The digit exchanges in the rotational beta expansions of algebraic numbers

Hajime Kaneko* and Makoto Kawashima

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Abstract

In this article, we investigate the $\beta$-expansions of real algebraic numbers. In particular, we give new lower bounds for the number of digit exchanges in the case where $\beta$ is a Pisot or Salem number. Moreover, we define a new class of algebraic numbers, quasi-Pisot numbers and quasi-Salem numbers, which gives a generalization of Pisot numbers and Salem numbers.

Our method is applicable also to the digit expansions of complex algebraic numbers, which gives a new estimate. In particular, we investigate the digits of rotational beta expansion considered by Akiyama and Caalim [3] and zeta-expansion by Surer [21], where the base is a quasi-Pisot or quasi-Salem number.

1 Introduction

Let $\beta > 1$ be a real number. In [20], Rényi introduced the representations of real numbers in base $\beta$, so called $\beta$-expansions. Little is known on the digits of $\beta$-expansions of algebraic numbers. For instance, if $\beta = b \geq 2$ is an integer, then the $\beta$-expansion coincides with the usual base-$b$ expansion. Borel [3] conjectured that all algebraic irrational numbers are normal numbers in base-$b$. However, if $b \geq 3$, then it is still unknown whether the digit 1 appears infinitely many times in the base-$b$ expansions of algebraic irrational numbers. In this article, we investigate the complexity of the digit expansions of real and complex algebraic numbers. In particular, we consider the digits of $\beta$-expansions in the case where $\beta$ is a Pisot or Salem number. We now recall the definition of Pisot and Salem numbers. Let $\beta$ be an algebraic

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integer. We call $\beta$ a Pisot number (resp. Salem number) if its conjugates over $\mathbb{Q}$, except $\beta$ itself have moduli less than 1 (resp. if its conjugates over $\mathbb{Q}$, except $\beta$ itself have absolute values not greater than 1 and there exists a conjugate of $\beta$ with absolute value 1).

We introduce the notation throughout this article as follows. We denote the set of nonnegative integers (resp. positive integers) by $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{N}$). We denote the integral and fractional parts of a real number $x$ by $[x]$ and $\{x\}$, respectively. We denote by $[x]$ the minimal integer not less than $x$ and use the Landau symbol $O$ and the Vinogradov symbols $\ll, \gg$ with their usual meaning. We denote the algebraic closure of the rational number field by $\overline{\mathbb{Q}}$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For an algebraic number $\beta$, we denote the conjugates of $\beta$ by $\beta_i$ for $1 \leq i \leq |\mathbb{Q}(\beta) : \mathbb{Q}|$ with $\beta_1 = \beta$. We assume that $\beta_2$ is the complex conjugate of $\beta$ if $\beta \notin \mathbb{R}$. Moreover, let $\mathbb{Z}(\subset \overline{\mathbb{Q}})$ be the set of algebraic integers.

For an algebraic number $\alpha$, we denote by $\alpha[0]$ the value $\max_\sigma |\sigma(\alpha)|$, where $\sigma$ runs throughout the embeddings of $\mathbb{Q}(\alpha)$ to $\mathbb{C}$.

Let $\beta > 1$ be a real number. The $\beta$-transformation $T_\beta : [0, 1] \rightarrow [0, 1)$ is defined by

$$T_\beta(x) := \{\beta x\},$$

for $x \in [0, 1]$. Let $\xi$ be a real number with $0 \leq \xi \leq 1$. If $\beta = b \in \mathbb{Z}$, we also assume $\xi < 1$. For $n \in \mathbb{N}$, we put $t_n(\beta; \xi) := [\beta T_\beta^{n-1}(\xi)]$. Then we have $t_n(\beta; \xi) \in \mathbb{Z} \cap [0, \beta)$. The $\beta$-expansion of $\xi$ is defined by

$$\xi = \sum_{n=1}^{\infty} t_n(\beta; \xi)\beta^{-n}.$$}

In the case where $\xi$ is a general nonnegative real number, then using a suitable integer $R \geq 0$ with $\beta^{-R} \in [0, 1)$, we define the $\beta$-expansion of $\xi$ by

$$\xi = \sum_{n=1}^{\infty} t_n(\beta; \xi)\beta^{-n} := \beta^R \sum_{n=1}^{\infty} t_n(\beta; \beta^{-R}\xi)\beta^{-n}.$$

Note that the choice of $R$ is not unique (although its choice does not affect the subsequent digit asymptotics). For a positive integer $N$, the number of digit exchanges $\gamma(\beta, \xi; N)$ and the number of nonzero digits $\nu(\beta, \xi; N)$ are defined by

$$\gamma(\beta, \xi; N) := \text{Card}\{n \in \mathbb{N} | n \leq N, t_{n}(\beta; \xi) \neq t_{n+1}(\beta; \xi)\},$$

$$\nu(\beta, \xi; N) := \text{Card}\{n \in \mathbb{N} | n \leq N, t_{n}(\beta; \xi) \neq 0\},$$

respectively, where Card denotes the cardinality. It is easily seen that we have the following relations among $\gamma(\beta, \xi; N)$ and $\nu(\beta, \xi; N)$:

$$\nu(\beta, \xi; N) \geq \frac{1}{2} \gamma(\beta, \xi; N) + o(1).$$

Rényi [20] showed for any $\beta > 1$ that there exists a unique $T_\beta$-invariant measure $p_\beta$ on $[0, 1]$ which is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$. In particular, $p_\beta$ is ergodic. We recall the $\beta$-normality of $\xi \in [0, 1]$.

Let $S := \mathbb{Z} \cap [0, \beta)$. Let $1 \leq k < \ell$. For $\xi \in [0, 1]$, we define the finite word $w_{k,\ell}(\beta; \xi)$ by

$$w_{k,\ell}(\beta; \xi) := t_k(\beta; \xi)t_{k+1}(\beta; \xi)\cdots t_\ell(\beta; \xi),$$

for any $\xi \in [0, 1]$. For any word $v = v_1 \cdots v_k$ of length $k$, we define the cylinder set $[v]$ by

$$[v]_\beta := \{\xi \in [0, 1] | w_{1,k}(\beta; \xi) = v\}.$$
A word $v$ is called admissible if $[v]_{\beta}$ is not empty. Recall that $\xi \in [0,1]$ is $\beta$-normal if

$$\lim_{N \to \infty} \frac{1}{N} \text{Card}\{n \leq N \mid w_{n,n+k-1}(\beta;\xi) = v\} = p_{\beta}([v]_{\beta}).$$

for any admissible finite word $v$ of arbitrary length $k$. Adamczewski and Bugeaud \cite{1} introduced a hypothesis on $\beta$-normality as follows: Let $\beta > 1$ and $\xi \in [0,1]$ be algebraic numbers. Then $\xi$ is $\beta$-normal or $\xi$ has ultimately periodic $\beta$-expansion.

Suppose that $\xi$ is $\beta$-normal. Then the sequences $(N^{-1}\gamma(\beta,\xi;N))_{N\geq1}$ and $(N^{-1}\nu(\beta,\xi;N))_{N\geq1}$ converge to positive values. The lower bounds for the number of digit exchanges of algebraic numbers were studied in \cite{6} \cite{7} \cite{8} \cite{14} \cite{15}, which gives partial results on the $\beta$-normality of algebraic numbers. In particular, Bugeaud \cite{7} proved the following: Let $\beta$ be a Pisot or Salem number and $\xi$ an algebraic number with $t_n(\beta;\xi) \neq t_{n+1}(\beta;\xi)$ for infinitely many $n$. Then there exist effectively computable positive numbers $C_1(\beta,\xi)$ and $C_2(\beta,\xi)$, depending only on $\beta$ and $\xi$, such that

$$(3) \quad \gamma(\beta,\xi;N) \geq C_1(\beta,\xi) \frac{(\log N)^{3/2}}{(\log \log N)^{1/2}},$$

for any $N \geq C_2(\beta,\xi)$. In particular, combining \cite{2} and \cite{3}, we have

$$(4) \quad \nu(\beta,\xi;N) \geq \frac{C_1(\beta,\xi)}{3} \frac{(\log N)^{3/2}}{(\log \log N)^{1/2}},$$

for any sufficiently large $N$. Lower bound \cite{4} was improved in \cite{16} and \cite{17} as follows:

**Theorem 1.1.** \cite{17} Theorem 2.2 \cite{10} Let $\beta$ be a Pisot or Salem number and $\xi$ an algebraic number with $[Q(\beta,\xi) : Q(\beta)] = D$. Suppose there exists a sequence $t = (t_n)_{n \in \mathbb{Z}_{\geq1}}$ of integers satisfying the following two assumptions:

(i) There exists a positive integer $B$ such that, for any $n \in \mathbb{Z}_{\geq1}$,

$$0 \leq t_n \leq B.$$

Moreover, there exist infinitely many $n$ such that $t_n > 0$.

(ii) We have

$$\xi = \sum_{n=1}^{\infty} t_n \beta^{-n}.$$

Then there exist effectively computable positive constants $C_3 = C_3(\beta,\xi,B)$ and $C_4 = C_4(\beta,\xi,B)$, depending only on $\beta, \xi$ and $B$, such that, for any integer $N$ with $N \geq C_4$,

$$\lambda(\Gamma(t;N)) \geq C_3 \frac{N^{1/D}}{(\log N)^{1/D}},$$

where $\Gamma(t) := \{n \in \mathbb{Z}_{\geq1} \mid t_n \neq 0\}$ and $\lambda(\Gamma(t;N)) := \text{Card}([1,N] \cap \Gamma(t))$.

We note that the theorem above is also applicable to general representations of algebraic real numbers $\xi$ by infinite series in base-$\beta$.

It is natural to conjecture that a counterpart of Theorem \cite{17} holds also for the number of digit exchanges in beta expansion. Consider the case where $\beta = b$ is a integer greater than 1. If the minimal polynomial of algebraic irrational $\xi$ satisfies certain assumptions, then it is known for any sufficiently large $N$ that $\gamma(\beta,\xi;N) \gg N^{1/d}$, where $d = [Q(\xi) : Q]$ \cite{14} and \cite{15}.

The main results of this article give a counterpart of Theorem 1.1 for more general Pisot and Salem numbers $\beta$. Moreover, our method is also applicable to a broader class of algebraic numbers, that we
call quasi-Pisot numbers and quasi-Salem numbers, which we define in Section 2. Thus, our main results also give new lower bounds for the number of digit exchanges and the number of nonzero digits of more general numerical representation. In fact, we also consider asymptotic behaviour of the digits in negative beta expansion and rotational beta expansion in Section 3. We prove our main results in Section 4.

2 Main results

To state our main results, we introduce quasi-Pisot and quasi-Salem numbers as follows: For a complex number $z$, we denote its complex conjugate by $\overline{z}$. Let $\beta$ be an algebraic integer with $|\beta| > 1$. We say $\beta$ is a quasi-Pisot number (resp. quasi-Salem number) if $|\beta_i| < 1$ for any $\beta_i \notin \{\beta, \overline{\beta}\}$ (resp. $|\beta_i| \leq 1$ for any $\beta_i \notin \{\beta, \overline{\beta}\}$) and there exists $1 \leq j \leq |\mathbb{Q}(\beta) : \mathbb{Q}|$ satisfying $|\beta_j| = 1$. For instance, any rational integer $b$ with $|b| \geq 2$ is a quasi-Pisot number. Any quadratic algebraic integer $\beta$ with $|\beta| > 1$ and $\beta \notin \mathbb{R}$ is a quasi-Pisot number. If $\beta$ is a negative real number such that $-\beta$ is a Pisot number (resp. Salem number), then $\beta$ is a quasi-Pisot number (resp. quasi-Salem number). See also example 3.3 for another example of quasi-Pisot numbers. For examples of complex quasi-Pisot and quasi-Salem numbers, see [9] and [12] Tables 6.3 and 6.4. For instance, two zeros $\beta, \overline{\beta}$ of $X^8 - X^7 + X^6 - X^4 + X^2 - X + 1$ with $|\beta| > 1$ are quasi-Salem numbers.

We give lower bounds for the digit exchanges in the representations of complex algebraic numbers by infinite series in base-$\beta$ in the case where $\beta$ is a quasi-Pisot or quasi-Salem number.

Theorem 2.1. Let $\beta$ be a quasi-Pisot or quasi-Salem number and $\xi$ an algebraic number with $D = [\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)]$. Let $S$ be a finite subset of $\mathbb{Z}[\beta]$ with $0 \in S$. Moreover, if $S \notin \mathbb{Z}$, then suppose that $\beta \notin \mathbb{R}$ and there exists an imaginary quadratic algebraic integer $\alpha \in \mathbb{Q}(\beta)$ such that $S$ is a finite subset of the ring of integers of $\mathbb{Q}(\alpha)$.

Let $R \geq 0$ and $t = (t_n)_{n \geq 1-R}$ be a sequence of elements of $S$ satisfying $\xi = \sum_{n=1-R}^{\infty} t_n \beta^{-n}$. Assume there exist $\pi, A_0, A_1, \ldots, A_D \in \mathbb{Z}[\beta]$ with $\pi \neq 0, A_D \neq 0$ satisfying the following:

(i) $A_D \xi^D + A_{D-1} \xi^{D-1} + \cdots + A_0 = 0$,

(ii) $\frac{(\beta - 1)^D \pi A_0}{\pi} \in \mathbb{Z}[\beta]$ for $1 \leq k \leq D$,

(iii) $\frac{(\beta - 1)^D \beta^n A_0}{\pi} \notin \mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$.

Then there exist effectively computable positive numbers $C_5$ and $C_6$ such that

$$\gamma(t; N) := \text{Card}\{n \in \mathbb{N} | n \leq N, t_n \neq t_{n+1}\} \geq C_5 \left( \frac{N}{\log N} \right)^{1/D},$$

for all $N \geq C_6$.

Remark 2.2. Let $\xi$ be an algebraic number such that $\sum_{n=0}^{D} A_n \xi^n = 0$, where $A_0, \ldots, A_D \in \mathbb{Z}[\beta]$ and $A_D \neq 0$. Assume that (i), (ii), and (iii) in Theorem 2.1 holds with some $\pi \in \mathbb{Z}[\beta] \setminus \{0\}$. Let $\rho$ be any element in $\mathbb{Z}[\beta]$. Then $\xi + \rho$ also satisfies the same assumptions. In fact, putting $P(X) = \sum_{n=0}^{D} A_n X^n := \sum_{n=0}^{D} A_n (X - \rho)^n$, we see that $P(\xi + \rho) = 0$. Moreover, $\pi, \overline{A_0}, \ldots, \overline{A_D}$ fulfill (i), (ii), and (iii). Hence, it suffices to prove Theorem 2.1 in the case of $R = 0$, by considering $\xi + \rho$ with $\rho = -\sum_{n=1-R}^{0} t_n \beta^{-n} \in \mathbb{Z}[\beta]$ when $R \geq 1$. 
Remark 2.3. Suppose that the assumption on Theorem 2.1 holds and \( R = 0 \). Then we have \( \operatorname{Card}\{n \in \mathbb{N} \mid t_n \neq t_{n+1}\} = \infty \). Suppose on the contrary that we have \( \operatorname{Card}\{n \in \mathbb{N} \mid t_n \neq t_{n+1}\} < \infty \). Then there exist \( t \in S \) and \( N_1 \in \mathbb{Z}_{\geq 0} \) satisfying \( t_n = t \) for all \( n > N_1 \). Then we have

\[
\xi = \sum_{n=1}^{N_1} t_n \beta^{-n} + \sum_{n=N_1+1}^{\infty} t \beta^{-n} = \sum_{n=1}^{N_1} t_n \beta^{-n} + \frac{t \beta^{-N_1}}{\beta - 1}.
\]

By equality (5), we have \( D = [\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = 1 \). Then by assumptions (i) and (ii), we obtain \( A_0 = -A_1 \xi \in \mathbb{Z}[\beta], \quad \frac{A_1}{\pi} \in \mathbb{Z}[\beta] \).

Combining (5) and the above relations, we have

\[
\frac{(\beta - 1)\beta^{N_1} A_0}{\pi} = -\frac{A_1}{\pi} \cdot (\beta - 1)\beta^{N_1} \xi
\]

\[
= -\frac{A_1}{\pi} \cdot (\beta - 1)\beta^{N_1} \left( \sum_{n=1}^{N_1} t_n \beta^{-n} + \frac{t \beta^{-N_1}}{\beta - 1} \right) \in \mathbb{Z}[\beta],
\]

which contradicts assumption (iii).

In the theorem below, we treat the case where \( \eta \) is a complex number of the form \( \sum_{n=0}^{\infty} s_n \beta^{-n} \) for the technical reason of the proof.

Theorem 2.4. Let \( \beta \) be a quasi-Pisot or quasi-Salem number and \( \eta \) an algebraic number with \( D = [\mathbb{Q}(\beta, \eta) : \mathbb{Q}(\beta)] \). Let \( S \) be a finite subset of \( \mathbb{Z}[\beta] \) with \( 0 \in S \). Moreover, if \( S \not\subset \mathbb{Z} \), then suppose that \( \beta \not\in \mathbb{R} \) and there exists an imaginary quadratic algebraic integer \( \alpha \in \mathbb{Q}(\beta) \) such that \( S \) is a finite subset of the ring of integers of \( \mathbb{Q}(\alpha) \).

Let \( s = (s_n)_{n \geq 0} \) be a sequence of elements of \( S \) satisfying \( \eta = \sum_{n=0}^{\infty} s_n \beta^{-n} \). Assume there exist \( B_0 \in \mathbb{Q}(\beta) \) and \( B_1, \ldots, B_D \in \mathbb{Z}[\beta] \) with \( B_D \neq 0 \) satisfying the following:

\[
\sum_{k=0}^{D} B_k \eta^k = 0,
\]

\[
B_0 \beta^n \notin \mathbb{Z}[\beta] \text{ for all } n \in \mathbb{N}.
\]

Then there exist effectively computable positive numbers \( C_7 \) and \( C_8 \) such that

\[
\lambda(s; N) := \operatorname{Card}\{n \in \mathbb{Z}_{\geq 0} \mid n < N, s_n \neq 0\} \geq C_7 \left( \frac{N}{\log N} \right)^{1/D},
\]

for all \( N \geq C_8 \).

In Example 2.5 and examples in Section 3, the implied constants in the symbol \( \gg \) are positive and effectively computable.

Example 2.5. Let \( \beta \) be a Pisot or Salem number. Let \( p \) be a prime number which is coprime to \( \beta(\beta-1) \) and unramified in \( \mathcal{O}_{\mathbb{Q}(\beta)} \). Let \( D \) be a positive integer. We consider the \( \beta \)-expansion \( \sum_{n=-\infty}^{\infty} t_n(\beta; \xi) \beta^{-n} \) of the number \( \xi := p^{-1/D} \). We see that \( [\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D \). In fact, let \( \mathcal{P} \subset \mathcal{O}_{\mathbb{Q}(\beta)} \) be a prime ideal over \( p \) and \( \mathcal{R} \) the local ring of \( \mathcal{O}_{\mathbb{Q}(\beta)} \) at \( \mathcal{P} \), using Eisenstein irreducibility criterion for polynomials, we get that the polynomial \( X^D - p \) is irreducible in \( \mathbb{Q}(\beta)[X] \). Thus, \( pX^D - 1 \) is irreducible in \( \mathcal{R}[X] \), and
so irreducible in \( \mathbb{Q}(\beta)[X] \) by Gauss’s lemma. By putting \( A_D X^D + \cdots + A_0 := p X^D - 1 \) and \( \pi := p \), assumptions (i), (ii) and (iii) in Theorem 2.4 are satisfied. Thus, we obtain

\[
\gamma(\beta, \xi; N) \gg \left( \frac{N}{\log N} \right)^{1/D},
\]
for any sufficiently large \( N \).

3 Application to negative and rotational beta expansion

Let \((X, \mathcal{B}, \mu, T)\) be an ergodic measure-preserving system on a compact metric space \( X \) with sigma-algebra \( \mathcal{B} \) of Borel sets in \( X \). Then \( x \in X \) is called \( T \)-generic if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f d\mu,
\]
for any continuous function \( f \) on \( X \). If \((X, \mathcal{B}, \mu, T) = ([0, 1], \mathcal{B}, p \beta, T_\beta)\), then \( x \) is \( T_\beta \)-generic if and only if \( x \) is \( \beta \)-normal.

Ito and Sadahiro [13] introduced the negative beta expansions of real numbers, which gives a numeration system where the base is a negative real number. The negative beta expansion is defined in terms of the iteration of the map \( \tilde{T}_\beta : [-\beta/(\beta + 1), 1/(\beta + 1)] \to [-\beta/(\beta + 1), 1/(\beta + 1)] \) defined by

\[
\tilde{T}_\beta(x) = \left\{ -\beta x + \frac{\beta}{\beta + 1} \right\} - \frac{\beta}{\beta + 1},
\]
where \( \beta > 1 \) is a real number. Applying the theorem by Li and Yorke [19], Ito and Sadahiro [13] verified that there exists a unique \( \tilde{T}_\beta \)-invariant measure \( \tilde{p}_\beta \) which is absolutely continuous with respect to the Lebesgue measure, and so \( p_{-\beta} \) is ergodic.

We now introduce a modified negative beta expansion studied by Liao and Steiner [18]. Let \( T_{-\beta} : [0, 1] \to [0, 1] \) be defined by \( T_{-\beta}(x) := 1 - \{\beta x\} \), where \( \tilde{T}_{-\beta} \) is conjugate to \( T_{-\beta} \) through the conjugacy function \( f(x) = (\beta + 1)^{-1} - x \). The \( (-\beta) \)-expansion of \( \xi \in [0, 1] \) is defined as

\[
x = \sum_{n=1}^{\infty} t_n(-\beta; \xi)(-\beta)^{-n},
\]
where \( t_n(-\beta; \xi) = \lfloor \beta T_{-\beta}^{n-1}(\xi) \rfloor + 1 \in \mathbb{Z} \cap [1, 1 + \beta] \). In the case where \( \xi \) is a general real number, using a suitable integer \( R \geq 0 \) with \((-\beta)^{-R} \in [0, 1]\), we define the \( (-\beta) \)-expansion of \( \xi \) in the same way as [11]. Note that the choice of \( R \) is not unique (although its choice does not affect the subsequent digit asymptotics). For more general numeration systems of real numbers related to beta expansion, see for instance [10] [11].

As a counterpart of the hypothesis on \( \beta \)-normality stated in Section 1, it is natural to conjecture that if \( \beta > 1 \) and \( \xi \in \mathbb{R} \) are algebraic numbers, then \( \xi \) is \( T_{-\beta} \)-generic or the \( (-\beta) \)-expansion of \( \xi \) is ultimately periodic. We consider the number of digit exchanges \( \gamma(-\beta, \xi; N) \) defined by

\[
\gamma(-\beta, \xi; N) := \text{Card}\{n \in \mathbb{N} \mid n \leq N, t_n(-\beta; \xi) \neq t_{n+1}(-\beta; \xi)\}.
\]

Example 3.1. Let \( \beta \) be a Pisot or Salem number. Let \( p \) be a prime number which is coprime to \( \beta(\beta - 1) \) and unramified in \( \mathbb{Q}(\beta) \). Let \( D \) be a positive integer. We consider the \( (-\beta) \)-expansion \( \sum_{n=1}^{\infty} t_n(\beta; \xi)\beta^{-n} \) of \( \xi \), where \( \xi \) is a unique zero of the polynomial \( p X^D + p X^{D-1} + \cdots + p X - 1 \) with \( 0 < \xi < 1 \). In the same way as Example 2.5 we see that

\[
\gamma(-\beta, \xi; N) \gg \left( \frac{N}{\log N} \right)^{1/D},
\]
for any sufficiently large \( N \).
Akiyama and Caalim [4] defined a rotational beta expansion, which is a natural generalization of beta expansion for the complex plane. We introduce a special version of this expansion. Let \( \beta \) be a complex number with \( \beta \not\in \mathbb{R} \) and \( |\beta| > 1 \). Let \( \tau_1, \tau_2 \in \mathbb{C}\setminus\{0\} \) with \( \tau_1/\tau_2 \not\in \mathbb{R} \). Denote \( F := \{ x\tau_1 + y\tau_2 \mid x \in [-1/2, 1/2), y \in [-1/2, 1/2) \} \). Let \( \overline{F} \) be the closure of \( F \). Define a map \( T = T_{\beta, \tau_1, \tau_2} : \overline{F} \to F \) by
\[
T(z) := \beta z - \delta(z),
\]
where \( \delta(z) \) is a unique element in \( \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2 \) satisfying \( \beta z - \delta(z) \in F \). We denote by \( S = S(\beta, \tau_1, \tau_2) \) the set of digits \( \delta(z) \) with \( z \in \overline{F} \). Note that \( S \) is a finite set. Then the rotational \( \beta \)-expansion, or simply \( \beta \)-expansion, of \( \xi \in F \) is defined by
\[
\xi = \sum_{n=1}^{\infty} d_n \beta^{-n},
\]
where \( d_n = d_n(\beta, \tau_1, \tau_2; \xi) = \delta(\beta^{n-1}(z)) \in S \). In the case where \( \xi \) is a general complex number, using a suitable nonnegative integer \( R \) with \( \beta^{-R} \xi \in F \), we define the rotational \( \beta \) expansion of \( \xi \) in the same way as (1).

We define the number of digit exchanges in the rotational beta expansion of \( \xi \) by
\[
\gamma(\xi; N) = \gamma(\beta, \tau_1, \tau_2; \xi; N) := \text{Card}\{n \in \mathbb{N} \mid n \leq N, d_n(\beta, \tau_1, \tau_2; \xi) \neq d_{n+1}(\beta, \tau_1, \tau_2; \xi)\}.
\]

Akiyama and Caalim [3] gave a sufficient condition for \( \beta, \tau_1, \tau_2 \) which guarantee the uniqueness of absolutely continuous invariant probability measure \( p_\beta = p_{\beta, \tau_1, \tau_2} \) on \( F \), and \( p_\beta \) is equivalent to the Lebesgue measure on \( F \).

Surer [21] also investigated a numerical system of complex numbers called zeta-expansion. We introduce a special version of this numerical system. Let again \( \beta \) be a complex number with \( \beta \not\in \mathbb{R} \) and \( |\beta| > 1 \). Set \( \tau_1 := 1 \) and \( \tau_2 := -\overline{\beta} \). Then we have \( F = \{ x - y\overline{\beta} \mid x \in [-1/2, 1/2), y \in [-1/2, 1/2) \} \) and \( \beta F = \{ -|\beta|^2 y + \beta x \mid x \in [-1/2, 1/2), y \in [-1/2, 1/2) \} \). It is remarkable that if \( z \in F \), then we have \( \delta(z) \in \mathbb{Z} \) because the imaginary parts of \( -\overline{\beta} \) and \( \beta \) coincide, where \( \delta(z) \) is defined by (9). Then the zeta-expansion of \( \xi \in \mathbb{C} \) is defined by
\[
\xi = \sum_{n=1-R}^{\infty} d_n(\beta, 1, -\overline{\beta}; \xi) \beta^{-n}.
\]

**Example 3.2.** Let \( \beta \) be a quasi-Pisot or quasi-Salem number. Let \( p \) be a prime number which is coprime to \( \beta(\beta-1) \) and unramified in \( \mathbb{Q}(\beta) \). Let \( D \) be a positive integer. We consider the zeta-expansion of \( \xi \), where \( \xi \) is a zero of the polynomial \( pX^D + pX^{D-1} + \cdots + pX - 1 \). In the same way as Example 2.4, we see that
\[
\gamma(\beta, 1, -\overline{\beta}; \xi) \gg \left( \frac{N}{\log N} \right)^{1/D},
\]
for any sufficiently large \( N \) because the digits of zeta-expansions are rational integers.

We give an example of the digit exchanges for a rotational beta expansion whose digits are not generally rational integers.

**Example 3.3.** Let \( \zeta_7 := e^{2\pi i/7} \) be the primitive 7-th root of unity. For an integer \( a \geq 2 \), we put
\[
\xi_a := \zeta_7^{(1-a)/2} \zeta_7^a - 1.
\]
Then we have \( \xi_a = \pm \frac{\sin(\pi a/7)}{\sin(\pi/7)} \in \mathbb{R} \). Define the multiplicative group \( C := \{ \zeta_7^{m_1} \zeta_7^{m_2} \zeta_7^{m_3} \mid m_1, m_2, m_3 \in \mathbb{Z} \} \). Note that \( C \) is called the group of cyclotomic units of \( \mathbb{Q}(\zeta_7) \) and \( C \) is a finite index subgroup of the units group of \( \mathbb{Z}[\zeta_7] \) (see Section 8 in [22]). Let \( m_1, m_2, m_3 \) be positive integers with \( 1 \leq m_1 \leq 6 \) and
\[
m_2 \log(1.247) + m_3 \log(0.554) < 0.
\]
Put $\beta := \xi_1^{m_1} \xi_2^{m_2} \xi_3^{m_3} \in C$. Let $\sigma_i \in \text{Gal}(\mathbb{Q}(\zeta_i)/\mathbb{Q})$ with $\sigma_i(\zeta_i) = \xi_i$ for $1 \leq i \leq 6$. Remark $\sigma_i(\xi_a) = \xi_a \xi_i^{-1}$ and

$$|\xi_2| = 1.8019377358 \ldots, \quad |\xi_3| = 2.24697960372 \ldots,$$

$$|\sigma_2(\xi_2)| = 1.24697960372 \ldots, \quad |\sigma_2(\xi_3)| = 0.55490813208 \ldots,$$

$$|\sigma_3(\xi_2)| = 0.445041867911 \ldots, \quad |\sigma_3(\xi_3)| = 0.8019377358 \ldots.$$

By the above equalities and $(10)$, we obtain

$$|\beta| = |\sigma_6(\beta)| > 1, \quad |\sigma_3(\beta)| = |\sigma_4(\beta)| < |\sigma_2(\beta)| = |\sigma_5(\beta)| < 1.$$

Thus the number $\beta$ is a quasi-Pisot number and $\mathbb{Q}(\zeta_7) = \mathbb{Q}(\beta)$. Using the Legendre symbol $(\alpha)$, we put $\alpha := \sum_{a=1}^7 (\alpha) \zeta_a^2$ (a Gauss sum). Then $\alpha$ is an imaginary quadratic integer. Let $O_{\mathbb{Q}(\alpha)}$ (resp. $O_{\mathbb{Q}(\zeta_7)}$) be the ring of integers of $\mathbb{Q}(\alpha)$ (resp. $\mathbb{Q}(\zeta_7)$). We consider rotational $\beta$-expansion, where $\tau_1, \tau_2$ are elements of $O_{\mathbb{Q}(\alpha)} \cap \mathbb{Z}[\beta]$ with $\tau_1/\tau_2 \notin \mathbb{R}$. Then the set $S$ of the digits in the rotational $\beta$ expansion satisfies the assumptions of Theorem $2.1$

Let $p$ be a prime number which is coprime to $\beta(\beta - 1)$ and unramified in $O_{\mathbb{Q}(\zeta_7)}$. Let $D$ be a positive integer. Then, by the same arguments in Example 2.3, the polynomial $pX^D - 1$ is irreducible in $\mathbb{Q}(\zeta_7)[X]$. Put $\xi := p^{-1/D}, \pi := p, A_D := p, A_{D-1} = \ldots = A_1 = 0$ and $A_0 := -1$. Then the numbers $\xi, \pi, A_D, \ldots, A_0$ satisfy assumptions (i), (ii) and (iii) in Theorem $2.1$. Hence, we obtain that

$$\gamma(\beta, \tau_1, \tau_2; \xi) \gg \left(\frac{N}{\log N}\right)^{1/D},$$

for any sufficiently large $N$.

Akiyama and Caalim [2] introduced a rotational beta expansion in $\mathbb{R}^m$, which is a natural generalization of the rotational expansion in $\mathbb{C}$. The rotational beta expansion in $\mathbb{R}^m$ is defined in terms of a map $T(z) = \beta M z$ for $z \in \mathbb{R}^m$, where $\beta > 1$ is a real number and $M$ is an orthogonal matrix of order $m$. It is a future work to investigate the uniformity of the digits in the rotational beta expansion of elements of $\mathbb{R}^m$.

4 Proof of main results

4.1 Reduction of Theorem 2.1 to Theorem 2.4

By Remark 2.2, we may assume that $R = 0$. Firstly, we reduce Theorem 2.1 to Theorem 2.4. Define the sequence $v = (v(m))_{m \in \mathbb{Z}_{\geq 0}}$ of nonnegative integers by $v(0) = -1 + N_0, \ v(1) < v(2) < \ldots$ and

$$\{n \in \mathbb{N} \mid t_n \neq t_{n+1}\} = \{v(m) \mid m \in \mathbb{N}\}.$$

Denote $t_{v(m)}(= t_{1+v(m)-1})$ by $x(m)$ for any $m \in \mathbb{N}$. Note that $x(1) = t_{N_0} \neq 0$. We see

$$\xi = \sum_{n=1}^{\infty} t_n \beta^{-n} = \sum_{m=1}^{\infty} \sum_{n=1+v(m)-1}^{v(m)} t_n \beta^{-n}$$

$$= \frac{1}{\beta - 1} \left(x(1) \beta^{-v(0)} + \sum_{m=1}^{\infty} (x(m+1) - x(m)) \beta^{-v(m)}\right) = \frac{1}{\beta - 1} \sum_{n=0}^{\infty} s_n \beta^{-n},$$
where the sequence \((s_n)_{n \in \mathbb{Z}_{\geq 0}}\) of integers is defined by
\[
s_n = \begin{cases} 
  x(1) & \text{if } n = v(0), \\
  x(m + 1) - x(m) & \text{if there exists } m \in \mathbb{N} \text{ satisfying } n = v(m), \\
  0 & \text{otherwise.}
\end{cases}
\]

Putting \(\tilde{S} := \{a - b \mid a, b \in S\}\), we see that \(\beta\) and \(\tilde{S}\) satisfy the assumptions of Theorem 2.4. In fact, if \(S \not\subset \mathbb{Z}\), then \(\tilde{S}\) is a finite subset in the ring of integers of \(\mathbb{Q}(\alpha)\), where \(\alpha\) is denoted in the assumption of Theorem 2.1. Note that \((s_n)_{n \in \mathbb{Z}_{\geq 0}}\) is bounded and \(s_n \in \tilde{S}\) for any \(n \geq 0\). Putting \(\eta = (\beta - 1)\xi\) and \(B_k = A_k(\beta - 1)^{D-k}\pi^{-1}\) for \(0 \leq k \leq D\), we get that \(\eta, B_k (0 \leq k \leq D)\) satisfy (i) and (ii) by assumptions (i), (ii) and (iii) of Theorem 2.1. Since \(x(1) \neq 0\), we see \(\{n \in \mathbb{N} \mid s_n \neq 0\} = \{v(0)\} \cup \{n \in \mathbb{N} \mid t_n \neq t_{n+1}\}\), and so we reduced the proof of Theorem 2.4 to the proof of Theorem 2.3.

### 4.2 Preliminaries for the proof of Theorem 2.4

In what follows, the implied constants in the symbols \(\gg, \ll\) and the constants \(C_9, C_{10}, \ldots\) are effectively computable positive ones. If necessarily, changing \(\eta \beta^n\) with suitable nonnegative integer \(N\) by \(\eta\), we may assume that \(s_0 \neq 0\). Let \(\Gamma := \{n \in \mathbb{Z}_{\geq 0} \mid s_n = 0\} \) and \(\lambda(\Gamma; N) := \text{Card}\{n \in \mathbb{Z}_{\geq 0} \mid s_n \neq 0, n < N\} = \lambda(s; N)\) for \(N \in \mathbb{N}\). For a positive integer \(k\), we have
\[
\eta^k = \left(\sum_{n=0}^{\infty} s_n \beta^{-n}\right)^k = \sum_{m=0}^{\infty} \left(\sum_{m_1 + \cdots + m_k = m} s_{m_1} \cdots s_{m_k}\right) \beta^{-m}.
\]

For \(m \in \mathbb{Z}_{\geq 0}\), we denote the complex number \(\sum_{m_1 + \cdots + m_k = m} s_{m_1} \cdots s_{m_k}\) by \(\rho(k; m)\). If \(S \subset \mathbb{Z}\), then put \(O := \mathbb{Z}\). If \(S \not\subset \mathbb{Z}\), then let \(O\) be the ring of integers of \(\mathbb{Q}(\alpha)\). By \(s_m \in S \subset O \cap \mathbb{Z}[\beta]\) for any \(m \geq 0\), we see \(\rho(k; m) \in O \cap \mathbb{Z}[\beta]\) and
\[
|\rho(k; m)| \leq T^k (m+1)^k,
\]
where \(T = \max\{|\alpha| \mid \alpha \in S\}\). Moreover, if \(\rho(k; m) \neq 0\), then
\[
|\rho(k; m)| \geq C_9 := \inf\{|z| \mid z \in O \setminus \{0\}\} > 0,
\]
by \(\rho(k; m) \in O\).

For any integer \(k\) with \(0 \leq k \leq D\), we define the set \(k\Gamma\) of integers by
\[
k\Gamma = \begin{cases} 
  \{0\} & \text{if } k = 0, \\
  \{m_1 + \cdots + m_k \mid m_1, \ldots, m_k \in \Gamma\} & \text{if } 1 \leq k \leq D.
\end{cases}
\]
Note that if \(m \notin k\Gamma\) we have \(\rho(k; m) = 0\). Moreover, \(1\Gamma = \Gamma, 0 \in \Gamma\) and \(0\Gamma \subset \Gamma \subset \cdots \subset D\Gamma\). For a positive integer \(N\), put \(\lambda(k\Gamma; N) = \text{Card}(k\Gamma \cap [0, N])\). By the definition of \(k\Gamma\), we have
\[
\text{Card}\{m \in \mathbb{Z}_{\geq 0} \mid m < N, \rho(k; m) \neq 0\} \leq \lambda(k\Gamma; N) \leq \text{Card}(\Gamma \cap [0, N])^k = \lambda(\Gamma; N)^k.
\]
By the above equality, we shall estimate the lower bounds for \(\text{Card}\{m \in \mathbb{Z}_{\geq 0} \mid m < N, \rho(k; m) \neq 0\}\). Since it is difficult to estimate the lower bounds for them directly, we consider the complex number \(Y_R\) defined by
\[
Y_R = \sum_{k=1}^{D} B_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R),
\]
where \( R \in \mathbb{Z}_{\geq 0} \). We prove that \( Y_R \neq 0 \). By equality (8), we have

\[
0 = \sum_{k=0}^{D} B_k \eta^k = B_0 + \sum_{k=1}^{D} B_k \sum_{m=0}^{\infty} \beta^{-m} \rho(k;m).
\]

Multiplying the above equality by \( \beta^R \), we obtain

\[
0 = B_0 \beta^R + \sum_{k=1}^{D} B_k \sum_{m=0}^{\infty} \beta^{-m+R} \rho(k;m) = B_0 \beta^R + \sum_{k=1}^{D} B_k \sum_{m=-R}^{\infty} \beta^{-m} \rho(k;m+R).
\]

By the above equalities,

\[
Y_R = -B_0 \beta^R - \sum_{k=1}^{D} B_k \sum_{m=-R}^{0} \beta^{-m} \rho(k;m+R).
\]

(13)

In particular, \( Y_R \) is an algebraic number. If \( Y_R = 0 \) then, by (13) and \( B_k \in \mathbb{Z}[^{[\beta]}] \) for \( 1 \leq k \leq D \), we have \( B_0 \beta^R \in \mathbb{Z}[\beta] \), which contradicts (7). Hence, we conclude \( Y_R \neq 0 \).

**Lemma 4.1.** There exist positive integers \( C_{10}, C_{11} \) satisfying \( |Y_R| > R^{-C_{10}} \) for all \( R \geq C_{11} \).

**Proof.** Put \( d = \deg \beta \) and denote the set of embeddings of \( \mathbb{Q}(\beta) \) into \( \mathbb{C} \) by \( \{\sigma_1, \ldots, \sigma_d\} \). We describe \( \sigma_1(x) = x \) for all \( x \in \mathbb{Q}(\beta) \) and denote the complex conjugate of \( \sigma_1 \) by \( \sigma_2 \) if \( \beta \notin \mathbb{R} \).

Let \( 2 \leq i \leq d \) if \( \beta \in \mathbb{R} \) (resp. \( 3 \leq i \leq d \) if \( \beta \notin \mathbb{R} \)). Recall that \( \rho(k;m) \in \mathbb{Z}[\beta] \) for any \( 1 \leq k \leq D \) and \( m \in \mathbb{Z}_{\geq 0} \). By (13), we have

\[
|\sigma_i(Y_R)| \leq |\sigma_i(B_0 \beta^R)| + \sum_{k=1}^{D} |\sigma_i(B_k)| \sum_{m=-R}^{0} |\sigma_i(\beta^{-m})| |\sigma_i(\rho(k;m+R))|
\]

\[
\leq |B_0| + \sum_{k=1}^{D} |\sigma_i(B_k)| \sum_{m=-R}^{0} T^k(R + m + 1)^k
\]

(14)

\( \ll (R + 1)^{D+1} \).

Note that in the above second inequality, we use \( |\sigma_i(\beta)| \leq 1 \) and (11). Take a positive integer \( J \) satisfying \( JB_0 \in \mathbb{Z} \). By equality (13) and \( B_k \in \mathbb{Z}[\beta] \) for \( k = 1, \ldots, D \), we have \( JY_R \in \mathbb{Z} \).

If \( \beta \in \mathbb{R} \), then we have

\[
1 \leq |JY_R| \prod_{i=2}^{d} |\sigma_i(JY_R)| \ll |Y_R|(R + 1)^{(d-1)(D+1)},
\]

by (14). In the case of \( \beta \notin \mathbb{R} \), also using (13), we have

\[
1 \leq |JY_R|^2 \prod_{i=3}^{d} |\sigma_i(JY_R)| \ll |Y_R|^2(R + 1)^{(d-2)(D+1)}.
\]

In both cases, there exist positive integers \( C_{10}, C_{11} \) satisfying \( |Y_R| > R^{-C_{10}} \) for all \( R \geq C_{11} \), which completes the proof of Lemma 4.1. \( \square \)

Let \( N \) be a positive integer. We put

\[
C_{12} = \frac{|B_D|}{2|\beta|} C_9,
\]

and \( y_N := \text{Card}\{R \in \mathbb{Z}_{\geq 0} \mid R < N, |Y_R| \geq C_{12}\} \).
Lemma 4.2. For all sufficiently large integer \( N \), we have

\[ y_N \ll \log N + \lambda(\Gamma; N)^D. \]

Proof. Put \( K = [(D + 1)\log_{|\beta|} N] \), where \( \log_{|\beta|} x = (\log x) / (\log |\beta|) \). Then by the definition of \( y_N \), we have

\[ y_N \leq K + y_{N-K+1} = K + \sum_{0 \leq R \leq N-K} \frac{1}{Y_R} \leq K + \frac{1}{C_{12}} \sum_{R=0}^{N-K} |Y_R|. \]

We estimate the upper bound for \( \sum_{R=0}^{N-K} |Y_R| \). By the definition of \( Y_R \), we obtain

\[ \sum_{R=0}^{N-K} |Y_R| \leq \sum_{R=0}^{N-K} \sum_{k=1}^{D} \sum_{m=1}^{\infty} |B_k\beta^{-m} \rho(k; m + R)| \]

\[ = \sum_{k=1}^{D} |B_k| \sum_{m=1}^{\infty} \sum_{R=0}^{N-K} |\beta^{-m} \rho(k; m + R)| \]

\[ = \sum_{k=1}^{D} |B_k| z_N(k), \]

where \( z_N(k) = \sum_{m=1}^{\infty} \sum_{R=0}^{N-K} |\beta^{-m} \rho(k; m + R)| \). To obtain the assertion of Lemma 4.2, it is enough to prove

\[ z_N(k) \ll \lambda(\Gamma; N)^D \text{ for all } 1 \leq k \leq D. \]

Put \( S_1(k) = \sum_{m=1}^{K-1} |\beta|^{-m} \sum_{R=0}^{N-K} |\rho(k; m + R)| \) and \( S_2(k) = \sum_{m=K}^{\infty} |\beta|^{-m} \sum_{R=0}^{N-K} |\rho(k; m + R)| \). Note that by the definition of \( z_N(k) \), we have \( z_N(k) = S_1(k) + S_2(k) \). First, we estimate the upper bound for \( S_1(k) \) as follows:

\[ S_1(k) \leq \sum_{m=1}^{K-1} |\beta|^{-m} \sum_{R=0}^{N-1} |\rho(k; R)| \leq \sum_{m=1}^{K-1} |\beta|^{-m} \sum_{k=0}^{N-1} |\rho(k; R)| \]

\[ \ll \sum_{R=0}^{N-1} |\rho(k; R)| \leq \sum_{R=0}^{N-1} \sum_{m_1, \ldots, m_k \in \Gamma, m_1 + \cdots + m_k = R} |s_{m_1} \cdots s_{m_k}| \]

\[ = \sum_{m_1, \ldots, m_k \in \Gamma, m_1 + \cdots + m_k < N} |s_{m_1} \cdots s_{m_k}| \leq T^k \sum_{m_1, \ldots, m_k \in \Gamma, m_1 + \cdots + m_k < N} 1 \]

\[ \leq T^k \lambda(\Gamma; N)^k \ll \lambda(\Gamma; N)^D. \]

Note that (16) is obtained in a similar way as inequality (11). Second, we estimate the upper bound for \( S_2(k) \) as follows:

\[ S_2(k) \ll \sum_{m=K}^{\infty} |\beta|^{-m} \sum_{R=0}^{N-K} (m + R + 1)^D \]

\[ \leq \sum_{m=K}^{\infty} |\beta|^{-m} \sum_{R=0}^{N-K} (m + N)^D \]

\[ \leq \sum_{m=K}^{\infty} |\beta|^{-m} N(m + N)^D. \]
Note that in inequality (18), we use (11). If \( N \gg 1 \), we have
\[
\left( \frac{m + 1 + N}{m + N} \right)^D \leq \frac{1 + |\beta|}{2} \quad \text{for any } m \geq 1.
\]
Then combining (19) and (20), we obtain
\[
S_k(k) \ll |\beta|^{-K} N(K + N)^D \sum_{m=0}^{\infty} |\beta|^{-m} \left( \frac{1 + |\beta|}{2} \right)^m \ll N^{D+1}|\beta|^{-K} \leq 1.
\]
Combining (17) and the above relation, we obtain (15), which completes the proof of Lemma 4.2.

4.3 Relation between \( Y_R \) and \( Y_{R-1} \)

For a positive integer \( N \), we put \( \tau(N) = \tau := \text{Card}((D - 1)\Gamma; N) \) and define the sequence of integers \( (i(h))_{1 \leq h \leq \tau + 1} \) by \( [0, N) \cap (D - 1)\Gamma =: \{0 = i(1) < \cdots < i(\tau)\} \) and \( i(\tau + 1) := N \). Note that we have
\[
\tau(N) \leq \lambda(\Gamma; N)^{D-1}.
\]
For \( 1 \leq h \leq \tau \), we put \( I_h = [i(h), i(h + 1)) \cap \mathbb{Z} \) and \( y_N(h) = \text{Card}\{R \in I_h \mid |Y_R| \geq C_{12}\} \). Then by the definition of \( I_h \) and \( y_N(h) \), we have
\[
\sum_{h=1}^{\tau} \text{Card} I_h = N,
\]
\[
\sum_{h=1}^{\tau} y_N(h) = y_N.
\]

**Lemma 4.3.** Let \( h, R \) be integers satisfying \( 1 \leq h \leq \tau \) and \( R \in (i(h), i(h + 1)) \). Then we have
\[
Y_{R-1} = B_D \beta \rho(D; R) + \frac{1}{\beta} Y_R.
\]

**Proof.** Firstly, since \( (i(h), i(h + 1)) \cap (D - 1)\Gamma = \emptyset \), we have \( (i(h), i(h + 1)) \cap k\Gamma = \emptyset \) for \( 1 \leq k \leq D - 1 \) and
\[
\rho(k; R) = 0 \quad \text{for } 1 \leq k \leq D - 1.
\]
By the definition of \( Y_{R-1} \), we have
\[
Y_{R-1} = \sum_{k=1}^{D} B_k \sum_{m=1}^{\infty} \beta^{-m} \rho(k; m + R - 1)
\]
\[
= \sum_{k=1}^{D} B_k \beta^{-1} \rho(k; R) + \sum_{k=1}^{D} B_k \sum_{m=2}^{\infty} \beta^{-m} \rho(k; m + R - 1)
\]
\[
= \frac{B_D}{\beta} \rho(D; R) + \sum_{k=1}^{D} B_k \sum_{m=1}^{\infty} \beta^{-m-1} \rho(k; m + R)
\]
\[
= \frac{B_D}{\beta} \rho(D; R) + \beta^{-1} Y_R.
\]
Note that in above third equality, we use equality (24), which completes the proof of Lemma 4.3. \( \square \)
Lemma 4.4. Let $N \gg 1$. Put $C_{13} = 1 + D + C_{10}$. Let $h$ and $R$ be integers with $1 \leq h \leq \tau$ and

$$(25) \quad i(h) + 3C_{13}\log|\beta|N < R < i(h + 1).$$

Then we have

$$(26) \quad R - \max\{R' \mid R' < R, |Y_{R'}| \geq C_{12}\} \leq 2C_{13}\log|\beta|N.$$

Proof. Let $N$ be a sufficiently large integer such that (20) holds. Taking $R$ with (25), we see by the definition of $Y_R$ that

$$|Y_R| \leq \sum_{k=1}^{D} |B_k| \sum_{m=1}^{\infty} |\beta^{-m}\rho(k; m + R)|$$

$$\ll \sum_{k=1}^{D} |B_k| \sum_{m=1}^{\infty} |\beta^{-m}(m + R + 1)^D$$

$$\leq \sum_{k=1}^{D} |B_k| \sum_{m=1}^{\infty} |\beta^{-m}(m + N)^D$$

$$\ll \sum_{m=1}^{\infty} |\beta^{-m}(m + N)^D$$

$$\leq |\beta|^{-1}(N + 1)^D \sum_{m=0}^{\infty} |\beta|^{-m}\left(\frac{1 + |\beta|}{2}\right)^m.$$ 

Then by the above inequalities, we have for any $N \gg 1$ that

$$(27) \quad |Y_R| < N^{D+1}.$$ 

Put $S = \lceil C_{13}\log|\beta|N \rceil$. Assume that we have

$$\rho(D; R - m) = 0 \text{ for all } 0 \leq m \leq S.$$ 

Since $i(h) < R - S < \cdots < R - 1 < R < i(h + 1)$, using Lemma 4.3 and the assumption above, we obtain

$$|\beta^{S+1}Y_{R-S-1}| = \cdots = |\beta^2Y_{R-2}| = |\beta Y_{R-1}| = |Y_R| < N^{D+1}.$$ 

Recall that $C_{10}$ and $C_{11}$ are the positive integers defined in Lemma 4.1. By the above inequality and Lemma 4.1, we have

$$|\beta|^{S+1} < N^{D+1}|Y_{R-S-1}|^{-1} < N^{D+1}(R - S - 1)^{C_{10}} \leq N^{D+1+C_{10}} = N^{C_{13}}.$$ 

Take $N$ satisfying (27) and $R - S - 1 \geq \log|\beta|N \geq C_{11}$. Thus, we get

$$S + 1 = \lceil C_{13}\log|\beta|N \rceil + 1 < C_{13}\log|\beta|N,$$

a contradiction. Hence, there exists $m'$ with $0 \leq m' \leq S$ satisfying $\rho(D; R - m') \neq 0$, and so

$$(28) \quad |\rho(D; R - m')| \geq C_9,$$

by (12).

Under the above preparation, we shall prove inequality (20). For any integers $h$ and $R$ with $1 \leq h \leq \tau$ and (25), we take $m'$ as above. Put

$$R_1 := \max\{R' \mid R' < R, |Y_{R'}| \geq C_{12}\}.$$
Then we have $R - m' \in (i(h), i(h + 1))$ and
\[ Y_{R-m'-1} = \frac{B_D}{\beta} \rho(D; R - m') + \frac{1}{\beta} Y_{R-m'}. \]

First we assume that
\[ |Y_{R-m'}| \geq C_{12} = \frac{|B_D|}{2|\beta|} C_9, \]
Then we have $R_1 \geq R - m'$ and $R - R_1 \leq m' \leq 2C_{13}\log_{|\beta|} N$, which implies (23).

In the case of $|Y_{R-m'}| < C_{12}$, using $|\beta| > 1$ and (28), we get that
\[ |Y_{R-m'}| \geq \frac{|B_D|}{|\beta|} C_9 - C_{12} = C_{12}. \]
Therefore, we deduce
\[ R - R_1 \leq m + 1 \leq 2C_{13}\log_{|\beta|} N, \]
which completes the proof of Lemma 4.

4.4 Completion of the proof of Theorem 2.1

We shall show by Lemma 4.4 that there exists a constant $C_{14}$ satisfying the following: if $N \gg 1$, then
\begin{equation}
(29) \quad y_N(h) \geq \frac{\text{Card } I_h}{C_{14}\log_{|\beta|} N},
\end{equation}
for any $1 \leq h \leq \tau$. In fact, take $C_{14}$ with $C_{14} > 4C_{13}$. If $\text{Card } I_h \geq 4C_{13}\log_{|\beta|} N$, then (29) follows from Lemma 4.4. In the case of $\text{Card } I_h < 4C_{13}\log_{|\beta|} N$, we get (29) because the right-hand side is equal to 0.

Using relations (23), (29), (22) and (21), we obtain
\[ y_N = \sum_{h=1}^{\tau} y_N(h) \geq \sum_{h=1}^{\tau} \left( \frac{\text{Card } I_h}{C_{14}\log_{|\beta|} N} - 1 \right) \geq \frac{N}{C_{14}\log_{|\beta|} N} - \tau \geq \frac{N}{C_{14}\log_{|\beta|} N} - \lambda(\Gamma; N)^{D-1}. \]
Then by Lemma 4.2 we have
\[ C_{15} \left( \log N + \lambda(\Gamma; N)^D \right) \geq y_N \geq \frac{N}{C_{14}\log_{|\beta|} N} - \lambda(\Gamma; N)^{D-1}. \]
By the inequality above, we conclude for any $N \gg 1$ that
\[ (1 + C_{15})\lambda(\Gamma; N)^D \geq \frac{N}{2C_{14}\log_{|\beta|} N}, \]
which completes the proof of Theorem 2.1.

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Hajime KANEKO  
Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, 305-8571, JAPAN;  
Research Core for Mathematical Sciences  
University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, 305-8571, JAPAN  
kanekoha@math.tsukuba.ac.jp

Makoto KAWASHIMA  
Faculty of Production Engineering, Nihon University, 2-11-1 Shinsakae, Narashino,  
Chiba, 275-, 8576, Japan  
kawashima.makoto@nihon-u.ac.jp