On analytic properties of Meixner-Sobolev orthogonal polynomials of higher order difference operators

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Abstract

In this contribution we consider sequences of monic polynomials orthogonal with respect to Sobolev-type inner product

\[ \langle p, q \rangle_{\lambda} = \sum_{x \geq 0} p(x)q(x)\frac{\mu^x \Gamma(x + \gamma)}{\Gamma(x)\Gamma(x + 1)} + \lambda D_1^i p(\alpha)D_2^i q(\alpha), \quad i = 1, 2, \]

where \( \lambda \in \mathbb{R}^+, j \in \mathbb{N}, \alpha \leq 0, \gamma > 0, 0 < \mu < 1 \) and \( D_1, D_2 \) denote the forward and backward difference operators, respectively. We derive an explicit representation for these polynomials. The ladder operators associated with these polynomials are obtained, and the linear difference equation of second order is also given. In addition, for these polynomials we derive a \((2j + 3)\)-term recurrence relation. Finally, we find the Mehler-Heine type formula for the particular case \( \alpha = 0 \).

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The study of the sequences of polynomials orthogonal with respect to the Sobolev-type inner product

\[ \langle p, q \rangle_{\lambda} = \int_{\Omega} p(x)q(x)\rho(x)\,dx + \lambda p^{(j)}(c)q^{(j)}(c), \quad (1) \]

where \( \Omega = \mathbb{R}, \lambda \in \mathbb{R}^+, c \in \mathbb{R} \) and \( \rho(x) \) is a weight function being introduced by Marcellán and Ronveaux in [24]. In this work, they obtained the second order linear differential equation that such polynomials satisfy. Thenceforwards, many researchers have achieved remarkable results in this area. For example, the zero distribution of the polynomials orthogonal with respect to the inner product [11] when \( \Omega = (0, \infty) \) and \( c = 0 \) was studied by Meijer in [25]. In [16], the authors analyzed the asymptotic behavior of polynomials orthogonal with respect to inner
product expressed in \([1]\) when \(\Omega = (0, \infty), \lambda \in \mathbb{R}^+\), \(c = 0\) and \(\rho(x) = x^\alpha e^{-x}\) with \(\alpha > -1\). Moreover, the location of the zeros of polynomials orthogonal with respect to the previous inner product is analyzed in \([23]\). A particular case when \(j = 1\) is considered in \([18]\). Recently, in \([21]\) the authors established the asymptotic behavior of sequences of polynomials orthogonal with respect to the inner product

\[
\langle p, q \rangle_{s,n} = \int_{-1}^{1} p(x)q(x)(1-x)^\alpha (1+x)^\beta \, dx + \lambda_n p^{(j)}(1) \, q^{(j)}(1),
\]

where \(\alpha, \beta > -1\), \(j \geq 0\) and

\[
\lim_{n \to \infty} \lambda_n n^\gamma = \lambda > 0,
\]

with \(\gamma\) a fixed real number. They also deduced the Mehler–Heine type formula for these polynomials.

Indeed, the study of orthogonal polynomials with respect to the inner product involving differences instead of derivatives

\[
\langle p, q \rangle_{\lambda} = \int_{\mathbb{R}} p(x)q(x) \, d\psi(x) + \lambda \Delta p(c) \Delta q(c),
\]

where \(\lambda \in \mathbb{R}^+, c \in \mathbb{R}\) and \(\psi\) is a distribution function with infinite spectrum, was introduced by H. Bavinck in \([11, 12]\). Moreover, in these works Bavinck obtained algebraic properties and some results connected to the location of the zeros of the orthogonal polynomials with respect to the inner product \([2]\). On the other hand, in \([12]\) he proved that the orthogonal polynomials with respect to inner product defined in equation \([2]\) satisfy a five term recurrence relation. Furthermore, in \([13]\) he considered the inner product

\[
\langle p, q \rangle = (1 - \mu)^\gamma \sum_{x \geq 0} p(x)q(x) \frac{\mu^\gamma \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(x + 1)} + \lambda p(0)q(0),
\]

where \(\gamma > 0\), \(0 < \mu < 1\), \(\lambda > 0\) and \(\mathbb{P}\) denote the linear space of all polynomials with real coefficients. Here, he obtained a second order difference equation with polynomial coefficients, which the orthogonal polynomials with respect to \([3]\) satisfy. Then, in \([14]\) he showed that the Sobolev type Meixner polynomials orthogonal with respect to the inner product

\[
\langle p, q \rangle = (1 - \mu)^\gamma \sum_{x \geq 0} p(x)q(x) \frac{\mu^\gamma \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(x + 1)} + M p(0)q(0) + N \Delta p(0) \Delta q(0),
\]

where \(\gamma > 0\), \(0 < \mu < 1\) and \(M, N \geq 0\), are eigenfunctions of a difference operator. Other results connected with the Sobolev type Meixner polynomials can be found in \([7, 8, 9, 22, 26]\).

In this contribution we will focus our attention on the sequence \(\{Q_n^{\alpha, \lambda}\}_{n \geq 0}\) of monic orthogonal polynomials with respect to the following inner product on \(\mathbb{P}\) involving differences of higher order

\[
\langle p, q \rangle_{\lambda} = \sum_{x \geq 0} p(x)q(x) \frac{\mu^\gamma \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(x + 1)} + \lambda \mathcal{D}_i^j p(\alpha) \mathcal{D}_j^i q(\alpha), \quad i = 1, 2,
\]

where \(\lambda \in \mathbb{R}_+, j \in \mathbb{N}, \alpha \leq 0\), \(\gamma > 0\), \(0 < \mu < 1\) and \(\mathcal{D}_i\) with \(i = 1, 2\), denotes the forward difference operator \(\mathcal{D}_1 \equiv \Delta\) and backward difference operator \(\mathcal{D}_2 \equiv \nabla\) defined by \(\mathcal{D}_1 f(x) = f(x + 1) - f(x)\) and \(\mathcal{D}_2 f(x) = f(x) - f(x - 1)\), respectively. In this work we get the connection
1 Preliminary results

Let \( \{M_n^{\gamma,\mu}\}_{n \geq 0} \) be the sequence of monic Meixner polynomials \([1,28]\), orthogonal with respect to the inner product on \( P \)

\[
\langle p, q \rangle = \sum_{x \geq 0} p(x)q(x) \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(x + 1)}, \quad \gamma > 0, \quad 0 < \mu < 1,
\]

which can be explicitly given by

\[
M_n^{\gamma,\mu}(x) = (\gamma)_n \left( \frac{\mu}{\mu - 1} \right)^n \ _2F_1 \left( \begin{array}{c} -n, -x \\ \gamma \end{array} \right| 1 - \mu^{-1} \right), \tag{5}
\]

where \( \gamma > 0 \) and \( 0 < \mu < 1 \), please refer to \([1,28,20]\). Here, \(_rF_s\) denotes the ordinary hypergeometric series defined by

\[
_rF_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right| x \right) = \sum_{k \geq 0} (a_1, \ldots, a_r)_k x^k \left( b_1, \ldots, b_s \right)_k k!, \tag{6}
\]

where

\[
(a_1, \ldots, a_r)_k := \prod_{1 \leq i \leq r} (a_i)_k,
\]

and \((\cdot)_n\) denotes the Pochhammer symbol \([10,17]\), also called the shifted factorial, defined by

\[
(x)_n = \prod_{0 \leq j \leq n-1} (x + j), \quad n \geq 1, \quad (x)_0 = 1.
\]

Moreover, \( \{a_i\}_{i=1}^r \) and \( \{b_j\}_{j=1}^s \) are complex numbers subject to the condition that \( b_j \neq -n \) with \( n \in \mathbb{N} \setminus \{0\} \) for \( j = 1, 2, \ldots, s \).
Theorem 1 [6, p. 62] The series $\sum$ converges absolutely of all $x$ if $r \leq s$ and for $|x| < 1$ if $r = s + 1$, and it diverges for all $x \neq 0$ if $r > s + 1$ and the series does not terminate.

Next, we summarize some basic properties of Meixner orthogonal polynomials to be used in the sequel.

Proposition 1 Let $\{M_n^{\gamma,\mu}\}_{n \geq 0}$ be the classical Meixner sequence monic orthogonal polynomials. The following statements hold.

1. Three term recurrence relation
   \[ xM_n^{\gamma,\mu}(x) = M_{n+1}^{\gamma,\mu}(x) + \alpha_n^{\gamma,\mu} M_n^{\gamma,\mu}(x) + \beta_n^{\gamma,\mu} M_{n-1}^{\gamma,\mu}(x), \quad n \geq 0, \]  
   where
   \[ \alpha_n^{\gamma,\mu} = \frac{n(1+\mu) + \mu \gamma}{1-\mu}, \quad \beta_n^{\gamma,\mu} = \frac{n \mu (n+\gamma-1)}{(1-\mu)^2}, \]  
   with initial conditions $M_{-1}^{\gamma,\mu}(x) = 0$, and $M_0^{\gamma,\mu}(x) = 1$.

2. Structure relations. For every $n \in \mathbb{N}$,
   \[ (x + \gamma \delta_{i,1}) D_i M_n^{\gamma,\mu}(x) = n M_n^{\gamma,\mu}(x) + \frac{n \mu \delta_{i,2} (n+\gamma-1)}{1-\mu} M_{n-1}^{\gamma,\mu}(x), \]  
   where $i = 1, 2$, and $\delta_{i,j}$ denotes the Kronecker delta function.

3. Squared norm. For every $n \in \mathbb{N}$,
   \[ \|M_n^{\gamma,\mu}\|_2^2 = \frac{n! \binom{\gamma}{n} \mu^n}{(1-\mu)^{\gamma+2n}}. \]  

4. Orthogonality relation. For $a < 0$
   \[ \sum_{x \geq 0} M_n^{\gamma,\mu}(x) M_m^{\gamma,\mu}(x) \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(\gamma) \Gamma(x+1)} = \|M_n^{\gamma,\mu}\|_2^2 \delta_{m,n}, \]  
   where by $\delta_{i,j}$ we denote the Kronecker delta function.

5. Value in the initial extreme of the orthogonality interval,
   \[ M_n^{\gamma,\mu}(0) = \binom{\gamma}{n} \left( \frac{\mu}{\mu - 1} \right)^n. \]  

6. Forward and backward shift operators.
   \[ D_i^k M_n^{\gamma,\mu}(x) = \left[ n \right]_k M_{n-k}^{\gamma+k,\mu}(x - \delta_{i,2}k), \quad i = 1, 2. \]
   Here, $\left[ \cdot \right]_n$ denotes the Falling Factorial [11, p. 6], defined by
   \[ \left[ x \right]_n = (-1)^n (-x)_n, \quad n \geq 1, \quad \left[ x \right]_0 = 1. \]
7. Mehler–Heine type formula \([15, \text{eq. 35}]\)

$$
\lim_{n \to \infty} \frac{(\mu - 1)^n M_{n}^{\gamma, \mu} (x)}{\Gamma (n - x)} = \frac{1}{(1 - \mu)^{1/2} \Gamma (-x)}. \quad (12)
$$

Furthermore, we denote the \(n\)-th reproducing kernel by

$$
K_n(x, y) = \sum_{0 \leq k \leq n} \frac{M_k^{\gamma, \mu} (x) M_k^{\gamma, \mu} (y)}{\| M_k^{\gamma, \mu} \|_2^2}. \quad (13)
$$

Then, for all \(n \in \mathbb{N}\),

$$
K_n(x, y) = \frac{1}{\| M_n^{\gamma, \mu} \|_2^2} \frac{M_{n+1}^{\gamma, \mu} (x) M_{n+1}^{\gamma, \mu} (y) - M_n^{\gamma, \mu} (x) M_n^{\gamma, \mu} (y)}{x - y}. \quad (14)
$$

Provided \(D_k f(x) = D_i \left( D_k^{-1} f(x) \right)\) with \(i = 1, 2\), for the partial finite difference of \(K_n(x, y)\) we will use the following notation

$$
K^{(i,j)}_n(x, y) = D_{i,x} \left( D_{j,y} \left( K_n(x, y) \right) \right) = \sum_{0 \leq k \leq n} \frac{D_{j,y} M_k^{\gamma, \mu} (x) D_{i,x} M_k^{\gamma, \mu} (y)}{\| M_k^{\gamma, \mu} \|_2^2}. \quad (15)
$$

For abbreviation we denote

$$
\langle x \rangle_n^i = \begin{cases} [x]_n, & i = 1, \\ (x)_n, & i = 2. \end{cases}
$$

**Proposition 2** Let \(\{M_n^{\gamma, \mu}\}_{n \geq 0}\) be the sequence of monic Meixner orthogonal polynomials. Then, the following statement holds, for all \(n \in \mathbb{N}\),

$$
K^{(i,j)}_{n-1}(x, y) = \frac{j^j}{\| M_{n-1}^{\gamma, \mu} \|_2^2} \langle x - y \rangle_i^j + 1 
\times \left( M_n^{\gamma, \mu} (x) \sum_{0 \leq k \leq j} \frac{D_{k,y} M_{n-1}^{\gamma, \mu} (y)}{k!} \langle x - y \rangle_i^k - M_{n-1}^{\gamma, \mu} (x) \sum_{0 \leq k \leq j} \frac{D_{k,y} M_n^{\gamma, \mu} (y)}{k!} \langle x - y \rangle_i^k \right), \quad i = 1, 2. \quad (16)
$$

**Proof.** In fact, applying the \(j\)-th finite difference to (15) with respect to \(y\) we obtain

$$
K^{(0,j)}_{n-1}(x, y) = \frac{1}{\| M_{n-1}^{\gamma, \mu} \|_2^2} \left[ M_n^{\gamma, \mu} (x) D_{i,y} \left( \frac{M_{n-1}^{\gamma, \mu} (y)}{x - y} \right) - M_{n-1}^{\gamma, \mu} (x) D_{i,y} \left( \frac{M_n^{\gamma, \mu} (y)}{x - y} \right) \right]. \quad (17)
$$

Using a analogue of the Leibnitz’s rule \([11, 13]\)

$$
D_i^n [f(x) g(x)] = \sum_{0 \leq k \leq n} \binom{n}{k} D_i^k [f(x)] D_i^{n-k} [g(x \pm k)], \quad i = 1, 2. \quad (18)
$$
since

\[ x \leq k \leq n \] we deduce

\[ \frac{\langle x \rangle^n_i}{\langle x \rangle^k_i} = \langle x + k \rangle^{n-k}_i \quad \text{if} \quad n \geq k, \]

we deduce

\[ \langle x - y \rangle^{j+1}_i = \frac{\langle x - y \rangle^j_i}{\langle x \rangle^k_i}. \]

Thus,

\[ \mathcal{D}_i^j \left( \frac{M_{n-1}^{\gamma,\mu} (x)}{x-y} \right) = \frac{j!}{\langle x - y \rangle^j_i} \sum_{0 \leq k \leq j} \frac{\mathcal{D}_i^j M_{n-1}^{\gamma,\mu} (y)}{k!} (x-y)^k_i. \]

Therefore, from the above and \[17\] we get \[16\]. Evidently

\[ \mathcal{H}^{(0,j)}_{n-1} (x,y) = \frac{j!}{\|M_{n-1}^{\gamma,\mu}\|^2} \left[ M_{n-1}^{\gamma,\mu} (x) \mathcal{M}_j (x,y,M_{n-1}^{\gamma,\mu}) - M_{n-1}^{\gamma,\mu} (x) \mathcal{M}_j (x,y,M_{n-1}^{\gamma,\mu}) \right], \]

where \( \mathcal{M}_j (x,y,M_{n-1}^{\gamma,\mu}) \) and \( \mathcal{M}_j (x,y,M_{n-1}^{\gamma,\mu}) \) denote the Taylor polynomials of degree \( j \) \[5, 11\], around the point \( x = y \), of the polynomials \( M_{n-1}^{\gamma,\mu} (x) \) and \( M_{n-1}^{\gamma,\mu} (x) \), respectively. \( \blacksquare \)

**Proposition 3** Let \( \{M_{n}^{\gamma,\mu}\}_{n \geq 0} \) be the sequence of monic Meixner orthogonal polynomials. Then, the following statement holds, for all \( n \in \mathbb{N} \),

\[ \mathcal{H}^{(j,j)}_{n-1} (0,0) = \frac{j!}{\mu^j (\gamma)_j} \sum_{0 \leq k \leq n-j-1} (j+1)_k (\gamma+j)_k \frac{\mu^k}{(1)_k k!}. \]

**Proof.** In fact, having \[10\] and \[13\] into account, as well as

\[ \frac{\langle x \rangle^n_i}{\langle x \rangle^m_m} = (x + m)_{n-m}, \quad \text{if} \quad n \geq m, \quad (19) \]

we deduce

\[ \mathcal{H}^{(j,j)}_{n-1} (0,0) = \frac{(1 - \mu)^{\gamma+2j}}{\mu^{2j} (\gamma)_j^2} \sum_{1 \leq k \leq n-1} [k]^2_j (\gamma)_k \frac{\mu^k}{k!} \]

\[ = \frac{(1 - \mu)^{\gamma+2j}}{\mu^{2j-1} (\gamma)_j^2} \sum_{0 \leq k \leq n-2} [k+1]^2_j (\gamma)_{k+1} \frac{\mu^k}{(k+1)!}. \]
Thus, applying again (19) and using the identity
\[
\frac{a + k}{a} = \frac{(a + 1)_k}{(a)_k},
\] (20)
we arrived to the desired result then a straightforward by tedious verification. ■

**Corollary 1** Let \(\{M_n^{\gamma,\mu}\}_{n \geq 0}\) be the sequence of monic Meixner orthogonal polynomials. Then, the following limit
\[
\lim_{n \to \infty} \mathcal{K}_{n-1}^{(j,j)}(0,0) = j! (1 - \mu)^{\gamma + 2j} \frac{\gamma + 2j}{\mu} \Gamma(j + \gamma + 2j),
\]
holds.

## 2 Connection formula and hypergeometric representation of \(\mathcal{M}_n^{i,\lambda}(x)\)

In this section, we first express the Meixner-Sobolev type orthogonal polynomials \(\mathcal{M}_n^{i,\lambda}(x)\) in terms of the monic Meixner orthogonal polynomials \(M_n^{\gamma,\mu}(x)\) and the Kernel polynomial (16).

Taking into account the Fourier expansion we have
\[
\mathcal{M}_n^{i,\lambda}(x) = M_n^{\gamma,\mu}(x) + \sum_{0 \leq k \leq n-1} a_{n,k} M_k^{\gamma,\mu}(x).
\]

Then, from the properties of orthogonality of \(M_n^{\gamma,\mu}\) and \(\mathcal{M}_n^{i,\lambda}\) respectively, we arrived to
\[
a_{n,k} = -\frac{\lambda \mathcal{D}_i^j M_n^{i,\lambda}(\alpha) \mathcal{D}_i^j M_k^{\gamma,\mu}(\alpha)}{\|M_k^{\gamma,\mu}\|^2}, \quad 0 \leq k \leq n - 1.
\]

Thus, we get
\[
\mathcal{M}_n^{i,\lambda}(x) = M_n^{\gamma,\mu}(x) - \frac{\lambda \mathcal{D}_i^j \mathcal{M}_n^{i,\lambda}(\alpha) \mathcal{K}_{n-1}^{(j,j)}(x,\alpha)}{1 + \lambda \mathcal{K}_{n-1}^{(j,j)}(\alpha,\alpha)},
\]
After some manipulations, we deduce
\[
\mathcal{D}_i^j \mathcal{M}_n^{i,\lambda}(\alpha) = \frac{\mathcal{D}_i^j M_n^{\gamma,\mu}(\alpha)}{1 + \lambda \mathcal{K}_{n-1}^{(j,j)}(\alpha,\alpha)}.
\] (21)

Consequently, from the previous we have
\[
\mathcal{M}_n^{i,\lambda}(x) = M_n^{\gamma,\mu}(x) - \frac{\lambda \mathcal{D}_i^j M_n^{\gamma,\mu}(\alpha)}{1 + \lambda \mathcal{K}_{n-1}^{(j,j)}(\alpha,\alpha)} \mathcal{K}_{n-1}^{(0,j)}(x,\alpha), \quad i = 1, 2.
\] (22)

Next we will focus our attention in the representation of \(\mathcal{M}_n^{i,\lambda}(x)\) as hypergeometric functions. Clearly, from (16) and (22) we have
\[
\mathcal{M}_n^{i,\lambda}(x) = A_{1,n}^{(i)}(x) M_n^{\gamma,\mu}(x) + B_{1,n}^{(i)}(x) M_{n-1}^{\gamma,\mu}(x),
\] (23)
where
\[ A^{(i)}_{1,n}(x) = 1 + \frac{1}{(x - \alpha)^{j+1}} \sum_{0 \leq k \leq j} a^{(k)}_{i,n}(x - \alpha)^k, \]
and
\[ B^{(i)}_{1,n}(x) = \frac{1}{(x - \alpha)^{j+1}} \sum_{0 \leq k \leq j} b^{(k)}_{i,n}(x - \alpha)^k, \]
with
\[ a^{(k)}_{i,n} = -\frac{\lambda j! \mathcal{D}_i^j M^{\gamma,\mu}_{n-1}(\alpha) \mathcal{D}_i^k M^{\gamma,\mu}_{n-1}(\alpha)}{||M^{\gamma,\mu}_{n-1}||^2 (1 + \lambda \mathcal{X}^{(j,j)}_{n-1}(\alpha,\alpha)) k!}. \]
and
\[ b^{(k)}_{i,n} = \frac{\lambda j! \mathcal{D}_i^j M^{\gamma,\mu}_{n-1}(\alpha) \mathcal{D}_i^k M^{\gamma,\mu}_{n-1}(\alpha)}{||M^{\gamma,\mu}_{n-1}||^2 (1 + \lambda \mathcal{X}^{(j,j)}_{n-1}(\alpha,\alpha)) k!}. \]

**Theorem 2** The monic Meixner-Sobolev orthogonal polynomials \( \mathcal{M}^{i,\lambda}_n(x) \) have the following hypergeometric representation for \( i = 1, 2, \)
\[ \mathcal{M}^{i,\lambda}_n(x) = (\gamma)_{n-1} \left(\frac{\mu}{\mu - 1}\right)^{n-1} H^{(i)}_n(x) \binom{\alpha}{\gamma} \binom{-n, -x, f^{(i)}_n(x)}{\gamma, f^{(i)}_n(x) - 1} 1 - \mu^{-1}, \]
where \( f^{(i)}_n(x) \) is given in (24) and
\[ h^{(i)}_n(x) = -\left(\frac{\mu (\gamma + n - 1)}{1 - \mu} A^{(i)}_{1,n}(x) - B^{(i)}_{1,n}(x)\right). \]

**Proof.** In fact, having into account
\[ (-x)_k = 0, \quad \text{if} \quad x < k, \]
as well as (24) and (23) we deduce
\[ \mathcal{M}^{i,\lambda}_n(x) = (\gamma)_n \left(\frac{\mu}{\mu - 1}\right)^n A^{(i)}_{1,n}(x) \sum_{0 \leq k \leq n} \frac{(-n)_k (-x)_k (1 - \mu^{-1})^k}{(\gamma)_k k!} \]
\[ + (\gamma)_{n-1} \left(\frac{\mu}{\mu - 1}\right)^{n-1} B^{(i)}_{1,n}(x) \sum_{0 \leq k \leq n-1} \frac{(1 - n)_k (-x)_k (1 - \mu^{-1})^k}{(\gamma)_k k!}. \]
Then, using the identity
\[ \frac{a + k}{a} = \frac{(a + 1)_k}{(a)_k}, \]
we get
\[ \mathcal{M}^{i,\lambda}_n(x) = (\gamma)_n \left(\frac{\mu}{\mu - 1}\right)^n A^{(i)}_{1,n}(x) \sum_{0 \leq k \leq n} \frac{(-n)_k (-x)_k (1 - \mu^{-1})^k}{(\gamma)_k k!} \]
\[ + (\gamma)_{n-1} \left(\frac{\mu}{\mu - 1}\right)^{n-1} B^{(i)}_{1,n}(x) \sum_{0 \leq k \leq n} \frac{(n - k)_k (-x)_k (1 - \mu^{-1})^k}{(\gamma)_k k!}. \]
Thus, we have
\[ \mathcal{M}_{n}^{i,\lambda}(x) = (\gamma)_{n-1} \left( \frac{\mu}{\mu - 1} \right)^{n-1} \sum_{0 \leq k \leq n} g_{n}^{(i)}(x) \frac{(-n)_{k}(-x)_{k}(1 - \mu^{-1})^{k}}{(\gamma)_{k} k!}, \]
where
\[ g_{n}^{(i)}(x) = - \frac{B_{1,n}^{(i)}(x)}{n} \left( f_{n}^{(i)}(x) + k - 1 \right), \]
with
\[ f_{n}^{(i)}(x) = \frac{n(\gamma + n - 1)(1 - \mu)^{-1} A_{1,n}^{(i)}(x) - n + 1}{B_{1,n}^{(i)}(x)} \tag{29} \]
A trivial verification shows that
\[ g_{n}^{(i)}(x) = - \frac{B_{1,n}^{(i)}(x)}{n} \left( f_{n}^{(i)}(x) - 1 \right) \frac{\left( f_{n}^{(i)}(x) \right)_{k}}{\left( f_{n}^{(i)}(x) - 1 \right)_{k}} \]
\[ = - \left( \mu (\gamma + n - 1)(1 - \mu)^{-1} A_{1,n}^{(i)}(x) - B_{1,n}^{(i)}(x) \right) \frac{\left( f_{n}^{(i)}(x) \right)_{k}}{\left( f_{n}^{(i)}(x) - 1 \right)_{k}}. \]
Therefore
\[ \mathcal{M}_{n}^{i,\lambda}(x) = - \left( \mu (\gamma + n - 1)(1 - \mu)^{-1} A_{1,n}^{(i)}(x) - B_{1,n}^{(i)}(x) \right) \times (\gamma)_{n-1} \left( \frac{\mu}{\mu - 1} \right)^{n-1} \sum_{0 \leq k \leq n} \frac{(-n)_{k}(-x)_{k}(1 - \mu^{-1})^{k}}{(\gamma)_{k} k!}, \]
which coincides with (28). This completes the proof. 

3 Linear difference equation of second order

In this section, we will obtain a second order linear difference equation that the sequence of monic Meixner-Sobolev type orthogonal polynomials \( \{\mathcal{M}_{n}^{i,\lambda}\}_{n \geq 0} \) satisfies. In order to do that, we will find the ladder (creation and annihilation) operators, using the connection formula (23), the three term recurrence relation (7) satisfied by \( \{\mathcal{M}_{n}^{i,\mu}\}_{n \geq 0} \) and the structure relation (8).

From (23) and recurrence relation (7) we deduce the following result
\[ \mathcal{M}_{n-1}^{i,\lambda}(x) = A_{2,n}^{(i)}(x) M_{n}^{i,\mu}(x) + B_{2,n}^{(i)}(x) M_{n-1}^{i,\mu}(x), \tag{30} \]
where
\[ A_{2,n}^{(i)}(x) = - \frac{B_{1,n-1}^{(i)}(x)}{\beta_{n-1}^{i,\mu}}, \]
and
\[ B_{2,n}^{(i)}(x) = A_{1,n-1}^{(i)}(x) + A_{2,n}^{(i)}(x) (\alpha_{n-1}^{i,\mu} - x). \]
Applying the $\mathcal{D}_i$ operator to (31) and using (18) we have

$$\mathcal{D}_i M_{n}^{i,\lambda} (x) = M_{n}^{\gamma,\mu} (x) \mathcal{D}_i A_{1,n}^{(i)} (x) + A_{1,n}^{(i)} (x \pm 1) \mathcal{D}_i M_{n}^{\gamma,\mu} (x)$$

$$M_{n-1}^{\gamma,\mu} (x) \mathcal{D}_i B_{1,n}^{(i)} (x) + B_{1,n}^{(i)} (x \pm 1) \mathcal{D}_i M_{n-1}^{\gamma,\mu} (x).$$

Then, multiplying the previous expression by $(x + \gamma \delta_{i,1})$ and using the structure relation (8) as well as the recurrence relation (7) we deduce

$$(x + \gamma \delta_{i,1}) \mathcal{D}_i M_{n}^{i,\lambda} (x) = C_{1,n}^{(i)} (x) M_{n}^{\gamma,\mu} (x) + D_{1,n}^{(i)} (x) M_{n-1}^{\gamma,\mu} (x),$$

and

$$(x + \gamma \delta_{i,1}) \mathcal{D}_i M_{n-1}^{i,\lambda} (x) = C_{2,n}^{(i)} (x) M_{n}^{\gamma,\mu} (x) + D_{2,n}^{(i)} (x) M_{n-1}^{\gamma,\mu} (x),$$

respectively, where

$$C_{1,n}^{(i)} (x) = (x + \gamma \delta_{i,1}) \mathcal{D}_i A_{1,n}^{(i)} (x) + n A_{1,n}^{(i)} (x \pm 1) - \frac{(n - 1)(n + \gamma - 2) \mu \delta_{i,2} B_{1,n}^{(i)} (x \pm 1)}{\beta_{n-1}^{\gamma,\mu} (1 - \mu)},$$

$$D_{1,n}^{(i)} (x) = (x + \gamma \delta_{i,1}) \mathcal{D}_i B_{1,n}^{(i)} (x) + (n - 1) B_{1,n}^{(i)} (x \pm 1) + \frac{(n - 1)(n + \gamma - 2) (x - \alpha_{n-1}^{\gamma,\mu}) \mu \delta_{i,2} B_{1,n}^{(i)} (x \pm 1) + n (n + \gamma - 1) \mu \delta_{i,2} A_{1,n}^{(i)} (x \pm 1)}{\beta_{n-1}^{\gamma,\mu} (1 - \mu)},$$

and

$$C_{2,n}^{(i)} (x) = - \frac{D_{1,n-1}^{(i)} (x)}{\beta_{n-1}^{\gamma,\mu}},$$

$$D_{2,n}^{(i)} (x) = C_{1,n-1}^{(i)} (x) + C_{2,n}^{(i)} (x) (\alpha_{n-1}^{\gamma,\mu} - x).$$

Moreover, from (31)-(30) we have

$$M_{n}^{\gamma,\mu} (x) = \frac{B_{2,n}^{(i)} (x) \mathcal{M}_{n}^{i,\lambda} (x) - B_{1,n}^{(i)} (x) \mathcal{M}_{n-1}^{i,\lambda} (x)}{\Theta_{n} (x; i)},$$

and

$$M_{n-1}^{\gamma,\mu} (x) = \frac{A_{1,n}^{(i)} (x) \mathcal{M}_{n-1}^{i,\lambda} (x) - A_{2,n}^{(i)} (x) \mathcal{M}_{n}^{i,\lambda} (x)}{\Theta_{n} (x; i)},$$

where

$$\Theta_{n} (x; i) = \det \begin{pmatrix} A_{1,n}^{(i)} (x) & B_{1,n}^{(i)} (x) \\ A_{2,n}^{(i)} (x) & B_{2,n}^{(i)} (x) \end{pmatrix}. $$

Thus, replacing the above in (31)-(32) we conclude

$$\tilde{\Theta}_{n} (x; i) \mathcal{D}_i M_{n}^{i,\lambda} (x) + \Lambda_{1,n}^{(i)} (x; i) \mathcal{M}_{n}^{i,\lambda} (x) = \Lambda_{1,n}^{(i)} (x; i) \mathcal{M}_{n-1}^{i,\lambda} (x),$$

and

$$\tilde{\Theta}_{n} (x; i) \mathcal{D}_i M_{n-1}^{i,\lambda} (x) + \Lambda_{1,n}^{(i)} (x; i) \mathcal{M}_{n-1}^{i,\lambda} (x) = \Lambda_{2,n}^{(i)} (x; i) \mathcal{M}_{n}^{i,\lambda} (x),$$
respectively, where
\[ \tilde{\Theta}_n (x; i) = (x + \gamma \delta_{i1}) \Theta_n (x; i), \]
and
\[ \Lambda_j^{(k)} (x; i) = (-1)^k \det \begin{pmatrix} C_{j,n}^{(i)} (x) & A_j^{(i)} (x) \\ D_{j,n}^{(i)} (x) & B_j^{(i)} (x) \end{pmatrix}, \quad j = 1, 2, \quad k = 1, 2. \] (34)

**Proposition 4** Let \( \{ \mathcal{M}_{n}^{\lambda} \}_{n \geq 0} \) be the sequence of monic Meixner-Sobolev orthogonal polynomials defined by (28) and let \( I \) be the identity operator. Then, the ladder (destruction and creation) operators \( a, a^\dagger \) are defined by
\[ a = \tilde{\Theta}_n (x; i) \partial_i + \Lambda_{2,n}^{(1)} (x; i) I, \]
\[ a^\dagger = \tilde{\Theta}_n (x; i) \partial_i + \Lambda_{1,n}^{(2)} (x; i) I, \]
which verify
\[ a \left( \mathcal{M}_{n}^{\lambda} (x) \right) = \Lambda_{1,n}^{(1)} (x; i) \mathcal{M}_{n-1}^{\lambda} (x), \]
\[ a^\dagger \left( \mathcal{M}_{n-1}^{\lambda} (x) \right) = \Lambda_{2,n}^{(2)} (x; i) \mathcal{M}_{n}^{\lambda} (x), \] (35)
where \( \tilde{\Theta}_n (x; i) \) and \( \Lambda_j^{(k)} (x; i) \) with \( i, j, k = 1, 2 \) are given in [23, 24].

**Theorem 3** Let \( \{ \mathcal{M}_{n}^{\lambda} \}_{n \geq 0} \) be the sequence of monic polynomials orthogonal with respect to the inner product (4). Then, the following statement holds. For all \( n \geq 0 \)
\[ F_n (x; i) \partial_i^2 \mathcal{M}_{n}^{\lambda} (x) + G_n (x; i) \partial_i \mathcal{M}_{n}^{\lambda} (x) + H_n (x; i) \mathcal{M}_{n}^{\lambda} (x) (x) = 0, \] (36)
where
\[ F_n (x; i) = \frac{\tilde{\Theta}_n (x; i) \tilde{\Theta}_n (x; i + 1; i)}{\Lambda_{1,n}^{(1)} (x \pm 1; i)}, \]
\[ G_n (x; i) = \frac{\tilde{\Theta}_n (x; i) \partial_i \tilde{\Theta}_n (x; i) - \tilde{\Theta}_n (x; i) \partial_i \Lambda_{1,n}^{(1)} (x; i)}{\Lambda_{1,n}^{(1)} (x \pm 1; i)} \]
\[ \quad + \frac{\tilde{\Theta}_n (x; i) \Lambda_{2,n}^{(1)} (x \pm 1; i)}{\Lambda_{1,n}^{(1)} (x \pm 1; i)} = \frac{\tilde{\Theta}_n (x; i) \Lambda_{2,n}^{(1)} (x; i)}{\Lambda_{1,n}^{(1)} (x; i)} + \frac{\tilde{\Theta}_n (x; i) \Lambda_{1,n}^{(2)} (x; i)}{\Lambda_{1,n}^{(1)} (x; i)}. \]

and
\[ H_n (x; i) = \frac{\tilde{\Theta}_n (x; i) \partial_i \Lambda_{2,n}^{(1)} (x; i)}{\Lambda_{1,n}^{(1)} (x \pm 1; i)} - \frac{\tilde{\Theta}_n (x; i) \Lambda_{2,n}^{(1)} (x; i) \partial_i \Lambda_{1,n}^{(1)} (x; i)}{\Lambda_{1,n}^{(1)} (x \pm 1; i)} \]
\[ \quad + \frac{\Lambda_{2,n}^{(2)} (x; i) \Lambda_{1,n}^{(1)} (x; i)}{\Lambda_{1,n}^{(1)} (x; i)} - \Lambda_{2,n}^{(2)} (x; i). \]
where \( \tilde{\Theta}_n (x; i) \) and \( \Lambda_j^{(k)} (x; i) \) with \( i, j, k = 1, 2 \) are given in [23, 24].
On the other hand, we have

\[
\frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) = \mathcal{M}_{n-1}^{i,\lambda}(x).
\]

Thus, applying the operator \(\mathcal{D}_{i}\) to both members of the previous expression, we get

\[
\mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) \right] = \Lambda_{2,n}^{(2)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x).
\]

Proof. From (35) we have

\[
\frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) = \mathcal{M}_{n-1}^{i,\lambda}(x).
\]

Next, applying the operator \(\mathcal{D}_{i}\) to both members of the previous expression, we get

\[
\mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) \right] = \Lambda_{2,n}^{(2)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x).
\]

Thus

\[
\tilde{\Theta}_{n}(x; i) \mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) \right] = \Lambda_{2,n}^{(2)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x).
\]

On the other hand, we have

\[
\mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) \right] =
\mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \left( \tilde{\Theta}_{n}(x; i) \mathcal{D}_{i}\mathcal{M}_{n}^{i,\lambda}(x) + \Lambda_{n}^{(1)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x) \right) \right]
= \mathcal{D}_{i}\tilde{\Theta}_{n}(x; i) \mathcal{D}_{i}\mathcal{M}_{n}^{i,\lambda}(x) + \mathcal{D}_{i}\frac{\Lambda_{n}^{(1)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x)}{\Lambda_{1,n}^{(1)}(x; i)}.
\]

Then, using

\[
\mathcal{D}_{i}\left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \mathcal{D}_{i} f(x) - f(x) \mathcal{D}_{i} g(x)}{g(x) g(x + 1)},
\]

we deduce

\[
\mathcal{D}_{i}\left[ \frac{1}{\Lambda_{1,n}^{(1)}(x; i)} \mathcal{M}_{n}^{i,\lambda}(x) \right] =
\mathcal{D}_{i}\tilde{\Theta}_{n}(x; i) \mathcal{D}_{i}\mathcal{M}_{n}^{i,\lambda}(x) - \tilde{\Theta}_{n}(x; i) \mathcal{D}_{i}\mathcal{M}_{n}^{i,\lambda}(x) \mathcal{D}_{i}\Lambda_{1,n}^{(1)}(x; i)
- \frac{\Lambda_{n}^{(1)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x)}{\Lambda_{1,n}^{(1)}(x; i) \Lambda_{1,n}^{(1)}(x + 1; i)} + \mathcal{D}_{i}\frac{\Lambda_{n}^{(1)}(x; i) \mathcal{M}_{n}^{i,\lambda}(x)}{\Lambda_{1,n}^{(1)}(x; i) \Lambda_{1,n}^{(1)}(x + 1; i)}.
\]
Therefore, applying (18) we conclude

\[
\frac{\Theta_n(x; i) \Theta_n(x + 1; i)}{\Lambda^{(1)}_{1,n}(x + 1; i)} \mathcal{D}^2 M^{i,\lambda}_{n}(x) + \left[ \frac{\Theta_n(x; i) \mathcal{D} \Theta_n(x; i)}{\Lambda^{(1)}_{1,n}(x + 1; i)} - \frac{\Theta_n(x; i)^2 \mathcal{D} \Lambda^{(1)}_{1,1}(x; i)}{\Lambda^{(1)}_{1,n}(x; i)} \right] \mathcal{D} M^{i,\lambda}_{n}(x)
\]

\[
+ \left[ \frac{\Theta_n(x; i)}{\Lambda^{(1)}_{1,n}(x + 1; i)} \right] \mathcal{D} \Lambda^{(1)}_{2,n}(x; i) - \frac{\Theta_n(x; i) \Lambda^{(2)}_{1,n}(x; i)}{\Lambda^{(1)}_{1,n}(x; i)} \mathcal{D} \Lambda^{(1)}_{1,n}(x; i)
\]

\[
+ \left[ \frac{\Lambda^{(2)}_{1,n}(x; i)}{\Lambda^{(1)}_{1,n}(x; i)} \mathcal{D} \Lambda^{(1)}_{2,n}(x; i) - \Lambda^{(2)}_{2,n}(x; i) \right] M^{i,\lambda}_{n}(x) = 0,
\]

which coincide with (30).

4 The \((2j + 3)\)-term recurrence relation

In this section we find the \((2j + 3)\)-term recurrence relation that the sequence of monic Meixner-Sobolev type orthogonal polynomials \((28)\) satisfies. For this purpose, we will use the remarkable fact, which is a straightforward consequence of \((4)\), that the multiplication operator by \((x - \alpha)^{j+1}_i\) is a symmetric operator with respect to such a discrete Sobolev inner product. Indeed, for any \(p, q \in \mathbb{P}\) we have

\[
\left\langle (x - \alpha)^{j+1}_i p(x) , q(x) \right\rangle = \left\langle p(x) , (x - \alpha)^{j+1}_i q(x) \right\rangle = \left\langle (x - \alpha)^{j+1}_i p(x) , q(x) \right\rangle = \left\langle p(x) , (x - \alpha)^{j+1}_i q(x) \right\rangle.
\]

(37)

Notice that, having the expressions \((23), (24)\) and \((25)\) into account, we deduce the following result.

**Lemma 1** Let \(\left\{ M^{i,\lambda}_{n} \right\}_{n \geq 0}\) be the sequence of monic Meixner-Sobolev type orthogonal polynomials defined by \((28)\). Then, the following holds.

\[
\langle x - \alpha \rangle^{j+1}_i M^{i,\lambda}_{n}(x) = A_n(x; i) M^{\gamma,\mu}_{n-1}(x) + B_n(x; i) M^{\gamma,\mu}_{n-1}(x),
\]

(38)

where

\[
A_n(x; i) = \langle x - \alpha \rangle^{j+1}_i + \sum_{0 \leq k \leq j} a^{(k)}_{i,n} \langle x - \alpha \rangle^k_i,
\]

and

\[
B_n(x; i) = \sum_{0 \leq k \leq j} b^{(k)}_{i,n} \langle x - \alpha \rangle^k_i,
\]

with \(a^{(k)}_{i,n}\) and \(b^{(k)}_{i,n}\) given in \((29)\) and \((27)\), respectively.
Theorem 4 Let \( \{ M_n^{i,\lambda} \}_{n \geq 0} \) be the sequence of monic Meixner-Sobolev type orthogonal polynomials defined by (28). Then, for every \( \lambda \in \mathbb{R}_+, j \in \mathbb{N}, \alpha \leq 0, \gamma > 0 \) and \( 0 < \mu < 1 \) the norm of these polynomials, orthogonal with respect to (4) is

\[
\| M_n^{i,\lambda} \|_\lambda^2 = \| M_n^{\gamma,\mu} \|_\lambda^2 + b_{i,n}^{(j)} \| M_n^{-1} \|_\lambda^2,
\]

where \( b_{i,n}^{(j)} \) is the polynomial coefficient defined by (27).

Proof. Clearly

\[
\| M_n^{i,\lambda} \|_\lambda^2 = \langle M_n^{i,\lambda} (x), (x - \alpha)^{j+1} \pi_{n-j-1} (x) \rangle_\lambda,
\]

for every monic polynomials \( \pi_{n-j-1} \) of degree \( n-j-1 \). From (37) we have

\[
\langle M_n^{i,\lambda} (x), (x - \alpha)^{j+1} \pi_{n-j-1} (x) \rangle_\lambda = \langle (x - \alpha)^{j+1} M_n^{i,\lambda} (x), \pi_{n-j-1} (x) \rangle_\lambda = \langle (x - \alpha)^{j+1} M_n^{i,\lambda} (x), \pi_{n-j-1} (x) \rangle_\lambda.
\]

Next we use the connection formula (38). Taking into account that \( A_n (x; i) \) is a monic polynomial of degree exactly \( j+1 \) and \( B_n (x; i) \) is a polynomial of degree exactly \( j \) with the leading coefficient \( b_{i,n}^{(j)} \) we deduce

\[
\| M_n^{i,\lambda} \|_\lambda^2 = \langle A_n (x; i) M_n^{\gamma,\mu} (x), \pi_{n-j-1} (x) \rangle + \langle B_n (x; i) M_n^{\gamma,\mu} (x), \pi_{n-j-1} (x) \rangle,
\]

which coincides with (39). □

Theorem 5 ((2j + 3)-term recurrence relation) For every \( \lambda \in \mathbb{R}_+ \), \( j \in \mathbb{N}, \alpha \leq 0, \gamma > 0 \) and \( 0 < \mu < 1 \), the monic Meixner-Sobolev type polynomials \( \{ M_n^{\lambda} \}_{n \geq 0} \), orthogonal with respect to (4) satisfy the following \((2j + 3)\)-term recurrence relation

\[
(x - \alpha)^{j+1} M_n^{i,\lambda} (x) = M_{n+j}^{i,\lambda} (x) + \sum_{n-j-1 \leq k \leq n+j} c_{n,k}^{(i)} M_k^{i,\lambda} (x),
\]

where

\[
c_{n,n+j}^{(i)} = \frac{\langle (x - \alpha)^{j+1} M_n^{\gamma,\mu} (x), M_{n+j}^{\lambda} (x) \rangle_\lambda}{\| M_{n+j}^{i,\lambda} \|_\lambda^2} + a_{i,n}^{(j)}.
\]
\[
c_{n,k}^{(i)} = \frac{\langle x - \alpha \rangle_i^{j+1} M_n^{\gamma,\mu} (x), M_k^{\lambda,\delta}(x) \rangle_{\lambda}}{\| M_k^{\lambda,\delta} \|_{\lambda}^2} + \sum_{k-n+1 \leq l \leq j} a_{i,n}^{(l)} \frac{\langle x - \alpha \rangle_i^{j} M_n^{\gamma,\mu} (x), M_k^{\lambda,\delta}(x) \rangle_{\lambda}}{\| M_k^{\lambda,\delta} \|_{\lambda}^2} + \sum_{k-n+2 \leq l \leq j} b_{i,n}^{(l)} \frac{\langle x - \alpha \rangle_i^{j} M_n^{\gamma,\mu} (x), M_k^{\lambda,\delta}(x) \rangle_{\lambda}}{\| M_k^{\lambda,\delta} \|_{\lambda}^2} + a_{i,n}^{(k-n)} + b_{i,n}^{(k-n+1)} , \quad k = n - 1, \ldots, n + j - 1 , \quad a_{i,n}^{(-1)} = 0 , \quad (41)
\]

\[
c_{n,n-j}^{(i)} = \frac{\langle M_n^{\alpha,\beta}(x), (x - \alpha)^{j+1} M_n^{\gamma,\mu}(x) \rangle_{\lambda} + a_{i,n}^{(j)} \| M_n^{\gamma,\mu} \|_{\lambda}^2}{\| M_n^{\alpha,\beta} \|_{\lambda}^2} , \quad (42)
\]

and

\[
c_{n,n-j-1}^{(i)} = \frac{\| M_n^{\alpha,\beta} \|_{\lambda}^2}{\| M_n^{\alpha,\beta} \|_{\lambda}^2} . \quad (43)
\]

**Proof.** Let consider the Fourier expansion of \( (x - \alpha)^{j+1} M_n^{\alpha,\beta}(x) \) in terms of \( \{ M_n^{\alpha,\beta} \}_{n \geq 0} \)

\[
\langle x - \alpha \rangle_i^{j+1} M_n^{\alpha,\beta}(x) = M_n^{\alpha,\beta}(x) + \sum_{0 \leq k \leq n+j} c_{n,k} M_k^{\alpha,\beta}(x) ,
\]

Thus

\[
c_{n,k}^{(i)} = \frac{\langle x - \alpha \rangle_i^{j+1} M_n^{\alpha,\beta}(x), M_k^{\alpha,\beta}(x) \rangle_{\lambda}}{\| M_k^{\alpha,\beta} \|_{\lambda}^2} , \quad k = 0, \ldots, n + j .
\]

Evidently, from the properties of orthogonality of \( M_n^{\gamma,\mu} \), we deduce that \( c_{n,k}^{(i)} = 0 \) for \( k = 0, \ldots, n - j - 2 \). In order to compute \( (40)-(43) \) it is sufficient to use \( (37) \) and \( (38)-(27) \) as well as the orthogonality conditions of \( \[4] \).

## 5 Mehler-Heine type formula

The main result of this section will be to establish Mehler–Heine type formula of \( \{ Q_n^{\lambda,\delta} \}_{n \geq 0} \) for the case where \( \alpha = 0 \). Let us see the following result.
Lemma 2 For $\gamma > 0$, $0 < \mu < 1$, $0 \leq k \leq j$ and $\alpha = 0$ the following limits

$$\lim_{n \to \infty} \frac{(\gamma)_n [n]_j [n-1]_k \mu^n}{(n-1)!} = 0,$$  \hspace{1cm} (44)

and

$$\lim_{n \to \infty} \frac{(\gamma)_n [n]_j [n]_k (n + \gamma - 1) \mu^{n+1}}{(n-1)!} = 0,$$  \hspace{1cm} (45)

hold.

Proof. In fact, firstly let us prove (44). Taking $\mu = p - 1$ with $p > 1$ and using the relation [27] eq. 5, p. 23

$$\Gamma (z) = \lim_{n \to \infty} \frac{(n-1)!}{n^z},$$

we deduce

$$\lim_{n \to \infty} \frac{(\gamma)_n [n]_j [n-1]_k \mu^n}{(n-1)!} = \frac{1}{\Gamma (\gamma)} \lim_{n \to \infty} \frac{N_{j,k} (n; \gamma)}{p^n} = \frac{1}{\Gamma (\gamma)} \lim_{n \to \infty} \frac{n^{\gamma + k + j}}{p^n}. \hspace{1cm} (46)$$

Since

$$N_{j,k} (n; \gamma) = n^\gamma [n]_j [n-1]_k = n^{\gamma + 1} (n-1)^2 \cdots (n-k)^2 (n-k-1) \cdots (n-j-1)$$

$$= n^{\gamma + 2k + j - k - 1 + 1} \left(1 - \frac{1}{n}\right)^2 \cdots \left(1 - \frac{k}{n}\right)^2 \left(1 - \frac{k + 1}{n}\right) \cdots \left(1 - \frac{j + 1}{n}\right) \sim n^{\gamma + k + j}.$$ 

Therefore, applying to (46) L’Hospital’s rule several times we obtain the desired result. In order to prove (45) one proceeds analogously.

Lemma 3 For $\gamma > 0$, $0 < \mu < 1$, $0 \leq k \leq j$ and $\alpha = 0$ the following limits

$$\lim_{n \to \infty} A_{1,n}^{(1)} (x) = 1 \quad \text{and} \quad \lim_{n \to \infty} B_{1,n}^{(1)} (x) = 0,$$  \hspace{1cm} (47)

hold.

Proof. For such purpose it is enough to check

$$\lim_{n \to \infty} a_{1,n}^{(k)} = \lim_{n \to \infty} b_{1,n}^{(k)} = 0.$$

From (3), (10) and (11) we have

$$a_{1,n}^{(k)} = \frac{\lambda_j! (1 - \mu)^{\gamma - 1}}{(\gamma)_j \left(1 + \lambda \chi_{n-1}^{(j,j)} (0,0)\right)} \frac{(\gamma)_n [n]_j [n-1]_k \mu^n}{(n-1)!} \frac{1}{(\gamma)_k} \frac{(\mu - 1)_k}{(\mu)_k} \frac{1}{k!},$$

and

$$b_{1,n}^{(k)} = \frac{\lambda_j! (1 - \mu)^{\gamma - 2}}{(\gamma)_j \left(1 + \lambda \chi_{n-1}^{(j,j)} (0,0)\right)} \frac{(\gamma)_n [n]_j [n]_k (n + \gamma - 1) \mu^{n+1}}{(n-1)!} \frac{1}{(\gamma)_k} \frac{(\mu - 1)_k}{(\mu)_k} \frac{1}{k!}.$$

Then, having into account the Theorem and the Corollary and the previous Lemma we deduce.
Theorem 6 Let be $\gamma > 0$, $0 < \mu < 1$, $0 \leq k \leq j$ and $\alpha = 0$. Then, we have

$$
\lim_{n \to \infty} \frac{(\mu - 1)^n Q_{n}^{1,\lambda}(x)}{\Gamma(n - x)} = \frac{1}{(1 - \mu)^{\gamma + x} \Gamma(-x)}, \quad (48)
$$

uniformly on compact subsets of the complex plane.

Proof. In fact, multiplying (23) by the factor $(\mu - 1)^n / \Gamma(n - x)$ we have

$$
\frac{(\mu - 1)^n Q_{n}^{1,\lambda}(x)}{\Gamma(n - x)} = A_{1,n}^{(1)}(x) \frac{(\mu - 1)^n M_{n}^{\gamma,\mu}(x)}{\Gamma(n - x)} + B_{1,n}^{(1)}(x) \frac{(\mu - 1)^n M_{n-1}^{\gamma,\mu}(x)}{\Gamma(n - x)}.
$$

Then, applying the previous Lemma as well as the (12) we arrived to the desired result.

Finally, we show some graphical experiments of the limit function in (48) for several values of $n$ using Mathematica software, see Figures 1–4.
Figure 3: Limit function in (48) for $n = 100$, (red color) left member and (green color) right member. Data: $\gamma = 7$, $\mu = 1/5$, $\lambda = 10^{-21}$ and $j = 177$.

Figure 4: Limit function in (48) for $n = 150$, (red color) left member and (green color) right member. Data: $\gamma = 7$, $\mu = 1/5$, $\lambda = 10^{-21}$ and $j = 177$.

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