The planar Tree Lagrange Inversion Formula

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Introduction
A planar tree power series over a field $K$ is a formal expression

$$\sum c_T \cdot T$$

where the sum is extended over all isomorphism classes of finite planar reduced rooted trees $T$ and where the coefficients $c_T$ are in $K$. Multiplications of these power series is induced by planar grafting of trees and turns the $K$-vectorspace $K\{x\}_\infty$ of those power series into an algebra, see [G].

If $f \in K\{x\}_\infty$ there is a unique $g(x) \in K\{x\}_\infty$ of order $> 0$ such that

$$g(x) = x \cdot f(g(x))$$

where $f(g(x))$ is obtained by substituting $g(x)$ for $x$ in $f(x)$. Formulas for the coefficients of $g$ in terms of the coefficients of $f$ are obtained by the use of the planar tree Lukaciewicz language. This result generalizes the classical Lagrange inversion formula, see [C],[R],[Sch].

1 Lukaciewicz languages
Let $Y$ be a $\mathbb{N}$-graded set. It is given as a disjoint union of subsets $Y_i$ for $i \in \mathbb{N}$. Assume that $Y_0$ is not empty.

Let $\mathbb{W}(Y)$ be the monoid of words over $Y$ and let

$$\delta: \mathbb{W}(Y) \rightarrow \mathbb{N}$$

be the monoid morphism from $\mathbb{W}(Y)$ into additive monoid $\mathbb{N}$ of natural numbers for which

$$\delta(y) = i - 1$$

whenever $y \in Y_i$. 

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Definition 1.1. \( \text{Luk}(Y) \) is the set of all words \( w \in \mathbb{W}(Y) \) such that \( \delta(w) = -1 \) and such that if \( w' \) is a proper left factor of \( w \), then \( \delta(w') \geq 0 \).

We call \( \text{Luk}(Y) \) the Lukaciewicz language over the graded set \( Y \).

In case \( \#Y_i = 1 \) for all \( i \), then \( \text{Luk}(Y) \) is the classical Lukaciewicz language, for instance used in [C], (11.3). We will show that the results in [C], (11.3) remain correct in the more general situation.

Let \( M(Y) \) be the submonoid of \( \mathbb{W}(Y) \) generated by \( \text{Luk}(Y) \).

Proposition 1.2. (i) \( M(Y) \) is a freely generated by \( \text{Luk}(Y) \)

(ii) Let \( w \in \mathbb{W}(Y) \) with \( \delta(w) = -r, r \in \mathbb{N} \), then \( w \in M(Y) \) if and only if \( r > 0 \) and if \( w' \) is a proper left factor of \( w \), then \( \delta(w') > -r \).

Proof. By definition \( \text{Luk}(Y) \) has the prefix property and therefore \( \text{Luk}(Y) \) is a code which means that \( M(Y) \) is freely generated by \( \text{Luk}(Y) \). \( \square \)

Example

\[
\bigcup_{i=1}^{\infty} xi
\]

then

\[
\#Y_0 = 1
\]

\[
Y_0 \in Y_0
\]

\[
\mathbb{W}(Y+) \subseteq \text{Luk}
\]

\[
w \rightarrow wr + 1_0
\]

If \( w_1, ..., w_m \in \text{Luk}(Y) \) and \( y_m \in Y_m, m \geq 0 \), then \( v := y_m \cdot w_1 \cdot ... \cdot w_m \in \text{Luk}(Y) \)

Obviously

\[
\delta(v) = \delta(y_m) + \sum_{i=1}^{m} \delta(w_i) =
\]

\[
(m - 1) + m(-1) = -1.
\]

Let \( v' \) be a proper left factor of \( v \), say \( v' = v'' \). Then \( v' = y_m w_1 \cdot ... \cdot w_k \cdot w_{k+1}' \) with \( 0 \leq k < m \), \( w_{k+1}' \) is a proper factor of \( w_{k+1} \).

Then

\[
\delta(v') = (m - 1) + \sum_{i=1}^{k} \delta(w_{k+1}') =
\]

\[
= (m - k - 1) + \delta(w_{k+1}) \geq m - k - 1 \geq 0
\]
Proposition 1.3. Let \( w \in \text{Luk}(Y) \) and let \( w(1) \) be the first letter of \( w \). If \( w(1) \in Y_m, m \geq 1 \), then there are unique \( w_1, \ldots, w_m \in \text{Luk}(Y) \) such that

\[
w = w(1) \cdot w_1 \cdot w_2 \cdot \ldots \cdot w_m
\]

Proof. If \( m = 0 \), then \( w = w(1) \), because otherwise \( w(1) \) would be a proper left factor of \( w \) contained in \( \text{Luk}(Y) \).

Let now \( m > 0 \). Then \( \delta(w') = \delta(w) - \delta(w(1)) = (-1) - (m - 1) = -m \).

There is a left factor \( w_1' \) of \( w' \) of smallest length relative to \( Y \) such that

\[
\delta(w_1') = -1
\]

Then \( w_1' \in \text{Luk}(Y) \). In a similar way \( w_2', \ldots, w_m' \in \text{Luk}(Y) \) are determined.

Next we define the height \( ht(v) \) of an element \( v \in \text{Luk}(Y) \) by induction on the length of \( v \).

If \( v \) is a letter in \( Y_0 \), then \( ht(v) := 0 \).

Let now \( v \in \text{Luk}(Y) \) and assume that the length \( l_Y(v) = n > 1 \). Then

\[
v = v(1) \cdot v_1' \cdot \ldots \cdot v_m'
\]

if

\[
v(1) \in Y_m, \ m > 0.
\]

and \( l_Y(v_i') < n \) for all \( i \). By induction hypothesis we may assume that \( ht(v_i') \) is already defined for all \( i \).

Let \( ht(v) := 1 + \max_{i=1}^m ht(v_i') \)

Definition 1.4. \( ht(v) \) is called the height of \( v \in \text{Luk}(Y) \).

A classical result about Lukaciewicz languages is described now.

Let \( \text{Luk}(N) \) be the Lukaciewicz language relative to the decomposition of \( N \) into subsets \( N_i = \{i\} \) for \( i \in N \). For \( n \in N \) we denote by \( \lambda(n) \) the letter in \( W \in (N) \) represented by \( n \) in order that no confusion occurs between \( \lambda(n) \cdot \lambda(m) \) and \( \lambda(n \cdot m) \).

Let \( PT \) be the set of isomorphism classes of finite planar rooted trees, see [G 1] and \( PT' = PT \cup \{1\} \) where \( 1 \) is the empty tree. The map

\[
w : PT' \to \text{Luk}(N)
\]

is defined inductively with respect to the number \( \#T^0 \) of vertices of \( T \in PT' \).

If \( T = 1 \), let \( w(T) := \lambda(0) \in \text{Luk}(N) \).

If \( T = x \) is the tree in \( PT \) with a single vertex, then

\[
w(x) = \lambda(1)\lambda(0) \in \text{Luk}(N).
\]
If \( \#T^0 > 1 \) and \( \rho_T \) is the root of \( T \), then the subgraph \( T - \rho_T \) obtained by deleting the vertex \( \rho_T \) and all the edges in \( T \) incident with \( \rho_T \) is a forest. Any component \( V_i \) of \( T - \rho_T \) is a rooted tree whose root is the vertex in \( V_i \) incident with \( \rho_T \) in \( T \).

Also the components \( 1, ..., V_m, \ m \geq 1 \), are ordered and \( \#V^0 < \#T \).

We may assume that \( w(V_1), ..., w(V_m) \) are already defined. Let

\[
w(T) := \lambda(m) \cdot w(V_1) \cdot \ldots \cdot w(V_m).
\]

Clearly \( w(T) \in \text{Luk}(\mathbb{N}) \)

**Proposition 1.5.** The map \( w : \mathcal{P}T' \to \text{Luk}(\mathbb{N}) \) is bijective.

Moreover \( \text{ht}(w(T)) = \max_{b \in \mathcal{L}(T)}(\text{dist}(\rho_T, b)) \) where \( \text{dist}(\rho_T, b) \) is the distance of the root \( \rho_T \) of \( T \) to a leaf \( b \) in \( T \) which is the length of a smallest path in \( T \) connecting \( \rho_T \) and \( b \).

**Proof.** 1) Let \( S, T \in \mathcal{P}T \) and assume that \( w(S) = x(T) \).

Then

\[
ar(S) = a(T) = m
\]

If

\[
m = 0
\]

then \( S = T \) is the empty tree.

If \( m > 0 \), then

\[
T = T_1 \cdot \ldots \cdot T_m
\]

\[
S = S_1 \cdot \ldots \cdot S_m
\]

and \( \#T^0_i < \#T^0, \#S^0_i < \#S^0 \). If we proceed by induction on \( \#T^0 + \#S^0 \) we see that \( T_i = S_i \) for all \( i \). Thus \( S = T \) and the map \( w \) is proved to be injective.

2) Let now \( v \in \text{Luk}(\mathcal{P}T)' \). If \( v \in Y_0 \), then \( v = w(1) \), \( 1 \) empty tree. If the length of \( v \) is greater than 1,

\[
v = v(1) \cdot v'_1 \cdot \ldots \cdot v'_n
\]

according to Proposition 1.4. Then \( v(1) = \lambda(S), S \in \mathcal{P}T \) and

\[
w(T_i) = v'_i
\]

for

\[
T_1, ..., T_m \in \mathcal{P}T.
\]

Then \( \text{deg} \ (S) = m \) and \( w(\ast_S(T_1, ..., T_m)) = v \) where \( \ast_S(T_1, ..., T_m) \) is the planar grafting of \( T_1, ..., T_m \) over \( S \).

**Example 1.6.** \( v = \lambda(3)\lambda(0)\lambda(1)\lambda(2)\lambda(0)\lambda(1)\lambda(0) \in \text{Luk}(\mathbb{N}) \)

and \( w(T) = v \) for
Let $\mathbf{PRT}$ be the set of isomorphism classes of finite planar reduced rooted trees and $\mathbf{PRT}' = \mathbf{PRT} \cup \{1\}$ where 1 denotes the empty tree, see [G1]. Let $\deg : \mathbf{PRT}' \to \mathbb{N}$ be the degree map defined by $\deg(1) = 0$, $\deg(T) = \#L(T)$ where $L(T)$ is the set of leaves of $T$. We denote by Luk ($\mathbf{PRT}'$) the Lukaciewicz language relative to the grading by $\deg$.

In the following we will give an interpretation of the elements in Luk ($\mathbf{PRT}'$) by right-sided decompositions of trees. For any tree $T \in \mathbf{PRT}$ and any vertex $a$ of $T$, we denote by $T_a$ the closed subtree in $T$ generated by $a$.

There is a unique edge $k$ in $T$ incident with $a$ such that $T_a$ is a connected component of $T - k$ and such that the root $\rho_T$ of $T$ is the root in $T_a$. This gives another characterization of $T_a$.

Let $T \in \mathbf{PRT}$ and $x$ be the tree in $\mathbf{PRT}$ with the single vertex.

**Definition 2.1.** $T$ is called right-sided, if there is $T' \in \mathbf{PRT}'$ such that

$$T = x \cdot T'$$

The set of all right-sided trees in $\mathbf{PRT}$ is denoted by $x \cdot \mathbf{PRT}$!

Then

$$x = x \cdot 1, \ x^2 = x \cdot x, \ x \cdot x^2, \ x^3, \ x \cdot (x^2 \cdot x), \ x \cdot (x \cdot x^2) \in x \cdot \mathbf{PRT}!$$

**Definition 2.2.** $S$ is called completely right-sided in $T$, if $\deg(S) > 1$ and for any leaf $b \in L(S)$ the closed subtree $T_b$ in $T$ generated by $b$ is right-sided.

If $S$ is completely right-sided in $T$, then the forest

$$T - \text{In}(S)$$

obtained by removing all the inner vertices of $S$ from $T$ is a disjoint union of right-sided trees.

Let $S, S'$ be completely right-sided open subtrees of $T$ and $S \subseteq S'$.
Definition 2.3. \( S \) is strictly contained in \( S' \), if relative to \( T \) a leaf \( b \) of \( S \) which is also a leaf of \( S' \) is already a leaf of \( T \).

Let \( T \in x \cdot \text{PRT} \) and \( r \geq 1 \).
A right-sided open flag on \( T \) of length \( r \), is a sequence
\[
S = (S_1, S_2, ..., S_r)
\]
with:
\[
S_r = T
\]
\( S_i \) is a completely right-sided open subtree of \( T \) for all \( i < r \)
\( S_i \) is strictly contained in \( S_{i+1} \) for all \( 1 \geq i < r \).
Denote by \( \Lambda_r(T) \) the set of all proper right-sided open flags on \( T \).
Let \( \Lambda_r = \bigcup \Lambda_r(T) \) where the union is extended over all trees in \( T \) for any \( r \geq 1 \).
Let
\[
\Lambda = \bigcup_{r=0}^{\infty} \Lambda_r
\]
with
\[
\Lambda_0 = \{x\}
\]
We are going to define a map
\[
w : \Lambda \longrightarrow \text{Luk}(\text{PRT}')
\]
If \( S \in \Omega_0 \), then \( w(S) := \lambda(1) := \text{letter in } \mathbb{W}(\text{PRT})' \) associated to the empty tree.
Let now
\[
S \in \Lambda_r(T), r \geq 1, T \in x \cdot \text{PRT}
\]
If \( r = 1 \), then \( S = T \) and \( w(T) := \lambda(T') \cdot \lambda(1)^m \), if \( T = x \cdot T' \), \( \text{deg}(T') = m \)
Let now \( r \geq 2 \), \( S = (S_1, ..., S_r) \). We proceed by induction on \( \text{deg}(T) \).
Consider the planar forest
\[
T - \text{In}(S_1) = V_0 \cup V_1 \cup ... \cup V_m
\]
and denote the connected components of \( T - \text{In}(S_1) \) by \( V_0, V_1, ..., V_m \). Then \( V_0 = x \) is the leaf of the left factor of \( T \).
Let
\[
S|V_i = (S_2 \cap V_i, ..., S_r \cap V_i).
\]
Then
\[
S_r \cap V_i = V_i
\]
and
\[
S|V_i \in \Lambda_{r-1}(V_i)
\]
for all $i$. Define

$$w(S) := \lambda(S'_1) \cdot w(S|V_1)w(S|V_2) \cdot \ldots \cdot w(S|V_m)$$

where

$$S_1 = x \cdot S'_1, \ S'_1 \in \text{PRT}.$$ As $w(S,V_i) \in \text{Luk PRT}'$ and $\text{deg}(S'_1) = m$ we get $w(S) \in \text{Luk(PRT)'}$

**Theorem 2.4.** The map

$$w : \Lambda \rightarrow \text{Luk(PRT)'}$$

is bijective.

### 3 Right-sided decompositions

Let $T \in \text{PRT}$ and $S$ be a non-empty subtree of $T$. The vertex in $S$ closest to the root of $T$ is defined to be the root $\rho_S$ of $S$ which turns $S$ into a rooted tree.

**Definition 3.1.** $S$ is relatively open in $T$, if $S$ is an open subtree in the closure $T(\rho_S)$ of $\rho_S$ in $T$.

It is easy to show that $S$ is relatively open in $T$ of and only if $ar_S(a) = ar_T(a)$ for all vertices $a$ of $S$ with $ar_S(a) > 0$. There $ar_T(a)$ denotes the number of outgoing edges of the vertex $a$ in $T$.

Let $T$ be a right-sided tree in $\text{PRT}$ of degree $> 1$, $T = xT'$, and let $Q$ be a system of relatively open subtrees $U$ of $T, T = x \cdot T'$

**Definition 3.2.** $Q$ is called right-sided decomposition of $T$, if

1. Each $U \in Q$ is right-sided
2. $\bigcup_{U \in Q} U = T$
3. If $b$ is a leaf of $T$, then $\{b\} \in Q$ if and only if $b$ is different from the first leaf of $T$.
4. If $U \cap U' \neq \phi, U \neq U'$, then $U \cap U' = \{a\}$, a vertex of $T$, and if $a$ is not the root of $U$, then it is a leaf in $T$ and the root of $U'$.
Denote by \( \Gamma(T) \) the set of all right-sided decompositions of \( T \). Recall that \( \Lambda(T) \) denotes the set of all proper open flags of \( T \). For \( S = (S_1, ..., S_r) \in \Lambda(T) \) let \( Q(S) \) denote the systems of subtrees of \( T \) which contains \( S_1 \), all the connected components of \( S_i - \text{In}(S_{i-1}) \) for \( 1 < i \leq r \) and \( \{ b \in L(T) : b \neq \text{first leaf of } T \} := L(T') \), if \( T = x \cdot T' \).

Then
\[
Q(S) \in \Gamma(T)
\]

**Proposition 3.3.** The map
\[
S \mapsto Q(S)
\]

is a bijection
\[
\Lambda(T) \to \Gamma(T)
\]

for all
\[
T \in x \cdot PRT.
\]

**Example**

\[
T = x \cdot (x \cdot x^2 \cdot x \cdot x^3)
\]

\[
S = \text{subgraph of } T' \text{ vertices indexed by 1. } S \text{ is open completely right-sided subtree of } T \text{ and } T - \text{In}(S) = V_2 \cup ... \cup V_5 \text{ where } V_i = \text{subgraph of } T \text{ of vertices indexed by } i.
\]

### 4 Lagrange inversion formula for planar tree power series

Let \( K \) be a field and \( A = K\{x\}_\infty \) the K-algebra of planar tree power series in \( x \) over \( K \).

**Proposition 4.1.** Let
\[
f \in A, \text{ ord}(f) = 0
\]

Then there is a unique power series
\[
1/f \in A
\]
such that
\[ f \cdot (1/f) = 1. \]

Then \( \text{ord}(x \cdot (1/(f))) = 1 \) and there is a unique \( g \in A \) such that
\[ \text{ord}(g) = 1 \]
\[ \varphi_g(Sx1/f) = x \]

Proof. 1) The coefficients \( \gamma(T) \) of \( 1/f \) satisfy the following system of equations:
\[ c_1(f) \cdot \delta(1) = 1 \]
\[ c_x(f) \cdot \gamma(1) + c_1(f) \cdot \gamma(x) = 0 \]

If
\[ ar(T) = m \geq 2, T = T_1 \cdot T_2 \cdot ... \cdot T_m \]
Then
\[ c_T(F) \gamma(1) + c_1(f) \gamma(T) = 0 \]
if
\[ m > 2. \]

If
\[ m = 2 \]
Then
\[ c_T(f) \gamma(1) + c_1(f) \cdot \gamma(T) + c_{T_1}(f) \gamma(T_2) = 0 \]

There is a unique solution of this system of equation and \( \sum \gamma(T) \cdot T \) is equal to \( 1/f \).

2) \( \varphi_{(x \cdot 1/f)} \) is an automorphism of \( A \) as linear term of \( x \cdot 1/f \neq 0 \). Thus the inverse of \( \varphi(x \cdot 1/(f)) \) exists and is equal to \( \varphi_g \) for some \( g \in A \) Then \( \varphi_g \) for some \( g \in A \) Then
\[ \varphi_g \circ \varphi_{(x \cdot 1/f)} = \varphi_x \]
and
\[ \varphi_{g(x \cdot 1/f)} = x. \]

\[ \square \]

Proposition 4.2. Let \( f \in A \). Then there is a unique \( g \in A \) such that \( g(x) = x \cdot f(g(x)) \). It follows that \( \text{ord}(g) \geq 1 \).

Proof. 1) Let \( a(t) = c_T(f) \) Now there is a unique system \( b(T) \) of coefficients such that
\[ b(T) = 0 \text{ if } T \neq x \cdot T' \]
\[ b(x \cdot T') = \sum_{S' \in \Omega(T')} a(S')b(T' - \text{In}(S')) \]
Then
\[ g = \sum b(T) \cdot T \]
satisfies the equation of the Proposition. 2) Let \( b(T) = c_T(g) \). Then
\[ c_T(x \cdot f(g(x))) = 0 \]
if
\[ T \neq x \cdot T' \]
and
\[ c_{x \cdot T'}(x \cdot f(g(x))) = c_{T'}(f(g(x)) \]
\[ = \sum_{S' \in \Omega(T')} b(T' - \text{In}(S')) \]
\[ = \sum_{S \in \Omega(T)} a(S') b(T' - \text{In}(S)) \]

Let
\[ a(T) = c_T(f). \]

\[ \square \]

**Corollary 4.3.**
\[ g(x) = x \cdot f(g(x)) \]

**Proof.**
\[ \varphi_g(x \cdot 1/f) = g(x) \cdot (1/f)(g(x)) \]
but
\[ (1/f(g(x)) = 1/f(g(x)) \]
because
\[ \text{ord}(f(g(x)) = 0. \]

\[ \square \]

**Theorem 4.4.** Let \( f, g \) as in the Proposition above and
\[ a(T) = c_T(f), b(T) = c_T(g). \]

Then
\[ b(x) = a(1) \]
and if
\[ T \in x \cdot \text{PRT}, \ deg(T) > 1, \]
then
\[ b(T) = \sum_{Q \in \Gamma(T)} a(Q) \]
where
\[ a(Q) := \prod_{U \in Q} a(U'). \]

**Proof.** We proceed by induction on \( n = \text{deg}(T) \). If \( n = 1 \), it is clear.
If \( n > 1 \), then
\[ b(T) = \sum a(S) \cdot b(T - \text{In}(S)) \]
where the summation is extended over all proper completely right-sided open subtrees \( S \) in \( T \). By induction hypothesis
\[ b(T - \text{In}(S)) = \prod_{i=1}^{m} b(V_i) \]
if
\[ T - \text{In}(S) = V_1 \cup ... \cup V_m \]
and
\[ b(V_i) = \sum_{Q \in \Gamma(V_i)} a(Q) \]
The map \( Q(T) \to Q(V_1) \times ... \times Q(V_m) \) which maps \( S \) onto \( (S(V_1), ..., SL(V_m)) \) is a bijection. This proves the formula. \( \Box \)

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