BOREL COHOMOLOGY AND THE RELATIVE GORENSTEIN CONDITION FOR CLASSIFYING SPACES OF COMPACT LIE GROUPS

J.P.C.GRLEES

Abstract. For a compact Lie group $G$ we show that if the representing spectrum for Borel cohomology generates its category of modules if $G$ is connected. For a closed subgroup $H$ of $G$ we consider the map $C^*(BG) \to C^*(BH)$ and establish the sense in which it is relatively Gorenstein. Throughout, we pay careful attention to the importance of connectedness of the groups.

Contents

1. Introduction 1
2. The K"unneth-Eilenberg-Moore spectral sequence 3
3. Generating the category of $b$-modules 4
4. The K"unneth spectral sequence 5
5. Tales of the torus 7
6. The relative Gorenstein condition 9
References 10

1. Introduction

1.A. Context. In this paper we consider a compact Lie group $G$ and continue the study of $C^*(BG) = C^*(BG;k)$ for a commutative ring $k$ as in [4]. We wish to use methods of commutative algebra, so we view $C^*(BG)$ as a commutative ring spectrum as in [4]. More precisely, $C^*(BG)$ is the function spectrum $F(BG_+, Hk)$ from the based space $BG_+$ to the Eilenberg-MacLane spectrum $Hk$ representing ordinary cohomology with coefficients in $k$; the name is justified by the fact that it is a commutative model for the cochains on $BG$ in the sense that $\pi_*(C^*(BG)) = H^*(BG)$.

To describe the result it is convenient to say that $G$ is $k$-connected if it is connected or if there is a prime $p$ so that $p^nk = 0$ and $\pi_0(G)$ is a $p$-group. We say that a $d$-dimensional real representation $V$ of $G$ is $k$-orientable if the action of $G$ on $H_d(S^V;k)$ is trivial. We note that if $k$ is a field, every representation is $k$-orientable if $G$ is $k$-connected or in general if $k$ of characteristic 2.

1.B. The absolute case. It is established in [4] that the map $C^*(BG) \to k$ is Gorenstein in the sense that if the adjoint representation $LG$ is $k$-orientable then there is an equivalence $\text{Hom}_{C^*(BG)}(k, C^*(BG)) \simeq L_{LG}k$ of $C^*(BG)$-modules. The suspension by $LG$ is simply

I am grateful for to the Isaac Newton Institute for providing the environment to do this work and to EPSRC Grant Number EP/P031080/1 for support.
suspension by \( g = \dim(G) \), but writing it in this way gives a statement that is natural for automorphisms of \( G \).

The statement is definitely false in general. The paper [1] gives the example \( G = O(2n) \), and one easily checks that
\[
\text{Hom}_{C^*(BO(2))}(\mathbb{Q}, C^*(BO(2))) \simeq \Sigma^3 \mathbb{Q}.
\]

1.C. The relative case. The present paper begins the investigation of the relative case. We suppose given a closed subgroup \( H \), and consider the associated map
\[
C^*(BG) \longrightarrow C^*(BH)
\]
of commutative ring spectra. If \( H \) is the trivial group this reduces to the absolute case considered above. We will show that the map is relatively Gorenstein, but in a sense that involves certain twisting.

If the representation \( LG \) of \( G \) is \( k \)-orientable then Theorem 6.1 states that there is an equivalence
\[
\text{Hom}_{C^*(BG)}(C^*(BH), C^*(BG)) \simeq C^*(BH^{-L(G,H)}),
\]
of \( C^*(BG) \)-modules, where \( L(G, H) = LG/LH \) is the representation of \( W_G(H) = N_G(H)/H \) given by the conjugation action on the tangent space to the identity coset of \( G/H \). If in addition \( LH \) is \( k \)-orientable then this shows that \( C^*(BG) \longrightarrow C^*(BH) \) is relatively Gorenstein of shift \( \dim(G/H) \) as one might hope.

If we remove the condition that \( LG \) is \( k \)-orientable then there is an equivalence
\[
\text{Hom}_{C^*(BG)}(C^*(BH^{LG}), C^*(BG^{LG})) \simeq C^*(BH^{-L(G,H)}).
\]
Note that this applies also in the case when \( H = 1 \) where it gives an appropriate statement when the adjoint representation is not \( k \)-orientable. The reader is encouraged to see how this works for the case \( G = O(2) \) and \( k = \mathbb{Q} \) mentioned above.

1.D. Relationship to previous results. Equivalence (1) was asserted as Theorem 6.8 of [2] in general, but even when \( G \) is \( k \)-connected, the proof given there is incomplete. Indeed, there are two significant omissions. First, the argument given there relied on the assertion that the Borel spectrum generates its category of modules. It is explained in Sections 3 and 4 that this follows from the convergence of an Eilenberg-Moore spectral sequence when \( G \) is \( k \)-connected and that it is not true in general. This completes the proof of Theorem 6.1 in the case that \( G \) is \( k \)-connected. We then observe in Section 6 that Equivalence (2) follows by embedding \( G \) in a connected group \( U \) and then using the results for \( G \subset U \) and \( H \subset U \). Finally, Theorem 6.8 of [2] is not true in general when \( LG \) is not orientable, even when \( H = 1 \), as pointed out above.

The author observed the gap in the earlier proof when considering the general Gorenstein properties of the map \( C^*(BG) \longrightarrow C^*(BH) \). The formulation of the Gorenstein property in Equation (1) is rather straightforward. However for many purposes, one is more interested in the Gorenstein duality property which leads to a suitable local cohomology spectral sequence. There is a well understood equivariant approach to this (the local cohomology theorem for the family of subconjugates of \( H \)), but the commutative algebra, Morita theory and orientability involves significant new complications that the author intends to return to elsewhere.
1.E. **Borel cohomology.** We are concerned with Borel cohomology with coefficients in a commutative ring \( k \), defined on based \( G \)-spaces \( X \) by

\[
b^*_G(X) = H^*(EG \times_G X, EG \times x_0; k) = H^*(EG_+ \land_G X; k).
\]

From the isomorphisms

\[
b^*_G(X) = H^*(EG_+ \land_G X) \cong [EG_+ \land_G X, Hk]^* \cong [EG_+ \land X, Hk]^*_G \cong [X, F(EG_+, Hk)]^*_G
\]

it is apparent that the representing spectrum is given by \( b = F(EG_+, Hk) \). As first pointed out in [6] this can be realized as a commutative orthogonal spectrum, since \( Hk \) is a commutative ring and \( EG \) is a space with a diagonal.

The main substance of this paper is the proof of some basic facts that are useful in studying the homotopy category of \( b \)-module \( G \)-spectra. The results generally have counterparts in more classical approaches, many of which are familiar.

1.F. **Conventions.** We say that a space \( B \) is \( k \)-simply connected if \( B \) is simply connected or if \( p^n k = 0 \) for some prime \( p \) and \( n \geq 1 \), the space \( B \) is connected and \( \pi_1(B) \) is a finite \( p \)-group.

We also say that a group \( G \) is \( k \)-connected if it is connected or if \( p^n k = 0 \) for some prime \( p \) and \( n \geq 1 \) and \( \pi_0(G) \) is a finite \( p \)-group. If \( G \) is \( k \)-connected then \( BG \) is \( k \)-simply connected.

2. **THE KÜNNETH-EILENBERG-MOORE SPECTRAL SEQUENCE**

Ordinary derived homological algebra lifts to non-equivariant spectra very well [5] to give a Künneth spectral sequence

\[
\text{Tor}^{R_*}_*(M_*, N_*) \Rightarrow (M \otimes_R N)_*
\]

for any commutative ring spectrum \( R \) and \( R \)-modules \( M \) and \( N \). This works because we can realize a \( R_* \) resolution by the ring \( R \) (we will run through the argument in Section 4 below); because \( R \) is a generator of the category of \( R \)-modules, the spectral sequence always converges.

Equivariantly, we can make the same construction, but usually the spectral sequence will not converge. This is because if \( R \) is a \( G \)-spectrum we need all the modules \( G/H_+ \land R \) to generate the category of \( R \)-modules. The point of the following theorem is that the ring \( G \)-spectrum \( b \) does generate the category of all \( b \)-modules provided \( G \) is \( k \)-connected. This is a very special feature of Borel cohomology.

**Theorem 2.1.** If \( b \) represents Borel cohomology with coefficients in \( k \) and \( G \) is \( k \)-connected then for any \( b \)-modules \( M, N \) there is a strongly convergent spectral sequence

\[
\text{Tor}^{H^*_*(BG)}_{*,*}(\pi_*^GM, \pi_*^GN) \Rightarrow \pi_*^G(M \otimes_b N),
\]

and a conditionally convergent spectral sequence

\[
\text{Ext}^{H^*_*(BG)}_{*,*}(\pi_*^GM, \pi_*^GN) \Rightarrow \pi_*^G(\text{Hom}_b(M, N)).
\]

A case of particular interest is \( N = F(G/K_+, b) \), where we have

\[
M \otimes_b N = M \otimes_b F(G/K_+, b) \cong F(G/K_+, M)
\]

since \( G/K_+ \) is small as a \( G \)-spectrum.
Corollary 2.2. If the group $G$ is $k$-connected then for any subgroup $K \subseteq G$, there is a strongly convergent spectral sequence

$$\text{Tor}^{H^*(BG)}_*(b_G^*(X), H^*(BK)) \Rightarrow b_K^*(X).$$

Remark 2.3. (i) The hypothesis that $G$ is $k$-connected is necessary to get a general statement. For example, if $K = G_e$ is the identity component of $G$, and $k$ is a field of characteristic 0 then $b_G^*(X) = b_K^*(X)^{G_d}$ where $G_d = G/G_e$ is the discrete quotient. Working rationally, we may realise a non-zero simple $G_d$-module $V$ as a Moore spectrum $MV$ and hence get a zero $E_2$-term, whilst $b_G^*(MV) \neq 0$.

(ii) The corollary takes the form of an Eilenberg-Moore spectral sequence in the sense that it has the expected $E_2$-term and end point. In fact the proof will show that the entire spectral sequence coincides with the classical Eilenberg-Moore spectral sequence for the pullback square

$$\begin{array}{ccc}
E K \times_K Z & \longrightarrow & EG \times_G Z \\
\downarrow & & \downarrow \\
BK & \longrightarrow & BG
\end{array}$$

where $Z$ is a $G$ space and $X = Z_+$.

3. Generating the category of $b$-modules

Just as every $G$-spectrum is built out of cells $G/K_+$ as $K$ varies through closed subgroups, so any $b$-module is built out of the extended $b$-modules $G/K_+ \wedge b$.

We note that $[X, F(G/K_+, b)]^*_G = b_K^*(X)$ so that if $F(G/K_+, b)$ is built from $b$ then $b_G^*(X) = 0$ implies $b_K^*(X) = 0$ if $X$ is finite. Such an implication does not hold in general, but from the Eilenberg-Moore theorem recalled in Remark 2.3 (ii), we see

$$C^*(EK \times_K Z) \simeq C^*(EG \times_G Z) \otimes_{C^*(BG)} C^*(BK)$$

provided $G$ is connected. This may make the following statement plausible.

Theorem 3.1. If $G$ is $k$-connected and $b$ represents Borel cohomology with coefficients in $k$ then $b$ is a generator of the category of $b$-modules.

Remark 3.2. Note that there is no finiteness requirement. The question of which $b$-modules are finitely built by $b$ is more subtle. In [7] the notion of finite generation is introduced, which is obviously a necessary condition for a module to be finitely built by $b$. It is shown that for finite groups $b$ finitely builds every finitely generated $b$-module precisely when $G$ is $p$-nilpotent.

The proof of the theorem is by comparison with classical Eilenberg-Moore spectral sequence, and the agreement of the spectral sequences may be of interest in itself. In Section 5 we outline an alternative approach via reduction to unitary groups and then the maximal torus.

3.A. Consequence of $b$ generating. As noted above there are Künneth and Universal Coefficient spectral sequences for the (non-equivariant) fixed point ring spectrum $b^G$:

$$\text{Ext}^{*,*}_{H^*(BG)}(M^G_*, N^G_*) = \text{Ext}^{*,*}_{BG}(M_*^G, N_*^G) \Rightarrow \pi_*(\text{Hom}_G(M^G, N^G)).$$

$$\text{Tor}^{H^*(BG)}_{*,*}(M_*^G, N_*^G) = \text{Tor}^{b^G}_{*,*}(M_*^G, N_*^G) \Rightarrow \pi_*(M^G \otimes_{BG} N^G).$$
The point is that when \( b \) is a generator of the category of \( b \)-module \( G \)-spectra, these also calculate the homotopy of the \( G \)-spectra \( \text{Hom}_b(M, N) \) and \( M \otimes_b N \) respectively, since in that case the following two elementary equivalences apply to all modules.

**Lemma 3.3.** (i) The map
\[
\text{Hom}_b(M, N)^G \longrightarrow \text{Hom}_b(G^G, N^G)
\]
is an equivalence if \( M \) is built by \( b \). Accordingly, there is then a spectral sequence
\[
\text{Ext}^{*,*}_{H^*_b(BG)}(M^G, N^G) \Rightarrow \pi_*^G(\text{Hom}_b(M, N)).
\]
(ii) The natural map
\[
M^G \otimes_b N^G \longrightarrow (M \otimes_b N)^G
\]
is an equivalence if \( M \) is built by \( b \). Accordingly, there is then a spectral sequence
\[
\text{Tor}^{*,*}_{H^*_b(BG)}(M^G, N^G) \Rightarrow \pi_*^G(M \otimes_b N). \quad \square
\]

**Proof:** In both cases, the map is obviously an equivalence when \( M = b \). The class of objects for which it is an equivalence is closed under coproducts, integer suspensions and mapping cones. \quad \square

### 4. The K"unneth spectral sequence

In this section we prove Theorems 2.1 and 3.1. The main point of the proof of the Eilenberg-Moore-type spectral sequence is that the usual construction gives a *convergent* spectral sequence. Applying the spectral sequence in the form of Corollary 2.2 shows that if \( M_\infty^G = 0 \) then \( M_\infty^K = 0 \) for all subgroups \( K \subseteq G \) and hence that \( b \) generates all \( b \)-modules.

**4.A. The construction.** First, we construct a resolution of \( \pi_*^G(M) \) by free \( H^*(BG) = \pi_*^G(b) \)-modules
\[
0 \leftarrow M_*^G \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots
\]
We then realize it by a diagram
\[
\begin{array}{cccccc}
M_0 & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & M_3 \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0 & \rightarrow & \Sigma^1P_1 & \rightarrow & \Sigma^2P_2 & \rightarrow & \Sigma^3P_3
\end{array}
\]
where \( \pi_*^G(P_s) \cong P_s \). Defining \( M^s \) by the cofibre sequence
\[
M^s \longrightarrow M \longrightarrow M_s
\]
we see that \( M^s \) and \( M_\infty = \lim M^s \) are built from \( b \), and that there is a map \( \alpha_M : M_\infty \longrightarrow M \) which is a \( \pi_*^G \)-isomorphism. Furthermore the map is unique over \( M \). This induces a map
\[
\alpha_M \otimes_b N : M_\infty \otimes_b N \longrightarrow M \otimes_b N,
\]
which we will prove is also a \( \pi_*^G \)-isomorphism. The point is that the spectral sequence obtained by filtering \( M_\infty \) then reads
\[
\text{Tor}^{*,*}_{H^*_b(BG)}(\pi_*^G(M_\infty), \pi_*^G(N)) \Rightarrow \pi_*^G(M_\infty \otimes_b N) = \pi_*^G(M \otimes_b N)
\]
as required.

It remains to show that $\alpha_M \otimes_b N : M^\infty \otimes_b N \rightarrow M \otimes_b N$ is a $\pi_*^G$-isomorphism. Define

\[ C_N := \{M \mid \alpha_M \otimes_b N \text{ is a } \pi_*^G \text{ iso} \} \]

\[ D_M := \{N \mid \alpha_M \otimes_b N \text{ is a } \pi_*^G \text{ iso} \} \]

We note that $C_N$ is a localizing subcategory for any $N$ and $D_M$ is a localizing subcategory for any $M$.

We will show in Subsection 4.B below that $\alpha_M \otimes N$ is a $\pi_*^G$-isomorphism for $M = F(X, b)$ and $N = F(G/K_+, b)$. Since the spectra $F(G/K_+, b)$ as $K$ varies generate all $b$-modules, this shows that the category $D_F(X, b)$ is the whole category of $b$-modules. It follows that for each $N$, the category $C_N$ contains $F(X, b)$. Since the modules $F(X, b)$ generate all $b$-modules, we infer $\alpha_M \otimes_b N$ is a $\pi_*^G$ isomorphism for all $M, N$.

This completes the formal part of the proof, and leaves us with the substance. It remains to show that $M^\infty \otimes_b N \rightarrow M \otimes_b N$ is a $\pi_*^G$-isomorphism when $M = F(G/K_+, b), N = F(X, b)$. Indeed, this is the map $F(G/K_+, M^\infty) \rightarrow F(G/K_+, M)$ which in $G$ fixed points reads $\pi_*^K(M^\infty) \rightarrow \pi_*^K F(X, b) = H_*^K(X)$. We will observe that we can choose the resolution so that the filtration on $(M^\infty)^K$ is the Eilenberg-Moore filtration, and then the isomorphism follows from the convergence of the Eilenberg-Moore spectral sequence, which is the main result of [3].

4.B. Comparison with the classical Eilenberg-Moore spectral sequence. We observe that suitable constructions of the Eilenberg-Moore and Künneth spectral sequences coincide.

To start with, if we are given maps $X \xrightarrow{f} B \xleftarrow{g} Y$ of spaces, we may form the two-sided geometric cobar complex

\[ CB^n(X, B, Y) = X \times B^{\times n} \times Y \]

with coface maps

\[ d^0(x, b_1, \ldots, b_n, y) = (x, f(x), b_1, \ldots, b_n, y) \]

\[ d^i(x, b_1, \ldots, b_n, e) = (x, b_1, \ldots, *, \ldots, b_n, e) \text{ for } 1 \leq i \leq n \]

\[ d^{n+1}(x, b_1, \ldots, b_n, e) = (x, b_1, \ldots, b_n, g(y), y). \]

The $i$th codegeneracy map is given by projection away from the $i + 1$st factor of $B$.

The Eilenberg-Moore spectral sequence of the pullback square

\[
\begin{array}{ccc}
P & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
Y & \xleftarrow{g} & B
\end{array}
\]

where $f$ or $g$ is a fibration, is obtained by taking cochains of $CB(X, B, Y)$, which is equivalent to the bar construction $B(C^*(X), C^*(B), C^*(Y))$ if $X, B$ and $Y$ are locally finite.

More precisely, if $A$ is a strictly commutative ring spectrum, we write $C^*(X; A) = F(X_+, A)$, and note that we may form the simplicial spectrum $B(C^*(X; A), C^*(B; A), C^*(Y; A))$ and there is a map

\[ B(C^*(X; A), C^*(B; A), C^*(Y; A)) \rightarrow C^*(CB(X, B, Y), A) \]

which is a weak equivalence if $B, X, Y$ are finite complexes, or if they are locally finite and $A$ represents ordinary cohomology.
Now consider the special case

\[
\begin{array}{ccc}
EG \times_K Z & \longrightarrow & EG \times_G Z \\
\downarrow & & \downarrow \\
BK & \longrightarrow & BG
\end{array}
\]

This is an instance of the above, so we can form the cosimplicial space \(CB(EG \times_G Z, BG, BK)\). We can also form the cosimplicial \(G\)-space \(CB(EG \times Z, BG, BK)\). It is then immediate that we have an isomorphism

\[
F(CB(EG \times_G Z, BG, BK)_+, Hk) \simeq F(CB(EG \times Z, BG, BK)_+, b)^G
\]

of simplicial spaces.

Finally, we observe that the cosimplicial filtration of \(CB(EG \times_G Z, BG, EG)\) gives a filtration of \(F(CB(EG \times_G Z, BG, EG)_+, b)\) which is a particular example of a filtration giving rise to the Künneth spectral sequence.

Now the resolution is \(B(C^*(X_{hG}), C^*(BG), b)\), so the spectral sequence is obtained by applying tensoring with \(N = F(G/K_+, b)\)

\[
B(C^*(X_{hG}), C^*(BG), b) \otimes_b F(G/K_+, b) \simeq B(C^*(X_{hG})), C^*(BG), F(G/K_+, b)).
\]

The spectral sequence is obtained by passing to fixed points to obtain \(B(C^*(X_{hG}), C^*(BG), C^*(BK))\) and then taking homotopy. This is precisely the Eilenberg-Moore spectral sequence as required.

5. Tales of the torus

In this section we describe a method based on the properties of the torus. It is standard practice in transformation groups to have a statement \(P(G)\) for each compact Lie group \(G\). We then attempt a proof of \(P(G)\) in general by the strategy

- \(P(G)\) is true with \(G = T(n)\) an \(n\)-torus
- \(P(G)\) for \(G = U(n)\) follows from \(P(T(n))\) where \(T(n)\) is the maximal torus of \(U(n)\).
- If \(G\) is a subgroup of \(U(n)\) then \(P(G)\) follows from \(P(U(n))\).

In our case we are interested in a statement that is not true for all groups, so the reduction cannot work in generality. However the first and second steps work, and we can investigate the third.

**Theorem 5.1.** If \(G\) is a torus, then for any coefficient ring \(k\), the category of \(b\)-modules is generated by \(b\).

**Proof:** Suppose \(G\) is a torus. We simply need to show if \(M\) is a \(b\)-module with \(\pi_*^HM = 0\) then in fact \(\pi_*^HM = 0\) for all \(H\) (and hence \(M \simeq 0\)).

For any subgroup \(H\) of codimension \(c\) we may find one dimensional representations \(\alpha_1, \ldots, \alpha_c\) so that \(H = \ker(\alpha_1) \cap \ldots \cap \ker(\alpha_c)\), and then

\[
G/H = S(\alpha_1) \times \cdots \times S(\alpha_c).
\]

Now observe that \(S(\alpha) = G/\ker(\alpha)\), and we have a cofibre sequence

\[
S(\alpha)_+ \rightarrow S^0 \rightarrow S^\alpha.
\]
Smashing with a $b$-module $M$ this gives

$$S(\alpha)_+ \wedge M \to S^0 \wedge M \to S^\alpha \wedge M \simeq S^2 \wedge M,$$

where the last equivalence is a Thom isomorphism. This shows that if $\pi^G_*(M) = 0$ then $\pi^G_*(G/\ker(\alpha)_+ \wedge M) = 0$.

Repeating, we see $\pi^H_*(M) = 0$. $\square$

**Theorem 5.2.** If $G = U(n)$, then for any coefficient ring $k$, the category of $b$-modules is generated by $b$.

The following observation will be useful.

**Lemma 5.3.** If $G \supseteq H \supseteq K$ and $b^*_H$ is a retract of $b^*_K$ then $M^*_H$ is a retract of $M^*_K$ for all $b$-modules $M$.

**Proof:** The hypothesis is that the $H$-spectrum $b$ is a retract of $F(H/K_+, b)$, and hence $F(G/H_+, b)$ is a retract of $F(G/K_+, b)$. Tensoring with $M$ and using the equivalence $F(G/H_+, b) \otimes b M \simeq F(G/H_+, M)$ we obtain the desired statement. $\square$

It is well known that $b^*_{T(n)} = k[x_1, \ldots, x_n]$ is free over $b^*_{U(n)} = k[c_1, \ldots, c_n] = k[x_1, \ldots, x_n]^\Sigma_n$.

The Künneth Theorem (a special case of Theorem 2.1) therefore takes an elementary form, which admits a simple proof.

**Lemma 5.4.** There is a natural isomorphism

$$b^*_{T(n)}(X) = b^*_{U(n)}(X) \otimes b^*_{U(n)} b^*_{T(n)}$$

**Proof:** There is a natural transformation of equivariant cohomology theories from the right to the left. The classical Eilenberg-Moore Theorem states that it is an isomorphism for spaces $X$. This includes the cells $G/K_+$, so the natural transformation is an isomorphism in general. $\square$

**Remark 5.5.** It would be nice to have a proof based on a general calculation that the $U(n)$-space $U(n)/K \times U(n)/T(n)$ has a nice cell structure for all $K$.

We are finally ready to prove Theorem 5.2

**Proof:** If $M$ is a $U(n)$-equivariant $b$-module with $M^*_{U(n)} = 0$ then by Lemma 5.4 we have $M^*_{T(n)} = 0$. By Theorem 5.1 it follows that $M^*_K = 0$ for all $K \subseteq T(n)$.

Now if $e$ is the idempotent of the Burnside ring supported on subgroups of the maximal torus we have $eM \simeq 0$ and $eb \simeq b$ so

$$M \simeq b \otimes b M \simeq eb \otimes b M \simeq b \otimes b eM \simeq 0$$

as required. $\square$
Remark 5.6. We now consider the deduction of the statement for a general group $G$, which we know is true if $G$ is $k$-connected. First, we may embed $G$ in $U(n)$ for some $n$.

Now suppose $\pi^G_*(M) = 0$ and note that $\pi^U(n)_*(F_G(U(n)_+, M)) = \pi^G_*(M) = 0$. From the case of $U(n)$ we infer that $F_G(U(n)_+, M) \simeq \ast$ (first $U(n)$-equivariantly and then $G$-equivariantly by restriction).

There is always a $G$-map $F_G(U(n)_+, M) \to F_G(G_+, M)$, but it need not be split. We can attempt to find a splitting by considering a $G$-cell structure of $U(n)$. However if $M = b$ we see that a first obstruction is the surjectivity of the restriction map $H_\ast(BU(n)) \to H_\ast(BG)$, and of course it often happens that $H_\ast(BG)$ is not generated by Chern classes.

6. The relative Gorenstein condition

We are ready to give the proof of the relative Gorenstein property. To obtain a natural statement we note that $L(G, H) = LG/LH$ is a representation of $N_G(H)$, typically non-trivial on $H$.

Theorem 6.1. For any $k$-connected compact Lie group $G$ and any closed subgroup $H$ we have an equivalence of $C_\ast(BG)$-modules with an action of $\pi_0(W_G(H))$:

$$\text{Hom}_{C_\ast(BG)}(C_\ast(BH), C_\ast(BG)) \simeq C_\ast(BH^{-L(G,H)})$$

The equivalence holds for arbitrary $G$ if $LG$ is $k$-orientable and $LH$ is $k$-orientable. Without the $k$-connectedness or orientability hypotheses, we have the equivalence

$$\text{Hom}_{C_\ast(BG)}(C_\ast(BH^{LG}), C_\ast(BG^{LG})) \simeq C_\ast(BH^{-L(G,H)})$$

Remark 6.2. The first equivalence does not hold in general. This is familiar from the case $H = 1$ where \cite{1} gives the example $G = O(2n)$. The first case is very easy to check:

$$\text{Hom}_{C_\ast(BO(2))}(k, C_\ast(BO(2))) \simeq \Sigma^3 k.$$
Taking fixed points we have
\[ \text{Hom}(F(G/H_+, b), b)^G \simeq (G/H_+ \wedge b)^G \simeq (\Sigma^{L(G,H)}b)^H \simeq C^*(BH^{-L(G,H)}). \]

We may now complete the proof if \( G \) is connected. In that case Theorem 3.1 gives equivalences
\[
\text{Hom}_b(F(G/H_+, b), b)^G \overset{\cong}{\rightarrow} \text{Hom}_{bG}(F(G/H_+, b)^G, b^G) \simeq \\
\text{Hom}_{bG}(b^H, b^G) \simeq \text{Hom}_{C^*(BG)}(C^*(BH), C^*(BG)),
\]
as in Lemma 3.3 (i).

Finally, we note that this permits us to deduce the general case.

Choose a faithful representation of \( G \) in \( U = U(n) \) for some \( n \). From the connected case we have
\[ \text{Hom}_{C^*(BU)}(C^*(BG), C^*(BU)) \simeq C^*(BG^{-L(U,G)}) \]
and
\[ \text{Hom}_{C^*(BU)}(C^*(BH), C^*(BU)) \simeq C^*(BH^{-L(U,H)}). \]

Now we have
\[
\text{Hom}_{C^*(BG)}(C^*(BH), C^*(BG^{-L(U,G)})) \simeq \text{Hom}_{C^*(BG)}(C^*(BH), \text{Hom}_{C^*(BU)}(C^*(BG), C^*(BU))) \\
\simeq \text{Hom}_{C^*(BU)}(C^*(BH), C^*(BU)) \\
\simeq C^*(BH^{-L(U,H)}).
\]
as required.

This completes the proof if \( LG \) and \( LH \) are \( k \)-orientable. Because \( L(U,G) \) need not be trivial for \( G \), some additional work is necessary to go further. To start with, note that \( L(U,H) \) does admit an action of \( N_U(H) \) and hence \( N_G(H) \).

We start with the equivalence of \( U \)-spectra
\[ U_+ \wedge_H (\Sigma^{-L(U,G)}b) \simeq \text{Hom}_b(F_H(U_+, \Sigma^{L(U,G)}b), b), \]
and take \( U \)-fixed points. Since \( U \) is connected, by 3.1 and 3.3 this gives the first equivalence of the following:
\[
(\Sigma^{L(U,H)-L(U,G)b})^H \simeq \text{Hom}_{b^U}(\Sigma^{L(U,G)b})^H, b^U) \\
\simeq \text{Hom}_{bG}(\Sigma^{L(U,G)b})^H, \text{Hom}_{b^U}(b^G, b^U)) \\
\simeq \text{Hom}_{bG}(\Sigma^{L(U,G)b})^H, (\Sigma^{L(U,G)b})^G)
\]
Since \( L(U,G) = LU/LG, L(U,H) = LU/LH \), Lemma 6.3 permits us to deduce the second equivalence of the theorem. \( \square \)

**References**

[1] D. J. Benson and J. P. C. Greenlees. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. *J. Pure Appl. Algebra*, 122(1-2):41–53, 1997.

[2] D.J. Benson and J.P.C. Greenlees. Stratifying the derived category of cochains on \( BG \) for \( G \) a compact Lie group. *J. Pure Appl. Algebra*, 218(4):642–650, 2014.

[3] W. G. Dwyer. Strong convergence of the Eilenberg-Moore spectral sequence. *Topology*, 13:255–265, 1974.

[4] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006.
[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory, volume 47 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[6] A. D. Elmendorf and J. P. May. Algebras over equivariant sphere spectra. J. Pure Appl. Algebra, 116(1-3):139–149, 1997. Special volume on the occasion of the 60th birthday of Professor Peter J. Freyd.

[7] J. P. C. Greenlees and G. Stevenson. Morita theory and singularity categories. pages 1–44. arXiv:1702.07957.

Mathematical Institute, Zeeman Building Coventry CV4 7AL, UK.
E-mail address: john.greenlees@warwick.ac.uk