SCALAR FIELD COSMOLOGIES WITH
PERFECT FLUID IN
ROBERTSON-WALKER METRIC

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Abstract
Several isotropic, homogeneous cosmological models containing a self-interacting minimally coupled scalar field, a perfect fluid source and cosmological constant are solved. New exact, asymptotically stable solutions with an inflationary regime or a final Friedmann stage are found for some simple, interesting potentials. It is shown that the fluid and the curvature may determine how these models evolve for large times.

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1 Introduction

A self-interacting scalar field has been introduced in cosmological models as a matter source to the Einstein equations because, when dominated by the potential energy, it violates the strong energy condition and drives the universe into an inflationary period \[1\],\[2\],\[3\]. Scalar fields also play a fundamental role in unified theories of the strong, weak and electromagnetic interactions through the mechanism of spontaneous symmetry breaking. Of particular interest for cosmology are those symmetries which spontaneously broken today were restored in the hot early universe. Thus, several kinds of topological defects may have been produced \[4\],\[5\]. Further, a scalar field has been proposed as a kind of dark matter \[6\].

Currently, there is no underlying principle that uniquely specifies the potential for the scalar field and many proposals have been considered. Some were based in new particle physics and gravitational theories \[1\]. Others were postulated ad-hoc to obtain the desired evolution. For instance, Ellis et al. \[7\] have proposed a scheme to find the potential function for a given expansion of a Robertson-Walker universe. Also, a formalism has been proposed to reconstruct the potential from knowledge of tensor gravitational spectrum or the scalar density fluctuation spectrum \[8\], \[9\].

Notwithstanding the number of papers devoted to understand the dynamics of the self-interacting scalar field in curved spacetimes, very little is known yet about exact solutions of these cosmological models. Most of these solutions correspond to models in a spatially flat Robertson-Walker metric which have no other source besides the scalar field \[1\], \[10\], \[11\], \[12\], \[13\], \[14\], \[15\], \[16\], \[17\], \[18\], \[19\]. Few exact solutions are known with spatial curvature \[20\], \[21\], \[22\], \[23\], \[24\]; or a perfect fluid \[6\], \[23\], \[24\]; and we are aware of none with both terms. The problem arises in the non-linearity of the system of differential equations for the scalar and gravitational fields. Usually, in studies of inflationary dynamics, the evolution is divided in several steps, and during each of them some kind of approximation is assumed to simplify the system of equations \[1\], \[2\].

In section 2 of this paper we show a procedure to reduce to quadratures the Einstein-scalar field equations in a Robertson-Walker metric for an arbitrary potential, a perfect baryotropic fluid source and a cosmological constant. This allows us to address the issue of whether the scalar field is always the dominating force in driving the evolution of the universe or the fluid
and curvature terms may also play a role. We analyse in section 3 general properties of the solutions and we study their asymptotic stability by means of the method of Lyapunov [25]. We consider also their structural stability. Our procedure is illustrated in section 4 with several examples. First, we use it to rederive and generalize a few solutions for the well known exponential potential, and then we present some new exact solutions for several simple, interesting potentials. Some of them are suitable for cosmological models based on the chaotic inflationary scenario. No a priori assumption like a slowly varying field is required to perform the calculations, so that we can check the validity of this assumption. Finally, the conclusions are stated in section 5.

2 The Model

We wish to investigate the evolution of a universe filed with a scalar field and a perfect fluid. The scalar field $\phi$ has a self-interaction potential $V(\phi)$ and is minimally coupled to gravity, so that it obeys the Klein-Gordon equation

$$\Box \phi + \frac{dV}{d\phi} = 0$$

The perfect fluid has four-velocity $u_i$ and its pressure $p$ and energy density $\rho$ are related by the equation of state

$$p = (\gamma - 1)\rho$$

with a constant adiabatic index $0 \leq \gamma \leq 2$. Thus, we must solve equation (1) together with the Einstein equations

$$R_{ik} - \frac{1}{2}g_{ik}R + \Lambda g_{ik} = T_{ik}^\phi + T_{ik}^f$$

We are using units such that $c = 8\pi G = 1$, $\Lambda$ is the cosmological constant and

$$T_{ik}^\phi = \phi_{,i}\phi_{,k} - g_{ik}\left[\frac{1}{2}\phi_{,m}\phi^{,m} - V(\phi)\right]$$

$$T_{ik}^f = (\rho + p)u_iu_k - pg_{ik}$$

are the stress-energy tensors of the field and the fluid.
In a Robertson-Walker metric

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  

(6)

with scale factor \( a(t) \) and curvature parameter \( k = 0, \pm 1 \), equations (1) and (3) become

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0
\]

(7)

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho - 3\frac{k}{a^2} + \Lambda
\]

(8)

where the dot means \( d/dt \), \( H = \dot{a}/a \) and \( \phi = \phi(t) \). Also, from the conservation of (3), \( \rho = \rho_0/a^3 \gamma \) where \( \rho_0 \geq 0 \) is a constant.

To integrate this system we make the change of variables \( dt = a^3 d\eta \) in (7)

\[
\frac{d^2\phi}{d\eta^2} + a^6 \frac{dV}{d\phi} = 0
\]

(9)

and we write

\[
V[\phi(a)] = \frac{F(a)}{a^6}
\]

(10)

Thus we obtain the first integral of (7)

\[
\frac{1}{2} \dot{\phi}^2 + V(\phi) - \frac{6}{a^6} \int da \frac{F}{a} = \frac{C}{a^6}
\]

(11)

where \( C \) is an arbitrary integration constant. Then, using the scale factor as the independent variable and equations (8) and (11), we have reduced the problem to quadratures:

\[
\Delta t = \sqrt{3} \int \frac{da}{a} \left[ \frac{6}{a^6} \int da \frac{F}{a} + \frac{C}{a^6} + \rho - 3\frac{k}{a^2} + \Lambda \right]^{-1/2}
\]

(12)

\[
\Delta \phi = \sqrt{6} \int \frac{da}{a} \left[ \frac{-F + 6 \int da F/a + C}{6 \int da F/a + C + \rho a^6 - 3ka^4 + \Lambda a^6} \right]^{1/2}
\]

(13)
where $\Delta t \equiv t - t_0$, $\Delta \phi \equiv \phi - \phi_0$ and $t_0$, $\phi_0$ are two other arbitrary integration constants.

Equations (10), (12) and (13) allows us to obtain a solution $a(t), \phi(t)$ for the potential $V(\phi)$, given a function $F(a)$, the parameters $\rho_0$, $\gamma$, $k$ and $\Lambda$, and a choice of the primitive $\int daF/a$ and the integration constant $C$.

3 General Behavior of Solutions

In general the set of solutions that arise from (12), (13) contains complex solutions without physical sense that must be rejected. So, to obtain physical solutions we require that the functions

$$G(a) = \frac{6}{a^6} \int da F/a + C/a^6 + \rho - 3 \frac{k}{a^2} + \Lambda \quad (14)$$

$$L(a) = -\frac{F}{a^6} + \frac{6}{a^6} \int da F/a + C/a^6 \quad (15)$$

are nonnegative on an interval $(a_1, a_2)$, $0 \leq a_1 < a_2 \leq \infty$. Since $G(a) = 3H^2$ and $L(a) = \dot{\phi}^2/2$, a sufficient condition is that $a > 0$, $\dot{a} \neq 0$ and $\dot{\phi} \neq 0$. To relax these conditions we need to study the behavior of the scale factor, which is determined by the function $G(a)$. Thus, a solution $a(t)$ is monotonic ($\dot{a} \neq 0$) when $G(a) > 0$ but it is bounded if $G(a_0) = 0$ for $0 < a_0 < \infty$. A growing monotonic solution starts from $a = 0$ and expands without bound. A solution is singular when $a(t)$ vanishes at a finite time, that is when the integral

$$T(a_1, a_2) = \int_{a_1}^{a_2} \frac{da}{a \sqrt{G(a)}} \quad (16)$$

converges in the limit $a_1 \to 0$. Let us assume that in this limit the leading behaviors of $F(a)$ and $\int daF/a$ are $\tilde{K}a^n$ and $\tilde{K}a^n/n$, where $\tilde{K} > 0$ and $n \neq 0$ are constants. So $G(a) \sim K/a^\alpha$ where $K > 0$ is another constant and $\alpha$ is the maximum of the exponents in (14), namely $(6 - n, 6, 3\gamma, 2)$. Then solutions are singular when $\alpha > 0$, which occurs when any of the following conditions are met: $C > 0$, $\rho_0 > 0$, $k = -1$ or $n < 6$. Singular solutions have particle horizons when $\alpha > 2$, that is when $C > 0$ or $n < 4$ or $\gamma > 2/3$. Similarly,
assuming \( F(a) \sim a^n \) for \( a \to \infty \), we find that no solution diverges at a finite time.

For bounded solutions, we observe that the scale factor reaches \( a_0 \) at a finite time \( t_0 \) when \( T(a_1, a_2) \) converges in the limit \( a_1 \to a_0 \). Let us assume that \( G(a) \sim K(a^\delta - a_0^\delta)^\beta, \beta > 0 \) and \( \delta \neq 0 \) constants, in a neighborhood of \( a_0 \). If \( \beta < 2 \) then \( a_0 \) is a local extremum of \( a(t) \) and the evolution is time-symmetric about \( t_0 \). Otherwise, for \( \beta \geq 2 \), \( a(t) \to a_0 \) when \( |t| \to \infty \).

Provided that the requirements for a physical solution are met, the freedom to select \( F(a) \) reflects the freedom to choose the potential. However, only in restricted cases the functions \( a(t), \phi(t) \) and \( V(\phi) \) arise in closed explicit form.

Though the solution (12) (13) of the system of equations (7) (8) has three constants of integration, it does not provide the general solution for a given potential. Rather it provides a special solution for each member of a family of potentials. So, for each potential we need to study the stability of this solution.

For open or flat models such that \( V(\phi) \) has a local minimum at \( \phi_m \) and \( \Lambda_{eff} \equiv V(\phi_m) + \Lambda \geq 0 \), we can study the stability of solutions with asymptotic behavior \( \phi(t) \to \phi_m \). First we note that \( \phi = \phi_m \) and \( a(t) \) given by

\[
\Delta t = \sqrt{3} \int \frac{da}{a} \left( \rho - \frac{3k}{a^2} + \Lambda_{eff} \right)^{-1/2}
\]  

(17)

is a classical solution of (7) and (8) with the effective cosmological \( \Lambda_{eff} > 0 \) ([20]). \( a(t) \) may not be given by (17) when \( \Lambda_{eff} = 0 \).

To study the asymptotic stability of a solution it becomes convenient to turn it into a fixed point. For the models under consideration \( a(t) \) is monotonic we may use \( a \) as the independent variable. Then we can restrict the analysis to the phase space \( (\phi, \phi') \) and study the stability of the point \( (\phi_m, 0) \). We find that the energy density of the field

\[
\rho_\phi = V(\phi) + \frac{1}{2} \phi'^2
\]  

(18)

is a suitable a Lyapunov function as it satisfies \( \rho_\phi(\phi, \phi', a) > -\Lambda \) and \( \rho_\phi' = -3aH^2\phi^2 < 0 \) in a neighborhood of \( (\phi_m, 0) \). So this point is an attractor and any solution such that \( \phi \to \phi_m \) for \( a \to \infty \) (equivalently \( t \to \infty \)) is asymptotically stable. In this limit, when \( \Lambda_{eff} > 0 \), equation (17) gives the leading behavior of \( a(t) \). Since
\[ \dot{H} = -\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \gamma \rho + \frac{k}{a^2} \tag{19} \]

is negative we note that these solutions have a deflationary behavior.

On the other hand, in closed models some solutions are non-monotonic. However, for any solution such that the scale factor is monotonic starting from a given time, we may apply the same analysis as before about its stability. Also, for \( k = 1 \) we have the scalar field analog of the Einstein Universe [26]. In effect using (4), (8) and (19) we find for a nonconstant potential a unique static solution \( a = a_E > 0 \) and \( \phi = \phi_E \), provided that \( V'(\phi_E) = 0 \),

\[ V(\phi_E) = \frac{3\gamma - 2}{\gamma a_E^2} - \Lambda \tag{20} \]

and \( \gamma \rho_0 = 2a_E^{3\gamma - 2} \). In the case \( \gamma = 2/3 \) there is no particular equilibrium density and there is a static solution for any \( a_E \). To study the stability of the static solution we consider small perturbations about it. In the linear regime the system decouples and we find that the Einstein solution is stable when \( \gamma < 2/3 \) and \( V''(\phi_E) > 0 \), and it is unstable otherwise.

Recently it has been considered the question whether small variations in the form of the potential induce small variations in the scale factor and vice versa. A model with this property is called structurally stable or "rigid" [27]. In the case that \( k = \Lambda = \rho = 0 \), the rigidity of inflationary models has been shown [28]. We wish to investigate whether this property is also valid without these restrictions. Let us consider a family of functions \( F(a) \) labeled by a parameter \( r \). If \( F(a,r) \) is a continuous function and \( a > 0 \), using (10) and (13), a small change \( \delta r \) induces a small change \( \delta V \) in \( V(\phi) \). Also, using (12), it induces a small change \( \delta a \) in \( a(t) \). Clearly, small neighbourhoods in the space of scale factors and in the space of potentials map continuously into each other. Thus we conclude that models where \( F \) is continuous are structurally stable.

### 4 Examples

In this section we obtain several new exact solutions for simple interesting potentials and we concentrate mainly on models with spacial curvature and a fluid source since very few exact solutions are known for them.
4.1 Exponential and exponential-like potentials

We consider first potentials of the form

\[ V(\phi) = B \exp(-\sigma \Delta \phi) \]  

(21)

Exponential potentials like (21) have been extensively considered by many authors, mainly in the context of “power-law” inflationary models. We rederive and generalize exact solutions for this potential. For this we take

\[ F(a) = B a^s \]  

(22)

where \( B \) and \( s \) are constants, we choose \( \int daF/a = Ba^s/s \) with \( C = 0 \) and we obtain a real solution when \( 0 < s \leq 6 \) for \( B > 0 \). Since \( s = 6 \) corresponds to a constant potential, we consider this case no further. For \( s < 6 \), the energy density of the scalar field redshifts as a perfect fluid with equation of state \( p_\phi = (\gamma_\phi - 1)\rho_\phi / a^n \), where \( \rho_0 = 6B/s \) and \( n = 3\gamma_\phi = 6 - s \) [6]. Using (11) and (12) we find a power-law solution of the form

\[ a(t) = (K \Delta t)^\lambda \quad \Delta \phi = M \ln a \]  

(23)

with some constants \( K \), \( M \) and \( \lambda \) which are functions of the parameters of the system.

Case 1. \( \rho_0 = k = \Lambda = 0 \).

\[ \sigma = (6 - s)^{1/2} \quad K = \left( \frac{B}{2s} \right)^{1/2} \quad (6 - s) \quad \lambda = \frac{2}{6 - s} \quad M = (6 - s)^{1/2} \]

This solution has been found by Lucchin et al. [10], its asymptotic stability has been analysed by Halliwell [20] and the general solution for this model has been found by Salopek et al. [15].

Case 2. \( \rho = \Lambda = 0 \), \( s = 4 \), \( k = 0 \), \(-1 \) or \( k = 1 \) with \( B > 2 \).

\[ \sigma = \left[ \frac{2}{B}(B - 2k) \right]^{1/2} \quad K = \left( \frac{B}{2} - k \right)^{1/2} \]

\[ \lambda = 1 \quad M = \left( \frac{2B}{B - 2k} \right)^{1/2} \]
This solution has been found by Ellis et al. [7] (see also [20] for $k = -1$).

Case 3. $k = \Lambda = 0$, $s = 6 - 3\gamma$. 

$$\sigma = \left[ (6 - s) \left( 1 + \frac{\rho_0 s}{6B} \right) \right]^{1/2} \quad K = \left( \frac{6B + s \rho_0}{3s} \right)^{1/2} \frac{6 - s}{2}$$

$$\lambda = \frac{2}{6 - s} = \frac{2}{3\gamma} \quad M = \left[ \frac{6B(6 - s)}{6B + \rho_0 s} \right]^{1/2}$$

This solution has been found by Ratra et. al [6] in the case $\gamma = 1$, and we extend it here to arbitrary $\gamma$. It generalizes case 1 for a fluid source and for $\gamma > 2/3$ it is not inflationary and has particle horizons. It is usually assumed that an exponential potential with $\sigma$ small enough leads to a power-law inflationary evolution ($\lambda > 1$), so that any matter term becomes rapidly dominated by the scalar field energy density. However in our exact solution, which is asymptotically stable, the energy densities of the field and the fluid keep a constant ratio along the evolution.

Another interesting generalization of case 1 arises from a nonvanishing constant $C$, which yields a family of exponential-like potentials. First we note that $C \geq 0$ for a real solution. Then, using (22), (13) and (10) we find the potential in parametric form

$$\Delta \phi(a) = \frac{\sqrt{6}}{s} \left\{ \ln a^s + 2 \left( \frac{6 - s}{6} \right)^{1/2} \ln \left[ (a^s + c_1)^{1/2} + (a^s + c_2)^{1/2} \right] - 2 \ln \left[ \left( \frac{6 - s}{6} \right)^{1/2} (a^s + c_1)^{1/2} + (a^s + c_2)^{1/2} \right] \right\}$$

$$V(a) = B a^{s-6}$$

where $c_1 = sC/((6 - s)B)$ and $c_2 = sC/(6B)$. The leading behavior of this potential is (21) with $\sigma = -(6 - s)^{1/2}$ for $\phi \to \infty$, while $\sigma = -(6 - n)/\sqrt{6}$ for $\phi \to -\infty$.

When $C > 0$ we find that

$$\Delta t = \frac{1}{3} \left( \frac{s}{6Bc_1} \right)^{1/2} a^3 F_1 \left( \frac{1}{2}, 1 + \frac{3}{s}, 1, -a^s \right)$$

where $F_1$ is the hypergeometric function.
where $\binom{2}{1}$ is the hypergeometric function \([29]\). These solutions also begin at a singularity, this time like $a \sim \Delta t^{1/3}$ for $\Delta t \to 0$, while their asymptotic behavior is like case 1 for $t \to \infty$. Though these solutions have some initial conditions fixed, we note that they are asymptotically stable.

A common framework to solve equations (7), (8) in discussions of inflation is the "slow-roll" approximation. However, as it has not been required by our procedure of integration, we can use it to investigate the limitations imposed by this approximation. Following Copeland et al. \([8]\), and using equations (10) (11) we calculate the slow-roll parameters

$$
\epsilon \equiv \frac{\dot{\phi}^2/2}{V + \phi^2/2} = 1 - \frac{F}{C + 6 \int dF/a} = \left(\frac{6 - s}{6}\right) \frac{a^s + c_1}{a^s + c_2} \tag{27}
$$

$$
\eta \equiv \frac{\ddot{\phi}}{H \dot{\phi}} = \frac{12 F - a F' - 36 \int dF/a - 6 C}{2 (C + 6 \int dF/a - F)} = \frac{(s - 6) a^s - 6 c_1}{2 (a^s + c_1)} \tag{28}
$$

The parameter $\epsilon$ measures the relative contribution of the field’s kinetic energy to its total energy density and $\eta$ measures the ratio of the field’s acceleration relative to the friction term. The slow-roll approximation is valid when $|\epsilon| \ll 1$ and $|\eta| \ll 1$. These requirements impose constrains on the form of the potential and the value of the initial conditions.

In this case, for $C = 0$, the slow-roll constrain reads $6 - s \ll 1$, that is, this approximation is valid only when the potential has a very mild slope and $\lambda \gg 1$. Moreover, for $C > 0$, the evolution near the singularity is dominated by the kinetic energy. Thus, in general, these solutions cannot be obtained under the slow-roll approximation.

### 4.2 Cosh and cosh-like potentials

We consider now potentials of the form

$$V(\phi) = V_0 [\cosh (\sigma \Delta \phi)]^q \tag{29}$$

with $V_0 > 0$. This potential has a minimum at the origin when $q > 0$ or a maximum when $q < 0$ with a value $V_0$. For large $|\phi|$ it grows or vanishes exponentially. The case $q < 0$ appears when reconstructing the potential in the assumption of small amplitude of the tensor gravitational wave spectrum.
For $q > 0$ we have the stable solution $\phi = \phi_0$ and $a(t)$ given by (17) with $\Lambda_{\text{eff}} = V_0$. We consider below other solutions for this potential.

**Case 1.** $F(a) = B a^s (b + a^s)^n$, where $B > 0$, $b > 0$, $s$ and $n$ are constants such that $s(n + 1) = 6$, $s \neq 0$, $s \neq 6$, and $\rho_0 = k = \Lambda = C = 0$.

$$V_0 = B \quad \sigma = \frac{s}{2 \sqrt{6}} \quad q = 2 \left( \frac{6}{s} - 1 \right)$$

We obtain the time evolution for arbitrary $s$ in implicit form:

$$\Delta t = \frac{a^3}{\sqrt{3} B b^{3/s}} 2 F_1 \left( \frac{3}{s}, \frac{3}{s}, 1 + \frac{3}{s}, -\frac{a^s}{b} \right)$$

$$\Delta \phi = -\frac{2 \sqrt{6}}{s} \text{arccoth} \left( 1 + \frac{a^s}{b} \right)^{1/2}$$

It is suitable for further analysis to consider explicit expressions for the leading behaviour of $a(t)$ and $\phi(t)$ in the limit of small and large $a^s/b$

$$a(t) \simeq \left( \sqrt{3} B b^{3/s} \Delta t \right)^{1/3}, \quad a^s \ll b$$

$$a(t) \simeq \exp \left[ \frac{1}{2} \sqrt{\frac{B}{3}} \Delta t \right], \quad a^s \gg b$$

$$\Delta \phi \simeq \left( \frac{2}{3} \right)^{1/2} \ln \Delta t, \quad a^s \ll b$$

$$\Delta \phi \simeq -\frac{2 \sqrt{6} b}{s} \exp \left[ -\frac{s}{2} \left( \frac{B}{3} \right)^{1/2} \Delta t \right], \quad a^s \gg b$$

The calculation of the slow-roll parameters

$$\epsilon = \frac{b}{b + a^s}$$

$$\eta = -\frac{1}{2} \left( \frac{6 b + s a^s}{b + a^s} \right)$$

shows that the slow-roll approximation is valid only for $a^s \gg b$ if $|s| \ll 1$. So these are new solutions which cannot be obtained under such an approximation.
We may describe the behavior of these solutions as follows. When \(0 < s < 6\), \(a(t)\) begins at a singularity with the behavior (32) for \(\Delta t \to 0\) and is asymptotically de Sitter like (33) for \(\Delta t \to \infty\). The evolution of the field \(\phi(t)\) starts dominated by a diverging kinetic energy and settles down at the minimum of the potential for large times driving the exponential expansion. When \(s < 0\) the evolution begins like (33) in the remote past and then deflates into the asymptotic Friedmann behavior (32) in the far future. The field begins rolling down very slowly from the maximum of the potential. Then it gains kinetic energy that decays much more slowly than the potential energy. As it becomes dominant, it drives the deflationary era and makes the field grow without bound. We provide in this way an exact realization of the deflationary universe scenario proposed recently by Spokoiny [30]. We note also that for either sign of \(s\), these solutions are stable. Finally, when \(s > 6\), the scale factor behaves the same as the case \(0 < s < 6\), but the field begins its evolution dominated by the kinetic energy and approaches very slowly to the maximum of the potential for large times. This is an unstable solution.

Case 2. \(F(a) = Ba^s, \Lambda = C = 0, s = 6 - 3\gamma, 0 < \gamma < 2, k = 1\).

\[
V_0 = B \left[ \frac{1}{3} \left( \frac{2B}{2 - \gamma} + \rho_0 \right) \right]^{3\gamma-2} 
\]

\[
\sigma = \frac{3\gamma - 2}{2} \left\{ \frac{1}{6\gamma} \left[ 2 + \frac{\rho_0}{B} (2 - \gamma) \right] \right\}^{1/2} \quad q = \frac{6\gamma}{3\gamma - 2}
\]

The time evolution is given by

\[
\Delta t = \frac{2}{3\gamma} \left( \frac{\omega}{B} \right)^{1/2} a^{3\gamma/2} F_1 \left( \frac{1}{2}, \frac{3\gamma}{6\gamma - 4}; \frac{3}{2} - \frac{1}{2 - 3\gamma}; \omega a^{3\gamma - 2} \right) \quad (38)
\]

\[
\Delta \phi = -\frac{1}{\sigma} \arctanh \left( 1 - \omega a^{3\gamma - 2} \right)^{1/2} \quad (39)
\]

where

\[
\omega = 3 \left( \frac{2}{2 - \gamma} + \frac{\rho_0}{B} \right)^{-1}
\]

For \(0 < \gamma < 2/3\) \((q < 0)\), the scale factor has a bounce and its asymptotic behavior for large times is \(t^{2/(3\gamma)}\). For \(2/3 < \gamma < 2\) \((q > 0)\), the evolution
starts from a singularity with the leading behavior $\Delta t^{2/(3\gamma)}$, it reaches a maximum and then collapses again. We note that for the physically interesting case of radiation $\gamma = 4/3$, the solution takes a simple closed form:

$$a(t) = \left( B + \frac{\rho_0}{3} - \Delta t^2 \right)^{1/2} \quad (40)$$

Case 3. $F(a) = Ba^s$, $\rho_0 = \Lambda = C = 0$, $0 < s < 6$, $s \neq 4$, $k = 1$.

$$V_0 = B \left( \frac{s}{2B} \right)^{\frac{6-s}{4-s}} \quad \sigma = \frac{4 - s}{2\sqrt{6-s}} \quad q = \frac{2(6-s)}{4-s}$$

The time evolution is given by

$$\Delta t = \left( \frac{2s}{B} \right)^{1/2} a^{3-s/2} 6-s \quad _2F_1 \left( \frac{1}{2}, \frac{s-6}{2(s-4)}; \frac{3s-14}{2(s-4)}; \frac{sa^{4-s}}{2B} \right) \quad (41)$$

$$\Delta \phi = -\frac{1}{\sigma} \text{arctanh} \left( 1 - \frac{sa^{4-s}}{2B} \right)^{1/2} \quad (42)$$

For $4 < s < 6$ ($q < 0$), the scale factor has a bounce and its asymptotic behavior for large times is $t^{2/(6-s)}$. For $0 < s < 4$ ($q > 0$), the scale factor starts from a singularity with the leading behavior $\Delta t^{2/(6-s)}$, it reaches a maximum and then collapses again.

We show here another potential that have qualitatively the form (29):

$$V(\phi) = B \left[ \frac{\epsilon \exp(\sigma \Delta \phi)}{\left( 2+3\gamma \right)} (1+3\epsilon kB \exp(\sigma \Delta \phi) \right)^{\frac{3\gamma-4}{2-3\gamma}} - \frac{\rho c}{B} \exp(2\sigma \Delta \phi) \right]^{\frac{3\gamma-4}{2-3\gamma}} \quad (43)$$

where $\epsilon = \pm 1$ and

$$\sigma = \frac{2-3\gamma}{\sqrt{(4-3\gamma)}} \quad (44)$$

They arise in the case that $F(a) = Ba^s$, $\Lambda = C = 0$, $s = 2 + 3\gamma$ and $k = 1$. There is a critical energy density coefficient $\rho_c = 3(2 + 3\gamma)/(8B)$ and for $\rho_0 < \rho_c$ we find a cosh-like potential with $q > 0$ when either $\gamma < 2/3$.
(\epsilon = 1), \ 2/3 < \gamma < 4/3 (\epsilon = -1) or 2/3 < \gamma < 4/3 (\epsilon = 1), this time \ q < 0. The scale factor starts from a singularity with the leading behavior \ \Delta t^{2/(4-3\gamma)}, it reaches a maximum and then recollapses again.

We see from our examples that we cannot discard a priori the influence of a curvature term in the determination of the evolution, since a potential that leads in the flat case to an asymptotically expanding scale factor yields recollapsing solutions when \ k = 1.

4.3 Sinh and sinh-like potentials

We consider potentials of the form

\[ V(\phi) = V_0 |\sinh(\sigma \Delta \phi)|^q \]

with \ V_0 > 0. This potential has a vanishing minimum at the origin when \ q > 0 or diverges when \ q < 0. For large \ |\phi| it grows or vanishes exponentially. This kind of potentials satisfy, for suitable values of \ q > 0, the requirements usually imposed in the chaotic inflationary scenario \ [4]. We study several exact solutions for this potential.

Case 1. \ F(a) = Ba^s, \ \Lambda = C = 0, \ s = 6 - 3\gamma, \ 0 < \gamma < 2, \ k = -1.

\[ V_0 = B \left[ \frac{1}{3} \left( \frac{2B}{2 - \gamma} + \rho_0 \right) \right]^{\frac{3\gamma}{2}} \]

\[ \sigma = \frac{3\gamma - 2}{2} \left\{ \frac{1}{6\gamma} \left[ 2 + \frac{\rho_0}{B} (2 - \gamma) \right] \right\}^{1/2} \quad q = \frac{6\gamma}{3\gamma - 2} \]

The time evolution is given by

\[ \Delta t = a_{2F1} \left( \frac{1}{2}, \frac{1}{2 - 3\gamma}, 1 + \frac{1}{2 - 3\gamma}, -\frac{a^{2-3\gamma}}{\omega} \right) \]

\[ \Delta \phi = -\frac{1}{\sigma} \arccoth \left( 1 + \omega a^{3\gamma - 2} \right)^{1/2} \]

The evolution of the scale factor for these models starts from a singularity as \ \Delta t^{\lambda_1} for \ \Delta t \to 0 and grows monotonically with asymptotic behavior \ t^{\lambda_2} for \ t \to \infty. In this case, for \ 0 < \gamma < 2/3, \ \lambda_1 = 1 \ and \ \lambda_2 = 2/(3\gamma) > 1, while for \ 2/3 < \gamma < 2, \ \lambda_1 = 2/(3\gamma) < 1 \ and \ \lambda_2 = 1.
Case 2. $F(a) = B a^s, k = \Lambda = C = 0, 0 < s < 6, 6 - 3\gamma - s \neq 0$.

$$V_0 = B \left( \frac{s \rho_0}{6B} \right)^\frac{6 - s}{6 - 3\gamma - s} \sigma = \frac{6 - 3\gamma - s}{2\sqrt{6 - s}} \quad q = \frac{2(6 - s)}{6 - 3\gamma - s}$$

The time evolution is given by

$$\Delta t = \left( \frac{2s}{B} \right)^{1/2} \frac{a^{3-s/2}}{6 - s} \left( 2F_1 \left( -\frac{6 - s}{2}, -\frac{3(6 - 2\gamma - s)}{2(6 - 3\gamma - s)}, -\frac{s \rho_0}{6B a^{6-3\gamma-s}} \right) \right)$$

$$\Delta \phi = -\frac{1}{\sigma} \arccoth \left( 1 + \frac{s \rho_0}{6B a^{6-3\gamma-s}} \right)^{1/2}$$

For $6 - 3\gamma - s > 0, q > 0$, $\lambda_1 = 2/(6 - s)$ and $\lambda_2 = 2/(3\gamma)$. Then, $\lambda_1 < \lambda_2$ but this is not an inflationary solution unless $\gamma < 2/3$. For $6 - 3\gamma - s < 0, q < 0$, $\lambda_1 = 2/(3\gamma)$ and $\lambda_2 = 2/(6 - s)$. So $\lambda_1 < \lambda_2$, and this model may describe the passage from a primordial radiation-dominated era to a power-law inflationary stage.

Case 3. $F(a) = B a^s, \Lambda = C = 0, s = 4, k = 0, -1$ or $k = 1$ if $B > 2$.

$$V_0 = B \left[ \frac{3}{\rho_0} - \frac{B}{2} - k \right]^{1/2} \sigma = \frac{3\gamma - 2}{2} \left( \frac{1}{2} - \frac{k}{B} \right)^{1/2} \quad q = \frac{4}{2 - 3\gamma}$$

The time evolution is given by

$$\Delta t = \frac{a}{\left( \frac{B}{2} - k \right)^{1/2}} \left( 2F_1 \left( -\frac{1}{2}, -\frac{1}{2 - 3\gamma}, -\frac{3(1 - \gamma)}{2 - 3\gamma}, -\omega a^{2-3\gamma} \right) \right)$$

$$\Delta \phi = -\frac{1}{\sigma} \arccoth \left( 1 + \omega a^{2-3\gamma} \right)^{1/2}$$

For $\gamma < 2/3, q > 0$ and we find $\lambda_1 = 1$ and $\lambda_2 = 2/(3\gamma) > 1$. For $\gamma > 2/3, q < 0$, $\lambda_1 = 2/(3\gamma)$ and $\lambda_2 = 1$.

Case 4. $F(a) = B a^s, \rho_0 = \Lambda = C = 0, 0 < s < 6, s \neq 4, k = -1$.

$$V_0 = B \left( \frac{s}{2B} \right)^{\frac{6 - s}{6 - s}} \sigma = \frac{4 - s}{2\sqrt{6 - s}} \quad q = \frac{2(6 - s)}{4 - s}$$
The time evolution is given by

$$\Delta t = a_2 F_1 \left( \frac{1}{2}, \frac{1}{s-4}, \frac{s-3}{s-4}, -\frac{2B}{s} a^{s-4} \right)$$  \hspace{1cm} (52)$$

$$\Delta \phi = -\frac{1}{\sigma} \text{arccoth} \left( 1 + \frac{s}{2B} a^{4-s} \right)^{\frac{1}{2}}$$  \hspace{1cm} (53)$$

For $0 < s < 4$ $q > 0$, $\lambda_1 = 2/(6 - s) < 1$ and $\lambda_2 = 1$. For $4 < s < 6$, $q < 0$, $\lambda_1 = 1$ and $\lambda_2 = 2/(6 - s) > 1$, so that the evolution is inflationary.

The expression (43) ($\epsilon = 1$) also yields potentials of the sinh form, with $q > 0$ for $\rho_0 > \rho_c$. They occur when $k = 1$ and $\gamma < 2/3$, or $k = -1$ and $0 < \gamma < 4/3$, $\gamma \neq 2/3$. The scale factor starts from a singularity and grows monotonically. For $0 < \gamma < 2/3$ $\lambda_1 = 2/(4 - 3\gamma) < 1$ and $\lambda_2 = 2/(3\gamma) > 1$, while for $2/3 < \gamma < 4/3$ $\lambda_1 = 2/(3\gamma) < 1$ and $\lambda_2 = 2/(4 - 3\gamma) > 1$. In either case there are particle horizons and the evolution is inflationary.

We note that all the solutions presented for potential (43) are asymptotically stable. Many of them show the influence of the fluid source on their behavior, either near the singularity or for large times.

5 Conclusions

We have shown that the Einstein equations with a self-interacting minimally coupled scalar field, a perfect fluid source and cosmological constant can be reduced to quadratures in the Robertson-Walker metric. For that, the scale factor is taken as the independent variable and the potential is expressed in terms of it. This is fully justified as long as $a(t)$ and $\phi(t)$ are monotonic functions. For this class of cosmological models we have reobtained in a unified manner many exact solutions presented by other authors, and found several new exact solutions for some simple interesting potentials. We have investigated their behavior in detail, and shown that most of the solutions that do not recollapse in a finite time are asymptotically and structurally stable.

For a given potential, the influence of a fluid source or the spatial curvature may dominate the evolution of the early universe. In effect, an asymptotically inflationary behavior cannot occur when the energy density of the scalar field decays faster than the energy density of a fluid with $\gamma > 2/3$. We
have shown asymptotically stable solutions such that both densities keep a constant ratio, or one of them dominates for large times. All the same, a negative spatial curvature may lead to a linearly expanding scale factor. On the other hand, a positive curvature has frequently as a consequence bouncing solutions or solutions that undergo recollapse to a second singularity. In such cases, the energy density of the scalar field may increase, instead of rolling down the potential.

We have studied the limitations imposed by the slow-roll approximation and most of our solutions cannot be obtained by means of this approximation. Calculations of the primordial spectrum of perturbations, based on recent measurements of the cosmological background radiation show that tensorial perturbations may have played an important role in the formation of cosmic structures. However, the amount of gravitational perturbations predicted by means of the slow-roll approximation is very small [31]. Thus exact solutions which lay outside the slow-roll regime may lead to an improved understanding of the evolution of the early universe.
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