A DIFFERENTIAL EQUATION WITH MONODROMY GROUP $2.J_2$

STEFAN REITER

Abstract. We construct a sixth order differential equation having the central extension of $C_2$ by the Hall-Janko group $J_2$ as monodromy group. Moreover, it arises from an iterated application of tensor products and convolution operations from a first order differential equation.

1. Introduction

According to [13, 5.6.1] there are two important constructions of the sporadic simple Hall-Janko group $J_2$ of order 604800, namely as permutation group on 100 points by Marshall Hall and as a quaternionic reflection group in 3 dimensions in connection with the Leech lattice. It is well known that $2.J_2$, the central extension of $C_2$ by the Hall-Janko group $J_2$, is an irreducible subgroup of $\text{Sp}_6(\mathbb{C})$ [3, p. 42-43]. Generators of the six dimensional representation of $2.J_2$ over $\mathbb{Q}(\zeta_5)$ were already determined by Lindsey II [6]. Here we show that the group $2.J_2$ appears as a monodromy group of a sixth order differential equation that can be constructed by an iterated application of tensor products and convolution operations from a first order differential equation.

Throughout the article, let $\partial = \frac{d}{dx}$, $\vartheta = x\partial$ and $L = \sum_{i=0}^{n} a_i(x)\partial^i \in \mathbb{C}(x)[\partial]$ be a differential operator. Recall that the adjoint $L^*$ of $L$ is defined as $L^* = \sum_{i=0}^{n} (-\partial)^i a_i(x)$ and that $L$ is called self adjoint if $L = (-1)^n L^*$. If $L$ is self adjoint then the differential Galois group of $L$, and hence the monodromy group of $L$, is contained in the symplectic group $\text{Sp}_n(\mathbb{C})$ if $n$ is even [7].

Theorem. 1.1. The formally self adjoint fuchsian operator

$$L_{2.J_2} = 250000 (6 \vartheta + 5) (6 \vartheta - 1) (3 \vartheta - 1) (3 \vartheta + 1) (6 \vartheta + 1) (6 \vartheta - 5) - 125 x (6 \vartheta + 1) (6 \vartheta + 5) \cdot (1296000 \vartheta^4 + 2592000 \vartheta^3 + 2578320 \vartheta^2 + 1282320 \vartheta + 213703) + 11664 x^2 (10 \vartheta + 17) (5 \vartheta + 7) (10 \vartheta + 11) (10 \vartheta + 9) (5 \vartheta + 3) (10 \vartheta + 3)$$

has the Riemann scheme

$$\mathcal{R}(L_{2.J_2}) = \begin{cases} 
0 & 1 & \infty \\
5/6 & 3 & 17/10 \\
1/3 & 5/2 & 7/5 \\
1/6 & 2 & 11/10 \\
-1/6 & 1 & 9/10 \\
-1/3 & 1/2 & 3/5 \\
-5/6 & 0 & 3/10 
\end{cases}$$

and $2.J_2$ as monodromy group. Moreover, its monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0) \to \text{Sp}_6(\mathbb{C})$ is uniquely determined by the local monodromy, i.e. the Jordan forms.
Note that if $P = \sum x^i p_i(\vartheta)$ is any differential operator then it follows from the identities $\vartheta x^i = x^i (\vartheta + i)$ and $\vartheta^n = -\vartheta - 1$ in the ring $\mathbb{C}[x][\vartheta]$ that $P^* = \sum x^i p_i(-\vartheta - i - 1)$. Hence we obtain for the above differential operator $L_{2, J_2}$ that
\[(x^{-1}L_{2, J_2})^* = x^{-1}L_{2, J_2},\]
meaning that the operator $L_{2, J_2}$ is formally self adjoint. Necessary conditions that a tuple of matrices $T = (T_1, \ldots, T_{r+1})$, $T_1 \cdots T_{r+1} = 1$, $T \in \text{GL}_n(\mathbb{C})^{r+1}$, appears as a monodromy tuple of an irreducible fuchsian differential equation are given by the Scott formula \cite[Theorem 1]{Scott}: 
\[\sum_{i=1}^{r+1} \text{rk}(T_i - 1) \geq 2n,\]
\[\sum_{i=1}^{r+1} \dim C_{\text{GL}_n(\mathbb{C})}(T_i) \leq (r - 1)n^2 + 2,\]
where $\dim(C_{\text{GL}_n(\mathbb{C})}(T_i))$ denotes the dimension of the centralizer of $T_i$ in $\text{GL}_n(\mathbb{C})$. On the other hand the algorithm of Katz \cite{Katz} provides a simple tool to check whether to a given tuple of Jordan forms $(J(T_1), \ldots, J(T_{r+1})) \in \text{GL}_n(\mathbb{C})^{r+1}$ satisfying
\[\sum_{i=1}^{r+1} \dim C_{\text{GL}_n(\mathbb{C})}(J(T_i)) = (r - 1)n^2 + 2\]
there exists a corresponding irreducible monodromy tuple $T$. Such a tuple $T$ is called linearly rigid and it is uniquely determined by its tuple of Jordan forms up to simultaneous conjugation in $\text{GL}_n(\mathbb{C})$. A look in the Atlas of finite groups \cite[p. 43]{Atlas} allows to read off the Jordan forms of the elements of $2.J_2$ in their six-dimensional representation. However applying the Scott formula and the Katz-algorithm it turns out that there is no linearly rigid irreducible tuple that generates a subgroup of $2.J_2 \subset \text{Sp}_6(\mathbb{C})$. One can weaken the notion of linear rigidity in the following way. We say that an irreducible monodromy tuple $T \in \text{Sp}_n(\mathbb{C})^{r+1}$ is symplectically rigid if the dimension formula holds:
\[(1.1) \quad \sum_{i=1}^{r+1} \dim(C_{\text{Sp}_n(\mathbb{C})}(T)) = (r - 1)\dim(\text{Sp}_n(\mathbb{C})),\]
where $C_{\text{Sp}_n(\mathbb{C})}(T_i)$ denotes the centralizer of local monodromy generator $T_i$ in the algebraic group $\text{Sp}_n(\mathbb{C})$. Symplectic rigidity is also a necessary condition for the existence of only finitely many equivalence classes of irreducible tuples in the symplectic group with given local monodromy \cite[Corollary 3.2]{Symplectic}. In the six dimensional representation of $2.J_2$ we find elements of order $6, 2, 10$ with Jordan forms
\[(\zeta_6, \zeta_6, \zeta_3, \zeta_3^{-1}, \zeta_6^{-1}, \zeta_6^{-1}), \quad (-1, -1, 1, 1, 1), \quad (\zeta_{10}, \zeta_{10}^3, \zeta_5^3, \zeta_5^{-3}, \zeta_5^{-3}, \zeta_5^{-1}),\]
where $\zeta_k$ denotes a primitive $k$-th root of unity and $(\zeta_6, \zeta_6, \zeta_3, \zeta_3^{-1}, \zeta_6^{-1}, \zeta_6^{-1})$ denotes an element having Jordan form with two Jordan blocks of size one with eigenvalues $\zeta_6$, $\zeta_6^{-1}$ resp., and one Jordan block of size one with eigenvalues $\zeta_3$, $\zeta_3^{-1}$ resp.. Since the centralizer dimensions in $\text{Sp}_6(\mathbb{C})$ of the elements are $5, 13, 3$ resp., this triple satisfies the symplectic dimension formula. The computation of the corresponding normalized structure constant $n(T)$, cf. \cite[Chapter I, Theorem 5.8]{Structure}, yields $n(T) = 1$. Hence, such a triple $T$ in $(2.J_2)^3$ with these Jordan forms and product $1$ exists. Moreover, this triple in $\text{Sp}_6(\mathbb{C})^3$ obviously generates an irreducible subgroup of $2.J_2$, since there is no invariant subspaces of dimension $1$, $2$ or $3$. Thus $T$ is uniquely determined in $(2.J_2)^3$ up to simultaneous conjugation. Since the finite linear quasi-primitive groups generated by bi-reflections have been classified in \cite{Reflection} it turns out that the generated group is $2.J_2$. It remains to construct the differential operator $L_{2, J_2}$. For this we apply the Katz-Existence Algorithm to the monodromy triple $T$. Then we end up with a symplectically rigid triple in dimension $4$ containing a bi-reflection. This arises from a monodromy triple of a hypergeometric differential equation of order $4$ by taking the wedge product and applying a suitable middle convolution as shown in \cite[Theorem 3.3]{Convolution}. Thus we are
in the linearly rigid (hypergeometric) case. Therefore this triple $\mathcal{T} \in \text{Sp}_0(\mathbb{C})^3$ is uniquely determined by its triple of Jordan forms and can be explicitly constructed from a rank one triple together with the differential equation using an iterated sequence of tensor products and convolution operations, see Section 2.

2. Middle convolution

We review some of the properties of the middle convolution for monodromy tuples and differential operators, cf. [3, Section 2] and [2]. The following result is a consequence of the numerology of the middle convolution $\text{MC}_\lambda$ (cf. [3]):

Proposition 2.1. Let $A = (A_1, \ldots, A_{r+1}) \in \text{GL}_n(\mathbb{C})^{r+1}$ be an irreducible monodromy tuple of rank $n$ with at least two non trivial elements. Further let $\lambda \in \mathbb{C}^\times \setminus \{1\}$. Let $(\tilde{B}_1, \ldots, \tilde{B}_{r+1})$ be the monodromy tuple $\text{MC}_\lambda(A)$. Then the following hold:

(i) The monodromy tuple $\text{MC}_\lambda(A)$ is again irreducible of rank $\text{rk}(\text{MC}_\lambda(A)) = \sum_{i=1}^r \text{rk}(A_i - 1) + \text{rk}(A_{r+1} \lambda^{-1} - 1) - \text{rk}(A)$.

(ii) Every Jordan block $J(\alpha, l)$ occurring in the Jordan decomposition of $A_i$ contributes a Jordan block $J(\alpha \lambda, l')$ to the Jordan decomposition of $\tilde{B}_i$, where

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l - 1, & \text{if } \alpha = 1, \\ l + 1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of $\tilde{B}_i$ are blocks of the form $J(1, 1)$.

(iii) Every Jordan block $J(\alpha^{-1}, l)$ occurring in the Jordan decomposition of $A_{r+1}$ contributes a Jordan block $J(\alpha^{-1} \lambda^{-1}, l')$ to the Jordan decomposition of $\tilde{B}_{r+1}$, where

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l - 1, & \text{if } \alpha = 1, \\ l + 1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of $\tilde{B}_{r+1}$ are blocks of the form $J(\lambda^{-1}, 1)$.

(iv) $\text{MC}_\lambda$ preserves linear rigidity.

We have the following explicit construction over any field $K$ for $\text{MC}_\lambda(A)$ in Proposition 2.1. cf. [3]:

The convolution $C_\lambda(A)$ of $A$ with $\lambda \in K^*$ is given by the following $r + 1$ tuple of matrices $B$ in $\text{GL}_n^r(K)^{r+1}$, where $B_i - 1_{rn}$ is a block-matrix that is zero outside the $i$-th block row, given by

$$((A_1 - 1)\lambda, (A_2 - 1)\lambda, \ldots, (A_i - 1)\lambda, A_{i+1} \lambda - 1, A_{i+1} - 1, \ldots, A_r - 1)$$

for $i = 1, \ldots, r$. Moreover $B_1 \cdots B_r - \lambda$ is the block matrix

$$\text{diag}(A_2 \cdots A_r, A_3 \cdots A_r, \ldots, 1) \cdot \lambda : \begin{pmatrix} A_1 - 1 & \ldots & A_r - 1 \\ \vdots & & \vdots \\ A_1 - 1 & \ldots & A_r - 1 \end{pmatrix}.$$

There are the following invariant subspaces:

$$K = \oplus \ker(A_i - 1), \quad \mathcal{L} = \cap_{i=1}^r \ker(B_i - 1) = \ker(B_{r+1} - 1).$$

If $\lambda \neq 1$ then $\mathcal{L} = \{\text{diag}(A_2 \cdots A_r v, A_3 \cdots A_r v, \ldots, v) \mid v \in \ker(A_1 \cdots A_r \lambda - 1)\}$ and $K \cap \mathcal{L} = 0$. The tuple $\text{MC}_\lambda(A)$ corresponding to the middle convolution is given by the action of $B$ on $K^n/(K + \mathcal{L})$. Furthermore if $U \leq K^n$ is an $A$ invariant subspace then $U^r$ is $B$ invariant.
Let \((\lambda_1, \ldots, \lambda_{r+1}) \in K^{r+1}\) be monodromy tuple of rank one. Then we denote by
\[
\text{MT}_{(\lambda_1, \ldots, \lambda_{r+1})}(A) := (\lambda_1 A_1, \ldots, \lambda_{r+1} A_{r+1}).
\]
Let \(\lambda_1, \lambda_2 \in K^*, \lambda = \lambda_1 \lambda_2, \) and
\[
F := \text{MT}_{(\lambda_1^{-1}, \ldots, 1, \lambda_2)} \circ C_\lambda \circ \text{MT}_{(\lambda_1^{-1} \lambda_2, 1, \ldots, 1, \lambda_2)} \circ C_{\lambda^{-1}} \circ \text{MT}_{(\lambda_2, 1, \ldots, 1, \lambda_2)},
\]
\[
\tilde{F} := \text{MT}_{(\lambda_1^{-1}, 1, \ldots, 1, \lambda_2)} \circ \text{MC}_\lambda \circ \text{MT}_{(\lambda_1^{-1} \lambda_2, 1, \ldots, 1, \lambda_2)} \circ \text{MC}_{\lambda^{-1}} \circ \text{MT}_{(\lambda_2, 1, \ldots, 1, \lambda_2)}.
\]

In [H] Cor. 5.15 a) it is already shown that \(\tilde{F}\) preserves autoduality, i.e. if \(A\) is contained in a symplectic or an orthogonal group then the same holds for \(\tilde{F}(A)\). But here we show the refined statement that \(\tilde{F}\) preserves a symmetric bilinear form, resp. an antisymmetric bilinear form.

**Theorem. 2.2.** Let \(A \in \text{GL}_n(K)^{r+1}\) be an irreducible monodromy tuple, such that \(A_i^* X A_i = X, i = 1, \ldots, r, \) for some \(0 \neq X \in \text{Mat}_n(K)\).

(i) Then \(\det(X) \neq 0\) and \(X^t = X, X^t = -X \) resp.

(ii) Let \(B := F(A)\). Then there exists \(0 \neq Y \in \text{Mat}_{nr}(K)\), \(Y^t = Y, Y^t = -Y \) resp., such that
\[
B_i^t Y B_i = Y, \quad i = 1, \ldots, r.
\]

Moreover the matrix \(Y\) is defined via block-matrices as follows:
\[
Y = D_1 Y_0 D_1,
\]
where
\[
D_1 = \text{diag}(D_{11}, \ldots, D_{1r}),
\]
\[
D_{1i} = \text{diag}(A_1 \lambda_1 - 1, A_2 - 1, \ldots, A_r - 1),
\]
\[
D_{11} = \text{diag}(A_1 \lambda_1 - 1) \lambda^{-1}, (A_2 - 1) \lambda^{-1}, \ldots, (A_i - 1) \lambda^{-1}, A_i \lambda - 1, A_{i+1} - 1, \ldots, A_r - 1),
\]
\[
i > 1,
\]
\[
Y_0 = (y_{ij})_{i,j=1,\ldots,r},
\]
\[
y_{1j} = D_{11} H D_{2j} + D_{1j} H (1 - \lambda_1 / \lambda_2), \quad 1 < j,
\]
\[
y_{1j} = D_{1j} H D_{3j} + H D_{4j} (1 - \lambda_1 / \lambda_2), \quad 1 < j,
\]
\[
y_{ij} = D_{2j} H D_{2j} \lambda_2, \quad 2 \leq i, j \leq r,
\]
\[
D_{2j} = \text{diag}(A_1 / \lambda_2 - 1, A_1 \lambda_1 - 1, \ldots, A_1 \lambda_1 - 1),
\]
\[
D_{2i} = \text{diag}(A_1 - 1, \ldots, A_i - 1), \quad i > 1,
\]
\[
D_{31} = \text{diag}(A_1 / \lambda_1 / \lambda_2, A_1 \lambda_1 - 1, \ldots, A_1 \lambda_1 - 1),
\]
\[
D_{4i} = \text{diag}(0, A_i^{-1} - 1, 1, A_i^{-1} \lambda - 1, (A_i^{-1} - 1) \lambda, \ldots, (A_i^{-1} - 1)), \quad i > 1,
\]
\[
H = (h_{ij})_{i,j=1,\ldots,r} \in \text{Mat}_{nr}(K), \quad h_{ij} = X.
\]

Further, if \(\text{char}(K) \neq 2\) then \(\tilde{F}\) preserves both symmetric and antisymmetric bilinear forms.

**Proof.** (i) Since \(\ker(X)\) is \(A\) invariant the irreducibility gives \(\det(X) \neq 0\). Further, the equality \(A_i^* X A_i = X^t\) implies \(X^t = \gamma X\) for some \(\gamma \in K\). Hence \(X = (X^t)^t = \gamma^2 X\) gives (i).

(ii) The claim for \(Y\) and \(\tilde{F}(A)\) is a straightforward computation. It remains to show the claim for \(\tilde{F}(A)\). Let
\[
\tilde{A} = \text{MT}_{(\lambda_1^{-1} \lambda_2, 1, \ldots, 1, \lambda_2^{-1})} \circ C_{\lambda^{-1}} \circ \text{MT}_{(\lambda_1, 1, \ldots, 1, \lambda_1^{-1})}(A)
\]
and \(U = \oplus_{i=1}^r (\ker(A_i - 1) + K + \mathcal{L})\), where
\[
K = \ker(A_1 \lambda_1 - 1) \oplus \ker(A_2 - 1) \oplus \ldots \oplus \ker(A_r - 1)
\]
\[
\mathcal{L} = \{\text{diag}(A_2 \cdots A_r v, \ldots, v) \mid v \in \ker(A_1 A_2 \cdots A_r \lambda^{-1} - 1)\}
\]
are the \(B\) invariant subspaces that arise in the convolution process. Further,
\[
\ker(A_i - 1) = (\ker(A_1 \lambda_1^{-1} - 1), 0, \ldots, 0)
\]
is in the kernel of $D_{13} D_{24} H D_{21} D_{11}$ since $(A_1 \lambda_1 - 1)^4 H = -H \lambda_1 A_1^{-1} (A_1 \lambda_1^{-1} - 1)$. Hence it is straightforward to check that $Y(U) = 0$. Since $\mathcal{B} \cong \mathcal{F}(A)$ is the irreducible tuple induced by the action of $\mathcal{B}$ on $K^{nr}/U$ the claim follows.

By the Riemann-Hilbert correspondence, each monodromy tuple $T \in \text{GL}_n(\mathbb{C})^{\times}$ corresponds to an ordinary Fuchsian differential equation (or, equivalently, an operator $L = \sum_{i=0}^{m} x^i P_i(\vartheta) \in \mathbb{C}[x, \vartheta]$ in the Weyl algebra $\mathbb{C}[x, \vartheta = x \frac{d}{dx}]$) with regular singularities $x_1, \ldots, x_r, x_{r+1} = \infty$). Let $f$ be a solution of $L$, viewed as a section of the local system $\mathcal{L}$ of solutions of $L$, and let $a \in \mathbb{Q} \setminus \mathbb{Z}$.

For two simple loops $\gamma_p, \gamma_q$, based at $x_0 \in A^1 \setminus \{x_1, \ldots, x_r\}$, and moving counterclockwise around $p$, resp. $q$, we define the Pochhammer contour

$$\left[\gamma_p, \gamma_q\right] := \gamma_p^{-1} \gamma_q^{-1} \gamma_p \gamma_q.$$

For $y \in A^1 \setminus \{x_1, \ldots, x_r\}$, the integral

$$C_a^p(f)(y) := \int_{[\gamma_p, \gamma_q]} f(x)(y - x)^a \frac{dx}{y - x}$$

is called the convolution of $f$ and $x^a$ with respect to the Pochhammer contour $[\gamma_p, \gamma_q]$. In [2] Prop. 4.10 it is shown that $C_a^p(f)$ is a solution of

$$C_a(L) := \sum_{i=0}^{m} x^i \prod_{j=0}^{i-1} (\vartheta + i - a - j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a) \in \mathbb{C}[y, \vartheta]$$

for each $p \in \mathbb{P}^1$. In general $C_a(L)$ is not irreducible but the factor that coincides with the differential operator associated to the middle convolution $MC_{\lambda}(T)$, $\lambda = \exp(2\pi ia)$, via the Riemann-Hilbert correspondence can be often easily determined, cf. [2] Cor. 4.16.

Note further that if $T$ is a monodromy tuple of $L(\vartheta)$, where $T$ is the local monodromy at 0, then the tensor product $MT_{(\lambda, 1, \ldots, 1, \lambda^{-1})}(T)$ changes $L(\vartheta)$ to $L(\vartheta - a)$, $\exp(2\pi ia) = \lambda$.

3. Proof of Theorem 1.1

Proof. Let $T \in \text{Sp}_6(\mathbb{C})^3$ be a monodromy triple with Jordan forms

$$(\zeta_0, \zeta_6, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_6^{-1}, \zeta_6^{-1}), \quad (-1, -1, 1, 1, 1), \quad (\zeta_{10}, \zeta_{10}, \zeta_5^{-3}, \zeta_5^{-3}, \zeta_5^{-3}, \zeta_5^{-3}).$$

Applying the sequence

$$MT_{(\zeta_0, 1, \zeta_6^{-1})} \circ MC_{\zeta_6^{-3}} \circ MT_{(\zeta_0^{-3}, \zeta_6^{-1}, 1, \zeta_6^{-1})} \circ MC_{\zeta_6^{-3}} \circ MT_{(\zeta_0, 1, \zeta_6^{-1})}$$

we get a monodromy triple $\tilde{T} \in \text{Sp}_6(\mathbb{C})^3$ with Jordan forms

$$(\zeta_3^{-3}, \zeta_3^{-3}, \zeta_3^{-3}), \quad (-1, -1, 1, 1), \quad (\zeta_{10}, \zeta_{10}, \zeta_{10}, \zeta_{10}, \zeta_{10}, \zeta_{10})$$

by Theorem 2.2 and Proposition 2.4. This triple is again symplectically rigid by 1.1.

Due to [2] Theorem 3.3 we know that $\tilde{T}$ is uniquely determined by its Jordan forms. This also shows the existence and uniqueness of $\tilde{T}$ since the middle convolution $MC_{\lambda}$ is invertible, i.e. $MC_{\lambda} \circ MC_{\lambda^{-1}} \cong \text{id}$ by [3] Theorem 3.5 and Proposition 3.2.

Moreover it is also shown in [2] Theorem 3.3 that $\tilde{T}$ arises from a monodromy triple of a hypergeometric differential operator $L_4$ of order 4 by taking the wedge product and applying the middle convolution $MC_{-\lambda}$.

This allows us to construct the operator $L_{2, 2}$ by applying the corresponding operations for the fuchsian differential operators as explained in [2] Section 4 and indicated in Section 2. We start with the hypergeometric differential operator

$$L_4 = \sum_{\vartheta = 20} (15 \vartheta - 3) (15 \vartheta - 8) (15 \vartheta - 2) - 81 \vartheta (20 \vartheta - 11) (20 \vartheta + 13) (20 \vartheta - 3) (20 \vartheta + 1)$$
Thus, \( J(2) \) gives finally \( T \) beginning of the proof (\( \mathbf{L} \) gives).

The corresponding formally self adjoint operator is then equivalent operators of degree six.

One should note that the two differential operators, namely \( L_4 \) and its dual \( L_4^* \), give rise to equivalent operators of degree six.

Hence \( M \) yields a symplectically rigid triple of rank 4 by Proposition 5.15 and Cor. 5.15 with local monodromy at 0, 1 and \( \infty \)

\( \langle \zeta_{10}, \zeta_6, \zeta_6^{-1}, \zeta_6^{-1} \rangle, \quad (-1, -1, 1, 1), \quad (\zeta_5, \zeta_5^2, \zeta_5^2, \zeta_5^{-1}) \).

The corresponding formally adjoint operator is then

\[
P = 900 \left( 6 \vartheta + 5 \right) \left( 10 \vartheta + 1 \right) \left( 10 \vartheta + 9 \right) \left( 6 \vartheta + 1 \right) \cdot
\]

\[
-x (6480000 \vartheta^4 + 25920000 \vartheta^3 + 42051600 \vartheta^2 + 32263200 \vartheta + 952215) + x^2 5184 (5 \vartheta + 11) (5 \vartheta + 7) (5 \vartheta + 8) (5 \vartheta + 4).
\]

Applying \( MT_{(-1,-1)} \) we get by uniqueness \( \tilde{T} \). Further, the inverse sequence of the beginning of the proof

\[
MT_{(\zeta_6^{-1}, \zeta_6)} \circ MC_{\zeta_6^{-1} \zeta_6} \circ MT_{(\zeta_5 \zeta_6^{-1}, \zeta_6^{-1})} \circ MC_{\zeta_5 \zeta_5^{-1}} \circ MT_{(\zeta_5^{-1}, \zeta_5)}
\]

gives finally \( T \) and \( L_{2, J_2} \). Note that applying this sequence to \( P \) gives \( L_{2, J_2} \) as an irreducible factor of \( L_3 (\vartheta + 1/6) \), where by (2.2)

\[
L_3 (\vartheta) := C_{2/5 + 1/6}(L_2 (\vartheta - 3/5 - 1/6 + 2)), \quad L_2 := C_{3/5 + 1/6}(P(\vartheta - 9/10)).
\]

Thus

\[
L_3 (\vartheta) = 6750000 \vartheta (\vartheta - 1) (3 \vartheta + 2) (3 \vartheta + 1) (6 \vartheta + 1) (2 \vartheta - 1) (30 \vartheta - 17) (30 \vartheta + 7) - 1125x \vartheta (3 \vartheta + 2) (30 \vartheta + 37) (30 \vartheta + 13) \cdot
\]

\[
(4320000 \vartheta^4 + 5760000 \vartheta^3 + 499440 \vartheta^2 + 204960 \vartheta + 20201) + 16x^2 (15 \vartheta + 2) (15 \vartheta + 23) (30 \vartheta + 67) (30 \vartheta + 37) (15 \vartheta + 14) (15 \vartheta + 11) \cdot (30 \vartheta + 43) (30 \vartheta + 13)
\]

and we get the factorization

\[
L_3 (\vartheta) = (30 \vartheta - 17)(30 \vartheta + 7) L_{2, J_2}(\vartheta - 1/6) .
\]

Since there is a triple in \((2, J_2)^3\) with product 1 and same Jordan forms as \( T \), which can be verified by computing the corresponding normalized structure constant, cf. [9] Chap. I, Theorem 5.8 and [3] p. 43, the uniqueness implies that the monodromy group of \( L_{2, J_2} \) is a subgroup of \( 2.J_2 \). Since the finite linear quasi-primitive groups generated by bi-reflections have been classified in [12] Main Theorem one gets that the generated group is \( 2.J_2 \) since the trace of the element of order 10 is in \( Q(\zeta_5) \) but not in \( Q \).

Starting with the monodromy group generators of the generalized hypergeometric differential equation of order 4, e.g. taken from [1] Theorem 3.5, and performing the operations in the above proof on the level of monodromy tuples, cf. Section 2 or [4], one can also construct the corresponding monodromy tuple for \( L_{2, J_2} \) and the invariant hermitian matrix \( H \) over the ring of integers \( \mathbb{Z}[i, \frac{1}{2\sqrt{5}}] \).
Remark. 3.1. Starting with the hypergeometric operator

\[
L_4 = 16(2\theta + 2a_1 - c_1 - 1)(2\theta - 2a_1 + c_1 - 1)(2\theta - 2a_1 - c_1 - 1)(2\theta + 2a_1 + c_1 - 1)
- x(4\theta + 2(c_3 + c_2) + 1)(4\theta + 2(c_2 - c_3) - 1)(4\theta - 2(c_3 + c_2) + 1)(4\theta + 2(c_3 - c_2) - 1)
\]

we get analogously to the proof of Theorem 1.1 the formally adjoint operator of degree 6

\[
L = 64(\theta - a_1)(\theta + a_1)(\theta - 2a_1)(\theta + 2a_1)(\theta + 1 + a_1)(\theta - 1 - a_1)
- x(\theta + 1 + a_1)(\theta - a_1).
\]

\[
(128\theta^4 + 256\theta^3 + \theta^2(-64v_1 + 304) + \theta(-64v_1 + 176) - 32v_2 + 16v_1^2 - 24v_1 + 39)
+ x^264(\theta + 1 + c_3)(\theta + 1 - c_3)(\theta + 1 + c_2)(\theta + 1 - c_2)(\theta + 1 + c_1)(\theta + 1 - c_1),
\]

where

\[
v_1 = (2a_1)^2 + c_1^2 + c_2^2 + c_3^2, \quad v_2 = (2a_1)^4 + c_1^4 + c_2^4 + c_3^4,
\]

with Riemann scheme

\[
\mathcal{R}(L) = \begin{cases}
0 & 1 & \infty \\
1 + a_1 & 3 & c_3 + 1 \\
2a_1 & 5/2 & c_2 + 1 \\
a_1 & 2 & c_1 + 1 \\
-a_1 & 1 & -c_1 + 1 \\
-2a_1 & 1/2 & -c_2 + 1 \\
-1 - a_1 & 0 & -c_3 + 1
\end{cases}.
\]

Specializing the parameters \((a_1, c_1, c_2, c_3)\) we get further examples of operators having monodromy group 2.J_2:

\[
\begin{array}{ccc}
a_1 & (c_1, c_2, c_3) & \text{group} \\
-1/6 & (19/20, 9/20, 3/4) & 2^{3+4} : (3 \times S_3) \\
(13/20, 3/20, 3/4) & (6/7, 5/7, 4/7) & 2 \times U_3(3) \\
(9/10, 7/10, 3/5) & (17/20, 13/20, 3/4) & 2 \times U_3(3) \\
(9/10, 7/10, 4/5) & (7/12, 1/12, 2/3) & 2 \times U_3(3) \\
(14/15, 11/15, 5/3) & (13/15, 8/15, 2/3) & 2 \times U_3(3)
\end{array}
\]

We also get the following irreducible operators having finite monodromy group contained in 2.J_2, cf. [3] p. 42:

\[
\begin{array}{ccc}
a_1 & (c_1, c_2, c_3) & \text{group} \\
-1/6 & (11/12, 7/12, 3/4) & 2^{3+4} : (3 \times S_3) \\
(13/14, 11/14, 9/14) & (7/8, 5/8, 3/4) & 2 \times U_3(3)
\end{array}
\]

The group \(2^{3+4} : (3 \times S_3)\) is an imprimitive subgroup of \(2.J_2 \subseteq Sp_6(\mathbb{C})\) of order \(2^8 \cdot 3^2\). It is a transitive group on 24 points and has the transitive group identification number 5045.

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(S. Reiter) Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany

E-mail address: stefan.reiter@uni-bayreuth.de