notation that \( G = (V, E) \) and \( \sigma := \{\sigma_v\}_{v \in V} \in S^V \). It can be easily verified that the above \( F \) is additive and \( M \)-Lipschitz under edge-addition where \( M = \max_{s, t \in S} J(s, t) \).
Statistics such as above arise often in statistical physics where $\sigma$ is said to be the configuration of a system on the graph $G$, $J$ is interpreted to encode pairwise interaction between vertices, $h$ is considered as the external field and the statistics $F$ is called as the log-partition function.

Two particular case of special interest that arise from statistical mechanics are: (i) Ising model : $S = \{+1, -1\}$ and $J(s,t) = e^{-\beta s t}$ for $\beta \geq 0$ and (ii) Potts model : $S = \{1, \ldots, q\}$ and $J(s,t) = 1_{s \neq t} + e^{-\beta} 1_{s = t}$ for $\beta \geq 0$. Again, in statistical physics terminology, $\beta$ is known as the inverse temperature.

For Lipschitz functions, we now present some corollaries to our main theorem. The first is an easy consequence of Lemma 3.2 and our main theorem 2.1.

Corollary 3.1. Let $\{d_i^n\}_{i=1}^n$ be a degree sequence and $G_n := G(n, \{d_i^n\}_{i=1}^n)$ be the corresponding configuration model. Let $F$ be invariant under graph isomorphisms, additive and $M$-Lipschitz under switchings. Suppose that for some $\kappa \geq 0$,

(i) $d_{n \max}^n = O(n^\beta)$ for some $0 \leq \beta \leq \kappa$

(ii) $\forall \gamma > 2\kappa - 2\beta - 1 + 1$, and

\[ \Pr(\left| C^{\max}_n \right| > An^{1/(\gamma - 1)}) = o(n^{2\kappa - 2\beta - 1}) \tag{3.1} \]

(iii) $\Var(F_n) = \Omega(n^{\frac{\gamma}{4} + \kappa})$.

then

\[ \frac{F_n - \mathbb{E}(F_n)}{\sqrt{\Var((F_n))}} \rightarrow Z. \]

Corollary 3.2. Let $\{d_i^n\}_{i=1}^n$ be a bounded degree sequence and let $G_n := G(n, \{d_i^n\}_{i=1}^n)$ be the corresponding configuration model. Let $F$ be invariant under graph isomorphisms, additive and $M$-Lipschitz under switchings. Let the $D_n$ be the degree of a randomly chosen vertex in $G(n, \{d_i^n\}_{i=1}^n)$. Suppose that

(i) $D_n \overset{d}{\rightarrow} D, \mathbb{E}[D_n] \rightarrow \mathbb{E}[D] \in (0, \infty), \frac{\mathbb{E}[D_n(D_n - 1)]}{\mathbb{E}[D_n]} \rightarrow \frac{\mathbb{E}[D(D - 1)]}{\mathbb{E}[D]} \in [0, 1)$.

(ii) $\Var(F(G_n)) = \Omega(n)$

then

\[ \frac{F_n - \mathbb{E}(F_n)}{\sqrt{\Var(F_n)}} \rightarrow Z, \]

with $Z$ being a standard Normal random variable.

Proof. If $\{d_i^n\}_{i=1}^n$ satisfies the assumptions of Lemma 3.1 and $\sup_{n \geq 1} d_{n \max}^n < \infty$, then Lemma 3.1 implies that (3.1) holds with $\kappa = \frac{1}{2}$ and for any $\gamma > 3$. Hence (G1) holds with $\kappa = \frac{1}{2}$ and $\alpha_n = An^{\frac{1}{\gamma - 1}}$ for any $\gamma > 3$. In other words, central limit theorem for Lipschitz functionals of bounded degree graphs follows if we show that $\Var(F_n) = \Omega(n)$. This completes the proof. ∎

An important example of a non-Lipschitz additive statistics for a random graph is Susceptibility. Namely, for a graph $G$, with $n$ vertices and $K$ connected components, define for $\rho \geq 0$

\[ S_p(G) = \sum_{i=1}^K \text{Size of $i$-th component}^p. \]
Note that $S_0(G)$ is the number of connected components, $S_1(G) = n$, $S_2(G)$ is called Susceptibility and $S_p(G)$ is not in general Lipschitz for $p \geq 2$. As mentioned before strong law for $S_2(G(n, \{d^n_i\}_{i=1}^n))$ (with $G(n, \{d^n_i\}_{i=1}^n)$ being the configuration model) has been shown in [15] and a central limit theorem for subcritical Erdős-Rényi graphs has been shown in [14]. We now present a corollary that provides assumptions under which a central limit theorem holds for $S_p(G(n, \{d^n_i\}_{i=1}^n))$ when $p \geq 2$.

**Corollary 3.3.** Let $p \geq 2$. Let $\{d^n_i\}_{i=1}^n$ satisfy assumptions of Lemma 3.1 and $G_n := G(n, \{d^n_i\}_{i=1}^n)$ be the corresponding configuration model. Assume that

1. $\gamma > 4p + 4$.

2. $\mathbb{P}(|C^n_{\max}| > A n^{1/(\gamma-1)}) = o(n^{-\frac{a}{\gamma-1}})$ for some $A > 0$ and $a > (4p - 1)(\gamma - 1) + 3$.

3. $\text{Var}(S_p(G_n)) = \Omega(n)$

Then,

$$
\frac{S_p(G_n) - \mathbb{E}(S_p(G_n))}{\sqrt{\text{Var}(S_p(G_n))}} \xrightarrow{d} Z,
$$

with $Z$ being a standard Normal random variable.

**Proof.** It is easy to see using (ii) and definition of $C_n$ for Theorem 2.1 that

$$
C_n \leq c_p E([C^n_{\max}]^{4p}) \leq c_p [n^{4p} + n^{4p - a(\gamma-1)}] \leq c_p [n^{4p} + n^{4p(\gamma-1) - a}]
$$

Note that $d^n_{\max} = O(n^{\frac{1}{\gamma-1}})$ and with $\alpha_n = An^{\frac{1}{\gamma-1}}$. We have by the assumptions on $a$ and $\gamma$ that

$$
C_n(d^n_{\max})^2 \alpha_n \leq c_2 [n^{\frac{4p+3}{\gamma-1}} + n^{\frac{4p(\gamma-1) - 1 - a}{\gamma-1}}] = o(n)
$$

and

$$
C_n(d^n_{\max})^2 \mathbb{P}(|C^n_{\max}| > \alpha_n) \leq c_3 [n^{\frac{4p+2-a}{\gamma-1}} + n^{\frac{4p(\gamma-1) + 2 - 2a}{\gamma-1}}] = o(1).
$$

Hence the conditions of Theorem 2.1 hold with $\kappa = 1/2$ and the normal convergence follows.

**Remark 3.1.**

1. Suppose that $F$ is $M$-Lipschitz and the degree sequence satisfies the conditions in Lemma 3.1. Then, from (2.2), we obtain the bounds that

$$
\frac{3}{2(\gamma-1)} \leq \kappa \leq \frac{1}{2} + \frac{1}{\gamma-1}.
$$

Note that such bounds are possible only when $\gamma > 2$.

2. Let us assume again that $F$ is a Lipschitz function under switchings. Suppose the assumptions of Lemma 3.1 hold with some $\gamma > 3$. Then we have that $d^n_{\max} = O(n^{\frac{1}{\gamma-1}})$ (see [26, Section 3.4]). Thus, we get that Lemma 3.2 implies that (G1) holds with $\alpha_n = An^{\frac{1}{\gamma-1}}$ and for some $\kappa \leq 1/2 + 1/(\gamma - 1)$ provided we have that

$$
\mathbb{P}(|C^n_{\max}| > A n^{1/(\gamma-1)}) = o(n^{2\kappa-2/(\gamma-1)-1})
$$

The upper bound on $\kappa$ is justified due to the above remark. Thus, central limit theorem for Lipschitz functionals of such graphs hold if $\text{Var}(F_n) = \Omega(n^{1/2 + \kappa})$ with $\kappa$ as above.
4 Proof of the main result

Proof of Theorem 2.1: Let $X_{1,n}, \ldots, X_{m_n,n}$ be the sequence of matchings of half-edges generated by the edge-exploration process as defined in Section 2. For the exploration process at step $t$, define the unexplored vertex set as $U_t = \{ k \in [n] : W_k \cap U_t \neq \emptyset \}$. Denote the explored vertex-set as $E_t := [n] \setminus U_t$. Observe that the sequence $X_{1,n}, \ldots, X_{m_n,n}$ is not identically distributed but have the following independence property that will be used crucially by us.

Independence property: Conditioned on $A_t = 0$, we have that $G(n, \{d^n_i\}_{i=1}^n)$ is a disjoint union of $G^1_n := G(|U_t|, \{d^n_i\}_{i \in U_t})$ and $G^2_n := G(|E_t|, \{d^n_i\}_{i \in E_t})$ and $G^1_n$ and $G^2_n$ are independent.

We refer the reader to [4, Lemma 3.3] for a proof of the above. In other words, when there are no active half-edges, the configuration model becomes a union of two independent configuration models - one on the connected vertex set and the other on unexplored vertex set.

Clearly, $F_n \in \sigma\{X_{1,n}, \ldots, X_{m_n,n}\}$. For $1 \leq k \leq m_n, n \geq 1$ let $F_{k,n} = \sigma\{X_{1,n}, X_{2,n}, \ldots, X_{k-1,n}, X_{k,n}\}$ and set $F_{0,n} = \emptyset$. Observe that

$$F_n - \mathbb{E}(F_n) = \sum_{k=1}^{m_n} \mathbb{E}(F_n | F_{k,n}) - \mathbb{E}(F_n | F_{k-1,n}).$$

and $\Delta_{k,n} = \mathbb{E}(F_n | F_{k,n}) - \mathbb{E}(F_n | F_{k-1,n})$ is a martingale difference sequence. Thus to prove the central limit theorem, we shall verify the conditions of the central limit theorem for martingale difference arrays due to McLeish. Namely, if $D_{k,n} = \frac{\Delta_{k,n}}{\sqrt{\mathbb{Var}(F_n)}}$ and

$$\sup_{n \geq 1} \mathbb{E}\left(\max_{1 \leq k \leq m_n} |D_{k,n}|^2\right) < \infty \quad (4.1)$$

$$\max_{1 \leq k \leq m_n} D_{k,n} \xrightarrow{p} 0, \quad (4.2)$$

$$\sum_{k=1}^{n} D_{k,n}^2 \xrightarrow{p} 1 \quad (4.3)$$

then it follows from [18, Theorem 2.3] that

$$\sum_{k=1}^{m_n} D_{k,n} \xrightarrow{d} Z \quad (4.4)$$

with $Z$ being a standard Normal random variable. As $\frac{F_n - \mathbb{E}(F_n)}{\sqrt{\mathbb{Var}(F_n)}} = \sum_{k=1}^{n} D_{k,n}$ we would have the result. To complete the proof we will verify (4.1), (4.2), and (4.3).

Verifying (4.1): By orthogonality of martingale differences, we have that

$$\mathbb{Var}(F_n) = \sum_{k=1}^{m_n} \mathbb{E}(\Delta_{k,n}^2) \quad (4.5)$$

and this implies

$$\sup_{n \geq 1} \mathbb{E}\left(\max_{1 \leq k \leq m_n} |D_{k,n}|^2\right) \leq \sup_{n \geq 1} \sum_{k \leq m_n} \mathbb{E}(|D_{k,n}|^2) = \sup_{n \geq 1} \sum_{k \leq m_n} \frac{\mathbb{E}(\Delta_{k,n}^2)}{\mathbb{Var}(F_n)} = 1.$$
Verifying (4.2): By the trivial bound that $m_n \leq n d_{\text{max}}^n$, we have that for any $\epsilon > 0$,

$$\mathbb{P}(\max_{k \leq m_n} |\Delta_{k,n}| \geq \epsilon \sqrt{\text{Var}(F_n)}) \leq \sum_{k=1}^{m_n} \frac{\epsilon^{-4} \mathbb{E}(|\Delta_{k,n}|^4)}{\text{Var}(F_n)^2} \leq C_n m_n n^{-1-2\epsilon^{-4}} \leq C_n d_{\text{max}}^n n^{-2\epsilon^{-4}}.$$

By (G1) and the fact that $\alpha_n \geq 1$ the above implies that

$$\max_{k \leq m_n} |D_{k,n}| \xrightarrow{p} 0.$$

Verifying (4.3): We are left to verify is the convergence in probability of squared martingale differences. For $1 \leq k \leq m_n$, define

$$E_{n,k} = \{A_t = \emptyset \text{ for some } t \in [k - 2d_{\text{max}}\alpha_n, k]\}$$

$$W_{k,n} = D_{k,n}^2 1_{E_{n,k}} \text{ and } Z_{k,n} = D_{k,n}^2 - W_{k,n}.$$ 

By (4.5),

$$\sum_{k=1}^{m_n} \mathbb{E}(W_{k,n}) + \sum_{k=1}^{m_n} \mathbb{E}(Z_{k,n}) = 1.$$

Using the above, the triangle inequality and non-negativity of $Z_{k,n}$'s, we have

$$|\sum_{k=1}^{m_n} D_{k,n}^2 - 1| \leq |\sum_{k=1}^{m_n} W_{k,n} - 1| + |\sum_{k=1}^{m_n} Z_{k,n}|$$

$$\leq |\sum_{k=1}^{m_n} W_{k,n} - \sum_{k=1}^{m_n} \mathbb{E}(W_{k,n})| + \sum_{k=1}^{m_n} \mathbb{E}(Z_{k,n}) + \sum_{k=1}^{m_n} Z_{k,n}$$

$$= I + II + III. \tag{4.6}$$

We shall now show that each of the terms I and III goes to zero in mean. The latter fact will imply that II goes to zero. We begin with I. By Cauchy-Schwarz inequality,

$$[\mathbb{E}(I)]^2 \leq \mathbb{E}(I^2) = \text{Var} \left( \sum_{k=1}^{m_n} W_{k,n} \right)$$

$$= \sum_{k=1}^{m_n} \text{Var}(W_{k,n}) + 2 \sum_{k=1}^{m_n} \sum_{h=k+1}^{m_n} \text{Cov}(W_{k,n}, W_{h,n}) + 2 \sum_{k=1}^{m_n} \sum_{h=k+2d_{\text{max}}^n\alpha_n}^{m_n} \text{Cov}(W_{k,n}, W_{h,n}).$$

Now, by the independence property stated at the beginning of the proof and additivity of $F$, we have that $W_{k,n} \in \sigma\{X_{t,n} : t \in [k - 2d_{\text{max}}\alpha_n, k]\}$. Thus, $W_{k,n}$ is independent of $W_{h,n}$ for all $h > k + 2d_{\text{max}}\alpha_n$. 


This along with the Cauchy-Schwarz inequality will imply
\[
\mathbb{E}(I)^2 \leq \sum_{k=1}^{m_n} \text{Var}(W_{k,n}) + 2 \sum_{k=1}^{m_n} \sum_{h=k+1}^{k+2d_{\text{max}}\alpha_n} \text{Cov}(W_{k,n}, W_{h,n}) + 0
\]
\[
\leq \sum_{k=1}^{m_n} \mathbb{E}(W_{k,n}^2) + 2 \sum_{k=1}^{m_n} \sum_{h=k+1}^{k+2d_{\text{max}}\alpha_n} \sqrt{\mathbb{E}(W_{k,n}^2) \mathbb{E}(W_{h,n}^2)}
\]
\[
\leq \sum_{k=1}^{m_n} \frac{\mathbb{E}(\Delta_{k,n}^4)}{\text{Var}(F_n)^2} + 2 \sum_{k=1}^{m_n} \sum_{h=k+1}^{k+2d_{\text{max}}\alpha_n} \frac{\sqrt{\mathbb{E}(\Delta_{k,n}^4) \mathbb{E}(\Delta_{h,n}^4)}}{\text{Var}(F_n)^2}
\]
\[
\leq \frac{m_n C_n + 4C_n m_n d_{\text{max}}^n \alpha_n}{\text{Var}(F_n)^2}
\]
\[
\leq \frac{nd_{\text{max}}^n C_n + 4n(d_{\text{max}}^n)^2 C_n \alpha_n}{\text{Var}(F_n)^2}
\]
Using the first assumption of (G1) and (F1), we have that
\[\mathbb{E}(I) \to 0 \text{ as } n \to \infty.\] (4.7)

For term II, again by Cauchy-Schwarz inequality
\[\mathbb{E}(II) = \mathbb{E} \left( \sum_{k=1}^{m_n} Z_{k,n} \right) = \sum_{k=1}^{m_n} \mathbb{E} \left( \frac{\Delta_{k,n}^2 1_{(E_{n,k})^c}}{\text{Var}(F_n)} \right) \]
\[\leq \sqrt{\frac{m_n^2 C_n \mathbb{P}(E_{n,k}^c)}{\text{Var}(F_n)^2}} \]
\[\leq \sqrt{\frac{n^2 (d_{\text{max}}^n)^2 C_n \mathbb{P}(C_{\text{max}}^n \geq \alpha_n)}{\text{Var}(F_n)^2}} \]
Therefore, by the latter assumption of (G1) and (F1) we have
\[\mathbb{E}(II) \to 0 \text{ as } n \to \infty.\] (4.8)

As noted earlier this implies that
\[III \to 0 \text{ as } n \to \infty.\] (4.9)

By (4.6, 4.7, 4.8, 4.9) we have that
\[\sum_{k=1}^{n} D_{k,n}^2 \overset{L^1}{\to} 1.\]
This completes the proof. \(\square\)

We conclude with the following remark on the possible class of models for which Theorem 2.1 holds.

**Remark 4.1.** As seen from the proof of Theorem 2.1 the key tool was the martingale central limit theorem. For this we used two key properties from the construction of the model:

(a) The filtration \(\mathcal{F}_{k,n} = \sigma\{X_{1,n}, X_{2,n}, \ldots, X_{k-1,n}, X_{k,n}\}\) generated by the sequence of matchings \(X_{1,n}, \ldots, X_{m_n,n}\) has appropriate ‘independence’ property as stated in the beginning of the proof.

(b) The statistic is additive and the graph is subcritical to ensure ‘fast enough decoupling’ of the martingale-difference array sequence induced by the above filtration and the statistic.

Thus if a random graph model can be constructed using matchings that satisfy (a) and (b) above then for any additive statistic satisfying (F1) and (G1) we can prove a central limit theorem.
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References

[1] D. Aldous and J. M. Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In Probability on discrete structures, 1–72. Springer, 2004.

[2] O. Angel, R. van der Hofstad, and C. Holmgren. Limit laws for self-loops and multiple edges in the configuration model. arXiv:1603.07172, 2016.

[3] F. Ball and P. Neal. The asymptotic variance of the giant component of configuration model random graphs. Ann. Appl. Probab., 27(2):1057–1092, 2017.

[4] A. D. Barbour and A. Röllin. Central limit theorems in the configuration model. arXiv:1710.02644, 2017.

[5] D. Barraez, S. Boucheron, and W. F. De La Vega. On the fluctuations of the giant component. Comb., Probab. Comput., 9(4):287–304, 2000.

[6] B. Bollobás. Random Graphs. Cambridge Studies in Advanced Mathematics., vol. 73. Cambridge University Press, Cambridge, 2001.

[7] B. Bollobás and O. Riordan. Asymptotic normality of the size of the giant component via a random walk. J. Comb. Th., Series B, 102(1):53–61, 2012.

[8] C. Bordenave and P. Caputo. Large deviations of empirical neighborhood distribution in sparse random graphs. Prob. Th. Rel. Fields, 163(1-2):149–222, 2015.

[9] J. Ding, A. Sly, and N. Sun. Maximum independent sets on random regular graphs. Acta Math., 217(2):263–340, 2016.

[10] V. Féray. Weighted dependency graphs. arXiv:1605.03836, 2016.

[11] S. Janson. A functional limit theorem for random graphs with applications to subgraph count statistics. Rand. Struct. Alg., 1(1):15–37, 1990.

[12] S. Janson. The largest component in a subcritical random graph with a power law degree distribution. Ann. Appl. Probab., 1651–1668, 2008.

[13] S. Janson and M. J. Luczak. Asymptotic normality of the k-core in random graphs. Ann. Appl. Prob., 18(3):1085–1137, 2008.

[14] S. Janson and M. J. Luczak. Susceptibility in subcritical random graphs. J. Math. Phy., 49(12):125–207, 2008.

[15] S. Janson. Susceptibility of random graphs with given vertex degrees. J. Comb., 1, 3-4, 357–387, 2010.

[16] S. Janson, T. Łuczak, and A. Ruciński. Random graphs, volume 45. John Wiley & Sons, 2011.
[17] K. Krokowski, A. Reichenbachs, and C. Thäle. Discrete Malliavin-Stein method: Berry-Esseen bounds for random graphs and percolation. *Ann. Probab.* 45, no. 2, 1071–1109, 2017.

[18] D. L. McLeish. Dependent central limit theorems and invariance principles. *Ann. Probab.*, 620–628, 1974.

[19] A. Nachmias and Y. Peres. Component sizes of the random graph outside the scaling window. *Latin Amer. J. Prob. Math. Stat.*, 3:133–142, 2007.

[20] M. D. Penrose. A central limit theorem with applications to percolation, epidemics and boolean models. *Ann. Probab.*, 1515–1546, 2001.

[21] M. D. Penrose and J. E. Yukich. Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.*, 1005–1041, 2001.

[22] B. Pittel. On tree census and the giant component in sparse random graphs. *Rand. Struct. Alg.*, 1(3):311–342, 1990.

[23] A. Ruciński. When are small subgraphs of a random graph normally distributed? *Prob. Th. Rel. Fields*, 78(1):1–10, 1988.

[24] J. Salez. The interpolation method for random graphs with prescribed degrees. *Comb. Probab. Comput.*, 25(3):436–447, 2016.

[25] R. van der Hofstad. *Random graphs and complex networks: Volume 1*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2016.

[26] R. van der Hofstad. *Random graphs and complex networks: Volume 2*. [http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf](http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf), 2017.

[27] N. C. Wormald. Models of random regular graphs. *London Mathematical Society Lecture Note Series*, 239–298, 1999.

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