A PLEASANT EXERCISE

J.-P. LABESSE

Abstract. In the Proceedings of the AMS Boulder conference in 1965 Langlands states a combinatorial lemma involving families of characteristic functions attached to ordered partitions of an obtuse basis in a finite dimensional euclidean vector space. Langlands does not give any indication about the proof of the lemma which is said to be a “pleasant exercise”. Since we did not find a proof in the literature we decided to give one. We believe it of some interest for the history of the subject.

1. Introduction

A combinatorial lemma. Some months ago Bill Casselman observed that there does not seem to exist any proof in the literature for a combinatorial lemma stated by Langlands in [7, Section 8] we shall call the Boulder lemma. This is the first occurrence of a family of combinatorial statements that became, after Langlands article [8] and Arthur’s early contributions ([1], [2], [3]), a ubiquitous tool for harmonic analysis on reductive groups over local fields and over adeles, in particular for the Trace Formula. In [7, Section 9] the Boulder lemma is used by Langlands to define and control avatars, denoted $E''$, of Eisenstein series arising from cuspidal forms and he gives a formula for the scalar product of two such $E''$-series. This is a generalization of a formula due to Selberg in rank one. It allows Langlands to outline a more direct proof than in [6] of the meromorphic continuation of Eisenstein series in arbitrary rank. Langlands also anticipated that the scalar product formula would be an essential tool to establish the Trace Formula for reductive groups of arbitrary rank. In [5, Lemma 4.2] Arthur proves a formula for the scalar product of two truncated Eisenstein series (arising from cuspidal forms as above) and attributes the formula to Langlands. This implicitly assumes that series $E''$ are nothing but truncated Eisenstein series. This is not immediate since ordered partitions used by Langlands never occur in Arthur’s statement. At least from an historical point of view, it seems useful to give a proof for the Boulder lemma, to relate it to the modern treatment of the combinatorics and, in particular, to establish the equivalence alluded to above.

The contents. We state the Boulder lemma with Langlands’ notation as Proposition 2.1. Our proof, given in section 4, relies on matrix equations established in section 3 that are variants of combinatorial identities mainly due to Arthur inspired by Langlands work. Our Proposition 3.4 generalizes the well known “Langlands’ combinatorial lemma” (see [1, Lemma 2.3] and [2, Lemma 6.3]). This combinatorial lemma appears here as Corollary 3.6, we observe that a particular case of it (when the parameter $\Lambda$ is in the positive chamber) is a key ingredient for the Langlands classification of admissible irreducible representations of reductive groups over local fields. It appears, I believe for the first time, in [8, page 156]. In fact this paper was circulated, as a preprint, a long time before its actual publication (see also §6 of Chapter IV in [4]). Now, Corollary 3.5 is essentially equivalent to Proposition 4.1 which together with 3.6 yields a proof of the Boulder lemma. As another corollary of Proposition 4.1 we show in 1
Section 5 that Langlands’ series $E''$ do coincide with Arthur’s truncated Eisenstein series and hence that Langlands’ formula for the scalar product of $E''$-series is equivalent to the formula for the scalar product of truncated Eisenstein series proved by Arthur. We believe that Arthur knew about Proposition 4.1 since this is implicit in his attribution to Langlands of the scalar product formula.

2. The Boulder lemma

Ordered partitions. Consider a euclidean vector space $V$ of dimension $p \geq 1$. The euclidean structure allows to identify $V$ with its dual $V'$. Let $\Delta$ be a basis and $\hat{\Delta}$ the dual basis. We denote by $D : \lambda \in \Delta \mapsto D(\lambda) = \mu \in \hat{\Delta}$ the canonical bijection between $\Delta$ and $\hat{\Delta}$. Let $p$ be an ordered partition of $\Delta$ built out of $r = r(p)$ non empty disjoint subsets $F_p^u$ with cardinals $a_p^u$ with $1 \leq u \leq r$ so that

$$\Delta = \bigcup_{u=1}^{u=r} F_p^u \quad \text{and} \quad \sum_{u} a_p^u = p .$$

This is equivalent to be given an increasing filtration of $\Delta$ by subsets $E_p^u$

$$\emptyset = E_p^0 \subset E_p^1 \subset E_p^2 \subset \cdots \subset E_p^r = \Delta \quad \text{where} \quad E_p^u = \bigcup_{v \leq u} F_p^v .$$

This defines an increasing filtration of $V$ by subspaces $U_p^v$ generated by $E_p^v$

$$\{0\} = U_p^0 \subset U_p^1 \subset U_p^2 \subset \cdots \subset U_p^r = V$$

and a decreasing filtration by subspaces $V_p^v$ generated by $\Delta - E_p^v$

$$V = V_p^0 \supset V_p^1 \supset V_p^2 \supset \cdots \supset V_p^r = \{0\} .$$

For any $v \in \{0, 1 \cdots, r\}$ we have an orthogonal decomposition: $V = U_p^v \oplus V_p^v$. Let $W_p^u$ be the orthogonal supplement to $U_p^{u-1}$ in $U_p^u$ or, equivalently to $V_p^{u-1}$ in $V_p^u$:

$$U_p^{u-1} \oplus W_p^u = U_p^u \quad \text{and} \quad V_p^{u-1} \oplus W_p^u = V_p^u .$$

One has an orthogonal direct sum decomposition of $V$:

$$V = \bigoplus_{u=1}^{u=r} W_p^u .$$

For $\lambda \in F_p^u$ denote by $\lambda_p$ the orthogonal projection of $\lambda$ on $W_p^u$; the $\lambda_p$ build a basis $\Delta_p^u$ of it. Similarly for $\mu \in F_p^u := D(E_p^u)$ denote by $\mu_p$ the orthogonal projection of $\mu$ on $W_p^u$, they build a basis $\hat{\Delta}_p^u$. Since $U_p^u$ is orthogonal to $V_p^u$ one has

$$\langle \mu, \lambda \rangle = \langle \mu, \lambda_p \rangle = \langle \mu_p, \lambda \rangle = \langle \mu_p, \lambda_p \rangle .$$

This shows that the sets $\Delta_p^u$ and $\hat{\Delta}_p^u$ are dual basis. Denote by $\Delta_p$ the union of the $\Delta_p^u$ (resp. $\hat{\Delta}_p$ the dual basis) and $D_p : \Delta_p \rightarrow \hat{\Delta}_p$ the natural bijection.
A PLEASANT EXERCISE

Functions \( \phi_p^\Lambda \) and \( \psi_p^\Lambda \). Let \( H \mapsto \phi_p^\Lambda(H) \) be the characteristic function of the cone in \( V \) defined by

\[
\lambda_p(H) \leq 0 \text{ if } \mu_p(\Lambda) > 0 \text{ or } \lambda_p(H) > 0 \text{ if } \mu_p(\Lambda) \leq 0 \text{ where } \mu_p = D_p(\Lambda_p)
\]

and \( H \mapsto \psi_p^\Lambda(H) \) the characteristic function of the cone \( \lambda_p(H) > 0 \) for \( \lambda \in F_p^1 \) and

\[
\lambda_p(H) \leq 0 \text{ if } \mu_p(\Lambda) > 0 \text{ or } \lambda_p(H) > 0 \text{ if } \mu_p(\Lambda) \leq 0 \text{ for } \lambda \notin F_p^1
\]

where \( \mu_p = D(\lambda_p) \). Let \( b_p^\Lambda \) be the number of \( \mu \in \hat{\Delta} \) such that \( \mu_p(\Lambda) \leq 0 \) and \( c_p^\Lambda \) the number of those \( \mu \) such that moreover \( \mu \notin F_p^1 \). Finally Langlands defines integers

\[
\alpha_p^\Lambda = b_p^\Lambda + \sum_{u=1}^{u=r} (a_p^u + 1) \quad \text{and} \quad \beta_p^\Lambda = 1 + c_p^\Lambda + \sum_{u=2}^{u=r} (a_p^u + 1) .
\]

An element \( \Lambda \in V \) is said to be “in the positive chamber” (with respect to \( \Delta \)) if \( \lambda(\Lambda) > 0 \) for all \( \lambda \in \Delta \). Let \( d_\Delta(\Lambda) = 1 \) iff \( \Lambda \) is in the positive chamber. We may now state the Boulder lemma \([7, \text{Section 8}]\).

**Proposition 2.1.**

\[
\sum_{p \in \mathcal{P}(\Delta)} (-1)^{\alpha_p^\Lambda} \phi_p^\Lambda(H) = d_\Delta(\Lambda) + \sum_{p \in \mathcal{P}(\Delta)} (-1)^{\beta_p^\Lambda} \psi_p^\Lambda(H)
\]

where \( \mathcal{P}(\Delta) \) is the set of ordered partitions of \( \Delta \).

**Proof.** Langlands says that the proof is a “pleasant exercise”. We observe that when \( \Lambda = 0 \) then \( \phi_p^\Lambda = \psi_p^\Lambda \) and \( \alpha_p^\Lambda = \beta_p^\Lambda + 2a_p^1 \) for all \( p \); now since \( p \geq 1 \) we have \( d(\Lambda) = 0 \) and the proposition is trivially true. For arbitrary \( \Lambda \) a proof is given at the end of section \([3] \) as an immediate consequence of Corollaries \([3, 3] \) and \([3, 0] \).

3. Matrix equations

A slightly more general setting. Consider a euclidean vector space \( V_0 \), a basis \( \Delta_0 \) and two subsets \( P \subset Q \subset \Delta_0 \). Let \( \Delta_0^Q \) be the projection of \( Q \) on the orthogonal of \( P \) and \( V_0^P \) the subspace generated by \( \Delta_0^Q \). We denote by \( D_P^Q \) the bijection between \( \Delta_0^Q \) and its dual basis \( \hat{\Delta}_0^Q \). For \( H \in V \) we denote by \( H_0^P \) the orthogonal projection of \( H \) on \( V_0^P \). With no loss of generality, we may assume that \( \Delta = \Delta_0^Q \) and \( V = V_0^P \) for a pair of subsets \( P \subset Q \subset \Delta_0 \) for some \( \Delta_0 \). This allows us to use Arthur’s notation introduced in \([1, 2] \) and \([3] \) where \( \Delta_0 \) is the set of simple roots attached to a minimal parabolic subgroup \( P_0 \) in a reductive group \( G \). Using that standard parabolic subgroups are bijectively attached to subsets (that may be empty) of \( \Delta_0 \) and that this bijection is compatible with inclusion then, by abuse of notation, using the same letter for a subset \( P \) of \( \Delta_0 \) and the standard parabolic subgroup it defines, we recover Arthur’s setting and notation. Hence one may forget about \( G \), and consider \( P_0 \) as the empty subset in \( \Delta_0 \). We have to give new proof of some of the classical results since we shall not assume \( \Delta_0 \) obtuse.

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1. Although we have identified \( V \) with its dual we shall look at elements in \( \Delta_p \) and \( \hat{\Delta}_p \) as linear forms rather than vectors. Langlands does not indentify \( V \) with \( V' \); he considers \( H \) as an element of \( V \) and \( \Lambda \) as an element in \( V' \). Here we emphasis a kind of symmetry between \( H \) and \( \Lambda \).

2. We observe that Langlands defines \( \phi_p^\Lambda(H) \) and \( \psi_p^\Lambda(H) \) only for \( H \) and \( \Lambda \) regular i.e. for \( H \) outside the walls defined by the \( \lambda_p \) and \( \Lambda \) outside the walls defined by the \( \mu_p \) for some \( p \). Non regular elements are excluded in his formulation of the Boulder lemma. Moreover Langlands assumes \( \Delta \) is obtuse.
Functions $\theta^\Lambda_{P,Q}$ and $\tilde{\theta}^\Lambda_{P,Q}$. For $P \subset Q$ let $\theta^\Lambda_{P,Q}$ be the characteristic function of the set of $H \in V_0$ such that
\[
\lambda(H) \leq 0 \text{ if } \mu(\Lambda) > 0 \text{ or } \lambda(H) > 0 \text{ if } \mu(\Lambda) \leq 0 \quad \text{for } \lambda \in \Delta^Q_P, \quad \mu = D^Q_P(\lambda).
\]
Similarly let $\tilde{\theta}^\Lambda_{P,Q}$ be the characteristic function of the set of $H$ such that
\[
\mu(H) \leq 0 \text{ if } \lambda(\Lambda) > 0 \text{ or } \mu(H) > 0 \text{ if } \lambda(\Lambda) \leq 0 \quad \text{for } \lambda \in \Delta^Q_P, \quad \mu = D^Q_P(\lambda).
\]
By convention $\theta^\Lambda_{P,Q} = \tilde{\theta}^\Lambda_{P,Q} = 0$ when $P \not\subset Q$ and $\theta^\Lambda_{P,P} = \tilde{\theta}^\Lambda_{P,P} = 1$.

When $-\Lambda$ is in the closure of the positive chamber with respect to an obtuse basis $\Delta^Q_P$ (which implies $\mu(\Lambda) \leq 0$ for all $\mu \in \widehat{\Delta}^Q_P$) and in all cases when $\Lambda = 0$, we have
\[
\theta^\Lambda_{P,Q} = \tau^Q_P \quad \text{and} \quad \tilde{\theta}^\Lambda_{P,Q} = \tilde{\tau}^Q_P
\]
where $\tau^Q_P$ and $\tilde{\tau}^Q_P$ are Arthur’s functions. More generally, one can express the $\theta^\Lambda_{P,Q}$ and $\tilde{\theta}^\Lambda_{P,Q}$ in terms of the $\tau^Q_P$ and $\tilde{\tau}^Q_P$. To do this we need some notation. Let $P_\Lambda$ (or $P^Q_\Lambda$ if some confusion may arise) be the subset of $\Delta_0$ such that $\Delta^Q_{P,\Lambda}$ is the set of $\mu \in \widehat{\Delta}^Q_P$ with $\mu(\Lambda) > 0$. Similarly, let $Q^\Lambda$ (or $Q_\Lambda$) if some confusion may arise) be the subset of $\Delta_0$ such that $\Delta^Q_{P,\Lambda}$ is the set of $\lambda \in \Delta^Q_P$ with $\lambda(\Lambda) > 0$.

Lemma 3.1.
\[
\theta^\Lambda_{P,Q} = \sum_{P_\Lambda \subset S \subset Q} (-1)^\delta^\Lambda_{P,Q} \tau^S_P \quad \text{and} \quad \tilde{\theta}^\Lambda_{P,Q} = \sum_{P \subset S \subset Q^\Lambda} (-1)^\delta^\Lambda_{S,Q} \tilde{\tau}^S_P
\]

Proof. This is an immediate consequence of the binomial identity. \qed

Let $a^\Lambda_P$ be the cardinal of $\Delta^Q_P$ (so that $a^P_\Lambda$ is the cardinal of $P$), $b^\Lambda_{P,Q}$ the cardinal of the set of $\mu \in \widehat{\Delta}^Q_P$ such that $\mu(\Lambda) \leq 0$ and $\tilde{\theta}^\Lambda_{P,Q}$ the cardinal of the set of $\lambda \in \Delta^Q_P$ such that $\lambda(\Lambda) \leq 0$. Let
\[
\eta^\Lambda_{P,Q} = a^P_\Lambda + b^\Lambda_{P,Q} \quad \text{and} \quad \tilde{\eta}^\Lambda_{P,Q} = a^P_\Lambda + b^\Lambda_{P,Q}.
\]
Consider the matrix $\theta^\Lambda$ (resp. $\tilde{\theta}^\Lambda$) with entries
\[
(-1)^\delta^\Lambda_{P,Q} \theta^\Lambda_{P,Q} \quad \text{resp.} \quad (-1)^\delta^\Lambda_{P,Q} \tilde{\theta}^\Lambda_{P,Q}.
\]
We shall be interested in the products $A^{\Lambda_1,\Lambda_2} = \theta^{\Lambda_1} \tilde{\theta}^{\Lambda_2}$ and $B^{\Lambda_1,\Lambda_2} = \tilde{\theta}^{\Lambda_1} \theta^{\Lambda_2}$:
\[
A^{\Lambda_1,\Lambda_2}_{P,R} := \sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \eta^\Lambda_{P,Q} \tilde{\eta}^\Lambda_{Q,R} \text{ and } B^{\Lambda_1,\Lambda_2}_{P,R} := \sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \tilde{\eta}^\Lambda_{P,Q} \eta^\Lambda_{Q,R}.
\]

Lemma 3.2. Assume $\Lambda_1 = \Lambda_2 = 0$. Then,
\[
A^{\Lambda_1,\Lambda_2}_{P,R} = \sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \tau^Q_P \tilde{\tau}^R_P = \delta_{P,R}
\]
where $\delta_{P,R}$ is the Dirac symbol (it is zero if $P \neq R$ and 1 if $P = R$).

Proof. The assumption on the $\Lambda_i$ imply that
\[
A^{\Lambda_1,\Lambda_2}_{P,R} = \sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \eta^\Lambda_{P,Q} \tilde{\eta}^\Lambda_{Q,R} = \sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \tau^Q_P \tilde{\tau}^R_P.
\]
We have to show that
\[
\sum_{P \subset Q \subset R} (-1)^\delta_{P,Q-R} \tau^Q_P \tilde{\tau}^R_P = \delta_{P,R}.
\]

3Our $\tilde{\theta}^\Lambda_{P,Q}(H)$ is denoted $\phi^Q_P(\Lambda, H)$ by Arthur in [2] page 940. Unfortunately Arthur’s notation is in conflict with Langlands one in [3] and since we stick to Langlands’ notation we had to introduce a different one.
When $\Delta^R_P$ is an obtuse basis this is \cite[Proposition 1.7.2]{2} which itself is a special case of \cite[Lemma 6.1]{2}. We give a proof which does not use this assumption \footnote{Casselman has already observed, some years ago, that the obtuse assumption is not necessary (private communication).}. Fix $H \in V_0$. As above, but with $H$ in place of $\Lambda$, we introduce subsets $P_H \subset \Delta_0$ and $R^H \subset \Delta_0$. Then
\[
\tau^Q_P(H)\tau^R_Q(H) = 1
\]
if and only if $P_H \subset Q \subset R^H$ and hence
\[
\sum_{P \subset Q \subset R} (-1)^{a^Q_P} \tau^Q_P(H)\tau^R_Q(H) = \sum_{P_H \subset Q \subset R^H} (-1)^{a^Q_P} .
\]
The binomial identity tells us this alternating sum vanishes unless $P_H = R^H$. Assume it does not vanish, then $P^R_H \neq 0$ if $P \neq R$. Now consider $\lambda \in \Delta^P_R$ and $\mu \in \Delta^R_P$ such that $\mu = D^P_R(\lambda)$ then, since $P_H = R^H = Q$, we have
\[\lambda(H) > 0 \quad \text{and} \quad \mu(H) \leq 0 \quad \text{for} \quad \lambda \in \Delta^Q_P\]
while
\[\lambda(H) \leq 0 \quad \text{and} \quad \mu(H) > 0 \quad \text{for} \quad \mu \in \Delta^R_Q\]
and hence
\[(H^R_P, H^R_P) = \sum_{\{\lambda \in \Delta^P_R \mid \mu = D^P_R(\lambda)\}} \lambda(H)\mu(H) \leq 0\]
which contradicts the non vanishing of $H^R_P$ unless $P = R$. \hfill \Box

**Corollary 3.3.** If $\Delta_0$ is an obtuse basis and if the $-\Lambda_i$ are in the closure of the positive chamber, then $A^{\Lambda_1, \Lambda_2} = 1$ which in turn implies $B^{\Lambda_1, \Lambda_2} = 1$.

**The key ingredient.** The following proposition and its proof is a generalization of Arthur’s Lemma 6.3 in \cite{2} (which appears below as Corollary 3.6).

**Proposition 3.4.**
\[
B_{P,R}^{A_1, A_2} = (-1)^{a^R_P} \delta_{P_A, R^A} .
\]

**Proof.** By definition
\[
B_{P,R}^{A_1, A_2} = \sum_{P \subset Q \subset R} (-1)^{a^Q_P} \tau^Q_P \hat{\theta}_{P,Q}^A \theta^A_{Q,R} .
\]

In view of \cite{3} we have
\[
B_{P,R}^{A_1, A_2} = \sum_{P \subset Q \subset R} (-1)^{a^Q_P} \tau^Q_P \sum_{P \subset S \subset Q^{R^A}} (-1)^{a^{R^A} S} \tau^S_S \sum_{Q_A \subset S \subset R} (-1)^{a^R_{Q_A}} \tau^R_{Q_A} .
\]

Since $a^Q_P = b^R_{Q,R}$ and $a^Q_{P,R} = \hat{b}_P^A$ we get
\[
B_{P,R}^{A_1, A_2} = \sum (-1)^{a^Q_P} (-1)^{a^{R^A} S} \tau^Q_P \tau^S_S .
\]

where the sum runs over triples $(S, Q, S')$ verifying the inclusions
\[(*) \quad P \subset S \subset Q^{R^A} \subset Q \subset Q^{R^A}_P \subset S' \subset R .
\]

Observe that $Q^R_{P_A} \supset P^R_{A_1}$ and $Q^R_{R_A} \supset R^R_{A_1}$. This shows that given any pair $(S, S')$ with $S \subset Q^{R^A}$ and $S' \supset R^R_{A_1}$ we have the inclusions $(*)$ for any $Q$ with $S \subset Q \subset S'$. Then
\[
B_{P,R}^{A_1, A_2} = \sum (-1)^{a^{R'}_{Q_A}} \sum_{Q \subset Q \subset S'} (-1)^{a^Q_P} \tau^Q_P \tau^S_S .
\]
It follows from [3.2] that the sum over $Q$ vanishes unless $S = S'$ and hence we have

$$B_{P,R}^{\Lambda_1,\Lambda_2} = \sum_{\{S \mid P_{\Lambda_1}^0 \subseteq S \subseteq R_{\Lambda_2}^0\}} (-1)^{\hat{\alpha}_P^S}.$$  

This in turn, by the binomial identity, vanishes unless $P_{\Lambda_1} = R_{\Lambda_2}$. 

**Corollary 3.5.** The matrices $\theta^\Lambda$ and $\hat{\theta}^\Lambda$ are inverse of each other. 

**Proof.** Proposition 3.4 shows that $B_{P,R}^{\Lambda,\Lambda}$ vanishes unless $P_{\Lambda} = R^\Lambda$. But, as in the proof of 3.2 we see this condition implies $\langle \Lambda^R, \Lambda^R \rangle \leq 0$. Then, if it does not vanish we may assume $\Lambda = 0$ which implies $P = R^\Lambda$ and $P_{\Lambda} = R$ which yields $P = R$. 

**Corollary 3.6.**

$$\sum_{P \subseteq Q \subseteq R} (-1)^{\hat{\phi}_{P,Q}^\Lambda} \theta_{P,Q}^\Lambda \tau_Q^R = d_{\Delta_P^R}(\Lambda).$$

**Proof.** Consider the particular case $\Lambda_1 = 0$ and $\Lambda_2 = \Lambda$. Then $P_{\Lambda_1} = R$ and 3.4 shows that $B_{P,R}^{0,\Lambda} = 0$ unless $R = R^\Lambda$ which holds if and only if $\Lambda$ is in the positive chamber with respect to $\Delta_P^R$ so that

$$B_{P,R}^{0,\Lambda} = (-1)^{\alpha_P^R} d_{\Delta_P^R}(\Lambda).$$

It remains to observe that

$$\eta_{P,Q}^\Lambda - \eta_{Q,R}^\Lambda = a_P^P + \hat{b}_{P,Q}^\Lambda - a_Q^Q - \hat{b}_{Q,R}^\Lambda = a_P^R + \hat{b}_{P,Q}^\Lambda \pmod{2}.$$ 



4. **APPLICATION TO THE BOULDER LEMMA**

Where ordered partitions disappear. Consider an ordered partition $\mathfrak{p}$ of $\Delta = \Delta_P^R$ and $Q$ with $P \subseteq Q \subseteq R$ such that $F_{\mathfrak{p}}^1 = \Delta_Q^R$. Let $\mathfrak{q}$ be the ordered partition of $\Delta_Q^R$ such that $F_{\mathfrak{q}}^1 = F_{\mathfrak{p}}^{i+1}$. Then

\[
(\ast) \quad \phi_{\mathfrak{p}}^\Lambda = \theta_{Q,R}^\Lambda \psi_{\mathfrak{q}}^\Lambda \quad \text{and} \quad \psi_{\mathfrak{p}}^\Lambda = \tau_Q^R \phi_{\mathfrak{q}}^\Lambda
\]

moreover

\[
(\ast\ast) \quad r(\mathfrak{p}) = r(\mathfrak{q}) + 1 \quad b_{\mathfrak{p}}^\Lambda = b_{Q,R}^\Lambda + b_{\mathfrak{q}}^\Lambda \quad \text{and} \quad \alpha_{\mathfrak{p}}^\Lambda = \alpha_{\mathfrak{q}}^\Lambda + a_{\mathfrak{p}}^R + 1 + b_{Q,R}^\Lambda.
\]

The next proposition shows how to replace, up to a sign, alternating sums over ordered partitions of functions $\phi_{\mathfrak{p}}^\Lambda$ by a single characteristic function.

**Proposition 4.1.** Let $\hat{\theta}_\Lambda^\Lambda$ be the characteristic function of the $H$ such that

$$\mu(H) \leq 0 \text{ if } \lambda(\Lambda) > 0 \text{ or } \mu(H) > 0 \text{ if } \lambda(\Lambda) \leq 0$$

for all $\lambda \in \Delta$ and $\mu = D(\lambda)$. Let $\hat{\theta}_\Lambda^\Lambda$ be the number of $\lambda$ such that $\lambda(\Lambda) \leq 0$. Then

$$\sum_{\mathfrak{p} \in \mathcal{P}(\Delta)} (-1)^{\alpha_{\mathfrak{p}}^\Lambda} \phi_{\mathfrak{p}}^\Lambda = (-1)^{\hat{\phi}_\Lambda^\Lambda} \hat{\theta}_\Lambda^\Lambda.$$  

**Proof.** The above remarks $(\ast)$ and $(\ast\ast)$ show that

$$\sum_{\mathfrak{p} \in \mathcal{P}(\Delta_P^R)} (-1)^{\alpha_{\mathfrak{p}}^\Lambda} \phi_{\mathfrak{p}}^\Lambda = \sum_{P \subseteq Q \subseteq R} (-1)^{\alpha_{\mathfrak{p}}^Q+1+b_{Q,R}^\Lambda} \theta_{Q,R}^\Lambda \sum_{\mathfrak{q} \in \mathcal{P}(\Delta_Q^R)} (-1)^{\alpha_{\mathfrak{q}}^\Lambda} \phi_{\mathfrak{q}}^\Lambda.$$  

The equivalence of Proposition 4.1 and Corollary 3.5 now follows by induction on the cardinal of $\Delta_P^R$. 

Proof of the Boulder lemma. The Proposition 2.1 claims an identity that in view of (**) can be written
\[
\sum_{p \in \mathcal{P}(\Delta)} (-1)^a_p \phi_p^\Lambda(H) = d_{\Delta_p}^\Lambda - \sum_{P \subset Q \leq R} \frac{\tau^R_Q(H)}{\tau^P_Q(H)} \sum_{q \in \mathcal{P}(\Delta_q^R)} (-1)^{a^\Lambda_q} \phi_q^\Lambda(H).
\]
To prove it we have to show that, thanks to Proposition 4.1,
\[
(-1)^{b^\Lambda_R} \hat{\theta}^\Lambda_{P,R}(H) = d_{\Delta_p}^\Lambda - \sum_{P \subset Q \leq R} (-1)^{b^\Lambda_Q} \frac{\tau^R_Q(H)}{\tau^P_Q(H)} \hat{\theta}^\Lambda_Q(H)
\]
but this nothing than Corollary 3.6.

5. Truncated Eisenstein series

Series $E''$ versus truncated Eisenstein series. Here we borrow the notation from [5]. Let $G$ be a reductive group over a number field $F$ with minimal parabolic subgroup $P_0$. Let $P$ be a standard parabolic subgroup with Levi $M$, $S$ an associate standard parabolic subgroup and $\Phi$ a cuspidal form on $X_P$ and $T$ some parameter in $a_0$. Let $\Delta = \Delta_Q^R$ and consider $\Lambda \in V \otimes \mathbb{C}$ such that $\Re(\Lambda)$ is far enough inside the positive Weyl chamber (with respect to $\Delta$). Langlands introduces in [7] §9 functions
\[
F''_S(x, \Phi, \Lambda) = \sum_{s \in W(a_P, a_S) \cap \mathcal{P}(\Delta)} \sum_{p \in \mathcal{P}(\Delta)} (-1)^{a^\Lambda_p R(s)} \phi_p^R(s)(H_S(x) - T_S)(M(s, \Lambda)\Phi)(x, s\Lambda)
\]
where, by definition,
\[
\Phi(x, \Lambda) = e^{<\Lambda + P, H_P(x)>} \Phi(x).
\]
The reader is warned that the dependence on $T$ (denoted $H_0$ by Langlands in [7]) does not appear explicitly on the left hand side. Then, Langlands introduces
\[
E''(x, \Phi, \Lambda) = \sum_{S \in \mathcal{S}(F) \setminus G(F)} \sum_{\xi \in \mathcal{S}(F) \setminus G(F)} F''_S(\xi, \Phi, \Lambda)
\]
where the sum over $S$ runs over parabolic subgroups associated to $P$. On the other hand Arthur has defined in [3] a truncation operator denoted $\Lambda^T$. When applied to the Eisenstein series
\[
E(x, \Phi, \Lambda) = \sum_{\gamma \in \mathcal{P}(F) \setminus G(F)} \Phi(\gamma x, \Lambda)
\]
then, according to [3] Proposition 5.4.1, one obtains the following expression
\[
\Lambda^T E(x, \Phi, \Lambda) = \sum_{S \in \mathcal{S}(F) \setminus G(F)} \sum_{\xi \in \mathcal{S}(F) \setminus G(F)} (-1)^{a(s)} \phi_{M,s}(s^{-1}(H_S(\xi x) - T_S))(M(s, \Lambda)\Phi)(\xi x, s\Lambda)
\]
where, by definition,
\[
\phi_{M,s}(H) = \hat{\theta}^{R(s)}_{P,G}(sH)
\]
and the integer $a(s)$ is the number of simple roots $\lambda \in \Delta_S$ such that $\lambda(s\Lambda) < 0$.

Proposition 5.1.
\[
E''(x, \Phi, \Lambda) = \Lambda^T E(x, \Phi, \Lambda).
\]

Proof. It suffices to observe that if $\Re(\Lambda)$ is in the positive chamber, then for $H \in a_P$ and $s \in W(a_P, a_S)$ one has
\[
\sum_{p \in \mathcal{P}(\Delta)} (-1)^{a^\Lambda_p R(s)} \phi_p^R(s)(H) = (-1)^{a(s)} \phi_{M,s}(H).
\]
This assertion is a particular case of Proposition 4.1 where $\Delta_0$ is the set of simple roots of $G$ with respect to the minimal parabolic subgroup $P_0$ and $\Delta = \Delta^R$.
On scalar products. A formula for the scalar product of two series $E''$ is stated by Langlands [7, Section 9] without proof. Arthur proves a formula [3, Lemma 4.2] for the scalar product of two truncated Eisenstein series which is said (again without proof) to be equivalent to Langlands' one. As already observed this equivalence is not obvious since ordered partitions that show up in Langlands' formula never occur in Arthur’s statement. Now, this follows from 5.1. This equivalence of the formulas for scalar products can also be proved directly using again Proposition 4.1. We leave it to the reader. We observe that a proof of the scalar product formula simpler than in [3] is given in [5, Chapter 5].

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Institut Mathématique de Luminy, UMR 7373