Killing–Yano symmetry of Kaluza–Klein black holes in five dimensions

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Abstract
Using a generalized Killing–Yano equation in the presence of torsion, spacetime metrics admitting a rank-2 generalized Killing–Yano tensor are investigated in five dimensions under the assumption that its eigenvector associated with the zero eigenvalue is a Killing vector field. It is shown that such metrics are classified into three types and the corresponding local expressions are given explicitly. It is also shown that they cover some classes of charged, rotating Kaluza–Klein black hole solutions of minimal supergravity and Abelian heterotic supergravity.

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1. Introduction

Killing–Yano tensors which were introduced by Yano [1] as a generalization of Killing vectors to higher rank antisymmetric tensors have attracted the interests of many authors in the study of black hole physics as their presence endows black hole spacetimes with remarkable mathematical properties. For instance, Carter [2] studied a certain class of metrics including the Kerr metric as well as Kerr–Newman metric,

$$\text{d}s^2 = \frac{r^2 + p^2}{Q(r)}\text{d}r^2 + \frac{r^2 + p^2}{P(p)}\text{d}p^2 - \frac{Q(r)}{r^2 + p^2}(\text{d}r + p\text{d}\sigma)^2 + \frac{P(p)}{r^2 + p^2}(\text{d}r - r^2\text{d}\sigma)^2,$$

(1.1)

which is called Carter’s class. He demonstrated that for all the metrics of this class, both the Hamilton–Jacobi and Klein–Gordon equations can be solved by the separation of variables. These separability structures are known to be deeply related to the existence of rank-2 Killing–Yano tensors [3, 4]. Moreover, Dietz and Rudiger [5] have shown that any four-dimensional
metric admitting a rank-2 Killing–Yano tensor can be always written in the form of Carter’s class. Namely, in four dimensions, spacetime metrics admitting a rank-2 Killing–Yano tensor involve many rotating black hole solutions of Einstein’s equations, e.g., Kerr and Kerr–Newman metrics, and due to the Killing–Yano symmetry, some test field equations such as Hamilton–Jacobi and Klein–Gordon equations are solvable by the separation of variables. One of our tasks remaining in this direction is clarifying what happens in higher dimensions.

The investigation of Killing–Yano symmetry in higher dimensional black hole spacetimes has just started in the last decade, e.g., see reviews [6–8]. It is known that in higher dimensional vacuum solutions describing rotating black holes with spherical horizon topology [9–12], rank-2 closed conformal Killing–Yano (CKY) tensors explain the separability of the Hamilton–Jacobi and Klein–Gordon equations on those spacetimes. Here, conformal Killing–Yano tensors are antisymmetric tensors which were introduced by Tachibana [13] and Kashiwada [14] as a generalization of conformal Killing vectors. However, CKY tensors are no longer useful to explain the separability structure in higher dimensional charged, rotating black hole spacetimes. Accordingly, a further generalization of CKY tensors was introduced by the authors of [15, 16]. The generalized CKY tensors can be thought of as CKY tensors on spacetimes with a skew-symmetric torsion $T$. The torsion is usually (not necessarily) identified with matter fluxes appearing in the theories. For instance, the five-dimensional gauged minimal supergravity black hole [17] admits a rank-2 generalized CKY tensor when the torsion is identified with the Hodge dual of the Maxwell field, $T = *F/\sqrt{3}$. It was also shown that the Abelian heterotic supergravity black holes as well as their generalization to higher dimensions [18–20] admit a rank-2 generalized CKY tensor. In this case, the torsion is identified with the 3-form field strength, $T = H$.

Compared to the asymptotically flat black holes, the separability structure of black strings or Kaluza–Klein black holes has not been studied very well. Since the inheritance of separability by the uplift which is obvious for vacuum solutions is also expected in the presence of flux along the extra dimension, we are lead to the investigation of the generalized Killing–Yano symmetry in those spacetimes. In this paper, we thus elaborate the relationship between Killing–Yano symmetry and separability of the Hamilton–Jacobi and Klein–Gordon equations in charged, rotating Kaluza–Klein black hole spacetimes, especially in the five-dimensional minimal supergravity and Abelian heterotic supergravity. In fact, it is shown that Killing–Yano symmetry of the Kaluza–Klein black holes is described by rank-2 generalized Killing–Yano tensors. While Killing–Yano and generalized Killing–Yano tensors have been discussed in relation to the separation of Dirac equations as well [21–28], we will not discuss it in this paper.

The presence of Killing–Yano symmetry itself is a strong enough restriction so that one can obtain explicit expressions of the metrics before imposing the dynamical equations. For example, some geometrical properties of spacetimes admitting Killing–Yano tensors were discussed in four dimensions [5, 29, 30]. The most general metric admitting a rank-2 closed CKY tensor was obtained in arbitrary dimension [31, 32]. Even when it is not possible to write down the most general metric, this approach enables us to understand the separability structure of various black hole spacetimes in a wider and unified framework [33, 34]. By making a suitable simplification, we indeed derive a class of five-dimensional metrics admitting a rank-2 generalized Killing–Yano tensor, which include and generalize the known examples of black strings and Kaluza–Klein black holes.

This paper is organized as follows. In section 2, we first attempt to classify spacetime metrics admitting a rank-2 generalized Killing–Yano tensor in five dimensions under the assumption that there exists a particular Killing vector field. A large family of metrics is obtained. We find that resulting metrics are classified into three types, which we call types
A, B and C, and some local expressions of the metrics are given explicitly. In section 3, we consider the solution in the five-dimensional minimal supergravity, which is obtained as an uplift of the Kerr spacetime, and see that the metric falls into type A of the classified metrics. It is shown that a rank-2 generalized Killing–Yano tensor is responsible for the separation of variables in the Hamilton–Jacobi and Klein–Gordon equations. In section 4, we review the charged, rotating black string solution discovered by Mahapatra [35] in heterotic supergravity. We will see that the metric falls into type A again. The separability of the Hamilton–Jacobi and Klein–Gordon equations is also associated with a rank-2 generalized Killing–Yano tensor. Adopting the generalized Killing–Yano symmetry, we construct a class of charged, rotating Kaluza–Klein black hole solutions in theory. Section 5 is devoted to summary and discussion.

2. Metrics admitting a rank-2 generalized Killing–Yano tensor in five dimensions

In this section, we attempt to classify spacetime metrics admitting a rank-2 generalized Killing–Yano tensor in five dimensions. Although we are interested in Lorentzian manifolds, for simplicity, the calculation in this section is carried out in Euclidean signature \((+, +, \ldots, +)\).

Keeping applications to Kaluza–Klein spacetimes in mind, we assume that there exists a particular Killing vector field that is an eigenvector of the generalized Killing–Yano tensor with zero eigenvalue, leaving more general investigations for a future work. In section 2.1, we begin with reviewing the basics of the rank-2 generalized Killing–Yano tensors. Introducing canonical frames associated with such tensors, we derive the general forms of connection 1-forms in terms of the canonical frame in section 2.2. The computation there is performed by exploiting the technique of [34]. Furthermore, we proceed to restrict the forms of the connection 1-forms by imposing integrability conditions, so that commutation relations among canonical basis vectors are obtained in section 2.3. Finally, solving the commutation relations in sections 2.4–2.6, some local expressions of the metrics are given explicitly. In the process of solving the commutation relations, we find that resulting metrics are classified into three types, which we call types A, B and C.

2.1. Basics

Let \((M, g)\) be a five-dimensional Riemannian manifold and \(\{e_a\}\) be an orthonormal frame. Throughout the paper, Latin indices \(a, b, \ldots\) range from 1 to 5. Greek letters \(\mu, \nu \ldots\) will later be used to denote two-dimensional eigenspaces of non-zero eigenvalues of a 2-form. The dual frame \(\{e^a\}\) satisfies \(e^a \cdot e^b = \delta^a_b\) where \(\cdot\) represents the inner product. A \(p\)-form \(k\) is written as

\[
k = \frac{1}{p!} k_{a_1 \ldots a_p} e^{a_1} \wedge e^{a_p}, \quad k_{[a_1 \ldots a_p]} = k_{a_1 \ldots a_p}.
\]

(2.1)

For a 2-form \(f\), a rank-2 Killing–Yano tensor, introduced by [1], is subject to the equation

\[
\nabla_a f_{bc} + \nabla_b f_{ca} = 0,
\]

(2.2)

where \(\nabla_a\) is the Levi-Civita connection. In the presence of a skew-symmetric torsion, \(T_{[abc]} = T_{abc}\), a connection \(\nabla^T_a\) is defined by

\[
\nabla^T_a \epsilon^b = \nabla_a \epsilon^b + \frac{1}{2} T_{abc} \epsilon^c.
\]

(2.3)

By replacing the connections \(\nabla_a\) in (2.2) with \(\nabla^T_a\), rank-2 generalized Killing–Yano (GKY) tensors are defined [15] by

\[
\nabla^T_a f_{bc} + \nabla^T_b f_{ca} = 0.
\]

(2.4)
For a rank-\(p\) GKY tensor, the Hodge dual gives a rank-\((D - p)\) generalized closed conformal Killing–Yano (GCCKY) tensor in \(D\) dimensions \cite{33}. In \(D = 5\), the Hodge dual \(h = *f\) of a rank-2 GKY tensor \(f\) is a rank-3 GCCKY tensor obeying
\[
\nabla^T_a h_{bcd} = g_{ab} \xi_{cd} + g_{ac} \xi_{db} + g_{ad} \xi_{bc},
\]
(2.5)
where
\[
\xi_{ab} = \frac{1}{3} \nabla^T_c h_{cab}
\]
(2.6)
is called an associated 2-form of \(h\). Equation (2.5) implies that
\[
\nabla^T_a [h_{abcd}] = 0, \quad \nabla^T_b \xi_{ba} = 0.
\]
(2.7)

From a rank-2 GKY tensor \(f\), one can construct a rank-2 Killing–Stäckel tensor
\[
K_{ab} = f_{ac} f_{b}^c,
\]
(2.8)
which is characterized by
\[
\nabla^T_a K_{bc} = 0.
\]
(2.9)
In general, the existence of a rank-2 Killing–Stäckel tensor guarantees the separation of the Hamilton–Jacobi equations for geodesics with the corresponding separation constant \(\kappa^{(H)}\) given by
\[
\kappa^{(H)} = K_{ab} \Pi^a \Pi^b,
\]
(2.10)
where \(\Pi^a\) is the canonical momentum associated with the geodesic. When \(K_{ab}\) is written in terms of a rank-2 Killing–Yano tensor as (2.8), the separation constant of the Klein–Gordon equation \(\kappa^{(KG)}\) appears as an eigenvalue of a symmetry operator \([\hat{K}, \Box] = 0\), where \(\Box\) is the scalar wave operator \(\Box \equiv g^{ab} \nabla_a \nabla_b\). That is,
\[
\hat{K} \Psi = \kappa^{(KG)} \Psi.
\]
(2.11)
However, this is not always true for rank-2 GKY tensors. In fact, we will see that the separation of variables occurs in a deformed Klein–Gordon equation in section 4.

2.2. General forms of the connection 1-forms

Let us consider a rank-3 GCCKY tensor \(h\) in five dimensions, which is equivalent to considering a rank-2 GKY tensor \(f\). Then, one can always find an orthonormal frame \(\{e^\mu\} = \{e^\mu, e^\hat{\mu}\} = \{e^1, e^2\}\), such that a metric \(g\) and a rank-3 GCCKY tensor \(h\) are simultaneously written in the form
\[
g = \sum_{\mu=1}^{2} (e^\mu \otimes e^\mu + e^{\hat{\mu}} \otimes e^{\hat{\mu}}) + e^0 \otimes e^0,
\]
(2.12)
\[
h = \sum_{\mu=1}^{2} x_\mu e^\mu \wedge e^{\hat{\mu}} \wedge e^0,
\]
(2.13)
where \(x_\mu\) are called the eigenvalues of \(h\). The rank-3 GCCKY tensor is said to be non-degenerate if its eigenvalues \(x_\mu\) are non-vanishing functions with \(x_1 \neq x_2\). Since there are still degrees of freedom under rotation in each \((e^\mu, e^{\hat{\mu}})\)-plane, the orthonormal frame is fixed completely by introducing a 1-form \(\eta\) as
\[
\eta = -e_{0 \mu} \xi^\mu = \sqrt{Q_1} e^1 + \sqrt{Q_2} e^2,\]
(2.14)
where \(\xi\) is the associated 2-form introduced in (2.6), and \(Q_1\) and \(Q_2\) are unknown functions. This means that we used the remaining rotations so as to set \(\xi_{10} = \xi_{20} = 0\). The fixed
orthonormal frame is called a canonical frame. With respect to the canonical frame, the rank-2 GKY tensor $\mathbf{f}$ and rank-2 Killing–Stäckel tensor $\mathbf{K}$, given by (2.8), are written as

$$\mathbf{f} = x_2 \mathbf{e}^1 \wedge \mathbf{e}^1 + x_1 \mathbf{e}^2 \wedge \mathbf{e}^2,$$

$$\mathbf{K} = x_2^2 (\mathbf{e}^1 \otimes \mathbf{e}^1 + \mathbf{e}^1 \otimes \mathbf{e}^1) + x_1^2 (\mathbf{e}^2 \otimes \mathbf{e}^2 + \mathbf{e}^2 \otimes \mathbf{e}^2).$$

The GCCKY equation (2.5) can be thought of as relating components of connection 1-forms $\omega^\mu_a$ to the eigenvalues $\mu_a$ of the GCCKY tensor $\mathbf{h}$ and their derivatives. Exploiting the technique of [34] which was developed for rank-2 GCCKY tensors, we now apply it to rank-3 GCCKY tensors in five dimensions. Thus, a similar calculation leads us to the following results:

$$\omega^\mu_a = -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu - \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu + \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu - \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu + \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu - \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu + \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu - \frac{x_1}{x_2} \mathbf{e}^a$$

$$= -\frac{x_1 x_2}{x^2} \mathbf{e}^\mu - \frac{x_1}{x_2} \mathbf{e}^a$$

where the symbols $\kappa_{a\mu}$ are defined as

$$\kappa_{a\mu} = e_{b\mu} \nabla^b \Psi.$$

### 2.3. Integrability conditions

To obtain the information about second derivatives $\kappa_{a\mu}$, we consider the integrability conditions of the Killing–Yano equation (2.5),

$$- R^f_{a \mu \nu} h_{f \gamma} - R^f_{a \mu \nu} h_{f \gamma} - R^f_{a \mu \nu} h_{f \gamma} = g_{a \nu} \nabla^f_\mu \xi_a + g_{b \nu} \nabla^f_\mu \xi_a + g_{a \nu} \nabla^f_\mu \xi_a - (d \leftrightarrow e)$$

$$+ T_{a \mu \nu \xi_a} + T_{a \mu \nu \xi_a} + T_{a \mu \nu \xi_a},$$

where $R^f_{a \mu \nu}$ are components of the Riemann tensor with respect to $\nabla^f_\mu$ defined by

$$(\nabla^f_a \nabla^f_b - \nabla^f_b \nabla^f_a + T_{a \mu \nu \xi_a}) = - R^d_{a \mu \nu} Z_d.$$
The general form of the integrability conditions is too complicated to solve analytically since it contains many coupled, nonlinear partial differential equations. Therefore, for simplicity, we impose an assumption that \( e_0 \) is a Killing vector field. This is motivated by the fact that the known example which admits a rank-3 GCCKY tensor in the literature \([36]\) satisfies this assumption. It leads us to vanishing of many components of the associated 2-form \( \xi \) and the torsion \( T \). In fact, the Killing equation \( \nabla_a(e^0_b) + \nabla_b(e^0_a) = 0 \) implies
\[
\xi_{\mu \nu} = \xi_{\mu \nu} = \xi_{\mu \nu} = 0, \tag{2.28}
\]
and hence it follows from the integrability conditions \((2.25)\) that
\[
T_{\nu \nu \bar{\nu}} = T_{\mu \nu \bar{\nu}} = T_{\mu \nu \bar{\nu}} = 0,
\]
\[
T_{\mu \nu \bar{\nu}} = -\frac{\kappa_{\mu \nu}}{\sqrt{Q_\mu}}, \quad T_{\mu \nu \bar{\nu}} = -\frac{\kappa_{\mu \nu}}{\sqrt{Q_\mu}}. \tag{2.29}
\]
The commutators now simplify to
\[
[e_\mu, e_\nu] = -x_\nu \sqrt{Q_\mu} e_\mu - x_\mu \sqrt{Q_\nu} e_\nu,
\]
\[
[e_\mu, e_{\bar{\nu}}] = K_\mu e_\mu + L_\mu e_{\bar{\nu}} + M_\mu e_{\bar{\nu}} - T_{\mu \nu \bar{\nu}} e_0,
\]
\[
[e_\mu, e_{\nu}] = -x_\nu \sqrt{Q_\mu} e_\mu - x_\mu \sqrt{Q_\nu} e_\nu,
\]
\[
[e_{\bar{\mu}}, e_{\bar{\nu}}] = 0, \quad [e_\mu, e_0] = 0, \quad [e_{\bar{\mu}}, e_0] = 0, \tag{2.30}
\]
where
\[
K_\mu = \frac{\kappa_{\mu \nu}}{\sqrt{Q_\mu}}, \quad L_\mu = -\frac{1}{\sqrt{Q_\mu}} \left( \frac{x_\mu Q_\mu}{x_\mu^2 - x_\nu^2} - \kappa_{\mu \nu} \right),
\]
\[
M_\mu = \frac{2x_\nu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} - T_{\mu \nu \bar{\nu}}. \tag{2.31}
\]
The classification problem under the assumption that \( e_0 \) is a Killing vector field reduces to solving the commutators \((2.30)\). The integrability conditions, e.g., the Jacobi identities, which have not been satisfied yet, give rise to two algebraic equations
\[
M_1 K_2 = 0, \quad M_2 K_1 = 0, \tag{2.32}
\]
and a system of coupled, nonlinear partial differential equations
\[
e_\nu(K_\mu) = \frac{x_\nu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} K_\mu,
\]
\[
e_\nu(L_\mu) = \frac{x_\nu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} L_\mu - M_\mu M_\nu - \frac{2x_\nu x_\rho \sqrt{Q_\mu} \sqrt{Q_\nu}}{(x_\mu^2 - x_\nu^2)^2},
\]
\[
e_\nu(M_\mu) = \left( \frac{2x_\nu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} L_\nu \right) M_\mu,
\]
\[
e_\nu(T_{\mu \nu \bar{\nu}}) = \frac{2x_\nu \sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} T_{\mu \nu \bar{\nu}} - M_\mu T_{\nu \bar{\nu}}. \tag{2.33}
\]
and
\[
e_{\bar{\mu}}(K_\mu) = 0, \quad e_{\bar{\mu}}(L_\mu) = 0, \quad e_{\bar{\mu}}(M_\mu) = 0, \quad e_{\bar{\mu}}(T_{\mu \nu \bar{\nu}}) = 0,
\]
\[
e_0(K_\mu) = 0, \quad e_0(L_\mu) = 0, \quad e_0(M_\mu) = 0, \quad e_0(T_{\mu \nu \bar{\nu}}) = 0. \tag{2.34}
\]
There exist orthonormal frames satisfying \((2.30)\) at least if its integrability conditions hold. Namely, the algebraic equations \((2.32)\) imply that there are three types of the solutions: \((A) \ K_1 = K_2 = 0, \ (B) \ M_1 = M_2 = 0 \) and \((C) \ K_1 = M_1 = 0 \) or \( K_2 = M_2 = 0 \). For each type, we are able to find large families of solutions of the remaining partial differential equations.
2.4. Type A metric

2.4.1. Type A. Firstly, we consider $K_1 = K_2 = 0$ case. In this case, we find the canonical frame $\{e_\mu\}$ satisfying the commutators (2.30) as

$$e_\mu = \frac{\partial}{\partial x^\mu}, \quad e_0 = \frac{\partial}{\partial \psi},$$

$$e_\tilde{\nu} = \frac{1}{f_\mu \sqrt{(x^\mu - x^\lambda X_\mu)}} \left( (x^\mu + N_\mu) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_\mu \frac{\partial}{\partial \psi} \right),$$  \hspace{1cm} (2.35)

where $X_\mu, N_\mu$ and $f_\mu$ are unknown functions of one variable $x_\mu$ and $\lambda$ is an arbitrary constant.

Rewriting $(x_1, x_2)$ as $(x, y)$, the corresponding metric is written by

$$g = \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 + (d\psi + \lambda W_1)^2 + \frac{f_1(x)^2 X(x)}{x^2 - y^2} (d\tau + y^2 d\sigma - W_1)^2$$

$$+ \frac{f_2(y)^2 Y(y)}{y^2 - x^2} (d\tau + x^2 d\sigma - W_1)^2,$$  \hspace{1cm} (2.36)

where

$$W_1 = \frac{N_1(x)}{(x^2 - y^2)} (d\tau + y^2 d\sigma) + \frac{N_2(y)}{(y^2 - x^2)} (d\tau + x^2 d\sigma),$$

$$\Phi = 1 + \frac{N_1(x)}{x^2 - y^2} + \frac{N_2(y)}{y^2 - x^2}.$$  \hspace{1cm} (2.37)

The metric contains six unknown functions of one variable, $X(x), Y(y), N_1(x), N_2(y), f_1(x)$ and $f_2(y)$. Since the metric components are independent of the coordinates $\tau, \sigma$ and $\psi$, $\partial/\partial \tau$, $\partial/\partial \sigma$ and $\partial/\partial \psi$ are three Killing vector fields. Then, the torsion is given by

$$T = \left[ \frac{2x}{x^2 - y^2} - \frac{f_2(y)}{f_1(x)} \left( \frac{2x}{x^2 - y^2} + \frac{\partial \ln \Phi}{\partial x} \right) \right] \sqrt{\frac{Y(y)}{y^2 - x^2}} e^1 \wedge e^1 \wedge e^2$$

$$+ \left[ \frac{2y}{y^2 - x^2} - \frac{f_1(x)}{f_2(y)} \left( \frac{2y}{y^2 - x^2} + \frac{\partial \ln \Phi}{\partial y} \right) \right] \sqrt{\frac{X(x)}{x^2 - y^2}} e^2 \wedge e^2 \wedge e^1$$

$$+ \frac{\lambda}{f_1(x)} \frac{\partial \ln \Phi}{\partial x} e^3 \wedge e^1 \wedge e^0 + \frac{\lambda}{f_2(y)} \frac{\partial \ln \Phi}{\partial y} e^2 \wedge e^3 \wedge e^0.$$  \hspace{1cm} (2.38)

This type of metrics includes the charged, rotating black strings discovered by Mahapatra [35] and Kaluza–Klein black holes in Abelian heterotic supergravity, as we shall see in section 4.

2.4.2. Type $A_\infty$. An interesting special case arises in the limit $\lambda \rightarrow \infty$, keeping fixed the product $\lambda N_\mu$. To be explicit, if one scales $\lambda$ and $N_\mu$ in (2.35) as follows,

$$\lambda \rightarrow \lambda/\epsilon, \quad N_\mu \rightarrow \epsilon N_\mu,$$  \hspace{1cm} (2.39)

and then takes the scaling limit $\epsilon \rightarrow 0$, one obtains precisely

$$e_\mu = \frac{X_\mu}{x^\mu - x^\lambda X_\mu}, \quad e_0 = \frac{\partial}{\partial \psi},$$

$$e_\tilde{\nu} = \frac{1}{f_\mu \sqrt{(x^\mu - x^\lambda X_\mu)}} \left( x^\mu \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - N_\mu \frac{\partial}{\partial \psi} \right).$$  \hspace{1cm} (2.40)
where $X_\mu, N_\mu$ and $f_\mu$ are functions of single variable $x_\mu$ again. Thus, the corresponding metric is given by

$$g = \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 + (d\psi + W_2)^2 + \frac{f_1(x)^2 X(x)}{x^2 - y^2} (d\tau + y^2 d\sigma)^2$$

$$+ \frac{f_2(y)^2 Y(y)}{y^2 - x^2} (d\tau + x^2 d\sigma)^2,$$  

(2.41)

where

$$W_2 = \frac{N_1(x)}{x^2 - y^2} (d\tau + y^2 d\sigma) + \frac{N_2(y)}{y^2 - x^2} (d\tau + x^2 d\sigma).$$  

(2.42)

The metric contains six unknown functions of one variable, $X(x), Y(y), N_1(x), N_2(y), f_1(x)$ and $f_2(y)$ and has three Killing vector fields $\partial/\partial \tau$, $\partial/\partial \sigma$ and $\partial/\partial \psi$. The torsion is given by

$$T = \left(1 - \frac{f_2(y)}{f_1(x)}\right) \frac{2x}{x^2 - y^2} \frac{Y(y)}{y^2 - x^2} e^1 \wedge e^1 \wedge e^2$$

$$+ \left(1 - \frac{f_1(x)}{f_2(y)}\right) \frac{2y}{y^2 - x^2} \frac{X(x)}{x^2 - y^2} e^2 \wedge e^3 \wedge e^1$$

$$+ \frac{1}{f_1(x)} \frac{\partial \Phi}{\partial x} e^1 \wedge e^1 \wedge e^0 + \frac{1}{f_2(y)} \frac{\partial \Phi}{\partial y} e^2 \wedge e^3 \wedge e^0.$$  

(2.43)

This type of metrics includes the charged, rotating Kaluza–Klein black holes in five-dimensional minimal supergravity discussed in the following section.

### 2.5. Type B metric

Let us consider the next case $M_1 = M_2 = 0$. This is an exceptional type appearing only when the torsion is present. In this case, we find a solution

$$e^\mu = \sqrt{X_\mu} \left( \frac{\partial}{\partial x_\mu} - Z_\mu \frac{\partial}{\partial \psi} \right),$$

$$e^\nu = \sqrt{Y_\nu} \frac{\partial}{\partial x_\nu}, \quad e^0 = \frac{\partial}{\partial \psi},$$  

(2.44)

where $X_\mu, Y_\nu$ and $Z_\mu$ are functions of two variables $x_\mu$ and $y_\nu$. Rewriting $(x_1, x_2, y_1, y_2) = (x, y, \tau, \sigma)$, we obtain the corresponding metric by

$$g = \frac{x^2 - y^2}{X_1(x, \tau)} dx^2 + \frac{y^2 - x^2}{X_2(y, \sigma)} dy^2 + \frac{x^2 - y^2}{Y_1(x, \tau)} dx^2 + \frac{y^2 - x^2}{Y_2(y, \sigma)} dy^2$$

$$+ (d\psi + Z_1(x, \tau) dx + Z_2(y, \sigma) dy)^2.$$  

(2.45)

The metric contains six unknown functions of two variables: $X_1(x, \tau), Y_1(x, \tau), Z_1(x, \tau), X_2(y, \sigma), Y_2(y, \sigma)$ and $Z_2(y, \sigma)$. The torsion is given by

$$T = \frac{2x}{x^2 - y^2} \frac{X_2(y, \sigma)}{y^2 - x^2} e^1 \wedge e^1 \wedge e^2 + \frac{2y}{y^2 - x^2} \frac{X_1(x, \tau)}{x^2 - y^2} e^2 \wedge e^3 \wedge e^1$$

$$+ \frac{X_1(x, \tau) Y_1(x, \tau)}{(x^2 - y^2)^2} \frac{\partial Z_1(x, \tau)}{\partial \tau} e^1 \wedge e^1 \wedge e^0$$

$$+ \frac{X_2(y, \sigma) Y_2(y, \sigma)}{(y^2 - x^2)^2} \frac{\partial Z_2(y, \sigma)}{\partial \sigma} e^2 \wedge e^3 \wedge e^0.$$  

(2.46)
2.6. Type C metric

In the last case, by virtue of symmetry between $x_1$ and $x_2$, we can take $K_2 = 0$ and $M_2 = 0$ without loss of generality. A solution we found is given by

$$
\begin{align*}
\mathbf{e}_1 &= \sqrt{\frac{x^2 - y^2}{x^2}} \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \sqrt{\frac{y}{x^2}} \frac{\partial}{\partial y}, \quad \mathbf{e}_0 = \frac{\partial}{\partial \psi}, \\
\mathbf{e}_1 &= \frac{1}{\sqrt{x^2 - y^2}} \Psi_1(x) \left( \frac{\partial}{\partial \tau} + \Omega_1(x) \frac{\partial}{\partial \sigma} + \lambda \frac{\partial}{\partial \psi} \right), \\
\mathbf{e}_2 &= \frac{1}{\sqrt{x^2 - y^2}} \Psi_2(y) \left( \frac{\partial}{\partial \sigma} + \Omega_2(y) \frac{\partial}{\partial \psi} \right),
\end{align*}
$$

(2.47)

which contains five functions of single variable $Y(y)$, $\Psi_1(x)$, $\Psi_2(y)$, $\Omega_1(x)$, $\Omega_2(y)$, and a function of two variables $X(x, \tau)$. Since $\lambda$ is a gauge parameter, we set $\lambda = 0$. The metric is

$$
g = \frac{x^2 - y^2}{X(x, \tau)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 + [d\psi + \Omega_1(x) \Omega_2(y) d\tau - \Omega_2(y) d\sigma]^2
+ (x^2 - y^2) [\Psi_1(x)^2 d\tau^2 + \Psi_2(y)^2 (-\Omega_1(x) d\tau + d\sigma)^2].
$$

(2.48)

The torsion is given by

$$
T = \left( \begin{array}{c}
\frac{2x}{x^2 - y^2} \frac{\Psi_2(y)}{\Psi_1(x)} \frac{\partial \Omega_1(x)}{\partial x} \sqrt{\frac{X(x, \tau)}{x^2}} e^1 \wedge e^1 \wedge e^5 \\
+ \frac{2y}{x^2 - y^2} \frac{\Psi_2(y)}{\Psi_1(x)} \frac{\partial \Omega_2(y)}{\partial x} \sqrt{\frac{X(x, \tau)}{x^2}} e^1 \wedge e^1 \wedge e^4 \\
+ \frac{1}{\Psi_2} \frac{\partial \Omega_2(y)}{\partial y} \sqrt{\frac{X(x, \tau)}{(x^2 - y^2)^2}} e^2 \wedge e^5 \wedge e^0
\end{array} \right).
$$

(2.49)

3. Killing–Yano symmetry of Kaluza–Klein black holes in Einstein–Maxwell–Chern–Simons theory

In this section, we investigate Killing–Yano symmetry of Kaluza–Klein black holes in five-dimensional Einstein–Maxwell–Chern–Simons theory. The action of the theory consists of a (Lorentzian) metric $g_{\mu \nu}$ and a Maxwell field $A_{\mu}$,

$$
S = \int *(R + \Lambda) - \frac{1}{2} \star F \wedge F + \frac{\lambda_{cs}}{3\sqrt{3}} F \wedge F \wedge A,
$$

(3.1)

where $F = dA$ is a field strength of the Maxwell field, $\Lambda$ is a cosmological constant and $\lambda_{cs}$ is coupling constant of the Chern–Simons term. This is said to be the pure Einstein–Maxwell theory when $\lambda_{cs} = 0$ and the minimal supergravity when $\lambda_{cs} = 1$. The equations of motion are given by

$$
R_{ab} + \frac{\Lambda}{3} g_{ab} = \frac{1}{2} \left( F_{ac} F^c_b - \frac{1}{6} g_{ac} F_{de} F^{de} \right),
$$

(3.2)

$$
d \star F = \frac{\lambda_{cs}}{\sqrt{3}} F \wedge F = 0.
$$

(3.3)
3.1. Uplift of the Kerr–Newman solution

Many exact solutions of five-dimensional Einstein–Maxwell–Chern–Simons theory have already been discovered in the literature. Of them, we focus especially on Kaluza–Klein type metrics,

$$g = e^{-2\phi/\sqrt{3}} g^{(4)} + e^{\phi/\sqrt{3}}(d\psi + \mathcal{W}_i dx^i)^2,$$

(3.4)

with a Maxwell field written in the form

$$A = A_i dx^i + \rho d\psi,$$

(3.5)

where $\mathcal{W}_i, A_i, \phi$ and $\rho$ are functions of the coordinates $x^i$ in four dimensions. Then, we find that by setting

$$\phi = \rho = 0, \quad *G = \frac{1}{\sqrt{3}} F,$$

(3.6)

where $*$ represents the Hodge star with respect to $g^{(4)}$, $G = d\mathcal{W}$ and $F = dA$ are field strengths in four dimensions, the action (3.1) consistently reduces to the four-dimensional Einstein–Maxwell theory (e.g., see [37]). According to [38], this identification leads to an uplift of the Reissner–Nordström solution for an arbitrary value of the coupling $\lambda_{cs}$. This fact motivates us to consider $S^1$ bundle over the Kerr–Newman spacetime. That is, the ansatz is the following:

$$g = -\Delta_1 (dt - a \sin^2 \theta d\phi)^2 + \Sigma_1 dr^2 + \Sigma_1 d\theta^2 + \frac{\sin^2 \theta}{\Sigma_1} (adr - (r^2 + a^2) d\phi)^2 + \left(\frac{L}{2} d\psi - \frac{\alpha qr}{\Sigma_1} (dt - a \sin^2 \theta d\phi) - \frac{\beta q \cos \theta}{\Sigma_1} (adr - (r^2 + a^2) d\phi)\right)^2,$$

(3.7)

$$A = -\frac{\gamma qr}{\Sigma_1} (dt - a \sin^2 \theta d\phi) - \frac{\delta q \cos \theta}{\Sigma_1} (adr - (r^2 + a^2) d\phi),$$

(3.8)

and

$$\Delta_1 = r^2 + a^2 + q^2 - 2mr, \quad \Sigma_1 = r^2 + a^2 \cos^2 \theta.$$

(3.9)

If the Einstein–Maxwell–Chern–Simons theory is imposed, then the equations of motion require the parameters $\alpha, \beta, \gamma$ and $\delta$ to satisfy some algebraic relations. The Einstein equation (3.2) yields

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 4,$$

(3.10)

$$3(\alpha^2 - \beta^2) + \gamma^2 - \delta^2 = 0,$$

(3.11)

$$3\alpha \beta + \gamma \delta = 0,$$

(3.12)

which can be solved by

$$\alpha = \frac{\delta}{\sqrt{3}}, \quad \beta = -\frac{\gamma}{\sqrt{3}}, \quad \alpha^2 + \beta^2 = 1.$$

(3.13)

These relations actually imply conditions (3.6). Although the Maxwell equation (3.3) impose additional conditions

$$\alpha \gamma - \beta \delta - \frac{2\lambda_{cs}}{\sqrt{3}} \gamma \delta = 0,$$

(3.14)

$$\alpha \delta + \beta \gamma + \frac{\lambda_{cs}}{\sqrt{3}} (\gamma^2 - \delta^2) = 0,$$

(3.15)
relations (3.13) automatically guarantee them when $\lambda_{\alpha\beta} = 1$. Otherwise, we have the trivial solution $\alpha = \beta = \delta = \gamma = 0$. If we were to set $a = 0$ which corresponds to non-rotating (four-dimensional) spacetimes, equation (3.12) would have been absent and dynamical solutions could have been obtained for $\lambda_{\alpha\beta} \neq 1$ [38]. In the presence of rotation $a \neq 0$, only the minimal supergravity can accommodate the uplift within the present setup.

The metric describes charged, rotating Kaluza–Klein black holes when $\beta \neq 0$ and black strings when $\beta = 0$. The constant $L$ represents the size of the extra dimension. The parameters $m, a, q, \alpha, \gamma$ and $\delta$ are related to five charges: mass $M$, angular momenta $J^\psi$ and $J^\phi$ that are associated with the $\psi$ and $\phi$ directions, electric charge $Q$ and magnetic flux $\Psi$. Thus, obtained solution is trivial because this is just an uplift of the four-dimensional Kerr–Newman solution. However, as we will see below, this is an interesting example from the viewpoint of generalized Killing–Yano symmetry.

### 3.2. Hidden symmetry

In this subsection, we show that the solutions obtained above fall into the general class of metrics derived in the previous section. Then, by writing down the Hamilton–Jacobi equation and Klein–Gordon equation on those spacetimes, we explicitly demonstrate the connection between the separability of these equations and the underlying Killing–Yano symmetry.

#### 3.2.1. Killing–Yano symmetry

The metric (3.7) admits three Killing vectors $\partial/\partial \tau, \partial/\partial \sigma$ and $\partial/\partial \psi$. Besides them, we can find a rank-2 Killing–Stäckel tensor and a rank-2 generalized Killing–Yano tensor. To see this, it is helpful to use the coordinates

$$
\begin{align*}
p &= a \cos \theta, \\
\tau &= t - a \phi, \\
\sigma &= \frac{\phi}{a},
\end{align*}
$$

which Carter introduced in [2] to study the separability of the Hamilton–Jacobi equation for geodesics in the Kerr spacetime. The coordinate transformation (3.16) brings the metric (3.7) to a simple algebraical form

$$
g = \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 - \frac{f_1^2 Q}{r^2 + p^2} (d\tau + p^2 d\sigma)^2 + \frac{f_2^2 P}{r^2 + p^2} (d\tau - r^2 d\sigma)^2
$$

$$
+ \left( d\psi + \frac{N_1}{r^2 + p^2} (d\tau + p^2 d\sigma) + \frac{N_2}{r^2 + p^2} (d\tau - r^2 d\sigma) \right)^2,
$$

where

$$
Q = r^2 + a^2 + q^2 - 2mr, \\
P = a^2 - p^2, \\
N_1 = -\alpha qr, \\
N_2 = -q p, \\
f_1 = 1, \\
f_2 = 1.
$$

The inverse metric is given by

$$
\left( \frac{\partial}{\partial s} \right)^2 = -\frac{1}{f_1^2(r^2 + p^2)Q} \left( r^2 \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - N_1 \frac{\partial}{\partial \psi} \right)^2
$$

$$
+ \frac{1}{f_2^2(r^2 + p^2)P} \left( r^2 \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - N_2 \frac{\partial}{\partial \psi} \right)^2
$$

$$
+ \frac{Q}{r^2 + p^2} \left( \frac{\partial}{\partial \sigma} \right)^2 + \frac{P}{r^2 + p^2} \left( \frac{\partial}{\partial p} \right)^2 + \left( \frac{\partial}{\partial \psi} \right)^2,
$$

Looking at (3.17), we note that the metric is a Lorentzian counterpart of the type A∞ metric (2.41) obtained in section 2.4. This means that (3.17) admits a rank-2 GKY tensor. We may consider the metric with the functions of (3.18) replaced by arbitrary one-variable functions.
\( Q(r), P(p), N_1(r), N_2(p), f_1(r) \) and \( f_2(p) \), which we call off-shell metric. By considering such an off-shell metric, we can deal with the metric (3.7) in a more algebraically general framework. For the off-shell metric (3.17), the canonical orthonormal frame is introduced as

\[
\begin{align*}
    e^1 &= \sqrt{\frac{Q}{r^2 + p^2}} dr,
    & e^i &= f_i \sqrt{\frac{Q}{r^2 + p^2}} (dr + p^2 d\sigma), \\
    e^2 &= \sqrt{\frac{r^2 + p^2}{P}} dp,
    & e^5 &= f_5 \sqrt{\frac{P}{r^2 + p^2}} (dr - r^2 d\sigma), \\
    e^0 &= d\psi + \frac{N_1}{r^2 + p^2} (dr + p^2 d\sigma) + \frac{N_2}{r^2 + p^2} (dr - r^2 d\sigma).
\end{align*}
\]

With respect to this canonical frame, we can easily write down the metric and the rank-2 GKY tensor as

\[
g = e^i e^i - e^i e^1 + e^e e^2 + e^5 e^5 + e^0 e^0. \tag{3.21}
\]

\[
f = p e^i \wedge e^1 + r e^2 \wedge e^5. \tag{3.22}
\]

From (2.8), we obtain a rank-2 Killing–Stäckel tensor

\[
K = p^2 (e^i e^i - e^i e^1) - r^2 (e^e e^2 + e^5 e^5), \tag{3.23}
\]

which is given in terms of the coordinate basis by

\[
\begin{align*}
K^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} &= - \frac{p^2}{f^2_1 (r^2 + p^2) Q} \left( r^2 \frac{\partial}{\partial r} + \frac{\partial}{\partial \sigma} - N_1 \frac{\partial}{\partial \psi} \right)^2 \\
&+ \frac{r^2}{f^2_1 (r^2 + p^2) P} \left( p^2 \frac{\partial}{\partial r} - \frac{\partial}{\partial \sigma} - N_2 \frac{\partial}{\partial \psi} \right)^2 \\
&+ \frac{p^2 Q}{r^2 + p^2} \left( \frac{\partial}{\partial \sigma} \right)^2 - \frac{r^2 P}{r^2 + p^2} \left( \frac{\partial}{\partial \tau} \right)^2.
\end{align*}
\]

### 3.2.2. Separation of variables in the Hamilton–Jacobi equation.

The existence of rank-2 Killing–Stäckel tensors is in general related to the separation of variables in Hamilton–Jacobi equations for geodesics

\[
g^{ab} \partial_a \partial_b S = -m^2. \tag{3.25}
\]

For the off-shell metric (3.17), since the inverse metric is given by (3.19), the Hamilton–Jacobi equation for geodesics (3.25) can be solved by the separation of variables with a function

\[
S = R(r) + \Theta(p) + \pi_\tau \tau + \pi_\sigma \sigma + \pi_\psi \psi, \tag{3.26}
\]

where \( \pi_\tau, \pi_\sigma \) and \( \pi_\psi \) are arbitrary constants, and the functions \( R(r) \) and \( \Theta(p) \) satisfy the ordinary differential equations

\[
\begin{align*}
\left( \frac{dR}{dr} \right)^2 - \left( \frac{W_r}{f_1 Q} \right)^2 - \frac{V_r}{Q} &= 0, \\
\left( \frac{d\Theta}{dp} \right)^2 + \left( \frac{W_p}{f_2 P} \right)^2 - \frac{V_p}{P} &= 0
\end{align*}
\]

with the potentials including a separation constant \( \kappa \),

\[
\begin{align*}
W_r &= r^2 \pi_\tau + \pi_\sigma - N_1 \pi_\psi, \\
V_r &= - (\pi_\psi^2 + m^2) r^2 + \kappa, \\
W_p &= p^2 \pi_\tau - \pi_\sigma - N_2 \pi_\psi, \\
V_p &= - (\pi_\psi^2 + m^2) p^2 - \kappa.
\end{align*}
\]
The constant $\kappa$ is given by eliminating $-m^2$, as
\[
\kappa = \frac{p^2}{r^2 + p^2} \left[ \frac{d}{dr} \left( f_1 Q \frac{dR}{dr} \right) - \frac{W_2^2}{f_1^2 Q} \right] - \frac{r^2}{r^2 + p^2} \left( \mathcal{P} \Pi_\rho - \frac{W_\rho^2}{f_2^2 \mathcal{P}} \right),
\]
(3.29)
where $\Pi_\sigma = \partial_\sigma S$ is the canonical momentum given by
\[
\Pi_\tau = \pm \sqrt{\left( \frac{W_\rho}{f_1 Q} \right)^2 + V_\rho^2}, \quad \Pi_\rho = \pm \sqrt{\left( \frac{W_\rho}{f_2 \mathcal{P}} \right)^2 + V_\rho^2}
\]
(3.30)
and $\Pi_\sigma = \pi_\sigma$, $\Pi_\tau = \pi_\tau$, and $\Pi_\phi = \pi_\phi$.

We have now four constants of motion. Three of them $\Pi_\tau$, $\Pi_\sigma$ and $\Pi_\phi$ are associated with the Killing vectors $\partial/\partial \tau$, $\partial/\partial \sigma$ and $\partial/\partial \phi$, respectively. The Killing–Stäckel tensor $K_{ab}$ is responsible for the other one $\kappa$. In fact, from equation (3.24), one can easily confirm that
\[
\kappa = K^{ab} \Pi_a \Pi_b,
\]
(3.31)
which is precisely the general relation (2.9). Its constancy is in general guaranteed by the commutativity with the Hamiltonian $H = \chi^{ab} \Pi_a \Pi_b$ under the Poisson bracket, namely $\{H, \kappa\} = 0$. Thus, we have found that the charged, rotating Kaluza–Klein black hole spacetime (3.7) does posses the separable structure for the Hamilton–Jacobi equation and its separation constant is indeed related to the underlying Killing–Yano symmetry that generates the rank-2 Killing–Stäckel tensor necessary for the separation of variables.

### 3.2.3. Separation of variables in the Klein-Gordon equation.

The massive Klein–Gordon equation is given by
\[
(\Box + m^2) \Psi = 0.
\]
(3.32)
With the help of $\Box \Psi = (1/\sqrt{-g}) \partial_a (\sqrt{-g} g^{ab} \partial_b \Psi)$, we have
\[
\left[ \frac{1}{f_1} \frac{d}{dr} \left( f_1 Q \frac{dR}{dr} \right) + \frac{1}{f_2} \frac{d}{dp} \left( f_2 \mathcal{P} \frac{d\Theta}{dp} \right) - \frac{1}{f_1^2 Q} \left( r^2 \frac{d}{d\tau} + \frac{d}{d\sigma} - N_1 \frac{d}{d\phi} \right)^2 \right.

+ \left. \frac{1}{f_2^2 \mathcal{P}} \left( p^2 \frac{d}{d\tau} - \frac{d}{d\sigma} - N_2 \frac{d}{d\phi} \right)^2 + (r^2 + p^2) \left( \frac{d^2}{d\phi^2} + m^2 \right) \right] \Psi = 0.
\]
(3.33)
This equation is separable with a function
\[
\Psi = R(r) \Theta(p) e^{i(\tau + \sigma + \phi)}
\]
(3.34)
where $\pi_\tau$, $\pi_\sigma$, and $\pi_\phi$ are constants again, and the functions $R(r)$ and $\Theta(p)$ satisfy the ordinary differential equations
\[
\frac{1}{f_1} \frac{d}{dr} \left( f_1 Q \frac{dR}{dr} \right) - \frac{W_2^2}{f_1^2 Q} = V_\tau = 0,
\]
\[
\frac{1}{f_2} \frac{d}{dp} \left( f_2 \mathcal{P} \frac{d\Theta}{dp} \right) + \frac{W_\rho^2}{f_2^2 \mathcal{P}} = V_\rho = 0
\]
(3.35)
with the potentials given by (3.28) including the separation constant $\kappa$. By eliminating $m^2$, the constant $\kappa$ is this time given by
\[
\kappa = \frac{p^2}{r^2 + p^2} \left[ \frac{1}{f_1} \frac{d}{dr} \left( f_1 Q \frac{dR}{dr} \right) - \frac{W_2^2}{f_1^2 Q} \right] + \frac{r^2}{r^2 + p^2} \left[ \frac{1}{f_2} \frac{d}{dp} \left( f_2 \mathcal{P} \frac{d\Theta}{dp} \right) + \frac{W_\rho^2}{f_2^2 \mathcal{P}} \right].
\]
(3.36)
Again, it is straightforward to check that this constant coincides with the one following from the general property of the symmetry operator (2.10) together with the Killing–Stäckel tensor (3.23), that is, we have
\[
\mathcal{K} \Psi \equiv \nabla_a K^{ab} \nabla_b \Psi = \kappa \Psi.
\]
(3.37)
This is the relationship between the Killing–Stäckel tensor and separability of the Klein–Gordon equation.
3.3. More general solution

Our purpose here is to construct the most general solution of the Einstein–Maxwell–Chern–Simons theory, namely equations (3.2) and (3.3), for the type $A_\infty$ off-shell metric (3.17). We adopt the ansatz that identifies torsion with the flux by

$$T = \frac{1}{\sqrt{3}} \ast F.$$  \hfill (3.38)

Under the assumption (3.38), the Maxwell–Chern–Simons equation (3.3) and the Bianchi identity, $dF = 0$, are written as

$$dT - \lambda_{\text{cs}} (\ast T) \wedge (\ast T) = 0, \quad d \ast T = 0.$$ \hfill (3.39)

Substituting the expression for the torsion (2.43), the first equation requires both $f_1 = f_2$ and $\lambda_{\text{cs}} = 1$. Since $f_1$ and $f_2$ are one-variable functions of only $r$ and $p$, respectively, we find that $f_1 = f_2$ must be constant and then it can be absorbed in $N_\mu$ via rescaling of $r$ and $\sigma$. Thus, we may set $f_1 = f_2 = 1$. The restriction to $\lambda_{\text{cs}} = 1$ corresponds to the minimal supergravity. The second equation then solves as

$$N_1 = \tilde{a} r^2 + b_1 r, \quad N_2 = \tilde{a} p^2 + b_2 p,$$ \hfill (3.40)

where $\tilde{a}$, $b_1$, and $b_2$ are constants. Note that $\tilde{a}$ is a gauge parameter which can be eliminated by gauge transformation of $\psi$. We set it to be zero. These conditions render many components of the Einstein equations trivial. We need $(0, 0)$ component to derive $\Lambda = 0$. Then, (3, 3) and (4, 4) components determine $Q$ and $P$ as

$$Q = \tilde{c} r^2 + m_1 r + q_1, \quad P = -\tilde{c} p^2 + m_2 p + q_2,$$ \hfill (3.41)

where $\tilde{c}$, $m_1$, $m_2$, $q_1$, and $q_2$ are constants which satisfy

$$q_1 - q_2 = b_1^2 + b_2^2.$$ \hfill (3.42)

Again the gauge freedom enables us to rescale $\tilde{c}$ to be 1 so that we have obtained a five-parameter family of solutions.

In order to compare it with the Kaluza–Klein black hole solution in section 3.1, we make the coordinate transformation (3.16) and set $\mu = m_2/a$ and $q_0 = q_2/a^2$. The obtained solution is written as

\begin{align*}
g &= \frac{\Delta}{\Sigma} (dr - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} (\Sigma \cos^2 \theta - \mu \cos \theta + q_0 - 1) d\phi^2 \\
&\quad + \left( \frac{\Sigma}{\Sigma} \sin^2 \theta + \mu \cos \theta + q_0 - 1 \right) (adr - (r^2 + a^2) d\phi)^2 \\
&\quad + \left( dr + \frac{b_1 r}{\Sigma} (dr - a \sin^2 \theta d\phi) + \frac{b_2 \cos \theta}{\Sigma} (adr - (r^2 + a^2) d\phi) \right)^2. \\
A &= -\frac{\sqrt{3} b_1 r}{\Sigma} (dt - a \sin^2 \theta d\phi) + \frac{\sqrt{3} b_2 \cos \theta}{\Sigma} (adr - (r^2 + a^2) d\phi),
\end{align*} \hfill (3.43)

where

$$\Delta = r^2 + m_1 r + a^2 q_0 + b_1^2 + b_2^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta.$$ \hfill (3.44)

It can be seen that the metric is the uplift of the Kerr–Newman–NUT solution [2]. The result of section 3.1 is included as a special case $\mu = 0$ and $q_0 = 1$.

---

4 It was shown in [39] that a five-dimensional spacetime admitting a rank-2 closed CKY tensor which is a generalized closed CKY tensor at the same time with the torsion satisfying (3.39) is uniquely given by the Chong–Cvetiˇc–L¨u–Pope solution [17] in five-dimensional minimal supergravity.
4. Construction of Kaluza–Klein black holes in heterotic supergravity

In this section, we consider Abelian heterotic supergravity in five dimensions, which is the low-energy effective theory of heterotic string theory. The string-frame action consists of a (Lorentzian) metric \( g_{\mu\nu} \), scalar field \( \phi \), \( U(1) \) gauge potential \( A_\mu \) and 2-form potential \( B_{\mu\nu} \),

\[
S = \int e^\phi \left( R + *d\phi \wedge d\phi - *F \wedge F - \frac{1}{2} *H \wedge H \right),
\]

where \( F = dA \) and \( H = dB - A \wedge dA \). The equations of motion are

\[
R_{ab} - \nabla_a \nabla_b \phi - F_a^c F_{bc} - \frac{1}{4} H_{a}^{cd} H_{bc} = 0,
\]

\[
d(e^\phi * F) - e^\phi * H \wedge F = 0,
\]

\[
d(e^\phi * H) = 0,
\]

\[
R - (\nabla \phi)^2 - 2\nabla^2 \phi \frac{1}{2} F^2 - \frac{1}{16} H^2 = 0.
\]

We start from reviewing a known black string solution of this theory, revealing its Killing–Yano symmetry. Then, using the general form of metric obtained in section 2, we construct a class of charged, rotating Kaluza–Klein black holes.

4.1. Hidden symmetry of the Mahapatra solution

Using the technique of Hassan and Sen [40], a charged, rotating black string solution with four parameters \((m, a, \delta_1, \delta_2)\) was constructed by Mahapatra [35]. The solution is given by

\[
g = \frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi - W)^2 + \frac{a^2 \sin^2 \theta}{\Sigma} \left( dt - \frac{r^2 + a^2}{a} d\phi - W \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + (d\psi - \frac{\beta}{1 - \alpha} W)^2,
\]

\[
A = - \frac{\gamma}{1 - \alpha} W,
\]

\[
B = \left(-dt + \frac{\beta}{1 - \alpha} d\psi\right) \wedge W,
\]

\[
e^\phi = 1 + \frac{-mr(1 - \alpha)}{\Sigma},
\]

where

\[
\Delta = r^2 + a^2 - 2mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta,
\]

\[
W = \frac{-mr(1 - \alpha)}{\Sigma - mr(1 - \alpha)} (dt - a \sin^2 \theta d\phi).
\]

The parameters \(\alpha, \beta, \gamma\) are required to satisfy the relation \(\alpha^2 = 1 + \beta^2 + \gamma^2\), which can be written by two parameters \(\delta_1\) and \(\delta_2\) as

\[
\alpha = \cosh \delta_1 \cosh \delta_2, \quad \beta = \cosh \delta_1 \sinh \delta_2, \quad \gamma = \sinh \delta_1.
\]
The field strengths $F$ and $H$ are easily computed as
\[ F = -\frac{\gamma}{1-\alpha} dW, \quad H = \eta \wedge dW, \tag{4.13} \]
where
\[ \eta = dr - \frac{\beta}{1-\alpha} d\psi - \frac{\gamma^2}{(1-\alpha)^2} W \]
\[ = \frac{\Delta}{\Sigma} (dr - a \sin^2 \Theta d\phi - W) - \frac{a^2 \sin^2 \Theta}{\Sigma} \left( dr - \frac{r^2 + a^2}{a} d\phi - W \right) - \frac{\beta}{1-\alpha} \left( d\psi - \frac{\beta}{1-\alpha} W \right). \tag{4.14} \]

4.1.1. Killing–Yano symmetry. Let us unveil the Killing–Yano symmetry of Mahapatra’s spacetime. Performing the coordinate transformation (3.16) again, the metric (4.6) can be written in a simple algebraical form
\[ g = -\frac{f_1^2}{r^2 + p^2} (dr + p^2 d\sigma - W)^2 + \frac{f_2^2}{r^2 + p^2} (dr - r^2 d\sigma - W)^2 + \frac{r^2 + p^2}{Q} d\tau^2 \]
\[ + \frac{r^2 + p^2}{\mathcal{P}} dp^2 + (d\psi + \lambda W)^2, \tag{4.15} \]
where
\[ W = \frac{1}{\Phi} \left( \frac{N_1}{\Sigma} (dr + p^2 d\sigma) + \frac{N_2}{\Sigma} (dr - r^2 d\sigma) \right), \]
\[ \Phi = 1 + \frac{N_1}{\Sigma} + \frac{N_2}{\Sigma}, \quad \Sigma = r^2 + p^2, \]
\[ Q = r^2 - 2mr + a^2, \quad \mathcal{P} = a^2 - p^2, \quad N_1 = -mr(1-\alpha), \]
\[ N_2 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad \lambda = -\frac{\beta}{1-\alpha}. \tag{4.16} \]
We observe that this metric falls into the family $A_2$ (2.36) in the general classification of section 2. Similarly to the previous section, we consider hidden symmetry for the off-shell metric (4.15) with $Q, \mathcal{P}, N_1, N_2, f_1, f_2$ and $\lambda$ replaced by unknown functions $Q(r), \mathcal{P}(p), N_1(r), N_2(p), f_1(r)$ and $f_2(p)$ and an arbitrary constant $\lambda$, respectively. For the off-shell metric (4.15), the canonical frame is introduced as
\[ e^1 = \sqrt{\frac{r^2 + p^2}{Q}} dr, \quad e^1 = f_1 \sqrt{\frac{Q}{r^2 + p^2}} (dr + p^2 d\sigma - W), \]
\[ e^2 = \sqrt{\frac{r^2 + p^2}{\mathcal{P}}} dp, \quad e^3 = f_2 \sqrt{\frac{\mathcal{P}}{r^2 + p^2}} (dr - r^2 d\sigma - W), \]
\[ e^0 = d\psi + \lambda W. \tag{4.17} \]
With respect to this orthonormal frame, the metric and the rank-2 generalized Killing–Yano tensor are written as (3.21) and (3.22), respectively. Furthermore, we also obtain a Killing–Stäckel tensor of the form (3.23). Since the inverse metric is given by
\[ \left( \frac{\partial}{\partial s} \right)^2 = -\frac{1}{f_1^2 (r^2 + p^2) Q} \left( (r^2 + N_1) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - \lambda N_1 \frac{\partial}{\partial \psi} \right)^2 \]
\[ + \frac{1}{f_2^2 (r^2 + p^2) \mathcal{P}} \left( (p^2 + N_2) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_2 \frac{\partial}{\partial \psi} \right)^2 \]
\[ + \frac{Q}{r^2 + p^2} \left( \frac{\partial}{\partial r} \right)^2 + \frac{\mathcal{P}}{r^2 + p^2} \left( \frac{\partial}{\partial \phi} \right)^2 + \left( \frac{\partial}{\partial \psi} \right)^2, \tag{4.18} \]
the contravariant Killing–Stückel tensor can be written as
\[
K^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} = -\frac{p^2}{f_1^2(r^2 + p^2)Q} \left( (r^2 + N_1) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - \lambda N_1 \frac{\partial}{\partial \psi} \right)^2
+ \frac{r^2}{f_2^2(r^2 + p^2)\mathcal{P}} \left( (p^2 + N_2) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_2 \frac{\partial}{\partial \psi} \right)^2
+ \frac{p^2 Q}{r^2 + p^2} \left( \frac{\partial}{\partial \tau} \right)^2 + \frac{r^2 \mathcal{P}}{r^2 + p^2} \left( \frac{\partial}{\partial \psi} \right)^2.
\]

(4.19)

4.1.2. Separation of variables in the Hamilton–Jacobi equation. For the off-shell metric (4.15), the geodesic equation (3.25) can be solved by the separation of variables with a function (3.26) and obtain the ordinary differential equations (3.27) with the potentials
\[
W_\tau = (r^2 + N_1)\pi_\tau + \pi_\sigma - \lambda N_1 \pi_\psi, \quad V_\tau = -(\pi_\psi^2 + m^2)r^2 + \kappa,
\]
\[
W_\sigma = (p^2 + N_2)\pi_\tau - \pi_\sigma - \lambda N_2 \pi_\psi, \quad V_\sigma = -(\pi_\psi^2 + m^2)p^2 - \kappa.
\]

(4.20)

where the separation constant \(\kappa\) is given by (3.29) with the above potentials. Indeed, since this constant is related to the Killing–Stückel tensor (4.19) with the general formula (3.31), we find that the separation of variables for the geodesic equation in Mahapatra’s spacetime is underwritten by the existence of the generalized Killing–Yano symmetry.

4.1.3. Separation of variables in the Klein–Gordon equation. In contrast to the case of minimal supergravity black holes, the Klein–Gordon equation (3.32) for the off-shell metric (4.15) does not separate. On the other hand, the deformed Klein–Gordon equation
\[
(\Box + m^2)\psi - (\nabla_\psi)(\nabla^\nu\psi) = 0
\]

(4.21)

does with \(\psi = \ln \Phi\). Equation (4.21) is equivalent to Klein–Gordon equation in Einstein frame,
\[
(\Box_E + m^2)\psi = 0,
\]

(4.22)

where \(\Box_E\) is the d’Alembertian with respect to the Einstein-frame metric \(g_E = \psi^{2/3}g\). Namely, separation of variables for the Klein–Gordon equation naturally occurs in the Einstein frame. The similar situation was seen in the Kerr–Sen black holes [33].

4.2. Kaluza–Klein black hole solutions

We attempt to find the general solutions of the equations of motion (4.2)–(4.5) which take the form of type \(A_3\) metric (4.15), under the ansatz for the matter fields
\[
\psi = \ln \Phi, \quad F = c_F dW, \quad H = c_H T,
\]

(4.23)

where \(T\) is the type \(A_3\) torsion (2.38). For a non-trivial solution, equation (4.3) requires
\[
c_H = 1.
\]

(4.24)

Under this condition, the rest of (4.3) become dependent on the dynamical equations of \(H\) (4.4). Combined with the diagonal part of the Einstein equations (4.2), one can derive
\[
H_1 = f_2, \quad \text{or} \quad f_1 = \tilde{f} x^2, \quad f_2 = \tilde{f} y^2.
\]

(4.25)

where \(\tilde{f}\) is constant. Since the latter is not consistent with the off-diagonal terms of the Einstein equations, we focus on the former and set \(f_1 = f_2 = 1\) by redefining the coordinates as before. The remaining component of (4.4) and the (3, 4)-component of (4.2) lead to
\[
N_1 = \tilde{a} r^2 + b_1 r, \quad N_2 = \tilde{a} p^2 + b_2 p.
\]

(4.26)
Finally, the consistency condition $dH = -F \wedge F$ determines $P$ and $Q$ as

$$P(p) - Q(r) = -(c_F^2 + \lambda^2)(r^2 + p^2) + c_0(r^2 + p^2)\Phi.$$  \hfill (4.27)

Equations (4.25)–(4.27) are sufficient to guarantee that all the remaining equations are satisfied. By means of the physically irrelevant rescaling of $\Phi$, one can choose $\tilde{a} = 0$ (note our definition of $\Phi$) contains the normalized constant term) and derive

$$Q = (c_F^2 + \lambda^2 - c_\phi)r^2 - b_1 c_\phi r + c_0, \quad \text{and} \quad P = -(c_F^2 + \lambda^2 - c_\phi)p^2 + b_2 c_\phi p + c_0.$$  \hfill (4.28)

The overall factor of $P$ and $Q$ can be gauged away, leaving a family of solutions with five parameters $(c_0, c_F, c_\phi, b_1, b_2)$, describing charged, rotating black holes and black strings.

Finally, by performing the coordinate transformation (3.16) and setting $\mu = b_2/a$ and $q_0 = c_0/a^2$, the solution is written in the original coordinates as

$$g = \frac{\Delta}{\Sigma} (dt - a \sin^2 \theta \, d\phi - W)^2 + \frac{a^2 Y}{\Sigma} \left( \frac{dr}{r} - \frac{r^2 + a^2}{a} \, d\phi - W \right)^2 + \frac{\Sigma}{\Delta} \, dr^2$$

$$+ \frac{\Sigma \sin^2 \theta}{\Sigma} \, d\theta^2 + (d\psi + \lambda W)^2,$$  \hfill (4.29)

$$A = c_F W,$$  \hfill (4.30)

$$B = (-dt - \lambda d\psi) \wedge W,$$  \hfill (4.31)

$$e^\nu = 1 + \frac{b_1 r + \mu a^2 \cos \theta}{\Sigma},$$  \hfill (4.32)

where

$$\Delta = (c_F^2 + \lambda^2 - c_\phi)r^2 - b_1 c_\phi r + a^2 q_0, \quad \Sigma = r^2 + a^2 \cos^2 \theta,$$

$$Y = (c_F^2 + \lambda^2 - c_\phi) \sin^2 \theta + \mu c_\phi \cos \theta + q_0 - c_F^2 - \lambda^2 + c_\phi,$$

$$W = \frac{b_1 r}{\Sigma + b_1 r + \mu a^2 \cos \theta} \left( dr - a \sin^2 \theta \, d\phi \right) + \frac{\mu \cos \theta}{\Sigma + b_1 r + \mu a^2 \cos \theta} \left( a \, dt - (r^2 + a^2) \, d\phi \right).$$  \hfill (4.33)

It is easy to check that the Mahapatra solution (4.6)–(4.11) is recovered when we take $\lambda = -\beta/(1 - \alpha)$, $c_F = -\gamma/(1 - \alpha)$, $c_\phi = -2/(1 - \alpha)$, $b_1 = -m(1 - \alpha)$, $\mu = 0$ and $q_0 = c_F^2 + \lambda^2 - c_\phi = 1$. It should be commented that the present metric can be regarded as an uplift of the Kerr–Sen–NUT solution obtained by [33].

5. Summary and discussions

Initially, we have classified five-dimensional metrics admitting a rank-2 generalized Killing–Yano tensor under the assumption that its eigenvector associated with the zero eigenvalue is a Killing vector field. The metrics have been classified into three types A, B and C in general, and local expressions of the corresponding metrics have been obtained explicitly. One of the open problems is to classify them without assuming the Killing vector, in the presence of torsion. In this case, the large number of unknown variables arising from the components of $\xi$ makes it difficult to solve the integrability conditions. However, even if it is not possible to
obtain a general classification, it would be of a great interest as an attempt to find spacetimes of less homogeneity.

We also have demonstrated separability structures of the Hamilton–Jacobi and Klein–Gordon equations for some known charged, rotating Kaluza–Klein black hole and black string solutions in the five-dimensional minimal supergravity as well as heterotic supergravity. We have found that those spacetimes fall into the class of the type A metric and the separability structure is indeed related to the underlying generalized Killing–Yano symmetry. As the Killing–Yano tensor by itself provides a constant of motion for the Dirac equation [4, 21–28], it would be also interesting to study its separability in the presence of torsion. In that case, one might need to consider a deformed Dirac equation with torsion, as discussed in [33, 25]. The separability might also be extended to other test field equations such as Maxwell’s equations. In our calculation, the torsion tensors associated with the generalized Killing–Yano symmetry have been identified with the matter fields as $T = \ast F / \sqrt{3}$ in the five-dimensional minimal supergravity and as $T = H$ in the Abelian heterotic supergravity, which is analogous to the asymptotically flat black hole spacetimes. In the limit of vanishing torsion, both of them reduce to the Kerr string solution.

The obtained classification provides an alternative to various approaches for finding new exact solutions. In fact, using the type A metrics derived in section 2, we have constructed the general solutions describing charged, rotating Kaluza–Klein black holes in the five-dimensional minimal supergravity and Abelian heterotic supergravity. Although we have concentrated on the type A metrics in this paper since our primary aim has been the application to Kaluza–Klein black holes, types B and C metrics also offer possibilities to find novel exact solutions, which can be interesting since they exist only when the torsion is present. It should also be commented that all the calculations in this paper can be generalized to odd dimensions higher than 5. It would enable us to seek uplift of charged, rotating black hole solutions to higher dimensions.

In [41], the existence of the ordinary Killing–Yano tensors was investigated on nearly Kähler manifolds and manifolds with a weak $G_2$-structure. The Killing–Yano equations on manifolds with a G-structure were investigated in the absence [42] and presence [43] of torsion. The generalized Killing–Yano tensors are also related to Kähler manifolds in even dimensions and Sasakian manifolds in odd dimensions. It was demonstrated in [34] that Kähler manifolds studied by [45] admit rank-2 generalized conformal Killing–Yano tensors. In odd dimensions, a concrete example of the Killing–Yano tensor was constructed [44] on Sasakian manifolds studied by [46]. Moreover, a notion of deformed Sasakian manifolds in the presence of torsion was introduced by [36], and the authors have shown an example admitting a rank-3 generalized closed conformal Killing–Yano tensor. It can be shown that the Sasakian manifolds with torsion discussed in [36] also take the form of the type A metrics in our classification. This work might be useful to obtain further examples of Sasakian manifolds with torsion.

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