Multipartite Entanglement Detection Via Projective Tensor Norms

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Abstract. We introduce and study a class of entanglement criteria based on the idea of applying local contractions to an input multipartite state, and then computing the projective tensor norm of the output. More precisely, we apply to a mixed quantum state a tensor product of contractions from the Schatten class $S_1$ to the Euclidean space $\ell_2$, which we call entanglement testers. We analyze the performance of this type of criteria on bipartite and multipartite systems, for general pure and mixed quantum states, as well as on some important classes of symmetric quantum states. We also show that previously studied entanglement criteria, such as the realignment and the SIC POVM criteria, can be viewed inside this framework. This allows us to answer in the positive two conjectures of Shang, Asadian, Zhu, and Gühne by deriving systematic relations between the performance of these two criteria.

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1. Introduction

Determining whether a multipartite quantum system is in a separable or an entangled state is of prime importance in quantum information theory. Indeed, if such a quantum system is in a separable state, it means that there are no intrinsically quantum correlations between its subsystems, so that it is not providing any advantage compared to a classical system in information processing tasks. However, the problem of deciding if a multipartite quantum state is separable or entangled (and even only approximate versions of it) is known to be computationally hard [8]. Standard solutions to overcome this practical difficulty consist in looking for necessary conditions to separability that would be easier to check than separability itself. These are usually dubbed entanglement criteria, and various families of such criteria have already been extensively studied in the past. From a mathematical point of view, a quantum state is described by a positive semidefinite operator on a complex Hilbert space having unit Schatten 1-norm. And as we will see in more details later, for a quantum state on a multipartite system (i.e., on a tensor product Hilbert space), being entangled is characterized by having a so-called projective Schatten 1-norm which is strictly larger than 1 [22]. But there is no efficient way of estimating such tensor norm in general [19]. An alternative consists in looking at other tensor norms, whose values are easier to compute and always smaller than the tensor norm characterizing entanglement (so that if they are strictly larger than 1, then the state is guaranteed to be entangled).

This is the approach that we take in this work. We define and study a class of entanglement criteria based on the idea of applying local contractions to an input multipartite state, and then computing the projective tensor norm of the output. More precisely, the local contractions that we consider are from the Schatten 1-norm to the $\ell_2$-norm, i.e., from a non-commutative space to a commutative one. This is what makes such entanglement criteria interesting in practice: They can be seen as reducing the study of mixed state entanglement to that of pure state entanglement, which is an easier task.

Another advantage of our entanglement criteria is that their definition is independent from the number of subsystems. Several aspects are, admittedly, simpler to understand in the bipartite case, but they remain equally well suited to the case where more than two parties are involved. In fact, one of the main issues with most well-known entanglement criteria is that they are specifically designed for bipartite systems, and generalizations to systems with more parties are not fully satisfying. Indeed, they usually consist in applying the bipartite criterion across all bipartitions, which certifies entanglement across
bipartitions of the subsystems, but not genuinely multipartite entanglement [12].

Well-studied entanglement criteria, such as the realignment criterion [4, 23] and the SIC POVM criterion [27], are important examples in the framework we consider. Our work provides natural generalizations of these criteria to the multipartite setting, going beyond the biseparable case already discussed in the literature. Moreover, we establish an exact relation between the performance of the realignment and the SIC POVM criteria, solving in the positive two conjectures from [27].

The remainder of the paper is organized as follows. In Sect. 2, we first recall basic facts about tensor norms on Banach spaces and then relate them with the characterization of entanglement. With these observations at hand, we can define in Sect. 3 the main objects of interest in this work, which we dub entanglement testers and establish some of their first key properties. Section 4 is dedicated to providing explicit examples of testers, by showing that several well-known entanglement criteria (such as the celebrated realignment criterion or one based on SIC POVMs introduced more recently) can actually be seen as corresponding to a tester. In Sect. 5, we define and characterize an important sub-class of testers: that of perfect testers, which detect all entangled pure states. This naturally brings us to Sect. 6, where we show that the examples of Sect. 4 are in fact all special instances inside an important sub-class of perfect testers: that of symmetric testers. From then on we focus for a while on these symmetric testers. We quantify how they perform in detecting the entanglement of several classes of bipartite states: pure states (Sect. 7), Werner and isotropic states (Sect. 8), pure states with white noise (Sect. 9). On all these examples, we observe a systematic relation between the performance of the realignment and SIC POVM testers, proving on the way a conjecture from [27]. So in Sect. 10 we ask whether it would hold more generally, for any bipartite state. This allows us to answer in the positive another conjecture from [27]. After this time spent on studying the specific case of symmetric testers, we go back to a more general question in Sect. 11, namely: is our family of entanglement criteria complete, i.e., in other words, can any bipartite entangled state be detected by a tester? In Sect. 12, we take a look at what can be shown in the multipartite case. Finally, we present in Sect. 13 an overview of the main results, as well as a list of the problems we have left open and some directions for future work.

2. Tensor Products of Banach Spaces and Quantum Entanglement

We gather in this section basic definitions and facts about the different natural norms one can put on the algebraic tensor product of finite dimensional Banach spaces. Studying the different tensor norms of multipartite pure and mixed quantum states, and relating these norms to quantum entanglement, is the main theme of our work. In this sense, the current section contains
the mathematical foundation on which the practical applications to quantum information are built upon.

Let us start by recalling the definitions of the projective and the injective tensor norms for (finite dimensional) Banach spaces.

**Definition 2.1.** Consider $m$ Banach spaces $A_1, \ldots, A_m$. For a tensor $x \in A_1 \otimes \cdots \otimes A_m$, we define its **projective tensor norm**

$$\|x\|_\pi := \inf \left\{ \sum_{k=1}^r \|a_k^1\| \cdots \|a_k^m\| : r \in \mathbb{N}, \ a_k^i \in A_i, \ x = \sum_{k=1}^r a_k^1 \otimes \cdots \otimes a_k^m \right\},$$

(1)

and its **injective tensor norm**

$$\|x\|_\varepsilon := \sup \left\{ \left| \langle \alpha^1 \otimes \cdots \otimes \alpha^m | x \rangle \right| : \alpha^i \in A_i^*, \ \|\alpha^i\| \leq 1 \right\}.$$  

(2)

It is immediate to see that the projective tensor norm can be equivalently defined as:

$$\|x\|_\pi := \inf \left\{ \sum_{k=1}^r |\lambda_k| : r \in \mathbb{N}, \ x = \sum_{k=1}^r \lambda_k a_k^1 \otimes \cdots \otimes a_k^m, \ a_k^i \in A_i, \ |a_k^i| \leq 1 \right\}.$$  

(3)

The projective and injective norms are examples of tensor norms (also known as reasonable cross-norms) [25]. We recall that a tensor norm is a norm which factorizes on simple tensors, and its dual norm as well. Now, it is clear from the definitions that, for all $a_1 \in A_1, \ldots, a_m \in A_m$,

$$\|a_1 \otimes \cdots \otimes a_m\|_\pi = \|a_1 \otimes \cdots \otimes a_m\|_\varepsilon = \|a_1\| \cdots \|a_m\|.$$  

And the same factorization property holds for their dual norms on the tensor product of dual spaces $A_1^* \otimes \cdots \otimes A_m^*$. Indeed, the projective and the injective norms are actually dual to one another: for all $x \in A_1 \otimes \cdots \otimes A_m$,

$$\|x\|_\pi = \sup_{\|\alpha\|_{A_1^* \otimes \cdots \otimes A_m^*} \leq 1} \langle \alpha | x \rangle,$$

$$\|x\|_\varepsilon = \sup_{\|\alpha\|_{A_1^* \otimes \cdots \otimes A_m^*} \leq 1} \langle \alpha | x \rangle.$$  

The last property of the projective and the injective tensor norms that we would like to mention is that they are extremal among tensor norms [25, page 127]: for any other tensor norm $\| \cdot \|$ on $A_1 \otimes \cdots \otimes A_m$, we have

$$\forall \ x \in A_1 \otimes \cdots \otimes A_m, \ |\|x\|_\varepsilon \| \leq \|x\| \leq \|x\|_\pi.$$  

We denote by $A_1 \otimes_\pi \cdots \otimes_\pi A_m$, resp. $A_1 \otimes_\varepsilon \cdots \otimes_\varepsilon A_m$, the space $A_1 \otimes \cdots \otimes A_m$ equipped with the projective, resp. injective, norm.

The following fact will be crucial to the main definition from the next section.
**Proposition 2.2.** Consider \( m \) linear operators \( T_i : A_i \rightarrow B_i \) between Banach spaces \( A_i, B_i, 1 \leq i \leq m \). Then, for any tensor norm on \( B_1 \otimes \cdots \otimes B_m \),

\[
\left\| \bigotimes_{i=1}^{m} T_i \right\|_{A_1 \otimes \cdots \otimes A_m \rightarrow B_1 \otimes \cdots \otimes B_m} = \prod_{i=1}^{m} \| T_i \|_{A_i \rightarrow B_i}.
\]

**Proof.** For the sake of clarity, let us consider the case \( m = 2 \), the proof in the general case being similar. Set \( T := T_1 \otimes T_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \). The extremal points of the unit ball of the projective tensor product \( A_1 \otimes A_2 \) are products of the extremal points of the unit balls of the factors \( A_1, A_2 \). Now, on such a product input \( a = a_1 \otimes a_2 \), the output \( T(a) \) factors as \( T_1(a_1) \otimes T_2(a_2) \). And the maximal value of such a product tensor with an element of \((B_1 \otimes B_2)^*\) is attained on a product tensor. The result thus follows. \( \square \)

We shall now relate the different tensor norms discussed above to quantum entanglement. First, let us recall that a pure multipartite quantum state is the projection onto the direction given by a unit vector \( \psi \) (sometimes called a state vector) in a tensor product of complex Hilbert spaces \( H_1 \otimes \cdots \otimes H_m \). Here, we shall always assume \( m \geq 2 \). Since we only consider finite-dimensional Hilbert spaces, we make the identification \( H_i \cong \mathbb{C}^{d_i} \) for \( 1 \leq i \leq m \). The state vector \( \psi \) is said to be separable if it is a product tensor:

\[
|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_m\rangle.
\]

The Banach space structure we consider for each factor is \((H_i, \| \cdot \|_2)\), where each space comes equipped with its Euclidean norm. We write \( \ell_2^d := (\mathbb{C}^d, \| \cdot \|_2) \). In the case of state vectors, the relation between entanglement and tensor norms is obvious, and formally stated below.

**Proposition 2.3.** A state vector \( \psi \in H_1 \otimes \cdots \otimes H_m \), \( \| \psi \|_2 = 1 \), is separable if and only if

\[
\| \psi \|_\pi = \| \psi \|_\pi = 1.
\]

Actually, the injective norm is closely related to a fundamental multipartite entanglement measure, the so-called geometric measure of entanglement [28, 33, 34], defined as

\[
G(\psi) := -\log \sup \left\{ \| \langle \varphi_1 \otimes \cdots \otimes \varphi_m | \psi \rangle \|^2 : \varphi_i \in H_i, \| \varphi_i \| = 1 \right\} = -2 \log \| \psi \|_\varepsilon.
\]

Let us now move to the more general case of mixed multipartite quantum states, i.e., of operators \( \rho \) on \( H_1 \otimes \cdots \otimes H_m \) which are positive semidefinite and of unit trace. The Banach space structure we consider on each of the spaces \( B(H_i) \cong \mathcal{M}_{d_i}(\mathbb{C}) \), \( 1 \leq i \leq m \), is that given by the Schatten 1-norm (or nuclear norm)

\[
\| X \|_1 = \text{Tr} \sqrt{X^*X}.
\]

We write \( S_1^d := (\mathcal{M}_{d}(\mathbb{C}), \| \cdot \|_1) \) for the complex Banach space. Since mixed quantum states are self-adjoint operators, we shall also consider the real Banach space \( S_{1,sa}^d := (\mathcal{M}_{d}^{sa}(\mathbb{C}), \| \cdot \|_1) \). Note that, in general, we have \( S_{1,sa}^d = S_1^d \cap \mathcal{M}_{d}^{sa}(\mathbb{C}) \) (see, e.g., [2, Section 1.3.2]).
We recall the following fact, relating the separability problem for mixed quantum states to projective tensor norms. Although this is a well-known fact, we give the proof for the sake of completeness and in order to showcase the relation between the positivity properties of $\rho$ and its tensor norms.

**Theorem 2.4.** [22, Theorem 5] or [19, Theorem 1.1] For a multipartite mixed quantum state $\rho \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$, $\rho \geq 0$, $\text{Tr} \rho = 1$, the following assertions are equivalent:

(i) $\rho$ is separable,

(ii) $\|\rho\|_{S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}} = 1$,

(iii) $\|\rho\|_{S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}} = 1$.

**Proof.** Let us first show the implication $(i) \implies (ii)$. Given a separable quantum state $\rho$, we have

$$\rho = \sum_{k=1}^{r} p_k |x_k^1 \rangle \langle x_k^1| \otimes \cdots \otimes |x_k^m \rangle \langle x_k^m|,$$

for a probability distribution $(p_k)_{k=1}^{r}$ and unit vectors $x_k^i \in \mathbb{C}^{d_i}$, $k \in [r]$, $i \in [m]$. Obviously, $\|x_k^i \rangle \langle x_k^i|\|_{S_{d_i,sa}^{d_i}} = \|x_k^i\|^2 = 1$, for every $k$ and $i$. So using the separable decomposition, we have

$$\|\rho\|_{S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}} \leq \sum_{k=1}^{r} p_k \prod_{i=1}^{m} \|x_k^i \rangle \langle x_k^i|\|_{S_{d_i,sa}^{d_i}} = 1.$$

Recall that the projective tensor norm $S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}$ is the largest cross norm on $\mathcal{M}_{sa}^{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{sa}^{d_m}(\mathbb{C})$. So in particular it is larger than the norm $S_{1,sa}^{d_1} \cdots S_{1,sa}^{d_m}$ on this space, i.e.,

$$\|\rho\|_{S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}} \geq \|\rho\|_{S_{1,sa}^{d_1} \cdots S_{1,sa}^{d_m}} = \text{Tr} \rho = 1.$$

The implication $(ii) \implies (iii)$ is trivial:

$$1 = \|\rho\|_{S_{d_1,sa}^{d_1} \otimes \cdots \otimes S_{d_m,sa}^{d_m}} \geq \|\rho\|_{S_{1,sa}^{d_1} \cdots S_{1,sa}^{d_m}} \geq \|\rho\|_{S_{1,sa}^{d_1} \cdots S_{1,sa}^{d_m}} = \text{Tr} \rho = 1,$$

where we have used the fact that the infimum in (1) is taken over a smaller set of possible decompositions in the self-adjoint case.

Finally for the implication $(iii) \implies (i)$, consider a decomposition

$$\rho = \sum_{k=1}^{s} a_k^1 \otimes \cdots \otimes a_k^m$$

achieving the minimum in (1), and such that each term is nonzero. (Because of the equivalence between definitions (1) and (3), the infimum is indeed attained in our case, as the unit balls for $\|\cdot\|_{S_{1,sa}^{d_1}}, \ldots, \|\cdot\|_{S_{1,sa}^{d_m}}$ are compact.) Above, we have $a_k^i \in \mathcal{M}_{d_i}(\mathbb{C})$, not necessarily self-adjoint. We argue in the same way as before:
\[
1 = \|\rho\|_{S_1^d \otimes \cdots \otimes S_1^m} = \sum_{k=1}^{s} \prod_{i=1}^{k} \|a_k^i\|_{S_1^d_i} \geq \sum_{k=1}^{s} \prod_{i=1}^{k} |\text{Tr} a_k^i| \\
\geq \sum_{k=1}^{s} \prod_{i=1}^{k} \text{Tr} a_k^i = \text{Tr} \rho = 1, \tag{5}
\]
where we have used the inequality \(\|X\|_{S_1^d} \geq |\text{Tr} X|\). Hence, for each \(k \in [s]\), \(i \in [m]\), we have \(\|a_k^i\|_{S_1^d_i} = |\text{Tr} a_k^i|\), and thus \(a_k^i = \omega_k^i b_k^i\) for some phase factor \(\omega_k^i \in \mathbb{C}, |\omega_k^i| = 1\), and positive semidefinite operators \(b_k^i \in \mathcal{M}_{d_i}(\mathbb{C})\). Let us define \(\omega_k := \prod_{i=1}^{m} \omega_k^i\) (satisfying \(|\omega_k| = 1\)) and

\[
\tau_k := \left| \prod_{i=1}^{m} \text{Tr} a_k^i \right| = \prod_{i=1}^{m} \text{Tr} b_k^i > 0.
\]
From (5), we have

\[
1 = \sum_{k=1}^{s} \tau_k = \sum_{k=1}^{s} |\omega_k \tau_k|,
\]
and thus the \(\omega_k\)'s are simultaneously equal to \(\pm 1\). Since \(\rho\) is positive semidefinite, they all must be equal to \(+1\), and thus

\[
\rho = \sum_{k=1}^{s} b_k^1 \otimes \cdots \otimes b_k^m
\]
for positive semidefinite operators \(b_k^i\), finishing the proof. \(\Box\)

To summarize, we have seen that the entanglement of pure, resp. mixed, quantum states is related to the projective tensor product of \(\ell_2\), resp. \(S_1\), Banach spaces. This connection will be discussed at length in the next section.

### 3. Entanglement Testers

We introduce in this section the main tool developed in this work, *entanglement testers*. Mathematically, these are linear applications from the space of matrices (physically, mixed quantum states) to the space of vectors (physically, vector quantum states). We shall study these maps and their properties from the point of view of Banach spaces, so we shall endow the space of matrices with the Schatten 1-norm \(S_1\) and the set of vectors with the Euclidean norm \(\ell_2\). In the context of quantum information theory, these are the natural norms for the vector spaces, when studying mixed and pure quantum states respectively.

To a \(n\)-tuple of matrices \((E_1, \ldots, E_n) \in (\mathcal{M}_{d}(\mathbb{C}))^n\), we associate the linear map

\[
\mathcal{E} : X \in \mathcal{M}_{d}(\mathbb{C}) \mapsto \sum_{k=1}^{n} \text{Tr}(E_k^* X)|k\rangle \in \mathbb{C}^n,
\]
where \(\{|k\rangle\}_{k=1}^{n}\) is some fixed orthonormal basis of \(\mathbb{C}^n\) (see Fig. 1). Note that any linear map between these spaces can be written in this way. In a similar
manner, to a \( n \)-tuple of matrices \((F_1, \ldots, F_n) \in (\mathcal{M}_d(\mathbb{C}))^n \) we associate the \( \mathbb{R} \)-linear map \( \mathcal{F} : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^n \) defined in the obvious manner. Note that if the matrices \( F_i \) are themselves self-adjoint, the map \( \mathcal{F} \) takes values in \( \mathbb{R}^n \). We introduce now the main definition of this paper.

**Definition 3.1.** A \( \mathbb{C} \)-linear map \( \mathcal{E} \) as above is called a \( \mathbb{C} \)-tester if \( \| \mathcal{E} \|_{S^d_1 \to \ell^2_n} = 1 \). Similarly, an \( \mathbb{R} \)-linear map \( \mathcal{F} \) is called an \( \mathbb{R} \)-tester if \( \| \mathcal{F} \|_{S^d_{1,sa} \to \ell^2_n} = 1 \).

In the definition above, we distinguish between the real (self-adjoint) and the complex (general) cases. The following lemma shows that one can extend an \( \mathbb{R} \)-tester to a \( \mathbb{C} \)-tester.

**Lemma 3.2.** Given an \( \mathbb{R} \)-linear map \( \mathcal{F} : S^d_{1,sa} \to \ell^2_n \), one can define its complexification

\[
\mathcal{F}' : X + iY \in S^d_1 \mapsto \mathcal{F}(X) + i\mathcal{F}(Y) \in \ell^2_n.
\]

We have \( \| \mathcal{F}' \|_{S^d_1 \to \ell^2_n} = \| \mathcal{F} \|_{S^d_{1,sa} \to \ell^2_n} \). In particular, if \( \mathcal{F} \) is an \( \mathbb{R} \)-tester, then \( \mathcal{F}' \) is a \( \mathbb{C} \)-tester.

**Proof.** The inequality \( \| \mathcal{F}' \| \geq \| \mathcal{F} \| \) is clear. For the converse, consider an extreme point \( |a\rangle\langle b| \) of \( S^d_1 \), for which \( \| \mathcal{F}' \| = \| \mathcal{F}'(a\langle b|)\| \). We have

\[
\| \mathcal{F}' \|^2 = \| \mathcal{F}'(a\langle b|)\|^2 = \| \mathcal{F}'\left(\frac{|a\rangle\langle b| + |b\rangle\langle a|}{2}\right)\|^2 + \| \mathcal{F}'\left(\frac{|a\rangle\langle b| - |b\rangle\langle a|}{2i}\right)\|^2 \\
\leq \frac{\| \mathcal{F} \|^2}{4} \left(2\|a\rangle\langle b|\|^2 + 2\|b\rangle\langle a|\|^2\right) = \| \mathcal{F} \|^2.
\]

In this paper, we shall focus mainly on the theory of \( \mathbb{C} \)-testers, to which we refer simply as (entanglement) testers. In some sections (e.g., Sect. 5), we shall want to differentiate between the self-adjoint case and the general one. To do this, we shall explicitly use the more precise notions of \( \mathbb{R} \)-testers and \( \mathbb{C} \)-testers.
We now look at tensor products of testers. Given $m$ sets of operators $E_i = \{E_{i;k}\}_{k=1}^{n_i}$, $1 \leq i \leq m$, consider the respective maps

$$E_i : X \in \mathcal{M}_{d_i}(\mathbb{C}) \mapsto \sum_{k=1}^{n_i} \text{Tr}(E_{i;k}^* X) |k\rangle \in \mathbb{C}^{n_i}.$$ 

The tensor product of these $m$ maps acts on multipartite matrices $X \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$ as:

$$E_1 \otimes \cdots \otimes E_m(X) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \text{Tr}(E_{1;k_1}^* \otimes \cdots \otimes E_{m;k_m}^* X) |k_1 \cdots k_m\rangle.$$

Note that, from a physical perspective, the application $E_1 \otimes \cdots \otimes E_m$ maps mixed quantum states to pure quantum states, of possibly different dimensions (see Fig. 2). This brings us to the main theoretical insight of this section, the following corollary of Proposition 2.2.

**Corollary 3.3.** Let $E_i = \{E_{i;k}\}_{k=1}^{n_i}$, $1 \leq i \leq m$, be $m$ sets of operators as above, and let $E_1, \ldots, E_m$ be the corresponding linear maps. Then, for any $X \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$, we have

$$\|E_1 \otimes \cdots \otimes E_m(X)\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_m}} \leq \|E_1\|_{\mathcal{S}_{d_1}^{n_1}} \cdot \|E_m\|_{\mathcal{S}_{d_m}^{n_m}} \cdot \|X\|_{\mathcal{S}_{d_1}^{n_1} \otimes \cdots \otimes \mathcal{S}_{d_m}^{n_m}}.$$

In particular, if the $E_i$'s are testers (real or complex), then for any multipartite quantum state $\rho$, the following implication holds:

$$\rho \text{ separable} \quad \Rightarrow \quad \|E_1 \otimes \cdots \otimes E_m(\rho)\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_m}} \leq 1.$$
Reciprocally, we have the following entanglement criterion: if the $E_i$’s are testers, then
\[ \|E_1 \otimes \cdots \otimes E_m(\rho)\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_m}} > 1 \implies \rho \text{ is entangled}. \]

In the rest of the paper, we shall study the power of the entanglement criterion formulated above. We shall investigate which entangled states can be detected by a given family of testers, and which testers are best at detecting entanglement. Moreover, we shall see in the following sections that many known entanglement criteria fall into this framework.

Let us now mention that in the same way that the map $E_1 \otimes \cdots \otimes E_m$ gives an entanglement criterion, its inverse (assuming it exists) gives a separability criterion. Indeed, using the same notation as above, and assuming that each map $E_i$ is invertible, we have
\[ \|\rho\|_{S_{d_1}^{d_1} \otimes \cdots \otimes S_{d_n}^{d_n}} \leq \|E_1^{-1}\|_{\ell_2^{n_1} \rightarrow S_{d_1}^{d_1}} \cdots \|E_m^{-1}\|_{\ell_2^{n_m} \rightarrow S_{d_m}^{d_m}} \|E_1 \otimes \cdots \otimes E_m\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_m}}. \]
Hence, for any multipartite mixed quantum state $\rho$, the following implication holds:
\[ \|E_1 \otimes \cdots \otimes E_m(\rho)\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_m}} \leq \frac{1}{\|E_1^{-1}\|_{\ell_2^{n_1} \rightarrow S_{d_1}^{d_1}} \cdots \|E_m^{-1}\|_{\ell_2^{n_m} \rightarrow S_{d_m}^{d_m}}} \implies \rho \text{ is separable}. \quad (6) \]

We postpone the discussion of these separability criteria to Sect. 4, where we shall see that they can only certify trivial separable states, so they are not useful in practice.

### 3.1. Entanglement Testers and Their Associated Test Operator

Given a set of operators $E = \{E_k\}_{k=1}^n$ on $\mathbb{C}^d$, let $E : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathbb{C}^n$ be the corresponding linear map, i.e.,
\[ E : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{k=1}^n \text{Tr}(E_k^*X)|k\rangle \in \mathbb{C}^n. \]

We impose that $E$ is a tester, as defined in Definition 3.1. We recall that this means that $\|E\|_{S_{d_1}^{d_1} \rightarrow \ell_2^{n_1}} = 1$, i.e.
\[ \max_{\|X\|_1 \leq 1} \|E(X)\|_2 = 1. \]

Now, given $X \in \mathcal{M}_d(\mathbb{C})$, we have
\[ \|E(X)\|_2 = \left( \sum_{k=1}^n |\text{Tr}(E_k^*X)|^2 \right)^{1/2} = \left( \sum_{k=1}^n \text{Tr}(E_k^* \otimes E_k X \otimes X^*) \right)^{1/2}. \]
Hence, setting
\[ T_E := \sum_{k=1}^n E_k \otimes E_k^*, \quad (7) \]
we want that
\[
\max_{\|X\|_1 \leq 1} \text{Tr}(T_E X \otimes X^*) = \max_{\|X\|_1 \leq 1} \langle T_E, X \otimes X^* \rangle = 1.
\] (8)

We call the operator \( T_E \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \), defined in Eq. (7), the \textit{test operator} associated with the tester \( \mathcal{E} \).

We would now like to characterize the set of test operators on \( \mathbb{C}^d \otimes \mathbb{C}^d \). With this aim in view, given a tester \( E : S_1^d \rightarrow \ell_2^n \), let us define the completely positive map \( T_E : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C}) \) having the \( E_k \)'s as Kraus operators, i.e.,
\[
T_E : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{k=1}^n E_k X E_k^* \in \mathcal{M}_d(\mathbb{C}).
\]

Then, denoting by \( \Theta_E \) the Choi operator associated with \( T_E \), i.e.,
\[
\Theta_E := \sum_{i,j=1}^d T_E(|i\rangle\langle j|) \otimes |i\rangle\langle j|,
\]

it is easy to check that we actually have
\[
\Theta_E = \sum_{k=1}^n |e_k\rangle\langle e_k|,
\] (9)

where, for each \( 1 \leq k \leq n \), \( e_k \in \mathbb{C}^d \otimes \mathbb{C}^d \) is the vector version of \( E_k \in \mathcal{M}_d(\mathbb{C}) \), i.e., \( |e_k\rangle = \sum_{i,j=1}^d \langle i| E_k |j\rangle |ij\rangle \). Another way of writing this is, in terms of the operator \( T_E \) defined in (7), as
\[
T_E = \Theta_E F,
\] (10)

where \( \Gamma \) stands for the partial transposition and \( F \) for the flip operator
\[
F : \mathbb{C}^d \otimes \mathbb{C}^d \rightarrow \mathbb{C}^d \otimes \mathbb{C}^d
\]
\[
x \otimes y \mapsto y \otimes x.
\] (11)

Yet another way of relating the operator \( \Theta_E \) to the linear map \( \mathcal{E} \) is via the relation
\[
\Theta_E = \mathcal{E}^* \mathcal{E},
\]
since we can rewrite the application \( \mathcal{E} \) as:
\[
\mathcal{E} = \sum_{i=1}^k |k\rangle\langle e_k|,
\]
once we identify (as vector spaces) \( \mathcal{M}_d(\mathbb{C}) \) with \( \mathbb{C}^{d^2} \). Graphical representations of the operators \( \Theta_E \) and \( T_E \) are provided in Fig. 3.

\textbf{Lemma 3.4.} \textit{The set of test operators on} \( \mathbb{C}^d \otimes \mathbb{C}^d \) \textit{is}
\[
\left\{ \Theta^F F : \Theta \geq 0, \|\Theta\|_{S_{\infty,d,s} \otimes \varepsilon S_{\infty,d,s}} = 1 \right\}.
\]
Proof. We know from Eq. (10) that $T$ is a non-normalized test operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ if and only if $T = \Theta^\Gamma F$, where $\Theta$ is the Choi operator associated to a completely positive map on $\mathcal{M}_d(\mathbb{C})$. By the Choi–Jamiołkowski isomorphism, this is the same as saying that $\Theta$ is a positive semidefinite operator on $\mathbb{C}^d \otimes \mathbb{C}^d$.

Let us now turn to the normalization condition given by (8), i.e.,

$$\max \left\{ \text{Tr}(T^* X \otimes X^*) : X \in S^d_1 \right\} = 1,$$

By extremality in $S^d_1$ of rank one operators of the form $|x\rangle\langle y|$, where $x, y$ are unit vectors in $\mathbb{C}^d$, the latter is equivalent to

$$\max \left\{ \langle x \otimes y | T | y \otimes x \rangle : x, y \in \mathbb{C}^d, \|x\|_2 = \|y\|_2 = 1 \right\} = 1.$$

In terms of $\Theta$, this reads

$$\max \left\{ \langle x \otimes y | \Theta | x \otimes y \rangle : x, y \in \mathbb{C}^d, \|x\|_2 = \|y\|_2 = 1 \right\} = 1.$$

By extremality in $S^d_{1,sa}$ of rank one operators of the form $\pm |x\rangle\langle x|$, where $x$ is a unit vector in $\mathbb{C}^d$, and because $\Theta$ is additionally positive semidefinite, the condition above is simply

$$\max \left\{ \text{Tr}(\Theta^* X \otimes Y) : X, Y \in S^d_{1,sa} \right\} = 1,$$

i.e., by definition $\|\Theta\|_{S^d_{\infty,sa} \otimes S^d_{\infty,sa}} = 1$. \hfill \Box

Note that Lemma 3.4 also provides a canonical way of constructing operators $\{E_k\}_{k=1}^n$ on $\mathbb{C}^d$ corresponding to a given test operator $T$ on $\mathbb{C}^d \otimes \mathbb{C}^d$. The strategy is to look at $\Theta = (TF)^\Gamma$, and diagonalize it as

$$\Theta = \sum_{k=1}^n \lambda_k |x_k\rangle\langle x_k|,$$

with $1 \leq n \leq d^2$, $\lambda_1, \ldots, \lambda_n > 0$, $\{x_1, \ldots, x_n\}$ orthonormal family in $\mathbb{C}^d \otimes \mathbb{C}^d$. Then, we just have to define for each $1 \leq k \leq n$, $E_k$ as being the matrix version of $|e_k\rangle := \sqrt{\lambda_k}|x_k\rangle$. By construction, we have

$$T = \sum_{k=1}^n E_k \otimes E_k^*.$$
This means that any test operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ can be decomposed into a sum of at most $d^2$ terms of the form $E_k \otimes E_k^*$, where the $E_k$'s are orthogonal operators on $\mathbb{C}^d$.

### 3.2. Equivalent Testers

We consider now the notion of equivalent testers, characterizing pairs of testers which detect the same sets of entangled states. The definition below is motivated by the fact that the projective tensor norm on a tensor product of $\ell_2^n$ spaces is invariant by local unitary operators. So, applying such an operator to the output of a tensor product of testers does not change the outcome of the entanglement test.

**Definition 3.5.** Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are called *equivalent* if there exists a unitary operator $U \in U(n)$ such that, for all $X \in \mathcal{M}_d(\mathbb{C})$, we have

$$\mathcal{F}(X) = U\mathcal{E}(X).$$

A simple calculation shows that the operators $(F_j)_{j=1}^n$ defining the tester $\mathcal{F}$ are related to the operators $(E_k)_{k=1}^n$ defining $\mathcal{E}$ by the relation

$$\forall 1 \leq j \leq n, \quad F_j = \sum_{k=1}^n \bar{U}_{jk} E_k. \quad (12)$$

**Proposition 3.6.** Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are equivalent if and only if they have the same test operator.

**Proof.** One direction is immediate: assuming that the operators $F_j$ are given by (12), we have

$$T_\mathcal{F} = \sum_{j=1}^n F_j \otimes F_j^* = \sum_{j,k,l=1}^n \bar{U}_{jk} U_{jl} E_k \otimes E_l^* = \sum_{k=1}^n E_k \otimes E_k^* = T_\mathcal{E}.$$

For the other direction, note that $T_E = T_F$ implies $\Theta_E = \Theta_F$, hence the completely positive maps associated to the testers are identical: $T_E = T_F$. The conclusion follows from the fact that two different Kraus decompositions of a completely positive map are related by a unitary transformation as in (12) (see [16, Theorem 8.2] or [31, Corollary 2.23]).

### 3.3. Practical Interest

Note that, in the bipartite case, an $\ell_2^n \otimes \pi \ell_2^n$ norm can be simply seen as an $S_1^n$ norm. Indeed, let $\mathcal{E}, \mathcal{F} : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^n$ be defined by

$$\mathcal{E}(X) = \sum_{k=1}^n \mathrm{Tr}(E_k^* X) |k\rangle \rangle \text{ and } \mathcal{F}(X) = \sum_{l=1}^n \mathrm{Tr}(F_l^* X) |l\rangle \rangle.$$

We then have, for any $X \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$,

$$\mathcal{E} \otimes \mathcal{F}(X) = \sum_{k,l=1}^n \mathrm{Tr}(E_k^* \otimes F_l^* X) |kl\rangle \rangle \in \mathbb{C}^n \otimes \mathbb{C}^n,$$
which we can identify with
\[
\sum_{k,l=1}^{n} \text{Tr}(E_k^* \otimes F_l^* X) |k\rangle \langle l| \in \mathcal{M}_n(\mathbb{C}).
\]
In this way, the $\ell_2^n \otimes \pi \ell_2^n$ norm of $\mathcal{E} \otimes \mathcal{F}(X)$ seen as an element of $\mathbb{C}^n \otimes \mathbb{C}^n$ is nothing else than the $S^n_1$ norm of $\mathcal{E} \otimes \mathcal{F}(X)$ seen as an element of $\mathcal{M}_n(\mathbb{C})$. Now, computing an $S^n_1$ norm is much cheaper than computing an $S^n_1 \otimes \pi S^n_1$ norm. The practical interest of our approach is thus clear in the bipartite case.

But what about the difference in computational cost in the multipartite case? Assume that we have a multipartite system with $m$ subsystems. In this case deciding entanglement consists in computing an $(S^d_1)^{\otimes m}$ norm, i.e., as we have just explained, an $(\ell_2^d \otimes \pi \ell_2^d)^{\otimes m}$ norm. Now, an important property of the projective norm is that it is associative: given Banach spaces $X_1, X_2, X_3$, $X_1 \otimes \pi X_2 \otimes \pi X_3 = (X_1 \otimes \pi X_2) \otimes \pi X_3 = X_1 \otimes \pi (X_2 \otimes \pi X_3)$. Hence, an $(\ell_2^d \otimes \pi \ell_2^d)^{\otimes m}$ norm can actually be seen as an $(\ell_2^d)^{\otimes 2m}$ norm. On the other hand, deciding whether maps $\mathcal{E}_1, \ldots, \mathcal{E}_m$ detect entanglement here consists in computing an $(\ell_2^m)^{\otimes m}$ norm. This means that what we gain with our approach is a factor 2 in the number of tensor products.

In addition to being associative, the projective norm is also commutative: given Banach spaces $X_1, X_2, X_3$, $X_1 \otimes \pi X_2 \otimes \pi X_3 = (X_1 \otimes \pi X_2) \otimes \pi X_3 = X_1 \otimes \pi (X_2 \otimes \pi X_3)$. Hence, an $(\ell_2^d \otimes \pi \ell_2^d)^{\otimes m}$ norm can actually be seen as an $(\ell_2^d)^{\otimes 2m}$ norm. On the other hand, deciding whether maps $\mathcal{E}_1, \ldots, \mathcal{E}_m$ detect entanglement here consists in computing an $(\ell_2^m)^{\otimes m}$ norm. This means that what we gain with our approach is a factor 2 in the number of tensor products.

In addition, the projective norm is also commutative: given Banach spaces $X_1, X_2, X_1 \otimes \pi X_2 = X_2 \otimes \pi X_1$. This means that, for any state $\rho$ on $(\mathbb{C}^d)^{\otimes m}$, for any permutation $\sigma \in S_{2m}$, denoting by $\rho_{\sigma}$ the matrix obtained from the matrix $\rho$ by permuting its indices according to $\sigma$, we have
\[
\|\rho\|_{(\ell_2^d)^{\otimes 2m}} = \|\rho_{\sigma}\|_{(\ell_2^d)^{\otimes 2m}}.
\]
We could thus enlarge even further our family of entanglement criteria to: If a state $\rho$ on $(\mathbb{C}^d)^{\otimes m}$ is separable, then for any testers $\mathcal{E}_1, \ldots, \mathcal{E}_m : S^d_1 \to \ell_2^n$ and any permutation $\sigma \in S_{2m}$,
\[
\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho_{\sigma})\|_{(\ell_2^d)^{\otimes m}} \leq 1.
\]
Considering arbitrary permutations has, however, one important drawback: One loses the local aspect of the tester maps from Definition 3.1. It is in this extended sense that we shall prove the completeness of entanglement criteria defined by testers for mixed bipartite states in Sect. 11.

It is important to mention the case of the permutations corresponding to partial transpositions. Indeed, if $I \subseteq [m]$ is the set of indices that are partially transposed and $\rho_{\Gamma I}$ is the corresponding matrix, we have the following equality:
\[
\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho_{\Gamma I})\|_{(\ell_2^d)^{\otimes m}} = \|\mathcal{E}_1' \otimes \cdots \otimes \mathcal{E}_m'(\rho)\|_{(\ell_2^d)^{\otimes m}},
\]
where $\mathcal{E}_1', \ldots, \mathcal{E}_m'$ are defined as
\[
\mathcal{E}_i' := \begin{cases} \mathcal{E}_i^\circ & \text{if } i \in I \\ \mathcal{E}_i & \text{if } i \notin I \end{cases},
\]
with $\mathcal{E}_i^\circ$ the tester whose operators are the transposition of those of $\mathcal{E}_i$, so that $\mathcal{E}_i^\circ$ acts as $\mathcal{E}_i^\circ(X) = \mathcal{E}_i(X^\top)$ (see Fig. 4).
4. Important Examples of Testers

We now show that several well-known entanglement criteria can actually be seen as part of our framework, i.e., as being associated with an entanglement tester $E$. The way they are usually presented, all these examples appear as being designed for bipartite systems only. However, with our point of view, it is clear that they can naturally be extended to any number of subsystems (see Sect. 12).

4.1. Maps Defined from Matrix Bases

The first important example that enters into our framework is the celebrated realignment criterion [4,23]. Given an orthonormal basis $\{ |i \rangle \}_{i=1}^{d}$ of $\mathbb{C}^d$, the set $R = \{ R_{ij} \}_{i,j=1}^{d} = \{ |i \rangle \langle j | \}_{i,j=1}^{d}$ defines a map

$$\mathcal{R} : X \mapsto \sum_{i,j=1}^{d} \langle i | X | j \rangle | i j \rangle .$$

The $\ell_2^d$ norm of $\mathcal{R}(X)$ is thus the $S_2^d$ norm of $X$, hence $\| \mathcal{R}(X) \|_{\ell_2^d} = \| X \|_{S_2^d} \leq \| X \|_{S_1^d}$, which implies that $\| \mathcal{R} \|_{S_1^d \rightarrow \ell_2^d} \leq 1$. Moreover, considering $X = |x \rangle \langle x |$ for a unit vector $x \in \mathbb{C}^d$ shows that actually $\| \mathcal{R} \|_{S_1^d \rightarrow \ell_2^d} = 1$. Thus, $\mathcal{R}$ is a tester. And conversely,

$$\mathcal{R}^{-1} : x \mapsto \sum_{i,j=1}^{d} x_{ij} | i \rangle \langle j | ,$$

so that $\| \mathcal{R}^{-1} \| = \sqrt{d}$. In other words, this means that, with the proper identification $\mathcal{R} = \text{id}$, these norm estimates read $\| \text{id} \|_{S_1^d \rightarrow S_2^d} = 1$ and $\| \text{id} \|_{S_2^d \rightarrow S_1^d} = \sqrt{d}$ (see Fig. 6).

Note that the test operator $T_R$ associated to the realignment map $\mathcal{R}$ is the flip operator $F$ from Eq. (11). Indeed,

$$T_R = \sum_{i,j=1}^{d} R_{ij} \otimes R_{ij}^* = \sum_{i,j=1}^{d} |i \rangle \langle j | \otimes | i \rangle \langle i | = F .$$

One can generalize the previous example to so-called cross-norm criteria in other local matrix bases than the one of matrix units. Given any orthonormal basis $G = \{ G_k \}_{k=1}^{d^2}$ of $\mathcal{M}_d(\mathbb{C})$, the corresponding map is defined as

$$\mathcal{G} : X \mapsto \sum_{k=1}^{d^2} \text{Tr}(G_k^* X) | k \rangle .$$
Just as $\mathcal{R}$, it is such that the $\ell^2_2$ norm of $G(X)$ is the $S^d_2$ norm of $X$, so that $\|G\| = 1$. And conversely,

\[ G^{-1} : x \mapsto \sum_{k=1}^{d^2} x_k G_k, \]

so that $\|G^{-1}\| = \sqrt{d}$. Hence, exactly as for the map $\mathcal{R}$, the map $G$ can be seen as the identity map from $S^d_1$ to $\ell^d_2 \cong S^d_2$. The testers $\mathcal{R}$ and $G$ are equivalent in the sense of Definition 3.5.

In [29], deformed versions of these criteria, based on observed correlations in local matrix bases, were studied. This family of criteria actually enters in our framework as well. Let us briefly explain how. Let $G = \{G_k\}_{k=1}^{d^2}$ be an orthonormal basis of $\mathcal{M}_d(\mathbb{C})$, fix $x \geq 0$, and define

\[ \tilde{G}_x : X \mapsto x \text{Tr}(G^*_1 X) \langle 1 \rangle + \sum_{k=2}^{d^2} \text{Tr}(G^*_k X) \langle k \rangle. \]

Assume now that $G = \{G_k\}_{k=1}^{d^2}$ is a canonical orthonormal basis, i.e., with one of its elements proportional to the identity, here $G_1 = I/\sqrt{d}$, and the others traceless. We then have, for any $X$ such that $\|X\|_1 \leq 1$,

\[ \|\tilde{G}_x(X)\|_2 = \left( \sum_{k=1}^{d^2} |\text{Tr}(G^*_k X)|^2 - (1 - x^2)|\text{Tr}(G^*_1 X)|^2 \right)^{1/2} \]
\[ = \left( \text{Tr}(XX^*) - \frac{1 - x^2}{d}|\text{Tr}(X)|^2 \right)^{1/2} \]
\[ \leq \left( 1 - \frac{1 - x^2}{d} \right)^{1/2}. \]

Hence, the map

\[ G_x := \left( \frac{d}{d - 1 + x^2} \right)^{1/2} \tilde{G}_x \]

is such that $\|G_x\| = 1$, and thus provides an entanglement criterion. This is in fact nothing else than a rephrasing of [29, Theorem 1].

Let us point out that, already with this first family of testers, the partial transposition criterion [11,18] can be seen as part of the (generalized) tester-based entanglement criterion developed here. Indeed, we have (see Fig. 5)

\[ \|\mathcal{R} \otimes \mathcal{R}(\rho F)\|_{\ell^2_2 \otimes \ell^2_2} = \|\rho^\Gamma\|_{S^d_1}. \]

Hence, the realignment criterion applied to the permuted matrix $\rho F$ detects the entanglement in $\rho$ if and only if the partial transposition criterion applied to $\rho$ detects it.
4.2. Maps Defined From 2-Designs

We discuss in this section testers coming from spherical 2-designs; an important special case corresponds to the entanglement criterion based on SIC POVMs introduced in [27, Sect. IV] (see also [15]). Given a spherical 2-design \( \{|x_k\rangle\}_{k=1}^{d^2} \) of \( \mathbb{C}^d \) with \( d^2 \) elements (see Eq. (14) below for the definition), the set \( S = \{ S_k := \sigma |x_k\rangle\langle x_k| \}_{k=1}^{d^2} \) defines a map

\[
S : X \mapsto \sigma \sum_{k=1}^{d^2} \langle x_k | X | x_k \rangle |k\rangle.
\]

It is such that

\[
\|S(X)\|_2 = \sigma \left( \sum_{k=1}^{d^2} |\langle x_k | X | x_k \rangle|^2 \right)^{1/2} = \sigma \left( \sum_{k=1}^{d^2} \text{Tr} \left( |x_k\rangle\langle x_k| \otimes^2 X \otimes X^* \right) \right)^{1/2} = \sigma \left( \frac{2d}{d+1} \text{Tr} \left( \frac{I + F}{2} X \otimes X^* \right) \right)^{1/2},
\]

where the last equality is because, by definition of a spherical 2-design,

\[
\frac{1}{d^2} \sum_{k=1}^{d^2} |x_k\rangle\langle x_k| \otimes^2 = \frac{I + F}{d(d+1)}.
\]

This implies that \( \|S\| = \sigma \sqrt{2d/(d+1)} \), so in order to obtain the correct normalization for the map \( S \), one needs to fix

\[
\sigma = \sqrt{\frac{d+1}{2d}}.
\]

Let us now compute \( \|S^{-1}\| \). We have \( S^{-1} = S^*(SS^*)^{-1} \), with

\[
S = \sum_{k=1}^{d^2} \text{Tr}(S_k^*|k\rangle\langle k|),
\]
\[ S^* = \sum_{k=1}^{d^2} S_k \langle k \rangle. \]

Using the symmetry of the \( S_k \)'s, the Gram matrix \( G = SS^* \) is easily computed as

\[
G = \sum_{k,l=1}^{d^2} \text{Tr}(S_k S_l) |k\rangle \langle l| = \frac{1}{2d} J + \frac{1}{2} I,
\]

where \( J \) is the all ones \( d^2 \times d^2 \) matrix. The inverse of \( G \) is thus

\[
G^{-1} = 2I - \frac{2d}{d+1} |v\rangle \langle v|,
\]

where the unit vector \( v \in \mathbb{C}^{d^2} \) is defined as

\[ |v\rangle = \frac{1}{d} \sum_{k=1}^{d^2} |k\rangle. \] (15)

We thus have

\[
S^{-1} = 2S^* - \sqrt{\frac{2d}{d+1}} |I\rangle \langle v|,
\]

where \( |I\rangle \) is the vectorization of the \( d \times d \) identity matrix. So for a given \( y \in \mathbb{C}^{d^2} \), we have

\[
S^{-1}(|y\rangle) = 2 \sum_{k=1}^{d^2} y_k S_k - \sqrt{\frac{2}{d(d+1)}} \left( \sum_{k=1}^{d^2} y_k \right) I = \sum_{k=1}^{d^2} y_k M_k,
\]

where we have defined the \( d \times d \) matrices \( M_k \) as

\[ M_k := \sqrt{\frac{2}{d}} \left( \sqrt{d+1}|x_k\rangle \langle x_k| - \frac{1}{\sqrt{d+1}} I \right). \]

Hence,

\[
\max_{\|y\|_2 \leq 1} \left\| \sum_{k=1}^{d^2} y_k M_k \right\|_1 = \max_{\|y\|_2 \leq 1} \max_{\|Y\|_\infty \leq 1} \left\| \sum_{k=1}^{d^2} y_k \text{Tr}(M_k Y) \right\|_1
\]

\[
= \max_{\|y\|_2 \leq 1} \max_{\|Y\|_\infty \leq 1} \left( \sum_{k=1}^{d^2} [\text{Tr}(M_k Y)]^2 \right)^{1/2}
\]

\[
= \max_{\|Y\|_\infty \leq 1} \left( \sum_{k=1}^{d^2} \text{Tr}(M_k \otimes M_k^* Y \otimes Y^*) \right)^{1/2}.
\]
Now by definition,
\[ M_k \otimes M_k^* = \frac{2}{d} \left( (d+1)|x_k\rangle\langle x_k| \otimes |x_k\rangle\langle x_k| \otimes I - I \otimes |x_k\rangle\langle x_k| + \frac{1}{d+1}I \right). \]

And therefore,
\[ \sum_{k=1}^{d^2} M_k \otimes M_k^* = \frac{2}{d} \left( d(I + F) - 2dI + \frac{d^2}{d+1}I \right) = 2 \left( F - \frac{1}{d+1}I \right). \]

This implies that
\[
\max_{\|Y\|_\infty \leq 1} \left( \sum_{k=1}^{d^2} \text{Tr}(M_k \otimes M_k^* Y \otimes Y^*) \right)^{1/2} = \max_{\|Y\|_\infty \leq 1} \left( 2\text{Tr} \left( \left( F - \frac{1}{d+1}I \right) Y \otimes Y^* \right) \right)^{1/2} = \sqrt{2} \max_{\|Y\|_\infty \leq 1} \left( \text{Tr}(YY^*) - \frac{1}{d+1}|\text{Tr}(Y)|^2 \right)^{1/2} = \sqrt{2d}.
\]

And we have thus eventually shown that \( \|S^{-1}\| = \sqrt{2d} \).

Note that the test operator \( T_S \) associated with the 2-design map \( S \) is the projector on the symmetric subspace \( (I + F)/2 \). Indeed, by definition of a 2-design, as recalled in Eq. (14), we have
\[ T_S = \sum_{k=1}^{d^2} S_k \otimes S_k^* = \frac{d+1}{2d} \sum_{k=1}^{d^2} |x_k\rangle\langle x_k| \otimes |x_k\rangle\langle x_k| = \frac{I + F}{2}, \]

which is the projection on the symmetric subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \) (see also Fig. 6).

### 4.3. Separability Criteria

Recall from Sect. 3 that linear maps \( \mathcal{E} \) also provide separability criteria (see Eq. (6)). It follows that maps \( \mathcal{E} : S_1^d \to \ell_2^n \) with minimal \( \|\mathcal{E}\| \cdot \|\mathcal{E}^{-1}\| \) should give the best combined (entanglement + separability) criterion.
The two examples that we will be mostly focusing on in this paper are the matrix unit map \( R \) and the 2-design map \( S \). Note from the above discussion that, for \( R, S \) normalized to have norm 1, we have \( \|S^{-1}\| = \sqrt{2d} > \sqrt{d} = \|R^{-1}\| \). So the 2-design map gives a ‘worse’ combined (entanglement + separability) criterion than the matrix unit map, in the sense that
\[
\|S\| \cdot \|S^{-1}\| > \|R\| \cdot \|R^{-1}\|.
\]
Note, however, that this goes in the opposite direction as the one suggested by the observations in [27].

\[\text{Remark 4.1.}\] The Banach–Mazur distance between Banach spaces \( X, Y \) is defined as the infimum over all maps \( T : X \to Y \) of \( \|T\| \cdot \|T^{-1}\| \). In [30, Theorem 45.2], it is shown that the Banach–Mazur distance between \( S_{1}^{d} \) and \( \ell_{2}^{d} \approx S_{2}^{d} \) is \( \sqrt{d} \). This means that the realignment criterion is the ‘best’ possible, when one is looking for simultaneous entanglement and separability criteria.

The separability criterion defined by Eq. (6) has no interest when used with the realignment and SIC POVM maps. Indeed, by the computations of \( \|R^{-1}\| \) and \( \|S^{-1}\| \), we have that a quantum state \( \rho \in \mathcal{M}_{d}(\mathbb{C}) \otimes \mathcal{M}_{d}(\mathbb{C}) \) is certified separable by the realignment, resp. SIC POVM, criterion whenever
\[
\|R^\otimes 2(\rho)\|_{1} \leq \frac{1}{d}, \text{ resp. } \|S^\otimes 2(\rho)\|_{1} \leq \frac{1}{2d}.
\]
Note, however, that we have
\[
\|R^\otimes 2(\rho)\|_{1} \geq \|R^\otimes 2(\rho)\|_{2} = \|\rho\|_{2} \geq \frac{1}{d} \|\rho\|_{1} = \frac{1}{d},
\]
with equality if and only if the spectrum of \( \rho \) is flat, i.e. \( \rho = I/d^2 \). Hence, the only bipartite state which is certified separable by the realignment criterion is the maximally mixed state. And similarly,
\[
\|S^\otimes 2(\rho)\|_{1} \geq \|S^\otimes 2(\rho)\|_{2} = \left( \frac{1}{2} \left( \|\rho\|_{1}^2 + \|\rho\|_{2}^2 \right) \right)^{1/2}
\]
\[
\geq \left( \frac{1 + 1/d^2}{2} \right)^{1/2} \|\rho\|_{1}
\]
\[
= \left( \frac{1 + 1/d^2}{2} \right)^{1/2}.
\]
So no bipartite state is certified separable by the SIC POVM criterion.

\section{Perfect Testers}
In this section, we introduce and study a special class of entanglement testers, called perfect testers, which are strong enough to detect any pure entangled state.
5.1. Definition and Characterization of Perfect Testers

**Definition 5.1.** Let $d \geq 2$. A $\mathbb{C}$-tester $E : S_1^d \to \ell_2^n$ is called $\mathbb{C}$-perfect if, for any state vectors $\varphi, \chi \in \mathbb{C}^d \otimes \mathbb{C}^d$, at least one of them entangled,
\[
\|E \otimes 2 (|\varphi\rangle \langle \chi|)\|_{\ell_2^2} > 1.
\]

An $\mathbb{R}$-tester $F : S_{1,sa}^d \to \ell_2^n$ is called $\mathbb{R}$-perfect if, for any entangled state vector $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$,
\[
\|F \otimes 2 (|\varphi\rangle \langle \varphi|)\|_{\ell_2^2} > 1.
\]

We state and prove below the main result of this section, a characterization of perfect testers, both in the real and the complex cases.

**Theorem 5.2.** Consider a $\mathbb{C}$-linear map $E : S_1^d \to \ell_2^n$. The following statements are equivalent:

1. $E$ is a $\mathbb{C}$-perfect tester,
2. The norm $\|E\|_{S_1^d \to \ell_2^n} = 1$ is attained at all the extremal points of the unit ball of $S_1^d$: for all unit vectors $x, y \in \mathbb{C}^d$, we have $\|E(|x\rangle \langle y|)\|_2 = 1$,
3. $E$ is an isometry $S_2^d \to \ell_2^n$.

Similarly, for an $\mathbb{R}$-linear map $F : S_{1,sa}^d \to \ell_2^n$, the following are equivalent:

1. $F$ is an $\mathbb{R}$-perfect tester,
2. The norm $\|F\|_{S_{1,sa}^d \to \ell_2^n} = 1$ is attained at all the extremal points of the unit ball of $S_{1,sa}^d$: For all unit vector $x \in \mathbb{C}^d$, we have $\|E(|x\rangle \langle x|)\|_2 = 1$.

**Proof.** Let us start with the complex case, and prove the implication $(3) \implies (1)$. Writing the Schmidt decompositions
\[
|\varphi\rangle = \sum_i \sqrt{p_i} |a_i\rangle \otimes |b_i\rangle,
\]
\[
|\chi\rangle = \sum_j \sqrt{q_j} |c_j\rangle \otimes |d_j\rangle,
\]
we have
\[
E \otimes 2 (|\varphi\rangle \langle \chi|) = \sum_{ij} \sqrt{p_i q_j} E(|a_i\rangle \langle c_j|) \otimes E(|b_i\rangle \langle d_j|).
\]
Since $E$ is an isometry, one recognizes above the Schmidt decomposition of the left-hand side, so
\[
\|E \otimes 2 (|\varphi\rangle \langle \chi|)\|_{\ell_2^2} \geq \sum_{ij} \sqrt{p_i q_j} = \left( \sum_i \sqrt{p_i} \right) \left( \sum_j \sqrt{q_j} \right),
\]
which implies the claim, since at least one of the last two factors is strictly larger than 1.
Let us now move to the implication (1) \( \Rightarrow \) (2). Consider any four unit vectors \( x, y, x', y' \in \mathbb{C}^d \), and pick orthogonal unit vectors \( x_\perp, y_\perp, x'_\perp, y'_\perp \in \mathbb{C}^d \) (we assume here \( d \geq 2 \)). Define, for \( k \in \mathbb{N} \), the entangled vectors

\[
|\varphi_k\rangle = \sqrt{1 - \frac{1}{k}} |x\rangle \otimes |x'\rangle + \sqrt{\frac{1}{k}} |x_\perp\rangle \otimes |x'_\perp\rangle
\]

\[
|\chi_k\rangle = \sqrt{1 - \frac{1}{k}} |y\rangle \otimes |y'\rangle + \sqrt{\frac{1}{k}} |y_\perp\rangle \otimes |y'_\perp\rangle.
\]

Using the hypothesis that \( \mathcal{E} \) is a \( \mathbb{C} \)-perfect tester, we have

\[
\|\mathcal{E} \otimes_2 (|\varphi_k\rangle \langle \chi_k|)\|_{\ell^2_2 \otimes \pi_{\ell^2_2}} > 1,
\]

hence

\[
\|\mathcal{E}(|x\rangle \langle y|)\|_{\ell^2_2} \leq 1 \quad \text{and} \quad \|\mathcal{E}(|x'\rangle \langle y'|)\|_{\ell^2_2} \leq 1,
\]

so actually both norms are equal to 1, proving the claim that \( \mathcal{E} \) preserves the Euclidean norms of unit rank operators.

Finally, the implication (2) \( \Rightarrow \) (3) follows from Lemma 5.4 applied to \( \mathcal{E}^* \mathcal{E} \), in which we identify \( \mathcal{M}_d(\mathbb{C}) \cong \mathbb{C}^d \otimes \mathbb{C}^d \).

The equivalence in the real case can be proven in a similar manner. \( \square \)

**Remark 5.3.** Amongst the examples of testers presented in Sect. 4, we see that the ones defined from matrix bases are \( \mathbb{C} \)-perfect testers, while the ones defined from deformed canonical matrix bases and the ones defined from 2-designs are only \( \mathbb{R} \)-perfect testers.

Note also that the two conditions above corresponding to the real case are not equivalent to the stronger condition

\[
(3') \quad \mathcal{F} \text{ is an isometry } S^d_{2,sa} \to \ell^2_2.
\]

Indeed, there exist \( \mathbb{R} \)-linear maps preserving the Euclidean norm of any unit rank self-adjoint matrix \(|x\rangle \langle x|\) which are not isometries. An example is the SIC POVM tester \( \mathcal{S} \) described in Sect. 4. Indeed, from Eq. (13), we have

\[
\|\mathcal{S}(|x\rangle \langle x|)\|_2 = 1 \quad \text{for any unit vector } x \in \mathbb{C}^d.
\]

However, we have

\[
\|\mathcal{S}(I)\|_2 = \left\| \sqrt{\frac{d+1}{2d}} I \right\|_2 = \sqrt{\frac{d+1}{2}} < \sqrt{d} = \|I\|_2.
\]

The statement of the following lemma is very similar to [14, Lemma 2.1]. Also, its proof can be seen to follow from [20, Proposition 3].

**Lemma 5.4.** Let \( A \in \mathcal{M}_{d_1d_2}(\mathbb{C}) \) be such that for all unit vectors \( x \in \mathbb{C}^{d_1}, y \in \mathbb{C}^{d_2}, \)

\[
\langle x \otimes y|A|x \otimes y \rangle = 1.
\]

Then, \( A = I_{d_1d_2} \).

**Proof.** Rewrite (16) as

\[
\text{Tr} \left( (A - I)|x\rangle \langle x| \otimes |y\rangle \langle y| \right) = 0.
\]

Since the \( \mathbb{C} \)-linear span of \(|x\rangle \langle x| \otimes |y\rangle \langle y|\) is the whole matrix algebra \( \mathcal{M}_{d_1d_2}(\mathbb{C}) \), we get \( A - I = 0 \). \( \square \)
5.2. Test Operators Associated to Perfect Testers

We have seen in the previous subsection that $\mathbb{C}$-perfect testers are precisely those for which the map $\mathcal{E}$ is an isometry $S^d_2 \to \ell^n_2$.

**Theorem 5.5.** Let $\mathcal{E} : S^d_1 \to \ell_2^d$ be a $\mathbb{C}$-perfect tester. The test operator of $\mathcal{E}$ is then $T_\mathcal{E} = F$, the flip operator. Hence, by Proposition 3.6, $\mathcal{E}$ is equivalent to the realignment tester $\mathcal{R}$, in the sense of Definition 3.5.

**Proof.** Since $\mathcal{E}$ is an isometry, we have (identifying $\mathcal{M}_d(\mathbb{C})$ with $\mathbb{C}^d$ as vector spaces) $\Theta_{\mathcal{E}} = \mathcal{E}^* \mathcal{E} = I_d^2$, and thus $T_\mathcal{E} = F$. \hfill $\square$

Let us now analyze the case of $\mathbb{R}$-perfect testers. Having characterized them in Theorem 5.2, we would now like to re-express what this means at the level of the associated test operator. In other words, given an $\mathbb{R}$-linear map $\mathcal{E} : S^d_{1,sa} \to \ell_2^n$ such that the norm $\|\mathcal{E}\|_{S^d_{1,sa} \to \ell_2^n} = 1$ is attained at all the extremal points of the unit ball of $S^d_{1,sa}$, how is its associated test operator $T_\mathcal{E}$ characterized? We state the answer below.

**Theorem 5.6.** If an $\mathbb{R}$-linear map $\mathcal{E} : S^d_{1,sa} \to \ell_2^n$ is an $\mathbb{R}$-perfect tester, then its associated test operator $T_\mathcal{E}$ is of the form

$$T_\mathcal{E} = \frac{I + F}{2} + T',$$

with $T'$ orthogonal to $(I + F)/2$ (i.e., the Hilbert–Schmidt inner product between $T'$ and $(I + F)/2$ is equal to zero).

Let $\mathcal{E} : S^d_{1,sa} \to \ell^n_2$ be such that, for all unit vector $x \in \mathbb{C}^d$, $\|\mathcal{E}(|x\rangle\langle x|)\| = 1$. As we have already seen before, this means that, for all unit vector $x \in \mathbb{C}^d$, $\langle x \otimes x|T_\mathcal{E}|x \otimes x \rangle = 1$. The statement in Theorem 5.6 is thus an immediate consequence of Lemma 5.7.

**Lemma 5.7.** Let $T$ be an operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ such that, for any unit vector $x \in \mathbb{C}^d$, $\langle x \otimes x|T|x \otimes x \rangle = 1$. Then,

$$T = \frac{I + F}{2} + T',$$

with $T'$ orthogonal to $(I + F)/2$.

**Proof.** Decompose $T$ into its symmetric and anti-symmetric parts as

$$T = \Pi_S T \Pi_S + \Pi_S T \Pi_A + \Pi_A T \Pi_S + \Pi_A T \Pi_A =: \Pi_S T \Pi_S + T',$$

where $\Pi_S = (1 + F)/2$ and $\Pi_A = (1 - F)/2$ are the projectors onto the symmetric and anti-symmetric subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$. Now, for any unit vector $x \in \mathbb{C}^d$, $x \otimes x$ belongs to the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$. Hence, $\langle x \otimes x|T'|x \otimes x \rangle = 0$ and $\langle x \otimes x|\Pi_S|x \otimes x \rangle = 1$, so that by assumption $\langle x \otimes x|\Pi_S T \Pi_S - \Pi_S|x \otimes x \rangle = 0$. And since the span of the $x \otimes x$’s is actually the whole symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$, this means that $\Pi_S T \Pi_S = \Pi_S$, as announced. \hfill $\square$
The operators $T'$ satisfying the condition of Theorem 5.6 are all operators which are supported on the anti-symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$. As particular examples we have all multiples of $(I - F)/2$, the projector on this subspace. This gives the following family of corresponding operators $T$:

$$T_\delta := \frac{1}{2} \left( 1 + \delta \right) I + \frac{1}{2} \left( 1 - \delta \right) F. \quad (17)$$

In order for these to actually be a test operator, we further need to impose that

$$\forall X \in \mathcal{M}_d(\mathbb{C}), \quad \text{Tr}(T_\delta^* X \otimes X^*) \geq 0.$$ 

Now, given $X \in \mathcal{M}_d(\mathbb{C})$,

$$\text{Tr}(T_\delta^* X \otimes X^*) = \frac{1}{2} \left( (1 + \delta) |\text{Tr}(X)|^2 + (1 - \delta) |\text{Tr}(|X|^2)| \right),$$

and the latter quantity is always non negative if and only if $-1 \leq \delta \leq 1$.

Note that the operator Schmidt rank of $T_\delta$, $-1 \leq \delta < 1$, is simply the rank of $d^2$. This means that a tester $\mathcal{E}$ having as associated test operator $T_\delta$, $-1 \leq \delta < 1$, needs to be composed of at least $d^2$ operators $\{E_1, \ldots, E_n\}$.

Testers having a test operator of the form described by Eq. (17) play a central role in our paper. The two main examples of testers that we are considering, namely the realignment tester $R$ and the SIC POVM tester $S$, defined in Sect. 4, enter in this category. More specifically, we have $T_R = T_{-1}$ and $T_S = T_0$. In Sect. 6, we explain a general approach to construct testers of this kind.

6. Construction of Testers from Symmetric Families of Operators

In this section, we investigate what are the conditions on a set of operators $\{E_k\}_{k=1}^n$ on $\mathbb{C}^d$ so that its associated operator $T_E$ on $\mathbb{C}^d \otimes \mathbb{C}^d$ is a linear combination of $I$ and $F$, i.e.,

$$\sum_{k=1}^n E_k \otimes E_k^* = \alpha F + \beta I.$$ 

In [1], this problem was studied in the case where the $E_k$’s are Hermitian, but the result obtained there easily generalizes to the non-Hermitian case. The equivalent of [1, Theorem 1] reads as follows.

**Theorem 6.1.** Let $\{E_k\}_{k=1}^{d^2}$ be a basis of operators on $\mathbb{C}^d$. Then, the following statements are equivalent

$$\sum_{k=1}^{d^2} E_k \otimes E_k^* = (\beta + \alpha) \frac{I + F}{2} + (\beta - \alpha) \frac{I - F}{2} = \alpha F + \beta I, \quad (18)$$

In [1], this problem was studied in the case where the $E_k$’s are Hermitian, but the result obtained there easily generalizes to the non-Hermitian case. The equivalent of [1, Theorem 1] reads as follows.
\[ \forall 1 \leq k, l \leq d^2, \quad \text{Tr}(E_k^* E_l) = \alpha \delta_{kl} + \gamma \text{Tr}(E_k^*) \text{Tr}(E_l). \] (19)

In this case, we have \( \alpha > 0, \alpha + d\beta > 0 \) and \( \gamma = \beta / (\alpha + d\beta) \).

Explicitly, the constants \( \alpha \) and \( \beta \) are thus given by the following formulas:

\[
\alpha = \frac{1}{d^3 - d} \left( d \sum_{k=1}^{d^2} \text{Tr}(E_k^* E_k) - d \sum_{k=1}^{d^2} \text{Tr}(E_k^*) \text{Tr}(E_k) \right),
\]

\[
\beta = \frac{1}{d^3 - d} \left( - d \sum_{k=1}^{d^2} \text{Tr}(E_k^* E_k) + d \sum_{k=1}^{d^2} \text{Tr}(E_k^*) \text{Tr}(E_k) \right).
\]

A family of operators \( \{E_k\}_{k=1}^{d^2} \) satisfying the equivalent conditions given in Theorem 6.1 can be seen as a generalized minimal 2-design. As we will show next, this framework actually encompasses the three examples that we mentioned in Sect. 4.

In the case of the realignment map \( R \), where the associated set of operators is the basis of matrix units \( R = \{R_{ij}\}_{i,j=1}^{d} = \{|i\rangle\langle j|\}_{i,j=1}^{d} \), we have

\[ \forall 1 \leq i, j \leq d, \quad \text{Tr}(R_{ij}) = \delta_{ij}, \]

\[ \forall 1 \leq i, j, k, l \leq d, \quad \text{Tr}(R_{ij}^* R_{lk}) = \delta_{il} \delta_{jk}. \]

Consequently, the parameters from Theorem 6.1 are \( \alpha = 1 \) and \( \beta = \gamma = 0 \), so that \( T_R = F \).

In the case of the map \( G \), where the associated set of operators is a canonical matrix basis \( G = \{G_k\}_{k=1}^{d^2} = \{\sigma|x_k\rangle\langle x_k|\}_{k=1}^{d^2} \), with \( \sigma = \sqrt{(d+1)/(2d)} \), we have

\[ \forall 1 \leq k \leq d^2, \quad \text{Tr}(G_k) = \sqrt{d} \quad \text{and} \quad \forall 2 \leq k \leq d^2, \quad \text{Tr}(G_k) = 0, \]

\[ \forall 1 \leq k, l \leq d^2, \quad \text{Tr}(G_k^* G_l) = \delta_{kl}. \]

Consequently, as in the previous example, the parameters from Theorem 6.1 are \( \alpha = 1 \) and \( \beta = \gamma = 0 \), so that \( T_G = F \).

In the case of the SIC POVM map \( S \), where the associated set of operators is a renormalized symmetric 2-design \( S = \{S_k\}_{k=1}^{d^2} = \{\sigma|x_k\rangle\langle x_k|\}_{k=1}^{d^2} \), with \( \sigma = \sqrt{(d+1)/(2d)} \), we have

\[ \forall 1 \leq k \leq d^2, \quad \text{Tr}(S_k) = \sqrt{\frac{d+1}{2d}}, \]

\[ \forall 1 \leq k, l \leq d^2, \quad \text{Tr}(S_k^* S_l) = \frac{d+1}{2d} \left( \delta_{kl} + \frac{1}{d+1} (1 - \delta_{kl}) \right). \]

Consequently, the parameters from Theorem 6.1 are \( \alpha = \beta = 1/2 \) and \( \gamma = 1/(d+1) \), so that \( T_S = (I + F)/2 \).

The latter example actually generalizes the situation where the \( E_k \)'s are coming from a so-called non-degenerate symmetric family of operators, as defined in [13]. This means that

\[ \forall 1 \leq k \leq d^2, \quad \text{Tr}(E_k) = t, \]

\[ \forall 1 \leq k, l \leq d^2, \quad \text{Tr}(E_k^* E_l) = a \delta_{kl} + b(1 - \delta_{kl}). \] (20)
In this case, Eq. (19) holds with \( \alpha = a - b \) and \( \gamma = b/|t|^2 \), so that Eq. (18) holds with \( \alpha = a - b \) and \( \beta = (a - b)b/(|t|^2 - db) \). Note that the operators composing the SIC POVM map \( S \) do satisfy Eqs. (20).

In [7], sets of operators satisfying Eq. (20) are studied as well. There, only Hermitian families are considered, but with any number \( n \) (not necessarily equal to \( d^2 \)) of operators. Such families are dubbed conical 2-designs. And it is shown that the class of conical 2-designs includes several sub-classes of operators which are relevant for quantum information, such as arbitrary rank symmetric informationally complete measurements (SIMs) or full sets of arbitrary rank mutually unbiased measurements (MUMs).

**Lemma 6.2.** Let \( \mathcal{E}: S^d_1 \to \ell^2_n \) be a \( \mathbb{C} \)-linear map such that \( T_E = \alpha F + \beta I \), with \( \alpha \geq 0 \) and \( \beta \geq -\alpha/d \). Then,

\[
\|\mathcal{E}\|_{S^d_1 \to \ell^2_n} = \begin{cases} 
\sqrt{\alpha + \beta} & \text{if } \beta \geq 0 \\
\sqrt{\alpha} & \text{if } \beta < 0
\end{cases}
\]

**Proof.** We have

\[
\|\mathcal{E}\|_{S^d_1 \to \ell^2_n}^2 = \sup_{\|X\|_1 \leq 1} \text{Tr}(T_E^* X \otimes X^*) = \sup_{\|X\|_1 \leq 1} \alpha \|X\|^2 + \beta |\text{Tr} X|^2.
\]

If \( \beta \geq 0 \), the above supremum is equal to \( \alpha + \beta \). Indeed, for any \( X \), \( \|X\|_2 \leq \|X\|_1 \) and \( |\text{Tr} X| \leq \|X\|_1 \), with both inequalities being saturated by rank one projections. While if \( \beta < 0 \), the above supremum is equal to \( \alpha \). This is because, for any \( X \), \( \|X\|_2 \leq \|X\|_1 \) and \( |\text{Tr} X| \geq 0 \), with both inequalities being saturated by rank one operators of the form \( |x\rangle\langle y| \) for orthogonal unit vectors \( x, y \).

As a consequence of Lemma 6.2, we have that if \( \alpha, \beta \geq 0 \) and \( \alpha + \beta = 1 \), or \( \alpha = 1 \) and \( -1/d \leq \beta < 0 \), then any \( \mathbb{C} \)-linear map \( \mathcal{E}: S^d_1 \to \ell^2_n \) such that \( T_E = \alpha F + \beta I \) is a tester. Conversely, if \( T = \alpha F + \beta I \) with \( \alpha, \beta \) satisfying the above conditions, then we can exhibit operators \( \{ E_k \}_{k=1}^{d^2} \) such that \( T \) is the test operator associated to the corresponding map \( \mathcal{E} \), i.e., such that \( T = \sum_{k=1}^{d^2} E_k \otimes E_k^* \). The construction follows the strategy described in Sect. 3.1. Set \( \Theta = (TF)^1 \), i.e.,

\[
\Theta = \alpha I + \beta d |\psi\rangle\langle \psi|.
\]

\( \Theta \) can be diagonalized as:

\[
\Theta = (\alpha + \beta d) |x_1\rangle\langle x_1| + \alpha |x_2\rangle\langle x_2| + \cdots + \alpha |x_{d^2}\rangle\langle x_{d^2}|,
\]

where \( x_1 = \psi \) and \( x_2, \ldots, x_{d^2} \) are such that \( \{ x_1, \ldots, x_{d^2} \} \) forms an orthonormal basis of \( \mathbb{C}^d \otimes \mathbb{C}^d \). Defining \( |e_1\rangle = \sqrt{\alpha + \beta d} |x_1\rangle \) and \( |e_k\rangle = \sqrt{\alpha} |x_k\rangle \) for \( 2 \leq k \leq d^2 \), we then have

\[
T = \sum_{k=1}^{d^2} E_k \otimes E_k^*,
\]

where the \( E_k \)'s are the matrix versions of the \( e_k \)'s. Concretely, this means that \( E_1 = \sqrt{\alpha + \beta d} G_1 \) and \( E_k = \sqrt{\alpha} G_k \) for \( 2 \leq k \leq d^2 \), with \( \{ G_1, \ldots, G_{d^2} \} \) a
canonical basis of $\mathcal{M}_d(\mathbb{C})$ (i.e., $G_1$ is a multiple of the identity and $G_2, \ldots, G_{d^2}$ are traceless).

It is interesting to note that in the case $\alpha = 1, \beta = 0$, this canonical construction yields a tester of the form of the map $\mathcal{G}$ (introduced in Sect. 4). More generally, we see that any symmetric test operator with $0 < \alpha \leq 1$ and $\beta = 1 - \alpha$ has a corresponding canonical tester of the form of one of the maps $\mathcal{G}_x$ with $x \geq 1$ (also introduced in Sect. 4). Specifically, the test operator $T = \alpha F + (1 - \alpha)I$ is associated with the tester $\mathcal{G}_x$ with $x = \sqrt{(1/\alpha - 1)d + 1}$. Note finally that, in the case $\alpha = \beta = 1/2$, the canonical construction described here provides an alternative map having the same test operator as the map $S$ (i.e., being an equivalent tester to $S$).

7. Entanglement Detection of Dipartite Pure States by Symmetric Testers

In this section, we compute the norm of the action of a given symmetric tester, as defined in Sect. 6, on an arbitrary pure bipartite quantum state. We focus next on the realignment and SIC POVM testers, establishing an equality between the corresponding norms which was conjectured in [27].

Let $E : S_1^d \rightarrow \ell_2^n$ be a linear map, defined by

$$E : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{k=1}^n \text{Tr}(E_k^* X) |k\rangle \in \mathbb{C}^n.$$ 

Denote by $T_E$ its associated test operator, as defined by Eq. (7), which we assume to be symmetric, in the sense of Eq. (20), i.e.,

$$T_E = \alpha F + \beta I,$$

for some parameters $\alpha \geq 0$ and $\beta \geq -\alpha/d$. Note that, for now, we do not ask that $E$ should be normalized to be an entanglement tester.

Consider an arbitrary bipartite unit vector $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$, with Schmidt decomposition

$$|\varphi\rangle = \sum_{i=1}^r \sqrt{\lambda_i} |e_i f_i\rangle,$$

where $\lambda_1, \ldots, \lambda_r > 0$ are such that $\sum_{i=1}^r \lambda_i = 1$ and $\{e_1, \ldots, e_r\}, \{f_1, \ldots, f_r\}$ are orthonormal families in $\mathbb{C}^d$.

**Proposition 7.1.** Let $E : S_1^d \rightarrow \ell_2^n$ be a linear map as above, which is symmetric in the sense of Eq. (20) with corresponding parameters $(\alpha, \beta)$. Then, for any bipartite unit vector $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$ as above, we have

$$\|E \otimes 2(|\varphi\rangle \langle \varphi|)\|_1 = \alpha + \beta + 2\alpha \sum_{i<j} \sqrt{\lambda_i \lambda_j}.$$
Proof. Start from the Schmidt decomposition (21), and set $|u_{ij}\rangle := \sum_{k=1}^{n} \langle e_{j}|E_{k}^{*}|e_{i}\rangle |k\rangle$, $|v_{ij}\rangle := \sum_{k=1}^{n} \langle f_{j}|E_{k}^{*}|f_{i}\rangle |k\rangle$, $1 \leq i, j \leq r$. We then have, viewing $\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|)$ as belonging to $\mathcal{M}_{n}(\mathbb{C})$ rather than $\mathbb{C}^{n} \otimes \mathbb{C}^{n}$,

$$\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|) = \sum_{i,j=1}^{r} \sqrt{\lambda_{i} \lambda_{j}} |u_{ij}\rangle\langle v_{ij}|.$$ 

Let us begin with considering the situation where the $f_{i}$'s are equal to the $\bar{e}_{i}$'s in (21). This implies that the $\bar{v}_{ij}$'s are equal to the $u_{ij}$'s, and therefore that $\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|)$ is positive semidefinite. Hence,

$$\|\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_{1} = \text{Tr} (\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|)) = \sum_{i,j=1}^{r} \sqrt{\lambda_{i} \lambda_{j}} \|u_{ij}\|^{2}.$$ 

Observe next that, for all $1 \leq i, j \leq r$, the symmetry of the map $\mathcal{E}$ implies that

$$\langle u_{ij}|u_{i'j'}\rangle = \langle e_{i}e_{j'}|T_{E}|e_{i'}e_{j'}\rangle = \alpha \delta_{ii'}\delta_{jj'} + \beta \delta_{ij}\delta_{i'j'}.$$ 

(22)

Therefore, $\|u_{ij}\|^{2} = \alpha + \beta \delta_{ij}$ and the conclusion follows for the special case where $u_{ij} = v_{ij}$.

To conclude, we are just left with understanding what happens when the $f_{i}$'s are not equal to the $\bar{e}_{i}$'s. Note that the vectors $v_{ij}$ also satisfy Eq. (22), so the families $\{u_{ij}\}_{i,j=1}^{r}$ and $\{v_{ij}\}_{i,j=1}^{r}$ have the same Gram matrix. Thus, there exists a unitary operator $W_{E}$ on $\mathbb{C}^{n}$ mapping the $u_{ij}$'s to the $\bar{v}_{ij}$'s. Hence, using the invariance of $\| \cdot \|_{1}$ under multiplication by a unitary,

$$\left\| \sum_{i,j=1}^{r} \sqrt{\lambda_{i} \lambda_{j}} |u_{ij}\rangle\langle v_{ij}| \right\|_{1} = \left\| \left( \sum_{i,j=1}^{r} \sqrt{\lambda_{i} \lambda_{j}} |u_{ij}\rangle\langle u_{ij}| \right) W_{E}^{*} \right\|_{1}$$

$$= \left\| \sum_{i,j=1}^{r} \sqrt{\lambda_{i} \lambda_{j}} |u_{ij}\rangle\langle u_{ij}| \right\|_{1},$$

finishing the proof. \qed

Let us consider now the special cases of the realignment tester $\mathcal{R}$ and the SIC POVM tester $S$. These maps are symmetric testers, with respective parameters $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (1/2, 1/2)$. In both cases, $\alpha + \beta = 1$ and $\alpha > 0$. Hence, the tester provides a necessary and sufficient condition for separability of bipartite pure states, namely: for any bipartite state vector $\varphi$, $\|\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_{1} \leq 1$ if and only if $\varphi$ is separable. More precisely, we have

$$\|\mathcal{R}^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_{1} = \sum_{1 \leq i, j \leq r} \sqrt{\lambda_{i} \lambda_{j}} = 1 + \sum_{1 \leq i \neq j \leq r} \sqrt{\lambda_{i} \lambda_{j}},$$

$$\|S^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_{1} = \sum_{1 \leq i, j \leq r} \sqrt{\lambda_{i} \lambda_{j}} = 1 + \sum_{1 \leq i < j \leq r} \sqrt{\lambda_{i} \lambda_{j}}.$$
Note that we also know from [17, Proof of Theorem 0.1] that
\[ \| \varphi \rangle \langle \varphi \|_{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}} = \sum_{1 \leq i, j \leq r} \sqrt{\lambda_{i} \lambda_{j}}, \]
so that we actually have
\[ \| R_{\otimes 2}^{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}(\varphi) \|_{1} = \| \varphi \rangle \langle \varphi \|_{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}. \]

Moreover, the realignment map is always ‘better’ than the SIC POVM map on pure states, in the sense that: for any state vector \( \varphi \),
\[ \| R_{\otimes 2}^{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}(\varphi) \|_{1} \geq \| S_{\otimes 2}^{\otimes s}^{\otimes} \langle \varphi \rangle \|_{1}, \]
with strict inequality as soon as \( \varphi \) is entangled. More precisely: for any state vector \( \varphi \),
\[ \| R_{\otimes 2}^{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}(\varphi) \|_{1} = 1 = \| S_{\otimes 2}^{\otimes s}^{\otimes} \langle \varphi \rangle \|_{1} - 1 = 2 \left( \| S_{\otimes 2}^{\otimes s}^{\otimes} \langle \varphi \rangle \|_{1} - 1 \right). \]  
This proves the conjectured equality in [27, equation (21)].

8. Entanglement Detection of Bipartite Isotropic and Werner States by the Realignment and the SIC POVM Testers

The goal here is to determine when the realignment and SIC POVM testers detect the entanglement of isotropic and Werner states. These are defined, respectively, as
\[ \tau_{\mu} := \mu |\psi\rangle \langle \psi| + (1 - \mu) \frac{I}{d}, \quad 0 \leq \mu \leq 1, \]
\[ \sigma_{\mu} := \mu \frac{I + F}{d(d + 1)} + (1 - \mu) \frac{I - F}{d(d - 1)}, \quad 0 \leq \mu \leq 1. \]

For that, let us start with computing the action of \( S_{d}^{\otimes s} \otimes s_{t}^{\otimes} \) on \( I, |\psi\rangle \langle \psi| \) and \( F \).

First of all,
\[ R_{\otimes 2}^{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}(I) = \sum_{i,j,k,l=1}^{d} \text{Tr}(|ij\rangle \langle kl|) = \sum_{i,k=1}^{d} |ii\rangle \langle kk| = d|\psi\rangle \langle \psi|, \]
\[ S_{\otimes 2}^{\otimes s}^{\otimes}(I) \]
\[ = \frac{d + 1}{2d} \sum_{i,j=1}^{d} \text{Tr}(|x_{i}x_{j}\rangle \langle x_{i}x_{j}|) = \frac{d + 1}{2d} \sum_{i,j=1}^{d} |i\rangle \langle j| = \frac{d + 1}{2d}. \]

Then observe that, for any operators \( A, B \) on \( \mathbb{C}^{d} \), \( \text{Tr}(A \otimes B |\psi\rangle \langle \psi|) = \text{Tr}(AB^T)/d \). Hence,
\[ R_{\otimes 2}^{S_{d}^{\otimes s} \otimes s_{t}^{\otimes}}(|\psi\rangle \langle \psi|) = \frac{1}{d} \sum_{i,j,k,l=1}^{d} \text{Tr}(|ij\rangle \langle kl|) = \frac{1}{d} \sum_{i,j=1}^{d} |ij\rangle \langle ij| = I, \]
\[ S_{\otimes 2}^{\otimes s}^{\otimes}(|\psi\rangle \langle \psi|) = \frac{d + 1}{2d^{2}} \sum_{i,j=1}^{d} \text{Tr}(|x_{i}x_{j}\rangle \langle x_{i}x_{j}|) = \frac{d + 1}{2d^{2}}. \]
\[
\frac{d+1}{2d^2} \left( \left( 1 - \frac{1}{d+1} \right) \sum_{i=1}^{d^2} |i\rangle \langle i| + \frac{1}{d+1} \sum_{i,j=1}^{d^2} |i\rangle \langle j| \right) \\
= \frac{1}{2d} \left( I + \frac{1}{d} J \right).
\]

Finally observe that for any operators \( A, B \) on \( \mathbb{C}^d \), \( \text{Tr}(A \otimes BF) = \text{Tr}(AB) \). Hence,

\[
\mathcal{R}^{\otimes 2}(F) = \sum_{i,j,k,l=1}^{d} \text{Tr}(|j\rangle \langle i| |k\rangle \langle l|) = \sum_{i,j=1}^{d} |ij\rangle \langle ji| = F,
\]

\[
\mathcal{S}^{\otimes 2}(F) = \frac{d+1}{2d} \sum_{i,j=1}^{d^2} \text{Tr}(|x_i\rangle \langle x_i| |x_j\rangle \langle x_j|) = \frac{1}{2} \left( I + \frac{1}{d} J \right).
\]

### 8.1. Isotropic States

With these preliminary computations at hand, let us start with understanding the entanglement detection of isotropic states. For any \( 0 \leq \mu \leq 1 \), we have

\[
\mathcal{R}^{\otimes 2}(\tau_{\mu}) = \frac{1}{d} \left( \mu I + (1 - \mu) |\psi\rangle \langle \psi| \right),
\]

\[
\mathcal{S}^{\otimes 2}(\tau_{\mu}) = \frac{1}{2d} \left( \mu I + \frac{d+1-\mu}{d^2} J \right).
\]

Since \( \mathcal{R}^{\otimes 2}(\tau_{\mu}) \) and \( \mathcal{S}^{\otimes 2}(\tau_{\mu}) \) are positive semidefinite, we then simply have

\[
\| \mathcal{R}^{\otimes 2}(\tau_{\mu}) \|_1 = \text{Tr} \left( \mathcal{R}^{\otimes 2}(\tau_{\mu}) \right) = \frac{1}{d} \left( (d^2 - 1)\mu + 1 \right),
\]

\[
\| \mathcal{S}^{\otimes 2}(\tau_{\mu}) \|_1 = \text{Tr} \left( \mathcal{S}^{\otimes 2}(\tau_{\mu}) \right) = \frac{d+1}{2d} \left( (d-1)\mu + 1 \right).
\]

Hence,

\[
\| \mathcal{R}^{\otimes 2}(\tau_{\mu}) \|_1 > 1 \iff \| \mathcal{S}^{\otimes 2}(\tau_{\mu}) \|_1 > 1 \iff \mu > \frac{1}{d+1}.
\]

As a comparison, we know that we also have \( \tau_{\mu} \) entangled iff \( \mu > 1/(d+1) \) \cite{10}. So both the realignment and the SIC POVM maps detect all entangled isotropic states.

What is more, we know from \cite[Theorem 11]{24} that

\[
\| \tau_{\mu} \|_{S_1^d \otimes_s S_1^d} = \begin{cases} 
1 & \text{if } \mu \leq 1/(d+1) \\
((d^2 - 1)\mu + 1)/d & \text{if } \mu > 1/(d+1)
\end{cases}.
\]

So we actually have that, for any entangled isotropic state \( \tau_{\mu} \),

\[
\| \mathcal{R}^{\otimes 2}(\tau_{\mu}) \|_1 = \| \tau_{\mu} \|_{S_1^d \otimes_s S_1^d}.
\]
8.2. Werner States

Let us now turn to understanding the entanglement detection of Werner states. For any $0 \leq \mu \leq 1$, we have

$$R^{\otimes 2}(\sigma_\mu) = \frac{\mu}{d(d+1)}(d|\psi\rangle\langle\psi| + F) + \frac{1-\mu}{d(d-1)}(d|\psi\rangle\langle\psi| - F)$$

$$= \frac{d+1-2\mu}{d^2-1}|\psi\rangle\langle\psi| + \frac{2\mu d - d - 1}{d(d^2-1)}F,$$

$$S^{\otimes 2}(\sigma_\mu) = \frac{\mu}{2d(d+1)} \left( \frac{d+2}{d}J + I \right) + \frac{1-\mu}{2d(d-1)}(J-I)$$

$$= \frac{2\mu d - d - 1}{2d(d^2-1)}I + \frac{d^2 + d - 2\mu}{2d^2(d^2-1)}d^2 = \mu,$$

which are both smaller than 1. While if $\mu \leq (d+1)/2d$, then

$$\|R^{\otimes 2}(\sigma_\mu)\|_1 = \frac{d+1-2\mu}{d^2-1} + \frac{2\mu d - d - 1}{d(d^2-1)} \times d^2 = 2\mu - 1,$$

$$\|S^{\otimes 2}(\sigma_\mu)\|_1 = \frac{2\mu d - d - 1}{2d(d^2-1)} \times d^2 + \frac{d^2 + d - 2\mu}{2d^2(d^2-1)} \times d^2 = \frac{d+1}{d} - \mu.$$

In general, we can see that

$$\|R^{\otimes 2}(\sigma_\mu)\|_1 > 1 \iff \|S^{\otimes 2}(\sigma_\mu)\|_1 > 1 \iff \mu < \frac{1}{d}.$$ 

As a comparison, we know that we have $\sigma_\mu$ entangled iff $\mu < 1/2$ [32]. So as soon as $d > 2$, both the realignment and the SIC POVM maps do not detect all entangled Werner states (and they perform increasingly poorly as $d$ grows).

What is more, we know from [24, Theorem 9] that

$$\|\sigma_\mu\|_{S_1^d \otimes_s S_1^d} = \begin{cases} 1 & \text{if } \mu \geq 1/2 \\ 2(1-\mu) & \text{if } \mu < 1/2 \end{cases}.$$ 

So for $d=2$ we have that, for any entangled Werner state $\sigma_\mu$,

$$\|R^{\otimes 2}(\sigma_\mu)\|_1 = \|\sigma_\mu\|_{S_1^2 \otimes_s S_1^2}.$$ 

But the two norms do not coincide for $d > 2$, even in the regime $\mu < 1/d$ where the map $R$ detects the entanglement of $\sigma_\mu$.

In Fig. 7, we plot the values of the tensor norms corresponding to the testers $R$ and $S$ applied to Werner states, as functions of the parameter $\mu$. It appears clearly on these graphs that both norms are simultaneously either above or below the value 1, i.e., that the two criteria are simultaneously either detecting or not detecting entanglement. The comparison between the cases
$d = 3$ and $d = 10$ highlights how the performance of both criteria decreases as $d$ grows.

As a final comment, let us point out that, for both isotropic and Werner states, the same equality (23) as the one established for pure states, relating the norms of the realignment and SIC POVM maps, holds:

$$\forall 0 \leq \mu \leq 1, \quad \|S^{\otimes 2}(\tau_\mu)\|_1 = \frac{\|R^{\otimes 2}(\tau_\mu)\|_1 + 1}{2},$$
$$\|S^{\otimes 2}(\sigma_\mu)\|_1 = \frac{\|R^{\otimes 2}(\sigma_\mu)\|_1 + 1}{2}.$$

9. Entanglement Detection of Bipartite Pure States with White Noise by the Realignment and the SIC POVM Testers

We now look at states which are the mixture of a pure state and the maximally mixed state, i.e., given $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$ a unit vector,

$$\rho_\mu := \mu |\varphi\rangle \langle \varphi| + (1 - \mu) \frac{I}{d^2}, \quad 0 \leq \mu \leq 1.$$

We wonder when the realignment and SIC POVM testers detect the entanglement of such states.

Let us write $\varphi$ in its Schmidt decomposition

$$|\varphi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |e_i f_i\rangle.$$

Note that we can assume without loss of generality that the $f_i$’s are equal to the $e_i$’s, since any unitary transformation of basis leaves $I/d^2$ invariant. If this

![Figure 7. 1-norm of realignment and SIC POVM transformations of Werner states, as functions of the parameter $\mu$, for $d = 3$ (on the left) and $d = 10$ (on the right) (color figure online)]
is so, we have shown before, in Sect. 7, that
\[ R^\otimes 2(|\varphi\rangle\langle \varphi|) = \sum_{i,j=1}^{r} \sqrt{\lambda_i \lambda_j} |u_{ij}\rangle \langle u_{ij}| \quad \text{and} \quad S^\otimes 2(|\varphi\rangle\langle \varphi|) = \sum_{i,j=1}^{r} \sqrt{\lambda_i \lambda_j} |v_{ij}\rangle \langle v_{ij}|, \]
where for any \( 1 \leq i, j, i', j' \leq r \),
\[ \langle u_{ij}|u_{i'j'}\rangle = \delta_{ii'}\delta_{jj'} \quad \text{and} \quad \langle v_{ij}|v_{i'j'}\rangle = \frac{\delta_{ii'}\delta_{jj'} + \delta_{ij}\delta_{i'j'}}{2}. \]

What is more, we have also shown before, in Sect. 8, that
\[ R^\otimes 2(I_d^2) = \frac{1}{d} |\psi\rangle \langle \psi| \quad \text{and} \quad S^\otimes 2(I_d^2) = \frac{d + 1}{2d^3} J. \]
As a consequence, we have that, for any \( 0 \leq \mu \leq 1 \),
\[ R^\otimes 2(\rho_{\mu}) = \mu \sum_{i,j=1}^{r} \sqrt{\lambda_i \lambda_j} |u_{ij}\rangle \langle u_{ij}| + (1 - \mu) \frac{1}{d} |\psi\rangle \langle \psi|, \]
\[ S^\otimes 2(\rho_{\mu}) = \mu \sum_{i,j=1}^{r} \sqrt{\lambda_i \lambda_j} |v_{ij}\rangle \langle v_{ij}| + (1 - \mu) \frac{d + 1}{2d^3} J. \]
Since both are positive semidefinite, we then simply have
\[ \| R^\otimes 2(\rho_{\mu}) \|_1 = \text{Tr} (R^\otimes 2(\rho_{\mu})) = \mu (1 + 2f(\varphi)) + \frac{1 - \mu}{d}, \]
\[ \| S^\otimes 2(\rho_{\mu}) \|_1 = \text{Tr} (S^\otimes 2(\rho_{\mu})) = \mu (1 + f(\varphi)) + \frac{(1 - \mu)(d + 1)}{2d}, \]
where we have set
\[ f(\varphi) := \sum_{i<j=1}^{r} \sqrt{\lambda_i \lambda_j}. \]

From these expressions, it is easy to see that
\[ \| R^\otimes 2(\rho_{\mu}) \|_1 > 1 \iff \| S^\otimes 2(\rho_{\mu}) \|_1 > 1 \iff \mu > \frac{d - 1}{(1 + 2f(\varphi))d - 1}. \]
And that also
\[ \| R^\otimes 2(\rho_{\mu}) \|_1 > \| S^\otimes 2(\rho_{\mu}) \|_1 \iff \mu > \frac{d - 1}{(1 + 2f(\varphi))d - 1}. \]
This means that the realignment and SIC POVM maps detect the entanglement of \( \rho_{\mu} \) below the same amount \( 1 - \mu \) of white noise. And in this range of \( \mu \), the realignment map is ‘better’ than the SIC POVM, in the sense that
\[ \| R^\otimes 2(\rho_{\mu}) \|_1 > \| S^\otimes 2(\rho_{\mu}) \|_1. \]
As special cases, we recover the previous results on the entanglement detection of pure states (\( \mu = 1 \)) and isotropic states (\( \varphi = \psi \)). Indeed, for any state vector \( \varphi \in \mathbb{C}^d \otimes \mathbb{C}^d \),
\[ 1 \leq 1 + 2f(\varphi) \leq d, \]
with equality in the first, resp. second, inequality iff \( \varphi \) is separable, resp. maximally entangled. Hence, for the case \( \mu = 1 \) we have
\[
\frac{d - 1}{(1 + 2f(\varphi))d - 1} < 1 \iff 1 + 2f(\varphi) > 1 \iff \varphi \text{ entangled,}
\]
while for the case \( \varphi = \psi \) we have
\[
\mu > \frac{d - 1}{d^2 - 1} \iff \mu > \frac{1}{d + 1} \iff \tau_\mu \text{ entangled.}
\]

Note furthermore that this is yet another class of states for which the same equality (23) as the one established for pure states, relating the norms of the realignment and SIC POVM maps, holds:
\[
\forall \ 0 \leq \mu \leq 1, \quad \| \mathcal{S} \otimes^2 (\rho_\mu) \|_{\ell^2_2} = \frac{\| \mathcal{R} \otimes^2 (\rho_\mu) \|_{\ell^2_2} + 1}{2}.
\]

10. Entangled States Which are Detected by the Realignment Tester are Detected by the SIC POVM Tester

In [27], the following was conjectured: Given an entangled state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \), if its entanglement is detected by the matrix unit tester \( \mathcal{R} : S_1^d \to \ell^2_2 \), then it is necessarily detected by the SIC POVM tester \( \mathcal{S} : S_1^d \to \ell^2_2 \) as well, i.e.,
\[
\| \mathcal{R} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} > 1 \implies \| \mathcal{S} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} > 1.
\] (24)

Here we answer this conjecture in the positive, by showing the following inequality, which clearly implies (24).

**Theorem 10.1.** For any quantum state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \), we have
\[
\| \mathcal{S} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} \geq \frac{\| \mathcal{R} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} + 1}{2}. \tag{25}
\]

Note that inequality (25) was proven to be an equality for several classes of states in Sects. 7, 8 and 9. We show next that it is not the case in general. To this end, consider a product state \( \rho = \rho_1 \otimes \rho_2 \), for quantum states \( \rho_1, \rho_2 \) having respective purities \( p_{1,2} := \text{Tr}(\rho_{1,2}^\otimes) \). We then have
\[
\| \mathcal{S} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} = \sqrt{\text{Tr} \left( \frac{1 + F}{2} \rho_1^\otimes \right) \text{Tr} \left( \frac{1 + F}{2} \rho_2^\otimes \right)} = \frac{\sqrt{(1 + p_1)(1 + p_2)}}{2},
\]
\[
\| \mathcal{R} \otimes^2 (\rho) \|_{\ell^2_2 \otimes \ell^2_2} = \sqrt{\text{Tr} \left( F \rho_1^\otimes \right) \text{Tr} \left( F \rho_2^\otimes \right)} = \sqrt{p_1 p_2}.
\]

Hence, (25) is saturated if and only if \( p_1 = p_2 \).

Before proving Theorem 10.1, we show a key lemma.

**Lemma 10.2.** Let \( \{a_1, \ldots, a_n\} \), \( \{b_1, \ldots, b_n\} \) be two orthonormal bases of \( \mathbb{C}^n \). For complex numbers \( \gamma_1, \ldots, \gamma_n \) such that \( |\gamma_i| \geq 1 \) for all \( 1 \leq i \leq n \), define the matrix
\[
S := \sum_{i=1}^{n} \gamma_i a_i \langle b_i |.
\]
Then, for any $X \in \mathcal{M}_n(\mathbb{C})$, we have

$$\|SXS^*\|_1 \geq \|X\|_1 + \sum_{i=1}^{n} (|\gamma_i|^2 - 1) \langle b_i | X | b_i \rangle.$$  \hspace{1cm} (26)

\textbf{Proof.} First, note that the matrix $S$ is invertible, with inverse

$$S^{-1} = \sum_{i=1}^{n} \gamma_i^{-1} |b_i\rangle \langle a_i|.$$

Writing $Y := SXS^*$, Eq. (26) is equivalent to

$$\|Y\|_1 \geq \|S^{-1}Y(S^*)^{-1}\|_1 + \sum_{i=1}^{n} \left(1 - |\gamma_i|^{-2}\right) \langle a_i | Y | a_i \rangle.$$  \hspace{1cm} (27)

Note that the right-hand side of the inequality above is equal to $\|\Phi(Y)\|_1$, where the map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{2n}(\mathbb{C})$ is given by

$$\Phi(Y) = \left(S^{-1}Y(S^*)^{-1}\right) \oplus \left( \bigoplus_{i=1}^{n} \left(1 - |\gamma_i|^{-2}\right) \langle a_i | Y | a_i \rangle \right).$$

Hence, Eq. (27) reads $\|\Phi(Y)\|_1 \leq \|Y\|_1$, which is true if $\Phi$ is a quantum channel (this is a simple consequence of the Russo–Dye theorem, as explained in [21]). Let us prove next that $\Phi$ is indeed a quantum channel. We have $\Phi(X) = KKK^* + \sum_{i=1}^{n} L_i XL_i^*$, where

$$K = \begin{bmatrix} S^{-1} \\ 0_n \end{bmatrix} \quad \text{and} \quad L_i = \begin{bmatrix} 0_n \\ \sqrt{1 - |\gamma_i|^{-2}} \langle a_i | \end{bmatrix}, \quad 1 \leq i \leq n.$$

So the fact that $\Phi$ is completely positive is clear. And the trace preserving condition is also easily shown to be true, since

$$K^*K + \sum_{i=1}^{n} L_i^*L_i = (S^{-1})^*S^{-1} + \sum_{i=1}^{n} (1 - |\gamma_i|^{-2}) |a_i\rangle \langle a_i| = \sum_{i=1}^{n} |a_i\rangle \langle a_i| = I_n.$$  \hspace{1cm} \Box

We can now give the proof of the main result of this section.

\textbf{Proof of Theorem 10.1.} Setting $X := \mathcal{R}^\otimes 2(\rho)$, where $X$ is viewed as belonging to $\mathcal{M}_{d^2}(\mathbb{C})$, it is easy to see that we have

$$\mathcal{S}^\otimes 2(\rho) = \frac{d+1}{2d} \sum_{k,l=1}^{d^2} \langle x_k \otimes x_l | \rho | x_k \otimes x_l \rangle |kl\rangle$$

$$= \frac{d+1}{2d} \sum_{k,l=1}^{d^2} \langle x_k \otimes \bar{x}_k | X | \bar{x}_l \otimes x_l \rangle |kl\rangle$$

$$= \frac{d+1}{2d} \sum_{k,l=1}^{d^2} \langle x_k \otimes \bar{x}_k | XF | x_l \otimes \bar{x}_l \rangle |kl\rangle$$
where $F$ is the flip operator. We now have $\|R^{\otimes 2}(\rho)\|_\ell^2_2 \otimes_\pi \ell^2_2 = \|X\|_1$ and $\|S^{\otimes 2}(\rho)\|_\ell^2_2 \otimes_\pi \ell^2_2 = \|\hat{S}(XF)\hat{S}^*\|_1$ \footnote{Note that this norm relation holds in general for any tester $E$: $\|E^{\otimes 2}(\rho)\|_\ell^2_2 \otimes_\pi \ell^2_2 = \|\hat{E}X\hat{E}^*\|_1$.}, where $\hat{S}$ is the matrix of the operator $S$, i.e.,

$$\hat{S} = \sqrt{\frac{d+1}{2d}} \sum_{k=1}^{d^2} |k\rangle \langle x_k \otimes \bar{x}_k|.$$ 

Noticing that $1 = \text{Tr} \rho = d\langle \psi | X | \psi \rangle$, where $|\psi\rangle := \sum_{i=1}^{d} |ii\rangle / \sqrt{d}$ is the maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$, the inequality in the statement reads

$$\|\hat{S}(XF)\hat{S}^*\|_1 \geq \frac{\|X\|_1 + d\langle \psi | X | \psi \rangle}{2} = \frac{\|XF\|_1 + d\langle \psi | XF | \psi \rangle}{2}.$$ 

In order to conclude, we need to show that $S := \sqrt{2} \hat{S}$ can be written as in Eq. (26) from Lemma 10.2, with $b_1 = \psi$, $\gamma_1 = \sqrt{d+1}$, and $\gamma_2 = \cdots = \gamma_{d^2} = 1$.

Indeed, let us set

$$|y_k\rangle := \sqrt{\frac{d+1}{d}} |x_k \otimes \bar{x}_k\rangle - \sqrt{\frac{d+1}{d}} - 1 \frac{1}{d} |\psi\rangle.$$ 

One can show, using the fact that the $|x_k\rangle/|x_k\rangle$'s form a SIC POVM, that the $|y_k\rangle$'s form an orthonormal basis of $\mathbb{C}^{d^2}$ (for similar ideas, see [9,13]). We have thus

$$S = \sqrt{\frac{d+1}{d}} \sum_{k=1}^{d^2} |k\rangle \langle x_k \otimes \bar{x}_k| = (\sqrt{d+1} - 1)|v\rangle \langle \psi| + \sum_{k=1}^{d^2} |k\rangle \langle y_k|,$$

where $|v\rangle$ is the normalized all-ones vector from (15), and $U$ is a unitary operator. One can see by direct computation that $U|\psi\rangle = |v\rangle$, so we can write

$$S = \sqrt{d+1}|v\rangle \langle \psi| + V,$$

where $V$ is a partial isometry mapping $(\mathbb{C}|\psi\rangle) \perp$ to $(\mathbb{C}|v\rangle) \perp$. We have thus shown that $S$ can be written as in (26), finishing the proof. \hfill $\square$

11. Completeness of the Family of Criteria in the Bipartite Case

Our goal in this section is to prove that, in the bipartite case, the family of entanglement criteria that we are looking at is complete. What we mean by this is that, given an entangled bipartite state, there always exist testers detecting its entanglement. We have already seen that this is the case for bipartite pure states, and we shall prove a similar result for multipartite pure states in Sect. 12. To be fully rigorous, what we are able to show in the case of bipartite mixed states is that our family of entanglement criteria is complete at least when extended to allow for a permutation of the indices before applying the
testers, as described in Sect. 3.3. More precisely, we will prove the following result.

**Theorem 11.1.** Let $\rho$ be an entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then, there exists a tester $E : S_1^d \rightarrow \ell_2^d$ such that
\[
\|\mathcal{E}^\sharp \otimes \mathcal{E} (F\rho^\Gamma)\|_{\ell_2^d \otimes \ell_2^d} > 1,
\]
where $\mathcal{E}^\sharp : S_1^d \rightarrow \ell_2^d$ is the tester whose operators are the adjoints of those of $\mathcal{E}$.

Concretely, $F\rho^\Gamma$ is the following permutation of indices of $\rho$:
\[
\rho = \sum_{i,j,k,l=1}^d \rho_{ij,kl} |ij\rangle \langle kl| \implies F\rho^\Gamma = \sum_{i,j,k,l=1}^d \rho_{ij,kl} |li\rangle \langle kj|.
\]

Before we launch into the proof of Theorem 11.1, let us make two basic but useful observations.

By duality we know that if $\rho$ is entangled, i.e., $\|\rho\|_{S_1^d \otimes \pi S_1^d} > 1$, then there exists $\Theta$ such that $\|\Theta\|_{S_\infty^d \otimes \epsilon S_\infty^d} \leq 1$ and $\text{Tr}(\Theta \rho) > 1$. Now, we can assume without loss of generality that $\Theta$ is Hermitian. This is because $\hat{\Theta} := (\Theta + \Theta^*)/2$, which is Hermitian, is also such that $\|\hat{\Theta}\|_{S_\infty^d \otimes \epsilon S_\infty^d} \leq 1$ and $\text{Tr}(\hat{\Theta} \rho) > 1$. Indeed, on the one hand,
\[
\text{Tr}(\hat{\Theta} \rho) = \frac{1}{2} (\text{Tr}(\Theta \rho) + \text{Tr}(\Theta^* \rho))
\]
\[
= \frac{1}{2} \left( \text{Tr}(\Theta \rho) + \text{Tr}(\Theta^* \rho) \right)
\]
\[
= \frac{1}{2} \left( \text{Tr}(\Theta \rho) + \overline{\text{Tr}(\Theta \rho)} \right)
\]
\[
= \text{Tr}(\Theta \rho)
\]
\[
> 1.
\]
And on the other hand, for all $X, Y$ such that $\|X\|_1, \|Y\|_1 \leq 1$,
\[
\left| \text{Tr}(\hat{\Theta} X \otimes Y) \right| = \left| \frac{1}{2} (\text{Tr}(\Theta X \otimes Y) + \text{Tr}(\Theta^* X \otimes Y)) \right|
\]
\[
= \left| \frac{1}{2} \left( \text{Tr}(\Theta X \otimes Y) + \text{Tr}(\Theta X^* \otimes Y^*) \right) \right|
\]
\[
\leq \frac{1}{2} (|\text{Tr}(\Theta X \otimes Y)| + |\text{Tr}(\Theta X^* \otimes Y^*)|)
\]
\[
\leq \|\Theta\|_{S_\infty^d \otimes \epsilon S_\infty^d}
\]
\[
\leq 1.
\]

What is more, we can also assume without loss of generality that $\Theta$ is positive semidefinite. This is because $\Theta_\lambda := \lambda I + (1 - \lambda)\Theta$, which is positive semidefinite for $0 \leq \lambda \leq 1$ large enough, is also such that $\|\Theta_\lambda\|_{S_\infty^d \otimes \epsilon S_\infty^d} \leq 1$ and $\text{Tr}(\Theta_\lambda \rho) > 1$. Indeed, on the one hand,
\[
\text{Tr}(\Theta_\lambda \rho) = \lambda \text{Tr}(\rho) + (1 - \lambda) \text{Tr}(\Theta \rho) > \lambda + (1 - \lambda) = 1.
\]
And, on the other hand, for all $X, Y$ such that $\|X\|_1, \|Y\|_1 \leq 1$,

$$|\text{Tr}(\Theta X \otimes Y)| = |\lambda \text{Tr}(X \otimes Y) + (1 - \lambda) \text{Tr}(\Theta X \otimes Y)|$$

$$\leq \lambda |\text{Tr}(X)\text{Tr}(Y)| + (1 - \lambda) |\text{Tr}(\Theta X \otimes Y)|$$

$$\leq \lambda \|X\|_1 \|Y\|_1 + (1 - \lambda) \|\Theta\|_{S^d \otimes S^d}$$

$$\leq \lambda + (1 - \lambda) = 1.$$

We are now ready to prove Theorem 11.1.

**Proof.** Let $\Theta$ be such that $\|\Theta\|_{S^d \otimes S^d} \leq 1$ and $\text{Tr}(\Theta \rho) > 1$. As justified above, we can assume without loss of generality that $\Theta^* = \Theta$ and $\Theta \geq 0$. From Lemma 3.4, we thus know that $T := \Theta F$ is a test operator. This means that there exist operators $\{E_k\}_{k=1}^{d^2}$ such that $T = \sum_{k=1}^{d^2} E_k \otimes E_k^*$ and $E : X \in M_d(\mathbb{C}^d) \mapsto \sum_{k=1}^{d^2} \text{Tr}(E_k^* X) |k\rangle \in \mathbb{C}^{d^2}$ is a tester.

Let us now prove that $\|E^* \otimes T \rho\|_{\ell_2^d \otimes \ell_2^d} > 1$. We have

$$E^* \otimes T \rho = \sum_{k,l=1}^{d^2} \text{Tr} (E_k \otimes E_l^* T \rho) |kl\rangle.$$

Next, observe that $|u\rangle := \sum_{k=1}^{d^2} |kk\rangle$ is such that $\|u\|_{\ell_2^d \otimes \ell_2^d} = 1$. Hence by duality,

$$\|E^* \otimes T \rho\|_{\ell_2^d \otimes \ell_2^d} \geq \langle u|E^* \otimes T \rho\rangle = \sum_{k=1}^{d^2} \text{Tr} (E_k \otimes E_k^* T \rho)$$

$$= \text{Tr} (T \rho)$$

$$= \text{Tr}((TF)^\rho)$$

$$> 1,$$

which is exactly what we wanted to show. \hfill \Box

**Remark 11.2.** Let us briefly comment on what would be needed in order to get a stronger version of Theorem 11.1, namely one where no pre-rewiring of the entangled state is required. In the proof of Theorem 11.1, we would like to write the entanglement witness $\Theta$ itself, and not $\Theta F$, as $\Theta = \sum_{k=1}^n E_k \otimes F_k$, with the maps $E$ and $F$ (associated to the $E_k$’s and $F_k$’s) being testers (i.e., such that $\|E\|_{S^d_1 \otimes \ell_2^d} \leq 1$ and $\|F\|_{S^d_1 \otimes \ell_2^d} \leq 1$). This means that we are looking for a tensor decomposition of $\Theta$ which satisfies this extra normalization assumption. Now, we know that the only condition that $\|\Theta\|_{S^d_1 \otimes S^d_\infty} \leq 1$ is not enough to guarantee the existence of such normalized tensor decomposition [3]. But this is not ruling out neither the statement that we are seeking, that is: for any entangled state, there exists an entanglement witness $\Theta$ which admits such normalized tensor decomposition.
An idea could be to further assume that $\Theta$ is separable, in addition to being Hermitian and positive semidefinite (which can always be done without loss of generality, by mixing it with a high enough weight of the identity). This would allow to study the tensor decompositions of $\Theta$ into positive semidefinite elements (sometimes also referred to as separable tensor decompositions) and see whether the normalization condition could be imposed on those.

Another path to explore would be to at least try and prove completeness of the tester-based criterion for a restricted class of states. One could, for instance, look at states having a particular symmetry, which would guarantee the existence of an entanglement witness having the same symmetry, which might in turn guarantee that the latter admits an appropriately normalized (symmetric) tensor decomposition.

12. Entanglement Testers in the Multipartite Setting

In this section, we discuss the power of the realignment entanglement tester, when used on multipartite pure quantum states. In the first subsection, we show that the criterion obtained by applying several copies of the realignment tester detects all pure entangled multipartite states. In the following two subsections, we show that the multipartite realignment criterion is, in a sense, optimal in the case of the so-called $W$ state and in the case of pure multipartite states admitting a generalized Schmidt decomposition.

12.1. Entanglement Detection of Multipartite Pure States by the Realignment Tester

We consider the entanglement criterion on $(\mathbb{C}^d)^\otimes m$ defined by the realignment map $R^\otimes m$, namely: for any state $\rho$ on $(\mathbb{C}^d)^\otimes m$,

$$\|R^\otimes m(\rho)\|_{(\ell_2^d)^\otimes \pi_m} > 1 \implies \rho \text{ entangled.}$$

We want to show that the above implication is an equivalence on the set of pure states. More precisely, we will establish the following result.

**Theorem 12.1.** For any unit vector $\varphi \in (\mathbb{C}^d)^\otimes m$,

$$\|R^\otimes m(|\varphi\rangle\langle \varphi|)\|_{(\ell_2^d)^\otimes \pi_m} \geq \frac{1}{\|\varphi\|_{(\ell_2^d)^\otimes \pi_m}}. \tag{28}$$

If in addition the injective $\ell_2$-norm of $\varphi$ is multiplicative under tensor product, i.e., $\|\varphi \otimes \varphi\|_{(\ell_2^d)^\otimes \pi_m} = \|\varphi\|_{(\ell_2^d)^\otimes \pi_m}^2$, then

$$\|R^\otimes m(|\varphi\rangle\langle \varphi|)\|_{(\ell_2^d)^\otimes \pi_m} \geq \frac{1}{\|\varphi\|_{(\ell_2^d)^\otimes \pi_m}^2}. \tag{29}$$

As an immediate consequence of Theorem 12.1, we have by Proposition 2.3

$$\varphi \text{ entangled } \implies \|\varphi\|_{(\ell_2^d)^\otimes \pi_m} < 1 \implies \|R^\otimes m(|\varphi\rangle\langle \varphi|)\|_{(\ell_2^d)^\otimes \pi_m} > 1.$$
So we indeed have shown that

\[ \varphi \text{ entangled} \iff \| \mathcal{R} \otimes m(|\varphi\rangle\langle\varphi|) \|_{(\ell_2^d)^\otimes \pi m} > 1. \]

**Remark 12.2.** An important class of unit vectors whose injective \( \ell_2 \)-norm is multiplicative under tensor product is the class of non-negative unit vectors, meaning unit vectors whose coefficients in the canonical basis are all non-negative. Indeed, we know by [34, Theorem 5] that the geometric measure of entanglement (as defined in Eq. (4)) of such a non-negative unit vector \( \varphi \) is additive: \( G(\varphi \otimes \varphi) = 2G(\varphi) \). So for those, the stronger inequality (29) in Theorem 12.1 holds. However, we also know by [34, Theorem 24] that most unit vectors \( \varphi \) exhibit extreme non-additivity of their geometric measure of entanglement, in the sense that \( G(\varphi \otimes \varphi) \) is close to \( G(\varphi) \), and not at all to twice this quantity. This means that there are very few unit vectors satisfying the condition for inequality (29) to hold (even just approximately).

**Proof.** Define \( \hat{\varphi} \in (\mathbb{C}^d^2)^\otimes m \) as

\[ |\hat{\varphi} \rangle := \mathcal{R} \otimes m(|\varphi\rangle\langle\varphi|) = \sum_{1 \leq i_1, \ldots, i_m \leq d} \sum_{1 \leq j_1, \ldots, j_m \leq d} \langle j_1 \cdots j_m | \varphi \rangle \langle \varphi | i_1 \cdots i_m \rangle |i_1 j_1 \cdots i_m j_m \rangle. \]

By duality, we know that

\[ \| \hat{\varphi} \|_{(\ell_2^d)^\otimes \pi m} \geq \frac{\langle \hat{\varphi} | \hat{\varphi} \rangle}{\| \hat{\varphi} \|_{(\ell_2^d)^\otimes \pi m}}. \tag{30} \]

First observe that, for any unit vectors \( a^1, \ldots, a^m \in \mathbb{C}^d \),

\[ \langle \hat{\varphi} | a^1 \cdots a^m \rangle = \sum_{1 \leq i_1, \ldots, i_m \leq d} \sum_{1 \leq j_1, \ldots, j_m \leq d} \varphi_{j_1 \cdots j_m} a_{i_1 j_1} \cdots a_{i_m j_m} = \langle \varphi \varphi | a^1 \cdots a^m \rangle. \]

Therefore,

\[ \| \hat{\varphi} \|_{(\ell_2^d)^\otimes \pi m} = \max \left\{ \langle \varphi | a^1 \cdots a^m \rangle, a^1, \ldots, a^m \in \mathbb{C}^d, \| a^1 \|, \ldots, \| a^m \| \leq 1 \right\} \]

\[ = \max \left\{ \langle \varphi \varphi | a^1 \cdots a^m \rangle, a^1, \ldots, a^m \in \mathbb{C}^d, \| a^1 \|, \ldots, \| a^m \| \leq 1 \right\} \]

\[ = \| \varphi \otimes \varphi \|_{(\ell_2^d)^\otimes \pi m}. \]

Second we have

\[ \langle \hat{\varphi} | \varphi \rangle = \sum_{1 \leq i_1, \ldots, i_m \leq d} \sum_{1 \leq j_1, \ldots, j_m \leq d} \varphi_{j_1 \cdots j_m} \varphi_{i_1 \cdots i_m} \varphi_{i_1 \cdots i_m} = | \langle \varphi | \varphi \rangle |^2 = 1. \]

Hence, inserting the two above equalities into Eq. (30), we get

\[ \| \hat{\varphi} \|_{(\ell_2^d)^\otimes \pi m} \geq \frac{1}{\| \varphi \otimes \varphi \|_{(\ell_2^d)^\otimes \pi m}}. \tag{31} \]

In the case where the injective \( \ell_2 \)-norm of \( \varphi \) is multiplicative, we have

\[ \| \varphi \otimes \varphi \|_{(\ell_2^d)^\otimes \pi m} = \| \varphi \varphi \|_{(\ell_2^d)^\otimes \pi m} = \| \varphi \|_{(\ell_2^d)^\otimes \pi m}. \]

Inequality (29) is thus proven.
To deal with the general case, let us define, for any state $\rho$ on $H_1 \otimes \cdots \otimes H_m$,
\[
h_{\text{sep}(H_1;\ldots;H_m)}(\rho) := \max \{ \text{Tr}(\rho \sigma^1 \otimes \cdots \otimes \sigma^m), \ \sigma^k \text{ state on } H_k, \ 1 \leq k \leq m \} \]
\[
= \max \{ \text{Tr}(\rho |a^1\rangle\langle a^1| \otimes \cdots \otimes |a^m\rangle\langle a^m|), \ a^k \in H_k, \ \|a^k\| = 1, \ 1 \leq k \leq m \},
\]
where the last equality is by extremality of pure product states amongst product states. We thus see that, for any unit vector $\chi, \chi'$
\[
\|\chi\|_{H_1 \otimes \cdots \otimes H_m} = \sqrt{h_{\text{sep}(H_1;\ldots;H_m)}(|\chi\rangle\langle \chi|)}.
\]
Now, let $\rho, \rho'$ be states on $H_1 \otimes \cdots \otimes H_m$, and assume that $\sigma^1, \ldots, \sigma^m$ are states on $H_1 \otimes H_1, \ldots, H_m \otimes H_m$ such that
\[
h_{\text{sep}(H_1;H_1;\ldots;H_m;H_m)}(\rho \otimes \rho') = \text{Tr}(\rho \otimes \rho' \sigma^1 \otimes \cdots \otimes \sigma^m).
\]
We then have, denoting by $\tilde{\sigma}^k := \text{Id} \otimes \text{Tr}(\sigma^k)$ the reduced state of $\sigma^k$ on $H_k$,
\[
h_{\text{sep}(H_1;\ldots;H_m)}(\rho) \geq \text{Tr}(\rho \otimes \tilde{\sigma}^1 \otimes \cdots \otimes \tilde{\sigma}^m)
\]
\[
= \text{Tr}(\rho \otimes I \sigma^1 \otimes \cdots \otimes \sigma^m)
\]
\[
\geq \text{Tr}(\rho \otimes \rho' \sigma^1 \otimes \cdots \otimes \sigma^m)
\]
\[
= h_{\text{sep}(H_1;H_1;\ldots;H_m;H_m)}(\rho \otimes \rho').
\]
This implies that, for any unit vectors $\chi, \chi' \in H_1 \otimes \cdots \otimes H_m$,
\[
\|\chi\|_{H_1 \otimes \cdots \otimes H_m} \geq \|\chi'\|_{H_1 \otimes H_1 \otimes \cdots \otimes H_m}.
\]
Coming back to Eq. (31), we eventually get using the above observation that
\[
\|\tilde{\varphi}\|_{(\ell_2^d)^{\otimes m}} \geq \frac{1}{\|\varphi\|_{(\ell_2^d)^{\otimes m}}},
\]
which is exactly inequality (28). \qed

Remark 12.3. Note that, in the case of a not necessarily normalized vector $\varphi \in (\mathbb{C}^d)^{\otimes m}$, Eq. (31) would actually be
\[
\|\tilde{\varphi}\|_{(\ell_2^d)^{\otimes m}} \geq \frac{\|\varphi\|_{2}^{\frac{4}{2}}}{\|\varphi \otimes \tilde{\varphi}\|_{(\ell_2^d)^{\otimes 2m}}^{\frac{2}{2}}}.\]
Now, in the case where the vectors $\chi, \chi' \in H_1 \otimes \cdots \otimes H_m$ are not necessarily normalized, Eq. (32) would read instead
\[
\|\chi\|_{H_1 \otimes \cdots \otimes H_m} \geq \frac{\|\chi \otimes \chi'\|_{H_1 \otimes \cdots \otimes H_m}^{2}}{\|\chi'\|_{2}^{2}}.
\]
This implies that, in the case of a not necessarily normalized vector $\varphi \in (\mathbb{C}^d)^{\otimes m}$, Eq. (28) would take the following, homogeneous, form:
\[
\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle \varphi|)\|_{(\ell_2^d)^{\otimes m}}^{4} \geq \frac{\|\varphi\|_{2}^{4}}{\|\varphi\|_{2}^{2}} \|\varphi\|_{(\ell_2^d)^{\otimes m}}^{2} = \|\varphi\|_{(\ell_2^d)^{\otimes 2m}}^{3}.\]
It is instructive to see what the lower bound (28) gives in the bipartite case. Indeed, in this case we can compute the exact values of the quantities on the left- and right-hand sides of the inequality. Namely, if a unit vector \( \varphi \in (\mathbb{C}^d)^{\otimes 2} \) has Schmidt decomposition
\[
|\varphi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |e_i f_i\rangle,
\]
then we have, on the one hand,
\[
\| R^{\otimes 2}(|\varphi\rangle\langle \varphi|) \|_{(\ell_2^d)^{\otimes 2}} = \left( \sum_{i=1}^{r} \sqrt{\lambda_i} \right)^2,
\]
and, on the other hand,
\[
\| \varphi \|_{(\ell_2^d)^{\otimes 2}} = \sqrt{\lambda_1}.
\]
So the largest gap in inequality (28) is when \( \varphi \) has uniform Schmidt coefficients, in which case the left-hand side is equal to \( r \), while the right-hand side is equal to \( \sqrt{r} \).

12.2. The Example of the \( W \) state

As an illustration of the power of the entanglement criterion based on the realignment tester on multipartite pure states, let us see what it yields when applied to the famous \( W \) state, known to be the maximally entangled three-qubit pure state [5]. We recall that the latter is defined as:
\[
|w\rangle := \frac{1}{\sqrt{3}} (|112\rangle + |121\rangle + |211\rangle) \in (\mathbb{C}^2)^{\otimes 3}.
\]
It is entangled, and we know from [6, Lemma 6.2] that
\[
\| |w\rangle\langle w| \|_{(S_1^2)^{\otimes 3}} = \| w \|_{(\ell_2^2)^{\otimes 3}}^2 = \left( \frac{3}{2} \right)^2 = \frac{9}{4}.
\]

We would now like to compare the above value to the value of the \((\ell_2^3)^{\otimes 3}\) norm of \( R^{\otimes 3}(|w\rangle\langle w|) \). Since \( w \) is non-negative, inequality (29) tells us that
\[
\| R^{\otimes 3}(|w\rangle\langle w|) \|_{(\ell_2^3)^{\otimes 3}} \geq \frac{1}{\| w \|_{(\ell_2^2)^{\otimes 3}}^2}.
\]
Now, we know from [6, Lemma 6.2] again that
\[
\| w \|_{(\ell_2^3)^{\otimes 3}} = \frac{2}{3}.
\]
We thus have
\[
\| R^{\otimes 3}(|w\rangle\langle w|) \|_{(\ell_2^3)^{\otimes 3}} \geq \frac{1}{(2/3)^2} = \frac{9}{4},
\]
which is actually an equality since, on the other hand,
\[
\| R^{\otimes 3}(|w\rangle\langle w|) \|_{(\ell_2^3)^{\otimes 3}} \leq \| w \|_{(S_1^2)^{\otimes 3}} = \frac{9}{4}.
\]
To summarize, we have shown that the realignment tester $R$ optimally detects the entanglement of the $W$ state, in the sense that
\[
\|R^\otimes 3(|w\rangle\langle w|)\|_{(L^2_2)^\otimes 3} = \|w\rangle\langle w\|_{(S^2_1)^\otimes 3} = \frac{9}{4} > 1.
\]
What is more, this is an example where our lower bound (29) is tight.

**Remark 12.4.** In general, we know from Theorem 12.1 that, for any unit vector $\varphi \in (\mathbb{C}^d)^\otimes m$, the following inequalities hold:
\[
\|\varphi\|_{(L^2_2)^\otimes m} = \|\varphi\|_{(S^1_1)^\otimes m} \geq \|R^\otimes m(|\varphi\rangle\langle \varphi|)\|_{(L^2_2)^\otimes m} \geq \frac{1}{\|\varphi\|_{(L^2_2)^\otimes m}}.
\]
This shows that for all unit vectors $\varphi \in (\mathbb{C}^d)^\otimes m$ for which $\|\varphi\|_{\pi} = 1$, the realignment criterion is exact. Similarly we also have that the realignment criterion is exact for all non-negative unit vectors $\varphi$ such that $\|\varphi\|_{\pi} = 1$.

Note that the unit vector $|v\rangle = \frac{1}{2}(|112\rangle + |121\rangle + |211\rangle - |222\rangle) \in (\mathbb{C}^2)^\otimes 3$, studied in [6, Lemma 6.1], saturates the duality relation $\|v\|_{\pi} = 1$. But it has negative coefficients, so we cannot guarantee the exactness of the realignment criterion in this case.

**12.3. The Case of Multipartite Pure States having a Generalized Schmidt Decomposition**

We now focus on the particular case where the unit vector $\varphi \in (\mathbb{C}^d)^\otimes m$ admits a generalized Schmidt decomposition [26], i.e.,
\[
|\varphi\rangle = \sum_{k=1}^{r} \sqrt{\lambda_k} |e^1_k \cdots e^m_k\rangle,
\]
where $\lambda_1, \ldots, \lambda_r > 0$ are such that $\sum_{k=1}^{r} \lambda_k = 1$ and $\{e^1_k\}_{k=1}^{r}, \ldots, \{e^m_k\}_{k=1}^{r}$ are orthonormal families in $\mathbb{C}^d$.

For such multipartite pure state $\varphi$, setting again $|\hat{\varphi}\rangle := R^\otimes m(|\varphi\rangle\langle \varphi|)$, we have
\[
|\hat{\varphi}\rangle = \sum_{k,l=1}^{r} \sqrt{\lambda_k \lambda_l} \sum_{1 \leq i_1, \ldots, i_m \leq d} \langle j_1 \cdots j_m |e^1_k \cdots e^m_k\rangle \langle e^1_l \cdots e^m_l |i_1 \cdots i_m\rangle |i_1 j_1 \cdots i_m j_m\rangle
\]
\[
= \sum_{k,l=1}^{r} \sqrt{\lambda_k \lambda_l} \sum_{i_1=1}^{d} \langle e^1_k |i_1\rangle \sum_{j_1=1}^{d} (j_1|e^1_l\rangle |j_1\rangle \cdots \cdots \sum_{i_m=1}^{d} \langle e^m_k |i_m\rangle \sum_{j_m=1}^{d} (j_m|e^m_l\rangle |j_m\rangle.
\]
Now, we just have to observe that, for each $1 \leq k, l \leq r$ and $1 \leq q \leq m$,
\[
\sum_{i_q=1}^{d} \langle e_i^q | i_q \rangle | i_q \rangle = |\bar{e}_i^q \rangle \quad \text{and} \quad \sum_{j_q=1}^{d} \langle j_q | e_j^q \rangle | j_q \rangle = |e_j^q \rangle.
\]
Therefore, we actually have
\[
|\hat{\varphi} \rangle = \sum_{k, l=1}^{r} \sqrt{\lambda_k \lambda_l} |\bar{e}_k^1 e_1^1 \cdots \bar{e}_m^m e_m^m \rangle.
\]
We recognize in the expression above a generalized Schmidt decomposition in $(\mathbb{C}^{d^2})^{\otimes m}$. Hence,
\[
\|\hat{\varphi}\|_{(\ell_2^d)^{\otimes m}} = \sum_{k, l=1}^{r} \sqrt{\lambda_k \lambda_l}.
\]
Note that, for such multipartite state $\varphi$, we know from [17, Proof of Theorem 0.1] that
\[
|||\varphi\rangle\langle\varphi|||_{(S_1^d)^{\otimes m}} = \sum_{k, l=1}^{r} \sqrt{\lambda_k \lambda_l}.
\]
So, just as in the bipartite case, this is a situation where
\[
\|\mathcal{R}^{\otimes m} (|\varphi\rangle\langle\varphi|)\|_{(\ell_2^d)^{\otimes m}} = |||\varphi\rangle\langle\varphi|||_{(S_1^d)^{\otimes m}}.
\]

13. Conclusions and Open Problems

We have introduced in this work a new paradigm for entanglement detection in bipartite and multipartite quantum systems, based on entanglement testers. The main idea is to reduce the characterization of entanglement, based on computing the projective norm in a tensor product of Banach spaces (matrices equipped with the Schatten 1-norm), to that of computing the projective norm in a tensor product of Hilbert spaces (vectors equipped with the $\ell_2$-norm). In other words, using entanglement testers, one reduces the entanglement problem of mixed quantum states to that of pure quantum states (which is known to be much simpler), at the cost of obtaining only a sufficient criterion for entanglement. The most symmetric entanglement testers correspond to the realignment criterion and to the SIC POVM criterion, which have been studied extensively in the literature, in the bipartite case. Our work provides a natural generalization of these criteria to the multipartite setting.

We analyze the performance of entanglement testers, identifying the important subclass of perfect testers (the ones which detect every pure entangled state). We compare the realignment tester with the SIC POVM tester, showing an exact relation between them in the bipartite case, which allows us to prove two recent conjectures. We then show that every entangled bipartite mixed quantum state can be detected, after index permutation, by a pair of specifically tailored testers, and that the realignment tester can detect any multipartite pure entangled state.
The main question that remains unanswered at this point is whether our family of entanglement criteria is complete, without allowing for permutation of indices before applying the testers. Concretely, we would like to know if, for any entangled state $\rho$ on $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_m}$, there exist testers $E_i : S_{d_i}^1 \rightarrow \ell_{n_i}^2$, $1 \leq i \leq m$, such that $\|E_1 \otimes \cdots \otimes E_m(\rho)\|_{\ell_{n_1}^2 \otimes \cdots \otimes \ell_{n_m}^2} > 1$. Even for bipartite states we are only able to show a weaker version of this statement (see Sect. 11). Note that, in the bipartite case, this problem can be seen as a factorization through $\ell_2$ problem. Indeed, given an entangled state $\rho$ on $\mathbb{C}^d \otimes \mathbb{C}^d$, there exists by definition an operator $T : S_1^d \rightarrow S_2^d$ with norm at most 1 witnessing its entanglement. And in order to exhibit testers detecting its entanglement, we would need to find operators $E, F : S_1^d \rightarrow \ell_2^n$ with norms at most 1 such that $T = E^* F$. It can be shown that not every $T : S_1^d \rightarrow S_2^d$ can be factorized in this way with constant 1 [3]. However, this does not tell us if there exist entangled states that do not have any factorizable entanglement witness.

In a different direction, it would be worth investigating further the performance of our entanglement criteria in the multipartite setting. Indeed, the only quantitative results that we establish in this work when more than two parties are involved are for pure states. But what about the case of mixed states? Are there interesting classes of multipartite mixed states whose entanglement can be detected by the realignment or SIC POVM testers? And can we, in general, compare the respective performances of these two testers?

Finally, it could be interesting to probe the efficiency of entanglement testers $E : S_1^d \rightarrow \mathbb{C}^n$ in the case where the output dimension $n$ is (much) smaller than the dimension of the input space $d^2$. Although such testers cannot be perfect, computing the projective norm of the output tensor is easier when the dimension $n$ is smaller, so the trade-off between the computational efficiency and the performance of these testers needs to be assessed.

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References

[1] Appleby, D.M., Fuchs, C., Zhu, H.: Group theoretic, Lie algebraic and Jordan algebraic formulations of the SIC existence problem. Quantum Inform. Comput 15, 12 (2013)

[2] Aubrun, Gu., Szarek, S.: Alice and bob meet banach: the interface of asymptotic geometric analysis and quantum information theory, volume 223. Am. Math. Soc. (2017)
[3] Aubrun, G.: Personal communication, (2020)
[4] Chen, K., Ling-An, W.: A matrix realignment method for recognizing entanglement. Quantum Inform. Comput. 3, 193–202 (2003)
[5] Derksen, H., Friedland, S., Lim, L.-H., Wang, L.: Theoretical and computational aspects of entanglement. arXiv preprint: arXiv:1705.07160v1, (2017)
[6] Friedland, S., Lim, L.-H.: Nuclear norm of higher-order tensors. Math. Comput. 87(311), 1255–1281 (2018)
[7] Graydon, M.A., Appleby, D.M.: Quantum conical designs. J. Phys. A: Math. Theor. 49(8), 085301 (2016)
[8] Gharibian, S.: Strong NP-hardness of the quantum separability problem. Quantum Inf. Comput. 10(3), 343–360 (2010)
[9] Gour, G., Kalev, A.: Construction of all general symmetric informationally complete measurements. J. Phys. A: Math. Theor. 47(33), 335302 (2014)
[10] Horodecki, M., Horodecki, P.: Reduction criterion of separability and limits for a class of distillation protocols. Phys. Rev. A 59, 4206 (1999)
[11] Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A 223(1), 1–8 (1996)
[12] Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed quantum states: linear contractions and permutation criteria. Open Syst. Inform. Dynam. 13(1), 103–111 (2006)
[13] Jivulescu, M.A., Nechita, I., Găvruţa, P.: On symmetric decompositions of positive operators. J. Phys. A: Math. Theor. 50(16), 165303 (2017)
[14] Johnston, N.: Characterizing operations preserving separability measures via linear preserver problems. Linear and Multilinear Algebra 59(10), 1171–1187 (2011)
[15] Lai, L.-M., Li, T., Fei, S.-M., Wang, Z.-X.: Entanglement criterion via general symmetric informationally complete measurements. Quantum Inf. Process. 17(11), 314 (2018)
[16] Nielsen, M.A., Chuang, I.L.: Quantum computation and quantum information. Cambridge University Press, UK (2010)
[17] Palazuelos, C.: On the largest Bell violation attainable by a quantum state. J. Funct. Anal. 267(7), 1959–1985 (2014)
[18] Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77(8), 1413 (1996)
[19] Pérez-García, D.: Deciding separability with a fixed error. Phys. Lett. A 330(3–4), 149–154 (2004)
[20] Puchała, Z., Gawron, P., Miszczak, J.A., Skowronek, Ł, Choi, M.-D., Žyczkowski, K.: Product numerical range in a space with tensor product structure. Linear Algebra Appl. 434(1), 327–342 (2011)
[21] Pérez-García, D., Wolf, M.M., Petz, D., Ruskai, M.B.: Contractivity of positive and trace-preserving maps under $L_p$ norms. J. Math. Phys. 47(8), 083506 (2006)
[22] Rudolph, O.: A separability criterion for density operators. J. Phys. A: Math. Gen. 33(21), 3951 (2000)
[23] Rudolph, O.: Computable cross-norm criterion for separability. Lett. Math. Phys. 70, 57–64 (2004)
[24] Rudolph, O.: Further results on the cross norm criterion for separability. Quantum Inf. Process. 4(3), 219–239 (2005)

[25] Ryan, R.A.: Introduction to tensor products of banach spaces. Springer, Berlin (2002)

[26] Sokoli, F., Alber, G.: Generalized Schmidt decomposability and its relation to projective norms in multipartite entanglement. J. Phys. A: Math. Theor. 47(32), 325301 (2014)

[27] Shang, J., Asadian, A., Zhu, H., Gühne, O.: Enhanced entanglement criterion via symmetric informationally complete measurements. Phys. Rev. A 98(2), 022309 (2018)

[28] Shimony, A.: Degree of entanglement. Ann. N. Y. Acad. Sci. 755(1), 675–679 (1995)

[29] Sarbicki, G., Scala, G., Chruściński, D.: Family of multipartite separability criteria based on a correlation tensor. Phys. Rev. A 101, 012341 (2020)

[30] Tomczak-Jaegermann, N.: Banach-Mazur distances and finite-dimensional operator ideals, 38. Longman Sc & Tech, (1989)

[31] Watrous, J.: The Theory of Quantum Information. Cambridge University Press, UK (2018)

[32] Werner, R.F.: Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable mode. Phys. Rev. A 40, 4277 (1989)

[33] Wei, T.-C., Goldbart, P.M.: Geometric measure of entanglement and applications to bipartite and multipartite quantum states. Phys. Rev. A 68(4), 042307 (2003)

[34] Zhu, H., Chen, L., Hayashi, M.: Additivity and non-additivity of multipartite entanglement measures. New J. Phys. 12(8), 083002 (2010)

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