APPENDIX TO DISCRETE LOCALITIES II:

\textit{p-local compact groups as localities}

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Our aim in this appendix is to show that the \textit{p-local} compact groups, introduced in [BLO2] and further developed in [BLO3], [JLL] and elsewhere, may be viewed as proper localities of a certain kind - to be called compact localities. As a corollary to a result of Ran Levi and Assaf Libman [LL] (and with an improvement due to Remi Molinier [M]), we will show that if \( \mathcal{F} \) is the fusion system of a compact locality \((\mathcal{L}, \Delta, S)\) then, up to an isomorphism of partial groups which restricts to an automorphism of \( S \), \( \mathcal{L} \) is the unique compact locality on \( \mathcal{F} \) having \( \Delta \) as its set of objects. This result, and the formalism of compact localities, provide a bridge between the theory being developed in this series and a more well-established theory having a much more homotopy-theoretic flavor.

We shall be closely following the arguments in the appendix to [Ch1], where an equivalence was established between the “\textit{p-local finite groups}” introduced in [BLO1], and proper, finite, centric localities.

Recall from the Appendix A in Part I (or from any of the above references) that a \textit{p-torus} \( T \) is by definition the direct product of a finite number of copies of the Prüfer group \( \mathbb{Z}/(p^{\infty}) \). A group \( G \) is \textit{virtually p-toral} if there exists a \textit{p-torus} of finite index in \( G \). A virtually \textit{p-toral} \textit{p-group} is a \textit{discrete p-toral group}.

\textbf{Definition A.1.} The discrete locality \((\mathcal{L}, \Delta, S)\) on \( \mathcal{F} \) is \textit{compact} if:

1. \( \mathcal{L} \) is proper, and
2. \( N_{\mathcal{L}}(P) \) is virtually \textit{p-toral} for each subgroup \( P \in \Delta \).

We will show via A.7, A.22, and A.23 below that there is an equivalence between the notions of compact locality and \textit{p-local} compact group. We then obtain an existence/uniqueness theorem for compact localities, as a corollary to [LL].

The set of elements \( x \) in a \textit{p-torus} \( T \) such that \( x^p = 1 \) is an elementary abelian \textit{p-group} of finite order \( p^k \) where \( k \) is equal to the number of factors in any decomposition of \( T \) as a direct product of Prüfer groups. We refer to \( k \) as the \textit{rank} of \( T \), and write \( rk(T) = k \). The identity group is the \textit{p-torus} of rank 0.
Lemma A.2. Let $G$ be a virtually $p$-toral group and let $L$ be a discrete $p$-toral group.

(a) There is a unique $p$-torus $T$ such that $T$ has finite index in $G$; and then $T$ contains every $p$-toral subgroup of $G$.
(b) All subgroups and homomorphic images of $G$ are virtually $p$-toral, and all subgroups and homomorphic images of $P$ are discrete $p$-toral.
(c) If $X$ and $Y$ are subgroups of $P$ with $X < Y$ ($X$ is a proper subgroup of $Y$) then $X < N_{Y}(X)$.

Proof. Let $X$ be a $p$-torus contained in $G$. Then $T \cap X$ has finite index in $X$, and then $T \cap X = X$ since $X$ is $p$-divisible. This proves (a). Now let $H$ be a subgroup of $G$. As $T \leq G$ by (a), there is an isomorphism $HT/T \cong H/(H \cap T)$, and thus $H$ is discrete $p$-toral. Let $N \leq G$. Then $TN/N$ has finite index in $G/N$, and $TN/N \cong T/(N \cap T)$ where $T/(N \cap T)$ by $p$-divisibility. This proves (b). Point (c) is given by Lemma A.3(a) in the Appendix to Part I. □

We refer to the unique $p$-torus of finite index in the virtually $p$-toral group $G$ as the maximal torus of $G$. If $P$ is a discrete $p$-toral group with maximal torus $T$ then (following [BLO2]) the order of $P$ is defined to be the pair

$$|P| = (rk(P), |P/T|).$$

If $Q$ is a discrete $p$-toral group with maximal torus $U$ then we write

$$|P| < |Q|$$

if $(rk(P), |P/T|) < (rk(Q), |Q/U|)$ lexicographically (i.e. $rk(P) < rk(Q)$, or $rk(P) = rk(Q)$ and $|P/T| < |Q/T|$).

Corollary A.3. Let $(\mathcal{L}, \Delta, S)$ be a compact locality on $\mathcal{F}$, and let $(\mathcal{L}', \Delta', S)$ be an expansion or a restriction of $\mathcal{L}$ to a proper locality. Then $\mathcal{L}'$ is a compact locality on $\mathcal{F}$.

Proof. By 7.2(a) $\mathcal{L}'$ is a proper locality on $\mathcal{F}$. Thus we need only verify that the condition (2) in the preceding definition holds with $(\mathcal{L}', \Delta', S)$ in the place of $(\mathcal{L}, \Delta, S)$. Here (2) is obvious if $\Delta' \subseteq \Delta$. By Zorn’s Lemma it suffices to consider the case where $(\mathcal{L}', \Delta', S)$ is an elementary expansion of $(\mathcal{L}, \Delta, S)$. That is, with $\mathcal{F} = S_{\mathcal{L}}(\mathcal{L})$, we may assume that $\Delta' = \Delta \cup R^{\mathcal{F}}$ for some $R \leq S$, and we may then take $R$ to be fully normalized with respect to the stratification on $\mathcal{F}$ induced from $\mathcal{L}$. Then $N_{\mathcal{L}'}(R)$ is a subgroup of $N_{\mathcal{L}}(Q)$ for some $Q \in \Delta$ by 7.1, and then A.2(b) shows that $N_{\mathcal{L}'}(R)$ is virtually $p$-toral, as required. □

Lemma A.4. Let $P$ and $Q$ be discrete $p$-toral groups.

(a) If $P \cong Q$ then $|P| = |Q|$.
(b) If $P \leq Q$ then $|P| \leq |Q|$, with equality if and only if $P = Q$.
(c) If $P \leq Q$ and $Q$ is a $p$-torus, then either $P = Q$ or $rk(P) < rk(Q)$.
(d) Assume $P \leq Q$, set $P_{0} = P$, and recursively define $P_{k}$ for $k > 0$ by $P_{k} = N_{Q}(P_{k-1})$. Set $B = \bigcup\{P_{k}\}_{k \geq 0}$. Then either $B = Q$ or $rk(B) < rk(P)$. 

2
Proof. Points (a) and (b) are straightforward. Now let $P \leq Q$ and assume that $Q$ is a $p$-torus. For any abelian $p$-group $A$ and any $m > 0$ let $A_m$ be the subgroup of $A$ generated by elements of order dividing $p^m$. Assuming that $P$ is a proper subgroup of $Q$, we find $Q_m \not\leq P$ for $m$ sufficiently large, and then the rank of $P_m$ is less than the rank of $Q_m$. This yields $rk(P) < rk(Q)$, and thus (c) holds.

In proving (d) let $V$ be the maximal torus of $B$, and let $V_0$ the maximal torus of $P_0$. Then $V_0 \leq V$. Suppose that $rk(B) = rk(P)$. Then $V_0 = V$, so $|B : P|$ is finite, and so there exists $k$ with $P_k = P_{k+1} = B$. Then $B = P$ by A.2(c), and thus (d) holds. □

Before we can begin working towards a correspondence between compact localities and $p$-local compact groups (and before we can state the definition of $p$-local compact group), we have first to sort out some potentially conflicting terminology concerning fusion systems. The difficulty here stems from our having already established some terminology in sections 2 and 8 of Part II, relating to stratified fusion systems. This terminology will be shown to agree with that of the cited references on $p$-local compact groups once the context has been narrowed down to the fusion systems associated with linking systems, but until that point has been reached there is a very real problem of confusion. For example, if $(\mathcal{F}, \Omega, \ast)$ is a stratified fusion system on a $p$-group $S$ then we have the notion (II.2.7) of a subgroup $P \leq S$ being fully normalized or fully centralized in $\mathcal{F}$, but there is a quite different definition of these terms in [BLO2]; and there is a similar difficulty concerning saturation of fusion systems. Our solution is to slightly alter the terminology from [BLO2], in the manner of the following definition.

Definition A.5. Let $\mathcal{F}$ be a fusion system over the discrete $p$-toral group $S$.

- A subgroup $P \leq S$ is fully order-centralized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^\mathcal{F}$.
- A subgroup $P \leq S$ is fully order-normalized in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^\mathcal{F}$.
- $\mathcal{F}$ is order-saturated if the following three conditions hold.

(I) For each $P \leq S$ the group $Out_\mathcal{F}(P)$ is finite. Moreover, if $P$ is fully order-normalized in $\mathcal{F}$, then $P$ is fully order-centralized in $\mathcal{F}$, and

$$Out_S(P) \in Syl_p(Out_\mathcal{F}(P)).$$

(II) If $P \leq S$ and $\phi \in Hom_\mathcal{F}(P, S)$ are such that $P\phi$ is fully order-centralized in $\mathcal{F}$, and if we set

$$N_\phi = \{g \in N_S(P) \mid \phi^{-1} \circ c_g \circ \phi \in Aut_S(P\phi)\},$$

then there exists $\overline{\phi} \in Hom_\mathcal{F}(N_\phi, S)$ such that $\phi = \overline{\phi} \mid p$.

(III) If $P_1 < P_2 < P_3 < \cdots$ is an increasing sequence of subgroups of $S$, with $P_\infty = \bigcup \{P_n\}_{n=1}^\infty$, and if $\phi : P_\infty \to S$ is a homomorphism such that $\phi \mid p_n \in Hom_\mathcal{F}(P_n, S)$ for all $n$, then $\phi \in Hom_\mathcal{F}(P_\infty, S)$.

By [BLO2, Lemma 1.6], if $\mathcal{F}$ is a fusion system over a discrete $p$-toral group $S$, then for any subgroup $P \leq S$ there are upper bounds for $|N_S(Q)|$ and for $|C_S(Q)|$ taken over $Q \in P^\mathcal{F}$. Thus $P$ has at least one fully order-normalized $\mathcal{F}$-conjugate and at least one fully order-centralized $\mathcal{F}$-conjugate.
Lemma A.6. Let $(\mathcal{L}, \Delta, S)$ be a compact locality on the order-saturated fusion system $\mathcal{F}$, and let $P \leq S$ be fully normalized in $\mathcal{F}$ with respect to the stratification induced from $\mathcal{L}$.

(a) $P$ is fully order-normalized in $\mathcal{F}$.
(b) $\text{Inn}(P) = O_p(\text{Aut}_\mathcal{F}(P))$ if and only if $P = O_p(N_\mathcal{F}(P))$.

Proof. As remarked above, there exists $Q \in P^\mathcal{F}$ such that $Q$ is fully order-normalized in $\mathcal{F}$. Set $U = N_S(P)$ and $V = N_S(Q)$. As $\mathcal{F}$ is order-saturated there then exists an $\mathcal{F}$-homomorphism $\phi : U \to V$ with $P\phi = Q$.

Set $\Omega = \Omega_S(\mathcal{L})$ and let $(\Omega, \ast)$ be the stratification on $\mathcal{F}$ induced from $\mathcal{L}$. For $X \leq S$ write $\text{dim}(X)$ for $\text{dim}_\Omega(X)$. As $P$ is fully normalized we have $\text{dim}(U) \geq \text{dim}(V)$, and then equality holds since $U\phi \leq V$. Let $\psi$ be an extension of $\phi$ to an $\mathcal{F}$-homomorphism $U^* \to V^*$. Then $U^*\psi = V^*$ since $\text{dim}(U^*) = \text{dim}(U)$ and $\text{dim}(V^*) = \text{dim}(V)$. Then also $U\phi = V$ since $U = N_{U^*}(U)$ and $V = N_{V^*}(V)$. Thus (a) holds.

As $\mathcal{F}$ is order-saturated there exists a subgroup $\overline{P}$ of $N_S(P)$ such that $P \leq \overline{P}$ and $\text{Aut}_\mathcal{F}(P) = O_p(\text{Aut}_\mathcal{F}(P))$. Condition (II) in definition A.6 then yields $\overline{P} \trianglelefteq N_\mathcal{F}(P)$.

So, if $P = O_p(N_\mathcal{F}(P))$ then $P = \overline{P}$, and thus $\text{Inn}(P) = O_p(\text{Aut}_\mathcal{F}(P))$. Now set $R = O_p(N_\mathcal{F}(P))$ and notice that $\text{Aut}_R(P) \leq \text{Aut}_\mathcal{F}(P)$. Then $P = R$ if $P = \overline{P}$, and this completes the proof of (b). □

From now on, when speaking of the fusion system $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ of a compact locality, if we say that a subgroup $P \leq S$ is fully normalized in $\mathcal{F}$ then we mean that $P$ is fully normalized with respect to the stratification induced from $\mathcal{L}$; and similarly for “fully centralized”.

Lemma A.7. Let $\mathcal{F}$ be a fusion system on the discrete $p$-toral group $S$. Assume:

1. $\mathcal{F}$ is saturated (as defined in 8.2),
2. $\text{Out}_\mathcal{F}(P)$ is finite for all $P \leq S$.

Then $\mathcal{F}$ is order-saturated. Further:

(a) A subgroup $P \leq S$ is fully order-normalized in $\mathcal{F}$ if and only if $P$ is fully normalized in $\mathcal{F}$.

(b) A subgroup $P \leq S$ is fully order-centralized in $\mathcal{F}$ if and only if $P$ is fully centralized in $\mathcal{F}$.

(c) Let $T$ be the maximal torus of $S$. Let $R$ be any subgroup of $S$ such that $R$ is a $p$-torus, let $R'$ be an $\mathcal{F}$-conjugate of $R$, and let $\alpha : R \to R'$ be an $\mathcal{F}$-isomorphism. Then $R$ and $R'$ are subgroups of $T$, and $\alpha$ extends to an $\mathcal{F}$-automorphism of $T$.

Moreover, the conditions (1) and (2) obtain if $\mathcal{F}$ is the fusion system of a compact locality.

Proof. Let $P \leq S$ be fully order-normalized in $\mathcal{F}$, and let $Q \in P^\mathcal{F}$ be fully normalized. Then there exists an $\mathcal{F}$-homomorphism $\phi : N_S(P) \to N_S(Q)$ with $P\phi = Q$. As $|N_S(P)| \geq |N_S(Q)|$ it follows from A.4 that $N_S(P)\phi = N_S(Q)$. Thus $P$ is fully normalized in $\mathcal{F}$, and $Q$ is fully order-normalized in $\mathcal{F}$. This yields (a), and (b) follows in similar fashion.

We next show that $\mathcal{F}$ satisfies the condition (1) in A.5. In view of the hypothesis (2) it need only consider the case where $P$ is fully order-normalized in $\mathcal{F}$. Then $P$ is fully
Let $P(=F)$ with $L$ for $F$ (by (b)). By definition 8.2, $P$ is fully automized, so $Out_{S}(P) \in Syl_p(Out_{F}(P))$ as required. The condition (II) in A.5 is satisfied since (by 8.2) $F$ is receptive.

Before verifying condition (III) in A.5, we shall need to prove (c). Let $R$ be any $p$-torus contained in $S$, let $T$ be the maximal torus of $S$, and let $\alpha : R \to R'$ be an $\mathcal{F}$-isomorphism. Then $R$ and $R'$ are subgroups of $T$ by A.2(a). Let $V \in R^F$ be fully centralized in $\mathcal{F}$ and since $T \leq C_{S}(R)$ there exists an $\mathcal{F}$-automorphism $\beta : T \to T$ with $R'\beta = V$. Then $\alpha \circ \beta$ is an $\mathcal{F}$-isomorphism $R \to V$, and $\alpha \circ \beta$ extends to an $\mathcal{F}$-automorphism $\gamma$ of $T$ since $V$ is receptive. Then $\gamma \circ \beta^{-1}$ is an extension of $\alpha$ to $T$, and so (c) holds.

We now turn to (III) in A.5. Let $\sigma = (P_i)_{i=1}^{\infty}$ be an increasing sequence of subgroups of $S$ and let $\tau = (\phi_i)_{i=1}^{\infty}$ be a sequence of $\mathcal{F}$-homomorphisms, $\phi_i : P_i \to S$, such that $\phi_i = \phi_{i+1} |_{P_i}$ for all $i$. Let $P$ be the union of the groups $P_i$ and let $\phi : P \to S$ be the union of the mappings $\phi_i$. For each infinite subset $I$ of natural numbers, $\phi$ is then the union of the mappings $\phi_i$ for $i \in I$; so in order to prove that $\phi$ is an $\mathcal{F}$-homomorphism we are free to replace $\sigma$ and $\tau$ by the sequences corresponding to $I$, and then to assume that $I = \mathbb{N}$. As $\mathcal{F}$ is stratified we have $(P_i)^* = P^*$ for $i$ sufficiently large, and so we may assume that $(P_i)^* = P^*$ for all $i$. Then each $\phi_i$ extends to an $\mathcal{F}$-homomorphism $P^* \to S$, which then restricts to an $\mathcal{F}$-homomorphism $\psi_i : P \to S$.

Let $U$ be the maximal torus of $P$. Then $P = U \Gamma$ for $n$ sufficiently large, and so we may assume $P = UP_1$. As $Aut_{\mathcal{F}}(T) = Out_{\mathcal{F}}(T)$ is finite, (c) implies that there are only finitely many $\mathcal{F}$-conjugates of $U$, and indeed that there exists an $\mathcal{F}$-homomorphism $\xi : U \to S$ and an infinite set $I$ of natural numbers such that $\psi_i |_{U} = \xi$ for all $i \in I$. As $P = UP_1$ the set $\{\psi_i\}_{i \in I}$ then has a single element $\psi$, and $\psi$ is then the union of the homomorphisms $\phi_i$. Thus $\psi = \phi$, so $\phi$ is an $\mathcal{F}$-homomorphism, and (III) holds.

Finally, assume that we are given a compact locality $(\mathcal{L}, \Delta, S)$. Then $\mathcal{L}$ is proper, so $\mathcal{F}_{S}(\mathcal{L})$ is saturated by 8.3(c). Thus, it only remains to show that the condition (2) holds for $\mathcal{F} = \mathcal{F}_{S}(\mathcal{L})$. By Theorem 7.2 there exists a proper expansion $(\mathcal{L}^{+}, \Delta^{+}, S)$ on $\mathcal{F}$, with $\mathcal{F}^c \subseteq \Delta^{+}$; and $\mathcal{L}^{+}$ is then compact by A.3. Thus, we may assume $\mathcal{F}^c \subseteq \Delta$.

As $(\mathcal{L}, \Delta, S)$ is proper we may appeal to Theorem 7.2 to obtain a proper expansion $(\mathcal{L}^{+}, \Delta^{+}, S)$ on $\mathcal{F}$, with $\mathcal{F}^c \subseteq \Delta^{+}$. By construction, subgroups of $\mathcal{L}^{+}$ are conjugates of subgroups of $\mathcal{L}$, and thus (2) holds and $\mathcal{L}^{+}$ is compact. We may therefore replace $\mathcal{L}$ with $\mathcal{L}^{+}$ in the remainder of the proof. That is, we may assume that we have $\mathcal{F}^c \subseteq \Delta$. Let $P \leq S$ be fully normalized in $\mathcal{F}$. As $\mathcal{F}$ is inductive, $P$ is then fully centralized in $\mathcal{F}$. Set $X = C_{S}(P)P$. Then $X$ is $\mathcal{F}$-centric, and so $X \in \Delta$. Set $H = N_{\mathcal{L}}(X)$. Every $\mathcal{F}$-automorphism of $P$ extends to an $\mathcal{F}$-automorphism of $X$ by receptivity, so

$$Aut_{\mathcal{F}}(P) = Aut_{N_{\mathcal{L}}(P)}(P).$$

Thus $Aut_{\mathcal{F}}(P)$ is a homomorphic image of the virtually $p$-toral group $N_{\mathcal{L}}(P)$. In particular every element of $Aut_{\mathcal{F}}(P)$ is of finite order. By [BLO2, Lemma 1.5(b)] every torsion subgroup of $Out(P)$ is finite, so we conclude that $Out_{\mathcal{F}}(P)$ is finite. That is, (2) holds, and the proof is complete. $\Box$

This completes the preliminaries concerning fusion systems. We next show how a compact locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$ gives rise to a “transporter system” of a certain kind,
and then that this transporter system is a $p$-local compact group provided that $\Delta$ is the set $F^c$ of $F$-centric subgroups of $S$. The definition of transporter system will be taken from [BLO3].

Let $(\mathcal{L}, \Delta, S)$ be any locality. For each $(P, Q) \in \Delta \times \Delta$ let $N_\mathcal{L}(P, Q)$ be the set of all $g \in \mathcal{L}$ such that $P \leq S_g$ and $P^g \leq Q$. There is then a category $\mathcal{T} = \mathcal{T}_\Delta(\mathcal{L})$ whose set of objects is $\Delta$, and whose morphisms $g : P \to Q$ are triples $(g, P, Q)$ such that $g \in N_\mathcal{L}(P, Q)$, with composition defined by

$$(g, P, Q) \circ (h, Q, R) = (gh, P, R).$$

In practice, the role of the objects $P$ and $Q$ will always be clear from the context, and we may therefore identify $\text{Mor}_\mathcal{T}(P, Q)$ with $N_\mathcal{L}(P, Q)$.

Let $F$ be the fusion system $F_S(\mathcal{L})$, i.e. the fusion system on $S$ generated by the conjugation maps $c_g : P \to Q$ with $P, Q \in \Delta$ and with $g \in N_\mathcal{L}(P, Q)$. There is then a functor

$$\rho : \mathcal{T} \to F$$

such that $\rho$ is the inclusion map $\Delta \to \text{Sub}(S)$ on objects, and such that $\rho_{P, Q} : N_\mathcal{L}(P, Q) \to \text{Hom}_F(P, Q)$ is the map which sends $g$ to the conjugation homomorphism $c_g : P \to Q$. We write $\rho_p$ for the homomorphism $\rho_{P, P} : \text{Aut}_\mathcal{T}(P) \to \text{Aut}_F(P)$.

Since $(S, \Delta, S)$ is a locality we have the category $\mathcal{T}_\Delta(S)$; and there is a functor

$$\epsilon : \mathcal{T}_\Delta(S) \to \mathcal{T}$$

which is the identity map $\Delta \to \Delta$ on objects, and where $\epsilon_{P, Q}$ is the inclusion map $N_S(P, Q) \to N_\mathcal{L}(P, Q)$. We write $\epsilon_P$ for the inclusion $N_S(P) \to N_\mathcal{L}(P)$. Also, for $P, Q \in \Delta$ with $P \leq Q$, write $\iota_{P, Q}$ for $(1)_{\epsilon_P} \epsilon_{P, Q}$.

The definition from [BLO3] of a transporter system over a discrete $p$-toral group is embedded in the statement of the following result. We remind the reader that we always understand composition of morphisms in a category to be taken from left to right.

**Proposition A.8.** Let $(\mathcal{L}, \Delta, S)$ be a compact locality on $\mathcal{F}$ and let

$$\mathcal{T}_\Delta(S) \xrightarrow{\epsilon} \mathcal{T} \xrightarrow{\rho} \mathcal{F}$$

be the pair of functors defined above. Then the following hold.

(A1) $\epsilon$ is the identity on objects and $\rho$ is the inclusion on objects.

(A2) For each $P, Q \in \Delta$ the group $\ker(\rho_P)$ acts freely on $\text{Mor}_\mathcal{T}(P, Q)$ from the left (by composition), and $\rho_{P, Q}$ is the orbit map for this action. Also, $\ker(\rho_Q)$ acts freely on $\text{Mor}_\mathcal{T}(P, Q)$ from the right.

(B) For each $P, Q \in \Delta$ the map $\epsilon_{P, Q}$ is injective, and $\epsilon_{P, Q} \circ \rho_{P, Q}$ sends $g \in N_S(P, Q)$ to $c_g \in \text{Hom}_F(P, Q)$. 

6
(C) For all \( \phi \in \text{Mor}_\mathcal{T}(P, Q) \) and all \( x \in P \), the following square commutes in \( \mathcal{T} \).

\[
\begin{array}{ccc}
P & \xrightarrow{\phi} & Q \\
(x) \in_P & \downarrow & \downarrow (x)(\rho(\phi) \circ \epsilon_Q) \\
P & \xrightarrow{\phi} & Q
\end{array}
\]

(I) Each \( \mathcal{F} \)-conjugacy class of subgroups in \( \Delta \) contains a subgroup \( P \) such that the image of \( N_S(P) \) under \( \epsilon_P \) is a Sylow \( p \)-subgroup of \( \text{Aut}_\mathcal{T}(P) \) \( \text{(i.e. a subgroup of finite index relatively prime to } p \text{)} \).

(II) Let \( \phi : P \to Q \) be a \( \mathcal{T} \)-isomorphism, and regard conjugation by \( \phi \) as a mapping \( c_\phi : \text{Aut}_\mathcal{T}(P) \to \text{Aut}_\mathcal{T}(Q) \). Let \( P \leq \overline{P} \leq S \) and \( Q \leq \overline{Q} \leq S \), be given, and suppose that \( c_\phi \) maps \( (P) \epsilon_P \) into \( (Q) \epsilon_Q \). Then there exists \( \phi \in \text{Mor}_\mathcal{T}(\overline{P}, \overline{Q}) \) such that \( \iota_{P, \overline{P}} \circ \overline{\phi} = \phi \circ \iota_{Q, \overline{Q}} \).

(III) Let \( P_1 \leq P_2 \leq P_3 \leq \cdots \) be an increasing sequence of members of \( \Delta \), and for each \( i \) let \( \psi_i : P_i \to S \) be a \( \mathcal{T} \)-homomorphism. Assume that \( \psi_i = \iota_{P_i, P_{i+1}} \circ \psi_{i+1} \) for all \( i \), and set \( P = \bigcup \{ P_i \}_{i=1}^{\infty} \). Then there exists \( \psi \in \text{Mor}_\mathcal{T}(P, S) \) such that \( \psi_i = \iota_{P_i, S} \circ \psi \) for all \( i \).

Proof. The condition (A1) is immediate from the definition of the functors \( \epsilon \) and \( \rho \). Under the identification of \( \text{Aut}_\mathcal{T}(P) \) with \( N_\mathcal{L}(P) \) we have composition in \( \text{Aut}_\mathcal{T}(P) \) given by group multiplication in \( N_\mathcal{L}(P), \text{Ker}(\rho_P) = C_\mathcal{L}(P) \), and similarly \( \text{Ker}(\rho_Q) = C_\mathcal{L}(Q) \). The actions defined in (A2) are then obviously free, and since \( \text{Aut}_\mathcal{F}(P) \cong N_\mathcal{L}(P) / C_\mathcal{L}(P) \) we obtain the conclusion of (A2).

The condition (B) is again immediate from the definition of \( \epsilon \). Now let \( g \in N_\mathcal{L}(P, Q) \) and let \( x \in P \). Regard \( g \) as a \( \mathcal{T} \)-homomorphism \( \phi : P \to Q \). Then \( (x) \epsilon_P \circ \phi \) is simply the product \( xg \), while the composition \( \phi \circ ((x)(\rho(\phi) \circ \epsilon_Q)) \) is the product \( gx^g \). As \( gx = gx^g \) we have the required commutativity of the diagram in (C).

Each \( \mathcal{F} \)-conjugacy class of subgroups in \( \Delta \) contains a subgroup \( P \) such that \( N_S(P) \) is a Sylow \( p \)-subgroup of \( N_\mathcal{L}(P) \), by I.3.10. Thus (I) holds.

Again let \( g \in N_\mathcal{L}(P, Q) \) be a \( \mathcal{T} \)-isomorphism. Then \( P^g = Q \), and \( c_g \) is an isomorphism \( N_\mathcal{L}(P) \to N_\mathcal{L}(Q) \). If \( P \leq \overline{P} \leq S \) and \( Q \leq \overline{Q} \leq S \) with \( \overline{P} \leq \overline{Q} \leq S \), then \( g \in N_\mathcal{L}(\overline{P}, \overline{Q}) \), and in this way \( g \) is a \( \mathcal{T} \)-homomorphism \( \overline{\phi} : \overline{P} \to \overline{Q} \). That is, (II) holds.

Let \( \{ P_i \}_{i=1}^{\infty}, \{ \psi_i \}_{i=1}^{\infty} \), and \( P \) be given as in (III). Then \( \psi_i \), written in full detail, is a triple \( (g_i, P_i, S) \) where \( P_i \leq S_{g_i} \). The “inclusion morphism” \( \iota_{P_i, P_{i+1}} \) is the triple \( (1, P_i, P_{i+1}) \), and thus

\[
\iota_{P_i, P_{i+1}} \circ \psi_{i+1} = (g_{i+1}, P_i, S).
\]

The hypothesis of (III) therefore translates into the statement that the sequence \( (g_i) \) is a constant sequence \( (g) \) where \( P \leq S_g \). Taking \( \psi = (g, P, S) \) then yields the conclusion of (III). □
Definition A.9. Let $\mathcal{F}$ be a fusion system over the discrete $p$-toral group $S$ and let $\Delta$ be any $\mathcal{F}$-closed set of subgroups of $S$. A transporter system associated to $\mathcal{F}$ consists of a category $T$ with $\text{Ob}(T) = \Delta$, together with a pair of functors

$$\mathcal{T}_\Delta(S) \xleftarrow{\epsilon} T \xrightarrow{\rho} \mathcal{F},$$

which satisfy the conditions (A1), (A2), (B), (C), (I), (II), and (III) from the preceding proposition (where $\rho_P : \text{Aut}_T(P) \to \text{Aut}_\mathcal{F}(P)$ is an abbreviation for $\rho_{P,P}$, and where $\iota_{P,Q}$ is an abbreviation for $(1)\epsilon_{P,Q}$ if $P,Q \in \Delta$ with $P \leq Q$). Write $\iota_P$ for $\iota_{P,P}$.

The definitions of linking system and of $p$-local compact group may now be given as follows, by [BLO3, Corollary A.5].

Definition A.10. The transporter system

$$\mathcal{T}_\Delta(S) \xleftarrow{\epsilon} T \xrightarrow{\rho} \mathcal{F}$$

is a linking system associated with $\mathcal{F}$ if the following conditions hold.

1. $\mathcal{F}$ is order-saturated.
2. We have $P \in \Delta$ for each $\mathcal{F}$-centric subgroup $P \leq S$ such that $O_p(\text{Out}_\mathcal{F}(P)) = 1$.
3. For each $P$ in $\Delta$ the kernel of the homomorphism $\rho_P : \text{Aut}_T(P) \to \text{Aut}_\mathcal{F}(P)$ is discrete $p$-toral.

In the special case that $\Delta$ is the set $\mathcal{F}^c$ of all $\mathcal{F}$-centric subgroups of $S$ we say that $(\epsilon, \rho)$ is a $p$-local compact group.

Proposition A.11. Let $(\mathcal{L}, \Delta, S)$ be a compact locality on $\mathcal{F}$, with $\Delta = \mathcal{F}^c$. Then the transporter system $(\epsilon, \rho)$ given by proposition A.8 is a $p$-local compact group.

Proof. We need only verify the conditions (1) and (3) in the preceding definition, since the hypothesis that $\Delta = \mathcal{F}^c$ yields the remaining requirements. Condition (1) is given by A.7. Now let $P \in \Delta$. There is then an isomorphism $\alpha : N_\mathcal{L}(P) \to \text{Aut}_T(P)$ given by $g \mapsto (g, P, P)$, and the kernel of $\rho_P$ is then the image under $\alpha$ of $C_\mathcal{L}(P)$. As $\mathcal{L}$ is proper and $P \in \mathcal{F}^c$, 6.9 yields $C_\mathcal{L}(P) = Z(P)$, and so (3) holds. \qed

Our goal now is to proceed in the opposite direction from that of the preceding result. Thus, starting with a $p$-local compact group, we aim now to construct a compact locality $(\mathcal{L}, \Delta, S)$.

In what follows we fix the $p$-local finite group $(\mathcal{T}_\Delta(S) \xleftarrow{\epsilon} T \xrightarrow{\rho} \mathcal{F})$, with the abbreviations $\rho_P$ and $\iota_{P,Q}$ as earlier. Write $\iota_P$ for the identity morphism $\iota_{P,P}$ in $\text{Aut}_T(P)$. Condition A.9(B) implies that the image of $\iota_{P,P'}$ under $\rho$ is the inclusion map $P \to P'$, so $\iota_{P,P'}$ is referred to as an inclusion morphism of $T$. This leads to the following definition.

Definition A.12. Let $P, Q, P', Q' \in \Delta$ with $P \leq P'$ and $Q \leq Q'$, and further let $\phi \in \text{Mor}_T(P, Q)$ and $\phi' \in \text{Mor}_T(P', Q')$. Then $\phi$ is an extension of $\phi'$, and $\phi'$ is a restriction of $\phi$ if

$$\iota_{P,P'} \circ \phi = \phi \circ \iota_{Q,Q'}. $$

The following result collects the basic properties concerning the transporter system $(\epsilon, \rho)$.
Lemma A.13.

(a) Let $P, Q, R \in \Delta$, and let

\[ P \xrightarrow{\phi} Q \quad \text{and} \quad Q \xrightarrow{\psi} R \]

be $\mathcal{F}$-homomorphisms. Further, let $\psi \in \text{Mor}_\mathcal{T}(Q, R)$ with $\rho(\psi) = \overline{\psi}$, and let $\lambda \in \text{Mor}_\mathcal{T}(P, R)$ with $\rho(\lambda) = \overline{\phi} \circ \overline{\psi}$. Then there exists a unique $\phi \in \text{Mor}_\mathcal{T}(P, R)$ such that $\rho(\phi) = \overline{\phi}$ and such that $\lambda = \phi \circ \psi$.

(b) Let $\psi : P \to Q$ be a $\mathcal{T}$-morphism and let $P_0, Q_0 \in \Delta$ with $P_0 \leq P$ and with $Q_0 \leq Q$. Suppose that $\rho(\psi)$ maps $P_0$ into $Q_0$. There is then a unique $\mathcal{T}$-morphism $\psi_0 : P_0 \to Q_0$ such that $\psi$ is an extension of $\psi_0$.

(c) A $\mathcal{T}$-homomorphism $\phi$ is a $\mathcal{T}$-isomorphism if and only if $\rho(\phi)$ is an $\mathcal{F}$-isomorphism.

(d) All morphisms of $\mathcal{T}$ are both monomorphisms and epimorphisms in the categorical sense. That is, we have left and right cancellation for morphisms in $\mathcal{T}$.

(e) Let $\phi_0 : P_0 \to Q_0$ be a $\mathcal{T}$-morphism and let $P_0 \leq P \leq S$ and $Q_0 \leq Q \leq S$. Then there exists at most one extension of $\phi_0$ to a $\mathcal{T}$-homomorphism $P \to Q$.

(f) Let $P, \overline{P}, Q, \overline{Q}$ be objects of $\mathcal{T}$, with $P \leq \overline{P}$ and with $Q \leq \overline{Q}$. Suppose that we are given a $\mathcal{T}$-isomorphism $\phi : P \to Q$ and an extension of $\phi$ to a $\mathcal{T}$-homomorphism $\overline{\phi} : \overline{P} \to \overline{Q}$. Then for each $x \in \overline{P}$ there is a commutative square:

\[
\begin{array}{ccc}
P & \xrightarrow{\phi} & Q \\
\downarrow x_{\delta_{P,P}} & & \downarrow y_{\delta_{Q,Q}} \\
\overline{P} & \xrightarrow{\overline{\phi}} & \overline{Q}
\end{array}
\]

where $y$ is the image of $x$ under $\rho(\overline{\phi})$.

(g) Every $\mathcal{T}$-morphism $\psi : P \to Q$ is the composite of a $\mathcal{T}$-isomorphism $\phi : P \to Q_0$ followed by an inclusion morphism $\iota_{Q_0, Q}$, where $Q_0$ is the image of $P$ under $\rho(\psi)$.

Proof. Points (a) through (d) constitute [BLO3, Proposition A.2], and (e) follows from the left cancellation in (d). For (f) one may appeal to the proof of [OV, Lemma 3.3(d)], as that proof does not depend on the finiteness of $S$. Finally, let $\phi : P \to Q$ be a $\mathcal{T}$-homomorphism and let $Q_0$ be the image of $P$ under $\rho(\phi)$. Then there exists a restriction of $\phi$ to a $\mathcal{T}$-homomorphism $\phi_0 : P \to Q_0$ by (b), and $\phi_0$ is then a $\mathcal{T}$-isomorphism by (c). This yields (g). □

(A.14) By [BLO2, Section 3] it is a feature of an order-saturated fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$ that there is a mapping $P \mapsto P^\bullet$ from $\text{Sub}(S)$ into $\text{Sub}(S)$ having the following properties.

1. $\{P^\bullet \mid P \leq S\}$ is $\mathcal{F}$-invariant, and is the union of a finite number of $S$-conjugacy classes of subgroups of $S$.
2. For subgroups $P \leq Q \leq S$ we have $P^\bullet \leq Q^\bullet$ and $(P^\bullet)^\bullet = P^\bullet$.
3. For all $P, Q \leq S$ we have $N_S(P, Q) \subseteq N_S(P^\bullet, Q^\bullet)$.
4. For all $P, Q \leq S$, each $\mathcal{F}$-homomorphism $\alpha : P \to Q$ extends to an $\mathcal{F}$-homomorphism $\alpha^\bullet : P^\bullet \to Q^\bullet$. 

9
In fact, we will not need (4) here. Rather, what we require is the following result concerning $p$-local finite groups.

**Lemma A.15.** Let $\mathcal{T}^\bullet$ be the full subcategory of $\mathcal{T}$ whose set of objects is $\{P^\bullet \mid P \in \Delta\}$. Then there is a functor

$$(-)^\bullet : \mathcal{T} \to \mathcal{T}^\bullet,$$

having the following properties.

(a) $(-)^\bullet$ is the mapping $P \mapsto P^\bullet$ on objects $P \in \Delta$.

(b) $(-)^\bullet$ restricts to the identity functor on $\mathcal{T}^\bullet$.

(c) For all $P, Q \in \Delta$ and all $\phi \in \text{Mor}_\mathcal{T}(P, Q)$, the image $\phi^\bullet$ of $\phi$ under $(-)^\bullet$ is an extension of $\phi$.

(d) If $\alpha : X \to Y$ and $\phi : P \to Q$ are $\mathcal{T}$-morphisms such that $\phi$ is an extension of $\alpha$, then $\phi^\bullet$ is an extension of $\alpha^\bullet$.

**Proof.** Points (a) through (c) are given by [JLL, Proposition 1.12]. By the same reference we have also the result that for all $X, P \in \Delta$ and all $g \in N_S(X, P)$ we have (in accord with A.14(3)) $\epsilon_{X,P}^\bullet = (g)\epsilon_{X^\bullet,P^\bullet}$. In particular, by taking $X \leq P$ and $g = 1$ we obtain $\iota_{X,P}^\bullet = \iota_{X^\bullet,P^\bullet}$. If $\alpha$ and $\phi$ are given as in (d), so that $\iota_{X,P} \circ \phi = \alpha \circ \iota_{Y,Q}$, the functoriality of $(-)^\bullet$ now yields $\iota_{X^\bullet,P^\bullet} \circ \phi^\bullet = \alpha^\bullet \circ \iota_{Y^\bullet,Q^\bullet}$. Thus (d) holds. □

**Lemma A.16.** Let $\phi_0 : P_0 \to Q_0$, $\phi : P \to Q$, and $\phi' : P' \to Q'$ be $\mathcal{T}$-isomorphisms, and suppose that both $\phi$ and $\phi'$ are extensions of $\phi_0$.

(a) If $P = P'$ or if $Q = Q'$, then $\phi = \phi'$.

(b) There is a unique extension of $\phi_0$ to an isomorphism $\psi : P \cap P' \to Q \cap Q'$, and both $\phi$ and $\phi'$ are extensions of $\psi$.

**Proof.** Assume that (a) is false. We may take $P = P'$, since the case where $Q = Q'$ will then follow by considering the inverses of the given $\mathcal{T}$-isomorphisms. Note that if $\phi^\bullet = (\phi')^\bullet$ then $\phi = \phi'$ by restriction. Since $\phi^\bullet$ and $(\phi')^\bullet$ are extensions of $(\phi_0)^\bullet$ by A.15(b), it therefore suffices to consider the case where $\phi_0, \phi,$ and $\psi$ are $\mathcal{T}^\bullet$-isomorphisms. The finiteness condition A.14(1) then yields the existence of a counter-example $(\phi_0, \phi, \phi')$ to (a) in which $|P_0|$ is maximal.

Let $x \in N_P(P_0)$, let $y$ be the image of $x$ under $\rho(\phi)$, and let $y'$ be the image of $x$ under $\rho(\phi')$. We appeal to A.14(f) with $(P_0, Q_0, N_P(P_0), N_Q(Q_0))$ in the role of $(P, Q, \bar{P}, \bar{Q})$, and obtain

$$\phi_0^{-1} \circ (x) \epsilon_{P_0,P_0} \circ \phi_0 = (y) \epsilon_{Q_0,Q_0} = (y') \epsilon_{Q_0,Q_0}.$$  

As $\epsilon_{Q_0,Q_0}$ is injective (by A.9(B)) we get $y = y'$, and thus $\rho(\phi)$ and $\rho(\phi')$ agree on $P_1 := N_P(P_0)$. Let $Q_1$ be the image of $P_1$ under $\rho(\phi)$. By A.13(b) there is a restriction $\phi_1 : P_1 \to Q_1$ of $\phi$ and a restriction $\phi'_1 : P_1 \to Q_1$ of $\phi'$, and then $\phi_1 = \phi'_1$ by A.13(e). Now $(\phi_1, \phi, \phi')$ is a counter-example to (a) with $|P_1| > |P_0|$, in violation of the maximality of $|P_0|$. This contradiction completes the proof of (a).

Set $X = P \cap P'$ and $Y = Q \cap Q'$. Then $\phi$ and $\phi'$ have restrictions $\psi : X \to (X)(\rho(\phi))$ and $\psi' : X \to (X)(\rho(\phi'))$ which, in turn, restrict to $\phi_0$. Then (a) yields $\psi = \psi'$, and this establishes (b). □
Define a relation $\uparrow$ on the set $\text{Iso}(\mathcal{T})$ $\mathcal{T}$-isomorphisms by $\phi \uparrow \phi'$ if $\phi'$ is an extension of $\phi$. We may also write $\phi' \downarrow \phi$ to indicate that $\phi$ is a restriction of $\phi'$.

**Lemma A.17.** The following hold.

(a) The relation $\uparrow$ induces a partial order on $\text{Iso}(\mathcal{T})$.

(b) The relation $\uparrow$ respects composition of morphisms. That is, if $\phi \uparrow \phi'$ and $\psi \uparrow \psi'$, and the compositions $\phi \circ \psi$ and $\phi' \circ \psi'$ are defined, then $(\phi \circ \psi) \uparrow (\phi' \circ \psi')$.

(c) For each $\mathcal{T}$-isomorphism $\alpha$ there exists a unique $\mathcal{T}$-isomorphism $\phi$ such that $\phi$ is maximal with respect to $\uparrow$ and such that $\alpha \uparrow \phi$.

**Proof.** For points (a) and (b) we repeat the proof of [Che1, Lemma X.7]. The transitivity of $\uparrow$ is immediate. Suppose that both $\phi \uparrow \phi'$ and $\phi \downarrow \phi''$, where $\phi \in \text{Iso}_{\mathcal{T}}(P,Q)$ and $\phi' \in \text{Iso}_{\mathcal{T}}(P',Q')$. Then $P = P'$, $Q = Q'$, $\iota_{P,P'} = \iota_P$, and $\iota_{Q,Q'} = \iota_Q$. Further, $\iota_P \phi' = \phi \circ \iota_Q$ and then $\phi' = \phi$ since $\iota_P$ and $\iota_Q$ are identity morphisms in $\mathcal{T}$. Thus (a) holds.

Suppose that we are given $\phi \uparrow \phi'$ and $\psi \uparrow \psi'$, with $\phi \circ \psi$ and $\phi' \circ \psi'$ defined on objects $P$ and $P'$ respectively. Set $Q = P\phi$ and $R = Q\psi$, and set $Q' = P'\phi'$ and $R' = Q'\psi'$. The following diagram, in which the vertical arrows are inclusion morphisms, demonstrates that $\phi \circ \psi \uparrow \phi' \circ \psi'$.

\[
\begin{array}{ccc}
P' & \rightarrow & Q' \\
\uparrow & & \uparrow \\
P & \rightarrow & Q
\end{array}
\begin{array}{c}
\phi' \downarrow \\
\phi \uparrow \\
\psi' \downarrow \\
\psi \uparrow \\
R' & \rightarrow & R
\end{array}
\]

This yields (b).

Let $\alpha \in \text{Iso}(\mathcal{T})$. The finiteness condition A.14(1), together with A.15(c,d), yields the existence of at least one $\mathcal{T}$-isomorphism $\phi$ such that $\alpha \uparrow \phi$ and such that $\phi$ is maximal with respect to $\uparrow$. Assuming now that $\alpha$ is a counter-example to (c), there then exists an $\uparrow$-maximal $\mathcal{T}$-isomorphism $\phi'$ with $\alpha \uparrow \phi'$ and with $\phi \neq \phi'$. Write $\alpha : X \rightarrow Y$, $\phi : P \rightarrow Q$, and $\phi' : P' \rightarrow Q'$. We may again apply A.14(1) in conjunction with A.15(c,d) in order to obtain such a triple $(\alpha, \phi, \phi')$ in which $|X|$ has been maximized.

Set $P_1 = N_P(X)$ and $Q_1 = N_Q(Y)$, and similarly define $P'_1$ and $Q'_1$. Set $X_1 = \langle P_1, P'_1 \rangle$ and $Y_1 = \langle Q_1, Q'_1 \rangle$. Let $\lambda : \text{Aut}_{\mathcal{T}}(P_0) \rightarrow \text{Aut}_{\mathcal{T}}(Q_0)$ be the isomorphism induced by conjugation by $\alpha$. Then A.13(f) implies that $\lambda$ maps $(X_1)_{\epsilon_{P_0}}$ to $(Y_1)_{\epsilon_{Q_0}}$. Condition (II) in the definition of transporter system then yields the existence of an extension of $\alpha$ to an isomorphism $\alpha_1 : X_1 \rightarrow Y_1$. Let $\phi_1$ be the restriction of $\phi$ to an isomorphism $P_1 \rightarrow Q_1$. Then $\phi \downarrow \phi_1 \uparrow \alpha_1$. Let $\psi$ be an $\uparrow$-maximal extension of $\alpha_1$. If $\phi = \psi$ then $X_1 = X$, whence $P = X = P'$, and then A.16(a) yields $\phi = \phi'$. Thus $\phi \neq \psi$, so $(\phi_1, \phi, \psi)$ provides a counter-example to (c). The maximality of $|X|$ then yields $X = N_P(X)$, and we again obtain $X = P$. Then $\phi = \alpha$, $\alpha$ is $\uparrow$-maximal, and again $\alpha = \phi'$. Thus we have a contradiction, proving (c). □

Let $\equiv$ be the equivalence relation on $\text{Iso}(\mathcal{T})$ generated by $\uparrow$, and let

\[
\mathcal{L} = \text{Iso}(\mathcal{T})/\equiv
\]
Lemma A.18. Let $f \in \mathcal{L}$.

(a) There is a unique $\phi \in f$ such that $\psi$ is $\uparrow$-maximal in the poset $\text{Iso}(\mathcal{T})$. Moreover, we then have $\alpha \uparrow \phi$ for all $\alpha \in f$, and $\phi^{-1}$ is the unique $\uparrow$-maximal member of $[\phi^{-1}]$.

(b) The unique maximal $\phi \in f$ is a $\mathcal{T}^*$-isomorphism.

(c) $f \cap \text{Iso}_\mathcal{T}(P,Q)$ has cardinality at most 1 for any $P,Q \in \text{Ob}(\mathcal{T})$.

Proof. There exists at least one $\uparrow$-maximal member $\phi \in f$ by A.14(1) with A.15(c,d). Suppose that $\phi$ and $\phi'$ are distinct $\uparrow$-maximal members of $f$. As $\phi \equiv \phi'$ there is a sequence $\sigma = (\psi_0, \cdots, \psi_n)$ of members of $f$ with $\phi = \psi_0$, $\phi' = \psi_n$, and such that for all $i$ with $1 \leq i \leq n$ we have either $\psi_{i-1} \uparrow \psi_i$ or $\psi_{i-1} \downarrow \psi_i$. Assume that the pair $(\phi, \phi')$ has been chosen so that $n$ is as small as possible. The maximality of $\phi$ implies $\phi \downarrow \psi_1$ and $\phi \uparrow \psi_2$ is not a restriction of $\phi$. Since $\psi_2$ is the restriction of some maximal isomorphism (necessarily in $f$), we obtain $n = 2$. Thus $\phi \downarrow \psi_1 \uparrow \phi'$. As this violates A.17(c) we obtain the uniqueness asserted in (a). Moreover, A.17(c) then implies that each $\alpha \in f$ extends to $\phi$. The inverse of any extension of $\alpha$ is an extension of $\alpha^{-1}$, so (a) holds. Point (b) then follows from A.15(c).

In order to prove (c), let $\psi, \psi' \in f \cap \text{Iso}_\mathcal{T}(P,Q)$. Then both $\psi$ and $\psi'$ are restrictions of a single $\phi \in f$, by (a). Now A.13(b) yields $\psi = \psi'$. □

Define $D$ to be the set of words $w = (f_1, \cdots, f_n) \in \mathcal{W}(\mathcal{L})$ such that there exists a sequence $(\phi_1, \cdots, \phi_n)$ of $\mathcal{T}$-isomorphisms with $\phi_i \in f_i$, and a sequence $(P_0, \cdots, P_n)$ of members of $\Delta$, such that each $\phi_i$ is a $\mathcal{T}$-isomorphism $P_{i-1} \rightarrow P_i$. As in section I.2 we may say also that $w \in D$ via $(P_0, \cdots, P_n)$, or via $P_0$. Define

$$\Pi : D \rightarrow \mathcal{L}$$

by $\Pi(w) = f$, where $f$ is the unique maximal element of $[\phi_1 \circ \cdots \circ \phi_n]$ given by A.18(a). That $\Pi$ is well-defined follows from A.17(b). Set $1 = [\iota_S]$, and for any $f \in \mathcal{L}$ let $f^{-1}$ be the equivalence class of $\phi^{-1}$, where $\phi$ is the unique maximal member of $f$.

Proposition A.19. $\mathcal{L}$ with the above structures is a partial group. Moreover, the following hold.

(a) For any $x \in S$, the $\equiv$-class $[(x)\epsilon_S]$ is the set of all $(x)\epsilon_{P,Q}$ such that $P^x = Q$, and $(x)\epsilon_S$ is the maximal member of $[(x)\epsilon_S]$.

(b) $[\iota_S] = \{\iota_P \mid P \in \text{Ob}(\mathcal{T})\}$, and $\iota_S$ is the maximal member of its class.

(c) For any $\phi \in \text{Iso}(\mathcal{T})$, $[\phi^{-1}]$ is the set of inverses of the members of $[\phi]$.

Proof. We first check via definition I.1.1 that $\mathcal{L}$ is a partial group. For any $f \in \mathcal{L}$ the word $(f)$ of length 1 is in $D$ since $f$ is represented by a $\mathcal{T}$-isomorphism. If $w \in D$ and $w = u \circ v$ then it is immediate from the definition of $D$ that both $u$ and $v$ are in $D$. Thus the condition I.1.1(1) in the definition of partial group is satisfied. By definition of
II we have \( \Pi(f) = f \) for \( f \in \mathcal{L} \), so I.1.1(2) holds. Condition I.1.1(3) is a straightforward consequence of associativity of composition of isomorphisms in \( \mathcal{T} \).

That the inversion map \( f \mapsto f^{-1} \) is an involutory bijection follows from A.18(a). Now let \( u = (f_1, \ldots, f_n) \in \mathcal{D} \) via \((P_0, \ldots, P_n)\), and set \( u^{-1} = (f_n^{-1}, \ldots, f_1^{-1}) \). Then \( u^{-1} \in \mathcal{D} \) via \((P_n, \ldots, P_0)\), so \( u^{-1} \circ u \in \mathcal{D} \). One obtains a representative in the class \( \Pi(u^{-1} \circ u) \) via a sequence of cancellations \( \phi_k^{-1} \circ \phi = \iota_{P_k} \), for representatives \( \phi_k \in f_i \), so \( \Pi(u^{-1} \circ u) \) is the equivalence class containing \( \iota_{P_0} \). Since \( \iota_{P_0} \uparrow \iota_S \), and since \( 1 = [\iota_S] \) by definition, we get \( \Pi(u^{-1} \circ u) = 1 \). Thus I.1.1(4) holds in \( \mathcal{L} \), and \( \mathcal{L} \) is a partial group.

We now prove (a). Let \( P \leq P' \) and \( Q \leq Q' \) in \( \Delta \), and let \( x \) be an element of \( S \) such that \( P^x = Q \) and \( (P')^x = Q' \). The functoriality of \( \epsilon \) yields

\[(1) \epsilon_{P,P'} \circ (x) \epsilon_{P',Q'} = (x) \epsilon_{P,Q'} = (x) \epsilon_{P,Q} \circ (1) \epsilon_{Q,Q'},\]

which means that \( (x) \epsilon_{P,Q}(g) \uparrow (x) \epsilon_{P',Q'} \). In particular, we get \( (x) \epsilon_{P,Q}(g) \uparrow (x) \epsilon_{S} \).

In order to complete the proof of (a), it now suffices to show that for any \( \phi \in Iso_{\mathcal{T}}(P,Q) \) with \( (x) \epsilon_{S} \equiv \phi \), we have \( \phi = (x) \epsilon_{P,Q} \). Suppose false, and let \( \sigma = (\phi_1, \ldots, \phi_n) \) be a sequence of \( \mathcal{T} \)-isomorphisms with \( \phi = \phi_1 \), \( (x) \epsilon_{S} = \phi_n \), and with either \( \phi_i \uparrow \phi_{i+1} \) or \( \phi_i \downarrow \phi_{i+1} \) for all \( i \) with \( 1 \leq i < n \). Among all \( (\phi, P, Q) \) with \( \phi \neq (x) \epsilon_{P,Q} \) and \( (x) \epsilon_{S} \equiv \phi \), choose \( (\phi, P, Q) \) so that the length of such a chain \( \sigma \) is as small as possible. Set \( \psi = \phi_2 \).

Then \( \psi = (x) \epsilon_{X,Y} \), where \( X \) and \( Y \) are objects of \( \mathcal{T} \) with \( X^x = Y \). Suppose \( \phi \uparrow \psi \).

Applying the functor \( \rho \) to the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(x) \epsilon_{X,Y}} & Y \\
\downarrow \iota_{P,X} & & \downarrow \iota_{Q,Y} \\
P & \xrightarrow{\phi} & Q \\
\end{array}
\]

and applying condition (B) in the definition of transporter system to \( \rho(\epsilon_{X,Y}(g)) \), we conclude that \( \rho(\phi) \) is the restriction of \( c_0 \) to the homomorphism \( \rho(\phi) : P \to Q \). In particular, we get \( P^x = Q \), so that also \( (x) \epsilon_{P,Q} \) is a restriction of \( (x) \epsilon_{X,Y} \). Then \( \phi = (x) \epsilon_{P,Q} \) by A.13(d), and contrary to hypothesis. On the other hand, if \( \phi \downarrow \psi \), then \( \phi = (x) \epsilon_{P,Q} \) by A.16(a), again contrary to hypothesis. This completes the proof of (a), and then (b) is the special case of (a) given by \( x = 1 \).

Let \( f = [\phi] \) be an equivalence class, with \( \phi \) maximal in \( f \). One checks (by reversing pairs of arrows in the appropriate diagrams) that if \( \psi \) is a \( \mathcal{T} \)-isomorphism, and \( \psi \) is a restriction of \( \phi \), then the \( \mathcal{T} \)-isomorphism \( \psi^{-1} \) is a restriction of \( \phi^{-1} \). Point (c) follows from this observation. \( \square \)

**Remark.** In view of A.19(a) there will be no harm in writing \( x \) to denote the equivalence class \([x] \epsilon_S\), for \( x \in S \). That is to say that from now on we shall identify \( S \) with the image of \( S \) under the composition of \( \epsilon_S \) with the projection \( Iso(\mathcal{T}) \to \mathcal{L} \).
Lemma A.20. Let $\phi : Z \to W$ be a $\mathcal{T}$-isomorphism, maximal in its $\equiv$-class. Let $X$ and $Y$ be objects of $\mathcal{T}$ contained in $Z$, and let $U$ and $V$ be the images of $X$ and $Y$, respectively, under $\rho(\phi)$. Suppose that there exist elements $x$ and $x'$ in $S$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\phi|_{X \to U}} & U \\
\downarrow_{(x)\epsilon_{X,Y}} & & \downarrow_{(x')\epsilon_{U,V}} \\
Y & \xrightarrow{\phi|_{Y \to V}} & V
\end{array}
$$

(*)

Then $x \in Z$, and $x'$ is the image of $x$ under $\rho(\phi)$.

Proof. Let $\phi'$ be the composition (in right-hand notation)

$$
\phi' = (x^{-1})\epsilon_{Zx} \circ \phi \circ (x')\epsilon_{Wx'}.
$$

Thus, $\phi' \in Iso_\mathcal{T}(Z^x, W^{x'})$, and the commutativity of (*) yields $\phi \equiv \phi'$. The maximality of $\phi : Z \to W$ implies that $Z^x \subseteq Z$ and $W^{x'} \subseteq W$. That is, $x \in N_S(Z)$ and $x' \in N_S(W)$. There is then a commutative diagram as follows.

$$
\begin{array}{ccc}
Z & \xrightarrow{\phi} & W \\
\downarrow_{(x)\epsilon_{Z}} & & \downarrow_{(x')\epsilon_{W}} \\
Z & \xrightarrow{\phi} & W
\end{array}
$$

Condition (II) in the definition of transporter system implies that there is an extension of $\phi$ to a $\mathcal{T}$-isomorphism $\langle Z, x \rangle \to \langle W, x' \rangle$, and the maximality of $\phi$ then yields $x \in Z$ and $x' \in W$. Condition (C) in the definition of transporter system implies that $x'$ is the image under $\rho(\phi)$ of $g$. \qed

Corollary A.21. Let $f \in \mathcal{L}$ and let $P \in \Delta$ with the property that, for all $x \in P$, $(f^{-1}, x, f) \in \mathcal{D}$ and $\Pi(f^{-1}, x, f) \in S$. Let $Q$ be the set of all such products $\Pi(f^{-1}, x, f)$. Then $Q \in \Delta$ and there exists $\psi \in f$ such that $\psi \in Iso_\mathcal{T}(P, Q)$.

Proof. As $(f^{-1}, x, f) \in \mathcal{D}$ there exist $U, X, Y, V \in \Delta$ and representatives $\psi$ and $\overline{\psi}$ of $f$ such that

$$
\begin{array}{c}
U \xrightarrow{\overline{\psi}^{-1}} X \xrightarrow{(x)\epsilon_{X,Y}} Y \xrightarrow{\psi} V
\end{array}
$$

is a chain of $\mathcal{T}$-isomorphisms, and where the middle arrow in the diagram is given by A.19(a). As $\Pi(f^{-1}, x, f) \in S$ there exists $x' \in S$ such that $\overline{\psi}^{-1} \circ (x)\epsilon_{X,Y} \circ \psi = (x')\epsilon_{U,V}$. Let $\phi : Z \to W$ be the maximal element of $f$. Then A.20 implies that $x \in Z$, and $x'$ is the image of $x$ under $\rho(\phi)$. In particular, we have $P \leq Z$ and $Q \leq W$, and we may therefore take $X = Y = P$ and $U = V = Q$, obtaining $\psi \in Iso_\mathcal{T}(P, Q)$. \qed
Lemma A.22. Let $\psi : P \to Q$ be a $\mathcal{T}$-isomorphism, and let $f = [\psi]$ be the equivalence class of $\psi$. Then $(f^{-1}, x, f) \in D$ for all $x \in P$, and $P^f = Q$ in the partial group $\mathcal{L}$. Moreover, the conjugation map $(f^{-1}, x, f) \mapsto \Pi(f^{-1}, x, f)$ is equal to $\rho(\psi)$.

Proof. For any $x \in P$, we have the composable sequence

$$Q \xrightarrow{\psi^{-1}} P \xrightarrow{(x)\epsilon_P} P \xrightarrow{\psi} Q$$

of $\mathcal{T}$-isomorphisms, so $(f^{-1}, x, f)$ is in $D$. By Condition (C) in the definition of transporter system then yields $\psi^{-1} \circ (x)\epsilon_P \circ \psi = (x')\epsilon_Q$, where $x' = (x)(\rho(\psi)) \in Q$. The class $[(x')\epsilon_Q]$ is the same as $[(x')\epsilon_S]$ by A.19(a); and we recall that we have introduced the convention to denote this class simply as $x'$. Thus $x^f = x'$, and so $P^f \subseteq Q$. Similarly $Q^{f^{-1}} \subseteq P$, from which one deduces that the conjugation map $c_f : P \to Q$ is surjective. Injectivity of $c_f$ follows from left and right cancellation in the partial group $\mathcal{L}$, so $P^f = Q$. The final assertion of the lemma is given by the observation, made above, that $x' = (x)(\rho(\psi))$. □

Theorem A.23. Let $(\mathcal{T}_\Delta(S) \xrightarrow{\xi} \mathcal{T} \xrightarrow{\rho} \mathcal{F})$ be a p-local compact group, and let $\mathcal{L} = \text{Iso}(\mathcal{T})/\equiv$ be the partial group given by A.19. For $x \in S$, identify $x$ with the $\equiv$-class of the $\mathcal{T}$-isomorphism $(x)\epsilon_S$ of $S$. Then $(\mathcal{L}, \Delta, S)$ is a compact locality on $\mathcal{F}$.

Proof. We have seen in A.19 that $\mathcal{L}$ is a partial group, and the remark following A.19 shows how to identify $S$ with its image under the composition of $\epsilon_S$ with the quotient map $\text{Iso}(\mathcal{T}) \to \mathcal{L}$. In order to show that $(\mathcal{L}, \Delta)$ is objective, begin with $w = (f_1, \cdots, f_n) \in D$. By definition, there exist representatives $\psi_i$ of the classes $\xi_i$ and a sequence $(P_0, \cdots, P_n)$ of objects in $\Delta$, such that each $\psi_i$ is a $\mathcal{T}$-isomorphism $P_{i-1} \to P_i$. Then $P_{i-1}^{f_i} = P_i$ for all $i$, by A.22. Conversely, given $w = (f_1, \cdots, f_n) \in \mathcal{W}(\mathcal{L})$, and given $(P_0, \cdots, P_n) \in \mathcal{W}(\Delta)$ with $P_{i-1}^{f_i} = P_i$ for all $i$, it follows from A.21 that $w \in D$. Thus, $(\mathcal{L}, \Delta)$ satisfies the condition (O1) in the definition I.2.1 of objective partial group. Since $\Delta = \mathcal{F}^c$ is $\mathcal{F}$-closed we also have $w \in D$, and thus $(\mathcal{L}, \Delta)$ is objective. As $\Delta$ is a set of subgroups of $S$, $(\mathcal{L}, \Delta, S)$ is then a pre-locality (as defined in I.2.6).

The conjugation maps $c_g : S_g \to S$ for $g \in \mathcal{L}$ are $\mathcal{F}$-homomorphisms, by A.21 and A.9(C). Thus $\mathcal{F}_S(\mathcal{L})$ is a subsystem of $\mathcal{F}$. Assuming now that $\mathcal{F} \neq \mathcal{F}_S(\mathcal{L})$, there exists an $\mathcal{F}$-isomorphism $\beta : X \to Y$ such that $\beta$ is not an $\mathcal{F}_S(\mathcal{L})$-homomorphism. By A.14(4) we may take $X = X^* \ast$ and $Y = Y^* \ast$, and by the finiteness condition in A.14(1) we may then assume that from among all $\mathcal{F}$-isomorphisms which are not $\mathcal{F}_S(\mathcal{L})$-homomorphisms, $\beta$ has been chosen so that $|X|$ is as large as possible. If $X \in \Delta$ then $Y \in \Delta$ and the surjectivity of $p_{X,Y}$ (condition (A2) in definition A.9) implies that $\beta$ is an $\mathcal{F}_S(\mathcal{L})$-homomorphism, so in fact $X \notin \Delta$. In particular $X < S$, and so $X < N_S(X)$. Similarly $Y < N_S(Y)$.

As $\mathcal{F}$ is order-saturated there exists a fully order-normalized $\mathcal{F}$-conjugate $Z$ of $X$, and there then exist $\mathcal{F}$-homomorphisms $\eta_1 : N_S(X) \to N_S(Z)$ and $\eta_2 : N_S(Y) \to N_S(Z)$ such that $X \eta_1 = Z = Y \eta_2$. Each $\eta_i$ is an $\mathcal{F}_S(\mathcal{L})$-homomorphism by the maximality in the choice of $X$, and it then suffices to show that the $\mathcal{F}$-automorphism $\alpha = \eta_1^{-1} \circ \beta \circ \eta_2$ of $Z$ is an $\mathcal{F}_S(\mathcal{L})$-homomorphism. As $\mathcal{F}$ is order-receptive, $\alpha$ extends to an $\mathcal{F}$-automorphism
\( \pi \) of \( C_S(Z)Z \). But \( C_S(Z)Z \) is centric in \( \mathcal{F} \), so \( C_S(Z)Z \in \Delta \), and \( \pi \) is then an \( \mathcal{F}_S(\mathcal{L}) \)-homomorphism. The same is then true of \( \alpha \), and so we have shown that \( \mathcal{F} = \mathcal{F}_S(\mathcal{L}) \).

Set \( \Gamma = \{ P^* \mid P \leq S \} \). For \( P, Q \in \Gamma \) we have (by A.14(2))

\[
(P \cap Q)^* \leq P \cap Q^* = P \cap Q,
\]

and thus \( \Gamma \) is closed under finite intersections. Let \( g \in \mathcal{L} \) and let \( \psi : P \to Q \) be the unique \( \uparrow \)-maximal representative of \( g \). Then \( P = S_g \) by A.22, and then \( S_g \in \Gamma \) by A.15(c). We may prove by induction on the length of \( w \in \mathcal{W}(\mathcal{L}) \) that \( S_w \in \Gamma \). Namely, write \( w = (g) \circ v \) where \( g \in \mathcal{L} \) and \( v \in \mathcal{W}(\mathcal{L}) \) with \( S_v \in \Gamma \). Then \( S_w = (S_{g^{-1}} \cap S_v)^{g^{-1}} \in \Gamma \) since, as we have seen, \( \Gamma \) is closed with respect to finite intersections and since (by A.14(1)) \( \Gamma \) is \( \mathcal{F} \)-invariant. The finiteness condition in A.14(1) now implies that \( \bigcap \{ S_w \mid w \in X \} \in \Gamma \) for each non-empty subset \( X \) of \( \mathcal{W}(\mathcal{L}) \). This shows that the poset \( \Omega_S(\mathcal{L}) \) defined in I.2.11 is finite-dimensional.

Let \( P \in \Delta \) and let \( \alpha_P : \text{Aut}_T(P) \to N_\mathcal{L}(P) \) be the mapping \( \psi \mapsto [\psi] \). Then \( \alpha_P \) is a homomorphism by A.17(b), \( \alpha_P \) is injective by A.19(c), and \( \alpha_P \) is surjective by A.20. Thus \( \alpha_P \) is an isomorphism, and then condition (I) in definition A.9 implies that \( N_\mathcal{L}(P) \) is virtually \( p \)-toral. As \( \Omega_S(\mathcal{L}) \) is finite-dimensional, all subgroups of \( \mathcal{L} \) are then virtually \( p \)-toral by I.2.17. In particular, if \( \mathcal{S} \) is a \( p \)-subgroup of \( \mathcal{L} \) containing \( S \) then \( \mathcal{S} \) is discrete \( p \)-toral. If \( S < \mathcal{S} \) then A.2(c) yields \( S < N_\mathcal{S}(S) \), which is contrary to condition (I) in A.5. Thus \( S \) is a maximal \( p \)-subgroup of \( \mathcal{L} \), and we have established that \( \mathcal{L} \) is a locality on \( \mathcal{F} \). Notice that A.22 implies that \( \alpha_P \) restricts to an isomorphism \( \text{Ker}(\rho_P) \to C_\mathcal{L}(P) \). As \( (\epsilon, \rho) \) is a \( p \)-local compact group, \( \text{Ker}(\rho_P) \) is a \( p \)-group, and thus \( N_\mathcal{L}(P) \) is of characteristic \( p \). That is, \( \mathcal{L} \) satisfies the condition (PL2) in the definition (6.7) of proper locality. Condition (PL1), that \( \mathcal{F}^{cr} \) be contained in \( \Delta \), is given by \( \Delta = \mathcal{F}^c \). Condition (PL3), that \( S \) has the normalizer-increasing property, is given by A.2(c). Thus \( \mathcal{L} \) is proper, and the proof is complete.

**Theorem A.24.** Let \( (\mathcal{L}, \Delta, S) \) be a compact locality on \( \mathcal{F} \), such that \( \Delta \) is the set \( \mathcal{F}^c \) of \( \mathcal{F} \)-centric subgroups of \( S \). Let \( (\mathcal{T}_\Delta(S) \xrightarrow{\imath} \mathcal{T} \xrightarrow{\pi} \mathcal{F}) \) be the \( \epsilon \)-local compact group constructed from \( \mathcal{L} \) as in A.8 and A.11, and let \( (\mathcal{L}', \Delta, S) \) be the compact locality constructed from \( (\epsilon, \rho) \) as in A.19 and A.23. Then the mapping

\[
\Phi : \mathcal{L} \to \mathcal{L}',
\]

which sends \( g \in \mathcal{L} \) to the \( \equiv \)-class of the \( \mathcal{T} \)-isomorphism \( (g, S_g, S_{g^{-1}}) \) (and with the identifications given by the remark following A.19) is an isomorphism of partial groups which restricts to the identity map on \( S \).

**Proof.** Let \( \Phi^* : \mathcal{W}(\mathcal{L}) \to \mathcal{W}(\mathcal{L}') \) be the mapping induced by \( \Phi \), and let \( w = (g_1, \ldots, g_n) \in \mathcal{D}(\mathcal{L}) \) via \( (P_0, \ldots, P_n) \), with \( P_0 = S_w \). We shall denote the \( \equiv \)-class of a \( \mathcal{T} \)-isomorphism \( (g, P, Q) \) by \( [g, P, Q] \). Then \( w\Phi^* = ([g_1, P_0, P_1], \ldots, [g_n, P_{n-1}, P_n]) \), and \( w\Phi^* \in \mathcal{D}(\mathcal{L}') \) via \( (P_0, \ldots, P_n) \) by A.22. The definition of the product \( \Pi' \) in \( \mathcal{L}' \) then yields

\[
\Pi'(w\Phi^*) = [\pi(w), P_0, P_n] = (\Pi(w))\Phi,
\]
and thus \( \Phi \) is a homomorphism of partial groups.

Recall that for \( P, P' \in \Delta \) with \( P \leq P' \), we have \( \iota_{P, P'} = (1, P, P') \). It follows that the extensions of a \( T \)-isomorphism \( (f, P, Q) \) are of the form \( (f, P', Q') \), and this implies that \( \Phi \) is injective. Since \( L' = \text{Im}(\Phi) \) by definition, \( \Phi \) is a bijection, and we now leave it to the reader to verify that \( \Phi^{-1} \) is a homomorphism. For \( x \in S \) we have identified \( x \) with \([x, S, S]\), so \( \Phi \) restricts to the identity map on \( S \).

**Theorem A.25.** Let \((L, \Delta, S)\) and \((L', \Delta, S)\) be compact localities on \( F \), having the same set of objects. Then there exists an isomorphism \( \alpha : L \to L' \) of partial groups, such that \( P\alpha = P \) for all \( P \in \Delta \cap F^c \). In particular, \( \alpha \) restricts to an automorphism of \( S \).

**Proof.** By Theorem A1 we may assume without loss of generality that \( \Delta \) is equal to the set \( F^c \) of \( F \)-centric subgroups of \( S \). Let \((T_\Delta(S) \xrightarrow{\xi} T \xrightarrow{\rho} F)\) be the \( p \)-local compact group constructed from \((L, \Delta, S)\) via A.8 and A.11, and let \((T_\Delta(S) \xrightarrow{\xi'} T' \xrightarrow{\rho'} F)\) be the \( p \)-local compact group similarly constructed from \((L', \Delta, S)\).

Following [BLO2] (but with notation which reflects our preference for right-hand composition of morphisms), we define the **orbit category** \( \mathcal{O} = \mathcal{O}^c(F) \) to be the category whose set of objects is \( \Delta \), with \( \text{Mor}_{\mathcal{O}}(P, Q) = \text{Hom}_F(P, Q)/\text{Inn}(Q) \).

That is, the \( \mathcal{O} \)-morphisms \( P \to Q \) are the sets \([\phi] = \{ \phi \circ c_x \mid x \in Q \} \), where \( \phi \) is an \( F \)-homomorphism \( P \to Q \). If also \( \psi : Q \to R \) is an \( F \)-homomorphism then one has the well-defined composition \([\phi] \circ [\psi] = [\phi \circ \psi] \).

There is a contravariant functor
\[
\mathcal{Z} : \mathcal{O}^{\text{op}} \to \text{Ab}
\]
(where \( \text{Ab} \) is the category of abelian groups), given by \( \mathcal{Z}(P) = Z(P) \) on objects, and defined in the following way on \( \mathcal{O} \)-morphisms. If \( \phi : P \to Q \) is an \( F \)-homomorphism (with \( P \) and \( Q \) centric in \( F \)), then \( \mathcal{Z} \) sends the \( \mathcal{O} \)-homomorphism \([\phi]\) to the homomorphism \( Z(Q) \to Z(P) \) obtained as the composition of the inclusion map \( Z(Q) \to Z(P\phi) \) followed by the map \( \phi_0^{-1} : Z(P\phi) \to Z(P) \), where \( \phi_0 \) is the \( F \)-isomorphism \( P \to P\phi \) induced by \( \phi \). The main result of [LL] is: If \( F \) is saturated then the higher limit functors \( \lim_k^{\leftarrow}(Z) \) are trivial for all \( k \geq 2 \).

There is a direct analogy with the theory of group extensions, which establishes that the vanishing of \( \lim^3(-)(Z) \) implies the existence of a \( p \)-local compact group \((T_\Delta(S) \xrightarrow{\xi} T \xrightarrow{\rho} F)\). Such a \( p \)-local compact group can be viewed as an “extension” of \( \mathcal{O} \), in the sense that there is a functor
\[
T \xrightarrow{\sigma} \mathcal{O},
\]
such that $\sigma$ induces the identity map $\text{Ob}(\mathcal{T}) \to \text{Ob}(\mathcal{O})$ (i.e. the identity map on $\Delta$), and such that the image of a $\mathcal{T}$-morphism $\psi : P \to Q$ under $\sigma$ is equal to $[\phi]$, where $\phi$ is the $\mathcal{F}$-homomorphism $(\psi)\rho$. The vanishing of $\lim_\leftarrow^2(\mathbb{Z})$ yields the uniqueness of this extension, up to isomorphism. That is, if $(\mathcal{T}_\Delta(S) \xrightarrow{\iota'} \mathcal{T}' \xrightarrow{\varphi'} \mathcal{F})$ is another $p$-local compact group, then there is an isomorphism $\beta : \mathcal{T} \to \mathcal{T}'$ of categories, such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\sigma} & \mathcal{O} \\
\downarrow{\beta} & & \downarrow{\quad} \\
\mathcal{T}' & \xrightarrow{\sigma'} & \mathcal{O}
\end{array}
\]

(and where $\sigma'$ is the functor defined by obvious analogy with $\sigma$).

It is immediate from the commutativity of the diagram that $\beta$ is the identity map on objects. Then for each $P \in \Delta$, $\beta$ restricts to a group isomorphism $\beta_P : \text{Aut}_{\mathcal{T}}(P) \to \text{Aut}_{\mathcal{T}'}(P)$.

In particular, $\beta$ maps the identity element $\iota_P$ to $\iota'_P$, where $\iota_P = (1)\epsilon_P$ and where $\iota'_P = (1)\epsilon'$. Now let $P \leq Q$ in $\Delta$. Then $\iota_{P,Q}$ is the unique (by A.13(d)) $\mathcal{T}$-morphism $\gamma$ having the property that $\iota_P \circ \gamma = \iota_Q$. Similarly, letting $\iota'_{P,Q}$ denote the image of $1$ under $\epsilon_{P,Q}$, then $\iota'_{P,Q}$ is the unique $\mathcal{T}'$-morphism $\gamma'$ such that $\iota'_P \circ \gamma' = \iota'_Q$. Thus $(\iota_{P,Q})\beta = \iota'_{P,Q}$. Now let $\psi : P \to Q$ and $\bar{\psi} : \overline{P} \to \overline{Q}$ be $\mathcal{T}$-isomorphisms such that $\psi \uparrow \overline{\psi}$. That it, assume that $\iota_{P,\overline{P}} \circ \overline{\psi} = \psi \circ \iota_{Q,\overline{Q}}$. Applying $\beta$ then yields $(\psi)\beta \uparrow (\overline{\psi})\beta$, and thus $\beta$ induces a mapping $\alpha : \mathcal{L} \to \mathcal{L}'$ (on equivalence classes of isomorphisms). The product $\Pi(g_1, \cdots, g_n)$ in $\mathcal{L}$ is the equivalence class of a composite of a sequence of representatives for $(g_1, \cdots, g_n)$, so $\alpha$ is a homomorphism of partial groups. As $\beta$ is invertible, $\alpha$ is then an isomorphism, as required. $\square$

**Corollary A.26.** Let $\mathcal{F}$ be a fusion system on the discrete $p$-toral group $S$. Then the following conditions are equivalent.

1. $\mathcal{F}$ is order-saturated (as defined in A.5).
2. $\mathcal{F}$ is saturated (as defined in 8.2), and $\text{Out}_\mathcal{F}(P)$ is finite for every subgroup $P \leq S$.
3. $\mathcal{F}$ is the fusion system of a compact locality.

**Proof.** If (3) holds then also (1) and (2) hold, by A.7. Thus it now suffices to show that (1) implies (3). Assume (1). By [LL] (and [M]) there exists a $p$-local compact group $(\mathcal{T}, \epsilon, \rho)$ over $\mathcal{F}$, and A.23 then yields (3). $\square$

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