Depinning as a coagulation process

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Abstract – We consider a one-dimensional model that describes the depinning of an elastic string of particles in a strongly pinning, phase-disordered periodic environment under a slowly increasing force. The evolution towards depinning occurs by the triggering of avalanches in regions of activity which are at first isolated, but later grow and merge. For large system sizes the dynamically critical behavior is dominated by the coagulation of these active regions. Our analysis and numerical simulations show that the evolution of the sizes of active regions is well described by a Smoluchowski coagulation equation, allowing us to predict correlation lengths and avalanche sizes in terms of certain moments of the size distribution.

Introduction. – Chains of particles connected by springs, where each particle experiences a randomly shifted periodic potential and an external driving force, have served as phenomenological models for charge density waves (CDWs) \cite{1,2}. Of particular interest is the depinning transition, from a pinned to a sliding state, which has been called a dynamic critical phenomenon \cite{3–9}. Such phenomena arise in a variety of areas: flux lines in type-II superconductors \cite{10}; fluid invasion in porous media \cite{11}; propagation of cracks \cite{12,13}; friction and earthquakes \cite{14,15}; and plastic flows in solids, where dislocational structures depin under shear load \cite{16,17}. These systems being far from equilibrium, the mechanisms leading to critical behavior and universal features are not yet sufficiently well understood. This is due, at least in part, to a lack of simple models admitting detailed analysis.

The articles \cite{18,19} discussed a one-dimensional automaton model \cite{20} with sandpile-like update rules emerging in the strong pinning limit of a CDW system driven along a transverse axis, with the external force increased slowly compared with relaxation times. As the spatially uniform driving force is raised, segments of the CDW interface depin via the triggering of avalanches. In this way the interface evolves through a sequence of increasingly distorted static configurations until a threshold force is reached. Raising the driving force beyond this threshold causes the CDW to depin and to enter a stationary dynamic sliding regime. The dynamics of the depinned interface has been related to sandpile models \cite{21,22} and in particular to their stationary states \cite{23–25}. We are interested in the subthreshold evolution towards depinning, which is an evolutionary process and thus nonstationary in time.

While refs. \cite{18,19} give a rigorous scaling limit for the final shape of the chain at threshold, these articles provide an analytic treatment of the evolution towards depinning only in the case of a special, threshold initial condition (TIC): we take as IC the final shape of the chain at depinning when a force is applied in the (−)-direction (the negative threshold configuration), and then evolve with forcing in the (+)-direction. In the TIC case the entire evolution turns out to be governed by a stochastic record-breaking process that can be described explicitly: using the terminology developed later in this article, at any time there is only one nonsingleton active region, which is arrested from sliding only by its pinned endpoints. For TIC the scaling exponents describing the criticality of the evolution to threshold can be obtained analytically, but (due to the special choice of IC) differ from the RG-based predictions of Narayan \textit{et al.} \cite{20,26}.

Here we consider a macroscopically flat initial condition (FIC), which gives rise to evolution involving multiple, spatially separated regions of avalanche activity and
hence is expected to be more generic then the TIC case. Moreover, our simulations show that evolution under FIC yields critical behavior with scaling exponents that do match [20,26]. Our aim – characterizing analytically the intermediate stages of the evolution from FIC – is more challenging. In this article we report progress toward this goal: an unexpectedly clean connection of the nonstationary growth of the interface with coagulation phenomena, supported by extensive simulation results. This provides a new viewpoint on macroscopic features of the evolution toward depinning and sheds light on the emergence of universality in dynamical critical phenomena. Our observations are not the first that involve both avalanches and a kinetic equation [27–29], but a connection between coagulation, (10), and depinning seems to be novel.

The model. – We begin by recalling the toy model of [18,19]. Fix a large integer \( L \), the system size, and consider \( L \)-periodic vectors \( \mathbf{z} \), \( \mathbf{m} \), and \( \rho \), related as follows:

\[
z_i = \rho_i + \Delta m_i = \rho_i + m_{i-1} - 2m_i + m_{i+1}. \tag{1}
\]

Here the real-valued vector \( \rho \) represents the quenched phase disorder, the integer vector \( \mathbf{m} \), playing the role of the interface height, counts the number of potential wells through which the particles are displaced, and \( \mathbf{z} \) corresponds to a suitable rescaling of the displacements of the particles from the centers of their wells. The process of raising the external force until a particle crosses wells, and letting the system relax to a new static configuration, is equivalent [19] to applying the following avalanche algorithm:

A1) Record the critical height \( h_c = \max_i z_i \).

A2) While there exists \( i \) such that \( z_i \geq h_c \), replace

\[
m_i \rightarrow m_i + 1, \quad z_i \rightarrow z_i - 2, \quad z_{i\pm1} \rightarrow z_{i\pm1} + 1,
\]

repeating as necessary until \( \max_i z_i < h_c \).

The change to \( \mathbf{m} \) and \( \mathbf{z} \) resulting from the application of this algorithm is called an avalanche. Corresponding to the assumption that the driving force is increased on a much slower time scale than the relaxation, we view the complete execution of A2 as occurring instantaneously in time.

Step A2) above is precisely sandpile [30] toppling at critical height \( h_c \). A standard argument [31] can be adapted to show that the result of toppling is independent of the order in which the indices \( i \) are chosen in step A2). However, unlike the standard sandpile [30–32], the dynamics are deterministic and extremal [33] and, moreover, the heights \( z_i \) have fractional parts from \( \rho_i \) which persist, since the \( m_i \) take only integer values.

More significantly, the evolution also differs from standard sandpiles in that it finishes after finitely many avalanches: it can be shown [19] that there is a unique number \( z_{\text{max}}^+ \), the threshold height, such that the above algorithm terminates if and only if \( h_c > z_{\text{max}}^+ \). Observe that \( \max_i z_i \) decreases under repeated application of the algorithm. The algorithm describes the evolution of the static configurations \( \mathbf{m} \) towards depinning as follows: let \( \tau = 0, 1, 2, \ldots \) index the observed configurations after \( \tau \) complete executions of the algorithm, and define the control parameter \( X \), the reduced driving force, as

\[
X(\tau) = \max_i z_i(\tau) - z_{\text{max}}^+ \geq 0. \tag{3}
\]

When after \( \tau^+ \) avalanches \( X(\tau^+) = 0 \), we have reached threshold and \( \mathbf{m}(\tau^+) \) is the (essentially unique [19]) threshold configuration, the final static shape of the chain prior to entering the sliding regime.

We can characterize the change to the configuration due to a single avalanche as follows. Define \( \delta m = \arg \max_j z_j \), when this is unique.1 Upon termination of the algorithm, ([19], Proposition 4.3) shows that \( \mathbf{z} \) has changed by

\[
z_{i_{\text{max}}} \rightarrow z_{i_{\text{max}}} - 1, \quad z_{i_m} \rightarrow z_{i_m} - 1, \quad z_{i_l} \rightarrow z_{i_l} + 1, \quad z_{i_r} \rightarrow z_{i_r} + 1, \tag{4}
\]

where \( i_l, i_r \) are the first sites to the left and right of \( i_{\text{max}} \) satisfying \( z_l < h_c - 1 \), and \( i_m = i_r + i_l - i_{\text{max}} \) is the reflection2 of \( i_{\text{max}} \) across the midpoint of the interval \( i_l, i_r \). Above and henceforth all addition and subtraction of indices are to be understood modulo \( L \), with results in \([0, L)\).

Figure 1(a) shows the change in \( \mathbf{m} \) resulting from an avalanche. Since \( \Delta \mathbf{m} \) changes exactly as \( \mathbf{z} \) changes (see (1)), \( \mathbf{m} \) is modified by adding a non-negative trapezoidal bump with slopes \( \{0, \pm 1\} \) and corners at \( i_l, i_{\text{max}}, i_m, i_r \). We define the avalanche size \( S = \sum_j \delta m_j \) as the total number of well jumps in the avalanche; this is just the (discrete) area of the trapezoid in fig. 1(a),

\[
S = (i_{\text{max}} - i_l)(i_r - i_{\text{max}}). \tag{5}
\]

\(^1\)This always holds in the case of absolutely continuous disorder.

\(^2\)When \( i_{\text{max}} = i_m \), the change is \( z_{\text{max}} \rightarrow z_{\text{max}} - 2 \).
We refer to the interval \([i_r, i_r]\) = \([i_l, i_l+1, \ldots, i_r]\) (possibly wrapping around the periodic boundary) as an avalanche segment, and
\[\xi = i_r - i_l + 1\] (6)
furnishes a correlation length.

Consider the case of a flat initial configuration (FIC), \(m(0) = 0\), so that \(z(0) = \rho\), taking the disorder \(\rho_i\) i.i.d. uniform on \([-1, +1]\). The evolution to threshold is illustrated in fig. 2. The left panel shows the finite-size scaling behavior of the disorder-averaged avalanche size \(S\) and correlation length \(\xi\) obtained from our numerical simulations, indicating that in the limit of large \(L\), the depinning transition is critical:
\[S \sim X^{-\gamma}\text{ and } \xi \sim X^{-\nu}\text{ with } \gamma = 4, \nu = 2,\] (7)
as predicted in [20,26]. The two panels on the right illustrate details of the evolution.

Coagulation. – Starting from the FIC, the system responds to the increasing driving force, evolving through a sequence of static configurations given by repeated application of the avalanche algorithm. As evident from fig. 2 (bottom right), the avalanche activity is initially localized to several spatially separated segments of the system and these can be regarded as locally depinned. It will turn out that the size statistics of avalanches are related to the length statistics of these segments and their evolution. We therefore inductively define a \(\tau\)-parametrized family of partitions \(\Pi(\tau)\) of \([0, \ldots, L - 1]\) into active regions (ARs), i.e., regions of past avalanche activity. Each region is an interval of sites, possibly wrapping around the periodic boundary; we write \([a, b]\) = \([a, a+1, \ldots, b]\). Initially \(\Pi(0)\) consists of singleton intervals \([a, a]\). To obtain \(\Pi(\tau)\) from \(\Pi(\tau - 1)\), we take the interval \([i_l, i_r]\) corresponding to the \(\tau\)-th avalanche (see (4)), and merge into a single interval all those members of \(\Pi(\tau - 1)\) which intersect with \([i_l, i_r]\) (see footnote 3). Note that the set of endpoints of intervals in \(\Pi(\tau)\) is exactly \(\{i : m = 0\}\). The evolving partition of ARs is illustrated in the green shaded areas of fig. 2 (bottom right).

To address the concern that by defining \(\Pi(\tau)\) as above, we have imposed coagulation on the problem, i.e., the setup is contrived to yield the desired result, we emphasize the following points:

P1) We do not require the coagulation to be binary, and yet will find that binary events are macroscopically dominant over a large portion of the evolution.

P2) The relative rates of coagulation events are not evident in the setup but rather will emerge, in one of the nicest possible forms, in both the numerics and a heuristic calculation.

P3) Each AR has at most one stop site which is pinned strongly enough to arrest subsequent avalanches.

P4) The various avalanches that have occurred within the set of current ARs have not yet interacted across the boundaries of the current ARs. We see that conditionally given \(\Pi(\tau)\),
\[(z, m : i \in [a, b], \text{ where}\ [a, b] \in \Pi(\tau),\] (8)
are statistically independent, with distributions depending only on the length \(\ell = b - a + 1\) (see footnote 4).

We proceed to explain P3 in detail. Suppose we have determined \(h_c\) and completed an avalanche, resulting in the changes (4). The new configuration has \(z\) satisfying
\[h_c - 2 < z_{m} < h_c - 1 < z_i < h_c, \forall i \in [i_l, i_r] \setminus \{i_m\}.\] (9)
The first two inequalities hold because \(i_m\) either did not initiate the avalanche and received \(-1\), or \(i_m = i_{\max}\) did initiate but received \(-2\). The third holds because sites strictly between \(i_l\) and \(i_r\) were not capable of arresting the avalanche, and the sites \(i_l\) and \(i_r\) were capable but each received \(+1\) (see footnote 5). We call \(i_m\) a stop site because this site is capable of halting (one side of) a subsequent avalanche\(^3\) at critical heights above \(z_{m} + 1\). This stop site may expire, before having the opportunity to stop an avalanche, if the critical height drops below \(z_{m} + 1\) before an avalanche hits this AR. When an avalanche joins two or more ARs, \(i_l\) and \(i_r\) must land on unexpired stop sites, using them by adding \(+1\) and creating a single new stop site.

The corresponding physical picture in terms of the redistribution of elastic and pinning forces as a result of an avalanche follows from fig. 1. When a segment \([i_l, i_r]\) depins, both the elastic and the opposing pinning forces increase at the boundary, while at sites \(i_m\) and \(i_{\max}\) the elastic forces decrease and thus the pinning forces there are relieved. In particular, owing to (9), the slack that \(i_m\) receives can be sufficient to pick up additional load from a subsequent avalanche without toppling. This is not the case for the remaining sites (including \(i_{\max}\)) in \([i_l, i_r]\), which – according to (9) – will topple with the next avalanche to reach this region. Stop sites thus arise as a result of the formation of elastic forces directed opposite to the direction of external forcing. They are not an artifact of this model, since in general a change to \(\Delta m\) cannot be non-negative everywhere. A more generic situation is sketched in fig. 1(b) where stop sites are replaced by localized regions where load is relieved. Let us emphasize that by establishing (9), an avalanche conditions the response of an AR to future avalanches.

\(^3\)For example, at some time \(\tau\), consider the restrictions of \(m\) and \(z\) to two distinct ARs with identical lengths. These will have identical statistics.

\(^4\)If the distribution for \(\rho_i\) were wider, then the third inequality of (9) would require modification for \(i_l, i_r\).

\(^5\)In the context of sandpiles such sites are referred to as troughs [27] or trapping sites [34,35] and they play a similar role in controlling the size of avalanches [27,36,37].

\(^6\)In the case of equality, \(i_m = i_{\max}\), the new value is \(z_{m} = h_c - 2\), and this is a stop site.
This response is markedly different in pristine regions (PRs), i.e., regions where no avalanches have yet occurred and hence \( z_i = \rho_i \); after one side of an avalanche enters a PR, it will continue to propagate for a number of sites \( \ell \) per unit volume of clusters of size \( \ell \) in the universe \([42]\), and aggregation of algae cells \([43]\). Our purpose is to compare with an exact solution in the additive case \( \alpha(\ell, \ell') = \ell + \ell' \),

\[
f(t, \ell) = e^{-t}B(1-e^{-t}, \ell), \quad B(\lambda, \ell) = \frac{(\lambda \ell)^{t-1}e^{-\lambda \ell}}{\ell!}
\]

for \( \ell = 1, 2, \ldots [39,44] \).

Associated with a realization of the toy model and its random partitions \( \Pi(\tau) \) into ARs we have an empirical size distribution

\[
N(\tau, \ell) = \sum_{[a,b] \in \Pi(\tau)} \delta_{b-a+1, \ell},
\]

normalized so that \( L^{-1}N(0, \ell) = f(0, \ell) = \delta_{1,1} \), using Kronecker \( \delta \) notation. We give numerical evidence that \( f \) approximates a law of large numbers for \( N \). For this we take many realizations, which we synchronize in time not by the number of steps \( \tau \) but rather the reduced driving force \( X(\tau) \) defined in (3). Given a realization and \( x \in [0,1] \), define \( \tau(x) \) by

\[
X(\tau) \leq x < X(\tau - 1).
\]

Using \( R \) independent realizations indexed by \( k \), we define

\[
F_R(x, \ell) = (LR)^{-1} \sum_{k=1}^{R} N_k(\tau_k(x), \ell).
\]
in fig. 3:

E1) The probability that the next avalanche begins with a site inside an AR of length \( \ell \) is similar to \( \ell/L \).

E2) Since large ARs will initiate avalanches most often, their stop sites will tend to be used rather than expire, meaning that one side of the avalanche will tend to be stopped in the triggering AR, while the other side exits the AR.

E3) On the side where the avalanche exits the triggering AR, it is very likely to stop if it hits another macroscopically sized AR. We thus expect to join to the triggering AR some small number of tiny ARs, which is not macroscopically observable, and (probably) at most one macroscopic AR.

Combining E2) and E3), we expect that assuming binary coagulation yields a reasonable approximation when we care primarily about large ARs, as we will for the correlation length and avalanche size.

E4) The model is spatially ordered, with nearest-neighbor interactions – the dynamics are not mean-field. Nevertheless the dynamics preserve mean-field statistics. In particular, the length of the second AR in the avalanche is selected uniformly from the list of all remaining AR lengths.

To further explain E4), recall that the Smoluchowski equation arises as a law of large numbers for the Marcus-Lushnikov [46,47] stochastic coalescent as the number of clusters tends to infinity [39,40,48–50]. These models are well mixed in the sense that any pair of clusters may interact, which contrasts with the toy model where ARs can interact only consecutively with some number of neighboring ARs to their left and right. Nonetheless it is possible to remain well mixed statistically with aggregating nearest-neighbor interactions\(^8\). In the case of our toy model it can be shown that the partitions \( \Pi(\tau) \), \( \tau = 0, 1, 2, \ldots \), are exchangeable in the sense that the vector of lengths

\[
(b - a + 1 : [a, b] \in \Pi(\tau))
\]

has a distribution which is invariant under permutations. This is essentially due to the fact that a permutation of the ARs in \( \Pi(\tau) \) lifts to a permutation of the finer partition \( \Pi(\tau - 1) \), which keeps consecutive those ARs that interact in step \( \tau \). Thus, exchangeability at step \( \tau - 1 \) implies exchangeability at step \( \tau \). More intuitively, if we shuffle the arrangement of the ARs between applying avalanche steps,
this will change the evolution of a fixed initial condition but will not affect the disorder-averaged statistics.

The Smoluchowski eq. (10) emerges by assuming that E1–E4 hold. If the system is large enough so that \( N(\tau, \ell) \) is effectively self-averaging, the change \( N(\tau + 1, \ell) - N(\tau, \ell) \) is approximated by

\[
\sum_{\ell'=1}^{\ell-1} \frac{\ell' N(\tau, \ell')}{L} \frac{N(\tau, \ell' - \ell)}{M_0(\tau)} - \frac{\ell N(\tau, \ell)}{L} \frac{N(\tau, \ell)}{M_0(\tau)} \tag{17}
\]

having written

\[
M_p(\tau) = \sum_{\ell=1}^{L} \ell^p N(\tau, \ell) \tag{18}
\]

for the \( p \)-th moment of \( N \). The summation in (17) is over those sizes which sum to \( \ell \), and reflects choosing a triggering AR with probability like \( \ell'/L \) and then a second AR uniformly from those which remain. The loss terms correspond to selection as a triggering AR or as a secondary AR. Symmetrizing the summation of (17) in the variables \( \ell' \), \( \ell - \ell' \) and factoring \( LM_0(\tau) \) yields an equation matching (10), up to a change in time scale.

**Observables and moments.** – Evaluating the evolutionary towards depinning as a coagulation process, we show next how this connection can be exploited to relate the statistics of the avalanche size and correlation length to the moments of the size distribution of ARs.

In simulations we have observed that in large ARs, the locations of avalanche triggering sites and stop sites are distributed approximately uniformly and independently within the AR\textsuperscript{11}. Assuming this, and that we have a binary aggregation event, we can relate the expectations of \( S \) and \( \xi \) to the moments (18):

\[
\frac{\xi}{2} = \frac{M_2}{3 M_1} + \frac{1}{2} \frac{M_1}{M_0} \quad \text{and} \quad S = \frac{1}{12} \frac{M_3}{M_1} + \frac{1}{6} \frac{M_2}{M_0}. \tag{19}
\]

The blue curves in fig. 2 (left) plot the moment relations for \( S \) and \( \xi \) from (19) using the statistics of the AR lengths obtained from our simulations. The black curve in the main panel is obtained by evaluating the moments \( M_p \) using the exact solution (11). The agreement with simulations over the scaling regime \( XL^{1/2} > 1 \) is quite good, deteriorating close to threshold for the result based on the exact Smoluchowski solution. The main reason for this discrepancy is the finite-size effect: the toy model admits clusters only as large as the system size \( L \), whereas no such restriction exists for the Smoluchowski equation. (The transition where finite-size effects become important is visible in the video provided as supplemental material SUPPLEMENT, ARdistribution, smol.mp4). Using (11) and (19) in the scaling regime, it is readily shown that \( S = \frac{2}{3} \xi^2 \), which implies the scaling relation \( \gamma = 2\nu \); cf. (7).

**Conclusion.** – We have presented theoretical arguments and numerical evidence connecting depinning phenomena with coagulation. As our analysis of the toy model clearly shows, the dynamics of avalanches in a given portion of the system differ considerably depending on whether this portion has seen past avalanche activity, or is pristine. Avalanches occurring in pristine areas remain bounded in size and extent, while previously active areas have no such restriction, permitting the avalanche activity to grow in scale. The macroscopic evolution towards depinning is therefore governed by the coagulation dynamics of the active regions.

We finish by giving several reasons why the relationship between depinning and coagulation deserves further exploration. First, although we started with a specific microscopic model, the critical behavior of the evolution to threshold turned out to be governed by a macroscopic coagulation process, retaining only few features of the underlying model: aggregation of depinned segments (our ARs), avalanches which relieve load in a few localized interior areas (our stop sites) while increasing it at the boundaries. It is thus conceivable that models with different microscopic evolution rules, such as those of CDW type with different periodic disorder or elastic couplings, might still be governed on macroscopic length scales by a coagulation process. In particular, one would expect the moment relations (19) for the avalanche size and correlation length to hold, since they emerged under rather mild assumptions. Second, combining the Brownian scaling limit result for the threshold configuration of [19] with the observations made here may lead to a stochastic process describing the macroscopic limit of depinning in these models. Together, these two points can provide an explanation for the emergence of universal features in such transitions.

Finally, the toy model discussed in this article is sufficiently tractable that we expect further analytical results. For instance, it may be possible, given the statistics of the quenched disorder \( \rho \), to explicitly relate the various time scales \( \tau \) and \( X \) for the toy model with time \( \ell \) for the Smoluchowski equation. This would provide direct access to \( \xi \) and \( S \) in terms of \( X \), and an analytical approach to deriving (7). The disorder dependence (or lack thereof) of the scaling exponents \( \gamma, \nu \) could be examined in this way.

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