Jensen’s Inequality for Backward SDEs Driven by \( G \)-Brownian motion

Ze-Chun Hu* and Zhen-Ling Wang
Nanjing University

Abstract  In this note, we consider Jensen’s inequality for the nonlinear expectation associated with backward SDEs driven by \( G \)-Brownian motion (\( G \)-BSDEs for short). At first, we give a necessary and sufficient condition for \( G \)-BSDEs under which one-dimensional Jensen inequality holds. Second, we prove that for \( n > 1 \), the \( n \)-dimensional Jensen inequality holds for any nonlinear expectation if and only if the nonlinear expectation is linear, which is essentially due to Jia (Arch. Math. 94 (2010), 489-499). As a consequence, we give a necessary and sufficient condition for \( G \)-BSDEs under which the \( n \)-dimensional Jensen inequality holds.

Keywords  \( G \)-BSDE, nonlinear expectation, Jensen’s inequality

MSC(2000): 60H10

1 Introduction

It’s well known that backward stochastic differential equations (BSDEs in short) play a very important role in stochastic analysis, finance and etc. We refer to a survey paper of Peng [20] for more details of the theoretical studies and applications to, e.g., stochastic controls, optimizations, games and finance.

Peng [13]–[19] defined the \( G \)-expectations, \( G \)-Brownian motions and built Itô’s type stochastic calculus. As to the classic setting, it’s important to study BSDEs under \( G \)-expectation, i.e. BSDEs driven by \( G \)-Brownian motions (\( G \)-BSDE for short). By Hu et al. [7], a general \( G \)-BSDE is to find a triple of processes \((Y, Z, K)\), where \( K \) is a decreasing \( G \)-martingale, satisfying

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).
\]

(1.1)
When the generator \( f \) in (1.1) is independent of \( z \) and \( g = 0 \), the above problem can be equivalently formulated as

\[
Y_t = \hat{\mathbb{E}}_t [\xi + \int_t^T f(s, Y_s) ds].
\]

The existence and uniqueness of such fully nonlinear BSDE was obtained in Peng \[14, 16, 19\]. Soner, Touzi and Zhang \[22\] have proved the existence and uniqueness for a type of fully nonlinear BSDE, called 2BSDE, whose generator can contain \( Z \)-term.

For the general \( G \)-BSDE (1.1), Hu et al. proved the existence and uniqueness in \[7\], and studied comparison theorem, nonlinear Feynman-Kac formula and Girsanov transformation in \[8\]. He and Hu \[5\] obtained a representation theorem for the generators of \( G \)-BSDEs and used the representation theorem to get a converse comparison theorem for \( G \)-BSDEs and some equivalent results for the nonlinear expectations generated by \( G \)-BSDEs. Peng and Song \[21\] introduced a new notion of \( G \)-expectation-weighted Sobolev spaces (\( G \)-Sobolev space for short), and proved that \( G \)-BSDEs are in fact path dependent PDEs in the corresponding \( G \)-Sobolev spaces.

In this note, we study Jensen’s inequality for \( G \)-BSDEs. For Jensen’s inequality for \( g \)-expectation associated classical BSDEs, we refer to Briand et al. \[1\], Chen et al. \[2\], Jiang and Chen \[12\], Hu \[6\], Jiang \[11\], Fan \[3\], Jia \[9\], Jia and Peng \[10\] and the references therein.

Recently, Guessab and Schmeisser \[4\] considered the \( d \)-dimensional Jensen inequality

\[
T[\psi(f_1, \cdots, f_d)] \geq \psi(T[f_1], \cdots, T[f_d]),
\]

where \( T \) is a functional, \( \psi \) is a convex function defined on a closed convex set \( K \subset \mathbb{R}^d \), and \( f_1, \cdots, f_d \) are from some linear space of functions. Among other things, the authors showed that if we exclude three types of convex sets \( K \), then Jensen’s inequality holds for a sublinear functional \( T \) if and only if \( T \) is linear, positive, and satisfies \( T[1] = 1 \), i.e. \( T \) is a linear expectation.

The rest of this note is organized as follows. In Section 2, we give some preliminaries about \( G \)-expectation and \( G \)-BSDEs. In Section 3, we consider Jensen’s inequality for the nonlinear expectation driven by \( G \)-BSDEs. In Subsection 3.1, we follow the method of Hu \[6\] and apply the comparisson theorem, the converse comparison theorem in He and Hu \[5\] to give a necessary and sufficient condition for \( G \)-BSDEs under which one-dimensional Jensen inequality holds. In Subsection 3.2, we prove that for \( n > 1 \), the \( n \)-dimensional Jensen inequality holds for any nonlinear expectation if and only if the nonlinear expectation is linear, which is essentially due to Jia \[9\], and as a consequence, we give a necessary and sufficient condition for \( G \)-BSDEs under which the \( n \)-dimensional Jensen inequality holds.

## 2 Preliminaries

In this section, we review some basic notions and results of \( G \)-expectation, the related spaces of random variables, and \( G \)-BSDE. The readers may refer to \[19, 7, 8\] for more details.

**Definition 2.1** Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a linear space of real valued function defined on \( \Omega \), and satisfy: (i) for each constant \( c, c \in \mathcal{H} \); (ii) if \( X \in \mathcal{H} \), then \( |X| \in \mathcal{H} \). The space \( \mathcal{H} \) can be
where $S \in \phi$ that considered as the space of random variables. A sublinear expectation $\hat{E}$ is a functional $\hat{E}: \mathcal{H} \to \mathbb{R}$ satisfying
(i) Monotonicity: $\hat{E}[X] \geq \hat{E}[Y]$, if $X \geq Y$;
(ii) Constant preserving: $\hat{E}[c] = c$, for $c \in \mathbb{R}$;
(iii) Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$, for each $X, Y \in \mathcal{H}$;
(iv) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, for $\lambda \geq 0$.
The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\hat{E}$ is called a nonlinear expectation and the triple $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space.

**Definition 2.2** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined in sublinear expectation spaces $(\Omega, \mathcal{H}, \hat{E}_1)$ and $(\Omega, \mathcal{H}, \hat{E}_2)$ respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{b,Lip}(\mathbb{R}^n)$, where $C_{b,Lip}(\mathbb{R}^n)$ denotes the space of all bounded and Lipschitz functions on $\mathbb{R}^n$.

**Definition 2.3** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent of another random vector $X \in \mathcal{H}^m$ under $\hat{E}[:]$, denoted by $Y \perp X$, if for all $\varphi \in C_{b,Lip}(\mathbb{R}^{n+m})$ one has $\hat{E}[\varphi(X,Y)] = \hat{E}[\hat{E}[\varphi(x,y)]|y=x]$.

**Definition 2.4** (G-normal distribution) A $d$-dimensional random vector $X = (X_1, \cdots, X_d)$ in sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normally distributed if for each $a$, $b \geq 0$, one has $aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X$, where $\bar{X}$ is an independent copy of $X$, i.e. $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$. Here, the letter $G$ denotes the function

$$G(A) := \hat{E}[\frac{1}{2}(AX, X)] : \mathcal{S}_d \to \mathbb{R},$$
where $\mathcal{S}_d = \{A|A \text{ is } d \times d \text{ symmetric matrix}\}$.

Peng [18] proved that $X = (X_1, \cdots, X_d)$ is $G$-normally distributed if and only if for each $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following $G$-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi.$$  

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The function $G(\cdot) : \mathcal{S}_d \to \mathbb{R}$ is a monotonic, sublinear mapping on $\mathcal{S}_d$ and $G(A) := \hat{E}[\frac{1}{2}(AX, X)] \leq \frac{1}{2}[A][E[|X|^2]]$, which implies that there exists a bounded, convex, and closed subset $\Gamma \subset \mathcal{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} tr[\gamma A],$$
where $\mathcal{S}_d^+$ denotes the collection of nonnegative elements in $\mathcal{S}_d$. In this note, we only consider nondegenerate $G$-normal distribution; that is, there exists some $\sigma^2 > 0$ such that $G(A) - G(B) \geq \sigma^2 tr[A - B]$ for any $A \geq B$.  

3
**Definition 2.5** (i) Let $\Omega = C^d_0(\mathbb{R}^+)$ denote the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ and $B_t(\omega) = \omega$ be the canonical process. For each fixed $T \in [0, \infty)$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, \ldots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, \infty), \varphi \in C_b.Lip(\mathbb{R}^{d \times n})\}.$$  

It is clear that $L_{ip}(\Omega_T) \subseteq L_{ip}(\Omega_{T'})$ for $t \leq T$. We also set $L_{ip}(\Omega) := \bigcup_{n=1}^\infty L_{ip}(\Omega_n)$. Let $G : \mathcal{S}_d \to \mathbb{R}$ be a given monotonic and sublinear function. $G$-expectation is a sublinear expectation defined by  

$$\mathbb{E}[X] = \mathbb{E}[\varphi(\sqrt{t_1-t_0}\xi_1, \ldots, \sqrt{t_m-t_{m-1}}\xi_m)]$$  

for all $X \in L_{ip}(\Omega)$ with $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \ldots, \xi_m$ is identically distributed $d$-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\bar{\Omega}, \mathcal{H}, \mathbb{E})$ such that $\xi_{i+1}$ is independent of $(\xi_1, \ldots, \xi_i)$ for every $i = 1, \ldots, m - 1$. The corresponding canonical process $B_t = (B_t^i)_{t=1}^m$ is called a $G$-Brownian motion.

(ii) For each fixed $t \in [0, \infty)$, the conditional $G$-expectation $\mathbb{E}_t[\cdot]$ for $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega)$, where without loss of generality we suppose $t = t_i$, $1 \leq i \leq m$, is defined by

$$\mathbb{E}_t[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \psi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),$$  

where $\psi(x_1, \ldots, x_i) = \mathbb{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].$

We denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_{p,G} = (\mathbb{E}[|X|^p])^{1/p}$. Similarly, we can define $L_{ip}^q(\Omega_T)$. It is clear that $L_G^q(\Omega) \subseteq L_G^p(\Omega)$ for $1 \leq p \leq q$ and $\mathbb{E}[\cdot]$ can be extended continuously to $L_G^1(\Omega)$.

For each fixed $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, $B_t^a = \langle a, B_t \rangle$ is a 1-dimensional $G_a$-Brownian motion on $(\Omega, \mathcal{H}, \mathbb{E})$, where $G_a(\alpha) = \frac{1}{2}(\sigma_{a\alpha}^2\alpha^+ - \sigma_{-a\alpha}^2\alpha^-)$, $\sigma_{a\alpha}^2 = 2G(aa^T) = \mathbb{E}[\langle a, B_1 \rangle^2]$, $\sigma_{-a\alpha}^2 = -2G(-aa^T) = -\mathbb{E}[-\langle a, B_1 \rangle^2]$. In particular, for each $t, s \geq 0$, $B_{t+s}^a - B_t^a \overset{d}{=} N(0 \times [s\sigma_{a\alpha}^2, s\sigma_{-a\alpha}^2]).$

Let $\pi_T^N = \{t_0^N, t_1^N, \ldots, t_N^N\}$, $N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_T^N) = \max\{|t_{i+1} - t_i| : i = 0, 1, \ldots, N - 1\} \to 0$. The quadratic variation process of $\langle B^a \rangle$ is defined by

$$\langle B^a \rangle_t := \lim_{\mu(\pi_T^N) \to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}}^a - B_{t_k}^a)^2 = (B^a_t)^2 - 2\int_0^t B^a_s dB^a_s.$$  

For each fixed $a, \bar{a} \in \mathbb{R}^d$, the mutual variation process of $B^a$ and $B^{\bar{a}}$ is defined by

$$\langle B^a, B^{\bar{a}} \rangle_t := \frac{1}{4}(\langle B^a + B^{\bar{a}} \rangle_t - \langle B^a - B^{\bar{a}} \rangle_t) = \frac{1}{4}(\langle B^{a+\bar{a}} \rangle_t - \langle B^{a-\bar{a}} \rangle_t).$$

**Definition 2.6** For fixed $T \geq 0$, let $M^0_G(0, T)$ be the collection of process in the following form: for a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t),$$

where $\xi_k(\omega) = \mathbb{E}[\varphi(\sqrt{t_1-t_0}\xi_1, \ldots, \sqrt{t_m-t_{m-1}}\xi_m)]$ for all $X \in L_{ip}(\Omega)$ with $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \ldots, \xi_m$ is identically distributed $d$-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\bar{\Omega}, \mathcal{H}, \mathbb{E})$ such that $\xi_{i+1}$ is independent of $(\xi_1, \ldots, \xi_i)$ for every $i = 1, \ldots, m - 1$. The corresponding canonical process $B_t = (B_t^i)_{t=1}^m$ is called a $G$-Brownian motion.
There exists some $L > \omega, y, z$ denote by $S$ satisfy the following properties: where $\xi \in L_{G}^{p}(\Omega_{t_{k}})$, $k = 0, 1, 2, \cdots, N - 1$. For $p \geq 1$, we denote by $H_{G}^{p}(0, T)$, $M_{G}^{p}(0, T)$ the completion of $M_{G}^{0}(0, T)$ under the norms $\|\eta\|_{H_{G}^{p}} = \{E[(\int_{0}^{T} |\eta_{t}|^{2})^{p/2}]\}^{1/p}$, $\|\eta\|_{M_{G}^{p}} = \{E[\int_{0}^{T} |\eta|^{p}dt]\}^{1/p}$, respectively.

Let $S_{G}^{0}(0, T) = \{h(t, B_{t_{1}}\cap, \cdots, B_{t_{n}}\cap) : t_{1}, \cdots, t_{n} \in [0, T], h \in C_{b,\text{Lip}}(\mathbb{R}^{n+1})\}$. For $p \geq 1$, denote by $S_{G}^{p}(0, T)$ the completion of $S_{G}^{0}(0, T)$ under the norm $\|\eta\|_{S_{G}^{p}} = \{E[\sup_{t \in [0, T]} |\eta_{t}|^{p}]\}^{1/p}$.

We consider the following type of $G$-BSDEs (in this note we always use Einstein convention):

$$
Y_{t} = \xi + \int_{t}^{T} f(s, Y_{s}, Z_{s})ds + \int_{t}^{T} g_{ij}(s, Y_{s}, Z_{s})d\langle B^{i}, B^{j}\rangle_{s} - \int_{t}^{T} Z_{s}dB_{s} - (K_{T} - K_{t}),
$$

where

$$
f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_{T} \times \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R},
$$
satisfy the following properties:

(H1) There exists some $\beta > 1$ such that for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_{G}^{\beta}(0, T)$;

(H2) There exists some $L > 0$ such that

$$
|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^{d} |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).
$$

Denote by $\mathcal{G}_{G}^{0}(0, T)$ the completion of processes $(Y, Z, K)$ such that $Y \in S_{G}^{0}(0, T)$, $Z \in H_{G}^{0}(0, T; \mathbb{R}^{d})$, $K$ is a decreasing $G$-martingale with $K_{0} = 0$ and $K_{T} \in L_{G}^{2}(\Omega_{T})$.

**Definition 2.7** Let $\xi \in L_{G}^{\beta}(\Omega_{T})$ and $f$ and $g_{ij}$ satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes $(Y, Z, K)$ is called a solution of (2.2) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y, Z, K) \in \mathcal{G}_{G}^{\alpha}(0, T)$;

(b) $Y_{t} = \xi + \int_{t}^{T} f(s, Y_{s}, Z_{s})ds + \int_{t}^{T} g_{ij}(s, Y_{s}, Z_{s})d\langle B^{i}, B^{j}\rangle_{s} - \int_{t}^{T} Z_{s}dB_{s} - (K_{T} - K_{t})$.

**Theorem 2.8** ([7]) Assume that $\xi \in L_{G}^{\beta}(\Omega_{T})$ and $f$ and $g_{ij}$ satisfy (H1) and (H2) for some $\beta > 1$. Then, equation (2.2) has a unique solution $(Y, Z, K)$. Moreover, for any $1 < \alpha < \beta$, one has $Y \in S_{G}^{\alpha}(0, T)$, $Z \in H_{G}^{\alpha}(0, T; \mathbb{R}^{d})$ and $K_{T} \in L_{G}^{2}(\Omega_{T})$.

In this note, we also need the following assumptions for $G$-BSDE (2.2) (see He and Hu [5]).

(H3) For each fixed $(\omega, y, z) \in \Omega_{T} \times \mathbb{R} \times \mathbb{R}^{d}$, $t \rightarrow f(t, \omega, y, z)$ and $t \rightarrow g_{ij}(t, \omega, y, z)$ are continuous.
(H4) For each fixed \((t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d\), \(f(t, y, z)\), \(g_{ij}(t, y, z) \in L^\beta_G(\Omega_t)\), and
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} \left( |f(u, y, z) - f(t, y, z)|^\beta + \sum_{i,j=1}^d |g_{ij}(u, y, z) - g_{ij}(t, y, z)|^\beta \right) du \right] = 0. \quad (2.3)
\]

(H5) For each fixed \((t, \omega, y) \in [0, T] \times \Omega_T \times \mathbb{R}\), \(f(t, \omega, y, 0) = g_{ij}(t, \omega, y, 0) = 0\).

3 Jensen’s inequality for \(G\)-BSDEs

We consider the following \(G\)-BSDE:
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g_{ij}(s, Y_s, Z_s)d\langle B^i, B^j \rangle_s - \int_t^T Z_s d\langle B \rangle_s - (K_T - K_t), \quad (3.1)
\]
where \(g_{ij} = g_{ji}\), and \(f\) and \(g_{ij}\) satisfy the conditions (H1)-(H5). Define \(\tilde{E}_t[\xi] = Y_t\).

3.1 One-dimensional Jensen inequality

**Theorem 3.1** The following two statements are equivalent:

(i) Jensen’s inequality holds, i.e., for each \(\xi \in L^2_G(\Omega_T)\), and any convex function \(h: \mathbb{R} \to \mathbb{R}\), if \(h(\xi) \in L^2_G(\Omega_T)\), then
\[
\tilde{E}_t[h(\xi)] \geq h(\tilde{E}_t[\xi]), \quad \forall t \in [0, T]. \quad (3.2)
\]

(ii) \(\forall \lambda, \mu \in \mathbb{R}, \lambda \neq 0, \forall (t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d,\)
\[
\lambda f(t, y, z) - f(t, \lambda y + \mu, \lambda z) + 2G((\lambda g_{ij}(t, y, z) - g_{ij}(t, \lambda y + \mu, \lambda z))_{i,j=1}^d) \leq 0, \quad \text{q.s.} \quad (3.3)
\]

**Proof.** The idea of the proof comes from Theorem 3.1 of [6].

\((i) \Rightarrow (ii):\) For fixed \(\lambda \neq 0\) and \(\mu\), we define a convex function \(h(x) = \lambda x + \mu\). Let \((Y_t, Z_t, K_t)\) be the unique solution of the \(G\)-BSDE (3.1). Define \(Y_t' = \lambda Y_t + \mu, Z_t' = \lambda Z_t, K_t' = \lambda K_t\). Then \((Y_t', Z_t', K_t')\) is the unique solution of the following \(G\)-BSDE:
\[
Y_t' = h(\xi) + \int_t^T f'(s, Y_s', Z_s')ds + \int_t^T g_{ij}'(s, Y_s', Z_s')d\langle B^i, B^j \rangle_s - \int_t^T Z_s' dB_s - (K_T' - K_t'), \quad (3.4)
\]
where \(f'(t, y, z) = \lambda f(t, \frac{y-\mu}{\lambda}, \frac{z}{\lambda}), g_{ij}'(t, y, z) = \lambda g_{ij}(t, \frac{y-\mu}{\lambda}, \frac{z}{\lambda})\).
Denote $\tilde{E}_t[h(\xi)] = Y'_t$. By (3.2), we get
\[
\tilde{E}_t[h(\xi)] \geq h(\tilde{E}_t[\xi]) = \lambda Y_t + \mu = Y'_t = \tilde{E}'_t[h(\xi)].
\] (3.5)
For any $\eta \in L^2_G(\Omega_T)$, put $\xi = h^{-1}(\eta)$. Then we have by (3.5)
\[
\tilde{E}_t[\eta] \geq \tilde{E}'_t[\eta].
\]
By the converse comparison theorem [5, Theorem 15], we obtain that
\[
(f' - f)(t, y', z') + 2G((g_{ij} - g_{ij})_{i,j=1})^d(t, y', z') \leq 0 \text{ q.s.},
\]
which implies
\[
f'(t, y', z') - f(t, y', z') + 2G((g_{ij} - g_{ij})_{i,j=1})(t, y', z') = \lambda f(t, \frac{y' - \mu}{\lambda}, \frac{z'}{\lambda}) - f(t, y', z') + 2G((\lambda g_{ij} - g_{ij})(t, \frac{y' - \mu}{\lambda}, \frac{z'}{\lambda}) - g_{ij}(t, y', z'))_{i,j=1}
\]
\[
= \lambda f(t, y, z) - f(t, \lambda y + \mu, \lambda z) + 2G((\lambda g_{ij} - g_{ij})(t, y, z))_{i,j=1}
\]
\[
\leq 0, \quad \text{q.s.}
\]
Hence (ii) holds.

(ii) $\Rightarrow$ (i) : First, take a linear function $h(x) = \lambda x + \mu$ where $\lambda \neq 0$. Let $(Y_t, Z_t, K_t)$ be the unique solution of G-BSDE (3.1), and denote $Y'_t = \lambda Y_t + \mu$, $Z'_t = \lambda Z_t$, $K'_t = \lambda K_t$. Then $(Y'_t, Z'_t, K'_t)$ is the unique solution of G-BSDE (3.4). Let $f', g'_{ij}$ be defined as in (3.4). Then by (ii), we have
\[
(f' - f)(t, y, z) + 2G((g_{ij} - g_{ij})_{i,j=1})(t, y, z) \leq 0 \text{ q.s.},
\]
which together with the comparison theorem [5, Proposition 13] implies that
\[
\tilde{E}_t[h(\xi)] \geq \tilde{E}'_t[h(\xi)] = Y'_t = \lambda Y_t + \mu = \lambda \tilde{E}_t[\xi] + \mu = h(\tilde{E}_t[\xi]).
\] (3.6)
For any convex function $h$, there exists a countable set $D$ in $\mathbb{R}^2$, such that
\[
h(x) = \sup_{(\lambda, \mu) \in D} (\lambda x + \mu).
\] (3.7)
By (3.6) and (3.7), we have
\[
\tilde{E}_t[h(\xi)] = \tilde{E}_t[\sup_{(\lambda, \mu) \in D} (\lambda x + \mu)] \geq \sup_{(\lambda, \mu) \in D} (\lambda \tilde{E}_t[\xi] + \mu) = h(\tilde{E}_t[\xi]),
\]
i.e. (i) holds.

\[
\begin{align*}
(\text{Remark 3.2})
\text{(i) If } f \text{ and } g_{ij} \text{ are independent of } y, \text{ then the condition of (3.3) becomes}
\lambda f(t, z) - f(t, \lambda z) + 2G((\lambda g_{ij} - g_{ij})(t, \lambda z))_{i,j=1} \leq 0, \text{ q.s.}
\end{align*}
\]
\[
(\text{(ii) If } g_{ij} \equiv 0, \text{ then the condition of (3.3) becomes}
f(t, \lambda y + \mu, \lambda z) \geq \lambda f(t, y, z), \text{ q.s.}
\] (3.8)
Taking $\lambda = 1$, then $f(t, y + \mu, z) \geq f(t, y, z)$, q.s., which implies that $f$ is independent of $y$. Thus (3.8) becomes $f(t, \lambda z) \geq \lambda f(t, z)$, q.s. This is just the condition in Hu [6, Theorem 3.1].

7
3.2 Multi-dimensional Jensen inequality

At first, we prove a result for any nonlinear expectation, which is essentially due to Jia (see [9, Theorem 3.3]).

**Theorem 3.3** Assume that \( n > 1 \) and \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is a nonlinear expectation space defined by Definition 2.1. Then the following two claims are equivalent:

(a) \( \hat{\mathbb{E}} \) is linear, i.e., for any \( \lambda, \gamma \in \mathbb{R} \), \( X, Y \in \mathcal{H} \),

\[
\hat{\mathbb{E}}[\lambda X + \gamma Y] = \lambda \hat{\mathbb{E}}[X] + \gamma \hat{\mathbb{E}}[Y];
\]  

(3.9)

(b) the \( n \)-dimensional Jensen inequality for nonlinear expectation \( \hat{\mathbb{E}} \) holds, i.e. for each \( X_i \in \mathcal{H}(i = 1, \cdots, n) \) and convex function \( h : \mathbb{R}^n \to \mathbb{R} \), if \( h(X_1, \cdots, X_n) \in \mathcal{H} \), then

\[
\hat{\mathbb{E}}[h(X_1, \cdots, X_n)] \geq h(\hat{\mathbb{E}}[X_1], \cdots, \hat{\mathbb{E}}[X_n]).
\]

**Proof.** The proof of [9, Theorem 3.3] can be moved to this case. For the reader’s convenience, we spell out the details.

(\( b \Rightarrow a \): For any \((\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \), by \( b \) we have that

\[
\hat{\mathbb{E}} \left[ \sum_{i=1}^{n} \lambda_i X_i \right] \geq \sum_{i=1}^{N} \lambda_i \hat{\mathbb{E}}[X_i].
\]  

(3.10)

Taking \( \lambda_1 > 0, \lambda_j = 0, j = 2, \cdots, n \), we get that

\[
\hat{\mathbb{E}} [\lambda_1 X_1] \geq \lambda_1 \hat{\mathbb{E}}[X_1] \geq \lambda_1 \cdot \frac{1}{n} \hat{\mathbb{E}}[\lambda X_1] = \hat{\mathbb{E}} [\lambda X_1],
\]

which together with \( \hat{\mathbb{E}}[0] = 0 \) (by (ii) in Definition 2.1) implies that \( \hat{\mathbb{E}} \) is positively homogeneous. Put \( \lambda_1 = 1, \lambda_2 = -1 \) and \( \lambda_1 = \lambda_2 = 1 \) respectively, and put \( \lambda_j = 0 \) for \( j > 2 \) in (3.10), we get

\[
\hat{\mathbb{E}}[X_1 - X_2] \geq \hat{\mathbb{E}}[X_1] - \hat{\mathbb{E}}[X_2], \quad \hat{\mathbb{E}}[X_1 + X_2] \geq \hat{\mathbb{E}}[X_1] + \hat{\mathbb{E}}[X_2].
\]

It follows that \( \hat{\mathbb{E}}[X_1] \leq \hat{\mathbb{E}}[X_2] + \hat{\mathbb{E}}[X_1 - X_2] \leq \hat{\mathbb{E}}[X_2 + (X_1 - X_2)] = \hat{\mathbb{E}}[X_1] \). Thus we have \( \hat{\mathbb{E}}[X_1 - X_2] = \hat{\mathbb{E}}[X_1] - \hat{\mathbb{E}}[X_2] \) and \( \hat{\mathbb{E}}[X_1 + X_2] = \hat{\mathbb{E}}[(X_1 + X_2) - X_2] + \hat{\mathbb{E}}[X_2] = \hat{\mathbb{E}}[X_1] + \hat{\mathbb{E}}[X_2] \). Hence \( \hat{\mathbb{E}} \) is homogeneous and thus it’s linear.

(\( a \Rightarrow b \): For any \((\lambda_1, \cdots, \lambda_n, \mu) \in \mathbb{R}^{n+1} \), by \( a \) and (ii) in Definition 2.1, we have

\[
\hat{\mathbb{E}} \left[ \sum_{i=1}^{n} \lambda_i X_i + \mu \right] = \hat{\mathbb{E}} \left[ \sum_{i=1}^{n} \lambda_i X_i \right] + \mu = \sum_{i=1}^{n} \lambda_i \hat{\mathbb{E}}[X_i] + \mu.
\]  

(3.11)

For any convex function \( h : \mathbb{R}^n \to \mathbb{R} \), there exists a countable set \( D \subset \mathbb{R}^{n+1} \) such that

\[
h(x) = \sup_{(\lambda_1, \cdots, \lambda_n, \mu) \in D} \left( \sum_{i=1}^{n} \lambda_i x_i + \mu \right).
\]  

(3.12)
By (3.11) and (i) in Definition 2.1, for any \((\lambda_1, \ldots, \lambda_n, \mu) \in D\), we have
\[
\hat{E}[h(X_1, \ldots, X_n)] \geq \hat{E}\left[\sum_{i=1}^{n} \lambda_i X_i + \mu\right] = \sum_{i=1}^{n} \lambda_i \hat{E}[X_i] + \mu,
\]
which together with (3.12) implies (b).

**Proposition 3.4** Assume that \(n > 1\) and \(t \in [0, T]\). Then the following two claims are equivalent:
(i) \(\hat{E}_t\) is linear, i.e., for any \(\lambda, \gamma \in \mathbb{R}, X, Y \in \mathcal{H}\),
\[
\hat{E}_t[\lambda X + \gamma Y] = \lambda \hat{E}_t[X] + \gamma \hat{E}_t[Y];
\]
(ii) the \(n\)-dimensional Jensen inequality for \(\hat{E}_t\) holds, i.e. for each \(X_i \in \mathcal{H}(i = 1, \ldots, n)\) and convex function \(h : \mathbb{R}^n \to \mathbb{R}\), if \(h(X_1, \ldots, X_n) \in \mathcal{H}\), then
\[
\hat{E}_t[h(X_1, \ldots, X_n)] \geq h(\hat{E}_t[X_1], \ldots, \hat{E}_t[X_n]).
\]

**Proof.** By \(\mathfrak{N}\) Theorem 5.1 (1)(2)], we know that \(\hat{E}_t\) satisfies monotonicity and constant preserving. Then all the proof of the above theorem can be moved to this case.

**Corollary 3.5** Assume that \(n > 1\). Then the following two claims are equivalent:
(i) for any \(t \in [0, T]\), the \(n\)-dimensional Jensen inequality for \(\hat{E}_t\) holds, i.e. for each \(X_i \in \mathcal{H}(i = 1, \ldots, n)\) and convex function \(h : \mathbb{R}^n \to \mathbb{R}\), if \(h(X_1, \ldots, X_n) \in \mathcal{H}\), then
\[
\hat{E}_t[h(X_1, \ldots, X_n)] \geq h(\hat{E}_t[X_1], \ldots, \hat{E}_t[X_n]);
\]
(ii) for any \(t \in [0, T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, \lambda \geq 0,\)
\[
f(t, y + y', z + z') - f(t, y, z) - f(t, y', z')
= -2G (\{g_{ij}(t, y + y', z + z') - g_{ij}(t, y, z) - g_{ij}(t, y', z')\})_{i,j=1}^d,
\]
and
\[
f(t, \lambda y, \lambda z) - \lambda f(t, y, z) = 2G (\{\lambda g_{ij}(t, y, z) - g_{ij}(t, y, \lambda z)\})_{i,j=1}^d
= -2G (\{g_{ij}(t, \lambda y, \lambda z) - \lambda g_{ij}(t, y, z)\})_{i,j=1}^d.
\]

**Proof.** By Proposition 3.4 we know that (i) holds if and only if for any \(t \in [0, T], \hat{E}_t\) is linear. Then by \(\mathfrak{M}\) Proposition 17 (2)(4)], we obtain that (i) and (ii) are equivalent.

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