On exact solutions of Maxwell-Bloch system for two-level medium with degeneracy

S. B. Leble*,
Wydzial Fizyki Technicznej i Matematyki Stosowanej
Politechnika Gdan’ska, ul. G.Narutowicza, 11/12 80-952, Gdan’sk-Wrzeszcz, Poland,
email leble@mifgate.pg.gda.pl

N.V.Ustinov
Kaliningrad State University, Theoretical Physics Department,
Al.Nevsky st.,14, 236041 Kaliningrad
email ustinov.leble@theor.phys.ksu.kern.ru
*permanent adress

Abstract

Maxwell–Bloch system describing the resonant propagation of electromagnetic pulses in both two–level media with degeneracy in angle moment projection and three–level media with equal oscillator forces is considered. The inhomogeneous broadening of energy levels is accounted. Binary Darboux Transformation generating the solutions of the system is constructed. Pulses corresponding to the transition between levels with largest population difference are shown to be stable. The solution describing the propagation of pulses in the medium exited by the periodic wave is obtained. The hierarchy of infinitesimal symmetries is obtained by means of Darboux transformation.
1 Introduction

Investigations of laser emission interaction with resonant media sufficiently promoted the understanding and mathematical description of this phenomena. The important kick for it gave the invention of the self-induced transparency (SIT) phenomena that is a passage of a powerful and ultrashort pulse without loss of a form and energy at a resonant medium. The group velocity of such pulses depends on their amplitude and duration: shorter pulse has larger velocity. It was found that while the propagation of pulses of different duration there may be situation when a quick pulse reaches more slow one, after "collision" both go out without change of their form and velocities only phase shifts appear. From mathematical point of view this wonderful stability of SIT pulses is the corollary of the complete integrability of Maxwell-Bloch (MB) equations that describe SIT phenomenon by two-level medium approximation with nondegenerate levels, recently developed for the case of two-photon processes. The SIT pulses correspond to soliton solutions of the equations and their collision properties reflect the asymptotic properties of multi- soliton states with respect to one-soliton ones.

For real media the energy levels are usually degenerate that contribute to peculiarities of polarized pulses propagation. In the Ref. the integrability of MB equations for arbitrary polarization of light being resonant with quantum transitions $j_b = 0 \rightarrow j_a = 1$ have been established. It was shown that solitons with circular polarization are stable.

This problems of interest relates to the interaction of many-frequency laser pulses with a resonant many-level medium. For example such many-frequency action widen possibilities of isotop dividing or simulation of chemical reactions as well as spectroscopy investigations. The peculiarities of coherent pulses that propagate in many-level media may be used in the frequency transformation effective.

Let’s consider a medium that may be described in the two-level atoms approximation with energy levels $E_{a,b}$ that are degenerate by projections $m$ and $\mu$ of total momenta $j_{a,b}$. Let plane electromagnetic wave propagates along z and its electric field has the form

$$E = E_0 \exp[i(kz - \omega t)] + c.c.$$  \hspace{1cm} (1.1)

where $t$ is time. The carrier frequency $\omega = kc$ of the pulse is close to $\omega_a = 2\pi(E_b - E_a)/\hbar$ of the atom transition $j_a \rightarrow j_b$.

The evolution of the pulse (1) in the semiclassical approach and coherent approximation is described by generalized MB equations that may be represented in the dimensionless form as

$$\epsilon_{q,\zeta} = \sum_{\mu,m} R_{\mu,m} < J^q_{\mu,m} >$$

$$R_{\mu,m;\tau} = i \sum_q (\epsilon_q \sum_{m'} J^q_{\mu,m'} R_{m',m} - \sum_{\mu'} R_{\mu,\mu'} J^q_{\mu',m})$$

$$R_{m,m';\tau} = i \sum_{q,m} (\epsilon_q^* J^q_{\mu,m} R_{\mu,m'} - \epsilon_q R_{\mu,\mu'} J^q_{\mu,m'})$$

$$R_{\mu,\mu';\tau} = i \sum_{q,m} (\epsilon_q J^q_{\mu,m} R_{\mu,m'} - \epsilon_q^* R_{\mu,\mu'} J^q_{\mu',m})$$
In those equations the amplitude $E_o$ of a light pulse, coefficients of the density matrix $\rho_{\mu,\mu}$, are connected with the functions $\epsilon_q, R_{\mu,\mu'}, R_{m,m'}, R_{\mu,m}$:

$$E_q = \hbar \epsilon_q / 2\pi t_o d,$$

$$\rho_{\mu,\mu'} = N_0 f(\eta) R_{\mu,\mu'},$$

$$\rho_{m,m'} = N_0 f(\eta) R_{m,m'},$$

$$\rho_{\mu,m} = N_0 f(\eta) R_{\mu,m} \exp[i(kz - \omega t)]$$

where $d$ is the reduced dipole moment of the transition $j_b - j_a$, $N_0$ - density of resonant particles, $t_o = (3\hbar / 2\pi \omega d^2 N_0)^{1/2}$ is time dimension constant. $\Delta = (\omega - \omega_o) t_o$ - resonance shift. $\tau = (t - z/c) / t_o$, $\zeta = z / ct_o$.

$$J^{\mu,m}_q = (-1)^{j_b-m} \begin{bmatrix} j_a & 1 & j_b \\ -m & q & \mu \end{bmatrix} 3^{1/2}$$

by the index $q$ the spherical components of the vector $\epsilon$ are labelled.

Let’s consider the transition $j_b = 0 - j_a = 1$ and introduce

$$e_+ = -i \epsilon_{+1}, e_- = -i \epsilon_{-1};$$

$$n_b = R_{\mu\mu}, |\mu=\mu'=0,\nu=-1;$$

$$n_{a+} = R_{mm'}|m=m'=1,\nu_a = R_{mm'}|m=m'=-1,$$

$$\nu_a = R_{mm'}|m=m'=-1,\nu_+ = R_{\mu\mu}|\mu=0,m=1,\nu_- = R_{\mu m}|\mu=0, m=-1$$

(1.2)

The quantization axis is choosen along the light propagation direction (axis $\zeta$).

As the interaction of a quantum system with the transverse electromagnetic waves is accompanied by transitions with angular momentum change (-1 or +1), the MB system may be simplified.

$$e_{-\zeta} = - < \nu_- >$$

$$e_{+\zeta} = - < \nu_+ >$$

$$n_{a-,\tau} = -\nu_- e^*_+ - \nu^*_+ e_-$$

$$n_{a+,\tau} = - \nu_+ e^*_+ - \nu^*_+ e_+$$

$$n_{b,\tau} = \nu_+ e^*_+ - \nu^*_+ e_+ + \nu_- e^*_+ + \nu^*_+ e_-$$

(1.3b)

$$\nu_{-\tau} = -i(\eta - \Delta)\nu_- + (n_{a-} - n_b) e_- + \nu_a e_+$$

$$\nu_{+\tau} = -i(\eta - \Delta)\nu_+ + (n_{a+} - n_b) e_+ + \nu_a e_-$$

$$\nu_{a,\tau} = - \nu_+ e^*_+ - \nu^*_+ e_+$$

(1.3c)

Here the angle brackets denote the inhomogeneous broadening averaging with normalized partition function $f(\eta)$

$$< F > = \int_{-\infty}^\infty f(\eta) F d\eta.$$
If the resonant atoms ensemble is in pure state, one may rest only five equations in the system (1.3).

\[
e_{-\zeta} = - < a_3 a_1^* > \\
e_{+\zeta} = - < a_3 a_2^* > \\
a_{1,\tau} = i(\eta - \Delta) a_1/2 - a_3 e_- \\
a_{2,\tau} = i(\eta - \Delta) a_2/2 - a_3 e_+ \\
a_{3,\tau} = i(\eta - \Delta) a_3/2 + a_1 e_- + a_2 e_+ 
\]

\begin{equation}
(1.4a)
\end{equation}

where \( a_{1,2,3} \) are amplitudes of the probability density (population numbers) of the lower and two upper levels.

The same system of nonlinear equations describes the resonant propagation of two-frequency radiation with equal shifts from resonance in three-level medium where one of the transitions is forbidden and the oscillator forces of the rest two transitions are equal.

\[2\text{ Zero curvature representation, reductions of potentials and Darboux transformation}\]

The Maxwell - Bloch equations (1.2) represents a complicated system that is not solved analytically in the case of arbitrary angular momenta. However in the case of resonant transitions \( j_b \rightarrow j_a = 1; j_b \rightarrow j_a = 0 \) the equations (1.2) may be presented as the compatibility condition of some linear system (zero curvature representation) with a spectral parameter (SP) \( \lambda \). For the case of equations (1.3) it is the following system

\[
\psi_{\tau} = V \psi \\\n\psi_{\zeta} = < \alpha(\lambda) A > \psi
\]

\begin{equation}
(2.1)
\end{equation}

where \( V = U - \lambda J, J = \text{diag}\{1,1,-1\}, \alpha(\lambda) = (2\lambda + i(\eta - \Delta))^{-1}, \)

\[
U = \begin{pmatrix}
0 & 0 & -e_-^* \\
0 & 0 & -e_+^* \\
e_- & e_+ & 0
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
n_{a-} & \nu_a & \nu_a^* & 0 \\
\nu_a^* & n_{a+} & \nu_a^* & 0 \\
\nu_- & \nu_+ & n_b & 0
\end{pmatrix}
\]

From the obvious equality \( \psi_{\zeta\tau} = \psi_{\zeta\tau} \) we obtain the system

\[
U_{\zeta} = [J, < A >]/2, \\
A_{\tau} = [U, A] - i(\eta - \Delta)[A, J]/2,
\]

\begin{equation}
(2.2)
\end{equation}

from which it follows that the elements of U and A matrices should satisfy MB system (1.3). If matrix elements of A are representable in the form of \( A_{kj} = a_k a_j^*(k,j = 1,2,3) \) (the pure quantum states) then from (1.3) the system (1.4) follows.
The equations (1.3,4) appear as the compatibility conditions of the conjugate Lax pair as well

\[ \xi_x = -\xi W \]
\[ \xi_z = -\xi <\alpha(\kappa)A> \]  

where \( W = U - \kappa J \).

As the matrices \( U \) and \( A \) satisfy the following reduction

\[
\begin{cases}
U + U^+ = 0 \\
A - A^+ = 0,
\end{cases}
\]

one may check that there in the space of matrix solutions of ZS equations exists the automorphism in the sense of the Ref. 7,

\[
\Psi(\lambda) \rightarrow [\Psi(-\lambda^*)^+]^{-1} \in \Psi(\lambda).
\]

By the way there is the coupling between the WF spaces of right and conjugate (left) zero curvature representation

\[
\kappa = -\lambda^*, \xi = \psi^+
\]

Such functions are referred as coupled by an automorphism.

The following statement is valid: The equations (1.3) are the corollary of (2.2) when the additional condition (2.4) is posed.

The DT technique that is developed in the Refs. 8,9 may be applied for the construction of solutions hierarchy for the MB system (1.3). We specialize the scheme here. Let \( \phi \) and \( \chi \) are functions of the right and left (conjugate) problems (2.1,3). Spectral parameters (SP) are \( \mu \) and \( \nu \) correspondingly. We also suppose that those functions are connected by the automorphism and

\[
\nu = -\mu^*, \chi = \phi^*.
\]

The transformed WFs by the binary\(^8\) DT

\[
\psi[1] = [1 - (\mu + \mu^*)P/(\lambda + \mu^*)]\psi
\]
\[\xi[1] = \xi[1 - (\mu + \mu^*)P/(\kappa - \mu)] \]  

are also the solutions of right and conjugate ZS zero curvature representations with new potentials \( V[1] = U[1] - \lambda J, W[1] = U[1] - \kappa J \) and

\[
A[1] = A - 2(\mu + \mu^*)[\alpha(\mu)AP + \alpha(\mu)^*PA - (\alpha(\mu)A + \alpha(\mu)^*)PA],
\]

where

\[
U[1] = U - (\mu + \mu^*)[J, P].
\]

The matrix \( P \) at the equations (2.6) is defined in analogy with Ref. 8 by

\[
P_{kj} = \phi_k\phi^*_j/\langle\phi^+, \phi\rangle
\]

As \( P^+ = P \) the reduction (2.4) is conserved. It means that elements of matrices \( U[1] \) and \( A[1] \) give new solutions of MB equations (1.3). For the transformed potential coefficients that have the sense of dimensionless electromagnetic field components the following expressions are valid

\[
e_-[1] = e_- + 2(\mu + \mu^*)\phi_0\phi^*_0/\langle\phi^+, \phi\rangle
\]
\[ e_+[1] = e_+ + 2(\mu + \mu^*)\phi_3\phi_2^*/(\phi^+, \phi) \] (2.7b)

If the initial state of the quantum system was pure the matrix \( A_{kj} = a_k a_j^* \) and the result of the transformation is pure as well with

\[ a[1] = a - 2(\mu + \mu^*)\alpha(\mu)^* Pa, \]

the normalization is conserved too.

As by the map (2.6) the transforms of WF's coupled by an authomorphism should also be connected by \( \xi[1] = \psi[1]^+ \) the reduction (2.4) will be valid by iterations of BDT. Final result may be expressed via \( \phi^{(q)} \) and \( \chi^{(q)} \) that are WF's of initial zero curvature representation to be coupled by authomorphisms in pairs\(^9\).

### 3 Solutions over the zero and periodic backgrounds.

In this section the construction of solutions of MB (and NS) systems by the transformation (2.6,7) will be considered.

**3.1** Starting from zero functions \( e_- = e_+ = 0, n_{a_+,-} = \nu_a = \nu_{+,-} = 0, \) but \( n_b = 1 \) we obtain after first iteration the one-soliton solution that generalize \( 2\pi \) pulse of two-state system. This solutions have been built in the paper\(^5\) within the IST method. However a construction of two-soliton solution met difficulties and the analysis of soliton collisions was made by asymptotic methods\(^6\). By the way in the Ref. 10 the algebraic approach that have features of Bäcklund and "Dressing" methods was developed and the breather two-soliton (with equal velocities) solution have been found.

After a second iteration preserving the reduction (the WF have SP coupled by \( \mu_{(2)} = \mu_{(1)}^* \)) we obtain the solution of MB system with potential elements

\[ e_-[2] = -2(a_1 c_1^* \Delta_2 \exp[-n\eta] - a_2 c_2^* \Delta_1 \exp[n\eta] - a_1 c_2^* \Delta^* - a_2 c_1^* \Delta)/\Delta[2] \]
\[ e_+[2] = -2(b_1 c_1^* \Delta_2 \exp[-n\eta] - b_2 c_2^* \Delta_1 \exp[n\eta] - b_1 c_2^* \Delta^* - b_2 c_1^* \Delta)/\Delta[2] \]
\[ \Delta[2] = \Delta_1 \Delta_2 - |\Delta|^2 \]
\[ \Delta_1 = [(|a_1|^2 + |b_1|^2) \exp[-\vartheta] + |c_1|^2 \exp[\vartheta]]/(2\mu_R) \]
\[ \Delta_2 = [(|a_2|^2 + |b_2|^2) \exp[-\vartheta] + |c_2|^2 \exp[\vartheta]]/(2\mu_R) \]
\[ \Delta = [(a_1 a_2^* + b_1 b_2^*) \exp[-\vartheta - i\theta] + c_1 c_2^* \exp[\vartheta + i\theta]]/(2\mu) \]

where

\[ \vartheta = 2\mu_R(\tau + <|\alpha(\mu)|^2> \xi), \]
\[ \theta = 2\mu_I \tau - (2\mu_I + \eta - \Delta) |\alpha(\mu)|^2 > \xi, \]

where \( \mu = \mu_R + \mu_I, a_1, 2, b_1, 2, c_{1,2} \) are complex constants. The transformation of such a structure we may also name the binary one\(^{1,8}\). This solution generalize the solution from Ref. 10. It is characterized by eight real parameters: \( \mu_R, \mu_I \), distance between pulse centers, electric field components ratio of the pulses and phase shifts of the same component of both pulses. At Ref. 10 the solution contain four parameters with fixed amplitude and phase shift ratios.
3.2 Let us consider the case of arbitrary level populations ("nonzero dipole temperature" of the medium): \( n_{a+} n_{a-} n_b \neq 0 , e_- = e_+ = \nu_- = \nu_+ = 0 \). WFs of the sistem (2.1) have the form

\[
\psi_1 = C_1 \exp(-\lambda \tau + <\alpha(\lambda) n_{a-} > \zeta) \\
\psi_2 = C_2 \exp(-\lambda \tau + <\alpha(\lambda) n_{a+} > \zeta) \\
\psi_3 = C_3 \exp(\lambda \tau + <\alpha(\lambda) n_{b} > \zeta)
\]

Transforming by (2.6) with WF \( \phi \) and spectral parameter \( \mu \) one get

\[
e_-[1] = 4 \mu R \phi_3 \phi_1^* / (\phi_1 + \phi_3), \\
e_+[1] = 4 \mu R \phi_3 \phi_2^* / (\phi_2 + \phi_3),
\]

where

\[
\phi_1 = C_1 \exp(-\mu \tau + <\alpha(\mu)(n_{a+} - n_{a-}) > \zeta/2) \\
\phi_2 = C_2 \exp(-\mu \tau + <\alpha(\mu)(n_{a+} - n_{a-}) > \zeta/2) \\
\phi_3 = C_3 \exp(\mu \tau + <\alpha(\mu)(2n_{b} - n_{a-} - n_{a+}) > \zeta/2)
\]

Let \( n_{b} > n_{a+} > n_{a-} \). Then while \( \tau \to -\infty , e_+[1] \to 0 \) at arbitrary \( \zeta \) and if \( \tau \to +\infty \), then \( e_-[1] \to 0 \). So the solution describes a transformation of a pulse resonant with the transition which has the minor population difference to one that corresponds to the larger population difference. It means that such pulses are not stable. In the paper 6 that process the more cumbersome IST technique in terms of Riemann - Hilbert problem we see the opposite statement . Our result seems to be more physical: the transition from the basic state to more populated one would be more transparent.

3.3 Now we shall construct solutions of the MB system starting from a periodic background. Let the plane electromagnetic wave is propagated along one of transitions

\[
e_+ = E \exp i(k \zeta + \omega \tau), \quad e_- = 0,
\]

where \( E \) is the a complex constant. As in MB equations the shift from resonance is taken into account we may put \( \omega = 0 \). Let also the second level is empty \( n_{a-} = 0 , n_{b} + n_{a+} = 1 \). Then from the equations (1.3) one have

\[
\nu_- = \nu_a = 0 \\
\nu_+ = i(2n_{b} - 1)(\omega + \eta - \Delta)^{-1} E \exp i(k \zeta + \omega \tau), \\
k = - < (2n_{b} - 1)(\omega + \eta - \Delta)^{-1} > , \\
n_{b} = (1+(\omega + \eta - \Delta)(4 \mid E \mid^2 + (\omega + \eta - \Delta)^{-1/2})/2.
\]

During the derivation of the lower level population \( n_{b} \) it was supposed that the quantum system had gone in the considered state from one with free upper levels. This condition fix the trace of the matrix \( A^2 \) that does not depend on \( \tau \). It may be seen from the second equation of the system (2.2). The case of the arbitrary higher level population may be considered analoguesly. Relaxation terms from spontaneous transitions may be neglected if the dimensionless electric field amplitude \( \mid E \mid \) is big enough. The solutions of the linear system are:

\[
\psi_1 = C_1 \exp(-\lambda \tau),
\]
\[ \psi_2 = (C_+ \exp(\vartheta) + C_- \exp(-\vartheta)) \exp((< \alpha(\lambda) > - ik) \zeta/2) \]
\[ \psi_3 = -(C_+ (\lambda + \sigma)) \exp(\vartheta) + C_- (\lambda - \sigma) \exp(-\vartheta)) \exp((< \alpha(\lambda) > + ik) \zeta/2)/E^*, \]
where
\[ \vartheta = \sigma(\tau + t < (2n_b - 1)\alpha(\lambda)(\eta - \Delta)^{-1} > \zeta) \]
\[ \sigma = (\lambda^2 - |E|^2)^{-1/2}. \]

Transforming by (2.6) and introducing a new parameter \( \gamma = \gamma_R + i\gamma_I \) instead of the SP \( \mu \) by
\[ \cosh(\gamma) = \mu/|E| \]

one obtains
\[ e_-[1] = -4E \cosh(\gamma_R) \cos(\gamma_I) \tilde{\phi}_3 \tilde{\phi}_1 \exp(i k \zeta/2)/(\tilde{\phi}^+, \tilde{\phi}), \]
\[ e_+[1] = -E(1 - 4E \cosh(\gamma_R) \cos(\gamma_I) \tilde{\phi}_3 \tilde{\phi}_2) \exp(i k \zeta)/(\tilde{\phi}^+ \tilde{\phi}), \]
where
\[ \tilde{\phi}_1 = C_1 \exp(-|E| \cosh(\gamma_R) \cos(\gamma_I)(\tau + D > \zeta) + \]
\[ i(|E| \sinh(\gamma_R) \sin(\gamma_I) \tau - D(\eta - \Delta) > \zeta/2)). \]
\[ \tilde{\phi}_2 = C_+ \exp(\vartheta) + C_- \exp(-\vartheta). \]
\[ \tilde{\phi}_3 = -E(C_+ \exp(\vartheta + \gamma) + C_- \exp(-\vartheta - \gamma))/|E|, \]
and \( \vartheta = Re \vartheta + Im \vartheta, \)
\[ Re \vartheta = |E| \cos(\gamma_I) \sinh(\gamma_R) \tau + \]
\[ < (2n_b - 1)(\eta - \Delta)^{-1}((\eta - \Delta) \sinh(\gamma_R) - 2 |E| \sin(\gamma_I))D > \zeta), \]
\[ Im \vartheta = |E| \cosh(\gamma_R) \sin(\gamma_I) \tau + < (2n_b - 1)(\eta - \Delta)^{-1}((\eta - \Delta) \sin(\gamma_I) + \]
\[ + 2 |E| \sinh(\gamma_R)D > \zeta, \]
\[ D = 2 |E|^2 (\cosh(2\gamma_R) + \cos(2\gamma_I)) + (\eta - \Delta)^2 \]

This solution is parametrized by four complex parameters \( E, \gamma, C_1, C_+/C_- \). The obtained solutions describe processes of fission and fusion of radiation pulses that are resonant with different transitions. It shows the existence of pulses with different propagation velocities over the periodic wave background. The propagation of SIT pulses at arbitrary level poulauion also demonstrate processes of frequency transformation. In order to verify this it is necessary to consider real parts of exponents indices in the expressions for \( \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3 \). The parameter \( \gamma \) is introduced namely for a convenience of this operation. The expressions that determine the character of pulses behaviours are however complicated but allow to plot electric field components. It is seen that the amplitude \( e_+ \) tends to \( |E| \) at large \( \zeta \) and the phase of the periodic wave undergo a shift.

The technique developed here may be applied for many-level systems in the cases when the corresponding MB equations are integrable. The studying of more complicated reductions of the ZS problem potencials seems to be interesting looking for more weak constraints on quantum systems parameters. Another possibility for applications is investigation of the conditions of pulse propagation in thin films as well as rejection from them.
4 Infinitesimal parameters of DT.

In this section the solution for a linearization of the equations (2.2) to be obtained. The dependence of this solution on an arbitrary function give the possibility to widen classes of solutions of initial-boundary problems, looking for sequence of infinitesimal symmetries and for the stability of soliton solutions analysis. The main observation that allows to develop the technique, is that the nontrivial binary transformations (2.6) may generate a new potential that coincide with the seed one. In this article BDT (not Matveev one) and the method of keeping reduction were used. It means that the DT is infinitesimal if $U[1] = U$, if $\nu = \mu$ and $(\chi, \varphi) \neq 0$.

Linearizing the system (1.7) with respect to perturbations $U^{(1)}$ and $A^{(1)}$ in relations to solutions $U$ and $A$ we get

$$
\begin{align*}
U_c^{(1)} &= \frac{1}{2} [J, <A^{(1)}>] \\
A^{(1)} &= \left[ \frac{i}{2} (\eta - \Delta) J + U, A^{(1)} \right] + [U^{(1)}, A]
\end{align*}
$$

(4.1)

We shall suppose that $\nu = \mu + \delta$, and $\delta$ is small.

Expanding (2.6) in Taylor series by the parameter $\delta$ we get

$$
\begin{align*}
U[1] &= U + \delta [J, P^{(0)}] + ... \\
A[1] &= A + 2\delta \alpha(\mu) [A, P^{(0)}] + ... 
\end{align*}
$$

where $P^{(0)} = \varphi \otimes \chi$, $\varphi$ and $\chi$ are solutions of the direct and conjugate problems (2.8,9) with the spectral parameter $\mu$, such that $(\chi, \varphi) = 1$.

From these results the following formulas for small disturbances are derived

$$
\begin{align*}
U^{(1)} &= [J, P^{(0)}] \\
A^{(1)} &= 2\alpha(\mu) [A, P^{(0)}]
\end{align*}
$$

(4.2)

After iterations of binary DT with the conditions

$$
\nu^{(q)} = \mu^{(q)} + \delta^{(q)},
$$

where

$$
\delta^{(q)} = O(\delta^{(1)}) \ (q = 1, N)
$$

and

$$
|\delta^{(1)}| << |\mu^{(1)}|,
$$

one get

$$
\begin{align*}
U^{(1)} &= \left[ J, \sum_{i=1}^{N} \beta_i \varphi(i) \chi(i) \right] \\
A^{(1)} &= 2 \left[ A, \sum_{i=1}^{N} \alpha(\mu) \alpha(i) \varphi(i) \chi(i) \right]
\end{align*}
$$

where $\beta_i$ — are constants.

Going from sums to integrals we obtain the following representations of the solutions of the system (2.2)

$$
\begin{align*}
U^{(1)} &= \left[ J, \int C \Psi(\lambda) \Phi(\lambda) d\lambda \right] \\
A^{(1)} &= 2 \left[ A, \int C \alpha(\lambda) \Psi(\lambda) \Phi(\lambda) d\lambda \right]
\end{align*}
$$

(4.3)
where $\Psi(\lambda)$ and $\Phi(\lambda)$ — are matrix solutions of the basic and conjugate problems (2.1,3) with a spectral parameter $\lambda$.

The validity of the formulas (4,2,3) one may check by the direct substitution in the equation (4.1).

We notice also that the reduction (2.5) is compatible with the equation (3.3). By means of equality (3.5) it is possible to obtain solutions that satisfy the reduction, if integrals in the right-hand-side would be presented as a sum by contour integrals that may be parametrized in the following way

$$\int_C \Psi(\lambda)\Phi(\lambda)d\lambda = \int_{t_1}^{t_2} \Psi(g(t))\Phi(g(t))g'(t)dt +$$

$$+ \int_{t_1}^{t_2} \Phi^+(g(t))\Psi^+(g(t))g^∗′(t)dt,$$

$$\int_C \alpha(\lambda)\Psi(\lambda)\Phi(\lambda)d\lambda = \int_{t_1}^{t_2} \alpha(g(t))\Psi(g(t))\Phi(g(t))g'(t)dt -$$

$$- \int_{t_1}^{t_2} \alpha^∗(g(t))\Phi^+(g(t))\Psi^+(g(t))g^∗′(t)dt.$$

The formulas of this paragraph are novel ones. As these expressions for small perturbations contain arbitrary matrix solutions of both problems (2.1,3) they may be applied for studying of stability of soliton solutions of the MB equation. It is also possible to construct infinitesimal symmetries of the equations. and widen classes of initial-boundary problems solutions as well.

## 5 Conclusion

We would repeat that in the infinitesimal binary DT was introduced and exploited to extract hierarchies of symmetries The technique of DT with reductions described above may also be applied to the case of NSS (Manakov equation) as the first equation and the reduction of the potential are the same. The only difference is in formulas (2.6a,b) and (2.7): the vector WF should be replaced by solutions of the consistent Lax problem. Analoguesly one can built solutions of the NSS over the periodic background (the soliton case is considered at Ref. 6). Let

$$e_- = 0, \quad e_+ = E \exp(i(k\zeta + \omega\tau))$$

Then from Lax equations it follows that $k = \omega^2 - |E|^2$. Solving the Lax system one get expressions for WF

$$\psi_1 = C_1\exp(-\lambda\tau + i\lambda^2\zeta),$$

$$\psi_2 = (C_+\exp(\vartheta) + C_-\exp(-\vartheta))\exp(-i(k\zeta + \omega\tau))/2 +$$

$$+ (C_+\vartheta + C_-\vartheta)\exp(\vartheta)exp(-i(k\zeta + \omega\tau))/2, \quad \vartheta = \vartheta - i(k\zeta + \omega\tau),$$

$$\psi_3 = -(C_+\lambda + \sigma - ik/2)\exp(\vartheta) + C_-\lambda - \sigma - ik/2\exp(-\vartheta)e^{i(k\zeta + \omega\tau)},$$

where

$$\vartheta = \sigma(\tau + i(\lambda + ik/2)\zeta)$$

$$\sigma = ((\lambda - ik/2)^2 - |E|^2)^{-1/2}. $$

If you put here $\lambda = \mu$ and substitute the obtained formula for $\phi$ in the equation (2.7) you’ll get the desired solution. Its difference from the solution of MB system is only in other dependence on the variable $\zeta$. 

It is possible to obtain the integrable equation that combine systems MB and NS. This new integrable system may describe a propagation of polarized waves in Kerr nonlinear media with resonant atoms admixture similarly to linear polarized case of Ref. 14. The similar approach to perturbation theory based on Riemann - Hilbert problem is presented at the recent paper 15.

6 Acknowledgement

One of us (S. Leble) thanks H.Steudel and N.Sasa for discussions and some useful references information.

7 References

1 R.K.Boullough, P.M.Jack, P.W.Kitchenside and R.Saunders, Physica Scripta, 20, 364 (1979).
2 S.L. McCall, E.L. Hahn, Phys. Rev. Lett., 181, 908 (1967).
3 G.L. Lamb, Rev.Mod.Phys. 43, 99 (1979).
4 H. Steudel, D.J. Kaup J.Mod.Opt. 43, 1851 (1966). H. Steudel, R. Meinel, D.J. Kaup. Solutions of degenerate two-photon propagation from Bäcklund transformations, J.Mod.Phys., to appear.
5 A.M. Basharov , A.I. Maimistov . JETP 87, 1595 (1984).
6 L.A. Bolshov , N.N. Elkin, V.V. Likhansky, M.I.Persiantsev . JETP 94 101 (1988).
7 A. Mikhailov The reduction problem and the inverse scattering method. Physica D 3 73 (1981).
8 Leble S.B., Ustinov N.V.Solitons of Nonlinear Equations Associated with Degenerate Spectral Problem of the Third Order.In: International Symposium on Nonlinear Theory and its Applications (NOLTA 93) Hawaii, U.S.A., December 5-10, 1993, 4.8-1,p.547-550.
9 S.B. Leble , N.V. Ustinov .: Deep reductions for matrix Lax system, invariant forms and elementary Darboux transforms. in Proceeding of NEEDS-92 Workshop, World Scientific, Singapore (1993), p.34-41. J. Phys. A: Math.Gen. 26 5007 (1993).
10 H. Steudel in Proceedings of 3d International Workshop on Nonlinear Processes in Physics. Eds. V.G. Bar’yakhtar et al. Kiev, Naukova Dumka (1988) Vol.1, p. 144.
11 N.R. Sibgatullin Dokl. AN SSSR, 291 p.302 (1986).
12 Ustinov N.V. ”The reduced self–dual Yang–Mills equation, binary and infinitesimal Darboux Transformations.” Journal of Mathematical Physics, , 39, pp.976-985 (1998).
13 S.V. Manakov JETP 65, 505 (1973).
14 S. Kakei J. Satsuma Multi-Soliton Solutions of a Coupled System of the Nonlinear Schrödinger Equation and the Maxwell-Bloch Equations, to be published.
15 V.S. Shchesnovich, Chaos, Solitons and Fractals 5 p.2121-2133 (1997).