Error bounds for sparse classifiers in high-dimensions

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Abstract

We prove an $L^2$ recovery bound for a family of sparse estimators defined as minimizers of some empirical loss functions – which include hinge loss and logistic loss. More precisely, we achieve an upper-bound for coefficients estimation scaling as $(k^*/n) \log(p/k^*)$: $n \times p$ is the size of the design matrix and $k^*$ the dimension of the theoretical loss minimizer. This is done under standard assumptions, for which we derive stronger versions of a cone condition and a restricted strong convexity. Our bound holds with high probability and in expectation and applies to an $L_1$-regularized estimator and to a recently introduced Slope estimator, which we generalize for classification problems. Slope presents the advantage of adapting to unknown sparsity. Thus, we propose a tractable proximal algorithm to compute it and assess its empirical performance. Our results match the best existing bounds for classification and regression problems.

1 Introduction

Motivated by the increasing availability of very large-scale datasets, high-dimensional statistics has focused on analyzing the performance of sparse estimators. An estimator is said to be sparse if the response of an observation is given by a small number of coefficients: sparsity delivers better interpretability and often leads to computational efficiency. Statistical performance and $L^2$ consistency for high-dimensional linear regression have been widely studied. For two polynomial-time sparse estimators, a Lasso [Tibshirani, 1996] and a Dantzig selector [Candes and Tao, 2007], Bickel et al. [2009] proved a $(k^*/n) \log(p)$ rate for the $L^2$ estimation of the coefficients: $n \times p$ is the dimension of the input matrix and $k^*$ the degree of sparsity of the vector used to generate the model. The optimality of this bound is essential for a theoretical understanding of the method performance. Candes and Davenport [2013] and Raskutti et al. [2011] proved a $(k^*/n) \log(p/k^*)$ lower bound for estimating the $L^2$ norm of a sparse vector, regardless of the input matrix and estimation procedure. This optimal minimax rate is known to be achieved by a sparse but theoretically intractable BIC estimator [Bunea et al., 2007] which considers an $L_0$ regularization. This BIC estimator adapts to unknown sparsity, that is, the degree $k^*$ does not have to be specified. Recently, Bellec et al. [2016] reached this optimal minimax bound for a Lasso estimator with knowledge of the sparsity $k^*$. They also proved that a recently introduced and polynomial-time Slope estimator [Bogdan et al., 2013] achieves this optimal rate while adapting to unknown sparsity.

Little work has been done on deriving (theoretical) upper bounds for the estimation error on high-dimensional classification problems. Indeed, the literature has essentially focused on analysis of convergence [Tarigan et al., 2006, Zhang et al.]. Recently, Peng et al. [2016] proved a $(k^*/n) \log(p)$ upper-bound for $L^2$ coefficients estimation of a $L_1$-regularized Support Vector Machines (SVM): $k^*$ is now the sparsity of the theoretical minimizer to estimate. They recovered the rate proposed by Van de Geer [2008], which
considered a weighted L1 norm for linear models. Ravikumar et al. [2010] obtained a similar bound for a L1-regularized Logistic Regression estimator in a binary Ising graph. Their frameworks and bounds are similar to the model proposed by [Belloni et al. 2011] for L1-regularized Quantile Regression; this inspired us to include this problem in our framework. However, this rate of $(k^*/n) \log(p)$ is not the best known for a classification estimator: [Plan and Vershynin 2013] proved a $k \log(p/k^*)$ error bound for estimating a single vector through sparse models – including 1-bit compressed sensing and Logistic Regression – over a bounded set of vectors. Contrary to this work, our approach does not assume a generative vector and applies to a larger class of problems (SVM, Quantile Regression) and regularizations (Slope). Finally, Pierre et al. 2017 studied a similar class of loss functions and regularization that the ones proposed herein. However, their proof technique is quite different than ours, leading to an estimation error rate of the order of $k^* \log(p)/n$, which is higher than the one we derive. The authors do not discuss any computational algorithms for the Slope estimator, which we do.

**What this paper is about:** In this paper, we propose a theoretical framework to analyze the properties of a general class of sparse estimators for classification problems – including SVM and Logistic Regression – with different regularization schemes. Our approach draws inspiration from the least squares regression case and illustrates the distinction between the regression and classification studies. Our main results are first presented for a family of L1-regularized estimators. We achieve a $(k^*/n) \log(p/k^*)$ upper-bound for coefficients estimation, which holds with high probability and in expectation. In addition, we introduce a version of the Slope estimator for classification problems: we propose a proximal algorithm to compute the solution, and we prove that a tractable Slope estimator achieves a similar upper-bound while adapting to unknown sparsity. To the best of our knowledge, it is the first time any of these bounds is reached for the estimators considered.

The rest of this paper is organized as follows. Section 2 introduces and discusses assumptions common in the literature, and builds the framework of our study in the particular case of L1-regularized estimators. Section 3 proves two essential results and derive our upper-bounds in Theorem 1 and Corollary 1. Finally, Section 4 defines and computes the Slope estimator for our class of problems and proves its statistical performance.

## 2 General assumptions with an L1 regularization

We consider a set of training data $\{(x_i, y_i)\}_{i=1}^n$, $(x_i, y_i) \in \mathbb{R}^p \times \mathcal{Y}$ from an unknown distribution $\mathbb{P}(X, y)$. We note our loss $f$ and define the theoretical loss $L(\beta) = \mathbb{E}(f((x, \beta); y))$. We consider a theoretical minimizer $\beta^*$:

$$\beta^* \in \arg\min_{\beta \in \mathbb{R}^p} \{\mathbb{E}(f((x, \beta); y))\}. \quad (1)$$

In the rest of this section, we denote by $k = \|\beta^*\|_0$ the number of non-zeros of the theoretical minimizer and $R = \|\beta^*\|_1$ its L1 norm. We assume $R \geq 1$. We study the L1-regularized L1-constrained problem defined as:

$$\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f((x_i, \beta); y_i) + \lambda \|\beta\|_1. \quad (2)$$

We consider an empirical minimizer $\hat{\beta}$, solution of Problem (2). The constraint $2R$ in Problem (2) is somewhat arbitrary: it enforces the empirical minimizer to be close enough to the theoretical minimizer $\beta^*$: $\|\hat{\beta} - \beta^*\|_1 \leq 3R$. The L1 regularization in Lagrangian form is known to induce sparsity in the coefficients of $\hat{\beta}$. Note that Problem (2) is fully tractable.

For a given $\lambda$, we fix a solution $\hat{\beta}(\lambda, R)$ of Problem (2) – $R$ is fixed throughout the paper. Our main result is an error bound – achieved for a certain $\lambda$ – for the L2 norm of the difference between the empirical
and theoretical minimizers $\|\hat{\beta}(\lambda, R) - \beta^*\|_2$. When no confusion can be made, we drop the dependence upon the parameters $\lambda, R$. Our bound is reached under standard assumptions in the literature. In particular, it is similar to those proposed by Peng et al. [2016], Ravikumar et al. [2010], Belloni et al. [2011]. Our framework of study is presented in the rest of this section.

### 2.1 Lipschitz loss function

Our first assumption concerns the Lipschitz-continuity of the loss $f$.

**Assumption 1** The loss $f(., y)$ is non-negative, convex and Lipschitz continuous with constant $L$, that is, $|f(t_1, y) - f(t_2, y)| \leq L|t_1 - t_2|, \forall t_1, t_2$. In addition, there exists $\partial f(., y)$ such that $f(t_2, y) - f(t_1, y) \geq \partial f(t_1, y)(t_2 - t_1), \forall t_1, t_2$.

$\partial f(., y)$ is said to be a sub-gradient of the loss: if $f(., y)$ is differentiable, we simply consider its gradient. It trivially holds $\|\partial f(., y)\|_\infty \leq L, \forall y$. We list three main examples that fall into this framework.

**Example 1: Support Vectors Machines** We assume $\mathcal{Y} = \{-1, 1\}$ and consider the L1-regularized L1-constrained Support Vector Machines (SVM) problem. It learns a classification rule of the data of the form $\text{sign}(\langle x, \beta \rangle)$ by solving the problem:

$$
\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^{n} \left(1 - y_i \langle x_i, \beta \rangle\right)_+ + \lambda \|\beta\|_1.
$$

(3)

The hinge loss $f(\langle x, \beta \rangle; y) = \max(0, 1 - y \langle x, \beta \rangle)$ admits as a subgradient $\partial f(., y) = 1(1 - y \geq 0)y$. and satisfies Assumption $\square$ for $L = 1$.

**Example 2: Logistic Regression** Here, $\mathcal{Y} = \{-1, 1\}$ and we consider the additional usual assumption that $\log(\mathbb{P}(y_i = 1|X = x_i)) - \log(\mathbb{P}(y_i = -1|X = x_i)) = \langle x_i, \beta \rangle, \forall i$. The L1-regularized L1-constrained Logistic Regression estimator is a solution of the problem:

$$
\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-y_i \langle x_i, \beta \rangle)) + \lambda \|\beta\|_1.
$$

(4)

The logistic loss $f(\langle x, \beta \rangle; y) = \log(1 + \exp(-y \langle x, \beta \rangle))$ has a derivative with respect to its first variable $|\partial_t f(t, y)| = \frac{1}{1 + e^{yt}} \leq 1$, hence it satisfies Assumption $\square$ for $L = 1$.

**Example 3: Quantile Regression** We now consider a class of parametric quantile estimation problems. Following [Buchinsky 1998], we assume that for $\theta \in (0, 1)$ the conditional quantile of $y$ given $X$ is given by $Q_\theta(y|X = x) = \langle x, \beta_\theta \rangle$, where the model is of the form $y = \langle x, \beta_\theta \rangle + u_\theta$, and $u_\theta$ is unknown. The L1-regularized L1-constrained $\theta$-Quantile Regression estimator is defined as a solution of:

$$
\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^{n} \rho_\theta(y_i - \langle x_i, \beta \rangle)) + \lambda \|\beta\|_1,
$$

(5)

where $\rho_\theta(t) = (\theta - 1(t \leq 0))t$ is the quantile regression loss. $\rho_\theta$ satisfies Assumption $\square$ for $L = \max(1 - \theta, \theta)$. Note that the hinge loss is a simple translation of the quantile regression loss for $\theta = 0$. 

3
2.2 Differentiability of the theoretical loss

The following assumption ensures the unicity of $\beta^*$ and the twice differentiability of the theoretical loss $L$. Equation (6) is equivalent to saying that the gradient of the theoretical loss is equal to the theoretical sub-gradient of the loss – defined in Assumption 1.

**Assumption 2** The theoretical minimizer is unique. In addition, the theoretical loss is twice-differentiable: we denote its gradient $\nabla L(\beta)$ and its Hessian matrix $\nabla^2 L(\beta)$. We also assume:

$$\nabla L(.) = \mathbb{E}(\partial f (\langle x, .\rangle; y) x) .$$

**Support Vectors Machines:** Koo et al. [2008] studied specific conditions under which Assumption 2 holds for SVM. In particular, if $f$ and $g$ denote the respective conditional densities of $x_i$ given $y_i = 1$ and $y_i = -1$; they proved that if the densities $f$ and $g$ are continuous with common support $S \subset \mathbb{R}^p$ and have finite second moments, then the gradient $\nabla L(\beta) = \mathbb{E}(1 \{1 - y \langle x, \beta \rangle \geq 0\}) y x \delta(\cdot)$ and the Hessian matrix $\nabla^2 L(\beta) = \mathbb{E}(\delta(1 - y \langle x, \beta \rangle) y x)$ ( $\delta(\cdot)$ is the Dirac function) are defined and continuous.

**Logistic and Quantile Regression:** The regularity of $\nabla L$ and $\nabla^2 L$ are trivial for the logistic regression loss. Equation (6) holds as the sub-gradient is simply the gradient of the loss. For the quantile regression loss, a study similar to the case of the hinge loss – using Assumption D.1 by Belloni et al. [2011] – can be applied to obtain Assumption 2.

2.3 Sub-Gaussian columns

We denote $X$ the design matrix, with rows $x_1, \ldots, x_n$. The following assumption guarantees that some random variable built from the columns $(X_1, \ldots, X_p)$ of $X$ have tails bounded by a sub-Gaussian random variable with variance proportional to $n$. We first recall the definition of a sub-Gaussian random variable [Rigollet, 2015]:

**Definition 1** A random variable $Z$ is said to be sub-Gaussian with variance $\sigma^2 > 0$ if $\mathbb{E}(Z) = 0$ and $\mathbb{P}(|Z| > t) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2} \right)$, $\forall t > 0$.

A sub-Gaussian variable will be noted $Z \sim \text{subG}(\sigma^2)$. We would like here to notice another important aspect of our contribution. Our next Theorem 3 derives a cone condition, a necessary step to prove our main results. Our approach draws inspiration from the regression case with Gaussian noise. However, it relies on a new study of sub-Gaussian random variables – such analysis is not needed in the regression case. Our results are derived under the following Assumption 3:

**Assumption 3** There exists $M > 0$ such that with the notations of Assumption 1:

$$\sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij} \sim \text{subG}(n L^2 M^2), \forall j.$$ (7)

$\beta^*$ minimizes the theoretical loss. Thus, from Assumption 2 $\mathbb{E} [\partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij}] = 0, \forall i, j.$ The next lemma gives more insight about Assumption 3. The proof is presented in Appendix A.

**Lemma 1** If the rows of the design matrix are independent and if all the entries $\partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij}, \forall i, j$ are sub-Gaussian with variance $L^2 M^2$, then $\sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij} \sim \text{subG}(n L^2 M^2), \forall j.$
In particular, if \( |x_{i,j}| \leq M, \forall i, j \), then if Assumption 1 is satisfied, Hoeffding’s lemma guarantees that 
\[ \partial f ((x_1, \beta^*) ; y_i) x_{i,j} \sim \text{subG}(L^2 M^2), \forall i, j. \] 
Thus Assumption 3 is satisfied. Assumption 3 is also satisfied if the design matrix \( \mathcal{X} \) consists of \( n \) independent samples of a multivariate centered Gaussian random variable. Hence, Assumption 3 is rather mild. It is considerably much weaker than Assumption (A1) [Peng et al., 2016] which imposes a finite bound on the L2 norm of each column of \( \mathcal{X} \).

### 2.4 Restricted eigenvalue conditions

The next assumption draws inspiration from the restricted eigenvalue conditions defined in regression problems [Belleg et al., 2016, Bickel et al., 2009]. In particular, for an integer \( k \), Assumption 4.1 ensures that a certain random variable is upper-bounded on the set of all \( k \) sparse vectors. Similarly, Assumption 4.2 ensures that the quadratic form associated to the Hessian \( \nabla^2 \mathcal{L}(\beta^*) \) is lower-bounded on a cone of \( \mathbb{R}^p \).

**Assumption 4** Let \( k \in \{1, \ldots, p\} \). Assumption 4.1 \((k)\) is satisfied if there exists a nonnegative constant \( \mu(k) \) such that almost surely:

\[
\mu(k) \geq \sup_{z \in \mathbb{R}^p: \|z\|_0 \leq k} \frac{\sqrt{k}\|\mathcal{X} z\|_1}{\sqrt{n}\|z\|_1} > 0.
\]

Let \( \gamma_1, \gamma_2 \) be two non-negative constants. Assumption 4.2 \((k, \gamma)\) holds if there exists a nonnegative constant \( \kappa(k, \gamma) \) which almost surely satisfies:

\[
0 < \kappa(k, \gamma) \leq \inf_{|S| \leq k} \inf_{z \in \Lambda(S, \gamma_1, \gamma_2)} \frac{\|z^T \nabla^2 \mathcal{L}(\beta^*) z\|_2}{\|z\|_2},
\]

where \( \gamma = (\gamma_1, \gamma_2) \) and for every subset \( S \subset \{1, \ldots, p\} \), the cone \( \Lambda(S, \gamma_1, \gamma_2) \subset \mathbb{R}^p \) is defined as:

\[
\Lambda(S, \gamma_1, \gamma_2) = \{z \in \mathbb{R}^p: \|z_{S^c}\|_1 \leq \gamma_1 \|z_S\|_1 + \gamma_2 \|z_S\|_2\}. \]

We refer to Assumption 4 \((k, \gamma)\) when both Assumptions 4.1 \((k)\), 4.2 \((k, \gamma)\) are assumed to hold.

In the SVM framework, Peng et al. [2016] defined Assumption (A4): it is similar to our Assumption 4.2 but considers a different cone of \( \mathbb{R}^p \). In addition, their Assumption (A3) define a certain \( \mu(k) \) as an upper bound of the quadratic form associated to \( n^{-1/2} \mathcal{X}^T \mathcal{X} \) restricted to the set of \( k \) sparse vectors. That is, under their definition, \( \|\mathcal{X} z\|_2/\sqrt{n} \leq \mu(k)\|z\|_2, \forall z: \|z\|_0 \leq k \). Our Assumption 4.1 is stronger: when satisfied, we can recover Assumption (A3) since that

\[
\forall z \in \mathbb{R}^p: \|z\|_0 \leq k, \|\mathcal{X} z\|_2/\sqrt{n} \leq \mu(k)\|z\|_2/\sqrt{k} \leq \mu(k)\|z\|_2
\]

where we have used Cauchy-Schwartz inequality on the \( k \) sparse vector \( z \). However, Assumption 4.1 uses an L1 norm, more naturally associated to the class of L1-regularized estimators studied here.

In the logistic regression case, Ravikumar et al. [2010] defined similar dependency and incoherence conditions for the population Fisher information matrix (Assumption A1 and A2). Finally, Assumption D.4 for Quantile Regression [Belloni et al., 2011] is equivalent to a uniform Restricted Eigenvalue condition.

### 2.5 Growth condition

Since \( \beta^* \) minimizes the theoretical loss, it holds \( \nabla \mathcal{L}(\beta^*) = 0 \). In particular, under Assumption 4.2 \((k^*, \gamma)\), the theoretical loss evaluated on the family of cones \( \Lambda(S, \gamma_1, \gamma_2) \) with \( |S| \leq k^* \) is lower-bounded by a quadratic form around \( \beta^* \). By continuity, we define the maximal radius on which the following lower-bound holds:

\[
r(k^*) = \max \left\{ r: \mathcal{L}(\beta^* + z) \geq \mathcal{L}(\beta^*) + \frac{\kappa(k^*, \gamma)}{4}\|z\|_2^2 \forall S \subset (p): |S| \leq k^*, \forall z \in \Lambda(S): \|z\|_1 \leq r \right\}
\]
where the notations $r(k^*)$, $\kappa(k^*)$ and $\Lambda(S)$ are shorthands for $r(k^*, \gamma_1, \gamma_2)$, $\kappa(k^*, \gamma_1, \gamma_2)$ and $\Lambda(S, \gamma_1, \gamma_2)$. This definition is similar to the one proposed by Belloni et al. [2011] in the proof of Lemma (3.7). We now define a growth condition which gives a relation between the number of samples $n$, the dimension space $p$, our constants introduced in Assumption 4, and a parameter $\delta$. We will refer to Assumption 5 when both Assumptions 5.1($k^*$) and 5.2($k^*, \gamma, \delta$) hold.

Assumption 5 is similar to Equation (17) [Ravikumar et al., 2010] for Logistic Regression. Belloni et al. [2011] also requires a growth condition for Theorem 2 to hold for Quantile Regression. Consequently, as we discussed, Assumptions 15 are common assumptions or similar to existing ones in the literature. The next section uses our framework to derive upper bounds for L2 coefficients estimation scaling with the size parameters $n, p, k^*$.

3 Main results

This section establishes the following theorem:

**Theorem 1** Let $\delta \in (0, 1)$, $\alpha > 1$ and assume Assumptions 13, 2($k^*, \gamma$) and 5($k^*, \gamma, \delta$) hold – where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_1 := \frac{\alpha}{\alpha-1}$, $\gamma_2 := \frac{\sqrt{\alpha}}{\alpha-1}$.

Then, the empirical estimator $\hat{\beta}$, defined as a solution of the learning Problem (2) for the regularization parameter $\lambda = 12\alpha LM \sqrt{\frac{\log(p/k^*)}{n}} \log(2/\delta)$, satisfies with probability at least $1 - \delta$:

$$
\|\hat{\beta} - \beta^*\|_2 \leq \frac{\alpha L\mu(k^*)}{\kappa(k^*)} \sqrt{\frac{k^* \log(Rp/k^*) + \log(2/\delta)}{n}} + \frac{\alpha LM}{\kappa(k^*)} \sqrt{\frac{k^* \log(p/k^*) \log(2/\delta)}{n}}. \tag{8}
$$

This upper bound scales as $((k^*/n) \log(p/k^*))^{1/2}$. It strictly improves over existing results. Note that our estimator is not adaptive to unknown sparsity: the regularization parameter $\lambda$ depends upon $k^*$.

The proof of Theorem 1 is presented in Appendix E. It relies on two essential steps: a cone condition and a restricted strong convexity condition: these results are respectively derived in Theorems 2 and 4. The two terms of the sum in Equation (8) are related to the two parameters $\lambda$ and $\tau$ respectively introduced in these theorems.

In addition, Theorem 1 holds for any $\delta \leq 1$. Thus, we obtain by integration the following bound in expectation. The proof is presented in Appendix F.

**Corollary 1** If the assumptions presented in Theorem are satisfied for a small enough $\delta$, then:

$$
E\|\hat{\beta} - \beta^*\|_2 \leq \frac{\alpha L}{\kappa(k^*)} \mu(k^*) + M \sqrt{\frac{k^* \log(Rp/k^*)}{n}}. \tag{9}
$$

The rest of this section follows through the steps required to prove Theorem 1 and Corollary 1.
3.1 Cone condition

Similarly to the regression case [Bickel et al., 2009, Bellec et al., 2018], we first derive a cone condition which applies to the difference between the empirical and theoretical minimizers. That is, by selecting a suitable regularization parameter, we show that this difference belongs to the family of cones \( \Lambda(S, \gamma_1, \gamma_2) \) of \( \mathbb{R}^p \) – defined in Assumption 2.

**Theorem 2** Let \( \delta \in (0, 1) \) and assume that Assumptions 1 and 3 are satisfied. Let \( \alpha \geq 2 \).

Let \( \hat{\beta} \) be a solution of Problem 2 with parameter \( \lambda = 12\alpha LM \sqrt{\frac{\log(2p/\kappa^*)}{n}} \log(2/\delta) \).

Then it holds with probability at least \( 1 - \frac{\delta}{2} \):

\[
\mathbf{h} := \hat{\beta} - \beta^* \in \Lambda \left( S_0, \gamma_1 := \frac{\alpha}{\alpha - 1}, \gamma_2 := \frac{\sqrt{\kappa^*}}{\alpha - 1} \right),
\]

where \( \alpha_0 = \frac{\alpha - 1}{\alpha} \) and \( S_0 \) is the subset of indices of the \( k^* \) highest coefficients of \( \mathbf{h} \).

The regularization parameter \( \lambda \) is selected so that it dominates the sub-gradient of the loss \( f \) evaluated at the theoretical minimizer \( \beta^* \). The proof is presented in Appendix B: it uses a new result to control the maximum of independent sub-Gaussian random variables. As a result, our cone condition is stronger than the ones proposed by Peng et al. [2016] and Ravikumar et al. [2010]: their value of \( \lambda^2 \) is of the order of \( (k^*/n) \log(p) \) whereas ours scales as \( (k^*/n) \log(p/k^*) \).

3.2 A supremum result

The next Theorem 3 is an essential step to obtain our main Theorem 1. It derives a control of the supremum of the difference between an empirical random variable and its expectation over a bounded set of sequences of \( k \) sparse vectors with disjoint supports. The restricted strong convexity condition derived in Theorem 2 is a consequence of Theorem 3.

To motivate this theorem, it helps considering the difference between the usual regression framework and our framework for classification problems. The linear regression case assumes the generative model \( y = X \beta^* + \epsilon \). Therefore, with the notations of Theorem 3, \( \Delta(\beta^*, z) = \frac{1}{n} \| Xz \|_2^2 - \frac{2}{n} \epsilon^T Xz \). By combining a cone condition (similar to Theorem 1) with an upper-bound of the term \( \epsilon^T Xz \), we can obtain a restricted strong convexity similar to Theorem 4. However, in the classification case, \( \beta^* \) is defined as the minimizer of the theoretical risk. Two major differences appear: (i) we cannot simplify \( \Delta(\beta^*, z) \) with basic algebra, (ii) we need to introduce the expectation \( \mathbb{E}(\Delta(\beta^*, z)) \) and to control the quantity \( |\mathbb{E}(\Delta(\beta^*, z)) - \Delta(\beta^*, z)| \).

**Theorem 3** We define \( \forall \mathbf{w}, \mathbf{z} \in \mathbb{R}^p: \)

\[
\Delta(\mathbf{w}, \mathbf{z}) = \frac{1}{n} \sum_{i=1}^n f(\langle x_i, \mathbf{w} + \mathbf{z} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle x_i, \mathbf{w} \rangle; y_i).
\]

Let \( k \in \{1, \ldots, p\} \) and \( S_1, \ldots, S_q \) be a partition of \( \{1, \ldots, p\} \) with \( q = \lfloor p/k \rfloor \) and \( |S_j| \leq k, \forall j \).

Let \( \tau(k) = 14L \mu(k) \sqrt{\frac{\log(Rp/k)}{n}} + \frac{\log(2/\delta)}{kn} \) and assume that Assumptions 4, 2, 1(k) and 3, 1(k) are satisfied. Then, for any \( \delta \in (0, 1) \), it holds with probability at least \( 1 - \frac{\delta}{2} \):

\[
\sup_{\mathbf{z}_{S_1}, \ldots, \mathbf{z}_{S_q} \in \mathbb{R}^p: \text{Supp}(\mathbf{z}_{S_j}) \subset S_j \forall j, \|\mathbf{z}_{S_j}\|_1 \leq 3R, \forall j} \left\{ \sup_{\ell=1, \ldots, q} \left\{ \Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_j}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_j})) - \tau(\|\mathbf{z}_{S_j}\|_1) \right\} \right\} \leq 0
\]
Supp(.) refers to the support of a vector and we have defined \( w_\ell = \beta^* + \sum_{j=1}^{\ell} z_{S_j}, \forall \ell. \)

The proof is presented in Appendix C. It uses Hoeffding’s inequality to obtain an upper bound of the inner supremum for any sequence of \( k \) sparse vectors. The result is extended to the outer supremum with an \( \epsilon \)-net argument.

### 3.3 Restricted strong convexity

Theorem 3 applies for a sequence of \( k^* \) sparse vectors with disjoint supports, built from \( h = \hat{\beta} - \beta^* \). In addition, we can exploit the minimality of \( \beta^* \) and the cone condition proved in Theorem 2. By pairing these results, we derive the next Theorem 4. It says that the loss \( f \) satisfies a restricted strong convexity [Negahban et al., 2009] with curvature \( \kappa(k^*)/4 \) and L2 tolerance function.

**Theorem 4** Let \( h = \hat{\beta} - \beta^* \) and \( \delta \in (0,1) \). Under the notations of Theorem 3 if Assumptions \( 4(k^*, \gamma) \) and \( 5(k^*, \gamma, \delta) \) are satisfied, then it holds with probability at least \( 1 - \delta \):

\[
\Delta (\beta^*, h) \geq \frac{1}{4} \kappa(k^*) \{ \|h\|_2^2 \wedge \tau(k^*)\|h\|_2 \} - \tau(k^*)\|h\|_2.
\]

The proof of the theorem is presented in Appendix D. Let us note an important remark: our parameter \( \tau(k^*)^2 \) scales as \( n^{-1}k^* \log(p/k^*) \), whereas Peng et al. [2016] and Ravikumar et al. [2010] both proposed a parameter scaling as \( n^{-1}k^* \log(p) \). Our restricted strong convexity is then stronger.

### 3.4 Proving Theorem 1 and Corollary 1

Our main bounds – presented in Theorem 1 and Corollary 1– follow from the two preceding Theorems 2 and 4. The proofs are respectively presented in Appendix E and F. Our family of L1-regularized L1-constrained estimators reach a bound that strictly improve over existing results. Our rate is the best known for the classification problems considered here, and it holds both with high probability and in expectation.

### 4 Algorithm and upper bounds for Slope estimator

This section introduces the Slope estimator – originally presented for the linear regression case [Bogdan et al., 2013, 2015] – to our class of problems. We propose a tractable algorithm to compute the estimator and study its statistical properties.

#### 4.1 Introducing Slope for classification

We consider a sequence \( \lambda \in \mathbb{R}^p \) such that \( \lambda_1 \geq \ldots \geq \lambda_p > 0 \), and we note \( S_p \) the set of permutations of \( \{1, \ldots, p\} \). The Slope regularization is defined as:

\[
|\beta|_S = \max_{\phi \in S_p} \sum_{j=1}^{p} |\lambda_j||\beta_{\phi(j)}| = \sum_{j=1}^{p} \lambda_j|\beta_{(j)}|,
\]

where \( |\beta_{(1)}| \geq \ldots \geq |\beta_{(p)}| \) is a non-increasing rearrangement of \( \beta \). Consequently for \( \eta > 0 \), we define the Slope estimator \( \hat{\beta} \) as the solution of the convex minimization problem:

\[
\min_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f (\langle x_i, \beta \rangle; y_i) + \eta |\beta|_S.
\]
The approach presented herein uses a proximal gradient algorithm – with Nesterov smoothing [Nesterov, 2005] in the case of the hinge loss and quantile regression loss – to solve Problem (11), extending the original definition of Slope [Bogdan et al., 2013] to a larger class of loss functions.

4.2 Smoothing the hinge loss

The method described in Section 4.3 to solve Problem (11) requires \( f(. , y) \) to be differentiable with Lipschitz-continuous gradient. Among the examples presented in Section 2, the logistic regression loss is the only case where this condition is satisfied.

To handle the non-smooth hinge loss, we use the smoothing scheme pioneered by Nesterov [2005]. We construct a convex function \( g_\tau \) with continuous Lipschitz gradient, which approximates the hinge loss for \( \tau \approx 0 \). Let us first note that \( \max(0, x) = \frac{1}{2}(x + |x|) = \max_{|w| \leq 1} \frac{1}{2}(x + wx) \) as this maximum is achieved for \( \text{sign}(x) \). Consequently the hinge loss can be expressed as a maximum over the \( L_\infty \) unit ball:

\[
\frac{1}{n} \sum_{i=1}^{n} \max(z_i, 0) = \max_{\|w\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^{n} [z_i + w_i z_i],
\]

where \( z_i = 1 - y_i x_i^T \beta, \forall i \). We apply the technique suggested by Nesterov [2005] and define for any \( \tau > 0 \) the smooth hinge loss:

\[
g_\tau(\beta) = \max_{\|w\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^{n} [z_i + w_i z_i] - \frac{\tau}{2n} \|w\|^2_2. \tag{12}\]

Let \( w_i^\tau(\beta) \in \mathbb{R}^n : w_i^\tau(\beta) = \min \left( 1, \frac{1}{\tau} |z_i| \right) \text{sign}(z_i), \forall i \) be the optimal solution of the right-hand side of Equation (12). The gradient of \( g_\tau \) is expressed as:

\[
\nabla g_\tau(\beta) = -\frac{1}{2n} \sum_{i=1}^{n} (1 + w_i^\tau(\beta)) y_i x_i \in \mathbb{R}^p, \tag{13}\]

and its associated Lipschitz constant is derived from the next theorem.

**Theorem 5** Let \( \mu_{\max}(n^{-1}X^T X) \) be the highest eigenvalue of \( n^{-1}X^T X \). Then \( \nabla g_\tau \) is Lipschitz continuous with constant \( C\tau = \mu_{\max}(n^{-1}X^T X) / 4\tau \).

The proof is presented in Appendix G. It follows Nesterov [2005] and uses first order necessary conditions for optimality. An adaption to the case of quantile regression loss is mentioned.

4.3 Thresholding operator for Slope

We note \( g(\beta) = \frac{1}{n} \sum_{i=1}^{n} f(\langle x_i, \beta \rangle; y_i) \). The smooth equivalent formulation of Problem (11) is:

\[
\min_{\beta \in \mathbb{R}^p} g(\beta) + \eta |\beta|_S, \tag{14}\]

with \( g \) being a differentiable loss with \( C \)-Lipschitz continuous gradient. If \( f \) is the hinge or quantile regression loss we use \( g_\tau \) as introduced in Section 4.2. For \( D \geq C \), we upper-bound \( g \) around any \( \alpha \in \mathbb{R}^p \) with the quadratic form \( Q_D(\alpha, .) \) defined as the right-hand side of the equation:

\[
g(\beta) \leq g(\alpha) + \nabla g(\alpha)^T (\beta - \alpha) + \frac{D}{2} \|\beta - \alpha\|^2_2. \tag{15}\]
We approximate the solution of Problem (11) by considering the loss $Q_D$ and solving the problem:

$$\arg\min_{\beta} Q_D(\alpha, \beta) + \eta |\beta|_S = \arg\min_{\beta} \frac{1}{2} \left\| \beta - \left( \alpha - \frac{1}{D} \nabla g(\alpha) \right) \right\|^2 + \frac{\eta}{D} |\beta|_S$$

$$= \arg\min_{\beta} \frac{1}{2} \left\| \beta - \left( \alpha - \frac{1}{D} \nabla g(\alpha) \right) \right\|^2 + \eta \|D\|_2 \left| \frac{1}{\|D\|_2} \sum_{j=1}^{p} \tilde{\eta}_j |\beta|_{(j)} \right|,$$

(16)

where $\gamma = \alpha - \frac{1}{D} \nabla g(\alpha)$ and $\tilde{\eta}_j = \frac{\eta}{\|D\|_2} \lambda_j$, $\forall j$. Problem (16) can be solved if the proximal operator for the sorted L1 norm is known. Note that the sign of each of the quantities $\beta_j$ and $\gamma_j$ are identical. The next Lemma provides a solution to this thresholding operator.

**Lemma 2** Let us assume that $\tilde{\gamma}_1 \geq \ldots \geq \tilde{\gamma}_p \geq 0$. Since $\tilde{\eta}_1 \geq \ldots \geq \tilde{\eta}_p \geq 0$, the solution of Problem (16) can be derived from the solution of the problem:

$$\min_{\beta \in \mathbb{R}^p} \frac{1}{2} \|\beta - \tilde{\gamma}\|^2 + \sum_{j=1}^{p} \tilde{\eta}_j \beta_j$$

s.t. $\beta_1 \geq \ldots \geq \beta_p \geq 0$.

(17)

Bogdan et al. [2015] proposed an efficient proximal algorithm to solve Problem (17) called FastProxSL1: it is guaranteed to terminate in at most $p$ iterations. We denote by $T_{\{\tilde{\eta}_j\}}(\gamma)$ a solution to Problem (16).

### 4.4 First order algorithm

The following algorithm applies the accelerated gradient descent method [Beck and Teboulle, 2009] on the smooth Slope Problem (14) using the thresholding operator for Slope. The iterations continue till the algorithm converges or a maximum number of iterations $T_{\text{max}}$ is reached.

**Input:** $X$, $y$, a sequence of Slope coefficients $\{\lambda_j\}$, a regularization parameter $\eta$, a stopping criterion $\epsilon$, a maximum number of iterations $T_{\text{max}}$.

**Output:** An approximate solution $\beta$ of the smooth Slope Problem (14).

1. Initialize $T = 1$, $q_1 = 1$, $\beta_1 = \delta_0 = 0$.

2. : While $\|\beta_T - \beta_{T-1}\|_2^2 > \epsilon$ and $T < T_{\text{max}}$ do:

   a. Compute $\delta_T = T_{\{\eta \lambda_j / C\}} \left( \beta_{T-1} - \frac{1}{C} \nabla g(\beta_{T-1}) \right)$.

   b. Define $q_{T+1} = \frac{1+\sqrt{1+4q_T^2}}{2}$ and compute $\beta_{T+1} = \delta_T + \frac{q_T-1}{q_{T+1}} (\delta_T - \delta_{T-1})$.

### 4.5 Error bounds for Slope

We extend our previous case and study under our framework the theoretical properties of a Slope estimator for a particular value of $\eta$. In particular, we consider the L1-constrained Slope estimator:

$$\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^{n} f \left( (x_i, \beta); y_i \right) + \eta |\beta|_S.$$  

(18)

Studying Slope is very similar to our previous work for L1-regularized estimators. First, we derive the following cone condition:
Theorem 6 Let \( \delta \in (0, 1) \) and \( \alpha \geq 2 \). We fix the Slope coefficients \( \lambda_j = \sqrt{\log(2pe/j)} \), \( \forall j \), and assume Assumptions 7 and 3 hold. Then the Slope estimator defined as a solution of Problem (18) for the regularization parameter \( \eta = 14\alpha L M n^{-1} \sqrt{\log(6/\delta)} \) satisfies with probability at least \( 1 - \frac{\delta}{2} \):

\[
\hat{\beta} - \beta^* \in \Gamma \left( k^*, \omega^* = \frac{\alpha + 1}{\alpha - 1} \right),
\]

where for every \( k \in \{1, \ldots, p\} \) and \( \omega > 0 \), the cone \( \Gamma(k, \omega) \) is defined as:

\[
\Gamma(k, \omega) = \left\{ z \in \mathbb{R}^p : \sum_{j=k+1}^{p} \lambda_j |z(j)| \leq \omega \sum_{j=1}^{k} \lambda_j |z(j)| \right\}
\]

with \( |z(1)| \geq \ldots \geq |z(p)| \), \( \forall z \).

The proof is presented in Appendix H. We consequently adapt Assumption 6 to the new family of cones \( \Gamma(k, \omega) \) introduced in Theorem 6.

Assumption 6 Let \( k \in \{1, \ldots, p\} \) and \( \omega > 0 \). Assumption 6 is said to hold if there exists a nonnegative constant \( \kappa(k, \omega) \) such that:

\[
0 < \kappa(k, \omega) \leq \inf_{z \in \Gamma(k, \omega)} \frac{\|z^T \nabla^2 \mathcal{L}(\beta^*) z\|_2}{\|z\|_2}.
\]

Similarly, we define a new growth condition, as an adaptation of Assumption 6 where we replace \( \kappa(k) \) by \( \kappa(k, \omega) \) (above). We call it Assumption 8 \( (k, \omega, \delta) \).

The following result holds for Slope. The proof is presented in Appendix I.

Corollary 2 Assume Assumptions 7, Assumptions 6 \( (k^*, \omega^*) \) and 8 \( (k^*, \omega^*, \delta) \) hold for a small enough \( \delta \), where \( \omega^* \) is defined in Theorem 6. In addition assume that \( \mu(k^*)^2 \log(Re) \leq \alpha^2 M^2 \).

Then the bounds presented in Theorem 7 and Corollary 7 are achieved by a Slope estimator, defined as a solution of Problem (18) for the coefficients \( \lambda_j = \sqrt{\log(2pe/j)} \), \( \forall j \) and the regularization parameter \( \eta = 12\alpha L M n^{-1} \sqrt{\log(4/\delta)} \) where \( \alpha \geq 2 \).

This Slope estimator adapts to unknown sparsity while achieving the same bound than the L1-regularized estimator studied in Theorem 1 and Corollary 1.

4.6 Simulations

We finally compute a family of Slope estimators and demonstrate its empirical performance – for L2 coefficients estimations and misclassification accuracy – we compare to L1 and L2-regularized-estimators.

Data Generation: We consider \( n \) independent realizations of a \( p \) dimensional multivariate normal centered distribution, with only \( k^* \) dimensions being relevant for classification. Half of the samples are from the +1 class and have mean \( \mu_+ = (1_{k^*}, 0)_{p-k^*} \). The other half are from the −1 class and have mean \( \mu_- = -\mu_+ \).

We consider a covariance matrix \( \Sigma_{ij} = p \) if \( i \neq j \) and 1 otherwise. The data of both \( \pm 1 \) classes respectively have the distribution: \( \forall i, x_i^+ \sim \mathcal{N}(\mu_+, \Sigma) \).

Competitors: Table 1 compares the performance of 3 approaches – each associated to a different regularization – for both the SVM and the Logistic Regression problems. Method (a) computes a family of L1-regularized estimators for a decreasing geometric sequence of regularization parameters \( \eta_0 > \ldots > \eta_M \). We start from a high enough \( \eta_0 \) so that the solution of Problem (2) is the 0 estimator and we fix \( \eta_M < 10^{-4} \eta_0 \).
For the hinge loss, we solve the Linear Programming L1-SVM problem with the commercial LP solver Gurobi version 6.5 with Python interface. The L1-regularized Logistic Regression is solved with scikit-learn Python package. In addition, method (b) returns a family of L2-regularized estimators with scikit-learn package: we start from $\eta_0 = \max_i \{\|x_i\|_2\}$ as suggested by Chu et al. [2015]. Finally, method (c) computes a family of Slope-regularized estimators, using the first order algorithm presented in Section 4.4 for $\tau = 0.2$. The Slope coefficients $\{\lambda_j\}$ are the ones proposed in Theorem 6; the set of parameters $\{\eta_i\}$ is identical to method (a).

**Metrics:** Following our theoretical results, we want to find the estimator which minimizes the L2 estimation error defined as:

$$\frac{\|\hat{\beta} - \beta^*\|_2}{\|\beta^*\|_2},$$

where $\beta^*$ is the theoretical minimizer. $\beta^*$ is computed on a large test set with 10,000 samples: we solve the SVM or Logistic Regression problem with a very small regularization coefficient on the $k^*$ columns relevant for classification. We also study the misclassification performances on this same test set. For each family returned by the methods (a), (b) and (c), we only keep the estimator with lowest misclassification error on an independent validation set of size 10,000.

Table 1 compares the L2 estimation error (L2-E), and the test misclassification error (Misc) – of these 3 estimators selected on the validation set. The results are averaged over 10 simulations.

| n = 100, p = 1k | n = 100, p = 10k | n = 1k, p = 1k | n = 1k, p = 10k |
|-----------------|-----------------|----------------|-----------------|
| L2-E Misc(%)    | L2-E Misc(%)    | L2-E Misc(%)   | L2-E Misc(%)    |
| L1 SVM          | 0.57 1.67       | 0.52 1.54      | 1.12 1.17       |
| L2 SVM          | 0.54 1.73       | 0.52 1.54      | 1.11 0.18       |
| Slope SVM       | 0.34 1.24       | 0.37 1.15      | 0.94 0.13       |
| L1 LR           | 0.48 1.40       | 0.46 1.37      | 1.04 0.18       |
| L2 LR           | 0.92 3.2        | 1.25 0.18      | 0.82 0.12       |
| Slope LR        | 0.22 1.14       | 0.18 1.12      | 0.81 0.12       |

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Appendices

A Usefull properties of sub-Gaussian random variables

This section presents useful preliminary results satisfied by sub-Gaussian random variables. As a consequence, we derive the proofs of Lemmas 1 and 4. This last result provides a probabilistic upper-bound on the maximum of independent sub-Gaussian variables.

A.1 Preliminary results

Under Assumption \( \mathcal{A} \) the random variables \( \sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle, y_i) x_{ij}, \forall j \) are sub-Gaussian. Each of them consequently satisfy the next Lemma 3:

**Lemma 3** Let \( Z \sim \text{subG}(\sigma^2) \) for a fixed \( \sigma > 0 \). Then for any \( t > 0 \) it holds

\[
\mathbb{E} (\exp(tZ)) \leq e^{4\sigma^2 t^2}.
\]

In addition, for any positive integer \( l \geq 1 \) we have:

\[
\mathbb{E} \left( |Z|^l \right) \leq (2\sigma^2)^{l/2} l \Gamma(l/2)
\]

where \( \Gamma \) is the Gamma function defined as \( \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \forall t > 0 \).

Finally, let \( Y = Z^2 - \mathbb{E}(Z^2) \) then we have

\[
\mathbb{E} \left( \exp \left( \frac{1}{16\sigma^2} Y \right) \right) \leq \frac{3}{2},
\]

(19)

and as a consequence \( \mathbb{E} \left( \exp \left( \frac{1}{16\sigma^2} Z^2 \right) \right) \leq 2 \).

**Proof:** The two first results correspond to Lemmas 1.4 and 1.5 \cite{Rigollet2015}.

In particular \( \mathbb{E} (|Z|^2) \leq 4\sigma^2 \).

In addition, using the proof of Lemma 1.12 we have:

\[
\mathbb{E} (\exp(tY)) \leq 1 + 128t^2 \sigma^4, \forall |t| \leq \frac{1}{16\sigma^2}.
\]

Equation (19) holds by selecting \( t = 1/16\sigma^2 \). We finally combine our precedent results with the fact that \( \frac{3}{2} e^{1/4} \leq 2 \) for the last part of the lemma. \( \square \)

A.2 Proof of Lemma 1

As a first consequence of Lemma 3 we easily derive the proof of Lemma 1 – stated in Section 2.3.

**Proof:** We note \( S_i = \partial f (\langle x_i, \beta^* \rangle, y_i), \forall i \).

Since \( \beta^* \) minimizes the theoretical loss, we have \( \mathbb{E}(S_i x_{ij}) = 0, \forall i, j \).

We fix \( M > 0 \) such that: \( \forall t > 0, \)

\[
\mathbb{P} (|S_i x_{ij}| > t) \leq 2 \exp \left( -\frac{t^2}{2L^2M^2} \right), \forall i, j.
\]
Then from Lemma 3 it holds:

$$\mathbb{E}(\exp(tS_{ij})) \leq e^{4L^2M^2t^2}, \ \forall t > 0, \forall i, j.$$ 

As a consequence, using Lemma 3 for the independent random variables $S_{1x_{1,j}}, \ldots, S_{nx_{n,j}}$, it holds $\forall t > 0,$

$$\mathbb{E} \left( \exp \left( \frac{t}{\sqrt{n}} \sum_{i=1}^{n} S_{i}x_{i,j} \right) \right) = \prod_{i=1}^{n} \mathbb{E} \left( \exp \left( \frac{t}{\sqrt{n}} S_{i}x_{i,j} \right) \right) \leq \prod_{i=1}^{n} e^{4L^2M^2t^2/n} = e^{4L^2M^2t^2}.$$

Let $M_1 = 2\sqrt{2}M$, then with a Chernoff bound:

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} S_{i}x_{i,j} > t \right) \leq \min_{s>0} \exp \left( \frac{M_1^2 L^2 s^2}{2} - st \right) = \exp \left( - \frac{t^2}{2L^2M_1^2} \right), \ \forall t > 0,$$

which concludes the proof. □

### A.3 A bound for the maximum of independent sub-Gaussian variables

The next two technical lemmas lead to an upper-bound for the maximum of sub-Gaussian random variables. Lemma 4 is an extension for sub-Gaussian random variables of Proposition E.1 [Bellec et al., 2016].

**Lemma 4** Let $g_1, \ldots, g_p$ be independent sub-Gaussian random variables with variance $\sigma^2$. Denote by $(g^{(1)}, \ldots, g^{(p)})$ a non-increasing rearrangement of $(|g_1|, \ldots, |g_p|)$. Then $\forall t > 0$ and $\forall j \in \{1, \ldots, p\}$:

$$\mathbb{P} \left( \frac{1}{j\sigma^2} \sum_{k=1}^{j} g_{(k)}^2 > t \log \left( \frac{2}{j}p \right) \right) \leq \left( \frac{2}{j}p \right)^{1 - \frac{1}{2\pi}}.$$

**Proof:** We first apply a Chernoff bound:

$$\mathbb{P} \left( \frac{1}{j\sigma^2} \sum_{k=1}^{j} g_{(k)}^2 > t \log \left( \frac{2}{j}p \right) \right) \leq \mathbb{E} \left( \exp \left( \frac{1}{16j\sigma^2} \sum_{k=1}^{j} g_{(k)}^2 \right) \right) \left( \frac{2}{j}p \right)^{1 - \frac{1}{2\pi}}.$$

Then we use Jensen inequality to obtain

$$\mathbb{E} \left( \exp \left( \frac{1}{16j\sigma^2} \sum_{k=1}^{j} g_{(k)}^2 \right) \right) \leq \frac{1}{j} \sum_{k=1}^{j} \mathbb{E} \left( \exp \left( \frac{1}{16\sigma^2} g_{(k)}^2 \right) \right) \leq \frac{1}{j} \sum_{k=1}^{p} \mathbb{E} \left( \exp \left( \frac{1}{16\sigma^2} g_{k}^2 \right) \right) \leq \frac{2}{j}p$$

with Lemma 3. □

From Lemma 4, we prove the following bound holding with high probability:

**Lemma 5** We consider the assumptions and notations of Lemma 4. In addition, we define the coefficients $\lambda_j = \sqrt{\log(2p/j)}$, $j = 1, \ldots, p$. Then for $\delta \in \left(0, \frac{1}{2}\right)$, it holds with probability at least $1 - \delta$:

$$\sup_{j=1,\ldots,p} \left\{ \frac{g_{(j)}}{\sigma \lambda_j} \right\} \leq 12\sqrt{\log(1/\delta)}.$$
Proof: We fix $\delta \in (0, \frac{1}{2})$ and $j \in \{1, \ldots, p\}$. We upper-bound $g^2_{(j)}$ by the average of all larger variables:

$$g^2_{(j)} \leq \frac{1}{j} \sum_{k=1}^{j} g^2_{(k)}.$$ 

Applying Lemma 4 gives, for $t > 0$:

$$\mathbb{P}\left( \frac{g^2_{(j)}}{\sigma^2 \lambda^2_j} > t \right) \leq \mathbb{P}\left( \frac{1}{j} \sum_{k=1}^{j} g^2_{(k)} > t \lambda^2_j \right) \leq \left( \frac{j}{2p} \right)^{9 \log(1/\delta) - 1}.$$ 

We fix $t = 144 \log(1/\delta)$ and use an union bound to get:

$$\mathbb{P}\left( \sup_{j=1,\ldots,p} \frac{g_{(j)}}{\sigma \lambda_j} > 12 \sqrt{\log(1/\delta)} \right) \leq \left( \frac{1}{2p} \right)^{9 \log(1/\delta)} - 1 \sum_{j=1}^{p} j^{9 \log(1/\delta) - 1}.$$ 

Since $\delta < \frac{1}{2}$ it holds that $9 \log(1/\delta) - 1 \geq 9 \log(2) - 1 > 0$, then the map $t > 0 \mapsto t^{9 \log(1/\delta) - 1}$ is increasing. An integral comparison gives:

$$\sum_{j=1}^{p} j^{9 \log(1/\delta) - 1} \leq \frac{1}{2} (p + 1)^{9 \log(1/\delta)} = \frac{1}{2} \delta^{-9 \log(p+1)}.$$ 

In addition $9 \log(1/\delta) - 1 \geq 7 \log(1/\delta)$ and

$$\left( \frac{1}{2p} \right)^{9 \log(1/\delta) - 1} \leq \left( \frac{1}{2p} \right)^{-7 \log(\delta)} = \delta^{7 \log(2p)}.$$ 

Finally, by assuming $p \geq 2$, then we have $7 \log(2p) - 9 \log(p + 1) > 1$ and we conclude:

$$\mathbb{P}\left( \sup_{j=1,\ldots,p} \frac{g_{(j)}}{\sigma \lambda_j} > 12 \sqrt{\log(1/\delta)} \right) \leq \delta,$$

which concludes the proof. \(\square\)

B Proof of Theorem 2

We use the minimality of $\hat{\beta}$ and Lemma 4 to derive the cone condition.

Proof: We assume without loss of generality that $|h_1| \geq \ldots \geq |h_p|$. We define $S_0 = \{1, \ldots, k^*\}$ as the set of the $k^*$ highest coefficients of $h = \hat{\beta} - \beta^*$.

$\hat{\beta}$ is the solution of Problem (2) hence:

$$\frac{1}{n} \sum_{i=1}^{n} f \left( \langle x_i, \hat{\beta} \rangle; y_i \right) + \lambda \| \hat{\beta} \|_1 \leq \frac{1}{n} \sum_{i=1}^{n} f \left( \langle x_i, \beta^* \rangle; y_i \right) + \lambda \| \beta^* \|_1.$$ 

(20)

Using the definition of $\Delta (\beta^*, h)$ as introduced in Theorem 3 Equation (20) can be written in a more compact form as:

$$\Delta (\beta^*, h) \leq \lambda \| \beta^* \|_1 - \lambda \| \hat{\beta} \|_1.$$ 

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Introducing the support $S^*$ of $\beta^*$ we have
\[
\Delta (\beta^*, h) \leq \lambda \|\beta_{S^*}\|_1 - \lambda \|\beta_{S^*}\|_1 - \lambda \|\beta_{(S^*)^c}\|_1 \\
\leq \lambda \|h_{S^*}\|_1 - \lambda \|h_{(S^*)^c}\|_1 \\
\leq \lambda \|h_{S_0}\|_1 - \lambda \|h_{(S_0)^c}\|_1,
\]
where this last relation holds by definition of $S_0$. We now want to lower bound $\Delta (\beta^*, h)$. Exploiting the existence of a bounded sub-Gradient $\partial f$ we obtain
\[
\Delta (\beta^*, h) \geq S (\beta^*, h) := \frac{1}{n} \sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle; y_i) \langle x_i, h \rangle.
\]
Since $f$ is $L$-Lipschitz, it holds $\|\partial f (., y)\|_\infty \leq L$, $\forall y$ and we have
\[
|S (\beta^*, h) | = \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{p} \partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij} h_j \right| \\
\leq \frac{1}{\sqrt{n}} \sum_{j=1}^{p} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij} \right| \right) |h_j|.
\]
Let us define the independent random variables $g_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \partial f (\langle x_i, \beta^* \rangle; y_i) x_{ij}$, $j = 1, \ldots, p$.

Assumption\textsuperscript{3} guarantees that $g_1, \ldots, g_p$ are sub-Gaussian with variance $L^2 M^2$. A first upper-bound of the quantity $|S (h)|$ could be obtained by considering the maximum of the sequence $\{g_j\}$. However Lemma\textsuperscript{5} gives us a stronger result.

Indeed, since $\delta \leq 1$ we introduce a non-increasing rearrangement $(g_{(1)}, \ldots, g_{(p)})$ of $(|g_1|, \ldots, |g_p|)$. We recall that $S_0 = \{1, \ldots, k^*\}$ denotes the subset of indexes of the $k^*$ highest elements of $h$ and we use Lemma\textsuperscript{5} to get, with probability at least $1 - \frac{2}{n}$:
\[
|S (\beta^*, h) | \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{p} g_j |h_j| = \frac{1}{\sqrt{n}} \sum_{j=1}^{p} g_{(j)} |h_{(j)}| = \frac{1}{\sqrt{n}} \sum_{j=1}^{p} g_{(j)} \frac{L M \lambda_j}{L M \lambda_j} |h_{(j)}| \\
\leq \frac{1}{\sqrt{n}} \sup_{j=1, \ldots, p} \left\{ \frac{g_{(j)}}{L M \lambda_j} \right\} \sum_{j=1}^{p} L M \lambda_j |h_{(j)}| \\
\leq 12 L M \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^{p} \lambda_j |h_{(j)}| \text{ with Lemma\textsuperscript{5}}
\]
\[
\leq 12 L M \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^{p} \lambda_j |h_j| \text{ since } \lambda_1 \geq \ldots \geq \lambda_p \text{ and } |h_1| \geq \ldots \geq |h_p| \\
\leq 12 L M \sqrt{\frac{\log(2/\delta)}{n}} \left( \sum_{j=1}^{k^*} \lambda_j |h_j| + \lambda_{k^*} \sum_{j=k^*}^{p} |h_j| \right) \\
= 12 L M \sqrt{\frac{\log(2/\delta)}{n}} \left( \sum_{j=1}^{k^*} \lambda_j |h_j| + \lambda_{k^*} \|h_{(S_0)^c}\|_1 \right).
\]

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Cauchy-Schwartz inequality leads to:

\[ k^* \sum_{j=1}^{k^*} \lambda_j |h_j| \leq \sqrt{\sum_{j=1}^{k^*} \lambda_j^2 \|h_{S_0}\|_2} \leq \sqrt{k^* \log(2pe/k^*)} \|h_{S_0}\|_2, \]

where we have used the Stirling formula to obtain

\[ \sum_{j=1}^{k^*} \lambda_j^2 = \sum_{j=1}^{k^*} \log(2p/j) = k^* \log(2p) - \log(k^*) \]

\[ \leq k^* \log(2p) - k^* \log(k^*/e) = k^* \log(2pe/k^*). \]

In the statement of Theorem 2, we have defined \( \lambda = 12\alpha LM \sqrt{n^{-1} \log(2pe/k^*) \log(2/\delta)}. \)

Because \( \lambda_k \leq \sqrt{\log(2pe/k^*)} \), Equation (22) leads to:

\[ |S(\beta^*, h)| \leq \frac{1}{\alpha} \lambda \left( \sqrt{k^* \|h_{S_0}\|_2} + \|h_{(S_0)^c}\|_1 \right) \]

Combined with Equation (21), it holds with probability at least \( 1 - \frac{\delta}{2} : \)

\[ -\frac{\lambda}{\alpha} \left( \sqrt{k^* \|h_{S_0}\|_2} + \|h_{(S_0)^c}\|_1 \right) \leq \lambda \|h_{S_0}\|_1 - \lambda \|h_{(S_0)^c}\|_1, \]

which immediately leads to:

\[ \|h_{(S_0)^c}\|_1 \leq \frac{\alpha}{\alpha - 1} \|h_{S_0}\|_1 + \frac{\sqrt{k^*}}{\alpha - 1} \|h_{S_0}\|_2. \]

We conclude that \( h \in \Lambda \left( S_0, \frac{\alpha}{\alpha - 1}, \sqrt{k^*/\alpha - 1} \right) \) with probability at least \( 1 - \frac{\delta}{2}. \)

C Proof of Theorem 3:

**Proof:** Let \( k \in \{1, \ldots, p\} \) and \( S_1, \ldots, S_q \) be a partition of \( \{1, \ldots, p\} \) such that \( q = \lceil p/k \rceil \) and \( |S_j| \leq k, \forall j. \)

We divide the proof of the theorem in 3 steps. We first upper-bound the inner supremum for any sequence of \( k \) sparse vectors \( z_{S_1}, \ldots, z_{S_q}. \) We then extend this bound for the supremum over a compact set of sequences through an \( \epsilon \)-net argument.

**Step 1:** Let us fix a sequence \( z_{S_1}, \ldots, z_{S_q} \in \mathbb{R}^p : \) \( \text{Supp}(z_{S_j}) \subset S_j \), \( \forall j \) and \( \|z_{S_j}\|_1 \leq 3R, \forall j. \)

In particular, \( \|z_{S_j}\|_0 \leq k, \forall j. \) In the rest of the proof, we define \( z_{S_0} = 0 \) and:

\[ w_\ell = \beta^* + \sum_{j=1}^\ell z_{S_j}, \forall \ell, \quad (23) \]

In addition, we introduce \( Z_{i\ell}, \forall i, \ell \) as follows

\[ Z_{i\ell} = f \left( \langle x_i, w_\ell \rangle; y_i \right) - f \left( \langle x_i, w_{\ell-1} \rangle; y_i \right) = f \left( \langle x_i, w_{\ell-1} + z_{S_{\ell}} \rangle; y_i \right) - f \left( \langle x_i, w_{\ell-1} \rangle; y_i \right). \]
In particular, let us note that:

\[
\Delta (w_{\ell-1}, z_{S_\ell}) = \frac{1}{n} \sum_{i=1}^{n} f \left( (x_i, w_{\ell-1} + z_{S_\ell}); y_i \right) - \frac{1}{n} \sum_{i=1}^{n} f \left( (x_i, w_{\ell-1}); y_i \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \{ f \left( (x_i, w_{\ell-1} + z_{S_\ell}); y_i \right) - f \left( (x_i, w_{\ell-1}); y_i \right) \}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} Z_{i\ell}.
\]

Assumption\[1\]guarantees that \( f(., y) \) is \( L \)-Lipschitz \( \forall y \) then:

\[
|Z_{ij}| \leq L |(x_i, z_{S_\ell})|.
\]

Then using Assumption\[4\] \( 1(k) \) on the \( k \) sparse vector \( z_{S_\ell} \) it holds:

\[
|\Delta (w_{\ell-1}, z_{S_\ell})| \leq \frac{1}{n} \sum_{i=1}^{n} |Z_{i\ell}| \leq \frac{1}{n} \sum_{i=1}^{n} L |(x_i, z_{S_\ell})| = \frac{L}{n} \|Xz_{S_\ell}\|_1 \leq \frac{L \mu(k)}{\sqrt{n} k} \|z_{S_\ell}\|_1.
\]

Consequently, using Hoeffding’s lemma, the centered bounded random variable \( \Delta (w_{\ell-1}, z_{S_\ell}) - \mathbb{E} (\Delta (w_{\ell-1}, z_{S_\ell})) \) is sub-Gaussian with variance \( \frac{L^2 \mu(k)^2}{4nk} \|z_{S_\ell}\|_1^2 \). It thus hold \( \forall t > 0 \),

\[
\mathbb{P} (|\Delta (w_{\ell-1}, z_{S_\ell}) - \mathbb{E} (\Delta (w_{\ell-1}, z_{S_\ell}))| \geq t \|z_{S_\ell}\|_1) \leq 2 \exp \left( -\frac{2knt^2}{L^2 \mu(k)^2} \right).
\]

Equation\[25\] holds for all values of \( \ell \). Thus, an union bound immediately gives:

\[
\mathbb{P} \left( \sup_{t=1, \ldots, q} \{|\Delta (w_{\ell-1}, z_{S_\ell}) - \mathbb{E} (\Delta (w_{\ell-1}, z_{S_\ell}))| - t \|z_{S_\ell}\|_1 \geq 0\} \right) \leq 2 \left[ \frac{p}{k} \right] \exp \left( -\frac{2knt^2}{L^2 \mu(k)^2} \right).
\]

\[
(26)
\]

\textbf{Step 2: } We extend the result to any sequence of vectors \( z_{S_1}, \ldots, z_{S_q} \in \mathbb{R}^p : \text{Supp}(z_{S_j}) \subset S_j, \forall j \) and \( \|z_{S_j}\|_1 \leq 3R, \forall j \) through an \( \epsilon \)-net argument.

We recall that an \( \epsilon \)-net of a set \( I \) is a subset \( N \) of \( I \) such that each element of \( I \) is at a distance at most \( \epsilon \) of \( N \). We know from Lemma 1.18 \[\text{Rigollet, 2015}\], that for any \( \epsilon \in (0,1) \), the ball \( \{ z \in \mathbb{R}^d : \|z\|_1 \leq R \} \) has an \( \epsilon \)-net of cardinality \( |N| \leq \left( \frac{2R+1}{\epsilon} \right)^d \) – the \( \epsilon \)-net is defined in term of \( L^1 \) norm. In addition, by following the proof of the lemma, we can create this set such that it contains \( 0 \).

Consequently, we use Equation\[26\] for a union of \( \epsilon \)-nets \( N_{k,R} = \bigcup_{\ell=1}^{q} N_{\ell,k,R} \). Each \( N_{\ell,k,R} \) is an \( \epsilon \)-net of the bounded sets of \( k \) sparse vectors \( I_{\ell,k,R} = \{ z_{S_j} \in \mathbb{R}^p : \text{Supp}(z_{S_j}) \subset S_j; \|z_{S_j}\|_1 \leq 3R \} \) (and containing \( 0_{S_\ell} \)). We note \( I_{k,R} = \bigcup_{\ell=1}^{q} I_{\ell,k,R} \). It then holds:

\[
\mathbb{P} \left( \sup_{z_{S_1}, \ldots, z_{S_q} \in N_{k,R}} \sup_{t=1, \ldots, q} \{|\Delta (w_{\ell-1}, z_{S_\ell}) - \mathbb{E} (\Delta (w_{\ell-1}, z_{S_\ell}))| - t \|z_{S_\ell}\|_1 \geq 0\} \right) \leq 2 \left[ \frac{p}{k} \right] \left( \frac{6R+1}{\epsilon} \right)^k \frac{p}{k} \exp \left( -\frac{2knt^2}{L^2 \mu(k)^2} \right) \leq 2 \left( \frac{2p}{k} \right)^2 \left( \frac{6R+1}{\epsilon} \right)^k \exp \left( -\frac{2knt^2}{L^2 \mu(k)^2} \right).
\]

(27)
Step 3: We now extend Equation (27) to control any vector in $I_{k,R}$. For $z_{S_1}, \ldots, z_{S_q} \in I_{k,R}$, there exists $\tilde{z}_{S_1}, \ldots, \tilde{z}_{S_q} \in N_{k,R}$ such that $\|z_{S_\ell} - \tilde{z}_{S_\ell}\|_1 \leq \epsilon_L, \forall \ell$. Similarly to Equation (23), we define:

$$\tilde{w}_\ell = \beta^* + \sum_{j=1}^q \tilde{z}_{S_j}, \forall \ell.$$ 

For a given $t$, let us define

$$f_t (w_{\ell-1}, z_{S_\ell}) = |\Delta (w_{\ell-1}, z_{S_\ell}) - \mathbb{E} (w_{\ell-1}, z_{S_\ell})| - t \|z_{S_\ell}\|_1, \forall \ell.$$ 

We fix $\ell_0(t)$ such that $\ell_0 \in \arg\max_{\ell=1,\ldots,q} \{f_{7t} (w_{\ell-1}, z_{S_\ell})\}$. The choice of $7t$ will be justified later. We fix $t$ and will just note $\ell_0 = \ell_0(t)$ when no confusion can be made.

With Assumption 1, we obtain:

$$\left| \Delta (w_{\ell_0-1}, z_{S_{\ell_0}}) - \Delta (\tilde{w}_{\ell_0-1}, \tilde{z}_{S_{\ell_0}}) \right|$$

$$= \frac{1}{n} \sum_{i=1}^n f (\langle x_i, w_{\ell_0-1} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f (\langle x_i, \tilde{w}_{\ell_0-1} \rangle; y_i) + \frac{1}{n} \sum_{i=1}^n f (\langle x_i, z_{S_{\ell_0}} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f (\langle x_i, \tilde{z}_{S_{\ell_0}} \rangle; y_i)$$

$$\leq \frac{1}{n} \sum_{i=1}^n L |\langle x_i, w_{\ell_0-1} - \tilde{w}_{\ell_0-1} \rangle| + \frac{1}{n} \sum_{i=1}^n L |\langle x_i, z_{S_{\ell_0}} - \tilde{z}_{S_{\ell_0}} \rangle|$$

$$= \frac{1}{n} \sum_{i=1}^n \sum_{\ell=1}^{q} L |\langle x_i, z_{S_{\ell}} - \tilde{z}_{S_{\ell}} \rangle|$$

$$\leq \frac{1}{n} \sum_{i=1}^n \sum_{\ell=1}^{q} L \|X (z_{S_{\ell}} - \tilde{z}_{S_{\ell}})\|_1$$

$$\leq \frac{1}{\sqrt{n}} \sum_{\ell=1}^{q} \frac{L}{\sqrt{k}} \mu(k) \|z_{S_{\ell}} - \tilde{z}_{S_{\ell}}\|_1$$

$$\leq \frac{1}{\sqrt{n}} \sum_{\ell=1}^{q} \frac{L}{\sqrt{k}} \mu(k) \epsilon \leq \eta L \mu(k) \epsilon \leq \eta \epsilon.$$ 

(28)

where $\eta = 2L \mu(k) \sqrt{\frac{\log(p/k)}{n}}$ and we have used Assumption 1. It then holds:

$$f_t (\tilde{w}_{\ell_0-1}, \tilde{z}_{S_{\ell_0}}) \geq f_t (w_{\ell_0-1}, z_{S_{\ell_0}}) - |\Delta (w_{\ell_0-1}, z_{S_{\ell_0}}) - \Delta (\tilde{w}_{\ell_0-1}, \tilde{z}_{S_{\ell_0}})|$$

$$- |\mathbb{E} (\Delta (w_{\ell_0-1}, z_{S_{\ell_0}}) - \Delta (\tilde{w}_{\ell_0-1}, \tilde{z}_{S_{\ell_0}}))| - t \|z_{S_{\ell_0}} - \tilde{z}_{S_{\ell_0}}\|_1$$

$$\geq f_t (w_{\ell_0-1}, z_{S_{\ell_0}}) - 2t \epsilon - t \epsilon.$$ 

**Case 1:** Let us assume that $\|z_{S_{\ell_0}}\|_1 \geq \epsilon/2$ and that $t \geq \eta$, then we have:

$$f_t (\tilde{w}_{\ell_0-1}, \tilde{z}_{S_{\ell_0}}) \geq f_t (w_{\ell_0-1}, z_{S_{\ell_0}}) - 2(2\eta + t) \|z_{S_{\ell_0}}\|_1 \geq f_{7t} (w_{\ell_0-1}, z_{S_{\ell_0}}).$$ 

(29)
Case 2: We now assume $\|z_{S_{i_0}}\|_1 \leq \epsilon/2$. Since $0_{S_{i_0}} \in \mathcal{N}_{k,R}$ we derive similarly to Equation (28):

$$\left| \Delta \left( w_{\ell_0-1}, z_{S_{i_0}} \right) - \Delta \left( w_{\ell_0-1}, 0_{S_{i_0}} \right) \right| \leq \frac{L\mu(k)}{\sqrt{nk}} \left\| z_{S_{i_0}} \right\|_1,$$

which then implies that:

$$f_{t_0} \left( w_{\ell_0-1}, z_{S_{i_0}} \right) \leq f_{t_0} \left( w_{\ell_0-1}, 0_{S_{i_0}} \right) + \frac{2L\mu(k)}{\sqrt{nk}} \left\| z_{S_{i_0}} \right\|_1 - 7t \left\| z_{S_{i_0}} \right\|_1,$$

and this quantity is smaller than $f_t \left( w_{\ell_0-1}, 0_{S_{i_0}} \right)$ as long as $7t \geq \frac{2L\mu(k)}{\sqrt{nk}}$. The latter condition is satisfied if $t \geq \eta$.

In this case, we can define a new $\tilde{\ell}_0$ for the sequence $z_{S_1}, \ldots, z_{S_{t_0}}, 0_{S_{i_0}}, z_{S_{i_0}+1}, \ldots, z_{S_q}$. After a finite number of iteration, by using the result in Equation (29) and the definition of $\ell_0$, we finally get that $f_{t_0} \left( w_{\ell_0-1}, z_{S_{i_0}} \right) \leq f_t \left( \tilde{w}_{\ell_0-1}, \tilde{z}_{S_q} \right)$ for some $\tilde{z}_{S_1}, \ldots, \tilde{z}_{S_q} \in \mathcal{N}_{k,R}$.

As a consequence of Cases 1 and 2, we obtain: $\forall t \geq \eta, \forall z_{S_1}, \ldots, z_{S_q} \in \mathcal{I}_{k,R}, \exists z_{S_1}, \ldots, z_{S_q} :$

$$\sup_{\ell=1,\ldots,q} f_{t_0} \left( w_{\ell-1}, z_{S_{\ell}} \right) = f_{t_0} \left( w_{\ell_0-1}, z_{S_{i_0}} \right) \leq f_t \left( \tilde{w}_{\ell_0-1}, \tilde{z}_{S_q} \right) \leq \sup_{\ell=1,\ldots,q} f_t \left( \tilde{w}_{\ell-1}, \tilde{z}_{S_{\ell}} \right).$$

This last relation is equivalent to saying that $\forall t \geq 7\eta$:

$$\sup_{z_{S_1}, \ldots, z_{S_q} \in \mathcal{I}_{k,R}} \left\{ \sup_{\ell=1,\ldots,q} f_t \left( w_{\ell-1}, z_{S_{\ell}} \right) \right\} \leq \sup_{z_{S_1}, \ldots, z_{S_q} \in \mathcal{N}_{k,R}} \left\{ \sup_{\ell=1,\ldots,q} f_{t/7} \left( \tilde{w}_{\ell-1}, \tilde{z}_{S_{\ell}} \right) \right\}. \quad (30)$$

As a consequence, we have $\forall t \geq 7\eta$:

$$\mathbb{P} \left( \sup_{z_{S_1}, \ldots, z_{S_q} \in \mathcal{I}_{k,R}} \left\{ \sup_{\ell=1,\ldots,q} \left\{ \left| \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) - \mathbb{E} \left( \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) \right) \right| - t \left\| z_{S_{\ell}} \right\|_1 \right\} \geq 0 \right\} \right)$$

$$\leq \mathbb{P} \left( \sup_{z_{S_1}, \ldots, z_{S_q} \in \mathcal{N}_{k,R}} \left\{ \sup_{\ell=1,\ldots,q} \left\{ \left| \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) - \mathbb{E} \left( \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) \right) \right| - \frac{t}{7} \left\| z_{S_{\ell}} \right\|_1 \right\} \geq 0 \right\} \right)$$

$$\leq \left( \frac{14R}{ek} \right)^k \exp \left( - \frac{2knt^2}{49L^2\mu(k)^2} \right) \text{ if } k \geq 2 \text{ and since } R \geq 1$$

$$\leq \left( \frac{R}{ek} \right)^k \exp \left( - \frac{2knt^2}{49L^2\mu(k)^2} \right) \text{ by fixing } \epsilon = k/(14p).$$

Thus we select $t$ such that $t \geq 7\eta$ and that the condition $t^2 \geq \frac{49L^2\mu(k)^2}{2kn} \left[ 2k \log \left( \frac{R}{ek} \right) + \log \left( \frac{2}{\delta} \right) \right]$ holds. To this end, we define:

$$\tau = 14L\mu(k) \sqrt{\frac{\log \left( \frac{R}{ek} \right)}{n} + \frac{\log \left( 2/\delta \right)}{nk}} \geq 7\eta.$$

We conclude that with probability at least $1 - \frac{\delta}{2}$:

$$\sup_{z_{S_1}, \ldots, z_{S_q} \in \mathcal{I}_{k,R}} \left\{ \sup_{\ell=1,\ldots,q} \left\{ \left| \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) - \mathbb{E} \left( \Delta \left( w_{\ell-1}, z_{S_{\ell}} \right) \right) \right| - \tau \left( \left\| z_{S_{\ell}} \right\|_1 + \eta \right) \right\} \right\} \leq 0.$$
Proof of Theorem 4:

Proof: The proof is divided in two steps. First, we lower-bound the quantity $\Delta (\beta^*, h)$ by using a decomposition of $\{1, \ldots, p\}$ and applying Theorem 4. Second, we consider the cone condition Theorem 2 and use the restricted eigenvalue condition presented in Assumption 4.2.

Step 1: Let us fix the partition $S_1 = \{1, \ldots, k^\ast\}, S_2 = \{k^\ast + 1, \ldots, 2k^\ast\}, \ldots, S_q$ of $\{1, \ldots, p\}$ with $q = \lceil p/k^\ast \rceil$. Thus it holds $|S_j| \leq k^\ast$, $\forall j$ and we can use Theorem 3. We define the corresponding sequence $h_{S_1}, \ldots, h_{S_q}$ of $k^\ast$ sparse vectors corresponding to the decomposition of $h = \beta - \beta^*$ on the partition. Thus, it holds with probability at least $1 - \frac{\delta}{2}$:

$$
\Delta (\beta^*, h) = \frac{1}{n} \sum_{i=1}^{n} f ((\mathbf{x}_i, \beta^* + h); y_i) - \frac{1}{n} \sum_{i=1}^{n} f ((\mathbf{x}_i, \beta^*); y_i)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} f \left( \sum_{j=1}^{q} h_{S_j} \right) - \frac{1}{n} \sum_{i=1}^{n} f \left( \sum_{j=1}^{q} h_{S_j} \right)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=1}^{q} h_{S_j} \right] - \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=1}^{q} h_{S_j} \right]
$$

$$
= \sum_{\ell=1}^{q} \Delta (w_{\ell-1}, h_{S_{\ell}}).
$$

where we have defined $w_{\ell} = \beta^* + \sum_{j=1}^{\ell} z_{S_j}, \forall \ell$ and $z_{S_0} = 0$ as in the proof of Theorem 3. Consequently, since $\|h_{S_{\ell}}\|_0 \leq k^\ast$ and $\|h_{S_{\ell}}\|_1 \geq R, \forall \ell$, it holds with probability at least $1 - \frac{\delta}{2}$:

$$
|\Delta (w_{\ell-1}, h_{S_{\ell}}) - E (w_{\ell-1}, h_{S_{\ell}})| \geq \tau \|h_{S_{\ell}}\|_1, \forall \ell,
$$

where $\tau = 14L\mu(k^\ast)\sqrt{\frac{\log(R p/k^\ast)}{n}} + \frac{\log(2/\delta)}{nk^\ast}$ is fixed in the rest of the proof.

As a result, following Equation (32), we have:

$$
\Delta (\beta^*, h) \geq \sum_{\ell=1}^{q} \left\{ E (w_{\ell-1}, h_{S_{\ell}}) - \tau \|h_{S_{\ell}}\|_1 \right\}
$$

$$
= E \left( \sum_{\ell=1}^{q} \Delta (w_{\ell-1}, h_{S_{\ell}}) \right) - \sum_{\ell=1}^{q} \tau \|h_{S_{\ell}}\|_1
$$

$$
= E (\Delta (\beta^*, h)) - \tau \|h\|_1.
$$

In addition, we have:

$$
E (\Delta (\beta^*, h)) = \frac{1}{n} \sum_{i=1}^{n} f ((\mathbf{x}_i, \beta^* + h); y_i) - f ((\mathbf{x}_i, \beta^*); y_i) = L(\beta^* + h) - L(\beta^*).
$$

Consequently, we conclude that with probability at least $1 - \frac{\delta}{2}$:

$$
\Delta (\beta^*, h) \geq L(\beta^* + h) - L(\beta^*) - \tau \|h\|_1.
$$
Step 2: We now lower-bound the right-hand side of Equation (34). Since $L$ is twice differentiable, a Taylor development around $\beta^*$ gives:

$$L(\beta^* + h) - L(\beta^*) = \nabla L(\beta^*)^T h + \frac{1}{2} h^T \nabla^2 L(\beta^*)^T h + o(\|h\|_2).$$

The optimality of $\beta^*$ implies $\nabla L(\beta^*) = 0$. In addition, Theorem 2 states that $h \in \Lambda(S_0, \gamma_1, \gamma_2)$ with probability at least $1 - \frac{\delta}{2}$. Consequently, we can use the restricted eigenvalue condition defined in Assumption 4.2($k, \gamma$). However, we do not want to keep the term $o(\|h\|_2)$ as it can hide non-trivial dependencies.

**Case 1:** If $\|h\|_2 \leq r(k^*)$ – where $r(k^*, \gamma)$ is shorthanded $r(k^*)$ and is the maximum radius introduced in the growth condition Assumption 5.2 – then by the result of Theorem 2 and Assumption 4.2($k, \gamma$), it holds with probability at least $1 - \frac{\delta}{2}$:

$$L(\beta^* + h) - L(\beta^*) \geq \frac{1}{4} \kappa(k^*) \|h\|_2^2. \tag{35}$$

**Case 2:** If now $\|h\|_2 \geq r(k^*)$, then using the convexity of $L$ thus of $t \to L(\beta^* + th)$, we similarly obtain with the same probability:

$$L(\beta^* + h) - L(\beta^*) \geq \frac{\|h\|_2}{r(k^*)} \left\{ L\left(\beta^* + \frac{r(k^*)}{\|h\|_2} h\right) - L(\beta^*) \right\} \text{ by convexity}$$

$$\geq \frac{\|h\|_2}{r(k^*)} \inf_{\|z\|_2 = r(k^*)} \{ L(\beta^* + z) - L(\beta^*) \}$$

$$\geq \frac{\|h\|_2}{r(k^*)} \frac{1}{4} \kappa(k^*) r(k^*)^2 = \frac{1}{4} \kappa(k^*) r(k^*) \|h\|_2^2. \tag{36}$$

Combining Equations (34), (35) and (36), we conclude that with probability at least $1 - \delta$ the following restricted strong convexity holds:

$$\Delta(h) \geq \frac{1}{4} \kappa(k^*) \|h\|_2^2 \wedge \frac{1}{4} \kappa(k^*) r(k^*) \|h\|_2 - \tau \|h\|_1. \tag{37}$$

□


**E Proof of Theorem 1**

**Proof:** We now prove our main Theorem 1. We recall that $S_0$ has been defined as the subset of the $k^*$ highest elements of $h$. Following Equation (20) and the restricted strong convexity derived in Theorem 4 (Equation (37)), it holds with probability at least $1 - \delta$:

\[
\frac{1}{4} \kappa(k^*) \left\{ \|h\|_2^2 \wedge r(k^*) \|h\|_2 \right\} \\
\leq \tau \|h\|_1 + \lambda \|h_{S^*}\|_1 - \lambda \|h_{(S^*)^c}\|_1 \\
= \tau (\|h_{S_0}\|_1 + \|h_{(S_0)^c}\|_1) + \lambda \sqrt{k^*} \|h_{S_0}\|_2 \\
\leq \tau \left( \|h_{S_0}\|_1 + \frac{\alpha}{\alpha - 1} \|h_{S_0}\|_1 + \frac{\sqrt{k^*}}{\alpha - 1} \|h_{S_0}\|_2 \right) + \lambda \sqrt{k^*} \|h_{S_0}\|_2 \\
\leq \frac{2\alpha - 1}{\alpha - 1} \tau \sqrt{k^*} \|h_{S_0}\|_2 + \frac{\tau \sqrt{k^*}}{\alpha - 1} \|h_{S_0}\|_2 + \lambda \sqrt{k^*} \|h_{S_0}\|_2 \\
\text{from Cauchy-Schwartz inequality on the } k^* \text{ sparse vector } h_{S_0} \\
\leq \left( \frac{2\alpha}{\alpha - 1} \tau + \lambda \right) \sqrt{k^*} \|h\|_2. \tag{38}
\]

With the definitions of $\tau$ and $\lambda$ as in the Theorems 2 and 3, Equation (38) leads to:

\[
\frac{1}{4} \kappa(k^*) \left\{ \|h\|_2 \wedge r(k^*) \right\} \leq \frac{28\alpha}{\alpha - 1} \mu(k^*) \sqrt{\frac{k^* \log(Rp/k^*)}{n}} + \frac{\log(2/\delta)}{kn} \\
+ 12\alpha LM \frac{\sqrt{\log(2pe/k^*) \log(2/\delta)}}{n}.
\]

Exploiting Assumption 5($k^*, \gamma, \delta$), and using that $\alpha \geq 2$, we obtain with probability at least $1 - \delta$:

\[
\|h\|_2^2 \lesssim \left( \frac{\alpha \mu(k^*)}{\kappa(k^*)} \right)^2 \frac{k^* \log(Rp/k^*)}{n} + \frac{\log(2/\delta)}{n} + \left( \frac{\alpha \mu(k^*)}{\kappa(k^*)} \right)^2 \frac{k^* \log(p/k^*) \log(2/\delta)}{n}.
\]

which concludes the proof.

**F Proof of Corollary 1**

**Proof:** In order to derive the bound in expectation, we define the bounded random variable:

\[
Z = \frac{\kappa(k^*)^2}{\alpha^2 L^2} \|\hat{\beta} - \beta^*\|_2.
\]

Since Assumption 5($k^*, \gamma, \delta_0$) is satisfied for a small enough $\delta_0$, we can fix $C$ such that $\forall \delta \in (0, 1)$, it holds with probability at least $1 - \delta$:

\[
Z \leq CH \left\{ \mu(k^*)^2 + M^2 \log(2/\delta) \right\} + C \mu(k^*)^2 \frac{\log(2/\delta)}{n} \text{ where } H = \frac{k^* \log(Rp/k^*)}{n}.
\]

Then it holds $\forall t \geq t_0 = \log(4)$:

\[
P \left( Z/C \geq H \mu(k^*)^2 + HM^2 t + \frac{\mu(k^*)^2}{n} \right) \leq 2e^{-t}.
\]
Let $q_0 = HM^2 t_0 + \mu(k^*)^2 t_0$, then $\forall q \geq q_0$

$$\mathbb{P}(Z/C \geq H \mu(k^*)^2 + q) \leq 2 \exp\left(-\frac{n}{nHM^2 + \mu(k^*)^2} q\right) \leq 2 \exp\left(-\frac{q}{HM^2}\right).$$

Consequently, by integration we have:

$$\mathbb{E}(Z) = \int_{0}^{+\infty} C \mathbb{P}(\frac{1}{C} \geq q) dq$$

$$= \int_{0}^{+\infty} C \mathbb{P}(\frac{1}{C} \geq H \mu(k^*)^2 + q) dq + CH \mu(k^*)^2$$

$$\leq \int_{q_0}^{+\infty} 2Ce^{-\frac{q}{nHM^2}} dq + Cq_0 + CH \mu(k^*)^2$$

$$\leq 2CHM^2 e^{-\frac{q_0}{nM\alpha}} + Cq_0 + CH \mu(k^*)^2$$

$$\leq 2CHM^2 + CHM^2 \log(4) + C \mu(k^*)^2 \log(4) + CH \mu(k^*)^2$$

$$\leq C_1 H(\mu(k^*)^2 + M^2)$$

for some universal constant $C_1$, since $H \gg n^{-1} \mu(k^*)$. Hence since $R \geq 1$ we conclude:

$$\mathbb{E}(\|\hat{\beta} - \beta^\star\|^2) \lesssim \left(\frac{\alpha L}{\kappa(k^*)}\right)^2 (\mu(k^*)^2 + M^2) \frac{k^* \log (R/p/k^*)}{n}.$$  

\[\Box\]

**G Proof of Theorem 5**

**Proof:** We fix $\tau > 0$ and denote $X = (X_1, \ldots, X_p) \in \mathbb{R}^{n \times p}$ the design matrix.

For $\beta \in \mathbb{R}^p$, we define $w^{\tau}(\beta) \in \mathbb{R}^n$ by:

$$w^{\tau}_i(\beta) = \min\left(1, \frac{1}{2\tau}|z_i|\right) \text{sign}(z_i), \ \forall i$$

where $z_i = 1 - y_i x_i^T \beta$, $\forall i$. We easily check that

$$w^{\tau}(\beta) = \arg\max_{\|w\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^{n} (z_i + w_i z_i) - \frac{\tau}{2n} \|w\|_2^2.$$  

Then the gradient of the smooth hinge loss is

$$\nabla g^{\tau}(\beta) = -\frac{1}{2n} \sum_{i=1}^{n} (1 + w^{\tau}_i(\beta)) y_i x_i \in \mathbb{R}^p.$$  

For every couple $\beta, \gamma \in \mathbb{R}^p$ we have:

$$\nabla g^{\tau}(\beta) - \nabla g^{\tau}(\gamma) = \frac{1}{2n} \sum_{i=1}^{n} (w^{\tau}_i(\gamma) - w^{\tau}_i(\beta)) y_i x_i.$$  

(40)
For $a, b \in \mathbb{R}^n$ we define the vector $a \ast b = (a_i b_i)_{i=1}^n$. Then we can rewrite Equation (40) as

$$\nabla g^T(\beta) - \nabla g^T(\gamma) = \frac{1}{2n} X^T [y \ast (w^T(\gamma) - w^T(\beta))].$$ (41)

The operator norm associated to the Euclidean norm of the matrix $X$ is $\|X\| = \max_{\|z\|_2=1} \|Xz\|_2$. Let us recall that $\|X\|^2 = \|X^T\|^2 = \|X^T X\| = \mu_{\text{max}}(X^T X)$ corresponds to the highest eigenvalue of the matrix $X^T X$.

Consequently, Equation (41) leads to:

$$\|\nabla L^T(\beta) - \nabla L^T(\gamma)\|_2 \leq \frac{1}{2n} \|X\| \|w^T(\gamma) - w^T(\beta)\|_2.$$ (42)

In addition, the first order necessary conditions for optimality applied to $w^T(\beta)$ and $w^T(\gamma)$ give

$$\sum_{i=1}^n \left\{ \frac{1}{2n} (1 - y_i x_i^T \beta) - \frac{\tau}{n} w_i^T(\beta) \right\} \{w_i^T(\gamma) - w_i^T(\beta)\} \leq 0,$$

and

$$\sum_{i=1}^n \left\{ \frac{1}{2n} (1 - y_i x_i^T \gamma) - \frac{\tau}{n} w_i^T(\gamma) \right\} \{w_i^T(\beta) - w_i^T(\gamma)\} \leq 0.$$ (44)

Then by adding Equations (43) and (44) and rearranging the terms we have:

$$\tau \|w^T(\gamma) - w^T(\beta)\|^2_2$$

$$\leq \frac{1}{2} \sum_{i=1}^n y_i x_i^T (\beta - \gamma) (w_i^T(\gamma) - w_i^T(\beta))$$

$$\leq \frac{1}{2} \|X(\beta - \gamma)\|_2 \|w^T(\gamma) - w^T(\beta)\|_2$$

$$\leq \frac{1}{2} \|X\| \|\beta - \gamma\|_2 \|w^T(\gamma) - w^T(\beta)\|_2,$$

where we have used Cauchy-Schwartz inequality. We easily derive:

$$\|w^T(\gamma) - w^T(\beta)\|_2 \leq \frac{1}{2\tau} \|X\| \|\beta - \gamma\|_2.$$ (45)

We conclude the proof by combining Equations (42) and (45):

$$\|\nabla L^T(\beta) - \nabla L^T(\gamma)\|_2 \leq \frac{1}{4\tau^2} \|X\|^2 \|\beta - \gamma\|_2$$

$$= \mu_{\text{max}}(n^{-1} X^T X) \|\beta - \gamma\|_2.$$ (46)

**The case of Quantile Regression:** For the quantile regression loss, the same smoothing method applies. Let us simply note that:

$$\rho_\theta(x) = \max \left( (\theta - 1)x, \theta x \right) = \frac{1}{2} ((2\theta - 1)x + |x|)$$

$$= \max_{|w| \leq 1} \frac{1}{2} ((2\theta - 1)x + wx).$$
Hence we can immediately use the same steps than for the hinge loss – which is a particular case of the quantile regression loss – and define the smooth quantile regression loss $g_\theta^\tau$. Its gradient is:

$$\nabla g_\theta^\tau(\beta) = -\frac{1}{2n} \sum_{i=1}^n (2\theta - 1 + w_i^\tau(\beta))y_i x_i \in \mathbb{R}^p,$$  \hspace{1cm} (46)

where we still have $w_i^\tau = \min\left(1, \frac{1}{\tau^2} |z_i|\right) \text{sign}(z_i)$ but now $z_i = y_i - x_i^T \beta$, $\forall i$. The Lipschitz constant of $\nabla g_\theta^\tau$ is still given by Theorem 5.

**H Proof of Theorem 6**

**Proof:** We still assume $|h_1| \geq \ldots \geq |h_p|$. Following Equation (21) it holds:

$$S(h) \leq \Delta(h) \leq \eta |\beta^*|_S - \eta |\hat{\beta}|_S.$$  \hspace{1cm} (47)

We want to upper-bound the right-hand side of Equation (47). We define a permutation $\phi \in \mathcal{S}_p$ such that $|\beta^*|_S = \sum_{j=1}^{k^*} \lambda_j |\beta_{\phi(j)}^*|$ and $|\hat{\beta}_{\phi(k^*+1)}| \geq \ldots \geq |\hat{\beta}_{\phi(p)}|:

$$\frac{1}{\eta} \Delta(h) \leq \sum_{j=1}^{k^*} \lambda_j |\beta_{\phi(j)}^*| - \max_{\psi \in \mathcal{S}_p} \sum_{j=1}^p \lambda_j |\hat{\beta}_{\psi(j)}|$$

$$\leq \sum_{j=1}^{k^*} \lambda_j \left(|\beta_{\phi(j)}^*| - |\hat{\beta}_{\phi(j)}|\right) - \sum_{j=k^*+1}^p \lambda_j |\hat{\beta}_{\phi(j)}|$$

$$= \sum_{j=1}^{k^*} \lambda_j |h_{\phi(j)}| - \sum_{j=k^*+1}^p \lambda_j |\hat{\beta}_{\phi(j)}|$$

$$\leq \sum_{j=1}^{k^*} \lambda_j |h_{\phi(j)}| - \sum_{j=k^*+1}^p \lambda_j |h_{\phi(j)}|. \hspace{1cm} (48)$$

Since $\lambda$ is monotonically non-decreasing: $\sum_{j=1}^{k^*} \lambda_j |h_{\phi(j)}| \leq \sum_{j=1}^{k^*} \lambda_j |h_j|$. Because $|h_{\phi(k^*+1)}| \geq \ldots \geq |h_{\phi(p)}|$: $\sum_{j=k^*+1}^p \lambda_j |h_{\phi(j)}| \leq \sum_{j=k^*+1}^p \lambda_j |h_{\phi(j)}|$. In addition, Equation (22) from Appendix 3 leads to, with probability at least $1 - \frac{\delta}{2}$:

$$|S(h)| \leq 14LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^p \lambda_j |h_j| \leq 12LM \sqrt{\frac{\log(6/\delta)}{n}} \sum_{j=1}^p \lambda_j |h_j| = \frac{\eta}{\alpha} |h|_S,$$

where $\eta$ in defined in the statement of the theorem. Thus, combining this last equation with Equation (48), it holds with probability at least $1 - \frac{\delta}{2}$:

$$\frac{1}{\alpha} |h|_S \leq \sum_{j=1}^{k^*} \lambda_j |h_j| - \sum_{j=k^*+1}^p \lambda_j |h_j|,$$

which is equivalent to saying that with probability at least $1 - \frac{\delta}{2}$:

$$\sum_{j=k^*+1}^p \lambda_j |h_j| \leq \frac{\alpha + 1}{\alpha - 1} \sum_{j=1}^{k^*} \lambda_j |h_j|,$$  \hspace{1cm} (49)

that is $h \in \Gamma \left(k^*, \frac{\alpha + 1}{\alpha - 1}\right)$. \hspace{1cm} \Box
I Proof of Corollary \[2\]

**Proof:** We follow the same path than in the proof of Theorem \[1\]. The results of Theorem \[3\] still hold for the value of \(\tau\) defined as:

\[
\tau = 14L\mu(k^*)\sqrt{\frac{\log (R_p/k^*)}{n} + \frac{\log (6/\delta)}{n}}.
\]

As a consequence, the restricted strong convexity derived in Lemma \[4\] can be applied. We consequently obtain with probability at least \(1 - \delta\):

\[
\frac{1}{4}k(k^*) \{||h||_2^2 \wedge r(k^*)||h||_2\} \leq \tau \cdot ||h||_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| - \eta \sum_{j=k^*+1}^{p} \lambda_j |h_j| \\
\leq \tau \cdot ||h_{S_0}||_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| + \tau \cdot ||h_{(S_0)c}||_1 - \eta \sum_{j=k^*+1}^{p} \lambda_j |h_j|. \tag{50}
\]

We want \(\tau \leq \eta \lambda_{k^*}\), that is \(14L\mu(k^*)\sqrt{\frac{\log (R_p/k^*)}{n} + \frac{\log (6/\delta)}{n}} \leq 12\alpha LM \sqrt{\frac{\log (2eR_p/k^*)}{n} \log (6/\delta)}\).

This will be satisfied if \(2\mu(k^*) \{\log (2eR_p/k^*) + \log (6/\delta)\} \leq \alpha^2 M^2 \{\log (2eR_p/k^*) \log (6/\delta)\}\).

And because \(\log (2eR_p/k^*) \geq 2\) and \(\log (6/\delta) \geq 2\) a necessary condition is:

\[
\mu(k^*)^2 \log (2eR_p/k^*) \log (6/\delta) \leq \alpha^2 M^2 \log (2eR_p/k^*) \log (6/\delta),
\]

which is true is the following condition is satisfied:

\[
\mu(k^*)^2 \log (Re) \leq \alpha^2 M^2. \tag{51}
\]

Equation \[51\] has been assumed Corollary \[2\]. Hence by plugging the result in Equation \[50\] we obtain, similarly to Section \[3\]

\[
\frac{1}{4}k(k^*) \{||h||_2^2 \wedge r(k^*)||h||_2\} \leq \tau \cdot ||h_{S_0}||_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| \\
\leq \tau \sqrt{k^*} ||h_{S_0}||_2 + \eta \sqrt{k^*} \log (2eR_p/k^*) ||h_{S_0}||_2 \\
\leq 2\eta \sqrt{k^*} \log (2eR_p/k^*) ||h_{S_0}||_2 \text{ since } \tau \leq \eta \lambda_{k^*} \\
\leq 24\alpha LM \frac{\sqrt{k^*} \log (2eR_p/k^*)}{n} \log (6/\delta) ||h||_2.
\]

This last equation is very similar to Equation \[38\] in the proof of Theorem \[1\]. We conclude the proof identically, and obtain the bound in expectation similarly than in the proof of Corollary \[1\]. \(\square\)