Multipliers of an Antiphase Solution in a System of Two Coupled Nonlinear Relaxation Oscillators

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Abstract. We consider a system of two non-linear differential-difference equations that models a behaviour of two coupled neural oscillators with a synaptic connection of a threshold type. To find a solution of the system means to find normalized membrane potentials of neurons. We prove that there exists an antiphase solution. It is a special relaxation periodic regime such that one membrane potential is a half-period shift of the other membrane potential. We prove that the antiphase solution is stable and construct its asymptotic. We show that there exists a solution with a bursting-effect by choosing a specific connection chain delay. That is, for any natural $n$ there is a connection chain delay such that an antiphase solution has exactly $n$ asymptotically high spikes on a period after a refractive segment.

1. Problem statement

We consider a mathematical model of synaptic coupled neuron type oscillators that is based on an idea of fast threshold modulation:

\begin{align*}
\dot{u}_1 &= \left(\lambda f(u_1(t - 1)) + bg(u_2(t - h)) \ln(u_*/u_1)\right)u_1, \\
\dot{u}_2 &= \left(\lambda f(u_2(t - 1)) + bg(u_1(t - h)) \ln(u_*/u_2)\right)u_2.
\end{align*}

(1)

This model was suggested in [1] and considered in [2, 3]. Here, $u_1(t)$, $u_2(t) > 0$ are normalized membrane potentials of neurons. Every neuron is modelled by a singularly perturbed differential-difference equation with delay

\[ \dot{u} = \lambda f(u(t - 1))u. \]

(2)

This equation was suggested in [4] (see also [5]). A parameter $\lambda \gg 1$ is large and characterizes the rate of electric processes in the system. A function $f(u) \in C^2(R_+)$ satisfies:

\[ f(0) = 1; \quad f(u) + a, uf'(u), u^2f''(u) = O(u^{-1}) \text{ as } u \to +\infty, \quad a = \text{const} > 0. \]

(3)

The term $bg(u_{j-1}(t-h)) \ln(u_*/u_j)u_j$ models a synaptic connection. We consider a connection of a threshold type with a time delay $h > 1$. Parameter $b = \text{const} > 0$, $u_* = \exp(c\lambda)$ is a threshold value, where $c = \text{const} \in R$. A function $g(u) \in C^2(R_+)$ satisfies the following conditions:

\[ \forall u > 0 \ g(u) > 0, \ g(0) = 0; \quad g(u) - 1, \ ug'(u), u^2g''(u) = O(u^{-1}) \text{ as } u \to +\infty. \]

(4)

A mathematical model of fast threshold modulation was studied in articles [6, 7, 8].
We study the existence and stability of relaxation periodic regimes of system (1). We are interested in a so-called *antiphase* solution (see [7]). In addition, the solution has a special neurodynamic system property — a *bursting-effect* (see [9, 10, 11, 12, 13, 14]).

**Definition 1.** A periodic solution of (1) is called antiphase if

(i) \( u_2(t) \) is a time shift of \( u_1(t) \): \( u_1(t) = u(t), \ u_2 = u(t + \Delta) \),

(ii) \( \Delta = T/2 \), where \( T \) is a period of \( u(t) \).

**Definition 2.** A bursting-effect is an alternation of several consecutive intensive spikes with refractory period of a membrane potential.

To define a limit object correctly we rewrite (1) in a suitable form. Using the following exponential substitution

\[ u_j = \exp(\lambda x_j), \quad j = 1, 2, \]

we get

\[ \dot{x}_1 = F(x_1(t - 1), \varepsilon) + b(c - x_1)G(x_2(t - h), \varepsilon), \]
\[ \dot{x}_2 = F(x_2(t - 1), \varepsilon) + b(c - x_2)G(x_1(t - h), \varepsilon), \]  

where \( \varepsilon = 1/\lambda \ll 1 \), and

\[ F(x, \varepsilon) = f(\exp(x/\varepsilon)), \quad G(x, \varepsilon) = g(\exp(x/\varepsilon)). \]

(7)

By (3) and (4), we define the limit functions:

\[ R(x) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} F(x, \varepsilon) = \begin{cases} 1 & \text{as } x < 0, \\ -a & \text{as } x > 0, \end{cases} \quad H(x) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} G(x, \varepsilon) = \begin{cases} 0 & \text{as } x < 0, \\ 1 & \text{as } x > 0. \end{cases} \]

(8)

In order to find an antiphase solution of (1) with a bursting-effect, we need to find a relaxation antiphase regime of (6). More precisely, we are interested in a solution with a given number of relatively short alternating segments of positivity and negativity of the solution which go after a long enough segment where the solution values are negative. By (5), the existence of such a solution of system (6) means that (1) has an antiphase regime with bursting-effect.

2. Analysis of the auxiliary equation

We are looking for a solution of (6) in the following form

\[ x_1(t) \equiv x(t), \quad x_2(t) \equiv x(t + \Delta). \]

(9)

Such a solution corresponds to discrete travelling waves. This approach was suggested in [15] and considered in several works (for example, [2, 16, 17]). Substituting (9) in system (6), we obtain the following:

\[ \dot{x} = F(x(t - 1), \varepsilon) + b(c - x)G(x(t - h + \Delta), \varepsilon). \]

(10)

For a fixed natural \( n \), we are interested in the existence of a relaxation cycle in (10) which contains \( n \) disjoint segments with positive values. We suppose that the parameters satisfy the following inequalities

\[ (n - 1)T_0 + t_0 + 1 < h - \Delta < nT_0, \quad c < -a - \frac{1}{b} + a + 1, \quad 1 - \xi, \]

(11)

where

\[ \xi \overset{\text{def}}{=} \exp(-bt_0), \quad t_0 \overset{\text{def}}{=} 1 + 1/a, \quad T_0 \overset{\text{def}}{=} 2 + 1/a + a. \]

(12)
We explain a meaning of the restrictions (11) below.

Considering a limit case for (10) as $\varepsilon \to 0$ and by (8), we obtain the limit relay equation

$$\dot{x} = R(x(t-1)) + b(c - x)H(x(t-h + \Delta)).$$

(13)

As in [1, 2, 14, 15], we construct a solution of (13) using step-by-step method. Fix constants $\sigma > 0$, $q_1 \in (0, \sigma)$, $q_2 > \sigma$, and define

$$S \overset{\text{def}}{=} \{ \varphi(t) \in C[-h+\Delta-\sigma,-\sigma], -q_1 \leq \varphi(t) \leq -q_2 \ \forall \ t \in [-h+\Delta-\sigma,-\sigma], \varphi(-\sigma) = -\sigma \}. \quad (14)$$

We use $x_\varphi(t)$ to denote a solution of (13) for $t \geq -\sigma$ with an arbitrary function $\varphi$ from (14).

We prove that there exists a $T^*$-periodic function $x^*(t)$ such that the solution $x_\varphi(t)$ coincides with it. Let us describe the function $x^*(t)$.

$$x^*(t) \overset{\text{def}}{=} \begin{cases} x_0(t) & \text{as } 0 \leq t \leq 1, \\ x_e(h - \Delta + kT_0, \alpha_k, t) & \text{as } t \in [h - \Delta + kT_0, h - \Delta + t_0 + kT_0], \\ t - h + \Delta - t_0 - kT_0 + \beta_k & \text{as } t \in [h - \Delta + t_0 + kT_0, h - \Delta + (k+1)T_0], \\ t - T^* & \text{as } t_0 + 1 \leq t \leq T_0, \end{cases} \quad (15)$$

where $x_0$ is a $T_0$-periodic solution of an equation $\dot{x} = R(x(t-1))$ (see [4]) defined as follows

$$x_0(t) \overset{\text{def}}{=} \begin{cases} t & \text{as } 0 \leq t \leq 1, \\ 1 - a(t-1) & \text{as } 1 \leq t \leq t_0 + 1, \\ t - T_0 & \text{as } t_0 + 1 \leq t \leq T_0, \end{cases} \quad (16)$$

and $x_e$ is a solution of an auxiliary problem

$$\dot{x} = 1 + b(c - x), \quad x|_{t=\tilde{t}} = \tilde{x};$$

(17)

for given constants $\tilde{t}$, $\tilde{x}$. Therefore, $x_e$ has the following form

$$x_e(\tilde{t}, \tilde{x}, t) \overset{\text{def}}{=} (\tilde{x} - 1/b - c) \exp(-b(t-\tilde{t})) + 1/b + c.$$  

(18)

The period $T^*$ is defined by the following identities

$$\alpha_0 \overset{\text{def}}{=} h - \Delta - nT_0, \quad \alpha_k \overset{\text{def}}{=} \xi^k(\alpha_0 - \eta) + \eta, \quad k = 1, \ldots, n - 1, \quad \eta \overset{\text{def}}{=} \frac{1}{b} + c + \frac{a + 1}{1 - \xi}; \quad (19)$$

\[ \text{Figure 1. Solution (15) of relay equation (13)} \]
\[
\beta_k \overset{\text{def}}{=} x_e(h - \Delta + kT_0, \alpha_k, h - \Delta + t_0 + kT_0), \quad k = 0, \ldots, n - 1.
\]

\[
T_\ast \overset{\text{def}}{=} -\alpha_n + h - \Delta + nT_0.
\]  

Note that restriction (11) means that the value of the delay \( h - \Delta \) lies on a segment of period where the function \( x_0(t) \) is negative and increasing.

Sequences \( \{\alpha_k\} \), \( \{\beta_k\} \) are values of function \( x_0(t) \) (see pic. 1). We claim that the sequence \( \{\alpha_k\} \) decreases. (As corollary, \( \{\beta_k\} \) decreases too.) Indeed, \( \xi \) is positive constant smaller than 1 and the coefficient \( (a_0 - \eta) \) under \( \xi^k \) is negative. The last follows from restriction (11) on \( c \).

It also follows from (11) that \( \alpha_0 < 0 \). This implies that all elements of \( \{\alpha_k\} \) are negative. We obtain that \( x_\ast(t) \) has "long" enough segment with negative value.

Let us formulate a statement about relation between periodic solutions of (10) and (13).

**Theorem 1.** Fix a natural \( n \). Let the parameters \( a, b, c, h, \Delta \) meet restrictions (11). Let \( x_\ast(t) \) be a \( T_\ast \)-periodic solution (15) of (13) with exactly \( n \) segments on a period where it is positive. Then there exists a sufficiently small \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0] \), equation (10) with initial function from (14) admits a unique exponentially orbitally stable cycle \( x^{\varepsilon}_\ast(t) \) of period \( T^{\varepsilon}_\ast \) satisfying the limit relations

\[
\lim_{\varepsilon \to 0} \max_{0 \leq t \leq T^{\varepsilon}_\ast} |x^{\varepsilon}_\ast(t) - x_\ast(t)| = 0, \quad T^{\varepsilon}_\ast = T_\ast + O(\varepsilon).
\]

A proof scheme is the same as, for example, in [1]. We give only a short plan here. Let us introduce some notation for its presentation. For an arbitrary function \( \varphi \) from (14), denote by \( x^\varepsilon(t, \varphi) \) a solution of (10) with initial condition \( \varphi(t), t \in [-h + \Delta - \sigma, -\sigma] \). Note that set (14) is closed, bounded and convex. Suppose that the equation \( x^\varepsilon(t - \sigma, \varphi) = -\sigma \) has 2n or more positive roots. We denote one with number 2n by \( T^\varepsilon(\varphi) \). Finally, we define the Poincaré operator \( \Pi^\varepsilon : C[-1 - \sigma, -\sigma] \to C[-1 - \sigma, -\sigma] \) by the formula \( \Pi^\varepsilon(\varphi) = x^\varepsilon(t + T^\varepsilon(\varphi), \varphi), -1 - \sigma \leq t \leq -\sigma \).

A proof plan is as follows. We find asymptotic (uniform with respect to \( \varepsilon \) and \( t \)) formulas for the solution \( x^\varepsilon(t, \varphi) \) on different time intervals (see similar constructions in [18]). As a corollary, we get that \( x^\varepsilon(t, \varphi) \) is close to \( x_\ast(t) \) on the segment \([-\sigma, T^\varepsilon(\varphi)]\). The formulas imply that for an appropriate choice of the parameters \( \sigma, q_1, q_2 \) the operator \( \Pi^\varepsilon \) is defined on the set \( S \) and takes it into itself. It allows us to use the Schauder principle to prove that there exists a fix point of \( \Pi^\varepsilon - \) a periodic solution of (6) with asymptotic \( x^\varepsilon(t, \varphi) \). Then we estimate Fréchet derivative of \( \Pi^\varepsilon \) on the fix point. From this exponential estimation it follows that the map is compressive and the fix point is unique. Moreover, the estimation gives us an information about multipliers. One of them equals 1, and all other are exponentially small. It means that periodic solution exponentially orbitally stable.

3. **Theorem about an existence of an antiphase regime of system (1)**

From Theorem 1 it follows that equation (10) admits a cycle \( x^{\varepsilon}_\ast(t) \) which is close to \( x_\ast(t) \), described by (15). Since a period \( T^{\varepsilon}_\ast = T^{\varepsilon}_\ast(\Delta) \) of the cycle depends on \( \Delta \), to prove that antiphase solution of (6) exist we need to prove that the equation

\[
T^{\varepsilon}_\ast(\Delta) = 2\Delta
\]

has a solution.

By (19), (21), (22), we obtain that restriction (23) has the following form:

\[
-\xi^n(h - \Delta - nT_0 - \eta) - \eta + h - \Delta + nT_0 = 2\Delta.
\]

This implies

\[
\Delta = \frac{1}{3 - \xi^n}(h - \eta)(1 - \xi^n) + nT_0(1 + \xi^n).
\]
Put (25) to (11). We get the following restriction on $h$:

$$t_0 + 1 + (2n-1)T_0 - \frac{1}{2}(1 - \xi^n)(\eta + a) < h < 2nT_0 - \frac{1}{2}(1 - \xi^n)\eta. \quad (26)$$

From above and Theorem 1 we obtain the following.

**Theorem 2.** Fix a natural $n$. Let the parameters $a, b, c, h$ meet restrictions (11), (26), $\Delta$ is described by (25). Then system (6) admits antiphase solution $(x^*_n(t), x^*_n(t+\Delta))$ with exactly $n$ segments on a period where it is positive.

Taking into account (5), we obtain that (1) admits an antiphase solution with a bursting.

4. **Theorem about a stability of the antiphase regime of system (1)**

Investigate stability of the antiphase solution in a phase space $C[-h - \sigma, -\sigma] \times C[-h - \sigma, -\sigma]$ of system (6). Let us proof the following.

**Theorem 3.** The antiphase solution $(x^*_n(t), x^*_n(t+\Delta))$ of (6), satisfying conditions of Theorem 2, is exponentially obitally stable.

We linearise system (6) on the antiphase solution. We get

$$\dot{\gamma}_1 = -bG(x^*_n(t+\Delta - h), \varepsilon)\gamma_1 + A(x^*_n(t-1), \varepsilon)\gamma_1(t-1) + b(c - x^*_n(t))B(x^*_n(t + \Delta - h), \varepsilon)\gamma_2(t-h), \quad (27)$$

$$\dot{\gamma}_2 = -bG(x^*_n(t-h), \varepsilon)\gamma_2 + A(x^*_n(t - \Delta - 1), \varepsilon)\gamma_2(t-1) + b(c - x^*_n(t - \Delta))B(x^*_n(t-h), \varepsilon)\gamma_1(t-h), \quad (28)$$

where $G(x, \varepsilon)$ is defined by (7),

$$A(x, \varepsilon) \overset{def}{=} f'(\exp(x/\varepsilon))\exp(x/\varepsilon)/\varepsilon, \quad B(x, \varepsilon) \overset{def}{=} g'(\exp(x/\varepsilon))\exp(x/\varepsilon)/\varepsilon. \quad (29)$$

Since (27), (28) is linear system with periodic coefficients (of period $T_\ast$), we can apply Lyapunov-Floquet theory. To study an antiphase solution stability we estimate its multipliers. To prove Theorem 3 we show that system (27), (28) has unit multiplier and all other its multipliers are less than 1.

Note that if $(\gamma_1(t), \gamma_2(t))$ is a solution of (27), (28) then $(\gamma_2(t+\Delta), \gamma_1(t+\Delta))$ is also a solution. It means that there exists a constant $\kappa$ such that, for any $t > -\sigma$, we have

$$\kappa\gamma_1(t) = \gamma_2(t + \Delta), \quad \kappa\gamma_2(t) = \gamma_2(t + \Delta). \quad (30)$$

By this and $T_\ast = 2\Delta$, for any $t > -\sigma$, we get $\kappa^2\gamma_1(t) = \gamma_1(t + T_\ast)$, $\kappa^2\gamma_2(t) = \gamma_2(t + T_\ast)$. It means that $\kappa^2$ is a multiplier. Thus, we have the following.

**Claim 1.** Any $\kappa$ is correspondence a multiplier $\nu = \kappa^2$ of linearised system (27), (28).

Furthermore, (30) implies that functions $\gamma_1$ and $\gamma_2$ have the following form:

$$\gamma_1(t) = \gamma(t), \quad \gamma_2(t) = \kappa\gamma(t - \Delta), \quad (31)$$

where $\gamma(t)$ satisfies the equation

$$\dot{\gamma} = -bG(x^*_n(t + \Delta - h), \varepsilon)\gamma + A(x^*_n(t-1), \varepsilon)\gamma(t-1) + \kappa b(c - x^*_n(t))B(x^*_n(t + \Delta - h), \varepsilon)\gamma(t - \Delta - h). \quad (32)$$
Lemma 1. Let us find how arguments of solutions to the equation 
\[ \gamma(x) \text{ function. Their values have an order } \mathcal{O}(\epsilon) \]
depend on \( \kappa \). Claims 1 and 2 allow us to find multipliers of (27), (28) in a following way. At first, we construct formulas \( \mu = \mu(\kappa) \) of multiplier of equation (32) (depending on \( \kappa \)). Secondly, we find \( \kappa \) from the equation \( \mu(\kappa) = \kappa^2 \). As a result we find desired multipliers.

Consider equation (32). Definition of \( A(x, \epsilon) \) and \( B(x, \epsilon) \) imply that they are close to delta-function. Their values have an order \( \mathcal{O}(e^{-q/\epsilon}) \) whenever \( x \) is not zero. Here and below \( q \) is a positive constant (precise value of which is not important). The function \( \gamma \) switch its value as arguments \( x_1^* + (t + \Delta - h) \) of functions \( A \) and \( B \) equal to zero and \( \epsilon \to 0 \). Denote roots of equation \( x_1^*(t) = 0 \) by \( t_{k,l} \) and \( t_0^{k,l} \), \( k = 0, \ldots, n - 1, l \in \{0\} \cup N \). By (15), (22), we have

\[ t_{k,l} \overset{\text{def}}{=} kT_0 + lT_\gamma + \mathcal{O}(e^{-q/\epsilon}), \quad t_0^{k,l} \overset{\text{def}}{=} t_0 + kT_0 + lT_\gamma + \mathcal{O}(e^{-q/\epsilon}) \]

Then \( \gamma \) has the following swich point

\[ 1 + t_{k,l}, \quad 1 + t_0^{k,l}, \quad h - \Delta + t_{k,l}, \quad h - \Delta + t_0^{k,l}. \]

Let us define multipliers of equation (32). As in [16], we call an eigenvalue of a special monodromy operator by a multiplier of a differential-difference equation. To define the operator let us fix real number \( \gamma \), and consider the space

\[ E = \{ \hat{\gamma}(t) \in C[-h - \Delta - \sigma, -\sigma], \quad \hat{\gamma}(-\sigma) = \gamma \}. \]

Definition 3. A monodromy operator of \( \gamma \) is an operator \( \mathcal{V} : E \to E \) such that

\[ \mathcal{V}\hat{\gamma} = \gamma(t + T^*_\gamma, \hat{\gamma}) \in E, \quad -h - \Delta - \sigma \leq t \leq -\sigma, \quad \text{(33)} \]

where \( \gamma(t, \hat{\gamma}) \) is a solution of (32) with the initial condition \( \hat{\gamma}(t), -h - \Delta - \sigma \leq t \leq -\sigma; T^*_\gamma \) is a period of antiphase solution.

By (31), we have the following.

Claim 1. Multipliers of system (27), (28) and multipliers of equation (32) are the same.

Let us prove the following fact.

Lemma 1. \( \gamma + (1 + k) = -a \gamma + (1 + k) + \mathcal{O}(e^{-q/\epsilon}) \), \( \gamma (1 + t_0^{k,l} + 0) = -\gamma (1 + kT_0 + 0) + \mathcal{O}(e^{-q/\epsilon}) \), \( \gamma (h - \Delta + T_{k,l} - 0) + \mathcal{O}(e^{-q/\epsilon}) \), \( \gamma (h - \Delta + t_0^{k,l} - 0) + \mathcal{O}(e^{-q/\epsilon}) \), \( \gamma (h - \Delta - t_0^{k,l} - 2\Delta) + \mathcal{O}(e^{-q/\epsilon}) \).

The proof of Lemma 1 is technical and use an asymptotic (15), (22) of an antiphase solution. Consider a point set

\[ M \overset{\text{def}}{=} \{ -\sigma, t_{k,0} - T^*_\gamma, t_0^{k,0} - T^*_\gamma : \quad k = 0, 1, \ldots, n - 1 \}, \quad \text{(39)} \]

embeded to the segment \([-h - \Delta - \sigma, -\sigma]\). Let us prove the following fact.

Lemma 2. Let \( \hat{\gamma}(t) \in E \) and value of \( \hat{\gamma} \) at points of \( M \) are known:

\[ \hat{\gamma} \overset{\text{def}}{=} \hat{\gamma}(-\sigma), \quad \hat{\gamma}_k \overset{\text{def}}{=} \hat{\gamma}(t_{k,0} - T^*_\gamma), \quad \hat{\gamma}_k^0 \overset{\text{def}}{=} \hat{\gamma}(t_0^{k,0} - T^*_\gamma) \].

\[ \hat{\gamma}_k - \hat{\gamma} \in \mathcal{O}(e^{-q/\epsilon}), \quad \text{for } k = 0, 1, \ldots, n - 1, \quad \hat{\gamma}(t_{k,0} - T^*_\gamma) \in \mathcal{O}(e^{-q/\epsilon}). \]
Then, across period $T^*_t$ of $x^*_t(t)$, the values maps to numbers
\[
\gamma(t_{k,0}) = \tilde{\gamma} + O(e^{-g/\varepsilon}), \quad \gamma(t_{0,0}) = -a \tilde{\gamma} + O(e^{-g/\varepsilon}),
\]
(41)
\[
\gamma(-\sigma + T^*_t) = \xi^0 \tilde{\gamma} + \sum_{k=0}^{n-1} c_k \tilde{\gamma} + \sum_{k=0}^{n-1} \xi^0 c_k \tilde{\gamma} + O(e^{-g/\varepsilon}),
\]
(42)
\[
c_k \overset{\text{def}}{=} \kappa b (c - \alpha_k) \xi^{n-k}, \quad c_k \overset{\text{def}}{=} \kappa b (c - \beta_k) \xi^{n-k-1}/a.
\]
(43)

The function $G(x^*_t(t - h + \Delta))$ is exponentially small on a segment $[-\sigma, t_{0,0} + h - \Delta]$. It implies that the derivative $\dot{\gamma}(t)$ close to zero on this segment excepting points $1 + t_{k,0}, 1 + t_{k+1,0}$.

Thus, by (36), we get $\gamma(t) = \tilde{\gamma} + O(e^{-g/\varepsilon})$ as $t \in [-\sigma, 1 + t_{0,0}]$, $[1 + t_{k,0}, 1 + t_{k+1,0}]$, $k \in 0, \ldots, n-2$, $[1 + t_{n-1,0}, h - \Delta + t_{0,0}]$, $\gamma(t) = -a \tilde{\gamma} + O(e^{-g/\varepsilon})$ as $t \in [1 + t_{k,0}, 1 + t_{k+1,0}]$, $k = 0, \ldots, n-1$. Therefore, (41) is proved.

The function $G(x^*_t(t - h + \Delta))$ is exponentially close to 1 on the segments $[h - \Delta + t_{0,0}, h - \Delta + t_{n-1,0}], k = 0, \ldots, n - 1$. This means that going through the intervals $(h - \Delta + t_{k,0}, h - \Delta + t_{k+1,0}), k = 0, \ldots, n - 1$, whose length is $t_0$, the $\gamma(t)$ is multiplied by $\xi = \exp(-bt_0)$. Finally, using Lemma 1, we get (42).

Lemma 2 is proved.

A main result of this section is the following.

**Lemma 3.** Monodromy operator (33) has multipliers $\mu_k, k \in N$, with the following properties:

(i) $\mu_1 = 1$,

(ii) $\mu_2, \mu_3 : |\mu_2| = |\mu_3| < 1$,

(iii) $\mu_4, \mu_5, \ldots : |\mu_k| = O(e^{-g/\varepsilon})$.

Let us prove it. For every function $\gamma \in E$, we have $\dot{\gamma}(t) = \gamma_1(t) + \gamma_0(t)$,
\[
\gamma_1(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{as } t \in M, \\ 0 & \text{as } t \in [-h - \Delta - \sigma, -\sigma] \setminus M, \end{cases} \quad \gamma_0(t) \overset{\text{def}}{=} \begin{cases} 0 & \text{as } t \in M, \\ \dot{\gamma}(t) & \text{as } t \in [-h - \Delta - \sigma, -\sigma] \setminus M. \end{cases}
\]
(44)

Since $V$ is linear, we obtain
\[
V \dot{\gamma} = V \gamma_1 + V \gamma_0 = \gamma(t, \gamma_1) + \gamma(t, \gamma_0) = \gamma(t, \gamma_1) + O(e^{-g/\varepsilon}).
\]

Thus, using Lemma 2, we have that an operator multipliers consist of two groups. The first one contains eigenvalues of $(2n + 1)$-dimensional linear operator
\[
\begin{pmatrix}
\xi^m & c_0 & c_0^0 & \cdots & c_{n-1} & c_0^{n-1} \\
1 & 0 & 0 & \cdots & 0 & 0 \\
-a & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 \\
-a & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
(45)

The second one contains infinity count of exponentially small multipliers. The characteristic polynomial of matrix (45) has the form
\[
\mu^{2n-1}(-\mu^2 + \xi \mu + \kappa(1 - \xi^{k+1})) = 0
\]
(46)

Solving system (47), (46) we find multipliers:
\[
\mu_1 = 1, \quad \mu_2, 3 = -1/2 - \xi^n \pm i \sqrt{3 - 4 \xi^n}/2.
\]
(47)

Modules of $\mu_{2,3}$ are close to 1, but less than 1.

Lemma 3 is proved. It means that antiphase solution is exponentially orbitally stable.
5. Conclusion
As an example, show (pic. 2) an antiphase solution of (6). We use a program Tracer3.70 developed by D. S. Glyzin. The function and parameters are as follows:

\[ f(u) = \frac{a(1-u)}{a+u}, \quad g(u) = \frac{u}{1+u}, \quad h = 16, \quad a = 1.5, \quad b = 2, \quad c = -4.5, \quad \lambda = 100. \]

![Figure 2. Antiphase solution of (6) that has two segments with positive values.](image)

To sum up, note that the delay \( h \) of connection chain can be chosen longer than period in model (2) of alone oscillation. We prove that this allows us to get an antiphase solution of (1) with a bursting-effect.

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References
[1] Glyzin S D, Kolesov A Yu and Rozov N Kh 2013 Diff. Eq. 49 1227-44.
[2] Glyzin S D, Kolesov A Yu and Rozov N Kh 2015 Rus. Math. Surveys 70 383-452
[3] Preobrazhenskaia M M 2017 Model. Anal. Inform. Sist. 24 186-204
[4] Kolesov A Yu, Mishchenko E F and Rozov N Kh 2010 Computat. Math. and Math. Physics 50 1990-2002
[5] Kashchenko S 2015 Models of Wave Memory (Berlin: Springer) p 239
[6] Somers D and Kopell N 1993 Biol. Cybern. 68 393-407
[7] Somers D and Kopell N 1995 J. Math. Biol. 33 261-80
[8] Terman D 2005 Tutorials in Math. Biosciences I, Lecture Notes in Math. 1860 21-68
[9] Chay T R and Rinzel J 1985 Biophys. J. 47 357-66
[10] Ermentrout G B and Kopell N 1986 SIAM J. Appl. Math. 46 233-53
[11] Izhikevich E 2000 Inter. J. of Bifurc. and Chaos. 10 1171-266
[12] Rabinovich M I, Varona P, Selverston A I and Abarbanel H D I 2006 Rev. Mod. Phys. 78 1213-65
[13] Coombes S and Bressloff P C 2005 Bursting: the Genesis of Rhythm in the Nervous System (Singapore: World Scientific Publishing Company) p 418
[14] Glyzin S D, Kolesov A Yu and Rozov N Kh 2013 Math. Notes 93 676-90
[15] Glyzin S D, Kolesov A Yu and Rozov N Kh 2011 Diff. Eq. 47 1697-713
[16] Glyzin S D, Kolesov A Yu and Rozov N Kh 2013 Theor. and Math. Physics 175 409-517
[17] Glyzin S D, Kolesov A Yu and Rozov N Kh 2012 Comput. Math. and Math. Physics 52 702-19
[18] Preobrazhenskaia M M 2017 Model. Anal. Inform. Sist. 24 550-66