ON THE GONCHAROV DEPTH CONJECTURE
AND POLYLOGARITHMS OF DEPTH TWO

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Abstract. We prove the surjectivity part of Goncharov’s depth conjecture. We also show that the depth conjecture implies that multiple polylogarithms of depth $d$ and weight $n$ can be expressed via a single function $\text{Li}_{n-d+1,1,\ldots,1}(a_1, a_2, \ldots, a_d)$, and we prove this latter statement for $d = 2$.

1. Introduction

Multiple polylogarithms are multivalued functions of variables $a_1, \ldots, a_d \in \mathbb{C}$ depending on positive integer parameters $n_1, \ldots, n_d \in \mathbb{N}$. In the polydisc $|a_1|, |a_2|, \ldots, |a_d| < 1$ polylogarithms are defined by power series

$$\text{Li}_{n_1,n_2,\ldots,n_d}(a_1, a_2, \ldots, a_d) = \sum_{0 < m_1 < m_2 < \cdots < m_d} \frac{a_1^{m_1} a_2^{m_2} \cdots a_d^{m_d}}{m_1^{n_1} m_2^{n_2} \cdots m_d^{n_d}}.$$

The number $n = n_1 + \cdots + n_d$ is called the weight of the multiple polylogarithm, and the number $d$ is called its depth. Goncharov suggested an ambitious conjecture, giving a necessary and sufficient condition for a sum of polylogarithms to have certain depth. In §3 we show that the Goncharov depth conjecture would have the following remarkable corollary.

Conjecture 1. Any multiple polylogarithm of weight $n \geq 2$ and depth $d \geq 2$ can be expressed as a linear combination of multiple polylogarithms $\text{Li}_{n-d+1,1,\ldots,1}$ and products of polylogarithms of lower weight.

We expect that there exists a presentation where all the arguments are Laurent monomials in $\sqrt[\sqrt[d]{N}]{a_1}, \ldots, \sqrt[\sqrt[d]{N}]{a_d}$ for sufficiently large $N$. We show that Conjecture 1 is true for $d = 2$.

Theorem 2. For every $0 < k < n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that the multiple polylogarithm

$$\text{Li}_{k,n-k}(x, y)$$

can be expressed as a linear combination of functions

$$\text{Li}_{n-1,1}(\sqrt[\sqrt[d]{N}]{x} \sqrt[\sqrt[d]{N}]{y}, \sqrt[\sqrt[d]{N}]{x^r} \sqrt[\sqrt[d]{N}]{y^s}) \quad \text{for } r, s, t, u \in \mathbb{Z}$$

and products of classical polylogarithms, where each appearance of $\sqrt[\sqrt[d]{N}]{z}$ denotes any $N$th root of $z$.

Here is an example of this type of identity in weight four and depth two

$$\text{Li}_{2,2}(x, y) = -4 \text{Li}_{3,1} \left( -\frac{\sqrt{x}}{\sqrt{y}} y \right) - 4 \text{Li}_{3,1} \left( \frac{\sqrt{x}}{\sqrt{y}} y \right) + 4 \text{Li}_{3,1} \left( -\frac{\sqrt{y}}{\sqrt{x}} x \right) + 4 \text{Li}_{3,1} \left( \frac{\sqrt{y}}{\sqrt{x}} x \right)$$

$$+ \text{Li}_{3,1}(x, y) - \text{Li}_{3,1}(y, x) - \text{Li}_{1,1} \left( \frac{y}{x} x \right) - \frac{1}{2} \text{Li}_4(xy) + \text{Li}_1(x) \text{Li}_3(y).$$

In §2 we give an elementary proof of Theorem 2. In §3 we recall the statement of Goncharov’s depth conjecture and prove a part of it (Theorem 5). Next, we show that the depth conjecture implies Conjecture 1.

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2. Proof of Theorem 2

We define $L(x, y \mid t_1, t_2)$ to be the following generating function

$$L(x, y \mid t_1, t_2) := \sum_{k, l > 0} \text{Li}_{k,l}(x/y, y)t_1^{k-1}t_2^{l-1} = \sum_{m, n > 0} \frac{x^my^n}{(m - t_1)(m + n - t_2)}.$$  

The key observation used in the proof of Theorem 2 is the following identity.

Proposition 3. For any integers $\alpha, \beta, \gamma > 0$ with $\gamma = \alpha + \beta$ and any $x, y$ with $|x|, |y| < 1$ we have

$$\sum_{X^\alpha = x, Y^\beta = y, Z^\gamma = xy} \frac{1}{(m - \beta t)(ma + n\beta)} - \frac{1}{(m - \beta t)(ma + n\beta)} + \frac{1}{(n + \alpha t)(ma + \alpha \beta)}$$

Expanding both sides of (1) as a power series in $t$ and comparing the coefficients of $t^{n-2}$ we see that for any integers $\alpha, \beta > 0$ the function

$$U_n^{\alpha, \beta}(x, y) := \sum_{k, l = n, k, l > 0} \text{Li}_{k,l}(y, y)(-\alpha)^{k-1}\beta^{l-1}$$

is expressible in terms of $\text{Li}_{n-1, 1}$ and $\text{Li}_n$. Since the matrix $((-i)^{k-1}(n - i)^{n-d-1})_{i,k=1}^{n-1}$ is invertible (its determinant is of Vandermonde type), each individual function $\text{Li}_{k,l}(y, y)$ for $k + l = n$ can be written as a rational linear combination of the functions $U_n^{1, n-1}(x, y), U_n^{2, n-2}(x, y), \ldots, U_n^{n-1, 1}(x, y)$, and hence it also can be expressed in terms of $\text{Li}_{n-1, 1}$ and $\text{Li}_n$, as claimed.

3. Surjectivity part of the Goncharov depth conjecture

To state the Goncharov depth conjecture we recall the definition of the Lie coalgebra $\mathcal{L}(F)$ of (formal) polylogarithms with values in a field $F$ (see also [4 §2.1]). The Lie coalgebra $\mathcal{L}(F)$ is positively graded by weight; the component of weight $n$ is generated over $\mathbb{Q}$ by formal symbols $\text{Li}_{n_0, \ldots, n_d}(a_1, \ldots, a_d)$ for $n_0 \in \mathbb{Z}_{\geq 0}, n_1, \ldots, n_d \in \mathbb{N}$ with $n_0 + n_1 + \cdots + n_d = n$ and $a_1, \ldots, a_d \in F^\times$, which are subject to (mostly unknown) functional equations for polylogarithms. The cobracket $\Delta: \mathcal{L}(F) \to \bigwedge^2 \mathcal{L}(F)$ was discovered by Goncharov ([1], [2], [3]); the definition was inspired by properties of mixed Hodge structures related to multiple polylogarithms. The Lie coalgebra $\mathcal{L}(F)$ is filtered by depth; denote by $\mathcal{D}_d\mathcal{L}(F)$ the subspace spanned by polylogarithms of depth not greater than $d$; let $\text{gr}_d^\mathcal{L}(F)$ be the associated graded space. The subspace $\mathcal{D}_1\mathcal{L}(F)$ spanned by classical polylogarithms $\text{Li}_{n_i}^{\mathbb{C}}(a)$ is denoted by $\mathcal{B}_n(F)$. 

Assume that $\Delta = \sum_{1 \leq i \leq j} \Delta_{ij}$ for $\Delta_{ij} : L_i(F) \to L_i(F) \wedge L_j(F)$. The truncated cobracket is a map $\Delta : L(F) \to \wedge^2 L(F)$ defined by the formula $\Delta = \sum_{2 \leq i \leq j} \Delta_{ij}$. In other words, $\Delta$ is obtained from $\Delta$ by omitting the component $L_1(F) \wedge L_{n-1}(F)$ from the cobracket. Denote by $\text{coLie}_d(V)$ the cofree graded Lie coalgebra cogenerated by a graded vector space $V$.

By [13, Proposition 4.1], the iterated truncated cobracket $\Delta^{[d-1]}$ vanishes on $D_{d-1}L_\ast(F)$ and defines a map

$$\Delta^{[d-1]} : \text{gr}_d L_{\geq 2}(F) \to \text{coLie}_d \left( \bigoplus_{n \geq 2} B_n(F) \right).$$

Conjecture 4 (Goncharov, [13, Conjecture 7.6]). A linear combination of multiple polylogarithms has depth less than or equal to $d$ if and only if its $d$-th iterated truncated cobracket vanishes. Moreover, the map $\Delta^{[d-1]}$ for $d \geq 1$ is an isomorphism.

We prove the surjectivity part of Conjecture 4.

Theorem 5. Assume that the field $F$ is quadratically closed. Then the map

$$\Delta^{[d-1]} : \text{gr}_d L_{\geq 2}(F) \to \text{coLie}_d \left( \bigoplus_{n \geq 2} B_n(F) \right)$$

is surjective.

Proof. It is easy to see that

$$\Delta^{[d-1]}(L^\mathbb{C}_{n-d+1,\ldots,1}(a_1,\ldots,a_d)) = \sum_{n_1+n_2+\cdots+n_d=n} \text{Li}^\mathbb{C}_{n_1}(a_1) \otimes \cdots \otimes \text{Li}^\mathbb{C}_{n_d}(a_d).$$

Recall that if $F$ contains all degree $r$ roots of unity then classical polylogarithms $\text{Li}_n(a)$ satisfy the following distribution relations:

$$\text{Li}_n^\mathbb{C}(a^r) = r^{n-1} \sum_{\zeta=1}^{r} \text{Li}_n^\mathbb{C}(\zeta a).$$

It follows that for any $s \in \mathbb{N}$

$$\Delta^{[d-1]} \left( \sum_{x^2 = a_d} L^\mathbb{C}_{n-d+1,\ldots,1}(a_1,\ldots,a_{d-1},x) \right)$$

$$= \sum_{n_1+n_2+\cdots+n_d=n} 2^{-s(n_d-1)} \text{Li}^\mathbb{C}_{n_1}(a_1) \otimes \cdots \otimes \text{Li}^\mathbb{C}_{n_d}(a_d)$$

$$= \sum_{2 \leq n_d \leq n-2d+2} \left( \sum_{n_1+n_2+\cdots+n_{d-1}=n-n_d} \text{Li}^\mathbb{C}_{n_1}(a_1) \otimes \cdots \otimes \text{Li}^\mathbb{C}_{n_{d-1}}(a_{d-1}) \right) \otimes 2^{-s(n_d-1)} \text{Li}^\mathbb{C}_{n_d}(a_d).$$

From the properties of the Vandermonde determinant it follows that for every $n_d \geq 2$ the element

$$\left( \sum_{n_1+n_2+\cdots+n_{d-1}=n-n_d} \text{Li}^\mathbb{C}_{n_1}(a_1) \otimes \cdots \otimes \text{Li}^\mathbb{C}_{n_{d-1}}(a_{d-1}) \right) \otimes \text{Li}^\mathbb{C}_{n_d}(a_d)$$

lies in the image of $\Delta^{[d-1]}$. Continuing in a similar fashion, we conclude that for every $n_1,\ldots,n_d \in \mathbb{N}$ the element

$$\text{Li}^\mathbb{C}_{n_1}(a_1) \otimes \cdots \otimes \text{Li}^\mathbb{C}_{n_d}(a_d)$$

lies in the image of $\Delta^{[d-1]}$. From here the statement follows. \qed
Assume that Goncharov’s depth conjecture holds. It follows from the proof of Theorem 5 that $L^\ast(F)$ is generated by functions $\text{Li}_{n-d,1,...,1}(a_1,...,a_d)$. Shuffle relations for multiple polylogarithms imply that $\text{Li}^\ast_{n-d,1,...,1}$ can be expressed via $\text{Li}^\ast_{n-d+1,1,...,1}$ (corresponding to the function $\text{Li}_{n-d+1,1,...,1}$) and functions of lower depth, so Conjecture 4 implies Conjecture 1.

Theorem 5 has the following striking corollary.

**Corollary 6.** Let $F$ be a quadratically closed field. Assume that Conjecture 4 holds for $d = 1$. Then it holds for all $d \geq 1$ and the Lie coalgebra $L_{\geq 2}(F)$ with cobracket $\Delta$ is cofree.

**Proof.** It is sufficient to prove that $\bigwedge\Delta^{d-1}$ is an isomorphism; Conjecture 4 and cofreeness of $L_{\geq 2}(F)$ would follow from the spectral sequence of the filtered complex $\bigwedge(D_{\geq 2}(F))$. We argue by induction on $d$; the base case $d = 1$ is a tautology. Suppose that for $k \leq d - 1$ the map $\Delta^{k-1}$ is an isomorphism. By Theorem 5 it is sufficient to show that $\Delta^{d-1}$ is injective. Consider an element $x \in D_dL_{\geq 2}(F)$ such that $\Delta^{d-1}(x) = 0$.

The map

$$\Delta^*: \text{gr}^D L_{\geq 2}(F) \to \text{coLie} \left( \bigoplus_{n \geq 2} B_n(F) \right)$$

is a morphism of Lie coalgebras, so

$$\sum_{i+j=d} \Delta^{[i-1]} \wedge \Delta^{[j-1]}(\Delta(x)) = 0.$$  

By the induction assumption, this implies that $\Delta(x)$ vanishes in $\bigwedge^2 \text{gr}^D L_{\geq 2}(F) = \text{gr}^D \left( \bigwedge^2 L_{\geq 2}(F) \right)$ so $\Delta(x) \in D_{d-1}(\bigwedge^2 L_{\geq 2}(F))$. The proof follows. The spectral sequence of the filtered complex implies that Lie coalgebra $D_{d-1}L_{\geq 2}(F)$ with cobracket $\Delta$ is cofree. Thus there exists $y \in D_{d-1}L_{\geq 2}(F)$ such that $\Delta(x - y) = 0$, so $x - y \in D_1L_{\geq 2}(F)$ by the assumption that Conjecture 4 holds for $d = 1$. It follows that $x \in D_{d-1}L_{\geq 2}(F)$. □

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