Wick Calculus

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Abstract

In quantum field theory, physicists routinely use “normal ordering” of operators, which just amounts to shuffling all creation operators to the left. Potentially confusing, then, is the occurrence in the literature of normal-ordered functions, sometimes called “Wick transforms”. We aim to introduce the reader to some basic results and ideas around this theme, if not the mathematical subtleties; our intended audience are instructors who want to add something to their quantum field theory course, or researchers who are interested but not specialists in mathematical physics. For rigorous proofs and careful mathematical discussions we only give references.

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1 Introduction

Normal ordering was introduced in quantum field theory by G.C. Wick in 1950, to avoid some infinities in the vacuum expectation values of field operators expressed in terms of creation and annihilation operators [1]. The simplest example of such infinities can be discussed based on non-relativistic quantum mechanics and the simple harmonic oscillator; an infinite number of harmonic oscillators make up a free quantum field. (The reader who needs a quick reminder of some basic quantum field-theoretical concepts may find comfort in appendix A). We will use the harmonic oscillator to exhibit the connection between Wick-ordered polynomials and the familiar Hermite polynomials. Then we turn to Wick transforms in the functional integral formalism of field theory, where we show that there is again a connection with Hermite polynomials. Several different approaches to Wick transforms that can be found in the literature are compared and we show why they are equivalent. In passing, we observe how the standard quantum field theory result known as “Wick’s theorem” follows rather directly in this framework, from well-known properties of Hermite polynomials. Finally, we provide one brief example of how the Wick transform can be utilized in a physical application.

2 Wick operator ordering

2.1 Simple Harmonic Oscillator

The Hamiltonian operator for the simple harmonic oscillator in non-relativistic quantum mechanics has the form

\[ H = \frac{1}{2}(P^2 + Q^2) , \]

where we have, as usual, hidden Planck’s constant \( \hbar \), the mass \( m \), and the angular frequency \( \omega \) in definitions of dimensionless momentum and position operators

\[ P := \frac{1}{\sqrt{\hbar m \omega}} \hat{p} , \quad Q := \sqrt{\frac{m \omega}{\hbar}} \hat{q} , \]

so that

\[ [P, Q] = -i . \]

If one defines the creation and annihilation operators \( a^\dagger \) and \( a \) by

\[ a^\dagger = \frac{1}{\sqrt{2}} (Q - iP) , \quad (1) \]

\[ a = \frac{1}{\sqrt{2}} (Q + iP) , \quad (2) \]
so that
\[ [a, a^\dagger] = 1 , \]
one finds that
\[ H = \frac{1}{2}(P^2 + Q^2) = \frac{1}{2} (a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2} . \]
This means, as proved in introductory quantum mechanics books (see e.g. Sakurai [2]), that the eigenvalues of the Hamiltonian operator come in the sequence
\[ E_n = n + \frac{1}{2} , \quad n = 0, 1, 2, \ldots . \]
In particular, the ground state energy (or zero-point energy), which is the lowest eigenvalue of the Hamiltonian, is non-zero:
\[ H|0\rangle = \frac{1}{2} |0\rangle . \]
This is in agreement with Heisenberg’s uncertainty principle: It is the smallest energy value that saturates the uncertainty relation (again, see e.g. Sakurai [2] for the explicit calculation).

This zero-point energy has observable physical consequences; as an illustration, it is possible to measure zero-point motion (which leads to the zero-point energy) of atoms in a crystal by studying dispersion of light in the crystal. Classical theory predicts that any oscillations of the atoms in the crystal, and therefore also dispersion effects, cease to exist when the temperature is lowered towards absolute zero. However, experiments demonstrate that dispersion of light reaches a finite, non-zero value at very low temperature.

In quantum field theory, a free scalar field can be viewed as an infinite collection of harmonic oscillators, as described in appendix A. If we proceed as before, each oscillator will give a contribution to the zero-point energy, resulting in an infinite energy, which seems like it could be a problem.

One way to remedy the situation is to define the ground state as a state of zero energy. We can achieve this by redefining the Hamiltonian: We subtract the contribution of the ground state and define a so-called Wick-ordered (or normal-ordered) Hamiltonian, denoted by putting a colon on each side, by
\[ :H: = : \frac{1}{2} (a^\dagger a + aa^\dagger) : = \frac{1}{2} (a^\dagger a + aa^\dagger) - \frac{1}{2} (0|a^\dagger a + aa^\dagger |0) \equiv a^\dagger a . \]
Hence, in this example the definition of Wick ordering can be thought of as a redefinition of the ground state of the harmonic oscillator.

On the other hand, we also see that in the last equality in (5) all creation operators end up on the left. A general prescription for Wick ordering in quantum field theory in a creation/annihilation operator formalism is then: “Permute all the \( a^\dagger \) and \( a \), treating them as if they commute, so that in the end all \( a^\dagger \) are to the left of all \( a \).” The resulting expression is, of course, the same:
\[ :H: = a^\dagger a . \]
2.2 Wick ordering and Hermite polynomials

The first connection between Wick ordering and Hermite polynomials arises if we study powers of the (dimensionless) position operator \( Q \). Physically, in the harmonic oscillator, the eigenvalue of \( Q^2 \) gives the variance (squared standard deviation) of the oscillator from rest. We have \( Q = (a^\dagger + a) / \sqrt{2} \), but to avoid cluttering the equations with factors of \( \sqrt{2} \), we will study powers of just \((a^\dagger + a)\).

\[
(a^\dagger + a)^2 = a^{\dagger 2} + a^\dagger a + aa^\dagger + a^2 \\
= a^{\dagger 2} + 2a^\dagger a + a^2 + [a, a^\dagger] \\
= : (a^\dagger + a)^2: + [a, a^\dagger] \\
= : (a^\dagger + a)^2: + 1 \quad \text{by (5)}.
\]

Arranging terms in a similar way for higher powers of \((a^\dagger + a)\) we find

\[
(a^\dagger + a)^3 = : (a^\dagger + a)^3: + 3(a^\dagger + a) \\
(a^\dagger + a)^4 = : (a^\dagger + a)^4: + 6 : (a^\dagger + a)^2: + 3.
\]

We can summarize the results as follows, with the notation \( a^\dagger + a = q, \)

\[
q^2 = :q^2: + 1, \\
q^3 = :q^3: + 3 : q: , \\
q^4 = :q^4: + 6 : q^2: + 3.
\]

Since we can recursively replace normal-ordered terms on the right by expressions on the left which are not normal-ordered (e.g. \( :q^2: \) can be replaced by \( q^2 - 1 \)), we can also invert these relations:

\[
:q^2: = q^2 - 1 = He_2(q), \\
:q^3: = q^3 - 3q = He_3(q), \\
:q^4: = q^4 - 6q^2 + 3 = He_4(q),
\]

where the polynomials \( He_n(q) \) are a scaled version of the more familiar form of the Hermite polynomials \( H_n \):

\[
He_n(x) = 2^{-n/2}H_n(x/\sqrt{2}).
\]

Confusingly, in some mathematical physics literature, the \( He_n \) are often just called \( H_n \). Some of the many useful properties are collected in appendix B for easy reference (a more complete collection is given in e.g. Abramowitz & Stegun [3]).

Because of this relation between operator Wick-ordering and Hermite polynomials, the mathematical physics literature sometimes defines “Wick ordering” in terms of Hermite polynomials:

\[
: q^n : \overset{\text{def}}{=} He_n(q).
\]
Although $q$ is an operator composed of noncommuting operators $a$ and $a^\dagger$, this alternative definition naturally generalizes to Wick-ordering of functions. As promised, we will explore this idea in the next section.

One reason that the connection to Hermite polynomials is not mentioned in standard quantum field theory literature is the fact (which was also Wick's motivation) that the normal-ordered part is precisely the part that will vanish when one takes the vacuum expectation value. Indeed, the traditional way to define normal ordering, the one given at the end of section 2.1 ("put $a^\dagger$ to the left of $a"$), yields for powers of $q$ simply

$$:q^n: = \sum_{i=1}^{n} \binom{n}{i} (a^\dagger)^{n-i} a^i,$$

which vanishes for any nonzero power $n$ when applied to the vacuum state $|0\rangle$.

In other words, since one knows that normal ordered terms vanish upon taking the vacuum expectation value, one may not be interested in their precise form.

However, when the expectation value is not taken in the vacuum (for example, in a particle-scattering experiment), this part does of course not vanish in general, and there are in fact many instances where the actual normal-ordered expression itself is the one of interest.

# 3 Functional integrals and the Wick transform

Most modern courses on quantum field theory discuss functional integrals (sometimes called "path" integrals, however only in nonrelativistic quantum mechanics does one really integrate over paths). In a functional-integral setting, the counterpart of the Wick ordering in the operator formalism is the Wick transform. This transform applies to functions and functionals. It can, like its quantum-mechanics counterpart eq. (7), be defined by means of Hermite polynomials. But first, let us briefly skip ahead and explain why such a transform will prove to be useful.

## 3.1 Integration over products of fields

In the functional formalism, physical quantities like scattering cross sections and decay constants are computed by integrating over some polynomial in the fields and their derivatives. The algorithmic craft of such computations is described in textbooks such as Ryder [4] and Peskin & Schroeder [5]. Even though there are examples of physical effects that can be studied with functional-integral methods but not with ordinary canonical quanti-
within the scope of this paper we can only give examples of some things that can be derived more quickly or transparently using functional integrals.

In fact, we will also be concerned with some rather basic questions that are usually glossed over in introductory treatments: what does the functional integral itself really mean? While a complete answer is not even known, and certainly beyond the scope of this short article, we intend to give some flavor of the first steps towards addressing this question and how the Wick transform has been put to work in this regard.

First, a restriction: the polynomials considered in this section are polynomials of Euclidean fields (fields defined on four-dimensional Euclidean space $\mathbb{R}^4$). Similar formalisms exist for Minkowski fields (fields defined on spacetime) with minor changes in the equations (see e.g. the aforementioned textbook [5]). Unfortunately, functional integrals over Minkowski fields are less mathematically developed than integrals over Euclidean fields, therefore we shall restrict attention to Wick transforms of functions and functionals of Euclidean fields — primarily polynomials and exponentials. The Wick transform, like the Hermite polynomials, has orthogonality properties that turn out to be useful in quantum field theory, as we shall see. First, we have to introduce a few mathematical concepts.

### 3.1.1 Gaussian measures

Here, our aim is to fix the notation, and to briefly remind the reader how to integrate over Euclidean fields, without going into too much detail. The standard mathematical framework to perform such integration is the theory of Gaussian measures in Euclidean field theory, for which details can be found in the mathematical physics literature, such as Glimm & Jaffe [6] and Janson [7].

As a first try, one would define a field in the functional integral as a function $\phi$ on $\mathbb{R}^4$. Fields in the functional integral, however, may seem like functions at first glance, but can produce divergences that cannot (for instance) be multiplied in the way that functions can. A more useful way to regard a quantum field in the functional integral formalism is as a distribution $\Phi$ acting on a space of test functions $f$:

$$\Phi(f) \overset{\text{def}}{=} \langle \Phi, f \rangle,$$

where the bracket $\langle \ , \ \rangle$ denotes duality, i.e. $\Phi$ is such that it yields a number when applied to a smooth test function $f$. In many situations the distribution $\Phi$ is equivalent

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1 In particular, contributions to the $S$-matrix that have an essential singularity at zero coupling constant cannot be found by standard perturbative expansion around zero coupling, yet these “nonperturbative” contributions can be studied using functional integrals. Rates for decays that would be strictly forbidden without these effects can be computed, see for example the book by Ryder [8, Ch. 10.5].
to a function $\phi$, which means this number is the ordinary integral

$$\langle \Phi, f \rangle = \int_{\mathbb{R}^4} d^4x \, \phi(x) f(x) .$$

(9)

A familiar example of a distribution is the one-dimensional Dirac distribution $\Phi = \delta$, for which we have

$$\Phi(f) = \langle \delta, f \rangle = f(0) .$$

Now, just like a function, $\Phi$ in general belongs to an infinite-dimensional space. To be able to integrate over this space (not to be confused with the integral in equation (9), which is an ordinary integral over spacetime) we need a measure, some generalization of the familiar $dx$ in the ordinary integral above. Here, a useful generalization will actually depend on the Green’s function, called covariance in this context and denoted $C$. In general, the covariance is a positive, continuous, non-degenerate bilinear form on the space of test functions. In the following, we will often encounter the covariance at coincident test functions, here denoted $C(f, f)$.

To get to the point, a Gaussian measure $d\mu_C$ is defined by its covariance $C$ as

$$\int_{Y} d\mu_C(\Phi) \exp \left( -i \langle \Phi, f \rangle \right) = \exp \left( -\frac{1}{2} C(f, f) \right)$$

(10)

over a space $Y$ of distributions $\Phi$. For comparison, the usual Gaussian measure on $\mathbb{R}^d$ is defined by

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{d^d x}{(\det D)^{1/2}} \, e^{-\frac{1}{2} Q(x)} \, e^{-i \langle x', x \rangle} = e^{-\frac{1}{2} W(x')}$$

(11)

where

$$\langle x', x \rangle_{\mathbb{R}^d} = x'_\mu x^\mu$$

is the duality in $\mathbb{R}^d$, $x, x' \in \mathbb{R}^d$,

$$Q(x) \text{ is a quadratic form on } \mathbb{R}^d, \quad Q(x) = D_{\mu\nu} x'^\mu x^\nu = \langle Dx, x \rangle_{\mathbb{R}^d},$$

$$W(x') \text{ is a quadratic form on } \mathbb{R}^d, \quad W(x') = x'_\mu C^{\mu\nu} x'_\nu = \langle x', C x' \rangle_{\mathbb{R}^d},$$

such that

$$DC = CD = 1 .$$

(12)

To avoid dimension-dependent numerical terms (powers of $\pi$, powers of 2) in the definition (11) of the Gaussian measure one can, alternatively, define it by

$$\int_{\mathbb{R}^d} \frac{d^d x}{(\det D)^{1/2}} \, e^{-\pi Q(x)} e^{-2\pi i \langle x', x \rangle} = e^{-\pi W(x')} .$$

In fact, this can even be convenient in the simplest Fourier transforms for those who forget where the $2\pi$ goes: $\hat{f}(p) = \int dx \, e^{-2\pi i px} f(x)$ yields an inverse $f(x) = \int dp \, e^{2\pi i px} \hat{f}(p)$, without forefactor.
In the more familiar $\mathbb{R}^d$ case, it is the combination of standard measure and kinetic term that corresponds to the measure $d\mu_C$ we introduced above:

$$\frac{d^d x}{(\det D)^{1/2}} e^{-\frac{1}{2} Q(x)}$$

is analogous to $d\mu_C(\Phi),$ (13)

but going back from $d\mu_C$ to an explicit separation as on the left will not turn out to be necessary for our discussion. In fact, by defining the measure $d\mu_C$ through eq. (10), we have not even specified what such a separation would mean.

With the above expressions in mind, the covariance at incident points is expressed as the following integral, obtained by expanding eq. (10):

$$C(f, f) = \int d\mu_C(\Phi) \langle \Phi, f \rangle^2.$$ (14)

In fact, the integral on the left-hand side of eq. (10) is the generating function of the Gaussian measure; let us denote this integral by $Z(f).$ This means that by successive expansion of eq. (10), the $n$-th moment of the Gaussian measure can be compactly written as

$$\int d\mu_C(\Phi) \langle \Phi, f \rangle^n \quad = \quad \left( -i \frac{d}{d\lambda} \right)^n Z(\lambda f) \bigg|_{\lambda=0}$$

$$\quad = \quad \left\{ \begin{array}{ll} (n-1)!! C(f, f)^{n/2} & n \text{ even} \\ 0 & n \text{ odd}, \end{array} \right.$$ (15)

where $n!! = n(n-2)(n-4)\cdots$ is the semifactorial. For convenience, we introduce the following notation for the average with respect to the Gaussian measure $\mu_C$:

$$\langle F[\Phi(f)] \rangle_{\mu_C} := \int d\mu_C(\Phi) F[\Phi(f)].$$ (16)

Note the difference between the brackets $\langle \rangle_{\mu_C}$ used for average and the brackets $\langle , \rangle$ used for duality. Armed with this set of definitions, we can define a Wick transform of functionals of fields.

### 3.1.2 Wick transforms, definitions

The goal here is to provide some idea of how to address the difficult mathematical problem of making sense out of products of distributions, and integrals of such products, which (as argued above) are ubiquitous in quantum field theory, although their exact meaning is not usually discussed in standard introductory textbooks. In order to simplify quantum field theory calculations, one defines the Wick transform of a power $\Phi(f)^n := \langle \Phi^n, f \rangle$ so
as to satisfy an orthogonality property with respect to Gaussian integration. Recalling the orthogonality properties of Hermite polynomials (appendix B) and the definition of a simple idea is to define the Wick transform in terms of Hermite polynomials:

\[
: \Phi(f) _n :_C = C(f, f)^{n/2} He_n \left( \frac{\Phi(f)}{\sqrt{C(f, f)}} \right).
\]

(17)

Notice that this depends on the covariance \(C\), and that there is no analogous dependence in the analogous harmonic-oscillator definition (7).

The orthogonality of two Wick-transformed polynomials is then expressed by

\[
\int d\mu_C(\Phi) : \Phi(f) _n :_C : \Phi(g) _m :_C = \delta_{m,n} n! (\langle \Phi(f) \Phi(g) \rangle _\mu)_n.
\]

(18)

An entertaining exercise is to show this, which we will do in section 3.1.3 (paragraph 2).

The Wick transform can also be defined recursively by the following equations

\[
\begin{align*}
: \Phi(f) _0 :_C &= 1 \\
\frac{\delta}{\delta \Phi} : \Phi(f) _n :_C &= n : \Phi(f) _{n-1} :_C & n = 1, 2, ... \\
\int d\mu_C(\Phi) : \Phi(f) _n :_C &= 0 & n = 1, 2, ...
\end{align*}
\]

(19)

where the functional derivative with respect to a distribution is simply

\[
\frac{\delta}{\delta \Phi} \Phi(f) = f.
\]

Let us check that the Wick transform \( : \Phi(f) _n :_C \) defined by eq. (19) is the same as the Wick transform given in terms of Hermite polynomials in eq. (17). This is of course to be expected, since Hermite polynomials themselves satisfy similar recursion relations, but it is a useful exercise to check that it works. To begin with, we establish a property of Wick exponentials.

\textit{Wick exponentials}

Let \( \exp(\alpha \Phi(f))_C \) be the formal series

\[
: \exp(\alpha \Phi(f)) :_C \equiv 1 + \alpha : \Phi(f) :_C + \frac{1}{2} \alpha^2 : \Phi(f)^2 :_C + ...
\]
where normal-ordering is defined by eq. \((19)\).

**Exercise:** Show that
\[
: \exp(\alpha \Phi(f)) C : = \frac{\exp(\alpha \Phi(f))}{\langle \exp(\alpha \Phi(f)) \rangle_{\mu_C}}. \tag{20}
\]

**Solution:**
One can evaluate the right-hand side of the equation by expanding numerator and denominator into a power series and dividing one power series by the other.

\[
\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k
\]

where \(c_n + \frac{1}{a_0} \sum_{k=1}^{n} c_{n-k} a_k - b_n = 0\). Comparing the resulting series, term by term, to the power series expansion of the left side proves equation \((20)\).

**Equivalence of Hermite polynomial and recursive definitions**

**Exercise:** Show the equivalence of \((19)\) and \((17)\).

**Solution:** We can explicitly calculate the denominator in eq. \((20)\):

\[
\langle \exp(\alpha \Phi(f)) \rangle_{\mu_C} = \int d\mu_C(\Phi) \exp(\alpha \Phi(f))
\]

\[
= \int d\mu_C(\Phi) \sum_n \frac{\alpha^n}{n!} \Phi(f)^n
\]

\[
= \sum_n \frac{\alpha^{2n}}{n!} \int d\mu_C(\Phi) \Phi(f)^{2n} \quad \text{by eq. } (13)
\]

\[
= \exp\left(\frac{1}{2} \alpha^2 C(f,f)\right) \quad \text{by eq. } (14).
\]

Thus, from eq. \((20)\) we find

\[
: \exp(\alpha \Phi(f)) C : = \exp \left( \alpha \Phi(f) - \frac{1}{2} \alpha^2 C(f,f) \right). \tag{21}
\]

Multiplying the power series expansions\(^3\) of \(\exp(\alpha \Phi(f))\) and \(\exp(\frac{1}{2} \alpha^2 C(f,f))\) and comparing the result term by term to the series expansion of the left side of eq. \((21)\) yields

\[
: \Phi(f)^n C : = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{m!(n-2m)!} \Phi(f)^{n-2m} \left(-\frac{1}{2} C(f,f)\right)^m. \tag{22}
\]

\(^3\)Similarly to the division of power series mentioned above, the multiplication of power series is simply

\[
\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{n=0}^{\infty} d_n x^n
\]

where \(d_n = \sum_{m=0}^{n} a_m b_{n-m}\).
Rewriting this expression as

\[ : \Phi(f)^n :_{\mathcal{C}} = C(f, f)^{n/2} \sum_{m=0}^{[n/2]} (-1)^m \frac{n!}{2^m m! (n-2m)!} \left( \frac{\Phi(f)}{\sqrt{C(f, f)}} \right)^{n-2m}, \quad (23) \]

and using the formula for the defining series of the Hermite polynomials given in appendix B, one recovers eq. (17).

### 3.1.3 Wick transforms, properties

Many properties of Wick ordered polynomials can be conveniently derived using the formal exponential series. For simplicity, we assume that all physical quantities that one may wish to compute (scattering cross sections, etc.) are written with normalization factors of 1/C(f, g), which in effect lets us set the coincident-point covariance to unity: C(f, f) = 1. It can be restored by comparison with eq. (23). The properties we are interested in are useful exercises to show:

1. Show that

   \[ \langle : \Phi(f) + \Phi(g) :_{\mathcal{C}} \rangle = \exp (-\langle \Phi(f) \Phi(g) \rangle_{\mu_{\mathcal{C}}}) : \exp(\Phi(f)) :_{\mathcal{C}} : \exp(\Phi(g)) :_{\mathcal{C}} \]

   \textit{Solution:}

   \[ : \exp(\Phi(f)) :_{\mathcal{C}} : \exp(\Phi(g)) :_{\mathcal{C}} \]

   \[ = \exp (\Phi(f) + \Phi(g)) \exp \left( -\frac{1}{2} \left[ \langle \Phi(f)^2 \rangle_{\mu_{\mathcal{C}}} + \langle \Phi(g)^2 \rangle_{\mu_{\mathcal{C}}} \right] \right) \]

   \[ : \exp (\Phi(f) + \Phi(g)) :_{\mathcal{C}} \exp (\langle (\Phi(f) \Phi(g)) \rangle_{\mu_{\mathcal{C}}}) , \]

   where we have used eq. (21) in the first line, and, after completing the square in the second factor, again in the second line. Dividing both sides of the equation by the second factor completes the demonstration.

2. Show that

   \[ \langle : \Phi(f)^n :_{\mathcal{C}} : \Phi(g)^m :_{\mathcal{C}} \rangle_{\mu_{\mathcal{C}}} = \delta_{nm} n! \langle \Phi(f) \Phi(g) \rangle_{\mu_{\mathcal{C}}}^n \]

   \textit{Solution:}

   If we take the expectation value of both sides of the last line in the proof above, we find

   \[ \langle : \exp(\Phi(f)) :_{\mathcal{C}} : \exp(\Phi(g)) :_{\mathcal{C}} \rangle_{\mu_{\mathcal{C}}} = \exp (\langle (\Phi(f) \Phi(g) \rangle_{\mu_{\mathcal{C}}}) \]

   using eq. (19). Expanding the exponentials on both sides and comparing term by term completes the proof.
3. Show that
\[ : \Phi(f)^{n+1} :_{C} = n : \Phi(f)^{n-1} :_{C} - \Phi(f) : \Phi(f)^{n} :_{C} \]

_Solution:_
This is a consequence of the equivalence of Wick ordered functions and Hermite polynomials. The expression follows from the recursion relation for Hermite polynomials given in appendix B.

4. The definition of Wick transforms given above can be generalized to several fields in a very straightforward manner. We quote here some results without proof (details can be found in e.g. the book by Simon [4]). The reader may find it interesting to check that it works:
\[ : \Phi(f_{1}) \ldots \Phi(f_{n+1}) : = : \Phi(f_{1}) \ldots \Phi(f_{n}) : \Phi(f_{n+1}) \]
\[ - \sum_{k=1}^{n} C(f_{k}, f_{n+1}) : \Phi(f_{1}) \ldots \Phi(f_{k-1}) \Phi(f_{k}) \Phi(f_{k+1}) \ldots \Phi(f_{n}) : \]
\[ \int d\mu_{C}(\Phi) : \Phi(f_{1}) \ldots \Phi(f_{n}) : = 0 \]
\[ \int d\mu_{C}(\Phi) : \Phi(f_{1}) \ldots \Phi(f_{n}) : : \Phi(g_{1}) \ldots \Phi(g_{m}) : = 0 \quad \text{for } n \neq m \]

These latter multi-field expressions reproduce, within this functional framework, what is usually referred to as “Wick’s theorem” in the creation/annihilation-operator formalism. In that formalism, it takes some effort to show this theorem; here we find it somewhat easier, relying on familiar properties of the Hermite polynomials.

### 3.2 Wick transforms and functional Laplacians

#### 3.2.1 Definition

We can also define Wick transforms of functions by the following exponential operator expression, which is convenient in many cases (e.g. in two-dimensional quantum field theory settings, such as in references [8] and [12]):
\[ : \phi^{n}(x) :_{C} \overset{\text{def}}{=} e^{-\frac{1}{2}\Delta_{C}} \phi^{n}(x) \]  \hspace{1cm} (24)

where the functional Laplacian is defined by
\[ \Delta_{C} = \int d^{4}x \int d^{4}x' \ C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} \]  \hspace{1cm} (25)

which, again, depends on the covariance $C$. 

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Instead of proving eq. (24), which is straightforward, we will just illustrate the equivalence of definition (24) and definition (17) in the common example of a \(\phi^4\) power, when the definition reads:

\[
: \phi^4(y) : = \exp \left( -\frac{1}{2} \int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} \right) \phi^4(y) .
\]

Expanding the exponential we find

\[
: \phi^4(y) : = \phi^4 + \left( -\frac{1}{2} \right) \left( \int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} \right) \phi^4(y) \\
+ \frac{1}{2!} \left( -\frac{1}{2} \right)^2 \left( \int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} \right)^2 \phi^4(y) .
\]

All higher terms in the expansion are zero. We can now evaluate each term separately; the second term is the integral

\[
\int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} \phi^4(y) \\
= \int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} 4\phi^3(y) \delta(x' - y) \\
= \int d^4x C(x, y) 12\phi^2(y) \delta(x - y) \\
= 12\phi^2(y)
\]

if the covariance is normalized to unity. We use this result in the third term:

\[
\int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x')} 12\phi^2(y) \\
= \int d^4x d^4x' C(x, x') \frac{\delta}{\delta \phi(x)} 24\phi(y) \delta(x' - y) \\
= 24 \int d^4x C(x, y) \delta(x - y) \\
= 24
\]

with the same normalization of the covariance. We collect these results with the appropriate coefficients from the expansion:

\[
: \phi^4(y) : = \phi^4(y) - \frac{1}{2} \cdot 12 \phi^2(y) + \frac{1}{2!} \cdot \frac{1}{2} \cdot 24 \\
= \phi^4(y) - 6 \phi^2(y) + 3 \\
= He_4(\phi(y)) ,
\]

which completes the illustrative example.
3.3 Further reading

Although we hope to have given some flavor of some of the techniques and ideas of quantum field theory mathematical-physics style, we have of course really only given a few illustrative examples and demonstrated some simple identities. For more on the mathematical connection between Wick transforms on function spaces and Wick ordering of annihilation and creation operators, we recommend textbooks such as [3] and [4].

3.4 An application: specific heat

In this last section, we discuss one example of a physical application of some of the results above. By using the connection (17) between the Wick transform and Hermite polynomials, we show how one can exploit standard properties of those polynomials to simplify certain calculations.

Consider the familiar generating function of Hermite polynomials (but for the scaled polynomials (R)):
\[
e^{x \alpha - \alpha^2/2} = \sum He_n(x) \alpha^n/n!
\]  
(26)

This generating function gives a shortcut to computing some quantum effects in two-dimensional quantum field theory, where normal ordering is often the only form of renormalization necessary. An important issue is the scaling dimension of the normal-ordered exponential \( e^{ip \phi} \), where \( p \) is a momentum. (The real part of this operator can represent the energy of a system where \( \phi \) is the quantum field.) In other words, the question is: if we rescale our momentum \( p \rightarrow \Lambda p \), equivalent to a rescaling \( x \rightarrow \Lambda^{-1} x \) in coordinate space, how does this operator scale? Since an exponential is normally expected to be dimensionless, one might guess that the answer is that it does not scale at all, i.e. that the scaling dimension is zero. In fact, this is not so due to quantum effects induced by the normal ordering. Passing to the functional integral, we can easily compute the effect of the normal ordering (here, the Wick transform):

\[
:e^{ip \phi}:C = \sum_{n=0}^{\infty} \frac{1}{n!} (ip)^n \phi^n :C \\
= \sum_{n=0}^{\infty} \frac{1}{n!} (ip)^n C^{n/2} He_n \left( \phi / \sqrt{C} \right) \\
= e^{\frac{1}{2} p^2 C} e^{ip \phi},
\]  
(27)

where we used the definition (17) and the previously given generating function (20). Now, the covariance (Green’s function) \( C \) is a logarithm in two dimensions, i.e. the solution of the two-dimensional Laplace equation is a logarithm. To regulate divergences when
\( p \to \infty \), one introduces a cutoff \( \Lambda \) on the momentum, which makes \( C = \ln \Lambda \). Substituting this into (27) yields the answer
\[
e^{ip\phi} = e^{p^2/2} e^{ip\phi}.
\]
Thus, the anomalous scaling dimension, usually denoted by \( \gamma \), is \( \gamma = p^2/2 \) for the exponential operator. This is an important basic result in conformal field theory (see e.g. [11], p. 451). Here, the \( p^2/2 \) just comes from the \( \alpha^2/2 \) in the generating function (26).!

How could such a quantum effect be measured? Consider the "two-dimensional Ising model with random bonds" (p. 719, [12]). This is just the familiar Ising model, but one allows the coupling between spins to fluctuate, i.e. the coupling becomes a space-dependent Euclidean field in the spirit of previous sections. The energy of the system is described by (the real part of) an exponential operator :\( e^{ip\phi} : \) as stated above. Using renormalization group methods, the anomalous dimension (28) leads to a formula for the specific heat in this system (eq. (356) in [12], where the derivation is also given). The specific heat is in principle directly measurable as a function of the temperature, or more conveniently, as a function of \( \theta = (T - T_c)/T_c \), the dimensionless deviation from the critical temperature. The renormalization group description predicts a certain double logarithm dependence on \( \theta \) that could not have been found by simple perturbation theory, and it uses as input the result (28).

Admittedly, the telegraphic description in the previous paragraph does not do justice to the full calculation of the specific heat in the two-dimensional Ising model with random bonds. Our purpose here was simply to show how the Wick transform reproduces the quantum effect (28), and then to give some flavor of how this effect is measurable.

4 Conclusion

In this article we have shown that the scope of normal ordering has expanded to settings beyond the original one of ordering operators. Several different definitions of Wick ordering of functions have been discussed and their equivalence established.

For deeper understanding and further applications of these ideas, the interested reader is invited to consult the quoted literature, which is a selection of texts we found particularly useful. Specifically, for the physics of functional integrals we enjoy [6]. For a more mathematically oriented treatment we find [4] quite useful.
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A Wick ordering of operators in QFT

In this appendix, we briefly remind the reader how the need for Wick ordering arises in the operator formulation of quantum field theory. All of this is standard material and can be found in any introductory book on quantum field theory (e.g. [5]), albeit in lengthier and more thorough form. We set $\hbar = 1$ throughout.

Consider a real scalar field $\phi(t, x)$ of mass $m$ defined at all points of four-dimensional Minkowski spacetime and satisfying the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(t, x) = 0.$$ 

The differential operator in the parenthesis is one instance of what we call $D$ in the text. The classical Hamiltonian of this scalar field is

$$H = \frac{1}{2} \sum_x \left[ (\pi(t, x))^2 + (\nabla \phi(t, x))^2 + m^2 \phi^2(t, x) \right] \quad (29)$$

where $\pi$ is the variable canonically conjugate to $\phi$, in fact it is simply $\pi = \partial \phi / \partial t$. Here we can think of the first term as the kinetic energy, and the second as the shear energy. This classical system is quantized in the canonical quantization scheme by treating the field $\phi$ as an operator, and imposing equal-time commutation relations

$$[\phi(t, x), \phi(t, x')] = 0,$$

$$[\pi(t, x), \pi(t, x')] = 0,$$

$$[\phi(t, x), \pi(t, x')] = i\delta^3(x - x').$$

The plane-wave solutions of the Klein-Gordon equation are known as the field modes, $u_k(t, x)$. Together with their respective complex conjugates $u_k^*(t, x)$ they form a complete orthonormal basis, so the field $\phi$ can be expanded as

$$\phi(t, x) = \sum_k [a_k u_k(t, x) + a_k^\dagger u_k^*(t, x)].$$

The equal time commutation relations for $\phi$ and $\pi$ are then equivalent to

$$[a_k, a_{k'}^\dagger] = 0,$$

$$[a_k^\dagger, a_{k'}^\dagger] = 0,$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$
These operators are defined on a Fock space, which is a Hilbert space made of \( n \)-particle states \( (n = 0, 1, \ldots) \). The normalized basis ket vectors, denoted by \( | \rangle \), can be constructed starting from the vector \( |0\rangle \), the vacuum. The vacuum state \( |0\rangle \) has the property that it is annihilated by all the \( a_k \) operators:

\[
a_k |0\rangle = 0, \quad \forall k.
\]

In terms of the frequency \( \omega_k = c\sqrt{|k|^2 + m^2} \), the Hamiltonian operator obtained from (29) is

\[
\hat{H} = \frac{1}{2} \sum_k (a_k^\dagger a_k + a_k a_k^\dagger) \omega_k = \sum_k (a_k^\dagger a_k + \frac{1}{2} \omega_k),
\]

where in the last step we used the commutation relations from above. Calculating the vacuum energy reveals a potential problem:

\[
\langle 0| \hat{H} |0\rangle = \langle 0|0 \rangle \sum_k \frac{1}{2} \omega_k = \sum_k \frac{1}{2} \omega_k \rightarrow \infty,
\]

where we have used the normalization condition \( \langle 0|0 \rangle = 1 \). This infinite constant can be removed as described in the text.

Propagation amplitudes in quantum field theory (and hence scattering cross sections and decay constants) are given in terms of expectation values of time-ordered products of field operators. These time-ordered products arise in the interaction Hamiltonian of an interacting quantum field theory. The goal is to compute propagation amplitudes for these interactions using essentially time-dependent perturbation theory, familiar from quantum mechanics. At leading order in the coupling constant, these products can be simplified, and the zero-point constant energy removed by using Wick’s theorem.

The way Wick ordering is applied in practice to calculations in QFT is through “Wick’s theorem”, which gives a decomposition of time-ordered products of field operators into sums of normal-ordered products of field operators (again, we refer to e.g. [5]). In this paper, in section 3.1.3, Wick’s theorem appears in the functional-integral formulation of the theory.

## B Properties of \( \text{He}_n(x) \)

Here we list a few useful properties of the scaled Hermite polynomials. More can be found in [3].

Defining series:

\[
\text{He}_n(x) = \sum_{m=0}^{\left[ \frac{n}{2} \right]} (-1)^m \frac{n!}{m! 2^m (n - 2m)!} x^{n-2m},
\]
where \([n/2]\) is the integer part of \(n/2\).

Orthogonality:
\[
\int_{-\infty}^{\infty} dx \ e^{-x^2/2} \ He_n(x) \ He_m(x) = \delta_{mn} \sqrt{2\pi} n!.
\]

Generating function:
\[
\exp \left( x\alpha - \frac{1}{2} \alpha^2 \right) = \sum_{n=0}^{\infty} He_n(x) \frac{\alpha^n}{n!}.
\]

Recursion relation:
\[
He_{n+1}(x) = x \ He_n(x) - n \ He_{n-1}(x).
\]

The first five:
\[
\begin{align*}
He_0(x) &= 1, \\
He_1(x) &= x, \\
He_2(x) &= x^2 - 1, \\
He_3(x) &= x^3 - 3x, \\
He_4(x) &= x^4 - 6x^2 + 3.
\end{align*}
\]

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