Coarse-graining of the Einstein-Hilbert Action rewritten with the two-dimensional Fisher information metric

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Abstract

In this study, considering the Fisher information metric given by the statistical average, we rewrite the Einstein-Hilbert (EH) action. Then, determining the transformation rules of the Fisher metric, etc under the coarse-graining, we perform the coarse-graining toward that rewritten EH action. We finally obtain the fixed-point. The space-time we consider in this study is two-dimensional.
1 Introduction

Behaviors of the space-time at the scale where the quantum effect becomes dominant is very important to understand the dynamics of our space-time and get the uniform understanding of the matters and the gravities. To this purpose, the quantum theory of gravity is imperative, and one of the recent momenta for this is to reconsider the thermodynamic \cite{1,2,3,4} and holographic nature \cite{5,6,7,8} originally inhered in the gravity. We here would like to focus on the two works in this direction:

1. One is \cite{9}, which illustrates the derivation of the Einstein equation from the first law of thermodynamics by considering the local Rindler horizon at each point of the space-times.

2. Another one is \cite{10}, which proposes that the space-times are composed of layers. Each layer is analog of the event-horizon, so the area of which is proportional to the amount of entropy associated with the inside of that layer. This idea sheds light on the entropy to be the origin of the gravitational force.

Since thermodynamics and entropy are the result of the statistical average, above works lead us to a notion that the gravitational theory is an effective theory given from some statistical averages.

Based on the notion above, in the former part in this paper, we will rewrite the Einstein-Hilbert (EH) action with the Fisher information metric (Fisher metric). This is because the metrices are given as a statistical average in the theory of the Fisher metric, so considering the Fisher metric goes along with the notion above and rewriting the EH action in terms of the Fisher metric is the work to perform first.

Next, we mention the issue we try to address after rewriting EH action. The problem to come first when we consider the Fisher metric is the confirmation of the physical rightness of the Fisher metric. One way for this is to check whether we can understand or not the distant gravitational phenomena such as dark energy, dark matter, the accelerating expansion of the universe and so on based on the obtained theory. This is the confirmation of the theory based on the actual observational data, and from such a viewpoint, after we rewrite the EH action, determining the transformation rules of the ingredients in the rewritten theory under the coarse-graining, we perform the coarse-graining toward our rewritten EH action. The theory obtained by this coarse-graining can be considered as the effective theory written in terms of the Fisher metric, observed from far.

Since the space-time we consider in this study is not four-dimensional but two-dimensional for technical reason, and the distant gravitational phenomena mentioned above are beyond the author’s knowledge, we take the comparison with the distant gravitational phenomena to future work after we accomplish what we do in this study in the four-dimensional space-time, and what we will do in this study are finally the following three:

1. rewriting the EH action in terms of the ingredients in the Fisher metric,

2. determining the transformation rules of the Fisher metric, etc under the coarse-graining, performing the coarse-graining toward that rewritten EH action,
In this study, we use the Fisher information metric, however we do not consider the theories of information. In what follows, we mention the organization of this paper.

In Sec. 2, we define the Fisher metric with the probability distribution \( p(x, \theta) = e^{-\gamma(x, \theta)} \) (\( x \) and \( \theta \) mean labels of states and parameters).

In Sec. 3, we specify the form of \( p \) as \( p(x, \theta) = e^{-(\theta^\mu F_\mu(x) - \phi(\theta))} \), and then give the expression of the Ricci tensor in terms of that \( p \).

In Sec. 4, specifying \( \theta^\mu, F_\mu \) and \( \phi \) in the \( p \) given in Sec. 3, we obtain the expression of the Ricci tensor and then the EH action described by those.

In Sec. 5, giving the explain for the space in which we define our rewritten EH action and introducing the fixed-points, we give the transformation rules for the ingredients in the EH action under the coarse-graining.

In Sec. 6, we perform the coarse-graining toward our rewritten EH action. From that, we obtain the fixed-point in our rewritten EH action.

In Appendices A.1 and A.2, we derive the expression of the Ricci tensors in the case with \( p = e^{-\gamma} \). Then based on that expression, we derive the expression of the Ricci tensors when \( p \) is given as \( p = e^{-(\theta^\mu F_\mu - \phi)} \). The final result is (112), which leads to (18).

In Appendix B, we show that the Fisher metric obtained from the \( p \) with \( \theta^\mu, F_\mu \) and \( \phi \) in Sec. 4 can form the two-dimensional AdS metric in the Poincaré coordinate in some conditions.

In Appendices C.1 and C.2, we review the basic points in the renormalization transformation, then actually demonstrate the renormalization transformation and determination of the fixed-point by considering the Gaussian type action in the flat Euclidean space. In Appendix C.3, we overview what we should do if we perform the renormalization transformation toward our model. Appendix C.4 is devoted to showing a calculation for a equation appearing in Appendix C.1.

Lastly, a paper [12] rewrites the Einstein equation with the Fisher metric. Since this is the same topic with the former part of this paper, we should touch on the differences between [12] and our paper. It is true that a large part of the analysis in [12] is a very helpful, however it performs ill-justified analysis, which is written between (76) and (77) in [12]: It is a manipulation such as \( \langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle \) (\( O_{1,2} \) mean some observables). Although [12] mentions it is approximation, what sense it is approximation is unclear. Our analysis, specifically in Sec. 4, is performed without that manipulation.

2 Definition of the Fisher metric

In this section, we give the definition of the Fisher information metric (Fisher metric). Let us first consider a statistical theory with parameters \( \theta \equiv (\theta^1, \theta^2, \ldots, \theta^D) \). Then let us consider probability distribution \( p_x(\theta) \), where \( x \) means the label distinguishing physical states. The summation of \( p \) is generally written as \( \sum_{x=0}^{\infty} p_x(\theta) = 1 \). We can
change the labeling of $x$ without loss of generality as $\sum_{x=-\infty}^{\infty} p_x(\theta) = 1$. Let us consider the case that $x$ is continuous numbers. Then we can write this relation as

$$\int_{-\infty}^{\infty} dx \ p(x, \theta) = 1. \quad (1)$$

Note that now the 1 above is a constant never depending on $\theta$.

We here would like to give the definition of the statistical average. If we write $\langle \cdots \rangle$, which means

$$\langle \cdots \rangle \equiv \frac{\int_{-\infty}^{\infty} dx \ \cdots \ p(x, \theta)}{\int_{-\infty}^{\infty} dx \ p(x, \theta)} = \int_{-\infty}^{\infty} dx \ \cdots \ p(x, \theta), \quad (2)$$

where since $\int_{-\infty}^{\infty} dx \ p(x, \theta) = 1$ as in (1), we do not need to write the denominator explicitly. In what follows, we basically abbreviate to write the integral region of $x$.

Let us represent $p$ as

$$p(x, \theta) = e^{-\gamma(x, \theta)}. \quad (3)$$

Then, the Fisher metric is defined as

$$g_{\mu\nu}(\theta) = \int dx \ p \frac{\partial \gamma(x, \theta)}{\partial \theta^\mu} \frac{\partial \gamma(x, \theta)}{\partial \theta^\nu} = \langle \partial^\mu \gamma(x, \theta) \partial^\nu \gamma(x, \theta) \rangle, \quad \partial^\mu \equiv \frac{\partial}{\partial \theta^\mu}. \quad (4)$$

where $\mu, \nu = 1, 2, \cdots, D$. Therefore, based on the Fisher metric, we can consider an $n$-dimensional Riemannian geometry with $\theta$ as the coordinates.

In what follows, we denote $\gamma$ and $g_{\mu\nu}$ before taking the statistical average in boldface. Therefore, $g_{\mu\nu} = \partial^\mu \gamma \partial^\nu \gamma$ and $g_{\mu\nu} = \langle g_{\mu\nu} \rangle$.

One quantity prescribed with the Fisher metric is the difference of the probability distribution for infinitesimal variations of $\theta$. We show it as

$$\int dx \left( p(x, \theta + d\theta) - p(x, \theta) \right) = g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu + O(\theta^3). \quad (5)$$

In addition to (1), there is another expression for the Fisher metric as

$$g_{\mu\nu} = \langle \partial^\mu \partial^\nu \gamma \rangle. \quad (8)$$

† To show (8), let us start with

$$\partial^\mu(p \partial^\nu \gamma) = \partial^\mu p \partial^\nu \gamma + p \partial^\mu \partial^\nu \gamma. \quad (6)$$

Using $p \partial^\nu \gamma = -\partial^\nu p$, we can rewrite (6) as

$$-\partial^\mu \partial^\nu p = -p \partial^\mu \gamma \partial^\nu \gamma + p \partial^\mu \partial^\nu \gamma. \quad (7)$$

Since $\int dx \ p$ is unit, a constant, $\int dx \ \partial^\mu \partial^\nu p = 0$ and we can reach (8). Note that $\langle \partial^\mu \partial^\nu \gamma \rangle = \langle \partial^\mu \gamma \partial^\nu \gamma \rangle$ but $\partial^\mu \partial^\nu \gamma \neq \partial^\mu \gamma \partial^\nu \gamma$. 

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3 Form of $p$ and the Ricci tensor in terms of that

In the previous section, we have given the definition of the Fisher metric given by the statistical average with the probability distributions $p = e^{-\gamma}$ as in (4) and (8). In this section, we specify the form of $\gamma$ as in (10), and then give the expression of the Ricci tensor in that $p$ as in (18).

From the general statistical physics’s point of view, we can write $p(x, \theta)$ as

$$p(x, \theta) = e^{-\beta E(x, \theta) - \ln Z(\theta)}.$$

Generalizing this form, let us assume the form of $p(x, \theta)$ as

$$p(x, \theta) = e^{\theta \mu F_\mu(x) - \phi(\theta)}.$$  \hspace{1cm} (9)

Then, $\gamma$ in (3) can be written as

$$\gamma(x, \theta) = -\theta \mu F_\mu(x) + \phi(\theta).$$ \hspace{1cm} (10)

At this time, the differentials of $\gamma$ are given as

$$\partial_\mu \gamma(x, \theta) = -F_\mu(x) + \partial_\mu \phi(\theta),$$ \hspace{1cm} (11)

$$\partial_\mu \partial_\nu \gamma(x, \theta) = \partial_\mu \partial_\nu \phi(\theta).$$ \hspace{1cm} (12)

Note that when $p$ is assumed as in (9), as can be seen in (12), $g_{\mu\nu}(x)$ is independent of $x$ in the stage before the statistical average is taken. Therefore, in the case of (9), the following manipulation is possible:

$$\langle \partial_\mu \partial_\nu \gamma(x, \theta) \cdots \rangle = \partial_\mu \partial_\nu \phi(\theta) \langle \cdots \rangle = g_{\mu\nu}(\theta) \langle \cdots \rangle.$$ \hspace{1cm} (13)

Therefore, when $p$ is assumed as (9), we can express $g_{\mu\nu}(\theta)$ without $\langle \cdots \rangle$ as

$$\langle \partial_\mu \partial_\nu \gamma(x, \theta) \rangle = \partial_\mu \partial_\nu \phi(\theta) = g_{\mu\nu}(\theta).$$ \hspace{1cm} (14)

From (11), the statistical average of $F_\mu(x)$ can satisfy the following relation:

$$\langle F_\mu(x) \rangle = \partial_\mu \phi(\theta),$$ \hspace{1cm} (15)

where we have used $\int dx \partial_\mu \gamma p = -\partial_\mu \left( \int dx \ p \right) = 0$. Then, from (11), we can see

$$\partial_\nu \gamma \partial_\mu \gamma = F_\mu F_\nu - F_\mu \partial_\nu \phi - F_\nu \partial_\mu \phi + \partial_\mu \phi \partial_\nu \phi,$$

$$= F_\mu F_\nu - F_\mu F_\nu,$$ \hspace{1cm} (16)

where we have used (15). Therefore, when $p$ is given as in (9), in addition to (4) and (8), we have another expression of the Fisher metrices as

$$g_{\mu\nu} = \langle F_\mu F_\nu \rangle - \langle F_\mu \rangle \langle F_\nu \rangle = -\langle F_\mu \partial_\nu \gamma \rangle.$$ \hspace{1cm} (17)

In (17), we have given the expression of the Ricci tensor in $p = e^{-\gamma}$. Through the derivation we note in Appendix $\mathbf{A.2}$ we can obtain the expression of the Ricci tensor in $\gamma = \theta^\mu F_\mu - \phi$ as

$$R_{\mu\nu} = \frac{1}{4} g^{\sigma\zeta} g^{\rho\kappa} \left( \langle F_\mu \partial_\zeta \gamma \partial_\sigma \gamma \rangle \langle F_\nu \partial_\rho \gamma \partial_\tau \gamma \rangle - \langle F_\mu \rangle g_{\zeta\sigma} \langle F_\nu \rangle g_{\rho\tau} \right)$$

$$- \frac{1}{4} g^{\sigma\zeta} g^{\rho\kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle F_\mu F_\nu \partial_\zeta \gamma \rangle.$$ \hspace{1cm} (18)
4 Specifying of $\theta^\mu$, $F^\mu$ and $\phi$, and rewriting of the EH action

In the previous section, we have obtained the expression of the Ricci tensor when $p = e^{-\gamma}$ with $\gamma = -\theta^\mu F^\mu + \phi$, as in (18). In this section, specifying $\theta^\mu$, $F^\mu$ and $\phi$, we obtain the expression of the Ricci tensor and then the Einstein-Hilbert (EH) action described by those.

4.1 $\theta^\mu$, $F^\mu$ and $\phi$

We consider the following $\theta^\mu$, $F^\mu$ and $\phi$:

$$F(x) = (F_1(x), F_2(x)) = (x, x^2), \quad \theta = (\theta^1, \theta^2) = \left(\frac{\bar{x}}{\sigma_0^2}, \frac{1}{2\sigma_0^2}\right),$$

$$\phi(\theta) = \ln \left[\sqrt{2\pi \sigma_0}\right] + \frac{x^2}{2\sigma_0^2} = \frac{1}{2} \ln \left[-\frac{\pi}{\theta^2}\right] - \frac{(\theta^1)^2}{4\theta^2}. \quad (20)$$

At this time, $p$ in (9) can be written as

$$p = \exp \left[-\ln \left[\sqrt{2\pi \sigma_0}\right] - \frac{x^2}{2\sigma_0^2} + \frac{\bar{x}^2}{2\sigma_0^2} - \frac{x^2}{2\sigma_0^2} \right] = \frac{1}{\sqrt{2\pi \sigma_0}} \exp \left[-\frac{(x - \bar{x})^2}{2\sigma_0^2}\right]. \quad (22)$$

When $p$ is given as above, the statistical system behind the Fisher metric have the following two properties: 1) The realizing state as a result of the statistical average is a state labeled by one $x$ denoted as $\bar{x}$. 2) The frequency of the appearance of the states labeled by $x$ follows the Gaussian distribution around $\bar{x}$.

When $p$ is given as in (22), the Fisher metric is composed as

$$\begin{pmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \theta^1 \\ \theta^1 & \frac{(\theta^1)^2}{(\theta^2)^2} - \theta^2 & \frac{(\theta^1)^2}{(\theta^2)^3} \end{pmatrix}, \quad \begin{pmatrix} g^{00} & g^{01} \\ g^{01} & g^{11} \end{pmatrix} = 2 \begin{pmatrix} (\theta^1)^2 - \theta^2 & \theta^1 \theta^2 \\ \theta^1 \theta^2 & (\theta^2)^2 \end{pmatrix} \quad (23)$$

We can reach the one above if we calculate based on (11) or (14). In what follows, we consider the two-dimensional space with the metric above. In Appendix 13, we show that the Fisher metric obtained from $p$ in (22) can form the two-dimensional AdS space metric in a part of the whole space.
4.2 EH action

In this subsection, we obtain the EH action described by $\theta^\mu$, $F^\mu_\nu$ and $\phi$. We first rewrite a part, $\langle F^\mu_\nu \partial_\varsigma \gamma \partial_\varsigma \gamma \rangle$, in (13). To this purpose, we expand as

\[ F^\mu_\nu (x) = F^\mu_\nu (\bar{x}) + \partial_x F^\mu_\nu (x) \bigg|_{x = \bar{x}} (x - \bar{x}) + \frac{1}{2} \partial^2_x F^\mu_\nu (x) \bigg|_{x = \bar{x}} (x - \bar{x})^2, \]  

(24)

\[ \partial_\mu \gamma (x, \theta) = \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} + \partial_x \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} (x - \bar{x}) + \frac{1}{2} \partial^2_x \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} (x - \bar{x})^2. \]  

(25)

Note that there is no notation, $\mathcal{O} ((x - \bar{x})^3)$, in the ones above. We can understand this from the fact that more than 3rd-order derivative with respect to $x$ vanish as

\[ \partial^2_x \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} = -\partial^2_x F^\mu_\nu (x) \bigg|_{x = \bar{x}} = 0 \quad \text{for } n = 3, 4, 5, \cdots, \]  

(26)

when $\gamma$ and $F^\mu_\nu$ are given as in (10) and (19), respectively.

Substituting (25) into the condition:

\[ \langle \partial_\mu \gamma \rangle = -\partial_\mu \left( \int dx p(x, \theta) \right) = 0. \]  

(27)

we can obtain the following relation:

\[ \frac{1}{2} \partial^2_x \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} \sigma_0^2 = -\partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}}. \]  

(28)

In the process to obtain the one above, we have used the following calculation:

\[ \langle (x - \bar{x})^n \rangle = \begin{cases} 0 & \text{for } n = \text{odd numbers}, \\ (n - 1)!! \sigma_0^2 & \text{for } n = \text{even numbers}, \end{cases} \]  

(29)

where $\langle (x - \bar{x})^n \rangle = \int dx (x - \bar{x})^n p$. The l.h.s. in (28) is the coefficient of the third term in the r.h.s. of (25) (if divided by $\sigma_0^2$). In addition, we can also rewrite the coefficient of the second term in the r.h.s. of (25) as

\[ \partial_x \partial_\mu \gamma (x, \theta) \bigg|_{x = \bar{x}} = \partial_x \partial_\mu \left( -\theta^\nu F^\nu_\mu (x) + \phi (\theta) \right) = -\partial_x F^\mu_\nu (x). \]  

(30)

Using (24) and (25) with (28) and (30), it turns out as

\[ \langle F^\mu_\nu (x) \partial_\varsigma \gamma (x) \partial_\varsigma \gamma (x) \rangle = \left\langle \left( F^\mu_\nu (\bar{x}) + \partial_x F^\mu_\nu (x) \bigg|_{x = \bar{x}} (x - \bar{x}) + \frac{1}{2} \partial^2_x F^\mu_\nu (x) \bigg|_{x = \bar{x}} (x - \bar{x})^2 \right) \right. \]

\[ \times \left\{ \partial_\varsigma \gamma (x, \theta) \bigg|_{x = \bar{x}} \left( 1 - \frac{(x - \bar{x})^2}{\sigma_0^2} \right) - \partial_x F^\varsigma_\varsigma (x) \bigg|_{x = \bar{x}} (x - \bar{x}) \right\} \]

\[ \times \left\{ \partial_\varsigma \gamma (x, \theta) \bigg|_{x = \bar{x}} \left( 1 - \frac{(x - \bar{x})^2}{\sigma_0^2} \right) - \partial_x F^\varsigma_\varsigma (x) \bigg|_{x = \bar{x}} (x - \bar{x}) \right\} \left. \right\}. \]  

(31)
The one above can be calculated as

\[ (31) = (2F_\mu(\bar{x}) + 5\sigma_0^2\partial^2_x F_\mu(x)|_{x=\bar{x}}) \partial_\xi \gamma(\bar{x}, \theta) \partial_\xi \gamma(\bar{x}, \theta) \]
\[ + 2\sigma_0^2 \partial_x F_\mu(x)|_{x=\bar{x}} \left( \partial_x F_\xi(x)|_{x=\bar{x}} \partial_\xi \gamma(\bar{x}, \theta) + \partial_x F_\zeta(x)|_{x=\bar{x}} \partial_\zeta \gamma(\bar{x}, \theta) \right) \]
\[ + \partial_x F_\zeta(x)|_{x=\bar{x}} \partial_x F_\zeta(x)|_{x=\bar{x}} \left( \sigma_0^2 F_\mu(\bar{x}) + \frac{3}{2}\sigma_0^4 \partial^2_x F_\mu(x)|_{x=\bar{x}} \right) \]
\[ = (2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}) \left( -F_\zeta(\bar{x}) \partial_\xi \phi(\theta) - F_\zeta(\bar{x}) \partial_\zeta \phi(\theta) + \partial_\xi \phi(\theta) \partial_\zeta \phi(\theta) \right) \]
\[ + 2\sigma_0^2 \partial_x F_\mu(x)|_{x=\bar{x}} \left( \partial_x F_\xi(x)|_{x=\bar{x}} \partial_\xi \phi(\theta) + \partial_x F_\xi(x)|_{x=\bar{x}} \partial_\zeta \phi(\theta) \right) + \mathcal{F}_{0, \mu \zeta \xi}, \quad (32) \]

where we have performed the calculations like \( (x - \bar{x})^p \left( 1 - \frac{(x - \bar{x})^2}{2\sigma_0^2} \right)^q \), and when moving from the first to the second lines, we have used \( \text{(10)} \). Further, we have defined the \( \theta \)-independent constant part as

\[ \mathcal{F}_{0, \mu \zeta \xi} \equiv (2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}) \partial_\xi \zeta(\bar{x}) \partial_\xi \zeta(\bar{x}) \]
\[ - 2\sigma_0^2 \partial_x F_\mu(x)|_{x=\bar{x}} \left( \partial_x F_\xi(x)|_{x=\bar{x}} \partial_\xi \xi(\bar{x}) + \partial_x F_\zeta(x)|_{x=\bar{x}} \partial_\zeta \zeta(\bar{x}) \right) \]
\[ + \partial_x F_\xi(x)|_{x=\bar{x}} \partial_x F_\zeta(x)|_{x=\bar{x}} \left( \sigma_0^2 F_\mu(\bar{x}) + \frac{3}{2}\sigma_0^4 \partial^2_x F_\mu(x)|_{x=\bar{x}} \right). \quad (33) \]

We can see \( \mathcal{F}_{0, \mu \zeta \xi} = \mathcal{F}_{0, \mu \xi \zeta} \). Factorizing by \( \partial_\xi \phi(\theta) \), \( \partial_\xi \phi(\theta) \) and \( \partial_\xi \phi(\theta) \partial_\zeta \phi(\theta) \),

\[ (32) = \left\{ - (2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}) \partial_\xi \phi(\theta) \right\} + \left\{ - (2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}) \partial_\xi \phi(\theta) \partial_\zeta \phi(\theta) + \mathcal{F}_{0, \mu \zeta \xi} \right\} \]
\[ \equiv \mathcal{P}_{\mu \xi} \partial_\xi \phi(\theta) + \mathcal{P}_{\mu \zeta} \partial_\zeta \phi(\theta) + \mathcal{Q}_\mu \partial_\xi \phi(\theta) \partial_\zeta \phi(\theta) + \mathcal{F}_{0, \mu \zeta \xi}, \quad (34) \]

where \( \mathcal{P}_{\mu \xi} \) and \( \mathcal{Q}_\mu \) are the \( \theta \)-independent constant parts given as

\[ \mathcal{P}_{\mu \xi} \equiv - (2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}) F_\zeta(\bar{x}) + 2\sigma_0^2 \partial_x F_\mu(x)|_{x=\bar{x}} \partial_x F_\zeta(x)|_{x=\bar{x}}, \quad (35) \]
\[ \mathcal{Q}_\mu \equiv 2F_\mu(\bar{x}) + 5\sigma_0^2 \partial^2_x F_\mu(x)|_{x=\bar{x}}. \quad (36) \]
Then we can write the one part in \((18)\) as
\[
g^{\sigma\tau} g^{\rho\varsigma} \left\{ \langle F_\mu \partial_\varphi \gamma \partial_{\sigma} \gamma \rangle \langle F_\nu \partial_{\rho} \gamma \partial_{\tau} \gamma \rangle - \langle F_\mu \rangle g_{\varsigma\sigma} \langle F_\nu \rangle g_{\rho\tau} \right\} = g^{\sigma\tau} g^{\rho\varsigma} \left\{ -g_{\varsigma\sigma} g_{\rho\tau} \partial_{\mu} \phi(\theta) \partial_{\nu} \phi(\theta) \right. \\
+ \mathcal{F}_{0,\mu\varsigma} \left( \mathcal{P}_{\nu\rho} \partial_{\tau} \phi(\theta) + \mathcal{P}_{\nu\tau} \partial_{\rho} \phi(\theta) \right) + \mathcal{F}_{0,\nu\tau} \left( \mathcal{P}_{\mu\varsigma} \partial_{\sigma} \phi(\theta) + \mathcal{P}_{\mu\sigma} \partial_{\varsigma} \phi(\theta) \right) \\
+ \left( \mathcal{F}_{0,\mu\varsigma} \mathcal{Q}_\nu + \mathcal{F}_{0,\nu\varsigma} \mathcal{Q}_\mu \right) \partial_{\rho} \phi(\theta) \partial_{\tau} \phi(\theta) \\
+ \left( \mathcal{P}_{\mu\varsigma} \partial_{\sigma} \phi(\theta) + \mathcal{P}_{\mu\sigma} \partial_{\varsigma} \phi(\theta) \right) \left( \mathcal{P}_{\nu\rho} \partial_{\tau} \phi(\theta) + \mathcal{P}_{\nu\tau} \partial_{\rho} \phi(\theta) \right) \\
+ \mathcal{Q}_\nu \partial_{\rho} \phi(\theta) \partial_{\tau} \phi(\theta) \left( \mathcal{P}_{\mu\varsigma} \partial_{\sigma} \phi(\theta) + \mathcal{P}_{\mu\sigma} \partial_{\varsigma} \phi(\theta) \right) \\
+ \mathcal{Q}_\mu \partial_{\varsigma} \phi(\theta) \partial_{\sigma} \phi(\theta) \left( \mathcal{P}_{\nu\rho} \partial_{\tau} \phi(\theta) + \mathcal{P}_{\nu\tau} \partial_{\rho} \phi(\theta) \right) \\
+ \mathcal{Q}_\mu \mathcal{Q}_\nu \partial_{\rho} \phi(\theta) \partial_{\tau} \phi(\theta) \partial_{\varsigma} \phi(\theta) \partial_{\sigma} \phi(\theta) + \mathcal{F}_{0,\mu\varsigma} \mathcal{F}_{0,\nu\tau} \right\}, \tag{37}
\]
where we have used \((15)\). Next, let us look at the rest part in \((18)\).

First, when \(p\) is given as \((19)\), we can write as \(\partial_\gamma g_{\mu\nu} = \partial_\nu g_{\mu\nu} \partial_\gamma \phi\) according to \((12)\). On the other hand, we can also write as \(\partial_\gamma g_{\mu\nu} = -\langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle\). Therefore, the following equality is held\(^\dagger\)
\[
\langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle = -\langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle. \tag{38}
\]
Therefore, we can write the rest part as
\[
g^{\gamma\sigma} g^{\rho\varsigma} \langle \partial_{\rho} \gamma \partial_{\sigma} \gamma \partial_{\varsigma} \gamma \rangle \langle F_\mu \mathcal{F}_\nu \partial_{\gamma} \gamma \rangle = -g^{\gamma\sigma} g^{\rho\varsigma} \langle \partial_{\rho} \partial_{\sigma} \partial_{\varsigma} \phi \rangle \langle F_\mu \mathcal{F}_\nu \partial_{\gamma} \gamma \rangle. \tag{39}
\]
Let us evaluate \(\langle F_\mu \mathcal{F}_\nu \partial_{\gamma} \gamma \rangle\) in the same manner with \((31)\):
\[
\langle F_\mu \mathcal{F}_\nu \partial_{\gamma} \gamma \rangle = \left\langle \left( F_\mu (\hat{x}) + \partial_\mu F_\mu (x) \right) \bigg|_{x=\hat{x}} (x - \hat{x}) + \frac{1}{2} \partial_\mu^2 F_\mu (x) \bigg|_{x=\hat{x}} (x - \hat{x})^2 \right\rangle \\
\times \left\langle \left( F_\nu (\hat{x}) + \partial_\nu F_\nu (x) \right) \bigg|_{x=\hat{x}} (x - \hat{x}) + \frac{1}{2} \partial_\nu^2 F_\nu (x) \bigg|_{x=\hat{x}} (x - \hat{x})^2 \right\rangle \\
\times \left\{ \partial_{\gamma} \gamma (x) \bigg|_{x=\hat{x}} \left( 1 - \frac{(x - \hat{x})^2}{\sigma_0^2} \right) - \partial_\gamma F_\nu (x) \bigg|_{x=\hat{x}} (x - \hat{x}) \right\}, \nonumber
\]
\[
= -\sigma_0^2 \left\langle 2 \partial_\mu F_\mu (x) \bigg|_{x=\hat{x}} \partial_\gamma F_\nu (x) \bigg|_{x=\hat{x}} + 3 \sigma_0^2 \partial_\mu^2 F_\mu (x) \bigg|_{x=\hat{x}} \partial_\gamma^2 F_\nu (x) \bigg|_{x=\hat{x}} \right\rangle \partial_\gamma \phi (\theta) + \mathcal{F}_{1,\mu\varsigma} \tag{40}
\]
\(^\dagger\) We can calculate as \(\partial_\gamma g_{\mu\nu} = -\langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle + \langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle + \langle \partial_{\sigma} \phi \rangle \langle \partial_{\tau} \gamma \rangle\). Using \((27)\) and \((12)\), we can see \(\langle \partial_\gamma \partial_{\sigma} \gamma \partial_{\tau} \gamma \rangle + \langle \partial_{\sigma} \phi \rangle \langle \partial_{\tau} \gamma \rangle = 0\).
\(^\ddagger\) In \((39)\), using \((11)\) and the general relation \(\partial_\gamma g_{\mu\nu} = -g^{\mu\alpha} g^{\rho\beta} \partial_\gamma g_{\alpha\beta}\), it is possible to write the front factor as \(g^{\sigma\tau} g^{\rho\varsigma} \partial_\rho \partial_\tau \partial_{\gamma} \phi = g^{\sigma\tau} g^{\rho\varsigma} \partial_\gamma g_{\rho\tau} = -\partial_\gamma g^{\rho\varsigma} \).
where we have defined the \(\theta\)-independent constant part in the one above as

\[
\mathcal{F}_{1,\mu\nu\zeta} \equiv -\partial_{\bar{x}} F_{\zeta}(x)|_{x=\bar{x}} \left\{ \sigma_{0}^{2} \left( F_{\mu}(\bar{x}) \partial_{x} F_{\nu}(x)|_{x=\bar{x}} + F_{\nu}(\bar{x}) \partial_{x} F_{\mu}(x)|_{x=\bar{x}} \right) + \frac{3}{2} \sigma_{0}^{4} \left( \partial_{x} F_{\mu}(x)|_{x=\bar{x}} \partial_{x}^{2} F_{\nu}(x)|_{x=\bar{x}} + \partial_{x} F_{\nu}(x)|_{x=\bar{x}} \partial_{x}^{2} F_{\mu}(x)|_{x=\bar{x}} \right) \right\}. \tag{41}
\]

We can see \(\mathcal{F}_{1,\mu\nu\zeta} = \mathcal{F}_{1,\nu\mu\zeta}\). Therefore,

\[
\sigma_{0}^{2} g^{\sigma\tau} g^{\rho\zeta} \left( \partial_{\rho} \partial_{\sigma} \partial_{\tau} \phi(\theta) \right) \left\{ (2 \partial_{\bar{x}} F_{\mu}(x)|_{x=\bar{x}} \partial_{x} F_{\nu}(x)|_{x=\bar{x}} + 3 \sigma_{0}^{2} \partial_{x}^{2} F_{\mu}(x)|_{x=\bar{x}} \partial_{x} F_{\nu}(x)|_{x=\bar{x}}) \partial_{\zeta} \phi(\theta) + \mathcal{F}_{1,\mu\nu\zeta} \right\}. \tag{42}
\]

We can write \(R_{\mu\nu}\) in (18) just as (37) + (42). Then, we can obtain the Ricci scalar, and can obtain the EH action described by \(\theta^{\mu}, F_{\mu}\) and \(\phi\). We write it formally as

\[
S = \int d\theta^{D} \sqrt{-g} \mathcal{L}, \quad \mathcal{L} = g^{\mu\nu} \left( (37) + (42) \right), \tag{43}
\]

As such, although we have specified the Gaussian type of \(p\) giving a two-dimensional space-time and expanded around \(x = \bar{x}\), we have obtained the EH action in terms of the ingredients of the Fisher metric without the problem mentioned in the last of the introduction, which is one of results in this study. Lastly we list the points in the rewriting of the EH action and the problems in that:

- We have performed the expansions around \(\bar{x} = 0\) as in (24) and (25). At this time, for the form of \(p\) we have taken specifically in (22), the expansions can stop at the second order as in (26).
- We have used the condition: \(\int dx p(x, \theta) = 1\) as in (27).

As the problems in the rewritten EH action:

- Due to the way the derivatives appear, the form of the action is not the one that we can rewrite in the momentum space through the Fourier transformation. As a result, we cannot perform the renormalization transformation. This is because the renormalization transformation is generally performed in the momentum space as we describe in Appendix C.1.
- Since \(g_{\mu\nu}\) and \(g^{\mu\nu}\) are given by \(\phi\), there is no quadratic term in the action. Therefore, there is no two-point correlated functions, and the perturbative analysis with the Wick contraction is unavailable.
5 Coarse-grainings of the ingredients in our rewritten EH action

In the sections so far, we have considered the Fisher metric given by \( p = e^{-\gamma} \), where \( \gamma(x, \theta) = \theta^\mu F_\mu(x) - \phi(\theta) \), and the components of those have been given in Sec.(4.1). With those, we have obtained the rewritten EH action as in (43).

In this section, we define our system in Sec.5.1, then introducing the fixed-points, we give the transformation rules of the ingredients in our rewritten EH action under the coarse-graining and scale-down. In particular, the transformation rule of the Fisher metric is one of results in this study. (In what follows, if we say “under the coarse-graining”, it means “under the coarse-graining and scale-down”.)

The coarse-graining and scale-down compose the renormalization transformation [11], which links to various aspects of the theory arising from the difference of the scale and is very interesting. However, the renormalization transformation is generally performed in the momentum space and performing the Fourier transformation in our model is difficult. However since the renormalization transformation is very interesting, we give a brief review for the renormalization transformation in Appendix.C.1.

5.1 Space we put our action

We first define the space on which we put our action (43), which we refer to as \( \Lambda \). We assume \( \Lambda \) as a \( D \)-dimensional cubic lattice with the lattice spacing 1 and the length of each side is \( N \). Therefore,

- the lattice points exist every lattice spacing 1,
- the total number of the lattice points in \( \Lambda \) is \( N^D \),

where \( D \) is common in the one in Sec.2. We impose the periodic boundary condition in each direction, therefore the \( \Lambda \) can be considered as a \( D \)-dimensional torus. Correspondingly, \( N \) is assumed as even integers.

We denote the lattice points \( \theta \) in \( \Lambda \) as \((\theta^1, \theta^2, \cdots, \theta^D)\), where each component is integers satisfying \(|\theta^i| \leq N/2\). We consider \( \phi(\theta) \) on each lattice point, where \( \phi(\theta) \in \mathbb{R} \).

We have \( \phi(\theta) \) exist on each lattice point of \( \Lambda \), which is a curved space with the Fisher metric determined from \( \phi(\theta) \). Therefore, in this study we are not considering \( \phi(\theta) \) on some flat Euclidian space separately from \( \Lambda \) (in this case, using the information there, we will come to constitute the Fisher metric and EH action, but there is no grand for the connection between the theories in the flat Euclidian space and \( \Lambda \)), but considering only \( \Lambda \). \( F_\mu(x) \) is considered a kind of parameter).

In general, the systems in the statistical mechanics and field theories have degrees of the freedom more than the Avogadro constant. Therefore \( N \) is finally taken to infinity.
5.2 Coarse-graining of $\phi$ and fundamental definition of renormalization transformation

In this subsection, we define the fundamental points of our renormalization transformation. We first consider sufficiently large odd integers $L$, to which we refer as the “renormalization scale”. $N$ is assumed to be multiples of $L$. Let us denote what we have mentioned now as

- $L$ is some odd integers to which we refer as the “renormalization scale”;
- $N = N'L$, where $N'$ is some constant even integers.

Then, separately from $\Lambda$, we consider another $D$-dimensional cubic lattice with the length of each side is $N/L = N' \in \text{even } \mathbb{Z}$, where its lattice spacing is 1. We refer to this lattice space as $\Lambda'$. We consider that $\Lambda'$ is obtained from $\Lambda$ by dividing the each side by $L$, which we refer to as the “scale-down”.

For the lattice points $\eta = (\eta^1, \eta^2, \cdots, \eta^D)$ in $\Lambda'$ (We use $\eta^\mu$ as the notations of the coordinate in $\Lambda'$ as well as $\theta^\mu$ in $\Lambda$ in what follows.), we now define the region in $\Lambda$ that we refer to as $B_\eta$ as

$$B_\eta = \left\{ \theta = (\theta^1, \theta^2, \cdots, \theta^D) \in \Lambda \mid \eta \in \Lambda' \text{ and } |\theta^i - L\eta^i| \leq \frac{L-1}{2} \right\}. \quad (44)$$

Then if $\Lambda'$ can be obtained from $\Lambda$ through the scale-down by $1/L$, let us consider that the filed $\phi'(\eta)$ in $\Lambda'$ is a mass of $\phi(\theta)$ in the region $B_\eta$ in $\Lambda$. This consideration leads to write the relation between $\phi'(\eta)$ and $\phi(\theta)$ as

$$\phi'(\eta) = \frac{1}{L^\Delta} \sum_{\theta \in B_\eta} \phi(\theta) = \frac{1}{L^\Delta} \sum_{\theta \in B_0} \phi(L\eta + \theta), \quad (45)$$

where 0 means zero vector in $\Lambda'$. We fix $\Delta$ later such that the fixed-points which we introduce in the next subsection can exist. (From (45), we can get an interpretation that the coarse-graining is the manipulation to summarize fine dynamics in some regions to one local dynamics. Combining with the scale-down, we can interpret (45) as the effective description when we look at the system further away.)

The action $\int_{\Lambda'} d\eta^D \sqrt{-g} \mathcal{L}'(\phi'(\eta))$ in the $\Lambda'$ system with a configuration of $\phi' \equiv \{\phi'(\eta) \mid \eta \in \Lambda'\}$ can be defined from the ingredients in the $\Lambda$ system as

$$\exp \left[ -\int_{\Lambda'} d\eta^D \sqrt{-g} \mathcal{L}'(\phi'(\eta)) \right] = N_0 \int D\phi \left\{ \prod_{\eta \in \Lambda'} \delta \left( \phi'(\eta) - \frac{1}{L^\Delta} \sum_{\theta \in B_\eta} \phi(\theta) \right) \right\} \times \exp \left[ -\int_{\Lambda} d\theta^D \sqrt{-g} \mathcal{L}(\phi(\theta)) \right], \quad (46)$$

where $\int D\phi(\cdots) \equiv \prod_{\theta \in \Lambda} \left( \int_{-\infty}^{\infty} d\phi(\theta) \right)(\cdots)$. 

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In the one above, \( \phi(\theta) \) is real functions and \( \mathcal{N} \) is a constant by which the r.h.s. can become unit when \( \phi' = 0 \). Therefore, we can write \( \mathcal{N} \) as

\[
\mathcal{N}_0^{-1} = \int \mathcal{D}\phi \prod_{\eta \in \Lambda} \delta \left( \sum_{\theta \in B_i} \phi(\theta) \right) \exp \left[ -\int_{\Lambda} d\theta^D \sqrt{-g} \mathcal{L}(\phi) \right]. \tag{47}
\]

(47) is the definition of the “renormalization transformation”, which is composed of the coarse-graining (45) and scale-down (If some coefficients, etc also get modification, exchanges concerning those are also included in (46).).

Since higher momenta are integrated out in the renormalization transformation as shown in (139) or (146) in Appendix C.1 the renormalization transformation corresponds to, renormalizing fine behavior into ingredients in the theory, enlarge the scale when we look at the theory.

Although setting \( \mathcal{N}_0 \) as above is conventions, it is unclear whether \( \mathcal{N}_0 \) can be derived or not in the process to organize the path-integral once \( S \) is given. Since \( \mathcal{N}_0 \) will effect to the constant part in the renormalization-flow, if we concern the constant part how to determine \( \mathcal{N}_0 \) is a sensitive problem.

5.3 Fixed-points

Denoting \( \int_{\Lambda} d\theta^D \sqrt{-g} \mathcal{L}(\phi) \) and \( \int_{\Lambda'} d\eta^D \sqrt{-g'} \mathcal{L}'(\phi') \) in (46) as \( S^{(i)} \) and \( S^{(i+1)} \), let us consider a sequence, \( S^{(0)} \to S^{(1)} \to \ldots \to S^{(i)} \to S^{(i+1)} \to \ldots \), generated by \( S^{(i+1)} = R \cdot S^{(i)} \), where \( R \) means the renormalization transformation prescribed in (46).

Now we consider the action invariant with respect to the renormalization transformation \( R \). Then, denoting the invariant action as \( S^* \), we can write as

\[
R \cdot S^* = S^*. \tag{48}
\]

We refer to \( S^* \) as the “fixed-point”.

In this study, we determine \( \Delta \) in (45) such that the fixed-point can exist.

5.4 Coarse-gainings of other than \( \phi \)

In this subsection, we give the transformation rules under the coarse-grainings for the ingredients, \( \theta^\mu, \partial^\mu, F_\mu, \) and \( g_{\mu\nu} \) and so on. In particular, determining the transformation rule of the Fisher metric under the coarse-graining is one of the results in this study.

Regarding \( \theta^\mu \), those play the role of the coordinates in \( \Lambda \), which are not the target of the coarse-graining. As mentioned in Sec 5.2, those just get the scale-down in one renormalization transformation as

\[
\eta^\mu = \theta^\mu / L, \quad \eta_\mu = \theta_\mu / L, \tag{49}
\]

where \( L, \eta^\mu, \theta^\mu \) are defined in Sec 5.2 and \( \eta^\mu \) and \( \theta^\mu \) represent the coordinates after and before the scale-down, which we can denote as

\[
\eta^\mu \in \Lambda', \quad \theta^\mu \in \Lambda. \tag{50}
\]
We use those notations through this paper. Note that since the scale-down is the transformation irrelevant with the indices $\mu$ and $\nu$, there is no difference originating in contravariant and covariant vectors. Along with (49), in one scale-down, $\partial/\partial \eta^\mu$ and $\partial/\partial \eta^\mu$ are transformed as

$$\frac{\partial}{\partial \eta^\mu} = L \frac{\partial}{\partial \theta^\mu}, \quad \frac{\partial}{\partial \eta^\mu} = L \frac{\partial}{\partial \theta^\mu}. \quad (51)$$

Next, from (15), we can see that $F^\mu_\eta(x)$ transform in one coarse-graining as

$$F^\mu_\eta(x) = L^{-\Delta+1} \sum_{\theta \in B_\eta} F^\mu_\theta(x) = L^{D-\Delta+1} F^\mu_\phi(x). \quad (52)$$

In the one above, $\mu$ in the l.h.s. and r.h.s. indicate the coordinates $\eta^\mu$ and $\theta^\mu$ respectively, and $x$ is that appearing in Sec.2. We have evaluated the summation by using the fact that $F^\mu_\theta(x)$ is independent of $\theta^\mu$ and the number of the lattice points in $B_\eta$ is $L^D$.

$$\xi^\mu = L^{-1} \eta^\mu, \quad \partial/\partial \xi^\mu = L \partial/\partial \eta^\mu, \quad \frac{\partial}{\partial \xi^\mu} = L^2 \frac{\partial}{\partial \theta^\mu}, \quad \mathbf{F}^\prime_\mu(x) = L^{2(D-\Delta+1)} \mathbf{F}_\mu(x), \quad (53)$$

where $\xi^\mu$ mean the coordinates with the scale-down twice from $\Lambda''$ (so, we can write as $\xi^\mu \in \Lambda''$).

When $F^\mu_\phi(x)$ are transformed as (52), $F^\prime_0,^\mu_\sigma, P^\mu, Q^\mu$ and $F^\prime_1,^\mu_\nu_\zeta$ given in (33), (35), (36) and (41) are transformed by the following manners:

$$F^\prime_0,^\mu_\sigma = L^{3(D-\Delta+1)} F^\mu_0,^\mu_\sigma, \quad P^\prime_\mu = L^{2(D-\Delta+1)} P^\mu,$$

$$Q^\prime_\mu = L^{D-\Delta+1} Q^\mu, \quad \mathbf{F}^\prime_1,^\mu_\nu_\zeta = L^{3(D-\Delta+1)} \mathbf{F}_1,^\mu_\nu_\zeta. \quad (54)$$

When $\phi$ gets the coarse-graining as in (15), the Fisher metric gets the coarse-graining as can be seen from its definitions given in Sec.2 and 3. We have given the definitions of the Fisher metric in several ways. The coarse-grained Fisher metrics obtained from those should agree each other. For example, the results obtained from (56) and (51) should agree. However, as shown in what follows, those do not agree in fact.

If we follow (14), we can write the Fisher metric after one coarse-graining as

$$g^\prime_\mu_\nu(\eta) = \left\langle \frac{\partial}{\partial \eta^\mu} \frac{\partial}{\partial \eta^\nu} \phi^\prime(\eta) \right\rangle = \left\langle \frac{\partial}{\partial \eta^\mu} \frac{\partial}{\partial \eta^\nu} \phi^\prime(\eta) \right\rangle \langle 1 \rangle = L^{-\Delta+2} \frac{\partial}{\partial \eta^\mu} \frac{\partial}{\partial \theta^\nu} \sum_{\theta \in B_\eta} \phi(L \eta + \theta) \int dx e^{-\gamma'(x,\eta)}, \quad (56)$$

where $\langle \cdots \rangle$ is the statistical average with regard to $x$ as defined in (2).
Let us evaluate \( \int dx \, e^{-\gamma'(x, \eta)} \). We can write \( \gamma'(x, \eta) \) as

\[
\gamma'(x, \eta) = -\eta^\mu F'_\mu(x) + \phi'(\eta)
\]

\[
= \frac{1}{L^\Delta} \sum_{\theta \in B_\eta} \left( -\eta^\mu L F_\mu(x) + \phi(L\eta + \theta) \right)
\]

\[
= \frac{1}{L^\Delta} \sum_{\theta \in B_\eta} \left( -\eta^\mu L F_\mu(x) + \frac{1}{2} \ln \left[ -\frac{\pi}{L\eta^2 + \theta^2} \right] - \frac{(L\eta^1 + \theta^1)^2}{4(L\eta^2 + \theta^2)} \right),
\] (57)

where we have used (22). In order to make our analysis possible, we approximately assume that \( \phi(\theta) \) is unique in each region \( B_\eta \). More concretely, \( \phi(\theta) \) is a constant given by \( \phi(L\eta) \) in each region \( B_\eta \). Then, we can write the Fisher metric (56) as

\[
g'_{\mu \nu}(\eta) \sim L^{D-\Delta+2} \frac{\partial}{\partial \theta^\mu} \frac{\partial}{\partial \theta^\nu} \phi(\theta) \int dx \, e^{-\gamma'(x, \eta)},
\] (58)

where \( \gamma'(x, \eta) \sim L^{D-\Delta} \left( -L\eta^\mu F_\mu(x) + \frac{1}{2} \ln \left[ -\frac{\pi}{L\eta^2} \right] - \frac{(L\eta^1)^2}{4L\eta^2} \right) \)

\[
= \frac{L^{D-\Delta}}{2\sigma_0^2} \left( (x - \bar{x})^2 + \sigma_0^2 \ln \left[ 2\pi\sigma_0^2 \right] \right),
\] (59)

\( \sim \) means we have used the approximation mentioned between (57) and (58), and using \( L(\eta^1, \eta^2) = (\theta^1, \theta^2) \) we have exchanged \( \eta^\mu \) to \( \theta^\mu \), then rewritten with (19) and (20).

Then, \( g_{11} \) and \( g_{12} \) calculated based on (58) can agree with those calculated based on (1) using the same \( \gamma' \) in (59), however \( g_{22} \) cannot agree. Its reason is that \( \int dx \, e^{-\gamma'(x, \eta)} \) depends on the coordinates as

\[
\int_{-\infty}^{\infty} dx \, e^{-\gamma'(x, \eta)} = \sqrt{\frac{1}{L^{D-\Delta}} \left( -\frac{\theta^2}{\pi} \right)^{-1+L^{D-\Delta}}},
\] (60)

where we have used

\[
\gamma'(x, \eta) = -\frac{L^{D-\Delta}}{4\theta^2} \left( (\theta^1 + 2x\theta^2)^2 - 2\theta^2 \ln \left[ -\frac{\pi}{\theta^2} \right] \right).
\] (61)

This can be equivalently obtained from (59). Therefore, \( \partial_\mu \left( \int dx \, e^{-\gamma'(x, \eta)} \right) = 0 \) is not held. This is the condition written under (7), and due to this, we cannot rewrite (1) to (8). If \( L^{D-\Delta} = 1 \), (60) can be unit, and at this time we can confirm \( g_{22} \) can also agree.

As such, how to determine the coarse-grained Fisher metric is a problem. Although we can make logic for this variously, we here would like to give the one we can organize consistently to the end, which starts with the notion that if the coarse-grained scalar field is given as (15), also for the coarse-graining of the tensor field, we may write as

\[
g'_{\mu \nu}(\eta) = \sum_{\theta \in B_\eta} g_{\mu \nu}(L\eta + \theta) \sim L^{D-\Delta_s} g_{\mu \nu}(L\eta) = L^{D-\Delta_s} g_{\mu \nu}(\theta),
\] (62)
where $\Delta_g$ is intended to play the same role with $\Delta$ in (45), and "∼" has the same intention with the one in (58).

Since the coarse-graining is irrelevant with the contravariant and covariant vectors, if we can write like (62), we can also write as

$$g'_\mu\nu(\eta) \sim L^{D-\Delta_g} g_{\mu\nu}(L\eta) = L^{D-\Delta_g} g_{\mu\nu}(\theta).$$  \tag{63}$$

Then, for some vectors $V^\mu(\eta)$ in $\Lambda'$, we consider a relation:

$$V^\mu(\eta) = g^{\mu\nu}(\eta) g'_{\nu\lambda}(\eta) V^\lambda(\eta).$$  \tag{64}$$

The one above is a relation based on the fact that metrices in $\Lambda'$ are given by $g^{\mu\nu}(\eta)$ and $g'_{\nu\lambda}(\eta)$. Then, since $g^{\mu\nu}(L\eta) g_{\nu\lambda}(L\eta) = \delta^\mu_\lambda$,

$$V^\mu(\eta) = L^{2(D-\delta_g)} V^\mu(\eta).$$  \tag{65}$$

Therefore, $\Delta_g = D$ is led, and the transformation rule for $n$ times coarse-grainings is determined from (62) and (63) as

$$\cdot \ g^{(n)\mu\nu}(\zeta) = 2L^{2n} \left( \frac{(\zeta^1)^2 - \zeta^2/L^n}{\zeta^1 \zeta^2} \right) \equiv L^{2n} g^{(n)\mu\nu}(\zeta),$$  \tag{66}$$

$$\cdot \ g_{\mu\nu}^{(n)}(\zeta) = \frac{1}{2L^n} \left( \frac{1}{\zeta^1 \zeta^2} - \frac{\zeta^1}{(\zeta^2)^2} \right) \equiv L^{-n} g_{\mu\nu}^{(n)}(\zeta),$$  \tag{67}$$

where the superscripts "($n$)" ($n = 0, 1, 2, \cdots$) mean the number of the coarse-graining those got, and $\zeta^\mu$ mean the coordinates with $n$ times scale-down.

## 6 Coarse-graining and fixed-point of our rewritten EH action

In Sec 3 and 4 we have considered the Fisher metric, $p = e^{-\theta^m F^m(x) - \phi(\theta)}$ (as for the components of those, see Sec (4.1)), and have obtained the rewritten EH action as in (43). In Sec 5 introducing the fixed-points, we have given the transformation rules of the ingredients in our rewritten EH action under the coarse-graining. In this section, we perform the coarse-graining toward our rewritten EH action (43), then examine the fixed-points.

The coarse-graining we perform corresponds to the first term in r.h.s. of (162) although there is difference whether it is in the coordinate space or momentum space. We regard this as $R$ in Sec 5.2 and we consider the fixed-points by such a $R$.

Using the transformation rules (52), (53), (54), (55), (66) and (67), we perform the coarse-graining toward our rewritten EH action (43). We here give the transformation rule of $\phi(\theta)$ we employ based on (45) as

$$\phi'(\eta) \sim L^{D-\Delta} \phi(\theta),$$  \tag{68}$$

and...
where “∼” has the same intention with the one in (58). Summarizing the manipulation we perform in $n$ times coarse-grainings from $S^{(0)}$ on $\Lambda^{(0)}$ to $S^{(n)}$ on $\Lambda^{(n)}$,

$$
\begin{align*}
\theta^\mu & \rightarrow L^n \zeta^\mu, \quad \theta_\mu \rightarrow L^n \zeta_\mu, \\
\frac{\partial}{\partial \theta^\mu} & \rightarrow \frac{1}{L^n} \frac{\partial}{\partial \zeta^\mu}, \quad \frac{\partial}{\partial \theta_\mu} \rightarrow \frac{1}{L^n} \frac{\partial}{\partial \zeta_\mu}, \\
\phi^{(0)}(\theta) & \rightarrow L^{-n(D-\Delta)} \phi^{(n)}(\zeta), \\
F^{(0)}_\mu(x) & \rightarrow L^{-n(D-\Delta+1)} F^{(1)}_\mu(x), \\
J_{0,\mu\nu\sigma}^{(0)} & \rightarrow L^{-3n(D-\Delta+1)} J_{0,\mu\nu\sigma}^{(1)}, \\
P_\mu^{(0)} & \rightarrow L^{-2n(D-\Delta+1)} P_\mu^{(1)}, \\
Q^{(0)}_\mu & \rightarrow L^{-n(D-\Delta+1)} Q^{(1)}_\mu, \\
J_{1,\mu\nu\zeta}^{(0)} & \rightarrow L^{-3n(D-\Delta+1)} J_{1,\mu\nu\zeta}^{(1)}, \\
g^{(0)\mu\nu}(\theta) & \rightarrow g^{(n)\mu\nu}(\zeta), \\
g^{(0)}_{\mu\nu}(\theta) & \rightarrow g^{(n)}_{\mu\nu}(\zeta), \\
\sigma^2_0 & \rightarrow \sigma^2_0/L^{2n},
\end{align*}
$$

where the reason for (71) is given at (82) since we would like to determine it after $\Delta$ is determined.

Since $L$-dependences remain in $g^{(1)\mu\nu}(\eta)$ and $g^{(1)\mu\nu}(\eta)$ as can be seen in (66) and (67), we may consider we should consider not (70) but (165). However, since the necessary condition as the metrics, the each one is inverse matrix for each other, is held in $g^{(1)\mu\nu}(\eta)$ and $g^{(1)\mu\nu}(\eta)$, we consider the replacement in terms of $g^{(1)\mu\nu}(\eta)$ and $g^{(1)\mu\nu}(\eta)$ as in (70). In Appendix C.2, we note how the situation will be if we consider (165).

Depending on either of those, $\Delta$ will be different as in (78) and (171), and in the case of (78), the relations of (4) and (14) can be held for the coarse-grained $\phi^{(n)}(\zeta)$ and $g^{(n)\mu\nu}(\zeta)$, $g^{(n)}_{\mu\nu}(\zeta)$ for arbitrary $n$. For concrete things for this, see the last of this section.

Although this result means that the formulation system that the Fisher metric is given by $\phi^{(n)}(\zeta)$ can be held at the arbitrary $n$ times coarse-grainings, this can hold or not in the process of the coarse-graining is highly nontrivial, because the coarse-graining is performed independently of the formulation system of the Fisher metric, and $\Delta$ should be determined to the proper value such that the formulation system of the Fisher metric can be held despite that $\Delta$ is determined irrelevantly with the formulation system of the Fisher metric.

As shown in (78), $\Delta$ will be determined to the proper value if we consider (70). However, currently we have no idea what $\Delta$ can be determined to the proper is accident or not.

We can obtain our rewritten EH action with $n$ times coarse-grainings as (We have given an example of the renormalization transformation, which is in effect a coarse-graining, in Appendix C.2)

$$
S^{(n)}(\zeta) = L^D \int d\zeta^D \sqrt{-g^{(n)}(\zeta)} L^{(n)}(\zeta).
$$
where $S^{(n)}$ is the theory on $A^{(n)}$ with the coordinate $\zeta^\mu = \theta^\mu / L^n$ for any $n$ as defined under (66). $L^{nD}$ comes from $d\theta^D \sqrt{-g(\theta)}$, and $\mathcal{L}^{(n)}(\zeta)$ is given as

$$
\mathcal{L}^{(n)}(\zeta) = \left\{ \begin{array}{ll}
g^{(n)\sigma\tau}(\zeta) g^{(n)\rho\kappa}(\zeta) g^{(n)\mu\nu}(\zeta) & \\
- L^{-2n(D-\Delta+1)} g^{(n)}_{\zeta\sigma}(\zeta) g^{(n)}_{\rho\tau}(\zeta) \partial_{\mu} \phi^{(n)}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) & \\
+ L^{-6n(D-\Delta+1)} \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) + \mathcal{P}^{(n)}_{\nu\tau} \partial_{\rho} \phi^{(n)}(\zeta) & \\
+ L^{-6n(D-\Delta+1)} \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) + \mathcal{P}^{(n)}_{\nu\rho} \partial_{\mu} \phi^{(n)}(\zeta) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) + \mathcal{F}^{(n)}_{0,\nu\zeta\sigma}(\zeta) \right) \partial_{\rho} \phi^{(n)}(\zeta) + \mathcal{P}^{(n)}_{\nu\tau} \partial_{\rho} \phi^{(n)}(\zeta) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) \right. & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\rho\sigma}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) \right) & \\
+ L^{-6n(D-\Delta+1)} \left( \mathcal{F}^{(n)}_{0,\mu\nu\sigma}(\zeta) \partial_{\rho} \phi^{(n)}(\zeta) \right)
\end{array} \right\}
$$

In the one above, $\partial_{\mu} = \partial / \partial \zeta^\mu$.

In $S^{(n)}$ above, we can see there appear four kinds of the exponents, which we express as $\kappa_{1,2,3,4}$ as

$$
\begin{align}
\kappa_1 &= D - 2(D - \Delta + 1), \\
\kappa_2 &= D - 6(D - \Delta + 1), \\
\kappa_3 &= D - (4D - 4\Delta + 6), \\
\kappa_4 &= D - 4(D - \Delta + 2),
\end{align}
$$

where in the value of $\kappa_1$, we have take into account of the two facts: 1) $L^n g^{(n)\kappa\zeta}(\zeta) g^{(n)\kappa\zeta}(\zeta) = \delta_\zeta$, 2) We later take the contraction as mentioned under (79). As a result not 6 but 4 has been taken. Then, if we take $\Delta$ such that

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\[ \Delta = \frac{2+D}{2}, \text{ which leads to } \kappa_2 = -2D, \kappa_3 = -2 - D, \kappa_4 = -4 - D, \]

\[ \Delta = \frac{6+5D}{6}, \text{ which leads to } \kappa_1 = \frac{2D}{3}, \kappa_3 = -\frac{6+D}{3}, \kappa_4 = -4 + \frac{D}{3}, \]

\[ \Delta = \frac{3(2+D)}{4}, \text{ which leads to } \kappa_1 = \frac{2+D}{2}, \kappa_2 = 3 - \frac{D}{2}, \kappa_4 = -2, \]

\[ \Delta = \frac{4+D}{4}, \text{ which leads to } \kappa_1 = \frac{4+D}{2}, \kappa_2 = 6 - \frac{D}{2}, \kappa_3 = 2. \]

Therefore, when we take \( \Delta \) as

\[ \Delta = (2 + D)/2 = 2, \text{ where } D = 2 \text{ in this study.} \quad (78) \]

the fixed-point exists (in other cases, some terms diverge), which is

\[ \lim_{n \to \infty} S^{(n)}(\zeta) = -D \int d\zeta^D \sqrt{-g^{(\infty)}(\zeta)} g^{(\infty)\mu\nu}(\zeta) \partial_{\mu} \phi^{(\infty)}(\zeta) \partial_{\nu} \phi^{(\infty)}(\zeta), \quad (79) \]

where we have performed the contraction: \( g^{(n)\sigma\tau}(\zeta) g^{(n)\rho\xi}(\zeta) g_{\sigma\rho}^{(n)}(\zeta) g_{\tau\xi}^{(n)}(\zeta) = D. \)

Let us turn to \( \phi^{(n)} \) and \( p^{(n)} \), the \( p(n) \) getting \( n \) times coarse-grainings, to give \( g^{(n)\mu\nu} \) and \( g_{\mu\nu}^{(n)} \) in (66) and (67). To this purpose, let us note the two facts:

1). Components of \( g^{(n)\mu\nu} \) and \( g_{\mu\nu}^{(n)} \) are given just by exchanging \( \theta^\mu \) in the components of (23) with \( L\zeta^\mu \) as can be seen in (66) and (67).

2). \( \phi \) in (21) gives (23).

Form those, we can reach the following \( \phi^{(n)} \) to give \( g^{(n)\mu\nu} \) and \( g_{\mu\nu}^{(n)} \) in (66) and (67) as

\[ \phi^{(n)}(\zeta) = \frac{1}{2} \ln \left[ -\frac{\pi}{L^2} \right] \frac{L^n(\zeta)}{4\zeta^2}. \quad (80) \]

We can confirm that we can obtain \( g^{(n)\mu\nu} \) and \( g_{\mu\nu}^{(n)} \) in (66) and (67) from \( \partial/\partial \zeta^\mu \partial/\partial \xi^\nu \phi^{(n)}(\zeta) \) and \( g^{(n)\mu\nu} \partial_{\mu} \partial_{\nu} \phi^{(n)}(\zeta) \) according to (114).

Now we can see from (80) that \( \phi^{(n)}(\zeta) \sim -\frac{L^n(\zeta)}{4\zeta^2} + O(n \ln L) \) at \( n \to \infty \). Therefore, with (66) and (67), we can see (79) has the \( L \)-dependence, \( L^n(-D/2+4) \). Therefore, the value of (79) appears to get diverged at \( n \to \infty \). However, as can be seen from the fact that \( \theta^\mu = L^n \zeta^\mu \) is finite, when \( L^n \) is large, \( \zeta^\mu \) is small for that. Therefore, (79) never get diverged at \( n \to \infty \) even if it has \( L \)-dependence of \( L^n(-D/2+4) \).

\( \phi^{(n)}(\zeta) \) is linked with \( \phi^{(0)}(\xi) \) by the relation (69), and \( \phi^{(0)}(\xi) \) is given in (21). We can see that when the forms of \( \phi^{(n)}(\zeta) \) and \( \phi^{(0)}(\xi) \) are given as (69) and (21), only when \( \Delta = D \), (69) can be held; If the value of \( \Delta \) is some ones other than that, \( \phi^{(0)}(\xi) \) given as (21) and \( \phi^{(n)}(\zeta) \) given as (69) do not satisfy the relation (69). Whether \( \Delta \) can be determined to the proper value or not is highly nontrivial, as mention the reason under (71).
From the description in Sec.4.1, \( p^{(n)} \) can be also known, which is that with changing \( \sigma_0^2 \) with \( \sigma_0^2/L^n \) as

\[
p^{(n)} = \frac{1}{\sigma_0} \sqrt{\frac{L^n}{2\pi}} \exp \left[ -\frac{L^n}{2\sigma_0^2} (x - \bar{x})^2 \right].
\]  

(81)

Therefore, under \( n \) times coarse-grainings, \( \sigma_0^2 \) is considered to get the change as

\[
\sigma_0^2 \rightarrow \sigma_0^2/L^{2n}.
\]  

(82)

Lastly, we can see \( \gamma^{(n)} \) is given by the \( n \) times coarse-grainings toward \( \gamma^{(0)} \) as

\[
\gamma^{(n)} = -L^{-(D-\Delta)} \zeta^\mu \mathbf{F}_\mu^{(n)}(x) + \phi^{(n)}(\zeta).
\]  

(83)

Again, only when we consider \( \Delta = D \) given in (78), we can obtain (66) and (67) from (81) and (83) according to (4) for arbitrary \( n \).

\section{Summary}

We would like to summarize this study. First of all, we have been interested to consider the gravitational theory in terms of some statistical averages. From this viewpoint, we have employed the Fisher metric, \( g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma(x, \theta) \partial_\nu \gamma(x, \theta) \rangle \), where \( \gamma(x, \theta) = -\theta^\mu \mathbf{F}_\mu(x) + \phi(\theta) \).

In this study, considering \( \phi(\theta) \) on a space \( \Lambda \), we have considered \( \phi(\theta) \) as the under-lying entity of the metrics.

What we have done in this study are the following three:

1). Rewriting the EH action in terms of the ingredients in the Fisher metric,
2). determining the transformation rules of the Fisher metric, etc under the coarse-graining, performing the coarse-graining toward that rewritten EH action,
3). based on that, obtaining the fixed-point.

First, rewriting the EH action in terms of the ingredients in the Fisher metric is the work to perform first when we consider the Fisher metric. Therefore, the 1) would be natural.

When we consider the Fisher metric, the interesting problem to come first is the confirmation of the physical rightness of the Fisher metric. One way for this is to compare the theory with the distant gravitational phenomena such as dark energy, dark matter and the accelerating expansion of the universe. This is the confirmation based on the observational facts. From such a point of view, we have performed 2) and 3). The theory finally obtained is considered as the effective theory written in terms of the ingredients in the Fisher metric, observed from far.

The space-time this study has considered is two-dimensional. Besides, there are three problems in this study:
1). The action we have rewritten in terms of the Fisher metric has not been the form which we can rewrite into the momentum space using the Fourier transformation,

2). since the Fisher metric is given by $\phi(\theta)$, the quadratic part giving the two-point correlated function does not exist in the action if we look at the action in terms of $\phi(\theta)$,

3). how to determine the constant $N$ in \[139\].

1) and 2) are important upon performing the renormalization transformation, and for the reason that we are unable to resolve those this time, we have sufficed it to perform the coarse-graining (evaluating only the first term in r.h.s. of \[162\] and not evaluating the second term) in this study. 3) is also important because it effects to the constant part in the action, which are crucial in terms of the cosmological constant.

Upon performing the comparison with the distant gravitational phenomena, even if being unable to perform the renormalization transformation and we went only with the coarse-graining, at least we should consider the four-dimensional space-time. We would like to take this problem to future work.

**Acknowledgment.**— I would like to thank Atushi Nakamura for his advise. Although I have pointed out some problems in \[12\], it (and \[14\], \[15\]) has been technically a big helpful.

## A Expression of the Ricci tensor

In Appendix A.1, we derive the expression of the Ricci tensors when $p = e^{-\gamma}$ as in \[3\]. The final result is \[97\]. Based on that, in Appendix A.2 we obtain the expression of the Ricci tensor in \[13\] when $\gamma$ is given as $\gamma = -\theta^\mu F_\mu + \phi$ as in \[10\]. (A large part of the description in this Appendix is overlapped with \[12\].)

### A.1 Expression of the Ricci tensor when $p = e^{-\gamma}$

We first obtain the expression of the Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\tau} (\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}) \quad (84)$$

in terms of $\gamma$. Here, as mentioned in Sec 2, we write $\gamma$ and $g_{\mu\nu}$ before taking the statistical average in boldface. Then, from \[4\], we can see

$$\partial_\sigma g_{\mu\nu} = - (\partial_\sigma \gamma \partial_\mu \gamma \partial_\nu \gamma) + \langle (\partial_\sigma \partial_\mu \gamma) \partial_\nu \gamma \rangle + \langle \partial_\mu \gamma (\partial_\sigma \partial_\nu \gamma) \rangle. \quad (85)$$

Therefore,

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\tau} \left( \langle (\partial_\mu \partial_\nu \gamma) \partial_\tau \gamma \rangle - \frac{1}{2} \langle \partial_\mu \gamma \partial_\nu \gamma \partial_\tau \gamma \rangle \right). \quad (86)$$
Next, let us obtain the expression of the Ricci tensors:

\[ R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\sigma_{\rho\sigma} \Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} \]  

(87)

in terms of \( \gamma \). We first write \( R_{\mu\nu} \) as

\[ R_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} \]  

(88)

where

\[ A_{\mu\nu} = g^{\sigma\tau} \partial_\sigma \left( \left( (\partial_\mu \partial_\nu \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \partial_\tau \gamma \right) \right) \]

\[ - g^{\sigma\tau} \partial_\nu \left( \left( (\partial_\mu \partial_\nu \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \partial_\tau \gamma \right) \right) \]  

(89)

\[ B_{\mu\nu} = \left( \partial_\sigma g^{\sigma\tau} \right) \left( \left( (\partial_\mu \partial_\nu \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \partial_\tau \gamma \right) \right) \]

\[ - \left( \partial_\nu g^{\sigma\tau} \right) \left( \left( (\partial_\mu \partial_\nu \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \partial_\tau \gamma \right) \right) \]

(90)

\[ C_{\mu\nu} = g^{\sigma\tau} g^{\rho\xi} \left( \left( (\partial_\rho \partial_\sigma \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \right) \right) \]

\[ \left( \left( (\partial_\mu \partial_\nu \gamma) \partial_\xi \gamma \right) - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \partial_\xi \gamma \right) \right) \]

\[ - g^{\sigma\tau} g^{\rho\xi} \left( \left( (\partial_\rho \partial_\sigma \gamma) \partial_\tau \gamma \right) - \frac{1}{2} \left( \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \right) \right) \]

(91)

We can rewrite \( A_{\mu\nu} \) in (89) as

\[ A_{\mu\nu} = g^{\sigma\tau} \left\{ - \frac{1}{2} \left( \partial_\sigma \gamma \left( \partial_\mu \partial_\nu \gamma \right) \partial_\tau \gamma \right) + \left( \partial_\mu \partial_\nu \gamma \right) \left( \partial_\sigma \partial_\tau \gamma \right) \right\} \]

\[ + \frac{1}{2} \left( \partial_\nu \gamma \left( \partial_\mu \partial_\sigma \gamma \right) \partial_\tau \gamma \right) - \left( \partial_\mu \partial_\sigma \gamma \right) \left( \partial_\nu \partial_\tau \gamma \right) \]

\[ - \frac{1}{2} \left( \partial_\mu \gamma \partial_\nu \gamma \left( \partial_\sigma \partial_\tau \gamma \right) \right) + \frac{1}{2} \left( \partial_\mu \gamma \partial_\tau \gamma \left( \partial_\nu \partial_\sigma \gamma \right) \right) \]

(92)

where

\[ \omega \equiv g^{\sigma\tau} \left( \partial_\sigma \partial_\tau \gamma - \frac{1}{2} \partial_\sigma \gamma \partial_\tau \gamma \right) \]  

(93)

We here would like to note two points in the footnote\footnote{\textsuperscript{\textcopyright}}.
We can also rewrite $B_{\mu\nu}$ in (93) as

$$B_{\mu\nu} = -g^{\sigma\tau}g^{\rho\varsigma} \left( -\langle \partial_\sigma \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle + \langle \partial_\sigma \partial_\tau \gamma \partial_\rho \gamma \rangle + \langle \partial_\tau \gamma (\partial_\sigma \partial_\rho \gamma) \rangle \right)$$

$$\times \left( \langle \partial_\mu \partial_\nu \gamma \partial_\varsigma \gamma \rangle - \frac{1}{2} \langle \partial_\mu \gamma \partial_\sigma \gamma \partial_\varsigma \gamma \rangle \right)$$

$$+ g^{\sigma\tau}g^{\rho\varsigma} \left( -\langle \partial_\sigma \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle + \langle \partial_\nu \partial_\tau \gamma \partial_\rho \gamma \rangle + \langle \partial_\tau \gamma (\partial_\rho \partial_\nu \gamma) \rangle \right)$$

$$\times \left( \langle \partial_\mu \partial_\sigma \gamma \partial_\varsigma \gamma \rangle - \frac{1}{2} \langle \partial_\nu \gamma \partial_\sigma \gamma \partial_\varsigma \gamma \rangle \right),$$

(94)

where we have used a general relation $\partial_\tau g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\partial_\sigma \gamma^{\alpha\beta}$. Summing up $B_{\mu\nu}$ above with $C_{\mu\nu}$ in (91),

$$B_{\mu\nu} + C_{\mu\nu}$$

$$= g^{\sigma\tau}g^{\rho\varsigma} \left( \langle \partial_\mu \partial_\nu \gamma \partial_\varsigma \gamma \rangle - \frac{1}{2} \langle \partial_\mu \gamma \partial_\sigma \gamma \partial_\varsigma \gamma \rangle \right) \left( \langle \partial_\sigma \partial_\tau \gamma \partial_\rho \gamma \rangle - \frac{1}{2} \langle \partial_\sigma \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle \right)$$

$$- g^{\sigma\tau}g^{\rho\varsigma} \left( \langle \partial_\sigma \partial_\nu \gamma \partial_\rho \gamma \rangle - \frac{1}{2} \langle \partial_\sigma \gamma \partial_\nu \gamma \partial_\rho \gamma \rangle \right) \left( \langle \partial_\mu \partial_\tau \gamma \partial_\varsigma \gamma \rangle - \frac{1}{2} \langle \partial_\mu \gamma \partial_\tau \gamma \partial_\varsigma \gamma \rangle \right)$$

$$= g^{\sigma\tau}g^{\rho\varsigma} \left( \langle \partial_\nu \partial_\rho \gamma \partial_\mu \gamma \partial_\varsigma \gamma \rangle \right) \left( \langle \partial_\nu \partial_\tau \gamma \partial_\rho \gamma \rangle - \frac{1}{2} \langle \partial_\nu \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle \right)$$

$$- g^{\sigma\tau}g^{\rho\varsigma} \left( \langle \partial_\nu \partial_\rho \gamma \partial_\mu \gamma \partial_\varsigma \gamma \rangle \right) \left( \langle \partial_\nu \partial_\tau \gamma \partial_\rho \gamma \rangle - \frac{1}{2} \langle \partial_\nu \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle \right)$$

$$= g^{\sigma\tau}g^{\rho\varsigma} \left( \langle \partial_\nu \partial_\rho \gamma \partial_\mu \gamma \partial_\varsigma \gamma \rangle \right) \left( \langle \partial_\nu \partial_\tau \gamma \partial_\rho \gamma \rangle - \frac{1}{2} \langle \partial_\nu \gamma \partial_\tau \gamma \partial_\rho \gamma \rangle \right)$$

$$+ \left\langle \phi \left( \partial_\nu \partial_\rho \gamma - \frac{1}{2} \partial_\nu \gamma \partial_\rho \gamma \right) \right\rangle,$$

(95)

where

$$\phi \equiv -g^{\sigma\tau}g^{\rho\varsigma}\partial_\nu \gamma \left( \partial_\mu \partial_\tau \gamma - \frac{1}{2} \partial_\mu \gamma \partial_\tau \gamma \right) \right\rangle.$$

(96)

- We cannot deform $\omega$ above to $\frac{1}{2}g^{\sigma\tau}g^{\rho\varsigma}$, because $\partial_\sigma \partial_\tau \gamma$ and $\partial_\tau \gamma \partial_\sigma \gamma$ are different before taken the statistical average [2].

- Next, even if we could deform to $\frac{1}{2}g^{\sigma\tau}g^{\rho\varsigma}$, we could not deform as $g^{\sigma\tau}g^{\rho\varsigma} = n$ by performing the contraction.

The metrics before taken the statistical average which we denote in the bold face as $g^{\mu\nu}(x, \theta)$ would be always metrics of some spaces whatever $x$. However, $g_{\mu\nu}(x, \theta)$ and $g_{\mu\nu}(\theta)$ are associated with different spaces as the metric; $g_{\mu\nu}(x, \theta)$ and $g_{\mu\nu}(\theta)$ are one of some possible metrics and $g_{\mu\nu}(\theta)$ are the metrics of the space appearing after taken the statistical average. Therefore, $g_{\mu\nu}(x, \theta)$ and $g_{\mu\nu}(\theta)$ are not in the relation of the inverse matrix each other.

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From (92) and (93), we can obtain the expression of the Ricci tensor in \( p = e^{-\gamma} \) as

\[
R_{\mu\nu} = \langle (\omega + \phi) \partial_\mu \partial_\nu \gamma \rangle - \frac{1}{2} (\langle (g^{\sigma\tau}(\partial_\sigma \partial_\tau + \phi) \partial_\mu \gamma \partial_\nu \gamma \rangle) - g^{\sigma\tau} \langle (\partial_\nu \gamma \partial_\mu \partial_\nu \gamma \rangle) + \frac{1}{2} g^{\sigma\tau} \langle \partial_\sigma \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle + \frac{1}{2} g^{\sigma\tau} g^{\rho\xi} \langle \partial_\xi \gamma \partial_\mu \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \partial_\sigma \gamma \rangle.
\] (97)

A.2 Expression of the Ricci tensor when \( \gamma = -\theta^\mu F_\mu + \phi \)

In this appendix, we obtain the expression of the Ricci tensor when \( \gamma = -\theta^\mu F_\mu + \phi \) as in (9) based on (97). The final result is (112), which leads to (18). We exploit the relations in Sec.3.

Upon evaluating (97), we first calculate the statistical average of \( \omega \) and \( \phi \) in (93) and (96) as

\[
\langle \omega \rangle = g^{\sigma\tau} \left( \partial_\sigma \partial_\tau \phi - \frac{1}{2} \langle \partial_\sigma \gamma \partial_\tau \gamma \rangle \right) = \frac{D}{2},
\] (98)

\[
\langle \phi \rangle = 0.
\] (99)

We also calculate the term in the last line in (97) as

\[
\langle \partial_\xi \gamma \left( \partial_\mu \partial_\sigma \gamma - \frac{1}{2} \partial_\mu \gamma \partial_\sigma \gamma \right) \rangle \langle \partial_\rho \gamma \left( \partial_\nu \partial_\tau \gamma - \frac{1}{2} \partial_\nu \gamma \partial_\tau \gamma \right) \rangle
\]

\[
= \left( \langle \partial_\xi \gamma \rangle \langle \partial_\mu \partial_\sigma \phi \rangle - \frac{1}{2} \langle \partial_\xi \gamma \partial_\mu \gamma \partial_\sigma \gamma \rangle \right) \left( \langle \partial_\rho \gamma \rangle \langle \partial_\nu \partial_\tau \phi \rangle - \frac{1}{2} \langle \partial_\rho \gamma \partial_\nu \gamma \partial_\tau \gamma \rangle \right)
\]

\[
= \frac{1}{4} \langle \partial_\xi \gamma \partial_\mu \gamma \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \gamma \partial_\tau \gamma \rangle.
\] (100)

Then,

\[
(97) = \langle (\omega + \phi) \partial_\mu \partial_\nu \gamma \rangle - \frac{1}{2} \langle (g^{\sigma\tau}(\partial_\sigma \partial_\tau + \phi) \partial_\mu \gamma \partial_\nu \gamma \rangle) - g^{\sigma\tau} \langle (\partial_\nu \gamma \partial_\mu \partial_\nu \gamma \rangle) + \frac{1}{2} g^{\sigma\tau} \langle \partial_\sigma \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle + \frac{1}{2} g^{\sigma\tau} g^{\rho\xi} \langle \partial_\xi \gamma \partial_\mu \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \partial_\sigma \gamma \rangle
\]

\[
= \frac{D}{2} \partial_\mu \partial_\nu \gamma - \frac{1}{2} \langle g^{\sigma\tau} \partial_\nu \partial_\sigma \gamma \langle \partial_\mu \gamma \partial_\nu \gamma \rangle + \langle \phi \partial_\mu \gamma \partial_\nu \gamma \rangle \rangle
\]

\[
+ \frac{1}{4} g^{\sigma\tau} g^{\rho\xi} \langle \partial_\xi \gamma \partial_\mu \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \partial_\sigma \gamma \rangle,
\] (101)

where we have used the fact that \( \partial_\mu \partial_\nu \gamma \) is independent of \( x \) as in (12) when \( \gamma = -\theta^\mu F_\mu + \phi \) (we also proceed with the following calculation using this relation), and

\[
g^{\sigma\tau} \langle (\partial_\nu \gamma \partial_\mu \partial_\nu \gamma \rangle) = g_{\mu\nu},
\] (102)

\[
g^{\sigma\tau} \langle \partial_\nu \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle + \partial_\sigma \gamma \partial_\nu \gamma \partial_\mu \gamma \rangle = 2g_{\mu\nu}.
\] (103)
Continuing the evaluation,

\[
\tag{101}
\frac{D}{2} \partial_\mu \partial_\nu \gamma - \frac{1}{2} (n g_{\mu\nu} + \langle \phi \partial_\mu \gamma \partial_\nu \gamma \rangle) + \frac{1}{4} g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \gamma \partial_\tau \gamma \rangle
\]

\[= -\frac{1}{4} g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle + \frac{1}{4} g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\sigma \gamma \rangle \langle \partial_\rho \gamma \partial_\nu \gamma \partial_\tau \gamma \rangle,
\]

where, when \( \gamma = -\theta^\mu F_\mu + \phi \), \( \phi \) can be evaluated as

\[
\phi = -g^{\sigma \tau} g^{\rho \kappa} \partial_\zeta \gamma \left( \langle \partial_\mu (\partial_\sigma \gamma) \rangle - \frac{1}{2} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \right)
\]

\[= \frac{1}{2} g^{\sigma \tau} g^{\rho \kappa} \partial_\zeta \gamma \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle
\]

and then,

\[
\langle \phi \partial_\sigma \gamma \partial_\nu \gamma \rangle = \frac{1}{2} g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle.
\]

Evaluating each term in \( \langle 104 \rangle \) as

\[
g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle
\]

\[= g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma \langle \partial_\mu \phi - F_\mu \rangle \langle \partial_\nu \phi - F_\nu \rangle \rangle
\]

\[= g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle - \langle \partial_\zeta \gamma F_\mu \rangle \partial_\nu \phi - \langle \partial_\zeta \gamma F_\nu \rangle \partial_\mu \phi + \langle \partial_\zeta \gamma F_\mu F_\nu \rangle \rangle,
\]

\[
g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma \partial_\mu \gamma \partial_\nu \gamma \rangle
\]

\[= g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\zeta \gamma \langle \partial_\mu \phi - F_\mu \rangle \partial_\mu \rangle \langle \partial_\rho \gamma \langle \partial_\nu \phi - F_\nu \rangle \partial_\nu \rangle \partial_\tau \gamma
\]

\[= g^{\sigma \tau} g^{\rho \kappa} (g_{\rho \sigma} \partial_\mu \phi - \langle \partial_\zeta \gamma \partial_\mu \gamma F_\mu \rangle \langle g_{\rho \sigma} \partial_\nu \phi - \langle \partial_\zeta \gamma \partial_\nu \gamma F_\nu \rangle \rangle
\]

\[= n \partial_\mu \phi \partial_\sigma \phi - g^{\sigma \tau} \partial_\mu \phi \langle \partial_\rho \gamma \partial_\tau \gamma F_\nu \rangle - g^{\kappa \zeta} \langle \partial_\zeta \gamma \partial_\sigma \gamma F_\mu \rangle \partial_\nu \phi
\]

\[+ g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\zeta \gamma \partial_\sigma \gamma F_\mu \rangle \langle \partial_\rho \gamma \partial_\tau \gamma F_\nu \rangle \partial_\nu \phi \rangle,
\]

we can write as

\[
\tag{104}
\frac{D}{4} \partial_\mu \phi \partial_\nu \phi
\]

\[+ \frac{1}{4} \left\{ - g^{\sigma \tau} \partial_\mu \phi \langle \partial_\rho \gamma \partial_\tau \gamma F_\nu \rangle - g^{\sigma \tau} \partial_\mu \phi \langle \partial_\zeta \gamma \partial_\sigma \gamma F_\mu \rangle \partial_\nu \phi
\]

\[+ g^{\sigma \tau} g^{\rho \kappa} \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma F_\mu \rangle \partial_\nu \phi + \langle \partial_\zeta \gamma F_\nu \rangle \partial_\mu \phi \rangle
\]

\[+ \frac{1}{4} g^{\sigma \tau} g^{\rho \kappa} \left( \langle \partial_\zeta \gamma \partial_\sigma \gamma F_\mu \rangle \langle \partial_\rho \gamma \partial_\tau \gamma F_\nu \rangle - \langle \partial_\rho \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \langle \partial_\zeta \gamma F_\mu F_\nu \rangle \right).
\]
We can calculate the terms appearing in (109) as

\[-g^\rho\tau \partial_\mu \phi \langle \partial_\rho \gamma \partial_\tau \gamma \mathbf{F}_\nu \rangle - g^{\sigma \zeta} \langle \partial_\sigma \gamma \partial_\zeta \gamma \mathbf{F}_\mu \rangle \partial_\nu \phi \]

\[= -g^\rho\tau \partial_\mu \phi \langle \partial_\rho \gamma \partial_\tau \gamma (\partial_\nu \phi - \partial_\nu \gamma) \rangle \partial_\nu \phi \]

\[= -2D \partial_\mu \phi \partial_\nu \phi + g^\rho\tau \partial_\mu \phi \langle \partial_\rho \gamma \partial_\tau \gamma \partial_\nu \gamma \rangle + g^{\sigma \zeta} \partial_\nu \phi \langle \partial_\sigma \gamma \partial_\zeta \gamma \partial_\nu \gamma \rangle , \quad (110)\]

\[g^{\sigma \tau} g^{\rho \zeta} \langle \partial_\rho \gamma \partial_\zeta \gamma \partial_\tau \gamma \rangle \left( \langle \partial_\nu \gamma \mathbf{F}_\mu \rangle \partial_\nu \phi + \langle \partial_\nu \gamma \partial_\mu \gamma \rangle \partial_\nu \phi \right) \]

\[= -g^{\sigma \tau} (\langle \partial_\nu \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \partial_\nu \phi + \langle \partial_\nu \gamma \partial_\sigma \gamma \partial_\tau \gamma \rangle \partial_\nu \phi) , \quad (111)\]

where we have used (17) in (111). With those above, we can write as

\[(109) = -D^4 \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} \left\{ (110) + (111) \right\} + \frac{1}{4} g^{\sigma \tau} g^{\rho \zeta} (\cdots) \]

\[= -D^4 \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} g^{\sigma \tau} g^{\rho \zeta} (\cdots) . \quad (112)\]

Using (15) and so on, the one above can reach (18).

**B  AdS space in our Fisher metric**

By performing appropriate variable transformations, we can show that the Fisher metric obtained from \( p \) in (22) can be the metric of the two-dimensional AdS space in the Poincaré coordinate in a part of the whole space with the coordinates \( \theta \). In this Appendix, we show it. (This Appendix is written by referring [12], [14] and [15].)

We here write the \( p \) in (22) again as

\[p = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right] = \exp \left[ -\ln \left( \sqrt{2\pi} \sigma \right) - \frac{x^2}{2\sigma^2} - \frac{x\bar{x}}{2\sigma^2} - \frac{\bar{x}^2}{2\sigma^2} \right] . \quad (113)\]

At this time, comparing with (9), we can determine \( F \), \( \theta \) and \( \phi \) as

\[F(x) = (F_1(x), F_2(x)) = (x, x^2) , \quad (114)\]

\[\theta = (\theta^1, \theta^2) = \left( \frac{\bar{x}}{\sigma_0^2}, -1 - \frac{1}{2\sigma_0^2} \right) , \quad (115)\]

\[\phi(\theta) = \ln \left( \sqrt{2\pi} \sigma_0 \right) + \frac{\bar{x}}{2\sigma_0^2} = \frac{1}{2} \ln \left[ -\frac{\pi}{\theta^2} \right] - \frac{(\theta^1)^2}{4\theta^2} . \quad (116)\]

Let us start with those.

We can rewrite \( \phi \) above as

\[\phi = \frac{1}{2} \ln \left[ -\frac{\pi}{\theta^2} \right] - \ln \left[ 1 - \frac{(\theta^1)^2}{4\theta^2} \right] = \frac{1}{2} \ln \left[ -\frac{\pi}{\theta^2} \right] - \frac{1}{2} \ln \left[ 1 - \frac{(\theta^1)^2}{2\theta^2} \right] , \quad (117)\]
where we have assumed
\[
\frac{(\theta^1)^2}{\theta^2} \ll 1. \tag{118}
\]
Performing the rescaling: \((\theta^1, \theta^2) \rightarrow (\sqrt{\pi} \theta^1, \pi \theta^2)\) and using this assumption, we can finally rewrite \(\phi\) above into the following one:
\[
\phi = -\frac{1}{2} \ln \left[ -\left( \theta^2 - \frac{(\theta^1)^2}{2} \right) \right]. \tag{119}
\]
From (119),
\[
\partial_1 \phi = \frac{\theta^1}{2\theta^2 - (\theta^1)^2} = \frac{\sqrt{\kappa} y^1}{(y^2)^2}, \quad \partial_2 \phi = -\frac{1}{2\theta^2 - 2(\theta^1)^2} = -\frac{\kappa}{(y^2)^2}. \tag{120}
\]
where
\[
y^1 \equiv \sqrt{\kappa} \theta^1, \quad y^2 \equiv \left( \theta^2 - \frac{1}{2}(\theta^1)^2 \right)^{1/2}, \quad \kappa \equiv 1/2. \tag{121}
\]
Therefore,
\[
d(\partial_1 \phi) = \sqrt{\kappa} \frac{(y^2)^2 dy^1 - 2y^1 y^2 dy^2}{(y^2)^4}, \quad d(\partial_2 \phi) = \frac{2\kappa}{(y^2)^3} dy^2, \tag{122}
\]
\[
d\theta^1 = \frac{dy^1}{\sqrt{\kappa}}, \quad d\theta^2 = 2y^2 dy^2 + \frac{y^1}{\kappa} dy^1. \tag{123}
\]
With these, we can write as
\[
ds^2 = \partial_\mu \partial_\nu \phi d\theta^\mu d\theta^\nu = d(\partial_\nu \phi) d\theta^\nu = \frac{(dy^1)^2 + 4\kappa (dy^2)^2}{(y^2)^2}, \tag{124}
\]
where \(g_{\mu\nu} = \partial_\mu \partial_\nu \phi\) as in (14). From the result above, we can see that the Fisher metric obtained from \(p\) in (22) can be the metric of the two-dimensional AdS space with the coordinate \(\theta\) in the region where the assumption (118) is held.

**C Renormalization transformation**

As mentioned in the last of Sec 4.2 and beginning of Sec 5, we this time cannot rewrite our model into the momentum space. Therefore, we this time cannot perform the renormalization transformation. However, we would like to review the basic points in the renormalization transformation (Appendix C.1), then demonstrate the renormalization transformation and the fixed-point actually by taking the Gaussian type action in the flat Euclidean space (Appendix C.2). We finally review what we should do in the renormalization transformation toward our model if we did (Appendix C.3). In Appendix C.4, we show a calculation to obtain an equation in Appendix C.1. We write the description except for Appendix C.3 referring [13].
C.1 Short review for the renormalization in the momentum space

We start with considering the representation in the momentum space. To this purpose, we consider one more lattice space, which we denote as $\mathcal{H}_N$. Then, we consider the momentum vectors in the lattice $\mathcal{H}_N$ as

$$k = (k^1, k^2, \ldots, k^D) = \frac{2\pi}{N} (n^1, n^2, \ldots, n^D),$$

(125)

where $N$ is even integers common with that in Sec.5.1, and $n^i$ are integers satisfying $|n^i| \leq N/2$. Therefore,

- the values of $k^i$ are $\pi, \pm 2\pi(N/2 - 1)/N, \pm 2\pi(N/2 - 2)/N, \cdots, 0$,

- the length of each side and the lattice spacing in $\mathcal{H}_N$ are $2\pi$ and $2\pi/N$, respectively.

$\mathcal{H}_N$ and $\Lambda$ form a pair through the Fourier transformation.

From the definitions of $\Lambda$ and $\mathcal{H}_N$, the following relations can be held:

$$\sum_{\theta \in \Lambda} e^{-i k \theta} = N^D \delta_{k,0}, \quad \sum_{k \in \mathcal{H}_N} e^{i k \theta} = N^D \delta_{\theta,0},$$

(126)

where the numbers of the lattice points in $\Lambda$ and $\mathcal{H}_N$ are both $N^D$. Using the relations above, we can write the Fourier transformation between $\phi(\theta)$ in $\Lambda$ and $\varphi_k$ in $\mathcal{H}_N$ as

$$\phi(\theta) = \sum_{k \in \mathcal{H}_N} e^{i k \theta} \varphi(k),$$

(127)

$$\varphi(k) = \frac{1}{N^D} \sum_{\theta \in \Lambda} e^{-i k \theta} \phi(\theta).$$

(128)

Now, let us introduce the following notations:

$$\int_{k \in \left[ -\frac{\pi}{N}, \frac{\pi}{N} \right]^D} dk^D (\cdots) \equiv \left( \frac{2\pi}{N} \right)^D \sum_{k \in \mathcal{H}_N} (\cdots).$$

(129)

The intention in the notation above is that the values of $k$ take the points in $N$-divided interval, $-\pi < k \leq \pi$, in the $n$-dimensional lattice space $\mathcal{H}_N$. Since the lattice spacing of $\mathcal{H}_N$ is $2\pi/N$, in the limit: $N \to \infty$, the r.h.s. in the one above can become a $n$-dimensional Riemann integral. Incidentally, in the limit: $N \to \infty$, $\sum_{\theta \in \Lambda}$ can become $\int_{-\infty}^{\infty} d\theta^D$, because the lattice spacing in $\Lambda$ is 1.

Now, let us rewrite (126) as follows:

$$\frac{1}{(2\pi)^D} \sum_{\theta \in \Lambda} e^{-i k \theta} = \delta(k),$$

(130)

$$\frac{1}{(2\pi)^D} \int_{k \in \left[ -\frac{\pi}{N}, \frac{\pi}{N} \right]^D} dk^D e^{i k \theta} = \delta_{\theta,0}.$$
In (130), we have adopted the following relation: 
\[ \delta(k) \equiv \left( \frac{N}{\pi} \right)^D \delta_{k,0}, \]
which is the general relation between the Kronecker and Dirac delta functions. In (131), we have used (126) and (129) \[ \| \].

We write the fields in \( \mathcal{H}_N \) and \( \Lambda \) in the plain wave expansion as:
\[ \phi(\theta) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{k \in [-\pi,\pi]^D} dk \ e^{ik\theta} \varphi(k), \quad (134) \]
\[ \varphi(k) = \left( \frac{1}{2\pi} \right)^{n/2} \sum_{\theta \in \Lambda} e^{-ik\theta} \phi(\theta). \quad (135) \]

Next, in (135), let us consider \( \varphi'(\eta) \) in \( \Lambda' \) in its r.h.s instead of \( \phi(\theta) \) in \( \Lambda \), then write \( \varphi'(k) \) in its l.h.s. as some coarse-grained fields in the momentum space \( \mathcal{H}_{N/L} \). Here, \( \Lambda' \) and \( \mathcal{H}_{N/L} \) form a pair through the Fourier transformation, and \( N/L \) is common in both. Then, we can obtain the relation of the coarse-graining between \( \varphi'(k) \) in \( \mathcal{H}_{N/L} \) and \( \varphi(k) \) in \( \mathcal{H}_N \) as:
\[ \varphi'(k) = \left( \frac{1}{2\pi} \right)^{D/2} \sum_{\eta \in \Lambda'} e^{-ik\eta} \varphi'(\eta) = \frac{1}{L^D} g(k) \varphi(k/L), \quad (136) \]
where \( g(k) = \frac{1}{L^D} \prod_{j=1}^D \frac{\sin k^{(j)}/2}{\sin k^{(j)}/(2L)}. \]

For the calculation process above, see Appendix C.4.

\( \mathcal{H}_{N/L} \) is a cubic lattice with the length of each side, \( 2\pi \), and the number of its lattice point on one line is \( N/L \). Therefore, its lattice spacing is \( 2\pi/(N/L) \). We can write \( k \) in \( \varphi'(k) \), which are the lattice point in \( \mathcal{H}_{N/L} \), as \( k^i = \frac{2\pi}{N/L} n^i \), where \( n^i \) are integers satisfying \( |n^i| \leq (N/L)/2 \). Therefore, \( k^i \) and \( k^i/L \) in \( \varphi'(k) \) and \( \varphi(k/L) \) in (136) take

\[ \text{The r.h.s. of (131) is not the Dirac delta function, despite that the l.h.s. can become a D-dimensional Riemann integral with the limit: } N \to \infty. \text{ The reason we consider is that the length of each side in } \mathcal{H}_N \text{ is } 2\pi, \text{ finite, even if we take the limit: } N \to \infty. \text{ If we had defined } \mathcal{H}_N \text{ in such a way that the length of each side is infinitely stretching, we could have denoted the r.h.s. of (131) as the Dirac delta function.} \]

\[ \text{** We give the notations of usual Fourier transformations in the one-dimensional space with a period } 2l. \text{ Skipping fine explanation, those are usually written as} \]
\[ f(x) = \sum_{n=-\infty}^{\infty} g_n e^{i\pi nx}, \quad g_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-i\pi nx} \ dx. \quad (132) \]
When we take \( l \) to \( \infty \), treating \( \pi/l \) as a factor \( h \), \( \sum_{n=-\infty}^{\infty} h \to \int_{-\infty}^{\infty} dk \), where \( nh \equiv k \) and \( h = dk \). Then, in the limit: \( l \to \infty \), we can see that we can rewrite ones above into
\[ f(x) = \int_{-\infty}^{\infty} dk \ g(k) e^{ikx}, \quad g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ f(x) e^{-ikx}. \quad (133) \]
We would like to stress that \( k \) can run from \( -\infty \) to \( +\infty \), while our \( k \) can run from \( -\pi \) to \( \pi \).
the following values:

\[ k_i = \pi, \pm 2\pi \frac{(N/2L - 1)}{N/L}, \pm 2\pi \frac{(N/2L - 2)}{N/L}, \ldots, 0, \quad (137) \]

\[ \frac{k_i}{L} = \frac{\pi}{L}, \pm \frac{2\pi}{L} \frac{(N/2L - 1)}{N/L}, \pm \frac{2\pi}{L} \frac{(N/2L - 2)}{N/L}, \ldots, 0. \quad (138) \]

Of course \( k \) in (136) are common, however, \( k_i/L \) in \( \varphi(k/L) \) can run from \(-\pi\) to \(\pi\), so the values in (138) are a part of the whole values \( k_i/L \) can takes.

Based on the coarse-graining in the momentum space as in (136), as well as (46), we can prescribe the renormalization transformation to the action \( S' \) in \( H_{N/L} \) with a configuration \( \varphi' \equiv \{ \varphi'(k) \mid k \in \mathcal{H}_{N/L} \} \) as

\[
\exp[-S'(\varphi')] = \mathcal{N}_g \int \mathcal{D}\varphi \prod_{k \in \mathcal{H}_{N/L}} \delta\left(\varphi'(k) - \frac{1}{L^\Delta} g(k)\varphi(k/L)\right) \exp[-S(\varphi)], \quad (139)
\]

where

\[
\int \mathcal{D}\varphi (\cdots) \equiv \prod_{k \in \mathcal{H}_N} \left( \int_{-\infty}^{\infty} d\varphi(k) \right) (\cdots),
\]

\[
\mathcal{N}_g^{-1} = \int \mathcal{D}\varphi \prod_{k \in \mathcal{H}_{N/L}} \delta(g(k)\varphi(k/L)) \exp[-S(\varphi)]. \quad (140)
\]

If our Lagrangian were composed of only the quadratic term, we could write (139) as in the footnote (conversely, if the action includes interaction terms, the integral of momenta becomes multi-dimensional. As a result, the expression of the action becomes long for that).

Fully involving \( g(k) \) in (139) in our analysis is technically impossible. Therefore, assuming that dynamics in our system is moderate (hydrodynamic approximation), let us assume that among \( k \) in \( \mathcal{H}_{N/L} \) those affecting in effect are

\[
|k^i| \leq \frac{1}{L^\sigma} \quad (0 < \sigma < 1). \quad (142)
\]

\[
\exp\left[-\int_{k \in \mathcal{H}_{N/L}} dk^D \sqrt{-g'} \mathcal{L}'(\varphi'(k))\right] = \mathcal{N}' \int \mathcal{D}\varphi \prod_{k \in \mathcal{H}_{N/L}} \delta\left(\varphi'(k) - \frac{1}{L^\Delta} g(k)\varphi(k/L)\right)
\]

\[
\times \exp\left[-\int_{k \in \mathcal{H}_{N/L}} dk^D \sqrt{-g'} \mathcal{L}(\varphi(k))\right], \quad (141)
\]

\[
\mathcal{N}'^{-1} = \int \mathcal{D}\varphi \prod_{k \in \mathcal{H}_{N/L}} \delta(g(k)\varphi(k/L)) \exp\left[-\int_{k \in \mathcal{H}_{N/L}} dk^D \sqrt{-g'} \mathcal{L}(\varphi)\right],
\]

regarding \( \int \mathcal{D}\varphi (\cdots) \), it is common with that in (139). For the notation of the integral of \( k \), see (129).
Then, denoting as $k^{(j)} = \alpha^{(j)}/L^\sigma$, we can evaluate $g(k)$ as

$$g(k) = \prod_{i=1}^{D} \left\{ 1 - \frac{1}{24} \left( 1 - \frac{1}{L^2} \right) \left( \frac{\alpha_i}{L^\sigma} \right)^2 + \mathcal{O}(L^{-3\sigma}) \right\} \simeq 1. \quad (143)$$

If we employ the one above, the error of $g(k)$ is estimated as $1/L^{2\sigma}$. So, for example if $(L, \sigma) = (10, 0.5)$, the exact value of $g(k)$ is $1 \pm 1/L^{2\sigma} = 1 \pm 0.25$.

Therefore, assuming (142), we write the relation of the coarse-graining (136) approximately as

$$\varphi'(k) = \frac{1}{L^\Delta} \varphi(k/L) + \mathcal{O}(L^{-2\sigma}) \simeq \frac{1}{L^\Delta} \varphi(k/L). \quad (144)$$

In the one above we can read off that small momentum modes appear as large momentum modes in the process of the coarse-graining. This is consistent with the fact that the coarse-graining is the manipulation to enlarge the scale we use when we look at the theory.

Corresponding to the statement under (138), let us divide the cubic lattice $\mathcal{H}_N$ into two parts:

- one is the region satisfying $|k| L < \pi$, 
- another one is the region $\pi L \leq |k| L < \pi$.

The former is a small cubic lattice including the origin, and the later can be considered as an outer shell enclosing the former. We denote the former and later as $\mathcal{H}_<$ and $\mathcal{H}_\geq$ (therefore, $\mathcal{H}_< + \mathcal{H}_\geq = \mathcal{H}_N$). Correspondingly, let us denote $\int D\varphi$ in (139) as

$$\int D\varphi = \int D\varphi_< \int D\varphi_\geq, \quad (145)$$

where $\int D\varphi<$ and $\int D\varphi_\geq$ respectively correspond to the path-integral of $\varphi(k)$ in $\mathcal{H}_<$ and $\mathcal{H}_\geq$.

Based on (144), we can prescribe the renormalization transformation as

$$\exp[-S'(\varphi')] = \mathcal{N} \int D\varphi \prod_{k \in \mathcal{H}_N/L} \delta \left( \varphi'(k) - \frac{1}{L^\Delta} \varphi(k/L) \right) \exp[-S(\varphi)]$$

$$= \mathcal{N} \int D\varphi_\geq \exp \left[ -S(\varphi) \big|_{\varphi(p) = L^\Delta \varphi(Lp)} \right], \quad (146)$$

$$\text{where} \quad \mathcal{N}^{-1} = \int D\varphi \prod_{k \in \mathcal{H}_N/L} \delta \left( \varphi(k/L) \right) \exp[-S(\varphi)] = \int D\varphi_\geq \exp \left[ -S(\varphi(k)) \right].$$

$p (= Lk)$ in the second line of (146) are confined as $|p| < \pi/L$. If some coefficients, etc also get modification under the coarse-graining, exchanges of those are also included in the r.h.s. of (146).
C.2 Demonstration of the renormalization transformation and the fixed-points

In this appendix, by considering the Gaussian type action in the flat Euclidean space, we demonstrate the renormalization transformation and fixed-point, concretely. Since the second term given by the expectation value in the r.h.s. of (158) is not evaluated, it is subtle whether we can refer to what we will do in this appendix as the renormalization transformation or not, however we refer to what we will do in this appendix as the renormalization transformation.

The action we consider is

\[
S(\varphi) = \frac{1}{2} \int_{-\frac{\pi}{N}, \pi} d\mathbf{k}^D (k^2 + \mu^2) \varphi(k)\varphi(-k). \tag{147}
\]

Let us divide this into two parts as

\[
S(\varphi) = \frac{1}{2} \left( \int_{|k| < \frac{\pi}{L}} + \int_{\frac{\pi}{L} \leq |k| \leq \pi} \right) d\mathbf{k}^D (k^2 + \mu^2) \varphi(k)\varphi(-k) \equiv S_<(\varphi) + S_>(\varphi). \tag{148}
\]

Then, substituting this in (146), we can obtain

\[
\exp \left[ -S'(\varphi') \right] = N \int D\varphi \geq \exp \left[ -(S_<(\varphi) |_{\varphi(k) = L^\Delta \varphi'(Lk)} + S_>(\varphi)) \right] \\
= \exp \left[ -S_<(\varphi) |_{\varphi(k) = L^\Delta \varphi'(Lk)} \right]. \tag{149}
\]

Therefore, starting with (147), using (147) and (148) concretely, we can obtain the action with one renormalization transformation on \( \mathcal{H}_{N/L} \) as

\[
S'(\varphi') = \frac{1}{2} \int_{|k| < \frac{\pi}{L}} d\mathbf{k}^D L^{2\Delta} (k^2 + \mu^2) \varphi'(Lk)\varphi'(-Lk) \\
= \frac{1}{2} \int_{-\frac{\pi}{N/L}, \frac{\pi}{N/L}} dp^D L^{2\Delta - D} \left( \left( \frac{p}{L^\varphi} \right)^2 + \mu^2 \right) \varphi(p)\varphi'(-p), \tag{150}
\]

where \( p^\mu = Lk^\mu \). The number of the lattice points in \( \mathcal{H}_{N/L} \) is \( N/L \) as in (137) and (138). We take the limit: \( N \to \infty \), which leads to no physical difference concerning \( -\frac{\pi}{N/L}, \frac{\pi}{N/L} \) and \( -\pi, \pi \). At this time, \( \varphi'(k') \) and \( \varphi(k) \) linked by (144) can be also considered physically no difference, and regarded as \( \varphi'(k) \sim \varphi(k) \) in effect. Finally, the (150) at \( N \to \infty \) can be written as

\[
S'(\varphi) = \frac{1}{2} \int_{-\frac{\pi}{N}, \frac{\pi}{N}} dp^D L^{2\Delta - D} \left( \left( \frac{p}{L^\varphi} \right)^2 + \mu^2 \right) \varphi(p)\varphi(-p). \tag{151}
\]

\* We write this point referring the description between (3.98) and (3.99) in [13]. However, there is no detailed explanation for why \( \varphi'(k) \sim \varphi(k/L) \) except for that it becomes so at large \( N \) limit in [13]. However, if this one can hold at the large \( N \) limit, based on (144) we can suppose the factor out, \( \varphi(k/L) \sim L^\Delta \varphi(k) \), is occurring at \( N \to \infty \).
We can also obtain \( S^{(n)}(\varphi) \) starting with (147) as
\[
S^{(n)}(\varphi) = \frac{1}{2} \int_{-\pi}^{\pi} d^n p \left( \frac{p}{L^n} \right)^2 \varphi(p)\varphi(-p),
\]
where the superscripts \( \text{“}(n)\text{”} \) \( (n = 0, 1, 2, \cdots) \) mean the number of the coarse-graining those got, and we can write \( \varphi \) as \( \varphi^{(0)} \) and \( S(\varphi) \) as \( S^{(0)}(\varphi^{(0)}) \).

We can see there are two fixed-points as
1). For \( \Delta = \frac{D}{2} \), \( S^*(\varphi) = \frac{1}{2} \int_{-\pi}^{\pi} d^n p \mu^2 \varphi(p)\varphi(-p), \)
2). For \( \Delta = \frac{D + 2}{2} \), \( S^*(\varphi) = \frac{1}{2} \int_{-\pi}^{\pi} d^n p^2 \varphi(p)\varphi(-p), \)
where 2) is accompanied by the condition \( \mu = 0 \). Therefore, \( \Delta \) is given as above, the action (147) reaches to the one above after many renormalization transformations.

### C.3 Exterior of the renormalization in our model

Let us just denote our action in the momentum space as
\[
S = S_0 + S_2 + V + C,
\]
where \( S_{0,2} \) mean the quadratic term to give the two-point correlation function \( (S_0 \text{ and } S_2 \text{ mean the parts remaining and depressing under the renormalization transformations.}) \) \( V \) and \( C \) mean the interaction term and constant part.

Let us denote \( S_0 + S_2, V \) and \( S \) as
\[
S_0 + S_2 = \left( \int_{|k| < \frac{\pi}{L}} + \int_{\frac{\pi}{L} \leq |k| \leq \pi} \right) dk^D \sqrt{-g} \mathcal{L}_0 + \mathcal{L}_2 = (S_0 + S_2)_< + (S_0 + S_2)_>,
\]
\[
V = \left( \int_{\text{all } |k| < \frac{\pi}{L}} + \int_{\frac{\pi}{L} \leq |k| \leq \pi} \right) dk^D \sqrt{-g} \mathcal{V} = V_< + V_>,
\]
\[
S = ((S_0 + S_2)_< + V_>) + ((S_0 + S_2)_> + V_>) + C \equiv S_< + S_> + C.
\]

For the meanings of \( \int_{|k| < \frac{\pi}{L}} \) and \( \int_{\frac{\pi}{L} \leq |k| \leq \pi} \), see Sec.C.1 Then, we represent the renormalization transformed action in (146) as
\[
S' = -\ln \int \mathcal{D}\varphi_> \exp \left[ -\left( S_< + C \right) \bigg|_{\varphi(k) = L^\Delta \varphi'(Lk)} + S_>, \right] - \ln \mathcal{N}.
\]

\[
= (S_< + C) \bigg|_{\varphi(k) = L^\Delta \varphi'(Lk)} - \ln \int \mathcal{D}\varphi_> \exp [-S_> - \ln \mathcal{N},
\]
\[
= (S_< + C) \bigg|_{\varphi(k) = L^\Delta \varphi'(Lk)} - \ln \int \mathcal{D}\varphi_> \exp [- (S_0_> + S_2_> + V_>] - \ln \mathcal{N},
\]
\[
(157)
\]
where $\Omega$ means the replacements given between (68) and (72) (the replacement of $\phi$ is excluded). Continuing calculation,

$$
(157) = (S + C) \Omega_{\varphi} = \frac{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq} + V_{\geq}) \right]}{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]} - \ln \mathcal{N}'
$$

$$
= (S + C) \Omega_{\varphi} = \left. \left.\frac{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]}{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]} \right|_{\varphi(k) = L^2 \varphi'} - \ln \mathcal{N}'
$$

$$
= (S + C) \Omega_{\varphi} = \left. \left.\frac{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]}{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]} \right|_{\varphi(k) = L^2 \varphi'} - \ln \mathcal{N}'
$$

where $\mathcal{N}'$ and $\langle \cdots \rangle$ are defined as

$$
\ln \mathcal{N}' = \ln \mathcal{N} + \int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right] - \ln \mathcal{N}'
$$

$$
\langle \cdots \rangle = \frac{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]}{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]}
$$

$\mathcal{N}$ is defined under (146), and $\langle X_1; X_2; \cdots; X_n \rangle$ means the joint cumulant $\langle \cdots \rangle$.

In (158), if we expand to the first order with regard to $n$, we can write what should will evaluate as

$$
S' = (S + C) \Omega_{\varphi} = \left. \left.\frac{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]}{\int \mathcal{D} \varphi \exp \left[ -(S_{0, \geq} + S_{2, \geq}) \right]} \right|_{\varphi(k) = L^2 \varphi'} + \langle V_{\geq} \rangle - \ln \mathcal{N}',
$$

where the difference originated in the joint cumulant is irrelevant at $n = 1$.

There is no way to evaluate $\langle V_{\geq} \rangle$ except for performing the Wick contraction by performing the path-integral with regard to $\varphi_{\geq}$. However since $g_{\mu \nu}$ and $g^{\mu \nu}$ are given by $\varphi$, those are also the targets of the pass-integral of $\varphi_{\geq}$. As a result, the quadratic part does not exist in the action looking at the action in terms of $\varphi_{\geq}$. Therefore, the evaluation of $\langle V_{\geq} \rangle$ is impossible as long as we do not perform very rough treatment treating $g_{\mu \nu}$ and $g^{\mu \nu}$ separately from $\varphi$. This point is also a very difficult point in this study considering $\phi(\theta)$ as the underlying entity, in addition to the difficulty in performing the Fourier transformation.

\[\text{† For example,} \]

$$
\langle X; Y \rangle = \langle XY \rangle - \langle X \rangle \langle Y \rangle,
$$

$$
\langle X; Y ; Z \rangle = \langle XY Z \rangle - \langle X \rangle \langle Y Z \rangle - \langle Y \rangle \langle X Z \rangle - \langle Z \rangle \langle XY \rangle + 2 \langle X \rangle \langle Y \rangle \langle Z \rangle,
$$

$$
\langle X; Y ; Z; W \rangle = \langle XY Z W \rangle - \langle XY \rangle \langle Z W \rangle - \langle X Z \rangle \langle Y W \rangle - \langle X W \rangle \langle Y Z \rangle.
$$

(161)
C.4 Derivation of (136)

In this Appendix, we show the calculation process of (136). Since fine explanation would not be needed, we write only calculation process.

\[
\varphi'(k) = \left(\frac{1}{2\pi}\right)^{D/2} \sum_{\eta \in \Lambda'} e^{-ik\eta}\varphi'(\eta) = \left(\frac{1}{2\pi}\right)^{D/2} \frac{1}{L^\Delta} \sum_{\eta \in \Lambda'} e^{-ik\eta} \sum_{\theta \in B_0} \phi(L\eta + \theta) = \left(\frac{1}{2\pi}\right)^{D} \frac{1}{L^\Delta} \sum_{\eta \in \Lambda'} e^{-ik\eta} \sum_{\theta \in B_0} \int_{p \in [-\pi,\pi]^D} dp \ e^{i(L\eta + \theta)p} \varphi(p) = \left(\frac{1}{2\pi}\right)^{D} \frac{1}{L^\Delta} \sum_{\eta \in \Lambda'} e^{-ik\eta} \sum_{\theta \in B_0} \int_{p \in [-\pi,\pi]^D} dp \ e^{i\theta p} \sum_{\eta \in \Lambda'} e^{-i(k-Lp)\eta} \varphi(p) = \frac{1}{L^{\Delta + D}} \sum_{\theta \in B_0} \int_{p \in [-\pi,\pi]^D} d(Lp) \ e^{i\theta p} \delta^D(k - Lp) \varphi(p) = \frac{1}{L^\Delta} \frac{1}{L^D} \sum_{\theta \in B_0} \prod_{j=1}^{D} \exp \left[ i \frac{k^i}{L} \theta^j \right] \varphi(k/L) = \frac{1}{L^\Delta} g(k) \varphi(k/L). (163)
\]

\[
g(k) = \frac{1}{L^D} \sum_{\theta \in B_0} \prod_{i=1}^{D} \exp \left[ i \frac{k^i}{L} \theta^j \right] = \frac{1}{L^D} \prod_{i=1}^{D} \left( \sum_{l = -(L-1)/2}^{(L-1)/2} \exp \left[ i \frac{k^i l^j}{L} \right] \right) = \frac{1}{L^D} \prod_{i=1}^{D} \frac{\sin \left( \frac{k^i}{2} \right)}{\sin \left( \frac{k^i}{2L} \right)}. (164)
\]

D Coarse-graining in terms of \(g^{(n)}_{\mu\nu}(\zeta)\) and \(g^{(n)}_{\mu\nu}(\zeta)\)

In this appendix, considering the following replacements,

\[
g^{(0)}_{\mu\nu}(\theta) \rightarrow L^2 \ g^{(1)}_{\mu\nu}(\eta), \quad g^{(0)}_{\mu\nu}(\theta) \rightarrow L^{-2} \ g^{(1)}_{\mu\nu}(\eta), \quad (165)
\]

instead of (70) (the grounds of this is (66) and (67)), we show that in the case of (165), we cannot obtain the proper \(\Delta\), which means we cannot obtain the consistent results: the relations of (4) and (14) cannot be held for \(\phi^{(n)}(\zeta)\) and \(g^{(n)}_{\mu\nu}(\zeta)\) obtained by performing the coarse-graining.

In this section, we do not include the replacement (71), since determining the transformation rule of \(\sigma^2_0\) would not make sense as long as \(\Delta\) can be determined rightly, and the transformation rule of \(\sigma^2_0\) is not important in the purpose in this appendix.
By the replacements with but (70) not (165), we can obtain as
\[ S^{(n)}(\zeta) = L^{nD/2} \int d\zeta D \sqrt{-g^{(n)}(\zeta) \mathcal{L}^{(n)}(\zeta)}. \]  
(166)
where \( S^{(n)} \) is the theory on \( \Lambda^{(n)} \) with the coordinate \( \zeta^{\mu}/L^{n} \) for any \( n \) as defined under (66). \( L^{nD/2} \) comes from \( d\theta D \sqrt{-g(\theta)} \), and \( \mathcal{L}^{(n)}(\zeta) \) is given as
\[
\mathcal{L}^{(n)}(\zeta) = L^{6n} g^{(n)\sigma\tau}(\zeta) g^{(n)\rho\kappa}(\zeta) g^{(n)\mu\nu}(\zeta) \left\{ 
- L^{-2n(D-\Delta+2)} g^{(n)\zeta_\sigma}(\zeta) g^{(n)\nu_\mu}(\zeta) \partial_{\mu} \phi^{(n)}(\zeta) \partial_{\nu} \phi^{(n)}(\zeta) 
+ L^{-6n(D-\Delta+1)} F^{(n)}_{0,\nu_\mu_\sigma}(P_{\sigma\nu} \partial_\tau \phi^{(n)}(\zeta) + P_{\sigma\nu} \partial_\rho \phi^{(n)}(\zeta)) 
+ L^{-6n(D-\Delta+1)} F^{(n)}_{0,\nu_\mu_\sigma}(P_{\nu_\mu_\sigma} \partial_\tau \phi^{(n)}(\zeta) + P_{\nu_\mu_\sigma} \partial_\rho \phi^{(n)}(\zeta)) 
+ L^{-6n(D-\Delta+1)} F^{(n)}_{0,\nu_\mu_\sigma}(Q^{(n)}_{\nu_\mu_\sigma} + F^{(n)}_{0,\nu_\mu_\sigma} Q^{(n)}_{\mu_\nu_\sigma}) \partial_\rho \phi^{(n)}(\zeta) \partial_\tau \phi^{(n)}(\zeta) 
+ L^{-6n(D-\Delta+1)} \left( P^{(n)}_{\nu_\mu_\sigma} \partial_\mu \phi^{(n)}(\zeta) + P^{(n)}_{\nu_\mu_\sigma} \partial_\nu \phi^{(n)}(\zeta) \right) \left( P^{(n)}_{\nu_\mu_\sigma} \partial_\sigma \phi^{(n)}(\zeta) + P^{(n)}_{\nu_\mu_\sigma} \partial_\phi^{(n)}(\zeta) \right) 
+ L^{-6n(D-\Delta+1)} Q^{(n)}_{\mu_\nu_\sigma} \partial_\mu \phi^{(n)}(\zeta) \partial_\nu \phi^{(n)}(\zeta) \left( P^{(n)}_{\mu_\nu_\sigma} \partial_\sigma \phi^{(n)}(\zeta) + P^{(n)}_{\mu_\nu_\sigma} \partial_\phi^{(n)}(\zeta) \right) 
+ L^{-6n(D-\Delta+1)} Q^{(n)}_{\mu_\nu_\sigma} \partial_\mu \phi^{(n)}(\zeta) \partial_\nu \phi^{(n)}(\zeta) \partial_\sigma \phi^{(n)}(\zeta) \partial_\phi^{(n)}(\zeta) 
+ L^{-6n(D-\Delta+1)} H^{(n)}_{0,\nu_\mu_\sigma} F^{(n)}_{0,\nu_\mu_\sigma} \left\{ 
\left. 2\partial_x F^{(n)}_{\mu}(x) \right|_{x=x_\zeta} \partial_y F^{(n)}_{\nu}(x) \right|_{x=x_\zeta} + 3\sigma_0^2 \left. \partial^2_x F^{(n)}_{\mu}(x) \right|_{x=x_\zeta} \partial^2_y F^{(n)}_{\nu}(x) \right|_{x=x_\zeta} \partial_\tau \phi^{(n)}(\zeta) 
+ F^{(n)}_{1,\nu_\mu_\zeta} \right\}. 
\]  
(167)
In the one above, \( \partial_\mu = \partial/\partial \zeta^\mu \).

In \( S^{(n)} \) above, we can see there appear three kinds of the exponents, which we express as \( \kappa_{1,2} \) as
\[
\kappa_1 = D/2 + 4 - 2(D - \Delta + 2), 
\kappa_2 = D/2 + 6 - 6(D - \Delta + 1), 
\kappa_3 = D/2 - (4D - 4\Delta + 2),
\]  
(168)(169)(170)
where in the value of \( \kappa_1 \), we have take into account of the two facts: 1) \( L^n g^{(n)}_{\zeta_\sigma}(\zeta) g^{(n)\zeta_\rho}(\zeta) = \delta_\zeta^\xi \), 2) We later take the contraction as mentioned under (74). As a result not 6 but 4 has been taken. Then, if we take \( \Delta \) such that
\(
\kappa_1 \text{ vanishes; } \Delta = 3D/4, \text{ which leads } \kappa_2 = -D \text{ and } \kappa_3 = -2 - D/2.
\)
\(
\kappa_2 \text{ vanishes, } \Delta = 11D/12, \text{ which leads } \kappa_1 = -2 + D/6 \text{ and } \kappa_3 = -2 + D/6.
\)
\(
\kappa_3 \text{ vanishes, } \Delta = (4 + 7D)/8, \text{ which leads } \kappa_1 = (4 + D)/4 \text{ and } \kappa_2 = 3 - D/4.
\)

Therefore, when we take \( \Delta \) as
\[
\Delta = 3D/4, \quad \text{where } D = 2 \text{ in this study. (171)}
\]
the fixed-point exists, which is
\[
\lim_{n \to \infty} \mathcal{S}^{(n)}(\zeta) = -D \int d\zeta^D \sqrt{-g^{(\infty)}(\zeta) g^{(\infty)\mu\nu}(\zeta) \partial_\mu \phi^{(\infty)}(\zeta) \partial_\nu \phi^{(\infty)}(\zeta)}, \quad (172)
\]
where we have performed the contraction: \( L^{2n} g^{(n)\sigma\tau}(\zeta) g^{(n)\rho\sigma}(\zeta) g^{(n)}_{\sigma\tau}(\zeta) g^{(n)}_{\rho\sigma}(\zeta) = D. \)

However, if it comes to \( \phi^{(n)}(\zeta) \) with \( \Delta \) in \( (171) \), the relations of \( [1] \) and \( [12] \) for the coarse-grained \( \phi^{(n)}(\zeta) \) and \( g^{(n)\mu\nu}(\zeta) \) and \( \bar{g}^{(n)}_{\mu\nu}(\zeta) \) cannot be held. (Only when \( \Delta \) is given as \( D \) as in \( [7] \), it can be held.)

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