Is a Plasma Diamagnetic?

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Abstract

Classical plasmas in thermodynamic equilibrium should be neither para- nor diamagnetic due to the action of the Lorentz force. Magnetic confinement, however, is based on the observed diamagnetism of laboratory plasmas. The apparent paradox is investigated on the basis of the resistive magnetohydrodynamic equations. It is found that, at least in simple plasma configurations, these equations do not permit a solution, i.e. the paradox cannot be resolved. It seems that the Lorentz force is a test-particle approximation which is not suitable to describe the interaction of moving particles in agreement with the conservation of energy.

Résumé

Les plasmas classiques en équilibre thermodynamique ne devraient être ni para- ni diamagnétiques à cause de l’action de la force de Lorentz. Pourtant le confinement magnétique des plasmas en laboratoire est fondé sur leur diamagnétisme observé. Le paradoxe est exploré sur la base des équations de la théorie magnéto-hydrodynamique résistive. On trouve que ces équations ne permettent pas une solution unique, en tout cas pour des configurations simples; c’est à dire qu’il n’est pas possible de résoudre le paradoxe. Il semble que la force de Lorentz représente seulement une approximation pour des particules singulières. Si l’on veut observer la conservation d’énergie, il faut admettre que cette force n’est pas appropriée pour décrire exactement l’interaction des particules en mouvement.

Keywords:

1. magnetized plasmas
2. plasma diamagnetism and paramagnetism
3. Bohr-Van Leeuwen theorem
4. magnetohydrodynamics, ideal and resistive
5. magnetic plasma confinement, ideal and slowly diffusing equilibrium
6. theta pinch
7. magnetic interaction of moving particles
8. Lorentz force, Maxwell’s equations, and conservation of energy

I Introduction

Feynman claims in his lectures [1] that both dia- and paramagnetism are exclusively quantum mechanical effects. His argument is of a very general nature: Since the classical Lorentz force is perpendicular to the velocity of a charged particle, the energy \( \frac{mv^2}{2} \) of the particle does not depend on the magnetic field. Two boxes with the same number of particles and the same temperature must consequently contain the same energy even if one of the boxes is placed in a magnetic field. If the particles would alter the magnetic field, for example diamagnetically, the magnetic field energy would
change and the total energy could not be the same. For this reason neither para- nor diamagnetism can arise as long as it is only the Lorentz force which acts upon the particles.

On the other hand, hot fusion plasmas are governed by the Lorentz force alone, since quantum-mechanical effects are negligible. Because of the rotational direction of the gyrating particles these plasmas behave clearly diamagnetically which is the reason why they can be confined by magnetic fields. In tokamaks, for example, one observes magnetic field changes, when the confined plasma is heated by external sources. The change of the toroidal magnetic flux is regularly monitored with a so-called ‘diamagnetic loop’ [2]. It allows to determine the energy content of the plasma [3] and yields information that can be utilized to control the position of the plasma in the vacuum vessel [4].

It has been pointed out that an ideally confined plasma is not in any contact with material walls and, therefore, not in complete thermodynamic equilibrium. This seems to explain why diamagnetism may occur classically in special circumstances. A. Schlüter [5] quoting N. Bohr shows how the diamagnetism disappears in a plasma which is placed in a homogeneous field and surrounded by reflecting walls (Fig. 1).

![Figure 1 Gyration of particles in a box with reflecting walls (homogeneous field)](image)

Each gyrating particle constitutes a magnetic moment which by superposition would result in a magnetic field opposing the external field (diamagnetism). There is, however, an additional opposite magnetic moment created by the particles which are reflected at the walls such that the net effect is zero. This example confirms Feynman’s conjecture, but it is not a generally valid demonstration.

A counter-example may be produced by considering an inhomogeneous field: A straight wire carries a current and is surrounded by a toroidal vessel containing a plasma (Fig. 2).
In this case – in addition to the magnetic moment of the gyrating particles – we have also a current density in the plasma volume parallel to the external current which is due to the particle drift in the inhomogeneous field. This current will alter the externally applied field so that the field energy is changed by the presence of the particles. As a consequence the energetic state depends on whether the vessel is being placed in the inhomogeneous field or not, in contrast to Feynman’s general conclusion.

In this paper we analyze the apparent paradox by applying the ideal and resistive magneto-hydrodynamic (mhd) equations to simple plasma configurations, but we find ourselves unable to remove the contradiction. It turns out that there is an intrinsic inconsistency between Lorentz force, Maxwell's equations, and energy conservation. This leads us to the conclusion that the Lorentz force is a test-particle approximation which ignores the back-reaction on the field-producing magnet. In most instances this is justified to a high degree of accuracy, as the test-particle interacts with typically $10^{23}$ field-producing particles. When it comes to a plasma, however, the field produced by the gyrating particles cannot be neglected any longer and the Lorentz force turns out to be insufficient to describe the interaction of the particles in motion.

II The Resistive and Ideal Magneto-Hydrodynamic Model of a Plasma

The mhd-equations are derived from the Boltzmann-equation applied to an ionized gas which is subject to the action of the Lorentz force. Derivations are found in many text-books. Quoting from [6] we have from the momentum balance of electrons and ions in a fully ionized hydrogen plasma:

\[ \vec{j} \times \vec{B} = \nabla p + m_i n \frac{d\vec{v}}{dt} \]

\[ \vec{E} + \vec{v} \times \vec{B} = \frac{1}{en} \left( \vec{j} \times \vec{B} \right) - \frac{\nabla p_e}{en} + \eta \vec{j} \]

Here we have put $n = n_i = n_e$ and omitted terms of the order of the mass ratio $m_e/m_i$. Furthermore, heat conduction and viscosity are neglected. For sufficiently slow processes Ampère’s law holds in
the form:

\[ \text{rot } \vec{B} = \mu_0 \vec{j} \]  

(3)

Faraday’s law of induction is:

\[ \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  

(4)

We need the equation of continuity for the particle density:

\[ \text{div } (n\vec{v}) = -\frac{\partial n}{\partial t} \]  

(5)

and the equation for the internal energy:

\[ \frac{f}{2} \frac{dp}{dt} + \frac{f + 2}{2} p \text{div } \vec{v} = S + \eta \vec{j}^2 \]  

(6)

with \( p = n(T_i + T_e) \) and \( f = 3 \) for a plasma with three degrees of freedom. \( S \) is a power density which is added to the plasma, e.g., by electromagnetic radiation. Equation (6) does not contain the magnetic field which is a consequence of the Lorentz force acting presumably on the particles. Apart from the Joule heating term the equation is the same as the one for an ideal gas in the absence of a magnetic field.

If the electron temperature is sufficiently high, the terms in (2) and (6) containing the resistivity \( \eta \), which accounts for the momentum exchange between electrons and ions, may be dropped. The resulting system of equations is the model of ideal mhd being valid on a time-scale short compared to the electron-ion collision time.

III A Linear Theta-Pinch Heated by Radiation

We apply the ideal mhd-equations with \( \eta = 0 \) to a linear Theta-Pinch in equilibrium, the external field of which is produced by a superconducting coil (Fig. 3).
The straight field lines are parallel to the $z$-axis. From $11$ and $13$ follows for the internal and external field components:

$$p(r) + \frac{B_i^2(r)}{2\mu_0} = \frac{B_e^2}{2\mu_0}$$  \hspace{1cm} (7)

We assume that at time $t = 0$ the plasma is heated by switching on a radiation source so that the pressure is increased. Because of $(7)$ the magnetic field must change and the plasma radius $a$ defined by $p(a) = 0$ may be displaced. As long as the radiation source is sufficiently weak, the kinetic energy of the plasma motion is negligible compared to the thermal energy and $(7)$ still holds during the expansion of the plasma. The total magnetic flux inside the coil remains unchanged because of $(4)$ as the electric field vanishes at the surface of the superconductor:

$$\int_0^a \frac{\partial B_e}{\partial t} r \, dr + \int_a^b \frac{\partial B_e}{\partial t} r \, dr = 0$$  \hspace{1cm} (8)

We insert $(2)$ into $(11)$:

$$\frac{\partial}{\partial r} (r v B_i) + \frac{\partial B_e}{\partial t} r = 0$$  \hspace{1cm} (9)

where $v$ denotes the radial component of the plasma velocity, and integrate from $0$ to $a$. Together with $(8)$ we obtain an equation for the change of the external field due to the expansion velocity of the plasma edge:

$$\frac{1}{B_e} \frac{dB_e}{dt} = \frac{1}{b^2 - a^2} \frac{da^2}{dt}$$  \hspace{1cm} (10)

An equation for the divergence of the velocity field follows from $(11)$:

$$v \frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} = \frac{2 S}{f} - f + 2 \frac{p \partial (r v)}{r \partial r}$$  \hspace{1cm} (11)

by elimination of the time derivative of the pressure with $(7)$ and $(9)$:

$$\frac{1}{r} \frac{\partial}{\partial r} (r v) = \frac{2 \mu_0 S - f B_e (dB_e)/dt}{f B_e^2 - (f - 2) \mu_0 p}$$  \hspace{1cm} (12)

Integration from $0$ to $a$ results in a second equation for the boundary velocity

$$\frac{da^2}{dt} = \int_0^a \frac{4 \mu_0 S - f (dB_e^2)/dt}{f B_e^2 - (f - 2) \mu_0 p} r \, dr$$  \hspace{1cm} (13)

which yields together with $(10)$ an equation for the change of the external field due to the applied heating power:

$$\frac{dB_e^2}{dt} \left( b^2 - a^2 + \int_0^a \frac{2 f r \, dr}{f B_e^2 - (f - 2) \mu_0 p} \right) = \int_0^a \frac{8 \mu_0 S r \, dr}{f B_e^2 - (f - 2) \mu_0 p}$$  \hspace{1cm} (14)

The task is now to solve $(11)$ with the velocity as given by $(12)$ inside a moving boundary as described by $(10)$ and $(13)$. The boundary conditions are:

$$[\partial p/\partial r]_{r=0} = 0 , \quad p(a) = 0 , \quad v(0) = 0 .$$

As initial condition we may choose an arbitrary pressure profile $p(r, 0)$. If $p(a) = 0$ is to hold at all times, we must require that the heating source $S$ vanishes at the plasma boundary. For the sake of simplicity we choose

$$S(r, t) = \alpha p(r, t)$$  \hspace{1cm} (15)

where $\alpha$ is a constant.
We introduce dimensionless variables:

\[ r^2 = x a^2(\tau), \quad 0 \leq x \leq 1, \quad t = \frac{f}{2\alpha} \tau \]  

(16)

\[ p = \frac{B^2}{2\mu_0} \frac{\beta}{1 + \delta \beta}, \quad \delta = \frac{f - 2}{2f}, \quad v = \frac{\alpha a^2}{r f} \left( u(x, \tau) + \frac{x}{a^2} \frac{da^2}{d\tau} \right) \]

The transformation rules are:

\[ \frac{1}{r} \frac{\partial}{\partial r} = 2 \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = \frac{2\alpha}{f} \left( \frac{\partial}{\partial \tau} - \frac{x}{a^2} \frac{da^2}{d\tau} \frac{\partial}{\partial x} \right) \]  

(17)

Equations (11) and (12) transform into:

\[ u \frac{\partial \beta}{\partial x} + \frac{\partial \beta}{\partial \tau} = \beta (1 - (1 - \delta) \beta) (1 + \delta \beta) \left( 1 - \frac{2\delta dB_e}{B_e d\tau} \right) \]  

(18)

\[ \frac{\partial u}{\partial x} = \beta \left( \frac{1}{2} - \frac{\delta dB_e}{B_e d\tau} \right) - \frac{1}{a^2 B_e} \frac{d (a^2 B_e)}{d\tau} \]  

(19)

The boundary conditions following from (18) and (16) are:

\[ \frac{d\beta (0, \tau)}{d\tau} = \beta (0, \tau) \left[ 1 - (1 - \delta) \beta (0, \tau) \right] \left[ 1 + \delta \beta (0, \tau) \right] \left( 1 - \frac{2\delta dB_e}{B_e d\tau} \right) \]  

(20)

\[ \beta (1, \tau) = 0, \quad u (0, \tau) = u (1, \tau) = 0 \]

We combine (18) and (19) into a single equation by taking \( u \) as the independent variable instead of \( x \):

\[ u \frac{\partial \beta}{\partial u} + \frac{\partial \beta}{\partial \tau} = \beta (1 - (1 - \delta) \beta) (1 + \delta \beta) \left( 1 - \frac{2\delta dB_e}{B_e d\tau} \right) \]  

(21)

This is possible, since (19) is an ordinary differential equation which does not depend on \( x \) explicitly.

Upon substitution of (19) equation (21) is an inhomogeneous quasilinear equation of first order which has the characteristic system:

\[ \frac{1}{\beta} \frac{d\beta}{d\tau} = \beta (1 - (1 - \delta) \beta) (1 + \delta \beta) \left( 1 - \frac{2\delta dB_e}{B_e d\tau} \right) \]  

(22)

\[ \frac{1}{u} \frac{du}{d\tau} = \beta \left( \frac{1}{2} - \frac{\delta dB_e}{B_e d\tau} \right) - \frac{1}{a^2 B_e} \frac{d (a^2 B_e)}{d\tau} \]

The initial condition on \( u \) results from (19) and (20):

\[ u(x, 0) = \left( \frac{1}{2} - \frac{\delta}{B_e (0)} \right) \left[ \int_0^x \beta(x, 0) \, dx - x \int_0^1 \beta(x, 0) \, dx \right] \]  

(23)

By solving the system of ordinary differential equations (22) we obtain a relationship between \( u, \beta, \) and \( \tau \) when we eliminate the initial profiles \( u(x, 0) \) and \( \beta(x, 0) \) with (23). It may be inserted into (19) in order to express \( u \) and \( \beta \) as functions of \( x \) and \( \tau \) by further integration.

It turns out, however, that the solution of (22) does not satisfy the boundary conditions (20) in general. In order to demonstrate this we choose for simplicity the special case \( \delta = 0 \) corresponding to \( f = 2 \). It will become obvious that the difficulty remains in the more physical case \( f = 3 \). As an initial \( \beta \)-profile we take \( \beta(x, 0) = 1 - x^2 \). The initial \( u \)-profile becomes with (23):

\[ u(x, 0) = \frac{x}{2} - \frac{1}{B_e (0)} \left[ \int_0^x \beta(x, 0) \, dx - x \int_0^1 \beta(x, 0) \, dx \right] \]  

(24)

By elimination of \( x \) we have:

\[ u(x, 0) = \frac{1}{6} \beta(x, 0) \sqrt{1 - \beta(x, 0)} \]
Integration of (22) yields:

\[
\frac{\beta}{1 - \beta} = \frac{\beta(x, 0) e^\tau}{1 - \beta(x, 0)}
\]

\[u = u(x, 0) \frac{g(0)}{6g} \sqrt{1 - \beta(x, 0)(1 - e^\tau)}, \quad g = a^2 B_e
\]

Elimination of the initial profiles from (24 - 26) leads to the result:

\[u = g(0) \frac{\beta \sqrt{1 - \beta} e^{-\tau}}{(1 - \beta(1 - e^{-\tau}))^2}
\]

In order to determine the time evolution of \(g\), we differentiate (24) with respect to \(x\) and evaluate it at \(\beta(1, \tau) = 0\) with (19) substituted:

\[
\frac{\partial \beta}{\partial x} = - \frac{6 e^\tau}{g(0)} \frac{dg}{d\tau}
\]

We may also differentiate (18) with respect to \(x\) and evaluate it at \(x = 1\) imposing (20):

\[
\frac{\partial^2 \beta}{\partial x \partial \tau} = \frac{\partial \beta}{\partial x} \left(1 + \frac{1}{g(0)} \frac{dg}{d\tau}\right), \quad x = 1
\]

Integration with respect to time yields:

\[
\frac{\partial \beta}{\partial x} = - \frac{6 e^\tau}{g(0)} \frac{dg}{d\tau}
\]

Elimination of the slope of \(\beta\) at the boundary from (30) and (28) yields a differential equation for \(g\):

\[
\frac{1}{g} \frac{dg}{d\tau} = - \frac{1}{6} \left[ \frac{\partial \beta}{\partial x} \right]_{x=1}
\]

from which \(g(\tau)\) may be determined.

If we insert, however, \(31\) into \(19\) and integrate from the axis to the boundary we find with \(20\):

\[
\int_0^1 \beta dx = - \frac{1}{3} \left[ \frac{\partial \beta}{\partial x} \right]_{x=1} = - \frac{2}{3}
\]

Obviously, this integral equation is only satisfied at \(\tau = 0\). At later times \(\beta\) evolves according to (18) from the initial profile to \(\beta(x, \infty) \to 1\), so that \(32\) cannot hold at all times.

In view of this result we come to the conclusion that the set of equations (1 - 6) has no solution in general which would satisfy the boundary conditions. As the problem of heating a Theta-Pinch plasma in equilibrium is physically well posed, it must have a solution in reality. Evidently nature “uses equations” which are different from those formulated in (1 - 6).

IV A Slowly Diffusing Theta-Pinch Equilibrium

Inclusion of finite resistivity removes the conservation of flux inside the plasma, but it does not remedy the situation. Starting from an equilibrium (7) with a function \(p(r)\) the plasma should slowly diffuse and ultimately fill the entire volume inside the coil. Spitzer [6] gives an expression for the diffusion velocity:

\[
\vec{v}_{Dq} = - \frac{\eta \nabla p}{B^2}
\]

which is easily derived from (1) and (2) under the assumption of the magnetic field staying constant in time. He rightly remarks that this condition is only satisfied in the test-particle approximation \(\mu_0 p / B^2 \to 0\), but, nevertheless, he claims one paragraph below that (33) is of general validity restricted only by the exclusion of inertial terms.
Let us assume that there is no heating source $S$, but field energy is dissipated into internal energy by the diffusion process. Equation (34) remains unchanged as long as the plasma diffusion occurs inside a superconducting coil. We derive the total power balance by taking the scalar product of (4) with $\mathbf{j}$ and by eliminating the triple product with (1):

$$\mathbf{E} \cdot \mathbf{j} = \eta \mathbf{j}_x^2 + \mathbf{v} \cdot \nabla p \quad \text{(34)}$$

By comparison with (6) we find:

$$\frac{f}{2} \frac{\partial p}{\partial t} + \frac{f + 2}{2} \text{div} (p \mathbf{v}) = \mathbf{E} \cdot \mathbf{j} \quad \text{(35)}$$

Introducing the Poynting vector with (3) and (4) we have:

$$\frac{f}{2} \frac{\partial p}{\partial t} + \frac{f + 2}{2} \text{div} (p \mathbf{v}) + \frac{1}{\mu_0} \frac{\partial B^2}{\partial t} + \frac{1}{2\mu_0} \frac{\partial B_x^2}{\partial t} = 0 \quad \text{(36)}$$

Integration over the plasma volume up to the coil radius using Gauss’ theorem yields:

$$\int_0^a \left( \frac{f}{2} \frac{\partial p}{\partial t} + \frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} \right) r \, dr + \int_a^b \left( \frac{1}{2\mu_0} \frac{\partial B_x^2}{\partial t} \right) r \, dr = 0 \quad \text{(37)}$$

as the surface integrals arising from the divergence terms vanish. For simplicity we have omitted the term with the kinetic energy $m_i n v^2/2$ as the diffusion velocity is very small compared to the thermal speed. Furthermore, in accordance with (34) we have neglected the electrostatic field energy arising through the electric field component $E_x = (\partial p_i / \partial r) / e n$, which provides the confinement of the ions. This contribution is negligibly small in laboratory plasmas compared to thermal and magnetic field energy. The heating term $\eta \mathbf{j}_x^2$ does not appear in (37) explicitly. The increase in internal energy must come from a decrease of the field energy as the system is energetically closed by the condition $B = 0$ at the superconducting surface.

We insert (2) into (4):

$$\frac{\partial}{\partial r} (r v B_i + r \eta \mathbf{j}) + \frac{\partial B_i}{\partial t} = 0 \quad \text{(38)}$$

This equation must be solved together with the local power balance (4):

$$\frac{f}{2} \left( \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial r} \right) = \eta \mathbf{j}_x^2 - \frac{f + 2}{2} \frac{1}{r} \frac{\partial (r v)}{\partial r} \quad \text{(39)}$$

The solutions for the pressure and the internal magnetic field must satisfy the force balance (7). We write the velocity as the sum of Spitzer’s diffusion velocity (33) and a term accounting for the effect of a finite pressure which is not included in (33):

$$r v = - \frac{r \eta \mathbf{j}}{B_i} + r v_p = u_S + u_p \quad \text{(40)}$$

With the abbreviation $x = r^2$ equation (34) reads:

$$\frac{\partial}{\partial x} (u_p B_i) + \frac{1}{2} \frac{\partial B_i}{\partial x} = 0 \quad \text{(41)}$$

and (39) becomes together with (4):

$$\left( u_S + \frac{f}{f + 2} u_p \right) \frac{\partial p}{\partial x} + \frac{f}{(f + 2)} \frac{\partial p}{\partial t} + p \left( \frac{\partial u_S}{\partial x} + \frac{\partial u_p}{\partial x} \right) = 0 \quad \text{(42)}$$

Substituting the internal field with (7) into (41) yields a second differential equation for the pressure:

$$u_p \frac{\partial p}{\partial x} + \frac{1}{2} \frac{\partial p}{\partial t} + 2p \frac{\partial u_p}{\partial x} = \frac{B_x^2}{2\mu_0} \frac{\partial u_p}{\partial x} + \frac{1}{4\mu_0} \frac{dB_x^2}{dt} \quad \text{(43)}$$
It is quite obvious that (42) and (43) cannot lead to the same solution. The characteristic system of the inhomogeneous first order equation (42) is:

\[
\frac{dx}{u_S + f u_p/(f + 2)} - \frac{2}{f} \frac{dt}{p(\partial u_S/\partial x + \partial u_p/\partial x)} = \frac{dp}{f + 2}
\]

and that of (43):

\[
\frac{dx}{u_p} = 2 \frac{dt}{(B_e^2 - 2\mu_0 p) \partial u_p/\partial x + (dB_e^2)/(4 dt)}
\]

Suppose a solution \( p(x, t) \) exists and is known. One may then calculate the velocity \( u_p(x, t) \) by eliminating the time derivative of the pressure from (42) and (43):

\[
u_p = -\int_0^x \left( \frac{f + 2}{f B_e^2 - (f - 2) \mu_0 p} \right) dx
\]

Now the characteristic systems (44) and (45) may be integrated. The four resulting families of characteristics must lie entirely in the surface \( p(x, t) \). This is, however, not possible, unless at least \( u_S(x, t) = 0 \) which is excluded at finite resistivity.

The so called 'slowly diffusing equilibrium', which is intuitively expected in the case of finite resistivity, is not obtainable from the resistive mhd-equations when the pressure dependent term omitted by Spitzer in (33) is included. Only in the test-particle case \( \mu_0 p / B^2 \to 0 \) an approximate diffusion velocity may be obtained from (33) at constant magnetic field. In general, the momentum balance (1, 2) and the power balance (6) are inconsistent in conjunction with Maxwell’s equations (3, 4). Again, we must conclude that nature uses a different set of equations or, more properly speaking, at least one of the laws of nature as codified in (1 - 6) must be incomplete. It should be noted that the discrepancy encountered cannot be resolved by inclusion of heat conduction. Its dependence on temperature is different from that of the resistivity so that a cancellation of terms is generally not possible.

V Discussion and Conclusion

The mathematical model of resistive mhd (1 - 6) is, of course, not an exact description of reality. It leaves out not only a number of well known effects such as heat conduction, viscosity, thermo electricity, gravity, but it neglects also the finite mass of electrons, the difference of electron and ion density, the effects of quantum mechanics etc. The idealizations involved are common practice in the mathematical modelling of physical reality, but this should lead only to minor deviations of the predictions from the observations on basic features. In hydrodynamics similar approximations are made, but the results as derivable from the model equations are in reasonable agreement with observations.

In magneto-hydrodynamics the situation is different according to our analysis: The mathematical model does not permit a prediction in principle, as the equations are internally inconsistent and do not yield unique solutions. This is not acceptable even for an idealized model and one must find out the reason. The derivation of the hierarchy of equations follows the same principles as in hydrodynamics so that an inconsistency is not to be expected at first sight. If it arises nevertheless, it must have to do with an inconsistency in the basic interaction law between individual particles as it is described by Lorentz force and Maxwell’s equations.

The validity of Maxwell’s equations has been sufficiently confirmed and cannot be put into doubt in the present context. This is also true for the energy principle. The correctness of the Lorentz force, when applied to individual particles, seems also to be sufficiently verified. It is, however, practically impossible in these experiments to measure the back-reaction of the orbiting particles on the field producing magnet. We can, therefore, not exclude that a term in the elementary force law, which either cancels or is negligible in test-particle experiments, has escaped the attention. At least one observation raises doubts:

The angular momentum vector of a negatively charged particle gyrating in a homogeneous field is parallel to the field vector. If many particles in a circular conductor rotate in the same direction, they form a current which produces a magnetic moment such that the total field at the center of the
loop is decreased. In this case the conductor is unstable as it has a tendency to turn around an axis which lies in its plane when a perturbation occurs. This indicates that the particles in the conductor are in a higher energetic state than they would be, if the current of negative particles would flow opposite to their natural direction of gyration. It is, of course, well known that a current loop in a magnetic field has in addition to its self-energy an extra potential energy which is the negative scalar product of its magnetic moment with the field.

The Lorentz force, however, does not predict a difference in the energetic state of an individual particle regardless whether it rotates clockwise or counter-clockwise in a magnetic field. Suppose a charged particle is attached at the periphery of a rotatable disk, similar to ‘Feynman’s paradox’ [1]. As the Lorentz force points towards or away from the axis of the disk, no extra work is necessary to reverse the direction of rotation so that the particle’s energy – in contrast to the particles in a current loop – is independent of the sense of rotation. In the Appendix we show explicitly how the Lorentz force is at variance with the conservation of energy when a single particle interacts with a superconducting magnet.

The comparison of a gyrating particle with a current loop seems to point to an inconsistency which is probably at the root of the discrepancy which we have found when the Lorentz force is applied to a plasma. Because of the large amount of particles involved, we need to know the correct force law describing their interaction, not only a test-particle approximation in an external field. The back-reaction on the field producing magnet, which, at sufficiently high pressure, is the plasma itself, cannot be neglected any longer. If the Lorentz force would be complemented by a suitable term making the energetic state of a particle dependent on whether it is in a magnetic field or not, the equation of the internal energy (6) would be altered and, hopefully, the discrepancy could be removed.

Appendix
A charged particle is attached at the periphery of a rotatable disk (Fig. 4) which is turned by a motor.

![Figure 4 Interaction of a charged particle with a superconducting magnet](image)

In a concentric superconducting ring flows a current which produces a magnetic field perpendicular...
to the plane of the disk. The particle’s equation of motion is:

\[ m \frac{d\vec{v}}{dt} = \vec{F}_m + q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]  

(A.1)

The first term is the motor force acting on the particle via the mechanical fixture, the second term is the Lorentz force. The power balance is obtained by taking the scalar product of (A.1) with the velocity:

\[ m \frac{d\vec{v}^2}{2 dt} = \vec{v} \cdot \vec{F}_m + q \vec{v} \cdot \vec{E} \]  

(A.2)

An electric field at the position of the particle is present when the current in the ring changes in time. We derive its tangential component from the vector potential of the superconductor:

\[ E_{s\phi} = -\frac{\partial A_{s\phi}}{\partial t} = -\frac{\mu_0}{4\pi} \frac{dI}{dt} \int_0^{2\pi} \frac{b \cos \varphi d\varphi}{(b^2 + a^2 - 2ab \cos \varphi)^{\frac{3}{2}}} \]  

(A.3)

where \(a\) and \(b\) are the radii of the particle orbit and the superconductor, respectively.

The particle produces also a vector potential:

\[ \vec{A}_p = \frac{\mu_0 q}{4\pi} \frac{\vec{v}}{|\vec{x} - \vec{x}'|} \]  

(A.4)

and an electric field when it is accelerated:

\[ \vec{E}_p = -\frac{\partial \vec{A}_p}{\partial t} = -\frac{\mu_0 q}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \frac{d\vec{v}}{dt} \]  

(A.5)

The tangential component of this field is:

\[ E_{p\phi} = -\frac{\mu_0 q}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \left( -\frac{dv_x}{dt} \sin \varphi + \frac{dv_y}{dt} \cos \varphi \right) \]  

(A.6)

where \(\vec{x}'\) denotes the position of the particle. As it is compelled to move in tangential direction we have:

\[ E_{p\phi} = -\frac{\mu_0 q}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \left( \frac{dv_x}{dt} \cos (\varphi - \varphi') + v_x \frac{d\varphi'}{dt} \sin (\varphi - \varphi') \right) \]  

(A.7)

At the position of the superconducting ring the tangential field component of the particle is:

\[ E_{p\phi} = -\frac{\mu_0 q}{4\pi} \frac{dv_x}{dt} \frac{\cos (\varphi - \varphi') + v_x \frac{d\varphi'}{dt} \sin (\varphi - \varphi')}{(a^2 + b^2 - 2ab \cos (\varphi - \varphi'))^{\frac{3}{2}}} \]  

(A.8)

Integration over the angle \(\varphi\) from 0 to \(2\pi\) yields the loop voltage:

\[ U_p = -\frac{\mu_0 q}{4\pi} \int_0^{2\pi} \frac{dv_x}{dt} \cos \alpha + v_x \frac{d\varphi'}{dt} \sin \alpha}{(a^2 + b^2 - 2ab \cos \alpha)^{\frac{3}{2}}} \]  

\[ b d\alpha , \quad \alpha = \varphi - \varphi' \]  

(A.9)

In the limit \(a \ll b\) one obtains:

\[ U_p \simeq -\frac{\mu_0 q}{4\pi} \frac{dv_x}{dt} \frac{a}{b} \]  

(A.10)

Obviously, the particle works on the superconductor which carries a current:

\[ U_p I = -\frac{\mu_0 q}{4\pi} \frac{dv_x}{dt} \frac{a}{b} I \]  

(A.11)

Its energy is increased or decreased depending on the direction of rotation:

\[ W_{ps} = -\frac{\mu_0 q}{4\pi} \frac{a}{b} \frac{d\varphi}{dt} I \]  

(A.12)
The superconductor cannot sustain an electric field and must consequently compensate the applied voltage to zero by changing its current:

\[ U_p = L \frac{dI}{dt} \]  

(A.13)

\( L \) is its coefficient of self-induction which may be calculated by inserting into (A.13) the self-induced voltage \( 2\pi b \frac{\partial A}{\partial t} \):

\[
2\pi b \frac{\mu_0}{4\pi} \frac{dI}{dt} \left[ \int_0^{2\pi} \frac{\cos \varphi \, b \, d\varphi}{(r^2 + b^2 - 2r b \cos \varphi)^{3/2}} \right]_{r \to b} = L \frac{dI}{dt} \]  

(A.14)

or:

\[
L = \frac{\mu_0 b^2}{2} \left[ \int_0^{2\pi} \frac{\cos \varphi \, d\varphi}{(r^2 + b^2 - 2r b \cos \varphi)^{3/2}} \right]_{r \to b} \]  

(A.15)

\( L \) is proportional to the radius \( b \). The factor of proportionality depends on the cross-section of the current ring and diverges logarithmically for a thin filament.

Inserting now (A.10) into (A.13) we find the connection between acceleration of the particle and current change in the superconductor:

\[
\frac{dI}{dt} = -\frac{\mu_0 q a}{4bL} \frac{dv_\varphi}{dt} \]  

(A.16)

Substituting this into (A.3) we obtain the electric field at the position of the particle:

\[
E_{s\varphi} = -\frac{\mu_0}{4b} \frac{dI}{dt} \int_0^{2\pi} \frac{\cos \varphi \, b \, d\varphi}{(a^2 + b^2 - 2a b \cos \varphi)^{3/2}} \]  

\[
\ll \mu_0 \frac{q a \, dv_\varphi}{4bL} \frac{\mu_0 a}{4b}, \quad \text{for } a \ll b \]  

(A.17)

Inserting this into the power balance (A.2) of the motor yields finally:

\[
\vec{\dot{v}} \cdot \vec{F}_m = \frac{m}{2} \frac{dv_\varphi^2}{dt} - \left( \frac{\mu_0 q a}{4b} \right)^2 \frac{1}{2L} \frac{dv_\varphi^2}{dt} \]  

(A.18)

The work done by the motor is apparently independent of the direction of rotation. To achieve, however, the same kinetic energy of the particle, more energy is necessary when there is no superconductor present. The additional term arising from the Lorentz force is independent of the magnetic field and diminishes proportional to \( b^{-3} \) because of (A.15).

The interaction energy \( W_{ps} \) (A.12) which depends on the direction of rotation and on the magnetic field is not accounted for in (A.18). It corresponds quantitatively to the potential energy \( -\vec{\mu} \cdot \vec{B} \) of a magnetic moment in a magnetic field: The average current of the particle on its orbit multiplied with the enclosed area is \( qa \, v_\varphi/2 = |\vec{\mu}| \) and the magnetic field at the position of the particle as obtainable from \( \vec{B} = \text{rot} \, \vec{A} \) is \( \mu_0 I/2b \). One would expect that the motor must supply or gain this extra work, but the Lorentz force does not allow for it.

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Figure captions

Figure 1 Gyration of particles in a box with reflecting walls (homogeneous field)
Figure 2 Gyration of particles in a box with reflecting walls (inhomogeneous field)
Figure 3 Field and pressure distribution in a Theta-Pinch
Figure 4 Interaction of a charged particle with a superconducting magnet