Interactive Collaborative
Exploration using Incomplete Contexts

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Abstract A well-known knowledge acquisition method in the field of
Formal Concept Analysis (FCA) is attribute exploration. It is used to re-
veal dependencies in a set of attributes with help of a domain expert. In
most applications no single expert is capable (time- and knowledge-wise)
of exploring the knowledge domain alone. However, there is up to now
no theory that models the interaction of multiple experts for the task
of attribute exploration with incomplete knowledge. To this end, we to
develop a theoretical framework that allows multiple experts to explore
domains together. We use a representation of incomplete knowledge as
three-valued contexts. We then adapt the corresponding version of attribute
exploration to fit the setting of multiple experts. We suggest formalizations
for key components like expert knowledge, interaction and collaboration
strategy. In particular, we define an order that allows to compare the results
of different exploration strategies on the same task with respect to their
information completeness. Furthermore we discuss other ways of comparing
collaboration strategies and suggest avenues for future research.

Keywords: Formal Concept Analysis, Incomplete Context, Collaboration,
Attribute Exploration

1 Introduction

Nowadays information is generated and collected at unimaginable scales. Some
of it is published online, for example on Wikipedia or in knowledge bases such
as Wikidata or DBpedia. Collecting information from a domain is the first step
to acquiring knowledge. Often the next step is to structure the information and
extract conceptual knowledge, a task performed by experts of the domain. But
even for domains of reasonable size experts normally have incomplete knowledge
and collaboration is necessary to improve the results.

Authors are given in alphabetical order. No priority in authorship is implied.
For domains that can be represented as data-tables of objects and attributes, Formal Concept Analysis (FCA) [7] provides the well-known knowledge acquisition method of attribute exploration [4, 5]. This method helps an expert to systematically obtain knowledge about the structural dependencies between attributes.

In this paper, we present a theoretical framework for a cooperative attribute exploration that allows for a set of experts. Before diving into technical details, however, we illustrate in the next three paragraphs by an example the procedure of state-of-the-art attribute exploration with a single expert. To this end, imagine that we want to explore dependencies of properties of the sports disciplines of the Summer Olympics 2020 in Tokyo. Some properties of interest could be if a discipline has individual or team competitions, if the contestants of events are males, females or mixed, or how many events are held and how often the discipline was already part of the Olympic Games.

For the basic version of attribute exploration to work, we need all properties to be binary attributes. For this purpose, FCA provides a technique called scaling, cf. [7, Sec. 1.3 ff.], to transform each non-binary property to multiple binary properties. Here we use the three properties ‘has at least 5, 10 or 20 events’ for the number of events of a discipline and the three properties ‘was part of at least 8, 16 or 24 Olympic Games’ for the number of Olympic Games that a discipline was already part of, both of which are examples of scaling using an ordinal scale.

The basic attribute exploration algorithm developed by Ganter [4] works as follows: The algorithm systematically asks questions such as ‘Do all disciplines have events for males and for females?’ and ‘Have disciplines that hold at least ten events been part of at least eight Olympic Games?’. These questions are answered by the expert who conducts the exploration. More precisely, the expert confirms a question if it is true or rejects it with a counterexample if it is false. Here, the expert would reject the first question since ‘Artistic Swimming’ is an Olympic discipline that only has events for females. The expert would further report all properties that ‘Artistic Swimming’ has. More specifically, that it has no male events, no mixed events and no individual events but does have team events, and that the total number of events is less than five and that the discipline was part of at least eight but no more than fifteen Olympic Games. The expert would confirm the second question to be a valid implication. The attribute exploration systematically asks such questions until all possible questions can either be inferred from the set of accepted questions or rejected based on the counterexamples given by the expert.

Burmeister and Holzer have developed an extension of attribute exploration where the expert may have incomplete knowledge of the domain, cf. [2, 3, 9–11]. In this setting, the expert can also answer questions with “I don’t know”. Furthermore, the expert does not need to know the relations of an object to all attributes when providing a counterexample. This extension makes attribute exploration more viable in practice. In most applications, however, no single expert is capable (time- and knowledge-wise) of exploring a domain on her own. Yet, to the best of

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1 The information for this example was obtained from https://tokyo2020.org/, https://www.olympic.org/tokyo-2020 and https://en.wikipedia.org/wiki/Olympic_sports. The full example context can be found in the appendix.
our knowledge there presently exists no theory that allows multiple experts with incomplete knowledge to cooperatively perform attribute exploration.

The purpose of this paper is to provide such a theoretical framework, i.e., develop the attribute exploration with incomplete knowledge to work with multiple experts. To this end, we suggest a possible formalization of attribute exploration in a collaborative setting with experts that have incomplete knowledge. As a first step, we formalize expert knowledge. Then, we introduce an order relation that allows us to compare and combine expert knowledge. We develop formalizations for expert interaction and collaboration strategy. Furthermore, we define an order that enables the comparison of results of different exploration strategies on the same task with respect to their information completeness. In order to discuss methods to evaluate and compare collaboration strategies we use some examples of collaboration strategies. These examples also reveal some possible improvements to our approach.

We mainly build upon the results of Burmeister and Holzer about representing incomplete knowledge and the corresponding version of attribute exploration for a single expert. Note that, in this paper, we neither consider imprecise knowledge, e.g., the case where an expert is 80% sure that an object has an attribute or 90% sure an implication is valid, nor contradictory knowledge, e.g., the case where experts disagree whether an object has attribute.

We begin by giving a review of related work in Section 2. Then we recapitulate the existing theory needed to formulate attribute exploration for a single expert with incomplete knowledge in Section 3. Afterwards we develop our theory to handle multiple experts with incomplete knowledge in a collaborative setting in Section 4. Examples for collaboration strategies are given and used to discuss methods to compare different strategies. In Section 5 we give a conclusion and recollect some avenues for future research. Lastly, this paper has a running example where we provide the Sports Disciplines of the Summer Olympics 2020 context and an extensive example of three experts performing a collaborative exploration with incomplete knowledge. For better readability, this example is presented in one piece in an appendix (Section 6), but is cross-referenced throughout the whole paper wherever appropriate.

2 Related Work

Much work has already been done concerning the modelling of uncertainty and incomplete knowledge in particular, e.g., Bayesian statistics, modal logics, possibility theory and probabilistic logics. Of particular interest for us are the three-valued logics Kleene-Algebras [12] and Kripke-Semantics [14] as they have been used to model incomplete knowledge in FCA [3, 17].

In the realm of FCA, attribute exploration for incomplete knowledge has been around for about 30 years. The first attempts to model incomplete knowledge in the context of formal concept analysis were made by Burmeister using Kleene-logic in [3] where he already discussed attribute exploration with incomplete knowledge and strategies to deal with questions that can not be answered directly.

In later works different approaches were explored. For example, in [6] Ganter models incomplete knowledge using two formal contexts: One for attributes an
object certainly has and one for attributes an object possibly has. Another example
is [16] where Obiedkov discussed the evaluation of propositional formulas in incom-
plete contexts using a three-valued modal logic with a ‘nonsense’ value. In 2016 the
book ‘Conceptual Exploration’ [5] written by Ganter and Obiedkov was published
giving an extensive overview on the many variations of attribute exploration.

The paper ‘On the Treatment of Incomplete Knowledge in Formal Concept
Analysis’ [2] by Burmeister and Holzer gives a good overview on how to treat
incomplete knowledge in attribute exploration. It covers a wide range of topics
from an introduction of incomplete contexts, the definitions of possible and certain
intents and extents and attribute implications in incomplete contexts to reductions
of question-marks, three-valued Kleene-logic and an algorithm for attribute ex-
ploration that allows questions to be answered with ‘true’, ‘false’ or ‘unknown’.
However, attribute exploration remains focused on a single expert.

The thesis of Holzer [11] and the later adaptions as papers [9, 10] contain in-depth
results about incomplete contexts, their relationship to attribute implications and
a version of attribute exploration that allows the expert to answer questions with
“I don’t know”. As before, attribute exploration remains focused on a single expert.

The publications of Burmeister and Holzer can be considered the most elaborate
for dealing with incomplete knowledge in the realm of FCA. Hence, we use their
results as a foundation for our work on collaborative exploration with incomplete
knowledge.

There exist some publications that deal with ideas of collaboration in the realm
of FCA but not nearly as many as deal with incomplete knowledge. A publication
specifically addressing collaborative attribute exploration to help with ontology
construction is [15] by Obiedkov and Romashkin. Here, some issues arising with
collaborative exploration are identified, e.g., the need to allow for incomplete
elements and for having policies of collaboration that can deal with conflicting
information. However, these issues are merely stated and not examined in detail.
Consequently the task of further improving the theoretic foundations of attribute
exploration in a collaborative setting is raised.

Recently Hanika and Zumbrägel suggested an approach for collaborative ex-
ploration based on experts for attribute sets [8]. They employ the notion of a
consortium of experts and discuss its ability, i.e., how much of the domain can be
explored given certain experts for attribute sets, and the value of being able to
combine examples. However, the experts do not directly talk about objects in the
domain (which makes it impossible to merge partial knowledge about the same
object) and are not allowed to answer “I don’t know”.

Another obstacle of an efficient interactive collaborative attribute exploration
is the sequentiality of asking questions when utilizing the NextClosure algorithm,
cf. Ganter [4, 5, 7], to generate questions. In [13], Kriegel modified the NextClosure
algorithm to obtain a parallel version of attribute exploration with all-knowing
experts. This allows multiple questions to be generated at once and might help to
further improve the efficiency of collaborative attribute exploration with incomplete
knowledge.
3 Recollection of known Results

In this section, we recollect some basic definitions from FCA as introduced in [19] and recapitulated in [7] and recollect notions and results from [2, 9, 11] for incomplete contexts and attribute exploration for incomplete knowledge. We add some notation to make things more readable in the following sections and give a few examples to ease understanding of the core ideas.

3.1 Formal Context

Formal contexts are one of the most basic structures in FCA. Note that we consider the incidence relation as a function to better fit our needs later on.

Definition 3.1 (formal context, c.f. [19]). A (one-valued) formal context \( K = (G, M, I) \) consists of a set of objects \( G \), a set of attributes \( M \) and an incidence relation \( I \subseteq G \times M \) with \((g,m) \in I\) meaning the object \( g \) has the attribute \( m \).

There are several interpretations for \((g,m) \notin I\), cf. [2, 3], the standard one being, “\( g \) does not have the attribute \( m \) or it is irrelevant, whether \( g \) has \( m \)”. In the following we interpret \((g,m) \notin I\) as “\( g \) does not have \( m \)”, which is reasonable when modeling incomplete knowledge.

This interpretation can be equivalently modeled by a (two-valued) formal context \( K = (G, M, I) \) that consists of a set of objects \( G \), a set of attributes \( M \) and an incidence function \( I : G \times M \rightarrow \{ \times, \circ \} \). The incidence function describes whether an object \( g \) has an attribute \( m \): \( I(g,m) = \times \) means “\( g \) has \( m \)” and \( I(g,m) = \circ \) means “\( g \) does not have \( m \)”. Clearly we can use a one-valued formal context to define an equivalent two-valued formal context and vice versa using \((g,m) \in I \iff I(g,m) = \times\).

In the following we will use these two definitions interchangeably.

A formal context can be represented as a table with rows of objects and columns of attributes. The table entries signify the relation of objects and attributes. It is customary to put an “\( \times \)” to indicate that an object has an attribute and to either put an “\( \circ \)” or leave a table cell empty to indicate that an object does not have an attribute. For readability we will leave the cells empty. See Figure 1 for representations of a formal context using part of the introductory example about properties of the sports disciplines of the Summer Olympic Games 2020.

Definition 3.2. (derivation operators, c.f. [19]) Let \( K = (G, M, I) \) be a formal context. For a set of objects \( A \subseteq G \) we define the set of attributes common to the objects in \( A \) by
\[
A' := \{ m \in M | \forall g \in A : I(g,m) = \times \}. 
\]
Analogously, for a set of attributes \( B \subseteq M \) we define the set of objects that have all the attributes from \( B \) by
\[
B' := \{ g \in G | \forall m \in B : I(g,m) = \times \}. 
\]

Definition 3.3 (formal concept, intent, extent, c.f. [19]). Let \( K = (G, M, I) \) be a formal context. A formal concept of \( K \) is a pair \((A,B)\) with \( A \subseteq G \) and \( B \subseteq M \) such that \( A' = B \) and \( A = B' \). We call \( A \) the extent and \( B \) the intent of the formal concept \((A,B)\).
Figure 1: Example of two representations of a formal context about the sports disciplines of the Summer Olympic Games 2020, see Footnote 1, with attributes a) \( \geq 10 \text{ events} \), b) \( \geq 5 \text{ events} \), c) female only events, d) male only events, and e) part of \( \geq 8 \text{ Olympics} \).

Note that for any set \( A \subseteq G \) the set \( A' \) is the intent of a concept and for any set \( B \subseteq M \) the set \( B' \) is the extent of a concept.

### 3.2 Incomplete Context

In the following we introduce incomplete contexts as a means to model partial knowledge. Note that once again we use an incidence function.

**Definition 3.4** (incomplete context, c.f. [2]). An incomplete context is a three-valued context \( K = (G, M, \{\times, o, ?\}, I) \) consisting of a set of objects \( G \), a set of attributes \( M \), a set of values \( \{\times, o, ?\} \) and an incidence function \( I: G \times M \rightarrow \{\times, o, ?\} \).

For \( g \in G \) and \( m \in M \) we say that “it is known that \( g \) has \( m \)” if \( I(g, m) = \times \), “it is known that \( g \) does not have \( m \)” if \( I(g, m) = o \) and “it is not known whether \( g \) has \( m \)” if \( I(g, m) = ? \).

Like a formal context, an incomplete context can be represented as a table of objects and attributes with the entries signifying the relation. Here we use “\( \times \)” to indicate that object and attribute are known to be related, an empty cell to indicate they are known not to be related and “\( ? \)” to indicate that the relation is not known. See Figure 2 for an example of incomplete knowledge added to the example in Figure 1.

Figure 2: Example of an incomplete contexts with attributes a) \( \geq 10 \text{ events} \), b) \( \geq 5 \text{ events} \), c) female only events, d) male only events, and e) part of \( \geq 8 \text{ Olympics} \). Imagine someone saw the context in Figure 1 and further knew that ‘Taekwondo’ is an Olympic discipline but was unsure how many events there are and for how long it has been Olympic.
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Notation 3.5 \((I^x, I^o, I^?)\). Let \(K = (G, M, \{\times, o, ?\}, I)\) be an incomplete context. To refer to certain subsets of \(G \times M\) we define:

\[
I^x := \{(g, m) \in G \times M | I(g, m) = \times\}
\]

\[
I^o := \{(g, m) \in G \times M | I(g, m) = o\}
\]

\[
I^? := \{(g, m) \in G \times M | I(g, m) = ?\}
\]

Notation 3.6 \((K|_A)\). The restriction of an incomplete context \(K = (G, M, \{\times, o, ?\}, I)\) to a subset \(A \subseteq G\) is denoted by \(K|_A := (A, M, \{\times, o, ?\}, I|_{A \times M})\), where

\[
I|_{A \times M} : A \times M \to \{\times, o, ?\} \text{ with } I|_{A \times M}(g, m) = I(g, m).
\]

If an incomplete context \(K = (G, M, \{\times, o, ?\}, I)\) does not contain any “?”, i.e., all relations of objects and attributes are known, it can be identified with a formal context \(\tilde{K} = (G, M, I)\) with \(I = I^x\). We also call such a context \textit{complete} incomplete context.

Any formal context \(K = (G, M, I)\) can also be identified with an incomplete context where the incidence relation is completely known, i.e., as a context \(K = (G, M, \{\times, o, ?\}, J)\) where

\[
J(g, m) = \begin{cases} 
\times & \text{if } (g, m) \in I \\
o & \text{if } (g, m) \not\in I
\end{cases}
\]

Therefore complete incomplete contexts and formal contexts will be used synonymously, the particular representation will be mentioned if it is necessary and can not be inferred from the context.

Similar to the case of formal contexts we define derivation operators for incomplete contexts. Since it may be unknown if an object has an attribute we define two operators, one for the relations that are known and one for the relations that are possible.

Definition 3.7 (certain and possible derivation operators, c.f. [2, 9]). Given an incomplete context \(K = (G, M, \{\times, o, ?\}, I)\) we define the \textit{certain intent} for \(A \subseteq G\) by

\[
A^\square := \{m \in M | I(g, m) = \times \text{ for all } g \in A\}
\]

and the \textit{possible intent} by

\[
A^\Diamond := \{m \in M | I(g, m) \in \{\times, ?\} \text{ for all } g \in A\}
\]

For \(B \subseteq M\) we define the \textit{certain extent} \(B^\square\) and the \textit{possible extent} \(B^\Diamond\) in the same way. For \(g \in G\) and \(m \in M\) we use the abbreviations \(g^\square, g^\Diamond, m^\square\) and \(m^\Diamond\).

Example 3.8. Recall the incomplete context from Figure 2. Let us take a look at the \textit{possible extent} and the \textit{certain extent} of \(A = \{\text{Taekwondo, Badminton}\}\). This means we look at the set of attributes that both \textit{Taekwondo} and \textit{Badminton} certainly have and at the set of attributes that they possibly have. Here, the \textit{possible extent} \(\{\text{Taekwondo, Badminton}\}^\Diamond\) is \(\{b, c, d, e\}\), whereas the \textit{certain extent} \(\{\text{Taekwondo, Badminton}\}^\square\) is \(\{e, d\}\).
Remark 3.9. In the case of a formal context $K = (G, M, I)$ (or a complete incomplete context $K = (G, M, \{\times, o, ?\}, I)$, i.e., with $I' = \emptyset$) the certain and possible intent and extent are the same and are equivalent to the usual intent and extent for formal contexts, i.e., for $A \subseteq G$ and $B \subseteq M$ we have
\[
A' = A^\Box = A^\Diamond \\
B' = B^\Box = B^\Diamond
\]

3.3 Order on Incomplete Contexts

The values $\{\times, o, ?\}$ can be ordered in at least two ways that make sense semantically: We can order them according to their trueness, i.e., $o < ? < \times$ (trueness order, see Figure 3a), and we can order them according to the amount of information they represent, i.e., $? < \times$ and $? < o$ (information order, see Figure 3b). In the latter case the values $\times$ and $o$ are incomparable.

![Figure 3: Two orders on the values $\{\times, o, ?\}$](image)

Both of these orders are useful when thinking about how to evaluate formulas in an incomplete context, namely in terms of three-valued logics using Kleene-algebras and in terms of Kripke semantics, cf. Section 3.4.

With the Kripke semantics we think of formulas of being certainly valid (see, Definition 3.19) if they hold in every completion of an incomplete context $K$, i.e., in all contexts where every “?” in $K$ is replaced by an “$\times$” or an “$o$”. This is computationally very inefficient, since we need to evaluate the formulas in $2^{|I'(K)|}$ many contexts. However, we are mainly interested in a subset of formulas, the so called attribute implications (see, Definition 3.17) where the evaluation in the information order is equivalent to the evaluation in the trueness order (see, Lemmas 3.25 and 3.26), which is far more efficient in terms of computational complexity.

The information order is further useful to define an order on incomplete contexts (see, Definitions 3.10 and 4.6). This provides the basis to compare the knowledge of experts (see, Definition 4.14) and the results of collaborative attribute explorations (see, Section 4.6). Most importantly, it enables combining expert knowledge.

Now we define an order where we compare two incomplete contexts with regard to the information they contain using the information order, cf. Figure 3b.
Definition 3.10 (information order, c.f. [11]). Let $K_1 := (G, M, \{\times, o, ?\}, I_1)$ and $K_2 := (G, M, \{\times, o, ?\}, I_2)$ be two incomplete contexts defined on the same object and attribute sets. We say that $K_2$ contains at least as much information as $K_1$, abbreviated $K_1 \leq K_2$, if

$$\forall g \in G, \forall m \in M : I_1(g, m) = \times \Rightarrow I_2(g, m) = \times$$

and

$$I_1(g, m) = o \Rightarrow I_2(g, m) = o$$

which is equivalent to

$$\forall g \in G, \forall m \in M : I_1(g, m) \leq I_2(g, m)$$

where $\leq$ is the information order on $\{\times, o, ?\}$.

Remark 3.11. Note that we are overloading the notation $\leq$. It is used both on the value-level and the context-level. We also use the name information order for this order on the context-level.

Example 3.12. Imagine we asked four people which of the properties a) $\geq 10$ events, b) $\geq 5$ events, c) female only events, d) male only events, and e) part of $\geq 8$ Olympics of Taekwondo has, and the four incomplete contexts $K_1,...,K_4$ represent their answers.

We see that $K_4$ contains at least as much information as $K_1$, i.e., $K_1 \leq K_4$. Further, the contexts $K_2$ and $K_3$ are incomparable, since $K_2 \nleq K_3$ and $K_3 \nleq K_2$.

The information order on the set of incomplete contexts for fixed sets of objects and attributes, i.e., on the set $\{\times, o, ?\}^{G \times M}$, corresponds to the component-wise comparison of all object-attribute incidences $(g, m) \in G \times M$ in the information order on $\{\times, o, ?\}$. In this order the infimum for any two such incomplete contexts exists, it is the component-wise infimum of the contexts. Equally, the supremum of two incomplete contexts on the same objects and attributes is the component-wise supremum. However, the supremum only exists if the contexts do not contain any contradictory information.

Given two incomplete contexts $K_1$ and $K_2$ the contradictory information are all pairs $(g, m)$ where an object $g$ is known to have an attribute $m$ in one context and known not to have it in the other, i.e., $(I^x_1 \cap I^o_2) \cup (I^o_1 \cap I^x_2)$. Note that it is not necessary for the contexts to be defined on the same objects and attributes to determine the conflicting information.

The infimum and supremum of two incomplete contexts represent their shared and joint information.
Corollary 3.13 (c.f. [11]). Let $G$ and $M$ be fixed sets of objects and attributes. Let $\{\times, o, ?\}^{G \times M}$ be the set of all incomplete contexts with objects $G$ and attributes $M$. Let $K_1, K_2 \in \{\times, o, ?\}^{G \times M}$ with $K_1 := (G, M, \{\times, o, ?\}, I_1)$ and $K_2 := (G, M, \{\times, o, ?\}, I_2)$.

a) Then $\{\times, o, ?\}^{G \times M}$ together with the information order $\leq$ (see Definition 3.10) forms a $\wedge$-semilattice where the infimum of two incomplete contexts $K_1$ and $K_2$ is given by

\[ K_1 \wedge K_2 := (G, M, \{\times, o, ?\}, I_1 \wedge I_2) \]

with $I_1 \wedge I_2 : G \times M \rightarrow \{\times, o, ?\}$ where

\[ (I_1 \wedge I_2)(g, m) := \begin{cases} \times & \text{if } I_1(g,m) = \times = I_2(g,m) \\ o & \text{if } I_1(g,m) = o = I_2(g,m) \\ ? & \text{otherwise} \end{cases} \]

b) Further, if the two contexts $K_1$ and $K_2$ have no conflicting information the supremum $K_1 \vee K_2$ exists and is given by

\[ K_1 \vee K_2 := (G, M, \{\times, o, ?\}, I_1 \vee I_2) \]

with $I_1 \vee I_2 : G \times M \rightarrow \{\times, o, ?\}$ where

\[ (I_1 \vee I_2)(g, m) := \begin{cases} \times & \text{if } I_1(g,m) = \times \text{ or } I_2(g,m) = \times \\ o & \text{if } I_1(g,m) = o \text{ or } I_2(g,m) = o \\ ? & \text{otherwise} \end{cases} \]

Example 3.14. Recall the four incomplete contexts $K_1, ..., K_4$ from Example 3.12. We have $K_1 = K_2 \wedge K_3$. Further, the contexts $K_2$ and $K_3$ contain no conflicting information and their supremum is $K_4 = K_2 \vee K_3$.

Remark 3.15. We later show that given an incomplete context $K_0$ the set of $\{K \mid K \leq K_0\}$ is a bounded lattice with respect to the information order and that the operators $\land$ and $\lor$ coincide with the supremum and infimum induced by the information order on incomplete contexts, see Corollary 4.11.

### 3.4 Attribute Implications

For the rest of this paper the set of attributes $M$ is considered to be finite. One fundamental concept in FCA is that of attribute implications. They are used to describe dependencies between attributes of a formal or incomplete context and are defined as propositional formulas over an attribute set $M$ in the following way:
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Definition 3.16 (formulas, models, c.f. [9]). We define $F(M)$ as the set of propositional formulas over $M$ where $M$ is the set of propositional variables. Let $\alpha \in F(M)$ and $B \subseteq M$. We say $B$ is a model of $\alpha$ or equally $B$ respects $\alpha$ if the interpretation of $\alpha$ is true for the valuation $v_B : M \rightarrow \{true, false\}$ with $v_B(m) = true : \Leftrightarrow m \in B$.

For a set of formulas $P \subseteq F(M)$ define the set of models of $P$ by $\text{Resp}(P) := \{B \subseteq M \mid B \text{ respects each } \alpha \in P\}$.

For $A \subseteq M$ define $\langle A \rangle_P := \bigcap \{B \in \text{Resp}(P) \mid A \subseteq B\}$.

The set of attribute implications over a set of attributes $M$ is a specific subset of the propositional formulas $F(M)$:

Definition 3.17 (attribute implication, $A \rightarrow B$, c.f. [2, 9]). For $S \subseteq M$ we let $\bigwedge S := (s_1 \land \ldots \land s_n)$ if $S = \{s_1, \ldots, s_n\}$ and $\bigwedge S := \text{true}$ if $S = \emptyset$. For $A \subseteq M$ and $B \subseteq M$ we write $A \rightarrow B$ for $\bigwedge A \rightarrow \bigwedge B$ and call this formula attribute implication or short implication. If $A = \{a_1, \ldots, a_m\}$ we also write $a_1 \ldots a_m$ instead of $A$ and if $B = \{b_1, \ldots, b_n\}$ we also write $b_1 \ldots b_n$ instead of $B$, e.g., we write $a_1 \ldots a_m \rightarrow b_1 \ldots b_n$ instead of $A \rightarrow B$. Further, $A$ is referred to as premise and $B$ as conclusion of the implication. We abbreviate the set of all implications over the attribute set $M$ by $\text{Imp}_M := \{A \rightarrow B \mid A, B \subseteq M\}$.

Definition 3.18. For a set $L$ of attribute implications over an attribute set $M$ we define $\text{Cons}(L)$ as the set of all implications obtainable from $L$ by using the Armstrong rules [1] (for sets $A, B, C, D$)

\[
\frac{A \rightarrow A}{A \rightarrow A}, \quad \frac{A \rightarrow C}{A \cup B \rightarrow C}, \quad \frac{A \rightarrow B \quad B \cup C \rightarrow D}{A \cup C \rightarrow D}.
\]

Definition 3.19 (valid formula, c.f. [9]). Given a formal context $K = (G, M, I)$ we call a formula $\alpha \in F(M)$ valid for an object $g \in G$ if $g'$ is a model of $\alpha$. The formula is valid in $K$ if it is valid for all objects $g \in G$. An attribute implication $A \rightarrow B \in \text{Imp}_M$ is valid in $K$ if and only if every object $g \in G$ that has all the attributes in $A$ also has all the attributes in $B$. We then say $B$ follows from $A$ in $K$.

In the case of incomplete contexts, i.e., three-valued contexts, there exist many different logics to evaluate formulas, e.g., the Kleene-Logic [12] and other three-valued logics, cf. [2, 11, 17]. Here we use the Kripke-semantics.

Note that Holzer used a different but equivalent set of rules to define the consequence operator in [9–11], however, it is common to utilize the Armstrong rules, cf. [2, 7].
Definition 3.20 (certainly valid, satisfiable, c.f. [2, 9]). Given an incomplete context $\mathbb{K} = (G, M, \{\times, o, ?\}, I)$ and a formula $\alpha \in F(M)$. A formal context $\tilde{\mathbb{K}}$ is a completion of an incomplete context $\mathbb{K}$ if $\mathbb{K} \leq \tilde{\mathbb{K}}$. The formula $\alpha$ is Kripke-valid or certainly valid if it is valid in every completion of $\mathbb{K}$. Further the formula $\alpha$ is satisfiable or possibly valid if it is valid in at least one completion of $\mathbb{K}$.

Remark 3.21. For a complete context both certain and possible validity are equivalent and coincide with the usual formulation (as in Definition 3.19) of valid formulas for formal contexts.

Example 3.22. As an example recall the incomplete context in Figure 2. Here the implication $b \rightarrow d$ is certainly valid as it is valid in every completion of the context, whereas $c \rightarrow e$ is satisfiable but not certainly valid since ‘Taekwondo’ has ‘female only events’ but could or could not be ‘part of at least eight Olympic Games’.

Definition 3.23 ($\text{Imp}(\mathbb{K})$, $\text{Sat}(\mathbb{K})$, c.f. [9, 11]). Given an incomplete context $\mathbb{K}$ we denote the set of all certainly valid implications by

$$\text{Imp}(\mathbb{K}) := \{ A \rightarrow B \in \text{Imp}_M | A \rightarrow B \text{ is certainly valid in } \mathbb{K} \}$$

and the set of all satisfiable implications by

$$\text{Sat}(\mathbb{K}) := \{ A \rightarrow B \in \text{Imp}_M | A \rightarrow B \text{ is satisfiable in } \mathbb{K} \}.$$  

For the rest of this section we recollect some facts about attribute implications in the case of incomplete contexts. The set of certainly valid implications is closed with respect to the Armstrong rules.

Theorem 3.24 (see [7, 9, 11]). With $\text{Cons}(\cdot)$ and $\text{Imp}(\cdot)$ as defined in Definitions 3.18 and 3.23 we have $\text{Cons}(\text{Imp}(\mathbb{K})) = \text{Imp}(\mathbb{K})$ for every incomplete context $\mathbb{K}$.

The operators $\Box$ and $\Diamond$ can be used to efficiently compute whether an attribute implication is certainly valid or satisfiable. This corresponds to the evaluation in Kleene-Legic [2, 11, 17].

The following lemmas clarify that implicational formulas can be evaluated in Kripke semantics and Kleene-Logic giving the same result. Note that for arbitrary attribute formulas this is not true.

Lemma 3.25 (see [9] Lemma 5). Let $\mathbb{K} = (G, M, \{\times, o, ?\}, I)$ be an incomplete context and $A, B \subseteq M$. Then the following conditions are equivalent:

1. $A \rightarrow B \in \text{Imp}(\mathbb{K})$
2. $B \setminus A \subseteq A \Diamond \Box$
3. $A \Box \subseteq (B \setminus A) \Diamond$
4. For all $g \in G$ with $A \subseteq g \Diamond$ we have $B \setminus A \subseteq g \Box$
5. For all $g \in G$ holds: If $I(g, a) \neq o$ for all $a \in A$ then $I(g, b) = \times$ for all $b \in B \setminus A$.

Lemma 3.26 (see [9] Lemma 6). Let $\mathbb{K} = (G, M, \{\times, o, ?\}, I)$ be an incomplete context and $A, B \subseteq M$. Then the following conditions are equivalent:

1. $A \rightarrow B \in \text{Sat}(\mathbb{K})$
2. $B \subseteq A^{\Box}$
3. $A^{\Box} \subseteq B^{\Box}$
4. For all $g \in G$ with $A \subseteq g^{\Box}$ we have $B \subseteq g^{\Box}$
5. For all $g \in G$ holds: If $I(g,a) = \times$ for all $a \in A$ then $I(g,b) \neq o$ for all $b \in B$.

Further the operator $\cdot^{\Box}$ is useful to compute the maximal satisfiable conclusion for a premise.

**Lemma 3.27** (see [9] Corollary 6). Let $K = (G, M, \{\times, o, ?\}, I)$ be an incomplete context and $A \subseteq M$. Then

$$A^{\Box} := \{m \in M \mid A \rightarrow m \in \text{Sat}(K)\}.$$

### 3.5 Attribute Exploration

Let $K^{U} = (G^{U}, M, I^{U})$, $|M| < \infty$, be an (unknown) formal context called universe. This context represents the knowledge domain of interest. (We assume that the domain can be represented in such a way). The so called attribute exploration is an interactive algorithm that helps an expert gain maximum insight into the dependency structure of the domains attributes.

The following algorithm is taken from [11] and condensed to the main steps. It describes the process of exploring a knowledge domain modelled as an unknown formal context $K^{U}$ using the knowledge of an expert in an algorithmic fashion. Here we assume that the expert’s answers are always consistent with the domain, i.e., an accepted implication is valid in $K^{U}$ and given counterexamples are objects of the domain contradicting the implication in question. The attribute exploration produces a set of valid implications in the universe $K^{U}$ and a list of counterexamples against non-valid implications. The following algorithm describes the exploration:

(E1) At the beginning of the exploration algorithm the user enters the (finite) set of attributes $M$ whose dependencies are to be explored.

(E2) Let $j := 1$. The set of accepted implications is initialized with the empty set $P_{1} := \emptyset$. The context of examples is initialized with an empty incomplete context $K_{1} := (G_{1} = \emptyset, M, \{\times, o, ?\}, I_{1})$ with $I_{1} : \emptyset \times M \rightarrow \{\times, o, ?\}$.

(E3) The set $P_{j}$ contains the implications accepted as valid so far. In the $j$-th step the algorithm chooses an implication $A \rightarrow B$ that might be valid in $K^{U}$, such that the set $A \subseteq M$ is minimal (w.r.t. $\subseteq$), respects $P_{j}$ and $B := A^{\Box} := \{m \in M \mid A \rightarrow m \in \text{Sat}(K_{j})\} \neq A$. If the implication is derivable from $P_{j}$ it is accepted automatically. Otherwise the program asks the expert whether $A \rightarrow B$ is valid in the universe $K^{U}$. The expert can answer yes, no or unknown:

(YES) The implication $A \rightarrow B$ is accepted as valid and added to the set of accepted implications: $P_{j+1} := P_{j} \cup \{A \rightarrow B\}$. Let $K_{j+1} = K_{j}$.

(NO) The expert must give at least one counterexample $g \in G^{U}$ against the implication $A \rightarrow B$. For each counterexample she enters the context row of $g$ which may contain question marks, i.e., unknown relations between $g$ and some attributes. Let $P_{j+1} := P_{j}$ and $K_{j+1}$ be the context $K_{j}$ after adding the rows of all counterexamples $g$. 
The expert is asked for which attributes \( b \in B \) the implication \( A \rightarrow b \) is unknown. Let \( Z := \{ b \in B | A \rightarrow b \text{ is unknown} \} \). For \( b \in B \setminus Z \) the implication \( A \rightarrow b \) is valid in the universe \( K^U \), because every counterexample against \( A \rightarrow b \) would be a counterexample against \( A \rightarrow b \).

For \( b \in Z \) the algorithm checks whether \( A \rightarrow b \in \text{Cons}(P_j \cup \{ A \rightarrow B \setminus Z \}) \) holds. In case it holds \( b \) can be removed from \( Z \), since \( A \rightarrow b \) follows from implications known to be valid in the universe \( K^U \) and must therefore also be valid. In case it does not hold for \( b \), i.e., \( A \rightarrow b \notin \text{Cons}(P_j \cup \{ A \rightarrow B \setminus Z \}) \), fictitious objects are added to \( K_j \).

For each remaining \( b \in Z \) we add the fictitious object \( g^2_{A \rightarrow b} \) that contradicts the implication \( A \rightarrow b \), i.e., \( g^2_{A \rightarrow b} \) has all attributes in \( A \), does not have the attribute \( b \) and the relation to all other attributes is unknown. We assume that \( g^2_{A \rightarrow b} \) is a new object, i.e., \( g^2_{A \rightarrow b} \notin G^U \) and \( g^2_{A \rightarrow b} \notin G_j \).

Let \( K_{j+1} := (G_j \cup \{ g^2_{A \rightarrow b} | b \in Z \}, M, \{ \times, \#, ? \}, J) \) with \( J(g, m) = I_j(g, m) \) for all \( g \in G_j \), \( m \in M \) and for \( b \in Z \) let \( J(g^2_{A \rightarrow b}, a) = \times \) for \( a \in A \), \( J(g^2_{A \rightarrow b}, b) = \# \) and \( J(g^2_{A \rightarrow b}, m) = ? \) for \( m \in M \setminus (A \cup \{ b \} \).

Let \( P_{j+1} := P_j \cup \{ A \rightarrow B \setminus Z \} \) if \( B \setminus Z \neq A \) and \( P_{j+1} := P_j \) if \( B \setminus Z = A \).

(E4) If every set \( A \) that respects \( P_j \) and is not already a premise in \( P_j \) satisfies \( A = \{ m \in M | A \rightarrow m \in \text{Sat}(K_j) \} \) the algorithm ends. Otherwise increment \( j \), i.e., let \( j := j + 1 \), and repeat the steps (E3) and (E4).

This is a stripped-down version of attribute exploration for incomplete knowledge (without handling of background knowledge and reductions of question marks), because most of the exploration procedure itself is of no particular interest for the remainder of this paper. An example of how this algorithm works in detail is beyond the scope of this paper and we refer the reader to [2] and [11].

There already exists theory on more advanced techniques such as the use of background knowledge and reductions of question marks based on already accepted implications. For exploration with incomplete knowledge more information can, for example, be found in [2, 9, 11]. Other modifications such as allowing exceptions to attribute implications [18] and background knowledge in the form of implications and clauses [5, 6, 18] seem adaptable to attribute exploration with incomplete knowledge as well.

**Fact 3.28** (cf. [2, 9, 10]). At the end of the attribute exploration the result contains maximal information (with respect to the expert’s knowledge) about implications of the unknown universe \( K^U \). Assuming the exploration ended after \( j \) steps, the result consists of

1. a list of implications \( P_j \) that are known to be valid,
2. a list of fictitious counterexamples \( G^* := G_j \setminus G^U \) contradicting implications where the expert answered ‘unknown’,
3. a list of counterexamples \( G_j \setminus G^* \) contradicting the implications which are known not to be valid and
4. a list of implications \( P_j \cup \{ A \rightarrow b | g^2_{A \rightarrow b} \in G^* \} \) which are possibly valid.

To obtain the complete knowledge about the domain it now suffices to check all implications that were answered by ‘unknown’ before. If for each of these implications
it can be decided whether they are valid or have to be rejected, complete knowledge about the domain is received: An implication is valid in $K^U$ if and only if it is derivable from the implications accepted as valid and implications rejected as ‘unknown’ that in fact are valid in $K^U$.

So far we have presented known results. We have seen that attribute exploration works with a single (reliable) expert who can respond with partial knowledge to questions posed by the exploration algorithm. Figure 4 visualizes this part of the exploration process.

\[ A \rightarrow B \text{ valid?} \]

Figure 4: Visualization of exploration with one expert

Let us recap: The algorithm of the attribute exploration under incomplete knowledge generates questions that are to be answered by the domain expert. The expert is not omniscient but reliable, i.e., the answers she gives are consistent with the true knowledge in the domain. The result is dependent on the knowledge of the expert conducting the exploration. The answers ‘no’ and ‘unknown’ come with additional information provided by the expert, i.e., with counterexamples or the set of attributes for which is unknown if they follow from the premise. The result of such an attribute exploration is a set of attribute implications known to be valid, an incomplete context of counterexamples that contradicts implications that are known to be invalid in the domain and a set of fictitious counterexamples that contradict implications that were unknown to the expert.

4 Experts and Collaboration

Based on previous works on attribute exploration under incomplete knowledge (c.f. Section 3.5) we now modify the attribute exploration to work with multiple experts. The idea is that instead of a single expert who answers the questions directly we have a strategy to answer the questions with help of multiple experts – see Figure 5. Once again the answers ‘no’ and ‘unknown’ come with additional information. Namely, some counterexamples and the set of attributes $Z \subseteq B$ for which the implication $A \rightarrow Z$ is unknown and $A \rightarrow B \setminus Z$ is valid. Note that we slightly modify the attribute exploration to receive an answer that already contains all the additional information expected by the algorithm.
With this picture in mind we begin by formalizing the expert. We generalize the information order to work on different object sets to be able to compare and combine the knowledge of experts. We then adapt the idea of a consortium \cite{8} to a group of experts, formulate a notion of collaboration, give a few examples and proceed to discuss methods to compare different collaboration strategies.

An example of three experts with incomplete knowledge conducting a collaborative exploration of properties of the *Disciplines of the Summer Olympic Games 2020* can be found in the appendix (see Example 6.1).

### 4.1 Expert Knowledge

The expert, a key component of every attribute exploration, is often only described as an entity that correctly answers the posed questions, especially in many of the earlier works on the subject. Later works, e.g., \cite{5,8}, make an effort to also model the expert in a formal way, usually as a mapping from the set of attribute implications into the target domain represented as closure system over \( M \). In the following we suggest a model for experts in an incomplete knowledge setting where we encode the knowledge of an expert by both an incomplete context of examples and a set of valid implications. We then model a notion of interaction with the expert.

First, we formally introduce the *universe*, cf. Section 3.5, which represents complete (but unknown) knowledge of the domain of interest, i.e., the domain about which the expert has knowledge. We assume that the universe can be represented as a formal context.

**Definition 4.1 (universe).** In the following let \( K^U = (G,M,I) = (G,M,\{\times,\circ\},I) \), \(|M| < \infty\) always be a formal context which we call *universe*. The set \( \mathcal{L} := \text{Imp}(K^U) \) denotes the set of valid implications in the *universe*.

Note that the the universe could equally be defined as a formal or incomplete context. We chose to restrict ourselves to a universe represented by a formal context for the sake of readability.
Definition 4.2 (expert knowledge). Expert knowledge about the universe $\mathbb{K}^U$

$$E = (\mathbb{K}_E, \text{Cons}(\mathcal{L}_E))$$

consists of a context $\mathbb{K}_E := (G_E, M, \{x, o, ?, I_E\}, G_E \subseteq G)$, of objects that are (partially) known to the expert, i.e., $\mathbb{K}_E \leq \mathbb{K}^U|_{G_E}$, and a set of implications $\mathcal{L}_E \subseteq \mathcal{L}:= \text{Imp}(\mathbb{K}^U)$ that the expert knows to be valid.

Note that we also use the terms expert for a domain and expert for a universe to indicate that an expert has expert knowledge about a universe.

By definition the set of partial counterexamples and the set of known valid implications are compatible:

Lemma 4.3. Let $E = (\mathbb{K}_E, \text{Cons}(\mathcal{L}_E))$ be expert knowledge about a universe $\mathbb{K}^U$. Then every implication in $\text{Cons}(\mathcal{L}_E)$ is satisfiable in $\mathbb{K}_E$.

Proof. Let $A \rightarrow B \in \text{Cons}(\mathcal{L}_E)$. Since $\mathcal{L}_E \subseteq \mathcal{L}$ and $\text{Cons}(\mathcal{L}) = \mathcal{L}$ we know that $A \rightarrow B$ is valid in $\mathbb{K}^U$. Therefore $\mathbb{K}^U$ does not contain any counterexamples for $A \rightarrow B$ and $A \rightarrow B$ is valid in every subcontext $(T, M, I|_{T \times M})$ with $T \subseteq G$ of $\mathbb{K}^U$. Since $\mathbb{K}_E \leq \mathbb{K}^U|_{G_E}$ and $A \rightarrow B$ is valid in $\mathbb{K}^U|_{G_E}$ we have that $A \rightarrow B$ is satisfiable in $\mathbb{K}_E$.

One might be tempted to use the example knowledge to infer further implications, but much like in reality this is not justified. The certainly valid implications of the expert’s example context are not necessarily valid in the universe. Furthermore, the valid implications known to the expert are not necessarily certain valid with regard to the set of partial counterexamples known to the expert as shown in Remark 4.4.

Remark 4.4. Assume that we have an expert $(\mathbb{K}_E, \text{Cons}(\mathcal{L}_E))$ for the universe $\mathbb{K}^U$ then the implications in $\mathcal{L}_E$ are not necessarily certainly valid in $\mathbb{K}_E$. Consider for instance the following example (cf. Figure 1):

| $\mathbb{K}^U$ | a | b | c |
|----------------|---|---|---|
| Aquatics – Swimming | x | x | x |
| Badminton | x | x | |
| Gymnastics – Rhythmic | ? | ? | |

Let $\{a \rightarrow c\} = \mathcal{L}_E \subseteq \mathcal{L} = \text{Imp}(\mathbb{K}^U)$, then $a \rightarrow c$ is satisfiable but not certainly valid in $\mathbb{K}_E$. Further, the implication $b \rightarrow a$ is certainly valid in $\mathbb{K}_E$ but not valid in $\mathbb{K}^U$.

Remark 4.5. In our setting (and in contrast to [8]), multiple experts for a universe can know about different objects or different attribute object relations. However, they can not have conflicting knowledge, cf. lemma 4.20.

4.2 Generalized Information Order

We want to compare and combine the knowledge of different experts. To achieve this we need to be able to compare both the known examples and the known implications.
The known implications can easily be compared using the set inclusion order $\subseteq$. However, to compare the known examples of different experts we need to generalize the information order to allow comparing contexts with different object sets.

**Definition 4.6** (generalized information order). Given two incomplete contexts $K_1 = (G_1, M, \{\times, o, ?\}, I_1)$ and $K_2 = (G_2, M, \{\times, o, ?\}, I_2)$ on object sets $G_1, G_2 \subseteq G$, we say that $K_2$ contains at least as much information as $K_1$, abbreviated $K_1 \leq g K_2$, if

$$G_1 \subseteq G_2 \text{ and } K_1 \leq K_2|_{G_1}.$$ where $\leq$ is the information order on incomplete contexts, see Definition 3.10.

Obviously we have $\leq g = \leq$ if we compare incomplete contexts that have the same object and attribute sets, therefore we use contains at least as much information in both cases.

The infimum of two incomplete contexts on the same attribute set with respect to the generalized information order always exists:

**Definition 4.7** (generalized information infimum). Given two incomplete contexts $K_1 = (G_1, M, \{\times, o, ?\}, I_1)$ and $K_2 = (G_2, M, \{\times, o, ?\}, I_2)$ on object sets $G_1, G_2 \subseteq G$, the generalized information infimum $K_1 \land_g K_2$ is defined by

$$K_1 \land_g K_2 := (G_1 \cap G_2, M, \{\times, o, ?\}, I_1 \land I_2)$$

where

$$I_1 \land I_2 : (G_1 \cap G_2) \times M \to \{\times, o, ?\}$$

with $(I_1 \land I_2)(g, m) = I_1(g, m) \land I_2(g, m)$ defined as before, see Corollary 3.13 a).

Again, for incomplete contexts on the same attribute set with no conflicting information there exists a supremum with respect to the generalized information order:

**Definition 4.8** (generalized information supremum). Given two incomplete contexts $K_1 = (G_1, M, \{\times, o, ?\}, I_1)$ and $K_2 = (G_2, M, \{\times, o, ?\}, I_2)$ on object sets $G_1, G_2 \subseteq G$, with no conflicting information the generalized information supremum $K_1 \lor_g K_2$ is defined by

$$K_1 \lor_g K_2 := (G_1 \cup G_2, M, \{\times, o, ?\}, I_1 \lor I_2)$$

where

$$I_1 \lor I_2 : (G_1 \cup G_2) \times M \to \{\times, o, ?\}$$

with $(I_1 \lor I_2)(g, m) = I_1(g, m) \lor I_2(g, m)$ defined as before, see Corollary 3.13 b), with the addition that we extend the domains of $I_1$ and $I_2$ to $G_1 \cup G_2$, each by mapping previously undefined object-attribute combinations to ?.

The following lemma and corollaries allow us to compare and combine example knowledge.

**Lemma 4.9.** The set $S$ of all incomplete contexts that are derived from an incomplete context $K^U = (G, M, \{\times, o, ?\}, I)$, $S := \{K | K \leq g K^U\}$, ordered by the generalized information order constitutes a bounded lattice where $\land_g$ and $\lor_g$ define the infimum and supremum.
Proof. The infimum of any two contexts from $S$ exists and is the infimum on the restrictions of the contexts to the set of shared objects, cf. Corollary 3.13 a). There is no conflicting information for any two contexts in $S$. This directly follows from the definition of $S$ where every context contains partial information of the same context $K^U$. Therefore the supremum on the set of shared objects always exists, cf. Corollary 3.13 b), which can be extended to the set of combined objects by using the corresponding valuations of any object-attribute pair where the object only appears in exactly one of the contexts. Further the incomplete context that contains no objects $\emptyset,M,\{\times,?,\}]$ is in $S$ and is the lower bound for all infima of incomplete contexts in $S$. Equally the context $K^U$ is the upper bound for all suprema of incomplete contexts in $S$.

Now we show that for $K_1, K_2 \in S$ where $K_1 = (G_1, M, \{\times,?,\}, I_1)$ and $K_2 = (G_2, M, \{\times,?,\}, I_2)$ that

1) $K_1 \leq_g K_2 \Rightarrow K_1 \land_g K_2 = K_1$ and
2) $K_1 \leq_g K_2 \Rightarrow K_1 \lor_g K_2 = K_2$.

which shows that the infimum and supremum in $S$ with respect to $\leq_g$ coincide with $\land_g$ and $\lor_g$.

1) $\Rightarrow$: Let $K_1 \leq_g K_2$. Then $G_1 \subseteq G_2$ and $K_1 \subseteq K_2|G_1$. Hence $G_1 \cap G_2 = G_1$ and $\forall(g,m) \in G_1 \times M : I_1(g,m) = I_2(g,m)$. Therefore $K_1 \land_g K_2 = K_1$.

2) $\Leftarrow$: Let $K_1 \leq_g K_2$. Then $G_1 \subseteq G_2$ and $G_1 \cup G_2 = G_2$. Further $K_1 \subseteq K_2|G_1$ implies $\forall(g,m) \in G_1 \times M : I_1(g,m) \leq I_2(g,m)$. Therefore $I_1 \cup I_2 = I_2$ and $K_1 \lor_g K_2 = K_2$.

Corollary 4.10. Given an incomplete context $K^U = (G, M, \{\times,?,\}, I)$. Every subset of $\{K \mid K \leq_g K^U\}$ that contains $K^U$ and $\emptyset,M,\{\times,?,\},I_0$ is a bounded lattice with respect to the generalized information order.

Corollary 4.11. The set $S$ of all incomplete contexts that are derived from an incomplete context $K^U$, $S := \{K \mid K \leq K^U\}$, ordered by the information order constitutes a bounded lattice where $\land$ and $\lor$ define the infimum and supremum.

Proof. This follows from $\leq = \leq_g \land \land_g$ and $\lor = \lor_g$ for incomplete contexts on the same object sets together with lemma 4.9.

The following fact and corollary allow us to compare and combine implications and implicational knowledge by making use of lattice structures and the corresponding infimum and supremum operators.

Fact 4.12. Given a formal context $K^U$. The power set on the set of valid implications $\text{Imp}(K^U)$ with the subset inclusion as order relation forms a lattice where intersection and union define infimum and supremum.
Corollary 4.13. Given a formal context $K_U$ let $\mathcal{X} := \{\text{Cons}(X) \mid X \subseteq \text{Imp}(K_U)\}$. Then $(\mathcal{X}, \subseteq)$ is a lattice with infimum $\bigcap \mathcal{X}$ and supremum $\text{Cons}(\bigcup \mathcal{X})$ for all $\mathcal{X} \subseteq \mathcal{X}$.

Proof. By definition $\text{Cons}(\cdot)$ is a closure operator on $\text{Imp}(K_U)$, cf. [7, Prop. 21], and therefore $\mathcal{X} := \{\text{Cons}(X) \mid X \subseteq \text{Imp}(K_U)\}$ is a closure system. Using Prop. 3 from [7] we obtain this fact. \qed

Note that $\mathcal{X}$ is the set of possible implicational knowledge about a universe $K_U$.

4.3 Comparing and Combining Expert Knowledge

The general information order allows us to compare experts in terms of their example knowledge. Together with the set inclusion order on implications known by the experts we can now compare expert knowledge.

Definition 4.14. Given a formal context $K_U = (G, M, I)$, $\mathcal{L} = \text{Imp}(K_U)$ and two experts $E_1$ and $E_2$ on $K_U$ where $E_1 := (K_1, \text{Cons}(\mathcal{L}_1))$, $K_1 \leq g K_U$, $\mathcal{L}_1 \subseteq \mathcal{L}$ and $E_2 := (K_2, \text{Cons}(\mathcal{L}_2))$, $K_2 \leq g K_U$, $\mathcal{L}_2 \subseteq \mathcal{L}$ we say:

a) $E_2$ has at least as much example knowledge as $E_1$ if $K_1 \leq g K_2$,

b) $E_2$ has at least as much implication knowledge as $E_1$ if $\text{Cons}(\mathcal{L}_1) \subseteq \text{Cons}(\mathcal{L}_2)$,

c) $E_2$ knows at least as much as $E_1$ if $E_2$ has at least as much example and implication knowledge as $E_1$. We denote this by $E_1 \leq E_2$.

Further, we can combine the knowledge of experts using the component-wise infimum and supremum on the product lattice of incomplete example contexts and the implication knowledge lattice:

Definition 4.15. Given expert knowledge of two experts $E_1$ and $E_2$ of a domain $K_U$. The maximum combined knowledge, i.e., the supremum of example and implication knowledge, is defined by

$$E_1 \lor E_2 := (K_1 \lor g K_2, \text{Cons}(\mathcal{L}_1 \cup \mathcal{L}_2)).$$

The shared knowledge, i.e., the infimum of example and implication knowledge, is defined by

$$E_1 \land E_2 := (K_1 \land g K_2, \text{Cons}(\mathcal{L}_1 \cap \mathcal{L}_2)).$$

Note that $\text{Cons}(\mathcal{L}_1 \cup \mathcal{L}_2) = \text{Cons}(\text{Cons}(\mathcal{L}_1) \cup \text{Cons}(\mathcal{L}_2))$ by definition of $\text{Cons}(\cdot)$.

Definition 4.16. The maximum combined knowledge of a group of experts $\{E_1, \ldots, E_n\}$ is defined by $\bigvee_{i \in \{1, \ldots, n\}} E_i$.

Remark 4.17. The maximum combined knowledge serves as a reference point for strategies of experts working collaboratively. However there are some limitations. Consider two experts $E_1 = (K_1, \text{Cons}(\mathcal{L}_1))$ and $E_2 = (K_2, \text{Cons}(\mathcal{L}_2))$ for a formal context $K_U$ (cf. Figure 1) with $\text{Imp}(K_U) = \mathcal{L}$ given by:
Hence $L = \{a \rightarrow b\}$ and let $L_1 = L_2 = \emptyset$. Then $K_1 \lor_g K_2 = K^U$ and $E_1 \lor E_2 = (K^U, \emptyset)$. However, unless we know that $K_1 \lor_g K_2$ contains all objects from the domain (or at least one object for every distinct set of attributes that appears in the universe) we cannot use the context of counterexamples to obtain missing valid implications, cf. Remark 4.4: 
If we have that all objects are known we can improve the set of implications by combining it with the set of certainly valid implications from the combined incomplete context:

$$(K_1 \lor_g K_2, \text{Cons}(\text{Imp}(K_1 \lor_g K_2) \cup L_1 \cup L_2))$$

However, this is not true in general. Consider two experts $E_3 = (K_3, \text{Cons}(L_3))$ and $E_4 = (K_4, \text{Cons}(L_4))$ for the formal context $K^U$ given by:

$$
\begin{array}{c|c c}
\text{Aquatics – Swimming} & a & b \\
\text{Badminton} & \times & \\
\text{Gymnastics – Rhythmic} & \times & \\
\end{array}
\quad \begin{array}{c|c c}
\text{Aquatics – Swim.} & a & b \\
\text{Gym. – Rhy.} & ? & \\
\end{array}
\quad \begin{array}{c|c c}
\text{Badminton} & a & b \\
\text{Gym. – Rhy.} & ? & \\
\end{array}
$$

Once more $L = \{a \rightarrow b\}$ and let $L_3 = L_4 = \emptyset$. Then $K_3 \lor_g K_4$ is defined by

$$
\begin{array}{c|c c}
\text{Aquatics – Swimming} & a & b \\
\text{Badminton} & \times & \\
\end{array}
$$

and $\text{Cons}(L_1 \cup L_2) = \emptyset$. Therefore $E_3 \lor E_4 = (K_3 \lor_g K_4, \emptyset)$. If we tried to use the certainly valid implications of the example context we would obtain

$$\text{Cons}(\text{Imp}(K_3 \lor_g K_4) \cup L_3 \cup L_4) = \{\emptyset \rightarrow b, a \rightarrow b\} \not\subseteq L.$$ 

One way to deal with this problem could be to introduce an incomplete context $K_7 = (G, M, \{\times, a, b\}, I_7)$ where $I_7(g, m) = ?$ for all $g \in G$ and $m \in M$ and define

$$(K_1 \lor_g K_2, \text{Cons}(\text{Imp}(K_1 \lor_g K_2 \lor_g K_7) \cup L_1 \cup L_2)).$$

Still, for all practical purposes where we are unable to determine all of the objects belonging to a domain we cannot make use of the examples to expand the set of known implications.

### 4.4 Expert Interaction and Collaboration Strategy

Now that we have formalized expert knowledge we need to consider what it means to interact with an expert, i.e., how an expert can respond to a question.
Definition 4.18 (expert interaction). Given an expert $E = (K_E, \text{Cons}(L_E)) \in E$ from the set of all possible experts $E$ for a universe $K^U$, we view the answer given by the expert as a function

$$EI : \text{Imp}_M \times E \rightarrow \{(\text{true}, \emptyset)\} \cup \{\{\text{false}\} \times \{K|K \text{ an incomplete context over } M\}\}$$

$$\cup \{\{\text{unknown}\} \times P(M)\}$$

where the input $(A \rightarrow B, E) \in \text{Imp}_M \times E$ is considered as asking the expert “Does $A \rightarrow B$ hold in the universe?” and the answer given is consistent with the true knowledge, i.e., example and implication knowledge, of the domain. This means that an expert can only accept an implication if it is actually true, i.e., in $\text{Imp}(K^U)$, and can only reject questions with real examples from the universe.

This definition allows for experts to withhold knowledge. For example, because checking all examples an expert could think of might take too long the expert only checks the first couple of examples that come to mind. For now we employ an expert (analog to the expert used by Holzer, cf. [11]) that answers as fully as possible. The interaction with such an expert is formalized by the standard expert interaction. An expert with standard expert interaction is thought to give answers that contain all that the expert knows.

Definition 4.19 (standard expert interaction). Recall the attribute exploration in Section 3.5. Formally the expert $E = (K_E, \text{Cons}(L_E))$ answers in the following way.

$$EI_S(A \rightarrow B, E) = \begin{cases} (\text{true}, \emptyset) & \text{if } A \rightarrow B \in \text{Cons}(L_E) \\ (\text{false}, K_C) & \text{if } \{g \in G_E|A \subseteq g \uparrow \land (M \setminus g) \cap B \neq \emptyset\} =: C \neq \emptyset, \\
(\text{unknown}, Z) & \text{otherwise, where } Z := B \setminus \langle A \rangle_{L_E} \\
\end{cases}$$

Now, we adapt the idea of a consortium, cf. [8], which is basically a fixed group of experts that have a given way of responding to an implicational question. In our formalization the knowledge of an expert and the ways she can interact with the knowledge are separate.

Here, a group of experts is a set of experts that are all experts for the same universe $K^U$. A group of experts will usually be denoted by $E = \{E_1, \ldots, E_n\}$. For now, we also consider the group of experts that conduct an exploration fixed. Evaluating the implications of dynamically changing expert sets is a topic for future work. Note that every expert in a group of experts could have her own method of interaction with the expert knowledge. At present, we assume that all experts can be interacted with in the same way. Allowing and combining different modes of interaction is a topic for future research.

It easily follows that the knowledge of all experts is compatible:

Lemma 4.20. Let $\{E_1, \ldots, E_n\}$ be a group of experts for the domain $K^U$ where $E_i = (K_{E_i}, \text{Cons}(L_{E_i}))$ for $i \in \{1, \ldots, n\}$. Then the example knowledge of all experts has no conflicting information and all implications known by each expert are satisfiable implications for all experts in the pool.
Proof. That there is no conflicting information is a direct conclusion from Corollary 4.10. Further, for every combination of $i$ and $j$ with $i, j \in \{1, \ldots, n\}$ we have that $Y_{ij} := (K_{E_i}, \text{Cons}(L_{E_j}))$ is an expert for $K^U$. Since $Y_{ij}$ is an expert, using lemma 4.3, it follows that all implications in $\text{Cons}(L_{E_j})$ are satisfiable in the context of examples $K_{E_i}$. Hence, the know implications of each expert are satisfiable in the known example contexts of all experts. 

Now we have to think about how a group of experts for a domain can be used to answer a question posed by the exploration algorithm. In other words, we have to think about how a group of experts can cooperate. To this end we formalize the notion of a collaboration strategy.

Definition 4.21 (collaboration strategy). A collaboration strategy is an algorithm that, given a group of experts $E := \{E_1, \ldots, E_n\}$ for the universe $K^U = (G, M, I)$, an expert interaction strategy $EI$ and a question in form of an implication $A \rightarrow B$ ($A, B \subset M$), returns an answer that is consistent with the universe, i.e., that the algorithm only accepts valid implications and only rejects invalid implications with proper counterexamples. This means that the collaboration strategy can be seen as a function

$$
\varphi: \text{Imp}_M \times \{E\} \times \{EI\} \rightarrow \{(\text{true}, \emptyset)\} \cup \{\text{false}\} \times \{K | K \text{ an incomplete context over } M\} \cup \{\text{unknown}\} \times P(M)
$$

with the properties

1. $\varphi(A \rightarrow B, E, EI) = (\text{true}, \emptyset) \Rightarrow A \rightarrow B \in \text{Imp}(K^U)$
2. $\varphi(A \rightarrow B, E, EI) = (\text{false}, K) \Rightarrow \forall g \in G: A \subseteq g^{-1} \cap (M \setminus g^0) \cap B \neq \emptyset$
   where $K = (G, M, \{\times, o, ?\}, I) \leq_g K^U$
3. $\varphi(A \rightarrow B, E, EI) = (\text{unknown}, Z) \Rightarrow A \rightarrow (B \setminus Z) \in \text{Imp}(K^U)$ and $Z \subseteq B$.

A collaboration strategy takes the role of an intermediary between the attribute exploration algorithm and the group of experts. It takes the questions posed by the exploration and interacts with the experts to find an answer which it then reports back to the exploration algorithm, cf. Figure 5.

4.5 Discussion of Collaboration Strategies

We first present different collaboration strategies. To emphasize that our approach for a group of experts is a generalization and contains the classical attribute exploration, we begin with a strategy for the exploration with a single expert.

Strategy 4.22 (single expert). Given we only have one expert $E$ and the interaction strategy $EI$. The canonical strategy is to relay the questions to the expert and the answers back to the attribute exploration without any modifications. If we use the standard interaction strategy $EI_S$ then Fact 3.28 guarantees that this results in the maximal information obtainable with respect to the expert’s knowledge. The single expert strategy $\varphi_{\text{single}}$ is defined by:

$$
\varphi_{\text{single}}(A \rightarrow B, \{E\}, EI) := EI(A \rightarrow B, E).
$$
Clearly this strategy gives answers consistent with the domain.

We continue with two extreme cases of collaboration strategies, namely the ignorant strategy and the maximum knowledge strategy. The former does not bother interacting with any of the experts and instead always responds with ‘unknown’ and the latter asks the experts for everything they know before giving the best possible answer.

Strategy 4.23 (ignorant). Given a group of experts $\mathcal{E} = \{E_1, \ldots, E_n\}$ for a universe $\mathbb{K}^U$ and an interaction strategy $EI$. The ignorant strategy $\varphi_{\text{ignorant}}$ is defined by:

$$
\varphi_{\text{ignorant}}(A \rightarrow B, \mathcal{E}, EI) := (\text{unknown}, B).
$$

This strategy can be interpreted as modelling experts that are unwilling to participate in the exploration. Then the only valid option to give answers consistent with the universe is to use a strategy that always answers ‘unknown’. Note that this strategy also works if the set of experts $\mathcal{E}$ is empty.

The maximum knowledge strategy is an easy strategy that allow experts to obtain maximum knowledge about a domain.

Strategy 4.24 (maximum knowledge). Given a group of experts $\mathcal{E} := \{E_1, \ldots, E_n\}$, $n \geq 1$, for a universe $\mathbb{K}^U$. We first ask all experts for all of their example knowledge and all of their implicational knowledge. Then we define a new artificial expert that has the combined knowledge of all experts.

$$
E_{\text{max}} := \left( \bigvee_{i=1}^{n} \mathbb{K}_{E_i}, \text{Cons} \left( \bigcup_{i=1}^{n} \mathbb{L}_{E_i} \right) \right) = \bigvee_{i=1}^{n} E_i
$$

The maximum knowledge strategy $\varphi_{\text{max}}$ can now be described as relaying the question to the artificial maximal expert using the standard expert interaction.

$$
\varphi_{\text{max}}(A \rightarrow B, \mathcal{E}, EI) := \varphi_{\text{single}}(A \rightarrow B, \{E_{\text{max}}\}, EI_S) = EI_S(A \rightarrow B, E_{\text{max}}).
$$

This strategy can be interpreted as modelling the experts sitting together and discussing every question posed by the exploration algorithm until they find the best answer they can provide together.

Theorem 4.25. Given a group of experts $\mathcal{E} := \{E_1, \ldots, E_n\}$, $n \geq 1$, for a universe $\mathbb{K}^U$, the maximal expert $E_{\text{max}} = \bigvee_{i=1}^{n} E_i$ has the maximal knowledge of the group of experts. Attribute exploration with the strategy $\varphi_{\text{max}}$ results in the maximal obtainable knowledge from the experts $\mathcal{E}$.

Proof. The maximal expert represents the maximal knowledge of the group of experts by definition as their supremum in the set of possible experts for the universe $\{E | E \text{ expert for } \mathbb{K}^U\}$. The attribute exploration with a single expert results in the maximum of obtainable knowledge from the expert, cf. Fact 3.28. Therefore $\varphi_{\text{max}}$ results in the maximum of obtainable knowledge from the group of experts $\{E_1, \ldots, E_n\}$. \hfill $\square$
Even though Strategy 4.24 is a valid collaboration strategy it does not represent a realistic way of collaboration if the group of experts becomes large. Further, if we ignore the interpretation of this strategy as sitting together and discussing the questions, we realize that it is highly unlikely that an expert with reasonably large knowledge of any domain can reproduce all facts about it at once. Nonetheless, this collaboration strategy serves as a means to evaluate other collaboration strategies as the maximal expert represents the maximum of obtainable knowledge from the group of experts, cf. Theorem 4.25.

There are of course some more realistic collaboration strategies. In the following we present the broadcast strategy (Strategy 4.26), the iterative strategy (Strategy 4.27) and the random selection strategy (Strategy 4.29), each of which models one way of asking a group of experts. The idea of the broadcast strategy is to ask all experts in a group and combine the answers to form the best collaborative answer possible. This requires the strategy to interact with every expert for every question but the experts can be asked independently from each other and thus this can be done simultaneously. The time required to answer a question using this strategy is the longest time it takes for any of the experts to answer.

**Strategy 4.26** (broadcast). Let \( \{E_1, \ldots, E_n\} \) be a group of experts for the universe \( U \) with the expert interaction \( EI \) for all experts. Given a question ‘Does \( A \rightarrow B \) hold in the universe?’ the broadcast strategy asks all experts at once and collects all the answers, i.e., the results of all interactions \( EI(A \rightarrow B, E_i) \). Then it combines the answers to form the best possible response, i.e., the broadcast strategy combines the attributes known to follow from the premise and it combines all counterexamples.

A more sensible approach in terms of the number of required expert interactions is to order the experts randomly for every question and relay the question to one expert at a time until either one expert accepted or rejected the attribute implication or all experts have been consulted, cf. Strategy 4.27. Basically this strategy increases the average time required to answer a question in order to reduce the average number of expert interactions required. That way the amount of work required from every expert is reduced and we presume that the result in terms of how much knowledge is obtained stays about the same. Note that the amount of example knowledge might be reduced since in general not all known counterexamples to an implication will be found if not all experts are consulted. Further note that the implication knowledge should not be impacted by the iterative approach, since no implication knowledge of not consulted experts is lost if an implication is accepted. If we know about what the experts know it seems reasonable to order the experts in an optimal way for every question, again this is a subject for future research.

**Strategy 4.27** (iterative). Let \( \{E_1, \ldots, E_n\} \) be a group of experts for the universe \( U \) with the standard expert interaction \( EI_S \) for all experts. In Algorithm 1 we present the iterative strategy \( \varphi_{iter} \) that asks questions iteratively to reduce the number of interactions needed per question when exploring the domain.

The answers produced by \( \varphi_{iter} \) are consistent with the domain: An implication is only rejected if an expert knows a counterexample, it is accepted if either an
**Algorithm 1:** Collaboration Strategy $\phi_{\text{iter}}$ using iterative questions

**Input:** $A \rightarrow B, \{E_1, ..., E_n\}$

**Output:** Collaborative Answer

1. $Y := \emptyset$ /* collect the set of attributes known to follow from $A$ */
2. for $E$ in $\{E_1, ..., E_n\}$ do /* iterate the group of experts */
3.     $R := EI_S(A \rightarrow B, E)$ /* ask the expert and collect the response */
4.     if $R = (\text{true,} \emptyset)$ then return $(\text{true,} \emptyset)$
5.     if $R = (\text{false,} K_C)$ then return $(\text{false,} K_C)$
6.     if $R = (\text{unknown,} Z)$ then $Y := Y \cup (B \setminus Z)$
7. end
8. if $Y = B$ then /* if it is known conjointly that $B$ follows from $A$ */
9.     return $(\text{true,} \emptyset)$
10. end
11. return $(\text{unknown,} B \setminus Y)$

expert accepted it or the group of experts knows that the conclusion follows from the premise. Otherwise the response is ‘unknown’.

**Remark 4.28.** An extensive example of a collaborative exploration using the iterative strategy can be found in the appendix (see Example 6.1). Subject of the exploration is a subset of the attributes of the Disciplines of the Summer Olympic Games 2020 context from the introduction (see Section 1). The complete context can also be found in the appendix.

Another simple approach to adapt the attribute exploration to a group of experts is to randomly select a new expert to ask each time the exploration algorithm poses a question. The random selection strategy describes this form of collaboration where every expert is equally likely to be selected. This strategy is useful to balance the amount of interaction required from each expert. It further reduces the average time needed to answer a question compared to the broadcast strategy because here the expert that takes the longest to answer is not always asked. However, in general the random selection strategy is strictly worse than the broadcast in terms of obtained knowledge. An exception is the case where all experts have the same knowledge. Then the random selection strategy results in the same amount of obtained knowledge but, compared to the broadcast strategy, with fewer expert interactions and after a shorter period of time.

**Strategy 4.29 (random selection).** Given a group of experts $E := \{E_1, ..., E_n\}, n \geq 1$, for a universe $K^U$ and an interaction strategy $EI$. Consider the random selection strategy $\phi_{\text{rand}}$ that randomly asks one of the experts and simply gives the answer as response

$\phi_{\text{rand}}(A \rightarrow B, E, EI) := \phi_{\text{single}}(A \rightarrow B, \{E_i\}, EI)$ where $E_i \sim \text{Uniform}(\{E_1, ..., E_n\})$.

$\text{Uniform}(\{E_1, ..., E_n\})$ denotes randomly picking an expert with equal probability.

Note that instead of the Uniform distribution we could also employ other discrete probability distributions to assign weights to the experts.
Interactive Collaborative Exploration using Incomplete Contexts

Figure 6: An example of a context where implicational knowledge can be adversely distributed between experts.

Remark 4.30. The standard expert interaction exhibits an interaction complexity issue when multiple experts together know that an implication is valid but none of the experts alone knows this. Figure 6 gives an example of a context where two experts can have adversely distributed implication knowledge. When using the standard expert interaction it can happen that many interactions are required to confirm a valid implication.

The set of valid implications in $K$ (cf. Figure 6) is $\text{Imp}(K) = \{a \rightarrow bc, b \rightarrow c\}$. Consider two experts $\{E_1, E_2\} =: \mathcal{E}$ with implicational knowledge $L_1 = \{a \rightarrow b\}$ and $L_2 = \{b \rightarrow c\}$ and the standard expert interaction $EI_S$. If the question ‘Does $a$ imply $c$?’ is posed none of the experts could agree and even if they were allowed to work together using the standard expert interaction none of them could report anything since $EI_S$ only allows to report parts of the conclusion that are known to follow.

In comparison if the question ‘Does $a$ imply $bc$?’ is posed to both experts, at least expert $E_1$ would report that $b$ is known to follow from the premise. To also obtain that $c$ follows, the expert $E_2$ needs to be consulted a second time about the question whether the implication $ab \rightarrow c$ is true. Then $E_2$ would accept this implication to be valid and therefore confirm that the implication $a \rightarrow c$ is valid.

Clearly this poses a problem for defining collaboration strategies. The example shows that there are cases when an implication can be accepted by a group of experts $\mathcal{E}$ even though no single expert alone can accept it but that this requires potentially repeated questioning of the experts for every single question posed by the attribute exploration. The first part of the example shows that there exist implications that cannot be accepted using the standard expert interaction even though they would be accepted by the maximal expert $E_1 \lor E_2$. Furthermore, it usually requires every expert to be consulted if nothing about the experts knowledge is known.

A quick generalization of the previous example shows that there are cases where the number of required interactions with the experts $\mathcal{E}$ is at least $|\mathcal{E}| \cdot (|M| - 1)$:

Let $K$ be a context with $M = \{a_0, \ldots, a_n\}$ where the implications $a_0 \rightarrow a_1 \ldots a_n$ and $a_i \rightarrow a_{i+1}$ with $i \in \{0, \ldots, n-1\}$ are valid. Clearly there exist contexts where these implications are valid, for example, the empty context on the attribute set $M$ where every implication is valid or a sufficiently large ordinal scale, cf. [7]. Consider a group of experts $\mathcal{E} = \{E_1, \ldots, E_m\}$, $m \geq n$, for $K$ such that the expert $E_i$ knows the implication $a_{i-1} \rightarrow a_i$ for $1 \leq i \leq n$ and has no implicational knowledge for $n < i \leq m$.

To answer the question ‘Does $a_0$ imply $a_1 \ldots a_n$?’ every expert needs to be consulted $n$ times. The first iteration results in $a_0 \rightarrow a_1$ which is turned into the question ‘Does $a_0 a_1$ imply $a_1 \ldots a_n$?’ for the second iteration and so on. After $n$ iterations
it is finally known that the implication $a_0 \Rightarrow a_1 \ldots a_n$ is valid. In each iteration, all $m$ experts need to be consulted to assure that no information was missed. Hence the number of interactions needed to accept an implication can reach $|\mathcal{E}| \cdot (|M| - 1)$. Even in the best case when asking iteratively and always the right expert first it would require $n - 1$ interactions to accept the implication.

Note that this is not only a problem due to the definition of the standard expert interaction but a fundamental problem that arises when exploring distributed implication knowledge. From a theoretical point we could simply redefine the notion of expert interaction and let experts respond with all the implicational knowledge they have. However, from a real world perspective (and sticking to our Olympic Disciplines example from Section 1) it would be like asking “Do all disciplines that have female-only events also have male-only events?” and to receive an answer like “No, but if a discipline holds more than ten events it was part of at least eight Olympic Games” on the off-chance that those facts are somehow related. Clearly such an approach does not make much sense.

**Corollary 4.31.** Given a universe $\mathbb{K}^U = (G, M, I)$ with $|M| = n + 1$. Let $\{E_1, \ldots, E_m\}$ be a group of experts for $\mathbb{K}^U$ and the interaction with the experts is the standard expert interaction $EI_S$. The worst case number of interactions required to accept a single valid implication is at least $m \times n$.

**Proof.** This directly follows from Remark 4.30.

One possible solution to mitigate this problem is to allow the collaboration strategies to report with more than just ‘yes’, ‘no’ or ‘unknown’ back to the attribute exploration. Then the collaboration strategy could collect the known implications found during the interactions with the experts and add the implications to the reply. This might reduce repetitive interactions by preventing some questions where the answer can then be inferred directly by the exploration algorithm. However, this would require modifying the attribute exploration to allow using such additional information.

### 4.6 Comparing and Evaluating Collaboration Strategies

One important topic that we have touched before but not explicitly discussed yet is the question of how to evaluate and compare collaboration strategies. We have already mentioned three criteria that seem important: The portion of knowledge that is obtained, the (average) time needed and the experts effort required to obtain it.

First of all, note that the result of a collaborative exploration is very much dependent on the group of experts. Therefore, the maximal expert and the maximal collaboration strategy are useful tools to evaluate the portion of obtained knowledge relative to the possible maximum.

The result of an attribute exploration consists of the set of known valid implications and the incomplete context of counterexamples. Hence, the result of an exploration is an element of the product lattice of implications and examples and can be compared in the same way as expert knowledge. However, this also means that different exploration results may be incomparable in this order relation. A path to circumvent this problem could be to define some metric on the elements of
the product lattice to capture the relative knowledge in a single number and make it easily comparable. Though, how to define such a metric is not obvious to the authors.

Another difficulty inherent in the definition of collaboration strategy as algorithm is that random algorithms can be used, which implies that the answer to a question is not guaranteed to be deterministic.

An example for such a strategy is Strategy 4.29. It can be used to show that strategies which incorporate random elements are even more difficult to evaluate: Example 4.32. Assume that we want to explore a universe \( \mathbb{K}^U \) with two experts for the domain \( E_1 = (K_2, \text{Cons}(L_1)) \) and \( E_2 = (K_2, \text{Cons}(L_2)) \) utilizing the collaboration strategy \( \varphi_{\text{rand}} \), cf. Strategy 4.29. Let the expert \( E_1 \) be all-knowing, i.e., \( K_1 = \mathbb{K}^U \) and \( L_1 = \text{Imp}(\mathbb{K}^U) \), and let the expert \( E_2 \) be completely ignorant, i.e., \( K_2 = (\emptyset, M, \{'\times', 'o', '?\}', I_\emptyset) \) and \( L_2 = \emptyset \).

If we explore the universe using the random selection strategy then the exploration result can vary extremely depending on the randomness. It could be that no knowledge at all was discovered if the strategy always selected the expert \( E_2 \), it could be a completely explored domain if the strategy always selected the expert \( E_1 \) or it could be anything in between.

Another important consideration when evaluating a collaboration strategy is how much effort by the group of experts is needed. This can be measured, for example, by counting the required interactions per expert and comparing based on total, average, maximum or other suitable metrics. Note that minimizing expert effort alone is problematic. An optimal strategy in this regard is the ignorant strategy, cf. Strategy 4.23, which does not bother interacting with the experts, always answers ‘unknown’ and clearly is not what we want.

Now we give a brief ranking of the three more realistic collaboration strategies from Section 4.5 (broadcast, iterative and random selection). We compare them based on knowledge obtained (K), time needed (T) and number of expert interactions (I):

K: We presume that the broadcast and the iterative strategy result in about the same obtained knowledge (cf. Section 4.5). However, the broadcast strategy will result in a more extensive set of counterexamples. The random selection strategy is difficult to compare to the two other strategies because of its randomness (cf. Example 4.32), though we expect this strategy to obtain far less knowledge in general.

T: The random selection strategy takes the least time, followed by the broadcast strategy and the worst in terms of time needed is the iterative strategy due to its sequential expert interactions.

I: The random selection strategy requires the least number of expert interactions (one per question), the broadcast strategy requires the most number of interactions (one per expert per question) and the iterative strategy requires a number of interactions in between depending on the order in which the experts are asked and how knowledgable the experts are.

It has become apparent that comparing collaboration strategies is a complex task where further research is required. Defining what characterizes a ‘good’ collaboration strategy is hard. As a rule of thumb it balances the knowledge obtained, the time needed and the effort required from the experts.
5 Conclusions and Outlook

We have extended the theory of attribute exploration for incomplete knowledge to work in a setting of multiple experts with incomplete knowledge of a domain. To this end we have formalized expert knowledge as a tuple of (possibly incomplete) examples and valid implications and formalized a notion of interaction with expert knowledge. Further we have defined a collaboration strategy as an algorithm that takes an implicational question and a group of experts as input and returns an answer that fits the scheme required by the attribute exploration for incomplete knowledge in [11]. Orders on incomplete contexts and expert knowledge have been introduced to facilitate comparability of the results of attribute explorations by multiple experts. Some collaboration strategies and ways to compare such strategies in general have been discussed. Numerous questions and avenues for further research have been identified.

In particular, we will develop further characterizations of ‘good’ collaboration strategies. One problem that should be tackled in the future is to find a metric which allows for an easy comparison of the knowledge discovered as result of attribute exploration with some collaboration strategy. Basically this implies defining a metric on the lattice of possible exploration results. Other avenues for future research could deal with: considering changing sets of experts, different modes of interaction, more specifically defined experts (such as the experts in [8]) or assuming knowledge about the knowledge of experts to reduce the amount of interactions required.

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## 6 Appendix

### 6.1 Disciplines of the Summer Olympic Games 2020 Context

| Discipline                               | ≤ 5 events | ≤ 10 events | ≤ 20 events | Ball Game | Combat Sport | Female Only Events | Has Paralympic Equivalent | Indoor Event | Mixed Event | Open Event | Part of ≥ 8 Olympics | Part of ≥ 16 Olympics | Part of ≥ 24 Olympics | Team Competition | Water Sport |
|------------------------------------------|------------|-------------|-------------|-----------|--------------|-------------------|------------------------|--------------|-------------|-----------|--------------------|--------------------|--------------------|-------------------|------------|
| Aquatics – Artistic Swimming             |            |             |             |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Aquatics – Diving                        | ☑          | ☑           | ☑           |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Aquatics – Marathon Swimming             |            |             |             |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Aquatics – Swimming                      | ☑ ☑ ☑      |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Aquatics – Water Polo                    | ☑ ☑ ☑ ☑    |             |             | ☑ ☑ ☑ ☑    |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Archery                                  | ☑           |             |             |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Athletics                                | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑        |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Badminton                               | ☑           |             |             |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Baseball/Softball                        | ☑ ☑ ☑ ☑    |             |             | ☑ ☑ ☑ ☑    |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Basketball – 3x3                         | ☑ ☑ ☑ ☑    |             |             | ☑ ☑ ☑      |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Basketball – Basketball                  | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Boxing                                   | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Canoe – Slalom                           | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Canoe – Sprint                           | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Cycling – BMX Freestyle                  | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Cycling – BMX Racing                      | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Cycling – Mountain Bike                   | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Cycling – Road                           | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Cycling – Track                          | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Dressage                                 | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Eventing                                 | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Jumping                                  | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Fencing                                  | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Football                                 | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Golf                                     | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Gymnastics – Artistic                    | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Gymnastics – Rhythmic                    | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Gymnastics – Trampoline                  | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Handball                                 | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
| Hockey                                   | ☑ ☑ ☑ ☑    |             | ☑ ☑ ☑ ☑ ☑   |           |              |                   |                        |              |             |           |                   |                    |                    |                   |            |
The information for the Disciplines of the Summer Olympic Games 2020 context was obtained from https://tokyo2020.org/, https://www.olympic.org/tokyo-2020 and https://en.wikipedia.org/wiki/Olympic_sports.

Note that the number of events (≥ 5, ≥ 10 and ≥ 20) and the number of Olympics that a discipline was of (≥ 8, ≥ 16 and ≥ 24) are ordinally scaled attributes, cf. [7].

### 6.2 Example of a collaborative exploration with three experts

**Example 6.1.** In the following we give an example of attribute exploration with multiple experts using the iterative collaboration strategy (Strategy 4.27). We explore a subset of the attributes of the Olympic Disciplines 2020, cf. Section 6.1, namely the attributes ≥ 5 events, ≥ 10 events, female only events, male only events and part of ≥ 8 olympics. The collaboration strategy makes use of three (fictitious) experts.
The first expert $E_1 = (K_1, \text{Cons}(L_1))$ prefers Olympic Disciplines with a long tradition in the Olympic Games. She knows that all disciplines with more than ten events also have more than five events and are part of at least eight olympics, i.e., $L_1 = \{ \{ \geq 10 \text{ events} \} \rightarrow \{ \geq 5 \text{ events, part of } \geq 8 \text{ olympics} \} \}$.

The second expert $E_2 = (K_2, \text{Cons}(L_2))$ is a fan of water sport and likes to watch mixed events. She knows that all disciplines with more than five events also have male only events, i.e., $L_2 = \{ \{ \geq 5 \text{ events} \} \rightarrow \{ \text{male only events} \} \}$.

The third expert $E_3 = (K_3, \text{Cons}(L_3))$ likes combat sports. She only knows that all disciplines with more than ten events also have more than five events, i.e., $L_3 = \{ \{ \geq 10 \text{ events} \} \rightarrow \{ \geq 5 \text{ events} \} \}$.

The following contexts represent the example knowledge of the three experts:

| Expert 1 | Expert 2 |
|----------|----------|
| $K_1$    | $K_2$    |
| Aq. – Diving | × | × | × | × |
| Aq. – Swimming | × | × | × | × |
| Aq. – Water Polo | × | × | × | × |
| Athletics | × | × | × | × |
| Boxing | × | × | × | × |
| Cycling – Road | × | × | × | × |
| Cycling – Track | × | × | × | × |
| Equestrian – Dressage | × | × |
| Equestrian – Eventing | × | × |
| Equestrian – Jumping | × |
| Fencing | × | × | × | × |
| Football | × | × | × | × |
| Gymnastics – Artistic | × | × | × | × |
| Hockey | × | × | × | × |
| Modern Pentathlon | × | × | × | × |
| Rowing | × | × | × | × |
| Sailing | × | × | × | × |
| Shooting | × | × | × | × |
| Weightlifting | × | × | × | × |
| Wrestling – Freestyle | × | × | × | × |
| Wrestling – Greco Roman | × | × | × | × |

$L_1 = \{ \{ \geq 10 \text{ events} \} \rightarrow \{ \geq 5 \text{ events, part of } \geq 8 \text{ olympics} \} \}$

$L_2 = \{ \{ \geq 5 \text{ events} \} \rightarrow \{ \text{male only events} \} \}$
Expert 3

\[ \mathcal{K}_3 \]

| Disciplines          | 5 events | 10 events | female only events | male only events | part of ≥ 8 olympics |
|----------------------|----------|-----------|--------------------|------------------|---------------------|
| Archery              | ×        | ×         | ×                  | ×                | ×                   |
| Boxing               | × × × ×   | ×         | ×                  | ×                | ×                   |
| Fencing              | × × × ×   | ×         | ×                  | ×                | ×                   |
| Judo                 | × × × ×   | ×         | ×                  | ×                | ×                   |
| Karate – Kata        | ×         | ×         | ×                  | ×                | ×                   |
| Karate – Kumite      | ×         | ×         | ×                  | ×                | ×                   |
| Modern Pentathlon    | × × ×     | ×         | ×                  | ×                | ×                   |
| Shooting             | × × × ×   | ×         | ×                  | ×                | ×                   |
| Taekwondo            | ×         | ×         | ×                  | ×                | ×                   |
| Wrestling – Freestyle| × × × ×   | ×         | ×                  | ×                | ×                   |
| Wrestling – Greco Roman | ×   | ×         | ×                  | ×                | ×                   |

\[ \mathcal{L}_3 = \{ \geq 10 \text{ events} \rightarrow \geq 5 \text{ events} \} \]

For the purpose of this example the order in which the experts are asked is always the same: First \( E_1 \), then \( E_2 \) and last \( E_3 \). We list the questions posed by the attribute exploration algorithm, all interactions with the experts and the relevant parts taking place in the collaboration strategy.

**Question 1** posed by the attribute exploration: “Do all Disciplines have more than five events, more than ten events, female only events, male only events and have been part of at least eight Olympic Games?”.

The corresponding short implicational form of this question is:

\[ \emptyset \rightarrow \{ \geq 5 \text{ events}, \geq 10 \text{ events, female only events, male only events, part of } \geq 8 \text{ olympics} \} \]

From now on we use the short form of the questions to improve readability.

The *collaboration strategy* then poses **Question 1** to the Experts:

**Interaction with** \( E_1 \):
The expert knows this to be false and responds with (false ,\( \mathcal{K}_{Q1} \)).

The *collaboration strategy* returns the context of counterexamples provided by \( E_1 \) to the attribute exploration.
Counterexample Question 1

| $\mathcal{K}_{Q1}$ | $\geq$ 5 events | $\geq$ 10 events | female only events | male only events | part of $\geq$ 8 olympics |
|---------------------|-----------------|------------------|--------------------|------------------|--------------------------|
| Aq. – Diving        | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Aq. – Water Polo    | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Cycling – Road      | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Equestrian – Dressage | $\times$     |                  |                    |                  |                          |
| Equestrian – Eventing | $\times$    |                  |                    |                  |                          |
| Equestrian – Jumping | $\times$      |                  |                    |                  |                          |
| Football            | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Hockey              | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Modern Pentathlon   | $\times$        | $\times$         | $\times$           | $\times$         |                          |
| Wrestling – Greco Roman | $\times$        | $\times$         | $\times$           | $\times$         |                          |

Counterexample Question 2

| $\mathcal{K}_{Q2}$ | $\geq$ 5 events | $\geq$ 10 events | female only events | male only events | part of $\geq$ 8 olympics |
|---------------------|-----------------|------------------|--------------------|------------------|--------------------------|
| Aq. – Marathon Swimming | $\times$ | $\times$ |         |                  |                          |
| Surfing             | $\times$        | $\times$         |                    |                  |                          |
| Triathlon           | $\times$        | $\times$         |                    |                  |                          |

Question 2: $\emptyset \rightarrow \{$part of $\geq$ 8 olympics$\}$

**Interaction with $E_1$:**
This is unknown to $E_1$ and she responds with $\{$unknown$,$ part of $\geq$ 8 olympics$\}$.

**Interaction with $E_2$:**
The expert knows this to be false and responds with $\{$false$,$ $K_{Q2}$$\}$.

Question 3: $\{$$\geq$ 10 events$\}$ $\rightarrow$ $\{$$\geq$ 5 events, female only events, male only events, part of $\geq$ 8 olympics$\}$

**Interaction with $E_1$:**
This is unknown to $E_1$ and she responds with $\{$unknown$,$ \{$female only events$,$ male only events$\}$$\}$.

**Interaction with $E_2$:**
This is unknown to $E_2$ and she responds with $\{$unknown$,$ \{$female only events $\geq$ 5 events$,$ male only events$,$ part of $\geq$ 8 olympics$\}$$\}$.

**Interaction with $E_3$:**
This is unknown to $E_3$ and she responds with $\{$unknown$,$ \{$female only events$,$ male only events$,$ part of $\geq$ 8 olympics$\}$$\}$.

Here the iterative strategy collected the set of attributes known to follow and replies with $\{$unknown$,$ \{$female only events$,$ male only events$\}$$\}$. The attribute exploration introduces two fictitious counterexamples, one for each of the unknown attributes.

Question 4: $\{$$\geq$ 10 events$\}$ $\rightarrow$ $\{$$\geq$ 5 events, part of $\geq$ 8 olympics$\}$

**Interaction with $E_1$:** The expert knows this to be true and responds with $\{$true$,$ $\emptyset$$\}$.
Counterexample Question 5

| KQ5 | ≥ 5 events | male only events | part of ≥ 8 olympics |
|-----|------------|------------------|---------------------|
| Karate – Kumite | × | × | × |
| Taekwondo | × | × | × |

Counterexample Question 8

| KQ8 | ≥ 5 events | male only events | part of ≥ 8 olympics |
|-----|------------|------------------|---------------------|
| Aq. – Artistic Swimming | × | × | × |

Question 5: \(\{\geq 5 \text{ events}\} \rightarrow \{\text{male only events, part of } \geq 8 \text{ olympics}\}\) ?

**Interaction with E₁:**
This is unknown to E₁ and she responds with \((\text{unknown}, \{\text{male only events, part of } \geq 8 \text{ olympics}\})\).

**Interaction with E₂:**
This is unknown to E₂ and she responds with \((\text{unknown}, \{\text{part of } \geq 8 \text{ olympics}\})\).

**Interaction with E₃:**
The expert knows this to be false and responds with \((\text{false}, KQ₅)\).

Question 6: \(\{\geq 5 \text{ events}\} \rightarrow \{\text{male only events}\}\) ?

**Interaction with E₁:**
This is unknown to E₁ and she responds with \((\text{unknown}, \{\text{male only events}\})\).

**Interaction with E₂:**
The expert knows this to be true and responds with \((\text{true}, \emptyset)\).

Question 7: \(\{\geq 5 \text{ events, } \geq 10 \text{ events}, \text{male only events, part of } \geq 8 \text{ olympics}\} \rightarrow \{\text{female only events}\}\) ?

**Interaction with E₁:**
This is unknown to E₁ and she responds with \((\text{unknown}, \{\text{female only events}\})\).

**Interaction with E₂:**
This is unknown to E₂ and she responds with \((\text{unknown}, \{\text{female only events}\})\).

**Interaction with E₃:**
This is unknown to E₃ and she responds with \((\text{unknown}, \{\text{female only events}\})\).

Again the iterative strategy collected the set of attributes known to follow and replies with \((\text{unknown}, \{\text{female only events}\})\). The exploration algorithm introduces a fictitious counterexample for the attribute.

Question 8: \(\{\text{female only events}\} \rightarrow \{\text{male only events}\}\) ?

**Interaction with E₁:**
This is unknown to E₁ and she responds with \((\text{unknown}, \{\text{male only events}\})\).

**Interaction with E₂:**
The expert knows this to be false and responds with \((\text{false}, KQ₈)\).
Result of the collaborative attribute exploration:

The result of the exploration is the set of accepted implications $\mathcal{L}_{\text{result}}$ and the incomplete context of counterexamples $\mathcal{K}_{\text{result}}$ which contains the set of fictitious counterexamples

$$G^* = \{g_i \geq 5 \text{ events, male only events, part of } \geq 8 \text{ olympics, } \geq 10 \text{ events} \rightarrow \{\text{female only events}\},$$

$$g_i \geq 10 \text{ events} \rightarrow \{\text{female only events}\},$$

$$g_i \geq 10 \text{ events} \rightarrow \{\text{male only events}\}\}. $$

**Result of the Attribute Exploration**

| $\mathcal{K}_{\text{result}}$ | 5 events | 10 events | female only events | male only events | part of ≥ 8 olympics |
|-------------------------------|----------|-----------|--------------------|-----------------|---------------------|
| Aquatics – Artistic Swimming  | x        | x         | x                  | x               | x                   |
| Aquatics – Diving             |          | x         | x                  | x               | x                   |
| Aquatics – Marathon Swimming  |          | x         | x                  | x               | x                   |
| Aquatics – Water Polo         |          | x         | x                  | x               | x                   |
| Cycling – Road                |          | x         | x                  | x               | x                   |
| Equestrian – Dressage         |          |          |                    |                 | x                   |
| Equestrian – Eventing         |          |          |                    |                 | x                   |
| Equestrian – Jumping          |          |          |                    |                 | x                   |
| Football                      |          | x         | x                  | x               | x                   |
| Hockey                        |          | x         | x                  | x               | x                   |
| Karate – Kumite               | x        | x         | x                  | x               | x                   |
| Modern Pentathlon             |          | x         | x                  | x               | x                   |
| Surfing                       |          | x         | x                  | x               | x                   |
| Taekwondo                     | x        | x         | x                  | x               | x                   |
| Triathlon                     |          | x         | x                  | x               | x                   |
| Wrestling – Greco Roman       | x        | x         | x                  | x               | x                   |

We can see that each of the experts provided at least one example that none of the other experts could have provided to the context of counterexamples $\mathcal{K}_{\text{result}}$, namely *Aquatics – Artistic Swimming, Cycling – Road and Taekwondo*. Further the set $\mathcal{L}_{\text{result}}$ of accepted implications is larger than any of the individuals experts known implication sets.