NONNEGATIVELY CURVED HOMOGENEOUS METRICS IN LOW
DIMENSIONS

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Abstract. We consider invariant Riemannian metrics on compact homogeneous spaces $G/H$ where an intermediate subgroup $K$ between $G$ and $H$ exists. In this case, the homogeneous space $G/H$ is the total space of a Riemannian submersion. The metrics constructed by shrinking the fibers in this way can be interpreted as metrics obtained from a Cheeger deformation and are thus well known to be nonnegatively curved. On the other hand, if the fibers are homothetically enlarged, it depends on the triple of groups $(H, K, G)$ whether nonnegative curvature is maintained for small deformations.

Building on the work of L. Schwachhöfer and K. Tapp [ST], we examine all $G$-invariant fibration metrics on $G/H$ for $G$ a compact simple Lie group of dimension up to 15. An analysis of the low dimensional examples provides insight into the algebraic criteria that yield continuous families of nonnegative sectional curvature.

1. Introduction

The study of Riemannian manifolds of nonnegative or positive sectional curvature is one of the original questions of global Riemannian geometry. This is an area of geometry that has motivated deep and beautiful mathematical results, and is still characterized more by its open questions than by its known theorems.

We know few obstructions to nonnegative or positive curvature. There are some topological restrictions, most famously Bonnet-Myers and Synge’s Theorem. There are few known examples of spaces with positive curvature. When we relax the curvature condition to nonnegative curvature, we get more examples of manifolds with nonnegative sectional curvature, many of these discovered within the past ten years [W],[Z].

All known examples of compact irreducible manifolds of nonnegative curvature, homogeneous and inhomogeneous, come from constructions that begin with a compact Lie group and a biinvariant metric. Riemannian submersion metrics are a natural starting point, since, by O’Neill’s formula, we know that taking a quotient tends to increase sectional curvature. In light of the prevalence of quotients of Lie group actions, it makes sense to fully understand the basic setting. In this paper we consider a simple deformation of homogeneous metrics as our source for more examples of spaces of nonnegative sectional curvature.
To begin, we take a fibration of homogeneous spaces arising from a chain \((H, K, G)\) of nested compact Lie groups \(H \subseteq K \subseteq G\):

\[
K/H \rightarrow G/H \\
\downarrow \\
G/K.
\]

Let \(g\) be the Lie algebra corresponding to \(G\) and let \(\mathfrak{k}\) and \(\mathfrak{h}\) be the Lie subalgebras corresponding to the subgroups \(K\) and \(H\) of \(G\), respectively, such that \(\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}\). Let \(g_0\) be a biinvariant metric on \(G\). We use \(g_0\) to fix orthogonal complements: Let \(\mathfrak{m}\) be the subspace of \(\mathfrak{k}\) orthogonal to \(\mathfrak{h}\) such that \(\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}\), the tangent space in the direction of the fiber \(K/H\), and let \(\mathfrak{s}\) be the subspace of \(\mathfrak{g}\) orthogonal to \(\mathfrak{s}\) such that \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}\), the tangent space in the direction of the base \(G/K\). Notice, \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{s}\) and the tangent space to the total space \(G/H\) is \(\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s}\). For any element \(X\) in \(\mathfrak{p}\), we may write \(X = X^m + X^s\), where \(X^m\) in \(\mathfrak{m}\) denotes the vertical component of \(X\), and \(X^s\) in \(\mathfrak{s}\) denotes the horizontal component.

We define the following one-parameter family of metrics on \(G/H\):

\[
(1.1) \quad g_t(X, Y) = \frac{1}{1-t} \cdot g_0(X^m, Y^m) + g_0(X^s, Y^s).
\]

**Main Theorem.** Let \(G\) be a simple compact Lie group of dimension at most 15. Then the homogeneous space \(G/H\) with fibration metric \(g_t\) corresponding to a chain \((H, K, G)\) of nested compact Lie groups admits nonnegative sectional curvature for small \(t > 0\) if and only if one of the following holds:

(i) \((K, H)\) is a symmetric pair, or more generally, \([\mathfrak{m}, \mathfrak{m}]^m = 0\);

(ii) the chain \((H, K, G)\) is one of \((\text{SU}(2), \text{SO}(4), \text{SO}(5))\) or \((\text{SU}(2), \text{SO}(4), G_2)\) where in the second case the subgroup \(\text{SU}(2)\) is such that \(\text{SU}(2) \subset \text{SU}(3) \subset G_2\).

Our result explores and builds on the following result of Schwachhöfer and Tapp, who also showed that the chain \((\text{SU}(2), \text{SO}(4), G_2)\) satisfies the inequality \((\ast)\) below.

**Theorem 1.1.** [ST] (1) The metric \(g_t\) has nonnegative curvature for small \(t > 0\) if and only if there exists some \(C > 0\) such that for all \(X\) and \(Y\) in \(\mathfrak{p}\),

\[
([X^m, Y^m]^m] \leq C|[X, Y]|.
\]

(2) In particular, if \((K, H)\) is a symmetric pair, then \(g_t\) has nonnegative curvature for small \(t > 0\), and in fact for all \(t \in (-\infty, 1/4]\).

**Remark 1.2.** The first part of (2) above follows from the observation that when \((K, H)\) is a symmetric pair, we have \([\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}\). In this case the left hand side of the inequality in condition \((\ast)\) is always zero. Also, note that whenever \([\mathfrak{m}, \mathfrak{m}] = 0\), as when \(K\) is a torus, condition \((\ast)\) holds. We say that \(H\) is a symmetric subgroup of \(K\) if we have \([\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}\).

**Remark 1.3.** When \(H\) is trivial, our fibration is

\[
K \rightarrow G \rightarrow G/K
\]
and \(g_t\) is in fact a \textit{left-invariant} metric on \(G\). In this case, Schwachhöfer proved \(g_t\) has nonnegative curvature for small \(t > 0\) only if the semisimple part of \(\mathfrak{f}\) is an ideal of \(\mathfrak{g}\), see [S]. In particular, when \(\mathfrak{g}\) is simple and \(\mathfrak{f}\) is nonabelian, one does not get nonnegative curvature.

Condition (\(\ast\)) follows if the chain \((H, K, G)\) satisfies the hypothesis of the following Theorem of Schwachhöfer and Tapp. In this case, more arbitrary changes of the metric on \(G/H\) preserve nonnegative curvature.

\textbf{Theorem 1.4.} [ST] \textit{If there exists} \(C > 0\) \textit{such that for all} \(X, Y \in \mathfrak{p}\),

\[(**) \quad |X^m \wedge Y^m| \leq C |[X, Y]|,
\]

\textit{then any left-invariant metric on} \(G\) \textit{sufficiently close to} \(g_0\) \textit{which is} \(\text{Ad}_H\)-invariant and \textit{is a constant multiple of} \(g_0\) \textit{on} \(\mathfrak{s}\) \textit{and} \(\mathfrak{h}\) (\textit{but arbitrary on} \(\mathfrak{m}\)) \textit{has nonnegative sectional curvature on all planes contained in} \(\mathfrak{p}\); \textit{hence, the induced metric on} \(G/H\) \textit{has nonnegative sectional curvature}.

We are interested in the class of metrics on \(G/H\) obtained via scaling along the fibration above. The goal is a further exploration of the algebraic sources of condition (\(\ast\)) to find new spaces with metrics of nonnegative curvature within this framework. In this paper we consider all chains where \(G\) is a low-dimensional simple compact Lie group. In all cases where \((K, H)\) is not a symmetric pair, we find \(X = X^m + X^s\) and \(Y = Y^m + Y^s\) in \(\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s}\) for which \([X, Y] = 0\) yet \([X^m, Y^m]^m \neq 0\); hence, condition (\(\ast\)) fails. We do not know, however, whether the failure of condition (\(\ast\)) is equivalent to the existence of such a pair of vectors \(X, Y\).

It is clear that whenever there is such a pair of vectors \(X, Y \in \mathfrak{m} \oplus \mathfrak{s}\) for some chain \((H, K, G)\), then condition (\(\ast\)) fails as well for the chain \((H', K, G')\), for every closed subgroup \(H' \subset H\), and for every \(G'\) such that \(G \subset G'\), cf. Lemma 2.2.

\textbf{Example 1.5.} To illustrate the delicate relationship here between the algebra and the geometry, we give the following pair of fibrations. In \(G_2\), there are two subgroups in \(\text{SO}(4)\), each isomorphic to \(\text{SU}(2)\), but not conjugate in \(G_2\). There are two chains \((H, \text{SO}(4), G_2)\) with \(H \cong \text{SU}(2)\). We let \(\text{SU}(2)\) denote the \(\text{SU}(2)\) which is \textit{not} a subgroup of \(\text{SU}(3)\) in \(G_2\), cf. Table 2. The chain \((\text{SU}(2), \text{SO}(4), G_2)\) satisfies condition (\(\ast\)), while the chain \((\text{SU}(2), \text{SO}(4), G_2)\) does not. Notice in both of these examples, the base is \(G_2/\text{SO}(4)\) and the fibers are real projective spaces \(\mathbb{R}P^3\) with a symmetric metric. This pair of fibrations shows that a complete understanding of the algebraic criteria which govern the geometry here must go beyond notions like subalgebras, ideals, rank, etc.

\section{2. Low-dimensional examples}

In this section we analyze all simple compact Lie groups \(G\) of dimension up to 15. For each Lie group \(G\), we completely determine the chains of closed connected subgroups \(H \subset K \subset G\) for which condition (\(\ast\)) holds. Thus, we find all those homogeneous spaces \(G/H\) admitting a one-parameter family of fibration metrics with nonnegative curvature.
For each Lie group \( G \), whenever \( K \) is a torus, condition (*) holds trivially, by Remark 1.2. Thus we do not need to further consider the tori \( K \) in \( G \). Similarly, condition (*) holds when \( H \) is a symmetric subgroup of \( K \). Each Lie group \( G \) we consider here is simple, thus when \( K \) is nonabelian, we will not need to consider the case of trivial \( H \). Recall that when \( H \) is trivial, our fibration metrics are in fact left-invariant metrics on \( G \). In [S], Schwachhöfer proves they have nonnegative curvature only if the semi-simple part of \( \mathfrak{t} \) is an ideal of \( \mathfrak{g} \).

In what follows, we will need the following lemmas, included here for convenience.

**Lemma 2.1.** Let \((H,K,G)\) be a chain of compact groups such that two elements of \( \mathfrak{m} \) commute if and only if they are linearly dependent. If
\[
\{[U,V] \mid U,V \in \mathfrak{m}\} \cap \{[W,Z]^t \mid W,Z \in \mathfrak{s}\} = \{0\},
\]
then condition (**) holds. (Here \( \overline{\mathcal{S}} \) denotes the topological closure of a subset \( \mathcal{S} \).)

**Proof.** This is a reformulation of the method that is used to prove [ST, Proposition 4.2]. For the convenience of the reader we reproduce the proof of Schwachhöfer and Tapp here: Suppose condition (**) is not satisfied. Then there exist sequences \( \{X_r\} \) and \( \{Y_r\} \) in \( \mathfrak{p} \) such that for each \( r \) the pair of vectors \((X_r^m,Y_r^m)\) is orthonormal and \( \lim[X_r,Y_r] = 0 \). After passing to subsequences, we know orthonormal limits \( X^m := \lim X_r^m \) and \( Y^m := \lim Y_r^m \) exist. By hypothesis, \( B := [X^m,Y^m] \neq 0 \). Meanwhile, \( 0 = \lim[X_r,Y_r]^t = \lim[X_r^m,Y_r^m] + \lim[X_r^s,Y_r^s]^t \), so that \( B = -\lim[X_r^s,Y_r^s]^t \) is a nonzero element of \( \{[U,V] \mid U,V \in \mathfrak{m}\} \cap \{[W,Z]^t \mid W,Z \in \mathfrak{s}\} \). This yields our contradiction. Thus condition (**) must hold.

**Lemma 2.2.** Let \( H' \subseteq H \subseteq K \subseteq G \subseteq G' \) be a chain of nested compact groups such that condition (*) cannot hold for the chain \((H,K,G)\). Then condition (*) cannot hold for the chain \((H',K,G')\).

**Proof.** Let \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{s} \) such that \( \mathfrak{m} \) is the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{k} \) and \( \mathfrak{s} \) is the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g} \). By hypothesis, we can find sequences \( \{X_r\} \) and \( \{Y_r\} \) in \( \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s} \) such that \( \|[X_r^m,Y_r^m]^{m}\| = 1 \) and \( \lim[X_r,Y_r] = 0 \). Now let \( \mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}' \oplus \mathfrak{s}' \) be the decomposition of \( \mathfrak{g}' \) such that \( \mathfrak{m}' \) is the orthogonal complement of \( \mathfrak{h}' \) in \( \mathfrak{t} \) and \( \mathfrak{s}' \) is the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g}' \). We have \( \mathfrak{m} \subseteq \mathfrak{m}' \) and \( \mathfrak{s} \subseteq \mathfrak{s}' \). Then \( \mathfrak{p} \subseteq \mathfrak{p}' = \mathfrak{m}' \oplus \mathfrak{s}' \). The same sequences \( \{X_r\} \) and \( \{Y_r\} \) are in \( \mathfrak{p}' \) and we have
\[
\|[X_r^m,Y_r^m]^{m}\| = \|[X_r^m,Y_r^m]^{m}\| = 1.
\]
Thus condition (*) is not satisfied for the chain \((H',K,G')\). \(\square\)

**Notation 2.3.** By \( \text{A}_1^{pr} \), \( \text{SO}(3)^{pr} \), or \( \text{SU}(2)^{pr} \), we denote subgroups of certain simple Lie groups which correspond to principal three-dimensional subalgebras see [D, §9].

**Notation 2.4.** When we give specific choices of \( X \) and \( Y \) in \( \mathfrak{p} \), we will use the following standard Lie algebra notation. Let \( E_{ij} \) \((i \neq j)\) denote the skew-symmetric matrix with \( ij^{th} \) entry 1 and \( ji^{th} \) entry \(-1\), all other entries 0. Let \( F_{ij} \) \((i \neq j)\) denote the symmetric matrix with \( ij^{th} \) and \( ji^{th} \) entries 1, all other entries 0. Let \( F_{jj} \) denote the diagonal matrix with \( jj^{th} \) entry 1, all other entries 0.
2.1. \( G = \text{SU}(2) \). We note that \( \text{SU}(2) \) has no nonabelian subgroups \( K \). Since the biinvariant metric on \( G = \text{SU}(2) = S^3 \) has positive curvature, it is no surprise that the only chain, \((\text{Id}, \text{SO}(2), \text{SU}(2))\), fulfills condition (*)\). In fact, all left-invariant metrics with nonnegative sectional curvature on \( \text{SU}(2) \) are classified in [BFSTW].

2.2. \( G = \text{SU}(3) \). There are only three conjugacy classes of non-abelian subgroups of \( \text{SU}(3) \), which we will denote as \( S_2, S_3, S_5 \). Sectional curvature on \( \text{SU}(2) \) is abelian, \( \text{SU}(3) \) is symmetric. Furthermore, whenever \( G = \text{SU}(2) \) has no nonabelian subgroups \( K \). Since the biinvariant metric on \( G = \text{SU}(2) = S^3 \) has positive curvature, it is no surprise that the only chain, \((\text{Id}, \text{SO}(2), \text{SU}(2))\), fulfills condition (*)\). In fact, all left-invariant metrics with nonnegative sectional curvature on \( \text{SU}(2) \) are classified in [BFSTW].

\( K = \text{SU}(2) \times \text{U}(1) \). All closed subgroups \( H \subset K \) of rank 2 are symmetric.

(a) The subgroup \( H = \text{SU}(2) \subset K \) has a one-dimensional complement \( m \). Since \([m, m] = 0\), condition (*) is trivially satisfied.

(b) For the real circle subgroup \( H = \text{SO}(2) \subset \text{SU}(2) \subset K \), we see \([m, m] \subset h \). Condition (*) again is trivially satisfied.

(c) Consider the one-parameter family of circles \( H = \Delta_{p,q} \text{U}(1) \subset T^2 \subset K = \text{U}(2) \).

Recall, the Lie subalgebra is (for coprime integers \( p \) and \( q \))

\[
\Delta_{p,q} \text{U}(1) := \text{span} \left\{ \left( \begin{array}{ccc}
p i & 0 & 0 \\
0 & q i & 0 \\
0 & 0 & -(p+q)i \\
\end{array} \right) = i (p F_{11} + q F_{22} - (p+q) F_{33}) \right\}.
\]

Note that \( \Delta_{p,q} \text{U}(1) \) denote the same subgroup; thus, without loss of generality we will always take \( p \geq 0 \). We find that condition (*) is fulfilled if and only if \((p, q) = (1, -1)\). When \((p, q) = (1, -1)\), one sees that this circle is a symmetric subgroup: \([m, m] \subset h = \Delta_{(1, -1)} \text{U}(1) \). Thus condition (*) is fulfilled. Otherwise, we may take \( X^m = E_{12}, Y^m = i F_{12} \) (in \( m \) for all \((p, q)\)) and \( X^s = E_{13} + E_{23}, Y^s = i (F_{23} - F_{13}) \in s \). We see that \([X^m + X^s, Y^m + Y^s] = 0\), while \([X^m, Y^m] = 2i (F_{11} - F_{22}) \).

Provided \((p, q) \neq (1, -1)\), this has a nonzero \( m \)-component.

(2) \( K = \text{SU}(2) \) and \( K = \text{SO}(3) \). One-dimensional subgroups of \( \text{SU}(2) \) and \( \text{SO}(3) \) are symmetric.

Remark 2.5. Whenever \( K = \text{SU}(2) \) or \( K = \text{SO}(3) \), the only nontrivial subgroups \( H \) are circles, which are symmetric. Furthermore, whenever \( K \) is abelian, \([m, m] = 0\), and hence condition (*) always holds.

2.3. \( G = \text{SO}(5) \). In Table 1 we give the conjugacy classes of closed nonabelian connected subgroups of \( \text{SO}(5) \), together with inclusion relations. Note that the two normal subgroups of \( \text{SO}(4) = \text{SU}(2) \cdot \text{SU}(2) \) are conjugate by an inner automorphism of \( \text{SO}(5) \).

(1) \( K = \text{SO}(4) \). Here

\[
\text{SO}(4) = \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(4) \end{pmatrix} \subset \text{SO}(5).
\]

All nonabelian closed subgroups of \( \text{SO}(4) \) are symmetric except \( \text{SU}(2) \).

(a) Let us consider the chain \((\text{SU}(2), \text{SO}(4), \text{SO}(5))\). In this case, \( m \subset g \) is a subalgebra isomorphic to \( \text{su}(2) \). Since all nonzero matrices in \( \text{su}(2) \) are invertible, all nonzero matrices in the set of brackets \([U, V]\) where \( U, V \in m \) are of rank 4. On the other
Table 1. Conjugacy classes of nonabelian connected subgroups in SO(5)

| Hand, for elements $W, Z \in \mathfrak{s}$, brackets $[W, Z] = [W, Z]^t$ have rank at most 2. Thus we conclude that the chain $(SU(2), SO(4), SO(5))$ fulfills the hypothesis of Lemma 2.1, and hence of Theorem 1.4. Indeed, this chain here corresponds on the Lie algebra level to the chain $(Sp(1), Sp(1) \times Sp(1), Sp(2))$, which is exactly the Hopf fibration $S^7 \to \mathbb{HP}^1 = S^4$. Since $S^7$ with the normal homogeneous metric has positive sectional curvature, so has any sufficiently small deformation.

(b) The subgroup $H = T^2 \subset SO(4)$ is symmetric.

(c) Consider the one-parameter family of diagonal circles $H = \Delta_\theta SO(2) \subset T^2 \subset K = SO(4)$. On the Lie algebra level,

$$\mathfrak{h} = \text{span}\{\cos \theta E_{23} + \sin \theta E_{45}\} \subset \text{span}\{E_{23}, E_{45}\} = \mathfrak{t}.$$  

We show that condition $(\ast)$ fails for all $\theta$. If $\sin \theta \neq 0$, we may take $X^m = E_{34}, Y^m = E_{24}$, which lies in $\mathfrak{m}$ for all $\theta$, $X^s = E_{13}, Y^s = -E_{12}$. We see $[X^m, Y^m] = E_{23}$, which has a nonzero $\mathfrak{m}$-component, yet $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$. If instead $\sin \theta = 0$, we choose $X^m = E_{25}, Y^m = E_{24}$ (again, in $\mathfrak{m}$ for all $\theta$) and $X^s = E_{14}, Y^s = E_{15}$. We have $[X^m, Y^m] = E_{45}$, which also has a nonzero $\mathfrak{m}$-component, while $[X, Y] = 0$. Thus condition $(\ast)$ fails.

(2) $K = SO(3) \times SO(2)$. Here

$$SO(3) SO(2) = \begin{pmatrix} SO(3) & 0 \\ 0 & SO(2) \end{pmatrix} \subset SO(5).$$

(a) For the subgroups $H = SO(3) \times \text{Id}, H = SO(2) \times SO(2), H = SO(2) \times \text{Id}$ of $K$ we have $[\mathfrak{m}, \mathfrak{m}]^m = 0$.

(b) Consider the one-parameter family of one-dimensional subalgebras

$$\mathfrak{h} = \text{span}\{\cos \theta E_{13} + \sin \theta E_{45}\} \subset \text{span}\{E_{13}, E_{45}\} = \mathfrak{t}.$$  

When $\sin \theta = 0$ we have again $H = SO(2) \times \text{Id}$, with $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. But when $\sin \theta \neq 0$, we no longer have that $(K, H)$ is a symmetric pair and we show condition $(\ast)$ is not fulfilled. We take the following $X$ and $Y$ in $\mathfrak{p}$: $X^m = E_{13}, Y^m = E_{23}$, and $X^s = -E_{14}, Y^s = E_{24}$. Then $[X^m, Y^m] = [E_{13}, E_{23}] = -E_{12}$ has a nonzero $\mathfrak{m}$-component when $\sin \theta \neq 0$, yet $[X, Y] = 0$. Thus condition $(\ast)$ fails.
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inclusion relations are given by Table 2, cf. [K2, Proposition 15]. Note that the two normal

K

G

which can be described by the basis below for the Lie algebra (note there is no

the usual embedding of

The conjugacy classes of nonabelian compact connected subgroups of G2 are not conjugate in G2; we will distinguish these nonconjugate isomorphic subgroups of SO(4) by writing one of them with a tilde.

\( K = \mathbb{U}(2) \). Here

\[
\mathbb{U}(2) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \text{SO}(4) \mid A + iB \in \mathbb{U}(2) \right\} \subset \text{SO}(4) \subset \text{SO}(5).
\]

(a) The subgroups \( H = \mathbb{SU}(2) \) and \( H = T^2 \) are symmetric.

(b) We have a one-parameter family of circles

\[ H = \Delta_{p,q} \mathbb{U}(1) \subset T^2 \subset K = \mathbb{U}(2) \subset \text{SO}(5). \]

As in Example 2.2(1c), we always take \( p \geq 0 \). When \( (p,q) = (1,-1) \), \( H = \Delta_{1,-1} \mathbb{U}(1) = \mathbb{S}(\mathbb{U}(1)\mathbb{U}(1)) \) is a symmetric subgroup of \( K \) and \([m,m] \subset h\). Hence, condition \((*)\) holds trivially in this case. We show that condition \((*)\) fails in case \((p,q) \neq (1,-1)\). We have \( s = \text{so}(5) \oplus \mathbb{U}(2) \) and \( m = \mathbb{U}(2) \oplus u_{p,q}(1) \) (note that \( m \neq \text{su}(2) \)). When \( (p,q) \neq (1,-1) \), we exhibit a commuting pair. We take

\[ X^m = \frac{1}{2}(E_{25} + E_{34}) \quad \text{and} \quad Y^m = \frac{1}{2}(E_{23} + E_{45}), \]

these elements of \( u(2) \oplus t^2 \) are in \( \mathfrak{m} \) for any pair \((p,q)\). Furthermore, we take

\[ X^s = E_{14} + \frac{1}{2}(E_{23} - E_{45}) \quad \text{and} \quad Y^s = E_{12} + \frac{1}{2}(E_{25} - E_{34}). \]

We see that \([X,Y] = [X^m + X^s, Y^m + Y^s] = 0\) while \([X^m, Y^m] = -\frac{1}{2}(E_{24} - E_{35})\), which has nonzero \( m \)-component provided \((p,q) \neq (1,-1)\).

(4) \( K = \mathbb{SU}(2), \quad K = \text{SO}(3), \quad K = \text{SO}(3)\text{pr} \). Condition \((*)\) is always satisfied, see Remark 2.5.

2.4. \( G = G_2 \). We have \( G_2 \cong \text{Aut}(O) \), the automorphism group of the octonions. We use the usual embedding of \( G_2 \) in \( \text{SO}(7) \) given by the action on the purely imaginary octonions, which can be described by the basis below for the Lie algebra (note there is no \( X_3 \) or \( Y_3 \)).

\[
\begin{align*}
X_1 &= E_{46} - E_{57}, & Y_1 &= 2E_{13} + E_{46} + E_{57}, \\
X_2 &= E_{45} + E_{67}, & Y_2 &= 2E_{23} - E_{45} + E_{67}, \\
X_4 &= E_{16} + E_{25}, & Y_4 &= 2E_{34} - E_{25} + E_{16}, \\
X_5 &= E_{17} - E_{24}, & Y_5 &= 2E_{35} + E_{24} + E_{17}, \\
X_6 &= E_{14} + E_{27}, & Y_6 &= 2E_{36} + E_{27} - E_{14}, \\
X_7 &= E_{15} - E_{26}, & Y_7 &= 2E_{37} - E_{26} - E_{15}, \\
Z_1 &= 2E_{12} - E_{47} + E_{56}, & Z_2 &= E_{47} + E_{56}.
\end{align*}
\]

The conjugacy classes of nonabelian compact connected subgroups of \( G_2 \) together with inclusion relations are given by Table 2, cf. [K2, Proposition 15]. Note that the two normal subgroups of type \( A_1 \) in \( \text{SO}(4) \) in \( G_2 \) are not conjugate in \( G_2 \); we will distinguish these nonconjugate isomorphic subgroups of \( \text{SO}(4) \) by writing one of them with a tilde.

(1) \( K = \mathbb{SU}(3) \). This can be viewed as the group of elements of \( \text{SO}(7) \) which fix the purely imaginary octonion given by the third canonical basis vector of \( \mathbb{R}^7 \). At the Lie algebra level, \( \mathfrak{su}(3) = \text{span}\{Z_1, Z_2, X_1, \ldots, X_7\} \).
The subgroup $H = U(2) \cong S(U(2)U(1))$ is symmetric.

(b) When $H = SU(2)$, condition (*) fails. To see this, we use $su(2) = \text{span}\{X_1, X_2, Z_2\}$ and take $X^m = \sqrt{2}X_4 + X_5$, $Y^m = \sqrt{2}X_7 + X_6$, $X^s = -Y_6$, $Y^s = Y_5$. Then $[X^m, Y^m] = Z_1 + 3Z_2$ has a nonzero $m$-component, and yet $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$.

(c) When $H = T^2$ (where $t^2 = \text{span}\{Z_1, Z_2\}$), condition (*) fails. We take $X^m = (\sqrt{2} - 3)X_4 + X_6$, $Y^m = (2\sqrt{2} - 1)X_5 + (\sqrt{2} - 1)X_7$, and $X^s = (1 - \sqrt{2})Y_5 + Y_7$, $Y^s = (1 - \sqrt{2})Y_6 + Y_4$. Then we see $[X^m, Y^m] = (\sqrt{2} - 1)X_2$ and yet $[X, Y] = 0$.

(d) The subgroup $H = SO(3)$ is symmetric in $K = SU(3)$.

(e) When $H = SO(2)$ (SO(2) $\subset SU(2)$), condition (*) fails by Lemma 2.2.

(2) $K = SO(4)$. At the Lie algebra level, we have

$$so(4) = su(2) \oplus \widetilde{su}(2), \text{ where } su(2) = \text{span}\{X_1, X_2, Z_2\}, \widetilde{su}(2) = \text{span}\{Y_1, Y_2, Z_1\}.$$ 

Since all rank two subgroups $H$ of $SO(4)$ are symmetric, we need only consider the rank one subgroups of $SO(4)$.

(a) The subgroup $H = SO(3)$ is symmetric in $SO(4)$.

(b) By Theorem 1.4 in [ST], when $H = SU(2)$, condition (*) is satisfied.

(c) When $H = \widetilde{SU}(2)$, we show condition (*) fails. We take $X^m = X_2$, $Y^m = X_1$, and $X^s = X_4 + X_5$, $Y^s = X_6 + X_7$. Then we see $[X^m, Y^m] = -2Z_2$ and yet $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$.

(d) We consider the one-parameter family of circles $H = \Delta_\theta SO(2)$, where $\theta \in [0, 2\pi)$.

We show that condition (*) fails for all $\theta$. At the Lie algebra level,

$$\Delta_\theta so(2) = \text{span}\{\frac{1}{\sqrt{3}}(\cos \theta)Z_1 + (\sin \theta)Z_2\}.$$ 

If $\tan \theta \neq \sqrt{3}$ we take $X^m = X_2 + Y_2$, $Y^m = -3X_1 + Y_1$, and $X^s = X_4 + Y_4$, $Y^s = -3X_7 + Y_7$. We find that $[X, Y] = 0$ yet $[X^m, Y^m] = 2(Z_1 + 3Z_2)$, in $t^2$. This has a nonzero $m$-component except when $\tan \theta = \sqrt{3}$. And if $\tan \theta = \sqrt{3}$, we can instead take $X^m = \frac{1}{\sqrt{3}}(X_1 + Y_1)$, $Y^m = \frac{1}{\sqrt{3}}(-X_2 + Y_2)$, while $X^s = X_4$ and $Y^s = X_7$. This time $[X, Y] = 0$ and $[X^m, Y^m] = -Z_1 - Z_2$. This has a nonzero $m$-component except when $\tan \theta = \frac{1}{\sqrt{3}}$. We conclude that condition (*) fails for every choice of $H = \Delta_\theta SO(2)$. 

[Table 2. Conjugacy classes of nonabelian connected subgroups in $G_2$]
(3) \( K = U(2) \). In terms of the basis for \( g_2 \) above,
\[
u(2) = \text{span}\{X_1, X_2, Z_1, Z_2\}.
\]
(a) Both \( T^2 \) and \( SU(2) \) are symmetric subgroups of \( U(2) \).
(b) We consider the one-parameter family of circles \( H = \Delta_\theta U(1) \subseteq T^2 \subseteq U(2) \). Notice this is the same one-parameter family we saw in the case \( K = SO(4) \), but since here \( K = U(2) \) is different, \( m \) and \( s \) are also different. Here
\[
m = \text{span}\{X_1, X_2\} \oplus \text{span}\left\{-\frac{1}{\sqrt{3}}(\sin \theta)Z_1 + (\cos \theta)Z_2\right\}.
\]
When \( \theta = \frac{\pi}{4} \), \([m, m] \subseteq \mathfrak{h}\), and condition (\( \ast \)) holds. We show that condition (\( \ast \)) fails if \( \theta \neq \frac{\pi}{4} \): We may take \( X^m = X_2, Y^m = X_1 \in m \) and \( X^s = -\frac{1}{\sqrt{2}}(X_7 - Y_7), Y^s = \frac{1}{\sqrt{2}}(X_4 + Y_4) \). We find that \([X, Y] = 0\) and \([X^m, Y^m] = -2Z_2\), which has a nonzero \( m \)-component when \( \theta \neq \frac{\pi}{4} \).

(4) \( K = U(2) \). In terms of the basis for \( g_2 \) above,
\[
u(2) = \text{span}\{Y_1, Y_2, Z_1, Z_2\}.
\]
(a) As in the case above, both \( T^2 \) and \( SU(2) \) are symmetric subgroups of \( U(2) \).
(b) We again consider the one-parameter family of circles \( H = \Delta_\theta U(1) \subseteq T^2 \). This time,
\[
m = \text{span}\{Y_1, Y_2\} \oplus \text{span}\left\{-\frac{1}{\sqrt{3}}(\sin \theta)Z_1 + (\cos \theta)Z_2\right\}.
\]
We show that condition (\( \ast \)) holds only for \( \sin \theta = 0 \). When \( \sin \theta = 0 \), \([m, m] \subseteq \mathfrak{h}\) and condition (\( \ast \)) holds trivially. Otherwise, we take \( X^m = -Y_2, Y^m = Y_1 \) (in \( m \) regardless of \( \theta \)), and \( X^s = X_1 + \sqrt{2}X_6, Y^s = X_2 + \sqrt{2}X_5 \). For this pair, \([X, Y] = 0\) yet \([X^m, Y^m] = -2Z_1\), which has a nonzero \( m \)-component for \( \sin \theta \neq 0 \).

(5) \( K = SU(2), K = SU(2), K = SO(3)_\text{pr}, \) and \( K = SO(3) \). Condition (\( \ast \)) is always satisfied, see Remark 2.5.

2.5. \( G = SO(6) \). We note that at the Lie algebra level, \( so(6) \cong su(4) \). We will use the 6-dimensional orthogonal representation in the discussion that follows. We will need to consider the subgroups of \( SO(6) \) up to automorphisms.

Let \( SU(2) \Delta_\varphi = SU(2) \Delta_\varphi U(1) \) (equivalently, \( SU(2) \Delta_\varphi SO(2) \)) denote the one-parameter family of subgroups of \( U(2) \cap SO(4) \) where \( \varphi \in [0, 2\pi) \) determines the angle of the diagonal circle. We embed such that when \( \sin \varphi = 0 \), \( SU(2) \Delta_\varphi = U(2) \). In the following, we will assume \( \sin \varphi \neq 0 \).

In the table, there are three conjugacy classes of closed subgroups isomorphic to \( SO(3) \). They can be distinguished by their representation on \( \mathbb{R}^6 \) which is given by restricting the standard representation of \( SO(6) \): The group denoted by \( SO(3) \) acts on \( \mathbb{R}^6 \) by the adjoint plus a three-dimensional trivial representation, \( \Delta SO(3) \) acts by the direct sum of two copies of the adjoint representation and \( SO(3)_\text{pr} \) acts irreducibly on a 5-dimensional linear subspace of \( \mathbb{R}^6 \).
Table 3. Conjugacy classes of nonabelian connected subgroups in $\text{SO}(6)$

**Proposition 2.6.** All conjugacy classes of closed connected nonabelian subgroups w.r.t. inner and outer automorphisms of $\text{SO}(6)$ and their inclusion relations are given by Table 3.

**Proof.** To verify the table, it suffices to check at each node if the nonabelian maximal subalgebras are correctly represented, cf. [D, Theorem 15.1] or [K1, Theorem 2.1]. □

(a) Condition ($*$) holds if $H$ is one of the subgroups $U(2)\ U(1)$, $\text{SU}(3)$, $\text{SO}(3)\ U(1)$, $\text{SO}(3)\ U(1)$, and $\Delta\ \text{SO}(3)$, for which we have $[m, m] \subseteq h$.

(b) When $H = T^3$, we show condition ($*$) fails. Let $t^3 = \text{span}\{E_{14}, E_{25}, E_{36}\}$. We take $X^m = E_{12} + E_{45}$, $Y^m = E_{23} + E_{56}$ and $X^s = E_{12} - E_{45}$, $Y^s = -E_{23} + E_{56}$. Then $[X^m, Y^m] = E_{13} + E_{46} \in m$, and $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$. This also shows condition ($*$) fails for every abelian subgroup $H' \subset T^3 \subset K = U(3)$ by Lemma 2.2.

(c) When

$$H = U(2) = \left( \begin{array}{cc} \mathbb{U}(2) & 0 \\ 0 & 1 \end{array} \right) \subset U(3),$$

we show condition ($*$) fails. We can take $X^m = E_{13} + E_{46}$, $Y^m = E_{36}$, and $X^s = E_{26} - E_{35}$, $Y^s = E_{12} - E_{45}$. Then $[X^m, Y^m] = -[X^s, Y^s] = E_{16} + E_{34}$ and this is in $m$, while $[X^m + X^s, Y^m + Y^s] = 0$. This also shows condition ($*$) fails for $H = \text{SU}(2)$ by Lemma 2.2.

(d) When $H = \text{SU}(2)\Delta\varphi\ U(1)$, condition ($*$) fails, provided $\tan \varphi \neq -\sqrt{2}$. If $\tan \varphi = -\sqrt{2}$, then $H = S(\text{U}(2)\ U(1))$, a symmetric subgroup. If $\sin \varphi = 0$, we are in the case (c) above. At the Lie algebra level, $\mathfrak{su}(2) = \text{span}\{E_{12} + E_{45}, E_{15} + E_{24}, E_{14} - E_{25}\}$ and the diagonal $u(1)$ is

$$\text{span}\{\frac{1}{\sqrt{2}}\cos \varphi (E_{14} + E_{25}) + \sin \varphi E_{36}\}.$$
We take $X^m = E_{23} + E_{56}$, $Y^m = E_{26} + E_{35}$, and $X^s = E_{12} + E_{13} - E_{45} - E_{46}$, $Y^s = E_{15} - E_{16} - E_{24} + E_{34}$. Then $[X^m + X^s, Y^m + Y^s] = 0$ and $[X^m, Y^m] = 2(E_{25} - E_{36})$ has a nonzero $m$-component, provided $\tan \varphi \neq -\sqrt{2}$.

(2) $K = SO(5)$. Here we view

$$SO(5) = \begin{pmatrix} SO(5) & 0 \\ 0 & 1 \end{pmatrix} \subset SO(6).$$

(a) Condition (*) holds for the symmetric subgroups $H = SO(3)SO(2)$ and $H = SO(4)$.

(b) When $H = U(2) \subset SO(4)$, we show condition (*) fails. We take $X^m = E_{25} + E_{35}$, $Y^m = E_{15} + E_{45}$, and $X^s = E_{26} + E_{36}$, $Y^s = -E_{16} - E_{46}$. Then $[X^m, Y^m] = E_{12} + E_{13} - E_{24} - E_{34}$ has a nonzero $m$-component, yet $[X, Y] = 0$. Thus for $H = SU(2), H = T^2 \subset U(2)$, condition (*) fails as well, by Lemma 2.2.

(c) When $H = SO(3) \subset SO(4)$, we take $X^m = E_{15}, Y^m = E_{14}$, and $X^s = E_{46}, Y^s = E_{56}$. We see $[X^m + X^s, Y^m + Y^s] = 0$ yet $[X^m, Y^m] = -E_{45} \in m$.

(d) For the subgroup $H = SO(3)^{pr} \subset SO(5)$, we show condition (*) fails. We give a basis for this maximal Lie subalgebra:

$$so(3)^{pr} = \text{span} \left\{ \sqrt{3}E_{14} - E_{24} + E_{35}, \sqrt{3}E_{15} + E_{25} + E_{34}, 2E_{23} - E_{45} \right\}.$$ We take $X^m = E_{12}, Y^m = E_{13}$, and $X^s = E_{26}, Y^s = -E_{36}$, so that $[X^m, Y^m] = -E_{23}$ has a nonzero $m$ component, yet $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$.

(3) $K = SO(4)SO(2)$. Here we view

$$K = \begin{pmatrix} SO(4) & 0 \\ 0 & SO(2) \end{pmatrix} \subset SO(6).$$

(a) For each of the following subgroups $H$ we have $[m, m] \subseteq \mathfrak{h}$ and thus condition (*) holds:

$SO(4), SO(3)SO(2), SO(3) \subset SO(4), U(2)U(1), U(2), T^3 = SO(2)SO(2)SO(2), T^2 = SO(2)SO(2) \cdot \text{Id}.$

(b) When $H = SU(2)SO(2) \subset SO(4)SO(2)$, we show condition (*) fails. Here we use that $so(4) = su(2) \oplus su(2)$, so that one $su(2)$ factor is in $\mathfrak{h}$, and $m$ is the other $su(2)$ factor: thus, $[m, m] = m$. (Note the two $SU(2)$ factors are conjugates in $SO(6)$.) Take $X^m = E_{12} - E_{34}, Y^m = E_{14} + E_{23}, X^s = E_{56}, Y^s = \sqrt{2}(E_{36} + E_{45}).$ Here $m = \text{span}\{X^m, Y^m, [X^m, Y^m]\}$ and $[X, Y] = [X^m + X^s, Y^m + Y^s] = 0$.

(c) When $H = SU(2)\Delta_\varphi$, condition (*) fails. At the Lie algebra level, this is

$$\text{span}\{E_{12} + E_{34}, E_{14} + E_{23}, E_{13} - E_{24}\} \oplus \text{span}\{\frac{1}{\sqrt{2}} \cos \varphi (E_{13} + E_{24}) + \sin \varphi E_{56}\}.$$ We may take the same $X^m + X^s$, $Y^m + Y^s$ as in the previous example. Note that $[X^m, Y^m] = -2(E_{13} + E_{24})$ has a non-zero $m$-component provided $\sin \varphi \neq 0$.

(d) When $H = \Delta_\varphi U(1)SO(2)$ (where $\Delta_\varphi U(1) \subset U(2)$) it is easy to see that condition (*) fails. When $U(1)$ is entirely within one of the $SU(2)$ factors, we know condition (*) fails by the previous example. Otherwise, we take $X^m = E_{13} - E_{24}, Y^m = E_{23} + E_{14}$ (in $m$ regardless of angle $\theta$), and $X^s = \sqrt{2}(E_{25} + E_{36})$ and $Y^s = \sqrt{2}(E_{15} - E_{46}).$
We consider the family of subgroups $H = \text{SO}(3) \cdot \text{SO}(2)$ and $H = \Delta \text{SO}(3)$, as well as for $H = \text{SO}(2) \cdot \text{SO}(2)$ which has $[m,m] \subset \mathfrak{h}$.

(b) When $H = \text{SO}(3) \cdot \text{Id} \cong \text{Id} \cdot \text{SO}(3)$ we see condition $(\ast)$ holds. We use that $m$ is the other subalgebra $\mathfrak{so}(3)$, so that $[m,m] = m$. For $H = \text{SO}(3) \cdot \text{Id}$, we may take $X^m = E_{45}$, $Y^m = E_{46}$, so that $[X^m, Y^m] = -E_{56} \in \mathfrak{m}$. By choosing $X^s = E_{25}$, $Y^s = -E_{26}$ we get $[X^m + X^s, Y^m + Y^s] = 0$.

(c) In the case $H = \Delta \theta \text{SO}(2)$, condition $(\ast)$ fails. When $\text{SO}(2)$ lies entirely within one of the $\text{SO}(3)$ factors, condition $(\ast)$ fails by the previous example. Otherwise we exhibit a commuting pair: $X^m = E_{23}$, $Y^m = E_{13}$, and $X^s = E_{14}, Y^s = -E_{15}$. We see $[X^m + X^s, Y^m + Y^s] = 0$ while $[X^m, Y^m] = E_{12}$, which has a nonzero $m$-component as long as our $\text{SO}(2)$ does not lie entirely within either of the $\text{SO}(3)$ factors.

(5) $K = \text{U}(2) \cdot \text{U}(1) = \text{U}(2) \cdot \text{SO}(2)$. (This is a subset of $\text{SO}(4) \cdot \text{SO}(2)$.)

(a) For each of the subgroups $H = \text{U}(2), T^2 \text{SO}(2), \text{SU}(2) \cdot \Delta \varphi, \text{SU}(2) \cdot \text{SO}(2), \text{SU}(2) \cdot \text{Id}$, and $\text{SO}(2) \cdot \text{SO}(2)$, we have $[m,m] = 0$. Thus condition $(\ast)$ holds in all these cases.

(b) When $H = \text{U}(1) \cdot \text{SO}(2), m = \mathfrak{su}(2)$ and since $[m,m] = m$, we show condition $(\ast)$ fails. We take $X^m = E_{12} + E_{34}, Y^m = E_{14} + E_{23}$, and $X^s = \sqrt{2}(E_{15} + E_{26}), Y^s = \sqrt{2}(E_{35} - E_{46})$. We see $[X^m, Y^m] = 2(E_{13} - E_{24}) \in \mathfrak{m}$, while $[X^m + X^s, Y^m + Y^s] = 0$. By Lemma 2.2, condition $(\ast)$ fails for every abelian $H' \subset \text{U}(1) \cdot \text{SO}(2)$.

(6) $K = \text{SU}(3)$.

(a) Condition $(\ast)$ holds for symmetric subgroups $H = S(\text{U}(2) \cdot \text{U}(1)), H = \Delta \text{SO}(3)$.

(b) When $H = T^2$ we show condition $(\ast)$ fails. We take $X^m = E_{12} + E_{45}, Y^m = E_{13} + E_{46}$, and then we take $X^s = E_{23} + E_{34}, Y^s = E_{12} - E_{45}$. It is easy to see that $[X^m + X^s, Y^m + Y^s] = 0$ while $[X^m, Y^m]$ is a nonzero element of $\mathfrak{m}$.

(c) When $H = \text{SU}(2)$, condition $(\ast)$ cannot hold. We exhibit a commuting pair: We take $X^m = E_{23} + E_{56}, Y^m = E_{26} + E_{35},$ and $X^s = E_{12} + E_{13} - E_{45} - E_{46}, Y^s = E_{15} - E_{16} - E_{24} + E_{34}$. Then $[X^m + X^s, Y^m + Y^s] = 0$ and $[X^m, Y^m] = 2(E_{25} - E_{36})$ has a nonzero $m$-component.

(7) $K = \text{SO}(3) \cdot \text{U}(1)$. (Here, $\text{SO}(3) = \Delta \text{SO}(3) \subset \text{SO}(3) \cdot \text{SO}(3)$.)

(a) The subgroup $H = \text{SO}(2) \cdot \text{U}(1)$ is symmetric, and $H = \text{Id} \cdot \Delta \text{SO}(3)$ has $[m,m] = 0$.

(b) We consider the family of subgroups $H = \Delta \varphi \text{SO}(2)$, where when $\sin \theta = 0$, $H = \text{U}(1) \cdot \text{Id}$, and $\cos \theta = 0$ corresponds to $H = \text{Id} \cdot \text{SO}(2)$. Just as in the case above, when $\theta = 0$, $[m,m] = 0$ and condition $(\ast)$ holds. We prove that when $\sin \theta \neq 0$, condition $(\ast)$ fails by exhibiting a pair of commuting vectors: Let $X^m = E_{12} + E_{45}, Y^m = E_{13} + E_{46}$, and $X^s = E_{12} - E_{45}, Y^s = -E_{13} - E_{46}$. We see $[X^m + X^s, Y^m + Y^s] = 0$, while $[X^m, Y^m] = -(E_{23} + E_{56})$, which has a nonzero $m$-component, since $\sin \theta \neq 0$.

(8) $K = \text{SO}(4)$. 
(a) Both $H = \SO(2) \SO(2)$ and $H = \SO(3)$ are symmetric subgroups.

(b) When $H = \SU(2)$ (recall, $\so(4) = \su(2) \oplus \su(2)$ with conjugate factors $\su(2)$), we have $m = \su(2)$, so that $[m, m] = m$. In this case, condition ($\ast$) fails. We take $X^m = E_{12} - E_{34}$, $Y^m = E_{13} + E_{24}$, and $X^s = \sqrt{2}(E_{35} + E_{16})$, $Y^s = \sqrt{2}(E_{25} + E_{34})$. Then $[X^m + X^s, Y^m + Y^s] = 0$, while $[X^m, Y^m] = 2(E_{14} - E_{23})$.

(c) When $H = \Delta_\theta \SO(2)$, we show condition ($\ast$) fails for all choices of $\theta$. We may take $X^m = E_{23}, Y^m = E_{13}$ and $X^s = E_{25}, Y^s = -E_{15}$. For this pair, $[X^m + X^s, Y^m + Y^s] = 0$ and $[X^m, Y^m]^m \neq 0$ as long as $\sin \theta \neq 0$. In case $\sin \theta = 0$, we may take $X^m = E_{14}, Y^m = E_{13}$ and $X^s = E_{45}, Y^s = -E_{35}$, so that $[X, Y] = 0$ and $[X^m, Y^m] = E_{34} \in m$.

(9) $K = \SO(3) \SO(2)$. (This is a subset of $\SO(5)$.)

(a) The subgroup $H = \SO(2) \SO(2)$ is symmetric and $H = \SO(3) \cdot \Id$ has $[m, m] = 0$.

(b) We consider the family of subgroups $H = \Delta_\theta \SO(2)$, where when $\sin \theta = 0$, $H = \SO(2) \cdot \Id$, and $\cos \theta = 0$ corresponds to $H = \Id \cdot \SO(2)$. We see that in the case $\sin \theta = 0$, $[m, m]^m = 0$ and condition ($\ast$) holds. But when $\sin \theta \neq 0$, we show condition ($\ast$) fails: We take $X^m = E_{12}, Y^m = E_{13}, X^s = E_{24}, Y^s = -E_{34}$. Here we have $[X^m + X^s, Y^m + Y^s] = 0$, while $[X^m, Y^m] = -E_{23}$, with a nonzero $m$-component when $\sin \theta \neq 0$.

(10) $K = \SU(2) \Delta_\varphi \SO(2)$. Recall, $K \subset U(2) \SO(2)$ where $\varphi$ determines the circle angle in $U(2) \SO(2)$. On the Lie algebra level, $\mathfrak{k} = \text{span}\{E_{12} + E_{34}, E_{14} + E_{23}, E_{13} - E_{24}\} \oplus \text{span}\{\frac{1}{\sqrt{2}} \cos \varphi (E_{13} + E_{24}) + \sin \varphi E_{56}\}$.

(a) For the symmetric subgroups $H = \SU(2) \cdot \Id, H = \SO(2) \Delta_\varphi \SO(2)$, condition ($\ast$) holds.

(b) Consider $H = \Delta_\varphi \SO(2) \subset \SO(2) \Delta_\varphi \SO(2)$. Here the first $\SO(2)$ is the symmetric subgroup of $\SU(2)$; $H$ is the diagonally embedded circle. When $\sin \theta = 0$, $H = \SO(2) \cdot \Id$ and we have $[m, m]^m = 0$, thus condition ($\ast$) holds. But when $\sin \theta \neq 0$, then we find a commuting pair. We take $X^m = E_{12} + E_{34}, Y^m = E_{14} + E_{23}, X^s = E_{15} + E_{26}, Y^s = \sqrt{2}(E_{35} + E_{16})$. We have $[X^m + X^s, Y^m + Y^s] = 0$, while $[X^m, Y^m] = 2(E_{13} - E_{24})$, which has a nonzero $m$-component since $\sin \theta \neq 0$.

(11) $K = \SU(2) \SO(2)$. (This is a subset of $\SU(3)$.)

(a) The subgroups $H = T^3 \subset \SU(2)$ are symmetric; condition ($\ast$) holds.

(b) Consider the one-parameter family of diagonally embedded circles $H = \Delta_\theta \SO(2)$. When $\sin \theta = 0$, $H = \SO(2) \subset \SU(2)$ and $[m, m] \subset \mathfrak{h}$. But when $\sin \theta \neq 0$, we show condition ($\ast$) cannot hold, by finding a commuting pair. We take $X^m = E_{12} + E_{45}, Y^m = E_{15} + E_{24}$ (in $m$ regardless of $\theta$), and $X^s = \sqrt{2}(E_{16} + E_{35}), Y^s = \sqrt{2}(E_{25} - E_{46})$. We have $[X^m + X^s, Y^m + Y^s] = 0$, while $[X^m, Y^m] = 2(E_{14} - E_{23})$, which has a nonzero $m$-component provided $\sin \theta \neq 0$.

(12) $K = \SU(2)$. (This is the only case when condition ($\ast$) holds.)
Otherwise, the pair of vectors given above (in (11)(b)) serves as our commuting pair here with no modification needed.

\[(13) \quad K = SU(2) \subset SU(3), \quad K = \Delta SO(3) \subset SO(3)SO(3), \quad K = SO(3) \cdot \text{Id} \subset SO(3)SO(3), \quad K = SO(3)^{pr} \subset SO(5). \] Condition \((*)\) is always satisfied, see Remark 2.5.

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