PHYSICS BEYOND THE LIMITS OF UNCERTAINTY RELATIONS

MICHO ĐURĐEVICH

To see the world in a grain of sand,
And a heaven in a wild flower;
Hold infinity in the palm of your hand,
And eternity in an hour.

1. Introduction

Quantum mechanics is a physics of microworld. Its principal aim is to provide a mathematically coherent picture of physical reality at the deepest possible experimentally accessible level—the quantum level. This means understanding phenomenas involving elementary particles and quantas of interactions, and the study of the internal structure of matter and fields.

Quantum world is very different from the picture given by classical mechanics, which is a macroworld physics.

One of the principal purely quantum phenomenas is complementarity. If we consider all possible properties/attributes of a given quantum system then it turns out that for every state of the system there exists an infinite collection of properties that are ontically inapplicable to the system in this particular state. This means that in such situations an attribute from the mentioned collection does not hold, and also its negation does not hold.

This is very different from classical mechanics, where every possible attribute of a physical system has a definitive (0 or 1) value in every state. In other words, either the attribute holds or its negation holds in a given state of the system.

All the properties of a classical system form a Boolean algebra. All possible attributes of a quantum system form an essentially different structure—a non-distributive lattice.

We can also say that quantum systems are never completely understandable with the help of a single system of classical-type attributes. On the other hand, in every concrete experimental context, the subset consisting precisely of the system attributes actualized in this context necessarily forms a classical Boolean algebra. The situation is somehow similar to the relation between Euclidean geometry—where the space is covered by a single coordinate system, and general Riemannian geometry—where it is only possible to cover the whole space by the atlas of local coordinate systems, each describing a portion of the space.

Another fundamental difference between classical and quantum mechanics is that quantum mechanics is an intrinsically stochastical theory. In other words, probability concepts are incorporated in the very roots of the quantum theory. Even if we know everything about a quantum-mechanical system, it is still not

William Blake, Auguries of Innocence
possible to predict with certainty the outcomes of all measurements performed on this system.

Accordingly, physical quantities (observables) do not possess definitive values in quantum states. More precisely, if a quantum system is described by a Hilbert space $H$ then the possible states of the system are described by unit vectors $\psi \in H$, (and it is assumed that unit vectors that differ up to a phase factor describe the same physical state). Physical observables are described by selfadjoint operators $F$ acting in $H$. The mean value of quantum observables in states are then computed by the standard quantum-mechanical formula:

$$\langle F \rangle_\psi = \langle \psi, F\psi \rangle.$$

This situation is in a sharp contrast with classical mechanics, which is a completely causal theory. The states of a system are in one-to-one correspondence with points $\omega \in \Gamma$ of the phase space $\Gamma$, and physical observables are interpreted as the appropriate functions defined on this space. The values of observables $f: \Gamma \to \mathbb{R}$ in the states $\omega$ are simply the numbers $f(\omega)$. In classical mechanics, the probabilities appear as something secondary, whenever we do not know exactly the state of the system. A standard example is given by classical statistical mechanics, which studies very complicated physical systems and so it is effectively impossible to determine the state of the system. Accordingly in classical statistical mechanics the system is effectively described by a probability distribution over the phase space (for example—the canonical ensemble distribution).

It is natural to ask the following question—is it possible to explain the stochasticity of quantum mechanics as a simple consequence of an incompleteness of quantum theory? In other words, as a simple consequence of the fact that quantum theory does not include certain deeper parameters ($\Leftrightarrow$ hidden variables), which if included in the game would re-establish causality as in the classical physics.

In such a way we arrive to the idea of subquantum mechanics.

2. Subquantum Mechanics

By subquantum mechanics we understand a physical theory based on the following ideas:

**Individual-system interpretation**

The theory describes individual physical systems, and goes deeper than quantum mechanics in formulating the picture of physical reality.

**Causality property**

In the framework of the theory, a concept of a subquantum space is defined. The elements of this space are in one-to-one correspondence with possible subquantum states of a given physical system. It is assumed that the system is always in some subquantum state (at the subquantum level of description). If a subquantum state of the system is known, then the outcomes of all quantum measurements performed on the system are completely determined.

**Statistical compatibility with quantum mechanics**

The probabilistic nature of quantum mechanics (in other words the fact that predictions of quantum mechanics are generally not applicable to individual physical
systems, but to statistical ensembles) is interpreted as a consequence of an incompleteness of quantum mechanics. Accordingly, quantum states are interpreted as certain probability measures on the subquantum space. The probabilities of quantum events in quantum states, coincide with the probabilities of the appropriate subquantum counterpart-events, relative to the associated probability measures.

********

A theory based on the above mentioned ideas is logically possible. Classical examples of consistent subquantum theories are Bohm-De Broglie pilot-wave theory \[3\], Wiener-Siegel theory \[24\] and the theory of Bohm-Bub \[4\].

On the other hand, the structure of quantum mechanics implies strong restrictions to the structure of a possible subquantum theory. These restrictions are coming from the obstacles contained in various so-called ‘no-go’ theorems. Generally speaking, all such statements tell us that a subquantum theory satisfying certain extra conditions is not possible. Namely, as shown by von Neumann \[16\], Gleason \[13\] and Bell \[2\], the concept of a subquantum state is not compatible with the algebra quantum observables. Incompatibility problems of this kind can be resolved only in the framework of contextual subquantum theories \[23\]. The common and characteristic property of all such theories is that the value of a given quantum observable, in a given subquantum state depends also on an additional entity, that is physically interpretable as the corresponding measurement context. In general, one and the same quantum observable realized in two different contexts will have different values in the same subquantum state.

The contextuality property is a necessary consistency condition. Here, we shall discuss quantum and subquantum mechanics within the framework of the \(C^\ast\)-algebraic physics (see for example \[6\], \[14\], \[16\], \[22\]) where physical theories are described by the associated \(C^\ast\)-algebras, generated by physical observables. Physical properties are then intrinsically related to algebraic structure of the corresponding \(C^\ast\)-algebras of observables. In this algebraic framework, statistical states of the system (analogs of probability measures in the classical context) are defined as positive normalized linear functionals \(\rho: \Sigma \rightarrow \mathbb{C}\) on the \(C^\ast\)-algebra \(\Sigma\) of observables. For a given observable \(F = F^\ast \in \Sigma\) the number \(\rho(F)\) is interpreted as the mean value of \(F\) in the state \(\rho\). A very special role is played by the pure states. The set \(S(\Sigma)\) of all states is a convex compact (in the \(*\)-weak topology) subset of the dual space \(\Sigma^\ast\). Pure states are defined as extremal elements of this convex set. According to Krein-Millman theorem, the set \(S(\Sigma)\) is a closure of the convex linear combinations of pure states. Pure states are the analogs of points of the phase space in classical mechanics (the points viewed as maximally concentrated \(\delta\)-like probability distributions).

The attributes of the system are in one-to-one correspondence with orthogonal projectors (⇔ hermitian idempotents) of the appropriate enveloping von Neumann algebra \(M(\Sigma) \supseteq \Sigma\), containing \(\Sigma\) as an everywhere dense \(*\)-subalgebra. By definition, a property \(p\) holds in the state \(\rho\) if and only if \(\rho(p) = 1\). The negation of the property \(p\) is given by the projector \(1 - p\). A careful exposition of algebraic quantum mechanics, from the states-properties viewpoint, can be found in \[21\].

As far as simple quantum mechanics is concerned (systems with finitely many degrees of freedom, without superselection sectors) we may assume that the \(C^\ast\)-algebra \(\Sigma\) describing the system consists of compact operators acting in the (infinite-dimensional separable) Hilbert state space \(H\). The algebra \(M(\Sigma)\) consists of all bounded operators in \(H\). The statistical states \(\rho: \Sigma \rightarrow \mathbb{C}\) of such a system (in the
sense of the above mentioned algebraic definition) are in a natural correspondence with statistical operators \( \hat{\rho} : H \rightarrow H \). The correspondence is given by the formula

\[ \rho(F) = \text{Tr}(F \hat{\rho}). \]

Pure states then correspond to one-dimensional projectors, which are in a natural correspondence with unit vectors \( \psi \in H \), identified modulo phase factors.

### 3. Contextual Extensions

In the framework of a general approach I developed in my doctoral thesis [8], a subquantum theory is represented by \( C^* \)-algebraic extensions of the form:

\[ 0 \rightarrow K \rightarrow \hat{\Sigma} \xrightarrow{\pi} C(\Omega) \rightarrow 0 \]

where \( \hat{\Sigma} \) is a \( C^* \)-algebra of ‘subquantum variables’ and \( \Omega \) a compact topological space consisting of all subquantum states of a given physical system. The elements of \( \Omega \) are actually the characters of \( \hat{\Sigma} \). By definition, characters are multiplicative (hermitian) and non-trivial linear functionals. It is easy to prove that a state \( \rho \) on a \( C^* \)-algebra \( A \) is a character, if and only if its dispersion

\[ \Delta_\rho(a) = \rho(a^2) - \rho(a)^2 \quad a = a^* \in A \]

vanishes on the elements from a given generating set (consisting of hermitian elements).

The above mentioned map \( \pi \) is induced by the evaluation on characters, in other words

\[ \pi(u)[\omega] = \omega(u). \]

Furthermore, the ideal

\[ \ker(\pi) = K \]

is generated by all commutators (this is by definition the commutant \( \text{com}(\hat{\Sigma}) = K \) of \( \hat{\Sigma} \)).

For a given subquantum state \( \omega \) and a given subquantum variable \( u \in \hat{\Sigma} \), the number

\[ \omega(u) = \pi(u)[\omega] \]

is interpreted as the value of the variable \( u \) in the subquantum state \( \omega \).

The idea that the subquantum theory is deeper than quantum mechanics is incorporated in the concept of a quantum approximation. More precisely, this means that the \( C^* \)-algebra \( \Sigma \) of quantum observables is realized as a certain approximative image of the subquantum algebra \( \hat{\Sigma} \), with the help of a *-epimorphism \( \phi : \hat{\Sigma} \rightarrow \Sigma \) (or a more general completely positive surjection). In other words, the map is associating to subquantum variables their ‘quantum approximations’. In summary, we have the following ‘double’ extension diagram:

\[
\begin{array}{c}
\{ \text{HiddenVariables} \leftrightarrow A \} \\
\downarrow \\
\{ \text{Complementarity} \leftrightarrow K \} \rightarrow \{ \hat{\Sigma} \leftrightarrow \text{Subq}(\Sigma) \} \xrightarrow{\pi} \{ C(\Omega) \leftrightarrow \text{Classical World} \} \\
\downarrow \phi \\
\{ \Sigma \leftrightarrow \text{Quantum World} \}
\end{array}
\]
where $\Lambda = \ker(\phi)$ is interpretable as the space of ‘truly hidden’ subquantum variables—the entities invisible at the quantum level.

Now what about measurement contexts? The contexts are certain commutative $C^*$-subalgebras of $\Sigma$. From the physical viewpoint, every context consists of quantum observables that are simultaneously actualizable in a given experimental situation (defining this context). In the conceptual framework of the theory, it is assumed that specifying a quantum observable $F$ together with a measurement context $C$ defines a subquantum variable $u \leftrightarrow (F, C)$.

Our subquantum theory must be statistically compatible with quantum mechanics. The statistical compatibility is expressed as the requirement that for every quantum state $\rho : \Sigma \to \mathbb{C}$ there exists a probability measure $\mu_\rho$ defined on a subquantum space $\Omega$ such that for every quantum observable $F$ and the appropriate measurement context $C$ we have

$$\rho(F) = \int_{\Omega} \omega(u)d\mu_\rho(\omega) \quad u \leftrightarrow (F, C).$$

In other words, the stochasticity of quantum states $\rho$ is interpreted as a lack of knowledge of the elements of the subquantum space $\Omega$.

The contextuality of the subquantum theory manifests as follows: Let us assume that we know a quantum observable $F$, and let us assume that it is measured in two different contexts, corresponding to two different commutative $C^*$-subalgebras of $\Sigma$, say $C_1$ and $C_2$. From the subquantum point of view, this means that we actually measure two different physical quantities $u \leftrightarrow (F, C_1)$ and $v \leftrightarrow (F, C_2)$ obtained by specifying the quantum observable and the surrounding context. Generally, we will have $u \neq v$. However, a necessary consistency condition is that $\phi(u) = \phi(v) = F$ because the two subquantum variables manifest as the same quantum observable. Moreover, it turns out that for every subquantum state $\omega$ there necessarily exist a quantum observable $F$ and contexts $C_1, C_2 \ni F$ with the property:

$$\omega(u) \neq \omega(v) \quad u \leftrightarrow (F, C_1) \quad v \leftrightarrow (F, C_2).$$

In other words, the values of quantum observables in subquantum states are necessarily context-dependent.

It is important to stress that from the subquantum viewpoint there is no contextuality at all! The contextuality only appears if we try to understand subquantum phenomena in terms of quantum observables, which are homomorphic images of the ‘true’ subquantum variables. One of the main ideas of my contextual extensions approach is to understand contextuality as a manifestation of an inadequacy of the algebra $\Sigma$ of quantum observables, to describe completely physical systems.

It is also very important to observe that probability measures $\mu_\rho$ that correspond to quantum states $\rho$ are contextually invariant in the sense that they do not distinguish contextual refinements of the same quantum observable. This is a trivial consequence of the fact that there is no contextuality in quantum mechanics (and in particular quantum states do not know anything about contexts). In fact, it is possible to prove [8] the converse—every contextually invariant probability distribution on $\Omega$ naturally induces a quantum state on $\Sigma$, assuming that $\Sigma$ is sufficiently quantum-like (for example, if $\Sigma$ is a von Neumann algebra without type $I_2$-factors in its central decomposition). This follows from a von-Neumann algebra generalization [18] of the Gleason theorem [13].
4. Additional Remarks

We saw that quantum description is understandable as an approximation of the subquantum one. Subquantum states of the system do not distinguish subquantum variables that coincide modulo the elements from the commutant \( \text{com}(\Sigma) = K \) of \( \Sigma \). In other words, the description of a physical system in terms of subquantum space \( \Omega \) can be understood as another approximation of the complete description, obtained by ignoring all complementarity phenomenas. This is because complementarity (at the subquantum level) is based on non-commutativity of \( \Sigma \), and ignoring it we arrive to a classical type theory based on a commutative \( C^* \)-algebra

\[ C(\Omega) = \hat{\Sigma}/\text{com}(\hat{\Sigma}) \]

All phenomenas related to causality are expressible in terms of \( \Omega \). Nevertheless, the complete description is given by a non-commutative algebra \( \hat{\Sigma} \), not less non-commutative than the quantum one.

One of the principal questions a coherent subquantum theory must answer is how dynamics looks like. There exists an interesting class of subquantum theories \(^{11}\) where the subquantum space \( \Omega \) is equipped with a symplectic manifold structure, such that the quantum evolution can be obtained from one-parametric flow generated by a smooth function on \( \Omega \), playing the role of a subquantum hamiltonian. In this sense, Shrödinger equation can be viewed as a statistical version of classical Hamiltonian equations.

The algebra \( \Sigma \) of quantum observables does not admit dispersion-free states (characters). This is a purely algebraic formulation of the mentioned no-go theorems, and can be viewed as a consequence of uncertainty relations.

It is important to mention that \( C^* \)-algebraic extensions considered here are also very interesting from the point of view of non-commutative geometry.

At first, there exist powerful techniques \(^{1}\) to apply \( C^* \)-algebraic extensions (of the types similar to our subquantum extensions) to study topological properties of classical topological spaces \( \Omega \). Secondly, noncommutative \( C^* \)-algebras give us examples of quantum spaces—the main objects of study of non-commutative geometry. Let us observe that, from the non-commutative geometric viewpoint, subquantum states (as characters) play the role of points of the quantum space associated to the algebra \( \hat{\Sigma} \) of subquantum variables.

5. Locality & Wholeness

In developing a subquantum theory, it is of a crucial importance to consider the problematics of describing the composite systems. In connection with such systems, it seems natural to ask whether the theory possesses the following locality property:

**Locality Property**

Let us consider a composite system of two distant ‘particles’, and let us assume that a subquantum state of the composite system is given. The result of an arbitrary measurement performed on the first particle is independent of what (and if something) is simultaneously being measured on the second particle—assuming there are no interactions between two measurements.

The answer to the question of existence of a local subquantum theory is deeply connected with the assumption about the type of probability theory applied at the
subquantum level. Namely, if we consider a local and causal theory and assume in addition that probabilities of all experimentally accessible events are expressible by the classical (Kolmogorovian) statistics, then the probabilities associated to certain joint measurements will satisfy the Bell inequalities [1]. On the other hand, according to quantum mechanics the same probabilities are violating the Bell inequalities. In classical statistics, the domain of a probability measure is always a $\sigma$-field on the space of elementary events.

The violation of Bell’s inequalities implies that a local subquantum theory based on classical statistics is not possible.

However, classical statistics is not the only way to describe the diversity of lack-of-knowledge situations. Moreover, from the point of view of complementarity, it actually seems very unnatural to use classical probability theory as the base for the statistics on the subquantum space. This is because:

(i) It turns out that the family of experimentally accessible subquantum events (certain subsets of subquantum space $\Omega$) is not closed under unions and intersections, and therefore it is not a $\sigma$-field (in contrast to classical probability theory).

(ii) There is no any physical justification to attach a probability to experimentally non-realizable events. Accordingly, a good probability theory must operate exclusively with experimentally realizable events and tell us how to compute probabilities of such events.

Starting from the above observations, it is possible to develop a new probability theory which includes classical probability theory as a very special case. And in the framework of such a generalized probability theory, it is possible to unify quantum mechanics with the principles of causality and locality [19, 20, 14, 11, 9, 10].

As pointed out in [19], there exists a certain similarity between probability and geometry. Kolmogorovian probability theory corresponds to Euclidean geometry. However, Euclidean geometry is not the only possible geometrical scheme. Experimental evidence, combined with mathematical clarity, is what determines suitability of geometrical systems to describe properties of the physical space and time.

The same applies to probability.

On the other hand, there is no experimental justification to assume locality at the subquantum level. Physical reality could equally well be totally non-local. The idea of locality is closely related with the concepts of space and time. And it is not clear how the space-time looks like at the level of ultra-small distances, characterized by the Planck length. Furthermore, there are various arguments to believe that the space-time exhibits qualitatively new behaviors at the Planck scale, related to quantum fluctuations of geometry. Accordingly, the proper picture of the space-time should be given by non-commutative geometry ($\Leftrightarrow$ quantum geometry) [7]. Quantum geometry introduces a new concept of space, by unifying classical geometry, non-commutative C*-algebras and basic ideas and principles of quantum physics.

Quantum geometry deals with quantum spaces, which are very different from classical spaces—in particular quantum spaces may have no points at all. Moreover, there exist quantum spaces without any ‘regions’—and it seems that the classical concept of localizations is something very characteristic for classical spaces.

So it may happen that locality is just an illusion. At the deepest level, perhaps, everything is intrinsically connected with everything and the very concept of locality loses its standard meaning.
References

[1] Bell J S: On the Einstein-Podolsky-Rosen Paradox, Physics (NY) 1 195–200 (1964)
[2] Bell J S: On the Problem of Hidden Variables in Quantum Mechanics, Rev Mod Phys 38 447–452 (1966)
[3] Bohm D: A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, Phys Rev 85 165/180 (1952)
[4] Bohm D, Bub J: A proposed solution of the Measurement Problem in Quantum Mechanics by a Hidden Variable Theory, Rev Mod Phys 38 453–469 (1966)
[5] Brateli O, Robinson D: Operator Algebras and Quantum Statistical Mechanics, Volumes 1/2, Springer-Verlag (1979)
[6] Brown L G, Douglas R G, Fillmore P G: Extensions of C*-algebras and K-Homology, Ann Math, 105 265–324 (1977)
[7] Connes A: Noncommutative Geometry, Academic Press (1994)
[8] Đurđević M: Algebro-Geometric Constructions of Subquantum Theories, Doctoral Thesis, Faculty of Physics, University of Belgrade, Serbia [in Serbian :)] (1993)
[9] Đurđević M: Quantum Field Theory and Local Contextual Extensions, J Phys A: Math Gen 25 665–677 (1992)
[10] Đurđević M: A C*-algebraic formulation of Local Contextual Hidden Variables, J Phys A: Math Gen 25 5345–5364 (1992)
[11] Đurđević M, Vujićić M, Herbut F: Symplectic Hidden-Variables Theories–The Missing Link in Algebraic Contextual Approaches, J Math Phys 32 3088–3093 (1991)
[12] Emch E G: Algebraic Methods in Statistical Mechanics and Quantum Field Theory, Wiley-Interscience (1972)
[13] Gleason A M: Measures on the closed subspaces of a Hilbert space, J Math Mech 6 886–893 (1957)
[14] Gudder S: Reality, Locality, and Probability, Found Phys 14 997–1010 (1984)
[15] Gudder S: Probability Manifolds, J Math Phys 25 2397–2401 (1984)
[16] von Neumann J: Mathematical Foundations of Quantum Mechanics, Princeton University Press (1955)
[17] Kastler D: Equilibrium States of Matter and Operator Algebras, Symposia Matematica, 20 49–107 (1976)
[18] Maeda S: Probability Measures on Projections in von Neumann Algebras, Rev Math Phys 1 235–291 (1990)
[19] Pitowsky I: Resolution of the Einstein-Podolsky-Rosen and Bell Paradoxes, Phys Rev Lett 48 1299–1302 (1982)
[20] Pitowsky I: Deterministic Model of Spin and Statistics, Phys Rev D, 27, 2316–2326 (1982)
[21] Primas H: Foundations of Theoretical Chemistry, in "Quantum Dynamics of Molecules: The new Experimental Challenge to Theorists" NATO Advanced Study Series, 57 39–113 (1979)
[22] Robinson D: C*-algebras and Quantum Statistical Mechanics, in "C*-algebras and their applications to Statistical Mechanics and Quantum Field Theory" Varenna Lectures, Course LX, North-Holland (1976)
[23] Shimony A: Contextual Hidden Variables Theories and Bell’s Inequalities, Brit J Phil Sci 35 25–45 (1984)
[24] Wiener N, Siegel A: A new Form for the Statistical Postulate of Quantum Mechanics, Phys Rev 91 1551–1560 (1953)

Instituto de Matematicas, UNAM, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, Mexico DF, CP 04510, Mexico

E-mail address: micho@matem.unam.mx
http://www.matem.unam.mx/~micho