Quantum electrodynamic fluctuations of the macroscopic Josephson phase

H. Kohler¹, F. Guinea², and F. Sols¹

¹ Departamento de Física Teórica de la Materia Condensada e Instituto Nicolás Cabrera, Universidad Autónoma, E-28049 Madrid, Spain
² Instituto de Ciencia de Materiales, CSIC, Campus de Cantoblanco, E-28049 Madrid, Spain

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We study the equilibrium dynamics of the relative phase in a superconducting Josephson link taking into account the quantum fluctuations of the electromagnetic vacuum. The photons act as a superohmic heat bath on the relative Cooper pair number and thus, indirectly, on the macroscopic phase difference $\phi$. This leads to an enhancement of the mean square $\langle \phi^2 \rangle$ that adds to the spread due to the Coulomb interaction carried by the longitudinal electromagnetic field. We also include the interaction with the electronic degrees of freedom due to quasiparticle tunneling, which couple to the phase and only indirectly to the particle number. The simultaneous inclusion of both the radiation field fluctuations and quasiparticle tunneling leads to a novel type of particle–bath Hamiltonian in which the quantum particle couples through its position and momentum to two independent bosonic heat baths. We study the interplay between the two mechanisms in the present context and find interference contributions to the quantum fluctuations of the phase. We explore the observability of the QED effects discussed here.

I. INTRODUCTION

A proper understanding of decoherence as resulting from the dissipative effect on a quantum system of a complex environment is of paramount importance in diverse areas of modern physics and technology. The influence of many degrees of freedom on a given quantum variable has for long been considered a problem of fundamental interest, partly because of its connection to the polaron problem. However, it was only in the eighties when, because of its relevance to macroscopic quantum tunneling and coherence, the problem was addressed by a large community of physicists. The loss of quantum wave coherence -also referred to as decoherence or dephasing- appears in a wide range of physical contexts. It has been generally recognized that decoherence acts much faster than energy dissipation, since it only requires the excitation or destruction of a single quantum of the thermal oscillator bath, while many quanta are necessary to change the particle energy appreciably. In the context of mesoscopic electron systems, dephasing is responsible for the destruction of those quantum interference effects which characterize transport through ballistic nanostructures or disordered metals at low temperatures. Particle-bath models have been used to address such a problem. In disordered metals at sufficiently low temperatures, the dominant mechanism responsible for the loss of phase coherence of electrons near the Fermi surface is the Coulomb interaction with identical electrons in the metal. The question of whether or not the electron dephasing time diverges at zero temperature has been the subject of a recent controversy.

We wish to emphasize here that, under the generic term decoherence or dephasing, one may be contemplating physical problems of widely different conceptual and technical nature which may yield different answers to apparently similar questions. For instance, the general question ‘does zero-point decoherence exist?’ does not admit a universally valid response. The answer depends on the system under attention, the nature of its surrounding environment, the properties of the coupling, and the physical representation within which we frame the question. The latter remark refers to the importance of the basis set in whose representation the reduced density matrix of the quantum system tends to become diagonal as a result of the interaction with the environment. The choice of representation is related in turn to the observable to be measured, since the expectation value must be sensitive to the off-diagonal terms of the density matrix in the desired representation. As examples of distinct physical problems having to do with zero-point fluctuations, we may mention the problem of decoherence for a quantum particle coupled to a dissipative bath, which is qualitatively different from that posed by Fermi surface electrons interacting with other electrons in a disordered metal. An intermediate class of problems is that of conduction electrons interacting with an external environment, which has been shown to modify some Fermi liquid properties such as persistent currents in mesoscopic rings. The message is that possible answers to questions having to do with the loss of quantum wave coherence must be investigated case by case and cannot be translated from one physical context to another without a careful scrutiny.

The general problem of decoherence is receiving renewed attention due to its central role in the design of quantum information processing systems. In this context, it seems that all potential sources of decoherence must be explored. Due to its ubiquity, the quantum electrodynamic (QED) field provides the most basic decoherence mechanism that a charged particle may experience. The purpose of this paper is to investigate the role of the quantum electromagnetic (EM) field as a possible source of decoherence. Specifically, we aim at understanding the effect of the transverse...
component of the EM field. It must be distinguished from the longitudinal component, which in the radiation gauge is originated by the electrostatic Coulomb interaction among charged particles. The combined effect of the transverse and the longitudinal field in a many-body context has been recognized to be responsible for logarithmic departures from the Fermi liquid picture in two-dimensional electron systems and for the reduction of persistent currents in mesoscopic rings. In the context of particle-bath problems, it is fair to say however that, despite its fundamental character, the QED field has been little explored as a dissipative environment that may disturb electrons and, in general, charged particles. From the quantum dissipation viewpoint, Quantum Electrodynamics poses at least two new problems. First, in vacuum it is a superohmic environment and, as such, it has been much less studied than its ohmic counterparts. Second, it has no intrinsic upper cutoff. Thus, when needed, appropriate cutoffs must be introduced on the basis of sound physical arguments.

Some studies can be found in the literature dealing with the coherence properties of free charged particles in the presence of the background radiation field. To understand in greater depth the effect of the transverse photon field as a source of dissipation, we choose as our case study a system which is otherwise quite well understood: the phase-number variable of a Josephson link connecting two superconductors. Typically, such a weak link is achieved with a tunneling barrier or a point contact. The problem which we wish to address here may be viewed as the macroscopic quantum-mechanical version of the Lamb shift problem, since we aim at calculating the effect of the fluctuations of the transverse photon field on an otherwise conservative quantum-mechanical problem. In particular we will focus on the calculation of the uncertainty \( \Delta \phi \equiv \langle \phi^2 \rangle - \langle \phi \rangle^2 \), where \( \phi \) is the relative phase between the two superconductors. We will focus on the equilibrium dynamics, for which \( \langle \phi \rangle = 0 \), and on the harmonic limit \( \langle \phi^2 \rangle < 1 \). The effect of the transverse EM field on \( \langle \phi^2 \rangle \) can be potentially detected, for instance, as a Debye-Waller reduction of the Josephson critical current or as an intrinsic source of uncertainty in the fine measurement of tiny magnetic fields.

This paper builds on a preliminary study presented in Ref. where it was shown that, in the absence of a Josephson coupling, recurrent dynamics alternating between quantum collapses and revivals is robust against fluctuations of the QED field. Such collapses and revivals were later predicted and observed in atomic Bose-Einstein condensates. However, a number of theoretical questions, including the role of the upper frequency cutoff, were left unexplored. In the present work, we also investigate the effect of quasiparticles, which cause fluctuations of the longitudinal field and which we approximate by effective oscillators, adapting the work of Ref. to a Hamiltonian description. When computing the combined effect of both the QED and quasiparticle fields, we find that, in the denominators of the spectral weight contributing to \( \langle \phi^2 \rangle \), there are terms which are caused by the interference between the two baths. In Ref. current and voltage fluctuations in tunnel junctions were studied. There special attention was paid to the biased case and to the effect of fluctuations on the \( I - V \) characteristics. Here we have focussed on the calculation of the equilibrium values of \( \langle \phi^2 \rangle \) and \( \langle N^2 \rangle \) and have used a consistent particle-bath approach. Moreover, we have attempted to clearly separate the transverse and longitudinal electromagnetic fields and have modelled quasiparticles with effective oscillator baths. Our work presents a compact and somewhat involved treatment of the effect of the photon and quasiparticle fields on the phase of a superconducting weak link. Fortunately, some aspects of this complex problem may be understood in simpler terms for some particular cases. This observation motivates section II, which is devoted to provide a largely self-contained preview of some of the results to be derived more rigorously later. That discussion is inspired to some extent in simplified derivations of the atomic Lamb shift (see e.g. Ref.). Starting from the classical Langevin equation satisfied by \( \phi \), the EM field is introduced by invoking gauge invariance arguments. In the simplest cases, we are able to derive fourth-order differential equations which, in the absence of quasiparticles, reduce to Abraham-Lorentz equations for both \( N \) and \( \phi \). We argue conclusively that the coupling to the EM field is essentially that of a fluctuating dipole.

Section III deals in detail with the coupling of the macroscopic superconducting phase to the QED field, which is obtained invoking gauge invariance and introducing appropriate canonical transformations. The number and phase autocorrelation functions are calculated. This permits to compute the QED correction to the longitudinal field result for \( \langle \phi^2 \rangle \) in terms of the junction aspect ratio and the fine structure constant. In section IV, we study the coupling to the quasiparticle bath, i.e. to the many-body fluctuations of the longitudinal field. We adapt the work of Ref. to a Hamiltonian language where the dynamics is investigated through the Heisenberg equations of motion. Section V addresses the most complex issue, namely, the combined effect of the photon and quasiparticle fields. There we face a fundamental problem in quantum dissipation that has so far received little attention: the behavior of a quantum particle interacting with two different baths through its position and momentum. Finally, section VI is devoted to a concluding discussion.

II. GENERAL REMARKS ON THE EQUATIONS OF MOTION

The standard way to describe the dynamics of the macroscopic phase in Josephson links is the resistively and capacitively shunted junction (RCSJ) model, which contemplates an ideal Josephson junction shunted by a
resistor and a capacitor. The resistor models the dissipative effect of incoherent quasiparticle tunneling through the junction, while the capacitor accounts for the charging energy, which plays the role of a kinetic energy for the phase. In the absence of driving currents, the RSCJ model reads

\[
\dot{N}(t) = \frac{E_J}{\hbar} \sin \phi(t) - \frac{\hbar}{4e^2R} \dot{\phi}(t) + \frac{1}{2e} I(t), \tag{2.1}
\]

\[
\dot{\phi}(t) = \frac{E_C}{\hbar} N(t), \tag{2.2}
\]

where \(N\), the number of transferred Cooper pairs across the junction, is canonically conjugate to the relative phase \(\phi\). We have introduced the notation \(E_J = \hbar I_c/2e\) and \(E_C = 4e^2/C\) for the Josephson coupling energy and the charging energy respectively (\(I_c\) and \(C\) are the critical current and the capacitance of the junction). \(I(t)\) is a stochastic process with zero mean. At high temperatures it is related to the resistance by Einstein’s relation \((I(t)I(0)) = 2k_B T \delta(t)/R\), where \(R\) is the resistance of the junction in the normal state. Eq. \(2.2\) is recognized as the ac Josephson relation.

The effect of the EM vacuum fluctuations may be introduced through the following argumentation: Written in language of the Coulomb gauge (as is standard in these physical contexts\(^{28,29}\)), Eqs. \(2.1\) and \(2.2\) relate the phase to gauge invariant quantities. More specifically, they include the effect of the longitudinal electric field, which in its simplest form yields a circulation

\[
\int \mathbf{E}_\| \cdot dr = 2eN/C. \tag{2.3}
\]

Thus one may view the transverse EM modes as the cause of additional voltage fluctuations \(V(t) = \int \mathbf{E}_\perp \cdot dr\) with zero mean \((V(t)) = 0\). This interpretation is physically appealing but still insufficient to understand in depth the detailed nature of the effect of \(\mathbf{E}_\perp\) on \(\phi\).

For one thing, we may wonder whether the fluctuating transverse voltage may generate a “slow” contribution to the r.h.s. of \(2.2\) which would relate to the “fast” part through an appropriate expression of the fluctuation-dissipation theorem. This would be analogous to the already noted relation between the second and third terms on the r.h.s. of Eq. \(2.1\). In this regard we note that the slow term proportional to \(\dot{\phi}\) in \(2.1\) is but the Markovian limit of a more general retarded expression\(^{31}\),

\[
\int_{-\infty}^{t'} \Gamma_{qp}(t - t') \dot{\phi}(t') dt' \rightarrow (\hbar/4e^2R) \dot{\phi}(t). \tag{2.4}
\]

Clearly, the fluctuating current \(I(t)\) plays a role for \(\phi\) analogous to that which the fluctuating transverse potential \(V(t)\) represents for \(N\). This observation suggests that \(N\) and \(\phi\) satisfy formally similar retarded equations of motion, which must be the form

\[
\dot{N}(t) = -\frac{E_J}{\hbar} \sin \phi(t) - \int_{-\infty}^{t'} \Gamma_{qp}(t - t') \dot{\phi}(t') dt' + \frac{1}{2e} I(t), \tag{2.5}
\]

\[
\dot{\phi}(t) = \frac{E_C}{\hbar} N(t) + \int_{-\infty}^{t'} \Gamma_{EM}(t - t') \dot{N}(t') dt' + \frac{2e}{\hbar} V(t). \tag{2.6}
\]

A rigorous derivation and ultimate justification of this result is possible after the analysis given in Secs. \(\text{III} I\) and \(\text{IV} V\) where particle-bath couplings are derived which yield Eqs. \(2.5\) and \(2.6\) as the Heisenberg equations of motion for \(N\) and \(\phi\). The dissipation kernels \(\Gamma_{EM}(t)\) and \(\Gamma_{qp}(t)\) are given in terms of the spectral functions \(J_{EM}(\omega)\) and \(J_{qp}(\omega)\), respectively, through the relations

\[
\Gamma_{EM}(t) = \int_{0}^{\infty} J_{EM}(\omega) \cos(\omega t) d\ln(\omega) \tag{2.7}
\]

\[
\Gamma_{qp}(t) = \int_{0}^{\infty} J_{qp}(\omega) \cos(\omega t) d\ln(\omega) \tag{2.8}
\]
The current and voltage fluctuations are related to the spectral functions through the fluctuation-dissipation theorem,

\begin{equation}
\langle V(t)V(0) \rangle = \frac{\hbar^2}{8e^2} \int_{-\infty}^{\infty} J_{\text{EM}}(\omega) \frac{\exp(i\omega t)}{1 - \exp(-\hbar\beta\omega)} d\omega \tag{2.9}
\end{equation}

\begin{equation}
\langle I(t)I(0) \rangle = 2e^2 \int_{-\infty}^{\infty} J_{\text{qp}}(\omega) \frac{\exp(i\omega t)}{1 - \exp(-\hbar\beta\omega)} d\omega . \tag{2.10}
\end{equation}

As expressed in Eqs. (2.6) and (2.10), the symmetry between \( \phi \) and \( I(t) \) on the one hand and \( N \) and \( V(t) \) on the other hand, is most evident. Both stochastic “sources” obey two independent fluctuation–dissipation theorems. The study of this type of “mixed” dissipation has been pioneered by Leggett\textsuperscript{26}, for quantum systems. Dissipation due to coupling to the momentum variable has been addressed for an ohmic bath in Ref. \textsuperscript{28}. The formal symmetry between \( \phi \) and \( N \) is broken not only by the different potential terms, already present in Eqs. (2.5) and (2.6), but also by the different form of the spectral functions \( J_{\text{EM}}(\omega) \) and \( J_{\text{qp}}(\omega) \) describing the noise. \( J_{\text{qp}}(\omega) \) is usually taken to be ohmic\textsuperscript{26,31},

\begin{equation}
J_{\text{qp}}(\omega) = \frac{\hbar\omega}{2\pi e^2 R}, \tag{2.11}
\end{equation}

which through Eq. (2.4) makes Eq. (2.6) become Eq. (2.1). It also guarantees that Eq. (2.10) achieves the correct high temperature limit in the form of the Einstein relation mentioned above. By contrast, we expect that, in three-dimensional free space, the spectral function of the electromagnetic field is cubic. Actually, a detailed calculation performed in Sec. III yields

\begin{equation}
J_{\text{EM}}(\omega) = \frac{8d^2\alpha\omega^3}{3\pi e^2}, \tag{2.12}
\end{equation}

where \( d \) is the distance between the two electronic clouds and \( \alpha \simeq 1/137 \) is the fine structure constant. Moreover, the spectral function \( J_{\text{EM}}(\omega) \) has to be regularized by an upper cutoff frequency \( \omega_c \), which is the inverse response time of the electromagnetic field. This point is discussed in detail in Sec. III, where the cutoff frequency is shown to be determined by the surface area of the electrodes. Typical values for the diameter of the electrodes of Josephson point contacts are about 100 nm. This translates to a frequency of \( \omega_c \simeq 2\pi \times 10^{18} \) Hz which is higher than typical values of the Josephson plasmon frequency \( \omega_{Jp} = \sqrt{E_J/E_C} = \sqrt{2eI_C/\hbar C} \sim 2\pi \times 10^{10} \) Hz by several orders of magnitude.

Being the cutoff frequency high, we may approximate Eq. (2.6) by a form which is local in time:

\begin{equation}
\dot{\phi}(t) = \frac{E_C}{\hbar} N(t) + \frac{80d^2\omega_c}{3\pi e^2} \ddot{N}(t) - \frac{40d^2}{3e^2} \dddot{N}(t) + \frac{2e}{\hbar} V(t) . \tag{2.13}
\end{equation}

Eliminating \( \phi \) in favor of \( N \) and vice versa, Eqs. (2.1) and (2.13) become two decoupled fourth-order differential equations whose inhomogeneous, fast fluctuating parts are however related to each other. Specifically, if we approximate \( \sin \phi \simeq \phi \), we obtain

\begin{equation}
-\frac{h}{3e^2 RC} \dddot{N}(t) - \frac{4d^2}{3e^2} \left( \frac{E_J}{\hbar} - \frac{\hbar\omega_c}{2\pi e^2 R} \right) \dddot{N}(t) + \left( 1 + \frac{80d^2 E_J\omega_c}{3\pi \hbar e^2} \right) \dddot{N}(t) + \frac{1}{RC} \dddot{N}(t) + \omega_{Jp}^2 N(t) = \frac{1}{2e} I(t) - \frac{2e E_J}{\hbar} \dot{V}(t) , \tag{2.14}
\end{equation}

\begin{equation}
-\frac{h}{3e^2 RC} \dddot{\phi}(t) - \frac{4d^2}{3e^2} \left( \frac{E_J}{\hbar} - \frac{\hbar\omega_c}{2\pi e^2 R} \right) \dddot{\phi}(t) + \left( 1 + \frac{80d^2 E_J\omega_c}{3\pi \hbar e^2} \right) \dddot{\phi}(t) + \frac{1}{RC} \dddot{\phi}(t) + \omega_{Jp}^2 \phi(t) = \frac{E_C}{2e\hbar} I(t) + \frac{80d^2\omega_c}{3\pi e^2} \dddot{I}(t) - \frac{20d^2}{3e^2} \chi^{-1}(\omega) \dddot{\phi}(t) + \frac{2e}{\hbar} \dddot{V}(t) . \tag{2.15}
\end{equation}

Here we stress that \( I(t) \) and \( V(t) \) are two uncorrelated stochastic processes. Nevertheless, their simultaneous coupling to the junction generates interference terms. The most obvious signature of such interference is the fourth order time derivative in Eqs. (2.14) and (2.15), since that term vanishes whenever any of the two coupling constants \( R^{-1} \) or \( \alpha \) vanishes.

We observe that the left hand sides of Eqs. (2.14) and (2.15) are identical except for an interchange of \( N \) and \( \phi \). This means that the stochastic differential equations satisfied by \( N \) and \( \phi \) only differ in the fast fluctuating part. The structure of Eqs. (2.14) and (2.15) suggests the introduction of a generalized “response” function

\begin{equation}
\chi^{-1}(\omega) = \omega_{Jp}^2 - \omega^2 + \frac{E_J}{\hbar} \left( \frac{i\pi}{2} - \frac{\omega_c}{\omega} \right) J_{\text{EM}}(\omega) + \frac{E_C}{\hbar} \frac{i\pi}{2} J_{\text{qp}}(\omega) + \frac{i\pi}{2} J_{\text{qp}}(\omega) J_{\text{EM}}(\omega) \left( \frac{i\pi}{2} - \frac{\omega_c}{\omega} \right) . \tag{2.16}
\end{equation}
A more general form of the response function (2.14), valid for arbitrary spectral densities, will be given in Sec. V. In Fourier space, Eqs. (2.14) and (2.15) read

$$N(\omega) = \chi(\omega) \left[ \frac{i\omega}{2e} I(\omega) - \frac{2e}{\hbar} \left( \frac{E_C}{\hbar} + \frac{i\pi}{2} J(\omega) \right) V(\omega) \right] \quad (2.17)$$

$$\phi(\omega) = \chi(\omega) \left[ \frac{i\omega}{2e} V(\omega) + \left( \frac{E_C}{\hbar} + \frac{i\pi}{2} J(\omega) - \frac{\omega}{\omega} J(\omega) \right) \frac{1}{2e} I(\omega) \right]. \quad (2.18)$$

Ultimately we are interested in autocorrelation functions of the form

$$\langle \phi(t)\phi(t') \rangle = \frac{1}{2\pi} \int \langle |\phi(\omega)|^2 \rangle e^{i\omega(t-t')} d\omega ,$$

where interference terms must clearly play a role.

QED effects can be studied more clearly in Eqs. (2.14) and (2.15) if one neglects the effect of the electronic environment, i.e. if one eliminates in Eqs. (2.14) and (2.15) all terms depending on $R^{-1}$ or $I(t)$. One obtains

$$- \frac{4E_J\alpha d^2}{3c\hbar} \cdots N(t) + \left( 1 + \frac{8\alpha d^2 E_J\omega_c}{3\pi\hbar^2} \right) \tilde{N}(t) + \omega_{3P}^2 N(t) = - \frac{2\varepsilon E_J}{\hbar} V(t) \quad (2.20)$$

$$- \frac{4E_J\alpha d^2}{3c\hbar} \phi + \left( 1 + \frac{8\alpha d^2 E_J\omega_c}{2\pi\hbar^2} \right) \tilde{\phi}(t) + \omega_{3P}^2 \phi(t) = \frac{2e}{\hbar} V(t), \quad (2.21)$$

which are equations of the Abraham–Lorentz type\cite{23}. The fact that the EM field couples to the particle number and not (directly) to the phase is reflected in the different r.h.s. of Eqs. (2.20) and (2.21), which are proportional to $V(t)$ and $\dot{V}(t)$, respectively. This means that, although their slow parts obey identical equations, $\phi$ and $N$ experience different fluctuations terms and thus possess different power spectra. Close inspection reveals that (2.20) and (2.21) are the classical equations of motion for a dipole in a harmonic potential interacting with its own radiation. Indeed it can have been anticipated that the equation of motion for $N$ should be that of a radiating dipole, since the relative particle number $N$ couples to the EM vacuum through the electric dipole which it generates, $P = 2\varepsilon Nd$. In effect, neglecting the fluctuating term as well as the mass renormalization terms, Eq. (2.20) can be written as

$$\ddot{P}(t) + \omega_{3P}^2 P(t) - \tau \ddot{P}(t) = 0 , \quad (2.22)$$

with $\tau = 4E_J\alpha d^2/3\hbar^2 c^2$. For a radiating harmonic dipole of charge $e$ and mass $m$ the characteristic time $\tau$ is given by\cite{24} $\tau = 2e^2/3mc^3$. We conclude that the equation of motion for $N$ is that of a radiating dipole with effective mass $m = \hbar^2/3E_Jd^2$.

### III. COUPLING TO THE ELECTROMAGNETIC ENVIRONMENT

A general description of the electrodynamics in a Josephson junction with electrodes in the $x - y$ plane is given by a two dimensional sine–Gordon Hamiltonian for the position dependent phase\cite{25} $\phi(x, y)$. The characteristic length scale of the theory is the Josephson penetration depth $\lambda_J$. A small Josephson junction is characterized by an area $A$ of the electrodes which is much smaller than $\lambda_J$. In this limit spatial variations of the phase in the $x - y$ plane are energetically highly unfavorable. They are suppressed like $\sim \sqrt{A}/\lambda_J$. Then, the sine-Gordon model reduces to the Hamiltonian of a pendulum, first discussed by Anderson\cite{25}.

$$H_0 = E_J (1 - \cos \phi) + \frac{E_C}{2} N^2 + H_{EM} . \quad (3.1)$$

The phase difference across the junction and the number of transferred Cooper pairs are canonical conjugates: $[\phi, N]_c = i$. As indicated in Sec. II we introduce the coupling to the electromagnetic environment replacing $\phi$ by its gauge-invariant expression $\phi = \phi_1 - \phi_2 + \Lambda_{12}$, where $\Lambda_{12} \equiv (2e/\hbar c) \int_0^T d\tau \cdot A(r)$. The line integral connects two points 1 and 2 lying deep enough in the left and right superconductor, where phase fluctuations are completely suppressed. That means, it extends over a minimal length of $2\lambda_L$, where $\lambda_L$ is the penetration depth of the magnetic field. In dissipative quantum systems one usually assumes the influence of the system onto the bath to be small. However in our case the system is made by superconductors which of course alter drastically the low frequency behavior of the electromagnetic vacuum. The screening of the EM field out of the superconductor should be taken into
account by an appropriate choice of the wave functions $A_{k\lambda}(r)$ of the normal modes $k\lambda$ in terms of which the vector potential $A(r)$ is expanded. These must be the modes which diagonalise the EM Hamiltonian. Then we can write

$$A(r) = \sum_{k\lambda} \left( \frac{2\pi\hbar c^2}{\omega_k} \right)^{1/2} \tilde{e}_\lambda \left( A_{k\lambda}(r) \, a_{k\lambda} + \text{H.c.} \right),$$

$$H_{\text{EM}} = \sum_{k\lambda} \hbar \omega_k a_{k\lambda}^\dagger a_{k\lambda}. \quad (3.2)$$

The influence of the superconductor on the EM field is encoded in the set of orthonormal functions $A_{k\lambda}(r)$. They are solutions of a Helmholtz equation with boundary conditions imposed by the geometry of the junction and the nature of the superconducting state. $H_{\text{EM}}$ is expressed in its usual form. It is also convenient to expand

$$\Lambda_{12} = \sum_{k\lambda} \left( \frac{8\pi \varepsilon^2}{\hbar \omega_k} \right)^{1/2} \left( A_{k\lambda} a_{k\lambda} + A_{k\lambda}^* a_{k\lambda}^\dagger \right),$$

$$A_{k\lambda} = \int_1^2 A_{k\lambda}(r) \tilde{e}_\lambda \, \text{d}r \quad (3.3)$$

Shifting $\phi$ by $-\Lambda_{12}$ via a unitary transformation $H \rightarrow U^{-1}HU$ with $U = \exp(iN\Lambda_{12})$ one obtains

$$H = H_0 - N \sum_{k\lambda} (8\pi \varepsilon^2 \hbar \omega_k)^{1/2} \left( i A_{k\lambda} a_{k\lambda} - i A_{k\lambda}^* a_{k\lambda}^\dagger \right) + 8\pi^2 \sum_{k\lambda} |A_{k\lambda}|^2. \quad (3.4)$$

If the superconductors are equal on both sides one has a reflection symmetry with respect to the $x-y$ plane. This symmetry and the trivial observation that $\Lambda_{12} = -\Lambda_{21}$ are sufficient to conclude that all $A_{k\lambda}$ are antisymmetric in $z_2 - z_1$ and purely real. This allows us to further simplify Eq. (3.4). Finally, with $a_{k\lambda} \rightarrow i a_{k\lambda}$ we arrive at

$$H = E_f (1 - \cos \phi) + \left( \frac{E_c}{2} + \sum_{k\lambda} \mu_{k\lambda}^2 \hbar \omega_k \right) N^2 + N \sum_{k\lambda} \mu_{k\lambda} \left( a_{k\lambda} + a_{k\lambda}^\dagger \right) + \sum_{k\lambda} \hbar \omega_k a_{k\lambda}^\dagger a_{k\lambda}, \quad (3.5)$$

where $\mu_{k\lambda} \equiv (8\pi \varepsilon^2 \hbar \omega_k)^{1/2} A_{k\lambda}$. The Hamiltonian (3.5) is of the Caldeira–Leggett type with a periodic potential. However, there is a subtle but important difference. If, as is standard, we interpret $\phi$ as the position operator and $N$ as the momentum, then we must conclude that the EM field couples to the momentum operator instead of to the position. This type of coupling is often referred to as anomalous coupling.27,32

### A. QED spectral density

In order to obtain the spectral density $J_{\text{EM}}(\omega)$, defined as

$$J_{\text{EM}}(\omega) \equiv \frac{2}{\hbar^2} \sum_{k\lambda} \mu_{k\lambda}^2 \delta(\omega - \omega_k) \quad (3.6)$$

one must in principle solve for the normal modes $A_{k\lambda}(r)$, which are the solutions of a complicated boundary value problem. For simplicity, we approximate the EM normal modes as those of free space with appropriate cutoffs which account for the real geometry. We expect this to be a good approximation for point-contact Josephson links while it will provide only a qualitative description in the case of parallel plates separated by a dielectric. To obtain $J_{\text{EM}}(\omega)$ we substitute the sum by an integral, $\sum_{k\lambda} \rightarrow \left( V/8\pi^3 \right) \sum \int k^2 \text{d}k \text{d}(\cos \theta_k) \text{d}\phi_k$, and take advantage of the transversality of the electromagnetic field $\tilde{\varepsilon}_z \cdot e_k = 0$. We choose $\tilde{\varepsilon}_1 = e_\theta$ and $\tilde{\varepsilon}_2 = e_\phi$. With $e_\theta \cdot e_z = 0, e_\theta \cdot e_z = -\sin \theta_k$ and $e_k \cdot e_z = \cos \theta_k$ we arrive at

$$J_{\text{EM}}(\omega) = \frac{\alpha}{\pi^2} \omega \int_0^1 \text{d}y \left( \frac{1}{y^2} - 1 \right) \sin^2 \left( \frac{\omega d}{c} y \right) - \sin^2 \left( \frac{2\omega d}{c} \right) \sin \left( \frac{2\omega d}{c} \right) - \frac{1}{2} \right). \quad (3.7)$$
\( J_{\text{EM}}(\omega) \) has the cubic infrared behavior predicted by the electric dipole approximation\(^{33}\). In the ultraviolet regime it diverges quadratically. Specifically, we have

\[
\begin{align*}
J_{\text{EM}}(\omega) &= \frac{8\alpha d^2}{3\pi c^2} \omega^3 \quad \text{for} \quad \frac{\omega_d}{c} \ll 1, \\
J_{\text{EM}}(\omega) &= \frac{2\alpha d}{c} \omega^2 \quad \text{for} \quad \frac{\omega_d}{c} \gg 1.
\end{align*}
\]  

\( (3.8) \)

Despite being a sum of oscillating functions, \( J(\omega) \) is monotonically increasing. The crossover from cubic to quadratic behavior takes place on the scale of \( c/d \). Finally, as already mentioned above, the reduction of the two dimensional sine–Gordon model to Eq. \((3.1)\) requires a constant \( A_c/d \). This is taken into account by an ultraviolet cutoff function \( g(\omega/\omega_c) \). Any function \( g \) with \( g(0) = 1 \) whose modulus vanishes at infinity faster than any power-law decay serves as a cutoff function. One can choose also an algebraic cutoff function provided that \( \lim_{\omega \to \infty} J_{\text{EM}}(\omega) = 0 \). We take the cutoff frequency \( \omega_c = c/\sqrt{A} \), where \( A \) is the surface of the electrodes. For point contact structures one usually has \( \sqrt{A} \gg d \). Therefore, in order to obtain rough quantitative estimates one may use the simple form of \( J_{\text{EM}}(\omega) \) in the infrared limit given in Eq. \((3.8)\) multiplied by the cutoff function. We express all lengths of the structure relative to the cutoff frequency \( \omega_c \approx c/\sqrt{A} \). This leads to the introduction of the aspect ratio \( f = d/\sqrt{A} \). In a point-contact experiment, we usually have \( f \lesssim 1 \). Now the spectral density reads

\[
J_{\text{EM}}(\omega) = \gamma \frac{\omega^3}{\omega_c^2} g\left( \frac{\omega}{\omega_c} \right),
\]

\( (3.9) \)

where the dimensionless coupling constant

\[
\gamma = \frac{8\alpha f^2}{3\pi}
\]

\( (3.10) \)

has been introduced, which hereafter will be treated as independent of the cutoff. Eq. \((3.9)\) is the spectral function we will use in the remainder of this work. Strictly speaking the cutoff \( \omega_c = c/\sqrt{A} \) only applies to \( k_\parallel \). In the direction perpendicular to the junction the cutoff length is much smaller and of the order of the Thomas–Fermi screening length \( \lambda_{\text{TF}} \approx 10\AA \). However, since the perpendicular component of the mode wave vector does not couple to \( \phi \), we may assign to \( \int dk_z \) the cutoff which we wish. In particular, we choose an isotropic cutoff for convenience.

**B. Equilibrium correlation functions**

The Hamiltonian \((3.9)\) is still difficult to handle due to the nonlinearity of the \( \cos \phi \) term. It has been studied extensively in its different aspects and with a variety of different methods\(^{35,36}\). In our present study we are interested in the equilibrium dynamics when \( k_B T, E_C \ll E_J \). Then \( \langle \phi^2 \rangle \ll 1 \) and we can reasonably approximate \( 1 - \cos \phi \approx \phi^2/2 \). The problem reduces to Ullersmas model for a damped harmonic oscillator\(^{35,36}\). For thermal initial conditions the symmetrized autocorrelation function of the particle number \( C_{\text{NN}}^{(+)}(t) = \frac{1}{2} \langle [N(t)N(0)]^+ \rangle \) and of the phase difference \( C_{\phi\phi}^{(+)}(t) = \frac{1}{2} \langle [\phi(t)\phi(0)]^+ \rangle \) can be expressed in terms of the susceptibility\(^{35,36}\)

\[
\chi(\omega) = \chi' + i \chi''.
\]

\( (3.11) \)

In the last line of Eq. \((3.11)\) we have introduced the function

\[
\tilde{f}(\omega) = \omega^2 \int_0^\infty \frac{\omega' f(\omega')}{\omega' (\omega'^2 - \omega^2 - \omega'^2)} d\omega' + i \frac{\pi}{2} f(|\omega|),
\]

\( (3.12) \)

which is defined for any sufficiently well behaved function \( f(\omega) \). We note that \( \tilde{f}(\omega) \) is always symmetric, although \( f(\omega) \) is not necessarily so. One can consider \( \tilde{f}(\omega) \) as a kind of symmetrized Riemann transform of \( f(\omega) \).
To evaluate the quantities in Eq. (3.11) we employ the cubic spectral density given in Eq. (3.9), which scales with \( \omega_c^{-2} \). This property guarantees finite results in the Markovian limit \( \omega_c \to \infty \). By definition, the cutoff frequency is much larger than \( E_J \) and the Josephson–plasmon frequency \( \omega_{JP} \). Then, to leading order in \( \omega_c^{-1} \) the contribution of the principal value is just \( N_c \omega_c \), where \( N_c \equiv \int_0^\infty g(x) dx \). Below we take \( g(x) = \Theta(1-x) \), which implies \( N_c = 1 \). At zero temperature, the integrals in Eq. (3.11) can be expressed in terms of the roots of the equation

\[
-i\omega^3 - \left( \frac{2\hbar \omega_c^2}{\pi \gamma E_J} + \frac{2\omega_c}{\pi} \right) \omega^2 + \frac{2\hbar}{\pi \gamma E_J} \omega^2 \omega_{JP}^2 = 0 .
\]  
(3.13)

The polynomial has one purely imaginary root \( i\lambda \) and a pair of roots \( iz, iz^* \). We obtain

\[
C_{NN}^{(+)} (0) = \frac{1}{\pi^2 \gamma (\lambda - z)(\lambda - z^*)} \left[ \text{Re} \left( \ln \frac{\omega_c^2 + z^2}{z^2} - \ln \frac{\omega_c^2 + \lambda^2}{\lambda^2} \right) - \frac{\text{Re} (z - \lambda)}{\text{Im} (z)} \text{Im} \left( \ln \frac{\omega_c^2 + z^2}{z^2} \right) \right]
\]

\[
C_{\phi \phi}^{(+)} (0) = \frac{\hbar^2}{\pi^2 E_J^2 \gamma (\lambda - z)(\lambda - z^*)} \left[ \text{Re} \left( \lambda^2 \ln \frac{\omega_c^2 + \lambda^2}{\lambda^2} - z^2 \ln \frac{\omega_c^2 + z^2}{z^2} \right) - \frac{\text{Re} (z - \lambda)}{\text{Im} (z)} \text{Im} \left( z^2 \ln \frac{\omega_c^2 + z^2}{z^2} \right) \right]
\]

(3.14)

(3.15)

One calculates the parameters \( \lambda = \frac{2\hbar}{\pi E_J \gamma} \omega_c^2 + \frac{2}{\pi} \omega_c + \frac{\pi E_J \gamma \omega_{JP}^2}{2\hbar} \frac{1}{\omega_c^2} + O(\omega_c^{-3}) \)

\[
z_1 = -\frac{\pi E_J \gamma \omega_{JP}^2}{4\hbar} \frac{1}{\omega_c^2} + O(\omega_c^{-3}) \]

\[
z_2 = \omega_{JP} - \frac{E_J \gamma \omega_{JP}}{2\hbar} \frac{1}{\omega_c} + \frac{3E_J^2 \gamma^2 \omega_{JP}^2}{8\hbar^2} \frac{1}{\omega_c^2} + O(\omega_c^{-3}) .
\]

(3.16)

The Josephson plasmon frequency acquires an imaginary part \( (z_1) \) only to second order in \( \omega_c^{-1} \). On the other hand, \( z_2 = -\text{Re}(iz) \) decreases by an amount \( E_J \gamma \omega_{JP} N_c / 2\hbar \omega_c \), which may be viewed as a macroscopic Lamb shift. Unlike in the ohmic case, a crossover to overdamped oscillations is not possible if \( \omega_c \) remains large enough. Expanding Eq. (3.15) in powers of \( \omega_c^{-1} \) one obtains

\[
\langle \phi^2 \rangle = \frac{1}{2} \sqrt{\frac{E_C}{E_J}} \left[ 1 + \frac{E_J \gamma}{2\hbar \omega_{JP}} + \left( \frac{E_J \gamma}{\hbar \omega_{JP}} + \frac{3}{2} \right) \frac{E_J \gamma}{\hbar \omega_c} + O(\omega_c^{-2}) \right]
\]

\[
\langle N^2 \rangle = \frac{1}{2} \sqrt{\frac{E_C}{E_J}} \left[ 1 - \frac{E_J \gamma}{2\hbar \omega_c} + O(\omega_c^{-2}) \right].
\]

(3.17)

In the limit \( \gamma \to 0 \) the susceptibility of the unperturbed harmonic oscillator is recovered and we obtain the well-known results for \( \langle N^2 \rangle \) and \( \langle \phi^2 \rangle \). The sign of the correction reflects that the QED field tends to measure \( N \) and the expense of increasing the uncertainty in \( \phi \).

For finite temperatures it is convenient to introduce an algebraic cutoff function. We choose \( g(x) = (1 + x^4)^{-1} \). With this choice the number of zeros of \( \chi^{-1}(\omega) \) is finite and we are in a position to evaluate both integrals in Eq. (3.11) exactly. We obtain the following low temperature dependence.

\[
\langle N^2 \rangle = \langle N^2 \rangle_{T=0} + \frac{\pi^4}{30\hbar^4 E_C \omega_{JP}^2 \omega_c^2} (k_B T)^4 + O(T^6)
\]

\[
\langle \phi^2 \rangle = \langle \phi^2 \rangle_{T=0} + \frac{4\pi^4}{63\hbar^6 \omega_{JP}^4 \omega_c^2} (k_B T)^6 + O(T^8) .
\]

(3.18)

Note that for \( \omega_c \to \infty \) there are no finite \( T \) corrections. A sketch of the derivation of these results is given in Appendix A.

The results in Eqs. (3.17) and (3.18) deserve some discussion. First we concentrate on the mean square of the phase difference. One can write the leading contribution to \( \langle \phi^2 \rangle \) in Eq. (3.17) in the following way

\[
\langle \phi^2 \rangle = \frac{1}{2} \sqrt{\frac{E_C}{E_J}} + \frac{2\alpha}{3\pi} f^2 .
\]

(3.19)

This equation is one of the central results of this work. It clearly distinguishes between the uncertainty of the phase difference due to the longitudinal electromagnetic field and that due to the transverse electromagnetic field. The QED
correction depends on the details of the junction only through the aspect ratio $f$. Junction with $f$ close to unity are possible in the case of point contacts. However, the universal constant $2a/3\pi \approx 1.5 \times 10^{-3}$ renders the influence of the transverse field a tiny effect. The contribution of the longitudinal field contains several quantities which are experimentally relevant. First, we have the critical current in $E_J = h I_c/2e$. It may be approximated by the formula of Ambegaokar and Baratoff

$$I_c(T) = \frac{\pi \Delta(T)}{2eR} \tanh \frac{\Delta(T)}{2k_B T}$$

at zero temperature. It may be viewed as decreased by the QED field through and effective Debye-Waller factor. Second, we have the capacitance in $E_C = 4e^2/C$.

The case of a plate capacitance is characterized by $\sqrt{A} \gg d > k_F^{-1}$. This limit corresponds to an aspect ratio $f \ll 1$. The capacitance can be approximated by $C = eA/4\pi d$. The normal state resistance of a tunneling junction is given by $R^{-1} = e^2 N(E_F) A k_F^2 |T|^2/2$, where $N(E_F)$ is the density of states per spin and $|T|^2$ is the average transmission probability for electrons at the Fermi surface. For a rectangular barrier of height not much greater than $E_F$, one may approximate

$$|T|^2 \approx a e^{-b k_F d'},$$

where $a$ and $b$ are numbers of order unity and $d' \lesssim d$ is the barrier thickness. For qualitative estimates we may assume $d' = d$ which underestimates slightly the QED effects. For simplicity, we also assume $a \approx b \approx 1$ in the ensuing discussion. If we only include the leading QED correction, Eq. 3.11 becomes

$$\langle \phi^2 \rangle = \frac{2}{3\pi A_F} \left[ 12 \left( \frac{\pi^5 \xi_0}{e a_0} \right)^{1/2} \sqrt{d_F e^{d_F/2} + \alpha d_F^2} \right] $$

where $d_F \equiv k_F d$, $A_F \equiv A k_F^2$, $a_0$ is the Bohr radius, and $\xi_0 = \hbar v_F / \pi \Delta$ is the zero temperature coherence length. Since typically $\xi_0 \sim 2000 a_0$, the ratio between the two terms within square brackets is $\sim (150/\alpha) d_F^{3/2} \exp(d_F/2)$, which takes a minimum value of $\sim 10^4$ at $d_F = 3$. We conclude that, for plate capacitances and within the approximation of free space photons, EM vacuum fluctuations contribute negligibly to the quantum phase spread.

The situation is not much better if one considers the case $d \ll \sqrt{A}$, as might correspond to e.g. a tunneling point contact. The capacitance is then approximated as $C \sim \sqrt{A}$. Eq. 3.11 is replaced by

$$\langle \phi^2 \rangle = \frac{2}{3\pi A_F} \left[ 12 \left( \frac{\pi^5 \xi_0}{e a_0} \right)^{1/2} \alpha d_F + \alpha d_F^2 \right],$$

For the QED term to be comparable to the charging contribution, one would need unrealistically small areas which would effectively suppress the supercurrent.

The case of a planar junction which has been considered in the first place presents the unfortunate property that both the longitudinal and transverse contributions scale identically with the junction area. That makes it difficult to overturn the natural smallness of the QED correction. We note however that we have considered the spectrum of photons in free space, an approximation that was admitted to be less adequate for the case of broad Josephson junctions. In fact, the structure of Eq. 3.11 suggests the possibility of designing devices where the coupling of the phase to the photon field could be amplified.

IV. COUPLING TO THE ELECTRONIC BATH

A. Effective Hamiltonian

A complete treatment of the macroscopic phase dynamics in Josephson junctions must include the effect of quasi-particles and Cooper pairs. It was shown in Ref. 23 within a path-integral language that the electronic degrees of freedom can be taken into account by a coupling to two independent oscillator baths. Starting from a microscopic tunneling Hamiltonian, they derived an effective Lagrangian in which the phase difference couples to two heat baths which we label with superindices ($\pm$). Here we propose an alternative Hamiltonian–equation–of–motion approach to derive the coupling and spectra of these effective baths:

$$H = E_J (1 - \cos \phi) + \frac{E_C}{2} N^2 + \sum_i \hbar \omega_i \left[ b_i^{(+)} b_i^0 \cos \left( \phi \frac{\hbar \omega_i}{2} \right) \right]$$

where $b_i^{(+)}$ and $b_i^0$ are creation and destruction operators for the $i$th mode of the electronic bath. This Hamiltonian is quadratic in the phase, which implies that the phase will fluctuate independently of the electronic degrees of freedom. The electronic contribution to the Hamiltonian is given by

$$H_e = \sum_i \frac{\hbar \omega_i}{2} b_i^{(+)} b_i^0 + \frac{\lambda_i^{(+)}(\phi)}{\hbar \omega_i} \cos \left( \phi \frac{\hbar \omega_i}{2} \right)$$

where $\lambda_i^{(+)}(\phi)$ are the electronic contributions to the coupling to the $i$th mode of the electronic bath.
The order parameters \( \Delta(\mathbf{r}) \) we suppressed the dynamics of the electromagnetic field for momentary convenience. The operators \( \Psi \) given.

The coupling constants \( \lambda_i^{(\pm)} \) are defined through the spectral densities they must yield. Specifically, the expression

\[
J^{(\pm)}(\omega) = \hbar^{-2} \sum_i \lambda_i^{(\pm)2} \delta(\omega - \omega_i)
\]

must be equal to

\[
J^{(\pm)}(\omega) = \frac{|T|^2}{2\hbar^2} \int d\omega' N_{qp}(\omega') N_{qp}(\omega' - \omega) \left( 1 \pm \frac{\Delta^2/\hbar^2}{\omega'^2 - \omega^2/4} \right)
\]

\[
\left\{ \tanh \left[ \frac{\hbar\lambda}{2} (\omega + \omega') \right] - \tanh \left[ \frac{\hbar\lambda}{2} (\omega' - \omega) \right] \right\},
\]

where

\[
N_{qp}(\omega) = N(E_F) \Theta (\hbar\omega - \Delta) \frac{\hbar\omega}{\sqrt{\hbar^2\omega^2 - \Delta^2}}
\]

is the quasiparticle density of states. We derive this Hamiltonian in subsection IV B avoiding the path–integral formalism used in Ref. 23. Our criterion will be that the heat baths must yield the correct equations of motion for the phase \( \phi(t) \). The reader who is familiar with the subject or is not interested in this point might skip the next subsection an take the Hamiltonian for given.

### B. Derivation of the effective Hamiltonian including coupling to electronic degrees of freedom

Our starting point is the standard tunneling Hamiltonian for two coupled superconductors in the Bogoliubov–de Gennes mean–field approximation. We suppress the dynamics of the electromagnetic field for momentary convenience. The operators \( H(\mathbf{r}, \mathbf{r}') \) as well as \( \Delta(\mathbf{r}, \mathbf{r}') \) obey the self-consistency condition within the bulk of each superconductor. In Eq. (4.6) we suppressed the dynamics of the electromagnetic field for momentary convenience. The operators \( H(\mathbf{r}, \mathbf{r}') \) as well as \( \Delta(\mathbf{r}, \mathbf{r}') = |\Delta(\mathbf{r}, \mathbf{r}')| \exp(i\phi(\mathbf{r}, \mathbf{r}')) \) are local in space. After a further gauge transformation

\[
U = \exp \left\{ \frac{i}{2} \int d\mathbf{r} \, \Psi^\dagger(\mathbf{r}) \right. \delta \left[ \phi(\mathbf{r}_1), \phi(\mathbf{r}_1), \phi(\mathbf{r}_2), \phi(\mathbf{r}_2) \right] \left. \Psi(\mathbf{r}) \right\},
\]

and in the grand canonical ensemble, the entries of \( \Omega \) in Eq. (4.6) look explicitly as follows:

\[
H_{L,R}(\mathbf{r}, \mathbf{r}') = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{2\hbar}{m} \nabla \cdot \mathbf{v}_{L,R}(\mathbf{r}) \right] + \frac{e^2N}{C} \delta(\mathbf{r} - \mathbf{r}')
\]

\[
\Delta_{L,R}(\mathbf{r}, \mathbf{r}') = |\Delta_{L,R}(\mathbf{r})| \delta(\mathbf{r} - \mathbf{r}')
\]

\[
T_{\mathbf{r}_1, \mathbf{r}_2} = \exp(i\phi(\mathbf{r}_1, \mathbf{r}_2)/2) \tilde{T}_{\mathbf{r}_1, \mathbf{r}_2}.
\]
We remember here that \( N \) is the semi-difference of Cooper–pairs between the left and the right electrodes, \( N = (N_1 - N_2)/2 \). We have written the tunneling part of the Hamiltonian \( H_T \) in such a way that the gauge dependence becomes explicit.

\[
T_{r,r'} = \hat{T}_{r,r'} \exp i \Lambda(r,r') \quad \Lambda(r,r') = \frac{e}{\hbar c} \int_{r}^{r'} A(r) \cdot dr = -\Lambda(r',r) .
\] (4.10)

\( \hat{T}_{r,r'} \) is gauge invariant and, without loss of generality, can be assumed to be real symmetric. Thus the fermion–fields \( \psi_\sigma \) are the only quantities in Eq. (4.10) which break local gauge invariance. The notation above may look unnecessary, since in the standard descriptions of Josephson junctions 30 for simplicity that the tunneling matrix element is independent of the electron momenta, i.e. one sets \( \angle \psi_\sigma \) or \( \phi(r,r') = \phi(r) - \phi(r') + \Lambda(r,r') \).

In order to derive an effective theory for the phase one makes two further approximations. First, one assumes for simplicity that the tunneling matrix element is independent of the electron momenta, i.e. one sets \( \hat{T}(r,r') = T \delta(r_1 - r'_1) \delta(z') \)). Second, one approximates the unperturbed parts \( H_{L,R}(r) \) substituting \( \psi(r) \) and \( \Delta(r) \) by their mean field bulk values \( \psi(r) = 0, z \neq 0, \Delta(r) = \Delta \). This is a reasonable approximation in the tunneling limit 31,32,33. Using the above approximations we may write the tunneling Hamiltonian as

\[
H = (E_C/2)N^2 + H_L + H_R + H_T
\]

\[
H_{L(R)} = \sum_{l(r),\sigma} \varepsilon_{l(r)} c_{l(r),\sigma}^\dagger c_{l(r),\sigma} + \Delta \sum_{l(r)} \left( c_{l(r),\uparrow}^\dagger c_{l(r),\downarrow}^\dagger + H.c. \right)
\]

\[
H_T = T \cos(\phi/2) \sum_{l,r,\sigma} \left( c_{l\sigma}^\dagger c_{r\sigma} + H.c. \right) + iT \sin(\phi/2) \sum_{l,r,\sigma} \left( c_{l\sigma}^\dagger c_{r\sigma} - H.c. \right) .
\] (4.11)

In the third line we have used the definition of the Bogoliubov quasiparticle operators \( \gamma_{k,\uparrow} = u_k c_{k\uparrow} - v_k c_{-k\downarrow} \) and \( \gamma_{k,\downarrow} = v_k c_{-k\downarrow} + v_k c_{k\uparrow} \).

Now we adopt a bosonization procedure whereby the Hamiltonian (4.11) is mapped onto an effective Hamiltonian of the form

\[
H_{eff} = \frac{E_C}{2} N^2 + \cos(\phi/2) \sum_i \lambda_i^{(+)}(b_i^{(+)} + b_i^{(+)}\dagger) + \sin(\phi/2) \sum_i \lambda_i^{(-)}(b_i^{(-)} + b_i^{(-)}\dagger) + \sum_{(\pm),i} \omega_i b_i^{(\pm)}\dagger b_i^{(\pm)} ,
\] (4.12)

where the \( b_i^{(\pm)} \) are bosonic operators and the coupling strengths \( \lambda_i^{(\pm)} \) are yet undefined. To further specify the Hamiltonian (4.12) we use the following strategy: We derive the Heisenberg equations for the canonical conjugate variables \( N \) and \( \phi \) as well as for the electronic degrees of freedom of the Hamiltonian (4.11). Then the equations for the electronic degrees of freedom are solved in perturbation theory. Upon elimination of the electronic degrees of freedom one obtains two coupled integro–differential quantum Langevin equations 34,35 for the Heisenberg operators \( N \) and \( \phi \). It turns out that, up to second order in the hopping \( T \), these equations have the same structure as those obtained from Eq. (4.12). This allows us to adjust the coupling parameters \( \lambda_i^{(\pm)} \) in Eq. (4.12) in such a way that the equations of motion for \( N \) and \( \phi \) become identical. Below we implement this strategy.

From the Hamiltonian (4.11) we obtain the Heisenberg equations

\[
\dot{\phi}(t) = \hbar^{-1} E_C N(t)
\]

\[
\dot{N}(t) = \frac{T}{2\hbar} \sum_{l,r,\sigma} c_{l\sigma}^\dagger(t) c_{r\sigma}(t) e^{i\phi(t)/2} + H.c. .
\] (4.13)

To close these equations we evaluate the time evolution of the electronic part perturbatively in \( H_T \). To first order we obtain

\[
c_{l\sigma}(t)c_{r\sigma}(t) = c_{l\sigma}^\dagger(t)c_{r\sigma}^\dagger(t) + \frac{T}{\hbar} \int_{-\infty}^{t} dt' \sum_{l,r,\sigma} [c_{l\sigma}^\dagger(t)c_{r\sigma}(t), c_{r\sigma}^\dagger(t')c_{l\sigma}(t')] e^{-i\phi(t')/2} + ...
\[
\frac{T}{\hbar} \int_{-\infty}^{t} dt' \sum_{i,r,\sigma} [c_{\sigma}^I(t)c_{\sigma}^I(t') + c_{-\sigma}^I(t)c_{-\sigma}^I(t')] e^{i\phi(t')/2},
\]  
(4.14)

and proceed similarly with the other relevant fermionic operators. The superscript \(I\) denotes operators in the interaction picture. Introducing Eq. (4.14) into Eq. (4.13) yields

\[
\frac{\hbar}{E_C} \dot{\phi}(t) = F(t) \exp \left(\frac{i\phi(t)}{2}\right) - \int_{-\infty}^{t} dt' \dot{\alpha}(t-t') \exp \left(\frac{i\phi(t) - \phi(t')}{2}\right) - \int_{-\infty}^{t} dt' \dot{\beta}(t-t') \exp \left(\frac{i\phi(t) + \phi(t')}{2}\right) + \text{H.c.}.
\]
(4.15)

Since the uncoupled Hamiltonians \(H_{L,R}\) in Eq. (4.11) are quadratic forms, the corresponding grand canonical ensemble is a Gaussian ensemble. Therefore the effect of the operator

\[
\dot{F}(t) = \frac{T}{2\hbar} \sum_{i,r,\sigma} c_{\sigma}^I(t)c_{\sigma}^I(t)
\]
(4.16)

is determined by its second moment \(\langle \dot{F}(t)\dot{F}(t') \rangle_B\), where subscript \(B\) denotes average over bath coordinates. These quantities are related to the thermal averages of the operator valued dissipation kernels

\[
\dot{\alpha}(t) = \frac{|T|^2}{2\hbar^2} \sum_{i,r,\sigma} [c_{\sigma}^I(t)c_{\sigma}^I(t), c_{\sigma}^I(0)c_{\sigma}(0)] -
\]
\[
\dot{\beta}(t) = \frac{|T|^2}{2\hbar^2} \sum_{i,r,\sigma} [c_{\sigma}^I(t)c_{\sigma}^I(t), c_{-\sigma}^I(0)c_{-\sigma}(0)]
\]
(4.17)

by a fluctuation–dissipation theorem. In order to derive a differential equation for the the \(N\)-point function \(\langle \phi(t_1) \ldots \phi(t_N) \rangle_B\) from the Heisenberg equations Eq. (4.13), one must in principle evaluate averages of the form \(\langle \dot{\alpha}(t_1) \ldots \dot{\alpha}(t_N) \rangle_B\). However, since all averages are made over a Gaussian ensemble, Wick’s theorem applies and they reduce to products of \(\langle \dot{\alpha}(t_i) \rangle_B\) and \(\langle \dot{\beta}(t_i) \rangle_B\) respectively. Therefore, in order to completely determine the quantum Langevin equation Eq. (4.13), it is sufficient to calculate the averages \(\langle \dot{\alpha}(t) \rangle_B\) and \(\langle \dot{\beta}(t) \rangle_B\). These quantities are calculated in standard textbooks on superconductivity. The outcome is the well known result that \(\langle \dot{\alpha}(t) \rangle_B\) and \(\langle \dot{\beta}(t) \rangle_B\) are the sine–transforms of the sum and difference, respectively, of the two spectral functions \(J^{(\pm)}(\omega)\),

\[
\langle \dot{\alpha}(t) \rangle_B = \alpha(t) = \frac{1}{2} \int_{0}^{\infty} \left[ J^{(+)}(\omega) + J^{(-)}(\omega) \right] \sin(\omega t) d\omega
\]
\[
\langle \dot{\beta}(t) \rangle_B = \beta(t) = \frac{1}{2} \int_{0}^{\infty} \left[ J^{(+)}(\omega) - J^{(-)}(\omega) \right] \sin(\omega t) d\omega,
\]
(4.18)

where \(J^{(\pm)}(\omega)\) are the spectral functions defined in Eq. (4.15).

We can also derive a quantum Langevin equation for \(N\) and \(\phi\) from the bosonized Hamiltonian Eq. (4.12). Since the interaction here is linear in the bosonic operators, the procedure is exact (by contrast, fermionic correlators are computed up to second order in \(H_T\)). The resulting Heisenberg equation is identical to that in Eq. (4.12). However, in this case the response kernels are given by

\[
\alpha(t) = \frac{1}{2\hbar^2} \sum_{i} \sin(\omega_i t) \left( \lambda_i^{(+)2} + \lambda_i^{(-)2} \right)
\]
\[
\beta(t) = \frac{1}{2\hbar^2} \sum_{i} \sin(\omega_i t) \left( \lambda_i^{(+)2} - \lambda_i^{(-)2} \right).
\]
(4.19)

Since the coupling parameters \(\lambda_i^{(\pm)}\) have not yet been specified we are free to choose them in a way that adjusts the dissipation kernels in Eqs. (4.19) to the kernels (4.13) derived from the microscopic theory. Thus we may conclude that the two Hamiltonians (4.11) and (4.12), together with the spectral density (4.15), are equivalent to order \(O(T^2)\), at least as far as the dynamics of the phase is concerned. We stress that, due to the Gaussian nature of the averages, this equivalence holds not only for the dynamics of the mean \(\langle \phi(t) \rangle\) and \(\langle N(t) \rangle\) but also for the dynamics of the higher correlators.
In order to make contact with the Hamiltonian Eq. (4.1) we add and subtract in Eq. (4.12) the expression

\[ \sum_i \frac{\lambda^{(\pm)}_i \cos^2 \left( \frac{\phi}{2} \right)}{\hbar \omega_i} + \sum_i \frac{\lambda^{(-)}_i \sin^2 \left( \frac{\phi}{2} \right)}{\hbar \omega_i} = \]

\[ \frac{1}{2} \sum_i \frac{\lambda^{(\pm)}_i - \lambda^{(-)}_i}{\hbar \omega_i} \cos \phi + \frac{1}{2} \sum_i \frac{\lambda^{(\pm)}_i + \lambda^{(-)}_i}{\hbar \omega_i}, \]

which allows us to complete the square. The \( \phi \) dependent term in Eq. (4.20) can be calculated using the definitions of \( \lambda^{(\pm)}_i \) in terms of the spectral densities \( J^{(\pm)} \) given in Eq. (5.3). We obtain the result of Ambegaokar and Baratoff\(^{31}\), namely,

\[ \frac{1}{2} \sum_i \frac{\lambda^{(\pm)}_i - \lambda^{(-)}_i}{\omega_i} \cos \phi = \frac{1}{2} \int_0^\infty d\omega J^{(\pm)}(\omega) - J^{(-)}(\omega) \cos \phi \]

\[ = \frac{\hbar \pi \Delta(T)}{4 e^2 R} \tanh \left( \frac{\beta \Delta(T)}{2} \right) \cos \phi \]

\[ = E_J \cos \phi. \] (4.21)

The second term in the r.h.s. of Eq. (4.20) is divergent but, fortunately, it is independent of \( \phi \). Since we are allowed to add an arbitrary constant to a Hamiltonian, we subtract this infinity and keep, for convenience, only a constant \( E_J \) from this second term. This yields the Hamiltonian (4.1).

Of course the two Hamiltonians (4.1) and (4.12) are identical, but the physics is more transparent in Eq. (4.1). It is the Hamiltonian of a particle moving in a cosine potential coupled nonlinearly to two independent bosonic baths. Since the interaction is in the form of a complete square, the coupling to the heat bath does not change the potential landscape. This is a well known fact in the theory of dissipative quantum systems\(^{31}\).

V. SIMULTANEOUS COUPLING TO PHOTONS AND QUASIPARTICLES

A. Mixed coupling Hamiltonian

If we now introduce again the dynamics of the electromagnetic field as explained in Sec. III we arrive at a rather comprehensive description of the dynamics of the phase in a Josephson–junction. It is instructive to cast by a further unitary transformation the resulting Hamiltonian into a highly suggestive form. Applying \( U^\dagger H U \) with

\[ U = \exp \left[ \cos \left( \frac{\phi}{2} \right) \sum_i \frac{\lambda^{(\pm)}_i}{\hbar \omega_i} \left( b_i^{(\pm)} - b_i^{(\pm)\dagger} \right) + \sin \left( \frac{\phi}{2} \right) \sum_i \frac{\lambda^{(-)}_i}{\hbar \omega_i} \left( b_i^{(-)} - b_i^{(-)\dagger} \right) \right] \] (5.1)

onto \( H \) in Eq. (5.1) shifts the interaction of the “electronic” bosons with \( \phi \) into an apparently more complicated interaction with \( N \). However, we can write the transformed Hamiltonian as

\[ H = \frac{E_C}{2} (N + \delta N)^2 + E_J [1 - \cos (\phi + \delta \phi)] \]

\[ + \sum_i \hbar \omega_i b_i^{(\pm)\dagger} b_i^{(\pm)} + \sum_i \hbar \omega_i b_i^{(-)\dagger} b_i^{(-)} + \sum_{k\lambda} \hbar \omega_k a_{k\lambda}^\dagger a_{k\lambda}. \] (5.2)

This form is rather general and physically quite elucidating. We observe that the position degree of freedom of a one body Hamiltonian acquires a fluctuating part. So does its canonical momentum. These fluctuating parts obtain their own dynamics as bosonic excitations. The specific form of these fluctuating operators has either to be derived from a many–body Hamiltonian or it has to be modelled phenomenologically. In our case the fluctuating term of the particle number

\[ \delta N = \frac{i}{2} \sin \left( \frac{\phi + \delta \phi}{2} \right) \sum_i \frac{\lambda^{(\pm)}_i}{\hbar \omega_i} \left( b_i^{(\pm)} - b_i^{(\pm)\dagger} \right) - \frac{i}{2} \cos \left( \frac{\phi + \delta \phi}{2} \right) \sum_i \frac{\lambda^{(-)}_i}{\hbar \omega_i} \left( b_i^{(-)} - b_i^{(-)\dagger} \right) \] (5.3)

is due to the hopping of Cooper pairs and quasiparticles across the junction. It is phase dependent and was derived from a microscopic model. An interesting message from Eq. (5.2) is that the Coulomb interaction does not distinguish
between $N$ and $\delta N$. It is only sensitive to the overall particle number difference. This may be viewed as a form of minimal coupling, since it reflects the “gauge invariance” of the Coulomb term and is analogous to the criterion whereby the coupling to the EM field is introduced by replacing $\phi$ by its gauge invariant form. The fluctuating part of the phase is $\delta \phi = \lambda_1 \delta$, see Eq. (3.3).

Despite the appealing compactness of Eq. (3.3), for calculational purposes it is more convenient to shift both $N$ and $\phi$ by their fluctuating part $-\delta N$ and $-\delta \phi$ through a unitary transformation. We employ the inverse of (3.1) to shift $N$ and $U = \exp(-iNA_{12})$ to shift $\phi$. The resulting Hamiltonian is much closer to the standard Caldeira–Leggett form

$$
H = E_J (1 - \cos \phi) + \frac{E_C}{2} N^2 + \sum \hbar \omega_i \left[ b_i^{(+)} + \frac{\lambda_i}{\hbar \omega_i} \cos \left( \frac{\phi}{2} \right) \right] \left[ b_i^{(+)} + \frac{\lambda_i}{\hbar \omega_i} \cos \left( \frac{\phi}{2} \right) \right] + \sum \hbar \omega_i \left[ b_i^{(-)} \right] \left[ b_i^{(-)} + \frac{\lambda_i}{\hbar \omega_i} \sin \left( \frac{\phi}{2} \right) \right] + \sum \hbar \omega_k \left( a_k^{+} + \frac{\mu_{k\lambda}}{\hbar \omega_k} N \right) \left( a_k + \frac{\mu_{k\lambda}}{\hbar \omega_k} N \right),
$$

with the coupling constants $\lambda_i^{(\pm)}$ defined by $J^{(\pm)}(\omega)$ through Eqs. (4.2) and (4.3) and $\mu_{k\lambda}$ defined by $J_{EM}(\omega)$ through Eqs. (4.6) and (4.9). The nonlinear coupling of the phase to two different baths is rather difficult to handle. Following Ref. 23, one considerable simplification is achieved by keeping only a local contribution in the $\delta$ kernel integral in Eq. (4.15). This is a valid approximation if the phase varies slowly on the time scale in which decays $\beta(t)$, which is set by $\hbar / \Delta$. Yet another simplification is obtained if one keeps only the first non–vanishing term of the exponential in the $\hat{\delta}$ kernel integral in Eq. (4.15). With these approximations the nonlinear coupling of the phase to the electronic degrees of freedom has been replaced by a linear coupling to only one effective heat–bath of an “almost” ohmic character. These are the basic assumptions which underlie the RCSJ model. The limits of validity of these approximations have been discussed in the literature. In the RCSJ limit, the Hamiltonian (5.3) becomes

$$
H = E_J (1 - \cos \phi) + \frac{E_C}{2} N^2 + \phi \sum \lambda_i (b_i + b_i^{\dagger}) + N \sum \mu_{k\lambda} (a_k^{+} + a_k)
$$

$$
+ \sum \hbar \omega_i \lambda_i \lambda_i^{(\pm)} + \sum \hbar \omega_k \lambda_k \lambda_k^{(\pm)} + N^2 \left( \sum \mu_{k\lambda}^2 \sum \hbar \omega_k \lambda_k \lambda_k^{(\pm)} + \omega^2 \sum \hbar \omega_i \lambda_i^2 \right). \tag{5.5}
$$

The coupling parameters $\lambda_i$ are adjusted to satisfy $J_{qp}(\omega) = 2\hbar^{-2} \sum \lambda_i^2 \delta(\omega - \omega_i)$, where

$$
J_{qp}(\omega) = J^{(+)}(\omega) + J^{(-)}(\omega)
$$

$$
= \frac{\hbar}{4 \pi R e^2} \int d\omega' N_{qp} \left( \omega' + \frac{\omega}{2} \right) N_{qp} \left( \omega' - \frac{\omega}{2} \right) \left\{ \tanh \left[ \frac{\hbar \beta}{2} \left( \omega' + \frac{\omega}{2} \right) \right] - \tanh \left[ \frac{\hbar \beta}{2} \left( \omega' - \frac{\omega}{2} \right) \right] \right\}, \tag{5.6}
$$

$R$ being the normal resistance of the junction. This means that, in the RCSJ approximation, diffusive effects due to $\beta(t)$ are suppressed.

In the high temperature limit, Eq. (5.6) yields the ohmic spectral density for a normal conductor junction given in Eq. (2.11): $J_{qp}(\omega) \sim \omega / R_Q$, $R_Q \equiv 2 \pi R e^2 / \hbar$. This ultimately justifies the RSCJ model. A plot of $J_{qp}(\omega)$ for various temperatures has been given by Harris. One observes that, already for relatively high temperatures, the spectral density deviates from strictly Ohmic behavior. In particular for $k_B T \lesssim \Delta / \hbar$ the spectral density is better approximated by the zero temperature spectral density, which can be calculated analytically, the most prominent feature being the existence of a gap at $\omega = 2\Delta$. It is known that an infrared gap in the spectral function reduces the dissipative effect of the bath. Therefore, in the RCSJ model the effect of quasiparticle tunneling is overestimated.

A central physical feature is that the two baths in Eq. (5.5) have competing effects. On the one hand, the quasiparticle environment tends to “measure” $\phi$ (the position of the particle). On the other hand the environment “measures” $N$ (the momentum of the particle). Of course, due to the uncertainty principle, the two baths cannot be equally effective. In general, it is not clear a priori which mechanism will dominate. In this case, however, both because QED is weak and superohmic, one expects the quasiparticle measurement of the phase to be dominant.

Apart from its relevance in the present context, the model which we have constructed is worthwhile to be studied in its own right. We address this subject in a more general context in Ref. 46. Here, however, we restrict ourselves to the case in which one bath is ohmic (quasiparticles) and the other one is cubic (photons).

### B. Equilibrium correlations

Like in Sec. III B we use the harmonic oscillator approximation for the nonlinear potential terms. In contrast to the case of a harmonic oscillator coupled to a single bath, this model is not exactly solvable for an arbitrary initial
condition. Fortunately, we are mostly interested in the equilibrium quantities $C_{\phi\phi}^{(+)}(t)$ and $C_{\phi\phi}^{(+)}(t)$.

$$C_{\phi\phi}^{(+)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\chi(\omega)|^2 \cos(\omega t) \coth(\hbar \beta \omega / 2)$$

$$\{ \text{Im} \tilde{J}_{qp}(\omega) \left( \frac{E_C^2}{\hbar^2} - 2 \frac{E_C}{\hbar} \text{Re} \tilde{J}_{EM}(\omega) + |\tilde{J}_{EM}(\omega)|^2 \right) + \omega^2 \text{Im} \tilde{J}_{EM}(\omega) \} d\omega ,$$

$$C_{\phi\phi}^{(+)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\chi(\omega)|^2 \cos(\omega t) \coth(\hbar \beta \omega / 2)$$

$$\{ \text{Im} \tilde{J}_{EM}(\omega) \left( \frac{E_C^2}{\hbar^2} - 2 \frac{E_C}{\hbar} \text{Re} \tilde{J}_{qp}(\omega) + |\tilde{J}_{qp}(\omega)|^2 \right) + \omega^2 \text{Im} \tilde{J}_{qp}(\omega) \} d\omega .$$

(5.7)

Here, we have introduced the generalized “susceptibility”

$$\chi^{-1}(\omega) = \omega_Q^2 - \omega_J - \hbar^{-1} E_J \tilde{J}_{EM}(\omega) - \hbar^{-1} E_C \tilde{J}_{qp}(\omega) + \tilde{J}_{EM}(\omega) \tilde{J}_{EM}(\omega)$$

(5.8)

Again, $f(\omega)$ denotes the symmetrized Riemann transform as defined in Eq. (2.12). To leading order in $\omega_e^{-1}$, Eq. (5.8) reproduces Eq. (2.10), which was derived from a simplified discussion. Of course, these expressions reduce to Eqs. (3.11) for $J_{qp}(\omega) = 0$. In Eqs. (5.7) we used the spectral function $J_{EM}(\omega)$ is defined by Eq. (3.7), while for the quasiparticles we use the ohmic form $J_{qp}(\omega) = (\omega / R_Q)(g/\omega \omega_e)$ keeping in mind that it overestimates dissipation at low temperatures. Moreover we introduce formally an electronic cutoff $\Omega_c$, which we may choose as the conduction band width. For the smallest geometries available one has approximately $\Omega_c \gg \omega_e$. In the following we restrict ourselves to $T = 0$. As pointed out in Sec. IIIIB this should be a good approximation at sufficiently low temperatures $k_B T < \Delta \sim k_B T_c \ll \Omega_c$.

The evaluation of the integrals in Eq. (5.8) involves the solution of a fifth order polynomial. Three of its solution scale as $\sim \omega_e$ while the other two solutions shift the Josephson plasmon frequency into the complex plane. Eventually the real part disappears completely, leading to overdamped oscillations which are characteristic of ohmic and subharmonic spectral densities. For $\langle \phi^2 \rangle$ we find, to leading order in $\omega_e^{-1}$ and $\Omega_c^{-1}$

$$\langle \phi^2 \rangle = 1/2 \sqrt{E_J / E_J} \frac{\ln[(\kappa + \sqrt{\kappa^2 - 1})/(\kappa - \sqrt{\kappa^2 - 1})]}{\pi \sqrt{\kappa^2 - 1}} + f_{\phi}(\lambda_0, \lambda_1, \lambda_2) + O(\omega_e^{-1}, \Omega_c^{-1})$$

$$f_{\phi}(\lambda_0, \lambda_1, \lambda_2) = \frac{2R_Q^3}{\pi^{3/2}} \frac{\gamma}{\Delta(\lambda^2)} \left[ \begin{array}{ccc} g_{\phi}(\lambda_0) & \lambda_0^2 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array} \right]$$

$$g_{\phi}(\lambda_i) = \left( \frac{\pi \gamma + \pi - \pi^3 \lambda_i^2}{2R_Q + \pi^3 \lambda_i^2} \right) \ln \left( \frac{1 + \lambda_i^2}{\lambda_i^2} \right)$$

(5.9)

where we have used the notation $\kappa = (\pi/4R_Q) \sqrt{E_J / E_J}$. Moreover, we have introduced Vandermonde’s determinant $\Delta_N(t) \equiv \prod_{i,j} (x_i - x_j)$, $i, j = 1, ..., N$. The dimensionless quantities $\lambda_0, \lambda_1, \lambda_2$ are the coefficients of the leading order term in $\omega_e$ of the three solutions of the fifth order polynomial which scale with $\omega_e$. They are given by the solution of the third order polynomial

$$\lambda^3 + \left( \frac{2}{\pi} - \frac{2}{\pi} \right) \lambda^2 - \frac{2}{\pi} \lambda + \frac{2R_Q}{\pi \gamma} = 0 .$$

(5.10)

Remarkably, the roots $\lambda_i$ only depends on the ratio $R_Q / \gamma$. The expression for $\langle \phi^2 \rangle$ in the first line of Eq. (5.9) consists of two contributions. The first term is recognized as the mean square of the position operator of a harmonic oscillator with ohmic damping. It is the leading order contribution in $\Omega_c$ of quasiparticle tunneling. The second term, $f_{\phi}$, stems from the electromagnetic environment. Obviously the exact analytic form of $f_{\phi}$ stated in Eq. (5.9) is not too illuminating. To get further insight we consider the limiting cases $R_Q / \gamma \ll 1$ and $R_Q / \gamma \gg 1$, which allow for an expansion of the logarithm in $g_{\phi}$ and a perturbative solution of Eq. (5.10). For $R_Q / \gamma \gg 1$ we obtain for $f_{\phi}$

$$f_{\phi} = \frac{\gamma}{4} + \frac{\gamma^2}{4R_Q} \left( 1 + \frac{\pi^2}{4} \right) + O \left( \frac{\gamma^2}{R_Q} \right) .$$

(5.11)

As we have seen above, the parameter of the coupling to the electromagnetic field $\gamma = 8\alpha f^2 / 3\pi$ is always small. However, the dimensionless resistance can be either large or of order unity depending on the experiment. Assuming
$R_Q \gg 1$ or, equivalently, $\kappa \ll 1$, we can also expand the first relation in Eq. (5.9). We obtain, to first order in $R_Q^{-1}$:

$$
\langle \phi^2 \rangle = 1 \frac{1}{2} \sqrt{\frac{E_C}{E_J}} - \frac{E_C}{4E_J R_Q} + \frac{\gamma}{4} + \frac{\gamma^2}{4 R_Q} \left(1 + \frac{\pi^2}{4}\right) .
$$

(5.12)

This result illustrates nicely the competing character of the effects of the quasiparticle bath on the one hand and the photonic bath on the other hand. The second term in Eq. (5.12) is the dominant quasiparticle contribution for $\Omega \to \infty$. It leads to a reduction of $\langle \phi^2 \rangle$ or, equivalently, to an enhancement of coherence. The third term in Eq. (5.12) is the QED correction to first order in $\omega_c^{-1}$, which was already found in Eqs. (3.17) and (3.19) of Sec. III B. This term is always positive. One can estimate the quasiparticle term by using again the Ambegaokar-Baratoff formula and the capacitance of a plate capacitor. One finds that the QED contribution would dominate for an unrealistic thickness $d > 12 \hbar c / \Delta \approx 100 \mu$m.

For the sake of completeness we also discuss the opposite limit $R_Q / \gamma \ll 1$. After some algebra, we obtain that the leading order term of the “mixing” function $f_\phi$ is

$$
f_\phi = \frac{1}{4} R_Q \ln \left(\frac{\gamma}{R_Q}\right) .
$$

(5.13)

Expanding Eq. (5.13) for $R_Q \ll 1$ we obtain

$$
\langle \phi^2 \rangle = \frac{R_Q}{\pi^2} \left[4 \ln \left(\frac{2}{R_Q}\right) + \frac{\pi^2}{2} \ln \left(\frac{\gamma}{R_Q}\right)\right] .
$$

(5.14)

The result for $\langle N^2 \rangle$ is formally similar to that of $\langle \phi^2 \rangle$:

$$
\langle N^2 \rangle = \frac{1}{2} \sqrt{\frac{E_J}{E_C}} \ln \left[\frac{(\kappa + \sqrt{\kappa^2 - 1})/((\kappa - \sqrt{\kappa^2 - 1})}\right]

+ \frac{1}{2 R_Q} \ln \left(\frac{\Omega_c}{\Omega_{JP}}\right) + f_N(\lambda_0, \lambda_1, \lambda_2) + O(\omega_c^{-1}, \Omega_c^{-1})

f_N(\lambda_0, \lambda_1, \lambda_2) = \frac{2 R_Q}{\pi^3 \gamma^2} \Delta_s(\lambda^2)

\ln \left(\frac{\lambda_i}{\lambda_s}\right) + O\left(\frac{\gamma^2}{R_Q^2}\right) .
$$

(5.15)

For small EM coupling ($\gamma / R_Q \ll 1$) the mixing function $f_N$ reduces to

$$
f_N = \gamma \frac{\gamma}{4 R_Q} \left(1 + \frac{\pi^2}{4}\right) + O\left(\frac{\gamma^2}{R_Q^2}\right) .
$$

(5.16)

We reproduce in Eq. (5.16) the logarithmic dependence on the quasiparticle cutoff frequency, a well known feature of the coupling to an Ohmic heat bath.

**VI. SUMMARY**

We have explored the role of the quantum electrodynamic field as a possible agent of zero-point decoherence in some physical systems. We have focussed on the fundamental problem of the dynamics of the phase-number variable in a Josephson link. The transverse electromagnetic field has been shown to couple to the relative Cooper number through the electric dipole which it generates. The effect of quasiparticles has been approximated with an effective oscillator bath which couples to the phase. This poses a new quantum dissipation problem of a type which has been little studied: that of a quantum particle coupling through its position and momentum to two different baths. The resulting dissipative dynamics of the macroscopic phase-number variable has been investigated by means of a Hamiltonian equation-of-motion approach. We have found that the r.m.s. values of the phase and number contain interference contributions from the photon and quasiparticle bath. The effect of the QED field has been found to be quite small compared with charging effects due to Cooper pair and quasiparticle fluctuations. The unfortunate fact that the contribution of the longitudinal and transverse fields to the phase variance scale identically with the junction area precludes a simple observation of the effect of electromagnetic vacuum fluctuations in generic Josephson devices. However, the sensitivity to the photon and quasiparticle spectra which we have found suggests that the design of special structures, where the effect of electrodynamic zero-point fluctuations is amplified, cannot be ruled out.
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APPENDIX A: DERIVATION OF EQUATIONS (3.18)

With the Drude type cutoff \( g(x) = (1 + x^4)^{-1} \) the inverse susceptibility involves a fourth order polynomial in the numerator and a second order polynomial in the denominator

\[
\chi^{-1}(\omega) = \frac{1}{\omega^2 + \sqrt{2}\omega\omega_c - \omega_c^2} \left[ \omega^4 \pm i\sqrt{2}\omega\omega_c^3 \right. \\
+ \left. \left( \frac{\pi E_J}{2\sqrt{2}h} \gamma \omega_c - \omega^2_{JP} - \omega_c^2 \right) \omega^2 \mp i\sqrt{2}\omega^2_{JP}\omega_c\omega + \omega^2_{JP}\omega^2_c \right]
\] (A1)

The roots of the fourth order polynomial come in two pairs of complex numbers \( i\lambda = i\lambda_1 \pm i\lambda_2 \) and \( iz = iz_1 \pm iz_2 \). The roots of the second order polynomial are given by \( \lambda_0 = \omega_c \exp(\pm i\pi/4) \). One can evaluate Eqs. (3.11) by a contour integral similarly to the Drude model\(^{35,44} \). However, here we will take advantage of the general expressions for the mean squares \( \langle N^2 \rangle \) and \( \langle \phi^2 \rangle \) as derivatives of the partition function of the harmonic oscillator\(^{31} \). We have

\[
\langle N^2 \rangle = \frac{-E_J}{2\beta \sqrt{2} \hbar^2} \frac{d}{d\omega_{JP}} \ln Z(\beta)
\]
\[
\langle \phi^2 \rangle = \left. \frac{-1}{2E_J^2} \left( \omega_{JP} \frac{d}{d\omega_{JP}} + 2\gamma \frac{d}{d\gamma} \right) \ln Z(\beta) \right|_{\omega = \omega_{JP}}
\] (A2)

with the partition function of the damped harmonic oscillator

\[
Z(\beta) = \frac{1}{\hbar^2 \beta \omega_{JP}} \prod_{n=1}^{\infty} \nu_n^2 \chi(\nu_n)
\] (A3)

We have used the notation \( \nu_n = 2\pi n/\hbar \beta \) for the Matsubara frequencies.

At this point it proves useful to use an algebraic cutoff. That yields a finite number of roots which satisfy a series of algebraic relations, called Vieta relations\(^{47} \). This allows us to express the infinite product in Eq. (A3) compactly in terms of \( \Gamma \)-functions

\[
Z(\beta) = \frac{\hbar^2 \beta \omega_{JP}}{4\pi^2} \frac{|\Gamma(\lambda/\nu)|^2 |\Gamma(z/\nu)|^2}{|\Gamma(\lambda_0/\nu)|^2}
\] (A4)

The real and the imaginary parts of the roots \( \lambda, z \) scale quite differently with the cutoff frequency. They are approximately

\[
\lambda_1 = \frac{1}{\sqrt{2}} \omega_c + O(\omega_c^{-2})
\]
\[
\lambda_2 = \frac{1}{\sqrt{2}} \omega_c - \frac{E_J \pi \gamma}{8} \frac{\gamma^2}{\hbar^4} \frac{\sqrt{2}}{\omega_c} + O(\omega_c^{-2})
\]
\[
z_1 = -\frac{\pi E_J \gamma \omega_{JP}^2}{4 \hbar^2 \omega_c^3} + O(\omega_c^{-3})
\]
\[
z_2 = \omega_{JP} + \frac{\pi E_J \gamma \omega_{JP}}{4 \sqrt{2} \hbar \omega_c} + 3\pi^2 \frac{E_J^2 \gamma^2 \omega_{JP}}{64 \hbar^2 \omega_c^5} + O(\omega_c^{-3})
\] (A5)

Using the asymptotic expansion of the \( \Gamma \)-function we get an expansion of \( Z(\beta) \) in powers of \( k_B T \).

\[
Z(\beta) = Z^{(0)} + \sum_{n=1}^{\infty} Z^{(n)}(\beta) (k_B T)^n
\] (A6)
In zeroth order we recover Eqs. (3.17) with $N_c = \pi/2\sqrt{2}$. The next two orders in the expansion vanish identically due to the Vieta relations. The first non-vanishing correction term reads

$$Z^{(3)}(\beta) = \frac{1}{360\omega_0^3\omega_0^6} \left[ (\lambda^3 + \lambda^3) |z|^6 + (z^3 + z^3) |\lambda|^6 - \omega_0^6 \left( \lambda_0^3 + \lambda_0^3 \right) \right].$$

(A7)

Using Eq. (A2) and retaining only the highest order term in $\omega$, we finally arrive at the result for $\langle N^2 \rangle$ stated in Sec. [1113] with a quartic finite temperature correction. Moreover we observe that for $\langle \phi^2 \rangle$ the quartic contribution vanishes as $\omega_c \to \infty$. Therefore, in order to obtain the first finite temperature correction for $\langle \phi^2 \rangle$, we have to expand $Z(\beta)$ up to fifth order in $\beta$. The result with a sixtic finite temperature correction is given in Eq. (3.18).

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1. R. P. Feynman, F. L. Vernon, Jr., Ann. Phys. (New York) 24, 118 (1963); R. P. Feynman, A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw Hill, New York, 1965).
2. R. P. Feynman, Phys. Rev. B 97, 60 (1995).
3. A. O. Caldeira, A. J. Leggett, Ann. Phys. (New York) 149, 374 (1983).
4. A. J. Leggett et al., Rev. Mod. Phys. 59, 1 (1987).
5. W. G. Unruh, W. H. Zurek, Phys. Rev. D 40, 1071 (1989).
6. A. Stern, Y. Imry, Y. Aharonov, Phys. Rev. A 41, 3436 (1990).
7. I. Zapata, F. Sol, A. J. Leggett, Phys. Rev. A 67, 021603 (2003).
8. F. Sol, Ann. Phys. (New York) 214, 386 (1992).
9. B. L. Altshuler, A. G. Aronov, D. E. Khmelnitsky, J. Phys. C 15, 7367 (1982).
10. P. Mohanty et al., Phys. Rev. Lett. 78, 3366 (1997), Phys. Rev. B 55, R13452 (1997); D. S. Golubev et al., Phys. Rev. Lett. 81, 1074 (1998), Phys. Rev. B 59, 9195 (1999), id. 62, 14061 (2000); R. Raimondi et al., Phys. Rev. B 60, 5818 (1999); I. L. Aleiner et al., Wave Random Media 9, 201 (1999), J. Low Temp. Phys. 126, 1377 (2002); D. Cohen et al., Phys. Rev. B 59, 11143 (1999); M. H. Devoret et al., Phys. Rev. Lett. 64, 1824 (1990); A. Zawadowski et al., Phys. Rev. Lett. 83, 2632 (1999).
11. P. Cedraschi, M. Böttiker, Phys. Rev. B 63, 165312 (2001), Ann. Phys. (New York) 289, 1 (2001).
12. J. D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1977); C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons and Atoms. Introduction to Quantum Electrodynamics (Wiley, New York, 1989).
13. S. Chakravarty, R. Norton, O. F. Syljuåsen, Phys. Rev. Lett. 74, 1423 (1995).
14. D. Loss, A. Martin, Phys. Rev. B 47, 4619 (1993).
15. P. M. V. B. Barone, A. O. Caldeira, Phys. Rev. A 43, 57 (1991).
16. F. Sol, I. Zapata, in New Developments on Fundamental Problems in Quantum Dynamics, M. Ferrero, A. van der Merwe, eds. (Kluwer Academic, Dordrecht, 1997).
17. E. Joos, H. D. Zeh, Z. Phys. B 59, 223 (1985).
18. L. H. Ford, Phys. Rev. D 47, 5571 (1993).
19. Z. Haba, H. Kleinert, Euro. Phys. J. B 21, 4 (2001).
20. F. Sol, Physica B 194, 1389 (1994).
21. E. M. Wright et al., Phys. Rev. Lett. 77, 2158 (1996); Phys. Rev. A 56, 591 (1997); A. Imamoglu et al., Phys. Rev. Lett. 78, 2511 (1997); Y. Castin, J. Dalibard, Phys. Rev. A 55, 4330 (1997).
22. M. Greiner, O. Mandel, T. W. Hänsch, I. Bloch, Nature 419, 51 (2002).
23. U. Eckern, G. Schön, V. Ambegaokar, Phys. Rev. Lett. 48, 1745 (1982); Phys. Rev B 30, 6419 (1984).
24. D. Rogovin, D. J. Scalapino, Ann. Phys. (New York), 86, 1 (1974).
25. J. D. Bjorken, S. D. Drell, Relativistic Quantum Mechanics (McGraw Hill, New York, 1964).
26. G. Schön, A. D. Zaikin, Phys. Rep. 198, 237 (1990).
27. A. J. Leggett, Phys. Rev. B 30, 1208 (1984).
28. A. Barone, G. Paterno, Physics and Applications of the Josephson Effect (Wiley, New York, 1982).
29. M. Tinkham, Introduction to Superconductivity 2nd. ed. (McGraw-Hill, New York, 1996).
30. J. B. Ketterson, S. N. Song, Superconductivity (Cambridge University Press, Cambridge, 1999).
31. U. Weiss, Quantum Dissipative Systems, 2nd ed. (World Scientific, Singapore, 1999).
32. A. Cuccoli, A. Fubini, V. Tognetti, R. Vaia, Phys. Rev. E 64, 066124 (2001).
33. P. W. Anderson, in Lectures on the Many Body Problem, E. R. Caianiello, ed. (Academic, New York, 1964).
34. A. Schmidt, Phys. Rev. Lett. 51, 1506 (1983); M. Fisher, W. Zwerger Phys. Rev. B 32, 6190 (1985); F. Guinea, V. Hakim, A. Muramatsu, Phys. Rev. Lett. 54, 263 (1985); D. Loss, K. Mullen, Phys. Rev. A 43, 2129 (1991).
35. P. Ullersma, Physica (Utrecht), 32, 27 (1966); id. 56 (1966); id. 74 (1966); id. 90 (1966).
36. F. Haake, R. Reibold, Phys. Rev. A 32, 2462 (1985).
37. V. Ambegaokar, A. Baratoff, Phys. Rev. Lett. 10, 486 (1963).
38. B. D. Josephson, Phys. Lett. 1, 251 (1962).
39. M. H. Cohen, L. M. Falicov, J. C. Phillips, Phys. Rev. Lett. 8, 316 (1962).
40. A more physical local tunneling Hamiltonian, where $\delta(z)$ is replaced by $\delta'(z)$, has been proposed in Ref.[41]. The change has consequences on the energy dependence of the hopping matrix elements and thus on the average transmission $|T|^2$, which
is correctly given by the choice $\delta'(z)$. However, this is not a problem in the context of the Ambegaokar-Baratoff formula because the product $I_c R$ is independent of the actual value of the average transmission.

41 E. Prada, F. Sols, cond-mat/0307500, to be published.
42 The fact that the baths couple to $\sin(\phi/2)$ and $\cos(\phi/2)$ eliminates unwanted dependence on the initial conditions. 43 J. Sánchez-Cañizares, F. Sols, Physica A 212, 181 (1994).
44 R. E. Harris, Phys. Rev. B 11, 3329 (1975).
45 N. R. Werthamer, Phys. Rev. 147, 255 (1966).
46 H. Kohler, F. Sols, to be published.
47 M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972