AN EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE DIRICHLET PROBLEM WITH THE DATA IN MORREY SPACES

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Abstract. Let $n - 2 < \lambda < n$, $f$ be a function in Morrey spaces $L^{1,\lambda}(\Omega)$, and the equation

\[
\begin{cases}
Lu = f, \\
u \in W^{1,2}_0(\Omega),
\end{cases}
\]

be a Dirichlet problem, where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and $L$ is a divergent elliptic operator. In this paper, we prove the existence and uniqueness of this Dirichlet problem by directly using the Lax-Milgram Lemma and the weighted estimation in Morrey spaces.

Keywords: Morrey spaces, Dirichlet problem, elliptic equations.
1. INTRODUCTION

Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$, where $n \geq 3$, and $l$ be the diameter of $\Omega$. For every $a \in \Omega$ and $r > 0$, we define

$$B(a, r) = \{ y \in \mathbb{R}^n: |y - a| < r \},$$

and

$$\Omega(a, r) = \Omega \cap B(a, r) = \{ y \in \Omega: |y - a| < r \}.$$  

The Morrey spaces $L^{p, \lambda}(\Omega)$ is defined to be the set of all functions $f \in L^p(\Omega)$ which satisfy

$$\|f\|_{L^{p, \lambda}(\Omega)} = \sup_{a \in \Omega, r > 0} \left( \frac{1}{r^\lambda} \int_{\Omega(a, r)} |f(y)|^p \, dy \right)^{1/p} < \infty,$$

for $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. This Morrey spaces were introduced by C. B. Morrey [1] and still attracted the attention of many researcher to investigate its inclusion properties or application in partial differential equation [2, 3, 4, 5, 6, 7, 8].

Let $W^{1,2}(\Omega)$ be the Sobolev space equipped by the norm

$$\|u\|_{W^{1,2}(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{1/2} = \left( \int_\Omega |u|^2 + \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2}$$

for every $u \in W^{1,2}(\Omega)$.

Let $W_0^{1,2}(\Omega)$ be the Hilbert space, that is, the Sobolev norm is generated by an inner or scalar product on $W_0^{1,2}(\Omega)$.

We consider the following second order divergent elliptic operator

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{i,j} \frac{\partial u}{\partial x_i}),$$  \hspace{1cm} (1)

where $u \in W_0^{1,2}(\Omega)$,

$$a_{i,j} \in L^\infty(\Omega), \quad i,j = 1, \ldots, n,$$  \hspace{1cm} (2)

and there exists $\nu > 0$ such that

$$\nu |\xi|^2 \leq \sum_{i,j=1}^n a_{i,j}(x)\xi_i \xi_j,$$  \hspace{1cm} (3)

for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

Let $f \in L^{1,\lambda}(\Omega)$. In this paper, we will investigate the existence and uniqueness of the weak solution to the equation

$$\begin{cases}
    Lu = f, \\
    u \in W_0^{1,2}(\Omega),
\end{cases}$$  \hspace{1cm} (4)

where $L$ is defined by (1) and the $\lambda$ satisfies a certain condition. The Eq. (4) is called the Dirichlet problem.

The function $u \in W_0^{1,2}(\Omega)$ is called the weak solution of the Dirichlet problem (4) if

$$\int_\Omega \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} \, dx = \int_\Omega f(x)\phi(x) \, dx,$$  \hspace{1cm} (5)

for every $\phi \in W_0^{1,2}(\Omega)$.

Recently, Tumalun and Tuerah [9], continue the work of Di Fazio [10, 11], proved that the weak solution of gradient of (4) belongs to some weak Morrey spaces by assuming $f \in L^{1,\lambda}(\Omega)$ for $0 < \lambda < n - 2$. Notice that, the result in [9], generalized by themselves in [12]. In [9, 10, 11], the authors used a representation of the weak solution, which involves the Green function [13], and proved that this representation satisfies (5) to show the existence of the weak solution of (4). Cirmi et. al [14] proved that the weak solution of (4) exists and unique, and its gradient belongs to some Morrey spaces, where they assumed $f \in L^{1,\lambda}(\Omega)$ for $n - 2 < \lambda < n$. The proof of the existence and uniqueness of the weak solution, which is done by Cirmi et. al, used an approximation method.
By assuming \( f \in L^{1,\lambda}(\Omega) \), for \( n - 2 < \lambda < n \), in this paper we will give a direct proof of the existence and uniqueness of the weak solution of the Dirichlet problem (4). Our method uses a functional analysis tool, i.e. the Lax-Millgram lemma, combining with a weighted embeddings in Morrey and Sobolev spaces.

2. RESEARCH METHODS

The constant \( C = C(\alpha, \beta, \ldots, \gamma) \), which appears throughout this paper, denotes that it is dependent on \( \alpha, \beta, \ldots \), and \( \gamma \). The value of this constant may vary from line to line whenever it appears in the theorems or proofs.

Our method relies on functional analysis tools, that is Lax-Milgram lemma, that we will state in this section. We start by write down some properties related to Lax-Milgram lemma.

Let \( H \) be a Hilbert space with norm \( \| \cdot \| \) and \( B: H \times H \rightarrow \mathbb{R} \) be a bilinear mapping. The map \( B \) is called continuous if there exists a constant \( C_1 > 0 \) such that
\[
|B(u, w)| \leq C_1 \| u \| \| w \|,
\]
for all \( u, w \in H \), and called coercive if there exists a constant \( C_2 > 0 \) such that
\[
B(u, u) \geq C_2 \| u \|^2,
\]
for all \( u \in H \).

The following lemma is known as Lax-Milgram lemma and \( H \) is the Hilbert space with norm \( \| \cdot \| \). We refer to [15] for its proof.

**Lemma 1** (Lax-Milgram). Let \( B: H \times H \rightarrow \mathbb{R} \) be a continuous and coercive bilinear mapping. Then, for every bounded linear functional \( F: H \rightarrow \mathbb{R} \), there exists a unique element \( u \in H \) such that
\[
F(w) = B(u, w),
\]
for every \( w \in H \).

We associate the operator \( L \) with the mapping \( B: W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \rightarrow \mathbb{R} \) defined by the formula
\[
B(u, \phi) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} \, dx.
\]
(6)

For \( n - 2 < \lambda < n \) and \( f \in L^{1,\lambda}(\Omega) \), we define \( F_f: W_0^{1,2}(\Omega) \rightarrow \mathbb{R} \) by the formula
\[
F_f(\phi) = \int_{\Omega} f(x) \phi(x) \, dx.
\]
(7)

By the linearity of the weak derivative and integration, it is easy to show that \( B \), defined by (6), is a bilinear mapping. Notice that, according to (5), (6), and (7), \( u \in W_0^{1,2}(\Omega) \) is a weak solution of (4) if \( B(u, \phi) = F_f(\phi) \), (8)

for every \( \phi \in W_0^{1,2}(\Omega) \).

Now we state the following two theorems regarding to the estimation for any functions in \( W_0^{1,2}(\Omega) \), that we will need later. The first theorems called Poincaré’s inequality (see [15] for its proof) and the second theorem called sub representation formula (see [16] for its proof).

**Theorem 1** (Poincaré’s Inequality). If \( u \in W_0^{1,2}(\Omega) \), then there exists a positive constant \( C = C(l) \) such that
\[
\int_{\Omega} |u|^2 \leq C \int_{\Omega} |\nabla u|^2.
\]

**Theorem 2** (Sub Representation Formula). If \( u \in W_0^{1,2}(\Omega) \), then there exists a positive constant \( C = C(n) \) such that
\[
|u(x)| \leq C \int_{\Omega} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy,
\]
for a.e. \( x \in \Omega \).
We close this section by state the following Theorem which slightly modified from [17].

**Theorem 3.** Let $n - 2 < \lambda < n$. If $f \in L^{1, \lambda}(\Omega)$, then there exists a positive constant $C = C(n, \lambda, l)$ such that

$$\int_{\Omega} \frac{|f(x)|}{|z - x|^{n-2}} \, dx \leq C \|f\|_{L^{1, \lambda}},$$

for every $z \in \Omega$.

### 3. RESULTS AND DISCUSSION

To start our discussion, we prove that the bilinear mapping $B$ defined by (6) is continuous and coercive.

**Lemma 2.** The mapping $B$ defined by (6) is continuous and coercive.

**Proof.** Let $u \in W^{1,2}_0(\Omega)$. We first prove the coercivity property. By using (3) and then Poincaré’s inequality, we have

$$B(u, u) = \int_{\Omega} \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} \, dx \geq \nu \int_{\Omega} |\nabla u|^2$$

$$\geq \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} |u|^2 \geq C (\|u\|_{W^{1,2}(\Omega)})^2,$$

where $C = C(\nu, l)$ is a positive constant.

Now, we prove the continuity property. Let $u, \phi \in W^{1,2}_0(\Omega)$. Note that

$$A = \sum_{i,j=1}^{n} \|a_{i,j}\|_{L^{\infty}(\Omega)}$$

according to (2). By using Hölder’s inequality, we have

$$|B(u, \phi)| \leq \int_{\Omega} \sum_{i,j=1}^{n} \|a_{i,j}\|_{L^{\infty}(\Omega)} \left| \frac{\partial u(x)}{\partial x_i} \right| \left| \frac{\partial \phi(x)}{\partial x_j} \right| \, dx$$

$$\leq A \int_{\Omega} |\nabla u(x)| |\nabla \phi(x)| \, dx$$

$$\leq A \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \phi(x)|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq A \|u\|_{W^{1,2}(\Omega)} \|\phi\|_{W^{1,2}(\Omega)}.$$

This completes the proof. □

We need the theorem below to prove that the function $F_f$ defined by (7) is a bounded linear functional. This theorem states about a weighted estimation in Morrey spaces where the weight in Sobolev spaces. The proof of this theorem was given in [11]. However, the given proof did not complete. Here we give the complete proof.

**Theorem 4.** Let $n - 2 < \lambda < n$. If $f \in L^{1, \lambda}(\Omega)$, then there exists a positive constant $C = C(n, \lambda, l)$ such that

$$\int_{\Omega} |fu| \leq C \|f\|_{L^{1, \lambda}(\Omega)} \|u\|_{W^{1,2}(\Omega)}$$

for every $u \in W^{1,2}_0(\Omega)$.

**Proof.** Let $u \in W^{1,2}_0(\Omega)$. According to the sub representation formula of $u$ and Hölder’s inequality, we have
\[ \int_\Omega |f(x)u(x)| \, dx \leq C(n) \int_\Omega |f(x)| \left( \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy \right) \, dx \]

\[ = C(n) \int_\Omega |\nabla u(y)| \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \, dx \right) \, dy \]

\[ \leq C(n) \| \nabla u \|_{L^2(\Omega)} \left( \int_\Omega \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \, dx \right)^2 \, dy \right)^{\frac{1}{2}}. \] (9)

Notice that

\[ \int_\Omega |f(z)| \, dz \leq C(n, \lambda, l) \| f \|_{L^{1,\lambda}(\Omega)}. \] (10)

Hence

\[ \int_\Omega \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \, dx \right)^2 \, dy = \int_\Omega \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \right) \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \, dx \right) \, dy \]

\[ = \int_\Omega \int_\Omega \frac{|f(z)||f(x)|}{|x-y|^{n-1}} \, dx \, dz \]

\[ \leq C(n) \int_\Omega \int_\Omega |f(z)||f(x)| \frac{1}{|x-y|^{n-2}} \, dx \, dz \]

\[ = C(n) \int_\Omega |f(z)| \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-2}} \, dx \right) \, dz \]

\[ \leq C(n, \lambda, l) \| f \|_{L^{1,\lambda}(\Omega)} \int_\Omega |f(z)| \, dz \]

\[ \leq C(n, \lambda, l) \left( \| f \|_{L^{1,\lambda}(\Omega)} \right)^2. \] (11)

by virtue of Theorem 3 and (10). Combining (9) and (11), we obtain

\[ \int_\Omega |f(x)u(x)| \, dx \leq C(n) \| \nabla u \|_{L^2(\Omega)} \left( \int_\Omega \left( \int_\Omega \frac{|f(x)|}{|x-y|^{n-1}} \, dx \right)^2 \, dy \right)^{\frac{1}{2}} \]

\[ \leq C(n, \lambda, l) \| \nabla u \|_{L^2(\Omega)} \| f \|_{L^{1,\lambda}(\Omega)} \]

\[ \leq C(n, \lambda, l) \| u \|_{W^{1,2}(\Omega)} \| f \|_{L^{1,\lambda}(\Omega)}. \]

The theorem is proved. \[ \square \]

From Theorem 4, we obtain the following corollary.

**Corollary 1.** Let \( n - 2 < \lambda < n \). The mapping \( F_f: W^{1,2}_0(\Omega) \to \mathbb{R} \) defined by (7) is a bounded linear functional.

**Proof.** According to the linearity of integration, \( F_f \) is a linear functional on \( W^{1,2}_0(\Omega) \). For every \( u \in W^{1,2}_0(\Omega) \), Theorem 4 gives us

\[ |F_f(u)| \leq \int_\Omega |fu| \leq C \| u \|_{W^{1,2}(\Omega)}, \]

where the positive constant \( C = C(n, \lambda, l, \| f \|_{L^{1,\lambda}(\Omega)}) \). This means \( F_f \) is also bounded and the proof is complete. \[ \square \]
Combining Lemma 2, Corollary 1, and Lax-Milgram lemma, we now state the following existence and uniqueness of the weak solution of the Dirichlet problem (4).

**Theorem 5.** Let $n - 2 < \lambda < n$ and $f \in L^{1,\lambda}(\Omega)$ in Dirichlet problem (4). Then there exists a unique element $u \in W_0^{1,2}(\Omega)$ which is the weak solution of the Dirichlet problem (4).

4. **CONCLUSIONS**

The Dirichlet problem (4) has a unique weak solution by assuming the data belongs some Morrey spaces. This fact can be proved by using functional analysis tool, i.e. the Lax-Milgram lemma, combining with the weighted embedding in Morrey spaces where the weight in Soblev spaces.

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