Ergodicity of a collective random walk on a circle

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Abstract
We discuss the conditions for unique ergodicity of a collective random walk on a continuous circle. Individual particles in this collective motion perform independent (and different in general) random walks conditioned by the assumption that the particles cannot overrun each other. In addition to sufficient conditions for unique ergodicity, we discover a new and unexpected way for its violation due to excessively large local jumps. Necessary and sufficient conditions for the unique ergodicity of the deterministic version of this system are also obtained. Technically, our approach is based on the interlacing property of the spin function which describes the states of pairs of particles in the coupled processes being studied.

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1. Introduction
We consider a collective random walk of a configuration consisting of \( n \) particles on a unit continuous circle. Each particle that is not interacting with others, performs an independent random walk and the interaction between particles consists in the prohibition for particles to overrun each other. The \( i \)-th particle in the configuration at time \( t \in \mathbb{Z} \) is characterized by the position of its centre \( x_i^t \in S := [0, 1) \), the radius \( r_i \geq 0 \) of the ball (representing the particle), and the distribution of jumps \( P_i \) (i.e. the particle makes a jump equal to a random value \( \xi \) distributed according to \( P_i \)). In general our theory covers both positive and negative jumps, but to simplify the presentation we discuss in the Introduction only the case of non-negative jumps, i.e. \( P_i([0, 1]) = 1 \), leaving the general case until section 4.
The collective random walk under consideration is a close relative to exclusion processes introduced by Spitzer [9] and studied in a number of publications. One of the most prominent and detailed reviews of statistical properties of such processes considered on a lattice and in continuous time can be found in [7] (see further references therein and [1, 2, 6, 8] for more recent results).

We say that a particle configuration \( x^t := \{x^t_1, \ldots, x^t_n\} \) is admissible if the open balls corresponding to the particles in the configuration do not intersect (see figure 1), i.e. it satisfies the inequality:

\[
\sum_{i=1}^n \Delta x^t_i + \sum_{i=1}^{n+1} \Delta x^t_i - 1 \leq \sum_{i=1}^{n+1} \Delta x^t_i - r_i - r_{i+1}, \quad \forall i.
\]

The set of all admissible configurations is denoted by \( X \). Here and in the sequel (if the exception is not explicitly mentioned) arithmetic operations with metric elements (\( x_i, v_i, r_i \), etc.) are taken modulo 1, and the indices are taken modulo \( n \), i.e. \( x^t_{n+1} \equiv x^t_1 \), \( x^t_0 \equiv x^t_n \).

Finally the local dynamics (see figure 1) of an individual particle is defined by the relation

\[
x^t+1_i := \min\{x^t_i + v^t_i, x^t_{i+1} - r_i - r_{i+1}\},
\]

(1.1)

where the random variable (velocity) \( v^t_i \) is chosen according to the distribution \( P_i \), and the collection of multiplicators \( u^t := \{u^t_i\} \) with \( u^t_i \in \{0, 1\} \) represents the updating rule (see below). The random variables \( \{v^t_i\}_{i,t} \) are assumed to be mutually independent.

The moment of time when the \( i \)th particle is stopped by the \((i + 1)\)th particle (i.e. \( x^t_i + u^t_i v^t_i > x^t_{i+1} - r_i - r_{i+1} \)) will be referred to as the moment of interaction between these particles. Note that a homogeneous version of systems of this type (when \( r_i \) and \( P_i \) do not depend on \( i \)) was introduced and studied in [4].

Depending on the updating rule \( u^t \) discrete time processes under consideration may be classified into two types: with parallel and sequential updating. In the former case all particles are trying to move simultaneously which leads to an arbitrary number of simultaneous interactions. In the later case at each moment of time only one particle is chosen to jump according to a certain rule (e.g. by a random choice) and thus at most a single interaction may take place. In terms of \( u^t \) the parallel updating means that \( u^t_i \equiv 1 \) for all \( i, t \). The sequential updating may be realized in a number of ways and we shall consider the following two scenarios:

(a) random sequential updating: at time \( t \) the only positive entry in \( u^t \) is chosen at random according to a given distribution \( q := \{q_1, \ldots, q_n\} \) with \( \prod q_i > 0 \).
(b) deterministic sequential updating: we start by choosing a certain index \( i \) starting from which the particles are updated clockwise one at a time until we reach \( i \). Then we repeat the procedure, etc.

If the type of sequential updating is not specified explicitly we mean that it is either (a) or (b). Surprisingly conditions leading to the unique ergodicity in both these cases coincide.
The sequential updating in a sense is equivalent to a continuous time collective random walk in which a random alarm clock is attached to each particle and the particle moves only when the clock rings.

The local dynamics \( (1.1) \) together with a specific updating rule define a finite dimensional Markov chain in the phase space of admissible configurations. This local dynamics uniquely defines the dynamics of gaps

\[
\Delta_i^t := (x_{i+1}^t - r_{i+1}) - (x_i^t + r_i)
\]

between the particles. Naturally \( \Delta^t := \{\Delta_i^t\} \) is admissible if \( \Delta_i^t \geq 0 \) and \( \sum_i (\Delta_i^t + 2r_i) = 1 \).

In terms of gaps the local dynamics of particles may be rewritten as

\[
x_i^{t+1} := x_i^t + \min\{u_i^t v_i^t, \Delta_i^t\},
\]

while the actual dynamics of gaps is described by the relation

\[
\Delta_i^{t+1} := \Delta_i^t - \min\{u_i^t v_i^t, \Delta_i^t\} + \min\{u_i^{t+1} v_i^{t+1}, \Delta_i^{t+1}\}.
\]

(1.2)

This shows that the dynamics of gaps \( \Delta^t \) is a Markov chain as well.

Standard arguments about the compactness of the phase space and the continuity of the corresponding Markov operators imply the existence of invariant measures of the processes under study. The question about the uniqueness of the invariant measure is much more delicate. Our main results (theorems 1, 2) give sufficient conditions for the uniqueness of the invariant measure (i.e. for the unique ergodicity) in the true random setting when the distributions \( P_i \) are nontrivial. An unexpected counterexample in the case of parallel updating related to excessively large local velocities is constructed in proposition 1. Despite very weak assumptions made in these theorems they cannot be applied in the pure deterministic setting when each distribution \( P_i \) is supported by a single constant local velocity \( v_i > 0 \). Nevertheless we show that this deterministic process still might be uniquely ergodic (albeit due to different reasons) and give necessary and sufficient conditions for this (theorem 3). In comparison to our earlier note [5], sufficient conditions for the unique ergodicity are formulated in very different terms and become much weaker, especially in the general non-totally asymmetric case. Technically the main improvement is that instead of using the principle of ‘isolated interactions’ the present approach is based on the interlacing property of the spin function (see section 4, lemma 3), which describes the states of pairs of particles in the statically coupled processes being studied.

Let us discuss possible obstacles for the unique ergodicity in the simplest setting. Assume for a moment that instead of a system on the continuous circle we deal with a finite discrete time lattice system with \( L \) sites and periodic boundary conditions, i.e. \( x_i, v_i \in \mathbb{Z}/L, r_i = 1/(2L) \). Assume also that each particle jumps with probability \( p \in (0, 1) \) to its nearest right neighbouring site if it is not occupied or stays put otherwise. In other words we consider the simple discrete time totally asymmetric exclusion process with parallel updates. This Markov chain has an important property, that the probability of reaching one state from another in finite time is positive, which implies unique ergodicity of this process. On the other hand, a simple modification of this process allowing longer particle jumps, e.g. by two sites instead of one with even \( L \), breaks down this property. Nevertheless, as we shall show in this case and in the much more complicated case of the continuous circle, simple and not especially restrictive assumptions on the particle jumps guarantee the ergodicity of the dynamics of gaps. In some situations (see below) conditions for the unique ergodicity of the original system and of the dynamics of gaps coincide but typically this is not the case.

The paper is organized as follows. In section 2 we formulate our main results. Section 3 is dedicated to technical constructions allowing the analysis of synchronization-type phenomena and proofs of the main results in the totally asymmetric case. An alternative construction...
based on the dynamics of gaps only is discussed here as well. In section 4 we deal with a more
general situation, that of when the local velocities are taking both signs (and hence particles
may move towards each other), and discuss briefly the lattice version of the collective random
walk and a more strict version of the particles conflicts resolution. The last section is dedicated
to the analysis of how unique ergodicity may take place in the pure deterministic collective
walk.

2. Main results

We start with the totally asymmetric setting, i.e. \( v^i_t \geq 0 \) for all \( i, t \).

**Theorem 1 (Ergodicity).** Let the distributions of jumps \( v^i_t \) satisfy the following non-degeneracy condition
\[
P(v^i_t > v^j_t) > 0 \quad \forall i \neq j, \forall t. \tag{2.1}
\]
Then the Markov process \( \Delta^t := \{\Delta^t_i\} \) with sequential updating is uniquely ergodic, namely for
each admissible initial \( \Delta^0 \) the distributions of the random variables \( \Delta^t \) converge as \( t \to \infty \)
in Cesaro means to a limit which does not depend on \( \Delta^0 \). In the case of parallel updating the
same claim holds if additionally
\[
P(v^i_t < \varepsilon) > 0 \quad \forall \varepsilon > 0, \forall i, t. \tag{2.2}
\]

The non-degeneracy condition (2.1) is a rather distant generalization of the simplest law
\( P := (1 - p)\delta_0 + p\delta_\sigma \), when a particle makes a jump of length \( \sigma > 0 \) with the probability
\( p > 0 \) or stays put otherwise. The condition (2.2) implies that in the parallel updating scenario
not all particles interact all the time. See also the discussion of the necessity of this condition
after the proof of theorem 1. The following result shows that the presence of non-interacting
particles is ‘almost’ necessary for the unique ergodicity under parallel updating.

**Proposition 1.** Let there exist a collection of positive values \( v^0 := \{v^0_i\} \) such that
\[
\text{Var}(v^0) := \sum_i |v^0_i - v^0_{i+1}| > 0 \quad \text{and} \quad \sum_i v^0_i > 1 - 2 \sum_i r_i. \tag{2.3}
\]
Then the Markov process \( \Delta^t \) with parallel updating and \( P(v^i_t \geq v^0_i) = 1 \) for all \( i, t \) has
infinitely many ergodic invariant measures.

**Remark 2.** Using the static coupling construction developed in section 3 one can show that
each two \( x^i \)-invariant measures coincide up to a spatial shift, which together with the results
of theorem 1 implies the unique ergodicity of the original process under factorization upon
spatial shifts.

Now we formulate sufficient conditions of the unique ergodicity in the general non-totally
asymmetric case when the local velocities may take both positive and negative values.

**Theorem 2.** Let the distributions of jumps \( v^i_t \) satisfy one of the following non-degeneracy conditions
\[
P(v^i_t > v^j_t) > 0 \quad \forall i \neq j, \forall t, \tag{2.4}
\]
\[
P(v^i_t < v^j_t) > 0 \quad \forall i \neq j, \forall t. \tag{2.5}
\]
Then the Markov process \( \Delta^t := \{\Delta^t_i\} \) with sequential updating is uniquely ergodic. The same
claim holds true in the case of parallel updating if additionally at least for one index \( i \) we have
\[
P(v^i_t v^i_{i+1} < 0) > 0. \tag{2.6}
\]
Observe that if the condition (2.6) is violated we are back to the totally asymmetric case.

Very weak sufficient conditions of the unique ergodicity formulated in theorems 1 and 2 always include a version of a non-degeneracy assumption which gives an impression that in the deterministic setting (when the jump distributions $P_i$ are concentrated at a single point) the unique ergodicity is excluded. The following result addresses this question and shows that the corresponding deterministic dynamical system may possess a single nontrivial invariant measure.

**Theorem 3.** Let $P(v_t^i = v_i) = 1$ for some constant positive local velocities $v := \{v_i\}_{i=1}^n$ and $u_t^i \equiv 1$. Denote $v_{\text{min}} := \min_i \{v_i\}$ and $\alpha := 1 - 2 \sum_{i=1}^n r_i - n v_{\text{min}}$. Then the dynamical system defined by the relation (1.1) is uniquely ergodic if and only if

(a) $v_{\text{min}}$ is achieved at a single index $i_{\text{min}}$,

and one of the following assumptions holds true:

(b) $\alpha \geq 0$ and $v_{\text{min}}$ is irrational,

(c) $\alpha < 0$ and $\alpha + v_{\text{min}}$ is irrational.

The proofs of theorems 1 and 2 use a technical result which (especially due to its deterministic nature) is of interest by itself. Assume that the local particle velocities $v_t^i$ are given for all $i, t$. We say that the particle process satisfies the *chain-interacting* property if for any initial configuration $x^0$ subsequent particles will interact in finite time (see a more detailed version of this assumption in section 3).

**Theorem 4 (Synchronization).** Let the chain-interacting property hold. Then

(a) in the case of sequential updating for any initial admissible configurations $x^0, \hat{x}^0$ the processes $x^t, \hat{x}^t$ are getting synchronized with time, namely

$$\sum_{i=1}^n |\Delta_i^t - \hat{\Delta}_i^t| \xrightarrow{t \to \infty} 0.$$

(b) in the case of parallel updating the same claim holds if additionally for infinitely many moments of time $t$ there exists $j = j(t)$ such that either $v_t^j < 0 < v_t^{j+1}$ or $v_t^j = 0$.

The chain-interacting property holds in the case of sequential updating if

$$\inf_{i, t} [v_t^i] > \frac{1}{n}(1 - 2 \sum_{i} r_i) > 0,$$

while in the case of parallel updating it is enough to assume that

$$\forall t_0 \geq 0, \forall i \in \{1, \ldots, n\} \exists \tau_i < \infty : \sum_{t=t_0}^{t+t_\tau_i} (v_t^i - v_{t+1}^i) > 1.$$

3. Synchronization phenomenon and proofs in the totally asymmetric setting

Define a *static coupling* between the processes $x^t, \hat{x}^t$ satisfying the relation (1.1) as a pair-process $(x, \hat{x})^t$ in which all random choices related to particles with equal indices coincide, i.e. $v_t^i = \hat{v}_t^i$ for all $i, t$. In the statically coupled processes we say that the $i$th pair of particles *interacts* with the $(i+1)$th one if at least one of the $i$th particles in these processes is doing so.
3.1. Construction in terms of particle’s positions

Consider a ‘lifting’ of the process $x'$ acting on the circle $S$ to the real line $\mathbb{R}$ defined by the relation

$$
R(x, t, i) := \begin{cases} 
0 & \text{if } t = 0, i = 1 \\
\sum_{j=1}^{i-1} \Delta^0_j & \text{if } t = 0, i > 1 \\
R(x, t - 1, i) + \min\{u_i^{-1}v_{i-1}, \Delta_i^{-1}\} & \text{if } t > 0.
\end{cases}
$$

(3.1)

Rewriting the definition of a gap in terms of the lifting map we have

$$
\Delta_i = \begin{cases} 
R(x, t, i + 1) - R(x, t, i) & \text{if } 1 \leq i < n \\
R(x, t, 1) + 1 - R(x, t, n) & \text{if } i = n,
\end{cases}
$$

and the total distance covered by the $i$th particle during the time from 0 to $t$ is equal to $R(x, t, i) - R(x, 0, i)$.

Let the processes $x', \dot{x}'$ be statically coupled. For each $i$ we associate to the $i$th particle a new random variable

$$
st_i := R(x, t, i) - R(\dot{x}, t, i),
$$
to which we refer as a spin. It is useful to think about the pair of points $R(x, t, i), R(\dot{x}, t, i)$ as a dumbbell whose discs centres lie on two parallel straight lines. Then the spin $s_i$ describes the state of this dumbbell.

As we shall see, after the interaction between the $i$th and $(i + 1)$th particles in one of the processes $x'$ or $\dot{x}'$ (or in both of them) the corresponding spins become closer to each other in comparison to the situation just before the interaction (see figures 2 and 3). On the other hand, this might lead to the increase of the distinction with the spins of neighbouring $(i - 1)$th and $(i + 1)$th particles, i.e. either $|s_{i-1}' - s_i'|$ or $|s_i' - s_{i+1}'|$ may grow with time $t$. Nevertheless in the worst case the amount to which one of the distinctions was enlarged cannot be greater than the amount to which another distinction became smaller. The idea of our approach is to show that under dynamics the variation of the collection of spins $s'$, defined as

$$
\text{Var}(s') := \sum_{i=1}^{n} |s_i' - s_{i+1}'|,
$$
is a non-increasing function of the variable $t$ and converges to zero monotonically with time.

**Lemma 3 (Interlacing).** Let at time $t$ one (or both) of the $i$th particles in the processes $x', \dot{x}'$ interact with the $(i + 1)$th one. Then the interlacing property

$$
\min\{s_i', s_{i+1}'\} \leq s_i^{i+1} \leq \max\{s_i', s_{i+1}'\}
$$

(3.2)

takes place. Additionally, if $s_i' \neq s_{i+1}'$ then

$$
|s_i' - s_{i+1}'| > |s_i^{i+1} - s_{i+1}^{i+1}|.
$$

(3.3)
Figure 3. Change of the spin function under dynamics. $s'_i/s_{i+1}'$ are marked by open/closed circles.

**Proof.** Consider the interaction at time $t$ of the $i$th pair of particles with the $(i + 1)$th pair and assume that

$$\Delta'_i < v'_i \leq \Delta'_i.$$

(3.4)

This situation is depicted in figure 2. By definition of the spin, (3.4) implies that $s'_i > s'_{i+1}$.

We have $s'_i = R(x, t, i) - R(\hat{x}, t, i)$ and

$$R(x, t + 1, i) = \begin{cases} R(x, t, i + 1) - r_i - r_{i+1} < R(x, t, i) + v'_i & \text{if } 1 \leq i < n \\ R(x, t, 1) + 1 - r_n - r_1 < R(x, t, n) + v'_n & \text{if } i = n \end{cases}$$

and

$$R(\hat{x}, t + 1, i) = R(\hat{x}, t, i) + v'_i \leq R(\hat{x}, t, i + 1) - r_i - r_{i+1}.$$

Therefore

$$\min\{s'_i, s'_{i+1}\} \leq s'_{i+1} < s'_i = \max\{s'_i, s'_{i+1}\},$$

which additionally implies (3.3).

The situation $\Delta'_i \geq v'_i \geq \Delta'_i$ is considered similarly, exchanging the roles played by the processes $x'$ and $\hat{x}'$. It remains to study the case

$$v'_i > \max\{\Delta'_i, \Delta'_i\}.$$

This inequality means that after the interaction $s'_{i+1} = s'_i$, which implies (3.2), and if additionally $s'_i \neq s'_{i+1}$ we get (3.3) as well. \hfill \Box

**Lemma 4.** The variation of the spin function $\text{Var}(s')$ does not increase under dynamics.

**Proof.** We say that $[j, k] := [j, j + 1, \ldots, k]$ is the interval of positive monotonicity of the collection $s := \{s_i\}$ considered as a function of the variable $i$ if $s_j \leq s_{j+1} \leq \cdots \leq s_k$, and the interval of negative monotonicity if $s_j \geq s_{j+1} \geq \cdots \geq s_k$.

By the property (3.2) for each locally maximal interval of positive monotonicity $[i, j]$ of the function $s^{i+1}$ we have

$$\min\{s'_i, s'_{i+1}\} \leq s'_{i+1} \leq s'_i = \max\{s'_i, s'_{i+1}\}.$$

(3.5)

Similarly for each locally maximal interval of negative monotonicity $[i, j]$

$$\max\{s'_i, s'_{i+1}\} \geq s'_{i+1} \geq s'_i = \min\{s'_i, s'_{i+1}\}.$$

(3.6)
Consider two consecutive locally maximal intervals of positive monotonicity \([i^*_k, j^*_k]\) and \([i^*_{k+1}, j^*_{k+1}]\) of the function \(s^{t+1}\). Then by (3.5)
\[
\text{ind max}\{s^t_{i^*_{k+1}}, s^t_{j^*_{k+1}}\} \leq \text{ind min}\{s^t_{i^*_k}, s^t_{j^*_k}\},
\]
where
\[
\text{ind max}\{s_i, s_j\} := \begin{cases} i & \text{if } s_i \geq s_j \\ j & \text{otherwise} \end{cases} \quad \text{ind min}\{s_i, s_j\} := \begin{cases} i & \text{if } s_i \leq s_j \\ j & \text{otherwise} \end{cases}.
\]

The main difficulty in the analysis of the change of variation of the spin function is that the intervals of monotonicity of the functions \(s^t\) and \(s^{t+1}\) need not coincide and even may be very different from each other (see, e.g. figure 3). Additionally some individual slopes of the function \(s^{t+1}\) might be much larger than the corresponding slopes of the function \(s^t\).

Let \([i^-_k, j^-_k]\) and \([i^-_{k+1}, j^-_{k+1}]\) be locally maximal intervals of positive and negative monotonicity of the function \(s\) respectively, and let \([i_k, j_k]\) be any collection of non-intersecting intervals. (We say that integer intervals do not intersect if they have at most one common point.) Then by the triangle inequality
\[
\text{Var}(s) = \sum_k \left( (-s_{i^*_k} + s_{j^*_k}) + (s_{i^-_k} - s_{j^-_k}) \right) \\
= 2 \sum_k (-s_{i^*_k} + s_{j^*_k}) \geq 2 \sum_k (-s_{i_k} + s_{j_k}).
\]
(3.8)

Therefore, combining (3.5) and (3.6) and using (3.7) we obtain
\[
\text{Var}(s^{t+1}) = 2 \sum_k (-s^t_{i^*_{k+1}} + s^t_{j^*_{k+1}}) \\
\leq 2 \sum_k \left( -\text{min}\{s^t_{i^*_{k+1}}, s^t_{j^*_{k+1}}\} + \text{max}\{s^t_{i^*_k}, s^t_{j^*_k}\} \right) \\
\leq \text{Var}(s^t),
\]
which gives the desired inequality.

Since the result of lemma 4 seems to have an independent interest giving a comparison between the variation of two interlacing collections of points \(s^t\) and \(s^{t+1}\), we describe also a sketch of an alternative proof of this result based on the induction on the number of points \(n\). The base of induction – the case \(n = 2\) is trivial. Indeed, assume for definiteness that \(s^t_1 < s^t_2\). Then
\[
s^t_1 \leq s^t_{i^*_{k+1}} \leq s^t_{j^*_{k+1}} < s^t_2.
\]
Therefore \(\text{Var}(s^t) = 2|s^t_1 - s^t_2| \geq 2|s^t_{i^*_{k+1}} - s^t_{j^*_{k+1}}| = \text{Var}(s^{t+1})\).

It remains to show that the case of general \(n > 2\) can be reduced to the case of a smaller number of points. There are two possibilities; there exists at least one interval of monotonicity \(J\) of the function \(s^{t+1}\) of length greater than 1, or all locally maximal intervals of monotonicity of the function \(s^{t+1}\) are short of length 1. In the first case we may remove one of the particles in the middle of the interval \(J\). This will preserve the interlacing conditions (3.2), but not change the variation of \(s^{t+1}\) without the removed point, while the variation of \(s^t\) may only decrease. Thus we get the reduction to the smaller number of particles. In the second case we have only short intervals of monotonicity of the function \(s^{t+1}\) and the types (increasing or decreasing) of corresponding intervals of \(s^t\) are either opposite, or between two opposite type pairs of intervals there is a single pair of intervals with the same types of monotonicity. This situation is depicted in figure 4. The analysis of the case of short monotonicity intervals is straightforward if one recalls the inequality (3.7).
Now a nontrivial point is to show that under assumptions made above the variation $\text{Var}(s^t)$ vanishes with time a.s. To this end one needs to have some additional control over particles’ interactions.

We say that the particles with indices $i \leq j$ are clockwise chain-interacting if for any initial configuration $x^0$ a.s. there exists a sequence of (random) moments of interaction $t_i \leq t_{i+1} \leq \ldots \leq t_{j-1} < \infty$ between the corresponding particles. In other words, for each $k \in \{0, 1, \ldots, j - i - 1\}$ at time $t_{i+k}$

$$v_{i+k}^{t_{i+k}} > \Delta_{i+k}^{t_{i+k}}.$$  

(3.9)

Similarly one defines the anti-clockwise chain-interaction when $i > j$. If for any pair of indices $i, j$ the (anti-) clockwise chain-interacting property holds we say that the process satisfies the chain-interacting property.

**Lemma 5.** Let $x^t, \dot{x}^t$ be statically coupled copies of the same particle process with sequential updating satisfying the chain-interaction property and let $\text{Var}(s^0) > 0$. Then $\exists \tau = \tau(x^0, \dot{x}^0) < \infty : \text{Var}(s^{\tau+1}) < \text{Var}(s^{\tau})$.

**Proof.** By lemma 3

$$\text{Var}(s^{t+1}) \leq \text{Var}(s^t) \forall t \geq 0$$

and we need to show only that for some $t$ this inequality becomes strict. Assume from the contrary, that this is not the case, i.e.

$$\text{Var}(s^{t+1}) = \text{Var}(s^t) \forall t \geq 0.$$  

(3.10)

By the definition of sequential updating at the moments of time $t_i$ the particle’s interactions are isolated in the sense that the particles neighbouring the interacting ones make no interactions with other particles.

Let us show that if there exists an index $i$ such that

$$(s_{i-1}^t - s_i^t)(s_i^t - s_{i+1}^t) < 0, \quad s_i^{t+1} \neq s_i^t,$$  

(3.11)

then the variation strictly decreases at time $t$ (i.e. $\text{Var}(s^{t+1}) < \text{Var}(s^t)$). The condition (3.11) means that $s_i^t$ as a function on $i$ is non-monotone at $i$ and changes its value at this index at time $t$. This situation is depicted in figure 5.

By the interlacing property (3.2) if

$$\min[s_{i-1}^t, s_{i+1}^t] \leq s_i^t \leq \max[s_{i-1}^t, s_{i+1}^t]$$

then $\text{Var}(s^{t+1}) = \text{Var}(s^t)$, while the violation of this inequality leads by (3.3) to $\text{Var}(s^{t+1}) < \text{Var}(s^t)$. Therefore if the index $i$ is the position of a local extremum of the function $s^t$, then the variation becomes strictly smaller (see figure 5).
Now by the chain-interacting property the interaction occurs eventually between all neighbouring particles and thus the preservation of the variation implies that the spin function considered as a function on the index variable is monotone (otherwise by the argument above the equality (3.10) cannot hold). Finally the observation that the spin function is spatially periodic (is defined on a circle) shows that this function cannot be monotone, unless its variation vanishes (i.e. all spins are equal to each other). We came to a contradiction. □

Corollary 6. Let $x^t, \dot{x}^t$ be statically coupled copies of the same particle process with sequential updating satisfying the chain-interaction property. Then $\text{Var}(s^t) \to 0$.

Indeed, the monotonicity on $t$ (by lemma 4) of the nonnegative function $\text{Var}(s^t)$ implies its convergence to a limit, which in turn (by lemma 5) cannot differ from zero.

Lemma 7. $\text{Var}(s^t) \to 0$ implies $\sum |\Delta_i^t - \dot{\Delta}_i^t| \to 0$.

Proof. By the definition of the spin function

$$\Delta_i^t - \dot{\Delta}_i^t = s_{i+1}^t - s_i^t,$$

which implies the claim. □

Lemma 8. Let the updating be sequential and let the jump distributions satisfy the non-degeneracy condition (2.1). Then a.s. in the process $x^t$ each pair of particles is either clockwise or anti-clockwise chain-interacting.

Proof. The condition (2.1) implies that a.s. each particle interacts with one (or both) of its neighbours in finite time. Therefore since the total number of particles is finite it follows that a.s. after a finite time each particle will chain-interact with each other. □

Proof of theorem 1. We start by checking the sequential updating case. By lemma 8 the chain-interaction property is satisfied. Therefore by corollary 6 the functional $\text{Var}(s^t)$ vanishes with time. Thus the coupling time is finite and hence (see e.g. [3] for a suitable version of the corresponding statement) we get the desired claim.

In the parallel updating case the situation is somewhat more complicated. By lemma 4 the variation of the spin function cannot increase under dynamics and we only need to demonstrate that it strictly decreases with positive probability. To this end we make use of the condition (2.2), according to which with positive probability not all pairs of particles interact simultaneously.
Assume that at time $t$ the $i$th pair of particles in the statically coupled processes $x^t, \dot{x}^t$ does not interact with the $(i+1)$th pair. Then we can rewrite the parallel updating as $n$ deterministic sequential ones with the given local velocities defined in the parallel updating starting from the index $i+1$.

Note that in the absence of non-interacting particles the reduction to sequential updating cannot be done since otherwise the position of the $(i+1)$th particle will be different at the moment of the sequential updating of the $i$th particle, which will change its position at time $t+1$. This is the crucial observation in the proof of proposition 1 below.

Once we made the reduction to sequential updating, the results proven in that case imply that the variation of the spin function vanishes with time and hence we get the unique ergodicity of the corresponding gap process for parallel updating as well.

It is worth noting that in order to guarantee that not all particles interact simultaneously all the time it is enough to make an assumption

$$P\left(\sum_i v_i^t < 1 - 2 \sum_i r_i\right) > 0,$$

which is much weaker than (2.2). Unfortunately to make the simultaneous reduction of both coupled processes from parallel to sequential updating we need to find not only a single non-interacting particle but a non-interacting dumbbell—a pair of particles which do not interact with their right neighbours. Let us show that under the condition (3.12) this might not work.

Choose a sequence $0 < a_0 < a_1 < \cdots < 1/2$ and let $n = 2, r_1 = r_2 = 0$. Consider initial configurations of gaps $\Delta_0^t : = a_0, \Delta_0' : = 1 - a_0, \Delta_1^t : = 1 - a_0, \Delta_1' : = a_0$. Then under dynamics the gaps in the process $x^t$ are equal to \{a, 1 - a\}, while for the process $\dot{x}^t$ they are equal to \{1 - a, a\}. Thus the assumption (3.12) holds true for any moment of time, only one of the particles in each process does not interact with its right neighbour, but in both 1st and 2nd pairs (dumbbells) one of the particles makes the interaction with its right neighbour.

Proof of proposition 1. Set $a : = \frac{1}{n} (\sum_i v_i^0 - 1 + 2 \sum_i r_i)$ and choose some $0 < b \ll a$. The value $a$ is positive by (2.3). Consider a configuration of $n$ gaps $\tilde{\Delta} : = \{\max\{v_i^0 - a/2 + b, 0\}\}$. To construct an admissible configuration $\Delta$ we normalize $\tilde{\Delta}$ as follows:

$$\Delta : = \left\{\frac{\tilde{\Delta}_i (1 - 2 \sum_j r_j)}{\sum_j \tilde{\Delta}_j}\right\}.$$  

By the choice of the parameters $a, b$ for each $i$ we have $v_i^0 > \Delta_i$.

Therefore the application of the dynamics (1.2) to $\Delta$ is equivalent to the cyclic right shift: $\Delta_i \rightarrow \Delta_{i+1}$ for all $1 \leq i < n$ and $\Delta_n \rightarrow \Delta_1$. Thus the configuration $\Delta$ gives rise to an ergodic invariant measure (uniformly distributed on a finite set of points) of the process under consideration. Note that choosing different values of the 2nd parameter $b$ we are getting different invariant measures we get the claim.

Proof of theorem 4 in the case of non-negative local velocities. In the sequential updating case the claim about unique ergodicity follows from corollary 6, while in the parallel updating case we need additionally the condition that for infinitely many moments of time some particles remain in the same place to use it instead of the similar probabilistic assumption (2.2).

Therefore we need only to check sufficient conditions for the chain-interacting property. In the sequentially updating case condition (2.7) implies that during at most
(1−2∑ri)/min{|v′i|} < n iterations each particle will interact with its nearest right neighbour, which implies the property under question. In the parallel updating case the condition (2.8) plays the same role but does not give an explicit estimate of the interaction time.

3.2. Construction in terms of the gap process

During the discussion of an earlier version of this work a question was posed regarding whether it is possible to prove the unique ergodicity using the dynamics of gaps only. Here we give a positive answer to this question. Note that despite a certain simplification of arguments here we are losing important information about the original particle process and its geometric interpretation. Therefore we prefer to discuss both approaches rather than to choose only one of them.

Recall that the dynamics of gaps is defined by the relation (1.2). Similarly to the particle processes we say that two processes of gaps (with the same number of elements) Δt and Δt are statically coupled if v′i = vi for all i, t. Define a functional

\[ V(Δt, Δt') := \sum_{i=1}^{n} |Δt_i - Δt'_i|. \]

Lemma 9. For a pair of statically coupled processes of gaps Δt and Δt with sequential updating we have

\[ V(Δ^{t+1}, Δ^{t+1}) \leq V(Δ^{t}, Δ^{t}) \quad \forall t. \]  \hspace{1cm} (3.13)

Proof. In terms of gaps the interaction between the i-th and (i + 1)-th pair of particles at time t in the processes x^t, x^t takes place if and only if

\[ v'_i > \min\{Δ'_i, Δ'_i\}. \]  \hspace{1cm} (3.14)

There are 3 possibilities

\[ Δ'_i \geq v'_i > Δ'_i, \]  \hspace{1cm} (3.15)

\[ Δ'_i \geq v'_i > Δ'_i, \]  \hspace{1cm} (3.16)

\[ v'_i > \max\{Δ'_i, Δ'_i\}. \]  \hspace{1cm} (3.17)

We start with the case (3.15). Then

\[ Δ^{t+1}_{i-1} = Δ^{t}_{i-1} + Δ'_i, \quad Δ^{t+1}_{i} = 0, \]

\[ Δ^{t+1}_{i+1} = Δ^{t}_{i+1} + v'_i, \quad Δ^{t+1}_{i+1} = Δ'_i - v'_i. \]

Thus

\[ |Δ^{t+1}_{i-1} - Δ^{t}_{i-1}| + |Δ^{t+1}_{i} - Δ^{t}_{i}| = |Δ^{t}_{i-1} + Δ'_i - Δ^{t}_{i-1} - v'_i| + Δ'_i - v'_i \]

\[ = |(Δ^{t}_{i-1} - Δ^{t}_{i-1}) + (Δ'_i - v'_i)| + Δ'_i - v'_i \]

\[ \leq |Δ^{t}_{i-1} - Δ^{t}_{i-1}| + v'_i - Δ'_i + Δ'_i - v'_i \]

\[ = |Δ^{t}_{i-1} - Δ^{t}_{i-1}| - Δ'_i + Δ'_i \]

\[ \leq |Δ^{t}_{i-1} - Δ^{t}_{i-1}| + |Δ'_i - Δ'_i|. \]

Here the inequality in the third line follows from the triangle inequality. Note that this inequality becomes an equality if and only if Δ^{t}_{i-1} ≤ Δ^{t}_{i-1}. Together with the assumption (3.15) this
implies that two consecutive gaps in the process $\Delta'$ is less than or equal to the corresponding gaps in the process $\hat{\Delta}'$.

Similarly in the case (3.16) we get

$$|\Delta^{t+1}_{i-1} - \Delta^{t+1}_{i+1}| + |\Delta^{t+1}_{i} - \hat{\Delta}^{t+1}_{i}| \leq |\Delta^{t}_{i-1} - \Delta^{t}_{i+1}| + |\Delta^{t}_{i} - \hat{\Delta}^{t}_{i}|$$

and the inequality takes place if and only if two consecutive gaps in the process $\Delta'$ is larger than or equal to the corresponding gaps in the process $\hat{\Delta}'$.

In the remaining case (3.17) the calculation is even simpler:

$$\Delta^{t+1}_{i-1} = \Delta^{t}_{i-1} + \Delta^{t}_{i}, \quad \Delta^{t+1}_{i} = 0,$$

$$\hat{\Delta}^{t+1}_{i-1} = \hat{\Delta}^{t}_{i-1} + \hat{\Delta}^{t}_{i}, \quad \hat{\Delta}^{t+1}_{i} = 0.$$

Therefore

$$|\Delta^{t+1}_{i-1} - \hat{\Delta}^{t+1}_{i-1}| + |\Delta^{t+1}_{i} - \hat{\Delta}^{t+1}_{i}| = |(\Delta^{t}_{i-1} + \Delta^{t}_{i}) - (\hat{\Delta}^{t}_{i-1} + \hat{\Delta}^{t}_{i})| + 0$$

$$\leq |\Delta^{t}_{i-1} - \hat{\Delta}^{t}_{i-1}| + |\Delta^{t}_{i} - \hat{\Delta}^{t}_{i}|.$$ 
Again the equality takes place if and only if two consecutive gaps in the process $\Delta'$ are both larger or both smaller than the corresponding gaps in the process $\hat{\Delta}'$.

Observe now that, by the assumption that the process has sequential updating, during the interaction of the $i$th particle only the $(i-1)$th and the $i$th gaps may change. Thus the claim (3.13) follows. □

Using this result instead of lemma 3, and the functional $V(\Delta', \hat{\Delta}')$ instead of the variation of the spin function, one can follow arguments of the previous section to prove theorem 1. 

4. Local velocities of both signs and other generalizations

4.1. General (non-totally asymmetric) collective random walks

So far, to simplify the setting, we assumed that all particles move in the same direction, i.e. the local velocities $v'_i$ have the same (positive) sign. The presence of particles moving in opposite directions leads to a significant modification of the violation of the admissibility condition for local velocities. Now one needs to take into account not only the position of the succeeding particle, but also its velocity, as well as the corresponding quantities related to the preceding particle. In this more general case the $i$th local velocity does not break the admissibility condition if and only if

$$\max\{x^l_{i-1}, x^l_i + u^l_{i-1}v^l_{i-1}\} + r_{i-1} \leq \min\{x^l_i, x^l_i + u^l_i v^l_i\} - r_i$$

$$\max\{x^r_{i+1}, x^r_i + u^r_i v^r_i\} + r_i \leq \min\{x^r_{i+1}, x^r_{i+1} + u^r_{i+1} v^r_{i+1}\} - r_{i+1}.$$ 

If for some $i \in \{1, 2, \ldots, n\}$ and $j = i \pm 1$ the corresponding inequality is not satisfied we say that there is a conflict between the $i$th particle and the $j$th one and one needs to resolve it. In terms of gaps $\Delta'_i$ the inequalities above may be rewritten as follows:

$$\Delta'_j \geq \max\{u^l_jv^l_j, -u^l_{j+1}v^l_{j+1}, u^r_jv^r_j - u^r_{j+1}v^r_{j+1}\}, \quad j \in \{i - 1, i\} \quad (4.1)$$

In contrast to the case of particles moving in the same direction the resolution of the conflict between particles is not uniquely defined: to resolve a conflict between two mutually conflicting particles moving simultaneously in opposite directions (see figure 6) one needs to specify the positions of the particles after the conflict. This can be done in a number of ways and we shall consider a natural resolution of the conflict allowing each particle to move with the corresponding velocity as far as possible imitating a continuous time motion. Namely, in
Figure 6. Velocities of both signs.

the case of the mutual conflict between the \( i \)th and the \((i+1)\)th particles, i.e. \( v_i^f > 0 > v_{i+1}^f \), the natural resolution of the conflict leads to

\[
x_{i+1} := x_i + \frac{\Delta^f_i v_i}{v_i - v_{i+1}^f}
\]

\[
x_{i+1}^f := x_i + \frac{\Delta^f_{i+1} v_{i+1}^f}{v_i - v_{i+1}^f}.
\]

The difference between the condition (2.1) in the formulation of theorem 1 and the conditions (2.4, 2.5) in theorem 2 is that the latter are able to deal with particles jumping in both directions. To show that this is indeed necessary, consider an example with \( n = 4 \) particles whose velocity distributions satisfy the relations

\[
P_1((0, 1]) = P_2([-1, 0]) = P_3((0, 1]) = P_4([-1, 0]) = 1.
\]

In this example the particles will eventually meet in pairs (1+2 and 3+4), and the pairs will remain in two random places:

\[
\Delta_1^f, \Delta_1^{f \to \infty} = 0, \quad \text{while} \quad \sum_i^4 \Delta_i^f = 1.
\]

In the simplest case when the distributions of jumps \( P_i \) do not depend on the index \( i \) it is enough to assume that this common distribution is not supported by a single point.

**Proof of theorem 2.** Observe that the constructions developed during the analysis of the totally asymmetric setting remain valid in the general case as well. The only difference is that in addition to the interaction of a given particle with its right nearest neighbour one needs to take into account the interactions with the left nearest neighbour when the particle’s local velocity becomes negative. Fortunately only one of these interactions may take place at a given moment of time.

To apply the machinery developed in section 3 we need to check that in the case of the mutual conflict the interlacing property holds. All definitions made in section 3 remain valid except for the change in the last line of the definition of the lifting (3.1), where the term \( \min\{u_i^{-1} v_i, \Delta^{-1}_{i-1}\} \) should be changed to the actual distance covered by the \( i \)th particle during its jump at time \((t-1)\).

Let the processes \( x', x'' \) be statically coupled and the mutual conflict between at least one of the \( i \)th particles takes place. We restrict ourselves only to the situation \( \Delta_i^f \leq \Delta_i^f \) (and hence \( s_i^f \geq s_i^f + 1 \)) since the analysis of the alternative situation is completely similar. There are two possibilities.

(a) \( \Delta_i^f < v_i^f - v_{i+1}^f < \Delta_i^f \). Then denoting \( \ell := \frac{\Delta_i^f v_i}{v_i - v_{i+1}^f} < v_i^f \) we get

\[
R(x, t+1, i) = R(x, t, i) + \ell, \quad R(x, t+1, i+1) = R(x, t, i+1) - (\Delta_i^f - \ell),
\]

\[
R(\dot{x}, t+1, i) = R(\dot{x}, t, i) + v_i^f, \quad R(\dot{x}, t+1, i+1) = R(\dot{x}, t, i+1) + v_{i+1}^f.
\]
Therefore
\[ s_{i}^{t+1} = s_{i}^{t} + (\ell - v_{i}^{t}) < s_{i}^{t} = \max\{s_{i}^{t}, s_{i+1}^{t}\}, \]
\[ s_{i+1}^{t+1} = s_{i+1}^{t} + (\Delta_{i}^{t} - \ell + v_{i+1}^{t+1}) > s_{i+1}^{t} = \min\{s_{i}^{t}, s_{i+1}^{t}\} \]
since \( \ell - v_{i}^{t} < 0 \) and \( \Delta_{i}^{t} - \ell + v_{i+1}^{t+1} < 0 \).

The observation that \( \Delta_{i}^{t+1} = 0 < \Delta_{i}^{t+1} \) implies \( s_{i}^{t+1} > s_{i+1}^{t+1} \), which finishes the analysis of this possibility.

(b) \( v_{i}^{t} - v_{i+1}^{t+1} > \max\{\Delta_{i}^{t}, \Delta_{i+1}^{t}\} \). Denoting \( \ell := \frac{\Delta_{i}^{t}}{v_{i}^{t} - v_{i+1}^{t+1}} < v_{i}^{t} \) we get
\[ R(x, t+1, i) = R(x, t, i) + \ell, \quad R(x, t+1, i+1) = R(x, t, i+1) - (\Delta_{i+1}^{t} - \ell), \]
\[ R(\hat{x}, t+1, i) = R(\hat{x}, t, i) + \ell, \quad R(\hat{x}, t+1, i+1) = R(\hat{x}, t, i+1) - (\Delta_{i+1}^{t} - \ell). \]
Therefore
\[ s_{i}^{t+1} = s_{i}^{t} + (\ell - \ell) \leq s_{i}^{t}, \]
\[ s_{i+1}^{t+1} = s_{i+1}^{t} - (\Delta_{i}^{t} - \ell + \Delta_{i+1}^{t} - \ell), \]
\[ = s_{i+1}^{t} - (\Delta_{i}^{t} - \Delta_{i+1}^{t})(1 - \frac{v_{i}^{t}}{v_{i}^{t} - v_{i+1}^{t+1}}) \geq s_{i+1}^{t+1} \]
since \( \ell \leq \ell, \Delta_{i}^{t} \leq \Delta_{i+1}^{t} \) and \( \frac{v_{i}^{t}}{v_{i}^{t} - v_{i+1}^{t+1}} < 1 \). Eventually we obtain
\[ \min\{s_{i}^{t}, s_{i+1}^{t}\} \leq s_{i+1}^{t+1} = s_{i}^{t+1} \leq s_{i}^{t} = \max\{s_{i}^{t}, s_{i+1}^{t}\}. \]
Additionally, a close look at the calculations above shows that if \( s_{i}^{t} \neq s_{i+1}^{t+1} \) then
\[ |s_{i}^{t} - s_{i+1}^{t+1}| > |s_{i}^{t+1} - s_{i+1}^{t+1}|, \]
which gives the analogue of the inequality (3.3).

It remains to prove that under the assumptions of theorem 2 the chain interacting property holds true. Assume first that the process \( x^{t} \) has sequential updating. Then each of the non-degeneracy conditions obviously implies the chain-interacting property (in one of the directions). Namely the condition (2.4) implies the clockwise chain interaction, while the condition (2.5) implies the anti-clockwise chain interaction. Therefore the claim follows from the same arguments as in the proof of the totally asymmetric setting.

The situation with parallel updating is slightly more subtle. The point is that, in addition to the absence of interactions between the nearest particles (used in section 3), we get an additional way to make the reduction from parallel to consecutive updating. Indeed, if at time \( t \) two consecutive particles have opposite local velocities then there is at least another pair of consecutive particles satisfying this property. Moreover among such pairs there is at least one, say \( j \) and \( j + 1 \) such that \( v_{j}^{t} < 0 < v_{j+1}^{t+1} \). Therefore these two particles do not interact and hence in this situation one can make the reduction to sequential updating, which starts at the index \( j + 1 \) and goes up to the index \( j \).

Using the above trick we make the reduction to sequential updating if the event described in the assumption (2.2) takes place. The remaining part follows the same arguments as in the proof of theorem 1.

**Proof of theorem 4 in the case of local velocities taking both signs.** In addition to the already proven part relating to the local velocities of the same sign, we use the condition that for infinitely many moments of time \( t \) there exists \( j = j(t) \) such that \( v_{j}^{t} < 0 < v_{j+1}^{t+1} \) in order to make the reduction from parallel updating to sequential updating at these moments of time. After this reduction one applies the same arguments as in the probabilistic setting.
4.2. Strict exclusion

In all our previous constructions we have considered only those rules resolving particle conflicts allowing the particles to move as far as possible according to their local velocities. However, as we already mentioned the collective random walk under consideration is a generalization of the simple exclusion process, where a particle moves to the neighbouring site only if the latter is not occupied by another particle. From this point of view it seems to be natural to consider a similar conflict resolution rule. Namely we say that the particle process \( x^t \) satisfies the \textit{strict exclusion} rule if in the case of a conflict the corresponding particle remains in the same place, i.e.

\[ \text{if } v^t_i > \Delta^t_i \text{ or } -v^t_i > \Delta^t_{i-1} \text{ then } x^{t+1}_i = x^t_i. \]

It turns out that this ‘natural’ conflict resolution rule typically leads to a non-ergodic behaviour.

\textbf{Proposition 10.} Let \( P_i((-\varepsilon, \varepsilon])) = 0 \) for some \( \varepsilon > 0 \) and all \( i \). Then for \( n \) large enough the process \( \Delta^t \) is non-ergodic.

\textbf{Proof.} Let \( n > 1/\varepsilon \). Consider a configuration \( x := \{x_i\}_{i=1}^n \) such that \( \max_i \Delta_i < \varepsilon \). Then under the assumptions of the theorem, due to the strict exclusion interaction rule, all particles in the configuration \( x \) stay put under dynamics. Hence the Dirac measure \( \delta_x \) supported by the configuration \( x \) is invariant under dynamics as well as the Dirac measure supported by the sequence of the corresponding gaps \( \Delta_i \). Passing from the configuration \( x \) to close enough to configuration \( \tilde{x} \) such that \( \max_{\tilde{x}} \Delta_{\tilde{x}} < \varepsilon \) we are getting infinitely many different fixed points of the process \( x^t \) having different configurations of gaps. This proves the non-ergodicity of the process \( \Delta^t \). \( \square \)

In contrast to the non-strict exclusion case, here it is much more difficult to give sufficient conditions for the unique ergodicity. At present we can only formulate the following hypothesis.

\textbf{Hypothesis.} Let the assumptions (2.1, 2.2) hold true. Then the process of gaps \( \Delta^t \) with either sequential or parallel updating is uniquely ergodic.

4.3. Lattice exclusion process

Observe that ergodic type results for lattice versions of the problems under consideration being nontrivial as well may be derived from the present results.

In the lattice setting all elements of a configuration \( x \) belong to a finite set \( \{0, 1/n, 2/n, \ldots, (n-1)/n\} \) for some \( n \in \mathbb{Z}_+ \) which defines the number of lattice sites. The radius of a ball representing a particle satisfies the condition \( n r_j \in \{0, k + 1/2\} \) with \( k \in \mathbb{Z}_+ \), and the jump distribution is supported by the lattice points \( P_r((u^{r-1}_j, j/n)) = 1 \).

Despite an apparent significant difference between the behaviour of the lattice processes in the cases when \( r_j = 0 \) and \( r_j > 0 \) (in the former case an arbitrary number of particles may share the same lattice site, while in the latter case at most one particle is allowed per lattice site) the ergodicity conditions turn out to be the same.

5. Unique ergodicity in the deterministic setting

Now we address the question of unique ergodicity of the collective walk in the deterministic setting. This means that the jump distributions are supported by single points: \( P(v^t_i = v_i) = 1 \) for some constant local velocities \( v := \{v_i\}_{i=1}^n \).
We start the analysis from the case when the local velocities $v_i$ take both positive and negative values and consider the partition of the set of indices into groups of consecutive indices of three types corresponding to negative, positive and zero velocities, e.g. the configuration of signs $++00-+-++$ has five groups of all three types. Since there are oppositely signed velocities the number of groups is greater than one.

**Theorem 5.** The process of gaps $\Delta_i^t$ (with either sequential or parallel updating) is uniquely ergodic if and only if the number of different groups is at most three, and the only group of zero velocity particles (if it exists) consists of a single element which is located after the group of positive particles and before negative particles (i.e. $++0-0$).

**Proof.** If the conditions of the theorem are satisfied each initial configuration of gaps converges in finite time to the configuration having a single positive gap of length $(1 - 2 \sum_i \Delta_i)$, which proves the unique ergodicity. The presence of two zero velocities or more than two groups of signed velocities obviously contradicts the unique ergodicity. The observation that the wrong location of the unique zero velocity particle (i.e. $++-0$) leads to the presence of the invariant measure supported by two points and sensitively depending on the position of this particle finishes the proof. $\square$

Now we are ready to address a more interesting situation of local velocities of the same (say positive) sign. If updating is sequential, theorem 4 gives a sufficient condition of the unique ergodicity

$$v_{\min} := \min_i \{v_i\} > \frac{1}{2n} (1 - 2 \sum_i r_i).$$

This condition is probably non-optimal and one is tempted to weaken it to

$$\sum_i v_i > 1 - 2 \sum_i r_i.$$  \hspace{1cm} (5.1)

Unfortunately a simple example with two zero velocities $v_1 = v_2 = 0$, $v_3 = v_4 = \ldots = v_n = 1$, which definitely satisfies (5.1), leads to infinitely many invariant measures of the gap process.

In the case of parallel updating we are able to get both necessary and sufficient conditions for the unique ergodicity, formulated in theorem 3.

**Proof of theorem 3.** Assume that the assumption (a) holds true. Since the system is translationally invariant without any loss of generality we may assume that the only minimum is achieved at the index $i_{\text{min}} = n$. The key point to our argument is that for any initial admissible particles configuration $x^0$ there exists a finite time $t_n = t_n(x^0, v, \{r_i\})$ such that for each $t > t_n$ the particle configuration $x^t$ satisfies the property:

$$\Delta_i^t = \Delta_2^t = \ldots = \Delta_{i-1}^t = \beta, \quad \Delta_i^t \geq \beta,$$  \hspace{1cm} (5.2)

for a certain $0 < \beta < v_n$. If $\beta < v_n$, then $\Delta_i^t = \beta$.

Indeed, if this is the case, starting from the time $t_n$ our dynamical system is a direct product of $n$ identical irrational rotation maps

$$x_i^{t+1} := x_i^t + \beta \mod (1).$$
This direct product system possesses a number of invariant measures, but the property (5.2) defines a unique invariant measure uniformly distributed on the segment given in the coordinates \((x_1, \ldots, x_n)\) by the relations
\[ x_1 = x_n - (n - 1)\beta, \quad x_2 = x_n - (n - 2)\beta, \quad \ldots, \quad x_{n-1} = x_n - \beta, \]
provided the number \(\beta\) is irrational.

Let us prove that the property (5.2) holds true. Observe that after at most
\[ t_{n-1} := \left(1 - 2 \sum_{i=1}^{n} r_i \right) / (v_{n-1} - v_n) \]
iterations the \((n - 1)\)th particle will catch up with the \(n\)th one and thus for each \(t \geq t_{n-1}\) the corresponding gap \(\Delta_{n-1}'\) is exactly equal to the length of jump that the \(n\)th particle will perform at time \(t\), in particular \(\Delta_{n-1}' \leq v_n\).

Similarly
\[ \forall t \geq t_{n-2} := t_{n-1} + \left(1 - 2 \sum_{i=1}^{n} r_i \right) / (v_{n-2} - v_n) \]
the gap \(\Delta_{n-2}'\) will match the length of jump of the \(n\)th particle, etc. Eventually, after at most
\[ t_1 := \left(1 - 2 \sum_{i=1}^{n} r_i \right) \left(\frac{1}{v_{n-1} - v_n} + \frac{1}{v_{n-2} - v_n} + \ldots + \frac{1}{v_1 - v_n}\right) \]
iterations all particles will start moving synchronously.

If the assumption (b) holds true, then for all \(t \geq t_1\)
\[ \Delta_n' \geq 1 - 2 \sum_{i=1}^{n} r_i - (n - 1)v_n = \alpha + v_n \]
and hence the \(n\)th particle will stop interacting with others and will perform the pure rotation through the irrational angle \(v_n\), which guarantees the unique ergodicity. Note that the rationality of the rotation leads to the presence of infinitely many invariant measures.

If (c) holds true then \(\Delta_n' = \alpha + v_n < v_n\) for all \(t \geq t_1\) and due to the interactions with other particles the \(n\)th one will perform the pure rotation through another irrational angle \(\alpha + v_n\).

It remains to check that the violation of the assumption (a) implies the absence of unique ergodicity. If the minimum is achieved at several indices then each of the slowest particles will generate a ‘train’ of faster particles following it in the same manner as has been shown for the case of the single minimum. If
\[ 1 - 2 \sum_{i=1}^{n} r_i > n v_{\min} \]
then using the same argument as in the case of the single minimum one finds a partition of particles into groups following one of the slowest particles. In each group the particles are moving at the same speed as the leading one. By (5.3) each group may be slightly shifted, not perturbing its motion and the motion of the other particles. Therefore the system possesses an infinite number of invariant measures, corresponding to the trajectories of the perturbed configurations.

If the inequality (5.3) is violated, then there are infinitely many different particle configurations \(x^0\) such that \(\Delta^0 < v_{\min}\) and this property holds for any \(t > 0\). Thus under dynamics we get \(\Delta_{i+1}' := \Delta_{i+1}'\) for all \(i, t\). Therefore the obtained configuration is a periodic
point of the process $\Delta'$, which again implies the presence of an infinite number of invariant measures.

Note that the ‘most natural’ case of identical particles and equal constant local velocities, i.e. when $r_1 = \cdots = r_n$, $v_1 = \cdots = v_n$, does not satisfy the conditions of unique ergodicity.

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