Near equipartitions of colored point sets

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Abstract

Suppose that $nk$ points in general position in the plane are colored red and blue, with at least $n$ points of each color. We show that then there exist $n$ pairwise disjoint convex sets, each of them containing $k$ of the points, and each of them containing points of both colors.

We also show that if $P$ is a set of $n(d+1)$ points in general position in $\mathbb{R}^d$ colored by $d$ colors with at least $n$ points of each color, then there exist $n$ pairwise disjoint $d$-dimensional simplices with vertices in $P$, each of them containing a point of every color.

These results can be viewed as a step towards a common generalization of several previously known geometric partitioning results regarding colored point sets.

Keywords: colored point set; convex equipartition; colorful island; ham sandwich theorem

1 Introduction

In this note, we prove two results concerning partitions of colored point sets. We conjecture a common generalization of these results, as well as various other related results and conjectures [1, 2, 10]. First we establish some basic terminology.

Definitions. We say that a finite set in $\mathbb{R}^d$ is in general position if each of its subsets of size at most $d+1$ is affinely independent. A partition of a finite set $X$ into $m$ parts is called an $m$-coloring of $X$, the parts are called color classes, and we also say that $X$ is $m$-colored. We allow the color classes to be empty. A subset $Y \subseteq X$ is called $j$-colorful if $Y$ contains points from at least $j$ distinct color classes. Let $X$ be a subset of $\mathbb{R}^d$, and

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Figure 1: A 2-colored set of 50 points spanning 10 pairwise disjoint 2-colorful 5-islands.

$Y \subseteq X$. The convex hull of $Y$, denoted by conv $Y$, is called an island (spanned by $X$) if $X \cap \text{conv} Y = Y$. Equivalently, we say that the set $X$ spans $Y$. If conv $Y$ is an island spanned by $X$ and $|Y| = k$, then we also say that conv $Y$ is a $k$-island. If $Y \subseteq X$, we say that the island conv $Y$ is $j$-colorful if $Y$ is $j$-colorful. See Figure 1. Notice that when $X$ is in general position and $k \leq d + 1$, then a $k$-island spanned by $X$ is a $(k - 1)$-dimensional simplex with vertices in $X$.

**The results.** Our first result concerns partitions of 2-colored planar point sets into 2-colorful subsets of $k$ points with disjoint convex hulls.

**Theorem 1.** Let $k \geq 2$ and $n \geq 1$ be integers, and let $X$ be a 2-colored point set in general position in $\mathbb{R}^2$. Suppose that $|X| = kn$ and that there are at least $n$ points in each color class. Then $X$ spans $n$ pairwise disjoint 2-colorful $k$-islands.

Our second result concerns partitions of $d$-colored point sets in $\mathbb{R}^d$ into $d$-colorful subsets of $d + 1$ points with disjoint convex hulls.

**Theorem 2.** Let $d \geq 2$ and $n \geq 1$ be integers, and let $X$ be a $d$-colored point set in general position in $\mathbb{R}^d$. Suppose that $|X| = (d + 1)n$ and that there are at least $n$ points in each color class. Then $X$ spans $n$ pairwise disjoint $d$-colorful $(d + 1)$-islands.

Both theorems can be seen as particular cases of the following common generalization:

**Conjecture 3.** Let $d, k, m$ be integers satisfying $k, m \geq d \geq 2$. Let $X$ be an $m$-colored set of $kn$ points in general position in $\mathbb{R}^d$. Suppose that $X$ admits a partition into $n$ pairwise disjoint $d$-colorful $k$-tuples. Then $X$ spans $n$ pairwise disjoint $d$-colorful $k$-islands.

The condition that $X$ admits a partition into $n$ pairwise disjoint $d$-colorful $k$-tuples can be stated equivalently as the following Hall-type condition on the sizes of the color classes.
Lemma 4. Let $d, k, m$ be integers satisfying $k, m \geq d \geq 2$. Let $X$ be a set with $kn$ elements and let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be an $m$-coloring of $X$. The set $X$ admits a partition into $n$ pairwise disjoint $d$-colorful $k$-tuples if and only if for every $t \in [d-1]$ and every subset $I \subset [m]$ with $|I| = t$ we have

$$\sum_{i \in I} |X_i| \leq (k - d + t)n. \tag{1}$$

Lemma 4 is purely combinatorial, and not geometric in nature. We provide its proof in Section 2, where in Lemma 8 we also show how to reduce the problem in Conjecture 3 to the case of $2d - 1$ or $2d - 2$ colors by merging some color classes.

Note that by Lemma 4, the conditions on the sizes of the color classes stated in Theorem 1 and Theorem 2 are necessary for the existence of a partition into $n$ pairwise disjoint $d$-colorful $k$-tuples.

Theorem 1 confirms Conjecture 3 for $k \geq m = d = 2$. This, together with Lemma 8, implies Conjecture 3 for $d = 2$ and arbitrary $k, m \geq 2$. The proof of Theorem 1 is elementary and is based on a result by Kaneko, Kano, and Suzuki [9]. The proof is given in Section 3.

Theorem 2 confirms Conjecture 3 for $k - 1 = m = d \geq 2$. The proof of Theorem 2 is based on the continuous ham sandwich theorem and a special discretization argument [5, 10] and it is given in Section 4.

Relation to previous results. The classical ham sandwich theorem states that for any $d$ measures in $\mathbb{R}^d$ there is a hyperplane bisecting each of these $d$ measures simultaneously. The theorem is often used in two versions: a continuous version with “nice” measures (see Theorem 12) and a discrete version with discrete measures or point sets. The discrete ham sandwich theorem has been a source of influence for further developments related to a wide range of geometric partitioning results for discrete point configurations. The following result is a typical example.

Theorem 5 (Akiyama–Alon [2]). Let $d \geq 2$ and $n \geq 1$ be integers, and let $X$ be a $d$-colored point set in general position in $\mathbb{R}^d$. Suppose that $|X| = dn$ and that there are exactly $n$ points in each color class. Then $X$ spans $n$ pairwise disjoint $d$-colorful $d$-islands.

The planar case of Theorem 5 has the following generalization, conjectured by Kaneko and Kano [7], and proven independently by Bespamyatnikh et al. [3], Ito et al. [6] and Sakai [17].

Theorem 6 (Bespamyatnikh et al. [3], Ito et al. [6], Sakai [17]). Let $A$ and $B$ be disjoint finite sets in $\mathbb{R}^2$ such that $A \cup B$ is in general position, $|A| = an$, and $|B| = bn$. Then there exist $n$ pairwise disjoint convex sets $C_1, C_2, \ldots, C_n$ such that $|C_i \cap A| = a$ and $|C_i \cap B| = b$ for every $i \in [n]$.

Theorem 6 solves the case of Theorem 1 when the size of each color class is divisible by $n$. There is also a continuous version of Theorem 6 due to Sakai [17], which was generalized to
arbitrary dimension by Soberón [18]. Soberón’s proof of the continuous version relies on an ingenious application of power diagrams and Dold’s theorem. Even further generalizations were obtained by Karasev et al. [14] and independently by Blagojević and Ziegler [4]. However, going from the continuous version to the discrete version seems to require, in many cases, a non-trivial approximation argument, and we do not see how the continuous results [4, 14, 18] could be used to settle our Conjecture 3 for the case \( m = d \).

In the discrete setting, it is natural to try to relax the divisibility condition on the sets \(|A|\) and \(|B|\) in Theorem 6, and some partial results were obtained in [8, 9, 12, 13]. Another recent example in this direction is the following generalization of Theorem 5, due to Kano and Kynčl [10].

**Theorem 7** (Kano–Kynčl, [10]). Let \( d \geq 2 \) and \( n \geq 1 \) be integers, and let \( X \) be a \((d+1)\)-colored point set in general position in \( \mathbb{R}^d \). Suppose that \(|X| = dn\) and that there are at most \( n \) points in each color class. Then \( X \) spans \( n \) pairwise disjoint \( d \)-colorful \( d \)-islands.

Note that by Lemma 4, the conditions on the sizes of the color classes stated in Theorem 5 and Theorem 7 are necessary for the existence of a partition into \( n \) pairwise disjoint \( d \)-colorful \( d \)-tuples.

Theorem 5 proves the case \( k = m = d \) of Conjecture 3, while the case \( k = m - 1 = d \) is answered by Theorem 7. The case \( k = d \) and \( m \geq d \) was originally conjectured by Kano and Suzuki [10, Conjecture 3]. The case \( m \geq k = d = 2 \) was proved by Aichholzer et al. [1] and by Kano, Suzuki and Uno [11].

In this note we are mostly concerned with the case \( k \geq m = d \). For \( d = 2 \), Theorem 6 covers the subcase where the cardinality of each \( X_i \) is divisible by \( n \). Kaneko, Kano, and Suzuki [9] solved the subcase with \( d = 2 \), \( k \) odd and \( |X_1| - |X_2| \leq n \).

When \( n \) is a power of 2, the case \( m = d \) of Conjecture 3 can be obtained relatively easily by induction from the discrete ham sandwich theorem [15, Theorem 1.4.3], proceeding like in Case 1 in the proof of Theorem 13 in Section 4. Thus the main contribution of this paper and the main difficulty in it consists in removing the divisibility assumptions for \( n \) and the sizes of the color classes.

## 2 Auxiliary results

**Proof of Lemma 4.** Without loss of generality, we assume that \(|X_1| \geq |X_2| \geq \cdots \geq |X_m|\). Condition (1) can then be stated using only \( d - 1 \) inequalities as follows:

\[
\forall t \in [d-1] \quad \sum_{i=1}^{t} |X_i| \leq (k - d + t)n.
\]

(2)

The necessity of condition (2) follows from the fact that for every \( t \in [d] \), every \( d \)-colorful \( k \)-tuple has at most \( k - d + t \) elements in \( X_1 \cup X_2 \cup \cdots \cup X_t \), since it has at least \( d - t \) elements in \( X_{t+1} \cup X_{t+2} \cup \cdots \cup X_m \).
We now prove the sufficiency of (2). If $|X_1| = n$, then let $t = 0$. Otherwise let $t \in [m]$ be the largest index such that $|X_i| > n$. For each $i \in [d]$, let $Y_i$ be an arbitrary $n$-element subset of $X_i$ if $i \leq t$, otherwise let $Y_i = X_i$. Let $Y = \bigcup_{i=1}^m Y_i$; see Figure 2. We claim that $|Y| \geq dn$. Indeed, by condition (2) and by the assumptions that $|X| = kn$ and $k \geq d$ (which may be regarded as condition (2) for $t = 0$), we have

$$|Y| = |X| - |X \setminus Y| = kn - \sum_{i=1}^t (|X_i| - n) = kn + tn - \sum_{i=1}^t |X_i| \geq dn.$$ 

We construct a partition of $X$ into $n$ $d$-colorful $k$-tuples as follows. First we take a subset $Y'$ of exactly $dn$ elements of $Y$ and partition it into $n$ $d$-colorful $d$-tuples, by filling the elements of $Y'$ into an $n \times d$ grid, column by column, starting with the elements of $Y_1 \cap Y'$, then the elements of $Y_2 \cap Y'$, and so on. The rows of the grid then form the desired partition: since each $Y_i \cap Y$ has at most $n$ elements, no two elements in the same row get the same color, and hence the $d$-tuple in each row is $d$-colorful.

Finally, we extend each $d$-tuple of the partition to a $k$-tuple by adding arbitrary $k - d$ elements of the remaining set $X \setminus Y'$. The resulting $k$-tuples are automatically $d$-colorful.

**Merging colors**

In order to prove Conjecture 3 for $k = d$, it is enough to prove it for $m = 2d - 1$ [10]. Indeed, if the number of color classes is larger, we can merge two classes of small size into a single class of size at most $n$, which implies that condition (1) is still satisfied.

The next lemma implies that to prove Conjecture 3 for $k > d$, it is enough to prove it for $m = 2d - 2$. 

![Figure 2: Partition into parts $P_1, \ldots, P_6$ with parameters $k = 4$, $n = 6$, $m = 5$, and $d = 3$.](image)
Lemma 8. Let \( d, k, m \) be integers satisfying \( d \geq 2, k \geq d + 1, \) and \( m \geq 2d - 1. \) Let \( X \) be a set with \( kn \) elements and let \( X = X_1 \cup X_2 \cup \cdots \cup X_m \) be an \( m \)-coloring of \( X \) satisfying condition (1). There exist two color classes such that by merging them into a single color class, the resulting \((m - 1)\)-coloring of \( X \) still satisfies condition (1).

Proof. Assume that \( |X_1| \geq |X_2| \geq \cdots \geq |X_m| \). We merge \( X_{m-1} \) and \( X_m \) into a single class \( X'_{m-1} \). Since \( m \geq 2d - 1 \), we have \( |X'_{m-1}| = |X_{m-1}| + |X_m| \leq 2kn/(2d - 1) \). We now verify that the \((m - 1)\)-coloring \( \chi' = (X_1, X_2, \ldots, X_{m-2}, X'_{m-1}) \) satisfies (1).

Suppose that \( t \in [d - 1] \) is the smallest integer for which \( \chi' \) violates condition (1). The inequality \( 2kn/(2d - 1) \leq (k - d + 1)n \) follows from \((2d - 3)(k - d) \geq 1 \), thus \( t \geq 2 \). Then \( |X'_{m-1}| > |X_t| \), and so \( |X'_{m-1}| + \sum_{i=1}^{t-1} |X_i| > (k - d + t)n \) while \( \sum_{i=1}^{t-1} |X_i| \leq (k - d + t - 1)n \). By our assumption, we have \( |X'_{m-1}| \leq \frac{2}{m-t+1} \sum_{i=t}^{m} |X_i| \). Together, this gives

\[
(m - t + 1)(k - d + t)n < 2 \cdot \sum_{i=t}^{m} |X_i| + (m - t - 1) \cdot \sum_{i=1}^{t-1} |X_i| \\
\leq 2 \cdot \sum_{i=1}^{m} |X_i| + (m - t - 1) \cdot (k - d + t - 1)n \\
= 2kn + (m - t - 1)(k - d + t - 1)n
\]

and thus

\[
0 > (m - t + 1)(k - d + t - 2k - (m - t - 1)(k - d + t - 1) = m - t + 2k - 2d + 2t - 1 - 2k \geq t - 2 \geq 0;
\]

a contradiction.

The following examples show that it is not always possible to merge two color classes if \( m = 2d - 1 \) and \( k = d \) or if \( m = 2d - 2 \) and \( k \geq d + 1 \).

For \( m = 2d - 1 \) and \( k = d \), let \( n \) be a multiple of \( 2d - 1 \), let \( X \) be a set with \( dn \) elements and let \( X = X_1 \cup X_2 \cup \cdots \cup X_{2d-1} \) be a \((2d - 1)\)-coloring of \( X \) satisfying

\[
|X_1| = |X_2| = \cdots = |X_{2d-1}| = \frac{d}{2d - 1} \cdot n.
\]

Then by merging an arbitrary pair of color classes we get a class with \((2dn)/(2d - 1) > n\) elements, violating condition (1).

For \( m = 2d - 2 \) and \( k \geq d + 1 \), let \( n \) be a multiple of \( 2d - 3 \), let \( X \) be a set with \( kn \) elements and let \( X = X_1 \cup X_2 \cup \cdots \cup X_{2d-2} \) be a \((2d - 2)\)-coloring of \( X \) satisfying

\[
|X_1| = (k - d + 1)n \quad \text{and} \quad |X_2| = |X_3| = \cdots = |X_{2d-2}| = \frac{d - 1}{2d - 3} \cdot n.
\]

Let \( Y = X_{2d-3} \cup X_{2d-2} \). Now \( X = X_1 \cup Y \cup X_2 \cup X_3 \cup \cdots \cup X_{2d-4} \) is a \((2d - 3)\)-coloring of \( X \) where \( X_1 \) and \( Y \) are the two largest color classes. We have

\[
|X_1| + |Y| = \left( k - d + 1 + \frac{2d - 2}{2d - 3} \right) \cdot n > (k - d + 2)n,
\]

which violates condition (1).
3 Proof of Theorem 1

Our proof of Theorem 1 is a modification of the proof by Kaneko, Kano and Suzuki [9, Theorem 2] and relies on the following crucial result by Bespamyatnikh, Kirkpatrick and Snoeyink [3, Theorem 5], restated by Kaneko et al. [9, Theorem 6] in a more general form, which we also use. More precisely, the formulation by Bespamyatnikh et al. [3] assumes that $a_1/b_1 = a_2/b_2 = a_3/b_3$, but Kaneko et al. [9] observed that the proof can be easily modified so that the assumption can be omitted. See Figure 3.

Theorem 9 (3-cutting theorem [3, 9]). Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive integers. Let $A$ and $B$ be finite disjoint sets of points in the plane such that $A \cup B$ is in general position, $|A| = a_1 + a_2 + a_3$, and $|B| = b_1 + b_2 + b_3$. Suppose that any open halfplane containing exactly $a_i$ points from $A$ contains less than $b_i$ points from $B$. Then there exist disjoint convex sets $C_1, C_2, C_3$ such that $|C_i \cap A| = a_i$ and $|C_i \cap B| = b_i$ for every $i \in \{1, 2, 3\}$.

Our proof of Theorem 1 actually gives a slightly stronger conclusion. In particular, the $k$-islands form a “near-equipartition” in the sense that the numbers of points of a given color in distinct $k$-islands differ by at most 1.

Proof of Theorem 1. We denote the two color classes of $X$ by $A$ and $B$, so $X = A \cup B$. We proceed by induction on $n$. The statement is trivial for $n = 1$. If $|A|$ and $|B|$ are both divisible by $n$, then the result follows from Theorem 6. We may therefore assume that there are positive integers $a, b, s, t$ such that

$|A| = na + s, \quad |B| = nb + t, \quad k = a + b + 1, \quad s + t = n.$

We claim that there exist pairwise disjoint $k$-islands $C_1, C_2, \ldots, C_s, D_1, D_2, \ldots, D_t$ such that $|C_j \cap A| = a + 1$, $|C_j \cap B| = b$, $|D_i \cap A| = a$, and $|D_i \cap B| = b + 1$ for every $j \in [s], i \in [t]$.

For a fixed integer $i \in [t]$, consider an open halfplane $H_i$ containing precisely $ia$ points from $A$. If $|H_i \cap B| = i(b + 1)$, then the complement of $H_i$ contains $(n - i)a + s$ points from $A$ and $(n - i)b + (t - i)$ points from $B$, and we are done by induction. We may therefore
assume that $H_i$ contains strictly less than $i(b+1)$ points or strictly more than $i(b+1)$ points. The following observation is well-known (see for instance [3, Lemma 3]) and can be shown by a simple continuity argument.

**Observation 10.** Let $i \in [t]$ and let $H_i$ and $H'_i$ be two open halfplanes, each containing exactly $ia$ points from $A$. If $|H_i \cap B| < i(b+1)$ and $|H'_i \cap B| > i(b+1)$, then there exists a halfplane $H''_i$ satisfying $|H''_i \cap A| = ia$ and $|H''_i \cap B| = i(b+1)$.

In view of Observation 10 we may assume that either every open halfplane containing exactly $ia$ points from $A$ contains less than $i(b+1)$ points from $B$, or every open halfplane containing exactly $ia$ points from $A$ contains less than $i(b+1)$ points from $B$. We denote these two cases by $\sigma_a(i) = -1$ and $\sigma_a(i) = +1$, respectively.

By the same argument, for every fixed integer $j \in [s]$, either every open halfplane containing exactly $ja$ points from $A$ contains less than $jb$ points from $B$, or every open halfplane containing exactly $ja$ points from $A$ contains more than $jb$ points from $B$. We denote these two cases by $\sigma_{a+1}(i) = -1$ and $\sigma_{a+1}(i) = +1$, respectively.

Under the assumption that there is no open halfplane containing exactly $a$ points from $A$ and $(b+1)$ points from $B$, or $(a+1)$ points from $A$ and $b$ points from $B$, we observe the following.

**Observation 11.** $\sigma_a(1) = \sigma_{a+1}(1)$.

**Proof.** To see why this holds, consider a line $l$ passing through one point from $A$ and no point from $B$, and has precisely $a$ points from $A$ on its left side. Let $l'$ be a line parallel to $l$ slightly to the right of $l$ such that no points from $A \cup B$ are contained in the open strip bounded by $l$ and $l'$. Thus the two open left halfplanes bounded by $l$ and $l'$ contain the same number of points from $B$, which is either smaller than $b$ or greater than $b+1$. □

Without loss of generality, we may assume that $\sigma_a(1) = \sigma_{a+1}(1) = -1$. (Otherwise, we can just exchange the roles of $A$ and $B$.) We now claim that if the sequence $\sigma_a(1), \sigma_a(2), \sigma_a(3), \ldots$, changes signs, then we can find parameters satisfying the conditions of Theorem 9. To see this, suppose there exists a smallest integer $i \leq t$ such that

$$\sigma_a(1) = \sigma_a(2) = \cdots = \sigma_a(i-1) = -1 \quad \text{and} \quad \sigma_a(i) = +1.$$ 

Consider a line $l$ disjoint with $A \cup B$ that has exactly $ia$ points from $A$ on its left side. By the assumption $\sigma_a(i) = +1$, it follows that on the right side of $l$ there are exactly $|A| - ia = (n-i)a + s$ points from $A$ and less than $|B| - i(b+1) = (n-i)b + (t-i)$ points from $B$. Therefore, the hypothesis of Theorem 9 is satisfied with

$$a_1 = a, \quad a_2 = (i-1)a, \quad a_3 = (n-i)a + s$$

$$b_1 = b+1, \quad b_2 = (i-1)(b+1), \quad b_3 = (n-i)b + (t-i).$$

We can thus apply the inductive hypothesis in each of the resulting convex sets $C_1, C_2, C_3$ guaranteed by Theorem 9.

By the same argument applied to the sequence $\sigma_{a+1}(1), \sigma_{a+1}(2), \ldots, \sigma_{a+1}(s)$, we may assume that $\sigma_a(i) = \sigma_{a+1}(j) = -1$ for all $i \in [t]$ and $j \in [s]$. In particular, $\sigma_{a+1}(s) = \sigma_a(t)$, but this is a contradiction since these signs correspond to complementary halfplanes. □
4 Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 7 [10], but it is a bit easier, since here we can use directly the continuous ham sandwich theorem, instead of its generalization.

**Theorem 12** (The ham sandwich theorem [19], [16, Theorem 3.1.1]). Let $\mu_1, \mu_2, \ldots, \mu_d$ be $d$ absolutely continuous finite Borel measures on $\mathbb{R}^d$. Then there exists a hyperplane $h$ such that each open halfspace $H$ defined by $h$ satisfies $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ for every $i \in [d]$.

Theorem 2 follows by induction from the following special discrete version of the ham sandwich theorem, which is an analogue of the discrete hamburger theorem from [10].

We say that point sets $X_1, X_2, \ldots, X_d$ are balanced in a subset $S \subseteq \mathbb{R}^d$ if for every $i \in [d]$, we have

$$|S \cap X_i| \geq \frac{1}{d+1} \sum_{j=1}^d |S \cap X_j|.$$  

**Theorem 13.** Let $d \geq 2$ and $n \geq 2$ be integers. Let $X_1, X_2, \ldots, X_d \subset \mathbb{R}^d$ be $d$ disjoint point sets balanced in $\mathbb{R}^d$. Suppose that $X_1 \cup X_2 \cup \cdots \cup X_d$ is in general position and $\sum_{i=1}^{d+1} |X_i| = (d + 1)n$. Then there exists a hyperplane $h$ disjoint with each $X_i$ such that for each open halfspace $H$ determined by $h$, the sets $X_1, X_2, \ldots, X_d$ are balanced in $H$ and $\sum_{i=1}^d |H \cap X_i|$ is a positive integer multiple of $d + 1$.

4.1 Proof of Theorem 13

Let $X = \bigcup_{i=1}^d X_i$. Replace each point $p \in X$ by an open ball $B(p)$ of a sufficiently small radius $\delta > 0$ centered at $p$, so that no hyperplane intersects or touches more than $d$ of these balls. We will apply the ham sandwich theorem for suitably defined measures supported by the balls $B(p)$. Rather than taking the same measure for each of the balls, we use a variation of the trick used by Elton and Hill [5]. For each $p \in X$ and $k \geq 1$, we choose a small number $\varepsilon_k(p) \in (0, 1/k)$ so that for every $i \in [d]$ and for every subset $Y_i \subset X_i$, we have

$$\sum_{p \in Y_i} (1 - \varepsilon_k(p)) \neq \frac{1}{2} \cdot \sum_{p \in X_i} (1 - \varepsilon_k(p)). \quad (3)$$

Now let $k \geq 1$ be a fixed integer. For each $i \in [d]$, let $\mu_{i,k}$ be the measure supported by the closure of $\bigcup_{p \in X_i} B(p)$ such that it is uniform (that is, equal to a multiple of the Lebesgue measure) on each of the balls $B(p)$ and $\mu_{i,k}(B(p)) = 1 - \varepsilon_k(p)$.

We apply the ham sandwich theorem (Theorem 12) to the measures $\mu_{i,k}$, $i \in [d]$, and obtain a bisecting hyperplane $h_k$.

For each $i \in [d]$, let $\mu_i$ be the limit of the measures $\mu_{i,k}$ as $k$ tends to infinity; that is, $\mu_i$ is uniform on every ball $B(p)$ such that $p \in X_i$ and $\mu_i(B(p)) = 1$. Since the supports of all the measures $\mu_{i,k}$ are uniformly bounded, there is a sequence $\{k_m\}$ such that the subsequence of hyperplanes $h_{k_m}$ has a limit $h'$. More precisely, if $h_{k_m} = \{x \in \mathbb{R}^d; x \cdot v_m = c_m\}$ where $v_m \in S^{d-1}$, then $h' = \{x \in \mathbb{R}^d; x \cdot v = c\}$ where $v = \lim_{m \to \infty} v_m$ and $c = \lim_{m \to \infty} c_m$.
By the absolute continuity of the measures, each open halfspace $H$ defined by $h' \in \mathbb{R}^d$ satisfies $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ for every $i \in [d]$.

The condition (3) ensures that for every $m$, the hyperplane $h_{k_m}$ intersects the support of each measure $\mu_{i,k_m}$, $i \in [d]$. In particular, for each $i \in [d]$, there is a point $p_i \in X_i$ such that for infinitely many $m \geq 1$, the hyperplane $h_{k_m}$ intersects each of the balls $B(p_i)$. It follows that for each $i \in [d]$, the hyperplane $h'$ either touches $B(p_i)$ or intersects $B(p_i)$. In fact, $h'$ touches $B(p_i)$ if $|X_i|$ is even and $h'$ contains the point $p_i$, if $|X_i|$ is odd, and each open halfplane determined by $h'$ contains exactly $\lceil |X_i|/2 \rceil$ points of $X_i$.

We now rotate the hyperplane $h'$ slightly, to a hyperplane $h$ that touches each of the balls $B(p_i)$, so that the point sets $X_1, X_2, \ldots, X_d$ are balanced in each halfspace determined by $h$ and the number of points of $X$ in each halfspace is divisible by $d + 1$. Essentially, for each point $p_i$ we can decide independently on which side of $h$ it will end. We consider two cases according to the parity of $n$.

**Case 1.** Assume that $n = 2n'$ for some positive integer $n'$. Since the point sets $X_1, X_2, \ldots, X_d$ are balanced in $\mathbb{R}^d$, we have $|X_i| \geq 2n'$ for each $i \in [d]$. To satisfy the balancing condition for the two halfspaces, each halfspace must contain at least $n'$ points from each $X_i$. This is already satisfied for the original hyperplane $h'$ and each $X_i$ with an even number of points. If $|X_i|$ is odd for some $i$, then $|X_i| \geq 2n' + 1$ and thus moving the point $p_i$ to either side of $h$ will keep at least $n'$ points of $X_i$ in each halfspace. Since $|X|$ is even, there is an even number of $X_i$’s with odd cardinality, and therefore $|h' \cap X|$ is also even. Since $|X|/2 = n'(d + 1)$, the divisibility condition will be satisfied if each halfspace gets exactly $|X|/2$ points of $X$. This is easily achieved if we move half of the points from $h' \cap X$ to one halfspace and the remaining points of $h' \cap X$ to the other halfspace.

**Case 2.** Assume that $n = 2n' + 1$ for some positive integer $n'$. Then $|X| = (2n' + 1)(d + 1)$, and so the only way of satisfying the divisibility condition is having $(n' + 1)(d + 1)$ points in one halfspace and $n'(d + 1)$ points in the other halfspace determined by $h$. Since we can move $d$ points between the halfspaces, this determines the number of points $p_i$ that have to end in each halfspace determined by $h$.

Since $h'$ bisects each of the measures $\mu_i$ and $\mu_i(\mathbb{R}^2) = |X_i|$ for each $i$, we have to move $(d + 1)/2$ units of the total measure $\mu = \sum_{i=1}^d \mu_i$ from one halfspace to the other one. Moving a point $p_i$ contained in $h'$ to one halfspace corresponds to moving half unit of $\mu$, and moving a point $p_i$ from one open halfspace to the opposite open halfspace corresponds to moving one unit of $\mu$.

Assume without loss of generality that $h'$ is horizontal, so that $h$ will be almost horizontal. We will refer to the two halfspaces determined by $h'$ or $h$ as the halfspace above and below $h'$ or $h$, respectively.

Let $c = |h' \cap X|$; that is, $c$ is the number of sets $X_i$ with odd cardinality. Let $a$ be the number of the points $p_i$ above $h'$, and let $b$ be the number of the points $p_i$ below $h'$. Clearly, $a + b + c = d$. Assume without loss of generality that $a \geq b$. Since $c$ has the same parity as $|X| = (2n' + 1)(d + 1)$, then $d + 1 - c$ is even, and thus $a + b = d - c$ is odd. This
Figure 4: A ham sandwich cut by a hyperplane $h'$ for $m = d = 9$, $k = 10$ and $n$ odd. The hyperplane $h'$ intersects or touches $d = a + b + c$ of the balls, one from each $X_i$. We perturb $h'$ so that the $c$ balls with centers in $h'$ and $(d + 1 - c)/2$ of the balls tangent to $h'$ from above end up below the perturbed hyperplane.

implies that $a \geq b + 1$, and consequently $c/2 + a \geq (d + 1)/2$.

We move all the $c$ points of $h' \cap X$ below $h$, and an arbitrary set of $(d + 1 - c)/2$ points $p_i$ that are above $h'$ to the halfspace below $h$. After that, the halfspace below $h$ has exactly $(n' + 1)(d + 1)$ points of $X$. See Figure 4.

We now verify that the balancing condition is satisfied in both halfspaces. Since $|X_i| \geq 2n' + 1$ for every $i$, every set $X_i$ of odd cardinality has at least $n'$ points in the halfspace above $h$ and at least $n' + 1$ points below $h$. Every set $X_i$ of cardinality at least $2n' + 3$ has at least $n' + 1$ points in each halfspace determined by $h$. Every set $X_i$ of cardinality $2n' + 2$ with $p_i$ below $h'$ has exactly $n' + 1$ points in each halfspace determined by $h$. Finally, every set $X_i$ of cardinality $2n' + 2$ with $p_i$ above $h'$ has $n'$ or $n' + 1$ points above $h$, and $n' + 2$ or $n' + 1$ points below $h$.

5 Final remarks

Conjecture 3 is still open for $d \geq 3$ and $k, m \geq d + 1$. It is likely that generalizing the 3-cutting theorem [3, 9] to $\mathbb{R}^d$, up to $d + 1$ parts and $2d - 2$ color classes might be a fruitful approach to prove Conjecture 3 in full generality.

Several proofs of special cases of Conjecture 3 include a step where a partition theorem for measures is discretized into a corresponding partition theorem for point sets; see Sakai’s proof of the 3-cutting theorem [17], our Theorem 13 or the discrete version of the hamburger theorem [10]. However, there are difficulties with this approach already for $d = 3$ and $k = m = 4$: Figure 5 shows that cutting by a single hyperplane is not sufficient.

We were not able to prove Conjecture 3 even in the case when $X$ has the order type of the vertex set of the cyclic polytope, when one might expect the existence of a purely combinatorial solution.
Figure 5: An example for $d = 3$ and $k = m = 4$ showing that the discretization approach from the proof of Theorem 13 does not generalize easily. The figure represents a ham sandwich cut for four measures in $\mathbb{R}^3$. The cutting hyperplane, represented by the horizontal line, touches the supports of three different measures, but it cannot be locally modified to give a discrete balanced partition; that is, a partition satisfying the divisibility condition and the analogue of condition (1) simultaneously.

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