Flags of sheaves, quivers and symmetric polynomials

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Abstract

We study the representation theory of the nested instantons quiver presented in [2], which describes a particular class of surface defects in four-dimensional supersymmetric gauge theories. We show that the moduli space of its stable representations provides an ADHM-like construction for nested Hilbert schemes of points on $\mathbb{C}^2$, for rank one, and for the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$, for higher rank. We introduce a natural torus action on this moduli space and use equivariant localization to compute some of its (virtual) topological invariants, including the case of compact toric surfaces. We conjecture that the generating function of holomorphic Euler characteristics for rank one is given in terms of polynomials in the equivariant weights, which, for specific numerical types, coincide with (modified) Macdonald polynomials.

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Introduction

In [2] we introduced the moduli space of nested instantons as the moduli space of stable representations of a suitable quiver. This arises in the study of surface defects in supersymmetric gauge theory on $T^2 \times C_{g,k}$, where $T^2$ is a real two torus and $C_{g,k}$ a genus $g$ complex projective curve with $k$ marked points. Let us briefly describe some string theory motivations before presenting the content of the paper.

String theory motivations: The D-brane set-up engineering the surface defect is described in [2], and its analysis naturally led to a description in terms of representations of a quiver in the category of vector spaces, the D-branes being the objects and the open strings being the morphisms. Let us briefly resume the D-brane geometry and its relation with the relevant mathematical problems. One considers type IIB supersymmetric background given by $T^2 \times T^* C_{g,k} \times \mathbb{C}^2$, with $r$ D7-branes located at points of the fiber of the cotangent bundle and $n$ D3-branes along $T^2 \times C_{g,k}$. The low energy effective theory of the D7-branes is equivariant higher rank Donaldson-Thomas theory [12] on the four-fold $T^2 \times C_{g,k} \times \mathbb{C}^2$, while the low energy effective theory of the D3-branes is equivariant Vafa-Witten theory on $T^2 \times C_{g,k}$, [33]. In the chamber of small volume of $C_{g,k}$, the effective theory describing the surface defect is encoded in the theory of maps from $T^2$ to the moduli space of stable representations of the comet shaped quiver displayed in figure 1. For $k = 1$, this is described by the total space of a bundle $\mathcal{V}_g$ over the nested instanton moduli space, which in turn is the moduli space
of stable representations of the quiver displayed in figure 2. Let us remark that virtual invariants of \( \mathcal{V}_g \) have a connection to the cohomology of character varieties of punctured Riemann surfaces, and in particular to the conjecture proposed in [21] whose physical interpretation was provided in [8]. The interested reader can find the details in [2].

**Figure 1:** The comet-shaped quiver.

**Figure 2:** The nested instantons quiver.

**Content of the paper:** In this paper we concentrate on the study of representations of the nested instantons quiver with a single framing, namely we choose the dimension vector for the framing to be \( \mathbf{r} = (r, 0, \ldots, 0) \) where \( r \) is the dimension of the rightmost framing node. We also study its relation to flags of framed torsion-free sheaves on \( \mathbb{P}^2 \) and nested Hilbert schemes, and compute some relevant virtual invariants via equivariant localisation.
We want to point out that the moduli space we are studying seems to be analogous to the Filt-scheme studied in [26] in the case of smooth projective curves. The importance of studying these moduli spaces on (smooth projective) surfaces lies in their application to the computation of monopole contributions to Vafa-Witten invariants defined in [31, 32]. In fact these monopole contributions to Vafa-Witten invariants are expressed in terms of invariants of flags of sheaves, which in some cases reduce to nested Hilbert schemes, see [20, 24] for computations in this case. The deformation-obstruction theory and virtual cycle for the components of the monopole branch in Vafa-Witten theory giving rise to flags of higher rank sheaves were explicitly constructed in [30]. Nested Hilbert schemes on surfaces were interpreted in terms of degeneracy loci in [18, 17], where they are also shown to be equipped with a perfect obstruction theory. Similarly nested Hilbert schemes of points were also studied in [15], and a perfect obstruction theory and virtual cycles are explicitly constructed. Their application to reduced DT and PT invariants are also discussed in [15, 16, 11].

In the following we give a summary of the result we obtained in this paper. In section 1 we start our analysis by proving the following

**Theorem.** The moduli space \( N(r, n) \) of stable representation of the nested instantons quiver of numerical type \((r, n)\) is a virtually smooth quasi-projective variety over \( \mathbb{C} \) equipped with a natural action of \( T = T \times (\mathbb{C}^*)^r, \) \( T = (\mathbb{C}^*)^2, \) and a perfect obstruction theory.

We also prove that \( N(r, n) \) embeds into a smooth projective variety \( M(r, n) \), see Section 1.3.

In section 2, we construct the moduli space \( F(r, \gamma) \) of flags of framed torsion free sheaves on \( \mathbb{P}^2 \) and prove the existence of an isomorphism with \( N(r, n) \). As a particular case, we have

**Theorem.** The moduli space of nested instantons \( N(1, n) \) is isomorphic to the nested Hilbert scheme of points on \( \mathbb{C}^2 \), namely

\[
N(1, n) = X_0//G \simeq \text{Hilb}^0(\mathbb{C}^2).
\]

(0.1)

The moduli space of flags of sheaves is constructed by means of a functor

\[
F(r, \gamma) : \text{Sch}^{\text{op}} \rightarrow \text{Sets}
\]

parametrizing flags of torsion-free sheaves on \( \mathbb{P}^2 \) in

**Proposition.** The moduli functor \( F(r, \gamma) \) is representable. The (quasi-projective) variety representing \( F(r, \gamma) \) is the moduli space of flags of framed (coherent) torsion-free sheaves on \( \mathbb{P}^2 \), denoted by \( F(r, \gamma) \).

while its isomorphism with \( N(r, n) \) is proven in
Theorem. The moduli space of stable representations of the nested instantons quiver is a fine moduli space isomorphic to the moduli space of flags of framed torsion-free sheaves on \( \mathbb{P}^2 \): \( \mathcal{F}(r, \gamma) \cong \mathcal{N}(r, n) \), as schemes, where \( n_i = \gamma_1 + \cdots + \gamma_N \).

The ADHM construction of a particular class of flags of sheaves on \( \mathbb{P}^2 \) was given in [28], where their connection to shuffle algebras on \( K \)-theory is also studied. Moreover the construction of the functor \( F_{(r, \gamma)} \) shows that the moduli space of nested instantons is isomorphic to a relative Quot–scheme. Perfect obstruction theories on Quot–schemes and the description of their local model in terms of a quiver is discussed in [1, 29].

In section 3 we proceed to the evaluation of the relevant virtual invariants via equivariant localisation. The classification of the \( T \)-fixed locus of \( \mathcal{N}(r, n) \) is presented in the

Proposition. The \( T \)-fixed locus of \( \mathcal{N}(r, n_0, \ldots, n_{s-1}) \) can be described by \( s \)-tuples of nested coloured partitions \( \mu_1 \subseteq \cdots \subseteq \mu_s \subseteq \mu_0 \), with \( |\mu_0| = n_0 \) and \( |\mu_{i>0}| = n_0 - n_i \).

In 3.2 we compute the generating function of the virtual Euler characteristics of \( \mathcal{N}(1, n) \), see eq.(3.29) for the explicit combinatorial formula. We conjecture that, by summing over the nested partitions, this generating function is expressed in terms of polynomials:

Conjecture. The generating function \( \chi_{\text{vir}}(\mathcal{N}(1, n_0, \ldots, n_N); q_1^{-1}, q_2^{-1}) = \sum_{r_0} P_{r_0}(q, t)/N_{r_0}(q, t) \) is such that

\[
P_{r_0}(q, t) = \frac{Q_{r_0}(q, t)}{(1 - qt)^N},
\]

with \( Q_{r_0}(q, t) \in \mathbb{Z}[q, t] \).

For specific profiles of the nesting, these polynomials are conjectured to compute sums of \((q, t)\)–Kostka polynomials:

Conjecture. When \(|\mu_0| = |\mu_N| + 1 = |\mu_{N-1}| + 2 = \cdots = |\mu_1| + N\) we have

\[
Q_{\mu_0}(q, t) = \left< h_{\mu_0}(x), \tilde{H}_{\mu_0}(x; q, t) \right>
= \left< h_{\mu_0}(x), \sum_{\lambda \in \mathcal{P}(n_0)} \tilde{K}_{\lambda, \mu_0}(q, t) K_{\mu_0, \nu}(x) \right>
= \sum_{\lambda \in \mathcal{P}(n_0)} \tilde{K}_{\lambda, \mu_0}(q, t),
\]

where the Hall pairing \( \langle -, - \rangle \) is such that \( \langle h_{\mu}, m_{\lambda} \rangle = \delta_{\mu, \lambda} \) and \( \tilde{H}_{\mu}(x; q, t) \), \( \tilde{K}_{\lambda, \mu}(q, t) \) are the modified Macdonald polynomials and the modified Kostka polynomials, respectively.
In 3.3 we compute the generating function of the virtual $\chi_y$-genus of $\mathcal{N}(1,n)$, see eq.(3.49), and of $\mathcal{N}(r,n)$, see eq.(3.53).

We also show that, by specialising at $y = 1$, one gets that the generating function of nested partitions of arbitrary length is the Macmahon function as expected, see eq. (3.60).

In 3.4 we compute the generating function of the virtual elliptic genus of $\mathcal{N}(1,n)$, see eq.(3.72), and of $\mathcal{N}(r,n)$, see eq.(3.77).

Finally, in section 4, we extend our results to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in the case of $\chi_y$-genera, see formulae (4.14) and (4.19) respectively. Notice that the choice of computing $\chi_y$-genera was due to the expected simple polynomial dependence in $y$. Everything which was done in this context is however completely general and holds for any complex genus.

Acknowledgements: We thank U. Bruzzo, E. Diaconescu, L. G"ottsche, T. Hausel, M. Kool, A. Mellit, A. T. Ricolfi, F. Rodriguez-Villegas, A. Sheshmani, Y. Tanaka for useful discussion. The work of G.B. is partially supported by INFN - ST&FI and by the PRIN project ”Non-perturbative Aspects Of Gauge Theories And Strings”. The work of N.F. and A.T. is partially supported by INFN - GAST and by the PRIN project ”Geometria delle variet`a algebriche”.

1 The nested instantons quiver

1.1 Quiver representations and stability

In the following we will mainly be interested in studying the following quiver, which will be called the nested instantons quiver

$$
V_N \xrightarrow{\phi_N} \ldots \xrightarrow{\phi_2} V_1 \xrightarrow{\phi_1} V_0 \xleftarrow{\eta} W
$$

with relations

$$
[\alpha_0, \beta_0] + \xi \eta = 0, \quad [\alpha_i, \beta_i] = 0, \quad \alpha_i \phi_i - \phi_i \alpha_{i+1} = 0 = \beta_i \phi_i - \phi_i \beta_{i+1} \\
\gamma_i \alpha_i - \alpha_{i+1} \gamma_i = 0 = \gamma_i \beta_i - \beta_{i+1} \gamma_i, \quad \phi_i \gamma_i = 0, \quad \eta \phi_1 = 0, \quad \gamma_1 \xi = 0
$$

with relations

$$
[\alpha_0, \beta_0] + \xi \eta = 0, \quad [\alpha_i, \beta_i] = 0, \quad \alpha_i \phi_i - \phi_i \alpha_{i+1} = 0 = \beta_i \phi_i - \phi_i \beta_{i+1} \\
\gamma_i \alpha_i - \alpha_{i+1} \gamma_i = 0 = \gamma_i \beta_i - \beta_{i+1} \gamma_i, \quad \phi_i \gamma_i = 0, \quad \eta \phi_1 = 0, \quad \gamma_1 \xi = 0
$$
Given

\[ X = \text{End } V_0^2 \oplus \text{Hom}(V_0, W) \oplus \text{Hom}(W, V_0) \oplus \text{End}(V_1)^2 \oplus \text{Hom}(V_1, V_0) \oplus \text{Hom}(V_0, V_1) \oplus \cdots \oplus \text{End}(V_N)^2 \oplus \text{Hom}(V_N, V_{N-1}) \oplus \text{Hom}(V_{N-1}, V_N) \]

a representation of numerical type \((r, n)\) of (1.1) in the category of vector spaces will be given by the datum of \(X \ni h = (B_0^0, B_1^0, I, J, B_1^1, B_2^1, F^1, G^1, \ldots)\), satisfying

\[
\begin{align*}
[B_0^0, B_2^0] + IJ &= 0, \\
[B_1^0, B_2^1] &= 0, \\
B_1^1 F^i - F^i B_1^{i+1} &= 0 = B_2^i F^i - F^i B_2^{i+1} \\
G_i^1 B_1^i - B_2^{i+1} G_i &= 0 = G_i^1 B_1^i - B_2^{i+1} G_i, \\
F^i G^i &= 0, \\
F^1 I &= 0, \\
G^1 I &= 0
\end{align*}
\]

(1.2)

which we will call nested ADHM equations. In the following we need to address the problem of King stability for representations of the nested instantons quiver.

**Definition 1.** Let \(\Theta = (\theta, \theta_\infty) \in \mathbb{Q}^{s+1}\) be such that \(\Theta(X) = n \cdot \theta + r \theta_\infty = 0\). We will say that a framed representation \(X\) of (1.1) is \(\Theta\)-semistable if

\[ \forall 0 \neq \tilde{X} \subset X \text{ of numerical type } (0, \tilde{n}) \text{ we have } \Theta(\tilde{X}) = \theta \cdot \tilde{n} \leq 0; \]

\[ \forall 0 \neq \tilde{X} \subset X \text{ of numerical type } (\tilde{r}, \tilde{n}) \text{ we have } \Theta(\tilde{X}) = \theta \cdot \tilde{n} + \tilde{r} \theta_\infty \leq 0. \]

If strict inequalities hold \(X\) is said to be \(\Theta\)-stable.

In [4, 34] the two node case, namely \(N = 1\) was considered and we can here generalize their result to the more general nested instantons quiver (1.1).

**Proposition 1.1.** Let \(X\) be a representation of (1.1) of numerical type \((r, n) \in \mathbb{N}^{N+2}\), then choose \(\theta_i > 0, \forall i \geq 0\) and \(\theta_0 \) s.t. \(\theta_0 + n_1 \theta_1 + \cdots + n_{s-1} \theta_{s-1} < 0\). The following are equivalent:

(i) \(X\) is \(\Theta\)-stable;

(ii) \(X\) is \(\Theta\)-semistable;

(iii) \(X\) satisfies the following conditions:

\[ \begin{array}{l}
\text{S1 } F^i \in \text{Hom}(V_{i+1}, V_i) \text{ is injective, } \forall i \geq 1; \\
\text{S2 } \text{the ADHM datum } A = (W, V_0, B_0^0, B_2^0, I, J) \text{ is stable.}
\end{array} \]

**Proof.** (i) \(\Rightarrow\) (ii) This is obvious, as a \(\Theta\)-stable representation is also \(\Theta\)-semistable.

(ii) \(\Rightarrow\) (iii) Let us first take a \(\Theta\)-semistable representation \(X\) having at least one of the \(F^i\) not injective. Without loss of generality let \(F^k\) be the only one to be such a map. Then, if \(v_k \in \ker F^k \Rightarrow \)
\(B_{2}^{k+1}v_k \in \ker F^{k}\), due to the nested ADHM equations, and \(B_{2}^{k+1}(\ker F^{k}) \subset \ker F^{k}\) (the same is obviously true for \(B_{1}^{k+1}\)). Now

\[X = (0, \ldots, 0, \ker F^{k}, 0, \ldots, F^{k}, B_{1}^{k+1}|_{\ker F^{k}}, B_{2}^{k+1}|_{\ker F^{k}}, 0, \ldots, 0)\]

is a subrepresentation of \(X\) of numerical type \((0, \ldots, 0, \dim \ker F^{k}, 0, \ldots, 0)\). Thus

\[\tilde{n} \cdot \theta + r \theta_{\infty} = \theta_k \dim \ker F^{k} > 0,\]

which contradicts the hypothesis of \(X\) being \(\Theta\)--semistable.

If instead we take \(X\) to be \(\Theta\)--semistable and suppose \(S2\) to be false, then \(\exists 0 \subset S \subset V_1\) s.t. \(B_{1}^{0}(S), B_{2}^{0}(S), \text{Im}(I) \subseteq S\). In this case

\[X = (W, S, V_1, \ldots, B_{1}^{0}|_{S}, B_{2}^{0}|_{S}, I, J, \ldots)\]

is a subrepresentation of \(X\) of numerical type \((r, \dim S, n_1, \ldots)\) but, since \(n \cdot \theta + r \theta_{\infty} = 0\) having \(\theta_{i>0} > 0\) and \(\theta_0 - n_1 \theta_1 - \cdots < 0\), we have

\[\dim S \theta_0 + n_1 \theta_1 + \cdots + r \theta_{\infty} = (\dim S - n_0) \theta_0 > 0,\]

which again leads to a contradiction.

\((iii) \Rightarrow (i)\) If we take a proper subrepresentation \(\tilde{X}\) of numerical type \((\tilde{r}, \tilde{n})\), we just need to check the cases \(\tilde{r} = 0\) and \(\tilde{r} = r\).

- If \(\tilde{r} = r\) then \(\tilde{W} = W\), which in turn implies that \(I \neq 0\), otherwise the ADHM datum \((B_{1}^{0}, B_{2}^{0}, I, J)\) would not be stable. Since \(\tilde{X}\) is proper the following diagram commutes

\[
\begin{array}{ccc}
W & \xrightarrow{I} & V_0 \\
\downarrow{I_W} & & \downarrow{i} \\
W & \xrightarrow{I} & V_0
\end{array}
\implies i \circ I = I \circ I_W
\]

leading to a contradiction with the stability of \((W, V_0, B_{1}^{0}, B_{2}^{0}, I, J)\). Since we are interested in proper subrepresentations of \(X\), at least one \(\tilde{n}_i > 0\) is not zero, and at least one of these non-zero \(\tilde{n}_k < n_k\), so that \(\theta \cdot \tilde{n} + \theta_{\infty} r < 0\), and \(X\) is stable.
Let now \( \bar{r} = 0 \). Since we are interested in proper subrepresentations we must choose \( \bar{n}_0 > 0 \), otherwise \( \bar{V}_{k>0} = 0 \) by virtue of the injectivity of \( F_k \). In the same way as in the previous case the only option is \( \bar{n}_0 = n_0 \). Following the same steps we previously carried out \( \theta \cdot \bar{n} = \sum_{k>0} \theta_k (\bar{n}_k - n_k) - \theta_{\infty} r < 0 \).

\[\square\]

**Corollary 1.2.** If \( X \) is a stable representation of the nested instantons quiver, \( G^k = 0, \forall k \).

**Proof.** By the previous proposition, due to the injectivity of \( F^k \), \( F^k G^k = 0 \Rightarrow G^k = 0 \).

\[\square\]

### 1.2 The nested instantons moduli space

We want now to discuss the construction of the moduli space of stable representations of the quiver (1.1), and its connection to GIT theory and stability. First of all we define the space of the nested ADHM datum. On \( X \) ADHM datum. On \( X \) ADHM datum. On \( X \) ADHM datum. On \( X \) ADHM datum. On \( X \) ADHM datum.

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\[\square\]

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We want now to discuss the construction of the moduli space of stable representations of the quiver (1.1), and its connection to GIT theory and stability. First of all we define the space of the nested ADHM data to be the space \( X \) we defined previously, and an element \( X \in X \) is called an nested ADHM datum. On \( X \) we have a natural action of \( G = GL(V_0) \times \cdots \times GL(V_N) \) defined by

\[
\Psi : (g_0, g_1, \ldots, g_N, X) \mapsto (g_0 B_1^0 g_0^{-1}, g_0 B_2^0 g_0^{-1}, g_0 I, I g_0^{-1}, g_1 B_1^1 g_1^{-1}, g_1 B_2^1 g_1^{-1}, g_0 F g_1^{-1}, g_1 G g_1^{-1}, g_0^{-1}, g_1 B_1^N g_1^{-1}, g_1 B_2^N g_1^{-1}, g_0 N^{-1} g_1 g_1^{-1}, g_1 N^{-1} g_1 g_1^{-1})
\]

(1.5)

This action of \( G \) on \( X \) is free on the stable points of \( X \). In fact if \( g \in G \) is such that \( g \cdot X = X \), \( \forall X \in X \), we claim that \( g = (\mathbb{1}_{V_0}, \ldots, \mathbb{1}_{V_N}) \). In order to see this, let \( S = \ker(g_0 - \mathbb{1}_{V_0}) \). Since \( g \cdot X = X \) it follows that \( g_0 I = I \), which means \( \text{Im} I \subset S \). Moreover \( g_0 B_1^0 = B_1^0 g_0 \) and \( g_0 B_2^0 = B_2^0 g_0 \), but if \( v \in S = (g_0 - \mathbb{1}_{V_0}) v = 0 \Rightarrow g_0 v = v \), thus implying that \( B_1^0(S), B_2^0(S) \subset S \). The stability of \( (W, V_0, B_1^0, B_2^0, I, J) \) then forces \( S = V_0 \). Finally, \( g_0 = \mathbb{1}_{V_0}, F(1_{V_0} - g_1^{-1}) = 0 \Rightarrow g_1 = 1_{V_1} \) by the injectivity of \( F \). By using this procedure then one can prove by iteration that \( g_k = 1_{V_k}, \forall k \), thus \( g \cdot X = X \forall X \in X \Leftrightarrow g = \mathbb{1} \). This proves that the action of \( G \) is free on the stable points of \( X \), and it is easy to prove that it preserves \( X_0 \), which denotes the space of nested ADHM data satisfying the relations of quiver (1.1).

Now if \( x : G \to C^* \) is an algebraic character for the algebraic reductive group \( G \), we can produce the moduli space of \( x \)-semistable orbits following a construction due to [23], \( \mathcal{N}^{ss}(r, n) \), which is a quasi-projective scheme over \( C \) and is defined as

\[
\mathcal{N}^{ss}(r, n) = X_0 // G = \text{Proj} \left( \bigoplus_{n \geq 0} A(X_0(r, n))^{G / x^n} \right)
\]

9
with
\[
A(X_0(r,n))^{G,\chi^n} = \{ f \in A(X_0(r,n)) | f(h \cdot X) = \chi(h)^n f(X), \forall h \in G \}.
\]

The scheme \( N^{ss}_X(r,n) \) contains an open subscheme \( N^{is}_X(r,n) \subset N^{ss}_X(r,n) \) encoding \( \chi \)-stable orbits.

It turns out that also in this framed case there is a relation between \( \chi \)-stability and \( \Theta \)-stability, as it was shown in [23] in the non framed setting.

**Proposition 1.3.** Let \( \Theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \mathbb{Z}^{N+1} \) and define \( \chi_{\Theta} : G \to \mathbb{C}^* \) the character
\[
\chi_{\Theta}(h) = \det(h_0)^{-\theta_0} \cdots \det(h_N)^{-\theta_N}.
\]

A representation \( X \) of the nested ADHM quiver (1.1) is \( \chi \)-stable (semi)stable if it is \( \Theta \)-stable.

Since the proof for proposition 1.3 deeply relies on some known results about equivalent characterizations of \( \chi \)-stability, we will first recall them. In full generality, let \( V \) be a vector space over \( \mathbb{C} \) equipped with the action of a connected subgroup \( G \) of \( U(V) \), whose complexification is denoted by \( G^C \). Then if \( \chi : G \to U(1) \) is a character of \( G \), we can extend it to form its complexification \( \chi : G^C \to \mathbb{C}^* \). We then form the trivial line bundle \( V \times \mathbb{C} \), which carries an action of \( G^C \) via \( \chi \):

\[
g \cdot (x,z) = (g \cdot x, \chi(g)^{-1} z), \quad g \in G, (x,z) \in V \times \mathbb{C}.
\]

**Definition 2.** An element \( x \in V \) is

1. \( \chi \)-semistable if there exists a polynomial \( f \in A(V)^{G^C,\chi^n} \), with \( n \geq 1 \) such that \( f(x) \neq 0 \);
2. \( \chi \)-stable if it satisfies the previous condition and if
   
   (a) \( \dim(G^C \cdot x) = \dim(G^C/\Delta) \), where \( \Delta \subseteq G^C \) is the subgroup of \( G^C \) acting trivially on \( V \);
   
   (b) the action of \( G^C \) on \( \{ x \in V : f(x) \neq 0 \} \) is closed.

Given the previous definition, the next lemma due to King [23] gives an alternative characterization of \( \chi \)-(semi)stable points under the \( G^C \)-action.

**Lemma 1.4** (Lemma 2.2 and Proposition 2.5 in [23]). Given the character \( \chi : G^C \to \mathbb{C}^* \) for the action of \( G^C \) on the vector space \( V \), and the lift of this action to the trivial line bundle \( V \times \mathbb{C} \), a point \( x \in V \) is

1. \( \chi \)-semistable iff \( G^C \cdot (x,z) \cap (V \times \{ 0 \}) = \emptyset \), for any \( z \neq 0 \);
2. \( \chi \)-stable iff \( G^C \cdot (x,z) \) is closed and the stabilizer of \( (x,z) \) contains \( \Delta \) with finite index.

Equivalently, a point \( x \in V \) is

1. \( \chi \)-semistable iff \( \chi(\Delta) = \{ 1 \} \) and \( \chi(\lambda) \geq 0 \) for any 1-parameter subgroup \( \lambda(t) \subseteq G^C \) for which \( \lim_{t \to 0} \lambda(t) \cdot x \) exists;
2. \( \chi \)-stable iff the only \( \lambda(t) \) such that \( \lim_{t \to 0} \lambda(t) \cdot x \) exists and \( \chi(\lambda) = 0 \) are in \( \Delta \).

With these notations, if \( V^ss(\chi) \) denotes the set of \( \chi \)-semistable points of \( V, V ||_{\chi} G^C \) can be identified with \( V^ss(\chi)/\sim \), where \( x \sim y \) in \( V^ss(\chi) \) iff \( G^C \cdot x \cap G^C \cdot y \neq \emptyset \) in \( V^ss(\chi) \).

**Proof of proposition 1.3.** Take a \( \theta \)-semistable representation \( X \in X \) and assume it doesn’t satisfy \( \chi_\theta \)-semistability. Then there exists a 1–parameter subgroup \( \lambda(t) \) of \( G \) such that \( \lim_{t \to 0} \lambda(t) \cdot X \) exists and \( \chi(\lambda) < 0 \). However each such 1–parameter subgroup \( \lambda \) determines a filtration \( \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots \) of subrepresentations of \( X \), \([23]\), and

\[
\chi_\theta(\lambda) = -\sum_{n \in \mathbb{Z}} \theta(X_n) \geq 0, \tag{1.7}
\]

thus proving one side of the proposition, as the part concerning stability is obvious from the fact that trivial subrepresentations of \( X \) correspond to subgroups in \( \Delta \).

Conversely, if \( X \) is a \( \chi_\theta \)-semistable representation, we want to show that it is also a \( \theta \)-semistable one. We only need to consider two cases, corresponding to subrepresentations \( \bar{X} \) of \( X \) with \( \bar{r} = r \) or \( \bar{r} = 0 \). Each vector space in \( X \), say \( V_i \) will have then a direct sum decomposition \( V_i = \bar{V}_i \oplus \hat{V}_i \). We will then take a 1–parameter subgroup \( \lambda(t) \) such that

\[
\lambda_i(t) = \begin{bmatrix} t^{1} & 0 \\ 0 & t^{-1} \end{bmatrix}, \tag{1.8}
\]

Then one can easily compute

\[
\chi_\theta(\lambda(t)) \cdot z = \left[ \det(\lambda_0(t))^{-\theta_0} \cdots \det(\lambda_N(t))^{-\theta_N} \right]^{-1} \cdot z = t^{n} \theta z \tag{1.9}
\]

It is then a matter of a simple computation to verify that, if \( X \) wasn’t \( \theta \)-semistable, then one would have had \( \lim_{t \to 0} \lambda(t) \cdot X \in X \times \{0\} \), thus contradicting the \( \chi_\theta \)-semistability. A completely analogous computation can be carried over when \( \bar{r} = r \), taking

\[
\lambda_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix}, \quad \lambda_i(t) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix}, \quad i > 0, \tag{1.10}
\]

and since \( (\bar{n} - n) \cdot \theta > 0 \) if \( X \) is supposed not to be \( \theta \)-semistable, this would still lead to a contradiction.

Finally, if \( X \) was to be \( \chi_\theta \)-stable but not \( \theta \)-stable, the 1–parameter subgroups previously described would have stabilized the pair \( (X, z), z \neq 0 \), in the two different cases \( \bar{r} = 0 \) and \( \bar{r} = r \) respectively, thus again giving rise to a contradiction.
Corollary 1.5. Given a representation of the nested instantons quiver \((1.1)\) of numerical type \((r, n)\), there exists a chamber in \(Q^{N+1} \ni (\theta, \theta_\infty) = \Theta\) in which \(\theta_i > 0 > 0\) and \(\theta_0 + n_1 \theta_1 + \cdots + n_s \theta_{s-1} < 0\) such that the following are equivalent:

1. \(X\) is \(\Theta\)-semistable;
2. \(X\) is \(\Theta\)-stable;
3. \(X\) is \(\chi_\Theta\)-semistable;
4. \(X\) is \(\chi_\Theta\)-stable;
5. \(X\) satisfies \(S1\) and \(S2\) in proposition 1.1.

Because of the previous corollary, in the stability chamber defined by proposition 1.1 all notions of stability are actually the same, so that a representation satisfying anyone of the conditions in corollary 1.5 will be called stable, and the corresponding \(N^{ss}\) \(\chi_\Theta\)(\(r, n\)) \(\cong N\) \([r, 1]_{nr, \mu}\) (with the notations of \([2]\)) will be addressed to as the moduli space of stable representations of \((1.1)\) or, equivalently, as the moduli space of nested instantons. Altogether, the previous considerations prove the following theorem.

Theorem 1.6. The moduli space \(N\) \((r, n)\) of stable representation of the nested instantons quiver of numerical type \((r, n)\) is a virtually smooth quasi-projective variety equipped with a natural action of \(T = T \times (\mathbb{C}^*)^r\), \(T = (\mathbb{C}^*)^2\), and a perfect obstruction theory. The moduli space \(N\) \((r, n)\) can thus be identified in a suitable stability chamber with the moduli space of nested instantons \(N_{r,[r^1]_{n,\mu}}\).

Proof. The first part of the proof has already been proved. Consider then the following complex

\[
\begin{align*}
C(X) : \quad C^0(X) & \xrightarrow{d_0} C^1(X) \xrightarrow{d_1} C^2(X) \xrightarrow{d_1} C^3(X) \quad (1.11)
\end{align*}
\]

with

\[
\begin{align*}
C^0(X) &= \bigoplus_{i=0}^N \operatorname{End}(V_i), \\
C^1(X) &= \operatorname{End}(V_0)^{\otimes 2} \oplus \operatorname{Hom}(W, V_0) \oplus \operatorname{Hom}(V_0, W) \oplus \left[ \bigoplus_{i=1}^N \left( \operatorname{End}(V_i)^{\otimes 2} \oplus \operatorname{Hom}(V_i, V_{i-1}) \right) \right], \\
C^2(X) &= \operatorname{End}(V_0) \oplus \operatorname{Hom}(V_1, W) \oplus \left[ \bigoplus_{i=1}^N \left( \operatorname{Hom}(V_i, V_{i-1})^{\otimes 2} \oplus \operatorname{End}(V_i) \right) \right], \\
C^3(X) &= \bigoplus_{i=1}^N \operatorname{Hom}(V_i, V_{i-1}).
\end{align*}
\]

We thank Valeriano Lanza for pointing out to us a correction to the original proof for the two-nodes quiver found in \([34]\).
while the morphisms $d_i$ are defined as:

$$
d_0(h) = \begin{pmatrix}
[h_0, B_1^0] \\
[h_0, B_2^0] \\
h_0 I \\
[h_1, B_1^0] \\
h_0 F^1 - F^1 h^1 \\
\vdots
\end{pmatrix},
\quad
d_1 = \begin{pmatrix}
b_1^0 \\
b_2^0 \\
i \\
b_1^1 \\
b_2^1 \\
\vdots
\end{pmatrix},
\quad
d_2 = \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_{3N+3}
\end{pmatrix},
\quad
\begin{pmatrix}
[b_1^0, B_2^0] + [B_1^0, B_2^0] + i f + I j \\
F^1 + f^1 \\
B_2^0 f^1 + b_1^0 F^1 - F^1 b_1^1 - f^1 B_1^1 \\
B_2^0 f^1 + b_2^0 F^1 - F^1 b_2^1 - f^1 B_2^1 \\
\vdots \\
[b_1^1, B_2^1] + [B_1^1, B_2^1] \\
[b_1^2, B_2^2] + [B_1^2, B_2^2]
\end{pmatrix}
\begin{pmatrix}
c_1 F^1 + B_2^0 c_3 - c_3 B_2^2 + c_4 B_1^1 - B_1^0 c_4 - l c_2 - F^1 c_{2N+3} \\
\vdots \\
c_{2N+2+i} F^i + B_2^i c_{2+i} - c_{2+i} B_2^2 + c_{3+i} B_1^1 - B_1^0 c_{3+i} - f^i c_{2N+3+i} \\
\vdots
\end{pmatrix}.

(1.13)

Notice that the maps $d_0$ and $d_1$ are the linearisation of the action of $G$ on $X$ and of the nested instantons quiver relations (neglecting $G^i$, since $G^i = 0, \forall i$), respectively. The morphism $d_2$ is instead signalling the fact that the quiver relations are not all independent.

Our claim is then that $H^0(C(X)) = H^3(C(X)) = 0$, and that $C(X)$ is an explicit representation of the perfect obstruction theory complex, so $H^1(C(X))$ will be identified with the Zariski tangent to $N(r,n)$, while $H^2(C(X))$ will encode the obstructions to its smoothness. In fact elements of $H^1(C(X))$ parametrize infinitesimal displacements at given points, up to the $G$–action, so $H^1(C(X))$ provides a local model for the Zariski tangent space to $N(r,n)$. In the same way $H^2(C(X))$ is interpreted to be the local model for the obstructions as its elements encode the linear dependence of the nested ADHM equations. Actually one might explicitly determine the truncated cotangent complex $\mathcal{L}_{N(r,n)}^{\geq 1}$ by a standard computation in deformation theory along the lines of [10] and compute extension and obstruction classes in terms of the cohomology of the complex $C(X)$.

In order to see that indeed the 0–th and 3–rd cohomology of $C(X)$ does indeed vanish, we construct three other complexes $C(A)$, $C(B)$ and $C(A,B)$:

$$
\begin{array}{ccccccc}
\text{End}(V_0) \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
C(A) : & \text{End}(V_0) & \overset{d_0}{\rightarrow} & \text{Hom}(W, V_0) & \overset{d_1}{\rightarrow} & \text{End}(V_0) & \oplus \\
& & & \text{Hom}(V_0, W)
\end{array}
\quad
(1.14)
$$
with

\[
\begin{align*}
 d_0(h_0) &= \begin{pmatrix} h_0 & B_1^0 \\ h_0 & h_0 I \\ -Jh_0 \end{pmatrix}, &
 d_1 &= \begin{pmatrix} b_0^0 \\ b_2^0 \\ i \\ j \end{pmatrix} = [b_1^0, B_2^0] + [B_1^0, b_2^0] + Ij + ij; \\
\end{align*}
\]

\[C(B) : \bigoplus_{i=1}^N \text{End}(V_i) \xrightarrow{d_0} \bigoplus_{i=1}^N \text{End}(V_i)^{\otimes 2} \xrightarrow{d_1} \bigoplus_{i=1}^N \text{End}(V_i) \tag{1.15}\]

with

\[
\begin{align*}
 d_0(h) &= \begin{pmatrix} [h_1, B_1^1] \\ h_1 & B_1^1 \\ h_1 & h_1 I \end{pmatrix}, &
 d_1 &= \begin{pmatrix} b_1^1 \\ b_2^1 \\ b_1^N \\ b_2^N \end{pmatrix} = \begin{pmatrix} [b_1^1, B_2^1] + [B_1^1, b_2^1] \\ [b_1^N, B_2^N] + [B_1^N, b_2^N] \end{pmatrix}; \\
\end{align*}
\]

\[C(A, B) : \bigoplus_{i=1}^N \text{Hom}(V_i, V_{i-1})^{\otimes 2} \xrightarrow{d_0} \bigoplus_{i=1}^N \text{Hom}(V_i, V_{i-1}) \xrightarrow{d_1} \bigoplus_{i=1}^N \text{Hom}(V_i, V_{i-1}) \tag{1.16}\]

with

\[
\begin{align*}
 d_0(f) &= \begin{pmatrix} -B_1^0 f_1 + f_1 B_1^1 \\ -B_2^0 f_1 + f_1 B_2^1 \\ -B_1^{N-1} f_N + f_N B_1^N \\ -B_2^{N-1} f_N + f_N B_2^N \\ -Jf \end{pmatrix}, &
 d_1 &= \begin{pmatrix} c_3 \\ c_2 \\ c_2^{N+2} \end{pmatrix} = \begin{pmatrix} -B_2^0 c_3 + c_3 B_2^1 - c_4 B_1^1 + B_1^0 c_4 + Ic_2 \\ B_2^{N-1} c_{2N+2} - c_{2N+2} B_2^1 + c_{2N+3} B_1^1 - B_1^0 c_{2N+3} \end{pmatrix}. \\
\end{align*}
\]

Then one can prove that there exists a distinguished triangle

\[C(X) \xrightarrow{\rho} C(A) \oplus C(B) \xrightarrow{\rho} C(A, B), \tag{1.17}\]
coming from the fact that $C(X)[1]$ is a cone for $\rho = (\rho_0, \rho_1, \rho_2)$, where

$$
\rho_0 \begin{pmatrix} h_0 \\ \vdots \\ h_N \end{pmatrix} = \begin{pmatrix} -h_0 F^1 + F^1 h_1 \\ \vdots \\ -h_{N-1} F^N + F^N h_N \end{pmatrix}
$$

(1.18a)

$$
\rho_1 \begin{pmatrix} b_0^0 \\ b_2^0 \\ \vdots \\ b_1^1 \\ b_2^1 \\ \vdots \\ b_1^i \\ b_2^i \end{pmatrix} = \begin{pmatrix} -b_0^0 F^1 + F^1 b_1^1 \\ -b_2^0 F^1 + F^1 b_2^1 \\ \vdots \\ -b_1^N F^1 + F^1 b_1^N \\ -b_2^N F^1 + F^1 b_2^N \\ \vdots \\ -b_{1} F^1 \\ \vdots \\ -b_{N} F^N + F^N b_{N} \end{pmatrix}
$$

(1.18b)

$$
\rho_2 \begin{pmatrix} c_1 \\ \vdots \\ c_{2N+3} \\ \vdots \\ c_{3N+3} \end{pmatrix} = \begin{pmatrix} -c_1 F^1 + F^1 c_{2N+3} \\ \vdots \\ -c_{3N+2} F^N + F^N c_{3N+3} \end{pmatrix}
$$

(1.18c)

By the triangle (1.17) one gets the long sequence of cohomologies:

$$
0 \longrightarrow H^0(C(X)) \longrightarrow H^0(C(A) \oplus C(B)) \xrightarrow{H^0(\rho)} H^0(C(A, B)) \longrightarrow H^1(C(X)) \longrightarrow \\
\longrightarrow H^1(C(A) \oplus C(B)) \xrightarrow{H^1(\rho)} H^1(C(A, B)) \longrightarrow H^2(C(X)) \longrightarrow H^2(C(A) \oplus C(B)) \longrightarrow \\
\longrightarrow H^2(C(A, B)) \longrightarrow H^3(C(X)) \longrightarrow 0,
$$

(1.19)

and, since $A$ is a stable representation of the standard ADHM quiver, $H^0(C(A)) = H^2(C(A)) = 0$. Then $H^0(C(X)) = 0$ by the injectivity of $H^0(\rho) : H^0(C(B)) \to H^0(C(A, B))$. In fact we have

$$
H^0(\rho) \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} F^1 h_1 \\ \vdots \\ F^N h_N \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} = 0,
$$

(1.20)

since $F^i$ is injective. Moreover the stability of $X$ implies that $d_1 : C(A, B)^1 \to C(A, B)^2$ is surjective.
this in turn means that \( H^2(C(A,B)) = 0 \), which implies \( H^3(C(X)) = 0 \). In fact let’s take \( d'_i \):

\[
\begin{bmatrix}
B_1^i \phi_1 - \phi_1 B_2^i \\
- B_1^i \phi_1 + \phi_1 B_1^i \\
\vdots \\
B_N^i \phi_N - \phi_N B_2^i \\
- B_N^i \phi_N + \phi_N B_1^i \\
\phi_1 I
\end{bmatrix} = 0,
\]

and if \( \phi \in \ker(d'_i) \) then \( \ker(\phi_1) \) would be a \((B_1^0, B_2^0)\)-invariant subset of \( V_0 \) containing \( \text{Im}(I) \) which contradicts the stability of \( X \), by which we conclude that \( \ker(\phi_1) = V_0 \). Similar statements hold also for each other component of \( \phi \), which we then conclude to be \( \phi = 0 \).

The only thing left to prove is the virtual smoothness, namely that the moduli space \( \mathcal{N}(r,n) \) of stable representations of the nested instantons quiver is embedded in a smooth variety which is obtained as an hyperkähler quotient. We will leave this for section 1.3.

For future reference we want now to exhibit some morphisms between different nested instantons moduli spaces and between them and usual moduli spaces of instantons, which are moduli spaces of framed torsion-free sheaves on \( \mathbb{P}^2 \). We obviously have iterative forgetting projections \( \eta_i : \mathcal{N}(r, n_0, \ldots, n_i) \to \mathcal{N}(r, n_0, \ldots, n_{i-1}) \). Moreover we also have other morphisms to underlying Hilbert schemes of points on \( \mathbb{C}^2 \), which are summarized by the commutative diagram in figure 3. In order to see that these maps do indeed exist, take a stable representation \([X]\) of the nested instantons quiver. The fact that \([X]\) is stable implies that the morphisms \( F^i \) are injective, so that we can construct the stable ADHM datum \((W, \tilde{V}_i, \tilde{B}_1^i, \tilde{B}_2^i, \tilde{I}, \tilde{J})\) as follows. Let \( \tilde{V}_i \) be \( V_0 / \text{Im}(F^1 \cdots F^i) \) and choose a basis of \( \tilde{V}_i \) in such a way that

\[
F^1 \cdots F^i = \begin{pmatrix}
\mathbb{1} & \vdots \\
0 & 0
\end{pmatrix}, \quad F^1 \circ F^2 \circ \cdots \circ F^i : V_i \to V_0,
\]

whence \( V_0 = V_i \oplus \tilde{V}_i \). Then define the projections \( \pi_i : V_0 \to V_i \) and \( \tilde{\pi}_i : \text{Im}(I) \to \tilde{V}_i \) as \( \pi_i(v, \tilde{v}) = v \) and \( \tilde{\pi}_i(v, \tilde{v}) = \tilde{v} \), with \( v \in V_i, \tilde{v} \in \tilde{V}_i \). We can then show how \( \tilde{V}_i \) inherits an ADHM structure by its embedding in \( V_0 \). Indeed if we define \( \tilde{B}_1^i = B_1^0|_{\tilde{V}_i}, \tilde{B}_2^i = B_2^0|_{\tilde{V}_i}, \tilde{I}_i = \tilde{\pi}_i \circ I \) and \( \tilde{J} = J|_{\tilde{V}_i} \) the datum \((W, \tilde{V}_i, \tilde{B}_1^i, \tilde{B}_2^i, \tilde{I}_i, \tilde{J})\) satisfies the ADHM equation (1.22).

\[
[\tilde{B}_1^i, \tilde{B}_2^i] + \tilde{I}_i \tilde{J} = [B_1^0|_{\tilde{V}_i}, B_2^0|_{\tilde{V}_i}] + \tilde{\pi}_i \circ I \circ J|_{\tilde{V}_i} = \left( B_1^0, B_2^0 \right) + I|_{\tilde{V}_i} = 0.
\]

This new ADHM datum is moreover stable, as if it would exist \( 0 \subset \tilde{S}_i \subset \tilde{V}_i \) such that \( \tilde{B}_1^i = (\tilde{S}_i), \tilde{J}(W) \subset \tilde{S}_i \) it would imply that also the ADHM datum \((W, V_0, B_1^0, B_2^0, I, J)\) wouldn’t be stable. In fact in that
case we could take $0 \subset V_i \oplus \tilde{S}_i \subset V_0$ and it would be such that $B_0^0(V_i \oplus \tilde{S}_i), B_0^0(V_i \oplus \tilde{S}_i), I(W) \subset V_i \oplus \tilde{S}_i$.

In fact if we take any $(v, s) \in V_i \oplus \tilde{S}_i$ it happens that $B_0^0(v, s) = (B_0^0|_{V_i}(v), B_0^0|_{\tilde{S}_i}(s)) \in V_i \oplus \tilde{S}_i$, $B_2^0(v, s) = (B_2^0|_{V_i}(v), B_2^0|_{\tilde{S}_i}(s)) \in V_i \oplus \tilde{S}_i$ and $I(W) = I(W) \cap V_i \oplus I(W) \cap \tilde{V}_i = (\pi_i \circ I)(W) \oplus (\tilde{\pi}_i \circ I)(W) \subset V_i \oplus \tilde{S}_i$. Thus we constructed a map $\rho_{i}^{(N)}: \mathcal{N}(r, n_0, \ldots, n_N) \to M(r, n_0 - n_i)$.

### 1.3 Virtual smoothness

In this section we exhibit an embedding of the moduli space of nested instantons into a smooth projective variety, which is moreover hyperkähler. In the following vector space

$$X = \text{End}(V_0)^{\oplus 2} \oplus \text{Hom}(W, V_0) \oplus \text{Hom}(V_0, W) \bigoplus_{k=1}^{N} \left[ \text{End}(V_k)^{\oplus 2} \oplus \text{Hom}(V_{k-1}, V_k) \right] \oplus \text{Hom}(V_k, V_{k-1})$$

(1.23)
we will introduce a family of relations:

\[
B_0^1, B_0^2 + IJ + F^1G^1 = 0, \quad (1.24)
\]

\[
B_i^1, B_i^2 - G^iF^i + F^{i+1}G^{i+1} = 0, \quad i = 1, \ldots, N. \quad (1.25)
\]

Then an element \((B_0^1, B_0^2, I, J, \{B_i^1, B_i^2, F^i, G^i\})\) \(X \in X\) is called stable if it satisfies conditions \(S1\) and \(S2\) in proposition 1.1. With these conventions we will define \(M(r, n)\) to be the space of stable elements of \(X\) satisfying the relations (1.24)-(1.25):

\[
M(r, n) = \{X \in X : X \text{ is stable and satisfies (1.24), (1.25)}\}. \quad (1.26)
\]

Exactly in the same way as we did before we can easily see that there is a natural action of \(G = GL(V_0) \times \cdots \times GL(V_N)\) which is free on \(M(r, n)\) and preserves the equations (1.24)-(1.25): the same is then true for the natural \(U\)-action on \(M(r, n)\), with \(U = U(V_0) \times \cdots \times U(V_N)\). Thus a moduli space \(M(r, n)_{U}\) of stable \(U\)-orbits in \(M(r, n)\) can be defined by means of GIT theory, as it was the case for \(N(r, n)\) in the previous sections. It is moreover obvious that \(N(r, n) \hookrightarrow M(r, n)\), as any stable point of \(X\) satisfying the nested ADHM equations automatically satisfies (1.24) and (1.25).

Next let us point out that on each \(T \text{Hom}(V_i, V_k)\) we can introduce an hermitean metric by defining

\[
\langle X, Y \rangle = \frac{1}{2} \text{Tr} \left( X \cdot Y^\dagger + X^\dagger \cdot Y \right), \quad \forall X, Y \in \text{Hom}(V_i, V_k), \quad (1.27)
\]

which in turn can be linearly extended to a hermitean metric \(\langle -, - \rangle : T^2 M(r, n) \times T^2 M(r, n) \to \mathbb{C}\). Finally we can introduce some complex structures on \(T^2 M(r, n)\): given \(X \in T^2 M(r, n)\) these are defined as the following \(I, J, K \in \text{End}(T^2 M(r, n))\)

\[
I(X) = \sqrt{-1}X, \quad (1.28)
\]

\[
J(X) = (-b_0^{0\dagger}, b_0^{1\dagger}, -j^\dagger, i^\dagger, \{-b_2^{1\dagger}, b_1^{i\dagger}, -g^i, f^i\}), \quad (1.29)
\]

\[
K(X) = I \circ J(X), \quad (1.30)
\]

with \(X = (b_0^0, b_0^1, i, j, b_1^i, b_2^i, f^i, g^i)\). These three complex structures make the datum of

\[
(M(r, n), \langle -, - \rangle, I, J, K)
\]

a hyperkähler manifold, as one can readily verify. It is a standard fact that once we fix a particular complex structure, say \(I\), and its respective Kähler form, \(\omega_I\), the linear combination \(\omega_C = \omega_I + \sqrt{-1} \omega_K\) is a holomorphic symplectic form for \(M(r, n)\). The thing we finally want to prove is that the hyperkähler structure on \(M(r, n)\) induce a hyperkähler structure on the GIT quotient \(M(r, n)\), which will be moreover proven to be smooth. This is made possible by the fact that the natural \(U\)-action
on $\mathcal{M}(r,n)$ preserves the hermitean metric and the complex structures we introduced. Then, letting $u$ be the Lie algebra of the group $\mathcal{U}$, we need to construct a moment map

$$\mu : \mathcal{M}(r,n) \to u^* \otimes \mathbb{R}^3,$$

satisfying

1. $\mathcal{G}$–equivariance: $\mu(g \cdot X) = \text{Ad}_g^* \mu(X)$;

2. $\langle d\mu_i(X), \xi \rangle = \omega_i(\xi^*, X)$, for any $X \in T\mathcal{M}(r,n)$ and $\xi \in u$ generating the vector field $\xi^* \in T\mathcal{M}(r,n)$. If then $\xi \in u^* \otimes \mathbb{R}^3$ is such that $\text{Ad}_g^*(\xi) = \xi_i$ for any $g \in \mathcal{U}$, $\mu^{-1}(\xi)$ is $\mathcal{U}$–invariant and it makes sense to consider the quotient space $\mu^{-1}(\xi)/\mathcal{U}$. It is known, [22], that if $\mathcal{U}$ acts freely on $\mu^{-1}(\xi)/\mathcal{U}$, the latter is a smooth hyperkähler manifold, with complex structures and metric induced by those of $\mathcal{M}(r,n)$.

Our task of finding a moment map $\mu : \mathcal{M}(r,n) \to u^* \otimes \mathbb{R}^3$ then translates into the following. Define $(\mu^0_1, \ldots, \mu^N_1) = \mu_1 : \mathcal{M}(r,n) \to u$

$$\begin{cases}
\mu^0_1(X) = \frac{\sqrt{-1}}{2} \left( [B^0_1, B^0_1] + [B^0_2, B^0_2] + IF^1 - J^1 + F^1 F^1 - G^1 G^1 \right), \\
\mu^1_1(X) = \frac{\sqrt{-1}}{2} \left( [B^1_1, B^1_1] + [B^1_2, B^1_2] - F^1 F^1 + G^1 G^1 + F^2 F^2 - G^2 G^2 \right), \\
\vdots \\
\mu^N_1(X) = \frac{\sqrt{-1}}{2} \left( [B^N_1, B^N_1] + [B^N_2, B^N_2] - F^N F^N + G^N G^N \right),
\end{cases}$$

(1.31)

with $X = (B^0_1, B^0_2, I, J, [B^1_1, B^1_2, F^1, G^1]) \in \mathcal{M}(r,n)$. In addition to $\mu_1$ we also define a map $\mu_C : \mathcal{M}(r,n) \to g$, with $g = \mathfrak{gl}(V_0) \times \cdots \times \mathfrak{gl}(V_N)$:

$$\begin{cases}
\mu^0_C(X) = [B^0_1, B^0_2] + IF^1 G^1 \\
\mu^1_C(X) = [B^1_1, B^1_2] - G^1 F^1 + F^2 G^2 \\
\vdots \\
\mu^N_C(X) = [B^N_1, B^N_2] - G^N F^N,
\end{cases}$$

(1.32)

by means of which we define $\mu_{2,3} : \mathcal{M}(r,n) \to u$ as $\mu_C(X) = (\mu_2 + \sqrt{-1} \mu_3)(X)$. Notice that in absence of $B^1_i$ and $I, J$ the complex moment map we defined would reduce to the Crawley-Boevey moment map in [9]. We then claim that $\mu = (\mu_1, \mu_2, \mu_3)$ is a moment map for the $\mathcal{U}$–action on $\mathcal{M}(r,n)$. If this is true and $\chi$ is the algebraic character we introduced in section 1.2, the space

$$\widetilde{\mathcal{M}}(r,n) = \mu_1^{-1}(\sqrt{-1} \text{d}\chi) \cap \mu_C^{-1}(0) \cap \mathcal{M}(r,n) = \mu^{-1}(\sqrt{-1} \text{d}\chi, 0, 0) \cap \mathcal{M}(r,n)$$

(1.33)
is a smooth hyperkähler manifold which, by an analogue of Kempf-Ness theorem is also isomorphic to \( M(r, n) \). In fact it is known, due to a result of [23, 27] and the characterization of \( \chi \)–(semi)stable points we gave in the previous sections, that there exists a bijection between \( \mu_1^{-1}(\sqrt{-1}d\chi) \) and the set of \( \chi \)–(semi)stable points in \( \mu_C^{-1}(0) \). Then, in order to prove that \( \mu \) is actually a moment map, we will first compute the vector field \( \xi^* \) generated by a generic \( \xi \in u \). Let then \( X = (b_0^0, b_2^0, i, j, \{ b_i^1, b_j^1, f^1, g^1 \}) \) be a vector in \( TM(r, n) \) and \( \Psi_X : \mathcal{U} \to \mathbb{M}(r, n) \) the action of \( \mathcal{U} \) onto \( X \in \mathbb{M}(r, n) \): the fundamental vector field generated by \( \xi \in u \) is

\[
\xi^*_X = d\Psi_X(1_{\mathcal{U}})(\xi) = \frac{d}{dt} (\Psi_X \circ \gamma)|_{t=0},
\]

where \( \gamma \) is a smooth curve \( \gamma : (-\epsilon, \epsilon) \to \mathcal{U} \) such that \( \gamma(0) = 1_{\mathcal{U}} \) and \( \dot{\gamma}(0) = \xi \). Thus we can compute

\[
\xi^*_X = \left( [\xi_0, b_0^0], [\xi_0, b_2^0], \xi_0 i - j \xi_0, [\xi_1, b_1^1], [\xi_1, b_1^1], \xi_0 f^1 - f^1 \xi_1, \xi_1 g^1 - g^1 \xi_0, \ldots \right)
\]

\[
\ldots, [\xi_N, b_N^1], [\xi_N, b_N^N], \xi_{N-1} f^N - f^N \xi_N, \xi_N g^N - g^N \xi_{N-1}).
\]

Then if \( \pi_i : \mathbb{M}(r, n) \to \mathbb{M}(r, n) \) denotes the projection on the \( i \)-th component of the direct sum decomposition induced by (1.23) so that \( i \) runs over the index set \( I \), by inspection one can see that \( \omega_1 \) is exact, and in particular \( \omega_1 = d\lambda_1 \), with

\[
\lambda_1 = \frac{\sqrt{-1}}{2} \text{Tr} \left( \sum_{i \in I} \pi_i \wedge \pi_i^\dagger \right).
\]

This implies that

\[
\langle \mu_1(x), \xi \rangle = i_{\xi^*} \lambda_1,
\]

and it is easy to verify that \( \mu_1 : \mathbb{M}(r, n) \to u^* \) thus defined indeed matches with the definition (1.31). Similarly one can realize that

\[
\lambda_2 = \text{Re} \left[ \text{Tr} \left( \sum_{i \in \mathbb{Z} \setminus I} \pi_i \wedge \pi_{1+i} \right) \right],
\]

\[
\lambda_3 = -\sqrt{-1} \text{Im} \left[ \text{Tr} \left( \sum_{i \in \mathbb{Z} \setminus I} \pi_i \wedge \pi_{1+i} \right) \right].
\]

and the moment map components satisfying \( \langle \mu_i(x), \xi \rangle = i_{\xi^*} \lambda_i \) agree with the combination \( \mu_2 + \sqrt{-1}\mu_3 = \mu_C \) we gave previously in equation (1.32).

## 2 Flags of framed torsion-free sheaves on \( \mathbb{P}^2 \)

We give in this paragraph the construction of the moduli space of flags of framed torsion-free sheaves of rank \( r \) on the complex projective plane. We also show that there exists a natural isomorphism between the moduli space of flags of framed torsion-free sheaves on \( \mathbb{P}^2 \) and the stable
representations of the nested instantons quiver. In the rank \( r = 1 \) case our definition reduces to the nested Hilbert scheme of points on \( \mathbb{C}^2 \), as it is to be expected. By this reason we first want to carry out the analysis of the simpler \( r = 1 \) case, which also has the advantage of providing us with a new characterization of punctual nested Hilbert schemes on \( \mathbb{C}^2 \), analogous to that of [5].

### 2.1 Hilb^\hat{n}(\mathbb{C}^2) and \( \mathcal{N}(1, \mathbf{n}) \)

Before delving into the analysis of the relation between nested instantons moduli spaces and flags of framed torsion-free sheaves on \( \mathbb{P}^2 \), we want to show a special simpler case. In particular we will prove the existence of an isomorphism between the nested Hilbert scheme of points in \( \mathbb{C}^2 \) and the nested instantons moduli space \( \mathcal{N}(1, n_0, \ldots, n_N) \). This effectively gives us the ADHM construction of a general nested punctual nested Hilbert scheme on \( \mathbb{C}^2 \), which will also be the local model for more general nested Hilbert schemes of points on, say, toric surfaces \( S \). In order to see this, we first recall the definition of a nested Hilbert scheme of points.

**Definition 3.** Let \( S \) be a complex (projective) surface and \( n_1 \geq n_2 \geq \cdots \geq n_k \) a sequence of integers. The nested Hilbert scheme of points on \( S \) is defined as

\[
\text{Hilb}^{[n_1, \ldots, n_k]}(S) = \{ I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq O_S : \text{length}(O_S/I_i) = n_i \}. \tag{2.1}
\]

Alternatively, if \( X \) is a quasi-projective scheme over the complex numbers, we can equivalently define the nested Hilbert scheme \( \text{Hilb}^{[n_1, \ldots, n_k]}(X) \) as

\[
\text{Hilb}^{[n_1, \ldots, n_k]}(X) = \{ (Z_1, \ldots, Z_k) : Z_i \in \text{Hilb}^{n_i}(X), Z_i \text{ is a subscheme of } Z_j \text{ if } i < j \}, \tag{2.2}
\]

with \( Z_i \) being a zero-dimensional scheme, for every \( i = 1, \ldots, k \).

Before actually exhibiting the isomorphism we are interested in, we want to prove an auxiliary result, which gives an alternative definition for the nested Hilbert schemes over the affine plane, analogously to the case of Hilbert schemes studied in [27].

**Proposition 2.1.** Let \( \kappa \) be an algebraically closed field, and \( \mathbf{n} \) a sequence of integers \( n_0 \geq n_1 \geq \cdots \geq n_k \). Define \( \hat{\mathbf{n}} \) to be the sequence of integers \( \hat{n}_0 = n_0 \geq \hat{n}_1 = n_0 - n_k \geq \cdots \geq \hat{n}_k = n_0 - n_1 \), then there exists an isomorphism

\[
\text{Hilb}^{\hat{\mathbf{n}}} (\mathbb{A}^2) \cong \left\{ (b_1^0, b_2^0, i, b_1^1, b_2^1, f_1, \ldots, b_1^k, b_2^k, f_k) \left| \begin{array}{l}
(i) [b_1^0, b_2^0] = 0 \\
(ii) b_{1,2}^{-1} f_i - f_i b_{1,2} = 0 \\
(iii) \text{AS} \subset \kappa^{\hat{\mathbf{n}}_0} : b_{1,2}^0(S) \subset S \text{ and } \text{Im}(i) \subset S \\
(iv) f_i : \kappa^{\hat{\mathbf{n}}_i} \to \kappa^{\hat{\mathbf{n}}_{i-1}} \text{ is injective}
\end{array} \right. \right\} \mathcal{G}_{\mathbf{n}}, \tag{2.3}
\]
where $G_n = GL_{n_0}(k) \times \cdots \times GL_{n_k}(k)$, $b_{i,2}^1 \in \text{End}(k^{n_i})$, $i \in \text{Hom}(k, k^{n_0})$ and $f_i \in \text{Hom}(k^{n_i}, k^{n_{i-1}})$. The action of $G_n$ is given by

$$g \cdot (b_{i,1}^0, b_{i,2}^0, i, b_{i,1}^1, b_{i,2}^1, f_1, \ldots, b_{i,k}, b_{i,k}^k, f_k) = \left( g_0 b_{i,1}^0 g_0^{-1}, g_0 b_{i,2}^0 g_0^{-1}, g_0 i, \ldots, g_k b_{i,k}^k g_k^{-1}, g_k b_{i,k}^k g_k^{-1} \right).$$

Proof. Suppose we have a sequence of ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k \in \text{Hilb}^\mathbb{A}(\mathbb{A}^2)$. Let’s first define $V_0 = k[z_1, z_2]/I_0$, $b_{i,2}^0 \in \text{End}(V_0)$ to be the multiplication by $z_{i,2}$ mod $I_0$, and $i \in \text{Hom}(k, V_0)$ by $i(1) = 1 \text{ mod } I_0$. Then obviously $[b_{i,0}^0, b_{i,1}^0] = 0$ and condition (iii) holds since 1 multiplied by products of $z_1$ and $z_2$ spans the whole $k[z_1, z_2]$. Then define $\tilde{V}_i = k[z_1, z_2]/I_i$ and, since $I_0 \subseteq I_i$ for any $i > 0$, complete $\tilde{V}_i$ to $V_0$ as $V_0 = \tilde{V}_i \oplus V_i$, so that $V_i \cong k^{n_i}$. The restrictions of $b_{i,2}^0 \in V_i$ then yield homomorphisms $b_{i,2}^1 \in \text{End}(V_i)$ naturally satisfying $[b_{i,1}^0, b_{i,2}^1] = 0$, while the inclusion of the ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k$ implies the existence of an embedding $f_i : V_i \hookrightarrow V_{i-1}$ such that condition (ii) holds by construction.

Conversely, let $(b_{i,0}^0, b_{i,1}^1, b_{i,2}^1, b_{i,k}, f_k)$ be given as in the proposition. In the first place one can define a map $\phi_0 : k[z_1, z_2] \rightarrow k^{n_0}$ to be $\phi_0(f) = f(b_{i,0}^0, b_{i,0}^2)i(1)$. This map is surjective, so that $I_0 = \text{ker } \phi_0$ is an ideal for $k[z_1, z_2]$ of length $n_0$. Then, since $f_i \in \text{Hom}(k^{n_i}, k^{n_{i-1}})$ is injective we can embed $k^{n_i}$ into $k^{n_0}$ though $F_i = f_i \circ \cdots \circ f_{i-1} \circ f_i$ in such a way that $b_{i,2}^1 = b_{i,2}^0 \mathbb{A} \subseteq \mathbb{A}^{n_i}$, which is a simple consequence of condition (ii). Then we have the direct sum decomposition $k^{n_i} = k^{n_0-n_i} \oplus k^{n_i}$, the restrictions $b_{i,2}^1 = b_{i,2}^0|_{k^{n_0-n_i}}$ and the projection $\tilde{i} = \pi_i \circ i$, with $\pi_i = k^{n_i} \rightarrow k^{n_0-n_i}$ satisfying $[b_{i,1}^0, b_{i,2}^1] = 0$ and a stability condition analogous to (iii). Thus we define $\phi_i : k[z_1, z_2] \rightarrow k^{n_0-n_i}$ by $\phi_i(f) = f(b_{i,1}^0, b_{i,2}^1)i(1)$. This map is surjective, just like $\phi_0$, so that $I_j = \ker (\phi_j)$ is an ideal for $k[z_1, z_2]$ of length $n_0 - n_i$. Finally, due to the successive embeddings $k^{n_i} \hookrightarrow k^{n_{i-1}} \hookrightarrow \cdots \hookrightarrow k^{n_0}$ we have the inclusion of the ideals $I_j \subseteq I_{j-1}$.

One can readily notice that the description given by the previous proposition of the nested Hilbert scheme of points doesn’t really coincide with the quiver we were studying throughout this section. However we can very easily overcome this problem by using the fact that if $(b_{i,1}^0, b_{i,2}^2, i, j)$ is a stable ADHM datum with $r = 1$, then $j = 0$, [27]. This proves the following proposition.

**Proposition 2.2.** With the same notations of proposition 2.1, we have that

$$\text{Hilb}^\mathbb{A}(\mathbb{A}^2) \cong \left\{ (b_{i,1}^0, b_{i,2}^0, i, b_{i,1}^1, b_{i,2}^1, f_1, \ldots, b_{i,k}, b_{i,k}^k, f_k) : \begin{array}{l}
(a) [b_{i,1}^0, b_{i,2}^0] + ij = 0 \\
(a') [b_{i,1}^0, b_{i,2}^0] = 0, \ i > 0 \\
(b) b_{i,2}^1 f_i - f_i b_{i,2}^1 = 0 \\
(c) j f_i = 0 \\
(d) \mathbb{A} S \subseteq k^{n_0} \subseteq b_{i,2}^1(S) \subseteq S \text{ and } \text{Im}(i) \subseteq S \\
(e) f_i : k^{n_i} \rightarrow k^{n_{i-1}} \text{ is injective} \end{array} \right\} / G_n.$$
All the previous observations, together with corollary 1.5, immediately prove the following theorem.

**Theorem 2.3.** The moduli space of nested instantons $\mathcal{N}_{r,\lambda,n,\mu}$ is isomorphic as a scheme to the nested Hilbert scheme of points on $\mathbb{C}^2$ when $r = 1$ and $\lambda = [1^1]$.

$$\mathcal{N}(1,n) = \mathcal{X}_{\mathbb{C}/\mathbb{R}} \cong \text{Hilb}^n(\mathbb{C}^2).$$ (2.4)

**2.2 $\mathcal{F}(r, \gamma)$ and $\mathcal{N}(r, n)$**

A more general result relates the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$ to the moduli space of nested instantons. In the case of the two-step quiver this result was proved in [34], here we give a generalization of their theorem in the case of the moduli space $\mathcal{N}_{r,[r^1],n,\mu}$ represented by a quiver with an arbitrary number of nodes.

**Definition 4.** Let $\ell_\infty \subset \mathbb{P}^2$ be a line and $F$ a coherent sheaf on $\mathbb{P}^2$. A framing $\phi$ for $F$ is then a choice of an isomorphism $\phi : F|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$, with $r = \text{rk} F$. An $(N+2)$–tuple $(E_0,E_1,\ldots,E_N,\phi)$ is a framed flag of sheaves on $\mathbb{P}^2$ if $E_0$ is a torsion-free (coherent) sheaf on $\mathbb{P}^2$ framed at $\ell_\infty$ by $\phi$, and $E_{j>0}$ form a flag of subsheaves $E_N \subseteq \cdots \subseteq E_0$ of $E_0$ s.t. the quotients $E_j/E_i$, $i < j$, are supported away from $\ell_\infty$.

By the framing condition we get that $c_1(E_0) = 0$, while the quotient condition on the subsheaves of $E_0$ naturally implies that the quotients $E_j/E_N$ are 0–dimensional subsheaves and $c_1(E_{j>0}) = 0$.

Then a framed flag of sheaves on $\mathbb{P}^2$ is characterized by the set of integers $(r, \gamma)$, where $r = \text{rk} E_0 = \cdots = \text{rk} E_N$, $c_2(E_0) = \gamma_0$, $h^0(E_0/E_j) = \gamma_1 + \cdots + \gamma_j$ so that $c_2(E_{j>0}) = \gamma_0 + \cdots + \gamma_j$.

We now define the moduli functor

$$\mathcal{F}(r, \gamma) : \text{Sch}_\mathbb{C}^{\text{op}} \to \text{Sets},$$ (2.5)

by assigning to a $\mathbb{C}$–scheme $S$ the set

$$\mathcal{F}(r, \gamma)(S) = \{\text{isomorphism classes of } (2N+2)\text{–tuples } (F_S, \varphi_S, Q^1_S, g^1_S, \ldots, Q^N_S, g^N_S)\}$$

with

- $F_S$ a coherent sheaf over $\mathbb{P}^2 \times S$ flat over $S$ and such that $F_S|_{\mathbb{P}^2 \times \{s\}}$ is a torsion-free sheaf for any closed point $s \in S$, $\text{rk} F_S = r$, $c_1(F_S) = 0$ and $c_2(F_S) = \gamma_0$;

- $\varphi_S : F_S|_{\ell_\infty \times S} \to \mathcal{O}_{\ell_\infty \times S}^{\oplus r}$ is an isomorphism of $\mathcal{O}_{\ell_\infty \times S}$–modules;

- $Q^i_S$ is a coherent sheaf on $\mathbb{P}^2 \times S$, flat over $S$ and supported away from $\ell_\infty \times S$, such that $h^0(Q^i_S|_{\mathbb{P}^2 \times \{s\}}) = \gamma_1 + \cdots + \gamma_i$, for any closed point $s \in S$;
• $g_S^i : F_S \to Q_S^i$ is a surjective morphism of $\mathcal{O}_{\mathbb{P}^2 \times S}$–modules.

Two tuples $(F_S, q_S, Q_S^1, g_S^1, \ldots, Q_S^N, g_S^N)$ and $(F_S', q_S', Q_S'^1, g_S'^1, \ldots, Q_S'^N, g_S'^N)$ are said to be isomorphic if there exist isomorphisms of $\mathcal{O}_{\mathbb{P}^2 \times S}$–modules $\Theta_S : F_S \to F'_S$ and $\Gamma^i_S : Q_S^i \to Q'_S$ such that the following diagrams commute

$$
\begin{array}{ccc}
F_S|_{\ell_\infty \times S} & \xrightarrow{q_S} & \mathcal{O}_{\ell_\infty \times S} \\
\Theta_S|_{\ell_\infty \times S} & \Downarrow & \Theta_S \\
F'_S|_{\ell_\infty \times S} & \xrightarrow{q'_S} & \mathcal{O}_{\ell_\infty \times S}
\end{array} \quad \begin{array}{ccc}
F_S & \xrightarrow{g_S^i} & Q_S^i \\
\Theta_S & \Downarrow & \Theta_S \\
F'_S & \xrightarrow{g'_S^i} & Q'_S
\end{array}
$$

(2.6)

If this functor is representable, the variety representing it will called the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$.

What we want to show next is that the moduli space of flags of torsion free sheaves on $\mathbb{P}^2$ is a fine moduli space, and that it is indeed isomorphic (as a scheme) to the moduli space of nested instantons we defined previously. First of all we will focus our attention on proving the following statement.

**Proposition 2.4.** The moduli functor $F_{(r,\gamma)}$ is represented by a (quasi-projective) variety $\mathcal{F}(r,\gamma)$ isomorphic to a relative quot-scheme.

**Proof.** We base our proof on the concept of Quot functor, so let us recall its constuction and basic properties. First of all let us take the universal framed sheaf $(U^{(0)}, q_0)$ on $\mathbb{P}^2 \times \mathcal{M}(r,\gamma_0)$, with $q_0 : U^{(0)}|_{\ell_\infty \times \mathcal{M}(r,\gamma_0)} \xrightarrow{\cong} \mathcal{O}_{\ell_\infty \times \mathcal{M}(r,\gamma_0)}$ an isomorphism of $\mathcal{O}_{\ell_\infty \times \mathcal{M}(r,\gamma_0)}$–modules. We then define

$$
\text{Quot}_{(U^{(0)}, \gamma)} : \text{Sch}_{\mathcal{M}(r,\gamma_0)}^{\text{op}} \to \text{Sets}
$$

(2.7)

by

$$
\text{Quot}_{(U^{(0)}, \gamma)}(S) = \{\text{isomorphism classes of } (Q_S, q_S)\}
$$

(2.8)

where

• $Q_S$ is a torsion-free sheaf on $\mathbb{P}^2 \times S$, flat over $S$, supported away from $\ell_\infty \times S$ and such that $h^0(Q_S|_{\ell_\infty \times \{s\}}) = \gamma_1$, for any $s \in S$ closed;

• $q_S : U^{(0)} \to Q_S$, defined by $q_S := (\mathbb{1}_\mathbb{P}^2 \times \pi)^*$, where $\pi : S \to \mathcal{M}(r,\gamma_0)$, is a surjective morphism of $\mathcal{O}_{\mathbb{P}^2 \times S}$–modules.

By Grothendieck theory this is a representable functor and it was proved in [34] to be isomorphic to the moduli functor of flags of couples of framed torsion-free sheaves on $\mathbb{P}^2$. In fact there exist a natural forgetting map $F_{(r,\gamma_0, \gamma)} \to \text{Quot}_{(U^{(0)}, \gamma)}$ which act as $(F_S, q_S, Q_S^1, g_S^1) \mapsto (Q_S^1, g_S^1)$. This map
also has an inverse given by setting $F_S = \ker(g^1|_S)$, which has a framing $\varphi_S$ at $\ell_\infty \times S$ induced by the framing $\varphi_0$ of $U^{(0)}$ at $\ell_\infty \times \mathcal{M}(r, \gamma_0)$. The variety representing $F(r, \gamma_0, \gamma_1)$ is then the quot scheme Quot$^{\gamma_1}(U^{(0)})$ relative to $\mathcal{M}(r, \gamma_0)$. We can then construct a universal framed sheaf $(U^{(1)}_0, \varphi_0)$ on $\mathbb{P}^2 \times F(r, \gamma_1, \gamma_2)$ with $\varphi_0 : U^{(1)}_0 \to \mathcal{O}_{\mathbb{P}^2 \times F(r, \gamma_0, \gamma_1)}$ an isomorphism of $\mathcal{O}_{\ell_\infty \times F(r, \gamma_0, \gamma_1)}$-modules. One can then use the quot functor

$$\text{Quot}(U^{(1)}, \gamma_2) : \text{Sch}^{\text{op}}_{\mathcal{F}(r, \gamma_1, \gamma_2)} \to \text{Sets},$$

in order to show that $F(r, \gamma_0, \gamma_1, \gamma_2)$ is isomorphic to $\text{Quot}(U^{(1)}, \gamma_2)$, exactly in the same way as before and since the latter is representable so is the former. By iterating this procedure we can finally show that our moduli functor $F(r, \gamma)$ is indeed representable, being isomorphic to a quot functor $\text{Quot}(U^{(N-1)}, \gamma_N)$.

The moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$, $F(r, \gamma)$, is then a fine moduli space isomorphic to the relative quot-scheme Quot$^{\gamma_N}(U^{(N-1)})$.

**Remark 2.1.** The previous description of the moduli space of framed flags of sheaves on $\mathbb{P}^2$ suggests we could also take a slightly different perspective on $F(r, \gamma)$, namely as the moduli of the sequence of quotients

$$Z_N \curvearrowleft \cdots \curvearrowleft Z_1 \curvearrowleft F \twoheadrightarrow Q_1 \twoheadrightarrow \cdots \twoheadrightarrow Q_N,$$

where $F$ is a vector bundle. In this sense $F(r, \gamma)$ seems to be analogous to the Filt-scheme studied by Mochizuki in [26] in the case of curves.

Now that we proved that the definition of moduli space of framed flags of sheaves on $\mathbb{P}^2$ is indeed a good one we are ready to tackle the problem of showing that there exists an isomorphism between this moduli space and the space of stable representation of the nested instantons quiver we studied in the previous sections. First of all let us point out that our definition of flags of framed torsion-free sheaves reduce in the rank 1 case to the nested Hilbert scheme of points on $\mathbb{C}^2$, and the isomorphism we are interested in was showed to exist in theorem 2.3 of section 2.1. This is in fact compatible with the statement of the following theorem 2.5.

**Theorem 2.5.** The moduli space of stable representations of the nested ADHM quiver is a fine moduli space isomorphic to the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$: $F(r, \gamma) \simeq \mathcal{N}(r, n)$, as schemes, where $n_i = \gamma_i + \cdots + \gamma_N$.

**Proof.** We first want to show how, starting from an element of $\mathcal{N}(r, n_0, \ldots, n_N)$ one can construct a flag of framed torsion-free sheaves on $\mathbb{P}^2$. As we showed previously, to each $(V_i, B_1^i, B_2^i, F^i)$ in the datum of $X \in \mathcal{N}(r, n_0, \ldots, n_N)$ we can associate a stable ADHM datum $(W, V_i, B_1^i, B_2^i, I^i, F^i)$, fitting in
the diagram (2.10)

\[ \begin{array}{ccccccc}
V_1 & \xrightarrow{F^1} & V_0 & \xrightarrow{\tilde{\eta}} & V_1 \\
\uparrow & & \uparrow & & \uparrow \\
\{0\} & \longrightarrow & W & \longrightarrow & W \\
\uparrow & & \uparrow & & \uparrow \\
V_2 & \xrightarrow{F^2} & V_0 & \xrightarrow{\tilde{\eta}} & V_2 \\
\uparrow & & \uparrow & & \uparrow \\
\{0\} & \longrightarrow & W & \longrightarrow & W \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
V_N & \xrightarrow{F^N} & V_0 & \xrightarrow{\tilde{\eta}} & V_N \\
\uparrow & & \uparrow & & \uparrow \\
\{0\} & \longrightarrow & W & \longrightarrow & W \\
\end{array} \]

(2.10)

where we suppressed all of the endomorphisms $B^i_{1,2}$, $\tilde{B}^i_{1,2}$. We will then call $Z_i$, $S$ and $Q_i$ the representations of the ADHM data $([0], V_i, B^i_1, B^i_2)$, $(W, V_0, B^0_1, B^0_2, I, J)$ and $(W, \tilde{V}_i, \tilde{B}^i_1, \tilde{B}^i_2, \tilde{I}, \tilde{J})$, respectively. The diagram (2.10) can be restated in the following form:

\[ \begin{array}{ccccccc}
0 & \longrightarrow & Z_1 & \longrightarrow & S & \longrightarrow & Q_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z_2 & \longrightarrow & S & \longrightarrow & Q_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & Z_N & \longrightarrow & S & \longrightarrow & Q_N & \longrightarrow & 0 \\
\end{array} \]

(2.11)

Moreover, if $E^*_Z$, $E^*_S$ and $E^*_Q$ denotes the ADHM complex corresponding to $Z_i$, $S$ and $Q_i$ the diagram
(2.11) induces the following

\[ 0 \to E_{Z_1} \to E_S \to E_{Q_1} \to 0 \]
\[ \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \]
\[ 0 \to E_{Z_2} \to E_S \to E_{Q_2} \to 0 \]
\[ \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \]
\[ \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \]
\[ 0 \to E_{Z_N} \to E_S \to E_{Q_N} \to 0 \]

(2.12)

Then, since \( S \) and \( Q_i \) are stable one has that \( H^p(E^*_S) = H^p(E^*_Q) = 0 \), for \( p = -1, 1 \), so that for each line in (2.12) the long exact sequence for the cohomology associated to it reduces to:

\[ 0 \to H^0(E^*_S) \to H^0(E^*_Q) \to H^1(E^*_Z) \to 0, \]

(2.13)

and by the ADHM construction \((H^0(E^*_Q), \varphi)\) is a rank \( r \) framed torsion-free sheaf on \( \mathbb{P}^2 \), with framing \( \varphi : H^0(E^*_Q)|_{\ell_\infty} \cong W \otimes O_{\ell_\infty} \). Moreover \( H^0(E^*_S) \) is a subsheaf of \( H^0(E^*_Q) \), and \( H^1(E^*_Z) \) is a quotient sheaf

\[ H^1(E^*_Z) \cong H^0(E^*_Q)/H^0(E^*_S), \]

which is 0–dimensional and supported away from \( \ell_\infty \subset \mathbb{P}^2 \). Finally one can immediately see from (2.12) that \( H^0(E^*_Q) \) is a subsheaf of \( H^0(E^*_Q_{\text{fr}}) \). One can moreover check that the numerical invariants classifying flags of sheaves do agree with the statement of the theorem.

Conversely let \((E_0, \ldots, E_N, \varphi)\) be a flag of framed torsion-free sheaves on \( \mathbb{P}^2 \) such that \( \text{rk} E_j = r \), \( c_2(E_0) = \gamma_0 \), \( h^0(E_0/E_{j>0}) = \gamma_1 + \cdots + \gamma_j \). By definition each \((E_j, \varphi)\) defines a stable ADHM datum \( Q_j = (\tilde{W}_j, \tilde{V}_j, \tilde{B}_j, \tilde{I}_j, \tilde{J}_j) \) (with the convention of calling \( S = Q_N \)), since it can be identified with a framed torsion-free sheaf on \( \mathbb{P}^2 \), with \( \text{rk} E_j = r \), \( c_2(E_j) = \gamma_0 + \cdots + \gamma_j \). Moreover we have the inclusion \( E_0 \hookrightarrow E_j \), which induces an epimorphism \( \Psi_j : S \to Q_j \). In fact, we can construct vector spaces \( V_0, \tilde{V}_j, W \) and \( \tilde{W}_j \) as in [27], so that

\[ V_0 \cong H^0(E_N(-1)), \quad \tilde{V}_j \cong H^0(E_j(-1)), \quad W \cong H^0(E_N|_{\ell_\infty}), \quad \tilde{W}_j \cong H^0(E_j|_{\ell_\infty}), \]

(2.14)

and by the fact that the quotient sheaf \( E_j/E_N \) is 0–dimensional and supported away from \( \ell_\infty \) we can construct an isomorphism

\[ \Psi_{j,2} : H^0(E_N|_{\ell_\infty}) \cong H^0(E_j|_{\ell_\infty}). \]

(2.15)
Finally we have the exact sequence

\[ 0 \rightarrow E_N \rightarrow E_j \rightarrow E_j/E_N \rightarrow 0, \]

which induces the following exact sequence of cohomology, thanks to the fact that \( H^0(E_j(-1)) = 0 \), being that \( E_j \) is a framed torsion-free \( \mu \)-semistable sheaf with \( c_1(E_j) = 0 \) (due to the standard ADHM construction), and \( H^1(E_j/E_N(-1)) = 0 \), since the quotient sheaf \( E_j/E_N \) is 0–dimensional,

\[ 0 \rightarrow H^0(E_j/E_N(-1)) \rightarrow H^1(E_N(-1)) \xrightarrow{\Psi_j} H^1(E_j(-1)) \rightarrow 0. \] (2.16)

The morphism \( \Psi_j = (\Psi_{j,1}, \Psi_{j,2}) \) is then an epimorphism, since both \( \Psi_{j,1} \) and \( \Psi_{j,2} \) are surjective. Taking into account the flag structure of the datum \( (E_0, \ldots, E_N, \varphi) \), the sequences

\[ 0 \rightarrow \ker \Psi_{N-1} \rightarrow S \rightarrow Q_1 \rightarrow 0 \]
\[ \downarrow \quad \uparrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ 0 \rightarrow \ker \Psi_{N-2} \rightarrow S \rightarrow Q_2 \rightarrow 0 \]
\[ \downarrow \quad \uparrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \vdots \rightarrow \vdots \rightarrow \vdots \rightarrow \vdots \rightarrow \vdots \]
\[ \downarrow \quad \uparrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ 0 \rightarrow \ker \Psi_0 \rightarrow S \rightarrow Q_N \rightarrow 0 \] (2.17)
give us \((N+1)\) stable ADHM data fitting in the following diagram.

\[
\begin{align*}
V_1 & \xrightarrow{\psi_{N-1,1}} V_0 & \xrightarrow{\psi_{N-1,2}} \bar{V}_1 \\
\psi_{N-2,1} & & \psi_{N-2,2} \\
V_2 & \xrightarrow{\psi_{N-3,1}} V_0 & \xrightarrow{W} \bar{V}_2 \\
\psi_{N-3,2} & & \psi_{N-2,2} \\
V_N & \xrightarrow{\psi_{N,1}} V_0 & \xrightarrow{\psi_{N,2}} \bar{V}_N \\
\psi_{N,n_1} & & \psi_{N,n_2} \\
\{0\} & \xrightarrow{\{0\}} W & \xrightarrow{\{0\}} W
\end{align*}
\]

\((2.18)\)

\section{Virtual invariants}

In this section we study fixed points under the action of a torus on the moduli space of framed stable representations of fixed numerical type of the nested instantons quiver. By doing this we are then able to apply virtual equivariant localization and compute certain relevant virtual topological invariants. On the physics side this is equivalent to the computation of certain partition functions of some suitable quiver GLSM theory by means of the SUSY localization technique.

\subsection{Equivariant torus action and localization}

We begin the analysis of the fixed locus under a certain toric action on the moduli space of nested instantons with a brief recall of the results obtained in [34] and show how they enable us to fully characterize the \(T\)-fixed locus of the two-step nested instantons quiver. The main result we want to recall is the following:

**Theorem 3.1** (von Flach-Jardim, [34]). The moduli space \(\mathcal{N}(r, n_0, n_1) \simeq \mathcal{F}(r, n_0 - n_1, n_1)\) of stable representations of the nested ADHM quiver is a quasi-projective variety equipped with a perfect obstruction.
theory. Its $T$-equivariant deformation complex is the following

\[
\begin{align*}
Q \otimes \text{End}(V_0) & \oplus \Lambda^2 Q \otimes \text{End}(V_0) \\
\text{End}(V_0) & \oplus \text{End}(V_0) \\
\text{Hom}(W, V_0) & \oplus \text{Hom}(W, V_0) \\
\oplus & \Lambda^2 Q \otimes \text{End}(V_0) \oplus \text{End}(V_0) \\
& \oplus \Lambda^2 Q \otimes \text{End}(V_0) \oplus \text{End}(V_1) \oplus \Lambda^2 Q \otimes \text{End}(V_1) \\
& \oplus \Lambda^2 Q \otimes \text{End}(V_1) \oplus \text{End}(V_1) \\
& \oplus \Lambda^2 Q \otimes \text{End}(V_1) \oplus \text{End}(V_1) \\
& \oplus \Lambda^2 Q \otimes \text{End}(V_1) \oplus \text{End}(V_1) \\
& \oplus \Lambda^2 Q \otimes \text{End}(V_1) \oplus \text{End}(V_1) \\
\end{align*}
\]

with

\[
\begin{align*}
d_0(h_0, h_1) &= \left( [h_0, B_0^1], [h_0, B_2^0], h_0 I, h_0 F - F h_1 \right) \\
d_1(b_1^0, b_2^0, i, j, b_1^1, b_2^1, f) &= \left( [b_1^0, B_1^2] + [B_1^0, b_2^0] + iJ + IJ, \left[ b_1^0, b_2^0 \right] + f B_1^1, \right. \left. B_2^0 F + b_2^0 F - F b_2^1 - f B_2^1, \right. \left. \left[ b_2^0, b_1^0 \right] + b_2^0 B_1^0 - B_1^0 b_2^0 - Ic_4 - F c_5 \right) \\
d_2(c_1, c_2, c_3, c_4, c_5) &= c_1 F + B_2^0 c_2 - c_2 B_2^0 + c_3 B_1^0 - B_1^0 c_3 - Ic_4 - F c_5
\end{align*}
\]

Thus the infinitesimal deformation space and the obstruction space at any $X$ will be isomorphic to $H^1[\mathcal{C}(X)]$ and $H^2[\mathcal{C}(X)]$, respectively. $\mathcal{N}(r, n_1, n_2)$ is smooth iff $n_1 = 1$ ([7]).

Moreover, it turns out, [34], that there exists a surjective morphism $q: (B_1', B_2', I, J, B_1^1, B_2^1, F) \mapsto (B_1', B_2', I', J')$ mapping the nested ADHM data of type $(r, n_0, n_1)$ to the ADHM data of numerical type $(r, n_0 - n_1)$. Thus we have two different maps sending the moduli space of stable representations of the nested ADHM quiver to the moduli space of stable representations of ADHM data. The situation is depicted by the following commutative diagram

\[
\begin{align*}
\mathcal{N}(r, n_0, n_1) \xrightarrow{\eta} M(r, n_0) \\
\xrightarrow{q} \mathcal{M}(r, n_0 - n_1) \\
\end{align*}
\]

by means of which one can characterize $T$-fixed points of $\mathcal{N}(r, n_0, n_1)$ by means of fixed points of $\mathcal{M}(r, n_0)$ and $\mathcal{M}(r, n_0 - n_1)$. In particular we can first take the decomposition $V_0 = V \oplus V_1$, then decompose the vector spaces $V_0, V$ with respect to the action of $T$: if $\lambda_0: T \to U(V_0)$ and $\lambda: T \to$
$U(V)$ are morphisms for the toric action on $V_0$, $V$, we have

$$
\begin{cases}
V = \bigoplus_{k,l} V(k,l) = \bigoplus_{k,l} \{ v \in V | \lambda(t)v = t_1^k t_2^l v \} \\
V_0 = \bigoplus_{k,l} V_0(k,l) = \bigoplus_{k,l} \{ v_0 \in V_0 | \lambda_0(t)v_0 = t_1^k t_2^l v_0 \}
\end{cases}
$$

(3.2)

Thus if $X = (W, V, B'_1, B'_2, I', J')$, $X_0 = (W, V_0, B^0_1, B^0_2, I, J)$ are fixed points for this torus action, the very well known results about the classification of fixed points for ADHM data leads us to the following commutative diagram.

![Diagram](image)

(3.3)

**Proposition 3.2.** Let $X \in X_0$ be a fixed point of the toric action. The following statements hold:

1. If $k > 0$ or $l > 0$, then $V_0(k,l) = 0$, $V(k,l) = 0$;

2. $\dim V_0(k,l) \leq 1$, $\forall k,l$ and $\dim V(k,l) \leq 1$, $\forall k,l$;

3. If $k,l \leq 0$, then $\dim V_0(k,l) \geq \dim V_0(k-1,l)$, $\dim V_0(k,l) \geq \dim V_0(k,l-1)$, $\dim V(k,l) \geq \dim V(k-1,l)$, $\dim V(k,l) \geq \dim V(k,l-1)$ and $\dim V_0(k,l) \geq \dim V(k,l)$.

The previous propositions give us an easy way of visualizing fixed points of the $T$–action on the nested ADHM data. If we suitably normalize each non-zero map to 1 by the action of $\prod_{k,l} GL(V_0(k,l)) \times \prod_{k,l} GL(V(k',l'))$ each critical point can be put into one-to-one correspondence with nested Young diagrams $Y_\mu \subseteq Y_\nu$. Thus the fixed points of the original nested ADHM data are classified by couples $(\nu, \nu \setminus \mu)$, where $\mu \subset \nu$ and $\nu \setminus \mu$ is the skew Young diagram constructed by taking the complement of $\mu$ in $\nu$.

If we now take a fixed point $Z = (\nu, \mu)$ and define $\nu_i = \sum_k \dim V_0(k,1-i)$, $\nu'_j = \sum_l \dim V_0(1-j,l)$ and similarly $\mu_i = \sum_k \dim V(k,1-i)$, $\mu'_j = \sum_l \dim V(1-j,l)$, we can regard $V_0$ and $V$ as $T$–modules.
and write them as

\[
\begin{align*}
V_0 &= \bigoplus_{k,l} V_0(k,l) = \sum_{i=1}^{M_1} \sum_{j=1}^{v_i'} T_1^{-i+1} T_2^{-j+1} = \sum_{i=1}^{N_1} \sum_{j=1}^{\nu_i} T_1^{-i+1} T_2^{-j+1} \\
V &= \bigoplus_{k,l} V(k,l) = \sum_{i=1}^{M_2} \sum_{j=1}^{\mu_i'} T_1^{-i+1} T_2^{-j+1} = \sum_{i=1}^{N_2} \sum_{j=1}^{\mu_i} T_1^{-i+1} T_2^{-j+1}
\end{align*}
\]

(3.4)

with \( M_1 = \nu_1, M_2 = \mu_1, N_1 = \nu_1', N_2 = \mu_1' \). If we now take \( V_0 = V \oplus V_1 \), then

\[
V_1 = \sum_{(i,j) \in \nu \setminus \mu} T_1^{-i+1} T_2^{-j+1} = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i'} T_1^{-i+1} T_2^{-j+1} \tag{3.5}
\]

The virtual tangent space \( T^\text{vir}_Z \mathcal{N}(1, n_0, n_1) \) to \( \mathcal{N}(1, n_0, n_1) \) at \( Z \) can be regarded as a \( T^2 \)-module, so that

\[
T^\text{vir}_Z \mathcal{N}(1, n_0, n_1) = \text{End}(V_0) \otimes (Q - 1 - \Lambda^2 Q) + \text{End}(V_1) \otimes (Q - 1 - \Lambda^2 Q) + \text{Hom}(W, V_0) + \\
+ \text{Hom}(V_0, W) \otimes \Lambda^2 Q - \text{Hom}(V_1, W) \otimes \Lambda^2 Q + \text{Hom}(V_1, V_0)(1 + \Lambda^2 Q - Q) \\
= (V_1 \otimes V_0^* + V_1 \otimes V_1^* - V_1^* \otimes V_0) \otimes (Q - 1 - \Lambda^2 Q) + V_0 + V_0^* \otimes \Lambda^2 Q + \\
- V_1^* \otimes \Lambda^2 Q. \tag{3.6}
\]

In the first place we might recognize the term \( V_0^* \otimes V_0 (Q - \Lambda^2 Q - 1) + V_0 + V_0^* \otimes \Lambda^2 Q \) in the sum as being the tangent space at the moduli space of stable representation of the ADHM quiver \( T_2 \mathcal{M}(1, n_0) \), with \( \tilde{Z} = (\nu) \). Thus we have

\[
T^\text{vir}_Z \mathcal{N}(1, n_0, n_1) = T_2 \mathcal{M}(1, n_0) + (V_1 \otimes V_1^* - V_1^* \otimes V_0) \otimes (Q - 1 - \Lambda^2 Q) - V_1^* \otimes \Lambda^2 Q. \tag{3.7}
\]
We have

\[ V_1^* \otimes (Q - 1 - \Lambda^2 Q) = (T_1 - 1)(1 - T_2) \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} T_{2}^{j+1-i} \]

\[ = (T_1 - 1) \sum_{i=1}^{M_1} T_i^{j-i} (1 - T_2) \sum_{j=1}^{\nu_i' - \mu_i'} T_{2}^{j} \]

\[ = (T_1 - 1) \sum_{i=1}^{M_1} T_i^{j-i} (1 - T_2) \left( \frac{1 - T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} - 1 \right) \]

\[ = (T_1 - 1) \sum_{i=1}^{M_1} T_i^{j-i} (1 - T_2) \left( \frac{T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} \right) \]

\[ = (T_1 - 1) \sum_{i=1}^{M_1} T_i^{j-i} (1 - T_2) \left( T_2^{\nu_i' - \mu_i' + 1} \right) \]

so that

\[ V_1^* \otimes V_1 \otimes (Q - 1 - \Lambda^2 Q) = (T_1 - 1) \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \sum_{j'=1}^{\nu_i' - \mu_i'} T_{2}^{j'} \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \left( \frac{1 - T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} - 1 \right) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} \left( T_i^{j-i} - T_i^{j-i+1} \right) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \]

(3.8)

so that

\[ V_1^* \otimes V_1 \otimes (Q - 1 - \Lambda^2 Q) = (T_1 - 1) \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \sum_{j'=1}^{\nu_i' - \mu_i'} T_{2}^{j'} \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_i^{j-i} (1 - T_2) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \left( \frac{1 - T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} - 1 \right) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} \left( T_i^{j-i} - T_i^{j-i+1} \right) \left( T_2^{\nu_i' - \mu_i' + 1} - T_2^{j-i+1} \right) \]

(3.9)
As an immediate generalization of (3.12) we can easily see that

while we have

\[ V^*_1 \otimes V_0 \otimes (Q - 1 - \Lambda^2 Q) = (T_1 - 1) \sum_{j=1}^{N_1} \sum_{j'=1}^{N_1} T_1^{-j'+1} T_2^{-j+1} \sum_{i=1}^{M_1} T_1^{i-1} (T_2^{j'-} - T_2^{j'+}) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^i (T_2^{-j+\mu_1' + 1} - T_2^{-j+\nu_1' + 1})(T_1 - 1) \sum_{j'=1}^{N_1} T_1^{-j'} \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^i (T_2^{-j+\mu_1' + 1} - T_2^{-j+\nu_1' + 1})(T_1 - 1) \left( \frac{1}{1 - T_1^{-1}} - 1 \right) \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} (T_1^i - T_1^{-\nu_1'})(T_2^{-j+\mu_1' + 1} - T_2^{-j+\nu_1' + 1}), \]

and

\[ V^*_1 \otimes \Lambda^2 Q = T_1 T_2 \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^{i-1} T_2^{\mu_1' + j - 1} \]

\[ = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^i T_2^{\mu_1' + j}. \]

Putting everything together we finally get that

\[ T_Z^{\text{vir}} \mathcal{N}(1, n_0, n_1) = T_2 \mathcal{M}(1, n_0) + \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} (T_1^{i-\mu_1'} - T_1^i)(T_2^{-j+\mu_1' + 1} - T_2^{-j+\nu_1' + 1}) \]

\[ - \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^i T_2^{\mu_1'}. \]

As an immediate generalization of (3.12) we can easily see that

\[ T_Z^{\text{vir}} \mathcal{N}(r, n_0, n_1) = T_2 \mathcal{M}(r, n_0) + \sum_{a,b=1}^{r} \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} R_a R_b^{-1} \left( T_1^{i-\mu_1^{(b)}} - T_1^i \right) \left( T_2^{-j+\mu_1^{(b)} + 1} - T_2^{-j+\nu_1^{(b)} + 1} \right) \]

\[ - T_2^{-j+\nu_1^{(b)} + 1} \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^i T_2^{\mu_1^{(b)}}. \]

where \((T_1, T_2, R_a), a = 1, \ldots, r\) are the canonical generators of the representation ring of \( T \sim \mathcal{N}(r, n_0, n_1) \).
Remark 3.1. It turns out that the character representation for the virtual tangent $T^\text{vir}_Z N$ can be computed by exploiting deformation theory techniques. These techniques may also be employed to compute the virtual fundamental class and (T–character of) the virtual tangent bundle at fixed points of nested Hilbert schemes on surfaces, as it’s done in [15].

If in particular one takes $(\mathbb{C}^2)^{[N_0 \geq N_1]}$ to be the nested Hilbert scheme of points on $\mathbb{C}^2 = \text{Spec}(R)$, with $\mathbb{C}[x_0, x_1]$, by lifting the natural torus action on $\mathbb{C}^2$ to $(\mathbb{C}^2)^{[N_0 \geq N_1]}$, it is proved in [15] that the $T$–fixed locus is isolated and given by the inclusion of monomial ideals $I_0 \subseteq I_1$, which is equivalent to the assignment of couples of nested partitions $\mu \subseteq \nu$. Then the virtual tangent space at a fixed point is given by
\[
T^\text{vir}_{I_0 \subseteq I_1} = -\chi(I_0, I_0) - \chi(I_1, I_1) + \chi(I_0, I_1) + \chi(R, R),
\]
with $\chi(\cdot, \cdot) = \sum_{i=0}^2 (-1)^i \text{Ext}^i_R(\cdot, \cdot)$. Then the $T$–representation of $T^\text{vir}_{I_0 \subseteq I_1}$ can be explicitly written in terms of Laurent polynomials in the torus characters $t_1, t_2$ of $T$. Then in terms of the characters $Z_0, Z_1$ of the $T$–fixed 0–dimensional subschemes $Z_1 \subseteq Z_0 \subseteq \mathbb{C}^2$ corresponding to $I_0 \subseteq I_1$ one has (see eq. (29) in [15])
\[
\text{Tr} T^\text{vir}_{I_0 \subseteq I_1} = Z_0 + \frac{Z_1}{t_1 t_2} + (Z_0 Z_1 - Z_0 Z_0 - Z_1 Z_1) \left(1 - t_1\right) \left(1 - t_2\right) \frac{1}{t_1 t_2}.
\]
If we now make the necessary identifications $t_i = T_i^{-1}, Z_0 = V_0$ and $Z_1 = V$ we can see that equation (29) of [15] exactly agrees with our prescription for the character representation (3.7) of the virtual tangent space $T^\text{vir}_Z N(1, n_0, n_1)$, with $n_0 = N_0$ and $n_1 = N_0 - N_1$.

We now move on studying the fixed locus of the more general nested instantons moduli space $N(r, n_0, \ldots, n_N)$. However, similarly to the previous case we first want to show that the moduli space of stable representations of the nested ADHM quiver is equivalently described by the datum of $(N + 1)$ moduli spaces of framed torsion-free sheaves on $\mathbb{P}^2$, namely $\mathcal{M}(r, n_0), \mathcal{M}(r, n_0 - n_1), \ldots, \mathcal{M}(r, n_0 - n_{s-1})$. In order to do this we want to know if it is possible to recover the structure of the nested ADHM quiver given a set of stable ADHM data. First of all we can notice that, as $F^i$ is injective $\forall \nu_i$, we have the sum decomposition $V_0 = V_i \oplus \hat{V}_i$, but also $V_i = V_{i+1} \oplus \hat{V}_{i+1}$, with $\hat{V}_{i+1} = V_i/\text{Im} F_i$, so that $V_0 = V_i \oplus \hat{V}_i \oplus \hat{V}_{i-1}$, thus $\hat{V}_i = \hat{V}_i \oplus \hat{V}_{i-1}$.

Let us first focus on the vector spaces $V_0$ and $V_1$. It can be shown as in [4, 34] that once we fix a stable ADHM datum $(W, \hat{V}_1, \hat{B}_1^1, \hat{B}_2^1, \hat{I}_1, \hat{J}_1)$ and the endomorphisms $B_1^0, B_2^0 \in \text{End} V_1$ it is always possible to reconstruct the stable ADHM datum $(W, V_0, B_1^0, B_2^0, I, J)$ as
\[
B_1^0 = \begin{pmatrix} B_1^1 & B_1^2 \\ 0 & B_1^1 \end{pmatrix}, \quad B_2^0 = \begin{pmatrix} B_2^1 & B_2^2 \\ 0 & B_2^2 \end{pmatrix}, \quad I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \hat{J}_1 \end{pmatrix}
\]
(3.14)

together with the morphism $F^1 = 1_{V_1}$ such that $[B_1^0, B_2^0] = 0$, $B_1^0 F^1 - F^1 B_1^0 = B_2^0 F^1 - F^1 B_2^0 = 0$ and $J F^1 = 0$. The same can obviously be done for any of the stable ADHM data $(W, \hat{V}_i, \hat{B}_1^i, \hat{B}_2^i, \hat{I}_i, \hat{J}_i)$ we
constructed previously, and we would have

\[ B_1^0 = \begin{pmatrix} B_1^i & B_1^i \\ 0 & B_1^j \end{pmatrix}, \quad B_2^0 = \begin{pmatrix} B_2^i & B_2^i \\ 0 & B_2^j \end{pmatrix}, \quad I = \begin{pmatrix} I^i \\ I^j \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 \end{pmatrix} \]  

(3.15)

together with the morphism \( f^i = 1_{V_i} \) such that \([B_1^i, B_2^i] = 0, B_1^0 f^i - f^i A_1 = B_2^0 f^i - f^i B_2^i = 0\) and \( J f^i = 0 \). If we now fix

\[ F^i = \begin{pmatrix} 1_{V_i} \\ 0 \end{pmatrix}, \quad F^i : V_i \to V_{i-1}, \]  

(3.16)

which is clearly injective, then obviously \( f^i = F^1 F^2 \ldots F^i \), where \( F^i \) now stands for the linear extension to \( V_0 \), and \( B_0^0 f^i - f^i B_1^0 = 0 \) (resp. \( B_0^0 f^i - f^i B_2^0 = 0 \)) is equivalent to \( B_0^0 F^1 F^2 \ldots F^{i-1} F^i - F^{i-1} F^i F^1 \ldots F^i B_1^0 = B_0^0 F^1 F^2 \ldots F^{i-1} F^i - F^{i-1} F^i B_2^0 = 0 \) (resp. \( B_0^0 F^1 F^2 \ldots F^{i-1} F^i - F^{i-1} F^i B_2^0 = 0 \)), and \( J f^i = J F^1 F^2 \ldots F^i = 0 \). This construction makes it possible to us to classify the \( T \)-fixed locus of \( \mathcal{N}(r, n_0, \ldots, n_{s-1}) \) in terms of the \( T \)-fixed loci of \( \mathcal{M}(r, n_0) \) and \( \{\mathcal{M}(r, n_0 - n_i)\}_{i \geq 0} \). In particular the \( T \)-fixed locus of \( \mathcal{M}(r, k) \) is into \( 1 \)-1 correspondence with coloured partitions \( \mu = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^r \) such that \(|\mu| = |\mu^1| + \cdots + |\mu^r| = k\). This fact and the inclusion relations between the vector spaces \( V_i \) prove the following

**Proposition 3.3.** The \( T \)-fixed locus of \( \mathcal{N}(r, n_0, \ldots, n_{s-1}) \) can be described by \( s \)-tuples of nested coloured partitions \( \mu_1 \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0 \), with \(|\mu_0| = n_0\) and \(|\mu_{i>0}| = n_0 - n_i\).

In the same way as we did in a previous section, we can read the virtual tangent space to \( \mathcal{N}(r, n_0, \ldots, n_{s-1}) \) off the following equivariant lift of the complex (1.11)

\[
\begin{align*}
\bigoplus_{i=0}^{N} \text{End}(V_i) \\
\downarrow d_0 \\
Q \otimes \text{End}(V_0) \oplus \text{Hom}(W, V_0) \oplus \Lambda^2 Q \otimes \text{Hom}(V_0, W) & \oplus \left[ \bigoplus_{i=1}^{N} (Q \otimes \text{End}(V_i) \oplus \text{Hom}(V_i, V_{i-1})) \right] \\
\downarrow d_1 \\
\Lambda^2 Q \otimes (\text{End}(V_0) \oplus \text{Hom}(V_1, W)) & \oplus \left[ \bigoplus_{i=1}^{N} (Q \otimes \text{Hom}(V_i, V_{i-1}) \oplus \Lambda^2 Q \otimes \text{End}(V_i)) \right] \\
\downarrow d_2 \\
\bigoplus_{i=1}^{N} \Lambda^2 Q \otimes \text{Hom}(V_i, V_{i-1})
\end{align*}
\]

(3.17)
which gives us (3.18).

\[
T^\text{vir}_Z \mathcal{N}(1, n_1, n_2) = \text{End}(V_0) \otimes (Q - 1 - \Lambda^2 Q) + \text{Hom}(W, V_0) + \text{Hom}(V_0, W) \otimes \Lambda^2 Q + \\
+ \text{End}(V_j) \otimes (Q - 1 - \Lambda^2 Q) - \text{Hom}(V_j, W) \otimes \Lambda^2 Q + \\
+ \text{End}(V_1, V_0) \otimes (1 + \Lambda^2 Q - Q) + \\
+ \text{End}(V_2) \otimes (Q - 1 - \Lambda^2 Q) + \text{Hom}(V_2, V_1) \otimes (1 + \Lambda^2 Q - 1) + \\
\cdots \\
+ \text{End}(V_{s-1}) \otimes (Q - 1 - \Lambda^2 Q) + \text{Hom}(V_{s-1}, V_{s-2}) \otimes (1 + \Lambda^2 Q - Q) 
\]

By decomposing the vector spaces $V_j$ in terms of characters of the torus $T$ we can also rewrite the representation of (3.18) in $R(T)$ as (3.19)

\[
T^\text{vir}_Z \mathcal{N}(r, n) = T_2 M(r, n_0) + \sum_{a, b=1}^r \sum_{i=1}^r \sum_{j=1}^r R_b R_a^{-1} \left( T_1^{i-\mu^a_j} - T_1^i \right) \left( T_2^{-j+\mu^a_j+1} + \\
- T_2^{-j+\mu^a_j+1} \right) - \sum_{i=1}^r \sum_{j=1}^r T_1^i T_2^{-j+\mu^a_j} + \\
+ \sum_{k=2}^{s-1} \left[ \sum_{a, b=1}^r \sum_{i=1}^r \sum_{j=1}^r R_b R_a^{-1} \left( T_1^{i-\mu^a_k} - T_1^i \right) \left( T_2^{-j+\mu^a_k+1} + T_2^{-j+\mu^a_k+1} \right) + \\
+ (s - 1)(T_1 T_2), 
\right.
\]

where the fixed point $Z$ is to be identified with a choice of a sequence of coloured nested partitions $\mu_1 \subseteq \mu_{N-1} \subseteq \cdots \subseteq \mu_s \subseteq \mu_0$ as in proposition 3.3. $Z \leftrightarrow \mu_0$ and the last term, namely $(s - 1)(T_1 T_2)$, has been added in order to take into account the over-counting in the relations $[B_1^1, B_2^1] = 0$ due to the commutator being automatically traceless.

### 3.2 Virtual equivariant holomorphic Euler characteristic

The first virtual invariant we are going to study is the holomorphic virtual equivariant Euler characteristic of the moduli space of nested instantons. The fact that we can decompose the virtual tangent bundle as a direct sum of equivariant line bundles under the torus action we previously described greatly simplifies the computations.

In particular, given a scheme $X$ with a 1–perfect obstruction theory $E^*$, one can define a virtual structure sheaf $O_X^\text{vir}$. Moreover one can choose an explicit resolution of $E^*$ as $[E^{-1} \rightarrow E^0]$ a complex of vector bundles. If $[E_0 \rightarrow E_1]$ denotes the dual complex, then one can also define the virtual
tangent bundle $T_X^{\text{vir}} \in K^0(X)$ as the class $T_X^{\text{vir}} = [E_0] - [E_1]$. With these definitions, the virtual Todd genus of $X$ is defined as $td^{\text{vir}}(X) = td(T_X^{\text{vir}})$, and if $X$ is proper, given any $V \in K^0(X)$ one defines the virtual holomorphic Euler characteristic as

$$\chi^{\text{vir}}(X, V) = \chi(X, V \otimes O_X^{\text{vir}}), \quad (3.20)$$

and as a consequence of the virtual Riemann-Roch theorem [14] if $X$ is proper and $V \in K^0(X)$ the virtual holomorphic Euler characteristic admits an equivalent definition as

$$\chi^{\text{vir}}(X, V) = \int_{[X]^{\text{vir}}} \text{ch}(V) \cdot td(T_X^{\text{vir}}), \quad (3.21)$$

where $[X]^{\text{vir}}$ is the virtual fundamental class of $X$, $[X]^{\text{vir}} \in A^{vd}(X)$ and $vd$ denotes the virtual dimension of $X$, $vd = \text{rk} E_0 - \text{rk} E_1$. Clearly, if we are interested in $\chi^{\text{vir}}(X)$ then the previous formula reduces to

$$\chi^{\text{vir}}(X) = \int_{[X]^{\text{vir}}} td(T_X^{\text{vir}}), \quad (3.22)$$

whenever $X$ is proper.

Equations (3.21) and (3.22) can be made even more explicit. In fact if we take $n = \text{rk} E_0$, $m = \text{rk} E_1$, so that $vd = n - m$, and define $x_1, \ldots, x_n$ and $u_1, \ldots, u_m$ to be respectively the Chern roots of $E_0$ and $E_1$, then (3.22) becomes

$$\chi^{\text{vir}}(X) = \int_{[X]^{\text{vir}}} \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1-e^{-u_j}}{u_j}, \quad (3.23)$$

while for (3.21) we have

$$\chi^{\text{vir}}(X, V) = \int_{[X]^{\text{vir}}} \left( \sum_{k=1}^{r} e^{v_k} \right) \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1-e^{-u_j}}{u_j}, \quad (3.24)$$

since we can consider $V \in K^0(X)$ to be a vector bundle on $X$ with Chern roots $v_1, \ldots, v_r$.

Now, if we have a proper scheme $X$ equipped with an action of a torus $(\mathbb{C}^*)^N$ and an equivariant 1–perfect obstruction theory we can apply virtual equivariant localization in order to compute virtual invariants of $X$. We will now briefly recall how virtual localization works. First of all, for any equivariant vector bundle $B$ over a proper scheme $Z$ with a 1–perfect obstruction theory, which is moreover equipped with a trivial action of $(\mathbb{C}^*)^N$, we have the decomposition

$$B = \bigoplus_{k \in \mathbb{Z}^N} B^k, \quad (3.25)$$

where $B^k$ denotes the $(\mathbb{C}^*)^N$–eigenbundles on which the torus acts by $t_1^{k_1} \cdots t_N^{k_N}$. If we now give a set of variables $\epsilon_1, \ldots, \epsilon_N$, we identify $B$ with $B = \sum_k B^k e^{\epsilon_1} \cdots e^{\epsilon_N} \in K^0(Z)[[\epsilon_1, \ldots, \epsilon_N]]$. One then
defines $B^\text{fix} = B^0$ and $B^\text{mov} = \oplus_{k=0}B^k$. Then the Chern character $\text{ch} : K^0(Z) \to \Lambda^*(Z)$ can be extended by $\mathbb{Q}(\{\varepsilon_1, \ldots, \varepsilon_N\})$–linearity to

$$\text{ch} : K^0(Z)(\{\varepsilon_1, \ldots, \varepsilon_N\}) \to \Lambda^*(Z)(\{\varepsilon_1, \ldots, \varepsilon_N\}).$$

Since the Grothendieck group of equivariant vector bundles $K^0_{(\mathbb{C}^*)^N}(Z)$ is a subring of $K^0(Z)[[\varepsilon_1, \ldots, \varepsilon_N]]$, the restriction of the extension of $\text{ch}$ to $K^0_{(\mathbb{C}^*)^N}(Z)$ is naturally identified with the equivariant Chern character. Finally if one denotes by $p^\text{vir}_r$ the $\mathbb{Q}(\{\varepsilon_1, \ldots, \varepsilon_N\})$–linear extension of $\chi^\text{vir}(Z, -) : K^0(Z) \to Z$, and $p_*$ is the equivariant pushforward to a point, one can prove as in [14] that

$$p^\text{vir}_*(V) = p_*(\text{ch}(V)\text{td}(T^\text{vir}_Z) \cap [Z]^\text{vir}), \quad V \in K^0(Z)(\{\varepsilon_1, \ldots, \varepsilon_N\}). \quad (3.26)$$

Then, following [19], if we have a global equivariant embedding of a scheme $X$ into a nonsingular scheme $Y$ with $(\mathbb{C}^*)^N$ action, we can identify the maximal $(\mathbb{C}^*)^N$–fixed closed subscheme $X^f$ of $X$ with the scheme-theoretic intersection $X^f = X \cap Y^f$, where $Y^f$ is the nonsingular set-theoretic fixed point locus. By decomposing $Y^f$ into irreducible components $Y^f = \bigcup_i Y_i$ one can also define $X_i = X \cap Y_i$, which carry a perfect obstruction theory with virtual fundamental class $[X_i]^\text{vir}$. In this way, if $\tilde{V} \in K^0_{(\mathbb{C}^*)^N}(X)$ is an equivariant lift of the vector bundle $V$, $\tilde{V}_i$ is its restriction to $X_i$ and $p_i : X_i \to pt$ is the projection, one has that

$$\chi^\text{vir}(X, \tilde{V}; \varepsilon_1, \ldots, \varepsilon_N) = \sum_i p^\text{vir}_r(\tilde{V}_i/\Lambda_{-1}(N^\text{vir}_i)^\vee) = \sum_i p^\text{vir}_r(\tilde{V}_i/\Lambda_{-1}(T^\text{vir}_X)^\text{mov}_X)^\vee) \quad (3.27)$$

belongs to $\mathbb{Q}(\{\varepsilon_1, \ldots, \varepsilon_N\})$ and the virtual holomorphic Euler characteristic is $\chi^\text{vir}(X, V) = \chi^\text{vir}(X, \tilde{V} ; 0)$. Computations are now made very easy by the fact that we represented the virtual tangent space to the $T = (\mathbb{C}^*)^2$–fixed points to the moduli space of nested instantons in the representation ring $R(T)$ of the torus $(\mathbb{C}^*)^2$. In this way $T^\text{vir}_X$ is decomposed as a direct sum of line bundles which are moreover eigenbundles of the torus action. Then we can use the following properties

$$\text{ch}(E \oplus F) = \text{ch}E + \text{ch}F, \quad \Lambda_r(E \oplus F) = \Lambda_r(E) \cdot \Lambda_r(F), \quad S_r(E \oplus F) = S_r(E) \cdot S_r(F) \quad (3.28)$$

and equation (3.27) in order to compute the equivariant holomorphic Euler characteristic of the moduli space of nested instantons in terms of the fundamental characters $q_{1,2}$ of the torus $T$. These will be related to the equivariant parameters by $q_i = e^{\beta \varphi_i}$, with $\beta$ being a parameter having a very clear meaning in the physical framework modelling the moduli space of nested instantons as a low energy effective theory. In this framework it is very easy to explicitly compute the virtual equivariant holomorphic Euler characteristic of the moduli space of nested instantons as we already described the $T$–fixed locus of $N(r, n_0, \ldots, n_3)$ as being 0–dimensional and non-degenerate. As we saw in section 3.1 the fixed points of $N(r, n_0, \ldots, n_3)$ are completely described by $r$–tuples of nested coloured
partitions \( \mu_1 \subseteq \cdots \subseteq \mu_N \subseteq \mu_0 \), with \( \mu_j \in \mathcal{P}^r \), in such a way that \( |\mu_0| = \sum_j |\mu_j^0| = n_0 \) and \( |\mu_0 \setminus \mu_{i>0}| = n_{i>0} \). In the simplest case of \( r = 1 \) we get

\[
\chi^{\text{vir}}(\mathcal{N}(1,n_0,\ldots,n_N),\bar{V};q_1,q_2) = \sum_{\mu \subseteq \cdots \subseteq \mu_0 \atop |\mu_j| = n_j} \frac{T_{\mu_0,\mu_1}(q_1,q_2)W_{\mu_0,\ldots,\mu_N}(q_1,q_2)}{N_{\mu_0}(q_1,q_2)} \left[ \bar{V} \right]_{\mu_0,\ldots,\mu_N}, \tag{3.29}
\]

where \( a(s) \) and \( l(s) \) denote the arm length and the leg length of the box \( s \) in the Young diagram \( Y_\mu \) associated to \( \mu \), respectively. We moreover defined

\[
N_{\mu_0}(q_1,q_2) = \prod_{s \in Y_{\mu_0}} \left( 1 - q_1^{-1} - q_2^{-1} \right), \tag{3.30}
\]

\[
T_{\mu_0,\mu_1}(q_1,q_2) = \prod_{i=1}^{M_{\mu_0}} \prod_{j=1}^{M_{\mu_1}} \left( 1 - q_1^{-1} - q_2^{-1} \right), \tag{3.31}
\]

\[
W_{\mu_0,\ldots,\mu_N}(q_1,q_2) = \prod_{k=1}^{N} \prod_{i=1}^{M_{\mu_k}} \prod_{j=1}^{M_{\mu_{k+1}}} \left( 1 - q_1^{-1} - q_2^{-1} \right), \tag{3.32}
\]

A very interesting and surprising fact can be observed if we rearrange the expression the holomorphic virtual Euler characteristic of \( \mathcal{N}(1,n_0,\ldots,n_N) \). In fact if we perform the summation over the smaller partitions \( \mu_1 \subseteq \cdots \subseteq \mu_N \) and redefine \( q = q_1^{-1}, \; t = q_2^{-1} \), we get

\[
\chi^{\text{vir}}(\mathcal{N}(1,n_0,\ldots,n_N);q_1,q_2) = \sum_{\mu_0} \frac{P_{\mu_0}(q,t)}{N_{\mu_0}(q,t)} \tag{3.33}
\]

and the unexpected fact is that we think \( P_{\mu_0}(q,t) \) to be a polynomial in \( q,t \) except for a factor \( (1-qt)^{-1} \).

**Conjecture 1.** \( P_{\mu_0}(q,t) \) is a function of the form:

\[
P_{\mu_0}(q,t) = \frac{Q_{\mu_0}(q,t)}{(1-qt)^N}, \tag{3.34}
\]

with \( Q_{\mu_0}(q,t) \) a polynomial in the \( (q,t) \)-variables.

Sometimes the polynomials in (3.34) can be given an interpretations in terms of some known symmetric polynomials. In fact, let us define the following generating function

\[
Z_{\text{MD}}(q,t;x_0,\ldots,x_N) = \sum_{n_0 \geq \cdots \geq n_N} \chi^{\text{vir}}(\mathcal{N}(1,n_0,\ldots,n_N);q,t) \prod_{i=0}^{N} x_i^{n_i}, \tag{3.35}
\]

40
where \( m_i = n_i - n_{i+1} \) and the integers \( \tilde{n}_i \) form a sequence obtained from \( n_1 \) by asking the integers \( \tilde{n}_i = \tilde{n}_1 - \tilde{n}_{i+1} \) to be ordered. By construction \( Z_{MD}(q_1,t; x_0,\ldots ,x_N) \in \mathbb{Q}[q,t] \otimes \mathbb{Z} \Lambda(x), \) i.e. it is a symmetric function in \( \{x_i\}_{i=0}^N \) with coefficients in \( \mathbb{Q}[q,t] \). By conjecture 1 we have

\[
Z_{MD}(q_1,t; x_0,\ldots ,x_N) = \sum_{n_0 \geq \cdots \geq n_N} \sum_{\mu \in P(n_0)} \frac{Q_{\mu_0}(q,t)}{(1-qt)^N N_{\mu_0}(q,t)} \prod_{i=0}^N x_i^{m_i}.
\]

(3.36)

**Conjecture 2.** When \( |\mu_0| = |\mu_N| + 1 = |\mu_{N-1}| + 2 = \cdots = |\mu_1| + N \) we have

\[
Q_{\mu_0}(q,t) = \left\langle h_{\mu_0}(x), H_{\mu_0}^* (x;q,t) \right\rangle
\]

\[
= \left\langle h_{\mu_0}(x), \sum_{\lambda \in P(n_0)} \tilde{K}_{\lambda,\mu_0}(q,t) K_{\mu_0,\nu} m_{\nu}(x) \right\rangle
\]

\[
= \sum_{\lambda \in P(n_0)} \tilde{K}_{\lambda,\mu_0}(q,t),
\]

(3.37)

where the Hall pairing \( \langle -,- \rangle \) is such that \( \langle h_\mu, m_\lambda \rangle = \delta_{\mu,\lambda} \) and \( H_{\mu}(x;q,t), \tilde{K}_{\lambda,\mu}(q,t) \) are the modified Macdonald polynomials and the modified Kostka polynomials, respectively.

We checked the previous conjectures up to \( n_0 = 10 \).

If instead \( r > 1 \) we get a more complicated result, even though its structure is the same as we had previously

\[
\chi^\text{vir}(N(r,n_0,\ldots ,n_N), \mathcal{V}; q_1, q_2, [t_1]) = \sum_{\mu_0,\ldots ,\mu_N, |\mu_0| = n_0} \frac{T_{\mu_0,\mu_1}^{(r)}(q_1, q_2) W_{\mu_0,\ldots ,\mu_N}^{(r)}(q_1, q_2)}{N_{\mu_0}^{(r)}(q_1, q_2)} \bigg[ \mathcal{V} \bigg]_{\mu_0,\ldots ,\mu_N},
\]

(3.38)

with

\[
N_{\mu_0}^{(r)}(q_1, q_2) = \prod_{a,b = 1}^r \prod_{i = 1}^{N_0(a)} \left( 1 - t_{a b} q_1^{-\mu_{i,0}^{(a)} + 1} q_2^{-\mu_{i,0}^{(b)}} \right) \left( 1 - q_1^{-\mu_{i,0}^{(a)} + 1} q_2^{-\mu_{i,0}^{(b)}} \right),
\]

(3.39)

\[
T_{\mu_0,\mu_1}^{(r)}(q_1, q_2) = \prod_{a,b = 1}^r \prod_{i = 1}^{N_0(a)} \prod_{j = 1}^{N_0(b)} \left( 1 - t_{a b} q_1^{-j} q_2^{-j} \right),
\]

(3.40)

\[
W_{\mu_0,\ldots ,\mu_N}^{(r)}(q_1, q_2) = \prod_{k=1}^r \prod_{a,b = 1}^r \prod_{i = 1}^{N_0(b)} \prod_{j = 1}^{N_0(b)} \left( 1 - t_{a b} q_1^{\mu_{k,j}^{(a)} - 1} q_2^{\mu_{k,j}^{(b)}} \right) \left( 1 - t_{a b} q_1^{\mu_{k,j}^{(b)} - 1} q_2^{\mu_{k,j}^{(a)}} \right),
\]

(3.41)
where now \( t_{ab} = t_a t_b^{-1} \) and \( t_i \) are the fundamental characters of \( (\mathbb{C}^*)^r \) in \( G = (\mathbb{C}^*)^r \times T \), and \( a_b(s) \) denotes the arm length of the box \( s \) with respect to the Young diagram \( \mu^{(b)} \) associated to the partition \( \mu^{(b)} \) of \( \mu \) (with an analogous definition for the leg length).

### 3.3 Virtual equivariant \( \chi_{-y} \)-genus

The first refinement of the equivariant holomorphic Euler characteristic we are going to study is the virtual equivariant \( \chi_{-y} \)-genus, as defined in [14]. In order to exhibit the definition of virtual \( \chi_{-y} \)-genus let us first recall that if \( E \) is a rank \( r \) vector bundle on \( r \) one can define the antisymmetric product \( \Lambda_t E \) and the symmetric one \( S_t E \) as

\[
\Lambda_t E = \sum_{i=0}^{r} [\Lambda^i E] t^i \in K^0(X)[[t]], \quad S_t E = \sum_{i \geq 0} [S^i E] t^i \in K^0(X)[[t]].
\]

so that \( 1/\Lambda_t E = S_{-t} E \in K^0(X)[[t]] \). We can then define the virtual cotangent bundle \( \Omega_X^\text{vir} = (T_X^\text{vir})^\vee \) and the bundle of virtual \( n \)-forms \( \Omega_X^{n,\text{vir}} = \Lambda^n \Omega_X^\text{vir} \). If then \( X \) is a proper scheme equipped with a perfect obstruction theory of virtual dimension \( d \), the virtual \( \chi_{-y} \)-genus of \( X \) is defined by

\[
\chi_{-y}^\text{vir}(X) = \sum_{i \geq 0} (-y)^i \chi^\text{vir}(X, \Omega_X^{i,\text{vir}}), \tag{3.42}
\]

while, if \( V \in K^0(X) \), the virtual \( \chi_{-y} \)-genus of \( X \) with values in \( V \) is

\[
\chi_{-y}^\text{vir}(X, V) = \sum_{i \geq 0} (-y)^i \chi^\text{vir}(X, V \otimes \Omega_X^{i,\text{vir}}). \tag{3.43}
\]

Though in principle one would expect \( \chi_{-y}^\text{vir}(X, V) \) to be an element of \( \mathbb{Z}[[t]] \), it is in fact true that \( \chi_{-y}^\text{vir}(X, V) \in \mathbb{Z}[[t]], \tag{14} \).

By the form (3.22) and (3.21) of the holomorphic Euler characteristic it is easy to see that

\[
\chi_{-y}^\text{vir}(X) = \int_{[X]^{\text{vir}}} \text{ch}(\Lambda_{-y} T_X^\text{vir}) \cdot \text{td}(T_X^\text{vir}) = \int_{[X]^{\text{vir}}} \chi_{-y}(X), \tag{3.44}
\]

\[
\chi_{-y}^\text{vir}(X, V) = \int_{[X]^{\text{vir}}} \text{ch}(\Lambda_{-y} T_X^\text{vir}) \cdot \text{ch}(V) \cdot \text{td}(T_X^\text{vir}) = \int_{[X]^{\text{vir}}} \chi_{-y}(X) \cdot \text{ch}(V), \tag{3.45}
\]

which, in terms of the Chern roots of \( E_0, E_1 \) and \( V \) become

\[
\chi_{-y}^\text{vir}(X) = \int_{[X]^{\text{vir}}} \prod_{i=1}^{n} x_i \prod_{j=1}^{m} u_j \frac{1 - ye^{-x_i}}{1 - e^{-x_i}} \frac{1 - ye^{-u_j}}{1 - e^{-u_j}} \cdot \tag{3.46}
\]

\[
\chi_{-y}^\text{vir}(X, V) = \int_{[X]^{\text{vir}}} \left( \sum_{k=1}^{r} e^{r_k} \right) \prod_{i=1}^{n} x_i \prod_{j=1}^{m} u_j \frac{1 - ye^{-x_i}}{1 - e^{-x_i}} \frac{1 - ye^{-u_j}}{1 - e^{-u_j}}. \tag{3.47}
\]
Finally one can define the virtual Euler number \( e^\text{vir}(X) \) and the virtual signature \( \sigma^\text{vir}(X) \) of \( X \) as
\[
e^\text{vir}(X) = \chi^\text{vir}_- (X) \quad \text{and} \quad \sigma^\text{vir}(X) = \chi^\text{vir}_1 (X).
\]
Whenever \( y = 0 \) one recovers the holomorphic virtual Euler characteristic instead.

By extending the definition of \( \chi^\text{vir}_- \)-genus to the equivariant case in the obvious way and by making use of the equivariant virtual localization technique, one gets
\[
\chi^\text{vir}_-(X, \tilde{V}; \epsilon_1, \ldots, \epsilon_N) = \sum_i p^\text{vir}_i \left( \tilde{V} \otimes A_{-y} (\Omega^\text{vir}_X / \Lambda_{-1} (N^\text{vir}_1)) \right),
\]
whence \( \chi^\text{vir}_-(X, V) = \chi^\text{vir}_-(X, \tilde{V}; 0, \ldots, 0) \).

A simple computation in equivariant localization gives us the following result:
\[
\chi^\text{vir}_-(N(1, n_0, \ldots, n_N), \tilde{V}; q_1, q_2) = \sum_{\mu_0 \leq \cdots \leq \mu_N} T^{\text{vir}}_{\mu_0 \mu_1} (q_1, q_2) W^{\text{vir}}_{\mu_0, \ldots, \mu_n} (q_1, q_2) \left[ \tilde{V} \right]_{\mu_0, \ldots, \mu_N},
\]
with
\[
N^\text{vir}_{\mu_0} (q_1, q_2) = \prod_{s \in Y_{\mu_0}} \left( 1 - q_1^{-l(s)-1} q_2^{a(s)} \right) \left( 1 - y q_1^{-l(s)-1} q_2^{-a(s)-1} \right),
\]
\[
T^{\text{vir}}_{\mu_0 \mu_1} (q_1, q_2) = \prod_{i=1}^{M_0} \left( 1 - q_1^{-j-\mu'_i} q_2^{\mu_i} \right),
\]
\[
W^{\text{vir}}_{\mu_0, \ldots, \mu_n} (q_1, q_2) = \prod_{k=1}^{N} \prod_{i=1}^{M_k} \prod_{j=1}^{N_0} \left( 1 - q_1^{-\mu_k_i-j-\mu'_i, i} q_2^{\mu_k_i-j-\mu'_i, i} \right) \left( 1 - y q_1^{-\mu_k_i-j-\mu'_i, i} q_2^{-\mu_k_i-j-\mu'_i, i} \right) \left( 1 - q_1^{-\mu_k_i-j-\mu'_i, i} q_2^{\mu_k_i-j-\mu'_i, i} \right),
\]
The limit \( y \to 0 \) manifestly reverts to the case of the equivariant holomorphic Euler characteristic of the moduli space of nested instantons.

A similar result holds also for the general case \( r > 1 \):
\[
\chi^\text{vir}_-(N(r, n_0, \ldots, n_N), \tilde{V}; q_1, q_2, \{ t_i \}) = \sum_{\mu_0 \leq \cdots \leq \mu_N} T^{(r),\text{vir}}_{\mu_0 \mu_1} (q_1, q_2) W^{(r),\text{vir}}_{\mu_0, \ldots, \mu_n} (q_1, q_2) \left[ \tilde{V} \right]_{\mu_0, \ldots, \mu_N},
\]
with

\[
N_{(r)}^{(r,y)}(q_1, q_2) = \prod_{a,b=1}^{r} \prod_{a,b} Y_{\mu_{i,j}}^{(a)} \frac{(1-t_{ab} q_1^{l_z(s)-1} q_2^{a_b(s)})(1-t_{ab} q_1^{l_z(s)-a_b(s)-1})}{(1-y t_{ab} q_1^{l_z(s)-1} q_2^{a_b(s)})(1-y t_{ab} q_1^{l_z(s)-a_b(s)-1})},
\]

(3.54)

\[
T_{(r,y)}^{(r,y)}(q_1, q_2) = \prod_{a,b=1}^{r} \prod_{a,b} M_{\phi_{ab}}^{(a)} Y_{\mu_{i,j}}^{(b)} \frac{(1-t_{ab} q_1^{-j-\mu_{i,j}})(1-t_{ab} q_1^{-j-\mu_{i,j}})}{(1-y t_{ab} q_1^{-j-\mu_{i,j}})(1-y t_{ab} q_1^{-j-\mu_{i,j}})},
\]

(3.55)

\[
W_{(r,y)}^{(r,y)}(q_1, q_2) = \prod_{a,b=1}^{r} \prod_{a,b} M_{\phi_{ab}}^{(a)} Y_{\mu_{i,j}}^{(b)} \frac{(1-y t_{ab} q_1^{-k-\mu_{i,j}-1})(1-y t_{ab} q_1^{-k-\mu_{i,j}-1})}{(1-t_{ab} q_1^{-k-\mu_{i,j}-1})(1-t_{ab} q_1^{-k-\mu_{i,j}-1})},
\]

(3.56)

with the same notations of the previous section.

**Virtual Euler number and signature**

As we already pointed out previously, two specifications of the value of \( y \) in the \( \chi_{-y} \)-genus, namely \( y = \pm 1 \), give back two interesting topological invariants of a given nested instantons moduli space. Let us consider first the simpler case of rank 1. We can easily see, in the case of the virtual Euler number, that taking the specialization \( y = +1 \) amounts to counting the number of nested partitions of a given size. Then, if we assemble everything in a single generating function we have

\[
M(q_1, \ldots, q_N) = \sum_{f=0}^{\infty} \sum_{n_0, \ldots, n_f} e^{\text{vir}}(N(1, n_0, \ldots, n_f)) q_0^{n_0} \cdots q_f^{n_f} = \sum_{f=0}^{\infty} \sum_{n_0, \ldots, n_f} \# \{ \mu_1 \subseteq \cdots \subseteq \mu_f \subseteq \mu_0 \} q_0^{n_0} \cdots q_f^{n_f},
\]

(3.57)

for which an explicit expression is not available to our knowledge. Of course if we focus our attention on smooth nested Hilbert schemes only (i.e. \( n_{i_1} = 0 \) or \( n_1 = 1 \), \( n_{i_1-1} = 0 \)), the generating function of the virtual Euler number of smooth nested Hilbert schemes is easily expressed in terms of standard generating functions of partitions:

\[
\sum_{n \geq 0} e^{\text{vir}}(N(1, n, 1)) q^n = \prod_{k=0}^{\infty} \left( \frac{1}{1-q^k} \right) = (\phi(q))^{-1} = \sum_{n \geq 0} \chi(M(1, n)) q^n,
\]

(3.58)
which, in the case of higher rank becomes
\[
\sum_{n \geq 0} e^{vir}(\mathcal{N}(r, n, 1)) q^n = \prod_{k=0}^{\infty} \left( \frac{1}{1 - q^k} \right)^r = (\phi(1))^{-r} = \sum_{n \geq 0} \chi(\mathcal{M}(r, n)) q^n. \tag{3.59}
\]
We also notice that whenever \(q_0 = q_1 = \cdots = q_N\), generating function of Euler numbers is actually accounting for the enumeration of plane partition, whose generating function is known to be the Macmahon function \(\Phi(-q)\):
\[
M(q, \ldots, q) = \sum_{j=0}^{\infty} \sum_{n_0, \ldots, n_j} e^{vir}(\mathcal{N}(1, n_0, \ldots, n_j)) q^{n_0 + \cdots + n_j} = \Phi(-q). \tag{3.60}
\]

The case of the virtual signature is much easier instead. By taking \(y = -1\) we immediately see that \(\sigma^{vir}(\mathcal{N}(1, n)) = 0\).

### 3.4 Virtual equivariant elliptic genus

A further refinement of the virtual \(\chi_{-y}\)-genus is finally given by the virtual elliptic genus. In this case, if \(F\) is any vector bundle over \(X\), we define
\[
\mathcal{E}(F) = \bigotimes_{n \geq 1} \left( \Lambda_{-y} F^\vee \otimes \Lambda_{-y}^{-1} q^n F \otimes S_q(F \oplus F^\vee) \right) \in 1 + q \cdot K^0(X)[y, y^{-1}][[q]], \tag{3.61}
\]
so that the virtual elliptic genus Ell\(^{vir}\)(\(X; y, q\)) of \(X\) is defined by
\[
\text{Ell}^{vir}(X; y, q) = y^{-d/2} \chi^{vir}(X, \mathcal{E}(T^\text{vir}_X)) \in \mathbb{Q}(y^{1/2})[[q]], \tag{3.62}
\]
and also
\[
\text{Ell}^{vir}(X, V; y, q) = y^{-d/2} \chi^{vir}(X, \mathcal{E}(T^\text{vir}_X) \otimes V). \tag{3.63}
\]
By using virtual Riemann-Roch again one can see that Ell\(^{vir}\)(\(X; y, q\)) admits an integral form
\[
\text{Ell}^{vir}(X; y, q) = \int_{[X]^{vir}} \mathcal{E}_\ell(T^\text{vir}_X; y, q), \tag{3.64}
\]
\[
\text{Ell}^{vir}(X, V; y, q) = \int_{[X]^{vir}} \mathcal{E}_\ell(T^\text{vir}_X; y, q) \cdot \text{ch}(V), \tag{3.65}
\]
with
\[
\mathcal{E}_\ell(F; y, q) = y^{-rk F/2} \text{ch}(\Lambda_{-y} F^\vee) \cdot \text{ch}(\mathcal{E}(F)) \cdot \text{td}(F) \in A^*(X)[y^{-1/2}, y^{1/2}][[q]]. \tag{3.66}
\]
It is also interesting to study how the virtual elliptic genus is described in terms of the usual Chern roots $x_i, u_j, v_k$, as its formula involves the Jacobi theta function $\theta(z, \tau)$ defined as

$$\theta(z, \tau) = q^{1/8} \sum_{l=1}^{\infty} (1 - q^l)(1 - q^l y)(1 - q^l y^{-1}),$$

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$. In fact if $F$ is any vector bundle over $X$ with Chern roots $\{f_i\}$, one can prove [3] that

$$E \ell \ell(F; z, \tau) = \prod_{i=1}^{rk F} \frac{\theta(f_i/2\pi i - z, \tau)}{\theta(f_i/2\pi i, \tau)} \prod_{j=1}^{m} \frac{\theta(u_j/2\pi i, \tau)}{\theta(u_j/2\pi i - z, \tau)},$$

so that finally

$$\text{Ell}_{\text{vir}}(X; y, q) = \int \left[ X \right]_{\text{vir}} \prod_{i=1}^{n} x_i \frac{\theta(x_i/2\pi i - z, \tau)}{\theta(x_i/2\pi i, \tau)} \prod_{j=1}^{m} \frac{1}{u_j} \frac{\theta(u_j/2\pi i, \tau)}{\theta(u_j/2\pi i - z, \tau)},$$

$$\text{Ell}_{\text{vir}}(X, V; y, q) = \int \left[ X \right]_{\text{vir}} \left( \sum_{k=1}^{r} \epsilon^k \prod_{i=1}^{n} x_i \frac{\theta(x_i/2\pi i - z, \tau)}{\theta(x_i/2\pi i, \tau)} \prod_{j=1}^{m} \frac{1}{u_j} \frac{\theta(u_j/2\pi i, \tau)}{\theta(u_j/2\pi i - z, \tau)} \right),$$

Finally, by taking the same steps as in the previous paragraphs we can equivariantly extend the definition of the virtual elliptic genus, and by virtual localization find that

$$\text{Ell}_{\text{vir}}(X, \hat{V}; z, \tau; \epsilon_1, \ldots, \epsilon_N) = y^{-vd/2} \sum_{i} p_i^{\text{vir}} \left( \hat{V}_i \otimes \mathcal{E}(T_{X_{\theta}}^{\text{vir}} \otimes \Lambda_{-y}(\Omega_{X_{\theta}}^{\text{vir}} |_{X_i}/\Lambda_{-1}(N_i^{\text{vir}})^}) \right)$$

and $\text{Ell}_{\text{vir}}(X, V) = \text{Ell}_{\text{vir}}(X, \hat{V}; 0, \ldots, 0)$. In particular we get in rank 1

$$\text{Ell}_{\text{vir}}(\mathcal{N}(1, n_0, \ldots, n_N), \hat{V}; \epsilon, \epsilon_2) = \sum_{\mu_0 \leq \cdots \leq \mu_N \atop \mu_0 \backslash \mu_i = n_j} \frac{T_{p_0^{\mu_1}|}(\epsilon_1, \epsilon_2) W_{p_0^{\mu_1}|}(\epsilon_1, \epsilon_2)}{N_{p_0^{\mu_1}|}(\epsilon_1, \epsilon_2)} [\hat{V}]_{p_0^{\mu_1}| \cdots |p_N^{\mu_1}|},$$
with

\[ N_{\mu_0}^{z,\tau}(\epsilon_1, \epsilon_2) = \prod_{s \in \Omega_0} \left[ \frac{\theta(\epsilon_1 (l(s) + 1) - \epsilon_2 a(s), \tau)}{\theta(\epsilon_1 (l(s) + 1) - \epsilon_2 a(s) - z, \tau)} \right] \] (3.73)

and

\[ T_{\mu_0,\mu_1}^{z,\tau}(\epsilon_1, \epsilon_2) = \prod_{i=1}^{M_0} \prod_{j=1}^{M_i} \frac{\theta(\epsilon_1 i + \epsilon_2 (j + \mu'_{i,j}), \tau)}{\theta(\epsilon_1 i + \epsilon_2 (j + \mu_{i,j}), \tau)} \] (3.74)

and

\[ W_{\mu_0,\ldots,\mu_N}^{z,\tau}(\epsilon_1, \epsilon_2) = \prod_{k=1}^{N} \prod_{i=1}^{M_k} \prod_{j=1}^{N_0} \left[ \frac{\theta(\epsilon_1 (i - \mu_{k,j}) + \epsilon_2 (1 + \mu'_{0,i} - j), \tau)}{\theta(\epsilon_1 (i - \mu_{k,j}) + \epsilon_2 (1 + \mu_{0,i} - j) - z, \tau)} \right] \] (3.75)

with \( \epsilon_i = \epsilon_i / 2\pi i \). One can easily see that the virtual elliptic genus we just computed is indeed a Jacobi form, and that its limit \( \tau \to \infty \) reproduces the \( \chi_{-y} \)-genus. Moreover by taking the limit \( y \to 0 \) in the \( \chi_{-y} \)-genus one can recover the virtual equivariant holomorphic Euler characteristic.

Finally, if we study the virtual equivariant elliptic genus in the more general case of rank \( r \geq 1 \), we get

\[ \text{Ell}^{\text{vir}}(\mathcal{N}(r, n_0, \ldots, n_N), \tilde{\tau}; \epsilon_2, \{a_i\}) = \sum_{\mu_1, \ldots, \mu_N, \mu_0} \frac{T_{\mu_0,\mu_1}^{z,\tau}(\epsilon_1, \epsilon_2) W_{\mu_0,\ldots,\mu_N}^{z,\tau}(\epsilon_1, \epsilon_2)}{N_{\mu_0}^{z,\tau}(\epsilon_1, \epsilon_2)} [\tilde{\tau}]_{\mu_0,\ldots,\mu_N} \] (3.77)
with

\[
N_{\mu_0}^{x,\tau}(e_1,e_2) = \prod_{a,b=1}^{r} \prod_{s \in Y_{\mu_0}} \frac{\theta(a_{ab} + \epsilon_1(l(s) + 1) - \epsilon_2 a(s), \tau)}{\theta(a_{ab} + \epsilon_1(l(s) + 1) - \epsilon_2 a(s) - z, \tau)} \cdot \frac{\theta(a_{ab} - \epsilon_1 l(s) + \epsilon_2 (a(s) + 1), \tau)}{\theta(a_{ab} - \epsilon_1 l(s) + \epsilon_2 (l(s) + 1) - z, \tau)},
\]

(3.78)

\[
T_{\mu_0,\mu_1}^{x,\tau}(e_1,e_2) = \prod_{a,b=1}^{r} \prod_{s \in Y_{\mu_0}} \prod_{j=1}^{N_0(a)} \prod_{\bar{N}_0(b)} \frac{\theta(a_{ab} + \epsilon_1 i + \epsilon_2 (j + \mu_{1,j}^a), \tau)}{\theta(a_{ab} + \epsilon_1 i + \epsilon_2 (j + \mu_{1,j}^a)^r - z, \tau)},
\]

(3.79)

\[
W_{\mu_0,\mu_1,\ldots,\mu_N}^{x,\tau}(e_1,e_2) = \prod_{k=1}^{N} \prod_{a,b=1}^{r} \prod_{s \in Y_{\mu_0}} \prod_{i=1}^{M(a)} \prod_{j=1}^{N(b)} \frac{\theta(a_{ab} + \epsilon_1 (i - \mu_{b,j}^a) + \epsilon_2 (1 + \mu_{0,i} - j), \tau)}{\theta(a_{ab} + \epsilon_1 (i - \mu_{b,j}^a)^r + \epsilon_2 (1 + \mu_{0,i} - j)^r - z, \tau)} \cdot \frac{\theta(a_{ab} + \epsilon_1 (i - \mu_{k,j}^a) + \epsilon_2 (1 + \mu_{0,i} - j)^r, \tau)}{\theta(a_{ab} + \epsilon_1 (i - \mu_{k,j}^a)^r - \epsilon_2 (1 + \mu_{0,i} - j)^r - z, \tau)}. \]

(3.80)

Notice that by knowing the equivariant virtual elliptic genus one is able to recover both the virtual equivariant holomorphic Euler characteristic and $\chi_{-g}$ genus. In fact the limit $\tau \to i\infty$ of (3.77) recovers exactly the $\chi_{-g}$-genus found in (3.53) and a successive limit $y \to 0$ gives us back the virtual equivariant holomorphic Euler characteristic (3.38).

4 Toric surfaces

In this section we will generalize the results we got in the previous ones to the case of nested Hilbert schemes on toric surfaces, and in particular we will be interested in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. This is because one might expect any complex genus of $\text{Hilb}^{n}(S)$ to depend only on the cobordism class of $S$, as it was the case for $\text{Hilb}^n(S)$, $[13]$, and the complex cobordism ring $\Omega = \Omega^L \otimes \mathbb{Q}$ with rational coefficients was showed by Milnor to be a polynomial algebra freely generated by the cobordism classes $[\mathbb{P}^n]$, $n \geq 0$. Then in the case of complex projective surfaces any case can be reduced to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by the fact that $[S] = [\mathbb{P}^2] + b[\mathbb{P}^1 \times \mathbb{P}^1]$. The advantage given by having an ADHM-like construction for the nested punctual Hilbert scheme on the affine plane is that it provides us with
the local model of the more general case of smooth projective surfaces. In particular, whenever $S$ is toric, one can construct it starting from its toric fan by appropriately gluing the affine patches (e.g. fig. 4a for $\mathbb{P}^2$ and 4b for $\mathbb{P}^1 \times \mathbb{P}^1$), and computation of topological invariants can still be easily carried out by means of equivariant (virtual) localization. In general, given the toric fan describing the patches which glued together makes up a toric surface $S$, each patch $U_i$ will be $U_i \simeq \mathbb{C}^2$, with a natural action of $T = (\mathbb{C}^*)^2$. Moreover, if $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $Z \in \text{Hilb}^{(n)}(S)$ is a fixed point of the $T$–action, its support must be contained in $\{P_0, \ldots, P_{\chi(S)-1}\}$ (as a consequence of [6]) with $P_i$ corresponding to the vertices of the polytope associated to the fan, so that one can write in general that $Z = Z_0 \cup \cdots \cup Z_{\chi(S)-1}$, with $Z_i$ being supported in $P_i$. This also induces a decomposition of the representation in $R(T)$ of the virtual tangent space at the fixed points:

$$T^\text{vir}_Z(\text{Hilb}^{(n)}(S)) = \bigoplus_{\ell=0}^{\chi(S)-1} T^\text{vir}_{Z_\ell}(\text{Hilb}^{(n_\ell)}(U_\ell)).$$

Let us call then $\chi^\text{vir}_\ell(P_\ell) = \chi^\text{vir}_{\ell}(N(1, n^{(\ell)}_1, \ldots, n^{(\ell)}_N); q_{1,\ell}, q_{2,\ell})$: we will see how we will be able to compute $\chi^\text{vir}_\ell(\text{Hilb}^{(n)}(\mathbb{P}^2))$ and $\chi^\text{vir}_\ell(\text{Hilb}^{(n)}(\mathbb{P}^1 \times \mathbb{P}^1))$ in terms of $\chi^\text{vir}_{\ell}(P_\ell)$.
4.1 Case 1: \( S = \mathbb{P}^2 \)

We will be interested in the following generating function

\[
\sum_{n} \chi_{-\gamma}^\text{vir}(\text{Hilb}^n(\mathbb{P}^2)) q^n = \prod_{\ell=0}^{2} \left( \sum_{n_{\ell} \geq 0} \chi_{-\gamma}^\text{vir}(P_{\ell}) q^{n_{\ell}} \right),
\]

(4.2)

with \( \hat{n} \) defined as in section 2.1, and since the left-hand side doesn’t depend on \( q_{1,2} \), we can perform the computation by taking the iterated limits \( q_1 \to +\infty, q_2 \to +\infty \) or \( q_1 \to 0, q_2 \to 0 \). In each one of the three affine patches the weights of the torus action will be

\[
\begin{align*}
q_{1,(0)} &= q_1 & q_{2,(0)} &= q_2 \\
q_{1,(1)} &= 1/q_1 & q_{2,(1)} &= q_2/q_1 \\
q_{1,(2)} &= 1/q_2 & q_{2,(2)} &= q_1/q_2
\end{align*}
\]

(4.3)

We will study separately the three patches \( \ell = 0,1,2 \). First of all we notice that since the \( \chi_{-\gamma} \)-genus is multiplicative, the first contribution coming from \( N_{\mu_0 q_1 q_2} \) coincides with the same contribution arising in the context of standard Hilbert schemes. It was shown in [25] that

\[
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \frac{1}{N_{\mu_0 q_1 q_2}} = y^{\mu_0 - M_0},
\]

(4.4a)

\[
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \frac{1}{N_{\mu_0 q_1 q_2}} = y^{\mu_0},
\]

(4.4b)

\[
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \frac{1}{N_{\mu_0 q_1 q_2}} = y^{\mu_0 + s(\mu_0)}, \quad s(\mu_0) = \#\{s \in Y_{\mu_0} : a(s) \leq l(s) \leq a(s) + 1\},
\]

(4.4c)

so that we just need to evaluate the other contributions. Starting from \( T_{\mu_0 q_1 q_2} \) we get

\[
\begin{align*}
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \prod_{i=1}^{M_0} \prod_{j=1}^{M_0} \left[ \begin{array}{c}
1 - q_{1,0}^{-i} q_{2,0}^{-j} M_{0,i,j}
\end{array} \right] = 1,
\end{align*}
\]

(4.5a)

\[
\begin{align*}
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \prod_{i=1}^{M_0} \prod_{j=1}^{M_0} \left[ \begin{array}{c}
1 - q_{1,0}^{-i} q_{2,0}^{-j} M_{0,i,j}
\end{array} \right] = y^{-1},
\end{align*}
\]

(4.5b)

\[
\begin{align*}
\lim_{q_1 \to +\infty} \lim_{q_2 \to +\infty} \prod_{i=1}^{M_0} \prod_{j=1}^{M_0} \left[ \begin{array}{c}
1 - q_{1,0}^{-i} q_{2,0}^{-j} M_{0,i,j}
\end{array} \right] = 1,
\end{align*}
\]

(4.5c)
whence

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0, \mu_1}^{-y}(q_1, q_2, 0) = 1, \quad (4.6a) \]

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0, \mu_1}^{-y}(q_1, 1, q_2, 1) = y^{-|\mu_0 \setminus \mu_1|}, \quad (4.6b) \]

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0, \mu_1}^{-y}(q_1, 2, q_2, 2) = 1. \quad (4.6c) \]

Finally we need to take care of the limit involving \( W_{\mu_0 \cdots \mu_N}^{-y}(q_1, q_2) \) and in order to tackle let us first point out that we can rewrite \( W_{\mu_0 \cdots \mu_N}^{-y} \) in the following simpler form:

\[ W_{\mu_0 \cdots \mu_N}^{-y}(q_1, q_2) = \prod_{k=1}^{N} \prod_{s \in Y_{\mu_k}^{\text{rec}}} \left( \frac{1 - q_1^{l_k(s) - a_k(s) - 1}}{1 - y q_1^{l_k(s) - a_k(s) - 1}} \right) \left( \frac{1 - q_2^{l_k(s) - a_k(s) - 1}}{1 - y q_2^{l_k(s) - a_k(s) - 1}} \right), \quad (4.7) \]

where \( \mu_0^{\text{rec}} \) is the smallest rectangular partition containing \( \mu_0 \) and \( a_k(s) \) (resp. \( l_k(s) \)) denotes the arm length (resp. leg length) of the box \( s \) with respect to \( Y_{\mu_k} \). Then, by recalling that the partitions labelling the \( T \)-fixed points are included one into the other as \( \mu_1 \subseteq \cdots \subseteq \mu_N \subseteq \mu_0 \subseteq \mu_0^{\text{rec}} \), it’s easy to realize that, in the case \( \ell = 0 \), one gets

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1 - q_1^{l_k(s) - a_k(s) - 1}}{1 - y q_1^{l_k(s) - a_k(s) - 1}} = \begin{cases} 1 & \text{for } l_k(s) \leq 0 \\ y^{-1} & \text{for } l_k(s) > 0 \end{cases}, \quad (4.8) \]

and similarly in every other case:

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1 - q_1^{l_{k-1}(s) - a_{k-1}(s) - 1}}{1 - y q_1^{l_{k-1}(s) - a_{k-1}(s) - 1}} = \begin{cases} 1 & \text{for } l_{k-1}(s) \leq 0 \\ y^{-1} & \text{for } l_{k-1}(s) > 0 \end{cases}, \]

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1 - y q_1^{l_k(s) - a_k(s) - 1}}{1 - q_1^{l_k(s) - a_k(s) - 1}} = \begin{cases} 1 & \text{for } l_k(s) \leq 0 \\ y & \text{for } l_k(s) > 0 \end{cases}, \]

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1 - y q_1^{l_{k-1}(s) - a_{k-1}(s) - 1}}{1 - q_1^{l_{k-1}(s) - a_{k-1}(s) - 1}} = \begin{cases} 1 & \text{for } l_{k-1}(s) \leq 0 \\ y & \text{for } l_{k-1}(s) > 0 \end{cases}, \]

so that finally

\[ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0 \cdots \mu_N}^{-y}(q_1, 0, q_2, 0) = 1. \quad (4.10) \]
It is easy to see that the same holds true also for $\ell = 2$:

$$ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0, \ldots, \mu_N}^{-y}(q_{1,2}, q_{2,2}) = 1, \quad (4.11) $$

while the case $\ell = 1$ is more difficult, even though the analysis of the different cases can be carried out exactly in the same way. We then introduce the following notation:

$$ s(\mu_i, \mu_i) = \# \left\{ s \in Y_{\mu_0} : l_i(s) > a_i(s) + 1 \land l_i(s) = a_i(s) + 1, a_i(s) < -1 \right\}, \quad (4.12) $$

and we get

$$ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0, \ldots, \mu_N}^{-y}(q_{1,1}, q_{2,1}) = \prod_{k=1}^{N} y^{s(\mu_k, \mu_k)} + s(\mu_{k-1}, \mu_k) - s(\mu_k, \mu_k) + s(\mu_{k-1}, \mu_k). \quad (4.13) $$

Finally, by putting everything together, we have an explicit expression for (4.2).

$$ \sum_n \chi_{-y}^{\text{vir}}(\text{Hilb}^{(d)}(\mathbb{P}^2)) q^n = \left( \sum_n q^n \sum_{|\mu_i|} y^{|\mu_0|+M_{\ell}} \right) \left( \sum_n q^n \sum_{|\mu_i|} y^{|\mu_0|-|\mu_i|+M_{\ell}} \prod_{k=1}^{N} y^{s(\mu_k, \mu_k) + s(\mu_{k-1}, \mu_k) - s(\mu_k, \mu_k)} \right) \quad (4.14) $$

**4.2 Case 2: $S = \mathbb{P}^1 \times \mathbb{P}^1$**

Similarly to previous case, we are interested in studying the following generating function

$$ \sum_n \chi_{-y}^{\text{vir}}(\text{Hilb}^{(d)}(\mathbb{P}^1 \times \mathbb{P}^1)) q^n = \prod_{\ell=0}^{3} \left( \sum_{n_{\ell} \geq 0} \chi_{-y}^{\text{vir}}(P^1) q^{n_{\ell}} \right), \quad (4.15) $$

and we can still perform the computation by taking the successive limits $q_1 \to +\infty$, $q_2 \to +\infty$ or $q_1 \to 0$, $q_2 \to 0$. The four patches are now indexed by $\ell = (00), (01), (10), (11)$, and the characters $q_{i,\ell}$ can be identified to be in this case

$$ q_{1,00} = q_1, \quad q_{2,00} = q_2, \quad (4.16) $$

$$ q_{1,01} = q_1, \quad q_{2,01} = 1/q_2 $$

$$ q_{1,10} = 1/q_1, \quad q_{2,10} = q_2 $$

$$ q_{1,11} = 1/q_1, \quad q_{2,11} = 1/q_2 $$
An analysis similar to the one carried out in the previous section enables then us to conclude the following

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} 1/N_{\mu_0}^{-y}(q_1,00)q_2,00) = y^{\mu_0 - M_0},
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} 1/N_{\mu_0}^{-y}(q_1,00)q_2,01) = y^{\mu_0},
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} 1/N_{\mu_0}^{-y}(q_1,00)q_2,10) = y^{\mu_0},
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} 1/N_{\mu_0}^{-y}(q_1,00)q_2,11) = y^{\mu_0 + M_0},
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,00) = 1,
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,01) = 1,
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,10) = y^{-\mu_0 \mu_1},
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} T_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,11) = y^{-\mu_0 \mu_1},
\]

and

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,00) = 1,
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,01) = 1,
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,10) = 1,
\]

\[
\lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} W_{\mu_0,\mu_1}^{-y}(q_1,00)q_2,11) = 1,
\]

so that, by putting everything together, we have

\[
\sum_n x^{vir}(\text{Hilb}^n(\mathbb{P}^1 \times \mathbb{P}^1)) q^n = \left( \sum_n q^n \sum_{\mu_0} y^{\mu_0 - M_0} \right) \left( \sum_n q^n \sum_{\mu_1} y^{\mu_1} \right) \left( \sum_n q^n \sum_{\mu_0,\mu_1} y^{-\mu_0 \mu_1} \right) \left( \sum_n q^n \sum_{\mu_0,\mu_1} y^{\mu_0 + M_0} \right).
\]

References

[1] Sjoerd Viktor Beentjes and Andrea T. Ricolfi, Virtual counts on Quot schemes and the higher rank local DT/PT correspondence, accepted for publication in Math. Res. Lett., arXiv:1811.09859.

[2] Giulio Bonelli, Nadir Fasola, and Alessandro Tanzini, Defects, nested instantons and comet shaped quivers, arXiv e-prints (2019), arXiv:1907.02771.

[3] Lev A. Borisov and Anatoly Libgober, Elliptic genera of toric varieties and applications to mirror symmetry, Inventiones mathematicae 140 (2000), no. 2, 453–485.

[4] Ugo Bruzzo, Wu-Yen Chuang, Duiliu-Emanuel Diaconescu, Marcos Jardim, G. Pan, and Yi Zhang, D-branes, surface operators, and adhm quiver representations, Adv. Theor. Math. Phys. 15 (2011), no. 3, 849–911.
[5] Michael Bulois and Laurent Evain, Nested Punctual Hilbert Schemes and Commuting Varieties of Parabolic Subalgebras, Journal of Lie Theory 26 (2016), no. 2, 497–533, 43 pages.

[6] Pierre-Emmanuel Chaput and Laurent Evain, On the equivariant cohomology of Hilbert schemes of points in the plane, Annales de l’Institut Fourier 65 (2015), no. 3, 1201–1250 (en).

[7] Jan Cheah, Cellular decompositions for nested Hilbert schemes of points, Pacific J. Math. 183 (1998), no. 1, 39–90. MR 1616606

[8] Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Ron Donagi, and Tony Pantev, Parabolic refined invariants and Macdonald polynomials, Commun. Math. Phys. 335 (2015), no. 3, 1323–1379.

[9] William Crawley-Boevey, Geometry of the moment map for representations of quivers, Compositio Mathematica 126 (2001), no. 3, 257293.

[10] Duiliu-Emanuel Diaconescu, Moduli of ADHM Sheaves and Local Donaldson-Thomas Theory, J. Geom. Phys. 62 (2012), 763–799.

[11] Duiliu-Emanuel Diaconescu, Artan Sheshmani, and Shing-Tung Yau, Atiyah class and sheaf counting on local Calabi Yau fourfolds, arXiv e-prints (2018), arXiv:1810.09382.

[12] Simon K. Donaldson and Richard P. Thomas, Gauge theory in higher dimensions, The geometric universe: Science, geometry, and the work of Roger Penrose. Proceedings, Symposium, Geometric Issues in the Foundations of Science, Oxford, UK, June 25-29, 1996, 1996, pp. 31–47.

[13] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn, On the Cobordism Class of the Hilbert Scheme of a Surface, J. Algebraic Geom 10 (1999).

[14] Barbara Fantechi and Lothar Göttsche, RiemannRoch theorems and elliptic genus for virtually smooth schemes, Geom. Topol. 14 (2010), no. 1, 83–115.

[15] Amin Gholampour, Artan Sheshmani, and Shing-Tung Yau, Nested Hilbert schemes on surfaces: Virtual fundamental class, arXiv e-prints (2017), arXiv:1701.08899.

[16] Amin Gholampour, Artan Sheshmani, and Shing-Tung Yau, Localized Donaldson-Thomas theory of surfaces, American Journal of Mathematics (February 2020) 142 (2019), no. 1.

[17] Amin Gholampour and Richard P. Thomas, Degeneracy loci, virtual cycles and nested Hilbert schemes II, arXiv e-prints (2019), arXiv:1902.04128.

[18] ______, Degeneracy loci, virtual cycles and nested Hilbert schemes, I, Tunis. J. Math. 2 (2020), no. 3, 633–665. MR 4041285

[19] Tom Graber and Rahul Pandharipande, Localization of virtual classes, Inventiones mathematicae 135 (1999), no. 2, 487–518.
[20] Lothar Göttsche and Martijn Kool, *Refined SU(3) Vafa-Witten invariants and modularity*, Pure Appl. Math. Q. 14 (2018), no. 3-4, 467–513.

[21] Tams Hausel, Emmanuel Letellier, and Fernando Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties*, Duke Math. J. 160 (2011), no. 2, 323–400.

[22] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Communications in Mathematical Physics 108 (1987), no. 4, 535–589.

[23] Alastair D. King, *Moduli of representations of finite dimensional algebras*, The Quarterly Journal of Mathematics 45 (1994), no. 4, 515–530.

[24] Ties Laarakker, *Monopole contributions to refined Vafa-Witten invariants*, arXiv e-prints (2018), arXiv:1810.00385.

[25] Kefeng Liu, Catherine Yan, and Jian Zhou, *Hirzebruch X_y genera of the Hilbert schemes of surfaces by localization formula*, Science in China Series A: Mathematics 45 (2002), no. 4, 420–431.

[26] Takuro Mochizuki, *The structure of the cohomology ring of the filt schemes*, arXiv Mathematics e-prints (2003), math/0301184.

[27] Hiraku Nakajima and American Mathematical Society, *Lectures on Hilbert schemes of points on surfaces*, University lecture series, American Mathematical Society, 1999.

[28] Andrei Neguţ, *Moduli of flags of sheaves and their K-theory*, Algebr. Geom. 2 (2015), no. 1, 19–43. MR 3322196

[29] Andrea T. Ricolfi, *Virtual classes and virtual motives of Quot schemes on threefolds*, arXiv e-prints (2019), arXiv:1906.02557.

[30] Artan Sheshmani and Shing-Tung Yau, *Higher rank flag sheaves on Surfaces and Vafa-Witten invariants*, arXiv e-prints (2019), arXiv:1911.00124.

[31] Yuuji Tanaka and Richard Thomas, *Vafa-Witten invariants for projective surfaces I: stable case*, Journal of Algebraic Geometry (2017).

[32] Yuuji Tanaka and Richard P. Thomas, *Vafa-Witten invariants for projective surfaces II: semistable case*, Pure Appl. Math. Q. 13 (2017), no. 3, 517–562. MR 3882207

[33] Cumrun Vafa and Edward Witten, *A Strong coupling test of S duality*, Nucl. Phys. B431 (1994), 3–77.

[34] Rodrigo A. von Flach and Marcos Jardim, *Moduli spaces of framed flags of sheaves on the projective plane*, Journal of Geometry and Physics 118 (2017), 138–168.
