Degenerate three-level laser with parametric amplifier and squeezed vacuum

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Applying stochastic differential equations, we study the squeezing and statistical properties of the cavity and output modes of a degenerate three-level laser whose cavity contains a parametric amplifier and coupled to a squeezed vacuum reservoir. We consider the case in which the top and bottom levels of the three-level cascade atoms injected into the cavity are coupled by the pump mode emerging from the parametric amplifier. It turns out that the presence of the squeezed vacuum reservoir and the parametric amplifier contribute considerably to the mean photon number and the degree of squeezing of the cavity and output modes. It appears that almost perfect squeezing can be achieved at steady state and at threshold for a suitable choice of parameters.

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I. INTRODUCTION

The quantum properties of three-level lasers have been investigated by several authors\cite{1,2,3,4,5,6,7,8,9,10}. It is found that three-level lasers can generate squeezed light under certain conditions. The generation of squeezed light by three-level lasers could be realized when either the atoms are initially prepared in a coherent superposition of the top and bottom levels\cite{1,2,3,4,5,6,7,8,9,10} or when these levels are coupled by a strong coherent light\cite{2,3}. More recently a three-level laser with a parametric amplifier has been studied when either the three-level atoms are initially prepared in a coherent superposition of the top and bottom levels\cite{1} or when these levels are coupled by the pump mode emerging from the parametric amplifier\cite{2}. These studies show that the effect of the parametric amplifier is to increase both the mean photon number and the intracavity as well as the output mode squeezing significantly. All previous studies have been confined to the case for which the cavity mode of the three-level laser is coupled to a vacuum reservoir. Moreover, apart from the squeezing spectrum of the output mode, all attention has been paid to the calculation of the squeezing and statistical properties of the cavity mode. Since the output mode is accessible to measurement, it appears to be appropriate to study the quantum properties of this mode.

In this paper we analyze the squeezing and statistical properties of the cavity and output modes of a degenerate three-level laser whose cavity contains a degenerate parametric amplifier and coupled to a squeezed vacuum reservoir via a single-port mirror. We consider the case for which the top and bottom levels of the three-level atoms injected into the cavity are coupled by the pump mode emerging from the parametric amplifier. We obtain stochastic differential equations for the cavity mode variables employing the pertinent master equation. Using the solutions of these equations, we calculate the quadrature variance, the mean photon number, and the power spectrum of the cavity and output modes. In addition, we determine the squeezing spectrum of the output mode.

II. STOCHASTIC DIFFERENTIAL EQUATIONS

We consider a laser cavity containing a parametric amplifier and coupled to a squeezed vacuum reservoir. Three-level atoms in a cascade configuration are injected into the cavity at some constant rate $r_a$. We denote the upper level by $|a\rangle$, the middle level by $|b\rangle$, and the lower level by $|c\rangle$, as shown in Fig. 1. The dipole allowed transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ are resonant with the cavity mode. The direct transitions between levels $|a\rangle$ and $|c\rangle$ are dipole forbidden.

With the pump mode treated classically, a degenerate parametric amplifier is describable in the interaction picture by the Hamiltonian

$$\hat{H} = \frac{i\varepsilon}{2}(\hat{a}_2^\dagger - \hat{a}_2^2),$$

(1)
in which $\varepsilon = \lambda \mu$ with $\lambda$ and $\mu$ being respectively the coupling constant and the amplitude of the pump mode. The equation of evolution of the density operator associated with this Hamiltonian has the form

$$\frac{d}{dt} \hat{\rho} = \frac{\varepsilon}{2} (\hat{\rho} \hat{a}^2 - \hat{a}^2 \hat{\rho} + \hat{a}^2 \hat{\rho} - \hat{\rho} \hat{a}^2).$$

(2)

In addition, the Hamiltonian describing the interaction of a three-level atom with the cavity mode and with the pump mode emerging from the parametric amplifier has the form

$$\hat{H} = ig[\hat{a}^\dagger (|b\rangle \langle a| + |c\rangle \langle b|) - \hat{a}(|a\rangle \langle b| + |b\rangle \langle c|)] + i \frac{\Omega}{2} (|c\rangle \langle a| - |a\rangle \langle c|),$$

(3)

where $\Omega$ is proportional to the amplitude of the pump mode and $g$ is the atom-cavity mode coupling constant. We take the initial state of a single three-level atom to be

$$|\psi_A(0)\rangle = \frac{1}{\sqrt{2}} |a\rangle + \frac{1}{\sqrt{2}} |c\rangle$$

(4)

and hence the density operator for a single atom is

$$\hat{\rho}_A(0) = \frac{1}{2} |a\rangle \langle a| + \frac{1}{2} |c\rangle \langle c| + \frac{1}{2} |a\rangle \langle c| + \frac{1}{2} |c\rangle \langle a|.$$

(5)

Following the procedure developed in Ref. [9], we can show that the master equation for the laser cavity mode coupled to a squeezed vacuum reservoir to be

$$\frac{d}{dt} \hat{\rho} = A(2\hat{a}^\dagger \hat{\rho} \hat{a} - \hat{a} \hat{a}^\dagger \hat{\rho} - \hat{\rho} \hat{a} \hat{a}^\dagger) + B(2\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{\rho} \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{a})$$

$$+ C(\hat{a}^\dagger \hat{a} \hat{\rho} + \hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^2 \hat{\rho}) + D(\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger + \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} - \hat{a}^2 \hat{\rho}),$$

(6)

where

$$A = \frac{\kappa N}{2} + \frac{A}{4B} \left[ 1 - \frac{3\beta}{2} + \beta^2 \right],$$

(7a)

$$B = \frac{\kappa (N + 1)}{2} + \frac{A}{4B} \left[ 1 + \frac{3\beta}{2} + \beta^2 \right],$$

(7b)

$$C = \frac{\kappa M}{2} + \frac{A}{4B} \left[ -1 + \beta + \frac{\beta^2}{2} + \frac{\beta^3}{2} \right],$$

(7c)

$$D = \frac{\kappa M}{2} - \frac{A}{4B} \left[ -1 - \beta + \frac{\beta^2}{2} - \frac{\beta^3}{2} \right],$$

(7d)

$$B = (1 + \beta^2)(1 + \frac{\beta^2}{4}),$$

(7e)

$$\beta = \Omega / \gamma,$$

(7f)

$$N = \sinh^2 r,$$

(7g)

$$M = \sinh r \cosh r,$$

(7h)

$$A = \frac{2g^2 r_n}{\gamma^2}$$

(8)
is the linear gain coefficient, $\kappa$ is the cavity damping constant, $\gamma$ is the atomic decay rate assumed to be the same for all the three levels, and $r$ is the squeeze parameter. Therefore, on account of Eqs. (2) and (3) the master equation for the cavity mode of the quantum optical system under consideration becomes

$$\frac{d}{dt}\hat{\rho} = \frac{\varepsilon}{2}(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}\hat{a}^2\hat{\rho} - \hat{a}\hat{a}\hat{\rho}) + A(2\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{a}) + B(2\hat{a}\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{a}) + C(\hat{\rho}\hat{a}^2 + \hat{a}\hat{\rho} - \hat{a}\hat{\rho}^2 - \hat{a}\hat{\rho}^2) + D(\hat{\rho}\hat{a}^4 + \hat{a}\hat{\rho} - \hat{a}\hat{\rho}^2 - \hat{a}\hat{\rho}^2).$$

(9)

We next seek to determine, applying this master equation, stochastic differential equations for the cavity mode variables. To this end, applying (9) one readily finds

$$\frac{d}{dt}\langle \hat{a} \rangle = -(B - A)\langle \hat{a} \rangle + (C - D + \varepsilon)\langle \hat{a}^\dagger \rangle,$$

(10)

$$\frac{d}{dt}\langle \hat{a}^2 \rangle = -2(B - A)\langle \hat{a}^2 \rangle + 2(C - D + \varepsilon)\langle \hat{a}\hat{a}^\dagger \rangle + \varepsilon - 2D,$$

(11)

$$\frac{d}{dt}\langle \hat{a}\hat{a}^\dagger \rangle = -2(B - A)\langle \hat{a}\hat{a}^\dagger \rangle + (C - D + \varepsilon)(\langle \hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle) + 2A,$$

(12)

and the c-number equations corresponding to these normally ordered equations are

$$\frac{d}{dt}\langle \alpha \rangle = -(B - A)\langle \alpha \rangle + (C - D + \varepsilon)\langle \alpha^* \rangle,$$

(13)

$$\frac{d}{dt}\langle \alpha^2 \rangle = -2(B - A)\langle \alpha^2 \rangle + 2(C - D + \varepsilon)\langle \alpha^*\alpha \rangle + \varepsilon - 2D,$$

(14)

$$\frac{d}{dt}\langle \alpha^*\alpha \rangle = -2(B - A)\langle \alpha^*\alpha \rangle + (C - D + \varepsilon)(\langle \alpha^*\alpha \rangle + \langle \alpha^2 \rangle) + 2A.$$  

(15)

Based on Eq. (13), one can write

$$\frac{d}{dt}\alpha(t) = -(B - A)\alpha(t) + (C - D + \varepsilon)\alpha^*(t) + f(t),$$

(16)

where $f(t)$ is a noise force the properties of which remain to be determined. We note that Eq. (13) and the expectation value of Eq. (16) will have the same form provided that

$$\langle f(t) \rangle = 0.$$  

(17)

It is easy to see that

$$\frac{d}{dt}\langle \alpha^2(t) \rangle = 2\langle \alpha(t) \rangle \frac{d}{dt}\alpha(t)$$

(18)

and on substituting Eq. (16) into (18), we get

$$\frac{d}{dt}\langle \alpha^2(t) \rangle = -2(B - A)\langle \alpha^2(t) \rangle + 2\langle \alpha(t)f(t) \rangle + 2(C - D + \varepsilon)\langle \alpha^*\alpha(t) \rangle.$$  

(19)

It can also be verified in a similar manner that

$$\frac{d}{dt}\langle \alpha^*\alpha(t) \rangle = -2(B - A)\langle \alpha^*\alpha(t) \rangle + (C - D + \varepsilon)(\langle \alpha^*\alpha(t) \rangle + \langle \alpha^2(t) \rangle) + \langle \alpha(t)f^*(t) \rangle + \langle \alpha^*\alpha(t)f(t) \rangle.$$  

(20)

Comparison of Eqs. (14) and (19) as well as Eqs. (15) and (20) leads

$$\langle \alpha(t)f(t) \rangle = \frac{1}{2}(\varepsilon - 2D),$$

(21)
\[ \langle \alpha(t) f^*(t) \rangle + \langle \alpha^* f(t) \rangle = 2A. \] (22)

Furthermore, one can write a formal solution of Eq. (16) as

\[ \alpha(t) = \alpha(0)e^{-(B-A)t} + \int_0^t e^{-(B-A)(t-t')}[(C-D+\varepsilon)\alpha^*(t') + f(t')]dt'. \] (23)

Multiplying Eq. (23) on the left by \( f(t) \) and taking the expectation value, we have

\[ \langle \alpha(t) f(t) \rangle = \langle \alpha(0) f(t) \rangle e^{-(B-A)t} + \int_0^t e^{-(B-A)(t-t')}[(C-D+\varepsilon)\langle \alpha^*(t') f(t) \rangle + \langle f(t') f(t) \rangle]dt'. \] (24)

Assuming that the noise force at time \( t \) does not affect the cavity mode variables at earlier times and on taking into account (21), Eq. (24) can be put in the form

\[ \int_0^t e^{-(B-A)(t-t')} \langle f(t') f(t) \rangle dt' = \frac{1}{2}(\varepsilon - 2D). \] (25)

On the basis of this result, one can write [9]

\[ \langle f(t') f(t) \rangle = (\varepsilon - 2D)\delta(t-t'). \] (26)

It can also be established in a similar manner that

\[ \langle f(t) f^*(t') \rangle = 2A\delta(t-t'). \] (27)

We note that Eqs. (26) and (27) describe the correlation properties of the noise force \( f(t) \) associated with the normal ordering.

Now introducing a new variable defined by

\[ \alpha_\pm(t) = \alpha^*(t) \pm \alpha(t), \] (28)

we easily get with the help of (16) that

\[ \frac{d}{dt}\alpha_\pm(t) = -\lambda_\mp \alpha_\pm(t) + f^*(t) \pm f(t), \] (29)

where

\[ \lambda_\mp = (B-A) \mp (C-D+\varepsilon). \] (30)

The solution of Eq. (29) can be written as

\[ \alpha_\pm(t) = \alpha_\pm(0)e^{-\lambda_\mp t} + \int_0^t e^{-\lambda_\mp (t-t')} (f^*(t') \pm f(t'))dt'. \] (31)

It then follows that

\[ \alpha(t) = E_+(t)\alpha(0) + E_-(t)\alpha^*(0) + F(t), \] (32a)

in which

\[ E_\pm(t) = \frac{1}{2}(e^{-\lambda_- t} \pm e^{-\lambda_+ t}), \] (32b)

and

\[ F(t) = F_+(t) + F_-(t), \] (33a)

with

\[ F_\pm(t) = \frac{1}{2} \int_0^t e^{-\lambda_\pm (t-t')} (f^*(t') \pm f(t'))dt'. \] (33b)
III. QUADRATURE VARIANCE

In this section we seek to analyze the quadrature variance of the cavity and output modes.

A. Quadrature variance of the cavity mode

We define the quadrature operators for the cavity mode as

\[ \hat{a}^+ = \hat{a}^\dagger + \hat{a} \] (34)

and

\[ \hat{a}^- = i(\hat{a}^\dagger - \hat{a}). \] (35)

These quadrature operators satisfy the commutation relation \[ [\hat{a}^+, \hat{a}^-] = 2i. \] The variance of these quadrature operators is expressible in terms of c-number variables associated with the normal ordering as

\[ \Delta a^2 = 1 \pm \langle \alpha^\pm(t), \alpha^\pm(t) \rangle, \] (36)

where \( \alpha^\pm(t) \) is given by Eq. (28). Assuming the cavity mode to be initially in a vacuum state and taking into account (31) along with (17), we see that

\[ \langle \alpha^\pm(t) \rangle = 0. \] (37)

Thus in view of this result, Eq. (36) reduces to

\[ \Delta a^2 = 1 \pm \langle \alpha^2(t) \rangle. \] (38)

Furthermore, applying Eq. (29), one easily gets

\[ \frac{d}{dt} \langle \alpha^2(t) \rangle = -2\lambda_\mp \langle \alpha^2(t) \rangle + 2\langle \alpha(t) f(t) \rangle \pm 2\langle \alpha(t) f^*(t) \rangle. \] (39)

With the aid of Eq. (31) along with (26) and (27), we readily obtain

\[ \frac{d}{dt} \langle \alpha^2(t) \rangle = -2\lambda_\mp \langle \alpha^2(t) \rangle + 2(\varepsilon - 2D \pm 2A) \] (40)

and at steady state we have

\[ \langle \alpha^2(t) \rangle_{ss} = \frac{\varepsilon - 2D \pm 2A}{\lambda_\mp}. \] (41)

Hence on account of Eqs. (41), (30), and (7a)-(7h), the quadrature variance (38) takes at steady state the form

\[ \Delta a^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) e^{2r} + A(4 + \beta^2)}{2(\kappa - 2\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(4\beta - \beta^3)} \] (42a)

and

\[ \Delta a^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) e^{-2r} + 3A\beta^2}{2(\kappa + 2\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(4\beta + \beta^3)} \] (42b)

We next proceed to calculate the quadrature variance of the cavity mode at threshold. We note that Eq. (29) will not have a well-behaved solution if \( \lambda_\mp < 0 \). Hence we identify \( \lambda_\mp = 0 \) as the threshold condition. With the aid of Eq. (30) together with (7a)-(7h), the threshold condition can be written as

\[ \varepsilon = \frac{\kappa}{2} + \frac{A(2\beta - \beta^3)}{4(1 + \beta^2)(1 + \beta^2/4)}. \] (43)

Now introducing this into Eqs. (42a) and (42b), the quadrature variance at threshold takes the form

\[ \Delta a^2 \to \infty \] (44a)
\[ \Delta a^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4)e^{-2r} + 3A\beta^2}{4\kappa(1 + \beta^2)(1 + \beta^2/4) + 6A\beta}. \]  

(44b)

Fig. 2 indicates that the quadrature variance increases with \( \beta \) and decreases with the squeeze parameter \( r \). For \( r = 1, \kappa = 0.8, \) and \( A = 100 \), the minimum value of the quadrature variance is found to be 0.022 at \( \beta = 0.022 \). We immediately see that the intracavity squeezing is 97.8% below the vacuum level. We also note from Fig. 3 that the presence of both the parametric amplifier and the squeezed vacuum reservoir increases the intracavity squeezing over and above the squeezing achievable due to the coherently driven three-level laser \[2\]. Moreover, Fig. 3 clearly shows that the effect of the squeezed vacuum reservoir is significant for relatively small values of \( \beta \).

In the absence of the nonlinear crystal (NLC) the quantum optical system under consideration reduces to a coherently driven degenerate three-level laser coupled to a squeezed vacuum reservoir. The quadrature variance for this system takes upon setting \( \epsilon = \lambda\mu = 0 \) (with \( \mu \neq 0 \) in (42a) and (42b) the form

\[ \Delta a^2_{\pm} = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4)e^{-2r} + 3A\beta^2}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - \beta^3)}. \]  

(45a)

and

\[ \Delta a^2_{\mp} = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4)e^{-2r} + 3A\beta^2}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(2\beta + \beta^3)}. \]  

(45b)

As can be seen from Fig. 4 the degree of squeezing increases with the squeeze parameter \( r \) and decreases with \( \beta \). Fig. 5 also indicates that the presence of the squeezed vacuum reservoir increases the intracavity squeezing significantly for relatively small values of \( \beta \). For \( A=100, \kappa = 0.8, \) and \( r = 1.0, \) the minimum value of the quadrature variance given by (45b) turns out to be 0.035 and occurs at \( \beta = 0.023 \). We then note from this result that the intracavity squeezing is 96.5%.

### B. Quadrature variance of the output mode

The variance of the quadrature operators for the output mode, defined by

\[ \hat{a}_{+,\text{out}} = \hat{a}_{\text{out}}^\dagger + \hat{a}_{\text{out}} \]  

(46)

and

\[ \hat{a}_{-,\text{out}} = i(\hat{a}_{\text{out}}^\dagger - \hat{a}_{\text{out}}), \]  

(47)

can be expressed in terms of c-number variables associated with the normal ordering as

\[ \Delta a^2_{\pm,\text{out}} = 1 \pm \langle \alpha_{\pm,\text{out}}(t), \alpha_{\pm,\text{out}}(t) \rangle, \]  

(48)

in which

\[ \alpha_{\pm,\text{out}}(t) = \sqrt{\kappa}\alpha^*_\pm(t) - \alpha_{\pm,\text{in}}(t), \]  

(49)

and

\[ \alpha_{\pm,\text{in}}(t) = \frac{1}{\sqrt{\kappa}}(f_R^*(t) \pm f_R(t)), \]  

(50)

with \( f_R \) being the noise force associated with the squeezed vacuum reservoir. This noise force has the following correlation properties:

\[ \langle f_R(t) \rangle = 0, \]  

(51a)

\[ \langle f_R(t)f_R(t') \rangle = \kappa N\delta(t-t'), \]  

(51b)

\[ \langle f_R^*(t)f_R^*(t') \rangle = \langle f_R(t)f_R(t') \rangle = \kappa M\delta(t-t'). \]  

(51c)
In view of Eqs. (50) and (51a), one can write

\[ \langle \alpha_{\pm, in}(t) \rangle = 0. \] (52)

Thus on account of Eqs. (47) and (48), the quadrature variance (48) takes the form

\[ \Delta a_{\pm, out}^2 = 1 \pm \langle a_{\pm, out}^2(t) \rangle. \] (53)

Furthermore, with the aid of Eq. (49), we find

\[ \langle a_{\pm, out}^2(t) \rangle = \kappa \langle a_{\pm}^2(t) \rangle - 2\sqrt{\kappa} \langle \alpha_{\pm}(t) \alpha_{\pm, in}(t) \rangle + \langle a_{\pm, in}^2(t) \rangle. \] (54)

It can be readily verified that

\[ \langle \alpha_{\pm}(t) \alpha_{\pm, in}(t) \rangle = \sqrt{\kappa}(M \pm N) \] (55)

and

\[ \langle a_{\pm, in}^2(t) \rangle = 2(M \pm N). \] (56)

Upon substituting Eqs. (41), (55), and (56) into (54), we get

\[ \langle a_{\pm, out}^2(t) \rangle = \kappa \left[ \varepsilon - \frac{2D \pm 2A}{\lambda_F} \right] + 2(1 - \kappa)(M \pm N). \] (57)

Therefore, with the aid of this result and along with (7a)-(7h), the quadrature variance takes at steady state the form

\[ \Delta a_{\pm, out}^2 = 1 + \kappa \left[ \frac{2(1 + \beta^2)(1 + \beta^2/4)[2\varepsilon + \kappa(\varepsilon^2 - 1)] + A(4 - 2\beta + \beta^2 + \beta^3)}{2(\kappa - 2\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - \beta^3)} \right] + (1 - \kappa)(e^{2r} - 1) \] (58a)

and

\[ \Delta a_{\pm, out}^2 = 1 - \kappa \left[ \frac{2(1 + \beta^2)(1 + \beta^2/4)[2\varepsilon + \kappa(1 - e^{-2r})] + A(4\beta - 3\beta^2 + \beta^3)}{2(\kappa + 2\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(4\beta + \beta^3)} \right] - (1 - \kappa)(1 - e^{-2r}). \] (58b)

On account of Eq. (43) the quadrature variance reduces at threshold to

\[ \Delta a_{\pm, out}^2 \rightarrow \infty \] (59a)

and

\[ \Delta a_{\pm, out}^2 = 1 - \kappa \left[ \frac{2(1 + \beta^2)(1 + \beta^2/4)(2 - e^{-2r}) + A(6\beta - 3\beta^2)]}{4\kappa(1 + \beta^2)(1 + \beta^2/4) + 6A\beta} \right] - (1 - \kappa)(1 - e^{-2r}). \] (59b)

We note from Fig. 6 that the amount of squeezing of the output mode is less than that of the cavity mode for relatively small values of \( \beta \). The minimum value of the quadrature variance of the output mode is 0.045 at \( \beta = 0.022 \). This shows that the maximum squeezing of the output mode is 95.5%.

We next wish to consider the case for which the nonlinear crystal is removed from the cavity. To this end, upon setting \( \varepsilon = \lambda \mu = 0 \) (with \( \mu \neq 0 \)) in Eqs. (58a) and (58b), the quadrature variance for this case reduces to

\[ \Delta a_{\pm, out}^2 = 1 + \kappa \left[ \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4)(e^{2r} - 1) + A(4 - 2\beta + \beta^2 + \beta^3)}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - \beta^3)} \right] + (1 - \kappa)(e^{2r} - 1) \] (60a)

and

\[ \Delta a_{\pm, out}^2 = 1 - \kappa \left[ \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4)(1 - e^{-2r}) + A(4\beta - 3\beta^2 + \beta^3)}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(4\beta + \beta^3)} \right] - (1 - \kappa)(1 - e^{-2r}). \] (60b)

We notice that Eqs. (60a) and (60b) represent the quadrature variance of the output mode of a coherently driven three-level laser coupled to a squeezed vacuum reservoir. In Fig. 7, we plot Eqs. (55) and (60b) versus \( \beta \). These plots also show that the degree of squeezing of the output mode is less than that of the cavity mode. We have found that the minimum value of the quadrature variance of the output mode to be 0.055 at \( \beta = 0.023 \). This indicates that the maximum squeezing of the output mode for this case is 94.5% below the vacuum level. We observe that the degree of squeezing of the of the output mode in the presence of the NLC is greater by 1% than that without the NLC.
IV. SQUEEZING SPECTRUM

In this section we calculate the squeezing spectrum of the output mode. The squeezing spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as

\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2\Re \int_0^\infty \langle \alpha_\pm^{\text{out}}(t), \alpha_\pm^{\text{in}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau. \]  

(61)

Thus on account of Eqs. (37) and (52), the squeezing spectrum can be put in the form

\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2\Re \int_0^\infty \langle \alpha_\pm^{\text{out}}(t) \alpha_\pm^{\text{out}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau, \]  

(62)

so that introducing Eq. (49), we get

\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2\Re \int_0^\infty \langle \alpha_\pm(t) \alpha_\pm(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau \pm 2\sqrt{\kappa} \Re \int_0^\infty \langle \alpha_\pm(t) \alpha_\pm^{\text{in}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau \]

\[ + 2\sqrt{\kappa} \Re \int_0^\infty \langle \alpha_\pm^{\text{in}}(t) \alpha_\pm(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau \pm 2\Re \int_0^\infty \langle \alpha_\pm^{\text{in}}(t) \alpha_\pm^{\text{in}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau. \]  

(63)

Furthermore, the solution of Eq. (49) can also be written as

\[ \alpha_{\pm}(t + \tau) = \alpha_{\pm}(t)e^{-\lambda_{\pm}\tau} + \int_0^\tau e^{-\lambda_{\pm}(\tau-\tau')}(f^*(t+\tau') \pm f(t+\tau'))d\tau'. \]  

(64)

Now applying this equation along with the quantum regression theorem, one can establish that

\[ \langle \alpha_\pm(t) \alpha_\pm(t + \tau) \rangle_{ss} = \langle \alpha_\pm^2(t) \rangle_{ss} e^{-\lambda_{\pm}\tau}, \]  

(65)

\[ \langle \alpha_\pm(t) \alpha_\pm^{\text{in}}(t + \tau) \rangle_{ss} = 0, \]  

(66)

\[ \langle \alpha_\pm^{\text{in}}(t) \alpha_\pm(t + \tau) \rangle_{ss} = 2\sqrt{\kappa}(M \pm N)e^{-\lambda_{\pm}\tau}, \]  

(67)

\[ \langle \alpha_\pm^{\text{in}}(t) \alpha_\pm^{\text{in}}(t + \tau) \rangle_{ss} = 2(M \pm N)\delta(\tau). \]  

(68)

In view of these results, the squeezing spectrum takes the form

\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2\kappa \langle \alpha_\pm^2(t) \rangle_{ss} \Re \int_0^\infty e^{-(\lambda_{\pm} - i\omega)\tau} d\tau \pm 4\kappa(M \pm N)\Re \int_0^\infty e^{-(\lambda_{\pm} - i\omega)\tau} d\tau \]

\[ \pm 4(M \pm N)\Re \int_0^\infty e^{-i\omega\delta(\tau)} d\tau. \]  

(69)

Thus upon performing the integration and using (61), we easily find

\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm \frac{2\kappa(\varepsilon - 2D \pm 2A)}{\lambda_{\pm}^2 + \omega^2} \pm 2(M \pm N). \]  

(70)

Employing Eqs. (30) and (7a)-(7g), the squeezing spectrum can be written as

\[ S_{\pm}^{\text{out}}(\omega) = e^{2r} \left[ 1 - \frac{2\kappa\varepsilon + A(4\beta^2 + \beta)(1 + \beta^2)(4 + \beta^2) + \kappa A(4\beta^2 + \beta^2) e^{-2r}}{2\left(\frac{\lambda_{\pm}^2}{2} - \varepsilon + \kappa A(4\beta^2 + \beta^2)(1 + \beta^2)(4 + \beta^2)\right)^2 + \omega^2} \right], \]  

(71a)

and

\[ S_{\pm}^{\text{out}}(\omega) = e^{-2r} \left[ 1 - \frac{2\kappa\varepsilon + A(4\beta^2 + \beta)(1 + \beta^2)(4 + \beta^2) - \kappa A(4\beta^2 + \beta^2) e^{2r}}{2\left(\frac{\lambda_{\pm}^2}{2} + \varepsilon + \kappa A(4\beta^2 + \beta^2)(1 + \beta^2)(4 + \beta^2)\right)^2 + \omega^2} \right], \]  

(71b)
so that on account of \( (43) \) the squeezing spectrum reduces at threshold to

\[
S^\text{out}_+(\omega) = e^{2r} \left[ 1 + \frac{\kappa^2 + \frac{\kappa A(4 + \beta^2) e^{-2r}}{\omega^2}}{\omega^2} \right] \tag{72a}
\]

and

\[
S^\text{out}_-(\omega) = e^{-2r} \left[ 1 - \frac{\kappa^2 + \frac{3\kappa A\beta^2 e^{2r}}{2(1 + \beta^2)(1 + \beta^2/4)}}{\omega^2} \right]. \tag{72b}
\]

In Fig. 8, we plot the squeezing spectrum [Eq. (72b)] versus \( \omega \) and \( \beta \). This plot indicates that perfect squeezing is attainable for \( \omega = 0 \), \( \beta = 0 \) and for any values of \( A \), \( r \), and \( \kappa \). To see the effect of the squeezed vacuum reservoir on the squeezing spectrum, we plot Eq. (72b) versus \( \beta \) for \( \omega = 0 \), \( \kappa = 0.8 \), \( A = 100 \), and for different values of the squeeze parameter \( r \). It is easy to see from Fig. 9 that the presence of the squeezed vacuum reservoir increases the degree of squeezing significantly for \( \beta \neq 0 \).

It is interesting to consider once more the case for which the nonlinear crystal is removed from the laser cavity. Thus upon setting \( \varepsilon = \lambda \mu = 0 \) (with \( \mu \neq 0 \)) in Eqs. (71a) and (71b), we obtain the squeezing spectrum to be

\[
S^\text{out}_+(\omega) = e^{2r} \left[ 1 + \frac{2r[A(2^2 - \beta^2) - \frac{A}{2(2^2 + \beta^2)(1 + \beta^2/4)}]}{\omega^2} \right] \tag{73a}
\]

and

\[
S^\text{out}_-(\omega) = e^{-2r} \left[ 1 - \frac{2\kappa^2[A(2^2 - \beta^2) - \frac{A}{2(2^2 + \beta^2)(1 + \beta^2/4)}]}{\omega^2} \right]. \tag{73b}
\]

Expressions (73a) and (73b) represent the squeezing spectrum of a coherently driven three-level laser coupled to a squeezed vacuum reservoir. In Fig. 10, we plot Eq. (73b) versus \( \beta \) for \( \omega = 0 \), \( \kappa = 0.8 \), \( A = 100 \), and for different values of the squeeze parameter \( r \). These plots show that the degree of squeezing of the output mode increases with the squeeze parameter \( r \) and almost perfect squeezing occurs for small values of \( \beta \).

## V. PHOTON STATISTICS

We now proceed to calculate the mean photon number of the cavity and output modes. The mean photon number of the cavity mode can be written as

\[
\bar{n} = \langle a^\dagger(t)a(t) \rangle. \tag{74}
\]

Employing (52a) and its complex conjugate and assuming that the cavity mode is initially in a vacuum state, we get

\[
\langle a^\dagger(t)a(t) \rangle = \langle F^\dagger(t)F(t) \rangle \tag{75}
\]

It can be readily established using Eqs. (33a) and (33b) along with (20) and (27) that

\[
\langle F^\dagger(t)F(t) \rangle = \frac{2A - 2D + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_- t}) + \frac{2A + 2D - \varepsilon}{4\lambda_+} (1 - e^{-2\lambda_+ t}). \tag{76}
\]

On account of Eqs. (74), (75), (30), (75) and (76), the mean photon number turns out to be

\[
\bar{n} = \frac{[2\varepsilon + \kappa(e^{2r} - 1)](1 + \beta^2)(1 + \beta^2/4) + A(2 - \beta + \beta^2/2 + \beta^3/2)(1 - e^{-2\lambda_- t})}{4(1 + \beta^2)(1 + \beta^2/4)(\kappa - 2\varepsilon) + 2A(2\beta - \beta^3)} - \frac{[2\varepsilon + \kappa(e^{-2r} - 1)](1 + \beta^2)(1 + \beta^2/4) + A(2\beta - 3\beta^2/2 + \beta^3/2)(1 - e^{-2\lambda_- t})}{4(1 + \beta^2)(1 + \beta^2/4)(\kappa + 2\varepsilon) + 2A(4\beta + \beta^3)}. \tag{77}
\]

Fig. 11 shows that the presence of both the parametric amplifier and the squeezed vacuum reservoir increase the mean photon number significantly for relatively small values of \( \beta \). On the other hand, it can be readily established that the mean photon number of the output mode is expressible as

\[
\bar{n}_\text{out} = \kappa \bar{n} + (1 - \kappa)N, \tag{78}
\]

where \( \bar{n} \) is given by Eq. (77). Fig. 12 shows that the mean photon number of the cavity mode is greater than that of the output mode.
VI. POWER SPECTRUM

We finally calculate the power spectrum of the cavity and output modes.

A. Power spectrum of the cavity mode

The power spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as

\[ S(\omega) = 2Re \int_0^\infty \langle \alpha^*(t) \alpha(t+\tau) \rangle_{ss} e^{i\omega \tau} d\tau. \]  

(79)

The solution of Eq. (24) can also be written as

\[ \alpha(t+\tau) = E_+ (\tau) \alpha(t) + E_- (\tau) \alpha^*(t) + F(t+\tau), \]

(80a)

in which

\[ E_\pm (\tau) = \frac{1}{2} (e^{-\lambda_\pm \tau} \pm e^{-\lambda_+ \tau}) \]

(80b)

and

\[ F(t+\tau) = F_+ (t+\tau) + F_- (t+\tau), \]

(81a)

with

\[ F_\pm (t+\tau) = \frac{1}{2} \int_0^t e^{-\lambda_\mp (\tau-\tau')} (f^*(t+\tau') \pm f(t+\tau')) d\tau'. \]

(81b)

Upon multiplying Eq. (80a) by \( \alpha^* (t) \) and taking the expectation value of the resulting expression, we have

\[ \langle \alpha^*(t) \alpha(t+\tau) \rangle_{ss} = E_+ \langle \alpha^*(t) \alpha(t) \rangle_{ss} + E_- \langle \alpha^2 (t) \rangle_{ss}. \]

(82)

It can be readily established using Eqs. (32a), (32b), (33a), and (33b) together with the properties of the noise force

\[ f(t) \]

that the halfwidth decreases with \( \epsilon \).

We realize that halfwidth of the two Lorentzians does not depend on the squeeze parameter. These plots show that the halfwidth decreases with \( \epsilon \). When the value of \( \epsilon \) increases from 0.2 to 0.3 the halfwidth decreases from 0.80 to 0.75.
B. Power spectrum of the output mode

The power spectrum of the output mode is expressible in terms of c-number variables associated with the normal ordering as

\[ S^{\text{out}}(\omega) = 2Re \int_0^{\infty} \langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t + \tau) \rangle_{ss} e^{i\omega\tau} d\tau, \]  

(87)

where

\[ \alpha_{\text{out}}(t) = \sqrt{\kappa} \alpha(t) - \alpha_{\text{in}}(t). \]  

(88)

Now with the help of Eq. (88), we can write

\[ \langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t + \tau) \rangle_{ss} = \kappa(\alpha^*(t)\alpha(t + \tau))_{ss} - \sqrt{\kappa}(\alpha^*(t)\alpha_{\text{in}}(t + \tau))_{ss} 
- \sqrt{\kappa}(\alpha_{\text{in}}^*(t)\alpha(t + \tau))_{ss} + \langle \alpha_{\text{in}}^*(t)\alpha_{\text{in}}(t + \tau) \rangle_{ss}. \]  

(89)

It can be readily verified that

\[ \sqrt{\kappa}(\alpha^*(t)\alpha_{\text{in}}(t + \tau))_{ss} = 0, \]  

(90)

\[ \sqrt{\kappa}(\alpha_{\text{in}}^*(t)\alpha(t + \tau))_{ss} = \frac{\kappa}{2} [(M + N)e^{-\lambda_- \tau} + (M - N)e^{-\lambda_+ \tau}], \]  

(91)

\[ \langle \alpha_{\text{in}}^*(t)\alpha_{\text{in}}(t + \tau) \rangle_{ss} = N\delta(\tau). \]  

(92)

Thus on account of Eqs. (89), (90), (91), and (92), Eq. (87) takes the form

\[ \langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t + \tau) \rangle_{ss} = \kappa \left[ \frac{2(A - D) + \varepsilon}{4\lambda_-} - \frac{M + N}{2} \right] e^{-\lambda_- \tau} 
+ \kappa \left[ \frac{2(A + D) - \varepsilon}{4\lambda_+} - \frac{M - N}{2} \right] e^{-\lambda_+ \tau} + N\delta(\tau). \]  

(93)

On substituting this result into (87) and carrying out the integration, we get

\[ S^{\text{out}}(\omega) = \frac{A - D + \frac{\varepsilon}{2} - \lambda_-(M + N)}{\lambda_-^2 + \omega^2} + \frac{\kappa[A + D - \frac{\varepsilon}{2} - \lambda_+(M - N)]}{\lambda_+^2 + \omega^2} + N. \]  

(94)

With the aid of Eqs. (90) and (91), we find the power spectrum of the output mode to be

\[ S^{\text{out}}(\omega) = \frac{\kappa \left[ \left( \frac{A}{2} - \frac{A(2\beta - \beta^2)}{(1 + \beta^2)(8 + 2\beta^2)} \right) e^{2r} + \frac{A(4 + \beta^2)}{(1 + \beta^2)(8 + 2\beta^2)} \right]}{\left( \frac{\varepsilon}{2} - \varepsilon + \frac{A(2\beta - \beta^2)}{(1 + \beta^2)(4 + \beta^2)} \right)^2 + \omega^2} 
+ \frac{\kappa \left[ - \left( \frac{A}{2} + \frac{A(4\beta + \beta^2)}{(1 + \beta^2)(8 + 2\beta^2)} \right) e^{-2r} + \frac{3A\beta^2}{(1 + \beta^2)(8 + 2\beta^2)} \right]}{\left( \frac{\varepsilon}{2} + \varepsilon + \frac{A(4\beta + \beta^2)}{(1 + \beta^2)(4 + \beta^2)} \right)^2 + \omega^2} + \sinh^2 r. \]  

(95)

Expression (95) indicates that the power spectrum consists of two Lorentzians and a flat spectrum. Equations. (89) and (91) show that the width of the Lorentzians in the power spectrum of the cavity and the output modes are the same.

VII. CONCLUSION

We have considered a degenerate three-level laser containing a parametric amplifier and coupled to a squeezed vacuum reservoir. Applying the pertinent master equation, we have obtained stochastic differential equations associated with the normal ordering. Using the solutions of these equations, we have calculated the quadrature variance, the
mean photon number, and the power spectrum of the cavity and output modes of the system under consideration. We have also determined the squeezing spectrum of the output mode.

We have seen that the effect of the squeezed vacuum reservoir and the parametric amplifier is to increase the mean photon number and the degree of squeezing of the cavity and output modes. We have also found that the squeezed vacuum reservoir increases the degree of squeezing significantly over and above the squeezing attainable from a degenerate three-level laser with a parametric amplifier. It turns out that the degree of squeezing of the cavity mode is greater than that of the output mode for certain values of \( \beta \). On the other hand, the plots of the squeezing spectrum at threshold show that there is perfect squeezing of the output mode for \( \beta = \omega = 0 \) and for any values of \( A, \kappa \), and the squeeze parameter \( r \). We have also found that the mean photon number of the cavity mode is greater that of the output mode. Moreover, we have seen that the presence of the parametric amplifier leads to a decrease in the width of the power spectrum while the squeezed vacuum reservoir has no effect on the width.

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FIG. 1: A degenerate three-level laser with a parametric amplifier and a squeezed vacuum reservoir.
FIG. 2: A plot of the quadrature variance [Eq. (45b)] versus $\beta$ and $r$ for $\kappa = 0.8$ and $A = 100$.

FIG. 3: Plots of the quadrature variance versus $\beta$ for $\kappa = 0.8$ and $A = 100$, for $r = 0$ and $\epsilon = 0$ [Eq. (45b)] (dashed curve), for $r = 0$ in the presence of the parametric amplifier at threshold [Eq. (44b)] (solid curve), and for $r = 1.0$ and in the presence of the parametric amplifier at threshold [Eq. (44b)] (dotted curve).

FIG. 4: A plot of the quadrature variance [Eq. (45b)] versus $r$ and $\beta$ for $\kappa = 0.8$ and $A=100$. 
FIG. 5: Plots of the quadrature variance [Eq. (45b)] versus $\beta$ for $\kappa = 0.8$ and $A = 100$ in the absence of the squeezed vacuum reservoir (solid curve) and in the presence of the squeezed vacuum reservoir with $r = 1.0$ (dashed curve).

FIG. 6: (a) A plot of the quadrature variance of the output mode [Eq. (50b)] versus $\beta$ for $\kappa = 0.8$, $A = 100$, and $r = 1$. (b) A plot of quadrature variance of the cavity mode [Eq. (44b)] versus $\beta$ for $\kappa = 0.8$, $A=100$, and $r = 1$.

FIG. 7: (a) A plot of the quadrature variance of the output mode [Eq. (60b)] versus $\beta$ for $\kappa = 0.8$, $A = 100$, and $r = 1$. (b) A plot of quadrature variance of the cavity mode [Eq. (45b)] versus $\beta$ for $\kappa = 0.8$, $A=100$, and $r = 1$. 
FIG. 8: A plot of the squeezing spectrum [Eq. (72b)] versus $\beta$ and $\omega$ for $\kappa = 0.8$ and $A = 100$.

FIG. 9: Plots of the squeezing spectrum [Eq. (72b)] versus $\beta$ for $\omega = 0$, $\kappa = 0.8$, $A = 100$, and for different values of the squeeze parameter $r$.

FIG. 10: Plots of the squeezing spectrum [(73b)] versus $\beta$ for $\omega = 0$, $\kappa = 0.8$, $A = 100$, and for different values of the squeeze parameter $r$. 
FIG. 11: Plots of the mean photon number $\langle \hat{n} \rangle$ at steady state versus $\beta$ for $\kappa = 0.8$, $A = 25$ and (a) for $r = 0$ and $\varepsilon = \lambda \mu = 0$ with $\mu \neq 0$, (b) for $r = 0$ and $\varepsilon = 0.3$, (c) for $r = 1.0$ and $\varepsilon = 0.3$.

FIG. 12: (a) A plot of the mean photon number $\langle \hat{n} \rangle$ at steady state versus $\beta$ for $\kappa = 0.8$, $A = 25$, $r = 1.0$, and for $\varepsilon = 0.3$. (b) A plot of the mean photon number $\langle \hat{n} \rangle$ at steady state versus $\beta$ for $\kappa = 0.8$, $A = 25$, $r = 1.0$, and for $\varepsilon = 0.3$.

FIG. 13: Plots of the power spectrum [Eq. (86)] versus $\omega$ for $\Lambda = 100$, $\beta = 0.01$, $r = 1.0$, $\kappa = 0.8$, and for different values of $\varepsilon$. 