ON WEAK FANO VARIETIES WITH LOG CANONICAL SINGULARITIES

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ABSTRACT. We prove that the anti-canonical divisors of weak Fano 3-folds with log canonical singularities are semi-ample. Moreover, we consider semi-ampleness of the anti-log canonical divisor of any weak log Fano pair with log canonical singularities. We show semi-ampleness does not hold in general by constructing several examples. Based on those examples, we propose sufficient conditions which seem to be the best possible and we prove semi-ampleness under such conditions. In particular we derive semi-ampleness of the anti-canonical divisors of log canonical weak Fano 4-folds whose lc centers are at most 1-dimensional. We also investigate the Kleiman-Mori cones of weak log Fano pairs with log canonical singularities.

1. Introduction

Throughout this paper, we work over \( \mathbb{C} \), the complex number field. We start by some basic definitions.

Definition 1.1. Let \( X \) be a normal projective variety and \( \Delta \) an effective \( \mathbb{Q} \)-Weil divisor on \( X \). We say that \((X, \Delta)\) is a \textit{weak log Fano pair} if \(- (K_X + \Delta)\) is nef and big. If \( \Delta = 0 \), then we simply say that \( X \) is a \textit{weak Fano variety}.

Definition 1.2. Let \( X \) be a normal variety and \( \Delta \) an effective \( \mathbb{Q} \)-Weil divisor on \( X \) such that \( K_X + \Delta \) is a \( \mathbb{Q} \)-Cartier divisor. Let \( \varphi : Y \to X \) be a log resolution of \((X, \Delta)\). We set

\[
K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,
\]

where \( E_i \) is a prime divisor. The pair \((X, \Delta)\) is called

(a) \textit{kawamata log terminal} (klt, for short) if \( a_i > -1 \) for all \( i \), or
(b) \textit{log canonical} (lc, for short) if \( a_i \geq -1 \) for all \( i \).
Definition 1.3 (Lc center). Let \((X, \Delta)\) be an lc pair. We call that \(C \subset X\) is an \textit{lc center} of \((X, \Delta)\) if there exists a log resolution \(\varphi : Y \to X\) such that \(\varphi(E) = C\) for some prime divisor \(E\) on \(Y\) with \(a(E, X, \Delta) = -1\).

There are questions whether the following fundamental properties hold or not for a log canonical weak log Fano pair \((X, \Delta)\) (cf. [S, 2.6. Remark-Corollary], [P, 11.1]):

(i) Semi-ampleness of \(- (K_X + \Delta)\).
(ii) Existence of \(\mathbb{Q}\)-complements, i.e., existence of an effective \(\mathbb{Q}\)-divisor \(D\) such that \(K_X + \Delta + D \sim_{\mathbb{Q}} 0\) and \((X, \Delta + D)\) is lc.
(iii) Rational polyhedrality of the Kleiman-Mori cone \(\text{NE}(X)\).

It is easy to see that (i) implies (ii). In the case where \((X, \Delta)\) is a klt pair, the above three properties hold by the Kawamata-Shokurov base point free theorem and the cone theorem (cf. [KMM], [KoM]). Shokurov proved that these three properties hold for surfaces (cf. [S, 2.5. Proposition]).

Among other things, we prove the following:

Theorem 1.4 (=Corollaries 3.3 and 4.5). Let \(X\) be a weak Fano 3-fold with log canonical singularities. Then \(-K_X\) is semi-ample and \(\text{NE}(X)\) is a rational polyhedral cone.

Theorem 1.5 (=Corollary 3.4 and Theorem 4.4). Let \(X\) be a weak Fano 4-fold with log canonical singularities. Suppose that any lc center of \(X\) is at most 1-dimensional. Then \(-K_X\) is semi-ample and \(\text{NE}(X)\) is a rational polyhedral cone.

On the other hand, the above three properties do not hold for \(d\)-dimensional log canonical weak log Fano pairs in general, where \(d \geq 3\). Indeed, we give the following examples of plt weak log Fano pairs whose anti-log canonical divisors are not semi-ample in Section 5 (in particular, such examples of 3-dimensional weak log Fano plt pairs show the main result of [Kar1] does not hold). It is well known that there exists a \((d-1)\)-dimensional smooth projective variety \(S\) such that \(-K_S\) is nef and is not semi-ample (e.g. When \(d = 3\), we take a very general 9-points blow up of \(\mathbb{P}^2\) as \(S\)). Let \(X_0\) be the cone over \(S\) with respect to some projectively normal embedding \(S \subset \mathbb{P}^N\). We take the blow-up \(X\) of \(X_0\) at its vertex. Let \(E\) be the exceptional divisor of the blow-up. Then the pair \((X, E)\) is a weak log Fano plt pair such that \(- (K_X + E)\) is not semi-ample. Moreover we give an example of a log canonical weak log Fano pair without \(\mathbb{Q}\)-complements and an example whose Kleiman-Mori cone is not polyhedral.
We now outline the proof of semi-ampleness of $-K_X$ as in Theorem 1.4. First, we take a birational morphism $\varphi : Y \to X$ such that $\varphi^*(K_X) = K_Y + S$, $(Y, S)$ is dlt and $S$ is reduced. We set $C := \varphi(S)$, which is the union of lc centers of $X$. By an argument in the proof of the Kawamata-Shokurov base point free theorem (Lemma 2.11), it is sufficient to prove that $- (K_Y + S)|_S$ is semi-ample. Moreover we have only to prove that $-K_X|_C$ is semi-ample by the formula $K_X|_C = (\varphi|_S)^*([K_Y + S]|_S)$.

It is not difficult to see semi-ampleness of the restriction of $-K_X$ on any lc center of $X$. The main difficulty is how to extend semi-ampleness to $C$ from each 1-dimensional irreducible component $C_i$ of $C$ since the configuration of $C_i$’s may be complicated. The key to overcome this difficulty is the abundance theorem for 2-dimensional semi-divisorial log terminal pairs ([AFKM]). We decompose $C = C' \cup C''$, where

$$\Sigma := \{i \mid -K_X|_{C_i} \equiv 0\}, \ C' := \bigcup_{i \in \Sigma} C_i, \ \text{and} \ C'' := \bigcup_{i \notin \Sigma} C_i.$$ 

Let $S'$ be the union of the irreducible components of $S$ whose image on $X$ is contained in $C'$. We define the boundary $\text{Diff}_{S'}(S)$ on $S'$ by the formula $K_Y + S|_{S'} = K_{S'} + \text{Diff}_{S'}(S)$. The pair $(S', \text{Diff}_{S'}(S))$ is known to be semi-divisorial log terminal pair (sdlt, for short). Applying the abundance theorem to the pair $(S', \text{Diff}_{S'}(S))$, we see that $K_{S'} + \text{Diff}_{S'}(S)$ is $\mathbb{Q}$-linearly trivial, namely, there exists a non-zero integer $m_1$ such that $-m_1(K_Y + S)|_{S'} = -m_1(K_{S'} + \text{Diff}_{S'}(S)) \sim 0$. This shows that $-m_1K_X|_{C'} \sim 0$. On the other hand, since $-K_X|_{C''}$ is ample, we can take enough sections of $H^0(C'', -m_2K_X|_{C''})$ for a sufficiently large and divisible $m_2$ (Lemma 2.16). Thus, we can find enough sections of $H^0(C, -mK_X|_{C})$ for a sufficiently large and divisible $m$, and can conclude that $-K_X|_{C}$ is semi-ample.

To generalize this theorem to higher dimensional weak log Fano pairs, let us recall the following conjectures:

**Conjecture 1.6** (Abundance conjecture in a special case). Let $(X, \Delta)$ be a $d$-dimensional projective sdlt pair whose $K_X + \Delta$ is numerically trivial. Then $K_X + \Delta$ is $\mathbb{Q}$-linearly trivial, i.e., there exists an $n \in \mathbb{N}$ such that $n(K_X + \Delta) \sim 0$.

The abundance conjecture is one of the most famous conjecture in the minimal model program. This conjecture is true when $d \leq 3$ by the works of Fujita, Kawamata, Miyaoka, Abramovich, Fong, Kollár, McKernan, Keel, Matsuki, and Fujino.

By the same way as in the 3-dimensional case, we see the following theorem:
**Theorem 1.7** (=Theorem 3.1). Assume that Conjecture 1.6 in dimension $d - 1$ holds. Let $(X, \Delta)$ be a $d$-dimensional log canonical weak log Fano pair. Suppose that $M(X, \Delta) \leq 1$, where

$$M(X, \Delta) := \max \{ \dim P \mid P \text{ is an lc center of } (X, \Delta) \}.$$ 

Then $-(K_X + \Delta)$ is semi-ample.

Indeed, semi-ampleness of $-K_X$ as in Theorem 1.4 is derived from the above theorem since the singular locus of any normal 3-fold is at most 1-dimensional and Conjecture 1.6 for surfaces holds ([APKM]). We also derive semi-ampleness of weak Fano 4-folds such that $M(X, 0) \leq 1$ because Conjecture 1.6 for 3-folds holds ([Fj1]). We remark that by Examples 5.2 and 5.3, this condition for the dimension of lc centers is the best possible.

In Section 4 by the cone theorem for normal varieties by Ambro and Fujino (cf. Theorem 4.3), we derive the following:

**Theorem 1.8** (=Theorem 4.4). Let $(X, \Delta)$ be a $d$-dimensional log canonical weak log Fano pair. Suppose that $M(X, \Delta) \leq 1$. Then $\overline{NE}(X)$ is a rational polyhedral cone.

Note that rational polyhedrality of $\overline{NE}(X)$ as in Theorem 1.4 is a corollary of the above theorem. In Example 5.6, we also see that the Kleiman-Mori cone is not rational polyhedral in general when $M(X, \Delta) \geq 2$.

This paper is based on the minimal model theory for log canonical pairs developed by Ambro and Fujino ([A1], [A2], [A3], [Fj5], [Fj6], [Fj7]).

We will make use of the standard notation and definitions as in [KoM].

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2. Preliminaries and Lemmas

In this section, we introduce notation and some lemmas for the proof of Theorem 1.7 (=Theorem 3.1).

Definition 2.1. For a \(\mathbb{Q}\)-Weil divisor \(D = \sum_{j=1}^{r} d_j D_j\) such that \(D_j\) is a prime divisor for every \(j\) and \(D_i \neq D_j\) for \(i \neq j\), we define the round-up \(\lceil D \rceil = \sum_{j=1}^{r} \lceil d_j \rceil D_j\) (resp. the round-down \(\lfloor D \rfloor = \sum_{j=1}^{r} \lfloor d_j \rfloor D_j\)), where for every real number \(x\), \(\lceil x \rceil \) (resp. \(\lfloor x \rfloor \)) is the integer defined by \(x \leq \lceil x \rceil < x + 1\) (resp. \(x - 1 < \lfloor x \rfloor \leq x\)). The fractional part \(\{D\}\) of \(D\) denotes \(D - \lfloor D \rfloor\).

We call \(D\) a boundary \(\mathbb{Q}\)-divisor if \(0 \leq d_j \leq 1\) for every \(j\).

Definition 2.2 (Stratum). Let \((X, \Delta)\) be an lc pair. A stratum of \((X, \Delta)\) denotes \(X\) itself or an lc center of \((X, \Delta)\).

The following theorem is very important as a generalization of vanishing theorems (cf. [A2, Theorem 3.1], [Fj5, Theorem 2.2], [Fj6, Theorem 2.38], [Fj7, Theorem 6.3]).

Theorem 2.3 (Torsion-freeness theorem). Let \(Y\) be a smooth variety and \(B\) a boundary \(\mathbb{R}\)-divisor such that \(\text{Supp} B\) is simple normal crossing. Let \(f : Y \to X\) be a projective morphism and \(L\) a Cartier divisor on \(Y\) such that \(H \sim_{\mathbb{R}} L - (K_Y + B)\) is \(f\)-semi-ample. Then every associated prime of \(R^q f_* \mathcal{O}_Y(L)\) is the generic point of the \(f\)-image of some stratum of \((Y, B)\) for any non-negative integer \(q\).

The following theorem is proved by Fujino ([Fj7, Theorem 10.5]). We include the proof for the reader’s convenience.

Theorem 2.4. Let \(X\) be a normal quasi-projective variety and \(\Delta\) an effective \(\mathbb{Q}\)-divisor on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. Suppose that \((X, \Delta)\) is lc. Then there exists a projective birational morphism \(\varphi : Y \to X\) from a normal quasi-projective variety with the following properties:

(i) \(Y\) is \(\mathbb{Q}\)-factorial,
(ii) \(a(E, X, \Delta) = -1\) for every \(\varphi\)-exceptional divisor \(E\) on \(Y\),
(iii) for \(\Gamma = \varphi^{-1} \Delta + \sum_{E: \varphi\text{-exceptional}} E\),
it holds that \( Y, \Gamma \) is dlt and \( K_Y + \Gamma = \varphi^*(K_X + \Delta) \), and

(iv) Let \( \{ C_i \} \) be any set of lc centers of \((X, \Delta)\). Let \( W = \bigcup C_i \) with a reduced structure and \( S \) the union of the irreducible components of \( \Gamma \) which are mapped into \( W \) by \( \varphi \). Then \((\varphi|_S)_*\mathcal{O}_S \cong \mathcal{O}_W \).

**Proof.** Let \( \pi : V \to X \) be a resolution such that

1. \( \pi^{-1}(C) \) is a simple normal crossing divisor on \( V \) for every lc center \( C \) of \((X, \Delta)\), and

2. \( \pi^{-1}_- \Delta \cup \text{Exc}(\pi) \cup \pi^{-1}_-(\text{Nklt}(X, \Delta)) \) has a simple normal crossing support, where \( \text{Exc}(\pi) \) is the exceptional set of \( \pi \) and \( \text{Nklt}(X, \Delta) \) is the union of lc centers of \((X, \Delta)\).

By Hironaka’s resolution theorem, we can assume that \( \pi \) is a composite of blow-ups with centers of codimension at least two. Then there exists an effective \( \pi \)-exceptional Cartier divisor \( B \) on \( V \) such that \( -B \) is \( \pi \)-ample. We put

\[
F = \sum_{a(E, X, \Delta) > -1} E \quad \text{and} \quad G = \sum_{a(E, X, \Delta) = -1} E.
\]

Let \( H \) be a sufficiently ample Cartier divisor on \( X \) such that \(-B + \pi^*H\) is ample. We choose \( 0 < \varepsilon \ll 1 \) such that \( \varepsilon G - B + \pi^*(H) \) is ample. Since \(-B + \pi^*(H)\) and \( \varepsilon G - B + \pi^*(H) \) are ample, we can take effective \( \mathbb{Q} \)-divisors \( H_1 \) and \( H_2 \) on \( V \) with small coefficients such that \( G + F + \pi^{-1}_- \Delta + H_1 + H_2 \) has a simple normal crossing support and that \(-B + \pi^*H \sim_{\mathbb{Q}} H_1, \varepsilon G - B + \pi^*(H) \sim_{\mathbb{Q}} H_2 \). We take \( 0 < \nu, \mu \ll 1 \) such that every divisor in \( F \) has a negative coefficient in \( M := \Gamma_V - G - (1 - \nu)F - \pi^{-1}_- \Delta^{<1} + \mu B \),

where \( \Gamma_V \) is a \( \mathbb{Q} \)-divisor on \( V \) such that \( K_V + \Gamma_V = \pi^*(K_X + \Delta) \). Now we construct a log minimal model of \((V, G + (1 - \nu)F + \pi^{-1}_- \Delta^{<1} + \mu H_1)\) over \( X \). Since

\[
G + (1 - \nu)F + \mu H_1 \sim_{\mathbb{Q}} (1 - \varepsilon \mu)G + (1 - \nu)F + \mu H_2,
\]

it is sufficient to construct a log minimal model of \((V, (1 - \varepsilon \mu)G + (1 - \nu)F + \pi^{-1}_- \Delta^{<1} + \mu H_2)\) over \( X \). Because \((V, (1 - \varepsilon \mu)G + (1 - \nu)F + \pi^{-1}_- \Delta^{<1} + \mu H_2)\) is klt, we can get a log minimal model \( \varphi : Y \to X \) of \((V, (1 - \varepsilon \mu)G + (1 - \nu)F + \pi^{-1}_- \Delta^{<1} + \mu H_2)\) over \( X \) by [BCHM, Theorem 1.2].

We show this \( Y \) satisfies the conditions of the theorem. For any divisor \( D \) on \( V \) (appearing above), let \( D' \) denote its strict transform on \( Y \). We see the following claim:

**Claim 2.5.** \( F' = 0 \).
Proof of Claim 2.5. By the above construction,

\[ N := K_Y + G' + (1 - \nu)F' + \varphi^{-1}\Delta^{<1} + \mu H_1' \]

is \( \varphi \)-nef. Then

\[ -M' \sim_{\mathbb{Q}, \varphi} N - (K_Y + \Gamma_Y) \]

since \((\pi^*H)’ = \varphi^*H\), hence it is \( \varphi \)-nef. Since \( \varphi_\ast M' = 0 \), we see that \( M' \) is effective by the negativity lemma (cf. \[KoM, \text{Lemma 3.39}\]). Since every divisor in \( F \) has a negative coefficient in \( M' \), \( F \) is contracted on \( Y \). We finish the proof of Claim 2.5. \( \square \)

From Claim 2.5, the discrepancy of every \( \varphi \)-exceptional divisor is equal to \(-1\). We see that \( Y \) satisfies the condition (ii). By the above construction, \((Y, \Gamma)\) is a \( \mathbb{Q} \)-factorial dlt pair since so is \((Y, G' + \varphi^{-1}\Delta^{<1} + \mu H_1)\). We see the condition (i). Because the support of \( K_Y + \Gamma - \varphi^*(K_X + \Delta) \) coincide with \( F' \), we see the condition (iii).

Now, we show that \( Y \) and \( \varphi \) satisfy the condition (iv). Since we get \( Y \) by the log minimal model program over \( X \) with scaling of some effective divisor with respect to \( K_V + G + (1 - \nu)F + \pi^{-1}\Delta^{<1} + \mu H_1 \) (cf. \[BCHM\]), we see that the rational map \( f : V \to Y \) is a composition of \((K_V + G + (1 - \nu)F + \pi^{-1}\Delta^{<1} + \mu H_1)\)-negative divisorial contractions and log flips. Let \( \Sigma \) be an lc center of \((Y, \Gamma)\). Then it is also an lc center of \((Y, \Gamma + \mu H_1')\). By the negativity lemma, \( f : V \to Y \) is an isomorphism around the generic point of \( \Sigma \). Therefore, if \( \varphi(\Sigma) \subseteq W \), then \( \Sigma \subseteq S \) by the conditions (1) and (2) for \( \pi : V \to X \). This means that no lc centers of \((Y, \Gamma - S)\) are mapped into \( W \) by \( \varphi \). Let \( g : Z \to Y \) be a resolution such that

(a) \( \text{Supp } \Gamma_Z \) is a simple normal crossing divisor, where \( \Gamma_Z \) is defined by \( K_Z + \Gamma_Z = g^*(K_Y + \Gamma) \), and

(b) \( g \) is an isomorphism over the generic point of any lc center of \((Y, \Gamma)\).

Let \( S_Z \) be the strict transform of \( S \) on \( Z \). We consider the following short exact sequence

\(*\) \[ 0 \to \mathcal{O}_Z(\Gamma - (\Gamma_Z^{<1})^{-} - S_Z) \to \mathcal{O}_Z(\Gamma - (\Gamma_Z^{<1})^{-}) \]
\[ \to \mathcal{O}_{S_Z}(\Gamma - (\Gamma_Z^{<1})^{-}) \to 0. \]

We note that

\[ \Gamma - (\Gamma_Z^{<1})^{-} - S_Z - (K_Z + \{\Gamma_Z\} + \Gamma_Z^{<1} - S_Z) \sim_{\mathbb{Q}} -h^*(K_X + \Delta), \]
where $h = \varphi \circ g$. Then we obtain

$$0 \rightarrow h_* \mathcal{O}_Z(\Gamma_Z \setminus \Gamma_Z^{\leq 1} \setminus S_Z) \rightarrow h_* \mathcal{O}_Z(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \rightarrow h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \delta \rightarrow R^1 h_* \mathcal{O}_Z(\Gamma_Z \setminus \Gamma_Z^{\leq 1} \setminus S_Z) \rightarrow \cdots.$$ 

We claim the following:

**Claim 2.6.** $\delta$ is a zero map.

*Proof of Claim 2.6.* Let $\Sigma$ be an lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{\leq 1} - S_Z)$. Then $\Sigma$ is some intersection of components of $\Gamma_Z^{=1} - S_Z$. By the conditions (a) and (b), $\Gamma_Z^{=1} - S_Z$ is the strict transform of $\cup \Gamma - S$. By this, the image of $\Sigma$ by $g$ is some intersection of components of $\cup \Gamma - S$. In particular, $g(\Sigma)$ is an lc center of $(Y, \Gamma - S)$. Thus no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into $W$ by $h$. Assume by contradiction that $\delta$ is not zero. Then there exists a section $s \in H^0(U, h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}))$ for some non-empty open set $U \subseteq X$ such that $\delta(s) \neq 0$. Since $\text{Supp} \delta(s) \neq \emptyset$, we can take an associated prime $x \in \text{Supp} \delta(s)$. We see that $x \in W$ since $\text{Supp}(h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}))$ is contained in $W$. By Theorem 2.3, $x$ is the generic point of the $h$-image of some stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Since $h$ is a birational morphism, $x$ is the generic point of the $h$-image of some lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Because no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into $W$ by $h$, it holds that $x \notin W$. But this contradicts the way of taking $x$. 

Thus, we obtain

$$0 \rightarrow \mathcal{I}_W \rightarrow \mathcal{O}_X \rightarrow h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \rightarrow 0,$$

where $\mathcal{I}_W$ is the defining ideal sheaf of $W$ since $\Gamma_Z \setminus \Gamma_Z^{\leq 1}$ is effective and $h$-exceptional. This implies that $\mathcal{O}_W \simeq h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1})$. By applying $g_*$ to (2), we obtain

$$0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{O}_Y \rightarrow g_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \rightarrow 0,$$

where $\mathcal{I}_S$ is the defining ideal sheaf of $S$ since $\Gamma_Z \setminus \Gamma_Z^{\leq 1}$ is effective and $g$-exceptional. We note that

$$R^1 g_* \mathcal{O}_Z(\Gamma_Z \setminus \Gamma_Z^{\leq 1} \setminus S_Z) = 0$$

by Theorem 2.3 since $g$ is an isomorphism at the generic point of any stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Thus, $\mathcal{O}_W \simeq h_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \simeq \varphi_* g_* \mathcal{O}_{S_Z}(\Gamma_Z \setminus \Gamma_Z^{\leq 1}) \simeq \varphi_* \mathcal{O}_S$. We finish the proof of Theorem 2.4. □
Definition 2.7. Let $X$ be a normal variety and $D$ a $\mathbb{Q}$-Weil divisor. We define that
$$R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \lfloor mD \rfloor).$$

Definition 2.8 (semi-divisorial log terminal, cf. [Fj1]). Let $X$ be a reduced $S_2$-scheme. We assume that it is pure $d$-dimensional and is normal crossing in codimension 1. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \bigcup X_i$ be the decomposition into irreducible components, and $\nu : X' := \coprod X'_i \to X = \bigcup X_i$ the normalization. Define the $\mathbb{Q}$-divisor $\Theta$ on $X'$ by $K_X' + \Theta := \nu^*(K_X + \Delta)$ and set $\Theta_i := \Theta|_{X'_i}$.

We say that $(X, \Delta)$ is semi-divisorial log terminal (for short, $\text{sdlt}$) if $X_i$ is normal, that is, $X'_i$ is isomorphic to $X_i$, and $(X'_i, \Theta_i)$ is dlt for every $i$.

Definition and Lemma 2.9 (Different, cf. [C]). Let $(Y, \Gamma)$ be a dlt pair and $S$ a union of some components of $\lfloor \Gamma \rfloor$. Then there exists an effective $\mathbb{Q}$-divisor $\text{Diff}_S(\Gamma)$ on $S$ such that $(K_Y + \Gamma)|_S \sim_\mathbb{Q} K_S + \text{Diff}_S(\Gamma)$. The effective $\mathbb{Q}$-divisor $\text{Diff}_S(\Gamma)$ is called the different of $\Gamma$. Moreover it holds that $(S, \text{Diff}_S(\Gamma))$ is sdlt.

The following proposition is [Fk2 Proposition 2] (for the proof, see [Fk1], Proof of Theorem 3) and [Kaw Lemma 3]).

Proposition 2.10. Let $(X, \Delta)$ be a proper dlt pair and $L$ a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{N}$. If $Bs|mL| \cap \lfloor \Delta \rfloor = \emptyset$ for every $m \gg 0$, then $|mL|$ is base point free for every $m \gg 0$, where $Bs|mL|$ is the base locus of $|mL|$.

By this proposition, we derive the following lemma:

Lemma 2.11. Let $(Y, \Gamma)$ be a $\mathbb{Q}$-factorial weak log Fano dlt pair. Suppose that $-(K_S + \Gamma_S)$ is semi-ample, where $S := \lfloor \Gamma \rfloor$ and $\Gamma_S := \text{Diff}_S(\Gamma)$. Then $-(K_Y + \Gamma)$ is semi-ample.

Proof. We consider the exact sequence
$$0 \to \mathcal{O}_Y(-m(K_Y + \Gamma) - S) \to \mathcal{O}_Y(-m(K_Y + \Gamma)) \to \mathcal{O}_S(-m(K_Y + \Gamma)|_S) \to 0$$
for $m \gg 0$. By the Kawamata-Viehweg vanishing theorem (cf. [KMM], Theorem 1-2-5., [KoM] Theorem 2.70]), we have
$$H^1(Y, \mathcal{O}_Y(-m(K_Y + \Gamma) - S)) = H^1(Y, \mathcal{O}_Y(K_Y + \Gamma - S - (m+1)(K_X + \Gamma))) = \{0\},$$
since the pair \((Y, \Gamma - S)\) is klt and \(-(K_Y + \Gamma)\) is nef and big. Thus, we get the exact sequence

\[
H^0(Y, \mathcal{O}_Y(-m(K_Y + \Gamma))) \to H^0(S, \mathcal{O}_S(-m(K_Y + \Gamma)|_S)) \to 0.
\]

Therefore, we see that \(B_S| - m(K_Y + \Gamma)| \cap S = \emptyset\) for \(m \gg 0\) since \(-(K_S + \Delta_S)\) is semi-ample. Applying Proposition 2.10 we conclude that \(-(K_Y + \Gamma)\) is semi-ample. ☐

**Definition 2.12.** (cf. [GT 1.1. Definition], [KoS Definition 7.1])

Suppose that \(R\) is a reduced excellent ring and \(R \subseteq S\) is a reduced \(R\)-algebra which is finite as an \(R\)-module. We say that the extension \(i : R \hookrightarrow S\) is subintegral if one of the following equivalent conditions holds:

1. \((S \otimes_R k(p))_{\text{red}} = k(p)\) for all \(p \in \text{Spec}(R)\).
2. the induced map on the spectra is bijective and \(i\) induces trivial residue field extensions.

**Definition 2.13.** [KoS Definition 7.2]

Suppose that \(R\) is a reduced excellent ring. We say that \(R\) is semi-normal if every subintegral extension \(R \hookrightarrow S\) is an isomorphism.

A scheme \(X\) is called semi-normal at \(q \in X\) if the local ring at \(q\) is semi-normal. If \(X\) is semi-normal at every point, we say that \(X\) is semi-normal.

**Proposition 2.14.** [GT 5.3. Corollary] Let \((R, m)\) be a local excellent ring. Then \(R\) is semi-normal if and only if \(\hat{R}\) is semi-normal, where \(\hat{R}\) is \(m\)-adic completion of \(R\).

**Proposition 2.15.** (cf. [Ko 7.2.2.1], [KoS Remark 7.6]) Let \(C\) be a pure 1-dimensional proper reduced scheme of finite type over \(\mathbb{C}\), and \(q \in C\) a closed point. Then \(C\) is semi-normal at \(q\) if and only if \(\hat{\mathcal{O}}_{C,q}\) satisfies that

1. \(\hat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X]]\), or
2. \(\hat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X_1, X_2, \ldots, X_r]]/\langle X_iX_j | 1 \leq i \neq j \leq r \rangle\) for some \(r \geq 2\), i.e., \(q \in C\) is isomorphic to the coordinate axes in \(\mathbb{C}^r\) at the origin as a formal germs.

**Lemma 2.16.** Let \(C = C_1 \cup C_2\) be a pure 1-dimensional proper semi-normal reduced scheme of finite type over \(\mathbb{C}\), where \(C_1\) and \(C_2\) are pure 1-dimensional reduced closed subschemes. Let \(D\) be a \(\mathbb{Q}\)-Cartier divisor on \(C\). Suppose that \(D_1\) is \(\mathbb{Q}\)-linearly trivial and \(D_2\) is ample, where \(D_i := D|_{C_i}\). Then \(D\) is semi-ample.

**Proof.** Let \(C_1 \cap C_2 = \{p_1, \ldots, p_r\}\). We take \(m \gg 0\) which satisfies the following:
(i) \( mD_1 \sim 0 \),
(ii) \( \mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_{k \neq l} \mathfrak{m}_{p_k}) \) is generated by global sections for all \( l \in \{1, \ldots, r\} \), and
(iii) \( \mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_{k} \mathfrak{m}_{p_k}) \) is generated by global sections,
where \( \mathfrak{m}_{p_k} \) is the ideal sheaf of \( p_k \) on \( C_2 \). We choose a nowhere vanishing section \( s \in H^0(C_1, mD_1) \). By (ii), we can take a section \( t_l \in H^0(C_2, mD_2) \) which does not vanish at \( p_l \) but vanishes at all the \( p_k \) \( (k \in \{1, \ldots, r\}, k \neq l) \) for each \( l \in \{1, \ldots, r\} \). By multiplying suitable nonzero constants to \( t_l \), we may assume that \( t_l|_{p_l} = s|_{p_l} \). We set \( t := \sum t_l \in H^0(C_2, mD_2) \). Since \( C \) is semi-normal, Proposition 2.15 implies that \( \mathcal{O}_{C_1 \cap C_2} \simeq \bigoplus_{l=1}^r \mathbb{C}(p_l) \), where \( \mathbb{C}(p_l) \) is the skyscraper sheaf \( \mathbb{C} \) sitting at \( p_l \), by computations on \( \mathcal{O}_{C,p_l} \). Thus we get the following exact sequence:

\[
0 \to \mathcal{O}_C(mD) \to \mathcal{O}_{C_1}(mD_1) \oplus \mathcal{O}_{C_2}(mD_2) \to \bigoplus_{l=1}^r \mathbb{C}(p_l) \to 0,
\]

where the third arrow maps \((s', s'')\) to \((\{s' - s''\}|_{p_1}, \ldots, \{s' - s''\}|_{p_r})\).

Hence \( s \) and \( t \) patch together and give a section \( u \) of \( H^0(C, mD) \).

Let \( p \) be any point of \( C \). If \( p \in C_1 \), then \( u \) does not vanish at \( p \). We may assume that \( p \in C_2 \setminus C_1 \). By (iii), we can take a section \( t' \in H^0(C_2, mD_2) \) which does not vanish at \( p \) but vanishes at \( p_l \) for all \( l \in \{1, \ldots, r\} \). The zero section \( 0 \in H^0(C_1, mC_1) \) and \( t' \) patch together and give a section \( u' \) of \( H^0(C, mD) \). By construction, the section \( u' \) does not vanish at \( p \). We finish the proof of Lemma 2.16. \( \square \)

3. On semi-ampleness for weak Fano varieties

In this section, we prove Theorem 1.7 (=Theorem 3.1). As a corollary, we see that the anti-canonical divisors of weak Fano 3-folds with log canonical singularities are semi-ample. Moreover we derive semi-ampleness of the anti-canonical divisors of log canonical weak Fano 4-folds whose lc centers are at most 1-dimensional.

**Theorem 3.1.** Assume that Conjecture 1.6 in dimension \( d - 1 \) holds. Let \((X, \Delta)\) be a \( d \)-dimensional log canonical weak log Fano pair. Suppose that \( M(X, \Delta) \leq 1 \), where

\[
M(X, \Delta) := \max\{\dim P \mid P \text{ is an lc center of } (X, \Delta)\}.
\]

Then \(- (K_X + \Delta)\) is semi-ample.

**Proof.** By Theorem 2.4, we take a birational morphism \( \varphi : (Y, \Gamma) \to (X, \Delta) \) as in the theorem. We set \( S := \cup \Gamma \) and \( C := \varphi(S) \), where we consider the reduced scheme structures on \( S \) and \( C \). We have only to
prove that \(-(K_X + \Gamma_S) = -(K_Y + \Gamma)|_S\) is semi-ample from Lemma 2.11.
By the formula \((K_Y + \Gamma)|_S \sim_{\mathbb{Q}} (\varphi|_S)^*((K_X + \Delta)|_C)\), it suffices to show that \(-(K_X + \Delta)|_C\) is semi-ample. Arguing on each connected component of \(C\), we may assume that \(C\) is connected. Since \(M(X, \Delta) \leq 1\), it holds that \(\dim C \leq 1\). When \(\dim C = 0\), i.e., \(C\) is a closed point, then \(-(K_X + \Delta)|_C \sim_{\mathbb{Q}} 0\), in particular, is semi-ample.
When \(\dim C = 1\), \(C\) is a pure 1-dimensional semi-normal scheme by [A3 Theorem 1.1] or [Fj7 Theorem 9.1]. Let \(C = \bigcup_{i=1}^{r} C_i\), where \(C_i\) is an irreducible component, and let \(D := -(K_X + \Delta)|_C\) and \(D_i := D|_{C_i}\). We set
\[
\Sigma := \{i \mid D_i \equiv 0\}, \quad C' := \bigcup_{i \in \Sigma} C_i, \quad C'' := \bigcup_{i \notin \Sigma} C_i.
\]
Let \(S'\) be the union of irreducible components of \(S\) whose image by \(\varphi\) is contained in \(C'\). We see that \(K_{S'} + \Gamma_{S'} \equiv 0\), where \(\Gamma_{S'} := \text{Diff}(\Gamma_{C'})\).
Thus it holds that \(K_{S'} + \Gamma_{S'} \sim_{\mathbb{Q}} 0\) by applying Conjecture 1.6 to \((S', \Gamma_{S'})\). Since \((\varphi|_{S'})_* \mathcal{O}_{S'} \simeq \mathcal{O}_{C'}\), by the condition (iv) in Theorem 2.4 it holds that \(D|_{C'} \sim_{\mathbb{Q}} 0\). We see that \(D|_{C''}\) is ample since the restriction of \(D\) on any irreducible component of \(C''\) is ample. By Lemma 2.16 we see that \(D = -(K_X + \Delta)|_C\) is semi-ample. We finish the proof of Theorem 3.1
\[\Box\]

**Corollary 3.2.** Assume that Conjecture 1.6 in dimension \(d - 1\) holds. Let \((X, \Delta)\) be a \(d\)-dimensional log canonical weak log Fano pair. Suppose that \(M(X, \Delta) \leq 1\). Then \(R(X, -(K_X + \Delta))\) is a finitely generated algebra over \(\mathbb{C}\).

Conjecture 1.6 holds for surfaces and 3-folds by [AFKM] and [Fj1].
Thus we immediately obtain the following corollaries:

**Corollary 3.3.** Let \((X, \Delta)\) be a 3-dimensional log canonical weak log Fano pair. Suppose that \(\Delta, \frac{\Delta}{3} = 0\). Then \(-K_X + \Delta\) is semi-ample and \(R(X, -(K_X + \Delta))\) is a finitely generated algebra over \(\mathbb{C}\). In particular, if \(X\) is a weak Fano 3-fold with log canonical singularities, then \(-K_X\) is semi-ample and \(R(X, -K_X)\) is a finitely generated algebra over \(\mathbb{C}\).

**Corollary 3.4.** Let \((X, \Delta)\) be a 4-dimensional log canonical weak log Fano pair. Suppose that \(M(X, \Delta) \leq 1\). Then \(-K_X + \Delta\) is semi-ample and \(R(X, -(K_X + \Delta))\) is a finitely generated algebra over \(\mathbb{C}\). In particular, if \(X\) is a log canonical weak Fano 4-fold whose lc centers are at most 1-dimensional, then \(-K_X\) is semi-ample and \(R(X, -K_X)\) is a finitely generated algebra over \(\mathbb{C}\).

**Remark 3.5.** When \(M(X, \Delta) \geq 2\), \(-(K_X + \Delta)\) is not semi-ample and \(R(X, -(K_X + \Delta))\) is not a finitely generated algebra over \(\mathbb{C}\), in general (Examples 5.2 and 5.3).
Remark 3.6. Based on Theorem 3.1, we expect the following statement:

Let \((X, \Delta)\) be an lc pair and \(D\) a nef Cartier divisor. Suppose there is a positive number \(a\) such that \(aD - (K_X + \Delta)\) is nef and big. If it holds that \(M(X, \Delta) \leq 1\), then \(D\) is semi-ample.

However, there is a counterexample for this statement due to Zariski (cf. [KMM, Remark 3-1-2], [Z]).

4. On the Kleiman-Mori cone for weak Fano varieties

In this section, we introduce the cone theorem for normal varieties by Ambro and Fujino and prove polyhedrality of the Kleiman-Mori cone for a log canonical weak Fano variety whose lc centers are at most 1-dimensional. We use the notion of the scheme \(\text{Nlc}(X, \Delta)\), whose underlying space is the set of non-log canonical singularities. For the scheme structure on \(\text{Nlc}(X, \Delta)\), we refer [Fj7, Section 7] and [Fj4] in detail.

Definition 4.1. ([Fj7, Definition 16.1]) Let \(X\) be a normal variety and \(\Delta\) an effective \(\mathbb{Q}\)-divisor on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. Let \(\pi : X \to S\) be a projective morphism. We put

\[
\overline{NE}(X/S)_{\text{Nlc}(X, \Delta)} = \text{Im}(\overline{NE}(\text{Nlc}(X, \Delta)/S) \to \overline{NE}(X/S)).
\]

Definition 4.2. ([Fj7, Definition 16.2]) An extremal face of \(\overline{NE}(X/S)\) is a non-zero subcone \(F \subset \overline{NE}(X/S)\) such that \(z, z' \in F\) and \(z + z' \in F\) implies that \(z, z' \in F\). Equivalently, \(F = \overline{NE}(X/S) \cap H^\perp\) for some \(\pi\)-nef \(\mathbb{R}\)-divisor \(H\), which is called a supporting function of \(F\). An extremal ray is a one-dimensional extremal face.

1. An extremal face \(F\) is called \((K_X + \Delta)\)-negative if

\[
F \cap \overline{NE}(X/S)_{K_X + \Delta \geq 0} = \{0\}.
\]

2. An extremal face \(F\) is called rational if we can choose a \(\pi\)-nef \(\mathbb{Q}\)-divisor \(H\) as a support function of \(F\).

3. An extremal face \(F\) is called relatively ample at \(\text{Nlc}(X, \Delta)\) if

\[
F \cap \overline{NE}(X/S)_{\text{Nlc}(X, \Delta)} = \{0\}.
\]

Equivalently, \(H|_{\text{Nlc}(X, \Delta)}\) is \(\pi|_{\text{Nlc}(X, \Delta)}\)-ample for every supporting function \(H\) of \(F\).

4. An extremal face \(F\) is called contractible at \(\text{Nlc}(X, \Delta)\) if it has a rational supporting function \(H\) such that \(H|_{\text{Nlc}(X, \Delta)}\) is \(\pi|_{\text{Nlc}(X, \Delta)}\)-semi-ample.
Theorem 4.3. (Cone theorem for normal varieties, \[\text{[A2, Theorem 5.10], [Fj7, Theorem 16.5]}\]) Let $X$ be a normal variety, $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and $\pi : X \to S$ a projective morphism. Then we have the following properties.

1. $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X + \Delta \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X, \Delta)} + \sum R_j$, where $R_j$'s are the $(K_X + \Delta)$-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at $\text{Nlc}(X, \Delta)$. In particular, each $R_j$ is spanned by an integral curve $C_j$ on $X$ such that $\pi(C_j)$ is a point.

2. Let $H$ be a $\pi$-ample $\mathbb{Q}$-divisor on $X$. Then there are only finitely many $R_j$'s included in $(K_X + \Delta + H)_{< 0}$. In particular, the $R_j$'s are discrete in the half-space $(K_X + \Delta)_{< 0}$.

3. Let $F$ be a $(K_X + \Delta)$-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at $\text{Nlc}(X, \Delta)$. Then $F$ is a rational face. In particular, $F$ is contractible at $\text{Nlc}(X, \Delta)$.

By the above Theorem, we derive the following theorem:

Theorem 4.4. Let $(X, \Delta)$ be a $d$-dimensional log canonical weak log Fano pair. Suppose that $M(X, \Delta) \leq 1$. Then $\overline{NE}(X)$ is a rational polyhedral cone.

Proof. Since $-(K_X + \Delta)$ is nef and big, there exists an effective divisor $B$ satisfying the following: for any sufficiently small rational positive number $\varepsilon$, there exists a general $\mathbb{Q}$-ample divisor $A_\varepsilon$ such that $-(K_X + \Delta) \sim_\mathbb{Q} \varepsilon B + A_\varepsilon$.

We fix a sufficiently small rational positive number $\varepsilon$ and set $A := A_\varepsilon$. We also take a sufficiently small positive number $\delta$. Thus $\text{Supp}(\text{Nlc}(X, \Delta + \varepsilon B + \delta A))$ is contained in the union of lc centers of $(X, \Delta)$ and $-(K_X + \Delta + \varepsilon B + \delta A)$ is ample. By applying Theorem 4.3 to $(X, \Delta + \varepsilon B + \delta A)$, we get

$$\overline{NE}(X) = \overline{NE}(X)_{\text{Nlc}(X, \Delta + \varepsilon B + \delta A)} + \sum_{j=1}^{m} R_j$$

for some $m$.

Now we see that $\overline{NE}(X)_{\text{Nlc}(X, \Delta + \varepsilon B + \delta A)}$ is polyhedral since $\dim \text{Nlc}(X, \Delta + \varepsilon B) \leq 1$ by the assumption of $M(X, \Delta) \leq 1$. We finish the proof of Theorem 4.4.

□

Corollary 4.5. Let $X$ be a weak Fano 3-fold with log canonical singularities. Then the cone $\overline{NE}(X)$ is rational polyhedral.
Remark 4.6. When $M(X, \Delta) \geq 2$, $\overline{NE}(X)$ is not polyhedral in general (Example 5.6).

5. Examples

In this section, we construct examples of log canonical weak log Fano pairs $(X, \Delta)$ such that $-(K_X + \Delta)$ is not semi-ample, $(X, \Delta)$ does not have $\mathbb{Q}$-complements, or $\overline{NE}(X)$ is not polyhedral.

Basic construction 5.1. Let $S$ be a $(d-1)$-dimensional smooth projective variety such that $-K_S$ is nef and $S \subset \mathbb{P}^N$ some projectively normal embedding. Let $X_0$ be the cone over $S$ and $\phi : X \rightarrow X_0$ the blow-up at the vertex. Then the linear projection $X_0 \rightarrow S$ from the vertex is decomposed as follows:

This diagram is the restriction of the diagram for the projection $\mathbb{P}^{N+1} \rightarrow \mathbb{P}^N$:

Moreover, the $\phi_0$-exceptional divisor is the tautological divisor of $\mathcal{O}_{\mathbb{P}^N} \oplus \mathcal{O}_{\mathbb{P}^N}(-1)$. Hence $X \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(-H))$, where $H$ is a hyperplane section on $S \subset \mathbb{P}^N$, and the $\phi$-exceptional divisor $E$ is isomorphic to $S$ and is the tautological divisor of $\mathcal{O}_S \oplus \mathcal{O}_S(-H)$.

By the canonical bundle formula, it holds that

$$K_X = -2E + \pi^*(K_S - H),$$

thus we have

$$-(K_X + E) = \pi^*(-K_S) + \pi^*H + E$$

We see $\pi^*H + E$ is nef and big since $\mathcal{O}_X(\pi^*(H) + E) \simeq \phi^*\mathcal{O}_X(1)$ and $\phi$ is birational. Hence $-(K_X + E)$ is nef and big since $\pi^*(-K_S)$ is nef.

The above construction is inspired by that of Hacon and McKernan in Lazić’s paper (cf. [Lc, Theorem A.6]).

In the following examples, $(X, E)$ is the plt weak log Fano pair given by the above construction.
Example 5.2. This is an example of a $d$-dimensional plt weak log Fano pair such that the anti-log canonical divisors are not semi-ample, where $d \geq 3$.

There exists a variety $S$ such that $-K_S$ is nef and is not semi-ample (e.g. the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points). We see that $-(K_X + E)$ is not semi-ample since $-(K_X + E)|_E = -K_E$ is not semi-ample. In particular, $R(X, -(K_X + E))$ is not a finitely generated algebra over $\mathbb{C}$ by $-(K_X + \Delta)$ is nef and big.

Example 5.3. This is an example of a log canonical weak Fano variety such that the anti-canonical divisor is not semi-ample.

Let $T$ be a $k$-dimensional smooth projective variety whose $-K_T$ is nef and a $(d - k - 1)$-dimensional smooth projective manifold with $K_A \sim_{\mathbb{Q}} 0$, where $d$ and $k$ are integers satisfying $d - 1 \geq k \geq 0$. We set $S = A \times T$. Let $p_T : S \rightarrow T$ be the canonical projection. We see that $K_S = p_T^*(K_T)$. Let $A_p$ be the fiber of $p_T$ at a point $p \in T$, and $\varphi : X \rightarrow Y$ the birational morphism with respect to $|\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T)|$, where $H_T$ is some very ample divisor on $T$. We claim the following:

Claim 5.4. It holds that:

(i) $Y$ is a projective variety with log canonical singularities.

(ii) $\text{Exc}(\varphi) = E$ and any exceptional curve of $\varphi$ is contained in some $A_p$.

(iii) $\varphi^* K_Y = K_X + E$.

(iv) $\varphi(E) = T$ and $(\varphi|_E)^* K_T = K_E$.

Proof of Claim 5.4. We see (ii) easily. Because $-E|_E$ is ample, $E$ is not $\varphi$-numerical trivial. Set $\varphi^* K_Y = K_X + E + aE$ for some $a \in \mathbb{Q}$. Since $K_X + E$ is $\varphi$-numerical trivial, we see $a = 0$. Thus we obtain (iii). (i) follows from (iii). By (iii), $\varphi(E)$ is an lc center. By $|\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T)|_E \simeq p_T^* \mathcal{O}_T(H_T)$, it holds that $\varphi|_E = p_T$. Thus (iv) follows.

If $-K_T$ is not semi-ample, then $-K_Y$ is not semi-ample and $k \geq 2$. Thus we see that $Y$ is a log canonical weak Fano variety with $M(Y, 0) = k$ and $-K_Y$ is not semi-ample. In particular, $R(X, -K_X)$ is not a finitely generated algebra over $\mathbb{C}$ by $-K_X$ is nef and big (cf. [Ll, Theorem 2.3.15]).

Example 5.5. We construct an example of a weak log Fano plt pair without $\mathbb{Q}$-complements.

Let $S$ be the $\mathbb{P}^1$-bundle over an elliptic curve with respect to a non-split vector bundle of degree 0 and rank 2. Then $-K_S$ is nef and $S$ does not have $\mathbb{Q}$-complements (cf. [S, 1.1. Example]). Thus $(X, E)$ does not have $\mathbb{Q}$-complements by the adjunction formula $-(K_X + E)|_E = -K_E$. 


Example 5.6. We construct an example of a weak log Fano plt pair whose Kleiman-Mori cone is not polyhedral. Let $S$ be the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points. It is well known that $S$ has infinitely many $(-1)$-curves $\{C_i\}$. Then we see that the Kleiman-Mori cone $\overline{NE}(X)$ is not polyhedral.

Indeed, we have the following claim:

Claim 5.7. $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray with $(K_X + E).C_i = -1$. Moreover, it holds that $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$ $(i \neq j)$.

Proof of Claim 5.7. We take a semi-ample line bundle $L_i$ on $S$ such that $L_i$ satisfies $L_i.C_i = 0$ and $L_i.G > 0$ for any pseudoeffective curve $[G] \in \overline{NE}(S)$ such that $[G] \not\in \mathbb{R}_{\geq 0}[C_i]$. We identify $E$ with $S$. Let $\mathcal{L}_i$ be a pullback of $L_i$ by $\pi$ and $\mathcal{F}_i := \phi^*(O_X(1)) \otimes \mathcal{L}_i$. We show that $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. Since $(K_X + E)|_E \sim K_E$, it holds that $(K_X + E).C_i = -1$. By the cone theorem for dlt pairs, there exist finitely many $(K_X + E)$-negative extremal rays $R_k$ such that $[C_i] - [D] \in \sum R_k$ for some $[D] \in \overline{NE}(X)_{K_X+E=0}$. It holds that $\mathcal{F}_i.D = \mathcal{F}_i.R_k = 0$ for all $k$ since $\mathcal{F}_i.C_i = 0$ and $\mathcal{F}_i$ is a nef line bundle. We see that, if an effective 1-cycle $C$ on $X$ satisfies $\mathcal{F}_i.C = 0$, then $C = \alpha C_i$ for some $\alpha \geq 0$ by the construction of $\mathcal{F}_i$. Thus, any generator of $R_k$ is equal to $\alpha_k C_i$ for some $\alpha_k \geq 0$. Hence $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. It is clear to see that $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$. Thus the claim holds. □

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