Research Article

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Parabolic inequalities in Orlicz spaces with data in $L^1$

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Abstract: In this paper, we provide existence and uniqueness of entropy solutions to a general nonlinear parabolic problem on a general convex set with merely integrable data and in the setting of Orlicz spaces.

Keywords: nonlinear parabolic inequalities, entropy solution, Orlicz-Sobolev

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1 Introduction

In this paper, we deal with the boundary value problems

$$\begin{align*}
\frac{\partial u}{\partial t} + A(u) &= f & &\text{in } Q, \\
\frac{\partial u}{\partial t} &= 0 & &\text{on } \partial Q = \partial \Omega \times (0, T), \\
u(x, 0) &= u_0 & &\text{in } \partial \Omega,
\end{align*}$$

where

$$A(u) = -\operatorname{div}(a(, t, \nabla u)).$$

$Q = \Omega \times [0, T], \ T > 0$ and $\Omega$ is a bounded domain of $\mathbb{R}^N$, with the segment property. $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$a(x, t, \xi) \xi \geq a M(|\xi|),$$

$$[a(x, t, \xi) - a(x, t, \xi^*)][\xi - \xi^*] > 0,$$

$$|a(x, t, \xi)| \leq c(x, t) + k_i M^{-1} M(|\xi|),$$

where $c(x, t)$ belongs to $E_{\mathbb{R}}(Q), \ c \geq 0$ and $k_i (i = 1, 2)$ to $\mathbb{R}^+$ and $\alpha$ to $\mathbb{R}_+^*$. $f \in L^1(Q), \ f \geq 0$, $u_0 \in L^1(\Omega), \ u_0 \geq 0$.

There exists a real $\gamma > 1$ such that

$$\sup_{t > 0} \frac{t^{1+\frac{1}{\gamma}}}{M(t)} < +\infty.$$
The problem \((P)\) has several applications in engineering, game theory, finance, and economics. For example, one of the most important problems in finance is the optimal investment problem of a constant relative risk aversion. This problem leads to an obstacle parabolic problem with free boundaries (see [1]). Other important cases are the obstacle problem for parabolic minimizers studied in [2], where the obstacle is irregular, the pricing of American options (see [3]), as well as the models of pricing a double defaultable interest rate swap for which the solutions converge to a solution of a PDE coupled with two-obstacle problem.

On a physical area, the use of PDE in the convex set takes considerable importance, for example, the Boltzmann equation in a strictly convex domain with the specular, bounce-back, and diffuse (see [4]), dissipation inequalities for nonlinear PDEs which can be applied according to the choice of the so-called supply rate [5], Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP) equation, etc.

It is well known that \((P)\) admits at least one solution (see Leray and Lions [6], Browder and Brézis [7], and Puel [8]). In those papers, the function \(a(x, t, \xi)\) was assumed to satisfy a polynomial growth condition with respect to \(Vu\). When trying to generalize the last condition of \(a(\cdot, \xi)\) to the non-polynomial one, we are led to replace the space \(L^p(0, T; W^{1,p}(\Omega))\) by an inhomogeneous Sobolev space \(W^{1,X}_{LM}\) built from an Orlicz space \(L_M\) instead of \(L^p\), where the \(N\)-function \(M\), which defines \(L_M\), defines the new growth of the operator. Such type of extension of the growth condition is more realistic and appears in several physical phenomena.

Partial differential equations with data-only integrable received special attention. The cornerstone of the theory was initially developed by DiPerna and Lions [9], where they introduced the notion of renorma-

### 2 Preliminaries

Let us recall the following definitions of spaces and topologies that will be used later (for the detail, we refer the reader to the rich literature in [16–19]).

#### 2.1 \(N\)-function and Orlicz space

\(M : \mathbb{R}^+ \to \mathbb{R}^+\) be an \(N\)-function, i.e., \(M\) is continuous, convex, with \(M(t) > 0\) for \(t > 0\), \(\frac{M(t)}{t} \to 0\) as \(t \to 0\) and \(\frac{M(t)}{t} \to \infty\) as \(t \to \infty\).

The Orlicz space \(L_M(\Omega)\) is defined as the equivalence classes of real-valued measurable functions \(u\) on \(\Omega\) such that: \(\int_{\Omega} M\left(\frac{|u|}{\lambda}\right) dx < +\infty\) for some \(\lambda > 0\). Note that \(L_M(\Omega)\) is a Banach space under the norm \(\|u\|_{L_M(\Omega)} = \inf\{\lambda > 0 : \int_{\Omega} M\left(\frac{|u|}{\lambda}\right) dx \leq 1\}\). The closure in \(L_M(\Omega)\) of the set of bounded measurable functions with compact support in \(\Omega\) is denoted by \(E_M(\Omega)\).

Let us recall that two equivalent \(N\)-functions defined the same Orlicz space.
2.2 Inhomogeneous Orlicz-Sobolev spaces

The inhomogeneous Orlicz-Sobolev spaces are defined as follows: \( W^{1,s} L_0(Q) = \{ u \in L_0(Q) : D_1^s u \in L_0(Q) \} \) and \( W^{1,s} (E_0(Q)) = \{ u \in E_0(Q) : D_2^s u \in E_0(Q) \} \). The last space is a subspace of the first one, and both are Banach spaces under the norm \( \| u \| = \sum_{|\alpha| \leq s} \| D_1^s u \|_{s,Q} \). These spaces are considered as subspaces of the product space \( \Pi L_0(Q) \), which have as many copies as there are \( \alpha \)-order derivatives, \( |\alpha| \leq 1 \).

We shall also consider the weak topologies \( \sigma(\Pi L_0, E_0) \) and \( \sigma(\Pi L_0, \Pi E_0) \). The space \( W^{1,s} E_0(Q) \) is defined as the \( (\|\|) \) closure in \( W^{1,s} L_0(Q) \) of \( D(Q) \). We can easily show as in [17] that when \( \Omega \) has the segment property, then each element \( u \) of the closure of \( D(Q) \) with respect to the weak \( * \) topology \( \sigma(\Pi L_0, E_0) \) is limit, in \( W^{1,s} L_0(Q) \), of some subsequence \( (u_i) \in D(Q) \) for the modular convergence, i.e., there exists \( \lambda > 0 \) such that for all \( |\alpha| \leq 1 \), \( \int_\Omega M(\frac{Du_i}{\lambda}) \text{d}x \to 0 \) as \( i \to \infty \), and this implies that \( (u_i) \) converges to \( u \) in \( W^{1,s} L_0(Q) \) for the weak topology \( \sigma(\Pi L_0, \Pi E_0) \). Consequently, \( D(Q) = (\Pi E_0, \Pi E_0) = D(Q) = (\Pi E_0, \Pi E_0) \), and this space will be denoted by \( W^{1,s} L_0(Q) \). Furthermore, Poincaré’s inequality also holds in \( W^{1,s} L_0(Q) \).

The dual space of \( W^{1,s} L_0(Q) \) is defined as \( W^{-1,s} E_0(Q) = \{ f = \sum_{|\alpha| \leq s} D_1^s f : f \in E_0(Q) \} \) and equipped with the usual quotient norm. We also denote \( W^{-1,s} E_0(Q) = \{ f = \sum_{|\alpha| \leq s} D_1^s f : f \in E_0(Q) \} \).

3 Main results

The following lemmas will be of interest in the proof of our main results.

Let us denote \( X_\alpha = N C_\alpha^\alpha, C_\alpha \) is the measure of the unit ball of \( \mathbb{R}^N \), and for a fixed \( t \in [0, T] \), \( \mu(\theta) = \text{meas}\{x \in \Omega : |u(x, t)| > \theta\} \).

**Lemma 3.1.** [20] Let \( u \in W^{1,s} L_0(Q) \), and let fixed \( t \in [0, T] \), then we have

\[
-\mu'(\theta) \leq -\frac{1}{X_\alpha \mu(\theta)^\frac{1}{s}} \left(-\frac{1}{X_\alpha \mu(\theta)^\frac{1}{s}} \frac{d}{d\theta} \int_{|u| > \theta} M(|\nabla u|) \text{d}x \right), \quad \forall \theta > 0,
\]

where \( S \) is defined by \( \frac{1}{S(s)} = \sup \{ t : B(t) \leq s \}, B(s) = \frac{M(s)}{s} \).

**Lemma 3.2.** Under the hypotheses (1.1)–(1.3), if \( f, u_0 \) are regular functions and \( f, u_0 \geq 0 \), then there exists at least one positive weak solution of the problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div} a(x, t, u, \nabla u) &= f \quad \text{in } Q, \\
u(x, 0) &= u_0 \quad \text{on } \partial Q = \partial \Omega \times (0, T), \\
u(x, 0) &= u_0 \quad \text{in } \partial \Omega,
\end{aligned}
\]

such that

\[
\frac{\partial u}{\partial t} \geq 0, \quad \text{a.e. } t \in (0, T).
\]

**Proof.** Let \( u \) be a continuous function, we say that \( u \) satisfies \((\beta)\) condition if there exists a continuous and increasing function \( \beta \) such that \( \| u(t) - u(s) \|_s \leq \beta(\| u_0 \|_s) |t - s| \), where \( u_0(x) = u(x, 0) \).

Let \( X = \{ u \in W^{1,s} L_0(Q) \cap L^2(Q), u \text{ satisfies } (\beta) \text{ condition and } \frac{\partial u}{\partial t} \in L^\infty(0, T, L^2(\Omega)) \} \).

Let us consider the set \( C = \{ v \in X : v(t) \in C, \frac{\partial v}{\partial t} \geq 0 \text{ a.e. } t \in (0, T) \} \), where \( C \) is a closed convex of \( W^{1,s} L_0(\Omega) \). It is easy to see that \( C \) is a closed convex (since all its elements satisfy \((\beta)\) condition).
We claim that the problem
\[\begin{aligned}
\begin{cases}
    u \in C \cap L^2(Q), \\
    \frac{\partial u}{\partial t} - \text{div} \ a(x, t, u, \nabla u) = f & \text{in } Q, \\
    u = 0 & \text{on } \partial Q = \partial \Omega \times (0, T), \\
    u(x, 0) = u_0 & \text{in } \partial \Omega
\end{cases}
\end{aligned}\]

has a weak solution, which is unique in the sense defined in [21].

Indeed, let us consider the approximate problem:
\[\begin{aligned}
\begin{cases}
    \frac{\partial u_n}{\partial t} + A(u_n) + nT_n(\Phi(u_n)) = f & \text{in } Q, \\
    u_n(., 0) = u_0 & \text{in } \Omega,
\end{cases}
\end{aligned}\]

where the functional \( \Phi \) is defined by \( \Phi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) such that
\[\Phi(v) = \begin{cases} 0 & \text{if } v \in C, \\ +\infty & \text{otherwise}. \end{cases}\]

The existence of such \( u_n \in X \) was ensured by Kacur [22].

Following the same proof as in [21], we can prove the existence of a solution \( u \) of the problem \( (P') \) as limit of \( u_n \).

**Theorem 3.1.** Under hypotheses (1.1)–(1.6), the problem \( (P) \) has at least one entropy solution in the following sense:
\[\begin{aligned}
\begin{cases}
    u \in \mathcal{K}, T_k(u) \in W^{1,s}_{0}(\mathbb{R}) \cap L^q(Q), \ \forall k > 0, \\
    \int \mathcal{S}_k(u(T) - v(T))dx + \int \frac{\partial v}{\partial t} T_k(u - v)dxdt + \int a(., \nabla u)\nabla T_k(u - v)dxdt \leq \langle f, T_k(u - v) \rangle + \int S_k(u_0 - v(0))dx, \\
    \end{cases}
\end{aligned}\]

for all \( v \in \mathcal{K} \cap L^\infty(Q) \cap D, \) where
\[D = \{ v \in W^{1,s}_{0}(\mathbb{R}) : \frac{\partial v}{\partial t} \in W^{1,1}L^\infty(Q) + L^1(Q), \mathcal{S}_k(t) = \int T_k(s)ds \}.\]

**Proof.** Let us define the indicator functional, \( \Phi : M(Q) \rightarrow \mathbb{R} \cup \{+\infty\} \) such that:
\[\Phi(v) = \begin{cases} 0 & \text{if } v \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}\]

\( \Phi \) is weakly lower semicontinuous.

**I. A priori estimate**

Let us consider the following approximate problem:
\[\begin{aligned}
\begin{cases}
    \frac{\partial u_n}{\partial t} + A(u_n) + nT_n(\Phi(u_n)) = f_n & \text{in } Q, \\
    u_n(., 0) = u_{0n} & \text{in } \Omega,
\end{cases}
\end{aligned}\]

where \( (u_{0n}) \in \mathcal{D}(\Omega) \) such that \( u_{0n} \rightarrow u_0 \) strongly in \( L^q(\Omega) \), \( (f_n) \in \mathcal{D}(\Omega) \) such that \( |f_n| \leq |f| \) a.e., in \( Q \) and \( f_n \rightarrow f \) strongly in \( L^q(Q). \)

For the existence of a weak solution \( u_n \in W^{1,s}_{0}(\mathbb{R}) \), \( u_n \geq 0 \) of the aforementioned problem, see [23], and also \( (u_n) \) satisfies \( \frac{\partial u_n}{\partial t} \in W^{1,1}L^\infty(Q) + L^1(Q). \)
Let $v = T_k(u_n)$ be a test function in $(P_n)$, then
\begin{equation*}
\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle + \int_Q a(\nabla u_n, \nabla T_k(u_n))dxdt + \int_Q nT_k(\Phi(u_n))T_k(u_n)dxdt = \langle f, T_k(u_n) \rangle.
\end{equation*}

We deduce easily that
\begin{equation*}
\int_Q a(\nabla u_n, \nabla T_k(u_n))dxdt \leq Ck,
\end{equation*}
\begin{equation*}
\int_Q nT_k(\Phi(u_n))T_k(u_n)dxdt \leq Ck.
\end{equation*}

So, $T_k(u_n)$ is bounded in $W^{1,1}_0(Q)$. Then, there exist a subsequence (also denoted $(u_n)$) and a measurable function $u$ such that
\begin{equation*}
T_k(u_n) \rightharpoonup T_k(u), \text{ weakly in } W^{1,1}_0(Q) \text{ for } \sigma(\Pi_M, \Pi E_M), \text{ strongly in } E_M(Q) \text{ and a.e. in } Q. \tag{3.3}
\end{equation*}

Coming back to the inequality (3.2), we have \( \int_Q nT_k(\Phi(u_n)) \frac{T_k(u_n)}{k} dxdt \leq C, \) and by letting $k$ to infinity and using Fatou lemma, one has
\begin{equation*}
\int_Q T_k(\Phi(u_n)) dxdt \leq \frac{C}{n}.
\end{equation*}

Suppose there exists a subsequence $(u_n)$ such that $u_n \notin \mathcal{K}$ for all $n$, then
\begin{equation*}
n^2|Q| = \int_Q nT_k(\Phi(u_n))dxdt \leq C,
\end{equation*}
which is a contradiction. Then, there exists a subsequence that we denote as also $(u_n)$ such that $u_n \in \mathcal{K}$ for all $n$.

In what follows, we only consider this subsequence.

To prove that $u \in \mathcal{K}$, we need to prove that $u \in M(Q)$. For this reason, let us consider $\varphi$ as the truncation defined by
\begin{equation*}
\varphi(\zeta) = \begin{cases} 
0 & \text{if } 0 \leq \zeta \leq \theta, \\
\frac{1}{h}(\zeta - t) & \text{if } \theta < \zeta < \theta + h, \\
1 & \text{if } \zeta \geq \theta + h, \\
-\varphi(-\zeta) & \text{if } \zeta < 0.
\end{cases}
\end{equation*}

The function $\varphi(\zeta)$

for all $\theta, h > 0$.

Using $v = \varphi(u_n)$ as a test function in the approximate problem $(P_n)$, we obtain
\begin{equation*}
-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} M(\nabla u_n)dx \leq C \int_{\{|u_n| > \theta\}} \left( f_n - \frac{\partial u_n}{\partial t} \right)dx, \tag{3.4}
\end{equation*}
since $nT_k(\Phi(u_n)) = 0$.

By using Lemma 3.1, we obtained by following the same way as in [20], we have for a good $N$ -function $D$
\begin{equation*}
-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(\nabla u_n)dx \leq (-\mu'(\theta))D\left( \frac{1}{\lambda N \mu(\theta)^{\frac{1}{\alpha}}} \frac{d}{d\theta} \int_{\{|u_n| > \theta\}} M(\nabla u_n)dx \right), \tag{3.5}
\end{equation*}
Let us denote $k(t, s) = \int_0^t u_n(t, \rho) d\rho$, then $\frac{\partial k}{\partial t}(t, s) = \int_0^s \frac{\partial u_n(t, \rho)}{\partial \rho} d\rho$, and $F(t, \mu(\theta)) = \int_0^{\mu(\theta)} (f_n(\rho)) d\rho$. Using Lemma 3.1, one has
\[
1 \leq -\frac{\mu'(\theta)}{\lambda N(\mu(\theta))^{1 + \frac{1}{N}}} B^{-1} \left( \frac{1}{\lambda N(\theta)} \left[ F(t, \mu(\theta)) - \frac{\partial k}{\partial t}(t, \mu(\theta)) \right] \right).
\] (3.6)

Since $F(t, s) \geq \frac{\partial k}{\partial t}(t, s)$ and using Lemma 3.2, we have $\frac{\partial k}{\partial t}(t, s) \leq F(t, s)$. Combining (3.5)–(3.6) we obtain,
\[
\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(\nabla u_n) dx \leq (-\mu'(\theta)) DoB^{-1} \left( \frac{1}{\lambda N(\mu(\theta))^{1 + \frac{1}{N}}} \left[ F(t, \mu(\theta)) - \frac{\partial k}{\partial t}(t, \mu(\theta)) \right] \right),
\] (3.7)

\[
\int_{\Omega} D(\nabla u_n) dx = \int_{0}^{\infty} \left( -\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(\nabla u_n) dx \right) d\theta \leq \frac{C}{\lambda} \int_{0}^{\infty} DoB^{-1} \left( \frac{C}{\lambda^{1 + \frac{1}{N}}} \right) ds.
\]

**Case 1:** If there exists $D$ an $N$-function such that $\int_{0}^{\infty} DoB^{-1} \left( \frac{C}{\lambda^{1 + \frac{1}{N}}} \right) ds < +\infty$.

The sequence $(u_n)$ is bounded in $W_0^{1, 1} L_\delta(Q)$ and also $u \in W_0^{1, 1} L_\delta(Q)$.

**Case 2:** If $\int_{0}^{\infty} DoB^{-1} \left( \frac{C}{\lambda^{1 + \frac{1}{N}}} \right) ds < +\infty$.

Then, if we take $D(t) = t$, the sequence $(u_n)$ is bounded in $W_0^{1, 1} L_\delta(Q)$. Then, there exists a measurable function $u \in L^\gamma(Q) \left( \gamma = \frac{N}{N-1} \right)$ such that $u_n \rightharpoonup u$ in $L^\gamma(Q)$. By using [24] (see Proposition 2.3), it follows that
\[
\|\nabla u\|_M \leq \lim_n \|\nabla u_n\|_{L^\gamma(Q)} \leq C,
\]
where $\|\nabla u\|_M = \sup_\{ \int u \text{ div } v : v \in (Z(Q))^N \text{ and } \|v\|_{L^\gamma} \leq 1 \} < \infty$.

Then, $u \in BV_\gamma(Q) = \{ u \in L^\gamma(Q) : \text{ div } u \in (M(Q))^N \}$.

**Case 3:** General case.

Let $0 < \lambda \leq k$, then
\[
\text{meas}(|u_n| > \lambda) = \text{meas}(|T_k(u_n)| > \lambda) \leq \frac{1}{M(\lambda)} \int_{\Omega} M(|T_k(u_n)|) dx \leq \frac{Ck}{M(\lambda)}.
\]

Let us take $\lambda = k$, we have
\[
\lambda \text{meas}(|u_n| > \lambda)^{\frac{1}{\gamma}} \leq \frac{\lambda^{1 + \frac{1}{\gamma}}}{M(\lambda)^\gamma} \leq C \sup_{t > 0} \frac{t^1 + \frac{1}{\gamma}}{M(t)^\gamma} = H.
\]

Then, $(u_n)$ is bounded in $M^\gamma(Q) \subset L^\gamma(Q)$ (since $\gamma > 1$). Note that $M^\gamma(Q)$ is the Marcinkiewicz space. Then, $u \in M(Q)$.

Finally, since $\int_Q T_k(\varphi(u_n)) dx dt \leq \int_Q T_k(\varphi(u_n)) dx dt \leq \frac{C}{\lambda}$. Then, we deduce, $\int_Q \varphi(u) dx dt = 0$, which ensure $u \in \mathcal{K}$.

**II. Almost everywhere convergence of the gradients**

The main tool in this step proves
\[
\lim_{n \to \infty} \int_Q (a(\cdot, \nabla T_k(u_n)) - a(\cdot, \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)) dx dt = 0,
\]
which gives by the same argument as in [25] and adapted to the parabolic case, $\nabla u_n \rightharpoonup \nabla u$ a.e. in $Q$. 
This is possible by using the following regularization principle \( \omega_{\mu,j}^i = (T_k(v))_\mu + e^{-\mu T_k(\psi)} \), where \( v \in D(Q) \) such that \( v \rightarrow T_k(u) \) with the modular convergence in \( W^{1,1}_{0,0}L_M(Q) \), \( \psi \) is a smooth function such that \( \psi \rightarrow T_k(u_0) \) strongly in \( L^1(Q) \), and \( \omega_\mu \) is the mollifier function defined by Landes [26], \( \omega_\mu(x, t) = \mu \int_{-\infty}^t \omega(x, s) \exp(s - t)ds \), where \( \omega \) is the zero extension of \( \omega \) for \( s > T \). The function \( \omega_{\mu,j}^i \) have the following properties:

\[
\begin{align*}
\frac{\partial \omega_{\mu,j}^i}{\partial t} &= \mu(T_k(v_j) - \omega_{\mu,j}^i), \quad \omega_{\mu,j}^i(0) = \psi, \quad |\omega_{\mu,j}^i| \leq k, \\
\omega_{\mu,j}^i &\rightarrow T_k(u)_\mu + e^{-\mu T_k(\psi_j)} \quad \text{in} \quad W^{1,1}_{0,0}L_M(Q) \quad \text{for the modular convergence with respect to} \ j, \\
T_k(u)_\mu + e^{-\mu T_k(\psi_j)} &\rightarrow T_k(u) \quad \text{in} \quad W^{1,1}_{0,0}L_M(Q) \quad \text{for the modular convergence with respect to} \ \mu.
\end{align*}
\]

Consider, for \( m > k \), the following truncation

\[
\rho_m(s) = \begin{cases} 
1 & |s| \leq m, \\
1 - |s| & m < |s| < m + 1, \\
0 & |s| \geq m + 1,
\end{cases}
\]

and let \( R_m(s) = \int_s^\infty \rho_m(t)dt \).

Consider \( v = (T_k(u_m) - \omega_{\mu,j}^i)\rho_m(u_m) \) as a test function in the approximate problem \((P_n)\), then we have

\[
\left\langle \frac{\partial u_n}{\partial t}, v \right\rangle + \int_Q a(\nabla u_n)(\nabla T_k(u_n) - \nabla \omega_{\mu,j}^i)\rho_m(u_n) dxdy \\
+ \int_Q a(\nabla u_n)\nabla u_n(T_k(u_n) - \omega_{\mu,j}^i)\rho_m(u_n) = \int_{\partial Q} f_n v dxdy + \int_0^T \int_O f_n v dxdy dt.
\]

We will be interested to estimate the elements of the aforementioned equation.

Since \( u_m \in W^{1,1}_{0,0}L_M(Q) \), there exists a smooth function \( u_{\text{reg}} \) (see [23]) such that,

\[
u_{\text{reg}} \rightarrow u_m \quad \text{for the modular convergence in} \quad W^{1,1}_{0,0}L_M(Q) \quad \text{and} \\
\frac{\partial u_{\text{reg}}}{\partial t} \rightarrow \frac{\partial u_n}{\partial t} \quad \text{for the modular convergence in} \quad W^{1,1}_{0,0}L_M(Q) + L^1(Q).
\]

\[
\left\langle \frac{\partial u_m}{\partial t}, v \right\rangle = \lim_{\sigma \rightarrow 0} \int_Q (u_{\text{reg}})'(T_k(u_{\text{reg}}) - \omega_{\mu,j}^i)\rho_m(u_{\text{reg}})dxdy \\
= \lim_{\sigma \rightarrow 0} \left( \int_Q (R_m(u_m) - T_k(u_{\text{reg}}))'(T_k(u_{\text{reg}}) - \omega_{\mu,j}^i) + \int_Q (T_k(u_{\text{reg}}))'(T_k(u_m) - \omega_{\mu,j}^i)dxdy \right) \\
= \lim_{\sigma \rightarrow 0} \left( \int_Q (R_m(u_m) - T_k(u_{\text{reg}}))(T_k(u_{\text{reg}}) - \omega_{\mu,j}^i) + \int_Q (T_k(u_{\text{reg}}))'(T_k(u_{\text{reg}}) - \omega_{\mu,j}^i)'dxdy \right) \\
+ \int_Q (T_k(u_{\text{reg}}))'(T_k(u_m) - \omega_{\mu,j}^i)dxdy \\
= \lim_{\sigma \rightarrow 0} I_{\sigma}(\sigma) + I_{\sigma}(\sigma) + I_{\sigma}(\sigma).
\]

We deal now with the terms \( I_{\sigma}(\sigma), I_{\sigma}(\sigma), \) and \( I_{\sigma}(\sigma) \) to prove \( \left\langle \frac{\partial u_m}{\partial t}, v \right\rangle \geq \epsilon(n, j, i, \mu, s, m). \)
Claim 1: \( \lim_{\sigma \to 0} I_1(\sigma) \geq \varepsilon(n, j, i, \mu) \).

\[
I_1(\sigma) = \int_{\Omega} \left( (R_m(u_{n0})(T) - T_k(u_{n0})(T))(T_k(u_{n0})(T) - \omega_{\mu, j}(T)) \right) dx
- \int_{\Omega} \left( (R_m(u_{n0})(0) - T_k(u_{n0})(0))(T_k(u_{n0})(0) - \omega_{\mu, j}(0)) \right) dx
= \int_{u_{n0}(T) \leq k} \left( (R_m(u_{n0})(T) - T_k(u_{n0})(T))(T_k(u_{n0})(T) - \omega_{\mu, j}(T)) \right) dx
+ \int_{u_{n0}(T) > k} \left( (R_m(u_{n0})(T) - T_k(u_{n0})(T))(T_k(u_{n0})(T) - \omega_{\mu, j}(T)) \right) dx
- \int_{u_{n0}(0) \leq k} \left( (R_m(u_{n0})(0) - T_k(u_{n0})(0))(T_k(u_{n0})(0) - \omega_{\mu, j}(0)) \right) dx
- \int_{u_{n0}(0) > k} \left( (R_m(u_{n0})(0) - T_k(u_{n0})(0))(T_k(u_{n0})(0) - \omega_{\mu, j}(0)) \right) dx
= J_1 + J_2 + J_3 + J_4.
\]

If \( u_{n0} \leq k \), we have \( R_m(u_{n0}) = T_k(u_{n0}) \) (since \( m > k \)), and if \( u_{n0} \geq k \), we have \( R_m(u_{n0}) > k \geq |\omega_{\mu, j}| \). Then,

\[
J_1 = \int_{u_{n0}(T) \leq k} \left( (R_m(u_{n0})(T) - T_k(u_{n0})(T))(T_k(u_{n0})(T) - \omega_{\mu, j}(T)) \right) dx = 0,
\]

\[
J_2 = \int_{u_{n0}(T) > k} \left( (R_m(u_{n0})(T) - k)(k - \omega_{\mu, j}(T)) \right) dx \geq 0,
\]

\[
J_3 = -\int_{u_{n0}(0) \leq k} \left( (R_m(u_{n0})(0) - T_k(u_{n0})(0))(T_k(u_{n0})(0) - \omega_{\mu, j}(0)) \right) dx = 0.
\]

About \( J_4 \), we have \( \lim_{\sigma \to 0} J_4 = -\int_{u_{n0}(T) > k} \left( (R_m(u_{n0})(0) - u_{n0}(0))(T_k(u_{n0})(0) - \omega_{\mu, j}(0)) \right) dx dt = \varepsilon(n, i) \).

We conclude that

\[
\lim_{\sigma \to 0} I_1(\sigma) \geq \varepsilon(n, j, i, \mu).
\]

Claim 2: \( \lim_{\sigma \to 0} I_2(\sigma) \geq \varepsilon(n, j, i, \mu) \).

It is easy to remark that \( T_k(u_{n0})' = 0 \), if \( u_{n0} > k \), and \( (R_m(u_{n0}) - T_k(u_{n0}))(T_k(u_{n0}) - \omega_{\mu, j})' \chi_{u_{n0} > k} \geq 0 \).

Then,

\[
I_2(\sigma) = -\int_{u_{n0} \leq k} \left( (R_m(u_{n0}) - T_k(u_{n0}))(T_k(u_{n0}) - \omega_{\mu, j})' \right) dx dt + \int_{u_{n0} > k} \left( (R_m(u_{n0}) - T_k(u_{n0}))(\omega_{\mu, j})' \right) dx dt.
\]

For the first term and as for \( I_1 \), we have

\[
\int_{u_{n0} \leq k} \left( (R_m(u_{n0}) - T_k(u_{n0}))(\omega_{\mu, j})' \right) dx dt \geq \mu \int_{u_{n0} \leq k} \left( (R_m(u_{n0}) - T_k(u_{n0}))(T_k(v_j) - \omega_{\mu, j}) \right) dx dt.
\]

Then,

\[
\lim_{\sigma \to 0} I_2(\sigma) \geq \varepsilon(n, j, i, \mu).
\]

Claim 3: \( \lim_{\sigma \to 0} I_3(\sigma) \geq \varepsilon(n, j, i, \mu) \).

Since,

\[
I_3(\sigma) = \int_Q \left( (T_k(u_{n0}))'(T_k(u_{n0}) - \omega_{\mu, j}) \right) dx dt
= \int_Q \left( (T_k(u_{n0}) - \omega_{\mu, j})'(T_k(u_{n0}) - \omega_{\mu, j}) \right) dx + \int_Q \left( (\omega_{\mu, j})'(T_k(u_{n0}) - \omega_{\mu, j}) \right) dx dt.
\]
= \left[ \int_{\Omega} \frac{(T_k(u_{mn}) - \omega^j_{\mu,j})^2}{2} \, dx \right]^T + \mu \int_{Q} (T_k(v_j) - \omega^j_{\mu,j})(T_k(u_{mn}) - \omega^j_{\mu,j}) \, dx \, dt \\
\geq \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_{mn}(0)) - \psi^j_k)^2}{2} \, dx + \mu \int_{Q} (T_k(v_j) - \omega^j_{\mu,j})(T_k(u_{mn}) - \omega^j_{\mu,j}) \, dx \, dt.

Then,

\lim_{\sigma \to 0^+} l_{I_{\varepsilon}}(\sigma) \geq \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_{mn}) - \psi^j_k)^2}{2} \, dx + \mu \int_{Q} (T_k(v_j) - \omega^j_{\mu,j})(T_k(u_{mn}) - \omega^j_{\mu,j}) \, dx \, dt \\
= \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_{nn}) - \psi^j_k)^2}{2} \, dx + \mu \int_{Q} (T_k(v_j) - \omega^j_{\mu,j})(T_k(u_{nn}) - \omega^j_{\mu,j}) \, dx \, dt \\
= \varepsilon(n, j, i, \mu).

Finally, we have \( \left\{ \frac{\partial u_{mn}}{\partial t}, (T_k(u_{nn}) - \omega^j_{\mu,j}) \right\} \geq \varepsilon(n, j, i, \mu).

We will now treat the terms \( (3.8) \)–\( (3.9) \). Before that, we will give some convergence results.

Let \( s > 0, Q_s = \{(x, t) \in \Omega : |\nabla T_k(v_j)| \leq s\}, Q^s_j = \{(x, t) \in \Omega : |\nabla T_k(v_j)| \leq s\}, \) and \( \chi^s, \chi^s_j \) be their characteristic function, respectively.

Using \( (1.1) \) and \( (1.3) \), there exist some measurable function \( h_k \) such that

\[ a(\cdot, \nabla T_k(u_{mn})) \rightarrow h_k \text{ in } L^1(Q) \text{ for } a(\Pi L^1, \Pi E_1^1), \]

and we also have \( \nabla T_k(v_j) \chi^s_j \rightarrow \nabla T_k(v_j) \chi^s \) strongly in \( (L^1(Q))^N \) and \( a(\cdot, \nabla T_k(v_j) \chi^s_j) \rightarrow a(\cdot, \nabla T_k(v) \chi^s) \) strongly in \( (E_1(Q))^N \).

Concerning \( (3.9) \)

On the hand, we have

\[ \int_{Q} a(\cdot, \nabla u_{nn}) \nabla u_{nn}(T_k(u_{nn}) - \omega^j_{\mu,j}) \rho^j_m(u_{nn}) \, dx \, dt \geq 0. \]

On the other hand, since \( |f_n| \leq |f| \) and by using Lebesgue theorem with respect to \( n, j, i, \) and \( \mu, \) one get

\[ \int_{Q} f_n(T_k(u_{nn}) - \omega^j_{\mu,j}) \rho^j_m(u_{nn}) = \varepsilon(n, j, i, \mu). \]

Concerning the second term of \( (3.8) \)

\[ \int_{Q} a(\cdot, \nabla u_{nn}) (\nabla T_k(u_{nn}) - \nabla \omega^j_{\mu,j}) \rho^j_m(u_{nn}) \, dx \, dt \]

\[ = \int_{Q} a(\cdot, \nabla T_k(u_{nn})) - a(\cdot, \nabla T_k(v_j) \chi^s_j) \nabla T_k(u_{nn}) - \nabla \omega^j_{\mu,j} \rho^j_m(u_{nn}) \, dx \, dt + \int_{Q} a(\cdot, \nabla T_k(v_j) \chi^s_j) \nabla T_k(u_{nn}) - \nabla T_k(v_j) \chi^s_j \, dx \, dt \\
+ \int_{Q} a(\cdot, \nabla T_k(u_{nn})) \nabla \omega^j_{\mu,j} \rho^j_m(u_{nn}) \, dx \, dt + \int_{Q} a(\cdot, \nabla u_{nn}) \nabla \omega^j_{\mu,j} \rho^j_m(u_{nn}) \, dx \, dt \]

\[ = I_1 + I_2 + I_3 + I_4. \]

\[ I_2 = \int_{Q} a(\cdot, \nabla T_k(v_j) \chi^s_j) (\nabla T_k(u) - \nabla T_k(v_j) \chi^s_j) \, dx \, dt + \varepsilon(n) = \varepsilon(n, j). \]

For \( I_3, \) we have

\[ I_3 = \varepsilon(n) + \int_{Q} h_k \nabla T_k(v_j) \chi^s \, dx = \varepsilon(n, j) + \int_{Q} h_k \nabla T_k(u) \chi^s. \]
For $J_4$, recall that $\rho_m(s) = 0$ if $|s| \geq m + 1$, then

$$J_4 = \int_Q a(\cdot, \nabla u_n) \nabla \omega^{i_j}_\mu \rho_m(u_n) \, dx \, dt$$

$$= - \int_{|u_n| \leq m + 1} a(\cdot, \nabla u_n) \nabla \omega^{i_j}_\mu \rho_m(u_n) \, dx \, dt$$

$$= - \int_{|u_n| \leq k} a(\cdot, \nabla u_n) \nabla \omega^{i_j}_\mu \rho_m(u_n) \, dx \, dt - \int_{k < |u_n| \leq m + 1} a(\cdot, \nabla u_n) \nabla \omega^{i_j}_\mu \rho_m(u_n) \, dx \, dt$$

$$= \int h_k \nabla T_k(u) \, dx \, dt - \int_{k < |u_n| \leq m + 1} h_{m + 1} \nabla T_k(u) \rho_m(u) \, dx \, dt + \varepsilon(n, j, i, \mu)$$

$$= \int h_k \nabla T_k(u) \, dx \, dt + \varepsilon(n, j, i, \mu).$$

Using the aforementioned results, we obtain

$$\int_Q a(\cdot, \nabla u_n)(\nabla \nabla T_k(u_n) - \nabla \omega^{i_j}_\mu) \rho_m(u_n) \, dx \, dt$$

$$= \int_Q (a(\cdot, \nabla T_k(u_n)) - a(\cdot, \nabla T_k(v_j)\chi_j^s))(\nabla \nabla T_k(u_n) - \nabla \nabla T_k(v_j)\chi_j^s) \, dx \, dt + \varepsilon(n, j, i, \mu, s).$$

Combining (3.8)–(3.9), we obtain the almost everywhere convergence of the gradients.

**III. Modular convergence of the gradients**

For all $k > 0$, $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ for the modular convergence in $(L^\infty(Q))^N$. Indeed, we have proved that

$$\int_Q (a(\cdot, \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s))(\nabla \nabla T_k(u_n) - \nabla \nabla T_k(v_j)\chi_j^s) \, dx \, dt \leq \varepsilon(n, j, \mu, s, m).$$

Then,

$$\int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt$$

$$\leq \int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt + \int_Q a(\cdot, \nabla T_k(u)\chi^s)(\nabla \nabla T_k(u_n) - T_k(u)\chi^s) \, dx \, dt + \varepsilon(n, j, \mu, s, m),$$

$$\lim_{n} \int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt \leq \int_Q a(\cdot, \nabla T_k(u)) \nabla T_k(u) \, dx \, dt + \lim_{n} \varepsilon(n, j, \mu, s, m).$$

Also we have,

$$\lim_{n} \int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \leq \int_Q a(\cdot, \nabla T_k(u)) \nabla T_k(u) \leq \lim_{n} \int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n).$$

Then, we deduce that $a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(\cdot, \nabla T_k(u)) \nabla T_k(u) \chi^s$ in $L^1(Q)$.

As mentioned earlier, we obtain $a(\cdot, \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(\cdot, \nabla T_k(u)) \nabla T_k(u)$ in $L^1(Q)$.

Using Vitali’s theorem and (3.1) gives $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ for the modular convergence in $(L^\infty(Q))^N$. 
IV. Passage to the limit

The passage to the limit is an easy task by taking \( v \in K \cap L^\infty(Q) \cap D \) and \( T_k(u_n - v) \) as test function in \( (P_n) \).

V. Uniqueness

Following the same way as Theorem 5.1 [14] for the parabolic case, we obtain the uniqueness.

4 Conclusion

In this paper, we have focused on the existence, uniqueness, and regularity of a class of inequalities in a general convex set and in a nonstandard functional framework, which is the Sobolev Orlicz spaces. The techniques used are not standard and require a very particular handling of the test functions and the approximated problems.

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