Goldstone Theorem, Hugenholtz-Pines Theorem and Ward-Takahashi Relation in Finite Volume Bose-Einstein Condensed Gases

Hiroaki Enomoto\textsuperscript{a}, Masahiko Okumura\textsuperscript{b} and Yoshiya Yamanaka\textsuperscript{c}

\textsuperscript{a}Department of Physics, Waseda University, Tokyo 169-8555, Japan
\textsuperscript{b}Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan
\textsuperscript{c}Department of Materials Science and Engineering, Waseda University, Tokyo 169-8555, Japan

Abstract

We construct an approximate scheme based on the concept of the spontaneous symmetry breakdown, satisfying the Goldstone theorem, for finite volume Bose-Einstein condensed gases in both zero and finite temperature cases. In this paper, we discuss the Bose-Einstein condensation in a box with periodic boundary condition and do not assume the thermodynamic limit. When energy spectrum is discrete, we found that it is necessary to deal with the Nambu-Goldstone mode explicitly without the Bogoliubov’s prescription, in which zero-mode creation- and annihilation-operators are replaced with a $c$-number by hand, for satisfying the Goldstone theorem. Furthermore, we confirm that the unitarily inequivalence of vacua in the spontaneous symmetry breakdown is true for the finite volume system.

Key words: Bose-Einstein condensation, Spontaneous symmetry breakdown, Goldstone theorem, Ward-Takahashi relations, Hugenholtz-Pines theorem, Unitarily inequivalent vacua

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Email addresses: enomotohiroaki@akane.waseda.jp (Hiroaki Enomoto), okumura@aoni.waseda.jp (Masahiko Okumura), yamanaka@waseda.jp (Yoshiya Yamanaka).
1 Introduction

Various theoretical studies on weakly interacting dilute Bose-Einstein condensed systems have been done over many years. The excitation spectrum in a homogeneous system was first obtained by Bogoliubov in 1947 [1]. Afterwards the quantum correction in zero-temperature case was evaluated by Hugenholtz and Pines [2]. In course of their discussion, they showed the Hugenholtz-Pines (HP) theorem, which relates the chemical potential with the self-energy in a nonperturbative way. In these works one follows the Bogoliubov’s prescription [1], in which the zero-mode creation- and annihilation-operators $a_0$ and $a_0^\dagger$ are replaced with a c-number $\sqrt{N_c}$ by hand, where $N_c$ is the number of condensate particles, and the c-number is regarded as the order parameter. Though this prescription was simple and successful in deriving the excitation spectrum, its naive use is not consistent in quantum theory because the canonical commutation relations (CCRs) are not respected [3].

In modern quantum field theory, the order parameter is introduced as a vacuum expectation value of a quantum field. The self-consistent mechanism for creating the order parameter is the spontaneous symmetry breakdown (SSB). We have many examples of the SSB, appearing in ferromagnetic, superconducting, crystal orders, and so on [4]. The scenario of the SSB reads as follows: Suppose that an action or a Lagrangian is invariant under a continuous transformation but the vacuum does not show the original symmetry. Then the non-zero vacuum expectation value of the quantum field is induced, and is identified as the order parameter. In general, when a continuous symmetry is broken spontaneously, there must exist a gapless mode, called the Nambu-Goldstone (NG) mode, which is necessary to keep the original invariance of the action. This statement is the Goldstone theorem [5], and it is proven from the Ward-Takahashi (WT) relations [6,4]. The WT relations are in turn identities, derived from the CCRs, the Heisenberg equation of motion, and the transformation property.

The dilute gas systems of bosonic neutral atoms at recent experiments [7,8,9], forming Bose-Einstein condensates (BECs), are described by the model Lagrangian invariant under a global phase transformation. The appearance of BECs can be interpreted as a spontaneous breakdown of the global phase symmetry. For such a model in homogeneous situations, Hohenberg and Martin [10] showed that the HP theorem is a consequence of the WT relation with respect to the global phase transformation first, and generalized it to a finite temperature case. Recently, Boyanovsky et al [11] showed that the HP theorem holds at one-loop level in a homogeneous BEC system.

The concept of the SSB is inherent in quantum field theory but not in quantum mechanics because it is closely related to an infinite number of degrees of free-
dom, causing many unitarily inequivalent vacua. For spatially homogeneous systems of quantum field one finds inequivalent vacua under the thermodynamics limit, defined as that of infinite volume while its ratio to the number of particles is kept finite [4].

The whole story above applies in the case of homogeneous systems and cannot be extended to spatially inhomogeneous cases without due consideration. The BEC systems of neutral atoms at recent experiments have finite system size and spatial inhomogeneity due to trapping potentials. For such systems of finite size, the invariance of spatial translation is lost and the momentum is no longer a good quantum number, and the energy spectrum including the NG mode becomes discrete. The issue of existence of inequivalent vacua for such finite volume systems has not been considered thoroughly, so the occurrence of the SSB in the systems is not trivial. Meanwhile, it has been observed in the recent experiment on trapped Bose gases [12] that a condensate has a fixed phase. Furthermore, it is claimed that the Bogoliubov spectrum [13] and the Bogoliubov transformation [14] were observed in the two-photon Bragg scattering process. Even in the case of discrete spectrum, the theoretical formulation of quantum field with introduction of the order parameter is consistent with the experiment [15]. These experiments seem to support that as a result of the SSB the order parameter exists with a fixed phase, not only in homogeneous infinite volume system, but also in inhomogeneous system of finite size.

When an order parameter appears, a single vacuum should be selected among infinite possible ones and a quasi-particle picture should be established in a manner consistent with the Goldstone theorem. This is a vital but not trivial step in theoretical calculation. One needs to invent a selection procedure in theory. In the Bogoliubov prescription for a homogeneous system, the vacuum is selected through replacing $a_0$ and $a_0^\dagger$ with $\sqrt{N_c}$ by hand, and the Bogoliubov transformation of $p \neq 0$ modes diagonalizes the unperturbed Hamiltonian. It is remarked that the zero-mode operators are absent in the formulation so that the CCRs for the field operators and the Goldstone theorem cannot be satisfied exactly. But since the point of the zero-energy is embedded in a continuous spectrum, this fact is easily overlooked. In case of finite size system such as the trapped BEC systems with discrete spectrum, the Bogoliubov prescription clearly fails: The CCRs of the quantum field and the Goldstone theorem are violated due to the absence of the discrete zero-mode. Conventionally, in the SSB of quantum field theory, one adopts the systematic method to introduce an infinitesimal symmetry breaking term, known as Bogoliubov’s quasi-average [16], and one vacuum is selected among many. Note that the zero-mode is naturally included in this method in the symmetric limit. We have proposed the way of introducing the infinitesimal symmetry breaking term for the trapped BEC system and have explicitly shown that the WT relations and the Goldstone theorem hold at tree level [3]. It is also shown
that the vacua belonging to the order parameters with different phases are unitarily inequivalent to each other even for the trapped BEC [17].

Our goal is a formulation of a systematic and consistent treatment of inhomogeneous quantum field system of finite size, corresponding to the recent experiments of the trapped BEC. We mean by the word “consistent treatment” that the CCR, the WT relations and the Goldstone theorem are respected. As mentioned above, we have studied this subject at tree level [3], and should extend our discussions to any loop level. At present, such extension is not easy due to the complex structure of the unperturbed propagators.

Instead, in this paper, we consider a BEC system, without trapping potential but confined in a box with periodic boundary condition. We keep a finite volume size \( V \). To select a vacuum, we introduce an infinitesimal symmetry breaking term, characterized by an infinitesimal parameter \( \varepsilon \). What is important in our present study is that the limit \( \varepsilon \to 0 \) is taken but the limit \( V \to \infty \) is not, namely that the thermodynamic limit is not considered. We will see that inequivalent vacua emerge in the limit \( \varepsilon \to 0 \) with finite \( V \) (without the thermodynamic limit). While the energy spectrum is discrete, the Fourier (momentum) representation is available in this model, which makes calculations of loop expansions tractable. It is expected that a model with trapping potential mostly shares essential theoretical features with this model.

Explicitly we show that the HP theorem is derived from the WT relations even in this finite volume system. The quasi-particle picture in which the zero-mode (NG mode) is present is constructed at tree and one-loop levels. To extend the results at zero temperature to those at finite one, we employ thermofield Dynamics (TFD) [4,18] which is a real time canonical formalism of thermal field theory with doubled degrees of freedom.

This paper is organized as follows. In Section. 2, we give a formulation of the model Lagrangian density with an infinitesimal symmetry breaking term at zero temperature, and construct the quasi-particle picture consistently with the CCRs and the Goldstone theorem. The order parameter is introduced without using the Bogoliubov’s prescription. We give the quantum correction which keeps the HP theorem without taking the infinite volume limit. We also show that the vacuum in a broken phase is orthogonal to that in asymmetric phase. In Section. 3, we use TFD to extend the results in Section. 2 to finite temperature case. We review the TFD formalism briefly and investigate the WT relations at each loop level similarly to the zero temperature case. In this case, we also derive the HP theorem from the WT relations at arbitrary loop level. Section 4 is devoted to a summary and conclusion. In Appendix, we derive the relation between the vacua in the broken and symmetric phase.
2 Zero Temperature Case

We formulate a field-theoretical treatment for the BECs in a box with periodic boundary condition on the field variables. In this section, we discuss the zero temperature case. First we confine our discussion to tree level, and will see that the WT relation is satisfied and that the unitarily inequivalent vacua are realized. Next we derive the HP theorem from the WT relations at any loop level and show explicitly that our one-loop calculation satisfies the HP theorem.

2.1 Model Lagrangian density and Hamiltonian

We start with the following Lagrangian density, which describes a weakly interacting Bose gas in a box whose volume is denoted by $V$,

$$\mathcal{L} = \Psi^\dagger(x) \left( -\frac{i\hbar}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 + \mu \right) \Psi(x) - \frac{g}{2} \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x). \quad (1)$$

Here, $m$, $\mu$ and $g$ are the mass of a boson, the chemical potential and the coupling constant, respectively, and $x$ stands for the space-time coordinate $(x, t)$. The periodic boundary condition in each direction is imposed on the field variables:

$$\begin{aligned}
\Psi(x + L, y, z, t) &= \Psi(x, y, z, t) \\
\Psi(x, y + L, z, t) &= \Psi(x, y, z, t) \\
\Psi(x, y, z + L, t) &= \Psi(x, y, z, t)
\end{aligned} \quad (2)$$

where $L$ being the length of one side of the cube, $V = L^3$. It is easy to see that this Lagrangian density is invariant under the global phase transformation:

$$\begin{aligned}
\Psi(x) &\rightarrow e^{i\theta}\Psi(x) \\
\Psi^\dagger(x) &\rightarrow e^{-i\theta}\Psi^\dagger(x)
\end{aligned} \quad (3)$$

where $\theta$ is a real constant. When a uniform BEC is created, the global phase symmetry is spontaneously broken, and then the quantum field $\Psi(x)$ is divided into a classical constant field $v$ and a quantum field $\varphi(x)$ in the terminology of the canonical operator formalism:

$$\Psi(x) = v + \varphi(x). \quad (4)$$
We take a real value for \( v \), which does not affect generality of the following discussions. The classical field \( v \) is called the order parameter and is also expressed as \( v^2 = n_c \) in terms of the density of condensate particles \( n_c \). The classical field \( v \) may be defined as an expectation value of the Heisenberg field with respect to the vacuum \( |\Omega\rangle \),

\[
\langle \Omega | \Psi(x) | \Omega \rangle = v, \tag{5}
\]

or equivalently

\[
\langle \Omega | \varphi(x) | \Omega \rangle = 0. \tag{6}
\]

Let us introduce an artificial symmetry breaking term

\[
\mathcal{L}_\varepsilon = (\varepsilon \bar{\varepsilon}) v \left[ \Psi(x) + \Psi^\dagger(x) \right], \tag{7}
\]

and add it to the original Lagrangian density (1), then the total Lagrangian density \( \mathcal{L}_{\text{tot}} \) becomes

\[
\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_\varepsilon. \tag{8}
\]

Here, \( \varepsilon \) is an infinitesimal dimensionless parameter and \( \bar{\varepsilon} \) represents a typical energy scale of the system. Obviously the total Lagrangian \( \mathcal{L}_{\text{tot}} \) is not invariant under the global phase transformation (3). The parameter \( \varepsilon \) is taken to be vanishing at the final stage of the calculation, so that the original symmetry is restored.

We move to the canonical formalism in the interaction representation. The canonical conjugate of \( \varphi(x) \) is \( \pi(x) = i\hbar \varphi^\dagger(x) \). Then the CCRs are as follows:

\[
[\varphi(x, t), \varphi^\dagger(x', t)] = \delta^3(x - x'), \tag{9}
\]

\[
[\varphi(x, t), \varphi(x', t)] = [\varphi^\dagger(x, t), \varphi^\dagger(x', t)] = 0. \tag{10}
\]

Now, we have the total Hamiltonian

\[
H = H_0 + H_{\text{int}} + \text{const.}, \tag{11}
\]

where the unperturbed Hamiltonian \( H_0 \) is given as

\[
H_0 = \int d^3x \left[ \varphi^\dagger(x) \left( -\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \varphi(x) \right]
\]
\[ + \frac{gv^2}{2} \left\{ 4\varphi^\dagger(x)\varphi(x) + \varphi(x)\varphi(x) + \varphi^\dagger(x)\varphi^\dagger(x) \right\} \]. \quad (12)\]

and the perturbative Hamiltonian \( H_{\text{int}} \) is defined as

\[
H_{\text{int}} = v \left( -\mu + gv^2 - \varepsilon \bar{\varepsilon} \right) \int d^3x \left\{ \varphi(x) + \varphi^\dagger(x) \right\} \\
+ gv \int d^3x \left[ \varphi^\dagger(x)\varphi^\dagger(x)\varphi(x) + \varphi^\dagger(x)\varphi(x)\varphi(x) \right] \\
+ \frac{g}{2} \int d^3x \varphi^\dagger(x)\varphi^\dagger(x)\varphi(x)\varphi(x). \quad (13)\]

It is convenient to define the 2×2-matrix propagator for the quantum fields \( \varphi(x) \) and its Hermite conjugate \( \varphi^\dagger(x) \) by

\[
G(x - x') = \begin{pmatrix}
G_{11}(x - x') & G_{12}(x - x') \\
G_{21}(x - x') & G_{22}(x - x')
\end{pmatrix}
= \begin{pmatrix}
-\operatorname{i}\langle \Omega | T[\varphi(x)\varphi^\dagger(x')]|\Omega \rangle & -\operatorname{i}\langle \Omega | T[\varphi(x)\varphi(x')]|\Omega \rangle \\
-\operatorname{i}\langle \Omega | T[\varphi^\dagger(x)\varphi^\dagger(x')]|\Omega \rangle & -\operatorname{i}\langle \Omega | T[\varphi^\dagger(x)\varphi(x')]|\Omega \rangle
\end{pmatrix}, \quad (14)\]

where \( T \) is the symbol for time-ordered product. Its Fourier transform is given by

\[
G(p) = \begin{pmatrix}
G_{11}(p) & G_{12}(p) \\
G_{21}(p) & G_{22}(p)
\end{pmatrix} = \int \frac{d^4x}{(2\pi \hbar)^2} G(x) e^{\frac{\operatorname{i}}{\hbar} (p \cdot x - \omega t)}, \quad (15)\]

with the notation of \( p = (\omega, \mathbf{p}) \). It is mentioned that one has the following relations among the matrix elements:

\[
G_{11}(p) = G_{22}(-p), \quad G_{12}(p) = G_{21}(p). \quad (16)\]

### 2.2 Derivation of the Ward–Takahashi relation

It is well-known that the Goldstone theorem follows from the WT relations. We adopt the loop expansion, because the WT relations hold at each loop level in general.
First we review the WT relation in general case. Consider an infinitesimal transformation:

\[ \Psi(x) \rightarrow \Psi'(x) = \Psi(x) + \xi \delta \Psi(x) \]  

(17)

with the infinitesimal parameter \( \xi \). The change in the Lagrangian density is denoted by \( \xi \delta \mathcal{L} \):

\[ \xi \delta \mathcal{L} = \mathcal{L}[\Psi'(x)] - \mathcal{L}[\Psi(x)] . \]  

(18)

The Nöther theorem implies

\[ \delta \mathcal{L} = \partial^\mu N_\mu , \]  

(19)

where

\[ N_\mu = \frac{\partial \mathcal{L}}{\partial \Psi_\mu} \delta \Psi(x) . \]  

(20)

Now, we define the Nöther charge

\[ N(t) = \frac{1}{\hbar} \int \text{d}^3x \, N_0(x) , \]  

(21)

which generates the transformation in the commutation relation

\[ [\Psi(x), N(t)]_{t=t} = i \delta \Psi(x) . \]  

(22)

Then, the Nöther theorem (19) leads to

\[ \dot{N}(t) = \frac{1}{\hbar} \int \text{d}^3x \, \delta \mathcal{L} . \]  

(23)

One can easily derive the following relation,

\[ \frac{\partial}{\partial t} \langle \Omega | T[N(t)\Psi(x_1) \cdots \Psi(x_n)] | \Omega \rangle \\
= \sum_{a=1}^n \delta(t - t_a) \langle \Omega | T[\Psi(x_1) \cdots [N(t), \Psi(x_a)] \cdots \Psi(x_n)] | \Omega \rangle \\
+ \langle \Omega | T[\dot{N}(t)\Psi(x_1) \cdots \Psi(x_n)] | \Omega \rangle . \]  

(24)
Integrating both sides over the entire time domain and dropping the surface term at $t = \pm \infty$, we obtain
\[
\sum_{a=1}^{n} \frac{i\hbar}{\hbar} \langle \Omega | T[\Psi(x_1) \cdots \delta \Psi(x_a) \cdots \Psi(x_n)] | \Omega \rangle = \int d^3x \langle \Omega | T[\delta \mathcal{L}(x) \Psi(x_1) \cdots \Psi(x_n)] | \Omega \rangle.
\] (25)

This relation, derived independently of any approximate scheme, is called the WT relation.

In this paper, we are interested in the following infinitesimal global phase transformation:
\[
\delta \Psi(x) = i\Psi(x). \tag{26}
\]

The Nöther charge is given by
\[
N(t) = -\int d^3x \Psi(x) \Psi(x), \tag{27}
\]
indeed $[N(t), \Psi(x)]_{t_x=t} = \Psi(x) = -i\delta \Psi(x)$. We consider the following special form of the WT relation (25):
\[
\frac{i\hbar}{\hbar} \langle \Omega | \delta \delta \mathcal{L}_{\text{tot}}(x) | \Omega \rangle = \int d^4x' \langle \Omega | T[\delta \mathcal{L}_{\text{tot}}(x') \delta \mathcal{L}_{\text{tot}}(x)] | \Omega \rangle. \tag{28}
\]
Here, $\delta \mathcal{L}_{\text{tot}}(x)$ and $\delta \delta \mathcal{L}_{\text{tot}}(x)$ are defined as
\[
\delta \mathcal{L}_{\text{tot}}(x) = \delta \mathcal{L}_{\varepsilon}(x) = i[N(t), \mathcal{L}_{\varepsilon}(x)]_{t_x=t} = i(\varepsilon \bar{\varepsilon})v \left[ \Psi(x) - \Psi^\dagger(x) \right] \tag{29}
\]
and
\[
\delta \delta \mathcal{L}_{\text{tot}} = \delta \delta \mathcal{L}_{\varepsilon} = i[N(t), \delta \mathcal{L}_{\varepsilon}(x)]_{t_x=t} = -(\varepsilon \bar{\varepsilon})v \left[ \Psi(x) + \Psi^\dagger(x) \right]. \tag{30}
\]
Thus, the WT relation is written in the form of
\[
v = -\frac{(\varepsilon \bar{\varepsilon})v}{2\hbar} \int d^4x' \left[ G_{11}(x - x') + G_{22}(x - x') - G_{12}(x - x') - G_{21}(x - x') \right]
\]
\[
= -\frac{(\varepsilon \bar{\varepsilon})v}{\hbar} \left[ G_{11}(p = 0) - G_{12}(p = 0) \right], \tag{31}
\]
where we have used the relations (16) in the last line.
2.3 Diagonalization of unperturbed Hamiltonian and the Ward--Takahashi relation at tree level

In this subsection, we investigate the system at tree level. First let us determine the order parameter at tree level. Form the unperturbed Hamiltonian (12) and the first term in the perturbative Hamiltonian (13), we have the diagram at tree (zero-loop) level for the condition (6), corresponding to the expression

$$
\begin{pmatrix}
\langle \Omega | \varphi(x) | \Omega \rangle \\
\langle \Omega | \varphi^\dagger(x) | \Omega \rangle
\end{pmatrix} = i \int d^4x' G(x - x') (-\mu_0 + gv^2 - \varepsilon \bar{\epsilon}) = 0,
$$

(32)

where $\mu_0$ is the chemical potential at tree level. As this equation must be true for any $x$, $\mu_0$ is determined as

$$
\mu_0 = gv^2 - \varepsilon \bar{\epsilon}.
$$

(33)

Next, we construct the quasi-particle picture at tree level by diagonalizing the unperturbed Hamiltonian (12). The unperturbed quantum field $\varphi(x)$ is expanded in terms of annihilation-operators $a_p(t)$ as

$$
\varphi(x) = \frac{1}{\sqrt{V}} \sum_{p=-\infty}^{\infty} a_p(t) e^{i p \cdot x}.
$$

(34)

Here, $V$ is finite and the periodic boundary conditions on field operators (2) restrict $p$ as

$$
p = \frac{2\pi \hbar}{L} (n_x, n_y, n_z), \quad n_x, n_y, n_z: \text{integers}.
$$

(35)

The symbol $\sum_{p=-\infty}^{\infty}$ stands for $\sum_{n_x, n_y, n_z=-\infty}^{\infty}$ which includes $n_x = n_y = n_z = 0$. We emphasize that if one employs the Bogoliubov’s prescription, the quantum field does not contain $a_0(t)$ and breaks the CCRs, but our expansion contains it and the CCRs are held. The operators $a_p(t)$ and $a_p^\dagger(t)$ are subject to the commutation relations of creation- and annihilation-operators, which follow from the CCRs for quantum fields (9) and (10): $[a_p(t), a_p^\dagger(t)] = \delta_{pp'}$ and other commutations vanish. The temporal development of $\varphi(x)$ and $a_p(t)$ is generated by the unperturbed Hamiltonian (12). Substituting the expansion (34) and its Hermite conjugate into the unperturbed Hamiltonian (12), we obtain

$$
H_0 = \sum_{p=-\infty}^{\infty} \left[ (\epsilon_p + gv^2 + \varepsilon \bar{\epsilon}) a_p^\dagger a_p + \frac{gv^2}{2} (a_p a_{-p} + a_p^\dagger a_{-p}^\dagger) \right],
$$

(36)
where we use the relation (33) and the notation of

$$\epsilon_p = \frac{p^2}{2m}.$$  \hspace{1cm} (37)

As is well known, the Bogoliubov transformation

$$a_p = u_p b_p + v_p b_p^\dagger,$$ \hspace{1cm} (38)

where

$$u_p, v_p = \pm \sqrt{\frac{\epsilon_p + gv^2 + \epsilon \bar{\epsilon}}{2E_p}} \pm \frac{1}{2}$$ \hspace{1cm} (39)

and

$$E_p = \sqrt{\epsilon_p^2 + 2gv^2\epsilon_p + 2(\epsilon_p + gv^2)\epsilon \bar{\epsilon} + (\epsilon \bar{\epsilon})^2},$$ \hspace{1cm} (40)

diagonalizes the unperturbative Hamiltonian as

$$H_0 = \sum_{p=-\infty}^{\infty} E_p b_p^\dagger b_p + \text{const.}.$$ \hspace{1cm} (41)

We mention again that a crucial point here is that the mode with $p = 0$ is included in the transformation. The infinitesimal symmetry breaking term (7) makes the transformation between $a_0$ and $b_0$ well-defined as long as $\epsilon$ is kept finite.

The field operator $\varphi(x)$ is rewritten in terms of the creation- and annihilation-operators $b_p$ and $b_p^\dagger$ as

$$\varphi(x) = \frac{1}{\sqrt{V}} \sum_{p=-\infty}^{\infty} \left[ b_p u_p e^{i(p \cdot x - E_p t)} + b_p^\dagger v_p e^{-i(p \cdot x - E_p t)} \right].$$ \hspace{1cm} (42)

One can easily check that this quantum field and its Hermite conjugate satisfy the CCRs (9) and (10).

The unperturbed matrix propagator of the field $\varphi(x)$ on the vacuum $|\Omega_0\rangle$, which is defined by the relation $b_p|\Omega_0\rangle = 0$, is given by

$$G_0(x-x') = \begin{pmatrix} G_{0,11}(x-x') & G_{0,12}(x-x') \\ G_{0,21}(x-x') & G_{0,22}(x-x') \end{pmatrix}.$$
\[
\begin{pmatrix}
-i\langle \Omega_0 | T[\varphi(x)\varphi^\dagger(x')] | \Omega_0 \rangle & -i\langle \Omega_0 | T[\varphi(x)\varphi(x')] | \Omega_0 \rangle \\
-i\langle \Omega_0 | T[\varphi^\dagger(x)\varphi^\dagger(x')] | \Omega_0 \rangle & -i\langle \Omega_0 | T[\varphi^\dagger(x)\varphi(x')] | \Omega_0 \rangle 
\end{pmatrix}
\]  (43)

Its Fourier transform is defined by

\[
G_0(p) = \begin{pmatrix}
G_{0,11}(p) & G_{0,12}(p) \\
G_{0,21}(p) & G_{0,22}(p)
\end{pmatrix}
= \int \frac{d^4x}{(2\pi\bar{\hbar})^2} G_0(x)e^{\frac{i}{\bar{\hbar}}(p \cdot x - \omega t)},
\]  (44)

and the explicit forms of its matrix elements are

\[
G_{0,11}(p) = G_{0,22}(-p) = \frac{u_p^2}{\omega - \omega_p + i\delta} - \frac{v_p^2}{\omega + \omega_p - i\delta},
\]  (45)

\[
G_{0,12}(p) = G_{0,21}(p) = \frac{u_p v_p}{\omega - \omega_p + i\delta} - \frac{u_p v_p}{\omega + \omega_p - i\delta},
\]  (46)

where \(\bar{\hbar}\omega_p = E_p\) and \(\delta\) is an infinitesimal positive parameter.

Next, let us check the WT relation (31) at tree level, using the propagators (45) and (46). The right-hand side (RHS) of the relation (31) is manipulated as

\[
-\frac{(\varepsilon \bar{\varepsilon})}{\bar{\hbar}} [G_{0,11}(p = 0) - G_{0,12}(p = 0)] = -\frac{(\varepsilon \bar{\varepsilon})}{\bar{\hbar}} \left( \frac{-1}{\omega_{p=0}} \right) \left( \frac{g\nu^2 + \varepsilon \bar{\varepsilon}}{\bar{\hbar}\omega_{p=0}} + \frac{g\nu^2}{\bar{\hbar}\omega_{p=0}} \right)
= \frac{2g\nu^2(\varepsilon \bar{\varepsilon}) + (\varepsilon \bar{\varepsilon})^2\nu}{\bar{\hbar}^2\omega_{p=0}^2}
= v,
\]  (47)

where the last equality comes from the quasi-particle energy (40). This way the WT relation (31) at tree level is confirmed. We observe that the contribution of the zero-energy mode (the NG mode) is vital. In other words, for a finite volume system with a discrete spectrum, the Bogoliubov’s prescription without operators representing the NG mode can not preserve the WT relations.

### 2.4 Unitarily inequivalent vacua

In the previous subsection, we considered the vacuum \(|\Omega_0\rangle\) associated with the operator \(b_p\) as a physical one at tree level. There is another vacuum, denoted by \(|0\rangle\), which is annihilated by \(a_p\): \(a_p|0\rangle = 0\). We evaluate the inner product \(\langle 0|\Omega_0\rangle\) below, in the limit \(\varepsilon \to 0\) but keeping finite \(V\).
For this purpose, we will give explicit transformation to relate the two vacua, $|\Omega_0\rangle$ and $|0\rangle$, in Appendix A. We recapitulate Eq. (A.43) which is the conclusion of Appendix A:

$$|\Omega_0\rangle = \frac{1}{\sqrt{u_0}} \exp \left[ -\frac{1}{2} \sum_{p=-\infty}^{\infty} \ln u_p \right] \exp \left[ \frac{1}{2} \sum_{p=-\infty}^{\infty} \frac{v_p}{u_p} a_p a_p^\dagger \right] |0\rangle ,$$  \hspace{1cm} (48)

where the symbol $\sum_{p=-\infty}^{\infty}'$ means summation without $p = 0$. Under the limit $\varepsilon \to 0$, $u_p$ and $v_p$ with $p \neq 0$ are finite, $v_0/u_0$ becomes 1 and $u_0$ is divergent as $\varepsilon^{-\frac{1}{4}}$. Then one finds that

$$\langle 0|\Omega_0\rangle \sim \varepsilon^{-\frac{1}{8}} \to 0 \hspace{1cm} (\varepsilon \to 0).$$  \hspace{1cm} (49)

A conclusion in this subsection is that the vacua $|\Omega_0\rangle$ and $|0\rangle$ are orthogonal to each other in the limit $\varepsilon \to 0$ even for finite volume system. This means that the Fock space built on $|0\rangle$ is unitarily inequivalent to one built on $|\Omega_0\rangle$: a state $b_p^n|\Omega_0\rangle$ is never obtained by the superposition of $a_p^{n'}|0\rangle$, where $n$ and $n'$ are integers. We emphasize that the existence of inequivalent representations does not necessarily require the thermodynamic (infinite volume) limit in this model. This clearly shows that the choice of the unperturbed Hamiltonian and the associated vacuum is essential for the determination of physical quantities even in a finite Bose–Einstein condensed systems. For example, one can get the Bogoliubov spectrum only for the vacuum in a broken phase $|\Omega_0\rangle$, but never for that in a symmetric phase $|0\rangle$. The fact that the Bogoliubov spectrum and the Bogoliubov transformation are observed [13,14] forces us to choose the Bogoliubov vacuum $|\Omega_0\rangle$.

2.5 The Hugenholtz-Pines theorem at zero temperature

In this subsection, we derive the HP theorem from the WT relation at each loop level. First, we define $\delta \mu$ as the quantum correction to $\mu_0$ (the chemical potential at tree level),

$$\mu = \mu_0 + \delta \mu = g v^2 - \varepsilon \bar{\varepsilon} + \delta \mu ,$$  \hspace{1cm} (50)

and $\delta \mu$ will be determined later. We define the unperturbed Hamiltonian with $\mu_0$,

$$H_0 = \int d^3 x \left[ \phi^\dagger (x) \left( -\frac{\hbar^2}{2m} \nabla^2 - \mu_0 \right) \phi (x) \right]$$
while the perturbative Hamiltonian $H_{\text{int}}$ includes the $\delta \mu$ term,

$$H_{\text{int}} = v \left( -\mu_0 - \delta \mu + g v^2 - \varepsilon \bar{\epsilon} \right) \int d^3 x \left\{ \varphi(x) + \varphi^\dagger(x) \right\}$$

$$+ g v \int d^3 x \left[ \varphi^\dagger(x) \varphi^\dagger(x) \varphi(x) + \varphi^\dagger(x) \varphi(x) \varphi(x) \right]$$

$$+ \frac{g}{2} \int d^3 x \varphi^\dagger(x) \varphi^\dagger(x) \varphi(x) \varphi(x)$$

$$- \int d^3 x \delta \mu \varphi^\dagger(x) \varphi(x).$$

(52)

Since the unperturbed Hamiltonian $H_0$ is not changed from that at tree level, one can adopt the quasi-particle picture and the unperturbed propagator at tree level, given in Subsection 2.3.

Consider the Schwinger-Dyson equation,

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma(p) + \frac{\delta \mu}{\hbar} I,$$

(53)

where $G(p)$ and $\Sigma(p)$ are the full propagator and the self-energy from the loop diagrams, respectively, and $I$ is a unity $2 \times 2$ matrix. Both of $G(p)$ and $\Sigma(p)$ are matrices and can be calculated with the elements of the unperturbed matrix propagator (45) and (46) and the perturbative Hamiltonian (52). The matrix elements of $\Sigma(p)$ have the properties of

$$\Sigma_{11}(p) = \Sigma_{22}(-p), \quad \Sigma_{12}(p) = \Sigma_{21}(p),$$

(54)

where the matrix form of the self-energy is written as

$$\Sigma(p) = \begin{pmatrix} \Sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{22}(p) \end{pmatrix}.$$

(55)

Let us rewrite the RHS of the WT relation (31) at any loop level, we obtain that

$$- \frac{(\varepsilon \bar{\epsilon}) v}{\hbar} \left[ G_{11}(p = 0) - G_{12}(p = 0) \right]$$

$$= - \frac{(\varepsilon \bar{\epsilon}) v}{\hbar} \frac{G_{0,11}(p = 0) + G_{0,12}(p = 0) - \Sigma_{11}(p = 0) - \Sigma_{12}(p = 0) + \frac{\delta \mu}{\hbar}}{|G^{-1}(p = 0)|}$$

(56)
where $|G^{-1}(p)|$ represents the determinant of $G^{-1}(p)$. Since $G_0$ has the form in Eqs. (45) and (46), one can obtain the explicit form of $G_0^{-1}(p)$ as

$$
G_0^{-1}(p) = \begin{pmatrix}
(\omega - \omega_p)u_p^2 - (\omega + \omega_p)v_p^2 & (\omega + \omega_p)u_p v_p - (\omega - \omega_p)u_p v_p \\
(\omega + \omega_p)u_p v_p - (\omega - \omega_p)u_p v_p & (\omega - \omega_p)u_p^2 - (\omega + \omega_p)v_p^2
\end{pmatrix}. \tag{57}
$$

The determinant of $G^{-1}(p)$ is calculated as

$$
\left|G^{-1}(p)\right| = \left[\omega - \omega_p (u_p^2 + v_p^2) - \Sigma_{11}(p) + \frac{\delta \mu}{\hbar}\right] \left[-\omega - \omega_p (u_p^2 + v_p^2) - \Sigma_{22}(p) + \frac{\delta \mu}{\hbar}\right] \\
- [2\omega_p u_p v_p - \Sigma_{12}(p)]^2 \\
\simeq -\omega^2 + [\Sigma_{11}(p) - \Sigma_{22}(p)]\omega + \epsilon_p^2 + 2g v^2 \epsilon_p \\
+ \left(\frac{\epsilon_p}{\hbar} + \frac{g v^2}{\hbar}\right) \left[-\frac{2\delta \mu}{\hbar} + \Sigma_{11}(p) + \Sigma_{22}(p)\right] - \frac{2g v^2}{\hbar} \Sigma_{12}(p) \\
+ \frac{\epsilon \bar{\epsilon}}{\hbar} \left[\frac{2g v^2}{\hbar} - \frac{2\delta \mu}{\hbar} + \Sigma_{11}(p) + \Sigma_{22}(p)\right] + \frac{(\epsilon \bar{\epsilon})^2}{\hbar^2}. \tag{58}
$$

We dropped the quadratic terms of the self-energy and $\delta \mu$ in the second line of Eq. (58), because they are quantities of higher order. This way the RHS of the WT relation (31) at any loop level is organized as

$$
- \frac{(\epsilon \bar{\epsilon}) v}{\hbar} [G_{11}(p = 0) - G_{12}(p = 0)] \\
= \left[\frac{(\epsilon \bar{\epsilon}) v}{\hbar} \left\{\frac{2g v^2}{\hbar} - \frac{\delta \mu}{\hbar} + \Sigma_{11}(p = 0) + \Sigma_{12}(p = 0)\right\}\right] + \frac{(\epsilon \bar{\epsilon})^2 v}{\hbar^2} \\
\times \left[\frac{2g v^2}{\hbar} \left\{ -\frac{\delta \mu}{\hbar} + \Sigma_{11}(p = 0) - \Sigma_{12}(p = 0)\right\}\right] \\
+ 2\frac{\epsilon \bar{\epsilon}}{\hbar} \left\{\frac{g v^2}{\hbar} - \frac{\delta \mu}{\hbar} + \Sigma_{11}(p = 0)\right\} + \frac{(\epsilon \bar{\epsilon})^2}{\hbar^2} \left[\right]^{-1}. \tag{59}
$$

The WT relation (31) requires that this should be equal to $v$, or equivalently

$$
(2g v^2 + \epsilon \bar{\epsilon}) \left( -\frac{\delta \mu}{\hbar} + \Sigma_{11}(p = 0) - \Sigma_{12}(p = 0)\right) = 0 \tag{60}
$$

Since the nonvanishing order parameter $v \neq 0$ in the limit $\epsilon \rightarrow 0$ is under consideration, we finally attain the HP theorem
\[
\frac{\delta \mu}{\hbar} = \Sigma_{11}(p = 0) - \Sigma_{12}(p = 0). \tag{61}
\]

We again stress that the HP theorem is proven from the WT relation in the \(\varepsilon \to 0\) limit with finite \(V\) here, not in the thermodynamic (infinite volume) limit.

### 2.6 The Hugenholtz-Pines theorem at one-Loop level

In this subsection, let us check the HP theorem at one-loop level explicitly. According to the previous subsection, the HP theorem is a result of the WT relation. The loop expansion keeps the WT relation at each loop level. So, we naturally expect the HP theorem at one-loop level.

From the condition (6) at one-loop level, \(\delta \mu\) is given by

\[
\delta \mu = 2igG_{0,11}(x - x) + igG_{0,12}(x - x) = \frac{g}{V} \sum_{p = -\infty}^{\infty} \frac{2\epsilon_p + g\nu^2 + 2\varepsilon \bar{\epsilon}}{2E_p}. \tag{62}
\]

The elements of the matrix self-energy at one-loop level are obtained as

\[
\Sigma_{11}(p) = \Sigma_{22}(-p)
= \frac{g^2\nu^2}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{1}{2\omega_{p'}\omega_{p' - p'}} \left[ \frac{f(\omega_{p'}, \omega_{p' - p'})}{\omega - \omega_{p'} - \omega_{p' - p'} + i\delta} - \frac{f(-\omega_{p'}, -\omega_{p' - p'})}{\omega + \omega_{p'} + \omega_{p' - p'} - i\delta} \right]
+ \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{\epsilon_{p'} + g\nu^2 + \varepsilon \bar{\epsilon}}{\omega_{p'}}, \tag{63}
\]

\[
\Sigma_{12}(p) = \Sigma_{21}(p)
= \frac{g^2\nu^2}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{1}{2\omega_{p'}\omega_{p' - p'}} \left[ \frac{h(\omega_{p'}, \omega_{p' - p'})}{\omega - \omega_{p'} - \omega_{p' - p'} + i\delta} - \frac{h(-\omega_{p'}, -\omega_{p' - p'})}{\omega + \omega_{p'} + \omega_{p' - p'} - i\delta} \right]
- \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{g\nu^2}{2\omega_{p'}}, \tag{64}
\]

where we have introduced functions \(f(\omega_{p'}, \omega_{p' - p'})\) and \(h(\omega_{p'}, \omega_{p' - p'})\) as

\[
f(\omega_{p'}, \omega_{p' - p'}) = \frac{3\epsilon_{p'}\epsilon_{p' - p'}}{\hbar^2} - \omega_{p'}\omega_{p' - p'} + \frac{g\nu^2}{\hbar^2} (\epsilon_{p'} + \epsilon_{p' - p'}) + \frac{g^2\nu^4}{\hbar^2}
- \frac{g\nu^2}{\hbar} (\omega_{p'} + \omega_{p' - p'}) + \frac{1}{\hbar} (\epsilon_{p'}\omega_{p' - p'} + \epsilon_{p' - p'}\omega_{p'})
\]
Thus, the HP theorem has been confirmed at one-loop level.

We comment on a possible choice of the unperturbed Hamiltonian $H'_0$ which is obtained from replacing $\mu_0$ in Eq. (51) with $\mu$ given in Eq. (50). Then instead of $H_{\text{int}}$ we have the perturbative Hamiltonian $H'_0$ without the $\delta \mu$ term. One

\begin{align}
\Sigma_{11}(p = 0) &= \frac{g^2 v^2}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \left( -\frac{1}{4 \omega_{p'}} \right) \left( \frac{6 \epsilon_{p'}}{\hbar^2} - 2\omega_{p'}^2 + \frac{4g^2 v^2 \epsilon_{p'}}{\hbar^2} + \frac{2g^2 v^4}{\hbar^2} + 4\frac{\varepsilon \epsilon_{p'}}{\hbar} + \frac{6(\varepsilon)^2}{\hbar^2} \right) \\
&\quad + \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{\epsilon_{p'} + g v^2 + \varepsilon}{\omega_{p'}} \\
\Sigma_{12}(p = 0) &= \frac{g^2 v^2}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \left( -\frac{1}{4 \omega_{p'}} \right) \left( \frac{4 \epsilon_{p'}}{\hbar^2} - 4\omega_{p'}^2 + \frac{2g^2 v^4}{\hbar^2} + \frac{8(\varepsilon)^2}{\hbar^2} \right) \\
&\quad - \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{g v^2}{2\omega_{p'}}. \tag{68}
\end{align}

The following quantity, appearing in the RHS of Eq. (61), is simplified as

\begin{align}
\Sigma_{11}(p = 0) - \Sigma_{12}(p = 0) &= \frac{g^2 v^4}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \left( -\frac{1}{4 \omega_{p'}} \right) \left( \frac{2 \epsilon_{p'}}{\hbar^2} + 2\omega_{p'}^2 + 4\frac{g^2 v^2 \epsilon_{p'}}{\hbar^2} + \frac{4\epsilon \epsilon_{p'}}{\hbar^2} + \frac{4\varepsilon g v^2}{\hbar^2} + \frac{2(\varepsilon)^2}{\hbar^2} \right) \\
&\quad + \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{2\epsilon_{p'} + 3g v^2 + 2\varepsilon}{2\omega_{p'}} \\
&= \frac{g^2 v^2}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{1}{\omega_{p'}} + \frac{g}{\hbar^2 V} \sum_{p' = -\infty}^{\infty} \frac{2\epsilon_{p'} + 3g v^2 + 2\varepsilon}{2\omega_{p'}} \\
&= \frac{g}{\hbar V} \sum_{p' = -\infty}^{\infty} \frac{2\epsilon_{p'} + g v^2 + 2\varepsilon}{2E_{p'}} \\
&= \frac{\delta \mu}{\hbar}. \tag{69}
\end{align}

Thus, the HP theorem has been confirmed at one-loop level.

We comment on a possible choice of the unperturbed Hamiltonian $H'_0$ which is obtained from replacing $\mu_0$ in Eq. (51) with $\mu$ given in Eq. (50). Then instead of $H_{\text{int}}$ we have the perturbative Hamiltonian $H'_0$ without the $\delta \mu$ term. One
can repeat the same procedure of diagonalizing the unperturbed Hamiltonian as in Subsection 2.3. As a result, the unperturbed Hamiltonian is given by

\[ H_0 = \sum_{p=-\infty}^{\infty} E'_p \hat{b}_p^{\dagger} \hat{b}_p + \text{const.}, \]  

(70)

where

\[ E'_p = \sqrt{(\epsilon_p + gv^2 + \bar{\epsilon} - \delta \mu)^2 - g^2 v^4}. \]  

(71)

We find a difficulty because \( E'_p \) becomes complex in soft momentum region at one-loop level. Recalling \( \epsilon_p = p^2/2m \) and taking the limit of \( \epsilon \rightarrow 0 \), we have at \( p = 0 \),

\[ E'_{p=0} = \sqrt{\delta \mu (\delta \mu - 2gv^2)}. \]  

(72)

As is seen from Eq. (69), \( \delta \mu \) is determined to be positive for positive \( g \), which implies that \( E'_{p=0} \) is pure imaginary. Thus the choice of the unperturbed Hamiltonian \( H'_0 \) does not offer a consistent treatment.

3 Finite temperature case

We extend the discussions on the WT relations and the HP theorem at zero temperature to finite temperature case. We employ the TFD formalism to describe equilibrium situations. As will be seen, the TFD formalism is suitable for our purpose, because it is formulated as a canonical operator formalism of quantum field.

3.1 Thermofield dynamics

In this subsection, we give a brief review of TFD formalism. In TFD, thermal degrees of freedom are introduced by doubling each degree of freedom through the tilde conjugation. Thus, with an operator \( A \) we associate its tilde conjugate \( \tilde{A} \) according to the tilde conjugation rules [4,18]:

\[ (AB)^\dagger = \tilde{A} \tilde{B}; \]  

(73)

\[ (c_1A + c_2B)^\dagger = c_1^* \tilde{A} + c_2^* \tilde{B}; \]  

(74)

\[ (A^\dagger)^\dagger = \tilde{A}^\dagger, \]  

(75)
\[
\begin{align*}
(\tilde{A})^- &= \sigma A, \\
|\Omega\rangle^- &= |\Omega\rangle, \\
\beta\langle\Omega| &= \langle\Omega\rangle,
\end{align*}
\]
(76) (77) (78)

where \(c_1\) and \(c_2\) are complex \(c\)-numbers, \(|\Omega\rangle\) and \(\beta\langle\Omega|\) are the thermal vacua, and \(\sigma = 1\) for bosonic \(A\), while \(\sigma = -1\) for fermionic \(A\). The total Hamiltonian \(\hat{\tilde{H}}\) is given in terms of the non-tilde and tilde Hamiltonians as
\[
\hat{\tilde{H}} = \hat{H} - \hat{\tilde{H}},
\]
(79)

and the total Lagrangian \(\hat{\tilde{L}}\) is also given as the non-tilde Lagrangian minus the tilde one,
\[
\hat{\tilde{L}} = \hat{L} - \hat{\tilde{L}}.
\]
(80)

The thermal average is expressed as the thermal vacuum expectation value in TFD: \(\beta\langle\Omega|A|\Omega\rangle\). For simplicity, consider one bosonic degree of freedom whose creation and annihilation operators are represented by \(a\) and \(a^\dagger\), respectively. When the density matrix \(\rho\) is expressed in the form of \(\rho = f^{a^\dagger a}\), we have
\[
\beta\langle\Omega|A|\Omega\rangle = \frac{\text{tr}[\rho A]}{\text{tr}[\rho]},
\]
(81)

and
\[
n = \beta\langle\Omega|a^\dagger a|\Omega\rangle = \frac{f}{1 - f},
\]
(82)

using the properties of thermal vacua. If \(f = \exp(-\beta E)\) where \(\beta\) is the inverse temperature and \(E\) is an energy, \(\beta\langle\Omega|A|\Omega\rangle\) represents the thermal expectation value at finite temperature. One can easily generalize this argument to multimode cases.

For convenience for following calculation, let us introduce the matrix notation \(\phi^1_i\) and \(\tilde{\phi}^1_i\) which represent tilde and non-tilde quantum field \(\varphi\) and \(\tilde{\varphi}\) as
\[
\begin{align*}
\phi^1_i &= \phi_i, \quad \phi^2_i = \tilde{\phi}^\dagger_i, \\
\tilde{\phi}^1_i &= \phi^\dagger_i, \quad \tilde{\phi}^2_i = -\tilde{\phi}_i,
\end{align*}
\]
(83)

where
\[
\phi_1 = \varphi, \quad \phi_2 = \varphi^\dagger,
\]
\[
\begin{align*}
\phi_1^\dagger &= \varphi^\dagger, \quad \phi_2^\dagger &= \varphi, \\
\tilde{\phi}_1 &= \varphi, \quad \tilde{\phi}_2 &= \tilde{\varphi}^\dagger, \\
\tilde{\phi}_1^\dagger &= \tilde{\varphi}^\dagger, \quad \tilde{\phi}_2^\dagger &= \tilde{\varphi}.
\end{align*}
\] (84)

Then, the thermal propagators with four indices and with sixteen components are defined by

\[
G_{\beta,ij}^{\mu\nu}(x-x') = -i_\beta \langle \Omega | T[\phi_1^\mu(x)\bar{\phi}_2^\nu(x')] | \Omega \rangle_\beta,
\] (85)

where \(\mu, \nu = 1, 2\) and \(i, j = 1, 2\). The Fourier transformed thermal propagators are also defined as

\[
G_{\beta,ij}^{\mu\nu}(p) = \int \frac{d^4x}{(2\pi\hbar)^2} G_{\beta,ij}^{\mu\nu}(x) e^{ip \cdot (x-x') - \omega t}.
\] (86)

One can derive the following relations of the Fourier transformed thermal propagators by general properties of TFD:

\[
G_{\beta,11}^{\mu\nu}(p) = G_{\beta,22}^{\mu\nu}(-p), \quad G_{\beta,12}^{\mu\nu}(p) = G_{\beta,21}^{\mu\nu}(p), \quad G_{\beta}^{12}(p = 0) = 0.
\] (87)

### 3.2 The Ward-Takahashi relation at finite temperature

In TFD, the total Lagrangian \(\hat{\mathcal{L}}\) density is given by

\[
\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}}.
\] (88)

We consider following continuous infinitesimal transformations:

\[
\psi(x) \to \psi'(x) = \psi(x) + \xi \delta \psi(x) \\
\tilde{\psi}(x) \to \tilde{\psi}'(x) = \tilde{\psi}(x) + \xi \delta \tilde{\psi}(x).
\] (89)

The Nöther theorem at finite temperature is obtained from the infinitesimal transformations (89) as

\[
\frac{\partial}{\partial t} \hat{N}^0(x) + \nabla \cdot \hat{N}(x) = \delta \hat{\mathcal{L}},
\] (90)

where

\[
\hat{N}^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi(x))} \delta \Psi(x) - \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_\mu \tilde{\Psi}(x))} \delta \tilde{\Psi}(x)
\] (91)
\begin{align}
\delta \hat{\mathcal{L}} &= \delta \mathcal{L} - \delta \tilde{\mathcal{L}}. \quad (92)
\end{align}

Here, \( \delta \mathcal{L} \) was defined in Eq. (18) and \( \delta \tilde{\mathcal{L}} \) is defined as
\begin{align}
\xi \delta \hat{\mathcal{L}} &= \hat{\mathcal{L}}[\tilde{\Psi}'(x)] - \hat{\mathcal{L}}[\tilde{\Psi}(x)]. \quad (93)
\end{align}

One can easily obtain the WT relation at finite temperature by replacing \( \delta \mathcal{L} \) with \( \delta \hat{\mathcal{L}} \) in Eq. (25):
\begin{align}
\sum_{a=1}^{n} i\hbar_{\beta \frac{\langle x_{1} \cdots \Psi_{a} \cdots \Psi_{n} \rangle}{\langle x_{1} \cdots \Psi_{a} \cdots \Psi_{n} \rangle} \beta} = \int d^{4}x_{\beta} \langle \Omega \vert \hat{\mathcal{L}}(x_{1}) \cdots \Psi_{n} \rangle \vert \Omega \rangle_{\beta}. \quad (94)
\end{align}

Now we consider the following WT relation with respect to the global phase transformation, i.e., \( \delta \Psi(x) = i\Psi(x) \) and \( \delta \tilde{\Psi}(x) = -i\tilde{\Psi}(x) \) in Eq. (89). To deal with the spontaneous breakdown of the symmetry at finite temperature, we also introduce an infinitesimal symmetry breaking term with respect to the tilde field as
\begin{align}
\tilde{\mathcal{L}}_{\varepsilon} = (\varepsilon \bar{\varepsilon})v \left[ \tilde{\Psi}(x) + \tilde{\Psi}^{\dagger}(x) \right], \quad (95)
\end{align}

which is a tilde conjugate of \( \mathcal{L}_{\varepsilon} \) (7). Our starting total Lagrangian density at finite temperature is given by
\begin{align}
\hat{\mathcal{L}}_{\text{tot}} &= \mathcal{L}_{\text{tot}} - \tilde{\mathcal{L}}_{\text{tot}} , \quad (96)
\end{align}

where \( \mathcal{L}_{\text{tot}} \) was defined in Eq. (8) and \( \tilde{\mathcal{L}}_{\text{tot}} \) is defined as
\begin{align}
\tilde{\mathcal{L}}_{\text{tot}} = \tilde{\mathcal{L}} + \tilde{\mathcal{L}}_{\varepsilon}. \quad (97)
\end{align}

The Nöther charge of the total Lagrangian \( \hat{\mathcal{L}}_{\text{tot}} \) (96) with respect to the global phase transformations is obtained:
\begin{align}
\hat{N}(t) = N(t) - \tilde{N}(t) = -\int d^{3}x \left\{ \Psi^{\dagger}(x)\Psi(x) + \tilde{\Psi}^{\dagger}(x)\tilde{\Psi}(x) \right\}. \quad (98)
\end{align}

This generates the infinitesimal transformation, e.g.,
\[ \begin{align*}
\delta L_{\text{tot}} &= i[\hat{N}(t), L_{\text{tot}}(x)] = i(\varepsilon \bar{\varepsilon})v \left[ \Psi(x) - \Psi^\dagger(x) \right], \quad (99) \\
\delta \delta L_{\text{tot}} &= i[\hat{N}(t), \delta L_{\text{tot}}(x)] = -(\varepsilon \bar{\varepsilon})v \left[ \Psi(x) + \Psi^\dagger(x) \right], \quad (100)
\end{align*} \]

and
\[ \delta \hat{L}_{\text{tot}}(x) = i[\hat{N}(t), \hat{L}_\varepsilon(x)] = i(\varepsilon \bar{\varepsilon})v \left[ \Psi(x) - \Psi^\dagger(x) + \tilde{\Psi}(x) - \tilde{\Psi}^\dagger(x) \right]. \quad (101) \]

One can derive the following restricted relation from the WT relation at finite temperature (94), and from Eqs. (99), (100) and (101):
\[ i\hbar \beta \langle \Omega | \delta \delta L_{\text{tot}}(x) | \Omega \rangle_\beta = \int d^4x' \beta \langle \Omega | T[\delta \hat{L}_{\text{tot}}(x') \delta L_{\text{tot}}(x)] | \Omega \rangle_\beta. \quad (102) \]

We rewrite this relation in terms of the thermal propagators:
\[ v = -\frac{(\varepsilon \bar{\varepsilon})}{2\hbar} \int d^4x' \left[ G_{11}^{11}(x - x') + G_{12}^{11}(x - x') - G_{11}^{11}(x - x') - G_{12}^{11}(x - x') \right. \\
+ G_{12}^{12}(x - x') + G_{12}^{12}(x - x') - G_{12}^{12}(x - x') - G_{12}^{12}(x - x') \left. \right] \]
\[ \quad = -\frac{(\varepsilon \bar{\varepsilon})}{\hbar} \left[ G_{11}^{11}(p = 0) - G_{12}^{11}(p = 0) \right]. \quad (103) \]

3.3 The Ward-Takahashi relation at tree level in finite temperature case

Let us introduce the operators \( \{ \xi_p, \xi_p^\dagger \} \) and \( \{ \tilde{\xi}_p, \tilde{\xi}_p^\dagger \} \) related to the operators \( \{ b_p, b_p^\dagger \} \) and \( \{ \tilde{b}_p, \tilde{b}_p^\dagger \} \) by the following thermal Bogoliubov transformation:
\[ b_p = c_p \xi_p + s_p \tilde{\xi}_p^\dagger, \quad (104) \]

with
\[ c_p = \frac{1}{\sqrt{1 - e^{-\beta E_p}}}, \quad s_p = \frac{e^{-\beta E_p}}{\sqrt{1 - e^{-\beta E_p}}}, \quad (105) \]

where the energy \( E_p \) is given in Eq. (40). Since \( c_p^2 - s_p^2 = 1 \), we find that \( [\xi_p, \xi_p^\dagger] = [\tilde{\xi}_p, \tilde{\xi}_p^\dagger] = \delta_{pp'} \) and the other commutation relations vanish. With these operators, the unperturbed Hamiltonian at the finite temperature in TFD is given as
\[ \hat{H}_0 = \sum_{p=-\infty}^{\infty} E_p \left( b_p^\dagger b_p + \tilde{b}_p^\dagger \tilde{b}_p \right) = \sum_{p=-\infty}^{\infty} E_p \left( \xi_p^\dagger \xi_p - \tilde{\xi}_p^\dagger \tilde{\xi}_p \right). \quad (106) \]
The field operator \( \varphi(x) \) is rewritten in terms of the operators \( \{\xi_p, \xi_p^\dagger\} \) and \( \{\tilde{\xi}_p, \tilde{\xi}_p^\dagger\} \) as

\[
\varphi(x) = \frac{1}{\sqrt{V}} \sum_{p=-\infty}^{\infty} \left[ \left( c_p \xi_p + s_p \xi_p^\dagger \right) u_p e^{\frac{i}{\hbar}(p \cdot x - E_p t)} + \left( c_p \xi_p^\dagger + s_p \xi_p \right) v_p e^{-\frac{i}{\hbar}(p \cdot x - E_p t)} \right].
\] (107)

We construct the unperturbed propagators with this field operator and its Hermite and tilde conjugate:

\[
G_{\beta,0,ij}(x - x') = -i \beta \langle \Omega_0 | T [\phi_i^{\mu}(x) \phi_j^{\nu}(x')] | \Omega_0 \rangle_{\beta},
\] (108)

and its Fourier transformed one

\[
G_{\beta,0,ij}^{\mu\nu}(p) = \int \frac{d^4x}{(2\pi \hbar)^2} G_{\beta,0,ij}(x) e^{\frac{i}{\hbar}(p \cdot x - \omega t)}.
\] (109)

Here, the unperturbed thermal vacuum \( |\Omega_0\rangle_{\beta} \) is specified by \( \xi_p |\Omega_0\rangle_{\beta} = \tilde{\xi}_p |\Omega_0\rangle_{\beta} = 0 \). The explicit forms of the matrix elements, which are necessary to check the WT relation, are given as

\[
G_{\beta,0,11}^{11}(p) = G_{\beta,0,22}^{11}(-p) = \left( \frac{c_p^2}{\omega - \omega_p + i\delta} - \frac{s_p^2}{\omega - \omega_p - i\delta} \right) u_p^2 + \left( \frac{s_p^2}{\omega + \omega_p + i\delta} - \frac{c_p^2}{\omega + \omega_p - i\delta} \right) v_p^2.
\] (110)

\[
G_{\beta,0,12}(p) = G_{\beta,0,21}(p) = \left( \frac{c_p^2}{\omega - \omega_p + i\delta} + \frac{s_p^2}{\omega + \omega_p - i\delta} \right) u_p v_p.
\] (111)

At \( p = 0 \) they are reduced to

\[
G_{\beta,0,11}(p = 0) = G_{\beta,0,22}(p = 0) = \frac{1}{\omega_{p=0}} \left( c_{p=0}^2 - s_{p=0}^2 \right) \left( u_{p=0}^2 + v_{p=0}^2 \right) = -\frac{u_{p=0}^2 + v_{p=0}^2}{\omega_{p=0}} = G_{0,11}(p = 0) = G_{0,22}(p = 0),
\] (112)
\begin{align*}
G_{\beta,0,12}(p=0) &= G_{\beta,0,21}(p=0) \\
&= -\frac{2}{\omega_{p=0}} \left( c_{p=0}^2 - s_{p=0}^2 \right) u_{p=0} v_{p=0} \\
&= -\frac{2 u_{p=0} v_{p=0}}{\omega_{p=0}} = G_{0,12}(p=0) = G_{0,21}(p=0) .
\end{align*}
\quad (113)

Substituting Eqs. (112) and (113) into Eq. (103), we find that the WT relation at finite temperature holds at tree level:

\begin{align*}
- (\bar{\epsilon} \bar{\epsilon}) v &\left[ G_{\beta,11}(p=0) - G_{\beta,12}(p=0) \right] = - (\bar{\epsilon} \bar{\epsilon}) v [ G_{11}(p=0) - G_{12}(p=0) ] = v
\end{align*}
\quad (114)

In other words, at tree level, the Goldstone theorem also holds at finite temperature.

### 3.4 The Hugenholtz-Pines theorem at finite temperature

In Subsection 2.5, the HP theorem at \( T = 0 \) was derived from the WT relation. Now we will generalize this story to finite temperature case. We considered the quantum corrections of the chemical potential (50) at \( T = 0 \). At finite temperature one has to take account of thermal corrections as well as quantum ones. Explicitly \( \delta \mu \) in Eq. (50) now represents the thermal and quantum corrections to the chemical potential at tree level.

We consider the following Schwinger–Dyson equation at finite temperature:

\begin{align*}
G_{\beta}^{-1}(p) &= G_{\beta,0}^{-1}(p) - \Sigma_{\beta}(p) + \frac{\delta \mu}{\hbar} I_{4 \times 4} ,
\end{align*}
\quad (115)

where \( G_{\beta}(p) \) and \( \Sigma_{\beta}(p) \), having the indices \( (\mu, \nu) \) and \( (i, j) \) are the full propagator and self-energy at finite temperature, respectively, and \( I_{4 \times 4} \) is a unity \( 4 \times 4 \) matrix. In TFD, one can find the following relations in general,

\begin{align*}
G_{\beta,11}(p=0) &= G_{\beta,22}(p=0) ,
G_{\beta,12}(p=0) &= G_{\beta,21}(p=0) = 0 ,
\Sigma_{\beta,11}(p=0) &= \Sigma_{\beta,22}(p=0) ,
\Sigma_{\beta,12}(p=0) &= \Sigma_{\beta,21}(p=0) = 0 ,
\end{align*}
\quad (116)

Let us calculate the RHS of Eq. (103),
To that at

From Eqs. (112) and (113), we find that this expression is very similar in form to that at \( T = 0 \) in Eq. (56), except that the self-energy terms \( \Sigma_{\beta,11}(p = 0) \) and \( \Sigma_{\beta,12}(p = 0) \) replace \( \Sigma_{11}(p = 0) \) and \( \Sigma_{12}(p = 0) \) there. Thus the WT relation at finite temperature is equivalent to the following HP theorem at finite temperature:

\[
\frac{\delta \mu}{\hbar} = \Sigma_{\beta,11}(p = 0) - \Sigma_{\beta,12}(p = 0)
\]

which is very similarly as Eq. (61) has been derived at zero-temperature.

3.5 The Hugenholtz-Pines theorem at one-loop level in finite temperature case

Finally, in an explicit calculation at one-loop level in finite temperature case, we check the conclusion in the previous subsection, i.e., the HP theorem or the WT relation. The condition (6) fixes \( \delta \mu \) at one-loop level in finite temperature case (See Eq. (62) at \( T = 0 \)),

\[
\delta \mu = 2ig G_{\beta,0,11}(x - x) + ig G_{\beta,0,12}(x - x)
\]

\[
= g \sum_{p' = -\infty}^{\infty} \frac{1}{V} \coth \left( \frac{\beta E_{p'}}{2} \right) \frac{2\epsilon_{p'} + g v^2 + 2\epsilon \bar{\epsilon}}{2E_{p'}}.
\]

To investigate the HP theorem, it is necessary to have only the two matrix elements of the self-energy, \( \Sigma_{\beta,11}(p) \) and \( \Sigma_{\beta,12}(p) \). They read as

\[
\Sigma_{\beta,11}(p) = \frac{g^2 v^2}{\hbar^2} \sum_{p = -\infty}^{\infty} \frac{1}{V} \frac{1}{2\omega_{p'}\omega_{p - p'}}
\]

\[
\times \left[ \left( c_{p'}^2 c_p^2 - 2 c_{p'} c_p c_{p - p'} \right) \left\{ \frac{f(\omega_{p'}, \omega_{p - p'})}{\omega - \omega_{p'} - \omega_{p - p'} + i\delta} - \frac{f(-\omega_{p'}, -\omega_{p - p'})}{\omega + \omega_{p'} + \omega_{p - p'} - i\delta} \right\} \right.
\]

\[
+ \left( c_{p'}^2 s_p^2 - 2 c_{p'} s_p c_{p - p'} \right) \left\{ \frac{f(\omega_{p'}, \omega_{p - p'})}{\omega - \omega_{p'} + \omega_{p - p'} + i\delta} - \frac{f(-\omega_{p'}, -\omega_{p - p'})}{\omega + \omega_{p'} - \omega_{p - p'} - i\delta} \right\} \right.
\]
Using \( c_p^2 - s_p^2 = 1 \) and referring to the calculations in Subsection 2.6, we can easily find that the RHS of Eq. (118) is given by

\[
\Sigma_{\beta,12}(p) = \frac{g^2 v^2}{h^2} \sum_{p'=-\infty}^{\infty} \frac{1}{1 + \frac{2E_{p'}^2}{\omega_{p'}} + \frac{2\varepsilon_p^2 + 2\varepsilon_{\bar{p}}}{2E_{p'}}}.
\]

Using \( c_p^2 - s_p^2 = 1 \) and referring to the calculations in Subsection 2.6, we can easily find that the RHS of Eq. (118) is given by

\[
\Sigma_{\beta,11}(p = 0) - \Sigma_{\beta,12}(p = 0) = \frac{g}{h} \sum_{p'=-\infty}^{\infty} \frac{1}{1 + \frac{2E_{p'}^2}{\omega_{p'}} + \frac{2\varepsilon_p^2 + 2\varepsilon_{\bar{p}}}{2E_{p'}}} = \frac{\delta \mu}{h}.
\]
case of the SSB for finite volume system. Then it is found that the scheme for the trapped dilute Bose system satisfying the Goldstone theorem also need explicit treatment of the NG mode.

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A Bogoliubov transformation and vacua

In this appendix, we give explicit relation between \( |0\rangle \) and \( |\Omega_0\rangle \) which are the vacuum of \( a_p \) and \( b_p \) respectively [19].

The Bogoliubov transformation (38) is written as the following matrix form:

\[
\begin{pmatrix}
  b_p \\
  b_{-p}^\dagger
\end{pmatrix} = \begin{pmatrix}
  u_p & -v_p \\
  -v_p & u_p
\end{pmatrix} \begin{pmatrix}
  a_p \\
  a_{-p}^\dagger
\end{pmatrix}.
\] (A.1)

Next, we introduce an operator \( U_p \) which induces the transformation (A.1):

\[
b_p = U_p a_p U_p^\dagger, \quad b_{-p}^\dagger = U_p a_{-p}^\dagger U_p^\dagger,
\] (A.2)

where \( U_p \) must be unitary,

\[
U_p U_p^\dagger = U_p^\dagger U_p = \hat{1}.
\] (A.3)
Here, $\hat{1}$ is the identity operator. Eqs. (A.2) imply

$$b_p U_p |0\rangle = 0, \quad \langle 0 | U_p^\dagger \hat{1} U_p = 0. \quad (A.4)$$

We define the unitary operator $U$ as

$$U = \prod_{p=0}^{\infty} U_p, \quad (A.5)$$

where $\prod_{p=0}^{\infty}$ contains the suffix $p = 0$, and $U$ is obviously unitary. Then the relation between $|\Omega_0\rangle$ and $|0\rangle$ is given as

$$|\Omega_0\rangle = U |0\rangle. \quad (A.6)$$

First, we consider the $p \neq 0$ part of $U$. To determine an explicit representation of $U_p$ in terms of $a_p$ and $a_{-p}^\dagger$, we define the coherent state of $a_p$ and $a_{-p}^\dagger$,

$$|z(p)\rangle = \exp[z_p a_p^\dagger] \exp[z_{-p} a_{-p}^\dagger] |0\rangle = \left[ \sum_{\ell=0}^{\infty} \frac{(z_p a_p^\dagger)^\ell}{\ell!} \right] \left[ \sum_{\ell'=0}^{\infty} \frac{(z_{-p} a_{-p}^\dagger)^{\ell'}}{\ell'!} \right] |0\rangle, \quad (A.7)$$

where $z_p$ and $z_{-p}$ are eigenvalues of $a_p$ and $a_{-p}$, i.e.,

$$a_p |z(p)\rangle = z_p |z(p)\rangle, \quad a_{-p} |z(p)\rangle = z_{-p} |z(p)\rangle. \quad (A.8)$$

The coherent state has the properties of

$$a_p^\dagger |z(p)\rangle = \partial_{z_p} |z(p)\rangle, \quad a_{-p}^\dagger |z(p)\rangle = \partial_{z_{-p}} |z(p)\rangle, \quad (A.9)$$

$$\langle z(p) | a_p^\dagger = \langle z(p) | z_p^*, \quad \langle z(p) | a_{-p}^\dagger = \langle z(p) | z_{-p}^*, \quad (A.10)$$

$$\langle z(p) | a_p = \partial_{z_p} \langle z(p) |, \quad \langle z(p) | a_{-p} = \partial_{z_{-p}} \langle z(p) |, \quad (A.11)$$

and

$$\langle 0 | z(p) \rangle = 1. \quad (A.12)$$

The completeness condition of the coherent state reads

$$\int d\mu(z(p)) |z(p)\rangle \langle z(p)| = \int d\mu(z(p)) \ [|z_p \rangle \otimes |z_{-p}\rangle \langle z_{-p}|] = \hat{1}, \quad (A.13)$$

where the measure is defined as
\[
d\mu(z(p)) = \left[ e^{-|z_p|^2} \frac{dx_p \, dy_p}{\pi} \right] \left[ e^{-|z_{-p}|^2} \frac{dx_{-p} \, dy_{-p}}{\pi} \right], \quad (A.14)
\]
\[
z_{\pm p} = x_{\pm p} + iy_{\pm p}. \quad (A.15)
\]

We also mention the following formula of a generalized Gaussian integral:
\[
\int d\mu(z(p)) \exp \left[ -\frac{1}{2} \left( z^T(p)Mz(p) + z^*T(p)N^*z^*(p) \right) + u^Tz(p) + v^*Tz^*(p) \right] = \left[ \det(I - MN^*) \right]^{-\frac{1}{2}} \times \exp \left[ -\frac{1}{2} u^T(I - N^*M)^{-1}N^*u - \frac{1}{2} v^*T(I - MN^*)^{-1}Mv^* \right. \nonumber
\]
\[
\left. + v^*T(I - MN^*)^{-1}u \right]. \quad (A.16)
\]

Here, \( M \) and \( N^* \) are \( 2 \times 2 \) symmetric matrices,
\[
M^T = M, \quad N^{*T} = N^*, \quad (A.17)
\]

\( I \) is a unity \( 2 \times 2 \) matrix, and \( z, u \) and \( v \) are \( c \)-number \( 2 \)-components vectors,
\[
z(p) = \begin{pmatrix} z_p \\ z_{-p} \end{pmatrix}, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \quad v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}. \quad (A.18)
\]

Now, we are led to the operator \( U_p \) explicitly in terms of \( a_p \) and \( a_{\dagger p} \). One can easily derive the following relations from Eqs. (A.2)–(A.4) and (A.8):
\[
(b_p - z_p)|U_p|^z(p)\rangle = 0, \quad (b_{-p}^\dagger - \partial_{z_{-p}})|U_p|^z(p)\rangle = 0. \quad (A.19)
\]

Putting the expression (A.1) into Eqs. (A.19), we obtain
\[
(u_p a_p - v_p a_{\dagger p} - z_p)|U_p|^z(p)\rangle = 0, \quad (A.20)
\]
\[
(-v_p a_p + u_p a_{\dagger p} - \partial_{z_{-p}})|U_p|^z(p)\rangle = 0, \quad (A.21)
\]

whose inner product with the coherent state \( \langle w(p) \rangle \) are given by
\[
(u_p \partial_{w_p^*} - v_p w_{-p}^* - z_p)\langle w(p) | U_p | z(p) \rangle = 0, \quad (A.22)
\]
\[
(-v_p \partial_{w_p^*} + u_p w_{-p}^* - \partial_{z_{-p}})\langle w(p) | U_p | z(p) \rangle = 0. \quad (A.23)
\]

One finds that the following solution is admitted,
\[ \langle w(p)|U_p|z(p)\rangle = \frac{1}{u_p} \exp \left[ \frac{1}{2} w^*(p)X_p w(p) - \frac{1}{2} z^T(p)X_p z(p) + w^T(p)Y_p z(p) \right], \quad (A.24) \]

where

\[ X_p = \frac{v_p}{u_p} \sigma^1, \quad Y_p = \frac{1}{u_p} I, \quad (A.25) \]

and

\[ w(p) = \begin{pmatrix} w_p \\ w_{-p} \end{pmatrix}. \quad (A.26) \]

In the definition of the matrices \( X_p \) in Eq. (A.25), we used the Pauli matrix:

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (A.27) \]

The \( w \)- and \( z \)-independent normalization factor in Eq. (A.24) is fixed by the relation,

\[
\exp \left( w^*(p)w'(p) \right) = \langle w(p)|w'(p) \rangle \\
= \langle w(p)|U_p U_p^\dagger |w'(p) \rangle \\
= \int d\mu(z(p)) \langle w(p)|U_p z(p) \rangle \langle z(p)|U_p^\dagger |w'(p) \rangle \\
= \int d\mu(z(p)) \langle w(p)|U_p z(p) \rangle (\langle z'(p)|U_p z(p) \rangle)^*, \quad (A.28)
\]

where the formula (A.16) can be applied to the last expression of the integral. Thus we obtain an explicit operator representation of \( U_p \) with the help of the relation

\[ A(w^*_p, w^*_p, z_p, z_{-p}) = \langle w(p)|A(a_p^\dagger, a_{-p}^\dagger, a_p, a_{-p}): z(p) \rangle / \langle w(p)|z(p) \rangle \]

where the symbol \( : \cdots : \) implies the normal ordering to keep the creation operators to the left of the annihilation operators,

\[ U_p = \frac{1}{u_p} : \exp \left[ \frac{1}{2} a_p^\dagger T(p)X_p a^\dagger(p) - \frac{1}{2} a^T(p)X_p a(p) + a^T(p)Y_p a(p) \right] :. \quad (A.29) \]

where

\[ a(p) = \begin{pmatrix} a_p \\ a_{-p} \end{pmatrix}, \quad a^\dagger(p) = \begin{pmatrix} a_p^\dagger \\ a_{-p}^\dagger \end{pmatrix}. \quad (A.30) \]
Next, we consider the $p = 0$ part of $U$. The coherent state of the $p = 0$ mode as

$$ |z(0)\rangle = \exp[z_0 a_0^\dagger]|0\rangle = \left[ \sum_{\ell=0}^{\infty} \frac{(z_0 a_0^\dagger)^\ell}{\ell!} \right]|0\rangle, \quad (A.31) $$

where

$$ a_0|z(0)\rangle = z_0|z(0)\rangle, \quad a_0^\dagger|z(0)\rangle = \partial_{z_0}|z(0)\rangle, \quad (A.32) $$

$$ \langle z(0)|a_0 = \partial_{z_0^*}\langle z(0)|, \quad \langle z(0)|a_0^\dagger = z_0^*\langle z(0)|, \quad (A.33) $$

and

$$ \langle 0|z(0)\rangle = 1. \quad (A.34) $$

The completeness condition of the coherent state reads

$$ \int d\mu(z(0)) = \hat{1}, \quad (A.35) $$

where the measure is defined as

$$ d\mu(z(0)) = e^{-|z_0|^2} \frac{dx_0 \, dy_0}{\pi}, \quad (A.36) $$

$$ z_0 = x_0 + iy_0. \quad (A.37) $$

The following Gaussian integral is relevant here:

$$ \int d\mu(z(0)) \exp \left[ -\frac{1}{2} \left( M' z_0^2 + N' z_0^* z_0 \right) + u' z_0 + v' z_0^* \right] $n

$$ = (1 - M' N'^*)^{-1} \exp \left[ -\frac{1}{2} \frac{N' u'^2 + M' v'^2 - 2u' v'}{1 - M' N'^*} \right], \quad (A.38) $$

where $N'$, $M'$, $u'$, $v'$ are constants. In a manner similar to the manipulations of the $p \neq 0$ part of the $U$, we obtain the following equations,

$$ (u_0 \partial_{w_0^*} - v_0 w_0^* - z_0)\langle w(0)|U_0|z(0)\rangle = 0, \quad (A.39) $$

$$ (-v_0 \partial_{w_0^*} + u_0 w_0^* - \partial_{z_0})\langle w(0)|U_0|z(0)\rangle = 0. \quad (A.40) $$

One easily find the matrix elements of $U_0$ as

$$ \langle w(0)|U_0|z(0)\rangle = \frac{1}{\sqrt{u_0}} \exp \left[ \frac{v_0}{2u_0}(w_0^2 - z_0^2) + \frac{1}{u_0} w_0^* z_0 \right], \quad (A.41) $$
so the representation of $U_0$ in terms of $a_p$ and $a_0^\dagger$ is given as

$$U_0 = \frac{1}{\sqrt{u_0}} : \exp \left[ \frac{1}{2} \frac{v_0}{u_0} (a_0^\dagger a_0^2 - a_0^2) + \frac{1}{u_0} a_0^\dagger a_0 \right] : .$$  \hspace{1cm} (A.42)

As a summary of this appendix, the vacuum of the quasi-particle, described by the $b_p$ operators including the $p = 0$ case, is explicitly given by

$$|\Omega_0\rangle = U|0\rangle$$

$$= \frac{1}{\sqrt{u_0}} \exp \left\{ \frac{1}{2} \frac{v_0}{u_0} a_0^\dagger a_0 \right\} \left[ \prod_{p=0}^{\infty} \frac{1}{u_p} \exp \left\{ \frac{1}{2} a_p^{\dagger T}(p) X_p a_p(p) \right\} \right] |0\rangle$$

$$= \frac{1}{\sqrt{u_0}} \exp \left[ -\frac{1}{2} \sum_{p=0}^{\infty} \ln u_p \right] \exp \left[ \frac{1}{2} \sum_{p=0}^{\infty} \frac{v_p}{u_p} a_p^\dagger a_p^\dagger \right] |0\rangle ,$$  \hspace{1cm} (A.43)

where $\prod_{p=-\infty}^{-0}'$ and $\sum_{p=0}^{\infty}'$ mean product and summation without $p = 0$, respectively.

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