LUSZTIG INDUCTION AND $\ell$-BLOCKS
OF FINITE REDUCTIVE GROUPS

RADHA KESSAR AND GUNTER MALLE

To the memory of Robert Steinberg

Abstract. We present a unified parametrisation of $\ell$-blocks of quasi-simple finite groups of Lie type in non-defining characteristic via Lusztig’s induction functor in terms of $e$-Jordan-cuspidal pairs and $e$-Jordan quasi-central cuspidal pairs.

1. Introduction

The work of Fong and Srinivasan for classical matrix groups and of Schewe for certain blocks of groups of exceptional type exhibited a close relation between the $\ell$-modular block structure of groups of Lie type and the decomposition of Lusztig’s induction functor, defined in terms of $\ell$-adic cohomology. This connection was extended to unipotent blocks of arbitrary finite reductive groups and large primes $\ell$ by Broué–Malle–Michel [5], to all unipotent blocks by Cabanes–Enguehard [8] and Enguehard [11], to groups with connected center and primes $\ell \geq 7$ by Cabanes–Enguehard [9], to non-quasi-isolated blocks by Bonnafé–Rouquier [4] and to quasi-isolated blocks of exceptional groups at bad primes by the authors [13].

It is the main purpose of this paper to unify and extend all of the preceding results in particular from [9] so as to establish a statement in its largest possible generality, without restrictions on the prime $\ell$, the type of group or the type of block, in terms of $e$-Jordan quasi-central cuspidal pairs (see Section 2 for the notation used).

Theorem A. Let $H$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : H \to H$ endowing $H$ with an $\mathbb{F}_q$-rational structure. Let $G$ be an $F$-stable Levi subgroup of $H$. Let $\ell$ be a prime not dividing $q$ and set $e = e_\ell(q)$.

(a) For any $e$-Jordan-cuspidal pair $(L, \lambda)$ of $G$ such that $\lambda \in E(L^F, \ell')$, there exists a unique $\ell$-block $b_{G^F}(L, \lambda)$ of $G^F$ such that all irreducible constituents of $R_{L}^{G}(\lambda)$ lie in $b_{G^F}(L, \lambda)$.

(b) The map $\Xi : (L, \lambda) \mapsto b_{G^F}(L, \lambda)$ is a surjection from the set of $G^F$-conjugacy classes of $e$-Jordan-cuspidal pairs $(L, \lambda)$ of $G$ such that $\lambda \in E(L^F, \ell')$ to the set of $\ell$-blocks of $G^F$.

(c) The map $\Xi$ restricts to a surjection from the set of $G^F$-conjugacy classes of $e$-Jordan quasi-central cuspidal pairs $(L, \lambda)$ of $G$ such that $\lambda \in E(L^F, \ell')$ to the set of $\ell$-blocks of $G^F$.  

Date: June 9, 2015.

2010 Mathematics Subject Classification. 20C20, 20C15, 20C33.

Key words and phrases. finite reductive groups, $\ell$-blocks, $e$-Harish-Chandra series, Lusztig induction.

The second author gratefully acknowledges financial support by ERC Advanced Grant 291512.
(d) For $\ell \geq 3$ the map $\Xi$ restricts to a bijection between the set of $G^F$-conjugacy classes of $e$-Jordan quasi-central cuspidal pairs $(L, \lambda)$ of $G$ with $\lambda \in \mathcal{E}(L^F, \ell')$ and the set of $\ell$-blocks of $G^F$.

(e) The map $\Xi$ itself is bijective if $\ell \geq 3$ is good for $G$, and $\ell \neq 3$ if $G^F$ has a factor $3D_4(q)$.

The restrictions in (d) and (e) are necessary (see Remark 3.15 and Example 3.16).

In fact, part (a) of the preceding result is a special case of the following characterisation of the $\ell'$-characters in a given $\ell$-block in terms of Lusztig induction:

**Theorem B.** In the setting of Theorem A let $b$ be an $\ell$-block of $G^F$ and denote by $\mathcal{L}(b)$ the set of $e$-Jordan cuspidal pairs $(L, \lambda)$ of $G$ such that $\text{Irr}(b) \cap R^G_L(\lambda) \neq \emptyset$. Then

$$\text{Irr}(b) \cap \mathcal{E}(G^F, \ell') = \{ \chi \in \mathcal{E}(G^F, \ell') \mid \exists (L, \lambda) \in \mathcal{L}(b) \text{ with } (L, \lambda) \ll_e (G, \chi) \}.$$ 

Note that at present, it is not known whether Lusztig induction $R^G_L$ is independent of the parabolic subgroup containing the Levi subgroup $L$ used to define it. Our proofs will show, though, that in our case $b_{G^F}(L, \lambda)$ is defined unambiguously.

An important motivation for this work comes from the recent reductions of most long-standing famous conjectures in modular representation theory of finite groups to questions about quasi-simple groups. Among the latter, the quasi-simple groups of Lie type form the by far most important part. A knowledge and suitable inductive description of the $\ell$-blocks of these groups is thus of paramount importance for an eventual proof of those central conjectures. Our results are specifically tailored for use in an inductive approach by considering groups that occur as Levi subgroups inside groups of Lie type of simply connected type, that is, inside quasi-simple groups.

Our paper is organised as follows; in Section 2, we set up $e$-Jordan (quasi-central) cuspidal pairs and discuss some of their properties. In Section 3 we prove Theorem A (see Theorem 3.14) on parametrising $\ell$-blocks by $e$-Jordan-cuspidal and $e$-Jordan quasi-central cuspidal pairs and Theorem B (see Theorem 3.6) on characterising $\ell'$-characters in blocks. The crucial case turns out to be when $\ell = 3$. In particular, the whole Section 3.5 is devoted to the situation of extra-special defect groups of order 27, excluded in [9], which eventually turns out to behave just as the generic case. An important ingredient of Section 3 is Theorem 3.4 which shows that the distribution of $\ell$-characters in $\ell$-blocks is preserved under Lusztig induction from $e$-split Levi subgroups. Finally, in Section 4 we collect some results relating $e$-Jordan-cuspidality and usual $e$-cuspidality.

### 2. Cuspidal pairs

Throughout this section, $G$ is a connected reductive linear algebraic group over the algebraic closure of a finite field of characteristic $p$, and $F : G \to G$ is a Frobenius endomorphism endowing $G$ with an $F_q$-structure for some power $q$ of $p$. By $G^*$ we denote a group in duality with $G$ with respect to some fixed $F$-stable maximal torus of $G$, with corresponding Frobenius endomorphism also denoted by $F$.

#### 2.1. $e$-Jordan-cuspidality

Let $e$ be a positive integer. We will make use of the terminology of Sylow $e$-theory (see e.g. [5]). For an $F$-stable maximal torus $T$, $T_e$ denotes its Sylow $e$-torus. Then a Levi subgroup $L \leq G$ is called $e$-split if $L = C_G(Z^o(L)_e)$, and
\( \lambda \in \text{Irr}(L^F) \) is called \textit{e-cuspidal} if \(*R_{L^F < P}^L(\lambda) = 0\) for all proper \( e \)-split Levi subgroups \( M < L \) and any parabolic subgroup \( P \) of \( L \) containing \( M \) as Levi complement. (It is expected that Lusztig induction is in fact independent of the ambient parabolic subgroup. This would follow for example if the Mackey formula holds for \( R_L^G \), and has been proved whenever \( G^F \) does not have any component of type \( 2E_6(2), E_7(2) \) or \( E_8(2) \), see [3]. All the statements made in this section using \( R_L^G \) are valid independent of the particular choice of parabolic subgroup — we will make clarifying remarks at points where there might be any ambiguity.)

\textbf{Definition 2.1.} Let \( s \in G^*F \) be semisimple. Following [9, 1.3] we say that \( \chi \in \mathcal{E}(G^F, s) \) is \textit{\( e \)-Jordan-cuspidal}, or satisfies condition \((J)\) with respect to some \( e \geq 1 \) if

\begin{enumerate}[(J1)]
  \item \( \ell^e(C_{G^e}(s))_e = Z^e(G^e)_e \) and \\
  \item \( \chi \) corresponds under Jordan decomposition to the \( C_{G^e}(s)^F \)-orbit of an \( e \)-cuspidal unipotent character of \( C_{G^e}(s)^F \).
\end{enumerate}

If \( L \leq G \) is \( e \)-split and \( \lambda \in \text{Irr}(L^F) \) is \( e \)-Jordan-cuspidal, then \((L, \lambda)\) is called an \textit{\( e \)-Jordan-cuspidal pair}.

It is shown in [9, Prop. 1.10] that \( \chi \) is \( e \)-Jordan-cuspidal if and only if it satisfies the uniform criterion

\((\text{U}): \) for every \( F \)-stable maximal torus \( T \leq G \) with \( T_e \not\leq Z(G) \) we have \(*R_L^G(\chi) = 0.\)

\textbf{Remark 2.2.} By [9, Prop. 1.10(ii)] it is known that \( e \)-cuspidality implies \( e \)-Jordan-cuspidality; moreover \( e \)-Jordan-cuspidality and \( e \)-cuspidality agree at least in the following situations:

\begin{enumerate}[(1)]
  \item when \( e = 1 \);
  \item for unipotent characters (see [3] Cor. 3.13);
  \item for characters lying in an \( \ell' \)-series where \( \ell \) is good for \( G \) and either \( \ell \geq 5 \) or \( \ell = 3 \in \Gamma(G, F) \) (see [9, Thm. 4.2 and Rem. 5.2]); and \\
  \item for characters lying in a quasi-isolated \( \ell' \)-series of an exceptional type simple group for \( \ell \) a bad prime (this follows by inspection of the explicit results in [13]).
\end{enumerate}

To see the the first point, assume that \( \chi \) is 1-Jordan-cuspidal. Suppose if possible that \( \chi \) is not 1-cuspidal. Then there exists a proper 1-split Levi subgroup \( L \) of \( G \) such that \( *R_L^G(\chi) = 0 \). Then \( *R_L^G(\chi)(1) \neq 0 \) as \( *R_L^G \) is ordinary Harish-Chandra restriction. Hence the projection of \( *R_L^G(\chi) \) to the space of uniform functions of \( L^F \) is non-zero in contradiction to the uniform criterion \((\text{U})\).

It seems reasonable to expect (and that is formulated as a conjecture in [9, 1.11]) that \( e \)-cuspidality and \( e \)-Jordan-cuspidality agree in general. See Section 4 below for a further discussion of this.

We first establish conservation of \( e \)-Jordan-cuspidality under some natural constructions:

\textbf{Lemma 2.3.} Let \( L \) be an \( F \)-stable Levi subgroup of \( G \) and \( \lambda \in \text{Irr}(L^F) \). Let \( L_0 = L \cap [G, G] \) and let \( \lambda_0 \) be an irreducible constituent of \( \text{Res}_{L_0}^{L^F}(\lambda) \). Let \( e \geq 1 \). Then \((L, \lambda)\) is an \( e \)-Jordan-cuspidal pair for \( G \) if and only if \((L_0, \lambda_0)\) is an \( e \)-Jordan-cuspidal pair for \([G, G] \).
Proof. Note that $L$ is $e$-split in $G$ if and only if $L_0$ is $e$-split in $G_0$. Let $\iota : G \hookrightarrow \hat{G}$ be a regular embedding. It is shown in the proof of \cite{3} Prop. 1.10 that condition (J) with respect to $G$ is equivalent to condition (J) with respect to $\hat{G}$. Since $\iota$ restricts to a regular embedding $[G, G] \hookrightarrow \hat{G}$, the same argument shows that condition (J) with respect to $G$ is equivalent to that condition with respect to $[G, G]$. \hfill $\Box$

Proposition 2.4. Let $s \in G^{*F}$ be semisimple, and $G_1 \leq G$ an $F$-stable Levi subgroup with $G_1^*$ containing $C_{G^*}(s)$. For $(L_1, \lambda_1)$ an $e$-Jordan-cuspidal pair of $G_1$ below $E(G_1^F, s)$ define $L := C_G(Z^0(L_1)_e)$ and $\lambda := \epsilon_L \epsilon_{L_1} R^L_{L_1}(\lambda_1)$. Then $Z^0(L_1)_e = Z^0(L)_e$, and $(L_1, \lambda_1) \mapsto (L, \lambda)$ defines a bijection $\Psi_{G_1^G}$ between the set of $e$-Jordan-cuspidal pairs of $G_1$ below $E(G_1^F, s)$ and the set of $e$-Jordan-cuspidal pairs of $G$ below $E(G^F, s)$.

We note that the character $\lambda$ and hence the bijection $\Psi_{G_1^G}$ are above independent of the choice of parabolic subgroup. This is explained in the proof below.

Proof. We first show that $\Psi_{G_1^G}$ is well-defined. Let $(L_1, \lambda_1)$ be $e$-Jordan-cuspidal in $G_1$ below $E(G_1^F, s)$, so $s \in L_1^*$. Then $L^* := C_{G^*}(Z^0(L_1)_e)$ clearly is an $e$-split Levi subgroup of $G^*$. Moreover we have

$$L_1^* = C_{G_1^*}(Z^0(L_1^*)_e) = C_{G^*}(Z^0(L_1)_e) \cap G_1^* = L^* \cap G_1^*.$$

Now $s \in L_1^*$ by assumption, so

$$L_1^* = L^* \cap G_1^* \geq L^* \cap C_{G^*}(s) = C_{L^*}(s).$$

In particular, $L_1^*$ and $L^*$ have a maximal torus in common, so $L_1^*$ is a Levi subgroup of $L^*$. Thus, passing to duals, $L_1$ is a Levi subgroup of $L = C_G(Z^0(L_1)_e)$.

We clearly have $Z^0(L_1)_e \leq Z^0(L)_e$. For the reverse inclusion note that $Z^0(L)_e \leq L_1$, as $L_1$ is a Levi subgroup in $L$, so indeed $Z^0(L)_e \leq Z^0(L_1)_e$.

Hence $\lambda := \epsilon_L \epsilon_{L_1} R^L_{L_1}(\lambda_1)$ is irreducible since, as we saw above, $L_1^* \geq C_{L^*}(s)$. By \cite{10} Rem. 13.28, $\lambda$ is independent of the choice of parabolic subgroup of $L$ containing $L_1$ as Levi subgroup. Let’s argue that $\lambda$ is $e$-Jordan-cuspidal. Indeed, for any $F$-stable maximal torus $T \leq L$ we have by the Mackey-formula (which holds as one of the Levi subgroups is a maximal torus by a result of Deligne–Lusztig, see \cite{3} Thm. (2)) that $\epsilon_L \epsilon_{L_1} R^L_{L_1}(\lambda) = R^T_{L_1} R^L_{L_1}(\lambda_1)$ is a sum of $E$-conjugates of $R^L_{L_1}(\lambda_1)$. As $\lambda_1$ is $e$-Jordan-cuspidal, this vanishes if $T_e \not\subseteq Z^0(L_1)_e = Z^0(L)_e$. So $\lambda$ satisfies condition (U), hence is $e$-Jordan-cuspidal, and $\Psi_{G_1^G}$ is well-defined.

It is clearly injective, since if $(L, \lambda) = \Psi_{G_1^G}(L_2, \lambda_2)$ for some $e$-cuspidal pair $(L_2, \lambda_2)$ of $G_1$, then $Z^0(L_1)_e = Z^0(L)_e = Z^0(L_2)_e$, whence $L_1 = C_{G_1}(Z^0(L_1)_e) = C_{G_1}(Z^0(L_2)_e) = L_2$, and then the bijectivity of $R^L_{L_1}$ on $E(L_1^*, s)$ shows that $\lambda_1 = \lambda_2$ as well.

We now construct an inverse map. For this let $(L, \lambda)$ be an $e$-Jordan-cuspidal pair of $G$ below $E(G^F, s)$, and $L^* \leq G^*$ dual to $L$. Set

$$L_1^* := C_{G_1^*}(Z^0(L_1^*)_e) = C_{G^*}(Z^0(L_1)_e) \cap G_1^* = L^* \cap G_1^*,$$

an $e$-split Levi subgroup of $G_1^*$. Note that $s \in L^*$, so there exists some maximal torus $T^*$ of $G^*$ with $T^* \leq C_{G^*}(s) \leq G_1^*$, whence $L_1^*$ is a Levi subgroup of $L^*$. Now again

$$L_1^* = L^* \cap G_1^* \geq L^* \cap C_{G^*}(s) = C_{L^*}(s).$$
So the dual $L_1 := C_{G_1}(Z^\circ(L_1)e)$ is a Levi subgroup of $L$ such that $\epsilon_{L_1} \epsilon_L R^L_{L_1}$ preserves irreducibility on $E(L_1^F, s)$. We define $\lambda_1$ to be the unique constituent of $^*R^L_{L_1}(\lambda)$ in the series $E(L_1^F, s)$. Then $\lambda_1$ is $e$-Jordan-cuspidal. Indeed, for any $F$-stable maximal torus $T \leq L_1$ with $T_e \not\leq Z^\circ(L_1)e = Z^\circ(L_1)_e$ we get that $^*R^L_{L_1}(\lambda_1)$ is a constituent of $^*R^L_T(\lambda_1) = 0$ by $e$-Jordan-cuspidality of $\lambda$. Here note that the set of constituents of $^*R^L_T(\eta)$, where $\eta$ is a constituent of $^*R^L_{L_1}(\lambda_1)$ different from $\lambda_1$ is disjoint from the set of irreducible constituents of $^*R^L_{L_1}(\lambda_1)$.

Thus we have obtained a well-defined map $^*\Psi^G_{G_1}$ from $e$-Jordan-cuspidal pairs in $G$ to $e$-Jordan-cuspidal pairs in $G_1$, both below the series $s$. As the map $\Psi^G_{G_1}$ preserves the $e$-part of the center, $^*\Psi^G_{G_1} \circ \Psi^G_{G_1}$ is the identity. It remains to prove that $\Psi^G_{G_1}$ is surjective. For this let $(M, \mu)$ be any $e$-Jordan-cuspidal pair of $G$ below $E(G^F, s)$, let $(L_1, \lambda_1) = ^*\Psi^G_{G_1}(M, \mu)$ and $(L, \lambda) = \Psi^G_{G_1}(L_1, \lambda_1)$. Then we have $Z^\circ(M)_e \leq Z^\circ(L_1)_e = Z^\circ(L_1)e$, so $L = C_G(Z^\circ(L_1)e) \leq C_G(Z^\circ(M)_e) = M$ is an $e$-split Levi subgroup of $M$.

As $L_1 \leq L \leq M$ and $\epsilon_{L_1} \epsilon_M R^M_{L_1}$ is a bijection from $E(L_1^F, s)$ to $E(M^F, s)$, it follows that $\epsilon_L \epsilon_M R^M_{L_1}$ is a bijection between $E(L_1^F, s)$ and $E(M^F, s)$. As $\lambda$ and $\mu$ are $e$-Jordan-cuspidal, $(\lambda_1)$ implies that $Z^\circ(M)_e = Z^\circ(L_1)_e$, so $M = L$, that is, $(M, \mu)$ is in the image of $\Psi^G_{G_1}$.

The proof is complete.

The above bijection also preserves relative Weyl groups.

**Lemma 2.5.** In the situation and notation of Proposition 2.4 let $(L, \lambda) = \Psi^G_{G_1}(L_1, \lambda_1)$. Then $N_{G_1}^G(L_1, \lambda_1) \leq N_G(L, \lambda)$ and this inclusion induces an isomorphism of relative Weyl groups $W^G_{G_1}(L_1, \lambda_1) \cong W_G(L, \lambda)$.

**Proof.** Let $g \in N_{G_1}^G(L_1, \lambda_1)$. Then $g$ normalises $Z^\circ(L_1)_e$ and hence also $L = C_G(Z^\circ(L_1)_e)$. Thus,

$$^g\lambda = \epsilon_L \epsilon_{L_1} R^L_{L_1}(^g\lambda_1) = \epsilon_L \epsilon_{L_1} R^L_{L_1}(\lambda_1) = \lambda$$

and the first assertion follows.

For the second assertion let $g \in N_{G_1}^G(L_1, \lambda_1)$ and let $T$ be an $F$-stable maximal torus of $L_1$ and $\theta$ an irreducible character of $T^F$ such that $\lambda_1$ is a constituent of $R^L_{L_1}(\theta)$. Since $\lambda_1 \in E(L_1^F, s)$, $(T, \theta)$ corresponds via duality (between $L_1$ and $L_1^F$) to the $L_1^F$-class of $s$, and all constituents of $R^L_{L_1}(\theta)$ are in $E(L_1^F, s)$. Consequently, $R^L_{L_1}$ induces a bijection between the set of constituents of $R^L_{L_1}(\theta)$ and the set of constituents of $R^L_T(\theta)$. In particular, $\lambda$ is a constituent of $R^L_T(\theta)$. Since $g$ stabilises $\lambda$, $\lambda$ is also a constituent of $R^L_T(\theta)$. Hence $(T, \theta)$ and $^g(T, \theta)$ are geometrically conjugate in $L$. Let $l \in L$ geometrically conjugate $^g(T, \theta)$ to $(T, \theta)$. Since $C_G(s) \leq G_1$, $lg \in G_1$ (see for instance [13] Lemma 7.5]). Hence $F(l)^{-1} = F(lg)(lg)^{-1} \in G_1 \cap L = L_1$. By the Lang–Steinberg theorem applied to $L_1$, there exists $l_1 \in L_1$ such that $l_1l \in L^F$. Also, since $l_1 \in G_1$ and $g \in G^F$, $l_1lg \in G_1^F$. Thus, up to replacing $g$ by $l_1lg$, we may assume that $g \in G_1^F$.

Since $L_1 = C_{G_1}(Z^\circ(L_1)_e)$, it follows that $g \in N_{G_1}^G(L_1)$, and thus

$$\epsilon_L \epsilon_{L_1} R^L_{L_1}(\lambda_1) = \lambda = ^g\lambda = \epsilon_L \epsilon_{L_1} R^L_{L_1}(^g\lambda_1).$$

Since $R^L_{L_1}$ induces a bijection between the set of characters in the geometric Lusztig series of $L_1^F$ corresponding to $s$ (the union of series $E(L_1^F, t)$, where $t$ runs over the semisimple elements of $L_1^F$ which are $L_1$-conjugate to $s$) and the set of characters in the geometric

LUSZTIG INDUCTION AND $\epsilon$-BLOCKS 5
Let $\ell$ be a prime. We set $e_{\ell}(q) := \text{order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell \neq 2 \\ 4 & \text{if } \ell = 2. \end{cases}$

For a semisimple $\ell'$-element $s$ of $G^s$, we denote by $E_{\ell}(G^s, s)$ the union of all Lusztig series $E(G^s, st)$, where $t \in G^s$ is an $\ell'$-element commuting with $s$. We recall that the set $E_{\ell}(G^s, s)$ is a union of $\ell$-blocks. Further, if $G_1 \leq G$ is an $\ell'$-stable Levi subgroup such that $G_1^s$ contains $C_{G^s}(s)$, then $e_{G_1} \cdot \epsilon_G R_{G_1}^G$ induces a bijection, which we refer to as the Bonnafé–Rouquier correspondence, between the $\ell$-blocks in $E(G_1^s, s)$ and the $\ell$-blocks in $E(G^s, s)$.

**Proposition 2.6.** Let $\ell \neq p$ be a prime, $s \in G^s$ an $\ell'$-element and $G_1 \leq G$ an $\ell'$-stable Levi subgroup with $G_1^s$ containing $C_{G^s}(s)$. Assume that $b$ is an $\ell$-block in $E_{\ell}(G^s, s)$, and $c$ is its Bonnafé–Rouquier correspondent in $E_{\ell}(G_1^s, s)$. Let $e := e_\ell(q)$.

(a) Let $(L_1, \lambda_1)$ be $\ell$-Jordan-cuspidal in $G_1$ and set $(L, \lambda) = \Psi_{G_1}(L_1, \lambda_1)$. If all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in $c$, then all constituents of $R_{L}^{G}(\lambda)$ lie in $b$.

(b) Let $(L, \lambda)$ be $\ell$-Jordan-cuspidal in $G$ and set $(L_1, \lambda_1) = \Psi_{G_1}(L, \lambda)$. If all constituents of $R_{L}^{G}(\lambda)$ lie in $b$, then all constituents of $R_{L_1}^{G_1}(\lambda_1)$ lie in $c$.

**Proof.** Note that the hypothesis of part (a) means that for any parabolic subgroup $P$ of $G_1$ containing $L_1$ as Levi subgroup all constituents of $R_{L_1 \cap P}^{G_1}(\lambda_1)$ lie in $c$. A similar remark applies to the conclusion, as well as to part (b).

For (a), note that by the definition of $\Psi^G_{G_1}$ we have that all constituents of $\epsilon_{L_1} \cdot \epsilon_{L_1} R_{L}^{G}(\lambda) = R_{L_1}^{G_1}(\lambda_1) = R_{L}^{G_1}(\lambda_1)$ are contained in $R_{L_1}^{G_1}(c)$, hence in $b$ by Bonnafé–Rouquier correspondence.

In (b), suppose that $\eta$ is a constituent of $R_{L_1}^{G_1}(\lambda_1)$ not lying in $c$. Then by Bonnafé–Rouquier, $R_{G_1}^G(\eta)$ does not belong to $b$, whence $R_{L}^{G}(\lambda)$ has a constituent not lying in $b$, contradicting our assumption that all constituents of $R_{L_1}^{G_1}(\lambda_1) = R_{L_1}^{G_1}(\lambda_1)$ are in $b$.

2.3. $\ell$-quasi-centrality. For a prime $\ell$ not dividing $q$, we denote by $E(G^s, \ell')$ the set of irreducible characters of $G^s$ lying in a Lusztig series $E(G^s, s)$, where $s \in G^s$ is a semisimple $\ell'$-element. Recall from [13, Def. 2.4] that a character $\chi \in E(G^s, \ell')$ is said to be of central $\ell$-defect if the $\ell$-block of $G^s$ containing $\chi$ has a central defect group and $\chi$ is said to be of quasi-central $\ell$-defect if some (and hence any) character of $[G, G]^F$ covered by $\chi$ is of central $\ell$-defect.

**Lemma 2.7.** Let $L$ be an $\ell$-stable Levi subgroup of $G$, and set $L_0 = L \cap [G, G]$. Let $\ell \neq p$ be a prime.
(a) If $L_0 = C_{[G,G]}(Z(L_0)_{F}^\ell)$, then $L = C_G(Z(L)_{F}^\ell)$.

(b) Let $\lambda \in \mathcal{E}(L^F, \ell^e)$ and let $\lambda_0$ be an irreducible constituent of $\text{Res}_{L_0}^{L}(\lambda)$. Then $\lambda_0$ is of quasi-central $\ell$-defect if and only if $\lambda$ is of quasi-central $\ell$-defect.

**Proof.** Since $G = Z^\circ(G)[G,G]$ and $Z^\circ(G) \subseteq L$, we have that $L = Z^\circ(G)L_0$. Hence if $L_0 = C_{[G,G]}(Z(L_0)_{F}^\ell)$ then $L = C_G(Z(L_0)_{F}^\ell) \supseteq C_G(Z(L)_{F}^\ell) \supseteq L$. This proves (a). In (b), since $\lambda$ is in an $\ell^e$-Lusztig series, the index in $L^F$ of the stabiliser in $L^F$ of $\lambda_0$ is prime to $\ell$ and on the other hand, $\lambda_0$ extends to a character of the stabiliser in $L^F$ of $\lambda_0$. Thus, $\lambda(1)_{\ell} = \lambda_0(1)_{\ell}$. Since $[L_0, L_0] = [L, L]$, the assertion follows by [13, Prop. 2.5(a)].

**Remark 2.8.** The converse of assertion (a) of Lemma 2.7 fails in general, even when we restrict to $e\ell(q)$-split Levi subgroups: Let $\ell$ be odd and $G = GL_{\ell}$ with $F$ such that $G^F = GL_{\ell-1}(q)$ with $\ell|(q - 1)$. Let $L$ a 1-split Levi subgroup of type $GL_{\ell-1} \times GL_1$. Then $Z(L)_{F}^\ell \cong C_\ell \times C_\ell$ and $L = C_G(Z(L)_{F}^\ell)$. But $Z(L_0)_{F}^\ell \cong C_\ell \cong Z([G,G])_{F}^\ell$, hence $C_{[G,G]}(Z(L_0)_{F}^\ell) = [G,G]$.

One might hope for further good properties of the bijection of Proposition 2.7 with respect to (quasi)-centrality. In this direction, we observe the following:

**Lemma 2.9.** In the situation of Proposition 2.7, if $(L, \lambda)$ is of central $\ell$-defect for a prime $\ell$ with $e\ell(q) = e$, then so is $(L_1, \lambda_1) = \Phi_G^e(L, \lambda)$, and we have $Z(L_1)_{F}^\ell = Z(L_1)^F$.

**Proof.** By assumption, we have that $\lambda(1)_{\ell} = |L^F : Z(L)^F|_{\ell}$. Now $Z(L)$ lies in every maximal torus of $L_1$, hence in $L_1$, so we have that $Z(L_1)_{F}^\ell \leq Z(L_1)^F$. As $\lambda = \epsilon_\ell \epsilon_L R_{L_1}^L(\lambda_1)$ we obtain $\lambda(1)_{\ell} = \lambda_1(1)_{\ell} |L^F : L_1^F|_{\ell}$, whence

$$\lambda_1(1)_{\ell} = \lambda(1)_{\ell} |L^F : L_1^F|_{\ell}^{-1} = |L_1^F|_{\ell} |Z(L)^F|_{\ell}^{-1} \geq |L_1^F : Z(L_1)^F|_{\ell}.

$$

But clearly $\lambda_1(1)_{\ell} \leq |L_1^F : Z(L_1)^F|_{\ell}$, so we have equality throughout, as claimed.

**Example 2.10.** The converse of Lemma 2.9 does not hold in general. To see this, let $G = PGL_{\ell}$ with $G^F = PGL_{\ell}(q)$, $L = G$, and $G_1 \leq G$ an $F$-stable maximal torus such that $G_1^F$ is a Coxeter torus of $G^F$, of order $\Phi_\ell$. Assume that $\ell|(q - 1)$ (so $e = 1$). Then $L_1 = G_1$. Here, any $\lambda_1 \in \text{Irr}(L_1^F)$ is $e$-(Jordan-)cuspidal, and certainly of central $\ell$-defect, and $|Z(L_1)^F| = (\Phi_{\ell})_\ell = \ell$ for $\ell \geq 3$, while clearly $Z(L_1)_{F}^\ell = Z(G)_{F}^\ell = 1$. Furthermore

$$\lambda(1)_{\ell} = \lambda_1(1)_{\ell} |L^F : L_1^F|_{\ell} = |L^F : L_1^F|_{\ell}

$$

since $\lambda_1$ is linear. Since $|Z(L)^F|_{\ell} = 1$ and $|L_1^F|_{\ell} > 1$, it follows that $\lambda(1)_{\ell} |Z(L)^F|_{\ell} < |L^F|_{\ell}$

hence $\lambda$ is not of central $\ell$-defect (and even not of quasi-central $\ell$-defect).

**Example 2.11.** We also recall that $e$-(Jordan-)cuspidal characters are not always of central $\ell$-defect, even when $\ell$ is a good prime: Let $G^F = SL_{\ell}(q)$ with $\ell|(q - 1)$, so $e = 1$. Then for $\theta$ a Coxeter torus and $\theta \in \text{Irr}(T^F)$ in general position, $R_T^G(\theta)$ is $e$-(Jordan-)cuspidal but not of quasi-central $\ell$-defect.

For the next definition note that the property of being of (quasi)-central $\ell$-defect is invariant under automorphisms of $G^F$. 
Definition 3.12. Let $\ell \neq p$ be a prime and $e = e_\ell(q)$. A character $\chi \in \mathcal{E}(G^F, \ell')$ is called $e$-Jordan quasi-central cuspidal if $\chi$ is $e$-Jordan cuspidal and the $C_{G^s,(s)^F}$-orbit of unipotent characters of $C_{G_s,(s)^F}$ which corresponds to $\chi$ under Jordan decomposition consists of characters of quasi-central $\ell$-defect, where $s \in G^{e_\ell}$ is a semisimple $\ell'$-element such that $\chi \in \mathcal{E}(G^F, s)$. An $e$-Jordan quasi-central cuspidal pair of $G$ is a pair $(\mathcal{L}, \lambda)$ such that $\mathcal{L}$ is an $e$-split Levi subgroup of $G$ and $\lambda \in \mathcal{E}(L^F, \ell')$ is an $e$-Jordan quasi-central cuspidal character of $L^F$.

We note that the set of $e$-Jordan quasi-central cuspidal pairs of $G$ is closed under $G^F$-conjugation. Also, note that Lemma 2.3 remains true upon replacing the $e$-Jordan-cuspidal property by the $e$-Jordan quasi-central cuspidal property. This is because, with the notation of Lemma 2.3, the orbit of unipotent characters corresponding to $\lambda$ under Jordan decomposition is a subset of the orbit of unipotent characters corresponding to $\lambda_0$ under Jordan decomposition.

Finally we note that the bijection $\Psi_{G_1}$ of Proposition 2.6 preserves $e$-quasi-centrality since with the notation of the proposition $\lambda_1$ and $\lambda$ correspond to the same orbit of unipotent characters under Jordan decomposition.

3. Lusztig Induction and $\ell$-blocks

Here we prove our main results on the parametrisation of $\ell$-blocks in terms of $e$-Harish-Chandra series, in arbitrary Levi subgroups of simple groups of simply connected type. As in Section 2, $\ell \neq p$ will be prime numbers, $q$ a power of $p$ and and $e = e_\ell(q)$.

3.1. Preservation of $\ell$-blocks by Lusztig induction. We first extend [9, Thm. 2.5]. The proof will require three auxiliary results:

Lemma 3.1. Let $G$ be connected reductive with a Frobenius endomorphism $F$ endowing $G$ with an $\mathbb{F}_q$-rational structure. Let $M$ be an $e$-split Levi of $G^F$ and $c$ an $\ell$-block of $M^F$.

Suppose that

1. the set $\{d^{1,M^F}(\mu) \mid \mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')\}$ is linearly independent; and
2. there exists a subgroup $Z \leq Z(M)_\ell^F$ and a block $d$ of $C_{G_s}^e(Z)^F$ such that all irreducible constituents of $R_{M_s}^{G_s}(Z)(\mu)$, where $\mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')$, lie in the block $d$.

Then there exists a block $b$ of $G^F$ such that all irreducible constituents of $R_{M_s}^G(\mu)$, where $\mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')$, lie in the block $b$.

Proof. We adapt the argument of [13, Prop. 2.16]. Let $\chi \in \text{Irr}(G^F, \ell')$ be such that $\langle R_{M_s}^G(\mu), \chi \rangle \neq 0$ for some $\mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')$. Then $\langle \mu, R_{M_s}^G(\chi) \rangle \neq 0$. In particular, $c^* R_{M_s}^G(\chi) \neq 0$. All constituents of $s R_{M_s}^G(\chi)$ lie in $\mathcal{E}(M^F, \ell')$, so by assumption (1) it follows that $d^{1,M^F}(c^* R_{M_s}^G(\chi)) \neq 0$. Since $d^{1,M^F}(c^* R_{M_s}^G(\chi))$ vanishes on $\ell$-singular elements of $M^F$, we have that

$$\langle d^{1,M^F}(c^* R_{M_s}^G(\chi)), c^* R_{M_s}^G(\chi) \rangle = \langle d^{1,M^F}(c^* R_{M_s}^G(\chi)), d^{1,M^F}(c^* R_{M_s}^G(\chi)) \rangle \neq 0.$$  

If $\varphi$ and $\varphi'$ are irreducible $\ell$-Brauer characters of $M^F$ lying in different $\ell$-blocks of $M^F$, then $\langle \varphi, \varphi' \rangle = 0$ (see for instance [16, Ch. 3, Ex. 6.20(ii)]). Thus,

$$\langle d^{1,M^F}(c^* R_{M_s}^G(\chi)), c' R_{M_s}^G(\chi) \rangle = \langle d^{1,M^F}(c^* R_{M_s}^G(\chi)), d^{1,M^F}(c' R_{M_s}^G(\chi)) \rangle = 0$$

for all blocks $c'$ of $M^F$ different from $c$. So, $\langle d^{1,M^F}(c^* R_{M_s}^G(\chi)), R_{M_s}^G(\chi) \rangle \neq 0$ from which it follows that $\langle d^{1,M^F}(\mu'), R_{M_s}^G(\chi) \rangle \neq 0$ for some $\mu' \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')$. 

Continuing as in the proof of [13, Prop. 2.12] gives the required result. Note that Condition (1) of [13, Prop. 2.12] is not necessarily met as stated, since \( \mu' \) may be different from \( \mu \). However, \( \mu \) and \( \mu' \) are in the same block of \( M^F \) which is sufficient to obtain the conclusion of the lemma.

**Lemma 3.2.** Let \( G \) be connected reductive with a Frobenius endomorphism \( F \). Suppose that \( G \) has connected centre and \( [G, G] \) is simply connected. Let \( G = XY \) such that either \( X \) is an \( F \)-stable product of components of \([G, G]\) and \( Y \) is the product of the remaining components with \( Z(G) \), or vice versa. Suppose further that \( G^F/X^FY^F \) is an \( \ell \)-group, and let \( c \) be a \( \ell \)-block of \( M^F \) and let \( c' \) be an \( \ell \)-block of \( N^F \) covered by \( c \). Suppose that there exists a block \( b' \) of \( Y^F \) such that every irreducible constituent of \( R^N_M(\tau) \) for \( \tau \in Irr(c') \cap E(N^F, \ell') \) lies in \( b' \). Then there exists a block \( b \) of \( G^F \) such that every irreducible constituent of \( R^G_M(\mu) \) where \( \mu \in Irr(c) \cap E(M^F, \ell') \) lies in \( b \).

**Proof.** We will use the extension of Lusztig induction to certain disconnected groups as in [9, Sec. 1.1]. Let \( G_0 = [G, G] = [X, X] \times [Y, Y], \) \( M_0 = G_0 \cap M = [X, X] \times ([Y, Y] \cap N) \). Then, \( G_0^F \subseteq X^F Y^F \) and \( M_0^F \subseteq X^F N^F \). Let \( T \) be an \( F \)-stable maximal torus of \( M \). Since \( G \) and hence also \( M \) has connected centre, \( M = M_0^F T^F \) and \( G = G_0^F T^F \). Further, \( A := X^F Y^F \cap T^F = X^F N^F \cap T^F \) and \( X^F Y^F = G_0^F A = (G_0 A)^F, X^F N^F = M_0^F A = (M_0 A)^F \). As in [9, Sec. 1.1], we denote by \( E(X^F Y^F, \ell') \) the set of irreducible characters of \( X^F Y^F \) which appear in the restriction of elements of \( E(G^F, \ell') \) to \( X^F Y^F \).

Let \( \chi \in E(G^F, \ell') \). Since \( G^F/X^F Y^F \) is an \( \ell \)-group, by [9, Prop. 1.3(i)], \( Res^G_{X^F Y^F}(\chi) \) is irreducible. Now if \( \chi' \in Irr(G^F) \) has the same restriction to \( X^F Y^F \) as \( \chi \), then again since \( G^F/X^F Y^F \) is an \( \ell \)-group, either \( \chi' = \chi \) or \( \chi' \notin E(G^F, \ell') \). In other words, restriction from \( ZE(G^F, \ell') \) to \( ZE(X^F Y^F, \ell') \) is a bijection. Similarly, restriction from \( ZE(M^F, \ell') \) to \( ZE(X^F N^F, \ell') \) is a bijection.

In particular every block of \( G^F \) covers a unique block of \( X^F Y^F \). Since \( G^F/X^F Y^F \) is an \( \ell \)-group, there is a bijection (through covering) between the set of blocks of \( G^F \) and the set of blocks of \( X^F Y^F \). Hence, by the injectivity of restriction from \( ZE(G^F, \ell') \) to \( ZE(X^F Y^F, \ell') \), it suffices to prove that there is a block \( b_0 \) of \( X^F Y^F \) such that every irreducible constituent of \( Res^G_{X^F Y^F} R^M_G(\mu) \) with \( \mu \) ranges over \( Irr(c) \cap E(M^F, \ell') \) lies in \( b_0 \).

Following [9, Sec. 1.1], we have that \( Res^G_{X^F Y^F} R^G_M = R^G_{M(0)} \cdot Res^M_{X^F N^F} \) on \( Irr(M^F) \) (where here \( R^G_{M(0)} \) is Lusztig induction in the disconnected setting). Thus, it suffices to prove that there is a block \( b_0 \) of \( X^F Y^F \) such that every irreducible constituent of \( R^G_{M(0)} \cdot Res^M_{X^F N^F}(\mu) \) with \( \mu \) ranges over \( Irr(c) \cap E(M^F, \ell') \) is contained in \( b_0 \). By the above arguments applied to \( M^F \) and \( X^F N^F \), there is a unique block \( c_0 \) of \( X^F N^F \) covered by \( c \). The surjectivity of restriction from \( ZE(M^F, \ell') \) to \( ZE(X^F N^F, \ell') \) implies that it suffices to prove that there is a block \( b_0 \) of \( X^F Y^F \) such that every irreducible constituent of \( R^G_{M(0)}(\mu) \) for \( \mu \in Irr(c_0) \cap E(X^F N^F, \ell') \) is contained in \( b_0 \).

The group \( I := \{(x, x^{-1}) \mid x \in X^F \cap Y^F\} \) is the kernel of the multiplication map \( X^F \times Y^F \to X^F Y^F \). Identifying \( X^F Y^F \) with \( X^F \times Y^F / I \) through multiplication, \( Irr(X^F Y^F) \) is the subset of \( Irr(X^F \times Y^F) \) consisting of characters whose kernel contains \( I \). Since \( X^F \cap Y^F \leq X \cap Y = Z(G) \leq M, I \) is also the kernel of the multiplication map...
$X^F \times N^F \to X^F N^F$ and we may identify $\text{Irr}(X^F Y^F)$ with the subset of $\text{Irr}(X^F \times N^F)$ consisting of characters whose kernel contains $I$.

Any parabolic subgroup of $G_0$ containing $M_0$ as Levi subgroup is of the form $[X, X]P$, where $P$ is a parabolic subgroup of $[Y, Y]$ containing $N \cap [Y, Y]$ as Levi subgroup. Let $U := R_a(XP) = R_a(P) \leq [Y, Y]$ and denote by $\mathcal{L}^{-1}(U)$ the inverse image of $U$ under the Lang map $G \to G$ given by $g \mapsto g^{-1}F(g)$.

The Deligne–Lusztig variety associated to $R_{M_0 A}$ (with respect to $XP$) is $\mathcal{L}^{-1}(U) \cap G_0 A$. Since $T = T \cap M_0$ is normalised by $T$ and in particular by $A$. Hence,

$$\mathcal{L}^{-1}(U) \cap G_0 A = (\mathcal{L}^{-1}(U) \cap G_0) A = [X, X]^F(\mathcal{L}^{-1}(U) \cap [Y, Y]) A$$

$$= [X, X]^F(A \cap X^F)(\mathcal{L}^{-1}(U) \cap [Y, Y])(A \cap Y^F).$$

For the last equality, note that $A = X^F Y^F \cap T = (X^F \cap T)(Y^F \cap T) = (X^F \cap A)(Y^F \cap A)$.

Now, $\mathcal{L}^{-1}(U) \cap Y = (\mathcal{L}^{-1}(U) \cap [Y, Y])S^F$ for any $F$-stable maximal torus $S$ of $Y$. Applying this with $S = T \cap Y$, we have that $\mathcal{L}^{-1}(U) \cap (Y \cap Y') = \mathcal{L}^{-1}(U) \cap Y'$. Also, $[X, X]^F(A \cap X^F) = X^F$. Altogether this gives $\mathcal{L}^{-1}(U) \cap G_0 A = X^F(\mathcal{L}^{-1}(U) \cap Y')$.

Further, $\mathcal{L}^{-1}(U) \cap Y$ is the variety underlying $R_{N^F}^A$ (with respect to the parabolic subgroup $P Z(G)$). Hence, for any $\tau_1 \in \text{Irr}(X^F)$, $\tau_2 \in \text{Irr}(Y^F)$ such that $I$ is in the kernel of $\tau_1 \tau_2$, we have

$$R_{G_0 A}^{G^F}(\tau_1 \tau_2) = \tau_1 R_{N^F}^Y(\tau_2).$$

Further, $\tau_1 \tau_2 \in \mathcal{E}(X^F N^F, \ell')$ if and only if $\tau_1 \in \mathcal{E}(X^F, \ell')$ and $\tau_2 \in \mathcal{E}(\mathcal{N}, \ell')$.

To conclude note that $\ell'$ is the unique block of $N^F$ covered by $c_0$ and $c_0 = d \ell'$, where $d$ is a block $X^F$. Let $b'$ be the block of $Y^F$ in the hypothesis. Then, setting $b_0 = db'$ gives the desired result.

We will also make use of the following well-known extension of [12, Prop. 1.5].

**Lemma 3.3.** Suppose that $q$ is odd. Let $G$ be connected reductive with a Frobenius endomorphism $F$. Suppose that all components of $G$ are of classical type $A$, $B$, $C$ or $D$ and that $Z(G)/Z^o(G)$ is a 2-group. Let $s \in G^{* F}$ be semisimple of odd order. Then all elements of $\mathcal{E}(G^F, s)$ lie in the same 2-block of $G^F$.

**Proof.** Since $s$ has odd order and $Z(G)/Z^o(G)$ is a 2-group, $C_G^A(s)$ is connected. On the other hand, since all components of $G^*$ are of classical type and $s$ has odd order, $C_G^A(s)$ is a Levi subgroup of $G$. Thus, $C_G^A(s)$ is a Levi subgroup of $G^*$ and by Bonnafé–Rouquier the set of 2-blocks of $G^F$ which contain a character of $\mathcal{E}(G^F, s)$ is in bijection with the set of unipotent 2-blocks of $C^F$, where $C$ is a Levi subgroup of $G$ in duality with $C_G^A(s)$. Since all components of $C$ are of classical type, the claim follows by [12, Prop. 1.5(a)].

We now have the following extension of [11, Thm. 2.5] to all primes.

**Theorem 3.4.** Let $H$ be a simple algebraic group of simply connected type with a Frobenius endomorphism $F : H \to H$ endowing $H$ with an $\mathbb{F}_q$-rational structure. Let $G$ be an $F$-stable Levi subgroup of $H$. Let $\ell$ be a prime not dividing $q$ and set $e = e_{\ell}(q)$. Let $M$ be an $e$-split Levi subgroup of $G$ and let $c$ be a block of $M^F$. Then there exists a block $b$ of $G^F$ such that every irreducible constituent of $R_{M}^{G}(\mu)$ where $\mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell')$ lies in $b$. 

\[ \text{Proof.} \]
Proof. Suppose that \( \dim(G) \) is minimal such that the claim of the Theorem does not hold. Let \( s \in M^F \) be a semisimple \( \ell' \)-element such that \( \text{Irr}(c) \cap \mathcal{E}(M^F, \ell') \subseteq \mathcal{E}(M^F, s) \). Then all irreducible constituents of \( R^G_M(\mu) \) where \( \mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell') \) are in \( \mathcal{E}(G^F, s) \).

First suppose that \( s \) is not quasi-isolated and let \( G_1 \) be a proper \( F \)-stable Levi subgroup of \( G \) whose dual contains \( C_{G^*}(s) \). Let \( M^* \) be a Levi subgroup of \( G^* \) in duality with \( M \) and set \( M_1 = C_{G_1^*}(Z^\circ(M^*)_c) \). Then, as in the proof of Proposition 2.31, \( M_1^* \) is an \( \epsilon \)-split Levi subgroup of \( G_1^* \) and letting \( M_1 \) be the dual of \( M_1^* \) in \( G \), \( M_1 \) is an \( \epsilon \)-split Levi subgroup of \( G_1 \). Further, \( M_1 \geq C_M(\epsilon) \). Hence there exists a unique block say \( c_1 \) of \( M_1^F \) such that \( \text{Irr}(c_1) \cap \mathcal{E}(M_1^F, \ell') \subseteq \mathcal{E}(M_1^F, s) \) and such that \( c_1 \) and \( c \) are Bonnafé–Rouquier correspondents.

By induction our claim holds for \( G_1 \) and the block \( c_1 \) of \( M_1 \). Let \( b_1 \) be the block of \( G_1^F \) such that every irreducible constituent of \( R^G_{M_1}(\mu) \) where \( \mu \in \text{Irr}(c_1) \cap \mathcal{E}(M_1^F, \ell') \) lies in \( b_1 \) and let \( b \) be the Bonnafé–Rouquier correspondent of \( b_1 \) in \( G^F \).

Now let \( \mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, s) \) and let \( \chi \) be an irreducible constituent of \( R^G_M(\mu) \). Let \( \mu_1 \) be the unique character in \( \text{Irr}(M_1^F, s) \) such that \( \mu = \pm R^G_{M_1}(\mu_1) \). Then, \( \mu_1 \in \text{Irr}(c_1) \) and

\[
R^G_M(\mu) = R^G_{M_1}(R^M_{M_1}(\mu_1)) = R^G_{G_1}(R^G_{M_1}(\mu_1)).
\]

All irreducible constituents of \( R^G_{M_1}(\mu_1) \) lie in \( b_1 \). Hence, by the above equation and by the Bonnafé–Rouquier theorem, \( \chi \) lies in \( b \), a contradiction.

So, we may assume from now on that \( s \) is quasi-isolated in \( G^* \). By [9, Thm. 2.5], we may assume that \( \ell \) is bad for \( G \) and hence for \( H \). So \( H \) is not of type \( A \). If \( H \) is of type \( B, C \) or \( D \), then \( \ell = 2 \) and we have a contradiction by Lemma 3.3.

Thus \( H \) is of exceptional type. Suppose that \( s = 1 \). By [5, Thm. 3.2] \( G^F \) satisfies an \( \epsilon \)-Harish-Chandra theory above each unipotent \( \epsilon \)-cuspidal pair \( (L, \lambda) \) and by [11] Thms. A.A.bis], all irreducible constituents of \( R^G_L(\lambda) \) lie in the same \( \ell \)-block of \( G^F \).

So we may assume that \( s \neq 1 \). We consider the case that \( G = H \). Then by [13, Thm. 1.4], \( G^F \) satisfies an \( \epsilon \)-Harish-Chandra theory above each \( \epsilon \)-cuspidal pair \( (L, \lambda) \) below \( \mathcal{E}(G^F, s) \) and by [13, Thm. 1.2], all irreducible constituents of \( R^G_L(\lambda) \) lie in the same \( \ell \)-block of \( G^F \).

So, we may assume that \( G \) is proper in \( H \). If \( H \) is of type \( G_2, F_4 \) or \( E_6 \), then \( \ell = 2 \), all components of \( G \) are of classical type. For \( G_2 \) and \( F_4 \) we have that \( Z(H) \) and therefore \( Z(G) \) is connected. If \( H \) is of type \( E_6 \), since 2 is bad for \( G \), \( G \) has a component of type \( D_n \), \( n \geq 4 \). By rank considerations, \([G, G]\) is of type \( D_4 \) or \( D_5 \). Since \( |Z(H)/Z^\circ(H)| = 3 \) it follows again that \( Z(G) \) is connected. In either case we get a contradiction by Lemma 3.3.

So, \( H \) is of type \( E_7 \) or \( E_8 \). Since \( G \) is proper in \( H \), 5 is good for \( G \), hence \( \ell = 3 \) or 2. Also, we may assume that at least one of the two assumptions of Lemma 3.3 fails to hold for \( G, M \) and \( c \).

Suppose that \( \ell = 3 \). Since \( G \) is proper in \( H \) and 3 is bad for \( G \), either \([G, G]\) is of type \( E_6 \), or \( H \) is of type \( E_8 \) and \([G, G]\) is of type \( E_6 + A_1 \) or of type \( E_7 \). In all cases, \( Z(G) \) is connected (note that if \( H \) is of type \( E_7 \), then \([G, G]\) is of type \( E_6 \), whence the order of \( Z(G)/Z^\circ(G) \) divides both 2 and 3). If \( G = M \), there is nothing to prove, so we may assume that \( M \) is proper in \( G \). Let \( C := C_G(Z(M)^F_1) \geq M \).

We claim that there is a block, say \( d \) of \( C^F \) such that for all \( \mu \in \text{Irr}(c) \cap \mathcal{E}(M^F, \ell') \), every irreducible constituent of \( R^C_M(\mu) \) lies in \( d \). Indeed, since \( M \) is proper in \( G \) and since \( Z(G) \) is connected, by [7, Prop. 2.1] \( C \) is proper in \( G \). Also, by direct calculation either
C is a Levi subgroup of G or 3 is good for C. In the first case, the claim follows by the inductive hypothesis since M is also e-split in C. In the second case, we are done by [9, Thm. 2.5].

Thus, we may assume that assumption (1) of Lemma 3.1 does not hold. Hence, by [9, Thm. 1.7], 3 is bad for M. Consequently, M has a component of non-classical type. Since M is proper in G, this means that [G, G] is of type $E_6 + A_1$ or of type $E_7$ and [M, M] is of type $E_6$. Suppose that [G, G] is of type $E_6 + A_1$. Since [M, M] is of type $E_6$, and since 3 is good for groups of type A, the result follows from Lemma 3.2 applied with X being the component of G of type $E_6$, and [9, Thm. 2.5].

So we have [G, G] of type $E_7$ and [M, M] of type $E_6$. Suppose that $s$ is not quasi-isolated in M'. Then $c$ is in Bonnafé–Rouquier correspondence with a block, say $c'$ of a proper F-stable Levi subgroup, say $M'$ of M. The prime 3 is good for any proper Levi subgroup of M, hence by [9, Thm. 1.7] condition (1) of Lemma 3.1 holds for the group $M'$ and the block $c'$. By Bonnafé–Rouquier, this condition also holds for M and $c$, a contradiction. So, $s$ is quasi-isolated in $M^*$. Since as pointed out above, G has connected center, so does M whence $s$ is isolated in $M^*$. Also, note that since $s$ is also quasi-isolated in $G^*$, by the same reasoning $s$ is isolated in $E^*$. Inspection shows that the only possible case for this is when $s$ has order three with $C_{G^*}(s)$ of type $A_5 + A_2$, $C_{M^*}(s)$ of type $3A_2$.

Since $s$ is supposed to be a 3'-element, this case does not arise here.

Now suppose that $\ell = 2$. Since $Z(H)/Z^0(H)$ has order dividing 2, by Lemma 3.3 we may assume that G has at least one non-classical component, that is we are in one of the cases $[G, G] = E_6$, or $H = E_8$ and $[G, G] = E_6 + A_1$ or $E_7$. Again, in all cases, $Z(G)$ is connected and consequently $C_{G^*}(s)$ is connected and $s$ is isolated.

Suppose first that $[G, G] = E_7$. We claim that all elements of $E(G^F, s)$ lie in the same 2-block. Indeed, let $\bar{s}$ be the image of $s$ under the surjective map $G^* \to [G, G]^*$ induced by the regular embedding of $[G, G]$ in G. By [13, Table 4], all elements of $E([G, G]^F, \bar{s})$ lie in the same 2-block, say $d$ of $[G, G]^F$. So, any block of $G^F$ which contains a character in $E(G^F, s)$ covers $d$. By general block theoretical reasons, there are at most $|G^F/[G, G]^F|_{2'}$ 2-blocks of $G^F$ covering a given $d$. Now since $s$ is a 2'-element, $C_{[G, G]^*}(\bar{s})$ is connected. Thus, if $\mu \in E([G, G]^F, \bar{s})$, then there are $|G^F/[G, G]^F|_{2'}$ different 2'-Lusztig series of $G^F$ containing an irreducible character covering $\mu$. Since characters in different 2'-Lusztig series lie in different 2-blocks, the claim follows.

By the claim above, we may assume that either $[G, G] = E_6$ or $[G, G] = E_6 + A_1$. Since $s$ is isolated of odd order in $G^*$, by [13, Table 1] all components of $C_{G^*}(s)$ are of type $A_2$ or $A_1$. Consequently, all components of $C_{M^*}(s)$ are of type A. Suppose first that M has a non-classical component. Then $[M, M]$ is of type $E_6$, and $[G, G] = E_6 + A_1$. This may be ruled out by Lemma 3.2 applied with $X$ equal to the product of the component of type $E_6$ with $Z(G)$ and $Y$ equal to the component of type $A_1$.

So finally suppose that all components of M are of classical type. Then, $C_{M^*}(s) = C_{M^*}(s)$ is a Levi subgroup of M with all components of type A. Hence, the first hypothesis of Lemma 3.1 holds by the Bonnafé–Rouquier theorem and [9, Thm. 1.7]. So, we may assume that the second hypothesis of Lemma 3.1 does not hold. Let $C := C_G(Z(M^F)_2)$. Since M is a proper e-split Levi subgroup of G, and since $Z(G)$ is connected, by [7, Prop. 2.1] C is proper in G. By induction, we may assume that C is not a Levi subgroup of G. In particular, the intersection of C with the component of type $E_6$ of G is proper.
in that component and hence all components of \( C \) are of type \( A \) or \( D \). If all components of \( C \) are of type \( A \), then 2 is good for \( C \) and the second hypothesis of Lemma \( \text{(3.5)} \) holds by \( [9, \text{Thm. 2.5}] \). Thus we may assume that \( C \) has a component of type \( D \). Since all components of \( C \) are classical, by Lemma \( \text{(3.5)} \) we may assume that \( Z(C)/Z^0(C) \) is not a 2-group and consequently \( C \) has a component of type \( A_n \), with \( n \equiv 2 \pmod{3} \). But by the Borel–de Siebenthal algorithm, a group of type \( E_6 \) has no subsystem subgroup of type \( D_m + A_n \) with \( n \geq 1 \) and \( m \geq 4 \). \( \square \)

### 3.2. Characters in \( \ell \)-blocks

Using the results collected so far, it is now easy to characterise all characters in \( \ell \)-series inside a given \( \ell \)-block in terms of Lusztig induction.

**Definition 3.5.** As in \( [9, \text{1.11}] \) (see also \( [5, \text{Def. 3.1}] \)) for \( e \)-split Levi subgroups \( M_1, M_2 \) of \( G \) and \( \mu_i \in \text{Irr}(M_i^F) \) we write \( (M_1, \mu_1) \leq_e (M_2, \mu_2) \) if \( M_1 \leq M_2 \) and \( \mu_2 \) is a constituent of \( R_{M_2}^{M_1}(\mu_1) \) (with respect to some parabolic subgroup of \( M_2 \) with Levi subgroup \( M_1 \)). We let \( \ll_e \) denote the transitive closure of the relation \( \leq_e \).

As pointed out in \( [9, \text{1.11}] \) it seems reasonable to expect that the relations \( \leq_e \) and \( \ll_e \) coincide. While this is known to hold for unipotent characters (see \( [5, \text{Thm. 3.11}] \)), it is open in general.

We put ourselves in the situation and notation of Theorem A.

**Theorem 3.6.** Let \( b \) be an \( \ell \)-block of \( G^F \) and denote by \( \overline{\mathcal{L}}(b) \) the set of \( e \)-Jordan-cuspidal pairs \( (L, \lambda) \) of \( G \) such that \( \text{Irr}(b) \cap R_\ell^C(\lambda) \neq \emptyset \). Then

\[
\text{Irr}(b) \cap \mathcal{E}(G^F, \ell') = \{ \chi \in \mathcal{E}(G^F, \ell') \mid \exists (L, \lambda) \in \overline{\mathcal{L}}(b) \text{ with } (L, \lambda) \ll_e (G, \chi) \}.
\]

**Proof.** Let \( b \) be as in the statement and first assume that \( \chi \in \text{Irr}(b) \cap \mathcal{E}(G^F, \ell') \). If \( \chi \) is not \( e \)-Jordan-cuspidal, then it is not \( e \)-cuspidal, so there exists a proper \( e \)-split Levi subgroup \( M_1 \) such that \( \chi \) occurs in \( R_{M_1}^G(\mu_1) \) for some \( \mu_1 \in \mathcal{E}(M_1^F, \ell') \). Thus inductively we obtain a chain of \( e \)-split Levi subgroups \( M_r \leq \ldots \leq M_1 \leq M_0 := G \) and characters \( \mu_i \in \mathcal{E}(M_i^F, \ell') \) (with \( \mu_0 := \chi \)) such that \( (M_r, \mu_r) \) is \( e \)-Jordan cuspidal and such that \( (M_i, \mu_i) \leq_e (M_{i-1}, \mu_{i-1}) \) for \( i = 1, \ldots, r \), whence \( (M_r, \mu_r) \ll_e (G, \chi) \).

Let \( b_r \) be the \( \ell \)-block of \( M_r^F \) containing \( \mu_r \). Now Theorem \( \text{(3.3)} \) yields that for each \( i \) there exists a block, say \( b_i \), of \( M_i^F \) such that all constituents of \( R_{M_i}^{M_{i-1}}(\zeta_i) \) lie in \( b_{i-1} \) for all \( \zeta_i \in \text{Irr}(b_i) \cap \mathcal{E}(M_i^F, \ell') \). In particular, \( \chi \) lies in \( b_0 \), so \( b_0 = b \), and thus \( (M_r, \mu_r) \in \overline{\mathcal{L}}(b) \).

For the reverse inclusion, let \( (L, \lambda) \in \overline{\mathcal{L}}(b) \) and \( \chi \in \text{Irr}(G^F, \ell') \) such that \( (L, \lambda) \ll_e (G, \chi) \). Thus there exists a chain of \( e \)-split Levi subgroups \( L = M_r \leq \ldots \leq M_0 = G \) and characters \( \mu_i \in \text{Irr}(M_i^F) \) with \( (M_i, \mu_i) \leq_e (M_{i-1}, \mu_{i-1}) \). Again, application of Theorem \( \text{(3.3)} \) allows to conclude that \( \chi \in \text{Irr}(b) \). \( \square \)

### 3.3. \( \ell \)-blocks and derived subgroups

In the following two results, which will be used in showing that the map \( \Xi \) in Theorem A is surjective, \( G \) is connected reductive with Frobenius endomorphism \( F \), and \( G_0 := [G, G] \). Here, in the cases that the Mackey formula is not known to hold we assume that \( R_{L_0}^G \) and \( R_r^G \) are with respect to a choice of parabolic subgroups \( P_0 \geq L_0 \) and \( P \geq L \) such that \( P_0 = G_0 \cap P \).

**Lemma 3.7.** Let \( b \) be an \( \ell \)-block of \( G^F \) and let \( b_0 \) be an \( \ell \)-block of \( G_0^F \) covered by \( b \). Let \( L \) be an \( F \)-stable Levi subgroup of \( G \), \( L_0 = L \cap G_0 \) and let \( \lambda_0 \in \text{Irr}(L_0^F) \).

Suppose that every irreducible constituent of \( R_{L_0}^{G_0}(\lambda_0) \) is contained in \( b_0 \). Then there exists \( \lambda \in \text{Irr}(L^F) \) and...
\(\chi \in \text{Irr}(b)\) such that \(\lambda_0\) is an irreducible constituent of \(\text{Res}^{L_0}_{L_0^f}(\lambda)\) and \(\chi\) is an irreducible constituent of \(R_{L_0}^{G_f}(\lambda)\).

**Proof.** Since \(G = Z^e(G)G_0\), by [2] Prop. 10.10] we have that
\[
R_{L_0}^{G_f}\text{Ind}^{L_f}_{L_0}(\lambda_0) = \text{Ind}^{G_{G_0}}_{G_0}\text{Ind}^{G_f}_{L_0}(\lambda_0).
\]

Note that the result in [2] is only stated for the case that \(G\) has connected centre but the proof does not use this hypothesis. The right hand side of the above equality evaluated at 1 is non-zero. Let \(\chi' \in \text{Irr}(G_f)\) be a constituent of the left hand side of the equality. There exists \(\lambda \in \text{Irr}(L_f)\) and \(\chi_0\) in \(\text{Irr}(G_0_f)\) such that \(\lambda\) is an irreducible constituent of \(\text{Ind}^{L_f}_{L_0}(\lambda_0)\), \(\chi'\) is an irreducible constituent of \(R_{L_0}^{G_f}(\lambda)\), \(\chi_0\) is an irreducible constituent of \(R_{L_0}^{G_f}(\lambda_0)\) and \(\chi'\) is an irreducible constituent of \(\text{Ind}^{G_{G_0}}_{G_0}\text{Ind}^{G_f}_{L_0}(\lambda_0)\). Since \(\lambda_0 \in \text{Irr}(b_0)\), \(\chi'\) lies in a block, say \(b'\), of \(G_f\) which covers \(b_0\). Since \(b\) also covers \(b_0\) and since \(G_f/G_0_f\) is abelian, there exists a linear character, say \(\theta\) of \(G_f/G_0_f\) such that \(b = b' \otimes \theta\) (see [13] Lemma 2.2]).

Now the result follows from [2] Prop. 10.11] with \(\chi = \chi' \otimes \theta\).

**Lemma 3.8.** Let \(b\) be an \(\ell\)-block of \(G_f\) and let \(L\) be an \(F\)-stable Levi subgroup of \(G\) and \(\lambda \in \text{Irr}(L_f)\) such that every irreducible constituent of \(R_{L_0}^{G_f}(\lambda)\) is contained in \(b\). Let \(L_0 = L \cap G_0\) and let \(\lambda_0 \in \text{Irr}(L_0_f)\) be an irreducible constituent of \(\text{Res}^{L_0}_{L_0^f}(\lambda)\). Then there exists an \(\ell\)-block \(b_0\) of \(G_0_f\) covered by \(b\) and an irreducible character \(\chi_0\) of \(G_0_f\) in the block \(b_0\) such that \(\chi_0\) is a constituent of \(R_{L_0}^{G_f}(\lambda_0)\).

**Proof.** Arguing as in the proof of Lemma 3.7 there exists \(\chi \in \text{Irr}(G_f)\), \(\lambda' \in \text{Irr}(L_f)\) and \(\chi_0\) in \(\text{Irr}([G, G]^F)\) such that \(\lambda'\) is an irreducible constituent of \(\text{Ind}^{L_f}_{L_0}(\lambda_0)\), \(\chi\) is an irreducible constituent of \(R_{L_0}^{G_f}(\lambda')\), \(\chi_0\) is an irreducible constituent of \(R_{L_0}^{G_{G_0}}(\lambda_0)\) and \(\chi\) is an irreducible constituent of \(\text{Ind}^{G_{G_0}}_{G_0}\text{Ind}^{G_f}_{L_0}(\lambda_0)\). Now, \(\lambda = \theta \otimes \lambda'\) for some linear character \(\theta\) of \(L_f/L_0^f\). By [2] Prop. 10.11], \(\theta \otimes \chi\) is an irreducible constituent of \(R_{L_0}^{G_f}(\lambda)\), and therefore \(\theta \otimes \chi \in \text{Irr}(b)\). Further, \(\theta \otimes \chi\) is also a constituent of \(\text{Ind}^{G_f}_{G_0}\text{Ind}^{G_{G_0}}_{G_0}(\chi_0)\), hence \(b\) covers the block of \([G, G]^F\) containing \(\chi_0\).

**3.4. Unique maximal abelian normal subgroups.** A crucial ingredient for proving injectivity of the map in parts (d) and (e) of Theorem A is a property related to the non-failure of factorisation phenomenon of finite group theory, which holds for the defect groups of many blocks of finite groups of Lie type and which was highlighted by Cabanes [6]: For a prime \(\ell\) an \(\ell\)-group is said to be Cabanes if it has a unique maximal abelian normal subgroup.

Now first consider the following setting: Let \(G\) be connected reductive. For \(i = 1, 2\), let \(L_i\) be an \(F\)-stable Levi subgroup of \(G\) with \(\lambda_i \in \mathcal{E}(L_i^f, \ell')\), and let \(u_i\) denote the \(\ell\)-block of \(L_i^f\) containing \(\lambda_i\). Suppose that \(C_G(Z(L_i^f)_{\ell'}) = L_i\) and that \(\lambda_i\) is of quasi-central \(\ell\)-defect. Then by [13] Props. 2.12, 2.13, 2.16] there exists a block \(b_i\) of \(G_f\) such that all irreducible characters of \(R_{L_i}^{G_f}(\lambda_i)\) lie in \(b_i\) and \((Z(L_i), u_i)\) is a \(b_i\)-Brauer pair.

**Lemma 3.9.** In the above situation, assume further that for \(i = 1, 2\) there exists a maximal \(b_i\)-Brauer pair \((P_i, c_i)\) such that \((Z(L_i^f)_{\ell'}, u_i) \leq (P_i, c_i)\) and such that \(P_i\) is Cabanes. If \(b_1 = b_2\) then the pairs \((L_1, \lambda_1)\) and \((L_2, \lambda_2)\) are \(G_f\)-conjugate.
Proof. Suppose that $b_1 = b_2$. Since maximal $b_1$-Brauer pairs are $G^F$-conjugate it follows that $^g(Z(L_2^F), u_2) \leq ^g(P_2, c_2) = (P_1, c_1)$ for some $g \in G^F$. By transport of structure, $^gZ(L_2^F)$ is a maximal normal abelian subgroup of $P_1$, hence $^gZ(L_2^F) = Z(L_2^F)$. By the uniqueness of inclusion of Brauer pairs it follows that $^g(Z(L_2^F), u_2) = (Z(L_1^F), u_1)$. Since $L_i = C_G(Z(L_i^F) \leq M)$ this means that $^gL_2 = L_1$. Further, since $\lambda_i$ is of quasi central $\ell$-defect, by [14] Prop. 2.5(1), $\lambda_i$ is the unique element of $\mathcal{E}(L_1^F, \ell') \cap \text{Irr}(u_i)$. Thus $^g u_2 = u_1$ implies that $^g \lambda_2 = \lambda_1$ and $(L_1, \lambda_1)$ and $(L_2, \lambda_2)$ are $G^F$-conjugate as required.

By the proof of Theorems 4.1 and 4.2 of [9] we also have:

**Proposition 3.10.** Let $G$ be connected reductive with simply connected derived subgroup. Suppose that $\ell \geq 3$ is good for $G$, and $\ell \neq 3$ if $G^F$ has a factor $3D_4(q)$. Let $b$ be an $\ell$-block of $G^F$ such that the defect groups of $b$ are Cabanes. If $(L, \lambda)$ and $(L', \lambda')$ are $e$-Jordan-cuspidal pairs of $G$ such that $\lambda \in \mathcal{E}(L^F, \ell')$, $\lambda' \in \mathcal{E}(L'^F, \ell')$ with $b_{G^F}(L, \lambda) = b = b_{G^F}(L', \lambda')$, then $(L, \lambda)$ and $(L', \lambda')$ are $G^F$-conjugate.

Proof. This is essentially contained in Section 4 of [9]. Indeed, let $(L, \lambda)$ be an $e$-Jordan-cuspidal pair of $G$ such that $\lambda \in \mathcal{E}(L^F, \ell')$. Let $T^*, T, K = C_G(Z(L^F)), K^*, M$ and $M^*$ be as in the notation before Lemma 4.4 of [9]. Let $Z = Z(M)^\ell$ and let $\lambda_K$ and $\lambda_M$ be as in Definition 4.6 of [9], with $\lambda$ replacing $\zeta$. Then $Z \leq T$ and by Lemma 4.8, $M = C_G^o(Z)$. The simply connected hypothesis and the restrictions on $\ell$ imply that $C_G(Z) = C_G^o(Z) = M$. Let $b_2 = b_Z$ be the $\ell$-block of $M^F$ containing $M_\lambda$. Then by Lemma 4.13, $(Z, b_Z)$ is a self centralising Brauer pair and $(1, b_{G^F}(L, \lambda)) \leq (Z, b_Z)$. Further, by Lemma 4.16 of [9] there exists a maximal $b$-Brauer pair $(D, b_D)$ such that $(Z, b_Z) \leq (D, b_D)$. Then $Z$ is normal in $D$ and $C_D(Z) = Z$. Note that the first three conclusions of Lemma 4.16 of [9] hold under the conditions we have on $\ell$ (it is only the fourth conclusion which requires $\ell \in \Gamma(G, F)$). By Lemma 4.10 and its proof, we also have

$$(1, b_{G^F}(L, \lambda)) \leq (Z(L)^F, b_{K^F}(L, \lambda)) \leq (Z, b_Z).$$

Suppose that $N$ is a proper $e$-split Levi subgroup of $G$ containing $C_G^o(z) = C_G(z)$ for some $1 \neq z \in Z(D)G_a \cap G_b$. Then $N$ contains $L, M$ and $Z$ by Lemma 4.15(b). Since $L \cap G_b = K \cap G_b$, by Lemma 4.4(iii), it follows that $N$ also contains $K$ and $K = C_N(Z(L^F))$. Thus, replacing $G$ with $N$ in Lemma 4.13 we get that

$$(1, b_{N^F}(L, \lambda)) \leq (Z(L)^F, b_{K^F}(L, \lambda)) \leq (D, b_D).$$

Let $(L', \lambda')$ be another $e$-Jordan-cuspidal pair of $G$ with $\lambda' \in \mathcal{E}(L'^F, \ell')$ such that $b_{G^F}(L, \lambda) = b = b_{G^F}(L', \lambda')$. Denote by $K', M', D'$ etc. the corresponding groups and characters for $(L', \lambda')$. Up to replacing by a $G^F$-conjugate, we may assume that $(D', b_{D'}) = (D, b_D)$.

Suppose first that there is a $1 \neq z \in Z(D)G_a \cap G_b$. By Lemma 4.15(b), there is a proper $e$-split Levi subgroup $N$ containing $C_G(z)$. Moreover, $N$ contains $D, L', M', K'$ and $G_a$ and we also have

$$(1, b_{N^F}(L', \lambda')) \leq (Z(L')^F, b_{K^F}(L', \lambda')) \leq (D, b_D).$$

By the uniqueness of inclusion of Brauer pairs it follows that $b_{N^F}(L, \lambda) = b_{N^F}(L', \lambda')$. Also $D$ is a defect group of $b_{N^F}(L, \lambda)$. Thus, in this case we are done by induction.
So, we may assume that $Z(D) \leq G_a$ hence $D \leq G_a$. From here on, the proof of Lemma 4.17 of [9] goes through without change, the only property that is used being that $Z$ is the unique maximal abelian normal subgroup of $D$. \hfill \Box

We will also need the following observation:

**Lemma 3.11.** Let $P = P_1 \times P_2$ where $P_1$ and $P_2$ are Cabanes. Suppose that $P_0$ is a normal subgroup of $P$ such that $\pi_i(P_0) = P_i$, $i = 1, 2$, where $\pi_i : P_1 \times P_2 \to P_i$ denote the projection maps. Then $P_0$ is Cabanes with maximal normal abelian subgroup $(A_1 \times A_2) \cap P_0$, where $A_i$ is the unique maximal normal abelian subgroup of $P_i$, $i = 1, 2$.

**Proof.** Let $A = A_1 \times A_2$. The group $A \cap P_0$ is abelian and normal in $P_0$. Let $S$ be a normal abelian subgroup of $P_0$. Since $\pi_i(P_0) = P_i$, $\pi_i(S)$ is normal in $P_i$ and since $S$ is abelian, so is $\pi_i(S)$. Thus, $\pi_i(S)$ is a normal abelian subgroup of $P_i$ and is therefore contained in $A_i$. So, $S \leq (\pi_1(S) \times \pi_2(S)) \cap P_0 \leq (A_1 \times A_2) \cap P_0 = A \cap P_0$ and the result is proved. \hfill \Box

### 3.5. Linear and unitary groups at $\ell = 3$

Now assume we are in the exceptional case. Clearly $|P| = 27$. Let $P_1, P_2 \leq P$ be subgroups of order 9, and let $u_i \in P_i$ be non-central. Then $u_i$ is $G$-conjugate to $\text{diag}(1, \zeta, \zeta^2)$, where $\zeta$ is a primitive 3rd-root of unity in $\mathbb{F}_q$ (respectively $\mathbb{F}_{q^2}$). In particular, there exists $g \in G$ such that $g u_1 = u_2$. Let $\bar{g} : G \to G/Z(G)$ denote the canonical map. Then $\bar{g}(\bar{u}_1) = \bar{u}_2$. Since the Sylow 3-subgroup $P$ of $G$ is abelian, there exists $h \in N_G(P)$ with $h(\bar{u}_1) = \bar{u}_2$. Then $h \in N_G(P)$ and $h P_1 = P_2$ as $P_i = \langle Z(G), u_i \rangle$.

**Lemma 3.12.** Let $q$ be a prime power such that $3|(q-1)$ (respectively $3|(q+1)$). Let $G = \text{SL}_n(q)$ (respectively $\text{SU}_n(q)$) and let $P$ be a Sylow 3-subgroup of $G$. Then $P$ is Cabanes unless $n = 3$ and $3||(q-1)$ (respectively $3||(q+1)$). In particular, if $P$ is not Cabanes, then $P$ is extra-special of order 27 and exponent 3. In this case $N_G(P)$ acts transitively on the set of subgroups of order 9 of $P$.

**Proof.** Embed $P \leq \text{SL}_n(q) \leq \text{GL}_n(q)$. A Sylow 3-subgroup of $\text{GL}_n(q)$ is contained in the normaliser $C_{q-1} \triangleleft S_n$ of a maximally split torus. According to [6, Lemme 4.1], the only case in which $S_n$ has a quadratic element on $(C_{q-1})_3 \cap \text{SL}_n(q)$ is when $n = 3$ and $3||(q-1)$. If there is no quadratic element in this action, then $P$ is Cabanes by [6, Prop. 2.3]. In the case of $\text{SU}_n(q)$, the same argument applies with the normaliser $C_{q+1} \triangleleft S_n$ of a Sylow 2-torus inside $\text{GU}_n(q)$.

Now assume we are in the exceptional case. Clearly $|P| = 27$. Let $P_1, P_2 \leq P$ be subgroups of order 9, and let $u_i \in P_i$ be non-central. Then $u_i$ is $G$-conjugate to $\text{diag}(1, \zeta, \zeta^2)$, where $\zeta$ is a primitive 3rd-root of unity in $\mathbb{F}_q$ (respectively $\mathbb{F}_{q^2}$). In particular, there exists $g \in G$ such that $g u_1 = u_2$. Let $\bar{g} : G \to G/Z(G)$ denote the canonical map. Then $\bar{g}(\bar{u}_1) = \bar{u}_2$. Since the Sylow 3-subgroup $P$ of $G$ is abelian, there exists $h \in N_G(P)$ with $h(\bar{u}_1) = \bar{u}_2$. Then $h \in N_G(P)$ and $h P_1 = P_2$ as $P_i = \langle Z(G), u_i \rangle$.

**Lemma 3.13.** Suppose that $3||n$ and $3||(q-1)$ (respectively $3||(q+1)$). Let $\mathbf{G} = \text{GL}_n$, $\mathbf{G} = \text{SL}_n$ and suppose that $\mathbf{G}^F = \text{GL}_n(q)$ (respectively $\text{GU}_n(q)$). Let $s$ be a semisimple $3'$-element of $\mathbf{G}^F$ such that a Sylow 3-subgroup $D$ of $\text{CG}_F(s)$ is extra-special of order 27 and let $P_1, P_2 \leq D$ have order 9. There exists $g \in N_{\mathbf{G}^F}(D) \cap C_{\mathbf{G}^F}(\text{CG}_F(D))$ such that $g P_1 = P_2$.

**Proof.** Set $d = \frac{n}{3}$. Identify $\mathbf{G}$ with the group of linear transformations of an $n$-dimensional $\mathbb{F}_q$-vector space $V$ with chosen basis $\{e_{i,r} \mid 1 \leq i \leq d, 1 \leq r \leq 3\}$. For $g \in \mathbf{G}$, write $a(g)_{i,r,j,s}$ for the coefficient of $e_{i,r}$ in $g(e_{j,s})$. Let $w \in \mathbf{G}$ be defined by $w(e_{i,r}) = e_{i+1,r}$, $1 \leq i \leq d, 1 \leq r \leq 3$. For $1 \leq i \leq d$ let $V_i$ be the span of $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ and $\mathbf{G}_i = \text{GL}(V_i)$ considered as a subgroup of $\mathbf{G}$ through the direct sum decomposition $V = \bigoplus_{1 \leq i \leq d} V_i$. 
Up to conjugation in \( \hat{G} \) we may assume that \( F = \text{ad}_w \circ F_0 \), where \( F_0 \) is the standard Frobenius morphism which raises every matrix entry to its \( q \)-th power in the linear case, respectively the composition of the latter by the transpose inverse map in the unitary case. Note that then each \( \hat{G}_i \) is \( F_0 \)-stable.

Thus, given the hypothesis on the structure of \( D \), we may assume the following up to conjugation: \( s \) has \( d \) distinct eigenvalues \( \delta_1, \ldots, \delta_d \) with \( \delta_{r+1} = \delta_r^q \) (respectively \( \delta_i^q \)); \( V_i \) is the \( \delta_i \)-eigenspace of \( s \), and \( C_G(s) = \prod_{i=1}^d \hat{G}_i \). Further, \( F(\hat{G}_i) = \hat{G}_{i+1} \) and denoting by \( \Delta : \hat{G}_1 \to \prod_{i=1}^d \hat{G}_i \), \( x \mapsto x F(x) \cdots F^{d-1}(x) \), the twisted diagonal map we have \( C_{\hat{G}_i}(s) = \Delta(\hat{G}_1^{d^r}) \). Here, \( \hat{G}_1^{d^r} = \hat{G}_1^d \) is isomorphic to either \( \text{GL}_3(q^d) \) or \( \text{GU}_3(q^d) \). Note that \( \text{GU}_3(q^d) \) occurs only if \( d \) is odd.

Consider \( \hat{G}_1^{F_0} \leq \hat{G}_1^{d^r} \). Let \( U_1 \) be the Sylow 3-subgroup of the diagonal matrices in \( \hat{G}_1^{F_0} \) of determinant 1 and let \( \sigma_1 \in \hat{G}_1^{F_0} \) be defined by \( \sigma_1(e_{i,r}) = e_{i,r+1} \), \( 1 \leq r \leq 3 \). Then \( D_1 := \langle U_1, \sigma_1 \rangle \) is a Sylow 3-subgroup of \( \hat{G}_1^{F_0} \). Since by hypothesis the Sylow 3-subgroups of \( C_{\hat{G}_i}(s) \) have order 27, \( D := \Delta(D_1) \) is a Sylow 3-subgroup of \( C_{\hat{G}_i}(s) \) with \( \Delta(U_1) \cong U_1 \) elementary abelian of order 9. Note that \( \Delta(\sigma_1)(e_{i,r}) = e_{i,r+1} \) for \( 1 \leq i \leq d \) and \( 1 \leq r \leq 3 \).

Let \( \zeta \in \mathbb{F}_q \) be a primitive 3rd-root of unity. Let \( u_i \in U_1 \) be such that \( u_i(e_{i,r}) = \zeta^i e_{i,r} \), \( 1 \leq i \leq 3 \). For \( 1 \leq r \leq 3 \), let \( W_r \) be the span of \( \{ e_{1,r}, \ldots, e_{d,r} \} \). Then \( W_r \) is the \( \zeta^r \)-eigenspace of \( \Delta(u_1) \), whence

\[
C_{\hat{G}_i}(D) \leq C_{\hat{G}_i}(\Delta(U_1)) = C_{\hat{G}_i}(\Delta(u_1)) = \prod_{1 \leq r \leq 3} \text{GL}(W_r).
\]

Since \( \Delta(\sigma_1)(W_r) = W_{r+1} \), and \( \Delta(\sigma_1) \) acts on \( C_{\hat{G}_i}(\Delta(U_1)) \), it follows that \( C_{\hat{G}_i}(D) = \Delta'(\text{GL}(W_1)) \), where \( \Delta' : \text{GL}(W_1) \to \prod_{1 \leq r \leq 3} \text{GL}(W_r), x \mapsto x^s x^t x \), is the twisted diagonal.

We claim that \( \Delta(\hat{G}_1^{F_0}) \) centralises \( C_{\hat{G}_i}(D) \). Indeed, note that \( g \in \Delta(\hat{G}_1^{F_0}) \) if and only if \( a(g)_{i,r,j,s} = 0 \) if \( i \neq j \) and \( a(g)_{i,r,i,s} = a(F_0^{-1}(g))_{i,r,i,s} = a(g)_{i,r,i,s} \) for all \( i \) and all \( r, s \). Also, \( h \in C_{\hat{G}_i}(D) \) if and only if \( a(h)_{i,r,j,s} = 0 \) if \( r \neq s \) and \( a(h)_{i,r,j,s} = a(h)_{i,j,r,s} \) for all \( i, j \) and all \( r, s \). The claim follows from an easy matrix multiplication.

Let \( H = \langle \hat{G}_1^{F_0}, \hat{G}_1^{d^r} \rangle \) and note that \( D_1 \leq H \). By Lemma 3.12 applied to \( H \) any two subgroups of \( D_1 \) of order 9 are conjugate by an element of \( N_H(D_1) \). The lemma follows from the claim above.

3.6. Parametrising \( \ell \)-blocks. We can now prove our main Theorem A, which we restate. Recall the definition of e-Jordan (quasi-central) cuspidal pairs from the previous section.

**Theorem 3.14.** Let \( H \) be a simple algebraic group of simply connected type with a Frobenius endomorphism \( F : H \to H \) endowing \( H \) with an \( \mathbb{F}_q \)-rational structure. Let \( G \) be an \( F \)-stable Levi subgroup of \( H \). Let \( \ell \) be a prime not dividing \( q \) and set \( e = e_\ell(q) \).

(a) For any e-Jordan-cuspidal pair \( (L, \lambda) \) of \( G \) such that \( \lambda \in \mathcal{E}(L^F, \ell') \), there exists a unique \( \ell \)-block \( b_{G^F}(L, \lambda) \) of \( G^F \) such that all irreducible constituents of \( R_{L^F}^G(\lambda) \) lie in \( b_{G^F}(L, \lambda) \).

(b) The map \( \Xi : (L, \lambda) \mapsto b_{G^F}(L, \lambda) \) is a surjection from the set of \( G^F \)-conjugacy classes of e-Jordan-cuspidal pairs \( (L, \lambda) \) of \( G \) with \( \lambda \in \mathcal{E}(L^F, \ell') \) to the set of \( \ell \)-blocks of \( G^F \).

(c) The map \( \Xi \) restricts to a surjection from the set of \( G^F \)-conjugacy classes of e-Jordan quasi-central cuspidal pairs \( (L, \lambda) \) of \( G \) with \( \lambda \in \mathcal{E}(L^F, \ell') \) to the set of \( \ell \)-blocks of \( G^F \).
For \( \ell \geq 3 \) the map \( \Xi \) restricts to a bijection between the set of \( G^F \)-conjugacy classes
of \( e \)-Jordan quasi-central cuspidal pairs \((L, \lambda)\) of \( G \) with \( \lambda \in \mathcal{E}(L^F, \ell') \) and the set of
\( \ell \)-blocks of \( G^F \).

The map \( \Xi \) itself is bijective if \( \ell \geq 3 \) is good for \( G \), and \( \ell \neq 3 \) if \( G^F \) has a factor
\( 3D_4(q) \).

Remark 3.15. Note that (e) is best possible. See [11], [13] for counter-examples to the conclusion for bad primes, and [11] p. 348 for a counter-example in the case \( \ell = 3 \) and
\( G^F = 3D_4(q) \). Counter-examples in the case \( \ell = 2 \) and \( G \) of type \( A_n \) occur in the following
situation. Let \( G^F = \text{SL}_n(q) \) with 4\((q + 1) \). Then \( \ell = 2 \) and the unipotent 2-(Jordan-)
cuspidal pairs of \( G^F \) correspond to 2-cores of partitions of \( n - 1 \) (see [3, §3A]). On the
other hand, by [2, Thm. 13], \( G^F \) has a unique unipotent 2-block.

Also, part (d) is best possible as the next example shows.

Example 3.16. Consider \( G = \text{SL}_n \) with \( n > 1 \) odd, \( \tilde{G} = \text{GL}_n \), and let \( G^F = \text{SL}_n(q) \)
be such that \( q \equiv 1 \pmod{4} \) and \( 4 | (q + 1) \). Then for \( \ell = 2 \) we have \( e = e_{2q} = 2 \), and
\( \mathbb{F}_q \) contains a primitive \( n \)-th root of unity, say \( \zeta \). Let \( s = \text{diag}(1, \zeta, \ldots, \zeta^{n-1}) \in G^F \) and
let \( s \) be its image in \( G^* = \text{PGL}_n \). Then \( C_{G^*}(s) \) is the maximal 1-torus consisting of the
image of the diagonal torus of \( \tilde{G}^* \). Thus, \( (C_{G^*}(s))_2 = 1 = Z^e(G^*). \)
As \( |C_{G^*}(s)^F : C_{G^*}(s)^F| = n \), we have \( |\mathcal{E}(G^F, s)| = n \), and all of these characters are
2-Jordan quasi-central cuspidal. We claim that all elements of \( \mathcal{E}(G^F, s) \) lie in the same
2-block of \( G^F \), so do not satisfy the conclusion of Theorem 3.11(d).

Let \( T \) be a maximal torus of \( \tilde{G} \) in duality with \( C_{G^*}(s) \) and let \( \tilde{\theta} \in \text{Irr}(T^F) \) in duality
with \( s \). Let \( T = \tilde{T} \cap G \), and let \( \theta = \tilde{\theta}|_{T^F} \). Since \( s \) is regular, \( \tilde{\lambda} := R^G_{T}(\tilde{\theta}) \in \text{Irr}(G^F) \), and
\( \mathcal{E}(G^F, s) = \{\tilde{\lambda}\} \). Further, \( \tilde{\lambda} \) covers every element of \( \mathcal{E}(G^F, s) \). By [1, Prop. 10.10(b*)],
\[ R^G_T(\theta) = \text{Res}_{G^F}^G R^G_{T}(\tilde{\theta}) = \text{Res}_{G^F}^G (\tilde{\lambda}). \]
Thus, every element of \( \mathcal{E}(G^F, s) \) is a constituent of \( R^G_T(\theta) \). On the other hand, since \( \tilde{T} \) is the
torus of diagonal matrices, we have \( T = C_G(T^F_2) \) by explicit computation. Hence by
[13, Props. 2.12, 2.13(1), 2.16(1)], all constituents of \( R^G_T(\theta) \) lie in a single 2-block of \( G^F \).

Proof. Parts (a) and (b) are immediate from Theorem 3.3 and the proof of Theorem 3.6.
We next consider Part (e), where it remains to show injectivity under the given assumptions.
By [3, Thm. 4.1 and Rem. 5.2] only \( \ell = 3 \) and \( G \) of (possibly twisted) type \( A_n \)
remains to be considered. Note that the claim holds if \( 3 \in \Gamma(G, F) \) by [3, Sec. 5.2]. Thus
we may assume that the ambient simple algebraic group \( H \) of simply connected type is either
\( \text{SL}_n \) or \( E_6 \), and \( 3 \notin \Gamma(G, F) \). By Proposition 3.10 the claim holds for all blocks
whose defect groups are Cabanes.

Let first \( H = \text{SL}_m \) and \( G \leq H \) be an \( F \)-stable Levi subgroup. As \( 3 \notin \Gamma(G, F) \) we have
\( 3 | (q - 1) \) when \( F \) is untwisted. We postpone the twisted case for a moment. Embed
\( H \hookrightarrow H = \text{GL}_m \). Then \( \tilde{G} = GZ(H) \) is an \( F \)-stable Levi subgroup of \( \tilde{H} \), so has connected
center. Moreover, as \( H \) is self-dual, so is its Levi subgroup \( \tilde{G} \). In particular, \( 3 \in \Gamma(\tilde{G}, F) \).
Now let \( b \) be a 3-block of \( G^F \) in \( \mathcal{E}_3(G^F, s) \), with \( s \in G^* \) a semisimple 3'-element. Let
\( \tilde{b} \) be a block of \( \tilde{G} \) covering \( b \), contained in \( \mathcal{E}_3(\tilde{G}^F, \tilde{s}) \), where \( \tilde{s} \) is a preimage of \( s \) under
the induced map \( G^* \to \tilde{G}^* \). Since \( 3 | (q - 1) \), \( C_{G}(\tilde{s})^F \) has a single unipotent 3-block,
and so by [9 Prop. 5.1] a Sylow 3-subgroup \( \tilde{D} \) of \( C_\hat{G}(\hat{s})^F \) is a defect group of \( \bar{b} \). Thus, \( D := \tilde{D} \cap G = \tilde{D} \cap H \) is a defect group of \( b \).

Now \( C_\hat{G}(\hat{s}) \) is an \( F \)-stable Levi subgroup of \( \hat{G} \), so also an \( F \)-stable Levi subgroup of \( H = \text{GL}_m \). As such, it is a direct product of factors \( \text{GL}_m \) with \( \sum_i m_i = m \). Assume that there is more than one \( F \)-orbit on the set of factors. Then by Lemma 3.11 the Sylow 3-subgroup \( \tilde{D} \) of \( C_\hat{G}(\hat{s})^F \) has the property that \( D = \tilde{D} \cap H \) is 'Cabanes' and we are done. Hence, we may assume that \( F \) has just one orbit on the set of factors of \( C_\hat{G}(\hat{s}) \). But this is only possible if \( F \) has only one orbit on the set of factors of \( \hat{G} \). This implies that \( \hat{G}^F \cong \text{GL}_m(q^{m/n}) \) and \( G^F \cong \text{SL}_m(q^{m/n}) \) for some \( n/m \).

Exactly the same arguments apply when \( F \) is twisted, except that now \( 3(q + 1) \). So replacing \( q \) by \( q^{m/n} \) we may now suppose that \( G = \text{SL}_n \) with \( 3 \notin \Gamma(G, F) \). Assume that the defect groups of \( b \) are not Cabanes. Let \((L, \lambda)\) be an \( e \)-Jordan-cuspidal pair for \( b \) with \( \lambda \in \mathcal{E}(L^F, s) \) and let \( \tilde{L} = Z^2(G) \). There exists an irreducible constituent \( \tilde{\chi} \) of \( R^G_\lambda(\tilde{\lambda}) \) and an irreducible constituent, say \( \chi \) of \( R^G_\lambda(\lambda) \) such that \( \tilde{\chi} \) covers \( \chi \). By Lemma 2.3, \((L, \lambda)\) is an \( e \)-Jordan-cuspidal pair for \( \tilde{L} \). Thus, \( \tilde{b} \) covers \( b \).

As seen above \( C_\hat{G}(\hat{s})^F \) has a single unipotent 3-block and a Sylow 3-subgroup \( \tilde{D} \) of \( C_\hat{G}(\hat{s})^F \) is a defect group of \( \tilde{b} \) and \( D := \tilde{D} \cap G \) is a defect group of \( b \). Moreover \( F \) has a single orbit on the set of factors of \( C_\hat{G}(\hat{s}) \). By Lemma 3.12 \( C_\hat{G}(\hat{s})^F = \text{GL}_3(q^{2/3}) \) or \( \text{GU}_3(q^{2/3}) \), \( 3 \) does not divide \( \frac{q}{3} \) and \( D \) is extra-special of order 27 and exponent 3. Also, \( L \) is an \( e \)-split Levi subgroup isomorphic to a direct product of 3 copies of \( \text{GL}_2 \).

Let \( U = Z(L)^F \) and let \( c \) be the 3-block of \( L^F \) containing \( \lambda \). From the structure of \( \tilde{L} \) given above, \( |U| = 9 \) and \( L = C_G(U) \). Thus, by [9 Thm. 2.5] \((U, c)\) is a \( b \)-Brauer pair. Let \((D, f)\) be a maximal \( b \)-Brauer pair such that \((U, c) \leq (D, f) \).

Let \((L', \lambda')\) be another \( e \)-Jordan-cuspidal pair for \( b \) with \( \lambda' \in \mathcal{E}(L', s) \). Let \( U' = Z(L')^F \) and let \( c' \) be the 3-block of \( L'^F \) containing \( \lambda' \), so \( |U'| = 9 \) and \((U', c')\) is also a \( b \)-Brauer pair. Since all maximal \( b \)-Brauer pairs are \( G^F \)-conjugate, there exists \( h \in G^F \) such that \( h(U', c') \leq (D, f) \). Thus, \( U \) and \( h(U') \) are subgroups of order 9 of \( D \). By Lemma 3.13 there exists \( g \in N_{G^F}(D) \cap C_{G^F}(C_{G^F}(D)) \) such that \( g^h(U') = U \). Since \( g \) centralises \( C_{G^F}(D) \), \( g^h = f \) and since \( g \) normalises \( D, gD = D \). Hence \( (U, g^h c') = (U', c') \leq (D, f) = (D, f) \).

By the uniqueness of inclusion of Brauer pairs we get that \( g^h(U', c') = (U, c) \). Thus \( g^h L' = L \) and \( g^h c' = c \). Since \( U \) is abelian of maximal order in \( D \), \((U, c)\) is a self-centralising Brauer pair. In particular, there is a unique irreducible character in \( c \) with \( U \) in its kernel. Since \( \lambda \in \mathcal{E}(L^F, c') \), \( U \) is contained in the kernel of \( \lambda \). Hence \( g^h \lambda' = \lambda \) and injectivity is proved for type \( A \).

Finally suppose that \( H \) is of type \( E_6 \). By our preliminary reductions we may assume that \( G \) has only factors of type \( A \) and \( 3 \notin \Gamma(G, F) \). Thus \( G \) must have at least one factor of type \( A_2 \) or \( A_3 \). The remaining possibilities hence are: \( G \) is of type \( A_5 \), \( 2A_2 + A_1 \), or \( 2A_2 \). Note that for \( G \) of type \( 2A_2 + A_1 \), the \( A_1 \)-factor of the derived subgroup \([G, G] \) splits off, and that \( 2A_2 \) is a Levi subgroup of \( A_5 \). So it suffices to show the claim for Levi subgroups of this particular Levi subgroup \( G \) of type \( A_5 \). Since \( H \) is simply connected,
[G,G] ≃ SL_6 and thus virtually the same arguments as for the case of G = SL_n apply. This completes the proof of (e).

Part (d) follows whenever \( \ell \geq 3 \) is good for G, and \( \ell \neq 3 \) if \( G^F \) has a factor \( D_4(q) \), since then by (e) there is a unique e-Jordan-cuspidal pair for any \( \ell \)-block, and its (unipotent) Jordan correspondent has quasi-central \( \ell \)-defect by [8, Prop. 4.3] and Remark [2.2]. So now assume that either \( \ell \geq 3 \) is bad for G, or that \( \ell = 3 \) and \( G^F \) has a factor \( D_4(q) \).

Note that it suffices to prove the statement for quasi-isolated blocks, since then it follows tautologically for all others using the Bonnafé–Rouquier Morita equivalences, Proposition 2.4 and the remarks after Definition 2.12. Here note that by Lemma 2.5 the bijections of Proposition 2.4 extend to conjugacy classes of pairs. We first prove surjectivity. For this, by Lemma 3.7, Lemma 2.7 and by parts (a) and (b), we may assume that \( G = [G,G] \). Further, since \([G,G]\) is simply connected, hence a direct product of its components, we may assume that G is simple. Then surjectivity for unipotent blocks follows from [11, Thms. A, A.bis], while for all other quasi-isolated blocks it is shown in [13, Thm. 1.2] (these also include the case that \( G^F = D_4(q) \)).

Now we prove injectivity. If \( G = H \), then the claim for unipotent blocks follows from [11, Thms. A, A.bis], while for all other quasi-isolated blocks it is shown in [13, Thm. 1.2] (these also include the case that \( G^F = D_4(q) \)). Note that in Table 4 of [13], each of the lines 6, 7, 10, 11, 14 and 20 give rise to two e-cuspidal pairs and so to two e-Harish-Chandra series, but each e-Jordan cuspidal pair \((L, \lambda)\) which corresponds to these lines has the Cabanes property of Lemma [3.9] so they give rise to different blocks.

So, we may assume that \( G \neq H \), and thus \( \ell = 3 \). Suppose first that \( G^F \) has a factor \( D_4(q) \). Then \( H \) is of type \( E_6, E_7 \) or \( E_8 \), there is one component of \([G,G]\) of type \( D_4 \) and all other components are of type \( A \). Denote by \( G_2 \) the component of type \( D_4 \), and by \( G_1 \) the product of the remaining components with \( Z^0(G) \). We note that \( Z(G_1)/Z^0(G_1) \) is a 3'-group. Indeed, if \( H \) is of type \( E_7 \) or \( E_8 \), then \( Z(G)/Z^0(G) \) is of order prime to 3, hence the same is true of \( Z(G_1)/Z^0(G_1) \) and if \( H \) is of type \( E_6 \), then \( G_1 = Z^0(G) \).

Now, \( G^F = G_1^F \times G_2^F \). So, the map \((\{L_1, \lambda_1\}, \{L_2, \lambda_2\}) \to (L_1L_2, \lambda_1\lambda_2)\) is a bijection between pairs of e-Jordan cuspidal pairs for \( G_1^F \) and \( G_2^F \) and e-Jordan cuspidal pairs for \( G^F \). The bijection preserves conjugacy and quasi-centrality. All components of \( G_1 \) are of type \( A \) and as noted above 3 does not divide the order of \( Z(G_1)/Z^0(G_1) \), hence by [9, Sec. 5.2] we may assume that \( G = G_2 \), in which case we are done by [11] and [13].

Thus, \( G^F \) has no factor \( D_4(q) \). Set \( G_0 := [G,G] \). Since 3 is bad for \( G \), and \( G \) is proper in \( H \), we are in one of the following cases: \( G \) is of type \( E_7 \) and \( G_0 \) is simple of type \( E_6 \), or \( G \) is of type \( E_8 \) and \( G_0 \) is of type \( E_6, E_6 + A_1 \) or \( E_7 \). In all cases note that \( Z(G) \) is connected.

Let \( s \in G^F \) be a quasi-isolated semisimple 3'-element. Let \( \bar{s} \) be the image of \( s \) under the surjection \( G^* \to G_0^* \). Since \( Z(G) \) is connected, \( s \) is isolated in \( G^* \) and consequently \( \bar{s} \) is isolated in \( G_0^* \). In particular, if \( G_0 \) has a component of type \( A_1 \), then the projection of \( \bar{s} \) into that factor is the identity. Since \( s \) has order prime to 3, this means that if \( G_0 \) has a component of type \( E_6 \), then \( C_{G_0^*}(\bar{s}) \) is connected. We will use this fact later. Also, we note here that \( \bar{s} \neq 1 \) as otherwise the result would follow from [11] and the standard correspondence between unipotent blocks and blocks lying in central Lusztig series. Finally, we note that by [13, Thm. 1.2] the conclusion of Parts (a) and (d) of the
theorem holds for $G_{0}^{F}$ as all components of $G_{0}$ are of different type (so $e$ is the same for the factors of $G_{0}^{F}$ as for $G^{F}$).

Let $b$ be a 3-block of $G^{F}$ in the series $s$ and $(L, \lambda)$ be an $e$-Jordan quasi-central cuspidal pair for $b$ such that $s \in L^{F}$ and $\lambda \in E(L^{F}, s)$. Let $L_{0} = L \cap G_{0}$ and let $\lambda_{0}$ be an irreducible constituent of the restriction of $\lambda$ to $L_{0}^{F}$. By Lemma 3.8 there exists a block $b_{0}$ of $G_{0}^{F}$ covered by $b$, and such that all irreducible constituents of $R_{L_{0}}^{G_{0}}(\lambda_{0})$ belong to $b_{0}$. By Lemma 2.3 and the remarks following Definition 2.12 $(L_{0}, \lambda_{0})$ is an $e$-Jordan quasi-central cuspidal pair of $G_{0}^{F}$ for $b_{0}$.

First suppose that $C_{G_{0}}(\bar{s})$ is connected. Then all elements of $E(G_{0}^{F}, \bar{s})$ are $G^{F}$-stable and in particular, $b_{0}$ is $G^{F}$-stable. Now let $(L', \lambda')$ be another $e$-Jordan quasi-central cuspidal pair for $b$. Let $L_{0}' = L' \cap G_{0}$ and $\lambda_{0}'$ be an irreducible constituent of the restriction of $\lambda'$ to $L_{0}^{F}$. Then, as above $(L_{0}', \lambda_{0}')$ is an $e$-Jordan quasi-central cuspidal pair for $b_{0}$. But there is a unique $e$-Jordan quasi-central cuspidal pair for $b_{0}$ up to $G_{0}^{F}$-conjugacy. So, up to replacing by a suitable $G_{0}^{F}$-conjugate we may assume that $(L_{0}, \lambda_{0}) = (L_{0}', \lambda_{0}')$, hence $L = L'$, and $\lambda$ and $\lambda'$ cover the same character $\lambda_{0} = \lambda_{0}'$ of $L_{0}^{F} = L_{0}^{F}$.

If $\mu \in E(G_{0}^{F}, \bar{s})$, then there are $|G^{F}/G_{0}^{F}|\nu$ different $3'$-Lusztig series of $G^{F}$ containing an irreducible character covering $\mu$. Since characters in different $3'$-Lusztig series lie in different 3-blocks, there are at least $|G^{F}/G_{0}^{F}|\nu$ different blocks of $G^{F}$ covering $b_{0}$. Moreover, if $b'$ is a block of $G^{F}$ covering $b_{0}$, then there exists a linear character, say $\theta$ of $G^{F}/G_{0}^{F} \cong L^{F}/L_{0}^{F}$ of $3'$-degree such that $(L, \theta \otimes \lambda)$ is an $e$-Jordan quasi-central cuspidal pair for $b'$ and $\lambda_{0}$ appears in the restriction of $\theta \otimes \lambda$ to $L_{0}^{F}$. Since there are at most $|L^{F}/L_{0}^{F}||\nu = |G^{F}/G_{0}^{F}||\nu$ irreducible characters of $L^{F}$ in $3'$-series covering $\lambda_{0}$, it follows that $\lambda = \lambda'$.

Thus, we may assume that $C_{G_{0}}(\bar{s})$ is not connected. Hence, by the remarks above $G_{0}$ is simple of type $E_{7}$. Further $\bar{s}$ corresponds to one of the lines 5, 6, 7, 12, 13, or 14 of Table 4 of [13] (note that $\bar{s}$ is isolated and that $e$-Jordan (quasi)-central cuspidality in this case is the same as $e$-(quasi)-central cuspidality).

By [13] Lemma 5.2, $L_{0} = C_{G_{0}}(Z(L_{0}^{F})_{3})$. In other words, $(L_{0}, \lambda_{0})$ is a good pair for $b_{0}$ in the sense of [13] Def. 7.10. In particular, there is a maximal $b_{0}$-Brauer pair $(P_{0}, c_{0})$ such that $(Z(L_{0}^{F})_{3}, b_{L_{0}^{F}}(\lambda_{0})) \triangleright (P_{0}, c_{0})$. Here for a finite group $X$ and an irreducible character $\eta$ of $X$, we denote by $b_{X}(\eta)$ the $\ell$-block of $X$ containing $\eta$. By inspection of the relevant lines of Table 4 of [13] (and the proof of [13] Thm. 1.2)), one sees that the maximal Brauer pair $(P_{0}, c_{0})$ can be chosen so that $Z(L_{0}^{F})_{3}$ is the unique maximal abelian normal subgroup of $P_{0}$.

By [13] Thm. 7.11 there exists a maximal $b$-Brauer pair $(P, c)$ and $\nu \in E(L^{F}, \ell')$ such that $\nu$ covers $\lambda_{0}$, $P_{0} \leq P$ and we have an inclusion of $b$-Brauer pairs $(Z(L_{0}^{F})_{3}, b_{L_{0}^{F}}(\nu)) \triangleright (P, c)$. Since $\lambda$ also covers $\lambda_{0}$, $\lambda = \tau \otimes \nu$ for some linear character $\tau$ of $L_{0}^{F}/L_{0}^{F} \cong G_{0}^{F}/G_{0}^{F}$. Since tensoring with linear characters preserves block distribution and commutes with Brauer pair inclusion, replacing $c$ with the block of $C_{G_{0}}(P_{0})$ whose irreducible characters are of the form $\tau \otimes \varphi$, $\varphi \in \text{Irr}(c)$, we get that there exists a maximal $b$-Brauer pair $(P, c)$ such that $P_{0} \leq P$ and $(Z(L_{0}^{F})_{3}, b_{L_{0}^{F}}(\lambda)) \triangleright (P, c)$.

Being normal in $G^{F}$, $Z(G^{F})_{3}$ is contained in the defect groups of every block of $G^{F}$, and in particular $Z(G^{F})_{3} \leq P$. On the other hand, since $G_{0}$ has centre of order 2, $P_{0}Z(G^{F})_{3}$ is a defect group of $b$ whence $P$ is a direct product of $P_{0}$ and $Z(G^{F})_{3}$. Now, $Z(L_{0}^{F})_{3}$ is the unique maximal abelian normal subgroup of $P_{0}$. Hence, $Z(L_{0}^{F})_{3} = Z(G^{F})_{3} \times Z(L_{0}^{F})_{3}$ is
the unique maximal normal abelian subgroup of $P$ (see Lemma 3.11). Finally note that by Lemma 2.7, $\lambda$ is also of quasi-central $\ell$-defect. By Lemma 3.9 it follows that $(L, \lambda)$ is the unique $e$-Jordan quasi-central cuspidal pair of $G^F$ for $b$.

Finally, we show (c). In view of the part (d) just proved above, it remains to consider the prime $\ell = 2$ only. Suppose first that all components of $G$ are of simple type. Let $s \in G^{*F}$ be semisimple of odd order and let $b$ be a $2$-block of $G^F$ in series $s$. By Lemma 3.17 below there is an $e$-torus, say $S$ of $C_{G^e}^e(s)$ such that $T^e := C_{G^e}^e(s)(L)$ is a maximal torus of $C_{G^e}^e(s)$. Let $L^e = C_{G^e}(L)$ and let $L$ be a Levi subgroup of $G$ in duality with $L^e$. Then $L$ is an $e$-split subgroup of $G$ and $T^e = C_{L^e}^e(s)$. Let $\lambda \in \text{Irr}(L^e, s)$ correspond via Jordan decomposition to the trivial character of $T^e$. Then $(L, \lambda)$ is an $e$-Jordan quasi-central cuspidal pair of $G$.

Let $G \hookrightarrow \bar{G}$ be a regular embedding. By part (a), Lemmas 3.3 and 3.8 there exists $g \in \bar{G}^F$ such that $b = b_{G^F}^e(g, \lambda)$. Now since $(L, \lambda)$ is $e$-Jordan quasi-central cuspidal, so is $(g, \lambda)$. In order to see this, first note that, up to multiplication by a suitable element of $G^F$ and by an application of the Lang–Steinberg theorem, we may assume that $g$ is in some $F$-stable maximal torus of $C^G_e(s)$. Thus $g, \lambda$ and $g, \lambda$ correspond to the same $C_{L^e}(s)^e$ orbit of unipotent characters of $C_{L^e}^e(s)$. Now suppose that $G$ has a component of exceptional type. Then we can argue just as in the proof of surjectivity for bad $\ell$ in Part (d).

Lemma 3.17. Let $G$ be connected reductive with a Frobenius morphism $F : G \to G$. Let $e \in \{1, 2\}$ and let $S$ be a Sylow $e$-torus of $G$. Then $C_G(S)$ is a torus.

Proof. Let $C := [C_G(S), C_G(S)]$ and assume that $C$ has semisimple rank at least one. Let $T$ be a maximally split torus of $C$. Then the Sylow $1$-torus of $T$, hence of $C$ is non-trivial. Similarly, the reductive group $C'$ with complete root datum obtained from that of $C$ by replacing the automorphism on the Weyl group by its negative, again has a non-trivial Sylow $1$-torus. But then $C$ also has a non-trivial Sylow $2$-torus. Thus in any case $C$ has a non-central $e$-torus, which is a contradiction to its definition.

4. JORDAN DECOMPOSITION OF BLOCKS

Lusztig induction induces Morita equivalences between Bonnafé–Rouquier corresponding blocks. We show that this also behaves nicely with respect to $e$-cuspidal pairs and their corresponding $e$-Harish-Chandra series.

4.1. Jordan decomposition and $e$-cuspidal pairs. Throughout this subsection, $G$ is a connected reductive algebraic group with a Frobenius endomorphism $F : G \to G$ endowing $G$ with an $F_q$-structure for some power $q$ of $p$. Our results here are valid for all groups $G^F$ satisfying the Mackey-formula for Lusztig induction. At present this is known to hold unless $G$ has a component $H$ of type $E_6$, $E_7$ or $E_8$ with $H^F \in \{E_6(2), E_7(2), E_8(2)\}$, see Bonnafé–Michel [3]. The following is in complete analogy with Proposition 2.4.

Proposition 4.1. Assume that $G^F$ has no factor $2E_6(2), E_7(2)$ or $E_8(2)$. Let $s \in G^{*F}$, and $G_1 \leq G$ an $F$-stable Levi subgroup with $G_1^e$ containing $C_G^e(s)$. For $(L_1, \lambda_1)$ an $e$-cuspidal pair of $G_1$ below $\mathcal{E}(G^F, s)$ define $L := C_G(Z(L_1)), \lambda := \epsilon_{L_1}L_{L_1}R_{L_1}(L_1)$. Then $(L_1, \lambda_1) \mapsto (L, \lambda)$ defines a bijection $\Psi_{G_1}$ between the set of $e$-cuspidal pairs of $G_1$ below $\mathcal{E}(G^F, s)$ and the set of $e$-cuspidal pairs of $G$ below $\mathcal{E}(G^F, s)$.
Proof. We had already seen in the proof of Proposition 2.3 that $L$ is $e$-split and $Z_e(L_1)_e = Z_e(L)_e$. For the well-definedness of $\Psi^G_{G_1}$ it remains to show that $\lambda$ is $e$-cuspidal. For any $e$-split Levi subgroup $X \leq L$ the Mackey formula [3 Thm.] gives
\[
e_{L_1L_1}^{e_{L_1}} R_X^L(\lambda) = e_{L_1}^{L_1} R_X^{L_1}(\lambda_1) = \sum_g R_X^{L \cap g L_1} e_{L \cap g L_1}^{L_1}(\lambda_1^g)
\]
where the sum runs over a suitable set of double coset representatives $g \in L^F$. Here, $X \cap sL_1$ is $e$-split in $L_1$ since $L_1 \cap X^g = L_1 \cap C_L (Z_e(X^g)_e) = C_{L_1}(Z_e(X^g)_e)$. The $e$-cuspidality of $\lambda_1$ thus shows that the only non-zero terms in the above sum are those for which $L_1 \cap X^g = L_1$, i.e., those with $L_1 \leq X^g$. But then $Z_e(L)_e = Z_e(L_1)_e = Z_e(X^g)_e$, and as $X$ is $e$-split in $L$ we deduce that necessarily $X = L$ if $eR_X^{L_1}(\lambda_1) \neq 0$. So $\lambda$ is indeed $e$-cuspidal, and $\Psi^G_{G_1}$ is well-defined.

Injectivity was shown in the proof of Proposition 2.3 where we had constructed an inverse map with $L_1^* := L^* \cap G_1^*$ and $\lambda_1$ the unique constituent of $eR_X^{L_1}(\lambda)$ in $E(L^F, s)$. We claim that $\lambda_1$ is $e$-cuspidal. Indeed, for any $e$-split Levi subgroup $X \leq L_1$ let $Y := C_L(Z_e(X)_e)$, an $e$-split Levi subgroup of $L$. Then $eR_X^{L_1}(\lambda_1)$ is a constituent of
\[eR_X^{L}(\lambda) = eR_X^Y eR_Y^{L}(\lambda) = 0\]
by $e$-cuspidality of $\lambda$, unless $Y = L$, whence $X = Y \cap L_1 = L \cap L_1 = L$.

Thus we have obtained a well-defined map $e\Psi^G_{G_1}$ from $e$-cuspidal pairs in $G$ to $e$-cuspidal pairs in $G_1$, both below the series $s$. The rest of the proof is again as for Proposition 2.3. □

4.2. Jordan decomposition, $e$-cuspidal pairs and $\ell$-blocks. We next remove two of the three possible exceptions in Proposition 4.1 for characters in $\ell$-series:

**Lemma 4.2.** The assertions of Proposition 4.1 remain true for $G^F$ having no factor $E_8(2)$ whenever $s \in G^{*F}$ is a semisimple $\ell'$-element, where $e = e_\ell(q)$. In particular, $\Psi^G_{G_1}$ exists.

**Proof.** Let $s$ be a semisimple $\ell'$-element. Then by [9 Thm. 4.2] we may assume that $\ell \leq 3$, so in fact $\ell = 3$. The character table of $G^{*F} = E_6(2).3$ is known; there are 12 classes of non-trivial elements $s \in G^{*F}$ of order prime to 6. Their centralisers $C_{G^*}(s)$ only have factors of type $A$, and are connected. Thus all characters in those series $E(G^{*F}, s)$ are uniform, so the Mackey-formula is known for them with respect to any Levi subgroup. Thus, the argument in Proposition 4.1 is applicable to those series. For $G^F = E_7(2)$, the conjugacy classes of semisimple elements can be found on the webpage [11] of Frank Lübeck. From this one verifies that again all non-trivial semisimple $3'$-elements have centraliser either of type $A$, or of type $2D_4(q)A_1(q)\Phi_4$, or $3D_4(q)\Phi_1\Phi_3$. In the latter two cases, proper Levi subgroups are either direct factors, or again of type $A$, and so once more the Mackey-formula is known to hold with respect to any Levi subgroup. □

**Remark 4.3.** The assertion of Lemma 4.2 can be extended to most $\ell'$-series of $G^F = E_8(2)$. Indeed, again by [9 Thm. 4.2] we only need to consider $\ell \in \{3, 5\}$. For $\ell = 3$ there are just two types of Lusztig series for $3'$-elements which can not be treated by the arguments above, with corresponding centraliser $E_6(2)\Phi_3$ respectively $2D_6(2)\Phi_4$. For $\ell = 5$, there are five types of Lusztig series, with centraliser $E_6(2)^3A_2(2)$, $E_7(2)\Phi_2$, $2D_7(2)\Phi_2$, $E_6(2)\Phi_3$
and $2D_5(2)\Phi_2\Phi_6$ respectively. Note that the first one is isolated, so the assertion can be checked using [13].

**Proposition 4.4.** Assume that $G^F$ has no factor $E_8(2)$. Let $s \in G^*_F$, and $G_1 \leq G$ an $F$-stable Levi subgroup with $G_1^*$ containing $C_{G^*}(s)$. Assume that $b$ is an $\ell$-block in $E_{\ell}(G^F, s)$, and $c$ is its Bonnafé–Rouquier correspondent in $E_{\ell}(G_1^F, s)$. Let $e = e_{\ell}(q)$.

(a) Let $(L_1, \lambda_1)$ be $e$-cuspidal in $G_1$, where $(L, \lambda) = \Psi_{G_1^*}(L_1, \lambda_1)$. If all constituents of $R^{G_1}(\lambda_1)$ lie in $c$, then all constituents of $R^G_{L_1}(\lambda)$ lie in $b$.

(b) Let $(L, \lambda)$ be $e$-cuspidal in $G$, where $(L_1, \lambda_1) = *\Psi_{G_1^*}(L_1, \lambda)$. If all constituents of $R^G_{L}(\lambda)$ lie in $b$, then all constituents of $R^{G_1}_{L_1}(\lambda_1)$ lie in $c$.

The proof is identical to the one of Proposition 2.6, using Proposition 4.1 and Lemma 4.2 in place of Proposition 2.3.

**References**

[1] C. Bonnafé, Quasi-isolated elements in reductive groups. Comm. Algebra 33 (2005), 2315–2337.

[2] C. Bonnafé, Sur les caractères des groupes réductifs finis a centre non connexe : applications aux groupes spéciaux linéaires et unitaires. Astérisque 306 (2006).

[3] C. Bonnafé, J. Michel, Computational proof of the Mackey formula for $q > 2$. J. Algebra 327 (2011), 506–526.

[4] C. Bonnafé, R. Rouquier, Catégories dérivées et variétés de Deligne–Lusztig. Publ. Math. Inst. Hautes Études Sci. No. 97 (2003), 1–59.

[5] M. Broué, G. Malle, J. Michel, Generic blocks of finite reductive groups. Astérisque No. 212 (1993), 7–92.

[6] M. Cabanes, Unicité du sous-groupe abélien distingué maximal dans certains sous-groupes de Sylow. C. R. Acad. Sci. Paris 318 (1994), 889–894.

[7] M. Cabanes, M. Enguehard, Unipotent blocks of finite reductive groups of a given type. Math. Z. 213 (1993), 479–490.

[8] M. Cabanes, M. Enguehard, On unipotent blocks and their ordinary characters. Invent. Math. 117 (1994), 149–164.

[9] M. Cabanes, M. Enguehard, On blocks of finite reductive groups and twisted induction. Adv. Math. 145 (1999), 189–229.

[10] F. Digne, J. Michel, Representations of Finite Groups of Lie Type. LMS Student Texts, 21. Cambridge University Press, Cambridge, 1991.

[11] M. Enguehard, Sur les $\ell$-blocs unipotents des groupes réductifs finis quand $\ell$ est mauvais. J. Algebra 230 (2000), 334–377.

[12] M. Enguehard, Vers une décomposition de Jordan des blocs des groupes réductifs finis. J. Algebra 319 (2008), 1035–1115.

[13] R. Kessar, G. Malle, Quasi-isolated blocks and Brauer’s height zero conjecture. Annals of Math. 178 (2013), 321–384.

[14] F. Lübeck, Table at http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/index.html

[15] G. Malle, D. Testerman, Linear Algebraic Groups and Finite Groups of Lie Type. Cambridge Studies in Advanced Mathematics, 133, Cambridge University Press, Cambridge, 2011.

[16] H. Nagao, Y. Tsushima, Representations of Finite Groups. Academic Press, Boston, 1989.

Department of Mathematical Sciences, City University London, Northampton Square, London EC1V 01B, U.K.

E-mail address: Radha.Kessar.1@city.ac.uk

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.

E-mail address: malle@mathematik.uni-kl.de