Study on the stability of artificial vibration points with internal resonance in solar sail dynamics

Zhenbao Diao
Beijing University of Technology, Beijing, China
Email: diaozhenb@163.com

Abstract. In this paper, the linear and stability of solar sail spacecraft at the libration point of artificial balance is studied. The stability has been analyzed for the linear cases and nonresonance cases. The changing law of artificial balance point is found when the attitude angle of solar sail is fixed. The linear stability region of the solar sail spacecraft at L4 was found. Based on the circular restricted three-body problem, the dynamic model of the continuous solar sail spacecraft is established. The variation law of the position of manual translational point with the $\beta$ parameters is analyzed. The condition and existence range of linear stability of artificial translational point are given. The influence of thrust parameters on the system and manual translational point is analyzed. The dynamics equation of the solar sail spacecraft near the linearly stable artificial translational point and the nonlinear vibration theory, the range of the system mass parameters and thrust parameters when the system has 1:2 and 1:3 internal resonance is obtained. Using the multi-scale method, the stability of the long and short period motion of the detector is analyzed, and the boundedness of motion is proved.

1. Introduction
The libration point at which the gravity and centrifugal forces of two main bodies balance each other in the circular restricted three-body model (CR3BP). The parallel libration points L1 and L2 of the solar and terrestrial system are about 1.5 million kilometers away from the earth, located within a reasonable distance that can be reached by current space technology. Collinear translational points are not disturbed by near-earth magnetic field, atmosphere, space debris and other factors, and have a fairly stable dynamic and thermodynamic environment. At the same time, they are not blocked by the field of view of other celestial bodies. More importantly, because of its balanced nature, a probe located near the translational point needs very little fuel to stay in orbit for a long time.

Solar sail is another typical continuous small thrust propulsion technology besides electric propulsion. Although electric propulsion is much more efficient than conventional chemical propulsion, it is still limited by the amount of fuel the spacecraft can carry, which makes it difficult to achieve truly long-term deep space missions. The advent of solar sail technology has changed that situation completely. Solar sail voyages offer the possibility of low-cost long-range missions that are impossible for any other type of conventional spacecraft.

Solar sail technology was developed by NASA in the mid-1970s for the proposed Halley Comet Rendezvous mission. In this development work, Sauer conducted the study of time-optimal heliocentric orbits[1], and Sackett and Edelbaum conducted the study of time-optimal escape orbits[2]. Solar sails are capable of performing many tasks that conventional spacecraft cannot, such as chemical propulsion or even advanced solar-electric propulsion. These include constantly levitating at the sun's poles, using the sun's radiation pressure to balance the sun's gravity, or going into a circular orbit in sync with the sun's rotation. The solar sail light pressure is related to the mass ratio of the sail, the...
distance from the sun, the reflectivity of the sail and the attitude angle of the sail[3]. McInnes[4][5][6] made an in-depth study of solar sail orbit dynamics and calculated the position distribution of artificial translational points in the model. McInnes[7] studied the problem of constructing the Halo orbit of the collinear artificial translational point in the solar sail model, and based on the perturbation method, gave the expression construction method of the third-order periodic orbit of the solar sail. However, these analytical solutions obtained by linear approximation of the orbital nonlinear equations under given conditions are unstable. Solar sail collinear artificial translational point orbit mission needs to exert long-term orbit keeping control.

2. Solar Sail Circular Restricted Three-body Problem

2.1 Equation of motion
In the context of CRTBP, the Earth and the Sun are particles moving in circles around their common center of gravity due to mutual gravity. The solar sail spacecraft is assumed to be a massless particle that does not affect the motion of the two particles, but is influenced by their gravity as well as by SRP. We consider the solar sail spacecraft in a rotating coordinate system with the center of gravity of the sun and earth as the origin, and the angular velocity is $\omega=[0;0;1]$. The earth and the sun are fixed on the positive and negative axes of the X-axis respectively, the X-axis direction points to the earth, the z-axis is perpendicular to the plane of rotation of the sun and the earth, and the Y-axis is determined by the orthogonal reference frame.

In the system, the gravitational constant $G=1$ and the mass of the two main bodies are $m_{\text{earth}}=\mu=3.0034806\times10^{-6}$, $m_{\text{sun}}=1-\mu$ after dimensional normalization. The vector form of the motion equation of the solar sail in the sun-earth system is

$$\frac{d^2\mathbf{r}}{dt^2} + 2\omega \times \frac{d\mathbf{r}}{dt} + \omega \times (\omega \times \mathbf{r}) = \mathbf{a} - \nabla V$$

(1)

where $V$ is the three-body gravitational potential that can be defined by

$$V = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$$

(2)

Where $\mathbf{a}$ is the acceleration caused by the impact of solar photons on the solar sail. In this paper, we consider that the normal direction of the solar sail is in the same direction as the sun rays, so the acceleration can be written as:

$$\mathbf{a} = \beta \frac{1-\mu}{r_1^2} \mathbf{r}_1$$

(3)

where $\mathbf{r}_1$ is directed along the Sun-line and $r_1=|\mathbf{r}_1|$. $\beta$ is the solar sail lightness number. We have the position of the solar sail relative to the primary and secondary mass as

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

(4)

The classical case with no radiation pressure is obtained when $\beta=0$, and a value of $\beta=1$ would have radiation pressure equal to the gravitational force due to the Sun. As discussed typical values of $\beta$ for solar sails are fairly low, generally below 0.1 [8].

![Figure 1](image.png) Location and orientation of the solar sail in the CRTBP, shown in the x-z plane.
Based on the above assumptions, the motion equation of the solar sail in the sun-earth system can be obtained:

$$\begin{align*}
\ddot{x} - 2\dot{y} - x &= -\frac{1-\mu}{r_1^3} \dot{x} + \frac{1-\beta}{r_1^3} - \frac{\mu}{r_2^3} x - 1+\mu \\
\ddot{y} - 2\dot{x} - y &= -\frac{1-\mu}{r_1^3} \dot{y} + \frac{1-\beta}{r_1^3} y - \frac{\mu}{r_2^3} \frac{\mu}{r_2^3} \\
\ddot{z} &= -\frac{1-\mu}{r_1^3} - \frac{1-\beta}{r_1^3} z - \frac{\mu}{r_2^3} z 
\end{align*}$$

(5)

This system effective potential $\Omega$ can be written as:

$$\Omega = \frac{1}{2} \left( x^2 + y^2 \right) + \frac{(1-\mu)(1-\beta)}{r_1} + \frac{\mu}{r_2}$$

(6)

With the $\beta=0$, the classical Jacobi constant is recovered.

This new system differs from the CRTBP only by the modification to the Sun’s gravitational potential term and is still Hamiltonian. The mechanical energy constant of the system is

$$E = \frac{1}{2} \left( x^2 + y^2 + z^2 \right) - \frac{1}{2} \left( x^2 + y^2 \right) - \frac{(1-\mu)(1-\beta)}{r_1} - \frac{\mu}{r_2}$$

(7)

and the Jacobi constant of the system still can be defined as $C= -2E$. With the $\beta=0$, the classical Jacobi constant is recovered.

3. Artificial libration points linear stability and natural frequency

When $\beta<1$, the autonomous solar sail system is similarly to the CRTBP system. Although locations of five libration points are slightly modified from the CRTBP system depending on the value of the lightness parameter, they are still remain in the x–y plane [9]. The positions of the artificial libration points are found by making zero right sides of Eq.(5), that is, by solving $\Omega x=0$, $\Omega y=0$, that can be written as:

$$\begin{align*}
x - \frac{1-\mu}{r_1^3} x + \frac{1-\beta}{r_1^3} - \frac{\mu}{r_2^3} x - 1+\mu &= 0 \\
y - \frac{1-\mu}{r_1^3} y + \frac{1-\beta}{r_1^3} y - \frac{\mu y}{r_2^3} &= 0
\end{align*}$$

(8)

To find the collinear points which remain on the x-axis of the synodic system, we write $y = 0$ in Eqs.(8) and solve the first equation in Eqs.(8) for $x$, which leads to a fifth-order algebraic equation for each point. When we consider $y\neq0$, the solution of the non-collinear points is obtained. A variety of methods are available for the solution of those points with $\beta$ as parameter. Traditionally, an iteration process is used for finding the real root for the above algebraic equations similarly with Szebehely’s work [10]. In this study, the solutions of five libration points are numerically obtained and shown in Figure 2.
Figure 2 Locations of libration points with variation of $\beta$ from 0 to 1 ($\mu=0.00025$).

As we can conclude from the simulations, the locations of L1, L3, L4, L5 move towards the Sun, while L2 moves inwards towards the Earth with the increase of $\beta$.

To investigate the stability of artificial L4, it is necessary to move the origin of coordinate system from the barycenter to the stable libration point. For the L4-centric Synodic coordinate system, the translation transformation is carried out on the basis of the original coordinate system, which is denoted as L4-$\xi\eta\zeta$. In the new coordinate system, the motion of spacecraft can be denoted as $\rho=\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$. The relationship between L4-$\xi\eta\zeta$ and O-$xyz$ can be expressed as:

$$x = \xi + I_{4x}, \quad y = \eta + I_{4y}, \quad \zeta = z$$

(9)

And

$$\frac{1}{r_1} = \frac{1}{\sqrt{\left(\xi + I_{4x} + \mu\right)^2 + \left(\eta + I_{4y}\right)^2 + \zeta^2}} = m \sum_{n \geq 0} (m \rho)^n P \left( -m \left( I_{4x} + \mu \right) \xi + I_{4y} \eta \right) \rho,$$

$$\frac{1}{r_2} = \frac{1}{\sqrt{\left(\xi + I_{4x} + \mu - 1\right)^2 + \left(\eta + I_{4y}\right)^2 + \zeta^2}} = n \sum_{n \geq 0} (n \rho)^n P \left( -n \left( I_{4x} + \mu - 1 \right) \xi + I_{4y} \eta \right) \rho,$$

(10)

where

$$m = \frac{1}{\sqrt{(I_{4x} + \mu)^2 + I_{4y}^2}}, \quad n = \frac{1}{\sqrt{(I_{4x} + \mu - 1)^2 + I_{4y}^2}}, \quad \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

(11)

By this way, the equation of motion (5) to can be rewritten up to second order as:

$$\dot{\xi} - 2\eta - \Omega_{xx} \xi - \Omega_{xy} \eta = \alpha_{200} \xi^2 + \alpha_{110} \xi \eta + \alpha_{020} \eta^2 + \alpha_{002} \zeta^2 + O(3)$$

$$\dot{\eta} + 2\dot{\xi} - \Omega_{xy} \xi - \Omega_{yy} \eta = \beta_{200} \xi^2 + \beta_{110} \xi \eta + \beta_{020} \eta^2 + \beta_{002} \zeta^2 + O(3)$$

$$\dot{\zeta} + \zeta = \gamma_{101} \xi \zeta + \gamma_{011} \eta \zeta + O(3)$$

(12)

where $\alpha_{200}$, $\alpha_{110}$, $\alpha_{020}$, $\alpha_{300}$, $\alpha_{210}$, $\alpha_{120}$, $\beta$, $\gamma$ can be found in Appendix A. where $O(3)$ represents small quantities of order $\xi$, $\eta$, and $\zeta$ above order 3 (including order 3), the subscript indicates that the second derivatives of $\Omega$ with respect to x or y are evaluated at equilibrium point, and
From the linear part of Eq.(12), we can see that the motion in the z direction is decoupled from that in the x-y plane and is harmonic with frequency \( \omega_z = 1 \). For the x-y components, the characteristic equation of the linear system is

\[
\lambda^4 + 4 - \Omega_{xx} - \Omega_{yy} \lambda^2 + \Omega_{xy} \Omega_{yx} - \Omega_{xy}^2 = 0
\]  

(14)

The study of stability is important for understanding the dynamics around equilibrium points[3]. The triangular libration points are center-center-center and linearly stable under the modified Routh ratio. Below the modified Routh ratio, the motion is stable and can be represented as superposition of two oscillatory modes. The Linear stability natural frequencies with variation of \( \beta \) and \( \mu \) are shown in Figure 3.

**Figure 3** Linear stability \( \omega_1 \) with variation of \( \beta \) and \( \mu \).

**Figure 4** Linear stability \( \omega_2 \) with variation of \( \beta \) and \( \mu \).
4. Nonlinear Stability Analysis for L4 Point with 1:2 Resonances

In this study, we focus on the 1:2 resonance case. The method used in this study also can be applied to 1:3 resonance case. A perturbation solution for finite but small amplitude is obtained in this section using the method of multiple scales. According to this method, ξ and η are assumed to be functions of two time scales, a fast time, T0=εt, and two slower time, T1=εt, where ε is a small, dimensionless parameter the order of the amplitudes. We seek a third-order solution for small but finite amplitudes in the form.

\[ \varsigma(t;\varepsilon) = \varepsilon \varsigma^1(T_0, T_1) + \varepsilon^2 \varsigma^2(T_0, T_1) + 0(\varepsilon^3) \]
\[ \eta(t;\varepsilon) = \varepsilon \eta^1(T_0, T_1) + \varepsilon^2 \eta^2(T_0, T_1) + 0(\varepsilon^3) \]  

(15)

The time derivation becomes

\[ d/dt = D_0 + \varepsilon D_1 + \ldots, \quad D_n = \partial^n / \partial T^n \]

(16)

Substituting Eq.(16) into Eq.(12) and equating coefficients of like powers of ε, we obtain:

Order ε

\[ M_1 (\varsigma^1, \eta^1) = D^2_0 \varsigma^1_1 - 2D_0 \eta^1_1 - \Omega^0_\varsigma \varsigma^1_1 - \Omega^0_\eta \eta^1_1 = 0 \]
\[ M_2 (\varsigma^1, \eta^1) = D^2_0 \eta^1 + 2D_0 \varsigma^1 - \Omega^0_\varsigma \varsigma^1_1 - \Omega^0_\eta \eta^1_1 = 0 \]  

(17)

Order ε²

\[ M_1 (\varsigma^2, \eta^2) = D^2_0 \varsigma^2_1 - 2D_0 \eta^2_1 - \Omega^0_\varsigma \varsigma^2_1 - \Omega^0_\eta \eta^2_1 = 0 \]
\[ M_2 (\varsigma^2, \eta^2) = D^2_0 \eta^2 + 2D_0 \varsigma^2 - \Omega^0_\varsigma \varsigma^2_1 - \Omega^0_\eta \eta^2_1 = 0 \]

(18)

The solution of Eq.(17) can be expressed in the form

\[ \varsigma^1 = a_1 \cos (B_1) + a_2 \cos (B_2) \]
\[ \eta^1 = b_1 \sin (B_1) + c_1 \cos (B_1) + b_2 \sin (B_2) + c_2 \cos (B_2) \]

(19)

where \( \omega_n \) are the distinct natural frequencies, and

\[ b_1 = \frac{-2\omega_1 (\omega_2^2 + \Omega_{xy})}{4\omega_1^2 + \Omega_{xy}^2} a_1 = \frac{-2\omega_1}{\omega_1 + \Omega_{xy}} a_1 \]
\[ b_2 = \frac{-2\omega_2 (\omega_2^2 + \Omega_{xy})}{4\omega_2^2 + \Omega_{xy}^2} a_2 = \frac{-2\omega_2}{\omega_2 + \Omega_{xy}} a_2 \]
\[ c_1 = \frac{-\Omega_{xy} (\omega_2^2 + \Omega_{xy})}{4\omega_1^2 + \Omega_{xy}^2} a_1 = \frac{-\Omega_{xy}}{\omega_1 + \Omega_{xy}} a_1 \]
\[ c_2 = \frac{-\Omega_{xy} (\omega_2^2 + \Omega_{xy})}{4\omega_2^2 + \Omega_{xy}^2} a_2 = \frac{-\Omega_{xy}}{\omega_2 + \Omega_{xy}} a_2 \]  

(20)

Then, substituting \( \varsigma^1, \eta^1 \) into Eq.(18) yields

\[ M_1 (\varsigma^2, \eta^2) = P_{11} \sin (B_1) + Q_{11} \cos (B_1) + P_{12} \sin (B_2) + Q_{12} \sin (B_2) \]
\[ M_2 (\varsigma^2, \eta^2) = P_{21} \sin (B_1) + Q_{21} \cos (B_1) + P_{22} \sin (B_2) + Q_{22} \sin (B_2) \]  

(21)

where \( P_{11}, \ Q_{11}, \ P_{12}, \ Q_{12}, \ P_{21}, \ Q_{21}, \ P_{22}, \ Q_{22} \) can be found in Appendix B. In determining the solvability conditions of Eq.(21), we introduce the detuning parameter \( \sigma \) defined by

\[ \omega = 2\omega_1 + \varepsilon \sigma \]  

(22)

To determine the solvability conditions of Eq.(21), we seek a particular solution in the form
\[ \zeta_2 = 0 \]

\[ \eta_2 = d_1 \cos(B_1) + e_1 \sin(B_1) + d_2 \cos(B_2) + e_2 \sin(B_2) \]  

The particular solution contains secular terms which make \( \zeta_2 \) and \( \eta_2 \) become unbounded as \( T_0 \) goes infinitely. Hence, \( \zeta_2 \) and \( \eta_2 \) will dominate the lower-order terms, and the expansion will break down for large \( t \) unless the secular terms are eliminated.

Substituting Eq.(23) into Eq.(21), using Eq.(17), and equating the coefficients of cos and sin on both sides, we obtain

\[
\begin{align*}
2d_1 \omega_i - e_1 \Omega_{xy} &= P_{1i} \\
-2e_1 \omega_i - d_1 \Omega_{xy} &= Q_{1i} \\
-e_1 (\omega_1^2 + \Omega_{xy}) &= P_{2i} \\
-d_1 (\omega_1^2 + \Omega_{xy}) &= Q_{2i}
\end{align*}
\]  

The average equation is obtained based on the above derivation

\[
\begin{align*}
\alpha_i' &= \frac{1}{2} a_i a_2 \Gamma_i \cos(\gamma) \\
\beta_i' &= \frac{1}{2} a_i a_2 \Gamma_i \sin(\gamma) \\
\alpha_2' &= -\frac{1}{2} a_i^2 \Gamma_2 \cos(\gamma) \\
\alpha_2 \beta_2' &= \frac{1}{2} a_i^2 \Gamma_2 \sin(\gamma)
\end{align*}
\]  

Among them

\[
\Gamma_1 = \left| D_1 \right| \frac{(\omega_1^2 + \Omega_{xy})}{(\omega_1^2 - \omega_2^2)}
\]

\[
\Gamma_2 = \left| D_2 \right| \frac{(\omega_2^2 + \Omega_{xy})}{(\omega_2^2 - \omega_1^2)}
\]

Where \( D_1, D_2 \) can be found in Appendix A and \( \gamma = r - \varphi = \beta_2 - 2\beta_1 + \sigma T_1 - \varphi \). For the solar sail system, the natural frequencies are depending on solar sail lightness number \( \beta \), considering the internal resonance, the variation of \( \Gamma_1 \) and \( \Gamma_2 \) are shown in Figure 5.

\[ \text{Figure 5:} \text{1:2 Internal resonance } \Gamma_1 \text{ and } \Gamma_2 \text{ scope.} \]
As we can concluded from Figure 5 that $\Gamma_1$ is always negative and $\Gamma_2$ is always positive with variation of $\beta$ around the 1:2 internal resonance.

Eliminating $\beta_1$ and $\beta_2$ from Eq.(25) gives

$$a_2' = a_2\sigma + \frac{1}{2} a_1^2 \Gamma_2 \sin(\gamma) - a_2^2 \Gamma_1 \sin(\gamma)$$

(27)

The periodic orbits to second-order correspond to stationary solutions of Eqs.(25) and Eq.(27). They are given by

$$\cos \gamma_0 = 0, \quad \gamma_0 = \frac{n\pi}{2}$$

(28)

And

$$\left(\frac{\sigma}{\Gamma_2} a_1^2 - 2a_2^2 \Gamma_1 \right) \sin \gamma_0 = -2a_2\sigma$$

(29)

Since $\Gamma_1 < 0$, we introduce variables

$$\bar{a}_1 = \sqrt{\frac{\Gamma_2}{\Gamma_1}} a_1, \quad \bar{a}_2 = \sqrt{\frac{2\Gamma_1}{\Gamma_2}} a_2$$

(30)

Eq.(29) can be transformed as

$$\bar{a}_1^2 + (\bar{a}_2 - \rho)^2 = \rho^2, \quad \rho = \frac{-\sigma}{\sqrt{2\Gamma_1} \sin \gamma_0}$$

(31)

To determine the stability of those orbits, we let

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2$$

(32)

$$\bar{a}_1 = \bar{a}_1 + \Delta a_1 e^{i\epsilon}, \quad \bar{a}_2 = \bar{a}_2 + \Delta a_2 e^{i\epsilon}, \quad \gamma = \gamma_0 + \Delta \gamma e^{i\epsilon}$$

Substituting Eqs.(32) into Eqs.(25) and Eq.(27), and using Eqs.(28)-(29), expanding for small $\Delta a_1$, $\Delta a_2$ and $\Delta \gamma$, give

$$s\Delta a_1 = \frac{1}{2\sqrt{2}} \sqrt{\frac{\Gamma_2}{\Gamma_1}} \bar{a}_1 \bar{a}_2 \Delta \gamma \sin \gamma_0$$

$$s\Delta a_2 = \frac{1}{2\sqrt{2}} \sqrt{\frac{2\Gamma_1}{\Gamma_2}} \left(\bar{a}_1^2\right)^2 \left(\Delta \gamma \sin \gamma_0\right)$$

$$s\Delta \gamma = \left(\sigma + \bar{a}_2 \frac{\rho}{\bar{a}_2} \frac{3}{2}\right) \Delta a_2 + \bar{a}_1 \sqrt{\frac{2\Gamma_1}{\Gamma_2}} \sin \gamma_0 \Delta a_1$$

(33)

Eliminating $\Delta a_1$, $\Delta a_2$ and $\Delta \gamma$ in Eqs.(33), yields

$$s^2 = -\frac{1}{\Gamma_1} \left(\bar{a}_1^2\right)^2 \left(\sin \gamma_0\right)^2 \left(\frac{\rho}{\bar{a}_2} - \frac{3}{2}\right)$$

(34)

Thus, a periodic orbit is stable if

$$|\bar{a}_2| \leq \left|\frac{2}{3} \rho\right| \quad \text{or} \quad |a_2| \leq \left|\frac{\sigma}{3\Gamma_1 \sin \left(\frac{\pi n}{2}\right)}\right|$$

(35)

Otherwise, it is unstable.

5. Conclusion

In the same radiating planetary system, different light pressure parameters can be changed to obtain different stable orbits at L4, which provides more orbital possibilities. Through linearization analysis, the above range of $\mu$ and $\beta$ stable solution with amplitude-independent frequencies $\omega_1$ and $\omega_2$ can
be predicted. The long and short period stability of detector motion is analyzed when internal resonance occurs in the system. For the short period motion, there is a steady-state solution, and the system has a short period single mode orbital motion near the triangular translational point. The corresponding orbital frequency is the natural frequency of the linear system (quadratic nonlinearity does not affect the frequency). Although the initial motion state of the spacecraft is stable and periodic, the steady periodic motion will be broken by any small disturbance, making the motion no longer periodic. Therefore, the short period orbital family at L4 is unstable at 1:2 internal resonance.

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Appendix A

\[
\alpha_{200} = (1 - \mu)(1 - \beta)\left(-\frac{15}{2} m^7 (x_\gamma + \mu)^3 + \frac{9}{2} m^5 (x_\gamma + \mu) + \mu\left(-\frac{15}{2} n^7 (x_\gamma + \mu - 1)^3 + \frac{9}{2} m^5 (x_\gamma + \mu - 1)\right)\right)
\]

\[
\alpha_{410} = (1 - \mu)(1 - \beta)\left(-15 m^7 (x_\gamma + \mu)^2 y_0 + 3 m^5 y_0 + \mu\left(-15 n^7 (x_\gamma + \mu - 1)^2 y_0 + 3 n^5 y_0\right)\right)
\]

\[
\alpha_{020} = (1 - \mu)(1 - \beta)\left(-\frac{15}{2} m^7 (x_\gamma + \mu)^2 y_0^2 + \frac{3}{2} m^5 (x_\gamma + \mu) + \mu\left(-\frac{15}{2} n^7 (x_\gamma + \mu - 1)^2 y_0^2 + \frac{3}{2} m^5 (x_\gamma + \mu - 1)\right)\right)
\]

\[
\alpha_{002} = (1 - \mu)(1 - \beta) m^5 (x_\gamma + \mu) + \frac{3}{2} \mu m^5 (x_\gamma + \mu - 1)
\]

\[
\beta_{200} = (1 - \mu)(1 - \beta)\left(-\frac{15}{2} m^7 (x_\gamma + \mu)^2 y_0 + \frac{3}{2} m^5 y_0 + \mu\left(-\frac{15}{2} n^7 (x_\gamma + \mu - 1)^2 y_0 + \frac{3}{2} n^5 y_0\right)\right)
\]

\[
\beta_{410} = (1 - \mu)(1 - \beta)\left(-15 m^7 (x_\gamma + \mu) y_0^2 + 3 m^5 (x_\gamma + \mu) + \mu\left(-15 n^7 (x_\gamma + \mu - 1) y_0^2 + 3 n^5 (x_\gamma + \mu - 1)\right)\right)
\]

\[
\beta_{020} = (1 - \mu)(1 - \beta)\left(-\frac{15}{2} m^7 y_0^3 + \frac{9}{2} m^5 y_0 + \mu\left(-\frac{15}{2} n^7 y_0^3 + \frac{9}{2} n^5 y_0\right)\right)
\]

\[
\beta_{002} = \frac{3}{2}(1 - \mu)(1 - \beta) m^5 y_0 + \frac{3}{2} \mu m^5 y_0
\]

\[
\gamma_{011} = 3(1 - \mu)(1 - \beta) m^5 (x_\gamma + \mu) + 3 m^5 (x_\gamma + \mu - 1)
\]

\[
\gamma_{011} = 3(1 - \mu)(1 - \beta) m^5 y_0 + 3 m^5 y_0
\]

\[
|D_1| = \sqrt{-m_\beta b_\alpha a_\gamma c_\epsilon + a_\beta a_\epsilon a_\gamma} + (n_\beta b_\alpha c_\epsilon + a_\beta a_\epsilon a_\gamma)^2 \quad a_\alpha a_\beta
\]

\[
|D_2| = \sqrt{(b_\gamma m_\beta a_\beta a_\epsilon - c_\gamma n_\epsilon)^2 + (b_\gamma m_\beta a_\beta a_\epsilon + c_\gamma n_\epsilon)^2} \quad a_\alpha a_\beta
\]
Appendix B

\[ P_{11} = -m_{1a} \cos(r) - n_{1a} \sin(r) + 2a_1' \omega_1 + 2b_1' - 2c_1 \beta_1' \]
\[ Q_{11} = -m_{1a} \sin(r) + n_{1a} \cos(r) + 2a_1 \beta_1' \omega_1 + 2b_1 \beta_1' + 2c_1' \]
\[ P_{12} = m_{2a} \sin(r) + n_{2a} \cos(r) + 2a_2' \omega_2 + 2b_2' - 2c_2 \beta_2' \]
\[ Q_{12} = m_{2a} \cos(r) - n_{2a} \sin(r) + 2a_2 \beta_2' \omega_2 + 2b_2 \beta_2' + 2c_2' \]
\[ P_{21} = -m_{1b} \cos(r) - n_{1b} \sin(r) + 2a_1' \omega_1 + 2b_1' \omega_1 + 2c_1' \omega_1 \]
\[ Q_{21} = -m_{1b} \sin(r) + n_{1b} \cos(r) - 2a_1' - 2b_1' \omega_1 + 2c_1' \omega_1 \]
\[ P_{22} = m_{2b} \sin(r) + n_{2b} \cos(r) + 2a_2 \beta_2' + 2b_2 \beta_2' \omega_2 + 2c_2' \omega_2 \]
\[ Q_{22} = m_{2b} \cos(r) - n_{2b} \sin(r) - 2a_2' - 2b_2' \omega_2 + 2c_2 \beta_2' \omega_2 \]

\[ m_{1a} = \left( -\frac{1}{2} a_1 b_2 + \frac{1}{2} a_2 b_1 \right) a_{110} + (b_1 c_2 - c_1 b_2) a_{020} \]
\[ n_{1a} = a_1 a_2 a_{200} + \left( \frac{1}{2} a_1 c_2 + \frac{1}{2} a_2 c_1 \right) a_{110} + (b_1 b_2 + c_1 c_2) a_{020} \]
\[ m_{2a} = \frac{1}{2} a_1^2 a_{200} + \frac{1}{2} a_1 c_1 a_{110} + \left( \frac{1}{2} c_1^2 - \frac{1}{2} b_1^2 \right) a_{020} \]
\[ n_{2a} = \frac{1}{2} a_1 b_1 a_{110} + b_1 c_1 a_{020} \]
\[ m_{1b} = \left( -\frac{1}{2} a_1 b_2 + \frac{1}{2} a_2 b_1 \right) b_{110} + (b_1 c_2 - c_1 b_2) b_{020} \]
\[ n_{1b} = a_1 a_2 b_{200} + \left( \frac{1}{2} a_1 c_2 + \frac{1}{2} a_2 c_1 \right) b_{110} + (b_1 b_2 + c_1 c_2) b_{020} \]
\[ m_{2b} = \frac{1}{2} a_1^2 b_{200} + \frac{1}{2} a_1 c_1 b_{110} + \left( \frac{1}{2} c_1^2 - \frac{1}{2} b_1^2 \right) b_{020} \]
\[ n_{2b} = \frac{1}{2} a_1 b_1 b_{110} + b_1 c_1 b_{020} \]