DENSITY-CONSTRAINED CHEMOTAXIS AND HELE-SHAW FLOW

INWON KIM, ANTOINE MELLET, AND YIJING WU

Abstract. We consider a model of congestion dynamics with chemotaxis, where the density of cells follows the chemical signal it generates, while observing an incompressibility constraint. We show that when the chemical diffuses slowly and attracts the cells strongly, then the dynamics of the congested cells is well approximated by a surface-tension driven free boundary problem. More precisely, we show that in this limit the density of cell converges to the characteristic function of a set whose evolution is described by a Hele-Shaw free boundary problem with surface tension. Our problem is set in a bounded domain, which leads to an interesting analysis on the limiting boundary conditions for the density function. Namely, we prove that the assumption of Robin boundary conditions for the chemical potential leads to a contact angle condition for the free interface.

1. Introduction

1.1. A model for chemotaxis with density constraint. The classical parabolic-elliptic Patlak-Keller-Segel model for chemotaxis reads:

\[
\begin{aligned}
\partial_t \rho - \mu \Delta \rho + \chi \text{div} (\rho \nabla \phi) &= 0, \\
\eta \Delta \phi + \theta \rho - \sigma \phi &= 0,
\end{aligned}
\]

where \( \rho \) denotes the cell density and \( \phi \) the concentration of some chemical. The nonnegative parameters \( \mu \) and \( \eta \) are the cell and chemical diffusivity, \( \chi \) is the cell sensitivity, and \( \theta \) and \( \sigma \) describe the production and degradation of the chemical (see [10], [16], [7]).

In this model, the diffusion competes with the aggregating potential \( \phi \), leading to the well-known phenomena of concentration and finite time blow-up of the density (see e.g. [4], [8]). In order to investigate the behavior of the density \( \rho \) after saturation occurs we take into account the incompressibility of the cells by imposing a constraint \( \rho \leq \rho_M \). We replace \( \rho \) with \( \rho/\rho_M \) and \( \phi \) with \( \phi/(\rho_M \theta) \) so that \( \rho_M = \theta = 1 \) and denote \( \tilde{\chi} = \chi \rho_M \theta \). We are then led to the equation (see [18, 11] for details):

\[
\begin{aligned}
\partial_t \rho - \mu \Delta \rho + \tilde{\chi} \text{div} (\rho \nabla \phi) - \Delta p &= 0, \\
\rho \leq 1 \\
\sigma \phi - \eta \Delta \phi &= \rho,
\end{aligned}
\]

where the pressure \( p \) is a Lagrange multiplier for the constraint \( \rho \leq 1 \), and satisfies

\[ p \geq 0, \quad p(1 - \rho) = 0 \, \text{a.e.} \]

Similar models have been used in particular in the study of congested crowd motion (see [18]). The conditions on \( p \) can also be expressed by writing \( p \in P(\rho) \) with

\[
P(\rho) := \begin{cases} 
0 & 0 \leq \rho < 1 \\
[0, \infty) & \rho = 1
\end{cases}
\]

which is sometimes referred to as the Hele-Shaw graph.

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In a companion paper [11], we proved the existence and uniqueness of a weak solution for (1.1) and investigated its relation to some free boundary problems. In this paper, we will investigate the singular limit of strong attraction ($\chi \gg \mu$) and small chemical diffusion ($\eta \ll 1$) and prove that the model is asymptotically close to a Hele-Shaw free boundary problem with surface tension (1.8). This establishes the first rigorous link between a general Chemotaxis system and Hele-Shaw flow with surface tension (to the best of our knowledge).

We also aim to analyze the behavior of the solutions of (1.1) near a fixed boundary, by setting the problem in a bounded domain $\Omega \subset \mathbb{R}^d$. In particular we are interested in the effect on the dynamics of different absorption rates of the chemical at the boundary. For full generality, we will use Robin boundary conditions for $\phi$ with a fixed parameter for absorption rate. For the density, we impose Neumann boundary conditions which ensure the conservation of cell density.

Above discussions, by setting $\eta = \varepsilon^2$ and $\tilde{\chi} = \varepsilon^{-1}$ for small $\varepsilon > 0$, lead to the system (1.3)-(1.4):

$$
\begin{align*}
\begin{cases}
\partial_t \rho - \mu \nabla \rho + \text{div} (\varepsilon^{-1} \rho \nabla \phi - \nabla p) = 0, & \text{in } \Omega \times (0, \infty), \quad p \in P(\rho) \\
(\varepsilon^{-1} \rho \nabla \phi - \nabla p) \cdot n = 0, & \text{on } \partial \Omega \times (0, \infty) \\
\rho(x,0) = \rho_{\text{in}}(x) & \text{in } \Omega,
\end{cases}
\end{align*}
$$

with $\phi$ solving

$$
\begin{align*}
\begin{cases}
\sigma \phi - \varepsilon^2 \Delta \phi = \rho & \text{in } \Omega \\
\alpha \phi + \beta \varepsilon \nabla \phi \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

Note that the scaling of the continuity equation can also be obtained by rescaling the time variable so that we observe the evolution of $\rho$ at a time scale $t \sim \varepsilon^{-1}/\tilde{\chi}$, under the assumption that $\mu = O(\varepsilon \tilde{\chi})$.

Throughout the paper, we assume that $\alpha$, $\beta$ and $\sigma$ are constants satisfying

$$
\sigma > 0, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta > 0.
$$

The assumption $\sigma > 0$ is important. When $\sigma = 0$, the function $\tilde{\phi} = \varepsilon^2 \phi$ is the usual Newtonian potential (up to the boundary condition on $\partial \Omega$), which does not localize in the limit $\varepsilon \to 0$: see the discussion below (1.8). By contrast, when $\sigma > 0$, we have $\phi \sim \frac{1}{\varepsilon^2} p$ when $\varepsilon \ll 1$ and the effect of $\varepsilon^{-1} \nabla \phi$ on the dynamic of the saturated regions is akin to that of surface tension.

1.2. Relation to Hele-Shaw free boundary problems. When $\mu = 0$, (1.3)-(1.4) is a weak formulation for the free boundary problem

$$
\begin{align*}
\begin{cases}
\rho \in [0,1), \quad p = 0, \quad \partial_t \rho + \varepsilon^{-1} \text{div} (\rho \nabla \phi) = 0 & \text{in } \Omega \setminus \Omega_s(t) \\
\rho = 1, \quad p > 0, \quad \Delta p = \varepsilon^{-1} \Delta \phi & \text{in } \Omega_s(t),
\end{cases}
\end{align*}
$$

where $\Omega_s(t) = \{ \rho(t) = 1 \}$ denotes the saturated density set and the free boundary $\Sigma(t) = \partial \Omega_s(t) \cap \Omega$ moves according to the velocity law

$$
(1 - \rho|_{\partial \Omega^c_s})V = (\nabla \phi - \varepsilon^{-1} \nabla \phi) \cdot \nu|_{\Omega_s}.
$$

(Here $V$ denotes the outward normal velocity of $\Sigma(t)$ and $\nu$ denotes the outward normal of $\Omega_s(t)$). In particular, when the density is a characteristic function $\rho(x,t) = \chi_{\Omega_s(t)}(x)$, we recognize the usual one phase Hele-Shaw problem without surface tension, which we can write with the modified pressure $q = p + \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi \right)$, as

$$
\begin{align*}
\begin{cases}
\Delta q = 0 & \text{in } \Omega_s(t), \\
q = \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi \right) & \text{on } \Sigma(t) \\
V = -\nabla q \cdot \nu & \text{on } \Sigma(t).
\end{cases}
\end{align*}
$$
In other words, in this fully saturated regime, the chemotaxis system (1.3)-(1.4) can be seen as a free boundary problem describing the motion of the region occupied by the cell, driven by the chemical concentration $\phi$ and the pressure variable $p$. Since we obtained (1.3)-(1.4) by imposing the constraint $\rho \leq 1$, but without requiring $\rho \in (0,1)$, it is not clear that we should actually have $\rho(x,t) = \chi_{\Omega(t)}(x)$ in general. In [11], we proved that if $\rho$ is a characteristic function at $t = 0$, then this remains true at positive times for (1.3)-(1.4) when $\mu = 0$. On the other hand when $\mu > 0$ the density is never a characteristic function. Indeed in this case the saturated set interacts with the unsaturated part of the density by a Richards-type problem, as shown in [11].

Nevertheless, we will show in this paper that, in the limit $\varepsilon \to 0$, the effect of the attractive potential is strong enough to ensure the convergence of $\rho$ to a characteristic function $\chi_{\Omega(t)}(x)$ for all $\mu \geq 0$. We will then show that the asymptotic dynamic of $\Omega_s(t)$ is described by the Hele-Shaw free boundary problem with surface tension

$$\begin{cases} 
\Delta q = 0 \text{ in } \Omega_s(t), \\
v = \frac{\kappa}{4\varepsilon^2} \text{ on } \Sigma(t), \\
\n = -\nabla q \cdot \nu \text{ on } \Sigma(t),
\end{cases}$$

where $\kappa$ denotes the mean curvature of the free boundary $\Sigma(t)$ (taken to be positive when $\Omega_s(t)$ is convex). Formally, we can get (1.8) from (1.7) by proving that the quantity $\varepsilon^{-1} \left( \frac{1}{\sigma} - \phi \right)$ is an approximation of the mean-curvature of $\Omega_s$ when $\varepsilon \ll 1$.

Note that, as $\sigma$ tends to zero, the weight on surface tension grows to infinity in (1.8). Thus heuristically we expect that the limit density support will re-adjust itself into a ball at time scale of order $\sigma^{3/2}$ (and instantly when $\sigma = 0$). This is consistent with the convergence to radial solutions of (1.1) when $\sigma = \mu = 0$: see [9] and [17] for further discussions.

### 1.3. The presence of bounded domain.

Our result for $\mu = 0$ bears similarities with [9], where the emergence of surface tension and derivation of a Muskat problem is studied via a variational approximation. In that paper, the potential $\phi$ solves $\phi_t - \Delta \phi = \rho$ (in $\mathbb{R}^d$) and instead of the Keller-Segel system, the authors considers a discrete-time approximations constructed via a JKO scheme. A similar variational analysis is performed in [12] for the $L^2$-based thresholding scheme.

Note that both [9] and [12] consider the setting of periodic torus or entire $\mathbb{R}^d$ for the interaction energy, in which case $\phi$ can be written as a convolution with the heat kernel. Such a representation of $\phi$, as well as the symmetry of the heat kernel in space variables, played an important role in the analysis of the aforementioned papers, in particular when deriving the weak limit equation. The fact that our problem is set in a bounded domain presents an interesting challenge to this analysis. In particular this necessitates a more PDE-oriented proof of Proposition 5.2, replacing corresponding proofs in [12] and [9]. Our result appears to be the first that links a Keller-Segel system with a Hele-Shaw flow with surface tension, regardless of the choice of the domain. This connection was also suggested in the very recent paper [5], where the incompressible limit of a generalized version of Keller-Segel system is investigated. The model is a variant of Cahn-Hilliard equation, which can be seen as a diffuse-interface approximation of our Hele-Shaw flow with surface tension.

Another novel feature of our analysis, also related to the bounded domain, is the characterization of the free boundary behavior near the fixed boundary $\partial \Omega$. Of particular interest, in the context of the singular limit $\varepsilon \to 0$, is how the Robin boundary conditions imposed on $\phi$ plays a role on the dynamic of $\Omega_s(t)$. We will show that (1.8) must be supplemented by the contact angle condition

$$\cos(\theta) = \gamma := -\min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\sigma^2}} \right) \text{ on } \Sigma(t) \cap \partial \Omega,$$

where $\theta$ is the angle formed by the free surface $\Sigma(t)$ and the fixed boundary $\partial \Omega$, measured from inside of the set $\Omega_s(t)$, and the fixed boundary $\partial \Omega$ at the triple junction $\Sigma(t) \cap \partial \Omega$ (see Figure 1). In particular,
for Neuman condition $\alpha = 0$ (zero absorption of chemicals), the contact must be orthogonal, while for Dirichlet condition $\beta = 0$ (and whenever the absorption rate $\alpha$ is bigger or equal than $\sqrt{C_\beta}$), the contact must be tangential.

1.4. Notations and definitions. Throughout the paper $\Omega$ is a smooth bounded domain in $\mathbb{R}^d$.

We will use the following definition of weak solutions of (1.3)-(1.4), as in [11]:

Definition 1.1. The pair of functions $(\rho, p)$ is a weak solution of (1.3)-(1.4) if $\rho \in L^1(0, \infty; L^\infty(\Omega)) \cap C^{1/2}(0, \infty; H^{-1}(\Omega))$, $p \in L^2(0, \infty; H^1(\Omega))$ with

$$0 \leq \rho \leq 1, \quad p \geq 0, \quad (1 - \rho)p = 0 \quad \text{a.e. in } \Omega \times (0, \infty)$$

and the followings hold:

(1.10) $$\int_\Omega \rho_{\text{in}}(x)\zeta(x, 0) \, dx + \int_0^\infty \int_\Omega \rho \partial_t \zeta + \rho v \cdot \nabla \zeta \, dx \, dt = 0$$

for any function $\zeta \in C_c^\infty(\Omega \times [0, \infty))$ and for some $v \in L^2(\Omega \times (0, \infty), d\rho)$ satisfying

(1.11) $$\int_0^\infty \int_\Omega \rho v \cdot \xi - \frac{1}{\epsilon} \rho \nabla \phi \cdot \xi - \mu \rho \Delta \xi - p \nabla \xi \, dx \, dt = 0$$

for any vector field $\xi \in C_c^\infty(\Omega \times (0, \infty); \mathbb{R}^d)$ such that $\xi \cdot n = 0$ on $\partial \Omega$ and with $\phi$ given by (1.4).

Equality (1.10) is the usual weak formulation for the continuity equation $\partial_t \rho + \div (\rho v) = 0$ with Neumann boundary conditions and initial condition $\rho_{\text{in}}$. Equation (1.11) is equivalent to the equality $\rho v = \frac{1}{\epsilon} \rho \nabla \phi - \nabla p$ in $L^2(\Omega \times (0, \infty))$. It is written in this way to make it easy to compare with Definition 1.3 below (see (1.15)).

In [11] we prove the existence and uniqueness of a weak solution in the sense of Definition 1.1 using the fact that it is a gradient flow with respect to the Wasserstein metric. Here the free energy is given by

$$\mu \int_\Omega \rho \log \rho \, dx - \frac{1}{2\epsilon} \int_\Omega \rho \phi^2 \, dx,$$

with the constraint $\rho \leq 1$.

This energy structure of the equation will also play a key role in this paper. Because this energy does not behave well when $\epsilon \ll 1$, we will work instead with the functional

$$\mathcal{F}_\epsilon(\rho) = \begin{cases} 
\mu \int_\Omega \rho \log \rho \, dx + \frac{1}{2\epsilon} \int_\Omega \rho (1 - \sigma \phi^2) \, dx, & \text{if } 0 \leq \rho(x) \leq 1 \text{ a.e.} \\
\infty & \text{otherwise}.
\end{cases}$$

Since $\int_\Omega \rho \, dx$ is preserved by the equation we are only adding a constant to the energy, but this constant is important when $\epsilon \ll 1$ (it was proved in [15] that $\mathcal{F}_\epsilon(\rho)$ is bounded uniformly in $\epsilon$ when $\rho = \chi_E \in BV(\Omega; \{0, 1\})$). The following result was proved in [11]:

Theorem 1.2 ([11]). For any $\epsilon > 0$ and any initial condition $\rho_{\text{in}}$ satisfying

$$0 \leq \rho_{\text{in}} \leq 1 \quad \text{a.e. in } \Omega,$$

there exists a unique $(\rho^\epsilon, p^\epsilon)$ weak solution of (1.3)-(1.4) in the sense of Definition 1.1. Furthermore, $\rho^\epsilon$ satisfies the energy inequality

(1.12) $$\mathcal{F}_\epsilon(\rho^\epsilon(t)) + \int_0^t \int_\Omega |v^\epsilon|^2 \rho^\epsilon \, dx \, dt \leq \mathcal{F}_\epsilon(\rho_{\text{in}}) \quad \forall t > 0$$

with $v^\epsilon$ defined as in Definition 1.1.
The goal of this paper is to show that when \( \varepsilon \ll 1 \), the solution of (1.3)-(1.4) given by Theorem 1.2 converges to the solution of the following Hele-Shaw problem with surface tension:

\[
\begin{aligned}
\Delta q &= 0 & \text{in } \Omega_s(t), \\
q &= \frac{\kappa}{4\sigma^{3/2}} & \text{on } \Sigma(t) = \partial \Omega_s(t) \cap \Omega, \\
\nabla q \cdot n &= 0 & \text{on } \partial \Omega \cap \Omega_s(t), \\
V &= -\nabla q \cdot \nu & \text{on } \Sigma(t)
\end{aligned}
\]

(1.13)

together with the contact angle condition (1.9). Recall that \( n \) and \( \nu \) respectively denote the outward normal of \( \Omega \) and \( \Omega_s(t) \) at their boundary points.

The definition of a weak solution of (1.13)-(1.9) is parallel to the Definition 1.1.

**Definition 1.3.** The pair of functions \( (\rho, q) \) is a weak solution of (1.13)-(1.9) if

\[
\rho \in L^\infty(0, \infty; BV(\Omega; \{0, 1\})) \cap C^{1/2}(0, \infty; \mathcal{P}(\Omega)), \quad q \in L^2(0, \infty; (C^*(\Omega))^*)
\]

for some \( s > 0 \) and the followings hold:

\[
\int_\Omega \rho_0(x) \zeta(x, 0) \, dx + \int_0^\infty \int_\Omega \rho \partial_t \zeta + q \div \nabla \zeta \, dx \, dt = 0
\]

(1.14)

for any function \( \zeta \in C_c^\infty(\Omega \times [0, \infty)) \) and for some \( v \in L^2(\Omega \times (0, T), d\rho) \) satisfying

(1.15)

\[
\int_0^\infty \int_\Omega \rho v \xi - q \div \xi \, dx \, dt = -\frac{1}{4\sigma^{3/2}} \int_\Omega |\div \xi - \nu \otimes \nu : D\xi| |\nabla \rho| + \frac{\gamma}{4\sigma^{3/2}} \int_\partial\Omega |\div \xi - n \otimes n : D\xi| \rho \, d\mathcal{H}^{n-1}(x)
\]

for any vector field \( \xi \in C_c^\infty(\Omega \times (0, \infty); \mathbb{R}^d) \) such that \( \xi \cdot n = 0 \) on \( \partial \Omega \).

This definition, similar to the one given in [12] and [9], warrant several comments.

1. The condition \( \rho \in L^\infty(0, \infty; BV(\Omega; \{0, 1\})) \) implies that for a.e. \( t > 0 \) we have \( \rho(t) = \chi_{\Omega_s(t)} \) for a set \( \Omega_s(t) \subset \Omega \) with finite perimeter.
2. In (1.15), \( \nu = \frac{\nabla \rho}{|\nabla \rho|} \) stands for the \( L^\infty \) density of \( \nabla \rho \) with respect to the total variation \( |\nabla \rho| \) (which exists by Radon-Nikodym’s differentiation theorem). Since \( \rho(t) = \chi_{\Omega_s(t)} \in BV \), it is also the measure theoretic normal to the boundary \( \Sigma(t) = \partial \Omega_s(t) \). In particular, the term \( (\nu \otimes \nu : D\xi) |\nabla \rho| \) is of the form \( f(x, \lambda) |d\lambda| \) with \( f \) continuous and \( 1 \)-homogeneous and \( \lambda = \nabla \rho \). The integral in (1.15) thus makes sense (see for example [6]).
3. Note that \( q \) has very low regularity in this definition. Since \( q \sim \kappa \) along \( \Sigma(t) \), we cannot expect much more regularity on \( q \) without improving the regularity of the free boundary \( \Sigma(t) \).
4. As in Definition 1.1, (1.14) is simply the weak formulation for the continuity equation \( \partial_t \rho + \div (\rho v) = 0 \) with Neumann boundary conditions and initial data \( \rho_0 \). Since \( \rho = \chi_{\Omega_s(t)} \), it encodes the velocity law \( V = v \cdot \nu \), the incompressibility condition \( \div v = 0 \) in \( \Omega_s(t) \) and the Neumann condition \( v \cdot \nu = 0 \) on \( \partial \Omega \cap \Omega_s(t) \).
5. By taking test functions \( \xi \) supported in either \{\rho = 0\} or \{\rho = 1\}, we see that Equation (1.15) implies \( \nabla q = 0 \) in \( \rho(t) = 0 \) and \( v = -\nabla q \) in \( \Omega_s(t) \). Subtracting a constant if needed, we can in particular assume that \( q = 0 \) in \( \rho(t) = 0 \). For general test functions \( \xi \), and taking into account the right hand side of (1.15) we further get the surface tension condition \( q = \frac{\kappa}{4\sigma^{3/2}} \) on \( \Sigma(t) \) and the contact angle condition (1.9). This can be seen by using the classical formula (for a smooth interface \( \Sigma \)):

\[
\int_\Sigma \div \xi - \nu \otimes \nu : D\xi = \int_\Sigma \kappa \xi \cdot \nu + \int_\Gamma b \cdot \xi
\]

(1.16)
where \( \nu \) is the normal vector to \( \Sigma \), \( \kappa \) denotes the mean curvature of \( \Sigma \) and \( b \) is the conormal vector along \( \Gamma = \partial \Sigma \). Indeed, formally at least, the right hand side of (1.15) is (using the fact that \( \xi \cdot n = 0 \) on \( \partial \Omega \)):

\[
\frac{1}{4} \sigma^{3/2} \left[ -\int_{\Sigma} \kappa \xi \cdot \nu + \int_{\Gamma} \vec{b} \cdot \xi \right] + \frac{\gamma}{4\sigma^{3/2}} \left[ \int_{\partial \Omega \cap \Omega} \kappa \xi \cdot n + \int_{\Gamma} \vec{c} \cdot \xi \right]
\]

where \( \vec{b} \) and \( \vec{c} \) are unit conormal vectors along \( \Gamma = \partial \Sigma \cap \partial \Omega \): \( \vec{b} \) is tangent to \( \Sigma \) while \( \vec{c} \) is tangent to \( \partial \Omega \) (see Figure 1).

Integration by parts in (1.15) thus reveals that the jump of \( q \) across \( \Sigma \) must be equal to \( \frac{1}{4\sigma^{3/2}} \kappa \).

Since \( q = 0 \) in \( \{ \rho(t) = 0 \} \), we get \( q = \frac{1}{4\sigma^{3/2}} \kappa \) along \( \Sigma = \partial E \cap \Omega \). Finally, the cancellation of the lower dimensional integral requires that the component of the vector \( \vec{b} - \gamma \vec{c} \) that is tangential to \( \partial \Omega \) must vanish. In particular, we must have \( \vec{b} - \gamma \vec{c} \cdot \vec{c} = 0 \) and so \( \vec{c} \cdot \vec{b} = \gamma \), which gives the contact angle condition

\[ \cos \theta = \gamma. \]

(6) A simple computation show that (1.11) is equivalent to

\[
\int_{0}^{\infty} \int_{\Omega} \rho v \cdot \xi - q \text{div} \xi \, dx \, dt = \int_{\Omega} \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi \right) \xi \cdot \nabla \rho
\]

with \( q = p + \varepsilon^{-1} \rho (\frac{1}{2\sigma} - \phi) + \mu \rho \). Passing to the limit in the right hand side of this equation to derive (1.15) will be the main result of the second part of the paper (see Proposition 5.2) and is at the heart of the relation between the Hele-Shaw model with active potential (1.6) and the Hele-Shaw model with surface tension (1.13). We will see in particular that the contact angle \( \gamma \) depends on the boundary condition for the potential \( \phi \) and is given by

\[ \gamma = -\min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\sigma \beta}} \right). \]
1.5. Energy. The proof of the convergence will require some assumptions about the convergence of the energy. Before stating this assumption, we need to recall a few important facts about the singular part of the energy

\[ \mathcal{J}_\varepsilon(\rho) := \begin{cases} \frac{1}{2\varepsilon} \int_{\Omega} \rho (1 - \sigma \phi^\varepsilon) \, dx & \text{if } 0 \leq \rho(x) \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \]

The properties of \( \mathcal{J}_\varepsilon \) when \( \varepsilon \ll 1 \) were studied by two of the authors in [15]. The first important result is the following:

**Proposition 1.4.** [15] Let \( \Omega \) be a bounded open set with \( C^{1,\alpha} \) boundary. Given a set \( E \subset \Omega \) with finite perimeter \( P(E, \Omega) < \infty \), we have

\[ \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(\chi_E) = \frac{1}{4\sigma^{3/2}} \left[ \int_{\Omega} |\nabla \chi_E| + \int_{\partial \Omega} \frac{2\alpha}{\alpha + \sqrt{\sigma} \beta} \chi_E(x) \, dH^{n-1}(x) \right]. \]

We point out that while the result is proved only for \( \sigma = 1 \) in [15], but it can be easily extended to \( \sigma \neq 1 \) by scaling. More precisely with \( \bar{\phi} = \sigma \phi, \bar{\varepsilon} = \varepsilon / \sqrt{\sigma} \) and \( \bar{\beta} = \sqrt{\sigma} \beta \), equation (1.4) become

\[ \begin{cases} \bar{\phi} - \varepsilon^2 \Delta \bar{\phi} = \rho & \text{in } \Omega \\ \bar{\alpha} \bar{\phi} + \bar{\beta} \varepsilon \nabla \bar{\phi} \cdot n = 0 & \text{on } \partial \Omega, \end{cases} \]

which is the equation studied in [15].

Above proposition identifies the limit of \( \mathcal{J}_\varepsilon(\chi_E) \). However, this functional is not lower-semicontinuous when \( \frac{2\alpha}{\alpha + \sqrt{\sigma} \beta} > 1 \) and cannot be the \( \Gamma \)-limit of \( \mathcal{J}_\varepsilon \). We can in fact prove:

**Theorem 1.5.** Let \( \Omega \) be a bounded open set with \( C^{1,\alpha} \) boundary. The functional \( \mathcal{J}_\varepsilon \) \( \Gamma \)-converges, when \( \varepsilon \to 0 \) to

\[ \mathcal{J}_0(\rho) := \begin{cases} \frac{1}{4\sigma^{3/2}} \left[ \int_{\Omega} |\nabla \rho| + \int_{\partial \Omega} \min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\sigma} \beta} \right) \rho \, dH^{n-1}(x) \right] & \text{if } \rho \in BV(\Omega; \{0,1\}) \\ +\infty & \text{otherwise.} \end{cases} \]

This theorem is proved in [15] when \( \mathcal{J}_\varepsilon \) is restricted to characteristic functions, so we show in Appendix C how the proof can be generalized to our more general framework. This extension requires a new formulation of the energy \( \mathcal{J}_\varepsilon \), see (2.2).

This \( \Gamma \)-convergence result suggests that the solution of the gradient flow associated to the energy \( \mathcal{J}_\varepsilon \) (which corresponds to equation (1.3)) converges when \( \varepsilon \to 0 \) to a solution of the gradient flow associated to \( \mathcal{J}_0 \) which is formally a Hele-Shaw flow (1.13)-(1.9). This is indeed the result that we want to make precise in the present paper.

1.6. Main results. We are now able to state the main result:

**Theorem 1.6.** Given an initial data \( \rho_{\infty} = \chi_{E, \infty} \in BV(\Omega; \{0,1\}) \), \( \mu \geq 0 \) and a sequence \( \varepsilon_n \to 0 \), let \( (\rho^{\varepsilon_n}, p^{\varepsilon_n}) \) be the unique solution of (1.3)-(1.4) given by Theorem 1.2. Then along a subsequence the density \( \rho^{\varepsilon_n}(x, t) \) converges strongly in \( L^\infty((0,T); L^1(\Omega)) \) to

\[ \rho(x, t) \in L^\infty((0,T); BV(\Omega; \{0,1\})) \]

and the modified pressure variable \( q^{\varepsilon_n} \) (defined by (5.1)) converges to \( q \) weak-* in \( L^2((0,T); (C^s(\Omega))^*) \) for any \( s > 0 \). Furthermore, \( \rho \) satisfies the continuity equation (1.1) for some velocity function \( v(x, t) \) as well as the energy dissipation property

\[ \mathcal{J}_0(\rho(t)) + \int_0^t \int_\Omega |v|^2 \rho \, dx \, dt \leq \mathcal{J}_0(\rho_{\infty}). \]
Finally, if the following energy convergence assumption holds:

\[
\lim_{n \to \infty} \int_0^T J_{\epsilon_n}(\rho_{\epsilon_n}(t)) \, dt = \int_0^T J_0(\rho(t)) \, dt
\]

then the limit \((\rho, q)\) also satisfies the pressure equation \((1.15)\) on \((0, T)\). Thus it follows that \((\rho, p)\) is a weak solution of \((1.13) - (1.19)\) in the sense of Definition 1.3, with initial condition \(\rho_{in}\) and contact angle

\[
\gamma = -\min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\sigma \beta}} \right).
\]

The result also holds if we consider a sequence of initial data \(\rho_{in}^\epsilon = \chi_{E_{in}}\) bounded in \(BV(\Omega)\), converging strongly to \(\rho_{in} = \chi_{E_{in}}\) in \(L^1\) and satisfying \(\lim J_{\epsilon_n}(\rho_{\epsilon_n}) \to J_0(\rho_{in})\). The existence of such a sequence, for any finite perimeter set \(E_{in}\), is proved in [15] (Proposition 5.3) as part of the \(\Gamma\) convergence result.

We note that Theorem 1.6 is a conditional result, since it requires the energy convergence assumption \((1.20)\). The analysis of [15] implies that we always have

\[
\liminf_{n \to \infty} \int_0^T J_{\epsilon_n}(\rho_{\epsilon_n}(t)) \, dt \geq \int_0^T J_0(\rho(t)) \, dt,
\]

so \((1.20)\) ensures that there is no loss of boundary between phases in the limit. This assumption is rather natural and is similar to the one required for instance in [14, 12, 9]. It is likely that by proceeding as in [2] one could also obtain a weaker notion of solutions using the theory of varifolds without this assumption, but we do not pursue this direction here.

1.7. Outline of the paper. We begin with deriving two alternative formulas for the energy \(J_\epsilon\) in the next section, which play a crucial role in our analysis. Section 3 collects the main a priori estimates for the \(\epsilon\)-solutions. The proof of Theorem 1.6 is then split between sections 4 and 5. In Section 4, we prove Proposition 4.1 which gives the first part of the theorem, namely the strong convergence in \(L^1\) of the density toward a characteristic function which satisfies the continuity equation \((1.14)\) and the energy inequality \((1.19)\). Section 5 completes the proof of Theorem 1.6 by deriving equation \((1.15)\) under condition \((1.20)\). The main step is Proposition 5.2 which shows that the convergence of the energy \((1.20)\) implies the convergence of the first variation. In the last section, we briefly recall the construction of the JKO scheme used in [11] to prove the existence of weak solutions to \((1.3) - (1.4)\) (Theorem 1.2) and we state a convergence result similar to Theorem 1.6 for a discrete-time approximation: such a result is of independent interest for numerical applications.

2. Alternate formulas for \(J_\epsilon\)

A crucial tool in our analysis will be a couple of alternate formula for the energy \(J_\epsilon\). We recall that the total energy of the model, \(\mathcal{F}_\epsilon\), is given by

\[
\mathcal{F}_\epsilon(\rho) = \mu \int_\Omega \rho \log \rho \, dx + \mathcal{J}_\epsilon(\rho),
\]

where \(\mathcal{J}_\epsilon\) is defined by \((1.17)\) and plays key role in the analysis when \(\epsilon \ll 1\). For \(\rho\) satisfying the constraint \(0 \leq \rho \leq 1\), we have

\[
\mathcal{J}_\epsilon(\rho) = \frac{1}{2\sigma \epsilon} \int_\Omega \rho \, dx + \int_\Omega G_\epsilon(x, y) \rho(y) \, dy \, dx
\]

for some kernel \(G_\epsilon\). A similar energy functional is used in [12, 9] with \(G_\epsilon\) is the heat kernel in \(\mathbb{R}^d\). However, we will rely on some different formulations for \(J_\epsilon\) which make use of the particular equation
solved by the function \( \phi^\varepsilon \) in our model: First, we write
\[
J_\varepsilon(\rho) = \frac{1}{2\varepsilon} \int_\Omega (\rho - \rho^2) + (\rho^2 - 2\sigma \phi \rho + (\sigma \phi)^2) - (\sigma \phi) - \sigma \phi \rho \, dx
\]
and using equation (2.1) for \( \phi \) implies
\[
J_\varepsilon(\rho) = \frac{1}{2\varepsilon} \int_\Omega (\rho - \phi + (\rho - \sigma \phi) \phi - \sigma \phi (\phi - \rho)) \, dx
\]
when \( \beta \neq 0 \).

Alternatively, we can write the more symmetric formula (for any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta > 0 \)):
\[
J_\varepsilon(\rho) = \frac{1}{2\varepsilon} \int_\Omega (\rho - \phi + (\rho - \sigma \phi) \phi - \sigma \phi (\phi - \rho)) \, dx
\]
when \( \beta = 0 \) or
\[
J_\varepsilon(\rho) = \frac{1}{2\varepsilon} \int_\Omega (\rho - \phi + (\rho - \sigma \phi) \phi - \sigma \phi (\phi - \rho)) \, dx
\]
when \( \beta > 0 \).

This formula played a key role in the proof of Proposition 1.4 and Theorem 1.5 in [15]. Thanks to the constraint \( 0 \leq \rho \leq 1 \), all the terms in this formula are non-negative (without the constraint, the first term will favor values of \( \rho \) larger than 1). Furthermore, in the regime \( \varepsilon \ll 1 \), the first term will be bounded only for characteristic functions. This observation will be crucial when proving that the limiting density is a characteristic function (even though \( \rho^\varepsilon \) may not be). We also note that the scaling of the following two terms is consistent with the scaling of the classical Modica-Mortola regularization of the perimeter functional.

Our analysis will also require a slight variation of this formula: we can write
\[
\rho(1 - \phi) + (\rho - \sigma \phi) \phi = \rho + \sigma^2 \phi^2 - 2\rho \sigma \phi = (1 - \rho)(\sigma \phi)^2 + \rho(1 - \sigma \phi)^2
\]
leading to the formula
\[
J_\varepsilon(\rho) = \frac{1}{2\varepsilon} \int_\Omega (1 - \rho)(\sigma \phi)^2 + \rho(1 - \sigma \phi)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla \phi|^2 \, dx
\]
when \( \beta = 0 \).

Here also we note that all the terms are non-negative when \( 0 \leq \rho \leq 1 \). This formula will in particular be crucial in proving the strong convergence of \( \rho^\varepsilon \) and when deriving the pressure equation (1.15) (see Section 5).

### 3. A priori estimates

We now derive the a priori estimates that will be used to prove the convergence of \( \rho^\varepsilon \). We have:

**Lemma 3.1.** Let \( \rho_{in} \in BV(\Omega; \{0, 1\}) \) and \( (\rho^\varepsilon, \rho^\varepsilon) \) be the unique solution of (1.3)–(1.4) given by Theorem 1.4. There exists a constant \( C \) depending only on \( \int_\Omega |\nabla \rho_{in}| \) (and in particular independent of \( \varepsilon \)) such that for all \( \varepsilon > 0 \) we have:

(i) \( J_\varepsilon(\rho^\varepsilon(t)) \leq C \) for all \( t > 0 \).
(ii) \( \int_0^\infty \int_\Omega |v_\varepsilon|^2 \rho_\varepsilon \, dx \, dt \leq C \) and \( \| E_\varepsilon \|_{L^2(\Omega \times (0,\infty))} \leq C \)

(iii) \( \| \rho_\varepsilon(t) - \rho_\varepsilon(s) \|_{H^{-1}(\Omega)} \leq C\sqrt{t-s} \) for any \( 0 \leq s \leq t \).

Proof. We recall that
\[
\mathcal{F}_\varepsilon(\rho) = \mu \int_\Omega \rho \log \rho \, dx + \mathcal{J}_\varepsilon(\rho),
\]
where (since \( 0 \leq \rho \leq 1 \)) \( -C \leq \rho \log \rho \leq 0 \). The energy inequality \( (1.12) \) thus implies
\[
\mathcal{F}_\varepsilon(\rho_\varepsilon(t)) \leq \mathcal{F}_\varepsilon(\rho_{in}) \leq \mathcal{F}_\varepsilon(\rho_{in}) \forall t > 0.
\]
Using Proposition \( 1.4 \) we see that when \( \rho_{in} = \chi_{E_{in}} \in BV(\Omega; \{0,1\}) \), we have \( \mathcal{F}_\varepsilon(\rho_{in}) \leq C \) for some constant \( C \) independent on \( \varepsilon \). We deduce
\[
\mathcal{J}_\varepsilon(\rho_\varepsilon(t)) \leq \mathcal{J}_\varepsilon(\rho_{in}) - \mu \int_\Omega \rho \log \rho \, dx \leq C.
\]

The energy inequality also gives
\[
\int_0^\infty \int_\Omega |v_\varepsilon|^2 \rho_\varepsilon \, dx \, dt \leq \mathcal{F}_\varepsilon(\rho_{in}) \leq \mathcal{J}_\varepsilon(\rho_{in})
\]
and since \( \rho_\varepsilon \leq 1 \) (ii) follows immediately.

Finally, for a given test function \( \psi \in H^1(\Omega) \), the continuity equation \( (1.10) \) implies
\[
\int_\Omega \rho_\varepsilon(x,t) \psi(x) \, dx - \int_\Omega \rho_\varepsilon(x,s) \psi(x) \, dx = \int_s^t \int_\Omega \rho_\varepsilon \cdot \nabla \psi \, dx \, d\tau
\]
and so (since \( \rho_\varepsilon \leq 1 \)):
\[
\left| \int_\Omega (\rho_\varepsilon(x,t) - \rho_\varepsilon(x,s)) \psi(x) \, dx \right| \leq \left( \int_s^t \int_\Omega |v_\varepsilon|^2 \rho_\varepsilon \, dx \, d\tau \right)^{1/2} \left( \int_s^t \int_\Omega \rho_\varepsilon |\nabla \psi|^2 \, dx \, d\tau \right)^{1/2}
\]
\[
\leq \| \psi \|_{H^1(\Omega)} \left( \int_0^t \int_\Omega |v_\varepsilon|^2 \rho_\varepsilon \, dx \, d\tau \right)^{1/2} (t-s)^{1/2}
\]
and (iii) now follows from (ii).

We also need some estimates on \( \phi_\varepsilon \), solution of \( (1.4) \). The maximum principle applied to \( (1.4) \) immediately gives
\[
0 \leq \phi(x) \leq 1/\sigma \quad \text{in} \quad \Omega
\]
and multiplying \( (1.4) \) by \( \phi \) and integrating leads to the estimate
\[
\sigma \| \phi \|_{L^2(\Omega)}^2 + \varepsilon^2 \| \nabla \phi \|_{L^2(\Omega)}^2 \leq 1/\sigma.
\]

4. Strong Convergence of \( \rho_\varepsilon \) and Continuity Equation.

The main result of this section is the following proposition which proves the first part of Theorem 1.6.

**Proposition 4.1.** Let \( \rho_{in}(x) = \chi_{E_{in}} \in BV(\Omega; \{0,1\}) \) and \( \rho_\varepsilon(x,t) \) the unique solution of \( (1.3)-(1.4) \) given by Theorem 1.3. Consider a sequence such that \( \varepsilon_n \to 0 \). The followings hold:

(i) There exists a subsequence (still denoted \( \varepsilon_n \)) along which \( \rho_\varepsilon(t) \) converges uniformly with respect to \( t \) in \( H^{-1}(\Omega) \) to \( \rho(t) \) and \( E_\varepsilon(t) \) converges weakly in \( L^2(\Omega \times (0,\infty)) \) to \( E(t) \).

(ii) There exists \( v \in (L^2(\Omega \times (0,\infty)),d\rho)^d \) such that \( E = pv \) and the continuity equation \( (1.14) \) holds.
(iii) Up to another subsequence, $\rho^\varepsilon_n(t)$ converges to $\rho(t)$ strongly in $L^1(\Omega)$, uniformly in $t$. Furthermore, for all $t > 0$ we have

$$\rho(t) \in BV(\Omega; \{0, 1\})$$

(that is $\rho(t)$ is the characteristic function of a set of finite perimeter) and the energy inequality \eqref{eq1.19} holds.

Note that (i) and (ii) are classical. The most important statement is thus (iii).

Proof. The a priori estimates of Lemma 3.1 implies (i). Furthermore, we can pass to the limit in \eqref{eq1.10} to get

$$\int_\Omega \rho_{in}(x)\zeta(x, 0)\, dx + \int_0^\infty \int_\Omega \rho \partial_t \zeta + E \cdot \nabla \zeta \, dx = 0$$

for any function $\zeta \in C^\infty_c([0, \infty) \times \Omega)$. This is the continuity equation

\begin{equation}
\begin{aligned}
\partial_t \rho + \text{div} E &= 0, \\
\rho(x, 0) &= \rho_{in}(x).
\end{aligned}
\end{equation}

To complete the proof of (ii) and derive \eqref{eq1.14}, we just need to show that $E$ can be written in the form $\rho v$. This is a classical argument which we recall here: First recall that the function

$$\Theta : (\mu, F) \mapsto \begin{cases} 
\hat{T} \int_0^T \int_\Omega |F|^2 \mu & \text{if } F \ll \mu \text{ a.e. } t \in [0, T] \\
\hat{T} \int_0^T \int_\Omega |F|^2 \mu & \text{otherwise}
\end{cases}$$

is lower semi-continuous for the weak convergence of measure. Together with the uniform bound $\Theta(\rho^{\varepsilon_n}, E^{\varepsilon_n}) = \int_0^T \int_\Omega \rho^{\varepsilon_n} |\varepsilon_n|^2 \leq C$ (see Lemma 3.1 (ii)), it implies that $E$ is absolutely continuous with respect to $\rho$ and that there exists $v(t, \cdot) \in L^2(d\rho(t))$ such that $E = \rho v$. Inserting this in \eqref{eq4.1} yields \eqref{eq1.14}.

The rest of the proof is devoted to (iii). The fact that we can get stronger convergence for the density is non trivial and is due to the fact that the energy $\mathcal{J}_\varepsilon$ controls the BV norm of $\phi^\varepsilon$, which is close to $\rho^\varepsilon$ when $\varepsilon \ll 1$. To see this, we introduce the function

$$F(t) = \int_0^t 2 \min(\tau, 1 - \tau) \, d\tau = \begin{cases} 
t^2 & \text{for } 0 \leq t \leq 1/2 \\
2t - t^2 - \frac{1}{2} & \text{for } 1/2 \leq t \leq 1.
\end{cases}$$

We then have $F'(\sigma \phi) = 2 \min(\sigma \phi, 1 - \sigma \phi)$ and so

$$\frac{1}{\sigma^{3/2}} |\nabla F(\sigma \phi)| \leq 2 \frac{1}{\sqrt{\sigma}} |\nabla \phi| \min(\sigma \phi, 1 - \sigma \phi) \leq \frac{1}{\sigma \varepsilon} \min((\sigma \phi)^2, (1 - \sigma \phi)^2) + \varepsilon |\nabla \phi|^2 \leq \frac{1}{\sigma \varepsilon} \left[ (1 - \rho)(\sigma \phi)^2 + \rho(1 - \sigma \phi)^2 \right] + \varepsilon |\nabla \phi|^2$$

(as long as $0 \leq \rho \leq 1$). This inequality, together with the formula \eqref{eq2.2} for $\mathcal{J}_\varepsilon$ implies

\begin{equation}
\frac{1}{2\sigma^{3/2}} \int_\Omega |\nabla F(\sigma \phi^\varepsilon)| \, dx \leq \mathcal{J}_\varepsilon(\rho^\varepsilon)
\end{equation}

(note that in \cite{15} a similar inequality was derived when $\rho$ is a characteristic function. The computation above extends this important property of $\mathcal{J}_\varepsilon$ to the case where $0 \leq \rho \leq 1$ by using the formula \eqref{eq2.2}).

Inequality \eqref{eq4.2} shows that the boundedness of the energy $\mathcal{J}_\varepsilon(\rho^\varepsilon)$ implies some a priori estimates for the auxiliary function

$$\psi^n := 2F(\sigma \phi^\varepsilon)$$
More precisely, (4.2) and Lemma 3.1 (i) imply that
\[ \psi^n \text{ is bounded in } L^\infty((0, T); BV(\Omega)). \]

Next, we can write
\[ \psi^n = [2F(\sigma \phi^n) - 2F(\rho^n)] + [2F(\rho^n) - \rho^n] + \rho^n \]
and we are going to show that the first two terms in the right hand side go to zero (uniformly in t):
\begin{itemize}
  \item Formula (2.1) and the energy bound (Lemma 3.1 (i)) imply
    \[ ||\rho^n(t) - \sigma \phi^n(t)||_{L^2(\Omega)}^2 \leq 2\sigma \varepsilon_n J(\rho^n(t)) \leq 2\sigma \varepsilon_n J(\rho_n) \leq C \varepsilon_n \]
    and since \( F \) is Lipschitz, we deduce (with \( \delta = \sqrt{\varepsilon_n} \))
    \[ \int \Omega |2F(\rho^n(t)) - 2F(\sigma \phi^n(t))|^{2} \leq C \|\rho^n(t) - \sigma \phi^n(t)\|_{L^2(\Omega)}^2 \leq C \varepsilon_n \quad \forall t > 0. \]
  \item When \( \rho \) is a characteristic function, we have \( 2F(\rho) = \rho \) and so the second term in (4.4) vanishes. When \( \rho \in (0, 1) \), we can use the fact that \( |2F(\rho) - \rho| \leq C \delta \) whenever \( \rho < \delta \) or \( \rho > 1 - \delta \) and use the energy to control the set where \( \delta \leq \rho \leq 1 - \delta \). Indeed, formula (2.1) and the energy bound (Lemma 3.1 (i)) imply
    \[ |\{\delta \leq \rho^n \leq 1 - \delta\}| \leq \frac{1}{\delta(1 - \delta)} \int \Omega \rho^n(1 - \rho^n) \, dx \leq \frac{\varepsilon_n}{\delta(1 - \delta)} J(\rho^n(t)) \leq C \frac{\varepsilon_n}{\delta(1 - \delta)}. \]
    We deduce (with \( \delta = \sqrt{\varepsilon_n} \)):
    \[ \int \Omega |2F(\rho^n) - \rho^n| \, dx \leq \int \{\sqrt{\varepsilon_n} \leq \rho^n \leq 1 - \sqrt{\varepsilon_n}\} |2F(\rho^n) - \rho^n| \, dx + C|\Omega| \sqrt{\varepsilon_n} \]
    \[ \leq C|\{\sqrt{\varepsilon_n} \leq \rho^n \leq 1 - \sqrt{\varepsilon_n}\}| + C|\Omega| \sqrt{\varepsilon_n} \]
    \[ \leq C \sqrt{\varepsilon_n} \]
    and so
    \[ ||2F(\rho^n(t)) - \rho^n(t)||_{L^2(\Omega)} \leq C \varepsilon_n^{1/4} \quad \forall t > 0. \]
\end{itemize}

Since we already know that \( \rho^n \) converges uniformly in \( t \), with respect to the \( H^{-1}(\Omega) \) norm, to \( \rho \), we deduce from (4.4), (4.5), and (4.7) that
\[ \psi^n(t) \to \rho(t) \text{ in } H^{-1}(\Omega), \text{ uniformly in } t. \]

Using a Lions-Aubin compactness type result (see Lemma B.1), (4.3) and (4.8) imply
\[ \psi^n \to \rho \quad \text{strongly in } L^\infty((0, T); L^1(\Omega)). \]

In particular, the lower semicontinuity of the \( BV \) norm and (4.3) imply that \( \rho \in L^\infty((0, T); BV(\Omega)). \)

Finally, using (4.4) together with (4.5) and (4.7) we see that
\[ ||\rho^n(t) - \psi^n(t)||_{L^2(\Omega)} \leq C \varepsilon_n^{1/4} \]
so the strong convergence of \( \psi^n \) also implies that
\[ \rho^n \to \rho \quad \text{strongly in } L^\infty((0, T); L^1(\Omega)). \]

It remains to show that \( \rho \) is a characteristic function. We note that given \( t > 0 \), we can extract a subsequence which converges a.e. in \( \Omega \) and (4.8) implies that \( |\{\delta \leq \rho(t) \leq 1 - \delta\}| = 0 \) for all \( \delta > 0 \). We deduce that \( \rho(x, t) \in \{0, 1\} \) a.e. \( x \in \Omega \) (for all \( t > 0 \)).
Finally, we can pass to the limit in (1.12) to get the energy inequality (1.19): The lower-semicontinuity of $\Theta$ allows us to pass to the limit in the dissipation and Proposition 1.4 gives (since $F_\varepsilon(\rho_{in}) = J_\varepsilon(\rho_{in})$ when $\rho_{in}$ is a characteristic function)

$$\lim_{\varepsilon \to 0} F_\varepsilon(\rho_{in}) = J_0(\rho_{in}).$$

The liminf property of Proposition C.1 and the strong convergence of $\rho_\varepsilon$ then yield:

$$\lim_{\varepsilon \to 0} \liminf F_\varepsilon(\rho_{in}) \geq \mu \hat{\Omega} \rho_\varepsilon \log \rho_\varepsilon \, dx + \lim_{\varepsilon \to 0} \liminf J_\varepsilon(\rho_\varepsilon) \geq J_0(\rho_{in}).$$

This completes the proof of Proposition 4.1.

5. Convergence of the first variation

5.1. Convergence of the first variation and proof of Theorem 1.6

The only thing left to do to prove Theorem 1.6 is to pass to the limit in equation (1.11) (under the convergence assumption (1.20)) to derive (1.15). We recall (1.11) here:

$$\int_0^\infty \int_\Omega E^\varepsilon \cdot \xi \, dx \, dt = \int_0^\infty \int_\Omega \varepsilon^{-1} \rho^\varepsilon \nabla \phi^\varepsilon \cdot \xi + \mu \rho^\varepsilon \div \xi + p^\varepsilon \div \xi \, dx \, dt$$

and we note that when $\varepsilon \ll 1$, neither the term $\varepsilon^{-1} \rho^\varepsilon \nabla \phi^\varepsilon$ nor the function $p^\varepsilon$ (or its gradient $\nabla p^\varepsilon$) are bounded. As explained in the introduction, it is the modified pressure, defined by $p^\varepsilon + \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \rho^\varepsilon$ which is expected to converge to the pressure in the Hele-Shaw model with surface tension. We can also include the term $\mu \rho^\varepsilon$ in this modified pressure and normalize it to have zero average. We thus set:

$$(5.1) \quad q^\varepsilon = p^\varepsilon + \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \rho^\varepsilon + \mu \rho^\varepsilon + m^\varepsilon(\rho^\varepsilon)$$

where $m^\varepsilon(t)$ (constant in $x$) is chosen so that

$$(5.2) \quad \int_\Omega q^\varepsilon(x,t) \, dx = 0 \quad \forall t > 0.$$

After a straightforward computation, we can rewrite (1.11) as

$$(5.3) \quad \int_0^\infty \int_\Omega E^\varepsilon \cdot \xi \, dx \, dt = \int_0^\infty \int_\Omega \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho^\varepsilon + q^\varepsilon \div \xi \, dx \, dt.$$

Passing to the limit in (5.3) is the objective of this section. The main challenge being the term

$$(5.4) \quad \int_0^\infty \int_\Omega \varepsilon^{-1} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho^\varepsilon \, dx \, dt,$$

which gives rise to the mean-curvature and the contact angle condition in the limit $\varepsilon \to 0$. This term is related to the first variation of the energy $J_\varepsilon$ defined by (1.17). The key result of this section is Proposition 5.7 below, which states that the assumption on the convergence of the energy (1.20) implies the convergence of the first variation. Similar results have been proved for different energy functionals. In particular, a classical result of Reshetnyak [17] gives that if $\chi_{E_{\varepsilon}}$ converges to $\chi_E$ strongly in $L^1$, then the convergence of the perimeter

$$P(E_{\varepsilon}) := \int |\nabla \chi_{E_{\varepsilon}}| \to \int |\nabla \chi_{E}|.$$
implies the convergence of the first variation to
\[ \int (\nabla \xi - \nu \otimes \nu : D\xi)|D\chi_E|. \]
A similar result was proved by Luckhaus and Modica \[13\] for the Ginzburg-Landau functional
\[ E^1_\varepsilon(\rho) = \int_{\Omega} \varepsilon|\nabla \rho|^2 + \frac{1}{\varepsilon}(1 - \rho^2)^2 \, dx \]
and by Laux and Otto \[12\] with
\[ E^1_\varepsilon(\rho) = \varepsilon^{-1} \int \rho (1 - G_\varepsilon \ast \rho) \, dx \]
when \( G_\varepsilon \) is the Gaussian kernel of variance \( \varepsilon^2 \).

Crucially, both \( E^1_\varepsilon \) and \( E^2_\varepsilon \) are regularizations of the perimeter functional. In our case, we recall (see Theorem 1.5) that the energy functional \( J_\varepsilon \) \( \Gamma \)-converges to the perimeter functional as well, together with a boundary term. Furthermore, it is easy to check that (5.4) is the first variation of \( J_\varepsilon(\rho) \) for a perturbation defined by
\[
\begin{aligned}
\partial_s \rho_s + \nabla \rho_s \cdot \xi &= 0 \\
\rho_s|_{s=0} &= \rho.
\end{aligned}
\]

**Remark 5.1.** Note that (5.5) preserves the constraint \( \rho_s \leq 1 \) but not the condition \( \int \rho_s = 1 \) (unless \( \text{div} \xi = 0 \)). Alternatively, we could consider the first variation of \( J_\varepsilon(\rho) \) for a perturbation defined by
\[
\begin{aligned}
\partial_s \rho_s + \text{div}(\rho_s \xi) &= 0 \\
\rho_s|_{s=0} &= \rho.
\end{aligned}
\]
which leads to the integral \( \varepsilon^{-1} \int_{\Omega} \rho \xi \nabla \left( \frac{1}{2\varepsilon} - \phi^\varepsilon \right) \, dx \). The two integrals are the same when \( \text{div} \xi = 0 \), but this second integral does not, in general, converge when \( \varepsilon \to 0 \). This is due to the fact that (5.6) does not preserve characteristic functions and the energy \( J_\varepsilon \) blows up in that case for \( \varepsilon \ll 1 \).

The key result of this section, which will allow us to complete the proof of Theorem 1.6, is the following:

**Proposition 5.2.** Given a sequence of functions \( \rho^\varepsilon \in L^1(\Omega) \) satisfying \( 0 \leq \rho^\varepsilon \leq 1 \) and such that \( \rho^\varepsilon \to \rho \) strongly in \( L^1(\Omega) \) with \( \rho \in \text{BV}(\Omega; \{0,1\}) \) and
\[
\lim_{\varepsilon \to 0} J_\varepsilon(\rho^\varepsilon) = J_0(\rho),
\]
we have, for all \( \xi \in C^1(\Omega, \mathbb{R}^d) \) satisfying \( \xi \cdot n = 0 \) on \( \partial \Omega \),
\[
|\varepsilon^{-1} \int_{\Omega} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho^\varepsilon | \leq C \|D\xi\|_{L^\infty(\Omega)} J_\varepsilon(\rho^\varepsilon)
\]
and
\[
\lim_{\varepsilon \to 0} -\varepsilon^{-1} \int_{\Omega} \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho^\varepsilon
\]
\[
= \frac{1}{4\sigma^{3/2}} \int_{\Omega} |\nabla \xi - \nu \otimes \nu : D\xi| \, dx + \min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\sigma \beta}} \right) \int_{\partial \Omega} |\nabla \xi - n \otimes n : D\xi| \, d\mathcal{H}^{n-1}(x)
\]
where \( \nu = \frac{\nabla \rho}{|\nabla \rho|} \) and \( n \) denotes the outward normal unit vector to the fixed boundary \( \partial \Omega \).
Note that we can replace (5.7) with the equivalent condition
\[ \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon (\rho^\varepsilon) = \mathcal{F}_0 (\rho). \]
Indeed a bound on \( \mathcal{F}_\varepsilon (\rho^\varepsilon) \) or \( \mathcal{F}_\varepsilon (\rho^\varepsilon) \) implies that \( \rho \in BV(\Omega; \{0, 1\}) \) (see the proof of Proposition 4.1) so that \( \mathcal{F}_0 (\rho) = \mathcal{F}_0 (\rho) \) and \( \int \rho^\varepsilon \log \rho^\varepsilon \to 0 \).

Proceeding as in [12], we can check that this proposition imply the following time-dependent version:

**Corollary 5.3.** Given a sequence of characteristic functions \( \rho^\varepsilon (x, t) \) such that \( \rho^\varepsilon \to \rho \) in \( L^1 (\Omega \times (0, T)) \) and
\[ \lim_{\varepsilon \to 0} \int_0^T \mathcal{F}_\varepsilon (\rho^\varepsilon) \, dt = \int_0^T \mathcal{F}_0 (\rho) \, dt \]
we have, for all \( \xi \in C^1 (\Omega \times (0, T), \mathbb{R}^d) \) satisfying \( \xi \cdot n = 0 \) on \( \partial \Omega \),
\[ \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T \int_\Omega \left( \frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho^\varepsilon \, dt \]
\[ = \frac{1}{4\sigma^{3/2}} \left[ \int_0^T \int_\Omega \left[ \text{div} \xi - \nu \otimes \nu : D\xi \right] |\nabla \rho| \, dt \right] \]
\[ + \min \left( 1, \frac{2\alpha}{\alpha + \sqrt{\alpha \beta}} \right) \int_0^T \int_{\partial \Omega} [\text{div} \xi - n \otimes n : D\xi] \rho \, d\mathcal{H}^{n-1} (x) \, dt \]
(5.10)

With this corollary, we can now complete the proof of Theorem 1.6

*End of the proof of Theorem 1.6.* We can now pass to the limit in [5.3] and thus complete the proof of Theorem 1.6. We recall that \( E^\varepsilon \) converges weakly in \( L^2 \) and the convergence of the first term in the right hand side follows from Corollary 5.3, which we can use here since Proposition 4.1 gives the strong convergence of \( \rho^\varepsilon \) in \( L^1 \) and we are assuming condition (1.20). We thus only need to explain why the pressure \( q^\varepsilon \) converges.

We note that (5.3) and (5.8) imply
\[ \int_0^T \int_\Omega q^\varepsilon \text{div} \xi \, dx \, dt \leq \| E^\varepsilon \|_{L^2 (\Omega \times (0, T))} \| \xi \|_{L^2 (\Omega \times (0, T))} + C \mathcal{F}_\varepsilon (\rho (t)) \| D\xi \|_{L^1 ((0, T); L^\infty (\Omega))} \]
(5.11)

In particular, the a priori estimates of Lemma 3.1 implies that \( \nabla q^\varepsilon \) is bounded in \( L^2 ((0, T); (C^1 (\Omega))^*) \).

To get a bound on \( q^\varepsilon \), we proceed as in [9]: For \( \varphi \in C^4 (\Omega) \), we consider \( u \) solution of
\[ \begin{cases} \Delta u = \varphi - \int_\Omega \varphi \, dx & \text{in } \Omega \\ \nabla u \cdot n = 0 & \text{on } \partial \Omega \end{cases} \]
and take \( \xi = \nabla u \) as a test function in (5.11) to get (using classical Schauder estimates)
\[ \int_0^T \int_\Omega q^\varepsilon \Delta u \, dx \, dt \leq C \| D^2 u \|_{L^2 ((0, T); L^\infty (\Omega))} \leq C \| u \|_{L^2 ((0, T); C^{2, s} (\Omega))} \leq C \| \varphi \|_{L^2 ((0, T); C^{s} (\Omega))} \]
for any \( s > 0 \). Using (5.2) we deduce
\[ \int_0^T \int_\Omega q^\varepsilon \varphi \, dx \, dt \leq C \| \varphi \|_{L^2 ((0, T); C^{s} (\Omega))} \]
which implies that \( q^\varepsilon \) is uniformly bounded in \( L^2 ((0, T); (C^s (\Omega))^*) \) and has a weak-* limit \( q \). We can now pass to the limit in (5.3), for \( \xi \) smooth enough, and derive (1.15). \( \square \)
5.2. Proof of Proposition 5.2. As noted earlier, if we set \( \tilde{\phi} = \sigma \phi, \bar{\epsilon} = \epsilon / \sqrt{\sigma} \) and \( \bar{\beta} = \sqrt{\beta} \), equation (1.4) becomes

\[
\begin{cases}
\tilde{\phi} - \bar{\epsilon}^2 \Delta \tilde{\phi} = \rho & \text{in } \Omega \\
\alpha \tilde{\phi} + \bar{\beta} \bar{\epsilon} \nabla \tilde{\phi} \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It is therefore enough to prove the result when \( \sigma = 1 \) and use the fact that

\[
\epsilon^{-1} \int_\Omega \left( \frac{1}{2\sigma} - \bar{\epsilon}^2 \right) \xi \cdot \nabla \rho = \epsilon^{-1} \frac{1}{\sigma^{3/2}} \int_\Omega \left( \frac{1}{2} - \bar{\epsilon}^2 \right) \xi \cdot \nabla \rho^\sigma
\]
to get the result when \( \sigma > 0 \).

The proof of Proposition 5.2 makes use of the following lemma, the proof of which is elementary and presented at the end of this section:

**Lemma 5.4.** Given \( \rho(x) \) such that \( 0 \leq \rho \leq 1 \) and \( \phi \) solution of (1.4) with \( \sigma = 1 \), we have the following formulas, for all \( \xi \in C^1(\Omega, \mathbb{R}^d) \):

(i) If \( \alpha = 0 \) or \( \beta = 0 \) (Neumann or Dirichlet boundary condition for \( \phi \)) then

\[
- \epsilon^{-1} \int_\Omega (1 - 2\phi) \xi \cdot \nabla \rho = \epsilon^{-1} \int_\Omega \left[ (1 - \rho) \phi^2 + \rho (1 - \phi)^2 \right] \text{div} \xi \, dx + \epsilon \int_\Omega |\nabla \phi|^2 \text{div} \xi \, dx
\]

\[
- \epsilon^{-1} \int_\Omega (1 - 2\phi) \xi \cdot \nabla \rho = \epsilon^{-1} \int_\Omega \left[ (1 - \rho) \phi^2 + \rho (1 - \phi)^2 \right] \text{div} \xi \, dx
\]

\[
+ \epsilon \int_\Omega |\nabla \phi|^2 \text{div} \xi \, dx + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi|^2 \text{div} \xi \, dH^{n-1}(x)
\]

\[
- 2\epsilon \int_\Omega \nabla \phi \otimes \nabla \phi : D\xi \, dx - \int_{\partial \Omega} \frac{\alpha}{\beta} \phi^2 n \otimes n : D\xi \, dH^{n-1}(x).
\]

(ii) If \( \beta \neq 0 \), then

\[
- \epsilon^{-1} \int_\Omega (1 - 2\phi) \xi \cdot \nabla \rho = \epsilon^{-1} \int_\Omega \left[ (1 - \rho) \phi^2 + \rho (1 - \phi)^2 \right] \text{div} \xi \, dx
\]

\[
+ \epsilon \int_\Omega |\nabla \phi|^2 \text{div} \xi \, dx + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi|^2 \text{div} \xi \, dH^{n-1}(x)
\]

\[
- 2\epsilon \int_\Omega \nabla \phi \otimes \nabla \phi : D\xi \, dx - \int_{\partial \Omega} \frac{\alpha}{\beta} \phi^2 n \otimes n : D\xi \, dH^{n-1}(x).
\]

**Idea of the proof of Proposition 5.2.** While the proof of this Proposition 5.2 may appear long and technical, the idea is quite simple. We recall that when \( \alpha = 0 \) (Neumann boundary condition), we have (see (2.2)):

\[
\mathcal{J}_\epsilon(\rho^\sigma) = \frac{\epsilon^{-1}}{2} \int_\Omega (1 - \rho^\sigma)(\phi^\sigma)^2 + \rho^\sigma (1 - \phi^\sigma)^2 \, dx + \frac{\epsilon}{2} \int_\Omega |\nabla \phi^\sigma|^2 \, dx
\]

\[
= \frac{1}{2} \int_\Omega u_\epsilon^2 + v_\epsilon^2 \, dx.
\]

where we denoted \( u_\epsilon = \epsilon^{-1/2} \left[ (1 - \rho^\sigma)(\phi^\sigma)^2 + \rho^\sigma (1 - \phi^\sigma)^2 \right]^{1/2} \) and \( v_\epsilon = \epsilon^{1/2} |\nabla \phi^\sigma| \). The boundedness of the energy implies that \( u_\epsilon \) and \( v_\epsilon \) are bounded in \( L^2 \) and in order to pass to the limit in the first two terms in (5.12), we need to show the convergence of \( \int [u_\epsilon^2 + v_\epsilon^2] \text{div} \xi \, dx \).

Using the notations of Proposition 4.1 and in particular the function \( F \) such that \( F'(\phi) = 2 \min(\phi, 1 - \phi) \), we now write

\[
|\nabla F(\phi^\sigma)| = 2 \min(\phi^\sigma, 1 - \phi^\sigma) |\nabla \phi^\sigma| \leq 2u_\epsilon v_\epsilon \leq u_\epsilon^2 + v_\epsilon^2
\]

Since the limit \( \rho \) of \( \rho^\sigma \) is a characteristic function and we know that \( \phi^\sigma \to \rho \), we have \( F(\phi^\sigma) \to F(\rho) = \frac{1}{2} \rho \) in \( L^1 \) (see the proof of Proposition 4.1), and so

\[
\lim_{\epsilon \to 0} \inf \int_\Omega |\nabla F(\phi^\sigma)| \, dx \geq \frac{1}{2} \int_\Omega |\nabla \rho|.
\]
On the other hand, the assumption of convergence of the energy, (5.7), implies
\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon^2 + v_\varepsilon^2 \, dx = 2 J_0(\rho) = \frac{1}{2} \int |\nabla \rho|.
\]
Together, these inequalities imply that there is equality in (5.14) when \( \varepsilon \to 0 \), which means that \( u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \) in \( L^1 \) and that \( \lim_{\varepsilon \to 0} \int |\nabla F(\phi^\varepsilon)| \, dx = \frac{1}{2} \int |\nabla \rho| \). A Classical result (see Proposition A.1) now implies that
\[
\lim_{\varepsilon \to 0} \int_{\Omega} [u_\varepsilon^2 + v_\varepsilon^2] \, \text{div} \, \xi \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla F(\phi^\varepsilon)| \, \text{div} \, \xi \, dx = \frac{1}{2} \int_{\Omega} |\nabla \rho| \, \text{div} \, \xi.
\]
To pass to the limit in the last term in (5.12), we note that the (asymptotic) equality in Young’s inequality in (5.14) also implies that \( u_\varepsilon - v_\varepsilon \to 0 \) in \( L^2 \) and so \( 2 \varepsilon^{-1} |\nabla \phi^\varepsilon|^2 = 2 v_\varepsilon^2 \sim u_\varepsilon^2 + v_\varepsilon^2 \sim |\nabla F(\phi^\varepsilon)| \) which proves the convergence of \( 2 \varepsilon^{-1} |\nabla \phi^\varepsilon|^2 \). A simple (if somewhat technical) result (see Proposition A.2) then shows that the convergence of \( \varepsilon^{-1} |\nabla \phi^\varepsilon|^2 \) implies that of \( \varepsilon^{-1} \nabla \phi^\varepsilon \otimes \nabla \phi^\varepsilon \).

Additional care will be needed to take care of the boundary condition when \( \alpha \neq 0 \), which is why we will first give the detailed proof for Neumann boundary conditions, then Dirichlet conditions (which requires extending \( \phi^\varepsilon \) to \( \mathbb{R}^d \) by 0) and finally general Robin boundary conditions (which combine the Neumann and Dirichlet case).

**Proof of Proposition 5.2.** We note that (5.13) together with (2.1) immediately imply (5.8).

The difficult part of the proof is, of course, to establish the limit (5.9), and we will first give the proof in the simpler case of **Neumann boundary conditions** \( (\alpha = 0) \). We have to pass to the limit in (5.12). As above, we use the function \( F \) such that \( F'(\phi) = 2 \min(\phi, 1 - \phi) \) and note that
\[
F'(\phi) \leq 2 \left[ (1 - \rho) \phi^2 + \rho (1 - \phi)^2 \right]^{1/2}
\]
when \( 0 \leq \rho \leq 1 \). We thus have \( |\nabla F(\phi^\varepsilon)| \leq 2|\nabla \phi^\varepsilon| \left[ (1 - \rho^\varepsilon) \phi^2 + \rho^\varepsilon (1 - \phi^\varepsilon)^2 \right]^{1/2} \) and so
\[
|\nabla F(\phi^\varepsilon)| \leq 2 u_\varepsilon v_\varepsilon \leq u_\varepsilon^2 + v_\varepsilon^2 - (u_\varepsilon - v_\varepsilon)^2,
\]
where
\[
u_\varepsilon := \varepsilon^{-1/2} \left[ (1 - \rho^\varepsilon) (\phi^\varepsilon)^2 + \rho^\varepsilon (1 - \phi^\varepsilon)^2 \right]^{1/2} \quad \text{and} \quad v_\varepsilon = \varepsilon^{1/2} |\nabla \phi^\varepsilon|.
\]

Next, the strong convergence of \( \rho^\varepsilon \) and (4.5) imply that \( F(\phi^\varepsilon) \) converges to \( F(\rho) \) strongly in \( L^1 \). Since \( \rho \) is a characteristic function and \( F(0) = 0, F(1) = 1/2 \), we deduce \( F(\phi^\varepsilon) \to F(\rho) = \frac{1}{2} \rho \) strongly in \( L^1 \) and so:
\[
\liminf_{\varepsilon \to 0} \int_{\Omega} |\nabla F(\phi^\varepsilon)| \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla \rho|.
\]
On the other hand, the convergence assumption (5.7) implies
\[
\int_{\Omega} u_\varepsilon^2 + v_\varepsilon^2 \, dx = 2 J_\varepsilon(\rho^\varepsilon) \to \frac{1}{2} \int_{\Omega} |\nabla \rho|.
\]
Inequality (5.15) thus implies:
\[
\begin{align*}
(5.16) & \quad u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in} \ L^1(\Omega) \\
(5.17) & \quad \int_{\Omega} |\nabla F(\phi^\varepsilon)| \, dx \to \frac{1}{2} \int_{\Omega} |\nabla \rho| \\
(5.18) & \quad u_\varepsilon - v_\varepsilon \to 0 \quad \text{in} \ L^2(\Omega).
\end{align*}
\]
These facts allow us to pass to the limit in the first two terms of (5.12). Indeed, using first the definition of \( u^\varepsilon \) and \( v^\varepsilon \), then the limit (5.16) and finally (5.17) (together with Proposition A.1), we can write:

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\Omega} (\rho^\varepsilon - \phi^\varepsilon)^2 \text{div} \xi \, dx + \varepsilon \int_{\Omega} \frac{\nabla \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \cdot \nabla \xi \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} (u^\varepsilon + v^\varepsilon)^2 \text{div} \xi \, dx
\]

\[
= \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla F(\phi^\varepsilon)| \text{div} \xi \, dx
\]

\[
= \frac{1}{2} \int_{\Omega} \text{div} \xi |\nabla \rho|.
\]

(5.19)

Furthermore, (5.16) and (5.18) yields:

\[
2u^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in} \quad L^1(\Omega)
\]

which we use to pass to the limit in the term involving \( \nabla \phi^\varepsilon \otimes \nabla \phi^\varepsilon \). Indeed, we can write

\[
2\varepsilon \int_{\Omega} \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi \, dx = 2\varepsilon \int_{\Omega} \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi \, dx = \int_{\Omega} \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_j \phi^\varepsilon \partial_i \xi \, dx = \int_{\Omega} \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_j \phi^\varepsilon \partial_i \xi \, dx
\]

and since \( \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_j \phi^\varepsilon \partial_i \xi \) is bounded in \( L^\infty \), (5.20) implies that

\[
\lim_{\varepsilon \to 0} 2\varepsilon \int_{\Omega} \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_j \phi^\varepsilon \partial_i \xi \, dx.
\]

Using the fact that \( F'(\phi) \geq 0 \), we can also write

\[
\lim_{\varepsilon \to 0} 2\varepsilon \int_{\Omega} \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{\partial_i F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \partial_j F(\phi^\varepsilon) \partial_i \xi \, dx.
\]

and using (5.17) and Proposition A.2 we deduce

\[
\lim_{\varepsilon \to 0} 2\varepsilon \int_{\Omega} \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi \, dx = \frac{1}{2} \int_{\Omega} \frac{\partial_i \rho}{|\nabla \rho|} \partial_j \phi^\varepsilon \partial_i \xi \, dx.
\]

(5.21)

Using (5.12) together with (5.19) and (5.21) we get

\[
\lim_{\varepsilon \to 0} -\varepsilon^{-1} \int_{\Omega} (1 - 2\phi^\varepsilon) \xi \cdot \nabla \rho = \frac{1}{2} \int_{\Omega} \text{div} \xi |\nabla \rho| + \frac{1}{2} \int_{\Omega} \frac{\partial_i \rho}{|\nabla \rho|} \partial_j \phi^\varepsilon \partial_i \xi \, dx
\]

which gives (5.9) in the case of Neumann boundary condition \( \alpha = 0 \) (and when \( \sigma = 1 \)).

For the case of Dirichlet conditions \( (\beta = 0) \), we proceed similarly, but we first extend \( \phi^\varepsilon \) to \( \mathbb{R}^d \setminus \Omega \) by setting it equal to 0 in \( \mathbb{R}^d \setminus \Omega \). Denoting \( \tilde{\phi}^\varepsilon \) this extension, we find (using inequality (5.13) and the Dirichlet boundary condition for \( \phi^\varepsilon \)):

\[
\int_{\mathbb{R}^d} |\nabla F(\tilde{\phi}^\varepsilon)| \, dx = \int_{\Omega} |\nabla F(\phi^\varepsilon)| \, dx \leq \int_{\Omega} u^2 + v^2 \, dx = 2 J_\varepsilon(\rho^\varepsilon).
\]

The lower semicontinuity of the BV norm and assumption (5.7) give

\[
\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\nabla F(\tilde{\phi}^\varepsilon)| \, dx \geq \int_{\mathbb{R}^d} |\nabla F(\rho)| = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \rho| = \lim_{\varepsilon \to 0} 2 J_\varepsilon(\rho^\varepsilon).
\]

Indeed, when \( \beta = 0 \) (and \( \sigma = 1 \)), we have

\[
J_0(\rho) = \frac{1}{4} \int_{\Omega} |\nabla \rho| + \int_{\partial \Omega} \rho(x) d\mathcal{H}^{n-1}(x) = \frac{1}{4} \int_{\mathbb{R}^d} |\nabla \rho|.
\]
We can thus proceed as before to show that
\[ u_ε^2 + v_ε^2 - |\nabla F(φ^ε)| \to 0 \quad \text{in} \ L^1(Ω) \]
\[ \int_{R^d} |\nabla F(ϕ)| \, dx \to \frac{1}{2} \int_{R^d} |\nabla ρ| \]
\[ u_ε - v_ε \to 0 \quad \text{in} \ L^2(Ω). \]
These are the same convergences as (5.16)-(5.18), except for (5.17) which involves the extension \( \tilde{φ}^ε \). We can now write:
\[
\lim_{ε \to 0} \varepsilon^{-1} \int _{Ω} [(1 - ρ^ε)(φ^ε)^2 + ρ^ε(1 - φ^ε)^2] \, dV + \varepsilon \int _{Ω} |\nabla φ^ε|^2 \, dV
\]
\[
= \lim_{ε \to 0} \int _{Ω} \, dV (u_ε^2 + v_ε^2)
\]
\[
= \lim_{ε \to 0} \int _{Ω} \, dV |\nabla F(φ^ε)|
\]
\[
= \lim_{ε \to 0} \int _{Ω} \, dV |\nabla F(ϕ)|
\]
\[
= \frac{1}{2} \int _{R^d} \, dV |\nabla ρ| = \frac{1}{2} \left( \int _{Ω} \, dV |\nabla ρ| + \int _{∂Ω} ρ \, dH^{n-1}(x) \right).
\]
Similarly, we can show (as before)
\[
2ε \int _{Ω} \nabla φ^ε \otimes \nabla φ^ε : Dξ \, dx \to \frac{1}{2} \int _{R^d} \left( \frac{\nabla \tilde{ρ}}{|\nabla \tilde{ρ}|} \otimes \frac{\nabla \tilde{ρ}}{|\nabla \tilde{ρ}|} : Dξ \right) |\nabla ρ|
\]
where
\[
\int _{R^d} \left( \frac{\nabla \tilde{ρ}}{|\nabla \tilde{ρ}|} \otimes \frac{\nabla \tilde{ρ}}{|\nabla \tilde{ρ}|} : Dξ \right) |\nabla ρ| = \int _{Ω} (\nu \otimes \nu : Dξ) |\nabla ρ| + \int _{∂Ω} (n \otimes n : Dξ) ρ \, dH^{n-1}(x).
\]
This gives the result when \( β = 0 \) (Dirichlet boundary conditions).

For the case of Robin conditions, we need to combine the two cases above. We first choose \( φ : R^d \to [0, 1] \) continuous function such that
\[
\int _{∂Ω} |χ_E - φ| \, dH^{n-1} \leq δ.
\]
Introducing the function \( G(t) = \frac{2}{β} t^2 - F(t) \), we then write:
\[
\int _{R^d} |\nabla F(ϕ)| \, dx + \int _{∂Ω} G(φ^ε)φ^ε \, dH^{n-1} = \int _{R^d} |\nabla F(φ^ε)| \, dx + \int _{∂Ω} F(φ^ε)φ^ε \, dH^{n-1} + \int _{∂Ω} G(φ^ε)φ^ε \, dH^{n-1}
\]
\[
= \int _{Ω} |\nabla F(φ^ε)| \, dx + \int _{∂Ω} \frac{α}{β} |φ^ε|^2 \, dH^{n-1}
\]
which leads to
\[
\int _{Ω} |\nabla F(φ^ε)|(1 - φ) \, dx + \int _{R^d} |\nabla F(ϕ)| \, dx + \int _{∂Ω} G(φ^ε)φ^ε \, dH^{n-1} + \int _{∂Ω} \frac{α}{β} |φ^ε|^2 (1 - φ) \, dH^{n-1}
\]
\[
= \int _{Ω} |\nabla F(φ^ε)| \, dx + \int _{∂Ω} \frac{α}{β} |φ^ε|^2 \, dH^{n-1}.
\]
The right hand side satisfies (with the same notations as above):
\[
\int_\Omega |\nabla F(\phi')|\,dx + \int_{\partial \Omega} \frac{\alpha}{\beta} \phi'^2 d\mathcal{H}^{n-1} \leq \int_\Omega u^2 + v^2 - (u - v)^2\,dx + \int_{\partial \Omega} \frac{\alpha}{\beta} \phi'^2 d\mathcal{H}^{n-1} \\
= 2 \mathcal{J}_\varepsilon(\rho') - \int_\Omega (u - v)^2\,dx
\]
so in order to proceed as before, we need to show that the lim inf of the left hand side is greater than
\[
2 \mathcal{J}_0(\rho).
\]
For this, we notice that the function
\[
G(t) = \frac{\alpha}{\beta} t^2 - F(t) = \begin{cases} \left(\frac{\alpha}{\beta} - 1\right) t^2 & \text{for } 0 \leq t \leq 1/2 \\
\left(\frac{\alpha}{\beta} + 1\right) t^2 - 2t + \frac{1}{2} & \text{for } 1/2 \leq t \leq 1 \end{cases}
\]
satisfies
\[
\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi')|(1 - \varphi)\,dx + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi'|^2 d\mathcal{H}^{n-1} + \int_\Omega G(\phi') \varphi d\mathcal{H}^{n-1} + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi'|^2 (1 - \varphi) d\mathcal{H}^{n-1}
\geq \int_\Omega |\nabla F(\rho)|(1 - \varphi)\,dx + \int_{\partial \Omega} |\nabla F(\bar{\rho})|\varphi\,dx + \int_{\partial \Omega} \min \left\{ 0, \frac{\alpha}{\alpha + \beta} - \frac{1}{2} \right\} \varphi d\mathcal{H}^{n-1} + 0
\geq \int_{\partial \Omega} \frac{1}{2} |\nabla \rho|\,dx + \int_{\partial \Omega} \frac{1}{2} |\nabla \varphi| d\mathcal{H}^{n-1} + \int_{\partial \Omega} \min \left\{ 0, \frac{1}{2}, \frac{\alpha}{\alpha + \beta} \right\} \varphi d\mathcal{H}^{n-1}
\geq \int_{\partial \Omega} \frac{1}{2} |\nabla \rho|\,dx + \int_{\partial \Omega} \min \left\{ \frac{1}{2}, \frac{\alpha}{\alpha + \beta} \right\} \rho d\mathcal{H}^{n-1} - C\delta
\]
\[
= 2 \mathcal{J}_0(\rho) - C\delta
\]
thanks to our particular choice of \( \varphi \). Going back to (5.22), we see that we just showed that
\[
\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi')|\,dx + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi'|^2 d\mathcal{H}^{n-1} \geq \mathcal{J}_0(\rho) - C\delta
\]
and since this holds for any \( \delta > 0 \), we get
\[
\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi')|\,dx + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi'|^2 d\mathcal{H}^{n-1} \geq \mathcal{J}_0(\rho).
\]
We can now conclude as in the previous cases: Using (5.23) and the assumption that \( \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(\rho') = \mathcal{J}_0(\rho) \) to conclude that
\[
u^2 + v^2 - |\nabla F(\phi')| \to 0 \quad \text{in } L^1(\Omega)
\]
\[
\int_\Omega |\nabla F(\phi')|\,dx \to \frac{1}{2} \int_\Omega |\nabla \rho|
\]
\[
u - v \to 0 \quad \text{in } L^2(\Omega).
\]
Furthermore, using (5.24), we also get
\[
\limsup_{\varepsilon \to 0} \left| \int_\Omega |\nabla F(\phi')(1 - \varphi)|\,dx - \frac{1}{2} \int_\Omega |\nabla \rho|(1 - \varphi) \right| \leq C\delta,
\]
\[
\limsup_{\varepsilon \to 0} \left| \int_\mathbb{R}^d |\nabla F(\bar{\phi})|\varphi\,dx - \frac{1}{2} \int_\mathbb{R}^d |\nabla \rho|\varphi \right| \leq C\delta
\]
\[
\limsup_{\varepsilon \to 0} \left| \int_{\partial \Omega} G(\phi') \varphi d\mathcal{H}^{n-1} - \int_{\partial \Omega} \min \left\{ 0, \frac{\alpha}{\alpha + \beta} - \frac{1}{2} \right\} \varphi d\mathcal{H}^{n-1} \right| \leq C\delta
\]
and
\[
\limsup_{\varepsilon \to 0} \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi\varepsilon|^2 (1 - \varphi) d\mathcal{H}^{n-1} \leq C\delta.
\]

We then write (using (5.25))
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\Omega} [(1 - \rho^c) (\phi^c)^2 + \rho^c (1 - \phi^c)^2] \text{div} \xi d\Omega + \varepsilon \int_{\Omega} |\nabla \phi^c|^2 \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
= \lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} (u^c_x + v^c_y) \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[(5.28)\]
\[
= \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla F(\phi^c)| \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
and using the same function \(\varphi\) as above, we decompose the integral in (5.28) as follows:
\[
\int_{\Omega} |\nabla F(\phi^c)| \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
= \int_{\Omega} |\nabla F(\phi^c)| \varphi \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \varphi \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
+ \int_{\Omega} |\nabla F(\phi^c)| (1 - \varphi) \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 (1 - \varphi) \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
= \int_{\mathbb{R}^d} |\nabla F(\phi^c)| \varphi \text{div} \xi d\Omega + \int_{\partial \Omega} G(\phi^c) \varphi \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
+ \int_{\Omega} |\nabla F(\phi^c)| (1 - \varphi) \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 (1 - \varphi) \text{div} \xi d\mathcal{H}^{n-1}(x)
\]

We deduce
\[
\limsup_{\varepsilon \to 0} \left| \int_{\Omega} |\nabla F(\phi^c)| \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x) \right|
\]
\[
- \frac{1}{2} \int_{\Omega} |\nabla \rho|^2 (1 - \varphi) \text{div} \xi - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \rho| \varphi \text{div} \xi - \int_{\partial \Omega} \min \left\{ 0, \frac{\alpha}{\alpha + \beta} \right\} \varphi \text{div} \xi d\mathcal{H}^{n-1} \right| \leq C\delta
\]
that is
\[
\limsup_{\varepsilon \to 0} \left| \int_{\Omega} |\nabla F(\phi^c)| \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x) - \frac{1}{2} \int_{\Omega} |\nabla \rho| \text{div} \xi - \int_{\partial \Omega} \min \left\{ \frac{1}{2}, \frac{\alpha}{\alpha + \beta} \right\} \varphi \text{div} \xi d\mathcal{H}^{n-1} \right| \leq C\delta
\]
and the choice of \(\varphi\) implies
\[
\limsup_{\varepsilon \to 0} \left| \int_{\Omega} |\nabla F(\phi^c)| \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x) - \frac{1}{2} \int_{\Omega} |\nabla \rho| \text{div} \xi - \int_{\partial \Omega} \min \left\{ \frac{1}{2}, \frac{\alpha}{\alpha + \beta} \right\} \chi_E \text{div} \xi d\mathcal{H}^{n-1} \right| \leq C\delta.
\]
Since the left hand side is independent of \(\varphi\), we can take \(\delta \to 0\) and use (5.28) to get
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\Omega} [(1 - \rho^c) (\phi^c)^2 + \rho^c (1 - \phi^c)^2] \text{div} \xi d\Omega + \varepsilon \int_{\Omega} |\nabla \phi^c|^2 \text{div} \xi d\Omega + \int_{\partial \Omega} \frac{\alpha}{\beta} |\phi^c|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
\[
= \frac{1}{2} \int_{\Omega} |\nabla \rho| \text{div} \xi + \int_{\partial \Omega} \min \left\{ \frac{1}{2}, \frac{\alpha}{\alpha + \beta} \right\} \rho \text{div} \xi d\mathcal{H}^{n-1}
\]
When \( \alpha = 0 \), we have \( \nabla \phi^\varepsilon \cdot n = 0 \) on \( \partial \Omega \) so the last term vanishes. If \( \beta = 0 \), then \( \phi = 0 \) on \( \partial \Omega \) and since \( \xi \) is tangential to \( \partial \Omega \), we have \( \nabla \phi^\varepsilon \cdot \xi = 0 \) on \( \partial \Omega \) and the last term vanish also. In both cases, we get formula (5.12).

In the case of general Robin boundary conditions (with in particular \( \beta \neq 0 \)), we can write the last term as:

\[
2 \varepsilon \int_{\partial \Omega} \nabla \phi^\varepsilon \cdot n \nabla \phi^\varepsilon \cdot \xi d\mathcal{H}^{n-1}(x) = -2 \frac{\alpha}{\beta} \int_{\partial \Omega} \phi^\varepsilon \nabla \phi^\varepsilon \cdot \xi d\mathcal{H}^{n-1}(x) = -\frac{\alpha}{\beta} \int_{\partial \Omega} |\nabla \phi^\varepsilon|^2 \cdot \xi d\mathcal{H}^{n-1}(x)
\]
Formula (1.16) applied to $\Sigma = \partial \Omega$ (since $\xi \cdot n = 0$) gives
\[
\int_{\partial \Omega} \text{div} (\phi^2 \xi) d\mathcal{H}^{n-1}(x) = \int_{\partial \Omega} n \otimes n : D(\phi^2 \xi) d\mathcal{H}^{n-1}(x) \\
= \int_{\partial \Omega} \phi^2 n \otimes n : D\xi d\mathcal{H}^{n-1}(x).
\]
Writing $\text{div} (\phi^2 \xi) = \nabla(|\phi^2|^2) \cdot \xi + |\phi^2|^2 \text{div} \xi$, we deduce
\[
2\varepsilon \int_{\partial \Omega} \nabla \phi \cdot n \nabla \phi \cdot \xi d\mathcal{H}^{n-1}(x) = -\frac{\alpha}{\beta} \int_{\partial \Omega} \phi^2 n \otimes n : D\xi d\mathcal{H}^{n-1}(x) + \frac{\alpha}{\beta} \int_{\partial \Omega} |\phi|^2 \text{div} \xi d\mathcal{H}^{n-1}(x)
\]
and so
\[
-\varepsilon^{-1} \int_{\Omega} (1 - 2\phi^2) \xi \cdot \nabla \rho = \varepsilon^{-1} \int_{\Omega} [(1 - \rho)\phi^2 + \rho(1 - \phi^2)] \text{div} \xi dx + \varepsilon \int_{\Omega} |\nabla \phi|^2 \text{div} \xi dx + \frac{\alpha}{\beta} \int_{\partial \Omega} |\phi|^2 \text{div} \xi d\mathcal{H}^{n-1}(x) \\
- 2\varepsilon \int_{\Omega} \nabla \phi \otimes \nabla \phi : D\xi dx - \frac{\alpha}{\beta} \int_{\partial \Omega} \phi^2 n \otimes n : D\xi d\mathcal{H}^{n-1}(x).
\]
which is (5.13).

6. JKO Scheme and Convergence of the Discrete Time Approximation

The main result of this paper can also be proved at the level of the discrete time approximation constructed in [11]. Such a result can be relevant to some numerical applications so we will state it here. First, we briefly recall the construction of the JKO scheme: As usual, $\mathcal{P}(\Omega)$ denotes the set of probability measures on $\Omega$ and we denote
\[
K = \{\rho \in \mathcal{P}(\Omega), \rho(x) \leq 1 \text{ a.e. in } \Omega\}
\]
(in particular all $\rho \in K$ are absolutely continuous with respect to the Lebesgue measure and we can identify the measure with its density). The set $\mathcal{P}(\Omega)$ is equipped with the usual Wasserstein distance, defined by
\[
W^2_2(\rho_1, \rho_2) = \inf_{\pi \in \Pi(\rho_1, \rho_2)} \int_{\Omega \times \Omega} |x - y|^2 d\pi(x, y)
\]
where $\Pi(\rho_1, \rho_2)$ denotes the set of all probability measures $\pi \in \mathcal{P}(\Omega \times \Omega)$ with marginals $\rho_1$ and $\rho_2$.

The idea of the JKO scheme is to construct a time-discrete approximation of the solution by successive applications of a minimization problem: For a given initial data $\rho_{in} \in K$, we fix a time step $\tau > 0$ (destined to go to zero) and define the sequence $\rho^n$ by:
\[
(6.1) \quad \rho^n = \rho_{in}, \quad \rho^n \in \text{argmin} \left\{ \frac{1}{2\tau} W^2_2(\rho, \rho^{n-1}) + \mathcal{F}_\epsilon(\rho) : \rho \in K \right\}, \quad \forall n \geq 1.
\]

The fact that this problem has a minimizer is proved in [11] Proposition 2.1. Furthermore, if $T^n$ is the unique optimal transport map from $\rho^n$ to $\rho^{n-1}$ (that is $T^n \# \rho^n = \rho^{n-1}$ and $W^2_2(\rho^n, \rho^{n-1}) = \int |x - T^n(x)|^2 d\rho^n$), we define the velocity
\[
v^n(x) = \frac{x - T^n(x)}{\tau}
\]
and the pressure variable $p^n(x)$ such that
\[
\rho^n v^n = \varepsilon^{-1} \rho^n \nabla \phi^n - \nabla p^n, \quad p^n \in H^1_0.
\]
with $\phi^n$ solution of (1.4). The existence of $p^n$ is shown in [11] Proposition 2.6.
We can then define the piecewise constant function $\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon} : [0, T] \to P(\Omega)$ by
\[
(6.3) \begin{align*}
\rho^{\tau, \varepsilon}(t) & := \rho^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau) \\
p^{\tau, \varepsilon}(t) & := p^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau).
\end{align*}
\]

The main result of [11] is the convergence of $(\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon})$ when $\tau \to 0$ with $\varepsilon > 0$ fixed to a weak solution of (1.13). The proof of Theorem 1.6 can easily be adapted to establish the convergence of $(\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon})$ to a weak solution of (1.13) when $\tau$ and $\varepsilon$ both go to zero:

**Theorem 6.1** (Convergence when $\varepsilon, \tau \to 0$). Given $T > 0$, an initial data $\rho_{in} = \chi_{E_{in}} \in BV(\Omega; \{0, 1\})$ and $\mu \geq 0$. Consider a subsequence $(\varepsilon_n, \tau_n)$ with $\max\{\varepsilon_n, \tau_n\} \to 0$. Up to a subsequence (still denoted $(\varepsilon_n, \tau_n)$), the discrete time approximation $\rho^{n, \tau_n}(x, t)$ converges to $\rho(x, t)$ strongly in $L^\infty((0, T); L^1(\Omega))$ and $q^{n, \tau_n}$ converges to $q$ weakly-$*$ in $L^2((0, T); (C^\infty(\Omega))^*)$ (for any $s > 0$). Furthermore, if the following energy convergence assumption holds:
\[
\lim_{n \to \infty} \int_0^T \int_\Omega J_{\varepsilon_n}(\rho^{n, \tau_n}(t)) \, dt = \int_0^T \int_\Omega J(\rho(t)) \, dt,
\]
then $(\rho, q)$ is a weak solution of (1.13) in the sense of Definition 1.3 with initial condition $\chi_{E_{in}}$ and contact angle
\[
\gamma = -\min \left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma\beta}}\right).
\]

We will not provide the details of the proof of this result which is a straightforward adaption of the arguments presented in this paper to prove Theorem 1.6. The key is to recall that the discrete approximations $\rho^{\tau, \varepsilon}$ and $p^{\tau, \varepsilon}$ satisfy some approximation of equations (1.10)-(1.11). Indeed, in addition to $\rho^{\tau, \varepsilon}$ and $p^{\tau, \varepsilon}$, we can define the piecewise constant interpolations $\rho^{\tau, n}(x, t)$, $p^{\tau, n}(x, t)$, $v^{\tau, \varepsilon}(x, t)$ and $\phi^{\tau, \varepsilon}(x, t)$ by
\[
(6.4) \begin{align*}
v^{\tau, \varepsilon}(t) & := v^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau) \\
\phi^{\tau, \varepsilon}(t) & := \phi^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau).
\end{align*}
\]
and the momentum
\[
E^{\varepsilon, \tau}(x, t) = \rho^{\tau, \varepsilon}(x, t)v^{\tau, \varepsilon}(x, t).
\]
Then we have (see [11]):

**Proposition 6.2.** For any smooth test function $\zeta(x, t)$ compactly supported in $\Omega \times [0, T]$ and given $N$ such that $N \tau \geq T$, there holds:
\[
\int_0^\infty \int_\Omega E^{\varepsilon, \tau} \cdot \nabla \zeta \, dx \, dt = -\int_\Omega \rho_{in}(x)\zeta(x, 0) \, dx - \int_0^\infty \int_\Omega \rho^{\varepsilon, \tau}(x, t)\partial_t \zeta(x, t) \, dx \, dt
\]
\[
+ \mathcal{O}\left(\|D^2\zeta\|_{L^\infty(\Omega \times \mathbb{R}^+)} \sum_{k=0}^N W_2^2(\rho_k^\tau, p_{k-1}^\tau) + \tau\|\partial_t \zeta\|_{L^\infty} + \tau T\|\partial_t^2 \zeta\|_{L^\infty}\right)
\]
For any smooth vector field $\xi(x, t)$ satisfying $\xi \cdot n = 0$ on $\partial \Omega$, there holds:
\[
(6.6) \int_0^\infty \int_\Omega E^{\varepsilon, \tau} \cdot \partial_n \zeta \, dx \, dt = \int_0^\infty \int_\Omega (\varepsilon^{-1} \rho^{\varepsilon, \tau} \nabla \phi^{\varepsilon, \tau} \cdot \zeta + \mu \rho^{\varepsilon, \tau} \div \xi + p^{\varepsilon, \tau} \div \xi \zeta) \, dx \, dt
\]
Passing to the limit in the continuity equation (6.5) can be done exactly as in the case $\tau \to 0$ with $\varepsilon > 0$ fixed (see [11]), while equation (6.6) is exactly the same as our equation (1.11), so we can adapt the proof presented in Section 5 of the present paper to pass to the limit in (6.6) and prove Theorem 6.1.
Appendix A. A few facts about $BV$ functions

First we recall the classical result:

**Proposition A.1.** Let $f_k$ be a sequence of functions such that $f_k \to f$ in $L^1(\Omega)$ when $k \to \infty$. Then

$$\liminf_{k \to \infty} \int_{\Omega} |\nabla f_k| \varphi \, dx \geq \int_{\Omega} |\nabla f| \varphi$$

for all $\varphi \in C(\Omega)$. Furthermore, if $\int_{\Omega} |\nabla f_k| \, dx \to \int_{\Omega} |\nabla f|$, then

$$\lim_{k \to \infty} \int_{\Omega} |\nabla f_k| \varphi \, dx = \int_{\Omega} |\nabla f| \varphi$$

for all $\varphi \in C(\Omega)$.

We also need the following result:

**Proposition A.2.** Let $f_k$ be a sequence of functions such that $f_k \to f$ in $L^1(\Omega)$ and $\int_{\Omega} |\nabla f_k| \, dx \to \int_{\Omega} |\nabla f|$. Then

$$\lim_{k \to \infty} \int_{\Omega} \zeta(x) \frac{\partial_i f_k}{|\nabla f_k|} \frac{\partial_j f_k}{|\nabla f_k|} |\nabla f_k| = \int_{\Omega} \zeta(x) \frac{\partial_i f}{|\nabla f|} \frac{\partial_j f}{|\nabla f|} |\nabla f|$$

for all $\zeta \in C(\Omega)$.

It is likely that this proposition is well known, but since we could not find a reference for its proof, we include one below.

**Proof of Proposition A.2.** We denote $\nu_i^k = \frac{\partial_i f_k}{|\nabla f_k|}$ which can be understood as the Radon-Nikodym derivative of $\partial_i f_k$ with respect to the measure $|\nabla f_k|$. Similarly, we note $\nu_i = \frac{\partial_i f}{|\nabla f|}$. First, we note that it is enough to prove that under the conditions of the proposition we have

$$\lim_{k \to \infty} \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| = \int_{\Omega} |\nu_i|^2 |\nabla f| \quad i = 1, \ldots, n$$

since we can write $\nu_i \nu_j = \frac{1}{2}[(\nu_i + \nu_j)^2 - \nu_i^2 - \nu_j^2]$.

The idea of the proof is as follows: Given a vector field $g = (g_1, \ldots, g_n) \in C^1_0(\Omega, \mathbb{R}^d)$ such that $|g(x)| \leq 1$ for all $x$, we compute

$$\begin{align*}
- \int_{\Omega} \partial_i g_i f_k \, dx &= \int_{\Omega} g_i \partial_i f_k \, dx = \int_{\Omega} g_i \nu_i^k |\nabla f_k| \, dx \\
&\leq \frac{1}{2} \int_{\Omega} |g_i|^2 |\nabla f_k| \, dx + \frac{1}{2} \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx
\end{align*}$$

(A.2)

When we sum these inequalities over $i = 1, \ldots, n$, the quantity on the left converges to $- \int_{\Omega} f \text{div} \, g \, dx$ which is close to $\int_{\Omega} |\nabla f|$ for a proper choice of $g$, while the right hand side is bounded by $\int_{\Omega} |\nabla f_k|$ (since $|g| \leq 1$ and $|\nu_i^k|^2 = 1$) which converges to $\int_{\Omega} |\nabla f|$ by assumption. Equality in these inequalities will yields the result. More precisely, if we denote,

$$u_i^k = \frac{1}{2} \int_{\Omega} |g_i|^2 |\nabla f_k| \, dx + \frac{1}{2} \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx + \int_{\Omega} \partial_i g_i f_k \, dx = \frac{1}{2} \int_{\Omega} |g_i - \nu_i^k|^2 |\nabla f_k| \, dx \geq 0$$

then we get

$$\sum_{i=1}^d u_i^k \leq \int_{\Omega} |\nabla f_k| \, dx + \int_{\Omega} f_k \text{div} \, g \, dx$$
We now fix $\eta > 0$ and choose $g \in C^1_0(\Omega; \mathbb{R}^d)$ such that $|g(x)| \leq 1$ for all $x$ and
\[
\int_{\Omega} |\nabla f| - \eta \leq - \int_{\Omega} f \, \text{div} \, g \, dx \leq \int_{\Omega} |\nabla f|.
\]
Since $f_k \to f$ in $L^1$ and $\int_{\Omega} |\nabla f_k| \, dx \to \int_{\Omega} |\nabla f|$, there exists $k_0$ (depending on $g$) such that when $k \geq k_0$ then
\[
\sum_{i=1}^d u_i^k = \int_{\Omega} |\nabla f_k| \, dx + \int_{\Omega} f_k \, \text{div} \, g \, dx \leq \int_{\Omega} |\nabla f| \, dx + \int_{\Omega} f \, \text{div} \, g \, dx + \eta \leq 2\eta.
\]
In particular
\[
u_i^k \leq 2\eta \quad i = 1, \ldots, d \quad \forall k \geq k_0.
\]
First, this implies that
\[
\left| \int_{\Omega} |\nabla f_k| - \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx \right| \leq \int_{\Omega} |g_i|^2 - |\nu_i|^2 |\nabla f_k| \, dx \leq \int_{\Omega} (g_i - \nu_i^k) (g_i + \nu_i^k) ||\nabla f_k|| \, dx \leq \left( \int_{\Omega} (g_i - \nu_i^k)^2 |\nabla f_k| \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla f_k| \, dx \right)^{1/2} \leq C/\sqrt{\eta}.
\]
Using this bound, the inequality (A.2) and the convergence of $f_k$ to $f$, we deduce (for $k \geq k_0$):
\[
\int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx \geq \frac{1}{2} \int_{\Omega} |g_i|^2 |\nabla f_k| \, dx + \frac{1}{2} \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx - C\sqrt{\eta}
\]
\[
\geq - \int_{\Omega} \partial_i g_i f_k \, dx - C\sqrt{\eta}
\]
\[
\geq - \int_{\Omega} \partial_i g_i f \, dx - C\sqrt{\eta}
\]
and
\[
\int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx \leq \frac{1}{2} \int_{\Omega} |g_i|^2 |\nabla f_k| \, dx + \frac{1}{2} \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx + C\sqrt{\eta}
\]
\[
\leq u_i^k - \int_{\Omega} \partial_i g_i f_k \, dx + C\sqrt{\eta}
\]
\[
\leq - \int_{\Omega} \partial_i g_i f \, dx + C\sqrt{\eta} + C\eta.
\]
In order to conclude, we note that if we take the constant sequence $f_k = f$ (for all $\varepsilon > 0$), then the argument above yields (with the same choice of function $g$, since it only depended on the limit $f$):
\[
\int_{\Omega} |\nu_i|^2 |\nabla f| \, dx \geq - \int_{\Omega} \partial_i g_i f \, dx - C\sqrt{\eta}
\]
and
\[
\int_{\Omega} |\nu_i|^2 |\nabla f| \, dx \leq - \int_{\Omega} \partial_i g_i f \, dx + C\sqrt{\eta} + C\eta.
\]
We have proved that given $\eta > 0$ there exists $k_0$ such that for $k \geq k_0$ we have
\[
\int_{\Omega} |\nu_i|^2 |\nabla f| \, dx - C\sqrt{\eta} \leq \int_{\Omega} |\nu_i^k|^2 |\nabla f_k| \, dx \leq \int_{\Omega} |\nu_i|^2 |\nabla f| \, dx + C\sqrt{\eta}
\]
which implies (A.1) and complete the proof.
Appendix B. A Lions-Aubin compactness result

The following result is a simple adaptation of some standard result. We provide a proof for the sake of completeness.

**Lemma B.1.** Let \( u_n \) be a sequence of function bounded in \( L^\infty((0,T); BV(\Omega)) \) and \( u_n \rightarrow u \) in \( L^\infty((0,T); H^{-1}(\Omega)) \) Then
\[
\sup_{t \in [0,T]} \| u_n(t) - u(t) \|_{L^1(\Omega)} \rightarrow 0
\]

**Proof.** Since \( \int |\nabla u_n(t)| \leq C \) and \( u_n(t) \) converges to \( u(t) \) in \( H^{-1}(\Omega) \), we can show that \( u(t) \in BV(\Omega) \)

and the \( BV \) bound, together with the strong convergence in \( H^{-1}(\Omega) \) implies
\[
\limsup_{\epsilon \rightarrow 0} \| u_n(t) - u(t) \|_{L^\infty((0,T); L^1(\Omega))} \leq C\delta.
\]

Since this holds for all \( \delta > 0 \), we deduce
\[
\limsup_{\epsilon \rightarrow 0} \| u_n - u \|_{L^\infty((0,T); L^1(\Omega))} = 0
\]
and the result follows. \( \square \)

Appendix C. \( \Gamma \)-convergence of \( J_\epsilon \)

We wish to prove the following proposition which gives the \( \Gamma \) convergence of \( J_\epsilon \) to \( J_0 \):

**Proposition C.1.** The following holds:

(i) For any family \( \{\rho^\epsilon\}_{\epsilon>0} \) that converges to \( \rho \) in \( L^1(\Omega) \),
\[
\liminf_{\epsilon \rightarrow 0} J_\epsilon(\rho^\epsilon) \geq J_0(\rho).
\]

(ii) Given \( \rho \in L^1(\Omega) \), there exists a sequence \( \{\rho^\epsilon\}_{\epsilon>0} \) that converges to \( \rho \) in \( L^1(\Omega) \) such that
\[
\limsup_{\epsilon \rightarrow 0} J_\epsilon(\rho^\epsilon) \leq J_0(\rho).
\]

We recall that this proposition is proved in [15] (Proposition 5.3) when \( J_\epsilon \) is restricted to characteristic functions. We show below how the proof can be adapted to our more general case.

**Proof of Proposition C.1.** First, we note that the limsup properties (part (ii)) follows from the corresponding result in [15] Proposition 5.3. Indeed, if \( \rho \notin BV(\Omega; \{0,1\}) \), then \( J_0(\rho) = \infty \) and there is nothing to prove, while if \( \rho \in BV(\Omega; \{0,1\}) \) then \( \rho = \chi_E \) for some \( E \) satisfying \( P(E) < \infty \), so [15] Proposition 5.3 applies.

To prove the liminf property (part (i)), we need to slightly modify the proof of [15] Proposition 5.3 by using the formula 2.2 instead of 2.1 for \( J_\epsilon \). We only provide details in the case of Neumann and Dirichlet conditions (the Robin boundary condition is then proved combining both arguments, as in [15]).
Neumann boundary conditions ($\alpha = 0$). If $\liminf_{\varepsilon \to 0} J_\varepsilon(\rho^\varepsilon) = \infty$, there is nothing to prove, so we can assume (up to a subsequence) that $J_\varepsilon(\rho^\varepsilon) \leq C$ and $\liminf_{\varepsilon \to 0} J_\varepsilon(\rho^\varepsilon) < \infty$. Up to another subsequence, we can also assume that $\rho^\varepsilon \to \rho$ a.e. in $\Omega$.

Next, we recall that (2.2) gives
\[ J_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega (1 - \rho)(\sigma \phi)^2 + \rho(1 - \sigma \phi)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla \phi|^2 \, dx \]
and introducing the functions (both defined for $t \in [0, 1]$)
\[ f(t) = 2 \min(t, 1 - t), \quad F(t) = \int_0^t f(\tau) \, d\tau = \begin{cases} t^2 & \text{for } 0 \leq t \leq 1/2 \\ 2t - t^2 - 1/2 & \text{for } 1/2 \leq t \leq 1 \end{cases} \]
we find (see (4.2)):
\[ J_\varepsilon(\rho) = \frac{1}{2\sigma^{3/2}} \int_\Omega |\nabla F(\sigma \phi^\varepsilon)| \, dx \leq J_\varepsilon(\rho^\varepsilon). \]

Furthermore, (2.1) give
\[ J_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) \, dx + \frac{1}{2\sigma\varepsilon} \int_\Omega (\rho - \sigma \phi)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla \phi|^2 \, dx \]
which implies
\[ \int_\Omega (\rho^\varepsilon - \sigma \phi^\varepsilon)^2 \, dx \leq 2\sigma \varepsilon J_\varepsilon(\rho^\varepsilon) \leq C \varepsilon \]
and
\[ \int_\Omega \rho^\varepsilon (1 - \rho^\varepsilon) \, dx \leq 2\sigma \varepsilon J_\varepsilon(\rho^\varepsilon) \leq C \varepsilon. \]
The first inequality implies that $\sigma \phi^\varepsilon$ converges in $L^2$ to $\rho$. The second inequality implies that that $\rho = 0$ or $1$ a.e. in $\Omega$.

We deduce that $F(\sigma \phi^\varepsilon)$ converges (strongly in $L^1$ for example) to $F(\rho) = \frac{1}{2} \rho$ (since $F(0) = 0$ and $F(1) = 1/2$), and (C.2) gives
\[ \liminf_{\varepsilon \to 0} J_\varepsilon(\rho^\varepsilon) \geq \frac{1}{2\sigma^{3/2}} \int_\Omega |\nabla F(\rho)| \, dx = \frac{1}{4\sigma^{3/2}} \int_\Omega |\nabla \rho| \, dx = J_0(\rho). \]

Dirichlet boundary conditions. We still have (C.1) and thus (C.2) in this case, but since $\phi^\varepsilon = 0$ on $\partial \Omega$, we can extend the function $\phi^\varepsilon$ by zero outside $\Omega$. Denoting by $\phi^\varepsilon$ this extension, we find
\[ J_\varepsilon(\rho^\varepsilon) \geq \frac{1}{2\sigma^{3/2}} \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx = \frac{1}{2\sigma^{3/2}} \int_{\mathbb{R}^n} |\nabla F(\phi^\varepsilon)| \, dx \]
and so (proceeding as above)
\[ \liminf_{\varepsilon \to 0} J_\varepsilon(\rho^\varepsilon) \geq \frac{1}{2\sigma^{3/2}} \int_{\mathbb{R}^n} |\nabla F(\rho)| \, dx = \frac{1}{2\sigma^{3/2}} \int_{\mathbb{R}^n} |\nabla \rho| \, dx = \frac{1}{4\sigma^{3/2}} \left[ \int_\Omega |\nabla \rho| \, dx + \int_{\partial \Omega} \rho \, dH^{n-1}(x) \right] = J_0(\rho). \]
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