Application of the fractional-order model in the problems of identification and order reduction for the controlled processes

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Abstract. In the paper, the problems of an identification and order reduction of the dynamics models of controlled processes are solved by applying of fractional-order transfer functions of the form an aperiodic (non-oscillatory) system of the nth order with time delay, and an aperiodic system of the nth order with an integrator and delay, where n generally is a positive fractional constant. The solution of the problem is based on the method of moments. The problem is reduced to solving four polynomial equations with respect to the four coefficients of the model. In the symbolic computing environment Maple, an identification and order reduction program has been developed that implements the described algorithm. The efficiency of the algorithm was tested on test models.

1. Introduction

Recently, the interest of researchers in control systems of fractional order has not weakened [1, 2]. These are control systems with fractional order PI, PD, and PID controllers, and models of controlled processes with fractional order transfer functions. Applying of fractional order controllers, in particular, the PID controllers widely used in industry, allow improving the performance of a control system. The reason for this is the increase in the number of controller settings, which now include the fractional order of the integrator and differentiator. So in [1, 2] there is an extensive reference list devoted to the use of fractional order systems in various applications. Using fractional models to describe the dynamics of a controlled process improves the accuracy of approximation of the dynamic characteristics of the process. In this paper, we solve the problems of identifying and order reduction of the dynamics models of the controlled process by the static and astatic fractional-order models. In the first case, the dynamics of the process is approximated by the transfer function of the first-order lag block that raised to the power n and the time delay block; in the second case, an integrator is added (astatic model is obtained). Here n generally is a positive constant. The proposed algorithm is based on the method of moments [3]. In the reduction problem, the moments are defined as the coefficients of the Taylor series expansion at the point s = 0 of the transfer function of the reduced part of the model. In the identification problem, the moments are determined by the process response to the input test signal. For a model with an integrator, the test signal should be vanishing at infinity. The problem is reduced to solving four polynomial equations with respect to the four coefficients of the model. The method of moments, perhaps for the first time, was used to identify the dynamics of a...
controlled process in [3]. In the Russian-language sources, there is a method of “areas” based on the method of moments [4]. In [5] a modification of the “areas” method was proposed, in [6] the method of moments was used to identify models with time delay. The method of moments and the close to it the Padé approximation method are widely used in the problem of order reduction [7].

2. Algorithm for solving problems of order reduction and identification

We consider the linearized process model described by transfer functions of the form

\[ W_c(s) = \frac{1}{s^r} \tilde{W}_c(s) = \frac{1}{s^r} \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n} e^{-\tau s} = \frac{1}{s^r} (\mu_0 + \mu_1 s + \ldots + \mu_k s^k + \ldots) , \]  

(1)

where \( \tilde{W}_c(0) = \frac{b_0}{a_0} < \infty, \quad r = 0,1, \quad a_i, b_i \) are constant coefficients; \( m \leq n; \quad \tau \) is a time delay;

\[ \mu_k = \frac{1}{k! s^k} \tilde{W}_c(s) \big|_{s=0}, \quad k = 0,1, \ldots \text{ are the moments of the transfer function } \tilde{W}_c(s). \]

The dynamic properties of the processes are approximated by a fractional order model of the form

\[ W_m(s) = \frac{K}{s^r (Ts + 1)^n} e^{-\tau s} , \]  

(2)

where \( K \) is the gain of the model, \( T \) is the time constant, \( \tau \) is the time delay, \( n \) is in the general case a positive fractional constant, the order of the model.

If \( r = 0 \) in formula (2), then the integrator is absent, the model describes a stable process with self-leveling (called static). If \( r = 1 \) then processes without a steady state are described (called astatic). The main initial data for solving problems of reduction and identification are the values of the four moments of the process. The algorithms for calculating the moments for each of these two problems are different. Thus, we consider them separately

2.1. Order reduction problem

In this case, the transfer function of the reduced model is known. The moments of the process are calculated in accordance with formulas (1). Expanding the proportional part of the model (2) in a Taylor series at the point \( s = 0 \), we get the moments of the model as a function of its parameters. Equating the moments of the process to the moments of the model, we obtain a system of equations for determining the parameters of the model

\[ F_0 = K - \mu_0 = 0, \]
\[ F_1 = K(nT + \tau) + \mu_1 = 0, \]
\[ F_2 = \frac{1}{2} K \left( n(n+1)T^2 + 2nT\tau + \tau^2 \right) - \mu_2 = 0, \]
\[ F_3 = \frac{1}{6} K \left( (n^2 + 3n + 2)nT^3 + 3n(n+1)\tau T^2 + 3nT\tau^2 + \tau^3 \right) - \mu_3 = 0. \]  

(3)

From the first equation of system (3), we find \( K = \mu_0 \), substitute it into the rest equations, then from the second one we find \( \tau \)

\[ \tau = -\frac{\mu_0 nT + \mu_1}{\mu_0} \]  

(4)

and substitute in the other two. As a result, we obtain a system of two polynomials with respect to \( T \) with coefficients depending on the moments of the process and the fractional order \( n \).
\[ F_2 = a_2 T^2 + a_0 = 0, \]
\[ F_1 = b_1 T^3 + b_2 T^2 + b_1 T + b_0 = 0, \]  
(5)

where \( a_0 = \frac{\mu_0^2 - 2 \mu_0 \mu_2}{2 \mu_0^2}, \ a_2 = \frac{n}{2}, \ b_0 = \frac{\mu_3}{6 \mu_0^3} - \frac{\mu_4}{\mu_0}, \ b_1 = \frac{\mu_2^2 - 2 \mu_0 \mu_2}{2 \mu_0} n, \ b_2 = \frac{\mu_1 - n}{2 \mu_0}, \ b_3 = \frac{3 n^2 - 2 n}{6}. \)

The system \( (5) \) is consistent if its resultant (determinant of Sylvester matrix) is zero

\[
\Delta = \begin{vmatrix}
0 & 0 & a_2 & 0 & a_0 \\
0 & a_2 & 0 & a_0 & 0 \\
b_1 & b_2 & b_1 & b_0 & 0 \\
0 & b_3 & b_2 & b_1 & b_0 \\
\end{vmatrix} = -\frac{n^2}{72 \mu_0} \left[ (3 \mu_0^2 + \mu_4 - 3 \mu_1 \mu_2 \mu_0)^2 \cdot n + \left( 2 \mu_2 \mu_0 - \mu_1^2 \right)^3 \right] = 0 \]  
(6)

As a result, we find an expression for the order of the model \( n \)

\[
n = \frac{\left( 2 \mu_2 \mu_0 - \mu_1^2 \right)^3}{\left( 3 \mu_0^2 + \mu_4 - 3 \mu_1 \mu_2 \mu_0 \right)^2} \]  
(7)

It can be shown [8] that for weakly oscillating processes \( 2 \mu_2 \mu_0 - \mu_1^2 > 0 \), therefore \( n > 0 \).

Now, from the first equation \( (5) \), we determine two solutions for \( T \), one of them is positive

\[
T = \left( \frac{a_0}{a_2} \right)^{1/2} = \left( \frac{2 \mu_2 \mu_0 - \mu_1^2}{\mu_0 n} \right)^{1/2} = \frac{3 \mu_1 \mu_2 \mu_0 - 3 \mu_1 \mu_0^2 - \mu_1^3}{\mu_0 \left( 2 \mu_2 \mu_0 - \mu_1^2 \right)} \]  
(8)

Finally, the time delay \( \tau \) is determined by formula \( (4) \), and \( K = \mu_0 \).

2.2. Identification problem
The moments of the process are determined from the curves of the input and output signal obtained as a result of an active experiment. The input test signal must be limited in amplitude; in the case of an astatic object, it must be vanishing at infinity. Let us consider the second case as the most difficult one. The transfer function of the process has the form

\[
W(s) = \frac{\bar{W}(s)}{s}, \]  
(9)

where \( \bar{W}(s) \) is the transfer function of the proportional part.

Laplace image of the output signal of the process (operator equation of the process)

\[
Y(s) = \bar{W}(s) X(s)/s, \]  

where \( X(s) = L\{ x(t) \}, Y(s) = L\{ y(t) \} \) images of input \( x(t) \) and output signal \( y(t) \).

Since the input signal of the process is vanishing at infinity, the steady-state value of the output signal is constant \( y_{ss} = \lim_{t \to \infty} y(t) = \text{const} \)

\[
y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\bar{W}(s)X(s)/s = \bar{W}_0(0)X(0) = K \cdot X(0) \]  
(10)

Let us determine the image of the transient component of the output signal \( Y(s) = L\{ y(t) \} \)
\[ Y_t(s) = \frac{\hat{W}(s)}{s} X(s) - \frac{K}{s} X(0) = \frac{\hat{W}(s)X(s) - K \cdot X(0)}{s} \] (11)

It is easy to see that the steady-state value of the transient component \( y_{t,\text{st}} \) is equal to zero.

We multiply both sides of equation (11) by \( s \)

\[ sY_t(s) = \hat{W}(s)X(s) - K \cdot X(0) \] (12)

Since \( x(t) \) and \( y(t) \) are vanishing functions, \( X(s) \) and \( Y(s) \) expand into a series in moments

\[ X(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots \quad Y_t(s) = \beta_0 + \beta_1 s + \beta_2 s^2 + \ldots \] (13)

Substituting (13) into (12) taking into account (1) after the transformation, we obtain

\[ \sum_{i=0}^{\infty} \beta_i s^i = \sum_{i=0}^{\infty} \mu_i s^i \sum_{j=0}^{\infty} \alpha_j s^j - K \cdot X(0) \] (14)

Expanding the parentheses and equating in (14) the coefficients at the same degrees \( s \), we obtain recurrent formulas for the moments of the process \( \mu_k \)

\[ \mu_0 = y_{t,\text{st}} / \alpha_0, \quad \mu_1 = (\beta_0 - \mu_0 \alpha_1) / \alpha_0, \quad \mu_2 = (\beta_1 - \mu_0 \alpha_2 - \mu_2 \alpha_0) / \alpha_0, \quad \mu_3 = (\beta_2 - \mu_0 \alpha_3 - \mu_2 \alpha_1 - \mu_3 \alpha_0) / \alpha_0 \] (15)

The moments of the input and output signals are determined by the formulas

\[ \alpha_k \int_0^T x(t)dt = \beta_k \int_0^T y(t)dt, \quad k = 0,1,\ldots; \] (16)

where \( T_x, T_y \) is the decay time of the input and output signals.

Integrals (16) can be calculated by any method of numerical integration, and for simple signals analytically. So, if \( x(t) = A \cdot (l(t) - l(t-T_x)) \) is a rectangular pulse with amplitude \( A \) and duration \( T_x \), then

\[ \alpha_k = \frac{(-1)^k A \cdot T_x^{k+1}}{(k+1)!}, \quad k = 0,1,\ldots \] (17)

If the object is static, we introduce new vanishing at infinity variables

\[ x_t(t) = x_{t,\text{st}} - x(t), \quad y_t(t) = y_{t,\text{st}} - y(t) \] (18)

Finding Laplace transform, we get

\[ X_t(s) = x_{t,\text{st}} / s - X(s), \quad Y_t(s) = y_{t,\text{st}} / s - W_t(s)X(s) \] (19)

Eliminating \( X(s) \) from the last equations, after transformations we obtain

\[ sY_t(s) - y_{t,\text{st}} + W_t(s)(x_{t,\text{st}} - sX_t(s)) = 0 \] (20)

Substituting in (20) instead of \( W_t(s), X_t(s), Y_t(s) \), their expansion in moments, from (1) and (13), we obtain formulas for calculating the moments of the object.
\[
\mu_0 = \frac{y_0}{x_0} = K, \quad \mu_t = \frac{\mu_0 \alpha_0 - \beta_0}{x_0}, \quad \mu_2 = \frac{\mu_0 \alpha_1 + \mu_1 \alpha_0 - \beta_1}{x_0}, \\
\mu_3 = \frac{\mu_2 \alpha_2 + \mu_1 \alpha_1 + \mu_1 \alpha_0 - \beta_2}{x_0}, \quad \mu_k = \frac{\sum_{i=0}^{k-1} \mu_0 \alpha_{i+1} - \beta_{k-1}}{x_0}, \ldots \tag{21}
\]

If \( x(t) = A \cdot 1(t) \), a step function, and \( y_i(t) = y(t) - y_{out} \), then we get
\[
\mu_0 = W(0) = \frac{y_{out}}{A} = K, \quad \mu_k = -\frac{\beta_{k-1}}{A}, \quad k = 0, 1, \ldots
\tag{22}
\]

3. Examples of application of reduction and identification algorithms

3.1. Order reduction problem.

Let the transfer function of the test controlled process be
\[
W_c(s) = \frac{2\exp(-s)}{s(3s + 1)^2(s + 1)^2}.
\]

Therefore, the moments of the proportional part of the process are \( \mu_0 = 2.00, \quad \mu_1 = -24.00, \quad \mu_2 = 173.0, \mu_3 = -979.3 \).

Thus, we obtain the transfer function of the model
\[
W_m = \frac{K_m \cdot \exp(-\tau_m \cdot s)}{s \cdot (T_m \cdot s + 1)^3 e^{3.540285274}} = \frac{2e^{-1.867469847\tau_m}}{s(2.862068913s + 1)^{3.540285274}}.
\]

Figure 1 shows the Nyquist plot of the process (drawn by points) and of the model (solid curve) plotted in the frequency range \( \omega = 0.1 \ldots 1.8 \text{ min}^{-1} \). Figure 2 shows the responses of the process and the model to a rectangular pulse of unit amplitude with a duration of 1.3 min. The characteristics of the object and the model are practically the same. When constructing the characteristics of the model, the Padé approximation of its transfer function was used.

![Figure 1](image1.png)  ![Figure 2](image2.png)

**Figure 1.** Nyquist plots of the process and the model.  **Figure 2.** Step responses of the process and the model.

3.2. Identification problem.

As a result of an experiment on a laboratory plant, the step response of the control process was obtained, 22 sample values \( Yout[i], \quad i=1,2,\ldots,22 \).

\[
Yout = [0, 0, 0, 0.3, 0.9, 2.3, 2.9, 4.4, 4.8, 5.1, 5.3, 5.5, 5.7, 5.8, 5.85, 5.9, 5.93, 5.97, 5.99, 6.6].
\]

The sampling interval was 1, the input signal amplitude was 8. An auxiliary function \( Y_e = Y_{out} - Y_{out} \) is obtained, figure 3. The moments of the object were calculated using the trapezoidal rule according to the formulas (15), (21): \( \mu_0 = 0.75, \quad \mu_1 = -4.495, \quad \mu_2 = 13.556, \quad \mu_3 = -20.039 \).
Thus, we obtain the transfer function of the model

\[ W_m(s) = 0.75e^{-1.928s} (2.2093s + 1)^{-2.1795} . \]

![Figure 3. Auxiliary function.](image)

![Figure 4. Step responses of the process and the model.](image)

Figure 4 shows the step responses of the process (drawn by points) and of the model (solid curve). The difference is not significant. For constructing the model step response, the Padé approximation of its transfer function was used.

4. Conclusion

Fractional-order models considered in this paper effectively approximate the dynamic characteristics of aperiodic (non-oscillatory) high-order processes with time delay. They can be used in the tasks of reduction and identification of controlled processes. To solve these problems, four moments of the process are enough, which makes it possible to reduce errors associated with calculating high-order moments from experimental data. Appropriate software has been developed in the environment of Maple computer algebra system, its operability and efficiency have been tested on test examples.

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