COHOMOLOGICAL OBSTRUCTIONS AND WEAK CROSSED PRODUCTS OVER WEAK HOPF ALGEBRAS

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Abstract. Let $H$ be a cocommutative weak Hopf algebra and let $(B, \varphi_B)$ a weak left $H$-module algebra. In this paper, for a twisted convolution invertible morphism $\sigma : H \otimes H \to B$ we define its obstruction $\theta_\sigma$ as a degree three Sweedler 3-cocycle with values in the center of $B$. We obtain that the class of this obstruction vanish in third Sweedler cohomology group $H^3_{\varphi_B}(H, Z(B))$ if, and only if, there exists a twisted convolution invertible 2-cocycle $\alpha : H \otimes H \to B$ such that $H \otimes B$ can be endowed with a weak crossed product structure with $\alpha$ keeping a cohomological-like relation with $\sigma$. Then, as a consequence, the class of the obstruction of $\sigma$ vanish if, and only if, there exists a cleft extension of $B$ by $H$.

Keywords: Weak Hopf algebra, Sweedler cohomology, weak crossed products, cleft extension, obstruction.

MSC2020: 18M05, 16T05, 20J06.

Introduction

Crossed products of a Hopf algebra by an algebra have been widely studied in relation to extensions of algebras, generalizing classical results of semidirect products and extensions of groups, together with a generalization of group cohomology to the Hopf algebra setting. In [17] Sweedler defines the so-called Sweedler’s cohomology for a cocommutative Hopf algebra $H$ and a commutative $H$-module algebra $B$. In this paper he also shows that any cleft $H$-extension of algebras $B \hookrightarrow A$ (that is, roughly speaking, a split extension) can be realized as a crossed product given in terms of the action of $H$ on $B$ and a 2-cocycle $\sigma : H \otimes H \to B$. Moreover, cleft extensions of $B$ are classified by the second cohomology group $H^2(H, B)$. Several generalizations of these results were carried out by Doi and Takeuchi [7], Blattner, Cohen and Montgomery [11] and Blattner and Montgomery [13] by dropping out the conditions of cocommutativity and commutativity, and the associativity of the action $\varphi_B : H \otimes B \to B$ and thus, the use of Sweedler’s cohomology. However some of its formalism is preserved: for an arbitrary Hopf algebra $H$ and an arbitrary algebra $B$, a crossed product is given in terms of a measuring $\varphi_B : H \otimes B \to B$ and a formal 2-cocycle $\sigma : H \otimes H \to B$ that must also satisfy the twisted condition needed to substitute the associativity of $\varphi_B$. Moreover, two such crossed products are equivalent if the cocycles satisfy a cohomological-like equivalence. This last result was interpreted in an actual cohomological setting by Doi in [8], where he shows that cleft extensions of an algebra $B$ by a cocommutative Hopf algebra $H$ with the same action are classified by $H^2(H, Z(B))$, where $Z(B)$ denotes the center of $B$. All these
results can be interpreted in a symmetric monoidal category with base object \( K \) (see for example [1] and [11] for cleft extensions in a monoidal setting).

The next objective became to decide when an algebra \( B \) admits a cleft extension by \( H \). Following some classical results of obstructions to extensions of groups (see, for example, [13]), Schauenburg finds in [16] a relation between the third Sweedler’s cohomology group \( \mathcal{H}^3(H, Z(B)) \) and cleft extensions, where \( Z(B) \) denotes the center of \( B \). For a measuring \( \varphi_B \) and a twisted morphism \( \sigma \), he generalizes the notion of obstruction as Sweedler three cocycle \( \theta_{\sigma} \) on \( H \) with values on the center of \( B \) and shows that the class \( [\theta_{\sigma}] \in \mathcal{H}^3(H, Z(B)) \) vanish if, and only if, \( \varphi_B \) and \( \sigma \) give rise to a crossed product on \( H \otimes B \) and, at last, to a cleft extension.

With the apparition of weak Hopf algebras as generalizations of groupoid algebras (see [6]) all the theory of cleft extensions, Sweedler’s cohomology and crossed products needed a deep review. Recall that the main point of a algebra-coalgebra \( H \) to be weak is that its unit does not need to be coassociative, nor its counit associative. These apparently innocent generalizations conceptually imply the existence of two objects, different from the base object \( K \) in the ground monoidal category when \( H \) is actually weak, that somehow will play the role of \( K \). From a practical point of view, this lack of (co)associativity of the (co)unit forces to a change in the definition of regular morphisms, and thus to a change in the tackling of cleft extensions, cohomological interpretations of crossed products and a rethinking of cohomology and crossed products themselves. For the cocommutative case these problems were successfully solved in [2] and [3], where the authors explore the meaning of cleft extension and weak crossed product for a cocommutative weak Hopf algebra \( H \) weakly acting on an algebra \( B \), and define Sweedler’s cohomology in weak contexts. In order to achieve these objectives, they consider the unit in \( \text{Reg}(H, B) \) as \( \varphi_B \circ (H \otimes \eta_B) \) (and thus, regular morphisms depend on \( \varphi_B \) and we denote the set by \( \text{Reg}_{\varphi_B}(H, B) \), where \( \varphi_B \) is the weak action of \( H \) on \( B \), and \( \eta_B \) is the unit of \( B \)). Moreover, to study weak crossed products they consider a preunit instead of a unit, so they obtain an algebra as a subobject of \( H \otimes B \), whose product is given in terms of \( \varphi_B \) and a twisted formal 2-cocycle \( \sigma : H \otimes H \to B \). In such terms, they are able to define a cohomology theory for a cocommutative weak Hopf algebra \( H \) and a commutative \( H \)-module algebra \( B \). Moreover, they identify cleft \( H \)-extensions of a weak \( H \)-module algebra \( B \) with products with convolution invertible 2-cocycle and classify them by \( \mathcal{H}^2_{\varphi_Z(B)}(H, Z(B)) \), this is, the second cohomology group. The relation of weak crossed products and cleft extensions for the non-cocommutative case was also studied in [12] by Guccione, Guccione and Valqui.

So once we have the proper concepts of cleft extensions, weak crossed products and Sweedler’s cohomology for the weak setting, we just need to find out the role of obstructions in relation to cleft extensions and their cohomological meaning, and these are the main objectives of the present paper. In order to attain such objectives, we first make a wide review of weak crossed products, and we find that we just need a measuring \( \varphi_B : H \otimes B \to B \) together with a twisted morphism \( \sigma : H \otimes H \to B \) that does not need to be convolution invertible but a formal 2-cocycle to define a weak crossed product on \( H \otimes B \). Moreover we obtain necessary and sufficient conditions for weak crossed products to be equivalent that, in particular, are given in terms of morphisms in \( \text{Reg}_{\varphi_B}(H, B) \). We finally use these results in the particular case of a cocommutative weak Hopf algebra \( H \) and a weak \( H \)-module algebra \( (B, \varphi_B) \).
We consider a twisted convolution invertible morphism $\sigma : H \otimes H \rightarrow B$ and define its cohomological obstruction $\theta_\sigma$ through the center of $B$. We obtain that this obstruction vanish in $\mathcal{H}_{\varphi Z(B)}^3(H, \mathcal{Z}(B))$ if, and only if, there exists a twisted convolution invertible 2-cocycle $\alpha : H \otimes H \rightarrow B$ such that $H \otimes B$ can be endowed with a weak crossed product structure with $\alpha$ keeping a cohomological-like relation with $\sigma$. This result means, in terms of cleft extensions, that if $(B, \varphi_B)$ is a weak $H$-module algebra with $\sigma : H \otimes H \rightarrow B$ twisted and convolution invertible then its obstruction vanish if, and only if, there exists a cleft extension of $B$ by $H$.

Throughout this paper $C$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object $K$ and symmetry isomorphism $c$. There is no loss of generality in assuming that $C$ is strict because every monoidal category is monoidally equivalent to a strict one. Then, we may work as if the constrains were all identities. We also assume that in $C$ every idempotent morphism splits, i.e., for any morphism $q : M \rightarrow M$ such that $q \circ q = q$ there exists an object $N$, called the image of $q$, and morphisms $i : N \rightarrow M$, $p : M \rightarrow N$ such that $q = i \circ p$ and $p \circ i = id_N$ ($id_N$ denotes the identity morphism for $N$). The morphisms $p$ and $i$ will be called a factorization of $q$. Note that $N$, $p$ and $i$ are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with (co)equalizers.

Finally, given objects $M$, $N$, $P$ and a morphism $f : N \rightarrow P$, we write $M \otimes f$ for $id_M \otimes f$ and $f \otimes M$ for $f \otimes id_M$.

An algebra in $C$ is a triple $A = (A, \eta_A, \mu_A)$, where $A$ is an object in $C$ and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in $C$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. We say that an algebra $A$ is commutative if $\mu_A = \mu_A \circ c_{A,A}$.

Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \rightarrow B$ in $C$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

Also, if $A$, $B$ are algebras in $C$, the object $A \otimes B$ is an algebra in $C$, where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. Note that, if $A$ and $B$ are commutative algebras, so is $A \otimes B$.

A coalgebra in $C$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where $D$ is an object in $C$ and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in $C$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. We say that a coalgebra $D$ is cocommutative if $\delta_D = c_{D,D} \circ \delta_D$.

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, a morphism $f : D \rightarrow E$ in $C$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

If $D, E$ are coalgebras in $C$, the tensor product $D \otimes E$ is a coalgebra in $C$, where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. Note that, if $D$ and $E$ are cocommutative coalgebras, so is $D \otimes E$.

Finally, if $A$ is an algebra, $C$ is a coalgebra and $f : C \rightarrow A$, $g : C \rightarrow A$ are morphisms in $C$, we define the convolution product by $f \ast g = \mu_A \circ (f \otimes g) \circ \delta_C$.

1. Generalities on measurings and crossed products in a weak setting

In this section we resume some basic facts about the general theory of weak crossed products in $C$, introduced in [9], particularized for measurings over a weak Hopf algebra $H$. Firstly, we recall the notion
of weak Hopf algebra, introduced in [6], and summarize some basic properties of these algebraic objects in a monoidal setting.

**Definition 1.1.** A weak bialgebra $H$ is an object in $C$ with an algebra structure $(H, \eta_H, \mu_H)$ and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H \circ H$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

Moreover, if there exists a morphism $\lambda_H : H \to H$ in $C$ (called the antipode of $H$) satisfying

(a4) $id_H \ast \lambda_H = ((\varepsilon_H \otimes \mu_H) \otimes H) \circ (H \otimes c_{H,H} \otimes H)$,

(a5) $\lambda_H \ast id_H = (H \otimes (\varepsilon_H \otimes \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \otimes \eta_H))$,

(a6) $\lambda_H \ast id_H \ast \lambda_H = \lambda_H$,

we will say that the weak bialgebra is a weak Hopf algebra.

We say that $H$ is commutative, if it is commutative as algebra and we say that $H$ is cocommutative if it is cocommutative as coalgebra.

1.2. Let $H$ be a weak bialgebra. For $n \geq 1$, we denote by $H^{\otimes n}$ the $n$-fold tensor power $H \otimes \cdots \otimes H$. By $H^{\otimes 0}$ we denote the unit object of $C$, i.e., $H^{\otimes 0} = K$. If $n \geq 2$, $m^{\otimes n}_H$ denotes the morphism $m^{\otimes n}_H : H^{\otimes n} \to H$ defined by $m^{\otimes 2}_H = \mu_H$ and by $m^{\otimes 3}_H = m^{\otimes 2}_H \circ (H \otimes \mu_H), \ldots, m^{\otimes k}_H = m^{\otimes (k-1)}_H \circ (H^{k-2} \otimes \mu_H)$ for $k \geq 2$. On the other hand, with $\delta^{\otimes n}_H$ we denote the coproduct defined in the coalgebra $H^{\otimes n}$. Then by the coassociativity of $\delta_H$ and the naturality of $c$, for $k = 1, \ldots, n-1$,

$$\delta^{\otimes n}_H = \delta^{\otimes (n-k)}_H \otimes H^{\otimes k}$$

holds. By [2] Proposition 2.10 we have that

$$\delta_H \circ m^{\otimes n}_H = (m^{\otimes n}_H \otimes m^{\otimes n}_H) \circ \delta_H \circ H^{\otimes n}.$$

Finally, note that, if $H$ is cocommutative, then so is $H^{\otimes n}$.

1.3. For any weak bialgebra, if we define the morphisms $\Pi^L_H$ (target), $\Pi^R_H$ (source), $\Pi^L_H$ and $\Pi^R_H$ by

$$\Pi^L_H = \left( (\varepsilon_H \otimes \mu_H) \otimes H \right) \circ (H \otimes c_{H,H} \otimes H),$$

$$\Pi^R_H = \left( H \otimes (\varepsilon_H \otimes \mu_H) \right) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \otimes \eta_H)),$$

$$\Pi^L_H = (H \otimes (\varepsilon_H \otimes \mu_H)) \circ ((\varepsilon_H \otimes \delta_H) \otimes H),$$

$$\Pi^R_H = ((\varepsilon_H \otimes \mu_H) \otimes H) \circ (H \otimes (\delta_H \otimes \eta_H)),$$

it is straightforward to show that they are idempotent and the equalities

(1) $\Pi^L_H \circ \Pi^L_H = \Pi^L_H$, \quad $\Pi^R_H \circ \Pi^R_H = \Pi^R_H$, \quad $\Pi^R_H \circ \Pi^L_H = \Pi^L_H$, \quad $\Pi^L_H \circ \Pi^R_H = \Pi^R_H$,

(2) $\Pi^L_H \circ \Pi^L_H = \Pi^L_H$, \quad $\Pi^R_H \circ \Pi^R_H = \Pi^R_H$, \quad $\Pi^R_H \circ \Pi^L_H = \Pi^L_H$, \quad $\Pi^L_H \circ \Pi^R_H = \Pi^R_H$,
hold.

On the other hand, denote by $H_L$ the image of the target morphism $\Pi^L_H$ and let $p^L_H : H \to H_L$, $i^L_H : H_L \to H$ be the morphisms such that $i^L_H \circ p^L_H = \Pi^L_H$ and $p^L_H \circ i^L_H = id_{H_L}$. Then

$$(H_L, \eta_{H_L} = p^L_H \circ \eta_H, \mu_{H_L} = p^L_H \circ \mu_H \circ (i^L_H \otimes i^L_H))$$

is an algebra and

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i^L_H, \delta_{H_L} = (p^L_H \otimes p^L_H) \circ \delta_H \circ i^L_H)$$

is a coalgebra. The morphisms $\eta_{H_L}, \mu_{H_L}, \varepsilon_{H_L}$ and $\delta_{H_L}$ are the unique morphisms satisfying

$$(i^L_H \circ \eta_{H_L} = \eta_H, ~ i^L_H \circ \mu_{H_L} = \mu_H \circ (i^L_H \otimes i^L_H), ~ \varepsilon_{H_L} \circ p^L_H = \varepsilon_H, ~ \delta_{H_L} \circ p^L_H = (p^L_H \otimes p^L_H) \circ \delta_H)$$

respectively.

For the morphisms target and source we have the following identities:

$$(\Pi^L_H \circ \mu_H \circ (H \otimes \Pi^L_H) = \Pi^L_H \circ \mu_H, ~ \Pi^R_H \circ \mu_H \circ (\Pi^R_H \otimes H) = \Pi^R_H \circ \mu_H)$$

$$(\Pi^L_H \circ \mu_H \circ (H \otimes \Pi^L_H) = \Pi^L_H \circ \mu_H, ~ \Pi^R_H \circ \mu_H \circ (\Pi^R_H \otimes H) = \Pi^R_H \circ \mu_H)$$

$$(H \otimes \Pi^L_H) \circ \delta_H \circ \Pi^L_H = \delta_H \circ \Pi^L_H, ~ (\Pi^R_H \otimes H) \circ \delta_H \circ \Pi^R_H = \delta_H \circ \Pi^R_H)$$

$$(H \otimes \Pi^L_H) \circ \delta_H \circ \Pi^L_H = \delta_H \circ \Pi^L_H, ~ (\Pi^R_H \otimes H) \circ \delta_H \circ \Pi^R_H = \delta_H \circ \Pi^R_H)$$

$$(\mu_H \circ (H \otimes \Pi^L_H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), ~ (\Pi^R_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H))$$

$$(\Pi^L_H \circ H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H(\Pi^R_H \otimes H))$$

$$(\mu_H \circ (\Pi^R_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), ~ (\Pi^L_H \circ H) \circ \delta_H = (H \otimes \delta_H) \circ (\Pi^R_H \otimes H)$$

$$(\Pi^L_H \circ H) \circ \delta_H = (H \otimes \mu_H) \circ (\delta_H \otimes H), ~ (\Pi^R_H \circ H) \circ \delta_H = (H \otimes \delta_H \otimes H))$$

$$(\Pi^L_H \circ H) \circ \delta_H = (H \otimes \delta_H) \circ (\Pi^R_H \otimes H)$$

$$(\Pi^R_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H(\Pi^L_H \otimes H))$$

$$(\delta_H \circ \eta_H = (\Pi^R_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \Pi^L_H) \circ \delta_H \circ \eta_H = (H \otimes \Pi^R_H) \circ \delta_H \circ \eta_H$$

$$= (\Pi^L_H \otimes H) \circ \delta_H \circ \eta_H, ~ \varepsilon_H \circ \mu_H = \varepsilon_H \circ \mu_H \circ (\Pi^R_H \otimes H) = \varepsilon_H \circ \mu_H \circ (\Pi^R_H \otimes H)$$

$$= \varepsilon_H \circ \mu_H \circ (H \otimes \Pi^L_H),$$
If $H$ is a weak Hopf algebra in $C$, the antipode $\lambda_H$ is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant, i.e.:

\begin{equation}
\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,
\end{equation}

\begin{equation}
\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.
\end{equation}

Also, it is straightforward to show that $\Pi^L_H$, $\Pi^R_H$ satisfy the equalities

\begin{equation}
\Pi^L_H = id_H \ast \lambda_H, \quad \Pi^R_H = \lambda_H \ast id_H, \quad \Pi^L_H \ast id_H = id_H = id_H \ast \Pi^R_H,
\end{equation}

\begin{equation}
\Pi^L_H \ast \lambda_H = \lambda_H = \lambda_H \ast \Pi^L_H,
\end{equation}

\begin{equation}
\Pi^L_H \ast \Pi^L_H = \Pi^L_H, \quad \Pi^R_H \ast \Pi^R_H = \Pi^R_H
\end{equation}

and

\begin{equation}
\Pi^L_H = \lambda_H \circ \Pi^L_H = \Pi^R_H \circ \lambda_H, \quad \Pi^R_H = \Pi^L_H \circ \lambda_H = \lambda_H \circ \Pi^R_H.
\end{equation}

Finally we also have

\begin{equation}
\mu_H \circ c_{H,H} \circ (H \otimes \Pi^L_H) \circ \delta_H = id_H = \mu_H \circ c_{H,H} \circ (\Pi^R_H \otimes H) \circ \delta_H.
\end{equation}

Now we recall the notions of measuring, left weak $H$-module algebra, and left $H$-module algebra.

**Definition 1.4.** Let $H$ be a weak Hopf algebra and let $B$ be an algebra. We say that the morphism $\varphi_B : H \otimes B \rightarrow B$ is a measuring if

1. $\varphi_B \circ (H \otimes \mu_B) = \mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes B \otimes B)$.

Set $u_{1}^{\varphi_B} = \varphi_B \circ (H \otimes \eta_B)$. If $\varphi_B$ is a measuring satisfying

2. $\varphi_B \circ (\eta_H \otimes B) = id_B$,

3. $u_{1}^{\varphi_B} \circ \mu_H = \varphi_B \circ (H \otimes u_{1}^{\varphi_B})$,

we will say that $(B, \varphi_B)$ is a left weak $H$-module algebra. If we replace (b3) by

\begin{equation}
\varphi_B \circ (\mu_H \otimes B) = \varphi_B \circ (H \otimes \varphi_B)
\end{equation}

we will say that $(B, \varphi_B)$ is a left $H$-module algebra.

If $(B, \varphi_B)$ is a left weak $H$-module algebra the following equivalent conditions are satisfied:

\begin{equation}
\varphi_B \circ (\Pi^L_H \otimes B) = \mu_B \circ (u_{1}^{\varphi_B} \otimes B),
\end{equation}

\begin{equation}
\varphi_B \circ (\Pi^R_H \otimes B) = \mu_B \circ c_{B,B} \circ (u_{1}^{\varphi_B} \otimes B),
\end{equation}

\begin{equation}
u_{1}^{\varphi_B} \circ \Pi^L_H = u_{1}^{\varphi_B},
\end{equation}

\begin{equation}
u_{1}^{\varphi_B} \circ \Pi^R_H = u_{1}^{\varphi_B},
\end{equation}

\begin{equation}
u_{1}^{\varphi_B} \circ \mu_H = u_{1}^{\varphi_B} \circ \mu_H \circ (H \otimes \Pi^L_H),
\end{equation}

\begin{equation}
u_{1}^{\varphi_B} \circ \mu_H = u_{1}^{\varphi_B} \circ \mu_H \circ (H \otimes \Pi^R_H).
\end{equation}
Let $\varphi_B$ be a measuring and $n \geq 1$. With $\varphi_B^\otimes n$ we will denote the morphism

$$
\varphi_B^\otimes n : H^\otimes n \otimes B \to B
$$
defined as $\varphi_B^\otimes 1 = \varphi_B$ and $\varphi_B^\otimes n = \varphi_B \circ (H \otimes \varphi_B^{\otimes (n-1)})$.

Note that, by (b1) of Definition 1.4, proceeding by induction, it is easy to show that

$$
(\varphi_B^\otimes n \circ (H^\otimes n \otimes \mu_B) = \mu_B \circ (\varphi_B^\otimes n \otimes \varphi_B^\otimes n) \circ (H^\otimes n \otimes c_{H^\otimes n, B} \otimes B) \circ (\delta_{H^\otimes n} \otimes B \otimes B)
$$

holds.

For $n \geq 2$ we define the morphism

$$
u_n^\varphi_B = \varphi_B \circ (m_H^\otimes n \otimes \eta_B) = \varphi_B^{\otimes (n-1)} \circ (H^{\otimes (n-1)} \otimes u_1^\varphi_B).
$$

Note that

$$u_n^\varphi_B = u_1^\varphi_B \circ m_H^\otimes n
$$
and, by [2, Proposition 2.11], we have

$$u_n^\varphi_B * u_n^\varphi_B = u_n^\varphi_B.
$$

**Definition 1.5.** Let $H$ be a weak Hopf algebra and let $B$ be an algebra. For a measuring $\varphi_B$ and any morphism $\sigma : H \otimes H \to B$, we define the morphisms

$$P_{\varphi_B} : H \otimes B \to B \otimes H, \quad F_{\sigma} : H \otimes H \to B \otimes H, \quad G_{\sigma} : H \otimes H \to H \otimes B,
$$
by

$$P_{\varphi_B} = (\varphi_B \otimes H) \circ (H \otimes c_{H, B}) \circ (\delta_H \otimes B),
$$

$$F_{\sigma} = (\sigma \otimes \mu_H) \circ \delta_{H^\otimes 2}
$$
and

$$G_{\sigma} = (\mu_H \otimes \sigma) \circ \delta_{H^\otimes 2}.
$$

By [2, Proposition 3.3] and some easy computations we have the following result.

**Proposition 1.6.** Let $H$ be a weak Hopf algebra and let $\varphi_B : H \otimes B \to B$ be a measuring. The morphism $P_{\varphi_B}$ defined in (35) satisfies

$$
(\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (P_{\varphi_B} \otimes B) = P_{\varphi_B} \circ (H \otimes \mu_B).
$$
The morphisms $\nabla_{B \otimes H}^\varphi : B \otimes H \to B \otimes H$ and $\nabla_{H \otimes B}^\varphi : H \otimes B \to H \otimes B$ defined by

$$
\nabla_{B \otimes H}^\varphi = (\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (B \otimes H \otimes \eta_B)
$$
and

$$
\nabla_{H \otimes B}^\varphi = (H \otimes \mu_B) \circ ((H \otimes \varphi_A) \circ (\delta_H \otimes \eta_A)) \otimes A
$$
are idempotent. Also, we have the following identities:

$$P_{\varphi_B} \circ (H \otimes \eta_B) = (u_1^\varphi_B \otimes H) \circ \delta_H,
$$
(40) \[(B \otimes \varepsilon_H) \circ P_{\varphi_B} = \varphi_B,\]

(41) \[\nabla^{\varphi_B} \otimes P_{\varphi_B} = P_{\varphi_B},\]

(42) \[\nabla^{\varphi_B}_{B \otimes H} = (\mu_B \otimes H) \circ (B \otimes ((u_1^{\varphi_B} \otimes H) \circ \delta_H)),\]

(43) \[\nabla^{\varphi_B}_{H \otimes B} = (H \otimes \mu_B) \circ (((H \otimes u_1^{\varphi_B}) \circ \delta_H) \otimes B),\]

(44) \[\nabla^{\varphi_B}_{B \otimes H} \circ (\eta_B \otimes H) = (u_1^{\varphi_B} \otimes H) \circ \delta_H,\]

(45) \[\nabla^{\varphi_B}_{H \otimes B} \circ (H \otimes \eta_B) = (H \otimes u_1^{\varphi_B}) \circ \delta_H,\]

(46) \[(\mu_B \otimes H) \circ (B \otimes \nabla^{\varphi_B}_{B \otimes H}) = (B \otimes \nabla^{\varphi_B}_{B \otimes H}) \circ (\mu_B \otimes H),\]

(47) \[(H \otimes \mu_B) \circ (\nabla^{\varphi_B}_{H \otimes B} \otimes B) = (\nabla^{\varphi_B}_{H \otimes B} \otimes B) \circ (H \otimes \mu_B),\]

(48) \[(B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} = \mu_B \circ (B \otimes u_1^{\varphi_B}),\]

(49) \[(\varepsilon_H \otimes B) \circ \nabla^{\varphi_B}_{H \otimes B} = \mu_B \circ (u_1^{\varphi_B} \otimes B),\]

(50) \[(B \otimes \delta_H) \circ \nabla^{\varphi_B}_{B \otimes H} = (\nabla^{\varphi_B}_{B \otimes H} \otimes H) \circ (B \otimes \delta_H),\]

(51) \[(\delta_H \otimes B) \circ \nabla^{\varphi_B}_{H \otimes B} = (H \otimes \nabla^{\varphi_B}_{H \otimes B}) \circ (\delta_H \otimes B),\]

(52) \[\mu_B \circ (u_1^{\varphi_B} \otimes \varphi_B) \circ (\delta_H \otimes B) = \varphi_B,\]

(53) \[(\mu_B \otimes H) \circ (u_1^{\varphi_B} \otimes P_{\varphi_B}) \circ (\delta_H \otimes B) = P_{\varphi_B},\]

(54) \[(B \otimes \varepsilon_H) \circ P_{\varphi_B} \circ (H \otimes \eta_B) = u_1^{\varphi_B},\]

(55) \[(\mu_B \otimes H) \circ (u_1^{\varphi_B} \otimes c_{H,B}) \circ (\delta_H \otimes B) = (\mu_B \otimes H) \circ (H \otimes c_{B,B}) \circ ((P_{\varphi_B} \circ (H \otimes \eta_B)) \otimes B).\]

On the other hand, by a similar proof to the one used in \[2\] Proposition 3.4| we have the following proposition.

**Proposition 1.7.** Let \(H\) be a weak Hopf algebra, let \(\varphi_B : H \otimes B \to B\) be a measuring and let \(\sigma : H \otimes H \to B\) be a morphism. The morphisms \(F_\sigma\) and \(G_\sigma\) defined in (55) and (56) satisfy the identities:

(56) \[(B \otimes \delta_H) \circ F_\sigma = (F_\sigma \otimes \mu_H) \circ \delta_{H \otimes 2},\]

(57) \[(\delta_H \otimes B) \circ G_\sigma = (\mu_H \otimes G_\sigma) \circ \delta_{H \otimes 2}.\]

Moreover, we also have the proposition:
Proposition 1.8. Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \to B$ be a measuring. The equality
\begin{equation}
\mu_B \circ (B \otimes u_1^B) \circ P_{\varphi_B} = \varphi_B,
\end{equation}
holds.

Let $\sigma : H \otimes H \to B$ be a morphism and $u_2^B$ be the morphism defined in (52). If $\sigma \ast u_2^B = \sigma$, the equality
\begin{equation}
\mu_B \circ (B \otimes u_1^B) \circ F_\sigma = \sigma,
\end{equation}
holds and, as a consequence, we have the following identities:
\begin{align}
\nabla^{\varphi_B}_{B \otimes H} \circ F_\sigma &= F_\sigma, \\
(B \otimes \varepsilon_H) \circ F_\sigma &= \sigma.
\end{align}

Moreover, if $\sigma : H \otimes H \to B$ is a morphism satisfying (60) and (61), we have that $\sigma \ast u_2^B = \sigma$.

Proof. Note that, (58) holds because
\begin{equation}
\mu_B \circ (B \otimes u_1^B) \circ P_{\varphi_B} = (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ P_{\varphi_B} = (B \otimes \varepsilon_H) \circ P_{\varphi_B} = \varphi_B.
\end{equation}

Trivially,
\begin{equation}
\mu_B \circ (B \otimes u_1^B) \circ F_\sigma = \sigma \ast u_2^B
\end{equation}
and then we obtain (60). On the other hand,
\begin{equation}
\nabla^{\varphi_B}_{B \otimes H} \circ F_\sigma = ((\mu_B \circ (B \otimes u_1^B) \circ F_\sigma) \otimes \mu_H) \circ \delta_{H \otimes 2} = F_\sigma.
\end{equation}

Therefore,
\begin{equation}
(B \otimes \varepsilon_H) \circ F_\sigma = (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ F_\sigma = \mu_B \circ (B \otimes u_1^B) \circ F_\sigma = \sigma.
\end{equation}

Finally, let $\sigma : H \otimes H \to B$ be a morphism satisfying (60) and (61). Then,
\begin{equation}
\sigma = (B \otimes \varepsilon_H) \circ F_\sigma = (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ F_\sigma = \mu_B \circ (B \otimes u_1^B) \circ F_\sigma = \sigma \ast u_2^B.
\end{equation}

\begin{flushright}
\Box
\end{flushright}

1.9. Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \to B$ be a measuring and let $\sigma : H \otimes H \to B$ be a morphism. In a similar way to what was proven in the previous proposition, we can ensure that the equality
\begin{equation}
\mu_B \circ (u_1^B \otimes B) \circ G_\sigma = u_2^B \ast \sigma
\end{equation}
holds. As a consequence we can obtain the following result:
Proposition 1.10. Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \to B$ be a measuring and let $\sigma : H \otimes H \to B$ be a morphism. Let $u^\varphi_B$ be the morphism defined in (12). If $u^\varphi_B \ast \sigma = \sigma$, the equality
\begin{equation}
\mu_B \circ (u^\varphi_B \otimes B) \circ G_\sigma = \sigma,
\end{equation}
holds and, as a consequence, we have the following identities:
\begin{equation}
\nabla_B^\varphi \circ G_\sigma = G_\sigma,
\end{equation}
\begin{equation}
(\varepsilon \otimes B) \circ G_\sigma = \sigma.
\end{equation}
Moreover, if $\sigma : H \otimes H \to B$ is a morphism satisfying (65) and (66), we have that $u^\varphi_B \ast \sigma = \sigma$.

Remark 1.11. By the previous propositions, (12) Propositions 2.7 and 2.8 and (11), (2), if $\sigma : H \otimes H \to B$ satisfies that $\sigma \ast u^\varphi_B = \sigma$, we have
\begin{equation}
\sigma \circ (\mu_B \otimes H) \circ (H \otimes \Pi_H^B \otimes H) = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \Pi_H^B \otimes H),
\end{equation}
\begin{equation}
\sigma \circ (\mu_B \otimes H) \circ (H \otimes \Pi_H^B \otimes H) = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \Pi_H^B \otimes H)
\end{equation}
and, if $u^\varphi_B \ast \sigma = \sigma$, the equalities
\begin{equation}
\sigma \circ (\mu_H \otimes H) \circ (H \otimes \Pi_H^B \otimes H) = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \Pi_H^B \otimes H),
\end{equation}
\begin{equation}
\sigma \circ (\mu_H \otimes H) \circ (H \otimes \Pi_H^B \otimes H) = \sigma \circ (H \otimes \mu_H) \circ (H \otimes \Pi_H^B \otimes H)
\end{equation}
hold.

1.12. Let $H$ be a weak Hopf algebra and let $B$ be an algebra. By the previous results, if $\varphi_B : H \otimes B \to B$ is a measuring, and $\sigma : H \otimes H \to B$ is a morphism such that $\sigma \ast u^\varphi_B = \sigma$, we have a quadruple
\begin{equation}
\mathbb{B}_H = (B, H, \psi^B_H = P_{\varphi_B}, \sigma^B_H = F_\sigma)
\end{equation}
as the ones introduced in (9) to define the notion of weak crossed product. For the quadruple $\mathbb{B}_H$ there exists a product in $B \otimes H$ defined by
\begin{equation}
\mu_B \otimes_{\varphi_B} H = (\mu_B \otimes H) \circ (\mu_B \otimes F_\sigma) \circ (B \otimes P_{\varphi_B} \otimes H)
\end{equation}
and let $\mu_{A \times H}^\varphi_{\varphi_B}$ be the product
\begin{equation}
\mu_{B \times H}^\varphi = p_{B \otimes H}^\varphi \circ \mu_{B \otimes H} \circ (i_{B \otimes H}^\varphi \otimes i_{B \otimes H}^\varphi),
\end{equation}
where $B \times_{\varphi_B} H$, $i_{B \otimes H}^\varphi : B \otimes \varphi_B H \to B \otimes H$ and $p_{B \otimes H}^\varphi : B \otimes H \to B \times_{\varphi_B} H$ denote the image, the injection, and the projection associated to the factorization of $\nabla_B^\varphi$.

Following (9) we say that $\mathbb{B}_H$ satisfies the twisted condition if
\begin{equation}
(\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (F_\sigma \otimes B) = (\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B})
\end{equation}
and the cocycle condition holds if
\begin{equation}
(\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (F_\sigma \otimes H) = (\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes F_\sigma).
\end{equation}
Note that, if $\mathcal{B}_H$ satisfies the twisted condition, by [9] Proposition 3.4, and (60) we obtain

\begin{equation}
(\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes \nabla_{B \otimes H}^{\varphi_B}) = (\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi_B} \otimes H), \tag{73}
\end{equation}

\begin{equation}
(\mu_B \otimes H) \circ (A \otimes F_\sigma) \circ (\nabla_{B \otimes H}^{\varphi_B} \otimes H) = (\mu_B \otimes H) \circ (B \otimes F_\sigma). \tag{74}
\end{equation}

Then, composing with $B \otimes \varepsilon_H$ in (74), and applying (61) we have

\begin{equation}
\mu_B \circ (B \otimes \sigma) \circ (\nabla_{B \otimes H}^{\varphi_B} \otimes H) = \mu_B \circ (B \otimes \sigma). \tag{75}
\end{equation}

Therefore, if $\mathcal{B}_H$ satisfies the twisted condition, the following equality

\begin{equation}
\mu_B \circ (B \otimes \sigma) \circ ((\nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes H)) \otimes H) = \sigma. \tag{76}
\end{equation}

holds.

**Theorem 1.13.** Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \to B$ be a measuring, and let $\sigma : H \otimes H \to B$ be a morphism such that $\sigma \ast u_2^H = \sigma$. Let $\mathcal{B}_H$ be the associated quadruple. Then, the following assertions hold.

(i) The quadruple $\mathcal{B}_H$ satisfies the twisted condition (71) iff $\sigma$ satisfies the twisted condition

\begin{equation}
\mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) = \mu_B \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes B). \tag{77}
\end{equation}

(ii) The quadruple $\mathcal{B}_H$ satisfies the cocycle condition (72) iff $\sigma$ satisfies the cocycle condition

\begin{equation}
\mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes F_\sigma) = \mu_B \circ (B \otimes \sigma) \circ (F_\sigma \otimes B). \tag{78}
\end{equation}

**Proof.** The proof follows as in [2] Theorems 3.12 and 3.13 because the cocommutativity condition for $H$ is not necessary and (61) holds.

If the twisted and the cocycle conditions hold, the product $\mu_{B \otimes \varphi_B} \circ H$ is associative and normalized with respect to $\nabla_{B \otimes H}^{\varphi_B}$, i.e.,

\begin{equation}
\nabla_{B \otimes H}^{\varphi_B} \circ \mu_{B \otimes \varphi_B} = \mu_{B \otimes \varphi_B} = \mu_{B \otimes \varphi_B} \circ (\nabla_{B \otimes H}^{\varphi_B} \otimes \nabla_{B \otimes H}^{\varphi_B}) \tag{79}
\end{equation}

and we have

\begin{equation}
\mu_{B \otimes \varphi_B} \circ (\nabla_{B \otimes H}^{\varphi_B} \otimes B \otimes H) = \mu_{B \otimes \varphi_B} \circ (B \otimes H \otimes \nabla_{B \otimes H}^{\varphi_B}). \tag{80}
\end{equation}

Then, $\mu_{B \otimes \varphi_B} \circ H$ is associative as well (see [9] Proposition 3.7]). Hence we define:

**Definition 1.14.** If $\mathcal{B}_H$ satisfies (71) and (72) we say that $(B \otimes H, \mu_{B \otimes \varphi_B} \circ H)$ is a weak crossed product.

The next natural question that arises is if it is possible to endow $\mu_{B \otimes \varphi_B} \circ H$ with a unit, and hence with an algebra structure. As we recall in [2], we need to use the notion of preunit to obtain this unit. In our setting, if $\mu_{B \otimes \varphi_B} \circ H$ is an associative product, by [9] Remark 2.4, $\nu : K \to B \otimes H$ is a preunit if

\begin{equation}
\mu_{B \otimes \varphi_B} \circ (\nu \otimes B \otimes H), \quad \nu = \mu_{B \otimes \varphi_B} \circ (\nu \otimes B \otimes H). \tag{81}
\end{equation}

By [9] Corollary 3.12], we know that, if $\nu$ is a preunit for $(B \otimes H, \mu_{B \otimes \varphi_B} \circ H)$, the object $B \otimes \varphi_B \circ H$ is an algebra with product $\mu_{B \otimes \varphi_B} \circ H$ and unit $\eta_{B \otimes \varphi_B} \circ H = p_{B \otimes H} \circ \nu$. 

The following proposition is a tool to establish the conditions under which the morphism \( \nu = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \) is a preunit for \( \mu_{B \otimes \varphi_B} \).

**Proposition 1.15.** Let \( H \) be a weak Hopf algebra, let \( \varphi_B : H \otimes B \to B \) be a measuring, and let \( \sigma : H \otimes H \to B \) be a morphism such that \( \sigma \ast u_{2}^{\varphi_B} = \sigma \). Then, the following equalities hold.

\[
\sigma \circ (\eta_H \otimes H) = \sigma \circ c_{H,H} \circ (H \otimes \Pi_{H}^{L}) \circ \delta_H, \tag{82}
\]

\[
\sigma \circ (H \otimes \eta_H) = \sigma \circ (H \otimes \Pi_{H}^{R}) \circ \delta_H. \tag{83}
\]

**Proof.** The first equality follows from equalities (68) and (24), and the second one is consequence of (67) and (21). \( \square \)

**Definition 1.16.** Let \( H \) be a weak Hopf algebra, let \( \varphi_B : H \otimes B \to B \) be a measuring, and let \( \sigma : H \otimes H \to B \) be a morphism. We say that \( \sigma \) satisfies the normal condition if

\[
\sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_{1}^{\varphi_B}. \tag{84}
\]

Therefore, if \( \sigma \ast u_{2}^{\varphi_B} = \sigma \), by Proposition 1.15 \( \sigma \) is normal if and only if

\[
\sigma \circ c_{H,H} \circ (H \otimes \Pi_{H}^{L}) \circ \delta_H = \sigma \circ (H \otimes \Pi_{H}^{R}) \circ \delta_H = u_{1}^{\varphi_B}. \tag{85}
\]

**Theorem 1.17.** Let \( H \) be a weak Hopf algebra, let \( \varphi_B : H \otimes B \to B \) be a measuring such that

\[
\nabla_{B \otimes H}^{\varphi_B} \circ (B \otimes \eta_H) = P_{\varphi_B} \circ (\eta_H \otimes B), \tag{85}
\]

and let \( \sigma : H \otimes H \to B \) be a morphism such that \( \sigma \ast u_{2}^{\varphi_B} = \sigma \). Let \( \Xi_H \) be the associated quadruple and assume that \( \Xi_H \) satisfies the twisted and the cocycle conditions (71) and (72). Then, \( \nu = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \) is a preunit for the weak crossed product associated to \( \Xi_H \) if and only if

\[
F_{\sigma} \circ (\eta_H \otimes H) = F_{\sigma} \circ (H \otimes \eta_H) = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes H). \tag{86}
\]

**Proof.** By [9] Theorem 3.11] a morphism \( \nu \) is a preunit for the associated weak crossed if and only if

\[
(\mu_B \otimes H) \circ (B \otimes F_{\sigma}) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes \nu) = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes H), \tag{87}
\]

\[
(\mu_B \otimes H) \circ (B \otimes F_{\sigma}) \circ (\nu \otimes H) = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes H) \tag{88}
\]

and

\[
(\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (\nu \otimes B) = (\mu_B \otimes H) \circ (B \otimes \nu) \tag{89}
\]

hold. Then the theorem follows because, on the one hand

\[
(\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (\nu \otimes B) = (\mu_B \otimes H) \circ (B \otimes F_{\sigma}) \circ ((P_{\varphi_B} \circ (\eta_H \otimes \eta_B)) \otimes B) \quad \text{(by the unit properties)}
\]

\[
= P_{\varphi_B} \circ (\eta_H \otimes B) \quad \text{(by (85) and the unit properties)}
\]

\[
= \nabla_{B \otimes H}^{\varphi_B} \circ (B \otimes \eta_H) \quad \text{(by (85))}
\]

\[
= (\mu_B \otimes H) \circ (B \otimes \nu) \quad \text{(by the unit properties)}.
\]
and, on the other hand, by the unit properties and (71), we have
\[(\mu_B \otimes H) \circ (B \otimes F_\sigma) \circ (P_{\varphi B} \otimes H) \circ (H \otimes \nu) = F_\sigma \circ (H \otimes \eta_H).\]

Finally, by (74), the equality
\[(\mu_B \otimes H) \circ (B \otimes P_{\varphi B}) \circ (\nu \otimes B) = F_\sigma \circ (\eta_H \otimes H)\]
holds.

□

Remark 1.18. Note that if \( \nu \) is a preunit for the weak crossed product associated to \( B \otimes H \), by [9, Theorem 3.11], the equality (89) holds for \( \nu \). Then,
\[\nabla_{B \otimes H} \circ \nu = \nu\]
holds. Therefore the preunit of a weak crossed product, if it exists, is unique because if \( (B \otimes H, \mu_{B \otimes \varphi B} H) \) admits two preunits \( \nu_1, \nu_2 \), we have \( \eta_B \times \varphi_B H = \eta_{B \otimes H} = \eta_{B \otimes H} = \eta_{B \otimes H} \) and then
\[v_1 = \nabla_{B \otimes H} \circ \nu_1 = \nabla_{B \otimes H} \circ \nu_2 = \nu_2.\]

As a consequence of Theorem 1.17 and by Proposition 1.15 we have:

Corollary 1.19. Let \( H \) be a weak Hopf algebra, let \( \varphi_B : H \otimes B \to B \) be a measuring, let \( \sigma : H \otimes H \to B \) be a morphism and let \( B \otimes H \) be the associated quadruple such that the assumptions of Theorem 1.17 hold. Then, \( \nu = \nabla_{B \otimes H} \circ (\eta_B \otimes \eta_H) \) is a preunit for the weak crossed product associated to \( B \otimes H \) if and only if \( \sigma \) satisfies the normal condition (84).

Proof. Considering (71), the proof follows from the equalities
\[F_\sigma \circ (\eta_H \otimes H) = ((\sigma \circ c_{H,H} \circ (H \otimes \Pi^L_H) \circ \delta_H) \otimes H) \circ \delta_H,\]
and
\[F_\sigma \circ (H \otimes \eta_H) = ((\sigma \circ (H \otimes \Pi^R_H) \circ \delta_H) \otimes H) \circ \delta_H,\]
which hold by (15), (12) and the naturality of \( c \). □

Therefore, as a consequence of the previous results, we obtain the complete characterization of weak crossed products associated to a measuring.

Corollary 1.20. Let \( H \) be a weak Hopf algebra, let \( \varphi_B : H \otimes B \to B \) be a measuring, let \( \sigma : H \otimes H \to B \) be a morphism and let \( B \otimes H \) be the associated quadruple such that the assumptions of Theorem 1.17 hold. Then the following statements are equivalent:

(i) The product \( \mu_{B \otimes \varphi_B H} \) is associative with preunit \( \nu = \nabla_{B \otimes H} \circ (\eta_B \otimes \eta_H) \) and normalized with respect to \( \nabla_{B \otimes H} \).

(ii) The morphism \( \sigma \) satisfies the twisted condition (77), the cocycle condition (78) and the normal condition (84).

Remark 1.21. Let \( H \) be a weak Hopf algebra. If \( (B, \varphi_B) \) is a left weak \( H \)-module algebra the equality (85) holds because:
\[\nabla_{B \otimes H} \circ (B \otimes \eta_H)\]
2. Equivalent weak crossed products

The general theory of equivalent weak crossed products was presented in [10]. In this section we remember the criterion obtained in [10] that characterises the equivalence between two weak crossed products induced by measurings.

We shall start by introducing the notion of equivalence of weak crossed products induced by measurings.

**Definition 2.1.** Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \rightarrow B$ be measurings, and let $\sigma, \tau : H \otimes H \rightarrow B$ be morphisms such that $\sigma \ast u_2^B = \sigma$, $\tau \ast u_2^B = \tau$. Assume that $\sigma, \tau$ satisfy the twisted condition (77) and the 2-cocycle condition (78), and suppose that $\nu$ is a preunit for $\mu_{B \otimes e_B} H$, and $u$ is a preunit for $\mu_{B \otimes e_B} H$. We say that $(B \otimes H, \mu_{B \otimes e_B} H)$ and $(B \otimes H, \mu_{B \otimes e_B} H)$ are equivalent weak crossed products if there is an isomorphism

$$T : B \times_{\varphi_B} H \rightarrow B \times_{\varphi_B} H$$

of algebras, left $B$-modules and right $H$-comodules, where the left actions are defined by $\varphi_{B \times e_B} H = p_{B \otimes H}^B \circ (\mu_B \otimes H) \circ (B \otimes \iota_{B \otimes H})$, $\varphi_{B \times e_B} H = p_{B \otimes H}^B \circ (\mu_B \otimes H) \circ (B \otimes \iota_{B \otimes H})$, and the right coactions are $\rho_{B \times e_B} H = (p_{B \otimes H}^B \otimes H) \circ (B \otimes \delta_H) \circ \iota_{B \otimes H}$, $\rho_{B \times e_B} H = (p_{B \otimes H}^B \otimes H) \circ (B \otimes \delta_H) \circ \iota_{B \otimes H}$.

In our setting the general criterion [10] Theorem 1.7] that characterizes equivalent weak crossed products admits the following formulation:

**Theorem 2.2.** Let $H$ be a weak Hopf algebra, let $\varphi_B : H \otimes B \rightarrow B$ be measurings and let $\sigma, \tau : H \otimes H \rightarrow B$ be morphisms such that $\sigma \ast u_2^B = \sigma$, $\tau \ast u_2^B = \tau$. Assume that $\sigma, \tau$ satisfy the twisted condition (77), the 2-cocycle condition (78) and suppose that $\nu$ is a preunit for $\mu_{B \otimes e_B} H$ and $u$ is a preunit for $\mu_{B \otimes e_B} H$. Let $(B, H, \rho_B, F, \sigma)$ be the quadruple associated to $\varphi_B$, and $\sigma$ and let $(B, H, P_{\varphi_B}, F, \tau)$ be the one associated to $\varphi_B$ and $\tau$. The following assertions are equivalent:

(i) The weak crossed products $(B \otimes H, \mu_{B \otimes e_B} H)$ and $(B \otimes H, \mu_{B \otimes e_B} H)$ are equivalent.

(ii) There exist two morphisms $T, S : B \otimes H \rightarrow B \otimes H$, of left $B$-modules for the trivial action $\varphi_{B \otimes H} = \mu_B \otimes H$, and right $H$-modules for the trivial coaction $\rho_{B \otimes H} = B \otimes \delta_H$, satisfying the conditions

$$T \circ \nu = u,$$

$$T \circ \mu_{B \otimes e_B} H = \mu_{B \otimes e_B} H \circ (T \otimes T),$$
\( S \circ T = \nabla_{B \otimes H}^{\phi_B}, \quad T \circ S = \nabla_{B \otimes H}^{\phi_B}. \)

(iii) There exist two morphisms \( \theta, \gamma : H \rightarrow B \otimes H \) of right \( H \)-modules for the trivial coaction satisfying the conditions

\[
(\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma = \nabla_{B \otimes H}^{\phi_B} \circ (\eta_B \otimes H),
\]

\[
P_{\phi_B} = (\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\phi_B}) \circ (\theta \otimes B),
\]

\[
F_\tau = (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \mu_{B \otimes \varphi_B} \circ (\theta \otimes \theta),
\]

\[
u = (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nu.
\]

Proof. The proof of this theorem is the one developed in \([10, \text{Theorem 1.7}]\). It is not difficult to check the right \( H \)-comodule condition for the morphisms \( S, T, \theta \) and \( \gamma \). We leave the details of the proof to the reader. \( \square \)

Proposition 2.3. Let \( H \) be a weak Hopf algebra, let \( \varphi_B, \phi_B : H \otimes B \rightarrow B \) be measurings and let \( \sigma, \tau : H \otimes H \rightarrow B \) be morphisms such that \( \sigma * u_2^{\varphi_B} = \sigma, \tau * u_2^{\phi_B} = \tau \). Assume that \( \sigma, \tau \) satisfy the twisted condition \((77)\), the 2-cocycle condition \((78)\) and suppose that \( \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \) is a preunit for \( \mu_{B \otimes \varphi_B} \) and \( \nabla_{B \otimes H}^{\phi_B} \circ (\eta_B \otimes \eta_H) \) is a preunit for \( \mu_{B \otimes \phi_B} \). If \( (B \otimes H, \mu_{B \otimes \varphi_B}) \) and \( (B \otimes H, \mu_{B \otimes \phi_B}) \) are equivalent weak crossed products, there exists morphisms \( T, S : B \otimes H \rightarrow B \otimes H \) of left \( B \)-modules for the trivial action and right \( H \)-comodules for the trivial coaction such that

\[
\nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) = T \circ (\eta_B \otimes \eta_H), \quad \nabla_{B \otimes H}^{\phi_B} \circ (\eta_B \otimes \eta_H) = S \circ (\eta_B \otimes \eta_H).
\]

Proof. If \( (B \otimes H, \mu_{B \otimes \varphi_B}) \) and \( (B \otimes H, \mu_{B \otimes \phi_B}) \) are equivalent weak crossed products, there exists an isomorphism of algebras, left \( B \)-modules and right \( H \)-comodules \( \Upsilon : B \times_{\varphi_B} H \rightarrow B \times_{\phi_B} H \). By (i) \( \Rightarrow \) (ii) of the previous theorem there exists two morphisms of left \( B \)-modules and right \( H \)-comodules \( T, S : B \otimes H \rightarrow B \otimes H \), defined by

\[
T = i_{B \otimes H}^{\varphi_B} \circ \Upsilon \circ p_{B \otimes H}^{\varphi_B}, \quad S = i_{B \otimes H}^{\phi_B} \circ \Upsilon^{-1} \circ p_{B \otimes H}^{\phi_B}
\]

and satisfying the conditions \((93)\),

\[
(100) \quad T \circ S \circ T = T
\]

and

\[
(101) \quad S \circ T \circ S = S.
\]

Then,

\[
\nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \quad T \circ \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \quad T \circ S \circ T \circ (\eta_B \otimes \eta_H) \quad T \circ (\eta_B \otimes \eta_H).
\]
On the other hand, if \( \nu \) and \( u \) are the preunits of \((B \otimes H, \mu_{B \otimes T^B H})\) and \((B \otimes H, \mu_{B \otimes H^B H})\), by (91) we have that \( S \circ T \circ \nu = S \circ u \). Then, by (93) we have that \( \nabla_{B \otimes H}^\nu \circ \nu = S \circ u \) and applying (90) we obtain that

\[
S \circ u = \nu
\]

holds. Therefore, in our particular case, we have

\[
\nabla_{B \otimes H}^\nu \circ (\eta_B \otimes \eta_H) = S \circ \nabla_{B \otimes H}^\nu \circ (\eta_B \otimes \eta_H) = S \circ T \circ S \circ (\eta_B \otimes \eta_H) = S \circ (\eta_B \otimes \eta_H).
\]

**Theorem 2.4.** Let \( H \) be a weak Hopf algebra, let \( \varphi_B, \phi_B : H \otimes B \to B \) be measurings, and let \( \sigma, \tau : H \otimes H \to B \) be morphisms such that \( \sigma \ast u_2^{\varphi_B} = \sigma, \tau \ast u_2^{\phi_B} = \tau \). Assume that \( \sigma, \tau \) satisfy the twisted condition (77), the 2-cocycle condition (78) and suppose that \( \nu \) is a preunit for \( \mu_{B \otimes \omega_B^B H} \), and \( u \) is a preunit for \( \mu_{B \otimes \omega_B^B H} \). The following assertions are equivalent:

(i) The weak crossed products \((B \otimes H, \mu_{B \otimes \omega_B^B H})\) and \((B \otimes H, \mu_{B \otimes \omega_B^B H})\) are equivalent.

(ii) There exists two morphisms \( h, h^{-1} : H \to B \) such that

\[
\begin{align*}
\mu_B \circ (B \otimes h^{-1}) & \circ (h \otimes \varphi_B) \circ (\delta_H \otimes B), \\
\tau & = \mu_B \circ (B \otimes h^{-1}) \circ \mu_{B \otimes \omega_B^B H} \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H)).
\end{align*}
\]

Proof. First we will prove that (i) \( \Rightarrow \) (ii). By Theorem 2.2 there exists two morphisms \( T, S : B \otimes H \to B \otimes H \) of left \( B \)-modules for the trivial action and right \( H \)-comodules for the trivial coaction defined as in the proof of the previous proposition and satisfying the conditions (91), (92), (93), (100) and (101). Also, \( S \) preserves the preunit, i.e., (102) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (103) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (104) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (105) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (106) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (107) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (108) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (109) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (110) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (111) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \). Also, (112) holds, and \( S \) is multiplicative, i.e., \( S \circ \mu_{B \otimes \omega_B^B H} = \mu_{B \otimes \omega_B^B H} \).
\[
F_\sigma = (\mu_B \otimes H) \circ (B \otimes \theta) \circ \mu_{B \otimes \sigma_B} H \circ (\gamma \otimes \gamma),
\]
\[
\nu = (\mu_B \otimes H) \circ (B \otimes \theta) \circ u,
\]
also hold. Define
\[
h = (B \otimes \varepsilon_H) \circ \theta, \quad h^{-1} = (B \otimes \varepsilon_H) \circ \gamma.
\]

Then, by the condition of right \( H \)-comodule morphism for \( \theta \) and \( \gamma \), we have
\[
\theta = (h \otimes H) \circ \delta_H, \quad \gamma = (h^{-1} \otimes H) \circ \delta_H.
\]
The equality (103) holds because
\[
h^{-1} \ast h
= (B \otimes \varepsilon_H) \circ (\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma \quad \text{(by the comodule morphism condition for } \gamma \text{ and counit properties)}
= (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ (\eta_B \otimes H) \quad \text{(by (103))}
= u_1 \varphi_B \quad \text{(by counit properties)}.
\]
Also,
\[
h \ast h^{-1} \ast h
= \mu_B \circ (H \otimes \varepsilon_H \otimes \varphi_B) \circ (\theta \otimes \eta_B) \quad \text{(by the comodule morphism condition for } \theta \text{ and (103))}
= (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ \theta \quad \text{(by counit properties and naturality of } \varepsilon \text{)}
= (B \otimes \varepsilon_H) \circ \theta \quad \text{(by (111))}
= h \quad \text{(by counit properties)}
\]
and
\[
h^{-1} \ast h \ast h^{-1}
= (\mu_B \otimes \varepsilon_H) \circ (B \otimes \gamma) \circ P_{\varphi_B} \circ (H \otimes \eta_B) \quad \text{(by naturality of } \varepsilon \text{)}
= (\mu_B \otimes \varepsilon_H) \circ (B \otimes \gamma) \circ \nabla^{\varphi_B}_{B \otimes H} \circ (\eta_B \otimes H) \quad \text{(by properties of } \eta_B \text{)}
= (\mu_B \otimes \varepsilon_H) \circ (B \otimes \gamma) \circ (\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma \quad \text{(by (111))}
= (\mu_B \otimes \varepsilon_H) \circ (B \otimes (\nabla^{\varphi_B}_{B \otimes H} \circ (\eta_B \otimes H))) \circ \gamma \quad \text{(by the associativity of } \mu_B \text{ and (113)).}
= (B \otimes \varepsilon_H) \circ \nabla^{\varphi_B}_{B \otimes H} \circ \gamma \quad \text{(by (103) and properties of } \eta_B \text{)}
= h^{-1} \quad \text{(by (111)).}
\]
The equality (107) follows directly from (105) because \( \gamma \) is a morphism of right \( H \)-comodules. Moreover, composing in (96) with \( B \otimes \varepsilon_H \), by (116) we prove (115). Finally, (116) holds because
\[
\tau
= (B \otimes \varepsilon_H) \circ F_\tau \quad \text{(by (111) for } \tau \text{)}
= (\mu_B \otimes \varepsilon_H) \circ (B \otimes \gamma) \circ \mu_{B \otimes \varphi_B} H \circ (\theta \otimes \theta) \quad \text{(by (111))}
= (\mu_B \otimes h^{-1}) \circ \mu_{B \otimes \varphi_B} H \circ ((h \otimes H) \circ \delta_H) \circ ((h \otimes H) \circ \delta_H) \quad \text{(by (116)).}
\]
Conversely, to prove (ii) \( \Rightarrow \) (i), define
\[
\theta = (h \otimes H) \circ \delta_H, \quad \gamma = (h^{-1} \otimes H) \circ \delta_H.
\]
Then $\theta$ and $\gamma$ are morphisms of right $H$-comodules, $h = (B \otimes \varepsilon_H) \circ \theta$ and $h^{-1} = (B \otimes \varepsilon_H) \circ \gamma$. To prove the equivalence between $(B \otimes H, \mu_B \otimes \varepsilon_B \otimes H)$ and $(B \otimes H, \mu_B \otimes \varepsilon_B \otimes H)$, we must show that (101), (105), (90), (97) and (96) hold. First note that, (98) follows from (107). Also, (91) holds because:

$$\nabla^B_{B \otimes H} \circ \theta = ((h \ast u^B_1) \otimes H) \circ \delta_H \quad \text{(by the coassociativity of $\delta_H$ and (107))}$$

$$= ((h \ast h^{-1} \ast h) \otimes H) \circ \delta_H \quad \text{(by (105))}$$

$$= \theta \quad \text{(by (105)).}$$

On the other hand, (95) follows by

$$(\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma$$

$$= ((h^{-1} \ast h) \otimes H) \circ \delta_H \quad \text{(by coassociativity of $\delta_H$)}$$

$$= (u^B_1 \otimes H) \circ \delta_H \quad \text{(by (105))}$$

$$= \nabla^B_{B \otimes H} \circ (\eta_B \otimes H) \quad \text{(by (105)).}$$

and (96) follows by

$$P^B_{\varphi_B}$$

$$= ((\mu_B \circ (\mu_B \otimes h^{-1})) \otimes H) \circ (h \otimes ((P_{\varphi_B} \otimes H) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B))) \circ (\delta_H \otimes B) \quad \text{(by (105) and coassociativity of $\delta_H$)}$$

$$= (\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B}) \circ (\theta \otimes B) \quad \text{(by coassociativity of $\delta_H$ and the naturality of $c$).}$$

Finally, (97) holds because

$$F^\varphi_r$$

$$= ((\mu_B \circ (B \otimes h^{-1}) \circ \mu_B \otimes \varepsilon_B) \circ ((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H)) \otimes \mu_H) \circ \delta_{H \otimes 2} \quad \text{(by (105))}$$

$$= ((\mu_B \circ (\mu_B \otimes h^{-1})) \otimes H) \circ (\mu_B \otimes ((F_r \otimes \mu_H) \circ \delta_{H \otimes 2})) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta) \quad \text{(by coassociativity of $\delta_H$ and the naturality of $c$)}$$

$$= (\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (\mu_B \otimes F^\varphi_r) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta) \quad \text{(by (59))}$$

$$= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \mu_B \otimes \varepsilon_B \otimes H \circ (\theta \otimes \theta) \quad \text{(by the definition of $\mu_B \otimes \varepsilon_B \otimes H$).}$$

\[ \square \]

**Remark 2.5.** Note that, in the conditions of (ii) of Theorem 2.4, composing with $H \otimes \eta_B$ in (105), we obtain the identity

(117) \[ h \ast h^{-1} = u^B_1. \]

**Definition 2.6.** Let $H$ be a weak Hopf algebra and let $\varphi_B : H \otimes B \to B$ be a measuring. We will say that the pair of morphisms $h, h^{-1} : H \to B$ is a gauge transformation for $\varphi_B$ if they satisfy (103) and (105).

By the previous Theorem 2.4 we know that, under suitable conditions, equivalent weak crossed products are related by gauge transformations. After the next discussion, we should be able to secure that the converse is also true.

2.7. Let $H$ be a weak Hopf algebra and let $\varphi_B : H \otimes B \to B$ be a measuring. Let $(h, h^{-1})$ be a gauge transformation for $\varphi_B$ and let $\sigma : H \otimes H \to B \otimes H$ be a morphism satisfying the identity $\sigma \ast u^\varphi_B = \sigma$, the
twisted condition (17) and the 2-cocycle condition (18). Suppose that \( \nu \) is a preunit for the associated weak crossed product \( \mu_B \otimes \sigma_B \). Define \( \theta \) and \( \gamma \) as in (119), i.e., \( \theta = (h \otimes H) \circ \delta_H \) and \( \gamma = (h^{-1} \otimes H) \circ \delta_H \). Then \( \theta \) and \( \gamma \) are morphisms of right \( H \)-comodules. Also, by (122), the coassociativity of \( \delta_H \) and the condition of gauge transformation, we have that \( \nabla^B_{B \otimes H} \circ \theta = \theta \) and then (95) holds. By similar arguments and the associativity of \( \mu_B \) we obtain the equality

\[
(\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nabla^B_{B \otimes H} = (\mu_B \otimes H) \circ (B \otimes \gamma).
\]

Moreover, by the coassociativity of \( \delta_H \) and the condition of gauge transformation we have

\[
(\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma = (u_1^{\varphi_B} \otimes H) \circ \delta_H = \nabla^B_{B \otimes H} \otimes (\eta_B \otimes H)
\]

and then (55) holds. As a consequence, we obtain

\[
(\mu_B \otimes H) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma)) = \nabla^B_{B \otimes H}.
\]

Define \( \varphi_B^h : H \otimes B \to B \) by

\[
\varphi_B^h = \mu_B \otimes (\mu_B \otimes h^{-1}) \circ (B \otimes P_{\varphi_B}) \circ (\theta \otimes B)
\]

and \( \sigma^h : H \otimes H \to B \) by

\[
\sigma^h = \mu_B \circ (B \otimes h^{-1}) \circ \mu_B \otimes \sigma_B \circ (\theta \otimes \theta).
\]

Then, \( \varphi_B^h \) is a measuring because

\[
\mu_B \circ (\varphi_B^h \otimes \varphi_B^h) \circ (H \otimes c_{H,B} \otimes B) = \mu_B \circ (B \otimes \mu_B) \circ ((H \otimes c_{H,B} \otimes B) \circ (B \otimes B)) \circ (\delta_H \otimes B \otimes B)
\]

\[
= \mu_B \circ (B \otimes \mu_B) \circ (\mu_B \otimes (h \otimes \varphi_B) \circ (\delta_H \otimes B)) \circ (\mu_B \circ (u_1^{\varphi_B} \otimes \varphi_B) \circ (\delta_H \otimes B)) \circ (h^{-1}) \circ (H \otimes c_{H,B} \otimes c_{H,B})
\]

\[
\circ (H \otimes c_{H,B} \otimes B) = \mu_B \circ (B \otimes \mu_B) \circ (h \otimes (\mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes B \otimes B)) \circ (h^{-1}) \circ (H \otimes H \otimes c_{H,B} \otimes B)
\]

\[
\circ (H \otimes c_{H,B} \otimes B) = \mu_B \circ (B \otimes \mu_B) \circ (h \otimes (\mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes B \otimes B)) \circ (h^{-1}) \circ (H \otimes c_{H,B} \otimes B)
\]

\[
\circ (H \otimes h^{-1} \otimes B) = \mu_B \circ (B \otimes \mu_B) \circ (h^{-1} \otimes (\mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes B \otimes B)) \circ (h^{-1}) \circ (H \otimes c_{H,B} \otimes B)
\]

and trivially, by the coassociativity of \( \delta_H \) and the naturality of \( c \), we have that

\[
P_{\varphi_B^h} = (\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B}) \circ (\theta \otimes B)
\]

Therefore (96) holds. Also, by the associativity of \( \mu_B \), (122) and the condition of gauge transformation

\[
u_1^{\varphi_B^h} = h \otimes h^{-1}
\]

holds. Then, as a consequence of the previous identity, we have that

\[
(\mu_B \otimes H) \circ (B \otimes \gamma) \circ \theta = (u_1^{\varphi_B^h} \otimes H) \circ \delta_H
\]

and

\[
(\mu_B \otimes H) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \gamma) \circ \theta)) = \nabla^B_{B \otimes H}.
\]
On the other hand for $F_{\sigma^{h}}$ we have

$$F_{\sigma^{h}} = ((\mu_B \circ (\mu_H \otimes h^{-1})) \otimes H) \circ (\mu_B \otimes ((F_{\sigma} \otimes \mu_H) \circ \delta_{H \otimes z})) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)$$

(by the naturality of $\epsilon$ and the coassociativity of $\delta_H$)

$$= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \mu_{B \otimes \varphi_B H} \circ (\theta \otimes \theta)$$

(by (50))

and (57) also holds. Moreover, $\sigma^{h} \ast u_{2}^{\varphi_B h} = \sigma^{h}$, because

$$\sigma^{h} \ast u_{2}^{\varphi_B h}$$

$$= \mu_B \circ ((\mu_B \otimes (\mu_B \otimes h^{-1})) \otimes (\varphi_B \circ (H \otimes \eta_B))) \circ (\mu_B \otimes ((F_{\sigma} \otimes \mu_H) \circ \delta_{H \otimes z})) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)$$

(by the naturality of $\epsilon$, the coassociativity of $\delta_B$ and the condition of morphism of right $H$-comodules for $\theta$)

$$= \mu_B \circ ((\mu_B \otimes (\mu_B \otimes h^{-1})) \otimes (\varphi_B \circ (H \otimes \eta_B))) \circ (\mu_B \otimes ((B \otimes \delta_H) \circ F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)$$

(by (50))

$$= \mu_B \circ (\mu_B \otimes (h^{-1} \ast u_{2}^{\varphi_B h})) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)$$

(by the associativity of $\mu_B$)

$$= \mu_B \circ (\mu_B \otimes (h^{-1} \ast h^{-1})) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)$$

(by (22))

$$= \sigma^{h}$$

(by (104) and associativity of $\mu_B$).

By (i) of Theorem 1.13 to obtain that $(B, H, P_{\varphi_B}, F_{\sigma^{h}})$ satisfies the twisted condition is enough to prove that (77) holds. Indeed,

$$\mu_B \circ (B \otimes \sigma^{h}) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B})$$

$$= \mu_B \circ (B \otimes h^{-1}) \circ \mu_{B \otimes \varphi_B H} \circ ((\mu_B \otimes H) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma))) \otimes \theta) \circ (\mu_B \otimes H \otimes H)$$

$$\circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes ((\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B} \otimes (\theta \otimes B)))$$

(by the associativity of $\mu_B$)

$$= \mu_B \circ (B \otimes h^{-1}) \circ \mu_{B \otimes \varphi_B H} \circ ((\mu_B \otimes H) \circ (B \otimes (\varphi_B \otimes B) \otimes H)$$

$$\circ (\theta \otimes ((\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B} \otimes (\theta \otimes B)))$$

(by the associativity of $\mu_B$)

$$= \mu_B \circ (B \otimes h^{-1}) \circ \mu_{B \otimes \varphi_B H} \circ ((\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B} \otimes (\theta \otimes B)))$$

(by the definition of $\mu_{B \otimes \varphi_B H}$)

$$= \mu_B \circ (\mu_B \otimes h^{-1}) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes (\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma)) \circ P_{\varphi_B}) \circ (\theta \otimes B)$$

(by (55)

and the associativity of $\mu_B$)

$$= \mu_B \circ (\mu_B \otimes h^{-1}) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes (((\mu_B \otimes H) \circ (B \otimes (\varphi_B \otimes B) \otimes H) \circ (\theta \otimes B)))$$

(by (55)

and the associativity of $\mu_B$)

$$= \mu_B \circ (\mu_B \otimes h^{-1}) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes (\varphi_B \otimes B) \otimes H) \circ (\theta \otimes B)))$$

(by (55) and the condition of gauge transformation)
\[
\mu_B \circ (B \otimes P_{\varphi^h_B}) \circ (F_{\varphi^h_B} \otimes B) \quad (\text{by the associativity of } \mu_B \text{ and the coassociativity of } \delta_H).
\]

Also, by (ii) of Theorem 1.13 to obtain that \((B, H, P_{\varphi^h_B}, F_{\varphi^h_B})\) satisfies the cocycle condition is enough to prove that (13) holds. Indeed,

\[
\mu_B \circ (B \otimes \sigma^h) \circ (P_{\varphi^h_B} \otimes H) \circ (H \otimes F_{\varphi^h_B})
\]

\[
= \mu_B \circ (B \otimes h^{-1}) \circ \mu_B \circ (\mu_B \otimes H \otimes \theta) \circ (B \otimes \mu_B \otimes H \otimes H)
\]

\[
\circ (B \otimes B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma)) \circ P_{\varphi_B} \circ \delta_H \circ (B \otimes P_{\varphi_B} \otimes H)
\]

\[
\circ (\theta \otimes (\mu_B \otimes H) \circ (\theta \otimes \theta)) \quad (\text{by the associativity of } \mu_B)
\]

\[
= \mu_B \circ (B \otimes h^{-1}) \circ \mu_B \circ (\mu_B \otimes H \otimes \theta) \circ (\mu_B \otimes P_{\varphi_B} \otimes H) \circ (B \otimes P_{\varphi_B} \otimes H)
\]

\[
\circ (\theta \otimes (\mu_B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta)) \quad (\text{by (13) and the associativity of } \mu_B)
\]

\[
= \mu_B \circ (B \otimes h^{-1}) \circ (\mu_B \otimes \theta) \circ (\mu_B \otimes P_{\varphi_B} \otimes H) \circ (B \otimes \theta) \circ (\mu_B \otimes F_{\varphi_B} \otimes H)
\]

\[
\circ (B \otimes F_{\varphi_B} \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta) \circ (\mu_B \otimes F_{\varphi_B} \otimes H)
\]

\[
\circ (\theta \otimes (\mu_B \otimes H) \circ (\theta \otimes \theta)) \quad (\text{by (13)} \text{ and the associativity of } \mu_B)
\]

\[
= \mu_B \circ (B \otimes h^{-1}) \circ \mu_B \circ (\mu_B \otimes H) \circ (\mu_B \otimes (B \otimes H) \circ (\theta \otimes \theta)) \circ \theta
\]

\[
\circ (\mu_B \otimes P_{\varphi_B} \otimes H) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta) \circ (\mu_B \otimes F_{\varphi_B} \otimes H)
\]

\[
\circ (B \otimes F_{\varphi_B} \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes \theta) \circ (\mu_B \otimes F_{\varphi_B} \otimes H)
\]

\[
\circ (\theta \otimes (\mu_B \otimes H) \circ (\theta \otimes \theta)) \quad (\text{by (13) and the condition of gauge transformation})
\]

\[
= \mu_B \circ (B \otimes \sigma^h) \circ (F_{\varphi^h_B} \otimes B) \quad (\text{by the associativity of } \mu_B).
\]

If we define \(\nu^h\) by \(\nu^h = (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nu\) we have that (13) holds trivially. Moreover \(\nu^h\) is a preunit for \((B \otimes H, \mu_B \otimes \sigma^h_B)\) because (17), (18) and (19) hold. Indeed, (17) follows by

\[
\mu_B \circ (B \otimes F_{\varphi^h_B}) \circ (P_{\varphi^h_B} \otimes H) \circ (H \otimes \nu^h)
\]

\[
= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \mu_B \circ (\mu_B \otimes H \otimes \theta)
\]

\[
\circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma)) \circ P_{\varphi_B} \circ \delta_B \circ (B \otimes P_{\varphi_B} \otimes H)
\]

\[
\circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \gamma)) \quad (\text{by (13)} \text{ and the definition of } \mu_B \circ \sigma^h_B \circ \delta_B \text{ and the associativity of } \mu_B)
\]

\[
= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (\mu_B \otimes F_{\varphi_B}) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (B \otimes \theta)) \circ \nu)
\]

\[
\circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \nu)) \quad (\text{by (13) and the definition of } \mu_B \circ \sigma^h_B \circ \delta_B \text{ and the associativity of } \mu_B)
\]

\[
= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (\mu_B \otimes F_{\varphi_B}) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes P_{\varphi_B} \otimes B) \circ (B \otimes \theta)) \otimes \theta)
\]

\[
\circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \circ \nu)) \quad (\text{by (13)})
\]
\[(\mu_B \otimes H) \circ (\mu_B \otimes \gamma) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes ((\mu_B \otimes H) \circ (B \otimes (\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \otimes \nu)))
\]

(by \[15\] and the associativity of \(\mu_B\))

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (\mu_B \otimes F_{\sigma}) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (\theta \otimes (\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \nu))\]

(by \[119\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nabla_{B \otimes H}^\varphi \circ \mu_B \otimes (\mu_B \otimes H) \circ (\theta \otimes (\nabla_{B \otimes H}^\varphi) \circ (B \otimes \theta) \otimes \gamma) \circ \nu))\]

(by \[23\] and \[118\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \theta\]

(by the condition of unit for \(\mu_B \otimes H \circ \nu)\)

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \theta\]

(by \[15\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \theta\]

(by \[93\])

\[= \nabla_{B \otimes H}^\varphi \circ \eta_B \otimes H\]

(by \[23\] and the properties of \(\eta_B\)).

Also, \[88\] holds because

\[\mu_B \circ (B \otimes F_{\sigma h}) \circ (\nu^h \otimes H)\]

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \mu_B \otimes \sigma_{\varphi_B H} \circ ((\mu_B \otimes H) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \nu) \circ \theta)\]

(by the associativity of \(\mu_B\))

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nabla_{B \otimes H}^\varphi \circ \mu_B \otimes \sigma_{\varphi_B H} \circ ((\mu_B \otimes H) \circ (B \otimes (\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \nu) \circ \theta)\]

(by \[23\] and \[118\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ \nabla_{B \otimes H}^\varphi \circ \eta_B \otimes H\]

(by \[23\] and the properties of \(\eta_B\)).

Finally,

\[\mu_B \circ (B \otimes P_{\varphi_B}^\nu) \circ (\nu^h \otimes H)\]

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (B \otimes P_{\varphi_B}^\nu) \circ ((\mu_B \otimes H) \circ (B \otimes ((\mu_B \otimes H) \circ (B \otimes \theta) \otimes \gamma) \circ \nu) \circ \theta) \circ \nu) \circ B\]

(by the associativity of \(\mu_B\))

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (B \otimes P_{\varphi_B}^\nu) \circ ((\nabla_{B \otimes H}^\varphi \circ \nu) \circ B)\]

(by \[114\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (B \otimes P_{\varphi_B}^\nu) \circ (\nu \otimes B)\]

(by \[90\])

\[= (\mu_B \otimes H) \circ (B \otimes \gamma) \circ (B \otimes \nu)\]

(by \[89\])

\[= (\mu_B \otimes H) \circ (B \otimes \nu^h)\]

(by the associativity of \(\mu_B\)).

and \[89\] holds.

Therefore, as a consequence of the previous facts, we have a theorem that generalizes to the monoidal setting \[15\] Theorem 5.4.

**Theorem 2.8.** Let \(H\) be a weak Hopf algebra, let \(\varphi_B, \phi_B : H \otimes B \to B\) be measurings, and let \(\sigma, \tau : H \otimes H \to H\) be morphisms such that \(\sigma * u_2^\varphi = \sigma, \tau * u_2^\varphi = \tau\). Assume that \(\sigma, \tau\) satisfy the twisted condition \[77\], the 2-cocycle condition \[78\] and suppose that \(\nu\) is a preunit for \(\mu_B \otimes \varphi_B H\), and \(u\) is a preunit for \(\mu_B \otimes \varphi_B H\). The weak crossed products \((B \otimes H, \mu_B \otimes \varphi_B H)\) and \((B \otimes H, \mu_B \otimes \varphi_B H)\) are equivalent if and only if there exists a gauge transformation \((h, h^{-1})\) for \(\varphi_B\) such that \(\phi_B = \varphi_B^h, \tau = \sigma^h\) and \(u = \nu^h\).

### 3. Regular morphisms

**Definition 3.1.** Let \(H\) be a weak Hopf algebra and \(\varphi_B\) be a measuring. With \(\text{Reg}_{\varphi_B}(H, B)\) we will denote the set of regular morphisms between \(H\) and \(B\), i.e., a morphism \(h : H \to B\) is a regular
morphism if there exists a morphism $h^{-1} : H \to B$, called the convolution inverse of $h$, such that the pair $(h, h^{-1})$ is a gauge transformation for $\varphi_B$ and
\[(125) \quad h \ast h^{-1} = u_1^{\varphi_B},\]
holds. Then, $\text{Reg}_{\varphi_B}(H, B)$, with the convolution as a product, is a group with unit $u_1^{\varphi_B}$.

### 3.2. Let $\mathcal{P}_{\varphi_B}$ be the set of all pairs $(\phi_B, \tau)$, where:

(i) The morphism $\phi_B : H \otimes B \to B$ is a measuring satisfying $u_1^{\phi_B} = u_1^{\varphi_B}$.

(ii) The morphism $\tau : H \otimes H \to B$ is such that $\tau = \tau * u_2^{\phi_B}$ and the associated quadruple $(B, H, F_{\phi_B}, F_{\tau})$ satisfies the twisted condition and the cocycle condition.

(iii) The associated weak crossed product $(B \otimes H, \mu_{B \otimes \phi_B})$ admits a preunit $\nu$.

By the results proved in the previous section we know that $\text{Reg}_{\varphi_B}(H, B)$ acts on $\mathcal{P}_{\varphi_B}$. The action
\[R : \text{Reg}_{\varphi_B}(H, B) \times \mathcal{P}_{\varphi_B} \to \mathcal{P}_{\varphi_B},\]
is defined by
\[(126) \quad R(h, (\phi_B, \tau)) = (\phi^h_B, \tau^h).\]

**Proposition 3.3.** Let $(B, \varphi_B)$, $(B, \phi_B)$ be a left weak $H$-module algebra and let $h : H \to B$ be a morphism such that $h \ast u_1^{\phi_B} = h = u_1^{\phi_B} \ast h$. Then, the following assertions are equivalent:

(i) $h \circ \eta_H = \eta_B$.

(ii) $h \circ \Pi^L_H = u_1^{\phi_B}$.

(iii) $h \circ \Pi^H_L = u_1^{\varphi_B}$.

Moreover, if $h \ast \eta_H = \eta_B$, the identity
\[(127) \quad (\mu_B \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta_H \circ \eta_H)) = \nabla^{\varphi_B}_{B \otimes H} \circ (B \otimes \eta_H),\]
holds.

Also, if $g : H \to B$ is a morphism such that $g \ast u_1^{\phi_B} = g = u_1^{\phi_B} \ast g$, the following assertions are equivalent:

(iv) $g \circ \eta_H = \eta_B$.

(v) $g \circ \Pi^L_H = u_1^{\varphi_B}$.

(vi) $g \circ \Pi^H_L = u_1^{\phi_B}$.

Then, if $g \ast \eta_H = \eta_B$, the identity
\[(128) \quad (\mu_B \otimes H) \circ (B \otimes ((g \otimes H) \circ \delta_H \circ \eta_H)) = \nabla^{\phi_B}_{B \otimes H} \circ (B \otimes \eta_H)\]
holds.

Moreover, if there exists $h^{-1} : H \to B$ such that $(h, h^{-1})$ is a gauge transformation for $\varphi_B$ and $h \ast h^{-1} = u_1^{\phi_B}$ holds, we have $h \ast \eta_H = \eta_B$ if and only if $h^{-1} \ast \eta_H = \eta_B$.

**Proof.** By the properties of $\eta_H$ and (b2) of Definition 1.4, we obtain that (ii) $\Rightarrow$ (i), and (iii) $\Rightarrow$ (i). Also, (i) $\Rightarrow$ (ii) holds because,
\[h \circ \Pi^L_H \]
\[ (u_1^\phi B \ast h) \circ \Pi_H^L \quad (\text{by } h = u_1^\phi B \ast h) \]
\[ = \mu_B \circ (u_1^\phi B \ast h) \circ (((\varepsilon_H \circ \mu_H) \otimes H) \circ (\delta_H \otimes H)) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) \]
(by naturality of \( c \) and coassociativity of \( \delta_H \))
\[ = \mu_B \circ (u_1^\phi B \ast h) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) \]
(by coassociativity of \( \delta_H \))
\[ = \mu_B \circ (u_1^\phi B \ast h) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) \]
(by (13) of Definition 1.4 and (29))
\[ = \mu_B \circ (u_1^\phi B \ast h) \circ (u_1^\phi B) \]
(by associativity of \( \mu_B \))
\[ = u_1^\phi B \]
(by (i), and properties of \( \eta_B \)).

On the other hand, \((i)\Rightarrow (iii)\) because
\[ h \circ \Pi_H^L \]
\[ = (h \ast u_1^\phi B) \circ \Pi_H^L \quad (\text{by } h = h \ast u_1^\phi B) \]
\[ = \mu_B \circ (h \otimes u_1^\phi B) \circ (H \otimes ((\varepsilon_H \circ \mu_H) \circ \delta_H \otimes H)) \circ (\delta_H \circ \eta_H) \otimes H \]
(by the coassociativity of \( \delta_H \))
\[ = \mu_B \circ (h \otimes u_1^\phi B \ast \mu_B \circ (H \otimes \Pi_H^L)) \circ (\delta_H \circ \eta_H) \otimes H \]
(by (13))
\[ = \mu_B \circ (h \otimes \varphi_B) \circ (\delta_H \circ \eta_H \otimes u_1^\phi B) \]
(by (b3) of Definition 1.4 and (29))
\[ = \mu_B \circ (h \otimes \varphi_B \circ (\Pi_H \otimes B)) \circ (\delta_H \circ \eta_H \otimes u_1^\phi B) \]
(by (14))
\[ = \mu_B \circ (h \otimes \mu_B \circ (u_1^\phi B \otimes B)) \circ (\delta_H \circ \eta_H \otimes u_1^\phi B) \]
(by (29))
\[ = \mu_B \circ (h \circ \eta_H \otimes u_1^\phi B) \]
(by associativity of \( \mu_B \))
\[ = \mu_B \circ (h \circ \eta_H \ast h \circ \eta_H \otimes u_1^\phi B) \]
(by (i), and properties of \( \eta_B \)).

As a consequence of these equivalences, we obtain (127) because
\[ (\mu_B \otimes H) \circ (B \otimes (h \circ \eta_H \otimes \eta_H \circ \delta_H)) \]
\[ = (\mu_B \otimes H) \circ (B \otimes ((h \circ \Pi_H^L) \circ H) \circ \delta_H \circ \eta_H) \]
(by (17))
\[ = \nabla_{B \otimes \eta_H}^B \circ (B \otimes \eta_H) \]
(by (iii) and (28)).

The proof for the equivalences associated to \( g \) and (128) are similar and we leave the details to the reader.

Finally, assume that there exists \( h^{-1} : H \rightarrow B \) such that \((h, h^{-1})\) is a gauge transformation for \( \varphi_B \) and \( h \ast h^{-1} = u_1^\phi B \) holds. If \( h \circ \eta_H = \eta_B \), we have
\[ h^{-1} \circ \eta_H \]
\[ = (u_1^\phi B \ast h^{-1} \ast h) \circ \eta_H \]
(by \( u_1^\phi B \ast h^{-1} = h^{-1} \))
\[ = ((h \circ \Pi_H^L) \ast h^{-1}) \circ \eta_H \]
(by (ii))
\[ = u_1^\phi B \circ \eta_H \]
(by (14)) and \( u_1^\phi B = h \ast h^{-1} \)
\[ = \eta_B \]
(by (b2) of Definition 1.4).

Conversely, if \( h^{-1} \circ \eta_H = \eta_B \), by similar arguments (in this case \( g = h \)),
\[ h \circ \eta_H = (h \ast u_1^\phi B \ast h^{-1}) \circ \eta_H = (h \ast (h^{-1} \circ \Pi_H^L)) \circ \eta_H = u_1^\phi B \circ \eta_H = \eta_B. \]
As a particular instance of the previous proposition we have the following corollary.

**Corollary 3.4.** Let \((B, \varphi_B)\) be a left weak \(H\)-module algebra and let \(h : H \to B\) be a morphism such that \(h \ast u_1^{\varphi} = h = u_1^{\varphi} \ast h\). Then, the following assertions are equivalent:

(i) \(h \circ \eta_H = \eta_B\).

(ii) \(h \circ \Pi_H^L = u_1^{\varphi}\).

(iii) \(h \circ \Pi_H^R = u_1^{\varphi}\).

Moreover, if \(h \in \text{Reg}_{\varphi_B}(H, B)\) with convolution inverse \(h^{-1} : H \to B\), we have \(h \ast \eta_H = \eta_B\) iff \(h^{-1} \circ \eta_H = \eta_B\). Then, under these conditions, if \(h \ast \eta_H = \eta_B\), the following assertions hold and are equivalent:

(iv) \(h^{-1} \circ \eta_H = \eta_B\).

(v) \(h^{-1} \circ \Pi_H^L = u_1^{\varphi}\).

(vi) \(h^{-1} \circ \Pi_H^R = u_1^{\varphi}\).

**Definition 3.5.** Let \(H\) be a weak Hopf algebra and \(\varphi_B : H \otimes B \to B\) be a measuring. With \(\text{Reg}_{\varphi_B}^l(H, B)\) we will denote the set of morphisms \(h : H \to B\) in \(\text{Reg}_{\varphi_B}(H, B)\) such that

\[
h \circ \eta_H = \eta_B.
\]

(129)

Then, if \((B, \varphi_B)\) is a left weak \(H\)-module algebra, by Corollary 3.4, \(\text{Reg}_{\varphi_B}^l(H, B)\) is a subgroup of \(\text{Reg}_{\varphi_B}(H, B)\).

**Remark 3.6.** The set \(\text{Reg}_{\varphi_B}^l(H, B)\) also acts on \(\mathcal{P}_{\varphi_B}\), i.e., we have a map

\[
R' : \text{Reg}_{\varphi_B}^l(H, B) \times \mathcal{P}_{\varphi_B} \to \mathcal{P}_{\varphi_B},
\]

defined by

\[
R'(h, (\phi_B, \tau)) = R(h, (\phi_B, \tau)),
\]

where \(R\) is the action defined in (126).

Note that, if \((B, \varphi_B)\) is a left weak \(H\)-module algebra, the measuring \(\varphi_B^h\) defined in (120) satisfies (b2) of Definition 3.4 because we have

\[
\varphi_B^h \circ (\eta_H \otimes B) = \mu_B \circ ((\mu_B \circ ((h \circ \Pi_H^L \circ B)) \otimes h^{-1}) \circ (H \otimes P_{\varphi_B}) \circ ((\delta_H \circ \eta_H) \otimes B) \quad \text{by (127)}
\]

\[
= \mu_B \circ ((\mu_B \circ (u_1^{\varphi} \circ B)) \otimes h^{-1}) \circ (H \otimes P_{\varphi_B}) \circ ((\delta_H \circ \eta_H) \otimes B) \quad \text{(by (i) \Rightarrow (iii) of Corollary 3.4)}
\]

\[
= \mu_B \circ (\mu_B \circ (\Pi_H^L \circ B) \otimes h^{-1}) \circ (B \otimes P_{\varphi_B}) \circ ((P_{\varphi_B} \circ (\eta_H \otimes \eta_B)) \otimes B) \quad \text{(by (40)}
\]

\[
= \mu_B \circ (B \otimes h^{-1}) \circ P_{\varphi_B} \circ (\eta_H \otimes B) \quad \text{(by (29) and properties of \(\eta_B\))}
\]

\[
= \mu_B \circ (\varphi_B \otimes B) \circ (H \otimes c_{B,B}) \circ (((H \otimes (h^{-1} \circ \Pi_H^L)) \circ \delta_H \circ \eta_H) \otimes B) \quad \text{(by (27) and naturality of \(c\))}
\]

\[
= \mu_B \circ (B \otimes u_1^{\varphi}) \circ P_{\varphi_B} \circ (\eta_H \otimes B) \quad \text{(by (v) of Corollary 3.4 and the naturality of \(c\))}
\]

\[
= \varphi_B \circ (\eta_H \otimes B) \quad \text{(by (28) for \(\phi_H\))}
\]

\[
= id_B \quad \text{(by (b2) of Definition 3.4)}
\]
Theorem 3.7. Let \( H \) be a weak Hopf algebra, let \((B, \varphi_B), (B, \phi_B)\) be left weak \( H \)-module algebras and let \( \sigma, \tau : H \otimes H \to B \) be morphisms such that \( \sigma \ast u_2^B = \sigma \), \( \tau \ast u_2^B = \tau \). Assume that \( \sigma, \tau \) satisfy the twisted condition \((17)\), the 2-cocycle condition \((18)\) and the normal condition \((34)\). The following assertions are equivalent:

(i) The weak crossed products \((B \otimes H, \mu_B \otimes \varphi_B)\) and \((B \otimes H, \mu_B \otimes \gamma)\) hold. Therefore, by Remark \((2.5)\) we obtain that \((117)\) holds.

(ii) There exists a gauge transformation \((h, h^{-1})\) for \( \varphi_B \) such that \((117)\),

(130)

\[ h \circ \eta_H = \eta_B, \]

(131)

\[ \mu_B \circ (B \otimes h) \circ P_{\varphi_B} = \mu_B \circ (h \otimes \varphi_B) \circ (\delta_H \otimes B), \]

(132)

\[ \mu_B \circ (B \otimes h) \circ F_\tau = (\mu_B \otimes H) \circ (\mu_B \otimes \sigma) \circ (B \otimes P_{\varphi_B} \otimes H) \circ ((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H)), \]

hold.

Proof. We first prove \((i) \Rightarrow (ii)\). By Corollary \((120)\) we know that \((B \otimes H, \mu_B \otimes \varphi_B)\) and \((B \otimes H, \mu_B \otimes \gamma)\) are weak crossed products with preunits \( \nu = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H), u = \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \), respectively. As in the proof of Theorem \((2.4)\) define \( \theta, \gamma \) by \((103)\) and \( h, h^{-1} \) by \((115)\). Then, using that \( T, S \) are morphisms of left \( B \)-modules and \((116)\) we have the following identities:

(133)

\[ (B \otimes \varepsilon_H) \circ T = \mu_B \circ (B \otimes h^{-1}), \]

\[ (B \otimes \varepsilon_H) \circ S = \mu_B \circ (B \otimes h). \]

By \((i) \Rightarrow (ii)\) of Theorem \((2.4)\) the pair \((h, h^{-1})\) is a gauge transformation for \( \varphi_B \) and the identities \((105), (106) \) and \((107)\) hold. Therefore, by Remark \((2.5)\) we obtain that \((117)\) holds.

On the other hand, we obtain \((130)\) because

\[ h \circ \eta_H \]

\[ = (B \otimes \varepsilon_H) \circ S \circ (\eta_B \otimes \eta_H) \text{ (by } 133) \]

\[ = (B \otimes \varepsilon_H) \circ \nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes \eta_H) \text{ (by } 19) \]

\[ = u_1^{\varphi_B} \circ \eta_H \text{ (by naturality of } \epsilon \text{ and comult properties) } \]

\[ = \eta_B \text{ (by } 12) \text{ of Definition } 1.3. \]

Now, by the proof of \((i) \Rightarrow (ii)\) of Theorem \((2.4)\) we know that \( S \) is multiplicative, i.e.,

(134)

\[ S \circ \mu_{B \otimes \varphi_B} H = \mu_{B \otimes \varphi_B} H \circ (S \otimes S). \]

Then composing with \( \eta_B \otimes H \otimes B \otimes \eta_H \) in the previous identity we have

\[ S \circ \mu_{B \otimes \varphi_B} H \circ (\eta_B \otimes H \otimes B \otimes \eta_H) \]

\[ = (\mu_B \circ H) \circ (\mu_B \circ \theta) \circ (B \otimes (F_\tau \circ (H \otimes \eta_H))) \circ P_{\varphi_B} \text{ (by } 108 \text{ and properties of } \eta_B) \]

\[ = (\mu_B \circ H) \circ (\mu_B \circ \theta) \circ (B \otimes (\nabla_{B \otimes H}^{\varphi_B} \circ (\eta_B \otimes H))) \circ P_{\varphi_B} \text{ (by } 108) \]

\[ = (\mu_B \circ H) \circ (B \otimes \theta) \circ \nabla_{B \otimes H}^{\varphi_B} \circ P_{\varphi_B} \text{ (by the left } B \text{-linearity of } \nabla_{B \otimes H}^{\varphi_B} \text{ and properties of } \eta_B) \]

\[ = (\mu_B \circ H) \circ (B \otimes \theta) \circ P_{\varphi_B} \text{ (by } 11) \]

and

\[ \mu_{B \otimes \varphi_B} H \circ (S \otimes S) \circ (\eta_B \otimes H \otimes B \otimes \eta_H) \]

\[ = \mu_{B \otimes \varphi_B} H \circ (((h \otimes H) \circ \delta_H) \otimes (((\mu_B \circ H) \circ (B \otimes ((h \otimes H) \circ \delta_H \circ \eta_H)))) \text{ (by } 109, 110, \text{ and properties} \]

...
of \( \eta_B \)
\[
\mu_{B \otimes_{\varphi, B} H} \circ (((h \otimes H) \circ \delta_H) \otimes (\nabla_{B \otimes H}^\varphi \circ (B \otimes \eta_H))) \quad \text{(by 127)}
\]
\[
\mu_{B \otimes_{\varphi, B} H} \circ (((h \otimes H) \circ \delta_H) \otimes (B \otimes \eta_H)) \quad \text{(by 50)}
\]
\[
(\mu_B \otimes H) \circ (\mu_B \otimes (\nabla_{B \otimes H}^\varphi \circ \eta_B \otimes H)) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta_H) \otimes B) \quad \text{(by 50)}
\]
\[
(\mu_B \otimes H) \circ (B \otimes (\nabla_{B \otimes H}^\varphi \circ P_{\varphi_B})) \circ (((h \otimes H) \circ \delta_H) \otimes B) \quad \text{(by the left } B\text{-linearity of } \nabla_{B \otimes H}^\varphi \text{ and properties of } \eta_B)
\]
\[
(\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta_H) \otimes B) \quad \text{(by the left } B\text{-linearity of } \nabla_{B \otimes H}^\varphi \text{ and 111)}
\]

Therefore,
\[
(\mu_B \otimes H) \circ (B \otimes \theta) \circ P_{\phi_B} = (\mu_B \otimes H) \circ (B \otimes P_{\varphi_B}) \circ (((h \otimes H) \circ \delta_H) \otimes B)
\]

holds and, composing with \( B \otimes \varepsilon_H \) in (135), we obtain (131) by the naturality of \( c \) and (40).

Finally, composing with \( \eta_B \otimes H \otimes \eta_B \otimes H \) in (134) we have
\[
S \circ \mu_{B \otimes_{\varphi, B} H} \circ (\eta_B \otimes H \otimes \eta_B \otimes H) = (\mu_B \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta_H)) \circ F_\tau \quad \text{(by 74, 109, 110 and the properties of } \eta_B)\]

and
\[
\mu_{B \otimes_{\varphi, B} H} \circ (S \otimes S) \circ (\eta_B \otimes H \otimes \eta_B \otimes H) = \mu_{B \otimes_{\varphi, B} H} \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H)) \quad \text{(by 109, properties of } \eta_B, \text{ and 110)}.
\]

As a consequence,
\[
(\mu_B \otimes H) \circ (B \otimes ((h \otimes H) \circ \delta_H)) \circ F_\tau = \mu_{B \otimes_{\varphi, B} H} \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H))
\]

holds and, composing with \( B \otimes \varepsilon_H \), we obtain (132) by the counit properties and (61).

Conversely, consider that (ii) holds. To prove that (ii) ⇒ (i), following (ii) ⇒ (i) of Theorem 2.4, we only need to obtain the equalities (105), (106) and (107). Indeed, note that by Proposition 3.3, \( h^{-1} \circ \eta_H = \eta_B \) because (130) holds. Then, (106) holds because
\[
\mu_B \circ ((\mu_B \circ (h \otimes \varphi_B) \circ (\delta_B \otimes B)) \circ h^{-1}) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B)
\]
\[
\mu_B \circ (B \otimes (\mu_B \circ (h \otimes h^{-1}))) \circ (P_{\phi_B} \otimes H) \circ (H \otimes c_{H,B}) \circ (\delta_B \otimes B) \quad \text{(by 131 and the associativity of } \mu_B)
\]
\[
\mu_B \circ (B \otimes (h \otimes h^{-1})) \circ P_{\phi_B} \quad \text{(by the naturality of } c \text{ and coassociativity of } \delta_B)
\]
\[
\mu_B \circ (B \otimes u_1^{\phi_B}) \circ P_{\phi_B} \quad \text{(by 117)}
\]
\[
= \phi_B \quad \text{(by 58 for } \phi_B).
\]

On the other hand,
\[
\mu_B \circ (B \otimes h^{-1}) \circ \mu_{B \otimes_{\varphi, B} H} \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H))
\]
\[
= \mu_B \circ (((\mu_B \otimes H) \circ (\mu_B \otimes \sigma)) \circ (B \otimes P_{\varphi_B} \otimes H) \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H)) \otimes (h^{-1} \otimes \mu_H)) \circ \delta_{H \otimes 2}
\]
\[
= \mu_B \circ (B \otimes (\mu_B \circ (h \otimes h^{-1}))) \circ (F_\tau \circ \mu_B) \circ \delta_{H \otimes 2} \quad \text{(by 136 and associativity of } \mu_B)
\]
\[
= \mu_B \circ (B \otimes (h \otimes h^{-1})) \circ F_\tau \quad \text{(by 59)}
\]
\[
= \mu_B \circ (B \otimes u_1^{\phi_B}) \circ F_\tau \quad \text{(by 117)}
\]
\[
= \tau \quad \text{(by 59)}
\]

and then (106) holds. Finally, we obtain (107) because
((\mu_B \circ (B \otimes h^{-1})) \otimes H) \circ (B \otimes \delta_H) \circ \nabla_B^\phi \circ (\eta_B \otimes \eta_H) \\
= ((u_1^\phi \ast h^{-1}) \otimes H) \circ \delta_H \circ \eta_H \quad \text{(by (17) and by the coassociativity of \delta_H)} \\
= (h^{-1} \otimes H) \circ \delta_H \circ \eta_H \quad \text{(by the gauge transformation condition)} \\
= ((h^{-1} \circ \Pi_H^L) \otimes H) \circ \delta_H \circ \eta_H \quad \text{(by (17))} \\
= (u_1^\phi \otimes H) \circ \delta_H \circ \eta_H \quad \text{(by (vi) of Proposition 3.3)} \\
= \nabla_B^\phi \circ (\eta_B \otimes \eta_H) \quad \text{(by (14)).} \\
\]

\[ \square \]

3.8. As a consequence of the previous theorem, it is possible to define a groupoid, denoted by \( \mathcal{G}_H^B \) whose objects are pairs

\[(\varphi_B, \sigma),\]

where \((B, \varphi_B)\) is a left weak \(H\)-module algebra, \(\sigma : H \otimes H \to B\) is a morphism such that

\[ u_2^\varphi \ast \sigma = \sigma \ast u_2^\varphi \]

and the associated quadruple \( \Xi_H \) satisfies the twisted, cocycle and normal conditions.

A morphism between two objects \((\varphi_B, \sigma), (\phi_B, \tau)\) of \( \mathcal{G}_H^B \) is defined by a morphism \( h : H \to B \) for which there exists a morphism \( h^{-1} : H \to B \) such that \((h, h^{-1})\) is a gauge transformation for \( \varphi_B \) satisfying the conditions (ii) of Theorem 3.7. If \( h : (\varphi_B, \sigma) \to (\phi_B, \tau), g : (\phi_B, \tau) \to (\chi_B, \omega) \) are morphisms in \( \mathcal{G}_H^B \), the composition, denoted by \( g \circ h \), is defined by

\[ g \circ h = g \ast h. \]

The previous composition is well defined because, if \( l = g \ast h \) and \( l^{-1} = h^{-1} \ast g^{-1} \), it is easy to show that \((l, l^{-1})\) is a gauge transformation for \( \varphi_B \) and \( l \ast l^{-1} = u_1^\chi \). Also,

\[ l \circ \eta_H = (g \ast (h \circ \Pi_H^L)) \circ \eta_H \quad \text{(by (14))} \\
= (g \ast u_1^\phi) \circ \eta_H \quad \text{(by (ii) of Proposition 3.3)} \\
= g \circ \eta_H \quad \text{(by the condition of gauge transformation)} \\
= \eta_B \quad \text{(by (14))} \]

and then (130) holds for \( l \). The equality (131) for \( l \) follows by

\[ \mu_B \circ (B \otimes l) \circ P_{\chi_B} = \mu_B \circ ((\mu_B \circ (B \otimes g) \circ P_{\chi_B}) \otimes h) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B) \quad \text{(by the associativity of \mu_B, the coassociativity of \delta_H and the naturality of \chi)} \]

\[ = \mu_B \circ ((\mu_B \circ (g \otimes \phi_B) \circ (\delta_H \otimes B) \otimes h) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B) \quad \text{(by (131))} \]

\[ = \mu_B \circ (g \otimes ((\mu_B \circ (h \otimes \varphi_B) \circ (\delta_H \otimes B))) \circ (\delta_H \otimes B) \quad \text{(by the coassociativity of \delta_H)} \]

\[ = \mu_B \circ (g \otimes (\mu_B \circ (B \otimes h) \circ P_{\phi_B}) \circ (\delta_H \otimes B) \quad \text{(by (131))} \]

\[ = \mu_B \circ ((l \otimes \phi_B) \circ (\delta_H \otimes B) \quad \text{(by the associativity of \mu_B and the coassociativity of \delta_H)} \]

and (132) holds because

\[ \mu_B \circ (B \otimes l) \circ F_\omega = \mu_B \circ ((\mu_B \circ (B \otimes g) \circ F_\omega) \otimes (h \circ \mu_H) \circ \delta_{H \otimes 2} \quad \text{(by the associativity of \mu_B, the coassociativity \delta_{H \otimes 2})} \]
\[ = \mu_B \circ ((\mu_B \circ (\mu_B \otimes \tau) \circ (B \otimes P_{\phi_B} \otimes H)) \circ (((g \otimes H) \circ \delta_H) \otimes ((g \otimes H) \circ \delta_H))) \otimes (h \circ \mu_H) \circ \delta_H \otimes 2 \]

(by **1.34**)

\[ = \mu_B \circ ((\mu_B \circ (B \otimes \phi_B)) \circ (\mu_B \circ (B \otimes h) \circ F_\tau)) \circ (B \otimes H \otimes c_{H,B} \otimes H) \circ (((g \otimes \delta_H) \circ \delta_H) \otimes ((g \otimes H) \circ \delta_H)) \]

(by the associativity of \(\mu_B\), the coassociativity of \(\delta_H\) and the naturality of \(c\))

\[ = \mu_B \circ ((\mu_B \circ (B \otimes \phi_B)) \circ (\mu_B \circ (\mu_B \otimes \sigma) \circ (B \otimes P_{\varphi_B} \otimes H)) \circ (((h \otimes H) \circ \delta_H) \otimes ((h \otimes H) \circ \delta_H))) \]

\[ = \mu_B \circ ((\mu_B \circ (B \otimes \phi_B)) \circ (\mu_B \circ (B \otimes h) \circ P_{\phi_B})) \circ ((\mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H))) \]

\[ \circ (((g \otimes \delta_H) \circ \delta_H) \otimes (((g \otimes h) \circ \delta_H) \otimes H) \circ \delta_H)) \]

(by the associativity of \(\mu_B\), the coassociativity of \(\delta_H\) and the naturality of \(c\))

\[ = \mu_B \circ ((\mu_B \circ (B \otimes B) \circ \sigma) \circ (B \otimes P_{\varphi_B} \otimes H)) \circ (((l \otimes H) \circ \delta_H) \otimes ((l \otimes H) \circ \delta_H)) \]

(by the associativity of \(\mu_B\), the coassociativity of \(\delta_H\) and the naturality of \(c\))

The identity of \((\varphi_B, \sigma)\) is \(id_{(\varphi_B, \sigma)} = u_1^{\varphi_B}\) because \(h \ast u_1^{\varphi_B} = h\) and \((u_1^{\varphi_B}, u_1^{\varphi_B})\) is a gauge transformation for \(\varphi_B\) satisfying **1.25**. The equality **1.30** follows from (b2) of Definition **1.34** and **1.31** is a consequence of the naturality of \(c\) and (b1) of Definition **1.34** and **1.32** holds because:

\[ (\mu_B \otimes H) \circ (\mu_B \otimes \sigma) \circ (B \circ P_{\varphi_B}) \circ (((u_1^{\varphi_B} \otimes H) \circ \delta_H) \otimes ((u_1^{\varphi_B} \circ H) \circ \delta_H)) \]

\[ = \mu_B \circ ((\mu_B \circ (u_1^{\varphi_B} \otimes H) \circ \sigma) \circ \delta_{H \otimes 2}) \]

(by naturality of \(c\) and coassociativity of \(\delta_H\))

\[ = u_2^{\varphi_B} \ast \sigma \]

(by **1.33**)

\[ = \sigma \]

(by **1.25**)

\[ = \mu_B \circ (B \otimes u_1^{\varphi_B}) \circ F_\sigma \]

As a consequence, \(h\) is an isomorphism with inverse \(h^{-1}\) and \((g \otimes h) \circ \delta_H = h^{-1} \otimes g^{-1}\). Therefore, \(G_H^B\) is a groupoid.

**3.9.** Let \(H\) be a weak Hopf algebra, let \((B, \varphi_B)\) be a left weak \(H\)-module algebra and let \(\sigma : H \otimes H \to B\) be a morphism such that \(\sigma \ast u_2^{\varphi_B} = \sigma\). Assume that \(\sigma\) satisfies the twisted condition **1.27**, the 2-cocycle condition **1.8** and the normal condition **1.31**. Let \(h\) be a morphism in \(Reg_{\varphi_B}(H, B)\). Then \((h, h^{-1})\) is a gauge transformation for \(\varphi_B\) such that **1.25** and **1.30** hold. Define \(\varphi_B^h\) and \(\sigma h\) as in **1.20** and **1.21** respectively. Then, by **2.7**, \(\varphi_B^h\) is a measuring such that **1.22** holds. Therefore \(u_1^{\varphi_B^h} = u_1^{\varphi_B^h}\) and then

\[ = \gamma_{B \otimes H}^B = \gamma_{B \otimes H}^B. \]

Moreover, by Remark **3.6** we know that \(\varphi_B^h\) satisfies (b2) of Definition **1.34**. On the other hand, \(\sigma^h\) is such that \(\sigma^h \ast u_2^{\varphi_B^h} = \sigma^h\) and satisfies the twisted condition **1.27**, the cocycle condition **1.8** and

\[ \nu^h = (\mu_B \otimes H) \circ (B \circ ((h^{-1} \otimes H) \circ \delta_H)) \circ \gamma_{B \otimes H}^B \circ \eta \]

is a preunit for the associated weak crossed product \((B \otimes H, \mu_{B \otimes \varphi_B^h}^h, h)\). Note that
\[ \mu^h = ((u_1^{\varphi_B} * h^{-1}) \otimes H) \circ \delta_H \circ \eta_H \quad (\text{by the coassociativity of } \delta_H) \]
\[ = (h^{-1} \otimes H) \circ \delta_H \circ \eta_H \quad (\text{by the condition of gauge transformation}) \]
\[ = ((h^{-1} \circ \Pi_H^L) \otimes H) \circ \delta_H \circ \eta_H \quad (\text{by } (17)) \]
\[ = (u_1^{\varphi_B} \otimes H) \circ \delta_H \circ \eta_H \quad (\text{by } (14)) \]
\[ = \nabla_{B \otimes H}^\varphi (\eta_B \otimes \eta_H) \quad (\text{by } (13)) \]

Therefore,
\[ \mu^h = \nabla_{B \otimes H}^\varphi (\eta_B \otimes \eta_H) = \nabla_{B \otimes H}^{\varphi_B} (\eta_B \otimes \eta_H) = \nu. \]

Also, \( \varphi_B^h \) satisfies (20) because
\[ \varphi_B^h = \mu_B \otimes (\mu_B \otimes h^{-1}) \circ (B \otimes P_{\varphi_B}) \circ > \left( (h \circ \Pi_H^L) \otimes H \right) \circ \delta_H \circ \Pi_H^L \circ B \quad (\text{by } (3)) \]
\[ = \mu_B \otimes (\mu_B \otimes h^{-1}) \circ (u_1^{\varphi_B} \otimes P_{\varphi_B}) \circ (\delta_H \circ \Pi_H^L \circ B) \quad (\text{by } (ii) \text{ of Corollary } 3.4) \]
\[ = \mu_B \otimes (B \otimes h^{-1}) \circ P_{\varphi_B} \circ (\Pi_H^L \circ B) \quad (\text{by } (53)) \]
\[ = \mu_B \circ ((\varphi_B \circ (\Pi_H^L \otimes B)) \otimes h^{-1}) \circ (H \otimes c_{H,B}) \circ (\delta_H \circ \Pi_H^L \circ \otimes B) \quad (\text{by } (5)) \]
\[ = \mu_B \circ ((\mu_B \circ c_{B,B} \circ (u_1^{\varphi_B}) \otimes h^{-1}) \circ (H \otimes c_{H,B}) \circ (\delta_H \circ \Pi_H^L \circ B) \quad (\text{by } (28) \text{ for } \varphi_B) \]
\[ = \mu_B \circ (B \otimes (u_1^{\varphi_B} \otimes h^{-1})) \circ c_{B,B} \circ (\Pi_H^L \otimes B) \quad (\text{by the associativity of } \mu_B \text{ and the naturality of } c) \]
\[ = \mu_B \circ c_{B,B} \circ ((h^{-1} \circ \Pi_H^L) \otimes B) \quad (\text{by the naturality of } c \text{ and the condition of gauge transformation}) \]
\[ = \mu_B \circ c_{B,B} \circ (u_1^{\varphi_B} \otimes B) \quad (\text{by } (vi) \text{ of Corollary } 3.4) \]
\[ = \mu_B \circ c_{B,B} \circ (u_1^{\varphi_B} \otimes B) \quad (\text{by } u_1^{\varphi_B} = u_1^{\varphi_B}) \]

As a consequence, using the same proof than in Remark 1.21 we obtain that (83) holds for \( \varphi_B^h \).

Therefore, by Corollary 1.19 we have that \( \sigma^h \) satisfies the normal condition (84), i.e.,
\[ \sigma^h \circ (\eta_H \otimes H) = u_1^{\varphi_B^h} = \sigma^h \circ (H \otimes \eta_H). \]

Finally, if \( B \) is commutative and \( H \) is cocommutative, the equality
\[ (138) \quad \mu_B \circ (B \otimes h) \circ P_{\varphi_B} = \mu_B \circ (h \otimes \varphi_B) \circ (\delta_H \otimes B) \]
holds and then
\[ \varphi_B^h = \mu_B \circ ((\mu_B \circ (h \otimes \varphi_B) \circ (\delta_H \circ B)) \otimes h^{-1}) \circ (H \otimes c_{H,B}) \circ (\delta_H \circ B) \quad (\text{by the coassociativity of } \delta_H) \]
\[ = \mu_B \circ (\mu_B \circ (B \otimes h \circ P_{\varphi_B})) \circ h^{-1}) \circ (H \otimes c_{H,B}) \circ (\delta_H \circ B) \quad (\text{by } (135)) \]
\[ = \mu_B \circ (B \otimes (h \circ h^{-1})) \circ P_{\varphi_B} \quad (\text{by the coassociativity of } \delta_B, \text{ the associativity of } \mu_B \text{ and the naturality of } c) \]
\[ = \mu_B \circ (B \otimes u_1^{\varphi_B}) \circ P_{\varphi_B} \quad (\text{by } (124)) \]
\[ = \varphi_B \quad (\text{by } (8)) \]

4. Hom-products, invertible morphisms and centers

In this subsection, for a weak Hopf algebra \( H \) and an algebra \( B \), we will explore a product in \( Hom_{\mathcal{C}}(H \otimes^n B, B) \) that will permit us to extend some results about the factorization through the center of \( B \), given in [14] for Hopf algebras, to the weak Hopf algebra setting.
**Definition 4.1.** Let $H$ be a weak Hopf algebra and let $\varphi_B : H \otimes B \to B$ be a measuring. Let $\varphi$ and $\psi \in Hom_C(H^{\otimes n} \otimes B, B)$. We define the product

\[ \wedge : Hom_C(H^{\otimes n} \otimes B, B) \times Hom_C(H^{\otimes n} \otimes B, B) \to Hom_C(H^{\otimes n} \otimes B, B) \]

between $\varphi$ and $\psi$ as

\[ \varphi \wedge \psi = \varphi \circ (H^{\otimes n} \otimes \psi) \circ (\delta_{H^{\otimes n}} \otimes B). \]

Obviously, $\wedge$ is an associative product because $\delta_{H^{\otimes n}}$ is coassociative.

We say that a morphism $\varphi \in Hom_C(H^{\otimes n} \otimes B, B)$ is $\varphi_B$-invertible if there exists a morphism $\varphi^\dagger \in Hom_C(H^{\otimes n} \otimes B, B)$ such that

\[ \varphi \wedge \varphi^\dagger = \mu_B \circ (\nu_B \otimes B). \]

**Proposition 4.2.** Let $H$ be a weak Hopf algebra and let $\varphi_B : H \otimes B \to B$ be a measuring. For $\omega : H^{\otimes n} \to B$ define

\[ \overline{\varphi} = \mu_B \circ (\omega \otimes B), \quad \overline{\varphi}^p = \mu_B \circ c_{B,B} \circ (\omega \otimes B). \]

Then, if $\omega, \theta \in Hom_C(H^{\otimes n}, B)$ and $\gamma \in Hom_C(H^{\otimes n} \otimes B, B)$ the following equalities hold:

(i) $\overline{\varphi} \wedge \overline{\theta} = \overline{\omega \star \theta}$.

(ii) If $H$ is cocommutative, $\overline{\varphi}^p \wedge \gamma = \mu_B \circ (\gamma \otimes \omega) \circ (H^{\otimes n} \otimes c_{H^{\otimes n},B}) \circ (\delta_{H^{\otimes n}} \otimes B)$.

(iii) If $H$ is cocommutative, $\overline{\varphi}^p \wedge \overline{\gamma} = \overline{\theta \star \omega}^p$.

(iv) If $H$ is cocommutative, $\overline{\varphi}^p \wedge \overline{B} = \overline{\theta \wedge \varphi}$.

(v) If $(B, \varphi_B)$ is a left weak $H$-module algebra, $\overline{u_B}^{\otimes n} \wedge \varphi^{\otimes n}_B = \varphi^{\otimes n}_B$.

(vi) If $H$ is cocommutative and $(B, \varphi_B)$ is a left weak $H$-module algebra, $\overline{u_B}^{\otimes n} \wedge \varphi^{\otimes n}_B = \varphi^{\otimes n}_B$.

(vii) $\overline{\varphi_B} \circ (H \otimes \omega) \wedge (\varphi_B \circ (H \otimes \gamma)) = \varphi_B \circ (H \otimes (\overline{\varphi} \wedge \gamma))$.

(viii) If $H$ is cocommutative, $\overline{\varphi_B} \circ (H \otimes \omega)^p \wedge (\varphi_B \circ (H \otimes \gamma)) = \varphi_B \circ (H \otimes (\overline{\varphi}^p \wedge \gamma))$.

**Proof.** The proof of (i) follows directly from the associativity of $\mu_B$. If $H$ is cocommutative, so is $H^{\otimes n}$ and, by the naturality of $c$, we obtain (ii). By similar reasoning and using the associativity of $\mu_B$ we obtain (iii) and (iv). On the other hand,

\[ \overline{u_B}^{\otimes n} \wedge \varphi^{\otimes n}_B \]

\[ = \mu_B \circ (\varphi^{\otimes n}_B \otimes \varphi^{\otimes n}_B) \circ (H^{\otimes n} \otimes c_{H^{\otimes n},B} \otimes B) \circ (\delta_{H^{\otimes n}} \otimes \eta_B \otimes B) \]

\[ = \varphi^{\otimes n}_B \circ (H^{\otimes n} \otimes (\mu_B \circ (\eta_B \otimes B))) = \varphi^{\otimes n}_B, \]

and then (v) holds. Similarly, using that $H^{\otimes n}$ is cocommutative, the naturality of $c$ and (iv) we prove (vi).

The identity, (vii) follows from

\[ \overline{\varphi_B} \circ (H \otimes \omega) \wedge (\varphi_B \circ (H \otimes \gamma)) \]

\[ = \mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_B \otimes ((\omega \otimes \theta) \circ (\delta_{H^{\otimes n}} \otimes B))) \quad \text{(by the naturality of } c) \]

\[ = \varphi_B \circ (H \otimes (\overline{\varphi} \wedge \gamma)) \quad \text{(by (iv))} \]

and, similarly, using that $H$ is cocommutative, we obtain (viii).
Remark 4.3. The equivalence of measurings (or, in particular, of weak actions) in terms of gauge transformations acquires a new meaning in terms of this product. Actually, if $H$ is cocommutative, the action described in (120) on a measuring $\phi_B$ can be seen as a conjugation by gauge transformations in the following way:

$$\phi_B^h = \hbar \wedge \hbar^{-1}\op \wedge \phi_B.$$ 

Moreover observe that for a cocommutative weak Hopf algebra $H$ and measurings $\varphi_B$ and $\phi_B$ satisfying conditions of Theorem 2.4 we can re-write equality (105) using the Hom-product as

$$\phi_B = \hbar \wedge \hbar^{-1}\op \wedge \varphi_B.$$ 

Also in this way, equality (131) of Theorem 3.7 can be interpreted as $\hbar^{-p} \wedge \phi_B = \hbar \wedge \varphi_B$, in coherence with the action of gauge transformations as a conjugation given above.

Proposition 4.4. Let $H$ be a cocommutative weak Hopf algebra and let $(B, \varphi_B)$ be a left weak $H$-module algebra. A morphism $\sigma : H^{\otimes 2} \to B$ satisfies the twisted condition (77) if and only if

$$\sigma^{op} \wedge \varphi_B^{\otimes 2} = \sigma \wedge (\varphi_B \circ (\mu_H \otimes B))$$

holds.

Proof. The proof follows from the following facts. First, note that by definition of $F_{\sigma}$, we have that

$$\mu_B \circ (B \otimes \varphi_B) \circ (F_{\sigma} \otimes B) = \sigma \wedge (\varphi_B \circ (\mu_H \otimes B)).$$

On the other hand, if $H$ is cocommutative, by the naturality of $c$, we have

$$\mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) = \sigma^{op} \wedge \varphi_B^{\otimes 2}.$$ 

$\square$

Definition 4.5. Let $H$ be a weak Hopf algebra and $(B, \varphi_B)$ be a left weak $H$-module algebra. For $n \geq 1$, with $\text{Reg}_{\varphi_B}(H^{\otimes n}, B)$ we will denote the set of morphisms $\sigma : H^{\otimes n} \to B$ such that there exists a morphism $\sigma^{-1} : H^{\otimes n} \to B$ (the convolution inverse of $\sigma$) satisfying the following equalities:

$$\sigma \ast \sigma^{-1} = \sigma^{-1} \ast \sigma = u_{n}^{\varphi_B},$$

$$\sigma \ast \sigma^{-1} \ast \sigma = \sigma, \quad \sigma^{-1} \ast \sigma \ast \sigma^{-1} = \sigma^{-1}.$$ 

Note that, for $n = 1$, we recover the group $\text{Reg}_{\varphi_B}(H, B)$ introduced in Definition 3.1. For any $n$, $\text{Reg}_{\varphi_B}(H^n, B)$ is a group with unit element $u_{n}^{\varphi_B}$ because by (34) we know that $u_{n}^{\varphi_B} \ast u_{n}^{\varphi_B} = u_{n}^{\varphi_B}$. Also, if $B$ is commutative and $H$ is cocommutative, we have that $\text{Reg}_{\varphi_B}(H^{\otimes n}, B)$ is an abelian group.

We denote by $\text{Reg}_{\varphi_B}(H_L, B)$ the set of morphisms $g : H_L \to B$ such that there exists a morphism $g^{-1} : H_L \to B$ (the convolution inverse of $g$) satisfying

$$g \ast g^{-1} = g^{-1} \ast g = u_0^{\varphi_B}, \quad g \ast g^{-1} \ast g = g, \quad g^{-1} \ast g \ast g^{-1} = g^{-1},$$

where $u_0^{\varphi_B} = u_1^{\varphi_B} \circ i^*_H$. Then by (27) we have $u_1^{\varphi_B} = u_0^{\varphi_B} \circ p_H^L$. 


Definition 4.6. For an algebra $B$ we define the center of $B$ as a subobject $Z(B)$ of $B$ with a monomorphism $z_B : Z(B) \to B$ satisfying the identity

$$ \mu_B \circ (B \otimes z_B) = \mu_B \circ c_{B,B} \circ (B \otimes z_B) $$

and such that, if $f : A \to B$ is a morphism for which $\mu_B \circ (B \otimes f) = \mu_B \circ c_{B,B} \circ (B \otimes f)$ holds, there exists an unique morphism $f' : A \to Z(B)$ satisfying $z_B \circ f' = f$. As a consequence, we obtain that $Z(B)$ is a commutative algebra, where $\eta_{Z(B)}$ is the unique morphism such that

$$ z_B \circ \eta_{Z(B)} = \eta_B $$

and $\mu_{Z(B)}$ is the unique morphism such that

$$ z_B \circ \mu_{Z(B)} = \mu_B \circ (z_B \otimes z_B). $$

For example, if $C$ is a closed category with equalizers and $\alpha_B$ and $\beta_B$ are the unit and the counit, respectively, of the $C$-adjunction $B \otimes - \dashv [B,-] : C \to C$, the center of $B$ can be obtained by the following equalizer diagram:

$$ \begin{array}{ccc}
Z(B) & \xrightarrow{z_B} & B \\
\downarrow & & \downarrow \theta_B \\
\theta_B & & [B,B]
\end{array} $$

where $\theta_B = [B,\mu_B] \circ \alpha_B(B)$ and $\theta_B = [B,\mu_B \circ c_{B,B}] \circ \alpha_B(B)$. Then in the category of modules over a commutative ring the center is an equalizer object.

Finally, note that by (143), composing with the symmetry isomorphism we obtain

$$ \mu_B \circ (z_B \otimes B) = \mu_B \circ c_{B,B} \circ (z_B \otimes B). $$

Example 4.7. Let $H$ be cocommutative weak Hopf algebra and let $(B,\varphi_B)$ be a left weak $H$-module algebra. Then, $\Pi^L_H = \Pi^L_H$ and by (25) and (26) we have that $\mu_B \circ c_{B,B} \circ (u^B_1 \otimes B) = \mu_B \circ (u^B_1 \otimes B)$. Then, $u^B_1$ factors through $Z(B)$. Therefore, there exists an unique morphism $v^B_1 : H \to Z(B)$ such that

$$ z_B \circ v^B_1 = u^B_1. $$

Then, taking into account the equality (33), we obtain

$$ \mu_B \circ c_{B,B} \circ (u^B_n \otimes B) = \mu_B \circ c_{B,B} \circ ((u^B_1 \otimes m^H_n) \otimes B) = \mu_B \circ ((u^B_1 \otimes m^H_n) \otimes B) = \mu_B \circ (u^B_n \otimes B) $$

and, as a consequence, $u^B_n$ factors through $Z(B)$. Therefore, there exists an unique morphism $v^B_n : H \otimes H \to Z(B)$ such that

$$ z_B \circ v^B_n = u^B_n. $$

Remark 4.8. Let $H$ be a weak Hopf algebra. Let $\omega : H^\otimes n \to B$ be a morphism. Then, $\omega$ factors through the center of $B$ if and only if $\omega = \omega^{\otimes p}$. Therefore, if $H$ is cocommutative and $(B,\varphi_B)$ is a left weak $H$-module algebra, $u^B_n = u^{\otimes n}_{n}^{\text{op}}$ for all $n \geq 1$.

Also, if $\omega$ factors through the center of $B$, $\omega \ast \tau = \tau \ast \omega$ for all morphism $\tau : H^\otimes n \to B$.

Proposition 4.9. Let $H$ be a cocommutative weak Hopf algebra. Let $(B,\varphi_B)$ be a left weak $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2,B)$ satisfying the twisted condition (77). Then, $\varphi_B$ is $\varphi_B$-invertible.
Proof. Let $h_\sigma$ and $h_{\sigma^{-1}}$ be the morphisms defined by

$$h_\sigma = \sigma \circ (H \otimes \lambda_H) \circ \delta_H, \quad h_{\sigma^{-1}} = \sigma^{-1} \circ (H \otimes \lambda_H) \circ \delta_H.$$  

Then, $h_\sigma \in \text{Reg}_{\varphi_B}(H, B)$ and $h_{\sigma^{-1}} = h_{\sigma^{-1}}$. Indeed, first note that

$$h_\sigma \ast h_{\sigma^{-1}} = \mu_B \circ (\sigma \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H$$

(by the coassociativity and the cocommutativity of $\delta_H$ and the naturality of $c$)

$$= (\sigma \ast \sigma^{-1}) \circ (H \otimes \lambda_H) \circ \delta_H$$

(by the coassociativity of $\delta_H$ and 19)

$$= u_1^{\varphi_B} \circ \mu_H \circ (H \otimes \lambda_H) \circ \delta_H$$

(by $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$)

$$= u_1^{\varphi_B} \otimes \Pi^L_H$$

(by the definition of $\Pi^L_H$)

$$= u_1^{\varphi_B}$$

(by 24)

and similarly, $h_{\sigma^{-1}} \ast h_\sigma = u_1^{\varphi_B}$. Also, by the coassociativity and the cocommutativity of $\delta_H$, the naturality of $c$, 18, and $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$ we have that

$$h_\sigma \ast h_{\sigma^{-1}} \ast h_\sigma = (\sigma \ast \sigma^{-1} \ast \sigma) \circ (H \otimes \lambda_H) \circ \delta_H = h_\sigma$$

and $h_{\sigma^{-1}} \ast h_\sigma \ast h_{\sigma^{-1}} = h_{\sigma^{-1}}$ hold.

Now, let $\varphi_\sigma$ be the morphism defined by

$$\varphi_\sigma = \mu_B \circ (\mu_B \otimes B) \circ (h_\sigma \otimes B \otimes h_{\sigma^{-1}}) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

Then, $\varphi_\sigma$ is $\varphi_B$-invertible with inverse defined by

$$\varphi_\sigma^\dagger = \mu_B \circ (\mu_B \otimes B) \circ (h_{\sigma^{-1}} \otimes B \otimes h_\sigma) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

Indeed:

$$\varphi_\sigma \wedge \varphi_\sigma^\dagger = \mu_B \circ (B \otimes (\mu_B \otimes c_{B,B})) \otimes ((h_\sigma \ast h_{\sigma^{-1}}) \otimes (h_\sigma \ast h_{\sigma^{-1}}) \circ \delta_H \otimes B)$$

(by the coassociativity and the cocommutativity of $\delta_H$, the naturality of $c$ and the associativity of $\mu_B$)

$$= \mu_B \circ (u_1^{\varphi_B} \otimes (\mu_B \otimes c_{B,B} \otimes (u_1^{\varphi_B} \otimes B))) \circ (\delta_H \otimes B)$$

(by $\sigma \in \text{Reg}_{\varphi_B}(H, B))$

$$= \mu_B \circ ((u_1^{\varphi_B} \ast u_1^{\varphi_B}) \otimes B)$$

(by the factorization of $u_1^{\varphi_B}$ through the center of $B$)

$$= \mu_B \circ (u_1^{\varphi_B} \otimes B)$$

(by 24).

On the other hand, let be the morphism $\varphi_B \wedge (\varphi_B \circ (\lambda_H \otimes B))$. For this morphism we have the following

$$h_\sigma^{op} \wedge (\varphi_B \wedge (\varphi_B \circ (\lambda_H \otimes B)))$$

$$= \mu_B \circ (\varphi_B \circ \sigma) \circ (H \otimes c_{H,B} \otimes H) \circ (\delta_H \otimes (c_{H,B} \circ (H \otimes \varphi_B) \circ ((\delta_H \circ \lambda_H) \otimes B))) \circ (\delta_H \otimes B)$$

(by the coassociativity and the cocommutativity of $\delta_H$, the naturality of $c$ and 23)

$$= \mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes B)$$

(by the cocommutativity of $\delta_H$ and the naturality of $c$)

$$= \mu_B \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes B) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes B)$$

(by 23)

$$= \mu_B \circ (B \otimes \varphi_B) \circ ((h_\sigma \otimes \Pi^L_H) \circ \delta_H) \otimes B$$

(by the cocommutativity of $\delta_H$, the naturality of $c$ and 23)

$$= \mu_B \circ (h_\sigma \circ u_1^{\varphi_B}) \circ \delta_H \otimes B$$

(by 23)

$$= \mu_B \circ (h_\sigma \ast u_1^{\varphi_B}) \otimes B$$

(by the associativity of $\mu_B$)
Indeed: Define $\varphi$ and therefore $\varphi$ is true for $n$ by Proposition 4.10.
By assumption the assertion is true for algebra and suppose that $\varphi_B \circ (\lambda_B \otimes B)$ holds because, on the one hand, $\varphi_B \circ (\varphi_B \circ (\lambda_B \otimes B))$ holds by (146) we have
$\varphi_B = \varphi_B \wedge (\varphi_B \circ (\lambda_B \otimes B))$
holds because, on the one hand,
$\varphi_B \circ (\varphi_B \circ (\lambda_B \otimes B))$
holds by the induction hypothesis
$\varphi_B \circ (\varphi_B \circ (\lambda_B \otimes B))$ (by associativity of $\wedge$ and (iii) of Proposition 4.2)
$= (\mu_B \otimes \varphi_B \wedge (\varphi_B \circ (\lambda_B \otimes B))$ (by $h_\sigma \in \text{Reg}_B (H, B)$)
$= (\mu_B \otimes \varphi_B \wedge (\varphi_B \circ (\lambda_B \otimes B))$ (by associativity of $\wedge$)
$= \varphi_B \circ (\lambda_B \otimes B)$ (by (vi) of Proposition 4.2)

and, on the other hand, by the cocommutativity of $\delta_H$, the naturality of $c$ and the associativity of $\mu_B$
$\varphi_B \circ (\lambda_B \otimes B)$
Finally, define the morphism $\varphi_B^\dagger$ by
$\varphi_B^\dagger = (\varphi_B \circ (\lambda_B \otimes B)) \wedge \varphi_B^{-1}$. Then, by (146) we have
$\varphi_B \wedge \varphi_B^\dagger = \varphi_B \wedge \varphi_B^\dagger = \mu_B \circ (u_1^B \otimes B)$
and therefore $\varphi_B$ is $\varphi_B$-invertible with inverse $\varphi_B^\dagger$.

Proposition 4.10. Let $H$ be a cocommutative weak Hopf algebra. Let $(B, \varphi_B)$ be a left weak $H$-module algebra and suppose that $\varphi_B$ is $\varphi_B$-invertible. Then, $\varphi_B^{\otimes n}$ is $\varphi_B^{\otimes n}$-invertible.

Proof. By assumption the assertion is true for $n = 1$. We will proceed by induction assuming that it is true for $n - 1$, i.e., $\varphi_B^{\otimes (n-1)}$ is $\varphi_B^{\otimes (n-1)}$-invertible with inverse $\varphi_B^{\otimes (n-1)}$, and we will prove it for $n$. Indeed: Define $\varphi_B^{\otimes n}$ by
$\varphi_B^{\otimes n} = \varphi_B^{\otimes (n-1)} \circ (H \otimes (n-1) \otimes \varphi_B^\dagger) \circ (\varphi_B^{\otimes (n-1)} \otimes B)$.

Then,
$\varphi_B^{\otimes n} \wedge \varphi_B^{\otimes n}$
$= \varphi_B \circ (H \otimes (\varphi_B^{\otimes (n-1)} \wedge \varphi_B^{\otimes (n-1)})) \circ (H \otimes H \otimes (n-1) \otimes \varphi_B^\dagger) \circ (H \otimes c_{H, H \otimes (n-1)} \otimes B) \circ (\delta_H \otimes H \otimes (n-1) \otimes B)$
(by naturality of $c$)
$= \varphi_B \circ (H \otimes (\mu_B \circ (u_1^B \otimes B)) \circ (H \otimes H \otimes (n-1) \otimes \varphi_B^\dagger) \circ (H \otimes c_{H, H \otimes (n-1)} \otimes B) \circ (\delta_H \otimes H \otimes (n-1) \otimes B)$
(by the induction hypothesis)
$= \varphi_B \circ (H \otimes \mu_B \circ (H \otimes \varphi_B^\dagger \otimes B) \circ (\delta_H \otimes (c_{B, B} \circ (u_1^B \otimes B)))$ (by the factorization of $u_1^B$ through the center of $B$ and the naturality of $c$)
$= \mu_B \circ (\varphi_B \circ \varphi_B) \circ (H \otimes c_{H, H \otimes B} \circ (\delta_H \otimes \varphi_B^\dagger \otimes B) \circ (\delta_H \otimes (c_{B, B} \circ (u_1^B \otimes B)))$ (by (b1) of Definition 1.2)
$= \mu_B \circ (\varphi_B \circ \varphi_B^\dagger) \otimes (H \otimes c_{H, H \otimes B} \circ (\delta_H \otimes (c_{B, B} \circ (u_1^B \otimes B)))$ (by naturality of $c$ and the coassociativity and cocommutativity of $\delta_H$)
\[ \mu_B \circ ((\mu_B \circ (u_n^{\varphi_B} \otimes B)) \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes (c_{B,B} \circ (u_{n-1}^{\varphi_B} \otimes B))) \] (by the \(\varphi_B\)-invertibility of \(\varphi_B\))
\[ = \mu_B \circ (B \otimes (\mu_B \circ (u_n^{\varphi_B} \otimes \varphi_B)) \circ (\delta_H \otimes B)) \circ (c_{H,B} \otimes B) \circ (H \otimes (c_{B,B} \circ (u_{n-1}^{\varphi_B} \otimes B))) \] (by the naturality of \(c\), the associativity of \(\varphi_B\) and the factorization of \(u_n^{\varphi_B}\) through the center of \(B\))
\[ = \mu_B \circ (B \otimes \varphi_B) \circ (c_{H,B} \otimes B) \circ (H \otimes (c_{B,B} \circ (u_{n-1}^{\varphi_B} \otimes B))) \] (by (147))
\[ = \mu_B \circ c_{B,B} \circ (u_n^{\varphi_B} \otimes B) \] (by the naturality of \(c\))
\[ = \mu_B \circ (u_n^{\varphi_B} \otimes B) \] (by the factorization of \(u_n^{\varphi_B}\) through the center of \(B\))
and, therefore \(\varphi_B^\otimes_n\) is \(\varphi_B^\otimes_n\)-invertible.

\[ \square \]

**Proposition 4.11.** Let \(H\) be a cocommutative weak Hopf algebra. Let \((B, \varphi_B)\) be a left weak \(H\)-module algebra and suppose that \(\varphi_B\) is \(\varphi_B\)-invertible. Then, \(\omega \in \text{Reg}_{\varphi_B}(H^{\otimes n}, B)\) satisfies
\[ (147) \quad \varpi \wedge \varphi_B^\otimes_n = \varpi^{op} \wedge \varphi_B^\otimes_n \]
if and only if it factors through the center of \(B\).

**Proof.** Assume that (147) holds. Then, by the associativity of \(\mu_B\) and \(\omega \in \text{Reg}_{\varphi_B}(H^{\otimes n}, B)\), we have
\[ \varpi \wedge \varphi_B^\otimes_n \wedge \varphi_B^\otimes_n = \varpi \wedge (\mu_B \circ (u_n^{\varphi_B} \otimes B)) = \mu_B \circ ((\omega \ast u_n^{\varphi_B}) \otimes B) = \varpi. \]

On the other hand,
\[ \varpi^{op} \wedge \varphi_B^\otimes_n \wedge \varphi_B^\otimes_n \]
\[ = \varpi^{op} \wedge (\mu_B \circ (u_n^{\varphi_B} \otimes B)) \] (by the \(\varphi_B^\otimes_n\)-invertibility)
\[ = \mu_B \circ c_{B,B} \circ (\omega \circ (\mu_B \circ c_{B,B} \circ (u_n^{\varphi_B} \otimes B))) \circ (\delta_{H^{\otimes n}} \otimes B) \] (by the factorization of \(u_n^{\varphi_B}\) through the center of \(B\))
\[ = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B} \circ (u_n^{\varphi_B} \otimes B))) \circ (c_{B,B} \otimes B) \circ (B \otimes c_{B,B} \circ (((\omega \otimes u_n^{\varphi_B}) \otimes \delta_{H^{\otimes n}}) \otimes B) \] (by the naturality of \(c\) and the associativity of \(\mu_B\))
\[ = \mu_B \circ c_{B,B} \circ ((u_n^{\varphi_B} \ast \omega) \otimes B) \] (by the naturality of \(c\) and the cocommutativity of \(\delta_H\))
\[ = \varpi^{op} \) (by \(\omega \in \text{Reg}_{\varphi_B}(H^{\otimes n}, B)\)).

Therefore, \(\varpi = \varpi^{op}\) and, as a consequence, \(\omega\) factors through the center of \(B\).

Conversely, if \(\omega\) factors through the center of \(B\), by Remark 4.8, we have that \(\varpi = \varpi^{op}\) and then (147) holds trivially.

\[ \square \]

**Proposition 4.12.** Let \(H\) be a cocommutative weak Hopf algebra. Let \((B, \varphi_B)\) be a left weak \(H\)-module algebra and suppose that \(\varphi_B\) is \(\varphi_B\)-invertible. Then, if \(\omega \in \text{Reg}_{\varphi_B}(H^{\otimes n}, B)\) satisfies (147), \(\omega^{-1}\) also satisfies (147). Then, as a consequence, \(\omega^{-1}\) factors through the center of \(B\).

**Proof.** By the equalities of Proposition 4.12 Remark 4.8 and Proposition 4.11 we have the following:
\[ \varpi \wedge \omega^{-1} = \omega^{-1} \wedge u_n^{\varphi_B} = \omega^{-1} \ast \omega^{op} = \omega^{-1} \wedge \varphi_B^\otimes_n \]
Then, as a consequence, we have that
\[ \omega^{-1} \wedge \varphi_B^\otimes_n = \omega^{-1} \wedge u_n^{\varphi_B} \wedge \varphi_B^\otimes_n = \omega^{-1} \wedge u_n^{\varphi_B} \wedge \varphi_B^\otimes_n = \omega^{-1} \wedge u_n^{\varphi_B} \wedge \varphi_B^\otimes_n = \omega^{-1} \wedge \varphi_B^\otimes_n \]
Proposition 4.13. Let $H$ be a cocommutative weak Hopf algebra. Let $(B, \varphi_B)$ be a left weak $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$ satisfying the twisted condition $(\text{I})$. Then, $\varphi_B$ induces a left $H$-module algebra structure on the center of $B$, where the action $\varphi_{Z(B)} : H \otimes Z(B) \rightarrow Z(B)$ is the factorisation of $\varphi_B \circ (H \otimes z_B) : H \otimes Z(B) \rightarrow B$ through the center of $B$.

Proof. First note that, by $(\text{I})$, (b1) of Definition $(\text{I})$ and the naturality of $c$, we obtain that the identity

$$\mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_B \otimes z_B \otimes B) = \mu_B \circ c_{B,B} \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_B \otimes z_B \otimes B)$$

holds. Then, on the one hand,

$$\mu_B \circ (\varphi_B \otimes (\mu_B \circ (u_1^B \otimes B))) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_B \otimes z_B \otimes B) = \mu_B \circ (\mu_B \otimes B) \circ (\varphi_B \otimes c_{B,B}) \circ (H \otimes c_{B,B} \otimes B) \circ (((H \otimes u_1^B) \circ \delta_H) \otimes z_B \otimes B) \quad (\text{by (I)})$$

and the associativity of $\mu_B$,

$$= \mu_B \circ (B \otimes (\mu_B \circ c_{B,B} \circ (u_1^B \otimes B))) \circ ((c_{B,H} \circ (u_1^B \otimes H) \circ \delta_H) \otimes z_B \otimes B) \quad (\text{by the coassociativity of } \delta_H \text{ and the naturality of } c)$$

and, on the other hand,

$$\mu_B \circ (\varphi_B \otimes (\mu_B \circ (u_1^B \otimes B))) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes B) = \mu_B \circ (\varphi_B \otimes (\varphi_B \otimes \varphi_{B}^\dagger)) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes B) \quad (\text{by the } \varphi_B-\text{invertivility of } \varphi_B)$$

and the naturality of $c$,

$$= \mu_B \circ (\varphi_B \otimes (\varphi_B \otimes \varphi_{B}^\dagger)) \circ (H \otimes c_{H,B} \otimes \varphi_{B}^\dagger) \circ (\delta_H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes B) \quad (\text{by the coassociativity of } \delta_H \text{ and the naturality of } c)$$

and, on the other hand,

$$\mu_B \circ (\mu_B \otimes (\mu_B \circ (u_1^B \otimes B))) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes B) = \mu_B \circ (\varphi_B \otimes (\varphi_B \otimes \varphi_{B}^\dagger)) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes B) \quad (\text{by the } \varphi_B-\text{invertivility of } \varphi_B \text{ and the naturality of } c)$$

and the associativity of $\mu_B$.
Therefore, as a consequence of the previous equalities, we have that
\[
\mu_B \circ ((\varphi_B \circ (H \otimes z_B)) \otimes B) = \mu_B \circ (c_{B,B} \circ ((\varphi_B \circ (H \otimes z_B)) \otimes B)
\]
and this implies that there exists a unique morphism \( \varphi_{Z(B)} : H \otimes Z(B) \to Z(B) \) such that
\[
(149) \quad z_B \circ \varphi_{Z(B)} = \varphi_B \circ (H \otimes z_B).
\]

The pair \((Z(B), \varphi_{Z(B)})\) is a left \(H\)-module algebra. Indeed, using hat \(z_B\) is a monomorphism we have that \(\varphi_{Z(B)} \circ (\eta_H \otimes Z(B)) = id_{Z(B)}\) because by \(\text{(149)}\) \(z_B \circ \varphi_{Z(B)} \circ (\eta_H \otimes Z(B)) = z_B\) holds. Also, we have the identity
\[
(150) \quad \mu_B \circ ((\varphi_B \circ (H \otimes \varphi_B)) \otimes B) \circ (H \otimes (c_{B,B} \circ (\sigma \otimes B))) \circ (\delta_{H \otimes z_B}) = \mu_B \circ (B \otimes \varphi_B) \circ (G_\sigma \otimes z_B)
\]
since
\[
\mu_B \circ ((\varphi_B \circ (H \otimes \varphi_B)) \otimes B) \circ (H \otimes (c_{B,B} \circ (\sigma \otimes B))) \circ (\delta_{H \otimes z_B}) = \mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes P_{\varphi_B}) \circ (H \otimes H \otimes z_B) \quad \text{(by the naturality of c)}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (F_\sigma \otimes z_B) \quad \text{(by (149))}
\]
\[
= \mu_B \circ (B \otimes (z_B \otimes \varphi_{Z(B)})) \circ (F_\sigma \otimes Z(B)) \quad \text{(by (150) and (149))}
\]
\[
= \mu_B \circ (\varphi_B \otimes B) \circ (H \otimes c_{B,B}) \otimes ((\mu_H \otimes (\sigma \otimes B)) \circ (\delta_{H \otimes z_B}) \otimes z_B) \quad \text{(by the naturality of c)}
\]
\[
= \mu_B \circ (\varphi_B \otimes B) \circ (H \otimes c_{B,B}) \otimes (G_\sigma \otimes z_B) \quad \text{(by the cocommutativity of \(\delta_{H \otimes z_B}\)).}
\]

Then,
\[
\mu_B \circ ((\varphi_B \circ (H \otimes \varphi_B)) \otimes B) \circ (H \otimes (c_{B,B} \circ (\sigma \otimes B))) \circ (\delta_{H \otimes z_B}) = \mu_B \circ (B \otimes \varphi_B) \circ (G_\sigma \otimes z_B)
\]
\[
\circ (H \otimes (c_{B,B} \circ (\sigma^{-1} \otimes B))) \circ (\delta_{H \otimes z_B}) \quad \text{(by the coassociativity of \(\delta_{H \otimes z_B}\))}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (H \otimes (c_{B,B} \circ (\sigma^{-1} \otimes B))) \circ (\delta_{H \otimes z_B}) \quad \text{(by \(\sigma \in \text{Reg}_{\varphi_B}(H \otimes B)\))}
\]
\[
= \mu_B \circ (\varphi_B \otimes B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes ((\mu_B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_H \otimes \eta_B))) \quad \text{(by the naturality of c)}
\]
\[
= \varphi_B \circ (H \otimes (\varphi_B \circ (H \otimes (\sigma \otimes B)))) \quad \text{(by (b1) of Definition 1.3)}
\]
\[
= \varphi_B \circ (H \otimes \varphi_B) \circ (H \otimes z_B) \quad \text{(by the properties of \(\eta_B\))}
\]
\[
= z_B \circ \varphi_{Z(B)} \circ (H \otimes \varphi_{Z(B)}) \quad \text{(by (149))}
\]

and, on the other hand,
\[
\mu_B \circ ((\varphi_B \circ (B \otimes \varphi_B) \circ (G_\sigma \otimes B)) \otimes B) \circ (H \otimes (c_{B,B} \circ (\sigma^{-1} \otimes B))) \circ (\delta_{H \otimes z_B}) = \mu_B \circ (B \otimes \varphi_B) \circ (H \otimes (c_{B,B} \circ (\sigma^{-1} \otimes B))) \circ (\delta_{H \otimes z_B})
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (H \otimes (c_{B,B} \circ (\sigma^{-1} \otimes B))) \circ (\delta_{H \otimes z_B}) \quad \text{(by the coassociativity of \(\delta_{H \otimes z_B}\))}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (H \otimes (c_{B,B} \otimes (w_2^B \otimes B))) \circ (\delta_{H \otimes z_B}) \quad \text{(by \(\sigma \in \text{Reg}_{\varphi_B}(H \otimes B)\))}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes (\mu_H \otimes \varphi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes z_B \otimes \eta_B)) \quad \text{(by (a1) of Definition 1.3)}
\]
\[
= \varphi_B \circ (H \otimes \varphi_B) \circ (H \otimes \varphi_B) \circ (H \otimes z_B) \quad \text{(by (b1) of Definition 1.3)}
\]
\[
= \varphi_B \circ (H \otimes \varphi_B) \circ (H \otimes \varphi_B) \circ (H \otimes \varphi_B) \circ (H \otimes z_B) \quad \text{(by the properties of \(\eta_B\))}
\]
\[
= z_B \circ \varphi_{Z(B)} \circ (H \otimes \varphi_{Z(B)}) \quad \text{(by (149))}.
\]
Therefore, as a consequence of (150), we have that \( \varphi_{Z(B)} \circ (H \otimes \varphi_{Z(B)}) = \varphi_{Z(B)} \circ (\mu_H \otimes Z(B)) \) and this implies that \((Z(B), \varphi_{Z(B)})\) is a left \(H\)-module. Finally, it is a left \(H\)-module algebra because composing with the monomorphism \(z_B\) we have

\[
\begin{align*}
\varphi_{Z(B)} \circ (\mu_H \otimes Z(B)) & = \mu_B \circ (\varphi_B \otimes \delta_B) \circ (\mu_H \otimes Z(B)) \\
& = \varphi_B \circ (H \otimes (\mu_B \otimes (z_B \otimes \eta_B))) \\
& = \varphi_B \circ (H \otimes (z_B \otimes \mu_{Z(B)})) \\
& = z_B \circ \varphi_{Z(B)} \circ (H \otimes \mu_{Z(B)})
\end{align*}
\]

and

\[
\begin{align*}
z_B \circ \varphi_{Z(B)} \circ (H \otimes u_1^{\varphi_Z(B)}) & = \varphi_B \circ (H \otimes u_1^{\varphi_Z(B)}) \\
& = u_1^{\varphi_Z(B)} \circ \mu_H \\
& = z_B \circ u_1^{\varphi_Z(B)} \circ \mu_H
\end{align*}
\]

Remark 4.14. Note that, under the conditions of the previous proposition, the equality

(151) \[ z_B \circ u_1^{\varphi_Z(B)} = u_1^{\varphi_Z(B)} \]

holds.

4.15. Let \(H\) be a cocommutative weak Hopf algebra. By [3, Theorem 3.1] we know that, if \((A, \varphi_A)\) is a left weak \(H\)-module algebra and \(\sigma \in \text{Reg}_{\varphi_A}(H^{\otimes 2}, A)\) satisfies the twisted condition (77), \((A, \varphi_A)\) is a left \(H\)-module algebra if and only if the morphism \(\sigma\) factorizes through the center of \(A\). Moreover, by [3, Corollary 3.1], \((A, \varphi_A)\) is a left \(H\)-module algebra if and only if the morphism \(u_2^{\varphi_A}\) satisfies the twisted condition (77).

Proposition 4.16. Let \(H\) be a cocommutative weak Hopf algebra and let \((B, \varphi_B)\) be a left weak \(H\)-module algebra. Let \(\sigma \in \text{Reg}_{\varphi_B}(H^{\otimes 2}, B)\) satisfying the twisted condition (77). Then, \(\alpha \in \text{Reg}_{\varphi_B}(H^{\otimes 2}, B)\) satisfies the twisted condition (77) if, and only if, there exists \(\tau \in \text{Reg}_{\varphi_{Z(B)}}(H^{\otimes 2}, Z(B))\) such that

\[ \alpha = (z_B \circ \tau) \ast \sigma. \]

Proof. Suppose that \(\alpha\) satisfies the twisted condition (77). We will see that \(\sigma \ast \alpha^{-1}\) factors through the center of \(B\). Following Proposition 4.14 to prove it we will see that \(\alpha = \varphi_B \circ (\mu_H \otimes B)\) satisfying the twisted condition (77) if, and only if, there exists \(\tau \in \text{Reg}_{\varphi_{Z(B)}}(H^{\otimes 2}, Z(B))\) such that

\[ \alpha = (z_B \circ \tau) \ast \sigma. \]

First, note that

\[ \varphi_B \circ (\mu_H \otimes B) = \varphi_B \circ (\mu_H \otimes B) \circ (\mu_H \otimes B) \]

because

\[
\begin{align*}
& \varphi_B \circ (\mu_H \otimes B) \\
& = u_2^{\varphi_B} \otimes (\varphi_B \circ (\mu_H \otimes B)) \\
& = \sigma^{-1} \ast \varphi_B \circ (\mu_H \otimes B) \\
& = \varphi_B \circ (\mu_H \otimes B)
\end{align*}
\]
Thus, for \( \alpha \) we have the same identity and then
\[
\sigma^{-1} \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} = \alpha^{-1} \wedge \alpha^{op} \wedge \varphi_B^{\otimes 2}
\]
holds. As a consequence,
\[
\sigma^{op} \wedge \varphi_B^{\otimes 2} = \sigma^{op} \wedge \alpha^{-1} \wedge \varphi_B^{\otimes 2}
\]
also holds since
\[
\sigma^{op} \wedge \varphi_B^{\otimes 2}
\]
\[
= \sigma^{op} \wedge u_2^Z \wedge \varphi_B^{\otimes 2} \quad \text{by (v) of Proposition 1.24}
\]
\[
= u_2^Z \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by (iv) of Proposition 1.24}
\]
\[
= \sigma \wedge \sigma^{-1} \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by } \sigma \in \text{Reg}_B(H^{\otimes 2}, B) \text{ and (i) of Proposition 1.24}
\]
\[
= \sigma \wedge \alpha^{-1} \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by (153)}
\]
\[
= \sigma^{op} \wedge \sigma \wedge \alpha^{-1} \wedge \varphi_B^{\otimes 2} \quad \text{by (iv) and (i) of Proposition 1.24}
\]
Therefore,
\[
\sigma \wedge \alpha^{-1} \wedge \varphi_B^{\otimes 2}
\]
\[
= \alpha^{-1} \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by (iii) of Proposition 1.24}
\]
\[
= \alpha^{-1} \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by (iv) and (i) of Proposition 1.24}
\]
\[
= \sigma \wedge \alpha^{-1} \wedge \varphi_B^{\otimes 2} \quad \text{by Remark 1.25}
\]
\[
= \sigma \wedge \sigma \wedge \alpha^{-1} \wedge \varphi_B^{\otimes 2} \quad \text{by the factorization of } u_2^Z \text{ through the center of } B
\]
and this implies that \( \sigma \wedge \alpha^{-1} \) factors through the center of \( B \). Then, by Proposition 1.12, the morphism \((\sigma \wedge \alpha^{-1})^{-1} = \alpha \wedge \sigma^{-1} \) also factors through the center of \( B \). If \( \tau \) is the factorization, we have that \( z_B \circ \tau = \alpha \wedge \sigma^{-1} \). Then, (152) holds.

Conversely, if (152) holds for \( \tau \in \text{Reg}_{\varphi_Z(B)}(H^{\otimes 2}, Z(B)) \), we have that
\[
\sigma^{op} \wedge \varphi_B^{\otimes 2}
\]
\[
= (z_B \circ \tau) \wedge \sigma^{op} \wedge \varphi_B^{\otimes 2} \quad \text{by (152)}
\]
\[
= \sigma^{op} \wedge z_B \circ \tau \wedge \varphi_B^{\otimes 2} \quad \text{by (iii) of Proposition 1.24}
\]
\[
= \sigma^{op} \wedge z_B \circ \tau \wedge \varphi_B^{\otimes 2} \quad \text{by the factorization through the center of } B
\]
\[
= \sigma \wedge z_B \circ \tau \wedge \varphi_B^{\otimes 2} \quad \text{by (iv) of Proposition 1.24}
\]
\[
= \sigma \wedge z_B \circ \tau \wedge \varphi_B \circ (\mu_B \otimes B) \quad \text{by (152)}
\]
\[
= (z_B \circ \tau) \wedge \sigma \wedge (\varphi_B \circ (\mu_B \otimes B)) \quad \text{by (i) of Proposition 1.24}
\]

and, therefore, \( \alpha \) satisfies the twisted condition. \( \square \)

5. Cohomological Obstructions in a Weak Setting

In the beginning of this section we review the basic facts about the Sweedler cohomology in a weak setting. This cohomology was introduced in [2] as a generalization of the classical Sweedler cohomology for Hopf algebras [17]. The groups \( \text{Reg}_{\varphi_B}(H_L, B) \) and \( \text{Reg}_{\varphi_B}(H^{\otimes n}, B) \), introduced in the previous section, will be the objects of the corresponding cosimplicial complex.
5.1. Let \( H \) be a cocommutative weak Hopf algebra and let \((B, \varphi_B)\) be a left weak \( H \)-module algebra. Following [2] we define the coface operators as the group morphisms

\[
\partial_{0,i} : \Reg_{\varphi_B}(H_L, B) \to \Reg_{\varphi_B}(H, B), \quad i \in \{0, 1\}
\]

\[
\partial_{0,0}(g) = \varphi_B \circ (H \otimes (g \circ p_L \circ \Pi_H)) \circ \delta_H, \quad \partial_{0,1}(g) = g \circ p_L,
\]

\[
\partial_{k,i} : \Reg_{\varphi_B}(H^{\otimes k}, B) \to \Reg_{\varphi_B}(H^{\otimes (k+1)}, B), \quad k \geq 1, \quad i \in \{0, 1, \cdots, k+1\}
\]

\[
\partial_{k,i}(\sigma) = \begin{cases} 
\varphi_B \circ (H \otimes \sigma), & i = 0 \\
\sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i}), & i \in \{1, \cdots, k\} \\
\sigma \circ (H^{(k-1)} \otimes (\mu_H \circ (H \otimes \Pi_H^B))), & i = k + 1,
\end{cases}
\]

On the other hand, we define the codegeneracy operators by \( s_{1,0} : \Reg_{\varphi_B}(H, B) \to \Reg_{\varphi_B}(H_L, B) \),

\[
s_{1,0}(h) = h \circ i_L,
\]

and \( s_{k+1,i} : \Reg_{\varphi_B}(H^{\otimes (k+1)}, B) \to \Reg_{\varphi_B}(H^{\otimes k}, B), \quad k \geq 1, \quad i \in \{0, 1, \cdots, k\},
\]

\[
s_{k+1,i}(\sigma) = \sigma \circ (H^{\otimes 1} \otimes \eta_H \otimes H^{\otimes (k-i)}).
\]

Taking into account the codegeneracy operators, we define the groups

\[
\Reg_{\varphi_B}^+(H^{\otimes (k+1)}, B) = \bigcap_{i=0}^{k} \Ker(s_{k+1,i}),
\]

\[
\Reg_{\varphi_B}^+(H_L, B) = \{ g \in \Reg_{\varphi_B}(H_L, B) : g \circ \eta_{H_L} = \eta_B \}.
\]

Note that \( \Reg_{\varphi_B}^+(H, B) \) is the group \( \Reg_{\varphi_B}^+(H, B) \) introduced in Definition 3.5 because

\[
\Reg_{\varphi_B}^+(H, B) = \Ker(s_{1,0}) = \{ h \in \Reg_{\varphi_B}(H, B) : h \circ i_L = u_0^\varphi_B \}
\]

\[
= \{ h \in \Reg_{\varphi_B}(H, B) : h \circ \Pi_H^L = u_1^\varphi_B \} = \{ h \in \Reg_{\varphi_B}(H, B) : h \circ \eta_H = \eta_B \} = \Reg_{\varphi_B}^+(H, B).
\]

Also, \( \Reg_{\varphi_B}^+(H^{\otimes 2}, B) \) is the subgroup of \( \Reg_{\varphi_B}(H^{\otimes 2}, B) \) formed by the elements satisfying the normal condition [3] because

\[
\Reg_{\varphi_B}^+(H^{\otimes 2}, B) = \Ker(s_{2,0}) \cap \Ker(s_{2,1})
\]

\[
= \{ \sigma \in \Reg_{\varphi_B}(H^{\otimes 2}, B) : \sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_1^\varphi_B \}
\]

and finally

\[
\Reg_{\varphi_B}^+(H^{\otimes 3}, B) = \Ker(s_{3,0}) \cap \Ker(s_{3,1}) \cap \Ker(s_{3,2})
\]

\[
= \{ \sigma \in \Reg_{\varphi_B}(H^{\otimes 3}, B) : \sigma \circ (\eta_H \otimes H \otimes H) = \sigma \circ (H \otimes \eta_H \otimes H) = \sigma \circ (H \otimes H \otimes \eta_H) = u_2^\varphi_B \}.
\]
5.2. Let $H$ be a cocommutative weak Hopf algebra. If $(A, \varphi_A)$ is a left $H$-module algebra, by [2] the groups $Reg_{\varphi_A}(H_L, A)$ and $Reg_{\varphi_A}(H^{\otimes n}, A)$, $n \geq 1$ are the objects of a cosimplicial complex of groups with the previous coface and codegeneracy operators. In this case,

$$D^k_{\varphi_A} = \partial_{k,0} \ast \partial_{k,1}^{-1} \ast \cdots \ast \partial_{k,k+1}^{-1}$$

denote the coboundary morphisms of the cochain complex

$Reg_{\varphi_A}(H_L, A) \xrightarrow{D^0_{\varphi_A}} Reg_{\varphi_A}(H, A) \xrightarrow{D^1_{\varphi_A}} Reg_{\varphi_A}(H^{\otimes 2}, A) \xrightarrow{D^2_{\varphi_A}} \cdots$

$\cdots \xrightarrow{D^{k-1}_{\varphi_A}} Reg_{\varphi_A}(H^{\otimes k}, A) \xrightarrow{D^{k}_{\varphi_A}} Reg_{\varphi_A}(H^{\otimes (k+1)}, A) \xrightarrow{D^{k+1}_{\varphi_A}} \cdots$

associated to the cosimplicial complex $Reg_{\varphi_A}(H^{\bullet}, A)$.

5.3. Let $H$ be a weak Hopf algebra. If $(B, \varphi_B)$ is a left weak $H$-module algebra and if $\sigma \in Reg_{\varphi_B}(H^{\otimes 2}, B)$, by [12] Proposition 5.5, the morphism $E(\sigma) : H^{\otimes 3} \to B$ defined by $E(\sigma) = \sigma \otimes \varepsilon_H$ satisfies the following identities:

$E(\sigma) \ast u_3^{\varphi_B} = u_3^{\varphi_B} \ast E(\sigma) = \partial_{2,3}(\sigma)$.  

Then, using that $\partial_{2,3}$ is a group morphism, we have

$$u_3^{\varphi_B} = \partial_{2,3}(\sigma)^{-1} \ast u_3^{\varphi_B} \ast E(\sigma) = \partial_{2,3}(\sigma)^{-1} \ast E(\sigma).$$

Therefore,

$E(\sigma) \ast u_3^{\varphi_B} = u_3^{\varphi_B} \ast E(\sigma) = \partial_{2,3}(\sigma)^{-1} \ast E(\sigma)$.

Similarly,

$$\partial_{2,3}(\sigma^{-1}) = E(\sigma^{-1}) \ast u_3^{\varphi_B} = u_3^{\varphi_B} \ast E(\sigma) = E(\sigma^{-1}),$$

where $E(\sigma^{-1}) = \sigma^{-1} \otimes \varepsilon_H$.

Then, if $(B, \varphi_B)$ is a left $H$-module algebra and $H$ is cocommutative, by (155), the second coboundary morphism of the cosimplicial complex $Reg_{\varphi_B}(H^{\bullet}, B)$ admits the following form:

$$D^2_{\varphi_B}(\sigma) = \partial_{2,0}(\sigma) \ast \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,2}(\sigma) \ast E(\sigma^{-1}).$$

5.4. Let $H$ be a cocommutative weak Hopf algebra. If $(A, \varphi_A)$ is a commutative left $H$-module algebra, $(Reg_{\varphi_A}(H^{\bullet}, A), D^\bullet_{\varphi_A})$ gives the Sweedler cohomology of $H$ in $(A, \varphi_A)$. Therefore, the $k$th group will be defined by

$$H^k_{\varphi_A}(H, A) = \frac{\text{Ker}(D^k_{\varphi_A})}{\text{Im}(D^{k-1}_{\varphi_A})}$$

for $k \geq 1$.

The normalized cochain subcomplex of $A$, denoted by $(Reg^+_A(H^{\otimes \bullet}, B), D^{\bullet}_{\varphi_A})$, is defined by the groups $Reg^+_{\varphi_A}(H^{\otimes (k+1)}, A)$, $Reg^+_{\varphi_A}(H_L, A)$ with $D^+_{\varphi_A}$ the restriction of $D^k_{\varphi_A}$ to $Reg^+_{\varphi_A}(H^{\otimes \bullet}, A)$.

We have that $(Reg^+_{\varphi_A}(H^{\otimes \bullet}, A), D^+_{\varphi_A})$, is a subcomplex of $(Reg_{\varphi_A}(H^{\otimes \bullet}, A), D^\bullet_{\varphi_A})$ and the injection map between $(Reg^+_{\varphi_A}(H^{\otimes \bullet}, A), D^+_{\varphi_A})$ and $(Reg_{\varphi_A}(H^{\otimes \bullet}, A), D^\bullet_{\varphi_A})$ induces an isomorphism of cohomology

$I^k_{\varphi_A} : H^k_{\varphi_A}(H, A) \to H^k_{\varphi_A}(H, A)$. 

5.5. Assume that $H$ is a weak Hopf algebra, let $(B, \varphi_B)$ be a left weak $H$-module algebra and let $\sigma, \tau \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$ satisfying the twisted condition (158) and the 2-cocycle condition (159). Then by Theorem 3.7, $(B \otimes H, \mu_{B \otimes \varphi_B}^\tau H)$ and $(B \otimes H, \mu_{B \otimes \varphi_B}^\sigma H)$ are equivalent if, and only if, there exists $h \in \text{Reg}_{\varphi_B}^+(H, B)$ satisfying (155) and (159). Then, if $H$ is cocommutative, by [2, Corollary 4.8, Theorem 4.9], $(B \otimes H, \mu_{B \otimes \varphi_B}^\tau H)$ and $(B \otimes H, \mu_{B \otimes \varphi_B}^\sigma H)$ are equivalent if, and only if, there exists $h \in \text{Reg}_{\varphi_B}^+(H, B)$ such that the equalities (138) and (160)-(161) hold. Note that the equality (138) is always true if $B$ is commutative and $H$ is cocommutative. Then, under these conditions, if $(B, \varphi_B)$ is a left $H$-module algebra, the equivalence between two weak crossed products $(B \otimes H, \mu_{B \otimes \varphi_B}^\sigma H)$ and $(B \otimes H, \mu_{B \otimes \varphi_B}^\tau H)$ is determined by the existence of $h$ in $\text{Reg}_{\varphi_B}^+(H, B)$ satisfying the equality (159). In this case, if $\sigma$ and $\tau \in \text{Reg}_{\varphi_B}^+(H^\otimes 2, B)$, (159) is equivalent to say that

$$\sigma * \tau^{-1} \in \text{Im}(D_{\varphi_B}^1),$$

i.e., $[\sigma] = [\tau]$ in $\mathcal{H}_{\varphi_B}^2(H, B)$.

5.6. Let $H$ be a weak Hopf algebra, let $(B, \varphi_B)$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$. Then, using the coface operators, it is an easy exercise to prove that $\sigma$ satisfy the cocycle condition (78) if and only if

$$\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = \partial_{2,3}(\sigma) * \partial_{2,1}(\sigma)$$

holds. Then, by (155), we have that $\sigma$ satisfy the cocycle condition (78) if and only if $\sigma$ satisfies the equality

$$\partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = E(\sigma) * \partial_{2,1}(\sigma).$$

Definition 5.7. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_B)$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$. We define the pre-obstruction of $\sigma$ as the morphism $w_\sigma : H^\otimes 3 \to B$, where

$$w_\sigma = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) * \partial_{2,1}(\sigma)^{-1} * \partial_{2,3}(\sigma)^{-1}.$$ 

The using that $\partial_{2,1}(\sigma)^{-1} = \partial_{2,1}(\sigma^{-1})$ and $\partial_{2,3}(\sigma)^{-1} = \partial_{2,3}(\sigma^{-1})$, by the previous considerations, we have that $\sigma$ satisfies the cocycle condition (78) if and only if $w_\sigma = u_3^{\varphi_B}$. Also, note that by (157), we have

$$w_\sigma = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) * \partial_{2,1}(\sigma^{-1}) * E(\sigma^{-1}).$$

Note that $w_\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 3, B)$ and it is easy to show that

$$\omega_{\sigma} = S_{\sigma} * R_{\sigma},$$

where

$$S_{\sigma} = \partial_{2,0}(\sigma) * \partial_{2,2}(\sigma) = \mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes F_\sigma)$$

and

$$R_{\sigma} = \partial_{2,1}(\sigma^{-1}) * E(\sigma^{-1}) = \mu_B \circ (\sigma^{-1} \otimes B) \circ (H \otimes c_{B,H}) \circ (G_{\sigma^{-1}} \otimes H),$$
are morphisms in \( \text{Reg}_{\varphi_B}(H^{\otimes 3}, B) \), where
\[
S^{-1}_\sigma = \mu_B \circ (\sigma^{-1} \otimes \varphi_B) \circ (H \otimes c_{H,H} \otimes B) \circ (\delta_H \otimes G_{\sigma^{-1}})
\]
and
\[
R^{-1}_\sigma = \mu_B \circ (B \otimes \sigma) \circ (F_{\sigma} \otimes H).
\]

**Proposition 5.8.** Let \( H \) be a cocommutative weak Hopf algebra, let \((B, \varphi_B)\) be a left weak \( H \)-module algebra and let \( \sigma \in \text{Reg}_{\varphi_B}(H^{\otimes 2}, B) \). Let \( \omega_\sigma \) be the pre-obstruction of \( \sigma \). Then
\[
\partial_{3,4}(\omega_\sigma) = \omega_\sigma \otimes \varepsilon_H.
\]

**Proof.** Note that by (162) we know that \( \omega_\sigma = S_\sigma \ast R_\sigma \). Then, if we prove that
\[
\partial_{3,4}(S_\sigma) = S_\sigma \otimes \varepsilon_H, \quad \partial_{3,4}(R_\sigma) = R_\sigma \otimes \varepsilon_H
\]
hold, we obtain (103) since
\[
\partial_{3,4}(\omega_\sigma) = \partial_{3,4}(S_\sigma \ast R_\sigma) = \partial_{3,4}(S_\sigma) \ast \partial_{3,4}(R_\sigma) = (S_\sigma \otimes \varepsilon_H) \ast (R_\sigma \otimes \varepsilon_H) = (S_\sigma \ast R_\sigma) \otimes \varepsilon_H = \omega_\sigma \otimes \varepsilon_H.
\]

First note that
\[
\partial_{3,4}(S_\sigma) = \mu_B \circ (B \otimes \sigma) \circ (P_{\varphi_B} \otimes H) \circ (H \otimes ((\sigma \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (\mu_H \circ (H \otimes \Pi_H^3)) \otimes (\mu_H \\
\circ (H \otimes \Pi_H^3))) \circ \delta_{H^{\otimes 3}})) \quad \text{(by (i) of [2, Proposition 2.6], (a1) of Definition 1.1 and the naturality of \( c \) and the associativity of \( \mu_H \))}
\]
\[
= \mu_B \circ (B \otimes \partial_{2,3}(\sigma)) \circ ((P_{\varphi_B} \circ (H \otimes \partial_{2,3}(\sigma)) \circ (H \otimes H \otimes H)) \circ (H \otimes \delta_{H^{\otimes 3}}) \quad \text{(by the naturality of \( c \))}
\]
\[
= S_\sigma \otimes \varepsilon_H \quad \text{(by (103) and the naturality of \( c \)).}
\]

Finally, \( \partial_{3,4}(R_\sigma) = R_\sigma \otimes \varepsilon_H \) holds because:
\[
\partial_{3,4}(R_\sigma) = \mu_B \circ (\partial_{2,3}(\sigma^{-1}) \otimes B) \circ (H \otimes H \otimes c_{B,H}) \circ (H \otimes c_{B,H} \otimes H) \circ (G_{\sigma^{-1}} \otimes H \otimes H) \quad \text{(by the naturality of \( c \))}
\]
\[
= R_\sigma \otimes \varepsilon_H \quad \text{(by the naturality of \( c \) and (103) for \( \sigma^{-1} \)).}
\]
\]

**Proposition 5.9.** Let \( H \) be a cocommutative weak Hopf algebra, let \((B, \varphi_B)\) be a left weak \( H \)-module algebra and let \( \sigma \in \text{Reg}_{\varphi_B}(H^{\otimes 2}, B) \) satisfying the twisted condition (174). Then, the pre-obstruction of \( \sigma \) factors through the center of \( B \).

**Proof.** We will use Proposition 4.11 to obtain that \( \omega_\sigma \) factors through the center of \( B \). To prove that
\[
\overline{\omega_\sigma} \wedge \varphi_B^{\otimes 3} = \overline{\omega_\sigma^{op}} \wedge \varphi_B^{\otimes 3}
\]
we first see
\[
\overline{\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)}^{op} \wedge \varphi_B^{\otimes 3} = \overline{\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)} \wedge (\varphi_B \circ (m_{H}^{\otimes 3} \otimes B))
\]
and
\[
\overline{\partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma)}^{op} \wedge \varphi_B^{\otimes 3} = \overline{\partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma)} \wedge (\varphi_B \circ (m_{H}^{\otimes 3} \otimes B)).
\]

Indeed:
\[
\begin{align*}
&\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)^{op} \land \varphi_B^{\otimes 3} \\
&= \partial_{2,2}(\sigma)^{op} \land \partial_{2,0}(\sigma) \land \varphi_B^{\otimes 3} \quad \text{(by (iii) of Proposition 4.2)} \\
&= \partial_{2,2}(\sigma)^{op} \land (\varphi_B \circ (H \otimes (\sigma^{op} \land \varphi_B^{\otimes 2}))) \quad \text{(by the naturality of } c, \text{ (b1) of Definition 4.3\text{ and the cocommutativity of } \delta_H)} \\
&= -\partial_{2,2}(\sigma)^{op} \land (\varphi_B \circ (H \otimes (\sigma \land \varphi_B \circ (\mu_H \otimes B)))) \quad \text{(by (iii) of Proposition 4.2)} \\
&= \partial_{2,0}(\sigma) \land \partial_{2,2}(\sigma)^{op} \land (\varphi_B \circ (H \otimes (\varphi_B \circ (\mu_H \otimes B)))) \quad \text{(by (vi) of Proposition 4.2)} \\
&= \partial_{2,0}(\sigma) \land \partial_{2,2}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by (al) of Definition 4.3\text{, the naturality of } c \text{ and } 160)} \\
&= \partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by the naturality of } c) \\
\end{align*}
\]

and

\[
\begin{align*}
&\partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma)^{op} \land \varphi_B^{\otimes 3} \\
&= \partial_{2,1}(\sigma)^{op} \land \partial_{2,3}(\sigma)^{op} \land \varphi_B^{\otimes 3} \quad \text{(by (iii) of Proposition 4.2)} \\
&= \partial_{2,1}(\sigma)^{op} \land ((\sigma^{op} \land \varphi_B^{\otimes 3}) \circ (H \otimes H \otimes \varphi_B)) \quad \text{(by the naturality of } c, \text{ the counit properties and } 160) \\
&= \partial_{2,1}(\sigma)^{op} \land ((\sigma \land (\varphi_B \circ (\mu_H \otimes B))) \circ (H \otimes H \otimes \varphi_B)) \quad \text{(by (iii) of Proposition 4.2)} \\
&= \partial_{2,1}(\sigma)^{op} \land \partial_{2,3}(\sigma) \land (\varphi_B \circ (\mu_H \otimes \varphi_B)) \quad \text{(by the naturality of } c \text{ and the counit properties)} \\
&= \partial_{2,3}(\sigma) \land \partial_{2,1}(\sigma)^{op} \land (\varphi_B \circ (\mu_H \otimes \varphi_B)) \quad \text{(by (iv) of Proposition 4.2)} \\
&= \partial_{2,3}(\sigma) \land ((\sigma^{op} \land \varphi_B^{\otimes 2}) \circ (\mu_H \otimes H \otimes B)) \quad \text{(by the naturality of } c \text{ and (al) of Definition 4.3)} \\
&= \partial_{2,3}(\sigma) \land ((\sigma \land \varphi_B \circ (\mu_H \otimes B)) \circ (\mu_H \otimes H \otimes B)) \quad \text{(by (ii) of Proposition 4.2)} \\
&= \partial_{2,3}(\sigma) \land \partial_{2,1}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by the naturality of } c \text{ and (al) of Definition 4.3)} \\
&= \partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by (i) of Proposition 4.2)} \\
\end{align*}
\]

Also,

\[
\begin{align*}
&\partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma) \land \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \\
&= \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1}) \ast \partial_{2,1}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by (iv) of Proposition 4.2)} \\
&= \partial_{2,3}(\sigma^{-1}) \ast \partial_{2,1}(\sigma) \land \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land \varphi_B^{\otimes 3} \quad \text{(by (iii) of Proposition 4.2)} \\
&= u_3^{\otimes 3} \land \varphi_B^{\otimes 3} \quad \text{(by the property of group morphism for } \partial_{2,1} \text{ and } \partial_{2,3}) \\
&= \varphi_B^{\otimes 3} \quad \text{(by (v) of Proposition 4.2)} \\
\end{align*}
\]

and, as a consequence, the following identity holds:

\[
\begin{align*}
\partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) = \partial_{2,3}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1}) \land \varphi_B^{\otimes 3}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\varphi_B^{\otimes 3} &= \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land \partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)^{op} \land \varphi_B^{\otimes 3} \quad \text{(by (iii) of Proposition 4.2)} \\
&= \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land \partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma) \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by (i) of Proposition 4.2)} \\
&= \partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma) \land \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1})^{op} \land (\varphi_B \circ (m_H^{\otimes 3} \otimes B)) \quad \text{(by (iv) of Proposition 4.2)} \\
&= \partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma) \land \partial_{2,1}(\sigma^{-1}) \ast \partial_{2,3}(\sigma^{-1}) \land \varphi_B^{\otimes 3} \quad \text{(by (iv) of Proposition 4.2)} \\
&= \varphi_B^{\otimes 3} \quad \text{(by (i) of Proposition 4.2)}
\end{align*}
\]

\[\square\]
Definition 5.10. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_B)$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$ satisfying the twisted condition (174). The obstruction of $\sigma$ is defined as the unique morphism $\theta_{\sigma} : H^\otimes 3 \to Z(B)$ such that $z_B \circ \theta_{\sigma} = \omega_{\sigma}$, where $\omega_{\sigma}$ is the pre-obstruction of $\sigma$.

Note that, by the previous proposition, we can assure that $\theta_{\sigma}$ exists. Also, $\theta_{\sigma} \in \text{Reg}_{\varphi_Z(B)}(H^\otimes 3, Z(B))$.

Theorem 5.11. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_B)$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_B}(H^\otimes 2, B)$ satisfying the twisted condition (174). Then, the pre-obstruction of $\sigma$ is a 3-cocycle, i.e., the following equality holds:

$$\partial_{3,0}(\omega_{\sigma}) \ast \partial_{3,2}(\omega_{\sigma}) \ast \partial_{3,4}(\omega_{\sigma}) = \partial_{3,1}(\omega_{\sigma}) \ast \partial_{3,3}(\omega_{\sigma}).$$

Proof: In order to prove the theorem we will see some equalities. First of all observe that by the definition of the pre-obstruction $\omega_{\sigma}$ we have:

$$\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma) = \omega_{\sigma} \ast \partial_{2,3}(\sigma) \ast \partial_{2,1}(\sigma).$$

Now using that $\partial_{3,2}$ is group morphism we have that

$$\partial_{3,2}(\partial_{2,0}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\omega_{\sigma}) \ast \partial_{3,2}(\partial_{2,3}(\sigma)) \ast \partial_{3,2}(\partial_{2,1}(\sigma)).$$

But observe that, by the associativity of $\mu_H$ and (150), we have

$$\partial_{3,2}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,2}(\sigma)),$$

$$\partial_{3,3}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\partial_{2,2}(\sigma)),$$

and, trivially,

$$\partial_{3,2}(\partial_{2,0}(\sigma)) = \partial_{3,0}(\partial_{2,1}(\sigma)).$$

Then, as a consequence of (169), (170) and (172) we obtain

$$\partial_{3,0}(\partial_{2,1}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,2}(\omega_{\sigma}) \ast \partial_{3,4}(\partial_{2,2}(\sigma)) \ast \partial_{3,2}(\partial_{2,1}(\sigma)).$$

On the other hand, by (150), we have

$$\partial_{3,0}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,0}(\sigma)),$$

and, trivially,

$$\partial_{3,3}(\partial_{2,0}(\sigma)) = \partial_{3,0}(\partial_{2,2}(\sigma))$$

holds.

Also, (176)

$$\partial_{3,3}(\partial_{2,1}(\sigma^{-1})) \ast \partial_{3,3}(\partial_{2,3}(\sigma^{-1})) = \partial_{3,1}(\partial_{2,2}(\sigma^{-1})) \ast \partial_{3,4}(\partial_{2,3}(\sigma^{-1}))$$

holds, because

$$\partial_{3,3}(\partial_{2,1}(\sigma^{-1})) \ast \partial_{3,3}(\partial_{2,3}(\sigma^{-1}))$$

$$= \partial_{3,3}(\partial_{2,1}(\sigma^{-1}) \ast \partial_{3,3}(\sigma^{-1})) \text{ (by the condition of group morphism for } \partial_{3,3})$$

$$= \mu_B \circ (\sigma^{-1} \otimes B) \circ (H \otimes c_{B,H}) \circ (G_{\sigma^{-1}} \otimes \mu_H) \text{ (by } (157), \text{ the counit properties and the naturality of } \epsilon)$$
\[= \partial_{3,1}(\partial_{2,2}(\sigma^{-1})) \ast \partial_{3,4}(\partial_{2,3}(\sigma^{-1})) \text{ (by (11) of Definition 1.1, naturality of } c \text{ and (13)).}
\]

Then, as a consequence of (170), we have the identity
\[
(177) \quad \partial_{3,3}(\partial_{2,2}(\sigma)) \ast \partial_{3,3}(\partial_{2,1}(\sigma)) = \partial_{3,4}(\partial_{2,3}(\sigma)) \ast \partial_{3,1}(\partial_{2,2}(\sigma))
\]
and using that \(\partial_{3,3}\) is a group morphism, (175) and (171) we can assure that
\[
(178) \quad \partial_{3,0}(\partial_{2,2}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) = \partial_{3,3}(\omega_{\sigma}) \ast \partial_{3,4}(\partial_{2,3}(\sigma)) \ast \partial_{3,1}(\partial_{2,2}(\sigma))
\]
holds.

Moreover,
\[
(179) \quad \partial_{3,0}(\partial_{2,0}(\sigma)) \ast \partial_{3,4}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,3}(\sigma)) \ast \partial_{3,1}(\partial_{2,0}(\sigma))
\]
holds because
\[
\partial_{3,0}(\partial_{2,0}(\sigma)) \ast \partial_{3,4}(\partial_{2,3}(\sigma))
= \mu_B \circ ((\varphi_B \circ (H \otimes \varphi_B)) \otimes \sigma) \circ (H \otimes H \otimes c_{H,B} \otimes H) \circ (H \otimes H \otimes H \otimes c_{H,B}) \circ (\delta_H \otimes (B \otimes \varepsilon_H) \otimes F_{\sigma})
\]
(by (13) and the naturality of \(c\))
\[
= \mu_B \circ (B \otimes \sigma) \otimes (\varphi_B \otimes H) \circ (H \otimes (P_{\varphi_B} \circ (H \otimes \sigma))) \text{ (by the naturality of } c \text{ and (61))}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (F_{\sigma} \otimes \sigma) \text{ (by (24))}
\]
\[
= \mu_B \circ (B \otimes \varphi_B) \circ (F_{\sigma} \otimes (\varepsilon_H \otimes B) \circ G_{\sigma}) \text{ (by (56))}
\]
\[
= \partial_{3,1}(\partial_{2,3}(\sigma)) \ast \partial_{3,1}(\partial_{2,0}(\sigma)) \text{ (by (150), the naturality of } c \text{ and (18))}
\]
and by (150) and the associativity of \(\mu_B\) we obtain the equalities
\[
(180) \quad \partial_{3,1}(\partial_{2,3}(\sigma)) = \partial_{3,4}(\partial_{2,1}(\sigma)),
\]
\[
(181) \quad \partial_{3,1}(\partial_{2,1}(\sigma)) = \partial_{3,2}(\partial_{2,1}(\sigma)).
\]

Finally, observe that, as \(\omega_{\sigma}\) factors through the center of \(B\), for all \(i \in \{1, 2, 3, 4\}\) and \(\tau \in \text{Reg}_{\varphi_B}(H \otimes^4 B)\), we have
\[
(182) \quad \tau \ast \partial_{3,i}(\omega_{\sigma}) = \partial_{3,i}(\omega_{\sigma}) \ast \tau.
\]

Therefore, we conclude the proof by cancellation because in one hand
\[
\partial_{3,0}(\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma))
= \partial_{3,0}(\partial_{2,0}(\sigma)) \ast \partial_{3,0}(\partial_{2,2}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) \text{ (by the condition of group morphism for } \partial_{3,0})
= \partial_{3,0}(\omega_{\sigma}) \ast \partial_{3,0}(\partial_{2,3}(\sigma)) \ast \partial_{3,1}(\partial_{2,1}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) \text{ (by the condition of group morphism for } \partial_{3,0} \text{ and (163))}
\]
and on the other hand
\[
\partial_{3,0}(\partial_{2,0}(\sigma) \ast \partial_{2,2}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma))
= \partial_{3,0}(\omega_{\sigma}) \ast \partial_{3,2}(\omega_{\sigma}) \ast \partial_{3,4}(\partial_{2,3}(\sigma)) \ast \partial_{3,4}(\partial_{2,1}(\sigma)) \ast \partial_{3,2}(\partial_{2,2}(\sigma)) \text{ (by the condition of group morphism for } \partial_{3,4} \text{ and (165))},
\]
Then, \( \theta_\sigma \) the obstruction of \( \sigma \). Then, \( \theta_\sigma \in \text{Im}(D^2_{\varphi Z(B)}) \) if, and only if, there exists \( \alpha \in \text{Reg}_{\varphi B}(H^{\otimes 2}, B) \) that satisfies the twisted condition (77) and the cocycle condition (78).

Proof. If \( \theta_\sigma \in \text{Im}(D^2_{\varphi Z(B)}) \), there exists \( \tau \in \text{Reg}_{\varphi Z(B)}(H^{\otimes 2}, Z(B)) \) such that \( D^2_{\varphi Z(B)}(\tau) = \theta_\sigma \). Then, (183)

\[
\omega_\sigma = D^2_{\varphi B}(z_B \circ \omega_{\tau^{-1}}) \star \omega_\sigma \quad \text{(by the properties of } \partial_{i,j} \text{ and } \tau) \\
= z_B \circ D^2_{\varphi Z(B)}(\tau^{-1}) \star D^2_{\varphi B}(\tau^{-1}) \quad \text{(by the properties of } D^2_{\varphi B} \text{)} \\
= D^2_{\varphi B}(z_B \circ \tau^{-1}) \star \partial_{2,0}(\alpha) \star \partial_{2,2}(\alpha) \star \partial_{2,1}(\alpha^{-1}) \star \partial_{2,3}(\alpha^{-1}) \quad \text{(by (180))} \\
= D^2_{\varphi B}(z_B \circ \tau^{-1}) = z_B \circ D^2_{\varphi Z(B)}(\tau^{-1}).
\]

and, as a consequence, \( \alpha \) satisfies the cocycle condition (78).

Conversely, assume that there exists \( \alpha \in \text{Reg}_{\varphi B}(H^{\otimes 2}, B) \) that satisfies the twisted condition (77) and the cocycle condition (78). Then, by Proposition 4.16 there exists \( \tau \in \text{Reg}_{\varphi Z(B)}(H^{\otimes 2}, Z(B)) \) such that (152) holds, i.e., \( \alpha = (z_B \circ \tau^{-1}) \star \sigma \). As a consequence, \( \sigma = (z_B \circ \tau^{-1}) \star \alpha \) and \( \theta_\sigma \in \text{Im}(D^2_{\varphi Z(B)}) \) since

\[
\omega_\sigma = D^2_{\varphi B}(z_B \circ \tau^{-1}) \star \partial_{2,0}(\alpha) \star \partial_{2,2}(\alpha) \star \partial_{2,1}(\alpha^{-1}) \star \partial_{2,3}(\alpha^{-1}) \quad \text{(by (180))} \\
= D^2_{\varphi B}(z_B \circ \tau^{-1}) = z_B \circ D^2_{\varphi Z(B)}(\tau^{-1}).
\]

\( \Box \)

Proposition 5.13. Let \( H \) be a cocommutative weak Hopf algebra and let \((B, \varphi_B)\) be a left weak \( H \)-module algebra. Let \( \sigma \in \text{Reg}^+_{\varphi B}(H^{\otimes 2}, B) \) satisfying the twisted condition (77). Then, \( \alpha \in \text{Reg}^+_{\varphi B}(H^{\otimes 2}, B) \) satisfies the twisted condition (77) if, and only if, there exists \( \tau \in \text{Reg}^+_{\varphi Z(B)}(H^{\otimes 2}, Z(B)) \) satisfying (152).

Proof. First note that, if \( H \) is cocommutative, \((D, \varphi_D)\) is a left weak \( H \)-module algebra and \( \beta \in \text{Reg}_{\varphi D}(H^{\otimes 2}, D) \), using that \( \Pi^\beta_H = \Pi^\alpha_H \) and (82) we obtain that

\[
\beta \circ (\eta_H \otimes H) = \beta \circ (\Pi^\beta_H \otimes H) \circ \delta_H
\]
holds. Also, (83), holds for $\beta$ and therefore $\beta$ satisfies the normal condition (84), i.e., $\beta \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, D)$ if and only if

$$\beta \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H} = \beta \circ (H \otimes \Pi_{H}^{R}) \circ \delta_{H} = u_{1}^{\varphi_{D}}.$$  

Let $\alpha \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ satisfying the twisted condition (77). By Proposition 4.16 there exists $\tau \in \text{Reg}_{\varphi_{Z(B)}}(H^{\otimes 2}, Z(B))$ satisfying (152). Then, $z_{B} \circ \tau = \alpha \ast \sigma^{-1}$ and $\tau$ satisfies the normal condition (84) because, in one hand,

$$z_{B} \circ \tau \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H} = \mu_{B} \circ ((\alpha \circ (\Pi_{H}^{L} \otimes H)) \otimes (\sigma^{-1} \circ (\Pi_{H}^{L} \otimes H))) \circ \delta_{H} \otimes \delta_{H} \text{ (by the naturality of } c \text{ and (i) of Proposition 2.6)}$$

$$= (\alpha \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H}) \ast (\sigma^{-1} \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H}) \text{ (by the cocommutativity of } \delta_{H})$$

$$= u_{2}^{\varphi_{B}} \ast u_{2}^{\varphi_{B}} \text{ (by (185))}$$

$$= z_{B} \circ u_{2}^{\varphi_{Z(B)}} \text{ (by (151))}$$

and, on the other hand, using the same arguments we have $z_{B} \circ \tau \circ (H \otimes \Pi_{H}^{R}) \circ \delta_{H} = z_{B} \circ u_{2}^{\varphi_{Z(B)}}.$

Conversely, if there exists $\tau \in \text{Reg}_{\varphi_{Z(B)}}(H^{\otimes 2}, Z(B))$ satisfying (152), by the previous arguments, we obtain that

$$\alpha \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H} = (z_{B} \circ \tau \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H}) \ast (\sigma^{-1} \circ (\Pi_{H}^{L} \otimes H) \circ \delta_{H}) = (z_{B} \circ u_{2}^{\varphi_{Z(B)}}) \ast u_{2}^{\varphi_{B}} = u_{2}^{\varphi_{B}}$$

and similarly $\alpha \circ (H \otimes \Pi_{H}^{R}) \circ \delta_{H} = u_{2}^{\varphi_{B}}.$ Therefore, $\alpha \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B).$ $\square$

Remark 5.14. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_{B})$ be a left weak $H$-module algebra and let $\sigma, \beta \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ satisfying the twisted condition (77). Let $\theta_{\sigma}, \theta_{\beta}$ the corresponding obstructions of $\sigma$ and $\beta.$ Then, by the previous proposition, it is easy to show that $[\theta_{\sigma}] = [\theta_{\beta}]$ in $\mathcal{H}_{\varphi_{Z(B)}}^{3+}(H, Z(B))$, i.e., $\theta_{\sigma}$ and $\theta_{\beta}$ are cohomologous.

Corollary 5.15. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_{B})$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ satisfying the twisted condition (77). Let $\theta_{\sigma}$ the obstruction of $\sigma$. Then, $\theta_{\sigma} \in \text{Im}(D_{\varphi_{Z(B)}}^{2+})$ if, and only if, there exists $\alpha \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ that satisfies the twisted condition (77), the cocycle condition (78) and the normal condition (84).

Proof. The result is a direct consequence of Theorem 5.12 and Proposition 5.13. $\square$

Corollary 5.16. Let $H$ be a cocommutative weak Hopf algebra, let $(B, \varphi_{B})$ be a left weak $H$-module algebra and let $\sigma \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ satisfying the twisted condition (77). Let $\theta_{\sigma}$ the obstruction of $\sigma$. Then, $[\theta_{\sigma}] = 0$ in $\mathcal{H}_{\varphi_{Z(B)}}^{3+}(H, Z(B))$ if and only if there exists a morphism $\alpha \in \text{Reg}_{\varphi_{B}}^{+}(H^{\otimes 2}, B)$ that satisfies the twisted condition (77), the cocycle condition (78) and the normal condition (84).

Proof. The proof follows by the previous corollary and Corollary 1.20. $\square$

As a consequence of this corollary we can assure that the obstruction vanishes if and only if there exists a weak crossed product with preunit $\nabla_{B \otimes H}^{\varphi_{B}} \circ (\eta_{B} \otimes \eta_{H})$ and normalized with respect to $\nabla_{B \otimes H}^{\varphi_{B}}$. Equivalently, by [12, Theorem 6.17, Corollary 6.18], this is equivalent to say that $B$ admits a $H$-cleft extension (see also [3, Proposition 3.5]).
Funding

The authors were supported by Ministerio de Ciencia e Innovación of Spain. Agencia Estatal de Investigación. Unión Europea - Fondo Europeo de Desarrollo Regional. Grant: Homología, homotopía e invariantes categóricos en grupos y álgebras no asociativas.

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