Complete classification of purely magnetic, non-rotating and non-accelerating perfect fluids.

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Recently the class of purely magnetic non-rotating dust spacetimes has been shown to be empty (Wylleman, Class. Quant. Grav. 23, 2727). It turns out that purely magnetic rotating dust models are subject to severe integrability conditions as well. One of the consequences of the present paper is that also rotating dust cannot be purely magnetic when it is of Petrov type D or when it has a vanishing spatial gradient of the energy density. For purely magnetic and non-rotating perfect fluids on the other hand, which have been fully classified earlier for Petrov type D (Lozanovski, Class. Quant. Grav. 19, 6377), the fluid is shown to be non-accelerating if and only if the spatial density gradient vanishes. Under these conditions, a new and algebraically general solution is found, which is unique up to a constant rescaling, which is spatially homogeneous of Bianchi type I(3)∞ in the extended Arianrhod-McIntosh classification.

The metric and the equation of state are explicitly constructed and properties of the model are briefly discussed. We finally situate it within the class of normal geodesic flows with degenerate shear tensor.

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1. INTRODUCTION

For a given spacetime geometry, the electric and magnetic parts of the Weyl tensor $C_{abcd}$ w.r.t. some unit timelike congruence $u^a$ are pointwise defined by

$$E_{ab} = C_{abcd} u^c u^d,$$

$$H_{ab} = \frac{1}{2} \eta^{ac} \eta^{bd} C_{mnbd} u^c u^d,$$ (2)

$\eta_{abcd}$ being the spacetime permutation tensor. $E_{ab}$ and $H_{ab}$ are traceless and symmetric tensors satisfying $H_{ab} u^b = E_{ab} u^b = 0,$ and determine the Weyl tensor completely [1, 2, 3]. They were first introduced (for the vacuum Riemann tensor) by Matte [1] when searching for gravitational quantities playing an analogous role to the electric and magnetic field in classical electromagnetism. Using the decomposition [1], the Bianchi identities take a form analogous to Maxwell’s equations for the electromagnetic field [4]. A non-conformally flat spacetime for which $E_{ab}$, resp. $H_{ab}$, vanish identically w.r.t. some $u^a_0$ has therefore been called purely magnetic (PM), resp. purely electric (PE), and its Weyl tensor is said to be PM, resp. PE, w.r.t. $u^a_0$. As $E_{ab}$ and $H_{ab}$ (w.r.t. any $u^a$) are diagonalizable tensors, the Petrov type of PE or PM spacetimes is necessarily I or D, and in each point $u^a_0$ is a Weyl principal vector (which is essentially unique for Petrov type I, and which is an arbitrary timelike vector in the plane of repeated principal null directions for Petrov type D) [5, 6]. The PM and PE property may be characterized independently of $u^a$,

as follows. Defining

$$Q_{ab} = E_{ab} + i H_{ab},$$ (3)

the quadratic, cubic and 0-dimensional invariants $I, J$ and $M$ of the Weyl tensor [7, 8] can be written as [1, 9]

$$I = Q^a_b Q^b_a = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = -2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1)$$

$$= (E^a_b E^b_a - H^a_b H^b_a) + 2i E^a_b H^b_a,$$ (4)

$$J = Q^a_b Q^c_a Q^d_b Q^d_c = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3 \lambda_1 \lambda_2 \lambda_3$$

$$= E^a_b E^b_c E^c_a - 3 E^a_b H^b_c H^c_a$$

$$- i (H^a_b H^b_c H^c_a - 3 E^a_b E^b_c H^c_a),$$ (5)

$$M = I^3 - J^2 - 6 = \frac{2(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2}{9 \lambda_1^2 \lambda_2^2 \lambda_3^2},$$ (6)

where the solutions $\lambda_i$ of $\lambda^3 - \frac{2}{3} \lambda - \frac{1}{3} J = 0$ are the non-zero eigenvalues of $Q_{ab}$ for any $u^a$. Hence, a spacetime is PE (PM) if and only if all $\lambda_i$ are real (imaginary), or $\overline{2}$ if and only if $M$ is real non-negative or infinite and $I$ is real positive (negative). For PE or PM spacetimes, $M = 0$ if and only if the Petrov type is D; the respective types in the algebraically general case were logically denoted $I(M^+) \text{ and } I(M^{\infty})$ in the extended Petrov classification of Arianrhod and McIntosh [10], where $I(M^{\infty})$ corresponds to $J = 0$, i.e. to one of the $\lambda_i$ being identically zero.

Whereas large and physically important classes of examples exist for PE spacetimes (for example all the static spacetimes are purely electric [11]), only a few examples exist of the purely magnetic ones. This is
particularly true for the vacuum solutions (with or without \(\Lambda\) term), where no PM solutions are known at all. This has lead to the conjecture that PM vacua do not exist \(\mathbb{R}\), but so far this has only been proved for Petrov types D \(\mathbb{R}^2\) and I\((M^\infty, 1)\) \([11]\) or when the timelike congruence \(u^a\) is shear-free \([12]\), non-rotating \([13, 14]\), geodesic \([15]\), or satisfies certain technical generalizations of these conditions \([16, 17]\). In \([18, 19]\) the non-existence of shear-free or non-rotating PM models was generalized to spacetimes with a vanishing Cotton tensor. As a positive example on the other hand, the metric constructed in \([20]\) turns out to be a PM kinematic counterpart to the Gœdel metric \([21]\), but its source is unphysical in \([20]\).

In \([22]\) was determined and the most general metric forms were found, exhibiting one arbitrary function and three parameters.

In a cosmological context, perfect fluid models are studied, the metric \(g_{ab}\) being a solution of the Einstein field equation with perfect fluid source term

\[
R_{ab} - \frac{1}{2}R g_{ab} = (\mu + p)u_a u_b + pg_{ab}. \tag{7}
\]

Here \(R_{ab}\) is the Ricci tensor, which is assumed not to be of \(\Lambda\)-type (\(\mu + p \neq 0\)). In this case, the average 4-velocity field \(u^a = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^a}\) of the fluid plays the role of a geometrically preferred unit timelike vector field on the spacetime. A possible cosmological constant \(\Lambda\) has been absorbed in the fluid’s energy density and pressure by means of re-defining \(\mu = \mu' + \Lambda\) and \(p = p' - \Lambda\). The electric and magnetic parts \([11]\) w.r.t. \(u_{\mu'}\) then represent the locally free gravitational field (not pointwise determined by the matter content via the Einstein equations \(\mathbb{R}\)), while the vanishing of their spatial divergence together with the non-vanishing of the so called ‘curl’ and ‘distortion’ parts of their spatial derivatives form necessary conditions for the existence of gravitational wave perturbations of the homogeneous and isotropic FRW spacetime \([24, 25, 26, 27]\). Whereas the electric part \(E_{ab}\) is the general relativistic generalization of the tidal tensor in Newtonian theory \([28]\), the magnetic part \(H_{ab}\) has no Newtonian analogue and it’s role is still poorly understood (see however \([29, 30, 31]\)).

This motivates the attempt to construct purely magnetic perfect fluid models with \(u_0^a = u_{\mu}' e^a\) in the above. A solution to \(\mathbb{R}\) of this kind will then be called a PMpf. The only known perfect fluid which is purely magnetic in a certain region of spacetime, but which is not a PMpf in the above sense (\(u_0^a \neq u_{\mu}' e^a\)), is the Lozanovskii-McIntosh metric \([32]\).

The present study is restricted to PMpf models, with the exception of section 4.4. First however we conclude this introduction with an overview of all performed studies and known examples of PMpf’s, to the best of our knowledge.

In the case where the Petrov type is D, an exhaustive classification of PMpf’s has by now been obtained, essentially due to two results. Firstly, it was shown most recently \([33]\) that Petrov type D PMpf’s are necessarily LRS class I or III, whereas secondly the most general metric forms \([37]\) for each of these classes were derived in \([22]\). Four systematic studies historically preceded these results: on the one hand LRS space-times in general were investigated in \([35, 36]\) by two different methods, hereby also briefly discussing the purely magnetic case: on the other hand shear-free \([3]\), resp. non-rotating \([37]\) PMpf’s of Petrov type D were shown to be LRS class I, resp. III. Finally the following three distinct PMpf solutions are particular cases of the general LRS cases above: the \(p = \frac{1}{2} \mu\) Collins-Stewart space-time \([33]\) and Lozanovski-Aarons metric \([29]\), both LRS class III, and the LRS class I rigidly rotating axistationary model with circular motion of \([34]\). The last example was found within a study (regardless of the Petrov type) of axistationary perfect fluids, which concentrated mostly on the PMpf subclass. Concretely, it was shown that axistationary PMpf’s with circular motion necessarily have non-vanishing vorticity and spatial-3 gradient of the matter density, and that such spacetimes are LRS class I.

No algebraically general PMpf’s have been found yet, although they might be of relevance for cosmological modelling. The PMpf subclass of irrotational dust space-times consists of ‘silent’ universe models \([33, 40]\) and was investigated in \([21]\), but the appearance of chains of severe integrability conditions (analogously as for the widely studied purely electric subclass in the Petrov type I case \([11, 12]\) ) made the authors conjecture that this subclass might be empty. This was recently proved in \([42]\) for the case of dust, regardless of Petrov type and cosmological constant.

The present study continues this line of investigation. Whereas evidence is provided that the class of PM rotating dust must be severely restricted as well, we present as a main result a first example of an algebraically generic PMpf solution, which is both non-rotating and non-accelerating. Up to a constant rescaling, this solution is shown to be the unique PMpf with these properties. Moreover it is spatially homogeneous with degenerate shear tensor and, just as the related LRS class III models of \([37]\), it turns out to satisfy the energy conditions in an open subset of spacetime.

The structure of the paper is as follows. In section 2 we set up the basic variables and equations for PMpf’s, in a mixed 1+3 covariant/orthonormal tetrad approach. After pointing out why solution families of the PM rotating dust class should be rather poor in number, the subclasses of Petrov type D and of vanishing spatial 3-gradient of the energy density are shown to be empty in section 3. Section 4 provides a characterization of the new metric, discusses its mathematical and physical features and situates it within the broader context of \([43]\). We end with a conclusion and discussion in section 5.
2. BASIC EQUATIONS FOR PMPF’S

We use units such that $8\pi G = c = 1$, Einstein summation convention, the signature $(-, +, +, +)$ for spacetime metrics $g_{ab}$, and abstract (Latin) index notation for tensorial quantities (except for basis vector fields which are written in bold face notation): round (square) brackets around indices denote (anti-)symmetrization. The perfect fluid field equations (7) may be rewritten as

$$R_{ab} = \frac{1}{2}(\mu + 3p)u_a u_b + \frac{1}{2}(\mu - p)h_{ab}. \quad (8)$$

Here $h_{ab} = g_{ab} + u_a u_b$ projects orthogonally to the fluid’s 4-velocity field $u^a$, ‘spatializing’ indices of tensorial quantities by contraction; if this operation is the identity operation for all indices, the tensor is called spatial. For covariant differential operations orthogonal to $u^a$, the streamlined notation of [4] [21] [42] is the most transparent. The covariant spatial derivative $D$ and the associated curl and divergence (div) operators, acting on vectors and 2-tensors, are defined as:

$$D_a S^{c\cdots d} = h^a{}_b h^b_\cdots h^d_\cdots h^e_\cdots \nabla_c S^{p\cdots q\cdots r\cdots}. \quad (9)$$

$$\text{div} V = D_a V^a, \quad \text{curl} V_a = \epsilon_{abc} D^b V^c. \quad (10)$$

$$\text{div} S_a = D^b S_{ab}, \quad \text{curl} S_{ab} = \epsilon_{cd(a} D^c S_{b)} d. \quad (11)$$

Here $\nabla_c$ is the covariant derivative associated with the Levi-Civita connection and $\epsilon_{abc} = \eta_{abc}\epsilon^d$ is the spatial permutation tensor. The kinematics of the perfect fluid $u^a$-congruence are then described [10] by its acceleration $\dot{u}_a$, vorticity $\omega_a = -\frac{1}{2}\text{curl} u_a$ and expansion tensor $\theta_{ab} = D(a u_b)$, which are all spatial. Here and in general, a dot denotes covariant (‘time’) differentiation along $u^a$. For scalar functions $f$ and vectors $v^a$ one has [4] [43]

$$(D_a f) = D_a \dot{f} - \theta_a \nabla f - \epsilon_{abc} \omega^b D^c f + u_a \dot{f} + \epsilon_{abc} D^b f, \quad (12)$$

$$\text{curl}(D f) = -\dot{\omega}_a, \quad (13)$$

$$\text{curl}(f v) = f \text{curl} v_a + \epsilon_{abc} D^b f v^c. \quad (14)$$

The expansion tensor is further decomposed as $\theta_{ab} = \sigma_{ab} + \frac{2}{3}\theta h_{ab}$, with $\sigma_{ab} = D(a u_b) - \frac{1}{2}\theta h_{ab} \equiv D(u_{ab})$ the trace-free shear tensor of the fluid and $\theta = \theta_a^a = \text{div} u$ its scalar expansion rate. In general, $S_{(ab)} \equiv h^a_\cdots h_b^\cdots S_{(cd)} - \frac{1}{2}S_{cd} h^c_\cdots h^{d_\cdots}$ is the spatially projected, symmetric and trace-free part of $S_{ab}$, while $V_{(a)} = h^a_\cdots V_b$ denotes the spatial projection of $V_a$. [27]

In a 1+3 covariant ‘threading’ approach, the tensorial quantities $u_a, \omega_a, \sigma_{ab}, \theta$ and $E_{ab}, H_{ab}, \mu, \rho$ are taken as the fundamental dynamical fields. One focusses on the Ricci-identity for $u_a$ and the Bianchi-identities, wherein the Ricci-tensor is substituted for the right hand side of (3). They can be split in constraint equations (involving only spatial derivatives) and propagation equations. For a general perfect fluid, the Ricci-equations become, in our notation:

$$\begin{align*}
\text{div} \sigma_a - \frac{2}{3} D_a \theta &= \nabla \omega_a + 2\epsilon_{abc} \omega^b \dot{u}^c = 0, \quad (15) \\
\text{div} \omega - \dot{\omega}^a \omega_a &= 0, \quad (16) \\
\nabla \sigma_{ab} + D(a \omega_b) - H_{ab} + 2\dot{u}_a \omega_b = 0, \quad (17) \\
\dot{\sigma}_{(ab)} + \frac{2}{3} \theta \sigma_{ab} + \epsilon_{c(a} \sigma_{b) c} + \omega_{(a} \omega_{b)} &= 0, \quad (18) \\
-D(a \dot{u}_b) - \dot{u}_a \omega_b + E_{ab} &= 0, \\
\dot{\theta} + \frac{1}{3} \dot{\omega}^a \omega_a - \sigma_{ab} \omega^b - 2\omega^a \omega_a - \nabla \dot{u} &= 0, \quad (19) \\
\omega_{(a)} + \frac{2}{3} \dot{\theta} \omega_a - \sigma_{ab} \omega^b + \frac{1}{2} \text{curl} \dot{u} &= 0. \quad (20)
\end{align*}$$

The equations of conservation of momentum and energy (contracted Bianchi-identities) are:

$$D_a p = -(\mu + p) \dot{u}_a, \quad (21)$$

$$\dot{\mu} = -(\mu + p) \theta. \quad (22)$$

For PMPf’s $E_{ab} = 0$, the remaining Bianchi-identities are

$$[\sigma, H]_a - 3H_{ab} \omega^b + \frac{1}{3} D_a \rho = 0, \quad (23)$$

$$\text{div} H_a = -(\mu + p) \omega_a = 0, \quad (24)$$

$$\text{curl} H_{ab} - \frac{1}{2} (\mu + p) \sigma_{ab} + 2\dot{u}^c \epsilon_{cd(a} H^{b)} = 0, \quad (25)$$

$$H_{(ab)} + \theta H_{ab} - 3\sigma_{(a} \omega_{b)} + \omega^c \epsilon_{cd(a} H^{b)} = 0. \quad (26)$$

Here $[S, T]_a \equiv \epsilon_{abc} S^{bd} T^c_d$ is the vector dual to the commutator of spatial tensors $S_{ab}$ and $T_{ab}$. Note that in general, the term $E_{ab}$ in (18) couples the evolution of the kinematical quantities to that of the Weyl tensor, which is no longer the case for PMPf’s. The constraints (21) and (22) express div $E_a = 0$ and $E_{ab} = 0$, respectively.

In an orthonormal tetrad approach, a specific orthonormal basis of vector fields $B = \{e_0 = u, e_\alpha\}$ is taken. Here and below, Greek indices run from 1 to 3, expressions containing these have to be read modulo 3 (e.g. $X_{a+1} = X_{a+1}$ for $\alpha = 3$ and is written for the action of $e_\alpha$ on functions $f$). For spatial tensorial quantities, only the components with Greek indices survive; one has in particular $h_{\alpha\beta} = \delta_{\alpha\beta}$, and $\epsilon_{\alpha\beta\gamma}$ becomes the alternating symbol on three indices, where we take the convention $\epsilon_{123} = \epsilon_{123} = 1$. The commutator coefficients $\gamma^a_{bc} = -\gamma^a_{cb}$ and Ricci-rotation coefficients $\Gamma^a_{bc}$ of $B$ are defined by

$$[e_b, e_c] = \gamma^a_{bc} e_a, \quad \nabla e_b = \Gamma^a_{bc} e_a. \quad (27)$$

As for any rigid frame, the lowered coefficients $\Gamma_{abc} = g_{ad} \Gamma^d_{bc} = -\Gamma^d_{bac}$ and $\gamma_{abc} = g_{ad} \gamma^d_{bc}$ are one-one related by

$$\gamma_{abc} = -2\Gamma_{a[bc]}, \quad \Gamma_{abc} = \frac{1}{2}(\gamma_{bac} + \gamma_{cab} - \gamma_{abc}). \quad (28)$$
Combinations hereof, together with $\mu, p$ and the components $H_{\alpha\beta}$ play the role of basic variables. In the present paper, we will use

\begin{align}
  h_{\alpha} &\equiv \Gamma_{\alpha+1\alpha+1} - \Gamma_{\alpha-1\alpha-1} = \sigma_{\alpha+1\alpha+1} - \sigma_{\alpha-1\alpha-1} \quad (29) \\
  \theta &\equiv \Gamma_{\alpha0\beta} \text{ or} \\
  \theta_{\alpha\alpha} &\equiv \Gamma_{\alpha\alpha\alpha} = \sigma_{\alpha\alpha} + \frac{1}{3} \theta, \quad (30) \\
  \sigma_{\alpha+1\alpha-1} &\equiv \frac{1}{2} (\Gamma_{\alpha+1\alpha-1} + \Gamma_{\alpha-1\alpha+1}), \quad (31) \\
  \omega_{\alpha} &\equiv \frac{1}{2} (\Gamma_{\alpha+1\alpha-1} - \Gamma_{\alpha-1\alpha+1}), \quad (32) \\
  \dot{u}_{\alpha} &\equiv \Gamma_{\alpha000}, \quad (33) \\
  \Omega_{\alpha} &\equiv \Gamma_{\alpha-1\alpha+10}, \quad (34) \\
  q_{\alpha} &\equiv \gamma_{\alpha-1\alpha-1} = -\Gamma_{\alpha-1\alpha-1}, \quad (35) \\
  r_{\alpha} &\equiv \gamma_{\alpha+1\alpha+1} = \Gamma_{\alpha+1\alpha+1}, \quad (36) \\
  n_{\alpha} &\equiv \Gamma_{\alpha+1\alpha-1} \text{ or} \\
  n_{\alpha\alpha} &\equiv \gamma_{\alpha+1\alpha-1} = n_{\alpha+1} + n_{\alpha-1}. \quad (37)
\end{align}

In the line of the Cartan one-form formalism, the basic equations are the commutator relations $[27]$, the components of the Ricci-identities and of the (second) Bianchi-identities $[21]-[26]$ (see e.g. $[17]$ §7). The Ricci-identities may be further split into

(a) the Jacobi-identities (first Bianchi-identities)

\[ \frac{1}{6} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = [\epsilon_{[a}, [\epsilon_{b}, \epsilon_{c}]]^d = \partial_{[a} \gamma^{d}_{\ b c] + \gamma^{f}_{\ [b c} \gamma^{d}_{a f]} = 0, \quad (38) \]

(b) the defining equations $[17]$ and $[18]$ for $H_{\alpha\beta}$ and $E_{\alpha\beta}$, and

(c) the components of the Einstein equations $[3]$, where the (00) component is Raychaudhuri’s equation $[13]$ and the (0a) component is $[14]$, whereas the (a\beta) component (which is not covered by the Ricci-identity for $u_\alpha$) reads

\[ \partial_{\alpha} \Gamma^{c}_{(a\beta)} - \partial_{\beta} \Gamma^{c}_{(a\alpha)} + \Gamma^{e}_{\ d c} \Gamma^{d}_{(a\beta)} - \Gamma^{e}_{\ d f} \Gamma^{f}_{(a \beta c]} = \frac{1}{2} (\mu - p) \delta_{a\beta}. \quad (39) \]

Note that $[13]$-28 are equalities between spatial tensorial quantities, which are moreover symmetric and trace-free in the case where they are 2-tensors. The non-trivial components are readily deduced from $[13]$-11, 35-36 and $[19]$-24 and

\[ D_{\alpha} V_{\beta} = \partial_{\alpha} V_{\beta} - V_{\delta} \Gamma^{\delta}_{\beta \alpha}, \quad (39) \]

\[ D_{\alpha} S_{\beta\gamma} = \partial_{\alpha} S_{\beta\gamma} - S_{\delta\gamma} \Omega^{\delta}_{\beta \alpha}, \quad (40) \]

\[ D_{\alpha} V_{\beta} = \partial_{\alpha} V_{\beta} - V_{\delta} \Gamma^{\delta}_{\beta \alpha}, \quad (39) \]

\[ D_{\alpha} S_{\beta\gamma} = \partial_{\alpha} S_{\beta\gamma} - S_{\delta\gamma} \Omega^{\delta}_{\beta \alpha}, \quad (40) \]

\[ D_{\alpha} V_{\beta} = \partial_{\alpha} V_{\beta} - V_{\delta} \Gamma^{\delta}_{\beta \alpha}, \quad (39) \]

\[ D_{\alpha} S_{\beta\gamma} = \partial_{\alpha} S_{\beta\gamma} - S_{\delta\gamma} \Omega^{\delta}_{\beta \alpha}, \quad (40) \]

\[ \frac{1}{2} (\sigma_{13} + \omega_2), \quad (43) \]

\[ 6H_{22} q_2 + (\mu + p)(\sigma_{13} - 3_2) = 0, \quad (44) \]

\[ 6H_{22} r_3 + (\mu + p)(\sigma_{12} + 3_3) = 0. \quad (44) \]
Now from the (22), (33) and (23) components of $H_{ab}$ and from \((\begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}) + (\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}) \) one gets

\[
\begin{align*}
(\sigma_{12} + \omega_3)q_2 + (\sigma_{13} - \omega_2)r_3 &= 0, \\
(\sigma_{13} - \omega_2)q_2 - (\sigma_{12} + \omega_3)r_3 &= 0,
\end{align*}
\]

(45) such that either $\sigma_{13} = \omega_2$ and $\sigma_{12} = -\omega_3$, or $q_2 = r_3 = 0$; in the latter case one has $\sigma_{13} = 3\omega_2, \sigma_{12} = -3\omega_3$ by (18). In both cases the (22), (33) and (23) components of (48) imply $(\omega_3 - \omega_2)(\omega_3 + \omega_2) = 0$ and $\omega_2\omega_3 = 0$, such that $\sigma_{12} = \sigma_{13} = \omega_2 = \omega_3 = 0$. Under these conditions, (29) yields

\[
m_1 = -18h\omega_1, \quad m_2 = m_3 = 0,
\]

(46) while from (20) and (18) we obtain in the variables \(\omega_1, \theta \)

\[
\begin{align*}
\partial_0\omega_1 &= -2\omega_2\omega_1, \\
\partial_0\partial_2 = & \omega_1^2 - \frac{1}{6}(\mu + 3p) - \theta_2^3.
\end{align*}
\]

(47) However, with (44) and $\omega_1 \neq 0 \neq \mu + p$ one deduces from $[\partial_2, \partial_3]u$ and (20) that $\theta_2^2 = \frac{1}{3}(h_2 + \theta) = 0$. Taking two $\partial_\theta$-derivatives hereof and using (47), (22), $\rho = 0$ and $\mu + p \neq 0$, yields $\theta = h_2 = 0$ in contradiction with (12).

**Theorem 3.2** A non-accelerating PMpf with vanishing spatial 3-gradient of the energy density is non-rotating.

**Proof.** Let \((M, g_{ab}, u_a)\) be a PMpf with $u_a = D_a\rho = 0$. As non-accelerating perfect fluids are dust or non-rotating, we may assume the Petrov type is I, since for Petrov type D the result follows from Theorem 3.1. The algebraic vector constraint (28) and its covariant derivative along $u^a$ yield

\[
\begin{align*}
[\sigma, H]a - 3\mathcal{H}_{ab}\omega^b &= 0, \\
6\sigma_cD^c(\nabla^b\omega^b) + \text{tr}(\sigma H)\omega &= \epsilon_{abc}\omega^bH^{c}\omega^d.
\end{align*}
\]

(48) Projected onto an orthonormal eigenframe $B$ of $H_{ab}$, the components of these equations form a system of 6 linear equations in the variables $\sigma_{12}, \sigma_{13}, \sigma_{23}, h_2$ and $h_3$, parametrized by the $H_{ab}$ and $\omega_a$, which can only be consistent if the determinant of the so called extended system matrix vanishes. Writing $x_a = H_{a+1+a+1} - H_{a-1-a-1}$, this yields

\[
x_1(x_2^2 + x_3^2)\omega_2^2\omega_3^2 + \text{cyclic terms} = 0.
\]

Hence, as the Petrov type is I, at least two of the spatial components $\omega_a$ must be zero, say $\omega_2 = \omega_3 = 0$. Herewith (43), (44) imply $\sigma_{13} = \sigma_{12} = 0$. Then $W_2 = W_3 = 0$ follows from (29) and (49) reduces to

\[
6\sigma_{ab}H^{bc}\omega_c - \sigma_{bc}H^{bc}\omega_a = 0.
\]

Taking a further covariant time derivative hereof, one finds

\[
(\sigma_{cd}\sigma_{cd} - 3\omega_c\omega^c)H_{ab}\omega^b - 2\sigma_{cd}H^{cd}\sigma_{ab}\omega^b - \sigma_{bc}\sigma^{cd}H^{d}\omega_a = 0.
\]

(50) Suppose now $\omega_1 \neq 0$, such that $0 \neq \omega \propto e_1$, is an eigenvectorfield of $H_{ab}$. Eliminating $\sigma_{23}$ from the first components of (49) and (48) (the latter being given by $\sigma_{23}x_1 + \omega_1(x_3 - x_2) = 0$, and then eliminating $h_2$, resp. $h_3$ from the result by means of (3) (i.e. $h_2x_3 + h_3x_2 = 0$), one finds

\[
\begin{align*}
\omega^1x_2x_3(x_2 - x_3)(4w_1^2\omega_1^2 + H_{2}\omega_2^2) &= 0, \\
\omega^2x_2x_3(x_2 - x_3)(4w_1^2\omega_1^2 + H_{3}\omega_3^2) &= 0.
\end{align*}
\]

Hence $x_3 = x_2$ (i.e. $H_{11} = 0$), such that $h_1 = \sigma_{23} = 0$ by (43) and (48), while the first component of (20) yields $W_1 = -\omega_1/2$. Now the (11) and (1) components of (20) and (31) respectively read

\[
\begin{align*}
(n_{22} - n_{33})x_2 - \frac{h_3^2}{3}(\mu + p) &= 0, \\
(q_1 + r_1)x_2 + \omega_1(\mu + p) &= 0,
\end{align*}
\]

(51) (52) which can only be consistent for $x_2 \neq 0$ if $\omega_1(n_{22} - n_{33}) + \frac{1}{3}(q_1 + r_1)h_2 = 0$. On the other hand, the (23) component of (17) yields $\omega_1(n_{22} - n_{33}) - (q_1 + r_1)h_2 = 0$, such that $n_{22} = n_{33}$, whence $h_2 = 0$ from (51), again in contradiction with (42).

**Remark.** This generalizes the result of (14). The reasoning was similar, but shortcuts were available due to the non-existence of Petrov type D [2] and type I(M$^\infty$) vacua.

From (42) and theorem 3.2 it follows:

**Corollary.** PM dust spacetimes with vanishing spatial 3-gradient of the energy density do not exist.

4. **ALGEBRAICALLY GENERAL PMPF’S**

Looking at (11), (20) and (21), one sees that non-rotating (non-vacuum) perfect fluids in general obey $\epsilon_{abc}D^b\mu \dot{u}^c = \epsilon_{abc}D^b\mu D^c\rho = 0$, i.e., either (a) the acceleration $\dot{u}_a$ and the spatial 3-gradient of the pressure $D_a\rho$ are $0$ and proportional to the spatial 3-gradient of the energy density $D_a\mu$, say $D_a\rho = fD_a\mu, f \neq 0$, (b) $\dot{u}_a = D_a\rho = 0$ or (c) $D_a\mu = 0$. Hence, a non-rotating perfect fluid exhibits a barotropic equation of state $p = \rho(\mu)$ if and only if either (1) the fluid has property (a) with $f = \frac{\rho_\mu}{\rho} = \mu = \frac{\rho_\mu}{\rho} \neq 0$, (2) the fluid has the properties (b) and (c), or (3) it is dust (constant $p$, i.e. $\frac{\rho_\mu}{\rho} = 0$).

For non-rotating PMpf’s, it was shown in (42) that the dust case (3) is impossible. On the other hand, non-rotating type D PMpf’s were fully classified in (51). Such
spacetimes were shown to belong to class (2) above and
turned out to be spatially homogeneous and locally rota-
tionally symmetric (LRS) of class III in the Stewart-Ellis
classification. The general metric form as well as the cor-
responding equation of state were explicitly constructed,
and the solutions satisfied the energy conditions in an
open region of spacetime. This result motivates the investi-
gation in the present section of the existence of Petrov
type I non-rotating PMpf’s obeying (2). It is proved
that for non-rotating PMpf’s (b) and (c) are actually
equivalent, and that non-rotating and algebraically gen-
eral PMpf’s exist for which the spatial gradients of mat-
ter density and pressure vanish. The solution is unique
to a constant rescaling of the metric. This provides
an affirmative answer to question (ii) at the beginning of
the previous section. The spacetime is characterized
in Theorem 4.2: it is found to be of Petrov type I(M∞),
has a degenerate shear tensor and is spatially homoge-
neous of Bianchi type VI0. Both the metric and equa-
tion of state are constructed, and the physical behavior
of the model is briefly discussed. Finally, the solution
is situated within a larger class of perfect fluid models by
dropping the purely magnetic condition.

4.1. Characterization.

Suppose \((M, g_{\alpha \beta}, u_\alpha)\) is a Petrov type I non-rotating
PMpf, and project the basic equations w.r.t. an orthonor-
mal eigenframe \(B\) of \(\sigma_{\alpha \beta}\), i.e., \(\sigma_{\alpha+1 \alpha-1} = 0\). Then the
\((\alpha \alpha)\) components of (17) become algebraic and very
compact in the \(n_{\alpha}\)-variables:

\[ H_{\alpha \alpha} = n_{\alpha+1} h_{\alpha+1} - n_{\alpha-1} h_{\alpha-1}. \]  

On the other hand, one can solve the \((0 \alpha)\) field equations and the \((\alpha + 1 \alpha - 1)\) components of (17) for
\(\partial_\alpha h_{\alpha+1}, \partial_\alpha h_{\alpha-1}, \) giving

\[ \partial_\alpha h_{\alpha+1} = -\partial_\alpha \theta - h_{\alpha+1} - r_\alpha h_{\alpha-1} - 2q_\alpha h_{\alpha+1}, \]

\[ \partial_\alpha h_{\alpha-1} = \partial_\alpha \theta - h_{\alpha+1} + 2r_\alpha h_{\alpha-1} + q_\alpha h_{\alpha+1}, \]

while (18) and (19) combine to prescriptions for the evolu-
tion of the variables (18):

\[ \partial_\theta h_{\alpha} = -\theta^2 + \partial_\alpha u_\alpha + u_\alpha^2 + q_{\alpha+1} u_{\alpha+1} - r_{\alpha-1} u_{\alpha-1} - \frac{1}{2} (\mu + 3p). \]

As shown below in the appendix, we may take \(W_{\alpha} = 0\)
for the cases \(D_\alpha \rho = 0\) and \(D_\alpha \rho = \dot{u}_\alpha = 0\) under study.
Then the \((\alpha \alpha)\) components of (24) reduce to

Inserting (51)-(55) into (51) one gets

\[ \partial_\alpha q_\alpha = \frac{1}{3} (h_\alpha - h_{\alpha+1} - \theta)(q_\alpha - u_\alpha) - H_{\alpha+1 \alpha-1}, \]

\[ \partial_\alpha r_\alpha = \frac{1}{3} (h_{\alpha-1} - h_\alpha - \theta)(r_\alpha + u_\alpha) - H_{\alpha+1 \alpha-1} \]  

while \(0 \alpha + 1 \alpha = \) yields

\[ \partial_\alpha h_{\alpha} = -\frac{1}{3} q_\alpha - \frac{1}{3} (h_{\alpha-1} - h_{\alpha+1}) h_{\alpha-1} + h_{\alpha+1} n_{\alpha+1} - h_{\alpha-1} n_{\alpha-1}. \]

We will make use of the following

Lemma 4.1 W.r.t. a shear-eigenframe of a non-
rotating PMpf:
(a) at most one of the \(n_{\alpha}\) vanishes;
(b) at most one of the \(h_{\alpha}\) vanishes;
(c) if \(\dot{u}_\alpha = 0\), then \(ab_2 + bh_3 = 0\) for constant \(a, b\)
\((a, b) \neq (0, 0))\) implies \(h_1 h_2 h_3 = 0\), i.e. the shear
tensor is degenerate.

Proof. (a) Immediately follows from (60) and (53).
(b) This is the statement that for non-rotating PMpf’s the
shear tensor cannot vanish, as is immediately seen from
(17) in covariant form (or from (53) and (54) in tetrad form).
(c) For \(\dot{u}_\alpha = 0\), the difference of the \((\alpha + 1 \alpha + 1)\) and
\((\alpha - 1 \alpha - 1)\) components of (18) reads

\[ \partial_\theta h_{\alpha} = -\frac{1}{3} (h_{\alpha+1} - h_{\alpha-1} + 2\theta) h_{\alpha}. \]

From this and \(h_1 + h_2 + h_3 = 0\), one deduces
\(\partial_\theta (ab_2 + bh_3) + \frac{1}{2} (2\theta + h_1 - h_2)(ab_2 + bh_3) = ah_1 h_2,\)
from which the result follows.

Theorem 4.2 For algebraically general PMpf’s, any
two of the following three conditions imply the third:

(i) the fluid congruence is non-rotating;
(ii) the fluid is non-accelerating (i.e. the spatial gradient
of the pressure vanishes);
(iii) the fluid has a vanishing spatial gradient of the matter
density.

Proof.
(i), (ii) \(\Rightarrow\) (iii). See appendix.
(ii), (iii) \(\Rightarrow\) (i). This was the content of Theorem 3.2.
(iii), (i) \Rightarrow (ii). When \( \partial_\alpha \mu = 0 \) one deduces from [10] (or [12]) and [21] that also \( \partial_\alpha \theta = 0 \). With \( H_{\beta+1 \beta-1} = 0 \), [21] becomes algebraic:

\[
(H_{aa} - H_{a-1a-1})n_{a+1a+1} + (H_{aa} - H_{a+1a+1})n_{a-1a-1} - \frac{\mu + p}{6}(h_{a+1} - h_{a-1}) = 0
\]

(62)

and consists of two independent equations. Elimination of \( \mu + p \) yields

\[
(h_2 - h_3)(H_{22} - H_{33})n_1 + \text{cyclic terms} = 0
\]

(63)

From [58], one deduces

\[
n_a H_{aa} = n_a(n_{a+1}h_{a+1} - n_{a-1}h_{a-1}),
\]

(64)

\[
(n_1n_2 + n_2n_3 + n_3n_1)h_a = n_a - n_{a+1} - n_{a-1} + n_a + n_{a+1} + n_{a-1}.
\]

(65)

If \( X \equiv n_1n_2 + n_2n_3 + n_3n_1 \) vanishes, then \( Y \equiv n_1H_{11} = n_2H_{22} = n_3H_{33} \) from [58], which is contradictory to lemma 4.1 (a). Hence \( X \neq 0 \) and one can use [58] to eliminate the \( h_a \). Doing this for [58], one obtains \((n_1 + n_2 + n_3)F = 0\), with

\[
F \equiv n_1H_{11}(H_{22} - H_{33}) + \text{cyclic terms}.
\]

This leaves two cases

(I) \( F = 0 \):

Calculating \( \partial_\theta F + \frac{2}{9}hF \) and substituting for \( h_a \) one gets another polynomial equation \( G = 0 \) in \( n_a \) and \( H_{aa} \). Substituting \( H_{a+1a+1} = -H_{aa} - H_{a-1a-1} \) and taking the resultant of \( F \) and \( G \) w.r.t. \( H_{a-1a-1} \) yields

\[
H_{aa}(n_1 - n_2)(n_2 - n_3)(n_3 - n_1)\times(n_1n_2 + n_2n_3 + n_3n_1)^2 = 0.
\]

Hence e.g. \( n_3 = n_2 \). Inserting this in \( F = 0 \) yields \( H_{11}(H_{22} - H_{33})(n_1 - n_2) \equiv 0 \). If \( H_{11} \) were zero then \( G = 0 \) would read \( n_2^2n_3^2 = 0 \), hence \( n_2 = n_3 = 0 \), again in contradiction with lemma 4.1 (a): if \( n_1 - n_2 \) were zero then \( G = 0 \) would read \( n_2^2(H_{11} - H_{22})(H_{22} - H_{33})(H_{33} - H_{11}) = 0 \). Thus \( F = 0 \) leads to Petrov type D (and hence to the models found in [27]), contrary to our type I assumption.

(II) \( n_1 + n_2 + n_3 = 0 \):

Applying \( \partial_\theta \) to this equation one finds

\[
(h_2 - h_3)n_1 + (h_3 - h_1)n_2 + (h_1 - h_2)n_3 = 0
\]

(66)

or, by [58],

\[
P_1 \equiv n_1^3H_{11} + n_2^3H_{22} + n_3^3H_{33} = 0.
\]

(67)

Calculating \( \partial_\theta P_1 + \frac{3}{2}hP_1 \) and again eliminating the \( h_a \) by [58], one obtains a further polynomial equation \( P_2 \) in the \( n_a \) and \( H_{aa} \). Substituting \( H_{a+1a+1} = -H_{aa} - H_{a-1a-1} \) and eliminating \( H_{a-1a-1} \) from \( P_1 \) and \( P_2 \) now yields

\[
H_{aa}^2n_1n_2n_3(n_1n_2 + n_2n_3 + n_3n_1)^2 = 0.
\]

(68)

Hence e.g. \( n_1 = 0 \), such that \( n_2 + n_3 = n_{11} = 0 \), \( h_1 = 0 \) from [60] and \( H_{11} = 0 \) from [67]. From \( h_1 = 0 \) and [60], one deduces \( q_1 = -r_1 \) and \( q_2 = 0, r_3 = 0 \). Now \( \partial_\theta q_2 = \partial_\theta r_3 = 0 \) yields \( \dot{u}_2(2h_2 - \theta) = \dot{u}_3(2h_2 - \theta) = 0 \) by means of [64].

Suppose that \( \dot{u}_2 \) and \( \dot{u}_3 \) are non-zero, such that \( 2h_2 - \theta = -3\theta_1 = 0 \). Then \( \partial_\theta(2h_2 - \theta) = 0 \) yields \( h_2(r_1 - 2q_1) = 0 \) by [63] and \( \partial_\theta h_2 = 0 \), such that \( q_1 = r_1 = 0 \) by 4.1 (b). Now \( \partial_\theta r_1 = 0 \) yields \( h_2\dot{u}_1 = 0 \) by [63] and hence, again by lemma 4.1 (b), \( \dot{u}_1 = 0 \). Then however, by [60] for \( \alpha = 1 \), one would have \( \mu + 3p = 0 \), leading to \( \partial_\theta p = \dot{u}_1 = 0 \), contradictory to the assumption \( (\dot{u}_2, \dot{u}_3) \neq (0, 0) \). Hence \( \dot{u}_2 = \dot{u}_3 = 0 \). Finally, the (23) component of [61] becomes \( \dot{u}_1(n_2 - n_3) = 0 \), such that also \( \dot{u}_1 = 0 \) by lemma 4.1 (b), and the fluid is non-accelerating.

4.2. Existence and properties.

From the proof of theorem 4.2 it follows that for non-rotating, algebraically general PMFs with vanishing spatial gradients of matter density and pressure, one of the eigenvalues of \( H_{aa} \) is identically zero, such that the Petrov type is I(\( M^\infty \)) in the extended Arnaud-Henry classification. The \( H_{ab} \)-eigenframe \( \{u, e_a\} \) being also an eigenframe of the shear tensor, the latter is degenerate in the plane perpendicular to the 0-eigendirection of \( H_{ab} \). When \( e_1 \) spans this eigendirection, i.e. for \( H_{11} = h_1 = 0 \) (such that \( H_{33} = -H_{22} \neq 0, h_3 = -h_2 \neq 0 \)), the following equations have by now been established:

\[
\partial_\alpha p = \partial_\alpha \mu = \partial_\alpha \theta = 0,
\]

(69)

\[
\dot{u}_\alpha = \omega_\alpha = \sigma_{\alpha+1 \alpha-1} = 0,
\]

(70)

\[
W_\alpha = 0,
\]

(71)

\[
n_1 = n_2 + n_3 = 0 \quad (i.e. n^a_a = n_{11} = 0),
\]

(72)

\[
q_1 + r_1 = q_2 = r_3 = 0,
\]

(73)

while [63] and [62] reduce to

\[
H_{22} = n_2h_2,
\]

(74)

\[
n_2^2 = \frac{1}{6}(\mu + p).
\]

(75)

Now from [68] and [65] one sees that \( \partial_\alpha h_2 = \partial_\alpha h_2 = 0 \), while with [60], [70] and [64] one immediately deduces from \( [\partial_\theta, \partial_\theta] \dot{\eta} \) that also \( \partial_\alpha h_2 = 0 \), whence \( q_1 = r_1 = 0 \) by [73] and [51]. From the (\( \alpha + 1 \alpha - 1 \)) field equations and \( \{1 2 3\}, [\partial_1, \partial_2] n_2 \) and \( [\partial_3, \partial_1] n_2 \) one deduces \( q_3 = r_2 = 0 \) and \( \partial_\alpha n_2 = 0 \) (alternatively, \( \partial_\alpha n_2 = \partial_\alpha H_{22} = 0 \) follows from [64] and [61], whence \( r_2 = q_3 = 0 \) by [21], but the previous reasoning is more generally valid, cf. infra). Finally, the (\( \alpha \alpha \)) field equa-
easily checked that the normalized vector fields $B$ of type $I$ are consistent and imply that corresponding solutions (76) and (77) are consistent under propagation along the matter flow lines, whereas applying $\partial_0$ to (78) yields an expression for $p$ in $\theta_{11}$ and $\theta_{22}$:

$$p = -3\theta_{22}(\theta_{11} - 2\theta_{22}).$$

With the above specifications, all basic equations are satisfied and consistent, implying that corresponding solutions exist. Since all $\partial_\alpha$-derivatives of the commutator coefficients vanish the spacetimes will be spatially homogeneous, while $q_\alpha = r_\alpha = 0$ and (72) imply that the Bianchi type is $I_0$ (see [49]).

4.3. Metric, uniqueness and equation of state

With (70)-(72), $q_\alpha = r_\alpha = 0$ and $\theta_{22} = \theta_{33}$, it can be easily checked that the normalized vector fields

$$e_0 \equiv u, \quad e_1,$$

$$e_2 \equiv \frac{1}{\sqrt{2}}(e_2 + e_3), \quad e_3 \equiv \frac{1}{\sqrt{2}}(e_2 - e_3)$$

are hypersurface-orthogonal and by (72) obey

$$[e_0, e_1] = -\theta_{11} e_1,$$

$$[e_0, e_2] = -\theta_{22} e_2, \quad [e_0, e_3] = -\theta_{22} e_3,$$

$$[e_1, e_2] = -\theta_{22} e_2, \quad [e_1, e_3] = -\theta_{22} e_3,$$

$$[e_2, e_3] = 0.$$

Following §6 of [49] and noting that $n_2 \neq 0$ because of (74), we may choose coordinates $t', x', y'$ such that

$$e_0 = \frac{\partial}{\partial t'}, \quad e_1 = n_2(t') \frac{\partial}{\partial x'},$$

$$e_2 = B(t') e^{x'} \frac{\partial}{\partial y'},$$

$$e_3 = B(t') e^{-x'} \frac{\partial}{\partial z'}.$$  

(83)

Relations (81) and (82) are satisfied if and only if

$$\frac{d\ln n_2}{dt'} = -\theta_{11}, \quad \frac{d\ln B}{dt'} = -\theta_{22}.$$  

(84)

By (71), (78) and (85), the autonomous dynamical system for the fundamental variables $\theta_{11}$ and $\theta_{22}$ becomes

$$\frac{d\theta_{11}}{dt'} = (\theta_{22} - \theta_{11})(\theta_{11} + 2\theta_{22}),$$

$$\frac{d\theta_{22}}{dt'} = (\theta_{22} - \theta_{11})\theta_{22}.$$  

(85)

As $\theta_{11} = \theta_{22}$ and $\theta_{22} = 0$ are not allowed by Lemma 1 (b) and e.g. (77) + (78), one derives from (80) and (81) that

$$\theta_{11} = (2\ln \theta_{22} - C_0 + 1)\theta_{22} \quad \text{or}$$

$$u \equiv \frac{\theta_{11}}{\theta_{22}} - 1 = 2\ln \theta_{22} - C_0,$$

(87)

with $C_0$ an integration constant. Using $u$ as time variable, this yields

$$\theta_{22} = \frac{C}{2} \exp\left(\frac{u}{2}\right), \quad u_{11} = \frac{C}{2} \exp\left(\frac{u}{2}\right)(u + 1),$$

(88)

with

$$C \equiv 2\exp\left(\frac{C_0}{2}\right).$$

From (80) and (87) one obtains $du = -2\theta_{22}udt' = -Cu \exp\left(\frac{u}{2}\right)dt'$, such that (84) may be integrated to give

$$n_2(u) = C_1 e^{\frac{u}{2}}, \quad B(u) = C_2 u^{\frac{1}{2}},$$

(89)

with $C_1$ and $C_2$ non-zero constants. From (70) one finds $8C_1^2 = C_2$, and after a coordinate change $u = 2e^{-t}, x' = x/2, y' = \sqrt{2} C_2 y, z' = \sqrt{2} C_2 z$ the metric reads

$$C^2 ds^2 = \exp(-2e^{-t})(-dt^2 + e'^2 dx^2) + e'^2 dt^2 + e'' dx^2).$$

(90)

Thus we find a unique spacetime up to a constant rescaling. The only non-zero components of the projected Weyl and shear tensors are

$$H_{23'}(t) = 2\theta_{22}(t) = -\frac{C^2}{2} \exp\left(-\frac{3}{2} t + 2e^{-t}\right),$$

(91)

$$\sigma_{23'}(t) = \sigma_{33'}(t) = \frac{1}{2} \sigma_{11}(t) = -\frac{1}{3} C \exp(-t - e^{-t}),$$

(92)

while the scalar expansion rate, energy density and pressure are given by

$$\theta(t) = C \exp\left(-e^{-t}\right)(e^{-t} + \frac{3}{2}),$$

(93)

$$\mu(t) = \frac{3C^2}{4} \exp(-2e^{-t})(e^{-t} + 1),$$

(94)

$$p(t) = \frac{3C^2}{4} \exp(-2e^{-t})(e^{-t} - 1).$$

(95)

Note from (89) and (96) that constant $p$ is not allowed, which could be deduced more directly from (78) and (80). Together with theorem 4.2 this provides an alternative proof for the non-existence of irrotational PM dust [42]. One may eliminate $t$ from (81) and (96) via

$$\mu - p = \frac{3C^2}{4} \exp(-2e^{-t}) > 0,$$

which yields the following equation of state

$$\mu + p = \frac{1}{2} (\mu - p) \ln \left(\frac{2(\mu - p)}{3C^2}\right).$$

(96)
We have \( u^a = C \exp(e^{-t}) \frac{\partial}{\partial t} \), which is future directed for \( C > 0 \). We conclude that \( \mathcal{A} \) is the metric of a perfect fluid model, which starts off with a stiff matter-like big-bang singularity at a finite proper time in the past (corresponding to \( t = -\infty \), with \( \lim_{u \to -\infty} u/p = 1 \) and which expands indefinitely towards an Einstein space with \( \mu(\infty) = 3\mu^2, p(\infty) = -3\mu^2, \theta(\infty) = 3\theta \). Note that the dominant energy condition \( \mu > 0, -\mu < p < \mu \) is satisfied throughout spacetime, whereas at \( t = 0 \) (i.e. after a proper time) \( \int_1^\infty \exp(-u) \frac{du}{u} \) becomes negative.

We conclude with:

**Theorem 4.3.** Up to a constant rescaling of the metric, there exists a unique purely magnetic perfect fluid which satisfies any two of the three properties of Theorem 4.2. This fluid is non-rotating and has vanishing spatial gradients of energy density and pressure, with the line element given by (30) and the equation of state by (20). It is orthogonally spatially homogeneous of Bianchi type VI\(_0\). The Petrov type is I(\( M^\infty \)) in the extended Arianrhod-McIntosh classification, the shear tensor being degenerate in the plane perpendicular to the 0-eigendirection of the projected Weyl tensor.

4.4. Relaxed purely magnetic condition.

We want to indicate here how the algebraically general purely magnetic spacetime of theorem 4.2 (and at the same Lozanovski's type D class \([31]\)) naturally fits into a wider class of perfect fluid models. More precisely, we drop the purely magnetic condition and look at the class \( \mathcal{A} \) of non-vacuum, non-conformally flat, non-rotating perfect fluids (\( \mathcal{A}, g_{ab}, u^a \)) which have a degenerate shear tensor \( \sigma_{ab} \neq 0 \) and vanishing spatial gradients of energy density and pressure. As \( \omega_a = \dot{u}_a = D_a p = D_a \mu = 0 \), one derives \( D_a \theta = 0 \) from (18) and (19) with \( f = \mu \), after which \( D_a(\sigma_{ab} \sigma^{bc}) = 0 \) follows from (18) and (19) with \( f = \theta \). The shear being degenerate, \( \sigma_{ab} \sigma^{bc} \) is the only independent scalar which may be built from \( \sigma_{ab} \). As \( \dot{u}_a = \omega_a = 0 \), we may choose a time coordinate \( t \) such that \( u = \frac{\partial}{\partial t} \), \( \mu, p, \theta \) and \( \sigma_{ab} \sigma^{ab} \) depend then on \( t \) only and it follows that \( \mathcal{A} \) is part of the class of so called *kinematically homogeneous* perfect fluids, defined and studied in [13]. It follows from the analysis there that for any member of \( \mathcal{A} \), an eigenframe \( B = \{ \mathbf{u}, \mathbf{e}_a \} \) of \( \sigma_{ab} \) exists, for which \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \) hold, the shear being degenerate in the \( (\mathbf{e}_2, \mathbf{e}_3) \) plane (\( h_1 = h_2 + h_3 = 0 \)). Herewith, it follows from (18) and (19) that \( \sigma_{ab}, E_{ab} \), and \( H_{ab} \) diagonalize simultaneously in \( B \). Thus the fluid congruence \( u^a \) is Weyl principal and, as follows from the introduction, purely magnetic (purely electric) spacetimes of \( \mathcal{A} \) automatically have a PM (PE) Weyl tensor \( u \) r.t. \( u^a \). From the (22) and (33) components of (18) and (19) one obtains

\[
E_{22} = E_{33}, \quad H_{22} = -h_2 n_3, \quad H_{33} = -h_2 n_2.
\]

Also, the difference of the (22) and (33) components of the field equations reduces to \( n_1(n_2 - n_3) = 0 \).

For \( n_2 = n_3 \), the Petrov type is D, and it was further deduced in [13] that the corresponding models are spatially homogeneous and LRS III. Note from [97] that any PE model in \( \mathcal{A} \) is of Petrov type D (with \( n_2 = n_3 = 0 \) and was proved [50] to belong to the Szekeres-Szafron family \([51, 52, 53, 54]\). This family is here characterized as the PE subclass of \( \mathcal{A} \), whereas Lozanovski’s family \([37]\) forms precisely the PM D subclass of \( \mathcal{A} \).

For \( n_1 = 0 \) on the other hand, one reobtains [22] from

\[
\left( \begin{array}{ccc} 3 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{array} \right) - \left( \begin{array}{ccc} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 0 & 3 \end{array} \right) = 0,
\]

while \( H_{11} = 0, H_{33} = -H_{22} \neq 0 \) then follows from (27). For \( n_2 = n_3 = 0 \) one obtains a subclass of the PE type D models of \( \mathcal{A} \), while the Petrov type is I if and only if \( n_2 \neq 0 \). Moreover, the essentially unique spacetime of theorem 4.3 can now be characterized alternatively as the unique type I(\( M^\infty \)) member of \( \mathcal{A} \); in this respect, also note that the invariants (4) and (6) become

\[
J = -2E_{22}(E_{22} + H_{22}^2), \quad M = \frac{-2H_{22}(E_{22}^2 + H_{22}^2)}{E_{22}(E_{22}^2 + H_{22}^2)}.
\]

Further, one derives that \( q_a = r_a = \partial_a n_2 = 0 \) in this case, just as in section 4.2 (following [43]). Thus the corresponding spacetimes are spatially homogeneous (\( \partial_a \equiv 0 \)) of Bianchi type VI\(_0\). The surviving equations describe the so-called “degenerate shear” subclass \( S^1_2(VI_0) \) of the Bianchi VI\(_0\) family \([55]\).

The algebraic relations \([98]-[103]\) are consistent under evolution by \([104]-[108]\) and the Bianchi propagation equations. Hence for every choice of the free function \( p(t) \), there is a family of solutions to the Einstein equations corresponding with the dynamical system \([101]-[103]\). When a specific barotropic equation of state \( p = \rho(\mu) \) is assumed, this system becomes autonomous by \([100]\); the case \( p = -\Lambda \) is of particular interest since a family of irrotational dust spacetimes is then obtained, but an explicit integration is not known. Another possibility to extract a subfamily is to impose conditions on the Weyl tensor, e.g. \( E_{22} = bH_{22} \) for constant \( b \). By \([95] \) and \([99]\) this equation turns \([102]-[105]\) into an autonomous dynamical system in \( h_2 \) and \( n_2 \), parametrized...
by $b$, the exact solution of which can be given in terms of elliptic integrals. Alternatively, $E_{22} = bH_{22}$ and its time evolution ensures that $n_2$ and $p$ become algebraically dependent of $h_2$ and $\theta$, which turns $\frac{\theta}{h_2} = \frac{1}{b}$ into an autonomous system. The case $b = 0$ eventually yields the purely magnetic spacetimes $\frac{\theta}{h_2} = \frac{1}{b}$. Finally note that imposing both a specific equation of state and a relation of the form $E_{22} = bH_{22}$, would generically force the mentioned dynamical systems to be inconsistent, which clarifies again the result for PM irrotational dust $^{[42]}$.

5. CONCLUSION

Perfect fluid solutions of the Einstein field equations were considered, for which the Weyl tensor is purely magnetic with respect to the fluid velocity $u^a$. We first generalized the results $^{[42]}$ on the non-existence of the so-called anti-Newtonian non-rotating dust universes, to the case of rotating dust which is either of Petrov type D, or which has a vanishing spatial gradient of the matter density. These results, in combination with some ongoing work on rotating dust (for which the further subcase of degenerate shear has by now been dealt with) lead us to conjecture that purely magnetic dust spacetimes do not exist. Motivated by the existence of non-rotating purely magnetic perfect fluids of Petrov type D, for which both the spatial gradients of energy density and pressure vanish $^{[37]}$, we studied the algebraically general case and demonstrated that any two of the following conditions implies the third: (i) the fluid congruence is non-rotating, (ii) the fluid is non-accelerating, (iii) the fluid has a vanishing spatial gradient of the matter density. For purely magnetic perfect fluids satisfying these conditions it turns out that the magnetic part of the Weyl tensor has one 0 eigenvalue and that the shear tensor is degenerate in the plane orthogonal to the corresponding eigenvector. The unique (up to constant rescaling) solution satisfying these conditions is an orthogonally spatially homogeneous perfect fluid of Bianchi type $\text{VI}_0$ which has a big bang singularity, starts of as stiff fluid and asymptotically evolves towards an Einstein space.

Finally notice that purely magnetic spacetimes of Petrov type D or $I(M^\infty)$, as considered in the present work, cannot be conformally mapped to vacuum spacetimes, by the results of $^{[8]}$ and $^{[1]}$ and the fact that the Weyl tensor is preserved under such mappings. The question remains open whether type $I(M^\infty)$ purely magnetic spacetimes exist at all.

An overview of the results obtained in this paper for Petrov type I is given in table I.

Note added in proof: it has been brought to our attention that the same metric of section 4.3 has been independently found by C. Lozanovski.

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APPENDIX

We provide here the quite long and technical proof for the part $(i), (ii) \Rightarrow (iii)$ of proposition 4.2, stating that for non-rotating and non-accelerating type I PMFs the spatial 3-gradient of the energy density vanishes. Projection w.r.t. an orthonormal eigenframe $B$ of $\sigma_{ab}$ turns out to be favorable, and we actually prove the result regardless of the Petrov type. We will split the proof in two cases: degenerate and non-degenerate shear. For conciseness we will write $\rho \equiv \mu + p$ and we add $m_\alpha \equiv \partial_\alpha \mu = \partial_\alpha \rho, z_\alpha = \partial_\alpha \theta$ as variables. Note that we may exclude $\rho = 0$ by the result of $^{[12]}$ or $^{[13]}$.

With $\sigma_{a+1 \alpha - 1} = 0$, the $(a+1 \alpha - 1)$ component of $^{[18]}$ reads $W_a h_\alpha = 0$, such that $W_a = 0$ when the shear tensor is non-degenerate; for degenerate shear, say $h_1 = 0$, only $W_2 = W_3 = 0$ follows, but we may then partially fix the frame by taking $W_1 = 0$ (hereby leaving the freedom of performing rotations about an angle $x$ in the $(e_2, e_3)$-plane satisfying $\partial_\theta z_2 = 0$), such that also here the triad $\{e_\alpha\}$ is Fermi-propagated along the fluid flow $^{[48]}$. Hence the equations $^{[48]} - ^{[5]}$ are valid. Further, the Bianchi constraint $\text{div} E_\alpha$ translates to

$$Q_\alpha \equiv m_\alpha + 3h_\alpha H_{a+1 \alpha - 1} = 0. \quad (A.1)$$

On applying $\partial_0$ to $Q_\alpha$ and eliminating $H_{a+1 \alpha - 1}$ by means of $Q_\alpha$, one gets

$$\partial_0 m_\alpha = -\frac{5}{3}\theta m_\alpha + \frac{1}{6}(h_{a+1} - h_{\alpha - 1})m_\alpha. \quad (A.2)$$

Herewith, $[\partial_0, \partial_\alpha]_\rho$ becomes

$$R_\alpha \equiv \rho z_\alpha - \frac{1}{6}(2\theta + h_{a+1} - h_{\alpha - 1})m_\alpha = 0 \quad (A.3)$$

while $[\partial_0, \partial_\alpha]_\theta$ results in an expression for $\partial_0 z_\alpha$:

$$\partial_0 z_\alpha = -\theta z_\alpha + (h_{a+1} - h_{\alpha - 1})z_\alpha + \frac{1}{6}m_\alpha + 2\rho h_\alpha^2 - 2r_\alpha h_{\alpha - 1} \quad (A.4)$$

(I) Suppose the shear tensor is degenerate, say $h_1 = 0$. Then $h_3 = -h_2 \neq 0$, $m_1 = 0$ by $R_1$, while $\{Q_2, R_2, \partial_\theta h_1 = 0\}$ (resp. $\{Q_3, R_3, \partial_\theta(h_1 = 0)\}$) can be solved for $H_{13}, z_2, q_2$ (resp. $H_{12}, z_3, r_3$) to give:

$$H_{13} = -\frac{m_2}{3h_2} z_2 = \frac{m_2(2\theta - h_2)}{6\rho}, \quad (A.5)$$

$$q_2 = \frac{m_2(2h_2\theta - h_2^2 + 2\rho)}{6\rho h_2^2}, \quad (A.6)$$

$$H_{12} = \frac{m_3}{3h_2} z_3 = \frac{m_3(2\theta - h_2)}{6\rho}, \quad (A.5)$$

$$r_3 = -\frac{m_3(2h_2\theta - h_2^2 + 2\rho)}{6\rho h_2^2}. \quad (A.6)$$
Suppose the shear is non-degenerate, i.e. solutions are known. Then \([\partial_0, \partial_2]m_3\) implies
\[
X_1 \equiv 9h_1 h_2 h_3 m_3^2 p + [3h_2^2 h_3^2 (r_2 m_3 + q_3 m_2)] + 3h_2^2 h_3^2 n_1 m_1 - h_1 (h_2 - h_3) m_3 m_3 |\rho|^2 = 0 \quad (A.10)
\]
where in (A.9) and (A.10) we have eliminated the \(H_{a+1,a-1}\) by \(Q_0\) terms and the \(z_a\) by \(R_3\). By repeating the above for the index couples 31 and 12 instead of 23, one gets equations \(X_2, X_3\), which can formally be obtained by cyclic permutation of the indices of \(X_1\) (twice). Taking the combination \(m_2 X_2 - m_3 X_3\) one gets (with \(\rho \neq 0\)):
\[
A_0 \equiv 3h_2^2 h_3^2 n_2 m_2 - 3h_2^2 h_3^2 n_3 m_3 + 3h_1 h_2 h_3 (m_2 m_3 (h_2 q_1 - h_3 r_1) + m_1 m_2 h_2 r_3 - m_1 m_3 h_3 q_2) = m_1 m_2 m_3 (h_1^2 + 2 h_2 h_3). \quad (A.11)
\]
On examination of (A.9), (A.10) and (A.11) one sees that calculation of \(A_{i+1} \equiv [\partial_0 A_i + (\frac{19 + 22}{3}) \theta A_i, i = 0, \ldots, 5\) yields a system \(\{A_i, i = 0, 6\}\) of 7 linear equations in \(n_2, n_3, q_1, r_1, q_2, r_3\), parametrized by the \(m_a\) and \(h_a\), which can only be consistent if the determinant of the extended system matrix vanishes. This yields \(m_1 m_2 m_3 h_1^2 h_2^2 h_3^2 C_1(h_2, h_3) = 0\), where \(C_1(h_2, h_3)\) is a homogeneous polynomial of degree \(8\); thus if \(C_1(h_2, h_3)\) vanished the ratio \(h_2/h_3\) would be constant, in contradiction with lemma 1 (c) and the assumed non-degeneracy of the shear. Therefore \(m_1 m_2 m_3 = 0\). As a vanishing spatial gradient of the energy density (all \(m_i\) zero) implies degenerate shear (cf. section 4), we are left with two qualitatively different subcases: (a) \(m_1 = m_2 m_3 \neq 0\) and (b) \(m_2 = m_3 = 0, m_1 \neq 0\). In general, if \(m_a = 0\) for fixed \(a\), it follows that \(H_{a+1,a-1} = z_a = 0\) from (A.1) and (A.3). From (A.4), (A.9) and (A.10), one then deduces \(L_a \equiv h_2^2 (q_a - h_{a-1} r_a) = 0, \partial_0 L_a = (4h_{a+1} + 5 h_{a-1}) h_{a+1} q_a + (4h_{a+1} + 5 h_{a+1}) h_{a-1} r_a = 0\), which for non-degenerate shear can only be consistent if \(q_a = r_a = 0\).
(a) $m_1 = 0, m_2 m_3 \neq 0$:
As also $q_1 = r_1 = 0$, one deduces $n_{22} = n_{33} = 0$ from $X_2 = X_3 = 0$, i.e., $n_2 = n_3 = -n_1 = \frac{1}{2} n_{11}$.
Then e.g. the (31) component of the field equations together with
\[
\begin{pmatrix} 2 & 1 & 2 & 3 \\
\end{pmatrix}
\]
and the (31) component of
\[
\text{(13)}
\]
together with the (2) component of
\[
\text{(24)}
\]
respectively give
\[
\begin{align*}
\partial_2 n_1 &= -2 q_2 n_1, \\
\partial_2 H_{33} &= (H_{33} - H_{22}) r_2 - 3 H_{13} n_1. \tag{A.12}
\end{align*}
\]
while
\[
\text{(53)}
\]
further reduces to
\[
H_{22} = n_1 h_2, \quad H_{33} = -n_1 h_3. \tag{A.13}
\]
Propagation of the second equation of
\[
\text{(A.13)}
\]
along the $\partial_4$ integral curves, using
\[
\text{(55)}, \quad \text{(A.12)}
\]
and
\[
\text{(A.13)},
\]
yields $n_1 (4 H_{13} + z_2 + 4 h_3 q_2) = 0$. As $n_1 = 0$ is not allowed by
\[
\text{(A.13)},
\]
it follows that $B_0 \equiv 4 H_{13} + z_2 + 4 h_3 q_2 = 0$. Calculation of
\[
R_{j+1} = \partial_6 B_1 + \frac{1 + 2 q_2}{2} \partial_6 B_1, \quad i = 0, 1, 2
\]
gives a homogeneous linear system
\[
\{ Q_2, B_0, B_1, B_2, B_3 \}
\]
in the variables $q_2, r_2, m_2, z_2, H_{13}$ and parametrized by $h_3, h_3$, which can only be consistent with $m_2 \neq 0$ if the determinant, namely
\[
\frac{1}{2} h_{1}^{1} h_{2}^{3} h_{3}^{3} (21 h_3^2 + 64 h_3 h_2 + 16 h_2^2)
\]
vanishes, which is contradictory to lemma 4.1 (c).

(b) $m_2 = m_3 = 0, m_1 \neq 0$:
Apart from $q_2 = r_2 = q_3 = r_3 = 0$, one has now $n_{11} = 0$ from
\[
\text{(A.10)}
\]
One can solve the (23) component of the field equations together with its $\partial_0$ derivative and $[\partial_0, \partial_1] (n_{22} - n_{33})$ for $\partial_1 n_{22}, \partial_1 n_{33}$ to give
\[
\begin{align*}
\partial_1 n_{22} &= -\frac{1}{2 h_1} \left[ (4 H_{23} - z_1 + 4 h_2 r_1) n_{22} \right. \\
&\quad + (4 H_{23} + z_1 + 4 h_2 q_1) n_{33} \right], \tag{A.14}
\end{align*}
\]
\[
\begin{align*}
\partial_1 n_{33} &= -\frac{1}{2 h_1} \left[ (4 H_{23} - z_1 - 4 h_3 r_1) n_{22} \right. \\
&\quad + (4 H_{23} + z_1 - 4 h_3 q_1) n_{33} \right]. \tag{A.15}
\end{align*}
\]
Then $[\partial_0, \partial_1] (n_{22} + n_{33})$, eliminating $H_{23}$ by $R_1$, gives
\[
\begin{align*}
C_0 &= 12 \left[ h_2 n_{22} - (2 h_3 + 3 h_2) n_{33} \right] h_2 q_1 \\
&\quad + 12 \left[ h_3 n_{33} - (2 h_2 + 3 h_3) n_{22} \right] h_2 r_1 \\
&\quad + 13 (n_{22} - n_{33}) m_1 - 6 \left[ (h_2 + 3 h_3) n_{22} \right. \\
&\quad + (h_3 + 3 h_2) n_{33} \right] z_1 = 0. \tag{A.16}
\end{align*}
\]
Calculation of $C_{i+1} = \partial_0 C_i + \frac{6 + 2 q_2}{3} \partial_6 C_i, \quad i = 0, \ldots, 3$, eliminating in each step $H_{23}$ by means of $R_1$, gives two homogeneous linear systems $\{ C_0, C_1, C_2, C_3 \}$, $j = 3, 4$, in the variables $q_1, r_1, m_1, z_1$ and parametrized by $h_2, h_3, n_{22}, n_{33}$. Again this can only be consistent with $m_1 \neq 0$ if the determinants of the coefficient matrices vanish. However, on computing the resultant of these determinants w.r.t. $n_{22}$, resp. $n_{33}$, one gets equations $h_1^4 h_2^5 h_3^2 P(h_2, h_3) n_{22}^{16} = h_1^2 h_2^3 h_3^2 P(h_2, h_3) n_{33}^{16} = 0$, with $P(h_2, h_3)$ a homogeneous polynomial of its arguments, such that $n_{22} + n_{33} = 0$ by lemma 4.1 (c). Hence $n_1 = n_2 = n_3 = 0$ in contradiction with
\[
\text{(53)}.
\]
Hence case (II) is not allowed, and this concludes the proof.

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[56] However, for LRS class I a complicated third order ODE needs to be solved. For LRS class III the metric was shown to be explicitly constructable, but a simpler form than the one suggested in [22] can be obtained by directly taking the function $s(t)$ as a new time-variable $u$ instead of $t$; the result is $C^2d\sigma^2 = \frac{1}{4}\left(-\frac{2d\sigma^2}{u^2} + u^2 \times [d\sigma + q eB(zdy - ydz)]^2 + e^2 (dy^2 + dz^2)\right)$, where $C, c$ and $q$ are constants and $k$ equals -1, 0 or 1, $B = -\log(1 + k(u^2 + z^2))$ and $\Gamma = 2q u^4 - ku^2 + c$.
[57] Following [45], different from the convention in [47], where $\eta_{tt}\equiv-1$ is taken.
[58] The $\Omega_\alpha$ are the non-zero components of the angular velocity vector of the triad $\{e_\alpha\}$ w.r.t. the ‘inertial compass’, see [48] and references therein. The notation $n_{\alpha\beta}$ is also in agreement with [45], where the further decomposition $n_{\alpha\beta} = \delta_{\alpha\beta}n_{\alpha\beta} + \delta_{\alpha\beta}n_{\alpha\beta} - \delta_{\alpha\beta}n_{\alpha\beta}$, $\eta_{\alpha\beta} = n_{\alpha\beta}$ of the purely spatial coefficients is exploited.