Generalized characters of the symmetric group

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Abstract

Normalized irreducible characters of the symmetric group $S(n)$ can be understood as zonal spherical functions of the Gelfand pair $(S(n) \times S(n), \text{diag } S(n))$. They form an orthogonal basis in the space of the functions on the group $S(n)$ invariant with respect to conjugations by $S(n)$. In this paper we consider a different Gelfand pair connected with the symmetric group, that is an “unbalanced” Gelfand pair $(S(n) \times S(n-1), \text{diag } S(n-1))$. Zonal spherical functions of this Gelfand pair form an orthogonal basis in a larger space of functions on $S(n)$, namely in the space of functions invariant with respect to conjugations by $S(n-1)$. We refer to these zonal spherical functions as normalized generalized characters of $S(n)$. The main discovery of the present paper is that these generalized characters can be computed on the same level as the irreducible characters of the symmetric group. The paper gives a Murnaghan–Nakayama type rule, a Frobenius type formula, and an analogue of the determinantal formula for the generalized characters of $S(n)$.

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1. Introduction

1.1. Preliminaries and formulation of the problem

One of the central goals of the representation theory of finite groups is in computation of characters of irreducible representations. When a group under considerations is the symmetric group, $S(n)$, the irreducible characters can be computed using either the Frobenius formula, or...
the determinantal formula, or the Murnaghan–Nakayama rule (see, for example, Macdonald [10], Sagan [21], Stanley [22]).

Let $\Lambda$ denote the algebra of symmetric functions, which is a graded algebra, isomorphic to the algebra of polynomials in the power sums $p_1, p_2, \ldots$. If we define $p_\rho = p_{\rho_1} p_{\rho_2} \ldots$ for each partition $\rho = (\rho_1, \rho_2, \ldots)$, then the $p_\rho$ form a homogeneous basis in $\Lambda$. (As in Macdonald [10] we identify each partition with its Young diagram.) Another natural homogeneous basis in $\Lambda$ is formed by the Schur functions $s_\lambda$ indexed by Young diagrams $\lambda$. The Frobenius formula is

$$p_\rho = \sum_{\lambda \vdash n} \chi^\lambda_\rho s_\lambda,$$

where $\chi^\lambda_\rho$ is the value of the irreducible character $\chi^\lambda$ of the symmetric group $S(n)$ on the conjugacy class in $S(n)$ indexed by the partition $\rho$ of $n$. This formula is the key result in the classical theory of characters of the symmetric group $S(n)$. It shows that the character table is the transition matrix between two bases $p_\rho$ and $s_\lambda$ in the algebra of symmetric functions $\Lambda$. The Frobenius formula follows from the fact that the Schur functions $s_\lambda$ are images of $\chi^\lambda$ in $\Lambda$ under a certain map. This map is called the characteristic map, see [10, I, §7]. Thus, if we denote this map by ch, we have

$$s_\lambda = \text{ch}(\chi^\lambda).$$

Another available result on irreducible characters of $S(n)$ is the formula which represents an irreducible character, $\chi^\lambda$, as an alternating sum of the induced characters (i.e. the determinantal formula). Namely, denote by $\eta_k$ the identity character of $S(k)$. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is any partition of $n$, let $\eta_\lambda$ denote $\eta_{\lambda_1} \cdot \eta_{\lambda_2} \cdots$. Here the multiplication, $f \cdot g$, between two characters, $f$ and $g$, of, say, groups $S(k)$ and $S(m)$ is defined by the induction

$$f \cdot g = \text{ind}^{S(k+m)}_{S(k) \times S(m)}(f \times g).$$

With the above notation the irreducible character $\chi^\lambda$ is given by

$$\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1 \leq i, j \leq n}.$$ 

Since $\text{ch}(\eta_\lambda) = h_\lambda$, where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$, and $h_r$ is the $r$th complete symmetric function, the determinantal formula for irreducible characters is equivalent to the Jacobi–Trudi formula for the Schur symmetric functions,

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n}.$$ 

The Murnaghan–Nakayama rule is a recursive method to compute the irreducible characters of the symmetric groups. It can be formulated as follows. Let us say that a skew Young diagram is a border strip if it is connected and does not contain any $2 \times 2$ block of boxes. Suppose that $\pi \sigma$ is an element of the symmetric group $S(n)$, where $\sigma$ is a cycle of length $j$, and $\pi$ is a permutation of the remaining $n - j$ numbers of cycle-type $\rho$, $\rho$ is a partition of $n - j$. The Murnagahn–
