A nonstandard uniform functional limit law for the increments of the multivariate empirical distribution function

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Abstract

Let \((Z_i)_{i \geq 1}\) be an independent, identically distributed sequence of random variables on \(\mathbb{R}^d\). Under mild conditions on the density of \(Z_1\), we provide a nonstandard uniform functional limit law for the following processes on \([0, 1)^d\):

\[
\Delta_n(z, h_n, \cdot) := s \mapsto \sum_{i=1}^n 1_{[s_1, s_2] \times \ldots \times [s_{d-1}, s_d]} \left( \frac{Z_i - z}{h_n d} \right), \quad s \in [0, 1)^d,
\]

along a sequence \((h_n)_{n \geq 1}\) fulfilling \(h_n \downarrow 0\), \(nh_n \uparrow, \ n h_n / \log c \to c > 0\). Here \(z\) ranges through a compact set of \(\mathbb{R}^d\). This result is an extension of a theorem of Deheuvels and Mason \(\cite{5}\) to the multivariate, non uniform case.

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1 Introduction and statement of the result

In this paper, we consider an independent, identically distributed sequence of random vectors \((Z_i)_{i \geq 1}\) having a density \(f\) on an open set \(O \subset \mathbb{R}^d\). We make the following assumption on \(f\):

\(\text{(Hf)} \quad f\) is continuous and strictly positive on \(O\).

Throughout this article, \(s, s' \in \mathbb{R}^d\), we shall write \(s \prec s'\) when \(s_i \leq s'_i\) for each \(i = 1, \ldots, n\). Intervals and semi intervals are implicitly understood as product of intervals or semi intervals, namely

\[
[s, s'] := \{u \in \mathbb{R}^d, \ s \prec u \prec s'\} \\
= [s_1, s'_1] \times \ldots \times [s_d, s'_d], \ s = (s_1, \ldots, s_d), \ s' = (s'_1, \ldots, s'_d).
\] (1.1)
We shall also write \( a \prec s \) (resp. \( s \prec a \)) for \( s \in \mathbb{R}^d \) and \( a \in \mathbb{R} \) when \( a \leq s_i \) (resp. \( s_i \leq a \)) for each \( i = 1, \ldots, d \). For fixed \( 0 < h < 1 \) and \( z \in O \), we define the following process on \([0, 1)^d\):

\[
\Delta_n(z, h, s) := \frac{1}{n} \sum_{i=1}^{n} 1_{[0,s]} \left( \frac{Z_i - z}{h^{1/d}} \right), \quad s \in [0, 1)^d.
\]

These processes, usually called functional increments of the empirical distribution function, have been intensively investigated in the literature (see, e.g., Shorack and Wellner [11], Van der Vaart and Wellner [12], Deheuvels and Mason [5, 3], Einmahl and Mason [7], Mason [9]). A particular domain of investigation of these increments is when their almost sure behavior is studied along a sequence of bandwidths \((h_n)_{n \geq 1}\) satisfying the following conditions:

\begin{align*}
(HVE1) & \quad 0 < h_n < 1, \ h_n \downarrow 0, \ nh_n \uparrow \infty, \\
(HVE2) & \quad nh_n / \log n \rightarrow c.
\end{align*}

Here, \( c > 0 \) denotes a finite constant. Such conditions on the sequence \((h_n)_{n \geq 1}\) are called Erdős-Rényi conditions, since these two authors have given a pioneering result in this domain (see [?]). Deheuvels and Mason [5] showed that, whenever the \((Z_i)_{i \geq 1}\) are uniformly distributed on \([0, 1]\), and under \((HVE1) - (HVE2)\), the increments \( n \Delta_n(z, h_n) / (c \log n) \) have a nonstandard almost sure behaviour. Before citing their result, we need to introduce the following notations. Set \( B([0, 1)^d) \) as the cone of all bounded increasing functions \( g \) on \([0, 1)^d\) (implicitly with respect to the order \( \prec \)), satisfying \( g(0) = 0 \). We shall endow this cone with the topology spawned by the usual sup-norm \( ||g|| = \sup_{s \in [0,1)^d} |g(s)| \). Define the usually called Chernoff function \( h \) as

\[
h(x) := \begin{cases} 
  x \log x - x + 1, & \text{for } x > 0; \\
  1, & \text{for } x = 0; \\
  \infty, & \text{for } x < 0.
\end{cases} \tag{1.2}
\]

That function is known to play an important role in the large deviation of Poisson processes on \([0, 1]\) (see, e.g., [3]). Define the following (rate) function on \( B([0, 1)^d) \). Whenever \( g \in B([0, 1)^d) \) is absolutely continuous with respect to the Lebesgue measure on \([0, 1)^d\), we set

\[
I(g) := \int_{[0,1)^d} h(g'(s)) ds, \tag{1.3}
\]

g' denoting (a version of) the derivative of \( g \) with respect to the Lebesgue measure. Whenever \( g \) fails to be absolutely continuous, we set \( I(g) = \infty \). Also define, for any \( a > 0 \),

\[
\Gamma_a := \{ g \in B([0,1)^d), \ I(g) \leq 1/a \}. \tag{1.4}
\]

In a pioneering work, Deheuvels and Mason [5] established the following nonstandard uniform functional limit law for the \( \Delta_n(z, h_n, \cdot) \), when the \((Z_i)\) are uniform on \([0, 1]\).
Theorem 1 (Deheuvels, Mason, 1992) Assume that \( d = 1 \) and that the \((Z_i)_{i \geq 1}\) are uniformly distributed on \([0, 1]\). Let \( 0 \leq a < b < 1 \) be two real numbers, and let \((h_n)_{n \geq 1}\) be a sequence of positive constants satisfying \((HVE1) - \(HVE2)\) for some constant \( c > 0 \). Then we have almost surely
\[
\lim_{n \to \infty} \sup_{z \in [0, 1-h_n]} \inf_{g \in \Gamma_c} \left\| \frac{n}{c \log n} \Delta_n(z, h_n, \cdot) - g \right\| = 0,
\]
\[
\forall g \in \Gamma_c, \lim_{n \to \infty} \inf_{z \in [0, 1-h_n]} \left\| \frac{n}{c \log n} \Delta_n(z, h_n, \cdot) - g \right\| = 0.
\]

As a corollary, the authors showed that, when the sequence of bandwidth \((h_n)_{n \geq 1}\) satisfies \((HVE1) - \(HVE2)\), the Parzen-Rosenblatt kernel density estimator is not uniformly strongly consistent. They proved this non-consistency result by making use of some optimisation techniques on Orlicz balls (see Deheuvels and Mason [4]). The aim of the present paper is to provide a generalisation of the former result to the case where the \((Z_i)_{i \geq 1}\) take values in \(\mathbb{R}^d\). This generalisation can be stated as follows.

Theorem 2 Assume that the \((Z_i)_{i \geq 1}\) have a density \(f\) satisfying \((H f)\). Let \(H \subset O\) be a compact set with nonempty interior. Let \((h_n)_{n \geq 1}\) be a sequence of positive constants fulfilling \((HVE1)\) and \((HVE2)\). Then we have almost surely

(i) \(\forall z \in H, \forall g \in \Gamma_{cf(z)}, \lim_{n \to \infty} \inf_{z' \in H} \left\| \Delta_n(z', h_n, \cdot) - g \right\| = 0, \quad (1.5)\)

(ii) \(\lim_{n \to \infty} \sup_{z \in H} \inf_{g \in \Gamma_{cf(z)}} \left\| \Delta_n(z, h_n, \cdot) - g \right\| = 0. \quad (1.6)\)

Denote by \(f_n(K, z, h_n)\) the usual kernel density estimator with bandwidth \(h_n\) and kernel \(K\). A consequence of Theorem 2 is that, under \((HVE1) - \(HVE2)\), \(f_n(K, z, h_n)\) is not uniformly consistent (in a strong sense) over \(\text{say} (Hf)\) an hypercube of \(\mathbb{R}^d\).

Corollary: Let \(K\) be a kernel with compact support and bounded variation. Assume \((H f)\) and \((HVE1) - \(HVE2)\). Let \(H \subset O\) be a compact with nonempty interior. Then the following event holds with probability one:

\[
\exists \epsilon > 0, \exists n_0, \forall n \geq n_0, \sup_{z \in H} | f_n(K, z, h_n) - f(z) | > \epsilon.
\]

Proof: The proof follows exactly the lines of Deheuvels and Mason (see [2], Theorem 4.2) and is based on some optimisation results on Orlicz Balls that have been provided in Deheuvels and Mason [4]. □

From now on, we shall make use of the following notation
\[
\Delta_n(z, h_n, s) := \frac{\sum_{i=1}^{n} 1_{[0,s]}(\frac{z_i - z}{h_n})}{cf(z) \log n}, \quad s \in [0, 1)^d.
\]

Remark 1.0.1
Deheuvels and Mason [6] have already given a nonstandard functional limit law for a single increment $\Delta_{n}(z_0, h_n, \cdot)$ when \( HVE2 \) is replaced by \( nh_n/\log \log n \to c > 0 \). Their result is presented in a more general setting, considering the \( \Delta_{n}(z_0, h_n, \cdot) \) as random measures indexed by a class of sets.

The remainder of this paper is organised as follows. In §2 we provide some tools in large deviation theory, which are consequences of results of Arcones [1] and Lynch and Sethuraman [8]. In §3, a uniform large deviation principle for "poissonized" versions of the \( \Delta_{n}(z, h_n, \cdot) \) is established. In §4 and §5, we make use of the just-mentioned uniform large deviation principle to prove Theorem 2.

2 Uniform large deviation principles

The main tool we shall make use of in §4 and §5 is a uniform large deviation principle for a triangular array of compound Poisson processes. We must first remind some usual notions in large deviation theory. Let \((E, d)\) be a metric space. A real function \( J : E \to [0, \infty] \) is said to be a rate function (implicitly for \((E, d)\)) when the sets \( \{ x \in E : J(x) \leq a \} \), \( a \geq 0 \), are compact sets of \((E, d)\). We shall first show that \( I \) is a rate function on \((B([0,1]^d), \| \cdot \|)\) by approximating it by suitably chosen simple rate functions.

2.1 Approximations of \( I \)

Given \( g \in B([0,1]^d) \) and a Borel set \( A \), we shall write

\[
g(A) := \int_{[0,1]^d} 1_{A} \, dg, \quad (2.1)
\]

which is valid as soon as either \( g \) or \( 1_{A} \) has bounded variation. For any integer \( p \geq 1 \) and for each \( 1 \prec i \prec 2^{p} \) set

\[
A_{i}^{p} := 2^{-p} [i-1, i), \quad (2.2)
\]

with the notation \( i-1 := (i_1 - 1, \ldots, i_d - 1) \). Recall that \( h \) is given in (1.2), and that \( \lambda \) is the Lebesgue measure on \([0,1)^d\). The following functions will play the role of approximations of \( I \) (given in (1.3)), as \( p \to \infty \):

\[
I_p(g) := \sum_{1 \prec i \prec 2^{p}} 2^{-pd} h \left( 2^{pd} g(A_{i}^{p}) \right) \quad (2.3)
\]

\[
= \sum_{1 \prec i \prec 2^{p}} \lambda(A_{i}^{p}) h \left( \frac{g(A_{i}^{p})}{\lambda(A_{i}^{p})} \right), \quad g \in B([0,1)^d).
\]

We point out the following properties of the function \( I \).

**Proposition 2.1** For each \( g \in B([0,1)^d) \), we have

\[
\lim_{p \to \infty} I_p(g) = I(g). \quad (2.4)
\]

Moreover, \( I \) is a rate function on \( (B([0,1)^d), \| \cdot \|) \).
Proof: Choose \( g \in B([0,1]^d) \) arbitrarily and assume that \( I(g) > 0 \) (nontrivial case). In a first time, we suppose that \( g \) has bounded variation, so that it can be interpreted as a finite measure. Denote by \( \mathcal{T}_p \) the \( \sigma \)-algebra of \([0,1]^d\) spawned by the sets \( A^p_i, 1 < i < 2^p \). Clearly, for all \( p \geq 1 \), the measure \( g \) is absolutely continuous with respect to the (trace of the) Lebesgue measure \( \lambda \) on \( \mathcal{T}_p \). Furthermore, the corresponding Radon-Nicodym derivative is given by the following equality.

\[
L_p := \frac{dg}{d\lambda} \big|_{\mathcal{T}_p} = \sum_{1 < i < 2^p} 1_{A^p_i} \frac{g(A^p_i)}{\lambda(A^p_i)}.
\]  

(2.5)

Clearly the \( \sigma \)-algebra spawned by the (increasing) sequence \( \langle \mathcal{T}_p \rangle_{p \geq 1} \) is equal to the Borel \( \sigma \)-algebra of \([0,1]^d\). Assume first that \( g \) is absolutely continuous with respect to \( \lambda \). According to Dacunha-Castelle and Duflo [2], p. 63, the sequence \( L_p \) converges \( \lambda + g \) almost everywhere to a positive function \( L \) satisfying \( L = g' \) (\( \lambda + g \) almost everywhere). Now select \( 0 < l < I(g) \) arbitrarily. By definition of \( I \), there exists \( \epsilon > 0 \) satisfying

\[
\int_{\epsilon < L < 1/\epsilon} h(L) d\lambda > l.
\]

Since \( L_p \to L \) (\( \lambda + g \) almost everywhere as \( p \to \infty \)) and since \( h \) is continuous, we have

\[
\liminf_{p \to \infty} h(L_p) 1_{\{\epsilon < L_p < 1/\epsilon\}} \geq h(L) 1_{\{\epsilon < L < 1/\epsilon\}} \lambda + g \text{ almost everywhere}
\]

Hence by an application of Fatou’s lemma,

\[
\liminf_{p \to \infty} \int_{\epsilon < L_p < 1/\epsilon} h(L_p) d\lambda \geq \int_{\epsilon < L < 1/\epsilon} h(L) d\lambda > l.
\]

Since \( \sup_{p \geq 1} I_p(g) \leq I(g) \) by a straightforward use of Jensen’s inequality, and since \( l < I(g) \) was chosen arbitrarily, we readily infer that \( I_p(g) \to I(g) \) as \( p \to \infty \). Now assume that \( I(g) = \infty \) and that \( g \) is not absolutely continuous with respect to \( \lambda \). According to Dacunha-Castelle and Duflo [2], p. 63, the sequence \( L_p \) converges \( \lambda + g \) almost everywhere to a positive function \( L \) satisfying \( (\lambda + g)(\{L = \infty\}) =: \tau > 0 \). Define

\[
\ell(x) := x^{-1} h(x) = \log(x) - 1 + x^{-1}, \quad x > 0.
\]

Clearly, \( \ell(x) \to \infty \) as \( |x| \to \infty \). Now select \( l > 0 \) arbitrarily, and choose \( A > 0 \) satisfying

\[
\inf_{x > A} \ell(x) > \frac{2 l}{\tau}.
\]
Since $L_p \to L(\lambda + g$ almost everywhere as $p \to \infty$) we have $g(L_p > A) > \tau/2$ for all large $p$, whence

$$I_p(g) \geq \int_{L_p \in (A,\infty)} \ell(L_p)L_p d\lambda$$

$$= \int_{L_p \in (A,\infty)} \ell(L_p)dg$$

$$\geq \frac{2\ell}{T} g(L_p > A)$$

$$> I.$$  \hfill (2.6)

We have shown that (2.4) is true for each $g$ with bounded variation. Whenever $g$ has infinite variation, then it can be shown that $I_p(g) \to \infty$ by a discrete version of the argument that have just been invoked to obtain (2.6). We omit details for sake of briefness.

Since all the functions $I_p$ are $|| \cdot ||$-continuous and since $I_p(g) \uparrow I(g)$ for all $g \in B([0,1]^d)$, we conclude that $I$ is lower-semicontinuous for $|| \cdot ||$. Hence, $I$ is a rate function if and only if the set $\Gamma_a$ is totally bounded for each $a > 0$ (recall (1.4)). Since $x^{-1}h(x) \to \infty$ as $|x| \to \infty$, we have, for some constant $M > 0$,

$$|x| \leq |x| 1_{|x| \leq M} + h(x),$$  \hfill (2.7)

from where we readily infer that

$$\int_{[0,1]^d} |g'| d\lambda \leq M + 1/a \text{ for each } a > 0 \text{ and } g \in \Gamma_a.$$  \hfill (2.8)

Applying the Arzela-Ascoli criterion, we conclude that, for each $a > 0$, the closed set $\Gamma_a$ is totally bounded, which entails that $I$ is a rate function on $(B([0,1]^d), || \cdot ||)$. This concludes the proof of Proposition 2.1.  \hfill □

2.2 Uniform large deviations in $(B([0,1]^d), || \cdot ||)$

We shall now give a definition of a large uniform large deviation principle in the metric space $(B([0,1]^d), || \cdot ||)$. In the sequel, $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ will always denote a triangular array of positive numbers satisfying $\max_{i \leq m_n} \epsilon_{n,i} \to 0$ as $n \to \infty$. Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of random elements on probability space $(\omega, T', \mathbb{P})$, taking values in $B([0,1]^d)$. In order to handle carefully the notions of inner and outer probabilities, we shall that each $X_{n,i}$ is a suitable projection mapping from $(\Omega, T')$ to $E$, where

$$\Omega := \prod_{n=1}^{\infty} \prod_{i=1}^{p} B([0,1]^d), \quad T' := \bigotimes_{n=1}^{p} \bigotimes_{i=1}^{p} T,$$
and $\mathcal{T}$ is the Borel $\sigma$-algebra of $\left( B([0,1]^d), \| \cdot \| \right)$. From now on, outer and inner probabilities $P^*$ and $P_\star$ are understood with $(\Omega, \mathcal{T}')$ as the underlying probability space. We say that $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the Uniform Large Deviation Principle (ULDP) for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for a rate function $J$ whenever the two following conditions hold.

- For any $\| \cdot \|$-open set $O \subset B([0,1]^d)$ we have
  \[ \liminf_{n \to \infty} \min_{i \leq m_n} \epsilon_{n,i} \log \left( P^* \left( X_{n,i}(\cdot) \in O \right) \right) \geq -J(O). \]  
  (2.9)

- For any $\| \cdot \|$-closed set $F \subset B([0,1]^d)$ we have
  \[ \limsup_{n \to \infty} \max_{i \leq m_n} \epsilon_{n,i} \log \left( P^* \left( X_{n,i}(\cdot) \in F \right) \right) \leq -J(F). \]  
  (2.10)

**Remark 2.2.1**

The same definition holds for triangular arrays of random variables taking values in $\mathbb{R}^p$, $p \geq 1$. The norm $\| \cdot \|$ can then be replaced by any norm.

Arcones [1] provided a powerful tool to establish Large Deviation Principles for sequences of bounded stochastic processes. Some verifications lead to the conclusion that the just-mentioned tool can be used in our context. Recall that the sets $A^p_i$ have been define by (2.2). Consider the following finite grid, for $p \geq 1$:

\[ s_{i,p} := 2^{-p}(i - 1), \ 1 \prec i \prec 2^p. \]  
(2.11)

Given, $p \geq 1$ and $g \in B([0,1]^d)$, we write

\[ g^{(p)} = \sum_{1 \prec i \prec 2^p} 1_{A^p_i} g(s_{i,p}). \]

**Proposition 2.2** Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of random elements taking values in $B([0,1]^d)$ almost surely, and let $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of positive real numbers. Assume that the following conditions are satisfied.

1. The triangular array of stochastic process $(X^{(p)}_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the rate function $I_p$ on $B([0,1]^d), \| \cdot \|$.

2. For each $\tau > 0$ and $M > 0$ there exists $p \geq 1$ satisfying

\[ \limsup_{n \to \infty} \max_{1 \prec i \prec 2^p} \max_{s \in A^p_i} \left( P^* \left( X_{n,i}(t) - X_{n,i}(s^{(p)}_{i}) \geq \tau \right) \right) \leq -M. \]

Then $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.

\[ J(g) := \sup_{p \geq 1} I_p \left( g^{(p)} \right), \ g \in B([0,1]^d). \]
Proof: The proof follows exactly the same lines as in the proof of Theorem 3.1 of Arcones [1]. Using these arguments in our context remains possible since the cone $B([0,1]^d)$ is a closed subset of $L^\infty([0,1]^d)$ for the usual sup norm $|| \cdot ||$. We avoid writing the proof for sake of brevity. □

Another tool we shall make an intensive use of is a ULDP for random vectors with mutually independent coordinates.

Proposition 2.3 Let $(X_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ and $(Y_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ be two triangular arrays of random vectors taking values in $\mathbb{R}^d$ and $\mathbb{R}^d$ respectively, and satisfying $X_{n,i} \perp Y_{n,i}$ for each $n \geq 1$, $1 \leq i \leq m_n$. Assume that both $(X_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ and $(Y_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ satisfy the ULDP for a triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for two rate functions $J_1$ and $J_2$ respectively. Then the triangular array $(X_{n,i}, Y_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.

$$J(z_1, z_2) := J_1(z_1) + J_2(z_2), \quad z_1 \in \mathbb{R}^d, \quad z_2 \in \mathbb{R}^d.$$  

Proof: The proof follows the same lines as Lemma 2.6 and Corollary 2.9 in Lynch and Sethuraman [3]. In the just-mentioned article, the authors make use of the notions of Weak Large Deviation Principle and of LD-tightness for sequences of random variables in a Polish space. These notions can be easily extended to the frame of triangular arrays of random variables. □

The following proposition is nothing else than the contraction principle in the framework of ULDP (see, e.g., [1], Theorem 2.1 for the most general version of that principle).

Proposition 2.4 Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of $\mathbb{R}^p$ valued random vectors satisfying the ULDP for a triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for a rate function $J$. Let $\mathcal{R}$ be a continuous mapping from $\mathbb{R}^d$ to $(B([0,1]^d), || \cdot ||)$. Then $(\mathcal{R}(X_{n,i}))_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.

$$J_{\mathcal{R}}(g) := \inf\{J(x), \mathcal{R}(x) = g\}, \quad g \in B([0,1]^d),$$

with the convention $\inf\emptyset = \infty$.

Proof: Straightforward. □

The following proposition shall be useful in our the proof of our Lemma 3.1.

Proposition 2.5 Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of real random variables and let $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of positive real numbers. Assume that there exists a strictly convex positive function $J$ on $\mathbb{R}$ and a real number $\mu$ such that $J(\mu) = 0$ and

$$\forall a > \mu, \quad \lim_{n \to \infty} \max_{i \leq m_n} \epsilon_{n,i} \log \left( \mathbb{P}(X_{n,i} \geq a) \right) - J(a) = 0, \quad (2.12)$$

$$\forall a < \mu, \quad \lim_{n \to \infty} \max_{i \leq m_n} \epsilon_{n,i} \log \left( \mathbb{P}(X_{n,i} \leq a) \right) - J(a) = 0. \quad (2.13)$$

Then $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for $J$.

Proof: The proof is routine calculus. □
3 A ULDP for poissonised versions of the $\Delta_n(z, h_n, \cdot)$

Define the following process, for each integer $n \geq 1$.

$$\Delta \Pi_n(z, h_n, s) := \sum_{i=1}^{\eta_n} 1_{[0,s]} \left( \frac{Z_i - z}{h_n} \right) \frac{1}{c f(z) \log n}, \ s \in [0,1)^d. \quad (3.1)$$

Here $\eta_n$ is a Poisson random variable independent of $(Z_i)_{i \geq 1}$, with expectation $n$. These "poissonized" versions of the processes $\Delta_n(z, h_n, \cdot)$ can be identified to random (Poisson) measures by the following relation

$$\Delta \Pi_n(z, h_n, A) := \int_{(0,1)^d} 1_A(s) d\Delta \Pi_n(z, h_n, s), \ A \text{ Borel}. \quad (3.2)$$

The key of our proof of Theorem 2 is the following ULDP.

**Proposition 3.1** Let $(z_{i,n})_{n \geq 1, 1 \leq i \leq m_n}$ be a triangular array of elements of $H$. Under the assumptions of Theorem 2, the triangular array of processes $(\Delta \Pi_n(z_{i,n}, h_n, \cdot))_{n \geq 1, 1 \leq i \leq m_n}$ satisfies the ULDP in $(B([0,1)^d), \| \cdot \|)$ for the rate function $I$ and for the following triangular array

$$\epsilon_{n,i} := \frac{1}{c f(z_{i,n}) \log n}, \ n \geq 1, \ 1 \leq i \leq m_n. \quad (3.3)$$

**Remark 3.0.2**

Proposition 3.1 is true whatever the constant $c > 0$ appearing in assumption (HVE1). This remark will show up to be useful in Lemma 5.2 in §5.

**Proof:** To prove proposition 3.1 we shall make use of Proposition 2.2. We hence have to check conditions 1, 2 and 3 of the just-mentioned proposition. This will be achieved through several lemmas.

3.1 A preliminary lemma

Recall notation (2.1). To check condition 2 of Proposition 2.2 we need first to establish the following lemma.

**Lemma 3.1** Assume that the hypothesis of Theorem 2 are satisfied. Then, for each $p \geq 1$ and for each $1 < i_0 < 2^p$, the triangular array of random variables $(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p))_{n \geq 1, 1 \leq i \leq m_n}$ satisfies the ULDP in $[0,\infty)$ for the triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function:

$$\bar{I}_p(x) := 2^{-pd}h \left( \frac{x}{2^{-pd}} \right) = \lambda (A_{i_0}^p) h \left( \frac{x}{\lambda (A_{i_0}^p)} \right), \ x \geq 0. \quad (3.4)$$
Proof: Fix once for all $p \geq 1$ and $1 < i_0 < 2^d$. We shall make use of Proposition 2.5 with $J := \tilde{I}_p$ and $\mu := 2^{-pd}$. We give details only for the proof of (2.12), as proving (2.13) is very similar. Fix $a > 2^{-pd}$. For each integers $n \geq 1$ and $1 \leq i \leq m_n$, we set (recall (3.2))

$$V_{i,n,i_0} := cf(z_{i,n})(\log n)\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p),$$

$$p_{i,n,i_0} := \mathbb{P}( Z_1 \in z_{i,n} + h_n^{1/d} A_{i_0}^p ).$$

Clearly $V_{i,n,i_0}$ is a Poisson random variable with expectation $np_{i,n,i_0}$. Since the density $f$ satisfies $(Hf)$ and since $\lambda(A_{i_0}^p) = 2^{-pd}$, we have

$$\lim_{n \to \infty} \max_{1 \leq i \leq m_n} \left| \frac{p_{i,n,i_0}}{f(z_{i,n})} - 1 \right| = 0. \quad (3.5)$$

Hence according to (HVE2) we have, ultimately as $n \to \infty$,

$$\min_{1 \leq i \leq m_n} \frac{acf(z_{i,n}) \log n}{np_{i,n,i_0}} > 1. \quad (3.6)$$

We then make use of Chernoff’s inequality for Poisson random variables to get, for all large $n$ (satisfying (3.6)) and for all $1 \leq i \leq m_n$,

$$\mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \geq a) = \mathbb{P}(V_{i,n,i_0} \geq acf(z_{i,n}) \log n)
\leq \exp\left(-np_{i,n,i_0}h\left(\frac{acf(z_{i,n}) \log n}{np_{i,n,i_0}}\right)\right). \quad (3.7)$$

But (3.7) in combination with (3.5) entails

$$\limsup_{n \to \infty} \max_{1 \leq i \leq m_n} \frac{p_{i,n,i_0}}{f(z_{i,n})} \frac{h\left(\frac{acf(z_{i,n}) \log n}{np_{i,n,i_0}}\right)}{h_n} \leq 2^{-pd} h\left(\frac{a}{2^{-pd}}\right), \quad (3.8)$$

which, together with (3.7) leads to

$$\limsup_{n \to \infty} \max_{1 \leq i \leq m_n} \epsilon_{i,n,}\log\left(\mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \geq a)\right) \leq -\tilde{I}_p(a). \quad (3.9)$$

Now select $y > a$ arbitrarily. If we could show that

$$\liminf_{n \to \infty} \min_{1 \leq i \leq m_n} \epsilon_{i,n}\log\left(\mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \geq a)\right) \geq -\tilde{I}_p(y),$$

then, as $y > a$ was chosen arbitrarily, and since $\tilde{I}_p$ is increasing on $[a,\infty)$, we should be able to conclude the proof of (2.12) with $J = \tilde{I}_p$. Now set $\phi(t) := \exp(\exp(t) - 1), \quad t \in \mathbb{R}$ and notice that $h(z) = \max_{u \in \mathbb{R}} zu - \log(\phi(u))$ for each $z > 0$. Set $u_0 := \log(2^{pd}y)$, so as

$$h(2^{pd}y) = 2^{pd}yu_0 - \log(\phi(u_0)). \quad (3.10)$$
Here (3.15) is a consequence of (3.12), with \( L \) integer large enough to fulfill (recall (3.5))

\[
denote by \( F \) the distribution function of a Poisson random variable with expectation 1, and define \( F_0 \) by
\[
d F_0(x) := \phi(u_0)^{-1} \exp(u_0x) dF(x).
\] (3.11)

Let "*" be the convolution operator for infinitely divisible laws and notice that, for each \( L > 0 \), we have
\[
d F_0^L(\cdot) = \phi(u_0)^{-L} \exp(u_0^L) dF^*(\cdot),
\] (3.12)
\[
\mathbb{E}_{F_0^L}(X) = 2^{pd} Ly,
\] (3.13)
\[
\text{Var}_{F_0^L}(X) = L \text{Var}_{F_0}(X)
\] (3.14)

Here we have written \( E_F(X) \) as the expectation of a random variable with distribution \( F \). Now fix \( \delta > 0 \) satisfying \( [y - \delta, y + \delta] \subset [a, \infty] \) arbitrarily. Obviously, \( F^{*n_{p_i,n,k_0}} \) is the distribution function of \( cf(z_{i,n})(\log n) \Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \), whence
\[
\mathbb{P} \left( \Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \geq \alpha \right) \geq \mathbb{P} \left( \Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \in [y - \delta, y + \delta] \right)
\]
\[
= \int_{\Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \in [y - \delta, y + \delta]} d F^{*n_{p_i,n,k_0}}(x)
\]
\[
\geq \exp(-u_0(y + \delta)cf(z_{i,n})(\log n)) \times \int_{\Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \in [y - \delta, y + \delta]} \exp(u_0x) d F^{*n_{p_i,n,k_0}}(x)
\]
\[
\geq \exp(-cf(z_{i,n})(\log n)u_0(y + \delta) + n_{p_i,n,k_0} \log(\phi(u_0)))
\]
\[
\times \int_{\Delta \Pi_n(z_{i,n}, h_n, A^p_{h_0}) \in [y - \delta, y + \delta]} d F^{*n_{p_i,n,k_0}}(x)
\] (3.15)
\[
:= a_{i,n,k_0,\delta} \times b_{i,n,k_0,\delta}.
\]

Here (3.15) is a consequence of (3.12), with \( L : = n_{p_i,n,k_0} \). Now let \( n \geq 1 \) be an integer large enough to fulfill (recall (3.3))
\[
\max_{1 \leq i \leq m_n} \left| \frac{n_{p_i,n,k_0}}{2^{-pd} cf(z_{i,n})(\log n)} - 1 \right| \leq u_0 \log(\phi(u_0))^{-1} \delta,
\] (3.16)
which enables us to write the following chain of inequalities.
\[
cf(z_{i,n})(\log n)u_0(y + \delta) - n_{p_i,n,k_0} \log(\phi(u_0))
\]
\[
\leq 2^{-pd}(y + \delta) cf(z_{i,n})(\log n) (u_0 2^{pd} - \log(\phi(u_0)) + u_0 \delta)
\]
\[
\leq 2^{-pd} cf(z_{i,n})(\log n) (h (2^{pd}y) + u_0(2^{pd} + 1) \delta)
\]
\[
= cf(z_{i,n})(\log n) \left( \sum_{p=0}^{\infty} \frac{\cdots}{i} + 2^{-pd} (2^{pd} + 1) u_0 \delta \right)
\]
\[
\leq cf(z_{i,n})(\log n) \left( \overline{I}_p(y) + 2u_0 \delta \right).
\] (3.17)
Therefore we have, for all large $n$ and for all $1 \leq i \leq m_n$,
\[
a_{i,n,i_0,\delta} \geq \exp \left( -cf(z_{i,n}) \log n \left( \bar{I}_p(y) + 2u_0\delta \right) \right),
\]
(3.18)
where $u_0 = \log(2^{pd}y)$ depends on $y > a$ only. It remains to show that
\[
\lim_{n \to \infty} \min_{1 \leq i \leq m_n} b_{i,n,i_0,\delta} = 1.
\]
(3.19)
Consider $n$ large enough to fulfill (recall (3.5))
\[
y - \delta < \min_{1 \leq i \leq m_n} \frac{\nu_{p,i,n,i_0}}{2^{pd}c_f(z_{i,n}) \log n} \leq \max_{1 \leq i \leq m_n} \frac{\nu_{p,i,n,i_0}}{2^{pd}c_f(z_{i,n}) \log n} < y + \delta
\]
so as, for all $1 \leq i \leq m_n$,
\[
\frac{\nu_{p,i,n,i_0}}{2^{pd}c_f(z_{i,n}) \log n} \times [y - 2^{pd}\delta, y + 2^{pd}\delta] \subset [y - \delta, y + \delta],
\]
(3.20)
and hence
\[
b_{i,n,i_0,\delta} \geq \int_{\nu_{p,n,i_0} \in [2^{pd}y - \delta, 2^{pd}y + \delta]} dF^*_{\nu_{p,n,i_0}}(x).
\]
Recalling (3.13) and (3.14) we get, by the Bienaymé-Tchebychev inequality,
\[
1 - b_{i,n,i_0,\delta} \leq \frac{\text{Var}_{F_0}(X)}{\delta \nu_{p,i,n,i_0}}.
\]
(3.21)
By assumption $(H_f)$ we infer that the $h^{-1}_{n,p_i,n,i_0}$ are bounded away from zero, from where (3.15) follows. Then (3.15), (3.18) and (3.19) entail
\[
\lim_{n \to \infty} \min_{1 \leq i \leq m_n} \epsilon_{n,i} \log \left( \mathbb{P} \left( \Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \geq a \right) \right) \geq -\bar{I}_p(y) + 2u_0\delta.
\]
(3.22)
Assertion (2.12) is then proved by combining (3.9) with (3.22), as $\delta > 0$ is arbitrary. □

3.2 Verification of condition 2 of Proposition 2.2

For $n \geq 1$ and $1 \leq i \leq m_n$, define the following $\mathbb{R}^{2^{pd}}$ valued random vector:
\[
X_{n,i} := (X_{i_0,n,i})_{1 \leq i_0 < 2^p} := (\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p))_{1 \leq i_0 < 2^p}.
\]
Notice that the random variables $X_{i_0,n,i}$, $1 \leq i_0 < 2^p$ are mutually independent for fixed $n \geq 1$ and $1 \leq i \leq m_n$ by usual properties of Poisson random measures.
Hence, by Lemma 3.1 together with Proposition 2.3 we deduce that the triangular array \((X_{n,i})_{n \geq 1, i \leq m_n}\) satisfies the ULDP with \((\epsilon_{n,i})_{n \geq 1, i \leq m_n}\) and with the following rate function.

\[
I'_p(x) := \sum_{1 \leq i < 2^p} 2^{-pd} h \left( \frac{x_i}{2^{-pd}} \right), \quad x \in [0, \infty)^{2^d}. \tag{3.23}
\]

Here we have written \(x := (x_i)_{1 \leq i < 2^p}\). We now define the following mappings from \([0, \infty)^{2^d}\) to \((B([0,1]^d))\)

\[
\mathcal{R}_p(x) : (0,1)^d \mapsto [0, \infty) \ni \sum_{A_i^p \subset [0,s]} x_i.
\]

Denote by \([x]\) the integer part of a real number \(x\) \(([x] \leq x < [x] + 1)\), and write \([s] := ([s_1], \ldots, [s_d])\) for any \(s = (s_1, \ldots, s_d) \in \mathbb{R}^d\). We point out that with probability one (recall the notations of Proposition 2.2)

\[
\limsup_{n \to \infty} \max_{1 \leq i \leq m_n} |\epsilon_{n,i}| \leq \tau \implies \mathcal{R}_p(X_{n,i})(s) = \Delta \Pi_n (z_{i,n}, h_n, 2^{-p}[2^p s]) \\
= \Delta \Pi_n (z_{i,n}, h_n, s)^{(p)}, \quad s \in [0,1]^d.
\]

For fixed \(p \geq 1\), we make use of the contraction principle (Proposition 2.4) to conclude that \((\mathcal{R}_p(X_{n,i}))_{n \geq 1, i \leq m_n}\) satisfies the ULDP for \((\epsilon_{n,i})_{n \geq 1, i \leq m_n}\) and for the following rate function.

\[
\mathcal{T}_p(g) := \inf \left\{ I'_p(x), \ x \in [0, \infty)^{2^d}, \ \mathcal{R}_p(x) = g \right\}, \ g \in B([0,1]^d), \tag{3.24}
\]

with the convention \(\inf \emptyset = \infty\). Obviously, the set appearing in (3.24) is non void if and only if \(g\) is the cumulative distribution function of a purely atomic measure with atoms belonging to the grid \(\{s_{i,p}, 1 \prec i \prec 2^p\}\). In that case we have

\[
\mathcal{T}_p(g) = \sum_{1 \leq i < 2^p} 2^{-pd} h \left( g(A_i^p) \right) / 2^{-pd} = I_p(g).
\]

Here, we have identified \(g\) to a positive finite measure on \([0,1]^d\) (recall (2.1)). Assumption 2 of Proposition 2.2 is then satisfied.

### 3.3 Verification of condition 3 of Proposition 2.2

Fix \(\tau > 0\) and \(M > 0\). We have to prove that, provided that \(p\) is large enough,

\[
\limsup_{n \to \infty} \max_{1 \leq i \leq m_n} \epsilon_{n,i} \geq \log \left( \mathbb{P} \left( \max_{1 \leq i < 2^p} \sup_{s \in A_i^p} \left| \Delta \Pi_n (z_{i,n}, h_n, s) - \Delta \Pi_n (z_{i,n}, h_n, 2^{-p} i - 1) \right| \geq \tau \right) \right) \leq -M. \tag{3.25}
\]
For fixed $p \geq 1$, $n \geq 1$, $1 \leq i \leq m_n$, a rough upper bound gives

$$
\mathbb{P} \left( \max_{1 \leq i < 2^p} \sup_{s \in A_i} \left| \Delta \Pi_n \left( z_{i,n}, h_n, s \right) - \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}(i-1) \right) \right| \geq \tau \right)
\leq 2^{pd} \max_{1 \leq i < 2^p} \mathbb{P} \left( \sup_{s \in A_i} \left| \Delta \Pi_n \left( z_{i,n}, h_n, s \right) - \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}(i-1) \right) \right| \geq \tau \right)
\leq \mathbb{P} \left( \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}i \right) - \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}(i-1) \right) \geq \tau \right).
$$

(3.26)

We shall now write

$$
W_{i,n,i,p} := cf(z_{i,n}) \log n \left( \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}i \right) - \Delta \Pi_n \left( z_{i,n}, h_n, 2^{-p}(i-1) \right) \right),
$$

$$
\mu_{i,n,i,p} := \mathbb{P} \left( \frac{Z_i - z_{i,n}}{h_n^{1/2}} \in [0, 2^{-p}i) \cup [0, 2^{-p}(i-1)) \right), \text{ and}
$$

$$
\nu_{i,p} := \lambda \left( [0, 2^{-p}i) \cup [0, 2^{-p}(i-1)) \right) \leq d2^{-p}.
$$

(3.27)

Clearly, $W_{i,n,i,p}$ is a Poisson random variable with expectation $n\mu_{i,n,i,p}$. Moreover, by assumption ($H_f$) we have

$$
\lim_{n \to \infty} \min_{1 \leq i \leq m_n, 1 \leq i < 2^p} \frac{cf(z_{i,n})(\log n)\nu_{i,p}}{n\mu_{i,n,i,p}} = 1.
$$

(3.28)

Recall that $x^{-1}h(x) \to \infty$ as $x \to \infty$. We can then choose $A_{M,\tau} > 1$ large enough to satisfy

$$
\inf_{x \geq A_{M,\tau}} \frac{h(x)}{x} > \frac{8M}{\tau}.
$$

(3.29)

By (3.27) we can choose $p$ large enough to fulfill

$$
\min_{1 \leq i < 2^p} \frac{\tau}{2\nu_{i,p}} > A_{\tau,M}.
$$

(3.30)

Assertion (3.28) together with (3.30) leads to the following inequality, for all large $n$, for all $1 \leq i \leq m_n$ and for all $1 < i < 2^p$.

$$
\frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,i,p}} \geq \frac{\tau}{2\nu_{i,p}} > A_{\tau,M} > 1.
$$

(3.31)

Applying Chernoff’s inequality to the Poisson random variables $W_{i,n,i,p}$ we get, for all large $n$ and for all $1 \leq i \leq m_n$,

$$
\mathbb{P}_{i,n,i,p} = \mathbb{P} \left( W_{i,n,i,p} \geq \tau cf(z_{i,n}) \log n \right)
\leq \exp \left( -n\mu_{i,n,i,p}h \left( \frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,i,p}} \right) \right).
$$
Therefore, recalling (3.28) and (3.31), the following inequality holds for all large \( n \), for all \( 1 \leq i \leq m_n \) and for all \( 1 < i < 2^p \).

\[
\mathbb{P}_{i,n,1,p} \leq \exp \left( -\frac{1}{2} c f(z_{i,n}) \nu_{1,p}(\log n) h \left( \frac{\tau}{2\nu_{1,p}} \right) \right) \leq \exp \left( -c f(z_{i,n}) 2M \log n \right). \tag{3.32}
\]

Here, (3.32) is a consequence of (3.30). By combining (3.32) with (3.26) we get, for all large \( n \) and for each \( 1 \leq i \leq m_n \),

\[
\mathbb{P} \left( \max_{1 < i < 2^p} \sup_{s \in A} \left| \Delta \Pi_n (z_{i,n}, h_n, s) - \Delta \Pi_n (z_{i,n}, h_n, 2^{-p}(i-1)) \right| \geq \tau \right) \leq \exp \left( -2Mc f(z_{i,n}) \log n + \log(2pd) \right),
\]

which proves (3.26) and shows that condition 3 of Proposition 2.2 is satisfied, as \( f \) is bounded away from zero on \( H \). We can now make use of the just-mentioned proposition in combination with Proposition 2.1 to conclude the proof of Proposition 3.1. \( \square \)

4 Proof of part (i) of Theorem 2

Denote by \( \text{Int}(H) \) the interior of \( H \), and fix \( z \in \text{Int}(H) \), \( g \in \Gamma_{cf(z)} \), and \( \epsilon > 0 \). We set

\[
g' := \{ g' \in B([0,1]^d), \| g' - g \| < \epsilon \}. \tag{4.1}
\]

By lower semi continuity of \( I \) in \( B([0,1]^d), \| \cdot \| \) (recall Proposition 2.1), there exists \( \alpha_1 > 0 \) satisfying

\[
I (g') = \frac{1 - 3\alpha_1}{cf(z)}. \tag{4.2}
\]

Now choose an hypercube with nonempty interior \( H' := [a_1, b_1] \times \ldots \times [a_p, b_p] \) fulfilling \( H' \subset H \), \( \mathbb{P} (Z_1 \in H') \leq 1/2 \) and

\[
\inf_{z' \in H'} \frac{f(z')}{f(z)} > \frac{1 - 2\alpha_1}{1 - \alpha_1}. \tag{4.3}
\]

Such a choice is possible since \( H \) has a nonempty interior by assumption. We now divide \( H' \) into disjoint hypercubes \( z_{i,n} + h_n^{1/d} [0,1]^d, 1 \leq i \leq m_n \), where \( m_n \) is the maximal number of disjoint hypercubes we can construct without violating

\[
\bigcup_{i=1}^{m_n} \left\{ z_{i,n} + h_n^{1/d} [0,1]^d \right\} \subset H'. \tag{4.4}
\]

Notice that, as \( n \to \infty \),

\[
m_n = h_n^{-1+o(1)} = n^{(1+o(1))}. \tag{4.5}
\]

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Now recall (3.1). By making use of a well-known "poissonization" technique (see, e.g., Mason [10], Fact 6), we get the following upper bound for all large \( n \).

\[
P \left( \bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^f \} \right) \\
\leq P \left( \bigcap_{i=1}^{m_n} \{ \Delta_n(z_{i,n}, h_n, \cdot) \notin g^f \} \right) \\
\leq 2 P \left( \bigcap_{i=1}^{m_n} \{ \Delta \Pi_n(z_{i,n}, h_n, \cdot) \notin g^f \} \right) \\
= 2 \prod_{i=1}^{m_n} (1 - P(\Delta \Pi_n(z_{i,n}, h_n, \cdot) \in g^f)) \\
\leq 2 \exp \left( -m_n \min_{1 \leq i \leq m_n} P(\Delta \Pi_n(z_{i,n}, h_n, \cdot) \in g^f) \right)
\]

(4.6)

The transition between (4.6) and (4.7) is a classical property of Poisson random measures, while inequality (4.8) is a consequence of \( 1 - u \leq \exp(-u), u \geq 0 \).

We now make use of Proposition 3.1 (with the open ball \( g^\epsilon \)) to get, for all large \( n \) (recall (4.2)),

\[
P \left( \bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^f \} \right) \\
\leq 2 \exp \left( -m_n \min_{1 \leq i \leq m_n} n^{-\frac{f(z_{i,n})}{\epsilon}} (1 - 2\alpha_1) \right)
\]

(4.7)

(4.8)

which is a consequence of (4.3) and (4.5). Hence we conclude by the Borel-Cantelli lemma that, almost surely,

\[
\lim_{n \to \infty} \inf \{ \| \Delta_n(z', h_n, \cdot) - g \|, z' \in H \} \leq \epsilon.
\]

As \( \epsilon > 0 \) was chosen arbitrarily, the proof of part (i) of Theorem 2 is concluded for each \( z \in \text{Int}(H) \). Now the case where \( z \in H \) does not belong to \( \text{Int}(H) \) is treated by making use of the following argument: for each \( z_1 \in H, g_1 \in \Gamma_{cf(z_1)} \) and \( \epsilon > 0 \), there exists \( z_2 \in \text{Int}(H) \) and \( g_2 \in \Gamma_{cf(z_2)} \) satisfying \( \| g_1 - g_2 \| < \epsilon \).

Such an argument is valid by \((Hf)\) and by Lemma 5.1 (see below). \( \square \)

5 Proof of part (ii) of Theorem 2

We shall make use of somewhat usual blocking arguments along the following subsequence \( n_k := [\exp(k/\log k)], k \geq 3 \) and its associated blocks \( N_k := \{n_{k-1} + 1, \ldots, n_k\} \). Given \( A \subset B([0,1]^d) \) and \( \epsilon > 0 \) we shall write

\[
A^\epsilon := \left\{ g \in B([0,1]^d), \inf_{g' \in A} \| g - g' \| < \epsilon \right\}.
\]

(5.1)

The following lemma shall come in handy.
Lemma 5.1 For any $\epsilon > 0$ and $L > 0$ there exists $\eta > 0$ satisfying, for each, $L' \in [(1 + \eta)^{-1}L, L]$, $\Gamma_{L'} \subset \Gamma_L$.

Proof: The proof is routine analysis.□

Now fix $\epsilon > 0$. Since $I$ is lower-semi continuous on $(B([0,1]^d), \| \cdot \|)$ (recall Proposition 2.1) we deduce that, given $z \in H$, there exists $\alpha_z > 0$ satisfying

$$I \left( B([0,1]^d) - \Gamma_{\epsilon f(z)}^c \right) = \frac{1 + 3\alpha_z}{\epsilon f(z)}.$$  \hspace{1cm} (5.2)

By $(Hf)$ and Lemma 5.1 we can construct an hypercube $H_z$ with nonempty interior satisfying the following conditions.

\begin{align*}
  z &\in H_z, \quad H_z \subset O, \quad & (5.3) \\
  \inf_{z_1, z_2 \in H_z} \frac{f(z_1)}{f(z_2)} &\geq \frac{1 + \alpha_z}{1 + 2\alpha_z}, \quad & (5.4) \\
  \bigcup_{z' \in H_z} \Gamma_{\epsilon f(z')}^c &\subset \Gamma_{\epsilon f(z)}, \quad & (5.5) \\
  \mathbb{P} \left( Z_1 \in \bigcup_{z \in H_z} \left\{ z + [0, h_{n_k} 1/d]^d \right\} \right) &\leq 1/2. \quad & (5.6)
\end{align*}

The compact set $H$ is included in the union of the interiors of $H_z$, $z \in H$, from where we can extract a finite union, noted as

$$H \subset \bigcup_{l=1}^L \text{Int}H_{z_l} \subset \bigcup_{l=1}^L H_{z_l} \subset O. \quad (5.7)$$

Our problem is now reduced to showing that, for fixed $l = 1, \ldots, L$,

$$\limsup_{n \to \infty} \sup_{z \in H_{z_l}, g \in \Gamma_{\epsilon f(z)}} \inf \frac{\| \Delta_n(z, h_n, \cdot) - g \|}{\| f(z) \|} \leq 10\epsilon \text{ almost surely}. \quad (5.8)$$

We now fix $1 \leq l \leq L$, and we write $H_{z_l} = [a_1, b_1] \times \ldots \times [a_d, b_d]$. We now introduce a parameter $\delta > 0$ that will be chosen in function of $\epsilon$ in the sequel. For each $k \geq 1$, we cover $H_{z_l}$ by hypercubes

$$H_{z_l} \subset \bigcup_{1 \leq i \leq m_{n_k}} C_{i, n_k} \subset O_1, \quad (5.9)$$

with

$$C_{i, n_k} := z_{i, n_k} + [0, (\delta h_{n_k})^{1/d}]^d, \quad k \geq 1, \quad 1 \leq i \leq m_{n_k} \quad \text{and}$$

$$m_{n_k} := \prod_{p=1}^d \left( \left[ \frac{b_p - a_p}{(\delta h_{n_k})^{1/d}} \right] + 1 \right). \quad (5.10)$$
Now define, for each \( k \geq 1, n \in N_k, z \in H, \)

\[
\mathcal{H}_n(z, s) := \frac{1}{c \log n_k} \sum_{i=1}^{n} 1_{(s, a)} \left( \frac{Z_i - z}{h_{n_k}^{1/d}} \right), \quad s \in [0, 1)^d.
\]

We shall first show that, for any choice \( \delta > 0, \) we have almost surely

\[
\limsup_{n \to \infty} \sup_{1 \leq i \leq m n_k} \inf_{g \in \Gamma_{cf}(z_i)} || \mathcal{H}_n(z_{i,n_k}, \cdot) - g || \leq 2\epsilon. \tag{5.11}
\]

Consider the following probabilities for all large \( k. \)

\[
P_k := \mathbb{P} \left( \bigcup_{1 \leq i \leq m n_k} \bigcup_{n \in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right).
\]

We have, ultimately as \( k \to \infty, \)

\[
P_k \leq m_k \max_{1 \leq i \leq m n_k} \mathbb{P} \left( \bigcup_{n \in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right). \tag{5.12}
\]

We now make use of a well-known maximal inequality (see, e.g., Deheuvels and Mason [5], Lemma 3.4) to get, for all large \( k \) and for all \( 1 \leq i \leq m n_k, \)

\[
\mathbb{P} \left( \bigcup_{n \in N_k} \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right) \leq 2\mathbb{P} \left( \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right). \tag{5.13}
\]

We point out that the conditions of Lemma 3.4 in [5] are satisfied since, by a straightforward use of Markov's inequality we have, ultimately as \( k \to \infty, \)

\[
\sup_{z \in H} \max_{n \in N_k} \mathbb{P} (|| \mathcal{H}_n(z, \cdot) - \mathcal{H}_n(z, \cdot) || \geq \epsilon) \leq \frac{1}{2}.
\]

Making use of (5.13) in (5.12), we obtain, for all large \( k, \)

\[
P_k \leq 2m_k \max_{1 \leq i \leq m n_k} \mathbb{P} \left( \mathcal{H}_n(z_{i,n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right)
\]

\[
= 2m_n \max_{1 \leq i \leq m n_k} \mathbb{P} \left( \Delta_{n_k}(z_{i,n_k}, h_{n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right)
\]

\[
\leq 4m_n \max_{1 \leq i \leq m n_k} \mathbb{P} \left( \Delta\Pi_{n_k}(z_{i,n_k}, h_{n_k}, \cdot) \notin \Gamma_{cf}(z_i) \right). \tag{5.14}
\]

The last inequality is a consequence of usual poissonization techniques (see, e.g., Mason [10], Fact 6). We now make use of Proposition 3.1 which, together with (5.2) leads to the following inequality, ultimately as \( k \to \infty, \)

\[
P_k \leq 4m_n \max_{1 \leq m_n} \exp \left( -\frac{f(z_{i,n_k})}{f(z_i)} (1 + 2\alpha z_i) \log n_k \right).
\]
Moreover (5.11) entails $\mathbb{P}_k \leq 4m_{n_k} \exp\left(-\left(1 + \alpha z_i\right) \log n_k\right)$. Since $m_{n_k} = n_k^{-1+o(1)} = n_k^{-1+o(1)}$ as $k \to \infty$ (recall (5.10)), the sumability of $\mathbb{P}_k$ follows, which proves (5.11) by the Borel-Cantelli lemma. We point out that (5.11) is true whatever the choice of $\delta > 0$ (recall (5.9)). We now focus on showing that, for a small value of $\delta > 0$ we have

$$\limsup_{k \to \infty} \sup_{z \in \mathcal{H}_i} \min_{1 \leq t \leq m_{n_k}} \max_{n \in \mathbb{N}} \| H_n(z, n_k, k) - \Delta_n(z, h_n, k) \| \leq 7\epsilon \ a.s., \quad (5.15)$$

which will be achieved through two separate lemmas.

**Lemma 5.2** Assume that the conditions of Theorem 2 are fulfilled. There exists $\delta_\epsilon > 0$ such that, for any choice of $0 < \delta < \delta_\epsilon$ we have almost surely

$$\limsup_{k \to \infty} \max_{n \in \mathbb{N}} \sup_{z \in C_{i,n_k}} \left\| H_n(z, n_k, k) - \frac{f(z)}{f(z, n_k)} H_n(z, k) \right\| \leq \epsilon.$$

**Proof:** For all large $k$ we have

$$\mathbb{P}\left( \max_{n \in \mathbb{N}} \max_{1 \leq t \leq m_{n_k}} \sup_{z \in C_{i,n_k}} \left\| H_n(z, n_k, k) - \frac{f(z)}{f(z, n_k)} H_n(z, k) \right\| > \epsilon \right)$$

$$= \mathbb{P}\left( \bigcup_{1 \leq t \leq m_{n_k}} \bigcup_{n \in \mathbb{N}} \sup_{z \in C_{i,n_k}} \left\| H_n(z, n_k, k) - \frac{f(z)}{f(z, n_k)} H_n(z, k) \right\| > \epsilon \right)$$

$$\leq m_{n_k} \max_{1 \leq m_{n_k}} \mathbb{P}\left( \bigcup_{n \in \mathbb{N}} \sup_{z \in C_{i,n_k}} \left\| H_n(z, n_k, k) - \frac{f(z)}{f(z, n_k)} H_n(z, k) \right\| > \epsilon \right) \quad (5.16)$$

Fix $k \geq 1, 1 \leq i \leq m_{n_k}$ and $z = z_{i,n_k} + (\delta h_{n_k})^{1/d}(0,1)^d$. We write $z_{i,n_k} := (z_{1,n_k}, \ldots, z_{d,n_k})$, $z := (z_1, \ldots, z_d)$ and $Z_j := (Z_{j,1}, \ldots, Z_{j,d})$, $j \geq 1$. Notice that for each $p = 1, \ldots, d$ we have $z_{i,n_k}^p \leq z^p \leq z_{i,n_k}^p + (\delta h_{n_k})^{1/d}$. Hence, in virtue of the equality $|1_A - 1_B| = 1_{A-B} + 1_{B-A}$ we have, for each integer $j$ we have almost surely, for each $(s_1, \ldots, s_d) \in [0,1)^d$,

$$\left| \frac{1_{[0,s)}}{h_{n_k}^{1/d}} - 1_{[0,s)]} \left( \frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}} \right) \right|$$

$$= 1 \{ \left[z, z + h_{n_k}^{1/d}s\right] - \left[z_{i,n_k}, z_{i,n_k} + (\delta h_{n_k})^{1/d}s\right] \} \left(Z_j\right) + 1 \{ \left[z_{i,n_k}, z_{i,n_k} + (\delta h_{n_k})^{1/d}s\right] - \left[z, z + h_{n_k}^{1/d}s\right] \} \left(Z_j\right)$$

$$\leq \sum_{i=1}^{d} 1 \left[z_{i,n_k}^{p,s} + (\delta h_{n_k})^{1/d}(s_{i,1})^{1/d} \right] \left(Z_j^{i} \right) \prod_{1 \leq p \neq i \leq d} 1 \left[z_{i,n_k}^{p,s}, z_{i,n_k}^{p,s} + h_{n_k}^{1/d}(s_{p,1})^{1/d} \right] \left(Z_j^{p} \right)$$

$$+ \sum_{i=1}^{d} 1 \left[z_{i,n_k}^{p,s} + (\delta h_{n_k})^{1/d}s_{i,1} \right] \left(Z_j^{i} \right) \prod_{1 \leq p \neq i \leq d} 1 \left[z_{i,n_k}^{p,s}, z_{i,n_k}^{p,s} + h_{n_k}^{1/d} \right] \left(Z_j^{p} \right) \quad (5.17)$$

$$= : X_{j,k,i,\delta}(s). \quad (5.18)$$
Here (5.17) follows from \( z_{i,n_k} \leq z_i^l \leq z_{i,n_k}^l + \delta^{1/d} n_k^{1/d} \), \( l = 1, \ldots, d \). As the \( X_{j,k,i,\delta} (\cdot) \) are positive processes almost surely, (5.18) entails, for all large \( k \) and for all \( 1 \leq i \leq m_n \),

\[
\mathbb{P} \left( \bigcup_{n \in N_k} \sup_{z \in C_{l,n_k}} \left| \mathcal{H}_n (z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n (z, \cdot) \right| > \epsilon \right) 
\leq \mathbb{P} \left( \bigcup_{n \in N_k} \sup_{z \in C_{l,n_k}} \sup_{s \in (0,1)^d} \sum_{j=1}^n \left| 1_{[0,s)} \left( \frac{Z_j - z}{h_{n_k}^{1/d}} \right) - 1_{[0,s)} \left( \frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}} \right) \right| \geq \epsilon c f(z_{i,n_k}) \log n_k \right)
\leq \mathbb{P} \left( \sum_{j=1}^n X_{j,k,i,\delta} (s) \geq \epsilon c f(z_{i,n_k}) \log n_k \right)
\leq \mathbb{P} \left( \left| \sum_{j=1}^n X_{j,k,i,\delta} (\cdot) \right| \geq \epsilon c f(z_{i,n_k}) \log n_k \right). \quad (5.19)
\]

But a close look at (5.17) leads to the conclusion that, almost surely, for each \( s \in [0,1]^d \),

\[
0 \leq \sum_{j=1}^n X_{j,k,i,\delta} (s) \leq 2d c f(z_{i,n_k}) \log n_k \sup_{s,s' \in [0,2]^d, \|s' - s\|_d < \delta} \left| \Delta_{n_k} (z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k} (z_{i,n_k}, h_{n_k}, s) \right| \quad (5.20)
\]

Here we have written \( |s| := \max \{|s_j|, j = 1, \ldots, p\} \). Now (5.20) together with (5.19) entails

\[
\frac{1}{2} \mathbb{P} \left( \bigcup_{n \in N_k} \sup_{z \in C_{l,n_k}} \left| \mathcal{H}_n (z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n (z, \cdot) \right| > \epsilon \right) 
\leq \frac{1}{2} \mathbb{P} \left( \sup_{s,s' \in [0,2]^d, \|s' - s\|_d < \delta^{1/d}} \left| \Delta_{n_k} (z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k} (z_{i,n_k}, h_{n_k}, s) \right| > \frac{\epsilon}{2d} \right)
\leq \mathbb{P} \left( \sup_{s,s' \in [0,1]^d, \|s' - s\|_d < \delta^{1/d}} \left| \Delta_{n_k} (z_{i,n_k}, h_{n_k}, 2s') - \Delta_{n_k} (z_{i,n_k}, h_{n_k}, 2s) \right| > \frac{\epsilon}{2d} \right) \quad (5.21)
\]

Here (5.21) follows from poissonization techniques. Now consider the following sequence \( \theta_n := 2^d h_n \), \( n \geq 1 \). Clearly, \( (\theta_n)_{n \geq 1} \) satisfies (HVE1) and (HVE2),
replacing $c$ by $c := 2^dc$. Moreover, for each $k \geq 1$, $1 \leq i \leq m_k$ we have almost surely, for all $s \in [0,1)^d$,
\[
\Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s) = \Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, s).
\] (5.22)

Applying Proposition 3.1 we deduce that the triangular array of processes
\[
U_{k,i}(\cdot) := \Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2\cdot), k \geq 1, 1 \leq i \leq m_k
\]
satisfies the ULDP in $(B([0,1)^d), || \cdot ||)$ (see §2) for the rate function $I$ and for the following triangular array:
\[
\epsilon_{k,i} := (c2^d f(z_{i,n_k}) \log n_k)^{-1}k \geq 1, 1 \leq i \leq m_k.
\]

Now consider the following set
\[
\Gamma := \left\{ g \in \mathcal{M}([0,1)^d), I(g) \leq \frac{4}{2^d c \beta} \right\}.
\]

By proposition 2.1, there exists $\delta_0 > 0$ such that
\[
\sup_{g \in 2\mathcal{G}} \sup_{s,s' \in [0,2)^d, || s' - s ||_d \leq \delta_0 / 2} | g(s') - g(s) | < (4d)^{-1} \epsilon. \tag{5.23}
\]

Now choose $0 < \delta < \delta_0$ arbitrarily for the construction of the $z_{i,n_k}$, $k \geq 1, 1 \leq i \leq m_k$ (recall (5.9)). By lower-semicontinuity of $I$, the closed set
\[
F := \left\{ g \in \mathcal{M}([0,2)^d), \inf_{g' \in \Gamma} \| g - g' \|_{[0,2)^d} \geq \frac{2^{-d} \epsilon}{8d} \right\}
\]
satisfies $I(F) > 4/(2^d c \beta)$. Hence, (5.21) together with (5.23) leads to the following inequalities for all large $k$ and for each $1 \leq i \leq m_k$.

\[
\begin{align*}
\mathbb{P} \left( \bigcup_{n \in N_k} \sup_{z_{i,n_k} \in \mathcal{C}_{i,n_k}} \left| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right| > \epsilon \right) \\
\leq 2\mathbb{P} \left( \sup_{s,s' \in [0,1)^d, || s' - s ||_d < \delta / 2} \left| U_{k,i}(s') - U_{k,i}(s) \right| > \epsilon(2d)^{-1} \right) \\
\leq 2\mathbb{P} (\Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, \cdot) \in F) \\
\leq 2 \exp \left( - \frac{3}{4} I(F) \epsilon f(z_{i,n_k}) \log n_k \right) \\
\leq 2 \exp \left( - 3 \frac{c2^d f(z_{i,n_k}) \log n_k}{\beta 2^d} \right) \\
\leq 2 \exp \left( - 3 \log n_k \right). \tag{5.24}
\end{align*}
\]
Now (5.24) in combination with (5.16) entails, for all large $k$,

$$
\mathbb{P} \left( \max_{n \in N_k} \max_{1 \leq i \leq m_{n_k}} \max_{z \in C_{i,n_k}} \left\| H_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} H_n(z, \cdot) \right\| > \epsilon \right) \leq \frac{2m_{n_k}}{n_k^3} \quad (5.25)
$$

But for fixed $\delta > 0$ we have $m_{n_k} = n_k^{-1+o(1)} = n_k^{1+o(1)}$ as $k \to \infty$. The proof of Lemma 5.2 is concluded by applying the Borel-Cantelli lemma to (5.25). \[\Box\]

**Lemma 5.3** Under the assumptions of Theorem 2 for any choice of $\delta > 0$, we have almost surely

$$
\limsup_{k \to \infty} \max_{1 \leq i \leq m_{n_k}} \max_{z \in C_{i,n_k}} \left\| \Delta_n(z, h_n, \cdot) - \frac{f(z)}{f(z_{i,n_k})} H_n(z, \cdot) \right\| \leq 6\epsilon.
$$

**Proof:** For all large $k$ and for all $1 \leq i \leq m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$ we have almost surely, for each $s \in [0,1]^d$,

$$
\Delta_n(z, h_n, s) = T_{n,i,k} \frac{f(z)}{f(z_{i,n_k})} H_n(z, \rho_{n,k} s), \quad (5.26)
$$

with $T_{n,i,k} : f(z_{i,n_k}) \log n_k/f(z) \log n$ and $\rho_{n,k}^d := h_{n_k}/h_n$. First notice that

$$
\lim_{k \to \infty} \max_{1 \leq i \leq m_{n_k}} \max_{z \in C_{i,n_k}} |T_{n,i,k} - 1| = 0, \quad \lim_{k \to \infty} \max_{n \in N_k} |\rho_{n,k} - 1| = 0.
$$

Moreover, by Proposition 2.1 we have

$$
\lim_{T \to 1, \rho \to 1} \sup_{g \in \Gamma_{\epsilon f(z_{i,k})}} \| T g(\rho^1 \cdot) - g(\cdot) \| = 0. \quad (5.27)
$$

Finally, by (5.11) and by Lemma 5.2 we have, for all large $k$ and for all $1 \leq i \leq m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$,

$$
\inf_{g \in \Gamma_{\epsilon f(z_{i,k})}} \left\| \frac{f(z)}{f(z_{i,n_k})} H_n(z, \cdot) - g \right\| < 3\epsilon \quad \text{almost surely.} \quad (5.28)
$$

Hence, combining (5.26), (5.27), (5.27), (5.28) and the triangle inequality, we obtain almost surely, for all large $k$ and for all $n \in N_k$:

$$
\left\| \Delta_n(z, h_n, \cdot) - \frac{f(z)}{f(z_{i,n_k})} H_n(z, \cdot) \right\| \leq 6\epsilon,
$$

which proves Lemma 5.3 \[\Box\]

**End of the proof of part(ii) of Theorem 2** By combining Lemma 5.3 with Lemma 5.2 we conclude that (5.15) is true for $\delta > 0$ small enough. Now (5.15) together with (5.11) leads to

$$
\limsup_{n \to \infty} \sup_{z \in H_{z_1}} \inf_{g \in \Gamma_{\epsilon f(z_{i,k})}} \| \Delta_n(z, h_n, \cdot) - g \| \leq 9\epsilon \quad \text{almost surely.}
$$
Whence, recalling (5.5),
\[
\limsup_{n \to \infty} \sup_{z \in H_{l}} \inf_{g \in \Gamma_{f(z)}} \| \Delta_n(z, h_n, \cdot) - g \| \leq 10\epsilon \text{ almost surely.} \tag{5.29}
\]
Repeating (5.29) for each \( l = 1, \ldots, L \) (recall (5.7)) we get
\[
\limsup_{n \to \infty} \sup_{z \in H} \inf_{g \in \Gamma_{f(z)}} \| \Delta_n(z, h_n, \cdot) - g \| \leq 10\epsilon \text{ almost surely.}
\]
As \( \epsilon > 0 \) was chosen arbitrarily, the proof of part (ii) of Theorem 2 is concluded. \( \square \)

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