On the number of irreducible points in polyhedra

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Abstract

We prove that the number of irreducible integer points in \( n \)-dimensional polytope with radius \( k \) given by a system of \( m \) linear inequalities is at most \( O(m^{\lfloor \frac{n}{2} \rfloor} \log_2 n - \frac{1}{2} k) \) if \( n \) is fixed. Using this result we prove the hypothesis asserting that the teaching dimension in the class of threshold functions of \( k \)-valued logic in \( n \) variables is \( \Theta(\log_2 n - \frac{1}{2} k) \) for any fixed \( n \geq 2 \).

Keywords: polyhedron, irreducible points, vertex, integer lattice, threshold function, teaching set, teaching dimension

1 Introduction

In the paper we study the number of irreducible points in polyhedra. Let \( P \) be a (convex) polyhedron in \( \mathbb{R}^d \). A point \( x \in P \cap \mathbb{Z}^n \) is called irreducible in \( P \) (more precisely, in \( P \cap \mathbb{Z}^n \)) if \( x \) can not be represented as \( x = \frac{1}{2}(y + z) \), where \( y \) and \( z \) are different points in \( P \cap \mathbb{Z}^n \).

It is not hard to see that any vertex of \( P_I = \text{conv}(P \cap \mathbb{Z}^n) \) is irreducible in \( P \cap \mathbb{Z}^n \). The converse is not true, as the following example shows. Let \( P = \{ x \in \mathbb{R}^2 : x_1 + x_2 \geq 1, 2x_1 - x_2 \leq 2, -x_1 + 2x_2 \leq 2 \} \). The point \((1, 1)\) is irreducible in \( P \), but it is not a vertex of \( P_I \). Nevertheless these properties (irreducibility and being vertex) are similar, as evidenced by a nearness of bounds for the number of vertices and the number of irreducible points \[13\].

Suppose that a polyhedron \( P \) is given by a system of linear inequalities \( Ax \leq b \), where \( A = (a_{ij}) \in \mathbb{Z}^{m \times n}, b = (b_i) \in \mathbb{Z}^m, |a_{ij}| \leq \alpha, |b_i| \leq \beta, \gamma = \max\{\alpha, \beta\} \). Let \( \varphi \) be the sum of the sizes of all inequalities in \( Ax \leq b \), that is, \( \varphi = O(mn \log_2 \gamma) \).

Upper bounds for the number of vertices in \( P_I \) are proposed in \[13\], \[8\], \[7\], etc. In \[7\] it is proved that, for any fixed \( n \), \( P_I \) has at most \( O(m^n \varphi^{n-1}) \) vertices. More precise bound, \( O(m^{\lfloor \frac{n}{2} \rfloor} \log_2 \varphi^{n-1}) \), was found in \[5\] (see \[19\]). To obtain upper bounds an approach due to Shevchenko \[13\] based on a separation property is used. A method developed in \[8\] and \[7\] using reflecting sets is essentially equivalent to Shevchenko’s approach.

Lower bounds for the number of vertices in \( P_I \) are proposed in \[18\], \[2\], \[4\], etc. In \[18\] it was shown that, for any fixed \( n \), a knapsack polytope can have \( \Omega(\varphi^{n-1}) \) vertices. Another construction with \( \Omega(\varphi^{n-1}) \) lower bound was proposed in \[2\]. In \[4\] it was
proved that, for any $m$ and any fixed $n$, there exists a polyhedron with $\Omega(m^{1/2}\log^{n-1} \gamma)$ vertices.

For some other results and comments concerning bounds for the number of vertices in $P_i$ see [15], [21], [19].

The main result of the paper is to find a tight upper bound for the number of irreducible points in a polyhedron. When $n$ is fixed this bound is close to the lower bound and it is essentially (asymptotically) the same as the bound for the number of vertices.

Let $P, P_1, P_2, \ldots, P_s$ be polytopes (bounded polyhedra) in $\mathbb{R}^n$. If $P = \bigcup_{i=1}^s P_i$, then $\{P_1, P_2, \ldots, P_s\}$ is called a cover of the polytope $P$. If the intersection of any two polytopes in the cover is empty or it is their common face, then the cover is called a regular partition. If all polytopes in a regular partition are simplex then the partition is called a triangulation.

Our method to find the upper bounds for the number of irreducible points in a polytope consists of the following. First, we obtain a bound for the number of irreducible points in a parallelepiped (see Section 2). In Section 3 we construct a cover of a polytope $P$ by parallelepipeds $P_1, P_2, \ldots, P_s$. To do this we build a triangulation of the polytope, then each simplex in the triangulation is covered by parallelepipeds. It is not hard to see that if $x \in P_i$ is irreducible in $P$ then $x$ is irreducible in $P_i$. This property allows us (in Section 4) to find a bound for the number of irreducible points in $P$. Namely, Theorem 2 asserts that the number of irreducible points in a polytope $P$ is at most $O\left(m^{1/2}\log^{n-1} \alpha \beta \right)$ ($n$ is fixed). Theorem 3 asserts that if $P$ has radius $k$ then the number of its irreducible points is at most $O(m^{1/2}\log^{n-1} k)$ ($n$ is fixed).

These results are used to prove a hypothesis concerning the teaching dimension of the class of threshold functions of $k$-valued logic in Section 5.

Let $n \geq 1$, $k \geq 2$, $E_k = \{0, 1, \ldots, k-1\}$. A function $f : E_k^n \rightarrow \{0, 1\}$ is called a threshold function iff there exists a hyperplane separating the set $M_0(f)$ of points, in which $f$ is 0, and the set $M_1(f)$ of points, in which $f$ is 1, that is, there are real numbers $a_0, a_1, \ldots, a_n$, such that

$$M_0(f) = \left\{x = (x_1, x_2, \ldots, n) \in E_k^n : \sum_{j=1}^n a_j x_j \leq a_0 \right\}. \quad (1)$$

The inequality $\sum_{j=1}^n a_j x_j \leq a_0$ in (1) is called the threshold inequality. It is easy to see that its coefficients can be chosen integer. Denote by $\mathcal{T}(n, k)$ the set of all threshold functions defined on $E_k^n$.

A set $T$ is called a teaching set for $f \in \mathcal{T}(n, k)$, iff, for any function $g \in \mathcal{T}(n, k) \setminus \{f\}$, there is a point $z \in T$ such that $f(z) \neq g(z)$. Teaching set is appeared in the problem of deciphering threshold function (or “learning halfspaces” in Algorithmic Learning Theory terminology) [24], [10]. A teaching set $T$ of $f \in \mathcal{T}(n, k)$ is called a minimal teaching set, if no proper subset of it is teaching for $f$. It is known (see, for example, [10], [25]), that the minimal teaching set $T(f)$ of any threshold function $f$ is unique.
The maximum cardinality of minimal teaching set,
\[ \sigma(n, k) = \max_{f \in T(n, k)} |T(f)|, \]
is called the teaching dimension.

Bounds for \( \sigma(n, k) \) are constructed in [9], [16], [25], [22], [23], etc. A generalization is considered in [17]. It is known that \( \sigma(n, k) \) depends on \( n \) exponentially, in particularly, \( \sigma(n, 2) = 2^n \). In [9] it is proved using [14], [7] that for any fixed \( n \)
\[ \sigma(n, k) = O(\log^{n-1} k). \]
In [16], [25] a lower bound
\[ \sigma(n, k) = \Omega(\log^{n-2} k) \]
is obtained. See also [22]. In [16] it is proved that \( t(2, k) = 4 \). In [23] a big subclass \( T'(n, k) \subset T(n, k) \) is considered. For any function \( f \in T'(n, k) \) it holds that
\[ |T(f)| \leq 2n(1 + \log_2 n)\left(1 + \log_2(k + 1)\right)^{n-2} \]
and the following hypothesis put forward
\[ \sigma(n, k) = \Theta(\log^{n-2} k) \tag{2} \]
for any fixed \( n \geq 2 \). In Section 5 we prove Theorem 6 asserting that \( \sigma(n, k) = O(\log^{n-2} k) \) for any fixed \( n \geq 2 \). Thus, hypothesis (2) is true.

Note that the average cardinality of minimal teaching set is studied in [1], [20].

Suppose \( X \subseteq \mathbb{R}^n \). Let \( \text{conv} X \) be the convex hull of \( X \), \( \text{cone} X \) is the cone hull of \( X \) (the set of all linear combinations of vectors in \( X \) with nonnegative coefficients), \( \text{Vert} X \) is the set of vertices of \( X \). If \( X, Y \) are subsets of \( \mathbb{R}^n \) then by \( X + Y \) and \( X - Y \) we mean the set of all vectors of the form \( x + y \) and consequently \( x - y \), where \( x \in X \), \( y \in Y \).

## 2 Irreducible points in parallelepipeds

In this section we propose upper bounds for the number of irreducible points in a parallelepiped.

Let \( A \in \mathbb{Z}^{n \times n} \) be a non-singular matrix, \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n \), \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n \), \( b < c \). Consider a parallelepiped
\[ P(A, b, c) = \{ x \in \mathbb{R}^n : b \leq Ax \leq c \}. \]

Denote \( M(A, b, c) = P(A, b, c) \cap \mathbb{Z}^n \).

**Theorem 1** Let \( N \) be the set of all irreducible points in \( M(A, b, c) \), where \( A \in \mathbb{Z}^{n \times n} \) is a non-singular matrix, \( c \in \mathbb{Z}^n \), \( b \in \mathbb{Z}^n \), then
\[ |N| \leq 2 \prod_{i=1}^{n-1} \left(3 + 2\log_2 \left(1 + \frac{c_i - b_i}{3}\right)\right). \tag{3} \]
Proof. Denote

\[ s_i = \left\lceil \log_2 \left( 1 + \frac{c_i - b_i}{3} \right) \right\rceil \quad (i = 1, 2, \ldots, n - 1). \tag{4} \]

This implies that

\[ 3 \cdot 2^{s_i - 1} - 3 < c_i - b_i \leq 3 \cdot 2^{s_i} - 3. \tag{5} \]

Denote by \( a^{(1)}, a^{(2)}, \ldots, a^{(n)} \) the rows of \( A \). Let \( j_1, j_2, \ldots, j_{n-1} \) be numbers such that \( j_i \in \{0, \ldots, 2s_i\} \ (i = 1, \ldots, n - 1) \). Let \( P(j_1, j_2, \ldots, j_{n-1}) \) be the set of points in \( \mathbb{R}^n \) that satisfy the following conditions: for all \( i = 1, 2, \ldots, n - 1 \)

\[ \begin{align*}
    b_i + 2^{j_i} - 1 &\leq a^{(i)}x < b_i + 2^{j_i+1} - 1 \quad (j_i = 0, 1, \ldots, s_i - 1), \\
    b_i + 2^{j_i} - 1 &\leq a^{(i)}x \leq c_i - 2^{j_i} + 1 \quad (j_i = s_i), \\
    c_i - 2^{j_i-s_i} + 1 &< a^{(i)}x \leq c_i - 2^{j_i-1-s_i} + 1 \quad (j_i = s_i + 1, \ldots, 2s_i), \\
    b_n \leq a^{(n)}x \leq c_n.
\end{align*} \tag{6} \]

It is not hard to see that the set of parallelepipeds \( P(j_1, j_2, \ldots, j_{n-1}) \), where \( 0 \leq j_i \leq 2s_i \ (i = 1, \ldots, n - 1) \), is a cover of \( P(A, b, c) \). This is showed on Fig. 1, where all irreducible points are vertices of the convex hull of \( M(A, b, c) \).

We show that each \( P(j_1, j_2, \ldots, j_{n-1}) \) contains at most 2 different points from \( N \).

Assume the contrary: let \( x, y, z \) be pairwise different points, \( x, y, z \in P(j_1, j_2, \ldots, j_{n-1}) \), and

\[ b_n \leq a^{(n)}x \leq a^{(n)}y \leq a^{(n)}z \leq c_n. \tag{7} \]
We consider the two mutually exclusive cases when

\[ a^{(n)}y - a^{(n)}x \leq a^{(n)}z - a^{(n)}y \]  \hspace{1cm} (8)

and when

\[ a^{(n)}y - a^{(n)}x > a^{(n)}z - a^{(n)}y. \]  \hspace{1cm} (9)

In the case (8) we consider the point \( x' = 2y - x \), and show that \( x' \in P(A, b, c) \).

The conditions \( b_n \leq a^{(n)}x' \leq c_n \) hold, since from (7) it follows that

\[ a^{(n)}x' = 2a^{(n)}y - a^{(n)}x \geq a^{(n)}y \geq b_n, \]

\[ a^{(n)}x' = 2a^{(n)}y - a^{(n)}x \leq 2a^{(n)}z - a^{(n)}y \leq c_n. \]

Now we verify the conditions \( b_i \leq a^{(i)}x' \leq c_i \) \((i = 1, 2, \ldots, n - 1)\). If \( 0 \leq j_i \leq s_i - 1 \), then taking into account (5) and (6) we obtain

\[ a^{(i)}x' \leq 2(b_i + 2^j_i + 1 - 2) - (b_i + 2^j_i - 1) \leq b_i + 3 \cdot 2^{s_i-1} - 3 < c_i, \]

\[ a^{(i)}x' \geq 2(b_i + 2^j_i - 1) - (b_i + 2^j_i + 2) = b_i. \]

If \( j_i = s_i \), then

\[ a^{(i)}x' \leq 2(c_i - 2^{s_i} + 1) - (b_i + 2^{s_i} - 1) \leq c_i, \]

\[ a^{(i)}x' \geq 2(b_i + 2^{s_i} - 1) - (c_i - 2^{s_i} + 1) \geq b_i. \]

If \( s_i + 1 \leq j_i \leq 2s_i \), then

\[ a^{(i)}x' \leq 2(c_i - 2^{j_i-s_i} + 1) - (c_i - 2^{j_i-s_i} + 2) = c_i, \]

\[ a^{(i)}x' \geq 2(c_i - 2^{j_i-s_i} + 2) - (c_i - 2^{j_i-s_i} + 1) \geq c_i + 3 \cdot 2^{s_i-1} > b_i. \]

Thus, \( x' \in M(A, b, c) \) and \( y = \frac{1}{2}(x + x') \), hence \( y \notin N \). Contradiction.

In the case (9) we consider the point \( z' = 2y - z \). Using analogous arguments one can show that \( z' \in M(A, b, c) \) \( y = \frac{1}{2}(z + z') \notin N \).

Thus, each parallelepiped \( P(j_1, j_2, \ldots, j_{n-1}) \) contains at most 2 points from \( N \), hence \(|N| \leq 2 \prod_{i=1}^{n-1} (1 + 2s_i)\). Using (4) we get (3).

### 3 Cover of a polytope by parallelepipeds

Our method for covering a polytope (i.e., bounded convex polyhedron) by parallelepipeds consists of constructing a polytope triangulation and covering each simplex in the triangulations by parallelepipeds. Denote

\[ \xi_n(m) = \left( m - \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \right) + \left( m - \left\lfloor \frac{n}{2} \right\rfloor - 1 \right). \]
Lemma 1 \([\text{[6]}], \text{see } \text{[13], [19]}\) For any \(n\)-dimensional polytope with \(m\) facets (\(n\)-dimensional faces) there exists its triangulation with at most \(n!\xi_n(m)\) simplexes.

The following assertion is a refinement of the result \([6]\).

Lemma 2 \((\text{based upon } \text{[6]})\) Any \(n\)-dimensional simplex \(S\) can be covered by at most \((n + 1) \cdot \left(\frac{n^2 - 2}{n - 1}\right)\) \(n\)-dimensional parallelepipeds.

Proof. Without loss of generality we consider the simplex

\[
S = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j \leq n^2 - 1, \ x_j \geq 0 \ (j = 1, 2, \ldots, n) \right\}.
\]

Let

\[
Y = \left\{ x \in \mathbb{Z}^n : \sum_{j=1}^{n} y_j \leq n^2 - 1, \ y_j \geq 1 \ (j = 1, 2, \ldots, n) \right\}.
\]

For each vector \(y \in Y\) we consider the parallelepiped

\[
\Pi(y) = \left\{ x \in \mathbb{R}^n : 0 \leq x_j \leq y_j \ (j = 1, 2, \ldots, n) \right\}.
\]

Let

\[
S' = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j \leq n^2 - n, \ x_j \geq 0 \ (j = 1, 2, \ldots, n) \right\}.
\]

We prove that

\[
S' \subseteq \bigcup_{y \in Y} \Pi(y) \subseteq S. \tag{10}
\]

The second inclusion is obvious. To prove the first one let us consider arbitrary vector \(x \in S'\). Let \(z = \lceil x \rceil\). Then

\[
\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \lceil x_j \rceil < n^2 - n + n = n^2.
\]

Taking into account that all components of \(z\) are integer we get

\[
\sum_{j=1}^{n} z_j \leq n^2 - 1.
\]

Increasing (as appropriate) components of \(z\) we get a vector \(y \in Y\) such that \(z \in \Pi(y)\) and consequently \(x \in \Pi(y)\).

Thus, we have constructed the family of parallelepipeds \(\{\Pi(y) : y \in Y\}\) that satisfy to \((10)\) and contain a certain vertex (vertex 0) of the simplex \(S\). Performing the same construction for every vertex of the simplex we obtain its cover.
Indeed, let $v_0, v_1, \ldots, v_d$ be the vertices of $S$, and $v_0 = 0$, and, for $i \geq 1$, all components of $v_i$ equal 0, except the $i$-th component, which equals $n^2 - 1$. For a point $x \in S$ there exist numbers $\alpha_0, \alpha_1, \ldots, \alpha_d$ such that

$$x = \sum_{i=0}^{n} \alpha_i v_i, \quad \sum_{i=0}^{n} \alpha_i = 1, \quad \alpha_i \geq 0 \quad (i = 0, 1, \ldots, n).$$

Without loss of generality we suppose that $\alpha_0 \geq \frac{1}{n+1}$, hence

$$\sum_{i=1}^{n} \alpha_i \leq \frac{n}{n+1},$$

and consequently

$$\sum_{j=1}^{n} x_j = (n^2 - 1) \sum_{i=1}^{n} \alpha_i \leq n^2 - n.$$

Thus, $x \in S'$. Now from (10) it follows that the constructed family of $(n + 1)\left|Y\right| = (n + 1)\binom{n^2 - 2}{n-1}$ parallelepipeds is a cover of the simplex. \hfill \Box

Lemma 3 Suppose that a polytope $P$ is given as a set of solutions to a system of linear inequalities $Ax \leq b$, where $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$, $b = (b_i) \in \mathbb{Z}^n$, $|a_{ij}| \leq \alpha$, $|b_i| \leq \beta$, then there exists a cover of the polytope by at most

$$\eta_n(m) = n! \xi_n(m) (n + 1) \binom{n^2 - 2}{n-1} \tag{11}$$

parallelepipeds $\Pi_\mu = \{ x \in \mathbb{R}^n : b^{(\mu)} \leq A^{(\mu)}x \leq c^{(\mu)} \}$ ($\mu = 1, \ldots, \eta_n(m)$), where $A^{(\mu)} \in \mathbb{Z}^{n \times n}$, $b^{(\mu)} = (b_i^{(\mu)}) \in \mathbb{Z}^n$, $c^{(\mu)} = (c_i^{(\mu)}) \in \mathbb{Z}^n$, such that

$$|c_i^{(\mu)} - b_i^{(\mu)}| \leq 2\alpha^n \beta^n (\sqrt{n})^{n^2+2n+2}. \tag{12}$$

Proof. The required cover is constructed as follows. First, using Lemma 1 we construct the triangulation of the polytope $P$. Then, using Lemma 2 we construct the cover of each simplex by parallelepipeds. The upper bound (11) for the total number of parallelepipeds is obtained as a product of the upper bounds for the number of simplexes in the triangulation and the number of parallelepipeds in the cover of the simplex.

Now we obtain the inequality (12). First, we find a bound for the quantity of the coefficients in systems of inequalities, which can describe the simplexes in the triangulations. It is well known that the components of the each vertex $v$ of $P$ (and consequently of simplexes in its triangulation) can be obtained by turning corresponding $n$ inequalities of $Ax \leq b$ into equations. Using Cramer’s rule and Hadamard inequality we get that $v = 1/q \cdot (p_1, p_2, \ldots, p_n)$, where $p_j \in \mathbb{Z}$ ($j = 1, 2, \ldots, n$), $q \in \mathbb{Z}$.

$$|q| \leq \alpha^n (\sqrt{n})^n, \quad |p_j| \leq \alpha^{n-1} \beta (\sqrt{n})^n \quad (j = 1, 2, \ldots, n). \tag{13}$$
If $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ are some vertices of the simplex, $v^{(i)} = 1/q^{(i)} \cdot (p_1^{(i)}, p_2^{(i)}, \ldots, p_n^{(i)})$, then the coefficients of the equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = a_0$, which describes the hyperplane passing through these vertices, can be calculated using the following formulas:

$$a_0 = \det(p_1, p_2, \ldots, p_n),$$
$$a_j = (-1)^{j+1} q_j \det(1, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n) \quad (j = 1, 2, \ldots, n),$$

where $1$ is the column of ones. From Hadamard inequality, using (13), we get

$$|a_0| \leq \left((\alpha \sqrt{n})^{n-1} \beta \sqrt{n} \cdot \sqrt{n}\right)^n = \alpha^{n(n-1)} \beta^n (\sqrt{n})^{n^2+n},$$

$$|a_j| \leq (\alpha \sqrt{n})^n \sqrt{n} \left((\alpha \sqrt{n})^{n-1} \beta \sqrt{n} \cdot \sqrt{n}\right)^{n-1} = \alpha^{n^2-n+1} \beta^{n-1}(\sqrt{n})^{n^2+n}, \quad (14)$$

which gives bounds for the quantity of the coefficients in systems of inequalities describing the simplexes in the triangulation.

Now we obtain bounds for coefficients $c_i^{(\mu)}$, $b_i^{(\mu)}$ of system of inequalities describing parallelepipeds in the cover of simplexes. Note that the method used in Lemma 2 gives parallelepipeds with facets which are parallel to facets of the corresponding simplexes. Hence the coefficients in LHS of equations for these facets (i.e. the coefficients of the matrices $A^{(\mu)}$) satisfy the inequality (13). To obtain a bound for $|b_i^{(\mu)}|$, $|c_i^{(\mu)}|$ we put the components of the vertex $v$ of the simplex into the equation of the facet. From (13) and (14) it follows that

$$|b_i^{(\mu)}| \leq n \cdot \alpha^{n-1} \beta (\sqrt{n})^n \cdot \alpha^{n^2-n+1} \beta^{n-1} (\sqrt{n})^{n^2+n} = \alpha^n \beta^n (\sqrt{n})^{n^2+2n+2}.$$  

The same inequality holds for $|c_i^{(\mu)}|$, which gives us (12).

**4 Irreducible points in a polytope**

Here, using results from two previous sections, we get a bound for the number of irreducible integer points in a polytope.

**Theorem 2** Suppose that a polytope $P$ is given as the set of all solutions to a system of linear inequalities $Ax \leq b$, where $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$, $b = (b_j) \in \mathbb{Z}^m$, $|a_{ij}| \leq \alpha$, $|b_j| \leq \beta$. Let $N$ be the set of all irreducible points in $P \cap \mathbb{Z}^n$, then

$$|N| \leq 2n! \xi_n(m)(n+1) \left(\frac{n^2-2}{n-1}\right)^{3 + 2 \log_2 \left(1 + \frac{2}{3} \alpha \beta^2 (\sqrt{n})^{n^2+2n+2}\right)} \left(\frac{n^2-2}{n-1}\right)^{n-1}. \quad (15)$$

**Proof.** Using Lemma 3 we construct a cover of $P$ by parallelepipeds. Obviously, $N$ is contained in the union of the sets of all irreducible integer points in all parallelepipeds. To bound the number of irreducible points in a parallelepiped we use Theorem 1. Putting (12) in (3) and multiplying the result by $\eta_n(m)$ from (11), we get (15). 


Suppose \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^{m'} \), \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \), with \( P \cap \mathbb{Z}^n \subseteq E_k^n \). If \( N \) is the set of all irreducible points in \( P \cap \mathbb{Z}^n \), then for \(|N|\) the inequality \( (15) \) holds, where \( m = m' + 2n \),

\[
\alpha \leq \frac{(k - 1)^{n-1}(n + 1)^{n+1}}{2^n}, \quad \beta \leq \frac{(k - 1)^n(n + 2)^{n+2}}{2^{n+1}}.
\]

**Proof.** For the inequality \( ax \leq a_0 \), where \( a \in \mathbb{R}^n \), \( a_0 \in \mathbb{R} \), we consider the system of \( k^n + 1 \) homogeneous linear inequalities in the variables \( b_0 \in \mathbb{R}, b_1 \in \mathbb{R}, b_{n+1} \in \mathbb{R} \):

\[
\begin{align*}
&b_0 - bx \geq 0 \quad \text{for all } x, \text{ such that } ax \leq a_0, \\
&-b_0 + bx - b_{n+1} \geq 0 \quad \text{for all } x, \text{ such that } ax > a_0, \\
&b_{n+1} \geq 0.
\end{align*}
\]

The set \( K \) of all its solutions is a polyhedral cone in \( \mathbb{R}^{n+2} \). Obviously, any vector in \( K \), with \( b_{n+1} > 0 \), has components \( b_0, b, b_{n+1} \), such that \( \{ x \in E_k^n : ax \leq a_0 \} = \{ x \in E_k^n : bx \leq b_0 \} \).

We prove that the cone \( K \) is pointed, i.e. it does not contain nonzero subspaces. Suppose that both vectors \( \pm (b_0, b, b_{n+1}) \) belong to \( K \). In this case, from the last inequality in \( (16) \), we get that \( b_{n+1} = 0 \), then, from other inequalities, it follows that \( bx = b_0 \) for all \( x \in E_k^n \). Since \( k \geq 2 \), then the affine dimension of \( E_k^n \) is \( n \), hence \( b_0 = b = b_{n+1} = 0 \). Thus, \( K \) does not contain nonzero subspaces.

From the theory of linear inequalities (see, for example, \[12\]) it follows that the set of extreme vectors \( g^{(1)}, g^{(2)}, \ldots, g^{(s)} \) of \( K \) forms its generative system, that is, \( K = \text{cone}\{g^{(1)}, g^{(2)}, \ldots, g^{(s)}\} \). Moreover, for each \( i = 1, 2, \ldots, s \) there exists a subsystem of \( (16) \) that becomes a system of equalities on \( g^{(i)} \), with coefficients of the system forming a matrix \( T_i \) of rank \( n + 1 \). This implies that \( g^{(i)} \) can be chosen integer with its \( j \)-th component equal up to sign to the minor of order \( n + 1 \) cut down from \( T_i \) by removing its \( j \)-th column.

Let us bound the quantity of the minor. Multiplying the rows corresponding to \( x \) with \( ax > a_0 \) and columns corresponding to \( b \) by \(-1\), we get a minor with nonnegative entries. Using well-known bounds for a determinant with nonnegative entries (see, for example, \[11\]), we get the following bounds for the components of \( g^{(i)} = (g^{(i)}_0, g^{(i)}_1, \ldots, g^{(i)}_n, g^{(i)}_{n+1}) \):

\[
|g^{(i)}_0| \leq \frac{(k - 1)^n(n + 2)^{n+2}}{2^{n+1}}, \quad |g^{(i)}_j| \leq \frac{(k - 1)^{n-1}(n + 1)^{n+1}}{2^n} \quad (j = 1, \ldots, n).
\]

Among vectors \( g^{(1)}, g^{(2)}, \ldots, g^{(s)} \) there is a vector \( (b_0, b, b_{n+1}) \) with \( b_{n+1} > 0 \). We call the inequality \( bx \leq b_0 \) the approximation to the inequality \( ax \leq a_0 \).

We replace all inequalities describing \( P \) by those approximations and append \( 2n \) inequalities \( 0 \leq x_j \leq k - 1 \) \( (k = 1, 2, \ldots, n) \). The inequality to be proved follows now from Theorem \[2\].

Note that, for any fixed \( n \), the bounds for \(|N|\) in Theorems \[2\] and \[3\] take the form, respectively,

\[
|N| = O \left( m^{\frac{n}{2j}} \log_2^{n-1}(\alpha\beta) \right), \quad |N| = O(m^{\frac{n}{2j}} \log_2^{n-1} k).
\]
5 Bounds for the teaching dimension of threshold functions

Suppose $f \in \mathcal{T}(n,k)$. Let $K(f) = \text{cone}(M_1(f) - M_0(f))$, $F_0(f) = \text{conv} M_0(f) - K(f)$, $F_1(f) = \text{conv} M_1(f) + K(f)$.

In [23] a characterization of $T(f)$ in terms of $F_0(f)$, $F_1(f)$ is proposed. Denote $T_\nu(f) = T(f) \cap M_\nu(f) \ (\nu = 0, 1)$.

Theorem 4 [23] Let $f \in \mathcal{T}(n,k)$, then $T_\nu(f) = \text{Vert} F_\nu(f) \ (\nu = 0, 1)$.

Corollary 1 Let $f \in \mathcal{T}(n,k)$, $x, y \in T_\nu(f) \ (\nu = 0, 1)$, $x \neq y$, then

$$2x - y \notin F_0(f) \cup F_1(f). \quad (17)$$

Unfortunately, no convenient description of $F_0(f) \cup F_1(f)$ is known in the general case. Nevertheless we consider a set $\mathcal{T}'(n,k)$ of functions $f$, each of which can be given by a threshold inequality such that

$$a_0 \in \mathbb{Z}, \quad a_j \in \mathbb{Z}, \quad 0 < a_0 < a_j(k - 1) \quad (j = 1, 2, \ldots, n).$$

Denote by $\mathbb{Z}^n_+$ the set of all vectors in $\mathbb{Z}^n$ with nonnegative components. We say that a set $G \subset \mathbb{Z}^n_+$ has the separation property, iff from conditions $x, y \in G$, $x \neq y$ it follows that $2x - y \notin \mathbb{Z}^n_+$ [13]. One can verify [23] that if $f \in \mathcal{T}'(n,k)$, then $F_0(f) \cup F_1(f) = \mathbb{Z}^n_+$ and, consequently, the property (17) is equivalent to the separation property. From this we get the following result.

Theorem 5 [23] If $f \in \mathcal{T}'(n,k)$ and $n \geq 2$, then

$$|T_\nu(f)| \leq n(1 + \log_2 n)(1 + \log_2(k + 1))^{n-2} \quad (\nu = 0, 1).$$

Theorem 6 For any fixed $n \geq 2$

$$\sigma(n,k) = O((\log_2^{n-2} k) \quad (k \to \infty).$$

Proof. Without loss of generality, we suppose that the coefficients of the threshold inequality of the function $f \in \mathcal{T}(n,k)$ satisfy the conditions $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$.

If $a_0 \leq (k - 1)a_n$, then the bound to be proved follows from Theorem 5. Now we consider the case when $a_0 > (k - 1)a_n$. If $e_n \notin K(f)$, then form $x \in T_\nu(f)$ it follows that $x_n = 0$ or $x_n = k - 1$, hence $|T_\nu(f)| \leq 2\sigma(n - 1, k)$.

We denote by $e_j$ the vector with all components equal to 0 except the $j$-th component equal to 1. Suppose $e_n \in K(f)$. We let

$$T_0'(f) = \{x \in T_0(f) : \sum_{j=1}^{n-1} a_j x_j \leq a_0 - (k - 1)a_n\},$$
\[ T_0''(f) = \{ x \in T_0(f) : \sum_{j=1}^{n-1} a_j x_j > a_0 - (k-1)a_n \}, \]

\[ T'_1(f) = \{ x \in T_1(f) : \sum_{j=1}^{n-1} a_j x_j > a_0 \}, \]

\[ T''_1(f) = \{ x \in T_1(f) : \sum_{j=1}^{n-1} a_j x_j \leq a_0 \}. \]

If \( x \in T'_\nu(f) \) (\( \nu = 0, 1 \)), then \( x_n = 0 \) or \( x_n = k - 1 \), hence \( |T'_\nu(f)| \leq 2\sigma(n-1, k) \).

Let
\[ P = \left\{ x \in E^n_k : a_0 - (k-1)a_n < \sum_{j=1}^{n-1} a_j x_j \leq a_0, \ x_n = 0 \right\}. \]

If \( y \in P \), then, taking into account that \( e_n \in K(f) \), we get
\[ \{ y + \alpha e_n : \alpha \in Z \} \subset F_0(f) \cup F_1(f). \]

Therefore, using Lemma 1, we get that \( |T''_\nu(f)| \) does not exceed the number of irreducible points in \( P \). Since the dimension of \( \text{conv} P \) is at most \( n-1 \), then, by Theorem 3 the number of irreducible points in \( P \) is \( O(\log^{n-2} k) \) when \( n \) is fixed.

Taking into account the lower bound \( \sigma(n, k) = \Omega(\log^{n-2} k) \) (for fixed \( n \geq 2 \)), obtained in [16], [25], from Theorem 3 we get the following assertion.

**Corollary 2** For any fixed \( n \geq 2 \)

\[ \sigma(n, k) = \Theta(\log^{n-2} k) \quad (k \to \infty). \]

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