An acceleration of convergence to some generalized-Euler-constant function

V. Lampret
AN ACCELERATION OF CONVERGENCE TO SOME GENERALIZED-EULER-CONSTANT FUNCTION

V. LAMPRET

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Abstract. For the generalized-Euler-constant function \( \gamma(a) \),

\[
\gamma(a) := \lim_{n \to \infty} \left( \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n-1}{a} \right) \quad (a > 0),
\]

and for any positive integer \( q \geq 2 \), using the Bernoulli numbers \( B_{2m} \), the sequences \( n \mapsto \mathcal{A}_n(a,q), n \mapsto \mathcal{B}_n(a,q) \) and \( n \mapsto \mathcal{C}_n(a,q) \), having the properties

\[
\lim_{n \to \infty} n^{2q-2} \left[ \gamma(a) - \mathcal{A}_n(a,q) \right] = \frac{B_{2q-2}}{2q-2},
\]
\[
\lim_{n \to \infty} n^{2q-2} \left[ \gamma(a) - \mathcal{B}_n(a,q) \right] = -\left(1 - 2^{3-2q}\right) \frac{B_{2q-2}}{2q-2}
\]

and

\[
\lim_{n \to \infty} n^{2q-1} \left[ \gamma(a) - \mathcal{C}_n(a,q) \right] = \frac{1}{2} B_{2q-2},
\]

are determined.

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1. INTRODUCTION

The gamma-sequence

\[
y_n(a) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n-1}{a} \quad (n \in \mathbb{N}),
\]

considered in [2,3] is convergent for \( a > 0 \) and defines the generalized-Euler-constant function \( \gamma(a) \),

\[
\gamma(a) := \lim_{n \to \infty} y_n(a) \quad (1.2)
\]
The name generalized-Euler-constant function has its origin in the identity \( \gamma(1) = C \), where \( C \) is the Euler-Mascheroni constant. Several results on the rate of convergence of the sequence (1.1) have been established in the literature.

Recently, A. Sîntămărian [4] accelerated the convergence (1.2) using the Stolz-Cesaro limit theorem. In this reference the sequences

\[
\alpha_{n,2}(a) := \sum_{k=0}^{n-1} \frac{1}{a + k} - \frac{1}{2(a + n - 1)} + \frac{1}{12(a + n - 1)^2} - \ln\left(\frac{a + n - 1}{a} + \frac{1}{120a(a + n - 1)^3}\right)
\]

and

\[
\beta_{n,2}(a) := \alpha_{n,2}(a) + \frac{1}{252(a + n - 1)^6}
\]

were considered and in Theorem 2 the equalities

\[
\lim_{n \to \infty} n^6[\gamma(a) - \alpha_{n,2}(a)] = \frac{1}{252}
\]

and

\[
\lim_{n \to \infty} n^8[\beta_{n,2}(a) - \gamma(a)] = \frac{121}{28800}
\]

were derived. Similarly, in Theorem 3, were considered some sequences \( \alpha_{n,3}(a) \), \( \beta_{n,3}(a) \) and \( \delta_{n,3}(a) \) such that the following limits hold:

\[
\lim_{n \to \infty} n^8[\alpha_{n,3}(a) - \gamma(a)] = \frac{1}{240}
\]

\[
\lim_{n \to \infty} n^{10}[\gamma(a) - \beta_{n,3}(a)] = \frac{1}{132}
\]

and

\[
\lim_{n \to \infty} n^{12}[\delta_{n,3}(a) - \gamma(a)] = \frac{174197}{8255520}
\]

In [4] the equalities above were demonstrated using rather tedious calculations.

The goal of this article is to complement/improve the results and the method of derivation as presented in [4]. In our paper we present an approach of incessant acceleration of the convergence (1.1) to any degree. We will present three classes of sequences converging to \( \gamma(a) \) much faster than the original sequence \( y_n(a) \) does.
2. Preliminaries

Referring to (1.1), (1.2) and [1, Theorems 1–3], we have the following equalities\(^1\)

\[
\gamma(a) = S_n(a, q) + R_n(a, q) \quad (n, q \in \mathbb{N}) \quad (2.1)
\]

\[
\sigma_n(a, q) + \rho_n(a, q) \quad (n, q \in \mathbb{N}) \quad (2.2)
\]

\[
S_n^*(a, q) + R_n^*(a, q) \quad (n, q \in \mathbb{N}) \quad (2.3)
\]

with\(^2\)

\[
S_n(a, q) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n}{a} + \frac{1}{2(a+n)} + \sum_{j=1}^{q-1} \frac{B_{2j}}{2j(a+n)^{2j}}. \quad (2.4)
\]

\[
\sigma_n(a, q) = \sum_{k=0}^{n-1} \frac{1}{a+k} + \ln \left(\frac{a}{a+n-\frac{1}{2}}\right) - \sum_{i=1}^{q-1} \frac{1 - 2^{-2i}}{2i} \cdot \frac{B_{2i}}{(a+n-\frac{1}{2})^{2i}}. \quad (2.5)
\]

and

\[
S_n^*(a, q) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n}{a}
- 1 - \ln \left(1 - \frac{1}{a+n+1}\right)^{a+n+1} + \frac{1}{2(a+n)} - \frac{1}{2} \ln \left(1 + \frac{1}{a+n}\right)
- \sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)(2j-1)} \left[\frac{1}{(a+n+1)^{2j-1}} - \frac{1}{(a+n+1)^{2j-1}} - \frac{2j-1}{(a+n)^{2j}}\right]. \quad (2.6)
\]

The remainders are estimated as

\[
|R_n(a, q)| < \frac{|B_{2q}|}{q(a+n)^{2q}}, \quad (2.7)
\]

\[
|\rho_n(a, q)| < \frac{|B_{2q}|}{q(a+n-\frac{1}{2})^{2q}} \quad (2.8)
\]

\(^1\)The sequence \(\sigma_n(a, q)\) in the expression (2.5) is given in the corrected form appearing in the proof of [1, Theorem 2], where in the first sum the start “\(k = 1\)” should be replaced by “\(k = 0\)” and where the summands in the third sum of \(\sigma_n(a, q)\) are written incorrectly.

\(^2\)By definition \(\sum_{k=1}^{m} x_k = 0\) for \(m < 1\).
Here, the symbol $B_k$ means the $k$-th Bernoulli number,

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^j}{j!} \quad (x \in \mathbb{R}, |t| < 2),$$

$B_k \equiv B_k(0), B_k(x)$ is $k$-th Bernoulli polynomial.

3. AN ACCELERATION OF CONVERGENCE

Referring to (2.4)–(2.6) we make the following definition.

**Definition 1.** For any $a > 0$ and any integer $q \geq 2$ we consider the following sequences:

$$n \mapsto A_n(a,q) := S_n(a,q - 1), \quad (3.1)$$

$$n \mapsto B_n(a,q) := \sigma_n(a,q - 1) \quad (3.2)$$

and

$$n \mapsto C_n(a,q) := S_n^*(a,q - 1). \quad (3.3)$$

Now, we are in the position to formulate the following result.

**Theorem 1.** For any positive $a$ and any integer $q \geq 2$ we have the following limits:

$$\lim_{n \to \infty} n^{2q-2} \gamma(a) - A_n(a,q) = \frac{B_{2q-2}}{2q-2} =: L_{\mathcal{A}}(q), \quad (3.4)$$

$$\lim_{n \to \infty} n^{2q-2} \gamma(a) - B_n(a,q) = -\left(1 - 2^{-3-2q}\right) \frac{B_{2q-2}}{2q-2} =: L_{\mathcal{B}}(q) \quad (3.5)$$

and

$$\lim_{n \to \infty} n^{2q-1} \gamma(a) - C_n(a,q) = \frac{1}{2} B_{2q-2} =: L_{\mathcal{C}}(q). \quad (3.6)$$

Note that the limits are independent of $a$.

**Proof.** According to (2.1), (2.4) and (3.1), we have

$$\gamma(a) = A_n(a,q) + \frac{B_{2q-2}}{(2q-2)(a+n)^{2q-2}} + R_n(a,q).$$
for \( n \geq 1, \, a > 0 \) and \( q \geq 2 \). Consequently, using (2.7), the equality (3.4) follows. Similarly, referring to (2.2), (2.5) and (3.2), we get
\[
\gamma(a) = B_n(a, q) - \frac{1 - 2^{3-2q}}{2q-2} \cdot \frac{B_{2q-2}}{(a + n - \frac{1}{2})^{2q-2}} + \rho_n(a, q),
\]
for \( n \geq 1, \, a > 0 \) and \( q \geq 2 \). Thus, considering (2.8), we confirm (3.5). Finally, referring to (2.3), (2.6) and (3.3) we obtain
\[
\gamma(a) = C_n(a, q) + R_n^*(a, q)
\]
for \( n \geq 1, \, a > 0 \) and \( q \geq 2 \). Denoting \( a + n = b, \, 2q - 3 = m \) and using Taylor's formula of order 1 around \( b \) for the function \( f(x) = x^{-m}, \, (b + 1)^{-m} = b^{-m} - m b^{-(m+1)} + \frac{1}{2} m(m+1)(b + \theta)^{-(m+2)} \), we obtain the equality
\[
1 \cdot \frac{1}{(a + n + 1)^{2q-3}} = \frac{1}{(a + n)^{2q-3}} - \frac{2q-3}{(a + n)^{2q-2}} + \frac{(2q-3)(2q-2)}{2(a + n + \theta)^{2q-1}}, \tag{3.8}
\]
for some \( \theta = \vartheta_n(a, q) \in (0, 1) \). From (3.7) and (3.8) we get the expression
\[
\gamma(a) - C_n(a, q) = \frac{B_{2q-2}}{(2q-2)(2q-3)} \cdot \frac{(2q-3)(2q-2)}{2(a + n + \theta)^{2q-1}} + R_n^*(a, q),
\]
which, recalling (2.9), demonstrates the relation (3.6).

**Example 1.** Referring to (3.4)–(3.6) and using [5] we obtain the following tables:

| \( q \) | 2 | 3 | 4 | 5 | 6 | 7 |
| --- | --- | --- | --- | --- | --- | --- |
| \( L_A(q) \) | \( \frac{1}{12} \) | \( -\frac{1}{120} \) | \( \frac{1}{252} \) | \( -\frac{1}{240} \) | \( \frac{1}{132} \) | \( -\frac{1}{32760} \) |

**Table 1.** The type \( A \)–limits; Theorem 1, Eq. (3.4).

| \( q \) | 2 | 3 | 4 | 5 | 6 | 7 |
| --- | --- | --- | --- | --- | --- | --- |
| \( L_B(q) \) | \( -\frac{1}{24} \) | \( 960 \) | \( -\frac{1}{8064} \) | \( 127 \) | \( 30720 \) | \( -\frac{311}{67584} \) | \( 14144477/67002480 \) |

**Table 2.** The type \( B \)–limits; Theorem 1, Eq. (3.5).
| $q$ | $L_C(q)$ |
|-----|---------|
| 2   | $\frac{1}{12}$ |
| 3   | $-\frac{1}{60}$ |
| 4   | $\frac{1}{84}$ |
| 5   | $-\frac{1}{60}$ |
| 6   | $-\frac{5}{132}$ |
| 7   | $-\frac{5}{640}$ |

Table 3. The type $C$–limits; Theorem 1, Eq. (3.6).

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Author’s address

V. Lampret

University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, 1000 Ljubljana, Slovenia

E-mail address: vito.lampret@fgg.uni-lj.si