Differential Rings from Special Kähler Geometry

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Abstract. We study triples of graded rings defined over the deformation spaces for certain one-parameter families of Calabi-Yau three-folds. These rings are analogues of the rings of modular forms, quasi modular forms and almost-holomorphic modular forms. We also discuss some of their applications in solving the holomorphic anomaly equations.

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1. Introduction

Modular forms are very interesting quantities in mathematics and physics due to their nice behaviors under the transformations in the corresponding modular groups. Given a genus zero subgroup $\Gamma$ of finite index of the full modular group $\Gamma(1) = \text{PSL}(2, \mathbb{Z})$, the generators of the ring of modular forms for $\Gamma$ could be obtained starting from some $\theta$ or $\eta$ functions or Eisenstein series. Alternatively, one could parameterize the modular curve $X_\Gamma = \Gamma \backslash \mathcal{H}^*$ by a Hauptmodul $\alpha$ (generator of the rational functional field of the modular curve), and consider the periods
which are solutions to the Picard-Fuchs equation attached to the elliptic curve family \( \pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma \) constructed from the modular group \( \Gamma \). Knowing the relation between the Hauptmodul \( \alpha \) and the normalized period \( \tau \) of the elliptic curve family, one could then obtain the graded differential ring of quasi modular forms \( \tilde{M}_* (\Gamma) \) by taking successive derivatives of the periods with respect to \( \tau \). This ring \( \tilde{M}_* (\Gamma) \) includes the ring of modular forms \( M_* (\Gamma) \) and contains further elements which are not modular but quasi modular \([1]\). The quasi modular forms in \( \tilde{M}_* (\Gamma) \) could be completed to modular forms by adding some non-holomorphic parts. Then one gets the graded differential ring of almost-holomorphic modular forms \( \hat{M}_* (\Gamma) \). See \([1, 2]\) and references therein for details about the construction.

Similarly, as we shall show in this work, for certain one-parameter Calabi-Yau three-fold family \( \pi : \mathcal{X} \to \mathcal{M} \), we can define a triple of graded rings \((R, \tilde{R}, \hat{R})\). The analogue of the modular variable \( \tau \) is constructed using the special Kähler geometry \([3]\) on the base \( \mathcal{M} \), and the periods are solved from the Picard-Fuchs equation of the Calabi-Yau family. This triple share similar structures and operations with the triple \((M_* (\Gamma), \tilde{M}_* (\Gamma), \hat{M}_* (\Gamma))\) defined for the elliptic curve family \( \pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma \). In particular, they have gradings which play the role of modular weights.

These rings appeared in various forms in the studies of BCOV holomorphic anomaly equations \([1, 2]\) and topological string partition functions for Calabi-Yau three-fold families \([3, 6, 7, 8]\) (see also \([9, 10, 11, 12, 13]\) for related works). In particular, the ring \( \tilde{R} \) is the ring constructed by Yamaguchi-Yau in \([6]\) (see also \([7]\)). In the recent work \([8]\), it was shown that for some special one-parameter Calabi-Yau three-fold families, the normalized topological string partition functions are polynomials in the generators of the ring \( \hat{R} \) and have weight zero.

For some particular one-parameter families of non-compact Calabi-Yau three-folds with an elliptic curve sitting inside each fiber \([14, 15, 16, 17, 18]\), the base \( \mathcal{M} \) of the Calabi-Yau three-fold family \( \pi : \mathcal{X} \to \mathcal{M} \) could be identified with a modular curve \( X_\Gamma \) \([8]\) (see also \([19, 10, 20]\)). The triple of rings \((R, \tilde{R}, \hat{R})\) we shall define are closely related to the known graded rings \((M_* (\Gamma), \tilde{M}_* (\Gamma), \hat{M}_* (\Gamma))\). This then allows one to express the coefficients and derivatives in the holomorphic anomaly equations purely in the language of modular form theory. Using the polynomial recursion technique \([5, 6, 7]\), one can then prove \([8]\) by induction that the topological partition functions, as solutions to the holomorphic anomaly equations, are almost-holomorphic modular forms of weight 0. Moreover, for each of these particular non-compact Calabi-Yau three-fold families, the boundary conditions, used in e.g., \([3, 6, 21, 9]\) to solve the topological string partition functions, become regularity conditions \([8]\) at different cusps on the modular curve for the holomorphic limits of the topological string partition functions which are quasi modular forms. Then, by using the Fricke involution (also called Atkin–Lehner involution) which is an automorphism of the corresponding modular curve, one could translate the regularity conditions at different cusps to some conditions on the \( q_\tau = \exp 2\pi i \tau \) expansions of the quasi modular forms. In this way, solving the holomorphic anomaly equations becomes a purely mathematical problem in which one needs to solve for certain quasi modular forms from the recursive equations and the regularity conditions. In \([8]\), the first few topological string partition functions for these special Calabi-Yau three-fold families were obtained genus by genus, and were expressed in terms of
generators of the ring of almost-holomorphic modular forms.

Experiences from dealing with the non-compact Calabi-Yau three-folds suggest that studying the arithmetic properties of the rings $\mathcal{R}, \mathcal{R}, \hat{\mathcal{R}}$ might be useful in solving the holomorphic anomaly equations for more general Calabi-Yau families. This is the motivation of this work.

This paper is based on [8] and follows closely the lines of thoughts in [12]. In [12] some differential rings were constructed using special Kähler geometry, and the parallelism to the quasi modular forms and almost-holomorphic modular forms was made. The main difference between our present work and [12] is that in the latter work the rings were constructed from the rings of propagators [5, 6, 7], while we construct the rings from the periods and Yukawa couplings solved from the Picard-Fuchs equations in such a way that the analogue between these rings and the rings of modular objects is more explicit. Many of the discussions in the current work are inspired by the examples discussed in [12].

The structure of this paper is as follows. In section 2, we shall review the preliminaries about modular curves, modular forms, Picard-Fuchs equations, special Kähler geometry and BCOV holomorphic anomaly equations, and set up the notations. In section 3, first we reproduce the details in constructing the graded rings $M_{\tau}(\Gamma), \tilde{M}_{\tau}(\Gamma), \hat{M}_{\tau}(\Gamma)$ for certain elliptic curve families. Then we construct by analogue the graded rings $\mathcal{R}, \tilde{\mathcal{R}}$ for certain one-parameter Calabi-Yau three-fold families $\pi : X \to \mathcal{M}$. In section 4, we consider these rings using the properties of the special Kähler geometry on the deformation space $\mathcal{M}$. We make use of the canonical coordinates and holomorphic limits to lift the ring $\tilde{\mathcal{R}}$ to a non-holomorphic ring $\hat{\mathcal{R}}$, by relating them to the ring constructed by Yamaguchi-Yau [6]. The main results of this paper are summarized in section 4.5. Some of their applications in solving the BCOV holomorphic anomaly equations are also discussed in section 4. We conclude with some discussions and questions in section 5.

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2. Preliminaries and background

To make the paper self-contained, in this section we give a review of some basic facts about modular groups and modular curves, modular forms, Picard-Fuchs equations, special Kähler geometry and BCOV holomorphic anomaly equations.
2.1. Modular groups and modular curves. The generators and relations for the group \( \text{SL}(2, \mathbb{Z}) \) are given by the following:

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = -I, \quad (ST)^3 = -I.
\]

The groups we are interested in in this paper are the Hecke subgroups of \( \Gamma(1) = \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm \text{I}\} \):

\[
(2.1) \quad \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| c \equiv 0 \mod N \right\} < \Gamma(1),
\]

with \( N = 2, 3, 4 \). A further subgroup that we will consider is the unique normal subgroup in \( \Gamma(1) \) of index 2. It is generated by \( T^2, T^{-1}S \) and is often denoted by \( \Gamma_0(1)^* \). By abuse of notation, we write it as \( \Gamma_0(N) \) with \( N = 1^* \) when listing it together with the groups \( \Gamma_0(N) \).

The group \( \text{SL}(2, \mathbb{Z}) \) acts on the upper half plane \( \mathcal{H} = \{ \tau \in \mathcal{H} \big| \text{Im} \tau > 0 \} \) by fractional linear transformations:

\[
(2.2) \quad \tau \mapsto \gamma \tau = \frac{a \tau + b}{c \tau + d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).
\]

The quotient space \( Y_0(N) = \Gamma_0(N) \setminus \mathcal{H} \) is a non-compact orbifold with certain punctures corresponding to the cusps and orbifold points corresponding to the elliptic points of the group \( \Gamma_0(N) \). By filling the punctures, one then gets a compact orbifold \( X_0(N) \) which can be equipped with the structure of a Riemann surface. The signature for the group \( \Gamma_0(N) \) and the two orbifolds \( Y_0(N), X_0(N) \) could be represented by \( \{ p, \mu; \nu_2, \nu_3, \nu_\infty \} \), where \( p \) is the genus of \( X_0(N) \), \( \mu \) is the index of \( \Gamma_0(N) \) in \( \Gamma(1) \), and \( \nu_i \) are the numbers of \( \Gamma_0(N) \)-equivalent elliptic fixed points or parabolic fixed points of order \( i \). The signatures for the groups \( \Gamma_0(N), N = 1^*, 2, 3, 4 \) are listed in the following table (see e.g. [22]):

| \( N \) | \( \nu_2 \) | \( \nu_3 \) | \( \nu_\infty \) | \( \mu \) | \( p \) |
|-------|-------|-------|-------|-------|-------|
| 1^*  | 0     | 1     | 2     | 2     | 0     |
| 2     | 1     | 0     | 2     | 3     | 0     |
| 3     | 0     | 1     | 2     | 4     | 0     |
| 4     | 0     | 0     | 3     | 6     | 0     |

The space \( X_0(N) \) is called a modular curve. When \( N = 1^*, 2, 3, 4 \), it has three singular points corresponding to the above \( \Gamma_0(N) \)-equivalent fixed points. There is an action called Fricke involution

\[
(2.3) \quad W_N : \tau \mapsto -\frac{1}{N \tau}
\]

on the modular curve \( X_0(N) \). It exchanges the two cusp classes \( [i \infty] \) and \([0]\), while fixing the fixed point.

More details about the basic theory could be found in e.g. [22, 23, 2].

\[\text{We use the notation } [\tau] \text{ to denote the equivalence class of } \tau \in \mathcal{H}^* \text{ under the group action of } \Gamma \text{ on } \mathcal{H}^*.\]
2.2. Modular forms. We proceed by recalling some basic concepts in modular form theory following [1] (see also [2] for more details). In the following, we shall use the notation $\Gamma$ for a general subgroup of finite index in $\Gamma(1)$. In particular, we can take $\Gamma$ to be the modular groups $\Gamma_0(N)$ described above.

A modular form of weight $k$ with respect to the group $\Gamma$ is defined to be a holomorphic function $f$ on $\mathcal{H}$ satisfying
\[(2.4) \quad f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau), \quad \forall \tau \in \mathcal{H}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma.\]
and growing at most polynomially in $\frac{1}{\text{Im} \tau}$ as $\frac{1}{\text{Im} \tau} \to 0$. One can also define modular forms with characters, see for example [23] for details. The space of holomorphic modular forms for $\Gamma$ forms a graded ring and is denote by $M^*_\Gamma(\Gamma)$.

A quasi modular form of weight $k$ with respect to the group $\Gamma$ is a holomorphic function satisfying the same growth conditions but with the transformation properties replace by: there exist holomorphic functions $f_i, i = 0, 1, 2, 3, \ldots, k - 1$ such that
\[(2.5) \quad f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) + \sum_{i=0}^{k-1} c^i(c\tau + d)^{k-i} f_i(\tau), \quad \forall \tau \in \mathcal{H}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma.\]
The space of quasi modular forms for $\Gamma$ forms a graded differential ring and is denote by $\tilde{M}^*_\Gamma(\Gamma)$.

An almost-holomorphic modular form is a function $f(\tau, \bar{\tau})$ on $\mathcal{H}$ which satisfies the same growth conditions as above and transforms as a modular form of weight $k$:
\[(2.6) \quad f(\frac{a\tau + b}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau, \bar{\tau}), \quad \forall \tau \in \mathcal{H}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma.\]
It can be written in the form \[(2.7) \quad f(\tau, \bar{\tau}) = \sum_{m=-[k/2]}^{m=[k/2]} f_m(\tau) Y^m, \quad Y = \frac{1}{\text{Im} \tau} \]
where $f_m(\tau)$ are holomorphic functions on $\mathcal{H}$ for $m = 0, 1, 2, \ldots, [k/2]$. The space of almost-holomorphic modular forms for $\Gamma$ forms a graded differential ring and is denote by $\tilde{M}_\Gamma(\Gamma)$.

As shown in [1], one has the ring isomorphism $\tilde{M}_\Gamma(\Gamma) = M_\Gamma(\Gamma) \otimes \mathbb{C}[E_2]$, where $E_2$ is the Eisenstein series defined by
\[E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k) q^k, \quad q = e^{2\pi i \tau}, \quad \sigma_1(k) = \sum_{d|k} d.\]
Moreover, there is a ring isomorphism $\tilde{M}_\Gamma(\Gamma) \to \tilde{M}_\Gamma(\Gamma)$ defined by $f(\tau, \bar{\tau}) \mapsto f_0(\tau)$, where $f_0$ is the function in [24]. If one regards $Y$ as a formal variable, then this is the “constant term map” obtained by taking the limit $Y = \frac{1}{\text{Im} \tau} \to 0$ (which could be induced from $\bar{\tau} \to \infty$, by thinking of $\bar{\tau}$ as a complex coordinate independent of $\tau$). The inverse map takes a quasi modular form to an almost-holomorphic modular form, we shall call this map “modular completion” in this paper.
Example 2.2.1. Take the group $\Gamma$ to be the full modular group $\Gamma(1) = \text{PSL}(2, \mathbb{Z})$. Then $\mathcal{M}_*(\Gamma(1)) = \mathbb{C}[E_4, E_6]$, where $E_4, E_6$ are the familiar Eisenstein series defined by

\[
E_4(\tau) = 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k)q^k, \quad q = e^{2\pi i \tau}, \quad \sigma_3(k) = \sum_{d|k} d^3, 
\]

\[
E_6(\tau) = 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k)q^k, \quad q = e^{2\pi i \tau}, \quad \sigma_5(k) = \sum_{d|k} d^5.
\]

The Eisenstein series $E_2$ is a quasi modular form for $\Gamma(1)$ since it transforms according to

\[
E_2(a\tau + b \atop c\tau + d) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i} c (c\tau + d), \quad \forall \tau \in \mathcal{H}, \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(1).
\]

Recall that

\[
\frac{1}{\text{Im} (a\tau + b \atop c\tau + d)} = (c\tau + d)^2 \frac{1}{\text{Im} \tau} - 2 i c (c\tau + d), \quad \forall \tau \in \mathcal{H}, \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(1),
\]

we know the modular completion of the quasi modular form $E_2(\tau)$ is

\[
\hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \text{Im} \tau}.
\]

Then $\mathcal{M}_*(\Gamma(1)) = \mathbb{C}[E_2, E_4, E_6]$, $\hat{\mathcal{M}}_*(\Gamma(1)) = \mathbb{C}[\hat{E}_2, E_4, E_6]$. The latter two carry differential structures given by

\[
DE_2 = \frac{1}{12} (E_2^2 - E_4), \quad DE_4 = \frac{1}{3} (E_2 E_4 - E_6), \quad DE_6 = \frac{1}{2} (E_2 E_6 - E_4^2),
\]

\[
\hat{D}\hat{E}_2 = \frac{1}{12} (\hat{E}_2^2 - \hat{E}_4), \quad \hat{D}\hat{E}_4 = \frac{1}{3} (\hat{E}_2 \hat{E}_4 - \hat{E}_6), \quad \hat{D}\hat{E}_6 = \frac{1}{2} (\hat{E}_2 \hat{E}_6 - \hat{E}_4^2),
\]

where $D = \frac{1}{2\pi i} \frac{d}{d\tau} : \hat{\mathcal{M}}_k(\Gamma(1)) \rightarrow \hat{\mathcal{M}}_{k+2}(\Gamma(1))$ and $\hat{D} = \hat{D} = \frac{1}{12} : \pi_{k} : \hat{\mathcal{M}}_k(\Gamma(1)) \rightarrow \hat{\mathcal{M}}_{k+2}(\Gamma(1))$.

2.3. Picard-Fuchs equations for families of Calabi-Yau manifolds. Throughout this work, we do not aim to give a complete discussion of all Picard-Fuchs equations for Calabi-Yau families. Instead, the examples we shall study are the Picard-Fuchs equations for some special Calabi-Yau families given in (2.9), (2.10) and (2.11) below. These equations are considered in e.g., [14, 24, 25, 15, 16, 17] in which the arithmetic properties like the integrality of the mirror maps and modular relations are studied. More general Picard-Fuchs equations attached to Calabi-Yau families are considered in e.g., [24, 25, 26, 27, 28, 29].

Consider a family $\pi : \mathcal{X} \rightarrow \mathcal{M}$ of Calabi-Yau $n$-folds $\mathcal{X} = \{ \mathcal{X}_z \}$ over a variety $\mathcal{M}$ parametrized by the complex coordinate system $z = \{ z^i \}_{i=1,2,\ldots, \dim \mathcal{M}}$. For a generic $z \in \mathcal{M}$, the fiber $\mathcal{X}_z$ is a smooth Calabi-Yau $n$-fold. We also assume that $\dim \mathcal{M} = h^1(\mathcal{X}_z, T_{\mathcal{X}_z})$ for a smooth $\mathcal{X}_z$, where $T_{\mathcal{X}_z}$ is the holomorphic tangent bundle of $\mathcal{X}_z$. The periods are defined to be the integrals $\int_C \Omega_z$, where $C \in H_n(\mathcal{X}_z, \mathbb{Z})$ and $\Omega = \{ \Omega_z \}$ is a holomorphic section of the Hodge line bundle $\mathcal{L} = \mathcal{R}^n \pi_* \Omega^1_{\mathcal{M}} \otimes \mathcal{O}_{\mathcal{M}}$. They satisfy a differential equation called the Picard-Fuchs equation induced from the Gauss-Manin connection on the Hodge bundle $\mathcal{H} = \mathcal{R}^n \pi_* \mathbb{C} \otimes \mathcal{O}_{\mathcal{M}}$. In the following, we shall use the notation $X$ to denote a generic fiber $\mathcal{X}_z$ in the family without specifying the point $z$ and use $\Pi$ to denote a period. We will also call the
base \(\mathcal{M}\) the deformation space (of complex structures of \(X\)).

The families of Calabi-Yau one-folds that we shall focus on in this work are the elliptic curve families \(\pi_{\Gamma_0(N)} : E_{\Gamma_0(N)} \rightarrow X_0(N) = \Gamma_0(N)\backslash \mathcal{H}^*\) with \(N = 1^*, 2, 3, 4\), where \(E_{\Gamma_0(N)}\) is the surface \([30, 31]\) defined by

\[
(2.8) \quad E_{\Gamma_0(N)} := (\Gamma_0(N) \times \mathbb{Z}^2) \setminus (\mathcal{H}^* \times \mathbb{C}) ,
\]

for \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (m, n) \) : \((\tau, z) \mapsto \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}, \frac{z + m\tau + n}{c\tau + d}\), \(\forall \gamma \in \Gamma_0(N)\).

The explicit equations, \(j\)-invariants and Picard-Fuchs operators of these families could be found in e.g., \([14, 15]\). In the following we shall only display the Picard-Fuchs operators

\[
(2.9) \quad \mathcal{L}_{\text{elliptic}} = \theta^2 - \alpha(\theta + 1/r)(\theta + 1 - 1/r), \quad \theta = \alpha \frac{\partial}{\partial \alpha},
\]

where \(r = 6, 4, 3, 2\) for \(N = 1^*, 2, 3, 4\), respectively. The parameter \(\alpha\) is the complex coordinate on the deformation space \(\mathcal{M}\) in which the Picard-Fuchs equation takes the above particular form. Thinking of the base space \(\mathcal{M}\) as the genus zero modular curve \(X_0(N)\), it is then a modular function (called Hauptmodul) for the modular group \(\Gamma_0(N)\). Each of these Picard-Fuchs equations has three regular singularities located at \(\alpha = 0, 1, \infty\) on the corresponding modular curve. The two points \(\alpha = 0, 1\) are the cusp classes \([i\infty], [0]\) respectively, and are exchanged by the Fricke involution \(W_N\) given in \([2, 3]\).

We shall also consider the Picard-Fuchs equations for the mirror families of the \(K_{p_2}\) and \(K_{dP_n}, n = 5, 6, 7, 8\), where \(dP_n\) is the del Pezzo surface obtained from blowing up \(\mathbb{P}^2\) at \(n\) points. We take the Kahler structures of the non-compact Calabi-Yau three-folds \(K_{p_2}, K_{dP_n}, n = 5, 6, 7, 8\) to be the ones induced from the hyperplane classes on the \(\mathbb{P}^2\). The explicit equations and Picard-Fuchs equations of these families are given in \([15, 17]\). The Picard-Fuchs equations have the following form:

\[
(2.10) \quad \mathcal{L}_{\text{CY}} = \mathcal{L}_{\text{elliptic}} \circ \theta = (\theta^2 - \alpha(\theta + 1/r)(\theta + 1 - 1/r)) \circ \theta ,
\]

where \(\mathcal{L}_{\text{elliptic}}\) and \(\theta\) are the same as those described in \((2.9)\).

To make things more precise, we shall discuss the mirror Calabi-Yau three-fold family \(\pi : X \rightarrow \mathcal{M}\) of the \(K_{p_2}\) family, constructed in \([17, 18]\). We refer the interested readers to \([10, 8]\) and references therein for the detailed discussion on other families. For each \(\alpha\) on \(\mathcal{M}\), the fiber \(X_\alpha\) of the non-compact Calabi-Yau threefold family is a conic fibration given by

\[
X_\alpha : uv - H(x, y; \alpha) = 0, \quad (u, v, x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2, \]

where \(H(x, y, \alpha) = y^2 - (x + 1)y - \alpha x^3\). The degeneration locus of this conic fibration is the elliptic curve \(E_\alpha : H(x, y; \alpha) = 0\). Then as \(\alpha\) moves in \(\mathcal{M}\), one gets an elliptic curve family \(\pi : \mathcal{E} \rightarrow \mathcal{M}\). It turns out that this elliptic curve family is 3-isogenous to the Hesse cubic curve family \(\pi_{\Gamma_0(3)} : E_{\Gamma_0(3)} \rightarrow X_0(3)\), see e.g., \([22]\) for details. Then we see \(\mathcal{M} \cong X_0(3)\). Moreover, the Picard-Fuchs equation for the Calabi-Yau three-fold family is \([16, 17]\)

\[
\mathcal{L}_{\text{CY}} = \mathcal{L}_{\text{elliptic}} \circ \theta = (\theta^2 - \alpha(\theta + 1/3)(\theta + 2/3)) \circ \theta .
\]
Near the point $\alpha = 0$, one can find a basis of solutions to $L_{CY}$ given by $X^0 = 1$, $t \sim \log \alpha + O(\alpha^0)$, $F_t$, so that $\theta t, \frac{1}{2\pi i} \kappa^{-1} \theta F_t$ are the periods $\omega_0, \omega_1 = \tau \omega_0$ of the elliptic curve $E_\alpha$, respectively. Then

$$\tau = \frac{\omega_1}{\omega_0} = \frac{1}{2\pi i} \kappa^{-1} \theta F_t$$

where $\kappa = -\frac{1}{3}$ is the classical triple intersection of $K_{P^2}$. This quantity will prove to be the key of understanding the arithmetic properties of special Kähler geometry later.

The compact Calabi-Yau three-fold family that we shall study is the quintic mirror family \cite{33} whose Picard-Fuchs equation is given by

$$L = \theta^4 - \alpha (\theta + 1/5)(\theta + 2/5)(\theta + 3/5)(\theta + 4/5),$$

where $\alpha$ is related to the parameter $z$ in \cite{33} by $\alpha = \frac{5}{5} z$.

For the Picard-Fuchs equations in (2.10), (2.11), they all have three regular singularities located at $\alpha = 0, 1, \infty$ on the base $M$. The point $\alpha = 0$ on the deformation space $M$ plays a special role and is called the large complex structure limit of the Calabi-Yau three-fold family. The solutions to these equations could be obtained using the Frobenius method and are related to hypergeometric functions.

2.4. Special Kähler geometry on deformation spaces. The discussions on special Kähler geometry and holomorphic anomaly equations in this section apply to multi-parameter Calabi-Yau families.

Consider a family of Calabi-Yau three-folds $X$ given by $\pi : X \to M$ as above. The base (called deformation space above) is equipped with the Weil-Petersson metric whose Kähler potential $K$ is determined from

$$e^{-K(z, \bar{z})} = i \int_{X_z} \Omega_z \wedge \overline{\Omega}_z,$$

where $\Omega = \{\Omega_z\}$ is a section of the Hodge line bundle $L$. The metric $G_{ij} = \partial_i \partial_j K$ is the Hodge metric induced from the Hermitian metric $h(\Omega, \Omega) = i \int \Omega \wedge \overline{\Omega}$ on the Hodge line bundle $L$. This metric is called special Kähler metric \cite{3, 34}. Among its other properties, it satisfies the following “special geometry relation”

$$-R_{i j k l} = \delta_{i j} \Gamma_{k l}^{k} = \delta^{k}_{i} G_{i j} + \delta^{k}_{j} G_{i j} - C_{i m n} \bar{C}_{j k}^{m n k}, \quad i, j, k, l = 1, 2, \ldots \dim M,$$

where

$$C_{i j k}(z) = - \int_{X_z} \frac{\partial_i \partial_j \partial_k \Omega_z}{\partial_k \partial_i \partial_j \Omega_z}$$

is the so-called Yukawa coupling and $\bar{C}_{j k}^{m n} = e^{2K} G_{i j}^{k} G_{i j}^{m n} \bar{C}_{j k}^{m n k}$. There is a natural covariant derivative $D_i$ acting on sections of the Hodge bundle $\mathcal{H} = \mathcal{R}^3 \mathcal{P}_s \wedge \mathcal{O}_M$ which is the sum of the Chern connection associated to the Weil-Petersson metric and the connection on $L$ induced by the Hermitian metric $h$. See \cite{3, 34, 12} for details on this.

The Yukawa couplings will play a key role in the construction of the rings $(\mathcal{R}, \mathcal{R}, \bar{\mathcal{R}})$ in this paper.
2.5. **Holomorphic anomaly equations.** The genus $g$ topological string partition function $\mathcal{F}^{(g)}$ as defined in \[1\] is a section of the line bundle $\mathcal{L}^{2-2g}$ over $\mathcal{M}$, it is shown to satisfy the following holomorphic anomaly equation:

\[
\bar{\partial}_\tau \mathcal{F} = \frac{1}{2} C^i_{jk} \overline{\mathcal{F}}^i_k + (1 - \frac{\chi}{24}) G_{ji},
\]

\[
\bar{\partial}_\tau \mathcal{F}^{(g)} = \frac{1}{2} C^i_{jk} \left( \sum_{r=1}^{g-1} D_r \mathcal{F}^{(r)} D_{g-r} \mathcal{F}^{(g-r)} + D_j D_k \mathcal{F}^{(g-1)} \right), \quad g \geq 2,
\]

where $\chi$ is the Euler characteristic of the mirror manifold of the Calabi-Yau threefold $X$. Any higher genus $\mathcal{F}^{(g)}$ can be determined recursively from these up to addition by a holomorphic function.

A solution of these recursion equations is given in terms of Feynman rules \[5\]. The propagators $S^i$, $S^i$, $S$ for these Feynman rules are defined by the following:

\[
\partial_i S^{ij} = \overline{C}^i_k, \quad \partial_i S^j = G^i_{ij} S^{ij}, \quad \partial_i S = G^i_{ii} S^i.
\]

By definition, the propagators $S$, $S^i$ and $S^{ij}$ are sections of the bundles $\mathcal{L}^{-2} \otimes \text{Sym}^m T \mathcal{M}$ with $m = 0, 1, 2$, respectively. The vertices of the Feynman rules are given by the functions $\mathcal{F}^{(g)}_{i_1 \cdots i_n} = D_{i_1} \cdots D_{i_n} \mathcal{F}^{(g)}$.

In \[\[6\] \[7\] it was proved, using the special geometry relation (2.13) and the holomorphic anomaly equation for genus one (2.15), that the holomorphic anomaly equations for $g \geq 2$ can be put into the following form:

\[
\bar{\partial}_\tau \mathcal{F}^{(g)} = \bar{\partial}_\tau \mathcal{P}^{(g)}
\]

where $\mathcal{P}^{(g)}$ is a polynomial of the generators $S^{ij}, S^i, S, K_i$ with the coefficients being holomorphic functions. The non-holomorphicity of the topological string partition functions only comes from these generators and the anti-holomorphic derivative on the left-hand side of the holomorphic anomaly equations can be replaced by derivatives with respect to these generators. Moreover, the derivatives of the generators are given by \[7\]

\[
D_i S^{jk} = \delta^j_i S^k + \delta^k_i S^j - C_{imn} S^{mj} S^{nk} + h^i_{jk},
\]

\[
D_i S^j = -C_{imn} S^{m} S^{jn} + 2 \delta^j_i S + h^i_{jk} K_k + h^j_i,
\]

\[
D_i S = -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h^m_i K_m K_n + h^m_i K_m + h_i,
\]

\[
D_i K_j = -K_i K_j + C_{ijm} S^{mn} K_n - C_{ijm} S^m + h_{ij},
\]

where $h^i_{jk}, h^j_i, h_i, h_{ij}$ are holomorphic functions. These holomorphic functions cannot be uniquely determined since the above equations are derived by integrating some equations \[7 \[35\], hence (2.19) does not actually give a ring due to the existence of these holomorphic functions. To make it a real ring, one needs to include all of the derivatives of these holomorphic ambiguities \[12\]. In \[\[6\] \[8\], the holomorphic functions are packaged together by making use of the special Kähler geometry and are essentially parametrized by the Yukawa couplings. Then one gets a differential ring $\mathcal{R}$ with finite generators. It also has a nice grading serving as the “modular weights”.

Note that (2.18) only determines the quantity $\mathcal{F}^{(g)}$ up to addition by an holomorphic function $f^g$ called the holomorphic ambiguity

\[
\mathcal{F}^{(g)} = \mathcal{P}^g(S^{ij}, S^i, S, K_i) + f^g.
\]
Boundary conditions are needed to fix the holomorphic ambiguity $f^g$, these includes the boundary condition at the large complex structure limit and the gap condition at the conifold loci on $\mathcal{M}$ \cite{12, 13, 36, 9, 21}. Recently, in \cite{8} it was observed that for the one-parameter non-compact Calabi-Yau three-fold family given in (2.10), the deformation space $\mathcal{M}$ could be identified with certain modular curve $X_0(N)$, the large complex structure limit and conifold point are the two cusp classes $[i\infty]$ and $[0]$. Moreover, the ring $\hat{\mathcal{R}}$ of propagators is essentially equivalent to the ring of almost-holomorphic modular forms $\hat{\mathcal{M}}^*(\Gamma_0(N))$. It follows by polynomial recursion \cite{5, 6, 7} that the partition functions $F^g$ are almost-holomorphic modular functions (i.e., modular weights are zero). The boundary conditions then become regularity conditions of these modular objects at the cusps. This then allows one to get the topological string partition functions as almost-holomorphic modular functions by recursively solving the holomorphic anomaly equations with the boundary conditions genus by genus.

3. Differential rings from Picard-Fuchs equations

In this section, first we shall construct $M^*_\infty(\Gamma_0(N))$, $\tilde{M}^*_\infty(\Gamma_0(N))$ for the family $\pi_{\Gamma_0(N)} : E_{\Gamma_0(N)} \to X_0(N) = \Gamma_0(N)\backslash \mathcal{H}^*$ using its Picard-Fuchs equation. The procedure then motivates us to construct similar rings $\mathcal{R}, \tilde{\mathcal{R}}$ for one-parameter Calabi-Yau three-fold families $\pi : \mathcal{X} \to \mathcal{M}$.

3.1. Picard-Fuchs equations for elliptic curve families. For the elliptic curve families $\pi_{\Gamma_0(N)} : E_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$ whose Picard-Fuchs equations are given by \cite{20}, the rings of quasi modular forms can be constructed explicitly using periods of these elliptic curve families, see e.g., \cite{2, 37, 8} and references therein. More precisely, for the Picard-Fuchs equation \cite{20}, we define

$$\beta := 1 - \alpha,$$

and choose a basis of solutions to be

$$\omega_0 = 2F_1(1/r, 1 - 1/r, 1; \alpha), \quad \omega_1 = \frac{i}{\sqrt{N}} 2F_1(1/r, 1 - 1/r, 1; \beta),$$

then one has

$$\tau = \frac{\omega_1}{\omega_0} = \frac{i}{\sqrt{N}} \frac{2F_1(1/r, 1 + 1 - r, 1; \beta)}{2F_1(1/r, 1 + 1 - r, 1; \alpha)},$$

where $r = 6, 4, 3, 2$ for $N = 1^*, 2, 3, 4$ respectively. One then defines a triple following \cite{38, 39, 37},

$$A = \omega_0, \quad B = (1 - \alpha)^{\frac{1}{2}} A, \quad C = \alpha^{\frac{1}{2}} A.$$

Remark 3.1.1. These functions $A, B, C$ are possibly multi-valued on the modular curve $X_0(N)$ and have divisors being $\frac{1}{2}(\alpha = \infty), \frac{1}{2}(\alpha = 1), \frac{1}{2}(\alpha = 0)$, respectively. This fact is not used in this paper, but is useful \cite{8} in analyzing the singularities of the topological string partition functions as solutions to the holomorphic anomaly equations.

Note that the quantities $A, B, C$ satisfy the equation $A^\tau = B^\tau + C^\tau$. We now define further the quantity

$$E = \partial_\tau \log C^\tau B^\tau, \quad \partial_\tau := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}.$$
It turns out that the ring generated by $A, B, C, E$ is closed under the derivative $\partial_\tau$.

**Theorem 3.1.2.** For each of the elliptic curve families $\pi_{\Gamma_0(N)}: \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$ with $r = 6, 4, 3, 2$ respectively, the following identities hold:

$$
\begin{align*}
\partial_\tau A &= \frac{1}{2r} A(E + \frac{C^r - B^r}{A^r} A^2), \\
\partial_\tau B &= \frac{1}{2r} B(E - A^2), \\
\partial_\tau C &= \frac{1}{2r} C(E + A^2), \\
\partial_\tau E &= \frac{1}{2r} (E^2 - A^4).
\end{align*}
$$

(3.6)

**Proof.** It follows from (3.11), (3.12), (3.13) below. □

The ring generated by $A, B, C, E$ has an obvious grading denoted by $k$ below: the gradings assigned to $A, B, C, E$ are 1, 1, 1, 2, taking the derivative $\partial_\tau$ will increase the grading by 2. Similar to the full modular group case, one gets the following

**Theorem 3.1.3.** For each of the elliptic curve families $\pi_{\Gamma_0(N)}: \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$ with $r = 6, 4, 3, 2$ respectively, define $\hat{E} = E + \frac{\Delta E}{6 \Im \tau}$ and $\hat{\partial}_\tau = \partial_\tau + \frac{r}{12 \Im \tau}$, then the following identities hold:

$$
\begin{align*}
\hat{\partial}_\tau A &= \frac{1}{2r} \hat{A}(\hat{E} + \frac{C^r - B^r}{A^r} A^2), \\
\hat{\partial}_\tau B &= \frac{1}{2r} \hat{B}(\hat{E} - A^2), \\
\hat{\partial}_\tau C &= \frac{1}{2r} \hat{C}(\hat{E} + A^2), \\
\hat{\partial}_\tau \hat{E} &= \frac{1}{2r} (\hat{E}^2 - A^4).
\end{align*}
$$

(3.7)

**Proof.** Assume that the desired non-holomorphic quantity $\hat{E}$ is given by $\hat{E} = E + \Delta E$ with $\Delta E = \frac{\lambda - \frac{3}{N}}{1 \Im \tau}$ for some constant $\lambda$. Then it is easy to see with $\hat{\partial}_\tau = \partial_\tau + \frac{r}{12 \Im \tau} \Delta E$, the first three identities follow from Theorem 3.1.2. Solving the constant $\lambda$ from the last identity, we then get $\lambda = \frac{r}{6}$. Thus the conclusion follows. □

**Remark 3.1.4.** Theorems 3.1.2 is known in the literature, see e.g., [40] and references therein. In fact, for each of these cases, one can find the $\theta$ or $\eta$ expressions of the quantities $A, B, C, E$ and prove the formulas by checking the $\theta$ or $\eta$ expressions. The relations between these generators and the Eisenstein series $E_2, E_4, E_6$ are also known. One could then, for example, use the Eisenstein series expression of $E$ to obtain the almost-holomorphic modular form $\hat{E} = E + \Delta E$, with

$$
\Delta E = \frac{2}{N + 1 \Im \tau}, \quad N = 1^*, 2, 3, \quad \Delta E = \frac{1}{3 \Im \tau}, \quad N = 4.
$$

(3.8)

These agree with the above choices $\Delta E = \frac{r - 3}{6 \Im \tau}$. See e.g., [8] for a collection of these results.

But this method could not be generalized to Calabi-Yau three-fold families.

**Remark 3.1.5.** Strictly speaking, the ring generated by $A, B, C, E$ above do not form a differential ring due to the negative powers in the equations. However, it is easy to see by choosing suitable powers of these generators, one can indeed
get a ring. For example, in the \( r = 6, 4, 3, 2 \) cases, one can choose \( A^4, B^6 - C^6, E; A^2, B^4, E; A, B^3 \) and \( E; A, B^2, E \), respectively.

The ring generated by \( A, B, C, E \) is not the ring of quasi modular forms for \( \Gamma_0(N) \). For example, in the case \( N = 3 \), the ring of quasi modular forms with the nontrivial multiplier system \( \chi_{-3} \) is \( \tilde{M}_*(\Gamma_0(3), \chi_{-3}) = \mathbb{C}[A, B^3, E] \cong \mathbb{C}[A, F = C^3 - B^3, E] \) and the differential equations are

\[
\begin{align*}
\partial_x A &= \frac{1}{2}(EA + F), \\
\partial_x F &= \frac{1}{2}(EF + A^2), \\
\partial_x E &= \frac{1}{6}(E^2 - A^4).
\end{align*}
\]

One also has \( \tilde{M}_*(\Gamma_0(2)) = \mathbb{C}[A^2, B^4, E] \), \( \tilde{M}_*(\Gamma_0(4), \chi_{-4}) = \mathbb{C}[A, B^2, E] \), etc.

However, in the whole discussion of this work we shall not the transformations in the corresponding modular group \( \Gamma_0(N) \), but use only the differential equations they satisfy. Moreover, in the applications to the study of the holomorphic anomaly equations and the topological string partition functions, eventually only elements in the above rings of quasi modular forms (\( r = 6 \) case is exceptional) will be involved.

For the reasons mentioned above, by abuse of language, we shall call the rings \( \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}], \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E] \) and \( \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E] \) the ring of modular forms, quasi modular forms, almost-holomorphic modular forms for \( \Gamma_0(N) \), and denote them by \( \tilde{M}_*(\Gamma_0(N)) \), \( \tilde{M}_*(\Gamma_0(N)) \), \( \tilde{M}_*(\Gamma_0(N)) \) respectively. We shall also call the gradings “modular weights” which could be negative.

Since later we shall need to generalize the construction to some Calabi-Yau threefold families using the Picard-Fuchs equations, we shall reproduce below the details in constructing the graded differential rings (\( \tilde{M}_*(\Gamma_0(N)), \partial_x \)) using properties of the equations in \( \ref{2.9} \).

We start from the following observation.

**Proposition 3.1.6.** For each of the elliptic curve families \( \pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4 \), one has \( \partial_x \alpha = \alpha \beta A^2 \), where as before \( \beta : 1 - \alpha, \partial_x = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} \).

**Proof.** For ease of notation, first we write the Picard-Fuchs operator in \( \ref{2.9} \) as

\[
\mathcal{L} = \theta^2 - \alpha(\theta + c_1)(\theta + c_2), \quad \text{with} \quad c_1 = \frac{1}{r}, \ c_2 = 1 - \frac{1}{r},
\]

and define

\[
\tilde{\mathcal{L}} := ((\theta + \theta \log \omega_0)^2 - \alpha(\theta + \theta \log \omega_0 + c_1)(\theta + \theta \log \omega_0 + c_2)) \frac{\Pi}{\omega_0}.
\]

Then we have

\[
\omega_0 \frac{\tilde{\mathcal{L}} \Pi}{\omega_0} = \mathcal{L} \Pi = 0 \quad \text{for a period} \ \Pi.
\]

In particular, we have \( \tilde{\mathcal{L}} \Pi = 0 \). Subtracting \( \tilde{\mathcal{L}} \Pi = 0 \) from \( \tilde{\mathcal{L}} \tau \), one then gets

\[
\beta \theta^2 \tau + (2\beta \theta \log \omega_0 - \alpha(c_1 + c_2)) \theta \tau = 0.
\]

\[\text{2The author would like to thank Prof. Don Zagier for discussions on these.}\]
That is,
\[ \theta \log(\omega_2^2 \theta^\tau) - (c_1 + c_2) \frac{\alpha}{\beta} = \theta \log(\omega_2^2 \theta^\tau) + (c_1 + c_2) \theta \log \beta = 0. \]

Solving this first order equation for \( \omega_2^2 \theta^\tau \), we then get
\[ \theta^\tau = \frac{c}{\beta^{c_1 + c_2} \omega_0^2} = \frac{c}{\beta \omega_0^2} . \]
for some constant \( c \). By looking at the leading terms in \( \alpha \) of both sides as \( \alpha \to 0 \), we can then find that \( c = \frac{1}{2\pi i} \). Hence \( \partial_\tau \alpha = \alpha \beta A^2 \) as claimed.

In what follows, we shall call the modular function \( \alpha \) the algebraic modulus for the modular curve, while \( \tau \) the transcendental modulus for the modular curve. The above formula then gives a differential equation relating the algebraic and transcendental moduli.

Recall the definitions of \( B, C \) which implies that \( \alpha \beta = C_r B_r \), we then get
\[ \text{Corollary 3.1.7. For each of the elliptic curve families } \pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4, \text{ the following is true:} \]
\[ (3.10) \]
\[ A^2 = \frac{\partial_\tau \alpha}{\alpha \beta} = \partial_\tau \log \frac{\alpha}{\beta} = \partial_\tau \log \frac{C^r}{B^r} . \]

Using the definition \( E = D \log C^r B^r \), we have
\[ (3.11) \]
\[ \partial_\tau B = \frac{1}{2r} B(E - A^2), \quad \partial_\tau C = \frac{1}{2r} C(E + A^2) . \]

From \( A^r = B^r + C^r \), we can easily get
\[ (3.12) \]
\[ \partial_\tau A = \frac{1}{2r} A(E + \frac{C^r - B^r}{A^r} A^2) . \]

Using the Picard-Fuchs equation satisfied by \( A \):
\[ \beta(\theta A)^2 - (c_1 + c_2)\alpha \theta A - c_1 c_2 \alpha A = 0 , \]
we obtain
\[ \partial_\tau^2 \log A = (\partial_\tau \log A)^2 + (c_1 + c_2 - 1) \alpha \beta A^2 \partial_\tau \log A + c_1 c_2 \alpha \beta A^4 , \]
This second order differential equation of \( A \) will become a first order differential equation of \( E \) since \( \text{(3.12)} \) says that \( E = 2r \partial_\tau \log A - (\alpha - \beta) A^2 \), one then gets
\[ (3.13) \]
\[ \partial_\tau E = \frac{1}{2r} E^2 + (2c_1 c_2 r \alpha \beta - \frac{1}{2r} (\alpha - \beta)^2 - 2 \alpha \beta) A^4 = \frac{1}{2r} E^2 - \frac{1}{2r} (\alpha + \beta)^2 A^4 = \frac{1}{2r} (E^2 - A^4) . \]

“Special coordinate” on the deformation space of elliptic curves. In the previous discussions, we obtained the graded differential rings started from Prop. \( (3.1.6) \), which could be thought of as a differential equation satisfied by the normalized period of the elliptic curves. As we shall see, for a one-parameter Calabi-Yau three-fold family \( \pi : \mathcal{X} \to \mathcal{M} \), the “normalized period”, called special coordinate \( t \) below, does not satisfy an analogous relation. However, there is a parameter \( \tau \) on \( \mathcal{M} \) satisfying a similar identity (see Prop. \( (3.2.2) \) below). To get the identity, one computes the Yukawa couplings in different coordinates.

In the following, we shall explain how to derive Prop. \( (3.1.6) \) by computing the Yukawa couplings for the purpose of later generalization.
Proposition 3.1.8. The Yukawa coupling in the transcendental modulus $\tau$, defined by $C_{\tau} = \int \Omega \wedge \partial_{\tau} \Omega$, satisfies $C_{\tau} = 1$.

That is, there is no “quantum correction” \cite{14, 21, 25, 12} added to the classical intersection number $\kappa = 1$.

Proof. Take the local parameter of the (punctured) deformation space near the point $\alpha = 0$ to be $\tau$. At the base point $\tau_*$, the fiber of the family $\pi_{\tau_0(N)} : E_{\tau_0(N)} \rightarrow X_0(N)$ is the torus $T_{\tau_0} = \mathbb{C}/\Lambda_{\tau_*}$, where $\Lambda_{\tau_*} = \mathbb{Z} \oplus \mathbb{Z} \tau_*$. We take the holomorphic top form on $T_{\tau_*}$ to be $dz_{\tau_*}$ descended from $\mathbb{C}$. For any $\tau$ near the base point $\tau_*$, the diffeomorphism sending $T_{\tau_*}$ to $T_{\tau}$ is given by $z_{\tau} = \frac{\tau - \tau_*}{\tau_* - \tau} z_{\tau_*} + \frac{\tau - \tau_*}{\tau_* - \tau} \bar{z}_{\tau_*}$. From this, one can then see that $\partial_{\tau} \Omega_{\tau_0}|_{\tau_*} = \frac{1}{\tau_* - \tau_0} (dz_{\tau_*} - d\bar{z}_{\tau_*})$. It follows then that $C_{\tau_*} = \int_{T_{\tau_*}} \frac{1}{2\pi \iota} \partial_{\tau} \Omega_{\tau_0} \wedge d\bar{z}_{\tau_*} = 1$.

One can also compute the Yukawa coupling in the algebraic modulus $\alpha$ as follows.

Proposition 3.1.9. The Yukawa coupling in the algebraic modulus $\alpha$, defined by $C_{\alpha} = \int \Omega \wedge \partial_{\alpha} \Omega$, satisfies $C_{\alpha} = \frac{1}{\sigma_{\tau}}$.

Proof. Recall that the Picard-Fuchs equation \cite{3.9} tells that when integrated over cycles, one has $\theta^2 \Omega = (c_1 + c_2) \bar{\theta} \Omega + c_1 c_2 \bar{\theta}^2 \Omega$. Now we have

$$\theta(\alpha C_{\alpha}) = \theta \int \Omega \wedge \theta \Omega = \int \theta \Omega \wedge \theta \Omega + \int \Omega \wedge \theta^2 \Omega$$

$$= 0 + \int \Omega \wedge \left((c_1 + c_2) \bar{\theta} \Omega + c_1 c_2 \bar{\theta}^2 \Omega\right)$$

$$= 0 + (c_1 + c_2) \bar{\theta} \int \Omega \wedge \theta \Omega + 0$$

$$= (c_1 + c_2) \bar{\theta} (\alpha C_{\alpha}) = \frac{\alpha}{\beta} (\alpha C_{\alpha})$$

Solving $\alpha C_{\alpha}$ from this equation, we get $\alpha C_{\alpha} = \frac{\alpha}{\beta}$ from some constant $c$. We then fix this $c$ by looking at the behavior of both sides near $\alpha = 0$. This gives $c = 1$. Hence the conclusion follows.

By computing the Yukawa coupling in two different coordinates $\tau$ and $\alpha$ in Prop. 3.1.8 and Prop. 3.1.9 we can then derive the equation $\partial_{\tau} \alpha = (C_{\alpha})^{-1} \omega_{\tau}^0$ given in Prop. 3.1.6 between the transcendental modulus $\tau$ and the algebraic modulus $\alpha$, from the following relation

$$C_{\tau} = \frac{1}{2\pi \iota} \frac{1}{\omega_{\tau}^0} \frac{\partial_{\tau} \alpha}{\partial_{\tau}} C_{\alpha}.$$

3.2. Picard-Fuchs equations for Calabi-Yau three–fold families. In this section, motivated by the discussions in elliptic curve families, we shall work out similar rings $R$, $\bar{R}$ living on the deformation spaces of Calabi-Yau three–folds. As before we limit ourselves to the case $\dim \mathcal{M} = 1$. We shall start by computing the Yukawa couplings in different coordinates, then we derive an equation analogous to Prop. 3.1.6 between the complex coordinate $\alpha$ and a suitably chosen coordinate $\tau$ on the deformation space $\mathcal{M}$. After that we construct a ring out of special Kähler geometry quantities (connections, Yukawa couplings, etc.).
In the following we shall first consider slightly more general Picard-Fuchs equations before we specialize to the Picard-Fuchs \((2.10) (2.11)\) mentioned above.

Suppose the Picard-Fuchs operator for the family \(\pi : X \to \mathcal{M}\) of Calabi-Yau three-folds \(X\) is of the form

\[
L = \theta^3 - \alpha \prod_{i=1}^4 (\theta + c_i) = (1 - \alpha)\theta^3 - \alpha (\sigma_1 \theta^1 + \sigma_2 \theta^2 + \sigma_3 \theta + \sigma_4),
\]

where \(\theta = \frac{d}{dt}\) and \(\sigma_1, \sigma_2\) are the symmetric polynomials of the constants \(c_1, c_2, c_3, c_4\). For the quintic mirror family case, one has \((c_1, c_2, c_3, c_4) = (1/5, 2/5, 3/5, 4/5)\) and thus \(\sigma_1 = 2\). As before, we shall denote \(\beta := 1 - \alpha\).

The large complex structure limit is given by \(\alpha = 0\). Near this point, the solutions to the Picard-Fuchs equation \(L \Pi = 0\) could be obtained by the Frobenius method and have the following form

\[
(X^0, X^1, P_1, P_0) = X^0(1, t, F_t, 2F - tF_t)
\]

where \(X^0(\alpha) \sim 1 + \mathcal{O}(\alpha), t \sim \log \alpha + \cdots\) near \(\alpha = 0\), and \(F_t = \partial_t F(t)\). The function \(F(t)\), called prepotential, has the form from mirror symmetry

\[
F(t) = \frac{\kappa}{6!} t_3^3 + a t^2 + b t + c + \sum_{d=1}^\infty N_{0,d}^\text{GW} e^{d t},
\]

where \(\kappa\) is the classical triple intersection number of the mirror manifold \(\hat{X}\) of \(X\), \(a, b, c\) are some numbers which are not important in our discussions, and \(N_{0,d}^\text{GW}\) is the genus zero degree \(d\) Gromov-Witten invariant of \(\hat{X}\). This particular structure among the periods comes from the special Kähler geometry on \(\mathcal{M}\) and mirror symmetry.

The Yukawa coupling in the \(t\) coordinate is then given by \(C_{\text{ytt}} = F_{\text{ytt}} = \kappa + \mathcal{O}(q^1)\) with \(q = e^t\). In the complex coordinate \(\alpha\), we have

**Proposition 3.2.1.** The Yukawa coupling, defined by \(C_{\alpha\alpha\alpha} = -\int \Omega \wedge \partial^\alpha \Omega\), is given by \(C_{\alpha\alpha\alpha} = \frac{\kappa}{\alpha^{2\beta}}\).

**Proof.** First due to Griffiths transversality, we have \(\alpha^3 C_{\alpha\alpha\alpha} = -\int \Omega \wedge \theta^3 \Omega\). By integration by parts and Griffiths transversality, it follows that

\[
\theta(\alpha^3 C_{\alpha\alpha\alpha}) = -\int \theta \Omega \wedge \theta^3 \Omega - \int \Omega \wedge \theta^4 \Omega = -\left(\theta \int \Omega \wedge \theta^2 \Omega - \int \theta^2 \Omega \wedge \theta^2 \Omega\right) - \int \Omega \wedge (\sigma_1 \theta^3 \Omega + \cdots) = -\theta \left(\theta \int \Omega \wedge \theta^3 \Omega - \int \Omega \wedge \theta^3 \Omega\right) + \sigma_1 \frac{\alpha}{\beta} (\alpha^3 C_{\alpha\alpha\alpha}) = -\theta(\alpha^3 C_{\alpha\alpha\alpha}) + \sigma_1 \frac{\alpha}{\beta} (\alpha^3 C_{\alpha\alpha\alpha}).
\]

Solving \(\alpha^3 C_{\alpha\alpha\alpha}\) from this equation, we then get \(C_{\alpha\alpha\alpha} = \frac{\kappa}{\alpha^{2\beta}}\). Using the fact that

\[
\frac{1}{(X^0)^2} C_{\alpha\alpha\alpha} \left(\frac{\partial \alpha}{\partial t}\right)^3 = C_{\text{ytt}} = \kappa + \mathcal{O}(q),
\]

we know \(c = \kappa\). Hence the assertion follows. \(\Box\)
Now that we have computed the Yukawa coupling in the special coordinate $t$ and complex coordinate $\alpha$, we shall find the analogue of Prop. 3.1.6 by defining the following coordinate

$$\tau = \frac{1}{2\pi i} \kappa^{-1} F_{tt}.$$  

This definition was motivated [10] to establish the modularity for non-compact Calabi-Yau three-folds, as we shall explain later. According to this definition of $\tau$, near $\alpha = 0$ we have

$$\tau(\alpha) \sim \frac{1}{2\pi i} \log \alpha + \mathcal{O}(\alpha^0).$$

Therefore, from

$$\frac{1}{(X^0)^2} C_{\alpha\alpha\alpha}(\frac{\partial \alpha}{\partial t})^3 = C_{tut} = 2\pi i \kappa \frac{\partial \tau}{\partial t} = 2\pi i \kappa \frac{\partial \tau}{\partial \alpha} \frac{\partial \alpha}{\partial t},$$

we obtain the following assertion.

**Proposition 3.2.2.**

$$(3.21) \quad D\alpha = \alpha \cdot \kappa (\alpha^3 C_{\alpha\alpha\alpha})^{-1} \cdot (X^0 \theta^t)^2 = \alpha \beta (X^0 \theta^t)^2, \quad D := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}.$$  

Note that the only places in which we have used the special Kähler geometry are in the definition (3.19) of $\tau$ in terms of $F_{tt}$ and the limit of (3.18) as $\alpha$ goes to 0. But we could have defined $\tau$ as the quantity satisfying the equation

$$\frac{1}{(X^0)^2} C_{\alpha\alpha\alpha}(\frac{\partial \alpha}{\partial t})^3 = 2\pi i \kappa \frac{\partial \tau}{\partial t}$$  

and the condition $\lim_{\alpha \to 0} 2\pi i \frac{\partial \tau}{\partial t} = 1$ without referring to the prepotential $F(t)$ and Yukawa coupling $C_{ttt}$, thus only the periods and no special Kähler geometry are used.

We shall now take Prop. 3.2.2 as the starting point to construct the analogue of the ring of quasi modular forms. Motivated by the discussions of elliptic curve families, we define the following triple

$$(3.22) \quad A = X^0 \theta^t, \quad B = (1 - \alpha) \frac{\partial}{\partial \alpha} A, \quad C = \alpha \frac{\partial}{\partial \alpha} A,$$

where $r$ is some undetermined constant and does not show up in the final form of the ring $\tilde{R}$ we shall consider later. Similarly we define

$$(3.23) \quad E = D \log C^r B^r = D \log \alpha \beta A^{2r} = (\alpha - \beta) A^2 + D \log A^{2r}.$$  

Now thanks to Prop. 3.2.2, we get

$$(3.24) \quad A^2 = D\alpha = D \frac{\alpha}{\beta} = D \frac{C^r}{B^r}.$$  

We also have the following relations among these generators following from the definitions of $A, B, C, E$ and (3.24),

$$(3.25) \quad DB = \frac{1}{2r} B(E - A^2), \quad DC = \frac{1}{2r} C(E + A^2).$$  

To get a closed ring, we need to prove $A$ satisfies a differential equation with coefficients being holomorphic functions of $\alpha, \beta$. Define

$$(3.26) \quad A' = X^0, \quad A'' = \theta t,$$

it turns out after adding $D^i A', D^i A'', i = 1, 2, 3$, the ring will close under the derivative $D$. Note that the generator $E$ is already contained according to (3.23).

**Proposition 3.2.3.** The ring $\tilde{R}$ generated by $D^i A', i = 0, 1, 2, 3; D^i A'', j = 0, 1, 2$ and $B, C, B^{-1}, C^{-1}$, is closed under the derivative $D$.  

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Proof. The Picard-Fuchs equation tells that if one defines
\[ \tilde{L} = (\theta + \theta \log X^0)^4 - \alpha \prod_{i=1}^{4} (\theta + \theta \log X^0 + c_i), \]
then \( X^0 \tilde{L} \frac{\Pi}{X^0} = \tilde{L} \Pi = 0 \) for a period \( \Pi \). In particular, one has \( \mathcal{L} X^0 = 0 \) and \( \tilde{L} \frac{X^1}{X^0} = \tilde{L} t = 0 \). The first equation \( \mathcal{L} X^0 = 0 \) tells that \( \theta^4 X^0 \) could be expressed as a polynomial of \( \theta X^0 \), \( i = 0, 1, 2, 3 \) with coefficients being rational functions of \( \alpha, \beta \). Using the relation \( \theta = \beta^{-1} (X^0 \theta t)^{-2} D \) following from Prop. 3.2.2, we know that \( D^4 X^0 \) is a polynomial in \( D^i X^0, 0, i = 1, 2, 3; D^j \theta t, j = 0, 1, 2, 3 \) and \( B, C, B^{-1}, C^{-1} \). Similarly, by considering the second equation \( \tilde{L} t = 0 \), one sees that \( \theta \theta t \) and thus \( D^3 \theta t \) is also contained in the ring as claimed. \( \square \)

**Remark 3.2.4.** Note that when taking the derivative \( D \), negative powers of generators will appear. But as mentioned in Remark 3.1.5, to avoid them one only needs to choose a suitable set of generators carefully. In fact, in the final form of the graded ring \( \tilde{R} \) we shall consider below, we are going to make a specific choice of generators so that no negative powers will appear in the derivatives of the generators.

From Prop. 3.2.2 one can easily see that in fact the subring generated by \( D^i A^i, i = 0, 1, 2, 3; D^j A^j, j = 0, 1, 2; \alpha^\pm, \beta^\pm \) is also closed under \( D \). We shall denote this differential subring by \( (\tilde{R}^\text{sub}, D) \) in which the constant \( r \) does not show up.

**Picard-Fuchs equations for non-compact Calabi-Yau three-fold families.** Now we consider the non-compact Calabi-Yau three–fold families (2.10) whose Picard-Fuchs equations reduce to some third order differential equations of the form \( \mathcal{L} \ell\text{iptic} \circ \theta \).

For each of these families, as explained in [8], one can identify the base \( \mathcal{M} \) with a certain modular curve \( X_0(N) \).

Then one has \( X^0 = 1 \) and thus \( A = \theta t = \omega_0 \); moreover, by choosing the normalization for \( F_i \) suitably, we can make \( \theta F_i \) to be \( 2\pi i \omega_1 \), where \( \omega_0, \omega_1 \) are the periods of \( \mathcal{L} \ell\text{iptic} \) given in (3.2). Now the parameter \( \tau = \frac{2\pi i}{\omega_1} \), the parameter \( \tau \) is the transcendental modulus of the elliptic curve sitting inside the Calabi-Yau three-fold and lies in the upper half plane \( \mathcal{H} \). Therefore, in these cases, one has \( \mathcal{R} \cong \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}] = M_4(\Gamma_0(N)), \tilde{\mathcal{R}} \cong \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E] = \tilde{M}_4(\Gamma_0(N)) \). See [8] and references therein for details.

**Gradings.** There are two natural gradings, denoted by \( (k, m) \) on the ring \( \tilde{R} \). The grading \( m \) indicates that the element is a section of \( \mathcal{L}^m \) and will be called the degree. Recall that \( X^0 \) is a period of the form \( \int_{C} \Omega \) and \( C_{\alpha\alpha} = -i \int_{N} \Omega \wedge \partial_{\alpha} \Omega \), where \( \Omega \) is a section of the Hodge line bundle \( \mathcal{L} \rightarrow \mathcal{M} \), we can easily figure out the degree of the generators. The second grading, called the weight \( k \), is motivated by the studies of elliptic curve families and non-compact Calabi-Yau three-folds discussed above, in which \( \tau \) is really parametrizing the upper half plane \( \mathcal{H} \). We then defines the degrees and weights for the quantities \( X^0, \theta t, B, C, \alpha, (\alpha^3 C_{\alpha\alpha}) \) to be \( (1, 0), (0, 1), (1, 1), (1, 1), (0, 0), (0, 2) \) respectively. Taking the derivative \( D \) with respect to \( \tau \) will not change the degree, but raise the weight by 2. Then we have the decomposition \( \mathcal{R} = \oplus_{(k, m)} \mathcal{R}_{k,m} \). Similarly, there is a such decomposition for the graded differential ring \( (\tilde{R}, D) \).
The above discussions suggests that the rings \( \mathcal{R} = \mathbb{C}[X^0, (\theta t)^{\pm 1}, B^{\pm 1}, C^{\pm 1}] \), \( \tilde{\mathcal{R}} = \mathcal{R} \otimes \mathbb{C}[D^0X^0, i = 1, 2, 3; D^0\theta t, j = 1, 2] \), defined on the deformation space \( \mathcal{M} \), are the analogues of \( M_\ast(\Gamma), \tilde{M}_\ast(\Gamma) \) defined on the modular curve \( X_\Gamma \), and the weight \( k \) plays the role of modular weight. The generators \( D^0X^0, i = 1, 2, 3; D^0\theta t, j = 1, 2 \) should be considered as the analogue of quasi modular forms. We shall give more evidences for this later.

Similar to what was explained in Remark 3.1.5 one can get a smaller differential ring \( \mathcal{R}^{sub} \). It turns out that using special Kähler geometry of the deformation space \( \mathcal{M} \), one may further reduce the number of generators in \( \tilde{\mathcal{R}} - \mathcal{R} \). For example, for the quintic mirror family case considered, the sequence \( D^0\theta t, i = 0, 1, 2 \) could be reduced to \( D^0\theta t, i = 0, 1 \) as discussed in [13, 11, 6]. This is proved using the fact that \( t \) is the canonical coordinate on the deformation space \( \mathcal{M} \)(more than just being the ratio of two periods), as we shall discuss in the next section.

4. Differential rings from special Kähler geometry

In this section, we shall use properties of the special Kähler geometry on \( \mathcal{M} \) to reduce the number of generators in \( \tilde{\mathcal{R}} \), and more importantly to define \( \tilde{\mathcal{R}} \) as the “non-holomorphic completion” of \( \mathcal{R} \).

We first start by reviewing some basic properties about the canonical coordinates and holomorphic limits which will be important later. The discussions on these concepts apply to multi-parameter Calabi-Yau families.

4.1. Canonical coordinates and holomorphic limits. On a Kahler manifold \( M \), according to [5], the canonical coordinates \( t = \{ t^i \}_{i=1,2,\ldots,\dim M} \) around the base point \( p \) are defined to be the holomorphic coordinates such that

\[
\partial_i K_i|_p = 0 = \partial_i \Gamma^k_{ij} |_p
\]

where \( I \) is a multi-index and \( \partial_i = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_m}, m = |I| \geq 0 \). Note that the first equation is a condition on the choice of the Kähler potential which transforms under the rule \( K \mapsto K + f + \bar{f} \), where \( f \) is purely holomorphic.

These coordinates are studied elsewhere in different contexts, for example [42, 43, 44, 45]. They are the normal coordinates for the Kähler geometry [43, 44] and can be constructed using the holomorphic exponential map [42].

Exponential map and Gaussian normal coordinates. Now we shall recall some basic facts from Riemannian geometry. Given a Riemannian manifold \( M \) with the metric \( G_{ij} \), the Gaussian normal coordinates base at the point \( p \in M \) could be obtained in two ways: either as a coordinate system centered around \( p \) such that \( \text{Sym}(\partial I \Gamma_{ij}^k)|_p = 0, |I| \geq 0 \), where \( \text{Sym}(\partial I \Gamma_{ij}^k) \) means the symmetrization of \( \partial I \Gamma_{ij}^k \) with respect to the sub-indices \( I \cup \{i,j\} \); or as linear coordinates on the tangent vector space \( T_pM \) defined by the exponential map \( \exp_p : T_pM \to M \). Using the second viewpoint, we get the following description: suppose a point \( q \) in a small neighborhood of \( p \) on \( M \) is on the geodesic \( \gamma(s) = \exp_p(sv) \), where \( |v| = 1 \), and \( s \) is the arc-length parameter. Assume \( q = \exp_p(sv) \) for some \( s \) and fix a coordinate system \( x = \{ x^i \} \) near \( p \) on \( M \), then the Gaussian normal coordinates \( \xi = \{ \xi^i \} \) of \( q = \exp_p(sv) \) are
related to the coordinates $x = \{x^i\}$ by using the equations for the geodesic:

$$x^i(\exp_p(sw)) = x^i(p) + s\xi^i - \sum_{N=2}^{\infty} \frac{1}{N!} \Gamma_{N|p}^i \xi^N,$$

where $\Gamma_{N|p}^i := \nabla_{N-(i,i)} \Gamma_{i;i2}^i$ are computed in $x = \{x^i\}$ coordinates, and $N$ is a multi-index as before.

**Holomorphic exponential map and canonical coordinates on Kähler manifolds.** Now assume $M$ is a Kähler manifold whose Kähler potential is $K(z, \bar{z})$, where $z = \{z^i\}_{i=1,2\cdots \dim M}$ is a complex coordinate system. Suppose the base point $p$ is taken to be $(z, \bar{z})$. From the second equation in (1.1), one can solve [43] [44] [45] for $t$ and get the following expression similar to (4.2):

$$t^i(z; \bar{z}, \bar{z}_a) = K^i(j)(z; \bar{z}_a; \bar{z}_a)(K_j(z; \bar{z}_a; \bar{z}_a) - K_j(z; \bar{z}_a; \bar{z}_a)), $$

where a function $f$ defined near the base point $(z, \bar{z})$ is denoted by $f(z, \bar{z}; \bar{z}, \bar{z}_a)$. The holomorphic function $f(z, \bar{z}; \bar{z}, \bar{z}_a)$ means the degree zero part in the Taylor expansion of the function $f(z, \bar{z}; \bar{z}_a)$ in $\bar{z}$ centered at $\bar{z}_a$, where one thinks of $(z, \bar{z})$ as independent coordinates. This will be explained below using holomorphic exponential map.

The canonical coordinates can not be defined in terms of geodesics in the Riemannian geometry since the exponential map is in general not holomorphic. However, there is a nice construction of holomorphic exponential map which gives rise to these canonical coordinates. To define the holomorphic exponential map, we first regard the complex manifold $M$ as a Riemannian manifold and thus get the map $\exp^R_p : T^R_p M \to M$. This also defines the Gaussian normal coordinates $\xi$. Thinking of $T^R_p M$ as a complex vector space equipped with the complex structure induced by the complex structure on $M$, then in general the map $\exp^R_p : (\xi, \bar{\xi}) \to (z(\xi, \bar{\xi}), \bar{z}(\xi, \bar{\xi}))$ is not holomorphic. Now with the assumption that the metric $G_{ij}(z, \bar{z})$ is analytic in $z, \bar{z}$, we can analytically continue the map $\exp^R_p$ to the corresponding complexifications $T^C_p M, MC = M \times \bar{M}$, where $\bar{M}$ is the complex manifold with opposite complex structure as $M$.

The coordinates on the complexifications $T^C_p M, MC = M \times \bar{M}$ are given by $(\xi, \eta)$ and $(z, w)$ respectively, they are the analytic continuation of the coordinates $(\xi, \bar{\xi}), (z, \bar{z})$ from $T^R_p M \to T^C_p M, \Delta : M \to MC = M \times \bar{M}$ respectively, where $\Delta : M \to M \times \bar{M}, p \to (p, \bar{p})$ is the diagonal embedding. Here the underlying point of $\bar{p}$ is really the same as $p$, but we have used the barred notation to indicated that it is a point on the complex manifold $\bar{M}$.

Since the Christoffel symbols $\Gamma^k_{ij}(z, \bar{z})$ are analytic in $(z, \bar{z})$, we know that the map $\exp^C_p : (\xi, \eta) \to (z(\xi, \eta), w(\xi, \eta))$ is analytic, that is, holomorphic in $(\xi, \eta)$. Moreover, the map $\exp^C_p$ defines a local bi-holomorphism from a small neighborhood around the point $0$ inside $T^C_p M$ to a small neighborhood of the point $(p, \bar{p})$ inside $MC$. One claims that $\exp^C_p|_{T^C_p M}$ gives a holomorphic map $T^1_{\bar{p}} M \to M$ which is a local bi-holomorphism from a small neighborhood of $0 \in T^C_p M$ to a small neighborhood of $p \in M$. To show that it maps $T^1_{\bar{p}} M$ to $M$, we only need to show that $w \circ \exp^C_p|_{T^C_p M} = w(p)$, that is, $w(\xi, \eta)|_{\eta = 0} = w(p)$. Recall that $\bar{z}$ and thus $w$ satisfies the equation for the geodesic equation

$$\frac{d^2}{ds^2} \bar{z}^k + \Gamma^k_{ij} \frac{d\bar{z}^i}{ds} \frac{d\bar{z}^j}{ds} = 0, \quad \frac{d\bar{z}^k}{ds}(0) = \bar{\xi}^k = 0, \quad \bar{z}(0) = \bar{z}(\bar{p}).$$
It is easy to see that \( w(s) = w(\bar{p}) \) is one and thus the unique solution to the differential equation. Therefore, \( w \circ \exp_p^C(\xi, \eta = 0) = w(\bar{p}) \) as desired. Since \( z(\xi, \eta) \) is holomorphic in both \( \xi, \eta \), we know \( z(\xi, \eta = 0) \) is holomorphic in \( \xi \). The same reasoning for the exponential map \( \exp_p \) shows that it is locally a bi-holomorphism.

Hence one gets a holomorphic exponential map \( \exp_p^{\text{hol}} : T^{1,0}_p M \to M \). We now denote the coordinate \( \xi \) on \( T^{1,0}_p M \) by \( t \), this is then the canonical coordinates desired since the equation satisfied by \( t \) which is similar to \( (4.1) \) implies the second equation in \( (4.1) \). This can be checked by direct computations.

The exponential maps \( \exp_p^\mathbb{R} \) and \( \exp_p^{\text{hol}} \) are contrasted as follows:

\[
\begin{align*}
\exp_p^\mathbb{R} &= \exp_p^C | T^0_p M = \exp_p^C | T^{1,0}_p M \otimes T^{1,0}_p M, \\
\exp_p^{\text{hol}} &= \exp_p^C | T^{1,0}_p M = \exp_p^C | j(T^{1,0}_p M) = T^{1,0}_p M \otimes \{0\}.
\end{align*}
\]

where \( T^{1,0}_p M \otimes T^{1,0}_p M \) means the image of the map \( T^{1,0}_p M \to T^{1,0}_p M \otimes T^{0,1}_p M, v \mapsto (v, v^*) \), where \( v^* \) is the complex conjugate of \( v \); and \( j(T^{1,0}_p M) \) is the image of the map \( j : T^{1,0}_p M \to T^{1,0}_p M \otimes T^{0,1}_p M, v \mapsto (v, 0) \).

**Holomorphic limit.** The holomorphic limit of any function \( f(z, \bar{z}) \) based at \( z_* \) is defined as follows. First one analytically continues the map \( f \) to a map defined on \( M_\mathbb{C} \). Using the fact that \( \exp_p^C \) is a local diffeomorphism from \( T^{1,0}_p M \) to \( M_\mathbb{C} \), we get \( \hat{f} = f \circ \exp_p^C : T^{1,0}_p M \to \mathbb{C} \). The holomorphic limit of \( f(z, \bar{z}) \) is given by \( \hat{f} | j(T^{1,0}_p M) : T^{1,0}_p M \to \mathbb{C} \).

From now on, to maintain consistency with the notations used in the literature, we shall use \((z, \bar{z}), (t, \bar{t})\) for \((z, w), (\xi, \eta)\) when considering holomorphic limits, if no confusion arises. In the following, sometimes we shall drop the notations \( z_*, \bar{z}_* \) for the base point if it is clear from the context.

**Remark 4.1.1.** In the canonical coordinates \( t \) on the Kähler manifold \( M \), the holomorphic limit is described by \( f \circ \exp_{p, \text{hol}} = \hat{f} | j(T^{1,0}_p) : T^{1,0} M \times \{0\} \to \mathbb{C}, t \mapsto f \circ \exp_{p, \text{hol}}(t) \). In terms of an arbitrary local coordinate system \( z \) on \( M \), taking the holomorphic limit of the function \( f(z, \bar{z}) \) at the base point \( z_* \) is the same as keeping the degree zero part of the Taylor expansion of \( f(z, \bar{z}) \) with respect to \( \bar{z} \), where the center of the Taylor expansion is \( \bar{z}_* \). That is, it is the evaluation map \( ev_{\bar{z}_*} : f(\bullet, \bullet) \mapsto f(\bullet, \bar{z}_*) \). This is the limit that is used in the study of topological string theory in [4, 5].

One thing that needs to be taken extra care of is the holomorphic limit of \( \det G \) appearing in computing the topological string partition functions. One has

\[
G_{z^i \bar{z}^j} = G_{\nu^i \bar{\nu}^j} + \alpha^{jk} \frac{\partial}{\partial z^i} G_{\nu^k \bar{\nu}^j}, i, j, \nu, \bar{\nu} = 1, 2, \ldots \dim M \quad \text{and} \quad \log \det G_{z^i \bar{z}^j} = \log \det G_{\nu^i \bar{\nu}^j} + \log \det \frac{\partial}{\partial z^i} G_{\nu^j \bar{\nu}^j} + \log \det \frac{\partial}{\partial \bar{z}^j} G_{\nu^i \bar{\nu}^j}.
\]

Since only the holomorphic derivative of \( \log \det G_{z^i \bar{z}^j} \) will appear in the topological string partition functions (and also in the ring \( \mathcal{R} \) we shall construct below), the purely anti-holomorphic term will disappear. Moreover, from \( (4.1) \) one can see that \( \log \det G_{\nu^i \bar{\nu}^j}(t, \bar{t}) = \log \det G_{\nu^i \bar{\nu}^j}(t_*, \bar{t}_*) \) is independent of \( t \). Therefore, when computing \( \log \det G_{z^i \bar{z}^j} \) one can effectively extract the purely anti-holomorphic term and the term \( \log \det G_{\nu^i \bar{\nu}^j}(t, \bar{t}) \), then one only needs to take the holomorphic limit of the term \( \frac{\partial}{\partial \bar{z}^j} G_{\nu^i \bar{\nu}^j} \). This could also be seen from \( (4.3) \), which implies that

\[
(4.4) \quad \frac{\partial}{\partial \bar{z}^j}(z, \bar{z}_*) = K_{kj}(z_*, \bar{z}_*) K_{kj}(z, \bar{z}_*).
\]
Therefore, in the coordinate system $z$, the holomorphic limit of the metric $G_{kj}$, denoted by $\lim G_{kj}$, is given by

\begin{equation}
\lim G_{kj}(z, \bar{z}) = G_{kj}(z, \bar{z}) = \frac{\partial h_i}{\partial z^k}(z)G_{ij}(z, \bar{z}).
\end{equation}

**Variation of the holomorphic exponential map and canonical coordinates.** The holomorphic exponential map $\exp^\text{hol}_p$ does not depend holomorphically on the base point $z$.\footnote{\[42\]} The canonical coordinates thus also have non-holomorphic dependence, as we shall also see below in some examples. This is due to the fact that the space $T^{1,0}_z M$ changes non-holomorphically when $z_*$ moves in $M$; that is, $\frac{\partial}{\partial z_*} \pi_{J_*} \neq 0$, where $\pi_{J_*} = \frac{1}{2}(I - iJ_*)$ is the projection from $T^C_{z_*} M$ to $T^{1,0}_z M$. For a more precise discussion on this, see \[42\].

Take $M$ to be the base $\mathcal{M}$ of the Calabi-Yau three-fold family $\pi : \mathcal{X} \to \mathcal{M}$ and think of $T^{1,0}_z M$ as a Lagrangian in $T^C_{z_*} \mathcal{M}$, this then fits in the frame work of geometric quantization and is related to the basepoint independence of the total free energy $Z = \sum_{g=0}^{\infty} N^{2g-2}F(g)$ of the topological string theory for the family, as studied in \[10\]. The background (base point) independence of $Z$ tells that it satisfies some wave-like equations on $\mathcal{M}$ arising from geometric quantization. These equations are shown \[10\] to be equivalent to the master anomaly equations for $Z$ in \[5\] which are identical to the holomorphic anomaly equations for the topological string partition functions $\mathcal{F}(g)$.

### 4.2. Examples of canonical coordinates.

In this section we shall compute the canonical coordinates for some Kähler manifolds.

**Example 4.2.1** (Fubini-Study metric). Consider the Fubini-Study metric defined on $\mathbb{P}^1$

$$\omega_{FS} = \frac{i}{2} \frac{1}{(1 + |z|^2)^2} dz \wedge d\bar{z}$$

with Kähler potential $K = \ln(1 + |z|^2)$. It follows then

$$K_z = \frac{\bar{z}}{(1 + |z|^2)^2}, \quad K_{\bar{z}} = \frac{1}{(1 + |z|^2)^2}, \quad \partial_z^N K_{\bar{z}} = \frac{(-1)^N N! \bar{z}^{N-1}}{(1 + |z|^2)^{N+1}}, \quad N \geq 1.$$ 

At the point $p$ represented by $z_* = 0$, we can see that $\partial_z^N K_{\bar{z}}|_p = 0 = \partial_{\bar{z}}^N K_z|_p, N \geq 1$. Hence $z$ is the canonical coordinate based at $z_* = 0$. To find the canonical coordinate at a generic point $p$ represented by $z_*$, we adopt \[4.3\] and get

$$t(z; z_*, \bar{z}_*) = (1 + |z_*|^2)^2 \left( \frac{z}{(1 + z \bar{z}_*)} - \frac{z_*}{(1 + z_* \bar{z}_*)} \right).$$

In particular, at $z_* = 0$, this coincides with $z$. The non-holomorphic dependence on the base point can be easily seen from this formula.

**Example 4.2.2** (Poincaré metric). Consider the $\text{SL}(2, \mathbb{Z})$ invariant metric

$$\omega = \frac{i}{2} K_\tau d\tau \wedge d\bar{\tau} = \frac{1}{y^2} dx \wedge dy$$

on the Poincaré upper half plane $\mathcal{H}$, where $e^{-K} = \frac{\tau}{i}, \tau = x + iy$. Straightforward computations show that

$$K_\tau = \frac{1}{\tau - \bar{\tau}}, \quad K_{\bar{\tau}} = -\frac{1}{(\tau - \bar{\tau})^2}.$$
It follows that the canonical coordinate based at \( p \) given by \( \tau_a \) is
\[
t(t; \tau_a, \bar{\tau}_a) = -(\tau_a - \bar{\tau}_a)^2 \left( \frac{1}{\tau - \tau_a} - \frac{1}{\tau_a - \bar{\tau}_a} \right)
\]
In particular, if one takes the base point \( \tau_a = i\infty \), then the canonical coordinate \( t \)
coincides with the complex coordinate on \( \mathcal{H} \) from the embedding \( \mathcal{H} \hookrightarrow \mathbb{C} \).

**Example 4.2.3** (Weil-Petersson metric for elliptic curve family). Taking the elliptic
curves parametrized by \( \mathcal{H} \). As in the proof of Prop. 3.4.8 take the holomorphic
top form \( \Omega_t = dz_t \) on \( T_t \). Using the diffeomorphism from the fiber \( T_t \) to the fiber
\( T_a \),
\[
z_t = \frac{\tau - \bar{\tau}_a}{\tau_a - \bar{\tau}_a} z_{\tau_a} + \frac{\tau_a - \tau}{\tau_a - \bar{\tau}_a} \bar{z}_{\tau_a},
\]
one can compute the Kähler potential for the Weil-Petersson metric from
\[
e^{-K(t; \tau_a, \bar{\tau}_a)} = i \int_{T_t} \Omega_t \wedge \overline{\Omega_t} = \frac{\tau - \bar{\tau}_a}{i}.
\]
This is then the Poincare metric on the upper half plane considered in the above example.

**Example 4.2.4.** Suppose on the Kähler manifold \( M \) there exists complex coordinates \( z = \{z^i\} \) and a holomorphic function \( F(z) \), so that the Kähler metric is given by
\[
\omega = \frac{i}{2} \text{Im} \tau \, dz \wedge d\bar{z} = i\partial\bar{\partial}K,
\]
where \( K = \frac{1}{2} \text{Im} w, w_i(z) = \partial_{z^j} F(z), \tau_{ij}(z) = \partial_{z^i} \partial_{\bar{z}^j} F(z). \) Manifolds satisfying
these properties are studied in detail in [34]. The canonical coordinates are then
given by
\[
t_i(z; z_a, \bar{z}_a) = \frac{1}{\tau_{ij}(z_a) - \tau_{ij}(\bar{z}_a)} (w_j(z; z_a, \bar{z}_a) - w_j(z, z_a, \bar{z}_a) - \tau_{jk}(\bar{z}_a)(z^k - z_a^k)).
\]

### 4.3. Special Kähler metric on deformation spaces

Now we take \( M \) to be the base of the family \( \pi : X \rightarrow M \) of Calabi-Yau three-folds \( X \). Assume that
\( \dim M = h = h^{2,1}(X) \).

Fixing a section \( \Omega(z) \) of the of the Hodge line bundle \( \mathcal{L} \rightarrow M \) and choosing a symplectic basis \( \{A^I, B^J\}_{I,J=0,1,\cdots,h} \) for \( H^2(X, \mathbb{Z}) \), then the periods are given by
\[
(\int_{A^I} \Omega, \int_{B^J} \Omega, \int_{B^J} \Omega, \int_{B^J} \Omega) = (X^0, X^a, F_a, F_0) = X^0(1, t^a, F_a, 2F - t^a F_0),
\]
where \( a = 1, 2, \cdots h \) and \( F(X^I) \) is \( [3,3] \) a holomorphic homogeneous function of \( X \)
degree 2. Here the function \( F \) is defined by \( (X^0)^{-2} F \) and the sub-indices mean
derivatives with respect to corresponding coordinates.

Now assume that \( z_a \) is the large complex structure limit defined by \( z = 0 \) and
\( A^0 \) is the vanishing cycle at this point. Then near the base point \( z_a \), the quantities
\( t^a(z; z_a, \bar{z}_a) = X^a(z; z_a, \bar{z}_a)/X^0(z; z_a, \bar{z}_a) \sim \ln(z^a + O(z^0)), a = 1, 2, \cdots h \) gives a local
doordinate system on the manifold \( M \) due to local Torelli theorem which says that the period map \( \mathcal{P} : M \rightarrow \mathbb{P} H^2(X, \mathbb{C}), z \mapsto [X^I(z), F_J(z)] \) is a local isomorphism.

These coordinates, as ratios of the periods, are called special coordinates in the
literature. Then the Kähler potential of the Weil-Petersson metric is determined from
\[
e^{-K} = iX^0 X^a \left( 2F(t) - 2F(t) + (t^a - \bar{t}_a)(F_a + \bar{F}_a) \right),
\]

22
Using the fact that the prepotential $F(t)$ has the form $F(t) = \frac{1}{6} \sum \alpha_t t^a t^b t^c + Q(t) + \sum_i N_i e^{\xi_i}$, where $Q(t)$ is a quadratic polynomial of $\{t^a\}$, it can be shown that the special coordinates $t^a(z; \bar{z}_*, \bar{z})$, $a = 1, 2, \cdots h$ defined near the large complex structure limit $z_*$ are the canonical coordinates based at $z_*$, see [6] for details. Moreover, rewriting the above equation as

\[(4.7)\]

$$e^{-K(z, \bar{z})} = X^0 X^0 e^{-K(t, \bar{t})}, \quad e^{-K(t, \bar{t})} = i \left(2F(t) - 2F(t) + (t^a - \bar{t}^a)(F_a + \bar{F}_a)\right),$$

one then gets [47]

\[(4.8)\]

$$K_{z^i} = -\partial_{z^i} \log X^0 + K_{t^a} \frac{\partial t^a}{\partial z^i}, \quad \Gamma_{z^j z^k}^{z^i} = \frac{\partial z^k}{\partial t^a} \frac{\partial t^a}{\partial z^i} + \frac{\partial z^k}{\partial t^a} \frac{\partial t^a}{\partial z^j} \frac{\partial t^a}{\partial z^j},$$

where $\Gamma^{\alpha}_{\beta \gamma}$ is computed in the metric given by the Kähler potential $K(t, \bar{t})$. Then at the large complex structure limit $z_*$, since $\{t^a\}$ are the canonical coordinates, according to (4.1), one has the following holomorphic limits:

\[(4.9)\]

$$\lim K_{z^i} = -\partial_{z^i} \log X^0, \quad \lim \Gamma_{z^j z^k}^{z^i} = \frac{\partial z^k}{\partial t^a} \frac{\partial t^a}{\partial z^i} \frac{\partial t^a}{\partial z^j}.$$

In the remaining of this work we will only consider the holomorphic limit based at the large complex structure $z_* = 0$ which is given by $\bar{t} = i\infty$, and simply denote this limit by $\lim$ without specifying the base point. This limit is interesting since it is in this particular limit that the topological string partition functions on a Calabi-Yau three-fold $X$ are identical (under the mirror map) to the generating functions of Gromov-Witten invariants of its mirror manifold $\hat{X}$.

### 4.4. Ring of Yamaguchi-Yau and the construction of the triple.

In this section, we shall construct the ring $\widehat{R}$. We shall review the construction of a ring in [3] by Yamaguchi-Yau for the quintic mirror family. The purpose is to reduce the number of generators for the algebra $\widehat{R}$ defined above and also find its non-holomorphic completion $\widetilde{R}$.

The construction of Yamaguchi-Yau says that the antiholomorphic dependence of the normalized topological string partition functions $F^{(g)} = (X^0)^{2g-2} F^{(g)}$ are encoded in the generators

$$\theta^i \log e^{-K}, \quad \theta \log \det G,$$

while the coefficients are polynomials of

$$\theta \log \alpha \beta C_{\alpha \beta \gamma} = \theta \log \frac{K}{\beta} = \frac{\alpha}{\beta}.$$

More precisely, according to the Picard-Fuchs equation [2.11] and the definition [2.12], one has

\[(4.10)\]

$$\mathcal{L} e^{-K} = \left(\theta^4 - \alpha \prod_{i=1}^4 (\theta + c_i)\right) e^{-K} = 0,$$

where $c_i = i/5, i = 1, 2, 3, 4$. This then implies that $\theta^4 e^{-K}$ is a polynomial of $\theta^i e^{-K}, i = 1, 2, 3$ and $\frac{\alpha}{\beta}$. The special geometry relation [2.13] implies that

\[(4.11)\]

$$\partial_\alpha \bar{\partial}_\alpha \Gamma^{\alpha}_{\alpha \bar{\alpha}} = \partial_\alpha G_{\alpha \bar{\alpha}} - \bar{\partial}_\alpha (e^{2K} G^{\alpha \bar{\alpha}} G^{\alpha \bar{\alpha}} C_{\alpha \bar{\alpha}}).$$
It follows then
\[ \partial_{\alpha}[\partial_{\alpha}\Gamma_{\alpha}\beta] + (\Gamma_{\alpha}\beta)^2 - 2\Gamma_{\alpha}\beta \partial_{\alpha}K = 0 \]

Hence we know
\[ \partial_{\alpha}\Gamma_{\alpha}\beta + (\Gamma_{\alpha}\beta)^2 - 2\Gamma_{\alpha}\beta \partial_{\alpha}K = f_{\alpha} \]
for some holomorphic function \( f_{\alpha} \). Taking the holomorphic limit of the left hand side, according to (4.13), we get
\[ \partial_{\alpha}^2 \frac{\partial}{\partial \alpha} + (\partial_{\alpha} \log \theta x) \partial_{\alpha} \log X^0 + 4\partial_{\alpha} \partial_{\alpha} \log X^0 
+ 2(\partial_{\alpha} \log X^0)^2 + (\partial_{\alpha} \log C_{\alpha\alpha\alpha})(-2\partial_{\alpha} \partial_{\alpha} \log X^0 - \partial_{\alpha} \partial_{\alpha} \log X^0) = f_{\alpha} . \]

The holomorphic function was fixed in \([14, 6]\) (see also \([12]\)) to be \( \frac{\beta}{\alpha^2} \).

One can also replace the coordinate \( \alpha \) in (4.11) by \( x = \ln \alpha \) defined locally on the punctured deformation space, then we get
\[ \theta^2 \log G_{x\bar{x}} + (\theta \log G_{x\bar{x}})^2 - 2\theta \log G_{x\bar{x}} \theta K 
- 4\theta^2 K + 2(\theta K)^2 + (\theta \log C_{xxx})(2\theta K - \theta \log G_{x\bar{x}}) = f_x , \]
where \( \theta = \partial_x = \alpha \frac{\partial}{\partial \alpha}, C_{xxx} = \alpha^3 C_{\alpha\alpha\alpha} = \frac{\alpha}{\beta}, \theta \log C_{xxx} = \frac{\alpha}{\beta} \), and \( f_x \) is another holomorphic function. Now we take the holomorphic limit of the above identity and get
\[ \theta^2 \log \theta t + (\theta \log \theta t)^2 + 2\theta \log \theta t \theta \log X^0 
- 4\theta^2 \log X^0 + 2(\theta \log X^0)^2 + (\theta \log C_{xxx})(-2\theta \log X^0 - \theta \log \theta t) = f_x , \]
with
\[ f_x = \frac{2}{\beta} . \]

Therefore, as shown in \([6]\), one gets the following Yamaguchi-Yau ring
\[ \mathcal{R}_{YY} = \mathbb{C}[\theta^i \log e^{-K}, i = 1, 2, 3; C_{x\bar{x}} = \theta \log G_{x\bar{x}}, \theta \log C_{xxx} = \frac{\alpha}{\beta}] . \]

Note that
\[ \theta \theta \log C_{xxx} = \theta \frac{\alpha}{\beta} = \frac{\alpha}{\beta^2} = \theta \log C_{xxx}(\theta \log C_{xxx} + 1) , \]
then the ring \( \mathcal{R}_{YY} \) is closed under taking the derivative \( \theta \). The generators of this ring \( (\mathcal{R}_{YY}, \theta) \) are essentially \( K_x, K_{xx}, K_{xxx}, \theta \log C_{xxx} \).

However, it is not convenient to directly interpret this as the analogue of the ring of almost-holomorphic modular forms. For this reason, we connect this ring \( (\mathcal{R}_{YY}, \theta) \) to \((\mathcal{R}, D)\).

Due to (4.13), and the relation between the derivatives \( \theta \) and \( D \) given by
\[ \theta = \beta^{-1}(X \theta t)^{-2} D \], we know that the set of generators for \( \mathcal{R} \) could be reduced to \( D^i X^0, i = 0, 1, 2, 3; D^j \theta t, j = 0, 1; B, C \). Recall that \( \mathcal{R} = \mathbb{C}[(X^0)^{\pm1}, (\theta t)^{\pm1}, B^{\pm1}, C^{\pm1}] \), then one can see that
\[ \mathcal{R} = \mathbb{C}[D^i \log X^0, i = 1, 2, 3; D^j \log \theta t, j = 1; \alpha, \beta] \otimes \mathcal{R} . \]
This motivates us to define the non-holomorphic completion \( \hat{R} \) of the ring to get the last subsection we then used special Kähler geometry to refine the generators of the graded differential ring which is an analogue of the ring of quasi modular forms. In \( D \), that is, the generators \( D^i \log X^0, i = 1, 2, 3; D \log \theta t \) in \( \tilde{R}_{0,0} \) are equivalent to the holomorphic limits of the non-holomorphic generators in \( R_{YY} \). It follows then that

\[
\tilde{R} = \lim R_{YY} \otimes R.
\]

This motivates us to define the non-holomorphic completion \( \hat{R} \) of \( \tilde{R} \) as

\[
\hat{R} = R_{YY} \otimes R.
\]

Moreover, \( F^{(g)} \in R_{YY} \subseteq \tilde{R}_{0,0} \), where \( R_{YY} \) and \( \tilde{R}_{0,0} \) are only differed by the holomorphic generators of degree and weight zero.

### 4.5. Summary of results.

In summary, in section 3 we constructed \( (\tilde{R}, D) \) as a graded differential ring which is an analogue of the ring of quasi modular forms. In the last subsection we then used special Kähler geometry to refine the generators of the ring to get

\[
\begin{align*}
\mathcal{R} &= \mathbb{C}[X^0]_{\pm 1}, (\theta t)_{\pm 1}, B^{\pm 1}, C^{\pm 1}], \\
\tilde{R} &= \mathcal{R} \otimes \mathbb{C}[D^i \log X^0, i = 1, 2, 3; D \log \theta t], \\
\hat{R} &= \mathcal{R} \otimes \mathbb{C}[D^i \log e^{-K}, i = 1, 2, 3; D \log \det G_{xx}].
\end{align*}
\]

Recall the structure of the graded rings \( (M_*(\Gamma), \tilde{M}_*(\Gamma), \hat{M}_*(\Gamma)) \) defined for \( \pi_\Gamma : \mathcal{E}_\Gamma \rightarrow X_\Gamma \)

\[
\begin{align*}
\partial_r : M_*(\Gamma) &\rightarrow \tilde{M}_*(\Gamma), \\
& \text{"modular completion" : } \tilde{M}_*(\Gamma) \rightarrow \hat{M}_*(\Gamma) \subseteq \tilde{M}_*(\Gamma)[Y], \quad Y = \frac{1}{12} \frac{-3}{\text{Im}\tau}, \\
\partial_r &\text{ is constant term map" } Y \rightarrow 0 : \tilde{M}_*(\Gamma) \rightarrow \hat{M}_*(\Gamma), \\
\partial_r &\text{ is constant term map" } Y \rightarrow 0 : \hat{M}_*(\Gamma) \rightarrow \tilde{M}_*(\Gamma), \\
\partial_r &\text{ is constant term map" } Y \rightarrow 0 : \tilde{M}_*(\Gamma) \rightarrow \hat{M}_*(\Gamma).
\end{align*}
\]

From \( \text{(1.7)} \), we know

\[
\begin{align*}
D \log e^{-K(x, \bar{x})} &= D \log (X^0 \bar{X}^0 \bar{e}^{-K(t, \bar{t})}) = D \log X^0 + D \log e^{-K(t, \bar{t})} \\
D \log G_{x\bar{x}} &= D \log (\theta t \bar{\theta} G_{it}) = D \log \theta t + D \log G_{it}.
\end{align*}
\]

Define \( Y_1 = D \log G_{tt}, Y_2 = -D \log e^{-K(t, \bar{t})} \), then we have the following analogue between \( (\mathcal{R}, \tilde{R}, \hat{R}) \) defined for \( \pi : \mathcal{X} \rightarrow \mathcal{M} \) and \( (M_*(\Gamma), \tilde{M}_*(\Gamma), \hat{M}_*(\Gamma)) \) defined for
sections of Sym$^k \mathcal{T} \mathcal{M} \otimes \mathcal{L}^m$.

The above construction for $(\mathcal{R}, \mathcal{R}, \bar{\mathcal{R}})$ could also be formally applied to the elliptic curve families in (3.3), see [12]. The Weil-Petersson metric is determined from $e^{-K(\alpha, \alpha)} = i \omega_0 \bar{\omega}_0 (\tau - \bar{\tau}) = i \omega_0 \bar{\omega}_0 e^{-K(\tau, \bar{\tau})}$. The quantities $Y_1, Y_2$ are now computed to be $\frac{-3}{12 \pi \mathrm{Im} \tau}$ and $\frac{1}{12 \pi \mathrm{Im} \tau}$, respectively. The triple $(\mathcal{R}, \mathcal{R}, \bar{\mathcal{R}})$ coincides with the triple $(M_s(\Gamma), \bar{M}_s(\Gamma), \bar{M}_e(\Gamma))$, as well as the maps among the members in the triple.

For the non-compact Calabi-Yau three-fold families in (2.10), one has $X^0 = 1$ and $\theta t = A$. The rings $(\mathcal{R}, \bar{\mathcal{R}})$ coincide with $(M_s(\Gamma), \bar{M}_s(\Gamma))$, as mentioned earlier in this paper. But the explicit forms for $Y_1, Y_2$ are difficult to compute in these cases.

It is easy to see that one should be able to apply the same construction for the quintic mirror family to construct triples $(\mathcal{R}, \mathcal{R}, \bar{\mathcal{R}})$ for other one-parameter Calabi-Yau three-fold families whose Picard-Fuchs equation takes the form as (4.15) with $\sum_i c_i = 2$. The only thing that needs to be checked is that the function $f_x$ in (4.13) is contained in $\mathbb{C}[B^{\pm 1}, C^{\pm 1}]$. In fact, for many Calabi-Yau families [14, 41, 6], this holomorphic function is a rational function\(^3\). We shall not discuss the details in this work.

4.6. **Special geometry polynomial ring.** Most of the generators in $\bar{\mathcal{R}}$ obtained from the elements in $\mathcal{R}_{YY}$ have weight zero. In [8], a set of the non-holomorphic, positive weight generators for $\bar{\mathcal{R}}$ are chosen so that no negative powers of the generators appear upon taking the derivative $D$. The particular form of the ring $\bar{\mathcal{R}}$ is termed the special polynomial ring in [8]. For completeness, in the following we shall review the construction of the generators therein.

First notice that the set of generators in $\bar{\mathcal{R}}$ given by $X^0 D^i \log e^{-K}, i = 1, 2, 3; \theta t \log \det G_{xx}$ is equivalent to the set of generators $S^{xx}, S^x, S, K_x$ in (2.19). The reason is as follows. Integrating the special geometry relation (4.13), we then get (4.20)

$$\Gamma^x_{xx} = 2K_x - C_{xxx}S^{xx} + S^x_{xx}$$

then up to multiplication and addition by $K_x$ and holomorphic quantities, $S^{xx}$ is essentially $\Gamma^x_{xx} = \theta \log \det G_{xx}$. The first or last equation in (2.19) tells that $S^x$ is essentially $\partial_x K_x$, and the seconde tells that $S$ is $\partial^2_x K_x$. Moreover, the derivatives

---

\(^3\)This is because the Picard-Fuchs equation for a non-compact Calabi-Yau three-fold family has only three periods, and the Kähler potential of the Weil-Petersson metric cannot be computed as the compact cases. One needs to compactify [17] the non-compact Calabi-Yau three-fold to a compact Calabi-Yau geometry, and then do computations there, after that one takes the decompactification limit of corresponding quantities.

\(^4\)The author thanks Prof. Shinobu Hosono for email correspondences and telling him the references on this.
of the generators in $\mathcal{R}_{YY}$ coincide with those for the generators $S^x x, S^x, S, K \alpha$ in (2.10).

Now a nice set of generators for the special geometry polynomial ring $\hat{\mathcal{R}}$ can be chosen as follows. First one makes the following change of generators [7]

$$\hat{S}^{\alpha} = S^\alpha, \hat{T} = S - S^\alpha K_\alpha, \hat{S} = S - S^\alpha K_\alpha + \frac{1}{2} S^\alpha K_\alpha K_\alpha, \hat{K}_\alpha = K_\alpha$$

Then as before one defines $\tau = \frac{1}{2\pi} \kappa^{-1} \partial \tau_1$ which gives $\frac{\partial \tau}{\partial t} = \frac{1}{2\pi} \kappa^{-1} \partial t$. Then one forms the following quantities on the deformation space $\mathcal{M}$:

$$K_0 = \kappa C^{-1}_{\alpha\alpha}(\theta t)^{-3}, \quad G_1 = \theta t, \quad K_2 = \kappa C^{-1}_{\alpha\alpha} \hat{K}_\alpha,$$

$$T_2 = \tilde{S}^{\alpha}, \quad T_4 = C^{-1}_{\alpha\alpha} \tilde{S}^{\alpha}, \quad T_6 = C^{-1}_{\alpha\alpha} \tilde{S}^{\alpha},$$

where the propagators $\hat{S}^{\alpha}, \hat{T}, \hat{S}$ are normalized by suitable powers of $X^0$ so that they are sections of $\mathcal{L}^0$. That is, they have degree zero. The weights of these generators are the sub-indices they carry. It follows that the derivatives of the generators of $\hat{\mathcal{R}}$ given in (2.19) now become ($\partial_t := \frac{1}{2\pi} \partial t$)

$$\partial_t K_0 = -2K_0 K_2 - K_0^2 G_1^2 (\tilde{h}_{\alpha\alpha} + 3(\sigma_{\alpha\alpha} + 1)),$$

$$\partial_t G_1 = 2G_1 K_2 - \kappa G_1 T_2 + K_0 G_1 (\sigma_{\alpha\alpha} + 1),$$

$$\partial_t K_2 = 3K_2^2 - 3\kappa K_2 T_2 - \kappa^2 T_4 + K_2^2 G_1 k_{\alpha\alpha} - K_0 G_1 K_2 \tilde{h}_{\alpha\alpha},$$

$$\partial_t T_2 = 2K_2 T_2 - \kappa T_2^2 + 2\kappa T_4 + \kappa^{-1} K_2^2 G_1 \tilde{h}_{\alpha\alpha},$$

$$\partial_t T_4 = 4K_2 T_4 - 3\kappa T_2 T_4 + 2\kappa T_6 - K_0 G_2 T_4 \tilde{h}_{\alpha\alpha} - \kappa^{-1} K_2^2 G_1 T_2 k_{\alpha\alpha} + \kappa^{-2} K_0^3 G_1^6 \tilde{h}_{\alpha\alpha},$$

$$\partial_t T_6 = 6K_2 T_6 - 6\kappa T_2 T_6 + \frac{\kappa}{2} T_4^2 - \kappa^{-1} K_2^2 G_1 T_4 k_{\alpha\alpha} + \kappa^{-3} K_0^4 G_1^8 \tilde{h}_{\alpha\alpha} - 2 K_0^2 G_1^2 T_6 \tilde{h}_{\alpha\alpha}.$$ 

The quantities $\tilde{h}_{\alpha\alpha}, \sigma_{\alpha\alpha}, k_{\alpha\alpha}, \tilde{h}_{\alpha\alpha}, \tilde{h}_{\alpha}, \tilde{h}_{\alpha}$ are holomorphic functions. It turns out that they are polynomials of an additional generator $C_0 = \theta \log C_{xx} = \frac{A}{B}$ with

$$\partial_t C_0 = C_0 (C_0 + 1) G_1^2.$$ 

These explicit polynomials for the quintic mirror family could be found in [8] and are omitted here.

4.7. Holomorphic anomaly equations. As mentioned earlier in section 4.4, one has $F^{(s)} := (X^0)^{2g-2} F^{(s)} \in \mathcal{R}_{YY} \subseteq \hat{\mathcal{R}}_{0,0}.$

The holomorphic anomaly equations then become [8]

$$\frac{\partial F^{(s)}}{\partial T_2} - \frac{1}{\kappa} \frac{\partial F^{(s)}}{\partial T_4} K_2 + \frac{1}{\kappa^2} \frac{\partial F^{(s)}}{\partial T_6} K_2^2 = \frac{1}{2} \sum_{r=1}^{g-1} \partial_t F^{(s-r)} \partial_t F^{(r)} + \frac{1}{2} \partial_t^2 F^{(s-1)},$$

$$\frac{\partial F^{(s)}}{\partial K_2} = 0,$$

where $\partial_t = (X^0)^{-2}(C_0 + 1)(\theta t)^{-3} \partial_t.$

Example 4.7.1. Consider the Calabi-Yau three-fold family $\pi : X \to \mathcal{M}$ which is mirror to the $K_{xx}$ family (with a one-dimensional base parametrizing the complexified Kähler structures of $K_{xx}$). It is proved in [8] and also mentioned earlier in section 3.2 that $\mathcal{M} \cong X_0(3), \hat{\mathcal{R}} \cong \hat{\mathcal{M}}_x(\Gamma_0(3)) = \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E]$. In this case one
can consistently choose the generators so that $T_4 = T_6 = K_2 = 0, T_2 = \frac{F}{2} \kappa C$ with $\partial_t = \kappa^{-1} C t t \partial_r = B^{-3} \partial_r$. Then the holomorphic anomaly equations simplify greatly. In particular, the equation for the holomorphic limits of $F^{(g)} = (X_0)^{2g-2} F^{(g)}$ at the large complex structure, denoted by $F_g \in \tilde{R}_{0,0} \subseteq C[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E]$, becomes

$$\partial_E F_g = \frac{1}{4 B^6} \left( \sum_{r=1}^{g-1} \partial_r F_{g-r} \partial_r F_r - \frac{E - A^2}{2} \partial_r F_{g-1} + \partial_r \partial_r F_{g-1} \right).$$

The boundary conditions at the large complex structure $\alpha = 0$ limit and the gap condition at the conifold point $\alpha = 1$ for the topological string partition functions now translate to the regularity conditions for the quasi modular form $F_g$ at the two cusp classes $[i\infty], [0]$ on $X_0(3)$. The Fricke involution $W_N : \tau \mapsto -1 \ \tau$ translates further these conditions to some conditions on the $q_\tau = \exp 2 \pi i \tau$ expansion of $F_g, F_g|_{W_N}$ at the infinity cusp $[i\infty]$. This then allows one to solve $F_g$ and thus $F^{(g)}$ genus by genus recursively. One can also prove the existence and uniqueness of the solutions to the holomorphic anomaly equations with the provided boundary conditions.

5. Conclusions and discussions

We constructed the graded rings $(\mathcal{R}, \tilde{R}, \hat{R})$ on the deformation space $\mathcal{M}$ from the periods of the Picard-Fuchs equation and special Kähler geometry on the deformation space. A parallelism between these rings and the rings $M_\ast(\Gamma), \tilde{M}(\Gamma), \hat{M}(\Gamma)$ was made: the way they were constructed; non-holomorphic completion and modular completion; holomorphic limit and “constant term map”. We further showed that in some special cases the rings $(\mathcal{R}, \tilde{R})$ are equivalent to the rings of modular quantities $(M_\ast(\Gamma), \tilde{M}(\Gamma))$. These give some evidences that indeed the graded rings $(\mathcal{R}, \tilde{R}, \hat{R})$ are analogues of the rings of modular objects $M_\ast(\Gamma), \tilde{M}(\Gamma), \hat{M}(\Gamma)$. We also discussed some of their applications in solving the holomorphic anomaly equations.

$$\begin{align*}
\tilde{D} \cup \hat{M}(\Gamma) & \quad \hat{R} \cup \tilde{D} \\
\downarrow \pi_\Gamma : E \Gamma \rightarrow X_\Gamma & \quad \downarrow \pi : X \rightarrow M \\
\hat{M}(\Gamma) & \quad \hat{R}
\end{align*}$$

In the above construction of the triple of graded rings $(\mathcal{R}, \tilde{R}, \hat{R})$, the parameter $\tau = \frac{1}{2\pi i} C t t$ defined in (3.19) on the deformation space $\mathcal{M}$ was introduced to match the known modularity in the non-compact examples. There are a number of interesting questions about this quantity $\tau$ we would like to address here and wish to pursue in the future.
Variation of Hodge structure. For the family \( \pi: X \to M \) of non-compact Calabi-Yau three-folds discussed above, the parameter \( \tau \) is exactly the transcendental modulus for elliptic curve \( \mathcal{E}_\alpha \) sitting inside the non-compact Calabi-Yau three-fold \( X_\alpha \). It is the normalized period for the elliptic curve and lies in the upper half plane. This results from the fact that the vector space of the periods \((1, t, F_t)\) of \( X_\alpha \) is closed under the monodromy, and upon taking derivatives these periods become \((0, \omega_0, \omega_1)\), where the latter two are the two periods of \( \mathcal{E}_\alpha \). In other words, while the three periods \((1, t, F_t)\) characterizes the variation of complex structure of the Calabi-Yau three-fold, the quantities \((\theta_t, \theta F_t)\) characterizes the variation of complex structure of the elliptic curve sitting inside it.

However, for a general one-parameter compact Calabi-Yau three-fold family, e.g., the quintic mirror family, the vector space of periods \((X^0, X^0t, X^0F_t)\) is not invariant under the monodromy group. It is not clear what the geometric meaning of \( \tau = \frac{1}{2\pi i} \kappa^{-1} F_{tt} \) is.

Enumerative content of \( \tau \) and integrality. For the particular non-compact geometries \((2.10)\), the \( F_g \)'s solved [8] from the holomorphic anomaly equations are explicit quasi modular functions in \( \tau \) (see also [10] for related work). Whether the \( q_\tau \) expansions of the topological string partition functions have any enumerative content and how the \( q_t \) and \( q_\tau \) expansions are related beg an explanation.

Now we briefly recall how the partition functions are related to the generating functions Gromov-Witten variants under the mirror symmetry conjecture. For the Calabi-Yau three-fold family \( \pi: X \to M \) whose generic fiber is \( X \), suppose the mirror family is given by \( \tilde{\pi}: \tilde{X} \to \tilde{M} \) whose generic fiber is \( \tilde{X} \). Mirror symmetry predicts the holomorphic limit \( F_g = \lim_{\tau} (X^0)^{2g-2} F(g) \) at the large complex structure limit is identical to the generating function of genus \( g \) Gromov-Witten invariants of \( \tilde{X} \), that is,

\[
F_g(t) = \sum_{d=0}^{\infty} N_{g,d}^\text{GW} q_d^t, \quad q_t = e^t.
\]

Recall the prepotential \( F(t) \) is given by

\[
F(t) = \frac{\kappa}{6!} t^3 + \sum_{d=1}^{\infty} N_{g=0,d}^\text{GW} q_t^d,
\]

then \( \tau = \frac{1}{2\pi i} \kappa^{-1} F_{tt} \) is the function determined from

\[
2\pi i \tau = t + \kappa^{-1} \sum_{d=1}^{\infty} N_{g,d}^\text{GW} d^2 q_t^d, \quad q_t = e^t.
\]

This implies in particular that

\[
q_\tau = \exp 2\pi i \tau = q_t (1 + \mathcal{O}(q_t)).
\]

It is natural to expect that there should be an enumerative problem associated to \( \tau \) in the sense

\[
F_g(\tau) = \sum_{d=0}^{\infty} N_{g,d}^\text{hyp} q_\tau^d,
\]

\footnote{The author thanks Murad Alim, Yaim Cooper and Shing-Tung Yau for discussions on this.}
where like the Gromov-Witten invariants \( N_{g,d}^{GW} \), the numbers \( N_{g,d}^{hyp} \) may hypothetically counting certain kind of invariants. Comparing the (5.4) with (5.1) and using (5.3), we can then find the “multiple-cover formula” relating \( N_{g,d}^{GW} \) and \( N_{g,d}^{hyp} \).

For the cases [8] in which the topological string partition functions have nice expressions in terms of quasi modular forms in \( \tau \), the integrality with respect to \( q_{\tau} \) is almost automatic. One then hopes that according to (5.3), studying the enumerative meaning of \( q_{\tau} \) expansion will help understand the integrality in \( q_{\tau} \) expansion as well.

We don’t have answers to any of these questions, and shall only display some examples below.

**Example 5.0.2 (Resolved Conifold).** Consider the resolved conifold which is the total space of \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1 \) and is a Calabi-Yau 3-fold. The Picard-Fuchs equation of the mirror Calabi-Yau family is given by, see e.g. [49]

\[
\mathcal{L} = \theta \left( \frac{\alpha}{1 - \alpha} \right)^{-1} \theta^2.
\]

Near the large complex structure limit given by \( \alpha = 0 \), a basis of the periods could be chosen to be

\[
X^0 = 1, \quad t = \ln \alpha, \quad F_t \sim (\ln \alpha)^2 + O(\alpha^0).
\]

Therefore near \( \alpha = 0 \), one has \( t = \ln \alpha \) and thus \( q_t = \alpha \). Moreover, the genus zero Gromov-Witten invariants are [50, 51, 52]

\[
N_{0,d}^{GW} = \frac{1}{d^3},
\]

and the prepotential is

\[
F(t) = \frac{\kappa}{3!} t^3 + \sum_{d=1}^{\infty} N_{0,d}^{GW} q_t^d = \frac{\kappa}{3!} t^3 + \sum_{d=1}^{\infty} \frac{1}{d!} q_t^d = \frac{\kappa}{3!} t^3 + \text{Li}_3(q_t).
\]

This implies in particular that

\[
C_{ttt} = \kappa + \sum_{d=1}^{\infty} q_t^d = \kappa + \frac{q_t}{1 - q_t}.
\]

The function \( \tau \) then satisfies

\[
2\pi i \tau = \kappa^{-1} F_{tt} = t + \kappa^{-1} \sum_{d=0}^{\infty} \frac{1}{d^3} q_t^d = t - \kappa^{-1} \ln(1 - q_t).
\]

Note that \( \kappa \) can not be determined by studying the periods and is ambiguous. Consideration in physics [53] tells that a natural choice is \( \kappa = 1 \). In the following, we shall take this choice.

**Remark 5.0.3.** From (5.10) one can see that \( \tau \) is itself the generating function of the sequence of numbers \( \frac{1}{d^3} = d^2 N_{0,d}^{GW} \), \( d = 1, 2, \cdots \). These numbers appear in the study of the stable-quotient invariants defined in [54] with

\[
d^2 N_{0,d}^{GW} = \int_{[\mathcal{Q}_{0,2}(\mathbb{P}^1,d)]^{vir}} e(Ob) \cup ev_1^* H \cup ev_2^* H,
\]

where \( Ob \) is the obstruction bundle in the construction of stable-quotient invariants, and the two insertions which give rise to \( ev_1^* H \cup ev_2^* H \) are required for the stability in genus 0.
For higher genus partition functions, it is well known that

\[ N_{g,d}^{GW} = d^{2g-3} N_{g,1}^{GW} = \frac{|B_{2g}|}{2g(2g-2)!} d^{2g-3}, \]

\[ F_g = \frac{|B_{2g}|}{2g(2g-2)!} \ln(1-q_t) \]

in particular, \( F_1 = -\frac{1}{12} \log(1-q_t) \).

To extract the numbers \( N_{g,d}^{hyp} \) associated to \( \tau \), we make use of (5.10) which gives rise to

\[ 2\pi i \tau = t - \ln(1-q_t), \quad q_r = \frac{q_t}{1-q_t}, \quad q_t = \frac{q_r}{1+q_r}. \]

Now from (5.9), one gets

\[ N_{0,0}^{hyp} = 0, \quad d = 0, \quad N_{0,1}^{hyp} = 1, \quad d = 1, \quad N_{0,2}^{hyp} = (-1)^{d+1} \kappa, \quad d \geq 2. \]

For the generating function \( \partial_t F_1 \), we get

\[ \partial_t F_1 = \frac{1}{12} \frac{q_t}{1-q_t} = \frac{1}{12} q_r. \]

This then tells that

\[ N_{1,d}^{hyp} = \frac{1}{12} d = 1, \quad N_{1,d}^{hyp} = 0, \quad d \geq 2. \]

For higher genus partition functions, we have

\[ \sum_{d=1}^{\infty} N_{g,d}^{hyp} q_r^d = \frac{|B_{2g}|}{2g(2g-2)!} \ln(1-q_r(1+q_r)^{-1}) = \frac{|B_{2g}|}{2g(2g-2)!} q_r^{g^{2g-3}}. \]

Since \( \theta_q := q_t, \partial_q \theta_q = (1+q_r) \partial_q \theta_q \), one can then find \( N_{g,d}^{hyp} \) by direct computations.

For any \( g \geq 2 \), the first few invariants with \( d = 1,2,3,\ldots \) are listed as follows:

\[ N_{g,d}^{hyp} : \]
\[ 1, -2 + 4^{2-g}, 6 - 3 \cdot 2^{5-2g} + 2 \cdot 9^{2-g}, \]
\[ -24 + 3 \cdot 2^{3-4g} - 8 \cdot 3^{5-2g} + 9 \cdot 4^{3-g}, \]
\[ 120 \left(1 - 2^{8-4g} - 2^{5-2g} + 5^{3-2g} + 2 \cdot 9^{2-g}\right). \]

**Example 5.0.4** (Local \( \mathbb{P}^2 \)). Now we consider the Calabi-Yau 3-fold \( K_{\mathbb{P}^2} \). In [8], the holomorphic limits of the first few topological string partition functions are solved genus by genus in terms of quasi modular forms and have nice expansions in \( q_r \). For example,

\[ C_{\text{ttt}} = -\frac{1}{3} \frac{\eta(3\tau)^3}{\eta(\tau)^3}, \]
\[ = -\frac{1}{3} \left(1 + 9q_r + 54q_r^2 + 252q_r^3 + 1008q_r^4 + 3591q_r^5 + \cdots \right), \]
\[ \partial_t F_1 = -\frac{1}{12} DF_1 \cdot \kappa^{-1} C_{\text{ttt}} = -\frac{1}{12} \left(\frac{3E_2(3\tau) + E_2(\tau) \eta(3\tau)^3}{4 \eta(\tau)^3}\right) \]
\[ = -\frac{1}{48} \left(1 + 3q_r - 18q_r^2 - 276q_r^3 - 1896q_r^4 - 9675q_r^5 + \cdots \right). \]
From these expansions one can immediately read off the numbers $N_{g,d}^{hyp}, g = 0, 1, d = 1, 2, \cdots$. The canonical coordinate $t$ is the following function of $q_\tau$:

$$\frac{1}{2\pi i} \frac{\partial t}{\partial \tau} = \kappa C^{-1}_{ttt} = \frac{\eta(\tau)^9}{\eta(3\tau)^3}, \quad t = \int \frac{dq_\tau}{q_\tau} \frac{\eta(\tau)^9}{\eta(3\tau)^3},$$

The constant from integration is fixed by comparing the asymptotic behaviors of $t$ and $\tau$ as $\tau \to i\infty$. A numeric experiment using Mathematica shows that

$$q_\tau = q_\tau - 9q_\tau^2 + 54q_\tau^3 - 246q_\tau^4 + 909q_\tau^5 - 2808q_\tau^6 + 7299q_\tau^7 - 15705q_\tau^8 + \cdots$$

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