Fractal Dynamics and Control of the Fractional Potts Model on Diamond-Like Hierarchical Lattices

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The fractional Potts model on diamond-like hierarchical lattices is introduced in this manuscript, which is a fractional rational system in the complex plane. Then, the fractal dynamics of this model is discussed from the fractal viewpoint. Julia set of the fractional Potts model is given, and control items of this fractional model are designed to control the Julia set. To associate two different Julia sets of the fractional model with different parameters and fractional orders, nonlinear coupling items are taken to make one Julia set change to another. The simulations are provided to illustrate the efficacy of these methods.

1. Introduction

The model of a diamond hierarchical lattice in statistical physics shows that there is an important relation between the limit point set of zero points of the partition function and the Julia set of a class of rational functions. Julia set is one of the important sets in fractal theory, which is used extensively in many fields [1–4]. In the past few years, properties and graphs of Julia sets of various kinds of systems were studied [5–7], and applications of Julia sets are also discussed, such as the typical Langevin problem and the dynamics of the particle’s movement [8–11].

Hamiltonian for \( \lambda \)-state Potts models on diamond-like hierarchical lattices is

\[
H = -J \sum_{i,j > \langle i,j >} \delta (\sigma_i, \sigma_j), \quad \sigma_i = 1, 2, \ldots, \lambda, \quad (1)
\]

where \( \delta \) is the signal of the Kronecker delta, \( J \) is the nearest interaction constant of the spin interval, \( \sigma_i \) are the states of parameters, and \( \lambda \) is the given constant, and the sum is in the nearest neighbourhood. The partition function of Hamiltonian for \( \lambda \)-state Potts models on diamond-like hierarchical lattices is

\[
Z = \sum_{\{\sigma_i\}} \exp \left[ K \sum_{\langle i,j \rangle} \delta (\sigma_i, \sigma_j) \right], \quad (2)
\]

where \( K = \beta J, \quad \beta = (1/k_1t), \ k_1 \) is the Boltzmann constant, and \( t \) is the temperature [12–14]. By use of the Migdal–Kadanoff renormalization group method, the authors in [15] proved that the limit sets of the zero point of the partition function for \( \lambda \)-state Potts models on diamond-like hierarchical lattices are the Julia sets \( J(T_{\lambda}) \) of the following rational functions:

\[
\omega = T_{\lambda}(z) = \left( \frac{z^2 + \lambda - 1}{2z + \lambda - 2} \right)^n. \quad (3)
\]

Fractional calculus is a generalization of the ordinary differential and integral to an arbitrary order. The fractional dynamical systems are related with the past status and can reflect the situation of the system more realistically [16–18]. And, the fractional difference provides us a new powerful tool to depict the dynamics of discrete complex systems.

In the practical applications, the behaviors depicted by the Julia set need to show different forms or one behavior under control changes to be another. Therefore, it is necessary to study the control of Julia sets [19, 20]. In [21, 22],
fractal dynamics of the fractional systems are discussed, where the systems are real systems with the form of polynomial. In this paper, we investigate the fractal behaviors of the following discrete model of (3) which is the rational fractional model in the complex plane:

\[ C^\Delta_a^\nu z(t) = \left( \frac{z^2(t + \nu - 1) + \mu - 1}{2\nu(t + \nu - 1) + \mu - 2} \right)^p, \]

where \( C^\Delta_a^\nu \) is the left Caputo-like delta difference, \( t \in \mathbb{N}_{a+1,\nu} \), \( \mathbb{N}_a = \{a, a + 1, a + 2, \cdots\} \) (\( a \in \mathbb{R} \) fixed), and \( z(a) = c, 0 < \nu \leq 1 \). For the function \( f(n) \), the delta difference operator \( \Delta \) is defined as \( \Delta f(n) = f(n + 1) - f(n) \).

2. Preliminary

For convenience, we give some necessary definitions and results on the discrete fractional calculus.

**Definition 1** (see [23]). Let \( u : \mathbb{N}_a \rightarrow \mathbb{R} \) and \( 0 < \nu \) be given. Then, the fractional sum of \( \nu \) order is defined by

\[ \Delta_a^{-\nu}u(t) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} u(s), \quad t \in \mathbb{N}_{a+\nu}, \]

where \( a \) is the starting point, \( \sigma(s) = s + 1 \), and \( t(\nu) \) is the following function defined as

\[ t(\nu) = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)} \]

**Definition 2** (see [24]). For \( 0 < \nu, \nu \notin \mathbb{N}_a \), and \( u(t) \) defined on \( \mathbb{N}_a \), the Caputo-like delta difference is defined by

\[ C^\Delta_a^\nu u(t) = \Delta_a^{-(\nu-\nu)}\Delta^\nu u(t) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} \Delta_a^\nu u(s), \]

where \( t \in \mathbb{N}_{a+m-\nu} \) \( m = \lceil \nu \rceil + 1 \).

**Theorem 1** (see [25]). For the delta fractional difference equation,

\[ C^\Delta_a^\nu u(t) = f(t + \nu - 1, u(t + \nu - 1)), \]

\[ \Delta^k a u(a) = u_k, \]

where \( m = \lceil \nu \rceil + 1 \),

\[ k = 0, \ldots, m - 1, \]

the equivalent discrete integral equation can be obtained as

\[ u(t) = u_0(t) + \frac{1}{\Gamma(\nu)} \sum_{j=a+\nu}^{t-\nu} (t - \sigma(s))^{(\nu-1)} \times f(s + \nu - 1, u(s + \nu - 1)), \quad t \in \mathbb{N}_{a+m}, \]

where the initial iteration \( u_0(t) \) reads

\[ u_0(t) = \sum_{k=0}^{m-1} \frac{(t - a)^{(k)}}{k!} \Delta^k a u(a). \]

The domains of equations (7) and (9) are disparate. The former is \( N_{a+m-\nu} \), and the latter is \( N_{a+m} \). The function \( u(t) \) is defined on the isolated time scale \( N_{a} \). From this viewpoint, it is commendable to use the discrete fractional calculus to initialize fractional difference equations.

3. Julia Set of Discrete Fractional System (4)

The integer form can be obtained for \( a = 0 \) and \( \nu = 1 \) in the discrete model (4):

\[ \Delta z(n) = \left( \frac{z^2(t + \nu - 1) + \mu - 1}{2\nu(t + \nu - 1) + \mu - 2} \right)^p, \]

or

\[ z(n + 1) = \left( \frac{z^2(t + \nu - 1) + \mu - 1}{2\nu(t + \nu - 1) + \mu - 2} \right)^p - z(n). \]

From Theorem 1, the discrete integral form for \( 0 < \nu < 1 \) is obtained as follows:

\[ z(t) = z(a) + \frac{1}{\Gamma(\nu)} \sum_{j=a+\nu}^{t-\nu} \frac{\Gamma(t - q)}{\Gamma(t - q + 1 - \nu)} \]

\[ \times \left[ \frac{z^2(q + \nu - 1) + \mu - 1}{2\nu(q + \nu - 1) + \mu - 2} \right]^p - z(q + \nu - 1) \].

Consequently, the numerical equation can be proposed:

\[ z(n) = z(a) + \frac{1}{\Gamma(\nu)} \sum_{j=a+\nu}^{n-\nu} \frac{\Gamma(n - j + \nu)}{\Gamma(n - j + 1)} \]

\[ \times \left[ \frac{z^2(j - 1) + \mu - 1}{2\nu(j - 1) + \mu - 2} \right]^p - z(j - 1) \].

In particular, if \( a = 0 \) and the summation starts with \( j = 1 \), then the discrete fractional system can be presented as

\[ z(n) = z(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n-\nu} \frac{\Gamma(n - j + \nu)}{\Gamma(n - j + 1)} \]

\[ \times \left[ \frac{z^2(j - 1) + \mu - 1}{2\nu(j - 1) + \mu - 2} \right]^p - z(j - 1) \].

The fractional system (15) has a discrete kernel function, and \( z(n) \) depends on the past information \( z(0), z(1), \cdots, z(n - 1) \). Therefore, the memory effects mean that their present state depends on all past states.

Julia set is one of the important sets in fractal theory, which is generated by the iteration of the integer order system. We will give the definition of the Julia set for the discrete fractional system (15).

**Definition 4.** Let \( \{z(n)\}_{n=0}^{\infty} \) be the trajectory of system (15) in the complex plane. The set
\[ D = \{ z(0) | z(n)_{n=1}^{\infty} \text{ remains bounded} \} \]  
\[ (16) \]
is called the filled Julia set corresponding to the map \( H(z) \).
And, the boundary of \( D \) is called the Julia set of system (15), which is denoted by \( J \), i.e., \( J = \partial D \).

To give the simulations of the Julia set of system (15), the procedure of computer graphics is provided as follows:

Step 1: take the parameters in the equation.
Step 2: loop through the items about \( J \) in the fractional order equation. Because \( J \) is added from 1 to \( n \), we can firstly express as something after the connection number, then the sum of the \( J \) terms are obtained.
Step 3: reuse a loop about \( n \) in the fractional order equation, and according to the equation, \( x(1) \) is iterated to \( x(2) \), \( x(2) \) is iterated to \( x(3) \), \( \cdots \). Eventually, \( x(n), y(n), \cdots \) are obtained, and a series of pairs of points are obtained.
Step 4: take a boundary about the Julia set; these pairs of points are iterating to form a graph, and the boundary of the image is the Julia set.

Take some different values of \( \nu, \mu, \) and \( p \) in the discrete fractional system (15), so we can get various Julia sets, see Figure 1.

Figure 1(a) is the Julia set for \( \nu = 1, \mu = 2, \) and \( p = 2 \) in the discrete fractional system (15), which is also the case of the integer order system discussed in [26]. Figures 1(b) and 1(c) are Julia sets for \( \nu = 0.6 \) and \( \nu = 0.3 \), respectively, when \( \mu = 2 \) and \( p = 2 \) in the discrete fractional system (15).

Figures 1(d)–1(f) are Julia sets for different values of \( \nu, \mu, \) and \( p \). From Figure 1, we can see that the left side of the Julia set is shrinking and the right side changes slightly with the decreasing of the orders \( \nu \).

4. Control of Julia Sets of Discrete Fractional System (15)

Julia set control is a hot topic, and many control methods are introduced in recent years [27, 28]. Julia set is closely related to the boundedness of the trajectory of systems. Therefore, the stability of the fixed point is considered to realize the control of the Julia set.

Let \( z(i) = z^*, i = 0, 1, \cdots, n, \) in system (15), then we obtain

\[ z^* = z^* + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \Gamma(n-j+\nu) \times \left( \frac{(z^*)^j + \mu - 1}{2z^* + \mu - 2} - z^* \right) \]  
\[ (17) \]

The solutions of (17) are called the fixed points of system (15).

Introduce the control item into system (15), and we get the controlled system as follows:

\[ z(n) = z(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \Gamma(n-j+\nu) \times \left( \left( \frac{z^2(j-1) + \mu - 1}{2z(j-1) + \mu - 2} - z(j-1) \right) \right) \]  
\[ + k\left( z^p(j-1) - (z^*)^p \right) \]  
\[ (18) \]

where \( k \) is the control parameter.

Since 1 is the fixed point for any \( \nu, \mu, p \) in system (15), we take \( z^* = 1 \) in controlled system (17). So, the controlled system (17) becomes

\[ z(n) = z(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \Gamma(n-j+\nu) \times \left( \left( \frac{z^2(j-1) + \mu - 1}{2z(j-1) + \mu - 2} - z(j-1) \right) \right) \]  
\[ + k\left( z^p(j-1) - 1^p \right) \]  
\[ (19) \]

Figures 2 and 3 illustrate the changing of Julia sets of the discrete fractional system (15) with different parameters.

Figures 2(a)–2(d) and 3(a)–3(d) are the changing of Julia sets when the orders are integer numbers \( \nu = 1 \) and \( \mu = 2 \) and \( p = 2 \). Figures 2(e)–2(h) and Figures 3(e)–3(h) are the changing of Julia sets when the orders are fractional numbers \( \nu = 0.6 \) and \( \nu = 0.45 \), respectively, and \( \mu = 6 \) and \( p = 2 \). From Figures 2 and 3, we can see that Julia sets of the controlled system (19) are shrinking with the increasing of control parameters \( k \).

In fact, the control item can be added to the other locations in system (15). For example,

\[ z(n) = z(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \Gamma(n-j+\nu) \times \left( \left( \frac{z^2(j-1) + \mu - 1}{2z(j-1) + \mu - 2} - z(j-1) \right) \right) \]  
\[ + k\left( z^p(j-1) - (z^*)^p \right) \]  
\[ (20) \]
Though the shapes of the controlled Julia sets are different, the control results are similar under these three ways (18), (20), and (21). And, Julia sets are shrinking with the increasing of the control parameters $k$.

5. Synchronization of Julia Sets of Discrete Fractional System (15)

Synchronization of nonlinear systems is an interesting topic and is applied extensively in mechanics, communication, and so on [18, 19]. From the definition of the Julia set, we know there is one Julia set once the system parameters are given. Much work has been done on the Julia set, which deals with the structure, properties, and graphs of a single Julia set. However, we also need to consider the relations of two different Julia sets. In recent years, synchronization of Julia sets are discussed [24, 25], where the systems are in the integer order. In this section, nonlinear coupled items are designed to achieve the synchronization of Julia sets of the fractional system (15).

Consider a system with the same form as (15) but with different orders and parameters:

$$w(n) = w(0) + \frac{1}{\Gamma(\nu')} \sum_{j=1}^{\mu} \frac{\Gamma(n - j + \nu')}{\Gamma(n - j + 1)} \times \left[ \left( \frac{w^2(j - 1) + \mu' - 1}{2w(j - 1) + \mu' - 2} \right)^{\nu'} - w(j - 1) \right],$$

(22)

where at least one of $\nu', \mu'$, and $\rho'$ is different from $\nu, \mu$, and $\rho$ in (15). Coupled item is introduced into system (22), then we have

**Figure 1:** Julia sets of the discrete fractional system (15) with different values of orders $\nu$ and parameters $\mu$ and $p$. (a) $\nu = 1, \mu = 2, p = 2$; (b) $\nu = 0.6, \mu = 2, p = 2$; (c) $\nu = 0.3, \mu = 2, p = 2$; (d) $\nu = 1, \mu = 6, p = 2$; (e) $\nu = 0.45, \mu = 6, p = 2$; (f) $\nu = 0.3, \mu = 6, p = 2$; (g) $\nu = 1, \mu = 7, p = 5$; (h) $\nu = 0.6, \mu = 7, p = 5$; (i) $\nu = 0.3, \mu = 7, p = 5$. 

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Figure 2: Changing of Julia sets of the discrete fractional system (15) with different values of control parameter $k$ when $\mu = 2$ and $p = 2$. (a) $k = 0.2$, (b) $k = 0.6$, (c) $k = 0.65$, (d) $k = 0.9$, (e) $k = 0.1$, (f) $k = 0.4$, (g) $k = 0.6$, and (h) $k = 0.7$.

Figure 3: Changing of Julia sets of the discrete fractional system (15) with different values of control parameter $k$ when $\mu = 6$ and $p = 2$. (a) $k = 0.2$, (b) $k = 0.6$, (c) $k = 0.65$, (d) $k = 0.9$, (e) $k = 0.1$, (f) $k = 0.3$, (g) $k = 0.5$, and (h) $k = 0.7$. 
Figure 4: Synchronization of Julia sets of (15) and (23) with different values of \( v, \mu, \) and \( p. \) (a) \( t = 0.08, \) (b) \( t = 0.3, \) (c) \( t = 0.6, \) (d) \( t = 0.9, \) (e) \( t = 0.01, \) (f) \( t = 0.3, \) (g) \( t = 0.7, \) (h) \( t = 0.9, \) (i) \( t = 0.001, \) (j) \( t = 0.3, \) (k) \( t = 0.5, \) and (l) \( t = 0.9. \)

\[
\begin{align*}
\begin{bmatrix}
\n\end{align*}
\]

where \( t \) is the coupled strength. Then,

\[
\begin{align*}
\frac{\text{d}w(n) - z(n)}{\text{d}t} &= \left\{ \frac{z(0)}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n - j + \nu)}{\Gamma(n - j + 1)} \times \left[ \left( \frac{\mu^{2}(j - 1) + \mu - 1}{2\mu(j - 1) + \mu - 2} - z(j - 1) \right) \right] + \frac{1}{\Gamma(\nu')} \sum_{j=1}^{n} \frac{\Gamma(n - j + \nu')}{\Gamma(n - j + 1)} \times \left[ \left( \frac{\mu^{2}(j - 1) + \mu - 1}{2\mu(j - 1) + \mu - 2} - z(j - 1) \right) \right] \right\} + \frac{1}{\Gamma(\nu')} \sum_{j=1}^{n} \frac{\Gamma(n - j + \nu')}{\Gamma(n - j + 1)} \times \left[ \left( \frac{\mu^{2}(j - 1) + \mu - 1}{2\mu(j - 1) + \mu - 2} - w(j - 1) \right) \right] \\
&= \left\{ \frac{z(0)}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n - j + \nu)}{\Gamma(n - j + 1)} \times \left[ \left( \frac{\mu^{2}(j - 1) + \mu - 1}{2\mu(j - 1) + \mu - 2} - z(j - 1) \right) \right] + \frac{1}{\Gamma(\nu')} \sum_{j=1}^{n} \frac{\Gamma(n - j + \nu')}{\Gamma(n - j + 1)} \times \left[ \left( \frac{\mu^{2}(j - 1) + \mu - 1}{2\mu(j - 1) + \mu - 2} - w(j - 1) \right) \right] \right\}.
\end{align*}
\]
Figure 4 shows the synchronization process of Julia sets for different parameters and orders of systems. Figures 4(a)–4(d) illustrate the synchronization of Julia sets in Figures 1(h) and 1(d). Under the coupling synchronization, the Julia set in Figure 1(h) is changing to be the Julia set in Figure 1(d) with the increasing of the coupling strength $t$. For this case, the Julia set of the fractional system changes, and the objective is the Julia set of the integer system.

Figures 4(e)–4(h) illustrate the synchronization of Julia sets in Figures 1(a) and 1(e). Under the coupling synchronization, the Julia set in Figure 1(a) is changing to be the Julia set in Figure 1(e) with the increasing of the coupling strength $t$. For this case, the Julia set of the integer system changes, and the objective is the Julia set of the fractional system.

Figures 4(i)–4(l) illustrate the synchronization of Julia sets in Figures 1(b) and 1(i). Under the coupling synchronization, the Julia set in Figure 1(b) is changing to be the Julia set in Figure 1(i) with the increasing of the coupling strength $t$. For this case, both Julia sets in Figures 1(b) and 1(i) are generated from the fractional order systems.

6. Conclusion

The fractional Potts model on diamond-like hierarchical lattices is introduced in this manuscript, which is a generalization of the Potts model with the integer order. By using the fractal theory, the Julia set of the discrete fractional of the model is given. Then, control of the Julia set is discussed by use of the fixed point 1. Some original values not present in the Julia set can be seen in the Julia set by the control. Julia sets are determined and not associated with each other when the values of the system parameters are different. Coupling items are designed to make one Julia set change to another. The simulations illustrate the efficacy of these methods.

Data Availability

The figure data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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