**Abstract.** Given a finite-dimensional algebra \( \Lambda \) and \( A \geq 1 \), we construct a new algebra \( \tilde{\Lambda}_A \), called the stretched algebra, and relate the homological properties of \( \Lambda \) and \( \tilde{\Lambda}_A \). We investigate Hochschild cohomology and the finiteness condition \((F_g)\), and use stratifying ideals to show that \( \Lambda \) has \((F_g)\) if and only if \( \tilde{\Lambda}_A \) has \((F_g)\). We also consider projective resolutions and apply our results in the case where \( \Lambda \) is a \( d \)-Koszul algebra for some \( d \geq 2 \).

**Introduction**

Let \( K \) be a field and let \( \Lambda = KQ/I \) be a finite-dimensional algebra where \( I \) is an admissible ideal of \( KQ \). For each \( A \geq 1 \), we construct from \( \Lambda \) a new algebra \( \tilde{\Lambda}_A \), called the stretched algebra. The aim of the paper is to relate the homological properties of \( \Lambda \) and \( \tilde{\Lambda}_A \). In Section 2, the focus is on Hochschild cohomology and the finiteness condition \((F_g)\) of [5], and in Section 3 we look at projective resolutions and apply the results to construct examples of stretched algebras.

Section 2 studies the Hochschild cohomology of \( \Lambda \) and the stretched algebra \( \tilde{\Lambda}_A \). Our motivation here lies in the theory of support varieties. For a group algebra of a finite group, Carlson introduced a powerful theory of support varieties of modules [2], [4]. Support varieties were extended to finite-dimensional algebras by Snashall and Solberg in [15], using the Hochschild cohomology ring of the algebra. And, under the finiteness condition \((F_g)\) of [5] (see Definition 2.6), many of the properties known for the group situation were shown to have analogues in this more general setting. Subsequently, the condition \((F_g)\) has been widely studied. Our intention is to use Nagase’s result [13, Proposition 6] concerning \((F_g)\) and algebras with stratifying ideals. In Theorem 2.2 we give an idempotent element \( \varepsilon \) of the stretched algebra \( \tilde{\Lambda}_A \), proving that \( \langle \varepsilon \rangle \) is a stratifying ideal in \( \tilde{\Lambda}_A \). We then show in Corollary 2.5 that the projective dimension of \( \tilde{\Lambda}_A/\langle \varepsilon \rangle \) is 2 as a \( \tilde{\Lambda}_A-\tilde{\Lambda}_A \)-bimodule. Our main result is Theorem 2.8 where we show that \( \tilde{\Lambda}_A \) has \((F_g)\) if and only if \( \Lambda \) has \((F_g)\).

Section 3 considers projective resolutions. In Theorem 3.1 we start with a minimal projective resolution of \( \Lambda/\tau \) as a right \( \Lambda \)-module, and explicitly describe a minimal projective resolution of \( \tilde{\Lambda}_A/\tilde{\tau}_A \) as a right \( \tilde{\Lambda}_A \)-module, where \( \tau \) (resp. \( \tilde{\tau}_A \)) denotes the Jacobson radical of \( \Lambda \) (resp. \( \tilde{\Lambda}_A \)). We apply this in the case where \( \Lambda \) is a \( d \)-Koszul algebra for some \( d \geq 2 \). This connects with work of Leader [11] in which she considered a family of algebras which are seen to be stretched algebras in the special case where \( \Lambda \) is a \( d \)-Koszul algebra. Our approach is very different, but as a consequence and in the case where \( \Lambda \) is \( d \)-Koszul, we recover [11, Theorem 8.15] by showing that \( \tilde{\Lambda}_A \) is a \((D,A)\)-stacked algebra where \( D = dA \); this is Theorem 3.4. The class of \((D,A)\)-stacked algebras was introduced by Leader and

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2010 Mathematics Subject Classification. 16G20, 16E05, 16E30, 16E40, 16S37.

Key words and phrases. \( d \)-Koszul, projective resolution, Ext algebra, Hochschild cohomology, finiteness condition.

This work formed part of the first author’s PhD thesis at the University of Leicester, which was supported by The Higher Committee for Education Development in Iraq (HCED).
Snashall in [12, Definition 2.1] (see Definition 3.3) and provides a natural generalisation of Koszul and d-Koszul algebras. Thus Theorem 3.4 gives us examples of stretched algebras as well as a construction of \((D, A)\)-stacked algebras.

We keep the following notation throughout the paper. The set of vertices of a quiver \(Q\) is denoted by \(Q_0\). An arrow \(\alpha\) starts at \(s(\alpha)\) and ends at \(t(\alpha)\); arrows in a path are read from left to right. A path \(p = \alpha_1\alpha_2\cdots\alpha_n\), where \(\alpha_1, \alpha_2, \ldots, \alpha_n\) are arrows, is of length \(n\) with \(s(p) = s(\alpha_1)\) and \(t(p) = t(\alpha_n)\). We write \(\ell(p)\) for the length of the path \(p\).

An element \(x\) in \(KQ\) is uniform if there exist vertices \(v, v'\) in \(Q\) such that \(x = vx = xv'\). We then write \(s(x) = v\) and \(t(x) = v'\). If the ideal \(I\) is generated by paths in \(KQ\) then \(KQ/I\) is a monomial algebra. If \(I\) is length homogeneous, then \(\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots\) is a graded algebra with the length grading, and \(\Lambda_0 \cong \Lambda/\tau\). The Ext algebra of \(\Lambda\) is given by \(E(\Lambda) = \oplus_{n \geq 0} \text{Ext}_n^R(\Lambda, \Lambda, \Lambda/\tau)\) with the Yoneda product. The Hochschild cohomology ring of \(\Lambda\) is given by \(\text{HH}^*(\Lambda) = \text{Ext}_\Lambda^*(\Lambda, \Lambda) = \oplus_{n \geq 0} \text{Ext}_\Lambda^n(\Lambda, \Lambda)\) with the Yoneda product, where \(\Lambda^e = \Lambda^{op} \otimes_K \Lambda\) is the enveloping algebra of \(\Lambda\). All modules are finite-dimensional right modules. We write \(\dim\) for \(\dim_K\) and \(\otimes\) for \(\otimes_K\); in all other cases the subscripts are specified. We use \(\text{pdim}\) for the projective dimension, \(\text{idim}\) for the injective dimension and \(\text{gldim}\) for the global dimension.

1. Constructing the stretched algebra

Let \(\Lambda = KQ/I\) be a finite-dimensional algebra where \(I\) is generated by a minimal set \(\mathcal{g}^2\) of uniform elements in \(KQ\). Let \(A \geq 1\). We describe the construction of \(\tilde{\Lambda}_A\) by using the quiver \(Q\) and ideal \(I\) of \(KQ\) to define a new quiver \(\tilde{Q}_A\) and admissible ideal \(I_A\) of \(K\tilde{Q}_A\) giving \(\tilde{\Lambda}_A = K\tilde{Q}_A/I_A\). This construction builds on ideas in [11]. We begin with the quiver \(\tilde{Q}_A\).

**Definition 1.1.** Let \(Q\) be a finite quiver. Let \(A \geq 1\). We construct the new quiver \(\tilde{Q}_A\) as follows:

- All vertices of \(Q\) are also vertices in \(\tilde{Q}_A\).
- For each arrow \(\alpha\) in \(Q\) we have \(A\) arrows \(\alpha_1, \alpha_2, \ldots, \alpha_A\) in \(\tilde{Q}_A\) and additional vertices \(w_1, w_2, \ldots, w_{A-1}\) in \(\tilde{Q}_A\), such that:
  \[
  \begin{array}{ll}
  s(\alpha_1) &= s(\alpha) \\
  t(\alpha_1) &= s(\alpha_2) = w_1 \\
  t(\alpha_2) &= s(\alpha_3) = w_2 \\
  & \vdots \\
  t(\alpha_{A-1}) &= s(\alpha_A) = w_{A-1} \\
  t(\alpha_A) &= t(\alpha)
  \end{array}
  \]
  and the only arrows incident with the vertex \(w_j\) are \(\alpha_j\) and \(\alpha_{j+1}\).

In this way the arrow \(\alpha\) in \(Q\) corresponds to a path \(\alpha_1 \cdots \alpha_A\) of length \(A\) in \(\tilde{Q}_A\). For ease of notation, we identify the set of vertices \(Q_0\) of \(Q\) with the corresponding subset of the vertices of \(\tilde{Q}_A\).

**Definition 1.2.** Let \(\theta^*: KQ \to K\tilde{Q}_A\) be the \(K\)-algebra homomorphism which is induced from

\[
\begin{cases}
  v \mapsto v & \text{for each vertex } v \in Q, \\
  \alpha \mapsto \alpha_1\alpha_2\cdots\alpha_A & \text{for each arrow } \alpha \in Q.
\end{cases}
\]

Moreover, \(\theta^*\) is also a \(K\)-algebra monomorphism.
Definition 1.3. Suppose \( w \in (\mathcal{Q}_A)_0 \setminus \mathcal{Q}_0 \). Define \( \tilde{p}_w \) to be the unique shortest path in \( K\mathcal{Q}_A \) which starts at a vertex in \( \mathcal{Q}_0 \) and ends at \( w \). Define \( \tilde{q}_w \) to be the unique shortest path in \( K\mathcal{Q}_A \) which starts at the vertex \( w \) and ends at a vertex in \( \mathcal{Q}_0 \).

Remark 1.4. Let \( w \in (\mathcal{Q}_A)_0 \setminus \mathcal{Q}_0 \). Then there is a unique arrow \( \alpha \) in \( \mathcal{Q} \) such that \( \theta^*(\alpha) = \alpha_1 \cdots \alpha_A \) and \( w = w_i \) for some \( i = 1, \ldots, A - 1 \). Let \( v = \sigma(\alpha) \) and let \( v' = t(\alpha) \).

Then the quiver \( \mathcal{Q}_A \) contains the subquiver

\[
v \xrightarrow{\alpha_1} w_1 \xrightarrow{\alpha_2} w_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{A-1}} w_{A-1} \xrightarrow{\alpha_A} v'
\]

Thus \( \tilde{p}_w = \alpha_1 \cdots \alpha_i \) and \( \tilde{q}_w = \alpha_i+1 \cdots \alpha_A \). Moreover \( \sigma(\tilde{p}_w) = v, t(\tilde{q}_w) = v' \) and \( \tilde{p}_w, \tilde{q}_w = \alpha_1 \cdots \alpha_A \).

We may illustrate these paths by:

\[
v \xrightarrow{\tilde{p}_w} w \xleftarrow{\tilde{q}_w} v'
\]

We are now ready to define the algebra \( \tilde{\Lambda}_A \).

Definition 1.5. Let \( \Lambda = K\mathcal{Q}/I \) be a finite-dimensional algebra where \( I \) is generated by a minimal set \( g^2 \) of uniform elements in \( K\mathcal{Q} \). List the elements of \( g^2 \) as \( g_1^2, g_2^2, \ldots, g_m^2 \). Let \( A \geq 1 \). Let \( \mathcal{Q}_A \) be the quiver defined in Definition 1.3. For \( i = 1, \ldots, m \), define \( \tilde{g}_i^2 = \theta^*(g_i^2) \). Then each \( \tilde{g}_i^2 \) is a uniform element in \( K\mathcal{Q}_A \) and \( \theta(\tilde{g}_i^2) = g_i^2 \) and \( t(\tilde{g}_i^2) = t(g_i^2) \). We define \( I_A \) to be the ideal of \( K\mathcal{Q}_A \) generated by \( \tilde{g}^2 = \{ \tilde{g}_1^2, \ldots, \tilde{g}_m^2 \} \) and define \( \tilde{\Lambda}_A = K\mathcal{Q}_A/I_A \). We call \( \tilde{\Lambda}_A \) the stretched algebra of \( \Lambda \).

Example 1.6. Let \( \Lambda = K\mathcal{Q}/I \) where \( \mathcal{Q} \) is the quiver

\[
x \xleftarrow{w} v \xrightarrow{y}
\]

and \( I = \langle x^2, xy - yx, y^2 \rangle \). Then, for \( A = 2 \), the stretched algebra \( \tilde{\Lambda}_2 \) is given by \( \tilde{\Lambda}_2 = K\tilde{\mathcal{Q}}/\tilde{I} \) where \( \tilde{\mathcal{Q}} \) is the quiver

\[
w \xleftarrow{x_1} y_1 \xrightarrow{y_2} w'
\]

and \( \tilde{I} = \langle (x_1x_2)^2, x_1x_2y_1y_2 - y_1y_2x_1x_2, (y_1y_2)^2 \rangle \).

This construction has the following properties.

Proposition 1.7. Let \( m_0 \) be the number of vertices of \( \mathcal{Q} \) and \( m_1 \) be the number of arrows of \( \mathcal{Q} \). We have the following properties.

(1) The stretched algebra \( \tilde{\Lambda}_A \) is a finite-dimensional algebra.

(2) The quiver \( \mathcal{Q}_A \) has \( m_0 + m_1(A - 1) \) vertices and \( m_1 A \) arrows.

(3) The set \( \tilde{g}^2 = \{ \tilde{g}_1^2, \ldots, \tilde{g}_m^2 \} \) is a minimal generating set of uniform elements for \( I_A \).

(4) If \( I \) is generated by length homogeneous elements, then \( I_A \) is generated by length homogeneous elements.

(5) If \( I \) is generated by length homogeneous elements all of length \( d \), then \( I_A \) is generated by length homogeneous elements all of length \( dA \).

(6) If \( \Lambda \) is a monomial algebra, then \( \tilde{\Lambda}_A \) is a monomial algebra.

To avoid too many subscripts and where there is no confusion, we write \( \tilde{\Lambda} \) (resp. \( \tilde{\mathcal{Q}}, \tilde{I} \)) instead of \( \tilde{\Lambda}_A \) (resp. \( \tilde{\mathcal{Q}}_A, \tilde{I}_A \)).

Definition 1.8. Let \( \varepsilon = \sum_{v \in \mathcal{Q}_0} v \), which is considered as an element of \( \tilde{\Lambda} \).
Proposition 1.11. Let \( \Lambda = \mathbb{K}Q/I \) be a finite-dimensional algebra. Then \( \Lambda \cong \mathbb{E}\Lambda \mathbb{E} \).

Furthermore, if \( w \in \tilde{Q}_0 \setminus Q_0 \), then we observe from Remark 1.14 that any element of \( \mathbb{E}\tilde{Q}w \) can be written as \( (\varepsilon \tilde{s})v\tilde{p}_w \) for some \( \tilde{s} \in \tilde{Q} \). Similarly, any element of \( wK\tilde{Q} \mathbb{E} \) can be written as \( \tilde{q}_w v'(\varepsilon \tilde{s}) \) for some \( \tilde{s} \in \tilde{Q} \).

Proposition 1.10. Let \( w \in \tilde{Q}_0 \setminus Q_0 \). Let \( v = \sigma(\tilde{p}_w) \) and \( v' = \tilde{t}(\tilde{q}_w) \). Let \( \lambda \in \Lambda \) and \( \lambda \in \tilde{\Lambda} \).

(1) If \( 0 \neq \tilde{\lambda}v \in \Lambda v \), then \( 0 \neq \tilde{\lambda}_w \in \tilde{\Lambda} \).

(2) If \( 0 \neq v' \tilde{\lambda} \in v' \Lambda \), then \( 0 \neq \tilde{q}_w \lambda \in \tilde{\Lambda} \).

Proof. (1). Suppose that \( \tilde{\lambda}_w = 0 \in \tilde{\Lambda} \). By considering \( \tilde{\lambda}v \) as an element of \( \mathbb{E}\tilde{Q}_w \), we have that \( \tilde{\lambda}_w \in \tilde{I} \). Now, \( \tilde{I} \) is generated by the set \( \{\tilde{g}_1, \ldots, \tilde{g}_2\} \) of uniform elements in \( \mathbb{E}\tilde{Q}_w \), so write \( \tilde{\lambda}_w = \sum_k \tilde{r}_k \tilde{g}_k \tilde{e}_k w \) for some \( \tilde{r}_k, \tilde{e}_k \in \tilde{Q} \). As noted above, each term \( \varepsilon \tilde{e}_k w \) is of the form \( \varepsilon \tilde{f}_{k',w} \tilde{p}_w \) for some \( \tilde{f}_{k',w} \in \tilde{Q} \). So we have \( \tilde{\lambda}_w = (\sum_k \tilde{r}_k \varepsilon \tilde{g}_k \varepsilon \tilde{f}_{k',w}) \tilde{p}_w \in \tilde{Q} \mathbb{E} \) and hence \( \tilde{\lambda}v \in \tilde{I} \). Thus \( \tilde{\lambda}v = 0 \in \tilde{\lambda}v \) as required.

The proof of (2) is similar. \( \square \)

The fact that \( \tilde{I} \) is generated by uniform elements in \( \mathbb{E}\tilde{Q} \) is used again in the proofs of the next two propositions; they are straightforward and are left to the reader.

Proposition 1.12. Let \( w \in \tilde{Q}_0 \setminus Q_0 \). We use the notation of Remark 1.14, so \( w = w_i \) for some \( i = 1, \ldots, A - 1 \).

(1) An element of \( \tilde{\lambda}w_i \) is of the form

\[
\tilde{\lambda}w_i = \sum_{j=1}^{i} c_{j} w_{j} \alpha_{j+1} \cdots \alpha_{i} w_i + \tilde{\mu} \tilde{p}_w_i
\]

where \( c_{j} \in \mathbb{K}, \tilde{\mu} \in \tilde{\Lambda} \).

(2) An element of \( w_i \tilde{\lambda} \) is of the form

\[
w_i \tilde{\lambda} = \sum_{j=1}^{A-1} c_{j} w_{i} \alpha_{i+1} \cdots \alpha_{j} w_j + \tilde{q}_w \tilde{\mu}
\]

where \( c_{j} \in \mathbb{K}, \tilde{\mu} \in \tilde{\Lambda} \).

(3) \( \dim \tilde{\lambda}w_i = i + \dim \tilde{\lambda}v \).

(4) \( \dim w_i \tilde{\lambda} = (A - i) + \dim v' \tilde{\lambda} \).

(5) \( \varepsilon \tilde{\lambda}w = \varepsilon \mathbb{E} \tilde{p}_w \) and \( w \tilde{\lambda} \mathbb{E} = \tilde{q}_w \varepsilon \tilde{\lambda} \mathbb{E} \).

Theorem 1.13. Let \( \Lambda = \mathbb{K}Q/I \) and let \( \tilde{\Lambda} \) be the stretched algebra. Let \( B = \varepsilon \tilde{\Lambda} \mathbb{E} \). Then
(1) $\tilde{\Lambda} = \Lambda$ is projective as a right $B$-module.
(2) $\varepsilon \Lambda$ is projective as a left $B$-module.

Proof. (1). We have that $\tilde{\Lambda} = \varepsilon \Lambda \oplus (1 - \varepsilon) \tilde{\Lambda} = \varepsilon \tilde{\Lambda} \oplus (\oplus_{w \in \tilde{Q}_0 \setminus Q_0} \tilde{Q}_w \Lambda)$. From Proposition 1.12(5) and Proposition 1.11(1), we have that $w \tilde{\Lambda} = \tilde{q}_w \Lambda = \tilde{q}_w B \cong t(\tilde{q}_w) B$. Thus, $\tilde{\Lambda} \cong B \oplus (\oplus_{w \in \tilde{Q}_0 \setminus Q_0} t(\tilde{q}_w) B)$. Noting that each $t(\tilde{q}_w)$ is a vertex in $Q_0$, it follows that $\tilde{\Lambda}$ is a projective right $B$-module.

The proof of (2) is similar. □

2. Stratifying ideals and the $(\text{Fg})$ condition

We consider the finiteness condition $(\text{Fg})$ under which we have a rich theory of support varieties for modules over a finite-dimensional algebra. Our first result is Theorem 2.2, which shows that the ideal $\langle \varepsilon \rangle$ is a stratifying ideal of the stretched algebra $\Lambda$. We start by recalling the definition of a stratifying ideal.

Definition 2.1. Let $A$ be a finite-dimensional algebra and let $e$ be an idempotent in $A$. The two sided ideal $\langle e \rangle = AeA$ is a stratifying ideal if:

1. the multiplication map $Ae \otimes_{eAe} eA \rightarrow AeA$ is an isomorphism, and
2. Tor$_n^{eAe}(Ae, eA) = 0$ for all $n > 0$.

It is clear that if the multiplication map $Ae \otimes_{eAe} eA \rightarrow AeA$ is an isomorphism and $Ae$ is a projective right $eAe$-module, then $\langle e \rangle$ is a stratifying ideal.

Theorem 2.2. Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Recall that $\varepsilon = \sum_{v \in Q_0} v$ and $B = \varepsilon \tilde{\Lambda}$ for some $\tilde{\Lambda}$. Then $\langle \varepsilon \rangle$ is a stratifying ideal of $\tilde{\Lambda}$.

Proof. From Theorem 1.13, $\tilde{\Lambda}$ is projective as a right $B$-module. So it suffices to show that the multiplication map $\psi: \tilde{\Lambda} \otimes_B e\tilde{\Lambda} \rightarrow e\tilde{\Lambda}e\tilde{\Lambda}$ is an isomorphism. It is clear that $\psi$ is a $\Lambda$-$\Lambda$-bimodule homomorphism and is onto. We show that $\psi$ is one-to-one. Suppose that $\psi(\sum \tilde{\Lambda} \otimes_B e\tilde{\nu}) = 0$, with $\tilde{\lambda}, \tilde{\mu} \in \tilde{\Lambda}$. From Proposition 1.12(5), $\tilde{\Lambda} = \varepsilon \Lambda \oplus (\oplus_{w \in \tilde{Q}_0 \setminus Q_0} \tilde{Q}_w \Lambda) = \varepsilon \Lambda \oplus (\oplus_{w \in \tilde{Q}_0 \setminus Q_0} \tilde{q}_w \Lambda)$ so we may write

$$\sum \tilde{\lambda} \otimes_B e\tilde{\nu} = \varepsilon \otimes_B e\tilde{\nu} + \sum_{w \in \tilde{Q}_0 \setminus Q_0} \tilde{q}_w \Lambda \otimes_B e\tilde{\nu}_w$$

for some $\tilde{\nu}, \tilde{\nu}_w$. Then $0 = \psi(\sum \tilde{\lambda} \otimes_B e\tilde{\nu}) = \varepsilon \tilde{\nu} + \sum_{w \in \tilde{Q}_0 \setminus Q_0} \tilde{q}_w \tilde{\nu}_w$. Left multiplication by $\varepsilon$ gives that $\varepsilon \tilde{\nu} = 0$. For each $w \in \tilde{Q}_0 \setminus Q_0$, left multiplication by $w$ gives $\tilde{q}_w \tilde{\nu}_w = 0$; then from Proposition 1.10, we have that $t(\tilde{q}_w) \tilde{\nu}_w = 0$. Thus $\sum \tilde{\lambda} \otimes_B e\tilde{\nu} = 0$ and $\psi$ is one-to-one. Hence $\langle \varepsilon \rangle$ is a stratifying ideal. □

We now study the quotient $\tilde{\Lambda}/\langle \varepsilon \rangle$. We use the notation introduced in Remark 1.4. In addition, for each arrow $\alpha$ in $Q$, let $\Gamma_\alpha$ denote the following subquiver of $Q$:

$$w_1 \overset{\alpha_1}{\rightarrow} w_2 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_{A-1}}{\rightarrow} w_{A-1}$$

We have $\tilde{\Lambda}/\varepsilon \tilde{\Lambda} \cong \oplus_{\alpha \in Q_1} (\tilde{\Lambda} w_1 \tilde{\Lambda} + \tilde{\Lambda} w_2 \tilde{\Lambda} + \cdots + \tilde{\Lambda} w_{A-1} \tilde{\Lambda} + \tilde{\Lambda} \tilde{\Lambda})/\tilde{\Lambda} \tilde{\Lambda}$. Define $X_\alpha = (\tilde{\Lambda} w_1 \tilde{\Lambda} + \tilde{\Lambda} w_2 \tilde{\Lambda} + \cdots + \tilde{\Lambda} w_{A-1} \tilde{\Lambda} + \tilde{\Lambda} \tilde{\Lambda})/\tilde{\Lambda} \tilde{\Lambda}$ so

$$\tilde{\Lambda}/\varepsilon \tilde{\Lambda} \cong \oplus_{\alpha \in Q_1} X_\alpha.$$ 

Moreover, $X_\alpha \cong K \Gamma_\alpha$ as $K$-algebras. The following result is now immediate.
Proposition 2.3. Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Then $\dim X_\alpha = A(A - 1)/2$ and $\dim \tilde{\Lambda}/\langle \varepsilon \rangle = m_1 A(A - 1)/2$, where $m_1$ is the number of arrows of $\mathcal{Q}$.

Theorem 2.4. Let $\Lambda = K\mathcal{Q}/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Then $\tilde{\Lambda}/\langle \varepsilon \rangle$ has a minimal projective $\tilde{\Lambda}-\tilde{\Lambda}$-bimodule resolution

$$0 \to \tilde{R}^2 \to \tilde{R}^1 \to \tilde{R}^0 \to \tilde{\Lambda}/\langle \varepsilon \rangle \to 0.$$ 

Proof. The main part of this proof is in constructing a minimal projective $\tilde{\Lambda}-\tilde{\Lambda}$-bimodule resolution for each algebra $X_\alpha$.

Let $\alpha$ be an arrow in $\mathcal{Q}$. We keep the notation of this section and of Remark 1.4. Let $v = \phi(\alpha)$ and $v' = \tau(\alpha)$. Set $\dim \tilde{\Lambda} v = V$ and $\dim v' \tilde{\Lambda} = V'$.

Define the bimodule $\tilde{R}_\alpha^0 = \oplus_{i=1}^{A-1} \tilde{\Lambda} w_i \otimes w_i \tilde{\Lambda}$ and the bimodule homomorphism $\Delta^0_{\alpha} : \tilde{R}_\alpha^0 \to X_\alpha$ by $w_i \otimes w_i \mapsto w_i + \tilde{\Lambda} \varepsilon \Lambda$ for $i = 1, \ldots, A - 1$. Using Proposition 1.12 we have

$$\dim \tilde{R}_\alpha^0 = \sum_{i=1}^{A-1} \dim(\tilde{\Lambda} w_i) \dim(w_i \tilde{\Lambda})$$

$$= \sum_{i=1}^{A-1} (i + V)((A - i) + V')$$

$$= \sum_{i=1}^{A-1} i(A - i) + \sum_{i=1}^{A-1} i(V + V') + (A - 1)VV'$$

$$= \frac{1}{6}(A - 1)A(A + 1) + \frac{1}{2}(A - 1)A(V + V') + (A - 1)VV'.$$

So, with Proposition 2.3, we have

$$\dim \ker \Delta^0_{\alpha} = \dim \tilde{R}_\alpha^0 - \dim X_\alpha$$

$$= \frac{1}{6}(A - 2)(A - 1)A + \frac{1}{2}(A - 1)A(V + V') + (A - 1)VV'.$$

The next step is to find the generators of $\ker \Delta^0_{\alpha}$. Let $K$ be the $\tilde{\Lambda}-\tilde{\Lambda}$-bimodule generated by $\{\tilde{p}_{w_i} \otimes w_i, w_{A-1} \otimes \tilde{q}_{w_{A-1}}, w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1}, \text{ for } i = 1, \ldots, A - 2\}$. Clearly $K \subseteq \ker \Delta^0_{\alpha}$. For the reverse inclusion, suppose first that $A = 2$. Set $U_1 = \tilde{\Lambda} \tilde{p}_{w_1} \otimes w_1 \tilde{\Lambda}$ and $U_2 = \tilde{\Lambda} w_1 \otimes \tilde{q}_{w_1} \tilde{\Lambda}$, and note that both $\tilde{p}_{w_i}$ and $\tilde{q}_{w_i}$ are arrows in $\mathcal{Q}$. Then $K = U_1 + U_2$. So $\dim K = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. We see that $U_1 \cap U_2 = \tilde{\Lambda} \tilde{p}_{w_1} \otimes \tilde{q}_{w_1} \tilde{\Lambda}$. So, from Propositions 1.11 and 1.12,

$$\dim K = V(1 + V') + (1 + V)V' - VV' = V + V' + VV'.$$

So $\dim K = \dim K \ker \Delta^0_{\alpha}$ and thus $K = \ker \Delta^0_{\alpha}$.

Now suppose that $A \geq 3$. Here we set $U_1 = \tilde{\Lambda} \tilde{p}_{w_1} \otimes w_1 \tilde{\Lambda}$, $U_2 = \tilde{\Lambda} w_{A-1} \otimes \tilde{q}_{w_{A-1}} \tilde{\Lambda}$ and $U_3 = \sum_{i=1}^{A-2} \tilde{\Lambda}(w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1}) \tilde{\Lambda}$. Then $K = U_1 + U_2 + U_3$ so $\dim K = \dim(U_1 + U_2) + \dim U_3 - \dim(U_1 + U_2) \cap U_3$. Now, $U_1 \subseteq \tilde{\Lambda} w_1 \otimes \tilde{w}_1 \tilde{\Lambda}$ and $U_2 \subseteq \tilde{\Lambda} w_{A-1} \otimes w_{A-1} \tilde{\Lambda}$, so, since $A \geq 3$, we have $U_1 \cap U_2 = \{0\}$. Thus $\dim(U_1 + U_2) = \dim U_1 + \dim U_2$. Note also that $\tilde{\Lambda}(w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1}) \tilde{\Lambda} \cong \tilde{\Lambda}(w_i \otimes w_{i+1}) \tilde{\Lambda}$ so $U_3 \cong \oplus_{i=1}^{A-2} \tilde{\Lambda}(w_i \otimes w_{i+1}) \tilde{\Lambda}$. Then Propositions 1.11 and 1.12 give

$$\dim U_1 = V((A - 1) + V')$$

$$\dim U_2 = ((A - 1) + V)V'$$

$$\dim U_3 = \sum_{i=1}^{A-2} (i + V)((A - (i + 1)) + V').$$
Finally, we can write
\[
\tilde{p}_w \otimes \tilde{q}_w - \tilde{p}_{w, A-1} \otimes \tilde{q}_{w, A-1} = \sum_{j=1}^{A-2} \tilde{p}_j(w_j \otimes \alpha_{j+1} - \alpha_{j+1} \otimes w_{j+1}) \tilde{q}_{j+1}
\]
so that \( \tilde{p}_w \otimes \tilde{q}_w - \tilde{p}_{w, A-1} \otimes \tilde{q}_{w, A-1} \in (U_1 + U_2) \cap U_3 \). Indeed, this element generates \((U_1 + U_2) \cap U_3 = V V' \). Hence
\[
\dim K = V((A - 1) + V') + ((A - 1) + V)V' + \left( \sum_{i=1}^{A-2} (i + V)((A - (i + 1)) + V') \right) - VV'
\]
\[
= (A - 1)(V + V') + VV' + \sum_{i=1}^{A-2} i(A - i - 1) + \sum_{i=1}^{A-2} i(V + V') + \sum_{i=1}^{A-2} VV'
\]
\[
= \frac{1}{6}(A - 2)(A - 1)A + \frac{1}{2}(A - 1)A(V + V') + (A - 1)VV'
\]
\[
= \dim \ker \Delta_\alpha^0
\]
and so \( K = \ker \Delta_\alpha^0 \).

Next we define the bimodule \( \tilde{R}_\alpha^1 = \tilde{\Lambda} v \otimes w_1 \Lambda \oplus (\oplus_{i=1}^{A-2} \tilde{\Lambda} w_i \otimes w_{i+1} \Lambda) \oplus \tilde{\Lambda} w_{A-1} \otimes v' \Lambda \) and the bimodule homomorphism \( \Delta_\alpha^1 : \tilde{R}_\alpha^1 \to \tilde{R}_\alpha^0 \) by
\[
\begin{align*}
  v \otimes w_1 &\mapsto \tilde{p}_w \otimes w_1 \\
  w_i \otimes w_{i+1} &\mapsto w_i \otimes \alpha_{i+1} - \alpha_{i+1} \otimes w_{i+1} \\
  w_{A-1} \otimes v' &\mapsto \tilde{q}_{w, A-1}
\end{align*}
\]
where \( \tilde{p}_w \otimes w_1 \) lies in the \( w_1 \otimes w_1 \)-component of \( \tilde{R}_\alpha^0 \), \( w_{A-1} \otimes \tilde{q}_{w, A-1} \) lies in the \( w_{A-1} \otimes w_{A-1} \)-component of \( \tilde{R}_\alpha^0 \), and, for \( i = 1, \ldots, A - 2 \), \( w_i \otimes \alpha_{i+1} \) lies in the \( w_i \otimes w_i \)-component of \( \tilde{R}_\alpha^0 \), and \( \alpha_{i+1} \otimes w_{i+1} \) lies in the \( w_{i+1} \otimes w_{i+1} \)-component of \( \tilde{R}_\alpha^0 \). Then
\[
\dim \tilde{R}_\alpha^1 = V((A - 1) + V') + \sum_{i=1}^{A-2} (i + V)((A - (i + 1)) + V') + ((A - 1) + V)V'
\]
\[
= \sum_{i=1}^{A-1} i(V + V') + AVV' + \sum_{i=1}^{A-2} i(A - (i + 1))
\]
\[
= \frac{1}{2}A(A - 1)(V + V') + AVV' + \frac{1}{6}A(A - 1)(A - 2)
\]
and hence \( \dim \ker \Delta_\alpha^1 = \dim \tilde{R}_\alpha^1 - \dim \ker \Delta_\alpha^0 = VV' \). To find \( \ker \Delta_\alpha^1 \), let
\[
z = (v \otimes \tilde{q}_w, -\tilde{p}_w \otimes \tilde{q}_w, \ldots, -\tilde{p}_w \otimes \tilde{q}_{w, A-1}, -\tilde{p}_{w, A-2} \otimes \tilde{q}_{w, A-1}, \ldots, -\tilde{p}_{w, A-1} \otimes v')
\]
Then \( z \) is in \( \ker \Delta_\alpha^0 \) and generates a sub-bimodule of \( \ker \Delta_\alpha^1 \) of dimension \( VV' \). Hence \( \ker \Delta_\alpha^1 = (z) \).

Now define the bimodule \( \tilde{R}_\alpha^2 = \tilde{\Lambda} v \otimes v' \Lambda \) and the bimodule homomorphism \( \Delta_\alpha^2 : \tilde{R}_\alpha^2 \to \tilde{R}_\alpha^1 \) by \( v \otimes v' \mapsto z \). Then \( \dim \tilde{R}_\alpha^2 = VV' \) and so \( \dim \ker \Delta_\alpha^2 = 0 \). Thus, \( X_\alpha \) has minimal projective \( \Lambda-\Lambda \)-bimodule resolution
\[
0 \to \tilde{R}_\alpha^2 \xrightarrow{\Delta_\alpha^2} \tilde{R}_\alpha^1 \xrightarrow{\Delta_\alpha^0} \tilde{R}_\alpha^0 \to X_\alpha \to 0.
\]
The result now follows. \( \square \)
Corollary 2.5. Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Then $\text{pdim}_{\tilde{\Lambda}} \tilde{\Lambda}/\langle \varepsilon \rangle = 2$.

We are now in a position to compare the Hochschild cohomology rings of $\Lambda$ and the stretched algebra $\tilde{\Lambda}$. We assume for the remainder of this section that $K$ is an algebraically closed field, and recall, for a finite-dimensional $K$-algebra $A$, that we have the natural ring homomorphism $A/\mathfrak{r} \otimes_A \mathcal{E} : \text{HH}^*(A) \to E(A)$.

Definition 2.6. [5] Let $A$ be an indecomposable finite-dimensional algebra over an algebraically closed field $K$. Then $A$ has $(F_g)$ if $A$ satisfies the following two conditions:

$(F_g1)$ There is a commutative Noetherian graded subalgebra $H$ of $\text{HH}^*(A)$ such that $H_0 = \text{HH}_0(A)$.

$(F_g2)$ $E(A)$ is a finitely generated $H$-module.

As a consequence, if $A$ has $(F_g)$ then both $\text{HH}^*(A)$ and $E(A)$ are finitely generated as $K$-algebras. Moreover, it was shown in [5, Proposition 2.5(a)], that if $A$ has $(F_g)$ then $A$ is Gorenstein. In [13], Nagase studied the finiteness condition $(F_g)$ for Nakayama algebras, proving in [13, Corollary 10] that a Nakayama algebra is Gorenstein if and only if it satisfies $(F_g)$. Stratifying ideals played a key role in this work; we use the following result from [13].

Proposition 2.7. [13, Proposition 6] Let $A$ be a finite-dimensional algebra over an algebraically closed field $K$ with a stratifying ideal $\langle \varepsilon \rangle$. Suppose $\text{pdim}_{A/\langle \varepsilon \rangle} A/\langle \varepsilon \rangle < \infty$. Then we have:

$(1)$ $\text{HH}^{\geq n}(A) \cong \text{HH}^{\geq n}(eAe)$ as graded algebras, where $n = \text{pdim}_{A/\langle \varepsilon \rangle} A/\langle \varepsilon \rangle + 1$,

$(2)$ $A$ satisfies $(F_g)$ if and only if $eAe$ satisfies $(F_g)$,

$(3)$ $A$ is Gorenstein if and only if $eAe$ is Gorenstein.

Combining this with Corollary 2.5 gives the following result for stretched algebras.

Theorem 2.8. Let $K$ be an algebraically closed field. Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra, so that $\langle \varepsilon \rangle$ is a stratifying ideal of $\tilde{\Lambda}$. Then:

$(1)$ $\text{HH}^{\geq 3}(\Lambda) \cong \text{HH}^{\geq 3}(\tilde{\Lambda})$ as graded algebras.

$(2)$ $\tilde{\Lambda}$ satisfies $(F_g)$ if and only if $\Lambda$ satisfies $(F_g)$.

$(3)$ $\tilde{\Lambda}$ is Gorenstein if and only if $\Lambda$ is Gorenstein.

More recently, Psaroudakis, Skartsætherhagen and Solberg [14] considered this finiteness condition for recollements of abelian categories, introducing the concept of an eventually homological isomorphism. In particular, for a finite-dimensional algebra $A$ with an idempotent $e$ over an algebraically closed field $K$, they determine when the functor $\text{res}_e : \text{mod} A \to \text{mod} eAe$ in a recollement of abelian categories is an eventually homological isomorphism.

Definition 2.9. [14, Section 3] Given a functor $F : \mathcal{B} \to \mathcal{C}$ between abelian categories and an integer $t$, the functor $F$ is called a $t$-homological isomorphism if there is a group isomorphism

$$\text{Ext}^j_F(B, B') \cong \text{Ext}^j_F(F(B), F(B'))$$

for every pair of objects $B, B'$ in $\mathcal{B}$, and every $j > t$. Note that we do not require these isomorphisms to be induced by the functor $F$. If $F$ is a $t$-homological isomorphism for some $t$, then we say that $F$ is an eventually homological isomorphism.

Proposition 2.10. [14, Lemma 8.23(ii) and proof] Let $A$ be a finite-dimensional algebra over an algebraically closed field $K$. Suppose that $\langle \varepsilon \rangle$ is a stratifying ideal in $A$. Then the following are equivalent:
(1) $\text{pdim}_A A/e < \infty$.

(2) The functor $\text{res}_e : \text{mod } A \rightarrow \text{mod } eAe$ is an eventually homological isomorphism. Moreover, if $\text{pdim}_A A/e = t < \infty$ then the functor $\text{res}_e$ is a $t$-homological isomorphism.

We come to the final result of this section.

**Theorem 2.11.** Let $K$ be an algebraically closed field. Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra, so that $(e)$ is a stratifying ideal of $\Lambda$. Then the functor $\text{res}_e : \text{mod } \Lambda \rightarrow \text{mod } e\Lambda e$ is a 2-homological isomorphism and hence an eventually homological isomorphism. Moreover, $\text{idim}_\tilde{\Lambda} \tilde{\Lambda} \leq \sup \{\text{idim}_\Lambda \Lambda, 2\}$.

**Proof.** From Corollary 2.5 and Proposition 2.10, the functor $\text{res}_e : \text{mod } \tilde{\Lambda} \rightarrow \text{mod } e\tilde{\Lambda} e$ is a 2-homological isomorphism.

The inequality certainly holds if $\Lambda$ has infinite injective dimension, so assume $\text{idim}_\Lambda \Lambda = n < \infty$ and let $m = \max\{\text{idim}_\Lambda \Lambda, 2\} + 1$. Then

$$\text{Ext}^n_{\tilde{\Lambda}}(X, Y) \cong \text{Ext}^n_{e\tilde{\Lambda}e}(\text{res}_e(X), \text{res}_e(Y))$$

for all $X, Y \in \text{mod } \tilde{\Lambda}$. Setting $Y = \tilde{\Lambda}$ gives

$$\text{Ext}^n_{\tilde{\Lambda}}(X, \tilde{\Lambda}) \cong \text{Ext}^n_{e\tilde{\Lambda}e}(\text{res}_e(X), \text{res}_e(\tilde{\Lambda})) \cong \text{Ext}^n_{e\tilde{\Lambda}e}(\text{res}_e(X), \tilde{\Lambda}e).$$

From Theorem 1.13(1), $\tilde{\Lambda}e$ is projective as a right $e\tilde{\Lambda}e$-module, so $\text{idim}_{e\tilde{\Lambda}e} \tilde{\Lambda}e \leq n$ and thus $\text{Ext}^{n+1}_{e\tilde{\Lambda}e}(\text{res}_e(X), \tilde{\Lambda}e) = 0$. Hence $\text{Ext}^n_{\tilde{\Lambda}}(X, \tilde{\Lambda}) = 0$ and $\text{idim}_{\tilde{\Lambda}} \tilde{\Lambda} \leq m - 1 = \max\{\text{idim}_\Lambda \Lambda, 2\}$ as required. \(\square\)

**Example 2.12.**

(1) Let $\Lambda = K[x]/(x^n)$ for some $n \geq 2$. Let $A \geq 2$. Then the stretched algebra $\tilde{\Lambda}$ has quiver

$$\begin{array}{c}
1 \rightarrow \overset{\alpha_1}{\alpha} \rightarrow \overset{\alpha_2}{\alpha} \rightarrow \cdots \rightarrow \overset{\alpha_{A-1}}{\alpha} \\
\alpha_A
\end{array}$$

and $\tilde{I} = (\alpha_1 \cdots \alpha_A)^m$. This is the algebra of [14, Example 8.14] with $m = A$.

(2) Let $\Lambda = KQ/I$ where $Q$ is the quiver

$$\begin{array}{c}
1 \rightarrow \overset{\alpha}{\alpha} \\
\overset{\beta}{\beta} \rightarrow 2
\end{array}$$

and $I = (\alpha \beta \alpha, \beta \alpha \beta)$. Let $A = 2$. Then the stretched algebra $\tilde{\Lambda}$ has quiver

$$\begin{array}{c}
\beta_2 \rightarrow \overset{\alpha_1}{\alpha} \rightarrow \overset{\alpha_2}{\alpha} \\
\beta_1
\end{array}$$

and $\tilde{I} = (\alpha_1 \alpha_2 \beta_1 \beta_2 \alpha_1 \alpha_2, \beta_1 \beta_2 \alpha_1 \alpha_2 \beta_1 \beta_2)$. The stretched algebra $\tilde{\Lambda}$ is the algebra of [6, Example 3.2], where it was shown that $\tilde{\Lambda}$ has $(\text{Fg})$ and that $\text{idim}_{\tilde{\Lambda}} \tilde{\Lambda} = 2$. Thus we may use Theorem 2.11 to show that $\Lambda$ has $(\text{Fg})$. Moreover, it is immediate that $\Lambda$ is self-injective, so that the upper bound on $\text{idim}_{\tilde{\Lambda}} \tilde{\Lambda}$ in Theorem 2.11 is achieved.
3. Minimal projective resolutions and $d$-Koszul algebras

In this section we keep the original assumptions, so that $K$ is a field, but is not necessarily algebraically closed, $\Lambda = KQ/I$ is a finite-dimensional algebra, $\Lambda \cong 1$, and $\tilde{\Lambda}$ is the stretched algebra. Then $I$ is generated by a minimal set $g^2$ of uniform elements in $KQ$, and $\tilde{I}$ is generated by the minimal set $\tilde{g}^2$ of uniform elements in $K\tilde{Q}$.

With the notation of [1, Chapter I.6], in addition to the functor $\text{res}_\varepsilon$ used above, we also have functors $T_\varepsilon, L_\varepsilon : \mod \varepsilon \tilde{\Lambda} \to \mod \tilde{\Lambda}$ so that $(T_\varepsilon, \text{res}_\varepsilon, L_\varepsilon)$ is an adjoint triple connecting $\mod \varepsilon \tilde{\Lambda}$ and $\mod \tilde{\Lambda}$, namely:

\[
\begin{array}{c}
\text{mod} \tilde{\Lambda} \\
\text{res}_\varepsilon \\
\text{mod} \varepsilon \tilde{\Lambda} \to \text{mod} \tilde{\Lambda} \to \text{mod} \varepsilon \tilde{\Lambda}
\end{array}
\]

with $\text{res}_\varepsilon(\varepsilon) = (\varepsilon)\varepsilon$, $T_\varepsilon(\varepsilon) = - \otimes_{\varepsilon \tilde{\Lambda}} \varepsilon \tilde{\Lambda}$ and $L_\varepsilon(\varepsilon) = \text{Hom}_{\varepsilon \tilde{\Lambda}}(\tilde{\Lambda} \varepsilon, -)$. The functor $T_\varepsilon$ carries projectives to projectives, and is an exact functor by Proposition [1, 3.2]. Using Theorem [1.9] we identify $\Lambda$ with $\varepsilon \tilde{\Lambda}$, and $\tau$ with $\varepsilon \tilde{\varepsilon}$.

The main result of this section is Theorem 3.1 which takes a minimal projective resolution $(P^n, d^n)$ of $\Lambda/\tau$ as a right $\Lambda$-module as given by Green, Solberg and Zacharia in [10], and uses it to construct a minimal projective resolution $(\tilde{P}^n, \tilde{d}^n)$ of $\tilde{\Lambda}/\tilde{\tau}$ as a right $\Lambda$-module. We end the paper with an application to $d$-Koszul algebras.

We recall briefly the construction of [10]. Let $g^0$ be the set of vertices of $Q$, $g^1$ the set of arrows of $Q$, and $g^2$ the minimal generating for $I$ as above. In [10], the authors show that there are sets $g^n$ of uniform elements in $KQ$, for $n \geq 3$, such that for each $x \in g^n$ we have $x = \sum_i g_i^{n-1} r_i$, for unique $r_i \in KQ$, $s_j \in I$. The sets $g^n$ can be chosen so that $(P^n, d^n)$ is a minimal projective resolution of $\Lambda/\tau$ with the following properties:

- $P^n = \oplus_i t(g_i^n) \Lambda$, for all $n \geq 0$;
- $\tilde{d}^0 : P^0 \to \Lambda/\tau$ is the canonical surjection;
- for each $n \geq 1$ and $x \in g^n$ there are unique elements $r_i \in KQ$ with $x = \sum_i g_i^{n-1} r_i$;
- for each $n \geq 1$ and for $x \in g^n$, the $\Lambda$-homomorphism $d^n : P^n \to P^{n-1}$ is such that $d^n(t(x))$ has entry $t(g_i^{n-1}) r_i$ in the summand of $P^{n-1}$ corresponding to $t(g_i^{n-1})$.

We come now to Theorem 3.1. Note that the proof requires a technical result which we state and prove in Proposition 3.2 immediately following the theorem.

**Theorem 3.1.** Let $\Lambda = KQ/I$ and let $\tilde{\Lambda}$ be the stretched algebra. Let $(\tilde{P}^n, \tilde{d}^n)$ be a minimal projective resolution for $\tilde{\Lambda}/\tilde{\tau}$ given by sets $\tilde{g}^n$. Then $(\tilde{P}^n, \tilde{d}^n)$ is a minimal projective resolution for $\tilde{\Lambda}/\tilde{\tau}$ which is defined by sets $\tilde{g}^n$, where

- $\tilde{g}^0$ is the set of vertices of $\tilde{Q}$,
- $\tilde{g}^1$ is the set of arrows of $\tilde{Q}$,
- for $n \geq 2$, $\tilde{g}^n = \{ \tilde{g}_i^n := \theta^* (g_i^n) \mid g_i^n \in g^n \}$.

**Remark.** Note that this agrees with the definition of $\tilde{g}^2$ given in Definition [1.5]. Moreover, for $n \geq 2$, each $\tilde{g}_i^n$ is a uniform element which starts (resp. ends) at the vertex $\alpha(g_i^n)$ (resp. $t(g_i^n)$) in $Q_0$ and so $\tilde{g}_i^n = \varepsilon \tilde{g}_i^n$.

**Proof.** Let $\tilde{g}^0$ be the set of vertices of $\tilde{Q}$, let $\tilde{g}^1$ be the set of arrows of $\tilde{Q}$, and let $\tilde{g}^2$ be as given in Definition [1.5]. For $n = 0, 1, 2$, define $P^n$ to be the projective $\Lambda$-module $P^n = \oplus_i t(g_i^n) \Lambda$. Define $\Lambda$-homomorphisms $\tilde{d}^0, \tilde{d}^1, \tilde{d}^2$ as follows:

- $\tilde{d}^0 : \tilde{P}^0 \to \tilde{\Lambda}/\tilde{\tau}$ is the canonical surjection;
• $\bar{d}^1 : \bar{P}^1 \to \bar{P}^0$ is given by $t(\bar{a}) \mapsto \bar{a}$ (where $\bar{a}$ is an arrow in $\bar{Q}$) with $\bar{a}$ in the summand of $\bar{P}^0$ corresponding to $\sigma(\bar{a})$;

• write $g^2_\alpha = \sum_\alpha \bar{a} \bar{\beta}_\alpha$, where the sum is over all arrows $\bar{a}$ in $\bar{Q}$, and $\bar{\beta}_\alpha \in K \bar{Q}$. Then $\bar{d}^2 : \bar{P}^2 \to \bar{P}^1$ is such that $\bar{d}^2(t(g^2_\alpha))$ has entry $t(\bar{a})\bar{\beta}_\alpha$ in the summand of $\bar{P}^1$ corresponding to $t(\bar{a})$.

Then the sequence

$$
\bar{P}^3 \xrightarrow{\bar{d}^3} \bar{P}^2 \xrightarrow{\bar{d}^2} \bar{P}^1 \xrightarrow{\bar{d}^1} \bar{P}^0 \xrightarrow{\bar{d}^0} \Lambda / \bar{r} \xrightarrow{0}
$$

is the first part of a minimal projective resolution of $\Lambda / \bar{r}$ as defined by [10] so is exact.

Define $\bar{g}^3 = \{g^3_\alpha := t^*(g^3_\alpha) \mid g^3_\alpha \in g^3\}$ and $\bar{P}^3 = \oplus_i t(\bar{g}^3_\alpha)\Lambda$. Fix the labelling of the set $g^2$ so that for each $g^3_\alpha \in g^3$, there are elements $r_{i,j} \in K \bar{Q}$ with $g^3_\alpha = \sum_j g^2_{i,j} r_{i,j}$. Define $\bar{d}^3 : \bar{P}^3 \to \bar{P}^2$ to be the $\bar{\Lambda}$-homomorphism such that $\bar{d}^3(t(g^3_\alpha))$ has entry $t(\bar{g}^2_{i,j})\theta(r_{i,j})$ in the summand of $\bar{P}^2$ corresponding to $t(\bar{g}^2_{i,j})$. With these definitions, the next step is to show that the sequence

$$
\bar{P}^3 \xrightarrow{\bar{d}^3} \bar{P}^2 \xrightarrow{\bar{d}^2} \bar{P}^1 \xrightarrow{\bar{d}^1} \bar{P}^0 \xrightarrow{\bar{d}^0} \Lambda / \bar{r} \xrightarrow{0}
$$

is exact. We keep the following notation. Write $g^2_\alpha = \sum_\alpha \alpha \beta_{j,\alpha}$, where the sum is over all arrows $\alpha \in Q_1$, and $\beta_{j,\alpha} \in K \bar{Q}$.

Then $g^2_\alpha = \sum_{\alpha \in Q_1} \alpha \beta_{j,\alpha} \theta^*(\beta_{j,\alpha})$ where $\theta^*(\alpha) = \alpha \beta_{j,\alpha} \equiv \alpha_1 \alpha_2 \cdots \alpha_\Lambda = \alpha_1 q_{(\alpha)}$. Let $\bar{x} \in \bar{P}^2$ and write $\bar{x} = \sum_j t(\bar{g}^2_{i,j})\lambda_j$ for some $\lambda_j \in \bar{\Lambda}$. Then $\bar{d}^2(\bar{x})$ has entry $\sum_j \bar{q}_{(\alpha)}(\bar{\beta}_{j,\alpha})\bar{\lambda}_j$ in the summand of $\bar{P}^1$ corresponding to $t(\alpha)$.

First we show that $\text{Ker} \bar{d}^2 \subset \text{Im} \bar{d}^3.$ Let $\bar{x} = \sum_j t(\bar{g}^2_{i,j})\lambda_j \in \text{Ker} \bar{d}^2$ so that $\bar{d}^2(\bar{x}) = 0$. Then $\bar{q}_{(\alpha)} \sum \bar{\beta}_{j,\alpha} \lambda_j = 0$ for each arrow $\alpha \in Q_1$. We have $\bar{x} \in \sum \bar{\beta}_{j,\alpha} \lambda_j$ and $\epsilon \lambda_j \in \theta(\lambda_j)$ for some $\lambda_j \in \Lambda$. So $0 = \bar{q}_{(\alpha)} \sum \bar{\beta}_{j,\alpha} \lambda_j = \bar{q}_{(\alpha)}(\theta(\sum \bar{\beta}_{j,\alpha} \lambda_j)) = 0$. Hence from Proposition [10], we have $\theta(\sum \bar{\beta}_{j,\alpha} \lambda_j) = 0$ and so $\sum \bar{\beta}_{j,\alpha} \lambda_j = 0$ for each arrow $\alpha \in Q_1$ since $\theta$ is one-to-one. Let $\bar{x} \in \sum \bar{\beta}_{j,\alpha} \lambda_j \in \bar{P}^2$, so we have $\bar{x} \in \ker \bar{d}^2$. But $\text{Im} \bar{d}^3 = \text{Ker} \bar{d}^2$ since $(P^n, d^n)$ is a minimal projective resolution of $\Lambda / \bar{r}$, so $\bar{x} \in \text{Im} \bar{d}^3$. By Proposition [12], $\sum_j t(\bar{g}^2_{i,j})\theta(\lambda_j)$ is in $\text{Im} \bar{d}^3$, that is, $\bar{x} \in \text{Im} \bar{d}^3$.

Now let $\bar{w} \in \bar{Q}_0 \setminus \bar{Q}_0$. Then $\bar{w} \sum_j t(\bar{g}^2_{i,j}) \lambda_j = 0$ for each arrow $\alpha \in Q_1$. We have $\sum_j t(\bar{g}^2_{i,j}) \lambda_j = 0$ so $\sum_j t(\bar{g}^2_{i,j}) \lambda_j = 0$ for each arrow $\alpha \in Q_1$. Let $\bar{w} \sum_j t(\bar{g}^2_{i,j}) \lambda_j \in \text{Im} \bar{d}^3$. Right multiplication by $\bar{p}_w$ gives $\bar{w} \in \text{Im} \bar{d}^3$. Therefore by Proposition [12], we have shown that $\text{Ker} \bar{d}^2 \subset \text{Im} \bar{d}^3$.

Now we show that $\text{Im} \bar{d}^3 \subset \ker \bar{d}^2$. Let $\bar{w} \in \bar{Q}_0 \setminus \bar{Q}_0$. Then $\bar{w} \sum_j t(\bar{g}^2_{i,j}) \lambda_j = 0$ for each arrow $\alpha \in Q_1$.

Hence $\bar{q}_{(\alpha)} \sum \bar{\beta}_{j,\alpha} \lambda_j = 0$ for each arrow $\alpha \in Q_1$ and so $\bar{d}^2(\bar{x}) = 0$. Now let $\bar{w} \in \bar{Q}_0 \setminus \bar{Q}_0$. Then $\bar{w} \sum_j t(\bar{g}^2_{i,j}) \lambda_j \in \text{Im} \bar{d}^3$. Right multiplication by $\bar{p}_w$ gives $\bar{w} \in \text{Im} \bar{d}^3$. Thus $\bar{w} \in \text{Im} \bar{d}^3$. Therefore by Proposition [12], we have shown that $\text{Im} \bar{d}^3 \subset \ker \bar{d}^2$.
Finally, we recall that the functor $T_\varepsilon$ is exact and $(P^n, d^n)$ is a minimal projective resolution of $\Lambda/\bar{r}$ given by sets $g^n$, so the sequence

$$\cdots \longrightarrow T_\varepsilon(P^n) \xrightarrow{T_\varepsilon(d^n)} T_\varepsilon(P^{n-1}) \longrightarrow \cdots \longrightarrow T_\varepsilon(P) \xrightarrow{T_\varepsilon(d^0)} T_\varepsilon(P^0)$$

is exact and $T_\varepsilon(P^n)$ is a projective $\Lambda$-module for all $n \geq 0$. Identifying $\Lambda$ with $\varepsilon \hat{\Lambda}$, we have $T_\varepsilon(P^n) = \bigoplus_i \varepsilon t(\theta^i(g^n_i)) \varepsilon \Lambda \cong \varepsilon \Lambda \cong \bigoplus_i \varepsilon t(\theta^i(g^n_i)) \hat{\Lambda}$ for $n \geq 2$. In particular, we identify $T_\varepsilon(P^2)$ with $\hat{P}^2$ and $T_\varepsilon(P^3)$ with $\hat{P}^3$. It is easy to verify that $T_\varepsilon(d^3)$ is then identified with $d^3$. So the sequence

$$\cdots \longrightarrow T_\varepsilon(P^n) \xrightarrow{T_\varepsilon(d^n)} T_\varepsilon(P^{n-1}) \longrightarrow \cdots \longrightarrow T_\varepsilon(P^4) \xrightarrow{T_\varepsilon(d^4)} P^3 \xrightarrow{d^3} \hat{P}^2$$

is exact.

Hence we have a projective resolution

$$\cdots \longrightarrow \hat{P}^n \xrightarrow{\hat{d}^n} \hat{P}^{n-1} \longrightarrow \cdots \longrightarrow \hat{P}^3 \xrightarrow{\hat{d}^3} \hat{P}^2 \xrightarrow{\hat{d}^2} \hat{P}^1 \xrightarrow{\hat{d}^1} \hat{P}^0 \xrightarrow{\hat{d}^0} \Lambda/\bar{r} \longrightarrow 0$$

for $\Lambda/\bar{r}$. To complete the proof, let $n \geq 4$, and define $\hat{g}^n = \{\hat{g}^n_i := \theta^i(g^n_i) \mid g^n_i \in g^n\}$ and $\hat{P}^n = \bigoplus_i \varepsilon t(\hat{g}^n_i) \hat{\Lambda}$. Write $g^n_i = \sum_j g^n_i - 1 r_j$ for some $r_j \in K \hat{Q}$, so that $\hat{g}^n_i = \sum_j \theta^i(g^n_i - 1 r_j)$. Define $\hat{d}^n : P^n \to \hat{P}^{n-1}$ to be the $\Lambda$-homomorphism where $\hat{d}^n(t(\theta^i(g^n_i)))$ has entry $t(\theta^i(g^n_i - 1)) \theta(r_j)$ in the summand of $\hat{P}^{n-1}$ corresponding to $t(\hat{g}^{n-1}_i)$. Then the identification of $T_\varepsilon(P^n)$ with $\hat{P}^n$ (for $n \geq 2$) also identifies $T_\varepsilon(d^n)$ with $\hat{d}^n$ for $n \geq 3$. So we have a projective resolution

$$\cdots \longrightarrow \hat{P}^n \xrightarrow{\hat{d}^n} \hat{P}^{n-1} \longrightarrow \cdots \longrightarrow \hat{P}^3 \xrightarrow{\hat{d}^3} \hat{P}^2 \xrightarrow{\hat{d}^2} \hat{P}^1 \xrightarrow{\hat{d}^1} \hat{P}^0 \xrightarrow{\hat{d}^0} \Lambda/\bar{r} \longrightarrow 0$$

for $\Lambda/\bar{r}$ given by the sets $\hat{g}^n$. Minimality follows since $\text{Im} \hat{d}^n \subseteq \text{rad}(\hat{P}^{n-1})$ for all $n \geq 0$. □

**Proposition 3.2.** Let $z = \sum_j t(\theta^i(g^n_j)) \mu_j \in P^2$ and $\tilde{z} = \sum_j t(\theta^i(g^n_j)) \theta(\mu_j) \in \hat{P}^2$, where $\mu_j \in \Lambda$. Then the following are equivalent:

1. $z \in \text{Im} d^3$;
2. $\tilde{z} \in \text{Im} \hat{d}^3$;
3. $\tilde{z} \hat{p}_w \in \text{Im} \hat{d}^3$ for each $w \in \hat{Q}_0 \setminus Q_0$.

**Proof.** We keep the notation of the proof of Theorem 3.1. Let $z = \sum_j t(\theta^i(g^n_j)) \mu_j \in P^2$, $y = \sum_i t(\theta^i(g^n_i)) s_i \in P^3$, $\tilde{z} = \sum_j t(\theta^i(g^n_j)) \theta(\mu_j) \in \hat{P}^2$ and $\tilde{y} = \sum_i t(\theta^i(g^n_i)) \theta(s_i) \in \hat{P}^3$. For each $g^n_i \in g^n$, consider the summand of $P^2$ (resp. $\hat{P}^2$) corresponding to $t(\theta^i(g^n_i))$ (resp. $t(\theta^i(g^n_i))$).

By definition, $d^3(t(\theta^i(g^n_i)))$ has entry $t(g^n_i r_{i,j})$ in the summand of $P^2$ corresponding to $t(\theta^i(g^n_i))$, and $\hat{d}^3(t(\theta^i(g^n_i)))$ has entry $t(\theta^i(g^n_i) r_{i,j})$ in the summand of $\hat{P}^2$ corresponding to $t(\theta^i(g^n_i))$. So $d^3(y)$ has entry $\sum_i t(g^n_i r_{i,j}) t(g^n_i) s_i$ in the summand of $P^2$ corresponding to $t(\theta^i(g^n_i))$, and $\hat{d}^3(\tilde{y})$ has entry $\sum_i t(\theta^i(g^n_i) r_{i,j}) t(\theta^i(g^n_i)) \theta(s_i)$ in the summand of $\hat{P}^2$ corresponding to $t(\theta^i(g^n_i))$. Since $\theta$ is one-to-one, $t(g^n_i r_{i,j}) t(g^n_i) s_i = t(\theta^i(g^n_i)) \mu_j$ if and only if $t(\theta^i(g^n_i) r_{i,j}) t(\theta^i(g^n_i)) \theta(s_i) = t(\theta^i(g^n_i)) \theta(\mu_j)$. Hence $z = d^3(y)$ if and only if $\tilde{z} = \hat{d}^3(\tilde{y})$. By Proposition 1.10 (1), $t(\theta^i(g^n_i) r_{i,j}) t(\theta^i(g^n_i)) \theta(s_i) = t(\theta^i(g^n_i) r_{i,j}) t(\theta^i(g^n_i)) \theta(s_i) \tilde{p}_w = t(\theta^i(g^n_i) r_{i,j}) t(\theta^i(g^n_i)) \tilde{p}_w$ where $w \in \hat{Q}_0 \setminus Q_0$. Hence $\tilde{z} = \hat{d}^3(\tilde{y})$ if and only if $\tilde{z} \hat{p}_w = \hat{d}^3(\tilde{y} \hat{p}_w)$. The result follows. □

The rest of this section concerns the application of Theorem 3.1 to $d$-Koszul algebras, whereby we recover a result of Leader [11, Theorem 8.15]. Recall that a graded algebra $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is said to be Koszul if $\Lambda_0$ has a linear resolution, that is, if the $n$th projective module $P^n$ in a minimal graded projective resolution $(P^n, d^n)$ of $\Lambda_0$ is generated
in degree $n$. Berger then introduced $d$-Koszul algebras, for $d \geq 2$, in [2] motivated by certain cubic Artin-Schelter regular algebras and anti-symmetrizer algebras. For finite-dimensional algebras, these were further generalised, firstly to $(D, A)$-stacked monomial algebras by Green and Snashall ([8] Definition 3.1 and see [9]) and then by Leader and Snashall to $(D, A)$-stacked algebras in [12].

Definition 3.3. ([12] Definition 2.1) Let $\Lambda = KQ/I$ be a finite-dimensional algebra. Then $\Lambda$ is a $(D, A)$-stacked algebra if there is some $D \geq 2, A \geq 1$ such that, for all $0 \leq n \leq \text{gldim} \Lambda$, the projective module $P^n$ in a minimal projective resolution of $\Lambda/r$ is generated in degree $\delta(n)$, where

$$
\delta(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
\frac{n}{2} D & \text{if } n \text{ even, } n \geq 2 \\
\frac{n-1}{2} D + A & \text{if } n \text{ odd, } n \geq 3.
\end{cases}
$$

When $A = 1$ and $D = d$, the $(d, 1)$-stacked algebras are precisely the finite-dimensional $d$-Koszul algebras of Berger (with the case $A = 1, D = 2$ giving the finite-dimensional Koszul algebras). In all cases, $\Lambda$ is a graded algebra in which the $n$th projective module $P^n$ in a minimal graded projective resolution $(P^n, d^n)$ of $\Lambda_0$ is generated in a single degree, and for which the Ext algebra $E(\Lambda)$ is finitely generated. Specifically, it was shown in [7, Theorem 4.1], that the Ext algebra of a $d$-Koszul algebra is generated in degrees $0, 1$ and $2$, and, in [12] Theorem 2.4] that the Ext algebra of a $(D, A)$-stacked algebra is generated in degrees $0, 1, 2$ and $3$.

We now apply Theorem 3.1 to $d$-Koszul algebras.

Theorem 3.4. ([11] Theorem 8.15] Let $\Lambda = KQ/I$ be a $d$-Koszul algebra for some $d \geq 2$. Let $A \geq 1$ and set $D = dA$. Then the algebra $\tilde{\Lambda}_A$ is a $(D, A)$-stacked algebra.

Proof. Let $\Lambda = KQ/I$ be a $d$-Koszul algebra. Let $(P^n, d^n)$ be a minimal projective resolution for $\Lambda/r$ given by sets $g^n$. Then $P^n$ is generated in degree

$$
\ell(g^n_i) = \begin{cases} 
\frac{n}{2} d & \text{if } n \text{ even, } n \geq 0 \\
\frac{n-1}{2} d + 1 & \text{if } n \text{ odd, } n \geq 1
\end{cases}
$$

and each $g^n_i \in g^n$ is a uniform homogeneous element with

$$
\ell(g^n_i) = \begin{cases} 
\frac{n}{2} d & \text{if } n \text{ even, } n \geq 0 \\
\frac{n-1}{2} d + 1 & \text{if } n \text{ odd, } n \geq 1
\end{cases}
$$

Let $(\tilde{P^n}, \tilde{d^n})$ be the minimal projective resolution for $\tilde{\Lambda}_A/r_A$ given by sets $\tilde{g}^n_i$ from Theorem 3.1. For $n \geq 2$, and each $\tilde{g}^n_i \in \tilde{g}^n$, we have $\ell(\tilde{g}^n_i) = A \cdot \ell(g^n_i)$. Thus

$$
\ell(\tilde{g}^n_i) = \begin{cases} 
\frac{n}{2} d A & \text{if } n \text{ even, } n \geq 2 \\
\frac{n-1}{2} d A + A & \text{if } n \text{ odd, } n \geq 3.
\end{cases}
$$

Let $D = dA$. Then, for all $n \geq 0$, we have $\ell(\tilde{g}^n_i) = \delta(n)$ so $\tilde{P^n}$ is generated in degree $\delta(n)$. Thus $\tilde{\Lambda}_A$ is a $(D, A)$-stacked algebra. \hfill \square

Example 3.5. Let $\Lambda, \tilde{\Lambda}$ be the algebras of Example 2.12(2). The algebra $\Lambda$ is a $d$-Koszul monomial algebra with $d = 3$. It now follows from Theorem 3.4 that $\tilde{\Lambda}$ is a $(6, 2)$-stacked monomial algebra. Indeed $\tilde{\Lambda}$ was given in [8] as an example of a $(6, 2)$-stacked monomial algebra.
References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory, LMS Student Texts 65, CUP, 2006.

[2] R. Berger, Koszulity for nonquadratic algebras, J. Algebra 239 (2001), 705-734.

[3] J.F. Carlson, The complexity and varieties of modules, Integral representations and applications (Oberwolfach, 1980), Lecture Notes in Mathematics, 882, pp. 415-422, Springer, Berlin-New York, 1981.

[4] J.F. Carlson, Varieties and the cohomology ring of a module, J. Algebra 85 (1983), 104-143.

[5] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, Support varieties for selfinjective algebras, K-Theory 33 (2004), 67-87.

[6] T. Furuya, N. Snashall, Support varieties for modules over stacked monomial algebras, Comm. Alg. 39 (2011), 2926-2942.

[7] E.L. Green, E. Marcos, R. Martínez-Villa, P. Zhang, D-Koszul algebras, J. Pure Appl. Algebra 193 (2004), 141-162.

[8] E.L. Green, N. Snashall, The Hochschild cohomology ring modulo nilpotence of a stacked monomial algebra, Colloq. Math. 105 (2006), 233-258.

[9] E.L. Green, N. Snashall, Finite generation of Ext for a generalization of D-Koszul algebras, J. Algebra 295 (2006), 458-472.

[10] E.L. Green, Ø. Solberg, D. Zacharia, Minimal Projective Resolutions, Trans. Amer. Math. Soc. 353 (2001), 2915-2939.

[11] J. Leader, Finite generation of Ext and (D, A)-stacked algebras, PhD thesis, University of Leicester, 2014.

[12] J. Leader, N. Snashall, The Ext algebra and a new generalisation of D-Koszul algebras, Quart. J. Math. 68, (2017), 433-458.

[13] H. Nagase, Hochschild cohomology and Gorenstein Nakayama algebras, Proceedings of the 43rd Symposium on Ring Theory and Representation Theory, Symp. Ring Theory Represent. Theory Organ. Comm., Soja, 2011, pp. 37-41.

[14] C. Psaroudakis, Ø. Skartsæterhagen, Ø. Solberg, Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements, Trans. Amer. Math. Soc. Series B, 1, (2014), 45-95.

[15] N. Snashall, Ø. Solberg, Support varieties and Hochschild cohomology rings, Proc. London Math. Soc., (3) 88 (2004) 705-732.

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