Entanglement in non-equilibrium steady states and many-body localization breakdown through dissipative tunneling

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We model an electric-field or current driven interacting disorder system, coupled to environment at the boundaries, through an effective non-Hermitian Hamiltonian and study the entanglement properties of its eigenstates. In particular, we investigate whether a many-body localizable system undergoes a transition to a current-carrying non-equilibrium steady state under the drive and how the entanglement properties of the quantum states change across the transition. We also discuss the dynamics, entanglement growth, and long-time fate of a generic initial state under an appropriate time-evolution of the system governed by the non-Hermitian Hamiltonian. Our study reveals rich entanglement structures of the eigenstates of the non-Hermitian Hamiltonian. We find transition between current-carrying states with volume-law to area-law entanglement entropy, as a function of disorder and the strength of the non-Hermitian term, related to the dissipative tunneling to the environment.

Classification of many-body quantum states in terms of their entanglement properties has become a major endeavour in recent years in condensed matter physics. These activities have revealed intriguing connection of entanglement with dynamics and thermalization of quantum systems [1–3]. It has been found that typical high-energy eigenstates of a generic isolated many-body system can be classified as either ergodic or many-body localized (MBL) [4–6], based on whether the entanglement entropy of a subsystem scales as the volume [7] or the area of the subsystem, respectively. Starting with an un-entangled initial state, the MBL systems do not thermalize under the unitary dynamics, unlike the ergodic ones. However, the MBL systems still give rise to a slow growth of entanglement entropy approaching a state with sub-thermal volume law scaling [8–10]. One of the most natural realizations of MBL phases are found in strongly disordered systems where single-particle Anderson localization is stable to interaction [11–14], even at finite energy densities above the ground state.

However, condensed matter systems are seldom isolated. In particular, some very important experimental setups require the system to be connected with external environment. A prominent example is a system connected to leads and driven by current or a voltage bias or an electric field [see Fig.1(a)]. Generically, such systems are expected to attain an unique current-carrying non-equilibrium steady state (NESS). It is interesting to explore whether such NESSs could also be classified according to their entanglement content, e.g., a many-body localizable system undergoing an entanglement transition driven by a current or electric field. It is also important to ask whether the entanglement transition could be coincident with a transition between NESSs with and without current.

Recently there have been some studies of entanglement transition for the current carrying NESSs in non-interacting models using either scattering state [15, 16] or non-equilibrium Green’s functions[17, 18]. Nevertheless, no known framework exist for studying entanglement of driven states of a disordered interacting system connected to infinite leads at the boundaries. One possible way to access such boundary driven system is via Markovian Lindblad quantum master equation approximation [19, 20]. However, such Markovian evolution is destined to lead to a description of NESS in terms of mixed state having area law entanglement [21] and hence make the notion of entanglement transition obscure. An important avenue, at present, is to try to mimic such NESS in less microscopic toy models, e.g., boundary driven random unitary circuit model [15]. Here we take a different route, and try to address driven states of interacting disordered systems through an effective model which incorporates the boundary dissipation and the current drive via a non-Hermitian term.

We study the following non-Hermitian one-dimensional (1d) XXZ spin ($S = 1/2$) model with an uniform random field $h_i \in [-W,W]$, $W$ being the disorder strength,

$$\mathcal{H} = -\frac{J}{2} \sum_i (S_i^x S_{i+1}^x + e^{-\Psi} S_i^- S_{i+1}^+) - \sum_i h_i S_i^z$$

Here $S_i^\pm = (S_i^x \pm i S_i^y)$, $L$ is the number of sites, $\Psi$ is a real number (see below) and $J$, $J_z$ are the spin exchanges; the latter controls the interaction strength. We apply periodic boundary condition. The above model can be rewritten as $\mathcal{H} = \mathcal{H}_h + i \lambda \mathcal{J}$, where $\mathcal{H}_h$ is the usual Hermitian random field $XXZ$ model with $J \rightarrow \tilde{J} = J \cosh \Psi$, $\lambda = J \sinh \Psi$ and $\mathcal{J} = -(iJ/2) \sum_i (S_i^z S_{i+1}^z - S_i^- S_{i+1}^+)$ is the sum of spin current across the system.

The model can be mapped, via Jordan-Wigner transformation, to a model of fermions or hard-core bosons hopping on the 1d lattice with random disorder potential and nearest neighbour repulsion. In that case, $\Psi$ is an imaginary vector potential through a ring [Fig.1(b)] and the model is an interacting version of Hatano-Nelson model [22–24]. The latter describes non-Hermitian...
single-particle localization-delocalization transition. The fermion model is invariant under \(i = \sqrt{-1} \rightarrow -i\), i.e., time (\(T\)) reversal. However, as in the usual \(\mathcal{PT}\)-symmetric non-Hermitian models \([25, 26]\), this \(T\)- or pseudo \(\mathcal{PT}\)-symmetry can be broken by the eigenstates leading to complex eigenvalues. We refer to this real to complex transition as \(T\)-reversal breaking for brevity even in the spin model [Eq.(1)]. Intriguingly, in the original Hatano-Nelson model the \(T\)-symmetry breaking transition of the single-particle eigenstates coincides with the localization-delocalization transition and the delocalized states carry finite current. This leads to the question whether there is any localization-delocalization transition in the interacting model and whether the transition coincides with the \(T\)-reversal breaking. Such congruence of the symmetry breaking and localization transition might lead to an interesting possibility of describing a “MBL transition” in terms of symmetry breaking.

Similar imaginary gauge potential, as in Eq.(1), has been also used in fermionic Hubbard model to describe electric-field driven Mott transition \([27–29]\), where \(\Psi\) mimics the effect of dissipative tunneling to the environment \([27]\). In that case, the model has been shown \([28]\) to describe many-body Landau-Zener (LZ) \([30]\) quantum tunneling processes near field driven Mott transition within Dykhne \([31]\) formalism. In this approach, a model with real gauge potential, such as for an constant electric field, is analytically continued to imaginary gauge potential to describe LZ transition. It is an interesting question whether such many-body LZ processes could provide an effective description for field or current driven transition in a MBL system. Here we do not attempt to address this issue, and, instead, use the non-Hermitian model [Eq.(1)] as an effective model to generate current driven pure states and describe possible entanglement transitions among them.

We further extend the model of Eq.(1) to describe the approach to the long-time NESS through the following dynamical equation for density matrix \(\rho\),

\[
\frac{d\rho}{dt} = -i[H_h, \rho] + \lambda (\{J, \rho\} - 2Tr(\rho J)) \equiv \mathcal{L}\rho. \tag{2}
\]

Here \(\mathcal{L}\) defines a Liouvillian operator. The above model has been used previously in the context of \(\mathcal{PT}\)-symmetric quantum mechanics to describe system with gain and loss \([32]\). Similar model has been also used in the context of field-driven Mott transition \([29]\). Eq.(2) can describe the evolution of both pure and mixed states and keep \(Tr\rho(t) = 1\). Unlike in a Lindblad master equation, an initial pure state remains pure during the time evolution, and, in this case, Eq.(2) reduces to a complex Schrodinger equation \([32]\). From Eq.(2), \(\rho(t)\) can be formally written using the non-Hermitian Hamiltonian [Eq.(1)] as \(\rho(t) = e^{-iHt} \rho_0 e^{-iHt} / Tr(e^{-iHt} \rho_0 e^{-iHt})\), where \(\rho_0\) is the initial density matrix. In particular, for an initial pure state \(\rho_0 = |\psi\rangle \langle \psi|\), the state at time \(t\) is given by

\[
|\psi(t)\rangle = \frac{\sum_n e^{-iE_n t + \Lambda_n t} (n_L|\psi\rangle |n_R\rangle)}{\sum_n e^{-iE_n t + \Lambda_n t} (n_L|\psi\rangle |n_R\rangle)} \tag{3}
\]

Here \(|n_L\rangle\) and \(|n_R\rangle\) are the left and right eigenvectors of \(H\) with eigenvalue \(E_n = \mathcal{E}_n + i \Lambda_n\) and the left right eigenvectors form an orthonormal basis, \(\langle n_L|m_R\rangle = \delta_{nm}\) and \(\sum_n |n_L\rangle \langle n_R| = 1\). The real part of the eigenvalue \(\mathcal{E}_n = \langle n_R|H_h|n_R\rangle / \langle n_R|n_R\rangle\), i.e., the expectation of parent Hermitian Hamiltonian, and the imaginary part \(\Lambda_n \propto \mathcal{J}_n = \langle n_R|J|n_R\rangle / \langle n_R|n_R\rangle\). It can be easily shown from Eq.(3) that, for eigenspectrum with non-zero imaginary part of the eigenvalues, the NESS \(\langle \psi(\infty)\rangle \equiv \langle \psi(t \rightarrow \infty)\rangle\) is given by \(\langle \psi(\infty)\rangle = (\langle s_L|\psi\rangle / (|s_L\rangle \langle s_L|)) (|s_R\rangle / (|s_L\rangle \langle s_L|))\) where \(|s_L\rangle\) is the eigenvector with maximum imaginary part of the eigenvalue \(\Lambda\). We obtain the eigenvalues and the (left and right) eigenvectors of the non-Hermitian Hamiltonian [Eq.(1)] using numerical exact diagonalization for

FIG. 1. The model and the phase diagram: (a) Schematic of an interacting disordered system driven by current through a voltage bias, \(\mu_L - \mu_R\), applied between left and right leads with chemical potentials \(\mu_L\) and \(\mu_R\), respectively. (b) The non-Hermitian model [Eq.(1)] mapped to fermions hopping on a ring with an imaginary flux \(i\Psi L\). (c) The phase diagram for \(\Psi = 0.3\). Eigenstates have volume-law entanglement and are delocalized for \(W < W_c,\) whereas, for \(W > W_c\), states have area-law entanglement and are localized. The color indicates the slope of the participation entropy with the system size. A finite fraction of eigenstates breaks \(T\)-reversal for \(W < W_c\). The region, \(W_{c1} < W < W_{c2}\), have area-law localized states that break \(T\)-reversal. There is a separate transition at \(W_{c3}\) in terms of system-size scaling of current (see main text).
system sizes \( L = 10 - 16 \) and sample over many disorder realizations (10000 for \( L = 10 \), 6000 for \( L = 12 \), 1440 for \( L = 14 \), and 200 for \( L = 16 \)). The Hamiltonian \([\text{Eq.}(1)]\) and the dynamics \([\text{Eq.}(2)]\) conserves \( S_{\text{tot}}^2 = \sum \sigma_i^2 \). Hence we work in the \( S_{\text{tot}}^2 = 0 \) subspace. We take the interaction \( J_z = J = 1 \) and vary \( W \) and \( \Psi \).

We construct the phase diagram \([\text{Fig.1(c)}]\) for the model \([\text{Eq.}(1)]\) as a function of the disorder \( W \) and the quantity, \( \epsilon = (\mathcal{E} - \mathcal{E}_0)/\mathcal{E}_1 \), defined from the real part of the eigenvalues; \( \mathcal{E}_M \) and \( \mathcal{E}_0 \) are the maximum and the minimum values of \( \mathcal{E} \), respectively. For brevity, we refer to \( \epsilon \) as energy density. We characterize the phases based on the finite-size scaling of the entanglement entropy, \( \mathcal{T} \)-symmetry breaking of the eigenstates and the current \( j_n = J_n/L \). Based on the dynamics defined in \([\text{Eq.}(2)]\), we also look into the long-time steady state and the time-evolution of entanglement entropy starting from an initially un-entangled state. Our main results are as follows –

1. We find a transition, \( W_{c1}(\epsilon) \), from a volume-law to an area-law entangled phase as a function of \( W \) and \( \epsilon \). As in the usual MBL transition, we find the entanglement transition to coincide with Hilbert-space delocalization-transition. The entanglement transition moves to higher disorder with increasing strength of the non-Hermitian term \( (\Psi) \). In this sense, a non-zero \( \Psi \) causes a breakdown of the MBL states of the parent Hermitian model.

2. We obtain a separate boundary, \( W_{c2}(\epsilon) \), for \( \mathcal{T} \)-symmetry breaking transition. Hence, unlike the non-interacting case \([22, 24]\), the localization transition does not coincides with \( \mathcal{T} \)-reversal symmetry breaking.

3. We find yet another distinct transition \( W_{c3}(\epsilon) \), within the \( \mathcal{T} \)-reversal broken region in terms of the scaling of the current \( j_n \)'s with system size.

4. As in the usual MBL systems, the entanglement entropy in the volume-law phase grows linearly with \( t \).

5. However, the entanglement growth is followed by a decay at late times towards an unique NESS.

6. We find that the memory of the initial state, characterized in terms of an order parameter, gets eventually lost as the NESS is attained at long times.

**Hilbert-space delocalization-localization transition:** To characterize the eigenstates of the non-Hermitian Hamiltonian \([\text{Eq.}(1)]\), we first obtain a phase diagram in terms of a diagnostic of Hilbert-space localization, namely the participation entropy, \( S_P(|n_R\rangle) = -\sum_{\alpha} p_{n}^{(\alpha)} \ln p_{n}^{(\alpha)} \) of the eigenstates in the basis of the spin configurations, \( [\alpha = S_1^z, S_2^z, \ldots, S_N^z] \) \([33, 34]\) with \( p_{n}^{(\alpha)} = |\langle \alpha | n_R \rangle|^2 \). In the delocalized phase, \( S_P = a \ln D_H \), where \( a \approx 1 \) and \( D_H \) is the dimension of the Hilbert space. On the other hand, in the localized phase, \( a \approx 0 \) and instead, one expects \( S_P \sim \ln(\ln D_H) \) \([34]\). We obtain the disorder averaged \( S_P(\epsilon) \) as a function \( W \) and \( \epsilon \) [Fig.1] for \( \Psi = 0.3 \) and \( L = 16 \). Indeed, as in the Hermitian MBL case \([34, 35]\), we find a delocalization-transition. The transition shifts to higher disorder for increasing \( \Psi \) (see the Supplementary Material \([35, S1]\)).

**Entanglement transition:** Next, we obtain the von Neumann entanglement entropy \( S_{EE} \) for each the eigenstates, i.e. \( S_{EE}(n_R) = -\text{Tr}(\rho_A \ln \rho_A) \). Here \( \rho_A = \text{Tr}_B |n_R\rangle \langle n_R| \) is the reduced density matrix of the subsystem \( A \), for the real-space bipartition of the system into left half, \( A \), and right half, \( B \). For \( \Psi = 0.3 \), the disorder averaged \( S_{EE} \) is plotted for \( \epsilon = 0.5 \), at the middle of the spectrum, as a function of \( W \) for different system sizes in Fig.2(a)(inset). We find a clear crossing of the \( S_{EE}/L \) vs. \( W \) curves around \( W = W_{c1} \approx 3.6 \), implying an entanglement transition, similar to the Hermitian case \([34, 35]\). The dependence of \( S_{EE} \) on \( L \) is consistent with a volume-law scaling for \( W < W_{c1} \), and an area-law scaling for \( W > W_{c1} \) (SM \([35, S2]\)). The transition is further corroborated by a reasonably good data collapse [Fig.2(a)], obtained using the finite-size scaling ansatz \( S_{EE}(L) = L g(L^{1/\nu_1} (W - W_{c1})) \) \([34]\), where \( g(x) \) is the scaling function. We find a critical exponent \( \nu_1 \approx 1 \) and \( W_{c1} \approx 3.6 \) from the scaling collapse. The crossing of the curves and the data collapse are most prominent near the middle of the spectrum. For a larger value of \( \Psi = 0.6 \), we find \( W_{c1} \approx 4.2 \pm 0.1 \), and \( \nu_1 \approx 1.5 \), i.e the extracted critical exponent changes with \( \Psi \) (see SM \([35, S21]\)).

We obtain the phase boundary [Fig.1(b)] from the standard deviation, \( \sigma_E \), of the entanglement entropy over disorder realizations. At the transition, \( \sigma_E \) is expected to show a peak that diverges with \( L \) \([37]\). In Fig.2(b), we
obtain a data collapse of $\sigma_E / (L - c)$ vs. $W$ for $\Psi = 0.3$ and $\epsilon = 0.5$ with $c$ as a fitting parameter [34] and the values of $\nu_1$ and $W_{c1}$ obtained from the finite-size scaling of $S_{EE}$. We plot a phase boundary in the $W - \epsilon$ plane for the entanglement transition from the peak position of $\sigma_E$ for $L = 16$. The phase boundary is consistent with that obtained from the participation entropy [Fig.1(c)].

We now address the question whether the entanglement or the localization transition respectively, with $\nu$ obtain a data collapse of $\nu$ vs. $W$ curves for different $L$, as shown in Fig.2(c) (inset) for $\Psi = 0.3$ and $\epsilon = 0.5$. However, as evident, the crossing point is at $W = W_{c2} \approx 4.75$, clearly larger than $W_{c1}$ for the entanglement transition. A good scaling collapse can again be obtained with an exponent $\nu_2 \simeq 0.9$ and $W_{c2} = 4.75 \pm 0.1$ [Fig.2(c)]. The collapse of the data for $\phi$ vs. $W$ could not be obtained with $W_{c2} = W_{c1}$. This establishes the fact that $\mathcal{T}$-reversal breaking is distinct from the entanglement transition and occurs within the area-law phase. We find similar results for $\Psi = 0.6$ (see SM [35], S21), namely $W_{c2} = 5.6 \neq W_{c1}$. In this case, the critical exponent $\nu_2 = 1.5$. From the crossing points of $\phi(\epsilon)$ vs. $W$ curves we obtain the phase boundary for the $\mathcal{T}$-reversal breaking in Fig.1(c).

To further characterize the $\mathcal{T}$-reversal breaking eigenstates, we compute the current $j(\epsilon)$, obtained by averaging over the magnitude of the currents, $|j|$, carried by the eigenstates with imaginary eigenvalues, and disorder realizations. Again, we find a crossing ($W = W_{c3} \approx 2.3$) in the $J = L_j$ vs. $W$ plots for different $L$ [Fig.2(d)] at $\epsilon = 0.5$ for $\Psi = 0.3$. The transition, which we refer to as current transition for brevity, is seemingly distinct from both entanglement and $\mathcal{T}$-reversal breaking transitions. We can obtain a scaling collapse for $j$ with $\nu_3 = 1$ and $W_{c3} = 2.3$, as shown in Fig.2(d)(inset). We obtain a phase boundary for the current transition at other values of $\epsilon$ from the crossing points, as shown in Fig.1(c).

The scaling of $j$ with $W$ for $W \ll W_{c3}$ is consistent with $j$ approaching a constant for $L \rightarrow \infty$ (not shown). The scale-invariant crossing point indicates a diffusive scaling of the current at the transition, namely $j \sim 1/L$. In fact, for $W_{c3} < W \lesssim W_{c2}$, the scaling of $j$ with $W$ could be consistent with $j \sim 1/L^\gamma$ with $\gamma \gtrsim 1$, and we expect $j \sim e^{-L/\zeta}$ for $W \gg W_{c2}$, deep inside the $\mathcal{T}$-reversal unbroken localized phase; $\zeta$ is the characteristic localization length. However, this is hard to verify from the exact diagonalization numerics limited to such small system sizes. As discussed later [see Fig.3(d)], the current, $j_{\psi}$, carried by the long-time NESS also exhibits similar transition. For $W > W_{c1}$, in the area-law phase, the scaling of the current and the transition at $W_{c2}$ could be studied by a matrix-product operator (MPO) based implementation [19, 20] of the dynamical evolution in Eq.(2). It would be interesting to establish the existence of such current-carrying pure NESS with area-law entanglement in an interacting system, as discussed in ref.[15].

**FIG. 3. Time evolution and NESS:** (a)-(c) Semilog plots of entanglement entropy $S_{EE}$ vs. $t$ for $W = 2.15$, 4.3, 5.57, respectively, starting with the Ne’el state, for the interacting, $J_z = 1\left(S_{EE}^{j_z=1}\right)$, and the non-interacting cases, $J_z = 0\left(S_{EE}^{j_z=0}\right)$. Inset in (a) shows log-log plot of $S_{EE}^{j_z=1}(t) - S_{EE}^{j_z=0}(t \rightarrow \infty)$ vs. $t$. The boxes in (b), (c) indicate the initial logarithmic growths in the localized phases, with and without $\mathcal{T}$-reversal breaking, respectively. (d) Entanglement entropy, $S_{EE}$, of the long-time NESS, as a function of $L$ for different $W$; the arrow indicates increasing values of $W = 1.6 - 6$, equally spaced, and $W = 7$. (e) Finite-size scaling collapse of long-time steady state current $j_\infty$ with $W_c = 2.9 \pm 0.1$ and $\nu = 1.8$. Inset shows the transition in terms of crossing of $J_\infty$ vs. $W$ curves for different $L$, in a semilog plot. (f) Time evolution of Ne’el order parameter as a function of $W$ and $L$. The triangles, circles and squares are for $W = 0.43$, 4.3, 7, respectively, with $L = 10$ (blue), $L = 12$ (orange), $L = 14$ (yellow) and $L = 16$ (purple).

**$\mathcal{T}$-reversal breaking:** We now address the question whether the entanglement or the localization transition coincides with the $\mathcal{T}$-reversal breaking, as in the non-interacting model [22–24]. We define a $\mathcal{T}$-reversal order parameter, $\phi(\epsilon)$, the fraction of imaginary eigenvalues at $\epsilon$. We find a clear crossing of the $\phi$ vs. $W$ curves for different $L$, as shown in Fig.2(c) (inset) for $\Psi = 0.3$ and $\epsilon = 0.5$. However, as evident, the crossing point is at $W = W_{c2} \approx 4.75$, clearly larger than $W_{c1}$ for the entanglement transition. A good scaling collapse can again be obtained with an exponent $\nu_2 \simeq 0.9$ and $W_{c2} = 4.75 \pm 0.1$ [Fig.2(c)].
cial effect of interaction, we compare $S_{EE}(t)$ with that obtained for the non-interacting Hatano-Nelson model ($J_z = 0$). The prominent features of $S_{EE}(t)$ are, (a) an initial growth, (b) a broad peak followed by a decay or relaxation, and (c) an eventual approach to a steady state value corresponding to the NESS. In the volume-law phase [Fig.3(a)(inset)], $S_{EE}(t)$ grows linearly with time, whereas, $S_{EE}(t) \propto \ln t$ initially in the area-law phases [Fig.3(b)-(c)], as in the Hermitian MBL case [8].

Using Eq.(3), the growth and subsequent decay of entanglement entropy can be understood from the density matrix, $\rho(t) = |\psi(t)\rangle\langle\psi(t)| = e^{-L \phi_0} = \sum_{m,n} e^{-(\beta_{nm} + \bar{b}_{nm})} C^m_n(t)|n_R\rangle\langle m_R|$, where the coefficient $C^m_n(t)$ is obtained from Eq.(3). Here $-(\beta_{nm} + \bar{b}_{nm})$ could be thought of as the eigenvalues of the Liouvillean operator $\mathcal{L}$; $\beta_{nm} = 2\Delta_n - \Lambda_n - \Lambda_m \geq 0$ and $\delta_{nm} = \mathcal{E}_n - \mathcal{E}_m$. The real part, $-\beta_{nm}$, of eigenvalue of $\mathcal{L}$ leads to relaxation and $\beta_{nm} = 0$ corresponds to the long-time steady state. For weak-strength ($\Psi \ll 1$) of the non-Hermitian term, and for $t \ll e^{L}$ [38], $\delta_{\text{typ}} \gg \beta_{\text{typ}}$, and the initial growth of the entanglement entropy appears over a time window, $\delta_{\text{typ}}^{-1} \lesssim t \lesssim \beta_{\text{typ}}^{-1}$ due to the dephasing from the exponential factor $e^{\bar{b}_{nm} t}$ in $\rho(t)$. In this time window, the factor $e^{-\beta_{nm} t}$ is expected to have gap [39], $\beta_{nm} \sim O(1)$, above its minimum value, $\beta_{nm} = 0$ (see SM [35], S3.1). On the contrary, the spectrum of $\delta_{nm}$ is gapless, $\min(\delta_{nm}) \sim e^{-L}$. Hence, there could be interesting dephasing dynamics that goes on during the decay over $\beta_{\text{typ}}^{-1} \lesssim t \lesssim \beta_{\text{typ}}^{-1}$, even though $\delta_{\text{typ}} \gg \beta_{\text{typ}}$. For $t > \beta_{\text{typ}}^{-1}$, the entanglement entropy rapidly approaches the value for that of the NESS, dictated by the eigenstate with the maximum imaginary part for the eigenvalue, as discussed earlier.

We have also studied the system-size scaling of $S_{EE}^{\text{peak}}$, the maximum value of $S_{EE}(t)$ in Figs.3.(a)-(c), as well as that of $S_{EE}^{\infty} = S_{EE}(t \to \infty)$, the entanglement entropy of the NESS. As shown in Fig.3(d) and in the SM [35], S3.2, we find no volume-law to area-law transition in either $S_{EE}^{\text{peak}}$ or $S_{EE}^{\infty}$ with the disorder strength. In fact, $S_{EE}^{\text{peak}}$ and $S_{EE}^{\infty}$ neither scale as the volume nor the area, even though both of these increase with $L$ (see SM [35], S3.2). As already discussed, there is also a current transition for the NESS, as in Fig.2(d) for the eigenstates.

The transition is shown in terms of a finite-size scaling collapse for $L_{\infty}$ vs. $W$ curves in Fig.3(e). We also show the time-evolution of Ne’el order parameter $N(t) = (2/L) \sum_{i=1}^{N}(-1)^i \langle \psi(t) | \sigma_i^z | \psi(t) \rangle$ in Fig.3(f). This characterizes the memory of the initial state and approaches a finite value for $t \to \infty$ for the infinite system size in the MBL phase; $N(t)$ decays to zero with time in the ergodic phase. Here, for the non-Hermitian case, we find that $N_{\infty} = N(t \to \infty)$, i.e. the Neel order of the NESS, decreases with $L$, even for strong disorder, deep in the area-law phase. This is expected in a driven system, which loses its initial memory due to the drive. We also notice an interesting dynamical regime, presumably over the time window $\beta_{\text{typ}}^{-1} < t < \beta_{\text{typ}}^{-1}$, in the decay of $N(t)$ [Fig.3(f)]. The order parameter plateaus over a region as a function of $t$, as if trying to retain the initial memory. In contrast, such regime is absent in the Hermitian model (see SM [35], S3.3).

In conclusion, we have studied a non-Hermitian disordered model, the interacting version of Hatano-Nelson model [22]. We propose that the model can be used to generate various current-carrying states and study their entanglement properties. We have established a rich phase diagram based on the properties of the eigenstates and the long-time NESS, as well as, from the time evolution of entanglement entropy, as a function of disorder. In future, it would be interesting verify whether the non-Hermitian model can indeed describe some features of a many-body localizable system under an actual electric field or current drive.

Note added in the manuscript — During the writing of the manuscript we became aware of ref.40 which has studied non-Hermitian MBL transition in the same model. Our focus, i.e. to study current carrying NESS, is entirely different from that of ref.40. Our results and conclusions are also substantially different from those in ref.40. In particular, unlike ref.40, we find that the phase boundaries in $W - \epsilon$ plane for $T$ reversal-symmetry breaking and entanglement transitions are distinct.

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Supplemental Material
for
Entanglement in non-equilibrium steady states and many-body localization breakdown through dissipative tunneling
by Animesh Panda, and Sumilan Banerjee

S1: Evolution of phase diagram as a function of \( \Psi \)

![Figure S1](image)

**FIG. S1. Phase diagrams:** The phase boundaries as a function of energy density \( \epsilon \) (see main text) and disorder \( W \). (i) \( W_{c1} \), Hilbert space delocalization-localization and entanglement transitions, (ii) \( W_{c2} \), time-reversal breaking transition, and (iii) \( W_{c3} \), current transition (see the main text for the definitions). The values of the non-Hermitian strength are \( \Psi = 0 \) (left), \( \Psi = 0.3 \) (middle), same as Fig.1(c), and \( \Psi = 0.6 \) (right).

In this section we show the evolution of the phase diagram, the one shown in Fig.1(c) (main text), as a function of the strength of the non-Hermitian term \( \Psi \) [Eq.(1)]. As shown in Figs.S1, the three phase boundaries \( W_{c1} \), \( W_{c2} \) and \( W_{c3} \) naturally shift to higher disorder with increasing \( \Psi \). We obtain the phase boundary, \( W_{c1}(\epsilon) \), from the peak of the standard deviation of entanglement entropy \( \sigma_E \) [Fig.2(b)]. Similar, albeit slightly higher, values for \( W_{c1} \) are obtained from the crossing of \( S_{EE}/L \) vs. \( W \) curves [Fig.2(a)(inset)]. However, we do not use the crossing point to plot phase boundary as a function of \( \epsilon \), since clear crossings can only be detected over middle one-third and one-half of the spectra for \( \Psi = 0.3 \) and \( \Psi = 0.6 \), respectively. We note that the clear crossing point for \( S_{EE}/L \) vs. \( W \) curves could be obtained for the Hermitian case (\( \Psi = 0 \)) almost over the entire spectrum, except at the very edges.

S2: Transition from volume-law to area-law entanglement for \( \Psi = 0.3 \)

The MBL transition is defined from volume-law to area-law transition [34] in the Hermitian case, i.e. \( \Psi = 0 \) in Eq. 1. A similar transition is also observed for \( \Psi = 0.3 \) with increasing disorder, as evident from Fig.S2(a) for \( \epsilon = 0.5 \). This is further corroborated by clear crossing of \( S_{EE}/L \) vs. \( W \) curves for different \( L \) in Fig.S2(b) (also shown in the inset of Fig.2(a), main text). This crossing point is used to find out the critical disorder \( W_{c1} \).

1. Finite-size scaling collapses for different values of \( \Psi \)

In this section we show the finite-size scaling collapses, similar to the ones shown in Figs.2, for \( \Psi = 0 \) and \( \Psi = 0.6 \) at the middle of the spectrum \( \epsilon = 0.5 \). Fig.S3 shows the data collapses for \( \Psi = 0 \). The entanglement transition \( W_{c1} \) is extracted to be \( \sim 3.3 \pm 0.1 \). Somewhat higher values of critical disorder for the MBL transition is obtained in ref.[34], which considers larger system sizes up to \( L = 22 \). It is a known effect in exact diagonalization studies of MBL, namely the critical disorder strength tends to shift to higher values with increasing \( L \) for the limited system sizes accessible so far.

The finite-size data collapse for \( \Psi = 0.6 \) is shown in Fig.S4. All the three values of the critical disorders (\( W_c \)’s) for the three transitions, and the critical exponents (\( \nu \)’s) increase as we go from \( \Psi = 0.3 \) (Figs.2) to \( \Psi = 0.6 \) (Figs.S4). Surprisingly we obtain two peaks in the variance \( \sigma_E \) for \( L = 14 \) and \( L = 16 \) [Fig.S4(b)]. As evident, the two-peak structure becomes more prominent for the higher system size. The reason behind the two-peak structure is not clear at present and will be studied in a future work.
FIG. S2. **Volume-law to area-law transition for** $\Psi = 0.3$: (a) $S_{EE}$ as function of $L$ for different disorder strengths at $\epsilon = 0.5$; the arrow denotes the direction of increasing disorder ($W$), from 0.43 to 7. (b) $S_{EE}/L$ is plotted against $W$ for different $L$.

FIG. S3. **Entanglement transition for** $\Psi = 0$: (a) $S_{EE}/L$ and (b) $\sigma_E$ for $\epsilon = 0.5$. The transition is detected from the system size crossing of $S_{EE}/L$ in the inset of (a). The values of the critical exponent and critical disorder extracted for the transition is $\nu_1 \approx 0.8$, $W_{c1} = 3.3 \pm 0.1$.

We note that the average $S_{EE}$ and $\sigma_E$ shown in Figs.2, S3, S4 and S2 are obtained by averaging over the eigenstates with real eigenvalues. The averaging over the eigenstates with complex eigenvalues also gives similar results.
FIG. S4. **Entanglement and T-reversal breaking transitions for** $\Psi = 0.6$: Data collapses for (a) $S_{EE}/L$, (b) $\sigma_E$, (c) $\phi$ and (d) $L_j$, at $\epsilon = 0.5$. The transitions are detected from the system size crossings shown in the insets of (a), (c) and (d). The values of the critical exponents and critical disorder extracted for the three transitions are, (a) $\nu_1 \simeq 1.5$, $W_{c1} = 4.2 \pm 0.1$, (c) $\nu_2 \simeq 1.5$, $W_{c2} = 5.6 \pm 0.1$ and (d) $\nu_3 \simeq 1.5$, $W_{c3} = 2.95 \pm 0.10$; $c$ is a fitting parameter in (b) (see main text).

**S3: Time evolution for different values of $\Psi$**

1. **Spectrum of the Liouvillian operator**

FIG. S5. **Probability distribution of the real part of the eigenvalues of the Liouvillian operator**: The probability distribution $P(\bar{\beta})$ for (a) $W = 1$ and (b) $W = 2$ for $\Psi = 0.3$. The boxes in (a) and (c) highlight the gapped part of the spectra, which are zoomed in (b) and (d), respectively.

As discussed in the main text, the time evolution of the density matrix is controlled by the eigenspectrum of the Liouvillian operator $\mathcal{L}$ in Eq.(2). The relaxation of the system to the NESS is controlled by the real part of the
eigenvalues of $\mathcal{L}$, i.e. $-\beta_{nm} > 0$, where $\beta_{nm} = 2\lambda_s - \lambda_n - \Lambda_n$. In Figs.S5, we plot the disorder averaged distribution, $P(\beta)$, of the normalized quantity, $\beta = (\beta - \beta_0)/(\beta_M - \beta_0)$ for $\Psi = 0.3$. Here $\beta_0$ and $\beta_M$ are the minimum and the maximum values of $\beta_{nm}$, respectively. Since the imaginary eigenvalues of the non-Hermitian Hamiltonian in Eq.(1) appears in complex conjugate pairs, the probability distribution is symmetric about $\bar{\beta} = 1/2$. Also, there is a peak at $\bar{\beta} = 0$, which corresponds to the NESS, as mentioned in the main text. As shown in Figs.S5, the peak is separated from most of the spectrum by a long tail. In the limit $L \to \infty$, the tail is expected to tend to a gap, $\beta_g$, that separates the peak from the relaxation modes with $\beta > 0$.

2. Entanglement entropy of NESS

![Figure S6](image1.png)

**FIG. S6.** $S_{EE}^{\text{peak}}$ and $S_{EE}^{\infty}$ for $\Psi = 0.3$: (a) $S_{EE}^{\text{peak}}$ and (b) $S_{EE}^{\infty}$ as a function of $L$ for different $W$. The solid and dashed lines in (a) and (b) are for $W < W_m$ and $W > W_m$, respectively; $W_m = 2$ in (a) and $W_m = 1.2$ in (b) are the positions of the maxima in $S_{EE}^{\text{peak}}$ and $S_{EE}^{\infty}$, respectively, as function of $W$. In (a) and (b) the arrows indicate the direction of increasing $W$ for the dashed lines and direction of decreasing $W$ for the solid lines. The range of $W$ is from 0.43 to 7. As shown in (c) and (d) unlike average $S_{EE}/L$, $S_{EE}^{\text{peak}}/L$ and $S_{EE}^{\infty}/L$ do not show any crossing when plotted against $W$.

In this section we study $S_{EE}^{\infty}$, as well as the maximum value of $S_{EE}(t)$, i.e. $S_{EE}^{\text{peak}}$ [Figs.3(a)-(c)], as a function of $W$ and $L$. For a given $L$, unlike the eigenstate averaged $S_{EE}$ [Figs.S2(a)], the maxima in $S_{EE}^{\text{peak}}$ and $S_{EE}^{\infty}$ appear at a finite disorder, as shown in Figs.S6(a)-(d). Also, we do not see any system-size crossing for $S_{EE}^{\infty}/L$ ($S_{EE}^{\text{peak}}/L$) vs. $W$ in Fig.S6(c)/(Fig.S6(d)). We were also not able to obtain any finite-size data collapse for $S_{EE}^{\infty}$ and $S_{EE}^{\text{peak}}$.

3. Time evolution of Ne’el order parameter

As discussed in the main text, we characterize the memory of the initial state by the Ne’el order parameter $N(t)$. For the Hermitian case $\Psi = 0$, as shown in Fig.S7(a) for $W = 0.43 < W_c$, the long-time value $N_\infty$ goes to zero in the ergodic phase. This can be seen from the fact that $N_\infty$ decreases with $L$ tending to zero for $L \to \infty$. On the contrary, in the MBL phase, for $W = 4.3, 7$, $N_\infty$ approaches a constant value with increasing $L$. Hence, the Ne’el order parameter in this case serves as the MBL order parameter, i.e. it diagnoses the persistence of the initial memory at arbitrary long times. In the non-Hermitian model [Eq.(1)], the NESS is approached at long times. This is true even deep in the MBL phase, $W \gg W_c$, since there is always a finite number of eigenstates with complex eigenvalues, albeit with a vanishing fraction, for any finite-size system. Of course, for $W \gg W_c$, it is expected that the imaginary part of the eigenvalues $\sim e^{-L/\zeta}$, and, hence, the NESS will ensue only after a very long time $\sim e^{L/\zeta}$ for large systems. As shown in Fig.S7(b) for $\Psi = 0.6$, and in Fig.3(f) for $\Psi = 0.3$, the Ne’el order parameter for the NESS decreases rapidly with $L$, presumably approaching zero for $L \to \infty$. However, as mentioned in the main text,
FIG. S7. **Time evolution of Ne’el order parameter**: The Ne’el order parameter $N(t)$ is shown for (a), the Hermitian case, $\Psi = 0$, and (b) $\Psi = 0.6$, as a function of $W$ and $L$. The triangles, circles and squares are for $W = 0.43, 4.3, 7$ respectively, with $L = 10$ (blue), $L = 12$ (orange), $L = 14$ (yellow) and $L = 16$ (purple).

There is an interesting dynamical regime $\beta^{-1}_{typ} < t < \beta^{-1}_g$ over which $N(t)$ tends to palteau, retaining the memory of the initial state.