A multi-moment scheme for the two dimensional Maxwell’s equations.

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Abstract

We develop a numerical scheme for solving time-domain Maxwell’s equation. The method is motivated by CIP method which uses function values and its derivatives as unknown variables. The proposed scheme is developed by using the Poisson formula for the wave equation. It is fully explicit space and time integration method with higher order accuracy and CFL number being one. The bi-cubic interpolation is used for the solution profile to attain the resolution. It preserves sharp profiles very accurately without any smearing and distortion due to the exact time integration and high resolution approximation. The stability and numerical accuracy are investigated.

1 Introduction

Multi-moment methods for time dependent differential equations aim to increase the accuracy of the numerical solution, and to lower the dispersive and dissipative errors in the numerical solution. The most distinguishing characteristic of the method is that more than one moment per grid point or per cell, such as function value, its derivatives and the integral over the cell etc, are considered as unknown variables, and they are simultaneously updated by coupling the differential equation and derived differential equations for the derivatives. A Hermite polynomial is usually defined on each cell using such quantities to interpolate the numerical solution, and is used to update the solutions. Such polynomial interpolation, being defined on each cell individually, reduces the numerical stencil width. The compactness of the stencil makes it feasible to handle the boundary condition and the interface condition numerically.

Such schemes were first proposed for one dimensional hyperbolic equations by Van-Leer [9] in a framework of a finite difference method, and by Takewaki, Nishiguchi and Yabe [15] on the basis of the characteristic equation for the first order PDE. The latter method is referred to as CIP method. Their works triggered off development of multi moment methods, and it is increasingly becoming an active area of research and have even been applied to various equations. There is a vast literature developing CIP methods and the following is a partial list of papers: nonlinear hyperbolic equations [17], multi dimensional hyperbolic equations [16, 18], a multi-dimensional the Maxwell’s equations [13], a numerical simulation for solid, liquid and gas [12, 19, 20], a new mesh system applicable to non-orthogonal coordinate system [21], a variant of CIP method [2].

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CIP method uses the cubic interpolation constructed via solution values and its derivative at two end points of a cell to approximate the solution in the cell. Explicit time integration formulas for both the exact solution and its derivative of a one dimensional transport equation with the constant velocity field are coupled with the cubic interpolation to obtain an explicit time integration numerical scheme (CIP scheme). In [8] we have developed and analyzed a CIP scheme for one dimensional hyperbolic equations with variable and discontinuous coefficients.

Most of the proposed CIP methods so far, when it is applied to higher dimensional equations, relies on the Strang splitting technique: It reduces the equation of higher dimension to a sequence of simpler one dimensional equations, repeatedly applies a CIP method for the reduced one dimensional equations.

In this paper we develop a multi-moment method for two dimensional Maxwell’s equations, which does not employ the directional splitting technique at all. We apply the exact integration method in time by the Poisson formula and use the bi-cubic interpolation. We refer [3] [10] [7], [14], [1] for numerical integration methods for the Maxwell and acoustic wave equations in general. The original second order Yee scheme using the time and space staggered grid is developed in [22] and a fourth order time and space variant of Yee scheme is developed in [4, 6].

Our contributions are as follows. The bi-cubic interpolation is combined with the Poisson formula to develop a fully explicit time and space numerical method for Maxwell, acoustic wave equations as well as the second order wave equation. We analyze the von-Neumann stability of the proposed method and we establish CFL number is one for the proposed method. The one step of the method involves updating four moments at each grid point and the symbolic formulation and the symmetry of the update is used to reduce operation counts. The method offers highly efficient to attain the desired accuracy due to the relaxed CFL number limitation and accurate space resolution by the bi-cubic profile. The method is compared with the fourth order time-space Yee’s scheme in terms of numerical accuracy at nodes and required operation counts given accuracy. The numerical convergence rate test shows the method is nearly fourth order and the operation counts are comparable with those for the fourth order Yee’s scheme in conventional computing. With distributed computing implementation the method becomes much efficient. Also, the method has the build-in bi-cubic interpolation and thus provides super-grid resolutions at each cell.

If we use the method with CFL= 1, as shown in Section 4 the method preserves sharp profiles in the solution very accurately without any smearing and distortion. The stability and numerical accuracy are analyzed. Also, the method computes directly the physical quantities e.g., current and electric field gradient, very accurately. The building block of our method is the exact integration rule for the Poisson formula against the polynomials. Thus, one can use various polynomial approximations locally, including the one using the solution values only. In the paper we only implement the methods for the periodic boundary condition but it can be extended to the various boundary conditions and class of absorbing boundary conditions.

Also, we can also extend the interface treatment in [11] for piecewise cubic interpolation at the material discontinuity and develop the immersed interface method for discontinuous media.

An outline of our presentation is: in Section 2 a CIP scheme is proposed, in Section 3 the stability and error analysis is presented. Finally in Section 4 we present our numerical tests and numerical convergent rate.
2 Derivation of the multi-moment scheme

Consider the two dimensional (TE) Maxwell’s equation for magnetic field $H = (0, 0, H_z)$ and electric field $E = (E_x, E_y, 0)$:

\[
\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \frac{\partial H_z}{\partial y}, \quad \frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \frac{\partial H_z}{\partial x}, \quad \frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \tag{1}
\]

where the material coefficients $\epsilon, \mu$ are constants. Let $c = \frac{1}{\sqrt{\varepsilon \mu}}$ be the speed of light. Our method uses the fact that (1) is equivalent to second order wave equations:

\[
\frac{\partial^2}{\partial t^2} H - c^2 \Delta H_z = 0, \quad \frac{\partial^2}{\partial t^2} E_x - c^2 \Delta E_x = 0, \quad \frac{\partial^2}{\partial t^2} E_y - c^2 \Delta E_y = 0. \tag{2}
\]

under the assumption that $\text{div}(E_x, E_y) = 0$. Similarly, the two dimensional acoustic wave equation for pressure $p$ and velocity $v = (v_x, v_y)$:

\[
\rho \frac{\partial v_x}{\partial t} = \frac{\partial p}{\partial x}, \quad \rho \frac{\partial v_y}{\partial t} = \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial t} = K \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right),
\]

can be treated in exactly the same manner.

2.1 Poisson’s formula and the multi-moment scheme

Let $u(t, x_1, x_2)$ be a solution of the wave equation:

\[
\frac{\partial^2}{\partial t^2} u - c^2 \Delta u = 0, \text{ with } u(0) = g, \ u_t(0) = h.
\]

By the Poisson’s formula [5], we have

\[
u(t, x) = \frac{1}{2\pi c t} \int_{B(x,ct)} g(\xi) + \nabla g(\xi) \cdot (\xi - x) + t h(\xi) \frac{1}{((ct)^2 - |\xi - x|^2)^{\frac{1}{2}}} \, d\xi.
\]

Using change of variable $\xi - x$ to $\xi$, the solution $u(t + \Delta t, x)$ is

\[
u(t + \Delta t, x) = \frac{1}{2\pi c \Delta t} \int_{B} \frac{u(t, x + \xi) + \xi \cdot \nabla u(t, x + \xi)}{((c\Delta t)^2 - |\xi|^2)^{\frac{1}{2}}} \, d\xi d\eta + \frac{1}{2\pi c} \int_{B} \frac{\partial u(t, x + \xi)}{((c\Delta t)^2 - |\xi|^2)^{\frac{1}{2}}} \, d\xi,
\]

where $B = B(0, c\Delta)$ and

\[
L[u(t, \cdot)|B] = \frac{1}{2\pi c \Delta t} \int_{B} \frac{u(t, \xi)}{((c\Delta t)^2 - |\xi|^2)^{\frac{1}{2}}} \, d\xi,
\]

for a function $u(t, \xi)$ and $\xi = (\xi_1, \xi_2)$.

The derivatives $\partial^\alpha_x u(t, x)$, $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{N}$, also satisfy the wave equation

\[
\frac{\partial^2}{\partial t^2} (\partial^\alpha_x u) - c^2 \Delta (\partial^\alpha_x u) = 0,
\]
as long as the solution is smooth, and thus the higher order derivatives of the solution is advanced via
\[ \partial_x^\alpha u(t + \Delta t, x) = L[(1 + \xi \cdot \nabla)\partial_x^\alpha u(t, x + \cdot)|B] + \Delta tL[\partial_x^\alpha \partial_t u(t, x + \cdot)|B]. \]

We obtain the exact integration formula for the solutions and its derivatives of \( H_z(t, x) \), \( E_x(t, x) \) and \( E_y(t, x) \) at \( x = 0 \): For \( \alpha = (0, 0) \) and \( \alpha \in \mathbb{N}^2 \),
\[
\partial_x^\alpha H_z(t + \Delta t, x) = L[(1 + \xi \cdot \nabla)\partial_x^\alpha H_z(t, x + \cdot)|B] + \Delta tL[\partial_x^\alpha \partial_t H_z(t, x + \cdot)|B]
\]
\[
= L[(1 + \xi \cdot \nabla)\partial_x^\alpha H_z(t, x + \cdot)|B] + \frac{\Delta t}{\mu}L[\partial_x^\alpha(\partial_{\xi_z} E_x(t, x + \cdot) - \partial_{\xi_z} E_y(t, x + \cdot))]|B],
\]
\[
\partial_x^\alpha E_x(t + \Delta t, x) = L[(1 + \xi \cdot \nabla)\partial_x^\alpha E_x(t, x + \cdot)|B] + \frac{\Delta t}{\mu}L[\partial_x^\alpha \partial_{\xi_z} H_z(t, x + \cdot)|B],
\]
\[
\partial_x^\alpha E_y(t + \Delta t, x) = L[(1 + \xi \cdot \nabla)\partial_x^\alpha E_y(t, x + \cdot)|B] - \frac{\Delta t}{\mu}L[\partial_x^\alpha \partial_{\xi_z} H_z(t, x + \cdot)|B].
\]

(3)

Here we use \([2]\) to exchange the time derivative and spatial derivatives.

Let us define a grid of points in the \((t, x)\) space. Let \( \Delta t \) and \( \Delta x \) be positive numbers. The grid is the set of points \((t_n, x_{ij}) = (n\Delta t, i\Delta x, j\Delta x)\) for arbitrary integers \((n, i, j)\). We let \( \partial_x^\alpha H_{z,ij}^n \), \( \partial_x^\alpha E_{x,ij}^n \) and \( \partial_x^\alpha E_{y,ij}^n \) stand respectively for the approximation to the solution \( \partial_x^\alpha H_z(t_n, x_i, y_j) \), \( \partial_x^\alpha E_x(t_n, x_i, y_j) \) and \( \partial_x^\alpha E_y(t_n, x_i, y_j) \) for \( \alpha \in (0, 0) \cup \mathbb{N} \).

The basis idea of the multi-moment scheme is to define a higher order polynomials \( P(x, y) \) on each cell \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\) using grid values including spatial derivatives at four corners of the cell, and substitute them to the exact time integration formula \([3]\): We evaluate the integrals of the polynomials over the ball \( B(0, c\Delta t) \). Henceforth we assume that \( c\Delta t \leq \Delta x \), and thus the four polynomials are involved in the integration over the ball, and thus the method uses variables at 9 nearest grid points.

We can derive various multi-moment schemes on the basis of \([3]\). The resulted scheme depends on the number of unknown variables we employ at each grid and the order of interpolation polynomials; for instance, if we take the grid values, the firs-order derivatives and their second order mixed derivatives as unknown variables, we use the bi-cubic Hermite interpolation, known as Boger-Fox-Schmit element in finite element methods,
\[
\sum_{k=0}^{3} \sum_{\ell=0}^{3} c_{k,\ell} x^k y^\ell.
\]

The coefficients of the polynomial are defined by the usual interpolation condition. The resulted scheme is written in terms of the bi-cubic polynomial. Let \( H_{z,ij} \), \( E_{x,ij} \) and \( E_{y,ij} \) denote the bi-cubic polynomials defined in the cell \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\) by the interpolation condition:
\[
\partial_x^\alpha H_{z,ij}(x_{\ell m}) = \partial_x^\alpha H_{z,\ell m}^n, \quad \partial_x^\alpha E_{x,ij}(x_{\ell m}) = \partial_x^\alpha E_{x,\ell m}^n, \quad \partial_x^\alpha E_{y,ij}(x_{\ell m}) = \partial_x^\alpha E_{y,\ell m}^n,
\]
for \( \alpha \in \{(0, 0), (1, 0), (0, 1), (1, 1)\} \) and \( x_{\ell m} \in \{x_{ij}, x_{i-1,j}, x_{i-1,j-1}, x_{i,j-1}\} \). Let us number four cells surrounding a grid \( x_{ij} \) counter clockwise: \( C_1 = [x_{i,j-1}, x_{i+1,j}] \times [y_j, y_{j+1}] \), \( C_2 = [x_{i-1,j}, x_i] \times [y_j, y_{j+1}] \), \( C_3 = [x_{i-1,j}, x_i] \times [y_{j-1}, y_{j}] \) and \( C_4 = [x_{i,j-1}, x_{i+1,j}] \times [y_{j-1}, y_{j}] \). We also number the polynomial defined on each cell accordingly, i.e., \( H_{z,ij} = H_{z,1} \), etc. We let \( B_k \) stand for
\[
B_1 = B(0, c\Delta t) \cap \{x \geq 0\} \cap \{y \geq 0\}, \quad B_2 = B(0, c\Delta t) \cap \{x \leq 0\} \cap \{y \geq 0\},
\]
\[
B_3 = B(0, c\Delta t) \cap \{x \leq 0\} \cap \{y \leq 0\}, \quad B_4 = B(0, c\Delta t) \cap \{x \geq 0\} \cap \{y \leq 0\}.
\]
With these notation, we obtain the CIP scheme:

\[
\begin{aligned}
\partial_x^2 H_{z_{ij}}^{n+1} &= \sum_{k=1}^{4} L[(1 + \xi \cdot \nabla)\partial_x^2 \partial_n H_{z,k}(x_{ij} + \cdot) | B_k] \\
&+ \frac{\Delta t}{\varepsilon} \sum_{k=1}^{4} L[\partial_x^2 E_{x,k}(x_{ij} + \cdot) - \partial_x E_{y,k}(x_{ij} + \cdot)] | B_k, \tag{4}
\end{aligned}
\]

\[
\begin{aligned}
\partial_x E_{x_{ij}}^{n+1} &= \sum_{k=1}^{4} L[(1 + \xi \cdot \nabla)\partial_x E_{x,k}(x_{ij} + \cdot) | B_k] + \frac{\Delta t}{\varepsilon} \sum_{k=1}^{4} L[\partial_x E_{y,k}(x_{ij} + \cdot) | B_k],
\partial_x E_{y_{ij}}^{n+1} &= \sum_{k=1}^{4} L[(1 + \xi \cdot \nabla)\partial_x E_{y,k}(x_{ij} + \cdot) | B_k] - \frac{\Delta t}{\varepsilon} \sum_{k=1}^{4} L[\partial_x E_{x,k}(x_{ij} + \cdot) | B_k],
\end{aligned}
\]

for \( \alpha = (0,0) \) (function value update), \( \alpha = (1,0) \) (x1 derivative update), \( \alpha = (0,1) \) (x2 derivative update) and \( \alpha = (1,1) \) \((x_1, x_2 \) second order mixed derivative update). As detailed in the following sections, (4) develops the moments update at time step \( t_{n+1} \) based on the 9 nearest grid moments at time step \( t_n \), i.e., update (10).

One can reduce the number of unknowns at a grid point; for example, one uses the grid values, the first-order derivatives as unknowns. The number of unknowns to be determined at each grid point becomes 9: each component \( H_z, E_x, E_y \) has 3 unknowns at a grid point. Possible choices for the interpolation are

\[
\sum_{k=0}^{3} \sum_{\ell=0}^{k} c_{k,\ell} x^{k-\ell} y^\ell + c_{3,4} x^2 y^3 + c_{4,3} x^3 y^2, \quad \text{or} \quad \sum_{k=0}^{3} \sum_{\ell=0}^{k} c_{k,\ell} x^{k-\ell} y^\ell + c_{4,1} x^3 y + c_{1,4} x y^3.
\]

The coefficients \( c_{k,\ell} \) are determined by using the grid values, the first-order derivatives at the four corners of a cell. The second mixed derivatives being not used, the resulted schemes have less complexity than the bi-cubic Hermite polynomial based scheme, however, they produce less accurate numerical solution, and the CFL number is less than 1. As for the other choice, we consider the bi-linear interpolation:

\[
c_{0,0} + c_{1,0} x + c_{0,1} y + c_{1,1} x y.
\]

We then obtain a derivative free nine point scheme.

### 2.2 Bi-cubic interpolation and the integration

In this section, we examine the details for computing the integrals in (4) when the bi-cubic interpolation is used for the interpolation, i.e., the number of unknowns for \( H_z, E_x \) and \( E_y \) is 4 respectively at a grid point; the grid value, the first-order derivatives and the second order mixed derivative.

#### 2.2.1 Notation

Let us introduce some notation. For the numerical quantities \( f_{i,j}, \partial_x f_{i,j}, \partial_x^2 f_{i,j} \) and \( \partial_x \partial_x^2 f_{i,j} \) given at the node \( x_{ij} \), we define a 4 × 1 vector (a multi-moment vector):

\[
f_{i,j} = [f_{i,j}, \partial_x f_{i,j}, \partial_x^2 f_{i,j}, \partial_x \partial_x^2 f_{i,j}] \top \in \mathbb{R}^4,
\]

Let \( f_{i,j} \) denote a vector composed of the multi-moment vector assigned at the four corner of the cell \( C_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]:

\[
f_{i,j} = [f_{i,j}^\top, f_{i+1,j}^\top, f_{i+1,j+1}^\top, f_{i,j+1}^\top] \top \in \mathbb{R}^{16}.
\]
Denote \( \frac{x-x_{ij}}{d_{ij}} := \left( \frac{x_1-x_{1i}}{d_{1i}}, \frac{x_2-x_{2j}}{d_{2j}} \right) \) where \( d_{1i} = x_{1,i+1} - x_{1,i} \) and \( d_{2j} = x_{2,j+1} - x_{2,j} \). We construct a bi-cubic polynomial

\[
F_{i,j}(x) = \sum_{k=0}^{3} \sum_{\ell=0}^{3} q_{k,\ell} \left( \frac{x_1-x_{1i}}{d_{1i}} \right)^k \left( \frac{x_2-x_{2j}}{d_{2j}} \right)^\ell = e \left( \frac{x-x_{ij}}{d_{ij}} \right) q^{i,j},
\]

on the cell \( C_{i,j} \), where the coefficient vector \( q^{i,j} \) are ordered as

\[
q^{i,j} := (q_{0,0}, q_{0,1}, q_{0,2}, q_{0,3}, q_{1,0}, q_{1,1}, q_{1,2}, q_{1,3}, q_{2,0}, q_{2,1}, q_{2,2}, q_{2,3}, q_{3,0}, q_{3,1}, q_{3,2}, q_{3,3})^\top.
\]

and \( e(x) \) denotes \( 1 \times 16 \) row vectors

\[
e(x) = \begin{bmatrix} 1, x_1, x_1^2, x_1^3 \end{bmatrix} \otimes \begin{bmatrix} 1, x_2, x_2^2, x_2^3 \end{bmatrix},
\]

i.e., the components of \( e(x) \) are

\[
e_{\ell,m}(x_1,x_2) = x_1^\ell x_2^m,
\]

and are ordered as in \((5)\).

The coefficient \( q^{i,j} \) of the bi-cubic polynomial \( F_{i,j} \) is determined by 16 interpolation conditions at four corners \( x_{i,j}, x_{i+1,j}, x_{i,j+1} \) and \( x_{i+1,j+1} \) of the cell:

\[
\begin{align*}
F_{i,j}(x_{i,j}) &= f_{i,j}, & \partial_x F_{i,j}(x_{i,j}) &= \partial_x f_{i,j}, & \partial_{x_1} F_{i,j}(x_{i,j}) &= \partial_{x_1} f_{i,j}, & \partial_{x_2} F_{i,j}(x_{i,j}) &= \partial_{x_2} f_{i,j}, & \partial_{x_1}^2 F_{i,j}(x_{i,j}) &= \partial_{x_1}^2 f_{i,j}, & \partial_{x_2}^2 F_{i,j}(x_{i,j}) &= \partial_{x_2}^2 f_{i,j}, \\
F_{i,j}(x_{i+1,j}) &= f_{i+1,j}, & \partial_x F_{i,j}(x_{i+1,j}) &= \partial_x f_{i+1,j}, & \partial_{x_1} F_{i,j}(x_{i+1,j}) &= \partial_{x_1} f_{i+1,j}, & \partial_{x_2} F_{i,j}(x_{i+1,j}) &= \partial_{x_2} f_{i+1,j}, & \partial_{x_1}^2 F_{i,j}(x_{i+1,j}) &= \partial_{x_1}^2 f_{i+1,j}, & \partial_{x_2}^2 F_{i,j}(x_{i+1,j}) &= \partial_{x_2}^2 f_{i+1,j}, \\
F_{i,j}(x_{i,j+1}) &= f_{i,j+1}, & \partial_x F_{i,j}(x_{i,j+1}) &= \partial_x f_{i,j+1}, & \partial_{x_1} F_{i,j}(x_{i,j+1}) &= \partial_{x_1} f_{i,j+1}, & \partial_{x_2} F_{i,j}(x_{i,j+1}) &= \partial_{x_2} f_{i,j+1}, & \partial_{x_1}^2 F_{i,j}(x_{i,j+1}) &= \partial_{x_1}^2 f_{i,j+1}, & \partial_{x_2}^2 F_{i,j}(x_{i,j+1}) &= \partial_{x_2}^2 f_{i,j+1}, \\
F_{i,j}(x_{i+1,j+1}) &= f_{i+1,j+1}, & \partial_x F_{i,j}(x_{i+1,j+1}) &= \partial_x f_{i+1,j+1}, & \partial_{x_1} F_{i,j}(x_{i+1,j+1}) &= \partial_{x_1} f_{i+1,j+1}, & \partial_{x_2} F_{i,j}(x_{i+1,j+1}) &= \partial_{x_2} f_{i+1,j+1}, & \partial_{x_1}^2 F_{i,j}(x_{i+1,j+1}) &= \partial_{x_1}^2 f_{i+1,j+1}, & \partial_{x_2}^2 F_{i,j}(x_{i+1,j+1}) &= \partial_{x_2}^2 f_{i+1,j+1}.
\end{align*}
\]

We obtain then the coefficients \( q^{i,j} \) of \( F_{i,j} \):

\[
q^{i,j} = Q R_{ij} f^{i,j},
\]

where \( Q = [Q_1, Q_2, Q_3, Q_4] \) is the interpolation matrix:

\[
Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
-3 & 0 & 2 & 0 \\
2 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-2 & 0 & 0 & 1 \\
0 & 0 & -3 & 2 \\
-3 & -2 & 0 & 1 \\
9 & 6 & 4 & 0 \\
-6 & -4 & -3 & 2 \\
2 & 1 & 0 & 0 \\
-6 & -3 & -4 & 2 \\
4 & 2 & 2 & 1 \end{bmatrix},
Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix},
Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix},
Q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-9 & 0 & 0 & 0 \\
-9 & 0 & 0 & 0 \\
-9 & -6 & 3 & 2 \\
6 & 4 & -3 & -2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
6 & 3 & -2 & -1 \\
-4 & -2 & 2 & 1 \end{bmatrix}.
\]

And \( R_{ij} \) is a tensor product of the 4 by 4 identity matrix \( I \) and the diagonal matrix with the diagonal entries \( [1, \Delta x_{1,i}, \Delta x_{2,j}, \Delta x_{1,i} \Delta x_{2,j}] \):

\[
R_{ij} = I \otimes \begin{bmatrix} 1 \\
d_{1,i} \\
d_{2,j} \\
d_{1,i} d_{2,j} \end{bmatrix}.
\]
Thus we obtain the bi-cubic polynomial:

\[ F_{i,j}(x) = e \left( \frac{x - x_{ij}}{d_{ij}} \right) Q R_{ij} f^{i,j} = e \left( \frac{x - x_{ij}}{d_{ij}} \right) Q_1 R_{ij} f_{i,j} + e \left( \frac{x - x_{ij}}{d_{ij}} \right) Q_2 R_{ij} f_{i+1,j} + e \left( \frac{x - x_{ij}}{d_{ij}} \right) Q_3 R_{ij} f_{i,j+1} + e \left( \frac{x - x_{ij}}{d_{ij}} \right) Q_4 R_{ij} f_{i,j+1}. \]

Next, let us introduce some matrices for basic operations. For a cubic polynomial \( p(x) = e_0(x)a \) for \( x \in \mathbb{R} \), where \( e_0(x) = (1, x, x^2, x^3) \) and \( a = (a_0, a_1, a_2, a_3)^\top \), we have

\[
\frac{d}{dx} p(x) = e_0(x) Da, \quad \frac{dx}{dx} p(x) = e_0(x) M x Da, \quad p(ax) = e_0(x) D_a a, \quad p(x - s) = e_0(x) T_s a,
\]

where

\[
D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_s = \begin{bmatrix} 1 & 0 & -x & x^2 - x^3 \\ 0 & 0 & 1 & -3x \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Below, we will use the commutative properties:

\[
D D_a = \alpha D_a D, \quad M D_a = \alpha^{-1} D_a M, \quad T_s D_a = D_a T_s.
\]

### 2.2.2 Computation of \( L \)

Now we express the integral in (4) in terms of the grid values. We compute the integrals

\[
L(AF_k(x_{i,j} + \cdot)|B_k),
\]

for \( k = 1, 2, 3, 4 \), where \( A \) denotes one of the operators

\[
(1 + \xi \cdot \nabla) \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}, \quad \partial_{\xi_1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}, \quad \partial_{\xi_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2},
\]

for \( \alpha_1 = 0, 1, \alpha_2 = 0, 1 \), and we renumber the polynomial; \( F_1 = F_{i,j}, F_2 = F_{i-1,j}, F_3 = F_{i-1,j-1}, \) and \( F_4 = F_{i,j-1} \). Let us denote the corresponding matrix representation for \( A \) by \( T_A \), i.e.,

\[
T_{(1 + \xi \cdot \nabla)} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} = (I \otimes I + MD \otimes I + I \otimes MD)(D^{\alpha_1} \otimes D^{\alpha_2}),
\]

\[
T_{\partial_{\xi_1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}} = (D \otimes I)(D^{\alpha_1} \otimes D^{\alpha_2}), \quad T_{\partial_{\xi_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}} = (I \otimes D)(D^{\alpha_1} \otimes D^{\alpha_2}).
\]

For the compact expression, we also number the multi-moment vector accordingly, i.e., \( f^1 = f^i_{-j}, f^2 = f^{i-1}j, f^3 = f^{i-1,j-1}, \) and \( f^4 = f^i_{j-1} \). Then from the representations

\[
F_1(x_{i,j} + \xi) = e \left( \frac{\xi + x_{i,j} - x_{i,j}}{d_{ij}} \right) Q R_{ij} f^1 = e(\xi) \left( D^{\frac{1}{\eta_{1,i}}} \otimes D^{\frac{1}{\eta_{2,j}}} \right) Q R_{ij} f^1,
\]

\[
F_2(x_{i,j} + \xi) = e(\xi) \left( D^{\frac{1}{\eta_{1,i-1}}} \otimes D^{\frac{1}{\eta_{2,j}}} \right) (T_{-1} \otimes I) Q R_{i-1,j} f^2,
\]

\[
F_3(x_{i,j} + \xi) = e(\xi) \left( D^{\frac{1}{\eta_{1,i-1}}} \otimes D^{\frac{1}{\eta_{2,j-1}}} \right) (T_{-1} \otimes T_{-1}) Q R_{i-1,j-1} f^3,
\]

\[
F_4(x_{i,j} + \xi) = e(\xi) \left( D^{\frac{1}{\eta_{1,i}}} \otimes D^{\frac{1}{\eta_{2,j-1}}} \right) (I \otimes T_{-1}) Q R_{i,j-1} f^4,
\]

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and hence we have
\[ L(AF_1(x_{i,j} + \cdot)|B_1) = L(e|B_1) T_A \left( D_{\frac{1}{x_{i,j}}} \otimes D_{\frac{1}{x_{j,i}}} \right) QR_{i,j} f^1, \]
\[ L(AF_2(x_{i,j} + \cdot)|B_2) = L(e|B_2) T_A \left( D_{\frac{1}{x_{i-1,j}}} \otimes D_{\frac{1}{x_{j,i-1}}} \right) (T_{-1} \otimes I)QR_{i-1,j} f^2 \]
\[ L(AF_3(x_{i,j} + \cdot)|B_3) = L(e|B_3) T_A \left( D_{\frac{1}{x_{i,j-1}}} \otimes D_{\frac{1}{x_{j-1,i}}} \right) (T_{-1} \otimes T_{-1})QR_{i,j-1} f^3 \]
\[ L(AF_4(x_{i,j} + \cdot)|B_4) = L(e|B_4) T_A \left( D_{\frac{1}{x_{i-1,j}}} \otimes D_{\frac{1}{x_{j,i-1}}} \right) (I \otimes T_{-1})QR_{i,j-1} f^4. \]

Thus the computation of the integrals is reduced to the computations of
\[ L(e_{\ell,m}|B_1) = \frac{1}{2\pi e\Delta t} \int_{B_1} \frac{e_{\ell,m}(\xi)}{(c\Delta t)^2 - |\xi|^2} \, d\xi. \]

Using change of variable \( \xi = (\xi_1, \xi_2) = c\Delta t (\cos \theta, \sin \theta) \), the integrals are evaluated as the function of \( d_c := c\Delta t \):

\[
\begin{align*}
\mathbf{d}_{1,c} &:= L(e|B_1) = \left[ \frac{1}{4} \frac{d_c}{8} \frac{d_c^2}{12} \frac{d_c^3}{16} \frac{d_c^4}{8} \frac{d_c^5}{6\pi} \frac{d_c^6}{32} \frac{d_c^7}{15\pi} \frac{d_c^8}{32} \frac{d_c^9}{60} \frac{d_c^{10}}{96} \frac{d_c^{11}}{16} \frac{d_c^{12}}{15\pi} \frac{d_c^{13}}{96} \frac{d_c^{14}}{105\pi} \right], \\
\mathbf{d}_{2,c} &:= L(e|B_2) = \left[ \frac{1}{4} \frac{d_c}{8} \frac{d_c^2}{12} \frac{d_c^3}{16} \frac{d_c^4}{8} \frac{d_c^5}{6\pi} \frac{d_c^6}{32} \frac{d_c^7}{15\pi} \frac{d_c^8}{32} \frac{d_c^9}{60} \frac{d_c^{10}}{96} \frac{d_c^{11}}{16} \frac{d_c^{12}}{15\pi} \frac{d_c^{13}}{96} \frac{d_c^{14}}{105\pi} \right], \\
\mathbf{d}_{3,c} &:= L(e|B_3) = \left[ \frac{1}{4} \frac{d_c}{8} \frac{d_c^2}{12} \frac{d_c^3}{16} \frac{d_c^4}{8} \frac{d_c^5}{6\pi} \frac{d_c^6}{32} \frac{d_c^7}{15\pi} \frac{d_c^8}{32} \frac{d_c^9}{60} \frac{d_c^{10}}{96} \frac{d_c^{11}}{16} \frac{d_c^{12}}{15\pi} \frac{d_c^{13}}{96} \frac{d_c^{14}}{105\pi} \right], \\
\mathbf{d}_{4,c} &:= L(e|B_4) = \left[ \frac{1}{4} \frac{d_c}{8} \frac{d_c^2}{12} \frac{d_c^3}{16} \frac{d_c^4}{8} \frac{d_c^5}{6\pi} \frac{d_c^6}{32} \frac{d_c^7}{15\pi} \frac{d_c^8}{32} \frac{d_c^9}{60} \frac{d_c^{10}}{96} \frac{d_c^{11}}{16} \frac{d_c^{12}}{15\pi} \frac{d_c^{13}}{96} \frac{d_c^{14}}{105\pi} \right].
\end{align*}
\]

Using these vectors, the integrals are expressed in terms of the vectors and the matrices, i.e., we obtain
\[ L(AF_1(x_{i,j} + \cdot)|B_1) = \mathbf{d}_{1,c} T_A \left( D_{\frac{1}{x_{i,j}}} \otimes D_{\frac{1}{x_{j,i}}} \right) QR_{i,j} f^1, \]
\[ L(AF_2(x_{i,j} + \cdot)|B_2) = \mathbf{d}_{2,c} T_A \left( D_{\frac{1}{x_{i-1,j}}} \otimes D_{\frac{1}{x_{j,i-1}}} \right) (T_{-1} \otimes I)QR_{i-1,j} f^2 \]
\[ L(AF_3(x_{i,j} + \cdot)|B_3) = \mathbf{d}_{3,c} T_A \left( D_{\frac{1}{x_{i,j-1}}} \otimes D_{\frac{1}{x_{j-1,i}}} \right) (T_{-1} \otimes T_{-1})QR_{i,j-1} f^3 \]
\[ L(AF_4(x_{i,j} + \cdot)|B_4) = \mathbf{d}_{4,c} T_A \left( D_{\frac{1}{x_{i-1,j}}} \otimes D_{\frac{1}{x_{j,i-1}}} \right) (I \otimes T_{-1})QR_{i,j-1} f^4. \]

From (9), we have for \( d_1, d_2 > 0 \)
\[ T_{(1+\xi, \nu)} \partial_{\xi_1} \partial_{\xi_2} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) = \frac{1}{d_1^{\xi_1} d_2^{\xi_2}} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) T_{(1+\xi, \nu)} \partial_{\xi_1} \partial_{\xi_2} \]
\[ T_{\partial_{\xi_1}} \partial_{\xi_2} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) = \frac{1}{d_1^{\xi_1} d_2^{\xi_2}} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) T_{\partial_{\xi_1}} \partial_{\xi_2} \]
\[ T_{\partial_{\xi_2}} \partial_{\xi_1} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) = \frac{1}{d_1^{\xi_1} d_2^{\xi_2}} (D_{d_1^{-1}} \otimes D_{d_2^{-1}}) T_{\partial_{\xi_2}} \partial_{\xi_1}. \]

We compute
\[ \mathbf{d}_{1,c} \left( D_{\frac{1}{x_{i,j}}} \otimes D_{\frac{1}{x_{j,i}}} \right) = \left[ \frac{1}{4} \frac{\mu}{8} \frac{\mu^2}{12} \frac{\mu^3}{16} \frac{\lambda}{8} \frac{\lambda \mu}{6\pi} \frac{\lambda \mu^2}{32} \frac{\lambda \mu^3}{15\pi} \frac{\lambda \mu^4}{12} \frac{\lambda \mu^5}{32} \right]. \]
where $\lambda = \frac{c\Delta t}{d_{1,i}}$ and $\mu = \frac{c\Delta t}{d_{2,j}}$. We denote the right hand side by $\Lambda(\lambda, \mu)$, i.e.,

$$
\Lambda(\lambda, \mu) = \begin{bmatrix}
1 & \mu & \mu^2 & \mu^3 & \lambda & \lambda \mu & \lambda^2 \mu & \lambda^2 \mu^2 & \lambda^3 \mu & \lambda^3 \mu^2 & 2\lambda^3 \mu^3
\end{bmatrix}.
$$

Let $\lambda_1 = \lambda_4 = \frac{c\Delta t}{d_{1,i}}$, $\lambda_2 = \lambda_3 = \frac{c\Delta t}{d_{1,i-1}}$, $\mu_1 = \mu_2 = \frac{c\Delta t}{d_{2,j}}$ and $\mu_3 = \mu_4 = \frac{c\Delta t}{d_{2,j-1}}$. Then

$$
d_{1,c} \left( \begin{bmatrix} D_{\pi,1} & D_{\pi,2} \end{bmatrix} \right) = \Lambda(\lambda_1, \mu_1), \\
d_{2,c} \left( \begin{bmatrix} D_{\pi,1} & D_{\pi,2} \end{bmatrix} \right) = \Lambda(\lambda_2, \mu_2), \\
d_{3,c} \left( \begin{bmatrix} D_{\pi,1} & D_{\pi,2} \end{bmatrix} \right) = \Lambda(\lambda_3, -\mu_3), \\
d_{4,c} \left( \begin{bmatrix} D_{\pi,1} & D_{\pi,2} \end{bmatrix} \right) = \Lambda(\lambda_4, -\mu_4)
$$

Therefore we have for $A = (1 + \xi \cdot \nabla)\frac{\partial^3 \alpha_1}{\partial \xi_1 \partial \xi_2}$,

$$
\sum_{k=1}^{4} L(AF_k(x_{i,j} + \cdot) B_k) = \frac{\Lambda(\lambda_1, \mu_1) T_A Q R_{i,j} f^1 + \Lambda(\lambda_2, \mu_2) T_A (T_{-1} \otimes I) Q R_{i,-1,j} f^2}{d_{1,i-1}^2 d_{2,j}^2} + \frac{\Lambda(\lambda_3, \mu_3) T_A (T_{-1} \otimes T_{-1}) Q R_{i,-1,j-1} f^3 + \Lambda(\lambda_4, \mu_4) T_A (I \otimes T_{-1}) Q R_{i,-1,j-1} f^4}{d_{1,i-1}^2 d_{2,j-1}^2}.
$$

Let us define 16 by 16 matrices $A_1$, $A_2$, $A_3$, $A_4$:

$$
A_1 = \begin{bmatrix}
\Lambda(\lambda_1, \mu_1) T_A Q R_{i,j} \\
\Lambda(\lambda_2, \mu_2) T_A (T_{-1} \otimes I) Q R_{i,-1,j} \\
\Lambda(\lambda_3, \mu_3) T_A (T_{-1} \otimes T_{-1}) Q R_{i,-1,j-1} \\
\Lambda(\lambda_4, \mu_4) T_A (I \otimes T_{-1}) Q R_{i,-1,j-1}
\end{bmatrix},
A_2 = \begin{bmatrix}
\Lambda(\lambda_1, \mu_1) T_A Q R_{i,j} \\
\Lambda(\lambda_2, \mu_2) T_A (T_{-1} \otimes I) Q R_{i,-1,j} \\
\Lambda(\lambda_3, \mu_3) T_A (T_{-1} \otimes T_{-1}) Q R_{i,-1,j-1} \\
\Lambda(\lambda_4, \mu_4) T_A (I \otimes T_{-1}) Q R_{i,-1,j-1}
\end{bmatrix},
A_3 = \begin{bmatrix}
\Lambda(\lambda_1, \mu_1) T_A Q R_{i,j} \\
\Lambda(\lambda_2, \mu_2) T_A (T_{-1} \otimes I) Q R_{i,-1,j} \\
\Lambda(\lambda_3, \mu_3) T_A (T_{-1} \otimes T_{-1}) Q R_{i,-1,j-1} \\
\Lambda(\lambda_4, \mu_4) T_A (I \otimes T_{-1}) Q R_{i,-1,j-1}
\end{bmatrix},
A_4 = \begin{bmatrix}
\Lambda(\lambda_1, \mu_1) T_A Q R_{i,j} \\
\Lambda(\lambda_2, \mu_2) T_A (T_{-1} \otimes I) Q R_{i,-1,j} \\
\Lambda(\lambda_3, \mu_3) T_A (T_{-1} \otimes T_{-1}) Q R_{i,-1,j-1} \\
\Lambda(\lambda_4, \mu_4) T_A (I \otimes T_{-1}) Q R_{i,-1,j-1}
\end{bmatrix},
$$

and let us define 4 by 4 matrices $a_{k}^{ij}$, $k = 1, \ldots, 9$:

$$
t_1 = 1 : 4, \\ t_2 = 5 : 8, \\ t_3 = 9 : 12, \\ t_4 = 13 : 16,
$$

$$
a_{1}^{ij} = A_3(:, t_1), \\ a_{2}^{ij} = A_3(:, t_2) + A_4(:, t_1), \\ a_{3}^{ij} = A_4(:, t_2), \\ a_{4}^{ij} = A_3(:, t_4) + A_2(:, t_1), \\ a_{5}^{ij} = A_1(:, t_1) + A_2(:, t_2) + A_3(:, t_3) + A_4(:, t_4), \\ a_{6}^{ij} = A_1(:, t_2) + A_4(:, t_3), \\ a_{7}^{ij} = A_2(:, t_4), \\ a_{8}^{ij} = A_1(:, t_4) + A_2(:, t_3), \\ a_{9}^{ij} = A_1(:, t_3).
$$

Here we use the Matlab notation to express a sub matrix of a given matrix, for instance, for 4 by 16 matrix $A$ and the index $t_1 = 1 : 4$, we denote by $A(:, t_1)$ the sub matrix $[A_{i,j}], i = 1, \ldots, 4, j = 1, \ldots, 4$ of the matrix $A$.

Let us label the multi-moment vectors at nearest nine grid points:

$$
f_1 = f_{i-1,j-1}, f_2 = f_{i,j-1}, f_3 = f_{i+1,j-1}, f_4 = f_{i-1,j}, f_5 = f_{i,j}, f_6 = f_{i+1,j}, f_7 = f_{i-1,j+1}, f_8 = f_{i,j+1}, f_9 = f_{i+1,j+1},
$$

Then the left hand side of (7) for $(\alpha_1, \alpha_2) = (0, 0), (1, 0), (0, 1), (1, 1)$ are expressed using the
nine vectors and the matrices $a_k^{i,j}$:

$$
\begin{aligned}
&= \sum_{k=1}^{4} L((1 + \xi \cdot \nabla) F_k(x_{i,j} + \cdot)|B_k) \\
&= \sum_{k=1}^{4} L((1 + \xi \cdot \nabla) \partial_{\xi_1} F_k(x_{i,j} + \cdot)|B_k) \\
&= \sum_{k=1}^{4} L((1 + \xi \cdot \nabla) \partial_{\xi_2} F_k(x_{i,j} + \cdot)|B_k) \\
&= \sum_{k=1}^{4} L((1 + \xi \cdot \nabla) \partial_{\xi_1 \xi_2} F_k(x_{i,j} + \cdot)|B_k)
\end{aligned}
$$

As for $A = \partial_{\xi_1} \partial_{\xi_1} \partial_{\xi_2}$ and $A = \partial_{\xi_2} \partial_{\xi_1} \partial_{\xi_2}$, we obtain

$$
c\Delta t \sum_{k=1}^{4} L(AF_k(x_{i,j} + \cdot)|B_k) = \lambda_1 \Lambda(\lambda_1, \mu_1) T_A Q R_i \varphi_1 + \frac{\lambda_2 \Lambda(\lambda_2, \mu_2)}{d_{1,i-1}^2 d_{2,j}^2} T_A(T_{-1} \otimes I) Q R_{i-1,j} f^2 + \frac{\lambda_3 \Lambda(\lambda_3, \mu_3)}{d_{1,i-1}^2 d_{2,j}^2} T_A(T_{-1} \otimes T_{-1}) Q R_{i-1,j-1} f^3 + \frac{\mu_4 \Lambda(\lambda_4, \mu_4)}{d_{1,i}^2 d_{2,j-1}^2} T_A(I \otimes T_{-1}) Q R_{i,j-1} f^4,
$$

and

$$
c\Delta t \sum_{k=1}^{4} L(AF_k(x_{i,j} + \cdot)|B_k) = \mu_1 \Lambda(\lambda_1, \mu_1) T_A Q R_i \varphi_1 + \frac{\mu_2 \Lambda(\lambda_2, \mu_2)}{d_{1,i-1}^2 d_{2,j}^2} T_A(T_{-1} \otimes I) Q R_{i-1,j} f^2 + \frac{\mu_3 \Lambda(\lambda_3, \mu_3)}{d_{1,i-1}^2 d_{2,j}^2} T_A(T_{-1} \otimes T_{-1}) Q R_{i-1,j-1} f^3 + \frac{\mu_4 \Lambda(\lambda_4, \mu_4)}{d_{1,i}^2 d_{2,j-1}^2} T_A(I \otimes T_{-1}) Q R_{i,j-1} f^4.
$$

Similarly, one has

$$
\begin{aligned}
&c\Delta t \sum_{k=1}^{4} L(\partial_{\xi_1} F_k(x_{i,j} + \cdot)|B_k) = \sum_{k=1}^{9} b_k^{i,j} f_k, \\
&c\Delta t \sum_{k=1}^{4} L(\partial_{\xi_2} F_k(x_{i,j} + \cdot)|B_k) = \sum_{k=1}^{9} c_k^{i,j} f_k.
\end{aligned}
$$

with 4 by 4 matrices $b_k^{i,j}, c_k^{i,j}, k = 1, \ldots, 9$ which are defined in the same manner.

### 2.3 The multi-moment scheme

Suppose the numerical approximations to the exact solutions $H^2_{x}(x_{i,j})$ and its first derivatives $\partial_{x_1} H^2_{x}(x_{i,j})$ and $\partial_{x_2} H^2_{x}(x_{i,j})$, and the second derivative $\partial_{x_1 x_2}^2 H^2_{x}(x_{i,j})$ are known at all grid
points \(x_{i,j} = (x_i, y_j)\) at time step \(t_n\), which we denote by

\[
h_{ij}^n, \quad \partial_x h_{ij}^n, \quad \partial_y h_{ij}^n, \quad \partial_{x\cdot y} h_{ij}^n.
\]

Similarly, for numerical approximations to \(E_x\) and \(E_y\) and the derivatives, we use symbols \(e_x^n_{x,i,j}\) and \(e_y^n_{y,i,j}\). Let \(h_{ij}^n, e_x^n_{x,i,j}\) and \(e_y^n_{y,i,j}\) denote the multi moment vectors at grid:

\[
\begin{align*}
\mathbf{h}_{ij}^n &= [h_{ij}^n, \partial_x h_{ij}^n, \partial_y h_{ij}^n, \partial_{x\cdot y} h_{ij}^n]^\top, \\
\mathbf{e}_{x_{ij}}^n &= [e_{x_{ij}}^n, \partial_x e_{x_{ij}}^n, \partial_y e_{x_{ij}}^n, \partial_{x\cdot y} e_{x_{ij}}^n]^\top, \\
\mathbf{e}_{y_{ij}}^n &= [e_{y_{ij}}^n, \partial_x e_{y_{ij}}^n, \partial_y e_{y_{ij}}^n, \partial_{x\cdot y} e_{y_{ij}}^n]^\top.
\end{align*}
\]

We number the numerical solutions in the way as above. Based on the exact integration formula (3), and (8), (9), we arrive at the multi-moment scheme for the Maxwell’s equations:

\[
\begin{align*}
\mathbf{h}_{5}^{n+1} &= \sum_{k=1}^{9} a_k^{i,j} \mathbf{h}_k^n + \frac{1}{c\mu} a_k^{i,j} \mathbf{e}_x_k^n - \frac{1}{c\mu} h_{i,j}^n \mathbf{e}_y_k^n, \\
\mathbf{e}_{x5}^{n+1} &= \sum_{k=1}^{9} \frac{1}{c\epsilon} a_k^{i,j} \mathbf{h}_k^n + a_k^{i,j} \mathbf{e}_x_k^n \\
\mathbf{e}_{y5}^{n+1} &= \sum_{k=1}^{9} -\frac{1}{c\epsilon} a_k^{i,j} \mathbf{h}_k^n + a_k^{i,j} \mathbf{e}_y_k^n.
\end{align*}
\] (10)

Thus, the method uses 9 nearest neighbor points for 12 components (4 moments) for \(H_z, E_x,\) and \(E_y.\) Each \(4\times36\) matrices \([a_k^{i,j}, \ldots, a_k^{i,j}]\), \([b_k^{i,j}, \ldots, b_k^{i,j}]\) and \([c_k^{i,j}, \ldots, c_k^{i,j}]\) has 100 nonzero entries. Hence the total cost for the update (10) amounts to 700.

We would like to emphasis that the multi-moment scheme provides with the first and the second derivatives as we as the function value at each grid. When displaying the numerical solution, one can construct the bi-cubic polynomial and can evaluate any spatial point with using the interpolation.

### 3 Stability

We analyze the stability of the multi-moment scheme. For simplicity, we assume that the grid length is uniform, i.e.,

\[
\Delta x := x_i - x_{i-1} = y_j - y_{j-1}
\]

for all \(i,j\). In this case, the 4 by 4 matrices \(a_k^{i,j}, b_k^{i,j}\) and \(c_k^{i,j}\) in (10) remain the same throughout of \((i,j),\) and thus we omit the subscript. Suppose that \(\mathbf{f} = \{f_{\ell,j}\}_{\ell,j}\) is \(N\) periodic with respect to \(\ell, j,\) i.e.,

\[
f_{0,j} = f_{N,j}, \quad f_{\ell,0} = f_{\ell,N},
\]

for all \(0 \leq \ell, j \leq M.\) Let us consider the discrete Fourier transform:

\[
\hat{\mathbf{f}}[k] := \sum_{i=0}^{N} \sum_{j=0}^{N} f_{i,j} e^{-2\pi ik \cdot x_{i,j}} = \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ f_{i,j}, \partial_x f_{i,j}, \partial_y f_{i,j}, \partial_{x\cdot y} f_{i,j} \right]^\top e^{-2\pi ik \cdot x_{i,j}},
\]

where \(k \cdot x_{i,j} = (k_1, k_2) \cdot (x_i, y_j) = k_1 x_i + k_2 y_j.\) The discrete Fourier transform of the sequence

\[
\mathbf{F}_{i,j} := a_1 f_{i-1,j-1} + a_2 f_{i,j-1} + a_3 f_{i+1,j-1} + a_4 f_{i-1,j} + a_5 f_{i,j} + a_6 f_{i+1,j} + a_7 f_{i-1,j+1} + a_8 f_{i,j+1} + a_9 f_{i+1,j+1}
\]

becomes:

\[
\sum_{\ell=0}^{N} \sum_{j=0}^{N} \mathbf{F}_{\ell,j} e^{-2\pi ik \cdot x_{\ell,j}} = g(\{a_k\}, 2\pi k_1 \Delta x, 2\pi k_2 \Delta x) \hat{\mathbf{f}}[k],
\] (11)
where \( g(\{a_k\}, 2\pi k_1 \Delta x, 2\pi k_2 \Delta x) \) is a 4 \times 4 matrix depending on \( \lambda, k_1 \Delta x \) and \( k_2 \Delta x \):

\[
g(\{a_k\}, 2\pi k_1 \Delta x, 2\pi k_2 \Delta x) = e^{2\pi i(-k_1-k_2)}\Delta x a_1 + e^{-2\pi i k_2 \Delta x} a_2 + e^{2\pi i(k_1-k_2)} \Delta x a_3 + e^{-2\pi i k_1 \Delta x} a_4 + a_5 + e^{2\pi i k_1 \Delta x} a_6 + e^{2\pi i(k_1+k_2)} \Delta x a_7 + e^{2\pi i k_2 \Delta x} a_8 + e^{2\pi i(k_1+k_2)} \Delta x a_9.
\]

Now let us assume \( H_x, E_x, E_y \) is periodic. Let \( h^n = \{h^n_{i,j}\}_{i,j}, e^n_x = \{e^n_{x,i,j}\}_{i,j} \) and \( e^n_y = \{e^n_{y,i,j}\}_{i,j} \). From (10) and (11), the amplification factor \( \Phi(\lambda, 2\pi k_1 \Delta x, 2\pi k_2 \Delta x) \) (12 by 12 matrix) of the proposed scheme is given by

\[
\Phi(\lambda, \theta_1, \theta_2) = \begin{bmatrix}
ge(\{a_k\}, \theta_1, \theta_2) & \frac{1}{c_e} g(\{c_k\}, \theta_1, \theta_2) & -\frac{1}{c_p} g(\{b_k\}, \theta_1, \theta_2) \\
\frac{1}{c_e} g(\{c_k\}, \theta_1, \theta_2) & g(\{a_k\}, \theta_1, \theta_2) & 0 \\
-\frac{1}{c_p} g(\{b_k\}, \theta_1, \theta_2) & 0 & g(\{a_k\}, \theta_1, \theta_2)
\end{bmatrix},
\]

where \( \theta_1 = 2\pi k_1 \Delta x, \theta_2 = 2\pi k_2 \Delta x \), i.e., \( \Phi(\lambda, 2\pi k_1 \Delta x, 2\pi k_2 \Delta x) \) maps \( \hat{h}^n, \hat{e}_x^n, \hat{e}_y^n \) to the next step \( h^{n+1}, e_{x,n+1}, e_{y,n+1} \).

Let \( \rho(\lambda, \theta_1, \theta_2) \) denote the set of eigenvalues of \( \Phi(\lambda, \theta_1, \theta_2) \) (9 eigenvalues). Figure 4 shows the maximum absolute value of the eigenvalues, \( \max\{|\rho(\lambda, \theta_1, \theta_2)|\} \), against \( |\theta_1, \theta_2| \in [-\pi, 0] \times [-\pi, 0] \) for \( \lambda = 0.2, \lambda = 0.5 \) and \( \lambda = 1 \), respectively. Numerically we find that all eigenvalues have the magnitude equal to or less than 1 for arbitrary \( (\theta_1, \theta_2) \). The magnitude is close to 1 in a wide range of \( |\theta_1, \theta_2| \), which indicates that the numerical scheme is less dissipative.

### 4 Numerical test

In this section, we show the numerical performance of the multi-moment scheme through some numerical tests.

**Example 1. Plane waves.**

In this example, we compute the numerical solutions for plane waves. We compare the numerical solutions with those produced by the fourth order in time and space FDTD (Yee’s scheme). The Yee’s scheme computes E field and H field at different time level, and thus one must provide the initial condition at time \( t = -\frac{N}{4} \) for E field and \( t = 0 \) for H field to obtain an accurate numerical solution. The plane wave solution is suitable to avoid the issue with the initial condition since the exact solution is easily obtained.

Let \( f_\sigma(x, y) = \exp(-\frac{x^2 + y^2}{\sigma^2}) \) for \( \sigma > 0 \). Let \( f_\sigma \) also denote its periodic extension to \( \mathbb{R}^2 \). The Yee’s scheme computes E field and H field at different time level, and thus one must provide the initial condition at time \( t = -\frac{N}{4} \) for E field and \( t = 0 \) for H field to obtain an accurate numerical solution. The plane wave solution is suitable to avoid the issue with the initial condition since the exact solution is easily obtained.

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If we set \( L = \cos \theta \) with \( \theta = \tan^{-1} m \) for \( m \in \mathbb{N} \), they are the periodic solution of the Maxwell’s equation for \( \epsilon = \mu = 1 \) in the domain \( \Omega = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \). In this numerical test we consider the solutions

\[
H_z(t, x, y) = \sum_{m=0}^{3} f_\sigma(x \cos \theta_m - y \sin \theta_m - t),
\]

\[
E_x(t, x, y) = \sum_{m=0}^{3} H_z(t, x, y) \sin \theta_m,\quad E_y(t, x, y) = \sum_{m=0}^{3} H_z(t, x, y) \cos \theta_m,
\]
where $\theta_m = \tan^{-1} m$. We test the case $\sigma^{-2} = 500, 1500$. Figure 1 shows the initial profile of $H_z(0, x, y)$, and the four plane waves at $t = 0$, $f_\lambda(x \cos \theta_m - y \sin \theta_m)$ for $m = 0, 1, 2, 3$, with $\sigma^{-2} = 500$, in the domain. The arrow in each plot shows the direction of wave propagation.

We report the accuracy of the multi moment scheme. The time step size is fixed to be $\Delta t = \lambda \Delta x$ for each mesh size $\Delta x = N^{-1}$, $N \in \{50, 100, 150, 200\}$. We report the numerical solutions for $\lambda = 1$ and $\lambda = \frac{1}{\sqrt{2}}$. The numerical solutions at time $T = 1$ are produced by the multi moment scheme and compared to the exact solution. The number of iteration is $N$ for $\lambda = 1$, and $1.4N$ for $\lambda = \frac{1}{\sqrt{2}}$ where the numerical solution approximates the solution at time $T = \frac{1}{4\sqrt{2}} \approx 0.98995$.

The initial value for $h_{i,j}^0$ is provided by the exact solution:

$$h_{i,j}^0 = H_z(0, x_i, y_j), \quad \partial_x h_{i,j}^0 = \partial_y H_z(0, x_i, y_j), \quad \partial_y h_{i,j}^0 = \partial_x H_z(0, x_i, y_j), \quad \partial_{x,y} h_{i,j}^0 = \partial_{x,y} H_z(0, x_i, y_j).$$

Similarly, $e_{x,i,j}^0$ and $e_{y,i,j}^0$ are given by the exact solution. In a practical situation, the initial condition in function form may not be available, and we only have the function value at each grid point. In that case we employ finite difference of the initial grid function to provide the initial condition for derivatives.

For each mesh size $\frac{1}{\sqrt{2}}$, the error in the numerical solutions is measured by $\ell^\infty$ norm:

$$\epsilon_1 = \max_{i,j} \{|h_{i,j}^n - H_z(T, x_i, y_j)|, |(e_x)_{i,j}^n - E_x(T, x_i, y_j)|, |(e_y)_{i,j}^n - E_y(T, x_i, y_j)|\}.$$ 

The multi-moment scheme computes the first derivatives and the second mix derivative as numerical solutions. We report the relative error in the first derivatives:

$$\epsilon_2 = \max_{i,j,\alpha} \left\{ \frac{\partial_{\alpha} h_{i,j}^n - \partial_{\alpha} h_{i,j}^n(T, x_i, y_j)}{|\partial_{\alpha} H_z(T, x_i, y_j)|}, \frac{|(\partial_{\alpha} e_x)_{i,j}^n - \partial_{\alpha} E_x(T, x_i, y_j)|}{|\partial_{\alpha} E_x(T, x_i, y_j)|}, \frac{|(\partial_{\alpha} e_y)_{i,j}^n - \partial_{\alpha} E_y(T, x_i, y_j)|}{|\partial_{\alpha} E_y(T, x_i, y_j)|} \right\},$$

where $\alpha = x, y$.

For comparison, we also report the numerical solution produced by fourth-order in time and space FDTD (see [4]). In the FDTD numerical simulation, we employ the CFL number to be $\frac{1}{\sqrt{2}}$ with which the FDTD method provides the best performance, i.e., FDTD with $\text{CFL} = \frac{1}{\sqrt{2}}$ produces most accurate numerical solutions among the other CFL. In this test, the initial values for $h_{i,j}^0$, $e_{x,i,j}^{-\frac{1}{2}}$ and $e_{y,i,j}^{-\frac{1}{2}}$ are given exactly:

$$h_{i,j}^0 = H_z(0, x_i, y_j), \quad (e_x)_{i,j}^{-\frac{1}{2}} = E_x(-\frac{Nt}{2}, x_i, y_j), \quad (e_y)_{i,j}^{-\frac{1}{2}} = E_y(-\frac{Nt}{2}, x_i, y_j).$$

When the analytic solutions are not available, one must solve the Maxwell’s equations backward in time to provide an accurate initial condition for $E_x(-\frac{Nt}{2}, x_i, y_j)$ and $E_y(-\frac{Nt}{2}, x_i, y_j)$ to start the FDTD scheme, by employing another time marching method such as Runge-Kutta schemes. We compute the error in the numerical solution:

$$\epsilon_1 = \max_{i,j} \{|h_{i,j}^n - H_z(T, x_i, y_j)|, |(e_x)_{i,j}^n - E_x(T, x_i, y_j)|, |(e_y)_{i,j}^n - E_y(T, x_i, y_j)|\},$$

$$\epsilon_2 = \max_{i,j,\alpha} \left\{ \frac{|\partial_{\alpha} h_{i,j}^n - \partial_{\alpha} H_z(T, x_i, y_j)|}{|\partial_{\alpha} H_z(T, x_i, y_j)|}, \frac{|(\partial_{\alpha} e_x)_{i,j}^n - \partial_{\alpha} E_x(T, x_i, y_j)|}{|\partial_{\alpha} E_x(T, x_i, y_j)|}, \frac{|(\partial_{\alpha} e_y)_{i,j}^n - \partial_{\alpha} E_y(T, x_i, y_j)|}{|\partial_{\alpha} E_y(T, x_i, y_j)|} \right\},$$

where $\partial_{\alpha} h_{i,j}^n$, $(\partial_{\alpha} e_x)_{i,j}^n$, and $(\partial_{\alpha} e_y)_{i,j}^n$ are computed by employing third order finite difference which is used in the fourth order FDTD scheme.
Figure 3 shows the error $\epsilon_1$ and $\epsilon_2$ of the solutions generated by the multi moment method with $\lambda = 1$ and $\lambda = \frac{1}{\sqrt{2}}$, and the fourth order FDTD with $\lambda = \frac{1}{\sqrt{2}}$ for the initial profile with $\sigma^{-2} = 500$ against the grid number $N$. The order of accuracy of the numerical solutions for each method is shown in Table 1. And Figure 4 and Table 2 are numerical results when $\sigma^{-2} = 1500$.

Let us consider the total computational cost in the multi-moment scheme to obtain the numerical solution at $T = 1$. For each time step, $300N^2$ operation is required to update $H_{z,i,j}$, and $200N^2$ for each $E_x$ and $E_y$. Since the number of time integration in the multi-moment scheme with $\lambda = 1$ is $\frac{1}{\Delta t}$, the total cost (operation count) amounts to $700N^2 \times N = 700N^3$.

Let us focus on the numerical solutions by the multi-moment scheme and Yee’s scheme when $N = 50$. Figure 3 shows that the error in the numerical solution produced by the multi-moment scheme with $\lambda = 1$ is $10^{-2}$ while the error by FDTD is $10^{-1}$. So, the multi-moment scheme is 10 times more accurate than the FDTD with $\lambda = \frac{1}{\sqrt{2}}$ for this mesh size. To obtain the same accurate numerical solution by FDTD, we must take the half mesh size $1/(2N) = 1/100$. For the fourth order Yee’s scheme, the cost for the one step map is 51 and the number of time integration is $\frac{1}{\Delta t} = 2\sqrt{2}N$, thus the total cost for the numerical solution at $T = 1$ amounts to $2\sqrt{2}N \times 51(2N)^2 \sim 577N^3$.

**Example 2. Sharp profile.**

Next we solve (1) with $\epsilon = \mu = 1$. The initial condition is

$$H(0, x, y) = \begin{cases} 1, & x \in D, \\ 0, & x \in D^c, \end{cases} \quad E_x(0, x, y) = E_y(0, x, y) = 0,$$

where $D = [0.25, 0.75] \times [0.25, 0.75]$. The mesh size is 0.01. The initial condition is approximated by bi-cubic polynomial with the first and second derivatives being 0. We do not use any other techniques to approximate the initial discontinuous profile. We use our algorithm (10) with CFL$= 1$. The initial condition and the numerical solution for $H$ at time $T = 0.15$ and $T = 0.25$ are displayed in Figure 5. No oscillation is observed in the numerical solution. We also employed the fourth order FDTD with the same initial condition. Numerical oscillations were found near the sharp profile.

**Example 3. Hidden resolution**

We solve (1) with $\epsilon = \mu = 1$ with the initial condition is

$$H(0, x, y) = \exp \left( -\frac{x^2}{1000} \right), \quad E_x(0, x, y) = E_y(0, x, y) = 0.$$

The mesh size is $\frac{1}{10}$ and CFL$= 1$, and so $\Delta t = \frac{1}{30}$. We compute the numerical solutions at $t = 10\Delta t$. We denote the numerical solutions by $\{h_{z,i,j}^{10}\}_{i,j}$, $\{\partial_x h_{z,i,j}^{10}\}_{i,j}$, $\{\partial_y h_{z,i,j}^{10}\}_{i,j}$, $\{\partial_{x,y} h_{z,i,j}^{10}\}_{i,j}$. When visualizing the numerical solutions, we usually construct the bi-linear interpolation in each cell using the numerical solution at the grid. For instance, for the visualization of the solution $\{h_{z,i,j}^{10}\}_{i,j}$, we plot the bi-linear interpolation constructed by using the grid value $\{h_{z,i,j}^{10}\}_{i,j}$, and for visualizing the numerical solution $\{\partial_x h_{z,i,j}^{10}\}_{i,j}$, we plot the bi-linear interpolation constructed by the grid value $\{\partial_x h_{z,i,j}^{10}\}_{i,j}$. These two bi-linear interpolations are unrelated. In the left column in Figure 6, we plot the piece wise bi-linear interpolation for $\{h_{z,i,j}^{10}\}_{i,j}$ (top) and the one for $\{\partial_x h_{z,i,j}^{10}\}_{i,j}$ (bottom). One can observe the spiky peaks and dips in the plots.

As have been mentioned repeatedly, the multi-moment scheme produces the derivatives as well as the function value at each grid. This is equivalent to state that the multi-moment scheme
computes the bi-cubic polynomial in each cell as a numerical solution. So when plotting the numerical solution, we should use the computed bi-cubic interpolation instead of bi-linear interpolation. Let us construct the bi-cubic polynomial \( h_{z,i,j}(x, y) \) in each cell \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\) using the numerical solutions, and let \( h_z(x, y) \) denote the piece wise bi-cubic polynomial defined in the domain \([-0.5, -0.5] \times [-0.5, -0.5]\), i.e.,

\[
h_z(x, y) = h_{z,i,j}(x, y), \quad (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j].
\]

In the right column of Figure 6, we show the surface plot of the bi-cubic interpolations \( h_z(x, y) \) (top) and \( \partial_x h_z(x, y) \) (bottom). The numerical solutions are depicted as smooth functions.

**Derivative free method**

We have also implemented the method using the bi-linear interpolation at each cell \( C_{i,j} \):

\[
F(x, y) = F(0, 0)(1 - x)(1 - y) + F(\Delta x, 0)x(1 - y) + F(0, \Delta y)(1 - x)y + F(\Delta x, \Delta y)xy
\]

In this way we obtain a derivative free nine point scheme:

\[
\begin{bmatrix}
\frac{h_{i,j}^{n+1}}{(e_x)_{i,j}^{n+1}} \\
\frac{h_{i,j}^{n+1}}{(e_y)_{i,j}^{n+1}}
\end{bmatrix} = \begin{bmatrix}
\frac{L_1}{c^2} L_2 & \frac{L_1}{c^2} L_2 & \frac{L_1}{c^2} L_3 \\
L_2 & L_1 & 0 \\
-L_3 & 0 & L_1
\end{bmatrix} \begin{bmatrix}
\{h_{i,j}^n\} \\
\{(e_x)_{i,j}^n\} \\
\{(e_y)_{i,j}^n\}
\end{bmatrix},
\]

where

\[
L_1 = \begin{bmatrix}
\frac{\lambda^2}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi} \\
\frac{\lambda^2}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi} \\
\frac{\lambda^2}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi} & \frac{(\pi - 2\lambda)\lambda}{2\pi}
\end{bmatrix}, \\
L_2 = \begin{bmatrix}
\frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} \\
\frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} \\
\frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8} & \frac{\lambda}{8}
\end{bmatrix}, \\
L_3 = \begin{bmatrix}
-\frac{\lambda}{8} & 0 & \frac{\lambda}{8} & -\frac{\lambda}{4} & 0 & \frac{\lambda}{8} & -\frac{\lambda}{8} & 0 & \frac{\lambda}{8}
\end{bmatrix}.
\]

This method is also stable with \( CFL = 1 \) but is second order accurate. Our numerical tests show that if we let \( \lambda = 1 \), (CFL=1) then there is no significant dissipation but \( \lambda = 0.5 \) it has 30% dissipation at \( T = 1 \) with speed one. For oblique plane waves there is no significant phase error with \( CFL = 1 \). An advantage of this method is that it is simple to be implemented and to be extended to the three dimension case.

5 Conclusion

We developed a numerical method for solving time-domain Maxwell’s equation. It is fully explicit space and time integration method with higher order accuracy and CFL number being one. The bi-cubic interpolation is used for the solution profile to attain the resolution. It preserves sharp profiles very accurately without any smearing and distortion due to the exact time integration and high resolution approximation. The stability of the method were analyzed, and the nearly forth order accuracy were observed.

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**Figure 1:** The maximum absolute value of the eigenvalues of the amplification matrix against $(\theta_1, \theta_2)$. $\lambda = 0.2$ (left), $\lambda = 0.5$ (center), $\lambda = 1$, (right).

**Figure 2:** Initial profiles $H_z(0, x, y) = \sum_{m=0}^{3} f_\sigma(x \cos \theta_m - y \sin \theta_m)$ (left) and $f_\sigma(x \cos \theta_m - y \sin \theta_m)$ for $m = 0, 1, 2, 3$. The inverse of the variance $\sigma^{-2}$ is 500. The arrow in each plot shows the direction of wave propagation.

**Table 1:** Numerical errors $\epsilon_1$ and $\epsilon_2$ and the order of convergence. $\sigma$ in the initial profile is $\sigma^{-2} = 500$.

| Multi moment (\(\lambda = 1\)) | Multi moment (\(\lambda = \frac{1}{\sqrt{2}}\)) | FDTD (\(\lambda = \frac{1}{\sqrt{2}}\)) |
|-------------------------------|---------------------------------|---------------------------------|
| $N$                           | $\epsilon_1$                    | $\epsilon_2$                    |
| 50                            | 9.86e-3                         | 1.72e-2                         |
| 100                           | 8.93e-4                         | 1.50e-3                         |
| 150                           | 6.06e-5                         | 3.20e-4                         |
| 200                           | 1.07e-1                         | 1.30e-1                         |
| 150                           | 1.72e-2                         | 2.20e-2                         |
| 200                           | 5.45e-3                         | 6.86e-3                         |
| 200                           | 2.32e-3                         | 2.95e-3                         |
| 200                           | 1.04e-1                         | 1.29e-1                         |
| 200                           | 7.15e-3                         | 1.07e-2                         |
| 200                           | 1.37e-3                         | 2.32e-3                         |
| 200                           | 4.39e-4                         | 7.33e-4                         |
| $\sigma^{-2}$                 |                                 |                                 |
Figure 3: Numerical error $\epsilon_1$ (left) and $\epsilon_2$ (right) in the numerical solutions for the initial condition $\sigma^{-2} = 500$. The order of accuracy is shown in Table 1.

Figure 4: Numerical error $\epsilon_1$ (left) and $\epsilon_2$ (right) in the numerical solutions for the initial condition $\sigma^{-2} = 1500$. The order of accuracy is shown in Table 2.

Table 2: Numerical errors $\epsilon_1$ and $\epsilon_2$ and the order of convergence. $\sigma$ in the initial profile is $\sigma^{-2} = 1500$.

| $N$ | Multi moment ($\lambda = 1$) | Multi moment ($\lambda = \frac{1}{\sqrt{2}}$) | FDTD ($\lambda = \frac{1}{\sqrt{2}}$) |
|-----|----------------------------|---------------------------------------------|----------------------------------|
|     | $\epsilon_1$               | $\epsilon_1$                               | $\epsilon_1$                     |
| 50  | 8.80e-2                    | 1.08e-2                                    | 9.10e-4                          |
| 100 | 3.02                        | 3.41                                        | 3.79                             |
| 150 | 1.69                        | 2.52                                        | 2.72                             |
| 200 | 2.50                        | 3.80                                        | 3.86                             |
|     | $\epsilon_2$               | $\epsilon_2$                               | $\epsilon_2$                     |
| 50  | 6.33e-1                    | 5.54e-2                                    | 6.48e-3                          |
| 100 | 2.06                        | 2.47                                        | 2.80                             |
| 150 | 1.85                        | 3.47                                        | 4.03                             |
| 200 |                             |                                             |                                  |
Figure 5: The initial condition for $H$ (the first column) and the numerical solution for $H$ at time $T = 0.15$ (the second column) and $T = 0.25$ (the third column).

Figure 6: Left column: 2D surface plot of $\{h^{10}_{i,j}\}_{i,j}$ (top) and $\{\partial_x h^{10}_{i,j}\}_{i,j}$ (bottom). Right column: The bi-cubic interpolation $h(x, y)$ (top) and $\partial_x h(x, y)$ (bottom).