Viability analysis of the first-order mean field games

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Abstract

The paper is concerned with the dependence of the solution of the deterministic mean field game on the initial distribution of players. The main object of study is the mapping which assigns to the initial time and the initial distribution of players the set of expected rewards of the representative player corresponding to solutions of mean field game. This mapping can be regarded as a value multifunction. We obtain the sufficient condition for a multifunction to be a value multifunction. It states that if a multifunction is viable with respect to the dynamics generated by the original mean field game, then it is a value multifunction. Furthermore, the infinitesimal variant of this condition is derived.

Keywords: mean field games, value multifunction, viability property, set-valued derivative.

MSC Classification (2010): 91A10, 91A23, 49J52, 49J53, 46G05, 49J21.

1 Introduction

The theory of mean field games aims to study noncooperative dynamical games of a large number similar players. The main idea of this approach is to examine the limit case when the number of players tends to infinity and each player becomes negligible.

The concept of mean field games was proposed by Lasry, Lions\textsuperscript{26, 27, 28} and by Huang, Caines, Malhamé\textsuperscript{20, 21}. Nowadays, there are several approaches to the mean field game theory. First one reduces the mean field game to the backward-forward system of fully-coupled nonlinear PDEs. The first equation is the Hamilton-Jacobi equation which describes the value function of the representative player. The second equation is the Chapman-Kolmogorov equation and it characterizes the distribution of all players. Within the framework of this approach the existence and uniqueness problems for mean field games were studied (see\textsuperscript{19, 22, 27, 28}, and reference therein). Moreover, one can construct an approximate Nash equilibrium for the finite player game given a solution of the mean field game\textsuperscript{22, 29}.

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The second approach to the mean filed games is called probabilistic. It involves the study of infinite player dynamical game with similar players and mean field interaction among them. The probabilistic approach considers the solution of mean field game as the symmetric Nash equilibrium in this game (see [10], [12], [13], [14], [15], [24]). This allows to prove that the open-loop equilibria of finite players games converge to the solution of mean field game when the number of players tends to infinity [10], [25].

The third approach is concerned with the study of the partial differential equation involving the derivatives in the space of probabilities called the master equation. It was proposed by Lions in his seminal lectures [29] (see also [8]). The master equation encapsulates the necessary information to describe the solution of mean field game. It is used to establish the convergence of feedback equilibria of finite player games to a solution of mean field game a nondegenerate stochastic game [9]. Moreover, Lions noticed that the classical mean field game system is the characteristic system of the master equation. The master equation was discussed in [6], [7], [11]. Note that in [11] the master equation was formally derived using the dynamic programming arguments applied to the optimization problem for the representative player. In [23] approximate equilibria in the finite player game with exogenous noise were constructed based on the solution of the mean field game with common noise.

Nowadays, the existence theorem for the master equation is obtained for the case of nondegenerate stochastic mean field games (possibly, with common noise) satisfying Lasry-Lions monotonicity condition [9]. The proof is based on the fact that the usual mean field game system provides characteristics for the master equation. Furthermore, the short-term existence theorem is proved in [13] and [17].

The master equation in particular describes the dependence of the solution to the mean field game on the initial distribution of players. In the paper we examine this dependence in the viewpoint of the probabilistic approach. The study is based on the viability theory arguments (see [2], [3]).

We restrict our attention to the deterministic mean field game. Moreover, for simplicity, we assume the periodic boundary conditions i.e. the phase space for each player is the \( d \)-dimensional torus \( \mathbb{T}^d \triangleq \mathbb{R}^d / \mathbb{Z}^d \). Recall that within the framework of the probabilistic approach the study of the deterministic mean field game is reduced to the study of the symmetric of equilibrium of the continuum player game where the dynamics of each agent is

\[
\frac{d}{dt} x(t) = f(t, x(t), m(t), u(t, x(t))). \tag{1}
\]

Here \( m(t) \) is a probability on \( \mathbb{T}^d \) describing the distribution of all players at time \( t \), \( t \in [0, T] \), \( x(t) \in \mathbb{T}^d \), \( u(t, x(t)) \in U \), \( U \) is a control space. We assume that the each player aims to maximize his/her payoff given by

\[
\sigma(x(T), m(T)) + \int_{t_0}^{T} g(t, x(t), m(t), u(t, x(t))) dt. \tag{2}
\]

To present the main objective of the paper let us consider the finite player game analogy. If one examine the \( N \) player non-cooperative differential game with weakly
coupled dynamics of agents, then the state space is \((\mathbb{T}^d)^N\), while the players’ outcome is a \(N\)-dimensional vector. In this finite player game the value function (multifunction) is the mapping which assigns to each initial position the set of Nash values.

Turning to the continuum player game, we get that the phase space should be \((\mathbb{T}^d)^c\), while the players’ outcome is an element of \(\mathbb{R}^c\). Here \(c\) is a continuum set. In the mean field game setting we can reduce the phase space of the game to the set of probabilities on \(\mathbb{T}^d\). Further, since we are seeking for the symmetric equilibrium, the players starting at the same point get the same outcome. Thus, we can index the players by the points of \(\mathbb{T}^d\) i.e. we put \(c = \mathbb{T}^d\). Simultaneously, given a solution of the mean field game, the expected reward of the representative player depends on his/her initial state continuously. Therefore, the mean field game analog of the value function is the mapping which assigns to an initial time \(t_0\) and an initial players’ distribution \(m_0\) a set of continuous functions from \(\mathbb{T}^d\) to \(\mathbb{R}\). Each element of this set is a value function for the representative player corresponding to a solution of the mean field game with the initial condition \(m(t_0) = m_0\).

The link between the master equation and the approach developed in the paper is follows. If the function \(v(t, x, m)\) solves the master equation for the mean field game, then the mapping \((t, m) \mapsto \{v(t, \cdot, m)\}\) is a value multifunction in the sense of the paper. However, in the general case we can not restrict our attention to single-valued functions due to the multiplicity of the solutions of the mean field game [5].

To examine the value multifunction we introduce a family of set-valued mappings \(\Psi^{r,s} : \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \rightrightarrows \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d)\) \((s, r \in [0, T], r \geq s)\). Here \(\mathcal{P}^1(\mathbb{T}^d)\) stands for the set of probabilities on \(\mathbb{T}^d\). Given \(m \in \mathcal{P}^1(\mathbb{T}^d), \phi \in C(\mathbb{T}^d)\), \(\Psi^{r,s}(m, \phi)\) is the set of pairs \((\mu, \psi)\) such that \(\phi\) is a reward of the representative player corresponding to a solution of the mean field game on \([s, r]\) with dynamics (1), the initial condition \(m(s) = m\) and the payoff functional of each agent given by

\[
\psi(x(r)) + \int_s^r g(t, x(t), m(t), u(t, x(t)))dt,
\]

while the probability \(\mu\) is the corresponding distribution of agents at the time \(r\). The family of transforms \(\{\Psi^{r,s}\}\) determines a forward dynamics on \(\mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d)\). Below we call it a mean field game dynamics. Apparently, the notion of mean field game dynamics is close to forward-forward mean filed games studied in [18].

We prove that if a multifunction is viable with respect to the mean field game dynamics, then this multifunction is a value multifunction. Furthermore, we study the infinitesimal form of the proposed viability condition. We introduce the set-valued derivative of the multifunction \(V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)\) by virtue of the mean field game dynamics. The viability theorem proved in the paper states that the multifunction \(V\) is viable with respect to the dynamics if and only if the set-valued derivative is nonempty at any point of the graph of \(V\).

The paper is organized as follows. General notation and assumption are introduced in Section 2. Moreover, in this section we give some properties of the dynamics of distribution of players. Section 3 is concerned with the definition of solution to the first order mean field game. In Section 4 we introduce the notion of...
the value multifunction and formulate the sufficient condition for a given multifunc-
tion of time and probability to be a value multifunction. The condition involves the
viability property with respect to the mean field game dynamics. The viability the-
orem which provides the infinitesimal form of the viability property is introduced
in Section 5. The subsequent sections are devoted to the proof of this theorem.
Auxiliary lemmas are given in Section 6. The sufficiency and necessity parts of the
viability theorem are proved in Sections 7 and 8 respectively.

2 Preliminaries

2.1 General notations

If \((X, \rho_X)\) is a separable metric space, \(Y \subset X, x \in X\), then put
\[
\text{dist}(x, Y) \triangleq \inf \{\rho_X(x, y) : y \in Y\}.
\]
Below \(B(X)\) stands for the Borel \(\sigma\)-algebra on \(X\). We denote by \(\mathcal{P}(X)\) the set of all
Borel probabilities on \(X\). Further, let \(\mathcal{P}^1(X)\) stand for the set of probabilities \(m\) on
\(X\) such that, for some \(x^* \in X\),
\[
\int_X \rho_X(x, x^*) m(dx) < \infty.
\]
We endow \(\mathcal{P}^1(X)\) with Kantorovich-Rubinstein metric (1-Wasserstein metric) de-
fined by the rule: for \(m_1, m_2 \in \mathcal{P}^1(X)\),
\[
W_1(m_1, m_2) \triangleq \inf \left\{ \int_{X \times X} \rho_X(x_1, x_2) \pi(dx_1, dx_2) : \pi \in \Pi(m_1, m_2) \right\}
= \sup \left\{ \int_X \phi(x)m_1(dx) - \int_X \phi(x)m_2(dx) : \phi \in \text{Lip}_1(X) \right\}.
\]
Here \(\Pi(m_1, m_2)\) is the set of probabilities \(\pi\) on \(X \times X\) such that its marginal distributions are \(m_1\) and \(m_2\) respectively, i.e.
\[
\Pi(m_1, m_2) = \{\pi \in \mathcal{P}(X \times X) : \pi(Y \times X) = m_1(Y), \ \pi(X \times Y) = m_2(Y)\};
\]
\(\text{Lip}_K(X)\) denotes the set of \(K\)-Lipschitz continuous functions \(\phi : X \to \mathbb{R}\). Note that,
if \(X\) is compact, then the Wasserstein distance metrizes the narrow convergence.
If \((\Omega^1, \Sigma^1)\) and \((\Omega^2, \Sigma^2)\) are measurable spaces, \(h : \Omega^1 \mapsto \Omega^2\) is measurable, \(m\) is
a probability of \(\Sigma^1\), then denote by \(h_\#m\) the probability on \(\Sigma^2\) given by the rule: for \(\Upsilon \in \Sigma^2\),
\[
(h_\#m)(\Upsilon) \triangleq m(h^{-1}(\Upsilon)).
\]
If \((X, \rho_X), (Y, \rho_Y)\) are separable metric spaces, \(\pi \in \mathcal{P}(X \times Y)\), then denote by
\(\pi(\cdot | x)\) a conditional distribution on \(Y\) given \(x\) i.e., for each \(x\), \(\pi(\cdot | x)\) is a probability
on \(Y\) and, for any \(\varphi \in C_b(X \times Y)\),
\[
\int_{X \times Y} \varphi(x, y) \pi(dx, dy) = \int_X \int_Y \varphi(x, y) \pi(dy | x) m(dx).
\]
Now, let \((X, \rho_X), (Y, \rho_Y)\) and \((Z, \rho_Z)\) be separable metric spaces, and let \(\pi_{1,2} \in \mathcal{P}(X \times Y)\) and \(\pi_{1,2} \in \mathcal{P}(Y \times Z)\) have the same marginal distribution on \(Y\) equal to \(m\). We define the probability \(\pi_{1,2} \ast \pi_{2,3} \in \mathcal{P}(X \times Z)\) by the following rule: for \(\varphi \in C_c(X \times Z)\),

\[
\int_{X \times Y} \varphi(x, z)(\pi_{1,2} \ast \pi_{2,3})(d(x, z)) \triangleq \int_Y \int_X \int_Z \varphi(x, z)\pi_{2,3}(dz|y)\pi_{1,2}(dx|y)m(dy).
\]

The probability \(\pi_{1,2} \ast \pi_{2,3}\) is the composition of \(\pi_{1,2}\) and \(\pi_{2,3}\). Note that in [1] it is denoted by \(\pi_{2,3} \circ \pi_{1,2}\).

If \(\mathcal{Y}\) is a multifunction from \([s, r]\) to finite dimensional euclidean space, then \(\int_s^r \mathcal{Y}(t)dt\) stands for the Aumann integral i.e.

\[
\int_s^r \mathcal{Y}(t)dt \triangleq \left\{ \int_s^r y(t)dt : y(\cdot) \text{ integrable, } y(t) \in \mathcal{Y}(t) \text{ a.e.} \right\}.
\]

### 2.2 Probabilities on state space and on space of motions

As it was mentioned above, we assume that the phase space is \(d\)-dimensional torus \(T^d = \mathbb{R}^d / \mathbb{Z}^d\). Further, \(T^d \times \mathbb{R}\) is an extended phase space; \(p\) denotes a natural projection from \(T^d \times \mathbb{R}\) onto \(T^d\). If \((x, z) \in T^d \times \mathbb{R}\), then

\[
\|(x, z)\| \triangleq \|x\| + |z|.
\]

Notice that the set \(C_{s,r} \triangleq C([s, r], T^d \times \mathbb{R})\) is the set of motions on \([s, r]\) in the extended space.

Let \(e_t : C_{s,r} \to T^d\) and \(\hat{e}_t : C_{s,r} \to T^d \times \mathbb{R}\) be evaluation operators defined by the rule: if \(w(\cdot) = (x(\cdot), z(\cdot)) \in C_{s,r}\), then

\[
e_t(w(\cdot)) \triangleq x(t), \quad \hat{e}_t(w(\cdot)) \triangleq w(t).
\]

Notice that \(e_t = p \circ \hat{e}_t\).

Furthermore, if \(s, r \in [0, T], s < r, t \in [s, r], \chi_1, \chi_2 \in \mathcal{P}^1(C_{s,r})\), then

\[
W_1(e_t \# \chi_1, e_t \# \chi_2) \leq W_1(\hat{e}_t \# \chi_1, \hat{e}_t \# \chi_2) \leq W_1(\chi_1, \chi_2).
\]

Denote the set of all measurable functions from \([s, t]\) to \(\mathcal{P}^1(T^d)\) by \(\mathcal{M}_{s,t}\). Analogously, let \(\mathcal{N}_{s,r}\) stand for the set of all measurable functions defined on \([s, r]\) taking values in \(\mathcal{P}^1(T^d \times \mathbb{R})\). We will call elements of both \(\mathcal{M}_{s,r}\) and \(\mathcal{N}_{s,r}\) flows of probabilities.

If \(\phi \in C(T^d)\) and \(\nu \in \mathcal{P}^1(T^d \times \mathbb{R})\), then denote by \([\phi, \nu]\) the averaging of the function \((x, z) \mapsto \phi(x) + z\) according to \(\nu\) i.e.

\[
[\phi, \nu] \triangleq \int_{T^d \times \mathbb{R}} (\phi(x) + z)\nu(d(x, z)).
\]
Remark 1. If \( \nu = \hat{\epsilon}_{s \# \chi} \), then
\[
[\phi, \nu] = \int_{C_{s, r}} (\phi(x(s)) + z(s)) \chi(d(x(\cdot), z(\cdot))).
\]

The definition of \([\phi, \nu] \) implies that, if \( \phi, \phi' \in \text{Lip}_K(\mathbb{T}^d) \), \( \nu, \nu' \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R}) \), then
\[
|[\phi, \nu] - [\phi', \nu']| \leq \|\phi - \phi'\| + K W_1(\nu, \nu'). \tag{5}
\]

Now let us introduce the notion of concatenations of the probabilities on the set of motions. First, we recall the notion of concatenation of motions. Let \( s < r < \theta \).

If \( w_1(\cdot) \in C_{s, r} \), \( w_2(\cdot) \in C_{r, \theta} \) are such that \( w_1(r) = w_2(t) \), then \( w_1(\cdot) \circ w_2(\cdot) \) is a motion \( w(\cdot) \in C_{s, \theta} \) given by
\[
w(t) = \begin{cases} w_1(t), & t \in [s, r], \\ w_2(t), & t \in [r, \theta]. \end{cases}
\]

Now, let \( \chi_1 \in \mathcal{P}^1(C_{s, r}) \), \( \chi_2 \in \mathcal{P}^1(C_{r, \theta}) \) be such that \( \hat{\epsilon}_{r \# \chi_1} = \hat{\epsilon}_{r \# \chi_2} \). Let \( \chi_2(d(w(\cdot))/w_0) \) denote the disintegration of \( \chi_2 \) along \( \hat{\epsilon}_{r \# \chi_2} \). Define the concatenation of probabilities \( \chi_1 \) and \( \chi_2 \) \( \chi_1 \circ \chi_2 \) by the following rule: for any \( \varphi \in C_0(C_{s, \theta}), \)
\[
\int_{C_{s, \theta}} \varphi(w(\cdot)) \chi(d(w(\cdot))) = \int_{C_{s, r}} \int_{C_{r, \theta}} \varphi(w_1(\cdot) \circ w_2(\cdot)) \chi_2(d(w_2(\cdot)|w_1(r))) \chi_1(d(w_1(\cdot))). \tag{6}
\]

### 2.3 Dynamics of distribution of players

We assume that the set \( U \) and the functions \( f, g, \sigma \) satisfy the following assumptions.

(M1) \( U \) is a metric compact;

(M2) functions \( f, g \) and \( \sigma \) are continuous;

(M3) there exists a function \( \alpha : \mathbb{R} \to [0, +\infty) \) such that \( \alpha(\delta) \to 0 \) as \( \delta \to 0 \) and, for any \( t', t'' \in [0, T] \), \( x \in \mathbb{T}^d \), \( m \in \mathcal{P}^1(\mathbb{T}^d) \), \( u \in U \),
\[
\|f(t', x, m, u) - f(t'', x, m, u)\| \leq \alpha(t' - t''),
\]
\[
|g(t', x, m, u) - g(t'', x, m, u)| \leq \alpha(t' - t'').
\]

(M4) \( f \) and \( g \) are Lipschitz continuous with respect to the space variable \( x \) and probability \( m \), i.e. there exists a constant \( L \) such that, for any \( t \in [0, T] \), \( x', x'' \in \mathbb{T}^d \), \( m', m'' \in \mathcal{P}^1(\mathbb{T}^d) \), \( u \in U \),
\[
\|f(t, x', m', u) - f(t, x'', m'', u)\| \leq L(\|x' - x''\| + W_1(m', m'')),
\]
\[
|g(t, x', m', u) - g(t, x'', m'', u)| \leq L(\|x' - x''\| + W_1(m', m''));
\]
(M5) there exists a constant $\kappa$ such that, for any $x', x'' \in \mathbb{T}^d$ and any $m \in \mathcal{P}(\mathbb{T}^d)$,

$$|\sigma(x', m) - \sigma(x'', m)| \leq \kappa \|x' - x''\|.$$ 

Conditions (M1)–(M5) imply the existence of a constant $R$ such that

$$\|f(t, x, m, u)\|, |g(t, x, m, u)| \leq R \tag{7}$$

Using the relaxation of the control problem for the representative player (see [31]), we get that his/her dynamics in the extended phase space obeys the following differential inclusion:

$$(\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)). \tag{8}$$

Here

$$F(t, x, m) \triangleq \text{co}\{(f(t, x, m, u), g(t, x, m, u)) : u \in U\}. \tag{9}$$

An equivalent approach to the relaxation of control problems is based on measure-valued controls [31]. A measure-valued control is a function $\xi : [s, r] \times \mathcal{B}(U) \to [0, 1]$ satisfying the following conditions:

- for each $t \in [s, r]$, $\xi(t, \cdot)$ is a probability on $U$;
- for any $\varphi \in C(U)$, the functions
  $$t \mapsto \int_U \varphi(u)\xi(t, du)$$
  is measurable.

We denote the set of measure-valued controls on $[s, r]$ by $\mathcal{U}_{s,r}$. Notice (see [31]) that under conditions (M1)–(M5), if $(x(\cdot), z(\cdot)) \in \mathcal{C}_{s,r}$ satisfies differential inclusion (8), then there exists $\xi \in \mathcal{U}_{s,r}$ such that, for a.e. $t \in [s, r]$,

$$\dot{x}(t) = \int_U f(t, x(t), m(t), u)\xi(t, du), \quad \dot{z}(t) = \int_U g(t, x(t), m(t), u)\xi(t, du). \tag{10}$$

If $s, r \in [0, T]$, $s < r$, $y \in \mathbb{T}^d$, $m(\cdot) \in \mathcal{M}_{s,r}$, then denote by $\text{Sol}(r, s, y, m(\cdot))$ the set of solution of (8) satisfying $x(s) = y$. Further, put

$$\text{SOL}(r, s, m(\cdot)) \triangleq \bigcup_{y \in \mathbb{T}^d} \text{Sol}(r, s, y, m(\cdot)).$$

Notice that, for any $s, r \in [0, T]$, $m(\cdot) \in \mathcal{M}_{s,r}$, each $(x(\cdot), z(\cdot)) \in \text{SOL}(r, s, m(\cdot))$ is absolutely continuous and

$$\|\dot{x}(t)\|, |\dot{z}(t)| \leq R \text{ a.e. } t \in [s, r]. \tag{11}$$

Integrating (8), we get that the dynamics of distribution of players in the extended space can be described by the following mean field type differential inclusion

$$\frac{d}{dt} \nu(t) \in \langle \hat{F}(t, \cdot, \nu(t)), \nabla \rangle \nu(t), \tag{12}$$
where $\hat{F}(t, w, \nu)$ is defined by the rule

$$\hat{F}(t, w, \nu) \triangleq \text{co}\{(f(t, p(w), p_\#\nu, u), f(t, p(w), p_\#\nu, u)) : u \in U\}. \quad (13)$$

In the general form the notion of solution to the mean field type differential inclusion can be introduced as follows.

**Definition 1.** Let $X$ be a finite dimensional Euclidean space. Further, let $G(t, w, \nu)$ be a multivalued function defined on $[0, T] \times X \times \mathcal{P}^1(X)$ with values in $X$. We say that the function $[s, r] \ni t \mapsto \nu(t) \in \mathcal{P}^1(X)$ solves the mean field type differential inclusion (shortly, MFDI)

$$\frac{d}{dt} \nu(t) \in (G(t, \cdot, \nu(t)), \nabla)\nu(t),$$

if there exists a probability $\chi \in \mathcal{P}^1(C[s, r], X)$ such that $\nu(t)$ is an evaluation of $\chi$ at time $t$, and $\chi$-a.e. $w(\cdot) \in C([s, r], X)$ satisfies the differential inclusion

$$\frac{d}{dt} w(t) \in G(t, w(t), \nu(t)).$$

**Remark 2.** Notice that $\nu(\cdot)$ solves mean field type differential inclusion (12) on $[s, r]$ if and only if there exists $\chi \in \mathcal{P}^1(C_{s, r})$ such that $\nu(t) = \hat{e}_t \# \chi$ and $\text{supp}(\chi) \subset \text{SOL}(r, s, m(\cdot))$ for $m(t) = p_\# \nu(t)$.

Using the same methods as in [30] one can prove the existence of at least one solution to (1) satisfying $\nu(s) = \nu_*$. Now let us list the properties of the solutions to mean field type differential inclusion (12). They are proved in the Appendix.

**Proposition 1.** Let $\nu(\cdot) \in \mathcal{N}_{s, r}$ solve (12), and let $\nu_* \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$ be such that $p_\# \nu(s) = p_\# \nu_*$. Then there exists a flow of probabilities $\bar{\nu}(\cdot) \in \mathcal{N}_{s, r}$ such that

1. $\bar{\nu}(\cdot)$ solves (12);
2. $p_\# \nu(t) = p_\# \bar{\nu}(t)$;
3. $\nu(s) = \nu*$;
4. for any $\phi \in C(\mathbb{T}^d)$, and any $t \in [s, r]$,

$$[\phi, \bar{\nu}(t)] = [\phi, \nu(t)] - \int_{\mathbb{T}^d \times \mathbb{R}} z\nu(s, d(x, z)) + \int_{\mathbb{T}^d \times \mathbb{R}} z\nu_*(d(x, z)).$$

**Proposition 2.** If $s_0, s_1, s_2 \in [0, T]$, $s_0 < s_1 < s_2$, $\nu_1(\cdot) \in \mathcal{N}_{s_0, s_1}$, $\nu_2 \in \mathcal{N}_{s_1, s_2}$ solve (12) and $\nu_1(s_1) = \nu_2(s_2)$, then

$$\nu(t) \triangleq \begin{cases} \nu_1(t), & t \in [s_0, s_1]; \\
\nu_2(t), & t \in [s_1, s_2] \end{cases}$$

is also a solution to (12).
Proposition 3. Let \( \{\nu_i(\cdot)\}_{i=1}^{\infty} \subset \mathcal{N}_{s,r} \). Assume that, for each \( i \), \( \nu_i(\cdot) \) solves (12) and satisfies \( \nu_i(s) = \nu^*_s \). Then there exist a sequence \( \{i_k\} \) and a flow of probabilities \( \nu^*(\cdot) \in \mathcal{N}_{s,r} \) such that

1. \( \nu^*(\cdot) \) solves (12);
2. \( \lim_{k \to \infty} \sup_{t \in [s, r]} W_1(\nu_{i_k}(t), \nu^*(t)) = 0. \)

3 Solution of the first-order mean field game

As it was mentioned in the Introduction, there are several methods to analyze the mean field game (1), (2). The approach based on PDEs reduces the original problem to mean field game system

\[
\frac{\partial V}{\partial t} + H(t, x, m(t), \nabla V) = 0, V(T, x) = \sigma(x, m(T)),
\]

\[
\frac{d}{dt} m(t) = \left\langle \frac{\partial H(t, x, m(t), \nabla V)}{\partial p}, \nabla \right\rangle m(t), \quad m(t_0) = m_0.
\]

Here, for \( t \in [0, T], x \in \mathbb{T}^d, m \in \mathcal{P}^1(\mathbb{T}^d), p \in \mathbb{R}^d, \)

\[
H(t, x, m, p) \triangleq \max_{u \in U} \langle p, f(t, x, m, u) \rangle + g(t, x, m, p).\]

Within the framework of this approach a solution of the mean field game is defined as a solution of this system. However, for our purposes it is convenient to use the probabilistic approach. The link between the mentioned approached is discussed in [24].

We adapt the probabilistic approach for the first order mean field game. The following definition is close to one proposed in [4].

Definition 2. We say that a pair \((V, m(\cdot))\), where \( V : [t_0, T] \times \mathbb{T}^d \to \mathbb{R} \) is a continuous function and \( m(\cdot) \in \mathcal{M}_{t_0, T} \), is a solution to mean field game (1), (2), if there exists a probability \( \chi \in \mathcal{P}^1(C_{t_0, T}) \) such that

1. \( m(t) = e_t \# \chi; \)
2. \( V(s, y) \) is a value of the optimization problem

\[
\text{maximize } [\sigma(x(T), m(T)) + z(T) - z(s)]
\]

subject to \( (x(\cdot), z(\cdot)) \in \text{Sol}(T, s, y, m(\cdot)); \)
3. \( \text{supp}(\chi) \subset \text{SOL}(T, t_0, m(\cdot)); \)
4. for any \( s, r \in [t_0, T], s < r, \) and any \( (x(\cdot), z(\cdot)) \in \text{supp}(\chi), \)

\[
V(s, x(s)) + z(s) = V(r, x(r)) + z(r).
\]
Given \( m_0 \in \mathcal{P}^1(\mathbb{T}^d) \), there exists at least one solution \((V, m(\cdot))\) to mean field game \((1), (2)\) satisfying \( m(t_0) = m_0 \) \(\square\). However, the uniqueness result is not valid in the general case \([5]\).

Now let us introduce the equivalent formulation of Definition \(2\) using the notion of solution to the differential inclusion. To this end, for \( s, r \in [0, T] \), \( s \leq r \), \( m(\cdot) \in \mathcal{M}_{s,r} \), define the operator \( B_{m(\cdot)}^{s,r} : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d) \) by the rule:

\[
(B_{m(\cdot)}^{s,r}) \psi(y) \triangleq \{ \psi(x(r)) + z(r) - z(s) : (x(\cdot), z(\cdot)) \in \text{Sol}(r, s, y, m(\cdot)) \}.
\]

The family \( \{B_{m(\cdot)}^{s,r}\}_{s \leq r} \) is a backward propagator. Indeed, if \( m(\cdot) \) is a flow of probabilities on \([s, \theta]\), \( r \in [s, \theta] \), then

\[
B_{m(\cdot)}^{s,r}B_{m(\cdot)}^{r,\theta} = B_{m(\cdot)}^{s,\theta},
\]

Moreover, \( B_{m(\cdot)}^{s,s} = \text{Id} \). Notice that, for any \( s, r \in [0, T] \), \( s \leq r \), the operator \( B_{m(\cdot)}^{s,r} \) is linear in max-plus algebra.

**Proposition 4.** The pair \((V, m(\cdot))\) is a solution to mean field differential game \((1), (2)\) if and only if there exists a solution to MFDI \((12)\) on \([t_0, T] \) \(\nu(\cdot)\) such that, for any \( s \in [t_0, T] \),

1. \( m(s) = p_\# \nu(s) \), \( s \in [t_0, T] \);
2. \( V(s, \cdot) = B_{m(\cdot)}^{s,T} \sigma(\cdot, m(T)) \);
3. \([\sigma(\cdot, m(T)), \nu(T)] \geq [V(s, \cdot), \nu(s)]\).

**Proof.** By Remark \(2\) we have that conditions 1 and 3 of Definition \(2\) are equivalent to the fact that \( \nu(\cdot) \) solves \((12)\) and \( m(s) = p_\# \nu(t) \). Condition 2 of Definition \(2\) can be rewritten in the form \( V(s, \cdot) = B_{m(\cdot)}^{s,T} \sigma(\cdot, m(T)) \). It remains to show that Condition 4 of Definition \(2\) and Condition 3 of the proposition are equivalent. To prove the first implication integrate condition 4 of Definition \(2\) for \( r = T \). Thus, using Remark \(2\) we get

\[
[\sigma(\cdot, m(T)), \nu(T)] = \int_{C_{s,T}} [\sigma(x(T), m(T)) + z(T)] \chi(d(x(\cdot), z(\cdot)))
\]

\[
= \int_{C_{s,T}} [V(s, x(s)) + z(s)] \chi(d(x(\cdot), z(\cdot))) = [V(s, \cdot), \nu(s)].
\]

Conversely, since \( V(s, \cdot) = B_{m(\cdot)}^{s,T} \sigma(\cdot, m(T)) \), we have that, for any \((x(\cdot), z(\cdot)) \in \text{SOL}(T, t_0, m(\cdot))\),

\[
V(s, x(s)) + z(s) \geq \sigma(x(T), m(T)) + z(T).
\]

Further, using Remark \(1\) one can rewrite Condition 3 of the proposition in the form

\[
\int_{C_{s,T}} [\sigma(x(T), m(T)) + z(T)] \chi(d(x(\cdot), z(\cdot)))
\]

\[
\geq \int_{C_{s,T}} [V(s, x(s)) + z(s)] \chi(d(x(\cdot), z(\cdot))).
\]
Therefore, we conclude that, for \( \chi \)-a.e. \((x(\cdot), z(\cdot)) \in \text{SOL}(T, t_0, m(\cdot))\),

\[
\sigma(x(T), m(T)) + z(T) = V(s, x(s)) + z(s).
\]

This and continuity of the functions \( \sigma \) and \( V \) imply Condition 4 of Definition 2.

4 Value multifunction

In this section we introduce the notion of the value multifunction that describes
the dependence of the solution of the mean field game on the initial distribution and
examine this dependence using the viability approach.

Definition 3. We say that a upper semicontinuous function multifunction \( V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \Rightarrow C(\mathbb{T}^d) \) is a value multifunction of mean field game (1), (2) if, for any \( t_0 \in [0, T] \), \( m_0 \in \mathcal{P}^1(\mathbb{T}^d) \), and \( \phi \in V(t_0, m_0) \), there exists a solution to mean field game (1), (2) \((V, m(\cdot))\) such that

\[
V(t_0, \cdot) = \phi(\cdot), \quad m(t_0) = m_0.
\]

Remark 3. Note that the value multifunction is not defied in the unique way. It is natural to say that \( \mathcal{W} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \Rightarrow C(\mathbb{T}^d) \) is the maximal value multifunction if it is a value multifunction and, for any value multifunction \( V \),

\[
\mathcal{W}(t, m) \subset \mathcal{W}(t, m), \quad t \in [0, T], \quad m \in \mathcal{P}^1(\mathbb{T}^d).
\]

The existence result for the maximal value function can be proved using the facts that the closure of union of value multifunctions is also a value multifunction.

We look for the sufficient condition for a multifunction \( V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \Rightarrow C(\mathbb{T}^d) \) to be a value multifunction. To this end we introduce the dynamical system on \( \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \).

Definition 4. For each \( s, r \in [0, T] \), \( s \leq r \), define a multifunction \( \Psi^{r,s} : \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \Rightarrow \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \) by the rule: \((\mu, \psi) \in \Psi^{r,s}(m, \phi)\) if and only if there exists a solution of MFDI (12) on \([s, r] \) \( \nu(\cdot) \), satisfying the following properties for \( m(t) = p\theta \nu(t) \):

1. \( m(s) = m, \quad m(r) = \mu \);
2. \( \phi = B_{m(\cdot)}^{r,s} \psi \);
3. \( [\psi, \nu(r)] \geq [\phi, \nu(s)] \).

We shall show that the family of set-valued mappings \( \Psi = \{\Psi^{r,s}\}_{s \leq r} \) provides a dynamics in the space \( \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \). First, notice that \( \Psi^{s,s}(m, \phi) = \{ (m, \phi) \} \).

Proposition 5. \( \Psi \) satisfies the semigroup property i.e., for any \( s_0, s_1, s_2 \in [0, T] \), \( s < r < \theta \),

\[
\Psi^{s_2, s_0} = \Psi^{s_2, s_1} \circ \Psi^{s_1, s_0}.
\]
Here we define the composition of multivalued maps in the usual manner, i.e. if \((X, \rho_X)\) is a metric space, then the composition of multivalued mappings \(\Phi_1, \Phi_2 : X \Rightarrow X\) is
\[
(\Phi_2 \circ \Phi_1)(x) \triangleq \bigcup_{y \in \Phi_1(x)} \Phi_2(y).
\]

**Proof of Proposition** Let us prove that \(\Psi^{s_2, s_1} \circ \Psi^{s_1, s_0} \subset \Psi^{s_2, s_0}\). Pick \((m^0, \phi^0), (m^1, \phi^1), (m^2, \phi^2) \in \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d)\) such that \((m^1, \phi^1) \in \Psi^{s_1, s_0}(m^0, \phi^0)\), \((m^2, \phi^2) \in \Psi^{s_2, s_1}(m^1, \phi^1)\). We are to prove that \((m^2, \phi^2) \in \Psi^{s_2, s_0}(m^0, \phi^0)\). For \(i = 1, 2\), let \(\nu_i(\cdot) \in \mathcal{N}_{s_i, s_{i-1}}, m_i(\cdot) \in \mathcal{M}_{s_i, s_{i-1}}\) be such that \(m_i(t) = \rho_{m_g} \nu_i(t), \nu_i(\cdot)\) solves MFDI \((12)\) on \([s_{i-1}, s_i]\), \(m_i(s_{i-1}) = m_i(s_i) = m^i, \phi^{i-1} = \mathcal{R}^{s_i-1, s_i} \phi^i, [\phi^i, \nu_i(s_i)] \geq [\phi^{i-1}, \nu_i(s_{i-1})]\). Due to Proposition \(\ref{P2}\), one may assume that \(\nu_1(s_1) = \nu_2(s_1)\). Let
\[

\nu(t) = \begin{cases}
\nu_1(t), & t \in [s_0, s_1], \\
\nu_2(t), & t \in [s_1, s_2],
\end{cases}
\]
\(m(t) \triangleq \rho_{m_g} \nu(t)\). Notice that \(m(t)\) coincides with \(m_1(t)\) on \([s_0, s_1]\) and with \(m_2(t)\) on \([s_1, s_2]\). By Proposition \(\ref{P2}\) we have that \(\nu(\cdot)\) is a solution to \((12)\) on \([s_0, s_2]\). Using the semigroup property for the family of operators \(B\) (see \((14)\)), we get
\[
\phi^0 = \mathcal{R}^{s_0, s_1} \phi^1.
\]

Further,
\[
[\phi^2, \nu(s_2)] = [\phi^2, \nu_2(s_2)] \geq [\phi^1, \nu_2(s_1)] = [\phi^1, \nu_1(s_1)] \geq [\phi^0, \nu_1(s_0)] = [\phi^0, \nu(s_0)].
\]

Thus, we obtain the inclusion \(\Psi^{s_2, s_1} \circ \Psi^{s_1, s_0} \subset \Psi^{s_2, s_0}\).

Now we turn to the opposite inclusion. Pick \((m^0, \phi^0), (m^2, \phi^2)\) such that \((m^2, \phi^2) \in \Psi^{s_2, s_0}(m^0, \phi^0)\). We shall prove that there exists \((m^1, \phi^1)\) such that \((m^1, \phi^1) \in \Psi^{s_1, s_0}(m^0, \phi^0)\) and \((m^2, \phi^2) \in \Psi^{s_2, s_1}(m^1, \phi^1)\). Let \(\nu(\cdot) \in \mathcal{N}_{s_0, s_2}, m(\cdot) \in \mathcal{M}_{s_0, s_2}\) satisfy conditions \((\Psi 1)-(\Psi 3)\) for \(s = s_0, r = s_2, m = m^0, \phi = \phi^0, \mu = m^2, \psi = \phi^2\). There exists \(\chi \in \mathcal{P}^1(\mathcal{C}_{s_0, s_2})\) such that \(\nu(t) = \hat{\epsilon}_t \# \chi\) and \(\text{supp}(\chi) \subset \text{SOL}(s_2, s_0, m(\cdot))\). Notice that the function \(\phi^0\) is a value of the problem
\[
\text{maximize } \phi^2(x(s_2)) + z(s_2) - z(s_0)
\]
subject to \((x(\cdot), z(\cdot)) \in \text{SOL}(s_2, s_0, m(\cdot))\). \((15)\)

Additionally, one can rewrite \((\Psi 3)\) in the form
\[
\int_{\mathcal{C}_{s_0, s_2}} (\phi^2(x(s_2)) + (z(s_2) - z(s_0))\chi(d(x(\cdot), z(\cdot))) \geq \int_{\mathcal{C}_{s_0, s_2}} \phi^0(x(s_0))\chi(d(x(\cdot), z(\cdot))).
\]

Using inclusion \(\text{supp}(\chi) \subset \text{SOL}(s_2, s_0, m(\cdot))\) we conclude that \(\chi\) is concentrated on the set of optimal motions to problem \((15), (16)\).

Now set \(m^1 = m(s_1), \phi^i = \mathcal{R}^{s_i-1, s_i} \phi^i\). Further, for \(t \in [s_{i-1}, s_i]\), put \(\nu_i(t)\) be equal to \(\nu(t)\), \(m_i(t) \triangleq \rho_{m_g} \nu_i(t)\). The dynamic programming gives that
\[
\phi^{i-1} = \mathcal{R}^{s_i-1, s_i} \phi^i, \quad i = 1, 2.
\]
It remains to prove that
\[ [\phi^i, \nu_i(s_i)] \geq [\phi^{i-1}, \nu_i(s_{i-1})]. \quad (17) \]

Let \( \chi_i \) be a projection of \( \chi \) on \( C_{s_{i-1}, s_i} \). We have that \( \nu_i(t) = \hat{e}_{t\#} \chi_i \supset \text{SOL}(s_i, s_{i-1}, m_i(\cdot)) \). Moreover, since \( \chi \) is concentrated on optimal motions to \( [15], [16] \), we have that each motion \( (x(\cdot), z(\cdot)) \in \text{SOL}(s_i, s_{i-1}, m_i(\cdot)) \) provides the solution to the problem
\[
\begin{align*}
\text{maximize} & \quad \phi^i(x(s_i)) + z(s_i) - z(s_{i-1}) \\
\text{subject to} & \quad (x(\cdot), z(\cdot)) \in \text{SOL}(s_i, s_{i-1}, m_i(\cdot)).
\end{align*}
\]

Integrating this and taking into account the property \( \text{supp}(\chi_i) \subset \text{SOL}(s_i, s_{i-1}, m_i(\cdot)) \), we get
\[
\int_{C_{s_{i-1}, s_i}} (\phi^i(x(s_i)) + z(s_i)) \chi_i(d(x(\cdot), z(\cdot))) 
\geq \int_{C_{s_{i-1}, s_i}} (\phi^{i-1}(x(s_{i-1})) + z(s_{i-1})) \chi_i(d(x(\cdot), z(\cdot))).
\]
This implies \( (17) \).

Recall (see the Introduction) that we call the dynamics generated by the family \( \{\Psi^{r,s}\}_{s \leq r} \) the mean field game dynamics.

**Definition 5.** We say that a upper semicontinuous multifunction \( V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \Rightarrow C(\mathbb{T}^d) \) is viable with respect to the mean field game dynamics if, for any \( s, r \in [0, T], s \leq r, m \in \mathcal{P}^1(\mathbb{T}^d), \phi \in V(s, m) \), there exist \( \mu \in \mathbb{T}^d \) and \( \psi \in C(\mathbb{T}^d) \) such that
\[
\begin{align*}
&\bullet \quad (\mu, \psi) \in \Psi^{r,s}(m, \phi); \\
&\bullet \quad \psi \in V(r, \mu).\n\end{align*}
\]

**Remark 4.** It is more accurate to say that the graph of the multifunction \( V \) is viable with respect to the mean field game dynamics. However, for the sake of shortness we will say that the multifunction \( V \) itself is viable.

**Remark 5.** One can prove that the maximal value function is viable with respect to the mean field game dynamics.

The link between viability property and value function is given in the following statement.

**Theorem 1.** Assume that a upper semicontinuous multifunction \( V : [0, T] \times \mathbb{T}^d \Rightarrow C(\mathbb{T}^d) \) is viable with respect to the mean field game dynamics and \( V(T, m) = \{\sigma(\cdot, m)\} \). Then \( V \) is a value multifunction.

The proof of Theorem 1 requires Lemmas 1, 2. This Lemmas are concerned with the continuity properties of the propagator \( B \). In the following the constant \( R \) satisfies \( (4) \).
**Lemma 1.** Assume that \( m(\cdot) \in \mathcal{M}_{0,r}, \ K > 0, \ \psi \in \text{Lip}_K(r) \) and \( \varphi \in C([0, r] \times \mathbb{T}^d) \) is such that, for all \( s \in [0, r] \), \( \varphi(s, \cdot) = B_{m(\cdot)}^{s,r}\psi \). Then

1. for \( s \in [0, T] \), \( \varphi(s, \cdot) \in \text{Lip}(K+1)e^{L(r-s-1)}(\mathbb{T}^d) \);
2. for any \( s, s' \in [0, r] \), \( |\varphi(s, y) - \varphi(s', y)| \leq R(K + 1)e^{L(r-\max(s,s'))}|s - s'| \).

**Proof.** Let \( y_1, y_2 \in \mathbb{T}^d \). Since \( \varphi(s, \cdot) = B_{m(\cdot)}^{s,r}\psi \), there exist \( (x_1(\cdot), z_1(\cdot)) \in C_{s,r} \) and \( \zeta \in \mathcal{U}_{s,T} \) such that \( (x_1(\cdot), z_1(\cdot)) \) solves differential inclusion \([8]\), \( x_1(s) = y_1 \),

\[ \phi(s, y_1) = \psi(x_1(r)) + z_1(r) - z_1(s), \]

and \([10]\) holds for \( x(\cdot) = x_1(\cdot), z(\cdot) = z_1(\cdot) \) and \( \zeta = \zeta_1 \).

Let \( (x_2(\cdot), z_2(\cdot)) \) solve initial the value problem for \([10]\) with \( \zeta = \zeta_1 \) and \( x_2(s) = y_2, z_2(s) = z_1(s) \). We have that

\[ \|x_1(t) - x_2(t)\| \leq \|y_1 - y_2\| + L \int_s^t \|x_1(\tau) - x_2(\tau)\| d\tau. \]

Using Gronwall’s inequality we get

\[ \|x_1(t) - x_2(t)\| \leq \|y_1 - y_2\|e^{L(t-s)}. \]

Further,

\[ |z_1(t) - z_2(t)| \leq \|y_1 - y_2\|(e^{L(t-s)} - 1). \]

Since \( (x_2(\cdot), z_2(\cdot)) \) solves \([8]\) and \( \varphi(s, \cdot) = B_{m(\cdot)}^{s,r}\psi \), we have that

\[ \varphi(s, y_2) \geq \varphi(x_2(r)) + z_2(r) - z_1(s). \]

Combining this, \([18]\) and \([19]\) we conclude that

\[ \varphi(s, y_2) - \varphi(s, y_1) \geq -[(K + 1)e^{L(r-s)} - 1]\|y_1 - y_2\|. \]

The opposite inequality is proved in the same way.

Now, we turn to the second statement of the Lemma. Without loss of generality assume that \( s' < s \). The semigroup property for the operator \( B \) (see \([14]\)) implies that \( \varphi(s', \cdot) = B_{m(\cdot)}^{s',s}\varphi(s, \cdot) \). Thus, for each \( y \in \mathbb{T}^d \), there exists a trajectory \( (x(\cdot), z(\cdot)) \in \text{Sol}(s, s', y, m(\cdot)) \) such that

\[ \varphi(s', y) = \varphi(s, x(s)) + z(s) - z(s'). \]

Hence,

\[ |\varphi(s', y) - \varphi(s, y)| = |\varphi(s, x(s)) - \varphi(s, y) + z(s) - z(s')| \]

\[ \leq |\varphi(s, x(s)) - \varphi(s, y)| + |z(s) - z(s')|. \]

By the first statement of the Lemma and inequality \([11]\) we have that

\[ |\varphi(s', y) - \varphi(s, y)| \leq R(K + 1)e^{L(r-\max(s,s'))}(s - s'). \]

This completes the proof of the second statement of the Lemma. \( \square \)
Lemma 2. If \( s, r, r' \in [0, T], s < r \leq r', m(\cdot), m'(\cdot) \in \mathcal{M}_{s, r'}, K > 0, \psi, \psi' \in \text{Lip}_K(\mathbb{T}^d) \), then

\[
\|B_{m(\cdot)}^{s, r}\psi - B_{m'(...)}^{s, r'}\psi'\| \leq \|\psi' - \psi\| + (K + 1)R(r' - r) + L(Ke^{LT} + LTe^{LT} + 1)(r - s) \sup_{t \in [s, r]} W_1(m'(t), m(t)).
\]

Proof. Let \( y \in \mathbb{T}^d \) and let \((x_1(\cdot), z_1(\cdot)) \in \text{Sol}(r, s, y, m(\cdot))\) be such that

\[
(B_{m(\cdot)}^{s, r'}(\cdot)) = \psi'(x_1(r')) + z_1(r') - z_1(s).
\]

There exists a relaxed control \( \xi \in \mathcal{U}_{s, r} \) such that \((10)\) holds true for \( x(\cdot) = x_1(\cdot), z(\cdot) = z_1(\cdot) \). Let \( x_2(\cdot) \) solve the initial value problem

\[
d\frac{d}{dt}x_2(t) = \int_U f(t, x_2(t), m'(t), u)\xi(t, du), \quad x_2(s) = y.
\]

Further, put

\[
z_2(t) \triangleq z_1(s) + \int_s^t \int_U g(t, x_2(t), m'(t), u)\xi(t, du)dt.
\]

We have that

\[
\|x_1(t) - x_2(t)\| \leq L \int_s^t \|x_1(\theta) - x_2(\theta)\|d\theta + L(t - s) \sup_{\theta \in [s, r]} W_1(m(\theta), m'(\theta)).
\]

Using Gronwall’s inequality, we get

\[
\|x_1(t) - x_2(t)\| \leq Le^{LT}(t - s) \sup_{\theta \in [s, r]} W_1(m(\theta), m'(\theta)). \tag{20}
\]

Further,

\[
|z_1(t) - z_2(t)| \leq L \int_s^t \|x_1(\theta) - x_2(\theta)\|d\theta + L(t - s) \sup_{\theta \in [s, r]} W_1(m(\theta), m'(\theta)).
\]

Using \((20)\), we get

\[
|z_1(t) - z_2(t)| \leq L(LTe^{LT} + 1)(t - s) \sup_{\theta \in [s, r]} W_1(m(\theta), m'(\theta)). \tag{21}
\]

Since \( \psi(x_2(r)) + z_2(r) \leq B_{m(\cdot)}^{s, r}\psi(y) \), from \((20)\) and \((21)\) we conclude that \( B_{m(\cdot)}^{s, r}\psi'(y) = \psi'(x_1(r')) + z_1(r') - z_1(s) \leq \psi(x_1(r')) + z_1(r') - z_1(s) + \|\psi - \psi'\| \leq \psi(x_1(r)) + z_1(r) - z_1(s) + (K + 1)R(r' - r) + \|\psi - \psi'\| \leq \psi(x_2(r)) + z_2(r) - z_2(s) + K\|x_1(r) - x_2(r)\| + |z_1(r) - z_2(r)| + (K + 1)R(r' - r) + \|\psi - \psi'\| \leq B_{m(\cdot)}^{s, r}\psi(y) + L(Ke^{LT} + LTe^{LT} + 1)(r - s) \sup_{\theta \in [s, r]} W_1(m(\theta), m'(\theta)) + (K + 1)R(r' - r) + \|\psi - \psi'\|.\]
Analogously, one can prove that

\[
B_{m\cdot}^{s,r}\psi(y) \leq B_{m\cdot}^{s,r'}\psi'(y) + L(Ke^{LT} + LTe^{LT} + 1)(r - s) \sup_{\theta \in [s,r']} W_1(m(\theta), m'(\theta)) \\
+ (K + 1)R(r' - r) + \|\psi - \psi'\|.
\]

These two inequalities yield the conclusion of the lemma.

**Proof of Theorem 7.** We shall prove that if \( V \) is viable with respect to the mean field game dynamics, then, for any \( t_* \in [0, T] \), \( m_* \in P^1(\mathbb{T}^d) \) and \( \phi_* \in V(t_*, m_*) \), there exists a solution to (1), (2) such that

\[
\psi(\cdot) \in V(t_*, \cdot). 
\]

By Proposition 1 we can assume without loss of generality that \( V(t_*, \cdot) = \phi_*(\cdot) \) and \( m(t_*) = m_* \).

Let \( N \) be a natural number, and let \( t_N^i \equiv t_* + (T - t_*)/N \). By assumption we have that there exists a sequence of pairs \( \{(\mu_N^i, \phi_N^i)\}_{t=0}^N \subset P^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \) satisfying the following conditions:

- \( \phi_N^0(\cdot) = \phi_*(\cdot) \), \( \mu_N^0 = m_* \);
- \( (\mu_N^i, \phi_N^i) \in \Psi^{t_N^i, t_N^{i-1}}(\mu_N^{i-1}, \phi_N^{i-1}) \), \( i = 1, \ldots, N \);
- \( \phi_N^N(\cdot) = \sigma(\cdot, \mu_N^0) \).

Since \( (\mu_N^i, \phi_N^i) \in \Psi^{t_N^i, t_N^{i-1}}(\mu_N^{i-1}, \phi_N^{i-1}) \), there exist \( \nu_N^i(\cdot) \in \mathcal{N}_{t_N^{i-1}, t_N^i} \) and \( m_N^i(\cdot) \in \mathcal{M}_{t_N^{i-1}, t_N^i} \) such that

- \( \nu_N^i(\cdot) \) is a solution of (12) on \( [t_N^{i-1}, t_N^i] \);
- \( m_N^i(t) = p\# \nu_N^i(t) \) when \( t \in [t_N^{i-1}, t_N^i] \);
- \( m_N^i(t_N^{i-1}) = \mu_N^{i-1} \), \( m_N(t_N^i) = \mu_N^i \);
- \( \phi_N^i = B_{m_N^i}^{t_N^i, t_N^{i-1}} \phi_N^{i-1} \);
- \( [\phi_N^i, \nu_N^i(t_N^i)] \geq [\phi_N^{i-1}, \nu_N^i(t_N^{i-1})] \).

By Proposition 1, we can assume without loss of generality that \( \nu_N^{i-1}(t_N^{i-1}) = \nu_N^i(t_N^{i-1}) \). Define \( \nu_N(\cdot) \in \mathcal{N}_{t_* T} \) by the rule: if \( t \in [t_N^{i-1}, t_N^i] \), then

\[
\nu_N(t) \equiv \nu_N^i(t).
\]

Further, let \( m_N(t) \equiv p\# \nu_N(t) \). By Proposition 2, we have that \( \nu_N(\cdot) \) solves (12). Moreover, \( m_N(t_*) = m_* \). Proposition 3 implies that there exists a flow of probabilities \( \nu(\cdot) \in \mathcal{N}_{t_* T} \) and a sequence \( \{N_k\}_{k=1}^\infty \) such that \( \nu(\cdot) \) is a solution of (12), and

\[
\lim_{k \to \infty} \sup_{t \in [t_*, T]} W_1(\nu_{N_k}(t), \nu(t)) = 0.
\]

Put

\[
m(t) \equiv p\# \nu(t). \quad (22)
\]
For \( s \in [t_*, T] \), set
\[
V_N(s, \cdot) = B_{m_N}^{s,T} \sigma(\cdot, m_N(T)).
\] (23)

By Lemma 1 the functions \( V_N \) are \( C_1 \)-Lipschitz continuous functions, where the constant \( C_1 \) does not depend on \( N \). Hence, without loss of generality, one can assume that the sequence \( \{ V_{N_k} \} \) converges to some function \( V \in C([t_*, T] \times \mathbb{T}^d) \).

Let us prove that \((V, m(\cdot))\) is a solution of mean field game (1), (2). To this end we check the conditions of Proposition 4. By construction we have that \( m(t) = p_\# \nu(t) \), where \( \nu(\cdot) \) solves MFDI (12). Notice that \( W_1(m_{N_k}(t), m(t)) \leq W_1(\nu_{N_k}(t), \nu(t)) \). Thus,
\[
\lim_{k \to \infty} \sup_{t \in [t_*, T]} W_1(\nu_{N_k}(t), m(t)) = 0.
\] (24)

It follows from (23) that, for any \( k \),
\[
\| V(s, \cdot) - B_{m(t)}^{s,T} \sigma(\cdot, m(T)) \| \leq \| V(s, \cdot) - V_{N_k}(s, \cdot) \| +
+ \| B_{m(t)}^{s,T} \sigma(\cdot, m_{N_k}(T)) - B_{m(t)}^{s,T} \sigma(\cdot, m(T)) \|.
\]

Using Lemma 2 we estimate the right-hand side of this inequality and get
\[
\| V(s, \cdot) - B_{m(t)}^{s,T} \sigma(\cdot, m(T)) \|
\leq \| V(s, \cdot) - V_{N_k}(s, \cdot) \| + \| \sigma(\cdot, m(T)) - \sigma(\cdot, m_{N_k}(T)) \|
+ L(\varepsilon e^{LT} + LT e^{LT} + 1)(T - s) \sup_{t \in [t_*, T]} W_1(\nu_{N_k}(t), m(t)).
\]

Passing to the limit when \( k \to 0 \) and using (24), we conclude that
\[
\| V(s, \cdot) - B_{m(t)}^{s,T} \sigma(\cdot, m(T)) \| = 0.
\] (25)

It remains to prove that
\[
[\sigma(\cdot, m(T)), \nu(T)] \geq [V(s, \cdot), \nu(s)].
\] (26)

To this end notice that \( \nu_N(t) = \nu_N^i(t) \), for \( t \in [t_N^{i-1}, t_N^i] \), \( V_N(t_N^i, \cdot) = \phi_N^i(\cdot) \) and \( V_N(T, \cdot) = \sigma(\cdot, m_N(T)) \). This and the inequalities \( [\phi_N^i, \nu_N(t_N^i)] \geq [\phi_N^i, \nu_N(t_N^{i-1})] \) yield that
\[
[\sigma(\cdot, m_N(T)), \nu_N(T)] \geq [V_N(t_N^{i-1}, \cdot), \nu_N(t_N^{i-1})], \quad i = 1, \ldots, N.
\]

Further, let \( i \) be such that \( s \in [t_N^{i-1}, t_N^i] \). We have that \( \| V_N(s, \cdot) - V_N(t_N^{i-1}, \cdot) \| \leq C_1/N \). Since \( \nu_N(\cdot) \) solves MFDI (12), \( W_1(\nu_N(s), \nu_N(t_N^{i-1})) \leq 2R/N \). Therefore, by (5)
\[
[\sigma(\cdot, m_N(T)), \nu_N(T)] \geq [V_N(s, \cdot), \nu_N(s)] - C_1(1 + 2R)/N.
\] (27)

Further, the functions \( x \mapsto \sigma(x, m(T)), x \mapsto V(s, x) \) are Lipschitz continuous with the constants \( \varepsilon \) and \( C_1 \) respectively. Hence, using (27) for \( N = N_k \) and (5), we conclude
\[
[\sigma(\cdot, m(T)), \nu(T)] \geq [V(s, \cdot), \nu(s)] - C_1(1 + 2R)/N_k - \| \sigma(\cdot, m(T)) - \sigma(\cdot, m_{N_k}(T)) \|
- \| V(s, \cdot) - V_{N_k}(s, \cdot) \| - \varepsilon W_1(\nu(T), \nu_{N_k}(T)) - C_1 W_1(\nu(s), \nu_{N_k}(s)).
\]
Passing to the limit, we get \((26)\).

The fact that \(\nu(\cdot)\) is a solution to \((12)\), equalities \((22)\), \((25)\) and inequality \((26)\) imply that \((V, m(\cdot))\) is a solution of mean field game \((1), (2)\) (see Proposition 1). Furthermore, by construction \(m(t_*) = m_*\) and \(V(t_*, m_*) = \phi_*\). Thus, \(V\) is a value multifunction. \(\square\)

## 5 Statement of the viability theorem

In this section we formulate the infinitesimal form of the viability property for mean field game dynamics.

To this end we use the probabilities on tangent bundles to \(T^d \times \{0\}\) and \(T^d \times \mathbb{R}\). Let \(m \in \mathcal{P}^1(T^d)\). Denote by \(\mathcal{L}(m)\) the set of probabilities \(\beta \in \mathcal{P}^1(T^d \times \mathbb{R}^{d+1})\) with the marginal on \(T^d\) equal to \(m\). Analogously, if \(\nu \in \mathcal{P}^1(T^d \times \mathbb{R})\), then \(\mathcal{L}_*(\nu)\) stands for the set of probabilities \(\gamma \in \mathcal{P}^1(T^d \times \mathbb{R} \times \mathbb{R}^{d+1})\) with the marginal on \(T^d \times \mathbb{R}\) equal to \(\nu\). If \(c > 0\), then put

\[
\mathcal{L}_c^*(m) \triangleq \{\beta \in \mathcal{L}(m) : \text{supp}\beta \subset T^d \times B_c \times [-c, c]\},
\]

\[
\mathcal{L}_c^*(m) \triangleq \{\gamma \in \mathcal{L}_*(m) : \text{supp}\gamma \subset T^d \times \mathbb{R} \times B_c \times [-c, c]\}.
\]

Here \(B_c\) stands for the closed ball of of the radius \(c\) centered at the origin.

If \(m \in \mathcal{P}^1(T^d)\), \(s, r \in [0, T]\), \(s \leq r\), then denote by \(A_{m}^{s, r}\) the operator on \(C(T^d)\) acting by the rule

\[
(A_{m}^{s, r} \phi)(x) \triangleq \sup \{\phi(x + (r-s)a) + (r-s)b : (a, b) \in F(s, x, m)\}.
\]

Here \(F\) is defined by \((9)\). For \(t \geq 0\), let the operator \(\Theta^r : T^d \times \mathbb{R}^{d+1} \to T^d \times \mathbb{R}\) be given by

\[
\Theta^r(x, a, b) \triangleq (x + \tau a, \tau b).
\]

Finally, for \(m \in \mathcal{P}^1(T^d)\), denote by \(\widehat{m}\) its lifting to \(\mathcal{P}^1(T^d \times \mathbb{R})\) defined by the rule: for \(\varphi \in C_0(T^d \times \mathbb{R})\),

\[
\int_{T^d \times \mathbb{R}} \varphi(x, z) \widehat{m}(dx, dz) \triangleq \int_{T^d} \varphi(x, 0)m(dx).
\]

Let \(\mathcal{V} : [0, T] \times \mathcal{P}^1(T^d) \to C(T^d)\) be upper semicontinuous, \(t \in [0, T]\), \(m \in \mathcal{P}^1(T^d)\), \(\phi \in C(T^d)\). Now, we turn to the definition of set-valued derivative of the multifunction \(\mathcal{V}\) at \(t, m, \phi\) by virtue of \(F\) under constraints determined by constant \(c\). We denote this derivative by \(\mathcal{D}_F^c \mathcal{V}(t, m, \phi)\).

**Definition 6.** A probability \(\beta \in \mathcal{L}_c^*(m)\) belongs to \(\mathcal{D}_F^c \mathcal{V}(t, m, \phi)\), if there exist sequences \(\{\tau_n\}_{n=1}^{\infty} \subset (0, +\infty)\), \(\{\beta_n\}_{n=1}^{\infty} \subset \mathcal{L}_c^*(m)\) and \(\{\phi_n\}_{n=1}^{\infty} \subset C(T^d)\) satisfying the following properties for \(\nu_n \triangleq \Theta^{\tau_n} \# \beta_n\) and \(m_n \triangleq p_\# \nu_n\):

1. \(\tau_n, W_1(\beta, \beta_n) \to 0\) as \(n \to \infty\);
2. \(\phi_n \in \mathcal{V}(t + \tau_n, m_n)\);
3. \[ \lim_{n \to \infty} \frac{\|A_{m}^{t+n} \phi_n - \phi\|}{\tau_n} = 0; \]
4. \[ \lim_{n \to \infty} \frac{[\phi_n, \nu_n] - [\phi, \hat{m}]}{\tau_n} \geq 0; \]
5. \[ \int_{\mathbb{T}^d \times \mathbb{R}^{d+1}} \text{dist}(v; F(t, x, m)) \beta(d(x, v)) = 0. \]

For \( M, C > 0 \), let \( \text{BL}_{M,C} \) denote the set of functions \( \phi \in \text{Lip}_{C}(\mathbb{T}^d) \) such that \( \| \phi \| \leq M \).

**Theorem 2.** Assume that the upper semicontinuous multifunction \( V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d) \) has nonempty values and there exist constants \( M \) and \( C \) such that, for any \( t \in [0, T] \), \( m \in \mathcal{P}^1(\mathbb{T}^d) \),

\[ V(t, m) \subset \text{BL}_{M,C}(\mathbb{T}^d). \]

Then, \( V \) is viable with respect to the mean field game dynamics if and only if, there exists a constant \( c > 0 \) such that, for any \( t \in [0, T] \), \( m \in \mathcal{P}^1(\mathbb{T}^d) \), \( \phi \in V(t, m) \),

\[ D_{c}^F(t, m, \phi) \neq \emptyset, \]

Theorems 1, 2 immediately implies the following.

**Corollary 1.** Let the upper semicontinuous multifunction \( V : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d) \) have nonempty values. Assume that, for any \( t \in [0, T] \), \( m \in \mathcal{P}^1(\mathbb{T}^d) \), \( \phi \in V(t, m) \),

- \( V(t, m) \subset \text{BL}_{M,C}(\mathbb{T}^d) \) where the constants \( M \) and \( C \) do not dependent on \( s \) and \( m \);
- \( V(T, m) = \{ \sigma(\cdot, m) \} \);
- \( D_{c}^F(t, m, \phi) \neq \emptyset \), where the constant \( c \) does not depend on \( t, m \) and \( \phi \).

Then \( V \) is a value multifunction of mean field game (1), (2).

**6 Properties of the “frozen” dynamics**

In this Section we present auxiliary statements those are used in the proof of Theorem 2.

For each \( \tau \geq 0 \), let \( \Xi^\tau \) be an operator from \( \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^{d+1} \) to \( \mathbb{T}^d \times \mathbb{R} \) defined by the rule

\[ \Xi^\tau(w, v) \triangleq w + \tau v. \]
Lemma 3. If \( \nu, \nu' \in P^1(T^d) \), \( \gamma \in \mathcal{L}_*(\nu) \), \( \tau, \tau' \geq 0 \), \( \pi^0 \in \Pi(\nu', \nu) \) is an optimal plan between \( \nu' \) and \( \nu \), then

1. \( W_1(\Xi^\tau \# \gamma, \Xi^{\tau'} \# \gamma) \leq |\tau - \tau'| \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \|v\| \gamma(d(w, v)) \); 

2. \( W_1(\Xi^\tau \# \gamma, \Xi^{\tau} (\pi^0 \# \gamma)) \leq W_1(\nu, \nu') \).

Proof. First, we have that

\[
W_1(\Xi^\tau \# \gamma, \Xi^{\tau'} \# \gamma) = \sup_{\varphi \in \text{Lip}_1(T^d \times \mathbb{R})} \left\{ \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \left[ \varphi(w + \tau v) - \varphi(w + \tau' v) \right] \gamma(d(w, v)) \right\}
\]

\[
\leq |\tau - \tau'| \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \|v\| \gamma(d(w, v)).
\]

This proves the first statement of the Lemma.

Further, we have that

\[
W_1(\Xi^\tau \# \gamma, \Xi^{\tau} (\pi^0 \# \gamma)) = \sup_{\varphi \in \text{Lip}_1(T^d \times \mathbb{R})} \left\{ \int_{(T^d \times \mathbb{R})^2} \int_{\mathbb{R}^{d+1}} \left[ \varphi(w' + \tau v) - \varphi(w + \tau v) \right] \gamma(dw|w) \pi^0(d(w', w)) \right\}
\]

\[
\leq \int_{(T^d \times \mathbb{R})^2} \|w' - w\| \pi^0(d(w', w)).
\]

This inequality yields the second statement of the Lemma.

The following statements are concerned with the continuity of the operator \( A \).

Lemma 4. If \( m \in P^1(T^d) \), \( s, r \in [0, T] \), \( s < r \), \( K > 0 \), \( \psi \in \text{Lip}_K(T^d) \), then \( A^{s,r}_m \psi \in \text{Lip}_{(K+1)e^{L(r-s)-1}}(T^d) \).

The lemma is proved in the same way as the first statement of Lemma 3.

Lemma 5. If \( m, m' \in P^1(T^d) \), \( K > 0 \), \( \psi, \psi' \in \text{Lip}_K(T^d) \), \( s, s', r \in [0, T] \), \( s < r \), \( s' < r \), then

\[
\|A^{s,r}_m \psi - A^{s',r}_m \psi'\| \leq \|\psi - \psi'\|
\]

\[
+ (K + 1)[L(r - \hat{s})W_1(m, m') + R|s - s'| + \alpha(s - s') \cdot (r - \hat{s})],
\]

where \( \hat{s} \triangleq \inf\{s, s'\} \).

Proof. First, recall that

\[
F(t, x, m) = \left\{ \int_U (f(t, x, m, u), g(t, x, m, u)) \xi(du) : \xi \in \text{rpm}(U) \right\}. \tag{28}
\]
Without loss of generality we can assume that \(s \leq s'\). Thus, we have that
\[
(A_{m}^{s,r} \psi)(x) = \sup_{\xi \in \text{rpm}(U)} \int_{U} (\psi(x + (r - s)f(s, x, m, u)) + (r - s)g(s, x, m, u))\xi(du)
\]
\[
\leq \sup_{\xi \in \text{rpm}(U)} \int_{U} (\psi(x + (r - s')f(s', x, m', u)) + (r - s')g(s', x, m', u))\xi(du)
\]
\[
+ (K+1)|L(r - s)W_1(m, m') + R|s - s'| + \alpha(s - s') \cdot (r - s)
\]
\[
= (A_{m}^{s',r} \psi')(x) + \|\psi - \psi'\|
\]
\[
+ (K+1)|L(r - s)W_1(m, m') + R|s - s'| + \alpha(s - s') \cdot (r - s).
\]
This proves the Lemma.

The following lemma provides the comparison of the original and “frozen” dynamics.

**Lemma 6.** Let \(m(\cdot)\) be a flow of probabilities on \([s, r]\), \(m_* \in \mathcal{P}^1(\mathbb{T}^d)\), and let \(\psi, \psi' \in \text{Lip}_K(\mathbb{T}^d)\). Assume that, for all \(t \in [s, r]\), \(W_1(m(t), m_*) \leq \delta_1\), then
\[
\|A_{m}^{s,r} \psi - B_{m}^{s,r} \psi'\| \leq (K+1)(\alpha(r - s) + LR(r - s) + L\delta_1)(r - s) + \|\psi - \psi'\|.
\]

**Proof.** First, by Lemma 5 we conclude that
\[
\|A_{m}^{s,r} \psi - A_{m}^{s',r} \psi'\| \leq \|\psi - \psi'\|. \tag{29}
\]

Let \(y \in \mathbb{T}^d\) and let \((a^*, b^*) \in F(s, y, m_*)\) be such that
\[
(A_{m}^{s',r} \psi')(y) = \psi'(y + (r - s)a^*) + (r - s)b^*.
\]

Using (29), we obtain that there exists a probability \(\xi \in \mathcal{P}(U)\) such that
\[
(a^*, b^*) = \int_{U} (f(t, x, m_*, u), g(t, x, m_*, u))\xi(du).
\]

Consider the motion \((x(\cdot), z(\cdot))\) satisfying
\[
\dot{x}(t) = \int_{U} f(t, x(t), m(t), u)\xi(du), \quad \dot{z}(t) = \int_{U} g(t, x(t), m(t), u)\xi(du),
\]
and \(x(s) = y, \ z(s) = 0\).

We have that \(\|x(t) - y\| \leq R(t - s)\). Thus, for every \(t \in [s, r]\),
\[
\|\dot{x}(t) - a^*\| \leq \alpha(t - s) + LR(t - s) + L\delta_1,
\]
\[
\|\dot{z}(t) - b^*\| \leq \alpha(t - s) + LR(t - s) + L\delta_1.
\]
Therefore,
\[\|x(r) - y - (r - s)a^*\| \leq (\alpha(r - s) + LR(r - s) + L\delta_1)(r - s),\]
\[|z(r) - (r - s)b^*| \leq (\alpha(r - s) + LR(r - s) + L\delta_1)(r - s).\]
This yields the inequality
\[A^{s,r}_{m,t} \psi' = \psi'(y + (r - s)a^*) + (r - s)b^* \leq \psi'(x(r)) + z(r) + (K + 1)(\alpha(r - s) + LR(r - s) + L\delta_1)(r - s) \leq B^{s,r}_{m,c} \psi'(K + 1)(\alpha(r - s) + LR(r - s) + L\delta_1)(r - s)\]
(30)
The opposite inequality
\[B^{s,r}_{m,c} \psi' \leq A^{s,r}_{m,t} \psi' + (K + 1)(\alpha(r - s) + LR(r - s) + L\delta_1)(r - s)\]
is proved in the same way. This, (29) and (30) imply the conclusion of the Lemma. 

7 Proof of the viability theorem. Sufficiency

In this section we assume that the upper semicontinuous multifunction \( V : [0, T] \times \mathcal{P}^1(T^d) \to C(T^d) \) has nonempty values, is bounded and, for any \( t \in [0, T] \), \( m \in \mathcal{P}^1(T^d), \phi \in V(t, m) \), the following properties holds true
\[ \phi \in BL_{M,C}(T^d), \mathcal{D}_{\phi} V(t, m, \phi) \neq \emptyset. \]
Here \( c, M \) and \( C \) are constants that do not dependent on \( s, m \) and \( \phi \).

We shall prove that \( V \) is viable with respect to the mean field game dynamics. The proof is a modification of the proof of the classic viability theorem presented in (2).

Denote by \( Z \) the set of triples \( (t, \nu, \phi) \in [0, T] \times \mathcal{P}^1(T^d \times \mathbb{R}) \times BL_{M,C}(T^d) \) such that \( \text{supp}(\nu) \subset T^d \times [-c, c] \) and \( \phi \in V(t, \mathcal{P}_#\nu) \). Further, for \( n \in \mathbb{N} \), set \( Z_n \triangleq Z \cap ([0, T - 1/n] \times \mathcal{P}^1(T^d \times \mathbb{R}) \times BL_{M,C}(T^d)) \). Clearly, the sets \( Z \) and \( Z_n \) are compacts.

**Lemma 7.** There exists a number \( \theta_n \in (0, 1/n) \) such that, for any \( (t, \nu, \phi) \in Z_n \), one can find \( (t^+, \nu^+, \phi^+) \in Z \) and \( \gamma \in L^s_*(\nu) \) such that the following properties holds true for \( m = \mathcal{P}_#\nu: \)

1. \( W_1(\Xi^{t^+-t}_#\gamma, \nu^+) < (t^+ - t)/n; \)
2. \( \|A^t_{m^+} \phi^+ - \phi\| < (t^+ - t)/n; \)
3. \( [\phi^+, \nu^+] > [\phi, \nu] - (t^+ - t)/n; \)
4. \( \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist} \left( v, \hat{F}(t, w, \nu) \right) \gamma(d(z, v)) < 1/n. \)
Here $\hat{F}$ defined by (13).

**Proof.** Let $(s, \eta, \psi) \in \mathcal{Z}_n$. Denote $\mu_\eta \triangleq p_\eta \eta$. Since $D^c_\beta(s, \mu_\eta, \psi) \neq \emptyset$ and the mapping $\beta \mapsto \int_{T^d \times R \times \mathbb{R}^{d+1}} \text{dist}(v, F(t, x, m))\beta(d(x, v))$ is continuous, there exist $\partial_{s, \eta, \psi} \in (0, 1/n)$, $\omega_{s, \eta, \psi} \in \mathcal{L}_c^2(\mu_\eta)$, $\eta_{s, \eta, \psi}^+ \in \mathbb{T}^d \times \mathbb{R}$ and $\psi_{s, \eta, \psi}^+ \in \text{Lip}_c(T^d)$ such that

- $\eta_{s, \eta, \psi}^+ = \Theta^{2\partial_{s, \eta, \psi}\#\omega_{s, \eta, \psi}}$;
- $\psi_{s, \eta, \psi}^+ \in \mathcal{V}(s + 2\partial_{s, \eta, \psi}, p_\#\eta_{s, \eta, \psi}^+)$;
- $|A_{\mu_\eta}^{s, s+2\partial_{s, \eta, \psi}^+} - \psi| \leq \partial_{s, \eta, \psi}/n$;
- $[\psi_{s, \eta, \psi}^+, \eta_{s, \eta, \psi}^+] > [\psi, \mu_\eta] - \partial_{s, \eta, \psi}/n$;
- $\int_{T^d \times \mathbb{R}^{d+1}} \text{dist}(v, F(t, x, m))\omega_{s, \eta, \psi}(d(x, v)) < 1/(2n)$.

Now define the probability $\zeta_{s, \eta, \psi} \in \mathcal{P}(T^d \times \mathbb{R} \times \mathbb{R}^{d+1})$ in the following way. If $\varphi \in C_b(T^d \times \mathbb{R} \times \mathbb{R}^{d+1})$,

$$
\int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \varphi(x, z, a, b)\zeta_{s, \eta, \psi}(d(x, z, a, b))
$$

$$
\triangleq \int_{T^d \times \mathbb{R}} \int_{\mathbb{R}^{d+1}} \varphi(x, z, a, b)\omega_{s, \eta, \psi}(d(a, b)|x)\eta(dz|x)\mu_\eta(dx).
$$

Clearly, the projection of $\zeta_{s, \eta, \psi}$ on $T^d \times \mathbb{R}^{d+1}$ is $\omega_{s, \eta, \psi}$, when its projection on $T^d \times \mathbb{R}$ is $\eta$. Thus, $\zeta_{s, \eta, \psi} \in \mathcal{L}_c^2(\eta)$. Let $\eta_{s, \eta, \psi}^+ \triangleq \Sigma^{2\partial_{s, \eta, \psi}^\#}\zeta_{s, \eta, \psi}$. We have that $p_\#\eta_{s, \eta, \psi} = p_\#\eta_{s, \eta, \psi}^+$ and

$$
\left[\psi_{s, \eta, \psi}^+, \eta_{s, \eta, \psi}^+\right] = \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} (\psi_{s, \eta, \psi}^+(x) + z)\eta_{s, \eta, \psi}^+(d(x, z))
$$

$$
= \int_{T^d} \int_{\mathbb{R}} \int_{\mathbb{R}^{d+1}} (\psi_{s, \eta, \psi}^+(x + 2\partial_{s, \eta, \psi}^+a) + z + 2\partial_{s, \eta, \psi}^+b)\omega_{s, \eta, \psi}(d(a, b)|x)\eta(dz|x)\mu_\eta(dx)
$$

$$
= \int_{T^d} \int_{\mathbb{R}^{d+1}} (\psi_{s, \eta, \psi}^+(x + 2\partial_{s, \eta, \psi}^+a) + 2\partial_{s, \eta, \psi}^+b)\omega_{s, \eta, \psi}(d(a, b)|x)\mu_\eta(dx)
$$

$$
+ \int_{T^d} \int_{\mathbb{R}} \eta(dz|x)\mu_\eta(dx)
$$

$$
= \left[\psi_{s, \eta, \psi}^+, \eta_{s, \eta, \psi}^+\right] + \int_{T^d \times \mathbb{R}} \eta(d(x, z)).
$$

Analogously,

$$
\left[\psi, \eta\right] = \left[\psi, \mu_\eta\right] + \int_{T^d \times \mathbb{R}} \eta(d(x, z)).
$$

Finally, let $s_{s, \eta, \psi}^+ \triangleq s + 2\partial_{s, \eta, \psi}^+$.

One can summarize the properties of $(s_{s, \eta, \psi}^+, \eta_{s, \eta, \psi}^+, \psi_{s, \eta, \psi}^+)$ and $\zeta_{s, \eta, \psi}$ as follows.

- $(s_{s, \eta, \psi}^+, \eta_{s, \eta, \psi}^+, \psi_{s, \eta, \psi}^+) \in \mathcal{Z}$;
\[
\eta^{+}_{s,\eta,\psi} = \Xi^{s,\eta,\psi}_{s,\eta,\psi};
\]
\[
\psi^{+}_{s,\eta,\psi} \in \mathcal{Y}(s^{+}_{s,\eta,\psi}, p_{#s,\eta,\psi});
\]
\[
|A^{s,\eta,\psi}_{s,\eta,\psi} \psi_{s,\eta,\psi} - \psi| < (s^{+}_{s,\eta,\psi} - s)/(2n);
\]
\[
|\psi^{+}_{s,\eta,\psi}; \eta^{+}_{s,\eta,\psi}] > [\psi, \eta] - (s^{+}_{s,\eta,\psi} - s)/(2n);
\]
\[
\int_{T^{d} \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \widehat{F}(t, w, \nu))\gamma(d(w, v)) < 1/(2n).
\]

Let \( \mathcal{E}_{s,\eta,\psi} \) be a set of triples \((t, \nu, \phi) \in \mathcal{Z}_{n}\) such that, for some \( \gamma \in \mathcal{L}_{\ast}(m) \), the following properties hold true:

\(\text{(E1)}\) \((t) \in (s - \eta, s, s + \eta, \psi);\)

\(\text{(E2)}\) \(W_{1}(\eta^{+}_{s,\eta,\psi}, \Xi^{s,\eta,\psi}_{s,\eta,\psi}) = (s^{+}_{s,\eta,\psi} - t)/n;\)

\(\text{(E3)}\) \(\|A^{t,\eta,\phi}_{s,\eta,\psi} \psi_{s,\eta,\psi} - \phi\| < (s^{+}_{s,\eta,\psi} - t)/n;\)

\(\text{(E4)}\) \([\psi^{+}_{s,\eta,\psi}, \eta^{+}_{s,\eta,\psi}] > [\phi, \nu] - (s^{+}_{s,\eta,\psi} - t)/n;\)

\(\text{(E5)}\) \[
\int_{T^{d} \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \widehat{F}(t, w, \nu))\gamma(d(w, v)) < 1/n.
\]

Now, let us prove that each set \( \mathcal{E}_{s,\eta,\psi} \) is open in \( \mathcal{Z}_{n} \). Let \((t, \nu, \phi) \in \mathcal{E}_{s,\eta,\psi}^{n}.\) There exists \( \gamma_{s} \in \mathcal{L}_{\ast}(\nu)\) such that conditions (E1)–(E5) are fulfilled for \((t, \nu, \phi)\) and \( \gamma_{s}.\)

Further, let \( \delta > 0,\) and let \((t, \nu, \phi) \in \mathcal{Z}_{n}\) be such that \|t - s\| < \delta, \( W_{1}(\nu, \nu) < \delta,\) \( \|\phi - \phi\| < \delta.\) We shall show that, if \( \delta \) is sufficiently small, then conditions (E1)–(E5) hold true for \((t, \nu, \phi)\) if \( \gamma = \pi_{\nu, \nu} \ast \gamma_{s}.\) Here \( \pi_{\nu, \nu} \) stands for an optimal plan between \( \nu \) and \( \nu_{\ast},\) while \( \ast \) denotes the composition of probabilities introduced by (3). This will imply that \( \mathcal{E}_{s,\eta,\psi} \) is open.

First, notice that, if \( \gamma_{s} \in \mathcal{L}_{\ast}(\nu_{s}),\) then \( \gamma = \pi_{\nu, \nu} \ast \gamma_{s}\) belongs to \( \mathcal{L}_{\ast}(\nu).\) Further, the condition (H1) holds for \((t, \nu, \phi)\) if \( \delta < \theta_{s,\eta,\psi} - |t - s|.\) Since \( \gamma_{s} \in \mathcal{L}_{\ast}(\nu_{s}),\) by Lemma 3 we have that

\[
|W_{1}(\Xi^{s,\eta,\psi}_{s,\eta,\psi, - t, \#\gamma, \Xi^{s,\eta,\psi}_{s,\eta,\psi, - t, \#\gamma_{s}}})| \leq 2c\delta.
\]

Therefore, condition (E2) is valid for \((t, \nu, \phi),\)

\[
(2c + 1/n)\delta < (s^{+}_{s,\eta,\psi} - t)/(2n) - W_{1}(\eta^{+}_{s,\eta,\psi}, \Xi^{s,\eta,\psi}_{s,\eta,\psi, - t, \#\gamma_{s}}).
\]

Analogously, by Lemma 3

\[
\|A^{t,\eta,\phi}_{s,\eta,\psi} \psi_{s,\eta,\psi} - \phi\| - \|A^{t,\phi}_{s,\eta,\psi} \psi_{s,\eta,\psi} - \phi_{\ast}\| \leq \delta + (C + 1)[L3d_{s,\eta,\psi} \delta + R\delta + 3d_{s,\eta,\psi} \alpha(\delta)].
\]
Therefore, in the case when
\[
(1 + L3\partial s_{\nu,\phi}(C + 1) + R + 1/n)\delta + 3\partial s_{\nu,\phi}(C + 1)\alpha(\delta)
\]
\[
< (s_{\nu,\phi}^+ - t_*)/n - \|A_{\nu,\phi}^{s_{\nu,\phi}^+ - \phi,\nu,\phi}^+ - \phi_*\|
\]
condition (E3) holds for \((t, \nu, \phi)\).

By (3) we have that \([|\phi_* - \nu_\delta|] - \|\phi, \nu\| \leq (C+1)\delta\). Therefore, condition (E4) holds true for \((t, \nu, \phi)\), if \((C + 1 + 1/n)\delta < [\psi_{s_{\nu,\phi}^+}, \eta_{s_{\nu,\phi}^+}] - [\phi_* - \nu_\delta] + (s_{\nu,\phi}^+ - t_*)/n\).

Finally, notice that \(|\text{dist}(v, \tilde{F}(t, w, \nu)) - \text{dist}(v, \tilde{F}(t, w, \nu_*))| \leq LW_1(\nu, \nu_*)\) and the function \((w, v) \mapsto \text{dist}(v, \tilde{F}(t, w, \nu))\) is \((L + 1)\)-Lipschitz continuous. Thus, by the second statement of Lemma 3, we get
\[
\left| \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \tilde{F}(t, w, \nu_*))\gamma_*(d(w, v)) - \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \tilde{F}(t, w, \nu))\gamma(d(w, v)) \right| \leq (2L + 1)\delta.
\]

Therefore, if one pick \(\delta\) less than
\[
\frac{1}{n} - \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \tilde{F}(t, w, \nu_*))\gamma_*(d(w, v)),
\]
then condition (E5) is valid for \((t, \nu, \phi)\) and \(\gamma\).

We have proved that if \((t_*, \nu_*, \phi_*) \in \mathcal{E}_{s,\nu,\phi}\), then its \(\delta\)-neighborhood in \(Z_n\) also lies in \(\mathcal{E}_{s,\nu,\phi}\) for sufficiently small \(\delta\). Furthermore, by construction \((s, \eta, \psi) \in \mathcal{E}_{s,\nu,\phi}\).

Thus, \(\{\mathcal{E}_{s,\nu,\phi}\}_{(s,\nu,\phi) \in Z_n}\) is an open cover of \(Z_n\). Since \(Z_n\) is compact, there exists a finite number of triples \(\{(s_i, \eta_i, \psi_i)\}_{i=1}^n\) such that
\[
Z_n \subset \bigcup_{i=1}^n \mathcal{E}_{s_i,\eta_i,\psi_i}.
\]
Put
\[
\theta_n \triangleq \min_{i=1, T^d} \partial(s_{\nu,\phi}^+).
\]
If \((t, \nu, \phi) \in Z_n\), then there exists \(i = 1, T^d\) such that
\[
(t, \nu, \phi) \in \mathcal{E}_{s_i,\eta_i,\psi_i}.
\]
Choose \(t^+ \triangleq s_{\nu,\phi}^+, \nu^+ \triangleq \eta_{\nu,\phi}^+, \phi^+ \triangleq \psi_{\nu,\phi}^+\). Finally, let \(\gamma\) be such that conditions (E1)–(E5) are fulfilled for \((t, \nu, \phi)\) and \(\gamma\).

For \(\tau, \theta \in [0, T]\), \(\tau < \theta\), define the mapping \(\Lambda^{\tau, \theta} : T^d \times \mathbb{R} \times \mathbb{R}^{d+1} \to C_{\tau, \theta}\) by the rule: if \(w \in T^d \times \mathbb{R}, v \in \mathbb{R}^{d+1}\), then put \(\Lambda^{\tau, \theta}(w, v)\) be equal to the function \(t \mapsto (w + (t - \tau)v)\).

Assume that \(s, r \in [0, T]\), \(s < r, m_s \in \mathcal{P}^1(T^d), \phi_s \in \mathcal{V}(s, m_s)\). Put \(r_n \triangleq r - 1/n\). Without loss of generality we assume that \(r_n > s\). Now, we construct a number \(J_n\), sequence of times \(\{t_j^{(n)}\}_{j=0}^{J_n}\), sequences of probabilities \(\{\nu_j^{(n)}\}_{j=0}^{J_n}, \{\eta_j^{(n)}\}_{j=0}^{J_n}, \{\gamma_j^{(n)}\}_{j=1}^{J_n}, \{\chi_j^{(n)}\}_{j=1}^{J_n}\) and sequence of functions \(\{\phi_j^{(n)}\}_{j=0}^{J_n}\) by the following rules.
Proof. First, we have that when $j \vdash e \bar{\gamma}_n \{ \chi_n \}$ we have that $\eta_n \equiv \nu_n \equiv \phi_n \equiv \phi_n$. If $t_n^{j-1} = \tau_n^j \{ \phi_n \}$ are already constructed and $t_n^{j-1} \leq \tau_n$, then apply Lemma 7 with $\tau = t_n^{j-1}$, $\nu = \nu_n^{j-1}$, $\phi = \phi_n^{j-1}$ and choose $t_n^j$ to be equal to $t^+$, $\nu_n^j$ be equal to $\nu^+$, $\phi_n^j$ be equal to $\phi^+$, $\gamma_n^j$ be equal to $\gamma$. Note that $\phi_n^j \in \mathcal{V}(t_n^{j-1}, p \# \nu_n^{j-1})$, $\gamma_n^j \in \mathcal{C}_n^s(\nu_n^{j-1})$. Let $\mathcal{P}_n^j$ be an optimal plan between $\eta_n^{j-1}$ and $\nu_n^{j-1}$. Set $\bar{\chi}_n^{j} \equiv \bar{\gamma}_n^{j-1} \# \gamma_n^{j}$. Obviously, $\bar{\gamma}_n^j \in \mathcal{C}_n^s(\eta_n^{j-1})$. Define the probability $\chi_n^j \equiv \mathcal{P}_n^j(\mathcal{C}_n^{j-1, j})$ by the rule:

$$\chi_n^j \equiv \Lambda^{t_n^{j-1}, t_n^j} \# \bar{\gamma}_n^j$$  \hspace{1cm} (31)

Finally, set $\eta_n^j \equiv \hat{e}_{t_n^j} \# \chi_n^j = \Xi_n^{t_n^j - t_n^{j-1}} \# \bar{\gamma}_n^j$.

If $t_n^j > \tau_n$, then choose $J_n$ to be equal to $j$.

Notice that $t_n^{j+1} - t_n^j \geq \tau_n$. Thus, $J_n$ is finite.

Let $\gamma_n^{j+1} \in \mathcal{C}_n^s(\nu_n^{j+1})$ be such that

$$\int_{\mathcal{T}^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \text{dist}(v, \hat{F}(t_n^{j+1}, w, \nu_n^{j+1})) \gamma_n^{j+1}(d(x, v)) = 0.$$  \hspace{1cm} (32)

We have that $\phi_n^{j+1} \in \mathcal{V}(t_n^j, p \# \nu_n^j)$.

Put

$\chi_n^j \equiv \chi_n^0 \circ \ldots \chi_n^{j-1} \# \chi_n^{j+1}$.  \hspace{1cm} (33)

Here $\circ$ stands for concatenation of probabilities (see [6]). Since $\hat{e}_{t_n^j} \# \chi_n^j = \hat{e}_{t_n^j} \# \chi_n^{j+1}$ when $j = 1, \ldots, J_n$, the probability $\chi_n$ is well-defined. Moreover, $\chi_n \in \mathcal{P}_n^j(\mathcal{C}_n^s)$, $\hat{e}_{t_n^j} \# \chi_n = \eta_n^j$.

Let us point out the properties of the constructed sequences.

**Lemma 8.** For $j = 0, J_n$,

$$W_1(\eta_n^j, \nu_n^j) \leq (t_n^j - \tau_n)/n.$$  \hspace{1cm} \text{Proof.} First, we have that $W_1(\eta_n^0, \nu_n^0) = 0$. Further, notice that $\eta_n^j = \Xi_n^{t_n^j - t_n^{j-1}} \# (\nu_n^{j-1} \# \gamma_n^j)$. Thus, if $W_1(\eta_n^{j-1}, \nu_n^{j-1}) \leq (t_n^{j-1} - \tau_n)/n$, then, using the second statement of Lemma 3 and Lemma 4, we conclude that

$$W_1(\eta_n^j, \nu_n^j) = W_1(\Xi_n^{t_n^j - t_n^{j-1}} \# (\nu_n^{j-1} \# \gamma_n^j), \nu_n^j) \leq W_1(\Xi_n^{t_n^j - t_n^{j-1}} \# (\nu_n^{j-1} \# \gamma_n^j), \Xi_n^{t_n^j - t_n^{j-1}} \# \gamma_n^j) + W_1(\Xi_n^{t_n^j - t_n^{j-1}} \# \gamma_n^j, \nu_n^j) \leq (t_n^{j-1} - \tau_n)/n + (t_n^j - t_n^{j-1})/n.$$ \hfill \blacksquare

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Let \( q_n(\cdot) \in \mathcal{M}_{s,r} \) be defined by the rule: \( q_n(t) \triangleq e_{t+\chi_n} \). Notice that
\[
q_n(t^n_j) = p_{n}^\# y^{(j)}_n. \tag{34}
\]

**Lemma 9.** There exist constants \( c_1 \) and \( c_2 \) such that the following estimate holds true:
\[
\|B_{q_n(\cdot)}^{n_j} \phi_n^J - \phi_*\| \leq c_1 \alpha(1/n) + c_2/n.
\]

**Proof.** To simplify designations put \( m_j^{\#} \triangleq p_{n}^\# v^{(j)}_n \).

Let us show that
\[
\|A_{m_j^{\#}}^{t^n_j - 1} t_{m_j^{\#}}^{n_j + 1} \phi_n^J - \phi_n^J\| \leq (t^n_j - t^n_{j-1})/n. \tag{35}
\]

We prove this inequality by the backward induction on \( j \). By Lemma 7 we have that
\[
\|A_{m_j^{\#}}^{t^n_j - 1} t_{m_j^{\#}}^{n_j + 1} \phi_n^J - \phi_n^{J-1}\| \leq (t^n_j - t^n_{j-1})/n.
\]

Now, assume that that inequality (35) holds true for some \( j \). We have that
\[
\|A_{m_j^{\#}}^{t^n_j - 1} t_{m_j^{\#}}^{n_j + 1} \phi_n^J - \phi_n^{J-1}\| \leq (t^n_j - t^n_{j-1})/n.
\]

This, Lemma 5 and assumption (35) imply the inequality:
\[
\|A_{m_j^{\#}}^{t^n_j - 1} t_{m_j^{\#}}^{n_j + 1} \phi_n^J - \phi_n^{J-1}\| \leq (t^n_j - t^n_{j-1})/n + (t^n_j - t^n_{j-1})/n.
\]

Hence (35) is fulfilled for any \( j = 0, \ldots, J_n - 1 \).

Definition of \( q_n(\cdot) \), (34) and Lemma 8 give, for each \( j \), the estimate
\[
\sup_{t \in [t^n_j, t^n_{j+1}]} W_1(q_n(t), m_j^{\#}) \leq (T + c)/n. \tag{36}
\]

Further, put
\[
C^*(s) \triangleq [(C + 1)e^{L(T-s)} - 1].
\]

Now, let us prove that, for any \( j = 0, \ldots, J_n - 1 \),
\[
\|B_{q_n(\cdot)}^{t^n_j - 1} A_{m_j^{\#}}^{t^n_j - 1} \phi_n^J - \phi_n^{J-1}\| \leq (C^*(t^n_{j+1}) + 1)(\alpha(1/n) + L(R + T + c)/n + T/n)(t^n_j - t^n_{j-1}). \tag{37}
\]

We prove this inequality by the backward induction. First, since \( C(t^n_j) = C \), by Lemma 6 and estimate (35) we have that
\[
\|B_{q_n(\cdot)}^{t^n_j - 1} A_{m_j^{\#}}^{t^n_j - 1} \phi_n^J - \phi_n^J\| \leq (C^*(t^n_{j+1}) + 1)(\alpha(1/n) + L(R + T + c)/n)(t^n_j - t^n_{j-1}).
\]
Further, assume that \((37)\) is fulfilled for some \(j\). We have that
\[
B_{q_n(t)}^{t_{m_n-1}} \phi_n^{J_n} = B_{q_n(t)}^{t_{m_n}} B_{q_n(t)}^{t_{m_n-1}} \phi_n^{J_n}.
\]
Additionally, by Lemmas \(\text{(1)}\) and \(\text{(4)}\) we have that
\[
B_{q_n(t)}^{t_{m_n} \cdot t_n^{J_n}} \phi_n^{J_n}, A_{m_n}^{t_{m_n} \cdot t_n^{J_n+1}} \ldots A_{m_n}^{t_{m_n-1} \cdot t_n^{J_n}} \phi_n^{J_n} \in \text{Lip}_{C^*(t_n)}(T^d).
\]
This and Lemma \(\text{(6)}\) imply that
\[
B_{q_n(t)}^{t_{m_n} \cdot t_n^{J_n}} \phi_n^{J_n} - A_{m_n}^{t_{m_n} \cdot t_n^{J_n}} A_{m_n}^{t_{m_n-1} \cdot t_n^{J_n+1}} \ldots A_{m_n}^{t_{m_n-1} \cdot t_n^{J_n}} \phi_n^{J_n}
\leq (C^*(t_n^0) + 1)(\alpha(1/n) + L(R + T + c)/n + T/n)(t_n^1 - t_n^{J_n-1})
+ (C^*(t_n^1) + 1)(\alpha(1/n) + L(R + T + c)/n + T/n)(t_n^{J_n} - t_n^1).
\]
Since the function \(s \mapsto C^*(s)\) decrease, we conclude that \((37)\) is fulfilled for all \(j\).
Since \(t_n^0 = s\), combining \((35)\) and \((37)\), we get
\[
\|B_{q_n(t)}^{t_{m_n} \cdot t_n^{J_n}} \phi_n^{J_n} - \phi_s\| \leq (C^*(0) + 1)(\alpha(1/n) + (LR + LT + Lc + T)/n).
\]
Finally, using Lemma \(\text{(2)}\) we have that
\[
\|B_{q_n(t)}^{t_{m_n} \cdot t_n^{J_n}} \phi_n^{J_n} - B_{q_n(t)}^{t_{m_n} \cdot t_n^{J_n}} \phi_n^{J_n}\| \leq (C + 1)R(r - t_n^{J_n}) \leq (C + 1)R/n.
\]
This and inequality \((38)\) yield the conclusion of the lemma. \(\square\)

**Lemma 10.** The following estimate holds true, for some constant \(c_3 > 0\):
\[
[\phi_n^{J_n}, \eta_n^{J_n}] > [\phi_s, \hat{m}_s] - c_3/n.
\]

**Proof.** First, notice that by construction
\[
[\phi_n^{J_n}, \nu_n^{J_n}] > [\phi_n^{J_n-1}, \nu_n^{J_n-1}] - (t_n^J - t_n^{J_n-1})/n.
\]
Therefore, since \(\phi_n^{J_n} = \phi_s, \nu_n^{J_n} = \hat{m}_s, s = t_n^0, r > t_n^{J_n}\), we get
\[
[\phi_n^{J_n}, \nu_n^{J_n}] > [\phi_s, \hat{m}_s] - (r - s)/n.
\]
Recall that \(\phi_n^{J_n} \in \text{Lip}_{C^*}(T^d)\). Hence, using \((5)\) and Lemma \(\text{(8)}\) we conclude that
\[
[\phi_n^{J_n}, \eta_n^{J_n}] > [\phi_s, \hat{m}_s] - (C + 1)(r - s)/n.
\]
\(\square\)

**Proof of Theorem 2.** Sufficiency. We will prove that there exists \((\mu, \psi)\) such that \(\psi \in \mathcal{V}(r, \mu)\) and \((\mu, \psi) \in \Psi^{x, s}(m_s, \phi_s)\). By construction we have that, for each \(n\), \(\text{supp}(\chi_n)\) lies in the compact set consisting of absolutely continuous trajectories \((x(\cdot), z(\cdot)) \in C_{s,r}\) such that \(\|\dot{x}(t)\|, |\dot{z}(t)| \leq c\) for a.e. \(t \in [s, r]\). Additionally, elements of \(\text{supp}(\chi_n)\) are uniformly bounded. Thus, by \([2\text{, Proposition 7.1.5}]\) we
conclude that \( \{ \chi_n \} \) is relatively compact. This means that there exists a sequence \( \{ n_k \} \) and a probability \( \chi \in \mathcal{C}_{0,T} \) such that

\[
W_1(\chi_{n_k}, \chi) \to 0 \quad \text{as} \quad k \to \infty.
\]  

(39)

Further, put

\[
\nu(t) \triangleq \hat{e}_{t\#} \chi, \quad m(t) \triangleq p_{\#} \nu(t), \quad \mu \triangleq m(r).
\]

Recall that \( \{ \phi_{n_k}^{J_n} \} \subset \operatorname{Lip}_C(\mathbb{T}^d) \) and \( \| \phi_{n_k}^{J_n} \| \leq M \). Thus, \( \{ \phi_{n_k}^{J_n} \} \) is relatively compact. We can assume without loss of generality that the sequence \( \{ \phi_{n_k}^{J_n} \} \) converges to a function \( \psi \in \operatorname{Lip}_C(\mathbb{T}^d) \). By (33) we have that

\[
W_1(m(t), q_{n_k}(t)) \leq W_1(\chi_{n_k}, \chi).
\]

(40)

Additionally, \( r - t_{n_k}^{J_n} \leq 1/n, W_1(q_{n_k}(t'), q_{n_k}(t'')) \leq cl(t' - t''), W_1(q_{n_k}(t_{n_k}^{J_n}), p_{\#} \nu_{n_k}^{J_n}) \leq W_1(\eta_{n_k}^{J_n}, \nu_{n_k}^{J_n}) \leq (r - s)/n \). Since \( \phi_{n_k}^{J_n} \in \mathcal{V}(t_{n_k}^{J_n}, p_{\#} \nu_{n_k}^{J_n}) \) and \( \mathcal{V} \) is upper semicontinuous, we have that

\[
\psi \in \mathcal{V}(r, \mu).
\]

(41)

To show that \( (\mu, \psi) \in \Psi^{r,s}(m_\ast, \phi_n) \) let us check that \( \nu(\cdot) \) solves (12) and \( \nu(\cdot), \psi \) satisfy conditions \( (\Psi_1) \) and \( (\Psi_3) \).

To prove that \( \nu(\cdot) \) is a solution of (12) pick \( \tau_0, \tau_1 \in [s, r] \). There exist numbers \( I_n^0 \) and \( I_n^1 \), such that \( \tau_0 \in [t_{n}^{I_n^0 - 1}, t_{n}^{I_n^0}] \), \( \tau_1 \in [t_{n}^{I_n^1 - 1}, t_{n}^{I_n^1}] \). Without loss of generality we can assume that \( I_n^0 < I_n^1 \). Put \( h_n^{I_n^0 - 1} \triangleq \tau_0 \). For \( k = I_n^0, \ldots, I_n^1 - 1 \), put \( h_n^k \triangleq t_n^k - t_n^{k - 1} \). Set \( h_n^0 \triangleq \tau_1 \). Further, denote \( \delta_n^k \triangleq h_n^k - h_n^{k - 1} \). Additionally, let \( \varpi_n(t) \triangleq \hat{e}_{t\#} \chi_n \). Notice that \( q_n(t) = p_{\#} \varpi_n(t) \). For any \( w(\cdot) \in \operatorname{supp}(\chi_n) \), we have that

\[
dist(w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \hat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta) \\
\leq \sum_{k = I_n^0}^{I_n^1} \dist(w(h_n^k) - w(h_n^{k - 1}), \int_{h_n^{k - 1}}^{h_n^k} \hat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta) \\
\leq \sum_{k = I_n^0}^{I_n^1} \dist(w(h_n^k) - w(h_n^{k - 1}), \delta_n^k \hat{F}(t_n^{k - 1}, w(t_n^{k - 1}), \varpi_n(t_n^{k - 1}))) \\
+ (\tau_n^1 - \tau_n^0)[\alpha(1/n) + 2Lc/n].
\]

Therefore, by the construction of \( \chi_n \) (see (31) and (32))

\[
\int_{C_{\tau_0, \tau_1}} \dist(w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \hat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta) \chi_n(d(w(\cdot))) \\
\leq \sum_{k = I_n^0}^{I_n^1} \int_{C_{\tau_0, \tau_1}} \dist(w(h_n^k) - w(h_n^{k - 1}), \delta_n^k \hat{F}(t_n^{k - 1}, w(t_n^{k - 1}), \varpi_n(t_n^{k - 1}))) \chi_n^k(d(w(\cdot))) \\
+ (\tau_n^1 - \tau_n^0)[\alpha(1/n) + 2Lc/n] \\
= \sum_{k = I_n^0}^{I_n^1} \delta_n^k \dist(v, \hat{F}(t_n^{k - 1}, w, \eta_n^{k - 1})) \chi_n^k(d(w, v)) \\
+ (\tau_n^1 - \tau_n^0)[\alpha(1/n) + 2Lc/n].
\]

29
Since functions \((w, v) \mapsto \text{dist}(v, \widehat{F}(t, w, \nu)), \nu \mapsto \text{dist}(v, \widehat{F}(t, w, \nu))\) are Lipschitz continuous with constants \((L + 1)\) and \(L\) respectively, by the second statement of Lemma 3, Lemma 8 and choice of \(\gamma^k_n, \gamma^k_n\) we have that
\[
\int_{C_{\tau_0, \tau_1}} \text{dist} \left( w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \widehat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta \right) \chi_n(d(w(\cdot))) \\
\leq \sum_{k=1}^{I^n} \int_{T^d \times \mathbb{R} \times \mathbb{R}^{d+1}} \delta^k_n \text{dist} \left( v, \widehat{F}(\nu^{k-1}_n, w, \nu^{k-1}_n) \right) \gamma^k_n(d(w, v)) \\
+ (\tau^1_n - \tau^0_n)[\alpha(1/n) + 2Lc/n + (1+2L)T/n] \\
\leq (\tau^1_n - \tau^0_n)[\alpha(1/n) + 2Lc/n + (1+2L)T/n + 1/n].
\]

Hence, using the fact that the function \(w(\cdot) \mapsto \text{dist}(w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \widehat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta)\) is \((2 + L(\tau_1 - \tau_0))\)-Lipschitz continuous and inequalities \((10)\), we get that
\[
\int_{C_{\tau_0, \tau_1}} \text{dist} \left( w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \widehat{F}(\theta, w(\theta), \nu(\theta))d\theta \right) \chi(d(w(\cdot))) \\
\leq \int_{C_{\tau_0, \tau_1}} \text{dist} \left( w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} \widehat{F}(\theta, w(\theta), \varpi_n(\theta))d\theta \right) \chi_n(d(w(\cdot))) \\
+ (2 + 2L(\tau_1 - \tau_0))W_1(\chi, \chi_n) \\
\leq (2 + L(\tau_1 - \tau_0))W_1(\chi, \chi_n) \\
+ (\tau^1_n - \tau^0_n)[\alpha(1/n_k) + 2Lc/n_k + (1+2L)T/n_k + 1/n_k].
\]

Passing to the limit when \(k \to \infty\), using the equality \(F(\theta, p(w(\theta)), p_{\#} \nu(\theta)) = \widehat{F}(\theta, w(\theta), \nu(\theta))\) and convergence \((39)\), we conclude that, for any \(\tau_0, \tau_1 \in [s, r]\),
\[
\int_{C_{\tau_0, \tau_1}} \text{dist} \left( w(\tau_1) - w(\tau_0), \int_{\tau_0}^{\tau_1} F(\theta, p(w(\theta)), m(\theta))d\theta \right) \chi(d(w(\cdot))) = 0.
\]

This means that, \(\chi\text{-a.e. } w(\cdot) \in C_{s, r}, w(\cdot) \in \text{SOL}(r, s, m(\cdot))\). This and definition of \(\nu(\cdot)\) imply that \(\nu(\cdot)\) solves \((12)\) on \([s, r]\).

Further, we have that property \((\Psi 1)\) is fulfilled by construction. Property \((\Psi 2)\) follows from Lemmas 2, 3 and convergences
\[
\|\psi - \phi_{m_k}\|, \sup_{t \in [s, r]} W_1(q_{m_k}(t), m(t)) \to 0 \text{ as } k \to \infty.
\]

Finally, convergence of \(\eta_{m_k}^n\) to \(\nu(r)\) and Lemma 10 yield \((\Psi 3)\).

Thus, we prove that \((\mu, \psi) \in \Psi^{r,s}(m, \phi)\). This and inclusion \(\psi \in \mathcal{V}(r, \mu)\) (see \((11)\)) implies the sufficiency part of Theorem 2.

8 Proof of the viability theorem. Necessity

Proof of Theorem 2. Necessity. Let \(\mathcal{V}\) be viable with respect to the mean field game dynamics. Choose \(s \in [0, T]\), \(m \in T^d, \phi \in \mathcal{V}(s, m)\). By Definition 5 for any \(r > s\),
there exist a flow of probabilities \( \nu^r(\cdot), m^r(\cdot) \), and a function \( \psi^r \in C(\mathbb{T}^d) \) such that 
\( \nu^r(\cdot) \) solves (12), \( m^r(t) = p^\# \nu^r(t) \) and conditions (11)–(13) hold true. Furthermore, 
\( \psi^r \in \mathcal{V}(r, m^r(r)) \). Without loss of generality one can assume that \( \nu^r(s) = \hat{m} \).

Let \( \chi^r \in \mathcal{C}_{s,r} \) be such that \( \nu^r(t) = \hat{\epsilon}_{t} \# \chi^r \) and \( \text{supp}(\chi^r) \subset \text{SOL}(r, s, m^r(\cdot)) \).

Let us introduce the operator \( \Delta^{s,r} : \mathcal{C}_{s,r} \to \mathbb{T}^d \times \mathbb{R}^{d+1} \) by the rule:

\[
\Delta^{s,r}(x(\cdot), z(\cdot)) = \left( x(s), \frac{x(r) - x(s)}{r - s}, \frac{z(r) - z(s)}{r - s} \right).
\]

Set

\[
\beta^r \triangleq \Delta^{s,r} \# \chi^r.
\]

Notice that \( \beta^r \in \mathcal{L}^c(m) \) for \( c = R \). Since \( \| B_{m^r(\cdot)}^{s,r} \psi^r - \phi \| = 0 \), using Lemma 6 we have that

\[
\| A_{m}^{s,r} \psi^r - \phi \| \leq (C + 1)(\alpha(r - s) + 2LR(r - s))(r - s).
\]  \( \text{(42)} \)

Notice that \( \Theta^{r-s} \# \beta^r = \nu^r(r) \). Furthermore, by construction we have that

\[
[\phi^r, \nu^r(r)] \geq [\phi, \hat{m}].
\]  \( \text{(43)} \)

For any \( w(\cdot) = (x(\cdot), z(\cdot)) \in \text{supp}(\chi^r) \), we have that

\[
0 = \text{dist} \left( w(r) - w(s), \int_{s}^{r} F(t, x(t), m^r(t))dt \right)
\]

\[
\geq \text{dist} (w(r) - w(s), (r - s)F(s, x(s), m(s))) - (r - s)[\alpha(r - s) + 2LR(r - s)].
\]

Thus,

\[
\text{dist} \left( \frac{w(r) - w(s)}{r - s}, F(s, x(s), m(s)) \right) \leq \alpha(r - s) + 2LR(r - s).
\]

Hence we have

\[
\int_{\mathbb{T}^d \times \mathbb{R}^{d+1}} \text{dist} (v, F(s, x(m))) \beta^r(d(x, v)) \leq \alpha(r - s) + 2LR(r - s). \tag{44}
\]

Since the set \( \mathcal{L}^c(m) \) is compact in \( \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R}^{d+1}) \) we have that there exists a probability \( \beta \) and a sequence \( \{r_n\}_{n=1}^{\infty} \) such that \( r_n \to s, W_1(\beta^{r_n}, \beta) \to 0 \) as \( n \to \infty \).

Let us show that the probability \( \beta \in \mathcal{D}_F \mathcal{V}(s, m, \phi) \). Letting \( \tau_n \triangleq r_n - s \) we get the first condition. We have that \( \psi^{r_n} \in \mathcal{V}(r^n, \Theta^{r^n-s} \# \beta^{r^n}) \). Thus, the second condition of Definition 6 is fulfilled. The third condition of Definition 6 follows from (12). Analogously, (43) implies Condition 4 for \( \nu^n \triangleq \nu^{r_n}(r_n) \). Finally, passing to the limit in (44) we get the last condition of Definition 6. \( \square \)
Appendix

Proof of Proposition 1. Let $\chi \in \mathcal{P}^1(C_{s,r})$ be such that $\nu(t) = \hat{e}_t\#\chi$, and let, for $m(t) \triangleq p_\#\nu(t) = e_t\#\chi$, $\text{supp}(\chi) \subset \text{SOL}(r, s, m(\cdot))$. Now, we define the probability $\bar{\chi} \in \mathcal{P}^1(C_{s,r})$ in the following way. Let $\{\chi_x\}$ be a disintegration of $\chi$ along $m(s)$ i.e. for every $\varphi \in C_b(C_{s,r}),$

$$\int_{C_{s,r}} \varphi(w(\cdot))\chi(d(w(\cdot))) = \int_{\mathbb{T}^d} \int_{C_{s,r}} \varphi(w(\cdot))\chi_x(d(w(\cdot)))m(s, dx).$$

Now, for $\varphi \in C_b(C_{s,r}),$ set

$$\int_{C_{s,r}} \varphi(w(\cdot))\bar{\chi}(d(w(\cdot)))$$

$$\triangleq \int_{\mathbb{T}^d} \int_{C_{s,r}} \int_{\mathbb{R}} \varphi(x(\cdot), z(\cdot) - z(s) + z_s)\chi_x(d(x(\cdot), z(\cdot)))\nu_s(dz_s|x)m(s, dx).$$

Put $\bar{\nu} \triangleq \hat{e}_t\#\chi$. Obviously, $p_\#\bar{\nu}(t) = m(t) = p_\#\nu(t), \bar{\nu}(s) = \nu_s$. Thus, statements 2 and 3 of the proposition are obtained.

Now, we shall prove that $\bar{\nu}(\cdot)$ solves MFDI (12). To this end let us show the inclusion $\text{supp}(\bar{\chi}) \subset \text{SOL}(r, s, m(\cdot))$. We have that $\bar{\chi}$ is concentrated on the set of motions $(x(\cdot), z(\cdot) + h)$, where $(x(\cdot), z(\cdot))$ solves (8) and $h$ does not depend on $t$. Therefore the motion $(x(\cdot), z(\cdot) + h)$ itself is a solution of (8). Thus, $\text{supp}(\bar{\chi}) \subset \text{SOL}(r, s, m(\cdot))$. This proves the first statement of the proposition.

Finally, for $\phi \in C(\mathbb{T}^d)$ and $t \in [s, r]$, we have

$$[\phi, \bar{\nu}(t)] = \int_{\mathbb{T}^d \times \mathbb{R}} (\phi(x) + z)\bar{\nu}(t, d(x, z))$$

$$= \int_{C_{s,r}} (\varphi(x(t)) + z(t))\bar{\chi}(d(x(\cdot), z(\cdot)))$$

$$= \int_{\mathbb{T}^d} \int_{C_{s,r}} \int_{\mathbb{R}} (\phi(x(t)) + z(t) - z(s) + z_s)\nu_s(dz_s|x)\chi_x(d(x(\cdot), z(\cdot)))m(s, dx)$$

$$= \int_{C_{s,r}} \int_{\mathbb{R}} (\phi(x(t)) + z(t))\chi_x(d(x(\cdot), z(\cdot))) - \int_{C_{s,r}} \int_{\mathbb{R}} z(s)\chi_x(d(x(\cdot), z(\cdot)))$$

$$+ \int_{\mathbb{T}^d \times \mathbb{R}} z\nu_s(d(x, z))$$

$$= \int_{\mathbb{T}^d \times \mathbb{R}} (\phi(x) + z)\nu(t, d(x, z)) - \int_{\mathbb{T}^d \times \mathbb{R}} z\nu(s, d(x, z)) + \int_{\mathbb{T}^d \times \mathbb{R}} z\nu_s(d(x, z)).$$

This gives the forth statement of the proposition. \hfill \blacksquare

Proof of Proposition 2. For $i = 1, 2$, pick $\chi_i \in \mathcal{P}^1(C_{s_i-1, s_i})$ such that $\nu_i(t) = \hat{e}_t\#\chi_i$, and, for $m_i(t) \triangleq p_\#\nu_i(t), \text{supp}(\chi_i) \subset \text{SOL}(s_i, s_i-1, m(\cdot))$. Put $\chi \triangleq \chi_1 \circ \chi_2$, $\nu(t) \triangleq \hat{e}_t\#\chi$. Further, denote $m(t) \triangleq p_\#\nu(t)$. Since

$$m(t) \triangleq \left\{ \begin{array}{ll}
m_1(t), & t \in [s_0, s_1];\\
m_2(t), & t \in [s_1, s_2],
\end{array} \right.$$
we have that $\text{supp}(\chi) \subset \text{SOL}(s_0, s_2, m(\cdot))$. This proves the proposition.

**Proof of Proposition** We pick $\chi_i \in \mathcal{P}^1(C_{s,r})$ such that $\nu_i(t) = \hat{\nu}_{t \# \chi_i}$ and $\text{supp}(\chi) \subset \text{SOL}(r, s, m_i(\cdot))$ where $m_i(t) = p_{\#} \nu_i(t)$. Notice that, for every $i$, $\chi_i$ is concentrated on the set of $2R$-Lipschitz continuous motions. Moreover, if $w(\cdot) \in \text{supp}(\chi_i)$, then

$$\|w(s)\| - 2R(r - s) \leq \|w(\cdot)\| \leq \|w(s)\| + 2R(r - s).$$

Hence, taking into account the equality $\hat{\nu}_{s \# \chi_i} = \nu_s(t \in \mathcal{P}^1(T^d \times \mathbb{R})$, we get that the sequence $\{\chi_i\}$ is tight and has uniformly integrable 1-moment. Thus, there exists a subsequence $\{\chi_{i_k}\}$ and a probability $\chi^* \in \mathcal{P}^1(C_{s,r})$ such that

$$\lim_{k \to \infty} W_1(\chi_{i_k}, \chi^*) = 0.$$

Put $\nu^*(t) \triangleq \hat{\nu}_{t \# \chi}, m^*(t) \triangleq p_{\#} \nu^*(t) = \epsilon_t \# \chi^*$. By ([1], Proposition 5.1.8) we have that

$$\lim_{k \to \infty} \sup_{t \in [s, r]} W_1(\nu_{i_k}(t), \nu^*(t)) = \lim_{k \to \infty} \sup_{t \in [s, r]} W_1(m_{i_k}(t), m^*(t)) = 0.$$

Now, let us prove that $\nu^*(\cdot)$ solves (12) for $m(\cdot) = m^*(\cdot)$. It suffices to prove that $\text{supp}(\chi^*) \subset \text{SOL}(r, s, m^*(\cdot))$. Let $w(\cdot) \in \text{supp}(\chi^*)$. By ([1] Proposition 5.1.8) there exists a sequence $\{w_k(\cdot)\} \in C_{s,r}$ such that $w_k(\cdot) \in \text{supp}(\chi_{i_k})$ and $\|w(\cdot) - w_k(\cdot)\| \to 0$ as $k \to \infty$. Furthermore, since $\text{supp}(\chi_{i_k}) \subset \text{SOL}(r, s, m_{i_k}(\cdot))$, we have that, for every $t', t'' \in [s, r], t' < t''$,

$$\text{dist} \left( w_k(t'') - w_k(t'), \int_{t'}^{t''} F(t, p(w_k(t)), m_{i_k}(t)) dt \right) = 0.$$

By condition (M4), we have that the functions

$$\left| \text{dist} \left( w(t'') - w(t'), \int_{t'}^{t''} F(t, p(w(t)), m^*(t)) dt \right) \right|$$

$$- \text{dist} \left( w_k(t'') - w_k(t'), \int_{t'}^{t''} F(t, p(w_k(t)), m_{i_k}(t)) dt \right)$$

$$\leq (2 + L)\|w(\cdot) - w_k(\cdot)\| + L \sup_{t \in [s, r]} W_1(m_{i_k}(t), m^*(t)).$$

Hence, we conclude that for any $t', t'' \in [s, r], t' < t''$,

$$\text{dist} \left( w(t'') - w(t'), \int_{t'}^{t''} F(t, p(w(t)), m^*(t)) dt \right) = 0.$$

This means that $w(\cdot)$ is a solution of (8) for $m(t) = m^*(t)$. Therefore, $\nu^*$ solves MFDI (12) for $m(\cdot) = m^*(\cdot)$.

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