NONLINEAR DIRICHLET PROBLEMS WITH DOUBLE RESONANCE

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Abstract. We study a nonlinear Dirichlet problem driven by the sum of a $p$–Laplacian ($p > 2$) and a Laplacian and which at $±∞$ is resonant with respect to the spectrum of $(-\Delta_p, W^{1,p}_0(\Omega))$ and at zero is resonant with respect to the spectrum of $(-\Delta, H^1_0(\Omega))$ (double resonance). We prove two multiplicity theorems providing three and four nontrivial solutions respectively, all with sign information. Our approach uses critical point theory together with truncation and comparison techniques and Morse theory.

1. Introduction. In this paper we study the following nonlinear nonhomogeneous Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0, \quad 2 < p < \infty. \quad (1)$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^2$– boundary $\partial \Omega$. Also $\Delta_p$ denotes the $p$–Laplace differential operator defined by

$$\Delta_p u = \text{div} \left( \frac{\|Du\|_{\mathbb{R}^N}^{p-2} Du} \right), \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

When $p = 2$, we have the usual Laplacian denoted by $\Delta$. The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous).

We assume that asymptotically at $±\infty$ the quotient $\frac{f(z, x)}{|x|^{p-2}}$ interacts with the principal eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$, while at zero it interacts with a nonprincipal eigenvalue of $(-\Delta, H^1_0(\Omega))$. So, we have a situation of "double resonance". Our aim under the above conditions, is to prove multiplicity theorems for problem

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providing sign information for all the solutions produced. We prove two multiplicity theorems producing three and four nontrivial solutions respectively, all with sign information. We stress that the differential operator \[ u \mapsto -\Delta_p u - \Delta u \] is nonhomogeneous and this fact is a source of difficulties in the study of problem (1).

We mention that problems driven by the sum of a \( p \)-Laplacian and a Laplacian \(( (p,2) \)-equations), arise in physical applications. We refer to the works of Benci-D'Avenia-Fortunato-Pisani [6] (quantum physics) and Cherfils-Ilyasov [8] (plasma physics). Recently, existence theorems for such equations were proved by Cingolani-Degiovanni [9], Cingolani-Vannella [10] and He-Li [15].

Multiplicity theorems can be found in Aizicovici-Papageorgiou-Staicu [2, 3], Gasinski-Papageorgiou [14], Papageorgiou-Radulescu [20, 21], Papageorgiou-Winkert [22] and Sun [25], under more restrictive conditions on the reaction which do not allow for double resonance to occur.

Our approach uses variational methods based on critical point theory together with suitable truncation and comparison techniques and Morse theory (critical groups).

Our main results are two multiplicity theorems producing three and four nontrivial smooth solutions, respectively. In the first multiplicity theorem, the reaction term \( f(z,x) \) is only a Carathéodory function, while in the second the reaction term \( f(z,x) \) is a measurable function with \( f(.,.) \in C^1(\mathbb{R}) \). The precise conditions on \( f(z,x) \) are given in hypotheses \( H_1 \) (see the beginning of Section 3, with a slightly stronger variant in hypotheses \( H_2 \) just before Proposition 15) for the Carathéodory case and in hypotheses \( H_3 \) (see the beginning of Section 4) for the differentiable case.

Next we state the two multiplicity theorems and provide the outline of their proofs. For the notation we refer to Section 2.

**Theorem A** If hypotheses \( H_1 \) hold, then problem (1) has at least three nontrivial solutions

\[ u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+ \quad \text{and} \quad y_0 \in [v_0, u_0] \cap C^1_0(\Omega), \quad \text{nodal}. \]

Moreover, under the stronger hypotheses \( H_2 \), we have that

\[ y_0 \in \text{int } C^1_0(\Omega) \setminus [v_0, u_0]. \]

The idea of the proof is the following. First we consider the positive and the negative truncations of the energy functional. Working with them and using variational methods, we produce two nontrivial smooth solutions of constant sign. Then we show that the problem has extremal constant sign solutions, that is, a smallest positive solution \( u_* \in C_+ \) and a biggest negative solution \( v_* \in -\text{int } C_+ \). Then we look at the order interval \([v_*, u_*]\). Using variational tools, we show that the problem has a solution \( y_0 \in [v_*, u_*] \), \( y_0 \not\in \{v_*, u_*\} \). Using critical groups we show that \( y_0 \) is nontrivial. Therefore due to the extremality of \( u_* \) and \( v_* \), \( y_0 \) must be nodal.

**Theorem B** If hypotheses \( H_3 \) hold, then problem (1) has at least four nontrivial solutions

\[ u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+ \quad \text{and} \quad y_0, \hat{y} \in \text{int } C^1_0(\Omega) \setminus [v_0, u_0], \quad \text{nodal}. \]
From Theorem A we already have the three solutions

\[ u_0 \in \text{int } C_+, \ v_0 \in -\text{int } C_+ \text{ and } y_0 \in \text{int } C^0_0(\partial \Omega) \{v_0, u_0\}, \ \text{nodal.} \]

The second nodal solution \( \tilde{y} \) is obtained by a careful calculation of the critical groups of the truncated at \( \{v_*, u_*\} \) energy functional (see (24)). Now the energy functional is \( C^2 \) (since \( f(z, \cdot) \in C^1(\mathbb{R}) \)) and so the results of Morse theory used in the calculation of the critical groups are sharper.

2. Mathematical background. Let \( (X, \|\cdot\|) \) be a Banach space and \( X^* \) its topological dual. By \( \langle \cdot, \cdot \rangle \) we denote the duality brackets for the pair \( (X^*, X) \), while \( \overset{\to}{\rightharpoonup} \) designates the weak convergence in \( X \). A map \( A : X \to X^* \) is said to be of type \( (S)_+ \), if for every sequence \( \{x_n\}_{n \geq 1} \subseteq X \) such that \( x_n \overset{\to}{\rightharpoonup} x \in X \) and

\[ \lim_{n \to \infty} \sup \langle A(x_n), x_n - x \rangle \leq 0, \]

one has

\[ x_n \to x \text{ in } X \text{ as } n \to \infty. \]

Given \( \varphi \in C^1(X) \), we say that \( c \) is a critical value of \( \varphi \), if there exists \( x^* \in X \) such that \( \varphi'(x^*) = 0 \) and \( \varphi(x^*) = c \). We say that \( \varphi \) satisfies the Palais-Smale condition (the PS-condition, for short), if the following is true:

"every sequence \( \{u_n\}_{n \geq 1} \subseteq X \) such that \( \{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R} \) is bounded and \( \varphi'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \) admits a strongly convergent subsequence."

This is a compactness-type condition on the functional \( \varphi \) which leads to a deformation theorem, from which one can derive the minimax theory of critical values of \( \varphi \). Prominent in that theory is the well known "mountain pass theorem" due to Ambrosetti-Rabinowitz [4].

**Theorem 2.1.** If \( (X, \|\cdot\|) \) is a Banach space, \( \varphi \in C^1(X) \) satisfies the PS-condition, \( u_0, u_1 \in X \) and \( \rho > 0 \) are such that \( \|u_1 - u_0\| > \rho \),

\[ \max \{\varphi(u_0), \varphi(u_1)\} < \inf \{\varphi(u) : \|u - u_0\| = \rho\} =: m_\rho, \]

and

\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \text{ where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}, \]

then \( c \geq m_\rho \) and \( c \) is a critical value of \( \varphi \).

In the study of problem (1), in addition to the Sobolev spaces \( W_0^{1,p}(\Omega) \) and \( H_0^1(\Omega) \) we will also use the Banach space \( C^0_0(\partial \Omega) \) defined by

\[ C^0_0(\partial \Omega) = \{u \in C^1(\partial \Omega) : u|_{\partial \Omega} = 0\}. \]

This is an ordered Banach space with positive cone

\[ C_+ = \{u \in C^0_0(\partial \Omega) : u(z) \geq 0 \text{ for all } z \in \Omega\}. \]

This cone has a nonempty interior, given by

\[ \text{int } C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial \Omega\right\}. \]
Here \( \frac{\partial u}{\partial n} \) denotes the outward normal derivative defined by

\[
\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},
\]

where \( n(\cdot) \) is the outward unit normal on \( \partial \Omega \).

The following notation will be used throughout the paper. By \( \| \cdot \| \) we will denote the norm of the Sobolev space \( W_0^{1,p}(\Omega) \). The Poincaré inequality implies that

\[
\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega),
\]

where \( \| \cdot \|_p \) stands for the \( L^p \)-norm. Also, by \( |\cdot|_N \) we denote the Lebesgue measure on \( \mathbb{R}^N \).

For every \( x \in \mathbb{R} \), we set

\[
x^\pm = \max \{ \pm x, 0 \}.
\]

Then given \( u \in W_0^{1,p}(\Omega) \), we define \( u^\pm (\cdot) = u(\cdot)^\pm \). We have

\[
u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^- \quad \text{and} \quad |u| = u^+ + u^-,
\]

If \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) is a measurable function (for example, a Carathéodory function), then we denote by \( N_h \) the corresponding Nemytskii map, i.e.,

\[
N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).
\]

Note that \( z \mapsto N_h(u)(z) = h(z, u(z)) \) is measurable.

Let \( f_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) be a Carathéodory function such that

\[
|f_0(z, x)| \leq a_0(z) \left( 1 + |x|^{r-1} \right) \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R}
\]

with \( a_0 \in L^\infty(\Omega)_+ \), and \( 1 < r < p^* \), where \( p^* \) is the critical Sobolev exponent, i.e.,

\[
p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}
\]

We set \( F_0(z, x) = \int_0^x f_0(z, s) \, ds \) and consider the \( C^1 \)-functional \( \varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
\varphi_0(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega F_0(z, u) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).
\]

The following is a particular case of a more general result that can be found in [2].

**Proposition 1.** If \( u_0 \in W_0^{1,p}(\Omega) \) is a local \( C_0^1(\overline{\Omega}) \) minimizer of \( \varphi_0 \), that is, there exists \( \rho_0 > 0 \) such that

\[
\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \quad \text{with} \quad \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0
\]

then \( u_0 \in C_0^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) and \( u_0 \) is also a local \( W_0^{1,p}(\Omega) \) minimizer of \( \varphi_0 \), that is, there exists \( \rho_1 > 0 \) such that

\[
\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \quad \text{with} \quad \|h\| \leq \rho_1.
\]

Let \( h_1, h_2 \in L^\infty(\Omega) \). We write \( h_1 \prec h_2 \) if, for any compact \( K \subset \Omega \), we can find \( \varepsilon = \varepsilon(K) > 0 \) such that

\[
h_1(z) + \varepsilon \leq h_2(z) \quad \text{for a.a. } z \in K.
\]

Note that if \( h_1, h_2 \in C(\Omega) \) and \( h_1(z) < h_2(z) \) for all \( z \in \Omega \), then \( h_1 \prec h_2 \).

The following strong comparison principle is essentially due to Arcoya-Ruiz [5] (see also Aizicovici-Papageorgiou-Staicu [2]).
Proposition 2. If $\xi \geq 0$, $h_1, h_2 \in L^\infty(\Omega)$, $h_1 \prec h_2$ and $u, v \in C^1_0(\Omega)$ are solutions of

\begin{align*}
-\Delta_p u(z) - \Delta u(z) + \xi |u(z)|^{p-2}u(z) &= h_1(z) \quad \text{in } \Omega \\
-\Delta_p v(z) - \Delta v(z) + \xi v(z)^{p-1} &= h_2(z) \quad \text{in } \Omega
\end{align*}

with $v \in \text{int } C_+$, then

$v - u \in \text{int } C_+.$

Let $A_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be the nonlinear map defined by

$$(A_p(u), h) = \int_\Omega \|Du\|_{R^N}^{p-2}(Du, Dh)_{R^N} \, dz \quad \text{for all } u, h \in W^{1,p}_0(\Omega).$$

When $p = 2$, we write $A_2 := A \in \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$. The following result is well known and can be found in Motreanu-Motreanu-Papageorgiou [18] (p.40):

**Proposition 3.** The map $A_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ ($1 < p < \infty$) defined by (3) is bounded (maps bounded sets to bounded sets), continuous, monotone (hence equation) or if $p$ known if this sequence exhausts $v$ with $\text{int } C_+$.

Next we recall some basic facts about the spectra of $(-\triangle_p, W^{1,p}_0(\Omega))$ and $(-\triangle, H^1_0(\Omega))$. So, let $1 < p < \infty$ and consider the following nonlinear eigenvalue problem:

\begin{align*}
-\Delta_p u(z) &= \hat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \\
|u|_{\partial\Omega} &= 0.
\end{align*}

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$ if problem (4) admits a nontrivial solution $\tilde{u}$ known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. We know that there exists a smallest eigenvalue $\hat{\lambda}_1(p)$ which has the following properties:

- $\hat{\lambda}_1(p) > 0$.
- $\hat{\lambda}_1(p)$ is isolated (that is, we can find $\varepsilon > 0$ such that $\left(\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \varepsilon\right) \cap \sigma(p) = \emptyset$ where $\sigma(p)$ is the set of eigenvalues of $(-\Delta_p, W^{1,p}_0(\Omega))$.
- $\hat{\lambda}_1(p)$ is simple (that is, if $\tilde{u}, \tilde{v}$ are both eigenvalues corresponding to $\hat{\lambda}_1(p)$, then $\tilde{u} = \xi\tilde{v}$ with $\xi \in \mathbb{R} \setminus \{0\}$.
- $\hat{\lambda}_1(p) = \inf \left\{ \frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W^{1,p}_0(\Omega), u \neq 0 \right\}.$

The infimum in (5) is attained on the corresponding one dimensional eigenspace. It is clear from (5) that the elements of this eigenspace have constant sign. Let $\tilde{u}_1(p)$ denote the $L^p$- normalized (that is, $\|\tilde{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p)$. The nonlinear regularity theory and the nonlinear maximum principle (see for example, Gasinski-Papageorgiou [13] (pp.737-738)) imply that $\tilde{u}_1(p) \in \text{int } C_+$.

The Ljusternik-Schnirelmann minimax scheme produces a whole increasing sequence $\left\{\hat{\lambda}_k(p)\right\}_{k \geq 1}$ of eigenvalues such that $\hat{\lambda}_k(p) \to \infty$. In general we do not know if this sequence exhausts $\sigma(p)$. This is the case if $N = 1$ (ordinary differential equation) or if $p = 2$ (linear eigenvalue problem). We mention that $\hat{\lambda}_1(p) > 0$ is
the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign changing) eigenfunctions.

If \( p = 2 \) (linear eigenvalue problem), then by \( E \left( \hat{\lambda}_k (2) \right) \) we denote the eigenspace corresponding to the eigenvalue \( \hat{\lambda}_k (2) \) and we have the following orthogonal direct sum decomposition

\[
H^1_0 (\Omega) = \bigoplus_{k \geq 1} E \left( \hat{\lambda}_k (2) \right).
\]

The eigenspaces \( E \left( \hat{\lambda}_k (2) \right) \) with \( k \in \mathbb{N} \) are finite dimensional and exhibit the so-called "unique continuation property" (UCP for short). This means that, if \( u \in E \left( \hat{\lambda}_k (2) \right) \) vanishes on a set of positive measure, then \( u \equiv 0 \).

We have the following variational characterizations of the eigenvalues \( \left\{ \hat{\lambda}_k (2) \right\}_{k \geq 1} : \)

\[
\hat{\lambda}_k (2) = \inf \left\{ \frac{\| Du \|_2^2}{\| u \|_2^2} : u \in \tilde{H}_k = \bigoplus_{i \geq k} E \left( \hat{\lambda}_i (2) \right), \; u \neq 0 \right\}
\]

\[
= \sup \left\{ \frac{\| Du \|_2^2}{\| u \|_2^2} : u \in \Pi_k = \bigoplus_{i = 1}^k E \left( \hat{\lambda}_i (2) \right), \; u \neq 0 \right\}.
\]  

(6)

In (6) both the infimum and supremum are achieved on \( E \left( \hat{\lambda}_k (2) \right) \). Using (6) and the UCP, we have the following result.

**Proposition 4.** (a) If \( \eta \in L^\infty (\Omega) \), \( \eta (z) \leq \hat{\lambda}_k \) for a.a. \( z \in \Omega \) and the inequality is strict on a set of positive measure, then

\[
\| Du \|_2^2 - \int_\Omega \eta (z) u^2 dz \geq C_1 \| u \|^2 \quad \text{for some} \quad C_1 > 0, \; \text{all} \; u \in \tilde{H}_k.
\]

(b) If \( \eta \in L^\infty (\Omega) \), \( \eta (z) \geq \hat{\lambda}_k \) for a.a. \( z \in \Omega \) and the inequality is strict on a set of positive measure, then

\[
\| Du \|_2^2 - \int_\Omega \eta (z) u^2 dz \leq -C_2 \| u \|^2 \quad \text{for some} \quad C_2 > 0, \; \text{all} \; u \in \Pi_k.
\]

Also exploiting (5), the simplicity of \( \hat{\lambda}_1 (p) \) and the fact that \( \hat{u}_1 (p) \in \text{int} C_+ \), we have (see Motreanu-Motreanu-Papageorgiou [18], p.305):

**Proposition 5.** If \( \theta \in L^\infty (\Omega) \), \( \theta (z) \leq \hat{\lambda}_1 (p) \) for a.a. \( z \in \Omega \) and the inequality is strict on a set of positive measure, then

\[
\| Du \|_p^p - \int_\Omega \theta (z) |u|^p dz \geq C_0 \| u \|_p^p \quad \text{for some} \quad C_0 > 0, \; \text{all} \; u \in W^{1,p}_0 (\Omega).
\]

We also recall some basic definitions and facts from Morse theory (critical groups).

So, let \( (Y_1, Y_2) \) be a topological pair with \( Y_2 \subset Y_1 \subset X \). For every integer \( k \geq 0 \), we denote by \( H_k (Y_1, Y_2) \) the \( k^{th} \)-relative singular homology group with integer coefficients for the topological pair \( (Y_1, Y_2) \). Let \( \varphi \in C^1 (X) \) and \( c \in \mathbb{R} \). We introduce the following sets:

\[
\varphi^c = \{ u \in X : \varphi (u) \leq c \},
\]

\[
K_\varphi = \{ u \in X : \varphi' (u) = 0 \};
\]

and

\[
K_\varphi^c = \{ u \in K_\varphi : \varphi (x) = c \}.
\]
Let $u \in K_\varphi$ be isolated. The critical groups of $\varphi$ at $u$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\})$$

for all $k \in \mathbb{N}_0$, where $U$ is a neighborhood of $u$ such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood $U$.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of $\varphi$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c)$$

for all $k \in \mathbb{N}_0$.

The second deformation theorem (see, for example, Gasinski-Papageorgiou [13]) (p. 628) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that $X = Y \oplus V$ with $\dim V < \infty$. We say that $\varphi$ has a local linking at the origin, if there exists $\rho > 0$ such that

$$\varphi(u) \leq 0 \quad \text{if} \quad u \in Y, \quad \|u\| \leq \rho$$

$$\varphi(u) > 0 \quad \text{if} \quad u \in V, \quad 0 < \|u\| \leq \rho.$$

Evidently $u = 0 \in K_\varphi$. The next result can be found, for example, in Motreanu-Motreanu-Papageorgiou [18], (p.171).

**Proposition 6.** If $\varphi \in C^1(X, \mathbb{R})$ has a local linking at the origin and $u = 0 \in K_\varphi$ is isolated, then $C_d(\varphi, 0) \neq 0$ where $d = \dim Y$.

3. **Three solutions.** In this section we prove a multiplicity result producing three nontrivial solutions, two of constant sign and a third one, nodal. The hypotheses on the reaction $f(z, x)$ are the following:

**H$_1$:** $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z) \left(1 + c|x|^{p-1}\right)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_{+}$, $p \leq r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s) \, ds$, then

$$\limsup_{x \to \pm \infty} \frac{pF(z, x)}{|x|^p} \leq \tilde{\lambda}_1(p)$$

uniformly for a.a. $z \in \Omega$ and there exists $\tilde{C} > 0$ such that

$$\tilde{C} \leq f(z, x) x - pF(z, x)$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$;

(iii) there exists $m \in \mathbb{N}$, $m \geq 2$, $\eta \in L^\infty(\Omega)$ and $\delta_0 > 0$ such that

$$\eta(z) \leq \tilde{\lambda}_{m+1}(2)$$

for a.a. $z \in \Omega$,

the above inequality is strict on a set of positive measure,

$$\limsup_{x \to 0} \frac{f(z, x)}{x} \leq \eta(z)$$

uniformly for a.a. $z \in \Omega$,

$$f(z, x) x \geq \tilde{\lambda}_m(2) x^2$$

for a.a. $z \in \Omega$, all $|x| \leq \delta_0$.

**Remarks:** Hypothesis **H$_1$ (ii)** implies that resonance is possible at $\pm \infty$ with respect to the first eigenvalue of $\left(-\Delta_p, W^{1, p}_0(\Omega)\right)$. Hypothesis **H$_1$ (iii)** implies that at $0$ we can have resonance with respect to any nonprincipal eigenvalue of $\left(-\Delta, H^1_0(\Omega)\right)$. We stress that no differentiability assumption is made on $f(z, .)$. In
particular, the limit as \( x \to 0 \) of the quotient \( \frac{f(z,x)}{x} \) need not exist. This is in contrast with the more restrictive conditions used by Aizicovici-Papageorgiou-Staicu [2] and Sun [25].

**Example:** The following map satisfies hypotheses (\( H_1 \)):

\[
f(x) = \begin{cases} \eta x & \text{if } |x| \leq 1 \\ \hat{\lambda}_m |x|^{p-2} x + \hat{\eta} x & \text{if } |x| > 1 \end{cases}
\]

with \( \eta \in \left[ \hat{\lambda}_m (2), \hat{\lambda}_{m+1} (2) \right], m \geq 2 \) and \( \hat{\eta} = \eta - \hat{\lambda}_1 > 0 \).

Let \( \varphi : W^{1,p}_0 (\Omega) \to \mathbb{R} \) be the energy functional for problem (1) defined by

\[
\varphi (u) = \frac{1}{p} \| Du \|_p^p + \frac{1}{2} \| Du \|_2^2 - \int \Omega F(z,u(z)) \, dz \quad \text{for all } u \in W^{1,p}_0 (\Omega).
\]

We know that \( \varphi \in C^1 \left( W^{1,p}_0 (\Omega) \right) \). First we will produce two constant sign solutions. To this end, we introduce the positive and the negative truncations of \( f(z,\cdot) \), that is, the Carathéodory functions

\[
f_\pm (z,x) = f(z,\pm x^\pm) \quad \text{for } (z,x) \in \Omega \times \mathbb{R}.
\]

We set

\[
F_\pm (z,x) = \int_0^x f_\pm (z,s) \, ds \quad \text{for all } (z,x) \in \Omega \times \mathbb{R},
\]

and consider the \( C^1 \) functionals \( \varphi_\pm : W^{1,p}_0 (\Omega) \to \mathbb{R} \) defined by

\[
\varphi_\pm (u) = \frac{1}{p} \| Du \|_p^p + \frac{1}{2} \| Du \|_2^2 - \int \Omega F_\pm (z,u(z)) \, dz \quad \text{for all } u \in W^{1,p}_0 (\Omega).
\]

**Proposition 7.** If hypotheses \( H_1 \) hold, then the functionals \( \varphi \) and \( \varphi_\pm \) are coercive.

**Proof.** We do the proof for the functional \( \varphi_+ \), the proofs for \( \varphi_- \) and \( \varphi \) being similar.

We argue by contradiction. So, suppose that \( \varphi_+ \) is not coercive. Then, we can find \( \{ u_n \}_{n \geq 1} \subseteq W^{1,p}_0 (\Omega) \) and \( C_3 > 0 \) such that

\[
\| u_n \| \to \infty \quad \text{and} \quad \varphi_+ (u_n) \leq C_3 \quad \text{for all } n \geq 1.
\]

For a.a. \( z \in \Omega \) and all \( x > 0 \), we have

\[
\frac{d}{dx} F_+ (z,x) x^p = f_+ (z,x) x^p - px^{p-1} F_+ (z,x) = f_+ (z,x) x - pF_+ (z,x)
\]

\[
\geq - \frac{\tilde{C}}{x^{p+1}} \quad \text{(see hypothesis } H_1 \text{ (ii))},
\]

hence

\[
\frac{F_+ (z,y)}{y^p} - \frac{F_+ (z,v)}{v^p} \geq \frac{\tilde{C}}{p} \left[ \frac{1}{y^p} - \frac{1}{v^p} \right] \quad \text{for a. a. } z \in \Omega, \text{ all } y > v > 0.
\]

We let \( y \to +\infty \) and use hypothesis \( H_1 \) (ii). Then

\[
\frac{\hat{\lambda}_1 (p)}{p} - \frac{F_+ (z,v)}{v^p} \geq \frac{\tilde{C}}{p} \frac{1}{v^p},
\]

hence

\[
pF_+ (z,v) \leq \frac{\tilde{C} (p)}{v^p} \quad \text{for a. a. } z \in \Omega, \text{ all } v \geq 0.
\]
From (8) we have
\[ \frac{1}{p} \| Du_n \|^p_p + \frac{1}{2} \| Du_n \|^2_2 - \int_{\Omega} F_+ (z, u_n (z)) \, dz \leq C_3. \]
Then
\[ \frac{1}{p} \| Du_n^+ \|^p_p + \frac{1}{p} \| Du_n^- \|^p_p - \frac{\lambda_1 (p)}{p} \| u_n^+ \|^p_p - \frac{C}{p} |\Omega|_N \leq C_3 \]
(see (9)), hence
\[ \frac{1}{p} \| Du_n^- \|^p_p \leq C_4 \text{ for all } n \geq 1, \]
with
\[ C_4 = C_3 + \frac{C}{p} |\Omega|_N > 0, \]
therefore
\[ \{ u_n \}_{n \geq 1} \subseteq W^{1,p}_0 (\Omega) \text{ is bounded.} \quad \text{(10)} \]
From (8) and (10) it follows that
\[ \| u_n^+ \| \rightarrow \infty \text{ as } n \rightarrow \infty. \quad \text{(11)} \]
Let \( y_n = \frac{u_n^+}{\| u_n^+ \|}, n \geq 1. \) Then \( \| y_n \| = 1 \) for all \( n \geq 1 \) and so by passing to a subsequence if necessary, we may assume that
\[ y_n \rightharpoonup y \text{ in } W^{1,p}_0 (\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p (\Omega), y \geq 0. \quad \text{(12)} \]
From (8) we have
\[ \frac{1}{p} \| Dy_n \|^p_p + \frac{1}{2} \| u_n^+ \|^{p-2} \| Dy_n \|^2_2 - \int_{\Omega} \frac{F_+ (z, u_n)}{\| u_n^+ \|^p} \, dz \leq \frac{C_3}{\| u_n^+ \|^p} \text{ for all } n \geq 1, \]
and so, we have
\[ u_n^+ (z) \rightarrow +\infty \text{ for a. a. } z \in \Omega. \quad \text{(14)} \]
From (8) we have
\[ \frac{1}{p} \| Du_n^+ \|^p_p + \frac{1}{2} \| Du_n^+ \|^2_2 - \int_{\Omega} F_+ (z, u_n) \, dz \leq C_3 \text{ for all } n \geq 1, \]
and so, we have
\[ u_n^- (z) \rightarrow +\infty \text{ for a. a. } z \in \Omega. \]
(see (5)). Then
\[ \frac{\lambda_1(2)}{2} \int_{\Omega} (u_n^+)^2 \, dz \leq C_5 \text{ for all } n \geq 1, \text{ with } C_5 = C_3 + \hat{C} |\Omega|_N \]
(see (12)), therefore
\[ +\infty = \int_{\Omega} \liminf_{n \to \infty} (u_n^+)^2 \, dz \leq \frac{2C_5}{\lambda_1(2)} \]
by Fatou’s lemma (see (14)), a contradiction. This proves the coercivity of \( \varphi_+ \).
Similarly for the functionals \( \varphi_- \) and \( \varphi \).

From Papageorgiou-Winkert [22], we have:

**Corollary 1.** If hypotheses \( H_1 \) hold, then the functionals \( \varphi \) and \( \varphi_\pm \) satisfy the PS-condition.

Now using the direct method, we can produce two constant sign solutions.

**Proposition 8.** If hypotheses \( H_1 \) hold, then problem (1) admits at least two constant sign solutions
\[ u_0 \in \text{int } C_+ \text{ and } u_0 \in \text{int } C_+. \]

**Proof.** First we produce a positive solution. From Proposition 7 we know that the functional \( \varphi_+ \) is coercive. Also using the Sobolev embedding theorem, we see that \( \varphi_+ \) is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find \( u_0 \in W^{1,p}_0(\Omega) \) such that
\[ \varphi_+(u_0) = \inf \left\{ \varphi_+(u) : u \in W^{1,p}_0(\Omega) \right\}. \]
(15)
Since \( \hat{u}_1(2) \in \text{int } C_+ \), we can find \( t \in (0,1) \) small such that \( t\hat{u}_1(2) \in [0,\delta_0] \) for all \( z \in \Omega \). Then we have
\[ \varphi_+(t\hat{u}_1(2)) = \frac{p}{p} \| Du_0(2) \|^p_p + \frac{\hat{\lambda}_2(2)}{2} t^2 - \int_{\Omega} F(z,t \hat{u}_1(2)) \, dz \text{ (recall } \| \hat{u}_1(2) \|_2 = 1) \]
\[ \leq \frac{p}{p} \| Du_0(2) \|^p_p - \frac{t^2}{2} \left( \hat{\lambda}_m(2) - \hat{\lambda}_1(2) \right) \text{ (see hypothesis } H_1(iii)). \]
(16)
Since \( p > 2 \) and \( m \geq 2 \), by choosing \( t \in (0,1) \) even smaller if necessary, from (16) we have
\[ \varphi_+(t\hat{u}_1(2)) < 0. \]
It follows that
\[ \varphi_+(u_0) < 0 = \varphi_+(0) \text{ (see (15))} \]

hence
\[ u_0 \neq 0. \]
From (15) we have
\[ \varphi'_+(u_0) = 0, \]

hence
\[ A_p(u_0) + A(u_0) = N_{f_+}(u_0). \]
(17)
On (17) we act with \( -u_0^- \in W^{1,p}_0(\Omega) \). Then
\[ \| Du_0^- \|^p_p + \| Du_0^- \|^2_2 = 0, \]
therefore
\[ u_0 \geq 0, \ u_0 \neq 0. \]
Then from (17) we have
\[ A_p(u_0) + A(u_0) = N_f(u_0), \]
hence
\[ -\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \text{ a.e. in } \Omega, \quad u_0|_{\partial \Omega} = 0. \quad (18) \]
From Ladyzhenskaya-Uraltseva [16] (p. 286) we have that \( u_0 \in L^\infty(\Omega) \) and then we can use Theorem 1 of Lieberman [17] and infer that \( u_0 \in C^+ \setminus \{0\}. \)

Hypotheses \( H_1(i), (iii), \) imply that given \( \rho > 0, \) we can find \( \hat{\xi}_\rho > 0 \) such that
\[ f(z, x) x + \hat{\xi}_\rho |x|^p \geq 0 \text{ for a.a. } z \in \Omega, \quad \text{all } |x| \leq \rho. \quad (19) \]

Let \( \rho = \|u_0\|_\infty \) and let \( \hat{\xi}_\rho > 0 \) be as postulated by (19). Then
\[ -\Delta_p u_0(z) - \Delta u_0(z) + \hat{\xi}_\rho u_0(z)^{p-1} \geq 0 \text{ for a.a. } z \in \Omega \text{ (see (18))} \]
hence
\[ \Delta_p u_0(z) + \Delta u_0(z) \leq \hat{\xi}_\rho u_0(z)^{p-1} \text{ for a.a. } z \in \Omega. \]

Then the strong maximum principle (p.111) and the boundary point theorem (p.120) of Pucci-Serrin [23] imply that
\[ u_0 \in \text{int } C^+. \]

In a similar fashion, working this time with the functional \( \varphi_- \), we produce a negative solution
\[ v_0 \in -\text{int } C^+. \]

\[ \square \]

In fact we can show that problem (1) has a smallest positive solution and a biggest negative solution (extremal constant sign solutions).

Let \( S_+ \) (resp. \( S_- \)) be the set of positive (resp. negative) solutions of problem (1). From Proposition 8 and its proof, we have
\[ \emptyset \neq S_+ \subseteq \text{int } C^+ \text{ and } \emptyset \neq S_- \subseteq -\text{int } C^+. \]

Moreover, as in Filippakis-Kristaly-Papageorgiou [12], exploiting the monotonicity of the map \( u \to A_p(u) + A(u) \), we have that
\( S_+ \) is downward directed
(that is, if \( u_1, u_2 \in S_+ \), then there exists \( u \in S_+ \) such that \( u \leq u_1, u \leq u_2 \)), and
\( S_- \) is upward directed
(that is, if \( v_1, v_2 \in S_- \), then there exists \( v \in S_- \) such that \( v_1 \leq v, v_2 \leq v \)).

Note that hypotheses \( H_1(i), (iii) \) imply that
\[ f(z, x) x \geq \hat{\lambda}_m(2) x^2 - C_6 |x|^r \text{ for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R}, \]
with \( C_6 > 0. \) This unilateral growth estimate on the reaction leads to the following auxiliary Dirichlet problem
\[ -\Delta_p u(z) - \Delta u(z) = \hat{\lambda}_m(2) u(z) - C_6 |u(z)|^{r-2} u(z) \text{ in } \Omega, \quad u|_{\partial \Omega} = 0. \quad (20) \]

From Proposition 3 and Lemma 2 of Aizicovici-Papageorgiou-Staicu [2], we have the following result:
Proposition 9. Problem (20) has a unique positive solution \( \bar{u} \in \text{int} C_+ \) and \( \bar{v} = -\bar{u} \in -\text{int} C_+ \) is the unique negative solution; moreover
\[
\bar{u} \leq u \text{ for all } u \in S_+ \text{ and } v \leq \bar{v} \text{ for all } v \in S_-.
\]

Using this proposition, we can produce extremal constant sign solutions for problem (1).

Proposition 10. If hypotheses \( H_1 \) hold, then problem (1) has a smallest positive solution \( u_* \in \text{int } C_+ \) and a biggest negative solution \( v_* \in -\text{int } C_+ \).

Proof. Let \( \{u_n\}_{n \geq 1} \subseteq S_+ \) be a decreasing sequence such that
\[
\inf S_+ = \inf_{n \geq 1} u_n \text{ (see [2]).}
\]
We have
\[
A_p(u_n) + A(u_n) = N_f(u_n), \quad \bar{u} \leq u_n \leq u_1 \text{ for all } n \geq 1 \tag{21}
\]
(see Proposition 9), hence \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_0(\Omega) \) is bounded (see hypothesis \( H_1(i) \)).

We may assume that
\[
u_n \rightharpoonup u_* \text{ in } W^{1,p}_0(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{22}
\]
On (21) we act with \( u_n - u_* \) and pass to the limit as \( n \to \infty \). Using (22), we have
\[
\lim_{n \to \infty} [(A_p(u_n), u_n - u_* - (A(u_n), u_n - u_*)) = 0,
\]
hence
\[
\limsup_{n \to \infty} [(A_p(u_n), u_n - u_* - (A(u_n), u_n - u_*)) \leq 0,
\]
(exploiting the monotonicity of \( A \)), therefore
\[
\limsup_{n \to \infty} (A_p(u_n), u_n - u_*) \leq 0,
\]
and by (22) and Proposition 3 we conclude that
\[
u_n \to u_* \text{ in } W^{1,p}_0(\Omega). \tag{23}
\]
Therefore, if in (21) we pass to the limit as \( n \to \infty \) and use (23), then
\[
A_p(u_*) + A(u_*) = N_f(u_*) \text{ and } \bar{v} \leq u_* \text{ (see (21)),}
\]
hence
\[
u_* \in S_+ \subseteq \text{int } C_+ \text{ and } u_* = \inf S_+.
\]
Similarly for the biggest negative solution \( v_* \in -\text{int } S_+ \), using this time the set \( S_- \).

Using these extremal constant sign solutions we will produce a nodal solution. To this end we introduce the following truncation of the reaction \( f(z,.) : \)
\[
g(z,x) = \begin{cases} f(z,v_*(z)) & \text{if } x < v_*(z) \\ f(z,x) & \text{if } v_*(z) \leq x \leq u_*(z) \\ f(z,u_*(z)) & \text{if } u_*(z) < x. \end{cases} \tag{24}
\]
This is a Carathéodory function. Let \( G(z,x) = \int_0^x g(z,s) \, ds \) and consider the \( C^1 \)-functional \( \psi : W^{1,p}_0(\Omega) \to \mathbb{R} \) defined by
\[
\psi(u) = \frac{1}{p} \|Du\|^p_p + \frac{1}{2} \|Du\|^2_2 - \int_{\Omega} G(z,u(z)) \, dz \text{ for all } u \in W^{1,p}_0(\Omega).
\]
Also, we consider the positive and negative truncations of $g(z,\cdot)$, namely the Carathéodory functions

$$g_\pm(z, x) = g(z, \pm x^\pm) \quad \text{for} \quad (z, x) \in \Omega \times \mathbb{R}.$$  

We set $G_\pm(z, x) = \int_0^x g_\pm(z, s) \, ds$ and consider the $C^1$--functionals $\psi_\pm: W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{p} \| Du \|^p_p + \frac{1}{2} \| Du \|^2_2 - \int_\Omega G_\pm(z, u(z)) \, dz \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

First we make an easy observation concerning the critical sets of the functionals $\psi$ and $\psi_\pm$. In what follows, given $u, v \in W_0^{1,p}(\Omega)$ with $v \leq u$, by $[v, u]$ we denote the order interval in $W_0^{1,p}(\Omega)$ defined by

$$[v, u] := \{ h \in W_0^{1,p}(\Omega) : v(z) \leq h(z) \leq u(z) \quad \text{for a.a.} \quad z \in \Omega \}.$$

In the next proposition $u_* \in \text{int} C_+$ and $v_* \in -\text{int} C_+$ are the two extremal constant sign solutions of (1) established in Proposition 10.

**Proposition 11.** If hypotheses $H_1$ hold, then

$$K_\psi \subseteq [v_*, u_*], \quad K_{\psi^+} = \{0, u_*\}, \quad K_{\psi^-} = \{0, v_*\}.$$

**Proof.** Let $u \in K_\psi$. Then we have

$$\psi'(u) = 0,$$

hence

$$A_p(u) + A(u) = N_g(u). \quad (25)$$

On (25) we act with $(u - u_*)^+ \in W_0^{1,p}(\Omega)$. We have

$$\langle A_p(u), (u - u_*)^+ \rangle + \langle A(u), (u - u_*)^+ \rangle = \int_\Omega g(z, u) (u - u_*)^+ \, dz = \int_\Omega f(z, u_*) (u - u_*)^+ \, dz \quad \text{(see (24))}$$

$$= \langle A_p(u_*), (u - u_*)^+ \rangle + \langle A(u_*), (u - u_*)^+ \rangle \quad \text{(since} \quad u_* \in S_+\text{)},$$

hence

$$\langle A_p(u) - A_p(u_*), (u - u_*)^+ \rangle + \langle A(u) - A(u_*), (u - u_*)^+ \rangle = 0,$$

therefore

$$\int_{\{u > u_*\}} \left( \|Du\|^p_{p-2} + \|Du_*\|^p_{p-2} \right) Du - \|Du_*\|^p_{p-2} Du_* \, dz \leq 0$$

It follows that $|\{u > u_*\}|_N = 0$, i.e.,

$$u \leq u_*.$$  

Similarly, if on (25) we act with $(v_* - u)^+ \in W_0^{1,p}(\Omega)$, we show that

$$v_* \leq u.$$  

So, we have proved that $u \in [v_*, u_*]$, hence

$$K_\psi \subseteq [v_*, u_*].$$
In a similar fashion we show that
\[ K_{\psi^+} \subseteq [0, u_*] \text{ and } K_{\psi^-} \subseteq [v_*, 0]. \]
The extremality of \( u_* \in \text{int } C_+ \) and \( v^* \in -\text{int } C_+ \) (see Proposition 10), implies that
\[ K_{\psi^+} \subseteq \{0, u_*\} \text{ and } K_{\psi^-} = \{0, v_*\}. \]

Because of (24), Proposition 11 and the extremality of the solutions \( u_* \in \text{int } C_+ \) and \( v^* \in -\text{int } C_+ \), we see that every nontrivial element of \( K_{\psi} \) is in fact a nodal solution of (1) and moreover, the nonlinear regularity theory of Lieberman [17] (Theorem 1) implies that \( K_{\psi} \subseteq C_1^0(\Omega) \).

We have the following result concerning the critical groups of \( \psi \) at the origin.

**Proposition 12.** If hypotheses \( H_1 \) hold and \( d_m = \dim \overline{\Pi}_m \), then
\[ C_{d_m}(\psi, 0) \neq 0. \]

**Proof.** We consider the homotopy
\[ h(t, u) = t\varphi(u) + (1 - t)\psi(u) \text{ for all } (t, u) \in [0, 1] \times W^{1,p}_0(\Omega). \]
Suppose we can find \( \{t_n\}_{n \geq 1} \subseteq [0, 1] \) and \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_0(\Omega) \) such that
\[ t_n \to t, \ u_n \to 0 \text{ in } W^{1,p}_0(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \]
From (27) we have
\[ A_p(u_n) + A(u_n) = t_n N_f(u_n) + (1 - t_n) N_g(u_n), \]
hence
\[-\lap_p u_n(z) - \lap u_n(z) = t_n f(z, u_n(z)) + (1 - t_n) g(z, u_n(z)) \text{ for a.a. } z \in \Omega.\]
As before, from Ladyzhenskaya-Uraltseva [16], it follows that we can find \( C_7 > 0 \) such that
\[ \|u_n\|_{\infty} \leq C_7 \text{ for all } n \geq 1. \]

Then Theorem 1 of Lieberman [17] implies that there exist \( \alpha \in (0, 1) \) and \( C_8 > 0 \) such that
\[ u_n \in C^{1,\alpha}_0(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}_0(\overline{\Omega})} \leq C_8 \text{ for all } n \geq 1. \]
Exploiting the compact embedding of \( C^{1,\alpha}_0(\overline{\Omega}) \) into \( C_1^0(\overline{\Omega}) \) and using (27), we have
\[ u_n \to 0 \text{ in } C_1^0(\overline{\Omega}), \]
hence
\[ u_n \in [v_*, u_*] \text{ for all } n \geq n_0 \]
(recall that \( u_* \in \text{int } C_+ \) and \( v^* \in -\text{int } C_+ \)), which contradicts (26). So, (27) cannot occur.

Note that \( \varphi \) satisfies the PS-condition (see Corollary 1), while \( \psi \) is coercive (see (24)) and so, \( \psi \) too satisfies the PS-condition (see Papageorgiou-Winkert [22]).
Choosing \( \varepsilon \) of \( M \) (see, for example, Motreanu-Motreanu-Papageorgiou \[18\], p.143), yields
\[
C_k (\psi, 0) = C_k (\varphi, 0) \quad \text{for all } k \geq 0.
\]
(29)

Let \( \sigma : H^1_0 (\Omega) \to \mathbb{R} \) be the \( C^1 \) - functional defined by
\[
\sigma (u) = \frac{1}{2} \| Du \|_2^2 - \int_{\Omega} F(z, u(z)) \, dz \quad \text{for all } u \in W^{1, p}_0 (\Omega).
\]

Recall that
\[
H^1_0 (\Omega) = \overline{H}_m \oplus \hat{H}_{m+1}
\]
where
\[
\overline{H}_m = \bigoplus_{k=1}^{m} E \left( \hat{\lambda}_k (2) \right) \quad \text{and} \quad \hat{H}_m = \bigoplus_{k \geq m+1} E \left( \hat{\lambda}_k (2) \right).
\]

We know that \( \overline{H}_m \subseteq C^0 (\overline{\Omega}) \) and it is finite dimensional. The finite dimensionality of \( \overline{H}_m \) implies that all norms are equivalent. So, we can find \( \delta_1 > 0 \) such that
\[
u \in \overline{H}_m, \ |u| \leq \delta_1 \text{ implies } |u(z)| \leq \delta_0 \text{ for all } z \in \overline{\Omega}.
\]
(30)

Then, for all \( u \in \overline{H}_m \) with \( |u| \leq \delta_1 \), we have
\[
\sigma (u) = \frac{1}{2} \| Du \|_2^2 - \int_{\Omega} F(z, u(z)) \, dz \leq \frac{1}{2} \| Du \|_2^2 - \left( \frac{\hat{\lambda}_m (2)}{2} \right) \| u \|_2^2 \leq 0
\]
(31)

(see (30), hypothesis \( \textbf{H}_1 \) (iii) and (6)). On the other hand, hypotheses \( \textbf{H}_1 \) (i), (iii) imply that given \( \varepsilon > 0 \), we can find \( C_9 = C_9 (\varepsilon) > 0 \) such that
\[
F (z, x) \leq \frac{\eta (z) + \varepsilon}{2} x^2 + \frac{C_9}{r} |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.
\]
(32)

Then for all \( u \in \hat{H}_{m+1} \), we have
\[
\sigma (u) = \frac{1}{2} \| Du \|_2^2 - \int_{\Omega} F(z, u(z)) \, dz
\geq \frac{1}{2} \left[ \| Du \|_2^2 - \int_{\Omega} \eta (z) u^2 (z) \, dz \right] - \frac{\varepsilon}{2 \lambda_1 (2)} \| u \|^2 - C_{10} \| u \|^r
\geq \frac{1}{2} \left[ C_1 - \frac{\varepsilon}{\lambda_1 (2)} \right] \| u \|^2 - C_{10} \| u \|^r \quad \text{(see Proposition 4)}.
\]

Choosing \( \varepsilon \in \left( 0, C_1 \lambda_1 (2) \right) \), we have
\[
\sigma (u) \geq C_{11} \| u \|^2 - C_{10} \| u \|^r \quad \text{for some } C_{11} > 0, \text{ all } u \in \hat{H}_{m+1}.
\]

Since \( r > 2 \), we can find \( \delta_2 \in (0, 1) \) small such that
\[
\sigma (u) > 0 \quad \text{for all } u \in \hat{H}_{m+1} \text{ with } 0 < \| u \| \leq \delta_2.
\]
(33)

Let \( \hat{\delta} = \min \{ \delta_1, \delta_2 \} \). Then from (31) and (33) we see that
\[
\sigma (u) \leq 0 \quad \text{for all } u \in \overline{H}_m \text{ with } \| u \| \leq \hat{\delta},
\]
\[
\sigma (u) > 0 \quad \text{for all } u \in \hat{H}_{m+1} \text{ with } 0 < \| u \| \leq \hat{\delta},
\]

hence \( \sigma (.) \) has a local linking at the origin, therefore
\[
C_{dm} (\sigma, 0) \neq 0 \quad \text{where } d_m = \dim \overline{H}_m \text{ (see Proposition 6)}.
\]
(34)
Let \( \hat{\sigma} = \sigma \mid_{W_0^{1,p}(\Omega)} \). Since \( W_0^{1,p}(\Omega) \) is dense in \( H_0^1(\Omega) \), we have
\[
C_k(\sigma,0) = C_k(\hat{\sigma},0) \quad \text{for all} \quad k \geq 0 \quad \text{(see Palais [19])},
\]
therefore
\[
C_{d_m}(\hat{\sigma},0) \neq 0 \quad \text{(see (34))}. \tag{35}
\]
Note that
\[
|\hat{\sigma}(u) - \varphi(u)| = \frac{1}{p} \|Du\|_p^p \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).
\]
Also, we have
\[
|\langle \sigma'(u) - \varphi'(u), h \rangle| = \left| \int_{\Omega} \|Du\|_{R\mathbb{N}}^{-2}(Du,Dh)_{R\mathbb{N}} \, dz \right| \\
\leq \|Du\|_{R\mathbb{N}}^{p-1} \|Dh\|_p \quad \text{for all} \quad u, h \in W_0^{1,p}(\Omega),
\]
hence
\[
\|\sigma'(u) - \varphi'(u)\|_* \leq \|Du\|_{R\mathbb{N}}^{p-1}.
\]
So, from Chang ([7], p.336) (see also Corvellec-Hantoute ([11], Theorem 5.1), we have
\[
C_k(\varphi,0) = C_k(\hat{\sigma},0) \quad \text{for all} \quad k \in \mathbb{N},
\]
therefore
\[
C_{d_m}(\varphi,0) \neq 0 \quad \text{(see (35))},
\]
\[
C_{d_m}(\psi,0) \neq 0 \quad \text{(see (29))}.
\]

The next proposition establishes the nature of the solutions \( u_* \in \text{int} \ C_+ \) and \( v_* \in \text{int} \ C_- \) with respect to the functional \( \psi \).

**Proposition 13.** If hypotheses \( H_1 \) hold, then \( u_* \in \text{int} \ C_+ \) and \( v_* \in \text{int} \ C_- \) are local minimizers of the functional \( \psi \).

**Proof.** From (24) it is clear that the functional \( \psi_+ \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find \( \hat{u}_* \in W_0^{1,p}(\Omega) \) such that
\[
\psi_+(\hat{u}_*) = \inf \left\{ \psi_+(u) : u \in W_0^{1,p}(\Omega) \right\}. \tag{36}
\]
As in the proof of Proposition 8, we choose \( t \in (0,1] \) small such that
\[
t\tilde{u}_1(2)(z) \in [0,\delta_0] \quad \text{for all} \quad z \in \overline{\Omega}, \quad \text{and} \quad t\tilde{u}_1(2) \leq u_*
\]
(see Lemma 3.3 of Filippakis-Kristaly-Papageorgiou [12]) and we obtain
\[
\psi_+(t\tilde{u}_1(2)) \leq 0 \quad \text{(see (24) and hypothesis \( H_1(iii) \))}
\]
It follows that
\[
\psi_+(\hat{u}_*) < 0 = \psi_+(0) \quad \text{(see (36))},
\]
hence
\[
\hat{u}_* \neq 0. \tag{37}
\]
From (36) and Proposition 11 we have
\[
\hat{u}_* \in K_{\psi_+} = \{0,u_*\},
\]
therefore
\[
\hat{u}_* = u_* \in \text{int} \ C_+ \text{ (see (37))}.
\]
Note that
\[ \psi_+ |_{C^+} = \psi |_{C^+}. \]
So, we conclude that
\[ u_* \text{ is a local } C^1_0(\Omega) \text{ minimizer of } \psi, \]
therefore
\[ u_* \text{ is a local } W^{1,p}_0(\Omega) \text{ minimizer of } \psi \text{ (see Proposition 1)}. \]
Similarly, for \( v_* \in -\text{int } C^+ \), using this time the functional \( \psi_- \).

Now we are ready to produce a nodal solution.

**Proposition 14.** If hypotheses \( H_1 \) hold, then problem (1) admits a nodal solution
\[ y_0 \in [v_*, u_*] \cap C^1_0(\Omega). \]

**Proof.** From (24) it is clear that \( \psi \) is coercive. Therefore
\[ C_k(\psi, \infty) = \delta_{k,0}Z \text{ for all } k \geq 0. \] (38)
From Proposition 12 we know that
\[ C_{d_m}(\psi, 0) \neq 0 \text{ with } d_m \geq 2 \text{ (recall } m \geq 2). \] (39)
From (38), (39) and Corollary 6.92 of Motreanu-Motreanu-Papageorgiou [18], p.173, it follows that we can find \( y_0 \in K_\psi \) such that
\[ \psi(y_0) < 0 \text{ and } C_{d_m-1}(\psi, y_0) \neq 0 \] (40)
or
\[ \psi(y_0) > 0 \text{ and } C_{d_m+1}(\psi, y_0) \neq 0. \] (41)
From (40) and (41) it is clear that in both cases we have \( y_0 \neq 0 \). From Proposition 13, we have
\[ C_k(\psi, u_*) = C_k(\psi, v_*) = \delta_{k,0}Z \text{ for all } k \geq 0. \] (42)
Since \( d_m \geq 2 \), from (40), (41) and (42) we see that
\[ y_0 \notin \{v_*, u_*\}. \]
Because \( K_\psi \subseteq [v_*, u_*] \) (see Proposition 11), \( y_0 \) is a nodal solution of (1) and the regularity theory of Lieberman [17] implies that \( y_0 \in C^1_0(\Omega) \).

In fact, if we strengthen the conditions on the reaction \( f(z,.) \), we can improve the conclusion of Proposition 14.

The new hypotheses on the reaction \( f(z,x) \), are the following:

**H\(_2\):** \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f(z,0) = 0 \) for a.a. \( z \in \Omega \),
hypotheses \( H_2(i), (ii), (iii) \) are the same as the corresponding hypotheses \( H_1(i), (ii), (iii) \) and
\( (iv) \) for every \( \rho > 0 \) there exists \( \xi_{\rho} > 0 \) such that for a.a. \( z \in \Omega \),
\[ x \mapsto f(z,x) + \xi_{\rho} |x|^{p-2} x \text{ is nondecreasing on } [-\rho, \rho]. \]

**Remark:** This new extra condition on \( f(z,.) \) is satisfied if for example, for a.a. \( z \in \Omega \), \( f(z,.) \in C^1(\mathbb{R}) \) and \( f'_z(z,.) \) is locally \( L^\infty(\Omega) \)-bounded.

**Example:** The map defined by (7) also satisfies hypotheses \( (H_2) \).
Proposition 15. If hypotheses $H_2$ hold, then problem (1) admits a nodal solution $$y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*].$$

Proof. By Proposition 14, we already have a nodal solution $y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega})$. Let

$$a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N$$

and note that

$$\text{div} a(Du) = \triangle u + \triangle u \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall $p > 2$) and

$$\nabla a(y) = |y|^{p-2} \left[ I + (p-2) \frac{y \otimes y}{|y|^2} \right] + I,$$

hence

$$(\nabla a(y) \xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2 \text{ for all } y, \xi \in \mathbb{R}^N. \quad (43)$$

Because of (43) and hypothesis $H_2 (iv)$ (recall that $p > 2$), we can use the tangency principle of Pucci-Serrin ([23], p.35) and infer that

$$y_0(z) < u_*(z) \text{ for all } z \in \Omega. \quad (44)$$

Let $\rho = \max \{||u_*||_{\infty}, ||v_*||_{\infty}\}$ and let $\tilde{\xi}_\rho > 0$ be as postulated by hypothesis $H_2 (iv)$. We choose $\tilde{\xi}_\rho > \tilde{\xi}_\rho$. Then we have

$$-\triangle_\rho y_0(z) - \triangle y_0(z) + \tilde{\xi}_\rho |y_0(z)|^{p-2} y_0(z)$$

$$= f(z, y_0(z)) + \tilde{\xi}_\rho |y_0(z)|^{p-2} y_0(z) + \left( \tilde{\xi}_\rho - \tilde{\xi}_\rho \right) |y_0(z)|^{p-2} y_0(z)$$

$$\leq f(z, u_*(z)) + \tilde{\xi}_\rho u_*^{p-1}(z) + \left( \tilde{\xi}_\rho - \tilde{\xi}_\rho \right) u_*^{p-1}(z) \quad (45)$$

(since $y_0 \leq u_*$, see $H_2 (iv)$)

$$= -\triangle_\rho u_*(z) - \triangle u_*(z) + \tilde{\xi}_\rho u_*(z)^{p-1} \text{ a.e. in } \Omega.$$ 

Let

$$h_1(z) = f(z, y_0(z)) + \tilde{\xi}_\rho |y_0(z)|^{p-2} y_0(z) + \left( \tilde{\xi}_\rho - \tilde{\xi}_\rho \right) |y_0(z)|^{p-2} y_0(z)$$

and

$$h_2(z) = f(z, u_*(z)) + \tilde{\xi}_\rho u_*^{p-1}(z) + \left( \tilde{\xi}_\rho - \tilde{\xi}_\rho \right) u_*^{p-1}(z).$$

Because of (44) and hypothesis $H_2 (iv)$, we have

$$h_1 < h_2$$

(see Section 2). Then from (45) and Proposition 2, we have

$$u_* - y_0 \in \text{int} C_+.$$ 

Similarly, we show that

$$y_0 - v_* \in \text{int} C_+.$$ 

Therefore we conclude that

$$y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*].$$

We can now state our multiplicity theorem, which is a direct consequence of Propositions 8, 14, 15.
Theorem 3.1. If hypotheses \( H_1 \) hold, then problem (1) admits at least three nontrivial solutions

\[ u_0 \in \text{int } C_+, \; v_0 \in -\text{int } C_+ \text{ and } y_0 \in [v_0, u_0] \cap C^1_0(\overline{\Omega}), \text{ nodal}. \]

Moreover, if instead, hypotheses \( H_2 \) hold, then we have

\[ y_0 \in \text{int } C^1_0(\overline{\Omega}) [v_0, u_0]. \]

4. Four solutions. In this section, we improve the regularity of \( f(z,.) \) and produce a second nodal solution for a total of four nontrivial solutions, all with sign information.

The new hypotheses on \( f(z,x) \) are the following:

\( H_3 \): \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a measurable function such that for a.a. \( z \in \Omega \), \( f(z,0) = 0 \), \( f(z,.) \in C^1(\mathbb{R}) \) and:

(i) \(|f_x^r(z,x)| \leq a(z) \left( 1 + |x|^{r-2} \right) \) for a.a. \( z \in \Omega \), all \( x \in \mathbb{R} \) with \( a \in L^\infty(\Omega)_+, \; p \leq r < p^* \);

(ii) if \( F(z,x) = \int_0^x f(z,s) ds \), then

\[ \limsup_{x \to \pm \infty} \frac{pF(z,x)}{|x|^p} \leq \lambda_1(p) \text{ uniformly for a.a. } z \in \Omega \]

and there exists \( \tilde{C} > 0 \) such that

\[ -\tilde{C} \leq f(z,x) - pF(z,x) \text{ for a.a. } z \in \Omega, \; \text{ all } x \in \mathbb{R}; \]

(iii) there exist \( m \in \mathbb{N}, m \geq 3, \) and \( \delta_0 > 0 \) such that

\[ \tilde{\lambda}_m(2) x^2 \leq f(z,x) \text{ for a.a. } z \in \Omega, \; \text{ all } |x| \leq \delta_0, \]

\[ f_x^r(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega, \]

\[ f_x^r(z,0) \leq \tilde{\lambda}_{m+1}(2) \text{ for a.a. } z \in \Omega, \text{ strictly on a set of positive measure}. \]

Remark: The continuous differentiability of \( f(z,.) \) and hypothesis \( H_3(i) \), imply that given \( \rho > 0 \), we can find \( \tilde{\xi}_\rho > 0 \) such that for a.a. \( z \in \Omega, \)

\[ x \mapsto f(z,x) + \tilde{\xi}_\rho |x|^{p-2} x \text{ is nondecreasing on } [-\rho, \rho]. \]

We have the following multiplicity result.

Theorem 4.1. If hypotheses \( H_3 \) hold, then problem (1) has at least four nontrivial solutions

\[ u_0 \in \text{int } C_+, \; v_0 \in -\text{int } C_+ \text{ and } y_0, \tilde{y} \in \text{int } C^1_0(\overline{\Omega}) [v_0, u_0], \text{ nodal}. \]

Proof. From Theorem 3.1 we already have three nontrivial solutions

\[ u_0 \in \text{int } C_+, \; v_0 \in -\text{int } C_+ \text{ and } y_0 \in \text{int } C^1_0(\overline{\Omega}) [v_0, u_0], \text{ nodal}. \]

We may assume that \( u_0 \) and \( v_0 \) are extremal constant sign solutions (that is \( u_0 = u_\ast \in \text{int } C_+ \) and \( v_0 = v_\ast \in -\text{int } C_+ \), see Proposition 10). From Proposition 13, we know that \( u_0 \) and \( v_0 \) are local minimizers of \( \psi \). So, we have

\[ C_k(\psi, u_0) = C_k(\psi, v_0) = \delta_{k,0} \mathbb{Z} \text{ for all } k \geq 0. \]
Also, from the proof of Proposition 14 we know that
\[ C_{d_m-1}(\psi, y_0) \neq 0, \ C_{d_m+1}(\psi, y_0) \neq 0 \] with \( d_m = \dim \bigoplus_{k=1}^{m} E \left( \hat{\lambda}_k(2) \right) \geq 3 \) (47)
(recall that \( m \geq 3 \)). Without any loss of generality, we may assume that \( \psi(v_0) \leq \psi(u_0) \) (the reasoning is similar if the opposite inequality holds). Since \( u_0 \) is a local minimizer of \( \psi \) (see Proposition 13), we can find \( \rho \in (0, 1) \) small such that
\[ \psi(v_0) \leq \psi(u_0) < \inf \left\{ \psi(u) : \|u - u_0\| = \rho \right\} =: m_\rho, \|v_0 - u_0\| > \rho \] (48)
(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29). Recall that \( \psi \) is coercive (see (24)). Hence it satisfies the PS-condition (see Papageorgiou-Winkert [22]). This fact and (48) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find \( \hat{y} \in W^{1,p}_0(\Omega) \) such that
\[ \hat{y} \in K_\psi \subseteq [v_0, u_0] \] (see Proposition 11) and \( m_\rho \leq \psi(\hat{y}) \).
(49)
So, \( \hat{y} \in \{u_0, v_0\} \) (see (48), (49)) and we have
\[ C_1(\psi, \hat{y}) \neq 0 \] (50)
(see Motreanu-Motreanu-Papageorgiou [18], p.176). Moreover, as in the proof of Proposition 15, using the nonlinear regularity theory of Lieberman [17] and the strong comparison principle from Proposition 2, we infer that
\[ \hat{y} \in \text{int}_{C_0^1(\Omega)}[v_0, u_0]. \] (51)
Recall that we assume that \( K_\psi \) is finite (see (26)) or otherwise we already have an infinity of nodal solutions. Then using the homotopy
\[ h(t, u) = t\varphi(u) + (1 - t)\psi(u) \quad \text{for all } (t, u) \in [0, 1] \times W^{1,p}_0(\Omega), \]
reasoning as in the proof of Proposition 12 (with \( 0 \) replaced by \( \hat{y} \)) and using (51), we show that
\[ C_k(\psi, \hat{y}) = C_k(\varphi, \hat{y}) \quad \text{for all } k \geq 0, \] (52)
and hence
\[ C_1(\varphi, \hat{y}) \neq 0 \quad \text{(see (50))}. \] (53)
But \( \varphi \in C^2 \left( W^{1,p}_0(\Omega) \right) \). So, by arguing as in the proof of Theorem 3 of [2] we conclude that
\[ C_k(\varphi, \hat{y}) = \delta_{k,1}Z \quad \text{for all } k \geq 0, \]
(54)
Let \( \sigma(\cdot) \) be as in the proof of Proposition 12. From that proof we know that
\[ C_k(\psi, 0) = C_k(\sigma, 0) \quad \text{for all } k \geq 0. \] (55)
But \( \sigma \in C^2 \left( H^1_0(\Omega) \right) \) and has a local linking at the origin. So, using Proposition 2.2 of Su [24] we have
\[ C_k(\sigma, 0) = \delta_{k,d_m}Z \quad \text{for all } k \geq 0 \] with \( d_m = \dim \bigoplus_{k=1}^{m} E \left( \hat{\lambda}_k(2) \right) \geq 3 \).
(56)
Then
\[ C_k(\psi, 0) = \delta_{k,d_m}Z \quad \text{for all } k \geq 0 \] with \( d_m \geq 3 \) (see (55)). From (46), (47), (54) and (56) it follows that
\[ \hat{y} \notin \{0, u_0, v_0, y_0\}. \]
Therefore $\tilde{y} \in \text{int}_C(\Omega) \setminus [v_0, u_0]$ is the second nodal solution of (1).

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