Parameter identification for nonlocal phase field models for tumor growth via optimal control and asymptotic analysis

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Abstract

We introduce the problem of parameter identification for a coupled nonlocal Cahn-Hilliard-reaction-diffusion PDE system stemming from a recently introduced tumor growth model. The inverse problem of identifying relevant parameters is studied here by relying on techniques from optimal control theory of PDE systems. The parameters to be identified play the role of controls, and a suitable cost functional of tracking-type is introduced to account for the discrepancy between some a priori knowledge of the parameters and the controls themselves. The analysis is carried out for several classes of models, each one depending on a specific relaxation (of parabolic or viscous type) performed on the original system. First-order necessary optimality conditions are obtained on the fully relaxed system, in both the two and three-dimensional case. Then, the optimal control problem on the non-relaxed models is tackled by means of asymptotic arguments, by showing convergence of the respective adjoint systems and the minimization problems as each one of the relaxing coefficients vanishes. This allows obtaining the desired necessary optimality conditions, hence to solve the parameter identification problem, for the original PDE system in case of physically relevant double-well potentials.

Key words: tumor growth, Cahn-Hilliard equation, parameter identification, inverse problem, optimal control, well-posedness, asymptotic analysis.

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1 Introduction

One of the main examples of complex systems studied nowadays in both the biomedical and the mathematical literature refers to tumor growth processes. In particular, there has been a recent surge in the development of phase field models for tumor growth. These models are one of the main examples of complex systems: they describe the evolution of a tumor mass surrounded by healthy tissues by taking into account biological mechanisms such as proliferation of cells via
nutrient consumption, apoptosis, chemotaxis, and active transport of specific chemical species. In this setting, the evolution of the tumor is described by means of an order parameter $\varphi$ which represents the local concentration of tumoral cells. The interface between the tumoral and healthy cells is considered to be a (narrow) layer separating the regions where $\varphi = \pm 1$, with $\varphi = 1$ being the tumor phase and $\varphi = -1$ being the healthy phase.

The representation of the tumor growth process is given here by a Cahn-Hilliard equation (cf. [5]) with non zero mass source for $\varphi$, coupled with a reaction-diffusion equation for the nutrient $\sigma$ (cf., e.g., [15, 39, 44, 45]). We just mention the fact that more sophisticated models may distinguish between different tumor phases (e.g., proliferating and necrotic), or, treating the cells as inertia-less fluids, include the effects of fluid flow into the evolution of the tumor, leading to (possibly multiphase) Cahn-Hilliard-Darcy or Cahn-Hilliard-Brinkman systems [16, 17, 22–24, 28, 33, 35, 36, 39, 61].

In this paper we study the inverse problem of parameters identification for a nonlocal version of a recently introduced phase field model of tumor growth (cf. [39, 53]). The nonlocality of the system consists in the presence of a convolution term in the Cahn-Hilliard equation (see (1.7) below), which comes from the variational derivative of the following nonlocal Helmholtz free energy functional (cf. [41–43])

$$E_{\text{nonloc}}(\varphi) := \frac{1}{4} \int_{\Omega \times \Omega} J(|x - y|)|\varphi(x) - \varphi(y)|^2 \, dx \, dy + \int_{\Omega} F(\varphi(x)) \, dx. \tag{1.1}$$

This replaces the standard local Ginzburg-Landau free energy

$$E_{\text{loc}}(\varphi) := \frac{1}{2} \int_{\Omega} |\nabla \varphi(x)|^2 \, dx + \int_{\Omega} F(\varphi(x)) \, dx. \tag{1.2}$$

Here $J$ stands for a spatial convolution kernel, such as the classical Bessel or Newtonian potentials, while typical choices for the non-convex interaction potentials $F$ are

$$F_{\text{pol}}(r) := \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{1.3}$$

$$F_{\text{log}}(r) := \frac{\vartheta}{2} \left[ (1 + r) \ln(1 + r) + (1 - r) \ln(1 - r) \right] - \frac{\vartheta_0}{2} r^2, \quad r \in (-1, 1), \quad 0 < \vartheta < \vartheta_0, \tag{1.4}$$

$$F_{\text{dob}}(r) := \begin{cases} c(1 - r^2) & \text{if } r \in [-1, 1], \\ +\infty & \text{otherwise,} \end{cases} \quad c > 0. \tag{1.5}$$

Let us refer to [32] and reference therein for a description of the state of the art on nonlocal Cahn-Hilliard equations and to the recent contributions [18–20, 49] for the results concerning the rigorous passage from nonlocal to local Cahn-Hilliard equations when the kernel suitably peaks around zero.

Here, $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is the space domain where the evolution takes place, and $T > 0$ is a fixed final time. We consider the following nonlocal state system, deduced starting from the nonlocal free energy functional (1.1), which has been recently studied from the well-posedness point of view in [53]:

$$\partial_t (\varepsilon \mu + \varphi) - \Delta \mu = (\mathcal{P} \sigma - \mathcal{A}) f(\varphi) \quad \text{in } Q := \Omega \times (0, T), \tag{1.6}$$

$$\mu = \tau \partial_t \varphi + a \varphi - J * \varphi + F'(\varphi) - \chi \sigma \quad \text{in } Q, \tag{1.7}$$

$$\partial_t \sigma - \Delta \sigma + \mathcal{B}(\sigma - \sigma_S) + \mathcal{C} \sigma f(\varphi) = -\eta \Delta \varphi \quad \text{in } Q, \tag{1.8}$$

$$\partial_n \mu = \eta \partial_n \varphi = \partial_n \sigma = 0 \quad \text{on } \Sigma := \partial \Omega \times (0, T), \tag{1.9}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{1.10}$$
Referring to [39] for more details on the physical meaning of the involved parameters, let us precise our notation here. The variable $\sigma$ denotes the the concentration of the surrounding nutrient, where $\sigma = 1$ stands for a nutrient-rich concentration while $\sigma = 0$ for a poor one. The variable $\mu$ represents the chemical potential associated to the tumor phase-field variable $\varphi$ and to the nutrient proportion $\sigma$. Moreover, we use the symbols $n$ and $\partial_n$ for the outer normal unit vector to $\partial \Omega$ and the outer normal derivative, respectively. The constants $P, A, B, C$ are nonnegative real parameters taking into account the proliferation rate of the tumor cells, the apoptosis rate, the consumption rate of the nutrient with respect to a pre-existing concentration $\sigma_S$, and the nutrient consumption rate, respectively. Moreover, the nonnegative constants $\lambda$ and $\eta$ model chemotaxis and active transport, respectively. The approximation coefficients $\tau$ and $\varepsilon$ represent a viscosity and a parabolic regularization of the Cahn-Hilliard equation, respectively.

In biological models, nonlocal interactions have been used to describe competition for space and degradation [60], spatial redistribution [4, 8–14, 22, 23, 26, 34, 35, 37–39, 45, 46, 51, 54–58] and also cell-to-cell adhesion [1, 7, 40]. The corresponding version is still at its beginning: we can indeed quote only the few very recent contributions [27, 29–31, 53].

For the nonlocal Cahn-Hilliard equation with source terms, analytic results such as well-posedness and long-time behavior have been obtained in [21, 50] for prescribed source terms or Lipschitz source terms depending on the order parameter. The aim of [53] was to extend the study of the nonlocal Cahn-Hilliard equation to the case where source terms are coupled with another variable: the nutrient concentration $\sigma$. In [53] the authors prove well-posedness and regularity of solutions of (1.6)–(1.10) when $\tau$ and $\varepsilon$ are positive, moreover they let them tend to zero and prove existence of solutions of the corresponding limit problem.

The contributions to the mathematical literature on local diffuse interface tumor growth models is quite wide (cf, e.g., [6, 8–14, 22, 23, 26, 34, 35, 37–39, 45, 46, 51, 54–58]) and it is devoted to different mathematical questions: not only are these related to well-posedness, but they also focus on long-time behavior of solutions and optimal control, possibly including more accurate effects in the model, like stochastic ones [52]. Nonetheless, the study of the nonlocal corresponding version is still at its beginning: we can indeed quote only the few very recent contributions [27, 29, 31, 53].

Here we aim to continue the study pursued in [53] investigating the inverse problem of parameter identification in (1.6)–(1.10) by means of optimal control theory. In order to do this we need to introduce a cost functional we are going to minimize. This technique has been already used in the framework of diffuse interface tumor growth models in [29, 46, 47], where the problem of parameters identification has been tackled with similar methods for the corresponding local tumor growth model (in [46, 47] and for a nonlocal model introduced in [44] (in [29]).

The parameters that we choose to identify are $P$, $\chi$, $\eta$, and $C$. The main idea is to introduce a cost functional of standard tracking-type form as

$$J(\varphi, P, \chi, \eta, C) := \frac{\beta_\Omega}{2} \| \varphi(T) - \varphi_\Omega \|_{L^2(\Omega)}^2 + \frac{\beta_Q}{2} \| \varphi - \varphi_Q \|_{L^2(Q)}^2$$

$$+ \frac{\alpha_\rho}{2} | P - P_* |^2 + \frac{\alpha_\chi}{2} | \chi - \chi_* |^2 + \frac{\alpha_\eta}{2} | \eta - \eta_* |^2 + \frac{\alpha_C}{2} | C - C_* |^2,$$  

for some prescribed target functions $\varphi_\Omega : \Omega \to \mathbb{R}$, $\varphi_Q : Q \to \mathbb{R}$, and some nonnegative weights $\beta_\Omega, \beta_Q, \alpha_\rho, \alpha_\chi, \alpha_\eta, \alpha_C$ (not all zero). The nonnegative constants $P$, $\chi$, $\eta$, $C$ represent instead some a priori knowledge for the parameters. Moreover, the set of admissible controls is defined as

$$U_{ad} := \{ (P, \chi, \eta, C) \in \mathbb{R}^4 : 0 \leq P \leq P_{\max}, 0 \leq \chi \leq \chi_{\max}, 0 \leq \eta \leq \eta_{\max}, 0 \leq C \leq C_{\max} \},$$

(1.12)
for some prescribed nonnegative constants \( P_{\text{max}}, \chi_{\text{max}}, \eta_{\text{max}}, C_{\text{max}} \). It is worth underlying that 
\[ \{1.12\} \] is a nonempty compact (closed and bounded) subset of \( \mathbb{R}^4 \).

The identification problem we are going to address in this work can be summarized as
\[
(CP)_{\varepsilon, \tau} \quad \text{Minimize} \; J(\varphi, P, \chi, \eta, C) \; \text{subject to:} \\
\quad (i) \; (\varphi, \mu, \sigma) \; \text{yields a solution to} \; (1.6)\text{–}(1.10); \\
\quad (ii) \; (P, \chi, \eta, C) \in \mathcal{U}_{\text{ad}}.
\]

More specifically, we would like to tackle the following problem: given a set of data \( P^*, \chi^*, \eta^*, C^* \) representing some a priori knowledge of the parameters, identify the optimal parameter values \( P, \chi, \eta, C \) so that the resulting model predictions and the data are close in the sense defined in the cost functional \[1.11\]. An alternative approach for parameter identification relies on the Bayesian calibration which has been recently used in [47] in the framework of local tumor growth models.

The second part of the paper concerns the asymptotic behavior of \((CP)_{\varepsilon, \tau}\) as \( \varepsilon \) and/or \( \tau \) approach zero in the state system above. In this direction, we will employ the symbols \((CP)_\varepsilon\), \((CP)_\tau\), and \((CP)\) for the corresponding optimal control problems in which \( \tau = 0 \), \( \varepsilon = 0 \) and \( \varepsilon = \tau = 0 \) in the order. For instance, we have
\[
(CP)_\varepsilon \quad \text{Minimize} \; J(\varphi, P, \chi, C) \; \text{subject to:} \\
\quad (i) \; (\varphi, \mu, \sigma) \; \text{yields a solution to} \; (1.6)\text{–}(1.10) \; \text{with} \; \tau = 0; \\
\quad (ii) \; (P, \chi, \eta, C) \in \mathcal{U}_{\text{ad}}.
\]

The problems \((CP)_\tau\) and \((CP)\) are defined analogously. Different requirements on the structural data are in order, depending on the asymptotic study under consideration. Precise assumptions are rigorously stated in the sections below.

The main idea to solve the limit identification problems \((CP)_\varepsilon\), \((CP)_\tau\), and \((CP)\) is the following. We know from [53] that the fully relaxed state system \[1.6\text{–}1.10\] converges to the corresponding limiting one, as \( \tau \) and/or \( \varepsilon \) vanishes. Based on this, we introduce a suitable approximating cost functional in such a way that also the respective adjoint systems converge in some sense with respect to \( \tau \) and/or \( \varepsilon \). This allows to pass to the limit in the first-order conditions for optimality by letting \( \tau \) and/or \( \varepsilon \) go to 0, and deduce the corresponding first-order conditions also for the limiting models. The main mathematical issue concerns the nonuniqueness of optimal controls: this is a consequence of the highly nonlinear nature of the problem, and may give some difficulties in the convergence of optimal controls. Indeed, it not necessarily true that optimal controls at \( \varepsilon, \tau > 0 \) converge to optimal controls at \( \tau = 0 \) and/or \( \varepsilon = 0 \). To overcome this issue, the cost functional at the approximated level has to be carefully adapted in order to recover, among the other things, convergence of the optimal controls.

**Plan of the paper.** In Section 2 we state the assumptions on the problem data and resume previous results. In Section 3 we study the optimization problem \((CP)_{\varepsilon, \tau}\) as \( \tau \) and \( \varepsilon \) are strictly positive and prove necessary optimality conditions, whereas in Section 4 we solve the optimization problems \((CP)_\varepsilon\), \((CP)_\tau\), and \((CP)\) through asymptotic arguments.

### 2 Assumptions and previous results

#### 2.1 Assumptions

Throughout the paper, \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) is a smooth bounded domain with boundary \( \Gamma \), and \( T > 0 \) is a fixed final time. For every \( t \in [0, T] \), we use the classical notation
\[
Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t), \quad Q^T_t := \Omega \times (t, T),
\]
and put for convenience
\[ Q := Q_T, \quad \Sigma := \Sigma_T. \]

For any Banach space \( X \), we denote its dual space by \( X^* \), the associated duality pairing by \( \langle \cdot, \cdot \rangle \) and if \( X \) is a Hilbert space, we denote its inner product by \( (\cdot, \cdot)_X \). For \( 1 \leq p \leq \infty \) and \( k \geq 0 \) we denote the usual Lebesgue and Sobolev spaces on \( \Omega \) by \( L^p(\Omega) \) and \( W^{k,p}(\Omega) \), along with the corresponding norms \( \| \cdot \|_p \) and \( \| \cdot \|_{W^{k,p}(\Omega)} \). For the case \( p = 2 \), these spaces become Hilbert spaces and we use the notation \( H^k(\Omega) = W^{k,2}(\Omega) \). Then, we define the functional spaces
\[ H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{ v \in H^2(\Omega) : \partial_n v = 0 \text{ a.e. on } \partial \Omega \}, \]
endowed with their natural norms \( \| \cdot \| := \| \cdot \|_2, \| \cdot \|_V, \) and \( \| \cdot \|_W \). The space \( H \) will be identified to its dual, as usual, so that we have the following continuous, dense, and compact embeddings:
\[ W \hookrightarrow V \hookrightarrow H \hookrightarrow V^*. \]

Our starting point to solve the above-mentioned optimal control problems are the well-posedness results of \([1.6] - [1.10]\) obtained in \([53]\), as well as the related addressed asymptotic analyses. In this direction, we postulate the following structural assumptions.

**A1** \( A, B \) and \( P_{\max}, \chi_{\max}, \eta_{\max}, C_{\max} \) are nonnegative constants.

**A2** \( f : \mathbb{R} \to [0, +\infty) \) is bounded and Lipschitz continuous.

**A3** \( \sigma_S \in L^\infty(Q) \) with \( 0 \leq \sigma_S(x, t) \leq 1 \) for a.e. \((x, t) \in Q\).

**A4** \( F : (-\ell, \ell) \to [0, +\infty) \) is of class \( C^4 \), where \( \ell \in (0, +\infty] \), and
\[ F'(0) = 0, \quad \lim_{r \to (\pm \ell)^+} \left[ F'(r) - \chi_{\max} \eta_{\max} r \right] = \pm \infty. \]

Note that the latter condition allows both for the logarithmic potential and for any polynomial super-quadratic potential. Nonetheless, potentials of double-obstacle type are excluded. Let us note also that the limiting condition at \( \pm \ell \) is satisfied in particular by any \( \eta \in [0, \eta_{\max}] \) and \( \chi \in [0, \chi_{\max}] \).

**A5** The kernel \( J \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) \) is even, i.e. \( J(x) = J(-x) \) for almost every \( x \in \mathbb{R}^d \). For any measurable \( v : \Omega \to \mathbb{R} \) we employ the notation
\[ (J * v)(x) := \int_\Omega J(x - y)v(y) \, dy, \quad x \in \Omega, \]
and set \( a := J * 1 \). Moreover, we assume that
\[ a_* := \inf_{x \in \Omega} \int_\Omega J(x - y) \, dy = \inf_{x \in \Omega} a(x) \geq 0, \]
\[ a^* := \sup_{x \in \Omega} \int_\Omega |J(x - y)| \, dy < +\infty, \quad b^* := \sup_{x \in \Omega} \int_\Omega |\nabla J(x - y)| \, dy < +\infty. \]

We set \( c_a := \max\{a^* - a_*, 1\} \) and we suppose that there is a constant \( C_0 > 0 \) such that
\[ a_* + F''(r) \geq C_0 \quad \forall r \in (-\ell, \ell). \]

Let us stress that this condition allows for non-convex potentials \( F \), as \( a_* \) might be strictly greater than \( C_0 \).
A6 The positive constants $\varepsilon$ and $\tau$ are such that $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (0, \tau_0)$, where $\varepsilon_0$ and $\tau_0$ are defined as

$$\varepsilon_0 := \min \left\{ \frac{1}{4c_0}, \frac{1}{\max\{1, a^* - \min\{a^*, C_0\}\}}, \frac{2C_0}{3(a^* + b^*)^2} \right\}, \quad \tau_0 := 1.$$

This is only a technical requirement on the “smallness” of the perturbation parameters.

A7 The convolution kernel satisfies the additional regularity property:

either $J \in W^{2,1}(B_R)$ or $J$ is admissible in the sense of Definition 2.1.

where $B_R$ is the open ball in $\mathbb{R}^d$ of radius $R := \text{diam}(\Omega)$ centred at $0$, and

**Definition 2.1.** A convolution kernel $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ is said to be admissible if it satisfies:

(i) $J \in C^3(\mathbb{R}^d \setminus \{0\})$.

(ii) $J$ is radially symmetric, i.e. $J(\cdot) = \tilde{J}(|\cdot|)$ for a non-increasing $\tilde{J} : \mathbb{R} \to \mathbb{R}$.

(iii) there exists $R_0 > 0$ such that $r \mapsto \tilde{J}''(r)$ and $r \mapsto \tilde{J}'(r)/r$ are monotone on $(0, R_0)$.

(iv) there exists $C_d > 0$ such that $|D^3J(x)| \leq C_d|x|^{d-1}$ for every $x \in \mathbb{R}^d \setminus \{0\}$.

Let us recall that the $W^{2,1}$ regularity condition above, despite being probably the most natural, prevents some relevant cases of kernels such as the Newtonian or the Bessel potential from being considered. By contrast, these can be included by using the notion of admissible kernel: see [25] and [3] Def. 1] for details.

When dealing with the aforementioned optimal control problems, we postulate that the cost functional $J$ and the space of admissible controls $U_{\text{ad}}$ are defined by (1.11), and (1.12) and that the following are fulfilled.

C1 The target functions $\varphi_\Omega : \Omega \to \mathbb{R}$ and $\varphi_Q : Q \to \mathbb{R}$ verify $\varphi_\Omega \in L^2(\Omega)$ and $\varphi_Q \in L^2(Q)$.

C2 $\beta_\Omega, \beta_Q, \alpha_P, \alpha_\chi, \alpha_\eta, \alpha_C$ are nonnegative constants not all zero.

C3 $P_*, \chi_*, \eta_*, C_*$ are nonnegative constants.

C4 $f \in C_0^2(\mathbb{R})$, i.e. $f$ is twice differentiable and $f, f'$ and $f''$ are bounded.

### 2.2 Well-posedness of the state system

The assumptions [A1, A7] above allow to prove the strong well-posedness of system (1.6)–(1.10) and to discuss the asymptotic behavior of the system as $\varepsilon$ and $\tau$ go to zero, as we will recall below. Let us mention that in [53] weak well-posedness of the state system (1.6)–(1.10) is also addressed, under less stringent assumptions on the data. However, since in this framework we are interested in solving the optimal control problem (CP)$_{\varepsilon, \tau}$, we are forced to work with strong solutions instead, which possess better stability properties with respect to the involved parameters. In particular, unlike weak solutions, strong solutions allow to consider also positive chemotaxis and active transport coefficients: for further details, we refer to [53].

Let us recall here the the well-posedness of the state system (1.6)–(1.10) in the strong sense.

**Theorem 2.2** (Strong well-posedness of the state system: $\varepsilon, \tau > 0$). Assume conditions [A1–A7] let $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (0, \tau_0)$, and suppose that the initial data $(\varphi_0, \mu_0, \sigma_0)$ verify

$$\varphi_0 \in H^2(\Omega), \quad \mu_0, \sigma_0 \in V \cap L^\infty(\Omega), \quad \exists r_0 \in (0, \ell) : \|\varphi_0\|_{L^\infty(\Omega)} \leq r_0.$$

(2.13)
Then, for every \((\mathcal{P}, \chi, \eta, \mathcal{C})\) \in \mathcal{U}_{ad} there exists a unique strong solution \((\varphi, \mu, \sigma)\) such that

\[
\varphi \in W^{1,\infty}(0,T;V) \cap H^1(0,T;H^2(\Omega)), \quad \partial_t \varphi \in L^\infty(Q), \quad \eta \varphi \in L^2(0,T;W), \quad (2.14)
\]

\[
\exists r^* \in (r_0, \ell) : \quad \sup_{t \in [0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \leq r^*, \quad (2.15)
\]

\[
\mu, \sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \quad (2.16)
\]

which fulfills equations (1.6)–(1.10) pointwise in \(Q\). Furthermore, there exists a constant \(K > 0\), depending only on the structural data in A1–A7, on the initial data, and on \(\varepsilon\) and \(\tau\), such that

\[
\|\partial_t \varphi\|_{L^\infty(Q)} + \|\varphi\|_{W^{1,\infty}(0,T;V) \cap H^1(0,T;H^2(\Omega))} + \|\eta \varphi\|_{L^2(0,T;W)}
\]

\[
+ \|\mu\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q)} + \|\sigma\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q)} \leq K \quad \forall (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{ad}. \quad (2.17)
\]

Moreover, for any pair of initial data \(\{(\varphi^1_0, \mu^1_0, \sigma^1_0)\}_{i=1,2}\) satisfying (2.13) there exists a constant \(K > 0\), depending on the structural data, \(\varepsilon\), and \(\tau\), such that, for every pair of parameters \(\{(\mathcal{P}^i, \chi^i, \eta^i, \mathcal{C}^i)\}_{i=1,2} \in \mathcal{U}_{ad}\), and for any respective strong solutions \(\{(\varphi^i, \mu^i, \sigma^i)\}_{i=1,2}\) satisfying (2.14)–(2.16) it holds that

\[
\|\varphi^1 - \varphi^2\|_{W^{1,\infty}(0,T;V) \cap H^1(0,T;H^2(\Omega))} + \|\mu^1 - \mu^2\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)}
\]

\[
+ \|\sigma^1 - \sigma^2\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq K \left(\|\varphi^1_0 - \varphi^2_0\|_{H^2(\Omega)} + \|\mu^1_0 - \mu^2_0\|_{V} + \|\sigma^1_0 - \sigma^2_0\|_{V} \right)
\]

\[
+ K \left(\|\mathcal{P}^1 - \mathcal{P}^2\| + |\chi^1 - \chi^2| + |\eta^1 - \eta^2| + |\mathcal{C}^1 - \mathcal{C}^2| \right). \quad (2.18)
\]

The (strong) well-posedness presented in the theorem above allows us to define the so-called control-to-state operator \(S\), assigning to every given admissible control \((\mathcal{P}, \chi, \eta, \mathcal{C})\) the unique corresponding state. Namely, we have

\[
S : (\mathcal{P}, \chi, \eta, \mathcal{C}) \mapsto (\varphi, \mu, \sigma),
\]

where \((\varphi, \mu, \sigma)\) is the unique solution to (1.6)–(1.10) obtained from Theorem 2.2. Moreover, let us draw a straightforward consequence of the separation results (2.15) established above which will be useful later on.

**Corollary 2.3.** In addition to the assumptions of Theorem 2.2 suppose that \(F \in C^k(-\ell, \ell)\) for some \(k \geq 4\). Then, there exists a positive constant \(K\), depending only on the structural data in A1–A7, on the initial data, and on \(\varepsilon\) and \(\tau\), such that

\[
\|\varphi\|_{L^\infty(Q)} + \max_{i=0,\ldots,k} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} \leq K \quad \forall (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{ad}. \quad (2.18)
\]

### 2.3 Asymptotic behavior of the state system

Let us recollect here the known results established in [53] concerning the asymptotic behavior of the state system (1.6)–(1.10) as \(\varepsilon, \tau \to 0\). Despite part of the following results work also under more general assumptions on the potential (see [53]), in some cases it will be useful to introduce also the following assumption on the potential \(F\).

**A8** There exist two positive constants \(c_F\) and \(C_F\) such that

\[
|F''(r)| \leq C_F(1 + |r|^2), \quad F(r) \geq c_F|r|^4 - C_F \quad \forall r \in \mathbb{R}.
\]

It is worth noting that these conditions are met by the classical regular potential (1.3), whereas prevent the singular choices (1.4) and (1.5) to be considered.
Theorem 2.4 (Asymptotics: $\varepsilon \to 0$). Assume $\text{A1 A8}$ let $\tau \in (0, \tau_0)$, and suppose that

$$
(P_{\tau}, x_{\tau}, \eta_{\tau}, C_{\tau}) \in U_{ad}, \quad \eta_{\tau} = 0, \quad \varphi_{0,\tau} \in V, \quad F(\varphi_{0,\tau}) \in L^1(\Omega), \quad \sigma_{0,\tau} \in H, \quad 0 \leq \sigma_{0,\tau}(x) \leq 1 \quad \text{for a.e. } x \in \Omega. \quad (2.19)
$$

For $\varepsilon \in (0, \varepsilon_0)$, let the coefficients $(P_{\varepsilon,\tau}, x_{\varepsilon,\tau}, \eta_{\varepsilon,\tau}, C_{\varepsilon,\tau})$ and the initial data $(\varphi_{0,\varepsilon,\tau}, \mu_{0,\varepsilon,\tau}, \sigma_{0,\varepsilon,\tau})$ satisfy (2.19) and (2.20). Suppose also that, as $\varepsilon \to 0$, it holds

$$(P_{\varepsilon,\tau}, x_{\varepsilon,\tau}, \eta_{\varepsilon,\tau}, C_{\varepsilon,\tau}) \to (P_{\tau}, x_{\tau}, \eta_{\tau}, C_{\tau}),$$

$$\varphi_{0,\varepsilon,\tau} \to \varphi_{0,\tau} \quad \text{weakly in } V, \quad \sigma_{0,\varepsilon,\tau} \to \sigma_{0,\tau} \quad \text{strongly in } H,$$

and that there exists a constant $M_0 > 0$, independent of $\varepsilon$, such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon^{1/2} \|\mu_{0,\varepsilon,\tau}\| + \|F(\varphi_{0,\varepsilon,\tau})\|_{L^1(\Omega)} + \varepsilon^{1/4} (\|\mu_{0,\varepsilon,\tau}\|_V + \|\sigma_{0,\varepsilon,\tau}\|_V + \|F'(\varphi_{0,\varepsilon,\tau})\|) \leq M_0. \quad (2.22)$$

Denote by $(\varphi_{\varepsilon,\tau}, \mu_{\varepsilon,\tau}, \sigma_{\varepsilon,\tau})$ the respective unique strong solution to the system (1.6)–(1.10) obtained from Theorem 2.4 with respect to the the coefficients $(P_{\varepsilon,\tau}, x_{\varepsilon,\tau}, \eta_{\varepsilon,\tau}, C_{\varepsilon,\tau})$ and the initial data $(\varphi_{0,\varepsilon,\tau}, \mu_{0,\varepsilon,\tau}, \sigma_{0,\varepsilon,\tau})$. Then, there exists a unique triplet $(\varphi_\tau, \mu_\tau, \sigma_\tau)$, with

$$\varphi_\tau \in H^1(0, T; H) \cap L^\infty(0, T; V), \quad \mu_\tau \in L^2(0, T; V), \quad \sigma_\tau \in H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q),$$

$$0 \leq \sigma_\tau(x, t) \leq 1 \quad \text{for a.e. } x \in \Omega, \quad \forall t \in [0, T],$$

such that

$$\langle \partial_t \varphi_\tau, v \rangle + \int_\Omega \nabla \mu_\tau \cdot \nabla v = \int_\Omega (P_{\tau} \sigma_\tau - A) f(\varphi_\tau) v,$$

$$\langle \partial_t \sigma_\tau, w \rangle + \int_\Omega \nabla \sigma_\tau \cdot \nabla w + \mathcal{B} \int_\Omega (\sigma_\tau - \sigma_S) w + \mathcal{C} \int_\Omega \sigma_\tau f(\varphi_\tau) w = 0,$$

for every $v, w \in V$, almost everywhere in $(0, T)$, and

$$\mu_\tau = \tau \partial_t \varphi_\tau + a \varphi_\tau - J * \varphi_\tau + F'(\varphi_\tau) - \chi_\tau \sigma_\tau \quad \text{a.e. in } Q,$$

$$\varphi_\tau(0) = \varphi_{0,\tau}, \quad \sigma_\tau(0) = \sigma_{0,\tau} \quad \text{a.e. in } \Omega.$$

Moreover, as $\varepsilon \to 0$ it holds that

$$\begin{align*}
\varphi_{\varepsilon,\tau} &\to \varphi_\tau \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V), \\
\mu_{\varepsilon,\tau} &\to \mu_\tau \quad \text{strongly in } L^2(0, T; V), \\
\sigma_{\varepsilon,\tau} &\to \sigma_\tau \quad \text{weakly-* in } H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q), \\
\varepsilon \mu_{\varepsilon,\tau} &\to 0 \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V),
\end{align*} \quad (2.23)$$

and

$$\begin{align*}
\varphi_{\varepsilon,\tau} &\to \varphi_\tau \quad \text{strongly in } C^0([0, T]; H), \\
\sigma_{\varepsilon,\tau} &\to \sigma_\tau \quad \text{strongly in } L^\infty(0, T; H) \cap L^2(0, T; V). \quad (2.27)
\end{align*}$$

In a similar fashion we have the following result.

Theorem 2.5 (Asymptotics: $\tau \to 0$). Assume $\text{A1 A7}$ let $\varepsilon \in (0, \varepsilon_0)$, and suppose that

$$(P_\varepsilon, x_\varepsilon, \eta_\varepsilon, C_\varepsilon) \in U_{ad}, \quad 0 \leq \chi_\varepsilon < \sqrt{c_a}, \quad (\chi_\varepsilon + \eta_\varepsilon + 4c_a \chi_\varepsilon)^2 < 8c_a C_0 + 4 \chi_\varepsilon \eta_\varepsilon, \quad (2.29)$$

$$\varphi_{0,\varepsilon}, \mu_{0,\varepsilon}, \sigma_{0,\varepsilon} \in H, \quad F(\varphi_{0,\varepsilon}) \in L^1(\Omega). \quad (2.30)$$
For all $\tau \in (0, \tau_0)$, let the coefficients $(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau})$ and the initial data $(\varphi_{0,\varepsilon, \tau}, \mu_{0,\varepsilon, \tau}, \sigma_{0,\varepsilon, \tau})$ satisfy (2.29)−(2.30) and (2.13). Suppose also that, as $\tau \to 0$, it holds

$$
(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau}) \to (P_{\varepsilon}, \chi_{\varepsilon}, \eta_{\varepsilon}, C_{\varepsilon}),
$$

$$
\varphi_{0,\varepsilon, \tau} \to \varphi_{0, \varepsilon}, \quad \mu_{0,\varepsilon, \tau} \to \mu_{0, \varepsilon}, \quad \sigma_{0,\varepsilon, \tau} \to \sigma_{0, \varepsilon} \quad \text{strongly in } H,
$$

(2.31) and that there exists a constant $M_0 > 0$, independent of $\tau$, such that, for all $\tau \in (0, \tau_0)$,

$$
\tau^{1/2} \|\varphi_{0,\varepsilon, \tau}\|_V + \|F(\varphi_{0,\varepsilon, \tau})\|_{L^1(\Omega)} \leq M_0.
$$

(2.32)

Denote by $(\varphi_{\varepsilon, \tau}, \mu_{\varepsilon, \tau}, \sigma_{\varepsilon, \tau})$ the respective unique strong solution to the system (1.6)−(1.10) obtained from Theorem 2.2 with respect to the coefficients $(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau})$ and the initial data $(\varphi_{0,\varepsilon, \tau}, \mu_{0,\varepsilon, \tau}, \sigma_{0,\varepsilon, \tau})$. Then, there exists a triplet $(\varphi_{\varepsilon}, \mu_{\varepsilon}, \sigma_{\varepsilon})$, with

$$
\varphi_{\varepsilon}, \mu_{\varepsilon} \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \varepsilon \mu_{\varepsilon} + \varphi_{\varepsilon} \in H^1(0, T; V^*) \cap L^2(0, T; V),
$$

$$
\sigma_{\varepsilon} \in H^1(0, T; V^*) \cap L^2(0, T; V),
$$

such that

$$
\langle \partial_t (\varepsilon \mu_{\varepsilon} + \varphi_{\varepsilon}), v \rangle + \int_\Omega \nabla \mu_{\varepsilon} \cdot \nabla v = \int_\Omega (P_{\varepsilon} \sigma_{\varepsilon} - \mathcal{A}) f(\varphi_{\varepsilon}) v,
$$

$$
\langle \partial_t \sigma_{\varepsilon}, w \rangle + \int_\Omega \nabla \sigma_{\varepsilon} \cdot \nabla w + B \int_\Omega (\sigma_{\varepsilon} - \sigma_{\varepsilon}^S) w + C_{\varepsilon} \int_\Omega \sigma_{\varepsilon} f(\varphi_{\varepsilon}) w = \eta_{\varepsilon} \int_\Omega \nabla \varphi_{\varepsilon} \cdot \nabla w,
$$

for every $v, w \in V$, almost everywhere in $(0, T)$, and

$$
\mu_{\varepsilon} = a \varphi_{\varepsilon} - J * \varphi_{\varepsilon} + F'(\varphi_{\varepsilon}) - \chi_{\varepsilon} \sigma_{\varepsilon} \quad \text{a.e. in } Q,
$$

$$
\varphi_{\varepsilon}(0) = \varphi_{0,\varepsilon}, \quad \sigma_{\varepsilon}(0) = \sigma_{0,\varepsilon} \quad \text{a.e. in } \Omega.
$$

Moreover, as $\tau \to 0$, along a non-relabelled subsequence it holds that

$$
\varphi_{\varepsilon, \tau} \to \varphi_{\varepsilon} \quad \text{weakly-* in } L^\infty(0, T; H) \cap L^2(0, T; V),
$$

(2.33)

$$
\mu_{\varepsilon, \tau} \to \mu_{\varepsilon} \quad \text{weakly-* in } L^\infty(0, T; H) \cap L^2(0, T; V),
$$

(2.34)

$$\varepsilon \mu_{\varepsilon, \tau} + \varphi_{\varepsilon, \tau} \to \varepsilon \mu_{\varepsilon} + \varphi_{\varepsilon} \quad \text{weakly in } H^1(0, T; V^*) \cap L^2(0, T; V),
$$

(2.35)

$$\sigma_{\varepsilon, \tau} \to \sigma_{\varepsilon} \quad \text{weakly in } H^1(0, T; V^*) \cap L^2(0, T; V),
$$

(2.36)

$$\tau \varphi_{\varepsilon, \tau} \to 0 \quad \text{strongly in } H^1(0, T; H) \cap L^\infty(0, T; V),
$$

(2.37)

and

$$\varphi_{\varepsilon, \tau} \to \varphi_{\varepsilon} \quad \text{strongly in } L^2(0, T; H), \quad \mu_{\varepsilon, \tau} \to \mu_{\varepsilon} \quad \text{strongly in } L^2(0, T; H),
$$

(2.38)

$$\sigma_{\varepsilon, \tau} \to \sigma_{\varepsilon} \quad \text{strongly in } C^0([0, T]; V^*) \cap L^2(0, T; H).
$$

(2.39)

Furthermore, if also

$$\eta_{\varepsilon, \tau} = 0, \quad 0 \leq \sigma_{0,\varepsilon, \tau}(x) \leq 1 \quad \text{for a.e. } x \in \Omega \quad \forall \tau \in (0, \tau_0),
$$

then the solution $(\varphi_{\varepsilon}, \mu_{\varepsilon}, \sigma_{\varepsilon})$ of the system (1.6)−(1.10) with $\tau = 0$ is unique, all the convergences hold along the entire sequence $\tau \to 0$, and

$$0 \leq \sigma_{\varepsilon}(x, t) \leq 1 \quad \text{for a.e. } x \in \Omega, \quad \forall t \in [0, T].
$$

Lastly, the joint asymptotics is given in the following statement.
Theorem 2.6 (Asymptotics: $\varepsilon, \tau \to 0$). Assume $\text{A1} \text{A8}$ and suppose that

\begin{align*}
(P, \chi, \eta, C) &\in \mathcal{U}_{\text{ad}}, \quad \eta = 0, \quad 0 \leq \chi < \sqrt{c_0}, \quad (\chi + \eta + 4c_0\chi)^2 < 8c_0C_0 + 4\chi \eta, \quad (2.40) \\
\varphi_0, \sigma_0 &\in H, \quad F(\varphi_0) \in L^1(\Omega). \quad (2.41)
\end{align*}

For every $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (0, \tau_0)$, let the coefficients $(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau})$ and the initial data $(\varphi_{0, \varepsilon, \tau}, \mu_{0, \varepsilon, \tau}, \sigma_{0, \varepsilon, \tau})$ satisfy (2.40)–(2.41) and (2.13). Suppose also that, as $(\varepsilon, \tau) \to (0, 0)$, it holds

\begin{align*}
(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau}) &\to (P, \chi, \eta, C), \\
\varphi_{0, \varepsilon, \tau} &\to \varphi_0 \quad \text{strongly in } H, \quad \sigma_{0, \varepsilon, \tau} \to \sigma_0 \quad \text{strongly in } H, \quad (2.42)
\end{align*}

and that there exists a constant $M_0 > 0$, independent of $(\varepsilon, \tau)$, such that

\begin{align*}
\tau^{1/2}\|\varphi_{0, \varepsilon, \tau}\|_V + \varepsilon^{1/2}\|\mu_{0, \varepsilon, \tau}\| + \|F(\varphi_{0, \varepsilon, \tau})\|_{L^1(\Omega)} &\leq M_0 \quad \forall (\varepsilon, \tau) \in (0, \varepsilon_0) \times (0, \tau_0). \quad (2.43)
\end{align*}

Denote by $(\varphi_{\varepsilon, \tau}, \mu_{\varepsilon, \tau}, \sigma_{\varepsilon, \tau})$ the respective unique strong solution to the system (1.6)–(1.10) obtained from Theorem 2.3 with respect to the coefficients $(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau})$ and the initial data $(\varphi_{0, \varepsilon, \tau}, \mu_{0, \varepsilon, \tau}, \sigma_{0, \varepsilon, \tau})$. Then, there exists a unique triplet $(\varphi, \mu, \sigma)$, with

\begin{align*}
\varphi &\in H^1(0, T; V^*) \cap L^2(0, T; V), \quad \mu \in L^2(0, T; V), \\
\sigma &\in H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q), \quad 0 \leq \sigma(x, t) \leq 1 \quad \text{for a.e. } x \in \Omega, \forall t \in [0, T],
\end{align*}

such that

\begin{align*}
\langle \partial_t \varphi, v \rangle + \int_\Omega \nabla \mu \cdot \nabla v &= \int_\Omega (P\sigma - A)f(\varphi)v, \\
\langle \partial_t \sigma, w \rangle + \int_\Omega \nabla \sigma \cdot \nabla w &= \int_\Omega (\sigma - \sigma_s)w + \mathcal{C} \int_\Omega \sigma f(\varphi)w = 0
\end{align*}

for every $v, w \in V$, almost everywhere in $(0, T)$, and

\begin{align*}
\mu &= a\varphi - J^\ast \varphi + F'(\varphi) - \chi \sigma \quad \text{a.e. in } Q, \\
\varphi(0) &= \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{a.e. in } \Omega.
\end{align*}

Moreover, as $(\varepsilon, \tau) \to (0, 0)$ it holds that

\begin{align*}
\varphi_{\varepsilon, \tau} &\to \varphi \quad \text{weakly-* in } L^\infty(0, T; H) \cap L^2(0, T; V), \quad (2.44) \\
\mu_{\varepsilon, \tau} &\to \mu \quad \text{weakly in } L^2(0, T; V), \quad (2.45) \\
\varepsilon \mu_{\varepsilon, \tau} + \varphi_{\varepsilon, \tau} &\to \varphi \quad \text{weakly in } H^1(0, T; V^*) \cap L^2(0, T; V), \quad (2.46) \\
\sigma_{\varepsilon, \tau} &\to \sigma \quad \text{weakly-* in } H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q), \quad (2.47) \\
\varepsilon^{1/2} \mu_{\varepsilon, \tau} &\to 0 \quad \text{weakly-* in } L^\infty(0, T; H), \quad (2.48) \\
\varepsilon \mu_{\varepsilon, \tau} &\to 0 \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V), \quad (2.49) \\
\tau \varphi_{\varepsilon, \tau} &\to 0 \quad \text{strongly in } H^1(0, T; H) \cap L^\infty(0, T; V), \quad (2.50)
\end{align*}

and

\begin{align*}
\varphi_{\varepsilon, \tau} &\to \varphi \quad \text{strongly in } L^2(0, T; H), \quad (2.51) \\
\sigma_{\varepsilon, \tau} &\to \sigma \quad \text{strongly in } C^0([0, T]; V^*) \cap L^2(0, T; H). \quad (2.52)
\end{align*}
3 The optimization problem \((CP)_{\varepsilon,\tau}\)

This section is focused on the analysis of the optimal control problem \((CP)_{\varepsilon,\tau}\), when \(\varepsilon,\tau > 0\) are given. More specifically, we show existence of optimal controls and necessary first-order condition for optimality. Throughout the whole Section 3, we work with \(\varepsilon \in (0,\varepsilon_0)\) and \(\tau \in (0,\tau_0)\) fixed. For this reason, we omit for brevity notation for the dependence on \(\varepsilon,\tau\) for the variables in play.

3.1 Existence of a minimizer

The first problem that we address concerns the existence of a minimizer of the optimal control problem \((CP)_{\varepsilon,\tau}\), with \(\varepsilon, \tau > 0\) being fixed. Its proof is rather standard and follows as a consequence of the direct method of calculus of variations.

Theorem 3.1 (Existence of a minimizer). Suppose that \(A_1\), \(A_7\) and \(C_1\), \(C_3\) are fulfilled. Then, the optimization problem \((CP)_{\varepsilon,\tau}\) admits a minimizer.

Proof. Without loss of generality we assume that all the constants \(\alpha_P,\alpha_\chi,\alpha_\eta,\alpha_C\) are positive. In fact, if this is not the case, we can consider the corresponding control \(P,\chi,\eta\) and/or \(C\) as a prescribed constant, redefine \(U_{ad}\) accordingly and argue in the same way. To begin with, notice that the cost functional is nonnegative so that we can consider a minimising sequence of elements of \(U_{ad}\). Namely, we take the minimising sequence \(\{(\varphi_n,\mu_n,\sigma_n)\}_n :\{S(P_n,\chi_n,\eta_n,C_n)\}_n \subset U_{ad}\) and the corresponding sequence of states \(\{(\varphi_n,\mu_n,\sigma_n)\}_n :\{S(P_n,\chi_n,\eta_n,C_n)\}_n\) all related to the same initial data \((\varphi_0,\mu_0,\sigma_0)\). Namely, we have

\[
0 \leq \lambda := \inf \left\{ J(\varphi, P, \chi, \eta, C) \mid (P, \chi, \eta, C) \in U_{ad}, \varphi = S(P, \chi, \eta, C) \right\},
\]

and

\[
J(\varphi_n, P_n, \chi_n, \eta_n, C_n) \searrow \lambda,
\]

where \(S_1\) denotes the first component of the solution operator \(S\). Noting that the bounds \(2.17\) are uniform in \(n\) thanks to the structure of \(U_{ad}\), invoking standard compactness results (cf., e.g., [59], Sec. 8, Cor. 4) it is straightforward to obtain the existence of limits \(\varphi, (P, \chi, \eta, C) \in U_{ad}\), and a not relabelled subsequence such that, as \(n \to \infty\),

\[
\varphi_n \to \varphi \quad \text{weakly-* in } W^{1,\infty}(0,T;V) \cap H^1(0,T;H^2(\Omega)),
\]

and strongly in \(C^0([0,T];C^0(\Omega))\),

\[
\mu_n \to \mu \quad \text{weakly-* in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q),
\]

\[
\sigma_n \to \sigma \quad \text{weakly-* in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q),
\]

\[
P_n \to P, \quad \chi_n \to \chi, \quad \eta_n \to \eta, \quad C_n \to C.
\]

It is then a standard matter to pass to the limit in the variational formulation of \(1.6)–(1.10)\) written for \(\{(\varphi_n,\mu_n,\sigma_n)\}_n\) to deduce that \((\varphi,\mu,\sigma) = S(P,\chi,\eta,C)\). Lastly, the weak sequential lower semicontinuity of \(J\) entails that \((P,\chi,\eta,C)\) is a minimizer for \((CP)_{\varepsilon,\tau}\) together with its corresponding state \((\varphi,\mu,\sigma)\). \(\square\)

3.2 Linearized system

We study here the linearized system, which can be formally obtained by differentiating the state system \(1.6)–(1.10)\) with respect to the control in a certain direction.

First, we fix some preliminary notation: let \((P,\chi,\eta,C) \in U_{ad}\) be fixed, set

\[
(\varphi,\mu,\sigma) := S(P,\chi,\eta,C),
\]

\[
(\varphi,\mu,\sigma) := S(P,\chi,\eta,C),
\]
and consider an arbitrary increment

\[ h := (h_P, h_X, h_\eta, h_\mathcal{C}) \in \mathbb{R}^4 \quad \text{such that} \quad (P, \chi, \eta, \mathcal{C}) + h \in \mathcal{U}_{ad}. \]

The variables of the linearized system are denoted by \((\xi, \nu, \zeta)\): of course, they depend on the increment \(h\), but we avoid keeping track of this explicitly for brevity of notation.

The linearized system reads:

\[
\begin{align*}
\partial_t (\varepsilon \nu + \xi) - \Delta \nu &= (P\sigma - A)f'(\overline{\sigma})\xi + P\zeta f(\overline{\sigma}) + h_P \sigma f(\overline{\sigma}) \quad &\text{in } Q, \\
\nu &= \tau \partial_t \xi + a\xi - J * \xi + F''(\overline{\sigma})\xi - \chi \xi - h_X \sigma \quad &\text{in } Q, \\
\partial_t \zeta - \Delta \zeta + B\zeta + C(\zeta f(\overline{\sigma}) + \sigma f'(\overline{\sigma})\xi) + h_c \overline{\sigma} f(\overline{\sigma}) &= -\Delta(\eta \xi + h_\eta \overline{\sigma}) \quad &\text{in } Q, \\
\partial_\mathbf{n} \nu &= \partial_\mathbf{n}(\eta \xi + h_\eta \overline{\sigma}) = \partial_\mathbf{n} \zeta = 0 \quad &\text{on } \Sigma, \\
\nu(0) &= \xi(0) = \zeta(0) = 0 \quad &\text{in } \Omega.
\end{align*}
\]

**Theorem 3.2** (Well-posedness of the linearized system). Assume A1–A7 and C1–C4. Then, the linearized system \((3.53) - (3.57)\) admits a unique solution \((\xi, \nu, \zeta)\) satisfying

\[
\xi \in H^1(0, T; H) \cap L^\infty(0, T; V), \quad \nu, \zeta \in H^1(0, T; V^*) \cap L^2(0, T; V).
\]

**Proof of Theorem 3.2.** In what follows we proceed formally by pointing out some a priori estimates. Anyway, it is a standard matter to perform the same computations within a Galerkin scheme and then pass to the limit as the discretization parameter approach infinity to deduce the same results at the continuous level. Moreover, let us notice that the symbols \(M\) and \(M(\delta)\) will denote generic constants depending only on structural data and possibly on an additional positive constant \(\delta\) and may change from line to line.

**First estimate:** To begin with we add to both sides of \((3.54)\) the term \((c_a + 2)\xi\). Next, we multiply \((3.53)\) by \(\nu\), the new \((3.54)\) by \(-\partial_t \xi\), the gradient of \((3.54)\) by \(-\nabla \xi\), \((3.55)\) by \(\zeta\), integrate over \(Q_t\) and by parts. Adding the resulting equalities we obtain that

\[
\begin{align*}
\frac{\varepsilon}{2} \|\nu(t)\|^2 + \int_{Q_t} |\nabla \nu|^2 &+ \tau \int_{Q_t} |\partial_t \xi|^2 + \frac{\tau}{2} \|\nabla \xi(t)\|^2 + \left(\frac{\varepsilon}{2} + 1\right)\|\xi(t)\|^2 \\
&\quad + \int_{Q_t} (a + F''(\overline{\sigma}))|\nabla \xi|^2 + \int_{Q_t} (a\xi - J * \xi) \partial_t \xi + \frac{1}{2} ||\xi(t)||^2 + B \int_{Q_t} |\xi|^2 + \int_{Q_t} |\nabla \zeta|^2
\end{align*}
\]

\[
\begin{align*}
= &\int_{Q_t} (P\sigma - A)f'(\overline{\sigma})\xi\nu + \int_{Q_t} P\zeta f(\overline{\sigma})\nu + \int_{Q_t} h_P \sigma f(\overline{\sigma})\nu - \int_{Q_t} F''(\overline{\sigma})\xi \partial_t \xi \\
&\quad + \int_{Q_t} \chi \xi \partial_t \xi + \int_{Q_t} h_X \sigma \partial_t \xi - \int_{Q_t} (c_a + 2) \xi \partial_t \xi \\
&\quad + \int_{Q_t} \xi \nabla \nu \cdot \nabla \xi - \int_{Q_t} (\nabla a) \xi \cdot \nabla \xi + \int_{Q_t} (\nabla J * \xi) \cdot \nabla \xi - \int_{Q_t} F''(\overline{\sigma})\xi \overline{\sigma} \cdot \nabla \xi \\
&\quad + \int_{Q_t} \chi \nabla \zeta \cdot \nabla \xi + \int_{Q_t} h_X \nabla \sigma \cdot \nabla \xi - \int_{Q_t} C(\zeta f(\overline{\sigma}) + \sigma f'(\overline{\sigma})\xi) \zeta - \int_{Q_t} h_c \overline{\sigma} f(\overline{\sigma}) \zeta \\
&\quad + \int_{Q_t} \eta \nabla \xi \cdot \nabla \zeta - \int_{Q_t} h_\eta \Delta \overline{\sigma} \zeta.
\end{align*}
\]

- \(I_1\)
The first two terms of the second line can be easily estimated by using (A5), which produces
\[
\int_{Q_t} (a + F''(\varphi))|\nabla \xi|^2 \geq C_0 \int_{Q_t} |\nabla \xi|^2, \quad (3.58)
\]
and similarly for the next term we have
\[
\int_{Q_t} (a \xi - J \ast \xi) \partial_t \xi \geq \frac{a_s - a^*}{2} \|\xi(t)\|^2 \geq -\frac{ca}{2} \|\xi(t)\|^2. \quad (3.59)
\]
Moreover, the terms on the right-hand side can be easily estimated using Young and H"older inequalities, the regularity of the state variables $\varphi, \sigma$ expressed in (2.17), the boundedness of $f$ and $f'$, and Corollary 2.3. In fact, for every $\delta > 0$, we obtain that
\[
|I_1| \leq M \int_{Q_t} (|\varphi| |\xi| + |\xi|) |\nabla \xi| + M \|F''(\varphi)\|_{L^\infty(Q)} \int_{Q_t} |\xi| |\partial_t \xi|
\leq \delta \int_{Q_t} |\partial_t \xi|^2 + M(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2 + |\xi|^2 + 1),
\]
\[
|I_2| \leq M \int_{Q_t} (|\xi| + |\varphi| + |\xi|) |\partial_t \xi| \leq \delta \int_{Q_t} |\partial_t \xi|^2 + M(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2 + 1).
\]
Similarly, using the continuous embedding $V \hookrightarrow L^6(\Omega)$, H"older’s inequality, the regularity $\varphi$ and again Corollary 2.3 we find that
\[
|I_3| \leq M \int_{Q_t} (|\nabla \nu| + |\xi|) |\nabla \xi| + M \|F''(\varphi)\|_{L^\infty(Q)} \int_{Q_t} |\xi| |\nabla \varphi| |\nabla \xi|
\leq \delta \int_{Q_t} |\nabla \nu|^2 + M(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2) + M \int_0^t \|\varphi(s)\|_{H^2(\Omega)} \|\xi(s)\|_V |\nabla \xi(s)| ds
\leq \delta \int_{Q_t} |\nabla \nu|^2 + M(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2).
\]
By analogous computations, we have that
\[
|I_4| \leq M \int_{Q_t} (|\nabla \zeta| + |\nabla \varphi|) |\nabla \xi| + M \int_{Q_t} (|\zeta| + |\varphi| |\xi| + |\varphi|) |\zeta|
\leq \delta \int_{Q_t} |\nabla \zeta|^2 + M(\delta) \int_{Q_t} (|\nabla \xi|^2 + |\xi|^2 + |\zeta|^2 + 1),
\]
\[
|I_5| \leq M \int_{Q_t} |\nabla \xi| |\nabla \varphi| + M \int_{Q_t} |\Delta \varphi| |\zeta| \leq \delta \int_{Q_t} |\nabla \zeta|^2 + M(\delta) \int_{Q_t} (|\zeta|^2 + |\nabla \xi|^2 + 1).
\]
Upon collecting the above estimates, we infer that
\[
\frac{\varepsilon}{2} |\nu(t)|^2 + (1 - \delta) \int_{Q_t} |\nabla \nu|^2 + (\tau - 2\delta) \int_{Q_t} |\partial_t \xi|^2 + \|\xi(t)\|^2 + \frac{\tau}{2} |\nabla \xi(t)|^2 + C_0 \int_{Q_t} |\nabla \xi|^2 + \frac{1}{2} |\zeta(t)|^2 + B \int_{Q_t} |\zeta|^2 + (1 - 2\delta) \int_{Q_t} |\nabla \zeta|^2
\leq M(\delta) \int_{Q_t} (|\xi|^2 + |\nabla \xi|^2 + |\zeta|^2 + 1).
\]
Then, we choose $\delta := \min\{\frac{\varepsilon}{4}, \frac{1}{4}\}$ so that Gronwall’s lemma yields that
\[
\|\xi\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\nu\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\zeta\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq M.
\]
Second estimate: In light of the above estimate, a comparison argument in (3.53) and (3.55) produces
\[ \|\partial_t (\varepsilon \nu + \xi)\|_{L^2(0,T;V^*)} + \|\partial_t \zeta\|_{L^2(0,T;V^*)} \leq M, \]
hence also
\[ \|\partial_t \nu\|_{L^2(0,T;V^*)} \leq M. \]

Conclusion: It is clear that these estimates are enough to pass to the limit in the linearized system. Furthermore, it is worth noting that the uniqueness directly follows from the linearity of the system and the estimates above. The proof of Theorem 3.2 is then concluded. \(\square\)

3.3 Fréchet differentiability of \(S\)

Theorem 3.3 (Fréchet differentiability). Assume \(A1–A7\) and \(C1–C4\) and let \((P, \chi, \eta, C) \in U_{ad}\) be fixed. Then, the control-to-state operator \(S : \mathbb{R}^4 \rightarrow X\) is Fréchet differentiable at \((P, \chi, \eta, C)\), where
\[ X := (H^1(0, T; H) \cap L^\infty(0, T; V)) \times (L^\infty(0, T; H) \cap L^2(0, T; V))^2. \]
Moreover, for every increment \(h \in \mathbb{R}^4\), we have \(D S(P, \chi, \eta, C)[h] = (\xi, \nu, \zeta)\), where \((\xi, \nu, \zeta)\) is the unique solution to (3.53)–(3.57) associated to \(h\), obtained by Theorem 3.2.

Proof of Theorem 3.3. To begin with, let us recall that \(h = (hP, h\chi, h\eta, hC)\) and let us set the corresponding control \((P^h, \chi^h, \eta^h, C^h) := (P + hP, \chi + h\chi, \eta + h\eta, C + hC)\). Next, we denote by
\[
(\varphi, \mu, \sigma) := S(P, \chi, \eta, C), \\
(\varphi^h, \mu^h, \sigma^h) := S(P^h, \chi^h, \eta^h, C^h), \\
(\xi, \nu, \zeta) := \text{Solution to the linearized system associated to } h,
\]
and set
\[
\phi := \varphi^h - \varphi - \xi, \quad \rho := \mu^h - \mu - \nu, \quad \omega := \sigma^h - \sigma - \zeta.
\]
By comparing Theorem 2.2 with Theorem 3.2, we deduce that the triplet \((\phi, \rho, \omega)\) satisfies
\[
\phi \in H^1(0, T; H) \cap L^\infty(0, T; V), \quad \rho, \omega \in H^1(0, T; V^*) \cap L^2(0, T; V).
\]
To prove the assertion, it is enough to show that
\[
\frac{\|S(P^h, \chi^h, \eta^h, C^h) - S(P, \chi, \eta, C) - (\xi, \nu, \zeta)\|_X}{|h|} \to 0 \quad \text{as } |h| \to 0,
\]
where \(|h| := |P^h| + |\chi^h| + |\eta^h| + |C^h|\). By using the notation above, this amounts to showing that
\[
\frac{\|(\phi, \rho, \omega)\|_X}{|h|} \to 0 \quad \text{as } |h| \to 0, \quad (3.60)
\]
so that it suffices to check that there exist two constants \(M > 0\) and \(\gamma > 1\), independent of \(h\), such that
\[
\|(\phi, \rho, \omega)\|_X \leq M|h|^\gamma. \quad (3.61)
\]
Taking the difference of the corresponding systems, we infer that the triplet \((\phi, \rho, \omega)\) solves the following system

\[
\begin{align*}
\partial_t (\varepsilon \rho + \phi) - \Delta \rho &= L_P & \text{in } Q, & \tag{3.62} \\
\rho &= \tau \partial_t \phi + a \phi - J * \phi + F''(\varphi)\phi + L_L & \text{in } Q, & \tag{3.63} \\
\partial_t \omega - \Delta \omega + B \omega + L_C &= L_\eta & \text{in } Q, & \tag{3.64} \\
\partial_n \rho &= \eta \partial_n \phi = \partial_n \omega = 0 & \text{on } \Sigma, & \tag{3.65} \\
\rho(0) &= \phi(0) = \omega(0) = 0 & \text{in } \Omega, & \tag{3.66}
\end{align*}
\]

where

\[
\begin{align*}
L_P &= \mathcal{P} (\varepsilon \rho + \phi) - \mathcal{P} (f(\varphi^h) - f(\varphi)) (\sigma^h - \sigma) + (\mathcal{P} \sigma - A) (f(\varphi^h) - f(\varphi) - f'(\varphi) \xi) \\
&\quad + h_P [(f(\varphi^h) - f(\varphi)) (\sigma^h - \sigma) + (f(\varphi^h) - f(\varphi)) \sigma + (\sigma^h - \sigma) f(\varphi)],
\end{align*}
\]

\[
\begin{align*}
L_L &= -\chi \omega - h_L (\sigma^h - \sigma) + F'(\varphi^h) - F'(\varphi) - F''(\varphi) (\varphi^h - \varphi),
\end{align*}
\]

and

\[
\begin{align*}
L_C &= C (f(\varphi) + (f(\varphi^h) - f(\varphi)) (\sigma^h - \sigma) + C (f(\varphi^h) - f(\varphi)) (\sigma^h - \sigma) + (f(\varphi^h) - f(\varphi)) \sigma + (\sigma^h - \sigma) f(\varphi).
\end{align*}
\]

As an easy consequence of Theorem 2.2, the following estimates hold:

\[
\begin{align*}
||\varphi||_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))} + ||\eta\varphi||_{L^2(0,T;W)} + ||\partial_t \varphi||_{L^\infty(Q)} \\
+ ||\mathcal{P} H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\cap L^\infty(Q)|| \leq K, \tag{3.67}
\end{align*}
\]

where the constant \(K\) is independent of \(h\). In order to prove (3.60), we multiply (3.62) by \(\rho\), (3.63) to which we add both sides \((c_n + 2) \phi\) by \(-\partial_t \phi\), the gradient of (3.63) by \(\nabla \varphi\), and (3.64) by \(\omega\). Using the same argument employed in (3.58)–(3.59) in the proof of the linearized system, we deduce that, upon integration over \(Q_t\) and addition of the resulting equalities,

\[
\begin{align*}
&\frac{\varepsilon}{2} ||\rho(t)||^2 + \int_{Q_t} |\nabla \rho|^2 + \tau \int_{Q_t} |\partial_t \phi|^2 + ||\phi(t)||^2 + C_0 \int_{Q_t} |\nabla \phi|^2 + \frac{\tau}{2} ||\nabla \phi(t)||^2 \\
&+ \frac{1}{2} ||\omega(t)||^2 + B \int_{Q_t} |\omega|^2 + \int_{Q_t} |\nabla \omega|^2 \\
&\leq \int_{Q_t} L_P \rho - \int_{Q_t} F''(\varphi) \partial_t \phi - \int_{Q_t} L_L \partial_t \phi - \int_{Q_t} (c_n + 2) \phi \partial_t \phi - \int_{Q_t} \nabla L_L \cdot \nabla \phi \\
&- \int_{Q_t} \phi \nabla a \cdot \nabla \phi + \int_{Q_t} (\nabla J) \ast \phi \cdot \nabla \phi - \int_{Q_t} F''(\varphi) \phi \nabla \varphi \cdot \nabla \phi + \int_{Q_t} (L_\eta - L_C) \omega. \tag{3.68}
\end{align*}
\]

The second term on the right-hand side can be estimated by using the separation property and Young’s inequality which lead to

\[
\int_{Q_t} F''(\varphi) \phi \partial_t \phi \leq M \int_{Q_t} |\phi||\partial_t \phi| \leq \delta \int_{Q_t} |\partial_t \phi|^2 + M(\delta) \int_{Q_t} |\phi|^2,
\]

for a positive constant \(\delta\) yet to be chosen. Let us recall Taylor’s formula with integral remainder for \(f\), which is well-defined owing to the required regularity:

\[
f(\varphi^h) - f(\varphi) - F''(\varphi) \xi = f'(\varphi) \phi + R_f^h (\varphi^h - \varphi)^2, \tag{3.69}
\]
where the remainder \( R_f^h \) is defined as
\[
R_f^h := \int_0^1 f''(\varphi + s(\varphi^h - \varphi))(1 - s) \, ds
\]
and it is uniformly bounded since \( f \in C^2_b(\mathbb{R}) \). Using the Young and Hölder inequalities, the boundedness and the Lipschitz continuity of \( f \) and \( f' \), the Taylor’s formula (3.69), the estimate (3.67), as well as the regularity of \( h \) which is bounded by a constant independent of \( \delta \), we infer that \( f \) and it is uniformly bounded since
\[
\int_{Q_t} L_P \rho - \int_{Q_t} L_C \omega \\
\leq M \int_{Q_t} (|\omega| + |\varphi^h - \varphi||\varphi^h - \varphi| + |\varphi + |\varphi^h - \varphi|^2)(|\rho| + |\omega|) \\
+ M|h| \int_{Q_t} (|\varphi^h - \varphi||\varphi^h - \varphi| + |\varphi^h - \varphi| + |\varphi - \varphi|)(|\rho| + |\omega|) \\
\leq M \int_{Q_t} (|\omega|^2 + |\rho|^2 + |\varphi|^2) + M \int_0^t ||\varphi^h - \varphi||_4^2 (||\rho|| + ||\omega||) \\
+ M(1 + |h|) \int_0^t ||\varphi^h - \varphi||_4 ||\varphi^h - \varphi||_4 (||\rho|| + ||\omega||) \\
+ M|h| \int_0^t (||\varphi^h - \varphi|| + ||\varphi^h - \varphi||)(||\rho|| + ||\omega||) \\
\leq M \int_{Q_t} (|\omega|^2 + |\rho|^2 + |\varphi|^2) + M(|h|^6 + |h|^4).
\]
Now, arguing again by Taylor’s formula with integral remainder, we have that
\[
F'(\varphi^h) - F'(\varphi) - F''(\varphi)(\varphi^h - \varphi) = R_{F'}^h(\varphi^h - \varphi)^2
\]
with remainder
\[
R_{F'}^h := \int_0^1 F'''(\varphi + s(\varphi^h - \varphi))(1 - s) \, ds
\]
which is bounded by a constant independent of \( h \) by Corollary 2.3. Taking these remarks into account, we infer that
\[
- \int_{Q_t} L_\chi \partial_t \phi + \int_{Q_t} L_\eta \omega \\
\leq M \int_{Q_t} |\omega| |\partial_t \phi| + M|h| \int_0^t ||\varphi^h - \varphi|| ||\partial_t \phi|| + M \int_{Q_t} ||\varphi^h - \varphi||^2 |\partial_t \phi| \\
+ M \int_{Q_t} |\nabla \phi||\nabla \omega| + M|h| \int_0^t ||\Delta(\varphi^h - \varphi)|| ||\omega|| \\
\leq \delta \int_{Q_t} (|\partial_t \phi|^2 + |\nabla \omega|^2) \\
+ M(\delta) \int_{Q_t} (|\omega|^2 + |h|^2 ||\varphi^h - \varphi||^2 + ||\varphi^h - \varphi||^4 + |\nabla \phi|^2 + |h|^2 ||\Delta(\varphi^h - \varphi)||^2) \\
\leq \delta \int_{Q_t} (|\partial_t \phi|^2 + |\nabla \omega|^2) + M(\delta) \int_{Q_t} (|\omega|^2 + |\nabla \phi|^2) + M(\delta)|h|^4.
\]
Furthermore, we have that

\[-\int_{Q_t} \nabla L \chi \cdot \nabla \phi = \chi \int_{Q_t} \nabla \omega \cdot \nabla \phi + h \int_{Q_t} \nabla (\sigma^h - \sigma) \cdot \nabla \phi
\]

\[-\int_{Q_t} \nabla (F'(\sigma^h) - F'(\sigma) - F''(\sigma)(\sigma^h - \sigma)) \cdot \nabla \phi,
\]

where, by the Young inequality,

\[\chi \int_{Q_t} \nabla \omega \cdot \nabla \phi + h \int_{Q_t} \nabla (\sigma^h - \sigma) \cdot \nabla \phi \leq \delta \int_{Q_t} |\nabla \omega|^2 + M(\delta) \int_{Q_t} (|\nabla \phi|^2 + |h|^2|\nabla (\sigma^h - \sigma)|^2)
\]

\[\leq \delta \int_{Q_t} |\nabla \omega|^2 + M(\delta) \int_{Q_t} |\nabla \phi|^2 + M(\delta)|h|^4,
\]

and

\[\int_{Q_t} \nabla (F'(\sigma^h) - F'(\sigma) - F''(\sigma)(\sigma^h - \sigma)) \cdot \nabla \phi
\]

\[= \int_{Q_t} (F''(\sigma^h)\nabla \sigma^h - F''(\sigma)\nabla \sigma - F''(\sigma^h)\nabla \sigma^h + F''(\sigma)\nabla \sigma + F''(\sigma^h) - F''(\sigma)) \nabla \sigma \cdot \nabla \phi + \int_{Q_t} (F''(\sigma^h) - F''(\sigma)) \nabla (\sigma^h - \sigma) \cdot \nabla \phi.
\]

Hence, using again the Taylor formula with integral remainder for $F''$, the separation property \([2.13]\), and the estimate \([3.67]\), it is straightforward to see that

\[-\int_{Q_t} \nabla (F'(\sigma^h) - F'(\sigma) - F''(\sigma)(\sigma^h - \sigma)) \cdot \nabla \phi \leq M \int_{Q_t} |\nabla \phi|^2 + M|h|^4.
\]

Finally, the last line of \([3.68]\) can be bounded from above using the H"{o}lder inequality, the continuous embedding $V \hookrightarrow L^4(\Omega)$, and the regularity of $\varphi$ as

\[\frac{\tau}{2} \int_{Q_t} |\partial_t \phi|^2 + M \int_{Q_t} |\phi|^2 + M \int_{Q_t} (|\phi|^2 + |\nabla \phi|^2) + M \int_0^t \|\nabla \varphi\|_2^2 \|\phi\|_2^2
\]

\[\leq \frac{\tau}{2} \int_{Q_t} |\partial_t \phi|^2 + M \int_0^t \|\phi\|_2^2.
\]

Therefore, upon collecting the above estimates, picking $\delta$ small enough, and invoking Gronwall’s lemma, we conclude the proof since \([3.61]\) has been shown with $\gamma = 2$. 

### 3.4 Adjoint system

In order to study first-order conditions for optimality for problem $(CP)_{\varepsilon, \tau}$, for a fixed admissible control $(\overline{P}, \overline{r}, \overline{\eta}, \overline{C}) \in \mathcal{U}_{ad}$ with corresponding state $(\varphi, \overline{r}, \overline{\sigma})$, we introduce and solve the auxiliary backward-in-time problem called adjoint system, in the new variables $(p, q, r)$. This system is formally obtained by taking the adjoint of the linearized system \([5.55] - [5.57]\), and reads

\[-\partial_t (p + \tau q) + aq - J * q + \overline{\eta} \Delta r + F''(\varphi)q
\]

\[+ \overline{C} \sigma f'(\varphi)r - (\overline{\varphi}^\sigma - A)f'(\varphi)p = \beta_Q(\varphi - \varphi_Q) \quad \text{in } Q, \quad (3.70)\]

\[-\varepsilon \partial_t p - \Delta p - q = 0 \quad \text{in } Q, \quad (3.71)\]

\[-\partial_t r - \Delta r + (B + \overline{C} f(\varphi))r - \overline{P} f(\varphi)p - \overline{\varphi} q = 0 \quad \text{in } Q, \quad (3.72)\]

\[\partial_n p = \partial_n r = 0 \quad \text{on } \Sigma, \quad (3.73)\]

\[\varepsilon p(T) = 0, \quad (p + \tau q)(T) = \beta_{\Omega}(\varphi(T) - \varphi_{\Omega}), \quad r(T) = 0 \quad \text{in } \Omega. \quad (3.74)\]

Here we state the corresponding well-posedness result.
Theorem 3.4 (Well-posedness of the adjoint system: $\varepsilon, \tau > 0$). Assume $A1$–$A7$ and $C1$–$C4$ and let $(\mathcal{P}, \overline{x}, \overline{n}, \overline{\alpha}) \in \mathcal{U}_{ad}$ be an admissible control, with corresponding state $(\overline{\varphi}, \overline{p}, \overline{\sigma})$. Then, the adjoint system (3.70)–(3.71) admits a unique solution $(p, q, r)$ such that

$$p, r \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \quad q \in H^1(0, T; H) \cap L^\infty(0, T; H).$$

Proof of Theorem 3.4. A rigorous proof has to be addressed within an approximation scheme. Anyhow, since the system is linear and the arguments are standard we just point out the formal a priori estimates, leaving the details to the reader.

First estimate: We multiply (3.70) by $q$, (3.71) by $-\partial_t p + p$, (3.72) by $-\Delta r$. After integrating over $Q_t^T$ and adding the resulting equalities, we obtain

$$\frac{\tau}{2} \|q(t)\|^2 + C_0 \int_{Q_t^T} |q|^2 + \varepsilon \int_{Q_t^T} |\partial_t p|^2 + \frac{\varepsilon}{2} \|p(t)\|^2 + \frac{1}{2} \|\nabla p(t)\|^2 + \int_{Q_t^T} |\nabla p|^2$$

$$+ \frac{B + 1}{2} \|r(t)\|^2 + \|\nabla r(t)\|^2 + B \int_{Q_t^T} |\nabla r|^2 + \int_{Q_t^T} |\partial_t r|^2 + \int_{Q_t^T} |\Delta r|^2$$

$$\leq \frac{1}{2\tau} \beta_\Omega(\overline{\varphi}(T) - \varphi_0) \|q\|^2 + \int_{Q_t^T} \beta q(\overline{\varphi} - \varphi_0) q - \int_{Q_t^T} \overline{\sigma} \nabla r q + \int_{Q_t^T} (J * q) q$$

$$- \int_{Q_t^T} C \sigma f'(\overline{\varphi}) r q + \int_{Q_t^T} (\mathcal{P} \sigma - A) f'(\overline{\varphi}) p q + \int_{Q_t^T} q p + \int_{Q_t^T} C f(\overline{\varphi}) r (\partial_t r + \Delta r)$$

$$- \int_{Q_t^T} \mathcal{P} f(\overline{\varphi}) p (\partial_t r + \Delta r) - \int_{Q_t^T} \nabla q (\partial_t r + \Delta r) - \int_{Q_t^T} r \partial_t r.$$

By virtue of the regularity of $\overline{\sigma}$, the boundedness of $f$ and $f'$ and Young’s inequality, we easily infer that

$$|I_1| \leq \delta \int_{Q_t^T} |\Delta r|^2 + M(\delta) \int_{Q_t^T} (|q|^2 + 1),$$

$$|I_2| + |I_3| \leq \delta \int_{Q_t^T} (|\partial_t r|^2 + |\Delta r|^2) + M(\delta) \int_{Q_t^T} (|p|^2 + |q|^2 + |r|^2)$$

for a positive constant $\delta$ yet to be chosen. Hence, we take $\delta$ small enough and Gronwall’s lemma along with elliptic regularity, produces

$$\|p\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|q\|_{L^\infty(0,T;H)} + \|r\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq M.$$

Second estimate: A comparison argument in (3.71) easily produce an $L^2(0,T;H)$ bound for $\Delta p$. Hence, using elliptic regularity theory we easily infer that

$$\|p\|_{L^2(0,T;W)} \leq M.$$ 

Third estimate: From the above estimate, a comparison argument in (3.70) leads us to obtain

$$\|\partial_t q\|_{L^2(0,T;H)} \leq M.$$


Fourth estimate: Notice that (3.71) and (3.72) have a parabolic structure in $p$ and $r$ with zero final condition and source term bounded in $L^\infty(0,T;H)$. Therefore, it easily follows from classical parabolic regularity theory that

$$
\|p\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|r\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \leq C.
$$

Arguing in a similar fashion as for the linearized system, due to the linearity of the adjoint system (3.70)–(3.74), the uniqueness directly follows from the above estimates and the proof is concluded. \(\square\)

### 3.5 Optimality conditions

This section is devoted to the study of necessary conditions for optimality for the optimization problem \((CP)_{\varepsilon, \tau}\).

First of all, we employ a classical tool to derive first-order necessary conditions for \((CP)_{\varepsilon, \tau}\). In fact, provided that \(J\) is sufficiently smooth and recalling the structure of \(U_{ad}\), a first-order necessary condition for \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C}) \in U_{ad}\) to be optimal is to verify the following variational inequality

$$
\langle D_{\text{red}}(\overline{P}, \overline{X}, \overline{\eta}, \overline{C}), (P, \chi, \eta, C) - (\overline{P}, \overline{X}, \overline{\eta}, \overline{C}) \rangle \geq 0 \text{ for every } (P, \chi, \eta, C) \in U_{ad},
$$

where \(D_{\text{red}}\) denotes the Gâteaux derivative of the reduced cost functional defined as

$$
D_{\text{red}}(P, \chi, \eta, C) := J(S_1(P, \chi, \eta, C), \overline{P}, \overline{X}, \overline{\eta}, \overline{C}), \quad (P, \chi, \eta, C) \in U_{ad}.
$$

Theorem 3.3 allows us to exploit this result to obtain an explicit expression in terms of the linearized variables.

**Theorem 3.5** (First-order necessary condition for optimality). Assume A1–A7 and C1–C4, and let \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C})\) be an optimal control for problem \((CP)_{\varepsilon, \tau}\), with corresponding state \((\overline{\varphi}, \overline{\mu}, \overline{\sigma})\). Then, \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C})\) necessarily satisfies

$$
\int_{\Omega} \beta_{\Omega}(\overline{\varphi}(T) - \varphi_0)\xi(T) + \int_{Q} \beta_{Q}(\overline{\varphi} - \varphi)\xi + \alpha_{P}(\overline{P} - P_*)(P - \overline{P})
$$

$$
+ \alpha_{X}(\overline{X} - X_*) (X - \overline{X}) + \alpha_{\eta}(\overline{\eta} - \eta_*) (\eta - \overline{\eta}) + \alpha_{C}(\overline{C} - C_*) (C - \overline{C}) \geq 0
$$

for every \((P, \chi, \eta, C) \in U_{ad},\) \hspace{1cm} (3.76)

where \(\xi\) is the first component of the unique solution \((\xi, \nu, \zeta)\) to the linearized system obtained by Theorem 3.2 associated to \(h = (P - \overline{P}, \chi - \overline{X}, \eta - \overline{\eta}, C - \overline{C}).\)

**Proof.** By Theorem 3.3 and the usual chain rule for Fréchet-differentiable functions, it follows immediately that the reduced cost functional \(D_{\text{red}}: U_{ad} \to \mathbb{R}\) is Fréchet-differentiable at \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C}).\) Hence, the optimality of \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C})\) yields directly (3.75), which in turn reads as (3.76). \(\square\)

The next step consists in simplifying the necessary conditions for the minimizer presented above, by using the adjoint system.

**Theorem 3.6** (Final first-order necessary conditions for optimality). Assume A1–A7 and C1–C4, let \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C}) \in U_{ad}\) be an optimal control for \((CP)_{\varepsilon, \tau}\), and let \((\overline{\varphi}, \overline{\mu}, \overline{\sigma})\) and \((p, q, r)\) be the corresponding state and adjoint variables, respectively. Then, \((\overline{P}, \overline{X}, \overline{\eta}, \overline{C})\) necessarily verifies

$$
\int_{Q} (P - \overline{P})\sigma f(\overline{\varphi}) p + \int_{Q} (\chi - \overline{X})\sigma q - \int_{Q} (\eta - \overline{\eta})\Delta\overline{\varphi} r - \int_{Q} (C - \overline{C})(\overline{\varphi} f(\overline{\varphi}) r
$$

$$
+ \alpha_{P}(\overline{P} - P_*)(P - \overline{P}) + \alpha_{X}(\overline{X} - X_*) (X - \overline{X})
$$

$$
+ \alpha_{\eta}(\overline{\eta} - \eta_*) (\eta - \overline{\eta}) + \alpha_{C}(\overline{C} - C_*) (C - \overline{C}) \geq 0 \text{ for every } (P, \chi, \eta, C) \in U_{ad}.
$$

(3.77)
Proof. We note that (3.77) directly follows from (3.76), provided to show the identity
\[
\int_Q \beta_Q(\bar{\varphi} - \varphi)\xi + \int_{\Omega} \beta_\Omega(\bar{\varphi}(T) - \varphi_\Omega)\xi(T)
= \int_Q h_p\sigma f(\bar{\varphi})p + \int_Q h_\chi\sigma q - \int_Q h_\eta\Delta \varphi r - \int_Q h_c\sigma f(\bar{\varphi}) r
\] (3.78)
with \( h = (\mathcal{P} - \bar{\mathcal{P}}, \chi - \bar{\chi}, \eta - \bar{\eta}, \mathcal{C} - \bar{\mathcal{C}}) \). To this end, we multiply (3.58)–(3.55) by \( p, q, \) and \( r \) in the order, integrate over \( Q \), and sum the equalities to obtain
\[
0 = \int_Q p[\partial_t(\varepsilon \nu + \xi) - \Delta \nu - (\bar{\mathcal{P}}\sigma - \mathcal{A}) f'(\bar{\varphi})\xi - \bar{\mathcal{P}}\zeta f'(\bar{\varphi}) - h_p\sigma f(\bar{\varphi})]
+ \int_Q q[-\nu + \tau \partial_t \xi + a \xi - J * \xi + F''(\bar{\varphi})\xi - \bar{\chi} \xi - h_\chi \sigma]
+ \int_Q r[\partial_t \xi - \Delta \xi + \mathcal{B} \xi + \bar{\mathcal{C}}(\xi f(\bar{\varphi}) + \bar{\sigma} f'(\bar{\varphi})\xi) + h_c\sigma f(\bar{\varphi}) + \bar{\eta} \Delta \xi + h_\eta \Delta \varphi].
\]
The terms involving the time derivatives can be easily handled by integrating by parts and using the initial conditions (3.57) and the terminal conditions (3.41) to obtain that
\[
\int_Q p[\partial_t(\varepsilon \nu + \xi) + \int_Q \tau q \partial_t \xi + \int_Q \partial_t \zeta r
= -\int Q [\varepsilon \partial_t p \nu - \int Q \partial_t p \xi - \int Q \tau \partial_t q \xi - \int Q \partial_t r \zeta + \int Q (p + \tau q)(T)\xi(T)]
= -\int Q [\varepsilon \partial_t p \nu - \int Q \partial_t (p + \tau q) \xi - \int Q \partial_t r \zeta + \int Q \beta_\Omega(\bar{\varphi}(T) - \varphi_\Omega)\xi(T)].
\]
Moreover, integrating by parts and rearranging the terms we get
\[
0 = \int_Q [\xi(\nu \partial_t (p + \tau q) + a q - J * q + \bar{\eta} \Delta r + F''(\bar{\varphi})q + \bar{\mathcal{C}} f'(\bar{\varphi}) r - (\bar{\mathcal{P}}\sigma - \mathcal{A}) f'(\bar{\varphi}) p)]
+ \int Q \zeta[-\partial_t r - \Delta r + (\mathcal{B} + \bar{\mathcal{C}} f(\bar{\varphi})) r - \bar{\mathcal{P}} f'(\bar{\varphi}) p - \bar{\chi} q]
- \int Q h_p\sigma f(\bar{\varphi}) p - \int Q h_\chi\sigma q + \int Q h_\eta\Delta \varphi r + \int Q h_c\sigma f(\bar{\varphi}) r + \int Q \beta_\Omega(\bar{\varphi}(T) - \varphi_\Omega)\xi(T).
\]
Hence, we recall the definition of the adjoint variables (3.70)–(3.72) to realize that the most part of the above terms simplify and the remaining equality is (3.78), as we claimed. \( \square \)

4 Asymptotic analysis

The goal of this section is to exploit the results established so far for \((CP)_{\varepsilon, \tau}\) in the case \( \varepsilon, \tau > 0 \) to show that we can solve the optimal controls \((CP)_{\varepsilon}, (CP)_{\tau}, (CP)_{\nu}\) through asymptotic arguments. In particular, we aim at passing to the limit as \( \varepsilon \) and \( \tau \) go to zero, both separately and jointly, in the optimality condition (3.77).

As the asymptotic analysis for the state system has already been recalled in Theorems 2.4, 2.5 and 2.6 (see also 56), the first novelty addressed here consists in understanding the asymptotic behavior of the adjoint system. In this direction, we show that the adjoint variables, depending on \( \varepsilon, \tau > 0 \), converge in some topology. To this end, we begin with obtaining some uniform estimates with respect to \( \varepsilon, \tau \) so to pass to the limit using classical weak and weak-star compactness arguments.
The second step consists in approximating the optimal controls of \((CP)_{\varepsilon}, (CP)_{\tau}, (\overline{CP})\) by means of sequences of optimal controls of \((CP)_{\varepsilon, \tau}\). A combination of these steps will allow us to rigorously pass to the limit in the optimality conditions \((3.77)\), recovering thus the corresponding ones for \((CP)_{\varepsilon}, (CP)_{\tau},\) and \((\overline{CP})\).

### 4.1 Uniform estimates on the adjoint problem

In this subsection, we assume to be in the setting of either Theorem 2.4 or 2.5 or 2.6. For every \(\varepsilon \in (0, \varepsilon_0)\) and \(\tau \in (0, \tau_0)\), let \((P_{\varepsilon, \tau}, x_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau}) \in u_{\text{ad}}\) be an admissible control, and let \((\overline{x}_{\varepsilon, \tau}, \overline{P}_{\varepsilon, \tau}, \overline{C}_{\varepsilon, \tau})\) and \((p_{\varepsilon, \tau}, q_{\varepsilon, \tau}, r_{\varepsilon, \tau})\) denote the unique solutions to the state system \((1.6)\)–\((1.10)\) and the adjoint system \((3.70)\)–\((3.71)\) with \(\varepsilon, \tau > 0\), respectively.

First of all, performing the same estimate as in the proof of Theorem 3.4 noting that \(\{\overline{x}_{\varepsilon, \tau}\}_{\varepsilon, \tau}\) is always bounded in \(C^0([0, T]; H)\) uniformly in both \(\varepsilon\) and \(\tau\) thanks to Theorems 2.4, 2.5, and 2.6 we have that

\[
\frac{\tau}{2} \|q_{\varepsilon, \tau}(t)\|^2 + C_0 \int_{Q_T^\varepsilon} |q_{\varepsilon, \tau}|^2 + \varepsilon \int_{Q_T^\varepsilon} |\partial_t p_{\varepsilon, \tau}|^2 + \frac{\varepsilon}{2} \|p_{\varepsilon, \tau}(t)\|^2 + \frac{1}{2} \|\nabla p_{\varepsilon, \tau}(t)\|^2 + \int_{Q_T^\varepsilon} |\nabla p_{\varepsilon, \tau}|^2 \\
+ \frac{B + 1}{2} \|r_{\varepsilon, \tau}(t)\|^2 + \|\nabla r_{\varepsilon, \tau}(t)\|^2 + B \int_{Q_T^\varepsilon} |\nabla r_{\varepsilon, \tau}|^2 + \int_{Q_T^\varepsilon} |\partial_t r_{\varepsilon, \tau}|^2 + \int_{Q_T^\varepsilon} |\Delta r_{\varepsilon, \tau}|^2 \\
\leq M \left(\frac{\beta_0^2}{2\tau} + 1\right) + \delta \int_{Q_T^\varepsilon} |q_{\varepsilon, \tau}|^2 - \int_{Q_T^\varepsilon} \eta_{\varepsilon, \tau} \Delta r_{\varepsilon, \tau} + p_{\varepsilon, \tau} + M(\delta) \int_{Q_T^\varepsilon} (|\overline{\varepsilon}_{\varepsilon, \tau} p_{\varepsilon, \tau}|^2 + |\overline{\sigma}_{\varepsilon, \tau} r_{\varepsilon, \tau}|^2) \\
+ \int_{Q_T^\varepsilon} (J + q_{\varepsilon, \tau}) q_{\varepsilon, \tau} + \delta' \int_{Q_T^\varepsilon} (|\partial_t r_{\varepsilon, \tau}|^2 + |\Delta r_{\varepsilon, \tau}|^2) + M(\delta, \delta') \int_{Q_T^\varepsilon} (|r_{\varepsilon, \tau}|^2 + |p_{\varepsilon, \tau}|^2) \\
- \int_{Q_T^\varepsilon} \chi_{\varepsilon, \tau} q_{\varepsilon, \tau} (\partial_t r_{\varepsilon, \tau} + \Delta r_{\varepsilon, \tau}),
\]

where the constants \(M, \delta, \delta', M(\delta)\), and \(M(\delta, \delta')\) are independent of \(\varepsilon\) and \(\tau\). The H"older inequality and the continuous inclusion \(V \hookrightarrow L^4(\Omega)\) yield also

\[
M(\delta) \int_{Q_T^\varepsilon} (|\overline{\varepsilon}_{\varepsilon, \tau} p_{\varepsilon, \tau}|^2 + |\overline{\sigma}_{\varepsilon, \tau} r_{\varepsilon, \tau}|^2) \leq M(\delta) \int_{Q_T^\varepsilon} (\|p_{\varepsilon, \tau}\|^2_V + \|r_{\varepsilon, \tau}\|^2_V).
\]

Secondly, taking the mean of \((3.70)\) we have

\[
- \partial_t ((p_{\varepsilon, \tau}) \Omega + \tau(q_{\varepsilon, \tau}) \Omega) + (F''(\overline{\varepsilon}_{\varepsilon, \tau}) q_{\varepsilon, \tau}) \Omega + C_{\varepsilon, \tau} (\overline{\sigma}_{\varepsilon, \tau} f'(\overline{\varepsilon}_{\varepsilon, \tau}) r_{\varepsilon, \tau}) \Omega \\
= ((P_{\varepsilon, \tau \varepsilon, \tau} - A) f'(\overline{\varepsilon}_{\varepsilon, \tau}) p_{\varepsilon, \tau}) \Omega + \beta Q((\overline{\varepsilon}_{\varepsilon, \tau} - \varphi Q)) \Omega,
\]

so that testing this latter by \((p_{\varepsilon, \tau}) \Omega + \tau(q_{\varepsilon, \tau}) \Omega\) we get

\[
\frac{1}{2} (p_{\varepsilon, \tau}(t)) \Omega + \tau(q_{\varepsilon, \tau}(t)) \Omega\|^2 \leq \frac{\beta_0^2}{2} (|\overline{\varepsilon}_{\varepsilon, \tau}(T) - \varphi Q|^2 + \int_{t}^{T} \beta_0^2 |\overline{\varepsilon}_{\varepsilon, \tau} - \varphi Q|^2_1) \\
+ M(\delta) \int_{t}^{T} (|p_{\varepsilon, \tau}) \Omega + \tau(q_{\varepsilon, \tau}) \Omega\|^2 (1 + \|F''(\overline{\varepsilon}_{\varepsilon, \tau})\|^2) + \delta \int_{t}^{T} \|q_{\varepsilon, \tau}\|^2 \\
+ M \int_{t}^{T} (\|\overline{\sigma}_{\varepsilon, \tau} r_{\varepsilon, \tau}\|_1^2 + \|\overline{\sigma}_{\varepsilon, \tau} p_{\varepsilon, \tau}\|_1^2 + \|p_{\varepsilon, \tau}\|_1^2)(4.80)
\]

where again the constants \(M\) and \(M(\delta)\) are independent of \(\varepsilon\) and \(\tau\). Now, since \(\tau \leq 1\), by the Jensen inequality we have

\[
\frac{1}{8} |(p_{\varepsilon, \tau}(t)) \Omega| \leq \frac{1}{4} |(p_{\varepsilon, \tau}(t)) \Omega + \tau(q_{\varepsilon, \tau}(t)) \Omega|^2 + \frac{\tau}{4 |\Omega|} \|q_{\varepsilon, \tau}(t)|^2,
\]
so that summing (4.79) and (4.80), using again the fact that \( \{ \varphi_{\varepsilon, \tau} \}_{\varepsilon, \tau} \) is always bounded in \( C^0([0, T]; H) \) uniformly in both \( \varepsilon \) and \( \tau \), and rearranging the terms, by the Poincaré-Wirtinger inequality we infer that

\[
\begin{align*}
    m\tau \| q_{\varepsilon, \tau}(t) \|^2 &+ C_0 \int_{Q_T^t} |q_{\varepsilon, \tau}|^2 + \varepsilon \int_{Q_T^t} |\partial_t p_{\varepsilon, \tau}|^2 + \frac{\varepsilon}{2} \| p_{\varepsilon, \tau}(t) \|^2 + m\| p_{\varepsilon, \tau}(t) \|_V^2 \\
    &+ \int_{Q_T^t} |\nabla p_{\varepsilon, \tau}|^2 + \frac{1}{2} \| r_{\varepsilon, \tau}(t) \|_V^2 + B \int_{Q_T^t} |\nabla r_{\varepsilon, \tau}|^2 + \int_{Q_T^t} |\partial_t r_{\varepsilon, \tau}|^2 + \int_{Q_T^t} |\Delta r_{\varepsilon, \tau}|^2 \\
    &\leq M \left( \frac{\beta^2}{2\tau} + 1 \right) + \delta \int_{Q_T^t} |q_{\varepsilon, \tau}|^2 + \delta' \int_{Q_T^t} \left( |\partial_t r_{\varepsilon, \tau}|^2 + |\Delta r_{\varepsilon, \tau}|^2 \right) + \int_{Q_T^t} \left( J \ast q_{\varepsilon, \tau} \right) q_{\varepsilon, \tau} \\
    &+ M(\delta, \delta') \int_t^T (\| r_{\varepsilon, \tau} \|^2 + \| p_{\varepsilon, \tau} \|^2) + M(\delta) \int_t^T \| F''(\varphi_{\varepsilon, \tau}) \|^2 (\| p_{\varepsilon, \tau} \|^2 + \tau \| q_{\varepsilon, \tau} \|^2) \\
    &+ M(\delta) \int_t^T \| \varphi_{\varepsilon, \tau} \|_V^2 (\| p_{\varepsilon, \tau} \|^2 + \| r_{\varepsilon, \tau} \|^2) \|_{V^2} - \int_{Q_T^t} \eta_{\varepsilon, \tau} \Delta r_{\varepsilon, \tau} q_{\varepsilon, \tau} \\
    &- \int_{Q_T^t} \chi_{\varepsilon, \tau} q_{\varepsilon, \tau} (\partial_t r_{\varepsilon, \tau} + \Delta r_{\varepsilon, \tau}),
\end{align*}
\]

where \( \delta, \delta' > 0 \) are arbitrary, and \( M, M(\delta), M(\delta, \delta') > 0 \) are independent of \( \varepsilon \) and \( \tau \).

### 4.2 The optimization problem \((CP)_\tau\)

Here, we solve \((CP)_\tau\) through an asymptotic approach by exploiting the proved results for \((CP)_{\varepsilon, \tau}\) by letting \( \varepsilon \to 0 \). Throughout the whole Section 4.2 we assume the following framework:

\[
\tau \in (0, \tau_0) \text{ fixed, } \qquad \eta_{\max} = \alpha_\eta = 0,
\]

This means that we neglect the term in \( \eta \) in the cost functional and in the state system, implying that all the admissible controls are in the form \((P, \chi, 0, C)\) and that (2.19) is automatically satisfied: with a slight abuse of notation, we look at \( \mathcal{U}_{ad} \) as a compact set in \( \mathbb{R}^3 \), and use the symbol \((P, \chi, C)\) for the generic admissible control in \( \mathcal{U}_{ad} \).

The first result that we present concerns existence of optimal controls for \((CP)_\tau\).

**Theorem 4.1.** Assume A1–A8 and C1–C3. Then, the optimization problem \((CP)_\tau\) admits a solution.

**Proof.** This result follows directly by adapting the direct method used in the proof of Theorem 3.1 taking into account the compactness of \( \mathcal{U}_{ad} \) and the convergence Theorem 2.4.

After the existence is established, our main goal is to provide some necessary conditions for optimality by letting \( \varepsilon \to 0 \) in (3.77) written with the subscripts \( \varepsilon, \tau \). Let then \((\bar{P}_\tau, \bar{\chi}_\tau, \bar{C}_\tau) \in \mathcal{U}_{ad}\) be an optimal control for problem \((CP)_\tau\), and let \((\bar{\varphi}_\tau, \bar{p}_\tau, \bar{r}_\tau)\) be the corresponding state variables solving (1.6)–(1.10) with \( \varepsilon = 0 \), in the sense of Theorem 2.4. Formally we expect that, as \( \varepsilon \to 0 \), the optimality condition reads

\[
\begin{align*}
    \int_Q (P - \bar{P}_\tau) \sigma_\tau f(\bar{\varphi}_\tau) p_\tau - \int_Q (\chi - \bar{\chi}_\tau) \sigma_\tau q_\tau - \int_Q (C - \bar{C}_\tau) \sigma_\tau f(\bar{\varphi}_\tau) r_\tau \\
    + \alpha_P (\bar{P}_\tau - P_\ast)(P - \bar{P}_\tau) + \alpha_\chi (\bar{\chi}_\tau - \chi_\ast)(\chi - \bar{\chi}_\tau) \\
    + \alpha_C (\bar{C}_\tau - C_\ast)(C - \bar{C}_\tau) \geq 0 \quad \text{for every } (P, \chi, C) \in \mathcal{U}_{ad},
\end{align*}
\]
where \((p_\tau, q_\tau, r_\tau)\) stands for some adjoint variables solving (3.70)–(3.74) with \(\varepsilon = 0\), whose meaning is yet to be defined. Unfortunately, the situation is slightly more delicate. In fact, even if we prove that the adjoint variables \((p_\varepsilon, q_\varepsilon, r_\varepsilon)\) converge to some limit \((p_\tau, q_\tau, r_\tau)\) in a suitable sense as \(\varepsilon \to 0\), it is not obvious that every optimal control \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau)\) can be recovered as the limit of a sequence of optimal controls \(\{(\overline{P}_{\varepsilon,\tau}, \overline{X}_{\varepsilon,\tau}, \overline{C}_{\varepsilon,\tau})\}_\varepsilon\) of \((CP)_{\varepsilon,\tau}\).

To overcome this issue we follow the same line of argument of \cite{2} (see also \cite{55, 57, 58} in the context of tumor growth models). We introduce a different cost functional, called adapted, depending on the fixed minimizer \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau)\) of \((CP)_{\tau}\), which is defined as

\[
\mathcal{J}_{ad}(\varphi, P, \chi, C) := \mathcal{J}(\varphi, P, \chi, C) + \frac{1}{2}|P - \overline{P}_\tau|^2 + \frac{1}{2}|\chi - \overline{X}_\tau|^2 + \frac{1}{2}|C - \overline{C}_\tau|^2.
\]

Keeping the optimal control \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau)\) of \((CP)_{\tau}\) fixed, note that \(\mathcal{J}_{ad} = \mathcal{J}\) on the minimizers of \((CP)_{\tau}\). The main idea behind this local perturbation concerns the fact that for the associated optimal control problem, which will be referred to as adapted, we can obtain a compactness-type property. Namely, we prove that every arbitrary minimizer \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau)\) of \((CP)_{\varepsilon,\tau}\) can be recovered as limit of a sequence of minimizers of \((CP)_{\varepsilon,\tau}^{ad}\), as \(\varepsilon \to 0\). The just mentioned adapted optimal control problem associated with \(\varepsilon, \tau\) reads as

\[
(CP)_{\varepsilon,\tau}^{ad} \text{ Minimize } \mathcal{J}_{ad}(\varphi, P, \chi, C) \text{ subject to: }
\]

(i) \((\varphi, \mu, \sigma)\) yields a solution to (1.6)–(1.10);

(ii) \((P, \chi, C) \in \mathcal{U}_{ad}\).

In a sense to be made rigorous later, we will prove that \((CP)_{\varepsilon,\tau}^{ad} \subseteq (CP)_{\tau}\) so that the passage to the limit as \(\varepsilon \to 0\) in the variational inequality (3.77) can be rigorously performed producing in turn the optimality condition of \((CP)_{\tau}\). Since \((CP)_{\varepsilon,\tau}^{ad}\) fulfills the same assumptions of \((CP)_{\varepsilon,\tau}\), for what we already proved in Section 3 we readily infer the following.

**Lemma 4.2.** Assume \(A1\) \(A8\) and \(C1\) \(C3\). Then, for every \(\varepsilon \in (0, \varepsilon_0)\) and for every optimal control \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau) \in \mathcal{U}_{ad}\) of \((CP)_{\tau}\), the optimization problem \((CP)_{\varepsilon,\tau}^{ad}\) admits a minimizer.

**Lemma 4.3.** Assume \(A1\) \(A7\) and \(C1\) \(C4\), and let \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau) \in \mathcal{U}_{ad}\) be an optimal control for \((CP)_{\tau}\). For every \(\varepsilon \in (0, \varepsilon_0)\), if \((\overline{P}_{\varepsilon,\tau}, \overline{X}_{\varepsilon,\tau}, \overline{C}_{\varepsilon,\tau}) \in \mathcal{U}_{ad}\) is an optimal control for \((CP)_{\varepsilon,\tau}^{ad}\), then the following first-order necessary condition holds

\[
\int_Q (P - \overline{P}_{\varepsilon,\tau}) \overline{p}_{\varepsilon,\tau} f(\overline{p}_{\varepsilon,\tau}) + \int_Q (\chi - \overline{X}_{\varepsilon,\tau}) \overline{q}_{\varepsilon,\tau} f(\overline{q}_{\varepsilon,\tau}) + \int_Q (C - \overline{C}_{\varepsilon,\tau}) \overline{r}_{\varepsilon,\tau} f(\overline{r}_{\varepsilon,\tau})
+
(P - \overline{P}_{\varepsilon,\tau}) (\alpha_p(\overline{p}_{\varepsilon,\tau}) - p_\tau) + (\overline{P}_{\varepsilon,\tau} - \overline{P}_\tau) + (\chi - \overline{X}_{\varepsilon,\tau}) (\alpha_\chi(\overline{X}_{\varepsilon,\tau} - \chi_\tau) + \overline{X}_{\varepsilon,\tau} - \overline{X}_\tau)
+
(C - \overline{C}_{\varepsilon,\tau}) (\alpha_C(\overline{C}_{\varepsilon,\tau} - C_\tau) + \overline{C}_{\varepsilon,\tau} - \overline{C}_\tau) \geq 0
\]

for every \((P, \chi, C) \in \mathcal{U}_{ad}\), where \((\overline{P}_{\varepsilon,\tau}, \overline{p}_{\varepsilon,\tau}, \overline{q}_{\varepsilon,\tau}, \overline{r}_{\varepsilon,\tau})\) denote the corresponding unique solutions to (1.6)–(1.10) and (3.70)–(3.74) with \(\varepsilon, \tau > 0\).

The sense in which the minimizers of \((CP)_{\varepsilon,\tau}^{ad}\) approximate the ones of \((CP)_{\tau}\) as \(\varepsilon \to 0\) is specified in the following theorem.

**Theorem 4.4.** Assume \(A1\) \(A8\) and \(C1\) \(C4\). Let \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau) \in \mathcal{U}_{ad}\) be an optimal control for \((CP)_{\tau}\), with corresponding state \((\overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau)\). Then, for every family \(\{(\overline{P}_{\varepsilon,\tau}, \overline{X}_{\varepsilon,\tau}, \overline{C}_{\varepsilon,\tau})\}_\varepsilon\) of optimal controls for \((CP)_{\varepsilon,\tau}^{ad}\), with corresponding states \(\{(\overline{P}_{\varepsilon,\tau}, \overline{X}_{\varepsilon,\tau}, \overline{C}_{\varepsilon,\tau})\}_\varepsilon\), as \(\varepsilon \to 0\) it holds that

\[
\overline{P}_{\varepsilon,\tau} \to \overline{P}_\tau \text{ weakly-}^* \text{ in } H^1(0, T; H) \cap L^\infty(0, T; V),
\]

and strongly in \(C^0([0, T]; H), \overline{P}_{\varepsilon,\tau} \to \overline{P}_\tau, \overline{X}_{\varepsilon,\tau} \to \overline{X}_\tau, \overline{C}_{\varepsilon,\tau} \to \overline{C}_\tau, \mathcal{J}_{ad}(\overline{P}_{\varepsilon,\tau}, \overline{P}_{\varepsilon,\tau}, \overline{X}_{\varepsilon,\tau}, \overline{C}_{\varepsilon,\tau}) \to \mathcal{J}(\overline{P}_\tau, \overline{P}_\tau, \overline{X}_\tau, \overline{C}_\tau).
\]
Proof. Since $\mathcal{U}_{ad}$ is a compact subset of $\mathbb{R}^3$, by virtue of (2.23), (2.27), and the Bolzano–Weierstrass theorem, we infer the existence of $\hat{\varphi} \in H^1(0, T; H) \cap L^\infty(0, T; V)$ and $(\hat{\mathcal{P}}, \hat{x}, \hat{c}) \in \mathcal{U}_{ad}$ such that, along a non-relabelled zero subsequence $\varepsilon_k$, as $k \to \infty$,

$$\nabla_{\varepsilon_k, \tau} \hat{\varphi} \rightarrow \hat{\varphi} \quad \text{weakly-}^* \text{ in } H^1(0, T; H) \cap L^\infty(0, T; V), \quad \text{and strongly in } C^0([0, T]; H),$$

$$\nabla_{\varepsilon_k, \tau} \hat{\mathcal{P}}, \quad \nabla_{\varepsilon_k, \tau} \hat{x}, \quad \nabla_{\varepsilon_k, \tau} \hat{c} \rightarrow \hat{\mathcal{P}}, \hat{x}, \hat{c}.$$

By Theorem 2.4 we infer that $\hat{\varphi}$ is actually the first component of the state system (1.6)–(1.10) with $\varepsilon = 0$ and parameters $(\hat{\mathcal{P}}, \hat{x}, \hat{c})$. Now, on the one hand the minimality of $(\nabla_{\varepsilon_k, \tau} \hat{\mathcal{P}}, \nabla_{\varepsilon_k, \tau} \hat{x}, \nabla_{\varepsilon_k, \tau} \hat{c})$ for $(CP)_{ad}$ entails that

$$J_{ad}(\nabla_{\varepsilon_k, \tau} \hat{\mathcal{P}}, \nabla_{\varepsilon_k, \tau} \hat{x}, \nabla_{\varepsilon_k, \tau} \hat{c}) \leq J_{ad}(\nabla_{\tau} \hat{\mathcal{P}}, \nabla_{\tau} \hat{x}, \nabla_{\tau} \hat{c}) = \mathcal{J}(\nabla_{\tau} \hat{\mathcal{P}}, \nabla_{\tau} \hat{x}, \nabla_{\tau} \hat{c}),$$

so that passing to the superior limit to both sides leads to

$$\limsup_{k \to \infty} J_{ad}(\nabla_{\varepsilon_k, \tau} \hat{\mathcal{P}}, \nabla_{\varepsilon_k, \tau} \hat{x}, \nabla_{\varepsilon_k, \tau} \hat{c}) \leq J(\nabla_{\tau} \hat{\mathcal{P}}, \nabla_{\tau} \hat{x}, \nabla_{\tau} \hat{c}).$$

On the other hand, by the lower semicontinuity of $J_{ad}$, we also have that

$$\liminf_{k \to \infty} J_{ad}(\nabla_{\varepsilon_k, \tau} \hat{\mathcal{P}}, \nabla_{\varepsilon_k, \tau} \hat{x}, \nabla_{\varepsilon_k, \tau} \hat{c}) \geq J(\hat{\varphi}, \hat{\mathcal{P}}, \hat{x}, \hat{c}) = \mathcal{J}(\hat{\mathcal{P}}, \hat{x}, \hat{c}).$$

Since $\hat{\varphi}$ is the first state component of the system with $\varepsilon = 0$, and $\mathcal{J}(\hat{\mathcal{P}}, \hat{x}, \hat{c})$, combining the above inequalities with the optimality of $(\nabla_{\tau} \mathcal{P}, \nabla_{\tau} \mathcal{x}, \nabla_{\tau} \mathcal{c})$ for $(CP)_{\tau}$ yields directly

$$\hat{\mathcal{P}} = \nabla_{\tau} \mathcal{P}, \quad \hat{x} = \nabla_{\tau} \mathcal{x}, \quad \hat{c} = \nabla_{\tau} \mathcal{c},$$

from which also $\hat{\varphi} = \nabla_{\tau} \mathcal{P}$ by uniqueness of the state system (1.7.0)–(1.7.10) with $\varepsilon = 0$. Also, we have the chain of equalities

$$\lim_{k \to \infty} J_{ad}(\nabla_{\varepsilon_k, \tau} \mathcal{P}, \nabla_{\varepsilon_k, \tau} \mathcal{x}, \nabla_{\varepsilon_k, \tau} \mathcal{c}) = \liminf_{k \to \infty} J_{ad}(\nabla_{\varepsilon_k, \tau} \mathcal{P}, \nabla_{\varepsilon_k, \tau} \mathcal{x}, \nabla_{\varepsilon_k, \tau} \mathcal{c}) = \limsup_{k \to \infty} J_{ad}(\nabla_{\varepsilon_k, \tau} \mathcal{P}, \nabla_{\varepsilon_k, \tau} \mathcal{x}, \nabla_{\varepsilon_k, \tau} \mathcal{c}) = \mathcal{J}(\nabla_{\tau} \mathcal{P}, \nabla_{\tau} \mathcal{x}, \nabla_{\tau} \mathcal{c}).$$

As the same argument holds along every arbitrary subsequence $\{\varepsilon_k\}_k$, by uniqueness of the limits the convergences actually hold along the whole sequence $\varepsilon$, and the proof is concluded. \(\square\)

4.2.1 Letting $\varepsilon \to 0$ in the adjoint system

This section is devoted to discuss and analyze the asymptotic behavior of the adjoint system (3.7.0)–(3.7.4) as $\varepsilon \to 0$, which will be a key ingredient to derive the optimality conditions of $(CP)_{\tau}$. To begin with, let us state the established result.

Theorem 4.5. Assume $A1$ $A8$ and $C1$ $C4$. Let $(\mathcal{P}_{\tau}, \mathcal{x}_{\tau}, \mathcal{c}_{\tau}) \in \mathcal{U}_{ad}, \{((\mathcal{P}_{\varepsilon, \tau}, \mathcal{x}_{\varepsilon, \tau}, \mathcal{c}_{\varepsilon, \tau}))\}_{\varepsilon} \subset \mathcal{U}_{ad}$ be such that $(\mathcal{P}_{\varepsilon, \tau}, \mathcal{x}_{\varepsilon, \tau}, \mathcal{c}_{\varepsilon, \tau}) \rightarrow (\mathcal{P}_{\tau}, \mathcal{x}_{\tau}, \mathcal{c}_{\tau})$ as $\varepsilon \to 0$. Let $(\nabla_{\tau} \mathcal{P}, \nabla_{\tau} \mathcal{x}, \nabla_{\tau} \mathcal{c})$ and $(\nabla_{\varepsilon, \tau} \mathcal{P}, \nabla_{\varepsilon, \tau} \mathcal{x}, \nabla_{\varepsilon, \tau} \mathcal{c})$ be the unique solutions to the state system (1.6)–(1.10) in the cases $\varepsilon = 0$ with coefficients $(\mathcal{P}_{\tau}, \mathcal{x}_{\tau}, \mathcal{c}_{\tau})$ and $\varepsilon \in (0, \varepsilon_0)$ with coefficients $(\mathcal{P}_{\varepsilon, \tau}, \mathcal{x}_{\varepsilon, \tau}, \mathcal{c}_{\varepsilon, \tau})$, as given by Theorems 2.4 and 2.8 respectively. Let also $(\mathcal{P}_{\tau}, q_{\tau}, r_{\tau})$ be the unique solution to the adjoint system (4.7.10)–(4.7.14) with $\varepsilon \in (0, \varepsilon_0)$ and coefficients $(\mathcal{P}_{\varepsilon, \tau}, \mathcal{x}_{\varepsilon, \tau}, \mathcal{c}_{\varepsilon, \tau})$, as given by Theorem 3.3. Then, there exists a triplet $(p_{\tau}, q_{\tau}, r_{\tau})$, with

$$p_{\tau} \in L^\infty(0, T; V) \cap L^2(0, T; W), \quad q_{\tau} \in L^\infty(0, T; H), \quad p_{\tau} + \tau q_{\tau} \in H^1(0, T; V^*),$$

$$r_{\tau} \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W),$$

such that the above property also holds along the whole sequence $\varepsilon$.
such that, for every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$, as $\varepsilon \to 0$ it holds

\[
\begin{align*}
  p_{\varepsilon, \tau} &\to p_{\tau} \quad \text{weakly-* in } L^\infty(0, T; V) \cap L^2(0, T; W), \\
  q_{\varepsilon, \tau} &\to q_{\tau} \quad \text{weakly-* in } L^\infty(0, T; H), \\
  r_{\varepsilon, \tau} &\to r_{\tau} \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W),
\end{align*}
\]

Moreover, for every $v, w, z$ such that, for every $\alpha$

\[
\begin{align*}
  p_{\varepsilon, \tau} + \tau q_{\varepsilon, \tau} &\to p_{\tau} + \tau q_{\tau} \quad \text{weakly in } H^1(0, T; V^*), \\
  \varepsilon p_{\varepsilon, \tau} &\to 0 \quad \text{strongly in } H^1(0, T; H).
\end{align*}
\]

Moreover, $(p_{\tau}, q_{\tau}, r_{\tau})$ is the unique weak solution to the adjoint system \((3.70) - (3.74)\) with $\varepsilon = 0$ and coefficients $(P_{\tau}, \chi_{\tau}, C_{\tau})$, in the sense that

\[
- \langle \partial_t (p_{\tau} + \tau q_{\tau}), v \rangle_V + \int_\Omega (aq_{\tau} - J * q_{\tau})v + \int_\Omega F''(\varphi_{\tau})q_{\tau}v
\]

\[
+ \int_\Omega C_{\tau}f'(\varphi_{\tau})r_{\tau}v - \int_\Omega P_{\tau}f'(\varphi_{\tau})p_{\tau}v = \int_\Omega \beta_Q(\varphi_{\tau} - \varphi_Q)v,
\]

\[
\int_\Omega \nabla p_{\tau} \cdot \nabla w - \int_\Omega q_{\tau}w = 0,
\]

\[
- \int_\Omega \partial_t r_{\tau}z + \int_\Omega \nabla r_{\tau} \cdot \nabla z + \int_\Omega C_{\tau}f(\varphi_{\tau})r_{\tau}z
\]

\[
- \int_\Omega P_{\tau}f(\varphi_{\tau})p_{\tau}z - \int_\Omega \chi_{\tau}q_{\tau}z = 0,
\]

for every $v, w, z \in V$, almost everywhere in $(0, T)$, and

\[
(p_{\tau} + \tau q_{\tau})(T) = \beta_\Omega(\varphi_{\tau}(T) - \varphi_\Omega), \quad r_{\tau}(T) = 0.
\]

**Proof of Theorem 4.2.** We use the estimate \((3.81)\). First of all, by \((A8)\) and Theorem 2.4, we have that \(\{F''(\varphi_{\varepsilon, \tau})\}_{\varepsilon} \) is uniformly bounded in \(L^\infty(0, T; L^3(\Omega))\) and \(\{\varphi_{\varepsilon, \tau}\}_{\varepsilon} \) is uniformly bounded in \(L^2(0, T; V)\); hence, recalling that $\tau$ is fixed and $\eta = 0$, by the Gronwall lemma along with elliptic regularity theory, there exists a positive constant $M_{\tau}$, which may depend on $\tau$ but is independent of $\varepsilon$, such that

\[
\varepsilon^{1/2}\|p_{\varepsilon, \tau}\|_{H^1(0, T; H)} + \|p_{\varepsilon, \tau}\|_{L^\infty(0, T; V)} + \|q_{\varepsilon, \tau}\|_{L^\infty(0, T; H)}
\]

\[
+ \|r_{\varepsilon, \tau}\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \leq M_{\tau}.
\]

In particular, by the Hölder inequality it follows that

\[
\|F''(\varphi_{\varepsilon, \tau})q_{\varepsilon, \tau}\|_{L^\infty(0, T; L^{6/5}(\Omega))} \leq M_{\tau}.
\]

Secondly, elliptic regularity theory and equation \((3.71)\) entail that

\[
\|p_{\varepsilon, \tau}\|_{L^2(0, T; W)} \leq M_{\tau}.
\]

Moreover, since $L^{6/5}(\Omega) \hookrightarrow V^*$, a comparison argument in \((3.70)\) yields, by the boundedness of \(\{\varphi_{\varepsilon, \tau}\}_{\varepsilon} \) in \(L^\infty(Q)\) and the estimates above, that

\[
\|p_{\varepsilon, \tau} + \tau q_{\varepsilon, \tau}\|_{H^1(0, T; V^*)} \leq M_{\tau}.
\]

The Banach–Alaoglu theorem and classical compact embedding results (see, e.g., \([59]\)) allow us to obtain from the above a priori estimates the existence of functions $(p_{\tau}, q_{\tau}, r_{\tau})$ such that as
We claim that these limit variables yield a weak solution to the adjoint system (3.70)--(3.74) in which we formally set \( \varepsilon = 0 \). In this direction, we just need to justify the passage to the limit as \( \varepsilon \to 0 \) in the variational formulation for system (3.70)--(3.74) written for the triplet \((p_{\varepsilon,\tau}, q_{\varepsilon,\tau}, r_{\varepsilon,\tau})\), which reads

\[
- \int_{\Omega} \partial_t(p_{\varepsilon,\tau} + \tau q_{\varepsilon,\tau})v + \int_{\Omega} (aq_{\varepsilon,\tau} - J * q_{\varepsilon,\tau})v + \int_{\Omega} F''(\varphi_{\varepsilon,\tau})q_{\varepsilon,\tau}v \\
+ \int_{\Omega} C_{\varepsilon,\tau} \varphi_{\varepsilon,\tau} f'(\varphi_{\varepsilon,\tau})r_{\varepsilon,\tau}v - \int_{\Omega} P_{\varepsilon,\tau} \varphi_{\varepsilon,\tau} f'(\varphi_{\varepsilon,\tau})p_{\varepsilon,\tau}v = \int_{\Omega} \beta Q(\varphi_{\varepsilon,\tau} - \varphi Q)v,
\]

(4.87)

\[
- \int_{\Omega} \varepsilon \partial_t p_{\varepsilon,\tau} w + \int_{\Omega} \nabla p_{\varepsilon,\tau} \cdot \nabla w - \int_{\Omega} q_{\varepsilon,\tau} w = 0,
\]

(4.88)

\[
- \int_{\Omega} \partial_t r_{\varepsilon,\tau} z + \int_{\Omega} \nabla r_{\varepsilon,\tau} \cdot \nabla z + \int_{\Omega} C_{\varepsilon,\tau} f(\varphi_{\varepsilon,\tau})r_{\varepsilon,\tau}z - \int_{\Omega} P_{\varepsilon,\tau} f(\varphi_{\varepsilon,\tau})p_{\varepsilon,\tau}z - \int_{\Omega} \chi_{\varepsilon,\tau} q_{\varepsilon,\tau}z = 0,
\]

(4.89)

for every \( v, w, z \in V \) and almost every \( t \in (0, T) \) and also in the terminal conditions

\[
\varepsilon p_{\varepsilon,\tau}(T) = 0, \quad (p_{\varepsilon,\tau} + \tau q_{\varepsilon,\tau})(T) = \beta_{\Omega}(\varphi_{\varepsilon,\tau}(T) - \varphi), \quad r_{\varepsilon,\tau}(T) = 0.
\]

(4.90)

It is worth noting that since \( \varepsilon > 0 \) the second condition of (4.90) reduces to \( \tau q_{\varepsilon,\tau}(T) = \beta_{\Omega}(\varphi_{\varepsilon,\tau}(T) - \varphi) \). Moreover, from the convergences (2.27)--(2.28), we also have that, possibly after another extraction,

\[
\varphi_{\varepsilon,\tau} \to \varphi \quad \text{strongly in } C^0([0,T];L^2(\Omega)), \text{ and a.e. in } Q, (4.91)
\]

\[
\varphi_{\varepsilon,\tau} \to \varphi \quad \text{strongly in } C^0([0,T];V^* \cap L^2(0,T;L^2(\Omega))), (4.92)
\]

for every \( \alpha \geq 1 \) if \( d = 2 \) and \( 1 \leq \alpha < 6 \) if \( d = 3 \). Therefore, most part of the above limits are easy consequence of (2.27)--(2.28), the above estimates and Lebesgue’s dominated convergence theorem as well. For instance, due to the boundedness and continuity of \( f' \) we have, e.g., that \( f'(\varphi_{\varepsilon,\tau}) \to f'(\varphi) \text{ a.e. in } Q \) so that we easily infer that

\[
\int_{\Omega} C_{\varepsilon,\tau} \varphi_{\varepsilon,\tau} f'(\varphi_{\varepsilon,\tau})r_{\varepsilon,\tau}v \to \int_{\Omega} C_{\varepsilon,\tau} \varphi f'(\varphi)r_{\varphi}v \quad \text{for every } v \in V,
\]

and the other terms can be handled in a similar fashion. The only term which has to be treated differently is the one involving the potential. It can be dealt with invoking the almost everywhere convergence (4.91), the weak-\(*\) convergence (2.23), the continuous embedding \( V \hookrightarrow L^6(\Omega) \), and the Severini-Egorov theorem. In fact, these properties imply in particular that, as \( \varepsilon \to 0 \),

\[
F''(\varphi_{\varepsilon,\tau}) \to F''(\varphi) \quad \text{strongly in } L^\beta(Q) \quad \text{for all } \beta \in [1,3)
\]

so that from the weak-strong convergence principle, we get

\[
F''(\varphi_{\varepsilon,\tau})q_{\varepsilon,\tau} \to F''(\varphi)q \quad \text{weakly in } L^2(0,T;L^\gamma(\Omega)) \text{ for all } \gamma \in [1,6/5).
\]
It follows then that
\[ \int_{\Omega} F''(\bar{\varphi}_{\varepsilon,\tau}) q_{\varepsilon,\tau} v \rightarrow \int_{\Omega} F''(\bar{\varphi}_{\tau}) q_\tau v \quad \forall v \in W. \]

This is enough to pass to the limit in the variational formulation (4.87–4.89) as \( \varepsilon \to 0 \) for every \( v \in W, w, z \in V \), and to obtain the required terminal conditions. Since at the limit \( F''(\bar{\varphi}_{\tau}) q_\tau \in L^\infty(0,T;L^6(\Omega)) \), by the density of \( W \) in \( V \) the variational formulation holds also for all \( v \in V \). Thus, we realize that the limit variables obtained above yield a weak solution to (3.70)–(3.74) in which \( p \) that (\( \varepsilon \) is set to zero. By linearity and the estimate (4.81), we deduce that \( (p_\tau, q_\tau, r_\tau) \) is the unique weak solution to (3.70)–(3.74) with \( \varepsilon = 0 \), hence also that the convergences above hold along the entire sequence \( \varepsilon \to 0 \), and the proof is concluded. \( \square \)

4.2.2 Letting \( \varepsilon \to 0 \) in the optimality condition

In this last step, we draw some consequences from the approximation of controls presented in Theorem 4.4 and Subsection 4.2.1 by passing to the limit in the variational inequality (4.83) as \( \varepsilon \to 0 \). This allows to prove the optimality conditions of \((CP)_\tau\) as follows:

**Theorem 4.6.** Assume \( \textbf{A1-A8 and C1-C4} \) Then, every optimal control \((\bar{P}_\tau, \bar{X}_\tau, \bar{C}_\tau)\) of \((CP)_\tau\) necessarily verifies

\[
\int_Q (P - \bar{P}_\tau) \bar{\sigma}_f(f(\bar{\varphi}_{\tau})p_\tau - \int_Q (X - \bar{X}_\tau)\bar{\sigma}_r q_\tau - \int_Q (C - \bar{C}_\tau)\bar{\sigma}_f(f(\bar{\varphi}_{\tau}) r_\tau
\]
\[
+ \alpha P(\bar{P}_\tau - \bar{P}_\tau)P(\bar{P}_\tau) + \alpha_\chi(\bar{X}_\tau - \chi_\tau)(X - \bar{X}_\tau) + \alpha_\sigma(\bar{C}_\tau - \bar{C}_\tau)(C - \bar{C}_\tau) \geq 0
\]

for every \((P, X, C) \in \mathcal{U}_{ad}\),

where \((\bar{\varphi}_{\tau}, \bar{P}_\tau, \bar{\sigma}_\tau)\) and \((p_\tau, q_\tau, r_\tau)\) are the unique solutions to (1.6)–(1.10) and (3.70)–(3.74) with \( \varepsilon = 0 \) in the sense of Theorems 2.4 and 4.3, respectively.

**Proof.** Let \( \{(\bar{\varphi}_{\varepsilon,\tau}, \bar{X}_{\varepsilon,\tau}, \bar{C}_{\varepsilon,\tau})\}_{\varepsilon} \) be an approximating sequence of minimizers for \((CP)_{ad}\) as given by Lemma 4.4. Then, by Lemma 4.3 the corresponding states \( \{(\varphi_{\varepsilon,\tau}, \varphi_{\varepsilon,\tau}, \sigma_{\varepsilon,\tau})\}_{\varepsilon} \) and adjoint variables \( \{(p_{\varepsilon,\tau}, q_{\varepsilon,\tau}, r_{\varepsilon,\tau})\}_{\varepsilon} \) satisfy (4.83). By the convergences in Theorems 2.3, 4.4 and 4.5 the thesis follows letting \( \varepsilon \to 0 \) in (4.83) using the dominated convergence theorem. \( \square \)

4.3 The optimization problem \((CP)_\varepsilon\)

Here, we continue the asymptotic analysis of the optimization problem \((CP)_{\varepsilon,\tau}\), focusing on the case \( \tau \to 0 \), and keeping \( \varepsilon \) fixed instead. Namely, throughout the whole Section 4.3 we assume the following framework:

\[ \varepsilon \in (0,\varepsilon_0), \quad \beta_\Omega = 0, \quad (2.30)\rightarrow(2.32) \]
\[ \chi_{\text{max}} < \sqrt{c_\alpha}, \quad (\chi_{\text{max}} + \eta_{\text{max}} + 4c_\alpha \chi_{\text{max}})^2 < 8c_\alpha C_0, \quad \eta_{\text{max}}^2 + \chi_{\text{max}}^2 < \frac{4}{9} C_0. \]

This implies that every admissible control \((P, X, \eta, C) \in \mathcal{U}_{ad}\) automatically satisfies (2.29). Moreover, let us remark that the assumption \( \beta_\Omega = 0 \) is rather unpleasant since it prevents us to control the tumor distribution at the terminal time \( T \). However, from (3.74) it is clear that this compatibility condition has to be imposed in the scenario \( \varepsilon > 0 \) and \( \tau = 0 \).

Existence of optimal controls for \((CP)_\varepsilon\) is given in the following result.

**Theorem 4.7.** Assume \( \textbf{A1-A7 and C1-C3} \). Then, the optimization problem \((CP)_\varepsilon\) admits a solution.
\textbf{Proof.} This result follows directly by adapting the direct method used in the proof of Theorem 3.1 taking into account the compactness of \( U_{\text{ad}} \) and the convergence presented in Theorem 2.5.

The main goal is to obtain now necessary conditions for optimality of \((CP)_\varepsilon\). The idea is to proceed on the same line of Section 4.2 by passing to the limit as \( \tau \to 0 \) in the adjoint problem and in the first order conditions for optimality for the adapted optimization problem \((CP)_{\text{ad}}^{\varepsilon, \tau}\). The main difference with respect to the case \( \varepsilon \to 0 \) is that the state system \((1.6)-(1.10)\) with \( \tau = 0 \) does not have a unique solution, as stated in Theorem 2.5. Notice that also uniqueness has been established in [53] as a consequence of a suitable error estimate between the solutions to the \( \varepsilon, \tau \) and \( \varepsilon \) problems. However, that result forces the authors to restrict to assume \( \eta = 0 \). This means that under no additional requirements on the data (in particular, if \( \eta_{\text{max}} > 0 \)), the control-to-state map is not even well-defined when \( \tau = 0 \). The main problem is that, despite Theorem 2.5, for a given minimizer \((\varphi_{\varepsilon}, P_{\varepsilon}, \chi_{\varepsilon}, \eta_{\varepsilon}, C_{\varepsilon})\) of \((CP)_{\varepsilon}\), it is not necessarily true that the state \( \varphi_{\varepsilon} \) can be approximated by some corresponding state solutions \( \{\varphi_{\varepsilon, \tau}\}_\tau \) of the state system with \( \varepsilon, \tau > 0 \) as \( \tau \to 0 \). For this reason, it is important that the adapted cost functional is modified accordingly, accounting also for the phase variable. Namely, in this section, given a certain minimizer \((\overline{\varphi}_{\varepsilon}, \overline{P}_{\varepsilon}, \overline{\chi}_{\varepsilon}, \overline{\eta}_{\varepsilon}, \overline{C}_{\varepsilon})\) for \((CP)_{\varepsilon}\), we consider the following adapted cost functional:

\[
J_{\text{ad}}(\varphi, P, \chi, \eta, C) := J(\varphi, P, \chi, \eta, C) + \frac{1}{2} \|\varphi - \overline{\varphi}\|^2_{L^2(Q)} + \frac{1}{2} \|P - \overline{P}\|^2 + \frac{1}{2} |\chi - \overline{\chi}|^2 + \frac{1}{2} |\eta - \overline{\eta}|^2 + \frac{1}{2} |C - \overline{C}|^2.
\]

The adapted optimization problem \((CP)_{\text{ad}}^{\varepsilon, \tau}\) is then defined exactly as in (4.82) with this new definition for the cost functional.

Arguing as in Section 4.2 we straightforwardly infer the corresponding of Lemmas 4.2–4.3 and Theorem 4.4 whose proofs are omitted since can be reproduced in the same fashion. These results concern solvability of \((CP)_{\text{ad}}^{\varepsilon, \tau}\), necessary conditions for \((CP)_{\text{ad}}^{\varepsilon, \tau}\), and approximation \((CP)_{\text{ad}}^{\varepsilon, \tau} \to (CP)_{\varepsilon}\) as \( \tau \to 0 \). Let us just point out that since the cost functional is corrected also with respect to the state variable, the forcing term of the corresponding adjoint system has to be corrected too, with no additional effort though.

\textbf{Lemma 4.8.} Assume A1–A7 and C1–C3. Then, for every \( \tau \in (0, \tau_0) \) and for every minimizer \((\overline{\varphi}_{\varepsilon}, \overline{P}_{\varepsilon}, \overline{\chi}_{\varepsilon}, \overline{\eta}_{\varepsilon}, \overline{C}_{\varepsilon})\) of \((CP)_{\varepsilon}\), the optimization problem \((CP)_{\text{ad}}^{\varepsilon, \tau}\) admits a minimizer.

\textbf{Lemma 4.9.} Assume A1–A7 and C1–C4, and let \((\overline{\varphi}_{\varepsilon, \tau}, \overline{P}_{\varepsilon, \tau}, \overline{\chi}_{\varepsilon, \tau}, \overline{\eta}_{\varepsilon, \tau}, \overline{C}_{\varepsilon, \tau})\) be a minimizer for \((CP)_{\varepsilon}\). For every \( \varepsilon \in (0, \tau_0) \), if \((\overline{\varphi}_{\varepsilon, \tau}, \overline{P}_{\varepsilon, \tau}, \overline{\chi}_{\varepsilon, \tau}, \overline{\eta}_{\varepsilon, \tau}, \overline{C}_{\varepsilon, \tau})\) is an optimal control for \((CP)_{\text{ad}}^{\varepsilon, \tau}\), then the following first-order necessary condition holds

\[
\int_Q (P - \overline{P}) \varphi_{\varepsilon, \tau} f(\varphi_{\varepsilon, \tau}) p_{\varepsilon, \tau} - \int_Q (\chi - \overline{\chi}) \varphi_{\varepsilon, \tau} q_{\varepsilon, \tau} - \int_Q (C - \overline{C}) \varphi_{\varepsilon, \tau} r_{\varepsilon, \tau} + (P - \overline{P}) (\alpha p(\overline{P}_{\varepsilon, \tau} - P_{\varepsilon, \tau}) + (\overline{P}_{\varepsilon, \tau} - \overline{P}_{\varepsilon})) + (\chi - \overline{\chi}) (\alpha \chi (\overline{\chi}_{\varepsilon, \tau} - \chi_{\varepsilon, \tau}) + (\overline{\chi}_{\varepsilon, \tau} - \overline{\chi}_{\varepsilon})) + \eta \varphi_{\varepsilon, \tau} (\alpha \varphi_{\varepsilon, \tau} - \alpha \varphi_{\varepsilon}) + (\overline{\varphi}_{\varepsilon, \tau} - \varphi_{\varepsilon, \tau}) \geq 0
\]

for every \((P, \chi, \eta, C) \in U_{\text{ad}}\), where \((\varphi_{\varepsilon, \tau}, P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, \eta_{\varepsilon, \tau}, C_{\varepsilon, \tau})\) are the corresponding unique solutions to the state system \((1.6)-(1.10)\) and the adjoint system \((8.70)-(8.74)\) with \( \varepsilon, \tau > 0 \) and with respect to the coefficients \((\overline{P}_{\varepsilon, \tau}, \overline{\chi}_{\varepsilon, \tau}, \overline{\eta}_{\varepsilon, \tau}, \overline{C}_{\varepsilon, \tau})\), the right-hand side of \((8.70)\) being modified as \( \beta Q(\overline{\varphi}_{\varepsilon, \tau} - \varphi_{\varepsilon})\) in \((\overline{\varphi}_{\varepsilon, \tau} - \varphi_{\varepsilon})\).

\textbf{Theorem 4.10.} Assume A1 A7 C1 C4 and let \((\overline{\varphi}_{\varepsilon}, \overline{P}_{\varepsilon}, \overline{\chi}_{\varepsilon}, \overline{\eta}_{\varepsilon}, \overline{C}_{\varepsilon})\) be a minimizer for \((CP)_{\varepsilon}\). Then, for every family of optimal controls \(\{\overline{P}_{\varepsilon, \tau}, \overline{\chi}_{\varepsilon, \tau}, \overline{\eta}_{\varepsilon, \tau}, \overline{C}_{\varepsilon, \tau}\}_\tau\) for \((CP)_{\text{ad}}^{\varepsilon, \tau}\), with corresponding states \(\{(\overline{\varphi}_{\varepsilon, \tau}, \overline{P}_{\varepsilon, \tau}, \overline{\chi}_{\varepsilon, \tau}, \overline{\eta}_{\varepsilon, \tau})\}_\tau\), as \( \tau \to 0 \) it holds that, for every \( \alpha \geq 1 \) if \( d = 2 \) and \( 1 \leq \alpha < 6 \) if
Proof. The proof is analogous to the one of Theorem 4.4, the only difference being that the identification of the limit $\hat{\phi} = \varphi_{\epsilon}$ follows from the additional correction in the cost functional (and not from the uniqueness of the state system with $\tau = 0$, which is indeed not true). The strong convergence of $\{\varphi_{\epsilon,\tau}\}$ is a consequence of the convergences in Theorem 2.5 and the compact inclusion $V \hookrightarrow L^\alpha(\Omega)$ for every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$. \hfill $\Box$

4.3.1 Letting $\tau \to 0$ in the adjoint system

In this section we study the passage to the limit as $\tau \searrow 0$ in the adjoint system (3.70)–(3.74), where the forcing term of (3.70) is modified as stated in Lemma 4.9.

**Theorem 4.11.** Assume $A1, A8$ and $C1, C4$. Let the parameters $(P_{\epsilon}, \chi_{\epsilon}, \eta_{\epsilon}, C_{\epsilon}) \in \mathcal{U}_{ad}$ and $\{(P_{\epsilon,\tau}, \chi_{\epsilon,\tau}, \eta_{\epsilon,\tau}, C_{\epsilon,\tau})\}_{\tau < \mathcal{U}_{ad}}$ be such that $(P_{\epsilon,\tau}, \chi_{\epsilon,\tau}, \eta_{\epsilon,\tau}, C_{\epsilon,\tau}) \to (P_{\epsilon}, \chi_{\epsilon}, \eta_{\epsilon}, C_{\epsilon})$ as $\tau \to 0$. Let $(\varphi_{\epsilon,\tau}, p_{\epsilon,\tau}, q_{\epsilon,\tau})$ be the solution to the state system (1.6)–(1.10) with $\tau > 0$ and coefficients $(P_{\epsilon,\tau}, \chi_{\epsilon,\tau}, \eta_{\epsilon,\tau}, C_{\epsilon,\tau})$ as given by Theorem 2.5, and let $(\varphi_{\epsilon,\tau}, p_{\epsilon,\tau}, q_{\epsilon,\tau})$ be the solution to the adjoint system (3.70)–(3.74) with $\epsilon, \tau > 0$, coefficients $(P_{\epsilon,\tau}, \chi_{\epsilon,\tau}, \eta_{\epsilon,\tau}, C_{\epsilon,\tau})$, and forcing term in (3.70) given by $\beta_{Q}(\varphi_{\epsilon,\tau} - \varphi Q) + (\varphi_{\epsilon,\tau} - \varphi_{\epsilon})$. Then, there exist a triplet $(p_{\epsilon}, q_{\epsilon}, r_{\epsilon})$, with

$$p_{\epsilon,\tau} \to p_{\epsilon} \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad q_{\epsilon} \in L^2(0, T; H),$$

such that as $\tau \to 0$, for every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$, it holds that

$$p_{\epsilon,\tau} \to p_{\epsilon} \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W),$$

$$q_{\epsilon,\tau} \to q_{\epsilon} \quad \text{weakly in } L^2(0, T; H),$$

$$r_{\epsilon,\tau} \to r_{\epsilon} \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W),$$

$$\tau q_{\epsilon,\tau} \to 0 \quad \text{strongly in } H^1(0, T; L^1(\Omega)) \cap L^2(0, T; H).$$

Moreover, $(p_{\epsilon}, q_{\epsilon}, r_{\epsilon})$ is the unique weak solution to the adjoint system (3.70)–(3.74) with $\tau = 0$ and coefficients $(P_{\epsilon}, \chi_{\epsilon}, \eta_{\epsilon}, C_{\epsilon})$, in the sense that

$$- \int_{\Omega} \partial_t p_{\epsilon} v - \int_{\Omega} (aq_{\epsilon} - J * q_{\epsilon}) v + \eta_{\epsilon} \int_{\Omega} \Delta r_{\epsilon} v + \int_{\Omega} F''(\varphi_{\epsilon}) q_{\epsilon} v$$

$$+ \int_{\Omega} C_{\epsilon} \varphi_{\epsilon} f'(\varphi_{\epsilon}) r_{\epsilon} v - \int_{\Omega} P_{\epsilon} \varphi_{\epsilon} f'(\varphi_{\epsilon}) p_{\epsilon} v = \int_{\Omega} \beta_{Q}(\varphi_{\epsilon} - \varphi Q) v,$$

$$- \varepsilon \int_{\Omega} \partial_t p_{\epsilon} w + \int_{\Omega} \nabla p_{\epsilon} \cdot \nabla w - \int_{\Omega} q_{\epsilon} w = 0,$$

$$- \int_{\Omega} \partial_t r_{\epsilon} z + \int_{\Omega} \nabla r_{\epsilon} \cdot \nabla z + \int_{\Omega} C_{\epsilon} f(\varphi_{\epsilon}) r_{\epsilon} z - \int_{\Omega} P_{\epsilon} f(\varphi_{\epsilon}) p_{\epsilon} z - \int_{\Omega} \chi_{\epsilon} q_{\epsilon} z = 0,$$

for every $v, w, z \in V$, almost everywhere in $(0, T)$, and

$$p_{\epsilon}(T) = 0, \quad r_{\epsilon}(T) = 0.$$
Proof of Theorem 3.1. We proceed by pointing out some a priori estimates on the adjoint variables uniformly with respect to $\tau$, using again the estimate (3.81) as a starting point. First of all, by Theorem 2.3 we have that $\{F(\varphi_{e,\tau})\}_\tau$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, so that by Assumption 8 we have that $\{F''(\varphi_{e,\tau})\}_\tau$ is uniformly bounded in $L^\infty(0,T;H)$. Secondly, by the assumption on the kernel $J$, the Young inequality, and by comparison in equation (3.71) we have

$$
\int_{Q_T^\prime} (J * q_{e,\tau}) q_{e,\tau} \leq \int_{Q_T^\prime} \int_t^T |J * q_{e,\tau}| \|q_{e,\tau}\| \, dt \leq (a^* + b^*) \int_{Q_T^\prime} \|q_{e,\tau}\| \|q_{e,\tau}\| = \delta \int_{Q_T^\prime} |q_{e,\tau}|^2 + \frac{(a^* + b^*)^2}{2\delta} \int_{Q_T^\prime} \|q_{e,\tau}\|^2 + \frac{(a^* + b^*)^2}{2\delta} \int_{Q_T^\prime} \|\partial_t p_{e,\tau}\|^2
$$

while the last two terms on the right-hand side of (3.81) can be bounded as

$$
- \int_{Q_T^\prime} \eta_{e,\tau} \Delta r_{e,\tau} q_{e,\tau} - \int_{Q_T^\prime} \eta_{e,\tau} q_{e,\tau} (\partial_t r_{e,\tau} + \Delta r_{e,\tau}) \leq \delta \int_{Q_T^\prime} |q_{e,\tau}|^2 + \frac{3(\eta_{e,\tau}^2 + \chi_{e,\tau}^2)}{4\delta} \int_{Q_T^\prime} |\Delta r_{e,\tau}|^2 + \frac{3\chi_{e,\tau}^2}{4\delta} \int_{Q_T^\prime} |\partial_t r_{e,\tau}|^2.
$$

Hence, recalling that $\{\varphi_{e,\tau}\}_\tau$ is uniformly bounded in $L^2(0,T;V)$, all the terms in (3.81) can be rearranged and treated by the Gronwall lemma, provided to fix $\delta, \delta' > 0$ such that

$$
3\delta < C_0, \quad \frac{(a^* + b^*)^2}{2\delta} \varepsilon < 1, \quad \frac{3(\eta_{e,\tau}^2 + \chi_{e,\tau}^2)}{4\delta} < 1, \quad \delta' < 1 - \frac{3(\eta_{e,\tau}^2 + \chi_{e,\tau}^2)}{4\delta}.
$$

An easy computation shows that this is possible if and only if

$$
\max \left\{ \frac{(a^* + b^*)^2}{2\delta} \varepsilon, \frac{3(\eta_{e,\tau}^2 + \chi_{e,\tau}^2)}{4\delta} \right\} < C_0
$$

which is indeed true by Assumption 6 and the fact that $\eta_{\max}^2 + \chi_{\max}^2 < \frac{4}{9} C_0$. Hence, (3.81) can be closed uniformly in $\tau$, and we obtain

$$
\|p_{e,\tau}\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \tau^{1/2} \|q_{e,\tau}\|_{L^\infty(0,T;H)} + \|r_{e,\tau}\|_{L^2(0,T;H)} + \|r_{e,\tau}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq M_e
$$

for a positive constant $M_e$ which may depend on $\varepsilon$ but it is independent of $\tau$. Then, elliptic regularity theory and (3.71) lead us to infer that

$$
\|p_{e,\tau}\|_{L^2(0,T;W)} \leq M_e.
$$

Moreover, noting that by the Hölder inequality we have that

$$
\|F''(\varphi_{e,\tau}) q_{e,\tau}\|_{L^2(0,T;L^1(\Omega))} \leq M_e,
$$

arguing as in the proof of Theorem 4.5 by a comparison argument in (3.70) we deduce that

$$
\tau \|q_{e,\tau}\|_{H^1(0,T;L^1(\Omega))} \leq M_e.
$$

Recalling the continuous embedding $W \hookrightarrow L^\infty(\Omega)$, Banach–Alaoglu theorem and standard compactness results allow us to obtain from the above a priori estimates that there exist functions $(p_{e,\tau}, q_{e,\tau}, r_{e,\tau})$ such that as $\tau \to 0$ it holds, along a non-relabeled subsequence,

- $p_{e,\tau} \to p_e$ weakly-* in $H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$,
- $q_{e,\tau} \to q_e$ weakly in $L^2(0,T;H)$,
- $r_{e,\tau} \to r_e$ weakly-* in $H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$, strongly in $C^0([0,T];L^2(\Omega)) \cap L^2(0,T;V)$,
- $\tau q_{e,\tau} \to 0$ weakly in $H^1(0,T;W^*)$ and strongly in $L^2(0,T;H)$,
for every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$. Arguing as in the proof of Theorem 4.5, we exploit the above convergences to pass to the limit in the variational formulation of the adjoint system given by (1.87)–(1.89) and in the terminal conditions (4.90). As a by-product, we obtain that the above limits are a weak solution to (3.70)–(3.74) with $\tau = 0$. To this end, note that by (2.33) and (2.38)–(2.39), along a non-relabelled subsequence, for every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$, we have that

$$\overline{\varphi}_{\varepsilon, \tau} \rightarrow \overline{\varphi}_{\varepsilon} \quad \text{a.e. in } Q, \quad \overline{\varphi}_{\varepsilon, \tau} \rightarrow \overline{\varphi}_{\varepsilon}, \quad \overline{\varphi}_{\varepsilon, \tau} \rightarrow \overline{\varphi}_{\varepsilon} \quad \text{strongly in } L^2(0, T; L^\alpha(\Omega)).$$

Consequently, all terms in (4.87)–(4.89) and (4.90) pass to the weak limit as $\tau \rightarrow 0$. The only delicate term to treat, as usual, is the one containing $F''$: let us spend a few words on this. By continuity of $F''$ it follows that, as $\tau \rightarrow 0$,

$$F''(\overline{\varphi}_{\varepsilon, \tau}) \rightarrow F''(\overline{\varphi}_{\varepsilon}) \quad \text{a.e. in } Q.$$

Now, since $\{F(\overline{\varphi}_{\varepsilon, \tau})\}_{\tau}$ is bounded in $L^\infty(0, T; L^1(\Omega))$, by A8 we know that $\{F''(\overline{\varphi}_{\varepsilon, \tau})\}_{\tau}$ is bounded in $L^\infty(0, T; H)$. Furthermore, the boundedness of $\{\varphi_{\varepsilon, \tau}\}_{\tau}$ in $L^2(0, T; L^\alpha(\Omega))$ and again A8 ensure also that $\{F''(\overline{\varphi}_{\varepsilon, \tau})\}_{\tau}$ is uniformly bounded in $L^1(0, T; L^3(\Omega))$. For any $\vartheta \in (0, 1)$, setting $\beta_{\vartheta} \in (2, 3)$ such that $\frac{1}{\beta_{\vartheta}} := \frac{\vartheta}{2} + \frac{1-\vartheta}{3}$, by interpolation we have that

$$\|F''(\overline{\varphi}_{\varepsilon, \tau})\|_{\beta_{\vartheta}} \leq \|F''(\overline{\varphi}_{\varepsilon, \tau})\|^{\vartheta}_{\beta} \|F''(\overline{\varphi}_{\varepsilon, \tau})\|_{3}^{1-\vartheta} \quad \text{a.e. in } (0, T),$$

from which it follows that

$$\|F''(\overline{\varphi}_{\varepsilon, \tau})\|_{L^1(0, T; L^{\beta_{\vartheta}}(\Omega))} \leq M_{\varepsilon}.$$

In particular, there exists $\overline{\vartheta} \in (0, 1)$ such that $\beta := \beta_{\overline{\vartheta}} = \frac{1}{1-\overline{\vartheta}} \in (2, 3)$: an easy computation yields $\overline{\vartheta} = \frac{4}{5}$ and $\beta = \frac{7}{3}$. This implies that

$$\|F''(\overline{\varphi}_{\varepsilon, \tau})\|_{L^{7/3}(Q)} \leq M_{\varepsilon}.$$

By the Severini-Egorov theorem we infer that, for all $\beta \in [1, \frac{7}{3})$,

$$F''(\overline{\varphi}_{\varepsilon, \tau}) \rightarrow F''(\overline{\varphi}_{\varepsilon}) \quad \text{weakly in } L^{7/3}(Q) \text{ and strongly in } L^\beta(Q).$$

In particular, since $\frac{7}{3} > 2$, this implies that

$$F''(\overline{\varphi}_{\varepsilon, \tau})q_{\varepsilon, \tau} \rightarrow F''(\overline{\varphi}_{\varepsilon})q_{\varepsilon} \quad \text{weakly in } L^1(Q).$$

Since $W \hookrightarrow L^\infty(\Omega)$, this allows to pass to the limit as $\tau \rightarrow 0$ in (1.87) for every test function $v \in W$. Since $F''(\overline{\varphi}_{\varepsilon}) \in L^1(0, T; L^3(\Omega))$, at the limit we have that $F''(\overline{\varphi}_{\varepsilon})q_{\varepsilon} \in L^{6/5}(\Omega) \hookrightarrow V^*$ almost everywhere in $(0, T)$, and the variational formulation holds also for all $v \in V$ by the density of $W$ in $V$.

Finally, by linearity of the system, the same estimates yield also uniqueness of $(p_{\varepsilon}, q_{\varepsilon}, r_{\varepsilon})$, and the convergences holds along the entire sequence $\tau \rightarrow 0$, as desired.

\[\square\]

### 4.3.2 Letting $\tau \rightarrow 0$ in the optimality condition

Lastly, we argue as in Theorem 4.6 to pass to the limit as $\tau \rightarrow 0$ to establish the necessary conditions for optimality of $(CP)_{\varepsilon}$. 

---
Theorem 4.12. Assume A1–A8 and C1–C4. Then, every minimizer $(\varphi_\varepsilon, \mathcal{P}_\varepsilon, \mathcal{X}_\varepsilon, \eta_\varepsilon, \sigma_\varepsilon)$ of $(CP)_\varepsilon$ necessarily satisfies

\[
\int_Q (P - \mathcal{P}_\varepsilon) \sigma f(\varphi_\varepsilon)p_\varepsilon - \int_Q (\chi - \mathcal{X}_\varepsilon) \sigma \eta q_\varepsilon - \int_Q (\mathcal{C} - \mathcal{C}_\varepsilon) \sigma f(\varphi_\varepsilon) r_\varepsilon
\]

\[
+ \alpha_p (\mathcal{P}_\varepsilon - \mathcal{P}_\ast)(\mathcal{P}_\varepsilon - \mathcal{P}_\ast) + \alpha_\sigma (\chi - \mathcal{X}_\ast)(\chi - \mathcal{X}_\ast)
\]

\[
+ \alpha_\eta (\eta - \eta_\varepsilon)(\eta - \eta_\varepsilon) + \alpha_\mathcal{C}(\mathcal{C}_\varepsilon - \mathcal{C}_\ast)(\mathcal{C}_\varepsilon - \mathcal{C}_\ast) \geq 0 \quad \text{for every } (\mathcal{P}, \chi, \eta, \mathcal{C}) \in \mathcal{U}_{ad},
\]

where $(\varphi_\varepsilon, \mathcal{P}_\varepsilon, \sigma_\varepsilon)$ is a solution to the state system \[(1.6) - (1.10)\] and and $(p_\varepsilon, q_\varepsilon, r_\varepsilon)$ is the unique weak solution to the adjoint system \[(3.70) - (3.74)\] with $\tau = 0$ and coefficients $(\mathcal{P}_\varepsilon, \mathcal{X}_\varepsilon, \eta_\varepsilon, \sigma_\varepsilon)$, in the sense of Theorems 2.5 and Theorem 4.11, respectively.

Proof. The proof is analogous to the one of Theorem 4.6 using Lemma 4.8 and 4.9 and Theorems 2.5, 4.10, and 4.11 instead.

4.4 The optimization problem $(\overline{CP})$

In this last section we deal with the optimization problem $(\overline{CP})$, by letting $\varepsilon, \tau \to 0$ jointly. Since most of the ideas have already been explained and motivated in detail in the previous Sections 4.2–4.3, we proceed here more quickly, avoiding technical details for brevity. Throughout the whole Section 4.4, we assume the following framework:

\[
\eta_{\max} = \alpha_{\max} = 0, \quad \beta_\Omega \varphi_\Omega \in V, \quad (2.30) - (2.32),
\]

\[
\chi_{\max} < \sqrt{c_a}, \quad (\chi_{\max} + \eta_{\max} + 4c_a \chi_{\max})^2 < 8c_a C_0, \quad \eta_{\max}^2 + \chi_{\max}^2 < \frac{4}{9} C_0.
\]

As in Section 4.2 since $\eta_{\max} = 0$, we shall consider $\mathcal{U}_{ad}$ as a subset of $\mathbb{R}^3$ instead, and write $(\mathcal{P}, \chi, \mathcal{C}) \in \mathcal{U}_{ad}$ for the generic admissible control.

As usual, existence of optimal controls for $(\overline{CP})$ is given in the following first result.

Theorem 4.13. Assume A1–A8 and C1–C3. Then, the optimization problem $(\overline{CP})$ admits a solution.

Proof. This result follows directly by adapting the direct method used in the proof of Theorem 3.1, taking into account the compactness of $\mathcal{U}_{ad}$ and the convergence pointed out in Theorem 2.6.

Now, we investigate necessary conditions for optimality. First of all, note that for every admissible control $(\mathcal{P}, \chi, \mathcal{C})$, the state system \[(1.6) - (1.10)\] with $\varepsilon = \tau = 0$ admits a unique solution by Theorem 2.6. Consequently, when introducing the adapted cost functional, by contrast with Section 4.3, it is not necessary here to use a perturbation with respect to the phase variable.

Nevertheless, looking at the final condition \[(3.74)\] and taking formally $\varepsilon = \tau = 0$, we immediately see that $p_{\varepsilon,\tau}(T) = 0$ for all $\varepsilon \in (0, \varepsilon_0)$ while at the limit $p(T) = \beta_\Omega (\mathcal{P}(T) - \varphi_\Omega)$. This immediately suggests that if $\beta_\Omega > 0$, we cannot pass to the joint limit $\varepsilon, \tau \to 0$ in the adjoint problem \[(3.70) - (3.74)\] as it is. At an intuitive level for the moment, the limit adjoint problem \[(3.70) - (3.74)\] with $\varepsilon = \tau = 0$ is still well-posed also when $\beta_\Omega > 0$. These heuristic considerations suggest that the right assumption is to keep a generic $\beta_\Omega \geq 0$, but to modify the final condition for $p_{\varepsilon,\tau}$ at the approximate level in a smart way, in order to recover the compatibility condition $p_{\varepsilon,\tau}(T) = \beta_\Omega (\mathcal{P}(T) - \varphi_\Omega)$ also when $\varepsilon, \tau > 0$. In order to do this, we introduce a correction in
the adapted cost functional, depending on a suitable combination of the terminal values of both the variables \( \varphi \) and \( \mu \).

For a given optimal control \((\overline{P}, \overline{\chi}, \overline{C})\) of \((\overline{CP})\) and for all \( \varepsilon \in (0, \varepsilon_0) \) and \( \tau \in (0, \tau_0) \), the idea is to set

\[
\mathcal{J}_{\text{ad}}(\varphi, \mu, P, \chi, C) := \mathcal{J}(\varphi, P, \chi, C) + (\varepsilon \mu(T) + \beta_\Omega(\varphi(T) - \varphi_\Omega))_H
+ \frac{1}{2}|P - \overline{P}|^2 + \frac{1}{2}|\chi - \overline{\chi}|^2 + \frac{1}{2}|C - \overline{C}|^2
\]

and define the adapted optimization problem \((CP)^{\text{ad}}_{\varepsilon, \tau}\) as in (4.82).

It is clear that the optimality condition for \((CP)\) follows similar lines of the previous sections, by firstly obtaining the approximating sequence of optimal controls of minimizers of \((CP)^{\text{ad}}_{\varepsilon, \tau}\) and then pass to the limit as \( \varepsilon, \tau \to 0 \). The major difference is the nature of the correction in the adapted cost functional, which yields a correction in the terminal values of the adapted adjoint system at \( \varepsilon, \tau > 0 \): for this reason, we recall such corrected adjoint system explicitly in Lemma 4.15 below. The proof of first-order conditions for optimality at the level \( \varepsilon, \tau > 0 \) follows, mutatis mutandis, the proof of Theorem 3.3 and is omitted for brevity.

The following results concern existence of optimal controls and first-order necessary conditions for \((CP)^{\text{ad}}_{\varepsilon, \tau}\), and the convergence \((CP)^{\text{ad}}_{\varepsilon, \tau} \searrow (CP)\) as \( \varepsilon, \tau \to 0 \).

**Lemma 4.14.** Assume \([A1] [A8] \) and \([C1] [C4] \). Then, for every optimal control \((\overline{P}, \overline{\chi}, \overline{C})\) for \((\overline{CP})\), for every \( \varepsilon \in (0, \varepsilon_0) \) and \( \tau \in (0, \tau_0) \) the optimization problem \((CP)^{\text{ad}}_{\varepsilon, \tau}\) admits a solution.

**Lemma 4.15.** Assume \([A1] [A8] \) and \([C1] [C4] \), and let \((\overline{P}, \overline{\chi}, \overline{C})\) be an optimal control for \((\overline{CP})\). For every \( \varepsilon \in (0, \varepsilon_0) \) and \( \tau \in (0, \tau_0) \), if \((P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau})\) is an optimal control of \((CP)^{\text{ad}}_{\varepsilon, \tau}\), then the following first-order necessary condition holds

\[
\int_Q (P - \overline{P}_{\varepsilon, \tau}) \sigma_{\varepsilon, \tau} f(\overline{\varphi}_{\varepsilon, \tau}) p_{\varepsilon, \tau} - \int_Q (\chi - \overline{\chi}_{\varepsilon, \tau}) \overline{\sigma}_{\varepsilon, \tau} q_{\varepsilon, \tau} - \int_Q (C - \overline{C}_{\varepsilon, \tau}) \overline{\sigma}_{\varepsilon, \tau} f(\overline{\varphi}_{\varepsilon, \tau}) r_{\varepsilon, \tau}
+ (P - \overline{P}_{\varepsilon, \tau})(\alpha P_{\varepsilon, \tau} + \Phi) + (\chi - \overline{\chi}_{\varepsilon, \tau})(\alpha C_{\varepsilon, \tau} - \Phi) + (\overline{C}_{\varepsilon, \tau} - \overline{C}) \geq 0 \quad \text{for every } (P, \chi, C) \in \mathcal{U}_{\text{ad}},
\]

where \((\overline{\varphi}_{\varepsilon, \tau}, P_{\varepsilon, \tau}, \overline{\varphi}_{\varepsilon, \tau})\) is the unique solution to (4.96)–(4.10) with \( \varepsilon, \tau > 0 \) and parameters \((P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau})\), and \((p_{\varepsilon, \tau}, q_{\varepsilon, \tau}, r_{\varepsilon, \tau})\) is the unique solution to the following adapted adjoint system

\[
\begin{align*}
- \partial_t(p_{\varepsilon, \tau} + \tau q_{\varepsilon, \tau}) + a q_{\varepsilon, \tau} - J * q_{\varepsilon, \tau} + F''(\overline{\varphi}_{\varepsilon, \tau}) q_{\varepsilon, \tau}
&\quad + \overline{C} f'(\overline{\varphi}_{\varepsilon, \tau}) r_{\varepsilon, \tau} - (\overline{P} \overline{\sigma}_{\varepsilon, \tau} - \mathcal{A}) f'(\overline{\varphi}_{\varepsilon, \tau}) p_{\varepsilon, \tau} = \beta Q(\overline{\varphi}_{\varepsilon, \tau} - \varphi Q) \quad \text{in } Q, \\
- \varepsilon \partial_t p_{\varepsilon, \tau} - \Delta p_{\varepsilon, \tau} - q_{\varepsilon, \tau} = 0 \quad &\text{in } Q, \\
- \partial_t r_{\varepsilon, \tau} - \Delta r_{\varepsilon, \tau} + (B + \overline{C} f'(\overline{\varphi}_{\varepsilon, \tau})) r_{\varepsilon, \tau} - \overline{P} f'(\overline{\varphi}_{\varepsilon, \tau}) p_{\varepsilon, \tau} - \overline{\chi} q_{\varepsilon, \tau} = 0 \quad &\text{in } Q, \\
\partial_n p_{\varepsilon, \tau} = \partial_n r_{\varepsilon, \tau} = 0 \quad &\text{on } \Sigma, \\
\varepsilon p_{\varepsilon, \tau}(T) = \varepsilon \beta_\Omega(\overline{\varphi}_{\varepsilon, \tau}(T) - \varphi_\Omega), \\
(p_{\varepsilon, \tau} + \tau q_{\varepsilon, \tau})(T) = \beta_\Omega(\overline{\varphi}_{\varepsilon, \tau}(T) - \varphi_\Omega) + \varepsilon \beta_\Omega \overline{p}_{\varepsilon, \tau}(T), \quad r_{\varepsilon, \tau}(T) = 0 \quad &\text{in } \Omega.
\end{align*}
\]

**Theorem 4.16.** Assume \([A1] [A8] \) and \([C1] [C4] \). Let \((\overline{P}, \overline{\chi}, \overline{C})\) be an optimal control for \((\overline{CP})\), with corresponding state \((\overline{\varphi}, \overline{p}, \overline{\chi})\). Then, for every family \(\{(P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau})\}_{\varepsilon, \tau}\) of optimal controls for \((CP)^{\text{ad}}_{\varepsilon, \tau}\), with corresponding states \(\{(\overline{\varphi}_{\varepsilon, \tau}, P_{\varepsilon, \tau}, \overline{\varphi}_{\varepsilon, \tau})\}_{\varepsilon, \tau}\), as \( \varepsilon, \tau \to 0 \) it holds that, for
every $\alpha \geq 1$ if $d = 2$ and $1 \leq \alpha < 6$ if $d = 3$,

\[ \varphi_{\varepsilon, \tau} \rightharpoonup \varphi \quad \text{weakly* in } L^\infty(0, T; H) \cap L^2(0, T; V), \]

\[ \text{strongly in } C^0([0, T]; V^*) \cap L^2(0, T; L^\alpha(\Omega)), \]

\[ \varphi_{\varepsilon, \tau} \rightharpoonup \varphi, \quad \chi_{\varepsilon, \tau} \rightharpoonup \chi, \quad C_{\varepsilon, \tau} \rightharpoonup C, \]

\[ \delta_{ad}(\varphi_{\varepsilon, \tau}, \varphi, \chi, C) \rightharpoonup \delta(\varphi, \varphi, \chi, C). \]

**Proof.** The proof is analogous to Theorem 4.3 by using the convergences of Theorem 2.6. \( \square \)

### 4.4.1 Letting \((\varepsilon, \tau) \to (0, 0)\) in the adjoint system

We focus here on the passage to the limit as \((\varepsilon, \tau) \to (0, 0)\) in the adjoint system (3.70)–(3.74).

**Theorem 4.17.** Assume \(\text{A1 A3 C1 C4}\) and \(\text{AI A8}\). Let \((P, \chi, C) \in U_{ad}, \{P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau}\}_{\varepsilon, \tau} \subset U_{ad}\) be such that \((P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau}) \to (P, \chi, C)\) as \(\varepsilon, \tau \to 0\). Let \((\varphi, \varphi, \varphi)\) and \((\varphi_{\varepsilon, \tau}, \varphi_{\varepsilon, \tau}, \varphi_{\varepsilon, \tau})\) be the unique solutions to the state system (1.0)–(1.10) in the cases \(\varepsilon = \tau = 0\) with coefficients \((P, \chi, C)\), and \(\varepsilon \in (0, \varepsilon_0)\) and \(\tau \in (0, \tau_0)\) with coefficients \((P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau})\), as given by Theorems 2.6 and 2.2 respectively. Let also \((P_{\varepsilon, \tau}, q_{\varepsilon, \tau}, r_{\varepsilon, \tau})\) be the unique solution to the adapted adjoint system (4.95)–(4.99) with \(\varepsilon \in (0, \varepsilon_0), \tau \in (0, \tau_0)\), and coefficients \((P_{\varepsilon, \tau}, \chi_{\varepsilon, \tau}, C_{\varepsilon, \tau})\), as given by Theorem 3.4. Then, there exists a triplet \((p, q, r)\) with

\[ p \in H^1(0, T; V^*) \cap L^2(0, T; W), \quad q \in L^2(0, T; H), \quad r \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \]

such that, for every \(\alpha \geq 1\) if \(d = 2\) and \(1 \leq \alpha < 6\) if \(d = 3\), along any double sequence

\[ (\varepsilon, \tau) \to (0, 0) \quad \text{such that} \quad \limsup_{(\varepsilon, \tau) \to (0, 0)} \frac{\varepsilon}{\tau} < +\infty, \]

it holds

\[ p_{\varepsilon, \tau} \rightharpoonup p \quad \text{weakly in } H^1(0, T; V^*) \cap L^2(0, T; W), \]

\[ q_{\varepsilon, \tau} \rightharpoonup q \quad \text{weakly in } L^2(0, T; H), \]

\[ r_{\varepsilon, \tau} \rightharpoonup r \quad \text{weakly-* in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \]

\[ \text{strongly in } C^0([0, T]; L^\alpha(\Omega)) \cap L^2(0, T; H), \]

\[ \varepsilon p_{\varepsilon, \tau} \to 0 \quad \text{strongly in } H^1(0, T; H) \cap L^\infty(0, T; V^*), \]

\[ \tau q_{\varepsilon, \tau} \to 0 \quad \text{strongly in } L^\infty(0, T; V). \]

Moreover, \((p, q, r)\) is the unique weak solution to the adjoint system (3.70)–(3.74) with \(\varepsilon = \tau = 0\) and coefficients \((P, \eta, C)\), in the sense that

\[ - (\partial_t p, v)_V + \int_\Omega (aq - J * q + F''(\varphi))v + \int_\Omega C \sigma f'(\varphi)rv - \int_\Omega P \sigma f'(\varphi) pv = \int_\Omega \beta_Q (\varphi - \varphi_Q)v, \]

\[ \int_\Omega \nabla p \cdot \nabla w - \int_\Omega qw = 0, \]

\[ - \int_\Omega \partial_t rz + \int_\Omega \nabla r \cdot \nabla z + \int_\Omega C f(\varphi) rz - \int_\Omega P f(\varphi) pz - \int_\Omega \chi qz = 0, \]

for every \(v, w, z \in V\), almost everywhere in \((0, T)\), and

\[ p(T) = \beta_\Omega (\varphi(T) - \varphi_\Omega), \quad r(T) = 0. \]
Proof of Theorem 4.17. We perform on the adapted adjoint system (4.95)–(4.99) the same first estimate of the proof of Theorem 3.4, getting

\[ \frac{\tau}{2} \| q_{\epsilon, \tau}(t) \|^2 + C_0 \int_{Q_t^f} |q_{\epsilon, \tau}|^2 + \epsilon \int_{Q_t^f} |\partial_t p_{\epsilon, \tau}|^2 + \frac{\epsilon}{2} \| p_{\epsilon, \tau}(t) \|^2 + \frac{1}{2} \| \nabla p_{\epsilon, \tau}(t) \|^2 + \int_{Q_t^f} |\nabla r_{\epsilon, \tau}|^2 \]

\[ + \frac{\mathcal{B} \tau}{2} + \frac{1}{2} \| r_{\epsilon, \tau}(t) \|^2 + \| \nabla r_{\epsilon, \tau}(t) \|^2 + \mathcal{B} \int_{Q_t^f} |\nabla r_{\epsilon, \tau}|^2 + \int_{Q_t^f} |\partial_t r_{\epsilon, \tau}|^2 + \int_{Q_t^f} |\Delta r_{\epsilon, \tau}|^2 \]

\leq \frac{\tau}{2} \| q_{\epsilon, \tau}(T) \|^2 + \frac{\epsilon}{2} \| p_{\epsilon, \tau}(T) \|^2 + \frac{1}{2} \| \nabla p_{\epsilon, \tau}(T) \|^2 + \int_{Q_t^f} \beta \| \varphi_{\epsilon, \tau} - \varphi \| q_{\epsilon, \tau} + \int_{Q_t^f} (J \ast q_{\epsilon, \tau}) q_{\epsilon, \tau} \]

\[ - \int_{Q_t^f} \mathcal{C}_{\epsilon, \tau} \sigma' f' \mathcal{P}_{\epsilon, \tau} r_{\epsilon, \tau} q_{\epsilon, \tau} + \int_{Q_t^f} \mathcal{P}_{\epsilon, \tau} \mathcal{P}_{\epsilon, \tau} - A f' \mathcal{P}_{\epsilon, \tau} p_{\epsilon, \tau} q_{\epsilon, \tau} + \int_{Q_t^f} q_{\epsilon, \tau} q_{\epsilon, \tau} \partial_t r_{\epsilon, \tau} \]

\[ + \int_{Q_t^f} \mathcal{C}_{\epsilon, \tau} f \mathcal{P}_{\epsilon, \tau} r_{\epsilon, \tau} \left( \partial_t r_{\epsilon, \tau} + \Delta r_{\epsilon, \tau} \right) - \int_{Q_t^f} \mathcal{P}_{\epsilon, \tau} f \mathcal{P}_{\epsilon, \tau} r_{\epsilon, \tau} \left( \partial_t r_{\epsilon, \tau} + \Delta r_{\epsilon, \tau} \right) \]

\[ - \int_{Q_t^f} \chi_{\epsilon, \tau} q_{\epsilon, \tau} \left( \partial_t r_{\epsilon, \tau} + \Delta r_{\epsilon, \tau} \right) - \int_{Q_t^f} \tau_{\epsilon, \tau} \partial_t r_{\epsilon, \tau} \]

We show only how to bound the first three terms on the right-hand side, as all the other terms on the right-hand side can be treated exactly in the same way as in Section 4.11 and in the proof of Theorem 1.11 using Theorem 2.6. To this end, taking into account the modified terminal conditions (4.99), we have

\[ p_{\epsilon, \tau}(T) = \beta \Omega (\varphi_{\epsilon, \tau}(T) - \varphi \Omega) \in V, \quad q_{\epsilon, \tau}(T) = \frac{\epsilon}{\tau} \beta \Omega p_{\epsilon, \tau}(T). \]

It follows that

\[ \frac{\tau}{2} \| q_{\epsilon, \tau}(T) \|^2 + \frac{\epsilon}{2} \| p_{\epsilon, \tau}(T) \|^2 + \frac{1}{2} \| \nabla p_{\epsilon, \tau}(T) \|^2 \]

\[ = \frac{\beta \Omega^2}{2} \| \varphi_{\epsilon, \tau}(T) \|^2 + \frac{\epsilon}{2} \beta \| \varphi_{\epsilon, \tau}(T) \|^2 + \frac{1}{2} \| \nabla \Omega (\varphi_{\epsilon, \tau}(T) - \varphi \Omega) \|^2. \]

Now, by Theorem 2.6 we have that \( \{ \epsilon^{1/2} \varphi_{\epsilon, \tau} \}_{\epsilon, \tau} \) is uniformly bounded in \( C^0([0, T]; H) \): hence, thanks also to the regularity \( \beta \Omega \varphi \in V \), we deduce that there exists \( M > 0 \), independent of \( \epsilon, \tau \), such that

\[ \frac{\tau}{2} \| q_{\epsilon, \tau}(T) \|^2 + \frac{\epsilon}{2} \| p_{\epsilon, \tau}(T) \|^2 + \frac{1}{2} \| \nabla p_{\epsilon, \tau}(T) \|^2 \leq M (1 + \epsilon + \frac{\epsilon}{\tau}). \]

Consequently, the scaling lim sup \( \frac{\tau}{\epsilon} < +\infty \) on \( (\epsilon, \tau) \) yields that

\[ \frac{\tau}{2} \| q_{\epsilon, \tau}(T) \|^2 + \frac{\epsilon}{2} \| p_{\epsilon, \tau}(T) \|^2 + \frac{1}{2} \| \nabla p_{\epsilon, \tau}(T) \|^2 \leq M. \]

As we anticipated above, all the remaining terms on the right-hand side can be treated exactly as in Section 4.11 and in the proof of Theorem 4.11 so that we infer that there exists a positive constant \( M \), independent of both \( \epsilon \) and \( \tau \), such that

\[ \epsilon^{1/2} \| p_{\epsilon, \tau} \|_{H^1(0, T; H)} + \| p_{\epsilon, \tau} \|_{L^\infty(0, T; V)} + \tau^{1/2} \| q_{\epsilon, \tau} \|_{L^\infty(0, T; H)} + \| q_{\epsilon, \tau} \|_{L^2(0, T; H)} \]

\[ + \| r_{\epsilon, \tau} \|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \leq M. \]

Moreover, elliptic regularity theory and (3.71)–(3.72) leads to

\[ \| p_{\epsilon, \tau} \|_{L^2(0, T; W)} \leq M, \]

while by comparison in (3.70), as in the proof of Theorem 4.11, we infer that

\[ \| p_{\epsilon, \tau} + \tau q_{\epsilon, \tau} \|_{H^1(0, T; L^1(\Omega))} \leq M. \]
By the usual (weak) compactness criteria, we infer the existence of functions \((p, q, r)\) such that as \(\varepsilon, \tau \to 0\) it holds, for every \(\alpha \geq 1\) if \(d = 2\) and \(1 \leq \alpha < 6\) if \(d = 3\), and along a non-relabelled subsequence,

\[
\begin{align*}
p_{\varepsilon, \tau} &\to p \quad \text{weakly in } L^2(0, T; W), \\
q_{\varepsilon, \tau} &\to q \quad \text{weakly in } L^2(0, T; H), \\
p_{\varepsilon, \tau} + \tau q_{\varepsilon, \tau} &\to p \quad \text{weakly in } H^1(0, T; W^*), \\
r_{\varepsilon, \tau} &\to r \quad \text{weakly-}^* \text{ in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\
\varepsilon p_{\varepsilon, \tau} &\to 0 \quad \text{strongly in } C^0([0, T]; L^\alpha(\Omega)) \cap L^2(0, T; W), \\
\tau q_{\varepsilon, \tau} &\to 0 \quad \text{strongly in } H^1(0, T; H) \cap L^2(0, T; W), \\
\end{align*}
\]

Moreover, since by Theorem 2.6 we have that, for every \(\alpha \geq 1\) if \(d = 2\) and \(1 \leq \alpha < 6\) if \(d = 3\),

\[
\varphi_{\varepsilon, \tau} \to \varphi \quad \text{strongly in } C^0([0, T]; V^*) \cap L^2(0, T; L^\alpha(\Omega)),
\]

we can pass to the limit as \(\varepsilon, \tau \to 0\) in the variational formulation (4.87)–(4.90), treating the term with \(F''\) as in the proof of Theorem 4.11, and conclude. The uniqueness of the weak solution \((p, q, r)\) follows from linearity and the estimates already performed.

\[\square\]

### 4.4.2 Letting \((\varepsilon, \tau) \to (0, 0)\) in the optimality condition

Here, we conclude the asymptotic analysis by letting \(\varepsilon, \tau \to 0\) in the optimality condition for \((CP)^{ad}_{\varepsilon, \tau}\), and proving the corresponding necessary conditions for \((CP)\).

**Theorem 4.18.** Assume \(A1\)–\(A8\) and \(C1\)–\(C4\). Then, every optimal control \((\overline{P}, \overline{X}, \overline{C})\) of \((CP)\) necessarily satisfies

\[
\int_Q (P - \overline{P}) \sigma f(\overline{\varphi}) p - \int_Q (\chi - \overline{X}) \sigma q - \int_Q (C - \overline{C}) \sigma f(\overline{\varphi}) r + \alpha_P (P - P_\tau)(P - \overline{P}) + \alpha_X (X - X_\tau)(X - \overline{X}) + \alpha_C (C - C_\tau)(C - \overline{C}) \geq 0
\]

for every \((P, X, C) \in U_{ad}\),

where \((\overline{\varphi}, \overline{p}, \overline{r})\) and \((p, q, r)\) are the unique solutions to (1.6)–(1.10) and (3.70)–(3.74) with \(\varepsilon = \tau = 0\) in the sense of Theorems 2.7 and 4.11 respectively.

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