Link invariants from finite Coxeter racks

Sam Nelson       Ryan Wieghard

Abstract

We study Coxeter racks over $\mathbb{Z}_n$ and the knot and link invariants they define. We exploit the module structure of these racks to enhance the rack counting invariants and give examples showing that these enhanced invariants are stronger than the unenhanced rack counting invariants.

Keywords: Knots and links, Coxeter racks, finite racks, knot and link invariants

2000 MSC: 57M25, 57M27, 17D99

1 Introduction

In 1982, Joyce introduced the term “quandle” for the algebraic structure defined by translating the Reidemeister moves into axioms for a binary operation corresponding to one strand crossing under another [2]. In 1992, Fenn and Rourke generalized the category of quandles to the larger category of “racks,” whose axioms derive from framed isotopy moves [3]. The same concepts appear under different names in the literature; quandles are called “distributive groupoids” in [4], a special case of quandle is called “kei” in [9] and racks are described as “automorphic sets” in [1].

One type of rack structure described in [3] is a Coxeter rack, the subset of an $\mathbb{R}$-vector space on which a symmetric bilinear form is nonzero, with rack operation defined as a kind of reflection. The operator groups of such racks are related to Coxeter groups; see [3] for more.

Replacing $\mathbb{R}$ with $\mathbb{Z}_n$ yields finite Coxeter racks, which are suitable for use as target racks for counting invariants of knots and links as described in [7]. In [5] and [8], quandle counting invariants are enhanced by making use of the module structure of the coloring quandles and biquandles, resulting in knot and link invariants which contain more information than the counting invariants alone. In this paper, we will apply the same idea to the counting invariants defined by finite Coxeter racks, obtaining a new family of enhanced rack counting invariants of knots and links.

The paper is organized as follows. In section 2 we review racks and the rack counting invariant. In section 3 we review Coxeter racks and make a few observations. In section 4 we define the enhanced Coxeter rack invariants and give examples demonstrating that the enhanced invariants are strictly stronger than the unenhanced rack counting invariants. In section 5 we collect some questions for future research.

2 Racks and quandles

Definition 1 A rack is a set $X$ with a binary operation $\triangleright : X \times X \to X$ such that

(i) for every pair $x, y \in X$ there is a unique $z \in X$ such that $x = z \triangleright y$, and

(ii) for every triple $x, y, z \in X$ we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Rack axiom (i) says that every element $y \in X$ acts on $X$ via a bijection $f_y : X \to X, f_y(x) = x \triangleright y$; the inverse defines a second operation $x \triangleright^{-1} y = f_y^{-1}(x)$, and we have $(x \triangleright y) \triangleright^{-1} y = x$ and $(x \triangleright^{-1} y) \triangleright y = x$ for all $x, y \in X$. Rack axiom (ii) says that the operation $\triangleright$ is self-distributive. A rack in which every element is idempotent, i.e., such that $x \triangleright x = x$ for all $x \in X$, is a quandle.
Rack structures abound in mathematics. Any algebraic structure which acts on itself by automorphisms is a rack: define \( f_y(x) = x \bowtie y \). Then
\[
f_z(x \bowtie y) = f_z(x) \bowtie f_z(y) \iff (x \bowtie y) \bowtie z = (x \bowtie z) \bowtie (y \bowtie z).
\]

Standard examples of racks include:

- A set \( S \) with \( x \bowtie y = \sigma(x) \) for some fixed bijection \( \sigma : S \to S \) (constant action rack or permutation rack)
- A group \( G \) with \( x \bowtie y = y^{-n}xy^n \) (conjugation rack)
- A group \( G \) with \( x \bowtie y = s(xy^{-1})y \) for a fixed automorphism \( s \in \text{Aut}(G) \)
- A module over \( \mathbb{Z}[t^\pm 1, s]/(s + t - 1) \) with \( x \bowtie y = tx + sy \) ((\( t, s \))-rack)

It is convenient to express a rack structure on a finite set \( \{x_1, x_2, \ldots, x_n\} \) by encoding its operation table as an \( n \times n \) matrix \( M \) whose \( i,j \) entry is \( k \) where \( x_k = x_i \bowtie x_j \). We call this matrix the rack matrix of \( T \), denoted \( M_T \).

**Example 1** Let \( R = \{1, 2, 3, 4\} \) and \( \sigma : R \to R \) be the permutation \((12)(34)\). Then the rack matrix of \((R, \sigma)\) is
\[
M_{(12)(34)} = \begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3
\end{bmatrix}.
\]

For defining invariants of framed oriented knots and links, we need the fundamental rack. By using the blackboard framing, we can consider link diagrams as framed link diagrams with framing numbers given by the self-writhe of each component; by choosing an order on the components, we can conveniently express the writhe of an \( n \)-component link diagram as a vector \( w \in \mathbb{Z}^n \).

The idea is then to think of arcs in an oriented link diagram as generators and the operation \( \bowtie \) as “crosses under from right to left” when looking in the positive direction of the overcrossing strand. The inverse operation \( \bowtie^{-1} \) then means “crosses under from left to right.”

Indeed, the rack axioms are simply the Reidemeister moves required for framed isotopy interpreted in light of this operation; see [5] or [7] for more.

Given a link diagram \( L \), we obtain a rack presentation with one generator for each arc and one relation at each crossing. That is, the fundamental rack \( FR(L) \) of the framed link specified by the diagram is the set of equivalence classes of rack words under the equivalence relation generated by the crossing relations together with the rack axioms. Note that changing the writhe of the diagram by Reidemeister I moves results in a generally different fundamental rack. Taking the quotient of \( FR(L) \) for any framing of \( L \) by setting \( a \sim a \bowtie a \) for all \( a \in FR(L) \) yields the knot quandle of \( L \), denoted \( Q(L) \).
Example 2 The pictured trefoil knot $3_1$ with writhe 3 has the listed fundamental rack presentation.

\[ FR(D) = \langle x, y, z \mid x \triangledown y = z, y \triangledown z = x, z \triangledown x = y \rangle \]

In \cite{7}, the quandle counting invariant $|\text{Hom}(Q(K), T)|$ is extended to include non-quandle racks as coloring objects. For a finite rack $T$, the rack rank of $T$, denoted $N(T)$, is the exponent of the permutation given by the diagonal of the rack matrix. If two ambient isotopic link diagrams $L$ and $L'$ have framing vectors with respect to an ordering on the components which are componentwise congruent mod $N(T)$, then there is a bijection between the sets of rack colorings of $L$ and $L'$ by $T$. We might say that as far as $T$ cares, the writhes of $L$ live in $(\mathbb{Z}_{N(T)})^n$. We are thus able to reduce the infinite set of fundamental racks of framings of $L$ to get a finitely computable ambient isotopy invariant of $L$.

Definition 2 Let $L$ be a link with $n$ components, $T$ a finite rack with rack rank $N(T)$ and $W = (\mathbb{Z}_{N(T)})^n$. The polynomial rack counting invariant of $L$ with respect to $T$ is

\[
PR(L, T) = \sum_{w \in W} |\text{Hom}(FR(D, w), T)| \prod_{i=1}^{n} q_i^{w_i},
\]

where $(D, w)$ is a diagram of $L$ with writhe vector $w \in W$ and $FR(D, w)$ is the fundamental rack of $(D, w)$.

Example 3 The Hopf link $H$ has rack counting polynomial $PR(H, T) = 4 + 4q_1 + 4q_2 + 8q_1q_2$ with respect to the rack $T$ with rack matrix

\[
M_T = \begin{bmatrix}
1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 \\
4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3
\end{bmatrix}
\]

as the tables of colorings show.

The two-component unlink $U_2$ has rack counting polynomial $PR(U_2, T) = 16 + 8q_1 + 8q_2 + 4q_1q_2$ with respect to this rack as the reader is invited to verify. Thus, the invariant detects the difference between the Hopf link and the two-component unlink.
3 Coxeter racks

In [3], a Coxeter rack is defined as the subset of $\mathbb{R}^n$ on which a symmetric bilinear form $\langle, \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is nonzero, with rack operation

$$x \triangleright y = x - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y.$$

Multiplying the right-hand side by $-1$ defines a quandle structure, called a Coxeter quandle.

We will study a family of slight generalizations of these Coxeter racks. First, we replace $\mathbb{R}$ with an arbitrary commutative ring $R$, choose a symmetric bilinear form $\langle, \rangle : R^n \times R^n \to R$ and consider the subset of $R^n$ given by

$$T = \{ x \in R^n \mid \langle x, x \rangle \in R^* \}$$

where $R^*$ is the set of units in $R$. Next, we note that replacing the $-1$ factor in the Coxeter quandle definition with any invertible scalar $\alpha \in R^*$ yields a valid rack structure, which gives us our generalized Coxeter rack definition. More formally, we have:

**Definition 3** Let $R$ be a commutative ring, $V$ an $R$-module and $\langle, \rangle : V \times V \to R$ a symmetric bilinear form. Let

$$T = \{ x \in V \mid \langle x, x \rangle \in R^* \},$$

where $R^*$ is the set of units in $R$. Note that $T \subseteq V$ since $0 \not\in T$. Define $\triangleright : T \times T \to T$ by

$$x \triangleright y = \alpha \left( x - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y \right)$$

where $\alpha \in R^*$. We call $(T, \triangleright)$ a generalized Coxeter rack and write $T = CR(V, \alpha, \langle, \rangle)$.

To verify that $(T, \triangleright)$ is a rack, we will find the following lemma useful:

**Lemma 1** Let $(T, \triangleright)$ be a generalized Coxeter rack. Then $\langle x \triangleright z, y \triangleright z \rangle = \alpha^2 \langle x, y \rangle$.

**Proof.**

$$\langle x \triangleright z, y \triangleright z \rangle = \left\langle \alpha x - \frac{2\alpha \langle x, z \rangle}{\langle z, z \rangle} z, \alpha y - \frac{2\alpha \langle y, z \rangle}{\langle z, z \rangle} z \right\rangle$$

$$= \alpha^2 \langle x, y \rangle \left( 1 - \frac{2\alpha^2 \langle y, z \rangle}{\langle z, z \rangle} \right) - 2\alpha^2 \frac{\langle x, z \rangle}{\langle z, z \rangle} \langle x, z \rangle - 2\alpha^2 \frac{\langle y, z \rangle}{\langle z, z \rangle} \langle y, z \rangle + 4\alpha^2 \frac{\langle x, z \rangle}{\langle z, z \rangle} \frac{\langle y, z \rangle}{\langle z, z \rangle} \langle z, z \rangle \langle z, z \rangle$$

$$= \alpha^2 \langle x, y \rangle + \frac{4\alpha^2 \langle x, z \rangle}{\langle z, z \rangle} \langle y, z \rangle \langle z, z \rangle$$

$$= \alpha^2 \langle x, y \rangle.$$
satisfies \((x \triangleright y) \triangleright^{-1} y = x:\)

\[
(x \triangleright y) \triangleright y = \alpha^{-1} \left( x \triangleright y - \frac{2}{\langle y, y \rangle} \langle x \triangleright y, y \rangle \right)
\]

\[
= \alpha^{-1} \left( \alpha \left( x - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y \right) - \frac{2}{\langle y, y \rangle} \langle \alpha \left( x - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y \right), y \rangle y \right)
\]

\[
= x - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y - \frac{2\langle x, y \rangle}{\langle y, y \rangle} y + \frac{4\langle x, y \rangle \langle y, y \rangle}{\langle y, y \rangle^2} y
\]

\[
= x.
\]

In particular, note that if \(\alpha = \alpha^{-1}\) then \(\triangleright = \triangleright^{-1}\) and \((M, \triangleright)\) is an involutory rack.

Finally, we check that \(\triangleright\) is self-distributive:

\[
(x \triangleright y) \triangleright z = \alpha(x \triangleright y) - 2\alpha \langle x \triangleright y, z \rangle z
\]

\[
= \alpha^2 x - 2\alpha \frac{\langle x, y \rangle}{\langle y, y \rangle} y + \left( -2\alpha^2 \frac{\langle x, z \rangle}{\langle z, z \rangle} + 4\alpha \frac{\langle x, y \rangle \langle z, z \rangle}{\langle y, y \rangle \langle z, z \rangle} \right) z
\]

On the other hand,

\[
(x \triangleright z) \triangleright (y \triangleright z) = \alpha(x \triangleright z) - 2\alpha \frac{\langle x, y \rangle}{\langle z, z \rangle} (y \triangleright z)
\]

\[
= \alpha^2 x - 2\alpha \frac{\langle x, z \rangle}{\langle y, y \rangle} z - 2\alpha \frac{\langle x, y \rangle}{\langle z, z \rangle} \left( \alpha y - 2\alpha \frac{\langle y, z \rangle}{\langle z, z \rangle} z \right)
\]

\[
= \alpha^2 x - 2\alpha^2 \frac{\langle x, z \rangle}{\langle y, y \rangle} y + \left( -2\alpha^2 \frac{\langle x, z \rangle}{\langle z, z \rangle} + 4\alpha \frac{\langle x, y \rangle \langle y, z \rangle}{\langle y, y \rangle \langle z, z \rangle} \right) z
\]

as required. \(\square\)

**Remark 1** In the classical case where \(V = \mathbb{R}^n, \langle , \rangle\) is the dot product and \(\alpha = -1\), the Coxeter quandle operation is the result of reflecting \(x\) through \(y\) in the plane spanned by \(x\) and \(y\):

![Diagram](image_url)

For the purpose of defining counting invariants, we need finite racks. Thus, we will consider the case \(V = (\mathbb{Z}_n)^m\) of generalized Coxeter racks which are subsets of free modules over the integers modulo \(n\). The input data required to construct such a rack consists of two integer parameters \(n\) and \(m\) (which determine \(V = (\mathbb{Z}_n)^m\)), a scalar \(\alpha \in \mathbb{Z}_n\) coprime to \(n\), and a symmetric matrix \(A \in M_m(\mathbb{Z}_n)\) which defines a symmetric bilinear form \(\langle , \rangle : (\mathbb{Z}_n)^m \times (\mathbb{Z}_n)^m \to \mathbb{Z}_n\) by

\[
\langle x, y \rangle = xAy^t
\]

where \(x\) is a row vector and \(y^t\) is a column vector.
Example 4 For a simple example, let us take $V = (\mathbb{Z}_3)^2$ with $\alpha = 1$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$. A straightforward computation then shows that $T = CR(V, \alpha, A) = \{x_1 = (1, 0), x_2 = (1, 1), x_3 = (2, 0), x_4 = (2, 2)\}$ and $T$ has rack matrix
\[
M_T = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & 1 & 3 \\ 4 & 2 & 4 & 2 \end{bmatrix}.
\]

We end this section with a few brief observations about finite Coxeter racks.

Proposition 3 If $R = \mathbb{Z}_2$, then every generalized Coxeter rack over $R$ has trivial rack operation.

Proof.
\[
x \triangleright y = 1 \left( x - \frac{2(x, y)}{\langle y, y \rangle} y \right) = x - 0 = x.
\]

\[\square\]

Corollary 4 Finite generalized Coxeter racks over $R = \mathbb{Z}_2$ are classified by the cardinality of the subset of $V$ on which $\langle \cdot, \cdot \rangle$ is nonzero.

Proposition 5 If $R$ has characteristic 2, then $CR(R^m, \alpha, A)$ is a constant action rack with permutation given by multiplication by $\alpha$.

Proof. If the characteristic of $R$ is 2, we have
\[
x \triangleright y = \alpha x + 0 = \alpha x
\]
for all $x \in T$.

\[\square\]

Proposition 6 Let $T = CR(R^m, \alpha, A)$ and $T' = CR(R^m, \alpha, \beta A)$ for an invertible scalar $\beta \in R^*$. Then $T = T'$.

Proof. First, $x Ax^t \in R^*$ if and only if $x \beta A x^t \in R^*$ so setwise $T = T'$. Moreover,
\[
x \triangleright_T y = \alpha \left( x - \frac{2x Ay^t}{y Ay^t - y} \right) = \alpha \left( x - \frac{2x \beta A y^t}{y \beta A y^t - y} \right) = (x \triangleright_{T'} y)
\]
so the rack structures are the same.

\[\square\]

Remark 2 Changing the scalar $\alpha$ in general does result in different rack structures: $\alpha = -1$ yields a quandle while $\alpha = 1$ yields a non-quandle rack, as we know. As another example, we note that the Coxeter racks $CR \left( (\mathbb{Z}_5)^2, 3, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$ and $CR \left( (\mathbb{Z}_5)^2, 1, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$ respectively have rack polynomials $16$ and $16t^4$ (see [6]), and hence cannot be isomorphic.
4 Coxeter enhanced rack counting invariants

Let \( R = \mathbb{Z}_n \), \( V = R^m \) and \( T = CR(V, \alpha, A) \). Let \( L \) be a link with \( c \) components and let \( W = (\mathbb{Z}_N(T))^c \). Consider a coloring \( f \in \text{Hom}(FR(D, w), T) \) of an oriented framed link diagram \((D, w)\). For each homomorphism \( f \), the image subrack \( \text{Im}(f) \) and hence the submodule \( \text{Span}(\text{Im}(f)) \subseteq V \) it spans are invariant under Reidemeister moves, so we can form an enhanced version of the counting invariant incorporating this extra information. Formally, we have

**Definition 4** Let \( L \) be a link with \( n \) components and \( T = CR(R, T, \alpha, A) \) a finite generalized Coxeter rack. Then the **Coxeter enhanced rack counting invariant** is

\[
\text{cp}(L, T) = \sum_{w \in W} \left( \sum_{f \in \text{Hom}(FR(D, w), T)} \prod_{i=1}^{n} q_i^{w_i} s^{\text{Span}(\text{Im}(f))} \right).
\]

We note from the definition that specializing \( s = t = 1 \) yields the rack counting polynomial. On the other hand, the Coxeter enhanced invariants are stronger than the corresponding unenhanced rack counting invariants, as the following example shows.

**Example 5** Let \( L_1 \) be the \((4, 2)\) torus link and \( L_2 \) the three-component link illustrated below. Let \( T \) be the Coxeter quandle \( T = CR((\mathbb{Z}_3)^2, 2, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}) \). Then \( T \) has quandle matrix

\[
M_T = \begin{bmatrix}
1 & 3 & 1 & 3 \\
4 & 2 & 4 & 2 \\
3 & 1 & 3 & 1 \\
2 & 4 & 2 & 4
\end{bmatrix}
\]

where we have \( x_1 = (1, 0), x_2 = (1, 1), x_3 = (2, 0) \) and \( x_4 = (2, 2) \). An easy computation then shows that while both \( L_1 \) and \( L_2 \) have quandle counting invariant \( |\text{Hom}(Q(L), T)| = 16 \), the Coxeter enhanced invariants tell the links apart, with \( \text{cp}(L_1, T) = 4s^3t + 4s^3t^2 + 8s^9t^4 \) while \( \text{cp}(L_2, T) = 4s^3t + 12s^3t^2 \).

![Diagram of links L1 and L2](image)

In light of proposition \( \text{6} \) we have

**Proposition 7** For any link \( L \), \( \text{cp}(L, CR(V, \alpha, A)) = \text{cp}(L, CR(V, \alpha, \beta A)) \) for any invertible scalar \( \beta \in R^* \).

**Example 6** The Coxeter enhanced rack counting invariants give us convenient information about the subracks of \( T = CR(V, \alpha, A) \) with surjective homomorphisms from the various fundamental racks of the framings of \( L \). The trefoil knot \( 3_1 \) has Coxeter enhanced rack invariant \( \text{cp} = 6s^4t^2 + 12s^8t^6 \) with respect to the Coxeter rack \( T = \left( (\mathbb{Z}_3)^2, 1, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \). We can immediately note several things: there are no homomorphisms from the fundamental rack of trefoil with odd framing into \( T \) since the coefficient of \( q^1 \) is zero; there are six surjective homomorphisms from the fundamental rack of any evenly-framed trefoil onto subracks of \( T \) with two elements spanning 1-dimensional subspaces of \((\mathbb{Z}_3)^2\), and there are twelve surjective homomorphisms from the fundamental rack of an evenly framed trefoil onto subracks of \( T \) with six elements spanning the whole space \((\mathbb{Z}_3)^2\) (as indeed six vectors in a two-dimensional subspace must).
Finally, we note that to define the Coxeter enhanced rack counting invariants, we need to know the vector space or \( R \)-module structure of the Coxeter rack in question, not just the rack structure. It seems possible \( \text{a priori} \) that the same Coxeter rack might embed in different modules or vector spaces or in different ways in the same module or vector space (indeed, see the next section), and in such a case we expect the resulting invariants to be different, although of course they specialize to the same rack counting invariant.

5 Questions

In this section we collect some questions for future research.

Given a finite Coxeter rack \( T \), to what degree is it possible to recover the module structure \((R, V, \alpha, A)\) from which \( T \) arises? We already know that changing \( A \) by an invertible scalar multiple yields an isomorphic (indeed, an \emph{identical}) rack; an easy computation shows that the Coxeter racks on \((\mathbb{Z}_3)^2\) with \( \alpha = 1 \) and bilinear forms defined by \( A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \) are not identical but are nonetheless isomorphic.

In [3] we also find \emph{Hermitian form} racks, which are defined on the subset of a vector space on which \( \langle x, x \rangle \in R^* \) for a form \( \langle , \rangle : V \times V \to R \) which is linear in the first variable and conjugate linear in the second for a conjugation on \( R \) (that is, a ring automorphism \( f : R \to R \) such that \( f(f(r)) = r \) for all \( r \in R \)). Our generalized Coxeter racks are just the Hermitian racks on \( \mathbb{Z}_n \)-modules with respect to the identity conjugation, the only conjugation on rings whose additive groups are cyclic. We have not considered Hermitian racks on modules over finite rings other than \( \mathbb{Z}_n \), but the same construction given in section [3] should give a \emph{Hermitian enhanced rack counting invariant} for any finite \( R \)-module with a nontrivial conjugation, e.g. vector spaces over the Galois field of four elements with the Frobenius automorphism.

What kinds of non-rack birack structures can be defined on finite vector spaces and modules with Coxeter/Hermitian type operations? Such structures should also give rise to enhanced counting invariants.

Our \texttt{python} code for computing Coxeter rack matrices and Coxeter enhanced rack counting invariants is available at \url{www.esotericka.org}.

References

[1] E. Brieskorn, Automorphic sets and braids and singularities. \textit{Contemp. Math.} \textbf{78} (1988) 45-115.
[2] D. Joyce. A classifying invariant of knots, the knot quandle. \textit{J. Pure Appl. Algebra} \textbf{23} (1982) 37-65.
[3] R. Fenn and C. Rourke. Racks and links in codimension two. \textit{J. Knot Theory Ramifications} \textbf{1} (1992), 343-406.
[4] S. V. Matveev. Distributive groupoids in knot theory. \textit{Math. USSR, Sb.} \textbf{47} (1984) 73-83.
[5] E.A. Navas and S. Nelson. On symplectic quandles. To appear in \textit{Osaka J. Math.}, \url{arXiv:math/0703727}
[6] S. Nelson. A polynomial invariant of finite racks. \textit{J. Alg. Appl.} \textbf{7} (2008) 263-273.
[7] S. Nelson. Link invariants from finite racks, \url{arXiv:0808.0029}
[8] S. Nelson and J.L. Rische. On bilinear biquandles. \textit{Colloq. Math.} \textbf{112} (2008) 279-289.
[9] M. Takasaki. Abstraction of symmetric transformation (in Japanese). \textit{Tohoku Math J.} \textbf{49} (1943) 145-207.