Some applications of the generalized Bernardi - Libera - Livingston integral operator on univalent functions

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Abstract

In this paper by making use of the generalized Bernardi - Libera - Livingston integral operator we introduce and study some new subclasses of univalent functions. Also we investigate the relations between those classes and the classes which are studied by Jin-Lin Liu.

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1 Introduction

Let $A$ be the class of functions of the form, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{ z : |z| < 1 \}$, also let $S$ denote the subclass of $A$ consisting of all univalent functions in $U$. Suppose $\lambda$ is a real number with $0 \leq \lambda < 1$, a function $f \in S$ is said to be starlike of order $\lambda$ if and only if $\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \lambda, z \in U$, also $f \in S$ is said to be convex of order $\lambda$ if and only if $\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \lambda, z \in U$, we denote by $S^*(\lambda), C(\lambda)$ the classes of starlike and convex functions of order $\lambda$ respectively. It is well known that $f \in C(\lambda)$ if and only if $zf' \in S^*(\lambda)$. Let $f \in A$ and $g \in S^*(\lambda)$ then $f \in K(\beta, \lambda)$ if and only if $\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \beta, z \in U$ where $0 \leq \beta < 1$. These functions are called close-to-convex functions of order $\beta$ type $\lambda$. A function $f \in A$ is called quasi-convex of order $\beta$ type $\lambda$
if there exists a function $g \in C(\lambda)$ such that $\text{Re} \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > \beta$. We denote this class by $K^*(\beta, \lambda)$ [10]. It is easy to see that $f \in K^*(\beta, \gamma)$ if and only if $zf' \in K(\beta, \gamma)$ [9]. For $f \in A$ if for some $\lambda(0 \leq \lambda < 1)$ and $\eta(0 < \eta \leq 1)$ we have

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \lambda \right) \right| < \frac{\pi}{2} \eta, \quad z \in U$$

then $f(z)$ is said to be strongly starlike of order $\eta$ and type $\lambda$ in $U$ and we denote this class by $S^*(\eta, \lambda)$. If $f \in A$ satisfies the condition

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \lambda \right) \right| < \frac{\pi}{2} \eta, \quad z \in U$$

for some $\lambda$ and $\eta$ as above then we say that $f(z)$ is strongly convex of order $\eta$ and type $\lambda$ in $U$ and we denote this class by $C(\eta, \lambda)$. Clearly $f \in C(\eta, \lambda)$ if and only if $zf' \in S^*(\eta, \lambda)$, specially we have $S^*(1, \lambda) = S^*(\lambda)$ and $C(1, \lambda) = C(\lambda)$.

For $c > -1$ and $f \in A$ the generalized Bernardi - Libera - Livingston integral operator $L_c f$ is defined as follows

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$  \hspace{1cm} (1.3)

This operator for $c \in N = \{1, 2, 3, \ldots \}$ was studied by Bernardi [1] and for $c = 1$ by Libera [5] (see also [8]). Now by making use of the operator given by (1.3) we introduce the following classes.

$$S^*_c(\lambda) = \{ f \in A : L_c f \in S^*(\lambda) \}$$

$$C_c(\lambda) = \{ f \in A : L_c f \in C(\lambda) \}$$

$$K_c(\beta, \lambda) = \{ f \in A : L_c f \in K(\beta, \lambda) \}$$

$$K^*_c(\beta, \lambda) = \{ f \in A : L_c f \in K^*(\beta, \lambda) \}$$

$$ST_c(\eta, \lambda) = \left\{ f \in A : L_c f \in S^*(\eta, \lambda), \frac{z(L_c f(z))'}{L_c f(z)} \neq \lambda, z \in U \right\}$$

$$CV_c(\eta, \lambda) = \left\{ f \in A : L_c f \in C(\eta, \lambda), \frac{(z(L_c f(z))')'}{(L_c f(z))'} \neq \lambda, z \in U \right\}.$$
Obviously $f \in CV_c(\eta, \lambda)$ if and only if $zf' \in ST_c(\eta, \lambda)$. J. L. Liu [6] and [7] introduced and investigated similarly the classes $S^*_c(\lambda), C_c(\lambda), K_c(\beta, \lambda), K^*_c(\beta, \lambda), ST_c(\eta, \lambda), CV_c(\eta, \lambda)$ by making use of the integral operator $I^\sigma f$ given by

$$I^\sigma f(z) = \frac{2\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t) dt, \sigma > 0, f \in A. \quad (1.4)$$

The operator $I^\sigma$ is introduced by Jung, Kim and Srivastava [3] and then investigated by Uralogaddi and Somanatha [12], Li [4] and Liu [6]. For the integral operators given by (1.3) and (1.4) we have easily verified following relationships.

$$I^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n \quad (1.5)$$

$$L_c f(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{n+c} a_n z^n \quad (1.6)$$

$$z(I^\sigma L_c f(z))' = (c+1)I^\sigma f(z) - cI^\sigma L_c f(z) \quad (1.7)$$

$$z(L_c I^\sigma f(z))' = (c+1)I^\sigma f(z) - cL_c I^\sigma f(z). \quad (1.8)$$

It follows from (1.5) that one can define the operator $I^\sigma$ for any real number $\sigma$. In this paper we investigate the properties of the classes $S^*_c(\lambda), C_c(\lambda), K_c(\beta, \lambda), K^*_c(\beta, \lambda), ST_c(\eta, \lambda), CV_c(\eta, \lambda)$, also we study the relations between these classes by the classes which are introduced by Liu in [6] and [7]. For our purposes we need the following lemmas.

**Lemma 1.1** [9]. Let $u = u_1 + iu_2, v = v_1 + iv_2$ and let $\psi(u, v)$ be a complex function $\psi : D \subset \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Suppose that $\psi$ satisfies the following conditions

(i) $\psi(u, v)$ is continuous in $D$

(ii) $(1, 0) \in D$ and $\text{Re}\{\psi(1, 0)\} > 0$

(iii) $\text{Re}\{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$.

Let $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ be analytic in $U$ so that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\text{Re}\{\psi(p(z), zp'(z))\} > 0, z \in U$ then $\text{Re}\{p(z)\} > 0, z \in U.$
Lemma 1.2 [11]. Let the function \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) be analytic in \( U \) and \( p(z) \neq 0, z \in U \) if there exists a point \( z_0 \in U \) such that \( |\arg(p(z))| < \frac{\pi}{2} \eta \) for \( |z| < |z_0| \) and \( |\arg p(z_0)| = \frac{\pi}{2} \eta \) where \( 0 < \eta \leq 1 \) then \( \frac{z p'(z_0)}{p(z_0)} = ik \eta \) and \( k \geq \frac{1}{2} (r + \frac{1}{r}) \) when \( \arg p(z_0) = \frac{\pi}{2} \eta \) also \( k \leq \frac{1}{2} (r + \frac{1}{r}) \) when \( \arg p(z_0) = -\frac{\pi}{2} \eta \), and \( p(z_0)^{1/\eta} = \pm ir (r > 0) \).

2 Main Results

In this section we obtain some inclusion theorems.

Theorem 2.1: (i) For \( f \in A \) if \( \Re \{ \frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)} \} > 0 \), then \( S_c^*(\lambda) \subset S_{c+1}^*(\lambda) \).

(ii) For \( f \in A \) if \( \Re \{ \frac{zf'(z)}{f(z)} - \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} \} > 0 \) then \( S_{c+1}^*(\lambda) \subset S_c^*(\lambda) \).

Proof: (i) Suppose that \( f \in S_c^*(\lambda) \) and set

\[
\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} - \lambda = (1 - \lambda)p(z) \tag{2.1}
\]

where \( p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n \). An easy calculation shows that

\[
\frac{zf'(z)}{f(z)} = \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} = 1 + \frac{z(L_{c+1} f(z))''}{(L_{c+1} f(z))'} = 1 + \frac{zH'(z)}{H(z)}.
\tag{2.2}
\]

By setting \( H(z) = \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} \) we have

\[
1 + \frac{z(L_{c+1} f(z))''}{(L_{c+1} f(z))'} = H(z) + \frac{zH'(z)}{H(z)}.
\tag{2.3}
\]

By making use of (2.3) in (2.2) since \( H(z) = \lambda + (1 - \lambda)p(z) \) so we obtain

\[
(1 - \lambda)p(z) + \frac{(1 - \lambda)z p'(z)}{\lambda + c + 1 + (1 - \lambda)p(z)} = \frac{zf'(z)}{f(z)} - \lambda.
\tag{2.4}
\]

If we consider \( \psi(u, v) = (1 - \lambda)u + \frac{(1 - \lambda)v}{\lambda + c + 1 + (1 - \lambda)u} \) then \( \psi(u, v) \) is a continuous function in \( D = \{ \lambda \in \frac{\lambda + c + 1}{\lambda + c + 1} \} \times \mathbb{C} \) and \( (0, 0) \in D \) also \( \psi(1, 0) > 0 \) and for all \( (iu_2, v_1) \in D \) with \( v_1 \leq -\frac{1 + u_2^2}{2} \) we have

\[
\Re \psi(iu_2, v_1) = \frac{(1 - \lambda)(\lambda + c + 1)v_1}{(1 - \lambda)^2 u_2^2 + (\lambda + c + 1)^2} \leq \frac{(1 - \lambda)(\lambda + c + 1)(1 + u_2^2)}{2[(1 - \lambda)^2 u_2^2 + (\lambda + c + 1)^2]} < 0.
\]
Therefore the function ψ(u, v) satisfies the conditions of Lemma 1.1 and since in view of the assumption by considering (2.4) we have \( \text{Re}\{\psi(p(z), zp'(z))\} > 0 \) therefore Lemma 1 implies that \( \text{Re} \ p(z) > 0, \ z \in U \) and this completes the proof.

(ii) For proving this part of theorem by the same method and using the easily verified formula similar to (2.2) by replacing \( c + 1 \) with \( c \) we get the desired result.

**Theorem 2.2**: (i) For \( f \in A \) if \( \text{Re}\left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)} \right\} > 0 \) then \( C_c(\lambda) \subset C_{c+1}(\lambda) \).

(ii) For \( f \in A \) if \( \text{Re}\left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)} \right\} > 0 \) then \( C_{c+1}(\lambda) \subset C_c(\lambda) \).

**Proof**: (i) In view of part (i) of Theorem 1 we can write

\[ f \in C_c(\lambda) \iff L_c f \in C(\lambda) \iff z(L_c f)' \in S^*(\lambda) \iff L_c z f' \in S^*(\lambda) \iff z f' \in S^*_c(\lambda) \iff f \in S^*_{c+1}(\lambda) \]

By the similar way we can prove the part (ii) of theorem.

**Theorem 2.3**: If \( c \geq -\lambda \) then \( f \in S^*(\lambda) \) implies \( f \in S^*_c(\lambda) \).

**Proof**: By differentiating logarithmically from both sides of (1.3) with respect to \( z \) we obtain

\[ \frac{z(L_c f(z))'}{L_c f(z)} + c = \frac{(c + 1)f(z)}{L_c f(z)}. \quad (2.5) \]

Again differentiating logarithmically from both sides of (2.5) we have

\[ p(z) + \frac{zp'(z)}{c + \lambda + p(z)} = \frac{zf'(z)}{f(z)} - \lambda \quad (2.6) \]

where \( p(z) = \frac{z(L_c f(z))'}{L_c f(z)} - \lambda \). Let us consider \( \psi(u, v) = u + \frac{v}{u + c + \lambda} \), then \( \psi \) is a continuous function in \( D = \{ \mathbb{C} - (-c - \lambda) \} \times \mathbb{C} \) and \( (1, 0) \in D \) also \( \text{Re} \ \psi(1, 0) > 0 \). If \( (iu_2, v_1) \in D \) with \( v_1 \leq \frac{-1 + u_2^2}{2} \) then \( \text{Re} \ \psi(iu_2, v_1) = \frac{v_1(c + \lambda)}{u_2^2 + (c + \lambda)^2} \leq 0 \), also since \( f \in S^*(\lambda) \) then (2.6) gives \( \text{Re}(\psi(p(z), zp'(z))) = \text{Re}\left\{ \frac{zf'(z)}{f(z)} - \lambda \right\} > 0 \). Therefore Lemma 1 concludes that \( \text{Re}\{p(z)\} > 0 \) and this completes the proof.

**Corollary 2.4**: If \( c \geq \lambda \) then \( f \in C(\lambda) \) implies \( f \in C_c(\lambda) \).

**Proof**: We have
Suppose that there exists \( z' \in S^*(\lambda) \implies z' \in S^*_e(\lambda) \implies L_e z' \in S^*(\lambda) \implies z(L_e f)' \in S^*(\lambda) \implies L_e f \in C(\lambda) \iff f \in C_e(\lambda).

**Theorem 2.5:** (i) For \( f \in A \) if \( \arg \left( \frac{zf'(z)}{f(z)} - \lambda \right) \leq \arg \left( \frac{z(L_e f)(z)'}{L_e f(z)} - \lambda \right), z \in U \) then \( ST_e(\eta, \lambda) \subset ST_{e+1}(\eta, \lambda), c > -1. \)

(ii) For \( f \in A \) if \( \arg \left( \frac{zf'(z)}{f(z)} - \lambda \right) \leq \arg \left( \frac{z(L_{e+1} f)(z)'}{L_{e+1} f(z)} - \lambda \right), z \in U \) then \( ST_{e+1}(\eta, \lambda) \subset ST_e(\eta, \lambda), c > -1. \)

**Proof:** (i) Let \( f \in ST_e(\eta, \lambda) \) and put

\[
\frac{z(L_{e+1} f(z))'}{L_{e+1} f(z)} = \lambda + (1 - \lambda)p(z) \tag{2.7}
\]

where \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) is analytic in \( U \) with \( p(z) \neq 0, z \in U \). It is easy to see that

\[
z(L_{e+1} f(z))' + (c + 1)L_{e+1} f(z) = (c + 2)f(z). \tag{2.8}
\]

Differentiating logarithmically with respect to \( z \) from both sides of (2.8) gives

\[
\frac{\frac{z(L_{e+1} f(z))'}{L_{e+1} f(z)}'}{z(L_{e+1} f(z))' + c + 1} + \frac{\frac{z(L_{e+1} f(z))'}{L_{e+1} f(z)}}{L_{e+1} f(z)} = \frac{zf'(z)}{f(z)}. \tag{2.9}
\]

Now by making use of (2.7) in (2.9) we have

\[
\frac{(1 - \lambda)zp'(z)}{\lambda + c + 1 + (1 - \lambda)p(z)} + (1 - \lambda)p(z) = \frac{zf'(z)}{f(z)} - \lambda. \tag{2.10}
\]

Suppose that there exists \( z_0 \in U \) in such a way \( \arg(p(z_0)) < \frac{\pi}{2} \eta \) for \( |z| < |z_0| \) and \( \arg(p(z_0)) = \frac{\pi}{2} \eta \), then by Lemma 1.2 we have \( \frac{\arg p'(z_0)}{p(z_0)} = i k \eta \) and \( p(z_0)^{1/\eta} = \pm i r (r > 0) \)

where \( k \geq \frac{1}{2}(r + \frac{1}{r}) \) when \( \arg(p(z_0)) = \frac{\pi}{2} \eta \) and \( k \leq \frac{1}{2}(r + \frac{1}{r}) \) when \( \arg(p(z_0)) = -\frac{\pi}{2} \eta \). If
\[ p(z_0)^{1/\eta} = i r \text{ then } \arg(p(z_0)) = \frac{\pi}{2} \eta \text{ and by considering (2.10) we have} \]
\[
\left| \arg \left( \frac{z_0 (L_c f(z_0))'}{L_c f(z_0)} - \lambda \right) \right| \geq \arg \left( \frac{z_0 f'(z_0)}{f(z_0)} - \lambda \right) = \arg \left\{ (1 - \lambda) p(z_0) \left[ 1 + \frac{ik\eta}{\lambda + c + 1 + (1 - \lambda)r^\eta e^{i\frac{\pi}{2}} \eta} \right] \right\} = \frac{\pi}{2} \eta \\
+ \tan^{-1} \left\{ \frac{k\eta [\lambda + c + 1 + r^\eta (1 - \lambda) \cos \frac{\pi}{2} \eta]}{(\lambda + c + 1)^2 + r^{2\eta} (1 - \lambda)^2 + (1 - \lambda)(\lambda + c + 1) \cos \frac{\pi}{2} \eta + k\eta r^\eta (1 - \lambda) \sin \frac{\pi}{2} \eta} \right\} \geq \frac{\pi}{2} \eta \text{ (Because } k \geq \frac{1}{2} (r + \frac{1}{r}) \geq 1) \]

which is a contradiction by \( f(z) \in ST_c(\eta, \lambda) \).

Now suppose \( p(z_0)^{1/\eta} = -i r \text{ then } \arg(p(z_0)) = -\frac{\pi}{2} \eta \text{ and we have} \)
\[
- \left| \arg \left( \frac{z_0 (L_c f(z_0))'}{L_c f(z_0)} - \lambda \right) \right| \leq \arg \left( \frac{z_0 f'(z_0)}{f(z_0)} - \lambda \right) = \arg \left\{ 1 + \frac{ik\eta}{\lambda + c + 1 + (1 - \lambda)r^\eta e^{-i\frac{\pi}{2}} \eta} \right\} = \frac{\pi}{2} \eta \\
+ \tan^{-1} \left\{ \frac{k\eta [\lambda + c + 1 + r^\eta (1 - \lambda) \cos \frac{\pi}{2} \eta]}{(\lambda + c + 1)^2 + r^{2\eta} (1 - \lambda)^2 + 2r^\eta (1 - \lambda)(\lambda + c + 1) \cos \frac{\pi}{2} \eta - k\eta r^\eta (1 - \lambda) \sin \frac{\pi}{2} \eta} \right\} \leq \frac{\pi}{2} \eta \text{ (Because } k \leq \frac{1}{2} (r + \frac{1}{r}) \leq -1) \]

which contradicts our assumption that \( f \in ST_c(\eta, \lambda) \), therefore \( |\arg(p(z))| < \frac{\pi}{2}, z \in U \) and finally \( |\arg \left( \frac{z (L_{c+1} f(z))'}{L_{c+1} f(z)} - \lambda \right) | < \frac{\pi}{2} \eta, z \in U \). However since for every \( \lambda (0 \leq \lambda < 1) \) we have \( \frac{z (L_{c+1} f(z))'}{L_{c+1} f(z)} \neq \lambda \) thus we have \( f \in ST_{c+1}(\eta, \lambda) \) and the proof is complete.

(ii) The proof of this part of theorem is similar with the proof of part (i) and we omit the proof.

Corollary 2.6 : (i) For \( f \in A \) if \( \left| \arg \left( \frac{z f'(z)}{f(z)} - \lambda \right) \right| \leq \left| \arg \left( \frac{z (L_c f(z))'}{L_c f(z)} - \lambda \right) \right|, z \in U \) then \( CV_c(\eta, \lambda) \subset CV_{c+1}(\eta, \lambda) \).

(ii) For \( f \in A \) if \( \left| \arg \left( \frac{z f'(z)}{f(z)} - \lambda \right) \right| \leq \left| \arg \left( \frac{z (L_{c+1} f(z))'}{L_{c+1} f(z)} - \lambda \right) \right|, z \in U \) then we have \( CV_{c+1}(\eta, \lambda) \subset CV_c(\eta, \lambda) \).
Proof: We give only the proof of part (i) and for this we have
\[ f \in CV_c(\eta, \lambda) \iff L_c f \in C(\eta, \lambda) \iff z(L_c f)' \in S^*(\eta, \lambda) \iff L_c z f' \in S^*(\eta, \lambda) \iff z f' \in ST_c(\eta, \lambda) \implies z f' \in ST_{c+1}(\eta, \lambda) \iff L_{c+1} z f' \in S^*(\eta, \lambda) \iff z(L_{c+1} f)' \in S^*(\eta, \lambda) \iff L_{c+1} f \in C(\eta, \lambda) \iff f \in CV_{c+1}(\eta, \lambda). \]

Theorem 2.7: For every \( c > -1 \) we have \( CV_c(\eta, \lambda) \subset ST_c(\eta, \lambda) \).

Proof: Let \( f \in CV_c(\eta, \lambda) \) then \( \left| \text{arg} \left( 1 + \frac{z(L_c f'(z))^n}{(L_c f(z))'} - \lambda \right) \right| < \frac{\pi}{2} \eta, z \in U \) and \( \left| \frac{z(L_c f(z))''}{(L_c f(z))'} - \lambda \right| = (1 - \lambda)p(z) \) for all \( \lambda, z \in U \). Suppose that
\[ \left| 1 + \frac{z(L_c f(z))''}{(L_c f(z))'} - \lambda \right| = (1 - \lambda)p(z) \]

where \( p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n \) is analytic in \( U \) with \( p(z) \neq 0 \) for all \( z \in U \). Differentiating both sides of (2.11) logarithmically with respect to \( z \) gives
\[ 1 + \frac{z(L_c f(z))''}{(L_c f(z))'} - \lambda = (1 - \lambda)p(z) + \frac{(1 - \lambda)z p'(z)}{\lambda + (1 - \lambda)p(z)}. \]

If there exists a point \( z_0 \in U \) such that \( |\text{arg}(p(z))| < \frac{\pi}{2} \eta \) and \( |\text{arg}(p(z_0))| = \frac{\pi}{2} \eta \) then by Lemma 2 we obtain \( \frac{z p'(z_0)}{p(z_0)} = ik \eta \) and \( p(z_0)^{1/\eta} = \pm ir \) where \( k \geq \frac{1}{2}(r + \frac{1}{r}) \) when \( \text{arg}(p(z_0)) = \frac{\pi}{2} \eta \) and \( k \leq -\frac{1}{2}(r + \frac{1}{r}) \) when \( \text{arg}(p(z_0)) = -\frac{\pi}{2} \eta \). Suppose that \( \text{arg}(p(z_0)) = -\frac{\pi}{2} \eta \) then
\[ \text{arg} \left\{ 1 + \frac{z_0(L_c f(z_0))''}{(L_c f(z_0))'} - \lambda \right\} = \text{arg} \left\{ (1 - \lambda)r^{\eta}e^{-i\frac{\pi}{2} \eta} \left[ 1 + \frac{ik \eta}{\lambda + (1 - \lambda)r^{\eta}e^{-i\frac{\pi}{2} \eta}} \right] \right\} = -\frac{\pi}{2} \eta + \text{arg} \left\{ 1 + \frac{ik \eta}{\lambda + (1 - \lambda)r^{\eta}e^{-i\frac{\pi}{2} \eta}} \right\} = -\frac{\pi}{2} \eta + \tan^{-1} \left\{ -\frac{k \eta \lambda + (1 - \lambda)r^{\eta} \cos \frac{\pi}{2} \eta}{\lambda^2 + 2\lambda(1 - \lambda)r^{\eta} \cos \frac{\pi}{2} \eta + (1 - \lambda)^2 r^{2n} - k \eta (1 - \lambda)r^{\eta} \sin \frac{\pi}{2} \eta} \right\} \leq -\frac{\pi}{2} \eta \] (Because \( k \leq -\frac{1}{2}(r + \frac{1}{r}) \leq -1 \))

which is a contradiction by \( f \in CV_c(\eta, \lambda) \). For the case \( \text{arg}(p(z_0)) = \frac{\pi}{2} \eta \) by the same way
and considering $k \geq \frac{1}{2}(r + \frac{1}{r}) \geq 1$ we obtain

$$\arg \left\{ 1 + \frac{z_0 (L_c f(z_0))''}{(L_c f(z_0))'} - \lambda \right\} \geq -\frac{\pi}{2} \eta.$$  

This also contradicts our assumption that $f \in CV_c(\eta, \lambda)$, thus we have $|\arg(p(z))| < \frac{\pi}{2} \eta$ (z ∈ U) and finally

$$\left| \arg \left( \frac{z (L_c f(z))'}{L_c f(z)} - \lambda \right) \right| < \frac{\pi}{2} \eta, \quad z \in U.$$

**Theorem 2.8**: (i) If for every $f \in A$ and $g \in S^*_c(\lambda)$ we have

$$\text{Re} \left\{ \frac{z}{L_c g(z)} \frac{(L_c f(z))'}{(L_c f(z))} \right\} > 0$$  \hspace{1cm} (2.12)

and

$$\text{Re} \left\{ \frac{z g'(z)}{g(z)} - \frac{z (L_c g(z))'}{L_c g(z)} \right\} > 0$$  \hspace{1cm} (2.13)

then $K_c(\beta, \lambda) \subset K_{c+1}(\beta, \lambda)$.

(ii) If for every $f \in A$ and $g \in S^*(\lambda)$ we have

$$\text{Re} \left\{ \frac{z}{L_{c+1} g(z)} \frac{(L_{c+1} f(z))'}{(L_{c+1} f(z))} \right\} > 0$$  \hspace{1cm} (2.14)

and

$$\text{Re} \left\{ \frac{z g'(z)}{g(z)} - \frac{z (L_{c+1} g(z))'}{L_{c+1} g(z)} \right\} > 0$$  \hspace{1cm} (2.15)

then $K_{c+1}(\beta, \lambda) \subset K_c(\beta, \lambda)$.

**Proof**: (i) Let $f \in K_c(\beta, \lambda)$ then there exists a function $\varphi(z) \in S^*(\lambda)$ such that

$$\text{Re} \left\{ \frac{z (L_c f(z))'}{\varphi(z)} \right\} > \beta, \quad z \in U.$$

There is a function $g$ in such a way $L_c g(z) = \varphi(z)$ therefore $g \in S^*_c(\lambda)$ and we have

$$\text{Re} \left\{ \frac{z (L_{c+1} f(z))'}{L_{c+1} g(z)} \right\} > \beta, \quad z \in U.$$  \hspace{1cm} (2.16)
where \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \). Now in view of (2.12) we can write

\[
0 > Re \left\{ \frac{\frac{d}{dz} \left( \frac{L_c z f'(z)}{L_c g(z)} \right)}{\frac{L_c g(z)}{L_c g(z)}} + c \right\} = Re \left\{ \frac{z L_c z f'(z)(L_c g(z))' - z(L_c z f'(z) L_c g(z))}{L_c g(z)[z(L_c g(z))' + c L_c g(z)]} \right\}
\]

\[
= Re \left\{ \frac{z(L_c f(z))'[z(L_c g(z))' + c L_c g(z)] - L_c g(z)[z(L_c z f'(z))' + c L_c z f'(z) + z(L_c g(z))']}{L_c g(z)[z(L_c g(z))' + c L_c g(z)]} \right\}
\]

Therefore we have

\[
Re \left\{ \frac{z(L_c f(z))'}{L_c g(z)} \right\} < Re \left\{ \frac{z(L_c z f'(z))' + c(L_c z f'(z))}{z(L_c g(z))' + c L_c g(z)} \right\}
\]  \hspace{1cm} (2.17)

Now by easy computation we obtain the following identities.

\[
z(L_c z f'(z))' + c(L_c z f'(z)) = \frac{c+1}{c+2} \left[ z(L_{c+1} z f'(z))' + (c + 1)(L_{c+1} z f'(z)) \right]
\]  \hspace{1cm} (2.18)

\[
z(L_c g(z))' + c(L_c g(z)) = \frac{c+1}{c+2} \left[ z(L_{c+1} g(z))' + (c + 1)(L_{c+1} g(z)) \right].
\]  \hspace{1cm} (2.19)

By making use of (2.18) and (2.19) in (2.17) we get

\[
Re \left\{ \frac{z(L_c f(z))'}{L_c g(z)} \right\} < Re \left\{ \frac{z(L_{c+1} z f'(z))' + (c + 1)(L_{c+1} z f'(z))}{z(L_{c+1} g(z))' + (c + 1) L_{c+1} g(z)} \right\}
\]

\[
= Re \left\{ \frac{z(L_{c+1} z f'(z))'}{L_{c+1} g(z)} \right\} + (c + 1) \frac{z(L_{c+1} f(z))'}{L_{c+1} g(z)}
\]

In view of (2.13) and considering Theorem 1 we have \( g \in S^*_c(\lambda) \) and \( \frac{z(L_{c+1} g(z))'}{L_{c+1} g(z)} = (1 - \lambda)Q(z) + \lambda \) where \( Re(Q(z)) > 0, z \in U, \) also according to (2.16) we have

\[
L_{c+1} z f'(z) = L_{c+1} g(z)[(1 - \beta)p(z) + \beta].
\]  \hspace{1cm} (2.20)

Differentiating logarithmically with respect to \( z \) from both sides of (2.20) gives

\[
\frac{z(L_{c+1} z f'(z))'}{L_{c+1} g(z)} = (1 - \beta)zp'(z) + [(1 - \lambda)Q(z) + \lambda][(1 - \beta)p(z) + \beta].
\]  \hspace{1cm} (2.21)
However,
\[
\begin{align*}
\Re \left\{ \frac{z(L_c f(z))'}{L_c g(z)} \right\} & < \Re \left\{ (1 - \beta)z p'(z) + [(1 - \lambda)Q(z) + \lambda][(1 - \beta)p(z) + \beta] + (c + 1)(1 - \beta)p(z) + \beta \right\} \\
& = \Re \{(1 - \beta)p(z) + \beta\} + \frac{(1 - \beta)z p'(z)}{(1 - \lambda)Q(z) + \lambda + c + 1}.
\end{align*}
\]
Equivalently
\[
\begin{align*}
\Re \left\{ \frac{z(L_c f(z))'}{L_c g(z)} - \beta \right\} & < \Re \left\{ (1 - \beta)p(z) + \frac{(1 - \beta)z p'(z)}{(1 - \lambda)Q(z) + \lambda + c + 1} \right\}.
\end{align*}
\]
By considering the function \( \psi(u, v) \) as
\[
\psi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{(1 - \lambda)Q(z) + \lambda + c + 1}
\]
and noting that \( \Re(Q(z)) > 0 \) we can easily verify that the function \( \psi \) is a continuous function in \( D = \mathbb{C} \times \mathbb{C} \) and \( \Re \{ \psi(1, 0) \} > 0 \), also if \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \) then we have
\[
\begin{align*}
\Re \{ \psi(iu_2, v_1) \} &= \Re \left\{ (1 - \beta)iu_2 + \frac{(1 - \beta)v_1}{(1 - \lambda)Q(z) + \lambda + c + 1} \right\} \\
& = \Re \left\{ \frac{(1 - \beta)v_1[\lambda + c + 1 + (1 - \lambda)\Re(Q(z))] - i(1 - \lambda)I_m(Q(z))}{[\lambda + c + 1 + (1 - \lambda)\Re(Q(z))]^2 + [(1 - \lambda)I_m(Q(z))]^2} \right\} \\
& = \Re \left\{ \frac{(1 - \beta)v_1[\lambda + c + 1 + (1 - \lambda)\Re(Q(z))] - i(1 - \lambda)I_m(Q(z))}{[\lambda + c + 1 + (1 - \lambda)\Re(Q(z))]^2 + [(1 - \lambda)I_m(Q(z))]^2} \right\} \\
& \leq \frac{\lambda + c + 1 + (1 - \lambda)\Re(Q(z))^2 + [(1 - \lambda)I_m(Q(z))]^2}{\lambda + c + 1 + (1 - \lambda)\Re(Q(z))^2 + [(1 - \lambda)I_m(Q(z))]^2} < 0.
\end{align*}
\]
Finally since in view of (2.22) we have \( \Re \{ \psi(p(z), z p'(z)) \} > 0 \) therefore Lemma 1.1 gives \( \Re(p(z)) > 0 \), \( z \in U \) and the proof is complete.

The proof of part (ii) is similar to part (i) and we omit it.

By the same method used in Theorem 6 we can prove the next theorem.

**Theorem 2.9** : (i) If for every \( f \in A \) and \( g \in C_c(\lambda) \) we have
\[
\Re \left\{ \frac{z \frac{d}{dz} \left( \frac{(L_c z f'(\lambda))'}{(L_c z g(z))'} \right)}{(L_c z g(z))'} + c + 1 \right\} > 0
\]

and
\[
\text{Re} \left\{ \frac{zg'(z)}{g(z)} - \frac{z(L_0g(z))'}{L_0g(z)} \right\} > 0
\]
then \(K^*_e(\beta, \lambda) \subset K^*_{e+1}(\beta, \lambda)\).

(ii) If for every \(f \in A\) and \(g \in C_{e+1}(\lambda)\) we have
\[
\text{Re} \left\{ \frac{z}{(L_0f(z))'} \left( \frac{z(L_0f(z))'}{(L_0f(z))'} \right) \right\} > 0
\]
and
\[
\text{Re} \left\{ \frac{zg'(z)}{g(z)} - \frac{z(L_{e+1}g(z))'}{L_{e+1}g(z)} \right\} > 0
\]
then \(K^*_{e+1}(\beta, \lambda) \subset K^*_e(\beta, \lambda)\).

**Theorem 2.10**: If \(-\lambda \leq c \leq 1 - 2\lambda\) then \(f \in S^*_\sigma(\lambda)\) implies \(I^e f \in S^*_\sigma(\lambda)\).

**Proof**: Suppose that \(f \in S^*_\sigma(\lambda)\) and set
\[
z(L_0I^e f(z))' - \frac{1 + (1 - 2\lambda)w(z)}{1 - w(z)}, \quad z \in U
\]
where \(w(z)\) is analytic or meromorphic in \(U\) with \(w(0) = 0\). By using (1.8) and (2.23) we obtain
\[
\frac{I^e f(z)}{L_0I^e f(z)} = \frac{c + 1 + (1 - c - 2\lambda)w(z)}{(c + 1)(1 - w(z))}.
\]
(2.24)

Differentiating logarithmically both sides of (2.24) with respect to \(z\) gives
\[
\frac{z(I^e f(z))'}{I^e f(z)} = \frac{1 + (1 - 2\lambda)w(z) + z w'(z)}{1 - w(z)} + \frac{(1 - c - 2\lambda)zw'(z)}{c + 1 + (1 - c - 2\lambda)w(z)}
\]
Now we assert that \(|w(z)| < 1, z \in U\), if not then there exists a point \(z_0 \in U\) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1\]
therefore by Jacks’ Lemma we have \(z_0 w'(z_0) = kw(z_0), k \geq 1\).

So we have
\[
\text{Re} \left\{ \frac{z_0(I^e f(z_0))'}{I^e f(z_0)} - \lambda \right\} = \text{Re} \left\{ \frac{1 + (1 - 2\lambda + k)e^{i\theta}}{1 - e^{i\theta}} + \frac{(1 - c - 2\lambda)k e^{i\theta}}{c + 1 + (1 - c - 2\lambda)e^{i\theta}} - \lambda \right\}
\]
\[
= \frac{-2k(1 - \lambda)(c + \lambda)}{(1 + c)^2 + 2(1 + c)(1 - c - 2\lambda) \cos \theta + (1 - c - 2\lambda)^2} \leq \frac{-k(c + \lambda)}{2(1 - \lambda)} \leq 0.
\]
This contradicts our hypothesis \( f \in S^*_c(\lambda) \) thus \(|w(z)| < 1, z \in U \) and by cosidering (2.23) we conclude that \( I^\sigma f \in S^*_c(\lambda) \).

**Corollary 2.11** : If \(-\lambda < c < 1 - 2\lambda \) and \( f \in C_\sigma(\lambda) \) then \( I^\sigma f \in C_c(\lambda) \).

**Proof** : We have

\[
f \in C_\sigma(\lambda) \Leftrightarrow zf' \in S^*_c(\lambda) \Leftrightarrow I^\sigma(zf') \in S^*_c(\lambda) \Leftrightarrow z(I^\sigma f)' \in S^*_c(\lambda) \Leftrightarrow I^\sigma f \in C_c(\lambda).
\]

**Theorem 2.12** : Let \(-\lambda \leq c, 0 \leq \lambda < 1\). If \( f \in A \) and \( \frac{z(L_cI^\sigma f(z))'}{L_c I^\sigma f(z)} \neq \lambda, z \in U \) then \( f \in ST_\sigma(\eta, \lambda) \) implies that \( I^\sigma f \in ST_c(\eta, \lambda) \).

**Proof** : Let \( f \in ST_\sigma(\eta, \lambda) \) and put

\[
\frac{z(L_c I^\sigma f(z))'}{L_c I^\sigma f(z)} = \lambda + (1 - \lambda)p(z) \tag{2.25}
\]

where \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) and \( p(z) \neq 0, z \in U \). By considering (1.8) and (2.25) we have

\[
(c + 1) \frac{I^\sigma f(z)}{L_c I^\sigma f(z)} = c + \lambda + (1 - \lambda)p(z) \tag{2.26}
\]

Differentiating logarithmically with respect to \( z \) from both sides of (2.26) gives

\[
\frac{z(I^\sigma f(z))'}{I^\sigma f(z)} - \lambda = (1 - \lambda)p(z) + \frac{(1 - \lambda)zp'(z)}{c + \lambda + (1 - \lambda)p(z)}.
\]

Suppose that there exisst a point \( z_0 \in U \) such that \(|arg(p(z_0))| < \frac{\pi}{2}\eta(|z_0| - |z_0|)\) and \(|arg(p(z_0)))| = \frac{\pi}{2}\eta\) then by Lemma 1.2 we have \( \frac{z_0 p'(z_0)}{p(z_0)} = ik\eta \) and \( p(z_0)^{1/\eta} = \pm ir (r > 0) \).

If \( p(z_0)^{1/\eta} = ir \) then

\[
\frac{z_0(I^\sigma f(z_0))'}{I^\sigma f(z_0)} - \lambda = (1 - \lambda)p(z_0) \left[ 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right] = (1 - \lambda)r^\eta e^{i\frac{\pi}{2}\eta} \left[ 1 + \frac{ik\eta}{c + \lambda + (1 - \lambda)r^\eta e^{i\frac{\pi}{2}\eta}} \right]
\]

\[
= \frac{\pi}{2}\eta + arg \left\{ 1 + \frac{ik\eta}{c + \lambda + (1 - \lambda)r^\eta e^{i\frac{\pi}{2}\eta}} \right\}
\]

\[
= \frac{\pi}{2}\eta + \tan^{-1} \left\{ \frac{kn[c + \lambda + (1 - \lambda)r^\eta \cos \frac{\pi}{2}\eta]}{(c + \lambda)^2 + 2(c + \lambda)(1 - \lambda)r^\eta \cos \frac{\pi}{2}\eta + (1 - \lambda)^2 r^{2\eta} + k\eta(1 - \lambda)r^\eta \sin \frac{\pi}{2}\eta} \right\}
\]

\[
\geq \frac{\pi}{2}\eta \quad \text{(Because} \quad k \geq \frac{1}{2}(r + \frac{1}{r}) \geq 1)\]
which contradicts our assumption \( f \in ST_\sigma(\eta, \lambda) \). By the same method we get a contradiction for the case \( p(z_0)^{1/\eta} = -ir(r > 0) \), therefore we have \( |arg(p(z))| < \frac{\pi}{2} \eta, z \in U \) and in view of (2.14) we conclude that \( I^\sigma f \in ST_c(\eta, \lambda) \).

**Corollary 2.13**: Let \( c \geq \lambda, 0 \leq \lambda < 1 \). If \( f \in A \) and \( \frac{[z(L_cI^\sigma f(z))]'}{(L_cI^\sigma f(z))'} \neq \lambda, z \in U \) then \( f \in CV_\sigma(\eta, \lambda) \) implies that \( I^\sigma f \in CV_c(\eta, \lambda) \).

We claim the similar results may be hold for meromorphic \( p \)-valent functions with alternating coefficient. For more information see [2].
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