Can Hawking temperatures be negative?

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Abstract

It has been widely believed that the Hawking temperature for a black hole is uniquely determined by its metric and positive. But, I argue that this might not be true in the recently discovered black holes which include the exotic black holes and the black holes in the three-dimensional higher curvature gravities. I argue that the Hawking temperatures, which are measured by the quantum fields in thermal equilibrium with the black holes, might not be the usual Hawking temperature but the new temperatures that have been proposed recently and can be negative. The associated new entropy formulae, which are defined by the first and second laws of thermodynamics, versus the black hole masses show some genuine effects of the black holes which do not occur in the spin systems. Some cosmological implications and physical origin of the discrepancy with the standard analysis are noted also.

PACS numbers: 04.70.Dy, 04.62.+v

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I. INTRODUCTION

A black hole is defined by the existence of the non-singular event horizon \( r_+ \), which is the boundary of the region of space-time which particles or photons can escape to infinity, classically. Bekenstein has shown that the black hole can be considered as a “closed” thermodynamical system with the temperature, proportional to the surface gravity \( \kappa_+ \), and the chemical potentials, proportional to the angular velocity \( \Omega_+ \) or electric potential \( \Phi_+ \), if there is, at the horizon \([1]\). The argument was based on the Hawking’s area (increasing) theorem \([2]\) and the black-hole analogue of the first law with the temperature \( T_+ \propto \kappa_+ \), which is “non-negative”, and the entropy \( S \propto A_+ \) for the horizon area \( A_+ \), which is “non-decreasing”, i.e., satisfying the second law of thermodynamics, due to the area theorem, as well as being non-negative. Later, Hawking found that the black hole can radiate, from the quantum mechanical effects, with the thermal temperature \( T_+ = \frac{\hbar \kappa_+}{2\pi} \) in accordance with the Bekenstein’s argument \([3]\) \( \text{[I am using units in which } c = k_B = 1] \); in this case, the black hole would not be a closed system anymore but interacting with its environments such as the generalized second law needs to be considered \([3, 4]\).

There is an alternative approach to compute the Hawking temperature by identifying \( \frac{\bar{\hbar}}{T_+} = 2\pi/\kappa_+ \) as the periodicity of the imaginary time coordinate which makes the metric regular at the horizon \([5]\) and this approach has been widely accepted; no counter examples for this approach have been known so far, as far as I know\(^1\). Now, since the surface gravity \( \kappa_+ \) at the horizon can be computed from the metric unambiguously, the Hawking temperature in this approach is uniquely determined also. This would be the origin of the widespread belief that the Hawking temperature be uniquely determined by the metric in any case. And also, it has been widely believed that the Hawking temperature be positive, as in the Bekenstein’s original argument \([1]\). Actually, this belief has been closely related to the “positive mass theorems” for black holes and the fact that the mass is grater than the modulus of the charge, if there is \([7]\).

In this Letter, I argue that this belief might not be true in the recently discovered black holes which include the exotic black holes and the black holes in the three-dimensional higher curvature gravities.

II. NEW HAWKING TEMPERATURES FROM THERMODYNAMICS

In the spin systems the temperature can be negative, due to the upper bound of the energy spectrum \([8]\). Recently, a number of black hole solutions which have similar upper bounds of the black hole masses have been discovered \([9, 10, 11, 12, 13]\). I have argued that the Hawking temperatures for these systems might not be given by the usual formula \( T_+ = \frac{\hbar \kappa_+}{2\pi} \) \([9, 10, 11]\), which is non-negative, but by new formulae which can be negative depending on the situations \([12, 13]\). The argument was based on the Hawking’s area theorem and the second law. This has been found to agree completely with CFT analysis, being related to the AdS/CFT correspondence, as far as the CFT analysis is available \([12, 13]\). In this section let me briefly introduce the black hole solutions and the thermodynamical arguments for the new Hawking temperatures which differ from the usual formula and can

\(^1\) Recently, it has been found that this approach does not work anymore in the smeared black holes in the quantum spacetime \([6]\).
be negative.

A. The exotic BTZ black holes

An exotic BTZ black hole is characterized by the following properties

a. The metric is formally the same as the BTZ black hole solution \([9, 11, 12, 13]\), which is given by \([14]\),

\[
\begin{align*}
    ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2 \\
    N^2 &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^\phi = \frac{-r_+ r_-}{lr^2},
\end{align*}
\]

or modulus a 2-sphere \([10]\). Here, \(r_+\) and \(r_-\) denote the outer and inner horizons, respectively.

b. The mass and angular momentum, computed from the standard Hamiltonian approach, are completely interchanged from the “bare” ones \(m\) and \(j\) as

\[
M = xj/l, \quad J = xlm
\]

with an appropriate coefficient \(x\); \(x = 1\) in Ref. \([9]\), \(x\) is a fixed value of \(U(1)\) field strength in Ref. \([10]\), and \(x\) is proportional to the coefficient of a gravitational Chern-Simons term in Refs. \([11, 12]\). Here, \(m\) and \(j\) are given by

\[
m = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad j = \frac{2r_+ r_-}{8Gl},
\]

which become the usual mass and angular momentum for the BTZ black hole, with a cosmological constant \(\Lambda = -1/l^2\), respectively \([10]\). The radii of the horizons are given by, in terms of \(m\) and \(j\),

\[
r_\pm = l\sqrt{4Gm \left[ 1 \pm \sqrt{1 - (j/ml)^2} \right]}
\]

and it is clear, from this, that the bare parameters, which are positive semi-definite, satisfy an inequality

\[
m \geq j/l
\]

But, there is a qualitative difference from the BTZ solution for Einstein gravity. This comes from the fact that the parameter \(l\) appears just as an integration constant in the black hole solution in Refs. \([9, 11, 12]\), though it is a parameter appearing in the action for Einstein gravity. Actually, the suitable distribution of matter replaces the cosmological constant in pure gravity for the case of Ref. \([9]\).

\[\beta\]

It has been claimed that this system, which goes beyond the physical bound to the coefficient of the Chern-Simons term, is not well defined \([11]\). But, the argument has been based on the results which are valid only for \(|\beta|/l < 1\) and it does not apply to our case \([12]\).
in order that the horizons exist (the equality for the extremal black hole with \( r_+ = r_- \)).

The remarkable result of (3) is that

\[
M^2 - J^2/l^2 = x^2[\frac{J^2}{l^2} - (m)^2] \leq 0
\]

for any non-vanishing \( x \), which shows an upper bound for the mass squared \( M^2 \) and a saturation for the extremal bare parameters, i.e., \( m = j/l \).

Now, given the Hawking temperature and angular velocity for the event horizon \( r_+ \) of the metric (1), following the usual approach \( [5] \),

\[
T_+ = \frac{\hbar \kappa}{2\pi} \bigg|_{r_+} = \frac{\hbar (r_+^2 - r_-^2)}{2\pi l^2 r_+}, \quad \Omega_+ = -\frac{N\phi}{l} \bigg|_{r_+} = \frac{r_-}{l r_+}
\]

with the surface gravity \( \kappa = \partial N^2 / \partial r \), the black hole entropy has been identified as

\[
S = x^2 \frac{2\pi r_-}{4G\hbar},
\]

which satisfies the first law

\[
\delta M = \Omega_+ \delta J + T_+ \delta S
\]

but depends on the inner horizon area \( A_- = 2\pi r_- \) \( [9, 10, 11] \), rather than the outer horizon’s \( A_+ = 2\pi r_+ \). But, there is no physical justification for this since the second law is not guaranteed \( [12, 13] \) (for an explicit demonstration, see Ref. \( [15] \)). Rather, I have recently proposed another entropy formula which does not have this problem

\[
S_{\text{new}} = |x|^2 \frac{2\pi r_+}{4G\hbar},
\]

in accordance with the Bekenstein’s original proposal \( [1] \). Then, it is quite easy to see that this does satisfy the second law since the metric (1) satisfies the Einstein equation in vacuum, regardless of the details of the actions,

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{l^2} g^{\mu\nu} = 0
\]

: The Raychaudhuri’s equation gives the Hawking’s area theorem for the outer horizon \( \delta A_+ \geq 0 \), i.e., \( \delta S_{\text{new}} \geq 0 \) since this vacuum equation satisfies the null energy condition trivially \( [2, 3, 4] \); this can be also proved by considering a “quasi-stationary” process which does not depend on the details of the gravity theory \( [15, 16] \). These results are closely related to the fact that \( dr_+/dm > 0, \ dr_-/dm \leq 0 \) for any (positive) \( m \) and \( j \) (equality for \( j = 0 \) ) since these describe the rates of the area changes under the positive energy matter accretion.

One interesting consequence of the new identification (11) is that I need to consider the rather unusual Hawking temperature and angular velocity ( \( \epsilon \equiv \text{sign}(x) \))

\[
T_- = \epsilon \frac{\hbar \kappa}{2\pi} \bigg|_{r_-} = \epsilon \frac{\hbar (r_+^2 - r_-^2)}{2\pi l^2 r_-}, \quad \Omega_- = -\frac{N\phi}{l} \bigg|_{r_-} = \frac{r_+}{l r_-},
\]
FIG. 1: The normalized entropies $S_{\text{new}}(\frac{|x|2\pi l}{4Gh})^{-1}$ vs. $M$ for various values of $|J|/l=1$ (red), 1.5 (green), 2 (blue) [bottom to top] ($l = G = |x| = 1$).

respectively $\frac{1}{2}, 1, \frac{1}{2}$, such as the first law, as well as the manifest second law, be satisfied also

$$\delta M = \Omega_+ \delta J + T_- \delta S_{\text{new}}.$$  \hspace{1cm} (14)

Here, I note that, with these correct values of $M$, $J$, and the entropy (11), which is proportional to the outer horizon area, there is no other choice in the temperature and angular velocity in the first law. The (positive) numerical coefficient in the temperature $T_-$ of (13) is not determined from the thermodynamical arguments but needs some other independent identifications: This has been confirmed indirectly in a CFT analysis in Refs. [12, 13]; however, in this Letter I support this, in a more traditional way, by identifying the Hawking temperature directly from the Green function analysis for a quantum field. But, it is important to note that, regardless of the numerical ambiguity, the temperature $T_-$ becomes "negative" always for $x > 0$. This can be easily understood from the existence of the upper bound of mass $M \leq J/l$ with positive $M$ and $J$, as in the spin systems $\frac{5}{2}$. Whereas, the temperature becomes positive for $x < 0$ due to the lack of an upper bound, i.e., $J/l \leq M$

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$^4$ This does not mean, of course, that one needs an observer sitting on the inner horizon $r_-$ to measure $T_-$ and $\Omega_-$, as it does not for $T_+, \Omega_+$.

$^5$ The mass bound and its resulting negative temperature might be related to the semiclassical instability that has been found, recently $\frac{17}{17}$. Actually, if one applies the first and second laws as in this Letter, the system of Ref. $\frac{17}{17}$ has a negative temperature also due to the negative mass, though not well explored in detail in the literatures $\frac{18}{18}$. So, I can suspect that the negative temperature might be a signal of the
with negative $M$ and $J$. These behaviors can be nicely captured in the entropy, as a function of $M$ and $J$ (Fig.1), using (3) and (5):

$$S_{\text{new}} = |x| \frac{2\pi l}{4G\hbar} \sqrt{(4GJ/xl)} \left[ 1 + \sqrt{1 - (Ml/J)^2} \right]. \quad (15)$$

Here, I note that the curves in Fig.1 are symmetric about $M = 0$, as in the spin systems: By the definition of the temperature $1/T = (\partial S/\partial M)_J$, I have $T_+ < 0$ on the right hand side ($x > 0$), whereas $T_- > 0$ on the left hand side ($x < 0$); the two temperatures $T_\pm = \pm \infty$ correspond to the same temperature for a vacuum with $M = 0$. But, note also that the entropy does not vanish at the energy boundary $M = J/l$, i.e., extremal black hole, and this would be inherent to black hole systems, which does not occur in spin systems [19].

It is also important to note the fact, which is crucial in the analysis of Sec. III, that the angular velocity has a lower bound $\Omega^- \geq 1/l$, due to the fact of $r_+ \geq r_-$; it is saturated by the extremal case $r_+ = r_-$ and divergent in the limit of $r_- \to 0$. This implies that this system is always rotating, as far as there is the event horizon $r_+$. And also, as $r_- \to 0$, this seems to be consistent with the fact of a non-vanishing angular momentum $J$ since it satisfies the conventional relation $J \propto \Omega M$, with the angular velocity $\Omega = \Omega^-$. 

B. The BTZ black hole with higher curvatures

The (2+1)-dimensional gravity with the higher curvature terms and a “bare” cosmological constant $\Lambda = -1/l^2$ can be generally described by the action [ omitting some boundary terms ]

$$I_g = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} \left( f(g^{\mu\nu}, R_{\mu\nu}, \nabla_{\mu}) + \frac{2}{l^2} \right), \quad (16)$$

where $f(g^{\mu\nu}, R_{\mu\nu}, \nabla_{\mu})$ is an arbitrary scalar function constructed from the metric $g^{\mu\nu}$, Ricci tensor $R_{\mu\nu}$, and the covariant derivatives $\nabla_{\mu}$ [20, 21]. The equations of motion are

$$\frac{\partial f}{\partial g_{\mu\nu}} - \frac{1}{2} g^{\mu\nu} f - \frac{1}{l^2} g_{\mu\nu} = t^{\mu\nu}, \quad (17)$$

where $t^{\mu\nu}$ is given by

$$t^{\mu\nu} = \frac{1}{2} (\nabla^\nu \nabla^\alpha P_\alpha^{\mu} + \nabla^\mu \nabla^\alpha P_\alpha^{\nu} - \Box P^{\mu\nu} - g^{\mu\nu} \nabla^\alpha \nabla^\beta P_{\alpha\beta} ) \quad (18)$$

with $P_{\alpha\beta} \equiv g_{\alpha\mu} g_{\beta\nu} (\partial f/\partial R_{\mu\nu})$.

In the absence of the higher curvature terms, the BTZ solution [14] is the unique black hole solution in vacuum. Whereas, even in the presence of the generic higher curvature terms, the BTZ solution can be still a solution since the local structure would be “unchanged” by the higher curvatures: Actually $t_{\mu\nu} = 0$ for the BTZ solution and the only effects are some

same instability as in Ref. [17] and the negative temperature spin systems in the ordinary surroundings with positive temperature [8]. But the detailed analysis may be beyond the scope of this Letter. I thank the referee for drawing my attention to this interesting problem.
“re-normalization” of the bare parameters \(l, r_{\pm}\), and the Newton’s constant \(G\), giving the Einstein equation

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{l_{\text{ren}}^2} g^{\mu\nu} = 0
\]  

(19)

in the renormalized frame \[13, 24\]. The renormalized cosmological constant \(\Lambda_{\text{ren}} = -1/l_{\text{ren}}^2\) depends on the details of the function \(f\), but the renormalized Newton’s constant\(^6\) is given by

\[
G_{\text{ren}} = \hat{\Omega}^{-1} G
\]

(20)

with

\[
\hat{\Omega} \equiv \frac{1}{3} g_{\mu\nu} \frac{\partial f}{\partial R^{\mu\nu}}.
\]

(21)

which is constant for any constant-curvature solution \[21\]. Now, due to the renormalization of the Newton’s constant, the original mass and angular momentum in (4) are modified as

\[
M = \hat{\Omega} m, \quad J = \hat{\Omega} j,
\]

(22)

respectively, by representing \(m\) and \(j\) as those in the renormalized frame \(m = \frac{r^2 + r_{\pm}^2}{8G\hat{\Omega}_{\text{ren}}}\), \(j = \frac{2r^2 + r_{\pm}^2}{8G\hat{\Omega}_{\text{ren}}}\), with the renormalized parameters \(l_{\text{ren}}, r_{\pm}\), but still with the bare Newton’s constant \(G\), such as \(m \geq j/l_{\text{ren}}\) is valid. Here, it is important to note that \(\hat{\Omega}\) is not positive definite \(^7\), such as the usual inequality for the mass and angular momentum would not be valid in general,

\[
M - J/l = \hat{\Omega}(m - j/l)
\]

(23)

but depends on the sign of \(\hat{\Omega}\): \(M \geq J/l\) for \(\hat{\Omega} > 0\), but \(M \leq J/l\) for \(\hat{\Omega} < 0\).

Regarding the black hole entropy, it has been computed as

\[
S_W = \hat{\Omega} \frac{2\pi r_{+}}{4G\hbar}
\]

(24)

from the Wald’s entropy formula \[20, 21, 24\]. But, this is problematic for \(\hat{\Omega} < 0\), though it satisfies the usual first law \[10\], since \(\delta S_W \leq 0\) from the area theorem which works in this case also due to the above Einstein equation \[19\] that satisfies the null energy condition in the renormalized frame also, trivially. So, I have recently proposed the modified entropy

\[
S_W' = |\hat{\Omega}| \frac{2\pi r_{+}}{4G\hbar},
\]

(25)

\(^6\) Recently, this idea has been generalized to more general class of black holes, including supergravity black holes \[22\].

\(^7\) This means a negative Newton’s constant, but there is no a priori reason to fix the sign in three dimensional gravities \[23\]. This does not affect the cosmic censorship condition in the Einstein frame either since the (three-dimensional) frame transformation \(g_{\mu\nu} = \hat{\Omega}^2 g_{\mu\nu}\) is insensitive to the sign of \(\hat{\Omega}\) \[10, 21\].
FIG. 2: The normalized entropies $S_{W^r}(|\hat{\Omega}|2\pi l_{ren}/4G\hbar)^{-1}$ vs. $M$ for various values of $|J|/l=0$ (grey), 1 (red), 1.5 (green), 2 (blue) [top to bottom] ($l_{ren} = G = |\hat{\Omega}| = 1$).

which agrees with CFT result as well [12, 13]. Then, I need to consider the modified temperature $T_+^\prime = \text{sign}(\hat{\Omega}) T_+$ in order to satisfy the first law

$$\delta M = \Omega_+ \delta J + T_+^\prime \delta S_{W^r}. \quad (26)$$

The negative temperature $T_+^\prime$ for $\hat{\Omega} < 0$ is consistent with the upper bound of mass $M \leq J/l$. The whole behaviors of the temperature can be easily captured in the entropy, as a function of $M$ and $J$ (Fig.2), using (5) and (22):

$$S_{W^r} = |\hat{\Omega}| \frac{2\pi l_{ren}}{4G\hbar} \sqrt{(4GM/\hat{\Omega}) \left[ 1 + \sqrt{1 - (J/Ml_{ren})^2} \right]} . \quad (27)$$

As can be observed in Fig.2, this system provides an unusual realization of the negative temperature, which does not occur in the usual spin systems: For $J \neq 0$, the two branches ($M > 0$ and $M < 0$) are disjointed, due to a gap in the mass spectrum; the left branch ($M < 0$) has an upper bound $M < J/l$ and negative temperature $T_+^\prime = -T_+$, whereas the right branch ($M > 0$) has no upper bound and so has the usual positive temperature $T_+$, following the usual definition. According to the statistical mechanics, the gap is natural because negative temperature is hotter than positive one with the inequality $T = 0_+ < T = \infty_+ < T = \infty_- < T = 0_-$: The left edge of $M > 0$ curve, which has $T = 0_+$, can not be smoothly connected to the right edge of $M < 0$ curve, which has $T = 0_-$; rather, the infinite right edge, which has $T = \infty_+$, may be connected to the infinite left edge, which has $T = \infty_-$. In this context, the $J = 0$ cases whose curves meet at $M = 0$ does not seem
to imply the un-bounded mass. Actually, the $M = 0$ case can not be considered as the black hole spectrum because there is no horizon either; there is a discontinuity in the mass spectrum at $M = 0$, which may be considered as an (open) upper bound for the left branch.

## III. HAWKING TEMPERATURES FROM THE GREEN FUNCTIONS

The Hawking temperature can be fundamentally determined by the periodicity of the thermal Green functions [26]. In the usual black hole systems this agrees with the periodicity for a regular Euclideanized metric at the event horizon $r_+$. Actually, the Hawking temperature for the BTZ metric has been determined in this way and found to be the same as $T_+$ of (8) [27]. So, according to the widespread belief that Hawking temperature be uniquely determined by the metric, the new Hawking temperatures which do not agree with the usual temperature might be considered as unphysical ones. But in this section I argue that this might not be true in general, like as in the systems that I have introduced in Sec. II: There were some “loopholes” in the usual analyses which were unimportant for the ordinary black holes. The results are consistent with the proposals of Sec. II.

To this end, I first note that the Hartle-Hawking Green function for a scalar field in the background metric (1) is given by [ I follow the approach of Ichinose-Satoh in Ref. [27] ]

$$-iG_{BH}(x, x') = \hbar (4\pi l)^{-1} \sum_{n=-\infty}^{\infty} (z_n^2 - 1)^{-1/2} [z_n + (z_n^2 - 1)^{1/2}]^{1-\lambda}, \quad (28)$$

where $x, x'$ are the points in the four dimensional embedding space and

$$z_n(x, x') - i\varepsilon = d_H^2 \left[ \sqrt{r^2 - r_+^2} \sqrt{r_-^2 - r_+^2} \cosh \left( r_- l^2 \Delta t - r_+ l^{-1} \Delta \phi \right) \right. \left. - \sqrt{r^2 - r_+^2} \sqrt{r_-^2 - r_+^2} \cosh \left( r_+ l^{-2} \Delta t - r_- l^{-1} \Delta \phi \right) \right], \quad (29)$$

with $d_H^2 = r_+^2 - r_0^2$, $\Delta t = t - t'$, $\Delta \phi_n = \phi - \phi' + 2n\pi$, and an infinitesimal positive imaginary part $i\varepsilon$ [ the number $\lambda$ is a positive number which depends on the scalar field’s mass and its coupling to the metric [27] ]. Here, it is important to note that $z_n$, and so $G_{BH}$, is symmetric under $r_+ \leftrightarrow r_-$ interchange; this would be a natural consequence of the symmetry in the metric (1) itself. Then, the Green function on the Euclidean black hole geometry with the Euclidean time $\tau = it$ and the “Euclidean” angle $\varphi = -i\phi$ for $r_- \neq 0$ is

$$G_{BH}^{Eucl}(\Delta \tau, \Delta \varphi; r, r') = iG_{BH}(\Delta t, \Delta \phi; r, r') \bigg|_{\Delta \tau = -i\Delta t, \Delta \varphi = -i\Delta \phi}. \quad (30)$$

The temperature, now, would be determined by comparing with the thermal Green function at temperature $\beta^{-1}$ and with a chemical potential $\Omega$ conjugate to angular momentum [ $T$ denotes the Euclidean time ordered product for scalar fields $\psi(x)$, and $\hat{H}$ and $\hat{J}$ are the generators of time translation and rotation, respectively ],

$$G_{\beta}^{Eucl}(x, x' ; \Omega) = \text{tr} \left[ e^{-\beta(\hat{H} - \Omega \hat{J})} T(\psi(x)\psi(x')) \right] / \text{tr} \left[ e^{-\beta(\hat{H} - \Omega \hat{J})} \right], \quad (31)$$

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8 For the system of Sec. III, the renormalized parameters, $l_{ren}, r_\pm$, are considered, instead.

9 The extra coordinates are frozen for the system of Ref. [10].
which has the following periodicity:

\[ G^{Eul}_\beta(\tau, \varphi, r, \tau', \varphi', r'; \Omega) = G^{Eul}_\beta(\tau + \beta h, \varphi - \Omega \beta h, r, \tau', \varphi', r'; \Omega). \]  

(32)

Because the Green function \( G_{BH} \) is a function of \( z_n \), one can find, from (29), that \( G^{Eul}_{BH} \) is periodic under the variation, with \((m, n \in \mathbb{Z})\),

\[ \delta(\tau/l) = 2\pi ld_H^2(-r_- m + r_+ n), \quad \delta(\varphi) = 2\pi ld_H^2(r_+ m - r_- n). \]  

(33)

If one requires that, as \( r_- \to 0 \), the chemical potential \( \Omega \), which being the angular velocity in a rotating black hole, vanishes, the fundamental period is determined uniquely as

\[ \tau \to \tau + 2\pi k_+^{-1} n, \quad \varphi \to \varphi - 2\pi k_+^{-1} \Omega_+ n \]  

(34)

with the angular velocity \( \Omega_+ \) and the temperature \( \beta^{-1} = \hbar k_+ / 2\pi \) as in (8); this is the usual result [27]. But, this does not apply to the exotic systems of Sec. IIA: The chemical potential \( \Omega_- \), which is defined basically by the first law (14) or (26) and also by the correct form of entropy (11) or (25), respectively, which would respect the second law, does not vanish as \( r_- \to 0 \) but actually it has a “lower” bound \( \Omega_- \geq 1/l \) from (13) [12]. Now, for the generality, let me just assume the existence of the lower bound only, regardless of its details. Then, it is easy to see that the fundamental period may be determined “uniquely” as

\[ \tau \to \tau + 2\pi k_-^{-1} m, \quad \varphi \to \varphi - 2\pi k_-^{-1} \Omega_- m, \]  

(35)

giving the angular velocity \( \Omega_- \) and the Hawking temperature \( \beta^{-1} = \hbar k_- / 2\pi \) as in (13), for \( x > 0 \). For \( x < 0 \), on the other hand, the positive temperature \( \beta^{-1} = -\hbar k_- / 2\pi \) may be also determined by considering \((\hat{H}, \hat{J}, \beta) \to (-\hat{H}, -\hat{J}, -\beta)\), in accordance with the negative \( M \) and \( J \), from (3). For the system of Sec. IIB with \( \hat{\Omega} < 0 \), in which the angular velocity \( \Omega_+ \) vanishes as \( r_- \to 0 \) though, the temperature may be determined as \( \beta^{-1} = -\hbar k_+ / 2\pi \), which being negative, with the ordinary angular velocity \( \Omega_+ \) as in (26), as well as the usual temperature \( \beta^{-1} = \hbar k_+ / 2\pi \) for \( \hat{\Omega} > 0 \). These results are consistent with the proposals of Sec. II and agree completely with CFT analyses [12, 13], as far as CFT analysis is available.

These systems also suggest that the temperature might not be uniquely determined by the metric.

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10 In the case of the BTZ black holes with a gravitational Chern-Simons term, the boundary stress tensor is different from the standard one, i.e., \( \tau_{ij} = c^2(2\beta_{kl}/l^2 \sqrt{\gamma(0)})(e^{jk\gamma(2)}_k + (i \leftrightarrow j)) \) (see Ref. [28] for the details) and this produces the mass and angular momentum [3] with \( x = 32\pi G \beta_{KL}/l \) in the standard definitions \( M = l \oint d\phi \tau_{tt}, \quad J = -l \oint d\phi \tau_{t\varphi} \). However, if we “naïvely” compute the corresponding CFT operators, in the standard definition, one finds \( L^{\pm}_0 - c^2/24 = (3M \pm J)/2 = \pm x(l \pm j)/2 \) with the central charge \( c^\pm = x(3l/2G) \) and the conformal weight \( L^\pm_0 \). This representation of the CFT can have negative central charge \( c \) in the standard definition as well defined also, by considering another representation of the Virasoro algebra with \( \hat{L}_n = -L_{-n}, \hat{c} = -c \). But, it is known to be well defined also, by considering another representation of the Virasoro algebra with \( \hat{L}_n = -L_{-n}, \hat{c} = -c \). This twist is not obvious in the stress tensor, but this is needed in order to have a well-defined Hilbert space without the negative-norm states. This looks to be also true in the higher curvature gravities though the explicit form of the boundary stress tensor is not known [12, 21]. But, the corresponding explanations in other cases are not available at present due to lack of the knowledge of the boundary CFTs.
IV. CONCLUDING REMARKS

So far, I have considered the cases which are described by the three-dimensional metric (1), up to extra sphere parts. But, there are also several other higher-dimensional black hole systems which show negative Hawking temperatures, though not well recognized in the literatures. The $AdS$ black holes in higher derivative gravities [29] and the phantom (haired) black holes [30] are the examples (see Ref. [15] for the details). The implications of these black holes to the evolution of the Universe filled with the phantom energy would be quite interesting: If I consider the accretion of the phantom energy onto a black hole with “negative” Hawking temperature, the black hole size increase [31], as in the wormhole cases [32] but in contrast to the ordinary black holes with positive Hawking temperatures [33], until a thermal equilibrium with an equilibrium temperature is reached. This equilibrium may be actually possible and can occur before the catastrophic situations in Ref. [32] if the phantom energy has the negative temperature as claimed in Ref. [31]. Furthermore, the generalized second law of the phantom Universe with a black hole can be satisfied also with the negative Hawking temperature [31]. The details will appear elsewhere.

Finally, I would like to note some physical origin of the discrepancy with the standard analysis as in Hawking’s original work [3] which yields the Hawking temperature $T_+$ independently of any details of the gravitational theory or assumptions about the first law holding. To this end, I first note that standard result is true only for the “Riemannian” or its equivalent “dual” in the teleparallel gravity [34]. Otherwise, the particle’s trajectory is “not” completely determined by the metric only, and actually this seems to be the case for the models of this paper. But, the details look different, depending on the models. (i). For example, for the model of Ref. [9], which is a teleparallel gravity with a vanishing curvature (and cosmological constant), the torsion is “not totally antisymmetric” (in other words, the quantity called “contortion” exits) such as this is not equivalent to the “Riemannian” geometry [35]. And, a non-vanishing angular velocity $\Omega_-$, which was crucial in the analysis of Sec. III, as well as a non-vanishing angular momentum $J$, as $r_- \to 0$, would be the result of the torsion, though the detailed relation is not explored here. (ii). For the model of Refs. [11, 12], a particle (or particle-like solution) would also undergo the mass/angular momentum interchange as in the black hole case (3) since one can “not” distinguish, basically, the black hole solution from the point particle solution, though its explicit form is not known, at the asymptotic infinity wherein the conserved ADM mass and angular momentum are computed [36]: It would behave as a “gravitational anyon”, similar to Deser’s for the asymptotically flat space [37]. And also, it seems that there is an intimate relation between the torsion (or contortion, more exactly) in Ref. [9] and the gravitational anyon in Refs. [11, 12], due to the relation between the torsion and spin [38]. (iii). For the model of Ref. [10], the explanation is not yet quite clear, but I suspect that the constant $U(1)$ flux on the 2-sphere would have a crucial role in the non-standard behavior of Hawking radiation also.

As an alternative aspect of the peculiarity of Hawking radiation for our models, I would like to note also that dynamical geometry responds differently under the emission of Hawking radiation, as I have emphasized in Ref. [12], recently. For example, the emission of energy $\omega$ would reduce the black holes’ mass $M$ from the conservation of energy, but this corresponds to the change of the angular momentum $j$ in the ordinary BTZ black hole context, due to the interchange of the roles of the mass and angular momentum as in (3). This is in sharp contrast to the case of ordinary black hole. This seems to be also a key point to understand the peculiar Hawking radiation in our system, and in this argument the conservations of
energy and angular momentum, which are not well enforced in the standard computation, have a crucial role. In this respect, the Parikh and Wilczek’s approach \cite{39}, which provides a direct derivation of Hawking radiation as a quantum tunneling by considering the global conservation law naturally, would be an appropriate framework to study the problem.

Finally, regarding the microscopic origin, it seems that the existence of the negative temperature might imply the *spin network model* of the quantum black holes \cite{40}, analogous to the ordinary spin systems which can have negative temperature. But it is not clear in that context how the negative temperature is activated in our exotic examples but not in the ordinary black holes.

**Acknowledgments**

I would like to thank Gungwon Kang and Segrei Odintsov for useful correspondences. This work was supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number R11 - 2005- 021.

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