Monodromic strings

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Abstract

We argue that apart from the standard closed and open strings one may consider a third possibility that we call monodromic strings. The monodromic string propagating on a target looks like an ordinary open string (a mapping from a segment to the target) but its space of states is isomorphic to that of a closed string. It is shown that the monodromic strings naturally appear in T-dualizing closed strings moving on simply connected targets. As a nontrivial topology changing example we show that the monodromic strings on a compact Poisson-Lie group are T-dual to the standard closed strings propagating on the noncompact dual PL group.
1. The D-brane example shows [1] that one should not apriori discard "non-standard" string boundary conditions from consideration. In fact, the powerful duality principle require their presence in string theory. In this paper, we shall introduce other non-standard string boundary conditions whose existence is required by Poisson-Lie T-duality [2]. We shall see a posteriori, that this new phenomenon exists also in the Abelian limit of the PL T-duality where it does not reduce to the standard momentum-winding change. In fact, it gives the version of the standard Abelian duality for the simply connected target. The reader will see that the developed formalism can be viewed as a sort of continuum version of the discrete orbifold construction [3] in the sense the twisted sectors are parametrized by a continuous parameter. The monodromic PL-T-duality (and its Abelian limit) simply exchanges the continuous families of the invariant and twisted sectors.

The problem addressed in this article is the old one: "How to include zero modes in the Poisson-Lie T-duality story"? A partial answer to this question was given in [4], where it was shown that the role of the Abelian momentum-winding lattice is in general played by the fundamental group of the underlying Drinfeld double. However, non-Abelian doubles have small fundamental groups in general, thus the phenomenon of the momentum-winding exchange does not have much content in the non-Abelian setting.

We are going to show here, that the non-Abelian momentum-winding exchange will become a much richer structure if we release the constraint that the string should be closed. In fact, the duality itself will tell us how to "tear up" a closed string. The disruption is measured by a certain monodromy and this monodromy is nothing but the (non-Abelian) momentum of the dual strictly closed string.

The plan of this paper is as follows. First we shall review the Poisson-Lie T-duality without the zero modes. In particular, we shall write down the corresponding duality invariant action in the Drinfeld double. Then we shall modify the action on the double by adding a new variable which will transform into the momentum zero modes of closed strings if we descend from the double to one of the Poisson-Lie group targets. But if we descend to another (dual) target that new variable becomes a monodromy that measures how the closed string got torn up.

We shall finish by a detailed description of this phenomenon in the context of the Lu-Weinstein-Soibelman [5] pair of the Poisson-Lie groups and also in the context of the ordinary Abelian T-duality [6].
2. The Poisson-Lie T-duality in its more modern version \[4\] relates two non-linear \(\sigma\)-models living in two different targets \(D/G\) and \(D/\tilde{G}\). Here \(D\) is a Lie group such that i) \(\dim D = 2\dim G = 2\dim \tilde{G}\); ii) \(G\) and \(\tilde{G}\) are both subgroups of \(D\); iii) it exists a symmetric non-degenerate invariant bilinear form \((.,.)\) on \(D = \text{Lie}(D)\) such that \((G, G) = (\tilde{G}, \tilde{G}) = 0\), where \(G = \text{Lie}(G)\) and \(\tilde{G} = \text{Lie}(\tilde{G})\). In other words, \(G\) and \(\tilde{G}\) are isotropic subalgebras of \(D\).

Let us state clearly, that not every group \(D\) satisfying the conditions i)-iii) is the so-called Drinfeld double. For this to be true, it is moreover required that iv) \(G \cap \tilde{G} = 0\). However, the modern version of the Poisson-Lie T-duality does not require iv). In our previous paper \[7\], we have somewhat abusively called \(D\) the ”Drinfeld double” even when the condition iv) was not satisfied.

In order to make this paper technically simpler, we shall study here an older less general version of the Poisson-Lie T-duality \[2, 8\] where on the top of the conditions i)-iii) two more things are required: the first is that \(D\) is indeed the Drinfeld double (the condition iv) is fulfilled) and the second is that \(D\) is the so-called perfect Drinfeld double, which means that \(D\) can be globally smoothly decomposed as \(D = GG = \tilde{G} \tilde{G}\).

If the double is perfect, then the Poisson-Lie T-duality exchanges the targets \(G\) and \(\tilde{G}\) because \(D/G\) can be identified with \(\tilde{G}\) and \(D/\tilde{G}\) with \(G\). Recall that a Poisson-Lie structure on \(G\) is characterized by a Poisson bivector \(\alpha \in \Lambda^2 TG\) (fulfilling the Jacobi identity) or, equivalently, by its right trivialization. The latter is a map \(\Pi : G \to \Lambda^2 G\) defined as follows

\[
\Pi(g) = R_{g^{-1}} \alpha_g, \quad (1)
\]

where \(R_g\) is the right transport on the group manifold \(G\). It is moreover required that a cocycle condition is fulfilled:

\[
\Pi(gh) = \Pi(g) + \text{Ad}_g \Pi(h). \quad (2)
\]

Now we can write down the actions of the corresponding pair of \(\sigma\)-models:

\[
S_\Pi = \frac{1}{8\pi} \int \langle (R + \Pi(g))^{-1}, dgg^{-1} \wedge *dgg^{-1} \rangle. \quad (3)
\]

\[
S_{\tilde{\Pi}} = \frac{1}{8\pi} \int \langle (R^{-1} + \tilde{\Pi}(\tilde{g}))^{-1}, d\tilde{g}\tilde{g}^{-1} \wedge *d\tilde{g}\tilde{g}^{-1} \rangle. \quad (4)
\]

Here \(\tilde{\Pi}(\tilde{g})\) is the Poisson-Lie structure on the dual group \(\tilde{G}\) and \(R \in G \otimes G\) is some nondegenerate bilinear form on the dual space \(G^\ast\) of the Lie algebra.
$\mathcal{G}$. Since $\mathcal{G}^*$ can be naturally identified with $\check{\mathcal{G}}$ via the bilinear form $(.,.)$ on the double $\mathcal{D}$, we may consider the inverse bilinear form $R^{-1}$ as an element of $\check{\mathcal{G}} \otimes \check{\mathcal{G}}$. The symbol $(.,.)$ denotes the pairing between a vector space and its dual. Finally, $dgg^{-1} \in T^*\Sigma \otimes \mathcal{G}$ is the pull-back of the right-invariant Maurer-Cartan form on $\mathcal{G}$ to the world-sheet $\Sigma$ and $*$ is the Hodge star on $\Sigma$.

The pair of models (3) and (4) was first introduced in [2] in somewhat disguised form. In the form (3) and (4), it was rewritten in [8]. It is very important to understand, in which sense these two models are dual to each other. We can study a closed string propagation on the group $\mathcal{G}$ governed by the action (3) and do the same thing for the target $\check{\mathcal{G}}$ and the action (4). However, there is no duality in this case. In other words, it is not true that the Poisson-Lie T-duality relates (3) and (4) as models of standard closed strings. The models (3) and (4) become dual only if we remove momentum zero modes from the closed strings. In the Abelian context this would mean that the strings may have only the oscillator excitations.

First of all, let us review [2, 4, 8] what we mean by the removing the momentum from the closed string in the non-Abelian context. For this, consider standard closed strings propagating in the simply connected group $\mathcal{G}$ according to the action (3). Corresponding field equations can be most easily expressed by introducing certain 1-form $\tilde{\lambda}$ on the world-sheet with values in the dual Lie algebra $\check{\mathcal{G}}$:

$$\tilde{\lambda} = -\pi_{\check{\mathcal{G}}} Ad_g(R + \Pi(g))^{-1}(\partial_+ gg^{-1}, .)d\xi^+ + \pi_{\check{\mathcal{G}}} Ad_g(R + \Pi(g))^{-1}(., \partial_- gg^{-1})d\xi^-.$$  

(5)

Here $\xi^\pm$ are the usual lightcone variables on the cylinder:

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \quad \tau \in \mathbb{R}, \quad \sigma \in [0, 2\pi];$$

\footnote{As we have already said, there exists a way of implementing some (discrete) momentum modes into the duality story if the groups $\mathcal{G}$ and/or $\check{\mathcal{G}}$ are not simply connected. If the double $\mathcal{D}$ is perfect then $\pi_1(\check{\mathcal{G}})$ parametrize the possible discrete momentum modes and $\pi_1(\mathcal{G})$ parametrizes the winding modes of the closed strings moving on $\mathcal{G}$. From the point of view of the target $\check{\mathcal{G}}$, the momentum-winding interpretation of the homotopy groups gets exchanged. This way covers the famous momentum-winding exchange in the Abelian T-duality context where $\mathcal{G}$ is a circle group and $\check{\mathcal{G}}$ is the dual circle. In this paper $\mathcal{G}$ and $\check{\mathcal{G}}$ are always simply connected groups; we are going to show that the momentum zero modes can be implemented into the duality story also for this special case at the price of "tearing up" the closed strings.}
\[ \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma} \]

and \( \pi_{\tilde{G}} \) is a projector to the subalgebra \( \tilde{G} \) with kernel \( G \). Note that the expression \( (R + \Pi(g))^{-1}(\partial_{\pm} gg^{-1}, \ldots) \) lies in \( \tilde{G} \). We view it as an element of \( \mathcal{D} \) and act in the adjoint way by the element \( g \in G \). The result is projected by \( \pi_{\tilde{G}} \) so that \( \tilde{\lambda} \) lies in \( \tilde{G} \). The field equations in terms of \( \tilde{\lambda} \) have a very simple form:

\[ d\tilde{\lambda} = \tilde{\lambda} \wedge \tilde{\lambda}, \quad (6) \]

or, in some basis \( \tilde{T}_a \) of \( \tilde{G} \):

\[ d\tilde{\lambda}^a = \frac{1}{2} \tilde{f}^{a}_{bc} \tilde{\lambda}^b \wedge \tilde{\lambda}^c. \quad (7) \]

Here \( \tilde{\lambda} = \tilde{\lambda}^a \tilde{T}_a \) and \( \tilde{f}^{a}_{bc} \) are the structure constants of \( \tilde{G} \). The only nontrivial fact needed for deriving the field equations (6) from the action (3) is the cocycle condition (2).

We see that every solution \( g(\tau, \sigma) \) of the field equations of the model (3) defines a flat \( \tilde{G} \)-valued connection \( \tilde{\lambda} \). Its monodromy is defined by the formula

\[ \tilde{M} = P \exp \int_{\gamma} \tilde{\lambda}(g), \quad (8) \]

where \( \gamma \) is a curve going around the cylinder. In particular, we can choose a curve \( \tau = \text{const.} \). This monodromy is called a noncommutative momentum \cite{2, 4, 8} and its conjugacy class does not depend on time if \( g \) is a solution of the field equations. In particular, if the noncommutative momentum \( \tilde{M} \) is the unit element \( \tilde{e} \) of the dual (by assumption also simply connected) group \( \tilde{G} \) at some time, then it will remain \( \tilde{e} \) for all times.

Suppose now that

\[ \tilde{M} = \tilde{e} \quad (9) \]

for some solution \( g(\tau, \sigma) \in G \). This means that it exists a single-valued function \( \tilde{h}(\tau, \sigma) \in \tilde{G} \) on the world-sheet such that

\[ \tilde{\lambda}(g) = d\tilde{h}\tilde{h}^{-1}. \quad (10) \]

Consider then the following \( D \)-valued function \( l(\tau, \sigma) \in D \) on the worldsheet:

\[ l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma). \quad (11) \]
This mapping can be decomposed as
\[ l(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma), \quad \tilde{g} \in \tilde{G}, \quad h \in G, \]  
(12)
because of the fact that we have two global decompositions of the double: \( D = G\tilde{G} = \tilde{G}G \). Then it turns out [2] that \( \tilde{g}(\tau, \sigma) \) is a solution of field equations of the dual model (4) and a dual \( \mathcal{G} \)-valued connexion \( \lambda \) is given by
\[ \lambda(\tilde{g}) = dhh^{-1}. \]  
(13)
Since the field \( h(\tau, \sigma) \) is evidently single-valued, the dual noncommutative momentum \( M \) also satisfies
\[ M = P \exp \int_{\gamma} \lambda(\tilde{g}) = e, \]  
(14)
where \( e \) is the unit element of \( G \).

It is well-known that the phase space of a field theoretical model can be viewed as the space of its classical solutions. Consider the phase space \( \mathcal{Y} \) of classical solutions corresponding to closed strings propagating according to (3) and perform a symplectic reduction by imposing the constraint (9) of the unit noncommutative momentum. We obtain in this way a reduced phase space \( \mathcal{Y}_e \). We do the same thing for the model (4) and we obtain a dual reduced phase space \( \tilde{\mathcal{Y}}_e \). Thus both reduced phase spaces \( \mathcal{Y}_e \) and \( \tilde{\mathcal{Y}}_e \) inherit symplectic structures from \( \mathcal{Y} \) and \( \tilde{\mathcal{Y}} \), respectively. Moreover, since the unit noncommutative momentum constraints commute with the time evolution, it follows that \( \mathcal{Y}_e \) and \( \tilde{\mathcal{Y}}_e \) inherit also certain Hamiltonians \( H_e \) and \( \tilde{H}_e \) from the closed string Hamiltonians \( H \) and \( \tilde{H} \).

The meaning of the usual statement that the models (3) and (4) are related by the Poisson-Lie T-duality is the following: There exists a symplectomorphism (preserving the Hamiltonian) between the dynamical systems (\( \mathcal{Y}_e, H_e \)) and (\( \tilde{\mathcal{Y}}_e, \tilde{H}_e \)). This symplectomorphism was found in [2, 8]. For its more algebraic description see [9].

There exists a duality invariant description [8] of the equivalent dynamical systems (\( \mathcal{Y}_e, H_e \)) and (\( \tilde{\mathcal{Y}}_e, \tilde{H}_e \)). It turns out that the phase space \( \mathcal{Y}_e \) can be identified with the coset \( LD/D \) where \( LD \) denoted the loop group of the Drinfeld double \( D \). In other words, \( LD \) is the set of smooth maps from a circle \( S^1 \) into \( D \) equipped with the pointwise multiplication. The symplectic form \( \Omega \) on this coset can be defined as the exterior derivative of certain 1-form \( \theta \) on \( LD/D \). The latter is most naturally defined in terms of its integral
along an arbitrary curve \( \Gamma \) in the phase space, parametrized by a parameter \( \tau \). This curve can be represented by a certain \( D \)-valued function \( l(\tau, \sigma) \in D \). We define

\[
\int_{\Gamma} \theta = \frac{1}{8\pi} \int (\partial_{\sigma}ll^{-1}, \partial_{\tau}ll^{-1}) + \frac{1}{48\pi} \int d^{-1}(dll^{-1}, [dll^{-1}, dll^{-1}]),
\]

where \((.,.)\) is the invariant bilinear form on \( D \) and we recognize also the well-known WZW term on the r.h.s.. Note that this definition of \( \theta \) is ambiguous since the choice of the inverse exterior derivative \( d^{-1} \) is too. However, this ambiguity disappears at the level of the symplectic form \( \Omega = d\theta \).

It may appear that (15) gives the action of the standard WZW model but this is not quite true, because \( \tau \) and \( \sigma \) are not the light cone variables \( \xi^\pm \). Nevertheless, the only difference between the ordinary WZW model and our expression (15) consists in the names of the variables. This means that our expression enjoys the formal mathematical properties of the standard WZW action. In particular, if we replace \( l(\tau, \sigma) \) by \( l(\tau, \sigma)l_0(\tau) \), the integral \( \int \theta \) does not change (the chiral invariance of the WZW model!) hence the symplectic form \( \Omega \) lives really on the coset \( LD/D \) and not on \( LD \) itself.

As it is well-known, a first order action of a dynamical system (\( \Omega = d\theta, H_{\tilde{e}} \)) is given by

\[
S = \int (\theta - H_{\tilde{e}}) dt,
\]

where the Hamiltonian \( H_{\tilde{e}} \) can be written as follows [8]:

\[
H_{\tilde{e}} = \frac{1}{8\pi} \int (\partial_{\sigma}ll^{-1}, R\partial_{\sigma}ll^{-1}).
\]

(17)

Here \( R \) is a linear idempotent self-adjoint map from the Lie algebra \( D \) to \( D \) itself. \( R \) has two equally degenerated eigenvalues +1 and -1 and the corresponding eigenspaces \( R_\pm \) are

\[
R_+ = \text{Span}(t + R(t, .)), t \in \tilde{G}, \quad R_- = \text{Span}(t - R(., t)), t \in \tilde{G}.
\]

(18)

Needless to say, \( R(., .) \) is the bilinear form appearing in (3) and (4). Putting (15) and (17) into (16), we obtain the explicit form of the action of the dynamical system \( (\Upsilon_{\tilde{e}}, H_{\tilde{e}}) = (\tilde{\Upsilon}_e, \tilde{H}_e) \):

\[
S(l) = \frac{1}{8\pi} \int (\partial_{\sigma}ll^{-1}, \partial_{\tau}ll^{-1}) + \frac{1}{48\pi} \int d^{-1}(dll^{-1}, [dll^{-1}, dll^{-1}])
\]
\[-\frac{1}{8\pi} \int (\partial_{\rho} l^{-1}, \mathcal{R} \partial_{\rho} l^{-1}). \]  

Note that this action is invariant with respect to the gauge transformation \( l(\tau, \sigma) \to l(\tau, \sigma) h_0(\tau) \), which means that the model lives rather on \( LD/D \) than on \( LD \).

We shall not review the derivation of the constrained models (3) and (4) from (19); actually such a derivation is a special case of a more general story that we are going to present in this paper.

3. We stress that the duality described so far takes place between \((\Upsilon, H)\) and \((\tilde{\Upsilon}, \tilde{H})\) and not between \((\Upsilon, H)\) and \((\tilde{\Upsilon}, \tilde{H})\). In other words: which is dual to the model (3) describing closed strings with \textit{arbitrary} noncommutative momentum. The crucial problem to face is the following: if the closed string solution \( g(\tau, \sigma) \) on \( G \) has a non-unit non-commutative momentum, then the configuration (cf. (11)) \( l(\tau, \sigma) \in D \) does not describe a propagation of a closed string in the double. The reason is that the map \( \tilde{h}(\tau, \sigma) \) defined by (10) is not single-valued on the world-sheet cylinder, but it develops some monodromy. If we restrict \( \tilde{h}(\tau, \sigma) \) to the interval \( \sigma \in [0, 2\pi] \), we obtain a strip propagating in \( \tilde{G} \) rather than a cylinder. The same thing happens for \( l(\tau, \sigma) = g(\tau, \sigma) \tilde{h}(\tau, \sigma) \), which does not correspond to a closed string world-sheet embedded in \( D \). If we project \( l(\tau, \sigma) \) to \( \tilde{g}(\tau, \sigma) \) according to (12), the string configuration \( \tilde{g}(\tau, \sigma) \in \tilde{G} \) will not be closed, i.e. \( \tilde{g}(\tau, \sigma+2\pi) \neq \tilde{g}(\tau, \sigma) \). It is precisely for this reason that the unit non-commutative momentum constraint was imposed in [2, 8].

Our point of view in this paper is very different: We say that it is not bad to tear up the closed strings but on the contrary, it is rather an interesting thing to do. The point is that the strings in the dual target get torn up in a controlled way dictated by duality. In particular, the duality predicts that the space of states (=the phase space at the classical level) of the torn up string on \( \tilde{G} \) must be identical to that of the standard closed string on \( G \). Thus we obtain a consistent dynamics of open-like strings whose space of states is that of the closed string! We shall do it in detail for the Lu-Weinstein-Soibelman (LWS) pair of Poisson-Lie groups.

4. The LWS double \( D \) is simply the complexification (viewed as the real group) \( \tilde{G}^C \) of a simple compact simply connected and connected group \( \tilde{G} \).
So, for example, the LWS double of $SU(2)$ is $SL(2, \mathbb{C})$. The invariant non-degenerate form $(\cdot, \cdot)$ on the Lie algebra $\mathcal{D}$ of $D$ is given by

$$(x, y) = \text{Im}K(x, y),$$

or, in other words, it is just the imaginary part of the Killing-Cartan form $K(\cdot, \cdot)$. Since $\tilde{G}$ is the real form of $\tilde{G}^\mathbb{C}$, clearly the imaginary part of $K(x, y)$ vanishes if $x, y \in \tilde{G}$. Hence, $\tilde{G}$ is indeed isotropically embedded in $\tilde{G}^\mathbb{C}$.

The dual subgroup $G$ coincides with the so called $AN$ group in the Iwasawa decomposition of $\tilde{G}^\mathbb{C}$:

$$\tilde{G}^\mathbb{C} = \tilde{G}.AN.$$  

(21)

For the groups $SL(n, \mathbb{C})$ the group $AN$ can be identified with upper triangular matrices of determinant 1 and with positive real numbers on the diagonal. In general, the elements of $AN$ can be uniquely represented by means of the exponential map as follows

$$g = e^\phi \exp[\Sigma_{\alpha>0} v_\alpha E_\alpha] \equiv e^\phi n.$$ (22)

Here $\alpha$’s denote the roots of $\tilde{G}^\mathbb{C}$, $v_\alpha$ are complex numbers and $\phi$ is an Hermitian element of the Cartan subalgebra of $\tilde{G}^\mathbb{C}$. Loosely said, $A$ is the "noncompact part" of the complex maximal torus of $\tilde{G}^\mathbb{C}$. The isotropy of the Lie algebra $\mathcal{G}$ of $G = AN$ follows from (20); the fact that $\mathcal{G}$ and $\tilde{G}$ generate together the Lie algebra $\mathcal{D}$ of the whole double is evident from (21).

5. The reason why we have chosen to work with the LWS double is simple: both isotropic subgroups $G$ and $\tilde{G}$ are non-Abelian and one of them ($\tilde{G}$) is compact and we have a very good control of the monodromy valued in the compact group. Indeed, the non-commutative momentum (8) of closed string propagating on the noncompact group $G = AN$ according to (3) takes values in $\tilde{G}$. As we have already remarked, the non-commutative momenta correspond to the conjugacy classes in the group $\tilde{G}$.

3Recall that the Hermitian element of any complex simple Lie algebra $\tilde{G}^\mathbb{C}$ is an eigenvector of the involution which defines the compact real form $\tilde{G}$; the corresponding eigenvalue is $(-1)$. This involution comes from the group involution $g \rightarrow (g^{-1})^\dagger$. The anti-Hermitian elements that span the compact real form are eigenvectors of the same involution with the eigenvalue equal to 1. For elements of $sl(n, \mathbb{C})$ Lie algebra, the Hermitian element is indeed a Hermitian matrix in the standard sense.
It is well-known [10], that if we choose a maximal torus \( T \) in \( \tilde{G} \), then every conjugacy class intersects \( T \). It is therefore enough to study when two elements of the maximal torus lie on the same conjugacy class. The maximal torus can be viewed as the quotient of the Cartan subalgebra \( \mathcal{T} \) by the coroot lattice \( Q^\vee \) (cf. [10, 11]). We know that if two elements of \( \mathcal{T} \) are on the same adjoint orbit of \( \tilde{G} \) iff they are related by the action of the Weyl group. Thus the fundamental domain of the joint actions of the Weyl group \( W \) and of the coroot lattice \( Q^\vee \) on \( \mathcal{T} \) can be identified with the space of conjugacy classes of \( \tilde{G} \). This fundamental domain is often referred to as the Weyl alcove. We shall denote it as \( \mathcal{T}_+ \).

Now we know that we need to add degrees of freedom corresponding to the non-commutative momenta into the duality invariant action \( S(l) \) in (19). It turns out that the way to do it is very simple; the new action reads

\[
S(l, \mu) = \frac{1}{8\pi} \int (\partial_\sigma ll^{-1}, \partial_\tau ll^{-1}) + \frac{1}{48\pi} \int d^{-1}(dll^{-1}, [dll^{-1}, dll^{-1}]) + \\
+ \frac{1}{4\pi} \int (\mu, l^{-1}\partial_\nu l) - \frac{1}{8\pi} \int (\partial_\sigma ll^{-1} + l\mu l^{-1}, R(\partial_\sigma ll^{-1} + l\mu l^{-1})).
\]

(23)

Here \( l(\tau, \sigma) \in D \) is a \( \sigma \)-periodic \( D \)-valued maps and \( \mu(\tau) \in \mathcal{T}_+(\subset \tilde{G}) \). This new action is gauge invariant with respect to the transformations \( l(\tau, \sigma) \rightarrow l(\tau, \sigma)t(\tau) \) where \( t(\tau) \in T \). This means that the phase space of this dynamical system is \((LD/T) \times \mathcal{T}_+ \).

We proceed by parametrizing the field configurations \( l(\tau, \sigma) \) according to the \( D = G\tilde{G} \) decomposition of the double. In other words, we parametrize \( l \) as \( l = g\tilde{h} \), where \( g \in AN \) and \( \tilde{h} \in \tilde{G} \). The Polyakov-Wiegmann formula [12] then says

\[
\frac{1}{8\pi} \int (\partial_\sigma ll^{-1}, \partial_\tau ll^{-1}) + \frac{1}{48\pi} \int d^{-1}(dll^{-1}, [dll^{-1}, dll^{-1}]) = \frac{1}{4\pi} \int (\partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g)
\]

and the whole action (23) becomes

\[
S(g, \tilde{h}, \mu) = \frac{1}{4\pi} \int \{ (\tilde{\Lambda}, g^{-1}(\partial_\sigma - \partial_\tau)g) + (g^{-1}\partial_\tau g + \tilde{\Lambda}, P(g)(g^{-1}\partial_\sigma g + \tilde{\Lambda})) \}.
\]

(24)

Here \( P \) is a projector on the subspace \( R_- \) with the kernel \( R_+ \) (cf. (18)). Moreover, we denote

\[
P(g) = Ad_{g^{-1}}PAd_g.
\]

(26)
Note that the dependence of the action (25) on $\tilde{h}$ and $\mu$ is completely contained in
\[
\tilde{\Lambda} = \partial_\sigma \tilde{h} \tilde{h}^{-1} + \tilde{h} \mu \tilde{h}^{-1}.
\] (27)

Now a crucial observation is as follows: In distinction to the case of the Poisson-Lie T-duality without the zero modes [8], the quantity $\tilde{\Lambda}$ is not constrained by the unit monodromy constraint. In fact, the quantity $\tilde{\Lambda}$ is not constrained by any constraint whatsoever because, by construction, it can have an arbitrary monodromy. This means that we can regard the action $S(g, \tilde{h}, \mu)$ as the action $S(g, \tilde{\Lambda})$ of two unconstrained periodic variables $g, \tilde{\Lambda}$, where, moreover, the dependence on $\tilde{\Lambda}$ is Gaussian. Thus we can solve away $\tilde{\Lambda}$ from (25) which gives
\[
\tilde{\Lambda} = \tilde{\lambda}_-(g) - \tilde{\lambda}_+(g),
\] (28)

where $\tilde{\lambda}_\pm(g)$ were defined in (5). Inserting $\tilde{\Lambda}$ from (28) into the action $S(g, \Lambda)$, we obtain the action (3):
\[
S_{\Pi}(g) = \frac{1}{8\pi} \int d\tau d\sigma (R + \Pi(g))^{-1}_{ij} [(\partial_\tau gg^{-1})^{i}(\partial_\tau gg^{-1})^{j} - (\partial_\sigma gg^{-1})^{i}(\partial_\sigma gg^{-1})^{j}].
\] (29)

Here $T_i$ is a basis in $\mathcal{G} = Lie(AN)$ and $\tilde{T}^i$ its dual basis of $\tilde{\mathcal{G}}$ so that
\[
(T_i, \tilde{T}^j) = \delta_i^j.
\] (30)

Thus
\[
\partial_\tau gg^{-1} = (\partial_\tau gg^{-1})^iT_i
\] (31)

and
\[
(R + \Pi(g))^{-1} = (R + \Pi(g))_{ij}^{-1}(\tilde{T}^i \otimes \tilde{T}^j).
\] (32)

The reader may easily check our derivation of the model (3) from the duality invariant action (23) by noting an explicit formula for the Poisson bivector $\Pi(g)$:
\[
\Pi(g) = b(g)a(g)^{-1},
\] (33)

where the matrices $a(g)$ and $b(g)$ are defined as
\[
g^{-1}T_i g = a(g)^i_j T_j;
\] (34)
\[
g^{-1}\tilde{T}^i g = b(g)^{ij}T_j + d(g)^{ij} \tilde{T}^j.
\] (35)
By using the definition (8) of the noncommutative momentum $\tilde{M}$ and formulas (27) and (28), we arrive at

$$\tilde{M} = \exp 2\pi \mu. \quad (36)$$

Thus the quantity $\mu$ in the duality invariant action (23) becomes indeed the noncommutative momentum of a closed string propagating on the target $G = AN$.

6. So far we have established that the action (3) describing (non-constrained) closed strings can be written in the first-order form (23). Now we are looking for the dual action. We shall find out that it is given by a slight but interesting modification of (4) that corresponds to a replacement of closed strings by monodromic strings.

Consider a dual decomposition $l = \tilde{g}h$, where $\tilde{g} \in \tilde{G}$ and $h \in G (= AN)$. The Polyakov-Wiegmann formula now reads

$$\frac{1}{8\pi} \int (\partial_\sigma ll^{-1}, \partial_\tau ll^{-1}) + \frac{1}{48\pi} \int dt^{-1}(dll^{-1}, [dll^{-1}, dll^{-1}]) = \frac{1}{4\pi} \int (\partial_\sigma hh^{-1}, \tilde{g}^{-1} \partial_\tau \tilde{g}). \quad (37)$$

It follows that the action (23) can be rewritten as

$$S(\tilde{g}, h, \mu) = \frac{1}{4\pi} \int (\mu, h^{-1} \partial_\tau h) + \frac{1}{4\pi} \int (\Lambda, \tilde{g}^{-1} (\partial_\tau - \partial_\sigma) \tilde{g} - \mu) +$$

$$+ \frac{1}{4\pi} \int (\tilde{g}^{-1} \partial_\sigma \tilde{g} + \mu + \Lambda, P(\tilde{g})(\tilde{g}^{-1} \partial_\sigma \tilde{g} + \mu + \Lambda)), \quad (38)$$

where we have set

$$\Lambda = \partial_\sigma hh^{-1} + h\mu h^{-1} - \mu. \quad (39)$$

In distinction to the variable $\Lambda$ of the previous case, the analogous quantity $\Lambda$ is now constrained. In order to understand the nature of this constraint, it is useful to decompose the variable $h(\tau, \sigma) \in AN$ as

$$h(\tau, \sigma) = e^{\phi(\tau, \sigma)} n(\tau, \sigma), \quad \phi \in \text{Lie}(A), \quad n \in \mathbb{N}. \quad (40)$$

Of course, the fields $\phi, n$ are also periodic in $\sigma$. The variable $\Lambda$ now becomes

$$\Lambda = \partial_\sigma \phi + e^{\phi}(\partial_\sigma nn^{-1} + n\mu n^{-1} - \mu)e^{-\phi} \equiv \Lambda_A + \Lambda_N. \quad (41)$$

We observe immediately, that $\Lambda_A$ is in $\text{Lie}(A)$ and $\Lambda_N$ is in $\text{Lie}(N)$. 

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By a straightforward study of adjoint orbits of Borel subgroups of $\tilde{G}^\mathbb{C}$, we arrive at conclusion that $\Lambda_N$ is not constrained. In other words, by varying $h(\tau, \sigma)$ and $\mu(t)$, one completely sweeps the space of all possible $\Lambda_N(\tau, \sigma)$. On the other hand, $\Lambda_A(\tau, \sigma)$ is clearly constrained, it misses the zero mode in the Fourier series in the variable $\sigma$. Such a constraint can be easily taken into account by adding to the action (38) a Lagrange multiplier term $\int(\nu(\tau), \Lambda_A(\tau, \sigma))$, where $\nu(\tau)$ is in the Cartan subalgebra $\mathcal{T}(\subset \tilde{G})$. Thus the action (38) gets transformed into the following equivalent one:

$$S(\tilde{g}, \Lambda, \nu, \mu, \phi_0) = -\frac{1}{4\pi} \int \{(\partial_\tau \mu, \phi_0) - (\Lambda, \nu)\} + \frac{1}{4\pi} \int (\Lambda, \tilde{g}^{-1}(\partial_\tau - \partial_\sigma)\tilde{g} - \mu) +$$

$$+ \frac{1}{4\pi} \int (\tilde{g}^{-1} \partial_\sigma \tilde{g} + \mu + \Lambda, \mathcal{P}(\tilde{g})(\tilde{g}^{-1} \partial_\sigma \tilde{g} + \mu + \Lambda)), \quad (42)$$

where

$$\phi_0(\tau) = \int d\sigma \phi(\tau, \sigma). \quad (43)$$

Recall that we wish to solve away the variables $\Lambda$ and $\phi_0$ (related to the variable $h$ in the decomposition $l = \tilde{g}h$). This can be done easily, since after the introducing the Lagrange multiplier $\nu$, those variables $\Lambda, \phi_0$ are unconstrained in (42). The resulting action is

$$S_{\Pi}(\tilde{g}, \nu, \mu_0) = \frac{1}{8\pi} \int d\tau d\sigma (R^{-1} + \Pi(\tilde{g}))_{ij}^{-1}$$

$$[(\partial_\tau \tilde{g}^{-1} + \tilde{g} \nu \tilde{g}^{-1})(\partial_\tau \tilde{g}^{-1} + \tilde{g} \nu \tilde{g}^{-1}) - (\partial_\sigma \tilde{g}^{-1} + \tilde{g} \mu_0 \tilde{g}^{-1})(\partial_\sigma \tilde{g}^{-1} + \tilde{g} \mu_0 \tilde{g}^{-1})]. \quad (44)$$

Note that in this final expression (44) for the dual action, we use the symbol $\mu_0$ instead of $\mu$. It is because $\mu_0$ is a constant not depending on the time $\tau$. This is dictated by integrating away the Lagrange multiplier $\phi_0$ in the action (42). On the other hand, $\nu$ still depends on $\tau$.

We would like to interpret our result. First of all, note that $\tilde{g}(\tau, \sigma)$’s are $\tilde{G}$-valued functions periodic in $\sigma$, so we could view (44) as a $\mu_0$-depending dynamical system describing closed string configurations interacting with some particle-like degrees of freedom $\nu$. Such a theory would not be anymore a $\sigma$-model in the standard sense of this word. For example, $\tilde{g}$-depending terms not containing derivatives of $\tilde{g}$ also appear in the action which would mean that our duality transformation has generated a tachyon potential.
We believe that the correct interpretation is the following: Introduce a new field variable
\[ \tilde{m}(\tau, \sigma) = \tilde{g}(\tau, \sigma)e^{\mu_0\sigma}. \] (45)
Such a configuration is referred to as the monodromic string for obvious reasons. Its image in the target \( \tilde{G} \) looks like an open string. The action (44) can be then rewritten as
\[ S_{\tilde{\Pi}}(\tilde{m}, \nu) = \frac{1}{8\pi} \int d\tau d\sigma (R^{-1} + \tilde{\Pi}(\tilde{m}))^{-1}_{ij} \]
\[ [(\partial_\tau \tilde{m}^{-1} + \tilde{m} \partial_\tau \tilde{m}^{-1})^i(\partial_\tau \tilde{m}^{-1} + \tilde{m} \partial_\tau \tilde{m}^{-1})^j - (\partial_\sigma \tilde{m}^{-1})^i(\partial_\sigma \tilde{m}^{-1})^j]. \] (46)
The reader may ask why we are allowed to replace \( \tilde{\Pi}(\tilde{g}) \) by \( \tilde{\Pi}(\tilde{m}) \). We have from (2)
\[ \tilde{\Pi}(\tilde{m}) = \tilde{\Pi}(\tilde{ge}^{\mu_0\sigma}) = \tilde{\Pi}(\tilde{g}) + Ad_{\tilde{g}}\tilde{\Pi}(e^{\mu_0\sigma}) = \tilde{\Pi}(\tilde{g}). \] (47)
The last equality follows from the fact that the Lu-Weinstein-Soibelman Poisson-Lie structure \( \tilde{\Pi}(\tilde{g}) \) on a compact simple connected and simply connected group \( \tilde{G} \) always vanishes on the maximal torus \( T \subset \tilde{G} \).

We can also naturally interpret the variable \( \nu(\tau) \in T \). In fact it is the gauge field. By using again the fact that \( \tilde{\Pi} \) vanishes on the maximal torus, we observe that the model (4) has a global symmetry \( \tilde{g} \to \tilde{g}t \), where \( t \in T \).

This global symmetry is present for the closed but also for the monodromic string. Its gauging amounts for introducing the gauge fields \( \nu \) into the action (4). This is precisely the action (46). The gauge transformation reads
\[ \tilde{m} \to \tilde{m}t(\tau), \quad \nu \to \nu - t^{-1}\partial_\tau t. \] (48)

Summarizing, we have obtained the following picture: The closed string model (3) on \( G = AN \) is dual to the monodromic string model (4) on the compact group \( \tilde{G} \), where the maximal torus momentum zero modes are gauged away. In some sense we may say, that the duality requires adding the monodromy zero modes to (4) but at the same time removing maximal torus momentum zero modes.

7. Abelian example. It turns out that one has a full control of the monodromy also in an almost trivial but instructive example where the Drinfeld double \( D \) is just a plane \( \mathbb{R}^2 \) viewed as the additive Abelian Lie group. Its Lie algebra
is again $\mathbb{R}^2$ and the exponential map is just the identity map. The invariant bilinear form is defined as
\[
((x_1, y_1), (x_2, y_2)) = x_1 y_2 + y_1 x_2.
\]
(49)
It is symmetric and non-degenerate. Then we define
\[
\mathcal{G} = \{(x, 0); x \in \mathbb{R}\}, \quad \tilde{\mathcal{G}} = \{(0, \tilde{x}); \tilde{x} \in \mathbb{R}\}.
\]
(50)
Both subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ are isotropic and $\mathcal{G} \cap \tilde{\mathcal{G}} = 0$. The double $D$ is moreover clearly perfect, i.e. $D = G + \tilde{G}$.

We can immediately write down the action (23) for our Abelian double, where $l(\tau, \sigma) = x(\tau, \sigma) + \tilde{x}(\tau, \sigma)$, $x(\tau, \sigma) \in G$, $\tilde{x}(\tau, \sigma) \in \tilde{G}$ and $\mu(\tau) \in \tilde{\mathcal{G}}$. The action reads
\[
4\pi S(x, \tilde{x}, \mu) = \int \partial_\sigma \tilde{x} \partial_\tau x + \int \mu \partial_\tau x - \frac{1}{2} R \int (\partial_\sigma \tilde{x} + \mu)^2 - \frac{1}{2} R^{-1} \int (\partial_\sigma x)^2 =
\]
\[
= \int \tilde{\Lambda} \partial_\tau x - \frac{1}{2} R \int \tilde{\Lambda}^2 - \frac{1}{2} R^{-1} \int (\partial_\sigma x)^2,
\]
(51)
where $R$ is a positive real number. We observe that the quantity $\tilde{\Lambda} = \partial_\sigma \tilde{x} + \mu$ is unconstrained, hence we may solve it away to obtain
\[
S(x) = \frac{1}{8\pi R} \int [(\partial_\tau x)^2 - (\partial_\sigma x)^2].
\]
(52)
The same action (51) can be rewritten from the dual point of view as
\[
4\pi S(x, \tilde{x}, \mu) = \int \partial_\sigma x \partial_\tau \tilde{x} - \int \partial_\tau \mu x_0 - \frac{1}{2} R \int (\partial_\sigma \tilde{x} + \mu)^2 - \frac{1}{2} R^{-1} \int (\partial_\sigma x)^2,
\]
where $x_0(\tau)$ is the zero Fourier component of the field $x(\tau, \sigma)$.

By repeating the same procedure as in the general case, we arrive at the following dual action
\[
\tilde{S}(\tilde{x}, \nu, \mu_0) = \frac{R}{8\pi} \int [(\partial_\tau \tilde{x} + \nu)^2 - (\partial_\sigma \tilde{x} + \mu_0)^2],
\]
(53)
where the field $\tilde{x}$ is still periodic. We can decompose $\tilde{x}$ as
\[
\tilde{x} = \tilde{x}_0 + \tilde{x}_{osc},
\]
(54)
where $\tilde{x}_0$ denotes the zero Fourier component and $\tilde{x}_{osc}$ the rest. Then define

$$\tilde{x}_{mon} = \tilde{x}_{osc} + \mu_0 \sigma.$$  \hfill (55)

Finally observe that the zero modes $\tilde{x}_0$ can be absorbed into $\nu$ by setting

$$\nu' = \nu + \partial_\tau \tilde{x}_0.$$  \hfill (56)

The resulting action reads

$$\tilde{S}(\tilde{x}_{mon}, \nu') = \frac{R}{8\pi} \int [(\partial_\tau \tilde{x}_{mon} + \nu')^2 - (\partial_\sigma \tilde{x}_{mon})^2].$$  \hfill (57)

The field $\nu'$ can be solved away to yield our final dual result

$$\tilde{S}(\tilde{x}_{mon}) = \frac{R}{8\pi} \int [(\partial_\tau \tilde{x}_{mon})^2 - (\partial_\sigma \tilde{x}_{mon})^2].$$  \hfill (58)

It is also instructive to calculate the noncommutative momentum $\tilde{M}$ of the standard closed strings living on $G$. For this, we have to solve the equations of motion of the model (51). The general solution $x_{sol}$ reads

$$x_{sol}(\tau, \sigma) = x_0 + p\tau + osc_L(\tau - \sigma) + osc_R(\tau + \sigma).$$  \hfill (59)

The quantity $\tilde{\lambda}$ then reads

$$\tilde{\lambda} = \partial_- x_{sol} d\xi^- - \partial_+ x_{sol} d\xi^+$$  \hfill (60)

and the momentum $\tilde{M}$

$$\tilde{M} = -\frac{p}{R}. $$  \hfill (61)

We observe that in the Abelian case, the noncommutative momentum becomes the standard momentum of the closed string.

8. Dressing cosets. There exists the generalization of the Poisson-Lie T-duality [17] relating models living on the double cosets $F \backslash D/G$ and $F \backslash D/\tilde{G}$, where $F$ is certain isotropic subgroup of $D$. We are going to show now that the story of the monodromic strings generalizes to this case. Recall first the basic ingredients of the dressing coset construction.
Consider now an $n$-dimensional linear subspace $\mathcal{R}_+ \subset \mathcal{D}$ such that it intersects with its orthogonal complement $\mathcal{R}_-$ in an isotropic Lie algebra $\mathcal{F}$, i.e.

$$
\mathcal{R}_+ \cap \mathcal{R}_- = \mathcal{F}; \quad [\mathcal{F}, \mathcal{F}] \subset \mathcal{F}.
$$

(62)

Moreover, both $\mathcal{R}_+$ and $\mathcal{R}_-$ should be invariant subspaces with respect to the adjoint action of $\mathcal{F}$:

$$
[\mathcal{F}, \mathcal{R}_+] \subset \mathcal{R}_+, \quad [\mathcal{F}, \mathcal{R}_-] \subset \mathcal{R}_-.
$$

(63)

It was shown in [17] that all these data define a pair of dual non-linear $\sigma$-models living respectively, on the targets $\mathcal{F}\setminus \mathcal{D}/G$ and $\mathcal{F}\setminus \mathcal{D}/\tilde{G}$. Their common dynamics is encoded in the first order Hamiltonian action [17] which can be obtained by setting $\mu = 0$ in the following more general action principle:

$$
S = \frac{1}{8\pi} \int d\sigma d\tau (\partial_\sigma ll^{-1}, \partial_\tau ll^{-1}) + \frac{1}{48\pi} \int d\sigma d\tau \{([\partial_\sigma ll^{-1}, ll^{-1}]_{l\mu l^{-1}}, [\partial_\tau ll^{-1}, ll^{-1}]_{l\mu l^{-1}})
$$

$$
+ \frac{1}{4\pi} \int (\mu, l^{-1}\partial_\tau l) - \frac{1}{8\pi} \int d\sigma d\tau \{((\partial_\sigma ll^{-1} + l\mu l^{-1})_0, (\partial_\sigma ll^{-1} + l\mu l^{-1})_0)
$$

$$
- ((\partial_\sigma ll^{-1} + l\mu l^{-1})_1, (\partial_\sigma ll^{-1} + l\mu l^{-1})_1)\}.
$$

(64)

The reader has certainly understood that the action with the incorporated variable $\mu(\tau) \in \mathcal{T}_+$ is the generalization of the dressing coset story to the monodromic case. Here as before $l(\sigma, \tau) = l(\sigma + 2\pi, \tau)$ is a mapping from a cylindrical worldsheet into the group manifold $\mathcal{D}$ and $\partial_\sigma ll^{-1} + l\mu l^{-1}$ is constrained to lie in $\mathcal{F}^\perp$:

$$
\partial_\sigma ll^{-1} + l\mu l^{-1} \in \mathcal{F}^\perp.
$$

(65)

Note that due to the non-degeneracy of the form $(.,.)$ we have

$$
\mathcal{F}^\perp = \text{Span}(\mathcal{R}_+, \mathcal{R}_-).
$$

(66)

We should also explain the meaning of the subscripts 0 and 1 in (64). We can write arbitrary element $x \in \mathcal{F}^\perp$ as

$$
x = x_0 + x_1, \quad x_0 \in \mathcal{R}_+, \quad x_1 \in \mathcal{R}_-.
$$

(67)

Of course, this decomposition is not unique, because the linear spaces $\mathcal{R}_+$ and $\mathcal{R}_-$ intersect at $\mathcal{F}$. The decomposition $x = x'_0 + x'_1$, where $x'_0 = x_0 + \phi$
and \( x'_1 = x_1 - \phi \), is equally good for an arbitrary \( \phi \in \mathcal{F} \). However, due to the fact that \((\mathcal{F}, \mathcal{R}_+) = (\mathcal{F}, \mathcal{R}_-) = 0\), the action (64) does not depend on this decomposition.

The action (64) possesses the following gauge symmetry \( l \rightarrow fl, f(\sigma, \tau) \in F \) which explains why we take the left coset \( F \backslash D/G \).

The way how to obtain the dual pair of the \( \sigma \)-models from the action (64) (for \( \mu = 0 \)) was described in [17]. The argument for the nonvanishing \( \mu \) goes in the similar way giving as the pair of the dual \( \sigma \)-models nothing but the actions (29) and (46). The reader may ask what is then the difference with the previous case (with no constraint of the form (65)). The answer is simple, if the condition (62) holds for \( \mathcal{R}_+ \), then the model (46) does not live on the target \( D/G \), because it develops additional gauge symmetry coming from \( l \rightarrow fl, f(\sigma, \tau) \in F \). Due to this gauge symmetry the target space of the model (46) is in fact \( F \backslash D/G \) instead of \( D/G \). Reasoning in the same way gives that the dual model (29) develops also the additional gauge symmetry and it lives on \( F \backslash D/G \).

9. Conclusions and outlook: The Poisson-Lie T-duality story [2] is the generalization of the traditional non-Abelian T-duality [14]. Apart from the papers already cited in the text, we may mention several other works who have developed various aspects of the construction [15]. A quantum version of the PL T-duality is in preliminary stage [16]. We believe that our present article will facilitate the work in the quantum direction since we did get finally rid of the highly nonlinear monodromy constraints. Thus we hope that the quantum monodromic strings may become a part of the standard superstring theories.

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