Cooling many-body systems using feedback control

Tarek A. Elsayed1,3,* and Boris V. Fine2,3,†

1Department of Physics, School of Science and Engineering, The American University in Cairo, AUC Avenue, P.O. Box 74, New Cairo, 11835, Egypt
2Laboratory for the Physics of Complex Quantum Systems, Moscow Institute of Physics and Technology, Institutsky per. 9, Dolgoprudny, Moscow region, 141700, Russia
3Institute for Theoretical Physics, University of Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany

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We propose a technique for polarizing and cooling finite many-body systems using feedback control. The technique requires the system to have one collective degree of freedom conserved by the internal dynamics. The fluctuations of other degrees of freedom are then converted into the growth of the conserved one. The proposal is validated using numerical simulations of classical and quantum spin systems in a setting representative of Nuclear Magnetic Resonance experiments. In particular, we were able to achieve 90 percent polarization for a lattice of 1000 classical spins starting from an unpolarized infinite temperature state.

Controlling the behavior of quantum systems has been the subject of ever increasing interest both on the theoretical and experimental levels in the context of nuclear magnetic resonance (NMR) [1–5], lasers [6, 7], nanomechanical resonators [8–11], trapped atoms and ions [12–14] and other applications of quantum technology [15–22], where concepts borrowed from the field of classical control are applied to the quantum domain [23].

The present work exploits the potential of the feedback control to cool a thermally isolated many-particle system. This amounts to implementing a practical Maxwell demon to overcome the fundamental trend of entropy growth imposed by the second law of thermodynamics. We propose a feedback scheme, according to which a system is periodically driven with an amplitude determined by a feedback loop from the measurement of a quantity that we want to steer to the desired value. Although periodic driving, normally, heats a many-body system [24], the proposed scheme does the opposite. While the idea behind the scheme is rather general, we focus specifically on spin systems in the context of NMR, where the experimental temperatures are, typically, very high on the energy scale of nuclear spins, which leads to very small nuclear polarizations and thus limits the use of NMR. In particular, nitrogen-vacancy centers diamond

\[ H_0 = \sum_{m<n} J_{mn}^z S_m^z S_n^z + \sum_{m} J_{mn}^+ (S_m^x S_n^x + S_m^y S_n^y) + \hbar z \sum_m S_m^z \]

(1)

is the internal dynamics part, with \( S_m^\alpha \) being either the classical variables or the quantum spin-1/2 operators for the respective spin projection of the \( m \)th spin \( S_m \), \( J_{mn}^\alpha \)

are the interaction constants, \( h_z \) is an external field and

\[ H_f = g(t) \sum_m S_m^z, \]

(2)

is the feedback control term with

\[ g(t) = g_0 \cos(\omega t) [f(t) - M_z]. \]

(3)

Here \( \omega \) is the driving frequency, \( g_0 \) is the amplitude prefactor, \( M_z = \sum_m S_m^z \) is the total \( z \)-polarization of the system monitored by the feedback loop, and \( f(t) \) is the function that steers \( M_z \) with the goal of maximizing it. In the simulations below, the steering function is simply linear: \( f(t) = ft \), where \( f \) is a time-independent parameter. When \( f(t) \) entrains \( M_z \), the latter also grows linearly on average, with small fluctuations near \( f(t) \). Important for our scheme is the fact that, without driving, the Hamiltonian \( H_0 \) conserves \( M_z \), which means that \( M_z \) stops changing whenever \( M_z = f(t) \), and, in general, the closer \( M_z \) to \( f(t) \), the weaker the driving. The system is assumed to be thermally isolated from the environment. When relevant, the gyromagnetic ratios and the Planck constant \( \hbar \) are set to 1.

The block diagram of the proposed feedback scheme is shown in Fig. 1(a). In the context of NMR, with a strong static magnetic field oriented along the \( z \)-axis, the first sum in Hamiltonian \( H_0 \) would represent the secular terms of nuclear spins-spin interaction in the Larmor rotated reference frame. The feedback control can then be implemented with the modulated radio-frequency (rf) magnetic field \( g = [g(t),0,0] \) acting on the spins along the \( x \)-axis [Fig. 1(b)], while the second sum in \( H_0 \) would represent the effect of a possible frequency offset from the resonance value. The implementation of the scheme conceptually requires two elements shown in Fig. 1(c): one generating the feedback rf field (coil \( C_1 \)) and one measuring the monitored quantity \( M_z \) (coil \( C_2 \)). The actual implementation may not require coils but rather use, e. g., magnetic resonance force microscope [2, 25, 26], nitrogen-vacancy centers diamond.
or superconducting quantum interference devices (SQUIDs) [30]. Feedback from monitored fluctuations of finite nuclear spin clusters was used in [2, 4], where the authors created average polarization per spin of order $1/\sqrt{N}$. Here, we aim at achieving the polarization per spin of order $1$.

We note that even though $M_z$ is not the energy of the system, its increase implies lowering the entropy, which, in turn, can be easily converted into lower temperature, once, e.g., the system is placed in an external magnetic field. Even without the external field, $M_z$ may be correlated with the energy and, hence, temperature, due to the $S^z_mS^z_n$ coupling in $\mathcal{H}_0$.

We now demonstrate that the scheme works for $10 \times 10 \times 10$ cubic lattice of classical spins of length $|\mathbf{S}_m| = 1$ with periodic boundary conditions, $h_z = 0$ and the interaction constants $J^z_{mn} = -2J^z_{nn} = \frac{(1-3\cos^2\theta_{mn})}{2(r_m-r_n)^3}$, where $r_m$ is the position of the $m$th lattice site and $\theta_{mn}$ is the angle between the $z$-axis and the vector $\mathbf{r}_m - \mathbf{r}_n$. The distance between the nearest lattice sites is equal to $1$. The above choice of $J^z_{nn}$ corresponds to the truncated magnetic dipole interaction between nuclear spins in solids [31, 32]. The parameters of the feedback control were: $\omega = 7$, $g_0 = 20$ and $\dot{f}(t) = -0.005$. The simulations were based on solving the equations of motion for spin vectors $\mathbf{S}_m = \mathbf{S}_m \times \mathbf{h}_m(t)$, where $\mathbf{h}_m(t)$ represents the local field at spin $m$, due to the interaction with the rest of the lattice and with the feedback field [33].

The results of the simulation are shown in Fig. 2. The value of $|M_z|$ of about 90 percent of the maximum polarization was achieved starting from an unpolarized infinite temperature state, while the feedback field $g(t)$ had a very low amplitude relative to the interaction coefficients in $\mathcal{H}_0$. The divergence of $g(t)$ at the end of the simulated time was an indication that the feedback scheme was about to become unstable.

Let us now turn to the quantum simulations. They are more challenging not only due to the dimension $2^N$ of the relevant Hilbert space, but also because the physical measurement of the monitored quantum observable is a process that destructs quantum superpositions, which, in turn, additionally complicates the simulations [23]. Here we bypass this complication by simulating a many-body pure quantum state $|\psi(t)\rangle$ with the feedback scheme based on the numerically computed quantum-mechanical expectation value $\langle M_z \rangle = \langle \psi(t) | \sum_m S^z_m | \psi(t) \rangle$. As a result, the evolution of $|\psi(t)\rangle$ remains unitary, but its dynamics becomes nonlinear. This simplification comes at a price — as explained later, it leads to the exponential suppression of the feedback efficiency.

The concrete simulation was performed for a periodic chain of 10 spins $1/2$ coupled by the nearest neighbor interaction with $J^z_{mn} = -2J^z_{nn} = -1$ and $h_z = 0.5$. The feedback parameters were $\omega = 7$, $g_0 = 20$ and $\dot{f}(t) = -10^{-6}$. In this case, the monitored variable $|M_z\rangle$ is correlated with the total energy of the system, so that the energy was decreasing as $|M_z\rangle$ was becoming large and negative, following $f(t)$. The initial (infinite-temperature) energy of the system was $0$. The ground-state energy was $E_0 = -5$. The time-dependent Schrödinger equation was propagated using the explicit numerical time integration by the 4th-order Runge-Kutta method as in Ref. [34].

The results of the above simulation are presented in Fig. 3. We started from a pure state $|\psi(0)\rangle$ representing the infinite temperature and having zero overlap with the ground state $|\psi_0\rangle$. As panels (a) and (b) respectively illustrate, we were able to drive the system to the final state with energy $E_f \approx 0.4E_0$ and about 4 percent overlap with $|\psi_0\rangle$. Panel (c) shows that the feedback field $g(t)$ remained very small throughout the simulations, which implies the smallness of $|f(t) - \langle M_z \rangle|$. As in the classical
The feedback field $g(t)$ is rotated by the driving field $g(t)$ until $M_z$ becomes equal to $f_0$. Identify three of them:

(i) The jump of the value $\Delta f$ should be smaller than the typical value of $|M_{yz}| - |M_z| \approx \frac{1}{2}|M_z|\Delta \phi^2$.

(ii) The time step $\Delta t$ should ideally be larger than the correlation time $T_2$ of the fluctuations of $\Delta M_y$. Otherwise, there will be no new statistically independent transverse fluctuation to capitalize on.

(iii) The feedback field $g(t)$ should be large enough to rotate $\mathbf{M}_{yz}$ by the above-defined angle $\Delta \phi$ during one half of the oscillation period $\pi/\omega$.

Let us apply the above conditions to classical spin lattices. In this case, $|\Delta M_y| \sim \sqrt{N}$, while we aim at achieving $|M_z| \sim N$. In such a regime, $\Delta \phi \sim 1/\sqrt{N}$, and hence, according to the condition (i), $\Delta f$ is less than a number of order $1$. In other words, each time step in the proposed scheme would, at most, increase the total spin polarization of the system by a number of order $1$. This constraint imposes the limitation on the size of the lattices where the relative polarization $\langle S_z \rangle \equiv |M_z|/N$ of the order of one can be practically achieved. The proposed method would not work for macroscopic systems, because the required number of time steps would be of the order of the Avogadro number. However, the systems consisting of thousands and even millions of spins can be realistically polarized by the method: the maximum number of time steps is then limited by the time $T_1$ characterizing the relaxation of $M_z$ due to the external environment. In the context of NMR, the relaxation time $T_1$ in pure dielectrics can reach $10^2$ s or more, while the transverse relaxation time $T_2$ can be as small as $10^{-4}$ s. We also note that, since $M_z = \langle S_z \rangle N$, the polarization increase after each step appearing in condition (i) is proportional to $1/\langle S_z \rangle$. Thus it is noticeably larger for weakly polarized states, which is particularly helpful for NMR. On the other hand, when $\langle S_z \rangle$ approaches 1, the amplitude of...
the transverse noise decreases. As this happens, any pre-
set value of Δf associated with a constant steering rate
would become too large, and hence Mz(t) would stop
following f(t), rendering the feedback loop unstable.

In terms of our actual simulations with continuous
f(t), the period of g(t), i.e. 2π/ω, can be identified as
the time step Δt for the discretized analog. The feed-
back field rotates Mz by half the period in one direction
and half the period in the opposite one, so that, with
the right value of g0, Mz(t) is supposed to reach f(t) during
one of the two half periods. We had 2π/ω ≈ 1, while
Tz ≈ 1/3, hence the condition (ii) requiring Δt ≫ Tz is
satisfied. Condition (i) requires the change of f(t) dur-
ing time interval Δt to be smaller than 1, which, given
that Δt ≈ 1 implies that |f| ≲ 1. This inequality was
conservatively satisfied by the actual value |f| = 0.005,
which helped us to reach the relative polarization of 90
percent. According to condition (iii) with the input from
Eq.(3), Δφ ∼ g0 ΔMz √ N, where ΔMz ∼ 1/N Mz Δφ is the
typical value of |f(t) − Mz| implied by condition (i).
Combining the latter two estimates with the assumption
Δφ ∼ 1/√N and dropping the numerical prefactors, we
obtain the relation 400√N/Δf ∼ 1, which is consistent with
our simulation parameters.

The role of the oscillating factor cos ωt in the feed-
back control function g(t) is to suppress the probability
that the feedback field rotates Mz by large angle in the
direction increasing |f(t) − Mz|. Without periodic sign
changes of g(t), the feedback field can accidentally drive
Mz sufficiently far from f(t), which in turn would lead
to the loss of the steering control. On the other hand,
for ω ≫ 1/Tz, the feedback scheme would not be able to
achieve the maximal conversion of the transverse fluctua-
tions ∆Mz into the the growth of |Mz|: either the scheme
would either suppress these fluctuations, or it would lose
the steering control over Mz. The former option would
then reduce the acceptable values of the steering rate ˙f.
Thus the choice of ω ∼ 1/Tz made in the simulations
appears to be close to the optimal one.

Let us turn to interpreting our quantum simulations.
Here, the preceding qualitative discussion still applies ex-
cept for the fact that the feedback scheme is not affected
by the measured value of Mz but rather by the quan-
tum mechanical expectation value ⟨Mz⟩ defined earlier and
also by ⟨ΔMz⟩ ≡ ⟨ψ(t)| ∑ m Sm |ψ(t)⟩. It is known [34] that the fluctuations of the quantum-mechanical ex-
petation values of physical observables in a randomly
chosen pure state are exponentially reduced in compar-
ison with the classically expected values by factor 1/√N,
where N = 2N is the dimension of system’s Hilbert
space. Qualitatively, this suppression is the consequence
of the so-called “quantum parallelism” [36]: the quan-
tum mechanical expectation value is an average char-
acteristic of a random superposition of N orthogonal ba-
sis states. Our feedback scheme can thus be thought of
as acting on N simultaneously evolving quantum states,
while monitoring only one average characteristic of all of
them. Obviously, such a scheme is exponentially less ef-
fficient than the one that monitors and drives only one of
those N states, which would be analogous to the clas-
sical simulations. Quantum parallelism implies that ⟨ΔMz⟩ ∼ 1/√N. Therefore, in the above-formulated con-
dition (i), Δφ ∼ ⟨ΔMz⟩/⟨Mz⟩ becomes exponentially
small. Yet, even with such an exponential suppression,
our simulations demonstrated that the feedback scheme
has nonzero efficiency, which should be viewed as a proof
of principle that the scheme can work for quantum dy-
namics. Moreover, these simulations should reasonably
represent the case of very weak quantum measurements
[37], in the sense that the latter would give only rather
limited information about the measured quantity, while
at the same time minimally disrupting the underlying
unitary quantum dynamics.

More importantly, according to the conjecture of Ref.
[34], when the fluctuations of one collective observable
such as ΔMz are continuously measured in a sufficiently
large system with resolution δMz of order √N, such mea-
surements would quickly destroy quantum parallelism
and thus make the amplitude of the monitored fluctua-
tions approach the value ∼ √N expected classically [38],
while, at the same time, affecting the dynamics of indi-
vidual spins only weakly [39]. The fragility of quantum
parallelism under random quantum measurements was
later demonstrated in Refs.[40, 41]. We further expect
that the considered feedback response belongs to the gen-
eral class of many-spin dynamical responses [33, 42, 43],
for which the known differences between the chaotic be-
havior of classical and quantum spin systems [44–46] does
not play a significant role. Thus, we conjecture that the
efficiency of the proposed feedback scheme for sufficiently
large physically monitored quantum clusters should be
comparable to that of our classical simulations.

Let us now discuss possible generalizations and the im-
provements of the proposed scheme. One obvious im-
provement would be to make the feedback parameters ˙f,
ω and g0 slowly dependent on time such that |f| is larger
when ⟨Sz⟩ is small, and smaller when ⟨Sz⟩ approaches
1. This would accelerate the initial polarization stage,
while allowing one to come closer to fully polarizing the
system.

Another more radical modification can involve moni-
toring ΔMz instead of Mz and then applying g(t) in the
form of short pulses [47] with the appropriate sign and
amplitude, such that Mz is rotated towards the z-axis.
One obvious advantage of such an approach is that the
required accuracy for measuring Mz is of order 1, while
the required accuracy for measuring ΔMz is of order √N.

Finally, while the proposed feedback scheme was pre-
sented for spin systems, it is conceptually applicable
to any many-body system having a collective variable,
which is conserved by the internal dynamics but can be changed by an external perturbation. In conclusion, we introduced and numerically verified a scheme for polarizing and cooling large but finite clusters of many particles, and we have also presented the initial analysis of the main physical factors that control the efficiency of the proposed scheme. The scheme is specifically tested in a setting representative of NMR experiments.

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