DIFFERENTIABILITY OF CONTINUOUS FUNCTIONS IN TERMS OF HAAR-SMALLNESS

ADAM KWELA AND WOJCIECH ALEKSANDER WOŁOSZYŃ

Abstract. One of the classical results concerning differentiability of continuous functions states that the set $SD$ of somewhere differentiable functions (i.e., functions which are differentiable at some point) is Haar-null in the space $C[0,1]$. By a recent result of Banakh et al., a set is Haar-null provided that there is a Borel hull $B \supset A$ and a continuous map $f : \{0,1\}^\mathbb{N} \to C[0,1]$ such that $f^{-1}[B + h]$ is Lebesgue's null for all $h \in C[0,1]$.

We prove that $SD$ is not Haar-countable (i.e., does not satisfy the above property with "Lebesgue's null" replaced by "countable", or, equivalently, for each copy $C$ of $\{0,1\}^\mathbb{N}$ there is an $h \in C[0,1]$ such that $SD \cap (C + h)$ is uncountable.

Moreover, we use the above notions in further studies of differentiability of continuous functions. Namely, we consider functions differentiable on a set of positive Lebesgue's measure and functions differentiable almost everywhere with respect to Lebesgue's measure. Furthermore, we study multidimensional case, i.e., differentiability of continuous functions defined on $[0,1]^k$. Finally, we pose an open question concerning Takagi's function.

1. Introduction

We follow the standard topological notation and terminology. By $|X|$ we denote the cardinality of a set $X$.

For a function $f \in C[0,1]$, by $D(f)$ we denote the set of all points $x \in [0,1]$ at which $f$ is differentiable. There are examples of continuous functions such that $D(f) = \emptyset$. One of the first and simplest examples is the famous Takagi's function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \text{dist}(2^i x, \mathbb{Z})$ (see [1], [2], or [15]). The size of the set of somewhere differentiable functions, i.e., functions $f$ such that $D(f) \neq \emptyset$, is a classical object of studies since Banach’s result stating that this set is meager in $C[0,1]$ (cf. [4]). One of the well-known results in this subject is Hunt’s theorem stating that the aforementioned set is Haar-null in the space $C[0,1]$ (see [12]).

This notion was first introduced in [8] by Christensen. He called a subset $A$ of an abelian Polish group $X$ Haar-null provided that there is a Borel hull $B \supseteq A$ and a Borel $\sigma$-additive probability measure $\lambda$ on $X$ such that $\lambda(B + x) = 0$ for all $x \in X$ (actually, in the original paper Christensen introduced this notion using universally measurable hulls instead of Borel hulls, however in many papers, e.g. [5] or [9], Haar-null sets are defined in the same way as above). A big advantage of this concept is that in a locally compact group it is equivalent to the notion of Haar measure zero sets and at the same time it can be used in a significantly larger class of groups. In [13] Hunt, Sauer and Yorke, unaware of Christensen's paper,
reintroduced the notion of Haar-null sets in the context of dynamical systems (in their paper Haar-null sets are called shy sets, and their complements are called prevalent sets).

Actually, since the set of somewhere differentiable functions is not Borel, Hunt had to show something more: the set of somewhere Lipschitz functions is Haar-null in $C[0,1]$ (a function $f \in C[0,1]$ is *somewhere Lipschitz* whenever there is an $x \in [0,1]$ and an $M \in \mathbb{N}$ such that $|f(x) - f(y)| \leq M|x - y|$ for each $y \in [0,1]$; observe that each somewhere differentiable function is somewhere Lipschitz).

In this paper, we are interested in the following notions of smallness. A subset $A$ of an abelian Polish group $X$ is:

- **Haar-countable** if there is a Borel hull $B \supseteq A$ and a copy $C$ of $\{0,1\}^\mathbb{N}$ such that $(C + x) \cap B$ is countable for all $x \in X$;
- **Haar-finite** if there is a Borel hull $B \supseteq A$ and a copy $C$ of $\{0,1\}^\mathbb{N}$ such that $(C + x) \cap B$ is finite for all $x \in X$;
- **Haar-$n$** for $n \in \mathbb{N}$, if there is a Borel hull $B \supseteq A$ and a copy $C$ of $\{0,1\}^\mathbb{N}$ such that $|(C + x) \cap B| \leq n$ for all $x \in X$.

Clearly,

$$\text{Haar-}n \implies \text{Haar-}(n+1) \implies \text{Haar-finite} \implies \text{Haar-countable}$$

for any $n \in \mathbb{N}$.

The choice of names in the above is due to Banakh et al., who recently unified the notions of Haar-null sets and Haar-meager sets (defined by Darji in [9]) in [5] by introducing the concept of Haar-small sets. A collection of subsets of a set $X$ is called a semi-ideal whenever it is closed under taking subsets. Following [5, Definition 11.1], for a semi-ideal $I$ on the Cantor cube $\{0,1\}^\mathbb{N}$, we say that a subset $A$ of an abelian Polish group $X$ is Haar-$I$ if there is a Borel hull $B \supseteq A$ and a continuous map $f : \{0,1\}^\mathbb{N} \to X$ such that $f^{-1}[B + x] \in I$ for all $x \in X$. It turns out that if $I$ is the $\sigma$-ideal $\mathcal{N}$ of subsets of $\{0,1\}^\mathbb{N}$ of Haar measure zero, then we obtain Haar-null sets (cf. [5, Theorem 11.5]). The same holds for the $\sigma$-ideal $\mathcal{M}$ of meager sets and Haar-meager sets (cf. [5, Theorem 11.5]), the $\sigma$-ideal of countable sets and Haar-countable sets, the ideal of finite sets and Haar-finite sets and the semi-ideal of sets of cardinality at most $n$ and Haar-$n$ sets (cf. [14, Proposition 1.2]).

Obviously, the collection of Haar-$I$ subsets of an abelian Polish group is a semi-ideal. Observe that $I \subseteq J$ implies that each Haar-$I$ set is Haar-$J$. Thus, Haar-null sets and Haar-meager sets only allow us to say that some properties are rare (and we cannot put them on a scale and compare with other rare properties), whereas Haar-$I$ sets allow us to develop a whole hierarchy of small sets.

Haar-countable, Haar-finite, and Haar-$n$ sets were profoundly studied by the first author in [14]. There are compact examples showing that none of the above implications can be reversed in $C(K)$ where $K$ is compact metrizable. Moreover, it is known that neither Haar-finite sets nor Haar-$n$ sets form an ideal (see [14, Corollary 5.2 and Theorem 6.1]). Zakrzewski considered Haar-small sets in [17] under the name of perfectly $\kappa$-small sets. A particular case of Haar-$I$ sets was investigated by Balcerzak in [3]. He introduced the so-called property (D) of a $\sigma$-ideal $I$, which says that there is a Borel Haar-$1$ set not belonging to $I$. Moreover, Banakh, Lyaskovska, and Repovš considered packing index of a set in [6]. Packing index is closely connected to Haar-$1$ sets (namely, a Borel set is Haar-$1$ if and only...
if its packing index is uncountable). In turn, Haar-countable sets were studied by Darji and Keleti in [10] and by Elekes and Steprāns in [11].

2. Nowhere differentiable functions

Hunt proved in [12] that the set \( \mathcal{SD} \) of somewhere differentiable functions is Haar-null in \( C[0, 1] \). Actually, a closer look at his proof gives us something more. Denote by \( \mathcal{E} (\mathcal{E}(\mathbb{R})) \) the σ-ideal generated by compact Haar null subsets of \( \{0, 1\}^\mathbb{N} \) (\( \mathbb{R} \), respectively) and recall that \( \mathcal{E} \) is a proper subfamily of \( \mathcal{N} \cap \mathcal{M} \) (see [7, Lemma 2.6.1]). Hunt’s proof shows that there is a continuous function \( \varphi : \mathbb{R}^2 \rightarrow C[0, 1] \) such that \( \varphi^{-1}[f + \mathcal{SD}] \in \mathcal{E}(\mathbb{R}) \) for all \( f \in C[0, 1] \). If \( \psi : \{0, 1\}^\mathbb{N} \rightarrow [0, 1]^2 \) denotes the map given by \( \psi(x_i)) = (\sum_{i \in \mathbb{N}} \frac{x_i}{2^i}, \sum_{i \in \mathbb{N}} \frac{x_i}{2^i}) \) then \( \varphi^{-1}[A] \in \mathcal{E} \) for all \( A \in \mathcal{E}(\mathbb{R}) \).

Thus, \( \psi^{-1}[\varphi^{-1}[f + \mathcal{SD}]] \in \mathcal{E} \) for all \( f \in C[0, 1] \), i.e., we have shown the following:

**Theorem 2.1** (Hunt, [12]). The set \( \mathcal{SD} \) is Haar-\( \mathcal{E} \) in \( C[0, 1] \).

In this Section we show that \( \mathcal{SD} \) is not Haar-countable in \( C[0, 1] \).

**Theorem 2.2.** The set of somewhere differentiable functions is not Haar-countable in \( C[0, 1] \).

**Proof.** Denote by \( C \) the ternary Cantor set (which is homeomorphic to \( \{0, 1\}^\mathbb{N} \)). Let \( \varphi : C \rightarrow C[0, 1] \) be continuous. We need to find a homeomorphic copy \( C' \) of \( C \) and a continuous function \( g : [0, 1] \rightarrow \mathbb{R} \) such that \( \varphi(c) - g \in \mathcal{SD} \) for all \( c \in C' \).

Define \( D_1 = \{0, 1\} \) and \( D_n = \left\{ \left\{ \frac{1}{3^i} : i \in \mathbb{N} \right\} \cap C \right\} \) for all \( n \in \mathbb{N}, n \neq 1 \). Let \( E_1 = D_1 \) and \( E_n = D_n \setminus D_{n-1} \), for \( n \in \mathbb{N}, n > 1 \). Observe that \( D = \bigcup_{n \in \omega} D_n \) is the set of all points from \( C \) with finite ternary expansion. For each \( n \in \mathbb{N} \) and \( d \in E_n \), let:

\[
U_d = [0, 1] \cap \left[ d - \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}}, d + \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} \right].
\]

Note that \( U_d \cap D_n = \{d\} \), for every \( d \in D_n \).

First, we need to shrink \( C \). Namely, we want to have a homeomorphic copy \( C' \subseteq C \) such that:

- \( \| f_c - f_{c'} \| \leq (c - c')^2 \), for all \( c, c' \in C \);
- \( \| f_d - f_{d'} \| < \inf_{x \in U_d \cup U_{d'}} ((d - x)^2 + (d' - x)^2) \), for all distinct \( d, d' \in D \) with \( U_d \cap U_{d'} \neq \emptyset \);
- \( \| f_d - f_{d'} \| < \frac{1}{2} (\text{dist}(d, U_{d'}))^2 = \frac{1}{2} (\text{dist}(d', U_d))^2 \), whenever \( d, d' \in E_n \) for some \( n, d \neq d' \) and \( U_d \cap U_{d'} \neq \emptyset \);
- \( \| f_d - f_{d'} \| < \frac{1}{2} (\text{dist}(d, U_{d'}))^2 \), whenever \( d' \in E_n \) and \( d \in D_n \setminus E_n \) for some \( n \) and \( U_d \cap U_{d'} \neq \emptyset \);

where \( f_x = \varphi(\psi(x)) \) and \( \psi : C \rightarrow C' \) is the aforementioned homeomorphism.

The construction of \( C' \) is rather standard. Nevertheless, we provide a short sketch of it.

Let \( c : C^2 \rightarrow \mathbb{R} \) be a continuous function such that \( \| f_c - f_{c'} \| \leq e(c, c') \) guarantees all of the above conditions and \( e(c, c') > 0 \) for all \( c, c' \in C \) with \( c \neq c' \) (which clearly exists). For a finite sequence \( s = (s_1, \ldots, s_n) \in \{0, 1\}^n \), let \( \bar{s} = \sum_{i=1}^n \frac{s_i}{3^i} \). In the first inductive step pick any basic clopen set \( W_{\bar{s}} \subseteq C \) such that:

\[
\text{diam}(\varphi[W_{\bar{s}}]) < \inf \left\{ e(c, c') : c \in C \cap \left(0, \frac{1}{3}\right] \land c' \in C \cap \left[\frac{2}{3}, 1\right) \right\}.
\]
If $W_s$ are already defined for all $s \in \{0, 1\}^n$, for each $s = (s_1, \ldots, s_n) \in \{0, 1\}^n$ find two disjoint basic clopen sets $W_s^{-}(0), W_s^{-}(1) \subseteq W_s$ such that $\text{diam}(\varphi[W_s^{-}(i)]) < \frac{1}{2n+1}$ and:

$$\text{diam}(\varphi[W_s^{-}(i)]) < \inf \left\{ \epsilon(c, c') : c \in C \cap \left[ s^{-}(i, 0), s^{-}(i, 0) + \frac{1}{3n+1} \right] \land c' \in C \cap \left[ s^{-}(i, 1), s^{-}(i, 1) + \frac{1}{3n+1} \right] \right\},$$

for $i = 0, 1$. It is easy to check that $C' = \bigcup_{x \in \{0, 1\}^n} \bigcap_{n \in \mathbb{N}} W_x(1, \ldots, n)$ is the required set.

Now, we want to construct a $g \in C[0, 1]$ such that $f_c - g$ has a derivative at $c$ equal to 0, for all $c \in C$. Inductively, we will define a sequence of continuous functions $(g_n) \subseteq C[0, 1]$ such that:

$$\forall d \in E_n \forall x \in U_d \left| f_d(x) - \sum_{i=1}^{n} g_i(x) \right| \leq \frac{1}{5} (x - d)^2;$$

$$\forall d \in D_n \forall x \in U_d \left| f_d(x) - \sum_{i=1}^{n} g_i(x) \right| \leq (x - d)^2.$$

At the end, we will put $g = \sum_{i=1}^{\infty} g_n$.

Start the construction with $g_1 \in C[0, 1]$ such that:

- $g_1 = f_0$ on $U_0 \setminus U_1$;
- $g_1 = f_1$ on $U_1 \setminus U_0$;
- $g_1(x)$ is between $f_0(x)$ and $f_1(x)$ for all $x \in U_0 \cap U_1$.

Note that we have $|f_0(x) - g_1(x)| = 0 \leq \frac{1}{5}(x - 0)^2$ for all $x \in U_1 \setminus U_0$ and $|f_0(x) - g_1(x)| \leq \|f_0 - f_1\| < \frac{1}{5}(\inf U_1 - 0)^2 \leq \frac{1}{5}(x - 0)^2$ for all $x \in U_0 \cap U_1$.

Similarly, $|f_1(x) - g_1(x)| \leq \frac{1}{5}(x - 1)^2$ for all $x \in U_1$. Thus, $g_1$ is as needed.

Once all $g_i$’s, for $i < n$, are defined, let $\tilde{g}_n : \bigcup_{d \in E_n} U_d \to \mathbb{R}$ be a continuous function such that:

- $\tilde{g}_n = f_d - \sum_{i=1}^{n-1} g_i$ on $U_d \setminus \bigcup_{d' \in E_n \setminus \{d\}} U_{d'}$ for each $d \in E_n$;
- if $d, d' \in E_n$ and $U_d \cup U_{d'} \neq \emptyset$, then $\tilde{g}_n(x)$ is between $f_d(x) - \sum_{i=1}^{n-1} g_i(x)$ and $f_{d'}(x) - \sum_{i=1}^{n-1} g_i(x)$ for all $x \in U_d \cap U_{d'}$.

Let $d \in E_n$ and notice that $|f_d(x) - \sum_{i=1}^{n-1} g_i(x) - \tilde{g}_n(x)| = 0 \leq \frac{1}{5}(x - d)^2$ for all $x \in U_d \setminus \bigcup_{d' \in E_n \setminus \{d\}} U_{d'}$ and $|f_d(x) - \sum_{i=1}^{n-1} g_i(x) - \tilde{g}_n(x)| \leq \|f_d - f_{d'}\| < \frac{1}{5}(\text{dist}(d, U_{d'}))^2 \leq \frac{1}{5}(x - d)^2$ for all $x \in U_d \cap U_{d'}$ whenever the latter intersection is non-empty. Moreover, if $d \in D_n \setminus E_n$ and $x \in U_d \cap U_{d'}$ for some $d' \in E_n$, then:

$$|d' - x| \leq \frac{1}{3^{n-1}} + \frac{1}{3^n} \leq \frac{1}{3^{n-1}} - \frac{1}{3^n} =$$

$$2 \left( \frac{1}{3^{n-1}} - \frac{1}{3^n} \right) \leq 2 \text{dist}(d, U_{d'}) \leq 2|d - x|.$$

Therefore,

$$\left| f_d(x) - \sum_{i=1}^{n-1} g_i(x) - \tilde{g}_n(x) \right| \leq \|f_d - f_{d'}\| + \|f_{d'}(x) - \sum_{i=1}^{n-1} g_i(x) - \tilde{g}_n(x)\| \leq$$
more, if 

Using \{\text{some notation}\} for enumeration of the set \(\mathbb{N}\):

\[f(x) = \sum_{j=1}^{n-1} g_i(x) - (d - x)^2 \leq g_n(x) \leq f_d(x) - \sum_{i=1}^{n-1} g_i(x) + (d - x)^2\]

for all \(d \in D_n\) such that \(x \in U_d\). Note that such extension exists. Indeed, we need to check that whenever \(x \in U_d \cap U_{d'}\), for some \(d,d' \in D_n \setminus E_n\) then

\[f_d(x) - \sum_{i=1}^{n-1} g_i(x) - (d - x)^2 \leq f_{d'}(x) - \sum_{i=1}^{n-1} g_i(x) + (d' - x)^2.\]

The above leads to \(|f_d(x) - f_{d'}(x)| \leq (d - x)^2 + (d' - x)^2\) which is satisfied as \(\|f_d - f_{d'}\| < \inf_{x \in U_d \cap U_{d'}} ((d - x)^2 + (d' - x)^2)\), for all distinct \(d,d' \in D\) with \(U_d \cap U_{d'} \neq \emptyset\).

Once the construction is completed, define \(g = \sum_{n=1}^{\infty} g_n\). For each \(d \in D\) and \(x \in U_d\), we have \(|f_d(x) - g(x)| \leq (x - d)^2\). Thus, \(f_d - g\) has a derivative at \(d\) equal to 0. Now, we want to show that \(f_d - g\) has a derivative at \(c\) equal to 0 for each \(c \in C\).

Fix \(c \in C \setminus D\) and \(h \in \mathbb{R}\) such that \(c + h \in [0,1]\). Assume that \(c + h \in U_d\) for some \(d \in D\) and \(|h| \geq \frac{1}{4}|c - d|\). Observe that:

\[
\left| \frac{f_d(c + h) - g(c + h) - f_d(c) + g(c)}{h} \right| \leq \left| \frac{f_d(c + h) - f_d(c)}{h} \right| + \left| \frac{f_d(c + h) - g(c + h) - f_d(c) + g(c)}{h} \right| \leq 2 \left| f_d - f_{d'} \right| + \left| \frac{f_d(c + h) - g(c + h) - f_d(c) + g(c)}{h} \right|.
\]

As \(\|f_d - f_d\| \leq (c - d)^2\) and \(|h| \geq \frac{1}{4}|c - d|\), we get that \(\left| f_d - f_{d'} \right| \leq \frac{4}{|c - d|}\). What is more, if \(d' \in D\) is such that \(c \in U_{d'}\) then \(\left| f_d(c) - g(c) \right| = \left| f_d(c) - f_{d'}(c) \right| + \left| f_{d'}(c) - g(c) \right| \leq (d - d')^2 + (d' - c)^2 \leq (d - c)^2\).

Using \(|h| \geq \frac{1}{4}|c - d|\) once again, we have:

\[
\left| \frac{f_d(c + h) - g(c + h) - f_d(c) + g(c)}{h} \right| \leq \frac{(c + h - d)^2 + (c - d)^2}{|h|} = \frac{2(c - d)^2 + 2h(c - d) + h^2}{|h|} \leq \frac{8(c - d)}{|h|} + 2|c - d| + |h|.
\]

Thus, \(f_d(c + h) = g(c + h) = f_d(c) \) tends to 0 as \(h \to 0\) and \(|c - d| \to 0\). Therefore, to finish the proof, it suffices to show that for each \(c \in C\) there are sequences \((d_n) \subseteq D \cap \{c,1\}\) and \((d'_n) \subseteq D \cap [0,c]\) converging to \(c\) and such that \((c,1) \subseteq \bigcup_{n \in \mathbb{N}} U_{d_n} \cap \{c + \frac{1}{2}(d_n - c),1\}\) and \([0,c] \subseteq \bigcup_{n \in \mathbb{N}} U_{d'_n} \cap [0,c - \frac{1}{4}(c - d_n)]\).

We will construct the sequence \((d_n)\). Construction of \((d'_n)\) is similar. Let \((c_n) \in \{0,2\}^{\mathbb{N}}\) be the ternary expansion of \(c \in C\) and \((i_n) \subseteq \mathbb{N}\) be the increasing enumeration of the set \(\{n \in \mathbb{N} : \ c_n = 0\}\). For all \(k \in \mathbb{N}\), define \(d_1 = 1, d_{2k} = 1 - \sum_{j=1}^{k-1} \frac{2}{3^j} - \frac{1}{3^k}\), and \(d_{2k+1} = 1 - \sum_{j=1}^{k-1} \frac{2}{3^j} + \frac{2}{3^k}\). Now, we show that \((d_n)\) is as required.
We need to show that sup $U_{d_{n+1}} > c + \frac{1}{4}(d_n - c)$ for all $n \in \mathbb{N}$. Clearly, sup $U_{d_{2k}} \geq d_{2k} + \frac{1}{3k} + \frac{1}{3k^2}$ and sup $U_{d_{2k+1}} \geq d_{2k+1} + \frac{1}{3k} + \frac{1}{3k^2}$, i.e., it suffices to show:

$$d_{2k} + \frac{1}{3k} + \frac{1}{3k^2} > c + \frac{1}{4}(d_{2k-1} - c);$$

$$d_{2k+1} + \frac{1}{3k} + \frac{1}{3k^2} > c + \frac{1}{4}(d_{2k} - c).$$

As $d_{2k} = c + (d_{2k} - c)$ and $d_{2k+1} = c + (d_{2k+1} - c)$, this is equivalent to:

$$d_{2k} - c + \frac{1}{2} + \frac{1}{3k} > \frac{1}{4} (d_{2k-1} - c);$$

$$d_{2k+1} - c + \frac{1}{2} + \frac{1}{3k} > \frac{1}{4} (d_{2k} - c).$$

Multiplying both sides by 4 and using $d_{2k-1} = d_{2k} + \frac{1}{3k}$ and $d_{2k} = d_{2k+1} + \frac{1}{3k}$, we get:

$$4(d_{2k} - c) + \frac{2 + \frac{1}{6}}{3k} > d_{2k} - c + \frac{1}{3k};$$

$$4(d_{2k+1} - c) + \frac{2 + \frac{1}{6}}{3k} > d_{2k+1} - c + \frac{1}{3k}.$$

Thus,

$$3(d_{2k} - c) > \frac{1 - 2 - \frac{2}{9}}{3k};$$

$$3(d_{2k+1} - c) > \frac{1 - 2 - \frac{2}{9}}{3k};$$

which is true since $d_{2k} > c$ and $d_{2k+1} > c$. This finishes the entire proof. \qed

By the above, the set of functions differentiable at some point is not Haar-countable. However, what about functions differentiable at more than one point? As for a given $\sigma$-ideal $I$ on $[0,1]$ the set $\{f \in C[0,1] : \emptyset \neq D(f) \in I\}$ is contained in the set of somewhere differentiable functions, this question is natural. The following slight strengthening of Theorem 2.2 gives only a partial answer to this problem.

**Corollary 2.3.** Let $I$ be a $\sigma$-ideal on $[0,1]$ containing some perfect set. The set of functions $f \in C[0,1]$ such that $D(f)$ is a nonempty set which belongs to $I$ is not Haar-countable.

**Proof.** First, assume that the ternary Cantor set $C$ belongs to $I$. We need to make two modifications of the proof of Theorem 2.2.

Since, by Hunt’s result, the set of somewhere Lipschitz functions is Borel and Haar-null (see [12]), the set $\mathcal{NL}$ of nowhere Lipschitz functions cannot be Haar-null. Thus, for $\varphi$ from the proof of Theorem 2.2, there is a $z \in C[0,1]$ such that $\varphi^{-1}[\mathcal{NL} - z]$ is not Lebesgue’s null. In particular, this is a Borel uncountable subset of $C$. Hence, it must contain a homeomorphic copy $P$ of $C$. Then, $\varphi(c) + z$ is nowhere differentiable for each $c \in P$. Thus, by performing the construction of $C'$ inside $P$ and defining $f_z = \varphi(\psi(x)) + z$ where $\psi$ is a homeomorphism from $C$ to $C'$, we may assume that $f_z$ is nowhere differentiable for each $c \in C$. Moreover, these changes do not affect the rest of the proof. If we find $g \in C[0,1]$ such that $f_z - g$ is somewhere differentiable for all $c \in C$, then $\varphi(c) + (z - g)$ is somewhere differentiable for uncountably many $c \in C$ (namely, for all $c \in C'$).
Now, we move to the second modification. We can ensure that \( g \) is differentiable outside \( C \). Indeed, for each connected component \( I \) of \([0, 1] \setminus C\) fix a sequence of closed (possibly empty) intervals \((I_n)\) such that \( \bigcup_n I_n = I \), \( I_n \subseteq I_{n+1} \) and \( I_n \cap \bigcup_{d \in E_n} U_d = \emptyset \) for all \( n \) (which is possible as \( \bigcup_n E_n \subseteq C \) and \( \sup_{d \in E_n} \text{diam}(U_d) \) tends to 0 as \( n \to +\infty \)). Now it suffices to require, additionally, in the inductive construction of \((g_n)\), that \( \sum_{i=1}^n g_i \) is differentiable on \( I_n \) and \( g_n \upharpoonright I_{n-1} = 0 \) (if \( n > 1 \)). This can be done as for \( x \in I_n \) the only requirement imposed on \( g_n \) resulting from the proof of Theorem 2.2 is:

\[
\sup_{d \in D_n, x \in U_d} (f_d(x) - (d - x)^2) \leq \sum_{i=1}^n g_i(x) \leq \inf_{d \in D_n, x \in U_d} (f_d(x) + (d - x)^2).
\]

Thus, there is no problem with the request \( g_n \upharpoonright I_{n-1} = 0 \) as \( (\bigcup_{d \in D_n} U_d) \cap I_{n-1} \subseteq (\bigcup_{d \in D_n} U_d) \cap I_n \subseteq \bigcup_{d \in D_n} U_d \) and in the \( n \)th induction step we already have:

\[
\sup_{d \in D_{n-1}, x \in U_d} (f_d(x) - (d - x)^2) \leq \sum_{i=1}^{n-1} g_i(x) \leq \inf_{d \in D_{n-1}, x \in U_d} (f_d(x) + (d - x)^2).
\]

What is more, \( \sum_{i=1}^{n-1} g_i(x) \) is differentiable on \( I_{n-1} \), hence so is \( \sum_{i=1}^n g_i(x) \). Since

\[
\sup_{d \in D_n, x \in U_d} (f_d(x) - (d - x)^2) < \inf_{d \in D_n, x \in U_d} (f_d(x) + (d - x)^2)
\]

(by the fact that \( \| f_d - f_d' \| < \inf_{x \in U_d \cap U_d'} ((d - x)^2 + (d' - x)^2) \), for all distinct \( d, d' \in D \) with \( U_d \cap U_d' \neq \emptyset \)), there is some freedom in the choice of \( g_n \) and we can ensure that \( \sum_{i=1}^n g_i \) is differentiable also on \( I_n \setminus I_{n-1} \).

After these modifications, as \( g \) is differentiable at each point \( x \in [0, 1] \setminus C \) while \( f_c \) is not, \( f_c - g \) is not differentiable at each point of \([0, 1] \setminus C\) and we get that

\[
D(f_c - g) \subseteq C \quad \text{for all } c \in C.
\]

The case where \( C \notin \mathcal{I} \) requires one additional modification. Since every perfect set contains a homeomorphic copy of the ternary Cantor set \( C \), we simply need to find such a copy \( R \) that belongs to \( \mathcal{I} \). Then, we can replace \( C \) with \( R \), modify sets \( D_n \) and \( U_d \), and perform similar reasoning as in the proof of Theorem 2.2.

It is known that \( D(f) \) is Borel (of type \( \mathcal{G}_{\delta_{\omega}} \)) for each \( f \in C[0, 1] \) (see [16]). Thus, we can consider Lebesgue’s measure of the set \( D(f) \). Since there are perfect sets of Lebesgue’s measure zero, the following is immediate.

**Corollary 2.4.** The set of functions \( f \in C[0, 1] \) such that \( D(f) \) is a nonempty set of Lebesgue’s measure zero is not Haar-countable.

As the \( \sigma \)-ideal of countable sets does not contain any perfect set, the following question arises.

**Problem 2.5.** Is the set of functions \( f \in C[0, 1] \) such that \( D(f) \) is countable but non-empty Haar-countable in \( C[0, 1] \)?

### 3. Differentiability and Lebesgue’s Measure

In this Section, we examine functions differentiable on a set of positive Lebesgue’s measure.
We will need the following notation. By the symbol \( \lambda \) we will denote the Lebesgue’s measure on \([0, 1]\). Moreover, for a function \( f \in C[0, 1] \) and \( M \in \mathbb{N} \), define
\[
L_M(f) = \{ x \in [0, 1] : \forall y \in [0, 1] |f(x) - f(y)| \leq M|x - y| \}.
\]
Then, \( f \) is somewhere Lipschitz if and only if the set \( L(f) = \bigcup_{M \in \mathbb{N}} L_M(f) \) is non-empty.

The next two rather folklore lemmas will be useful in our further considerations.

**Lemma 3.1.** For any \( f \in C[0, 1] \) and \( M \in \mathbb{N} \), the set \( L_M(f) \) is closed.

**Proof.** We will show that \([0, 1] \setminus L_M(f)\) is open. Fix any \( x \in [0, 1] \setminus L_M(f) \). Then, there is \( y \in [0, 1] \) such that \( |f(x) - f(y)| > M|x - y| \). Find \( \alpha > 0 \) such that \( |f(x) - f(y)| > M(|x - y| + \alpha) \). By continuity of \( f \) at \( x \), there is a \( \delta > 0 \) such that
\[
|f(x) - f(z)| < |f(x) - f(y)| - M(|x - y| + \alpha)
\]
whenever \( |x - z| < \delta \). Then, for each \( z \in [0, 1] \) such that \( |x - z| < \min\{\delta, \alpha\} \), we have:
\[
|f(z) - f(y)| \geq |f(x) - f(y)| - |f(x) - f(z)| > M(|x - y| + \alpha) > M|y - z|.
\]
Hence, \( z \notin L_M(f) \). \( \square \)

This result implies that \( L(f) \) is Borel (of type \( F_\sigma \)). Thus, we can consider Lebesgue’s measure of the sets \( L_M(f) \) and \( L(f) \).

**Lemma 3.2.** For each \( M \in \mathbb{N} \) and \( c \in (0, 1] \), the set of functions \( f \in C[0, 1] \) such that \( \lambda(L_M(f)) \geq c \) is closed in \( C[0, 1] \).

**Proof.** Fix a sequence \( (f_n) \subseteq C[0, 1] \) converging to some \( f \in C[0, 1] \) and such that \( \lambda(L_M(f_n)) \geq c \) for each \( n \). We need to show that \( \lambda(L_M(f)) \geq c \). Suppose, to the contrary, that \( \lambda(L_M(f)) < c \). Using regularity of Lebesgue’s measure, find an open set \( G \subseteq [0, 1] \) such that \( L_M(f) \subseteq G \) and \( \lambda(G) < c \).

For each \( n \), there is an \( x_n \in L_M(f_n) \setminus G \) (as \( \lambda(G) < \lambda(L_M(f_n)) \)). Since \([0, 1]\) is compact, without loss of generality we may assume that \((x_n)\) converges to some \( x \in [0, 1] \). Observe that \( x \notin L_M(f) \) as \( G \) is an open hull of \( L_M(f) \) and whole sequence \((x_n)\) is outside of \( G \). However,
\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_n)| + |f_n(x_n) - f_n(y)| + |f_n(y) - f(y)| \leq ||f - f_n|| + |f_n(x) - f_n(x_n)| + M|x_n - y| + ||f - f_n||
\]
for each \( y \in [0, 1] \). Convergence of \( (f_n) \) to \( f \) implies equicontinuity of \( (f_n) \) at \( x \). So, if \( n \) tends to infinity, we get that \( |f(x) - f(y)| \leq M|x - y| \). This contradicts \( x \notin L_M(f) \). \( \square \)

First, we want to focus on functions differentiable almost everywhere.

**Proposition 3.3.** The set of functions differentiable almost everywhere is Haar-1 in \( C[0, 1] \).

**Proof.** Let \( \mathcal{A} \) denote the set of functions \( f \in C[0, 1] \) such that \( \lambda(L(f)) = 1 \). Note that each function differentiable almost everywhere is in \( \mathcal{A} \). Thus, if we show that \( \mathcal{A} \) is Haar-1 in \( C[0, 1] \), the thesis will follow.
Recall that \( L(f) = \bigcup_{M \in \mathbb{N}} L_M(f) \) and \( L_M(f) \subseteq L_{M+1}(f) \). Thus, \( \lambda(L(f)) = \lim_{M \to \infty} \lambda(L_M(f)) \). Consequently,

\[
\mathcal{A} = \bigcap_{k \in \mathbb{N}} \bigcup_{M \in \mathbb{N}} \left\{ f \in C[0,1] : \lambda(L_M(f)) \geq \frac{k-1}{k} \right\}
\]

and \( \mathcal{A} \) is Borel by Lemma 3.2.

Observe that \( \mathcal{A} - \mathcal{A} \) is meager as a subset of the set of somewhere Lipschitz functions, which is meager by the Banach theorem [4] (see also the proof of Proposition 4.4). Indeed, if \( f \in \mathcal{A} - \mathcal{A} \) then there are \( g, h \in \mathcal{A} \) such that \( f = g - h \). Since \( \lambda(L(g)) = 1 = \lambda(L(h)) \), there is \( x \in \lambda(L(g)) \cap \lambda(L(h)) \). Thus, there are \( M_1, M_2 \in \mathbb{N} \) such that \( |g(x) - g(y)| \leq M_1|x - y| \) and \( |h(x) - h(y)| \leq M_2|x - y| \) for all \( y \in [0,1] \).

Thus, for each \( y \in [0,1] \) we have:

\[
|f(x) - f(y)| \leq |g(x) - g(y)| + |h(x) - h(y)| \leq (M_1 + M_2)|x - y|.
\]

The thesis follows now from [5, Corollary 16.3], which states that a Borel set \( B \) is Haar-1 if and only if \( B - B \) is meager.

Actually, this reasoning gives us something more.

**Corollary 3.4.** The set of functions \( f \in C[0,1] \) such that \( \lambda(D(f)) > \frac{1}{2} \) is Haar-1 in \( C[0,1] \).

**Proof.** Let \( \mathcal{A} \) denote the set of functions \( f \in C[0,1] \) such that \( \lambda(L(f)) > \frac{1}{2} \). Similarly as in the proof of Proposition 3.3, one can show that \( \mathcal{A} \) is Borel. If we show that \( \mathcal{A} \) is Haar-1 in \( C[0,1] \), the thesis will follow.

We will show that \( \mathcal{A} - \mathcal{A} \) is a subset of the set of somewhere Lipschitz functions. Indeed, if \( f \in \mathcal{A} - \mathcal{A} \) then there are \( g, h \in \mathcal{A} \) such that \( f = g - h \). Since \( \lambda(L(g)) > \frac{1}{2} \) and \( \lambda(L(h)) > \frac{1}{2} \), there is \( x \in \lambda(L(g)) \cap \lambda(L(h)) \). Proceeding in the same way as in the proof of Proposition 3.3, we can conclude that \( f \) is Lipschitz in \( x \).

From now on the proof is entirely the same as the proof of Proposition 3.3. \( \square \)

Now, we will study functions \( f \in C[0,1] \) such that \( \lambda(D(f)) \in (0,1) \).

**Proposition 3.5.** Let \( \mathcal{I} \) be a \( \sigma \)-ideal on \([0,1]\) containing no interval. The set of functions \( f \in C[0,1] \) such that neither \( D(f) \) nor \([0,1] \setminus D(f) \) belongs to \( \mathcal{I} \) is not Haar-finite in \( C[0,1] \).

**Proof.** Denote by \( \mathcal{A} \) the set of functions \( f \in C[0,1] \) such that \( D(f) \notin \mathcal{I} \) and \([0,1] \setminus D(f) \notin \mathcal{I} \).

Let \( \varphi : \{0,1\}^{\mathbb{N}} \to C[0,1] \) be a continuous map. We need to show that \( \varphi^{-1}[\mathcal{A} + h] \) is infinite for some \( h \in C[0,1] \).

If \( \varphi[\{0,1\}^{\mathbb{N}}] \) is finite, then let \( g \) be any element of \( \mathcal{A} \) and \( h \in C[0,1] \) be such that \( \varphi^{-1}[\{h\}] \) is infinite. Observe that \( h \in \mathcal{A} + h - g \). Hence, \( \varphi^{-1}[\mathcal{A} + h - g] \) is infinite as well.

Assume now that \( \varphi[\{0,1\}^{\mathbb{N}}] \) is infinite and take any injective convergent sequence \( (f_n) \in \varphi[\{0,1\}^{\mathbb{N}}] \). Denote \( f = \lim_n f_n \). For each \( n \), let \( a_n, b_n, c_n \in \left( \frac{1}{n+2}, \frac{1}{n+1} \right) \) be such that \( a_n < b_n < c_n \) and denote \( I_n^1 = [a_n, b_n] \) and \( I_n^2 = [b_n, c_n] \). For all \( n \) let also \( J_n = [c_{n+1}, a_n] \) and \( g_n : J_n \to \mathbb{R} \) be the linear function given by \( g_n(c_{n+1}) = 0 \) and \( g_n(a_n) = (f_n - f_{n+1})(a_n) \). Fix any nowhere differentiable function \( z \in C[0,1] \) such that \( \|z\| \leq 1 \) and \( z(b_n) = z(c_n) = 0 \) for each \( n \). Such function exists as given any nowhere differentiable function \( \tilde{z} \in C[0,1] \) with \( z(0) = z(1) = 0 \) (for instance, the Takagi function) we can define \( z \in C[0,1] \) in such a way that

\[
\varphi^{-1}[\mathcal{A} + h] \ni z(f_n) = z(g_n(a_n) - f_{n+1}(a_n)) \leq z(1) - z(0) = 0.
\]
\[ z \uparrow [b_n, c_n] = \frac{2\pi r_n}{|b_n - c_n|} \text{ and } z \uparrow [c_{n+1}, b_n] = \frac{2\pi r_n}{|c_{n+1} - c_n|} \text{ for each } n, \text{ where } r_n : [b_n, c_n] \to [0, 1] \]

and \( s_n : [c_{n+1}, b_n] \to [0, 1] \) are linear bijections. Define \( h : [0, 1] \to \mathbb{R} \) by:

- \( h \uparrow [c_1, 1] = f_1(c_1) \);
- \( h \uparrow I_n^1 = f_n \uparrow I_n^1 \) for each \( n \in \mathbb{N} \);
- \( h \uparrow I_n^2 = f_n - \frac{\pi}{n} \uparrow I_n^2 \) for each \( n \in \mathbb{N} \);
- \( h \uparrow J_n = (f_{n+1} + g_n) \uparrow J_n \) for each \( n \in \mathbb{N} \);
- \( h(0) = f(0) \).

Note that the function \( h \) is continuous (continuity in 0 follows from \( \lim_n f_n = f \), \( \lim_n \sup_{x \in I_n} g_n(x) = 0 \), and \( \lim_n \frac{\|x\|}{n} = 0 \)) and \( f_n \in A + h \) for each \( n \in \mathbb{N} \) (as \( (f_n - h) \uparrow I_n^1 \) is constant and \( (f_n - h) \uparrow I_n^2 \) is nowhere differentiable). Since \( (f_n) \) is injective, we conclude that \( \varphi^{-1}[A + h] \) is infinite. \( \square \)

The next Corollary is immediate.

**Corollary 3.6.** The set of functions \( f \in C[0, 1] \) such that \( \lambda(D(f)) \in (0, 1) \) is not Haar-finite in \( C[0, 1] \).

Now, we will show that the set of functions \( f \in C[0, 1] \) such that \( D(f) \) is of positive Lebesgue’s measure is Haar-countable. The next lemma is a straightforward modification of one of the proofs showing that the Weierstrass function is nowhere Lipschitz.

**Lemma 3.7.** There is an \( a \in (0, 1) \) and there is a \( b \in \mathbb{N} \) such that for any \( x \in (0, 1) \) and any increasing sequence \((s_j)\) of positive integers, and any \((r_j)\) in \((-1, 1)\), the expression

\[ \sum_{j=0}^{\infty} r_j a^{s_j} \left( \cos(b^{s_j} \pi y) - \cos(b^{s_j} \pi x) \right) \]

where \( y \) runs over \([0, 1]\), is unbounded.

**Proof.** Let \( a \in (0, 1) \) and \( b \in \mathbb{N} \) be such that \( b \) is odd, \( ab > 1 \), and \( \frac{2}{3} - \frac{4a}{1-a} - \frac{\pi}{ab-1} > 0 \) (for instance, any \( a < \frac{1}{2} \) and any odd \( b \in \mathbb{N} \) with \( b > \frac{3\pi+1}{a} \) are good as in this case we have \( \frac{4a}{1-a} < \frac{3}{2} \) and \( \frac{\pi}{ab-1} < \frac{1}{2} \)).

For each \( m \in \mathbb{N} \), let \( w_m \in \mathbb{Z} \) be such that \( x_m = xb^{s_m} - w_m \in (-\frac{1}{2}, \frac{1}{2}) \) and define \( y_m = (w_m - 1)/b^{s_m} \). As \( -\frac{1}{2} < xb^{s_m} - w_m \leq \frac{1}{2} \), we get \( xb^{s_m} - \frac{3}{2} \leq w_m - 1 < xb^{s_m} - \frac{1}{2} \) and \( x - \frac{3}{2b^{s_m}} \leq y_m < x - \frac{1}{2b^{s_m}} \). Thus, \( y_m \in [0, 1] \) for sufficiently large \( m \) (as \( x > 0 \)).

Fix any \( m \in \mathbb{N} \). Since \( |\sin(z)/z| \leq 1 \), using the formula for the difference of cosines, for each \( z \) we have:

\[ \left| \sum_{j=0}^{m-1} r_j a^{s_j} \left( \cos(b^{s_j} \pi y_m) - \cos(b^{s_j} \pi x) \right) \right| = \left| \sum_{j=0}^{m-1} r_j(ab)^{s_j} \pi \sin \left( \frac{b^{s_j} \pi (y_m + x)}{2} \right) \frac{\sin \left( \frac{(b^{s_j} \pi (y_m - x))}{2} \right)}{b^{s_j} \pi (y_m - x)} \right| \leq \pi \sum_{j=0}^{m-1} (ab)^{s_j} \leq \frac{\pi(ab)^{s_m}}{ab - 1} \]

We will need two observations: if \( j \geq m \), then \( \cos(b^{s_j} \pi y_m) = \cos(b^{s_j - s_m} \pi (w_m - 1)) = (-1)^{w_m - 1} = -(-1)^{w_m} \) (as \( b \) is odd and \( w_m \in \mathbb{Z} \)) and

\[ \cos(b^{s_j} \pi x) = \cos(b^{s_j - s_m} \pi w_m) \cos(b^{s_j - s_m} \pi x_m) - \sin(b^{s_j - s_m} \pi w_m) \sin(b^{s_j - s_m} \pi x_m) = \]

Proposition 3.8. The set of functions \( f \in C[0,1] \) such that \( \lambda(D(f)) > 0 \) is Haar-countable in \( C[0,1] \).

Proof. Denote by \( \mathcal{A} \) the set of functions \( f \in C[0,1] \) such that \( \lambda(L(f)) > 0 \) and note that each function \( f \in C[0,1] \) such that \( \lambda(D(f)) > 0 \) is in \( \mathcal{A} \). Moreover, analogously to the proof of Proposition 3.3, we get that \( \mathcal{A} \) is Borel, because \( \lambda(L(f)) = \lim_{M \to \infty} \lambda(L_M(f)) \) and

\[
\mathcal{A} = \bigcup_{k \in \mathbb{N}} \bigcup_{M \in \mathbb{N}} \left\{ f \in C[0,1] : \lambda(L_M(f)) \geq \frac{1}{k} \right\},
\]

since \( ab > 1 \) and \( \frac{2}{3} - \frac{4a}{1-a} - \frac{\pi}{ab-1} > 0 \). 

Recall that the \( \sigma \)-ideal of Lebesgue’s null sets is ccc, i.e., every family of pairwise disjoint Borel sets of positive Lebesgue’s measure is countable.
Let \( a \in (0, 1) \) and \( b \in \mathbb{N} \) be as in Lemma 3.7 and let \( \varphi : \{0, 1\}^\mathbb{N} \to C[0, 1] \) be given by:

\[
\varphi(\alpha)(x) = \sum_{j=0}^{\infty} (-1)^{\alpha(j)} a^j \cos(b^j \pi x)
\]

for each \( \alpha \in \{0, 1\}^\mathbb{N} \) and \( x \in [0, 1] \). Continuity of \( \varphi \) is obvious as \( \lim_m \sum_{j=m}^{\infty} a_j = 0 \) and, consequently,

\[
\lim_m \left( \sum_{j=m}^{\infty} (-1)^{\alpha(j)} a^j \cos(b^j \pi x) \right) = 0.
\]

Continuity of each \( \varphi(\alpha) \) follows from Weierstrass M-test and uniform limit theorem.

We claim that \( \varphi \) witnesses that \( A \) is Haar-countable. Suppose, to the contrary, that there is an \( h \in C[0, 1] \) and an uncountable set \( T \subseteq \{0, 1\}^\mathbb{N} \) such that \( \varphi(\alpha) \in A + h \) for each \( \alpha \in T \). Without loss of generality, we may assume that \( \{i \in \mathbb{N} : \alpha(i) \neq \beta(i) \} \) is infinite for each two distinct \( \alpha, \beta \in T \) (since for every \( \alpha \in \{0, 1\}^\mathbb{N} \) there are only countably many \( \beta \in \{0, 1\}^\mathbb{N} \) such that \( \{i \in \mathbb{N} : \alpha(i) \neq \beta(i) \} \) is finite).

For \( \alpha \in T \), consider the sets \( L(\varphi(\alpha) - h) \setminus \{0\} \). They are Borel (cf. Lemma 3.1) and of positive Lebesgue’s measure. By the ccc property, we conclude that there are \( \alpha, \beta \in T \) such that \( L(\varphi(\alpha) - h) \cap L(\varphi(\beta) - h) \setminus \{0\} \neq \emptyset \). Let \( x \) belong to that set (note that \( x \in (0, 1) \)).

Let \( (s_i) \) be the increasing enumeration of the set \( \{i \in \mathbb{N} : \alpha(i) \neq \beta(i) \} \) and denote \( r_j = \beta(s_j) - \alpha(s_j) \in \{-1, 1\} \). Observe that:

\[
\frac{\varphi(\alpha)(y) - \varphi(\beta)(y) - \varphi(\alpha)(x) - \varphi(\beta)(x)}{y-x} = 2 \sum_{j=0}^{\infty} r_j a^j \frac{(\cos(b^j \pi y) - \cos(b^j \pi x))}{y-x}
\]

for any \( y \in [0, 1] \). Therefore, by Lemma 3.7, the above expression is unbounded. On the other hand, there are \( M_\alpha, M_\beta \in \mathbb{N} \) witnessing that \( \varphi(\alpha) - h \) and \( \varphi(\beta) - h \) are Lipschitz at \( x \). Thus:

\[
\frac{|\varphi(\alpha)(y) - \varphi(\beta)(y) - \varphi(\alpha)(x) - \varphi(\beta)(x)|}{y-x} \leq \frac{|\varphi(\alpha)(y) - h(y) - \varphi(\alpha)(x) - h(x)|}{y-x} + \frac{|\varphi(\beta)(y) - h(y) - \varphi(\beta)(x) - h(x)|}{y-x} \leq M_\alpha + M_\beta
\]

for any \( y \in [0, 1] \). This is a contradiction. \( \square \)

The next table summarizes results of Sections 2 and 3.

| \( A \) | \( \{0\} \) | \( (0,1) \) | Haar-\( \mathcal{E} \) | Haar-countable | Haar-finite |
|---|---|---|---|---|---|
| \( \{f \in SD : \lambda(D(f)) \in A\} \) | Haar-countable | Haar-\( \mathcal{E} \) | \( \{1\} \) |
| \( \{f \in SD : \lambda(D(f)) \in A\} \notin \) | Haar-finite | Haar-countable | |

4. Multidimensional case

In this Section, we study nowhere differentiable functions on \( [0,1]^k \), i.e., functions defined on \( [0,1]^k \) which do not have a finite directional derivative at any point along any vector (at points from the boundary of \( [0,1]^k \) we require that there is no finite directional one-sided derivative along any vector). Such functions exist (actually, in Proposition 4.4 we even show that the set of such functions is comeager in \( C[0,1]^k \)), however, it is hard to find a suitable example in the literature. Therefore, below we provide one for \( k = 2 \) (with an informal proof).
Example 4.1. Let $T \in C[0,1]$ be the Takagi function, i.e.,

$$T(x) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \text{dist}(2^n x, \mathbb{Z}).$$

We define a new function on $[0,1]^2$:

$$F(x_1, x_2) = \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2}$$

(here $\tilde{T}(x) = T(\text{frac}(x))$, where $\text{frac}(x)$ is the fractional part of $x$). Choose a point $(0,0) \neq (\chi_0, \chi_1) \in [0,1]^2$ and a unit vector $\nu \in \mathbb{R}^2$. Denote by $\ell$ a line that goes through $(\chi_1, \chi_2)$ and that is parallel to $\nu$. For every point $(x_1, x_2)$ of $\ell \cap [0,1]^2$, consider the circle $O(x_1, x_2)$ centered at the origin that passes through $(x_1, x_2)$. Let $x$ be the $x$-intercept of $O(x_1, x_2)$ and define a one-to-one correspondence between $[\text{inf}(\sqrt{x_1^2 + x_2^2} : (x_1, x_2) \in \ell \cap [0,1]^2), \text{sup}(\sqrt{x_1^2 + x_2^2} : (x_1, x_2) \in \ell \cap [0,1]^2)]$ and $\ell \cap [0,1]^2$ by $x \mapsto (x_1, x_2)$. We will always denote by $x$ a point on $[0,1]$ that corresponds to the point $(x_1, x_2)$ on $\ell$. Note that $F(x_1, x_2) = F(x,0) = \tilde{T}(x)$. Denote by $d_x^2$ the standard euclidean metric on $[0,1]^2$. Moreover, denote by $a$ and $b$ the slope of $\ell$ and its $y$-intercept respectively. Choose two sequences $(u^n)$ and $(v^n)$ that tend to $\chi$ (recall that $\chi$ is the $x$-intercept of $O(\chi_1, \chi_2)$). Now, for every $n$, assume that all elements of $(u^k)_{k \geq n}$ and $(v^k)_{k \geq n}$ are within some interval $(a_n, b_n)$. Without loss of generality, we may assume that both $(a_n)$ and $(b_n)$ tend to $\chi$. Using elementary analytical geometry and a little bit of estimation, we can show that:

$$\frac{\alpha_n \sqrt{a^2+1}}{\sqrt{\beta_n^2(a^2+1) - b^2}} |u^n - v^n| \leq d_\ell^2((u^n_1, u^n_2), (v^n_1, v^n_2)) \leq \frac{\beta_n \sqrt{a^2+1}}{\sqrt{\alpha_n^2(a^2+1) - b^2}} |u^n - v^n|$$

(i.e., the euclidean metric on $(\alpha_n, \beta_n)$ is equivalent to $d_\ell^2$ on $x \in (\alpha_n, \beta_n)$). For simplicity, let $c_n = \frac{\alpha_n \sqrt{a^2+1}}{\sqrt{\beta_n^2(a^2+1) - b^2}}$ and $d_n = \frac{\beta_n \sqrt{a^2+1}}{\sqrt{\alpha_n^2(a^2+1) - b^2}}$. Observe that $(c_n)$ and $(d_n)$ have the same limit. Note that:

$$c_n \frac{|F(u^n_1, u^n_2) - F(v^n_1, v^n_2)|}{d_\ell^2((u^n_1, u^n_2), (v^n_1, v^n_2))} \leq \frac{\frac{|\tilde{T}(u^n) - \tilde{T}(v^n)|}{|u^n - v^n|}}{d_\ell^2((u^n_1, u^n_2), (v^n_1, v^n_2))} \leq d_n \frac{|F(u^n_1, u^n_2) - F(v^n_1, v^n_2)|}{d_\ell^2((u^n_1, u^n_2), (v^n_1, v^n_2))}.$$ 

Finally, it follows by the Squeeze Theorem that if $F$ is differentiable at $(\chi_1, \chi_2)$ in the direction $\nu$, then $T$ must be differentiable at $\chi$. This is impossible. Thus, $F$ is not differentiable at any point of $[0,1]^2 \setminus \{(0,0)\}$ in any direction. To complete the proof, proceed analogously for $(\chi_0, \chi_1) = (0,0)$ and use the fact that $T$ does not possess a finite one-sided derivative at 0.

Now, we want to show that, unlike the one-dimensional case, the set of somewhere differentiable functions on $[0,1]^k$ is not Haar-null. Actually, this follows from a more general fact. We will need the following notion.

A subset $A$ of an abelian Polish group $X$ is called thick if for any compact set $K \subseteq X$ there is an $x \in X$ such that $K + x \subseteq A$ (for more on thick sets see [5, Section 7]).

Remark 4.2. The following are equivalent for any Borel set $A$:

(a) $A$ is thick;
(b) $A$ is not $\text{Haar-}(\mathcal{P}(X) \setminus \{X\})$;
(c) $A$ is not $\text{Haar-}\mathcal{I}$ for any proper semi-ideal $\mathcal{I}$. 

Moreover, if $A$ is arbitrary, then (b) and (c) are equivalent and (a) implies both of them.

**Proof.** Indeed, (b) $\iff$ (c) is trivial. As $f([0,1]^N)$ is compact for every continuous $f$, thickness of $A$ ensures us that $A$ is not Haar-$\mathcal{P}(X) \setminus \{X\}$. Conversely, if $A$ is not thick, then the compact set $K$ witnessing it is a continuous image of $\{0,1\}^N$. This continuous map witnesses that $A$ is Haar-$\mathcal{P}(X) \setminus \{X\}$ provided that it is Borel. \hfill $\Box$

We are ready to prove the aforementioned general result.

**Proposition 4.3.** Let $k > 1$. The set of somewhere differentiable functions is thick in $C[0,1]^k$.

**Proof.** Firstly, we will assume that $k = 2$.

Let $C$ be the ternary Cantor set (which is homeomorphic to $\{0,1\}^N$) and $\varphi: C \to C[0,1]^2$ be continuous. We will define a continuous function $f: [0,1]^2 \to \mathbb{R}$ such that for all $c \in C$ the function $\varphi(c) - f$ is differentiable along $(0,1)$ in each point of the form $(c,x)$, $x \in [0,1]$.

Let $F: C \times [0,1] \to \mathbb{R}$ be given by $F \mid \{c\} \times [0,1] = \varphi(c) \mid \{c\} \times [0,1]$ for each $c \in C$.

We need to show that $F$ is continuous. Fix any $(c_0, y_0) \in C \times [0,1]$ and $\varepsilon > 0$. Since $\varphi(c_0)$ is continuous at $(c_0, y_0)$, there is an open neighborhood $(c_0, y_0) \in U \subseteq [0,1]^2$ such that $|\varphi(c_0)(x, y) - \varphi(c_0)(c_0, y_0)| < \frac{\varepsilon}{2}$ for every $(x, y) \in U$. Moreover, since $\varphi$ is continuous, there is an open neighborhood $c_0 \in V \subseteq [0,1]$ such that $|\varphi(c) - \varphi(c_0)| < \frac{\varepsilon}{2}$ for each $c \in V \cap C$. Thus, we have:

$$|F(c, y) - F(c_0, y_0)| = |\varphi(c)(c, y) - \varphi(c_0)(c_0, y_0)| \leq$$

$$\leq |\varphi(c)(c, y) - \varphi(c_0)(c, y)| + |\varphi(c_0)(c, y) - \varphi(c_0)(c_0, y_0)| \leq$$

$$\leq |\varphi(c) - \varphi(c_0)|| + |\varphi(c_0)(c, y) - \varphi(c_0)(c_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $(c, y) \in U \cap ((V \cap C) \times [0,1])$. Hence, $F$ is continuous.

Using Tietze extension theorem we get a continuous function $f: [0,1]^2 \to \mathbb{R}$ such that $f \mid C \times [0,1] = F$. Then, $\varphi(c) - f \mid \{c\} \times [0,1]$ is constantly equal to 0 for each $c \in C$. Hence, $\varphi(c) - f$ is somewhere differentiable and we are done.

If $k > 2$ then define $F: C \times [0,1] \times \{0\}^{k-2} \to \mathbb{R}$ by $F \mid \{c\} \times [0,1] \times \{0\}^{k-2} = \varphi(c) \mid \{c\} \times [0,1] \times \{0\}^{k-2}$ for each $c \in C$ and observe that $F$ is continuous for the same reason as above. Thus, using Tietze extension theorem we get a continuous function $f: [0,1]^k \to \mathbb{R}$ such that $\varphi(c) - f$ is somewhere differentiable. \hfill $\Box$

Although the set of somewhere differentiable functions on $[0,1]^k$ is not Haar-$\mathcal{I}$ for any semi-ideal $\mathcal{I}$, it is meager. In the one-dimensional case this was shown by Banach (see [4]). The authors find it surprising (and remain skeptical) that no multidimensional case of this theorem can be found in the literature, as the following reasoning is only a slight modification of the one-dimensional proof.

**Proposition 4.4.** The set of nowhere differentiable functions on $[0,1]^k$ is comeager in $C[0,1]^k$.

**Proof.** We will consider $k = 2$, but the argument works for other $k$’s as well. Let $E_n^{0,0}$ be the set of these functions $f \in C[0,1]^2$ such that there is a pair $(x, y) \in [0,1, 1 - \frac{1}{n+1}]$ and there is a unit vector $(v_1, v_2) \in [-1,1] \times [-1,1]$ such that for all
If and we can conclude that \( f((x, y) + h(v_1, v_2)) - f(x, y) \) \( < \varepsilon \). Define \( E_n^{0,1}, E_n^{1,0}, E_n^{1,1} \) analogously replacing \( R_n^{0,0} = [0, 1 - \frac{1}{n+1}] \times [0, 1 - \frac{1}{n+1}] \) with \( R_n^{0,1} = [0, 1 - \frac{1}{n+1}] \times [-1, 1] \), \( R_n^{1,0} = [-1, 1] \times [0, 1 - \frac{1}{n+1}] \) and \( R_n^{1,1} = [-1, 1] \times [-1, 1] \), respectively. Similarly, let \( E_n^{i,j} \) be the set of these functions \( f \in C[0, 1]^2 \) such that there is a pair \( (x, y) \in R_n^{i,j} \) such that for all \( h \in (0, \frac{1}{n}) \) it holds that \( |f((x, y) + h) - f(x, y)| < \varepsilon \). It suffices to prove that for each \( i, j \in \{0, 1\} \) and each \( n \in \mathbb{N} \):

a) \( E_n^{i,j} \) is closed;

b) \( E_n^{i,j} \) is nowhere dense;

c) \( E_n^{i,j} \) is nowhere dense;

d) \( E_n^{i,j} \) is nowhere dense;

and also that:

e) \( C[0, 1]^2 \setminus \bigcup_{i,j \in \{0, 1\}} (E_n^{i,j} \cup F_n^{i,j}) \) is a subset of the set of nowhere differentiable functions on \([0, 1]^2\).

For a) and b), it is enough to use the argument from the standard one-dimensional case. We will show that \( E_n^{0,0} \) is closed. The other cases are similar.

Let \( (f_n) \subseteq E_n^{0,0} \) and denote \( f = \lim_n f_n \). For each \( n \in \mathbb{N} \) there is \((x_n, y_n) \in [0, 1 - \frac{1}{n+1}]^2 \) and there is a unit vector \((v_1^n, v_2^n) \in [-1, 1] \times [-1, 1] \) such that for all \( h \in (0, \frac{1}{n+1}) \) it holds that \( |f_n((x_n, y_n) + h(v_1^n, v_2^n)) - f_n(x_n, y_n)| < \varepsilon h \). By the Bolzano-Weierstrass theorem, passing to a subsequence, if necessary, without loss of generality we can assume that \( \lim_n (x_n, y_n) = (x, y) \in [0, 1 - \frac{1}{n+1}]^2 \) and \( \lim_n (v_1^n, v_2^n) = (v_1, v_2) \in [-1, 1] \times [-1, 1] \). Fix \( h \in (0, \frac{1}{n+1}) \) and observe that:

\[
|f((x, y) + h(v_1, v_2)) - f(x, y)| \leq |f((x, y) + h(v_1, v_2)) - f((x_n, y_n) + h(v_1^n, v_2^n))| + |f((x_n, y_n) + h(v_1^n, v_2^n)) - f_n((x_n, y_n) + h(v_1^n, v_2^n))| + |f_n((x_n, y_n) + h(v_1^n, v_2^n)) - f_n((x_n, y_n))| + |f_n(x_n, y_n) - f((x_n, y_n))| + |f((x_n, y_n)) - f(x, y)| \leq |f((x_n, y_n) + h(v_1^n, v_2^n)) - f_n((x_n, y_n)))| + |f_n((x_n, y_n)) - f(x, y)| + \lim_{n \to +\infty} |f - f_n|| + \varepsilon h + \frac{\varepsilon h}{\frac{1}{n+1}} \text{dist}(x, y).
\]

If \( n \to +\infty \) then continuity of \( f \) at \((x, y)\) and at \((v_1, v_2)\) together with \( f = \lim_n f_n \) gives us that:

\[
|f((x, y) + h(v_1, v_2)) - f(x, y)| \leq \varepsilon h
\]

and we can conclude that \( f \in E_n^{0,0} \).

For c), we will prove that for every two-dimensional piecewise linear function \( g \) and every \( \varepsilon > 0 \) there is a function \( f \not\in E_n^{0,0} \) such that the norm of \( f - g \) is below \( \varepsilon \) (the cases of \( E_n^{0,1}, E_n^{1,0}, E_n^{1,1} \) are analogous). As the set of all two-dimensional piecewise linear functions is dense in \( C[0, 1]^2 \), the thesis will follow.

Let \( g \) and \( \varepsilon \) be defined as above and let \( M \) be equal to the maximal slope of \( g \). Subsequently, choose \( m \) such that \( \frac{m}{n+1} > M + \varepsilon \). Now, define a function \( f(x, y) = g(x, y) + \frac{\varepsilon}{2} \text{dist}(mx, Z) \). It is easy to see that \( |f(x, y) - g(x, y)| < \varepsilon \) for all \((x, y)\). Let \((x, y) \in [0, 1 - \frac{1}{n+1}]^2 \) and let \( v = (v_1, v_2) = (\frac{1}{n+1}, 1) \times [-1, 1] \) be a unit vector. Observe that:

\[
\left| \frac{\partial f}{\partial v} \right| = \left| \frac{\partial g}{\partial v} + \frac{\varepsilon}{2} m v_1 \frac{\partial \text{dist}(\cdot, Z)}{\partial x} \right| > \frac{\varepsilon}{n+1}
\]

as \( \left| \frac{\partial \text{dist}(\cdot, Z)}{\partial x} \right| = 1 \) and \( v_1 > \frac{1}{n+1} \).
For \( d \), we just repeat an argument we just gave: we choose \( m \) such that \( m^2 > M + n \), define \( f(x, y) = g(x, y) + \varepsilon \cdot \text{dist}(my, Z) \) and observe that:

\[
\left| \frac{\partial f}{\partial y} \right| = \left| \frac{\partial g}{\partial y} + \varepsilon \frac{m}{2} \frac{\partial \text{dist}(\cdot, Z)}{\partial y} \right| > n.
\]

Finally, for \( e \), let us suppose that \( f \in C[0, 1]^2 \setminus \left( \bigcup_{n \in \mathbb{N}} \bigcup_{i,j \in \{0,1\}} E_{n,i}^{i,j} \bigcup \bigcup_{n \in \mathbb{N}} F_{n,i}^{i,j} \right) \), choose \((x, y) \in (0, 1)^2\), and choose a unit vector \( v \). There are two cases to consider, but they essentially come down to the same argument. If the vector \( v \) is different than \((0, 1)\), we will use the sets \( E_{n,i}^{i,j} \), if not, we will use \( F_{n,i}^{i,j} \). Without loss of generality, we will continue under assumption that \( v = (0, 1) \). Let \( i, j \in \{0,1\} \) be such that \((x, y) \in \bigcap_{n \in \mathbb{N}} R_{n,i}^{i,j} \). Since \( f \in \bigcap_{n \in \mathbb{N}} C[0, 1]^2 \setminus F_{n,i}^{i,j} \), it follows that for all \( n \in \mathbb{N} \) there is an \( h_n \in (0, \frac{1}{n}) \) such that \(|f(x, y + h_n) - f(x, y)| \geq nh_n\). It is now easy to see that \( f \) is nowhere differentiable.

We end with an open problem which occurred during our studies.

Here, we examine the function \( F: [0, 1]^2 \to \mathbb{R} \) given by \( F(x, y) = T(x) + T(y) \), where \( T: [0, 1] \to \mathbb{R} \) is the Takagi’s function. We considered this function in the context of nowhere differentiable functions on \([0, 1]^2\) as \( F \) seemed to be a nice example of such a function. It is obvious that it is differentiable neither along \((1, 0)\) nor along \((0, 1)\). Moreover, we managed to obtain one partial result: \( F \) is differentiable at no point along \((1, 1)\). However, we were unable to solve the following.

**Problem 4.5.** Is \( F \) nowhere differentiable on \([0, 1]^2\)?

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