Overcoming the Obstacle of Fixed Eigenvalues in Decentralized Control

F. Liu and A. S. Morse

Abstract—Fixed eigenvalues (i.e., fixed modes) present obstacles to the decentralized stabilization and decentralized spectrum assignment of a multi-channel linear system, because they will appear in the closed-loop spectrum of the system with any given finite dimensional linear time-invariant decentralized control. This paper introduces a simple approach to fully eliminate the fixed eigenvalues of a jointly controllable and jointly observable multi-channel linear system by combining decentralized control with distributed control, that is, by allowing local communication between neighboring channels. It will be shown that if the neighbor graph associated with the channels is strongly connected, this approach can synthesize an augmented multi-channel linear system without fixed eigenvalues.

Index Terms—Fixed eigenvalues, fixed modes, multi-channel linear systems, decentralized control, distributed control.

I. INTRODUCTION

A “decentralized linear system”, or a multi-channel linear system is of the form

\[
\dot{x} = Ax + \sum_{i=1}^{k} B_i u_i, \quad y_i = C_i x
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times m_i} \), \( C_i \in \mathbb{R}^{p_i \times n} \), and \( k \geq 1 \) is the number of channels. Let \( k = \{1, 2, \ldots, k\} \). It is clear that each channel \( i \in k \) knows the system coefficient matrices \( A, B_i, \) and \( C_i \), has access to the control input \( u_i \), and is able to measure the output signal \( y_i \). The classical “decentralized control” problems \([1, 2]\) of \([1]\) impose the restriction that the measurement output \( y_i \) of each channel \( i \in k \) can be fed back to only its corresponding control input \( u_i \) possibly through a linear dynamic controller. Wang and Davison \([1]\) showed that no matter what these feedback controllers might be, as long as they are finite dimensional and linear time-invariant (LTI), the spectrum of the resulting closed-loop system contains a fixed subset of eigenvalues depending only on \( A \). The \( B_i \)’s and the \( C_i \)’s for all \( i \in k \), which they elected to called the set of “fixed modes” of the system. In the sequel, we will use the term “fixed eigenvalues”, because technically modes are not eigenvalues. Roughly speaking, the fixed eigenvalues of \([1]\), are the the eigenvalues of \( A \) that cannot be shifted with the decentralized output feedback laws \( u_i = F_i y_i, i \in k \). That is, a complex number \( \lambda \) is a fixed eigenvalue of \([1]\) if

\[
\lambda \in \bigcap_{F_i \in \mathbb{R}^{m_i \times r_i}} \sigma \left( A + \sum_{i=1}^{k} B_i F_i C_i \right)
\]

where \( \sigma(\cdot) \) denotes the spectrum. Since the \( F_i \)’s can be zero, a fixed eigenvalue of \([1]\) is clearly an eigenvalue of \( A \). If the right-hand side of \([2]\) is an empty set, the system is said to have no fixed eigenvalues.

Interestingly enough, the set of fixed eigenvalues (including multiplicity) of \([1]\) is proved to be the same as the subset of eigenvalues of \( A \) not shiftable by any finite dimensional LTI decentralized controller \([1]\). It means that the fixed eigenvalues are always contained in the resulting closed-loop spectrum, which presents obstacles to the decentralized stabilization and decentralized spectrum assignment problems treated in \([1, 2]\).

As a result, a necessary and sufficient condition for decentralized stabilization of \([1]\) with a decentralized LTI control is that the fixed eigenvalues are all stable \([1]\). Furthermore, a necessary and sufficient condition for “free” assignability of an overall closed-loop spectrum of \([1]\) with a decentralized LTI control is that the system has no fixed eigenvalues \([2]\). However, it should be noted that unlike the centralized case, free spectrum assignability in the decentralized case presumes that the overall spectrum admits a suitable partition into a finite number of symmetric sets, the partition being determined by the strongly connected components in a suitably defined directed graph of \([1, 2]\).

From the preceding there is no doubt that fixed eigenvalues play a central role in the decentralized control of multi-channel linear systems. However, it is computationally intractable to check by the definition in \([2]\) whether \( \lambda \in \mathbb{C} \) is a fixed eigenvalue of \([1]\). A convenient test for this is given in \([3]\). As it will be used a few times in this paper, we explicitly cite the test below. Let \( S = \{i_1, i_2, \ldots, i_s\} \subset k \) with \( i_1 < i_2 < \cdots < i_s \); the complement of \( S \) in \( k \) is denoted by \( k - S = \{j_1, j_2, \ldots, j_{k-s}\} \) with \( j_1 < j_2 < \cdots < j_{k-s} \). Let

\[
B_S = [B_{i_1} B_{i_2} \cdots B_{i_s}], \quad C_{k-S} = \begin{bmatrix} C_{j_1} \\ C_{j_2} \\ \vdots \\ C_{j_{k-s}} \end{bmatrix}
\]

Proposition 1: \([3]\) A \( k \)-channel linear system \( \{A, B_i, C_i; k\} \) given by \([1]\) has a fixed eigenvalue \( \lambda \in \sigma(A) \) if and only if \( \exists S \subset k \) such that

\[
\text{rank} \begin{bmatrix} A & B_S \\ C_{k-S} & 0 \end{bmatrix} < n
\]

It has been recognized that some fixed eigenvalues are less “fixed” than others. More specifically, some fixed eigenvalues of \([1]\) can be eliminated with appropriately defined linear time-varying decentralized controllers \([4, 5]\), with sampling strategies \([6, 7]\) or other techniques \([8, 9]\). The remaining fixed eigenvalues, called the firmly fixed eigenvalues, are characterized previously in \([10]\) where they are called the “quotient fixed modes”. The influence of the firmly fixed eigenvalues
cannot be eliminated with any decentralized control even if it is time-varying and/or nonlinear.

Fortunately, that is not the case when the decentralized restriction is relaxed a little bit by allowing local communication between neighboring channels. This paper shows that all the fixed eigenvalues of $P$ except those in the uncontrollable spectrum of $(A, B_k)$ or in the unobservable spectrum of $(C_k, A)$ can be eliminated using a simple method, provided that the neighbor graph associated with the channels is strongly connected.

II. Motivation and Problem Formulation

The popular distributed control of multi-agent systems and sensor networks studies a class of problems which usually request all agents or sensors in a network to cooperatively perform specific tasks based on the information each agent/sensor detects or receives from its “neighbors” in the network. It not only enhances the robustness of the systems in general, but also sometimes can solve problems that are difficult or impossible for an individual agent or sensor to solve. In a certain sense, distributed control lies somewhere in the middle of the “spectrum” with two extremes of centralized control and decentralized control at either end. Inspired by a recent result [11] which shows that the fixed eigenvalues of $P$ can be eliminated by means of a distributed state observer based feedback control, this paper identifies the key techniques for fixed eigenvalue elimination, namely state space extension and distributed control (or local communication), and introduces a simple approach to eliminate fixed eigenvalues without building a distributed observer. A detailed analysis of the two techniques is given below, which indicates that both of them are essential.

State space extension means to extend the state space of the original system $P$ by adding an appropriate number of integrators to each channel $P$. As noted in [11] and in Section VI of [12], the technique of state space extension alone does not help with the elimination of fixed eigenvalues.

On the other hand, without using integrators to extend the state space, a distributed control which only allows each channel $i \in k$ to share its measurement signal $y_i$ within a neighborhood may still result in a system with fixed eigenvalues. This point will be illustrated by an example.

Before giving the example, some terms need to be defined. A $k$-channel linear system $\{A, B_i, C_i; k\}$ given by $P$ is said to be jointly controllable and jointly observable if the matrix pairs $(A, B_k)$ and $(C_k, A)$ are controllable and observable, respectively. In the context of distributed control, each channel can receive information from its controllable neighbors, which are usually physically closed to it. For $i, j \in k$, channel $j$ is called a neighbor of channel $i$ if channel $i$ can receive information from channel $j$. Note that neighbor relations need not be symmetric. Let $N_i \subset k$ denote the set of labels of channel $i$’s neighbors excluding itself. In this paper, it is assumed that each channel’s neighbors do not change with time. The neighbor graph $N$ of the system is a directed graph defined on $k$ vertices, one for each channel, with an arc from vertex $j$ to vertex $i$ whenever channel $j$ is a neighbor of channel $i$. Thus $N$ is assumed to be fixed over time. It is also assumed throughout this paper that the neighbor graph $N$ is strongly connected. The question of what information should the channels send to each other in order to eliminate the fixed eigenvalues will be discussed in the following example.

Suppose we are given a jointly controllable and jointly observable 3-channel system

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

with a cyclic neighbor graph shown in Figure 1. If there is no state space extension, each channel $j \in k$ can share only its measurement signal $y_j$ within a neighborhood. A convenient way to model the case where channel $i \in k$ knows all of its neighbors measurement signals is to expand channel $i$’s output coefficient matrix to include $C_j$ for all $j \in N_i$. In this 3-channel system example, the expanded output coefficient matrices are respectively

$$\bar{C}_1 = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{C}_2 = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} C_3 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to check that

$$\text{rank} \begin{bmatrix} A - I & B_1 & B_3 \\ \bar{C}_2 & 0 & 0 \end{bmatrix} = 2 < 3$$

then it follows from Proposition [11] that system $\{A, B_i, \bar{C}_i; 3\}$ has a fixed eigenvalue of 1. This example demonstrates that the sole technique of distributed control cannot fully eliminate the fixed eigenvalues of a jointly controllable and jointly observable multi-channel linear system.

From the above discussion, it is clear that neither state space extension nor distributed control is sufficient for our purpose. Now we are ready to formulate the problem. Let $\{A, B_i, C_i; k\}$ given by $P$ be a jointly controllable and jointly observable $k$-channel linear system with a strongly connected and fixed neighbor graph $N$. Suppose each channel $i \in k$ can receive the current value of each of its neighbor’s extended state vector, the problem of interest is to construct a suitably defined family of integrators at each channel such that the augmented $k$-channel linear system has no fixed eigenvalues. It turns out that this problem can be resolved by a fairly simple design.
III. MAIN RESULT

In this session, it will be shown that the fixed spectrum of a jointly controllable and jointly observable multi-channel linear system can be fully avoided with a combination of state space extension and distributed control.

Suppose each channel of (1) has an extended state space of dimension $d \in \mathbb{N}$. Let $z_i \in \mathbb{R}^d$ be the extended state vector of channel $i$. The dimension $d$ will be discussed in Theorem 1.

In the setting of distributed control, each channel receives the value of the extended state vector $z_i$ from each of its neighbors $j \in \mathcal{N}_i$. Let

$$
z_i = \sum_{j \in \mathcal{N}_i} (z_j - g z_i) + v_i, \quad w_i = z_i$$

(4)

where $g \in \mathbb{R}$ is a scalar parameter, $v_i$ is the control input and $w_i$ is the measurement output for the extended state $z_i$. If the stability of the $z_i$’s is required, we can take $g > 1$ in (4). Otherwise, we can simply take $g = 0$. For economy, it is desirable to keep the dimension of the extended state space small. It will be shown that with a lower bound on $d$, this simple construction achieves the goals of fixed spectrum elimination (Theorem 1) and free spectrum assignment (Theorem 2).

Let $M$ be the $k \times k$ matrix associated with the neighbor graph $\mathcal{N}$ of (1) such that the $ij$th entry of $M$ is 1 whenever there is an arc from channel $j$ to channel $i$ for $i,j \in \mathcal{N} \triangleq \{1, 2, \ldots, k\}$, and is 0 otherwise. Let $|\mathcal{N}_i|$ be the number of neighbors of channel $i$. Let $D \triangleq \text{diag}\{|\mathcal{N}_1|, |\mathcal{N}_2|, \ldots, |\mathcal{N}_k|\}$ be a $k \times k$ diagonal matrix. Let matrix $H \triangleq M - gD$. Let $e_i, i \in \mathcal{k}$, denote the $i$th unit column vector in $\mathbb{R}^k$, which has 1 in the $i$th entry and has 0 in all other entries. Let

$$
\begin{align*}
\bar{x} &= \begin{bmatrix} x \\ z_1 \\ \vdots \\ z_k \end{bmatrix}, \quad \bar{u}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}, \quad \bar{y}_i = \begin{bmatrix} y_i \\ w_i \end{bmatrix}, & \text{for } i \in \mathcal{k} \\
\bar{A} &= \begin{bmatrix} A & 0 \\ 0 & H \otimes I_d \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i & 0 \\ 0 & e_i \otimes I_d \end{bmatrix}, \\
\bar{C}_i &= \begin{bmatrix} C_i & 0 \\ 0 & e_i \otimes I_d \end{bmatrix} 
\end{align*}
\quad (5)
$$

where $\otimes$ denotes the Kronecker product and $'$ denotes the matrix transposition. Combining systems (1) and (4), we get an augmented system

$$
\dot{x} = \bar{A} \bar{x} + \sum_{i=1}^{k} \bar{B}_i \bar{u}_i, \quad \bar{y}_i = \bar{C}_i \bar{x}
$$

(6)

A sufficient condition on the dimension $d$ of each channel’s extended state space will be derived, for which the augmented system (6) has no fixed spectrum. For $S = \{i_1, i_2, \ldots, i_s\} \subset \mathcal{k}$ with $i_1 < i_2 < \cdots < i_s$, let $E_S \triangleq \begin{bmatrix} e_{i_1} & e_{i_2} & \ldots & e_{i_s} \end{bmatrix}$.

Lemma 1: If the neighbor graph $\mathcal{N}$ of (1) is strongly connected,

$$
\text{rank} \begin{bmatrix} H - \lambda I_k & E_S \\ E_{k-S} & 0 \end{bmatrix} \geq k + 1
$$

for each nonempty proper subset $S \subset \mathcal{k}$ and each $\lambda \in \mathcal{C}$.

Proof of Lemma 1: It is claimed that for each nonempty proper subset $S \subset \mathcal{k}$ and each $\lambda \in \mathcal{C}$, the submatrix of $H - \lambda I_k$ obtained by deleting the $i$th row of $H - \lambda I_k$ for all $i \in S$ and deleting the $j$th column of $H - \lambda I_k$ for all $j \in \mathcal{k-S}$ is non-zero. It is because there is at least one arc from some channel $i \in S$ to some channel $j \in \mathcal{k-S}$, otherwise the neighbor graph $\mathcal{N}$ is not strongly connected. Thus

$$
\text{rank} \begin{bmatrix} H - \lambda I_k & E_S \\ E_{k-S} & 0 \end{bmatrix} \geq k + 1
$$

Theorem 1: Let $\{A, B_i, C_i; k\}$ given by (1) be a jointly controllable and jointly observable $k$-channel linear system of dimension $n$. Suppose the neighbor graph $\mathcal{N}$ of these $k$ channels is strongly connected and fixed over time, and for each channel $i \in \mathcal{k}$ the extended state vector $z_i$ of dimension $d$ is given by (4). If

$$
d \geq n - \min_{S \subseteq \mathcal{k}} \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix}
$$

(7)

where $\sigma(A)$ denotes the spectrum of $A$, the augmented system $\{A, B_i, C_i; k\}$ given by (5) and (6) has no fixed spectrum.

Proof of Theorem 1: First note that

$$
\begin{align*}
\text{rank} \begin{bmatrix} A - \lambda I_n + k & B_S \\ C_{k-S} & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} A - \lambda I_n + k & B_S \\ C_{k-S} & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \\
&= n + kd
\end{align*}
$$

Similarly, if $S = \emptyset$, as $\{A, B_i, C_i; k\}$ is jointly controllable, for any $\lambda \in \mathcal{C}$, we have

$$
\begin{align*}
\text{rank} \begin{bmatrix} A - \lambda I_n + k & B_S \\ C_{k-S} & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \\
&= n + kd
\end{align*}
$$

If $S$ is a nonempty proper subset of $\mathcal{k}$, it follows immediately from Lemma 1 that for any $\lambda \in \mathcal{C}$,

$$
\text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \geq kd + d
$$

Thus if $d$ satisfies (7),

$$
\begin{align*}
\text{rank} \begin{bmatrix} A - \lambda I_n + k & B_S \\ C_{k-S} & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} H \otimes I_d - \lambda I_{kd} & E_S \otimes I_d \\ E_{k-S} \otimes I_d & 0 \end{bmatrix} \\
&\geq \min_{S \subseteq \mathcal{k}} \text{rank} \begin{bmatrix} A - \lambda I_n & B_S \\ C_{k-S} & 0 \end{bmatrix} + kd + d \\
&\geq n + kd
\end{align*}
$$

By Proposition 1, the augmented system $\{A, B_i, C_i; k\}$ has no fixed spectrum.
If a multi-channel linear system has no fixed spectrum and its graph defined in [2] (not the neighbor graph!) is strongly connected, by Theorem 5 in [2], with controllers of appropriate dimensions the closed-loop spectrum can be freely assigned to any symmetric set of complex numbers rather than a partition of two or more symmetric sets. Next, it will be shown that this is indeed the case for the augmented system \( \{ A, B_i, C_i; k \} \) when the neighbor graph \( \mathbb{N} \) is strongly connected. To differentiate from the neighbor graph \( \mathbb{N} \), the graph in [2] will be called the transfer graph, because its arcs are determined by the transfer matrices between channels. The transfer graph \( T \) of a \( k \)-channel linear system \( \{ A, B_i, C_i; k \} \) is a directed graph which has \( k \) vertices, one corresponding to each channel, and has an arc from vertex \( i \) to vertex \( j \) whenever the transfer matrix \( C_j (\lambda I - A)^{-1} B_i \) from \( u_i \) to \( y_j \) is nonzero.

**Lemma 2:** If the neighbor graph \( \mathbb{N} \) of the \( k \)-channel linear system \( \{ A, B_i, C_i; k \} \) given by (1) is strongly connected, the transfer graph \( T \) of the \( k \)-channel linear system \( \{ H, e_i, e_j; k \} \) is also strongly connected for almost every \( g \in \mathbb{R} \).

**Proof of Lemma 2:** It is known that the transfer matrix \( C(\lambda I - A)^{-1} B \) of a \( k \)-dimensional system \( (C, A, B) \) is nonzero if and only if \( C [B \ AB \ldots A^{k-1} B] \neq 0 \). Thus it suffices to show that

\[
e_j^t e_i \quad H e_i \ldots \quad H^{k-1} e_i \neq 0, \quad \forall i, j \in k
\]

for almost every \( g \in \mathbb{R} \). Recall that \( H = M - g D \), so we only need to prove it for \( g = 0 \), i.e.,

\[
e_j^t e_i \quad M e_i \ldots \quad M^{k-1} e_i \neq 0, \quad \forall i, j \in k
\]  \( \text{(8)} \)

By the definition of matrix \( M \), it is clear that if the \( j \)th entry of \( M^t e_i \) is nonzero, where \( i, j \in k \) and \( t \in \mathbb{N} \), then there is a walk of length \( t \) from channel \( i \) to channel \( j \). As the neighbor graph \( \mathbb{N} \) is strongly connected, for each \( i, j \in k \), there is a path from channel \( i \) to channel \( j \) with a length no greater than \( k - 1 \). Therefore, (8) is true.

**Theorem 2:** Let \( \{ A, B_i, C_i; k \} \) given by (1) be a jointly controllable and jointly observable \( k \)-channel linear system of dimension \( n \). Suppose the neighbor graph \( \mathbb{N} \) of these \( k \) channels is strongly connected and fixed over time, and for each channel \( i \in k \) the extended state vector \( z_i \) of dimension \( d_i \) is given by (4). If (7) holds, the closed-loop spectrum of the augmented system \( \{ A, B_i, C_i; k \} \) given by (5) and (6) can be freely assigned to any symmetric set of complex numbers with controllers of appropriate dimensions for almost every \( g \in \mathbb{R} \).

**Proof of Theorem 2:** By Lemma 2 the transfer graph \( T \) of \( \{ H, e_i, e_j; k \} \) is strongly connected for almost every \( g \in \mathbb{R} \). So are the transfer graphs of \( \{ H \otimes I_d, e_i \otimes I_d, e_j \otimes I_d; k \} \) and thus of the augmented system \( \{ \bar{A}, \bar{B}_i, \bar{C}_i; k \} \). It follows that there exist matrices \( F_i \), \( i \in k \), of appropriate sizes such that the \( k \)-channel system \( \{ \bar{A} + \sum_{i \in k} \bar{B}_i F_i \bar{C}_i, \bar{B}_i, \bar{C}_i; k \} \) is controllable and observable through any channel [2], therefore the closed-loop spectrum of \( \{ \bar{A}, \bar{B}_i, \bar{C}_i; k \} \) can be freely assigned for almost every \( g \in \mathbb{R} \) using the technique in [13].

1 A walk in a graph allows for the possibility that a vertex in the walk is visited more than once.

**IV. Concluding Remarks**

This paper establishes a simple construction to fully eliminate the fixed eigenvalues of a jointly controllable and jointly observable multi-channel linear system by both techniques of state space extension and distributed control. It is worth pointing out that in this approach, the only centralized information for each channel to know is an integer, which is

\[
\min_{S \subset k} \forall \lambda \in \sigma(A) \quad \begin{bmatrix} A - \lambda I_n & B_S \\ C_k - S & 0 \end{bmatrix}
\]

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