Quasi-local formulation of the mirror TBA

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Abstract: We present a method of removing all infinite sums from the various forms of the mirror TBA equations and the energy formula of the AdS/CFT spectral problem. This new formulation of the TBA system is quasi-local because $Y$-functions that are connected by the TBA equations are at most next to nearest neighbors with respect to the $Y$-system diagram of AdS/CFT.
1. Introduction

One of the most important problems in testing the AdS/CFT correspondence in the planar limit is to determine the finite size spectrum of the AdS$_5 \times S^5$ superstring sigma model. After integrability was discovered in the string worldsheet theory, the mirror Thermodynamic Bethe Ansatz (TBA) technique was proposed to determine nonperturbatively the spectrum of the string theory. The TBA equations for AdS/CFT were first derived for the ground state and then using an analytic continuation trick TBA equations were conjectured for certain classes of states in the sl(2) sector of the theory.

An important property of the TBA equations is that the unknown functions (Y-functions) satisfy the so called Y-system functional relations. Recently an alternative derivation of the TBA equations appeared where it has been shown that the complicated ground state TBA equations can be derived from the Y-system if it is supplemented by discontinuity functional equations relating the square-root discontinuities of the various Y-functions. Later the derivation has been extended to excited states, too.

The main advantage of the Y-system based approach to the TBA problem is that it gives a deeper insight into the analytic properties of the Y-functions and opens the way to understand the analytic properties of the $T$ and $Q$ functions, which are the elementary building blocks of the underlying Y-system. This knowledge is indispensable if we want to construct an NLIE system formulated in terms of finitely many unknown functions for the AdS/CFT spectral problem.

So far the proposed TBA equations satisfied all tests in both the weak and strong coupling limits. Their correctness has been nicely demonstrated by the convincing agreement with gauge theory results in the weak coupling limit and with string theory results in the strong coupling limit.
In spite of the success of TBA technique in AdS/CFT, it has some obvious disadvantages as well. First of all, like all the TBA equations of known sigma-models it contains infinitely many unknown functions, which makes the study of their properties, both analytically and numerically, difficult. Secondly, contrary to the case of relativistic models, these TBA equations cannot be formulated in a local form, namely a $Y$-function associated to a node of the $Y$-diagram of the model cannot be expressed, in general, in terms of the $Y$-functions belonging to the neighboring nodes only.

This non-locality is related to the discontinuity structure of the $Y$-functions and the discontinuity relations \[15\]. As a consequence of non-locality all formulations\(^1\) of the mirror TBA equations presented so far contains infinite sums of certain $Y$ combinations convoluted with appropriate kernels on the right hand side of the equations. If expressed in terms of $Y$-functions, the TBA energy formula has the same property.

The purpose of this paper is to work out a method that enables us to replace all the infinite sums appearing in the TBA approach by expressions containing only a few terms and depending on a few $Y$-functions only.

Here we focus our attention to the hybrid formulation of the equations \[12\]. In this case all the infinite sums which appear in the TBA description (including the energy formula) are of the form:

\[
\sum_{Q=1}^{\infty} L_Q \ast K_Q, \tag{1.1}
\]

where $L_Q = \ln(1 + Y_Q)$ and $K_Q \equiv K_Q(u, v)$ is a kernel satisfying the relation:

\[
K_Q - s \ast (K_{Q-1} + K_{Q+1}) = \delta K_Q, \quad K_0 \equiv 0, \quad Q = 1, 2, \ldots \tag{1.2}
\]

The fact that enables us to transform (1.1) into a finite sum is that for the $K_Q$ kernels appearing in the TBA equations $\delta K_Q = 0$, with the exception of a few values of the index $Q$ only. See formulae (4.18)-(4.19).

In this paper we will derive finite expressions for the infinite sums

\[
\Omega = \sum_{Q=2}^{\infty} L_Q \ast K_Q. \tag{1.3}
\]

The simplification of $\Omega$ consists of two main steps. First we extend the definition of the convolution for functions of two variables in a natural way. Next with the help of the simplified TBA equations we can derive a linear integral equation for $\Omega$, which can be solved in Fourier space giving our final formula (2.33).

If we replace the infinite sums of type (1.3) with our new finite formulas (2.33) in the TBA equations and in the energy formula, the resulting formulation of the TBA equations becomes quasi-local\(^2\) as well. We think that such a quasi-local formulation of the TBA equations not only makes it possible to perform more accurate numerical simulations but

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\(^1\)Using the terminology of \[12\] canonical, simplified and hybrid versions of the TBA equations.

\(^2\)Here quasi-locality means that at most next to nearest neighbor $Y$-functions are connected by the TBA equations.
it is also an important step towards the NLIE formulation of the planar AdS/CFT spectral problem.

The paper is organized as follows: Section 2 contains the derivation of the finite expression for $\Omega$. In section 3 we discuss the energy formula and evaluate the finite formula for $\Omega$ when all $Y$-functions are replaced by their asymptotic counterparts. In section 4 we discuss the kernels appearing the TBA equations and present the complete set of quasi-local mirror TBA equations. The paper is closed by our conclusions.

Throughout the paper we use the notations, conventions and definitions of [12]. In this short paper we cannot reproduce all the formulae giving the definition of the various kernels appearing in the TBA equations and thus unfortunately the paper is not self-contained. The necessary definitions together with the list of the TBA equations in the form we are using them in this paper can also be found in [10].

2. TBA sums

Since the convolution operation between functions plays a central role in our considerations here we recall the relevant definitions. The objects appearing in these formulas are either “functions” (typically the logarithm of some $Y$-function) or “kernels”. “Functions” depend on one variable, kernels on two. Some kernels only depend on the difference of the two variables. This is the case for the most important kernel $s(u, v)$:

$$s(u, v) = s(u - v), \quad s(u) = \frac{g}{4 \cosh \frac{\pi u}{2}}.$$  \hspace{1cm} (2.1)

For the “function” $\star$ “kernel” and “kernel” $\star$ “function” type convolutions we have

$$(f \star F)(v) = \int_{-\infty}^{\infty} du \, f(u)F(u, v), \quad (F \star f)(v) = \int_{-\infty}^{\infty} du \, F(v, u)f(u)$$  \hspace{1cm} (2.2)

and for “kernel” $\star$ “kernel” type convolutions

$$(F_1 \star F_2)(u, v) = \int_{-\infty}^{\infty} dw \, F_1(u, w)F_2(w, v).$$  \hspace{1cm} (2.3)

The modified convolutions

$$(f \hat{\star} F)(v) = \int_{-2}^{2} du \, f(u)F(u, v), \quad (F \hat{\star} f)(v) = \left( \int_{2}^{\infty} + \int_{-\infty}^{-2} \right) du \, f(u)F(u, v)$$  \hspace{1cm} (2.4)

(and similarly for $F_1 \hat{\star} F_2$, $F_1 \hat{\star} F_2$) are also used in some of the TBA integral equations.

Note that the standard convolution\(^3\) definition between two functions of one variable

$$(f \star^\text{old} g)(u) = \int_{-\infty}^{\infty} dw \, f(u - w)g(w)$$  \hspace{1cm} (2.5)

is not always the same as our convolution, but

$$f \star s = f \star^\text{old} s,$$  \hspace{1cm} (2.6)

\(^3\)As is well known, $f \star^\text{old} g = g \star^\text{old} f$ and $\star^\text{old}$ reduces to ordinary product after Fourier transformation.
because $s$ is even.

In this paper we will use a new convolution definition, which always maps two functions of two variables into a function of two variables. Functions of one variable will be treated as functions of two variables by writing $f(u, v) \rightarrow f(u - v)$:

\[(F_1 \star F_2)(u, v) = \int_{-\infty}^{\infty} dw F_1(u, w)F_2(w, v), \quad (2.7)\]

\[(f \star F)(u, v) = \int_{-\infty}^{\infty} dw f(u - w)F(w, v), \quad (2.8)\]

\[(f \star g)(u, v) = \int_{-\infty}^{\infty} dw f(u - w)g(w - v) = (f \starold g)(u - v). \quad (2.9)\]

The new convolution $\star$ is an (in general) non-commutative, associative product of functions.

Finally we note that the prototype TBA equation

\[L = \Lambda \star s, \quad (2.10)\]

where $L$ and $\Lambda$ are “functions”, can be equivalently written

\[\hat{L} = \hat{\Lambda} \star s, \quad (2.11)\]

where

\[\hat{f}(u) = f(-u) \quad (2.12)\]

for any “function”.

The kernel identity (1.2) can be rewritten as

\[K_Q - s \star (K_{Q-1} + K_{Q+1}) = \delta K_Q, \quad K_0 \equiv 0, \quad Q = 1, 2, \ldots \quad (2.13)\]

and we also introduce the notation $\overline{K}_m = s \star K_m$. $\overline{K}_m$ satisfies an identity similar to (2.13) with $\delta \overline{K}_m = s \star \delta K_m$.

In this section we will present the simplification of the TBA sums working, for simplicity, in the $\mathfrak{sl}(2)$ sector of the theory$^4$. We will consider the infinite sums

\[\mathcal{A} = \sum_{Q=2}^{\infty} \hat{L}_Q \star K_Q \quad (2.14)\]

and

\[\mathcal{B} = \sum_{m=1}^{\infty} \hat{L}_m \star \overline{K}_{m+1} \quad (2.15)\]

After having computed $\mathcal{A}$ we can write our final result as $\Omega(v) = \mathcal{A}(0, v)$.

We recall that $L_Q = \ln(1 + Y_Q)$ and (see ref. 16)

\[\mathcal{L}_m = \ln \left( \tau_m \tau_{m+2} \left( 1 + \frac{1}{\tau_{m+2} Y_{m+2}} \right) \right), \quad m = 1, 2, \ldots \quad (2.16)\]

$^4$Performing the calculation for the general case is straightforward.
further \( r_m = \ln(1 + Y_{m|vw}) \) \( m = 1, 2, \ldots, r_0 = 0 \) and we define analogously
\[
R_Q = \ln \left( \tau^2_Q \left( 1 + \frac{1}{Y_Q} \right) \right), \quad Q = 2, 3, \ldots, \quad R_1 = \ln \left( 1 + \frac{1}{Y_1} \right).
\] (2.17)

Using the notation introduced above we write the simplified TBA integral equations \([4.1]\) and \([4.2]\) as \([16]\):
\[
L_m = r_m + L_{m+1} \ast s - (r_{m+1} + r_{m-1}) \ast s - \hat{f}_2 \delta_{m,1} \quad m = 1, 2, \ldots
\] (2.18)

and equivalently
\[
L_m = \hat{r}_m + \hat{L}_{m+1} \ast \ast s - \left( \hat{r}_{m+1} + \hat{r}_{m-1} \right) \ast \ast s - \hat{f}_2 \delta_{m,1} \quad m = 1, 2, \ldots
\] (2.19)

Here
\[
\hat{f}_2 = H \ast s, \quad H = \ln \left( \frac{1 - Y}{1 - Y_+} \right).
\] (2.20)

Similarly from \([1.5]\)
\[
\hat{L}_Q = \hat{R}_Q + 2 \hat{L}_{Q-1} \ast \ast s - \left( \hat{R}_{Q+1} + \hat{R}_{Q-1} \right) \ast \ast s - \hat{f}_1 \delta_{Q,2} \quad Q = 2, 3, \ldots
\] (2.21)

where
\[
\hat{f}_1 = \ln \tau^2_1 \ast s.
\] (2.22)

Now we take the convolution of \([2.21]\) with \( K_Q \) and sum over \( Q \) from 2 to infinity. After rearranging terms and using the kernel identity \([2.13]\) we get
\[
A = 2B + \sum_{Q=2}^{\infty} \hat{R}_Q \ast \ast \delta K_Q + \hat{R}_2 \ast \ast K_1 - \hat{R}_1 \ast \ast K_2 - \hat{f}_2 \ast \ast K_1.
\] (2.23)

After performing similar transformations we get from \([2.19]\)
\[
B = \sum_{m=1}^{\infty} \hat{r}_m \ast \ast \delta K_{m+1} + \hat{r}_1 \ast \ast K_1 + s \ast \ast s \ast \ast A - \hat{f}_2 \ast \ast K_2.
\] (2.24)

Eliminating \( B \) from \([2.23]\) and \([2.24]\) we get a linear integral equation of the form
\[
A - 2s \ast \ast s \ast \ast A = \omega
\] (2.25)

with
\[
\omega = \sum_{Q=2}^{\infty} \hat{R}_Q \ast \ast \delta K_Q + \hat{R}_2 \ast \ast K_1 - \hat{R}_1 \ast \ast K_2 - \hat{f}_2 \ast \ast K_2
\] (2.26)

+ 2 \sum_{m=1}^{\infty} \hat{r}_m \ast \ast \delta K_{m+1} + 2\hat{r}_1 \ast \ast K_1 - 2\hat{f}_2 \ast \ast K_2.

The linear equation \([2.25]\) can easily be solved since the second argument of the unknown \( A(u, v) \) is a “spectator” only and the essential part of the problem can be formulated in
terms of kernels depending on differences only and it can be algebraically solved in Fourier space. The solution is of the form
\[ A = M \bigotimes \omega \] (2.27)
with
\[ M = \delta + 2s_{1/2}, \quad s_{1/2}(u) = \frac{1}{2} \xi \left( \frac{u}{2} \right), \] (2.28)
where \( \delta \) is the Dirac delta function. This solution further satisfies
\[ M \bigotimes s = \sigma_{1/2}, \quad \sigma_{1/2} = s^+_1 + s^-_1, \quad \sigma_{1/2}(u) = \frac{g}{2\sqrt{2}} \cosh \frac{zu}{2}. \] (2.29)
A further useful identity is
\[ M \bigotimes s \bigotimes s = s_{1/2}. \] (2.30)
Putting everything together, \( A \) can be expressed explicitly:
\[ A = \sum_{Q=2}^\infty \hat{R}_Q \bigotimes (\delta K_Q + 2s_{1/2} \bigotimes \delta K_Q) + \hat{R}_2 \bigotimes \sigma_{1/2} \bigotimes K_1 - \hat{R}_1 \bigotimes \sigma_{1/2} \bigotimes K_2 \\
- \ln \tau_1 \bigotimes \sigma_{1/2} \bigotimes K_1 + 2 \sum_{m=1}^\infty \hat{r}_m \bigotimes \sigma_{1/2} \bigotimes \delta K_{m+1} \\
+ 2\hat{r}_1 \bigotimes s_{1/2} \bigotimes K_1 - 2\hat{r}_2 \bigotimes \sigma_{1/2} \bigotimes K_2. \] (2.31)
In all applications we need in this paper the kernel functions satisfy
\[ \delta K_Q = 0, \quad Q \geq 3 \] (2.32)
and furthermore we are interested in \( \Omega \) only and thus the original convolutions can be used.
Our final result is
\[ \Omega = \hat{R}_2 \ast (\delta K_2 + 2s_{1/2} \ast \delta K_2) + \hat{R}_2 \ast \sigma_{1/2} \ast K_1 - \hat{R}_1 \ast \sigma_{1/2} \ast K_2 \\
- \ln \tau_1 \ast \sigma_{1/2} \ast K_1 + 2r_1 \ast \sigma_{1/2} \ast \delta K_2 + 2r_1 \ast s_{1/2} \ast K_1 - 2H \ast s_{1/2} \ast K_2. \] (2.33)
The identity (2.30) was used to simplify the last term.

3. Energy formula and asymptotic form of the identity

One of the important applications of our method is the calculation of the infinite sum
\[ E = -\frac{1}{2\pi} \sum_{Q=1}^\infty \int_{-\infty}^{\infty} du L_Q(u) \frac{d\hat{p}^Q}{du} \] (3.1)
occurring in the expression for the energy of the state \[12\]. We can apply our formalism to calculate this sum with
\[ K_Q(u, v) \Rightarrow -\frac{1}{2\pi} \frac{d\hat{p}^Q}{du} = \frac{g}{2\pi} \left[ x^{[Q]}(u) - x^{[-Q]}(u) \right]. \] (3.2)
This expression satisfies the kernel identity (2.13), but approaches constant limits\textsuperscript{5} for large $|u|$. This last property of this kernel leads to a problem in applying (2.33) since the functions $\mathcal{R}_1$, $\mathcal{R}_2$ behave as $\ln |u|$ for large $|u|$ and the corresponding convolutions are divergent.

There are two possible solutions of this problem. First we can use the identity

$$ x^{[Q]'}(u) - x^{[-Q]'}(u) = i \int_{|v|>2} \mathrm{d}v \ K_Q(u-v) \frac{v + i\epsilon}{\sqrt{4 - (v + i\epsilon)^2}} $$

(3.3)

where

$$ K_Q(u) = \frac{Qg}{\pi} \frac{1}{Q^2 + g^2 u^2} $$

(3.4)

and use our formalism for $K_Q(u,v) \Rightarrow K_Q(u-v)$ to calculate

$$ t_1 = \sum_{Q=1}^{\infty} L_Q \ast K_Q $$

(3.5)

Since $K_Q$ is well-behaved at infinity, all convolutions are now convergent and after having computed $t_1(u)$ with our method we can calculate (3.1) by multiplying it with $\frac{j_0}{2\pi} \frac{u}{\sqrt{4-u^2}}$ and integrate the result for $|u| > 2$ (slightly above the real line).

An alternative, second solution (which is also applicable to other cases of interest) is to consider the asymptotic limit\textsuperscript{6} of our functions. The asymptotic limit is obtained from the Bethe Ansatz and these asymptotic solutions satisfy TBA integral equations that can be obtained from (4.1,4.2,4.5) by simply deleting all $(1 + Y_Q)$ factors occurring on the right hand side of these equations [14]. In our notation the asymptotic TBA equations become

$$ \hat{\mathcal{L}}^{(0)}_m = r^{(0)}_m - (r^{(0)}_{m+1} + r^{(0)}_{m-1}) \ast s - \hat{f}_1^{(0)} \delta_{m,1} $$

(3.6)

(for $m = 1, 2, \ldots$) and

$$ \hat{L}^{(0)}_Q = \hat{R}^{(0)}_Q + 2\hat{\mathcal{L}}^{(0)}_{Q-1} \ast s - (\hat{R}^{(0)}_{Q+1} + \hat{R}^{(0)}_{Q-1}) \ast s + (\hat{L}^{(0)}_{Q+1} + \hat{L}^{(0)}_{Q-1}) \ast s - \hat{f}_1^{(0)} \delta_{Q,2} $$

(3.7)

(for $Q = 2, 3, \ldots$) Since the particle rapidities $u_j$ occurring in $\tau_1$ can be treated simply as parameters, we will not distinguish between the asymptotic and “exact” versions of the function $f_1$ and will use

$$ \hat{f}_1^{(0)} = f_1 = \ln \tau_1^2 \ast s. $$

(3.8)

We can now apply the same steps to the asymptotic sums

$$ A^{(0)} = \sum_{Q=2}^{\infty} \hat{L}^{(0)}_Q \ast K_Q $$

$$ B^{(0)} = \sum_{m=1}^{\infty} \hat{L}^{(0)}_m \ast \mathcal{K}_{m+1} $$

(3.9)

as we did for their “exact” counterparts. We find that $A^{(0)}$ cannot be determined in this way (it cancels from the equations) but we obtain instead the following identity:

\textsuperscript{5}Note that $x(u) = (u - i\sqrt{4 - u^2})/2$.

\textsuperscript{6}This will be indicated by an upper index (0) on the corresponding function.
\[
0 = \sum_{Q=2}^{\infty} \left( (R_Q^{(0)} - L_Q^{(0)}) \ast (\delta K_Q + 2s_{1/2} \ast \delta K_Q) + (R_2^{(0)} - L_2^{(0)}) \ast \sigma_{1/2} \ast K_1 \right) \\
- (R_1^{(0)} - L_1^{(0)}) \ast \sigma_{1/2} \ast K_2 - \ln \tau_1^2 \ast \sigma_{1/2} \ast K_2 \\
+ 2 \sum_{m=1}^{\infty} r_m^{(0)} \ast \sigma_{1/2} \ast \delta K_{m+1} + 2r_1^{(0)} \ast s_{1/2} \ast K_1 - 2H^{(0)} \ast s_{1/2} \ast K_2.
\]

We can recognize that the asymptotic limit of the quantity defined by (2.26) appears here and by rearranging some of the terms it can be expressed as

\[
\omega^{(0)} = L_2^{(0)} \oplus s \oplus K_1 - L_1^{(0)} \oplus K_1 + \sum_{Q=1}^{\infty} L_Q^{(0)} \oplus \delta K_Q.
\]

From this result it is clear that \( \omega \), and using (2.25), also \( \Omega \) is exponentially small in the asymptotic limit, which of course must be true but is not obvious from the result (2.33).

Let us use the formula (3.10) for the calculation of the energy expression. In this case we have \( \delta K_Q = 0 \) for \( Q \geq 2 \) and we note that if we subtract it from (2.33) the \( \ln \tau_1^2 \) term cancels from the final result and we obtain

\[
t_1 = L_1 \ast K_1 + (R_2 - R_2^{(0)} + L_2^{(0)}) \ast \sigma_{1/2} \ast K_1 - (R_1 - R_1^{(0)} + L_1^{(0)}) \ast \sigma_{1/2} \ast K_2 \\
+ 2(r_1 - r_1^{(0)}) \ast s_{1/2} \ast K_1 - 2(H - H^{(0)}) \ast s_{1/2} \ast K_2.
\]

Here the difference \( R_2 - R_2^{(0)} \) decays fast enough for \( |u| \to \infty \) and this fact and the similar behavior of the other terms allow us to apply the identity (3.3) backwards, term by term, in this formula. The final expression for the energy sum (3.1) can be written as follows:

\[
\mathcal{E} = L_1 \ast \tilde{J}_1 + (R_2 - R_2^{(0)} + L_2^{(0)}) \ast \sigma_{1/2} \ast \tilde{J}_1 - (R_1 - R_1^{(0)} + L_1^{(0)}) \ast \sigma_{1/2} \ast \tilde{J}_2 \\
+ 2(r_1 - r_1^{(0)}) \ast s_{1/2} \ast \tilde{J}_1 - 2(H - H^{(0)}) \ast s_{1/2} \ast \tilde{J}_2,
\]

where we introduced the notation \( \tilde{J}_Q(u) = -\frac{1}{2\pi} \frac{d\delta_Q}{du} \).

4. The quasi-local TBA equations

In this section, for the sake of completeness, we present the quasi-local form of the \( AdS_5 \times S^5 \) mirror TBA equations. The form of the first group of the equations follows from the Y-system relations thus they are local and their form is the same as that of the simplified
version of the equations.

\[
Y_m^{(\alpha)} = t_m^{(\alpha)} \exp \left\{ \ln \left[ \frac{(1 + Y_{m+1|w}) (1 + Y_{m-1|w})}{(1 + Y_{m+1})} \right] * s \right\}, \quad m \geq 2, \tag{4.1}
\]

\[
Y_{1|w}^{(\alpha)} = t_{1|w}^{(\alpha)} \exp \left\{ \ln \left[ \frac{(1 + Y_{2|w})}{(1 + Y_2)} \right] * s + \ln \left[ \frac{1 - Y_{-}^{(\alpha)}}{1 - Y_{+}^{(\alpha)}} \right] * s \right\}, \tag{4.2}
\]

\[
Y_m^{(\alpha)} = t_m^{(\alpha)} \exp \left\{ \ln \left[ (1 + Y_{m+1|w}) (1 + Y_{m-1|w}) \right] * s \right\}, \quad m \geq 2, \tag{4.3}
\]

\[
Y_{1|w}^{(\alpha)} = t_{1|w}^{(\alpha)} \exp \left\{ \ln \left[ 1 + Y_{2|w} \right] * s + \ln \left[ \frac{1 - \frac{1}{Y_{-}^{(\alpha)}}}{1 - \frac{1}{Y_{+}^{(\alpha)}}} \right] * s \right\}, \tag{4.4}
\]

\[
Y_Q = t_Q \exp \left\{ \ln \left[ \frac{Y_{Q+1} Y_{Q-1} (1 + Y_{Q|w}) (1 + Y_{Q-1|w})}{Y_{Q+1|w} Y_{Q-1|w} (1 + Y_{Q+1}) (1 + Y_{Q-1})} \right] * s \right\}, \quad Q \geq 2. \tag{4.5}
\]

Here the source terms \( t_m^{(\alpha)} \) correspond to the singularities of the \( Y \)-functions within the physical strip (see below). The second group of equations contains the quantity \( \Omega \) which is a functional of the vector of kernels \( K_Q \). To emphasize this fact from now on \( \Omega \) in (4.3) will be denoted by \( \Omega(K_Q) \) to make this functional dependence explicit. On the right hand side of the subsequent equations \( \Omega(K_Q) \) always denotes the finite expression obtained for \( \Omega(K_Q) \) in our final formula (2.33). The quasi-local group of equations take the form:

\[
\frac{Y^{(\alpha)}}{Y^{(\alpha)}_+} = \frac{R_p B_m}{B_p R_m} \exp \left\{ -L_1 * K_{1y} - \Omega(K_{Qy}) \right\}, \tag{4.6}
\]

\[
Y^{(\alpha)}_+ Y^{(\alpha)}_- = \left( t_+^{(\alpha)} \right)^2 \exp \left\{ 2 \ln \left[ \frac{1 + Y^{(\alpha)}_{1|w}}{1 + Y_{1|w}} \right] * s + L_1 * \left[ -K_1 + 2 K^{11}_{xv} * s \right] - \Omega(K_Q) + 2 \Omega(K_{Q^1} * s) - \ln \left[ \frac{R_p^+ R^-}{R_{m^+} R_{m^-}} \right] * s \right\}, \tag{4.7}
\]

\[
\ln Y_1 = -L_1 + f_1 + 2 r_1 * s * K_{1y} + 2 \ln \left[ \frac{1 - Y_{-}}{1 - Y_{+}} \right] * s * K^{11}_{vwx} + 2 \mathcal{L}_{-} * K_{-}^{y} + 2 \mathcal{L}_{+} * K^{y1}_{+} \tag{4.8}
\]

\[
+ L_1 * K^{11}_{y(2)} + \Omega(K^{Q1}_{y(2)}) + 2 \Omega(s * K^{Q_{11}}_{vwx}),
\]

where

\[
f_1(u) = - \sum_{j=1}^{N} \ln S_{y(2)}^{11}(u, u) - 2 \left[ T^x * K^{11}_{vwx} \right](u)
\]

with

\[
T^\pm(u) = \sum_{j=1}^{N} \ln t(u - u_j \pm i \epsilon), \quad t(u) = \tanh \left( \frac{\pi q u}{4} \right). \tag{4.9}
\]
The source terms appearing in the TBA equations (4.1-4.8) contain the objects that are subject to the quantization conditions. In the \(\mathfrak{sl}(2)\) sector they can be expressed in terms of the \(\tau_m(u) = \prod_{j=1}^{N_m} t(u - \xi_{m,j})\) and \(\tilde{\tau}_m(u) = \prod_{j=1}^{\tilde{N}_m} t(u - \tilde{\xi}_{m,j})\) functions as follows:

\[
t_m|_{vw} = \tau_m \tau_{m+2}, \quad m = 1, 2, \ldots, \\
t_m|_{w} = \tilde{\tau}_m \tilde{\tau}_{m+2}, \quad m = 1, 2, \ldots, \\
t_Q = \tau_2^Q, \quad Q = 2, 3, \ldots, \\
t_1 = 1, \\
t_- = \tau_2/\tilde{\tau}_2.
\]  

Here the \(\xi_{m,j}\) and \(\tilde{\xi}_{m,j}\) are the objects to be determined by the quantization conditions:

\[
1 + Y_{m|vw}(\xi_{m+1,j}) = 0, \quad m = 1, 2, \ldots, \quad j = 1, \ldots, N_{m+1}, \\
1 + Y_{m|w}(\tilde{\xi}_{m+1,j}) = 0, \quad m = 1, 2, \ldots, \quad j = 1, \ldots, \tilde{N}_{m+1}.
\]  

The set \(\{\xi_{1,j}\}_{j=1,\ldots,N_1}\) is equal to the set of physical rapidities \(\{u_j\}_{j=1,\ldots,N}\) determined by the exact Bethe equations:

\[
1 + Y_1(u_j) = 0, \quad j = 1, \ldots, N
\]  

and the set \(\{\tilde{\xi}_{1,j}\}_{j=1,\ldots,\tilde{N}_1}\) is empty by definition. Here \(Y_1(u_j)\) denotes \(Y_1\) analytically continued to the physical sheet. For a detailed explanation see refs. [12] and [16]. We note that because of the left-right symmetry of the \(\mathfrak{sl}(2)\) sector we have omitted the wing index \((\alpha)\) in the formulae above.

In order to be able to use our result (2.33) in the equations (4.1-4.8) the vector \(\delta K_Q\) associated to the kernels \(K_Q\) appearing in the mirror TBA problem must be known. Since in the expression (2.33) for \(\Omega(K_Q)\), \(\delta K_Q\) with \(Q \geq 2\) appears only, we list the results for those \(\delta K_Q\) entering the mirror TBA equations (4.1-4.8) for \(Q \geq 2\). They are given by:

\[
\delta K_Q = 0, \quad \delta K_{Qy} = 0, \quad \delta K_{Q1}^{Q1} = 0, \quad Q \geq 2, \\
\delta(s \star K_{Qx}^{Qx,1}) = \delta Q, s \star s \star K_{y1}, \quad \delta K_{Q1}^{Q1} = -\delta Q, s, \quad Q \geq 2.
\]  

Using (4.18-4.19) and (2.33) all the \(\Omega(K_Q)\) terms can be explicitly evaluated. The substitution of these expressions into the equations (4.1-4.8) gives the quasi-local form of the mirror TBA equations.

5. Conclusion

In this paper we worked out a method that enabled us to remove all infinite sums from the mirror TBA equations and the energy formula of AdS/CFT and obtained a quasi-local formulation of the TBA problem. Quasi-locality means that in this form at most next to nearest neighbor\(^7\) \(Y\)-functions are coupled by the TBA equations.

\(^7\)with respect to the Y-system diagram of AdS/CFT
We think that the quasi-local formulation of the mirror TBA equations is an important step towards the NLIE formulation of the AdS/CFT spectral problem since using it the application of techniques worked out for the relativistically invariant models \[32, 33, 34, 35, 36\] becomes possible.

Indeed, the locality of the simplified version of the TBA equations made it possible to find a hybrid-NLIE formulation \[37\] of the AdS/CFT spectral problem, where the two SU(2) wings of the mirror TBA equations were resummed reducing remarkably the number of unknowns.

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