Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials

Viresh Patel∗ Guus Regts†

July 6, 2016

Abstract

In this paper we give a deterministic polynomial-time approximation algorithm for computing complex-valued evaluations of the Tutte polynomial and the independence polynomial on bounded degree graphs, as well for computing partition functions of complex-valued spin and edge-coloring models on bounded degree graphs.

Our work builds on a recent line of work initiated by Barvinok [2, 3, 4, 5], which provides a new algorithmic approach besides the existing correlation decay method for these types of problems.

Keywords: approximation algorithms, Tutte polynomial, independence polynomial, partition function, graph homomorphism, Holant problem.

1 Introduction and results

Several counting problems such as counting the number of proper vertex colorings of a graph, or counting the number of independent sets in a graph are known to be computationally hard in the sense of being #P-hard. These problems remain hard even when one restricts to graphs of maximum degree at most three [10, 19]. Although several efficient randomized approximation algorithms exist for some of these #P-hard problems, it was considered a major breakthrough when Weitz [43] found an efficient deterministic approximation algorithm for counting independent sets in graphs of maximum degree at most five. His technique is based on ideas from statistical physics (in particular on the absence of a phase transition, which in this setting corresponds to the uniqueness of the Gibbs measure) and is often called the correlation decay method. This idea was subsequently used and refined by Bayati, Gamarnik, Katz, Nair, and Tetali [1] to give an efficient deterministic approximation algorithm for counting matchings in bounded degree graphs. It was also applied by Gamarnik and Katz [22] to the problem of counting proper vertex colorings and computing partition functions of multi-spin systems (or Markov random fields as they are called in [22]). These results were extended and improved by Lu and Yin [31]. Several other applications and refinements of this method have been obtained see e.g. [37].

The counting problems mentioned correspond to evaluating certain graph polynomials (such as the chromatic polynomial or the independence polynomial) at certain
positive values. One of the drawbacks of the correlation decay method is that it is does not seem to apply to evaluations at negative or complex numbers, nor to partition functions of complex spin models and edge-coloring models (Holant problems). Several natural counting problems can in fact be stated as the partition function of a complex valued spin- or edge-coloring model.

In this paper we take a different approach, which does allow us to consider complex evaluations of certain graph polynomials and partition functions. The approach is based on a line of work initiated by Barvinok, yielding deterministic quasi-polynomial-time approximation algorithms to evaluate the permanent of complex matrices [2, 5], the partition functions of the number of cliques in a graph [3], the partition functions of polynomials on the Boolean cube [4], the partition functions of graph homomorphisms by Barvinok and Sobe on [7, 8], and the partition functions of edge-coloring models and the Tutte polynomial by the second author [34]. The approach is roughly as follows. First the problem of evaluating the partition function or graph polynomial is cast as the evaluation of a univariate polynomial. Next, a region is identified where this polynomial does not vanish, so in this region the logarithm of the polynomial is well-approximated by a low-order Taylor approximation (logarithmic in the degree of the polynomial). Finally computing this Taylor approximation amounts to finding the first few coefficients of the polynomial, which can be done in quasi-polynomial time. This approach is in fact also inspired by statistical physics via the Lee Yang theorem [29].

The main contribution of the present paper is a faster method for computing the low-order coefficients of graph polynomials whenever we work with bounded degree graphs. This allows us to give deterministic polynomial-time algorithms (FPTAS) rather than quasi-polynomial-time approximation algorithms for evaluating the independence polynomial, the Tutte polynomial, and computing partition functions of spin and edge-coloring models in the case of bounded degree graphs.

Below we will state and discuss our main results, but first we need a definition. Since we will approximate polynomials at complex values, we define what it means to be a good approximation.

**Definition 1.1.** Let \( q \) be a complex number and let \( \epsilon > 0 \). We call a complex number \( \xi \) a multiplicative \( \epsilon \)-approximation to \( q \) if

\[
e^{-\epsilon} \leq \frac{|q|}{|\xi|} \leq e^\epsilon,
\]

and if the angle between \( \xi \) and \( t \) (as seen as vector in \( \mathbb{C} = \mathbb{R}^2 \)) is at most \( \epsilon \).

**1.1 The independence polynomial**

The independence polynomial of a graph \( G = (V, E) \) is denoted by \( Z(G) \) and is defined as

\[
Z(G)(\lambda) := \sum_{I \subseteq V} \lambda^{|I|},
\]

In [43] Weitz proved, based on the correlation decay method, that if \( 0 \leq \lambda < \lambda_c \), where

\[
\lambda_c = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^\Delta},
\]

then there exists a deterministic algorithm, which given a graph \( G = (V, E) \) of maximum degree at most \( \Delta \) and \( \epsilon > 0 \), computes a multiplicative \( \epsilon \)-approximation to \( Z(G)(\lambda) \) in time \( (|V|/\epsilon)^{O(1)} \). Sly and Sun [38] proved this is tight by showing that, as \( \lambda \) as \( \lambda > \lambda_c \), one cannot efficiently approximate \( Z(G, \lambda) \) unless NP=RP.

In Section [4] we prove the following result.
Theorem 1.1. Let $\Delta \in \mathbb{N}$ and let $\lambda \in \mathbb{C}$ be such that $|\lambda| < \lambda^*(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$. Then there exists a deterministic algorithm, which, given a graph $G = (V, E)$ of maximum degree at most $\Delta$ and $\epsilon > 0$, computes a multiplicative $\epsilon$-approximation to $Z(G)(\lambda)$ in time $(|V|/\epsilon)^{O(1)}$.

For positive valued $\lambda$ our result is weaker than Weitz’s result since $\lambda_c > \lambda^*$. However our result works for negative and even complex $\lambda$, which in fact allows us to prove approximation results for ordinary counting problems. Let us give an example. We denote by $Z_{c}(G)(\lambda)$ the polynomial defined in the same way as the independence polynomial except that in the sum (2), we only allow independent sets whose cardinality is even. Fix $\Delta$ and fix $0 < \lambda < \lambda^*$. Then it is not difficult to see that by applying the algorithm from Theorem 1.1 to $\lambda$ and to $-\lambda$ we can find a multiplicative $\epsilon$-approximation to $Z_{c}(G)(\lambda)$ in time $(|V|/\epsilon)^{O(1)}$ for any graph $G = (V, E)$ whose maximum degree is at most $\Delta$. We give the short proof in Section 4.

The value $\lambda^*$ in Theorem 1.1 originates from a paper of Scott and Sokal [35]; they showed that for graphs of maximum degree $\Delta$, the independence polynomial does not vanish at any $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq \lambda^*$. Also the value of $\lambda^*$ is tight, as there exists a sequence of trees $T_n$ of maximum degree at most $\Delta$ and $\lambda_n < -\lambda^*$ with $\lambda_n \to -\lambda^*$ such that $Z(T_n, \lambda_n) = 0$, cf. [35, Example 3.6].

It would be very interesting to find out whether the result of Weitz can be proved using the approach we take here. This would in fact follow from a confirmation of a version of a conjecture of Sokal [39] (the conjecture is stated below Question 2.4 in [39]). See Question 5.2 below for the exact version that we would need. It would also be very interesting to know if it is hard to approximate $Z(G, \lambda)$ for $\lambda < -\lambda^*$. As far as we are aware, there are no results about the complexity of approximating the independence polynomial at non-positive values of $\lambda$.

As an extension to Theorem 1.1 for the special class of claw-free graphs, we are able to efficiently approximate the independence polynomial on almost the entire complex plane. We make use of a result of Chudnovsky and Seymour [16] stating that the independence polynomial of a claw-free graph has only negative real roots. This is combined with a technique of Barvinok [5] to extend the region where a polynomial can be efficiently approximated to beyond the zero-free region. We prove the following result in Section 4.

Theorem 1.2. Let $\Delta \in \mathbb{N}$ and let $\lambda \in \mathbb{C}$ be such that $\lambda$ is not a real negative number. Then there exists a deterministic algorithm, which, given a claw-free graph $G = (V, E)$ of maximum degree at most $\Delta$ and $\epsilon > 0$, computes a multiplicative $\epsilon$-approximation to $Z(G)(\lambda)$ in time $(|V|/\epsilon)^{O(1)}$.

Note that when $G$ is equal to the linegraph of some graph $H$ we have that $Z_G(\lambda)$ is equal to the matching polynomial of $H$. So in particular, Theorem 1.2 implies the result of Bayati, Gamarnik, Katz, Nair, and Tetali [1]. Our proof of it however is entirely different from the proof in [1].

1.2 The Tutte polynomial

The random cluster formulation of the Tutte polynomial of a graph $G = (V, E)$ is a two-variable polynomial, which is denoted by $Z_T(G)$ and is defined by

$$Z_T(G)(q, w) := \sum_{A \subseteq E} q^{k(A)} w^{|A|}$$

where $k(A)$ denotes the number of components of the graph $(V, A)$. In particular, if $w = -1$, $Z_T(G)(q, -1)$ is equal to the chromatic polynomial of $G$.

Jerrum and Sinclair [28] showed that when $q = 2$ and $w > 0$ there exists a randomized polynomial-time approximation algorithm for computing evaluations of the
For all $i \in \text{multiplicative}$ $A$ this dichotomy essentially says that computing the partition function of a complex spin model by Cai, Chen and Lu \cite{12} and Grohe \cite{11}, a full dichotomy theorem has been proved for the complexity of exactly computing the partition function.

Theorem 1.3. Let $\Delta \in \mathbb{N}$ and let $w \in \mathbb{C}$. Then there exists a constant $K$ (depending on $\Delta$ and $w$) such that if $q \in \mathbb{C}$ is such that $|q| > K$, then there exists a deterministic algorithm, which, given a loopless multigraph $G = (V, E)$ of maximum degree at most $\Delta$ and $\epsilon > 0$, computes a multiplicative $\epsilon$-approximation to $Z(G)(q, w)$ in time $(|V|/\epsilon)^{O(1)}$.

Remark 1.1. The constant $K$ in the theorem above comes from a paper of Jackson, Procopiuc and Sokal \cite{26} and unfortunately takes half a page to state exactly. However, when $w$ satisfies $|1 + w| \leq 1$ (this includes the chromatic polynomial), then the constant $C$ may be taken to be $6.91\Delta$.

1.3 Partition functions of spin models

Let $A \in \mathbb{C}^{k \times k}$ be a symmetric matrix. In the context of statistical physics $A$ is often called a spin model cf. \cite{18}. For a graph $G = (V, E)$, the partition function of $A$ is defined as

$$p(G)(A) = \sum_{\phi : V \rightarrow [k]} \prod_{(u, v) \in E} A_{\phi(u), \phi(v)}. \quad (4)$$

If $A$ is the adjacency matrix of some graph $H$, then $p(G)(H)$ is equal to the number of graph homomorphisms from $G$ to $H$. In \cite{27} $p(G)(A)$ is called the graph homomorphism partition function.

Building on a line of research started by work of Dyer and Greenhill \cite{24} and Bulatov and Grohe \cite{11}, a rich dichotomy theorem has been proved for the complexity of exactly computing the partition function of a complex spin model by Cai, Chen and Lu \cite{12}. This dichotomy essentially says that computing the partition function of $A$ exactly is #P hard unless the matrix $A$ has some special structure.

Lin, Liu and Lu \cite{30} proved, using the correlation decay approach, that for fixed $\Delta \in \mathbb{N}$, if a real matrix $A$ is sufficiently close to the all ones matrix (i.e. $|A_{i, j} - 1| \leq O(1)/\Delta$ for all $i, j = 1, \ldots, k$), then there exists a $(|V(G)|/\epsilon)^{O(1)}$-time algorithm for computing a multiplicative $\epsilon$-approximation to $P(G)(A)$ on graphs of maximum degree at most $\Delta$. Barvinok and Sobéron \cite{27} showed that their exists a $(|V(G)|/\epsilon)^{O(1)\ln|V(G)|}$-time algorithm for complex valued matrices $A$ that satisfy $|A_{i, j} - 1| \leq O(1)/\Delta$ for all $i, j = 1, \ldots, k$.

Building on the work of Barvinok and Sobéron we prove in Section 6 the following result.

Theorem 1.4. Let $\Delta, k \in \mathbb{N}$. Then there exists a deterministic algorithm, which, given a graph $G = (V, E)$ of maximum degree at most $\Delta$, a symmetric $k \times k$ matrix $A$ such that $|A_{i, j} - 1| \leq 0.34/\Delta$ for all $i, j = 1, \ldots, k$, and $\epsilon > 0$, computes a multiplicative $\epsilon$-approximation to $p(G)(A)$ in time $(|V|/\epsilon)^{O(1)}$. 

4
Remark 1.2. The constant 0.34 be replaced by 0.45 if $\Delta \geq 3$, and by 0.54 if $\Delta$ is large enough, cf. [7].

In [8] Barvinok and Soberón introduced partition functions of graph homomorphisms of $G$ with multiplicities and gave a quasi-polynomial-time algorithm for computing them for certain matrices. In Section 6 we will also show that our results also apply to these partition functions.

1.4 Partition functions of edge-coloring models

Edge-coloring models originate in statistical physics and their partition functions have been introduced to the graph theory community by de la Harpe and Jones [18] (where they are called vertex models). We call any map $h : \mathbb{N}^k \rightarrow \mathbb{C}$ a $k$-color edge-coloring model.

For a graph $G = (V, E)$, the partition function of $h$ is defined by

$$p(G)(h) := \sum_{\phi : E \rightarrow [k]} \prod_{v \in V} h(\phi(\delta(v))),$$

(5)

where $\delta(v)$ denotes the set of edges incident with the vertex $v$ and $\phi(\delta(v))$ denotes the multiset of colors that the vertex $v$ ‘sees’, which we identify with its incidence vector in $\mathbb{N}^k$ so that we can apply $h$ to it.

Partition function of edge-coloring models form a rich class of graph parameters including the number of matchings (take $h : \mathbb{N}^2 \rightarrow \mathbb{C}$ defined by $h(\alpha) = 1$ if $\alpha_1 \leq 1$ and 0 otherwise), as well as partition functions of spin models, as has been proved by Szegedy [40, 41]. These partition functions can be seen as Holant problems; see e.g. [14, 15, 13]. They can also be seen as tensor network contractions. We refer to [33] for more background.

Just as for partition functions for spin models much work has been done to establish a complexity dichotomy result for exactly computing Holant problems; see [14, 15, 13]. Not much is known about the complexity of approximating partition functions of edge-coloring models except for a few special cases. As already mentioned, Bayati, Gamarnik, Katz, Nair, and Tetali [11] found an efficient approximation algorithm for counting matchings in bounded degree graphs and Lin, Liu and Lu [30] found efficient approximation algorithms for counting edge covers. Both of these algorithms are based on the correlation decay method.

Building on work of the second author [34] we will prove the following result in Section 7.

**Theorem 1.5.** Let $\Delta, k \in \mathbb{N}$. Then there exists a deterministic algorithm, which, given a multigraph $G = (V, E)$ of maximum degree at most $\Delta$, a $k$-color edge-coloring model $h$ such that $|h(\phi) - 1| \leq 0.35$ for all $\phi \in \mathbb{N}^k$, and $\epsilon > 0$, computes a multiplicative $\epsilon$-approximation to $p(G)(h)$ in time $(|V|/\epsilon)^{O(1)}$.

**Remark 1.3.** The constant 0.35 may be replaced by 0.47 if $\Delta \geq 3$ and by 0.56 if $\Delta$ is large enough; see [34]. Moreover, for readers familiar with the orthogonal group invariance of these partition functions, it is interesting to note that one can use Corollary 6b from [34] to find a much larger family of edge-coloring models for which the partition function can be efficiently approximated.

1.5 Organization

The remainder of this paper is devoted to proving the results mentioned in this introduction. In the next section we shall consider an algorithm due to Barvinok [2] to approximate evaluations of polynomials. In Section 4 we will give an algorithm for computing coefficients of certain graph polynomials. These two algorithms (or variations of them) will then be combined in Sections 6–7 to prove our main theorems. These
sections can be read independent of one an other. Finally, we conclude in Section 8 with some remarks and questions.

2 Approximating evaluations of polynomials

In this section we present an algorithm due to Barvinok [2] to approximate evaluations of polynomials. We follow the approach in [34] and give full details for the sake of completeness.

Let \( p \in \mathbb{C}[z] \) be a polynomial of degree \( d \) and suppose that \( p(z) \neq 0 \) for all \( z \) in an open disk of radius \( M \). Define the univariate function \( f \) on this disk by

\[
f(z) := \ln p(z),
\]

where we fix a branch of logarithm by fixing the principal branch of the logarithm for \( p(0) \).

To approximate \( p \) at \( t \in \mathbb{C} \) for \(|t| < M\), we will find an additive approximation to \( f \) at \( t \) using the Taylor expansion around \( z = 0 \). This can then be transformed to give a multiplicative approximation to \( p \).

Let

\[
T_m(f)(t) := f(0) + \sum_{j=1}^{m} \frac{t^j}{j!} d^j f(z) \bigg|_{z=0}.
\]

(Note that \( \frac{d^j}{dz^j} f(z) \bigg|_{z=0} \) only depends on the values of \( \frac{d^j}{dz^j} p(z) \bigg|_{z=0} \) for \( j = 0, \ldots, m \).

(Here we agree that the 0th derivative is just the function itself.) To see this, note that \( f'(z) = p'(z)/p(z) \), that is, \( p'(z) = p(z)f'(z) \). So for \( m \geq 1 \) we have

\[
\frac{d^m}{dz^m} p(z) \bigg|_{z=0} = \sum_{j=0}^{m-1} \binom{m-1}{j} \left( \frac{d^j}{dz^j} p(z) \bigg|_{z=0} \right) \left( \frac{d^{m-j}}{dz^{m-j}} f(z) \bigg|_{z=0} \right).
\]

This implies that if we can compute the values of \( \frac{d^j}{dz^j} p(z) \bigg|_{z=0} \) for \( j = 0, \ldots, m \), then [34] provides a nondegenerate (as \( p(0) \neq 0 \)) triangular system of equations to compute \( \frac{d^m}{dz^m} f(z) \bigg|_{z=0} \) for \( j = 1, \ldots, m \), which can be done in time \( O(m^2) \). To summarise:

Lemma 2.1. Let \( p(z) \in \mathbb{C}[z] \) be such that \( p(0) \neq 0 \) and let \( f(z) = \ln p(z) \). If the values \( \frac{d^j}{dz^j} p \bigg|_{z=0} \) are given for \( j = 0, \ldots, m \), then the values \( \frac{d^m}{dz^m} f \bigg|_{z=0} \) for \( j = 0, \ldots, m \) can be computed in time \( O(m^2) \).

The quality of the approximation [7] depends on the location of the complex roots of \( p \).

Lemma 2.2. Given \( M > 0 \) and \( t \in \mathbb{C} \) satisfying \(|t| < M\), there exists a constant \( C = C(t, M) \) such that the following holds. Suppose \( p \) is a polynomial of degree \( d \) with no roots in the disk \( D \). Then for every \( \varepsilon > 0 \), \( \exp(T_m(f)(t)) \) is a multiplicative \( \varepsilon \)-approximation to \( p(t) \), where \( m = C \ln(d/\varepsilon) \).

Proof. Let \( q := |t| / M \). Then, as \(|t| < M \), we have \( q < 1 \). We will first show that

\[
|f(t) - T_m(f)(t)| \leq \frac{dq^{m+1}}{(m+1)(1-q)}.
\]

The proof of [9] is quite similar to the proof of Lemma 1.2 from [7], but for completeness we will give it here.
Since \( p(0) \neq 0 \) we may write
\[
p(z) = p(0) \prod_{i=1}^{d} \left( 1 - \frac{z}{\zeta_i} \right),
\]
where \( \zeta_1, \ldots, \zeta_d \in \mathbb{C} \) are the roots of \( p \). By assumption we know that \( |\zeta_i| \geq M \) for each \( i = 1, \ldots, d \). This implies that
\[
f(z) = \ln p(z) = \ln p(0) + \sum_{i=1}^{d} \ln \left( 1 - \frac{z}{\zeta_i} \right)\]
for any \( |z| \leq |t| \). Using the standard Taylor expansion for the principal branch of the logarithm, we obtain
\[
\ln \left( 1 - \frac{t}{z} \right) = -\sum_{j=1}^{m} \frac{t^j}{j} + R_m,
\]
where \( R_m \) satisfies
\[
|R_m| = \left| \sum_{j=m+1}^{\infty} \frac{1}{j!} \frac{t^j}{\zeta_i^j} \right| \leq \frac{1}{(m+1)!} \sum_{j=m+1}^{\infty} \frac{|t|^j}{M^j} \leq \frac{q^{m+1}}{(m+1)(1-q)}.
\]
since \( q = |t|/M < 1 \). Noting that
\[
-\frac{1}{j} \sum_{j=1}^{d} \frac{t^j}{\zeta_i^j} = \frac{d}{dz} f(z) \bigg|_{z=0},
\]
(9) now follows by combining (11) with (12) and using the bound on \( |R_m| \).

Take \( m = C(\ln d/\epsilon) \), where \( C \) is chosen such that \( C \geq (\ln 1/q)^{-1} \) and \( 1/m \leq 1 - q \) (so it is easy to check that e.g. \( C \geq (1-q)^{-1} \) suffices). Then the right-hand side of (9) is at most \( \epsilon \). Write \( z = T_m(f)(t) \). Then we have \( |e^{f(t) - 2}| \leq e^{f(t) - 2} \leq \epsilon^2 \) and similarly \( |e^{z-f(t)}| \leq \epsilon^2 \). This follows from the fact that for a complex number \( y = a + bi \), we have \( |e^y| = e^a \leq e^{\epsilon^2} \). Moreover, the angle between \( e^t \) and \( e^t \) is bounded by \( |\Im e^{z-f(t)}| \leq |\ln e^{z-f(t)}| \leq \epsilon \). This shows that \( e^t = \exp(T_m(f)(t)) \) is a multiplicative \( \epsilon \)-approximation to \( p(t) \).

Lemmas 2.1 and 2.2 imply that if we have an efficient way of computing the coefficients of \( x^j \) of \( p \) from \( j = 1 \) up to \( O(\ln(\deg(p))) \), then we have an efficient way of approximating evaluations of \( p \) at points in the disk around zero where \( p \) is nonvanishing. We formalise this in the corollary below. In the next section we will show that for certain types of graph polynomials we can compute the coefficients efficiently.

**Corollary 2.3.** Given \( M > 0 \) and \( t \in C \) satisfying \( |t| < M \), there exists a constant \( C = C(t, M) \) such that the following holds. Suppose \( p \in \mathbb{C}[z] \) is a polynomial given by \( p(z) = a_0 + a_1z + \cdots + a_dz^d \) with no roots in the disk \( D \). Suppose further we are able to compute the coefficients \( a_1, \ldots, a_d \) in time \( \tau(r) \) for each \( r = 1, \ldots, d \). Then we can compute a multiplicative \( \epsilon \)-approximation to \( p(t) \) in time \( O(m^2 + \tau(m)) \), where \( m = C(\ln(d/\epsilon)) \).

Note that the constant \( C \) is the same as in Lemma 2.2.

**Proof.** We can compute \( a_0, \ldots, a_m \) in time \( \tau(m) \) from which we obtain \( \frac{d}{dz} p |_{z=0} \) for \( j = 0, \ldots, m \). Lemma 2.1 implies that we can then compute \( \frac{d^j}{dz^j} f |_{z=0} \) for \( j = 0, \ldots, m \) in time \( O(m^2) \). Using these values we are able to compute \( T_m(f) \), as given in (7), in time \( O(m^2) \). Finally by Lemma 2.2 \( \exp(T_m(f)(t)) \) gives a multiplicative \( \epsilon \)-approximation to \( p(t) \).
3 Computing coefficients of graph polynomials

In this section we present our main technical contribution, which is an efficient way to compute the coefficients of certain graph polynomials for bounded degree graphs. Throughout, we will focus on graph polynomials whose coefficients can be expressed as linear combinations of induced subgraph counts and we give an efficient algorithm to compute these coefficients. The results in this section are just stated for graphs, but are in fact valid for multigraphs. So the reader could read multigraph instead of graph everywhere in this section. (The degree of a vertex in a multigraph is the number of edges incident with the vertex, where a loop is counted twice.)

We start with some definitions after which we state the main result of this section. By \( \mathcal{G} \) we denote the collection of all graphs and by \( \mathcal{G}_k \) for \( k \in \mathbb{N} \) we denote the collection of graphs with at most \( k \) vertices. A graph invariant is a function \( f : \mathcal{G} \to S \) for some set \( S \) that takes the same value on isomorphic graphs. For graphs \( H, G \) we denote by \( \text{ind}(H, G) \) the number of induced subgraphs of \( G \) that are isomorphic to \( H \). Note that if \( H \) is equal to the empty graph we have \( \text{ind}(H, G) = 1 \) for all \( G \). A graph polynomial is a graph invariant \( p : \mathcal{G} \to \mathbb{C}[z] \), where \( \mathbb{C}[z] \) denotes the ring of polynomials in the variable \( z \) over the field of complex numbers. Call a graph invariant \( f \) multiplicative if \( f(\emptyset) = 1 \) and \( f(G_1 \cup G_2) = f(G_1)f(G_2) \) for all graphs \( G_1, G_2 \) (here \( G_1 \cup G_2 \) denotes the disjoint union of the graphs \( G_1 \) and \( G_2 \)).

**Definition 3.1.** Let \( p \) be a monic and multiplicative graph polynomial defined by

\[
p(G)(z) := \sum_{i=0}^{d(G)} (-1)^i e_i(G) z^{d(G)-i}
\]

for \( G \in \mathcal{G} \). We call \( p \) a bounded induced graph counting polynomial (BIGCP) if there exists constants \( C_1, C_2 \in \mathbb{N} \) such that the following two conditions are satisfied:

(i) the coefficients \( e_i \) satisfy

\[ e_i(G) := \sum_{H \in \mathcal{G}_{C_1}^i} \lambda_{H,i} \text{ind}(H, G) \]

for certain \( \lambda_{H,i} \in \mathbb{C} \) for all graphs \( G \);

(ii) the coefficients \( \lambda_{H,i} \) for \( H \in \mathcal{G}_{C_1}^i \) can be computed in time \( O(\mathcal{C}_2^{\left|V(H)\right|}) \);

If, for example, for each \( i \), the coefficient \( e_i(G) \) in (13) is equal to \( (-1)^i \) times the number of independent sets of size \( i \) in \( G \), then it is easy to see that \( p \) (which is of course the independence polynomial) is a BIGCP. In this case the obvious brute force algorithm to compute the coefficient \( e_i(G) \) for an \( n \)-vertex graph \( G \) runs in time \( O(n^i) \) (by checking all \( i \)-subsets of \( V(G) \)) and if \( i = O(\ln n) \) then this is quasi-polynomial time. Our main result of this section is a general algorithm for computing coefficients of BIGCPs, which when applied to this example, computes \( e_i(G) \) in polynomial time even when \( i = O(\ln n) \) as long as the maximum degree of \( G \) is bounded.

**Theorem 3.1.** Let \( C > 0 \) and \( \Delta \in \mathbb{N} \) and let \( p(\cdot) \) be a bounded induced graph counting polynomial. Then there is a deterministic \( (n/\epsilon)^{O(1)} \)-time algorithm, which, given any \( n \)-vertex graph \( G \) of maximum degree at most \( \Delta \) and any \( \epsilon > 0 \), computes the coefficients \( e_0(G), \ldots, e_m(G) \) of \( p(G) \) for \( m = C \ln(n/\epsilon) \).

Before we prove Theorem 3.1 we will first gather some facts about induced subgraph counts and the number of connected induced subgraphs of fixed size that occur in a graph which we will need for the proof.
3.1 Induced subgraph counts

Define \( \text{ind}(H, \cdot) : \mathcal{G} \to \mathbb{C} \) by \( G \to \text{ind}(H, G) \). So we view \( \text{ind}(H, \cdot) \) as a graph invariant. We can take linear combinations and products of these invariants. In particular, for two graphs \( H_1, H_2 \) we have

\[
\text{ind}(H_1, \cdot) \cdot \text{ind}(H_2, \cdot) = \sum_{H \in \mathcal{G}} c^H_{H_1,H_2} \text{ind}(H, \cdot),
\]

where for a graph \( H \), \( c^H_{H_1,H_2} \) is the number of pairs of subsets of \( V(H) \), \((S, T)\), such that \( S \cup T = V(H) \) and \( H[S] = H_1 \) and \( H[T] = H_2 \). In particular, given \( H_1 \) and \( H_2 \), \( c^H_{H_1,H_2} \) is nonzero for only a finite number of graphs \( H \).

Computing the parameter \( \text{ind}(H, G) \) is generally difficult, but it becomes easier if \( H \) is connected (and \( V(H) \) is not too large) and \( G \) has bounded degree.

Lemma 3.2. Let \( H \) be a connected graph on \( k \) vertices and let \( \Delta \in \mathbb{N} \). Then

(i) there is an \( O(n\Delta^{k-1}) \)-time algorithm, which, given any \( n \)-vertex graph \( G \) with maximum degree at most \( \Delta \), checks whether \( \text{ind}(H, G) \neq 0 \);

(ii) there is an \( O(n^2\Delta^2(k-1)) \)-time algorithm, which, given any \( n \)-vertex graph \( G \) with maximum degree at most \( \Delta \), computes the number \( \text{ind}(H, G) \).

Note that Lemma 3.2 (i) enables us to test for graph isomorphism between bounded degree graphs when \( |V(G)| = |V(H)| \).

Proof. Let us list the vertices of \( V(H) \), \( v_1, \ldots, v_k \) in such a way that for \( i \geq 1 \) vertex \( v_i \) has a neighbour among \( v_1, \ldots, v_{i-1} \). Then to embed \( H \) into \( G \) we first select a target vertex for \( v_1 \) and then given that we have embedded \( v_1, \ldots, v_{i-1} \) with \( i \geq 2 \) there are at most \( \Delta \) choices for where to embed \( v_i \). After \( k \) iterations, we have a total of at most \( n\Delta^{k-1} \) potential ways to embed \( H \) and each possibility is checked in the procedure above. Hence we determine if \( \text{ind}(H, G) \) is zero or not in \( O(n\Delta^{k-1}) \) time.

The procedure above gives a list (of size at most \( n\Delta^{k-1} \)) of all sets \( S \subseteq V(G) \) such that \( G[S] = H \), although the list may contain repetitions. It takes time \( O(n\Delta^{k-1})^2 = O(n^2\Delta^2(k-1)) \) to eliminate repetitions, and the length of the resulting list gives the value of \( \text{ind}(H, G) \).

Next we consider how to enumerate all possible connected induced subgraphs of fixed size in a bounded degree graph. We will need the following result of Borgs Chayes Kahn and Lovász [5] Lemma 2.1:

Lemma 3.3. Let \( G \) be a graph of maximum degree \( \Delta \). Fix a vertex \( v_0 \) of \( G \). Then the number of connected induced subgraphs of \( G \) with \( k \) vertices containing the vertex \( v_0 \) is at most \( \frac{\Delta^{k-1}}{2} \).

As a consequence we can efficiently enumerate all connected induced subgraphs of logarithmic size that occur in a bounded degree graph \( G \).

Lemma 3.4. There is an \( O(n^2\Delta^2(e\Delta)^{2k}) \)-time algorithm which, given \( k \in \mathbb{N} \) and an \( n \)-vertex graph \( G = (V, E) \) of maximum degree \( \Delta \), outputs \( T_k \), the set of all \( S \subseteq V \) satisfying \( |S| \leq k \) and \( G[S] \) connected.

Proof. By the previous result, we know that \( |T_k| \leq nk(\Delta)^{k-1} \) for all \( k \).

We inductively construct \( T_k \). For \( k = 1 \), \( T_1 \) is clearly the set of singleton vertices and takes time \( O(n) \) to output.

Given that we have found \( T_{k-1} \) we compute \( T_k \) as follows. We first compute the multiset

\[
T_k^+ = \{ S \cup \{ v \} : S \in T_{k-1} \text{ and } v \in N_G(S) \}.
\]
Here \(|N_G(S)| \leq |S| \Delta \leq k\Delta\) and takes time \(O(k\Delta)\) to find (assuming \(G\) is given in adjacency list form). Therefore computing \(T'_k\) takes time \(O(|T_{k-1}|k\Delta) = O(nk^2(e\Delta)^k)\). Finally we compute the set \(T_k\) by removing the repetitions in \(T'_k\) (by comparing each element with all previous elements), which takes time \(O(n^2k^2(e\Delta)^{2k})\).

Starting from \(T_1\), we perform the above iteration \(k\) times, requiring a total running time of \(O(n^2k^2(e\Delta)^{2k})\).

It remains only to show that \(T_k\) contains all the sets we desire. Clearly \(T_{k-1} \subseteq T_k\) and assume by induction that \(T_{k-1}\) contains all \(T \subseteq V\) of size \(k-1\) with \(G[T]\) connected.

Given \(S \subseteq V\) such that \(|S| = k\) and \(G[S]\) is connected, take any tree of \(G[S]\), remove a leaf \(v\) and call the resulting set of vertices \(S'\). Then it is clear that \(S' \in T_{k-1}\) and this implies \(S = S' \cup \{v\} \in T_k\).

We call a graph invariant \(f : G \to \mathbb{C}\) additive if for each \(G_1, G_2 \in \mathcal{G}\) we have \(f(G_1 \cup G_2) = f(G_1) + f(G_2)\). The following lemma is a variation of a lemma due to Csikvári and Frenkel [17], it is fundamental to our approach.

**Lemma 3.5.** Let \(f : G \to \mathbb{C}\) be a graph invariant given by \(f(\cdot) := \sum_{H \in G} a_H \text{ind}(H, \cdot)\) (where only finitely many of the \(a_H\) are nonzero). Then \(f\) is additive if and only if only if \(a_H = 0\) for all graphs \(H\) that are disconnected.

**Proof.** Let \(H\) be connected. Then for \(G_1, G_2 \in \mathcal{G}\) we have \(\text{ind}(H, G_1 \cup G_2) = \text{ind}(H, G_1) + \text{ind}(H, G_2)\), as \(H\) is connected. Thus \(\text{ind}(H, \cdot)\) is additive. Clearly, linear combinations of additive graph parameters are again additive. This implies that if \(f\) is supported on connected graphs, then \(f\) is additive.

Suppose next that \(f\) is additive. We need to show that \(a_H = 0\) if \(H\) is disconnected. By the previous part of the proof, we may assume that \(a_H = 0\) for all connected graphs \(H\). Let now \(H = H_1 \cup H_2\) with both \(H_1\) and \(H_2\) nonempty. We may assume by induction that for all graphs \(H'\) of order strictly smaller than \(k := |V(H)|\) we have \(a_{H'} = 0\). Now, by additivity we have

\[
f(H) = f(H_1) + f(H_2) = \sum_{H' : |V(H')| \geq k} a_{H'} (\text{ind}(H', H_1) + \text{ind}(H', H_2)) = 0,
\]

since \(|V(H_i)| < k\) for \(i = 1, 2\). On the other hand we have

\[
f(H) = \sum_{H' : |V(H')| \geq k} a_{H'} \text{ind}(H', H) = a_H \text{ind}(H, H).
\]

As \(\text{ind}(H, H) \neq 0\), this implies that \(a_H = 0\) and finishes the proof. \(\square\)

### 3.2 Proof of Theorem 3.1

Recall that \(p(\cdot)\) is a bounded induced graph counting polynomial (BIGCP). Given an \(n\)-vertex graph \(G\) with maximum degree at most \(\Delta\), we must show how to compute the first \(m\) coefficients \(c_0(G), \ldots, c_m(G)\) of \(p(G)\) (from (13)) in time \((n/\epsilon)^{O(1)}\), where \(m = C \ln(n/\epsilon)\). To reduce notation, let us write \(p = p(G), d = d(G)\) for the degree of \(p\), and \(c_i = e_i(G)\) for \(i = 0, \ldots, d\).

Let \(\zeta_1, \ldots, \zeta_d \in \mathbb{C}\) be the roots of the polynomial \(p\) and for \(k \in \mathbb{N}\) define the \(k\)th power sum by

\[
p_k := \sum_{i=1}^d \zeta_i^k.
\]

Then, for \(i = 1, \ldots, d\), the \(e_i\) are given by the elementary symmetric functions evaluated at \((\zeta_1, \ldots, \zeta_d)\). In particular, the *Newton identities* give an explicit relation between the
all subsets for certain, yet unknown, coefficients. Using the algorithm of Lemma 3.4, we first compute the sets $m_1, \ldots, m_k$ lists all the sets $k$ graphs using Lemma 3.2 (i) to test for isomorphism. This takes time at most $k\lambda C_k$. This can be done for each $k$, so the total time to compute and list the $C_k(G)$ is bounded by $(n/e)^{O(1)}$.

To prove the lemma, let us fix $k \leq m$ and show how to compute the coefficients $a_{H,k}$ assuming that we have already computed and listed the coefficients $a_{H',k'}$ for all $k' < k$. Let us fix $H \in C_{C_k}(G)$. By (17), it suffices to compute the coefficient of $\lambda(H, \cdot)$ in $p_{k-1;e_1}$ for $i = 1, \ldots, k$ (where we set $p_0 = 1$). By (14), (15) and (18) we know that the coefficient of $\lambda(H, \cdot)$ in $p_{k-1;e_1}$ is given by

$$\sum_{H_1, H_2} c^H_{H_1, H_2} a_{H_2, (k-i)} \lambda_{H_1, i} = \sum_{(S,T): S \cup T = V(H)} a_{H[T],(k-i)} \lambda_{H[S],i}.$$ \hspace{1cm} (20)

As $|V(H)| \leq C_k k = O(\ln(n/e))$, the second sum in (20) is over at most $4^{C_k} = (n/e)^{O(1)}$ pairs $(S,T)$. For each such pair, we need to compute $\lambda_{H[S],j}$ and look up $a_{H[T],(k-i)}$. We can compute $\lambda_{H[S],j}$ in time bounded by $C_k^2 = (n/e)^{O(1)}$ since $p$ is a BIGCP.

Looking up $a_{H[T],(k-i)}$ in the given list requires us to test isomorphism of $H[T]$ with each graph in $C_{C_k(k-i)}(G)$ (noting that $a_{H[T],(k-i)} = 0$ if $H[T] \not\in C_{C_k(k-i)}(G)$ by Lemma 3.5). Using Lemma 3.2 (i) to test for graph isomorphism, this takes time at most

$$O(|C_{C_k(k-i)}(G)| C_k (k-i) \Delta^{C_k(k-i)-1}) = O(n/e)^{O(1)}.$$

Here we use Lemma 3.5 to bound $|C_{C_k(k-i)}(G)|$. Together, all this implies that the coefficient of $\lambda(H, \cdot)$ in $p_{k-1;e_1}$ can be computed in time bounded by $(n/e)^{O(1)}$, and so the coefficient $a_{H,k}$ can be computed in time $(n/e)^{O(1)}$. Thus all coefficients $a_{H,k}$ for $H \in C_{C_k}(G)$ can be computed and listed in time bounded by $|C_{C_k}(G)| (n/e)^{O(1)} = (n/e)^{O(1)}$. This can be done for each $k = 1, \ldots, m$ in time $(n/e)^{O(1)}$. \hfill $\blacksquare$
To finish the proof of the theorem, we compute \( p_k(G) \) for each \( k = 1, \ldots, m \) by adding all the numbers \( a_H \) ind \((H,G)\) over all \( H \in \mathcal{C}_{C_k}(G) \). This can be done in time \( O(m |\mathcal{C}_{C_k}(G)| |V|^2 \lambda^{2(C_1m-1)}) = (n/\epsilon)O(1) \), where we have used that computing ind \((H,G)\) with \( H \in \mathcal{C}_{C_k}(G) \) takes time \( O(\lambda^2 \lambda^{2(C_1m-1)}) \) by Lemma 5.2(ii). Using the Newton identities (17) we obtain the coefficients \( e_1, \ldots, e_m \) from \( p_1, \ldots, p_m \) in time \( O(m^2) \). This finishes the proof.

### 4 The independence polynomial

#### 4.1 The independence polynomial on bounded degree graphs

**Proof of Theorem [77]** First note that by a result of Shearer [36] and Scott and Sokal, cf. Corollary 5.7, we know that \( Z(G) \) \( (\lambda) \neq 0 \) for all all graphs \( G \) of maximum degree at most \( \Delta \) and all \( \lambda \in \mathcal{C} \) that satisfy \(|\lambda| \leq \lambda^* = \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}\).

We will show that for any \( n \)-vertex graph \( G \) of maximum degree at most \( \Delta \), we can compute the first \( m \) coefficients of \( Z(G) \) in time \( (n/\epsilon)O(1) \), where \( m = \text{Clm}(n/\epsilon) \) and \( \mathcal{C} = C(\lambda, \lambda^*) \) is the constant in Corollary 2.3. Noting that the degree of \( Z(G) \) is at most \( n \), Corollary 2.3 implies we can compute a multiplicative \( \epsilon \)-approximation to \( Z(G)(\lambda) \) in time \( (n/\epsilon)O(1) \).

It remains to show that Theorem 5.1 allows us to compute the first \( m \) coefficients of \( Z(G) \) in time \( O(n/\epsilon)O(1) \). Note that the \( k \)th coefficient of \( Z(G) \) is ind \((\bullet^k,G)\), where \( \bullet^k \) denotes the graph consisting of \( k \) isolated vertices.

To be able to apply Theorem 3.1 we will focus on the polynomial

\[
\hat{Z}(G)(\lambda) := \sum_{I \subseteq V(G) \atop \text{I independent}} \lambda^{a(I)-|I|},
\]

where \( a(I) \) denotes the maximum size of an independent set in \( G \). We have \( \hat{Z}(G)(\lambda) = \lambda^{a(G)}Z(G)(1/\lambda) \); thus \( \hat{Z}(G) \) is multiplicative as both \( Z(G) \) and \( \lambda^{a(G)} \) are multiplicative.

Since the coefficient of \( \lambda^{a(G)-k} \) is given by ind \((\bullet^k,G)\), we have that \( \hat{Z}(G) \) is a BIGCP (taking \( C_1 = C_2 = 1 \)). So we can apply Theorem 3.1 to see that for \( k = 1, \ldots, m \) we can compute the coefficients ind \((\bullet^k,G)\) of the independence polynomial for \( k = 1, \ldots, m \) in time \( (n/\epsilon)O(1) \). This concludes the proof.

Evaluating the independence polynomial at negative and complex values gives us new information about the distribution of independent sets in a graph, as illustrated by the following. Recall that we denote by \( \hat{Z}(G)(\lambda) \) the polynomial defined in the same way as the independence polynomial except that in the sum \( Z(I) \), we only allow independent sets whose cardinality is even.

**Theorem 4.1.** Let \( \Delta \in \mathbb{N} \) and let \( 0 \leq \lambda < \lambda^*(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta} \). Then there exists a deterministic algorithm, which, given a graph \( G = (V,E) \) of maximum degree at most \( \Delta \) and \( \epsilon > 0 \), computes a multiplicative \( \epsilon \)-approximation to \( \hat{Z}(G)(\lambda) \) in time \( (|V|/\epsilon)O(1) \).

**Proof.** We apply the algorithm of Theorem 3.1 to compute multiplicative \( \epsilon \)-approximations \( A(\lambda) \) and \( A(-\lambda) \) to \( \hat{Z}(G)(\lambda) \) and \( \hat{Z}(G)(-\lambda) \) respectively in time \( (|V|/\epsilon)O(1) \). We have

\[
e^{-\epsilon}Z(G)(\lambda) \leq A(\lambda) \leq e^\epsilon Z(G)(\lambda) \quad \text{and} \quad e^{-\epsilon}Z(G)(-\lambda) \leq A(-\lambda) \leq e^\epsilon Z(G)(-\lambda).
\]

Taking half the sum of these equations and noting that \( \hat{Z}(G)(\lambda) = \frac{1}{2}(Z(G)(\lambda) + Z(G)(-\lambda)) \), we see that \( \frac{1}{2}(A(\lambda) + A(-\lambda)) \) is a multiplicative \( \epsilon \)-approximation to \( \hat{Z}(G)(\lambda) \) provided both \( Z(G)(\lambda) \) and \( Z(G)(-\lambda) \) have the same sign.
Clearly $Z(G)(\lambda) > 0$ since the coefficients of $Z(G)$ are nonnegative real numbers. Also $Z(G)(-\lambda) > 0$ because we know by the result of Scott and Sokal \[35\] that $Z(G)$ does not vanish in the interval $[-\lambda^*, \lambda^*]$, and we know $Z(G)$ is positive in the interval $[0, \lambda^*]$ since all the coefficients of $Z(G)$ are nonnegative real numbers. Hence $Z(G)$ is positive on the whole interval $[-\lambda^*, \lambda^*]$ and in particular $Z(G)(\lambda) > 0$. \[\]

### 4.2 The independence polynomial on claw-free graphs

In this subsection, we illustrate a technique of Barvinok for approximating graph polynomials on larger regions of the complex plane by making careful polynomial transformations. We use this technique to prove Theorem \[12\] which shows that we can approximate the independence polynomial of claw-free graphs on almost the entire complex plane. First we require a few preliminary results.

**Proposition 4.2.** If $G$ is a claw-free graphs of maximum degree $\Delta$ and $\zeta$ is a root of the independence polynomial $Z(G)$ of $G$ then $\zeta \in \mathbb{R}$ with $\zeta < \frac{1}{\pi(\Delta - 1)}$.

**Proof.** The fact that $\zeta \in \mathbb{R}$ is a result of Chudnovsky and Seymour \[16\]. The fact that $\zeta$ must be negative follows because all the coefficients of $Z(G)$ are positive. Now the result of Sokal and Scott \[35\] states that

$$|\zeta| \geq \lambda^*(\Delta) = \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^\Delta} > \frac{1}{e(\Delta - 1)},$$

from which the proposition follows. \[\]

We also require the following lemma of Barvinok \[5\].

**Lemma 4.3.** For $\rho \in (0, 1)$ we define

$$\alpha = \alpha(\rho) = 1 - \exp(-\rho^{-1}), \quad \beta = \beta(\rho) = \frac{1 - \exp(-1 - \rho^{-1})}{1 - \exp(-\rho^{-1})} > 1,$$

$$N = N(\rho) = (1 + \rho^{-1}) \exp(1 + \rho^{-1}), \quad \sigma = \sigma(\rho) = \sum_{i=1}^{N} \frac{\alpha^i}{i}.$$

The polynomial

$$\phi(z) = \phi_\rho(z) = \frac{1}{\sigma} \sum_{i=1}^{N} \frac{(\alpha z)^i}{i}$$

has the following properties:

(i) $\phi(0) = 0$ and $\phi(1) = 1$ and $\phi$ has degree $N$;

(ii) If $z \in \mathbb{C}$ with $|z| \leq \beta$ then $\phi_\rho(z) \in S_\rho$, where

$$S_\rho := \{ z \in \mathbb{C} \mid -\rho \leq \Re(z) \leq 1 + 2\rho \quad \text{and} \quad -2\rho \leq \Im(z) \leq 2\rho \}.$$ 

**Proposition 4.4.** Fix $\lambda = re^{i\theta} \in \mathbb{C}$ with $\theta \in (-\pi, \pi)$. Let $S_\rho$ be as in the previous lemma, and let $\mathbb{R}^-$ denote the negative real line. Then

$$\lambda S_\rho \cap \mathbb{R}^- \subset \begin{cases} [-2\rho r, 0] & \text{if } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; \\ [-2\rho r/\sin \theta, 0] & \text{otherwise.} \end{cases}$$

**Proof.** $S_\rho$ is a bounded strip parallel to the real axis in the complex plane, so $\lambda S_\rho$ is the same strip enlarged by a factor $r$ and rotated by an angle $\theta$. The proposition then follows from elementary trigonometry. \[\]
Proof of Theorem 1.2. Recall that we are given a claw-free graph $G$ of maximum degree $\Delta$ and $\lambda \in \mathbb{C}$ that is not a negative real number and we wish to find a multiplicative $\varepsilon$-approximation to $Z(G)(\lambda)$.

Set $n := |V(G)|$ and let $\lambda = r e^{i\theta}$ with $\theta \in (-\pi, \pi)$. Set

$$\rho = \begin{cases} 1/6r(\Delta - 1) & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] ; \\ \sin \theta / 6r(\Delta - 1) & \text{otherwise}, \end{cases}$$

and consider the polynomial $g(z) = Z(G)(\lambda \phi_{\rho}(z))$. Note that the degree of $g$ is $O(n)$ since the degree of $Z(G)$ is at most $n$ and the degree of $\phi_{\rho}$ is a constant $N(\rho)$.

We will use Corollary 2.3 to find a multiplicative $\varepsilon$-approximation to $g(1) = Z(G)(\lambda)$ in time $(n/\varepsilon)^{O(1)}$. In order to apply Corollary 2.3 to draw this conclusion, it is enough to check that (i) $g$ has no roots in the disk $|z| \leq \beta := \beta(\rho)$ and that (ii) the first $m = C \ln(d/\varepsilon)$ coefficients of $g$ can be computed in time $(n/\varepsilon)^{O(1)}$, where $d = O(n)$ is the degree of $g$ and $C = C(\beta, 1)$ is the constant in the statement of Corollary 2.3.

It remains to check (i) and (ii). To see (i), note first that by Lemma 2.3 $\phi_{\rho}$ maps the disk $D = \{z \in \mathbb{C} \mid |z| \leq \beta\}$ into $S_{\rho}$. By Proposition 4.4 and our choice of $\rho$, we have $\lambda \phi_{\rho}(D) \cap \mathbb{R}^{-} \subseteq \left(-\frac{1}{3(\Delta - 1)}, 0\right)$. By Proposition 4.2 we know that if $Z(G)(\xi) = 0$ then $\xi \in \mathbb{R}$ with $\xi < -\frac{1}{r(\Delta - 1)}$. In particular this implies $g(\cdot) = Z(G)(\lambda \phi_{\rho}(\cdot))$ has no root in the disk $D$.

For (ii), given a polynomial $p(z) = \sum_{i=0}^{d} a_i z^i$, write $p_{|m|}(z) := \sum_{i=0}^{m} a_i z^i$. Then we note that $g_{|m|} = (Z(G) \circ (\lambda \phi_{\rho}))_{|m|} = (Z(G)_{|m|} \circ (\lambda \phi_{\rho}_{|m|}))_{|m|}$, where we crucially use the fact that $\phi$ has no constant term since $\phi(0) = 0$. In words, to obtain $g_{|m|}(z)$ we substitute $\lambda \phi_{\rho}_{|m|}(z)$ into $Z(G)_{|m|}(z)$ and keep the first $m$ terms. Thus, in $(m^3)$-time we can obtain the first $m$ coefficients of $g$ if we know the first $m$ coefficients of $Z(G)$, which we know can be obtained in time $(n/\varepsilon)^{O(1)}$, as in the proof of Theorem 1.1.

We remark that, for the $(n/\varepsilon)^{O(1)}$ running time in the algorithm above, the $O(1)$ in the exponent depends on $\lambda$ and grows exponentially fast in $r = |\lambda|$. However, this dependence can be brought down to $O(|\lambda|^{1/2})$ by adapting Lemma 4.3 as described by Barvinok [6].

5 The Tutte polynomial

Here we give a proof of Theorem 1.3.

Proof of Theorem 1.3. By a result of Jackson, Procaccia and Sokal, cf. [26, Theorem 1.2] (which is valid for loopless multigraphs) we know that there exists a constant $K > 0$ depending on $\Delta$ and $w$ such that for all $q$ with $|q| > K$ we have $Z_T(G)(q, w) \neq 0$ for all graphs $G$ of maximum degree at most $\Delta$. This is exactly opposite to what we need to apply Corollary 2.3 so let us define the graph polynomial $p_T$ by

$$p_T(G)(z) := z^{|V|} Z_T(G)(1/z, w), \tag{22}$$

for any graph $G = (V, E)$. Note that $p_T(G)$ has degree $n := |V|$ and that if $x$ is a multiplicative $\varepsilon$-approximation to $p_T(G)(1/q, w)$, then $q^n x$ is a multiplicative $\varepsilon$-approximation to $Z_T(G)(q, w)$, so it is sufficient to find the former.

We will show that for any $n$-vertex graph $G$ of maximum degree at most $\Delta$, we can compute the first $m$ coefficients of $p_T(G)$ in time $(n/\varepsilon)^{O(1)}$, where $m = C \ln(n/\varepsilon)$ and $C = C(1/q, 1/K)$ is the constant in Corollary 2.3. Corollary 2.3 then implies we can compute a multiplicative $\varepsilon$-approximation to $p_T(G)(1/q)$ and hence to $Z_T(G)(q, w)$ in time $(n/\varepsilon)^{O(1)}$. 

14
We will now show that Theorem 3.1 allows us to compute the first $m$ coefficients of $p_T(G)$ which are precisely the last $m$ coefficients of $Z_T(G)$ (with the second argument fixed) in time $O(n/e)^{O(1)}$, thereby proving the theorem. It suffices to show that $Z_T(G)(z,w)$ as a polynomial in $z$ is a BIGCP.

Since the Tutte polynomial $Z_T(G)(z,w)$ (as a polynomial in $z$) is a monic and multiplicative graph polynomial (of degree $n = |V(G)|$), we only need to show how the coefficient of $z^{n-k}$ can be expressed in terms of induced subgraph counts. By definition, the coefficient of $z^{n-k}$ in $Z_T(G)(z,w)$ is equal to the sum over all subsets $A$ of $E$ such that $A$ induces a graph with exactly $n-k$ components, where each subset is counted with weight $w^{|A|}$. Let us call a component of a graph nontrivial if it consists of more than one vertex. Suppose some subset of the edges $A \subseteq E$ induces $n-k$ components of which $c$ are nontrivial. Then we have $n-k-c$ isolated vertices and so the graph $H$, consisting of the union of these nontrivial components, has $n-(n-k-c) = k+c$ vertices. Every subset $S \subseteq V(G)$ such that $G[S]$ is isomorphic to $H$ gives the same contribution to the coefficient of $z^{n-k}$ as $A$; namely $w^{|E(H)|}$. Let us denote by $G^*$ the collection of all graphs that do not contain isolated vertices. Then the above implies that the coefficient of $z^{n-k}$ in $Z_T(G)$ can be expressed as follows

$$\sum_{H \in G^*} w^{|E(H)|} \text{ind}(H,G).$$

Since for each graph $H$ occurring in this sum we have

$$|V(H)| = k(H) + k \leq \frac{|V(H)|}{2} + k,$$

as $H$ has no isolated vertices, and hence $|V(H)| \leq 2k$. This implies that the Tutte polynomial is a BIGCP (taking $C_1 = 2$ and $C_2 = 1$). Then Theorem 3.1 implies that we can compute the coefficients of $z^{n-k}$ for $k = 0, \ldots, m$ in time bounded by $O(n/e)^{O(1)}$, proving the theorem.

\[\square\]

Remark 5.1. Csikvári and Frenkel [17] introduced graph polynomial of bounded exponential type and showed that these polynomials have bounded roots on bounded degree graphs. This was utilized in [34] to give quasi-polynomial-time approximation algorithm for evaluations of these polynomials. The Tutte polynomial with the second argument fixed is an example of such a polynomial. It would be interesting to remove the adjective ‘quasi’ in the results of [34] for these polynomials. It is however not clear how to do this in general using the methods developed in Section 3 of the present paper, as these polynomials do satisfy condition (i) of Definition 3.1 but is not clear that in general they also satisfy condition (ii).

6 Partition functions of spin models

In this section we will state and prove a generalization of Theorem 1.4 and we will indicate how our method applies to partition functions of graph homomorphisms with multiplicities.

6.1 Partition functions for edge-colored graphs

Suppose a graph $G = (V,E)$ has an edge-coloring $\psi : E \to [c]$, which need not be a proper coloring. Suppose also that for each $i = 1, \ldots, c$ we have a symmetric $k \times k$-matrix $A_i$. Let us write $A = (A^1, \ldots, A^c)$. Then we can extend the definition of the
partition function of a spin model as follows:

\[
p(G)(A) = \sum_{\phi: V \to |k|} \prod_{e=\{u,v\} \in E} A^{\phi(e)}_{\phi(u),\phi(v)}
\]  

(23)

We will refer to \(p(G)(A)\) as the partition function of \(A\). In [22] these is called a Markov random fields (if the \(A_i\) are nonnegative) and in [31] this is called a multi spin system. Clearly, if \(c = 1\) this just reduces to the partition function of a spin model. We have the following result, which implies Theorem 1.4.

**Theorem 6.1.** Let \(\Delta, k \in \mathbb{N}\). Then there exists a deterministic algorithm, which, given a \(c\)-edge-colored graph \(G = (V, E)\) of maximum degree at most \(\Delta\), symmetric \(k \times k\) matrices \(A_1, \ldots, A_c\) such that \(|A^i_{ij} - 1| \leq 0.34/\Delta\) for all \(i, j = 1, \ldots, k\) and \(s = 1, \ldots, c\), and \(\epsilon > 0\), computes a multiplicative \(\epsilon\)-approximation to \(p(G)(A)\) in time \((|V|/\epsilon)^{O(1)}\).

**Remark 6.1.** The implicit constant in the big \(O\) only depends on \(\Delta, k\) and the constant 0.34. In particular, the number of colors \(c\) does not play a role in the complexity of the algorithm. The constant 0.34 may be replaced by 0.45 if \(\Delta \geq 3\) and by 0.54 if \(\Delta\) is large enough. See [22].

**Proof.** Let \(J\) be the all ones matrix. For \(z \in \mathbb{C}\), let \(A'(z) := (J + z(A^1 - J), \ldots, J + z(A^c - J))\). Define two polynomials \(q\) and \(\overline{q}\) by,

\[
q(G)(z) = k^{-|V|}p(G)(A'(z)) \quad \text{and} \quad \overline{q}(G)(z) = z^{|E|}q(G)(1/z).
\]

Then \(q(G)(0) = 1\) and \(q(G)(1) = k^{-|V|}p(G)(A)\). We think of \(q\) as a homotopy from the easy-to-compute quantity \(q(G)(0)\) and the desired quantity \(q(G)(1)\). Barvinok and Soberón [22, Theorem 1.6] showed that there exists a constant \(\delta > 0\) such that \(q(G)(z) \neq 0\) for all \(z\) satisfying \(|z| \leq 1 + \delta\).

We will show that for any \(n\)-vertex graph \(G\) of maximum degree at most \(\Delta\), we can compute the first \(m\) coefficients of \(q(G)\) in time \((n/\epsilon)^{O(1)}\), where \(m = C \ln(n/\epsilon)\) and \(C = C(1, 1 + \delta)\) is the constant in Corollary 2.3. Noting that the degree of \(q(G)\) is at most \(|E| \leq n\Delta/2\), Corollary 2.3 implies we can compute a multiplicative \(\epsilon\)-approximation to \(q(G)(1)\) in time \((n/\epsilon)^{O(1)}\). So it remains to show that we can compute the first \(m\) coefficients of \(q(G)\), which are precisely the last \(m\) coefficients of \(\overline{q}(G)\), in time \((n/\epsilon)^{O(1)}\). (This can obviously be done in quasi-polynomial time by just going over all subsets of size \(m\) of \(E\) as is done in [71].)

By definition, \(q(G)(z)\)

\[
q(G)(z) = k^{-n} \sum_{\phi: V \to |k|} \prod_{e=\{u,v\} \in E} (J + (z(A^{\phi(e)} - J)))_{\phi(u),\phi(v)}
\]

\[
= k^{-n} \sum_{i=0}^{m} z^i \left( \sum_{F \subseteq E : |F| \leq 1} \sum_{\phi: V \to |k| \in \{u,v\} \in F} (A^{\phi(e)} - J)_{\phi(u),\phi(v)} \right).
\]

(25)

For a subset \(F\) of \(E\), define \(G[F]\) to be the edge-colored graph induced by the edges in \(F\). The vertex set of \(G[F]\) consists of those vertices incident with edges in \(F\) and hence has size at most \(2|F|\). Denote for \(\ell \in \mathbb{N}\) by \(G_\ell\) the collection of graphs on at most \(\ell\) vertices whose edges are colored with colors from \([c]\). For two edge-colored graphs \(H_1, H_2\) with colors from \([c]\) we denote by \(\text{ind}_c(H_1, H_2)\) the number of induced graphs of \(H_2\) that are isomorphic as edge-colored graphs to \(H_1\).
Then we see that the coefficient of \( z^i \) in (25) can be written as follows:

\[
\sum_{H \in \mathcal{G}_2(c)} k^{-|V(H)|} \prod_{\phi:V(H) \to [k]} \left( \sum_{e=\{u,v\} \in E(H)} (A^\phi(e) - 1) \phi(u) \phi(v) \right) \text{ind}_c(H, G).
\] (26)

If \( i \leq m \), the inner sum in (26) can be computed in time \( k^{O(2m)} = (n/e)^{O(1)} \). Since \( G \) has maximum degree at most \( \Delta \), the degree of \( q \) and \( \hat{q} \) is at most \( n\Delta/2 \) and \( \hat{q} \) is monic. In case \( c = 1 \), i.e., in case we are dealing with ordinary graphs, then this implies that \( \hat{q} \) is a BIGCP (with constant \( C_1 = 2 \) and \( C_2 = k \)) and so Theorem \ref{generalized_zeta} implies that we can compute the last \( m \) coefficients of \( \hat{q} \), and hence the first \( m \) coefficients of \( q \), in time bounded by \( O(n/e)^{O(1)} \).

In the general case we note that Theorem \ref{generalized_zeta} remains valid in case in Definition \ref{generalized_zeta} ind is replaced by \( \text{ind}_c \). To see this, it suffices to note that all lemmas and proofs in Section \ref{partition_functions} directly carry over to the edge-colored case. This finishes the proof. \( \square \)

### 6.2 Partition functions of graph homomorphisms with multiplicities

Let again \( G = (V, E) \) be equipped with an edge-coloring \( \psi : E \to [c] \), which need not be a proper coloring. Suppose also that for each \( i = 1, \ldots, c \) we have a symmetric \( k \times k \) matrix \( A_i \). Let us write \( A = (A_1, \ldots, A^c) \). Let \( n = |V| \) and let \( \mu = (\mu_1, \ldots, \mu_k) \) with \( \mu_i \in \mathbb{Z}_{\geq 1} \) for each \( i \) be such that that \( \sum_{i=1}^c \mu_i = n \). We call such \( \mu \) a composition of \( n \) in \( k \) parts. Barvinok and Soberón \cite{barvinok2013partition} define the partition function of graph homomorphisms with multiplicities \( \mu \) as

\[
p_\mu(G)(A) = \sum_{\phi:V \to [k]} \prod_{e=\{u,v\} \in E} A^\phi(e)_{\psi^{-1}(i)=\mu_i}.
\] (27)

We refer to \cite{barvinok2013partition} for more details and background on this type of partition function.

Building on a result from Barvinok and Soberón \cite{barvinok2013partition} Section 2] and using exactly same proof as above we directly establish the following:

**Theorem 6.2.** Let \( \Delta, k \in \mathbb{N} \). Then there exists a deterministic algorithm, which, given an \( c \)-edge colored graph \( G = (V, E) \) of maximum degree at most \( \Delta \), a composition \( \mu \) of \( |V| \) in \( k \) parts, symmetric \( k \times k \) matrices \( A_1, \ldots, A_c \), such that \( |A_{ij}^\phi - 1| \leq 0.1/ \Delta \) for all \( i, j = 1, \ldots, k \) and \( c = 1, \ldots, c \), and \( \epsilon > 0 \), computes an \( \epsilon \)-approximation to \( p_\mu(G)(A) \) in time \( (|V|/\epsilon)^{O(1)} \).

### 7 Partition functions of edge-coloring models

In this section we state and prove a generalization of Theorem \ref{partition_functions}. It is along the same lines as the generalization of Theorem \ref{partition_functions} in the previous section. The proof also goes along the same line, but as we will see below there are some details that are different.

#### 7.1 Partition functions for vertex-colored graphs

Let \( G = (V, E) \) be a graph that is equipped with a vertex coloring \( \psi : V \to [c] \) (\( \psi \) need not be a proper coloring). Suppose that we have \( k \)-color edge-coloring models \( h^1, \ldots, h^c \). Let us write \( \mathcal{H} = (h^1, \ldots, h^c) \). Often the pair \( (G, \{h^1, \ldots, h^c\}, \psi) \) is called a
signature grid, cf. \cite{14 15 13}. Then we can extend the definition of the partition function of an edge-coloring model as follows:

$$p(G)(\mathcal{H}) = \sum_{\phi: E \to k} \prod_{v \in V} h^{\phi(v)}(\phi(\delta(v))).$$

(28)

We will refer to \(p(G)(\mathcal{H})\) as the partition function of \(\mathcal{H}\). It is also called the Holant problem of the signature grid \((G, (h_1, \ldots, h^c), \psi)\) cf. \cite{14 15 13}. We have the following result, which implies Theorem 1.5.

**Theorem 7.1.** Let \(\Delta, k \in \mathbb{N}\). Then there exists a deterministic algorithm, which, given a \(c\)-vertex colored graph \(G = (V, E)\) of maximum degree at most \(\Delta\), \(k\)-color edge-coloring models \(h^1, \ldots, h^c\) that satisfy \(|h^\phi(\phi) - 1| \leq 0.35/(\Delta + 1)\) for all \(\phi \in \mathbb{N}^k\) and \(s = 1, \ldots, c\), and \(\varepsilon > 0\), computes a multiplicative \(\varepsilon\)-approximation to \(p(G)(\mathcal{H})\) in time \((|V|/\varepsilon)^{O(1)}\).

**Remark 7.1.** Just as for edge-colored graphs, the number of colors \(c\) does not play a role in the time complexity in the theorem above. Additionally, the constant 0.35 may be replaced by 0.47 if \(\Delta \geq 3\) and by 0.56 if \(\Delta\) is large enough; see \cite{34}. Moreover, for readers familiar with the orthogonal group invariance of these partition functions one can use Corollary 6b from \cite{34} to find a larger family of edge-coloring models for which the partition function can be efficiently approximated.

**Proof.** Let \(J\) denote the constant ones function \(J : \mathbb{N}^k \to \mathbb{C}\) (defined by \(J(\phi) = 1\) for all \(\phi \in \mathbb{N}^k\)). Let for \(z \in \mathbb{C}\), \(\mathcal{H}(z) := (J + z(h^1 - J), \ldots, J + z(h^c - J))\). Consider the following two polynomials:

$$q(G)(z) := k^{-|E|} p(G)(\mathcal{H}(z)) \quad \text{and} \quad \hat{q}(G)(z) := z^{|V|} q(G)(1/z).$$

(29)

Observe that \(q(G)(1) = k^{-|E|} p(G)(\mathcal{H}) = \hat{q}(G)(1)\) and that both \(q(G)\) and \(\hat{q}(G)\) are polynomials of degree at most \(n := |V|\). So, just as in the previous section, the problem of approximating the partition function \(p(G)(\mathcal{H})\) is replaced by approximating an evaluation of a univariate polynomial.

By Corollary 6a from \cite{34} (which is valid for multigraphs) there exists \(\delta > 0\) such that \(q(G)(z) \neq 0\) for all \(z \leq 1 + \delta\). We will show (in Theorem 7.2) that for any \(n\)-vertex graph \(G\) of maximum degree at most \(\Delta\), we can compute the first \(m\) coefficients of \(q(G)\) in time \((n/\varepsilon)^{O(1)}\), where \(m = C \ln(n/\varepsilon)\) and \(C = C(1+\delta)\) is the constant in Corollary 2.3. Noting that the degree of \(q(G)\) is at most \(n\), Corollary 2.3 implies we can compute a multiplicative \(\varepsilon\)-approximation to \(q(G)(1)\) in time \((n/\varepsilon)^{O(1)}\).

Ideally we would like to do this using Theorem 5.1 just as in the proof of Theorem 6.1. Since partition functions of edge-coloring models are multiplicative, the polynomial \(\hat{q}\) is also multiplicative. Moreover \(\hat{q}\) is monic of degree \(n\). So to be able to apply Theorem 5.1 we need only check that the coefficients of \(\hat{q}\), or equivalently the coefficients of \(q\), can be expressed as linear combinations of induced graph counts. This is in fact proved in \cite{34} for \(c = 1\), but in that expression it is not clear whether the coefficients \(\lambda_{H,i}\) in (14) can be computed efficiently. So instead of directly applying Theorem 5.1 we will have to do a little more work, which we postpone to the next section. \(\square\)

### 7.2 Computing coefficients of \(\hat{q}(G)(z)\)

By definition,

$$q(G)(z) = k^{-|E|} \sum_{\phi: E \to [k]} \prod_{v \in V} (J + z(h^{\phi(v)} - J)) (\phi(\delta(v)))$$

$$= k^{-|E|} \sum_{i=0}^{n} z^i \left( \sum_{|U|=i} \sum_{\phi: E \to [k]} \prod_{u \in U} (h^{\phi(u)} - J) (\phi(\delta(u))) \right).$$

(30)
We need the concept of a fragment, which is a pair \((H, \kappa)\), where \(H\) is a \(c\)-vertex colored graph and where \(\kappa\) is a map \(\kappa : V(H) \to \{0,1,\ldots,\Delta\}\). We think of \(\kappa(u)\) as a number of half edges incident with \(u\). Note that the graph \(G\) itself can be thought of as a fragment by taking the map \(\kappa : V(G) \to \{0,\ldots,\Delta\}\) to be \(\kappa(v) = 0\) for all \(v \in V(G)\).

For \(U \subseteq V\) we let \(G(U)\) be the fragment \((G[U], \kappa)\) where \(\kappa(u)\) is equal to the number of edges that connect \(u\) with \(V \setminus U\). Clearly, for each \(U\) of size \(i\) the second sum on the right in (30) only depends on the isomorphism class of the fragment \(G(U)\). (An isomorphism from a fragment \((H, \kappa)\) to a fragment \((H', \kappa')\) is an isomorphism \(\phi\) of the underlying graphs that preserves vertex colors and such that for each \(u \in V(H), \kappa(u) = \kappa'(\kappa(u))\).) For a fragment \(F = (H, \kappa)\) let \(E(F)\) denote the set of edges of \(F\) including half edges and let \(V(F)\) denote the vertex set of the underlying graph \(H\). Then define,

\[
p(F)(H) := \sum_{\phi : E(F) \to [k]} \prod_{v \in V(F)} h^\theta(v) (\phi(\delta(v))). \tag{31}
\]

Here we implicitly assume that \(\theta : V(F) \to [c]\) is the same as the vertex colouring \(\psi : V(H) \to [c]\) of \(H\). Define for a fragment \(F = (H, \kappa)\), \(\text{ind}^* (F, G)\) to be the number of sets \(U\) of size \([V(F)]\) such that \(G(U)\) is isomorphic to \(F\). Writing \(H = J = (h^1 - J, \ldots, h^c - J)\), we can rewrite (30) as

\[
q(G)(z) = k^{-|E|} \sum_{i=1}^n z^i \left( \sum_{\substack{F = (H, \kappa) \in [V(H)] = i \\text{of size } [V(F)]}} k^{|E(F)|} p(F)(H - J) \text{ind}^* (F, G) \right) = \sum_{i=1}^n z^i \left( \sum_{\substack{F = (H, \kappa) \in [V(H)] = i \\text{of size } [V(F)]}} k^{-|E(F)|} p(F)(H - J) \text{ind}^* (F, G) \right), \tag{32}
\]

where the sum runs over fragments. Let us denote the coefficient of \(z^i\) in (32) by \((-1)^i e_i\). Then

\[
\hat{q}(G)(z) = \sum_{i=0}^n (-1)^i e_i z^{n-i}.
\]

In [34] it is proved that in case \(c = 1\), \(\text{ind}^* (F, G)\) can be expressed as a linear combination of the parameters \(\text{ind}(H, G)\) for certain graphs \(H\). As mentioned above, the coefficients in this expression may not be easy to compute (at least we do not know how to do this). So will have to work with the parameters \(\text{ind}^* (F, \cdot)\) instead. This is not a severe problem since essentially if we replace \(\text{ind}\) in (14) by \(\text{ind}^*\), then Theorem 3.1 remains valid. Indeed, we have the following theorem.

**Theorem 7.2.** Let \(C > 0\) and \(\Delta \in \mathbb{N}\). Then there is a deterministic \((n/\varepsilon)^{O(1)}\)-time algorithm, which, given any \(n\)-vertex graph \(G\) of maximum degree at most \(\Delta\) and any \(\varepsilon > 0\), computes the coefficients \(e_1, \ldots, e_m\) of \(\hat{q}(G)\) for \(m = C \ln(n/\varepsilon)\).

The proof of Theorem 7.2 follows the same line as the proof of Theorem 3.1. Essentially we need to replace graphs by fragments in the proof and check that everything remains valid. For completeness we will give the proof.

We first need to note that for a fragment \(F_1 = (H_1, \kappa_1)\) the graph parameter \(\text{ind}^* (F_1, \cdot)\) can be extended to the collection of all fragments as follows: for a fragment \(F_2 = (H_2, \kappa_2)\) we let \(\text{ind}^* (F_1, F_2)\) denote the number of embeddings of \(H_1\) as induced subgraph in \(H_2\) preserving vertex colors, such that for each vertex \(v\) of \(H_1\) we have that the number of neighbours of \(v\) in \(V(H_2) \setminus V(H_1)\) is equal to \(\kappa_1(v) - \kappa_2(v)\). Then for two fragments \(F_1\) and \(F_2\) we have

\[
\text{ind}^* (F_1, \cdot) \cdot \text{ind}^* (F_2, \cdot) = \sum_F c_{F_1, F_2} \text{ind}^* (F, \cdot), \tag{33}
\]
where the sum runs over all fragments $F$ and where for a fragment $F$, $c_{F_1,F_2}^F$ denotes the number of pairs of subsets $(S,T)$ of $V(F)$ such that $S \cup T = V(F)$ and $F_1 = F(S)$ and $F_2 = F(T)$. (Here $F(S)$ is the fragment induced by $S$, i.e., if $F = (H, \kappa)$, then $F(S) = (H[S], \kappa)$ where for $s \in S$ we set $\alpha(s) = \deg_{H[S]}(s) - \deg_{H[S]}(s) + \kappa(s)$.) We call a fragment $F = (H, \kappa)$ connected if the graph $H$ is connected. We now adapt some of the statements and proofs of the results in Section 3 to include fragments.

We start with some definitions. By $\mathcal{F}$ we denote the collection of all fragments, and by $\mathcal{F}_k$ for $k \in \mathbb{N}$ we denote the collection of fragments with at most $k$ vertices. (Recall that we implicitly assume that the vertices of our fragments are colored with the colors $1, \ldots, \ell$.) For two fragments $F_1 = (H_1, \kappa_1)$ and $F_2 = (H_2, \kappa_2)$, $F_1 \cup F_2 := (H, \kappa)$, where $H = H_1 \cup H_2$ and $\kappa : V(H_1 \cup V(H_2) \rightarrow \{0, 1, \ldots, \ell\}$ is the map whose restriction to $V(H_1)$ is $\kappa_1$ and whose restriction to $V(H_2)$ is $\kappa_2$. An invariant of fragments is a function $f : \mathcal{F} \rightarrow \mathbb{C}$ for some set $S$ that takes the same value on isomorphic fragments. Call an invariant of fragments $f$ multiplicative if $f(\emptyset) = 1$ and $f(F_1 \cup F_2) = f(F_1)f(F_2)$ for all fragments $F_1, F_2$.

The maximum degree of a fragment $F = (H, \kappa)$ is equal to the maximum of $\deg(v) + \kappa(v)$ over $v \in V(G)$.

**Lemma 7.3.** Let $F = (H, \kappa)$ be a connected fragment on $k$ vertices and let $\Delta \in \mathbb{N}$. Then there is an $O(n^2 \Delta^{k-1})$-time algorithm, which, given any $n$-vertex fragment $\hat{F}$ with maximum degree at most $\Delta$, computes the number $\text{ind}^*(F, \hat{F})$.

Note that Lemma 7.3 enables us to test for isomorphism of fragments between bounded degree fragments when $|V(F)| = |V(\hat{F})|$.

**Proof.** This follows immediately from the proof of Lemma 3.2. We apply the proof of Lemma 3.2 to the underlying graphs and then remove any potential embedding that either violates the vertex coloring constraints or the constraints that $\kappa$ imposes.

We call an invariant of fragment $f : \mathcal{F} \rightarrow \mathbb{C}$ additive if for each $F_1, F_2 \in \mathcal{F}$ we have $f(F_1 \cup F_2) = f(F_1) + f(F_2)$. The following variation of a lemma due to Csikvári and Frenkel [17] has exactly the same proof as Lemma 3.5, one just needs to replace graph by fragment everywhere in the proof.

**Lemma 7.4.** Let $f : \mathcal{F} \rightarrow \mathbb{C}$ be an invariant of fragments given by $f(\cdot) := \sum_{F \in \mathcal{F}} a_F \text{ind}^*(F, \cdot)$ (where only finitely many of the $a_F$ are nonzero). Then $f$ is additive if and only if $a_F = 0$ for all fragments $F$ that are disconnected.

We now sketch the proof of Theorem 7.2.

### 7.2.1 Proof of Theorem 7.2

Let $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ be the roots of the polynomial $\hat{q}(G)$ and for $\ell \in \mathbb{N}$ let $p_\ell$ be the $\ell$th power sum, as defined in (16). By (32), for $i \geq 1$, the $e_i$ can be expressed as linear combinations of induced fragments counts of fragments with at most $\ell$ vertices. Since $e_1 = p_1$, this implies that the same holds for $p_1$. By induction, (33) and (17) we have that for each $\ell$

$$p_\ell = \sum_{F \in \mathcal{F}_\ell} a_{F,\ell} \text{ind}^*(F, G),$$

for certain, yet unknown, coefficients $a_{F,\ell}$.

Since $\hat{q}$ is multiplicative, the power sums are additive. Thus Lemma 7.4 implies that $a_{F,\ell} = 0$ if $F$ is not connected. Denote by $\mathcal{C}_\ell^r(G)$ the set of connected fragments $F$ of order at most $\ell$ such that $\text{ind}^*(F, G) \neq 0$. This way we can rewrite (34) as follows:

$$p_\ell = \sum_{F \in \mathcal{C}_\ell^r(G)} a_{F,\ell} \text{ind}^*(F, G).$$

20
The next lemma says that we can compute the coefficients \( a_{F, \ell} \) efficiently for \( \ell = 1, \ldots, m \), where \( m = C \ln(n/\varepsilon) \).

**Lemma 7.5.** There is an \( O(n/\varepsilon)^{O(1)} \)-time algorithm, which given an \( n \)-vertex graph \( G \) and \( \varepsilon > 0 \), computes and lists the coefficients \( a_{F, \ell} \) in \((35)\) for all \( F \in C'(\ell)(G) \) and all \( \ell = 1, \ldots, m = C \ln(n/\varepsilon) \).

**Proof.** Using the algorithm of Lemma \( 5.4 \), we first compute the sets \( \mathcal{T}_\ell \) consisting of all subsets \( S \) of \( V(G) \) such that \( |S| \leq \ell \) and \( G[S] \) is connected, for \( \ell = 1, \ldots, m \). This takes time bounded by \( (n/\varepsilon)^{O(1)} \). We next compute and list the fragments in \( C'(\ell)(G) \) by considering the set of fragments \( \{G(S) \mid S \in \mathcal{T}_\ell\} \) and removing copies of isomorphic fragments using Lemma \( 7.3 \) to test for isomorphism. This takes time at most \( (n/\varepsilon)^{O(1)} \) for each \( \ell \), so the total time to compute and list the \( C'(\ell)(G) \) is bounded by \( (n/\varepsilon)^{O(1)} \).

To prove the lemma, let us fix \( \ell \leq m \) and show how to compute the coefficients \( a_{F, \ell} \), assuming that we have already computed and listed the coefficients \( a_{F, \ell'} \) for all \( \ell' < \ell \). Let us fix \( F \in C'(\ell)(G) \). By the Newton identities \((17)\), it suffices to compute the coefficient \( \text{ind}^*(F, \cdot) \) in \( p_{\ell-\varepsilon} \) for \( i = 1, \ldots, \ell \) (where we set \( p_0 = 1 \)). By \((32), (33)\) and \((34)\) we know that the coefficient of \( (1)^{\ell-\varepsilon} \text{ind}^*(F, \cdot) \) in \( p_{\ell-\varepsilon} \) is given by

\[
\sum_{F_1 \in F_2 \in \mathcal{F}_{\ell-1} \atop |V(F_1)| = i} \frac{p(F_1)(J-H)}{k^{|E(F_1)|}} = \sum_{S, T \subseteq V(F) \atop |S| = \ell, |T| \leq \ell - i} \frac{a_{F(T), (\ell-1)}}{k^{|E(F(S))|}}. \tag{36}
\]

For each such pair \((S, T)\), we need to compute \( \frac{p(F(S))(J-H)}{k^{|E(F(S))|}} \) and look up \( a_{F(T), (\ell-1)} \). We can compute \( \frac{p(F(S))(J-H)}{k^{|E(F(S))|}} \) in time bounded by \( O(k^{\Delta_2}) = (n/\varepsilon)^{O(1)} \).

Looking up \( a_{F(T), (\ell-1)} \) in the given list requires us to test isomorphism of \( F(T) \) with each fragment in \( C'(\ell-1)(G) \) (noting that \( a_{F(T), (\ell-1)} = 0 \) if \( F(T) \notin C'(\ell-1)(G) \)) by Lemma \( 7.4 \). Using Lemma \( 7.3 \) to test for isomorphism, this takes time at most

\[
O(|C'(\ell-1)(G)|\ell \Delta^2 |\ell-1|) = O(n/\varepsilon)^{O(1)}.
\]

Here we use Lemma \( 5.3 \) to bound \( |C'(\ell-1)(G)| \leq |\mathcal{T}_{\ell-1}(G)| \). Together, all this implies that the coefficient of \( \text{ind}^*(F, \cdot) \) in \( p_{\ell-\varepsilon} \) can be computed in time bounded by \( (n/\varepsilon)^{O(1)} \), and so the coefficient \( a_{F, \ell} \) can be computed in time \( (n/\varepsilon)^{O(1)} \). Thus all coefficients \( a_{F, \ell} \) for \( F \in C'(\ell)(G) \) can be computed and listed in time bounded by \( |C'(\ell)(G)|/\varepsilon^2 = (n/\varepsilon)^{O(1)} \). This can be done for each \( \ell = 1, \ldots, m \) in time \( (n/\varepsilon)^{O(1)} \).

To finish the proof of the theorem, we compute \( p_\ell \) for each \( \ell = 1, \ldots, m \) by adding all the numbers \( a_{F, \ell} \text{ind}^*(F, G) \) over all \( F \in C'(\ell)(G) \). This can be done in time

\[
O(m|C'(m)(G)|n^2 \Delta^2 (m-1)) = (n/\varepsilon)^{O(1)},
\]

where we have used that computing \( \text{ind}^*(F, G) \) with \( F \in C'(\ell)(G) \) takes time \( O(n^2 \Delta^2 (m-1)) \) by Lemma \( 7.3 \) Using the Newton identities \((17)\) we obtain the coefficients \( e_1, \ldots, e_m \) from \( p_1, \ldots, p_m \) in time \( O(m^2) \). This finishes the proof.

### 8 Concluding remarks and open questions

In this paper we have presented a new method that yields deterministic polynomial-time approximation algorithms for evaluations of the Tutte polynomial, the independence polynomial and graph polynomials obtained from spin and edge-coloring models at complex numbers on bounded degree graphs. Our method could be crudely summarized by saying: *absence of zeros implies the existence of efficient approximation algorithms.*
This naturally leads to the question of how hard it is to approximate evaluations of
these graph polynomials when one is close to (complex) roots. Of course this question
is rather vague. So let us formulate a more concrete question.

Question 8.1. Recall the constant \( \lambda^*(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta-1}} \) for \( \Delta \in \mathbb{N} \). What is the complexity
of approximating \( Z_G(-\lambda) \) for \( \lambda > \lambda^*(\Delta) \) for graphs \( G \) of maximum degree at most \( \Delta \)?

As is noted in the introduction our result for the independence polynomial at posi-
tive \( \lambda \) does not allow us to efficiently approximate the independence polynomial at \( \lambda \)
for \( \lambda^* \leq \lambda < \lambda_c \). (Which can be done with the correlation decay approach cf. Weitz [43].) But using the approach in Section 4.2 it would follow f rom a positive answer to
the following question, which is a restatement of version of a conjecture of Sokal [39].

Question 8.2. Let \( \varepsilon > 0 \) and \( \Delta \in \mathbb{N} \). Does there exists \( \delta > 0 \) (possibly depending on \( \Delta \))
such that if \( \lambda \in \mathbb{C} \) satisfies

\[
|\Im(\lambda)| \leq \delta, \text{ and } 0 \leq \Re(\lambda) \leq (1 - \varepsilon) \frac{(\Delta - 1)^{\Delta-1}}{(\Delta - 1)^{\Delta}},
\]  \hfill (37)

then \( Z_G(\lambda) \neq 0 \) for all graphs \( G \) of maximum degree at most \( \Delta \).

Another question that arises naturally is the following. Barvinok [2, 5] found quasi-
polynomial-time approximation algorithms for computing the permanent of certain ma-
trices, based on absence of zeros. Our method for computing coefficients of certain
graph polynomials (i.e. BIGCP) on bounded degree graphs presented in Section 3 does
not seem to apply to permanents. It would be very interesting to find a more general
method that also applies to permanents.

Acknowledgements

We thank Alexander Barvinok for stimulating discussions, useful remarks and for shar-
ing the results in [6] with us.

References

[1] M. Bayati, D. Gamarnik, D. Katz, C. Nair and P. Tetali, Simple deterministic ap-
proximation algorithms for counting matchings. In Proceedings of the thirty-ninth
annual ACM symposium on Theory of computing (pp. 122–127), ACM, 2007.
[2] A. Barvinok, Computing the permanent of (some) complex matrices, Foundations
of Computational Mathematics (2014) 1–14.
[3] A. Barvinok, Computing the partition function for cliques in a graph, Theory of
Computing 11 (2015), Article 13 pp. 339–355
[4] A. Barvinok, Computing the partition function of a polynomial on the Boolean
cube, arXiv preprint, arXiv:1503.07463 (2015).
[5] A. Barvinok, Approximating permanents and hafnians, ArXiv preprint,
arXiv:1601.07518 (2016).
[6] A. Barvinok, Personal communication (2016).
[7] A. Barvinok and P. Soberón, Computing the partition function for graph homo-
morphisms, arXiv preprint, arXiv:1406.1771 (2014). To appear in Combinatorica, doi:
10.1007/s00493-016-3357-2.
[8] A. Barvinok and P. Soberón, Computing the partition function for graph homo-
morphisms with multiplicities, Journal of Combinatorial Theory, Series A 137 (2016)
1–26.
[9] C. Borgs, J. Chayes, J. Kahn and L. Lovász, Left and right convergence of graphs with bounded degree, Random Structures and Algorithms 42 (2013) 1–28.

[10] R. Bubley, M. Dyer, C. Greenhill and M. Jerrum: On approximately counting colorings of small degree graphs, SIAM Journal on Computing 29 (1999) 387–400.

[11] A. Bulatov and M. Grohe, The complexity of partition functions, Theoretical Computer Science 348 (2005) 148–186.

[12] J. Cai, X. Chen and P. Lu, Graph homomorphisms with complex values: A dichotomy theorem, SIAM Journal on Computing 42 (2013) 924–1029.

[13] J. Cai, H. Guo and T. Williams, A complete dichotomy rises from the capture of vanishing signatures, In: Proceedings of the forty-fifth annual ACM symposium on Theory of computing, pp. 635–644. ACM, 2013.

[14] J. Cai, S. Huang and P. Lu, From Holant to #CSP and Back: Dichotomy for Holant Problems. In ISAAC, pages 253–265, 2010.

[15] J. Cai, P. Lu and M. Xia, Computational complexity of Holant problems, SIAM Journal on Computing 40 (2011) 1101–1132.

[16] M. Chudnovsky and P. Seymour, The roots of the independence polynomial of a clawfree graph, Journal of Combinatorial Theory, Series B 97 (2007) 350–357.

[17] P. Csikvári and P. E. Frenkel, Benjamini–Schramm continuity of root moments of graph polynomials, European Journal of Combinatorics 52 (2016) 302–320.

[18] P. de la Harpe, and V.F.R. Jones, Graph invariants related to statistical mechanical models: examples and problems, Journal of Combinatorial Theory, Series B 57 (1993) 207–227.

[19] M. Dyer and C. Greenhill, On Markov chains for independent sets, Journal of Algorithms 35 (2000) 17–49.

[20] M. Dyer and C. Greenhill, The complexity of counting graph homomorphisms, Random Structures and Algorithms 17 (2000) 260–289.

[21] A. Galanis, D. Stefankovic, E. Vigoda and L. Yang: Ferromagnetic Potts model, Refined #BIS-hardness and related results. In RANDOM 2014, LNCS 6845, pp. 677–691 2014. Full version available at http://arxiv.org/abs/1311.4839.

[22] D. Gamarnik and D. Katz, Correlation decay and deterministic FPTAS for counting list-colorings of a graph, Journal of Discrete Algorithms 12 (2012) 29–47.

[23] L.A. Goldberg and H. Guo, The complexity of approximating complex-valued Ising and Tutte partition functions, arXiv preprint arXiv:1409.5627 (2014).

[24] L.A. Goldberg and M. Jerrum, Approximating the partition function of the ferromagnetic Potts model, Journal of the ACM 59 (2012) 1–25.

[25] L.A. Goldberg, and M. Jerrum, The complexity of computing the sign of the Tutte polynomial (and consequent #P-hardness of approximation) In Automata, Languages, and Programming, pp. 399–410, Springer Berlin Heidelberg, 2012.

[26] B. Jackson, A. Proucci and A.D. Sokal, Complex zero-free regions at large |q| for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights, Journal of Combinatorial Theory, Series B 103 (2013) 21–45.

[27] M. Jerrum, A very simple algorithm for estimating the number of k-colorings of a low-degree graph, Random Structures and Algorithms 7 (1995), 157–165.

[28] M. Jerrum and A. Sinclair, Polynomial-time approximation algorithms for the Ising model, SIAM Journal on computing 22 (1993) 1087–1116.

[29] T. Lee and T. Yang, Statistical theory of equations of state and phase transitions. I. Theory of condensation, Physical Review 87 (1952): 404.
[30] C. Lin, J. Liu and P. Lu, A simple FPTAS for counting edge covers, in: Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, pp341–348, SIAM, 2014.

[31] P. Lu and Y. Yin, Improved FPTAS for multi-spin systems, in: Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pp 639–654, Springer Berlin Heidelberg, 2013.

[32] D.G. Mead, Newton’s Identities, The American Mathematical Monthly, 99 (1992), 749–751.

[33] G. Regts, Graph Parameters and Invariants of the Orthogonal Group, PhD thesis, University of Amsterdam, 2013.

[34] G. Regts, Zero-free regions of partition functions with applications to algorithms and graph limits, arXiv preprint, arXiv.1507.02089 (2015). To appear in Combinatorica.

[35] A.D. Scott and A.D. Sokal, The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma, Journal of Statistical Physics 118 (2005) 1151–1261.

[36] J.B. Shearer, On a problem of Spencer, Combinatorica 5 (1998) 241–245.

[37] A. Sinclair, P. Srivastava and M. Thurley, Approximation algorithms for two-state anti-ferromagnetic spin systems on bounded degree graphs, Journal of Statistical Physics 155 (2014) 666–686.

[38] A. Sly and N. Sun, The computational hardness of counting in two-spin models on d-regular graphs, in Proceedings of the 53rd Annual Symposium on Foundations of Computer Science (FOCS), 2012 IEEE, pp. 361–369. IEEE, 2012.

[39] A. Sokal, A personal list of unsolved problems concerning lattice gases and anti-ferromagnetic Potts models, Markov Processes And Related Fields 7 (2001) 21–38.

[40] B. Szegedy, Edge-coloring models and reflection positivity, Journal of the American Mathematical Society 20 (2007) 969–988.

[41] B. Szegedy, Edge coloring models as singular vertex-coloring models, in: Fete of Combinatorics and Computer Science (G.O.H. Katona, A. Schrijver, T.Szönyi, editors), Springer, Heidelberg and János Bolyai Mathematical Society, Budapest (2010) 327–336.

[42] E. Vigoda, Improved bounds for sampling colorings, Journal of Mathematical Physics 41 (2000), 1555–1569.

[43] D. Weitz, Counting independent sets up to the tree threshold, in Proceedings of the thirty-eighth annual ACM symposium on Theory of computing, STOC 06, pages 140–149, New York, NY, USA, 2006. ACM.