THE SUBLEADING ORDER OF TWO DIMENSIONAL COVER TIMES

Abstract. The $\varepsilon$-cover time of the two dimensional torus by Brownian motion is the time it takes for the process to come within distance $\varepsilon > 0$ from any point. Its leading order in the small $\varepsilon$-regime has been established by Dembo, Peres, Rosen and Zeitouni [Ann. of Math., 160 (2004)]. In this work, the second order correction is identified. The approach relies on a multi-scale refinement of the second moment method, and draws on ideas from the study of the extremes of branching Brownian motion.

David Belius$^1$ Nicola Kistler$^2$

1. Introduction

A fundamental question one can ask about a Markov process concerns the time it takes to visit all of the state space. In this article we study this question for Brownian motion in the two dimensional Euclidean torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^2$, i.e. the box $[0,1)^2$ with periodic boundary. More precisely, we study the time it takes for the process to come within distance $\varepsilon > 0$ of every point, in the small $\varepsilon$-regime. This time is referred to as the $\varepsilon$-cover time, and is denoted by $C_\varepsilon$.

The $\varepsilon$-cover time (and its discrete version, the cover time) has been extensively studied over the past decades. For the two dimensional torus upper and lower bounds on the expected cover time were proven by Matthews [22] and Lawler [20]. The gap between these bounds was closed by Dembo, Peres, Rosen and Zeitouni [10], who proved the law of large numbers,

$$
\frac{C_\varepsilon}{\frac{1}{\pi} \log \varepsilon^{-1}} = \left(1 + o(1)\right) \log \varepsilon^{-2}.
$$

The question of lower order corrections, and, in general, fluctuations, was left open. By analogy to related models (for instance the two dimensional Gaussian free field) one may expect the presence of a “$\log \log \varepsilon$-correction term”, see [7, 12]. No suggestion for the exact form of this conjectured term (i.e. including multiplicative constant) appears in the literature. In this work we settle this issue by establishing the following asymptotics,

$$
\frac{C_\varepsilon}{\frac{1}{\pi} \log \varepsilon^{-1}} = \log \varepsilon^{-2} - \left(1 + o(1)\right) \log \log \varepsilon^{-1},
$$

in probability, as $\varepsilon \downarrow 0$.

The law of large numbers (1.1) is somewhat surprising. In fact, $C_\varepsilon$ is the maximum of all hitting times of balls in the torus of radius $\varepsilon$. To first approximation, these hitting times are exponentially distributed, with mean given by the denominator of (1.1). Now, since hitting times of highly overlapping balls should be roughly the same, one may take the maximum over a “packing” of $\sim \varepsilon^{-2}$ balls of radius $\varepsilon$ which do not overlap too much: assuming that these exponentials are independent, one indeed recovers (1.1). In other
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words, despite (what turn out to be) long-range correlations between hitting times of disjoint balls, the leading order of the maximum behaves as in the independent setting.

On the other hand, since the maximum of independent exponentials does not exhibit any correction, (1.2) is testament to the presence of these correlations. As it turns out, the field of hitting times is log-correlated, i.e. correlations decay (roughly) with the logarithm of the distance. The prototypical example of such a random field is branching Brownian motion, BBM for short. Our proof of (1.2) goes via a multi-scale analysis which is much inspired by the picture which has emerged in the study of the extremes of BBM [2, 5, 18]. The correction term in (1.2) corresponds to the well-known 3/2-correction first identified by Bramson [3] for the maximum of BBM (see the end of the introduction).

The above can be contrasted with the situation for discrete torii of higher dimensions, about which much more is known, see [4, 17]. For the cover time of the discrete torus [4] proves that in \( d \geq 3 \) there is no correction term to the leading order, and that the fluctuations follow the Gumbel distribution \( ^{3} \), just as for the maximum of independent exponentials. The reason for this behavior is the local transience of Brownian motion in \( d \geq 3 \), which leads to weak correlations among hitting times: weak enough for the extremes of the field to behave like the extremes of a field of independent random variables, even at the level of fluctuations.

In \( d = 2 \), the local recurrence of Brownian motion leads to intricate long-range correlations among hitting times, and these are responsible for a radically different process of covering. Perhaps more important than the numerical value of the subleading order identified in (1.2) is the description that our proof provides of this covering process: In short, at each scale the torus can be thought of as being tiled by neighbourhoods, where the scale corresponds to the neighbourhoods’ size. Because of ergodicity Brownian motion has a tendency to spend a similar amount of time in most neighbourhoods at each scale (the effect becomes weaker at smaller scales). But to leave an \( \varepsilon \)-ball unvisited until very late Brownian motion needs to spend atypically little time in that ball’s neighbourhoods (this effect becomes stronger at smaller scales). This “conflict” makes it harder to “miss” a small ball, thus making the cover time happen a little bit faster and giving rise to the subleading correction. Furthermore, the strategy\(^4\) needed to avoid a small ball up until right before the \( \varepsilon \)-cover time turns out to be to spend relatively more time in the intermediate scales. These phenomena can be considered instances of entropic repulsion.

As in [10], we control hitting times via excursions between concentric circles at different scales, relying on an implicit tree structure. Our main contribution is the identification of the mechanism by which this approximate tree structure gives rise to the covering behaviour described above, and the discovery of a concrete analogy to branching Brownian motion. Armed with this analogy we are able to apply methods from the study of branching Brownian motion to prove (1.2).

1.1. A sketch of the proof. In the following, \( F \) denotes a set of \( \varepsilon^{-2} \) points in the torus, scattered in such a way that the balls of radius \( \varepsilon \) centered at these points do not overlap too much.

1.1.1. (Failure of) vanilla second moment method. A classical approach for the study of extremes of random fields goes via the so-called second moment method, i.e. the

\(^3\) Although it does not appear in the literature, it is expected that the behaviour of the \( \varepsilon \)-cover time of the Euclidean torus in \( d \geq 3 \) is the same as in the discrete setting.

\(^4\) We do not prove that this is the only strategy, but [2] proves the analogous statement for Branching Brownian Motion, and this seems very likely to carry over to our setting.
This simulation shows the occupation times of balls in the torus at three different scales. Brownian motion is run up to time $10$ and the intensity of each pixel is given by the time spent in a ball centered at that pixel. The radius is indicated by the ball in the upper-right corner, and the occupation times are rescaled by a factor proportional to its area. The traversal process of a point (see (1.9)) can be thought of as a proxy for the occupation times of balls around that point. The picture hints at the approximate hierarchical structure: at the first scale all occupation times are essentially the same. At the second scale the torus has been split into regions of high, moderate and low occupation time. At the third scale these regions have been further subdivided.

**Figure 1.1.** Effect of approximate hierarchical structure in simulation of occupation times.

A comparative study of first and second moment of a suitably chosen quantity. In the case of cover times, a natural candidate is

$$Z(m) = \text{number of } y \in F \text{ that such that } B(y, \varepsilon) \text{ is hit after time } m.$$  

Assuming that hitting times are approximately exponential with mean $\left(1/\pi\right) \log \varepsilon^{-1}$, we have:

$$E[Z(m)] \approx \varepsilon^{-2} \exp\left(-\frac{m}{\pi \log \varepsilon^{-1}}\right).$$

Note that this is vanishing for $\varepsilon \downarrow 0$ if

$$m > \frac{1}{\pi} (1 + \delta) 2 \left(\log \varepsilon^{-1}\right)^2 \text{ and } \delta > 0.$$  

By the Markov inequality, one immediately obtains an upper bound on the leading order of the $\varepsilon$-cover time (under the exponentiality assumption). In hindsight, this bound is tight. The analysis of the second moment is however inconclusive: it does not yield a matching lower bound, due to strong correlations of hitting times. To overcome this obstacle one needs a more sophisticated multi-scale analysis [10]. At the level of the subleading order the situation is even more delicate, since already the analysis of the first moment is inconclusive. In fact, if we let

$$m(s) = \frac{1}{\pi} \log \varepsilon^{-1} \left\{\log \varepsilon^{-2} - s \log \log \varepsilon^{-1}\right\}, \quad s \in \mathbb{R},$$

one has (cf. (1.4) and (1.5))

$$E[Z(m(s))] \approx \left(\log \varepsilon^{-1}\right)^s,$$
which explodes if $s > 0$, although \([12]\) claims that $Z(m(s)) = 0$ with high probability also for $0 < s < 1$. The source of the problem is easily identified: by linearity of the expectation, we are completely dismissing the correlations of the field of hitting times, but these are severe enough to have an impact at the level of the subleading order.

To get around this $Z(m)$ should be replaced by a truncated version $\tilde{Z}(m)$, whose first moment already encodes information about the correlation structure. This approach can be used to derive the subleading order of branching Brownian motion (see \[18\]). The challenge is identifying the right truncation procedure for the cover time of the torus.

1.1.2. **Traversal processes.** As it turns out, a suitable truncation is formulated in terms of a traversal process associated to each of the $\varepsilon^{-2}$ points in $F$. This process captures the amount of time Brownian motion spends in the neighbourhood of the point $y$ at the different scales in an associated tower of concentric balls (see Figure 1 on page 3). The scales are represented by $L \approx \log \varepsilon^{-1}$ geometrically growing concentric balls,

\[
(1.8) \quad B(y, \varepsilon) = B(y, r_L) \subset B(y, r_{L-1}) \subset \ldots \subset B(y, r_1) \subset B(y, r_0),
\]

around each $y$, where $r_l = e \times r_{l+1} = e^{L-l-1}\varepsilon$ is the “size” of the $l$-th scale. We measure time spent in a ball $B(y, r_l)$ by the number of traversals made from scale $l$ to scale $l+1$, that is the number of times that Brownian motion moves from the exterior of $B(y, r_l)$ to the interior of $B(y, r_{l+1})$. More precisely, we count the number of such traversals that take place during the first $t$ excursions from $\partial B(y, r_1)$ to $\partial B(y, r_0)$. We call this number $T_l^{y,t}$ and view this as a process in $l$, thus obtaining for each $y \in \mathbb{T}$ and initial excursion count $t \geq 1$,

\[
(1.9) \quad \text{the traversal process } (T_l^{y,t})_{l \geq 0}, \text{ counting }
\]

the number of traversals from $\partial B(y, r_1)$ to $\partial B(y, r_{l+1})$.

Note that,

\[
(1.10) \quad T_{L-1}^{y,t} = 0 \iff B(y, \varepsilon) \text{ is not hit during } t \text{ excursions from } \partial B(y, r_1) \text{ to } \partial B(y, r_0),
\]

from $\partial B(y, r_1)$ to $\partial B(y, r_0)$, thus providing a connection between traversal processes, hitting times of balls, and ultimately the $\varepsilon$-cover time.

Since we have one traversal process for each $y \in \mathbb{T}$ there is no explicit hierarchical structure in our construction. However, the correlations of the processes have a crucial approximate hierarchical structure, which underlies the whole approach. If $y$ and $z$ are at distance of about $r_k$, then for $l$ slightly smaller than $k$ the balls of radius $r_l \gg r_k$ around $y$ and $z$ will have a very large overlap: they will be almost the same ball. Therefore, one would expect that the number of traversals around $y$ and $z$ at such scales essentially coincide, that is $T_l^{y,t} \approx T_l^{z,t}$ for $l \ll k$. On the other hand for $l$ larger than $k$ the balls of radius $r_l \ll r_k$ around $y$ and $z$ will be disjoint. By the strong Markov property the excursions of Brownian motion in disjoint balls are conditionally independent, and we may therefore expect that the traversal processes of $y$ and $z$ evolve essentially independently at such scales, conditionally on the number of traversals at scale $r_k$ (which should be roughly the same for both). This picture leads one to imagine a tree of depth $L$ where $y, z \in \mathbb{T}$ at distance of about $r_k$ roughly correspond to leaves whose most recent ancestor is in level $k$ of the tree (see Figure 6.1 on page 3).

The advantage of defining the traversal process in terms of excursions from $\partial B(y, r_1)$ to $\partial B(y, r_0)$ is that it then becomes a critical Galton-Watson process with geometric

\[5\)Or more accurately a forest of $\sim r_0^{-2}$ trees, the latter being the number of balls that can be “packed” into the highest scale.
offspring distribution, due to the “spacing” of the scales and a well-known result on the exit distribution of Brownian motion in two dimensions from an annulus (see Figure 3.1 on page 12). A concentration argument for excursion times shows that the time needed to make \( t \) excursions is very close to \( \frac{1}{\pi} t \), thus providing a way to “translate” between excursion counts and the actual time of Brownian motion.

1.1.3. Upper bound - barrier. The key idea, which eventually leads to the truncation, is based on the insight that a traversal process cannot die out too quickly. To formalize, consider the following “barrier” for the square root of the traversal count,

\[ \alpha = \alpha (l) = \left( 1 - \frac{l}{L} \right) \sqrt{t} - (\log L)^2 \quad \text{for} \quad l = 0, \ldots, L. \]

This barrier is the linear function interpolating between \( \sqrt{t} \) and 0, shifted downwards slightly (see Figure 7.1 on page 43). It turns out that with high probability,

\[ \text{no traversal process} \ T_{y,t}^L \text{ falls below} \ \alpha (l)^2, \quad \text{for} \ l = 0, \ldots, L - 1. \]

We prove this claim in two steps: first we reduce the “combinatorial complexity” at each scale by means of a packing argument, and, second, we use a Markov inequality over the scales (“multi-scale Markov”, cf. [18]). Roughly \( r_l^{-2} \) balls of radius \( r_l \) and at mutual distance roughly \( r_l \) can be “packed” into the torus. By the above intuition that \( T_{y,t}^L \approx T_{z,t}^L \) for \( y \) and \( z \) at distance smaller than \( r_l \) (see also Figure 4.1 on page 19), we expect the minimum of \( T_{y,t}^L \) over all \( y \in F \) to be essentially the minimum over this packing. A union bound then shows that the probability that the minimum of \( T_{y,t}^L \) over all \( y \in F \) drops below \( \alpha(l)^2 \) should be at most

\[ cr_l^{-2} \mathbb{P} \left[ T_{y,t}^L \leq \alpha(l)^2 \right]. \]

We derive a large deviation control on \( T_{y,t}^L \), which allows us to prove that with our choice of \( \alpha(l) \) the quantity in (1.13) tends to zero, and what’s more, the sum over \( l \) of (1.13) tends to zero. Therefore by a union bound over the scales \( l = 0, \ldots, L - 1 \) we will be able to show that no traversal process falls below \( \alpha(l)^2 \), i.e. derive (1.12). For the upper bound on the cover time, this “multi-scale” use of the Markov inequality is the only place where the correlation structure of the traversal processes is used.

1.1.4. Upper bound - matching. Now [11.12] suggests the following truncated version of the counting random variable \( Z(m) \), which also counts balls which are not hit (cf. (1.10)) but furthermore requires the traversal process to stays above \( \alpha(l)^2 \):

\[ \tilde{Z}(t) = \text{number of} \ y \in F \text{ such that} \ T_{y,t}^L = 0 \text{ and} \sqrt{T_{y,t}^L} \text{ never falls below} \ \alpha(l). \]

When written in terms of the number of scales \( L \approx (\log \varepsilon)^{-1} \), the number of excursions that typically take place up to the time \( m(s) \) from (1.6) turns out to be roughly

\[ t(s) = L \{2L - s \log L\}, \quad s \in \mathbb{R}. \]

Therefore to obtain the upper bound in (1.2) one has to show that \( \tilde{Z}(t(s)) = 0 \) with high probability for \( s < 1 \). The expectation of \( \tilde{Z}(t) \) can be written as

\[ E[\tilde{Z}(t)] = |F| \cdot \mathbb{P} \left[ T_{L-1}^t = 0 \right] \cdot \mathbb{P} \left[ \sqrt{T_{L-1}^t} \geq \alpha(l) \right. \text{ for} \ l = 0, \ldots, L - 1 \left| T_{L-1}^t = 0 \right]. \]
Note that when \( t = t(s) \) the product of the first two terms is essentially the expectation of the untruncated counting variable \( Z(m(s)) \), and as such will be equal to \( L^s \), cf. (1.7) and recall that \( L \approx \log \varepsilon^{-1} \). The gain comes from the conditional probability, which plays a fundamental role. It will become apparent that the mean of the process \( \sqrt{T_{l-1}^{y,t}} \) when conditioned on \( T_{l-1}^{y,t} = 0 \) is well approximated by \((1 - \frac{1}{L}) \sqrt{t} \). Furthermore, we will see that the fluctuations of \( \sqrt{T_{l-1}^{y,t}} \) around this mean behave roughly like a Brownian bridge on \([0, L] \) starting and ending in zero. Therefore the conditional probability in (1.16) roughly behaves as

\[
(1.17) \quad \mathbb{P} \left[ X_l \geq -(\log L)^2 \text{ for } l = 0, \ldots, L \right],
\]

where \( X_l \) is a Brownian bridge. It is well-known (e.g. by the Ballot theorem) that this probability is of order \( L^{-1} \) (ignoring unimportant \( \log \)-terms). Coming back to (1.16), this line of reasoning will lead to

\[
(1.18) \quad \mathbb{E} \left[ \tilde{Z}(t(s)) \right] \approx \frac{e^{2L} \cdot e^{-t(s)/L}}{L^s} \cdot L^{-1} \rightarrow \begin{cases} \infty & \text{if } s > 1, \\ 0 & \text{if } s < 1, \end{cases}
\]

where the first term arises because \( F \) has \( \varepsilon^{-2} \approx e^{2L} \) elements, and the second because the number of excursions until the ball \( B(y, \varepsilon) \) is hit turns out to be essentially exponentially distributed with mean \( L \). Matching to unity, we see that \( \mathbb{E} \tilde{Z}(t) \) is of order one for \( t \) close to \( t(1) \). In other words, the \( L^{-1} \approx 1/\log \varepsilon^{-1} \) at the end of (1.18) gives rise to the log log-correction of the cover time. For \( s < 1 \) the expectation tends to zero, giving the upper bound of (1.2). This will be formalized in Section 4.

1.1.5. **Lower bound.** As for a tight lower bound, the approach relies on a key idea related to (1.17). In fact, it can be proven that a Brownian bridge which is required to stay above the line \(-(\log L)^2\) for \( l = 0, \ldots, L \) stays well above that line, a phenomenon which is reminiscent of the entropic repulsion appearing in the statistical mechanics of random surfaces, see [2]. More precisely, it can be shown that with overwhelming probability such a Brownian bridge will typically lie higher than curves of the form \( \min\{l^\delta, (L - l)^\delta\} \) for any \( 0 < \delta < 1/2 \). Reformulating back in terms of the traversal process, this suggests that we do not lose any information by considering

\[
(1.19) \quad \sqrt{T_{l}^{y,t}} \text{ stays above } (1 - \frac{l}{L}) \sqrt{t} + \min\{l^{0.49}, (L - l)^{0.49}\}, \text{ for } 0 \ll l \ll L,
\]

(see Figure 7.1 on page 43). We count the number of balls which have not been hit during \( t \) excursions, and whose associated traversal process satisfies the constraint in (1.19):

\[
(1.20) \quad \tilde{Z}(t) = \text{number of } y \in F \text{ such that } T_{l}^{y,t} = 0 \text{ and } \sqrt{T_{l}^{y,t}} \text{ satisfies (1.19)}. \]

The expectation of \( \tilde{Z}(t(s)) \) turns out to be essentially that of the counting random variable \( \tilde{Z}(t(s)) \) used for the upper bound, and in particular it tends to infinity for \( s > 1 \) (see (1.18)). Furthermore the truncation turns out to reduce correlations sufficiently for the second moment to be asymptotically equivalent to the first moment squared. An application of the Payley-Zygmund inequality will therefore establish that, when \( s > 1 \), there will with high probability exist a \( y \in F \) whose ball \( B(y, \varepsilon) \) is not hit in \( t(s) \) excursions. By the aforementioned concentration of excursion times this will provide the lower bound on the \( \varepsilon \)-cover time from (1.2). This is formalized in Section 5.
Without the truncation the second moment explodes with respect to the first moment squared. When writing the second moment as a sum over \( y, z \in F \) of the probability that the traversal processes associated with both \( y \) and \( z \) satisfy the condition in (1.20), one sees that the source of the problem are those pairs of balls which lie at "mesoscopic" distance (smaller than \( r_0 \) but larger than \( \varepsilon \)). The truncation helps by penalising such pairs, since the "bump" on top of the line \( (1 - l/L)\sqrt{t} \) forces the square root traversal processes of \( y \) and \( z \) at distance roughly \( r_k \) to each make an atypically extreme jump from \((1 - k/L)\sqrt{t} + \min\{k^{0.49}, (L - k)^{0.49}\} \) to 0 between scales \( k \) and \( L - 1 \). One such jump turns out to be achievable, but two jumps in the same neighbourhood turn out to be too costly for such pairs to contribute significantly to the truncated second moment. Here it is crucial that the traversal processes decorrelate at scales \( l \gg k \), so that one really needs to make two essentially independent jumps if \( y \) and \( z \) are both to satisfy (1.20). Finally, the number of pairs at distance close to \( r_L \) is too small to contribute much to the second moment. Therefore the main contribution to the truncated second moment comes from pairs that are at distance at least \( r_0 \), and for these pairs the events of satisfying the condition in (1.20) turn out to be independent. We choose \( r_0 \) tending to zero slowly, which means that the overwhelming majority of the pairs are independent. This causes the second moment of \( \hat{Z} \) to be asymptotic to the first moment squared.

The rigorous implementation of this decoupling for scales \( l \gg k \) is arguably the most delicate and technically demanding step in our approach, and will be formalized in Section 6 (see also the statement Proposition 5.6 of the main bound and Remark 5.7).

1.1.6. Barrier estimates and excursion time concentration. The above sketch rests on being able to control the probability that the traversal process \( T_{l_{av}}^{y,t} \) avoids certain barriers. Proving these rigorously turns out to be delicate, and is carried out in Section 7 via a comparison of both a conditioned Galton-Watson process and a Brownian bridge to a Bessel bridge.

Furthermore, we have assumed that we are able to control the time needed to make \( t \) excursions. As the subleading correction term we are trying to establish is very small compared to the leading order, we need a very precise bound. The basic recipe for such bounds, used e.g. in [10] (a large deviation bound on excursion times obtained by estimating their exponential moments using Khasminskii’s lemma/Kac moment formula, together with a union bound), turns out to be insufficient. We must complement it with a packing argument to reduce combinatorial complexity in the union bound, and in our large deviation bound we need to exploit the Markovian structure of the excursions of Brownian motion. This is carried out in Section 8.

1.2. Relation to branching Brownian motion, or: “3/2 = 1”. The heuristics described above rests on the approximate hierarchical structure: it is absolutely fundamental that the traversal processes of two points at distance of about \( r_l \) for a given scale \( l \) essentially agree at higher scales, and decorrelate at lower scales. In other words, intuitively one starts with a small collection of traversal processes at the first scale which then branch in each subsequent scale, producing several offspring which evolve as (essentially) independent traversal processes. The situation is thus reminiscent of branching Brownian motion, one of the simplest models with an exact hierarchical structure. A similar procedure of truncation and matching to unity can be applied to establish the level of the maximum of BBM, see [18]. The key insight is that with suitably chosen scales (cf. (1.8)) the square roots of traversal processes correspond to the trajectories (or ‘profiles’) of particles in BBM, and that the truncations applied to profiles in BBM can successfully be applied to these processes.
A fundamental difference is that in case of BBM, the field consists of correlated Gaussian random variables, whereas in our case hitting times are (approximately) exponentially distributed. In particular, the tail of a Gaussian distribution has a polynomial term which exponentials do not have. This has a considerable impact when “matching to unity”: in the BBM version\(^6\) of (1.16) and (1.18) the middle term corresponds to the probability that a trajectory ends up close to the level of maximum, and has not only has the main part of the Gaussian tail \(e^{-t^2/(2L)}\), but also the polynomial term \(L^{-1/2}\) (when \(t\) is chosen to be of order \(L\), as it must be for the exponential part of the tail to match the combinatorial complexity \(e^{2L}\)). The last term in (1.16) and (1.18) is essentially the same barrier crossing probability also for BBM, but applied to the trajectory of a particle up to time \(L\), and it also has order \(\sim L^{-1}\), giving a total contribution from polynomial terms of \(L^{-3/2}\). This gives rise to well-known BBM correction involving \(3/2\) when “matching to unity”.

Thus the subleading correction for BBM (and by extension the Gaussian free field on the two dimensional torus \([7]\)) correctly “predicts” the correction term of the \(\varepsilon\)-cover time of the torus, once this small difference in the tail is taken into account. The subleading order for the cover time of the tree, which was established in \([14]\), can also be “predicted” in the same manner; in this case the subleading correction coincides numerically\(^7\) with our main result (1.2), since the tail of hitting times of leaves also lacks a polynomial term.

In short, the subleading correction in all of these models encodes the very same physical principle of entropic repulsion.

1.3. Open problems. Our main result is a necessary step towards the identification of the weak limit of the (suitably rescaled) \(\varepsilon\)-cover time. Based on the analogy with branching Brownian motion it is natural to expect the limiting law to be described by a mixture of Gumbel distributions, see e.g. \([2]\). Even more challenging would be the full description of the process of covering; also in this case, the analogy with BBM suggests that regions which are missed the longest form a Poisson cluster process of random intensity \([1, 3]\).

The extension of our main result to the discrete setting is also of interest. Here \(C_N\) is the time it takes for (discrete or continuous time) random walk to visit every vertex of the two dimensional discrete torus graph \((\mathbb{Z}/N\mathbb{Z})^2\), in the large \(N\)-limit. Dembo et. al. \([9]\) were able to deduce from (1.1) the corresponding law of large numbers for the discrete torus, namely:

\[
(1.21) \quad \frac{C_N}{\frac{2}{\pi} N^2 \log N} = \left(1 + o(1)\right) \log N^2,
\]

in probability, for \(N\) large. The deduction uses a strong “Hungarian” coupling of random walk and Brownian motion. As it turns out, this coupling is too coarse to deduce from our main result (1.2) the subleading order for \(C_N\). Nevertheless, the heuristic underlying the proof of (1.2) can be applied to the discrete setting as well, and leads to the following conjecture (see also Remark 8.11).

**Conjecture.**

\[
(1.22) \quad \frac{C_N}{\frac{2}{\pi} N^2 \log N} = \log N^2 - \left(1 + o(1)\right) \log \log N,
\]

in probability, as \(N \to \infty\).

\(^6\)To be precise, a version of BBM with branching at discrete integer times and average branching factor \(e^2\), run up to time \(L\).

\(^7\)To verify this one must rearrange (0.1) of \([14]\) appropriately, so that the cover time is rescaled by the expected hitting time of a leaf.
2. Notation and definitions

In this section we collect some important notations and definitions used throughout the article.

We write \( \mathbb{R} \) for the real numbers and define \( \mathbb{R}_+ = [0, \infty) \). For any real \( t \in \mathbb{R}_+ \), we let \( \lfloor t \rfloor \) denote the integer part of \( t \), and let \( \lceil t \rceil \) denote the smallest integer at least as large as \( t \). For any set \( A \) in a topological space, \( A^\circ \) denotes the interior of \( A \) and \( \bar{A} \) the closure. The boundary \( \partial A \) of the set \( A \) is defined by \( \bar{A} \setminus A^\circ \). For any \( a, b \in \mathbb{R} \) we denote the minimum of \( a \) and \( b \) by \( a \wedge b \). For any sequences \( a_t, b_t \), depending on some parameter \( t \) the notation \( a_t \asymp b_t \), means that there exist constants \( c \) and \( C \) (not depending on \( t \)) such that

\[
 cb_t \leq a_t \leq Cb_t \text{ for all } t \geq 0.
\]

Let \( \{0,1,\ldots\}^\infty \) be the space of integer sequences and let \( (T_t)_{t \geq 0} \) be the canonical process on this space. We let \( G_n \) denote the law on \( \{0,1,\ldots\}^\infty \) of a critical Galton Watson process with geometric offspring distribution and initial population \( n \in \{0,1,\ldots\} \). This offspring distribution has parameter \( \frac{1}{2} \) and is supported on \( \{0,1,\ldots\} \). For real \( t \geq 0 \), we take \( G_t \) to mean \( G_{\lfloor t \rfloor} \).

We write \( \mathbb{T} = (\mathbb{R}/\mathbb{Z})^2 \) for the two dimensional Euclidean torus. The map \( \pi \) is the natural projection of \( \mathbb{R}^2 \) onto \( \mathbb{T} \), and \( \pi^{-1}(x) \) the point in \( [0,1)^2 \subset \mathbb{R}^2 \) which maps to \( x \) under \( \pi \). The Euclidean metric on \( \mathbb{R}^2 \) induces a metric on \( \mathbb{T} \) which we denote by \( d(x,y), x,y \in \mathbb{T} \). The closed ball of radius \( r > 0 \) in \( \mathbb{T} \) or \( \mathbb{R}^2 \) centered at \( x \) is denoted by \( B(x,r) \).

For any interval \( I \subset \mathbb{R} \) and \( D = \mathbb{R}, \mathbb{T} \) or \( \mathbb{R}^2 \) we write \( C(I,D) \) for the space of continuous functions from \( I \) to \( D \) with the topology of uniform convergence. We let \( \mathcal{B}(I,D) \) denote the Borel sigma algebra on this space. The canonical process on \( C(I,\mathbb{R}) \) is denoted by \( X_t, t \geq 0 \), and the canonical processes on \( C(I,\mathbb{T}) \) and \( C(I,\mathbb{R}^2) \) are denoted by \( W_t, t \geq 0 \). We denote by \( \mathcal{F}_t = \mathcal{F}_t(I,D) \) the natural filtration of the canonical process. The canonical
shift on \( C(\mathbb{R}_+, D) \) is denoted by \( \theta_t, t \geq 0 \). To indicate “chunks” of the canonical process we write e.g. \( \{ W_{S+}, t \to W_{S+t} \} \) of \( W_t \) after time \( S \), where \( S \) is a random time. If \( S_1 \) and \( S_2 \) are two random times \( W_{(S_1+t) \wedge S_2} \) denotes the path \( t \to W_{(S_1+t) \wedge S_2} \) of \( W_t \) between times \( S_1 \) and \( S_2 \).

We let \( P^\mathbb{R} \), \( x \in \mathbb{R} \), be the law on \((C(\mathbb{R}_+, \mathbb{R}^2), B(\mathbb{R}_+, \mathbb{R}^2))\) which turns \( W_t, t \geq 0 \), into a standard Brownian motion starting at \( x \in \mathbb{R}^2 \). We let \( P_x, x \in \mathbb{T} \), be the law on \((C(\mathbb{R}_+, \mathbb{T}), B(\mathbb{R}_+, \mathbb{T}))\) of \((\pi(W_t))_{t \geq 0}\) under \( P^\mathbb{R} \); that is, \( P_x \) is the law of standard Brownian motion in \( \mathbb{T} \) started at \( x \). For any measure \( \nu \) on \( \mathbb{T} \) we let \( P_\nu \) be the law of Brownian motion with starting point distributed according to \( \nu \).

For any measurable set \( A \subset \mathbb{T} \) or \( \mathbb{R}^2 \) we define the hitting time \( H_A \) of \( A \) by

\[
H_A = \inf \{ t \geq 0 : W_t \in A \},
\]

and for \( A \subset \mathbb{R} \) we define \( H_A \) similarly but with \( W_t \) replaced by \( X_t \). For a singleton \( a \in \mathbb{R} \) we abbreviate \( H_a = H_{\{a\}} \). We write \( T_A \) for the exit time from \( A \subset \mathbb{T} \) or \( \mathbb{R}^2 \), that is

\[
T_A = \inf \{ t \geq 0 : W_t \notin A \}.
\]

Note that any set \( A \subset (0, 1)^2 \subset \mathbb{R}^2 \) can be identified with \( \pi(A) \subset \mathbb{T} \), and that the law of Brownian motion in \( \pi(A) \) and \( A \) coincide: formally speaking, for any \( a \in A \),

\[
(2.1) \quad \text{the } P_{\pi(a)} - \text{law of } \left( W_{\wedge T_{\pi(a)}} \right) \text{ is the } P^\mathbb{R} - \text{law of } W_{\wedge T_A}.
\]

In particular, when \( R < \frac{1}{2} \) the ball \( B(y, R) \subset \mathbb{T} \) can be identified with \( B\left(\pi^{-1}(y), R\right) \), and the laws of Brownian motion in these two balls coincide. Brownian motion in \( \mathbb{R}^2 \) is rotationally invariant, in the sense that for any rotation \( \rho : \mathbb{R}^2 \to \mathbb{R}^2 \) of the space \( \mathbb{R}^2 \) around a point \( y \in \mathbb{R}^2 \) we have that

\[
(2.2) \quad \text{the } P_{\rho(z)} - \text{law of } \left( \rho^{-1}(W_t) \right)_{t \geq 0} \text{ is } P^\mathbb{R}.
\]

The law of Brownian motion in a ball \( B(y, R) \subset \mathbb{T} \) in the torus is also rotationally invariant if \( R < \frac{1}{2} \), since if \( \rho : \mathbb{T} \to \mathbb{T} \) is a rotation of the ball \( B(y, R) \) around \( y \) (leaving \( B(y, R) \) invariant) then \((2.1)\) and \((2.2)\) imply that

\[
(2.3) \quad \text{the } P_{\rho(z)} - \text{law of } \left( W_{\wedge T_{B(y, R)}} \right) \text{ is the } P_\rho - \text{law of } W_{\wedge T_{B(y, R)}}.
\]

It is a standard fact that for any \( 0 < r < R \) the exit distribution of Brownian motion from the annulus \( B(y, R) \setminus B(y, r) \subset \mathbb{R}^2 \) satisfies

\[
P^\mathbb{R} \left[ T_{B(y, R)} < H_{B(y, r)} \right] = \frac{\log \left( \frac{|v-y|}{r} \right)}{\log \left( \frac{R}{r} \right)} \quad \text{for all } v \in B(y, R) \setminus B(y, r) \subset \mathbb{R}^2,
\]

(see Theorem 3.17 [23]). By \((2.1)\) the same also holds in the torus: for any \( 0 < r < R < \frac{1}{2} \)

\[
P_\nu \left[ T_{B(y, R)} < H_{B(y, r)} \right] = \frac{\log \left( d(v, y) / r \right)}{\log \left( R / r \right)} \quad \text{for all } v \in B(y, R) \setminus B(y, r) \subset \mathbb{T}.
\]

In this article we will make heavy use of departure and return times from concentric circles. For \( 0 < r < R < \frac{1}{2} \) and \( y \in \mathbb{T} \) the successive return times to \( B(y, r) \) are denoted by \( R_n(y, r), n \geq 1 \), and the successive departure times from \( B(y, R) \) are denoted by \( D_n(y, R), n \geq 1 \). Formally,

\[
\begin{align*}
(2.5) \quad R_1(y, R, r) &= H_{B(y, r)}, \\
R_n(y, R, r) &= H_{B(y, r)} \circ \theta_{D_{n-1}(y, R, r)} + D_{n-1}(y, R, r), n \geq 1, \\
D_n(y, R, r) &= T_{B(y, R)} \circ \theta_{R_{n-1}(y, R, r)} + R_{n-1}(y, R, r), n \geq 1.
\end{align*}
\]
Note that
\[ 0 \leq R_1(y, R, r) < D_1(y, R, r) < R_2(y, R, r) < D_2(y, R, r) < \ldots. \]
We will often refer to \( W_{R_n(y, R, r) + \ldots} \) as the \( n \)-th excursion or \( n \)-th traversal from \( \partial B(y, r) \) to \( \partial B(y, R) \); these (and other excursions) will play an important role in
the proofs.

Lastly a note on constants. The letter \( c \) represents a constant that is positive and
does not depend on any other parameters. It may represent different constants in different
formulas, and even within the same formula. Dependence on e.g. a parameters \( s \) is
denoted by \( c(s) \).

3. Statement of main theorem, construction of traversal processes and
first reduction

In this section we formally state the main theorem. We also start the proof by con-
structing the traversal processes around each point \( y \in T \) which where mentioned in the
introduction, and deriving their basic properties. We reduce the proof of the main theo-
rem to three main propositions, which will be proven in the subsequent sections. The first
two of these deal with the traversal processes. More precisely, Proposition 3.5 essentially
proves the upper bound of (1.2), and Proposition 3.6 essentially proves the lower bound.
However, they do this in terms of excursions, i.e. they determine how many excursions
around each point are needed to cover the torus. The third of the main proposition,
Proposition 3.7 relates this number of excursions to the actual time of Brownian motion,
thus allowing us to deduce the main result (1.2) from propositions 3.5 and 3.6.

To formally state our main result we define the \( \varepsilon \)-cover time \( C_\varepsilon \) as
\[
C_\varepsilon = \sup_{y \in T} H_B(y, \varepsilon),
\]
and (deviating slightly from the formulation in the introduction, cf. (1.6)) let
\[
m(\varepsilon, s) = \frac{1}{\pi} \log \varepsilon^{-1} (2 \log \varepsilon^{-1} - (1 - s) \log \log \varepsilon^{-1}).
\]
The formal statement of (1.2) is the following.

**Theorem 3.1.** For all \( s > 0 \) and all \( x \in T \)
\[
(3.3) \quad \lim_{\varepsilon \to 0} P_x[m(\varepsilon, -s) \leq C_\varepsilon \leq m(\varepsilon, s)] = 1.
\]

The proof of Theorem 3.1 (or rather, its reduction to propositions 3.5 and 3.7) will be given
at the end of this section.

We now construct the traversal processes that are the cornerstone of Theorem 3.1’s
proof. The construction will depend on an integer parameter \( L \geq 1 \), which represents the
number of scales that we consider. Let
\[
r_l = r_l(L) = e^{-\frac{3}{4} \log \log L - t}, t = 0, 1, \ldots,
\]
denote a sequence of radii, corresponding to the scales in the multiscale analysis described
in the introduction. Note that
\[
r_0 \to 0, \text{ as } L \to \infty,
\]
which is important for the proof of the lower bound Proposition 3.6 (since it means that
an overwhelming majority of all pairs of traversal processes depend on disjoint regions and
will therefore be completely independent, which helps in bounding the second moment of
the truncated counting random variable, see (5.29); if we were only proving the upper
Figure 3.1. There is a 50% probability that Brownian motion goes “up a scale” and 50% probability that it goes “down a scale”

bound of Theorem 3.1 we could have removed $\frac{3}{4} \log \log L$ from (3.4). We will show
Theorem 3.1 along the sequence $\varepsilon = r_L$ as $L \to \infty$. We will see that this easily implies
Theorem 3.1 in full generality.

The proof is based on tracking Brownian motion as it moves between the scales given
by the $r_l$. For this it is useful to note that since the $r_l$ shrink geometrically we have from
(2.4) that for any $y \in \mathcal{T}$ and $0 < l < L$

$$P_v [H_B(y,r_{l+1}) < T_B(y,r_{l-1})] = \frac{1}{2}, \quad \text{for } v \in \partial B(y,r_l).$$

That is Brownian motion essentially speaking flips an unbiased coin to decide whether to
move to a higher scale or a lower scale (see Figure 3.1 on page 12). More generally

$$P_v [H_B(y,r_{a}) < T_B(y,r_b)] = \frac{l_2 - l_1}{l_3 - l_1}, \quad \text{for } v \in \partial B(y,r_{a}), l_1 < l_2 < l_3.$$

It will be useful to introduce the following abbreviations for the time $R_n(y,r_l,r_{l+1})$
that the $n$–th traversal from scale $l$ and $l+1$ is completed and for the time $D_n(y,r_l,r_{l+1})$
that the $n$–th traversal from scale $l + 1$ to $l$ is completed (recall (2.5))

$$R_n^{y,l} = R_n^{y,l}(L) = R_n(y,r_l,r_{l+1}) \quad \text{and} \quad D_n^{y,l} = D_n^{y,l}(L) = D_n(y,r_l,r_{l+1}) \quad \text{for } n \geq 0, l \geq 0, y \in \mathcal{T}.$$

If $t \in \mathbb{R}_+$ we let $R^{y,l}_t = R^{y,l}_{[t]}$ and $D^{y,l}_t = D^{y,l}_{[t]}$. For $y \in \mathcal{T}$ and $t \in \mathbb{R}_+$ we can now formally
define the process of traversals $T^{y,t}_l, l \geq 0,$ by

$$T^{y,t}_l = T^{y,t}_l(L) = \sup \left\{ n \geq 0 : R^{y,l}_n \leq D^{y,0}_t \right\}, l \geq 0,$$

where we understand $R^{y,l}_0 = 0$. Note that $T^{y,t}_l$ is the number of traversals from $B(y,r_l)^c$
to $B(y,r_{l+1})$ made by Brownian motion during the first $|t|$ excursions from $\partial B(y,r_l)$
to $\partial B(y,r_0)$ (see Figure 3.2 on page 13 for an illustration). This means that the process
$T^{y,t}_l$ contains information about whether $B(y,r_L)$ has been hit by time $D^{y,0}_t$ or not, since
This illustration shows Brownian motion moving “in the first three scales” around a point $y$. The arrows indicate completed traversals. From this picture we can read off $T_{y, l}^2$, $l = 0, 1, 2$. The values are shown to the right.

Figure 3.2. Illustration of traversal process

(see (3.7) and (3.8), and cf. (1.10))

\[
T_{y, 2}^0 = 2, \quad T_{y, 1}^2 = 3, \quad T_{y, 2}^2 = 1.
\]

This gives a link between the $r_L$--cover time and the collection of process $T_{y, t}^l$. The traversal processes have the following simple characterisation.

Lemma 3.2. For all $y \in \mathbb{T}$ and $v \notin B(y, r_1)^\circ$ the $P_v$--law of $\{T_{y, t}^l\}_{l \geq 0}$ is the $\mathbb{G}_t$--law of $(T_l)_{l \geq 0}$, i.e. it is a critical Galton-Watson process with geometric offspring distribution.

Proof. Fix $y \in \mathbb{T}$. Consider the indicator functions

\[
I_{y, l, n}^t = \left\{ T_{B(y, r_{l-1})} \circ \theta_{D_{y, t}^l} < H_{B(y, r_{l+1})} \circ \theta_{D_{y, t}^l} \right\},
\]

which are one if Brownian motion next visits $\partial B(y, r_{l-1})$ after making a traversal $l+1 \rightarrow l$ and zero if it next visits $\partial B(y, r_{l+1})$. By (3.5) and the strong Markov property they are unbiased i.i.d. Bernoulli "coin flips" by (see also Figure 3.1 on page 12). We can reconstruct the traversal process from the $I_{y, l, n}$ recursively by setting $T_{y, t}^0 = \lfloor t \rfloor$ and

\[
T_{y, t}^{l+1} = \left\{ \text{the number of zeros among } I_{y, l+1, m}^t, m \leq n \right\}
\]

where $n$ is such that $I_{y, l+1, 1}^t + \ldots + I_{y, l+1, n}^t = T_{y, t}^l$, for $l \geq 0$.

Thus $T_{y, t}^{l+1}$ has a negative binomial distribution conditioned on $T_{y, t}^l$, $l' \leq l$, and since this distribution is also the sum of $T_{y, t}^l$ independent geometrics with support $\{0, 1, \ldots\}$ and mean 1 the claim follows. \hfill \Box

Remark 3.3. This can be seen as a discrete Ray-Knight theorem for the directed edge local times of a simple random walk $(Z_n)_{n \geq 0}$ on $\{0, 1, \ldots\}$. By (3.5) the process $Z_n$ can be constructed by letting it be the index of the successive scales around $y$ that $W_t$ visits, i.e. $Z_0 = 0, Z_1 = 1$ and $Z_2 = 0$ or 2 depending on if $W_t$ visits $\partial B(y, r_0)$ or $\partial B(y, r_2)$ first after $R_{y, 0}$, and so on. With this construction $T_{y, t}^l$ is $Z_n$’s edge local time at $l \rightarrow l + 1$
after \( t \) of \( Z_n \)'s excursions from 0. Also note that (3.6) can be seen as the standard result about the exit distribution of the simple random walk \( Z_n \) from the interval \([l_1, l_3]\).

As mentioned in the introduction, there are several instances when we will use “packings” of balls in the torus at different scales. The \( l \)-th scale packing will consist of balls centered at points of the following grid of “spacing” \( r_l/1000 \):

\[
F_l = \left\{ \left( \frac{i}{1000}, \frac{j}{1000} \right) : i, j \in \{0, 1, \ldots, \left\lfloor \frac{1000}{r_l} \right\rfloor \} \right\} \subset \mathbb{T}, l \geq 0.
\]

It will turn out that \( C_{r_L} \) is close to the maximum of the hitting times of balls centered in \( F_L \). We record for future use that (see (3.4))

\[
|F_l| \asymp cr_l^{-2} = c (\log L)^{3/2} e^{2l}, l \geq 0.
\]

Note that when comparing our method to the study of branching Brownian motion (or rather a version of BBM with branching at integer times; equivalently Gaussian Free Field on a tree) \( F_l \) corresponds to the vertices at distance \( l \) from the root. With this point of view we see that essentially speaking we have a “forest” of \( c (\log L)^{3/2} \) “pseudo-tree” with branching factor \( e^2 \).

We now state a second simple but crucial property of the traversal process, which essentially gives bounds on the probability the it “dies out” by generation \( L - 1 \) (using the Galton-Watson terminology), or equivalently that the ball \( B(y, r_L) \) does not get hit. For this we consider a number of traversals

\[
t_s = t_s(L) = L (2L - (1 - s) \log L) \quad \text{for} \quad s \in \mathbb{R},
\]

from scale 0 to scale 1, which we will see is roughly the number of traversals that take place up to time \( m(r_L, s) \) (cf. (1.15); note the slight difference). The bound is the following.

**Lemma 3.4.** For all \( y \in \mathbb{T}, x \notin B(y, r_1)^{\circ}, s \in \mathbb{R} \) and \( L \geq c(s) \)

\[
P_x \left[ T_{L-1}^{y, t_s} = 0 \right] \asymp e^{-2L} L^{1-s}.
\]

**Proof.** The event \( \{ T_{L-1}^{y, t_s} = 0 \} \) is the event that \( B(y, r_L) \) is not hit in \( |t_s| \) excursions from \( \partial B(y, r_1) \) to \( \partial B(y, r_0) \cup \partial B(y, r_L) \). By (3.6) one such excursion hits \( B(y, r_L) \) with probability \( 1/L \), regardless of where in \( \partial B(y, r_1) \) it starts. Thus using the strong Markov property

\[
P_x \left[ T_{L-1}^{y, t_s} = 0 \right] = \left( 1 - \frac{1}{L} \right)^{|t_s|} = e^{-\frac{t_s}{L}} \left( 1 + O \left( \frac{t_s}{L^2} + \frac{1}{L} \right) \right).
\]

Thus (3.13) follows since (recall (3.12))

\[
\frac{t_s}{L} = 2L - (1 - s) \log L.
\]

Using this bound and (3.11) one can roughly speaking compute the expectation of the simple untruncated random variable counting balls of radius \( r_L \) centered in \( F_L \) that are not hit in \( t_s \) excursions:

\[
\mathbb{E}_x \left[ \sum_{y \in F_L} 1\{ T_{L-1}^{y, t_s} = 0 \} \right] \asymp (\log L)^{3/2} L^{1-s},
\]

cf. (1.3) and (1.7) (here \( L \) corresponds to \( \log \varepsilon^{-1} \) and the \( \log L \) factor is an artifact of defining the \( r_l \) so that \( r_0 \downarrow 0 \)). This essentially proves that for \( s > 1 \) there are no balls
with center in $y \in F_L$ that avoid being hit in $t_s$ excursions from $\partial B(y, r_1)$ to $\partial B(y, r_0)$. For $s \leq 1$ we see that the expected number of balls that manage this tends to infinity. This does not correctly capture the actual number, as shown by our first main proposition which we now state. It essentially says that also for $0 < s \leq 1$ there will be no balls which avoid being hit in $t_s$ of “its” excursions from scale 1 to scale 0.

**Proposition 3.5.** $(x \in \mathbb{T})$

(3.16) \[ \lim_{L \to \infty} P_x \left[ T_{y, t_s}^{y, t_s} = 0 \text{ for some } y \in F_L \right] = 0, \text{ for all } s > 0. \]

This roughly speaking gives the upper bound of Theorem 3.1 “in terms of excursions at the highest scale”. Proposition 3.5 will be proven in Section 4 using a truncated first moment bound. We now state Proposition 3.6 which essentially says that for $s > 0$ there is (with high probability) some $y \in F_L$ which is not hit in $t_s$ of “its” excursions. This roughly speaking gives the lower bound of Theorem 3.1 “in terms of excursions at the highest scale”.

**Proposition 3.6.** $(x \in \mathbb{T})$

(3.17) \[ \lim_{L \to \infty} P_x \left[ T_{y, t_s}^{y, t_s} = 0 \text{ for some } y \in F_L \right] = 1, \text{ for all } s > 0. \]

Proposition 3.6 will be proved in Section 5 using a truncated second moment method. Finally we state Theorem 3.1 which essentially speaking says that at time $\frac{1}{\pi} t_s$ there will have been roughly $t_s$ excursions from scale 0 to scale 1 for all $y \in F_L$. This will allow us to deduce the main result (3.3) from the above two propositions.

**Proposition 3.7.** $(x \in \mathbb{T})$ For all $s > 0$

(3.18) \[ \lim_{L \to \infty} P_x \left[ D_{y, t_s}^{y, t_s} > \frac{1}{\pi} t_{2s} \text{ for some } y \in F_L \right] = 0 \text{ and, } \]

(3.19) \[ \lim_{L \to \infty} P_x \left[ D_{y, t_s}^{y, t_s} < \frac{1}{\pi} t_{-1} t_s \text{ for some } y \in F_L \right] = 0. \]

Proposition 3.7 will be proven in Section 8 using a packing argument and a large deviation bound for $D_{y, t_s}^{y, t_s}$. We now derive Theorem 3.1 from propositions 3.5-3.7.

**Proof of Theorem 3.1.** We first reduce the proof of the convergence in (3.3) to convergence along the subsequence $\varepsilon = r_L$. Assume we have shown that for all $s > 0$

(3.20) \[ \lim_{L \to 0} P_x \left[ C_{r_L} > m(r_L, s) \right] = 0 \text{ and, } \]

(3.21) \[ \lim_{L \to 0} P_x \left[ C_{r_L} < m(r_L, -s) \right] = 0. \]

Then for $\varepsilon > 0$ we may set

(3.22) \[ L_\pm = \log \varepsilon^{-1} - 3\varepsilon \log \log \varepsilon^{-1} \pm 1000, \]

so that $c\varepsilon \leq r_{L_+} \leq \varepsilon \leq r_{L_-} \leq c\varepsilon$ (see (3.4)). This in turn gives that $C_{r_{L_-}} \leq C_{r_{L_+}}$ and $m(r_{L_+}, s) = m(\varepsilon, s) \left( 1 + O \left( \frac{1}{\log \varepsilon^{-1}} \right) \right)$ (see (3.2)), so that $m(\varepsilon, s) \leq m(r_{L_-}, -s/2) \leq m(\varepsilon, s)$ for small enough $\varepsilon$. Therefore (3.3) follows from (3.20) with $L_+$ in place of $L$ and (3.21) with $L_-$ in place of $L$, and $s/2$ in place of $s$ in both instances.
We thus turn our attention to (3.20) and (3.21). For (3.20) we first reduce the a bound where the supremum in \(C_{r_L}\) (recall (3.1)) is taken over \(F_L\) and not all of \(\mathbb{T}\). We have that
\[
(C_{r_L} > m(r_L, s)) = \{H_{B(y, r_L)} > m(r_L, s) \text{ for some } y \in \mathbb{T}\}.
\]
Since \(r_{L+1}/1000 + r_{L+1} \leq 2r_{L+1} \leq r_L\) each ball of radius \(r_L\) in \(\mathbb{T}\) contains a ball of radius \(r_{L+1}\) centered at some \(z \in F_{L+1}\) (recall (3.10)). Also similarly to above \(m(r_L, s) \geq m(r_{L+1}, s/2)\) for \(L \geq c\). Therefore the probability in (3.20) is bounded above by
\[
P_x[H_B(z, r_{L+1}) > m(r_{L+1}, s/2) \text{ for some } z \in F_{L+1}],
\]
so that to show (3.20) it suffices to prove that for all \(s > 0\)
\[
\lim_{L \to \infty} P_x[H_B(z, r_{L+1}) > m(r_L, s) \text{ for some } z \in F_L] = 0.
\]
We have that (see (3.2) and (3.12))
\[
m(r_L, s) = \frac{1}{\pi} t_s (1 - O(\log \log L/L)) \geq \frac{1}{\pi} t_{s/2},
\]
for \(L\) large enough. Thus the probability in (3.24) is bounded above by
\[
P_x[H_B(z, r_{L+1}) > \frac{1}{\pi} t_{s/2} \text{ for some } z \in F_L],
\]
which in turn is bounded above by
\[
P_x[H_B(z, r_{L+1}) > D_{t_{s/4}}^{z,0} \text{ for some } z \in F_L] + P_x[D_{t_{s/4}}^{z,0} > \frac{1}{\pi} t_{s/2} \text{ for some } z \in F_L].
\]
Now since
\[
\{H_B(z, r_{L+1}) > D_{t_{s/4}}^{z,0} \text{ for some } z \in F_L\} \overset{(3.9)}{=} \{T_{t_{s/4}}^{z,0} L_{-1} = 0 \text{ for some } z \in F_L\},
\]
the two probabilities in (3.26) tend to zero when \(L \uparrow \infty\), by Proposition 3.5 and (3.18). This proves (3.24), and therefore also (3.20) and the upper bound of (3.3).

We now turn our attention to (3.21). We have
\[
(C_{r_L} < m(r_L, s)) \overset{(3.1)}{=} \{H_{B(y, r_L)} < m(r_L, s) \text{ for all } y \in \mathbb{T}\}.
\]
For \(L\) large enough we have that (cf (3.25)) \(m(r_L, -s) \leq \frac{1}{\pi} t_{-\frac{1}{2}s}\). Thus for such \(L\), (3.27) is included in
\[
\{H_{B(y, r_L)} < \frac{1}{\pi} t_{-\frac{1}{2}s} \text{ for all } y \in F_L\},
\]
which in turn is included in
\[
\{H_{B(y, r_L)} < D_{t_{-s}}^{y,0} \text{ for all } y \in F_L\} \cup \{D_{t_{-s}}^{y,0} < \frac{1}{\pi} t_{-\frac{1}{2}s} \text{ for some } y \in F_L\}.
\]
But
\[
\{H_{B(y, r_L)} < D_{t_{-s}}^{y,0} \text{ for all } y \in F_L\} \overset{(3.9)}{=} \{T_{t_{-s}}^{y,0} L_{-1} > 0 \text{ for all } y \in F_L\},
\]
so that we obtain for \(L\) large enough
\[
P_x[C_{r_L} < m(r_L, s)] \leq P_x[T_{t_{-s}}^{y,0} L_{-1} > 0 \text{ for all } y \in F_L] + P_x[D_{t_{-s}}^{y,0} < \frac{1}{\pi} t_{-\frac{1}{2}s} \text{ for some } y \in F_L].
\]
Taking the limit \(L \uparrow \infty\) we see that (3.21) follows from Proposition 3.6 and (3.19), so the lower bound of (3.3) follows. \(\square\)
In this section we have reduced the proof of the main result Theorem 3.1 to the three main propositions 3.5–3.7. The rest of the article is devoted to their derivation.

4. Upper bound on cover time in terms of excursions

In this section we prove Proposition 3.5, which is the first of the three main propositions used to prove the main result Theorem 3.1, and which gives the upper bound of that result “in terms of excursions from scale 1 to scale 0”. More precisely, recall the claim (3.16) of Proposition 3.5 that

\[ \lim_{L \to \infty} P_x \left[ T_{y,t}^{y,t} = 0 \text{ for some } y \in F_L \right] = 0 \text{ for all } s > 0, \]

(recall also the definitions from (3.8), (3.10) and (3.12)).

For technical reasons, we will consider rather than \( T_{y,t}^{y,t} \), a modified traversal process \( \tilde{T}_{y,t}^{y,t} \), which counts only traversals that take place after leaving \( B(y,r_0) \) for the first time (if the starting point of the Brownian motion lies inside \( B(y,r_1) \) then this modified traversal process and the original traversal process may not coincide). Formally we let (cf. (3.8))

\[ \tilde{T}_{y,t}^{y,t} = \sup \left\{ n \geq 0 : \mathbb{R}_{B(y,r_0)}^{T_{y,t}^{y,t}} \leq D_{y,0}^{T_{y,t}^{y,t}} \right\}, y \in F_L, l \geq 0, t \geq 0. \]

We will prove that

\[ \lim_{L \to \infty} P_x \left[ \tilde{T}_{y,t}^{y,t} = 0 \text{ for some } y \in F_L \right] = 0 \text{ for all } s > 0, \]

which is a slightly stronger statement than (3.16), because \( \tilde{T}_{y,t}^{y,t} \leq T_{y,t}^{y,t} \) almost surely for all \( y \). The modified traversal process is used because Lemma 3.2 and the strong Markov property imply that

\[ \text{the } P_x - \text{law of } \left( \tilde{T}_{y,t}^{y,t} \right)_{l \geq 0} \text{ is } \mathcal{G}_{t_s} \text{ for all } x, y \in T, \]

(this is not exactly true for \( T_{y,t}^{y,t} \), such that \( x \in B(y,r_1)^0 \)).

A previously discussed, a natural approach to proving (4.2) is the simple first moment upper bound using the counting random variable \( \sum_{y \in F_L} 1_{\{ \tilde{T}_{y,t}^{y,t} = 0 \}} \), but this however would yield (4.2) only for \( s < -1 \) (cf. (3.15)). We therefore introduce a truncation which is given in terms of the barrier

\[ \alpha(l) = \alpha(l, L, s) = \beta(l) - (\log L)^2, \]

where \( \beta(l) \) given by

\[ \beta(l) = \beta(l, L, s) = \left(1 - \frac{l}{L}\right)^{\sqrt{s}} \text{ for } l \in [0, L]. \]

The line \( \beta(l) \) turns out to essentially be the mean of the process \( \sqrt{T_{y,t}^{y,t}} \) conditioned on \( T_{L-1}^{y,t} = 0 \). See Figure 7.1 on page 43. We consider the truncated counting random variable which imposes an additional “barrier condition”

\[ \sum_{y \in F_L} 1_{\{ \tilde{T}_{y,t}^{y,t} = 0 \}} \cap \left\{ \sqrt{T_{y,t}^{y,t}} \geq \alpha(l) \text{ for } t = 0, \ldots, L \right\}. \]

Our claim (4.2) will follow from two main propositions: Proposition 4.2 and Proposition 4.7 below. Proposition 4.2 will show that the expectation of (4.6) goes to zero for all \( s > 0 \). Proposition 4.7 will show that with high probability there are no \( y \in F_L \) such that \( \sqrt{T_{y,t}^{y,t}} \) violates the barrier condition.
The key to bounding the expectation in (4.6) is bounding the conditional probability
\[
P_x \left[ \sqrt{\tilde{T}_{l}^{y,t_s}} \geq \alpha (l) \text{ for } l = 0, \ldots, L - 1 | \tilde{T}_{L-1} = 0 \right].
\]
By (4.3) this amounts to an estimate purely in terms of the Galton-Watson law \( \mathcal{G}_{t_s} \). For this law, Lemma 4.1 gives a bound on the probability of the form \( (\log L)^{4}/L \) (it is this extra factor that gives rise to the subleading correction). Lemma 4.1 will be proven together with other barrier estimates in Section 7.

To prove in Proposition 4.7 that no traversal process violates the barrier condition we will use a union bound over the scales: we aim to bound
\[
\sum_{l=1}^{L-1} P_x \left[ \sqrt{\tilde{T}_{l}^{y,t_s}} < \alpha (l) \text{ for some } y \in F_L \right].
\]
We then use a packing argument that defines a further modified traversal counter \( \hat{T}_{l}^{y,t_s} \) for each \( y \in F_{L+\log L} \) where the radii \( r_0, r_1, r_l \) and \( r_{l+1} \) have been slightly modified to ensure that if \( y \in F_{L} \) and \( y' \in F_{L+\log L} \) is the point in \( F_{L+\log L} \) closest to \( y \) then, roughly speaking,
\[
\hat{T}_{l}^{y',t_s} \leq \tilde{T}_{l}^{y,t_s} \text{ almost surely},
\]
(see Figure 4.1 on page 19). The only slightly modified radii will mean that \( \hat{T}_{l}^{y,t_s} \) has almost the same law as \( \tilde{T}_{l}^{y,t_s} \), and in particular in Lemma 4.6 we will derive a large deviation bound for \( \hat{T}_{l}^{y,t_s} \) which is almost the same as the corresponding bound for \( \tilde{T}_{l}^{y,t_s} \) (see Remark 4.5) essentially, both \( \tilde{T}_{l}^{y,t_s} \) and \( \hat{T}_{l}^{y,t_s} \) turn out to be compound binomial random variables with geometric compounding, so deriving a large deviation bound is straightforward). The domination in (4.9) will allow us to bound (4.8) by
\[
c \sum_{l=1}^{L-1} |F_{L+\log L}| P_x \left[ \sqrt{\hat{T}_{l}^{y,t_s}} < \alpha (l) \right]
\]
and the aforementioned large deviation bound on \( \tilde{T}_{l}^{y,t_s} \) will show that \( P_x \left[ \sqrt{\tilde{T}_{l}^{y,t_s}} < \alpha (l) \right] = o \left( L^{-1} |F_{L+\log L}|^{-1} \right) \), so that we will be able to conclude that (4.8) is \( o(1) \). Note that without the packing argument we would be bounding \( \sum_{l=1}^{L-1} |F_L| P_x \left[ \sqrt{\tilde{T}_{l}^{y,t_s}} < \alpha (l) \right] \), a quantity that can be shown to tend to infinity.

We now state the barrier crossing bound for the Galton-Watson law \( \mathcal{G}_{t} \) (see Figure 7.1 on page 43).

**Lemma 4.1.** For any \( s \in (-100, 100) \) we have that
\[
\mathcal{G}_{t_s} \left[ \sqrt{T_{l}} \geq \alpha (l) \right. \text{ for } l \in \{0, \ldots, L - 1\} | T_{L-1} = 0 \leq c \frac{(\log L)^{4}}{L}.
\]

Lemma 4.1 will be proven in Section 7 together with further barrier crossing bounds that will be needed in the proof of the lower bound in sections 5-7. We can now state and prove Proposition 4.2 (the first main ingredient in the proof of Proposition 3.5), which bounds the expectation of the counting random variable in (4.6). Note that the bound goes to zero for all \( s > 0 \).
Figure 4.1. An illustration of the packing used for the proof of Proposition 4.7. Each traversal counted by $\hat{T}_{y,t}^{s,l}$ (that is a traversal between circles of radii $r_l^+$ and $r_{l+1}^-$ before the $t_s$-th traversal from the circle of radius $r_l^+$ to the circle of radius $r_0^-$, dashed in the picture) gives rise to at least one traversal counted by $\tilde{T}_{y,t}^{s,l}$ (that is a traversal between circles of radii $r_l^+$ and $r_{l+1}^-$ before the $t_s$-th traversal from the circle of radius $r_l^+$ to the circle of radius $r_0^-$, solid in the picture). Therefore $\hat{T}_{y,t}^{s,l} \leq T_{y,t}^{s,l}$.

Proposition 4.2. \((x \in \mathbb{T})\) For all \(y \in \mathbb{T}, s \in (-100,100)\) and \(L \geq 1\)

\[(4.11) \quad E_x \left[ \sum_{y \in F_L} 1_{\{T_{y,t}^{s,l}=0\}} \cap \{\sqrt{T_{y,t}^{s,l}} \geq \alpha(l) \text{ for } l=0,\ldots,L\} \right] \leq c (\log L)^{11/2} L^{-s}.\]

Proof. The expectation in (4.11) equals (cf. (1.16))

\[|F_L| \cdot P_x \left[ T_{y,t}^{s,l} = 0 \right] \cdot P_x \left[ \sqrt{T_{y,t}^{s,l}} \geq \alpha(l) \text{ for } l \in \{0,\ldots,L-1\} \mid T_{y,t}^{s,l} = 0 \right],\]

for some arbitrary \(y \in \mathbb{T}\), where we have used that \(T_{y,t}^{s,l}, l \geq 0\), has the same law for all \(y\) (see (4.3)). By (3.11) and Lemma 3.4 the first two quantities are bounded by

\[c (\log L)^{3/2} e^{2L} \times e^{-2L} L^{1-s} \leq c (\log L)^{3/2} L^{1-s}\]

(for the latter we use the strong Markov property at time \(T_{B(y,r_1)}\) when \(y\) is such that \(x \in B(y,r_1)^0\), cf. (3.15)). The last probability equals, by (4.3),

\[G_{t_s} \left[ \sqrt{T_{y,t}^{s,l}} \geq \alpha(l) \text{ for } l \in \{0,\ldots,L-1\} \mid T_{y,t}^{s,l} = 0 \right].\]

Thus by Lemma 4.1 the expectation in (4.11) is bounded above by

\[c (\log L)^{3/2} L^{1-s} \times \frac{(\log L)^4}{L} = c (\log L)^{11/2} L^{-s}.\]

The next major step of this section is to prove Proposition 4.7 excluding the possibility that some traversal process violates the barrier condition. As mentioned at the start of
the section, we use a packing argument. To this end define modified radii $r_l^\pm$ by

\[
(4.12) \quad r_l^- = \left(1 - \frac{100}{L}\right) r_l \quad \text{and} \quad r_l^+ = \left(1 - \frac{100}{L}\right)^{-1} r_l \quad \text{for} \quad l \geq 0, 
\]

and count for each $y \in F_{l+\log L}$ the number of traversals from $\partial B(y, r_{l+1}^-)$ to $\partial B(y, r_l^+)$ during the first $t$ excursions from $\partial B(y, r_l^+)$ to $\partial B(y, r_0^+)$ as follows (cf. (4.1))

\[
(4.13) \quad \tilde{T}_{y,t}^l = \sup \left\{ n \geq 0 : R_n(y, r_l^+, r_{l+1}^-) \circ \theta_{T_B(y, r_0)} \leq D_{\tilde{\nu}_l} (y, r_0^+, r_l^+) \circ \theta_{T_B(y, r_0)} \right\}. 
\]

For each $y \in T$ let $y_l$ denote the point in $F_l$ closest to $y$ (breaking ties in some arbitrary way), and define

\[
\tilde{T}_{y,t}^l = \tilde{T}_{y,l+\log L,t}^l, \quad \text{for} \quad y \in T \setminus F_{l+\log L} \quad \text{for all} \quad t \geq 0, l \geq 1. 
\]

With this construction $\tilde{T}_{y,t}^l$ dominates $\tilde{T}_{l,t}^l$.

**Lemma 4.3.** For all $y \in T$, $l \geq 1$ and $t \geq 0$ we have that

\[
(4.14) \quad \tilde{T}_{y,t}^l \leq \tilde{T}_{l,t}^l. 
\]

**Proof.** By the definition (3.10) of $F_{l+\log L}$ we have

\[
(4.15) \quad d(y, y_{l+\log L}) \leq r_{l+\log L} \quad \text{for} \quad \frac{r_l}{L} = e^{r_{l+1}/L}. 
\]

Now because of the latter equality and the definition (4.12) of $r_l^\pm$ we have for $L$ large enough

\[
r_{l+1}^- + r_{l+\log L} \leq r_{l+1} \quad \text{and} \quad r_l + r_{l+\log L} \leq r_l^+, 
\]

so that for all $y \in T$

\[
(4.16) \quad B(y_{l+\log L}, r_{l+1}^-) \subset B(y, r_{l+1}) \subset B(y, r_l) \subset B(y_{l+\log L}, r_l^+), 
\]

and thus (recall (2.5) and (3.7))

\[
(4.17) \quad R_n \circ \theta_{T_B(y, r_0)} \leq R_n(y_{l+\log L}, r_l^+, r_{l+1}^-) \circ \theta_{T_B(y, r_0)} \quad \text{for all} \quad n \geq 0, 
\]

(see Figure 4.1 on page 19). Also (4.12) implies

\[
r_l + r_{l+\log L} \leq r_l^+ \quad \text{and} \quad r_0^- + r_{l+\log L} \leq r_0, 
\]

so that

\[
(4.18) \quad B(y, r_l) \subset B(y_{l+\log L}, r_l^+) \subset B(y_{l+\log L}, r_l^-) \subset B(y, r_0), 
\]

and therefore (similarly to (4.17))

\[
(4.19) \quad D_n(y_{l+\log L}, r_l^+, r_l^-) \circ \theta_{T_B(y, r_0)} \leq D_n \circ \theta_{T_B(y, r_0)} \quad \text{for all} \quad n \geq 0, 
\]

Now by the definitions of $\tilde{T}_{y,t}^l$ and $\tilde{T}_{l,t}^l$ (see (4.1) and (4.13)) the claim (4.14) now follows from (4.17) and (4.19). \[\square\]

We now show that $\tilde{T}_{l,t}^l$ has a binomial-geometric compound distribution.

**Lemma 4.4.** Let

\[
(4.20) \quad p = \frac{\log (r_l^+/r_{l+1}^-)}{\log (r_0^-/r_{l+1}^-)} \quad \text{and} \quad q = \frac{\log (r_0^-/r_1^+)}{\log (r_0^-/r_{l+1}^-)}, 
\]
Let $G_1, G_2, \ldots$ be geometric random variables with support $\{1, 2, \ldots\}$ and success probability $p$ and let $J_1, J_2, \ldots$ be Bernoulli random variables with success probability $q$, all mutually independent. We have for all $y \in F_{l+\log L}, t \geq 0$ and $l \geq 1$ that

\begin{equation}
\hat{T}_{l}^{y,t} \overset{\text{law}}{=} \sum_{i=1}^{[t]} J_i G_i.
\end{equation}

Proof. By (2.4) each excursion of Brownian motion from scale 1 to scale 0 (from $\partial B(y, r_{l+1}^1)$ to $\partial B(y, r_0^1)$) has probability $q$ of giving rise to at least one traversal from scale $l$ to $l+1$ (i.e. of hitting $B(y, r_{l+1})$ before leaving $\partial B(y, r_0^1)$). After hitting $B(y, r_{l+1})$ Brownian motion returns to $\partial B(y, r_l^1)$, and from there it has a probability $p$ to escape to $\partial B(y, r_0^1)$ (again by (2.4)) and end the traversal from scale 1 to scale 0; otherwise it returns to $B(y, r_{l+1})$ which gives rise to another traversal from scale $l$ to $l+1$. Each of these "coin flips" are independent by the strong Markov property, and thus (4.21) follows. \hfill \Box

Remark 4.5. Note that by (3.4) and (4.12)

\begin{equation}
p, q = \frac{1}{l+1} + O\left(L^{-1}\right) \text{ for } p, q \text{ as in (4.20),}
\end{equation}

and that the argument giving (4.21) applies equally well to $\hat{T}_{l}^{y,t}$ but with modified $p$ and $q$ given by

\begin{equation}
p = \frac{\log (r_l/r_{l+1})}{\log (r_0/r_{l+1})}; \quad q = \frac{1}{l+1} \quad \text{and } \quad q = \frac{\log (r_0/r_1)}{\log (r_0/r_{l+1})}.
\end{equation}

We will need a lemma on the large deviations of $\hat{T}_{l}^{y,t}$. We state it for a general geometric distribution with binomial compounding, and postpone the proof until the appendix.

Lemma 4.6. Let $G_1, G_2, \ldots, J_1, J_2, \ldots$ be as in Lemma 4.4 for $p \in (0, 1)$ and $q \in (0, 1)$. Then for all integers $n \geq 1$ and $\theta \leq n^{\frac{2}{p}}$

\begin{equation}
P \left[ \sum_{i=1}^{n} J_i G_i \leq \theta \right] \leq \exp \left( - \left( \sqrt{q\theta} - \sqrt{pq} \right)^2 \right).
\end{equation}

We can now use Lemma 4.3, Lemma 4.4 and Lemma 4.6 to deduce Proposition 4.7, proving that no traversal process violates the barrier condition.

Proposition 4.7. ($x \in \mathbb{T}$) For all $s \in (-100, 100)$

\begin{equation}
\lim_{L \to \infty} P_x \left[ \exists y \in F_L \text{ s.t. } \sqrt{\hat{T}_{l}^{y,t}} \leq \alpha (l) \text{ for } l \in \{0, \ldots, L - 1\} \right] = 0.
\end{equation}

Proof. By Lemma 4.3 and the definition (4.13) of $\hat{T}_{l}^{y,t}$ the probability in (4.24) is bounded above by

\begin{equation}
\sum_{l=1}^{L-1} \sum_{y \in F_{l+\log L}} P_x \left[ \sqrt{\hat{T}_{l}^{y,t}} \leq \alpha (l) \right] = \sum_{l=1}^{L-1} |F_{l+\log L}| P_x \left[ \sqrt{\hat{T}_{l}^{y,t}} \leq \alpha (l) \right]
\end{equation}

\begin{equation}
\overset{(3.11)}{\leq} c (\log L)^{3/2} L^2 \sum_{l=1}^{L-1} e^{2l} P_x \left[ \sqrt{\hat{T}_{l}^{y,t}} \leq \alpha (l) \right],
\end{equation}
for some arbitrary $y \in F_{l+\log L}$. Using Lemma 4.4, Lemma 4.6 and (4.22) it follows that

$$P_x \left[ \sqrt{T^y_{l,t}} \leq \alpha (l) \right]$$

is bounded above by

$$ce^{-\frac{(l \sqrt{t} + O(1))}{s}} \leq ce^{-\frac{(l \sqrt{t} + (\log L)^2 - c)}{s}} \leq ce^{-2l - c(\log L)^2},$$

where we have used that $t_s = 2L^2 (1 + O(\log L/L))$. Thus the probability in (4.24) is bounded above by

$$c (\log L)^{3/2} L^2 \sum_{l=1}^{L-1} e^{2l} \times e^{-2l - c(\log L)^2} \leq c (\log L)^{3/2} L^3 e^{-c(\log L)^2} = o(1).$$

We can now wrap up this section by proving Proposition 3.5.

Proof of Proposition 3.5. Since the probability in (3.16) is decreasing in $s$ it suffices to consider $s \in (0, 100)$. For such $s$ the statement (4.2) (and therefore the claim (3.16)) follows immediately from the Markov inequality, Proposition 4.2 (the right-hand side tends to zero for $s > 0$) and Proposition 4.7.

To complete the proof of our main result Theorem 3.1 it remains to show the lower bound in terms of excursions Proposition 3.6 and the concentration of excursion times Proposition 3.7, in addition to the barrier estimate Lemma 4.1 and the large deviation bound Lemma 4.6 used in this section.

5. Lower bound on cover time in terms of excursions

In this section we prove Proposition 3.6 which gives the lower bound of the main result Theorem 3.1 “in terms of excursions”, and was used in the proof of that result in Section 3. More precisely, our goal is to show the claim (3.17) from Proposition 3.6 that

$$\lim_{L \to \infty} P_x \left[ T^y_{L-1,t-s} = 0 \text{ for some } y \in F_L \right] = 1 \text{ for all } s > 0,$$

(see (3.8), (3.10) and (3.12) for the definitions).

As previously mentioned, a natural approach is to apply the second moment method to the counting random variable $\sum_{y \in F_L} 1 \{ T^y_{L-1,t-s} = 0 \}$, but this fails as the second moment of this sum is much larger than the first moment squared. To get around this we introduce a truncation, which takes the form of a barrier condition. The main point of the condition is that we require $\sqrt{T^y_{l}}$ to stay above $\gamma (l)$, where

$$\gamma (l) = \gamma (l, L, s) = \beta (l) + f (l),$$

$\beta (l)$ is the linear function from (4.5) (essentially the mean of the $\sqrt{T^y_{l,t-s}}$ when conditioned on $T^y_{L-1} = 0$) and $f (l)$ is a convex “bump” function given by

$$f (l) = f (l; L) = \min \left( t^{0.49}, (L - l)^{0.49} \right), l \in [0, L],$$

(see Figure 7.1 on page 43). It turns out that with this condition, the summands in the counting random variable decorrelate enough so that the variance should morally speaking not explode with respect to the first moment squared.
For technical reasons it turns out to help to introduce a second barrier
\[
\delta (l) = \delta (l, L, s) = \beta (l) + g (l),
\]
where \(g (l) \geq f (l)\) is the larger convex “bump”,
\[
g (l) = g (l; L) = \min \left( \left\{ 10.51, (L - l)^{0.51} \right\}, l \in [0, L] \right),
\]
and require also that the square root of the traversal processes stay below \(\delta (l)\), or in other words that they stay in the “tube” bounded by \(\gamma (l) = \delta (l)\). Furthermore it turns out to be too much to ask for the barrier condition to be satisfied for \(l\) close to 0 or \(L - 1\). We therefore introduce a cutoff
\[
l_0 = l_0 (L) = \left\lfloor \frac{1}{10} \log \log L \right\rfloor,
\]
and arrive at the final form of the summands
\[
I_y = \left\{ \gamma (l) \leq \sqrt{T_l^{y,l-s}} \leq \delta (l) \text{ for } l = l_0, \ldots, L - l_0 \text{ and } T_{l-l_0}^{y,l-s} = 0 \right\},
\]
for \(y \in F_L\). Finally, since the Lemma \(3.2\) gives the law of \(T_l^{y,l-s}\) technically speaking only applies when \(x \notin B (y, r_1)^c\) we sum not over \(F_L\) but over the smaller set
\[
\hat{F}_L = F_L \setminus B (x, r_0).
\]
Our truncated counting random variable is thus
\[
Z = \sum_{y \in \hat{F}_L} 1_{I_y}.\]
Note that \(\hat{F}_L\) is only marginaly smaller than \(F_L\) since \(|F_L \cap B (x, r_0)| / |F_L| \leq cr_0^2 \to 0\) by \(3.4\) and \(3.10\), so that with \(3.11\)
\[
\left| \hat{F}_L \right| = (1 - o (1)) |F_L| \asymp c (\log L)^{3/2} e^{2L}.
\]
Obviously
\[
\{ Z > 0 \} \subset \left\{ T_{l-l_0}^{y,l-s} = 0 \text{ for some } y \in F_L \right\}.
\]
We will show that in fact \(Z > 0\) with probability tending to one, giving our goal \(3.17\). This will be done in two steps. First we will show in Proposition \(5.2\) that for all \(s > 0\)
\[
E_x [Z] \asymp (\log L)^{3/2} l_0 L^s \to \infty, \text{ as } L \to \infty.
\]
For the second step (which is considerably more challenging) we show in Proposition \(5.6\) that for all \(s > 0\)
\[
E_x [Z^2] = (E_x [Z])^2 (1 + o (1)), \text{ as } L \to \infty.
\]
We will see that the lower bound Proposition \(3.6\) (i.e. \(3.17\)) is an easy consequence of \(5.12\), via the Paley-Zygmund inequality.

The proof of \(5.12\) is the heart of this section. The second moment \(E_x [Z^2] \) is a sum of “two point probabilities” \(P_x [I_y \cap I_z] \) for \(y, z \in \hat{F}_L\), and bounding \(E_x [Z^2] \) amounts to bounding these terms. Since \(r_0 \downarrow 0\) most pairs are at distance at least \(2r_0\), and it turns out that for such pairs the events \(I_y \) and \(I_z\) are exactly independent (essentially because they depend on the behaviour of Brownian motion in disjoint balls \(B (y, r_0) \) and \(B (z, r_0)\)). Because of this, the contribution of such terms to the second moment \(E_x [Z^2] \) will be
shown to be at most $E_x [Z]^2$. Thus the proof (5.12) is about showing that terms for pairs at distance less than $2r_0$ are negligible, or in other words

\[(5.13) \sum_{y,z \in \tilde{F}_L : d(y,z) < 2r_0} P_x [I_y \cap I_z] = o \left( E_x [Z]^2 \right).\]

Lemma 5.5 and Proposition 5.6 will provide bounds for $P_x [I_y \cap I_z]$ in this regime that will allow us to show (5.13). We state Lemma 5.5 and Proposition 5.6 in this section, but since their proofs are intricate (especially that of Proposition 5.6, the main bound) they are postponed until the next section.

Before starting the proofs we state a bound on the probability that a conditioned Galton-Watson process stays in the tube bounded by $\gamma (l)$ and $\delta (l)$. It will be proven (together with the barrier bound Lemma 4.1 from the previous section) in Section 7.

Lemma 5.1. For all $s \in (-1, 1)$ we have that

\[(5.14) G_t \left[ \gamma (l) \leq \sqrt{T_l} \leq \delta (l) \text{ for } l \in \{l_0, \ldots, L - l_0|T_L = 0\} \right] \approx \frac{l_0}{L}.\]

We now start the proofs of this section by proving the estimate (5.11) on $E_x [Z]$.

Proposition 5.2. $(x \in \mathbb{T})$ For all $s \in (-1, 1)$

\[(5.15) E_x [Z] \asymp c (\log L)^{3/2} l_0 L^s,\]

and for all $y, z \in \tilde{F}_L$,

\[(5.16) P_x [I_y] = P_x [I_z] \asymp c e^{-2L} L^s l_0.\]

Proof. By (5.9) the first claim (5.15) follows from (5.16). The equality in (5.16) holds since $T_{y,t}^{s,l-s}, l \geq 0$, and $T_{z,t}^{s,l-s}, l \geq 0$, have the same law by Lemma 3.2. For the bound in (5.16) note that (recall (5.6))

\[(5.17) P_x [I_y] = P_x [I_y | T_{y,t}^{s,l-s} = 0] P_x [T_{L-1}^{y,t-s} = 0].\]

By Lemma 3.4 we have

\[(5.18) P_x [T_{L-1}^{y,t-s} = 0] \asymp e^{-2L} L^{1+s},\]

and using Lemma 3.2

\[P_x [I_y | T_{L-1}^{y,t-s} = 0] = G_t \left[ \gamma (l) \leq \sqrt{T_l} \leq \delta (l) \text{ for } l = l_0, \ldots, L - l_0|T_L = 0\right].\]

Thus by (5.14) it follows that

\[(5.19) P_x [I_y | T_{L-1}^{y,t-s} = 0] \asymp \frac{l_0}{L}.\]

Plugging this into (5.17) together with (5.18) gives (5.16).

We now turn our attention to the main step of the proof of the lower bound Proposition 3.6 namely the second moment bound (5.12). For this we will need bounds on the two point probability $P_x [I_y \cap I_z]$.

We start with the case of $y$ and $z$ such that $B(y, r_0)$ and $B(z, r_0)$ are disjoint. The events will be independent in this case, and to show this we need the independence result (5.20) which now follows (we also include (5.21) since it will be used later in Section 6 and its proof is similar).
Lemma 5.3. \((x \in \mathbb{T})\) For all \(t \geq 0\) and \(y, z \in \widehat{F}_L\)

\[ (5.20) \quad \text{if } d(y, z) > 2r_0 \text{ then } \left(T_{l,t}^y, T_{l,t}^z\right)_{l \geq 0} \text{ and } \left(T_{l,t}^z, T_{l,t}^y\right)_{l \geq 0} \text{ are independent under } P_x. \]

Also

\[ (5.21) \quad \text{for } w, v \in \mathbb{T} \text{ the } P_w - \text{law of } \left(T_{l,t}^w\right)_{l \geq 0, t \geq 0} \text{ depends only on } d(w, v). \]

Proof. To see (5.21) note that \(T_{l,t}^w, l \geq 0, \text{ depends only on } T_{l,n}^w - T_{l,n-1}^w = T_{l,1}^w \circ \theta_{R_{n}^{w,0}}, l \geq 0, n \geq 1, \text{ where } T_{l,1}^w \circ \theta_{R_{n}^{w,0}} \text{ counts the traversals that happen during the } n\text{-th excursion from } \partial B(v, r_1) \text{ to } \partial B(v, r_0). \)

The traversal count \(T_{l,1}^w \circ \theta_{R_{n}^{w,0}}\) depends only on the excursion \(W_{(R_{n}^{w,0} \cup ...) \setminus D_{n}^{w,0}}, \text{ and furthermore it is a rotationally invariant function of that excursion.} \)

Let \(\tilde{w} \in \mathbb{T}\) be any point such that \(d(\tilde{w}, v) = d(w, v)\) and let \(u\) be any point in \(\partial B(v, r_1)\).

By the rotational invariance (2.3) of \(W_{l}\) in \(B(v, r_0)\) and the strong Markov property the \(P_w - \) and \(P_u - \) laws of \(T_{l,1}^w \circ \theta_{R_{n}^{w,0}}\) coincide (that is the law depends only on \(d(w, v)\)), and the \(P_w - \) and \(P_u - \) laws of \(T_{l,1}^w \circ \theta_{R_{n}^{w,0}}\) coincide for \(n \geq 2\) (that is the law does not even depend on \(d(w, v)\)).

Furthermore the strong Markov property implies that the \(T_{l,1}^w \circ \theta_{R_{n}^{w,0}}, n \geq 1, \) are independent. This gives (5.21).

To see (5.20) we similarly use that for \(v \in \{y, z\}\) the process \(T_{l,t}^v, l \geq 0, \text{ depends only on } T_{l,1}^v \circ \theta_{R_{n}^{v,0}}, l \geq 0, \text{ which in turn depend only on the excursions } W_{(R_{n}^{v,0} \cup ...) \setminus D_{n}^{v,0}}, n \geq 0. \)

The latter excursions refer to disjoint intervals of time for each \(n \geq 0\) and \(v \in \{y, z\}, \text{ since } B(y, r_0) \text{ and } B(z, r_0) \text{ are disjoint. Therefore, using rotational invariance and the strong Markov property as above, the processes } l \to T_{l,1}^v \circ \theta_{R_{n}^{v,0}}, v \in \{y, z\}, n \geq 1, \text{ are mutually independent. This implies (5.20).} \]

\[ \square \]

The two point probability for \(y\) and \(z\) such that \(B(y, r_0)\) and \(B(z, r_0)\) are disjoint can now be computed easily.

Corollary 5.4. \((x \in \mathbb{T})\) For all \(y, z \in \widehat{F}_L\) such that \(d(y, z) > 2r_0\)

\[ P_x[I_y \cap I_z] = P_x[I_y]^2. \]

Next we state a bound on the two point probability for \(y\) and \(z\) for which the largest non-overlapping balls \(B(y, r_k)\) and \(B(z, r_k)\) are of radius \(r_k\) for \(0 \leq k \leq l_0\). It will be proven in the next section. Note that the right-hand side is almost that of (5.16) squared.

Lemma 5.5. For all \(y, z \in \widehat{F}_L\) such that \(2r_{l_0} \leq d(y, z) < 2r_0\) and \(s \in (-1, 1)\) we have

\[ (5.22) \quad P_x[I_y \cap I_z] \leq c \left( e^{-(2L-2l_0) L s 0.51} L s 0.51 \right)^2. \]

Next we state the two point probability bound for the most important (and difficult) regime, which gives a bound for points \(y\) and \(z\) which are such that the largest non-overlapping ball is of radius \(r_k\) for \(l_0 < k < L - l_0\).

Proposition 5.6. \((x \in \mathbb{T})\) For all \(s \in (0, 1)\), \(l_0 < k < L - l_0\) and all \(y, z \in \widehat{F}_L\) such that \(2r_k < d(y, z) \leq 2r_{k-1}\) we have

\[ (5.23) \quad P_x[I_y \cap I_z] \leq c(s) e^{-(4L-2k) - cf(k) L s 1.02} L s 1.02 g(k)^2 (\log L)^{1.02}. \]

Remark 5.7. This bound is key to the whole approach. Since the proof (which is carried out in the next section) is involved, let us spend a few words on the heuristic which explains it. By (5.16) the claim (5.23) is equivalent to

\[ (5.24) \quad P_x[I_y \cap I_z] \leq e^{-2(L-k) - cf(k) L s 0.02} g(k)^2 (\log L)^{1.02}. \]
Recall that because of the approximate hierarchical structure, we expect that $T_{l}^{y,t-s}$ and $T_{l}^{z,t-s}$ roughly coincide for $l \leq k$ and “decouple” for $l \geq k$ (see Figure 6.1 on page 31). Therefore, avoiding the ball $B(z, r_L)$ in $t_s$ excursions from $\partial B(z, r_1)$ to $\partial B(z, r_0)$, when conditioning on $I_y$, is essentially equivalent to avoiding $B(z, r_L)$ in $\gamma(k)^2$ excursions from $\partial B(z, r_{k+1})$ to $\partial B(z, r_k)$. By (5.5), each such excursion avoids $B(z, r_L)$ with probability $1 - \frac{1}{T_{l}^{x,t-s}}$. We therefore expect that $P_x[I_y | I_y] = \sum_{\{y,z\} \in G_k} P_x[I_y \cap I_z]$ to correspond to the probability that a Brownian bridge starting at $\gamma(I)$ at time 0 and ending at 0 at time $L-k$ stays above the linear function with the same starting and ending points during the time interval $[0, L-k-l_0]$. This probability is of order $\sqrt{\gamma(l)/((L-k-l_0)^{(1+s)}-L s)}$, e.g. by the reflection principle. These considerations thus suggest that $P_x[I_y | I_y]$ should essentially be upper-bounded by $ce^{-2(L-k)-c\gamma(k)^2 s^2}L^s$, which is (marginally) better than the bound we derive rigorously.

Finally for the last case, that is when the largest non-overlapping balls $B(y, r_k)$ and $B(z, r_k)$ have radius $r_k$ for $k \geq L - l_0$, we have the following trivial bound which follows directly from (5.16).

\[
(5.26) \quad P_x[I_y \cap I_z] \leq P_x[I_y] \leq ce^{-2Lk L^s} \text{ for all } y, z \in \hat{F}_L.
\]

We have now arrived at the heart of this section, which is the bound on the second moment of the counting random variable $Z$.

**Proposition 5.8.** $(x \in T)$ For all $s > 0$

\[
E_x[Z^2] = (E_x[Z])^2 (1 + o(1)), \text{ as } L \to \infty.
\]

**Proof.** Write

\[
E_x[Z^2] = \sum_{y, z \in \hat{F}_L} P_x[I_y \cap I_z].
\]

Decompose the set of pairs of $y, z \in \hat{F}_L$ by setting

\[
G_0 = \left\{ (y, z) : y, z \in \hat{F}_L \text{ s.t. } d(y, z) > 2r_0 \right\},
\]

\[
G_k = \left\{ (y, z) : y, z \in \hat{F}_L \text{ s.t. } 2r_k < d(y, z) \leq 2r_{k-1} \right\} \text{ for } 1 \leq k < L,
\]

\[
G_L = \left\{ (y, z) : y, z \in \hat{F}_L \text{ s.t. } d(y, z) \leq 2r_{L-1} \right\}.
\]

We have that $U_{k=0}^L G_k = \hat{F}_L \times \hat{F}_L$ and therefore

\[
(5.28) \quad E_x[Z^2] = \sum_{\{y, z\} \in G_0} P_x[I_y \cap I_z] + \sum_{k=1}^L \sum_{\{y, z\} \in G_k} P_x[I_y \cap I_z].
\]

By Corollary 5.4 we have

\[
(5.29) \quad \sum_{\{y, z\} \in G_0} P_x[I_y \cap I_z] = \sum_{\{y, z\} \in G_0} P_x[I_y] P_x[I_z] \leq \sum_{y, z \in \hat{F}_L} P_x[I_y] P_x[I_z] = (E_x[Z])^2,
\]

\[
(5.30) \quad \sum_{\{y, z\} \in G_k} P_x[I_y \cap I_z] \leq \left( \frac{1 - \frac{1}{T_{l}^{x,t-s}}}{1 - \frac{1}{T_{l}^{x,t-s}}} \right)^{-2}.
\]

Therefore, we have

\[
(5.31) \quad E_x[Z^2] \leq \left( \frac{1 - \frac{1}{T_{l}^{x,t-s}}}{1 - \frac{1}{T_{l}^{x,t-s}}} \right)^{-2} (E_x[Z])^2.
\]
so that \((5.27)\) will follow once we have shown that

\[
(5.30) \quad \sum_{k=1}^{L} \sum_{\{y,z\} \in G_k} P_x [I_y \cap I_z] = o \left( E_x [Z]^2 \right).
\]

We first bound

\[
(5.31) \quad \sum_{k=1}^{L} \sum_{\{y,z\} \in G_k} P_x [I_y \cap I_z] \leq c \sum_{k=1}^{L} e^{4L-2k} \sup_{\{y,z\} \in G_k} P_x [I_y \cap I_z],
\]

where have used that for \(1 \leq k \leq L\)

\[
\begin{align*}
|G_k| & \leq \left| \tilde{F}_L \right| \sup_{v \in \tilde{F}_L} \left| \tilde{F}_L \cap B (v, 2r_{k-1}) \right| \\
& \leq c \left| \tilde{F}_L \right| \left( c \left| \tilde{F}_L \right| r_{k-1}^2 \right)^{3/2} \\
& \leq c (\log L)^{3/2} e^{4L-2k}.
\end{align*}
\]

Next we split the sum on the right-hand side of \((5.31)\) into three parts

\[
(5.33) \quad \sum_{k=1}^{L} \cdot \leq \sum_{1 \leq k \leq l_0} \cdot + \sum_{l_0 < k < L - l_0} \cdot + \sum_{L - l_0 \leq k \leq L} \cdot.
\]

For the first sum on the right-hand side we have by Lemma \((5.0)\) that it is at most,

\[
(5.34) \quad c (\log L)^{3/2} \sum_{1 \leq k \leq l_0} e^{4L-2k} (e^{-(2L-2l_0)} L^{s} g (l_0))^{2}.
\]

For the middle sum on the right-hand side of \((5.33)\) we use Proposition \((5.6)\) to obtain an upper bound of

\[
(5.35) \quad c (s) (\log L)^{3/2} \sum_{l_0 < k < L - l_0} e^{4L-2k} L^{2s} g (k)^2 \leq c (s) (\log L)^{3/2} e^{-cf (k)} \leq c (s) (\log L)^{3/2} L^{2s}.
\]

since

\[
\sum_{l_0 < k < L - l_0} g (k)^2 e^{-cf (k)} \to 0 \text{ by } (5.2), (5.4) \text{ and } (5.5).
\]

For the last sum on the right-hand side of \((5.33)\) we obtain from \((5.26)\) the following upper bound

\[
(5.36) \quad \sum_{L - l_0 \leq k \leq L} e^{4L-2k} e^{-2L} L^{s} = c (\log L)^{3/2} l_0 L^{s} \sum_{L - l_0 \leq k \leq L} e^{2L-2k}
\]

\[
\leq c (\log L)^{3/2} l_0 L^{s} e^{2L-2k} L^{s}.
\]

Combining \((5.34)-(5.36)\) we thus obtain this upper bound on the right-hand side of \((5.31)\):

\[
\begin{align*}
c L^{2s} (\log L)^{19/10} l_0^{1.02} g (l_0)^2 + c (s) (\log L)^{2.52} l_0^{1.02} L^{2s} & \leq c (s) L^{2s} (\log L)^{2.52} l_0^{1.02} g (l_0)^2 \\
& \leq c (s) (E_x [Z])^{2} (\log L)^{0.48} l_0^{-0.98} g (l_0)^2 o \left( E_x [Z]^2 \right).
\end{align*}
\]

This gives \((5.30)\), so the claim \((5.27)\) follows. \(\square\)
We have now reached the final goal of this section: the proof of Proposition 3.6.

Proof of Proposition 3.6. By (5.10) it suffices to show that \( P_x [Z > 0] \to 1 \), and by the Paley-Zygmund inequality we have

\[
P_x [Z > 0] \geq \frac{E_x [Z]^2}{E_x [Z^2]}.
\]

Thus the claim (3.17) follows by Proposition 5.8.

Of three main ingredients (propositions 3.5, 3.7) used to prove the main result Theorem 3.1 we have now derived most of the first two. Still missing are the proofs of the barrier estimate Lemma 4.1 and the large deviation result Lemma 4.6 (used but not proven in Section 4 for the proof of Proposition 3.5), the barrier estimates Lemma 5.1 used in this section to prove Proposition 3.6, and the two point probability bounds Lemma 5.5 and Proposition 5.6 also used in this section. The next section deals with these two point probability estimates.

6. Bounds on two point probabilities

In this section we will prove the crucial two point probability bounds Lemma 5.5 and Proposition 5.6, which were used to prove the lower bound Proposition 3.6 in the previous section. Recall these give a bounds on the probability \( P_x [I_y \cap I_z] \) where for \( v \in T \)

\[
(I_v) = \{ \gamma (l) \leq \sqrt{T_{y,t}^l} \leq \delta (l) \text{ for } l = l_0, \ldots, L - l_0 \text{ and } T_{L-1}^l = 0 \}.
\]

We will need to consider certain traversal processes that “start at lower scales”. For each \( k \geq 1 \) we define

\[
(T_{l}^{y,k,m}) = \sup \left\{ n \geq 0 : R_{n}^{y,l} \leq D_{m}^{y,k} \right\} \text{, } l \geq k, m \in \mathbb{R}_+,
\]

to be the number of traversals from scale \( l \) to \( l + 1 \) during the first \( t \) excursions from scale \( k - 1 \) to scale \( k \) (cf. the definition (3.8) of \( T_{l}^{y,t} \)). The definitions (3.8) and (6.1) imply the crucial “compatibility” property that

\[
(T_{l}^{y,k,m}) = (T_{l}^{y,t}) \text{ for } l \geq k, \text{ on } \left\{ m = T_{k}^{y,t} \right\},
\]

since on the latter event \( W_t \) does not visit \( B(y, r_k) \) between \( D_{m}^{y,k} \) and \( D_{m}^{y,0} \). Furthermore, the process \( T_{l}^{y,k,t} \) satisfies essentially the same properties as \( T_{l}^{y,t} \). In particular:

**Lemma 6.1.** If \( y \in T, k \geq 1, v \notin B(y, r_k)^c \) and \( t \geq 0 \), the \( P_v \)-law of \( (T_{k+l}^{y,k})_{l \geq 0} \) is \( \mathbb{G}_t \).

*Proof.* Almost identical to the proof of Lemma 3.2.

The \( T_{l}^{y,k,t} \) also satisfy a similar independence property.

**Lemma 6.2.** \( (x \in T) \) For all \( t \geq 0 \) and \( y, z \in \tilde{F}_L \) (see (5.7)) it holds that

\[
(T_{l}^{y,k,t})_{l \geq k} \text{ and } (T_{l}^{z,k,t})_{l \geq k} \text{ are independent under } P_x.
\]

*Proof.* Almost identical to the proof of (5.20).

We will need the following barrier crossing estimates for the Galton-Watson process \( T_l \), which will be proven (together with the previously used barrier estimates Lemma 4.1 and Lemma 5.1) in Section 7. The first corresponds to checking the barrier for \( l \geq k \), and the second to checking it for \( l \leq k \).
Lemma 6.3. For all \( l_0 < k < L - l_0 - 1, \) \( s \in (-1, 1), \) and \( \gamma(k)^2 \leq a \leq \delta(k)^2 \) we have

\[
\mathbb{E}_a \left[ \gamma(l) \right] \leq \sqrt{l L} \quad \text{for } l = k, \ldots, L - l_0 | T_{L-1-k} = 0 \leq c \frac{n_0^{0.51} g(k)}{L - k - l_0 - 1}.
\]

If \( l_0 + 1 < k < L - l_0 \) and \( s \in (-1, 1) \)

\[
\mathbb{E}_{l_0} \left[ \gamma(l) \right] \leq \sqrt{l} \leq \delta(l) \quad \text{for } l = l_0, \ldots, k | T_{L-1-l_0} = 0 \leq c \sqrt{\log(k+1)} - k - l_0 - 1.
\]

We are now ready to prove Lemma 5.5 from the previous section, which gives the bound \( \delta(x) \) on the two point probability \( P_x[I_y \cap I_z] \) for \( y \) and \( z \) such that \( B(y, r_k) \) and \( B(z, r_k) \) for \( 0 \leq k \leq l_0 \) are the largest non-overlapping balls around \( y \) and \( z \).

Proof of Lemma 5.5. By (6.6) and (6.2) with \( k = l_0 + 1 \) we have for \( v \in \{ y, z \}, \)

\[
I_v \leq \tilde{I}_v \text{ def } = \left\{ \gamma(l) \leq \sqrt{T^{y, l_0 + 1, m}_l} \quad \text{for } l = l_0 + 1, \ldots, L - l_0 \text{ and } T^{y, l_0 + 1, m}_{L-1} = 0 \right\}.
\]

Thus by Lemma 6.2 with \( k = l_0 + 1 \) (recall that \( d(y, z) \geq 2r_{l_0} \))

\[
P_x[I_y \cap I_z] \leq P_x[\tilde{I}_y \cap \tilde{I}_z] \leq \left( P_x[\tilde{I}_y] \right)^2.
\]

Thus to get (5.22) it suffices to show

\[
P_x[\tilde{I}_y] \leq c e^{-2(L-2l_0)} L^s \sqrt{\log(l_0)}.
\]

Now let

\[
\tilde{I}_{y,m} = \left\{ \gamma(l) \leq \sqrt{T^{y, l_0 + 1, m}_l} \quad \text{for } l = l_0 + 1, \ldots, L - l_0 \text{ and } T^{y, l_0 + 1, m}_{L-1} = 0 \right\},
\]

so that \( \tilde{I}_y = \bigcup_{\gamma(l_0+1) \leq m \leq \delta(l_0+1)} \tilde{I}_{y,m} \). If \( \tilde{I}_{y,m} \) holds and \( T^{y, l_0 + 1, m+1}_{L-1} = 0 \), then also \( \tilde{I}_{y,m+1} \) holds. Using this we obtain that

\[
\tilde{I}_y \leq \left( \bigcup_{\gamma(l_0+1) \leq m \leq \delta(l_0+1)} \tilde{I}_{y,m} \cap \left\{ T^{y, l_0 + 1, m+1}_{L-1} > 0 \right\} \right) \cup \tilde{I}_y, \delta^2(l_0+1),
\]

For \( y \notin B(x, r_0) \) we have by (6.1) that

\[
\left\{ T^{y, l_0 + 1, m}_{L-1} = 0 \text{ and } T^{y, l_0 + 1, m+1}_{L-1} = 0 \right\} = \left\{ D^{y, l_0 + 1, m}_{B(x, r_0)} < H^{y, l_0 + 1, m+1}_{B(y, r_0)} \right\}
\]

\[
= \left\{ H^{y, l_0 + 1, m+1}_{B(y, r_0)} < D^{y, l_0 + 1, m+1}_{B(x, r_0)} \right\} \circ \theta_{m+1}^{y, l_0 + 1, m+1},
\]

so that by the strong Markov property at time \( R^{y, l_0 + 1, m+1}_{m+1} \) we have

\[
P_x[\tilde{I}_{y,m} \cap \left\{ T^{y, l_0 + 1, m+1}_{L-1} > 0 \right\}] = P_x[\tilde{I}_{y,m} P_{W^{y, l_0 + 1, m+1}} R_{m+1} \left\{ H^{y, l_0 + 1, m+1}_{B(y, r_0)} < D^{y, l_0 + 1, m+1}_{B(x, r_0)} \right\}]
\]

\[
= P_x[\tilde{I}_{y,m}] \frac{1}{L - l_0 - 1},
\]

where we have used (3.6) and that \( \tilde{I}_{y,m} \) is \( \mathcal{F}^{m+1} \)-measurable. Thus

\[
P_x[\tilde{I}_y] \leq \frac{1}{L - l_0 - 1} \sum_{\gamma(l_0+1) \leq m \leq \delta(l_0+1)^2} P_x[\tilde{I}_{y,m}] + P_x[\tilde{I}_y, \delta^2(l_0+1)^2].
\]

Also \( P_x[\tilde{I}_{y,m}] \) equals

\[
q_m P_x[\tilde{I}_{y,m}] = 0,
\]

(6.9)
where we write
\[ q_m = P_x \left[ \gamma (l) \leq \sqrt{\frac{L}{l}} \right] \text{ for } l = l_0 + 1, \ldots, L - l_0 | T_{L-l_0}^{y,l_0+1,m} = 0 \].

Similarly to in the proof of Lemma 3.3, we have using (3.6),
\[ P_x \left[ T_{L-l_0}^{y,l_0+1,m} = 0 \right] = \left( 1 - \frac{1}{L - l_0 - 1} \right)^m. \]

We thus find that
\[ I_y \leq \left( 1 - \frac{1}{L - l_0 - 1} \right)^{\gamma (l_0+1)^2} \sum_{\gamma (l_0+1)^2 \leq m < \delta (l_0+1)^2} \left( 1 - \frac{1}{L - l_0 - 1} \right)^m q_m + \left( 1 - \frac{1}{L - l_0 - 1} \right)^{\delta (l_0+1)^2} q_{\delta (l_0+1)^2}, \]
\[ \leq \left( 1 - \frac{1}{L - l_0 - 1} \right)^{\gamma (l_0+1)^2} \sup_{\gamma (l_0+1)^2 \leq m < \delta (l_0+1)^2} q_m, \]
\[ \leq \left( 1 - \frac{1}{L - l_0 - 1} \right)^{\gamma (l_0+1)^2} e^{-\frac{\gamma (l_0+1)^2}{L - l_0 - 1}} \sum_{s \geq 0} \frac{\gamma (l_0+1)^2}{L - l_0 - 1} \leq ce^{-2(L-l_0)} L^{1+s}. \]

Also by Lemma 6.1
\[ q_m \leq \sup_{\gamma (l_0+1)^2 \leq a \leq \delta (l_0+1)^2} G_a \left[ \gamma (l) \leq \sqrt{\frac{L}{l}} \right] \text{ for } l = l_0 + 1, \ldots, L - l_0 | T_{L-l_0}^{y,l_0+1,m} = 0 \].

Thus (6.4) with \( k = l_0 + 1 \) gives that
\[ q_m \leq c \frac{g (l_0+1)}{L - l_0 - 2} \leq \frac{c}{L} \frac{g (l_0)^{0.51}}{L}. \]

Combining (6.11), (6.12) and (6.13) we obtain that
\[ P_x \left[ I_y \right] \leq ce^{-2(L-l_0)} L^{1+s} \times \frac{c \frac{g (l_0)^{0.51}}{L}}{L}, \]
which is equivalent to (6.6), so the proof of Lemma 5.5 is complete.

We now move to the more difficult bound, namely Proposition 5.6, which deals with \( y \) and \( z \) whose largest non-overlapping balls has radius \( r_k \) for \( l_0 < k < L - l_0 \). More precisely, Proposition 5.6 claims that for any \( s \in (-1, 1) \) and \( y, z \in \hat{F}_L \) such that
\[ 2r_k < d (y, z) \leq 2r_{k-1} \text{ for } l_0 < k < L - l_0, \]
we have
\[ P_x \left[ I_y \cap I_z \right] \leq ce^{-(4L-2k)-c_{f(k)} L^{2s} 2^{s_1} g (k)^2} (\log L)^{1.02}. \]

In the remainder of this section we consider \( y, z, k \) and \( s \) to be fixed. Since \( 2r_{k-1} + r_k \leq r_{k-2} \) (see (3.4) we have
\[ B (z, r_k) \subseteq B (y, r_{k-2}) \setminus B (y, r_{k}) \text{ and } B (y, r_k) \subseteq B (z, r_{k-2}) \setminus B (z, r_{k}), \]
(see Figure 6.1 on page 31) and by the definition (5.7) of \( \hat{F}_L \)
\[ x \notin B (y, r_0) \cup B (z, r_0). \]

We will consider separately the cases
\[ k \leq \left( 1 - \frac{8}{10} \right) L \text{ and } k \geq \left( 1 - \frac{8}{10} \right) L. \]
6.1. Main bound: late branching. Here we consider the case \((1 - \frac{s}{10}) L \leq k < L - l_0\). It turns out that for this regime we can ignore the contribution from the barrier condition on \(T^y_{L} \) and \(T^z_{L} \) for \(l \geq k - 2\) and still get a good enough bound. Therefore we let

\[
J^+_y = \left\{ \gamma(l) \leq \sqrt{T^y_{L}} \leq \delta(l) \text{ for } l = l_0, \ldots, k - 3 \right\},
\]

denote the barrier condition applied only up to \(k - 3\). We will bound the probability of

\[
J^+_y \cap \left\{ H_B(y,r_L) \geq D^y_{l-3} \right\} \cap \left\{ H_B(z,r_L) \geq D^z_{k-3} \right\},
\]

(which we will see contains the event \(I \cap I_z\)). We first bound the contribution from the part of (6.16) referring to \(y\).

**Lemma 6.4.** For all \((1 - \frac{s}{10}) L \leq k < L - l_0\),

\[
P_x \left[ J^+_y \cap \left\{ H_B(y,r_L) \geq D^y_{l-3} \right\} \right] \leq ce^{-2L^s \sqrt{\log(k - 2)}}.
\]

**Proof.** Since \(H_B(y,r_L) \geq D^y_{l-3}\) = \(T^y_{L-1} = 0\) (recall (3.9)) the probability in (6.17) is

\[
P_x \left[ T^y_{L-1} = 0 \right] P_x \left[ J^+_y \middle| T^y_{L-1} = 0 \right].
\]

By Lemma \[3.4\] the first of these is at most \(ce^{-2L^s L^1 + s}\). By Lemma \[3.2\] the second equals

\[
G_{l-3} \left[ \gamma(l) \leq \sqrt{T_l} \leq \delta(l) \text{ for } l = l_0, \ldots, k - 3 | T_{L-1} = 0 \right],
\]

and is thus bounded by \(c \sqrt{\log(k - 3)} / (k - 4 - l_0)\), by (6.5) with \(k - 3\) in place of \(k\). Since \(k \geq cL\) this gives the claim. \(\Box\)

It remains to bound the contribution from the part of (6.16) referring to \(z\). This should be roughly independent of the part referring to \(y\). To make this decoupling rigorous we must bound the probability of \( \left\{ H_B(z,r_L) \geq D^z_{k-3} \right\} \) conditioned on avoiding \(B(y,r_L)\).
The next lemma is a first step in this direction, and bounds the conditional probability of hitting \( B(z, r_L) \) from \( \partial B(z, r_{k+1}) \) before escaping to to \( \partial B(z, r_k) \).

**Lemma 6.5.** For any \( v \in \partial B(z, r_{k+1}) \)

\[
P_v \left[ H_{B(z,r_L)} < T_{B(z,r_k)} | H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right] \geq P_v \left[ H_{B(z,r_L)} < T_{B(z,r_k)} \right] \left( 1 - \frac{c}{L - k} \right).
\]

**Proof.** The right-hand is bounded below by

\[
P_v \left[ H_{B(z,r_L)} < T_{B(z,r_k)}, H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right].
\]

By the strong Markov property this equals

\[
P_v \left[ H_{B(z,r_L)} < T_{B(z,r_k)}, P_{T_{B(z,r_k)}} \left[ H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right] \right].
\]

Since \( \partial B(z, r_k) \subset A = B(y, r_{k-2}) \setminus B(r_k) \) we have that

\[
P_{T_{B(z,r_k)}} \left[ H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right] \geq \inf_{v \in A} P_v \left[ H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right] = P_w \left[ H_{B(y,r_L)} > T_{B(y,r_{k-2})} \right],
\]

for an arbitrary \( w \in \partial B(y, r_k) \). Now (6.18) follows since by (3.6) the latter probability is

\[
\frac{L - k}{L - k + 2} \geq 1 - \frac{c}{L - k}.
\]

\[ \square \]

We now aim to “decouple” the event \( J^+_y \) from the part of (6.16) that refers to \( z \). The main tool for this is a recursion which we now describe. Let

\[
J = \text{an arbitrary } \left( T^{y,l,s}_t \right)_{l \in \{0,...,k-3\}} - \text{measurable event},
\]

(6.19)

(here we will apply it with \( J = J^+_y \), but later we will use also \( J = C(\mathbb{R}_+, T) \), and

\[
A_n = J \cap \left\{ H_{B(y,r_L)} \geq D^y_{t-1} \right\} \cap \left\{ H_{B(z,r_L)} \geq D^z_{n-1} \right\}, n \geq 0.
\]

(6.20)

We have the following bound, which “extracts” the cost of an excursion from scale \( k+1 \) to \( k \) avoiding \( B(z, r_L) \), one at a time. The idea is that whether an excursion hits \( B(z, r_L) \) or not can only affect the event \( J \) through the end point of the excursion. But we will use (5.21) to show that the end point does not affect \( J \). Furthermore, we will use Lemma 6.5 to show that the cost of avoiding \( B(z, r_L) \) when conditioned on \( \left\{ H_{B(y,r_L)} \geq D^y_{t-1} \right\} \)

is almost the same as the unconditioned cost.

**Lemma 6.6.** For all \( n \geq 1 \)

\[
P_x \left[ A_n \right] \leq \left( 1 - \frac{1}{L - k} \right) \left( 1 - \frac{c}{L - k} \right) P_x \left[ A_{n-1} \right].
\]

(6.21)

**Proof.** Let

\[
B = 1 \left\{ H_{B(z,r_L)} \geq D^z_{n-1} \right\}, \text{ and let},
\]

(6.22)

\[
S = T_{B(y,k-2)} \circ \theta_{R^{z,k}_n} + R^{z,k}_n,
\]

(6.23)

be first time after \( R^{z,k}_n \) that \( W_t \) leaves \( B(y, k-2) \). By the assumption (6.19) the event \( J \) only depends on \( T^{y,l,s}_l \) for \( l \leq k-3 \), which depend only “on what \( W_t \) does in \( B(y, r_{k-2}) \). Therefore \( J \) is measurable with respect to \( W_t \backslash R^{z,k}_n \) and \( T^{y,l}_l (W_{S_+}), t \geq 0, l \geq 0 \) (where
The event \( \{ H_B(y, r_L) \geq D_{t-s}^{y,0} \} \) depends only on the same random variables together with \( \{ D_{l-s}^{y,0} \leq R_n^{z,k} \} \) and \( \{ D_{l-s}^{y,0} \geq R_n^{z,k} \} \), which gives encodes the dependence on \( W_{(R_n^{z,k} \wedge S)} \) (for \( t-s \) of \( y \)'s excursions from scale 1 to 0 have not been completed by time \( R_n^{z,k} \)). Thus there is a function \( g \) such that

\[
1 \{ H_B(y, r_L) \geq D_{t-s}^{y,0} \} = g \left( W_{(R_n^{z,k} \wedge S)} \right) C.
\]

Letting \( h = fg \) we have

\[
1_{A_{n-1}} = h \left( W_{(R_n^{z,k} \wedge S)} \right) BC.
\]

Furthermore

\[
1_{A_n} = h \left( W_{(R_n^{z,k} \wedge S)} \right) BCD,
\]

where

\[
D = 1 \{ H_B(z, r_L) \leq D^{y,0}_{l-s} \}.
\]

Now by (5.21) and the strong Markov property applied at time \( S \), the collection \( \left( T_{t}^{y,t} (W_{S^+}) \right)_{t \geq 0, l \geq 0} \) is independent of \( W_{\wedge S} \), since \( W_S \in \partial B(y, r_{k-2}) \). Thus letting

\[
\tilde{h} (w.) = E_{v} \left[ h \left( w., (T_{t}^{y,t})_{t \geq 0, l \geq 0} \right) \right] \text{ for } w. \in C(\mathbb{R}_+, T),
\]

for some arbitrary \( v \in \partial B(y, r_{k-2}) \), we have from (6.27)

\[
P_{x} \left[ A_{n-1} \right] = E_{x} \left[ \tilde{h} \left( W_{(R_n^{z,k})} \right) BC \right],
\]

and from (6.28)

\[
P_{x} \left[ A_{n} \right] = E_{x} \left[ \tilde{h} \left( W_{(R_n^{z,k})} \right) BCD \right].
\]

Using the strong Markov property, (6.23), (6.25) and (6.29),

\[
E_{x} \left[ CD | F_{R_n^{z,k}} \right] = 1 \{ D_{l-s}^{y,0} \leq R_n^{z,k} \} P_{W_{R_n^{z,k}}} \left[ H_B(z, r_L) > T_B(z, r_k) \right]
+ 1 \{ D_{l-s}^{y,0} \geq R_n^{z,k} \} P_{W_{R_n^{z,k}}} \left[ H_B(z, r_L) > T_B(z, r_k), H_B(y, r_L) > T_B(y, r_{k-2}) \right].
\]

We have that \( P_{W_{R_n^{z,k}}} \left[ H_B(z, r_L) > T_B(z, r_k) \right] = \frac{1}{1 - \frac{c}{L-k}} \) (recall (3.6) and \( W_{R_n^{z,k}} \in \partial B(z, r_{k+1}) \)), so that by Lemma 6.5

\[
P_{W_{R_n^{z,k}}} \left[ H_B(z, r_L) > T_B(z, r_k), H_B(y, r_L) > T_B(y, r_{k-2}) \right]
\leq \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) P_{W_{R_n^{z,k}}} \left[ H_B(y, r_L) > T_B(y, r_{k-2}) \right].
\]
Using this and the strong Markov property “in reverse” we have
\[ E_x \left[ CD|F^{x,k}_{R_x} \right] \]
\[ \leq \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) \left\{ 1 \left( D_{t-s}^{y,0} \leq R_x^{z,k} \right) + 1 \left( D_{t-s}^{y,0} \geq R_x^{z,k} \right) P_{W^{z,k}} \left[ H_B(y,r_L) > T_B(y,r_L-2) \right] \right\} \]
\[ \leq \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) \left[ 1 \left( D_{t-s}^{y,0} \leq R_x^{z,k} \right) \right] . \]

Using this with (6.30) and (6.31) yields (6.21) (note that \( B \) is \( F_{R_x^{z,k}} \)-measurable).

The above lemma gives the following corollary which fully “extracts” the cost of the part of (6.16) referring to \( z \).

**Corollary 6.7.** For any event \( J \) as in (6.19) we have that
\[ P_x \left[ J \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,0} \right\} \right] \]
\[ \leq ce^{-2(L-k)-cf(k)L(1+s)\left(1 - \frac{c}{L-k} \right)} P_x \left[ J \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,0} \right\} \right] . \]

**Proof.** The probability on the left hand-side is \( P_x \left[ A_{\gamma(k)^2} \right] \). Applying Lemma 6.6 recursively we have
\[ P_x \left[ A_{\gamma(k)^2} \right] \leq \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) \left[ \gamma(k)^2 \right] P_x \left[ J \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,0} \right\} \right] , \]

since \( A_0 = J \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,0} \right\} \). Now
\[ P_x \left[ A_{\gamma(k)^2} \right] \leq \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) \left[ \gamma(k)^2 \right] \leq ce^{-\gamma(k)^2} \leq ce^{-\gamma(k)^2 (1 - \frac{c}{L-k})} \leq ce^{-\gamma(k)^2} , \]

since \( \gamma(k)^2 \leq 2\beta(k)^2 = 2t_{-s} (1 - k/L)^2 \leq 4L^2 (1 - k/L)^2 \leq 4 (L - k)^2 \) (recall (4.12), (4.5) and (5.1)). Also
\[ \frac{t_{-s}}{L} \left( 1 - \frac{k}{L} \right) + cf(k) \leq \frac{\gamma(k)^2}{L - k} , \]

so that \( e^{-t_{-s}/L} = e^{-2L^1/s} \) (recall (3.14))
\[ \left( 1 - \frac{1}{L-k} \left( 1 - \frac{c}{L-k} \right) \right) \left[ \gamma(k)^2 \right] \leq ce^{-\gamma(k)^2} \leq ce^{-\gamma(k)^2 (1 - \frac{c}{L-k})} \].

Using (6.34) and (6.35) in (6.33) we obtain (6.32). □

We are now ready to prove the two point probability estimate for large \( k \) (we will see later that the event bounded below contains \( I_y \cap I_z \)). Recall the definition (6.15) of \( J_y^z \).

**Proposition 6.8.** If \( (1 - \frac{a}{10}) L \leq k < L - l_0 \) then
\[ P_x \left[ J_y^z \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,0} \right\} \cap \left\{ H_B(y,r_L) \geq D_{t-s}^{y,k} \right\} \right] \]
\[ \leq ce^{-\gamma(k)^2} \left( 1 - \frac{c}{L-k} \right) L^{2s} \sqrt{lo} \sqrt{g(k-2)} . \]

**Proof.** By Corollary 6.7 with \( J = J_y^z \) and Lemma 6.4 the probability in question is bounded by
\[ ce^{-2(L-k)-cf(k)L(1+s)\left(1 - \frac{c}{L-k} \right)} \times ce^{-\gamma(k)^2} \left( 1 - \frac{c}{L-k} \right) L^{2s} \sqrt{lo} \sqrt{g(k-2)} . \]

Thus (6.36) follows since \( (1 + s) \left( 1 - \frac{k}{L} \right) \leq s \) for \( k \geq \left( 1 - \frac{a}{10} \right) L \) and \( s \in (-1,1) \). □

We now turn to the bound for smaller \( k \).
6.2. Main bound: early branching. Here we consider the case \( l_0 < k \leq (1 - \frac{1}{10}) L \). It turns out that in this regime we can ignore the contribution from the barrier condition for \( l \leq k \). To deal with the condition for \( l \geq k \), we will need to decouple the contribution due to \( y \) and that due to \( z \). To do this we will need to “give ourselves a bit of space”, and we therefore define

\[
(6.37) \quad k^+ = k + \lceil 100 \log L \rceil,
\]

and let for \( v \in \{y, z\} \)

\[
(6.38) \quad J^v_i = \left\{ \gamma (l) \leq \sqrt{T_i^{y,t-s}} \leq \delta (l) \quad \text{for} \quad l = k^+, \ldots, L - l_0 \right\},
\]

be the barrier conditioned applied only for \( l \geq k^+ \). To obtain the two point bound for \( k \leq (1 - \frac{4}{10}) L \) we will bound the probability of

\[
(6.39) \quad J^y_i \cap \left\{ H_{B(y,r_l)} \geq D_{l-s}^{r_0} \right\} \cap J^z_i \cap \left\{ H_{B(z,r_l)} \geq D_{l-s}^{r_0} \right\},
\]

which we will see contains the event \( I_y \cap I_z \). When bounding \( J^v_i \) for \( v \in \{y, z\} \) we will compare the law of \( T_i^{y,t-s} \), \( l \geq k^+ \), conditioned on the other events of (6.39) to \( \mathbb{G}_a \) for \( \gamma (l)^2 \leq a \leq \delta (l)^2 \), so that we can apply the barrier crossing bound (6.4) for the law \( \mathbb{G}_a \).

As a first step in this direction we let, recalling the definition (2.3),

\[
(6.40) \quad X^i = X^i (v) = W(R_i(v,r_k,r_{k+1})) + D_i(v,r_k,r_{k+1}), i = 1, \ldots,
\]

be the excursions of \( W_t \) from \( \partial B (v, r_{k+1}) \) to \( \partial B (v, r_k) \). Let

\[
(6.41) \quad N = N (v) = \sup \left\{ n \geq 1 : D_n (v,r_k,r_{k+1}) < D_{l-s}^{r_0} \right\},
\]

be the number of excursions \( X^i \) that take place before time \( D_{l-s}^{r_0} \). Note that

\[
(6.42) \quad \sum_{i=1}^{N} T_i^{v,\infty} (X^i) = T_i^{v,t-s} \quad \text{for} \quad l \geq k^+,
\]

where \( T_i^{v,\infty} (X^i) \) counts traversals that take place during the excursion \( X^i \). Let

\[
J^v_{i,n} = \left\{ \gamma (l) \leq \sqrt{\sum_{i=1}^{n} T_i^{v,\infty} (X^i)} \leq \delta (l) \quad \text{for} \quad l = k^+, \ldots, L - l_0 \right\}, n \geq 0,
\]

and note that by (6.42) and (6.38)

\[
(6.43) \quad 1_{J^v_i} = 1_{J^v_{i,n}}.
\]

We thus aim to bound \( P_x \left[ J^v_{i,n} \right] \) and are therefore interested in the law of \( \sum_{i=1}^{n} T_i^{v,\infty} (X^i) \). This will be given by a Galton-Watson process with immigration: let \( \tilde{\mathbb{G}}_n \) denote the law such that \( (T_i)_{i \geq 0} \) is a critical branching process with \( T_{k-1} = 0 \) and immigration of \( n \) individuals in generations \( k, k+1, \ldots, k^+ \). That is,

\[
(6.44) \quad \text{let } \tilde{\mathbb{G}}_n \text{ be the law of } \left( \sum_{i=0}^{k^+} T_i^{p} \right)_{l \geq 0},
\]

where \( T_{k^+}^{k}, T_{k^+}^{k+1}, \ldots, T_{k^+}^{k^+} \) are iid with law \( \mathbb{G}_n \), and where we set \( T_i^{p} = 0 \) for \( l < p \).
To show that $\sum_{i=1}^{n} T_{l}^{v,\infty}(X_i)$ has this law the first step is the following lemma giving the law of an individual $T_{l}^{v,\infty}(X_i)$.

**Lemma 6.9.** For $v \in \{y, z\}$ and any $u \in \partial B(v, r_{k+1})$

the $P_u$ - law of $T_{l}^{v,\infty}(W_{S_{v,t} \wedge B(v,r_k)})_{l \geq k+}$ is $\mathcal{G}_1$.

**Proof.** Let

$$S_l = T_{B(v,r)}, l \geq 0,$$

and consider for $k \leq p \leq k^+$ the number of traversals at each scale which happen between $S_{p+1}$ and $S_p$,

$$T_{l}^{p} \overset{\text{def}}{=} T_{l}^{v,\infty}(W_{S_{p+1} \wedge S_p}), l \geq 0.$$

Since $T_{B(v,r_k)} = S_k > \ldots > S_{k^+} > S_{k+1} = 0$ we have

$$T_{l}^{v,\infty}(W_{S_{v,t} \wedge B(v,r_k)}) = \sum_{p=k}^{k^+} T_{l}^{p}.$$  

(6.45)

A proof similar to that of Lemma 3.2 shows that the law of $T_{l}^{p}$, is $\mathcal{G}_1$, and the strong Markov property shows that the $T_{l}^{p}$, $l \geq 0$, are independent. Thus the claim follows by (6.45) and the definition (6.44) of $\mathcal{G}_1$. \hfill \Box

From this we easily get the law of the sum $\sum_{i=1}^{n} T_{l}^{v,\infty}(X_i)$:

**Corollary 6.10.** ($v \in \{y, z\}$) We have

the $P_a$ - law of $\left(\sum_{i=1}^{n} T_{l}^{v,\infty}(X_i)\right)_{l \geq 0}$ is $\mathcal{G}_n$.

**Proof.** By the strong Markov property applied at times $R_i(v, r_k, r_{k+1}), i = 1, \ldots, n$, and Lemma 6.9 $(T_{l}^{v,\infty}(X_i))_{l \geq 0}$ are iid for $i = 1, \ldots, n$ with law $\mathcal{G}_1$. Thus clearly $\left(\sum_{i=1}^{n} T_{l}^{v,\infty}(X_i)\right)_{l \geq 0}$ has law $\mathcal{G}_n$ by the definition (6.44). \hfill \Box

We now provide a bound on the barrier crossing event corresponding to $J_{l}^{v,n}$ for the Galton-Watson law $\mathcal{G}_n$.

**Lemma 6.11.** For any $n \geq 0$ we have that

$$\mathcal{G}_n \left[ \gamma(l) \leq \sqrt{T_{l}} \leq \delta(l) \text{ for } l = k^+, \ldots, L - l_0|T_{L-1} = 0 \right] \leq c \frac{g(k^+)^{0.51}}{L - l_0 - k^+-1}.$$  

(6.46)

**Proof.** By definition of $\mathcal{G}_n$ the law of $(T_{l+k^+})_{l \geq 0}$ under $\mathcal{G}_n \left[ |T_{L-1} = 0, T_{k^+} = a \right]$ is the $\mathcal{G}_a \left[ |T_{L-k^+-1} = 0 \right]$ law of $(T_l)_{l \geq 0}$. Thus the probability in question is bounded above by

$$\sup_{\gamma(k)^{2} \leq \alpha \leq \delta(k)^{2}} \mathcal{G}_a \left[ \gamma(l) \leq \sqrt{T_{l-k^+}} \leq \delta(l) \text{ for } l = k^+, \ldots, L - l_0|T_{L-k^+-1} = 0 \right],$$

The required bound therefore follows by (6.4) with $k^+ \leq (1 - \frac{s}{10}) L$ in place of $k$. \hfill \Box

We now summarize our work so far for the regime $k \leq (1 - \frac{s}{10}) L$ in the form of a bound on the conditional probability of $J_{l}^{v,n}$. We will see that the conditioning essentially corresponds to conditioning on $\left\{ H_{B(v,r_L)} > D_{l-s}^{v,0} \right\}$. 
\textbf{Lemma 6.12.} \((v \in \{y, z\})\) For all \(n \geq 1,
\)
\begin{equation}
\sup_{n \geq 0} P_x \left[ J_{n}^{y,v} | H_{B(v,r_{L})} > D_{n}(v,r_{k},r_{k+1}) \right] \leq c \frac{g(k^+)^{10.51}}{L - l_{0} - k - 1}.
\end{equation}

\textit{Proof.} The event that we condition on in (6.47) can be rewritten as \(\{ \sum_{i=1}^{n} T_{L+1}^{w_{\infty}} (X^i) = 0 \}\). Therefore by Lemma 6.9 the probability in (6.47) equals that in (6.46), so that the required bound follows by Lemma 6.11 \(\Box\)

The above lemma will be used to give the contribution from \(J_{y,v}^{+}\) and \(J_{z,v}^{+}\) to our bound on the probability of (6.39). These contributions should be roughly independent, but to obtain a rigorous bound we will need a decoupling. Our approach is inspired by Lemma 7.4 \[11\]. The first step in obtaining the decoupling is the next lemma which essentially speaking shows that the exit distribution of Brownian motion from a ball (both unconditioned and conditioned to avoid a smaller ball) does not depend much on the starting point, as long as the starting point is not close to the boundary.

\textbf{Lemma 6.13.} \((v \in \{y, z\})\) Let \(\lambda\) be the uniform distribution on \(\partial B(v,r_{k})\). For any \(u \in \partial B(v,k^+) \cup \partial B(v,k^++1)\) and measurable \(B\),
\begin{equation}
P_u \left[ W_{H_{B(v,r_{k})}} \in B \right] = (1 + O(L^{-100})) \lambda (B),
\end{equation}
and for any \(u \in \partial B(v, k^+)\),
\begin{equation}
P_u \left[ W_{H_{B(v,r_{k})}} \in B | H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] = (1 + O(L^{-99})) \lambda (B).
\end{equation}

\textit{Proof.} A classical result on the harmonic measure of Brownian motion says that for \(R > 0\) and \(u \in B(0,R) \subset \mathbb{R}^2\)
\begin{equation}
P_u^{\mathbb{R}^2} \left[ W_{H_{B(0,R)}} \in \partial \right] = R^2 - |u|^2 \tilde{\lambda} (db),
\end{equation}
where \(\tilde{\lambda}\) is the uniform distribution on \(\partial B(0,R)\) (see Theorem 3.43 \[23\]). With \(R = r_{k}\) and \(|u| = r_{k^+}\) or \(|u| = r_{k++1}\) this implies (6.48), since \(r_{k+1}/r_{k} \leq c L^{-100}\) (by 3.4) and (6.37), and using also that \(B(0,r_{k}) \subset \mathbb{R}^2\) can be identified with \(B(v,r_{k}) \subset \mathbb{T}\); see (2.1). To get (6.49) note that \(P_u \left[ W_{H_{B(v,r_{k})}} \in B, H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] \)
eq \(P_u \left[ W_{H_{B(v,r_{k})}} \in B, H_{B(v,r_{k+1})} < T_{B(v,r_{k})} \right] \).

The first term equals \((1 + O(L^{-100})) \lambda (B)\) by (6.48). Also by the strong Markov property applied at time \(H_{B(v,r_{k+1})}\) and (6.48) the second term equals
\begin{equation}
P_u \left[ H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] = O(L^{-100}) \lambda (B).
\end{equation}
Thus \(P_u \left[ W_{H_{B(v,r_{k})}} \in B, H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] \)
eq \(P_u \left[ H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] \).

But by (3.6)
\begin{equation}
P_u \left[ H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] = \frac{1}{k^+ + 1 - k} \geq L^{-1},
\end{equation}
so that in fact \(P_u \left[ W_{H_{B(0,R)}} \in B, H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] \)
eq \(\lambda (B) P_u \left[ H_{B(v,r_{k+1})} > T_{B(v,r_{k})} \right] (1 + O(L^{-99}))\).
This gives (6.49).

As a step in the “decoupling” of \( J^+ \) and \( J^- \) we will now use the previous lemma to show that the part of an excursion from \( \partial B(v, r_{k+1}) \) to \( \partial B(v, r_k) \) that takes place within \( B(v, r_{k+1}) \) is almost independent from the end point of the excursion. This will be used to show that \( J^+ \), when conditioned to avoid \( B(y, r_L) \), is almost independent of the parts of \( J^- \) that refer to \( z \) (note that \( J^\pm \) only depends on the parts of the excursions that take place in \( B(y, r_{k+1}) \)), and vice versa with \( y \) and \( z \) swapped.

To this end, let

\[
S = S(v) = \sup \left\{ D_{n,k}^\pm : D_{n,k}^\pm < T_{B(v,r_k)} \right\},
\]

be the last excursion from scale \( k^+ + 1 \) to scale \( k^+ \) before \( T_{B(v,r_k)} \) ends. For \( a \in \partial B(v, r_{k+1}) \) and \( b \in \partial B(v, r_k) \) let

\[
(6.51) \quad \mu_{a,b} [\cdot] = P_a \left[ W_{\Lambda T_{B(v,r_k)}} \in \cdot | W_{T_{B(v,r_k)}} = b \right],
\]

be the law of an excursion starting in \( a \) conditioned to end in \( b \). Let

\[
(6.52) \quad \tilde{\mu}_{a,b} [\cdot] = \mu_{a,b} [\cdot | H_{B(v,r_L)} > T_{B(v,r_k)}] = P_a \left[ W_{\Lambda T_{B(v,r_k)}} \in \cdot | H_{B(v,r_L)} > T_{B(v,r_k)}, W_{T_{B(v,r_k)}} = b \right],
\]

be the law of an excursion conditioned to end in \( b \) and avoid \( B(v, r_L) \), and let

\[
(6.53) \quad \tilde{\mu}_a [\cdot] = P_a \left[ W_{\Lambda T_{B(v,r_k)}} \in \cdot | H_{B(v,r_L)} > T_{B(v,r_k)} \right],
\]

be the law of an excursion avoiding \( B(v, r_L) \), without conditioning on the end point. The result says that:

**Lemma 6.14.** \((v \in \{y,z\})\) For any \( a \in \partial B(v, k^+ + 1) \), \( w \in B(v, r_k) \) we have

\[
(6.54) \quad \tilde{\mu}_{u,w} [W_{\Lambda S} \in \cdot] = \tilde{\mu}_u [W_{\Lambda S} \in \cdot] \left( 1 + O(L^{-99}) \right).
\]

**Proof.** We will show that

\[
(6.55) \quad \mu_{u,w} [W_{\Lambda S} \in \cdot] = \left( 1 + O(L^{-99}) \right) \mu_{u,w} [W_{\Lambda S} \in \cdot].
\]

The claim then follows, since the left-hand side of (6.54) equals

\[
(6.56) \quad \frac{\mu_{u,w} [W_{\Lambda S} \in \cdot : H_{B(v,r_L)} > T_{B(v,r_k)}]}{\mu_{u,w} [H_{B(v,r_L)} > T_{B(v,r_k)}]},
\]

so that we can apply (6.55) to the denominator and numerator of (6.56) (note that \( H_{B(v,r_L)} > T_{B(v,r_k)} \) is \( W_{\Lambda S} \)-measurable) to get that (6.56) equals

\[
\frac{P_w [W_{\Lambda S} \in \cdot : H_{B(v,r_L)} > T_{B(v,r_k)}]}{P_w [H_{B(v,r_L)} > T_{B(v,r_k)}]} \left( 1 + O(L^{-99}) \right),
\]

which equals the right-hand side of (6.54).

To show (6.55) we note that \( P_u [W_{\Lambda S} \in A, S = D_{n,k}^{\pm}, W_{T_{B(v,r_k)}} \in B] \) equals

\[
P_u [W_{\Lambda D_{n,k}^{\pm}} \in A, T_{B(y,r_k)} \circ \theta_{D_{n,k}^{\pm}} < H_{B(y,r_{k+1})} \circ \theta_{D_{n,k}^{\pm}}, W_{T_{B(y,r_k)}} \in B].
\]

By the strong Markov property this probability can be written as

\[
(6.57) \quad P_u [W_{\Lambda D_{n,k}^{\pm}} \in A, P_{W_{D_{n,k}^{\pm}}} [T_{B(v,r_k)} < H_{B(y,r_{k+1})}, W_{T_{B(y,r_k)}} \in B]].
\]
But
\[
P_{\mathcal{W}^{v,k}_n} \left[ T_{B(v,r_k)} < H_{B(v,r_{k+1})} \right] P_{\mathcal{W}^{v,k}_n} \left[ W_{T_{B(v,r_k)}} \in B \right]
\]
\[
= P_{\mathcal{W}^{v,k}_n} \left[ W_{T_{B(v,r_k)}} \in B \mid T_{B(v,r_k)} < H_{B(v,r_{k+1})} \right] P_{\mathcal{W}^{v,k}_n} \left[ T_{B(v,r_k)} < H_{B(v,r_{k+1})} \right],
\]
by (6.49). Thus the probability (6.57) equals
\[
(1 + O(L^{-99})) \lambda(B) P_{\mathcal{W}^{v,k}_n} \left[ T_{B(v,r_k)} < H_{B(v,r_{k+1})} \right],
\]
and we get that
\[
P_u \left[ W_{\wedge A,S} \in A, S = D^{g,k}_n \right].
\]

Thus summing over \( n \) we obtain
\[
P_u \left[ W_{\wedge A,S} \in A, W_{T_{B(y,r_k)}} \in B \right] = (1 + O(L^{-99})) \lambda(B) P_u \left[ W_{\wedge A,S} \in A \right].
\]
Using (6.48) this gives
\[
P_u \left[ W_{\wedge A,S} \in A, W_{T_{B(y,r_k)}} \in B \right] = (1 + O(L^{-99})) P_u \left[ W_{T_{B(y,r_k)}} \in B \right] P_u \left[ W_{\wedge A,S} \in A \right],
\]
from which (6.55) follows (recall (6.51)). \( \square \)

We now prove a bound that deals with the contribution from the event \( J^i_u \), even when conditioned on “what goes on outside \( B(v,r_k) \)”. To this end let
\[
Y^i(v) = W_{(D_i(v,r_k,r_{k+1}))} A_{r+1}(v,r_{k+1}), i \geq 1,
\]
be the excursions from \( \partial B(v,r_{k+1}) \) to \( \partial B(v,r_k) \). Define the \( \sigma \)-algebra
\[
\mathcal{G} = \sigma(W_{\wedge A_i(v,r_k,r_{k+1}), Y^i : i \geq 1}).
\]
The bound says that:

**Proposition 6.15.** For any \( l_0 < k < L - l_0 \) and \( v \in \{ y, z \} \) we have that
\[
P_x \left[ J^i_v \cap \left( H_{B(v,r_k)} \geq D_{t_{-2}}^{v,0} \right) \right] \mathcal{G} \leq c \frac{g(k)}{L - l_0 - k - 1} P_x \left[ H_{B(v,r_k)} \geq D_{t_{-2}}^{v,0} \right].
\]

**Proof.** Recall the definitions (6.41) of \( N = N(v) \) and (6.40) of \( X^i \). We have that
\[
\{ H_{B(v,r_k)} \geq D_{t_{-2}}^{v,0} \} = \{ H_{B(v,r_k)} (X^i) \geq T_{B(v,r_k)} (X^i), i = 1, \ldots, N \} \text{ def } = A.
\]
Also \( N \leq T_{k+1}^{q,k} \) since each excursion \( X^i \) contains at least one traversal \( k^+ \rightarrow k^+ + 1 \) (recall (3.8)). Thus on the event \( J^i_v \) (see (6.38)) we have
\[
N(v) \leq \delta^2 (k^+) \leq 2L^2, \text{ by (3.12), (4.5), (5.3).}
\]
Let \( \rho_u, u \in B(v,r_k) \) denote the map that rotates \( B(v,r_k) \subset T \) around \( v \) so that \( u \) lies on the same horizontal line as \( v \), and for any path \( w \in C(\mathbb{R}_+, T) \) let \( \rho (w) = (\rho_{w_0} (w_t))_{t \geq 0} \).
We have that $T_{l}^{y,\infty} (X^{i}) = T_{l}^{y,\infty} \left( \rho \left( X^{i}_{\omega,S} \right) \right)$ (recall (6.23)), and therefore $J_{v}^{i}$ only depends only on

$$\bar{X}^{i} = \rho \left( X^{i}_{\omega,S} \right), \ i = 1, \ldots, n,$$

so that there exists a family of measurable functions $f_{1}, f_{2}, \ldots$, such that

$$1_{J_{v}^{i} \cap \{ H_{B(v,r_{L})} \geq D_{l-s}^{v,0} \}} = f_{n} \left( \bar{X}^{1}, \ldots, \bar{X}^{n} \right).$$

We thus have (recall (6.43) and (6.59))

$$P_{x} \left[ J_{v}^{i} \cap \{ H_{B(v,r_{L})} \geq D_{l-s}^{v,0} \} \right] = E_{x} \left[ f_{n} \left( X^{1}, \ldots, X^{N} \right) 1_{\mathcal{A}} | \mathcal{G} \right].$$

Now under $P_{x} \left[ \cdot | \mathcal{G} \right]$ the $X^{i}, i = 1, \ldots, N$, are independent and $X^{i}$ has the law $\mu_{X^{i}_{0}, X^{i}_{\infty}}$ from (6.51) (note that $X^{i}_{0}$ and $X^{i}_{\infty}$ are $\mathcal{G}$–measurable). Thus (6.61) in fact equals

$$1_{\{ N \leq 2L^{2} \}} \otimes_{i=1}^{N} \mu_{X^{i}_{0}, X^{i}_{\infty}} \left[ f_{n} \left( \bar{Z}^{1}, \ldots, \bar{Z}^{N} \right) 1_{\{ H_{B(v,r_{L})} > T_{B(y,r_{k})} \}} \right],$$

where the vector $(Z^{1}, \ldots, Z^{N})$ has law $\otimes_{i=1}^{N} \mu_{X^{i}_{0}, X^{i}_{\infty}}$ and

$$\bar{Z}^{i} = \rho \left( Z^{i}_{\omega,S} \right).$$

Now (recall (6.51) and (6.52))

$$\mu_{a,b} \left[ 1_{\{ H_{B(v,r_{L})} > T_{B(y,r_{k})} \}} \right] = \tilde{\mu}_{a,b} \left[ H_{B(v,r_{L})} > T_{B(y,r_{k})} \right],$$

so that in fact (6.62) equals

$$1_{\{ N \leq 2L^{2} \}} \left( \otimes_{i=1}^{N} \tilde{\mu}_{X^{i}_{0}, X^{i}_{\infty}} \left[ f_{n} \left( \bar{Z}^{1}, \ldots, \bar{Z}^{N} \right) \right] \right) P_{x} \left[ H_{B(v,r_{L})} \geq D_{l-s}^{v,0} | \mathcal{G} \right].$$

Using (6.54) together with $(1 + cL^{-99})2L^{2} \leq c$ this is bounded above by

$$\sup_{n \geq 0} \left( \otimes_{i=1}^{n} \tilde{\mu}_{u} \left[ f_{n} \left( \bar{Z}^{1}, \ldots, \bar{Z}^{n} \right) \right] \right) P_{x} \left[ H_{B(v,r_{L})} \geq D_{l-s}^{v,0} | \mathcal{G} \right],$$

for an arbitrary $u \in \partial B \left( v, r_{k+1} \right)$ (the law of $\rho \left( W_{\omega,S} \right)$ under $\tilde{\mu}_{u}$ is independent of $u$, see (2.3)). Now consider the law of $(\bar{X}^{1}, \ldots, \bar{X}^{n})$ under $P_{x} \left[ \cdot | H_{B(v,r_{k})} > D_{n} (v, r_{k}, r_{k+1}) \right]$. By the strong Markov property and the rotational invariance (2.3) this vector is iid with law $\tilde{\mu}_{u}$. Thus we have that

$$\otimes_{i=1}^{n} \tilde{\mu}_{u} \left[ f_{n} \left( \bar{Z}^{1}, \ldots, \bar{Z}^{n} \right) \right] = P_{x} \left[ f_{n} \left( \bar{X}^{1}, \ldots, \bar{X}^{n} \right) | H_{B(v,r_{L})} > D_{n} (v, r_{k}, r_{k+1}) \right]$$

$$= P_{x} \left[ J_{v}^{i} \cap \{ H_{B(v,r_{L})} > D_{n} (v, r_{k}, r_{k+1}) \} \right] \leq c \frac{(g^{+})^{0.51}}{L^{-L_{0}-k-1}},$$

by Lemma 6.12. Combining this with (6.61) and (6.64) gives the claim. \hfill \Box

We are now ready to prove the two point probability estimate for small $k$.

**Proposition 6.16.** If $k \leq (1 - \frac{3}{10}) L$ then

$$P_{x} \left[ J_{y}^{i} \cap \{ H_{B(y,r_{L})} \geq D_{l-s}^{v,0} \} \right] \leq c \left( \gamma \left( k \right) \leq \sqrt{\frac{L_{k}}{T_{k}^{y,\infty}}}, H_{B(z,r_{L})} \geq D_{l-s}^{v,0} \right) \leq c \left( s \right) e^{-(4L-2k)-(c(k) L^{2} a_{1}^{0.02} \left( g^{+} \right)^{2}}.$$
Proof. We first use Proposition 6.15 with \( v = y \). Note that with this choice of \( v \) we have
\[
J_z^v \cap \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}} , H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\} \in \mathcal{G} (y),
\]
since these events depend only “on what goes on outside \( B(y, r_L) \)” (recall (6.14) and (6.38)). Thus it follows by Proposition 6.15 that the probability in (6.66) is bounded above by
\[
c \frac{g (k^+)}{L - l_0 - k - 1} P_x \left[ \left\{ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\} \cup J_z^v \cap \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}} , H_{B(z, r_L)} \geq D^{0, 0}_{t, s} \right\} \right].
\]
Next we let \( v = z \) in Proposition 6.15. We now have that
\[
\left\{ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\} \cap \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}} , H_{B(z, r_L)} \geq D^{0, 0}_{t, s} \right\} \in \mathcal{G} (z),
\]
so that again by Proposition 6.15 the probability in (6.66) is bounded above by
\[
\left( \frac{c \frac{g (k^+)}{L - l_0 - k - 1} }{L - l_0 - k - 1} \right)^2 P_x \left[ \left\{ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\} \cap \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}} , H_{B(z, r_L)} \geq D^{0, 0}_{t, s} \right\} \right].
\]
Now \( \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}}, H_{B(z, r_L)} \geq D^{0, 0}_{t, s} \right\} \subset \{ H_{B(z, r_L)} \geq D^{2, k}_{\gamma (k)^2} \} \) (recall (3.7) and (3.8)) so that by Corollary 6.7 with \( J = C(\mathbb{R}_+, \mathbb{T}) \) the probability in (6.66) is at most
\[
(6.67) \left( \frac{c \frac{g (k^+)}{L - l_0 - k - 1} }{L - l_0 - k - 1} \right)^2 \times c e^{-2(L-k) - cf(k) L^{1+s}} \times P_x \left[ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right].
\]
The latter probability is bounded above by \( c e^{-2L L^{1+s}} \) by Lemma 3.4 so (6.67) simplifies to the right-hand side of (6.66) by noting that \( L - l_0 - k - 1 \geq c(s) L \). \( \square \)

We can now finish the section by deriving the two point probability estimate Proposition 3.6 using Proposition 6.8 and Proposition 6.16.

Proof of Proposition 3.6. We have (recall (3.8), (3.9), (5.6), (6.15) and (6.38))
\[
I_y \subset J_y^v \cap \left\{ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\} \text{ and } I_y \subset J_y^v \cap \left\{ H_{B(y, r_L)} \geq D^{0, 0}_{t, s} \right\}.
\]
Similarly
\[
I_z \subset \left\{ H_{B(z, r_L)} \geq D^{2, k}_{\gamma (k)^2} \right\} \text{ and } I_z \subset J_z^v \cap \left\{ \gamma (k) \leq \sqrt{T_k^{s, L}}, H_{B(z, r_L)} \geq D^{0, 0}_{t, s} \right\}.
\]
Thus by Proposition 6.8 and Proposition 6.16 we have
\[
P_x [I_y \cap I_z] \leq \left\{ \begin{array}{ll}
c e^{-2(L-k) - cf(k) L^{1+s}} \sqrt{l_0 g (k - 2)} & \text{if } k \geq (1 - \frac{s}{10}) L, \\
c (s) e^{-2(L-k) - cf(k) L^{1+s}} (g (k^+))^2 & \text{if } k \leq (1 - \frac{s}{10}) L.
\end{array} \right.
\]
Thus (5.23) follows, since \( g (k - 2) \leq cg (k) \) and \( g (k^+) \leq cg (k) (\log L)^{0.51} \). \( \square \)

This also completes the proof the lower bound Proposition 3.6 modulo the barrier crossing results Lemma 5.4 and Lemma 6.3 which we have as of yet only stated. Recall that the proof of the upper bound Proposition 3.5 was also completed in Section 4 modulo the barrier crossing result Lemma 4.1. The next section gives the proof of these results.
7. Barrier estimate proofs

In this section we prove the barrier crossing estimates Lemma 4.1, Lemma 5.1 and Lemma 6.3 for the Galton-Watson process \((T_l)_{l \geq 0}\) that were crucial in the proofs of the upper bound Proposition 3.3 in Section 4 and the lower bound Proposition 3.6 in sections 5 and 6.

The kind of barrier bounds we need appear in the literature for the Brownian bridge process (indeed they are an integral part of the analysis of branching Brownian motion that provides the inspiration for the proof of our main result, see [2,6,18]). Our approach is to derive the needed bounds for the Galton-Watson process from these Brownian bridge results, via a comparison to the Bessel bridge. Roughly speaking the squared Bessel is to derive the needed bounds for the Galton-Watson process from these Brownian bridge that provides the inspiration for the proof of our main result, see 

For the first step we exploit that \((T_l)_{l \geq 0}\) is the law of the discrete edge local times of a random walk on the path \(\{0,1,\ldots,L\}\), while the law of the continuous local times of the vertices is a squared Bessel process of dimension one. For the second step we use an explicit expression for the Radon-Nikodym derivative of law of the squared Bessel bridge.

For convenience, let us now restate Lemma 4.1, Lemma 5.1 and Lemma 6.3 as one proposition. Recall first the definitions of \(t_s = t_s (L)\) from (3.12) and of the straight line \(\beta (l)\) from (1.5) (giving, roughly speaking, the mean of the \(T_l\) when \(T_0 = t_s\) and conditioned on \(T_{L-1} = 0\)). Also recall the definitions of the barriers \(\alpha (l), \gamma (l)\) and \(\delta (l)\) from (4.4), (5.1), and (5.3) and the cut-off \(l_0 = l_0 (L)\) from (5.5) (see also Figure 7.1 on page 43). In the interest of brevity we introduce the following notation. For any \(I \subset [0,T]\) and function \(\eta : [0,T] \rightarrow \mathbb{R}\) we let \(B_\eta (I)\) denote the event that a process is above \(\eta (t)\) for all \(t \in I\). We let \(B^{0} (I)\) denote the event that a process is below \(\eta (t)\) for all \(t \in I\). For two functions \(\eta\) and \(\psi\) we let \(B^{\eta} \psi (I) = B_\eta (I) \cap B_\psi (I)\). With this notation, we can now restate Lemma 4.1 as (7.1), Lemma 5.1 as (7.2) and Lemma 6.3 as (7.3)-(7.4).

**Proposition 7.1.** For all \(L \geq 1\) and \(s \in (-100,100)\)

\[
\begin{align*}
G_{t_s} [B_{\alpha^2} (\{0,\ldots,L-1\}) | T_{L-1} = 0] & \leq c \frac{(\log L)^4}{L}, \\
G_{t_s} [B_{\gamma^2} (\{l_0,\ldots,L-l_0\}) | T_{L-1} = 0] & \leq l_0 \frac{L}{L}, \\
G_{t_s} [B_{\delta^2} (\{l_0,\ldots,k\}) | T_{L-1} = 0] & \leq \frac{\sqrt{\log (k)}}{k-l_0}.
\end{align*}
\]

If also \(l_0 + 1 < k < L - l_0\) then

\[
G_{t_s} [B_{\gamma^2} (\{l_0,\ldots,k\}) | T_{L-1} = 0] \leq \frac{\sqrt{\log (k)}}{k-l_0}.
\]

If \(l_0 < k < L - l_0 - 1\) and \(\gamma (k)^2 \leq a \leq \delta (k)^2\) then

\[
G_{t_s} [B_{\gamma^2} (\{0,\ldots,L-k-l_0\}) | T_{L-1-k} = 0] \leq \frac{c \gamma (k)^2 \gamma (k+1)}{L-k-l_0 - 1}.
\]

We start the proof of Proposition 7.1 by recalling and proving some barrier crossing bounds for the Brownian bridge. To state these we let \(P_x, x \in \mathbb{R}\), be the law on \((C(\mathbb{R}_+,\mathbb{R}), \mathcal{B}(\mathbb{R}_+,\mathbb{R}))\) which turns \(X_t, t \geq 0\), into a standard Brownian motion starting
In the proof of the upper bound Proposition 3.5 one shows that with high probability there is no point $y \in F_L$ such that $T_{L-1}^{y,ts} = 0$ and $\sqrt{T_{L}^{y,ts}}$ stays above $\alpha(l)$. In the proof of lower bound Proposition 3.6 one shows that with high probability there is a point $y \in F_L$ such that $T_{L-1}^{y,ts} = 0$ and $\sqrt{T_{L}^{y,ts}}$ stays in the aforementioned tube. This is done using bounds on the probability that $\sqrt{T_{L}}$ starting at $T_0 = t_s$ stays above $\alpha$, when conditioned on $T_{L-1} = 0$, and bounds on the probability that this process stays in the tube. Note that $\beta(l)$ is roughly speaking the mean of the conditioned process. (See Lemma 4.1, Lemma 5.1, and Proposition 7.1).

at $x \in \mathbb{R}$. For $T > 0$ and $a, b \in \mathbb{R}$ we write $\mathbb{P}_a \to_b^T$ for the law of Brownian bridge on $(C_0([0,T],\mathbb{R}),\mathcal{B}([0,T],\mathbb{R}))$ starting at $a \in \mathbb{R}$ and ending in $b \in \mathbb{R}$ at time $T$, that is

$$
\mathbb{P}_a \to_b^T[\cdot] \equiv \mathbb{P}_a[\cdot|X_T = b].
$$

Equivalently, $\mathbb{P}_a \to_b^T$ is the law of the Gaussian process on $[0,T]$ with

$$
E_x[X_t] = h(t) \quad \text{and} \quad \text{Cov}[X_t, X_s] = \frac{t(T-s)}{T} \quad \text{for } 0 \leq s \leq t \leq T,
$$

where $h$ is the linear function with $h(0) = a$ and $h(T) = b$. Recall that shifting a Brownian bridge by a linear function results in a Brownian bridge with a shifted starting and ending point, that is

$$
\mathbb{P}_a \to_b^T \text{ law of } X_t + h(t) \text{ is } \mathbb{P}_{a+h(0)} \to_{b+h(T)}^T \text{ for any linear } h : [0,T] \to \mathbb{R} \text{ and } a, b \in \mathbb{R}.
$$

We now recall some barrier estimates from the literature. The probability that Brownian bridge stays above (or below) a linear barrier throughout its lifetime can be explicitly
computed using the reflection principle; we have

\[ \mathbb{P}^T_{0\to 0} [B_h ([0, T])] = 1 - \exp \left( -\frac{2ab}{T} \right), \]  

for all \( T, a, b > 0 \) where \( h(t) \) is the linear function such that \( h(0) = -a < 0 \) and \( h(b) = -b < 0 \) (see Proposition 3 [26]). For a linear barrier that is “checked” only at integer times we have the following bound

\[ \mathbb{P}^T_{0\to 0} [B_h ([0, T] \cap \mathbb{N})] \leq c \frac{1 + a}{T} \frac{1 + b}{T - t_1 - t_2}, \]

for all \( T, a, b > 0 \) (see Lemma 6.2 [29]). Note that for \( T \) much larger than \( a \) and \( b \), the right-hand side of (7.7) and the right-hand side of (7.8) have the same order. Also note that (7.7) is trivially a lower bound for the probability in (7.8).

For a linear barrier \( h \) which is “checked” only during the interval \([t_1, T - t_2]\) for \( t_1 + t_2 < T \) we have the upper bound

\[ \mathbb{P}^T_{0\to 0} [B_h ([t_1, T - t_2])] \leq \frac{(a + \sqrt{t_1})(b + \sqrt{t_2})}{T - t_1 - t_2}, \]

where \( h(t_1) = -a \) and \( h(t_2) = -b \) (see Lemma 3.4 [2]). We now adapt the proof of (7.9) to give a version of that result for a barrier checked only at integer times.

**Lemma 7.2.** For any \( T > 0 \) and \( t_1, t_2 \geq 0 \) such that \( t_1 + t_2 < T \) and any \( a, b > 0 \)

\[ \mathbb{P}^T_{0\to 0} [B_h ([t_1, T - t_2] \cap \mathbb{N})] \leq c \frac{(c + a + \sqrt{t_1})(c + b + \sqrt{t_2})}{T - t_1 - t_2}, \]

where where \( h(t) \) is the linear function such that \( h(t_1) = -a \) and \( h(T - t_2) = -b \).

**Proof.** We may condition on \( X_{t_1}, X_{T-t_2} \) to get that the left-hand side of (7.10) equals

\[ \mathbb{P}^T_{0\to 0} \left[ \mathbb{E}^T_{X_{t_1} \to X_{T-t_2}} [B_h(t_1+,) ([0, T - t_1 - t_2] \cap \mathbb{N})] \right], \]

Now by (7.6) and (7.8) we have for \( u, v \in \mathbb{R} \),

\[ \mathbb{P}^T_{u \to v \to t_2} [B_h(t_1+,) ([0, T - t_1 - t_2] \cap \mathbb{N})] \leq c \frac{(1 + |u + a|)(1 + |v + b|)}{T - t_1 - t_2}. \]

We have (see (7.3))

\[ \text{Var} [X_t] = \text{Var} [X_{T-t}] = \frac{t(T-t)}{T}, \]

so that \( \mathbb{P}^T_{0\to 0} [|X_{t_1} + a|] \leq c \sqrt{t_1} + a, \mathbb{P}^T_{0\to 0} [|X_{T-t_2} + b|] \leq c \sqrt{t_2} + b \), and by Hölder’s inequality

\[ \mathbb{P}^T_{0\to 0} [|X_{t_1} + a| X_{T-t_2} + b|] \leq \sqrt{\mathbb{P}^T_{0\to 0} [X_{t_1} + a]^2} \mathbb{P}^T_{0\to 0} [X_{T-t_2} + b]^2 \leq \left( \sqrt{t_1 + a^2} \right) \left( \sqrt{t_2 + b^2} \right) \leq (\sqrt{t_1} + a) (\sqrt{t_2} + b). \]

Thus (7.10) follows by plugging (7.12) into (7.11) and taking the expectation. \[\square\]

Now consider the non-linear barrier \( h_\delta : [0, T] \to \mathbb{R} \) given by \( h_\delta (t) = \min \left( t^\delta, (T-t)^\delta \right) \) (note that with \( T = L \) we have \( f = h_{0.49} \) and \( g = h_{0.51} \), see (5.2) and (5.4)). Bramson shows that

\[ \mathbb{P}^T_{0\to 0} [B_{h_\delta} ([t, T - t]) | B_0 ([t, T - t])] \to 1 \text{ as } t \to \infty, \text{ for } \delta < \frac{1}{2}, \]
uniformly in $T$ (see Proposition 6.1 [3]) and

\begin{equation}
\mathbb{P}_{0 \to 0}^T \left[ B^{h_5} ([t, T - t]) | B_0 ([t, T - t]) \right] \to 1 \text{ as } t \to \infty, \text{ for } \delta > \frac{1}{2},
\end{equation}

uniformly in $T$ (see Lemma 2.7 [3]). Intuitively, (7.14) and (7.15) indicate that when conditioned on $B_0 ([t, T - t])$ Brownian bridge stays close to $h_{0.5}$. Also, they can be used to give the following lower bound on the probability that Brownian bridge manages to stay in a “tube” $[h_{1/2 - c}, h_{1/2 + c}]$ for small $c$, which will be needed for the lower bound of (7.2). For technical reasons related to how we later apply the result we let the starting point of Brownian bridge deviate somewhat from 0, and require that it also stays above $-\frac{T}{10000}$ during the initial time interval $[0, t]$.

**Lemma 7.3.** For any $T > 0$, $v \in (-1000, 1000)$ and $\frac{T}{v} > t \geq c$ we have that

\begin{equation}
\frac{c}{T} \leq \mathbb{P}_{v \to 0}^T \left[ B^{h_{0.501}}_\delta ([t, T - t]) \cap B_{-\frac{T}{10000}} ([0, t]) \right].
\end{equation}

**Proof.** Let $I = [t, T - t]$. We will show that

\begin{equation}
\frac{c}{T} \leq \mathbb{P}_{0 \to 0}^T \left[ B^{h_{0.501}}_\delta (I) \cap B_{-\frac{T}{10000}} ([0, t]) \right].
\end{equation}

This implies (7.16), since even if we shift the process and the barriers by $s \to v \frac{T}{T}$ (see (7.6)) we still have for $s \in I$ and $T$ and $t$ large enough

\begin{equation}
-\frac{T}{10000} \leq -\frac{T}{20000} + \frac{T - s}{T} \text{ and,}
\end{equation}

\begin{equation}
h_{0.499} (s) \leq h_{0.4999} (s) + v \frac{T - s}{T} \leq h_{0.5001} (s) + v \frac{T - s}{T} \leq h_{0.501} (s).
\end{equation}

From (7.7) we have that

\begin{equation}
\mathbb{P}_{0 \to 0}^T [B_0 (I)] = \mathbb{P}_{0 \to 0}^T \left[ \left( 1 - e^{-\frac{2X_t X_{T-t}}{T}} \right) 1_{\{X_t, X_{T-t} \geq 0\}} \right].
\end{equation}

Since $1 - e^{-x} \geq x/2$ for $x \in [0, 1]$ we thus have

\begin{equation}
\mathbb{P}_{0 \to 0}^T [B_0 (I)] \geq c \frac{T}{T - 2t} \mathbb{P}_{0 \to 0}^T \left[ \sqrt{t} \geq X_t, X_{T-t} \geq \frac{1}{1000} \sqrt{t} \right] \geq \frac{c \frac{T}{T - 2t}}{2},
\end{equation}

where in the last step we have used (7.13) and $\text{Cov} [X_t, X_{T-t}] = \frac{t^2}{2} > 0$ (see (7.5)).

Now since $X_t \geq 0$ on $B_0 (I)$ we have $\mathbb{P}_{0 \to 0}^T \left[ B_{-\frac{1}{20000}T} ([0, t]) | B_0 (I) \right] \geq \mathbb{P}_{0 \to 0}^T \left[ B_{-\frac{1}{20000}T} ([0, t]) \right]$, which equals $1 - e^{-\frac{2x^2}{T}}$ by (7.7). Thus

\begin{equation}
\mathbb{P}_{0 \to 0}^T \left[ \left( B_{-\frac{1}{20000}T} ([0, t]) \right) \cap B_0 (I) \right] \leq \frac{1}{4} \text{ for } t \geq c.
\end{equation}

Also by (7.14) and (7.15) we have for $t \geq c$,

\begin{equation}
\mathbb{P}_{0 \to 0}^T [B^{h_{0.4999}}_\delta (I) \cap B_0 (I)] \leq \frac{1}{4} \text{ and } \mathbb{P}_{0 \to 0}^T [B^{h_{0.5001}}_\delta (I) \cap B_0 (I)] \leq \frac{1}{4}
\end{equation}

Now using (7.20), (7.21) and a union bound we have that

\begin{equation}
\mathbb{P}_{0 \to 0}^T \left[ B_{-\frac{1}{20000}T} ([0, t]) \cap B^{h_{0.5001}}_\delta (I) | B_0 (I) \right] \geq \frac{3}{4},
\end{equation}

when $t \geq c$. Thus the claim (7.17) follows from (7.19). \qed
To use these results to prove Proposition 7.1 we must now compare the law of the conditioned Galton-Watson process to the law of a Brownian bridge. As mentioned above, this will go via squared Bessel bridges. Let us introduce the necessary notation. We let $Q_{x,0}^d, d \geq 0, x \geq 0$, be the law on $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}([0, \infty)))$ which turns $X_t, t \geq 0$, into a $d$–dimensional squared Bessel processes starting at $x$ (see Chapter XI.1 [24]). Recall that these are non-negative processes. For $d \geq 0, T > 0, a, b \in \mathbb{R}$ we denote the law of a $d$–squared Bessel bridge on $(C_0([0, T]), \mathcal{B}([0, T], \mathbb{R}))$ starting at $a$ and ending in $b$ by

\begin{equation}
Q_{a,b}^d \left[ \cdot \right] \overset{\text{def}}{=} Q_{a}^d \left[ |X_T = b| \right],
\end{equation}

(see Chapter XI.3 [25]). We will need some well-known facts about the Bessel bridge. For integer $d \geq 1$ the $d$–dimensional squared Bessel process is simply the norm squared of a Brownian bridge ending in zero, i.e. for any $T > 0$ and $a \geq 0$

\begin{equation}
Q_{a}^1 \left[ \cdot \right] = \mathbb{P}_a \left[ \left( |X_t|^2 \right)_{t \geq 0} \in \cdot \right] \quad \text{for } a \geq 0.
\end{equation}

Because of this, a 1–dimensional squared Bessel bridge ending in zero is the norm squared of a Brownian bridge ending in zero, i.e. for any $T > 0$ and $a \geq 0$

\begin{equation}
Q_{a,0}^{1,T} \left[ \cdot \right] = \mathbb{P}_{a}^{T} \left[ |X|^2 \in \cdot \right].
\end{equation}

The squared Bessel processes satisfy a well-known additivity property (see Theorem 1.2, Chapter XI.1 [25]):

\begin{equation}
\text{If } X^1 \text{ has law } Q_{a_1}^{d_1} \text{ and } X^2 \text{ is independent with law } Q_{a_2}^{d_2} \text{ then } X^1 + X^2 \text{ has law } Q_{a_1+a_2}^{d_1+d_2}, \text{ for all } d_1, d_2, a_1, a_2 > 0.
\end{equation}

A similar property holds for Bessel bridges (see (1.b)0 [24]). We will use the following special case:

\begin{equation}
\text{If } X^1 \text{ has law } Q_{x,0}^{0,T} \text{ and } X^2 \text{ is independent with law } Q_{a}^{1,T} \text{ then } X^1 + X^2 \text{ has law } Q_{x+a}^{1,T}, \text{ for all } T > 0.
\end{equation}

For the 0–dimensional Bessel bridge 0 is an absorbing state (see (5.3) [24]),

\begin{equation}
Q_{x,0}^{0,T} \left[ X_s = 0 \text{ for all } s \geq H_0 \right] = 1 \quad \text{for all } x, T > 0.
\end{equation}

Finally for $0 < S < T$ we can write down the Radon-Nikodym derivate of the laws under $Q_{x,0}^{0,T}$ and $Q_{x,0}^{1,T}$ of $(X_s)_{s \leq S}$ on the event $\{H_0 > S\}$.

**Lemma 7.4.** For all $0 < S < T$

\begin{equation}
\frac{dQ_{x,0}^{0,T}}{dQ_{x,0}^{1,T}}|_{\{X_s > S\}} = \left( \frac{1 - \frac{S}{T}}{X_S} \right)^2 \left( \frac{1}{X_S} \right)^{1/4} \exp \left( -\frac{3}{8} \int_0^S \frac{dt}{X_t} \right).
\end{equation}

\textbf{Proof.} A basic property of Bessel bridges ending in zero is that they can obtained from the Bessel process via (see (5.1) [24])

\begin{equation}
Q_{x,0}^{d,T} \text{ is the } Q_x^d \text{– law of } \left( \left( 1 - \frac{t}{T} \right)^2 X \frac{t}{T} \right)_{0 \leq t \leq T}.
\end{equation}

Also for all $R > 0$

\begin{equation}
\frac{dQ_{x}^{0}}{dQ_{x}^{1}}|_{\{X_R > R\}} = \left( \frac{x}{X_R} \right)^{1/4} \exp \left( -\frac{3}{8} \int_0^R \frac{dt}{X_t} \right),
\end{equation}

\textbf{Proof.}

\begin{equation}
\frac{dQ_{x,0}^{0,T}}{dQ_{x,0}^{1,T}}|_{\{X_s > S\}} = \left( \frac{1 - \frac{S}{T}}{X_S} \right)^2 \left( \frac{1}{X_S} \right)^{1/4} \exp \left( -\frac{3}{8} \int_0^S \frac{dt}{X_t} \right).
\end{equation}
by the first lemma of Section 12 \cite{author} (note that the index of a $d$-dimensional Bessel process is $d/2 - 1$). The claim \eqref{eq:7.28} now follows from \eqref{eq:7.29} and \eqref{eq:7.30} with $R = \frac{S}{1-S/T}$, using the substitution $\tilde{t} = \frac{t}{1-\tilde{t}^2}$ in the integral. \hfill \square

We are now ready to derive barrier bounds for the zero dimensional Bessel bridge. We first derive upper bounds for squares of linear barriers.

**Lemma 7.5.** Let $T > 0$ and $t_1, t_2 > 0$ be integers such that $t_1 < T - t_2$. For any linear function $h : [0, T] \to \mathbb{R}$ such that $h(t) > 0$ for $t \in [t_1, T - t_2]$ and any $u > 0$

$$
\mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} \left( \{t_1, \ldots, T - t_2\} \right) \right] 
\leq c \left( \frac{c + \sqrt{T} + \bar{u}(t_1) - h(t_1)}{c + \sqrt{T} + \bar{u}(T-t_2) - h(T-t_2)} \right)^{T-t_1-t_2},
$$

where $\bar{u}(t) = \frac{T-t}{t}$. If also $v \geq h(T-t_2)$ then

$$
\mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} \left( \{t_1, \ldots, T - t_2\} \right), \sqrt{X_{T-t_2}} \leq v \right]
\leq c \sqrt{\frac{\bar{u}(T-t_2)}{h(T-t_2)}} \sqrt{\frac{c + \sqrt{T} + \bar{u}(t_1) + v - h(t_1)}{c + \sqrt{T} + h(T-t_2)}}.
$$

**Proof.** Let $I = [t_1, T - t_2]$. By \eqref{eq:7.27}, the event $B_{h^2} (I \cap \mathbb{N})$ implies $\{H_0 > T - t_2\}$, since $h > 0$ throughout $I$ by assumption. Using this we obtain that

$$
\mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} (I \cap \mathbb{N}) \right] = \mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} (I \cap \mathbb{N}) \cap \{H_0 > T - t_2\} \right] \leq \mathbb{P}_{u \to 0}^{T} \left[ B_{h} (I \cap \mathbb{N}) \cap \{H_0 > T - t_2\} \right],
$$

where the inequality holds because adding a process with law $\mathbb{Q}_{0 \to 0}^{1,T}$ to $X$. only makes the barrier condition easier to satisfy, and the last equality holds because under $\mathbb{P}_{u \to 0}^{T}$ we have $|X_t| = X_t$ for $t \in I$ on $\{H_0 > T - t_2\}$. Using \eqref{eq:7.6} we thus have that

$$
\mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} (I \cap \mathbb{N}) \right] \leq \mathbb{P}_{u \to 0}^{T} \left[ B_{h} (I \cap \mathbb{N}) \cap \{H_0 > T - t_2\} \right],
$$

and by \eqref{eq:7.10} the right-hand side is bounded above by the bottom line of \eqref{eq:7.31}.

For \eqref{eq:7.32} note that similarly \eqref{eq:7.27} implies that the probability in question equals

$$
\mathbb{Q}_{u^2 \to 0}^{0,T} \left[ B_{h^2} (I \cap \mathbb{N}) \cap \{H_0 > T - t_2, \sqrt{X_{T-t_2}} \leq v \} \right].
$$

By Lemma \eqref{lem:7.4} with $S = T - t_2$ this is bounded above by

$$
C \sqrt{\frac{\bar{u}(T-t_2)}{h(T-t_2)}} \mathbb{Q}_{u^2 \to 0}^{1,T} \left[ B_{h^2} (I \cap \mathbb{N}) \cap \{H_0 > T - t_2, \sqrt{X_{T-t_2}} \leq v \} \right],
$$

since on $B_{h^2} (I \cap \mathbb{N})$ we have that \( \left( \left( 1 - \frac{T-t_2}{T} \right)^2 u^2 \right) / X_{T-t_2} \)^{1/4} = (\bar{u}(T-t_2) / \sqrt{X_{T-t_2}})^{1/2} \leq (\bar{u}(T-t_2) / h(T-t_2))^{1/2} \). But by \eqref{eq:7.24} the probability in \eqref{eq:7.33} equals

$$
\mathbb{P}_{u \to 0}^{T} \left[ B_{h} (I \cap \mathbb{N}), X_{T-t_2} \leq v \right] \leq \mathbb{P}_{u \to 0}^{T} [B_{h} (I \cap \mathbb{N})].
$$

Thus the required bound follows by \eqref{eq:7.6} and \eqref{eq:7.10}. \hfill \square

We now provide a lower bound on the probability that the zero dimensional squared Bessel bridge stays in a tube, cf. \eqref{eq:7.16}.

**Lemma 7.6.** If $T > 0$, $\frac{T}{3} > t \geq c$, $u \geq \frac{1}{1000}T$ and $v \in (-1000, 1000)$

$$
C \frac{t}{T - 2t} \leq \mathbb{Q}_{(u+v)^2 \to 0}^{0,T} \left[ B \left( \frac{u+h_0,501}{u+h_0,499} \right)^2 \left( \{t, T - t\} \right) \right], \text{ where } \bar{u}(t) = \frac{T-t}{T-u}.
$$
Proof. By (7.6), (7.16) and (7.24) we have
\[
(7.35) \quad c \frac{t}{L - 2t} \leq Q_{(u,v)}^{1,L} \left[ B_{(0,T-1)}^{(u+h_{0.501})} ([t, T - t]) \cap B_{(u-h_{0.499})}^{(u-h_{0.501})} ([0, t]) \right].
\]
By Lemma 7.4 with \( S = T - t \) the right-hand side of (7.35) is bounded above by
\[
(7.36) \quad Q_{(u,v)}^{0,L} \to 0 \left[ A; B_{(u+h_{0.501})}^{(u+h_{0.501})} ([t, T - t]) \cap B_{(u-h_{0.499})}^{(u-h_{0.501})} ([0, t]) \right]
\]
for \( A = \left( \left( -\frac{T-t}{T} \right)^2 \frac{u}{X_{T-t}} \right)^{\frac{1}{4}} \exp \left( -\frac{3}{8} \int_0^{T-t} X_s^{-1} ds \right) \). On the event in (7.36)
\[
\left( \frac{1 - \frac{T-t}{T}}{X_{T-t}} \right)^{\frac{1}{4}} \geq \left( \frac{\bar{u}(T-t)}{\bar{u}(T-t) + h_{0.501}(T-t)} \right)^{\frac{1}{2}} \geq c,
\]
provided \( t \geq c \) (recall the assumption \( u \geq \frac{1}{1000} T \)), and \( \sqrt{X_s} \geq \sqrt{u} (s) \) for \( s \in [0, T-t] \) (note that \( \bar{u}(s) \geq \frac{1}{1000} T \geq \bar{u}(s) \) for \( s \leq t \)) so that
\[
\int_0^{T-t} ds X_s \leq c \int_0^{T-t} \frac{ds}{\bar{u}(s)} \leq c \left( \frac{T}{u} \right) \int_0^{T-t} (T-s)^{-2} ds \leq c \int_t^\infty s^{-2} ds \leq c.
\]
Thus \( A \geq c \) on the event in (7.36), so the claim (7.34) follows. \( \square \)

It remains to derive our goal Proposition 7.1 from Lemma 7.5 and Lemma 7.6 by comparing the law of \( (T_l)_{l \geq 0} \) under \( \mathbb{G}_{l,T} \{ T_{L-1} = 0 \} \) and \( (X_l)_{l \geq 0} \) under \( Q_{l,T}^{0,L} \). To do this we exploit that that \( \mathbb{G}_{l,T} \{ T_{L-1} = 0 \} \) is essentially the law of the edge local time of the discrete simple random walk on \( \{0, \ldots, L\} \) when conditioned not to hit \( L \), while \( Q_{l,T}^{0,L} \) is the law of the vertex local time of the continuous time version of the same random walk. We can carry out the comparison using the natural coupling of discerete and continuous time random walk on \( \{0, \ldots, L\} \).

To this end, let \( Y_t, t \geq 0 \), be continuous time simple random walk on \( \{0, \ldots, L\} \) with jump rate 1, and let \( \tilde{P}_t \) be its law when starting from \( l \in \{0, \ldots, L\} \). Let
\[
(7.37) \quad d_l = \begin{cases} 
1 & \text{if } l = 0, \\
2 & \text{if } 0 < l < L, \\
1 & \text{if } l = L,
\end{cases}
\]
be the degree of the vertices in the path \( \{0, \ldots, L\} \) and let
\[
(7.38) \quad L_t = \frac{1}{d_l} \int_0^t 1_{\{Y_s = l\}} ds \quad \text{for } 0 \leq l \leq L, t \geq 0,
\]
be the local time of the random walk \( Y_t \). Define the inverse local time of 0 by
\[
\tau(t) = \inf \{ s \geq 0 : L_s^0 > t \}.
\]
The law of \( L_t^{\tau(t)} \) has a nice characterisation which can be derived from the Second Ray Knight Theorem (see the appendix for the derivation).

**Lemma 7.7.** For all \( L \in \{1, 2, \ldots\} \) and \( t \geq 0 \) the \( \tilde{P}_0 \)-law of \( \left( L_t^{\tau(t)} \right)_{l \in \{0, \ldots, L\}} \) is the \( Q_{2,L}^{0} \)-law of \( \left( \frac{1}{2} X_l \right)_{l \in \{0, \ldots, L\}} \).
Lemma 7.9. Provided by the following lemma, whose proof is also in the appendix.

Lemma 7.8. For all \( L \in \{1, 2, \ldots \} \), measurable \( A \subset \mathbb{R}^L \) and \( t \in \{1, 2, \ldots \} \)

\[
\bar{\mathbb{P}}_0 \left[ \left( L^{D_l}_l \right)_{l \in \{1, \ldots, L\}} \in A | L^{D_l}_L = 0 \right] = \bar{\mathbb{P}}_0 \left[ \left( \frac{1}{2} X_l \right)_{l \in \{0, \ldots, L-1\}} \in A \right] | L^{D_t}_L = 0.
\]

After applying Lemma 7.8 we will need a control on \( L^{D_t}_L \) conditioned on \( L^{D_t}_L = 0 \) provided by the following lemma, whose proof is also in the appendix.

Lemma 7.7. Where we also condition on the processes hitting zero, since this is what we do in Proposition 7.1. The proof is given in the appendix.

Lemma 7.10. For all \( t \in \{0, 1, \ldots \} \) the \( \bar{\mathbb{P}}_0 \)-law of \( \tilde{T}_l^t \), \( l \in \{0, \ldots, L-1\} \), is the \( \mathcal{G}_t \)-law of \( T_l \), \( l \in \{0, \ldots, L-1\} \).

Proof. The proof is omitted as it is very similar to that of Lemma 3.2.

To derive Proposition 7.1 from Lemma 7.3 and Lemma 7.6, we will have to “translate” between discrete and continuous local time. For this we will use the following lemma, which gives a large deviation bound for \( L^{D_t}_l \) conditioned on \( \tilde{T}_l^t \), \( l \geq 0 \).

Note that \( L^{\tau_l(t)} \) counts the local time accumulated at vertex \( l \) until local time \( t \) has accumulated at 0. In proving Proposition 7.1 we will in fact need the law of the local times accumulated during the first \( t \) excursions from zero, that is of \( L^{D_t}_l \) where,

\[
D_0 = 0, \quad D_1 = H_0 \circ \theta H_1 + H_1 \quad \text{and} \quad D_n = D_1 \circ \theta D_{n-1} + D_{n-1}, \quad n \geq 2.
\]

are the return times to 0 of \( Y_t \) (and \( H_1 \) and \( \theta_n \) are defined on the space \( C(\mathbb{R}^+, \{0, \ldots, L\}) \) in the natural manner). Next we state a description of the law of \( L^{D_t}_l \) that follows from Lemma 7.7 where we also condition on the processes hitting zero, since this is what we do in Proposition 7.1. The proof is given in the appendix.

(7.39) \( D_0 = 0, D_1 = H_0 \circ \theta H_1 + H_1 \) and \( D_n = D_1 \circ \theta D_{n-1} + D_{n-1}, n \geq 2 \), are the return times to 0 of \( Y_t \) (and \( H_1 \) and \( \theta_n \) are defined on the space \( C(\mathbb{R}^+, \{0, \ldots, L\}) \) in the natural manner). Next we state a description of the law of \( L^{D_t}_l \) that follows from Lemma 7.7 where we also condition on the processes hitting zero, since this is what we do in Proposition 7.1. The proof is given in the appendix.

(7.40) \( \bar{\mathbb{P}}_0 \left[ \sqrt{\frac{L-1}{L}} - 250 \leq \sqrt{L^{D_t}_l} \leq \sqrt{\frac{L-1}{L}} + 250 | L^{D_t}_L = 0 \right] \geq c \geq 0. \)

If \( l \in \{1, \ldots, L-1\} \) then with \( \bar{u}(l) = \sqrt{2L^{D_t}_L \frac{L-1}{L}} \) and \( u(l) = \sqrt{2L^{D_t}_L} \) we have

\[
\mathbb{E}_0 \left[ |\bar{u}(l) - v|^k | L^{D_t}_L = 0 \right] \leq c + |u(l) - v|^k \quad \text{for} \quad k \in \{1, 2\} \quad \text{and} \quad v \in \mathbb{R}.
\]

Next we exhibit the connection with the law \( \mathcal{G}_{t_n} \). Let \( J_1, J_2, \ldots \) be the jump times of \( Y_t \), and let \( J_0 = 0 \). Let

\[
Z_n = Y_{J_n}, \quad n \geq 0,
\]

be the discrete skeleton of the random walk \( Y_t \). Clearly \( Z_n \) is a discrete time simple random walk. Let

\[
\tilde{D}_n = \inf \left\{ m > \tilde{D}_{n-1} : Z_m = 0 \right\}, \quad n \geq 1, \quad \text{and} \quad \tilde{D}_0 = 0,
\]

be the successive returns to 0 of \( Z_n \). Finally let

\[
\tilde{T}_l^t = \sum_{m=1}^{\tilde{D}_{\{l\}}} 1_{\{Z_m = l+1, Z_{m-1} = l\}}, \quad l = 0, \ldots, L-1, t \geq 0,
\]

be the number of traversals from \( l \) to \( l + 1 \) up to time \( \tilde{D}_{\{l\}} \) (equivalently the edge local times of the edges \( l \to l + 1 \) up to time \( \tilde{D}_{\{l\}} \)). We have that

\[
\mathbb{P}_0 \left[ \mathcal{G}_{t_l} \right] \text{law of } \tilde{T}_l^t, l \in \{0, \ldots, L-1\}, \text{is the } \mathcal{G}_t \text{law of } T_l, l \in \{0, \ldots, L-1\}.
\]

Proof. The proof is omitted as it is very similar to that of Lemma 3.2.

To derive Proposition 7.1 from Lemma 7.3 and Lemma 7.6, we will have to “translate” between discrete and continuous local time. For this we will use the following lemma, which gives a large deviation bound for \( L^{D_t}_l \) conditioned on \( \tilde{T}_l^t, l \geq 0 \).
Letting define also (7.45) by Lemma 7.10 we have Lemma 7.12.

Proof. The continuous time random walk \( Y_t \) makes \( \tilde{T}^t_{l-1} + \tilde{T}^t_l \) discrete visits to the vertex \( l \) up to time \( D_t \). The holding times of the continuous time random walk \( Y_t \) are iid standard exponential random variables and are independent of the discrete skeleton of the random walk, so we have that the \( \tilde{\mathbb{P}}_0 \left[ |\sigma \left( \tilde{T}^t_l : l = 0, \ldots, L - 1 \right) \right] \) law of \( L^{D_t}_l \) is that of a sum of \( \tilde{T}^t_{l-1} + \tilde{T}^t_l \) iid exponentials with mean 1/2 (from the normalizing factor in (7.38)). Thus the claim follows by a standard large deviation bound.

We now state a similar result for \( \tilde{T}^t_l \) when conditioned on \( L^{D_t}_l, l = 0, \ldots, L \), whose proof will be given in the appendix.

Lemma 7.12. If \( l \in \{1, \ldots, L - 1\}, t > 0, \) and \( \theta > 0 \) then with \( \mu = \sqrt{D_t^{l} L^{D_t+1}_l} \),

\[
\tilde{\mathbb{P}}_0 \left[ \left| \tilde{T}^t_l - \mu \right| \geq \theta \sigma \left( L^{D_t}_l : l = 0, \ldots, L \right) \right] \leq ce^{-\frac{\theta^2}{\sigma^2} + \varepsilon}. 
\]

We are now ready to the main result Proposition 7.1 of this section.

Proof of Proposition 7.1. We start with the proof of (7.1). Let \( I = \{1, \ldots, L - \lfloor 3 (\log L)^2 \rfloor \} \).

By Lemma 7.10 we have

\[
G_{\alpha^2} \left( I \right) |T_{L-1} = 0 = \tilde{\mathbb{P}} \left[ A | \tilde{T}^{t_s}_{L-1} = 0 \right], \text{ where,} 
\]

\[
A = \left\{ \sqrt{\tilde{T}^{t_s}_l} \geq \alpha \left( l \right) \text{ for } l \in I \right\}.
\]

Define also

\[
B = \left\{ \sqrt{L^{D_t+1}_l} \geq \alpha_- \left( l \right) \text{ for } l \in I \right\}, \text{ where,}
\]

\[
\alpha_- \left( l \right) = \beta \left( l \right) - 2 (\log L)^2 = \alpha \left( l \right) - (\log L)^2, l \in \{0, 1, \ldots, L \}.
\]

Letting \( A = \sigma \left( \tilde{T}^{t_s}_l : l = 0, \ldots, L - 1 \right) \) we have by (7.43) that for \( l \in \{1, 2, \ldots, L - 1\} \)

\[
\tilde{\mathbb{P}}_0 \left[ \left| L^{D_t+1}_l - \mu_l \right| \geq \sqrt{\mu_l} (\log L)^2 \right] \leq ce^{-c(\log L)^4}, \text{ for } \mu_l = \frac{\tilde{T}^{t_s}_{l-1} + \tilde{T}^{t_s}_l}{2}.
\]

On the event \( A \) we have for \( l \in I \)

\[
\mu_l - \sqrt{\mu_l} (\log L)^2 = \left( \sqrt{\mu_l} - \frac{1}{2} (\log L)^2 \right)^2 - \frac{1}{4} (\log L)^4 \geq \left( \alpha \left( l \right) - \frac{1}{2} (\log L)^2 \right)^2 - \frac{1}{4} (\log L)^4 \geq \alpha_- \left( l \right)^2,
\]

where we have used that \( (\log L)^2 \leq \alpha \left( l \right) \leq \alpha \left( l - 1 \right) \) for \( l \in I \) (see (4.4), Figure 7.1 on page 43). Therefore (7.47) implies that

\[
\tilde{\mathbb{P}}_0 \left[ B | A \cap \left\{ \tilde{T}^{t_s}_{L-1} = 0 \right\} \right] \geq 1 - \sum_{l \in I} e^{-c(\log L)^4} \geq 1 - o(1),
\]

so that

\[
\tilde{\mathbb{P}}_0 \left[ A \cap \left\{ \tilde{T}^{t_s}_{L-1} = 0 \right\} \right] \leq c\tilde{\mathbb{P}}_0 \left[ A \cap B \cap \left\{ \tilde{T}^{t_s}_{L-1} = 0 \right\} \right] \leq c\tilde{\mathbb{P}}_0 \left[ B \cap \left\{ L^{D_t+1}_l = 0 \right\} \right],
\]
where \( I = \{0, \ldots, L - 3 (\log L)^2 \} \). Therefore the argument which gave (7.48) now gives

\[
\mathbb{E}_0 \left[ Q_{L^{-1}} \right] B \leq 2 \mathbb{E}_0 \left[ B \right] (I - 1) [L^D_{I^\alpha} = 0],
\]

where \( I - 1 = \{0, \ldots, L - \lfloor 3 (\log L)^2 \rfloor - 1 \} \). By the Bessel bridge barrier bound (7.31) with \( T = L - 1, t_1 = 0, t_2 = \lfloor 3 (\log L)^2 \rfloor \), \( h(\cdot) = \sqrt{2} \alpha_-(1 - \cdot) \) (which is positive on \([0, T - t_2]\)) and \( u = \sqrt{2 L^D_{I^\alpha}} \) the quantity in the expectation is bounded above by

\[
(c + |\hat{u}(1) - \sqrt{2} \alpha_-(1)|) \frac{(c + c \log L + \hat{u} \left( L - \lfloor 3 (\log L)^2 \rfloor \right) - \sqrt{2} \alpha_-(L - \lfloor 3 (\log L)^2 \rfloor)}{L - 1 - \lfloor 3 (\log L)^2 \rfloor},
\]

where \( \hat{u}(l) = \bar{u}(l - 1) = \sqrt{2 L^D_{I^\alpha}} \frac{L - l}{L - 1} \). Therefore using (7.41) and Hölder’s inequality (note that \( \sqrt{2} \beta(l) = \sqrt{2} [l, T] L^{-1} + O(1) \)) (7.49) is bounded above by

\[
\frac{(c + \sqrt{2} [\beta(l) - \alpha_-(1)]) (c + c \log L + \sqrt{2} \beta(L - \lfloor 3 (\log L)^2 \rfloor) - \alpha_-(L - \lfloor 3 (\log L)^2 \rfloor))}{L}.
\]

Thus by (7.46) in fact

\[
\mathbb{P}_0 [A|\hat{T}_{L-1} = 0] \leq \frac{(c + c \log L)^2}{L}.
\]

Now (7.1) follows by (7.45).

The proof of the upper bound of (7.2) and (7.4) are similar. For the upper bound of (7.2) we let

\[
I_1 = \{l_0, \ldots, L - l_0\} \quad \text{and} \quad I_2 = \{l_0 + 1, \ldots, L - l_0\},
\]

and note that similarly to (7.47) the large deviation bound (7.43) implies that

\[
\mathbb{P}_0 \left[ L^D_{I^\alpha} - \mu_l \geq \frac{1}{2} \sqrt{\mu_l} f(l) \right] \leq ce^{-c f(l)^2},
\]

where \( \mu_l \) as in (7.47). On the event \( \left\{ \sqrt{\hat{T}_{l^\alpha}^l} \geq \gamma(l) \right\} \) we have for \( l \in I_2 \)

\[
\frac{1}{2} \sqrt{\mu_l} f(l) = \left( \frac{1}{4} f(l) \right) - \frac{1}{4} f(l)^2 \geq \frac{1}{4} f(l) \geq \beta(l)^2,
\]

for \( L \) large enough, where we have used that \( f(l) \leq \gamma(l) \leq \gamma(l - 1) \) (see (5.1), Figure 7.1 on page 43). Therefore the argument which gave (7.48) now gives

\[
\mathbb{P}_0 \left[ \sqrt{\hat{T}_{l^\alpha}^l} \geq \gamma(l) \right] \leq \left( 1 - \sum_{l \in I_2} e^{-c f(l)^2} \right)^{-1} \mathbb{P}_0 \left[ L^D_{I^\alpha} \geq \beta(l) \right] \mathbb{P}_0 \left[ L^D_{I^\alpha} = 0 \right] = 0.
\]

We have \( \sum_{l \in I_2} e^{-c f(l)^2} = o(1) \), so that using Lemma 7.8 the bottom line equals

\[
(1 + o(1)) \mathbb{P}_0 \left[ \sum_{l \in I_2} e^{-c f(l)^2} \right] \mathbb{P}_0 \left[ B_{2 \beta(l)}^2 (I_2 - 1) \right] = 0.
\]
By (7.31) with $T = L - 1$, $t_1 = t_2 = l_0$, $h(\cdot) = \sqrt{2} \beta (1 + \cdot)$ and $u = \sqrt{2L_1^{D_{|t_s|}}}$ the quantity in the expectation is bounded above by

$$\frac{(c + c\sqrt{l_0} + |\tilde{u}(l_0 + 1) - \sqrt{2}\beta (l_0 + 1)|) (c + \sqrt{l_0} + |\tilde{u}(L - l_0) - \sqrt{2}\beta (L - l_0)|)}{L - 1 - \lfloor 3 \log L \rfloor},$$

where $\tilde{u}(l) = \tilde{u}(l - 1) = \sqrt{2L_1^{D_{|t_s|}}} \frac{L - l}{L - 1}$. Using (7.41) and Hölder’s inequality we get that

$$\tilde{P}_0 \left[ \sqrt{T_l^{t_s}} \geq \gamma (l) \text{ for } l \in I_1 | T_l^{t_s} = 0 \right] \leq c \frac{(c + \sqrt{l_0})^2}{L},$$

since $\sqrt{2L_1^{D_{|t_s|}}} \frac{L - l}{L - 1} = \sqrt{2}\beta (l) + O (1)$. Thus (7.2) follows by Lemma 7.10.

To show (7.4) we similarly use (7.43) and Lemma 7.10 to prove that

$$\mathbb{G}_a \left[ B_{\gamma(k+1)^2} (\{0, \ldots, L - k - l_0\}) | T_{L-1-k} = 0 \right] = \mathbb{G}_a \left[ \gamma (l + k) \leq \sqrt{T_l} \text{ for } l = 0, \ldots, L - k - l_0 | T_{L-k-1} = 0 \right]$$

$$\leq \left( 1 - \sum_{l=1}^{L-k-l_0} e^{-cf((l+k)^2)} \right)^{-1} \times \tilde{P}_0 \left[ \sqrt{L_l^{D_a}} \geq \beta (l + k) \text{ for } l = 0, \ldots, L - k - l_0 | L_{L-k}^{D_a} = 0 \right]$$

$$= (1 + o (1)) \tilde{P}_0 \left[ \sqrt{L_l^{D_a}} \geq \beta (l + k) \text{ for } l = 0, \ldots, L - k - l_0 | L_{L-k}^{D_a} = 0 \right].$$

By Lemma 7.8 with $L - k - 1$ in place of $L$ the last probability equals

$$\tilde{P}_0 \left[ Q_{L+1}^{0, L-k-1} (\{0, \ldots, L - k - l_0 - 1\}) | L_{L-k}^{D_a} = 0 \right],$$

so that by (7.31) with $T = L - k - 1$, $t_1 = 0$, $t_2 = l_0$, $h(\cdot) = \beta (k + 1 + \cdot)$ and $u = \sqrt{2L_1^{D_a}}$ the right-hand side of (7.52) is bounded above by

$$\tilde{E}_0 \left[ \frac{(c + |\tilde{u}(1) - \beta (k + 1)|) (c + \sqrt{l_0} + |\tilde{u}(L - k - l_0) - \beta (L - l_0)|)}{L - k - l_0} \right] | L_{L-k-1}^{D_a} = 0 \right]$$

where $\tilde{u}(l) = \tilde{u}(l - 1) = \sqrt{2L_1^{D_a}} \frac{L - l - l_0}{L - 1}$. Using (7.41) with $L - k$ in place of $L$ and Hölder’s inequality this is at most

$$c \frac{(c + |u(1) - \beta (k + 1)|) (c + \sqrt{l_0} + |u(L - k - l_0) - \beta (L - l_0)|)}{L - k - l_0 - 1},$$

where $u(l) = \sqrt{a \frac{L - l - l_0}{L - 1}}$. Now since $\gamma (k)^2 \leq a \leq \delta (k)^2$ we have we have that $\beta (k + l) \leq u(l) \leq \beta (k + l) + g (k + l)$, so that for $L \geq c$ this is at most

$$c \frac{(c + g(k + 1)) (c + \sqrt{l_0} + g(L - l_0))}{L - k - l_0 - 1} \leq c \frac{g(k + 1) l_0^{0.51}}{L - k - l_0 - 1},$$

(recall (5.4)), so (7.4) follows.

To show (7.3) we similarly use (7.43) prove that

$$\mathbb{G}_{t_s} \left[ \gamma (l) \leq \sqrt{T_l} \leq \delta (l) \text{ for } l = l_0, \ldots, k | T_{L-1} = 0 \right]$$

$$\leq c\tilde{P}_0 \left[ \sqrt{L_l^{D_{|t_s|}}} \geq \beta (l) \text{ for } l = l_0, \ldots, k, \sqrt{L_k^{D_{|t_s|}}} \leq \delta (l) + g (k) | L_{L-t_s}^{D_{|t_s|}} = 0 \right].$$
For this we use also that (cf. (7.47))
\[
\tilde{\mathbb{P}}_0 \left[ L_{k}^{D_{l+1}} \leq \mu_k + \sqrt{\mu_k \frac{1}{2}} g(k) | A \right] \leq ce^{-g(k)^2} \to 0, \text{ as } L \to \infty,
\]
and that on the event \( \{ \gamma (l) \leq \sqrt{T_{l}^{+}} \leq \delta (l) \text{ for } l \in \{l_0, \ldots, L - l_0\} \}\) we have
\[
\mu_k + \sqrt{\mu_k \frac{1}{2}} g(k) \leq \left( \delta (k - 1) + \frac{1}{4} g(k) \right)^2 \leq (\delta (k) + g(k))^2, \text{ see (5.3)}.
\]
This follows from (7.3). We then use Lemma 7.8 to obtain that
\[
\mathbb{G}_{L-1} \left[ \gamma (l) \leq \sqrt{T_{l}^{+}} \leq \delta (l) \text{ for } l = l_0, \ldots, k \right] \leq \mathbb{P}_0 \left[ l_{D_{l+1}} \leq \frac{c \sqrt{L_{l+1}^{D_{l+1}}} (c + \sqrt{L_{l+1}^{D_{l+1}}})}{k - l_0 - 1} \right],
\]
where \( \tilde{u} (l) = \tilde{u} (l - 1) = \sqrt{2L_{l+1}^{D_{l+1}}} \). By (7.41) and the Hölder inequality this bounded above by
\[
\frac{c \sqrt{2L_{l+1}^{D_{l+1}}} (1 - k/L) \gamma (l + 1)}{k - l_0 - 1},
\]
which is bounded above by the right-hand side of (7.3), so (7.3) follows.

It remains to show the lower bound of (7.2). For this we note that by Lemma 7.10
\[
\mathbb{G}_{l_1} \left[ B_{l_2}^{D_{l_2}} (I_1) | T_{L-1} = 0 \right] = \tilde{\mathbb{P}}_0 \left[ A | \tilde{T}_{l_2}^{L-1} = 0 \right],
\]
where \( I_1 = \{l_0, \ldots, L - l_0\} \) and
\[
A = \{ \gamma (l) \leq \sqrt{T_{l}^{+}} \leq \delta (l) \text{ for } l \in I_1 \}.
\]
Define also \( I_2 = \{ [l_2/l_0], \ldots, L - [l_2/l_0] \} \) and
\[
B = \{ \beta (l) + 2 f (l) \leq \sqrt{L_{l+1}^{D_{l+1}}} \leq \beta (l) + \frac{1}{2} g (l) \text{ for } l \in I_2 \}.
\]
By (7.44) we have, letting \( A = \sigma \left( L_{l}^{D_{l+1}} : l = 0, \ldots, L - 1 \right) \), that
\[
\tilde{\mathbb{P}}_0 \left[ \tilde{T}_{l_2}^{L-1} - \mu \right] \geq \sqrt{\mu_l^2 f (l) | A} \leq ce^{-c \gamma (l)^2},
\]
where \( \mu_l = \sqrt{L_{l+1}^{D_{l+1}}} L_{l+1}^{D_{l+1}} \). On the event \( B \) we have for \( l \in I_2 \)
\[
\mu_l - \sqrt{\mu_l^2 f (l)} = \left( \sqrt{\mu_l^2 - \mu_l^2 f (l)} \right)^2 - \frac{1}{16} f (l) \leq \left( \sqrt{\beta (l) + 2 f (l) \right)^2 - \frac{1}{16} f (l) \leq (\beta (l) + 2 f (l))^2 - \frac{1}{16} f (l) \leq 2 f (l) \}
\]
Furthermore on the event \( B \),
\[
\mu_l + \sqrt{\mu_l^2 f (l)} = \left( \sqrt{\mu_l^2 + \frac{1}{4} f (l)} \right)^2 - \frac{1}{16} f (l) \leq (\beta (l) + \frac{1}{2} g (l) - \frac{1}{4} f (l)^2 \leq \delta (l)^2.
\]
Therefore \( (7.54) \) implies that

\[
\tilde{\mathbb{P}}_0 \left[ B | L^{D_{ts}}_L = 0 \right] \leq \left( 1 - \sum_{l \in I_2} e^{\frac{-cf(k)}{2}} \right)^{-1} \tilde{\mathbb{P}}_0 \left[ B | L^{D_{ts}}_L = 0 \right] \leq c \tilde{\mathbb{P}}_0 \left[ A | \bar{T}^{s}_{L-1} = 0 \right],
\]

for \( L \) large enough. Using Lemma \( 7.8 \) the left-hand side equals

\[
c \tilde{\mathbb{P}}_0 \left[ \bigcap_{l = 0}^{L-1} \bigcap_{D_{ts}^{l+1} 
 \rightarrow 0} B^2(\beta(1+\frac{g(1+))}{2})^2 ([\frac{1}{2}l_0] - 1, \ldots, L - [\frac{1}{2}l_0] - 1) \right] \times \tilde{\mathbb{P}}_0 \left( (\beta (1) - 500)^2 \leq L^{D_{ts}}_L \leq (\beta (1) + 500)^2 | L^{D_{ts}}_L = 0 \right).
\]

By \( (7.40) \) the second probability is bounded below by \( c > 0 \), and by Lemma \( 7.6 \) with \( T = L - 1, t = \frac{1}{2}l_0 \) and \( u = \sqrt{2} \beta (1) \), the first is bounded below by \( c l_0 / (L - 1 - l_0) \geq c l_0 / L \).

Therefore the lower bound of \( (7.2) \) follows. \( \square \)

By completing the demonstration of the barrier crossing bounds, we have now proved all the “ingredients” that were used to prove the upper bound Proposition \( 3.5 \) and the lower bound Proposition \( 3.6 \) (except for the small proofs in the appendix). Thus of the tools that were used to deduce the main result Theorem \( 3.1 \) only the concentration result Proposition \( 3.7 \) remains to be proven.

8. Concentration of excursion times

In this section we will prove the concentration result Proposition \( 3.7 \) which bounds the total time \( D^{y,0}_n \) (recall \( 3.7 \)) needed to make \( t_s \) traversals from \( \partial B (y,r_0) \) to \( \partial B (y,r_1) \). We need the error in the bound to be smaller than the subleading correction term for \( C_\epsilon \), which is already small compared to the leading order (cf. \( 1.2 \)), and we therefore need a very precise estimate. Essentially, we must show that

\[
D^{y,0}_n = \frac{1}{\pi} t_s \left( 1 + o(\log L/L) \right) \text{ simultaneously for all } y \in F_L.
\]

The time \( D^{y,0}_n \) can be written as a sum of \( n \) random variables, namely the time each “trip” from \( \partial B (y,r_0) \) to \( \partial B (y,r_1) \) and back takes. Therefore the natural approach to get \( (8.1) \) - which we employ - is to derive a Cramer-type large deviation bound on

\[
P_x \left[ | D^{y,0}_n \left( 1 - \frac{1}{\pi} n \right) \geq \theta \right].
\]

However, several complications arise. Firstly, the typical way to obtain \( (8.1) \) from a large deviation bound on \( D^{n,0}_n \) for one \( y \), is to use a union bound over \( y \in F_L \). This fails in our case, because the best upper bound one can hope for is \( c e^{-c \theta^2 / n} \) (the bound one gets for sums of iid random variables), and to obtain \( (8.1) \) one needs to set \( n = t_s \approx L^2 \) and \( \theta = c (\log \epsilon L / L) n \) for some constant \( c \). This would give a bound of \( c e^{-c (\log \epsilon L) n} \) which does not “kill” \( | F_L | \geq e^{2L} \) (recall \( 3.11 \)). The issue is similar to that from the proof of Proposition \( 4.7 \) in Section \( 4 \) and the solution is also similar: we take the union bound instead over a packing of \( \sim r_0^{-2} \approx (\log L)^{3/2} \) circles of radius close to \( r_0 \), in such a way that the concentration of excursion times for all \( y \) in the packing implies the concentration of excursion times for all \( y \in F_L \).
Furthermore, the typical way to obtain a large deviation bound on \(D_n^{y,0}\) for one \(y\) is to write \(D_n^{y,0}\) as the sum
\[
D_n^{y,0} = \sum_{i=1}^{n} \left( D_i^{y,0} - R_i^{y,0} \right) + \sum_{i=0}^{n-1} \left( R_{i+1}^{y,0} - D_i^{y,0} \right),
\]
where \(D_0^{y,0} = 0\), of the lengths of each of the \(n\) excursions from \(\partial B(y, r_i)\) to \(\partial B(y, r_0)\) and the lengths of each of the \(n\) excursions from \(\partial B(y, r_0)\) to \(\partial B(y, r_1)\), and then use Khasminskii’s lemma/Kac’s moment formula and the strong Markov property to obtain large deviation bounds for each of the two sums, by bounding their exponential moments (cf. (8.14) and (8.16)). This turns out to work fine for the first sum, but a further complication arises when applying this recipe to the second sum. Essentially speaking, the recipe requires a bound on \(E_z [H_{B(y,r_1)}]\) (for appropriate random \(z\) this is the expectation of the summands) that is uniform over \(z \in \partial B(y, r_0)\) and whose error is at most as large as the \(\theta\) which we wish to use. Such a strong uniform bound turns out to unattainable. Instead, we employ a more sophisticated technique which involves considering the Markovian structure of the starting points \(W_{D_i^y}^i, i \geq 1\), of each excursion from \(\partial B(y, r_0)\) to \(\partial B(y, r_1)\), and computing exactly the expected length of an excursion when starting from the equilibrium distribution on starting points.

Let us now start the proof of Proposition 3.7. Recall (2.5) for the definition of \(D_n(y, R, r)\) and \(R_n(y, R, r)\). Most of the results of this section will be stated for general \(0 < r < R < \frac{1}{2}\). At the end, when we carry out the packing argument, we will use the results with \(R = r_0^\pm\) and \(r = r_i^\pm\), for \(r_i^\pm\) as in (4.12) (therefore it is good keep in mind that in the end we will have \(R/r \approx e\) and \(R \downarrow 0\) as \((\log L)^{-3/4}\)). When it does not cause confusion we will drop the arguments and write
\[
D_n = D_n(y, R, r) \quad \text{and} \quad R_n = R_n(y, R, r).
\]

We first introduce rigorously the equilibrium distribution mentioned above, which will be denoted by \(\mu^R\). By Lemma 2.1 of [28] there exists for all \(y \in \mathbb{T}\) a pair of probability measures \(\mu^R\) on \(\partial B(y, R)\) and \(\mu^R\) on \(\partial B(y, r)\) such that
\[
\mu^R(\cdot) = \int_{\partial B(y, R)} P_x [W_{\partial B(y, R)} \in \cdot] \mu^R (dv), \quad \text{and}
\mu^R(\cdot) = \int_{\partial B(y, r)} P_x [W_{\partial B(y, r)} \in \cdot] \mu^R (dv).
\]
(Actually these measures are the stationary distributions of the discrete time Markov chains \((W_{D_n})_{n \geq 1}\) and \((W_{R_n})_{n \geq 1}\)). Next we want to compute an exact formula for \(E_{\mu^R} [D_1]\). For this we will use Green functions. For any measurable \(A \subset \mathbb{T}\) let,
\[
p^A(t, x, y) = P_x [W_t \in dy, H_A > t],
\]
denote the transition density of \(W_t\) under \(P_x\) killed upon hitting \(A\). Recall that the killed Green functions \(G^A(\cdot, \cdot)\) is defined by
\[
G^A(x, y) = \int_0^\infty p^A(t, x, y) dt \text{ for } x, y \in \mathbb{T}.
\]
One can define a measure by
\[
G^A(x, B) = \int_B G^A(x, y) dy \text{ for } x \in \mathbb{T} \text{ and measurable } B \subset \mathbb{T}.
\]
Note that
\[
G^A(x, B) = E_x \left[ \int_0^{H_A} 1_{\{W_t \in B\}} dt \right] \text{ for } x \in \mathbb{T} \text{ and measurable } B \subset \mathbb{T}.
\]
A standard bound on killed Green functions for Brownian motion in \(\mathbb{R}^2\) imply the following bounds on the Green function \(G_{B(y,R)}^c(u,v)\) for \(u,v \in B(y,r)\) (see Lemma 3.36, [23] and note that \(B(y, R) \subset \mathbb{T}\) can be identified with a ball in \(\mathbb{R}^2\), cf. (2.1))

\[
G_{B(y,R)}^c(u,v) = -\frac{1}{\pi} \log d(u,v) + \frac{1}{\pi} E_u \left[ \log d(W_{T_{B(y,R)}}, v) \right].
\]

We are now ready to compute \(E_{\mu^R} [D_1]\).

**Lemma 8.1.** \((y \in \mathbb{T})\) For all \(0 < r < R < \frac{1}{2}\),

\[
E_{\mu^R} [D_1] = \frac{1}{\pi} \log \frac{R}{r}.
\]

**Proof.** Define a measure \(m\) on \(\mathbb{T}\) by

\[
m(\cdot) = \int_{\partial B(y,r)} \mu^R_R (dv) G_{B(y,R)}^c (v, \cdot) + \int_{\partial B(y,r)} \mu^R_r (dv) G_{B(y,r)} (v, \cdot).
\]

By a theorem of Maruyama and Tanaka (see (2.2), (2.13) and page 121 [28]; recall also (8.2)) we have that \(m\) is an invariant measure for \(P_x\), (an intuition for this result can be obtained by considering the corresponding statement for a Markov chain with discrete state space). By (8.3), the second line of (8.2) and the strong Markov property we have that

\[
m(\mathbb{T}) = E_{\mu^R} [H_{B(y,r)}] + E_{\mu^R} [T_{B(y,R)}] = E_{\mu^R} [D_1].
\]

Since clearly the only invariant measure for \(P_x\) is the uniform distribution \(\lambda\) on \(\mathbb{T}\) (up to multiplication by a constant), we have

\[
m = c \lambda.
\]

Thus

\[
E_{\mu^R} [D_1] = m(\mathbb{T}) = c.
\]

To determine the value of \(c\) we note that for \(\delta \in (0, r)\)

\[
m(B(y,\delta)) = \int_{\partial B(y,R)} \mu^R_R (du) \int_{B(y,\delta)} G_{B(y,R)}^c (u,v) \lambda (dv).
\]

By (8.4) we have that

\[
G_{B(y,R)}^c (u,v) = -\frac{1}{\pi} \log (r + O(\delta)) + \frac{1}{\pi} \log (R + O(\delta)).
\]

Thus for \(v \in \partial B(y,r)\) and \(w \in B(y,\delta)\) we get that

\[
m(B(y,\delta)) = \lambda(B(y,\delta)) \left( -\frac{1}{\pi} \log (r + O(\delta)) + \frac{1}{\pi} \log (R + O(\delta)) \right).
\]

Taking \(\delta \to 0\) we can now identify the constant in (8.6) as \(c = \frac{1}{\pi} \log \frac{R}{r}\), and thus (8.5) follows from (8.7). \(\square\)

We now start the proofs of the various large deviation bounds we need to prove Proposition 3.7. We will make the decomposition

\[
D_n = D_1 + \sum_{i=2}^{n} (D_i - R_i) + \sum_{i=1}^{n-1} (R_{i+1} - D_i),
\]

and derive bounds for these three terms separately (we consider \(D_1\) by itself since the first excursion to \(\partial B(y,r)\) might not actually start in \(\partial B(y, R)\) and vice versa).
Before we prove the required bounds on $D_1$ and $\sum_{i=2}^n (D_i - R_i)$, we recall a standard fact about the expected time to exit a ball. For any $0 < R < \frac{1}{2}$ we have for $y \in \mathbb{T}$ all $z \in B(y, R) \subset \mathbb{T}$ that

\begin{equation}
E_z \left[ T_{B(y,R)} \right] = \frac{R^2 - |z - y|^2}{2},
\end{equation}

since the ball $z \in B(y, R)$ can be identified with a ball in $\mathbb{R}^2$. Recall also Khasminskii’s lemma (a consequence of Kac’s moment formula, see (6) [16]), which implies that for any measurable $A \subset \mathbb{T}$ and any $n \geq 1$,

\begin{equation}
\sup_{z \in \mathbb{T}} E_z \left[ H_A^n \right] \leq n! \left( \sup_{z \in \mathbb{T}} E_z \left[ H_A \right] \right)^n.
\end{equation}

We have the following crude upper bound on $E_z \left[ H_{B(0,r)} \right]$ (see (2.1) [10])

\begin{equation}
\sup_{z \in \mathbb{T}} E_z \left[ H_{B(0,r)} \right] \leq c \log r^{-1}, \text{ for any } 0 < r < \frac{1}{2}.
\end{equation}

We now prove the large deviation bound for $D_1$.

**Lemma 8.2.** $(x, y \in \mathbb{T})$ For all $0 < r < R < \frac{1}{2}$ and $u \geq 0$

\begin{equation}
P_x \left[ D_1 \geq u \right] \leq ce^{-cu/\left(\log r^{-1}\right)}.
\end{equation}

**Proof.** By the exponential Chebyshev inequality we have for all $\lambda > 0$

\[ P_x \left[ D_1 \geq u \right] \geq E_x \left[ \exp(\lambda D_1) \right] e^{-\lambda u}. \]

By the strong Markov property applied at time $H_{B(y,r)}$ (recall (2.5))

\[ E_x \left[ \exp(\lambda D_1) \right] \leq \left( \sup_{z \in \mathbb{T}} E_z \left[ \exp(\lambda H_{B(y,r)}) \right] \right) \left( \sup_{z \in \mathbb{T}} E_z \left[ \exp(\lambda T_{B(y,R)}) \right] \right). \]

By (8.9) and (8.10) we have that

\begin{equation}
\sup_{z \in \mathbb{T}} E_z \left[ T_{B(y,R)}^m \right] \leq m! R^2 \text{ for all } m \geq 1.
\end{equation}

Thus using the series expansion of $e^x$ we have

\[ \sup_{z \in \mathbb{T}} E_z \left[ \exp(\lambda T_{B(y,R)}) \right] \leq \sum_{k \geq 0} (\lambda R^2)^k \leq 2, \]

provided $\lambda R^2 \leq \frac{1}{2}$. Similarly but using (8.11) instead of (8.9) we have that

\[ \sup_{z \in \mathbb{T}} E_z \left[ \exp(H_{B(y,r)}) \right] \leq \sum_{k \geq 0} (\lambda c \log r^{-1})^k \leq 2, \]

provided $c\lambda \log r^{-1} \leq \frac{1}{2}$, where $c$ is the constant from (8.11). Thus setting $\lambda = c \frac{1}{\log r}$ for a small enough constant $c$ we obtain (8.12). \hfill $\square$

The next lemma gives the large deviation bound the sum $\sum_{i=2}^n (D_i - R_i)$, that is on the time spent “going from $\partial B(y,r)$ to $\partial B(y,R)$”.

**Lemma 8.3.** $(x, y \in \mathbb{T})$ For any $0 < r < R < \frac{1}{2}$, $n \geq 2$ and $\delta \in (0, 1)$,

\[ P_x \left[ \frac{R^2 - r^2}{2} (n-1) (1-\delta) \leq \sum_{i=2}^n (D_i - R_i) \leq \frac{R^2 - r^2}{2} (n-1) (1+\delta) \right] \geq 1 - ce^{-c(n-1)\delta^2 \left( \frac{R^2 - r^2}{n-1} \right)^2}. \]
The equality (8.9) implies that
\[ E_z \left[ \exp \left( \lambda \sum_{i=2}^{n} (D_i - R_i) \right) \right] \leq \left( \sup_{z \in \partial B(y,r)} E_z \left[ \exp \left( \lambda T_{B(y,r)} \right) \right] \right)^n. \]

The equality \( \text{(8.9)} \) implies that
\[ E_z \left[ T_{B(y,r)} \right] = \frac{R^2 - r^2}{2} \text{ for } z \in \partial B(y,r). \]

Using the series expansion of \( e^x \), (8.13) and (8.15) one obtains a bound for \( E_z \left[ \exp \left( \lambda T_{B(y,r)} \right) \right] \) involving a geometric series, so that for \( \lambda \in \mathbb{R} \) such that \( |\lambda| R^2 \leq \frac{1}{2} \) (making the geometric series summable) one has
\[ E_z \left[ \exp \left( \lambda T_{B(y,r)} \right) \right] \leq 1 + \lambda \frac{R^2 - r^2}{2} + 2\lambda^2 R^2 \leq e^{\lambda \frac{R^2 - r^2}{2} + 2\lambda^2 R^2}. \]

Therefore by the exponential Chebyshev inequality have for all \( \lambda \in (0, \frac{1}{2}) \)
\[ P_x \left[ \sum_{i=2}^{n} (D_i - R_i) \geq \frac{R^2 - r^2}{2} (n - 1) (1 + \delta) \right] \leq e^{-\lambda(n-1)\left(\delta \frac{R^2 - r^2}{2} - 2\lambda R^2\right)}, \quad \text{and} \]
\[ P_x \left[ \sum_{i=2}^{n} (D_i - R_i) \leq \frac{R^2 - r^2}{2} (n - 1) (1 - \delta) \right] \leq ce^{-\lambda(n-1)\left(\delta \frac{R^2 - r^2}{2} - 2\lambda R^2\right)}. \]

Setting \( \lambda = \frac{\delta \frac{R^2 - r^2}{2}}{8} < \frac{1}{2} \) the claim follows. \( \square \)

Next we aim to prove a similar bound on the sum \( \sum_{i=1}^{n-1} (R_{i+1} - D_i) \), i.e. on the time spent “going from \( \partial B(y,R) \) to \( \partial B(y,r) \)”. This is much more delicate, essentially because \( E_z \left[ H_{B(y,r)} \right] \) is not constant over \( z \in \partial B(y,R) \). We consider the excursions
\[ W_{(D_i+\wedge R_{i+1}), i \geq 1}, \]
as a \( C_0 ([0, \infty), \mathbb{T}) \) -valued sequence. By the strong Markov property of \( W_t \) this sequence is a Markov chain with transition kernel
\[ K(\omega, A) = \int_{\partial B(y,r)} P_{\omega(n)} \left[ W_{H_{B(y,r)}} \in du \right] P_u \left[ W_{\wedge H_{B(y,r)}} \in A \right], \]
for \( \omega \in C_0 ([0, \infty), \mathbb{T}) \) and measurable \( A \subset C_0 ([0, \infty), \mathbb{T}) \).

We employ a renewal argument which consists in making successive attempts to replace the law \( P_{\omega(\infty)} \left[ W_{H_{B(y,r)}} \in du \right] \) of the transition from the previous excursion to the start of the next excursion by the law \( \mu_x^R \). We will see that we can make this succeed with a probability \( q \) given by
\[ q = q(y, R, r) \overset{\text{def}}{=} \inf_{u \in \partial B(y,R), v \in \partial B(y,r)} \frac{P_v \left[ W_{H_{B(y,r)}} \in du \right]}{\mu_v^R (du)}. \]
We have the following lower bound on \( q \).

**Lemma 8.4.** \( (y \in \mathbb{T}) \) For all \( 0 < r < R < \frac{1}{2}, \)
\[ q \geq \left( \frac{R - r}{R + r} \right)^2. \]

**Proof.** Because of \( \text{(8.2)} \)
\[ \mu_x^R (du) \leq \sup_{u \in \partial B(y,R), v \in \partial B(y,r)} P_v \left[ W_{H_{B(y,r)}} \in du \right]. \]
Also by \((6.50)\) we have
\[
\frac{R^2 - r^2}{(R + r)^2} \leq \inf P_v \left[ W_{H_B(y, R)} \in du \right] \leq \sup P_v \left[ W_{H_B(y, R)} \in du \right] \leq \frac{R^2 - r^2}{(R - r)^2},
\]
where \(\sup\) and \(\inf\) are over \(u \in \partial B(y, R), v \in \partial B(y, r)\). Recalling \((8.18)\), the claim follows.

When a renewal succeeds, the transition from the end \(\omega(\infty)\) of the previous path to start of the next will be given by \(\mu^R_r\). When it does not succeed, it will be given by \(\nu_{\omega(\infty)}\), where for each \(a \in \partial B(y, r)\) we define \(\nu_a\) by
\[
\nu_a(A) = \nu_a(y, R, r; A) = \frac{P_a \left[ W_{T_{B(y, R)}} \in A \right] - q \mu^R_r(A)}{1 - q},
\]
for measurable \(A \subset B(y, R)\). By \((8.18)\) this is a probability measure.

We now construct a chain with the law of \(W_{(D_1 + \cdots + R_i) \wedge R_i}, i \geq 1\), on a probability space \((\mathbb{P}, S, \mathcal{S})\) in a certain way that makes the renewal structure explicit. Define on \((\mathbb{P}, S, \mathcal{S})\) an iid sequence
\[I_1, I_2, \ldots,\]
of independent Bernoulli random variables (indicating whether a renewal takes place) with success probability \(q\), and define a sequence \(X^1, X^2, \ldots\) of random trajectories in \(C_0([0, \infty), \mathbb{T})\) such that
\[
X^1 \text{ has law } P_x \left[ W_{(D_1 + \cdots + R_2) \wedge R_2} \in d\omega \right],
\]
and \(X^{i+1}\) depends on \(X^1, \ldots, X^i\) and \(I_1, \ldots, I_i\) only through \(X^i_{\infty}\) and \(I_i\), in that
\[
X^{i+1} \text{ is sampled according to law } \begin{cases} P_{\mu^R_r} \left[ W_{\wedge H_B(y, r)} \in d\omega \right] & \text{if } I_i = 1, \\ P_{\nu_{X^{i}_{\infty}}} \left[ W_{\wedge H_B(y, r)} \in d\omega \right] & \text{if } I_i = 0. \end{cases}
\]
The reason for the previous construction is the following lemma.

**Lemma 8.5.** The \(\mathbb{P}-\text{law of } (X^i)_{i \geq 1}\) coincides with the \(\mathbb{P}_x-\text{law of } (W_{(D_1 + \cdots + R_i) \wedge R_i + 1})_{i \geq 1}\).

**Proof.** By construction \((X^i)_{i \geq 1}\) is a Markov chain on the space of excursions \(C_0([0, \infty), \mathbb{T})\), and it has transition kernel
\[
\tilde{K}(\omega, A) = q P_{\mu^R_r} \left[ W_{\wedge H_B(y, r)} \in A \right] + (1 - q) \int_{\partial B(y, R)} \nu_{\omega(\infty)}(d\omega) P_w \left[ W_{\wedge H_B(y, r)} \in A \right].
\]
By \((8.20)\) we see that \(\tilde{K}(\omega, A) = K(\omega, A)\) (recall \((8.17)\)), so \((W_{(D_1 + \cdots + R_i) \wedge R_i + 1})_{i \geq 1}\) and \((X^i)_{i \geq 1}\) share the same transition kernel. Furthermore by \((8.21)\) they share the same starting distribution. Thus Lemma 8.5 follows.

We can thus derive a large deviation bound for \(\sum_{i=1}^{n-1} (R_{i+1} - D_i)\) by deriving a bound for \(\sum_{i=1}^n H_B(y, r) \left( X^i \right) \). The latter will be facilitated by the built-in renewal structure provided by the \(I_1, I_2, \ldots\). To exploit this we let
\[
J_1 = 0 \text{ and } J_i = \inf \{ m > J_{i-1} : I_m = 1 \}, i \geq 2,
\]
be the renewal times. Define the total time spent “going from \(\partial B(y, R)\) to \(\partial B(y, r)\)” during the \(m\)-th renewal by
\[
G_m = \sum_{J_m < i \leq J_{m+1}} H_B(y, r) \left( X^i \right), m \geq 1.
\]
We have the following.
Lemma 8.6. Under $\mathbb{P}$

(8.24) \[ G_1, G_2, \ldots, \text{are independent}, \]
(8.25) \[ \text{and } G_2, G_3, \ldots \text{are iid}. \]

Proof. (8.24) and (8.24) both follow by the construction (8.22) of $(X_i^i)_{i \geq 1}$, since whenever $I_i = 1$ the starting point of the next trajectory is sampled according to $\mu_r^R$, i.e. “the past is forgotten”.

To be able to later compute a large deviation bound for $\sum_{i=1}^m G_i$ we now compute the mean of $G_i$ and a bound on its moments.

Lemma 8.7. For $m \geq 2$

(8.26) \[ \mathbb{E} [G_m] = \frac{E_{\mu_R^R} [H_{B(y,r)}]}{q}, \]
and for $m \geq 1$ and $k \geq 1$

(8.27) \[ \mathbb{E} \left[ G_m^k \right] \leq \frac{k!}{q'} (c \log r^{-1})^k . \]

Proof. To see (8.26) note that from the construction (8.23) of $G_m$ and (8.22) of $X^i$ we have for $m \geq 2$

\[ \mathbb{E} [G_m] = E_{\mu_R^R} [H_{B(y,r)}] + \sum_{j=1}^{\infty} (1 - q)^j E_{\mu_R^R} \left[ E_{\nu_{W_{R_j}}} [H_{B(y,r)}] \right] . \]

By (8.2) the $P_{\mu_R^R}$—law of $W_{R_j}$ is $\mu_R^R$. Thus in fact

\[ \sum_{j=1}^{\infty} (1 - q)^j E_{\mu_R^R} \left[ E_{\nu_{W_{R_j}}} [H_{B(y,r)}] \right] = E_{\nu_{\mu_R^R}} [H_{B(y,r)}] \frac{1 - q}{q} , \]

where $\nu_{\mu_R^R} (\cdot)$ denotes the measure $\int \mu_R^R (dz) \nu_z (\cdot)$. Now by (8.20) and (8.2)

\[ \nu_{\mu_R^R} (\cdot) = \frac{P_{\mu_R^R} [W_{TB(y,r)}] - q \mu_R^R (\cdot)}{1 - q} = \frac{\mu_R^R (\cdot) - q \mu_R^R (\cdot)}{1 - q} = \mu_R^R (dw) . \]

Thus (8.26) follows since for $m \geq 2$

\[ \mathbb{E} [G_m] = E_{\mu_R^R} [H_{B(y,r)}] + E_{\mu_R^R} \left[ H_{B(y,r)} \right] \frac{1 - q}{q} = E_{\mu_R^R} \left[ H_{B(y,r)} \right] . \]

To see (8.27) note that

\[ \mathbb{E} \left[ G_m^k \right] = \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} 1 \{ j_m + j \leq j_m+1 \} H_{B(y,r)} \left( X_j^{j_m+j} \right) \right)^k \right] \]

\[ = \sum_{i_1, i_2, \ldots, j = k} 1 \{ j_m + j \leq j_m+1 \} \prod_{j=1}^{\infty} \left( H_{B(y,r)} \left( X_j^{j_m+j} \right) \right)^{i_j} \]

\[ = \sum_{i_1, i_2, \ldots, j = k} (1 - q)^{\sup \{ j: i_j \neq 0 \} - 1} \mathbb{E} \left[ \prod_{j=1}^{\infty} \left( H_{B(y,r)} \left( X_j^{j_m+j} \right) \right)^{i_j} \right] . \]

By repeated application of Khasminskii’s lemma (8.10) and the strong Markov property we have

\[ \mathbb{E} \left[ \prod_{j=1}^{\infty} \left( H_{B(y,r)} \left( X_j^{j_m+j} \right) \right)^{i_j} \right] \leq (\sup \{ z \in \mathbb{T} \} E_z [H_{B(y,r)}])^k \prod_{j=1}^{\infty} i_j ! \leq (c \log r^{-1})^k \prod_{j=1}^{\infty} i_j ! . \]
Lemma 8.8.

Thus

\[
\mathbb{E}\left[ G_m^k \right] \leq (c \log r^{-1})^k \sum_{i_1, i_2, \ldots, i_j = k} (1 - q)^{\sup\{j: i_j \neq 0\} - 1} \prod_{j=1}^{\infty} i_j!.
\]

Now if \( E_1, E_2, \ldots \), are independent standard exponential random variables which are also independent of \( J_1 \), then since \( \mathbb{E} \left[ E_i^k \right] = k! \),

\[
\sum_{i_1, i_2, \ldots, i_j = k} (1 - q)^{\sup\{j: i_j \neq 0\} - 1} \prod_{j=1}^{\infty} i_j! = \mathbb{E} \left[ \left( \sum_{J_1 < i \leq J_2} E_i \right)^k \right] = k! q^k,
\]

where the last equality holds because \( \sum_{J_1 < i \leq J_2} E_i \) is an exponential random variable with mean \( q^{-1} \). Thus (8.27) follows.

Proof. For all \( \lambda > 0 \)

\[
\mathbb{E} \left[ \exp(\lambda G_2) \right] \leq 1 + \lambda \mathbb{E} [G_2] + \sum_{k \geq 2} \frac{\lambda^k}{k!} \mathbb{E} \left[ G_2^k \right].
\]

Using (8.27) this gives

\[
\mathbb{E} \left[ \exp(\lambda G_2) \right] \leq 1 + \lambda \mathbb{E} [G_2] + 2\lambda^2 \left( \frac{c \log r^{-1}}{q} \right)^2 \leq e^{\lambda \mathbb{E}[G_2]} + 2\lambda^2 (c \log r^{-1}/q)^2,
\]

provided

\[
|\lambda| \frac{c \log r^{-1}}{q} \leq \frac{1}{2}.
\]

Similarly (but more crudely) for such \( \lambda \) we have that

\[
\mathbb{E} \left[ \exp(\lambda G_1) \right] \leq \exp \left( \lambda \frac{c \log r^{-1}}{q} \right) \leq c.
\]

Thus using an exponential Chebyshev bound, (8.24) and (8.25) we have for all \( \lambda > 0 \) as in (8.29)

\[
\mathbb{P} \left[ \sum_{i=1}^{m} G_i \geq \mathbb{E} [G_2] m (1 + \delta) \right] \leq c \exp \left( - (m - 1) \lambda \left\{ \delta \mathbb{E} [G_2] - 2\lambda \left( c \frac{\log r^{-1}}{q} \right)^2 \right\} \right),
\]

and (using also that \( \lambda \mathbb{E} [G_2] (1 - \delta) \leq \lambda c \frac{\log r^{-1}}{q} \leq c)\)

\[
\mathbb{P} \left[ \sum_{i=2}^{m} G_i \leq \mathbb{E} [G_2] m (1 - \delta) \right] \leq c \exp \left( - (m - 1) \lambda \left\{ \delta \mathbb{E} [G_2] - 2\lambda \left( c \frac{\log r^{-1}}{q} \right)^2 \right\} \right).
\]

Setting \( \lambda = c \frac{\delta \mathbb{E}[G_2]}{(c \log r^{-1}/q)^2} \) for a small enough constant \( c \) (which we may since then (8.29) is satisfied by (8.27)) we get (8.28). □
We can now use Lemma 8.8 to derive a large deviation control on the sum \( \sum_{i=1}^{n-1} (R_{i+1} - D_i) \).
For this we essentially speaking need to control the number of renewals that take place in the first \( n - 1 \) steps of the Markov chain \( (X_i)_{i \geq 1} \).

**Proposition 8.9.** \((x, y \in \mathbb{T})\) If \( n \geq 1 \) and \( \delta \in (0, c) \) then with \( \mu = \mathbb{E}_{\mu^R} [H_{B(y,r)}] \)

\[
P_x \left[ \mu (n - 1) (1 - \delta) \leq \sum_{i=1}^{n-1} (R_{i+1} - D_i) \leq \mu (n - 1) (1 + \delta) \right] 
\geq 1 - \exp \left( -c (n - 1) q^2 \delta^2 \left( \frac{\mu}{\log r^{-1}} \right)^2 \right).
\]

**Proof.** By Lemma 8.5 it suffices to show that

\[
P \left[ \mu (n - 1) (1 - \delta) \leq \sum_{i=1}^{n-1} H_{B(y,r)} (X^i) \leq \mu (n - 1) (1 + \delta) \right] 
\geq 1 - \exp \left( -c n q^2 \delta^2 \left( \frac{\mu}{\log r^{-1}} \right)^2 \right).
\]

Let

\[
m^+ = \frac{(n - 1) q}{1 + \frac{\delta}{100}} \quad \text{and} \quad m^- = \frac{(n - 1) q}{1 - \frac{\delta}{100}}.
\]

We have

\[
P \left[ \sum_{m=1}^{m^-} G_m \leq \sum_{i=1}^{n-1} H_{B(y,r)} (X^i) \leq \sum_{m=1}^{m^+} G_m \right] 
\geq 1 - \mathbb{P} \left[ J_{m^-} < n - 1 \leq J_{m^+} \right] - \mathbb{P} \left[ \sum_{i=1}^{m^-} I_i \leq m^- \right] - \mathbb{P} \left[ \sum_{i=1}^{m^+} I_i \geq m^+ \right] \geq 1 - c e^{-c(n-1)q^2\delta^2},
\]

where the last inequality follows by Hoeffding’s large deviation inequality for the binomial distribution with parameters \( m^\pm \) and \( q \). Thus the complement of the probability in (8.31) is bounded above by

\[
P \left[ \sum_{i=1}^{m^+} G_i \geq \mu (n - 1) (1 + \delta) \right] + P \left[ \sum_{i=1}^{m^-} G_i \leq \mu (n - 1) (1 - \delta) \right] + c e^{-c(n-1)q^2\delta^2}.
\]

Now by (8.26)

\[
E_{\mu^R} [H_{B(y,r)}] (n - 1) (1 + \delta) = \mathbb{E} [G_2] m^+ \left( 1 - \frac{\delta}{100} \right) \geq (1 + \delta) \mathbb{E} [G_2] m^+ \left( 1 + \frac{\delta}{2} \right),
\]

and similarly

\[
E_{\mu^R} [H_{B(y,r)}] (n - 1) (1 - \delta) \leq \mathbb{E} [G_2] m^- \left( 1 - \frac{\delta}{2} \right).
\]

Thus using (8.28) the complement of the probability in (8.31) is bounded above by

\[
\exp \left( -c (m^- - 1) \delta^2 \left( \frac{\mu}{c \log r^{-1} / q} \right)^2 \right) + c e^{-c(n-1)q^2\delta^2},
\]

so (8.31) follows, since \( \mu \leq c \log r^{-1} \) by (8.11) and \( m^- \geq cn \). \( \square \)

We can now combine Lemma 8.2, Lemma 8.3 and Proposition 8.9 to obtain a large deviation bound for \( D_n \).

**Proposition 8.10.** \((x, y \in \mathbb{T})\) For all \( 0 < r < R < \frac{1}{2} \), \( n \geq 2 \) and \( \delta \in (0, c) \)

\[
P_x \left[ \frac{1}{\pi} \log \frac{R}{r} n (1 - \delta) \leq D_n \leq \frac{1}{\pi} \log \frac{R}{r} n (1 + \delta) \right] \geq 1 - c e^{-c n \delta^2 R (1 - \pi)^6 / (\log r^{-1})^2}.
\]
Proof. Using the decomposition \((8.8)\) the complement of the probability in \((8.34)\) is above by

\[
P_x \left[ D_1 \geq n \frac{\delta}{2} \frac{1}{\pi} \log \frac{R}{r} \right] + P_x \left[ \left\{ n \left( 1 - \frac{\delta}{2} \right) \frac{R^2 - r^2}{2} \leq \sum_{i=2}^{n} (D_i - R_i) \leq n \left( 1 + \frac{\delta}{2} \right) \frac{R^2 - r^2}{2} \right\} \right] \\
+ P_x \left[ \left\{ n \left( 1 - \frac{\delta}{2} \right) \mu \leq \sum_{i=1}^{n-1} (R_{i+1} - D_i) \leq n \left( 1 + \frac{\delta}{2} \right) \mu \right\} \right],
\]

with \(\mu\) as in Proposition\((8.9)\) since by \((8.5)\) and \((8.9)\),

\[
\frac{1}{\pi} \log \frac{R}{r} = E_{\mu^R} [D_1] = E_{\mu^R} [R_1] + E_{\mu^R} [D_1 - R_1] = \mu + \frac{R^2 - r^2}{2}.
\]

Using Lemma\((8.2)\), Lemma\((8.3)\) and Proposition\((8.9)\) this quantity is at most

\[
ce^{-c n \delta \log \frac{R}{r} / (\log r)} + cc^{-c n \delta^2} \left( \frac{R^2 - r^2}{R} \right)^2 + \exp \left( -c n q^2 \delta^2 \left( \frac{\mu}{\log r} \right) \right).
\]

By \((8.19)\) we have \(q = \left( \frac{1 - r/R}{r} \right)^2\), and simple calculus shows that

\[
\mu = \frac{1}{\pi} \log \frac{R}{r} - \frac{R^2 - r^2}{2} \geq c \left( 1 - \left( \frac{r}{R} \right)^2 \right),
\]

so \(q^2 \mu^2 \geq (1 - r/R)^6 / (1 + r/R)^2 \geq c (1 - r/R)^6\). A fortiori \(\log \frac{R}{r} \geq c \left( 1 - \left( \frac{r}{R} \right)^2 \right) \geq c (1 - r/R)^6\), and \(\left( \frac{R^2 - r^2}{R} \right)^2 = R \left( 1 - \left( \frac{r}{R} \right)^2 \right) \geq R (1 - r/R)^6\), so \((8.34)\) follows. \(\square\)

Finally we may now use Proposition\((8.10)\) to prove the main result of this section: Proposition\((3.7)\). For this we use a union bound over a “packing” of circles, similarly to in the proof of Proposition\((4.7)\).

Proof of Proposition\((3.7)\). For \(y \in F_L\) let \(y_l\) denote the point in \(F_l\) that is closest to \(y\) (breaking ties in some arbitrary way). We have (cf. \((4.15)\))

\[
d(y, y_{\log L}) \leq r_{\log L} \leq L^{-1}.
\]

Now setting

\[
(8.35) \quad r_l^- = \left( 1 - \frac{100}{L} \right) r_l \quad \text{and} \quad r_l^+ = \left( 1 - \frac{100}{L} \right)^{-1} r_l \quad \text{for} \quad l \in \{0, 1\}
\]

(as in \((4.12)\)) we have that

\[
r_l^- \leq r_1 - r_{\log L} \leq r_1 + r_{\log L} \leq r_1^+ \quad \text{and} \quad r_0^- \leq r_0 - r_{\log L} \leq r_0 + r_{\log L} \leq r_0^+.
\]

Thus for all \(y \in F_L\)

\[
(8.36) \quad B \left( y_{\log L}, r_1^- \right) \subset B \left( y, r_1 \right) \subset B \left( y_{\log L}, r_1^+ \right) \subset B \left( y_{\log L}, r_0^- \right) \subset B \left( y, r_0 \right) \subset B \left( y_{\log L}, r_0^+ \right).
\]

Because of \((8.36)\), each excursion from \(\partial B \left( y, r_1 \right)\) to \(\partial B \left( y, r_0 \right)\) happens during an excursion from \(\partial B \left( y_{\log L}, r_1^- \right)\) to \(B \left( y_{\log L}, r_1^+ \right)\). Thus for all \(y \in F_L\) and all \(s\)

\[
D_{\left[ t_s \right]}^{0,0} \leq D_{\left[ t_s \right]} \left( y_{\log L}, r_0^+, r_1^- \right).
\]

Also during each excursion from \(\partial B \left( y, r_1 \right)\) to \(\partial B \left( y, r_0 \right)\) at least one excursion from \(\partial B \left( y_{\log L}, r_1^+ \right)\) to \(B \left( y_{\log L}, r_0^- \right)\) takes place. Thus we have for all \(y \in F_L\) and all \(s\)

\[
D_{\left[ t_s \right]} \left( y_{\log L}, r_0^-, r_1^+ \right) \leq D_{\left[ t_s \right]}^{0,0}.
\]
Therefore the required bounds (3.18) and (3.19) follow from

\[\lim_{L \to \infty} P_x \left[ D_{\lfloor ts \rfloor} (y, r_0^+, r_1^-) > \frac{1}{\pi} t_2s \text{ for some } y \in F_{\log L} \right] = 0,\]

\[\lim_{L \to \infty} P_x \left[ D_{\lfloor ts \rfloor} (y, r_0^-, r_1^+) < \frac{1}{\pi} t_2s \text{ for some } y \in F_{\log L} \right] = 0.\]

We use a union bound to obtain that the probability in (8.37) is bounded above by

\[|F_{\log L}| \sup_{x,y \in T} P_x \left[ D_{\lfloor ts \rfloor} (y, r_0^+, r_1^-) > \frac{1}{\pi} t_2s \right].\]

If \(\delta = \frac{s \log L}{100 L}\) since \(t_2s/t_s \geq 1 + \frac{s \log r_0^+}{2L}\) and \(\log \frac{r_0^+}{r_1^-} = 1 + O\left(\frac{1}{L}\right)\) we have for \(L \geq c\)

\[\frac{1}{\pi} t_2s \geq \frac{1}{\pi} \log \frac{r_0^+}{r_1^-} t_s (1 + \delta)\]

Thus by Proposition 8.10 (note that \(\delta \to 0\), so for \(L\) large enough the proposition is applicable),

\[\sup_{y \in T} P_x \left[ D_{\lfloor ts \rfloor} (y, r_0^+, r_1^-) > \frac{1}{\pi} t_2s \right] \leq c e^{c|\delta|^2} \frac{1 - \frac{r_0^+}{r_1^-}}{\log r_1^-}^{3/2} / (\log L)^2 \leq c e^{c|\delta|^2} \frac{1 - \frac{r_0^+}{r_1^-}}{\log L}^{3/4} / (\log \log L)^2 \leq c e^{c|\delta|^2} (\log L)^{3/2} \leq c e^{c|\delta|^2} (\log L)^{1.01}.

Going back to (8.39) we have by (3.11) that the probability in (8.37) is bounded by

\[c (\log L)^{3/2} e^{2 \log L} \times c e^{c|\delta|^2} (\log L)^{1.01} = o(1).

Thus we have proved (8.37), and therefore (3.18). The claim (3.18) (and therefore (3.19)) follows similarly by a union bound, (3.11) and Proposition 8.10.

Having proven the concentration result Proposition 3.7, all three main propositions 3.5, 3.7, that went in to the proof of the main result Theorem 3.1 have been demonstrated. Thus the proof of Theorem 3.1 is complete (except for the small proofs in the appendix). Let us finish with a remark on the conjecture (1.22) about the cover time of the discrete two dimensional torus.

Remark 8.11. In the proof of Theorem 3.1 we have used the rotational invariance of Brownian motion in balls extensively. It is this invariance which gives us the exact formula (3.5) for the probability of going “up a scale or down a scale”, and the characterisation of the traversal process \(T_{y,l}^{p,l}, l \geq 0\), as a Galton-Watson process. A lattice random walk has no such invariance property. But for balls of large radius a discrete torus analogue of (3.5) still holds approximately, and therefore an analogue of our traversal processes should behave roughly as a Galton-Watson process. Our argument therefore provides a heuristic justification of (1.22). Since the discrete torus version of (3.5) comes with a quantitative error (see Proposition 1.6.7 and Exercise 1.6.8 [19]), it is conceivable that it can also be used to prove (1.22).

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9. Appendix

In the appendix we collect some less important proofs. We first give the proof of the large deviation bound Lemma 4.6 for sums of a binomial number of geometric random variables, which was used to prove the upper bound Proposition 3.3.

Proof of Lemma 4.6 Note that

\[ P \left[ \sum_{i=1}^{n} J_i G_i \leq \theta \right] = P \left[ J_1 + \ldots + J_n \right] \leq \theta \].

Now (since a sum of geometrics is a negative binomial distribution) we have

\[ P \left[ \sum_{i=1}^{m} G_i \leq \theta \right] = P \left[ I_1 + \ldots + I_\theta \geq m \right] \text{ for } m \geq 1, \]

where \( I_1, I_2, \ldots \) are iid Bernoulli random variables with success probability \( p \), which can be taken to be independent of the \( J_i - s \). Thus (by conditioning on \( J_1 + \ldots + J_n \) in (9.1) we have in fact

\[ P \left[ \sum_{i=1}^{n} J_i G_i \leq \theta \right] = P \left[ I_1 + \ldots + I_\theta \geq J_1 + \ldots + J_n \right]. \]

For any \( \lambda > 0 \) this probability is bounded above by

\[ \mathbb{E} \left[ \exp \left( \lambda \left( I_1 + \ldots + I_\theta - J_1 - \ldots - J_n \right) \right) \right] = \left( 1 + p \left( e^{\lambda} - 1 \right) \right)^\theta \left( 1 + q \left( e^{\lambda} - 1 \right) \right)^n \leq \exp \left( \theta p \left( e^{\lambda} - 1 \right) + qn \left( e^{\lambda} - 1 \right) \right), \]

where we have used that \( 1 + x \leq e^x \). Now (4.23) follows by setting \( \lambda = \frac{1}{2} \log \frac{q_m}{2p}. \)

□

Next we derive the characterisation Lemma 7.7 of local times of continuous time random walk on \( \{0, \ldots, L\} \) from the generalized second Ray-Knight theorem. Recall the definition of \( \tilde{P}_t \) and \( Y_t \) from above (7.38) and the definition of \( L^l_t \) from (7.38).

Proof of Lemma 7.7 Let \( \mathcal{L} = \{0, \ldots, L\} \). The generalized second Ray-Knight theorem (see [15] or Theorem 8.2.2 [21]) implies that \( \left( L^\tau_l + \frac{1}{2} \eta_l^2 \right)_{l \in \mathcal{L}} \overset{law}{=} \left( \frac{1}{2} \left( \theta_l + \sqrt{2t} \right)^2 \right)_{l \in \mathcal{L}} \), where \( \eta_l \) is a centered Gaussian process on \( \mathcal{L} \) with covariance \( \mathbb{E} [ \eta_l \eta_b] = \tilde{E}_b \left[ L^0_{\frac{a}{b}} \right] = a \) for \( b \leq a \), independent of \( L^\tau_l \). Thus \( \eta_l, l \in \mathcal{L} \), is in fact Brownian motion at the integer times \( l \in \mathcal{L} \). This in turn implies that \( \left( \frac{1}{2} \eta_l^2 \right)_{l \in \mathcal{L}} \) has the \( Q^0 \)-law of \( \left( \frac{1}{2} X_l \right)_{l \in \mathcal{L}} \) and \( \left( \frac{1}{2} \left( \eta_l + \sqrt{2t} \right)^2 \right)_{l \in \mathcal{L}} \) has the \( Q^1_{2t} \)-law of \( \left( \frac{1}{2} X_l \right)_{l \in \mathcal{L}} \) (recall (7.23)). By the additivity property (7.25) of Bessel processes we thus have that \( \left( L^\tau_l + \frac{1}{2} X_l \right)_{l \in \mathcal{L}} \overset{law}{=} \left( \frac{1}{2} X_l + \frac{1}{2} X^2_l \right)_{l \in \mathcal{L}} \)

where \( (X^1_l)_{l \geq 0} \) has law \( Q^1_0 \), \( X^2_l \) law \( Q^0_{2t} \). Now the claim follows because we may “cancel out” \( \frac{1}{2} X^2_l \) from this equality in law, since all random variables involved are non-negative (see (2.56) [27]).

□

Next we give the proof of Lemma 7.8 which describes the law of the local times \( L^{D_l}, l \in \{0, \ldots, L\} \), of continuous time random walk on \( \{0, 1, \ldots, L\} \) when conditioned on \( L^{D_l}_L = 0 \). Recall the definition of \( D_l \) from (7.39). For the proof let us denote by \( \Gamma \) the state space of \( (Y_l)_{l \geq 0}, \) that is the space of all piecewise constant cadlag functions from \( [0, \infty) \) to \( \{0, \ldots, L\} \).
Proof of Lemma 7.8. Define the successive returns to and departures from \( \{0, \ldots, L\} \setminus \{1\} \) of \( Y_t \) by \( \tilde{D}_0 = H_1 \), 
\[
\tilde{R}_n = H_{\{0,2\}} \circ \theta_{\tilde{D}_{n-1}} + \tilde{D}_{n-1}, n \geq 1, \quad \text{and} \quad \tilde{D}_n = H_1 \circ \theta_{\tilde{R}_n} + \tilde{R}_n, n \geq 1.
\]
Collect the excursions of \( Y_t \) into a marked point process \( \mu \) on \( [0, \infty) \times \Gamma \) defined by 
\[
\mu = \sum_{i \geq 1} \delta(L_{i}^{\delta}, Y_{(\tilde{R}_{i})^{\wedge} \wedge \tilde{D}_{i}}).
\]
The point process \( \mu \) is a Poisson point process on \( \mathbb{R}_+ \times \Gamma \) of intensity 
\[
\lambda \otimes \left( \frac{1}{2} \mathbb{P}_0 [Y_{\wedge} H_1 \in dw] + \frac{1}{2} \mathbb{P}_2 [Y_{\wedge} H_1 \in dw] \right),
\]
where \( \lambda \) is Lebesgue-measure normalized so that \( \lambda([0,1]) = 2 \). We can decompose this point process into 
\[
\mu_1 = 1_{\mathbb{R}_+ \times \{Y_0 = 0\}} \mu \quad \text{and} \quad \mu_2 = 1_{\mathbb{R}_+ \times \{Y_0 = 2, H_1 < H_3\}} \mu \quad \text{and} \quad \mu_3 = 1_{\mathbb{R}_+ \times \{Y_0 = 2, H_3 < H_1\}} \mu,
\]
where \( \mu_1 \) collects the excursions that start in 0, \( \mu_2 \) collects the excursions that start in 2 and hit \( L \), and \( \mu_3 \) has the excursions that start in 2 and avoid \( L \). Since we are restricting \( \mu \) to disjoint sets, \( \mu_1, \mu_2 \) and \( \mu_3 \) are independent Poisson point processes.

Let 
\[
\mu_1 = \sum_i \delta(S_i, w_i),
\]
for \( S_1 < S_2 < \ldots \), so that \( S_t \) is the local time at vertex 1 until the \( t \)-th jump to 0. Note that (recall (7.39)) 
\[
L_1^{D_t} = S_t, \quad \text{for} \ t \in \{1, 2, \ldots \}
\]
We have 
\[
L_1^{D_t} = \sum_{(s,w) \in \mu_2: s \leq t} L_1^{\infty}(w_i) \quad \text{for} \ l \in \{2, 3, \ldots \}
\]
where \( L_1^{\infty}(w) \) is the local time at \( l \) of the path \( w \), i.e. \( L_1^{\infty}(w) = d_t^{-1} \int_0^\infty 1_{\{w_s = t\}} ds \) for \( d_t \) as in (7.37). For any \( u \geq 0 \) define the vector 
\[
V_u = \left( u, \sum_{(s,w) \in \mu_2: s \leq t} L_1^{\infty}(w_i), \ldots, \sum_{(s,w) \in \mu_2: s \leq t} L_L^{\infty}(w_i) \right) \in \mathbb{R}^L,
\]
By (9.3) we have 
\[
(L_1^{D_t})_{t \in \{1, \ldots, L\}} = V_{S_t} \text{ on the event } \left\{ L_1^{D_t} = 0 \right\} = \{ \mu_3 ([0, S_t] \times \Gamma) = 0 \}.
\]
Furthermore note that \( L_1^{D_t} \) and \( \{ L_1^{D_t} = 0 \} \) only depend on \( \mu_1 \) and \( \mu_3 \), while \( V_u \) only depends on \( \mu_2 \), which is independent of \( \mu_1 \) and \( \mu_3 \). Therefore 
\[
\mathbb{P}_0 \left[ \left( L_1^{D_t} \right)_{t \in \{1, \ldots, L\}} \in A \mid L_1^{D_t} = 0 \right] = \mathbb{P}_0 \left[ f \left( L_1^{D_t} \right) \mid L_1^{D_t} = 0 \right],
\]
where \( f(u) = \mathbb{P}_0 [V_u \in A] \).

We are thus interested in the law of \( V_u \). Let \( \tilde{Y}_t \) be continuous time random walk on \( \{1, \ldots, L\} \) with local times and inverse local time at vertex 1 given by 
\[
\tilde{L}_1^u = \frac{1}{1 + 1_{\{1 < t < L\}}} \int_0^u 1_{\{\tilde{Y}_s = t\}} ds \quad \text{and} \quad \tilde{\tau}(t) = \inf \left\{ s \geq 0 : \tilde{L}_1^s > u \right\}.
\]
Sampling $\tilde{Y}_t$, $t \geq 0$, by “stiching together” the excursions in the point processes $\mu_2$ and $\mu_3$ we see that

$$
(9.6) \quad \left( \tilde{L}^{(u)}_{\tau} \right)_{l \in \{2, \ldots, L\}} \overset{\text{law}}{=} \left( \sum_{(s, w) \in \mu_2 \cup \mu_3; s \leq u} L^\infty_{l} (w) \right)_{l \in \{2, \ldots, L\}}.
$$

So by Lemma 7.7 (with $\{1, \ldots, L\}$ in place of $\{0, \ldots, L\}$) we have that

$$
(9.7) \quad \tilde{\mathbb{P}}_0 \left[ \left( \sum_{(s, w) \in \mu_2 \cup \mu_3; s \leq u} L^\infty_{l} (w) \right)_{l \in \{2, \ldots, L\}} \in \cdot \right] = \tilde{\mathbb{P}}_0 \left( \tilde{L}^{(u)}_{\tau} \in \cdot \right) = \mathbb{Q}^0_{2u} \left( \left( \frac{1}{2} X_l \right)_{l \in \{0, \ldots, L-1\}} \in \cdot \right).
$$

Now since

$$
\{ \mu_3 \left( [0, t] \times \Gamma \right) = 0 \} = \left\{ \sum_{(s, w) \in \mu_2 \cup \mu_3; s \leq t} L^\infty_{l} (w) = 0 \right\},
$$

and $V_u$ is independent of $\mu_3$ we have

$$
\tilde{\mathbb{P}}_0 [V_u \in A] = \tilde{\mathbb{P}}_0 [V_u \in A | \mu_3 \left( [0, t] \times \Gamma \right) = 0] = \tilde{\mathbb{P}}_0 \left[ \left( \tilde{L}^{(u)}_{L \tau} \right)_{l \in \{1, \ldots, L\}} | \tilde{L}^{(u)}_{L \tau} = 0 \right] \mathbb{Q}^0_{2u} \left( \left( \frac{1}{2} X_l \right)_{l \in \{0, \ldots, L-1\}} \in A | X_{L-1} = 0 \right) \mathbb{Q}^{0, L-1}_{2u-\tau} \left( \left( \frac{1}{2} X_l \right)_{l \in \{0, \ldots, L-1\}} \in A \right).
$$

Plugging this into (9.5) gives the claim. $\Box$

The same construction of $Y_t$ from the Poisson point processes $\mu_1, \mu_2$ and $\mu_3$ can be used to Lemma 7.9 which gives a control on the law of $L^D_{1t}$ conditioned on $L^D_{Lt} = 0$.

**Proof of Lemma 7.9.** We will first show that

$$
(9.8) \quad \text{the } \tilde{\mathbb{P}}_0 \left[ \left| L^D_{1t} = 0 \right| - \text{law of } L^D_{1t} \text{ is that of a sum of } t \text{ independent standard exponential random variables.}
$$

In the notation of the proof of Lemma 7.8 Since $L^D_{1t} = S_t$ and $\{ L^D_{Lt} = 0 \} = \{ \mu_3 \left( [0, S_t] \times \Gamma \right) = 0 \}$ we are interested in the law of $S_t$ given $\{ \mu_3 \left( [0, S_t] \times \Gamma \right) = 0 \}$. Since $\mu_3$ is independent of $S_t$ we have that

$$
\tilde{\mathbb{P}}_0 [S_t = ds, \mu_3 \left( [0, S_t] \times \Gamma \right) = 0] = \tilde{\mathbb{P}}_0 \left[ S_t = ds, \tilde{\mathbb{P}} \left[ \mu_3 \left( [0, s] \times \Gamma \right) = 0 \right] \right].
$$

The intensity of $\mu_3$ is the $\lambda \otimes \frac{1}{2} \tilde{\mathbb{P}}_2 [\cdot, H_L < H_1]$ (recall (9.2)), so that

$$
\tilde{\mathbb{P}}_0 \left[ \mu_3 \left( [0, s] \times \Gamma \right) = 0 \right] = e^{-s} \tilde{\mathbb{P}}_2 [H_L < H_1] = e^{-\frac{s}{\tau + 1}}, \text{ and } \tilde{\mathbb{P}}_0 \left[ S_t = ds, \mu_3 \left( [0, S_t] \times \Gamma \right) = 0 \right] = e^{-\frac{s}{\tau + 1}} \tilde{\mathbb{P}}_0 [S_t = ds].
$$

The $\tilde{\mathbb{P}}_0$–law of $S_t$ is the gamma distribution with shape $t$ and scale 1. Thus

$$
\tilde{\mathbb{P}}_0 [S_t = ds, \mu_3 \left( [0, S_t] \times \Gamma \right) = 0] = e^{-\frac{s}{\tau + 1}} s^{t-1} e^{-s} \frac{s^{t-1}}{(t-1)!} = e^{-\frac{s}{\tau + 1}} (\frac{1}{\tau + 1}\cdot s)^{t-1} \frac{1}{(t-1)!},
$$

so that the $\tilde{\mathbb{P}}_0 \left[ \left| \mu_3 \left( [0, S_t] \times \Gamma \right) = 0 \right| - \text{law of } S_t$ is the gamma distribution with shape $t$ and scale $(1 + 1/(L-1))^{-1} = (L-1)/L$. This proves (9.8).
Since \( t \leq 10L^2 \) the probability in (7.40) is bounded below by

\[
\tilde{p}_0 \left[ \sqrt{t \frac{L-1}{L}} - 100 \leq \sqrt{L_{D_t}^L} \leq \sqrt{t \frac{L-1}{L}} + 100 | L_{D_t}^L = 0 \right].
\]

By (9.8) and the Central Limit Theorem the \( \tilde{p}_0 \left[ |L_{D_t}^L = 0 \right] - \text{law of } \left( \frac{L_{D_t}^L}{\sqrt{t \frac{L-1}{L}}} - t \right)/\sqrt{t} \) converges to a normal random variable as \( t \to \infty \), uniformly in \( L \). This implies that (9.9) is bounded below, so (7.40) follows.

Also for \( k = 1, 2 \)

\[
| \tilde{u} (l) - v |^k \leq c \left\{ \left| \tilde{u} (l) - \sqrt{2t} \left( \frac{L-1}{L} \right)^{3/2} \right|^k + \left| \sqrt{2t} \left( \frac{L-1}{L} \right)^{3/2} - u (l) \right|^k + | u (l) - v |^k \right\}.
\]

We have

\[
\left| \sqrt{2t} \left( \frac{L-1}{L} \right)^{3/2} - u (l) \right| = u (l) \left| \sqrt{L_{D_t}^L} - 1 \right| \leq cu (l) L^{-1} \leq c \sqrt{L} L^{-1} \leq c.
\]

Also

\[
| \tilde{u} (l) - \sqrt{t} \left( \frac{L-1}{L} \right)^{3/2} | = L_{D_t}^L \left| \tilde{u} (l) - \sqrt{t} \left( \frac{L-1}{L} \right)^{3/2} \right| \leq c + \left| \tilde{u} (l) - \sqrt{t} \left( \frac{L-1}{L} \right)^{3/2} \right|,
\]

and

\[
| \tilde{u} (l) - \sqrt{t} \left( \frac{L-1}{L} \right)^{3/2} | \leq c \left| \sqrt{L_{D_t}^L} - \sqrt{t} \left( \frac{L-1}{L} \right)^{3/2} \right| \leq c \frac{L_{D_t}^L}{\sqrt{t} \sqrt{L_{D_t}^L}}.
\]

Taking the expectation in (9.10) and using

\[
\tilde{p}_0 \left[ \frac{L}{L-1} L_{D_t}^L - t \left| L_{D_t}^L = 0 \right] \right] \leq c \sqrt{t},
\]

(also Cauchy-Schwarz if \( k = 2 \)) we get (7.41).

We remains to prove Lemma 9.1, giving a large deviation bound for the number of traversals \( T_i^t \) (recall (7.42)) given the continuous local times \( L_{D_t}^L \). For this we will need the following computation of the conditional distribution of \( T_i^t \) (which can be seen as a special case of the results of Section 4 [13]). To prove it we use the following fact about the modified Bessel function of the first kind \( I_1 (\cdot) \):

\[
\sum_{m \geq 1} \frac{z^m}{m! (m-1)!} = \sqrt{z} I_1 (2 \sqrt{z}) \quad \text{for all } z \in \mathbb{R}.
\]

**Lemma 9.1.** For all \( u_0, u_1, u_2, \ldots, u_L \in [0, \infty) \) such that \( u_i = 0 \implies u_{i+1} = 0 \), and any \( l \in \{1, \ldots, L-1\} \) such that \( u_{i+1} > 0 \) we have for \( m \in \{1, 2, \ldots\} \)

\[
\tilde{p}_0 \left[ T_i^t = m | L_{D_t}^L = u_i, l = 0, \ldots, L \right] = \frac{(u_i u_{i+1})^m / (m! \cdot (m-1)!)}{\sqrt{u_i u_{i+1}} I_1 (2 \sqrt{u_i u_{i+1}})}.
\]

**Proof.** The law of \( T_1 \) under \( \mathbb{G}_a \) can be written down explicitly as

\[
\mathbb{G}_a [T_1 = b] = \binom{a+b-1}{a-1} \left( \frac{1}{2} \right)^{a+b} \quad \text{for } a \in \{1, 2, \ldots\}, \ b \in \{0, 1, 2, \ldots\},
\]

since there are \( \binom{a+b-1}{a-1} \) ways to write \( b \) as a sum of \( a \) non-negative integers, and since the probability that a geometric random variable with support \( \{0, 1, \ldots\} \) and mean 1 takes on
the value \( k \) is \( (\frac{1}{2})^{k+1} \). By Lemma 7.10, we therefore have for all \( t = t_0, t_1, t_2, \ldots, t_{L-1} \in \{0, 1, 2, \ldots\} \) such that \( t_i = 0 \implies t_{i+1} = 0 \)

\[
P_0 \left[ \tilde{T}_t^l = t_i, i = 0, \ldots, L - 1 \right] = \prod_{i \in \{1, \ldots, L-1\} : t_{i-1} > 0} \left( \frac{t_{i-1} + t_i - 1}{t_{i-1} - 1} \left( \frac{1}{2} \right)^{t_{i-1} + t_i} \right).
\]

Conditioned on the number of visits to each vertex the total holding times at the vertices are independent and gamma distributed, so we have for such \( t_i \) and any \( u_0, u_1, \ldots, u_L \in [0, \infty) \) such that \( t_{i-1} = 0 \implies u_l = 0 \) that

\[
P_0 \left[ \tilde{T}_t^l = t_i, l = 1, \ldots, L - 1, L_t^D = u_l, l = 0, \ldots, L \right] = \left\{ \prod_{i \in \{1, \ldots, L-1\} : t_{i-1} > 0} \left( \frac{t_{i-1} + t_i - 1}{t_{i-1} - 1} \left( \frac{1}{2} \right)^{t_{i-1} + t_i} \right) \right\} \times \left\{ \frac{e^{-u_1 u_0^{-1}}}{(t_0 - 1)!} \prod_{i \in \{1, \ldots, L-1\} : t_{i-1} > 0} \frac{e^{-2u_1 u_0^{i-1} + 1}}{u_i(t_{i-1} + t_i - 1)!} \frac{e^{-u_L u_L^L - 1}}{(t_{L-1} - 1)!} \right\}
\]

where the quantity in the last parenthesis is interpreted as 1 if \( t_{i-1} = 0 \) or \( u_L = 0 \). Exploiting two cancellations the right-hand side equals

\[
\left\{ \prod_{i \in \{1, \ldots, L-1\} : t_{i-1} > 0} \frac{1}{(t_{i-1} - 1)!} \right\} \times \left\{ \frac{e^{-u_1 u_0^{-1}}}{(t_0 - 1)!} \prod_{i \in \{1, \ldots, L-1\} : t_{i-1} > 0} \frac{e^{-2u_1 u_0^{i-1} + 1}}{u_i(t_{i-1} + t_i - 1)!} \frac{e^{-u_L u_L^L - 1}}{(t_{L-1} - 1)!} \right\}
\]

Considering only the terms that depend on \( t_i \) we have that if \( u_0, u_1, \ldots, u_{l+1} > 0 \)

\[
P_0 \left[ \tilde{T}_t^l = m|L_t^D = u_l, l = 0, \ldots, L \right] = \frac{1}{\tilde{Z} (m - 1)!} \frac{u_l^{m} u_{l+1}^{m}}{(m - 1)! m!} = \sqrt{u_l u_{l+1}} I_1 \left( 2 \sqrt{u_l u_{l+1}} \right)
\]

for a normalizing constant \( \tilde{Z} \) depending only on \( t_0, u_0, \ldots, u_L \). Using (9.11) we can identify the constant as

\[
\tilde{Z} = \sum_{m \geq 1} \frac{(u_l u_{l+1})^m}{(m - 1)! m!} = \sqrt{u_l u_{l+1}} I_1 \left( 2 \sqrt{u_l u_{l+1}} \right)
\]

We now prove the large deviation result Lemma 7.12 for the traversal process \( \tilde{T}_t^l \) conditioned on \( L_t^D, l = 0, \ldots, L \).

**Proof of Lemma 7.12** Denote \( \tilde{P}_0 \left[ l \mid \sigma \left( L_t^D : l = 0, \ldots, L \right) \right] \) by \( \tilde{Q} \). By Lemma 9.1

\[
\tilde{Q} \left[ \exp \left( \lambda \tilde{T}_t^l \right) \right] = \sum_{m \geq 1} \frac{(e^{\lambda} u^2)^{m} / (m! \cdot (m - 1)!)}{\mu I_1 \left( 2 \mu \right)} \frac{e^{\lambda/2} \mu I_1 \left( 2e^{\lambda/2} \mu \right)}{\mu I_1 \left( 2 \mu \right)}
\]

Thus for all \( \lambda > 0 \)

\[
\tilde{Q} \left[ \tilde{T}_t^l \geq \mu + \theta \right] \leq e^{\lambda/2} I_1 \left( 2e^{\lambda/2} \mu \right) I_1 \left( 2 \mu \right) \exp \left( -\lambda (\mu + \theta) \right)
\]

Using the standard estimate \( I_1 \left( z \right) = \frac{e^z}{\sqrt{2\pi z}} \left( 1 + O \left( z^{-1} \right) \right) \) we have that

\[
I_1 \left( 2e^{\lambda/2} \mu \right) / I_1 \left( 2 \mu \right) \leq ce^{\lambda/4} e^{2e^{\lambda/2 - 1}} \mu
\]
so that for all $\lambda > 0$
\[
\tilde{Q}\left[\tilde{T}_t^+ \geq \mu + \theta\right] \leq ce^{\lambda} \exp\left(2 \left\{ e^{\lambda/2} - 1 \right\} \mu - \lambda \{ \mu + \theta \} \right) \leq ce^{\lambda} \exp \left( c\lambda^2 \mu - \lambda \theta \right).
\]
Setting $\lambda = c\theta/\mu$ for a small enough $c$ the right-hand side is bounded above by $ce^{c\theta/\mu-c\theta^2/\mu}$, giving one half of (7.44). By estimating $\tilde{Q}\left[\exp \left(-\lambda \tilde{T}_t^+\right)\right]$ one can similarly show that $\tilde{Q}\left[\tilde{T}_t^+ \leq \mu - \theta\right] \leq ce^{c\theta/\mu-c\theta^2/\mu}$, giving the other half. 

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