Living in a world without imaginaries

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Abstract. We need complex numbers to compress and condense long mathematical formulae into shorter and more abstract ones. But we do not need them to describe the world we live in. Such a mathematical world is presented in this paper. It is a world which can be explained to and understood by learners much easier than the usual obscured world full of imaginaries. And it is a geometrical world, not an algebraic one.

1. Introduction
In his great book The Road to Reality Roger Penrose writes: “Yet, there are other kinds of number which, according to accepted theory, do appear to play a fundamental role in the workings of the universe. The most important and striking of these are the complex numbers, in which the seemingly mystical quantity \(\sqrt{-1}\), usually denoted by ‘i’, is introduced and adjoined to the real-number system. First encountered in the 16th century, but treated for hundreds of years with distrust, the mathematical utility of complex numbers gradually impressed the mathematical community to a greater and greater degree, until complex numbers became an indispensable, even magical, ingredient of our mathematical thinking. Yet we now find that they are fundamental not just to mathematics: these strange numbers also play an extraordinary and very basic role in the operation of the physical universe at its tiniest scales. This is a cause for wonder, and it is an even more striking instance of the convergence between mathematical ideas and the deeper workings of the physical universe than is the system of real numbers” [1].

Although everything written in this short description about the magical effects of the imaginary unit \(i\) is true, the imaginary unit \(i\) is not necessary to describe the world we live in. This magical tiny \(i\) might even be confusing and obscuring our picture of the world. The imaginary unit is surely helpful in compressing and condensing long mathematical formulae into shorter ones, which is a clear indication of high mathematical expertise.

But is it also helpful to present mathematical formulae full of this abstract symbol \(i\) to students at school, high school or universities who for the first time deal with these not always simple descriptions of space, time and the relations of physics? I am sure, it is not really helpful for learners.

Reading the lines of Penrose students learn that physics and mathematics contain mystical, even magical ingredients. But I want to live in a world which is understandable and not mystical. I like magicians and their shows. But I like these shows not because of the unphysical phenomena which are shown. I like them because I love to find out how the trick behind all this works.

Penrose and many others present the quantity \(\sqrt{-1}\) like magicians on the stage of physics and mathematics. But there is a trick behind complex numbers! Let’s find out this trick.
One way to understand imaginary numbers is to construct a world without them and look, which mathematical objects occupy the position of them. Such a world without an imaginary unit $i$ will be presented and discussed in this paper.

It is a world which consists of $+1$, $-1$, and matrices.

2. Historical background

In mathematics there are two contrasting, opposite tendencies: the tendency to explain and understand mathematical structures as algebraic relations and opposed to that as geometric relations. We can take a more geometric standpoint or we can take a more algebraic standpoint.

Historically de-geometrisation, i.e. leaving the geometric standpoint and approaching a more algebraic standpoint, is closely connected with the algebraisation of geometric constructions. And vice versa promoting geometrisation means de-algebraisation [2].

But these two tendencies not only influence the way we work in mathematics, it very deeply influences the way we see and interpret our world, as the following conversation between Dirac and Abdus Salam indicates. “Once Dirac asked me if I thought algebraically or geometrically,” Abdus Salam told and went on: “I did not know what he meant, but after further questioning, Dirac said ‘Precisely as I thought. You think algebraically as most people in the Indian subcontinent do.’ Dirac, it appeared, thought geometrically” [3].

Dirac even claimed, that he can “picture, without effort, the de Sitter space as a four-dimensional surface in a five-dimensional space” [4], which is something a convinced algebraist never would do, because “mathematics may be defined as the subject in which we never know what we are talking about” [5]. This explanation of Russell shows his clear focus on an algebraic world view.

But modern physics is deeply connected with a geometric world view. The geometrisation of the world which surrounds us is an elementary conceptual principle starting with the unification of space and time into a geometrically united spacetime.

Einstein explained this inherent connection between physics and geometry in his academy lesson in 1921: “Yet on the other hand it is certain that mathematics generally, and particularly geometry, owes its existence to the need which was felt of learning something about the relations of real things to one another. (...) It is clear that the system of concepts of axiomatic geometry alone cannot make any assertions as to the relations of real objects of this kind, which we will call practically-rigid bodies. To be able to make such assertions, geometry must be stripped of its merely logical-formal character by the co-ordination of real objects of experience with the empty conceptual frame-work of axiomatic geometry. (...) Geometry thus completed is evidently a natural science; we may in fact regard it as the most ancient branch of physics” [6]. Thus physics exerts a considerable stimulus on mathematics to geometrisation.

Hestenes even comes to the conclusion: “Mathematics is too important to be left to the mathematicians!” [8] He therefore supports the use of geometric algebra as a mathematical tool which unites algebraic and geometric ideas in a unified mathematical language.

Euler introduced the symbol $i$ for the imaginary unit $\sqrt{-1}$ in 1777 as an algebraic symbol [9]. Algebraic equations can be solved much easier using this symbol, and the story of $i$, this imaginary tale [10], surely is a story of algebra. The path the history of European mathematics had chosen with respect to imaginary and complex numbers, was an algebraic path.

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1 In this context Scriba and Schreiber use the German words “Entgeometrisierung” and “Algebraisierung” [2].
2 The complete German quotation is: “Andererseits ist es aber doch sicher, daß die Mathematik überhaupt und im speziellen auch die Geometrie ihre Entstehung dem Bedürfnis verdankt, etwas zu erfahren über das Verhalten wirklicher Dinge. (...) Es ist klar, daß das Begriffssystem der axiomatischen Geometrie allein über das Verhalten derartiger Gegenstände der Wirklichkeit, die wir als praktisch-starre Körper bezeichnen wollen, keine Aussagen liefern kann. Um derartige Aussagen liefern zu können, muß die Geometrie dadurch ihres nur logisch-formalen Charakters entkleidet werden, daß den leeren Begriffs schemata der axiomatischen Geometrie erlebbare Gegenstände der Wirklichkeit (Erlebnisse) zugeordnet werden. (...) Die so ergänzte Geometrie ist offenbar eine Naturwissenschaft; wir können sie geradezu als den ältesten Zweig der Physik betrachten” [7].
But is there a second possible path, not followed by our mathematical ancestors, which opens a geometric entrance to imaginary and complex numbers? I am convinced: a development of mathematics with a more geometrical focus could have led to a mathematical world without imaginaries. In this (till now not really appreciated) geometric mathematical world, geometric objects nearly automatically appear which take the function of the imaginary unit. But these objects are nothing artificial which has to be invented. They are objects which are there because they are inherent to geometry. They are as "real" as other real geometric objects.

Hamilton opened the door to this mathematical structures a little bit, when he discussed Pauli-like “quadruples of numbers that form 2 x 2 matrices” [10]. But it seems that the Quaternionists and Hamiltonians nearly immediately closed this door again by neglecting the geometric meaning of quaternion calculus and by “presenting their views in a rather one-sided and metaphysical bearing” only acknowledging algebraic definitions [11]3.

One can even conclude that history of mathematics suffered from a subvirus of the MV/K-type virus Hestenes identified [12]. This virus presses mathematicians to hallucinate that “complex Clifford algebras are more general than real Clifford algebras.” And this “suggests that the standard mathematical practice of regarding complex numbers as scalars is an egregious case of mistaken identity, a mistake which can be corrected by recognizing the geometric primacy of the real geometric algebra” [12].

3. Two-dimensional worlds
Two of the thee Pauli matrices do not include imaginary units and thus are quadrupoles of the real numbers +1, –1, and 0 only. These two Pauli matrices are the base vectors

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ (1)

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ (2)

of a two-dimensional space with signature (+, +). The other two basic elements of this space are a unit scalar

$$1^2 = \sigma_x^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (3)

and a base bivector

$$\sigma_{zx} = \sigma_z \sigma_x = -\sigma_x \sigma_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ (4)

which clearly can take the place of an imaginary unit because it squares to minus one.

$$\sigma_{zx}^2 = -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ (5)

3 The complete German quotation is: “Trotzdem diese Theorie nicht mehr jung ist, gehen die Ansichten über ihren Wert bis heute stark auseinander. Der Grund hierfür dürfte darin zu suchen sein, daß die Vertreter der Quaternionentheorie ihre Lehren meistens einseitig und mit einem metaphysischen Anfluge behaftet darstellen, wobei die einfache geometrische Deutung, welche man den Operationen ihres Kalkuls geben kann, nicht immer genügend hervortritt” [11].
This base bivector \( \sigma_x \sigma_z \) is in the first place a geometrical object: an oriented plane element in the direction of the vectors \( \sigma_x \) and \( \sigma_z \) with a unit area. But as we are able to calculate directly with geometrical objects in geometric algebra, it is an algebraic object too. This area is as real as a vector: it exists in our conceptual scheme of space. But it automatically takes the position of the now superfluous imaginary unit \( i \).

The two vectors and the bivector anticommute in this two-dimensional space. This feature is shown in the commutation table of these real (2x2) matrices (see table 1), where “com” means that the two elements commute while “A” means that the two elements anticommute.

### Table 1. Commutation table for real (2x2) matrices.

|  | \( \sigma_x \) | \( \sigma_z \) | \( \sigma_x \sigma_z \) |
|---|---|---|---|
| 1: \( 1 \) | \( \text{com} \) | \( \text{com} \) | \( \text{com} \) |
| 2: \( \sigma_z \) | \( \text{com} \) | \( +1 \) | \( \text{A} \) | \( \text{A} \) |
| 3: \( \sigma_x \) | \( \text{com} \) | \( \text{A} \) | \( +1 \) | \( \text{A} \) |
| 4: \( \sigma_x \sigma_z \) | \( \text{com} \) | \( \text{A} \) | \( \text{A} \) | \( -1 \) |

In the diagonal line the squares (3) and (5) of these elements are shown, which give the signature \((+, +)\). If \( \sigma_x \) and \( (\sigma_z \sigma_x) \) were employed as base vectors, the signature would be \((+, -)\). Of course, the basic elements \( \sigma_x \) and \( \sigma_z \) are in the first place geometric objects too in this two-dimensional world. Every vector \( r \) can be written as a linear combination of these two basic elements:

\[
r = x \sigma_x + z \sigma_z = \begin{pmatrix} z & x \\ x & -z \end{pmatrix}
\] (6)

Therefore \( \sigma_x \) and \( \sigma_z \) are correctly identified as base vectors. But they have a second, operational geometric meaning. They are basic reflections (see [1]), for a left- and right-sided multiplication of a vector \( r \) with \( \sigma_x \) or \( \sigma_z \) gives a vector \( r' \) which is reflected at the axes of the corresponding base vector:

\[
r' = \sigma_x \ \sigma_x \ = x \sigma_x - z \sigma_z
\] (7)

\[
r'' = \sigma_z \ \sigma_z \ = -x \sigma_x + z \sigma_z
\] (8)

Only the coordinates perpendicular to the reflection axes change the sign. A reflection at the \((\sigma_x \sigma_z)\)-plane is an identity operation given by

\[
r = \sigma_x \sigma_z \ \sigma_x \sigma_z = x \sigma_x + z \sigma_z
\] (9)

while a reflection at the origin

\[
r''' = -1 \ r \ 1 = -x \sigma_x - z \sigma_z
\] (10)

reverses all signs.

Of course the commutation relations are a consequence of the algebra of these matrices. The matrix representation chosen here is only one of an infinite number of possible matrix representations. For example one could chose the two base vectors

\[
e_1 = \sigma_x + \sigma_z + \sigma_x \sigma_z = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}
\] (11)

\[
e_2 = -\frac{1}{2} \sigma_x + \sigma_z + \frac{1}{2} \sigma_x \sigma_z = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}
\] (12)
with

\[ e_1^2 = e_2^2 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(13)

and

\[ e_1 e_2 = -\frac{1}{2} \sigma_x - \frac{3}{2} \sigma_z \sigma_x = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \]  

(14)

with

\[ (e_1 e_2)^2 = -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(15)

Again all vectors \( e_1, e_2 \) and the bivector \( e_1 e_2 \) anticommute, giving the same commutation table (see table 1). Therefore it is not necessary to use a special matrix representation. In some situations it might be helpful using a matrix representation, but in other situations it might be confusing. All information about the geometrical content is encoded in the algebra.

4. Four-dimensional worlds

Snygg already presented the base vectors of three-dimensional Euclidean space as real (4x4) matrices [13]. But real (4x4) matrices can be used to construct four linear independent base vectors, thus spanning a four-dimensional space.

The mathematics to construct (4x4) matrices using (2x2) matrices was already developed in 1858 by Zehfuss [14], although his direct product was named in a strange twist after Kronecker [15] who only much later mentioned this direct matrix product. The Zehfuss-Kronecker product now plays an important role in describing quantum computing (see [16]), then sometimes even more unsatisfactorily called tensor product (see e.g. [17]).

The Zehfuss-Kronecker product is defined as

\[ A \otimes B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \]  

(16)

in the special case of (2x2) matrices and generally

\[ A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \]  

(17)

The 16 new basic elements of a four-dimensional space can then be constructed using the four basic elements of the two-dimensional space (1) – (4) and multiplying them according to the direct Zehfuss-Kronecker product (17). The results are given in table 2, which of course show only one possible matrix representation. Taking (13), (11), (12), and (14) as starting point would give a set of 16 different matrices which nevertheless are subject to the same algebra.

These new basic elements must now be assigned to one scalar, four vectors, six bivectors, four trivectors and one tetravector.
The new scalar surely is the (4x4) identity matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

If we want a space with only positive signature elements, it is necessary to know the squares of the basic elements. They are:

\[
(\mathbf{1} \otimes \sigma_x)^2 = (\mathbf{1} \otimes \sigma_y)^2 = \mathbf{1}_{(4\text{-dim})}
\]
\[(\sigma_x \otimes 1)^2 = (\sigma_x \otimes 1)^2 = 1_{(4\text{-dim})}\]  
(20)

But these four Zehfuss-Kronecker products \((1 \otimes \sigma_x), (1 \otimes \sigma_y), (\sigma_x \otimes 1), \text{and} (\sigma_y \otimes 1)\) should not be chosen as base vectors because they do not form a set of anticommuting matrices.

\[(1 \otimes \sigma_y, \sigma_y) = (\sigma_x \sigma_y \otimes 1)^2 = -1_{(4\text{-dim})}\]  
(21)

\[(\sigma_x \otimes 1)^2 = (\sigma_x \otimes 1)^2 = 1_{(4\text{-dim})}\]  
(22)

\[(\sigma_y \otimes \sigma_x)^2 = (\sigma_y \otimes \sigma_x)^2 = 1_{(4\text{-dim})}\]  
(23)

\[(\sigma_x \otimes \sigma_x, \sigma_x) = (\sigma_x \otimes \sigma_x)^2 = -1_{(4\text{-dim})}\]  
(24)

\[(\sigma_y \otimes \sigma_y, \sigma_y) = (\sigma_y \otimes \sigma_y)^2 = -1_{(4\text{-dim})}\]  
(25)

\[(\sigma_y \otimes \sigma_x, \sigma_x) = (\sigma_y \otimes \sigma_x)^2 = 1_{(4\text{-dim})}\]  
(26)

All these relations can be found directly by evaluating the matrix representations. But didactically it makes more sense to derive (19) – (26) by using the multiplication rule of the Kronecker-Zehfuss product [18]

\[(A_1 \otimes B_1) (A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)\]  
(27)

because then it is clear that the results are a direct consequence of the underlying algebra and not of a special matrix representation.

The commutation rules can be found in an equally simple way using equation (27). For example the basic elements \((\sigma_x \sigma_x) \otimes \sigma_x\) (#14 in table A1, see appendix) and \(\sigma_x \otimes \sigma_x\) (#6 in table A1) anticommute because

\[((\sigma_x \sigma_x) \otimes \sigma_x) (\sigma_x \sigma_x) = ((\sigma_x \sigma_x) \otimes (\sigma_x \sigma_x) = \sigma_x \otimes 1\]  
(28)

and

\[(\sigma_x \otimes \sigma_x) ((\sigma_x \sigma_x) \otimes \sigma_x) = ((\sigma_x \sigma_x) \otimes (\sigma_x \sigma_x) = -\sigma_x \otimes 1\]  
(29)

thus giving

\[(\sigma_y \otimes \sigma_x) ((\sigma_x \sigma_x) \otimes \sigma_x) = - ((\sigma_x \sigma_x) \otimes \sigma_x) (\sigma_x \otimes \sigma_x)\]  
(30)

Again the commutation rules are a direct consequence of the underlying algebra and do not depend on the matrix representation. The commutation rules are shown in table A1 (see appendix) where “com” means that the two elements commute while “A” means that the two elements anticommute.

5. Some philosophy

Analyzing table A1 we now try to find sets of four basic elements which can be used as base vectors. Because each base vector has to anticommute with every other base vector, we need a set of four anticommuting base elements.

Astonishingly it is not possible to find a set with four anticommuting base vectors which all have a positive square of + 1. And it is not possible to find a set with four anticommuting base vectors which all have a negative square of –1.

We therefore fail to construct a four-dimensional space which has four spacelike base vectors. And we fail to construct a four-dimensional space which has four timelike base vectors. What is the philosophical meaning of this?

We all agree that we are living in a four-dimensional spacetime. But why isn’t it a four-dimensional space only? Why isn’t our world a four-dimensional time? Is there a reason for this?
We can not be sure that nature does not make use of an artificial imaginary unit. We can not be sure that nature only makes use of the geometrically based imaginary elements #4, #8, #12, #13, #14, and #15 of table A1.  

But we can conclude: If nature does not make use of an artificial imaginary unit, we will live in a world which automatically consists of space and time dimensions. Table A1 clearly shows that we are only able to construct spaces with a signature of (+, +, +, –) or (+, +, –, –). There are no pure spacelike four-dimensional worlds, and there are no pure timelike four-dimensional worlds in a totally geometrically based world.

We are not able to measure imaginary or complex values in physics. We can walk 7 km, but we never walk (7 + 3i) km. We use this imaginary unit called i because it simplifies our calculations. But it might be something artificial. Therefore it is not that stupid to construct a world without an artificial imaginary unit i. The world we live in might be structured like that, and the conclusion again is: If nature does not make use of an artificial imaginary unit, we will live in a world which automatically consists of space and time.

6. Dirac matrices

Dirac matrices are matrix representations of base vectors of four-dimensional spacetime [8], [19], [20]. As we are living in such a spacetime of signature (+, –, –, –) it makes sense to try to identify some of the basic elements of table 2 with Dirac matrices.

Comparing table 2 with the Dirac matrices given in [21] shows that the timelike Dirac matrix and base vector $\gamma_t$ can be identified with element #9:

$$\gamma_t = \sigma_z \otimes \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ (31)

The spacelike Dirac matrix and base vector in x direction $\gamma_x$ can be identified with element #14:

$$\gamma_x = (\sigma_z \sigma_x) \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$ (32)

Unfortunately the Dirac matrix and base vector in y direction $\gamma_y$ contains imaginary units i. But the spacelike Dirac matrix and base vector in z direction $\gamma_z$ can again be identified with element #15:

$$\gamma_z = (\sigma_z \sigma_x) \otimes \sigma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ (33)

The timelike Dirac matrix and base vector in v direction of cosmological relativity [22] can be identified with element #5:

$$\gamma_v = \sigma_x \otimes \mathbf{1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ (34)
Hence it is clear that real Dirac matrices without imaginary unit i can be interpreted as the base vectors of a spacetime with signature \((+, +, -, -)\) only.

It is even not possible to construct a space with signature \((+, -, -, -)\). But there are several possibilities to find adequate base vectors for a world with signature \((-+, +, +, +)\). When time is assigned to a negatively squaring base vector and the three space directions are assigned to positively squaring base vectors this mathematical model can reproduce the geometric structure of our special relativistic world we live in.

For example the basic elements #8, #2, #3, and #16 of table A1 show the necessary commutation relations and can be chosen as a set of base vectors.

Base vector \(e_1\) with \(e_1^2 = -1\):

\[
e_1 = \sigma_1 \otimes (\sigma_1 \sigma_0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\] (35)

Base vector \(e_2\) with \(e_2^2 = +1\):

\[
e_2 = 1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\] (36)

Base vector \(e_3\) with \(e_3^2 = +1\):

\[
e_3 = 1 \otimes \sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\] (37)

Base vector \(e_4\) with \(e_4^2 = +1\):

\[
e_4 = (\sigma_1 \sigma_0) \otimes (\sigma_1 \sigma_0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\] (38)

7. Outlook

“Are imaginary numbers real?” is asked by Gull, Lasenby, and Doran [19]. This question can be treated from epistemological totally different positions. And in many situations it is far from clear how we could or should or have to interpret imaginary and complex numbers.

There are still some secrets hidden behind the geometrical features of our world. We can try to find them. And the only way to do that is to follow geometry and algebra. Both fields should be set into proper and balanced relations. This means that we should not follow algebraic paths only. Perhaps algebra is the best instrument we possess to understand geometry, and perhaps geometry is the best instrument we possess to understand algebra.

The Zehfuss-Kronecker product can be applied again and again, as it is done in the theory of quantum computing. There must be a lot of geometry behind this important emerging field. It is surely promising to analyze higher-dimensional spaces which were constructed in a geometrical way. Even
quantum computing is a geometrical subject. Quantum algebra is geometric algebra. And it can be reformulated without this strange and sometimes superfluous symbol i.

**Appendix:**  
**Table A1.** Commutation table for constructed real (4x4) matrices.

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 1 | 1⊗1 | +1 | com | com | com | com | com | com | com | com | com |
| 2 | 1⊗G2 | +1 | com | com | com | com | com | com | com | com | com |
| 3 | 1⊗G3 | +1 | com | com | com | com | com | com | com | com | com |
| 4 | 1⊗(G2⊙G3) | +1 | com | com | com | com | com | com | com | com | com |
| 5 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 6 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 7 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 8 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 9 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 10 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 11 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 12 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 13 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 14 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 15 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |
| 16 | 1⊗(G2⊙G3)⊙1 | +1 | com | com | com | com | com | com | com | com | com |

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[1] Name of the reference 1.
[2] Name of the reference 2.
[3] Name of the reference 3.
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