Quantum Representation Theory and Manin matrices II: super case

Alexey Silantyev*

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow region, Russia
State University “Dubna”, Universitetskaya st. 19, 141980 Dubna, Moscow region, Russia

Abstract

We construct super-version of Quantum Representation Theory. The quadratic super-algebras and operations on them are described. We also describe some important monoidal functors. We proved that the monoidal category of graded super-algebras with Manin product is coclosed relative to the subcategory of finitely generated quadratic super-algebras. The super-version of the \((A, B)\)-Manin matrices is introduced and related with the quadratic super-algebras. We define a super-version of quantum representations and of quantum linear actions, relate them to each other and describe them by the super-Manin matrices. Some operations on quantum representations/quantum linear actions are described. We show how the classical representations lift to the quantum level.

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*aleksesilantjev@gmail.com
1 Introduction

It is well-known that in the usual differential/algebraic geometry the spaces contravariantly corresponds to the algebras of functions. The idea to generalise different kinds of spaces by substituting the commuting algebras of functions by non-commutative algebras is quite old. One source of such generalisation is quantisation in physics. By this reason such ‘non-commutative’ generalisation of a space are often called ‘quantum space’. A generalisation of Lie/algebraic group was called ‘quantum group’ by Drinfeld [Dr).

In [Man88] Yuri Manin proposed to consider quadratic algebras as quantum analogues of the vector spaces. This was a starting point to generalise classical representation theory to the quantum case in [S2]; namely, the notion of representation was generalised to the case of a quantum representation space. Such quantum representations are described by so-called Manin matrices.

In some form the Manin matrices appeared in [Man87, Man88] in connection with the quadratic algebras. The Manin matrices for polynomial algebras and their $q$-deformations were investigated and applied in the works [GLZ, CF, CM, CFR, RST, CFRS, IO, Molev]. For general quadratic algebras they were described in [S1]. Super-analogues of the Manin matrices was considered in [Man89, MR].

The main tool of Quantum Representation Theory in [S2] is the theory of monoidal categories. The correspondence between linear actions and representations in a closed monoidal category was generalised to a wider class of monoidal categories. This general representation theory was developed in [S3].

In the present paper we consider super-versions of quadratic algebras, of Manin matrices and of operations with the quadratic algebras. We apply the general approach of [S3] to an appropriate monoidal category and describe quantum representations by the super-Manin matrices.

The article is organised as follows. Section 2 is preliminary: we recall and introduce some terms and notations here. The quadratic super-algebras are investigated in Section 3. The binary operations (monoidal products) on quadratic super-algebras are described in Subsection 3.1. In Subsection 3.2 we collect useful functors and describe their properties. Subsection 3.3 is devoted to the coclosed structure on a monoidal category of the quadratic super-algebras given by so-called internal cohom-functor. In Subsection 4.1 we introduce super-Manin matrices for a pair of quadratic super-algebras in terms idempotent operators; also we introduce universal super-Manin matrices and use them to describe the internal cohom-functor. Quantum Representation Theory for the super-case is described in Subsection 4.2.
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2 Preliminaries

In this work we use the terms and notations introduced in §2 of [S2] and §2 of [S3]. First we briefly remind some of them. Then we introduce the structure of \(A\)-grading and consider the main category we use – the category of \(A\)-graded algebras, where \(A = \mathbb{Z}_2 \times \mathbb{Z}\).

2.1 Notations and conventions

2.1.1. Monoidal categories. We denote a monoidal category by \((C, \otimes)\) or \((C, \otimes, I)\), where \(C\) is a category with a bifunctor \(\otimes: C \times C \to C\) and a unit object \(I = I_C\). We suppose monoidal categories to be strict. The monoids and comonoids in this monoidal category form the categories \(\text{Mon}(C, \otimes)\) and \(\text{Comon}(C, \otimes) = (\text{Mon}(C^{\text{op}}, \otimes))^{\text{op}}\). For any \(M \in \text{Mon}(C, \otimes)\) and \(O \in \text{Comon}(C, \otimes)\) we have categories of (left) actions and coactions \(\text{Lact}(M)\) and \(\text{Lcoact}(O)\).

Let the monoidal category \((C, \otimes)\) be symmetric. Then \(\text{Mon}(C, \otimes)\) and \(\text{Comon}(C, \otimes)\) are monoidal and also symmetric (the monoidal product in this category is usually denoted by the same symbol). They have full subcategories of commutative monoids and cocommutative comonoids: \(\text{Mon}(C, \otimes) \subset \text{Mon}(C, \otimes)\), \(\text{cocomon}(C, \otimes) \subset \text{Mon}(C, \otimes)\). In this case one can also define bimonoids, which form the category

\[
\text{Bimon}(C, \otimes) = \text{Comon} \left(\text{Mon}(C, \otimes), \otimes\right) = \text{Mon} \left(\text{Comon}(C, \otimes), \otimes\right).
\]

Any bimonoid \(B \in \text{Bimon}(C, \otimes)\) gives the monoidal categories \(\text{Lact}(B)\) and \(\text{Lcoact}(B)\). They are symmetric if \(B\) is cocommutative or commutative respectively.

Any category \(C\) with finite products is a symmetric monoidal category \((C, \times)\), where \(\times\) is the categorical product and the unit object is the terminal object. An example of such category is the category of sets \(\text{Set}\).

2.1.2. Monoidal functors. A lax monoidal structure of a functor \(F: C \to D\) between two monoidal categories \((C, \otimes)\) and \((D, \odot)\) is given by morphisms \(\varphi: I_D \to FI_C\) and \(\phi_{X,Y}: FX \odot FY \to F(X \otimes Y)\) satisfying some conditions. A colax monoidal structure of \(F: C \to D\) is given by \(\varphi: FI_C \to I_D\) and \(\phi_{X,Y}: F(X \otimes Y) \to FX \odot FY\). Lax/colax monoidal functor is the triple \(F = (F, \varphi, \phi): (C, \otimes) \to (D, \odot)\). A monoidal functor between symmetric monoidal categories is called symmetric if it respects the symmetric structures.

Any (co)lax monoidal functor translates (co)monoids to (co)monoids and (co)actions to (co)actions, so we have the induced functors \(\text{Mon}(F): \text{Mon}(C, \otimes) \to \text{Mon}(D, \odot)\), \(F_M: \text{Lact}(M) \to \text{Lact} \left(\text{Mon}(F)M\right)\) for the lax case and \(\text{Comon}(F): \text{Comon}(C, \otimes) \to \text{Comon}(D, \odot)\), \(F^O: \text{Lcoact}(O) \to \text{Lcoact} \left(\text{Comon}(F)O\right)\) for the colax case.
If all $\varphi$ and $\phi_{X,Y}$ are isomorphisms, then the monoidal functor $F = (F, \varphi, \phi): (C, \otimes) \to (D, \odot)$ is called strong monoidal. A symmetric strong monoidal functor induces the functors $\text{Bimon}(F): \text{Mon}(C, \otimes) \to \text{Bimon}(D, \odot)$, $F_\mathbb{B}: \text{Lact}(\mathbb{B}) \to \text{Lact}(\text{Mon}(F)\mathbb{B})$ and $F^{\text{op}}_\mathbb{B}: \text{Lcoact}(\mathbb{B}) \to \text{Lcoact}(\text{Comon}(F)\mathbb{B})$.

A contravariant functor $F: C \to D$ is called (symmetric) lax/colax/strong monoidal iff the corresponding covariant functor $\tilde{F}: C^{\text{op}} \to D$ is (symmetric) lax/colax/strong monoidal (or, equivalently, the opposite functor $F^{\text{op}}: C \to D^{\text{op}}$ is (symmetric) colax/lax/strong monoidal).

### 2.1.3. Relative adjoints \[\text{[Ulms], [S3] \S \text{2.3}}\]. Consider categories $C$ and $D$. We say that a functor $F: C \to D$ has a right adjoint relative to a full subcategory $D' \subset D$ iff there exists a functor $G: D' \to C$ (called right adjoint for $F$ relative to $D'$) and a bijection

$$\text{Hom}(FX, Z) \cong \text{Hom}(X, GZ)$$

natural in $X \in C$, $Z \in D'$. We say that a functor $G: D \to C$ has a left adjoint relative to a full subcategory $C' \subset C$ iff there exists a functor $F: C' \to D$ (called left adjoint for $G$ relative to $C'$) and a bijection (2.1) natural in $X \in C'$ and $Z \in D$. The right/left adjoint with the relative adjunction (2.1) is unique up to an isomorphism. A functor $F: C \to D$ has a right adjoint relative to $D' \subset D$ iff there exist a universal morphism from $F$ to each object $Z \in D'$. A functor $G: D \to C$ has a left adjoint relative to $C' \subset C$ iff there exist a universal morphism from each object $X \in C'$ to $G$.

### 2.1.4. Relatively closed monoidal categories. In \[\text{[S3] \S \text{2.4}}\] we generalised the notions of closed and coclosed monoidal category in terms of relative adjoints. We call a monoidal category $(C, \otimes)$ closed/coclosed relative to a full subcategory $P \subset C$ or relatively closed/coclosed with parametrising subcategory $P$ iff for any $Y \in P$ the functor $F_Y = - \otimes Y$ has a right/left adjoint relative to $P$.

Fix parametrising subcategory $P$. Then $(C, \otimes)$ is relatively closed iff there exists a bifunctor $\text{hom}: P^{\text{op}} \times P \to C$ (called internal hom-functor) and a bijection

$$\theta: \text{Hom} \left( X, \text{hom}(Y, Z) \right) = \text{Hom}(X \otimes Y, Z)$$

natural in $X \in C$, $Y, Z \in P$. In this case the internal end-object $\text{end}(Y) = \text{hom}(Y,Y)$ has a structure of monoid in $(C, \otimes)$ for any $Y \in P$.

Dually, the monoidal category $(C, \otimes)$ is relatively coclosed iff there exists a bifunctor $\text{cohom}: P^{\text{op}} \times P \to C$ (called internal cohom-functor) and a bijection

$$\vartheta: \text{Hom} \left( \text{cohom}(Y, X), Z \right) = \text{Hom}(X, Z \otimes Y)$$

natural in $X, Y \in P$, $Z \in C$. In this case the internal coend-object $\text{coend}(Y) = \text{cohom}(Y,Y)$ has a structure of comonoid in $(C, \otimes)$ for any $Y \in P$.

The monoidal category is closed/coclosed iff it is closed/coclosed relative to the whole category $C$. 

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2.1.5. **Vector spaces and algebras.** Fix an infinite field \( \mathbb{K} \) of characteristic \( \text{char} \mathbb{K} \neq 2 \). We consider all vector spaces over \( \mathbb{K} \). The category of such vector spaces is denoted by \( \text{Vect} \). This is a monoidal category with the standard tensor product \( V \otimes W = V \otimes_\mathbb{K} W \). The monoids in this category form the category \( \text{Alg} = \text{Mon}(\text{Vect}, \otimes) \). We call its objects simply *algebras* (these are associative unital algebras over \( \mathbb{K} \)). An identity element of an algebra \( \mathcal{A} \) is denoted by \( 1_\mathcal{A} \) or simply 1.

The category \( \text{Alg} \) is also monoidal with respect to the tensor product of algebras. Its full monoidal subcategory consisting of the commutative algebras is opposite to the category of the affine schemes over \( \mathbb{K} \): \( (\text{CommAlg}, \otimes)^{\text{op}} = (\text{AffSch}, \times) \). Denote the categories of finite-dimensional vector spaces and algebras by \( \text{FVect} \) and \( \text{FAlg} = \text{Mon}(\text{FVect}, \otimes) \). These are monoidal subcategories of \( (\text{Vect}, \otimes) \) and \( (\text{Alg}, \otimes) \).

2.1.6. \( \mathcal{A} \)-**graded vector spaces.** Let \( \mathcal{A} \in \mathcal{C}\text{Mon}(\text{Set}, \times) \) be an Abelian monoid with the binary operation \( \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) written additively: \((a, b) \mapsto a + b\). Denote by \( \mathcal{A}\text{-GrVect} \) the category of \( \mathcal{A} \)-graded vector spaces \( V = \bigoplus_{g \in \mathcal{A}} V_g \). Elements of the component \( V_g \) are called homogeneous of degree \( g \). The morphisms in this category are graded linear maps \( f : V \to W \), i.e. \( f(V_g) \subset W_g \). Each such map is given by a family of arbitrary linear maps \( f_g : V_g \to W_g \). In other words, \( \mathcal{A}\text{-GrVect} \) is the category of functors \( \mathcal{A} \to \text{Vect} \).

The tensor product of two \( \mathcal{A} \)-graded vector spaces decomposed as

\[
V \otimes W = \bigoplus_{g \in \mathcal{A}} (V \otimes W)_g, \quad (V \otimes W)_g = \bigoplus_{h, h' \in \mathcal{A}, h + h' = g} V_h \otimes W_{h'}
\]  

(2.4)

(if \( \mathcal{A} \) is a group then \( (V \otimes W)_g = \bigoplus_{h \in \mathcal{A}} V_h \otimes W_{g-h} \)). In this way we obtain a monoidal category \( (\mathcal{A}\text{-GrVect}, \otimes) \). Its unit object is the vector space \( \mathbb{K} \) with the grading \( (\mathbb{K})_0 = \mathbb{K}, \mathbb{K}_g = 0 \) for \( g \neq 0 \), where 0 is the neutral element of \( \mathcal{A} \).

The monoidal category \( (\mathcal{A}\text{-GrVect}, \otimes) \) is closed. The component \( \text{hom}(W, Z)_g \) is the vector space consisting of the operators \( f : W \to Z \) such that \( f(W_h) = V_{g+h} \).

2.1.7. \( \mathcal{A} \)-**graded algebras.** A monoid \( \mathcal{A} \in \text{Mon}(\mathcal{A}\text{-GrVect}, \otimes) \) is an algebra with a grading \( \mathcal{A} = \bigoplus_{g \in \mathcal{A}} \mathcal{A}_g \) such that \( \mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{g+h} \) and \( 1_\mathcal{A} \in \mathcal{A}_0 \) (if \( \mathcal{A} \) is a group or a submonoid of a group, then the former condition implies the latter one). We obtain the category \( \mathcal{A}\text{-GrAlg} = \text{Mon}(\mathcal{A}\text{-GrVect}, \otimes) \), where the morphisms are graded homomorphisms of algebras. Its objects are called \( \mathcal{A} \)-graded algebras.

If \( (\mathcal{A}\text{-GrVect}, \otimes) \) is equipped with a symmetric structure, then it gives a monoidal structure on \( \mathcal{A}\text{-GrAlg} \) with the symmetric structure. Recall that the standard symmetric structure of \( (\text{Vect}, \otimes) \) is defined by the permutation operators \( \sigma_{V,W} : V \otimes W \to W \otimes V, \ v \otimes w \mapsto w \otimes v \). These operators are graded due to the commutativity of \( \mathcal{A} \), so it gives a simplest symmetric structure on \( (\mathcal{A}\text{-GrVect}, \otimes) \). However, it is not unique.

Let \( \mathcal{A}' \subset \mathcal{A} \) be a submonoid. Then \( \mathcal{A}'\text{-GrVect} \) and \( \mathcal{A}'\text{-GrAlg} \) are full monoidal subcategories of \( \mathcal{A}\text{-GrVect} \) and \( \mathcal{A}\text{-GrAlg} \), consisting of the \( \mathcal{A} \)-graded vector spaces (algebras).
2.2.1. Super-vector spaces and super-algebras.

2.2. Graded super-algebras

2.2.1. Super-vector spaces and super-algebras. Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, \overline{1}\}$. This is an Abelian additive group, which has a structure of ring given by the multiplication $\overline{k} \cdot \overline{l} = \overline{kl}$. Consider the category $\text{SVect} := \mathbb{Z}_2\text{-GrVect}$. Its objects $V \in \text{SVect}$ are called super-vector spaces. The homogeneous elements $v \in V_0$ are called even, while $v \in V_1$ are called odd. For $v \in V_\overline{k}$ we use the notation $[v] := \overline{k}$.

A subspace $\widetilde{V}$ of a super-vector space $V \in \text{SVect}$ is called super-subspace iff $\widetilde{V}$ has a structure of $\mathbb{Z}_2$-graded such that the embedding $\widetilde{V} \hookrightarrow V$ is $\mathbb{Z}_2$-graded, i.e. $\widetilde{V} = \widetilde{V}_0 \oplus \widetilde{V}_1$ where $\widetilde{V}_0 = \widetilde{V} \cap V_0$ and $\widetilde{V}_1 = \widetilde{V} \cap V_1$.

The objects of $\text{SAAlg} := \mathbb{Z}_2\text{-GrAlg}$ are monoids in the monoidal category $(\text{SVect}, \otimes)$ called super-algebras. The super-vector space $\text{end}(V) = \text{hom}(V,V)$ is equipped with a structure of super-algebra. The categories of finite-dimensional super-vector spaces and super-algebras are full monoidal subcategories $\text{FSVect}$ and $\text{FSAlg} = \text{Mon}(\text{FVect}, \otimes)$ of $(\text{SVect}, \otimes)$ and $(\text{SAAlg}, \otimes)$ respectively.

Fix the following symmetric structure on this monoidal category:

$$\sigma(v \otimes w) = (-1)^{|v||w|}w \otimes v,$$  

(2.5)

where $(-1)^k = (-1)^\overline{k}$ (the usage of the notation $[v]$ supposes that $v$ is homogeneous). We obtain the symmetric monoidal category $(\mathbb{Z}_2\text{-GrAlg}, \otimes)$. The product $\mathcal{R} \otimes \overline{\mathcal{R}}$ of two super-algebras $\mathcal{R}$ and $\overline{\mathcal{R}}$ is defined by the formula

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|}(a \cdot c) \otimes (b \cdot d), \quad a, c \in \mathcal{R}, \quad b, d \in \overline{\mathcal{R}}.$$  

(2.6)

The unit object of $(\mathbb{Z}_2\text{-GrAlg}, \otimes)$ is the algebra $\mathbb{K} \in \text{Alg} \subset \text{SAAlg}$, the symmetric structure is defined by the same formula (2.5). A commutative super-algebra is a super-algebra $\mathcal{R}$ satisfying the condition $ab = (-1)^{|a||b|}ba$, $a, b \in \mathcal{R}$. The commutative super-algebras form a full monoidal subcategory $\text{CommSAAlg} \subset \text{SAAlg}$, where the functor $\otimes$ coincides with categorical coproduct, so $(\text{CommSAAlg}, \otimes)$ is opposite to $(\text{SAffSch}, \times)$, where $\text{SAffSch} = \text{CommSAAlg}^{\text{op}}$ is the category of the affine super-schemes over $\mathbb{K}$. The objects of $\text{SAAlg}^{\text{op}}$ are the quantum analogues of the affine super-schemes.

Affine algebraic super-group is a group in $\text{SAffSch}$. More generally affine algebraic super-monoid is a monoid in $\text{SAffSch}$, i.e. a monoid in $(\text{CommSAAlg}^{\text{op}}, \otimes)$. The monoids in $(\text{SAAlg}^{\text{op}}, \otimes)$ are called quantum super-monoids. Their category is opposite to the category of super-bialgebras: $\text{Mon}(\text{SAAlg}^{\text{op}}, \otimes) = \text{Bimon}(\text{SVect}^{\text{op}}, \otimes) = \text{Bimon}(\text{SVect}, \otimes)^{\text{op}}$. The quantum super-groups correspond to the Hopf super-algebras.

The symmetric monoidal category $(\text{SVect}, \otimes)$ is closed. For $V, W \in \text{SVect}$ the internal hom-object $\text{hom}(V, W)$ is a super-vector space consisting of all the linear maps $f : V \to W$. 

$V$ such that $V_\overline{g} = 0$ for $g \in \mathbb{A}\backslash\mathbb{A}'$. In particular, $\text{Vect} = \{0\}\text{-GrVect} \subset \text{A-GrVect}$, $\text{Alg} \subset \text{A-GrAlg}$. 

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The elements $f \in \text{hom}(V, W)_0$ preserve the $\mathbb{Z}_2$-grading and we identify them with the morphisms $f \in \text{Hom}(V, W)$. The elements $f \in \text{hom}(V, W)_1$ are operators $f : V \to W$ such that $f(V_0) \subset W_1$ and $f(V_1) \subset W_0$.

Note that direct sum of vector spaces provides another monoidal product on the category $\text{SVect}$. The direct sum of $V, W \in \text{SVect}$ is a super-vector space $V \oplus W$ with the components $(V \oplus W)_0 = V_0 \oplus W_0$, $(V \oplus W)_1 = V_1 \oplus W_1$. The monoidal category $(\text{SVect}, \oplus)$ has a canonical symmetric structure.

2.2.2. $A$-graded super-algebras. Consider the product of an Abelian monoid $A$ and the group $\mathbb{Z}_2$. The $A \times \mathbb{Z}_2$-grading provides the categories $A$-$\text{GrSVect} := A \times \mathbb{Z}_2$-$\text{GrVect}$ and $A$-$\text{GrSAlg} := A \times \mathbb{Z}_2$-$\text{GrAlg}$; we call their objects $A$-graded super-vector spaces and $A$-graded super-algebras respectively. Each object $V \in A$-$\text{GrSVect}$ has decompositions

$$V = \bigoplus_{g \in A} (V_{g0} \oplus V_{g1}) = \bigoplus_{g \in A} V_g = V_0 \oplus V_1, \quad (2.7)$$

where $V_{gk} = V_{(g,k)} \in \text{Vect}$, $V_g = V_{g0} \oplus V_{g1} \in Z_2$-$\text{GrVect}$, $V_k = \bigoplus_{g \in A} V_{gk} \in A$-$\text{GrVect}$. Thus an $A$-graded super-vector space (super-algebra) is a vector space (an algebra) equipped with two gradings ($A$- and $\mathbb{Z}_2$-grading) which are compatible with each other. We have the following commutative diagrams of the forgetful functors:

$$\begin{array}{ccc}
A$-$\text{GrSVect} & \longrightarrow & \text{SVect} \\
\downarrow & & \downarrow \\
A$-$\text{GrVect} & \longrightarrow & \text{Vect}
\end{array} \quad \begin{array}{ccc}
A$-$\text{GrSAlg} & \longrightarrow & \text{SAlg} \\
\downarrow & & \downarrow \\
A$-$\text{GrAlg} & \longrightarrow & \text{Alg}
\end{array} \quad (2.8)

Note that $\text{Vect} = \{0\}$-$\text{GrVect}$. The embedding of the monoids gives the commutative diagrams of the embedding functors

$$\begin{array}{ccc}
\text{Vect} & \longrightarrow & \text{SVect} \\
\downarrow & & \downarrow \\
A$-$\text{GrVect} & \longrightarrow & A$-$\text{GrSVect}
\end{array} \quad \begin{array}{ccc}
\text{Alg} & \longrightarrow & \text{SAlg} \\
\downarrow & & \downarrow \\
A$-$\text{GrAlg} & \longrightarrow & A$-$\text{GrSAlg}
\end{array} \quad (2.9)

2.2.3. Monoidal products of $A$-graded super-algebras. Note that $(A$-$\text{GrSVect}, \otimes)$ and $(A$-$\text{GrSAlg}, \otimes)$ are symmetric monoidal categories with the standard tensor product of vector spaces and symmetric structure given by (2.5). The functors in (2.9) are symmetric strong monoidal with respect to $\otimes$. The functors in the left diagram (2.8) are also strong monoidal, but the vertical ones are not symmetric. By this reason the vertical functors in the right diagram (2.8) are not monoidal.

Let us introduce another monoidal structure on $A$-$\text{GrSAlg}$: for two $A$-graded super-algebras $A$ and $B$ we define Manin (white) product by the formula

$$A \circ B = \bigoplus_{g \in A} (A \circ B)_g, \quad (A \circ B)_g = A_g \otimes B_g; \quad (2.10)$$
as a super-graded vector space it has the components \((A \circ B)_0 = \bigoplus_{g \in A} (A_{g0} \otimes B_{g0} + A_{g1} \otimes B_{g1}),\)
\((A \circ B)_1 = \bigoplus_{g \in A} (A_{g0} \otimes B_{g1} + A_{g1} \otimes B_{g0}).\)
Note that \(A \circ B\) is a super-subalgebra in \(A \otimes B\), but the embedding is not graded. We also use the notation \(V \circ W\) for \(A\)-graded super-vector space with components \(V_g \circ W_g\), where \(V, W \in A\text{-GrSVect}\).

The cases we are interested in are \(A = \mathbb{N}_0\) and \(A = \mathbb{Z}\), where \(\mathbb{N}_0\) is a submonoid of \(\mathbb{Z}\), consisting of non-negative integers. The Manin product for \(\mathbb{N}_0\)- and \(\mathbb{Z}\)-graded algebras is defined in \([S2]\), \([S3]\). The embedding \(\mathbb{N}_0\text{-GrAlg} \hookrightarrow \mathbb{N}_0\text{-GrSAlg}\) is symmetric strong monoidal with respect to the Manin product \(\circ\). By this reason the super-version of Quantum Representation Theory we present here includes the corresponding results of \([S2]\).

Note that the unit objects of \((\mathbb{N}_0\text{-GrSAlg}, \circ)\) and \((\mathbb{Z}\text{-GrSAlg}, \circ)\) are the graded algebras of polynomials \(K[u] \in \mathbb{N}_0\text{-GrAlg}\) and of Laurent polynomials \(K[u, u^{-1}] \in \mathbb{Z}\text{-GrAlg}\) respectively. The embedding functor \(\mathbb{N}_0\text{-GrSAlg} \hookrightarrow \mathbb{Z}\text{-GrSAlg}\) is only colax monoidal: \(\varphi_{A, B} = \text{id}_{A \otimes B}\) are isomorphisms, but \(\varphi: K[u] \rightarrow K[u, u^{-1}]\) is not isomorphism.

### 3 Categories of quadratic super-algebras

The main ingredient of Quantum Representation Theory is the category of representation spaces. In the super-case this is the category of the (finitely generated) quadratic super-algebras (more precisely, its opposite category). Here we describe different types of operations on quadratic super-algebras: binary operations (monoidal products), non-binary operations realised as functors from the category of quadratic algebras. We also derive the internal (co)hom-functors, which play a key role in the theory.

#### 3.1 Monoidal structures

We use the standard tensor notations \(f^{(a)}\) for a morphism \(f\) putted to the \(a\)-th tensor (monoidal) factor. If \(f\) acts on a monoidal product of two objects, then we use the notation \(f^{(a,a+1)}\). For example,

\[
\sigma^{(23)} = \text{id} \otimes \sigma \otimes \text{id} \otimes \cdots \otimes \text{id}: V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes \cdots \otimes V_k \rightarrow V_1 \otimes V_3 \otimes V_2 \otimes V_4 \otimes \cdots \otimes V_k,
\]

where \(\sigma = \sigma_{V_2, V_3}: V_2 \otimes V_3 \rightarrow V_3 \otimes V_2\) is the symmetric structure. We need the following fact from the linear algebra (see e.g. [Lin] § 1, ex. 1.3)).

**Lemma 3.1.** For any vector spaces \(V, W \in \text{Vect}\) and subspaces \(V_0 \subset V\) and \(W_0 \subset W\) there is an isomorphism

\[
(V/V_0) \otimes (W/W_0) \cong (V \otimes W)/(V_0 \otimes W + V \otimes W_0) \quad (3.1)
\]

which sends \(\bar{v} \otimes \bar{w}\) to the class \(\overline{v \otimes w}\), where \(\bar{v} \in V/V_0\) is the class of \(v \in V\). If \(V, W\) have \(\mathbb{Z}_2\)-gradings and \(V_0, W_0\) are super-subspaces, then the isomorphism (3.1) is \(\mathbb{Z}_2\)-graded.
Proof. It is obvious that \( \bar{v} \otimes \bar{w} \mapsto \bar{v} \otimes w \) is a correctly defined epimorphism of super-vector spaces. By choosing appropriate bases in \( V \) and \( W \) one can show that its kernel vanishes. \( \square \)

3.1.1. Quadratic super-algebras. Let \( V \in SVect \). The tensor algebra \( TV \) has a structure of super-algebra compatible with the \( N_0 \)-grading: \( TV \in N_0 \text{-GrSAlg} \). Its components with respect to the \( N_0 \)-grading are \( (TV)_0 = \mathbb{K}, (TV)_1 = V, (TV)_2 = V \otimes V, (TV)_k = V^\otimes k, k \in N_0 \).

The quotient of \( TV \) over an ideal \( (R) \) generated by a super-subspace \( R \subset V \otimes V \) is also an \( N_0 \)-graded super-algebra. A quadratic super-algebra is an \( N_0 \)-graded super-algebra \( \mathcal{A} \) generated by elements of \( A_1 \) with quadratic commutation relations, that is \( \mathcal{A} \cong TV/(R) \) for some \( V \in SVect \) and super-subspace \( R \subset V \otimes V \). The category of quadratic super-algebras form a full subcategory \( QSA \subset N_0 \text{-GrSAlg} \). The quadratic super-algebra \( TV/(R) \) is finitely generated iff \( V \in FSVect \). Denote the full subcategory of the finitely generated quadratic super-algebras by \( FQSA \subset QSA \).

3.1.2. Super-version of Manin’s binary operations. Consider \( QSA \) as a full subcategory of \( N_0 \text{-GrSAlg} \). The following fact generalises [Man88, SS 3, Lemma 7] for the super-case.

Proposition 3.2. The subcategory \( QSA \subset N_0 \text{-GrSAlg} \) is monoidal with respect to the both monoidal products \( \otimes \) and \( \circ \). Let \( \mathcal{A} = TV/(R) \) and \( \mathcal{B} = TW/(S) \), where \( V, W \in SVect \) and \( R \subset V^\otimes 2, S \subset W^\otimes 2 \) are super-subspaces. Then we have isomorphisms

\[
\begin{align*}
\mathcal{A} \otimes \mathcal{B} & \cong T(V \oplus W)/(R'), \\
\mathcal{A} \circ \mathcal{B} & \cong T(V \otimes W)/(R_w),
\end{align*}
\]

where \( [V, W] \subset V \otimes W + W \otimes V \subset (V \oplus W)^\otimes 2 \) is spanned by \( v \otimes w - (-1)^{[v][w]} w \otimes v \) with homogeneous \( v \in V, w \in W \). The isomorphism (3.2) is induced by the embedding of \( N_0 \)-graded super-algebras

\[
TV \otimes TW = \bigoplus_{k,l \geq 0} V^\otimes k \otimes W^\otimes l \hookrightarrow T(V \oplus W).
\]

The isomorphism (3.3) is induced by the graded isomorphism

\[
\begin{align*}
\phi: TV \otimes TW & = \bigoplus_{k \geq 0} V^\otimes k \otimes W^\otimes k \cong \bigoplus_{k \geq 0} (V \otimes W)^\otimes k = T(V \otimes W), \\
\phi_k: v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_k & \mapsto (-1)^{\bar{p}}(v_1 \otimes w_1) \otimes \cdots \otimes (v_k \otimes w_k),
\end{align*}
\]

where \( \bar{p} = \sum_{1 \leq l < s \leq k} [v_s][w_l] \).

1More generally, one can define quadratic super-algebras over an algebra \( R \) as \( N_0 \)-graded super-algebras \( \mathcal{A} \) generated by \( \mathcal{A}_1 \) over \( \mathcal{A}_0 = R \). By default we mean the quadratic super-algebras over \( \mathbb{K} \) (connected quadratic super-algebras), cf. [S1], [S2].
Proof. First note that any element of $T(V \oplus W)$ is equal to a sum of elements of the spaces $V^{\otimes k} \otimes W^{\otimes l}$ modulo the ideal generated by $[V, W]$, so the embedding (3.3) gives the isomorphism $TV \otimes TW \cong T(V \oplus W)/([V, W])$ in $N_0\text{-GrVect}$. One can check that it respects the multiplication: e.g. $w \cdot v = (-1)^{|v||w|}v \otimes w \mapsto (-1)^{|v||w|}v \otimes w = w \otimes v \mod [V, W]$. Due to Lemma 3.1 the super-vector space $A \otimes B$ is isomorphic the quotient of $TV \otimes TW$ over $(R) \otimes TW + TV \otimes (S)$, so we obtain (3.2). Further, it is straightforward to prove that the formula (3.6) defines a morphism of $N_0$-graded super-algebras (3.5). By virtue of Lemma 3.1 the kernel of the epimorphism $TV \rightarrow A \otimes B$ is the ideal $(R) \otimes TW + TV \otimes (S)$. The isomorphism (3.5) maps this ideal to $(R_w)$. □

A quadratic super-algebra $A \cong TV/(R)$ is purely even or purely odd iff $V = V_0$ or $V = V_1$ respectively (see [S2, § 4.1.4]). The purely even ones form the subcategory $QA$ of the quadratic algebras $A = A_0$. The purely odd ones are such that $A_0 = \bigoplus_{k \geq 0} A_{2k}$, $A_1 = \bigoplus_{k \geq 0} A_{2k+1}$. Recall that Manin defined in [Man87], [Man88] four binary operations $\circ$, $\bullet$, $\otimes$, $\boxtimes$. The operations (3.2) and (3.3) generalises the Manin’s binary operations $\otimes$ and $\circ$; they respectively coincide for purely even algebras. The operation (3.2) for purely odd algebras coincides with the operation $\otimes$. Now let us generalise the black Manin product $\circ$ and ”odd” tensor product $\otimes$ (from the purely even case). For $A = TV/(R)$ and $B = TW/(S)$ as in Proposition 3.2 define

$$A \bullet B = T(V \otimes W)/(R_b), \quad R_b = \sigma^{(23)}(R \otimes S); \quad (3.7)$$

$$A \otimes B = T(V \oplus W)/(R''), \quad R'' = R \oplus S \oplus [V, W]_+; \quad (3.8)$$

where $[V, W]_+ \subset V \otimes W \oplus W \otimes V$ is spanned by $v \otimes w + (-1)^{|v||w|}w \otimes v$.

Proposition 3.3. The formula (3.7) gives a symmetric monoidal category $(QSA, \bullet)$ with unit object $\mathbb{K}[[\varepsilon]]/(\varepsilon^2) \in QA$ and symmetric structure (2.5). The functor $\bullet$ on morphisms is uniquely defined by the condition $(f \bullet g)_1 = f_1 \otimes g_1$.

Proof. Let $f: A \rightarrow \tilde{A}$ and $g: B \rightarrow \tilde{B}$, where $\tilde{A} = TV/(\tilde{R})$, $\tilde{B} = T\tilde{W}/(\tilde{S})$. Then we have $(f_1 \otimes g_1 \otimes f_1 \otimes g_1)(R_b) = \sigma^{(23)}(f_1 \otimes f_1 \otimes g_1 \otimes g_1)(R \otimes S) = \sigma^{(23)}(\tilde{R} \otimes \tilde{S})$. □

There is a natural epimorphism $A \bullet B \rightarrow A \circ B$ whose first degree component is the map id: $A_1 \otimes B_1 \rightarrow A_1 \otimes B_1$.

The subcategory $FQSA \subset QSA$ is monoidal with respect to all three products $\otimes$, $\circ$, $\bullet$.

3.1.3. Coproduct of quadratic super-algebras. Let us obtain the categorical coproduct of $A = TV/(R)$ and $B = TW/(S)$ in $QSA$.

Proposition 3.4. The coproduct of $A$ and $B$ is

$$A \amalg B = T(V \oplus W)/(R \oplus S). \quad (3.9)$$

It gives the symmetric monoidal categories $(QSA, \amalg)$ and $(FQSA, \amalg)$ with the unit object $\mathbb{K}$.
Proof. One can prove that \( A \amalg B \) is the coproduct in \( \mathsf{QSA} \) by a direct generalisation of [S2, § 4.1.9] to the super-case. The initial object is \( \mathbb{K} \). Since \( \mathbb{K} \in \mathsf{FQSA} \) and a coproduct of \( A, B \in \mathsf{FQSA} \) in \( \mathsf{QSA} \) belongs to \( \mathsf{FQSA} \), it coincides with the coproduct in \( \mathsf{FQSA} \). As any category with finite coproducts the categories \( \mathsf{QSA} \) and \( \mathsf{FQSA} \) are symmetric monoidal with respect to \( \amalg \).

We have the natural epimorphism \( A \amalg B \to A \otimes B \).

3.2 Some functors

3.2.1. Dualisation functor. Let \( V \in \mathsf{SVect} \). Its dual \( V^* = \text{hom}(V, \mathbb{K}) \) is the super-vector space with the components \( V_0^* = \{ \lambda : V \to \mathbb{K} \mid \lambda(V_1) = 0 \} \), \( V_1^* = \{ \lambda : V \to \mathbb{K} \mid \lambda(V_0) = 0 \} \). The dualisation can be considered as the contravariant functor \((-)^*: \mathsf{SVect} \to \mathsf{SVect} \), \( V \mapsto V^* \). It translates a morphism \( f: V \to W \) to \( f^*: W^* \to V^* \), where \( f^*(\mu)(v) = \mu(f(v)) \), \( \mu \in W^* \). The monoidal structure of this functor is defined by the embedding

\[
V^* \otimes W^* \hookrightarrow (V \otimes W)^* ,
\]

\[
(\lambda \otimes \mu)(v \otimes w) = (-1)^{[\mu][\lambda]} \lambda(v)\mu(w).
\]

This gives a contravariant symmetric lax monoidal functor \((-)^*: (\mathsf{SVect}, \otimes) \to (\mathsf{SVect}, \otimes) \). In the finite-dimensional case we obtain a contravariant symmetric strong monoidal functor \((-)^*: (\mathsf{FSVect}, \otimes) \to (\mathsf{FSVect}, \otimes) \). More generally, the formula (3.10) gives the isomorphism \( V^* \otimes W^* \cong (V \otimes W)^* \) natural in \( V \in \mathsf{FVect} \) and \( W \in \mathsf{Vect} \).

The dualisation operation has also a structure of contravariant symmetric strong monoidal functors \((-)^*: (\mathsf{SVect}, \oplus) \to (\mathsf{SVect}, \oplus) \) and \((-)^*: (\mathsf{FSVect}, \oplus) \to (\mathsf{FSVect}, \oplus) \).

Lemma 3.5. Let \( V_0 \) be a super-subspace of a super-vector space \( V \in \mathsf{SVect} \). Consider the projection \( \pi: V \to V/V_0 \). The image of \( \pi^*: (V/V_0)^* \to V^* \) equals

\[
V_0^\perp := \{ \lambda \in V^* \mid \lambda(v_0) = 0 \ \forall v_0 \in V_0 \}.
\]

It is a super-subspace \( V_0^\perp \subset V^* \) isomorphic to \((V/V_0)^* \) via \( \pi^* \). Suppose \( V, W \in \mathsf{FSVect} \) and let \( W_0 \subset W \) be also a super-subspace, then \( V_0^\perp \otimes W_0^\perp \) coincide with \((V_0 \otimes W + V \otimes W_0)^\perp \) as a super-subspace of \( V^* \otimes W^* = (V \otimes W)^* \).

Proof. It is obvious that \( \text{Ker}(\pi^*) = 0 \) and \( \text{Im}(\pi^*) \subset V_0^\perp \). Let \( t: V_0^\perp \to (V/V_0)^* \) be the map \( t(\lambda)(\bar{v}) = \lambda(v) \), where \( \lambda \in V_0^\perp \), \( v \in V \), \( \bar{v} \in V/V_0 \). Since \( \pi^* t(\lambda) = \lambda \) \( \forall \lambda \in V_0^\perp \), we obtain \( \text{Im}(\pi^*) = V_0^\perp \), so \( \pi^* \) induces the isomorphism \((V/V_0)^* \cong V_0^\perp \). By applying the functor \((-)^* \) to (3.11) we obtain \( V_0^\perp \otimes W_0^\perp = (V_0 \otimes W + V \otimes W_0)^\perp \).

Lemma 3.6. Let \( V_0 \) and \( W_0 \) be super-subspaces of \( V \in \mathsf{FVect} \) and \( W \in \mathsf{Vect} \) respectively. Identify the operators \( \xi \in \text{hom}(V, W) \) with the elements \( \xi \in V^* \otimes W \). Then the condition \( \xi(V_0) \subset W_0 \) is equivalent to \( \xi \in V_0^\perp \otimes W + V^* \otimes W_0 \).

Proof. The first condition means exactly that \( \xi \) belongs to the kernel of the epimorphism \( p: V^* \otimes W \to V_0^\perp \otimes (W/W_0) \). Note that \( V_0 = (V_0^\perp)^\perp \cong (V^*/V_0^\perp)^* \) and hence \( V_0^\perp \cong (V^*/V_0^\perp) \). By applying Lemma 3.5 for \( V_0^\perp \subset V^* \) and \( W_0 \subset W \) we see that \( \text{Ker} p = V_0^\perp \otimes W + V^* \otimes W_0 \).
3.2.2. Functors of parity change: $\Pi$ and $\hat{\Pi}$. Denote by $\Pi: \text{SVect} \to \text{SVect}$ the functor changing parity: $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$. This is an autoequivalence of $\text{SVect}$, it has symmetric strong monoidal structure: $(\text{SVect, } \oplus) \to (\text{SVect, } \oplus)$. It preserves the tensor product, but not as a monoidal functor: the isomorphism $(\Pi V) \otimes (\Pi W) \cong V \otimes W$ is natural in $V, W \in \text{SVect}$. It commutes with $(-)^*$ in the sense of natural isomorphism $(\Pi V)^* = \Pi(V^*)$.

Note that the composition $\Pi V^* \otimes \Pi W^* \cong (\Pi V \otimes \Pi W)^* \cong (V \otimes W)^* \cong V^* \otimes W^*$ coincides with $\Pi V^* \otimes \Pi W^* \cong \hat{V}^* \otimes \hat{W}^*$, where $\lambda \otimes \mu \mapsto -(-1)^{|\lambda| + |\mu|} \lambda \otimes \mu$, with $\lambda \in V^*$, $\mu \in W^*$.

Define the functor $\hat{\Pi}: \text{QSA} \to \text{QSA}$ by $\hat{\Pi}(TV/(R)) = T\Pi V/(R)$. This is an autoequivalance of $\text{QSA}$, more precisely, $\hat{\Pi}^2 = \text{id}$. It maps the purely even quadratic super-algebras to the purely odd ones and vice versa. By restricting it to the subcategory $\text{QSA} \subset \text{QSA}$ we obtain the functor $\hat{\Pi}: \text{QSA} \to \text{QSA}$, which gives the equivalence of the subcategories of purely even and odd quadratic super-algebras.

**Proposition 3.7.** The functor $\hat{\Pi}: \text{QSA} \to \text{QSA}$ has a structure of symmetric strong monoidal functor $\hat{\Pi}: (\text{QSA}, \Pi) \to (\text{QSA}, \Pi)$. The restricted functor $\hat{\Pi}: \text{QSA} \to \text{QSA}$ has symmetric strong monoidal structures $\hat{\Pi}: (\text{QSA}, \otimes) \to (\text{QSA}, \otimes)$ and $\hat{\Pi}: (\text{QSA}, \otimes) \to (\text{QSA}, \otimes)$, i.e. $\hat{\Pi}A \otimes \hat{\Pi}B = \hat{\Pi}(A \otimes B), \hat{\Pi}A \otimes \hat{\Pi}B = \hat{\Pi}(A \otimes B)$ for any $A, B \in \text{QSA}$. We also have $\hat{\Pi}A \otimes B = \hat{\Pi}(A \otimes \hat{\Pi}B), \hat{\Pi}A \otimes \hat{\Pi}B = \hat{\Pi}(A \otimes \hat{\Pi}B), \hat{\Pi}A \circ \hat{\Pi}B = A \circ \hat{\Pi}B, \hat{\Pi}A \circ \hat{\Pi}B = \hat{\Pi}(A \circ B)$, $\hat{\Pi}A \circ B = \hat{\Pi}(A \circ B)$, $\hat{\Pi}A \bullet B = \hat{\Pi}(A \bullet B) \forall A, B \in \text{QSA}$.

**Proof.** Let $A = TV/(R)$ and $B = TW/(S)$ be quadratic super-algebras. We have the identifications $\hat{\Pi}A \Pi \hat{\Pi}B = T(\Pi V \oplus \Pi W)/(R \oplus W) = \hat{\Pi}(A \Pi B)$. If $A, B \in \text{QSA}$, then $V, W \in \text{Vect}$ and hence $[\Pi V, \Pi W] = [V, W]$. This implies $\hat{\Pi}A \otimes \hat{\Pi}B = T(\Pi V \oplus \Pi W)/(R \oplus S \oplus [\Pi V, \Pi W]) = \hat{\Pi}(A \otimes B)$. The next three isomorphisms follows from $[\Pi V, \Pi W] = [V, W]$, $[\Pi V, W] = \Pi[V, W] = [V, \Pi W], [\Pi V, W] = \Pi[V, W] = [V, \Pi W]$. Further, we derive $\hat{\Pi}A \otimes \hat{\Pi}B = T(\Pi V \otimes \Pi W)/(\sigma_{\Pi V, \Pi W}(R \otimes S)) = T(V \otimes W)/(\sigma_{V, W}(R \otimes S)) = A \bullet B$, since $\sigma_{V, W}(r \otimes s) = -\sigma_{V, W}(r \otimes s)$ for any $r \in V^* \otimes 2$ and $s \in W^* \otimes 2$. The other isomorphisms are checked similarly. $\square$

3.2.3. Functors $(-)_1$ and $(-)_1$. For each $k \in \mathbb{Z}$ we have the functor $(-)_k: \text{Z-GrSAlg} \to \text{SVect}$ which extract the $k$-th component $A_k \in \text{SVect}$ from a $\mathbb{Z}$-graded super-algebra $A$. Its value on the graded homomorphism $f: A \to B$ is the component $f_k: A_k \to B_k$. The most interesting functor is $(-)_1: \text{Z-GrSAlg} \to \text{SVect}$. Its restrictions $(-)_1: \text{QSA} \to \text{SVect}$ and $(-)_1: \text{FQSA} \to \text{FSVect}$ have structures of symmetric strong monoidal functors

\begin{align*}
(-)_1: (\text{QSA, } \otimes) &\to (\text{SVect, } \oplus), & (-)_1: (\text{FQSA, } \otimes) &\to (\text{FSVect, } \oplus), \\
(-)_1: (\text{QSA, } \circ) &\to (\text{SVect, } \circ), & (-)_1: (\text{FQSA, } \circ) &\to (\text{FSVect, } \circ), \\
(-)_1: (\text{QSA, } \bullet) &\to (\text{SVect, } \bullet), & (-)_1: (\text{FQSA, } \bullet) &\to (\text{FSVect, } \bullet).
\end{align*}

The composition of $(-)_1: \text{QSA} \to \text{SVect}$ with the dualisation $(-)^*: \text{SVect} \to \text{SVect}$ gives the contravariant functor $(-)^*_1: \text{QSA} \to \text{SVect}, A \mapsto A^*_1$. These functors commutes
with the change of parity as follows: \((\hat{\Pi}\mathcal{A})_1 = \Pi\mathcal{A}_1\), \((\hat{\Pi}\mathcal{A})^*_1 = \Pi\mathcal{A}^*_1\), these isomorphisms are natural in \(\mathcal{A} \in \mathcal{QSA}\).

The restricted functor \((-): \mathcal{QSA} \rightarrow \mathcal{FSVect}\) has structures of contravariant symmetric strong monoidal functors

\[
(-)^*: (\mathcal{QSA}, \otimes) \rightarrow (\mathcal{FSVect}, \oplus), \quad (-)^*: (\mathcal{QSA}, \odot) \rightarrow (\mathcal{FSVect}, \otimes), \quad (\mathcal{QSA}, \bullet) \rightarrow (\mathcal{FSVect}, \odot).
\]  

3.2.4. Functors \(T\) and \(T^*\). Let us consider the functor \(T: \mathcal{SVect} \rightarrow \mathcal{QSA}\) making the tensor algebra \(TV \in \mathcal{QSA}\) for a super-vector space \(V\). For a \(\mathbb{Z}_2\)-graded linear map \(f: V \rightarrow W\) there exists a unique morphism \(Tf: TV \rightarrow TW\) with first degree component \((Tf)_1 = f\). This means exactly that \(T: \mathcal{SVect} \rightarrow \mathcal{QSA}\) is a left adjoint to \((-)_1: \mathcal{QSA} \rightarrow \mathcal{SVect}\). We have isomorphism \(T(\Pi V) = \hat{\Pi}(TV)\) natural in \(V \in \mathcal{SVect}\). The functor \(T: \mathcal{SVect} \rightarrow \mathcal{QSA}\) is symmetric strong monoidal as \((\mathcal{SVect}, \otimes) \rightarrow (\mathcal{QSA}, \oplus)\) and \((\mathcal{SVect}, \otimes) \rightarrow (\mathcal{QSA}, \odot)\) and symmetric colax monoidal as \((\mathcal{SVect}, \otimes) \rightarrow (\mathcal{QSA}, \oplus)\).

Denote the composition of the functors \((-)^*: \mathcal{SVect} \rightarrow \mathcal{SVect}\) and \(T: \mathcal{SVect} \rightarrow \mathcal{QSA}\) by \(T^*: \mathcal{SVect} \rightarrow \mathcal{QSA}\). This functor as well as its restriction \(T^*: \mathcal{FSVect} \rightarrow \mathcal{QSA}\) is a faithful contravariant functor. Note that a composition of a (symmetric) strong monoidal functor with a (symmetric) lax/colax/strong monoidal functor is a (symmetric) lax/colax/strong monoidal functor. Hence we obtain the following contravariant symmetric monoidal functors:

| colax monoidal | \(T^*: (\mathcal{FSVect}, \oplus) \rightarrow (\mathcal{QSA}, \otimes)\) |
|----------------|-------------------------------------------------------------|
| strong monoidal| \(T^*: (\mathcal{FSVect}, \oplus) \rightarrow (\mathcal{QSA}, \otimes)\) |
| strong monoidal| \(T^*: (\mathcal{FSVect}, \otimes) \rightarrow (\mathcal{QSA}, \odot)\) |

3.2.5. Functors \(S\) and \(S^*\). Recall two functors \(S: \mathcal{Vect} \rightarrow \mathcal{QA}\) and \(\Lambda: \mathcal{Vect} \rightarrow \mathcal{QA}\) which makes the symmetric algebra \(SV\) and external algebra \(\Lambda V\) from a vector space \(V\). The functor \(S\) can be extended to the super-case as follows: for \(V \in \mathcal{SVect}\) we define \(SV = SV_0 \otimes \hat{\Pi}(\Lambda V)\), where \(SV_0\) and \(\Lambda V\) are considered as purely even quadratic super-algebras. Note that \(SV\) is a quadratic super-algebra with the relations \(xy = (-1)^{[x][y]}yx\), where \(x, y \in V\) are homogeneous. We obtain the functor \(S: \mathcal{SVect} \rightarrow \mathcal{QSA}\), which translates a \(\mathbb{Z}_2\)-graded map \(f\) to \(SF_0 \otimes \hat{\Pi}(\Lambda f)\).

Consider the full subcategory \(\mathcal{CommQSA} \subseteq \mathcal{QSA}\) consisting of \(\mathcal{A} \in \mathcal{QSA}\), which are commutative as super-algebras: \(ab = (-1)^{[a][b]}ba\). Note that \(SV \in \mathcal{CommQSA}\) for any \(V \in \mathcal{SVect}\). Hence \(S\) can be considered as a functor \(\mathcal{SVect} \rightarrow \mathcal{CommQSA}\), which is a left adjoint to \((-)_1: \mathcal{CommQSA} \rightarrow \mathcal{SVect}\). Alternatively, one can say that \((-)_1: \mathcal{CommQSA} \rightarrow \mathcal{SVect}\) is a right adjoint for \(S: \mathcal{SVect} \rightarrow \mathcal{QSA}\) relative to the full subcategory \(\mathcal{CommQSA} \subseteq \mathcal{QSA}\).

By composing \((-)^*: \mathcal{SVect} \rightarrow \mathcal{SVect}\) and \(S: \mathcal{SVect} \rightarrow \mathcal{QSA}\) we obtain a contravariant faithful functors \(S^*: \mathcal{SVect} \rightarrow \mathcal{QSA}\) and \(S^*: \mathcal{FSVect} \rightarrow \mathcal{QSA}\).
Proposition 3.8. The functors $S$: $(\text{SVect}, \oplus) \to (\text{QSA}, \otimes)$, $S$: $(\text{FSVec}, \oplus) \to (\text{FQSA}, \otimes)$ are symmetric strong monoidal. Hence we have contravariant

**symmetric strong monoidal functor**

$S^*: (\text{FSVec}, \oplus) \to (\text{FQSA}, \otimes)$. \hfill (3.21)

The functors $S$: $(\text{SVect}, \otimes) \to (\text{QSA}, \circ)$, $S$: $(\text{FSVec}, \otimes) \to (\text{FQSA}, \circ)$ are symmetric colax monoidal. Hence we have contravariant

**symmetric colax monoidal functor**

$S^*: (\text{FSVec}, \otimes) \to (\text{FQSA}, \circ)$. \hfill (3.22)

**Proof.** Proposition 3.2 implies that the morphisms (3.4) and (3.5) induce the isomorphism $S(V \otimes W) \cong SV \otimes SW$ and epimorphism $S(V \otimes W) \to SV \circ SW$ for $V, W \in \text{SVect}$. \hfill \square

3.2.6. Koszul duality $(-)^!$ on FQSA. Let us consider a super-version of the Manin’s contravariant functor $(-)^!: \text{FQA} \to \text{FQA}$ introduced in [Man87]. For a quadratic super-algebra $A = TV/(R) \in \text{FQSA}$ we set

$$A^! = TV^*/(R^\perp), \quad R^\perp = \{\xi \in V^* \otimes V^* \mid \xi(r) = 0 \forall r \in R\},$$ \hfill (3.23)

where $\xi(r)$ is defined by the formula (3.10). Due to Lemma 3.5 the set $R^\perp$ is a super-subspace of $V^* \otimes V^* = (V \otimes V)^*$, so $A^! \in \text{FQSA}$. For any morphism $f: TV/(R) \to TW/(S)$ in FQSA we have $(f^! \circ f^!_1)S^\perp \subset R^\perp$, hence we obtain the contravariant functor $(-)^!: \text{FQSA} \to \text{FQSA}$ such that

$$(A^!)_1 = A^*_1, \quad (f^!)_1 = f^*_1, \quad \text{and} \quad (A^!)^! = A, \quad (f^!)^! = f.$$ \hfill (3.24)

It commutes with $\hat{\Pi}: \text{FQSA} \to \text{FQSA}$ in the sense that we have a natural isomorphism $(\hat{\Pi}A)^! \cong \hat{\Pi}(A^!)$. The right property (3.24) implies that the functor $(-)^!: \text{FQSA} \to \text{FQSA}$ is fully faithful.

Proposition 3.9. The contravariant functor $(-)^!$ has symmetric strong monoidal structures

$$\begin{align*}
(\text{FQSA}, \circ) & \to (\text{FQSA}, \bullet), & (\text{FQSA}, \bullet) & \to (\text{FQSA}, \circ), \\
(\text{FQSA}, \otimes) & \to (\text{FQSA}, \ast), & (\text{FQSA}, \ast) & \to (\text{FQSA}, \otimes)
\end{align*}$$

given by the identifications $\mathbb{K}[u]^\perp = \mathbb{K}[\varepsilon]/(\varepsilon^2)$, $\mathbb{K}[\varepsilon]/(\varepsilon^2)^! = \mathbb{K}[u]$, $\mathbb{K}^! = \mathbb{K}$ and the isomorphisms

$$\begin{align*}
(A \circ B)^! & \cong A^! \bullet B^!, & (A \bullet B)^! & \cong A^! \circ B^!, \\
(A \otimes B)^! & \cong A^! \otimes B^!, & (A \otimes B)^! & \cong A^! \otimes B^!
\end{align*}$$ \hfill (3.25, 3.26)

natural in $A, B \in \text{FQSA}$.

**Proof.** Let $A = TV/(R), B = TW/(S) \in \text{FQSA}$. By means of Lemma 3.5 we derive $R^\perp \otimes S^\perp = (R \otimes W^\otimes + V^\otimes \otimes S)^\perp = \sigma^{(23)}R^\perp_w$, where $R_w$ is defined by (3.3). Then
\( A^! \cdot B^! = T(V^* \otimes W^*)/(\sigma^{(23)}(R^1 \otimes S^1)) = T(V^* \otimes W^*)/(R^+_{\text{ev}}) \cong (A \circ B)^! \). The naturality is deduced by application of the faithful functor \((-\_\!)_!\), and using the properties \((3.21)\), \((A \circ B)_! = (A \cdot B)_! = A_! \otimes B_!\) (see \((3.11)\), \((3.15)\)). The third isomorphism is obtained from the formulae \((R \oplus S)^! = R^1 \oplus S^1\), \([V, W^!] = [V^*, W^*]^!\) and \((A \otimes B)_! = A_! \otimes B_!\). By substituting \(A \to A^!\) and \(B \to B^!\) and taking into account \((A^!)^! = A\) we obtain the right isomorphisms \((3.25)\), \((3.26)\).

3.2.7. Functors \(A\) and \(A^*\). For a super-vector space \(V \in \text{SVect}\) define the quadratic algebra \(\Lambda V = TV/(R)\), where \(R \subset V^\otimes 2\) is spanned by \(x \otimes y + (-1)^{|x||y|} y \otimes x\) with homogeneous \(x, y \in V\). We can decompose it to the purely even and purely odd parts as \(\Lambda V = \Lambda V_0 \oplus \Pi(SV_1)\). We obtain a functor \(\Lambda : \text{SVect} \to \text{QSA}\), whose composition with \((-\_)_!\) is the identical functor. Define \(\Lambda^* : \text{FSVect} \to \text{FQSA}\) as the composition of \((-)^*\) and \(\Lambda : \text{FSVect} \to \text{FQSA}\).

Proposition 3.10. The covariant functors \(\Lambda : (\text{SVect}, \oplus) \to (\text{QSA}, \otimes), \Lambda : (\text{FSVect}, \oplus) \to (\text{FQSA}, \otimes)\) and contravariant functor \(\Lambda^* : (\text{FSVect}, \oplus) \to (\text{FQSA}, \otimes)\) are fully faithful symmetric strong monoidal. The functors \(\Lambda : (\text{SVect}, \otimes) \to (\text{QSA}, \bullet), \Lambda : (\text{FSVect}, \otimes) \to (\text{FQSA}, \bullet)\), \(\Lambda^* : (\text{FSVect}, \otimes) \to (\text{FQSA}, \bullet)\) are symmetric lax monoidal. There are natural isomorphisms \((\Lambda V)^! \cong S^*(V), (SV)^! \cong \Lambda^*(V)\).

Proof. Straightforward.

3.2.8. Functor \((-)^{op}\). For a super-algebra \(A\) with the multiplication map \(\mu : A \otimes A \to A\) denote by \(A^{op}\) the algebra with the opposite multiplication: \(\mu^{op} = \mu \cdot \sigma : A \otimes A \to A\), \(a \otimes b \mapsto (-1)^{|a||b|} ba\). If \(A\) is \(A\)-graded, then the opposite super-algebra \(A^{op}\) is also \(A\)-graded. For a quadratic algebra \(A \cong TV/(R)\) we have \(A^{op} \cong TV/(\sigma R)\), so we get an autoequivalence \((-)^{op} : \text{QSA} \to \text{QSA}\). It commutes with the functor \(\Pi : \text{QSA} \to \text{QSA}\) and preserves the products \(\otimes, \otimes, \circ, \bullet\) and \(\Pi\) as symmetric strong monoidal functor. The restricted functor \((-)^{op} : \text{FQSA} \to \text{FQSA}\) commutes with \((-)!: \text{FQSA} \to \text{FQSA}\).

3.3 Internal cohom-functor

3.3.1. Internal hom and cohom in \(\text{FQSA}\). For finitely generated quadratic algebras the internal hom and cohom for the products \(\bullet\) and \(\circ\) respectively were obtained by Manin in \([\text{Man87}], [\text{Man88}]\). He derived a bijection

\[
\text{Hom}(A \bullet B, C) \cong \text{Hom}(A, B^! \circ C) \tag{3.27}
\]

natural in \(A, B, C \in \text{FQA}\). This implies that the symmetric monoidal categories \((\text{FQA}, \bullet)\) and \((\text{FQA}, \circ)\) are closed and cocomplete respectively. Let us generalise this fact for the quadratic super-algebras. First we consider two particular cases \(A = \mathbb{K} \langle \varepsilon \rangle/(\varepsilon^2)\) and \(C = \mathbb{K} \langle u \rangle\).

Lemma 3.11. For \(V, W \in \text{SVect}\) we have the identification \(\text{Hom}(\mathbb{K}, V^* \otimes W) = \text{Hom}(V, W)\) and \(\text{Hom}(W \otimes V^*, \mathbb{K}) = \text{Hom}(W, V)\). It induces the bijections

\[
\text{Hom}(\mathbb{K} \langle \varepsilon \rangle/(\varepsilon^2), A^! \circ B) \cong \text{Hom}(A, B), \quad \text{Hom}(B \bullet A^!, \mathbb{K} \langle u \rangle) \cong \text{Hom}(B, A) \tag{3.28}
\]
natural in $\mathcal{A} \in \text{FQSA}$, $\mathcal{B} \in \text{QSA}$.

**Proof.** By taking the even components of the natural isomorphisms of super-vector spaces $\text{hom}(\mathbb{K}, V^* \otimes W) = V^* \otimes W = \text{hom}(V, W)$, $\text{hom}(W \otimes V^*, \mathbb{K}) = W^* \otimes V = \text{hom}(W, V)$ and forgetful functor $\text{Vect} \to \text{Set}$ we obtain the identifications of the corresponding Hom-sets. Let $\mathcal{A} = TV/(R)$, $\mathcal{B} = TW/(S)$, where $V \in \text{FVect}$, $W \in \text{Vect}$. The elements of the Hom-sets in the left formula (3.28) are morphisms in QSA uniquely defined by their first degree components $\lambda: \mathbb{K} \to V^* \otimes W$ and $f: V \to W$ which satisfy the conditions $(\lambda \otimes \lambda) \in \sigma^{(23)}(R^1 \otimes W^{\otimes 2} + (V^*)^{\otimes 2} \otimes S)$ and $(f \otimes f)R \subset S$. If we identify $\lambda$ with $f$ via $\text{Hom}(\mathbb{K}, V^* \otimes W) = \text{Hom}(V, W)$, then we have $\lambda = \sum_i \lambda_i \otimes w_i$, $f = \sum_i w_i \lambda_i$ for some $\lambda_i \in V^*$, $w_i \in W$ such that $[\lambda_i] = [w_i]$. Note that the map $f \otimes f: V \otimes V \to W \otimes W$ acts on $v \otimes v' \in V \otimes V$ as $(f \otimes f)(v \otimes v') = f(v)f(v') = \sum_{i,j}(w_i \otimes w_j)\lambda_i(v)\lambda_j(v')$, hence $f \otimes f = \sum_{i,j}(-1)^{|\lambda_i||\lambda_j|}(w_i \otimes w_j)(\lambda_i \otimes \lambda_j)$. By applying Lemma 3.11 to the even operator $\xi = \sigma^{(23)}(\lambda \otimes \lambda) = \sum_i (-1)^{|\lambda_i||\lambda_j|}\lambda_i \otimes \lambda_j \otimes w_i \otimes w_j \in ((V^{\otimes 2})^* \otimes W^{\otimes 2})_0$ we see that these two conditions are equivalent to each other. The right isomorphism (3.28) is proved similarly (for $\mathcal{B} \in \text{FQSA}$ it follow from the left isomorphism (3.28), the isomorphism (3.25) and the fact that $(-)^1: \text{FQSA} \to \text{QSA}$ is fully faithful). \[ \Box \]

**Proposition 3.12.** The identification $\text{Hom}(\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}) = \text{Hom}(\mathcal{A}, \mathcal{B} \otimes \mathcal{C})$ induces the bijection $\text{Hom}(\mathcal{A} \bullet \mathcal{B} \bullet \mathcal{C}) \cong \text{Hom}(\mathcal{A}, \mathcal{B} \circ \mathcal{C})$ (or, equivalently, (3.27)) natural in $\mathcal{A}, \mathcal{B} \in \text{FQSA}, \mathcal{C} \in \text{QSA}$. It can be also regarded as a bijection natural in $\mathcal{A} \in \text{QSA}, \mathcal{B}, \mathcal{C} \in \text{FQSA}$. As consequence, the symmetric monoidal categories $(\text{QSA}, \bullet)$ and $(\text{QSA}, \circ)$ are closed and coclosed respectively relative to the subcategory $\mathcal{P} = \text{FQSA}$. The hom- and cohom functors have the form $\text{hom}_{\text{FQSA}}(\mathcal{B}, \mathcal{C}) = \mathcal{B} \circ \mathcal{C}$ and $\text{cohom}_{\text{FQSA}}(\mathcal{B}, \mathcal{A}) = \mathcal{A} \bullet \mathcal{B}$.

**Proof.** Lemma 3.11 and the isomorphisms (3.25) give the bijections $\text{Hom}(\mathcal{A} \bullet \mathcal{B} \bullet \mathcal{C}) \cong \text{Hom}(\mathcal{A} \bullet \mathcal{C}, \mathcal{B}) \cong \text{Hom}(\mathcal{A}, \mathcal{B} \circ \mathcal{C}) \cong \text{Hom}(\mathcal{A} \bullet \mathcal{C}, \mathcal{B}) \cong \text{Hom}(\mathcal{A}, \mathcal{B} \circ \mathcal{C})$ natural in $\mathcal{A}, \mathcal{B} \in \text{FQSA}, \mathcal{C} \in \text{QSA}$ and the bijections $\text{Hom}(\mathcal{A} \bullet \mathcal{B} \circ \mathcal{C}) \cong \text{Hom}(\mathcal{A} \bullet \mathcal{B} \bullet \mathcal{C}, \mathcal{A}) \cong \text{Hom}(\mathcal{A}, \mathcal{B} \circ \mathcal{C})$ natural in $\mathcal{A} \in \text{QSA}, \mathcal{B}, \mathcal{C} \in \text{FQSA}$. The hom- and cohom-functors are obtained from these bijections. \[ \Box \]

3.3.2. **Internal cohom for (FQSA, $\circ$) via the internal hom for SVect.** Let us derive more explicit expression for the cohom-functor generalising (3.2) Prop. 4.5. The first component is $\text{cohom}(\mathcal{B}, \mathcal{A})_1 = W^* \otimes V = \text{hom}(W, V)$, where $V = \mathcal{A}_1$, $W = \mathcal{B}_1$. There is a natural isomorphism

$$\text{hom}(W, V)^{\otimes 2} = W^* \otimes V \otimes W^* \otimes V \xrightarrow{\sigma^{(23)}} W^* \otimes W^* \otimes V \otimes V = \text{hom}(W^{\otimes 2}, V^{\otimes 2}). \quad (3.29)$$

**Proposition 3.13.** The cohom-functor for $\mathcal{A} = TV/(R)$ and $\mathcal{B} = TW/(S)$ has the form

$$\text{cohom}(\mathcal{B}, \mathcal{A}) = \mathcal{A} \bullet \mathcal{B}^1 = \widetilde{T}/(\widetilde{R}), \quad \text{where} \quad \widetilde{V} = \text{hom}(W, V) \quad (3.30)$$

and $\widetilde{R} \subset \widetilde{V}^{\otimes 2}$ is the preimage of \{ $\xi \in \text{hom}(W^{\otimes 2}, V^{\otimes 2}) \mid \xi(W^{\otimes 2}) \subset R, \xi(S) = 0$ \} under (3.22). For two morphisms $f$ in FQSA and $g$ in QSA the morphism $\text{cohom}(f, g)$ is defined by the component $\text{cohom}(f, g)_1 = \text{hom}(f_1, g_1) = f_1^* \cdot (g_1)_*$. \[ 16 \]
**Proof.** By the definitions (3.7) and (3.23) we have $A \bullet B' = T\tilde{V}/(\tilde{R})$ for the super-vector space $\tilde{V} = V \otimes W^\ast = \text{hom}(W, V)$ and super-subspace $\tilde{R} = \sigma^{(23)}(S^\perp \otimes R) \subset \tilde{V}^{\otimes 2}$. If $\xi$ belongs to $\sigma^{(23)} \tilde{R} = S^\perp \otimes R$, then $\xi(W^{\otimes 2}) \subset R$ and $\xi(S) = 0$. Conversely, due to Lemma 3.6 the latter two conditions imply $\xi \in (W^{\otimes 2})^\ast \otimes R$ and $\xi \in S^\perp \otimes V^{\otimes 2}$ respectively, hence $\xi$ belongs to the intersection $((W^{\otimes 2})^\ast \otimes R) \cap (S^\perp \otimes V^{\otimes 2}) = S^\perp \otimes R$.

Note that the map $\sigma: A_i \otimes B_i^\ast \cong B_i^\ast \otimes A_i$ gives the natural isomorphism

$$\text{cohom}(B, A) = A \bullet B' \cong B' \bullet A = \text{cohom}(A', B').$$

(3.31)

According to [S3, Th. 4.3] the monoidal categories $(\mathbb{N}_0\text{-GrAlg}, \circ)$ and $(\mathbb{Z}\text{-GrAlg}, \circ)$ are coclosed relative to $\text{FQA}$. This result can be generalised to the super-case. We prove this below by using Manin matrices (see p. 4.1.5).

### 4 Manin matrices and quantum representations

Now we generalise results on Manin matrices [S1] and on quantum representations [S2] to the super-case.

#### 4.1 Manin matrices for quadratic super-algebras

**4.1.1. Format.** Consider a basis $e: I \rightarrow W$ of a super-vector space $W \in \text{SVect}$. We always suppose that a basis is homogeneous: $e_i \in W_{k_i}$, where $e_i = e(i), i \in I$. The map $k: I \rightarrow \mathbb{Z}_2$, $i \mapsto \bar{k}_i$, is called format of the basis $(e_i)$. In other words, a format $k$ is given by the decomposition $I = I_0 \amalg I_1$ such that $[e_i] = 0$ for $i \in I_0$ and $[e_i] = 1$ for $i \in I_1$.

In the case $W \in \text{FSVect}$ one can take $I = \{1, 2, \ldots, d\}$. Then the format is a finite sequence $k = (\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_d)$. We can reorder a basis $(e_i)$ to the standard format $k = (0, \ldots, 0, 1, \ldots, 1)$, where the numbers of 0 and 1 are equal to $m = \dim W_0$ and $n = \dim W_1 = d - m$.

Let $k: I \rightarrow \mathbb{Z}_2$ and $l: \bar{I} \rightarrow \mathbb{Z}_2$ be maps from the sets $I$ and $\bar{I}$. Consider a map $I \times \bar{I} \rightarrow R$, $(i, j) \mapsto M_{ij} = M^j_i \in R$. It is called matrix $(M^j_i)$ of a format $k \times l$ over a super-vector space $R \in \text{SVect}$ if the entries $M_{ij}$ are homogeneous elements of degree $\bar{k}_i + \bar{l}_j$. For ordered $I$ and $\bar{I}$ (such as $I = \{1, 2, \ldots, d\}$, $\bar{I} = \{1, 2, \ldots, d\}$) the matrix is written as a table

$$\begin{pmatrix}
M^1_1 & \ldots & M^1_d \\
\vdots & \ddots & \vdots \\
M^n_d & \ldots & M^n_d
\end{pmatrix}.$$  

If the formats $k$ and $l$ are standard, then it has the block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \text{Mat}_{m \times m}(R_0)$, $B \in \text{Mat}_{\bar{m} \times m}(R_1)$, $C \in \text{Mat}_{\bar{m} \times n}(R_1)$, $D \in \text{Mat}_{\bar{n} \times n}(R_0)$.

**4.1.2. Matrix of an operator.** Let $W \in \text{SVect}$ and $\mathcal{R} \in \text{SAlg}$. The tensor product $W \otimes \mathcal{R}$ is a free right $\mathcal{R}$-module. We identify $w = w \otimes 1_{\mathcal{R}}$, so the element $w \otimes r \in \mathcal{R} \otimes W$.

---

2In [Man84], [Man91] Manin used the term "format" for a format of a matrix.
can be written as \( wr \). The isomorphism \( \sigma : W \otimes \mathcal{R} \xrightarrow{\cong} \mathcal{R} \otimes W \) gives the structure of the left module: \( r(wr') = (-1)^{|r| |r'|} wr', \ r, r' \in \mathcal{R}, \ w \in W. \)

Let \( \bar{W} \in \text{SVect} \). The \( \mathbb{Z}_2 \)-graded linear maps \( M : \bar{W} \to W \otimes \mathcal{R} \) are in one-to-one correspondence with the \( \mathbb{Z}_2 \)-graded morphisms of the right \( \mathcal{R} \)-modules \( \alpha : \bar{W} \otimes \mathcal{R} \to W \otimes \mathcal{R} \) as \( \alpha_M(\bar{w} \otimes r) = M(\bar{w})r, \ r \in \mathcal{R}, \ \bar{w} \in \bar{W} \). Similarly, such \( M \) uniquely defines a morphism of the left \( \mathcal{R} \)-modules \( \alpha_M : \mathcal{R} \otimes \bar{W} \to W \otimes \mathcal{R} \). We have \( \alpha_M = \alpha^{\text{st}} \) iff \( M(\bar{w})r = rM(\bar{w}) \) \( \forall \bar{w} \in \bar{W}, \ r \in \mathcal{R}. \)

Any morphism \( M \in \text{Hom}(\bar{W}, W \otimes \mathcal{R}) \) can be identified with an even element of the left free \( \mathcal{R} \)-module \( \mathcal{R} \otimes \text{hom}(\bar{W}, W) \), i.e. with an operator \( M \in (\mathcal{R} \otimes \text{hom}(\bar{W}, W))_\mathbb{Z}_2 \). Let \( (e_i)_{i \in I} \) and \( (\bar{e}_a)_{a \in I} \) be bases of \( W \) and \( \bar{W} \) with formats \( k: I \to \mathbb{Z}_2 \) and \( \bar{k}: \bar{I} \to \mathbb{Z}_2 \). For a fixed bases \( (e_i) \) and \( (\bar{e}_a) \) we identify the operator with its matrix: \( M = (M^a_i) \). The operator \( W \to W \otimes \mathcal{R} \) with entries \( \delta^i_j \) we denote by 1, it corresponds to the identity \( \mathcal{R} \)-module morphism \( \text{id} : W \otimes \mathcal{R} \to W \otimes \mathcal{R} \).

Let \( W' \in \text{SVect} \) has a basis \( (e'_i)_{i \in I'} \) of a format \( k' \). Let \( (N^i_j) \) be the matrix of an operator \( N : W' \to \bar{W} \otimes \mathcal{R} \), \( N^i_j = \sum_{a \in \bar{I}} \bar{e}_a N^a_i \), it has the format \( I \times k' \). Define the composition \( MN \) as the operator \( W' \to W \otimes \mathcal{R} \) corresponding to the composition \( W' \otimes \mathcal{R} \xrightarrow{\alpha_M} \bar{W} \otimes \mathcal{R} \xrightarrow{\alpha_N} W \otimes \mathcal{R} \), i.e.

\[
MN : W' \xrightarrow{N} \bar{W} \otimes \mathcal{R} \xrightarrow{M \otimes \text{id}_\mathcal{R}} W \otimes \mathcal{R} \otimes \mathcal{R} \xrightarrow{\text{id} \otimes \text{id} \otimes \mu_\mathcal{R}} W \otimes \mathcal{R}, \tag{4.1}
\]

where \( \mu_\mathcal{R} : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \), \( \mu_\mathcal{R}(r \otimes r') = rr' \), is the multiplication in \( \mathcal{R} \). The matrix of \( MN \) has the format \( k \times k' \), its entries are \( (MN)^i_j = \sum_{a \in \bar{I}} M^a_i N^a_j \).

If \( M \) and \( N \) are operators over some super-subalgebras of \( \mathcal{R} \), then we can regard them as operators over \( \mathcal{R} \), so the composition \( MN \) is defined as an operator over \( \mathcal{R} \) in the same way.

For a right \( \mathcal{R} \)-module \( R \) denote \( R^* = \text{hom}(\mathcal{R}, R) \). This is a contravariant functor from the category of right \( \mathcal{R} \)-modules to the category of left \( \mathcal{R} \)-modules. It translates free right modules to free left modules: \( (W \otimes \mathcal{R})^* = \mathcal{R} \otimes W^* \). The morphism \( \alpha_M : W \otimes \mathcal{R} \to W \otimes \mathcal{R} \) is translated to the morphism \( \alpha^*_M : \mathcal{R} \otimes W^* \to \mathcal{R} \otimes W^* \), \( \alpha^*_M(r \mu)(\bar{w} r') = r \mu(\alpha_M(\bar{w} r')) = r \mu(M(\bar{w})r'), \ r, r' \in \mathcal{R}, \ \mu \in W^*, \ \bar{w} \in \bar{W} \) and we identified \( \mu = 1_\mathcal{R} \otimes \mu \). We have \( \alpha^*_M = \alpha^{\text{st}}_M \) for the operator \( M^{\text{st}} \in \text{Hom}(W^*, \bar{W} \otimes \mathcal{R}) \) defined as \( M^{\text{st}}(\mu)(\bar{w}) = \mu(M(\bar{w})) \). Let \( (e^i) \) and \( (\bar{e}^a) \) be the bases of \( W^* \) and \( \bar{W}^* \) dual to \( (e_i) \) and \( (\bar{e}_a) \), they have the same formats \( k \) and \( I \). One can check that \( M^{\text{st}}(e^i) = \sum_{a \in \bar{I}} M^a_i \bar{e}^a = \sum_{a \in \bar{I}} \bar{e}^a M^{\text{st}}_a \), where \( (M^{\text{st}})^i_a = (-1)^{(k_i + I_a)} M^a_i \) are entries of the super-transposed matrix. Since the operation \( (-)^{\text{st}} \) is not involutive we need also the inverse of super-transposition. Define a matrix \( M^{\text{ist}} \) by the formulae \( (M^{\text{ist}})^{\text{st}} = M = (M^{\text{st}})^{\text{ist}} \), explicitly, \( (M^{\text{ist}})^i_a = (-1)^{(k_i + I_a)} M^a_i \).

Note that in the case \( \mathcal{R} = \mathbb{K} \) we have the operator \( M : \bar{W} \to W \). The operator \( M^{\text{st}} \) coincides with the image of \( M \) under the functor \( (-)^* : \text{SVect} \to \text{SVect} \), i.e. \( M^{\text{st}} = M^* : W^* \to W^* \).

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An element \( y \in \tilde{W} \otimes R \) has the unique decomposition \( y = \sum_{a \in T} c_a y^a, \) where \( y^a \in R \) are right coordinates of \( y \). The morphism \( \alpha_M : \tilde{W} \otimes R \to W \otimes R \) has the form \( \alpha_M(y) = \sum_{a \in T} M(\tilde{c}_a)y^a = \sum_{i \in I} e_i x^i \), where \( x^i = \sum_{a \in T} M_{i}^{a} y^a \) are right coordinates of \( x = \alpha_M(y) \). If we consider \( x \) and \( y \) as column-vectors with entries \( x^i \) and \( y^j \), then \( \alpha_M \) is equivalent to the action of the matrix \( M \) from the right: \( x = My \). Similarly the map \( \alpha_M^* : R \otimes W^* \to R \otimes \tilde{W}^* \) is equivalent to the action of the matrix \( M \) from the right: \( \psi \mapsto \psi M \), where \( \psi = \sum_{i \in I} \psi_i e^i \in R \otimes W^* \).

Consider the tensor product \( W \otimes \tilde{W} \). Its basis \( (e_i \otimes \tilde{c}_a) \) has the format \( k \oplus 1 : \{ (i, a) \mid i \in I, a \in T \} \to \mathbb{Z}_2 \), \((i, a) \mapsto \tilde{k}_i + \tilde{l}_a \). Introduce the operators \( M^{(1)} \) and \( M^{(2)} \) as

\[
M^{(1)} : \tilde{W} \otimes W' \xrightarrow{M \otimes id_{W'}} W \otimes R \otimes W' \xrightarrow{\sigma^{(23)}} W \otimes W' \otimes R, \\
M^{(2)} : W' \otimes \tilde{W} \xrightarrow{id_{W'} \otimes M} W' \otimes W \otimes R
\]

(thesnotations dependson \( W' \)). They have the entries

\[
(M^{(1)})^{ij}_{st} = (-1)^{|k_j + \bar{q}_a|} M_{i}^{k_j} \delta^i_{s}, \quad (M^{(2)})^{ij}_{ts} = \delta^i_{t} M_{i}^{j}.
\]

Let \( N : \tilde{W}' \to R \otimes W' \). The composition of the operators \( M^{(1)} : \tilde{W} \otimes \tilde{W}' \to W \otimes \tilde{W}' \otimes R \) and \( N^{(2)} : \tilde{W} \otimes \tilde{W}' \to \tilde{W} \otimes W' \otimes R \) is the operator

\[
M^{(1)}N^{(2)} : W \otimes \tilde{W}' \xrightarrow{M \otimes N} W \otimes R \otimes W' \otimes R \xrightarrow{\sigma^{(23)}} W \otimes W' \otimes R \xrightarrow{id \otimes id \otimes \mu_{R}} W \otimes W' \otimes R.
\]

Its entries are \((M^{(1)}N^{(2)})^{ij}_{st} = (-1)^{|k_i + \bar{q}_a|} M_{i}^{k_i} N_{s}^{l_j} \).

### 4.1.3. Quadratic algebras associated with idempotent operators.

Let \( B = TW/(R) \in QSA \). The super-subspace \( R \subset W \otimes W \) has a linear complement \( R_c \), i.e. a super-subspace \( R_c \subset W \otimes W \) such that \( W \otimes W = R \oplus R_c \). This decomposition defines the operator \( B \in \text{End}(W \otimes W) \) by the formula \( B(r + r_c) = r \) for \( r \in R, r_c \in R_c \). It is an idempotent: \( B^2 = B \). Thus we have \( B = TW/(\text{Im} B) \). Since \( R_c \) is not unique for a fixed algebra \( B \in QSA \), the idempotent \( B \) is also not unique.

The entries \( B^{st}_{ij} \) of the idempotent \( B \) in the basis \( (e_i \otimes e_j) \) are defined by the formula \( B(e_i \otimes e_j) = \sum_{a \in T} B^{st}_{ij}(e_a \otimes e_l) = \sum_{a \in T}(e_a \otimes e_l) B^{s}_{aj} \). Since the operator \( B \in \text{End}(W \otimes W) \) is even, \( B^{s}_{ij} = 0 \) if \( \tilde{k}_a + \tilde{k}_i + \tilde{k}_j \neq 0 \). The operator \( B^* \in \text{End}(W^* \otimes W^*) \) is also an idempotent. The dual basis is \( (1)^{|k_i + k_j|} (e^i \otimes e^j) \), hence we have \( B^*(e^s \otimes e^t) = (1)^{|k_i + k_j|} B^{st}_{ij} (e^i \otimes e^j) \).

Any quadratic super-algebra \( B \in FQSA \) has the form \( B \cong TW^*/(\text{Im} B^*) \) for an idempotent operator \( B \in \text{End}(W \otimes W) \). We use the notations \( \mathfrak{X}_B(\mathbb{K}) = TW^*/(\text{Im} B^*) \) and, more generally, \( \mathfrak{X}_B(R) = R \otimes \mathfrak{X}_B(\mathbb{K}) \cong \mathfrak{X}_B(\mathbb{K}) \otimes R \in \mathbb{N}_0\text{-GrSAlg} \) for \( R \in \text{SAlg} \). Denote by \( x^i \) the element of \( \mathfrak{X}_B(\mathbb{K}) \) corresponding to the basis element \( e^i \in W^* \). Then the super-algebra
\( \mathfrak{X}_B(\mathcal{R}) \) is an \( \mathcal{R} \)-algebra generated by \( x^i, i \in I \), with parities \([x^i] = \bar{k}_i \) and commutation relations \( \sum_{i,j \in I} (-1)^{k_i k_j} B^i_{ij} x^i x^j = 0 \).

Let \( S = 1 - B \) be the dual idempotent: \( \text{Im} S = R_c \). Denote \( \Xi_B(\mathbb{K}) = TW/(\text{Im} S) \), \( \Xi_B(\mathcal{R}) = \Xi_B(\mathbb{K}) \otimes \mathcal{R} \cong \mathcal{R} \otimes \Xi_B(\mathbb{K}) \). The latter is an \( \mathcal{R} \)-algebra generated by \( \psi_i, i \in I \), with parities \([\psi_i] = \bar{k}_i \) and commutation relations \( \sum_{i,j \in I} S^s_{ij} \psi_s \psi_t = 0 \), where \( \psi_i \in \Xi_B(\mathbb{K}) \) corresponds to \( e_i \in W \) and \( S^s_{ij} = \delta_i^s \delta_j^s - B^s_{ij} \) are entries of \( S \).

Consider the right \( \mathfrak{X}_B(\mathcal{R}) \)-module \( W \otimes \mathfrak{X}_B(\mathcal{R}) \) and denote \( X = \sum_{i \in I} e_i x^i \in W \otimes \mathfrak{X}_B(\mathcal{R}) \).

We can consider \( X \) as the column-vector \( X = \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix} \). The element \( \Psi = \sum_{i \in I} \psi_i e^i \in \Xi_B(\mathcal{R}) \otimes W^* \) of the left free \( \Xi_B(\mathcal{R}) \)-module \( \Xi_B(\mathcal{R}) \otimes W^* \) can be regarded as the row-vector \( \Psi = (\psi_1 \ldots \psi_N) \). By considering \( X \) as an operator \( K \rightarrow W \otimes \mathfrak{X}_B(\mathcal{R}) \) we can use the notations (4.2), (4.3), (4.5).

Let \( W^1 \) be a \( \mathcal{R} \)-algebra generated by \( \mu \). The latter implies \( \Psi(2) = \sum_{i,j \in I} (-1)^{k_i k_j} \psi_i \psi_j (e \otimes e^j) \in \Xi_B(\mathcal{R}) \otimes W^* \otimes W^* \). The operator \( B \) acts on \( X(1)X(2) \) and \( \Psi(1)\Psi(2) \) from the left and from the right respectively: \( B(X(1)X(2)) = \sum_{i,j \in I} (-1)^{k_i k_j} B(e_i \otimes e_j) x^i x^j = \sum_{i,j,s,t \in I} (-1)^{k_i k_j} B^s_{ij}(e_s \otimes e_t) x^i x^j \).

We see that the commutation relations of the \( \mathcal{R} \)-algebras \( \mathfrak{X}_B(\mathcal{R}) \) and \( \Xi_B(\mathcal{R}) \) are written in matrix form as

\[
B(X(1)X(2)) = 0, \quad \Psi(1)\Psi(2)S = 0. \tag{4.6}
\]

Note that the quadratic super-algebras \( \mathfrak{X}_B(\mathbb{K}) \) and \( \Xi_B(\mathbb{K}) \) are Koszul-dual to each other:

\[
\mathfrak{X}_B(\mathbb{K})^t = \Xi_B(\mathbb{K}), \quad \Xi_B(\mathbb{K})^t = \mathfrak{X}_B(\mathbb{K}).
\]

Let \( A_W = \frac{1 - \sigma_{W,W}}{2}, \quad S_W = \frac{1 + \sigma_{W,W}}{2} \) be super-antisymmetrizer and super-symmetrizer acting in \( W \otimes W \). Then we have

\[
SW^* = \mathfrak{X}_{A_W}(\mathbb{K}), \quad AW = \Xi_{A_W}(\mathbb{K}). \tag{4.7}
\]

**Lemma 4.1.** Let \( T: W \otimes W \rightarrow \tilde{W} \otimes \mathcal{R} \), then the condition \( T(X(1)X(2)) = 0 \) is equivalent to \( TS = 0 \). Let \( U: \tilde{W} \rightarrow W \otimes W \otimes \mathcal{R} \), then the condition \( (\Psi(1)\Psi(2))U = 0 \) is equivalent to \( BU = 0 \).

**Proof.** Define the entries \( T^a_{ij} \in \mathcal{R} \) as \( T(e_i \otimes e_j) = \sum_{a \in I} \tilde{e}_a T^a_{ij} \). If \( TS = 0 \), then \( T = TB \) and hence \( T(X(1)X(2)) = TB(X(1)X(2)) \). Conversely, let the element \( T(X(1)X(2)) \) vanish, then \( \sum_{i,j \in I} (-1)^{k_i k_j} T^a_{ij} x^i x^j = 0 \), hence \( T^a_{ij} = G^a_{st} B^s_{ij} \) for some \( G^a_{st} \in \mathcal{R} \). We get \( T = GB \), where \( G: W \otimes W \rightarrow \mathcal{R} \otimes \tilde{W} \) is the operator with the entries \( G^a_{ij} \). Thus \( TS = GBS = 0 \). The second equivalence is checked similarly.

\[\Box\]

**4.1.4.** \((B, \tilde{B})\)-Manin matrices. Let \( \tilde{B} \in \text{End}(\tilde{W} \otimes \tilde{W}) \) be another idempotent and \( \tilde{S} = 1 - \tilde{B} \). Their entries are defined by

\[
\tilde{B}(\tilde{e}_a \otimes \tilde{e}_b) = \sum_{a,b \in I} (\tilde{e}_a \otimes \tilde{e}_b) \tilde{B}^{ab}_{cd}, \quad \tilde{S}^{ab}_{cd} = \delta^a_c \delta^b_d - \tilde{B}^{ab}_{cd}.
\]
Denote the generators of $\mathfrak{X}_B(\mathbb{K})$ and $\Xi_B(\mathbb{K})$ by $\tilde{x}^a$ and $\tilde{\psi}_a$, where $a$ runs over $\tilde{I}$. They form the column-vector $\tilde{X} = \sum_{a \in \tilde{I}} \tilde{e}_a \tilde{x}^a$ and the row-vector $\tilde{\Psi} = \sum_{a \in \tilde{I}} \tilde{\psi}_a e^a$.

Any morphism of $\mathbb{N}_0$-graded super-algebras $f : \mathfrak{X}_B(\mathbb{K}) \to \mathfrak{X}_B(\mathbb{R})$ is uniquely determined by its first component $f_1 : W^* \to \mathcal{R} \otimes \tilde{W}^*$. In terms of the generators we have $f(x^i) = \sum_{a \in \tilde{I}} M^i_a \tilde{x}^a$ for some matrix $M = (M^i_a)$. This is a matrix of the operator $M : \tilde{W} \to \mathcal{R} \otimes \tilde{W}$ such that $f_1 = M^{st}$. Analogously, a morphism $g : \Xi_B(\mathbb{K}) \to \Xi_B(\mathbb{R})$ in $\mathbb{N}_0\text{-GrSAlg}$ has the form $g(\tilde{\psi}_a) = \sum_{i \in \tilde{I}} \psi_i M^i_a$.

**Theorem 4.2.** The following conditions are equivalent.

- The operator $M : \tilde{W} \to W \otimes \mathcal{R}$ satisfies
  $$BM^{(1)}M^{(2)}(1 - \tilde{B}) = 0. \tag{4.8}$$

- The entries $M^i_a \in \mathcal{R}$ have the parities $[M^i_a] = \bar{k}_i + \bar{l}_a$ and satisfy
  $$\sum_{i,j \in \tilde{I}, a,b \in \tilde{I}} (-1)^{(\bar{k}_i + \bar{l}_a)(\bar{k}_j + \bar{l}_b)} B^{ab}_{ij} M^i_a M^j_b \xi_{cd} = 0. \tag{4.9}$$

- The formula
  $$f_M(x^i) = \sum_{a \in \tilde{I}} M^i_a \tilde{x}^a \tag{4.10}$$
  defines a graded morphism $f_M : \mathfrak{X}_B(\mathbb{K}) \to \mathfrak{X}_B(\mathbb{R})$.

- The formula
  $$f^M(\tilde{\psi}_a) = \sum_{i \in \tilde{I}} \psi_i M^i_a \tag{4.11}$$
  defines a graded morphism $f^M : \Xi_B(\mathbb{K}) \to \Xi_B(\mathbb{R})$.

**Proof.** The first and second conditions are equivalent since the left hand side of (4.8) is the entries of the left hand side of (4.9). Let $y^i = \sum_{a \in \tilde{I}} M^i_a \tilde{x}^a$ and $Y = \sum_{i \in \tilde{I}} \tilde{e}_i y^i = M \tilde{X}$. The formula $f_M(x^i) = y^i$ defines the homomorphism iff $\sum_{i,j \in \tilde{I}, a,b \in \tilde{I}} (-1)^{k_i k_j} B^{ab}_{ij} y^i y^j = 0$, that is $B(Y^{(1)} Y^{(2)}) = 0$. Substitution $Y = M \tilde{X}$ yields $B(M^{(1)} M^{(2)}) (\tilde{X}^{(1)} \tilde{X}^{(2)}) = 0$. By means of Lemma 4.1 applied to $T = B(M^{(1)} M^{(2)})$ this is equivalent to (4.8). Similarly, the last condition means exactly that $(\Phi(1) \Phi(2)) \tilde{S} = 0$, where $\Phi = \Psi M$. By virtue of Lemma 4.1 applied to $U = (M^{(1)} M^{(2)}) \tilde{S}$ this is also equivalent to (4.8). \hfill \Box

**Definition 4.3.** The matrix $M = (M^i_a)$ satisfying the conditions from Theorem 4.2 is called super-Manin matrix for idempotents $B$ and $B$ or $(B, B)$-Manin matrix. If $B = B$, then we call it simply $B$-Manin matrix.
Thus we obtain a one-to-one correspondence between $(B, \tilde{B})$-Manin matrices, the morphisms $f_M : \mathcal{X}_B(K) \to \mathcal{X}_{\tilde{B}}(\mathcal{R})$ and the morphisms $f^M : \Xi_B(K) \to \Xi_B(\mathcal{R})$. These morphisms can be written in the matrix form:

$$f_M(X) = M\tilde{X}, \quad f^M(\tilde{\Psi}) = \Psi M. \quad (4.12)$$

4.1.5. Universal $(B, \tilde{B})$-Manin matrices. Let $U_{B, \tilde{B}}$ be the quadratic super-algebra with the generators $\mathcal{M}_a^i, i \in I$, $a \in \tilde{I}$, of parity $[\mathcal{M}_a^i] = k_i + \tilde{l}_a$; the commutation relations are $\sum_{i,j \in I} (-(1)^{k_i + \tilde{l}_a}) k_i B_{ij}^{st} M_a^i M_b^j = 0$. These generators form the $(B, \tilde{B})$-Manin matrix $\mathcal{M} = (\mathcal{M}_a^i)$. We call $\mathcal{M}$ the universal $(B, \tilde{B})$-Manin matrix by the following reason: any $(B, \tilde{B})$-Manin matrix $M$ over $\mathcal{R}$ is an image of $\mathcal{M}$ under a homomorphism of super-algebras $h : U_{B, \tilde{B}} \to \mathcal{R}$ in the sense that $M_a^i = h(\mathcal{M}_a^i)$. Theorem 4.2 gives the bijections

$$\text{Hom}_{\text{SAlg}}(U_{B, \tilde{B}}, \mathcal{R}) = \text{Hom}_{\text{FQSA}}(\mathcal{X}_B(K), \mathcal{X}_{\tilde{B}}(\mathcal{R})) = \text{Hom}_{\text{FQSA}}(\Xi_B(K), \Xi_B(\mathcal{R})). \quad (4.13)$$

Proposition 4.4. We have the isomorphisms of quadratic super-algebras

$$U_{B, \tilde{B}} \cong \text{cohom}(\mathcal{X}_B(K), \mathcal{X}_{\tilde{B}}(\mathcal{K})) \cong \text{cohom}(\Xi_B(K), \Xi_{\tilde{B}}(\mathcal{K})) \quad (4.14)$$

given by the identification $\mathcal{M}_a^i = x^i \otimes \tilde{\psi}_a = (-1)^{k_i + \tilde{l}_a} \tilde{\psi}_a \otimes x^i$.

Proof. Recall that $\text{cohom}(\mathcal{X}_B(K), \mathcal{X}_{\tilde{B}}(\mathcal{K}))$ has the form $\mathcal{X}_B(K) \otimes \mathcal{X}_{\tilde{B}}(\mathcal{K}) = \mathcal{X}_B(K) \otimes \Xi_{\tilde{B}}(\mathcal{K})$. Let $M_a^i = x^i \otimes \tilde{\psi}_a$. If $\sum_{i,j,a,b} T_{ij}^{ab} M_a^i M_b^j = 0$ for some $T_{ij}^{ab} \in K$, then $\sum_{i,j,a,b} T_{ij}^{ab} (-1)^{k_i + \tilde{l}_a} (e^i \otimes e^j \otimes \tilde{\psi}_a \otimes \tilde{\psi}_b) \in \text{Im}(B^a) \otimes \text{Im}(\tilde{B})$, so we have $T_{ij}^{ab} = \sum_{s,t,c,d} G_{st}^{cd} (-1)^{\tilde{k}_s + \tilde{k}_t} B_{ij}^{st} \tilde{\psi}_a \otimes \tilde{\psi}_b$ for some $G_{st}^{cd} \in K$, hence the relation $\sum_{i,j,a,b} T_{ij}^{ab} M_a^i M_b^j = 0$ is a linear combination of the relations (4.9). This implies the first isomorphism (4.14). Then, the second isomorphism (4.14) follows from (3.31). □

Let us generalise the result $[S3, \S 4.2.3, \text{Th. 4.3]}$ to the super-case by using the quadratic super-algebra $U_{B, \tilde{B}}$.

Theorem 4.5. The symmetric monoidal categories $(\mathbb{N}_0, \text{GrSAlg}, \circ)$ and $(\mathbb{Z}, \text{GrSAlg}, \circ)$ are coclosed relative to $P = \text{FQSA}$. The cohom-functor $\text{cohom} : \text{P}^{\text{op}} \times \text{P} \to \text{C}$ is the composition of the cohom-functor $\text{cohom} : \text{P}^{\text{op}} \times \text{P} \to \text{QSA}$ described in Propositions 3.1 and 4.4 with the embedding $\text{QSA} \hookrightarrow \text{C}$ for the both $\text{C} = \mathbb{N}_0, \text{GrSAlg}$ and $\text{C} = \mathbb{Z}, \text{GrSAlg}$.

Proof. If $R$ has a structure of $\mathbb{Z}$-graded super-algebra $\mathcal{R} = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}_k$, then the graded homomorphisms $h : U_{B, \tilde{B}} \to \mathcal{R}$ form a subset of $\text{Hom}_{\text{QSA}}(\mathcal{X}_B(K), \mathcal{X}_{\tilde{B}}(\mathcal{K}))$ corresponding to the $(B, \tilde{B})$-Manin matrices $M$ such that $M_a^i \in \mathcal{R}_1$. As a subset of $\text{Hom}_{\text{QSA}}(\mathcal{X}_B(K), \mathcal{X}_{\tilde{B}}(\mathcal{R}))$ it consists of the homomorphisms factorising as $f_M : \mathcal{X}_B(K) \mathcal{R} \circ \mathcal{X}_{\tilde{B}}(K) \to \mathcal{X}_{\tilde{B}}(\mathcal{R})$ through the graded homomorphism $f(x^i) = \sum_{a \in I} M_a^i \otimes \tilde{x}^a$. Consider the graded homomorphism $\eta : \mathcal{X}_B(K) \to U_{B, \tilde{B}} \circ \mathcal{X}_B(K)$,
\[ \eta(x^i) = \sum_{a \in I} M'_a \otimes x^a. \]

For any morphism \( f \) there exists a unique \( h \) making the diagram commute. Hence \((U_{B,B}, \eta)\) is a universal morphism from the object \( \mathcal{X}_B(K) \) to the functor \( G : \mathbb{Z} \text{-GrAlg} \to \mathbb{Z} \text{-GrAlg} \), \( G(R) = R \circ \mathcal{X}_B(K) \). Since any \( B, B' \in \mathcal{FQSA} \) are isomorphic to some \( \mathcal{X}_B(K) \) and \( \mathcal{X}_{B'}(K) \), the monoidal category \((\mathbb{Z} \text{-GrAlg}, \circ)\) is coclosed relative to \( \mathcal{FQSA} \) with the cohom-functor \((4.14)\), see [S3, § 2.3.2, Th. 3.13 (2')].

4.2 Quantum representations and quantum linear actions

4.2.1. (Co)representations. Let \((C, \otimes)\) be a relatively closed monoidal category with the parametrising subcategory \( P \subset C \). By following [S2], [S3] we define corepresentation of a comonoid \( O = (X, \Delta_X, \varepsilon_X) \in \text{Comon}(C, \otimes) \) on an object \( W \in P \) as a morphism \( \omega : \text{coend}(W) \to O \) in the category \( \text{Comon}(C, \otimes) \). Corepresentations of a fixed \( O \) form the category \( \text{Corep}_P(O) \). Its objects are \((W, \omega)\). A morphism \((W, \omega) \to (W', \omega')\) in \( \text{Corep}_P(O) \) is a morphism \( f : W \to W' \) in \( C \) such that the diagram

\[
\begin{array}{ccc}
\text{cohom}(W', W) & \xrightarrow{\text{cohom}(f, id_W)} & \text{cohom}(W, W) \\
\text{cohom}(id_{W'}, f) & & \omega \\
\text{cohom}(W', W') & & X
\end{array}
\]

commutes. The adjunction morphism

\[
\vartheta : \text{Hom} \left( \text{coend}(W), X \right) \cong \text{Hom}(W, X \otimes W)
\]

gives a one-to-one correspondence between the corepresentations \( \omega : \text{coend}(W) \to O \) and coactions \( \delta : W \to X \otimes W \) of \( O \) on \( W \). In this way we obtain the embedding of categories \( \text{Corep}_P(O) \hookrightarrow \text{Lcoact}(O) \).
Dually, if \((C, \otimes)\) is closed relative to \(P\), then one can define representations of a monoid \(M \in \text{Mon}(C, \otimes)\). They form the category \(\text{Rep}_P(M)\) consisting of \((W, \rho)\), where \(W \in P\) and \(\rho: M \to \text{end}(W)\). For example, if \((C, \otimes)\) is the closed symmetric monoidal category \((\text{SVect}, \otimes)\) defined in p. 2.2.1 (the case \(P = C = \text{SVect}\)), then we obtain the categories \(\text{Rep}_{\text{SVect}}(\mathbb{R})\) for \(\mathbb{R} \in \text{SAlg} = \text{Mon}(\text{SVect}, \otimes)\). This is a category of left super-modules over the super-algebra \(\mathbb{R}\).

**4.2.2. Quantum super-algebras.** Let us consider the cases of the relatively closed categories \((\mathbb{N}_0\text{-GrSAlg}, \circ)\) and \((\mathbb{Z}\text{-GrSAlg}, \circ)\). A comonoid in \((\mathbb{Z}\text{-GrSAlg}, \circ)\) is \(\mathbb{O} = (\mathbb{A}, \Delta, \varepsilon)\), where \(\mathbb{A} \in \mathbb{Z}\text{-GrSAlg}\) is a \(\mathbb{Z}\)-graded super-algebra \(\mathbb{A} = \bigoplus_{k \in \mathbb{Z}} \mathbb{A}_k\) and \(\Delta: \mathbb{A} \rightarrow \mathbb{A} \circ \mathbb{A}\) and \(\varepsilon: \mathbb{A} \rightarrow \mathbb{K}[u, u^{-1}]\) are graded homomorphisms with components \(\Delta_k: \mathbb{A}_k \rightarrow \mathbb{A}_k \otimes \mathbb{A}_k\), \(\varepsilon_k: \mathbb{A}_k \rightarrow \mathbb{K}u^k\). By composing them with non-graded homomorphisms \(\mathbb{A} \circ \mathbb{A} \rightarrow \mathbb{A} \otimes \mathbb{A}\) and \(\mathbb{K}[u, u^{-1}] \rightarrow \mathbb{K}, f(u) \mapsto f(1)\) we obtain morphisms \(\Delta_A: \mathbb{A} \rightarrow \mathbb{A} \otimes \mathbb{A}\) and \(\varepsilon_A: \mathbb{A} \rightarrow \mathbb{K}\) in \(\text{SAlg}\) such that \((\mathbb{A}, \Delta_{\mathbb{A}}, \varepsilon_{\mathbb{A}})\) is a super-bialgebra. Conversely, a super-bialgebra \((\mathbb{A}, \Delta_{\mathbb{A}}, \varepsilon_{\mathbb{A}})\) gives a comonoid \(\mathbb{O} = (\mathbb{A}, \Delta, \varepsilon)\) if \(\mathbb{A}\) is a \(\mathbb{Z}\)-graded super-algebra satisfying

\[
\Delta_{\mathbb{A}}(\mathbb{A}_k) \subset \mathbb{A}_k \otimes \mathbb{A}_k; \tag{4.20}
\]

the morphism \(\varepsilon: \mathbb{A} \rightarrow \mathbb{K}[u, u^{-1}]\) has the form \(\varepsilon(a_k) = \varepsilon_{\mathbb{A}}(a_k)u^k\), where \(a_k \in \mathbb{A}_k\).

Analogously, a comonoid \(\mathbb{O} = (\mathbb{A}, \Delta, \varepsilon) \in \text{Comon}(\mathbb{N}_0\text{-GrSAlg}, \circ)\) is given by \(\mathbb{A} = \bigoplus_{k \geq 0} \mathbb{A}_k\), \(\Delta: \mathbb{A} \rightarrow \mathbb{A} \circ \mathbb{A}\), \(\varepsilon: \mathbb{A} \rightarrow \mathbb{K}[u]\). Such comonoids correspond to bialgebras \((\mathbb{A}, \Delta_{\mathbb{A}}, \varepsilon_{\mathbb{A}})\) with \(\mathbb{N}_0\)-grading of \(\mathbb{A}\) making it an \(\mathbb{N}_0\)-graded super-algebra satisfying \((4.20)\). The colax monoidal embedding \((\mathbb{N}_0\text{-GrSAlg}, \circ) \rightarrow (\mathbb{Z}\text{-GrSAlg}, \circ)\) induces the categorical embedding \(\text{Comon}(\mathbb{N}_0\text{-GrSAlg}, \circ) \subset \text{Comon}(\mathbb{Z}\text{-GrSAlg}, \circ)\). The comonoids in \((\text{QSA}, \circ)\) (more precisely, monoids in \((\text{QSA}^{\text{op}}, \circ)\)) generalise the super-algebras for the quantum case. By **quantum super-algebra** we mean a general comonoid \(\mathbb{O} \in \text{Comon}(\mathbb{Z}\text{-GrSAlg}, \circ)\).

The comonoid \(\text{coend}(\mathbb{B}) \in \text{Comon}(\mathbb{N}_0\text{-GrSAlg}, \circ)\) described in p. 4.1.5 is embedded into \(\text{Comon}(\mathbb{Z}\text{-GrSAlg}, \circ)\) as the \(\mathbb{Z}\)-graded super-algebra \(\text{coend}(\mathbb{B}) = \mathcal{U}_B\) with the comultiplication \((4.16)\) and the counit \(\nu_B: \text{coend}(\mathbb{B}) \rightarrow \mathbb{K}[u, u^{-1}]\) defined by the formula \((4.17)\).

**4.2.3. Quantum representations and actions.** Let \(\mathbb{O} = (\mathbb{A}, \Delta, \varepsilon)\) be a quantum super-algebra.

**Definition 4.6.** Quantum representation of \(\mathbb{O}\) on \(\mathbb{B} \in \text{FQSA}\) is the corepresentation \(\omega: \text{coend}(\mathbb{B}) \rightarrow \mathbb{O}\). This is a graded homomorphism \(\omega: \text{coend}(\mathbb{B}) \rightarrow \mathbb{A}\) making the diagrams

\[
\text{coend}(\mathbb{B}) \xrightarrow{d_B} \text{coend}(\mathbb{B}) \circ \text{coend}(\mathbb{B}) \xrightarrow{\omega \circ \omega} \text{coend}(\mathbb{B}) \xrightarrow{\varepsilon} \mathbb{K}[u, u^{-1}]
\]

(4.21)
commute. Their morphisms are morphisms in $\text{Corep}_{FQSA}(\mathcal{O})$. Quantum linear action of $\mathcal{O}$ on $C \in \mathbb{Z}$-$\text{GrSAlg}$ is the coaction $\delta: C \to A \circ C$. This is a graded homomorphism such that the following diagrams commute:

\[
\begin{array}{c}
\begin{array}{ccc}
C & \xrightarrow{\delta} & A \circ C \\
\downarrow & & \downarrow \Delta_{\text{oid}} \\
\end{array}
\end{array}, \quad
\begin{array}{c}
\begin{array}{ccc}
A \circ C & \xrightarrow{\text{id} \circ \delta} & A \circ A \circ C \\
\downarrow & & \downarrow \Delta_{\text{oid}} \\
\end{array}
\end{array}, \quad
\begin{array}{c}
\begin{array}{ccc}
C & \xrightarrow{\delta} & A \circ C \\
\downarrow & & \downarrow \epsilon_{\text{oid}} \\
A \circ C & \xrightarrow{K[u, u^{-1}] \circ C} & K \\
\end{array}
\end{array}
\] (4.22)

Their morphisms are morphisms in $\text{Lcoact}(\mathcal{O})$.

For a quantum linear action $\delta$ consider a composition $\delta_A: C \to A \circ C \hookrightarrow A \otimes C$. This is a coaction of the super-bialgebra $(A, \Delta_A, \epsilon_A)$ on the super-algebra $C$. Conversely, a coaction $\delta_A: C \to A \otimes C$ of this super-bialgebra has this form for some $\delta$ if

\[
\delta_A(C_k) \subset A_k \otimes C_k. \quad (4.23)
\]

Thus we obtain the forgetful functor $\text{Lcoact}(\mathcal{O}) \to \text{Lcoact}(A, \Delta_A, \epsilon_A)$.

The bijection (4.19) for $(\mathbb{Z}$-$\text{GrSAlg}, \circ)$ has the form

\[
\vartheta: \text{Hom}(\text{coend}(B), A) \cong \text{Hom}(B, A \circ B). \quad (4.24)
\]

For any quantum super-algebra $\mathcal{O} = (A, \Delta, \epsilon)$ and quadratic super-algebra $B \in FQSA$ we have the one-to-one correspondence $\vartheta: \omega \leftrightarrow \delta$ between quantum representations $\omega$ and quantum linear actions $\delta$ of $\mathcal{O}$ on $B$. It defines the fully faithful functor $\text{Corep}_{FQSA}(\mathcal{O}) \hookrightarrow \text{Lcoact}(\mathcal{O})$, $(B, \omega) \mapsto (B, \vartheta(\omega))$.

**4.2.4. Multiplicative Manin matrices.** The matrix $M = (M^i_j)$ over a super-bialgebra $(A, \Delta_A, \epsilon_A)$ is called multiplicative iff

\[
\Delta_A(M^i_j) = \sum_l M^i_l \otimes M^l_j, \quad \epsilon_A(M^i_j) = \delta^i_j. \quad (4.25)
\]

Let $\mathcal{O} = (A, \Delta, \epsilon)$ be a quantum super-algebra and $(A, \Delta_A, \epsilon_A)$ be the corresponding super-bialgebra. The matrix $M$ with entries $M^i_j \in A$ is called multiplicative over $\mathcal{O}$ iff it is multiplicative over this super-bialgebra. It is called first order matrix iff $M^i_j \in A_1$. For instance, the universal $B$-Manin matrix is a first order multiplicative matrix over $U_B = \text{end}(\mathcal{X}_B(\mathbb{K}))$.

**Theorem 4.7.** Let $\mathcal{O} \in \text{Comon}(\mathbb{Z}$-$\text{GrSAlg}, \circ)$ and $B = \mathcal{X}_B(\mathbb{K})$ for some idempotent $B$. Let $\mathcal{M} = (M^i_j)$ be the universal $B$-Manin matrix. Any quantum representation of $\mathcal{O}$ on $B$ has the form $\omega(M^i_j) = M^i_j$ for some first order multiplicative $B$-Manin matrix $M = (M^i_j)$ over $\mathcal{O}$. Conversely, any such matrix defines a quantum representation $\omega: \text{coend}(B) \to \mathcal{O}$. The quantum linear action $\delta = \vartheta(\omega): B \to A \circ B$ has the form $\delta(x^i) = \sum_j M^i_j \otimes x^j$, i.e. $\delta_A = f_M$. Let $C = \mathcal{X}_C(\mathbb{K})$. Consider two quantum representations $\omega: \text{coend}(B) \to \mathcal{O}$
and $\nu$: $\text{coend}(C) \to \emptyset$ corresponding to the multiplicative first order $B$- and $C$-Manin matrices $M$ and $N$ over $\emptyset$. A homomorphism $f: B \to C$ is a morphism $(B, \omega) \to (C, \nu)$ in $\text{Corep}_{\text{FQSA}}(\emptyset) = \text{morph}(B, \vartheta(\omega)) \to (C, \vartheta(\nu))$ in $\text{Lcoact}(\emptyset)$ iff $f = f_K$ for a $(B, C)$-Manin matrix $K$ over $\mathbb{K}$ such that $MK = KN$.

This theorem follows from Theorems 4.2 and 4.5. The proof is similar to [S2, Th. 5.2, Prop. 5.3, 5.5, 5.6].

4.2.5. Opposite and coopposite quantum representations. Let $\mathcal{R} = (V, \mu_R, \eta_R)$ be a super-algebra with a multiplication $\mu_R: V \otimes V \to V$ and a unity map $\eta_R: V \to \mathbb{K}$, where $V \in \ SVect$. The opposite super-algebra is $\mathcal{R}^{op} = (V, \mu_R^{op}, \eta_R)$. If $\mathcal{R} \in \text{Z-GrSAlg}$, then $\mathcal{R}^{op} \in \text{Z-GrSAlg}$, see p. 3.2.8.

The opposite and coopposite to the comonoid $\emptyset = (A, \Delta, \varepsilon) \in \text{Comon}(\text{Z-GrSAlg}, \circ)$ are $\emptyset^{op} = (A^{op}, \Delta, \varepsilon) \in \text{Comon}(\text{Z-GrSAlg}, \circ), \emptyset^{cop} = (A, \Delta^{cop}, \varepsilon) \in \text{Comon}(\text{Z-GrSAlg}, \circ)$, where $\Delta^{cop} = \sigma \cdot \Delta$. The bialgebras corresponding to these comonoids are $(A^{op}, \Delta^{op}, \varepsilon_A)$ and $(A, \Delta^{cop}, \varepsilon_A)$.

Denote $B^{(21)} = \sigma \cdot B \sigma$. This is an operator $W \otimes W \to W \otimes W$ with the entries $(B^{(21)})_{st i j} = (-1)^{k_i k_j} B^{(21)}_{i j}$. The quadratic super-algebra $\mathcal{X}_{B^{(21)}}(\mathbb{K})$ is defined by the commutation relations $\sum_{i,j} B^{(21)}_{i j} x^i x^j = 0$, hence $\mathcal{X}_{B^{(21)}}(\mathbb{K}) = \mathcal{X}_{B(\mathbb{K})}^{op}$. Also, we obtain $\mathcal{U}_{B^{(21)}} = \mathcal{U}_{B}^{op}$, which is an isomorphism of quantum super-algebras: $\text{coend}(B^{op}) = \text{coend}(B)^{op}$.

Consider a quantum representation $\omega: \text{coend}(B) \to \emptyset$ of $\emptyset = (A, \Delta, \varepsilon)$ on $B = \mathcal{X}_B(\mathbb{K})$ corresponding to a multiplicative first order $B$-Manin matrix $M$ over $A$. It gives the comonoid morphism $\omega^{op}: \text{coend}(B^{op}) \to \emptyset^{op}$. This is a quantum representation of $\emptyset^{op}$ on $B^{op} = \mathcal{X}_{B^{(21)}}(\mathbb{K})$ defined by the multiplicative first order $B^{(21)}$-Manin matrix $M$ over $A^{op}$.

Due to [3.2.8] the quantum super-algebra $\text{coend}(B)$ coincides with $\text{coend}(B')$ as a graded super-algebra, but it has different comultiplication $d_{B'}$. Note that the matrix $M^{ist}$ is multiplicative over the comonoid $\text{coend}(B')$, i.e. $d_{B'}: (M^{ist})_{i j} \mapsto \sum_{l} (M^{ist})_{i l} \otimes (M^{ist})_{l j}$, where $(M^{ist})_{i j} = (-1)^{k_j} M_{i j}^{op}$ are entries of $M^{ist}$, see p. 4.1.2. Hence we have $\text{coend}(B') = \text{coend}(B)^{cop}$. The quantum representation $\omega: \text{coend}(B') \to \emptyset^{cop}$ of $\emptyset^{cop}$ on $B' = \Xi_B(\mathbb{K})$ is defined by the multiplicative first order $B'$-Manin matrix $M^{ist}$ over $\mathcal{A}^{cop}$. The corresponding coaction is $\delta^{cop}(\psi_i) = \sum_{j} (M^{ist})_{i j} \psi_j = \sum_{j} \psi_j M_{i j}^{op}$.

4.2.6. Parity change. Let us apply the functor $\hat{\Pi}: \text{FQSA} \to \text{FQSA}$ to the quantum representation space $B = \mathcal{X}_B(\mathbb{K})$. By definition we need to change $W$ by $\Pi W$, which is equivalent to the substitution $\tilde{k}_i \to \tilde{k}_i + \tilde{l}$. Denote by $\Pi B$ the operator $\Pi W \otimes \Pi W \to \Pi W \otimes \Pi W$ given by the same matrix, but with the changed format. Since $B^{ist}_{i j} \neq 0$ implies $\tilde{k}_i + \tilde{k}_j \neq \tilde{k}_i + \tilde{k}_j$ the relations $B^{ist}_{i j} (-1)^{\tilde{k}_i \tilde{k}_j} x^i x^j = 0$ keep the same, so that $\hat{\Pi} \mathcal{X}_B(\mathbb{K}) = \mathcal{X}_{\Pi B}(\mathbb{K})$.

We do not change the $\mathbb{Z}_2$-grading of $\mathcal{A}$. If $M = (M_{i j})$ is a multiplicative first order $B$-Manin matrix over $\mathcal{A}$, then the parity $[M_{i j}] = \tilde{k}_i + \tilde{k}_j$ does not change, hence $M$ is a $\Pi B$-Manin matrix. Thus we obtain a quantum representation of the same $\emptyset = (A, \Delta, \varepsilon)$ on the quadratic super-algebra $\hat{\Pi} B = \mathcal{X}_{\Pi}(\mathbb{K})$ defined by the multiplicative first order $\Pi B$-Manin matrix $M$.  

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4.2.7. Classical left modules over finite-dimensional super-algebras. Let $\mathcal{R}$ be a finite-dimensional super-algebra, that is $\mathcal{R} \in \text{Mon}(\text{FSVec}, \otimes)$. It can be lifted to the level of quantum super-algebras by the contravariant colax monoidal functor $S^*: (\text{FSVec}, \otimes) \to (\text{FQSA}, \circ)$. We obtain the quantum algebra $S^*\mathcal{R} = S\mathcal{R}^*$. This is a commutative super-algebra of polynomials on the super-vector space $\mathcal{R}$. The multiplication and unity of $\mathcal{R}$ induces the structure of super-bialgebra on $S\mathcal{R}^*$.

A structure of (left) $\mathcal{R}$-module on a super-vector space $W$ is given by a super-algebra morphism $\rho: \mathcal{R} \to \text{end}(W)$ or by an action $a: \mathcal{R} \otimes W \to W$. In the basis $(e_i)$ they have the form $\rho(r)e_j = a(r \otimes e_j) = \sum_i \rho^j_i(r)e_i$ for some linear functions $\rho^j_i \in \mathcal{R}^*$ such that $[\rho^j_i] = \bar{k}_i + \bar{k}_j$ and the matrix $M = (\rho^j_i)$ is multiplicative over $S\mathcal{R}^*$. Since $S\mathcal{R}^*$ is a commutative super-algebra this matrix is an $A_W$-Manin matrix.

4.2.8. Quantum representations of quantum super-monoids. A quantum super-monoid (super-group) is a super-bialgebra (Hopf super-algebra) $\mathbb{B} \in \text{Bimon}(\text{SVect}, \otimes) = \text{Comon}(\text{SAlg}, \otimes)$. The strong monoidal functor

$$(\text{SAlg}, \otimes) \to (\mathbb{Z} \text{-GrSAlg}, \circ), \quad \mathcal{R} \mapsto \mathcal{R} \otimes \mathbb{K}[u, u^{-1}] = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}$$

is symmetric and faithful. It induces the non-full categorical embedding $\text{Bimon}(\text{SVect}, \otimes) \to \text{Comon}(\mathbb{Z} \text{-GrSAlg}, \circ)$, which identifies $\mathbb{B} = (\mathcal{R}, \Delta_{\mathcal{R}}, \varepsilon_{\mathcal{R}})$ with the quantum algebra $\mathcal{O}_B := (\mathcal{R} \otimes \mathbb{K}[u, u^{-1}], \Delta_{\mathcal{R}} \otimes \mathbb{K}[u, u^{-1}], \varepsilon_{\mathcal{R}} \otimes \mathbb{K}[u, u^{-1}])$.

By quantum representation of a quantum monoid $\mathbb{B}$ on $B = \mathcal{X}_B(\mathbb{K})$ we mean a quantum representation of the corresponding comonoid $\mathcal{O}_B$, i.e. a graded homomorphism of bialgebras $\omega: \text{coend}(B) \to \mathcal{O}_B$. It is given by a multiplicative $B$-Manin matrix $M$ over $\mathbb{B}$.

Concluding remarks

We generalised Manin’s theory of quadratic algebras and Quantum Representation Theory introduced in [S2] to the super-case. To do the latter we applied the general approach described in [S3] to the monoidal category $(\mathbb{Z} \text{-GrSAlg}, \circ)$ with the parametrising subcategory $\text{FQSA}$. The abstract algebraic formulation and the language of monoidal categories allowed us to write all the formulae in the exactly the same form as for the non-super case [S2]. We expect that the approach [S3] works well for more general cases. The list of planned developments of the theory is written in the end of the article [S2].
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