SEMIGROUPS OF LOCALLY INJECTIVE MAPS AND TRANSFER OPERATORS

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ABSTRACT. We consider semigroups of continuous, surjective, locally injective maps of a compact metric space, and whether such semigroups admit a transfer operator.

1. INTRODUCTION

In this note we consider certain semigroups of continuous, surjective, locally injective maps acting on a compact metric space. In [2] R. Exel and J. Renault looked at crossed products arising from semigroups of local homeomorphisms acting on a compact metric space. In particular, the semigroups studied were assumed to satisfy an admissibility condition. Admissibility is equivalent to the existence of a transfer operator. The object here is to examine the question of admissibility of a slightly broader class of maps.

If \( P \) is a semigroup and \( \varphi : P \to CSLI(X), \ n \to \varphi_n, \) is an isomorphism of \( P \) into the continuous, surjective, locally injective maps of a compact metric space \( X \) to itself, then \( P \) can be viewed as a semigroup of unital endomorphisms \( \alpha_n \) of \( C(X) \), via \( \alpha_n(f) = f \circ \varphi_n \). A transfer operator is a linear map \( \mathcal{L}_n : C(X) \to C(X) \) which is a left inverse of \( \alpha_n, \ n \in P. \) Transfer operators have been studied in the contexts of both ergodic and topological dynamical systems (cf [1]). Here the maps we consider are locally injective, so that a transfer operator, if it exists, is of the form

\[
\mathcal{L}_n(f)(y) = \sum_{\varphi_n(x)=y} \omega(n,x)f(x)
\]

where \( \omega(n, \cdot) \) is a cocycle for the semigroup \( P. \) Local injectivity is neither necessary nor sufficient for the existence of a transfer operator.

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Unlike the situation for a local homeomorphism, where a cocycle, hence a transfer operator, can always be defined, for continuous, surjective, locally injective (CSLI) maps, it may happen that no cocycle exists, even if one relaxes the condition of strict positivity. Indeed, as we show, the existence of a strictly positive cocycle for a CSLI map \( \varphi \) implies that \( \varphi \) is a local homeomorphism. In our context, an admissible dynamical system is one which admits a nonnegative, continuous cocycle. (See Definition 1.) As we show, this weaker form of admissibility is not always satisfied, even in the case of a single CSLI map, that is where the semigroup is isomorphic with \( \mathbb{N} \). (See Example 1.) In the case of semigroups with a finite set of free generators (i.e., isomorphic to \((\mathbb{Z}^+)^k\)), we give necessary and sufficient conditions on the generators for admissibility.

We also consider another class of semigroups, divisible semigroups, which include examples such as the (additive) semigroup of positive dyadic rationals. There we give necessary (but not sufficient) conditions for a semigroup to be admissible, and an example where the cocycle is constructed.

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2. Preliminaries

All actions will take place on a compact metric space \( X \). The class of mappings studied here are continuous, surjective, and locally injective, which we abbreviate as CSLI. Note that if a CSLI map \( \varphi \) is also open, then it is a local homeomorphism. The class of CSLI maps on a metric space \( X \) is hereditary in the sense that if \( Y \subset X \) is a closed subset such that \( \varphi(Y) = Y \), then \( \varphi \) is also a CSLI map of \( Y \). By contrast, local homeomorphisms are not hereditary.

We will use \( P \) to denote a semigroup. For the semigroups \( P \) considered there is a group \( G \) such that \( P \subset G \), and the group operation on \( G \) maps \( P \times P \to P \). The semigroups considered here will be abelian, and the group operation will be written additively.

Thus, we are given a compact metric space \( X \), an abelian semigroup \( P \), and a map \( \varphi \) of \( P \) into the CSLI maps of \( X \), satisfying

\[
\varphi_{n+m} = \varphi_n \circ \varphi_m \quad \text{for all } n, m \in P.
\]

We will refer to the pair \((P, X)\) as a dynamical system. In case the semigroup \( P \) is the natural numbers \( \mathbb{N} \), there is a generator \( \varphi_1 \); we will write \( \varphi \) in place of \( \varphi_1 \).

If the semigroup \( P \) is contained in a torsion group \( G \), so that for each \( n \in P \) there is \( k \in \mathbb{N} \) such that \( kn = 0 \), then \( \varphi_n \) is a homeomorphism, since the \( k \)-fold composition with itself is a homeomorphism. Such
semigroups are not of interest here, where we want to study CSLI maps which are not homeomorphisms. The semigroups $\mathcal{P}$ will not necessarily be assumed to contain the zero element, in which case the corresponding subset $\{\varphi_n : n \in \mathcal{P}\}$ will not contain the identity.

Definition 1. We will say $\omega$ is a cocycle on a dynamical system $(\mathcal{P}, X)$ if

1. $\omega$ is a function from $\mathcal{P} \times X \to \mathbb{R}$, and $\omega(n, x) \geq 0$ for all $(n, x) \in \mathcal{P} \times X$;
2. for each $y \in X$, $n \in \mathcal{P}$, $\sum_{\varphi_n(x) = y} \omega(n, x) = 1$;
3. for each $n \in \mathcal{P}$, the map $x \in X \to \omega(n, x)$ is continuous;
4. $\omega$ satisfies the cocycle identity:

$$\omega(m + n, x) = \omega(m, x)\omega(n, \varphi_m(x)).$$

A dynamical system $(\mathcal{P}, X)$ will be called admissible if it admits a cocycle.

Our definition of admissibility differs from that in [2] as we do not include the requirements of strict positivity or coherence in the definition of admissibility. Indeed, for a singly generated semigroup, say with generator a CSLI map $\varphi$, we show there exists a strictly positive cocycle for the action if and only if $\varphi$ is a local homeomorphism (Corollary 5).

In the case of singly generated semigroups $\mathcal{P}$ (isomorphic with $\mathbb{N}$), admissibility depends on the existence of a cocycle for the generator, denoted by $\varphi$. Indeed, if a cocycle exists for $\varphi$ the cocycle for $\varphi^n$ (the $n$-fold composition of $\varphi$) is then determined by the cocycle identity (4). Thus, to simplify notation, for singly generated semigroups we write the cocycle as $\omega(\cdot)$ and omit the dependence on the semigroup.

If $F$ is a finite or infinite set, we use $|F|$ to denote the cardinality of $F$. We will denote the metric on the space $X$ by $\rho$.

3. Semigroups of CSLI maps

3.1. Generalities concerning CSLI maps. We begin with some elementary topological results for CSLI maps.

Lemma 1. Let $(\varphi, X)$ be a CSLI dynamical system. Then for all $x \in X$, $|\varphi^{-1}(x)| < \infty$.

Proof. If for some $x_0$, $|\varphi^{-1}(x_0)|$ is infinite, there is a sequence $\{u_n\} \subset \varphi^{-1}(x_0)$, with the $u_n$ distinct points, which converges to a point $u_0$. Thus, $\varphi(u_n) \to \varphi(u_0)$. As $\varphi(u_n) = x_0$, it follows $\varphi(u_0) = x_0$. But then $\varphi$ is not injective on any neighborhood of $u_0$. $\Box$
Let \( \phi \) be CSLI, \( y_0 \in X \), \( \phi^{-1}(y_0) = \{x_1, \ldots, x_N\} \). Given \( \epsilon > 0 \) let the compact neighborhoods \( N_j = \{x : \rho(x, x_j) \leq \epsilon\} \). Then there is a \( \delta > 0 \) so that if \( U = \{y : \rho(y, y_0) \leq \delta\} \) then \( \phi^{-1}(U) \subset \bigcup_{j=1}^{N} N_j \).

Proof. Assume that no such \( \delta \) exists, and let \( U_n \) be a nested neighborhood base at \( y_0 \) and \( x_n \in \phi^{-1}(U_n) \), \( x_n \notin \bigcup_{j=1}^{N} N_j \). Taking a subsequence, we may assume \( x_n \to x'_0 \), for some point \( x'_0 \in X \). Since \( \phi(x_n) \in U_n \), \( \phi(x'_0) = y_0 \). Hence \( \phi(x'_0) = y_0 \). But that is impossible, as \( \rho(x'_0, x_j) \geq \epsilon \) for \( j = 1, \ldots, N \).

Remark 1. Note that \( \epsilon > 0 \) can be taken sufficiently small so that the \( N_j \) are pairwise disjoint, and so that \( \phi \) is injective on \( N_j \).

Corollary 1. The set \( \{u \in X : |\phi^{-1}(u)| \leq N\} \) is open.

Proof. The neighborhoods \( U, N_j \) of Lemma 2 can be made small enough so that \( \phi \) is injective on each \( N_j \). Thus, for any \( u \in U \), \( |\phi^{-1}(u)| \leq N \).

Corollary 2. There exists \( N \in \mathbb{N} \) such that \( \sup_{y \in X} |\phi^{-1}(y)| = N \).

Proof. By Corollary 1 for each \( y \in X \) there is a neighborhood \( U_y \) of \( y \) and a minimal integer \( N_y \) so that for \( u \in U_y \), \( |\phi^{-1}(u)| \leq N_y \). Now by compactness of \( X \) there is a finite subcover \( U_{y_i} \), and \( N \) can be taken as the the maximum of the corresponding \( N_{y_i} \).

The next lemma is known, but we include it here for completeness.

Lemma 3. Let \( \phi \) be a local homeomorphism of a compact metric space \( X \). Let \( y_0 \in X \) and suppose \( |\phi^{-1}(y_0)| = N \). Then there is an open neighborhood \( U \) of \( y_0 \) for which \( |\phi^{-1}(y)| \geq N \), \( y \in U \).

Proof. Let \( \phi^{-1}(y_0) = \{x^0_{(j)} : j = 1, \ldots, N\} \). Let \( N_j \) be a neighborhood of \( x^0_{(j)} \) such that \( \phi(N_j) \) is a homeomorphism, and \( N_i \cap N_j = \emptyset \), for \( i \neq j \). Let \( U_j = \phi(N_j) \), \( j = 1, \ldots, N \), and set

\[ U = \bigcap_{j=1}^{N} U_j, \text{ and } W_j = \phi^{-1}(U) \cap N_j. \]

Set \( \phi^{(j)} = \phi|W_j \).

Let \( y \in U \) and \( x_{(j)} = \phi^{(j)}(y) \), \( j = 1, \ldots, N \). Then \( \{x_{(j)} : j = 1, \ldots, N\} \) is a set of cardinality \( N \), so that \( |\phi^{-1}(y)| \geq N \).

Corollary 3. Let \( (\phi, X) \) be a CSLI system, and let \( X_j = \{y \in X : |\phi^{-1}(y)| = j\} \). Then \( \phi \) is a local homeomorphism if and only if each \( X_j \) is both closed and open.
Proof. If \( \varphi \) is a local homeomorphism, it follows from Corollary 1 and Lemma 3 that each \( X_j \) is open. But then \( \bigcup_{i \neq j} X_i \) is open, so \( X_j \) is also closed.

Conversely suppose that each \( X_j \) is clopen. Then, referring to the proofs of Corollary 2 and Lemma 2, we may choose a sufficiently small neighborhood \( U \) of \( y_0 \) so that \( U \subset X_N \). Then each point \( y \) of \( U \) has exactly \( N \) inverse images. We can choose \( \epsilon \) sufficiently small so that \( \varphi|\mathcal{N}_j \) is injective. Set \( \mathcal{W}_j = \varphi^{-1}(U) \cap \mathcal{N}_j \). Then \( \varphi|\mathcal{W}_j \) is a homeomorphism from \( \mathcal{W}_j \) onto \( U \), \( j = 1, \ldots, N \). Thus, \( \varphi \) is a local homeomorphism. \( \square \)

Remark 2. Suppose \((\varphi, X)\) is a CSLI system, \( y_0 \in X \), \( |\varphi^{-1}(y_0)| = N \), and \( U \) is a compact neighborhood of \( y_0 \) such that \( \varphi^{-1}(U) = \bigcup_i \mathcal{N}_j \) where the \( \mathcal{N}_j \) are pairwise disjoint, and \( \varphi|\mathcal{N}_j \) is injective, \( j = 1, \ldots, N \). It does not follow that \( \varphi|\mathcal{N}_j \) is a homeomorphism of \( \mathcal{N}_j \) onto \( U \). Generally, \( \varphi|\mathcal{N}_j \) will be onto a proper subset of \( U \). The Baire Category Theorem asserts that for some \( j \), \( \varphi(\mathcal{N}_j) \) will have nonempty interior, but there is no reason that the point \( y_0 \) must lie in the interior.

3.2. Admissible CSLI systems.

Definition 2. Let \((\varphi, X)\) be a CSLI dynamical system, and \( x_1 \in X \). We say \( \varphi \) is \emph{locally open} at \( x_1 \) if there is an open neighborhood \( \mathcal{N} \) of \( x_1 \) such that the restriction \( \varphi|\mathcal{N} \) is an open map of \( \mathcal{N} \) into \( X \).

Lemma 4. Suppose \((\varphi, X)\) is a CSLI dynamical system, \( y_0 \in X \), \( \varphi^{-1}(y_0) = \{x_1, \ldots x_N\} \). Suppose the system admits a cocycle \( \omega \) and \( \omega(x_1) > 0 \). Then \( \varphi \) is locally open at \( x_1 \).

Proof. We can take a compact neighborhood \( U \) of \( y_0 \) so that \( \varphi^{-1}(U) = \bigcup_i \mathcal{W}_i \) where \( x_i \in \mathcal{W}_i \), \( 1 \leq i \leq N \) and the \( \mathcal{W}_i \) are pairwise disjoint (and compact), and so that the restriction of \( \varphi \) to \( \mathcal{W}_i \) is injective. We can also assume, by taking \( U \) sufficiently small and applying Lemma 2, that \( \eta := \min\{\omega(x) : x \in \mathcal{W}_1\} > 0 \), and for \( x \in \mathcal{W}_i \), \( |\omega(x) - \omega(x_i)| < \frac{\eta}{N} \), \( 1 \leq i \leq N \).

We claim \( \varphi(\mathcal{W}_1) = U \). If not, there is some \( y \in U \) such that \( \varphi^{-1}(y) = \{x'_i : x'_i \in \mathcal{W}_i, i \in I\} \) where \( I \) is a subset of \( \{2, \ldots, N\} \). But then

\[
\sum_{\varphi(x) = y} \omega(x) = \sum_{i \in I} \omega(x'_i) < \sum_{i \in I} [\omega(x_i) + \frac{\eta}{N}] < \sum_{i \in I} \omega(x_i) + \eta \leq 1
\]
contradicting the cocycle property. Thus the claim is verified.

Let \( V \) be an open set in \( X \) containing \( W_1 \) and disjoint from \( W_2, \ldots, W_N \). Let \( U^o \) be the interior of \( U \). Now \( \varphi^{-1}(U^o) \cap V =: \mathcal{V}_1 \) is an open neighborhood of \( x_1 \) disjoint from \( W_2, \ldots, W_N \), so is contained in \( W_1 \), and hence in the interior \( W_1^o \). We claim that the restriction \( \varphi|_{\mathcal{V}_1} \) is an open map of \( \mathcal{V}_1 \) into \( X \). Let \( O \subset \mathcal{V}_1 \) be open. Since \( \varphi|_{W_1} \) is a one to one continuous map of \( W_1 \) onto \( U \), it is a homeomorphism. Thus, \( \varphi(O) \) is open in \( U \). But \( \varphi(O) \subset U^o \), so that \( \varphi(O) \) is open in \( X \). Hence, \( \varphi \) is locally open at \( x_1 \).

\[ \square \]

**Corollary 4.** A necessary condition for a CSLI system \((X, \varphi)\) to admit a cocycle is: for every \( y \in X \) there exists a point \( x \in \varphi^{-1}(y) \) such that \( \varphi \) is locally open at \( x \).

**Corollary 5.** Let \((\varphi, X)\) be a CSLI system. Then the system admits a strictly positive cocycle if and only if \( \varphi \) is a local homeomorphism.

**Proof.** By Lemma 4 if the cocycle is positive at every point of \( X \), then \( \varphi \) is locally open at every point, hence it is an open map. Thus \( \varphi \) is a local homeomorphism.

Conversely, if \( \varphi \) is a local homeomorphism, let the sets \( X_j \) be as in Corollary 3 and set \( Z_j = \varphi(X_j) \). Then the sets \( Z_j \) are pairwise disjoint and clopen, and their union is \( X \). If the cocycle \( \omega \) is defined to be \( \frac{1}{j} \) on \( X_j \), then \( \omega \) is strictly positive.

Recall that a metric space is zero dimensional if it admits a basis for the topology which is both closed and open.

**Proposition 1.** Let \((\varphi, X)\) be CSLI, and suppose \( X \) is zero-dimensional. Then the necessary condition of Corollary 4 for a cocycle to exist is also sufficient.

**Proof.** By compactness, we can obtain a finite cover of disjoint, clopen sets \( U_i, 1 \leq i \leq m \), so that for each \( U_i , \varphi^{-1}(U_i) \) is a union of, say \( n_i \) disjoint clopen sets, \( W_{i,j} \), and the restriction of \( \varphi \) to each of them is injective. We may enumerate them so that \( \varphi|_{W_{i,1}} \) is an open map. Note that since the sets \( U_i \) are disjoint, we have that \( W_{i,j} \cap W_{i,j'} = \emptyset \) if \((i,j) \neq (i',j')\). Thus, the sets \( W_{i,j} \) constitute a finite, pairwise disjoint cover of \( X \) of clopen sets.

We now define a cocycle \( \omega \) on \( X \) as follows: for each \( i, 1 \leq i \leq m \) let \( \omega(x) = 1 \) for all \( x \in W_{i,1} \) and \( \omega(x) = 0 \) for \( x \in W_{i,j} \) for \( j > 1 \). Because the sets where \( \omega \) is 1 or 0 are clopen, \( \omega \) is continuous. And the cocycle condition, that for \( y \in X \),

\[ \sum_{\varphi(x) = y} \omega(x) = 1 \]
is satisfied, since by construction there is one $x$ for which $\omega(x) = 1$, and for the remaining $x$ for which $\varphi(x) = y, \omega(x) = 0$. \hfill \Box

Remark 3. Note that if for every $y \in X$ there exists a unique $x \in \varphi^{-1}(y)$ such that $\varphi$ is locally open in a neighborhood of $x$, then the cocycle $\omega$ is unique, and hence does not depend on the choice of the cover $\mathcal{U}_i$.

Example 1. This is an example of a semigroup $\mathcal{P} = \mathbb{N}$ of a CSLI dynamical system which is not admissible.

Let $\varphi : \Pi_{n \in \mathbb{Z}} \mathbb{T}_n \to \Pi_{n \in \mathbb{Z}} \mathbb{T}_n$ where $\mathbb{T}_n = \mathbb{T} = [0, 1)$, as follows: for a point $x = (x_n)$ in the product space, set

$$\varphi(x) = y \text{ where } y_n = x_n$$

for all $n \neq 1$ and $y_0 = 2x_1 \pmod{1}$. Note that $\varphi$ is a local homeomorphism of $\Pi_{n \in \mathbb{Z}} \mathbb{T}_n$.

Let $Z \subset \Pi_{n \in \mathbb{Z}} \mathbb{T}_n$ consist of those sequences $x = (x_n)$ satisfying: for all $n \geq 1$, $0 \leq x_n \leq \frac{1}{2}$. Clearly $Z$ is closed, and $\varphi(Z) \subset Z$. We take

$$X = \cap_{n=0}^{\infty} \varphi^n(Z),$$

where $\varphi^0$ is the identity, and for $n > 1$, $\varphi^n$ is the $n$-fold composition of $\varphi$ with itself. Then $\varphi(X) = X$.

Changing notation so $\varphi$ refers to the restriction of $\varphi$ to $X$, the dynamical system $(\varphi, X)$ is CSLI.

To show that $(\varphi, X)$ is not admissible, we suppose to the contrary that $\omega$ is a cocycle for $\varphi$. Set

$$y^0 = (\ldots, 0, 0, \frac{1}{2}, \frac{1}{2}, \ldots)$$

where the underzet 0 denotes the 0-th position in the array. Note that $\varphi^{-1}(y^0) = \{y^0, w^0\}$ where

$$w^0 = (\ldots, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \ldots).$$

First we show that $\omega(y^0) = 1$. To this end, define

$$y(t) = (\ldots, t, t, \frac{t}{2}, \frac{t}{2}, \ldots)$$

and note that, for $t > 0$, $\varphi^{-1}(y(t)) = \{y(t)\}$. Hence $\omega(y(t)) = 1$ for $t > 0$. Since $y(t) \to y^0$ as $t \to 1$ continuity of $\omega$ forces $\omega(y^0) = 1$.

Next we claim that $\omega(w^0) = 1$. To this end, we define

$$u(t) = (\ldots, t, \frac{1}{2} - t, \frac{1}{2} - t, \ldots)$$
for \( 0 < t < \frac{1}{2} \). To see that \( u(t) \in X \), note first that \( u(t) \in Z \). Let 
\[ n \in \mathbb{N}, \ n \geq 1 \] 
and set
\[ w(t) = (\ldots, t, t_0, \frac{t}{2}, \ldots, \frac{t}{2} - t, 1 - t, \ldots). \]
Then \( w(t) \in Z \) and \( \varphi^n(w(t)) = u(t) \). Thus, \( u(t) \in \varphi^n(Z) \) for every 
\( n \geq 0 \), so \( u(t) \in X \). Now set \( n = 1 \). The same argument shows 
that \( w(t) \in X \), and furthermore \( w(t) \) is the single inverse image of 
\( u(t) \). Thus, \( \omega(w(t)) = 1 \). As \( t \rightarrow 0 \), \( w(t) \rightarrow w^0 \). Continuity forces 
\( \omega(w^0) = 1 \). But the cocycle condition \( \omega(y_0) + \omega(w^0) = 1 \) is violated, 
so no cocycle exists and the system \( (\varphi, X) \) is not admissible, and in 
particular, \( \varphi \) is not a local homeomorphism.

**Example 2.** This is an example of an admissible CSLI system which is 
not a local homeomorphism. First we construct \( X \) as follows: let \( Z \) 
be the space obtained from \( \mathbb{R} \) by replacing each integer \( n \leq 0 \) by two 
points \( n^-, n^+ \) with \( n^- < n^+ \) so that \( n^- \) is the immediate predecessor 
of \( n^+ \). \( Z \) is an ordered set which is topologized by taking as a base 
for the open sets all sets of the form \( (a, b) \), \( a < b \in Z \) and \( (a, n^-] \) and 
\( [n^+, b) \) for \( n, \in Z, n \leq 0 \) and \( a < n < b \). Set \( X = Z \cup \{ -\infty, +\infty \} \) the 
two-point compactification of \( Z \).

Note that \( X \) is a compact metrizable space. Define \( \varphi : X \rightarrow X \) by 
taking the points \( \pm \infty \) to be fixed, and for \( x \neq \pm \infty \) setting
\[
\varphi(x) = \begin{cases} 
x + 1 & \text{if } x \leq 0^-, x \notin Z \\
(n + 1)^- & \text{if } x = n^-, n \leq -1 \\
(n + 1)^+ & \text{if } x = n^+, n \leq -1 \\
1 & \text{if } x = 0^- \\
x & \text{if } x \geq 0^+ 
\end{cases}
\]
Observe that \( \varphi \) is CSLI, but that \( \varphi \) is not a local homeomorphism 
because it is not an open map in a neighborhood of \( 0^- \).

Notice that the points \( y, 0^+ \leq y \leq 1 \) have two pre-images, and all 
other points have one pre-image. Define a cocycle on \( X \) by \( \omega(x) = 
1, 0^+ \leq x \leq +\infty, \ \omega(x) = 0, -1^+ \leq x \leq 0^-, \) and \( \omega(x) = 1, -\infty \leq 
x \leq -1^- \). Then \( \omega \) is a cocycle, but not strictly positive.

Note there does not exist a strictly positive cocycle. This follows 
from Corollary 5, but can also be seen directly. For \( y = 1 \) has two pre-
images, namely \( 0^- \) and \( 1 \), but any point \( y > 1 \) has only one preimage, 
\( \varphi^{-1}(y) = y \) so that for such \( y \) necessarily \( \omega(y) = 1 \). Continuity of \( \omega \) 
forces \( \omega(1) = 1 \), hence \( \omega(0^-) = 0 \).
Remark 4. Define the conditional expectation

\[ E(f)(x) = \alpha \circ \mathcal{L}(f)(x) = \sum_{\varphi(u) = \varphi(x)} \omega(u) f(u) \]

where \( \alpha(g) = g \circ \varphi, g \in C(X) \).

Then if \( \omega \) is not strictly positive, the conditional expectation can be degenerate. Indeed, suppose \( \omega(x) = 0 \) in a neighborhood \( U \) of a point \( x_0 \). Suppose \( f \) is a nonnegative function supported in \( U \) and that \( \varphi \) is injective on \( U \). Then for \( x \in X \)

\[ E(f)(x) = \sum_{\varphi(t) = \varphi(x)} f(t) \omega(t) = 0 \]

since \( \omega \) is zero where \( f \) is nonzero.

Thus, the conditional expectation associated to the cocycle in Example 2 is degenerate. However, it is possible to define a cocycle on the space \( X \) in that example so that it is nondegenerate, as we show.

Example 3. Let \((\varphi, X)\) be as in Example 2. Define a cocycle \( \omega \) as follows: \( \omega(0^-) = \omega(0^+) = 0 \). For \( 0^+ < x < 1 \), set \( \omega(x) = x \) and \( \omega(x-1) = 1 - x \). As before, for \( x \geq 1 \) we are forced to have \( \omega(x) = 1 \). Also as before, we must have \( \omega(x) = 1 \) for \( x \leq -1^{-} \). Since \( \varphi^{-1}(0^+) = \{ -1^+, 0^+ \} \) and \( \omega(0^+) = 0 \), necessarily \( \omega(-1^+) = 1 \). The cocycle vanishes on the set \( \{ 0^-, 0^+ \} \), which has no interior. Thus the resulting expectation \( E \) is nondegenerate.

Example 4. We take \( X \) as in Example 2 and define \( \varphi \) as it is there for \( x \leq 0^- \). For \( x \geq 0^+ \) we define \( \varphi \) by: \( \varphi(0^+) = 1 \) and \( \varphi(x) = x + 1 \) for \( x > 0^+ \). Then the only point with more than one inverse image is 1, and \( \varphi^{-1}(1) = \{ 0^-, 0^+ \} \). If the system were admissible, then necessarily \( \omega(x) = 1 \) for all \( x \in X \), \( x \neq 0^- \), \( 0^+ \). Then continuity would force \( \omega(0^-) = 1 = \omega(0^+) \), violating cocycle property (ii). Thus this provides another example of a non admissible system.

3.3. Finitely generated free semigroups. Next we want to consider finitely generated free semigroups; that is, semigroups isomorphic to \((\mathbb{Z}^+)^k \), where \( \mathbb{Z}^+ = \mathbb{N} \cup \{0\} \). Recall that (cf. [3]) in an abelian group \( G \), a finite set of elements \( \{ a_1, \ldots, a_k \} \) with \( a_i \neq a_j \) for \( i \neq j \) is independent if any relation of the form

\[ n_1 a_1 + \cdots + n_k a_k = 0, \quad (n_i \in \mathbb{Z}) \]

implies

\[ n_1 = \cdots = n_k = 0. \]
In our case, we are dealing with elements of an abelian semigroup, not a group. We could of course take recourse to the fact that the semigroup is embedded in a (smallest) abelian group, and make use of this definition in the ambient group. However, the semigroup is represented by maps which may not be invertible, and an ambient group is quite removed from the context of the dynamical system, so it is natural to want to express independence in the context of the semigroup. As it happens, it is easy to recast the definition of independence in the semigroup context.

**Definition 3.** Let $a_1, \ldots, a_k$ be elements of an abelian semigroup $\mathcal{P}$. The set $\{a_1, \ldots, a_k\}$ with $a_i \neq a_j$ for $i \neq j$ will be called *independent* if for any nonempty subset $E \subset \{1, \ldots, k\}$ and nonnegative integers $n_1, \ldots, n_k$ the relation

$$\sum_{j \in E} n_j a_j = \sum_{j \in E^c} n_j a_j$$

implies

$$n_1 = \cdots = n_k = 0.$$ 

In case the complement $E^c = \emptyset$, we interpret the right side of the equation to be zero.

Let $\mathcal{P}$ be an abelian semigroup with 0, and let $\{a_1, \ldots, a_k\}$ be a set of independent generators of $\mathcal{P}$.

**Proposition 2.** Let $\mathcal{P}$ act on the compact metric space $X$. Then the action is admissible iff each $\varphi_{a_j}$ is an admissible action, $1 \leq j \leq k$, and

$$\omega_i(1, x) \omega_j(1, \varphi_i(x)) = \omega_j(1, x) \omega_i(1, \varphi_j(x))$$

for all $i, j \in \{1, \ldots, k\}$, where $\omega_i$ is the cocycle associated with the subsemigroup $\mathbb{Z}^+ a_j$, $1 \leq j \leq k$, and we have written $\varphi_j$ for $\varphi_{a_j}$.

**Proof.** Note that $\mathcal{P}$ is isomorphic with $(\mathbb{Z}^+)^k$ under the isomorphism $a_j \rightarrow e_j$, where $e_j$ is the standard basis element $(0, \ldots, 0, 1, 0, \ldots, 0)$. For convenience, we work with $(\mathbb{Z}^+)^k$ in place of $\mathcal{P}$.

Suppose $\mathcal{P}$ is admissible with cocycle $\omega$. If we set $\omega_i(\ell, x) = \omega(\ell e_i, x)$, then the cocycle identity shows that the condition (†) is necessary.

Conversely, suppose cocycles $\omega_i$ are given which satisfy the conditions of the proposition. For $m \in (\mathbb{Z}^+)^k$, $m = (m_1, \ldots, m_k)$, let $|m| = m_1 + \cdots + m_k$. We will define a cocycle $\omega(m, x)$ on $\mathcal{P}$ by induction on $|m|$.

For $|m| = 0$, set $\omega(m, x) = 1$. For $|m| = 1$, $m = e_j$ for some $j$ and we define $\omega(m, x) = \omega_j(1, x)$. For $|m| = 2$, either $m = 2e_j$ for some $j$ or...
else $m = e_i + e_j$ for some $i \neq j$. In the first case, set $\omega(m, x) = \omega_j(2, x)$. This satisfies the cocycle identity because $\omega_j$ does. In the second case, define $\omega(m, x) = \omega_i(1, x)\omega_j(1, \varphi_i(x))$. Note that the cocycle identity is satisfied by assumption (‡).

Suppose now that $\omega(m, x)$ has been defined and satisfies the cocycle identity for $|m| \leq N$ for $N > 1$. We now define $\omega(m, x)$ for $|m| = N+1$.

Let $m$, $n$, $p$, $q \in P$ be such that $n + m = p + q$ and $|n|$, $|m|$, $|p|$, $|q|$ are all positive, and $|n| + |m| = |p| + |q| = N + 1$. We claim that

$$\omega(n, x)\omega(m, \varphi_n(x)) = \omega(p, x)\omega(q, \varphi_p(x)).$$

We do this first assuming $|n - p| = |m - q| = 1$. Thus, either there exists $j$ such that $n = p + e_j$ or $p = n + e_j$. The two cases are similar; we do the first case. Then, $q = m + e_j$. Thus,

$$\omega(n, x)\omega(m, \varphi_n(x)) = \omega(p + e_j, x)\omega(m, \varphi_n(x))$$

$$= \omega(p, x)\omega(e_j, \varphi_p(x))\omega(m, \varphi_n(x))$$

$$= \omega(p, x)\omega(e_j, \varphi_p(x))\omega(m, \varphi_{e_j+p}(x))$$

$$= \omega(p, x)\omega(e_j, \varphi_p(x))\omega(m, \varphi_{e_j+p}(x))$$

$$= \omega(p, x)\omega(n + e_j, \varphi_p(x))$$

$$= \omega(p, x)\omega(q, \varphi_p(x))$$

where we have used the induction hypothesis that the (partially defined) cocycle satisfies the cocycle identity where it is already defined. For the case where $|n - p| > 1$, we repeat the first step $|n - p|$ times.

Thus if $m + n = N+1$, if we set $\omega(n+m, x) = \omega(n, x)\omega(m, \varphi_n(x))$, the above calculation shows that $\omega$ is a well defined cocycle which satisfies the cocycle condition. Conditions (1) through (3) of Definition 1 are easily verified.

4. Divisible semigroups

An abelian group $G$ is divisible if the equation $mx = a$ ($m \in \mathbb{N}$, $a \in G$) has a solution $x \in G$ ([3]). One could use the same definition for semigroups. However, we want to consider examples such as the semigroup $P$ of positive dyadic rationals. Let $D = \{\frac{k}{2^n}, k \in \mathbb{Z}, n \in \mathbb{Z}\}$ be the group of dyadic rationals. This is not divisible, as the equation $mx = a$ is solvable for $x \in D$ only for $m$ a power of 2. Thus, for our purposes an alternative definition is appropriate.

**Definition 4.** A sequence $\{d_k\}$ in a semigroup $P$ will be called a fundamental sequence if

1. there exists a sequence of integers $n_k > 1$ such that $d_k = n_kd_{k+1}$, $k \geq 1$, and
(2) for every $d \in \mathcal{P}$ there exists $k \in \mathbb{N}$ such that $d_k$ divides $d$.

We say $\mathcal{P}$ is **divisible** if it contains a fundamental sequence.

**Proposition 3.** Let $\mathcal{P}$ be a divisible semigroup of CSLI maps on $X$. Then either all $\varphi_d, d \in \mathcal{P}$, are homeomorphisms, or else none is a homeomorphism.

**Proof.** Suppose for some $d \in \mathcal{P}$, $\varphi_d$ is a homeomorphism. Let $e$ be another element of $\mathcal{P}$.

If $d_k$ divides $d$ and $d_l$ divides $e$, taking $n = \max\{k, l\}$ we have that $d_n$ divides both $d$, $e$. Say $d = m d_n$; then $\varphi_d$ is the $m$–fold composition of $\varphi_{d_n}$ with itself. Since the composition is injective, $\varphi_{d_n}$ is injective, hence $\varphi_d$ is a homeomorphism. But since $e = m' d_n$, for some $m' \in \mathbb{N}$, $\varphi_e$ is a composition of homeomorphisms, hence is a homeomorphism. \qed

**Proposition 4.** Suppose $\mathcal{P}$ is a divisible semigroup of CSLI maps which are not homeomorphisms. Then there is an $x_0 \in X$ satisfying

$$|\varphi_d^{-1}(x_0)| > 1$$

for all $d \in \mathcal{P}$. Furthermore there exist $u_0 \neq v_0$ such that for all $d \in \mathcal{P}$, $\varphi_d(u_0) = \varphi_d(v_0)$.

**Proof.** By Proposition 3 either all the maps $\varphi_d (d \in \mathcal{P})$ are homeomorphisms or none is. Let $\{d_k\}$ be a fundamental sequence, and for each $k$ let $x_k$ satisfy $|\varphi_{d_k}^{-1}(x_k)| > 1$. Taking a subsequence, we may assume $x_k \to x_0$. Now if for some $d \in \mathcal{P}$ we had that $|\varphi_d^{-1}(x_0)| = 1$, then by Corollary 1 there is a neighborhood $U$ of $x_0$ such that $|\varphi_d^{-1}(x)| = 1$ for all $x \in U$. Let $d_k$ be such that $d_k$ divides $d$, say $d = m d_k$, with $k$ sufficiently large so that $x_k \in U$. But then

$$|\varphi_d^{-1}(x_k)| \geq |\varphi_{d_k}^{-1}(x_k)| > 1.$$ 

Thus, $|\varphi_d^{-1}(x_0)| > 1$. This proves the first assertion.

Next, let $\{u_k\}, \{v_k\}$ be sequences such that $u_k \neq v_k$ and $\varphi_{d_k}(u_k) = \varphi_{d_k}(v_k) = x_0$.

By taking subsequences, we may assume that $u_k \to u_0$, $v_k \to v_0$.

Fix $d \in \mathcal{P}$ and write $d = d_k + \epsilon_k$, for $k \geq N$ for some $N \in \mathbb{N}$. Then

$$\varphi_d(u_k) = \varphi_{\epsilon_k} \circ \varphi_{d_k}(u_k) = \varphi_{\epsilon_k}(x_0).$$

Now by taking a subsequence we may assume $\varphi_{\epsilon_k}(x_0) \to y_0$, say. Since $u_k \to u_0$, $\varphi_{d}(u_k) \to \varphi_{d}(u_0)$. But by the above, $\varphi_{d}(u_k) \to y_0$. Thus $\varphi_{d}(u_0) = y_0$. Similarly, $\varphi_{d}(v_0) = y_0$.

Now if $u_0 = v_0$, then $\varphi_d$ is not injective in any neighborhood of $u_0$. Thus, $u_0 \neq v_0$. \qed
Thus we can state our

**Theorem 1.** Let $\mathcal{P}$ be a divisible semigroup of CSLI maps acting on a compact metric space $X$. Suppose $\mathcal{P}$ separates the points of $X$. Then $\mathcal{P}$ consists of homeomorphisms.

**Proof.** By Proposition 3, either $\mathcal{P}$ consists of homeomorphisms, or else of CSLI maps which are not homeomorphisms. Suppose the latter is the case. Then with $u_0, v_0$ as in Proposition 4, let $d \in \mathcal{P}$.

By the Proposition, $u_0, v_0$ map to the same point under $\varphi_d$, for all $d \in \mathcal{P}$. But that contradicts the assumption that $\mathcal{P}$ separates the points of $X$. Thus the CSLI maps must in fact be homeomorphisms. □

It is not *a priori* obvious that divisible semigroups of CSLI maps which are not homeomorphisms exist. Before constructing the example, we remind the reader of a construction which has been used to “cut up” the real numbers to obtain a zero-dimensional space, $Z$. For each dyadic rational $d \in \mathbb{R}$, we replace $d$ by two points $d^-$ and $d^+$ so that $d^- < d^+$ and no point lies between $d^-$, $d^+$. Thus $Z$ is an ordered set.

Now we introduce a topology by taking as a base $\mathcal{B}$ for the topology the sets $[r^+, s^-]$ where $r < s$ are dyadic rationals. In this topology, every “open interval” $(a, b) = \{x \in Z, a < x < b\}$ is an open set in the topology of $Z$.

Observe that the complement of an interval $[r^+, s^-]$ is also open, so that $[r^+, s^-]$ is closed, hence clopen. One can show that the closed intervals $[a, b], a < b \in Z,$ are compact. Thus $Z$ is a locally compact Hausdorff space, which is metrizable, as the base $\mathcal{B}$ is countable.

**Example 5.** This is an example of a divisible semigroup. We construct a compact metric space $X = X_1 \cup X_2 \cup X_3$, the union of three disjoint sets. Take

$$X_1 = [0^+, +\infty]$$

the one-point compactification of the interval $z \in Z: z \geq 0^+$. Now we take $X_2, X_3$ both to be the one-point compactifications of copies of $(-\infty, 0^-]$ in $Z$. To distinguish them, we use superscripts hat and tilde.

Thus,$$X_2 = [-\infty, \hat{0}^-] \text{ and } X_3 = [-\infty, \tilde{0}^-].$$

The set $Z$ is not a group under addition, but there is an action of the group $\mathcal{D}$ of dyadic rationals on $Z$, as follows: let $d \in \mathcal{D}$ and define translation $\varphi_d$ on $Z$ by

$$\varphi_d(x) = \begin{cases} 
  d + x & \text{if } x \text{ is not a dyadic rational;} \\
  (d + x)^+ & \text{if } x = r^+ \text{ where } r \text{ is a dyadic rational} \\
  (d + x)^- & \text{if } x = r^- \text{ where } r \text{ is a dyadic rational.}
\end{cases}$$
It is easy to see that \( \varphi_d \) is continuous, as \( \varphi_d^{-1} \) maps basic open intervals to basic open intervals. Similarly, \( \varphi_d \) is seen to be an open map. Since it is both injective and surjective, it is a homeomorphism of \( Z \).

Let \( P \) denote the positive dyadic rationals, and, changing notation, let \( \varphi_d \ (d \in P) \) denote an action of \( X \), which we define as follows: \( \varphi_d \) leaves all three points at infinity fixed. For \( x \in X_1 \cup X_2 \) not a point at infinity, we let \( \varphi_d(x) \) be defined as follows: if \( x \in X_2 \), \( x = \hat{z} \) for some \( z \in Z \), and \( d \in P \),

\[
\varphi_d(x) = \begin{cases} 
\hat{d} + z \in X_2 & \text{if } d + z \leq 0^- \\
\hat{d} + z \in X_1 & \text{if } d + z \geq 0^+.
\end{cases}
\]

If \( x \in X_1 \), then \( \varphi_d(x) \) is defined exactly as on \( Z \). \( \varphi_d \) acts similarly on \( X_1 \cup X_3 \). Clearly, \( \varphi_d \) is surjective on \( X \). And in the same way as with \( Z \), one sees that \( \varphi_d \) is continuous and open. Note that if \( x \in [-d^+, 0^-] \subset Z \) that \( \varphi_d(x) = \varphi_d(\hat{x}) \), so that \( \varphi_d \) is not one-to-one. Thus, \( \{ \varphi_d : d \in P \} \) is a semigroup of local homeomorphisms on the compact space \( X \) which are not homeomorphisms. Note that the semigroup \( P \) has a fundamental sequence, namely \( \{ \frac{1}{2^n} \}_{n \in \mathbb{N}} \), so that \( P \) is a divisible semigroup.

Next we show that the semigroup is admissible. To simplify notation, when \( x \in X \) belongs to either \( X_2 \) or \( X_3 \), and there is no need to distinguish between \( X_2 \), \( X_3 \), we will omit the superscripts hat and tilde. Define the cocycle \( \omega \) on \( P \times X \) by

\[
\omega(d, x) = \begin{cases} 
1, & \text{if } x \leq -d^+ \text{ or } x \geq 0^+ \\
\frac{1}{2}, & \text{if } -d^+ \leq x \leq 0^-.
\end{cases}
\]

(1)

We do this for all \( d \in P \) and \( x \in X \). We need to show this is consistent with the cocycle identity. So suppose \( e, f \in P \) with \( e + f = d \). Suppose \( x \leq -d^+ \). Then \( \omega(e, x) = 1 \). Now we claim that \( \varphi_e(x) \leq -f^- \). For otherwise, we would have \( \varphi_e(x) \geq -f^+ \), hence \( \varphi_f(\varphi_e(x)) \geq \varphi_f(-f^+) \geq 0^+ \). But then \( \varphi_d(x) = \varphi_f(\varphi_e(x)) \geq 0^+ \) which contradicts that \( x \leq -d^- \).

Thus, both \( \omega(e, x) \) and \( \omega(f, \varphi_e(x)) \) equal 1, as does \( \omega(d, x) \).

The case were \( x \geq 0^+ \) is easier, for then it is clear that \( \omega(e, x) \) and \( \omega(f, \varphi_e(x)) \), and \( \omega(d, x) \) are all equal to 1.

Finally, let \( -d^+ \leq x \leq 0^- \). Consider two cases: if \( \varphi_e(x) \geq 0^+ \), then \( |\varphi_e^{-1}(\varphi_e(x))| = 2 \), so that \( \omega(e, x) = \frac{1}{2} \). As \( \varphi_e(x) \geq 0^+ \), \( \omega(f, \varphi_e(x)) = 1 \).

By definition \( \omega(d, x) = \frac{1}{2} \), so that the equality

\[
\omega(x, d) = \omega(e, x)\omega(f, \varphi_e(x))
\]

holds. In the other case, \( \varphi_e(x) \leq 0^- \), that is,

\[-d^+ \leq x \leq \varphi_e(x) \leq 0^- \].
Then \( \omega(e, x) = 1 \). But \( \varphi_f(\varphi_e(x)) = \varphi_d(x) \geq 0^+ \) so that both \( \omega(f, \varphi_e(x)) \), and \( \omega(d, x) = \frac{1}{2} \). So again the cocycle identity
\[
\omega(d, x) = \omega(e, x)\omega(f, \varphi_e(x))
\]
holds.

Finally we need to observe that for arbitrary \( d \in \mathcal{P} \), the map
\[
x \to \omega(d, x)
\]
is continuous. But observe that
\[
\{x : \omega(d, x) = \frac{1}{2}\} = [-\hat{d}^+, 0^-] \cup [-\tilde{d}^+, \tilde{0}^-]
\]
which is a clopen set. Thus, the set where the cocycle is 1 is also clopen, and so the cocycle is continuous.

Remark 5. Proposition 4 shows that some features of Example 5 are not arbitrary. The role of the point \( x_0 \) in the Proposition is played by \( 0^+ \), and the roles of \( u_0, v_0 \) are played by \( \hat{0}^-, \tilde{0}^- \) in the example.

Remark 6. We do not know of an example of a divisible semigroup of CSLI maps where the maps are not local homeomorphisms.

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