A NEW APPROACH ON HELICES IN PSEUDO-RIEMANNIAN MANIFOLDS

EVREN ZIPLAR, YUSUF YAYLI, AND İSMAIL GÖK

Abstract. In this paper, we give a definition of harmonic curvature functions in terms of \( V_n \) and define a new kind of slant helix which is called \( V_n \)-slant helix in \( n \)-dimensional pseudo-Riemannian manifold. Also, we give important characterizations about the helix.

1. Introduction

Curves theory is an important framework in the differential geometry studies. Helix is one of the most fascinating curves because we see helical structure in nature, science and mechanical tools. Helices arise in the field of computer aided design, computer graphics, the simulation of kinematic motion or design of highways, the shape of DNA and carbon nanotubes. Also, we can see the helical structure in fractal geometry, for instance hyperhelices \([8, 18]\).

Furthermore, helices share common origins in the geometries of the platonic solids, with inherent hierarchical potential that is typical of biological structures. The helices provide an energy-efficient solution to close-packaging in molecular biology, a common motif in protein construction, and a readily observable pattern at many size levels throughout the body. The helices are described in a variety of anatomical structures, suggesting their importance to structural biology and manual therapy \([14]\).

A curve of constant slope or general helix in Euclidean 3-space \( E^3 \) is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 \((10) \text{ and } (16)\) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. In \([12]\), Özdamar and Hacısalıhoğlu defined harmonic curvature functions \( H_i \) \((1 \leq i \leq n - 2)\) of a curve \( \alpha \) and generalized helices in \( E^3 \) to \( n \)-dimensional Euclidean space \( E^n \). Moreover, they gave a characterization for the inclined curves in \( E^n \):

\[ \text{“A curve is an inclined curve if and only if } \sum_{i=1}^{n} H_i^2 = \text{constant} \] (1.1)

Harmonic curvature functions have important role in characterizations of general helices in higher dimensions. Because, the notion of a general helix can be generalized to higher dimension in different ways. However, these ways are not easy to show which curves are general helices and finding the axis of a general helix is complicated in higher dimension. Thanks to harmonic curvature functions, we can easily obtain the axis of such curves. Moreover, this way is confirmed in 3-dimensional spaces.

Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean 3–space \( E^3 \) \([7]\). In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4–space \( E^4 \) which is called \( B_2 \)-slant helix and they gave some characterizations of these slant helices in Euclidean 4–space \( E^4 \) \([11]\). And then in 2009, Gök et al. generalized \( B_2 \)-slant helix in \( E^4 \) to \( E^n \), \( n > 3 \), called \( V_n \)-slant helix in Euclidean and Minkowski \( n \)-space \((5, 6)\). Lots of authors in their papers have investigated inclined curves and slant helices using the harmonic curvature functions in Euclidean and Minkowski \( n \)-space \((12, 2, 9, 13, 4)\). But, Zıplar et al. \((17)\) see for the first time that the characterization of inclined curves and slant helices in (1.1) is true only for the case necessity but not true for the case sufficiency in Euclian \( n \)-space. Then, they consider the pre-characterizations about inclined curves and slant helices and restructure them with the necessary and sufficient condition \([17]\).

Similar to the working in \([17]\), in this work, we define \( V_n \)-slant helix and give characterizations about the helix with necessary and sufficient condition in \( n \)-dimensional pseudo-Riemannian manifolds for the first time.
2. Preliminaries

In this section, we give some basic definitions from differential geometry.

**Definition 2.1.** A metric tensor \( g \) on a smooth manifold \( M \) is a symmetric non-degenerate \((0,2)\) tensor field on \( M \).

In other words, \( g(X,Y) = g(Y,X) \) for all \( X,Y \in TM \) (tangent bundle) and at the each point \( p \) of \( M \) if \( g(X_p,Y_p) = 0 \) for all \( Y_p \in T_p(M) \), then \( X_p = 0 \) (non-degenerate), where \( T_p(M) \) is the tangent space of \( M \) at the point \( p \) and \( g: T_p(M) \times T_p(M) \to \mathbb{R} \).

**Definition 2.2.** A pseudo-Riemannian manifold (or semi-Riemannian manifold) is a smooth manifold \( M \) furnished with a metric tensor \( g \). That is, a pseudo-Riemannian manifold is an ordered pair \((M,g)\).

**Definition 3.1.** We shall recall the notion of a proper curve of order \( n \) in a \( n \)-dimensional pseudo-Riemannian manifold \( M \) with the metric tensor \( g \). Let \( \alpha : I \to M \) be a non-null curve in \( M \) parametrized by the arclength \( s \), where \( I \) is an open interval of the real line \( \mathbb{R} \). We denote the tangent vector field of \( \alpha \) by \( V_1 \). We assume that \( \alpha \) satisfies the following Frenet formula:

\[
\begin{align*}
\nabla_{V_i} V_1 &= k_1 V_2, \\
\nabla_{V_i} V_i &= -\varepsilon_{i-2} \varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, & 1 < i < n \\
\nabla_{V_i} V_n &= -\varepsilon_{n-2} \varepsilon_{n-3} k_{n-1} V_{n-1},
\end{align*}
\]

where

\[
\begin{align*}
k_1 &= \|\nabla_{V_1} V_1\| > 0 \\
k_i &= \|\nabla_{V_i} V_i + \varepsilon_{i-2} \varepsilon_{i-1} k_{i-1} V_{i-1}\| > 0, & 2 \leq i \leq n-1 \\
\varepsilon_{j-1} &= g(V_j, V_j) \ (= \pm 1), & 1 \leq j \leq n, \text{on } I,
\end{align*}
\]

and \( \nabla \) is Levi-Civita connection of \( M \).

We call such a curve a proper curve of order \( n \), \( k_i \) \((1 \leq i \leq n-1)\) its \( i \)-th curvature and \( V_1, \ldots, V_n \) its Frenet Frame field.

Moreover, let us recall that \( \|X\| = \sqrt{g(X,X)} \) for \( X \in TM \), where \( TM \) is the tangent bundle of \( M \).

3. \( V_n \)-slant helices and their harmonic curvature functions

In this section, we give definition of a \( V_n \)-slant helix curve in a \( n \)-dimensional pseudo-Riemannian manifold. Furthermore, we give characterizations by using harmonic curvatures for \( V_n \)-slant helices.

**Definition 3.2.** Let \( M \) be a \( n \)-dimensional pseudo-Riemannian manifold and let \( \alpha(s) \) be a proper curve of order \( n \) (non-null) with the curvatures \( k_i \) \((i = 1, \ldots, n-1)\) in \( M \). Then, harmonic curvature functions of \( \alpha \) are defined by

\[
H_i^* : I \subset \mathbb{R} \to \mathbb{R}
\]

along \( \alpha \) in \( M \), where

\[
\begin{align*}
H_0^* &= 0, \\
H_1^* &= \varepsilon_{n-3}\varepsilon_{n-2} k_{n-2}, \\
H_i^* &= (k_{n-i} H_{i-2}^* - \nabla_{V_i} H_{i-1}^*) \varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)} k_{n-(i+1)}, & 2 \leq i \leq n-2.
\end{align*}
\]

Note that \( \nabla_{V_i} H_{i-1}^* = V_i (H_{i-1}^*) = H_{i-1}^{**} \).

**Definition 3.3.** Let \((M,g)\) be a \( n \)-dimensional pseudo-Riemannian manifold and let \( \alpha(s) \) be a proper curve of order \( n \) (non-null). We call \( \alpha \) as a \( V_n \)-slant helix in \( M \) if the function

\[
g(V_n,X)
\]

is a non-zero constant along \( \alpha \) and \( X \) is a parallel vector field along \( \alpha \) in \( M \) \((i.e. \, \nabla_{V_i} X = 0)\). Here, \( V_n \) is \( n \)-th Frenet Frame field and \( X \in TM \). Also, \( X \) is called the axis of \( \alpha \).
Lemma 3.1. Let \((M,g)\) be a \(n\)-dimensional pseudo-Riemannian manifold and let \(\alpha(s)\) be a proper curve of order \(n\) (non-null). Let us assume that \(H^*_n \neq 0\) for \(i = n - 2\). Then, \(\varepsilon_{n-3}H^*_1 + \varepsilon_{n-4}H^*_2 + \ldots + \varepsilon_0H^*_n\) is non-zero constant if and only if \(V_1(H^*_n) = H^*_n = k_1H^*_n, \) where \(V_1\) and \(\{H^*_1, \ldots, H^*_n\}\) are the \(i\)-th derivatives of \(\alpha\), respectively.

**Proof.** First, we assume that \(\varepsilon_{n-3}H^*_1 + \varepsilon_{n-4}H^*_2 + \ldots + \varepsilon_0H^*_n\) is non-zero constant. Consider the following functions given in Definition 3.1

\[
H^*_i = \left(k_{n-i}H^*_{i-2} - H^*_{i-1}\right) \frac{\varepsilon_{n-(i+2)}H^*_{n-i+1}}{k_{n-(i+1)}},
\]

for \(3 \leq i \leq n - 2\). So, from the equality, we can write

\[
k_{n-(i+1)}H^*_i = \varepsilon_{n-(i+2)}H^*_{n-i+1} \left(k_{n-i}H^*_{i-2} - H^*_{i-1}\right).
\]

Hence, in (3.1), if we take \(i + 1\) instead of \(i\), we get

\[
\varepsilon_{n-(i+3)}H^*_i = \varepsilon_{n-(i+2)}H^*_i - \varepsilon_{n-(i+1)}H^*_i - k_{n-(i+2)}H^*_{i+1}, \quad 2 \leq i \leq n - 3
\]

and

\[
H^*_i = -\frac{1}{\varepsilon_{n-4}H^*_i}k_{n-3}H^*_2
\]

or

\[
H^*_i = -\varepsilon_{n-4}k_{n-3}H^*_{2i}.
\]

On the other hand, since \(\varepsilon_{n-3}H^*_1 + \varepsilon_{n-4}H^*_2 + \ldots + \varepsilon_0H^*_n\) is constant, we have

\[
\varepsilon_{n-3}H^*_1 + \varepsilon_{n-4}H^*_2 + \ldots + \varepsilon_0H^*_n = 0
\]

and so,

\[
\varepsilon_0H^*_n = -\varepsilon_{n-3}H^*_1 + \varepsilon_{n-4}H^*_2 - \varepsilon_{n-5}H^*_3 - \ldots - \varepsilon_1H^*_n - \varepsilon_0H^*_{n-3}.
\]

By using (3.2) and (3.3), we obtain

\[
H^*_i H^*_{i+1} = -\varepsilon_{n-4}k_{n-3}H^*_2
\]

and

\[
\varepsilon_{n-(i+3)}H^*_i = \varepsilon_{n-(i+2)}H^*_i - \varepsilon_{n-(i+1)}H^*_i - k_{n-(i+2)}H^*_{i+1}, \quad 2 \leq i \leq n - 3
\]

Therefore, by using (3.4), (3.5) and (3.6), an algebraic calculus shows that

\[
\varepsilon_0H^*_n = \varepsilon_0k_1H^*_n - \varepsilon_0k_2H^*_n - \varepsilon_0k_3H^*_n - \ldots - \varepsilon_0k_nH^*_n - \varepsilon_0k_{n-2}H^*_n
\]

or

\[
H^*_n = k_1H^*_n - k_2H^*_n - k_3H^*_n - \ldots - k_{n-2}H^*_n
\]

Since \(H^*_n \neq 0\), we get the relation

\[
H^*_n = k_1H^*_n - 2
\]

Conversely, we assume that

\[
H^*_n = k_1H^*_n - 2
\]

By using (3.7) and \(H^*_n \neq 0\), we can write

\[
H^*_n = k_1H^*_n - 2
\]

From (3.6), we have the following equation system:

| \(i\) | \(n - 3\) | \(n - 4\) | \(n - 5\) | \(2\) |
|-----|-----|-----|-----|-----|
| \(\varepsilon_1H^*_n\) | \(\varepsilon_2H^*_n\) | \(\varepsilon_3H^*_n\) | \(\varepsilon_4H^*_n\) | \(\varepsilon_5H^*_n\) |
| \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) |
| \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) |
| \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) |
| \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) |
| \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) | \(H^*_n\) |

Moreover, from (3.5) and (3.8), we obtain

\[
\varepsilon_{n-3}H^*_1 = -\varepsilon_{n-4}k_{n-3}H^*_2
\]

and

\[
\varepsilon_0H^*_n = \varepsilon_0k_{n-2}H^*_n - \varepsilon_0k_{n-3}H^*_n
\]
So, by using the above equation system and considering (3.9) and (3.10), an algebraic calculus shows that
\[ \varepsilon_{n-3} H_1^2 H_1' + \varepsilon_{n-4} H_2^2 H_2' + \ldots + \varepsilon_0 H_{n-2}^2 H_{n-2}' = 0. \] (3.11)
And, by integrating (3.11), we can easily say that
\[ \varepsilon_{n-3} H_1^2 + \varepsilon_{n-4} H_2^2 + \ldots + \varepsilon_0 H_{n-2}^2 \]
is a non-zero constant. This completes the proof. \(\square\)

**Proposition 3.1.** Let \((M, g)\) be a \(n\)-dimensional pseudo-Riemannian manifold and let \(\alpha\) (s) be a proper curve of order \(n\) (non-null). If \(\alpha\) is an \(V_n\)-slant helix in \(M\), then we have
\[ g \left( V_{n-(i+1)}, X \right) = H_i^* g \left( V_n, X \right), \quad i = 0, 1, \ldots, n-2, \] (3.12)
where \(X\) is the axis of \(\alpha\). Here, \(\{V_1, V_2, \ldots, V_n\}\) denote the Frenet Frame of \(\alpha\) and \(\{H_1^*, H_2^*, \ldots, H_{n-2}^*\}\) denote the harmonic curvature functions of the curve \(\alpha\).

**Proof.** We will use the induction method. Let \(i = 1\). Since \(X\) is the axis of the \(V_n\)-slant helix \(\alpha\), we get
\[ X = \lambda_1 V_1 + \ldots + \lambda_n V_n. \]
From the definition of \(V_n\)-slant helix, we have
\[ g \left( V_n, X \right) = \lambda_n \varepsilon_{n-1} = \text{constant}. \] (3.13)
A differentiation in Eq. (3.13) and the Frenet formulas gives us that
\[ g \left( V_{n-1}, X \right) = 0. \] (3.14)
Again, differentiation in Eq. (3.14) and the Frenet formulas give
\[ g \left( \nabla V_1 V_{n-1}, X \right) = 0, \]
\[ -\varepsilon_{n-3} \varepsilon_{n-2} k_{n-2} g \left( V_{n-2}, X \right) + k_{n-1} g \left( V_n, X \right) = 0, \]
\[ g \left( V_{n-2}, X \right) = \varepsilon_{n-3} \varepsilon_{n-2} \frac{k_{n-1}}{k_{n-2}} g \left( V_n, X \right), \]
\[ g \left( V_{n-2}, X \right) = H_1^* g \left( V_n, X \right), \]
respectively. Hence, it is shown that the Eq. (3.12) is true for \(i = 1\).

We now assume the Eq. (3.12) is true for the first \(i - 1\). Then, we have
\[ g \left( V_{n-i}, X \right) = H_{i-1}^* g \left( V_n, X \right). \] (3.15)
A differentiation in Eq. (3.15) and the Frenet formulas give us that
\[ -\varepsilon_{n-i-2} \varepsilon_{n-i-1} k_{n-i-1} g \left( V_{n-i-1}, X \right) + k_{n-i} g \left( V_{n-i+1}, X \right) = \nabla V_1 H_{i-1}^* g \left( V_n, X \right). \]
Since we have the induction hypothesis, \(g \left( V_{n-i+1}, X \right) = H_{i-2}^* g \left( V_n, X \right)\), we get
\[ (k_{n-i} H_{i-2}^* - \nabla V_1 H_{i-1}^*) \frac{\varepsilon_{n-(i+2)} \varepsilon_{n-(i+1)}}{k_{n-(i+1)}} g \left( V_n, X \right) = g \left( V_{n-(i+1)}, X \right), \]
which gives
\[ g \left( V_{n-(i+1)}, X \right) = H_i^* g \left( V_n, X \right). \]
\(\square\)

**Theorem 3.1.** Let \((M, g)\) be a \(n\)-dimensional pseudo-Riemannian manifold and let \(\alpha\) (s) be a proper curve of order \(n\) (non-null). Then, \(\alpha\) is a \(V_n\)-slant helix in \(M\) if and only if it satisfies that
\[ \sum_{i=1}^{n-2} \varepsilon_{n-(i+2)} H_i^2 \]
is equal to non-zero constant and \(H_{n-2}^* \neq 0\).

**Proof.** Suppose \(\alpha\) be a \(V_n\)-slant helix. According to the Definition 3.2 and the proof of Proposition 3.1,
\[ g \left( V_n, X \right) = \lambda_n \varepsilon_{n-1} = \text{constant}, \] (3.16)
where \(X\) the axis of \(\alpha\). From Proposition 3.1., we have
\[ g \left( V_{n-(i+1)}, X \right) = H_i^* g \left( V_n, X \right) \] (3.17)
for \(1 \leq i \leq n-2\). Moreover, from (3.16) and Frenet formulas, we can write
\[ -\varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} g \left( V_{n-1}, X \right) = 0. \]
On the other hand, \( \nabla \) is a basis of \( \kappa \). Moreover, from (3.18), we get the system
\[
\begin{align*}
\varepsilon_0 \lambda_1 &= g(X, V_1) \\
\varepsilon_1 \lambda_2 &= g(X, V_2) \\
&\vdots \\
\varepsilon_{n-3} \lambda_{n-2} &= g(X, V_{n-2}) \\
\varepsilon_{n-2} \lambda_{n-1} &= g(X, V_{n-1}) = 0 \\
\varepsilon_{n-1} \lambda_n &= g(X, V_n)
\end{align*}
\]
by using the metric \( g \). Therefore, from Proposition 3.1 and the above system, we can see that the following system is true:
\[
\lambda_1 = g(X, V_1) = \varepsilon_0 H^*_n g(X, V_n) \\
\lambda_2 = g(X, V_2) = \varepsilon_1 H^*_1 g(X, V_n) \\
&\vdots \\
\lambda_{n-2} = g(X, V_{n-2}) = \varepsilon_{n-3} H^*_1 g(X, V_n) \\
\lambda_{n-1} = g(X, V_{n-1}) = 0 \\
\lambda_n = \varepsilon_{n-1} g(X, V_n)
\]
Thus, it can be easily obtained the axis of the curve \( \alpha \) as
\[
X = g(X, V_n) \left\{ \sum_{i=1}^{n-2} H^*_i V_{n-(i+1)} \varepsilon_{n-(i+2)} + (\varepsilon_{n-1} V_n) \right\} \quad (3.19)
\]
by making use of the equality (3.18) and the last system.

Therefore, from (3.19), we can write
\[
g(X, X) = [g(X, V_n)]^2 \left( \varepsilon_0^3 H^*_n + \ldots + \varepsilon_{n-3}^3 H^*_1 + \varepsilon_{n-1}^3 \right) \quad (3.20)
\]
Moreover, by the definition of metric tensor, we have
\[
|g(X, X)| = \|X\|^2.
\]
Since \( \alpha \) is a \( V_n \)-slant helix, \( \|X\| = \text{constant} \) and \( g(X, V_n) \) is non-zero constant along \( \alpha \). Hence, from (3.20), we obtain
\[
\varepsilon_0^3 H^*_n + \ldots + \varepsilon_{n-3}^3 H^*_1 + \varepsilon_{n-1}^3
\]
is constant. In other words,
\[
\varepsilon_0 H^*_n + \ldots + \varepsilon_{n-3} H^*_1 = \sum_{i=1}^{n-2} \varepsilon_{n-(i+2)} H^*_i
\]
is constant.

Now, we will show that \( H^*_n \neq 0 \). We assume that \( H^*_n = 0 \). Then, for \( i = n-2 \) in (3.17), we have
\[
g(V_1, X) = H^*_n g(X, V_n) = 0 \quad (3.21)
\]
If we take derivative in each part of (3.21) in the direction \( V_1 \) on \( M \), then we have
\[
g(\nabla V_1 V_1, X) + g(V_1, \nabla V_1 V_1, X) = 0 \quad (3.22)
\]
On the other hand, \( \nabla V_1 X = 0 \) since \( \alpha \) is a \( V_n \)-slant helix. Then, from (3.22), we have
\[
g(\nabla V_1 V_1, X) = k_1 g(V_2, X) = 0
\]
by using the Frenet formulas. Since $k_1$ is positive, it must be $g(V_2, X) = 0$. Now, for $i = n - 3$ in (3.17),
\[ g(V_2, X) = H_{n-3}^* g(V_n, X). \]
Since $g(V_2, X) = 0$ and $g(V_n, X) \neq 0$, it must be $H_{n-3}^* = 0$. Continuing this process, we get $H_i^* = 0$. Let us recall that $H_i^* = \varepsilon_{n-3} \varepsilon_{n-2} \frac{k_{n-1}}{k_{n-2}}$, thus we have a contradiction because all the curvatures are nowhere zero. Consequently, $H_{n-2}^* \neq 0$.

Conversely, we assume that $\sum_{i=1}^{n-2} \varepsilon_{n-(i+2)} H_i^2 = \text{constant}$ and $H_{n-2}^* \neq 0$. We take the vector field
\[ X = \lambda_n V_n + \sum_{i=1}^{n-2} \lambda_n \varepsilon_{n-1} \varepsilon_{n-(i+2)} H_i^* V_n - (i+1), \]
or
\[ X = \lambda_n V_n + \lambda_n \varepsilon_{n-1} \sum_{i=1}^{n-3} \varepsilon_{n-i} H_i^* V_n - (i-1), \]
where $\lambda_n$ is constant. We will show that it is parallel along $\alpha$, i.e. $\nabla V_i X = 0$. By direct calculation, we have
\[
\nabla V_i X = \nabla V_i (\lambda_n V_n) + \sum_{i=1}^{n-1} \varepsilon_{n-i} \nabla V_i \left( H_{n-i-2}^* V_n - (i-1) \right)
\]
\[ = \lambda_n \nabla V_i V_n + \lambda_n \varepsilon_{n-1} \sum_{i=1}^{n-1} \varepsilon_{n-i} \left[ H_{n-i-2}^* V_n - (i-1) + H_i^* \nabla V_i V_{n-(i-1)} \right]
\]
\[ = \lambda_n \varepsilon_{n-1} \left[ -\varepsilon_n k_{n-1} V_n - 1 + \sum_{i=3}^{n-1} \varepsilon_{n-i} H_{n-i-2}^* V_n - (i-1) - \varepsilon_{n-(i+1)} k_{n-i-1} V_n - i H_{n-i-2}^* \right]
\]
\[ = \varepsilon_n k_{n-1} V_n - 1 + \varepsilon_0 H_{n-2}^* V_1 + \varepsilon_0 k_1 H_{n-2}^* V_2. \]
Here, in the case $n = 3$, we omit the term of sum.

On the other hand, by using (3.2), we can write
\[ \varepsilon_{n-(i+1)} \varepsilon_{n-i} H_{i-2}^* = \varepsilon_{n-(i+1)} \varepsilon_{n-i} k_{n-(i+1)} H_{i-3}^* - k_{n-i} H_{i-1}^* \quad (3.23) \]
for $4 \leq i \leq n - 1$ together with (3.3). Moreover, from Lemma 3.1, we know that
\[ H_{i-2}^* = k_1 H_{i-3}^*. \quad (3.24) \]
Therefore, by using (3.3), (3.23), (3.24) and by the definition of $H_i^*$, algebraic calculus shows that $\nabla V_i X = 0$. Besides, $g(V_n, X) = \lambda_n \varepsilon_{n-1}$ is constant. Consequently, $\alpha$ is a $V_n$-slant helix in $M$.

**Corollary 3.1.** Let $(M, g)$ be a $n$-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order $n$ (non-null). Then, $\alpha$ is a $V_n$-slant helix in $M$ if and only if $H_{n-2}^* = k_1 H_{n-3}^*$ and $H_{n-2}^* \neq 0$, where $\{H_1^*, H_2^*, ..., H_n^*\}$ denote the harmonic curvature functions of the curve $\alpha$.

**Proof.** It is obvious by using Lemma 3.1. and Theorem 3.1. \qed

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Department of Mathematics, Faculty of Science, Cankırı University, Cankırı, Turkey
E-mail address: evrenziplar@karatekin.edu.tr

Department of Mathematics, Faculty of Science, University of Ankara, Tandoğan, Turkey
E-mail address: yayli@science.ankara.edu.tr

Department of Mathematics, Faculty of Science, University of Ankara, Tandoğan, Turkey
E-mail address: igok@science.ankara.edu.tr