Certain Grüss-type inequalities via tempered fractional integrals concerning another function

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**Abstract**

We study a generalized left sided tempered fractional (GTF)-integral concerning another function \(\Psi\) in the kernel. Then we investigate several kinds of inequalities such as Grüss-type and certain other related inequalities by utilizing the GTF-integral. Additionally, we present various special cases of the main result. By utilizing the connection between GTF-integral and Riemann–Liouville integral concerning another function \(\Psi\) in the kernel, certain distinct particular cases of the main result are also presented. Furthermore, certain other inequalities can be formed by applying various kinds of conditions on the function \(\Psi\).

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1 Introduction

The field of fractional calculus deals with the integrals and differentiation of an arbitrary non-integer order. In the last three centuries, this field has been considered as the most power tool in describing the anomalous kinetics and its wide applications in diverse domains. Numerous mathematical, statistical, engineering, physical, chemical, and biological phenomena can be modeled by utilizing ordinary differential equations involving fractional derivatives. Many a mathematician and physicist has contributed to the development of the theories of fractional calculus. The interesting reader is referred to [1–3] and the references therein. In practical applications, a numerous types of fractional integrals and derivatives operators, such as the Riemann–Liouville, Caputo, Riesz, Hilfer, Hadamard, Erdélyi–Kolber, Saigo, and Marichev–Saigo–Maeda operators, were extensively studied by various researchers. We refer the reader to [2–4].

Later, mathematicians introduced the notion of fractional conformable integrals and derivatives which are cited therein. Khalil et al. [5] introduced fractional conformable derivatives operators with some shortcoming. Abdeljawad [6] investigated the properties of the fractional conformable derivative operators. Jarad et al. [7] defined generalized fractional conformable integral and derivative operators. Anderson and Unless [8] presented...
the idea of a conformable derivative by employing local proportional derivatives. In [9],
Abdeljawad and Baleanu gave certain monotonicity results for fractional difference op-
erators with discrete exponential kernels. In [10], Abdeljawad and Baleanu have defined
a fractional derivative operator with exponential kernel and the discrete version. In [11],
Atangana and Baleanu defined a new fractional derivative operator with the non-local and
non-singular kernel. A fractional derivative without a singular kernel can be found in the
work of Caputo and Fabrizio [12]. Certain properties of fractional derivatives without a
singular kernel can be found in the work of Losada and Nieto [13]. In [14–16], the authors
studied stability analysis and a numerical scheme for fractional Klein–Gordon equations,
existence results in Banach space for a nonlinear impulsive system and results for mild
solutions of fractional coupled hybrid boundary value problems.

A variety of such types of distinguished operators led researchers to establish new ideas
and fractional integral inequalities by utilizing these new operators. In [17, 18], Hasib et
al. established various inequalities by using AB-fractional and Saigo fractional integral
operators. Recently Alzabut et al. and Rahman et al. [19, 20] studied generalized propor-
tional derivatives and integral operators and established a certain Gronwall inequality and
the Minkowski inequalities involving the said operators. Rahman et al. [21, 22] presented
fractional integral inequalities for a family of positive continuous and decreasing functions
and inequalities for convex functions by employing proportional Hadamard fractional in-
tegrals. Recently, researchers presented several various remarkable inequalities with prop-
erties and applications for the fractional conformable integrals and proportional integrals.
The interested reader may consult [23–33].

2 Preliminaries

In this section, we consider some well-known definitions and mathematical preliminaries.

Definition 2.1 ([34]) Suppose that the functions \( U, \quad \forall \in [a_1, b_1] \rightarrow \mathbb{R} \) are positive with \( A \leq U(\theta) \leq B \) and \( C \leq V(\theta) \leq D \), for all \( \theta \in [a_1, b_1] \), then the following inequality holds:

\[
\begin{align*}
&\left| \frac{1}{b_1-a_1} \int_{a_1}^{b_1} U(\theta) V(\theta) d\theta - \frac{1}{b_1-a_1} \int_{a_1}^{b_1} U(\theta) d\theta \int_{a_1}^{b_1} V(\theta) d\theta \right| \\
&\leq \frac{1}{4} (B-A)(D-C),
\end{align*}
\]

(2.1)

where the constants \( B, A, C, D \in \mathbb{R} \) and \( \frac{1}{4} \) is the sharp value of inequality (2.1).

Definition 2.2 ([35, 36]) The function \( U(\varphi) \) will be in the space \( L_{p,r}[0, \infty[ \) if

\[
L_{p,r}[0, \infty[ = \left\{ U : \|U\|_{L_{p,r}[0, \infty[} = \left( \int_{r}^{\infty} |U(\varphi)|^p \varphi^r d\varphi \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty, r \geq 0 \right\},
\]

(2.2)

If we consider \( r = 0 \), then (2.2) gives

\[
L_p[0, \infty[ = \left\{ U : \|U\|_{L_p[0, \infty[} = \left( \int_{0}^{\infty} |U(\varphi)|^p d\varphi \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty \right\}
\]

Definition 2.3 ([37]) Suppose that we have the function \( U \in L_1[0, \infty[ \) and assume that the
function \( \Psi \) is positive, monotone and an increasing on \([0, \infty[ \) and let \( \Psi' \) be continuous on
with \( \Psi(0) = 0 \). Then the Lebesgue real-valued measurable function \( \mathcal{U} \) defined on \([0, \infty[\) is said to be in the space \( \chi^p_\Psi(0, \infty) \), \((1 \leq p < \infty)\), for which

\[
\|\mathcal{U}\|_{\chi^p_\Psi} = \left( \int_a^\rho |\mathcal{U}(\varrho)|^p \Psi'(\varrho) \, d\varrho \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.
\]

When \( p = \infty \), then

\[
\|\mathcal{U}\|_{\chi^\infty_\Psi} = \text{ess sup}_{0 \leq \varrho < \infty} \left| \Psi'(\varrho) \mathcal{U}(\varrho) \right|.
\]

Note that the space \( \chi^p_\Psi(0, \infty) \) coincides with the space \( L^p[0, \infty[ \) if \( \Psi(\varrho) = \varrho \) for \( 1 \leq p < \infty \) and similarly with the space \( L^p[1, \infty[ \) if \( \Psi(\varrho) = \ln \varrho \) for \( 1 \leq p < \infty \).

The tempered fractional integral was first studied by Buschman [38], but Li et al. [39] and Meerschaert et al. [40] have described the associated tempered fractional calculus more explicitly. Fernandez and Ustaoğlu [41] investigated several analytic properties of tempered fractional integrals.

**Definition 2.4** ([39, 40]) Suppose that \([a, b]\) is a real interval and \( \kappa, \xi \in \mathbb{C} \) with \( \Re(\kappa) > 0 \) and \( \Re(\xi) \geq 0 \), then the left sided tempered fractional integral is defined by

\[
(a \mathcal{J}^{\kappa, \xi} \mathcal{U})(\rho) = \frac{1}{\Gamma(\kappa)} \int_a^\rho e^{-\xi(\rho-t)}(\varrho - t)^{\kappa-1} \mathcal{U}(t) \, dt, \quad a < \rho.
\]  

**Remark 2.1** Setting \( \xi = 0 \) in (2.3) yields the following Riemann–Liouville fractional integral:

\[
(a \mathcal{J}^\kappa \mathcal{U})(\rho) = \frac{1}{\Gamma(\kappa)} \int_a^\rho (\varrho - t)^{\kappa-1} \mathcal{U}(t) \, dt, \quad \rho > a.
\]  

The tempered fractional integral (2.3) satisfies the following semigroup property:

\[
a \mathcal{J}^{\kappa+\lambda, \xi} \left( a \mathcal{J}^{\lambda, \xi} \mathcal{U}(\varrho) \right) = a \mathcal{J}^{\kappa+\lambda, \xi} \mathcal{U}(\varrho).
\]

In [42], Fahad et al. defined the following general form of the generalized tempered fractional integral concerning another function.

**Definition 2.5** Let \( \mathcal{U} \) be an integrable function in the space \( \chi^p_\Psi(0, \infty) \) and assume that the function \( \Psi \) is positive, monotone and increasing on \([0, \infty[\) and let \( \Psi' \) be continuous on \([0, \infty[\) with \( \Psi(0) = 0 \). Then the left (GTF)-integral of a function \( \mathcal{U} \) concerning another function \( \Psi \) in the kernel is defined by

\[
\left( \Psi \mathcal{J}^{\kappa, \xi} \mathcal{U} \right)(\rho) = \frac{1}{\Gamma(\kappa)} \int_a^\rho e^{-\xi(\Psi(\rho)-\Psi(\varrho))}(\Psi(\rho) - \Psi(\varrho))^{\kappa-1} \Psi'(\varrho) \mathcal{U}(\varrho) \, d\varrho, \quad a < \rho,
\]  

where \( \xi > 0, \kappa \in \mathbb{C} \) with \( \Re(\kappa) > 0 \) and \( \Gamma(\cdot) \) is the well-known gamma function.
Remark 2.2 The left GTF-integral \((2.5)\) will reduce the following fractional integrals:

i. setting \(\Psi(\rho) = \rho\), then the left tempered fractional integral \((2.3)\) will be obtained,

ii. setting \(\xi = 0\), then the left generalized RL-fractional integral operator defined by [1] will be obtained,

iii. setting \(\Psi(\rho) = \ln \rho\), the left Hadamard GTF-integral defined by \((42)\) will be obtained,

iv. setting \(\Psi(\rho) = \rho^\kappa\), \(\kappa > 0\) and \(\xi = 0\), the left Katugampola \([36]\) fractional integral operator will be obtained,

v. setting \(\Psi(\rho) = \rho^\alpha + \beta\) and \(\xi = 0\) \((\text{where } \alpha \in (0, 1], \beta \in \mathbb{R} \text{ and } \mu + s \neq 0)\), then the generalized fractional conformable integral given in \([43]\) will be obtained.

In this manuscript, we will consider the following one sided GTF-integral.

Definition 2.6 Let \(U\) be an integrable function in the space \(\chi_p^\rho(0, \infty)\) and assume that the function \(\Psi\) is positive, monotone and increasing on \([0, \infty]\) and \(\Psi'\) is continuous on \([0, \infty]\) with \(\Psi(0) = 0\). Then the one sided (GTF) integral of a function \(U\) concerning another function \(\Psi\) in the kernel is defined by

\[
\left(\Psi J^{\alpha, \xi} U\right)(\rho) = \frac{1}{\Gamma(\kappa)} \int_0^\rho e^{-\xi\Psi(\rho) - \Psi(\omega)}(\Psi(\rho) - \Psi(\omega))^{\kappa-1}\Psi'(\omega)U(\omega) d\omega, \quad a < \rho, \tag{2.7}
\]

One can easily derive the following results.

Theorem 2.1 Let \(U : [0, \rho] \subseteq [0, \infty] \rightarrow \mathbb{R}\) be an GTF-integral operator concerning another function \(\Psi\), then we have

\[
\Psi J^{\kappa, \xi} \left(\Psi J^{\mu, \xi} U\right)(\rho) = \left(\Psi J^{\kappa+\mu, \xi} U\right)(\rho),
\]

where \(\kappa, \mu > 0\).

Theorem 2.2 The GTF-integral operator \(\Psi J^{\kappa, \xi} : L_1[0, \rho] \rightarrow L_1[0, \rho]\) will satisfy the following linearity property:

\[
\Psi J^{\kappa, \xi} (aU_1 + \beta U_2) = a\Psi J^{\kappa, \xi} U_1 + \beta\Psi J^{\kappa, \xi} U_2
\]

where \(U_1, U_2 \in L_1[0, \rho]\) and \(a, \beta \in \mathbb{R}\).

The main goal of this manuscript is to establish certain inequalities such as Grüss-type and several other inequalities by utilizing the GTF-integral \((2.1)\). Also, certain special and particular cases of the main result are presented.
3 Main results
We present generalizations of certain inequalities by utilizing the GTF-integral operator (2.7) containing another function $\Psi$ in its kernel in this section.

**Theorem 3.1** Suppose that we have the function $U \in \chi^k_{\nu}(0, \infty)$ and assume that the function $\Psi$ is positive, monotone and increasing on $[0, \infty]$ and its derivative $\psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Moreover, let $\phi_1$ and $\phi_2$ be two integrable functions defined on $[0, \infty]$ such that

$$\phi_1(\rho) \leq U(\rho) \leq \phi_2(\rho), \quad \rho \in [0, \infty].$$

(3.1)

Then, for $\rho > 0$, $\kappa, \lambda > 0$, we have

$$\Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) J^{\kappa, \lambda}_\rho U(\rho) + \Psi J^{\kappa, \lambda}_\rho U(\rho) J^{\kappa, \lambda}_\rho \phi_1(\rho) \geq \Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) J^{\kappa, \lambda}_\rho U(\rho) + \Psi J^{\kappa, \lambda}_\rho U(\rho) J^{\kappa, \lambda}_\rho \phi_1(\rho).$$

(3.2)

**Proof** Applying (3.1) for all $\rho \geq 0$ and $\zeta \geq 0$, we have

$$(\phi_2(\rho) - U(\rho))(U(\zeta) - \phi_1(\zeta)) \geq 0.$$  

It follows that

$$\phi_2(\rho)U(\zeta) + \phi_1(\zeta)U(\rho) \geq \phi_2(\rho)\phi_1(\zeta) + U(\rho)U(\zeta).$$

(3.3)

Multiplying both sides of (3.3) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\rho) - \Psi(\zeta))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\rho)$ and integrating the resulting estimate with respect to $\rho$ from 0 to $\rho$, we get

$$U(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\rho e^{-\xi(\Psi(\rho) - \Psi(\rho))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\rho) \phi_2(\rho) d\rho$$

$$\phi_1(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\rho e^{-\xi(\Psi(\rho) - \Psi(\rho))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\rho) U(\rho) d\rho$$

$$\geq \phi_1(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\rho e^{-\xi(\Psi(\rho) - \Psi(\rho))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\rho) \phi_2(\rho) d\rho$$

$$U(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\rho e^{-\xi(\Psi(\rho) - \Psi(\rho))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\rho) U(\rho) d\rho,$$

which with the aid of (2.7) becomes

$$U(\zeta)^\Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) + \phi_1(\zeta)^\Psi J^{\kappa, \lambda}_\rho U(\rho)$$

$$\geq \phi_1(\zeta)^\Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) + U(\zeta)^\Psi J^{\kappa, \lambda}_\rho U(\rho).$$

(3.4)

Again, multiplying both sides of (3.4) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\rho) - \Psi(\zeta))}(\Psi(\rho) - \Psi(\zeta))^{k-1}\psi'(\zeta)$ and integrating the resulting estimate with respect to $\zeta$ from 0 to $\zeta$, we obtain

$$\Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) J^{\kappa, \lambda}_\rho U(\rho) + \Psi J^{\kappa, \lambda}_\rho U(\rho) J^{\kappa, \lambda}_\rho \phi_1(\rho)$$

$$\geq \Psi J^{\kappa, \lambda}_\rho \phi_2(\rho) J^{\kappa, \lambda}_\rho \phi_1(\rho) + \Psi J^{\kappa, \lambda}_\rho U(\rho) J^{\kappa, \lambda}_\rho U(\rho),$$

(3.5)

which proves the desired assertion (3.2). \qed
**Theorem 3.2** Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0, \infty]$ and assume that the function $\Psi$ is positive, monotone and increasing on $[0, \infty]$ and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Assume that (3.1) holds and let $\psi_1, \psi_2$ be two integrable functions defined on $[0, \infty]$ such that

$$\psi_1(\rho) \leq \mathcal{V}(\rho) \leq \psi_2(\rho), \quad \rho \in [0, \infty]. \quad (3.6)$$

Then, for $\rho > 0$ and $\kappa, \lambda > 0$, the following four inequalities hold:

$$\begin{align*}
\psi \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_1(\rho) \\
\quad \geq \psi \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_1(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho), \quad (3.7) \\
\psi \mathcal{J}^{\kappa, \lambda} \phi_1(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \psi_2(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \\
\quad \geq \psi \mathcal{J}^{\kappa, \lambda} \phi_1(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_2(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho), \quad (3.8) \\
\psi \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_2(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho) \\
\quad \geq \psi \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_2(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho), \quad (3.9) \\
\psi \mathcal{J}^{\kappa, \lambda} \phi_1(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_1(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho) \\
\quad \geq \psi \mathcal{J}^{\kappa, \lambda} \phi_1(\rho) \psi \mathcal{J}^{\kappa, \lambda} \psi_1(\rho) + \psi \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \psi \mathcal{J}^{\kappa, \lambda} \mathcal{V}(\rho). \quad (3.10)
\end{align*}$$

**Proof** To prove inequality (3.7), using (3.1) and (3.6) for $\rho, \zeta \in [0, \infty]$ yields

$$(\phi_2(\rho) - \mathcal{U}(\rho))(\mathcal{V}(\zeta) - \psi_1(\zeta)) \geq 0.$$ 

It follows that

$$\phi_2(\rho) \mathcal{V}(\zeta) + \mathcal{U}(\rho) \psi_1(\zeta) \geq \phi_2(\rho) \psi_1(\zeta) + \mathcal{U}(\rho) \mathcal{V}(\zeta). \quad (3.11)$$

Multiplying both sides of inequality (3.11) with $\frac{1}{\Gamma(\kappa)} e^{-\xi(\Psi(\rho) - \Psi(\rho))} (\Psi(\rho) - \Psi(\rho))^{\kappa-1} \Psi'(\rho)$ and integrating the resulting estimate with respect to $\rho$ from 0 to $\varphi$, we get

$$\begin{align*}
\mathcal{V}(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\varphi e^{-\xi(\Psi(\rho) - \Psi(\rho))} (\Psi(\rho) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) \phi_2(\rho) \, d\rho \\
+ \psi_1(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\varphi e^{-\xi(\Psi(\rho) - \Psi(\rho))} (\Psi(\rho) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) \mathcal{U}(\rho) \, d\rho \\
\geq \psi_1(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\varphi e^{-\xi(\Psi(\rho) - \Psi(\rho))} (\Psi(\rho) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) \phi_2(\rho) \, d\rho \\
+ \mathcal{V}(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\varphi e^{-\xi(\Psi(\rho) - \Psi(\rho))} (\Psi(\rho) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) \mathcal{U}(\rho) \, d\rho,
\end{align*}$$

which with the aid of (2.7) becomes

$$\mathcal{V}(\zeta) \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) + \psi_1(\zeta) \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho) \geq \psi_1(\zeta) \mathcal{J}^{\kappa, \lambda} \phi_2(\rho) + \mathcal{V}(\zeta) \mathcal{J}^{\kappa, \lambda} \mathcal{U}(\rho). \quad (3.12)$$
Again, multiplying both sides of (3.12) with $\frac{1}{\Gamma(\beta)} e^{-(\Psi(q) - \Psi(\zeta))} (\Psi(q) - \Psi(\zeta))^{\lambda-1} \Psi'(\zeta)$, integrating the resulting estimate with respect to $\zeta$ from 0 to $\varnothing$ and using (2.7), we obtain

$$
\begin{align*}
\Psi J^{\mu, \lambda, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) \\
\geq \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q),
\end{align*}
$$

which completes the desired assertion (3.7). Inequalities (3.8)–(3.10) can be proved by using the following identities:

$$
\begin{align*}
(\psi_2(q) - \nu(q)) (\nu(q) - \phi_1(q)) & \geq 0, \\
(\phi_2(q) - \nu(q)) (\nu(q) - \psi_2(q)) & \leq 0,
\end{align*}
$$

and

$$
\begin{align*}
(\phi_1(q) - \nu(q)) (\nu(q) - \psi_1(q)) & \leq 0,
\end{align*}
$$

respectively.

\section{4 Certain other inequalities via GTF-integral concerning another function}

Certain other types of inequalities which involving generalized tempered fractional (GTF) integral (2.7) are presented in this section.

\textbf{Theorem 4.1} Suppose that the two positive functions $\nu$ and $\nu$ are defined on $[0, \infty]$ and assume that the function $\Psi$ is positive, monotone and increasing on $[0, \infty]$ and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. If $p_1, q_1 > 1$ are such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then, for $\varnothing > 0$, the following inequalities hold:

$$
\begin{align*}
\frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) \\
\geq \frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q),
\end{align*}
$$

(4.1)

$$
\begin{align*}
\frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) \\
\geq \frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q),
\end{align*}
$$

(4.2)

$$
\begin{align*}
\frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) \\
\geq \frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q),
\end{align*}
$$

(4.3)

and

$$
\begin{align*}
\frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) \\
\geq \frac{1}{p_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q) + \frac{1}{q_1} \Psi J^{\mu, \lambda, \psi}(q) J^{\lambda, \psi}(q).
\end{align*}
$$

(4.4)
Proof. Consider the well-known Young’s inequality [44]:

\[
\frac{1}{p_1} a^{p_1} + \frac{1}{q_1} b^{q_1} \geq ab, \quad a, b > 0, \quad \frac{1}{p_1} + \frac{1}{q_1} = 1.
\]  

(4.5)

Applying (4.5) for \(a = U(\rho) V(\zeta)\) and \(b = U(\zeta) V(\rho)\), \(\rho, \zeta > 0\), we have

\[
\frac{1}{p_1} \left( U(\rho) V(\zeta) \right)^{p_1} + \frac{1}{q_1} \left( U(\zeta) V(\rho) \right)^{q_1} \geq \left( U(\rho) V(\zeta) \right) \left( U(\zeta) V(\rho) \right).
\]  

(4.6)

Multiplying both sides of inequality (4.6) with \(\frac{1}{\Gamma(\kappa)} e^{-\xi(\Psi(\cdot)-\Psi(\rho))} (\Psi(\cdot) - \Psi(\rho))^{\kappa-1} \Psi'(\rho)\) and integrating the resulting estimate with respect to \(\rho\) from 0 to \(\varrho\), we get

\[
\frac{\gamma^{p_1}(\zeta)}{p_1 \Gamma(\kappa)} \int_{0}^{\varrho} e^{-\xi(\Psi(\cdot)-\Psi(\rho))} (\Psi(\cdot) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) U^{p_1}(\rho) \, d\rho
\]

\[+ \frac{\gamma^{q_1}(\zeta)}{q_1 \Gamma(\kappa)} \int_{0}^{\varrho} e^{-\xi(\Psi(\cdot)-\Psi(\rho))} (\Psi(\cdot) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) V^{q_1}(\rho) \, d\rho
\]

\[\geq \frac{\gamma^{q_1}(\zeta)}{\Gamma(\kappa)} \int_{0}^{\varrho} e^{-\xi(\Psi(\cdot)-\Psi(\rho))} (\Psi(\cdot) - \Psi(\rho))^{\kappa-1} \Psi'(\rho) U(\rho) V(\rho) \, d\rho,
\]

which in view of (2.7) becomes

\[
\frac{\gamma^{p_1}(\zeta) \Psi^{\kappa-\frac{1}{2}} U^{p_1}(\rho)}{p_1} + \frac{\gamma^{q_1}(\zeta) \Psi^{\kappa-\frac{1}{2}} V^{q_1}(\rho)}{q_1} \geq \frac{\gamma^{q_1}(\zeta)}{\Gamma(\kappa)} \Psi^{\kappa-\frac{1}{2}} U(\rho) V(\rho).
\]  

(4.7)

Now, multiplying both sides of (4.7) with \(\frac{1}{\Gamma(\kappa)} e^{-\xi(\Psi(\cdot)-\Psi(\rho))} (\Psi(\cdot) - \Psi(\rho))^{\kappa-1} \Psi'(\rho)\), integrating the resulting estimate with respect to \(\zeta\) from 0 to \(\varrho\) and using (2.7), we obtain

\[
\frac{\gamma^{p_1}(\rho) \Psi^{\kappa-\frac{1}{2}} U^{p_1}(\zeta)}{p_1} + \frac{\gamma^{q_1}(\rho) \Psi^{\kappa-\frac{1}{2}} V^{q_1}(\zeta)}{q_1} \geq \frac{\gamma^{q_1}(\rho)}{\Gamma(\kappa)} \Psi^{\kappa-\frac{1}{2}} U(\zeta) V(\rho),
\]

which is the desired assertion (4.1). The inequalities (4.2), (4.3) and (4.4) can be obtained by employing the similar procedure and setting the following parameters in (4.5), respectively:

\[
a = \frac{U(\rho)}{U(\zeta)}, \quad b = \frac{V(\rho)}{V(\zeta)}, \quad U(\rho), V(\zeta) \neq 0,
\]

(4.8)

\[
a = U(\rho) V^{\frac{2}{p_1}}(\zeta), \quad b = U^{\frac{2}{q_1}}(\zeta)V(\rho),
\]

(4.9)

and

\[
a = U^{\frac{2}{p_1}}(\rho)U(\zeta), \quad b = V^{\frac{2}{q_1}}(\rho)V(\zeta).
\]

(4.10)

Theorem 4.2 Suppose that the two positive functions \(U\) and \(V\) are defined on \([0, \infty]\) and assume that the function \(\Psi\) is positive, monotone and increasing on \([0, \infty]\) and its derivative \(\Psi'\) is continuous on \([0, \infty]\) with \(\Psi(0) = 0\). If \(p_1, q_1 > 1\) are such that \(\frac{1}{p_1} + \frac{1}{q_1} = 1\), then, for
Let $\varphi > 0$, the following inequalities hold:

\[
p_{1} \varphi J^{x,\xi} U(\varphi)^{\varphi} J^{x,\xi} V(\varphi) + q_{1} \varphi J^{x,\xi} V(\varphi)^{\varphi} J^{x,\xi} U(\varphi) \\
\geq \varphi J^{x,\xi} (U^{p_{1}}(\varphi)V^{q_{1}}(\varphi))^{\varphi} J^{x,\xi} (U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)),
\]

\[
p_{1} \varphi J^{x,\xi} U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} V^{q_{1}}(\varphi) + q_{1} \varphi J^{x,\xi} V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} U^{p_{1}}(\varphi) \\
\geq \varphi J^{x,\xi} V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} U^{p_{1}}(\varphi),
\]

\[
p_{1} \varphi J^{x,\xi} U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} V^{q_{1}}(\varphi) + q_{1} \varphi J^{x,\xi} V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} U^{p_{1}}(\varphi) \\
\geq \varphi J^{x,\xi} U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} V^{q_{1}}(\varphi).\]

\[
(4.13)
\]

and

\[
p_{1} \varphi J^{x,\xi} U^{2}(\varphi)V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} V^{q_{1}}(\varphi) + q_{1} \varphi J^{x,\xi} V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} U^{2}(\varphi) \\
\geq \varphi J^{x,\xi} U^{2}(\varphi)V^{q_{1}}(\varphi)^{\varphi} J^{x,\xi} V^{q_{1}}(\varphi).\]

\[
(4.14)
\]

**Proof** Consider the following well-known $A.M - G.M$ inequality:

\[
p_{1} a + q_{1} b \geq a^{p_{1}} b^{q_{1}}, \quad \forall a, b \geq 0, p_{1} + q_{1} = 1.
\]

Applying (4.15) for $a = U(\rho)V(\zeta)$ and $b = U(\zeta)V(\rho)$, $\rho, \zeta > 0$, we have

\[
p_{1} U(\rho)V(\zeta) + q_{1} U(\zeta)V(\rho) \geq (U(\rho)V(\zeta))^{p_{1}} (U(\zeta)V(\rho))^{q_{1}}.
\]

\[
(4.16)
\]

Multiplying both sides of inequality (4.16) with $\frac{1}{\Gamma(\zeta)} e^{-\xi(\varphi(\zeta) - \varphi(\rho))} (\varphi(\zeta) - \varphi(\rho))^{k-1} \varphi'(\rho)$ and integrating the resulting estimate with respect to $\rho$ from 0 to $\varphi$, we get

\[
p_{1} V(\zeta) \frac{1}{\Gamma(\kappa)} \int_{0}^{\varphi} e^{-\xi(\varphi(\zeta) - \varphi(\rho))} (\varphi(\zeta) - \varphi(\rho))^{k-1} \varphi'(\rho) U(\rho) d\rho \\
+ q_{1} U(\zeta) \frac{1}{\Gamma(\kappa)} \int_{0}^{\varphi} e^{-\xi(\varphi(\zeta) - \varphi(\rho))} (\varphi(\zeta) - \varphi(\rho))^{k-1} \varphi'(\rho) V(\rho) d\rho \\
\geq V^{p_{1}}(\zeta) U^{p_{1}}(\zeta) \frac{1}{\Gamma(\kappa)} \int_{0}^{\varphi} e^{-\xi(\varphi(\zeta) - \varphi(\rho))} (\varphi(\zeta) - \varphi(\rho))^{k-1} \varphi'(\rho) U^{p_{1}}(\rho)V^{q_{1}}(\rho) d\rho,
\]

which in view of (2.7) becomes

\[
p_{1} V(\zeta)^{\varphi} J^{x,\xi} U(\varphi) + q_{1} U(\zeta)^{\varphi} J^{x,\xi} V(\varphi) \geq V^{p_{1}}(\zeta) U^{p_{1}}(\zeta)^{\varphi} J^{x,\xi} (U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)).
\]

\[
(4.17)
\]

Again, multiplying both sides of (4.17) with $\frac{1}{\Gamma(\zeta)} e^{-\xi(\varphi(\zeta) - \varphi(\rho))} (\varphi(\zeta) - \varphi(\rho))^{k-1} \varphi'(\zeta)$, integrating the resulting estimate with respect to $\zeta$ from 0 to $\varphi$ and using (2.7), we obtain the desired assertion (4.11),

\[
p_{1} \varphi J^{x,\xi} U(\varphi)^{\varphi} J^{x,\xi} V(\varphi) + q_{1} \varphi J^{x,\xi} V(\varphi)^{\varphi} J^{x,\xi} U(\varphi) \\
\geq \varphi J^{x,\xi} (U^{p_{1}}(\varphi)V^{q_{1}}(\varphi))^{\varphi} J^{x,\xi} (U^{p_{1}}(\varphi)V^{q_{1}}(\varphi)).
\]
The inequalities (4.12), (4.13) and (4.14) can be easily obtained by following the same procedure and using the following parameters in (4.15), respectively. We have

\begin{align}
a &= \frac{U(\zeta)}{U(\rho)}, \quad b = \frac{V(\rho)}{V(\zeta)}, \quad U(\rho), V(\zeta) \neq 0, \\
&= \frac{U(\rho)}{U(\zeta)} \Psi^2(\zeta), \quad b = \frac{U^2(\zeta)}{V(\rho)}
\end{align}

(4.18)

and

\begin{align}
a &= \frac{U^2(\zeta)}{V(\rho)}, \quad b &= \frac{U^2(\zeta)}{V(\rho)}, \quad V(\rho), V(\zeta) \neq 0.
\end{align}

(4.20)

**Theorem 4.3** Suppose that the two positive functions \(U\) and \(V\) are defined on \([0, \infty]\) and assume that the function \(\Psi\) is positive, monotone and increasing on \([0, \infty]\) and its derivative \(\Psi'\) is continuous on \([0, \infty]\) with \(\Psi(0) = 0\). Let \(p_1, q_1 > 1\) be such that \(\frac{1}{p_1} + \frac{1}{q_1} = 1\). Suppose

\begin{align}
K &= \min_{0 \leq p \leq 0} \frac{U(\rho)}{V(\rho)} \quad \text{and} \quad H = \max_{0 \leq p \leq 0} \frac{U(\rho)}{V(\rho)}.
\end{align}

(4.21)

Then, for \(\varrho > 0\), the following inequalities hold:

\begin{align}
0 &\leq \varrho J^{x, \delta} U^2(\rho)^\varrho J^{x, \delta} V^2(\rho) - \left(\frac{\sqrt{K}}{4KH}\right)^2, \\
0 &\leq \varrho J^{x, \delta} U^2(\rho)^\varrho J^{x, \delta} V^2(\rho) - \left(\frac{\sqrt{H}}{4KH}\right)^2 \\
&\leq \varrho J^{x, \delta} U^2(\rho)^\varrho J^{x, \delta} V^2(\rho) - \left(\varrho J^{x, \delta} U(\rho)V(\rho)\right)^2.
\end{align}

(4.22)

and

\begin{align}
0 &\leq \varrho J^{x, \delta} U^2(\rho)^\varrho J^{x, \delta} V^2(\rho) - \left(\varrho J^{x, \delta} U(\rho)V(\rho)\right)^2 \\
&\leq \frac{H - K}{4KH} \left(\varrho J^{x, \delta} U(\rho)V(\rho)\right)^2.
\end{align}

(4.23)

(4.24)

**Proof** From (4.21), we have

\begin{align}
\left(\frac{U(\rho)}{V(\rho)} - K\right)\left(\varrho \frac{U(\rho)}{V(\rho)} \right)\varrho^2(\rho) \geq 0, \quad 0 \leq \varrho.
\end{align}

It follows that

\begin{align}
U^2(\rho) + KHJ^2(\rho) &\leq (K + H)U(\rho)V(\rho).
\end{align}

(4.25)

Multiplying both sides of inequality (4.25) with \(\frac{1}{\Gamma(x)}e^{-\varrho(\Psi(\rho)-\Psi(\rho))}(\Psi(\rho) - \Psi(\rho))^{x-1}\Psi'(\rho)\) and integrating the resulting estimate with respect to \(\rho\) from 0 to \(\varrho\) and using (2.7), we obtain

\begin{align}
\varrho J^{x, \delta} U^2(\rho) + KH\varrho J^{x, \delta} V^2(\rho) \leq (K + H)\varrho J^{x, \delta} U(\rho)V(\rho).
\end{align}

(4.26)
Now, since $\mathcal{K}\mathcal{H} > 0$ and
\[
(\sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u^2(\varrho) \sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} v^2(\varrho))^2 \geq 0,
\]
it follows that
\[
2\sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u^2(\varrho) \sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} v^2(\varrho) \leq \mathcal{K}\mathcal{H}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u^2(\varrho) + \mathcal{K}\mathcal{H}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} v^2(\varrho).
\]
Hence, by using (4.26) and (4.27), we have
\[
4\mathcal{K}\mathcal{H}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u^2(\varrho) \mathcal{H}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} v^2(\varrho) \leq (\mathcal{K} + \mathcal{H})^2 (\mathcal{K}\mathcal{H})^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u(\varrho) v(\varrho))^2,
\]
which gives the desired assertion (4.22).

Now, from (4.28), we have
\[
\sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u^2(\varrho) \mathcal{H}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} v^2(\varrho) \leq \mathcal{K} + \mathcal{H}\sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u(\varrho) v(\varrho).
\]
Subtraction of $(\mathcal{K} + \mathcal{H}\sqrt{\mathcal{K}\mathcal{H}}^\epsilon J^{\mathcal{K}\mathcal{H},\epsilon} u(\varrho) v(\varrho))$ from (4.29) yields the desired assertion (4.23). Similarly, we can easily prove the assertion (4.24) from (4.22). □

5 Special cases

This section is devoted to certain special cases of the main result obtained in Sects. 3 and 4.

(I) Applying Theorem 3.1 for $\Psi(\varrho) = \varrho$, we attain the following result for a one sided tempered fractional integral.

**Corollary 5.1** Suppose that we have the function $u \in L[0, \infty]$ and let $\phi_1$ and $\phi_2$ be two integrable functions defined on $[0, \infty]$ such that
\[
\phi_1(\varrho) \leq u(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0, \infty[.
\]
Then, for $\varrho > 0$, $\kappa, \lambda > 0$, we have
\[
J^{\kappa,\lambda}\phi_2(\varrho) J^{\kappa,\lambda} u(\varrho) + J^{\kappa,\lambda} u(\varrho) J^{\kappa,\lambda}\phi_1(\varrho) \\
\geq J^{\kappa,\lambda}\phi_2(\varrho) J^{\kappa,\lambda}\phi_1(\varrho) + J^{\kappa,\lambda} u(\varrho) J^{\kappa,\lambda} u(\varrho).
\]

(II) Applying Theorem 3.1 for $\xi = 1$, we attain the following result for the one sided generalized Riemann–Liouville fractional integral proved earlier by Kacar et al. [37].

**Corollary 5.2** Suppose that the function $u \in \chi_p^\mathcal{K}(0, \infty)$ and assume that the function $\Psi$ is positive, monotone and increasing on $[0, \infty]$ and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi'(0) = 0$. Moreover, let $\phi_1$ and $\phi_2$ be two integrable functions defined on $[0, \infty]$ such that
\[
\phi_1(\varrho) \leq u(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0, \infty[.
\]
Then, for \( q > 0, \kappa, \lambda > 0 \), we have

\[
\phi^* \mathcal{J}^* \phi_2(q) \phi^* \mathcal{J}^* \mathcal{U}(q) + \phi^* \mathcal{J}^* \mathcal{U}(q) \phi^* \mathcal{J}^* \phi_1(q) \\
\geq \phi^* \mathcal{J}^* \phi_2(q) \phi^* \mathcal{J}^* \phi_1(q) + \phi^* \mathcal{J}^* \mathcal{U}(q) \phi^* \mathcal{J}^* \mathcal{U}(q).
\]

(III) Applying Theorem 3.1 for \( \Psi(q) = 0 \) and \( \xi = 1 \), we attain the following result for one sided Riemann–Liouville fractional integral proved earlier by Tariboon et al. [45].

**Corollary 5.3** Suppose that the function \( \mathcal{U} \in L_1[0, \infty] \) and let \( \phi_1 \) and \( \phi_2 \) be two integrable functions on \([0, \infty]\) such that

\[
\phi_1(q) \leq \mathcal{U}(q) \leq \phi_2(q), \quad q \in [0, \infty].
\]

Then, for \( q > 0, \kappa, \lambda > 0 \), we have

\[
\mathcal{J}^* \phi_2(q) \mathcal{J}^* \mathcal{U}(q) + \mathcal{J}^* \mathcal{U}(q) \mathcal{J}^* \phi_1(q) \\
\geq \mathcal{J}^* \phi_2(q) \mathcal{J}^* \phi_1(q) + \mathcal{J}^* \mathcal{U}(q) \mathcal{J}^* \mathcal{U}(q).
\]

The special cases of Theorem 3.2 are presented by the following corollaries.

(I) Setting \( \Psi(q) = 0 \), we attain the following result for one sided tempered fractional integral.

**Corollary 5.4** Suppose that the two positive functions \( \mathcal{U} \) and \( \mathcal{V} \) are defined on \([0, \infty]\). Assume that (3.1) holds and \( \psi_1, \psi_2 \) are two integrable functions on \([0, \infty]\) such that

\[
\psi_1(q) \leq \mathcal{V}(q) \leq \psi_2(q), \quad q \in [0, \infty].
\]

Then, for \( q > 0 \) and \( \kappa, \lambda > 0 \), the following four inequalities hold:

\[
\mathcal{J}^{*\xi} \phi_2(q) \mathcal{J}^{*\xi} \mathcal{V}(q) + \mathcal{J}^{*\xi} \mathcal{V}(q) \mathcal{J}^{*\xi} \phi_1(q) \\
\geq \mathcal{J}^{*\xi} \phi_2(q) \mathcal{J}^{*\xi} \phi_1(q) + \mathcal{J}^{*\xi} \mathcal{V}(q) \mathcal{J}^{*\xi} \mathcal{V}(q),
\]

\[
\mathcal{J}^{*\lambda} \phi_1(q) \mathcal{J}^{*\lambda} \mathcal{V}(q) + \mathcal{J}^{*\lambda} \mathcal{V}(q) \mathcal{J}^{*\lambda} \phi_1(q) \\
\geq \mathcal{J}^{*\lambda} \phi_1(q) \mathcal{J}^{*\lambda} \phi_1(q) + \mathcal{J}^{*\lambda} \mathcal{V}(q) \mathcal{J}^{*\lambda} \mathcal{V}(q),
\]

\[
\mathcal{J}^{*\xi} \phi_2(q) \mathcal{J}^{*\lambda} \mathcal{V}(q) + \mathcal{J}^{*\lambda} \mathcal{V}(q) \mathcal{J}^{*\xi} \phi_2(q) \\
\geq \mathcal{J}^{*\xi} \phi_2(q) \mathcal{J}^{*\lambda} \phi_2(q) + \mathcal{J}^{*\lambda} \mathcal{V}(q) \mathcal{J}^{*\xi} \mathcal{V}(q),
\]

\[
\mathcal{J}^{*\lambda} \phi_1(q) \mathcal{J}^{*\xi} \mathcal{V}(q) + \mathcal{J}^{*\xi} \mathcal{V}(q) \mathcal{J}^{*\lambda} \phi_1(q) \\
\geq \mathcal{J}^{*\lambda} \phi_1(q) \mathcal{J}^{*\xi} \phi_1(q) + \mathcal{J}^{*\xi} \mathcal{V}(q) \mathcal{J}^{*\lambda} \mathcal{V}(q).
\]

(II) Applying Theorem 3.2 for \( \xi = 1 \), we attain the following result for the one sided generalized Riemann–Liouville fractional integral proved earlier by Kacar et al. [37].

**Corollary 5.5** Suppose that the two positive functions \( \mathcal{U} \) and \( \mathcal{V} \) are defined on \([0, \infty]\) and assume that the function \( \Psi \) is positive, monotone and increasing on \([0, \infty]\) and its derivative
\[ \Psi' \text{ is continuous on } [0, \infty[ \text{ with } \Psi(0) = 0. \text{ Assume that (3.1) holds and } \psi_1, \psi_2 \text{ are two integrable functions defined on } [0, \infty[ \text{ such that} \]

\[ \psi_1(\varphi) \leq \mathcal{V}(\varphi) \leq \psi_2(\varphi), \quad \varphi \in [0, \infty[. \]

Then, for \( \varphi > 0 \) and \( \kappa, \lambda > 0 \), the following four inequalities hold:

\[
\Psi J^\kappa \phi_2(\varphi) J^\lambda \mathcal{V}(\varphi) + \Psi J^\lambda \mathcal{U}(\varphi) J^\kappa \psi_1(\varphi)
\geq \Psi J^\kappa \phi_2(\varphi) J^\lambda \psi_1(\varphi) + \Psi J^\lambda \mathcal{U}(\varphi) J^\kappa \mathcal{V}(\varphi),
\]

\[
\Psi J^\lambda \phi_1(\varphi) J^\kappa \mathcal{V}(\varphi) + \Psi J^\kappa \psi_2(\varphi) J^\lambda \mathcal{U}(\varphi)
\geq \Psi J^\lambda \phi_1(\varphi) J^\kappa \psi_2(\varphi) + \Psi J^\kappa \mathcal{U}(\varphi) J^\lambda \mathcal{V}(\varphi),
\]

\[
\Psi J^\kappa \phi_2(\varphi) J^\lambda \mathcal{V}(\varphi) + \Psi J^\lambda \mathcal{U}(\varphi) J^\kappa \psi_1(\varphi)
\geq \Psi J^\kappa \phi_2(\varphi) J^\lambda \psi_1(\varphi) + \Psi J^\lambda \mathcal{U}(\varphi) J^\kappa \mathcal{V}(\varphi),
\]

\[
\Psi J^\lambda \phi_1(\varphi) J^\kappa \mathcal{V}(\varphi) + \Psi J^\kappa \psi_2(\varphi) J^\lambda \mathcal{U}(\varphi)
\geq \Psi J^\lambda \phi_1(\varphi) J^\kappa \psi_2(\varphi) + \Psi J^\kappa \mathcal{U}(\varphi) J^\lambda \mathcal{V}(\varphi).
\]

(III) Applying Theorem 3.2 for \( \Psi(\varphi) = \varphi \) and \( \xi = 1 \), we attain the following result for the one sided Riemann–Liouville fractional integral proved earlier by Tariboon et al. \[45\].

**Corollary 5.6** Suppose that the functions \( \mathcal{U} \) and \( \mathcal{V} \) are two positive functions defined on \([0, \infty[\). Assume that (3.1) holds and \( \psi_1, \psi_2 \) are two integrable functions defined on \([0, \infty[\) such that

\[ \psi_1(\varphi) \leq \mathcal{V}(\varphi) \leq \psi_2(\varphi), \quad \varphi \in [0, \infty[. \]

Then, for \( \varphi > 0 \) and \( \kappa, \lambda > 0 \), then the following four inequalities hold:

\[
J^\kappa \phi_2(\varphi) J^\lambda \mathcal{V}(\varphi) + J^\lambda \mathcal{U}(\varphi) J^\kappa \psi_1(\varphi)
\geq J^\kappa \phi_2(\varphi) J^\lambda \psi_1(\varphi) + J^\lambda \mathcal{U}(\varphi) J^\kappa \mathcal{V}(\varphi),
\]

\[
J^\lambda \phi_1(\varphi) J^\kappa \mathcal{V}(\varphi) + J^\kappa \psi_2(\varphi) J^\lambda \mathcal{U}(\varphi)
\geq J^\lambda \phi_1(\varphi) J^\kappa \psi_2(\varphi) + J^\kappa \mathcal{U}(\varphi) J^\lambda \mathcal{V}(\varphi),
\]

\[
J^\kappa \phi_2(\varphi) J^\lambda \mathcal{V}(\varphi) + J^\lambda \mathcal{U}(\varphi) J^\kappa \psi_1(\varphi)
\geq J^\kappa \phi_2(\varphi) J^\lambda \psi_1(\varphi) + J^\lambda \mathcal{U}(\varphi) J^\kappa \mathcal{V}(\varphi),
\]

\[
J^\lambda \phi_1(\varphi) J^\kappa \mathcal{V}(\varphi) + J^\kappa \psi_2(\varphi) J^\lambda \mathcal{U}(\varphi)
\geq J^\lambda \phi_1(\varphi) J^\kappa \psi_2(\varphi) + J^\kappa \mathcal{U}(\varphi) J^\lambda \mathcal{V}(\varphi).
\]

The following corollaries represent special cases of Theorem 4.1.

(I) Setting \( \Psi(\varphi) = \varphi \), we attain the following result for a one sided tempered fractional integral.
**Corollary 5.7** Suppose that the functions $U$ and $V$ are two positive functions defined on $[0, \infty)$ and let $p_1, q_1 > 1$ be such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then, for $q > 0$, the following inequalities hold:

\[
\frac{1}{p_1} J^{\alpha} U^{p_1}(q) J^\lambda V^{p_1}(q) + \frac{1}{q_1} J^{\alpha} V^{q_1}(q) J^\lambda U^{q_1}(q) \\
\geq J^{\alpha} U(q) V(q) J^\lambda U(q), \\
\frac{1}{p_1} J^{\alpha} U^{p_1}(q) J^\lambda V^{q_1}(q) + \frac{1}{q_1} J^{\alpha} V^{q_1}(q) J^\lambda U^{p_1}(q) \\
\geq J^{\lambda} U^{q_1}(q) J^{\alpha} U^{q_1}(q) J^\lambda U(q)V(q), \\
\frac{1}{p_1} J^{\alpha} U^{p_1}(q) J^\lambda V^{q_1}(q) + \frac{1}{q_1} J^{\alpha} V^{q_1}(q) J^\lambda U^{p_1}(q) \\
\geq J^{\lambda} U^{q_1}(q) J^{\alpha} U^{q_1}(q) J^\lambda U(q)V(q),
\]

and

\[
\frac{1}{p_1} J^\lambda U^{p_1}(q) J^{\alpha} V^{p_1}(q) + \frac{1}{q_1} J^\lambda V^{q_1}(q) J^{\alpha} U^{q_1}(q) \\
\geq J^{\lambda} U^{q_1}(q) J^{\alpha} U^{p_1}(q) J^\lambda U(q)V(q). 
\]

(II) Applying Theorem 4.1 for $\xi = 1$, we attain the following new result for a one sided generalized Riemann–Liouville fractional integral.

**Corollary 5.8** Suppose that the functions $U$ and $V$ are two positive functions defined on $[0, \infty)$ and assume that the function $\Psi$ is positive, monotone and increasing on $[0, \infty)$ and its derivative $\Psi'$ is continuous on $[0, \infty)$ with $\Psi(0) = 0$. If $p_1, q_1 > 1$ are such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then, for $q > 0$, the following inequalities hold:

\[
\frac{1}{p_1} \psi J^{\alpha} U^{p_1}(q)^\psi J^\lambda V^{p_1}(q) + \frac{1}{q_1} \psi J^{\alpha} V^{q_1}(q)^\psi J^\lambda U^{q_1}(q) \\
\geq \psi J^{\alpha} U(q)^\psi J^\lambda V(q) U(q), \\
\frac{1}{p_1} \psi J^{\alpha} U^{p_1}(q)^\psi J^\lambda V^{q_1}(q) + \frac{1}{q_1} \psi J^{\alpha} V^{q_1}(q)^\psi J^\lambda U^{p_1}(q) \\
\geq \psi J^{\lambda} U^{q_1}(q) J^{\alpha} U^{q_1}(q) J^\lambda U(q)V(q), \\
\frac{1}{p_1} \psi J^{\alpha} U^{p_1}(q)^\psi J^\lambda V^{q_1}(q) + \frac{1}{q_1} \psi J^{\alpha} V^{q_1}(q)^\psi J^\lambda U^{p_1}(q) \\
\geq \psi J^{\lambda} U^{q_1}(q) J^{\alpha} U^{q_1}(q) J^\lambda U(q)V(q),
\]

and

\[
\frac{1}{p_1} \psi J^\lambda U^{p_1}(q)^\psi J^{\alpha} V^{p_1}(q) + \frac{1}{q_1} \psi J^\lambda V^{q_1}(q)^\psi J^{\alpha} U^{q_1}(q) \\
\geq \psi J^{\alpha} U^{q_1}(q)^\psi J^\lambda U^{q_1}(q) J^\lambda U(q)V(q). 
\]
In a similar way, we can obtain the special cases of Theorems 4.2 and 4.3 by applying similar procedures.

6 Particular cases

Here, we present certain new particular cases of our main result by employing the connection of GTF-integral (2.7) with the classical Riemann–Liouville expression containing another function in the kernel.

Li et al. [39] defined the connection of a tempered fractional integral (2.3) with the Riemann–Liouville fractional integral by

\[
\mathcal{J}^\kappa_\phi U(\theta) = e^{\mathcal{J}^\kappa_\phi} \mathcal{J}_\phi \left[ e^{\mathcal{J}^\kappa_\phi} U(\theta) \right].
\]

Here, we propose the following connection of the GTF-integral (2.7) with the generalized Riemann–Liouville fractional integral as

\[
 \Psi \mathcal{J}^\kappa_\phi U(\theta) = e^{\Psi \mathcal{J}^\kappa_\phi} \mathcal{J}_\phi \left[ e^{\Psi \mathcal{J}^\kappa_\phi} U(\theta) \right],
\]

(6.1)

where \( \Psi \mathcal{J}^\kappa_\phi \left[ e^{\Psi \mathcal{J}^\kappa_\phi} U(\theta) \right] \) is the generalized Riemann–Liouville fractional integral concerning another function.

Applying the above connection (6.1) to Theorem 3.1, one can get the following new result in terms of the generalized Riemann–Liouville fractional integral in the sense of another function.

**Theorem 6.1** Suppose that the function \( U \in \mathcal{C}_0^\infty (0, \infty) \) and assume that the function \( \Psi \) is positive, monotone and increasing on \([0, \infty[\) and its derivative \( \Psi' \) is continuous on \([0, \infty[\) with \( \Psi(0) = 0 \). Moreover, let \( \phi_1 \) and \( \phi_2 \) be two integrable functions defined on \([0, \infty[\) such that

\[
\phi_1(\theta) \leq U(\theta) \leq \phi_2(\theta), \quad \theta \in [0, \infty[.
\]

Then, for \( \theta > 0, \kappa, \lambda > 0 \), we have

\[
 \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} U(\theta) \right) + \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_1(\theta) \right)
\]

\[
\geq \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) + \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_1(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_1(\theta) \right).
\]

Applying (6.1) to Theorem 3.2, one can obtain easily the following new results.

**Theorem 6.2** Suppose that the two positive functions \( U \) and \( V \) are defined on \([0, \infty[\) and assume that the function \( \Psi \) is positive, monotone and increasing on \([0, \infty[\) and its derivative \( \Psi' \) is continuous on \([0, \infty[\) with \( \Psi(0) = 0 \). Assume that (6.2) holds and \( \psi_1, \psi_2 \) are two integrable functions defined on \([0, \infty[\) such that

\[
\psi_1(\theta) \leq V(\theta) \leq \psi_2(\theta), \quad \theta \in [0, \infty[.
\]

Then, for \( \theta > 0 \) and \( \kappa, \lambda > 0 \), the following four inequalities hold:

\[
 \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} V(\theta) \right) + \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \psi_1(\theta) \right)
\]

\[
\geq \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \phi_2(\theta) \right) + \Psi \mathcal{J}^\lambda_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \psi_1(\theta) \right) \Psi \mathcal{J}^\kappa_\phi \left( e^{\Psi \mathcal{J}^\kappa_\phi} \psi_1(\theta) \right).
\]
\[ \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \phi_1 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) \]
\[ \geq \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \phi_1 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)), \]
\[ \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) \]
\[ \geq \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) + \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_1 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_2 (v)). \]

One can obtain the following new result of Theorem 4.1 in terms of the generalized Riemann–Liouville fractional integral in the sense of another function by utilizing (6.1).

**Theorem 6.3** Suppose that the two positive functions \( U \) and \( V \) are defined on \([0, \infty]\) and assume that the function \( \Psi \) is positive, monotone and increasing on \([0, \infty]\) and its derivative \( \Psi' \) is continuous on \([0, \infty]\) with \( \Psi (0) = 0 \). If \( p_1, q_1 > 1 \) is such that \( \frac{1}{p_1} + \frac{1}{q_1} = 1 \), then, for \( q > 0 \), the following inequalities hold:

\[ \frac{1}{p_1} \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_{p_1} (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_1 (v)) + \frac{1}{q_1} \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} \psi_1 (v)) \psi \mathcal{J}^{\lambda} (e^{\Psi (v)} U_{q_1} (v)) \]
with the classical Riemann–Liouville fractional integral and derived certain new results in terms of the Riemann–Liouville fractional integral concerning another function. One can easily obtain several other types of inequalities, such as Hadamard fractional integral inequalities and generalized fractional conformable inequalities by utilizing Remark 2.2. Moreover, certain new inequalities can be derived by utilizing the inequalities discussed in Sect. 6.

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