Abstract. In 2016, Yüce and Torunbalcı Aydın [17] defined dual Fibonacci quaternions. In this paper, we defined the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions. Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we gave some their quadratic properties, the summations, the Binet’s formulas and Cassini-like identities for these quaternions.

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Key words: Third-order Jacobsthal number, third-order Jacobsthal-Lucas number, third-order Jacobsthal quaternions, third-order Jacobsthal-Lucas quaternions, dual quaternion.

1. Introduction

The real quaternions are a number system which extends to the complex numbers. They are first described by Irish mathematician William Rowan Hamilton in 1843. In 1963, Horadam [8] defined the \( n \)-th Fibonacci quaternion which can be represented as

\[
Q_F = \{Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} : F_n \text{ is } n-\text{th Fibonacci number}\},
\]

where \( i^2 = j^2 = k^2 = ijk = -1 \).

In 1969, Iyer [13] derived many relations for the Fibonacci quaternions. In 1977, Iakin [11] introduced higher order quaternions and gave some identities for these quaternions. Furthermore, Horadam [9] extend to quaternions to the complex Fibonacci numbers defined by Harman [6]. In 2012, Halıcı [5] gave generating functions and Binet’s formulas for Fibonacci and Lucas quaternions.

In 2006, Majernik [15] defined a new type of quaternions, the so-called dual quaternions in the form \( Q_N = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\} \), with the following multiplication schema for the quaternion units

\[
i^2 = j^2 = k^2 = 0, \quad ij = -ji = jk = -kj = ik = -ki = 0.
\]
In 2009, Ata and Yaylı [1] defined dual quaternions with dual numbers coefficient as follows:

\[ Q_D = \{ A + B i + C j + D k : A, B, C, D \in \mathbb{D}, \ i^2 = j^2 = k^2 = i j k = -1 \}, \]

where \( \mathbb{D} = \mathbb{R}[\varepsilon] = \{ a + b \varepsilon : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \} \). It is clear that \( Q_N \) and \( Q_D \) are different sets. In 2014, Nurkan and Güven [16] defined dual Fibonacci quaternions as follows:

\[ D_F = \{ Q_n = \hat{F}_n + i \hat{F}_{n+1} + j \hat{F}_{n+2} + k \hat{F}_{n+3} : \hat{F}_n = F_n + \varepsilon F_{n+1} \}, \]

where \( i^2 = j^2 = k^2 = i j k = -1 \) and \( \hat{F}_n \) is the \( n \)-th dual Fibonacci number.

In 2016, Yüce and Torumbalci Aydin [17] defined dual Fibonacci quaternions as follows:

\[ N_F = \{ Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3} : F_n \text{ is } n\text{-th Fibonacci number} \}, \]

where \( i^2 = j^2 = k^2 = 0 \), \( i j = - j i \), \( j k = - k j \), \( i k = - k i \) = 0.

In the other hand, the Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [2]). The Jacobsthal numbers \( J_n \) are defined by the recurrence relation

\[ J_0 = 0, \ J_1 = 1, \ J_{n+1} = J_n + 2 J_{n-1}, \ n \geq 1. \]

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation \( j_{n+1} = j_n + 2 j_{n-1}, \ n \geq 1 \) and \( j_0 = 2, \ j_1 = 1 \). (see, [10]).

In [4] the Jacobsthal recurrence relation (1.6) is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [10] is expanded and extended to several identities for some of the higher order cases. In particular, third-order Jacobsthal numbers, \( \{ J_n^{(3)} \}_{n \geq 0} \), and third-order Jacobsthal-Lucas numbers, \( \{ j_n^{(3)} \}_{n \geq 0} \), are defined by

\[ J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2 J_n^{(3)}, \ J_0^{(3)} = 0, \ J_1^{(3)} = J_2^{(3)} = 1, \ n \geq 0, \]

and

\[ j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2 j_n^{(3)}, \ j_0^{(3)} = 2, \ j_1^{(3)} = 1, \ j_2^{(3)} = 5, \ n \geq 0, \]

respectively.

The following properties given for third-order Jacobsthal numbers and third-order Jacobsthal-Lucas numbers play important roles in this paper (for more, see [4]).

\[ 3 J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \]
(1.10) \[ j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}, \]

(1.11) \[ J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{if } n \not\equiv 1 \pmod{3}. \end{cases} \]

(1.12) \[ j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3}, \\ -3 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \]

(1.13) \[ j_n^{(3)} + j_{n+3}^{(3)} = 3J_{n+2}^{(3)}; \]

(1.14) \[ j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3}, \\ -1 & \text{if } n \equiv 1 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \]

(1.15) \[ \left( j_{n-3}^{(3)} \right)^2 + 3J_n^{(3)}j_n^{(3)} = 4^n, \]

(1.16) \[ \sum_{k=0}^{n} j_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if } n \not\equiv 0 \pmod{3}, \\ J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \pmod{3}. \end{cases} \]

and

(1.17) \[ \left( j_n^{(3)} \right)^2 - 9 \left( J_n^{(3)} \right)^2 = 2^{n+2}j_{n-3}^{(3)}. \]

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

\[ x^3 - x^2 - x - 2 = 0; \quad x = 2, \quad \text{and} \quad x = \frac{-1 \pm i\sqrt{3}}{2}. \]

Note that the latter two are the complex conjugate cube roots of unity. Call them \( \omega_1 \) and \( \omega_2 \), respectively. Thus the Binet formulas can be written as

(1.18) \[ J_n^{(3)} = \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21} \omega_1^n - \frac{3 - 2i\sqrt{3}}{21} \omega_2^n = \frac{1}{7} \left( 2^{n+1} - V_n^{(3)} \right) \]

and

(1.19) \[ j_n^{(3)} = \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7} \omega_1^n + \frac{3 - 2i\sqrt{3}}{7} \omega_2^n = \frac{1}{7} \left( 2^{n+3} + 3V_n^{(3)} \right), \]

respectively. Here \( V_n^{(3)} \) is the sequence defined by

(1.20) \[ V_n^{(3)} = \frac{3 + 2i\sqrt{3}}{3} \omega_1^n + \frac{3 - 2i\sqrt{3}}{3} \omega_2^n = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3}, \\ -3 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \]
Recently in [3], we have defined a new type of quaternions with the third-order Jacobsthal and third-order Jacobsthal-Lucas number components as

\[ JQ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)} i + J_{n+2}^{(3)} j + J_{n+3}^{(3)} k \]

and

\[ jQ_n^{(3)} = j_n^{(3)} + j_{n+1}^{(3)} i + j_{n+2}^{(3)} j + j_{n+3}^{(3)} k, \]

respectively, where \( i^2 = j^2 = k^2 = \text{ijk} = -1 \), and we studied the properties of these quaternions. Also, we derived the generating functions and many other identities for the third-order Jacobsthal and third-order Jacobsthal-Lucas quaternions.

In this paper, we define the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions as follows:

\[ JN_m^{(3)} = J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k \quad (m \geq 0) \]

and

\[ jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k \quad (m \geq 0), \]

respectively. Here \( i^2 = j^2 = k^2 = 0 \), \( \text{ij} = -\text{ji} = \text{jk} = -\text{kj} = \text{ik} = -\text{ki} = 0 \). Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we gave some their quadratic properties, the Binet’s formulas, d’Ocagne and Cassini-like identities for these quaternions.

### 2. Dual Third-order Jacobsthal Quaternions

We can define dual third-order Jacobsthal quaternions by using third-order Jacobsthal numbers. The \( n \)-th third-order Jacobsthal number \( J_n^{(3)} \) is defined by Eq. (1.7). Then, we can define the dual third-order Jacobsthal quaternions as follows:

\[ \mathbb{N}_j = \{ JN_m^{(3)} = J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k \}, \]

where \( J_m^{(3)} \) is the \( m \)-th third-order Jacobsthal number and \( \{1, j, k\} \) as in Eq. (1.2).

Also, we can define the dual third-order Jacobsthal-Lucas quaternion as follows:

\[ \mathbb{N}_j = \{ jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k \}, \]

where \( j_m^{(3)} \) is the \( m \)-th third-order Jacobsthal-Lucas number.
Then, the addition and subtraction of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by

\[ JN_m^{(3)} \pm jN_m^{(3)} = (J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k) \]
\[ \pm (J_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k) \]
\[ = (J_m^{(3)} \pm j_{m+1}^{(3)}) i + (J_{m+1}^{(3)} \pm j_{m+2}^{(3)}) j + (J_{m+2}^{(3)} \pm j_{m+3}^{(3)}) k \]
\[ + (J_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k) \]

and the multiplication of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by

\[ JN_m^{(3)} jN_m^{(3)} = (J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k)(j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k) \]
\[ = J_m^{(3)} j_m^{(3)} + (J_m^{(3)} j_{m+1}^{(3)} + J_{m+1}^{(3)} j_m^{(3)}) i + (J_m^{(3)} j_{m+1}^{(3)} + J_{m+2}^{(3)} j_m^{(3)}) j \]
\[ + (J_m^{(3)} j_{m+3}^{(3)} + J_{m+3}^{(3)} j_m^{(3)}) k. \]

Now, the scalar and the vector part of the \( JN_m^{(3)} \) which is the \( m \)-th term of the dual third-order Jacobsthal sequence \( \{JN_m^{(3)}\}_{m \geq 0} \) are denoted by

\[ (S_{JN_m^{(3)}}, V_{JN_m^{(3)}}) = (J_m^{(3)}, J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k). \]

Thus, the dual third-order Jacobsthal \( JN_m^{(3)} \) is given by \( S_{JN_m^{(3)}} + V_{JN_m^{(3)}} \). Then, relation (2.4) is defined by

\[ JN_m^{(3)} jN_m^{(3)} = S_{JN_m^{(3)}} S_{jN_m^{(3)}} + S_{JN_m^{(3)}} V_{jN_m^{(3)}} + S_{jN_m^{(3)}} V_{JN_m^{(3)}}. \]

The conjugate of dual third-order Jacobsthal quaternion \( JN_m^{(3)} \) is denoted by \( \overline{JN_m^{(3)}} \), and it is \( \overline{JN_m^{(3)}} = J_m^{(3)} - J_{m+1}^{(3)} i - J_{m+2}^{(3)} j - J_{m+3}^{(3)} k \). The norm of \( JN_m^{(3)} \) is defined as

\[ \|JN_m^{(3)}\|^2 = JN_m^{(3)} \overline{JN_m^{(3)}} = JN_m^{(3)} JN_m^{(3)} = \left( J_m^{(3)} \right)^2. \]

Then, we give the following theorem using statements (2.1) and (2.2).

**Theorem 2.1.** Let \( J_m^{(3)} \) and \( JN_m^{(3)} \) be the \( m \)-th terms of the third-order Jacobsthal sequence \( \{J_m^{(3)}\}_{m \geq 0} \) and the dual third-order Jacobsthal quaternion sequence \( \{JN_m^{(3)}\}_{m \geq 0} \), respectively. In this case, for \( m \geq 0 \) we can give the following relations:

\[ 2JN_m^{(3)} + JN_{m+1}^{(3)} + JN_{m+2}^{(3)} = JN_{m+3}^{(3)}, \]
\[ JN_m^{(3)} - JN_{m+1}^{(3)} i - JN_{m+2}^{(3)} j - JN_{m+3}^{(3)} k = J_m^{(3)}. \]
By using Eqs. (2.4) and (1.18), we get

\[ ||J_{N_m}^3||^2 + ||J_{N_{m+1}}||^2 + ||J_{N_{m+2}}||^2 = \frac{1}{t} \begin{pmatrix} 3 \cdot 2^{2(m+1)}(1 + 4i + 8j + 16k) \\ -2^{m+2}U_{N_m}^3 \\ -2^{m+3}U_m^3(i + 2j + 4k) \\ + 2(1 - i - j + 2k) \end{pmatrix}, \]

where \( U_{N_m}^3 = U_m^3 + U_{m+1}^3 i + U_{m+2}^3 j + U_{m+3}^3 k \) and \( U_m^3 = \frac{1}{t} \left( V_{m+1}^3 + 3V_{m+2}^3 \right). \)

Proof. (2.8): By the equations \( J_{N_m}^3 = J_m^3 + J_{m+1}^3 i + J_{m+2}^3 j + J_{m+3}^3 k \) and (1.7), we get

\[ 2J_{N_m}^3 + J_{N_{m+1}}^3 + J_{N_{m+2}}^3 = \begin{pmatrix} 2J_m^3 + 2J_{m+1}^3 i + 2J_{m+2}^3 j + 2J_{m+3}^3 k \\ + (J_{m+1}^3 + J_{m+2}^3 i + J_{m+3}^3 j + J_{m+4}^3 k) \\ + (J_{m+2}^3 + J_{m+3}^3 i + J_{m+4}^3 j + J_{m+5}^3 k) \end{pmatrix} = \begin{pmatrix} 2J_m^3 + J_{m+1}^3 + J_{m+2}^3 + (2J_{m+1}^3 + J_{m+2}^3 + J_{m+3}^3) i \\ + (2J_{m+2}^3 + J_{m+3}^3 + J_{m+4}^3 j + (2J_{m+3}^3 + J_{m+4}^3 + J_{m+5}^3) k) \end{pmatrix} = J_{m+3}^3 + J_{m+4}^3 i + J_{m+5}^3 j + J_{m+6}^3 k = J_{N_{m+3}}^3. \]

(2.9): By using \( J_{N_m}^3 \) in the Eq. (1.7) and conditions (1.2), we get

\[ J_{N_m}^3 - J_{N_{m+1}}^3 i - J_{N_{m+2}}^3 j - J_{N_{m+3}}^3 k = J_m^3 + J_{m+1}^3 i + J_{m+2}^3 j + J_{m+3}^3 k \]
\[ - (J_{m+1}^3 + J_{m+2}^3 i + J_{m+3}^3 j + J_{m+4}^3 k)i \\ - (J_{m+2}^3 + J_{m+3}^3 i + J_{m+4}^3 j + J_{m+5}^3 k)j \\ - (J_{m+3}^3 + J_{m+4}^3 i + J_{m+5}^3 j + J_{m+6}^3 k)k \]
\[ = J_{m+3}^3. \]

(2.10): By using Eqs. (2.4) and (1.18), we get

\[ (J_{N_m}^3)^2 = \left( J_m^3 \right)^2 + 2J_m^3 J_{m+1}^3 i + 2J_m^3 J_{m+2}^3 j + 2J_m^3 J_{m+3}^3 k, \]
and

\[
\left(J^3_m\right)^2 + \left(J^3_{m+1}\right)^2 + \left(J^3_{m+2}\right)^2 \\
= \frac{1}{49} \left(\left(2^m - V^3_m\right)^2 + \left(2^{m+2} - V^3_{m+1}\right)^2 + \left(2^{m+3} - V^3_{m+2}\right)^2\right) \\
= \frac{1}{49} \left(21 \cdot 2^{2(m+1)} - 2^{m+2} V^3_m + 2 V^3_{m+1} + 4 V^3_{m+2} + 14\right) \\
= \frac{1}{7} \left(3 \cdot 2^{2(m+1)} - 2^{m+2} U^3_m + 2\right),
\]

where \(U^3_m = \frac{1}{7}\left(V^3_m + 3 V^3_{m+2}\right)\). Finally, from the Eqs. (2.11) and (2.12), we obtain

\[
\left(J^N^3_m\right)^2 + \left(J^N^3_{m+1}\right)^2 + \left(J^N^3_{m+2}\right)^2 \\
= \left(J^3_m\right)^2 + \left(J^3_{m+1}\right)^2 + \left(J^3_{m+2}\right)^2 \\
+ 2 \left(J^3_m J^3_{m+1} + J^3_{m+1} J^3_{m+2} + J^3_{m+2} J^3_{m+3}\right) \cdot i \\
+ 2 \left(J^3_m J^3_{m+2} + J^3_{m+1} J^3_{m+3} + J^3_{m+2} J^3_{m+4}\right) \cdot j \\
+ 2 \left(J^3_m J^3_{m+3} + J^3_{m+1} J^3_{m+4} + J^3_{m+2} J^3_{m+5}\right) \cdot k \\
= \frac{1}{7} \left(3 \cdot 2^{2(m+1)}(1 + 4i + 8j + 16k) - 2^{m+2} U^N^3_m\right) \\
- 2^{m+3} V^3_m (i + 2j + 4k) + 2(1 - i - j + 2k),
\]

where \(U^N^3_m = U^3_m + U^3_{m+1}i + U^3_{m+2}j + U^3_{m+3}k\). □

**Theorem 2.2.** Let \(J^N^3_m\) and \(j^N^3_m\) be the \(m\)-th terms of the dual third-order Jacobsthal quaternion sequence \(\{J^N^3_m\}\)_{\(m \geq 0\}} and the dual third-order Jacobsthal-Lucas quaternion sequence \(\{j^N^3_m\}\)_{\(m \geq 0\)}, respectively. The following relations are satisfied

\[
\left(j^N^3_{m+3}\right)^2 + 3 J^N^3_{m+3} j^N^3_{m+3} = 4^m (1 + 4i + 8j + 16k).
\]
Proof. (2.13): From identities between third-order Jacobsthal number and third-order Jacobsthal-Lucas number (1.10) and (2.3), it follows that
\[ jN^{(3)}_{m+3} - 3JN^{(3)}_{m+3} = j^{(3)}_{m+3} + j^{(3)}_{m+4}i + j^{(3)}_{m+5}j + j^{(3)}_{m+6}k \]
\[ - 3(j^{(3)}_{m+3} + j^{(3)}_{m+4}i + j^{(3)}_{m+5}j + j^{(3)}_{m+6}k) \]
\[ = (j^{(3)}_{m+3} - 3j^{(3)}_{m+3}) + (j^{(3)}_{m+4} - 3j^{(3)}_{m+4})i \]
\[ + (j^{(3)}_{m+5} - 3j^{(3)}_{m+5})j + (j^{(3)}_{m+6} - 3j^{(3)}_{m+6})k \]
\[ = 2j^{(3)}_{m} + 2j^{(3)}_{m+1}i + 2j^{(3)}_{m+2}j + 2j^{(3)}_{m+3}k \]
\[ = 2jN^{(3)}_{m}. \]

The proof of (2.14) is similar to (2.13), using the identity (1.13).

(2.15): Now, using Eqs. (2.4), (2.11) and (1.15), we get
\[ (jN^{(3)}_{m})^{2} + 3JN^{(3)}_{m+3}jN^{(3)}_{m+3} \]
\[ = (j^{(3)}_{m})^{2} + 2j^{(3)}_{m}j^{(3)}_{m+1}i + 2j^{(3)}_{m}j^{(3)}_{m+2}j + 2j^{(3)}_{m}j^{(3)}_{m+3}k \]
\[ + 3j^{(3)}_{m+3}j^{(3)}_{m+3} + 3(j^{(3)}_{m+3}j^{(3)}_{m+4} + J^{(3)}_{m+4}j^{(3)}_{m+3})i \]
\[ + 3(j^{(3)}_{m+3}j^{(3)}_{m+5} + j^{(3)}_{m+5}j^{(3)}_{m+3})j + 3(j^{(3)}_{m+3}j^{(3)}_{m+6} + j^{(3)}_{m+6}j^{(3)}_{m+3})k \]
\[ = (j^{(3)}_{m})^{2} + 3j^{(3)}_{m+3}j^{(3)}_{m+3} \]
\[ + (2j^{(3)}_{m}j^{(3)}_{m+1} + 3(j^{(3)}_{m+3}j^{(3)}_{m+4} + j^{(3)}_{m+4}j^{(3)}_{m+3}))i \]
\[ + (2j^{(3)}_{m}j^{(3)}_{m+2} + 3(j^{(3)}_{m+3}j^{(3)}_{m+5} + j^{(3)}_{m+5}j^{(3)}_{m+3}))j \]
\[ + (2j^{(3)}_{m}j^{(3)}_{m+3} + 3(j^{(3)}_{m+3}j^{(3)}_{m+6} + j^{(3)}_{m+6}j^{(3)}_{m+3}))k \]
\[ = 4^{m+3}(1 + 4i + 8j + 16k). \]

\[ \square \]

**Theorem 2.3.** Let \( JN^{(3)}_{m} \) be the \( m \)-th term of the dual third-order Jacobsthal quaternion sequence \( \{JN^{(3)}_{m}\}_{m \geq 0} \). Then, we have the following identity
\[
(2.16) \quad \sum_{s=0}^{m} JN^{(3)}_{s} = JN^{(3)}_{m+1} - \frac{1}{21} \left( 7(1 + i + 4j + 7k) - 4V^{(3)}_{m+1} + V^{(3)}_{m} \right),
\]
where \( V^{(3)}_{m} = V^{(3)}_{m+1}i + V^{(3)}_{m+2}j + V^{(3)}_{m+3}k. \)
Proof. Since
\[
\sum_{s=0}^{m} J_s^{(3)} = J_{m+1}^{(3)} - \frac{1}{21} \left( 7 - 4V_{m+1}^{(3)} + V_{m}^{(3)} \right)
\]
(2.17)
\[
= \begin{cases} 
J_{m+1}^{(3)} & \text{if } m \not\equiv 0 \pmod{3} \\
J_{m+1}^{(3)} - 1 & \text{if } m \equiv 0 \pmod{3}
\end{cases}
\]
(see [4]), we get
\[
\sum_{s=0}^{m} J_{N_s}^{(3)} = \sum_{s=0}^{m} J_s^{(3)} + i \sum_{s=0}^{m+1} J_s^{(3)} + j \sum_{s=2}^{m+2} J_s^{(3)} + k \sum_{s=3}^{m+3} J_s^{(3)}
\]
\[
= J_{m+1}^{(3)} - \frac{1}{21} \left( 7 - 4V_{m+1}^{(3)} + V_{m}^{(3)} \right)
\]
\[
+ \left( J_{m+2}^{(3)} - \frac{1}{21} \left( 7 - 4V_{m+2}^{(3)} + V_{m+1}^{(3)} \right) \right) i
\]
\[
+ \left( J_{m+3}^{(3)} - \frac{1}{21} \left( 28 - 4V_{m+3}^{(3)} + V_{m+2}^{(3)} \right) \right) j
\]
\[
+ \left( J_{m+4}^{(3)} - \frac{1}{21} \left( 49 - 4V_{m+4}^{(3)} + V_{m+3}^{(3)} \right) \right) k
\]
\[
= J_{N_{m+1}}^{(3)} - \frac{1}{21} \left( 7(1 + i + 4j + 7k) - 4V_{m+1}^{(3)} + V_{m}^{(3)} \right),
\]
where \( V_{N_m}^{(3)} = V_m^{(3)} + V_{m+1}^{(3)} i + V_{m+2}^{(3)} j + V_{m+3}^{(3)} k \).
\[\square\]

Theorem 2.4. Let \( J_{N_m}^{(3)} \) and \( j_{N_m}^{(3)} \) be the \( m \)-th terms of the dual third-order Jacobsthal quaternion sequence \( \{J_{N_m}^{(3)}\}_{m \geq 0} \) and the dual third-order Jacobsthal-Lucas quaternion sequence \( \{j_{N_m}^{(3)}\}_{m \geq 0} \), respectively. Then, we have
\[
j_{N_m}^{(3)} J_{N_m}^{(3)} - J_{N_m}^{(3)} j_{N_m}^{(3)} = 2 (J_m^{(3)} j_{N_m}^{(3)} - j_m^{(3)} J_{N_m}^{(3)}),
\]
(2.18)
\[
j_{N_m}^{(3)} j_{N_m}^{(3)} + \overline{J_{N_m}^{(3)}} \overline{J_{N_m}^{(3)}} = 2 j_m^{(3)} J_{N_m}^{(3)}.
\]
(2.19)

Proof. (2.18): By the Eqs. (2.4) and (2.6), we get
\[
j_{N_m}^{(3)} J_{N_m}^{(3)} - J_{N_m}^{(3)} j_{N_m}^{(3)}
\]
\[
= (j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k)(J_m^{(3)} - J_{m+1}^{(3)} i - J_{m+2}^{(3)} j - J_{m+3}^{(3)} k)
\]
\[\quad - (j_m^{(3)} - j_{m+1}^{(3)} i - j_{m+2}^{(3)} j - j_{m+3}^{(3)} k)(J_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k)
\]
\[
= 2j_m^{(3)} (j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k) - 2j_m^{(3)} (j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k)
\]
\[
= 2 (J_m^{(3)} j_{N_m}^{(3)} - j_m^{(3)} J_{N_m}^{(3)}).
\]
\[ jN_m^{(3)} JN_m^{(3)} + \overline{jN_m^{(3)} JN_m^{(3)}} = (j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k)(J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k) + (J_m^{(3)} - J_{m+1}^{(3)} i - J_{m+2}^{(3)} j - J_{m+3}^{(3)} k)(j_m^{(3)} - j_{m+1}^{(3)} i - j_{m+2}^{(3)} j - j_{m+3}^{(3)} k) = J_m^{(3)} J_m^{(3)} + (j_m^{(3)} J_{m+1}^{(3)} + J_{m+1}^{(3)} J_m^{(3)} i + (j_m^{(3)} J_{m+2}^{(3)} + J_{m+2}^{(3)} J_m^{(3)} j) + (J_m^{(3)} J_{m+3}^{(3)} + J_{m+3}^{(3)} J_m^{(3)} k) + j_m^{(3)} J_m^{(3)} - (j_m^{(3)} J_{m+1}^{(3)} + J_{m+1}^{(3)} J_m^{(3)} i - (j_m^{(3)} J_{m+2}^{(3)} + J_{m+2}^{(3)} J_m^{(3)} j) - (J_m^{(3)} J_{m+3}^{(3)} + J_{m+3}^{(3)} J_m^{(3)} k) = 2j_m^{(3)} JN_m^{(3)}. \]

\[ \square \]

**Theorem 2.5 (Binet’s Formulas).** Let \( JN_m^{(3)} \) and \( jN_m^{(3)} \) be \( m \)-th terms of the dual third-order Jacobsthal quaternion sequence \( \{JN_m^{(3)}\}_{m \geq 0} \) and the dual third-order Jacobsthal-Lucas quaternion sequence \( \{jN_m^{(3)}\}_{m \geq 0} \), respectively. For \( m \geq 0 \), the Binet’s formulas for these quaternions are as follows:

\[ JN_m^{(3)} = \frac{2}{7} 2^m \alpha - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^m - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^m \]

(2.20)

and

\[ jN_m^{(3)} = \frac{8}{7} 2^m \alpha + \frac{3 + 2i\sqrt{3}}{7} \omega_1 \omega_1^m + \frac{3 - 2i\sqrt{3}}{7} \omega_2 \omega_2^m \]

(2.21)

respectively, where \( V_m^{(3)} \) is the sequence defined by

\[ V_m^{(3)} = \begin{cases} 
2 - 3i + j + 2k & \text{if } m \equiv 0 \pmod{3} \\
-3 + i + 2j - 3k & \text{if } m \equiv 1 \pmod{3} \\
1 + 2i - 3j + k & \text{if } m \equiv 2 \pmod{3}
\end{cases} \]

(2.22)

\[ \alpha = 1 + 2i + 4j + 8k \text{ and } \omega_{1,2} = 1 + \omega_{1,2} i + \omega_{1,2}^2 j + k. \]
Proof. Repeated use of (1.18) in (2.1) enables one to write for $\alpha = 1 + 2i + 4j + 8k$ and $\omega_{1,2} = 1 + \omega_{1,2}i + \omega_{1,2}^2j + k$

$$J_{N_1}^{(3)} = J_m^{(3)} + J_{m+1}^{(3)}i + J_{m+2}^{(3)}j + J_{m+3}^{(3)}k$$

$$= \frac{2}{7} 2^n - \frac{3 + 3i\sqrt{3}}{21} \omega_1^n - \frac{3 - 3i\sqrt{3}}{21} \omega_2^n$$

$$+ \left( \frac{2}{7} 2^{m+1} - \frac{3 + 3i\sqrt{3}}{21} \omega_1^{m+1} - \frac{3 - 3i\sqrt{3}}{21} \omega_2^{m+1} \right) i$$

$$+ \left( \frac{2}{7} 2^{m+2} - \frac{3 + 3i\sqrt{3}}{21} \omega_1^{m+2} - \frac{3 - 3i\sqrt{3}}{21} \omega_2^{m+2} \right) j$$

$$+ \left( \frac{2}{7} 2^{m+3} - \frac{3 + 3i\sqrt{3}}{21} \omega_1^{m+3} - \frac{3 - 3i\sqrt{3}}{21} \omega_2^{m+3} \right) k$$

$$= \frac{1}{7} \alpha^{2m+1} - \frac{3 + 3i\sqrt{3}}{21} \omega_1 \omega^m - \frac{3 - 3i\sqrt{3}}{21} \omega_2 \omega^m$$

(2.23)

and similarly making use of (1.19) in (2.2) yields

$$J_{N_2}^{(3)} = J_m^{(3)} + J_{m+1}^{(3)}i + J_{m+2}^{(3)}j + J_{m+3}^{(3)}k$$

$$= \frac{1}{7} \alpha^{2m+3} + \frac{3 + 3i\sqrt{3}}{7} \omega_1 \omega^m + \frac{3 - 3i\sqrt{3}}{7} \omega_2 \omega^m.$$

(2.24)

The formulas in (2.23) and (2.24) are called as Binet’s formulas for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions, respectively. Using notation in (2.22), we obtain the results (2.20) and (2.21). □

Theorem 2.6 (D’Ocagne-like Identity). Let $J_{N_1}^{(3)}$ be the $m$-th terms of the dual third-order Jacobsthal quaternion sequence $\{J_{N_1}^{(3)}\}_{m \geq 0}$. In this case, for $n \geq m \geq 0$, the d’Ocagne identities for $J_{N_1}^{(3)}$ is as follows:

$$J_{N_1}^{(3)} J_{N_1}^{(3)} - J_{N_{n+1}}^{(3)} J_{N_1}^{(3)} = \frac{1}{7} \left( \alpha (2^{n+1} U_{N_{m+1}}^{(3)} - 2^{m+1} U_{N_{n+1}}^{(3)}) + (1 - i - j + 2k) U_{n-m}^{(3)} \right),$$

(2.25)

$$\left( J_{N_{m+1}}^{(3)} \right)^2 - J_{N_{m+1}}^{(3)} J_{N_1}^{(3)} = \frac{1}{7} \left( 2^{m+1} \alpha (2 U_{N_{m+1}}^{(3)} - U_{N_{m+2}}^{(3)}) + (1 - i - j + 2k) \right),$$

(2.26)

where $U_{N_{m+1}}^{(3)} = \frac{1}{7} (2 V_{N_{m+1}}^{(3)} - V_{N_{m+1}}^{(3)})$ and $\alpha = 1 + 2i + 4j + 8k$. 


Proof. (2.25): Using Eqs. (2.20) and (2.22), we get

\[ JN_n^{(3)} JN_{m+1}^{(3)} - JN_{n+1}^{(3)} JN_m^{(3)} \]

\[ = \frac{1}{49} \left( (2^{n+1} - V N_n^{(3)})(2^{m+2} - V N_{m+1}^{(3)}) \right) \]

\[ - \frac{1}{49} \left( 2^{n+1} V N_{m+1}^{(3)} - 2^{m+2} V N_n^{(3)} + V N_n^{(3)} V N_{m+1}^{(3)} \right) \]

\[ = \frac{1}{7} \left( (1 + 2i + 4j + 8k)(2^{n+1} V N_{m+1}^{(3)} - 2^{m+1} V N_n^{(3)}) \right) \]

\[ + (1 - i - j + 2k) U_n^{(3)} \]

where \( UN_{m+1}^{(3)} = \frac{1}{7}(2V N_{m+1}^{(3)} - V N_m^{(3)}) \) and \( V N_m^{(3)} \) as in (2.22). In particular, if \( n = m + 1 \) in Eq. (2.27), we obtain for \( m \geq 0 \),

\[ (JN_{m+1}^{(3)})^2 - JN_{m+2}^{(3)} JN_m^{(3)} = \frac{1}{7} \left( (1 + 2i + 4j + 8k)2^{m+1}(2UN_{m+1}^{(3)} - UN_m^{(3)}) \right) + (1 - i - j + 2k) U_n^{(3)} \]

We will give an example in which we check in a particular case the Cassini-like identity for dual third-order Jacobsthal quaternions.

Example 2.7. Let \( JN_0^{(3)}, JN_1^{(3)}, JN_2^{(3)} \) and \( JN_3^{(3)} \) be the dual third-order Jacobsthal quaternions such that \( JN_0^{(3)} = i + j + 2k, JN_1^{(3)} = 1 + i + 2j + 5k, JN_2^{(3)} = 1 + 2i + 5j + 9k \) and \( JN_3^{(3)} = 2 + 5i + 9j + 18k \). In this case,

\[ \left( JN_1^{(3)} \right)^2 - JN_2^{(3)} JN_0^{(3)} = (1 + i + 2j + 5k)^2 - (1 + 2i + 5j + 9k)(i + j + 2k) \]

\[ = (1 + 2i + 4j + 10k) - (i + j + 2k) \]

\[ = 1 + i + 3j + 8k \]

\[ = \frac{1}{7} \left( 2(1 + 2i + 4j + 8k)(2UN_1^{(3)} - UN_2^{(3)}) \right) + (1 - i - j + 2k) \]

and

\[ \left( JN_2^{(3)} \right)^2 - JN_3^{(3)} JN_1^{(3)} = (1 + 2i + 5j + 9k)^2 - (2 + 5i + 9j + 18k)(1 + i + 2j + 5k) \]

\[ = (1 + 4i + 10j + 18k) - (2 + 7i + 13j + 28k) \]

\[ = -1 - 3i - 3j - 10k \]

\[ = \frac{1}{7} \left( 4(1 + 2i + 4j + 8k)(2UN_2^{(3)} - UN_3^{(3)}) \right) + (1 - i - j + 2k) \]
3. Conclusions

There are two differences between the dual third-order Jacobsthal and the dual coefficient third-order Jacobsthal quaternions. The first one is as follows: the dual coefficient third-order Jacobsthal quaternionic units are $i^2 = j^2 = k^2 = ijk = -1$ whereas the dual third-order Jacobsthal quaternionic units are $i^2 = j^2 = k^2 = 0$, $ij = -ji = jk = -kj = ik = -ki = 0$. The second one is as follows: the elements of the dual coefficient third-order Jacobsthal quaternion are $J^{(3)}_m + \varepsilon J^{(3)}_{m+1}$ ($\varepsilon^2 = 0$, $\varepsilon \neq 0$) whereas the elements of the dual third-order Jacobsthal quaternions are $m$-th third-order Jacobsthal number $J^{(3)}_m$.

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