A NOTE ON COMPACT MARKOV OPERATORS

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Abstract. The analytic properties of the Markov operator associated to a random walk are common tools in the study of the behaviour and some probabilistic features related to the walk. In this paper we consider a class of Markov operators which generalizes the class of compact Markov operators and we study some probabilistic properties of the associated random walk.

1. Basic definitions

Let $(X, P)$ be an irreducible, random walk on the state space $X$ which is at most countable. We suppose that the (usually infinite) stochastic matrix $P$ describes a Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with transition probabilities $p(x,y) := \mathbb{P}[Z_{n+1} = y | Z_n = x]$ homogeneous in time. Besides we consider the $n$-step transition probabilities $\{p^{(n)}(x,y)\}_{x,y \in X}$ which represent the stochastic matrix associated to the $n$-th convolution power of $P$.

The Markov operator associated to the random walk is defined as follows

$$D(P) := \left\{ f : X \to \mathbb{R} : \sum_{y \in X} p(x,y)|f(y)| < +\infty, \forall x \in X \right\},$$

(1) $$(Pf)(x) := \sum_{y \in X} p(x,y)f(y), \quad \forall f \in D(P), \forall x \in X;$$

note that $D(P) \supseteq l^\infty(X)$ and that $P|_{l^\infty(X)}$ is a bounded linear operator from $l^\infty(X)$ into itself.

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To explore the behaviour of the random walk \((X, P)\) and its main properties we introduce the two generating functions

\[ G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)z^n, \quad F(x, y|z) = \sum_{n=0}^{\infty} f^{(n)}(x, y)z^n \]

where \(\{f^{(n)}(x, y)\}_{x,y\in X}\) are the first time return probabilities, namely

\[ f^{(n)}(x, y) = \mathbb{P}(Z_n = y, Z_i \neq y, \forall i = 1, \ldots, n-1|Z_0 = x), \quad f^{(0)}(x, y) = 0. \]

Both the generating functions must be considered inside their circle of convergence in \(\mathbb{C}\).

An irreducible random walk \((X, P)\) is called transient if and only if there exists \((\Leftrightarrow \text{ for any})\) \(x \in X\) such that \(F(x, x) < 1\) and recurrent otherwise. Among the recurrent random walks we distinguish the class of positive recurrent and null recurrent depending on whether \(\tau_x := \sum_{n=1}^{\infty} n\mathbb{P}^{(n)}(x, x) < +\infty\) for some \((\Leftrightarrow \text{ for any})\) \(x \in X\) or not.

We note that positive recurrence is a strong assumption: for instance if \((X, P)\) is the simple random walk on an infinite, locally finite, non-oriented, connected graph \((X, E)\), then it is not positive recurrent. Indeed it is easily reversible with reversibility measure given by \(m(x) := \#\{y : (x, y) \in E\}\), which is clearly infinite. According to Theorem 1.18 of [1], if an irreducible Markov chain is recurrent, then it admits a unique (up to multiplication) stationary measure and this one is finite if and only if the walk is positive recurrent. Since a reversibility measure is stationary, if the walk were positive recurrent, then \(m\) should be finite.

The importance of this class of random walks is highlighted by Theorem 1.18 of [1] (see also Theorem 3.2 of [2]).

**Remark 1.1.** We note that \((X, P)\) is positive recurrent if and only if there exists \((\Leftrightarrow \text{ for all})\) \(x \in X\),

\[
\lim_{z \to 1^-} \frac{1 - F(x, x|z)}{1 - z} < +\infty.
\]

Just take in mind that the limit always exists (finite or infinite) due to the decomposition

\[
\lim_{z \to 1^-} \frac{1 - F(x, x|z)}{1 - z} = \lim_{z \to 1^-} \frac{1 - F(x, x|1)}{1 - z} + \lim_{z \to 1^-} \frac{F(x, x|1) - F(x, x|z)}{1 - z},
\]

and, using well-known arguments, we have

\[
\sum_{n=0}^{\infty} nf^{(n)}(x, x) = \lim_{z \to 1^-} \sum_{n=0}^{\infty} nf^{(n)}(x, x)z^n = \lim_{z \to 1^-} F'(x, x|z) = \lim_{z \to 1^-} \frac{F(x, x|1) - F(x, x|z)}{1 - z};
\]
2. Compact Markov operators

In this section we want to study the behaviour of a random walk whose associated Markov operator satisfying equation (2) below. In particular we study compact Markov operators.

We recall here the characterization of a compact operator (with non-negative matrix elements) defined by equation (1) (see [2], Theorem 2.2).

**Theorem 2.1.** Let $X$ be a countable set and let $P$ be a transition operator on $X$ with non-negative elements, satisfying the condition

$$
\sup_{x \in X} \sum_{y \in X} p(x, y) < +\infty.
$$

Then $P$ is a bounded, linear operator from $l^\infty(X)$ into itself; moreover $P : l^\infty \to l^\infty$ is compact if and only if for any given $\epsilon > 0$ there exists a finite subset $A_\epsilon \subset X$ such that

$$
\sup_{x \in X} \sum_{y \in X \setminus A_\epsilon} p(x, y) < \epsilon.
$$

The next Theorem is the main result of this section: its corollary enhances Proposition 2.3 of [2].

**Theorem 2.2.** Let $(X, P)$ be an irreducible Markov chain, and suppose that there exists $\epsilon \in (0, 1)$ and a finite subset $A \subset X$ such that

$$
\sup_{x \in X} \sum_{y \in X \setminus A} p(x, y) < \epsilon;
$$

then $(X, P)$ is positive recurrent.

Before looking at the proof, we want to understand what equation (2) implies from the point of view of the walker.

Let us consider the preadjoint map $P_* : l^1(X) \to l^1(X)$ acting as

$$
P_* \nu(y) \equiv \nu P(y) := \sum_{x \in X} \nu(x)p(x, y), \quad \forall y \in X.
$$

This is a mass-preserving map, indeed $\sum_{y \in X} \nu P(y) = \sum_{x \in X} \nu(x)$; moreover $\nu \geq 0$ implies $P_* \nu \geq 0$. If $\nu$ is the probability distribution of the position of the walker at a certain time, then $P_* \nu$ is the probability distribution after one step.

In term of this evolution map, equation (2) is equivalent to the existence of a finite subset $A$ such that, given any probability distribution $\nu$, the probability distribution after one step satisfies $P_* \nu(A) \geq 1 - \epsilon$ (or, equivalently, $P_*^n \nu(A) \geq 1 - \epsilon$ for any $n \in \mathbb{N}^*$).
Since from the Law of large numbers, for any given \( x \in X \), \( \mathbb{P} \)-a.c.

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x\}}(Z_n) = \begin{cases} 
0 & \text{in the transiente or null-recurrent case} \\
\frac{1}{\tau_x} & \text{in the positive recurrent case}
\end{cases}
\]

and since in the positive recurrent case, \( \frac{1}{\tau_x} \) represents the unique stationery probability measure (see [3], Section I.7, Theorem 1), hence the walker will pass (asymptotically) at least \( 1 - \epsilon \) of its time in \( A \).

**Proof. (of Theorem 2.2).** Let \( A \) and \( \epsilon \) satisfying equation (2).

Let us note that for any given \( x \in X \) we have

\[ \sup_{x \in X} \sum_{y \in X \setminus A} p^{(n)}(x, y) < \epsilon, \]

indeed for any given \( n \in \mathbb{N}^* \)

\[ \mathbb{P}(Z_n \in A | Z_0 = x) = \sum_{z \in X} \mathbb{P}(Z_n \in A | Z_{n-1} = z) \mathbb{P}(Z_{n-1} = z | Z_0 = x) \geq 1 - \epsilon. \]

Let \( x_0 \in X \setminus A \) be fixed and rewrite the previous equation as

\[ \sum_{x \in A} p^{(n)}(x_0, x) \geq 1 - \epsilon, \quad \forall n \in \mathbb{N}^*. \]

This implies

\[ \sum_{x \in A} G(x_0, x | z) \geq (1 - \epsilon) \left( \sum_{j=1}^{\infty} z^j \right) = \frac{z(1 - \epsilon)}{1 - z}. \]

Since \( G(x, y | z) = \delta_{xy} + F(x, y | z)G(y, y | z) = \delta_{xy} + F(x, y | z)/(1 - F(y, y | z)) \), the previous inequality becomes

\[ \sum_{x \in A} \frac{F(x_0, x | z)}{1 - F(x, x | z)} \geq \frac{z(1 - \epsilon)}{1 - z}. \]

Now, taking in mind the usual position \( 1/\infty = 0 \), since \( A \) is finite,

\[ 0 < 1 - \epsilon \leq \lim_{z \to 1^-} \sum_{x \in A} \frac{(1 - z)F(x_0, x | z)}{1 - F(x, x | z)} = \sum_{x \in A} \frac{F(x_0, x | 1)}{\lim_{z \to 1^-} (1 - F(x, x | z))/(1 - z)}. \]

This easily implies the existence of \( x_1 \in A \) such that

\[ \lim_{z \to 1^-} \frac{1 - F(x_1, x_1 | z)}{1 - z} < +\infty \]

hence, by Remark \( \Box \) \( (X, P) \) is positive recurrent. \( \Box \)

**Corollary 2.3.** Let \( (X, P) \) be an irreducible Markov chain, such that \( P \) is compact; then \( (X, P) \) is positive recurrent.
We emphasize that the Markov operator associated to a positive recurrent random walk needs not to be compact as the following example shows. Take $X = \mathbb{N}$ and define the transition probabilities as follows:

$$
\begin{align*}
    p(0, 1) &= 1, \\
    p(n, n + 1) &= 1 - p, \quad \forall n \in \mathbb{N}^*, \\
    p(n + 1, n) &= p, \quad \forall n \in \mathbb{N}, \\
    0 &\quad \text{otherwise},
\end{align*}
$$

where $p \in (0, 1)$. The first time return probabilities generator function $F$ can be easily calculated for $x = y = 0$ as

$$
F(0, 0|z) = \frac{2pz^2}{1 + \sqrt{1 - 4z^2p(1 - p)}};
$$

the corresponding random walk is transient if $p \in (0, 1/2)$, null recurrent if $p = 1/2$ and positive recurrent if $p \in (1/2, 1)$, but equation (2) does not hold, hence the Markov operator is always non compact.

3. SOME ESTIMATES

Let us define the time of the first return onto the vertex $x \in X$ as $T_x := \inf\{n \geq 1 : Z_n = x\}$; in the recurrent case we have

$$
\tau_x = \mathbb{E}[T_x|Z_0 = x] = \lim_{z \to 1^-} \frac{1 - F(x, x|z)}{1 - z} \equiv \lim_{z \to 1^-} (1 - z)G(x, x|z).
$$

Moreover if equation (2) holds, we have that $F(x, y) = 1$ for any $x, y \in X$ and

$$
1 \geq \sum_{x \in A} \frac{1}{\tau_x} \geq 1 - \epsilon
$$

which implies

$$
\min_{x \in A} \tau_x \leq \frac{\text{card}(A)}{1 - \epsilon}.
$$

Besides for the first time entrance in $A$, $T_A := \inf\{n > 0 : Z_n \in A\}$, the following hold

$$
\mathbb{P}(T_A \geq n|Z_0 = x) \leq \epsilon^{n-1}, \quad \mathbb{E}[T_A|Z_0 = x] \leq \frac{1}{1 - \epsilon}.
$$

In the reversible case, it is possible to find lower bounds for the $n$-step transition probabilities $p^{(n)}(x, y)$ and their generating function. In this case the reversibility measure $m$ satisfies $m(x) \propto 1/\tau_x$. 
Lemma 3.1. Let \((X, P)\) be a reversible random walk, \(m\) a reversibility measure and \(x \in \Gamma\) a fixed vertex. If there exists \(n \in \mathbb{N}^*\) and \(A \subseteq X\) such that
\[
\sum_{y \in X \setminus A} p^{(n)}(x, y) \leq \epsilon,
\]
then
\[
p^{(2n)}(x, x) \geq (1 - \epsilon)^2 \frac{m(x)}{m(A)}.
\]

The easy proof of this lemma is straightforward and we omit it. By using this result one see immediately that, for any \(x \in A\)
\[
G(x, x|z) \geq \frac{(1 - \epsilon)^2 m(x)}{1 - z^2 m(A)}, \quad z \in [0, 1).
\]

References

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