A new approach to fluctuations of reflected Lévy processes

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Abstract

We present a new approach to fluctuation identities for reflected Lévy processes with one-sided jumps. This approach is based on a number of easy to understand observations and does not involve excursion theory or Itô calculus. It also leads to more general results.

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1 Introduction

Let $X(t), t \geq 0$ be a Lévy process with no positive jumps. We consider $X(t)$ and $-X(t)$ reflected at 0, and study the times at which these reflected processes pass over a certain level $B > 0$. In fact, for each reflected process we are interested in the joint Laplace transform of the first passage time, the overshoot, and the corresponding value of the local time at 0. In addition, it is assumed that the initial value of $X(t)$ is shifted to an arbitrary $x_0 \in [0, B]$.

The main idea is to add an additional reflecting barrier at $B$. That is, we consider a process $W(t), t \geq 0$ with values in $[0, B]$ having the representation

$$W(t) = X(t) + L(t) - U(t),$$

where $L(t)$ and $U(t)$ are non-decreasing right-continuous functions called the local times at respectively the lower and the upper barriers (that is at 0 and at $B$). In addition, it is required that $L(0) = U(0) = 0$ and the points of increase of $L(t)$ and $U(t)$ are contained in the sets $\{t \geq 0 : W(t) = 0\}$ and $\{t \geq 0 : W(t) = B\}$ respectively. It is known that the triplet of functions $(W(t), L(t), U(t))$ exists and is unique, see e.g. [3], and is called the solution of the two-sided Skorokhod problem. Letting $B = \infty$ we obtain a one-sided reflection at 0, in which case $U(t) = 0$ and $L(t)$ can be given explicitly through $L(t) = -\min(X(t), 0)$, where $X(t) = \inf\{X(s) : 0 \leq s \leq t\}$.

Define the inverse local times through

$$\tau^L_x = \inf\{t \geq 0 : L(t) > x\}, \quad \tau^U_x = \inf\{t \geq 0 : U(t) > x\}. \quad (1)$$

Note that the one-sided reflection of $X(t)$ at 0 behaves as $W(t)$ up to time $\tau^U_x$, its first passage time over level $B$ is given by $\tau^U_x$, and the corresponding value of the local time at 0 is $L(\tau^U_x)$. Absence of positive jumps implies that $U(\tau^U_x) = 0$, in other words, there is no overshoot. One of our goals is to characterize the distribution of $(\tau^U_x, L(\tau^U_x))$ for an arbitrary starting point $x_0 \in [0, B]$. Similarly, looking down from the level $B$ we see that $(\tau^L_x, L(\tau^L_x), U(\tau^L_x))$ describes the first passage time, the overshoot, and the local time at 0 of the one-sided reflection of $-X(t)$ with the starting point $B - x_0$. Our second goal is to characterize this triplet.

In order to achieve the above goals we study the process $L(\tau^U_x), x \geq 0$, which turns out to be a compound Poisson process with some specific distribution of $L(\tau^U_x)$. We determine this initial

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distribution, the distribution of jumps and the jump arrival rate. Based on these key results we derive the Laplace transforms of \((τ_0^{\leftarrow}, L(τ_0^{\leftarrow}))\) and \((τ_b^+, L(τ_b^{\leftarrow}), U(τ_b^{\leftarrow}))\) for an arbitrary \(x_0\in [0,B]\). To our knowledge, these results are more general than the existing ones. We advise the reader to have a look at [6; Ch. 8.5], where the identities for reflected Lévy processes are presented and the relevant literature is discussed. The main references are [11; 7; 3]. The proofs in the cited papers are rather involved and are based on Itô’s excursion theory, stochastic calculus and martingale calculations. A proof based on direct excursion theory calculations is given in [2]. Our approach employs a number of easy to understand observations and does not require the use of above mentioned heavy machinery.

The following section contains some preliminary material. The main results under a tilted measure are derived in Section 3. Finally, Section 4 contains our results in their general form.

2 Preliminaries

Let \(X(t), t \geq 0\) be a Lévy process defined on \((Ω, F, \{F_t\}_{t \geq 0}, P^0)\). It is assumed that \(X(t)\) has no positive jumps, moreover, it is not a process with a.s. monotone paths. Then \(P^0 e^{αX(t)} = e^{φ(α)t}\) for some function \(φ(α)\), called Laplace exponent, and \(Re(α) \geq 0\). It is useful to note that \(φ(α)\) is analytic in the domain specified by \(Re(α) > 0\). Restricting ourselves to the real half line \(α \geq 0\), we note that \(φ(α)\) is convex, \(φ(α) \to ∞\) as \(α \to ∞\), and \(Ex(1) = φ'_++(0)\), where \(φ'_++\) denotes the right derivative of \(φ\).

Pick \(q > 0\) and denote the unique positive solution of \(φ(q) = q\) through \(Φ(q)\), which is sometimes called the right inverse of \(φ(α)\). Consider the Wald’s martingale \(e^{ϕ(q)X(t)−qt}, t \geq 0\) and the corresponding measure \(P\), that is, a measure with Radon-Nikodym derivative

\[
\frac{dP}{dP^0}|_{F_t} = e^{Φ(q)X(t)−qt}.
\]

It is easy to see that \(X(t)\) is a Lévy process under \(P\) and its Laplace exponent is

\[
ψ(α) = φ(α + Φ(q)) − q. \tag{3}
\]

Hence \(ψ(α)\) is analytic in the neighborhood of 0, and \(ψ'(0) = ϕ'(Φ(q)) > 0\) implying that \(X(t)\) has a positive drift under \(P\).

Let us proceed to so-called \(q\)-scale functions, which play a central role in the theory of fluctuations of one-sided Lévy processes. It is known that there exists a unique strictly increasing and continuous function \(W(q)(x), x \geq 0\) satisfying \(\int_0^{∞} e^{−αx}W(q)(x)dx = 1/(φ(α) − q)\) for \(α > Φ(q)\). Such a function can be given explicitly through

\[
W(q)(x) = e^{Ψ(q)x}W(x), \quad W(x) = \frac{1}{ψ'(0)}P(X ≥ −x), \tag{4}
\]

where \(X\) denotes the all-time infimum of \(X(t)\), see [3; Ch. 8.2]. In addition, \(W(0)(x)\) is defined as a limit of \(W(q)(x)\) as \(q \downarrow 0\).

Define the first passage times \(τ_α^± = \inf\{t \geq 0 : ±X(t) > x\}\) for \(x \geq 0\) and note using the strong Markov property that

\[
P(X ≥ −a) = P(τ_α^+ < τ_α^-)P(X ≥ −(a + b)), \quad a, b \geq 0.
\]

Hence \(P(τ_α^+ < τ_α^-) = W(a)/W(a + b)\) for \(a + b > 0\). In terms of the original measure \(P(τ_α^+ < τ_α^-)\) reads as \(E^0[e^{ϕ(q)X(τ_α^+−qτ_α^-); τ_α^+ < τ_α^-}]\) and hence

\[
E^0(e^{−qτ_α^-}; τ_α^+ < τ_α^-) = \frac{W(q)(a)}{W(q)(a + b)}.
\]

Finally, we note that \(W(x)\) and hence also \(W(q)(x)\) have right derivatives for \(x > 0\).
3 \ Results under the tilted measure $\mathbb{P}$

The main object of our study is the process $L(\tau^U_x), x \geq 0$, which has non-decreasing paths. Using the strong Markov property we see that $L(\tau^U_x)$ is a Lévy process. In fact, it is a compound Poisson process, because it does not jump in a fixed interval with positive probability. The following theorem identifies this process.

**Theorem 1.** The process $L(\tau^U_x), x \geq 0$ is a compound Poisson process characterized by

$$\psi^L(\alpha) = \frac{1}{x} \log \mathbb{E} e^{-\alpha [L(\tau^U_x) - L(\tau^U_{x'})]} = \frac{W(B)\psi(\alpha)}{Z(\alpha, B)} - \alpha,$$

$$\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_x)} = \frac{Z(\alpha, x_0)}{Z(\alpha, B)},$$

where $\text{Re} (\alpha) \geq 0$ and $Z(\alpha, x) = e^{\alpha x} (1 - \psi(\alpha)) \int_0^x e^{-\alpha y} e^{\alpha y} d\mathbb{P}(y)$. 

This result is based on the following lemma, which is closely related to the proof of [3, Theorem 4.1], where a Markov-modulated Brownian motion is considered. Similar ideas in a simpler form also appear in [3, Section 5].

**Lemma 2.** It holds for $\alpha < 0$ that

$$\int_0^\infty \mathbb{E}_{x_0} e^{\alpha X(\tau^U_{x'})} dx = \left( -e^{\alpha B/\alpha} + e^{\alpha (B+x)} \int_0^\infty e^{-\alpha y} \mathbb{P}(X \leq -y) dy \right) / (1 - \mathbb{P}(X \leq -B)).$$

It is noted that the expression in the lemma might not be finite.

**Proof.** Pick an arbitrary $y \in \mathbb{R}$ and consider the time points $t \geq 0$ such that $X(t) = y$ and $U(t-)<U(t+s), \forall s>0$ ($t$ is the point of increase of $U$). We denote this set through $T_y$. If $y \geq B$ then the first such point is $\tau_y$; if $y < B$ then it is $\inf\{t \geq 0 : X(t) \leq y \}$, $X(t) > y$. To justify the second statement, note that $X(t)$ should first hit level $y-B$ to guarantee that $W(t) = B$ at the time when $X(t)$ hits $y$; drawing a picture might be helpful here. Recall that $X(t)$ has a positive drift under $\mathbb{P}$, hence $T_y$ contains at least one point with probability 1 or $\mathbb{P}_{x_0}(X \leq y - B)$ corresponding to $y \geq B$ and $y < B$. Use the strong Markov property to see that there are at least $n \geq 1$ points in $T_y$ with probability $\mathbb{P}(X \leq -B)^{n-1}$ or $\mathbb{P}_{x_0}(X \leq y - B)\mathbb{P}(X \leq -B)^{n-1}$ depending on $y$. Hence the expected number of points in $T_y$ is

$$\mathbb{E}_{x_0}|T_y| = \begin{cases} (1 - \mathbb{P}(X \leq -B))^{-1}, & y \geq B \\ \mathbb{P}(X \leq y - B - x_0)(1 - \mathbb{P}(X \leq -B))^{-1}, & y < B. \end{cases}$$

Multiply both sides by $e^{\alpha y}$, where $\alpha < 0$, and integrate over $y$ to see that the right side is as stated in the lemma. It therefore remains to show that

$$\int \int_{-\infty}^\infty e^{\alpha y} \mathbb{E}_{x_0}|T_y| dy = \int_{0}^\infty \mathbb{E}_{x_0} e^{\alpha X(\tau^U_{x'})} dx.$$

But, $|T_y| = \sum_{x \geq 0} 1_{\{X(\tau^U_{x'}) = y\}}$ hence, in view of Fubini’s theorem, it is sufficient to show that

$$\int \int_{-\infty}^\infty e^{\alpha y} 1_{\{X(\tau^U_{x'}) = y\}} dy = \int_{0}^\infty e^{\alpha X(\tau^U_{x'})} dx$$

holds $\mathbb{P}$-a.s. For this note that $X(\tau^U_x) = -L(\tau^U_x) + x + B$ and $L(\tau^U_x)$ is piecewise constant. Let $L(\tau^U_{x'}) = C$ on the interval $(S, F)$ then

$$\int \int_{-\infty}^\infty \sum_{x \in [S, F]} e^{\alpha y} 1_{\{X(\tau^U_{x'}) = y\}} dy = \int_{S}^{F} e^{\alpha X(\tau^U_{x'})} dx.$$

Summing over all such intervals concludes the proof.  \( \square \)
Proof of Theorem 1. We start with generalized Pollaczek-Khinchine formula:

\[ \mathbb{E} e^{\alpha X} = \psi'(0) \alpha / \psi(\alpha), \quad \alpha > 0, \]  

see also [3, p. 217]. Let us show that this identity can be extended to some negative values of \( \alpha \).

Let

\[ r = \inf \{ \alpha \in \mathbb{R} : \mathbb{E} e^{\alpha X} < \infty \}, \]

which is non-positive. It is well known that \( \mathbb{E} e^{\alpha X} \) is analytic for \( \text{Re}(\alpha) > r \). Note that \( \mathbb{E} e^{\alpha X} \) is finite for \( \alpha > r \), and hence \( \psi(\alpha) \) is analytic in the same domain. Therefore \( \mathbb{E} e^{\alpha X} \psi(\alpha) - \psi'(0) \alpha \) is identically 0 here, and so (7) holds true for \( \text{Re}(\alpha) > r, \alpha \neq 0 \). Let us show that \( r < 0 \). If \( r = 0 \) then the right hand side of (7) should have a singularity at 0, see e.g. [2, Ch. II, Thm. 5b]. This is not the case, because \( \psi(\alpha) \) is analytic in the neighborhood of 0 and \( \psi'(0) > 0 \) according to representation (3).

Pick an arbitrary \( \alpha \in (r, 0) \) and apply Fubini’s theorem to see that

\[ \int_{0}^{\infty} e^{-\alpha y} \mathbb{P}(X \geq y) dy = \int_{0}^{\infty} \int_{0}^{x} e^{-\alpha y} dy \mathbb{P}(X \in dx) = \frac{1}{\alpha} (1 - e^{-\alpha x}), \]

which is further written as \( 1/\alpha - \psi'(0)/\psi(\alpha) \). Note also that \( \mathbb{P}(X \leq -y) = \mathbb{P}(X < -y) \) for \( y > 0 \), because \( X(t) \) can hit \(-y\) only if it has paths of unbounded variation [6, Theorem 7.11], but then it will pass over \(-y\) a.s. Then by the definition of \( W(y) \) we have \( \mathbb{P}(X \leq -y) = 1 - \psi'(0) W(y) \) for \( y > 0 \), and thus

\[ \int_{0}^{x_0} e^{-\alpha y} \mathbb{P}(X \leq -y) dy = \frac{1}{\alpha} (1 - e^{-\alpha x_0}) - \psi'(0) \int_{0}^{x_0} e^{-\alpha y} W(y) dy. \]

This yields an expression for \( \int_{x_0}^{\infty} e^{-\alpha y} \mathbb{P}(X \leq -y) dy \), which together with Lemma 2 imply

\[ \int_{0}^{\infty} \mathbb{E}_{x_0} e^{\alpha X(x_0^U)} dx = e^{\alpha(B+x_0)} \left( \int_{0}^{x_0} e^{-\alpha y} W(y) dy - \frac{1}{\psi(\alpha)} \right) / W(B). \]

Noting that \( X(x_0^U) = -L(x_0^U) + x + B \) we obtain

\[ \int_{0}^{\infty} \mathbb{E}_{x_0} e^{\alpha X(x_0^U)} dx = \mathbb{E}_{x_0} e^{-\alpha L(x_0^U)} e^{\alpha B} \int_{0}^{\infty} e^{\alpha x + \psi'(\alpha)} dx, \]

which is convergent as shown above. Therefore

\[ \frac{\mathbb{E}_{x_0} e^{-\alpha L(x_0^U)}}{\psi'(\alpha) + \alpha} = \frac{Z(\alpha, x_0)}{W(B) \psi(\alpha)} \]

for all \( \alpha \in (r, 0) \). The expressions in the statement of the theorem are obtained by noting that \( x_0^U = 0 \) a.s. given \( x_0 = B \). It remains to show that these identities hold true for \( \text{Re}(\alpha) > r \). Multiplied by \( Z(\alpha, B) \) they indeed hold true, because the Laplace transform \( \int_{0}^{\infty} e^{-\alpha y} W(y) dy \) is entire, see e.g. [4] Ch. II, Lem. 5], and hence \( Z(\alpha, x) \) is analytic in \( \text{Re}(\alpha) > r \). Finally, if \( Z(\alpha, B) = 0 \) for some \( \alpha \) in the domain of interest then \( \psi(\alpha) = 0 \) implying \( Z(\alpha, B) = e^{\alpha B} \neq 0 \), a contradiction.

Theorem 1 specifies the compound Poisson process \( L(x_0^U), x \geq 0 \) uniquely, hence it can potentially be used to obtain the jump arrival rate of this process. It is, however, easier to give a direct argument. Note that a jump of \( L(x_0^U) \) corresponds to an excursion from the maximum of height exceeding \( B \). Such excursions arrive with rate \( W'_+(B)/W(B) \), see e.g. [4] p. 220], thus we have the following result.

Lemma 3. The jump arrival rate of \( L(x_0^U) \) is \( \lambda_x = W'_+(B)/W(B) \).

Let us sketch an alternative proof of the above lemma without an explicit use of excursion theory. First, note that the probability that \( L(x_0^U) \) jumps in \([0, \epsilon]\) is bounded from below by \( \mathbb{P}(\tau_B^U < \tau_x^+) = 1 - W(B)/W(B + \epsilon) \). This shows that \( W'_+(B)/W(B) \) is the lower bound on the jump arrival rate. This lower bound is achieved if \( \mathbb{P}(\tau_{B-\epsilon} < \tau_x^+) \mathbb{P}(\tau_B^U < \tau_x^+)/\epsilon \) goes to 0 as \( \epsilon \downarrow 0 \), because of the strong Markov property. The alternative proof is completed after one observes that \( \mathbb{P}(\tau_{B-\epsilon} < \tau_x^+)/\epsilon \) is bounded for small enough \( \epsilon \).

Let us now present an important corollary of Theorem 1.
Corollary 4. It holds for $\alpha, \theta \geq 0$ that
\[
\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = Z(\alpha, x_0) + \frac{W(x_0)[W(B)\phi(\alpha) - (\alpha + \theta)Z(\alpha, B)]}{W(B) + \theta W(B)}.
\]

Proof. Let $\Delta = \inf \{x \geq 0 : L(\tau^U_x) > 0\}$ then according to Lemma 6, we have
\[
\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = \frac{\lambda L + \psi^L(\alpha)}{\lambda L + \theta},
\]
where $\alpha, \theta \geq 0$. Next we write
\[
\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0)}.
\]
But the strong Markov property implies that
\[
\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0)}.
\]
Hence
\[
\mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0)} = \left( \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0)} + \mathbb{P}(\tau_B > x_0) \frac{\psi^L(\alpha) - \theta}{\lambda L + \theta} \right) / \mathbb{E}_{x_0} e^{-\alpha L(\tau^U_0)}.
\]

Recall that $\mathbb{P}(\tau_B > x_0) = W(x_0)/W(B)$, and apply Lemma 5 and Theorem 4 to conclude the proof. \qed

4 Back to the original measure

In this section, we rewrite the results obtained in the previous section in terms of the original measure $P^0$.

Theorem 5. It holds for $q > 0, \alpha \geq \Phi(q), \theta \geq 0$ that
\[
\mathbb{E}_{x_0}^0 e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0) - q\tau^U_0} = Z(\alpha, x_0) + \frac{W(\alpha, x_0)[W(q)(B)\phi(\alpha) - (\alpha + \theta)Z(q, \alpha, B)]}{W(q)_+^+(B) + \theta W(q)(B)}.
\]

where
\[
Z(q)(\alpha, x) = e^{\alpha x} \left( 1 + (q - \phi(\alpha)) \int_0^x e^{-\alpha y} W(q)(y)dy \right).
\]
Moreover, the two-dimensional Lévy process $(L(\tau^U_x), \tau^U_x), x \geq 0$ is characterized by
\[
\log \mathbb{E}_{x_0}^0 e^{-\alpha L(\tau^U_0) - \Phi(q)(\alpha, x)} = \frac{W(q)(B)\phi(\alpha) - q}{Z(q)(\alpha, B)} = \alpha
\]
with $q > 0, \alpha \geq \Phi(q)$.

Proof. We only prove the first identity and note that the two other ones follow in a similar way from Theorem 4. Consider Corollary 4 with $\alpha', \theta' \geq 0$ and write the left hand side of the corresponding identity in terms of the original measure $P^0$:
\[
\mathbb{E}_{x_0}^0 e^{-\alpha' L(\tau^U_0) - \theta' U(\tau^U_0) - q\tau^U_0} \mathbb{E}_{x_0}^0 e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0) - q\tau^U_0} = e^{-\Phi(q)\alpha} \mathbb{E}_{x_0}^0 e^{-\alpha L(\tau^U_0) - \theta U(\tau^U_0) - q\tau^U_0},
\]
where $\alpha = \alpha' + \Phi(q)$ and $\theta = \theta' - \Phi(q)$. Moreover, it is easy to see that $e^{\Phi(q)\alpha} Z(\alpha', x) = Z(q)(\alpha, x)$ and $W(q)^+_+(B) = \Phi(q)W(q)(B) + e^{\Phi(q)B}W'_+(B)$. Simple algebraic manipulations complete the proof. \qed
We remark that the above identities also hold for $q = 0$. Let $q \downarrow 0$ to see this. The only technical difficulty arises when considering the first identity. Namely, one has to show that $\lim_{q \downarrow 0} W'(q)(B) = W'(0)^+(B)$. This difficulty is bypassed by noting that Lemma 3 and then also Corollary 4 hold under the original measure.

Appendix

Lemma 6. Let $X(t)$ be a compound Poisson process with rate $\lambda$ and Laplace exponent $\log \mathbb{E}e^{-\alpha X(t)} = \psi(\alpha)$. Let $J$ be the time of the first jump of $X(t)$ then

$$
\mathbb{E}e^{-\alpha X(J) - \theta J} = \frac{\lambda + \psi(\alpha)}{\lambda + \theta},
$$

where $\alpha, \theta \geq 0$.

Proof. Let $\eta$ be an exponential random variable of rate $\theta > 0$ independent of everything else. Clearly, $\mathbb{E}e^{-\alpha X(\eta)} = \theta/(-\psi(\alpha))$ and $\mathbb{P}(J > \eta) = \mathbb{E}e^{-\lambda \eta} = \theta/(\theta + \lambda)$. But we also have

$$
\mathbb{E}e^{-\alpha X(\eta)} = \mathbb{P}(J > \eta) + \mathbb{E}[e^{-\alpha X(J)}; J < \eta]\mathbb{E}[e^{-\alpha X(\eta)}],
$$

which results in the statement of the Lemma for $\theta > 0$. Taking limits as $\theta \downarrow 0$ proves the case of $\theta = 0$.

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