Quasinormal modes and thermodynamic properties of GUP-corrected Schwarzschild black hole surrounded by quintessence

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We study the Quasinormal Modes (QNMs) of the Schwarzschild black hole surrounded by a quintessence field after implementing the quantum corrections to its solution as required by the Generalised Uncertainty Principle (GUP). We analyse the dependence of the QNMs on the deformation parameters of GUP as well as on the quintessence parameter. In most cases the QNMs show the appreciable dependency on these parameters. For a better idea of the accuracy of calculations of QNMs, we compare the results of the QNMs obtained via Mashhoon method with the 6th order WKB method. A good agreement between these two methods of QNM calculations is seen depending on the different factors. Further, we study the thermodynamic properties of the GUP-corrected Schwarzschild black hole and check for any dependence with the deformation parameters and the quintessence parameter. In particular, we compute the Hawking temperature, heat capacity and entropy for the black hole and analyse the results graphically to show the dependency of the thermodynamic properties on the said parameters. We have seen that the thermodynamic properties of black holes also depend noticeably on the model parameters in most cases. Black hole remnants have been studied and it is shown that the possible existence of remnant radius as well as remnant temperature depends on the deformations introduced. However, it is observed that the GUP-corrected black hole constructed here can not become a remnant.

PACS numbers:
Keywords: Quasinormal Modes; Black holes; Gravitational Waves; Generalised Uncertainty Principle

I. INTRODUCTION

Black holes and Gravitational Waves (GWs) are two of the most fascinating predictions of General Relativity (GR). Black holes are among the most mysterious objects in our universe that have attracted the attention of scientific community for many decades. It is widely believed that when a sufficiently massive star runs out of fuel, the inward pull of gravity becomes dominant and there is a rapid collapse of matter towards the centre, which ultimately results in the formation of a black hole. A black hole interacts with its surrounding matter and is generally in a perturbed state. Such perturbations cause the black hole to undergo oscillation and that leads to the emission of GWs [1–5]. GWs are ripples in the spacetime fabric, generated by accelerating massive objects, which propagate at the speed of light. The LIGO-Virgo collaboration declared the first-ever detection of GWs on 14th of September 2015 [6], almost after 100 years of their prediction by prominent physicist A. Einstein in 1916. Since then, a number of detections of GWs has been reported by the collaboration [7–10]. It has generated a new era of gravitational astronomy, the likes of which has never been seen before.

Quasinormal Modes (QNMs) are some complex frequencies associated with the GWs that represent the reaction of a black hole, after some perturbations act on it [2, 3, 11–17]. There are different methods of calculations of QNMs from the black holes [2]. One of the simplest and elegant methods of finding out the QNMs was devised by B. Mashhoon, which is commonly called as the Mashhoon Method [2, 11, 18–20]. It is an analytical method which is easy to handle for a simple system. The most frequently used and trusted method of QNM analysis is the Wentzel-Kramers-Brillouin (WKB) method [2, 3, 21], which was initially utilized by Schutz and Will [22]. Improvements were made in this method by Konoplya who introduced corrections in WKB calculations upto 6th orders [14]. Recently, more higher order corrections are introduced in this method ([23] and references therein). In this work we consider both these methods of analyzing the QNMs for the sake of accuracy in the calculation. Apart from these, numerous works have been done on the analytical and numerical techniques to calculate the QNMs associated with the black hole perturbations in recent times [24–31]. Other works related to black hole physics may be found in [32, 33] and in the references therein.

Recently, the black hole physics and related thermodynamics have also attracted researcher’s attention and a lot of such works can be seen in recent times in literature [34–45]. It was after the ground-breaking works of Bekenstein and Hawking that cemented the idea of interpreting the black hole as a thermodynamic system, showing the corresponding properties like temperature and entropy. Hawking proposed that a black hole emits radiation from its event horizon [46–48] and Bekenstein

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showed that as the black hole engulfs matter, the information associated with it is not lost but is incorporated in the horizon area of the black hole [49–51]. These ideas led to a revolution in black hole thermodynamics.

The discovery of Rieß and Perlmutter, which showed that the universe is expanding with an acceleration [52, 53] led to a flood of theoretical models coming forward to resolve this unexpected observation. In the majority of such models the idea of dark energy was invoked to interpret this colossal expansion (see [54] for a review). One of the convenient ideas to deal with it was the ΛCDM model, in which Einstein’s cosmological constant was reintroduced as a homogeneous and isotropic fluid with the negative pressure and is considered to be the cause of this present state of expansion of the universe [55]. The cosmological constant was thought to originate out of quantum fluctuations of vacuum, but its theoretically predicted value could not match the observational value. To alleviate this problem, dynamical scalar field models were proposed, and the most common among them is the quintessence model of dark energy. In this model a scalar field minimally coupled to gravity is used to describe this late time accelerated expansion. Detailed about and current status of the quintessence model can be seen in Refs. [54–60].

Many works can be found in literature where the black hole thermodynamics is studied with a surrounding field. In 2003, a Schwarzschild black hole surrounded by a quintessence was studied and corresponding thermodynamic properties were examined by Kiselev [61]. He showed that presence of the surrounding field has a major impact on the properties of a black hole. Subsequently, Chen et al. [62] considered a 3-dimensional black hole with a quintessence matter surrounding it and examined thermodynamic properties of the black hole. Reissner-Nordström black holes were examined with a quintessential surrounding by Wei and Chu [63]. Thermodynamics of Nariai type black holes were considered by Fernando with a quintessential surrounding [64]. Recently, it is seen that many research works have considered the effects of quantum corrections via the Generalised Uncertainty Principle (GUP) on the thermodynamics of black hole [65–67]. One such work was carried out by Shahjalal in 2019 [65], where he compared the effects of the quantum deformations with and without the presence of quintessential surroundings. The case of rotating non-linear magnetically charged black holes was taken up by Ndongmo et al. [68], in which thermodynamics of the black hole was studied. Moreover, Anacleto et al. [66] studied the quantum-corrected Schwarzschild black holes and analysed the absorption and scattering processes. González et al. [69] studied a 3-dimensional Godel black hole and calculated the QNMs and Hawking radiation. Further, Lütfüoglu et al. [67] studied the thermodynamics of Schwarzschild black holes with the quintessential surrounding and GUP. They showed that the upper and lower bounds on various functions like temperature and entropy depend on the deformation parameters as well as on the quintessence coefficient, and also presented plots of P-V isotherms. It is notable that the combined linear and quadratic GUP approach was introduced in [70–75].

The study of black hole shadows has also been carried out in recent times as they provide useful insights into the black hole event horizon as well as into the optical properties of a black hole [76, 77]. The initial work in this direction was done by Synge [78] and Luminet [79] in the 1970s and for rotating Kerr black holes by Bardeen [80]. Lately, an interesting work on QNMs and shadows of Schwarzschild black hole was performed by Anacleto et al. [81], where they considered the GUP-modified Schwarzschild black hole solutions and calculated the QNMs and shadows of the black holes, and showed the dependency of these properties on the deformation parameters. In recent times, many works have been done in this field [82–94]. In this work, however, we shall not include the shadow analysis and it will be addressed in a future work.

Thus, inspired from the ongoing endeavours to explore these novel ideas, in this work we intend to study the various properties of a GUP-corrected Schwarzschild black hole surrounded by a quintessence field, such as the QNMs and thermodynamic properties like Hawking temperature, entropy, heat capacity and surface gravity. To the best of our information, GUP with both linear and quadratic terms has not been incorporated to Schwarzschild black hole surrounded by quintessence. The novelty of GUP that it introduces a minimum length scale, that is the Planck’s length, might play an important role in the properties of black holes [67, 81, 95, 96].

The rest of the paper is organized as follows. In section II, we compute the QNMs of the GUP-corrected Schwarzschild black hole surrounded by a quintessence field using the Mashhoon method and the 6th order WKB method, and analyse the results. In section III, we study the thermodynamic properties of the black hole and present a graphical analysis of dependency of the thermodynamic properties on the deformation parameters as well as on the quintessence coefficient. We summarise our results and present some concluding remarks in section IV.

II. QUASINORMAL MODES OF A GUP-CORRECTED SCHWARZSCHILD BLACK HOLE

The general form of the black hole metric as initially derived by Kiselev [61], in which he considered a Schwarzschild black hole surrounded by a quintessence dark energy with a particular energy density, can be expressed by

\[ ds^2 = -g(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\Omega^2, \tag{1} \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and the metric function \( g(r) \) has the form:

\[ g(r) = 1 - \frac{2M}{r} - \frac{e}{r^{3\omega+1}}. \tag{2} \]
In this function $M$ is the mass of the black hole, $\omega$ is the equation of state parameter of the quintessence field, and $e$ is the positive normalization coefficient that is dependent on the quintessence density. Since the recent past, a quantum correction to various black hole solutions, including the Schwarzschild one has been introduced via GUP in order to avoid singularities in such solutions by introducing a minimum length other than zero [97]. Under this correction, the normal metric of a black hole is modified, which gives a corresponding new horizon of the black hole. Thus, for example, in the case of Schwarzschild black hole, the original Schwarzschild horizon radius $r_h$ of the black hole has to be replaced with the GUP-corrected radius $r_{hGUP}$ for this purpose [81]. The basic steps of incorporation of GUP correction into the black hole metric (1) are the following.

Considering the modified Heisenburg algebra, we may write

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 - \frac{\alpha l_p}{\hbar} \Delta p + \frac{\beta l_p^2}{\hbar^2} (\Delta p)^2 \right),$$  \hspace{1cm} (3)$$

where $\alpha$ and $\beta$ are two dimensionless deformation parameters, and $l_p$ is the Planck’s length. Taking the unit system with $G = c = \hbar = l_p = 1$, equation (3) can be solved for $\Delta p$, which gives

$$\Delta p \approx \frac{4r_h + \alpha}{2\beta} \left[ 1 - \sqrt{1 - \frac{4\beta}{(4r_h + \alpha)^2}} \right].$$  \hspace{1cm} (4)$$

Here the uncertainty in position is taken as the horizon diameter $2r_h$ and we end up with the following expression [66]:

$$E_{GUP} \geq E \left[ 1 - \frac{4\alpha}{r_h} + \frac{16\beta}{r_h^2} + \ldots \right],$$  \hspace{1cm} (5)$$

where $E_{GUP}$ is the GUP-corrected energy of the black hole. Now considering the assumption that $E \sim M$, $E_{GUP} \sim M_{GUP}$ and calculating $r_h = \frac{2M}{1-e}$ from the function (2), we obtained the relation,

$$M_{GUP} \geq M \left[ 1 - \frac{4\alpha}{r_h} + \frac{16\beta}{r_h^2} \right] = M \left( 1 - \frac{2\alpha(1-e)}{M} + \frac{4\beta(1-e)^2}{M^2} \right).$$  \hspace{1cm} (6)$$

Thus, finally the line element of a GUP-corrected Schwarzschild black hole surrounded by a quintessence field ($\omega = -1/3$) takes the form:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2$$  \hspace{1cm} (7)$$

with the modified metric function

$$f(r) = 1 - \frac{2M}{r} \left( 1 - \frac{2\alpha(1-e)}{M} + \frac{4\beta(1-e)^2}{M^2} \right) - e \equiv 1 - \frac{2M_{GUP}}{r} - e.$$  \hspace{1cm} (8)$$

This metric function gives the GUP-corrected horizon radius as $r_{hGUP} = \frac{2M_{GUP}}{(1-e)}$. Fig. 1 shows the behaviours of the original metric and the GUP-corrected metric as functions of $r$ for various values of the related parameters. From the figure it is seen that there is only one event horizon for the black hole for different values of the parameters. There is no other horizon obtained for the black hole. The left plot shows that with the increasing values of $\alpha$, the horizon radius becomes smaller. Whereas the middle and the right plots show that with the increasing $\beta$ and $e$ values respectively, the horizon radius increases. It is also seen that the effect of parameters $\alpha$ and $\beta$ is identical and dominant than that of the parameter $e$.

At this stage it is necessary to mention that the small values of the GUP parameters is an obvious choice because any correction term that we introduce into our theory cannot be larger than the base term involved as these corrections are generally very minute in nature. In the literature, we found that the values of these parameters have been considered less than unity [65, 67, 81, 98]. This choice is well motivated as we are considering only small corrections to the original uncertainty relation and it is demanded for the derivation of the metric expression. Similarly, the quintessence parameter ($e$) has been constrained for the case of a Schwarzschild black hole surrounded by a quintessence field in Ref. [99], where the authors found a bound on the quintessential parameter as $10^{-21} \leq eM \leq 10^{-11}$. On the other hand in Refs. [62, 100], it can be seen that the quintessential parameter is taken of the order $\sim 0.1$. So, in our study we consider reasonably small values of these parameters.

### A. QNMs by Mashhoon Method

Mashhoon method is an analytical method of calculating the QNMs of a black hole by comparing its effective potential with a standard potential, such as the Poschl-Teller potential [18] or the Eckart potential [11]. In this work we consider the scalar field
perturbation for the calculation of QNMs from the black hole of our model. In this perturbation method the equation of motion for a test scalar field $\Phi$, which becomes perturbed according to the perturbation of the black hole spacetime, is considered and has the form:

$$\frac{1}{\sqrt{g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu})\Phi^{(\nu)} = 0. \quad (9)$$

It is appropriate to express the scalar field $\Phi$ as spherical harmonic waves in the form:

$$\Phi^{(\nu)}(t, r, \theta, \phi) = e^{-i\omega t} \frac{\psi(r)}{r} Y^{(\nu)}(\theta, \phi), \quad (10)$$

where $\psi(r)$ represents the radial part, $Y^{(\nu)}$ represents the spherical harmonics and $\omega$ represents the oscillation frequency for the time part of the wave. $\omega$ is actually the quasinormal frequency which we want to find out using two different methods. Using equation (10) in (9), we can conveniently transform the equation of motion into a Schrödinger-like wave equation given by

$$\frac{d^2\psi}{dx^2} + (\omega^2 - V_I)\psi = 0. \quad (11)$$

Here, the tortoise coordinate $x$ is defined as $dx = dr/f(r)$ and $V_I(r)$ is the effective black hole potential, which can be obtained from the formula $[16, 23]$

$$V_I(r) = f(r) \left[ \frac{f'(r)}{r} + \frac{l(l+1)}{r^2} \right]. \quad (12)$$

After obtaining the effective potential of the black hole, we compare it with the standard Poschl-Teller potential at their maxima. The Poschl-Teller potential has the form $[18]$

$$V_{PT} = \frac{V_0}{\cosh^2\alpha(x-x_0)}, \quad (13)$$

where the quantity $V_0$ denotes the height and $\alpha$ denotes the curvature of the potential at its maximum. On comparing the two potentials, we get the analytical form of the parameters $V_0$ and $\alpha$ (see the Appendix for the explicit form of their expressions). Then we utilize the formulae for calculating the QNMs, which is given by Mashhoon as $[18]$

$$\omega = \omega_R + i\omega_I = \pm \left( V_0 - \frac{\alpha^2}{4} \right)^{\frac{1}{2}} + i\alpha \left( n + \frac{1}{2} \right), \quad (14)$$

where $\omega_R$ and $\omega_I$ are the real and imaginary parts of the QNM frequency respectively. Now, using this formula (the explicit form of the formula can be found in the appendix) we have calculated the QNM frequencies of the GUP-corrected black hole surrounded by a quintessential dark energy field as shown in the Table I. In this calculation of QNMs, we have considered a small positive value of the quintessence parameter $e = 0.05$ laying well within the accepted range and mass of the black hole is considered as $M = 1$. The quantum deformation parameters $\alpha$ and $\beta$ are also assumed as small positive values within their well accepted range as shown in the table.
It is interesting to note that there is a striking dependence of the QNMs on the deformation parameters. As is seen from Table I, when \( \alpha \) is kept constant and \( \beta \) is increased, there is a noticeable decrease in the magnitude of the real part of the QNMs. That is, the amplitude of QNMs is inversely proportional to \( \beta \). Whereas, with a particular value of \( \beta \), it is seen that the amplitude increases with increase in \( \alpha \). On the other hand in the case of the imaginary part of the QNM representing the damping of the wave, for a fixed value of \( \alpha \), the damping increases with increase in \( \beta \). While, for a fixed \( \beta \), the damping decreases with an increase in \( \alpha \). We have also calculated the QNMs for the black hole with \( l = 2 \) as shown in Table II. Similar pattern is observed in this case also. But one notable feature that comes out is that with an increase in \( l \), the corresponding amplitudes of the QNMs increase noticeably, while the damping factor is not much affected although it decreases with increasing \( l \). Another feature that is apparent from all the calculated modes is that with the increase in \( n \) values the amplitude of QNMs remains the same but its damping increases. This is in fact already clear from equation (11).

It will also be interesting to observe any dependence of the QNMs with the quintessence parameter \( e \), which has been kept at a constant value in the previous analysis. For this purpose, as shown in the Table III, we have computed the QNMs for different values of the parameter \( e \) with some fixed values of the deformation parameters \( \alpha \) and \( \beta \). It is seen from the table that for a fixed value of \( l \), with increasing \( e \), the amplitude decreases, while the damping increases.

Figs. 2 – 4 give the visual representation of all these behaviours of QNMs of the black hole as discussed and presented in Tables I – III. Fig. 2 shows the variation of the amplitude and damping part of the QNMs with respect to the GUP parameter \( \alpha \) for a fixed value of the parameter \( \beta = 0.05 \) and for three different \( l \) values. It is seen that amplitude of QNMs increases slowly, whereas the damping decreases rapidly with the increasing values of \( \alpha \). In Fig. 3 the variations of amplitude and damping of QNMs with respect to \( \beta \) for a fixed value of \( \alpha = 0.05 \) and for three different \( l \) values are shown. This figure shows that the effect of \( \beta \) is totally opposite to that of \( \alpha \) on the QNMs. However, the trend of the effect of these two parameters on QNMs is almost similar. The first two plots of Fig. 4 show the behaviours of amplitude and damping respectively of QNMs of the black hole with respect to quintessence field parameter \( e \) for a constant value of \( \alpha = \beta = 0.05 \) and for two values \( l \). Similar to the case of the parameter \( \beta \) in this case also the amplitude decreases slowly, but the damping increases at a relatively faster step. The third plot of this figure shows the fact that for a fixed \( l \), the damping almost remains constant with increasing quintessence parameter \( e \) and the higher values of \( n \) give a much higher damping.

### B. QNMs by WKB method

The QNMs can be reliably calculated using the 6th order WKB method. It is a semi-analytical approximation method. The basics of the WKB method can be found extensively in literature ([2, 3, 16] and references therein). Here our basic intention is to make a comparative analysis of the QNM frequencies obtained by the Mashhoon method with that will be given by the WKB method.


TABLE II: QNMs of GUP-corrected black hole surrounded by a quintessence field for \( n = 0, n = 1, n = 2 \) and \( n = 3 \) modes, for multipole number \( l = 2 \), quintessence parameter \( e = 0.05 \) and various values of the deformation parameter \( \alpha \) and \( \beta \) obtained by using the Mashhoon method.

| \( \alpha \) | \( \beta \) | \( e \) | QNMs for \( n = 0 \) | QNMs for \( n = 1 \) | QNMs for \( n = 2 \) | QNMs for \( n = 3 \) |
|---|---|---|---|---|---|---|
| 0.00 | 0.00 | 0.05 | 0.447498 + 0.106271i | 0.447498 + 0.308031i | 0.447498 + 0.513384i | 0.447498 + 0.718738i |
| 0.00 | 0.01 | 0.05 | 0.430431 + 0.105325i | 0.430431 + 0.315974i | 0.428658 + 0.527870i | 0.430431 + 0.737273i |
| 0.00 | 0.02 | 0.05 | 0.414525 + 0.107389i | 0.414525 + 0.322167i | 0.414525 + 0.536946i | 0.414525 + 0.751724i |
| 0.00 | 0.03 | 0.05 | 0.399674 + 0.108966i | 0.399674 + 0.326897i | 0.399674 + 0.544828i | 0.399674 + 0.762759i |
| 0.02 | 0.00 | 0.05 | 0.468685 + 0.099131i | 0.468685 + 0.297394i | 0.466854 + 0.495657i | 0.466854 + 0.693919i |
| 0.02 | 0.01 | 0.05 | 0.448431 + 0.102519i | 0.448431 + 0.307558i | 0.448431 + 0.512596i | 0.448431 + 0.717635i |
| 0.02 | 0.02 | 0.05 | 0.431299 + 0.105201i | 0.431299 + 0.315602i | 0.431299 + 0.526004i | 0.431299 + 0.736406i |
| 0.02 | 0.03 | 0.05 | 0.415335 + 0.107293i | 0.415335 + 0.321880i | 0.415335 + 0.536467i | 0.415335 + 0.751054i |

| \( \alpha \) | \( \beta \) | \( e \) | QNMs for \( n = 0 \) | QNMs for \( n = 1 \) | QNMs for \( n = 2 \) | QNMs for \( n = 3 \) |
|---|---|---|---|---|---|---|
| 0.04 | 0.00 | 0.05 | 0.487794 + 0.094643i | 0.487794 + 0.283930i | 0.487794 + 0.473216i | 0.487794 + 0.662503i |
| 0.04 | 0.01 | 0.05 | 0.467862 + 0.098931i | 0.467862 + 0.296973i | 0.467862 + 0.494654i | 0.467862 + 0.692516i |
| 0.04 | 0.02 | 0.05 | 0.449367 + 0.102360i | 0.449367 + 0.307079i | 0.449367 + 0.511798i | 0.449367 + 0.716518i |
| 0.04 | 0.03 | 0.05 | 0.432170 + 0.105075i | 0.432170 + 0.315226i | 0.432170 + 0.525376i | 0.432170 + 0.735527i |
| 0.06 | 0.00 | 0.05 | 0.510501 + 0.089004i | 0.510501 + 0.267012i | 0.510501 + 0.445020i | 0.510501 + 0.623028i |
| 0.06 | 0.01 | 0.05 | 0.488885 + 0.094391i | 0.488885 + 0.283172i | 0.488885 + 0.471953i | 0.488885 + 0.660734i |
| 0.06 | 0.02 | 0.05 | 0.468874 + 0.098728i | 0.468874 + 0.296184i | 0.468874 + 0.493640i | 0.468874 + 0.691096i |
| 0.06 | 0.03 | 0.05 | 0.450307 + 0.102198i | 0.450307 + 0.306594i | 0.450307 + 0.510991i | 0.450307 + 0.715387i |

For consistency, the effective potential of the black hole, represented by equation (12) should satisfy some boundary conditions at the horizon and at infinity. Asymptotically flat spacetimes lead to the following quasinormal criteria:

\[
\psi(x) \rightarrow \begin{cases} 
P e^{i \omega x} & \text{if } x \to -\infty \\
Q e^{-i \omega x} & \text{if } x \to +\infty, 
\end{cases}
\]  

(15)

where \( P \) and \( Q \) denote constants of integration. Using these conditions we have calculated the QNM frequencies for our considered black hole. The results, i.e. the amplitude and damping of QNMs or the real and imaginary parts of QNM frequencies have been plotted against the GUP and the quintessence parameters as shown in Figs. 5 and 6. Fig. 5 shows the trends of variations of the real parts of the QNMs with respect to variations in \( \alpha, \beta \) and \( e \). It is clear that these trends are the same with the amplitude obtained by the Mashhoon method, but in this case the value of the amplitude is slightly greater than that obtained by the Mashhoon method. Whereas the damping of the QNMs shows the opposite behaviour with the Mashhoon method. However, it is to be noted that in WKB method, the variation of damping term is insignificant with respect to variation in all the model parameters.

Tables IV and V show a clear comparison between the Mashhoon method and the 6th order WKB method. Table IV is for the \( n = 0 \) case and Table V is for the \( n = 1 \). For \( n = 0 \) we found that the real part of the QNM calculated by the two methods agree to a good extent while the imaginary part of the modes vary to some extent. There is a far better agreement between two
FIG. 2: Behaviours of QNMs with respect to the GUP parameter $\alpha$ for three different values of $l$ with $n = 0$ and $\beta = 0.05$. The amplitude of QNMs increases with an increase in $\alpha$, while the damping decreases with increasing $\alpha$ (Mashhoon Method).

FIG. 3: Behaviours of QNMs with respect to the GUP parameter $\beta$ for three different values of $l$ with $n = 0$ and $\alpha = 0.05$. The amplitude decreases with an increase in $\beta$, while the damping increases with increasing $\beta$ (Mashhoon Method).

FIG. 4: Behaviours of QNMs with respect to the quintessence parameter $e$ for two different values of $l$ (first two plots with $n = 0$) and two different values $n$ (right plot with $l = 1$) with $\alpha = \beta = 0.05$. The amplitude decreases with an increase in $e$, while the damping increases with increasing $e$ (Mashhoon Method).
The notion of a minimal length scale does not exist in the normal Heisenberg algebra, but in the Planck energy scales, taking into consideration the effects of gravity, it becomes necessary. The introduction of the GUP is thus naturally motivated and leads to interesting results. The temperature of the Schwarzschild black hole is usually expressed in the form [101]:

$$T = \frac{k}{8\pi} \frac{dA}{dS},$$  \hspace{1cm} (16)
a change in entropy, which can have the smallest possible value of

\[ \Delta S = \begin{cases} \text{small} & \text{for large } \Delta \mathcal{H} \text{ and } \mathcal{H}_0, \\ \text{large} & \text{for small } \Delta \mathcal{H} \text{ and } \mathcal{H}_0. \end{cases} \]

\[ \lim_{r \to r_H} \int \frac{d^3 \mathbf{p}}{2\pi^2} \frac{1}{\sqrt{\mathcal{H}(r) - \mathcal{H}_0}} = \frac{1}{r_H} \left( 1 + \frac{3e_\omega}{r_H^{L+1}} \right), \]

(17)

where \( \kappa \) is the surface gravity of the black hole, \( A \) is the surface area and \( S \) is the entropy of the black hole. Calculations yield the expression for the surface gravity at the horizon in our case as

\[ \kappa = \lim_{r \to r_H} \sqrt{ -\frac{g_{11}}{g_{00}} \left( \frac{g^{00}}{g_{00}} \right)' } = \frac{1}{r_H} \left( 1 + \frac{3e_\omega}{r_H^{L+1}} \right), \]

where the GUP corrected horizon radius \( r_{h,GUP} \) of the black hole is denoted as \( r_H \). Liang [101] showed that the area of a black hole increases proportionately when it absorbs a particle of particular mass and size. A minimal change in area means a minimal change in entropy, which can have the smallest possible value of \( \ln 2 \) according to the information theory. So, we can express

| Multipole | \( \alpha \) | \( \beta \) | \( \epsilon \) | Mashhoon Method | 6th order WKB method | \( \Delta[\omega_M - \omega_{WKB}] \) |
|-----------|------------|------------|------------|-----------------|---------------------|---------------------|
| \( l = 1 \) | 0.01 | 0.01 | 0.03 | 0.274715 + 0.105993i | 0.274250 + 0.090298i | 5.7204 \times 10^{-3} |
| \( l = 1 \) | 0.01 | 0.01 | 0.05 | 0.263814 + 0.107754i | 0.265742 + 0.086685i | 5.4485 \times 10^{-3} |
| \( l = 1 \) | 0.01 | 0.03 | 0.05 | 0.240990 + 0.112803i | 0.248128 + 0.080939i | 5.0885 \times 10^{-3} |
| \( l = 1 \) | 0.05 | 0.01 | 0.05 | 0.291550 + 0.099263i | 0.287202 + 0.093685i | 5.8888 \times 10^{-3} |
| \( l = 2 \) | 0.01 | 0.01 | 0.03 | 0.454274 + 0.102489i | 0.453426 + 0.089402i | 3.5361 \times 10^{-3} |
| \( l = 2 \) | 0.01 | 0.01 | 0.05 | 0.439261 + 0.104011i | 0.439746 + 0.085840i | 3.3614 \times 10^{-3} |
| \( l = 2 \) | 0.01 | 0.03 | 0.05 | 0.407365 + 0.108192i | 0.410599 + 0.080150i | 3.1389 \times 10^{-3} |
| \( l = 2 \) | 0.05 | 0.01 | 0.05 | 0.478163 + 0.096792i | 0.475258 + 0.092772i | 3.6331 \times 10^{-3} |
| \( l = 3 \) | 0.01 | 0.01 | 0.03 | 0.634124 + 0.101454i | 0.633405 + 0.089164i | 2.5386 \times 10^{-3} |
| \( l = 3 \) | 0.01 | 0.01 | 0.05 | 0.614233 + 0.102908i | 0.614444 + 0.085861i | 2.4137 \times 10^{-3} |
| \( l = 3 \) | 0.01 | 0.03 | 0.05 | 0.571617 + 0.106832i | 0.573718 + 0.079941i | 2.2538 \times 10^{-3} |
| \( l = 3 \) | 0.05 | 0.01 | 0.05 | 0.666197 + 0.096067i | 0.664065 + 0.092530i | 2.6074 \times 10^{-3} |
| \( l = 4 \) | 0.01 | 0.01 | 0.03 | 0.814241 + 0.101021i | 0.813646 + 0.089066i | 1.9775 \times 10^{-3} |
| \( l = 4 \) | 0.01 | 0.01 | 0.05 | 0.789248 + 0.102448i | 0.789370 + 0.085524i | 1.8798 \times 10^{-3} |
| \( l = 4 \) | 0.01 | 0.03 | 0.05 | 0.735481 + 0.106263i | 0.737049 + 0.079855i | 1.7553 \times 10^{-3} |
| \( l = 4 \) | 0.05 | 0.01 | 0.05 | 0.854793 + 0.095764i | 0.853116 + 0.092430i | 2.0321 \times 10^{-3} |
looks insignificant. In all the cases, temperature seems to decrease with the increase in horizon radius. Thus, black holes with significant for black holes with small event horizon radii. Whereas for black holes with large event horizon radius influence and the surrounding quintessence field have some influence on the black hole temperature. In some cases the influence looks as mentioned. From this figure it is clear that introduction of quantum corrections due to the generalised uncertainty principle have some influence on the black hole temperature. This is illustrated by the plots of temperature vs horizon graphs shown in Fig. 7, which clearly shows the dependence on the horizon of the black hole depending on the values of $\alpha$ and $\beta$ [67], since the term inside the square root can not be negative. Thus, the factor $\epsilon$ is determined to be $4 \ln 2$ and we have the final form of the GUP-modified black hole temperature as

$$T_{GUP} = \frac{1}{2\pi} \left( 1 + \frac{3\epsilon w}{r_H^{3\omega+1}} \right) \left[ 1 - \sqrt{1 - \frac{4\beta}{(4r_H + \alpha)^2}} \right].$$

For $\omega = -\frac{1}{3}$, the expression for the GUP-corrected temperature can be simplified as

$$T_{GUP} = \frac{(4r_H + \alpha)}{2\pi \beta} \left[ 1 - \sqrt{1 - \frac{4\beta}{(4r_H + \alpha)^2}} \right].$$

It is interesting to note that the introduction of GUP corrections lead to the dependency of the temperature on the deformation parameters $\alpha$ and $\beta$, apart from the quintessence parameter $\epsilon$. In absence of the deformation parameters, the HUP-corrected temperature of the Schwarzschild black hole surrounded by quintessence is given by

$$T_{HUP} = \frac{1}{4\pi r_H} \left( 1 + \frac{3\epsilon w}{r_H^{3\omega+1}} \right).$$

Considering real-valued temperature, it can be seen from the above equation (20) that for $\omega = -\frac{1}{3}$, there exists some bounds on the horizon of the black hole depending on the values of $\alpha$ and $\beta$ [67], since the term inside the square root can not be negative. This is illustrated by the plots of temperature vs horizon graphs shown in Fig. 7, which clearly shows the dependence as mentioned. From this figure it is clear that introduction of quantum corrections due to the generalised uncertainty principle and the surrounding quintessence field have some influence on the black hole temperature. In some cases the influence looks significant for black holes with small event horizon radii. Whereas for black holes with large event horizon radius influence looks insignificant. In all the cases, temperature seems to decrease with the increase in horizon radius. Thus, black holes with
very large event horizon radii might have sub-zero temperatures associated with them. The temperature profiles obtained here are in good agreement with results presented in [67, 102].

Another interesting aspect of black hole physics is the study of the remnant formation, which is considered to be a stable state of the black hole that does not emit any heat and whose mass is reduced due to the evaporation. In this context the dependency of heat capacity of the black hole on the GUP parameters becomes an important feature that we want to analyse. For this purpose, we make use of the thermodynamic relation connecting the heat capacity $C$ of the black hole, its mass $M$ and temperature $T$ as given below:

$$C = \frac{dM}{dT}.$$  

From this relation, we derive the expression for the GUP-corrected heat capacity of the black hole as

$$C_{GUP} = -\frac{\pi \beta}{8 r_H} \left(4 \alpha (1 - e) + r_H \sqrt{\frac{(e-1)^2(4 \alpha + r_H)^2 - 64 \beta^2}{r_H^2} - e + 1}\right) \left((\alpha + 4 r_H)^2 - 4 \beta^2\right),$$  

(23)

where we have introduced a term $g$ defined as $g = \sqrt{1 - \frac{4 \beta^2}{(4 \alpha + r_H)^2}}$ and considered $\omega = -\frac{1}{2}$. Fig. 8 shows the variation of the heat capacity function (23) with the horizon radius for different values of $e$, $\alpha$ and $\beta$ considering $\omega = -\frac{1}{2}$ and $M = 1$. It is seen that the heat capacity is independent of the quintessence field, but depends heavily on the GUP-parameters $\alpha$ and $\beta$. This dependence is more pronounced on the parameter $\beta$, especially for small horizon radii black holes. In almost all the cases the heat capacity is significantly different from the case of the Schwarzschild black hole. The heat capacity is negative throughout, which is very large for the small horizon radii black holes as well as for the cases of large radii ones. Thus GUP-corrected black holes should lose more energy in the form of radiation than that of the Schwarzschild black hole, in particular, by small and large horizon radii black holes.

As already stated, when a black hole is not exchanging any heat with its surrounding, then we call this stable state as the remnant of the black hole. In this case, the heat capacity becomes zero. From the above expression, it can be shown that for the existence of remnant, the expression of the horizon radius of the remnant comes out as

$$r_{rem} = \frac{1}{4} \left(2 \sqrt{\beta} - \alpha\right).$$  

(24)

Thus, the remnant horizon radius depends on the deformation parameters $\alpha$ and $\beta$, and is independent of the behaviour of the surrounding field. It is interesting to note that in Ref. [67], this dependency was established with one parameter only. The expression for the remnant temperature is calculated as

$$T_{rem} = \frac{3 e \omega \left(\frac{\sqrt{\beta}}{2} - \frac{\alpha}{4}\right)^{-(3 \omega + 1)}}{\pi \sqrt{\beta}} + 1.$$  

(25)
For instance, the remnant temperature for a particular combination of $\alpha = 0.05$, $\beta = 0.05$, $e = 0.05$ and $\omega = -\frac{1}{2}$ comes out to be 1.352, which is above the upper limit of $T_{\text{GUP}} \leq 1.210$ for this case as obtained from the equation (21). This upper limit of $T_{\text{GUP}}$ is calculated from the condition that

$$\sqrt{1 - \frac{4\beta}{(4r_H + \alpha)^2}} \geq 0,$$

which gives minimum allowed horizon radius for this case as $\sim 0.1$. This implies that the GUP-corrected Schwarzschild black holes in our study can not reach the stage of the remnant, which is also clear from the heat capacity analysis above.

The entropy function of the black hole can be estimated from the thermodynamic relation,

$$S = \int \frac{dM}{T} \quad (26)$$

which, with the help of equations (8) and (21), can be expressed for the GUP-corrected black hole as

$$S_{\text{GUP}} = \frac{\pi M^2 ((a + 1)(\alpha + 4r_H)^2 - 4\beta \log((a + 1)(\alpha + 4r_H)))}{32(M^2 - 4\beta(e - 1)^2)} \quad (27)$$

Fig. 9 shows the variations of the entropy function (27) with respect to the horizon radius of the GUP-corrected black holes for different values of the model’s parameters together with the same for the Schwarzschild black holes. It is observed that the parameter $e$ and $\alpha$ have a very negligible impact on the entropy of the black holes. Whereas, the parameter $\beta$ has a significant impact on the entropy of black holes with sufficient horizon radii [67]. Moreover, the entropy of the GUP-corrected black holes is found to be higher than that of the Schwarzschild black holes for almost all cases and for sufficient horizon radius. This difference is substantial for a black hole with a larger horizon radius depending on the value of model parameter $\beta$.

![Variation of entropy of GUP-corrected black holes with respect to horizon radius for different values of $e$ with $\alpha = \beta = 0.05$ (left plot), for different values of $\alpha$ with $e = 0.1$ and $\beta = 0.05$ (middle plot), and for different values of $\beta$ with $e = 0.1$ and $\alpha = 0.05$ (right plot). In all three plots the solid red line represents the entropy of Schwarzschild black holes.](image)

IV. CONCLUSION

The primary objective of this work is to study the effects of the deformation parameters introduced by the GUP on the QNMs of oscillation of the Schwarzschild black holes, together with a brief review of the thermodynamic properties of such GUP-corrected black holes surrounded by a quintessence dark energy field. It has been observed that both the deformation parameters as well as the quintessence parameter play an important role on the behaviour of the QNMs of the black holes. We employed two methods for obtaining the QNMs, namely the Mashhoon method and the 6th order WKB method. Further we derive a GUP-modified temperature expression of the black holes and show its dependence on the deformation parameters as well as on the quintessence parameter. It is seen that there exist an upper bound on the temperature, which is impacted by the deformation parameters. Further, the heat capacity along with entropy have been evaluated for the GUP-corrected black holes and existence of black hole remnants has been studied. The existence of remnant radius and remnant temperature are certainly impacted by the deformation parameters. We observed that the GUP-corrected black holes can not reach the state of remnant. It is also seen
that the quintessence field and the first deformation parameter have no effective roles in the entropy of the black holes, which is dependent only on the second deformation parameter. It is quite remarkable that the introduction of small quantum corrections to the black hole metric can have notable influence on various properties of the black holes. This avenue has been investigated in the literature for many years but there are further scopes in this direction apart from the present study. It will be interesting to analyse the impact of these deformations on black hole properties, considering various Modified Theories of Gravity (MTGs).

It is to be noted that once perturbed, a black hole responds by radiating GWs, which evolve in time. This evolution process is divided into three phases, an instant outburst of radiation, a longer period of damped oscillations (QNMs) and at very late times, a suppressed power-law tail [2]. Since we have been able to detect the first phase of GW only, it remains a challenge for the physicists and engineers to develop and improve the sensitivity of modern day detectors so that the second phase of GW can be detected. Steps in this direction have already been undertaken in the means of ambitious future projects like the LISA [103] and the Einstein Telescope [104], which is believed to have far better sensitivity than the present detectors. The prospect of detection of QNMs by LISA has been analysed in Ref. [98]. The detection of QNMs of the black hole can have many useful implications. It can be used to constrain the GUP parameters and as testing grounds for various MTGs as well. These upcoming advanced detectors of GWs can shed more light into this field and provide data for validating various models available at present times, which is one of the primary objectives in this field of study.

Acknowledgement

UDG is thankful to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for the Visiting Associateship of the institute.

Appendix: Analytical form of expressions for QNM calculations by Mashhoon Method

The analytical formula for the QNM frequency calculation by Mashhoon method is as follows:

$$\omega(l, n) = \omega_R + i\omega_I = \left(V_0 - \frac{a^2}{4}\right)^{1/2} + ia\left(n + \frac{1}{2}\right),$$

where the expressions for the parameters $V_0$, $a$ and $x_0$ are found as

$$V_0 = -\frac{4A_1A_3(e - 1)^3\beta(l + 1)^3M^4}{A_2^4},$$

$$a = \sqrt{-\frac{A_7A_9\left(2\alpha A_8M^2 + 3H(e + l^2 + l - 1) + 4\beta(e - 1)^2MT + M^2T\right)}{A_5A_6}},$$

$$x_0 = \frac{1}{2\left(2elM^2 + 2eM^2 - 2l^2M^2 - 2lM^2\right)}\left[-24\beta e^3M - 24\beta e^2lM - 12\alpha e^2M^2 + 72\beta e^2M - \left[24\beta e^3M + 24\beta e^2lM + 24\beta e^2lM + 12\alpha e^2M^2 - 72\beta e^2M + 12\alpha e^2M^2 - 48\beta e^2M + 6eM^3 - 24\alpha eM^2 + 72\beta eM + 6l^2M^3 - 1024\beta e^2lM + 6l^2M^3 - 12\alpha M^2 + 24\beta M + 6l^2M - 12\alpha lM^2 + 24\beta lM^2 - 6M^3 + 12\alpha M^2 - 24\beta M^2 - 4(2lM^3 + 2elM^2 - 2l^2M^2 - 2lM^2)\left(256\beta^2 + 256\beta^2e^4 - 1024\beta^2e^3 + 256\alpha\beta e^3M + 1536\beta^2e^2 + 64\alpha^2e^2M^2 + 128\beta e^2M^2 - 768\alpha\beta e^2M - 1024\beta^2e + 64\alpha eM^3 - 128\alpha^2eM^2 - 256\beta eM^2 + 768\alpha\beta eM + 16M^4 - 64\alpha eM^3 + 64\alpha eM^2 + 128\beta M^2 - 256\alpha\beta M\right) - 12\alpha e^2M^2 + 48\beta e^2M^2 - 12\alpha M^2 + 48\beta e^2M - 6M^3 + 24\alpha eM^2 - 72\beta eM - 6l^2M^3 + 12\alpha lM^2 - 24\beta lM^2 - 6lM^3 + 12\alpha lM^2 - 24\beta lM + 6M^3 - 12\alpha M^2 + 24\beta M\right].$$
Here, we make use of various definitions as

\[
H = \left[(9 + 9e^2 + 14l + 23l^2 + 18l^3 + 9l^4 - 2e(9 + 3l + 7l^2))M^2 + 2(-1 + e)M\alpha + 4(-1 + e)^2\beta \right]^\frac{1}{2},
\]

\[
T = 9e^2 - 2e(7l^2 + 7l + 9) + 9l^4 + 18l^3 + 23l^2 + 14l + 9,
\]

\[
A_1 = (LM^3 + 2(3 + 3e^2 + l + l^2 - e(6 + l + l^2))M^2\alpha + 4(-1 + e)^2LM\beta + H),
\]

\[
A_2 = 6\alpha M^2\left(e^2 + e(l^2 + l - 2) - l^2 - l + 1\right) + 3M^3C + 12\beta(e - 1)^2MC + H,
\]

\[
A_3 = -2\alpha M^2\left(e^2 - e(3l^2 + 3l + 2) + 3l^2 + 3l + 1\right) + M^3J + 4\beta(e - 1)^2M
\]

\[
J + H,
\]

\[
A_4 = 4(e - 1)^3l^3(1 + l)^3M^4\left(-2\alpha M^2\left(e^2 - e(3l^2 + 3l + 2) + 3l^2 + 3l + 1\right) + 4\beta(e - 1)^2JM + H + JM^3\right)\left(2\alpha M^2(3e^2 - e(l^2 + l + 6) + l^2 + l + 3) + 4\beta(e - 1)^2ML + H + LM^3\right),
\]

\[
A_5 = \frac{2(-A_4)}{\left(6\alpha M^2\left(e^2 + e(l^2 + l - 2) - l^2 - l + 1\right) + 12\beta(e - 1)^2MP + H + 3M^3P\right)^4},
\]

\[
A_6 = \left(6\alpha M^2\left(e^2 + e(l^2 + l - 2) - l^2 - l + 1\right) + 3M^3P + 12\beta(e - 1)^2MP + H\right)^8,
\]

\[
A_7 = \left(6\alpha M^2\left(e^2 + e(l^2 + l - 2) - l^2 - l + 1\right) + M^3\left(e(4l^2 + 4l + 3) - l^2 - l - 3\right) + 1
\]

\[
2\beta(e - 1)^2M(e + l^2 + l - 1) + H\right)^2,
\]

\[
A_8 = 9e^3 - e^2(14l^2 + 14l + 27) + e(9l^4 + 18l^3 + 37l^2 + 28l + 27) - 9l^4 - 18l^3 - 23l^2 - 14l - 9,
\]

\[
A_9 = 64(e - 1)^5l^5(l + 1)^5M^5(4\beta(e - 1)^2 + 2\alpha(e - 1)M + M^2),
\]

\[
L = 3e - l^2 - l - 3,
\]

\[
P = c + l^2 + l - 1,
\]

\[
J = -c + 3l^2 + 3l + 1,
\]

\[
C = c + l^2 + l - 1.
\]

[1] For an introductory account, please refer to official LIGO website.
S. Masood et al., *The most general form of deformation of the Heisenberg algebra from the generalized uncertainty principle*, Phys. Lett. B 763, 218 (2016).

S. Pramanik et al., *Path integral quantization corresponding to the deformed Heisenberg algebra*, Annals of Physics 362, 24 (2015).

D.I. Kazakov and S. N. Solodukhin, *On quantum deformation of the Schwarzschild solution*, Nuclear Physics B 429, 1, 153 (1994).

D. J. Gogoi and U. D. Goswami, *Quasinormal Modes and Hawking Radiation Sparsity of GUP corrected Black Holes in Bumblebee Gravity with Topological Defects*, arXiv:2203.07594 [gr-qc] (2022).

V. H. Cáreanas et al., *Probing the parameters of a Schwarzschild black hole surrounded by quintessence and cloud of strings through four standard astrophysical tests*, EPJC 81, 866 (2021).

J. M. Todelo and V. B. Bezerra, *Black holes with quintessence in pure Lovelock gravity*, Gen. Relativ. Gravit 51, 41 (2019).

L. Xiang and X. Q. Wen, *A heuristic analysis of black hole thermodynamics with generalized uncertainty principle*, JHEP 10, 046 (2009).

R. J. Adler, P. Chen and D. I. Santiago, *The Generalized Uncertainty Principle and Black Hole Remnants*, Gen. Relativ. Gravit 33, 2101 (2001).

Official website of LISA.

Official website of ET.