Weak Gravity Conjecture, Central Charges and $\eta/s$

Shesansu Sekhar Pal

Center for Quantum Spacetime,
Sogang University, 121-742 Seoul, South Korea

Abstract

We correlate the weak gravity conjecture (WGC), the KSS conjecture with chemical potential at extremality and the central charges by going through a particular example in five dimensional AdS spacetime with two unknown coefficients $c_1$, $c_2$, assuming WGC exists in AdS spacetime. The result that follows from this example suggests that WGC makes the KSS conjecture to hold in the extremal limit but only when one of the coefficient vanishes ($c_1 = 0$, $c_2 \neq 0$ or $c_2 = 0$, $c_1 \neq 0$) and when both the coefficients are non zero it can respect and/or violate the KSS conjecture depending on the choice to $c_1$ at extremality, even though $\eta/s$ do not depend on $c_1$ at extremality. Moreover, WGC is not fully compatible with the calculation of central charges even though the bounds on coefficient $c_1$ that follows from demanding WGC stays within the bounds that central charges predict. As usual, the KSS conjecture is violated, of course, in the non-extremal limit.
1 Introduction

The weak gravity conjecture (WGC) in flat spacetime puts a very important restriction on the ratio of mass or energy density of particles or black holes to the charge it carries, in appropriate units it should be less than or equal to unity [1]. On assuming that there exists a similar kind of WGC in the AdS spacetime puts some interesting restrictions on the coefficients that appear in the low energy effective gravitational action in the finite ’t Hooft coupling limit. The application of the holographic correspondence [2] to systems which is described by that kind of action yields interesting connections between WGC and holography. In particular, we shall explore the connection between KSS conjecture [3] at extremality and the WGC.

The central charges of the dual field theory are connected to these coefficients that appear in the action of the effective bulk theory and hence appear in the computation of $\eta/s$ of the plasma.

Hence, it is natural to think all these three: WGC, KSS conjecture and the central charges are all correlated. In this paper we are going to demonstrate this by considering a specific example in the extremal limit. We expect the result, especially the connection between the WGC and the KSS conjecture is generic in the extremal limit.

1.1 Conjectures: WGC and KSS

The WGC [1], came out of the fact that gravity is the weakest force and says that a stable charged particle minimizes the ratio of energy density to charge density and is less than unity in some units\(^1\). Importantly, as emphasized in [1] the conjecture is not for a given a charge sector for which the masses are less then the charge but there could exists some states of this type.

It has also been suggested in [1] that WGC came out of the requirements of having finite number of stable particles which are not protected by any symmetry principle and fits in nicely with the absence of having any global symmetries in a consistent quantum gravitational theory.

The outcome of the WGC is that for extremal black hole with the appropriate ratio of energy density to charge density becomes unity and the ratio can go below unity for small charge corrections.

In [1], the authors suggested two different forms of the conjecture, one for the state with lightest charged particles and the other for the state with smallest mass to charge ratio. The combination of these two forms suggests the existence of lightest charged particle with mass to charge ratio should be less than that of the mass to charge ratio of the extremal black hole. In this paper we shall adopt this as our guiding principle and find out the consequence

\(^1\)Here we are using the terminology, energy density for mass and charge density for charge.
of this.

Probably, it is correct to say that WGC holds for arbitrary rank of the gauge group and \('t\) Hooft coupling, even though we will be restricting ourselves only to abelian gauge groups. Also, it does not depend on the nature of the asymptotic spacetime or the dimension of spacetime. We shall assume this and proceed further and apply it to situations where we are applying the holographic principle.

It is important to note that the holographic ways to calculate central charges, thermodynamics quantities or the transport coefficients always receive corrections from both the finiteness of the rank of the gauge group and \('t\) Hooft coupling.

The KSS conjecture [3] suggests that the ratio of coefficient of shear viscosity $\eta$ to entropy density $s$ must have a minimum value of $1/4\pi$ at zero chemical potential for theories that admits gravity dual. Of course, we have already witnessed several examples of the violation of it [10]-[20], at finite \('t\) Hooft coupling. But have not seen an example where it can be violated in the extremal limit at finite \('t\) Hooft coupling.

It is easy to convince oneself that $\eta/s$ attains the lower bound in the extremal limit for Einstein-Maxwell type of theories described by actions having only two derivatives [8]. This is due to the fact that any charged $AdS_d$ spacetime in the extremal limit becomes that of $AdS_2 \times R^{d-2}$ spacetime and we already know the result to $\eta/s$ that comes from pure AdS spacetime. But whenever, there is an interaction between the U(1) gauge fields and metric, which occurs for theories with more than two derivatives, then there is no reason to believe that the result to $\eta/s$ should obey the KSS conjecture even at extremality. However, due to WGC, one can show that $\eta/s$ obeys the KSS conjecture in the extremal limit in some cases and violates in some other cases.

The philosophy and the plan that we shall adopt in this paper consists of three steps.

Step 1: We shall look for the ratio of energy density to charge density for the given gravitational system in five spacetime dimension with unknown coefficients $c_i$'s and then find the restriction on these coefficients that follows from demanding this ratio to be smaller than unity in appropriate units.$^2$

Step 2: We shall use the result of the calculation of the central charges [5] to fix some of the coefficients $c_i$'s and use step one to put restriction on rest of the coefficients $c_j$'s.

Step 3: We shall calculate the ratio $\eta/s$ in the extremal limit and examine what happens to the KSS conjecture if we take the restriction that follows from WGC in the first step as well as those that follows from central charges in step two.

The result of this study can be summarized as follows. We consider a low energy effective gravitational theory with two unknown coefficients $c_1$ and $c_2$ and we fix one of the coefficient $c_1$ with the central charges of the corresponding dual field theory following the holographic anomaly calculation [6]. Then we use the result of [5] to get the bounds on this coefficient, and the result suggests it can take both positive and negative values. Now, demanding

$^2$This step was inspired by an analogous calculation done in asymptotically flat spacetime [22].
Let us consider an effective action of the following type

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ R + 12 - \frac{1}{4} F^{MN} F_{MN} + c_1 R_{MNKL} R^{MNKL} + c_2 R_{MNKL} F_{MN} F_{KL} \right],
\]

(1)

this action is a special case to the action considered in [4], but for our purpose this is good enough.

It admits the following form of the black hole solution

\[
ds^2 = -r^2 a(r)^2 dt^2 + r^2 (dx^2 + dy^2 + dz^2) + \frac{dr^2}{r^2 b(r)^2},
\]

\[A = h(r) dt,\]

(2)

with

\[
a(r)^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{q^2}{\ell^2 r^2} + \left( \frac{r_0}{r} \right)^2 \right] + c_2 \left[ - \frac{4q^4}{r^{12}} - \frac{22q^2}{r^6} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} \right] +
\]

\[\frac{8q^2 r_0^4}{r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) + c_1 \left[ \frac{17q^4}{6r^{12}} - \frac{158q^2}{3r^6} + \frac{23q^4}{2r^4} - \frac{2q^2}{r^2} + \frac{20q^2 r_0^4}{3r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) \right] +
\]

\[\frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2,
\]

\[
b(r)^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{q^2}{\ell^2 r^2} + \left( \frac{r_0}{r} \right)^2 \right] + c_2 \left[ - \frac{4q^4}{r^{12}} - \frac{22q^2}{r^6} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} \right] +
\]

\[\frac{20q^2 r_0^4}{3r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) + \frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2 + \left( \frac{2}{3} - \frac{10q^2}{3r^6} \right) \left( 1 - \frac{r_0^2}{r^2} \right) \left( 1 - \frac{q^2}{r^2} + \frac{r_0^2}{r^2} \right) +
\]

\[\frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2.
\]

The paper is organized by assuming a specific form of the low energy effective action. Then we study the thermodynamics and the transport properties of the black hole solution using the known recipes. Then we go through the three steps as mentioned above to find out the correlation among WGC, KSS conjecture and the central charges. In appendix, we calculate $\eta/s$ for another example and show that it is independent of the coefficient that appear in the higher derivative term to metric but depends on the coefficient that appear in the interaction term between the gauge field and the metric degrees of freedom.

## 2 A gravitational system

Let us consider an effective action of the following type

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ R + 12 - \frac{1}{4} F^{MN} F_{MN} + c_1 R_{MNKL} R^{MNKL} + c_2 R_{MNKL} F_{MN} F_{KL} \right],
\]

(1)

this action is a special case to the action considered in [4], but for our purpose this is good enough.

It admits the following form of the black hole solution

\[
ds^2 = -r^2 a(r)^2 dt^2 + r^2 (dx^2 + dy^2 + dz^2) + \frac{dr^2}{r^2 b(r)^2},
\]

\[A = h(r) dt,\]

(2)

with

\[
a(r)^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{q^2}{\ell^2 r^2} + \left( \frac{r_0}{r} \right)^2 \right] + c_2 \left[ - \frac{4q^4}{r^{12}} - \frac{22q^2}{r^6} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} \right] +
\]

\[\frac{8q^2 r_0^4}{r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) + c_1 \left[ \frac{17q^4}{6r^{12}} - \frac{158q^2}{3r^6} + \frac{23q^4}{2r^4} - \frac{2q^2}{r^2} + \frac{20q^2 r_0^4}{3r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) \right] +
\]

\[\frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2,
\]

\[
b(r)^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{q^2}{\ell^2 r^2} + \left( \frac{r_0}{r} \right)^2 \right] + c_2 \left[ - \frac{4q^4}{r^{12}} - \frac{22q^2}{r^6} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} + \frac{4q^4}{r^4} + \frac{2q^2}{r^2} \right] +
\]

\[\frac{20q^2 r_0^4}{3r^{10}} \left( 1 + \frac{q^2}{r_0^2} \right) + \frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2 + \left( \frac{2}{3} - \frac{10q^2}{3r^6} \right) \left( 1 - \frac{r_0^2}{r^2} \right) \left( 1 - \frac{q^2}{r^2} + \frac{r_0^2}{r^2} \right) +
\]

\[\frac{2r_0^8}{r^8} \left( 1 + \frac{q^2}{r_0^2} \right)^2.
\]
correction and when evaluated, it gives
the spatial coordinates, an inner horizon, as well. This solution is characterized by electric charge density, 2
horizon so as to have the vanishing norm at the horizon. The boundary speed is also set to unity as well as the charge density, which is set to 2√3q.

2.1 Thermodynamics
This is already calculated in [4] using background subtraction method, but for completeness, we shall record some of the formulae. The free energy density, 
free energy density or mass density ρ, is the inverse temperature associated to the charged black hole, \( V_3 \) is the volume of the spatial coordinates, \( S_{BH} \), is the action of the charged black hole and \( S_{AdS} \), is the action of the pure AdS spacetime with a constant gauge potential evaluated with higher derivative correction and when evaluated, it gives

\[
\begin{align*}
\rho &= \frac{q}{\pi r_0^3} \left[ 1 - \frac{\bar{c}_1}{3} + \frac{q^2}{r_0^6} \left( 1 + \frac{113}{3} c_1 - 32 c_2 \right) + \frac{q^4}{r_0^8} \left( \frac{23}{2} c_1 + 4 c_2 \right) \right] \\
\end{align*}
\]

The temperature, \( T \) and the chemical potential, \( \mu = lim_{r \to \infty} \frac{4\pi}{\pi} \) are

\[
\begin{align*}
T &= \frac{r_0}{\pi} \left[ 1 - \frac{5}{3} c_1 - \frac{q^2}{2r_0^6} \left( 1 + \frac{31}{3} c_1 + 16 c_2 \right) - \frac{q^4}{r_0^8} \left( 9c_1 - 4c_2 \right) \right], \\
\mu &= \frac{q\sqrt{3}}{\pi r_0^2} \left[ 1 - \frac{c_1}{3} + \frac{q^2}{r_0^6} \left( \frac{13}{3} c_1 - 8c_2 \right) \right] \\
\end{align*}
\]

Introducing a dimensionless parameter \( \bar{\mu} = \frac{\mu}{T} \), the charge density \( n_q = -\left( \frac{\partial w}{\partial \mu} \right)_T \), which is proportional to \( q \) and the horizon radius \( r_0 \) are related to \( T \) and \( \bar{\mu} \) as

\[
\begin{align*}
q &= \frac{\pi^3 T^3 \bar{\mu}}{\sqrt{3}} \left[ 1 + \frac{\bar{\mu}^2}{3} - \frac{\bar{\mu}^4}{36} + 8c_2 \bar{\mu}^2 + c_1 \left( \frac{11}{3} + \frac{26}{9} \bar{\mu}^2 + \frac{5\bar{\mu}^4}{12} \right) + O \left( \bar{\mu}^5, c_1^2, c_2^2 \right) \right], \\
r_0 &= \pi T \left[ 1 + \frac{\bar{\mu}^2}{6} - \frac{\bar{\mu}^4}{36} + \frac{\bar{\mu}^6}{108} + \frac{c_1}{3} \left( 5 + \frac{11}{2} \bar{\mu}^2 - \frac{5\bar{\mu}^4}{36} - \frac{23}{108} \bar{\mu}^6 \right) + \frac{c_2 \bar{\mu}^2}{3} \left( 8 - \frac{4}{3} \bar{\mu}^2 + \frac{4}{27} \bar{\mu}^4 \right) + O \left( \bar{\mu}^7, c_1^2, c_2^2 \right) \right] \\
\end{align*}
\]
After re-expressing the free energy density in terms of chemical potential and temperature

\[ w = -\frac{\pi^4 T^4}{2\kappa^2} \left[1 + 13c_1 + \left(1 + \frac{11}{3}\right)\bar{\mu}^2 + \frac{1}{6} \left(1 + \frac{26}{3} c_1 + 24c_2\right)\bar{\mu}^4 - \frac{1}{108} \left(1 - 15c_1\right)\bar{\mu}^6 + \mathcal{O}\left(\bar{\mu}^7, c_1^2, c_2^2\right)\right], \tag{7} \]

From which it just follows trivially using the identity for entropy density

\[ s = -\left(\frac{\partial w}{\partial T}\right)_\mu = \frac{4\pi^4 T^4}{2\kappa^2} \left[1 + \frac{\bar{\mu}^2}{2} + \frac{\bar{\mu}^6}{216} + c_1 \left(13 + \frac{11}{6} \bar{\mu}^2 - \frac{15}{216} \bar{\mu}^6\right) + \cdots \right] \tag{8} \]

The energy density

\[ \rho_E = w + T(s + \bar{\mu}n_q) = -3w, \]

\[ = \frac{3r_0^4}{2\kappa^2} \left[1 + \frac{19}{3} c_1 + \frac{q^2}{r_0^6} \left(1 - \frac{113}{3} c_1 - 32c_2\right) + \frac{q^4}{r_0^{12}} \left(\frac{23}{2} c_1 + 4c_2\right)\right] \tag{9} \]

The extremal limit corresponds to

\[ q^2 = 2r_0^6[1 - 48c_1], \tag{10} \]

which is independent of \(c_2\). On evaluating the energy density in the extremal limit gives

\[ \frac{2\pi(2\kappa^2)}{9} \frac{\rho_E}{q^4} = 1 - \frac{1}{3} \left(23c_1 + 48c_2\right) \tag{11} \]

### 2.2 shear viscosity and \(\eta/s\)

The coefficient of shear viscosity can be very easily computed using the prescription given in [9] and the result is same as in [4]

\[ \eta = \frac{r_0^3}{2\kappa^2} \left(1 - 24c_1\frac{q^2}{r_0^6}\right) \tag{12} \]

and the entropy density in terms of \(q/r_0^3\), can be calculated using the Wald’s entropy formula [23]

\[ s = \frac{4\pi r_0^3}{2\kappa^2} \left[1 + 8c_1 - \frac{q^2}{r_0^6} \left(28c_1 + 24c_2\right)\right] \tag{13} \]

The ratio, \(\eta/s\), simply reads to leading order in \(c_i\)’s as

\[ \frac{\eta}{s} = \frac{1}{4\pi} \left[1 - 8c_1 + \frac{q^2}{r_0^6} \left(4c_1 + 24c_2\right)\right], \tag{14} \]
which in the extremal limit gives
\[
\left( \frac{\eta}{s} \right)_{\text{extremal}} = \frac{1}{4\pi} \left[ 1 + 48c_2 \right],
\] (15)
which is independent of \( c_1 \), [4]. It looks like in the extremal limit, it is the coefficient of the interaction term between the gauge field and the metric, which is \( c_2 \), that appears in the computation of \( \eta/s \). This particular feature of \( \eta/s \) also appear in another example that is studied in the appendix.

3 WGC, KSS conjecture and central charges

Let us apply the WGC to eq(11), i.e. considering the left hand side as the ratio of mass density to charge density means the right hand side must obey
\[
1 - \frac{1}{3} \left( 23c_1 + 48c_2 \right) \leq 1,
\] (16)
which gives the constraint
\[
0 \leq \left( 23c_1 + 48c_2 \right) \leq 3.
\] (17)
We shall analyze this equation by considering different cases. First, let us consider a simpler case, where \( c_1 = 0 \), in this case we can rewrite the constraint as
\[
0 \leq 16c_2 \leq 1,
\] (18)
which simply means the coefficient \( c_2 \) is positive and the ratio, eq(15) obeys
\[
\frac{\eta}{s} \geq \frac{1}{4\pi},
\] (19)
which is nothing but respecting the KSS conjecture as the coefficient \( c_2 \) is positive. So, we just saw the imposition of WGC means respecting the KSS conjecture in the extremal limit but only when \( c_1 = 0 \).

For, \( c_1 \neq 0 \), a priori it is not clear why \( c_2 \) should be positive and hence respect the KSS bound?

There is one more ingredient that we have not yet taken into consideration and that is the central charges. We know from AdS/CFT correspondence that the central charges are related to the anomaly in the one point function of the trace of the energy momentum tensor [24].

For the gravity action described by
\[
S = \frac{1}{16\pi G} \int d^5x \left[ R + 12 + \alpha R^2 + \beta R^{MN} R_{MN} + \gamma R_{MNKL} R^{MNKL} \right],
\] (20)
have the central charges [6]

\[
\begin{align*}
\frac{c}{16\pi^2} &= \frac{1}{8\pi G} \left[ \frac{1}{16} + \left( -\frac{5\alpha}{2} - \frac{\beta}{2} + \frac{\gamma}{4} \right) \right], \\
\frac{a}{16\pi^2} &= \frac{1}{8\pi G} \left[ \frac{1}{16} + \left( -\frac{5\alpha}{2} - \frac{\beta}{2} - \frac{\gamma}{4} \right) \right],
\end{align*}
\]  

(21)

where the size of the AdS radius has been set to unity. With the choice for which the coefficients are set to \( \alpha = 0 = \beta \) and \( \gamma = c_1 \), relates to central charges as

\[
c_1 = \frac{1}{8} \left( 1 - \frac{a}{c} \right)
\]  

(22)

In an interesting study [5] have suggested bounds on the ratio of the central charges depending on the amount of supersymmetry preserved

\[
\begin{align*}
\text{For} \quad \mathcal{N} = 0, \quad &\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}, \\
\text{For} \quad \mathcal{N} = 1, \quad &\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}, \\
\text{For} \quad \mathcal{N} = 2, \quad &\frac{1}{2} \leq \frac{a}{c} \leq \frac{5}{4},
\end{align*}
\]  

(23)

Re-writing it in terms of \( c_1 \), we get the bounds as

\[
\begin{align*}
\text{For} \quad \mathcal{N} = 0, \quad &-\frac{13}{144} \leq c_1 \leq \frac{1}{12}, \quad \text{imply} \quad \left( -\frac{299}{432} \leq \frac{23}{3} c_1 \leq \frac{23}{36} \right), \\
\text{For} \quad \mathcal{N} = 1, \quad &-\frac{1}{16} \leq c_1 \leq \frac{1}{16}, \quad \text{imply} \quad \left( -\frac{23}{48} \leq \frac{23}{3} c_1 \leq \frac{23}{48} \right), \\
\text{For} \quad \mathcal{N} = 2, \quad &-\frac{1}{32} \leq c_1 \leq \frac{1}{16}, \quad \text{imply} \quad \left( -\frac{23}{96} \leq \frac{23}{3} c_1 \leq \frac{23}{48} \right),
\end{align*}
\]  

(24)

It is interesting to see that the magnitude of \( c_1 \), which is small and less than unity all the time and is consistent with our supergravity analysis, but can take both positive and negative values.

Let us use these constraint that we obtained from the calculation of central charges on the coefficient \( c_2 \), i.e. eq(24) in eq(17)

\[
-23c_1 \leq 48c_2 \leq 3 - 23c_1
\]  

(25)

and it just follows that \( c_2 \) is not always positive, and is independent of the amount of supersymmetry preserved, which implies from eq(15) that the KSS conjecture is not obeyed all the time even in the extremal limit. It would be very interesting to cross check this
constraint on $c_2$ from direct calculation of anomaly or from some other ways. See appendix for the full solution of eq(24) and eq(25).

It looks like the low energy effective action eq(1), may not be part of the swampland [7], as we have taken the WGC condition into account. However, it would be very interesting to consider an effective action which is part of the swampland and calculate $\eta/s$ in the extremal limit and see what it has got to say about the KSS conjecture in the extremal limit.

Let us look at the case for which $c_2 = 0$, then the resulting constraint that follows from eq(17) is not fully compatible with what follows from the calculation of the central charges i.e. from eq(24). However, it says that the restriction from WGC stays within the bounds resulting from the calculation of central charges. It is important to note that in the extremal limit $\eta/s$ do not depend on the coefficient $c_1$, so the KSS conjecture at extremality is safe. Moreover, in this limit, ($c_2 = 0$), WGC has got nothing to say on $\eta/s$.

4 Conclusion and discussion

In this paper we have studied the KSS conjecture with chemical potential in the extremal limit with higher derivative correction and showed by going through an example that it is the WGC [1] along with the restriction of central charges in four dimensional field theory [5], that follows from holography, makes the ratio of $\eta/s$ to stay above $1/4\pi$ in the extremal limit but only in certain cases. The bounds on the coefficient $c_1$ that results after imposing WGC do not completely agree with the bounds that results from central charge calculations. In particular, when $c_2 = 0$, WGC imply $c_1$ should always be positive but central charge calculation [5] suggests $c_1$ can take negative values.

We have also shown (also in [4]) that in the extremal limit the ratio $\eta/s$ depends only on the coefficients $c_2$ that appear in the interaction of the U(1) gauge fields and the metric. Moreover, the condition at extremality which is the ratio of charge density to the size of the horizon do not depends on the coefficient $c_2$.

It is certainly very interesting to study more examples in the Einstein-Maxwell sector with higher derivative terms as well as going beyond Einstein-Maxwell type of examples and examine what happens to the KSS conjecture at extremality. It would also be nice to have a bound on $c_2$ without invoking WGC and if it stays positive then the KSS conjecture holds at extremality.

In another context it is not a priori clear which kind of low energy effective action that one needed to consider and what are the criteria(s) to fix the form of such actions? Is it just the symmetry principle that is enough to fix the form? Or we need to take into account all: WGC, holography and symmetry principle as the guiding principle to fix the form, which

$^3$If we assume WGC is exact, then it says that the left hand side of eq(24) need to be corrected and should be set to zero, which would be very interesting to explore. However, if we assume eq(24) is not going to be corrected then it says that mass or energy density can be bigger than the charge density.
would certainly be very interesting to explore.

Note added: After submitting the paper to arXiv, we are informed of the paper [25], which also discusses the relation between WGC and $\eta/s$.

5 Acknowledgment

It is a pleasure to thank Ofer Aharony for going through the manuscript and giving few suggestions, also would like to thank Bum-Hoon Lee for a discussion and the members of CQUeST for their help.

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number R11-2005-021.

6 Appendix

In this appendix, we shall calculate the $\eta/s$ for a low energy effective action in five dimension, whose form is similar in nature to [5]

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ R + 12 \frac{1}{4} F^{MN} F_{MN} + c_1 C_{MNKL} C^{MNKL} + c_2 C^{MNKL} F_{MN} F_{KL} \right], \quad (26)$$

where $C_{MNKL}$ is the Weyl tensor. In order to calculate $\eta/s$ to leading order in $c_i$’s we do not need to know the full solution, as the metric components along the spatial directions do not receive any corrections to leading order in $c_i$, hence only the zeroth order solution in $(c_1)^0$ and $(c_2)^0$ is good enough.

Let us take the solution to have the form

$$ds^2 = -r^2 a^2(r) dt^2 + c^2(r) (dx^2 + dy^2 + dz^2) + \frac{1}{r^2 b^2(r)} dr^2, \quad (27)$$

with the lowest order solution

$$a^2(r) = b^2(r) = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{q^2}{r^2 r_0^2} + \left( \frac{r_0}{r} \right)^2 \right], \quad c^2(r) = r^2,$$

$$h(r) = \sqrt{3} q \left( \frac{1}{r^2} + \frac{1}{r_0^2} \right) \quad (28)$$

This solution has a horizon at $r = r_0$, the gauge potential vanishes at the horizon in order to have a vanishing norm there, the speed on the boundary has been set to unity and the charges is set to the value, proportional to $q$. 

10
The shear viscosity can be evaluated by computing the following quantity, following [9]

$$\eta = \lim_{k_a \to 0} \frac{\Pi(r, k_a)}{i \omega \phi(r, k_a)} \quad (29)$$

at the boundary, where $\Pi$ is the momentum associated to the field $\phi$, which is related to the metric fluctuation. Let us denote the metric fluctuation

$$h^y_x = \int [dk] \phi_k(r) e^{-i \omega t + ikz}, \quad (30)$$

where the graviton is moving along $z$ direction and we are using a short hand notation to write the appropriate measure factor for momentum integrals and factors of $2\pi$ in $[dk]$. The equation of motion can be derived from the following effective action

$$S = \frac{1}{2\kappa^2} \int dr [dk] \left( A(r) \phi''_k \phi_{-k} + B(r) \phi'_k \phi'_{-k} + C(r) \phi'_k \phi_{-k} + D(r) \phi_k \phi_{-k} + E(r) \phi''_k \phi_{-k} + F(r) \phi'_k \phi'_{-k} \right) + K,$$ 

where $K$ is the appropriately generalized Gibbons-Hawking boundary term [21].

Ignoring the details, the coefficient of shear viscosity, $\eta$ becomes

$$\eta = \lim_{\omega \to 0} \frac{\Pi}{i \omega \phi} = \frac{1}{\kappa^2} [\kappa_2(r) + \kappa_4(r)]_{r=\text{horizon}}, \quad (32)$$

where

$$\kappa_2 = \frac{1}{(r^2 a b)} \left[ A - B + \frac{F'}{2} \right], \quad \kappa_4 = \frac{d}{dr} \left[ E \frac{d}{dr} \left( \frac{1}{r^2 a b} \right) \right] \quad (33)$$

and on evaluating it

$$\eta = \frac{c^3}{2\kappa^2} + \frac{c_2 b^2 c^3 h'^2}{2\kappa^2 a^2} + \frac{c_1 c}{2\kappa^2 a^2} \left[ -2a^2 b^2 c^2 - 2ra^2 bc^2 b' + 2r^2 a^2 b^2 c^2 - 2rabc(2a' b' + b(3a' + ra c' + rca'')) + 2r^2 a^2 bc(b' c' + bc'') \right] \quad (34)$$

On evaluating the entropy density using Wald’s entropy formula [23]

$$s = \frac{2\pi c^3}{\kappa^2} - \frac{c_2 2\pi b^2 c^3 h'^2}{\kappa^2 a^2} - \frac{c_1 2\pi c}{\kappa^2 a^2} \left[ 2a^2 b^2 c^2 + 2ra^2 bc^2 b' + 2r^2 a^2 b^2 c^2 + rabc(2ra c' b' + b(-2ra c' + 2c(3a' + ra'')) - 2ra^2 bc(b' c' + b(2c' + ra''))) \right] \quad (35)$$

The temperature to the zeroth order in $(c_1)^0$ and $(c_2)^0$

$$T = \frac{r_0}{\pi} \left[ 1 - \frac{q^2}{2 r_0^2} \right]. \quad (36)$$
This implies extremality is at
\[ q^2 = 2 r_0^6 + O(c_1, c_2). \]  
(37)

Using the zeroth order solution, the coefficient of shear viscosity becomes
\[ \eta = \frac{r_0^3}{2\kappa^2} \left[ 1 + 4c_1 + \frac{q^2}{r_0^6}(-14c_1 + 12c_2) \right] \]  
(38)

and the entropy density becomes
\[ s = \frac{4\pi r_0^3}{2\kappa^2} \left[ 1 + 12c_1 - \frac{q^2}{r_0^6}(18c_1 + 12c_2) \right] \]  
(39)

giving the ratio
\[ \eta/s = \frac{1}{4\pi} \left[ 1 - 8c_1 + \frac{q^2}{r_0^6}(4c_1 + 24c_2) \right]. \]  
(40)

which in the extremal limit gives
\[ \left( \frac{\eta}{s} \right)_{\text{Extremal}} = \frac{1}{4\pi} [1 + 48c_2], \]  
(41)

independent of \( c_1 \) as in the previous example.

Using the holographic anomaly calculation, we can identify the coefficient \( c_1 \) with the central charges as
\[ c_1 = \frac{1}{8} \left( \frac{c_a}{\alpha} - 1 \right). \]  
(42)

Using eq(23), we get the constraint on \( c_1 \) as
For \( \mathcal{N} = 0 \), \[ -\frac{39}{248} \leq 3c_1 \leq \frac{3}{4}, \]
For \( \mathcal{N} = 1 \), \[ -\frac{1}{8} \leq 3c_1 \leq \frac{3}{8}, \]
For \( \mathcal{N} = 2 \), \[ -\frac{3}{40} \leq 3c_1 \leq \frac{3}{8}. \]  
(43)

It would be interesting to find the restriction on \( c_2 \) using the anomaly calculation [24].

7 Solution to constraints from WGC and Central charges

Solving eq(24) and eq(17), we get solutions, which has got both intervals and some isolated points. For \( \mathcal{N} = 0 \), it is
\[ c_1 = \frac{1}{12}, \ c_2 = \frac{-23}{576} \text{ and } c_1 = \frac{-13}{144}, \ c_2 = \frac{731}{6912}, \]
\[
\begin{align*}
-\frac{23}{576} & \leq c_2 \leq \frac{13}{576}, \quad -\frac{48c_2}{23} \leq c_1 \leq \frac{1}{12}, \\
\frac{13}{576} & \leq c_2 \leq \frac{299}{6912}, \quad -\frac{48c_2}{23} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{299}{6912} & \leq c_2 \leq \frac{731}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{731}{6912} & \leq c_2 \leq \frac{1488}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{1488}{6912} & \leq c_2 \leq \frac{299}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{299}{6912} & \leq c_2 \leq \frac{731}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{731}{6912} & \leq c_2 \leq \frac{1488}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}, \\
\frac{1488}{6912} & \leq c_2 \leq \frac{299}{6912}, \quad -\frac{13}{144} \leq c_1 \leq \frac{3 - 48c_2}{23}.
\end{align*}
\] (44)

For \( \mathcal{N} = 1 \), it is
\[
\begin{align*}
c_1 & = \frac{1}{16}, \quad c_2 = -\frac{23}{768} \quad \text{and} \quad c_1 = -\frac{1}{16}, \quad c_2 = \frac{71}{768}, \\
-\frac{23}{768} & \leq c_2 \leq \frac{23}{768}, \quad -\frac{48c_2}{23} \leq c_1 \leq \frac{1}{16}, \\
\frac{23}{768} & \leq c_2 \leq \frac{25}{768}, \quad -\frac{1}{16} \leq c_1 \leq \frac{1}{16}, \\
\frac{25}{768} & \leq c_2 \leq \frac{71}{768}, \quad -\frac{1}{16} \leq c_1 \leq \frac{3 - 48c_2}{23},
\end{align*}
\] (45)

Finally for \( \mathcal{N} = 2 \), it is
\[
\begin{align*}
c_1 & = \frac{1}{16}, \quad c_2 = -\frac{23}{768} \quad \text{and} \quad c_1 = -\frac{1}{32}, \quad c_2 = \frac{119}{1536}, \\
-\frac{23}{768} & \leq c_2 \leq \frac{23}{1536}, \quad -\frac{48c_2}{23} \leq c_1 \leq \frac{1}{16}, \\
\frac{23}{1536} & \leq c_2 \leq \frac{25}{768}, \quad -\frac{1}{32} \leq c_1 \leq \frac{1}{16}, \\
\frac{25}{768} & \leq c_2 \leq \frac{119}{1536}, \quad -\frac{1}{32} \leq c_1 \leq \frac{3 - 48c_2}{23},
\end{align*}
\] (46)

For each case of \( \mathcal{N} = 0, 1, 2 \), there exists solutions, where \( c_2 \) is both positive and negative, but it is not a priori clear which set to choose and which set to ignore.

References

[1] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, "The String Landscape, Black Holes and Gravity as the Weakest Force," JHEP 06 (2007) 060, [arXiv:hep-th/0601001].

[2] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2, 231 (1998), [arXiv:hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B428 (1998) 105, [arXiv:hep-th/9802109]; E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253, [arXiv:hep-th/9712201]; O. Aharony, S. S. Gubser, J. Maldacena H. Ooguri and Y. Oz, Large N field theories, String theory and gravity, Phys. Rept, 323 (2000) 183-386, [arXiv:hep-th/ 9905111].
[3] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” Phys. Rev. Lett. 94, 111601 (2005), [arXiv:hep-th/0405231].

[4] R. C. Myers, M. F. Paulos and A. Sinha, ”Holographic hydrodynamics with a chemical potential,” JHEP 06 (2009) 006, [arXiv: 0903.2834[hep-th]].

[5] D. M. Hofman and J. Maldacena, ”Conformal collider physics: Energy and charge correlations,” JHEP 05 (2008) 012, [arXiv:0803.1467]; D. M. Hofman, ”Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT,” Nucl. Phys. B823 (2009) 174, [arXiv:0907.1625].

[6] S. Nojiri and S. D. Odintsov, ”On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence,” Int. Jour. Mod. Phys. A 15, 413, 2000, [arXiv:hep-th/9903033]; M. Blau, K. S. Narain and E. Gava, JHEP 09 (1999) 018, [arXiv:hep-th/9904179].

[7] C. Vafa, ”The string landscape and the swampland,” [arXiv:hep-th/0509212].

[8] M. Edalati, J. I. Jottar and R. G. Leigh, ”Transport Coefficients at Zero Temperature from Extremal Black Holes,” [arXiv:hep-th/0910.0645]; M. F. Paulos, ”Transport coefficients, membrane couplings and universality at extremality,” arXiv:0910.4602 [hep-th]; R. G. Cai, Y. Liu and Y. W.Sun, ”Transport Coefficients from Extremal Gauss-Bonnet Black Holes,” [arXiv:0910.4705 [hep-th]].

[9] N. Iqbal and H. Liu, “Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm,” Phys. Rev. D 79, 025023 (2009), [arXiv:0809.3808 [hep-th]].

[10] S. S. Pal, ”$\eta/s$ at finite coupling,” Phys. Rev. D 81, 045005 (2010), [arXiv:0910.0101].

[11] A. Buchel, M. P. Heller and R. C. Myers, ”sQGP as hCFT,” Phys. Lett. B 680 (2009) 521, [arXiv:0908.2802[hep-th]].

[12] A. Buchel, R. C. Myers and A. Sinha, Beyond $\eta/s = 1/4\pi$, JHEP 03, 084 (2009), [arXiv:0812.2521 [hep-th]].

[13] A. Sinha and R. C. Myers, The viscosity bound in string theory, Nucl. Phys. A 830, 295C (2009) [arXiv:0907.4798 [hep-th]].

[14] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, “Viscosity Bound Violation in Higher Derivative Gravity,” Phys. Rev. D 77, 126006 (2008), [arXiv:0712.0805 [hep-th]]; Phys. Rev. Lett. 100, 191601 (2008), [arXiv:0802.3318 [hep-th]].
[15] Y. Kats and P. Petrov, "Effect of curvature squared correction in AdS on the viscosity of the dual gauge theory," JHEP 01 (2009) 044, [arXiv:0712.0743 [hep-th]].

[16] S. Cremonini, K. Hanaki, J. T. Liu and P. Szepietowski, “Higher derivative effects on \( \eta/s \) at finite chemical potential,” Phys. Rev. D 80, 025002 (2009), arXiv:0903.3244 [hep-th]; ” Black holes in five-dimensional gauged supergravity with higher derivatives,” JHEP 12 045, 2009, [arXiv:0812.3572 [hep-th]].

[17] N. Banerjee and S. Dutta, “Higher Derivative Corrections to Shear Viscosity from Graviton’s Effective Coupling,” [arXiv:0901.3848 [hep-th]]; ” Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity,” [arXiv:0903.3925 [hep-th]]; ”Near-Horizon Analysis of \( \eta/s \),” [arXiv: 0911.0557 [hep-th]].

[18] R. Brustein and A. J. M. Medved, ”Proof of a universal lower bound on the shear viscosity to entropy density ratio,” [arXiv: 0908.1473 [hep-th]].

[19] M. Alishahiha and A. Ghodsi, Phys. Rev. D 80 026004, 2009 [arXiv:0901.3431[hep-th]].

[20] X. H. Ge, Y. Matsuo, F. W. Shu, S. J. Sin and T. Tsukioka, “Viscosity Bound, Causality Violation and Instability with Stringy Correction and Charge,” JHEP 10, 009 (2008), [arXiv:0808.2354 [hep-th]]; X. H. Ge and S. J. Sin, Shear viscosity, instability and the upper bound of the Gauss-Bonnet coupling constant, JHEP 05, 051 (2009) [arXiv:0903.2527 [hep-th]]; R. G. Cai, Z. Y. Nie and Y. W. Sun, Shear Viscosity from Effective Couplings of Gravitons, Phys. Rev. D 78, 126007 (2008) [arXiv:0811.1665 [hep-th]]; R. G. Cai, Z. Y. Nie, N. Ohta and Y. W. Sun, Shear Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling, Phys. Rev. D 79, 066004 (2009) [arXiv:0901.1421 [hep-th]]; G. Koutsoumbas, E. Papantonopoulos, G. Siopsis, ”Shear Viscosity and Chern-Simons Diffusion Rate from Hyperbolic Horizons,” Phys. Lett. B 677, 74 (2009), [arXiv:0809.3388 [hep-th]].

[21] A. Buchel, J. T. Liu and A. O. Starinets, “Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory,” Nucl. Phys. B 707, 56 (2005), [arXiv:hep-th/0406264].

[22] Y. Kats, L. Motl and M. Padi, ”Higher-order corrections to mass-charge relation of extremal black holes,” JHEP 12 (2007) 068, [arXiv:hep-th/0606100].

[23] R. Wald ”Black hole entropy in the Noether charge,” Phys. Rev. D 48, 3427 (1993), [arxiv:gr-qc/9307038]; V. Iyer and R. Wald ”Some properties of Noether charge and a proposal for dynamical black hole entropy,” Phys. Rev. D 50, 846 (1994), [arxiv:gr-qc/9403028]; T. Jacobson, G. Kang and R. Myers ”On black hole entropy,” Phys. Rev. D 49, 6587 (1994), [arXiv:gr-qc/9312023].
[24] M. Henningson and K. Skenderis, "The holographic Weyl Anamoly," JHEP 07 (1998) 023, [arXiv:hep-th/9806087].

[25] S. Cremonini, J. T. Liu and P. Szepietowski, "Higher Derivative Corrections to R-charged Black Holes: Boundary Counterterms and the Mass-Charge Relation," [arXiv:0910.5159[hep-th]].