Rank 72 high minimum norm lattices

Robert L. Griess Jr.
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA
Abstract

Given a polarization of an even unimodular lattice and integer \( k \geq 1 \), we define a family of unimodular lattices \( L(M, N, k) \). Of special interest are certain \( L(M, N, 3) \) of rank 72. Their minimum norms lie in \( \{4, 6, 8\} \). Norms 4 and 6 do occur. Consequently, 6 becomes the highest known minimum norm for rank 72 even unimodular lattices. We discuss how norm 8 might occur for such a \( L(M, N, 3) \). We note a few \( L(M, N, k) \) in dimensions 96, 120 and 128 with moderately high minimum norms.

**Key words:** even unimodular lattice, extremal lattice, Leech lattice, fourvolution, polarization, high minimum norm.
Contents

1 Introduction 3
2 Integral sublattices of $\Upsilon^3$ 4
3 Minimum norms for rank 72 $L(M, N, 3)$ 6
4 Norm 6 vectors in rank 72 $L(M, N, 3)$ 7
5 Some higher dimensionss 10
6 Appendix: the index-determinant formula 10
7 Appendix: about fourvolution type sublattices and polarizations of Leech 11

1 Introduction

Integral positive definite lattices with high norm for a given rank and discriminant have attracted a lot of attention, due to their connections with modular forms, number theory, combinatorics and group theory. Especially intriguing are those even unimodular lattices which are extremal, i.e. their minimum norms achieve the theoretical upper bound $2\left(\left\lfloor \frac{n}{24} \right\rfloor + 1\right)$, where $n$ is the rank. The rank of an even unimodular lattices must be divisible by 8 (e.g., [16]). The rank of an even integral unimodular extremal lattice is bounded (see [1] or Chapter 7 of [4] and the references therein). Extremal lattices are known to exist in dimensions a multiple of 8 up through 80, except for dimension 72. An extremal rank 72 lattice would have minimum norm 8 [4, 1].

In this article, we construct a family of unimodular lattices $L(M, N, k)$ (2.6) for an integer $k$ and unimodular integral lattices $M, N$ which form a polarization (2.3). Estimates on the minimum norm of $L(M, N, k)$ give some new examples of lattices with moderately high minimum norms.

Of special interest are those $L(M, N, 3)$ of dimension 72 where we input Niemeier lattices for $M$ and $N$. Such a $L(M, N, 3)$ have minimum norm 4, 6 or 8. Norms 4 and 6 occur. According to [14], our result is the first proof that there exists a rank 72 even unimodular lattice for which the minimum norm
is at least 6. We indicate a specific criterion to be checked for such $L(M, N, 3)$ to have minimum norm 8. We conclude by noting certain $L(M, N, k)$ with moderately high norms in dimensions 96, 120 and 128.

This work was supported in part by National Cheng Kung University where the author was a visiting distinguished professor; by Zhejiang University Center for Mathematical Research; by the University of Michigan; and by National Science Foundation Grant NSF (DMS-0600854). We thank Alex Ryba for helpful discussions.

2 Integral sublattices of $\Upsilon^3$

**Definition 2.1.** <lattice> In this article, lattice means a rational positive definite lattice. The term even lattice means an integral lattice in which all norms are integral. For a lattice $L$, we define $\mu(L) := \min \{(x, x) \mid x \in L, x \neq 0\}$ and call it the minimum norm of $L$. If $L_1, L_2, \ldots$ is a set of lattices, we define $\mu(L_1, L_2, \ldots)$ to be the minimum of $\mu(L_1), \mu(L_2), \ldots$.

**Definition 2.2.** <polarization> Suppose that $E$ is an integral unimodular lattice. A polarization is a pair of sublattices $X, Y$ such that $(X, X) \leq 2\Z$, $(Y, Y) \leq 2\Z$, $X + Y = E$ and $X \cap Y = 2E$. It follows that $E$ is even. If $E$ is a lattice and $r > 0$ is a rational number such that $\sqrt{r}E$ is an integral unimodular lattice, a polarization of $E$ is a pair of sublattices $X, Y$ so that $\sqrt{r}M, \sqrt{r}N$ is a polarization of $\sqrt{r}E$.

**Remark 2.3.** <polarization2> If $Z$ is one of $X, Y$ as in (2.2) and $E$ is unimodular, then $\frac{1}{\sqrt{2}} Z$ is integral and unimodular, but may not be even. If $\frac{1}{\sqrt{2}} X$ and $\frac{1}{\sqrt{2}} Y$ are both even lattices we call the polarization an even polarization. If $E$ is not unimodular but $\sqrt{r}E$ is, the polarization $X, Y$ of $E$ is called even if the polarization $\sqrt{r}X, \sqrt{r}Y$ is even.

**Notation 2.4.** <ups> We let $\Upsilon$ be a lattice so that $U := \sqrt{2} \Upsilon$ is an even, integral unimodular lattice.

A polarization of $\Upsilon$ is therefore a pair of integral sublattices $M, N$ such that $M + N = \Upsilon$ and $M \cap N = 2\Upsilon$.

For the time being, $\text{rank}(\Upsilon) = \text{rank}(U)$ is an arbitrary multiple of 8. We know the complete list of possibilities for even, integral unimodular lattices only in dimensions 8, 16 and 24. The rank 24 lattices are called Niemeier lattices since they were first classified by Niemeier [15].
Lemma 2.5. \(<e8polar>\) The \(E_8\)-lattice has an even polarization.

Proof. This is a standard fact. It follows since the \(E_8\) lattice modulo 2 has a nonsingular form with maximal Witt index. One then quotes the characterization of \(E_8\) as the unique (up to isometry) rank 8 even unimodular lattice. Another proof uses the existence of a fourvolution \(\{7.1\}\) on \(E_8\) (one exists, for example, in a natural \(\text{Weyl}(D_8)\) subgroup; if one identifies \(E_8\) with \(BW_{2^3}\), the natural group of isometries \(BW_{2^3}\) contains lower fourvolutions). □

Notation 2.6. \(<gen1>\) We use the notation of \(\{2.4\}\) and let \(M, N\) be a polarization of \(\Upsilon\). Let \(k \geq 2\). Define these sublattices of \(\Upsilon^k\):

\[
L_M := \{(x_1, \ldots, x_k) \in M^k \mid x_1 + \cdots + x_k \in M \cap N\},
\]

\[
L_N := \{(y, y, \ldots, y) \mid y \in N\},
\]

\[
L(M, N, k) := L_M + L_N.
\]

Remark 2.7. \(<gen1.5>\) Because \(L(M, N, 1) = N\) and \(L(M, N, 2) \cong U \perp U\), the interesting case is \(k \geq 3\). If \(k = 2q\) is even, \(L(M, N, k)\) contains \(L^M + L^N\), a sublattice isometric to \(\sqrt{q} U\).

Proposition 2.8. \(<gen2>\) (i) The lattice \(L(M, N, k)\) is an integral lattice and the sublattice \(L_M\) is even.

(ii) If \(k\) is an even integer or \(N\) is an even lattice, \(L(M, N, k)\) is an even lattice. Otherwise, \(L(M, N, k)\) is odd.

(iii) \(L(M, N, k)\) is unimodular.

Proof. (i) To prove integrality, one shows that \(L_M\) and \(L_N\) are integral lattices and that \((L_M, L_N) \leq \mathbb{Z}\). The latter follows since for \((x_1, \ldots, x_k) \in L_M\), \(\sum_i x_i \in N\), an integral lattice. Finally, the evenness of \(L_M\) is obvious since it is integral and a set of generators is even (e.g., all vectors of the form \((x, x, 0^{k-1}), x \in M\) and \((y, 0^{k-1}), y \in 2\Upsilon\)).

(ii) This is obvious from the definition of \(L_N\).

(iii) To prove unimodularity, it suffices by \(\{6.1\}\) to show that \(|L : L_M|^2 = \text{det}(L_M)\). We have \(\text{det}(L_M) = \text{det}(M^k) |M^k : L_M|^2 = 1 \cdot 2^{\text{rank}(M)}\) and \(|L : L_M| = |L_M + L_N : L_M| = |L_N : L_N \cap L_M| = |L_N : L_N \cap M^k| |L_N \cap M^k : L_N \cap L_M| = 2^{\frac{1}{2} \text{rank}(M)} \cdot 1\). □
Theorem 2.9. \textit{<minlmn>} We use the notation $\mu(L_1, L_2, \ldots)$ \textit{(2.1)}.

(i) $\mu(L_M) = 2\mu(M, U)$ and $\mu(L^N) = k\mu(N)$.

(ii) $\mu(L) \leq \min\{k\mu(N), 2\mu(M, U)\}$.

(iii) $\mu(L) \geq \min\{\frac{k}{2}\mu(U), 2\mu(M, U)\}$.

Proof. (i) To determine $\mu(L_M)$, consider the possibility that all entries of $(x_1, \ldots, x_k) \in L_M$ are in $2\Upsilon$.

(ii) This follows from (i) since $L_M$ and $L^N$ are sublattices of $L$.

(iii) If a vector is in $L \setminus L_M$, all of its coordinates are nonzero. □

Notation 2.10. \textit{<leechdef>} We let $\Lambda$ be a Leech lattice, i.e., a Niemeier lattice without roots.

Uniqueness of a rootless Niemeier lattice was proved first in [3], then in different styles in [2] and [7].

We illustrate the use of (2.9) by constructing a Leech lattice. This argument comes from [17], [13]. An analogous construction of a Golay code was created earlier by Turyn [18]. The original existence proof of the Leech lattice [12] makes use of the Golay code (whereas (2.11) does not).

Corollary 2.11. \textit{<leech>} Leech lattices exist.

Proof. We take $M \cong N \cong E_8$ \textit{(2.5)}. From (2.9), $3 \leq \mu(L) \leq 4$. Since $L(M, N, 3)$ is even, $\mu(L(M, N, 3)) = 4$. □

Notation 2.12. \textit{<leechnota>} We use the standard notation $\Lambda$ for a Leech lattice.

3 Minimum norms for rank 72 $L(M, N, 3)$

Notation 3.1. \textit{<rank72nota>} In this section, $L(M, N, 3)$ is a rank 72 lattice for which $M$ and $N$ are Niemeier lattices.

The minimum norm of a Niemeier lattice is 2 unless it is the Leech lattice, for which the minimum norm is 4.

Corollary 3.2. \textit{<mul72>} (i) $\mu(L(M, N, 3)) \geq 4$.

(ii) If $M \not\cong \Lambda$, then $\mu(L(M, N, 3)) = 4$.

(iii) If $U \cong M \cong \Lambda$, then $\mu(L(M, N, 3)) \geq 6$.

(iv) If $U \cong M \cong \Lambda$, and $N \not\cong \Lambda$, then $\mu(L(M, N, 3)) = 6$. 

6
We now prove that situations (ii) and (iv) of the Corollary actually occur. This means proof that suitable polarizations of Υ exist.

**Proposition 3.3.** There exist \( L(M, N, 3) \) with minimum norms 4 and 6.

**Proof.** We take \( U \cong E^3_8 \) and \( M, N \leq U, M \cong N \cong \sqrt{2}E^3_8 \) such that \( M + N = U \) (for example, the orthogonal direct sum of three polarizations as in (2.11) will do). Then (ii) applies.

If \( U \cong \Lambda \), take in \( \Upsilon \) any sublattice \( M \cong \Lambda \) (see (7.2), (7.3)) and any \( N \cong E^3_8 \) (see [7] for existence). Then (iv) applies. □

**Corollary 3.4.** If \( \mu(L(M, N, 3)) = 8 \), \( M \cong N \cong \Lambda \).

The question remains whether there exists a polarization \( M, N \) so that \( \mu(L(M, N, 3)) = 8 \).

**Remark 3.5.** It would be useful to know more about embeddings of \( \sqrt{2}J \) into \( K \), where \( J, K \) are Niemeier lattices. For the case \( K \cong \Lambda \), see [5], Th. 4.1. Note also that embeddings of \( \sqrt{2}E^3_8 \) in \( \Lambda \) were used extensively in [7].

4 Norm 6 vectors in rank 72 \( L(M, N, 3) \)

**Notation 4.1.** Let \( L := L(M, N, 3) \), where \( M \cong N \cong \Lambda \) (by (7.3), there exists such a polarization).

From (3.2)(iii), \( \mu(L) \geq 6 \). We consider the possibility that \( L \) has vectors of norm 6 and derive some results about forms of norm 6 vectors.

We use parentheses both for inner products \( (x, y) \) and \( n \)-tuples \( (x_1, \ldots, x_n) \). We hope for no confusion when \( n = 2 \).

**Notation 4.2.** We call an ordered 4-tuple \( (w, x, y, z) \in N \times M \times M \times M \) admissible if \( x + y + z \in M \cap N \). The elements of \( L \) are the \( (x + w, y + w, z + w) \), for all admissible 4-tuples \( (w, x, y, z) \). We call admissible 4-tuples \( (x, y, z, w) \) and \( (x', y', z', w') \) equivalent if \( (x + w, y + w, z + w) = (x' + w', y' + w', z' + w') \). An offender is a 4-tuple \( (x, y, z, w) \) such that each of \( r_x := x + w, r_y := y + w, r_z := z + w \) has norm 2. Offenders are those admissible 4-tuples which give norm 6 vectors \( (x + w, y + w, z + w) \in L \) (since \( \mu(M) = 4 \), \( w \notin M \) or else \( M \) would contain roots). The set \( r_x, r_y, r_z \) is called a triple of offender roots.
If there are no offenders, \( L \) has minimum norm 8. We therefore study hypothetical offenders.

The rational lattice \( \Upsilon = M + N \) is not integral (in fact, \( (\Upsilon, \Upsilon) = \frac{1}{2} \mathbb{Z} \)). The next result asserts integrality of the sublattice of \( \Upsilon \) spanned by the components of an offender.

**Lemma 4.3.** \(<\text{offint}>\) For an offender, \((w, x, y, z)\), we define \( K \) to be the \( \mathbb{Z} \)-span of \( w, x, y, z \). Then

(i) The image of \( K \) in \((M + N)/M\) has order 2;

(ii) \( K \) is an even integral lattice.

**Proof.**

(i) The image of \( K \) in \((M + N)/M\) is spanned by the image of \( w \), and \( w \notin M, 2w \in M \).

(ii) Since \( x, y, z \) lie in an integral lattice \( M \) and \( w \in N \) is integral, it suffices to prove that each of \((w, x), (w, y), (w, z)\) is integral. We have \( 2 = (w + x, w + x) = (w, w) + 2(w, x) + (x, x) \). Since \( M \) and \( N \) are even lattices, \((w, w)\) and \((x, x)\) are even integers. So \((w, x)\) is integral. Similarly, we prove \((w, y), (w, z)\) are integral. \(\square\)

**Lemma 4.4.** \(<\text{shortmod}>\) Let \( Q \) be a sublattice of \( \Lambda \), \( Q \cong \sqrt{2} \Lambda \). The \( 2^{12} - 1 \) nontrivial cosets each contain exactly 48 norm 4 vectors, and such a set of 48 is an orthogonal frame: two members are proportional or orthogonal.

**Proof.** This may be proved by a rescaling of the argument that in \( \Lambda \), the norm 8 vectors which lie in the same coset of \( 2\Lambda \) constitute an orthogonal frame of 48 vectors. See [3, 6]. \(\square\)

**Lemma 4.5.** \(<\text{wnorm4}>\) Suppose that \( M \) has fourvolution type \([7,2]\). If \((w, x, y, z)\) is admissible and \( w \notin M \), there exists an equivalent admissible quadruple \((w', x', y', z')\) such that \( w' \) has norm 4.

**Proof.** This follows from \([4.4]\). There exists \( v \in \Upsilon \) so that \( w' := w - 2v \in N \) has norm 4 (recall that \( 2\Upsilon = M \cap N \)). Take \( x' := x + 2v, y' := y + 2v, z' := z + 2v \). These three vectors lie in \( M \). \(\square\)

**Lemma 4.6.** \(<\text{orthogoffenderroots}>\) A triple of offender roots is a pair-wise orthogonal set.

**Proof.** Suppose that two such roots are not orthogonal, say \( r = w + x \) and \( s = w + y \). Define \( J := \text{span}\{r, s\} \), an \( A_2 \)-lattice (note that \( J \) is integral, by \([4.3](ii)\)). Since \( M \cap J \) is contained in \( M \), it is rootless. However, \( M \cap J \) has index 2 in \( J \) gives a contradiction since every index 2 sublattice of \( J \) contains roots. \(\square\)
Lemma 4.7. Let $r, s, t$ be the three roots from an offender triple (in any order). The unordered set of inner products $(w, r), (w, s), (w, t)$ is $0, 0, \pm 1$. The unordered set of norms for $x, y, z$ is one of $6, 6, 4$ or $6, 6, 8$.

Proof. The second statement follows from the first, which we now prove. Let $r' \in \{r, -r\}$ satisfy $(w, r') \leq 0$. Similarly, let $s' \in \{s, -s\}$ satisfy $(w, s') \leq 0$ and $t' \in \{t, -t\}$ satisfy $(w, t') \leq 0$. Then $w + r' + s' + t' \in M \cap N$ and $w + r' + s' + t'$ has norm $4 + 2 + 2 + 2 + e$, where $e \leq 0$ and $e$ is even.

We observe that if $w + r' + s' + t'$ were 0, the pairwise orthogonality of $r, s, t$ would imply that $w$ has norm 6, which is not the case. Therefore, $w + r' + s' + t'$ has even norm at least 8. Consequently, $e = 0$ or $e = -2$. Since $M \cap N \cong \sqrt{2} \Lambda$, in which norms are divisible by 4 and nonzero norms are at least 8, $e = -2$. Therefore all but one of $(w, r), (w, s), (w, t)$ is 0 and the remaining one is $\pm 1$. □

Notation 4.8. An offender $(w, x, y, z)$ is a super offender if $w$ has norm 4 and the norms of $x, y, z$ in some order are 6, 6, 4.

Lemma 4.9. We may assume that an offender $(w, x, y, z)$ satisfies $(w, w) = 4$, $(w, t) = 1$ and $(z, z) = 4$. In other words, if an offender exists, a super offender exists.

Proof. Since $(w, t) = \pm 1$, $z = t - w$ has norm 4 or 8, respectively. Suppose the latter. Then $(-w, -x, -y, z + 2w)$ is admissible and its final component $z + 2w = t + w$ has norm 4. Therefore, $(-w, -x, -y, z + 2w)$ is a super offender. □

Theorem 4.10. Let $L := L(M, N)$, where $M \cong N$ are isometric to the Leech lattice. Then the minimum norm of $L$ is 6 if and only if there exists a super offender. Otherwise, the minimum norm is 8.

Remark 4.11. Given $M, N$, (4.10) indicates that checking a (very large) finite number of inner products will settle $\mu(L(M, N, 3))$.

There are finitely many polarizations $M, N$ of $\Upsilon$. Possibly some $L(M, N, 3)$ have minimum norm 6 and others have minimum norm 8. Use of isometry groups and other theory might reduce the number of computations significantly.
5 Some higher dimensionss

**Lemma 5.1.** There exist rank 32 even integral unimodular lattices $U, M, N$ so that $\mu(U) = \mu(M) = 4$, $\mu(N) \in \{2, 4\}$ and $\sqrt{2}M, \sqrt{2}N$ is a polarization of $U$.

**Proof.** We take $U$ to be $BW_{24}$. If $f$ is a fourvolution in $O(U)$, then $M := (f-1)U \cong \sqrt{2}U$. Therefore, the natural $F_2$-valued quadratic form on $U/2U$ is split (i.e., has maximal Witt index) and so there exists an even unimodular lattice $N$ so that $\sqrt{2}N$ is between $U$ and $2U$ and $\sqrt{2}N/2U$ complements $M/2U$ in $U/2U$. The extremal bound $\mu(N) \leq 4$ and evenness of $N$ imply the last statement. □

We now exhibit a few even unimodular lattices for which the minimum norm is moderately close to the extremal bound $2(1 + \lceil \frac{\text{rank}(L)}{24} \rceil)$.

**Proposition 5.2.** Let $U, M, N$ as in (5.1) and let $k = 3$. Then the minimum norm of the rank 96 lattice $L(M, N, 3)$ is 6 or 8.

**Proof.** The value of $\mu$ depends on whether there exists rank 32 even unimodular lattices $U, M, N$ as in (5.1) so that $\mu(N) = 4$. □

**Theorem 5.3.** There exists an even unimodular lattice $L(M, N, k)$ of rank $\ell$ and minimum norm $\mu$ for the following pairs $(\ell, \mu)$:

(i) $(96, 8)$ (the extremal bound is 10);

(ii) $(120, 8)$ (the extremal bound is 12).

(iii) $(128, 8)$ (the extremal bound is 12)

**Proof.** We use (2.9).

(i) Take $k = 4$ and $U, M, N \cong \Lambda (7.3)$.

(ii) Take $k = 5$ and $U, M, N \cong \Lambda (7.3)$.

(iii) Take $k = 4$ where $U, M, N$ are rank 32 lattices as in (5.1). □

6 Appendix: the index-determinant formula

**Theorem 6.1.** ("Index-determinant formula") Let $L$ be a rational lattice, and $M$ a sublattice of $L$ of finite index $|L : M|$. Then

$$\det(L) |L : M|^2 = \det(M).$$
Proof. This is a well-known result. Choose a basis \( x_1, \cdots, x_n \) for \( L \) and positive integers \( d_1, d_2, \cdots, d_n \), so that \( M \) has a basis \( d_1 x_1, d_2 x_2, \cdots, d_n x_n \). A Gram matrix for the lattice \( M \) is \( G_M = ((d_i x_i, d_j x_j)) = D G_L D \), where

\[
D = \begin{pmatrix}
  d_1 & & \\
  & d_2 & \\
  & & \ddots
\end{pmatrix},
\]

and \( G_L = ((x_i, x_j)) \) is a Gram matrix for \( L \). Thus \( \det(G_M) = \det(D)^2 \cdot \det(G_L) \). □

7 Appendix: about fourvolution type sublattices and polarizations of Leech

Definition 7.1. <fourvolution> A fourvolution \( f \) is a linear transformation whose square is \(-1\). If \( f \) is orthogonal, \( f - 1 \) doubles norms.

Definition 7.2. <fourvolutiontype> Let \( L \) be an integral lattice. A sublattice \( M \) of \( L \) is of fourvolution type if there exists a fourvolution \( f \) so that \( M = L(f - 1) \) (whence \( M \cong \sqrt{2}L \)). The same terminology applies to scaled copies of \( \Lambda \).

Lemma 7.3. <leechleechpolar> If \( U \cong \Lambda \), there are polarizations of \( \Upsilon \) by sublattices \( M \cong N \cong \Lambda \).

Proof. Here is one proof. We use a fact about \( O(\Lambda) \), that there are pairs of fourvolutions \( f, g \) so that \( \langle f, g \rangle \) is a double cover of a dihedral group of order \( 2k \) for which an element of odd order \( k > 1 \) has no eigenvalue 1 on \( \Lambda \). There exist examples of this for \( k = 3, 5 \), at least (for which \( C_{O(\Lambda)}(\langle f, g \rangle) \cong 2 \cdot G_2(4), 2 \cdot HJ \), respectively) [6]. We take \( M := \Lambda(f - 1) \) and \( N := \Lambda(g - 1) \). Since \( 2\Lambda = \Lambda(f - 1)^2 = \Lambda(g - 1)^2 \), \( M \cap N \geq 2\Lambda \). We argue that the pair \( M, N \) gives a polarization. Since \( (M \cap N)/2\Lambda \) consists of vectors fixed by \( \langle f, g \rangle \), it is 0. By determinant considerations, \( M + N = \Upsilon \). □

References

[1] Christine Bachoc and Gabriele Nebe, Extremal lattices of minimum 8 related to the Mathieu group \( M_{22} \), J. Reine Angew. Math. 494 (1998)
[2] Richard Borcherds, The Leech lattice, Proc. Royal Soc. London A398 (1985) 365-376.

[3] John Conway, A characterization of Leech’s lattice, Invent. Math. 7 (1969), 137-142.

[4] John Conway and Neil Sloane, Sphere Packings, Lattices and Groups, Springer-Verlag 1988.

[5] Dong, C.; Li, H.; Mason, G.; Norton, S. P., Associative subalgebras of the Griess algebra and related topics. The Monster and Lie algebras (Columbus, OH, 1996), 27–42, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.

[6] Robert L. Griess, Jr., Twelve Sporadic Groups, Springer Verlag, 1998.

[7] Robert L. Griess, Jr., Pieces of Eight, Advances in Mathematics, 148, 75-104 (1999).

[8] Robert L. Griess, Jr., Positive definite lattices of rank at most 8, Journal of Number Theory, 103 (2003), 77-84.

[9] Robert L. Griess, Jr., Involutions on the the Barnes-Wall lattices and their fixed point sublattices, I. Pure and Applied Mathematics Quarterly, vol.1, no. 4, (Special Issue: In Memory of Armand Borel, Part 3 of 3) 989-1022, 2005.

[10] Robert L. Griess, Jr., Pieces of 2^d: existence and uniqueness for Barnes-Wall and Ypsilanti lattices. Advances in Mathematics, 196 (2005) 147-192. math.GR/0403480

[11] Robert L. Griess, Jr., Corrections and additions to “Pieces of 2^d: existence and uniqueness for Barnes-Wall and Ypsilanti lattices.” [Adv. Math. 196 (2005) 147-192], Advances in Mathematics 211 (2007) 819-824.

[12] John Leech, Notes on sphere packings, Canadian Journal of Mathematics 19 (1967), 251-267.

[13] James Lepowsky and Arne Meurman, An E_8 approach to the Leech lattice and the Conway groups, J. Algebra 77 (1982), 484-504.
[14] Gabriele Nebe and Neil J. A. Sloane, Catalogue of Lattices, 
http://www.research.att.com/~njas/lattices/abbrev.html

[15] H. V. Niemeier, Definite Quadratische Formen der Diskriminante
1 und Dimension 24, Doctoral Dissertation, Göttingen, 1968.

[16] Jean-Pierre Serre, A Course in Arithmetic, Springer Verlag, Graduate Texts in Mathematics 7, 1973.

[17] Tits, Jacques: Four Presentations of Leech’s lattice, in Finite Simple Groups, II, Proceedings of a London Math. Soc. Research Symposium, Durham, 1978, ed. M. J. Collins, pp, 306-307, Academic Press, London, New York, 1980.

[18] E. F. Assmus, Jr., H. F. Mattson, Jr. and R. J. Turyn, Research to Develop the Algebraic Theory of Codes, Report AFCRL-67-0365, Air Force Cambridge Res. Labs., Bedford, Mass, June 1967.