Invariant Einstein metrics on three-locally-symmetric spaces

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In this paper, we classify three-locally-symmetric spaces for a connected, compact and simple Lie group. Furthermore, we study invariant Einstein metrics on these spaces.

1. Introduction

A Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is called Einstein if the Ricci tensor $\text{Ric}$ of the metric $\langle \cdot, \cdot \rangle$ satisfies $\text{Ric} = c\langle \cdot, \cdot \rangle$ for some constant $c$. The above Einstein equation reduces to a system of nonlinear second-order partial differential equations. But it is difficult to get general existence results. Under the assumption that $M$ is a homogeneous Riemannian manifold, the Einstein equation reduces to a more manageable system of (nonlinear) polynomial equations, which in some cases can be solved explicitly. There is a lot of progress in the study on invariant Einstein metrics of homogeneous manifolds, such as the articles [4–6, 10–12, 17–19, 21, 22, 24–28, 31–33, 35, 36], and the survey article [30] and so on.

Consider a homogeneous compact space $G/H$ with a semisimple connected Lie group $G$ and a connected Lie subgroup $H$. Denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of $G, H$ respectively. Assume that $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $B$, where $B$ is the Killing form of $\mathfrak{g}$. Every $G$-invariant metric on $G/H$ generates an $\text{ad}\mathfrak{h}$-invariant inner product on $\mathfrak{p}$ and vice versa [9]. This makes it possible to identify invariant Riemannian metrics on $G/H$ with $\text{ad}\mathfrak{h}$-invariant inner products on $\mathfrak{p}$. Note that the metric generated by the inner product $-B|_{\mathfrak{p}}$ is called standard. Furthermore, if $G$ acts almost effectively on the homogeneous space $G/H$, and $\mathfrak{p}$ is the

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direct sum of three ad\(\mathfrak{h}\)-invariant irreducible modules pairwise orthogonal with respect to \(B\), i.e.

\[
p = p_1 \oplus p_2 \oplus p_3,
\]

with \([p_i, p_i] \subset \mathfrak{h}\) for any \(i \in \{1, 2, 3\}\), then \(G/H\) is called a three-locally-symmetric space.

The notation of a three-locally-symmetric space is introduced by Nikonorov in [28]. There have been a lot of studies on invariant Einstein metrics for certain three-locally-symmetric spaces. For example, invariant Einstein metrics on the flag manifold \(SU(3)/T_{\text{max}}\) are given in [16], on

\[
Sp(3)/Sp(1) \times Sp(1) \times Sp(1) \text{ and } F_4/Spin(8)
\]

are obtained in [32], on the Kähler C-spaces

\[
SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3)),
\]
\[
SO(2n)/U(1) \times U(n - 1),
\]
\[
E_6/U(1) \times U(1) \times Spin(8)
\]

are classified in [22], another approach to

\[
SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))
\]

is given in [5]. The existence is proved in [28] of at least one invariant Einstein metric for every three-locally-symmetric space. Furthermore in [25], invariant Einstein metrics on

\[
Sp(l + m + n)/Sp(l) \times Sp(m) \times Sp(m),
\]
\[
SO(l + m + n)/SO(l) \times SO(m) \times SO(m)
\]

are studied. Recently in [1, 2], invariant Einstein metrics on three-locally-symmetric spaces are considered from the point of view of the normalized Ricci flows.

But the classification of three-locally-symmetric spaces is still incomplete, which leads to the incomplete classification of invariant Einstein metrics. In this paper, we complete the classification of three-locally-symmetric spaces for \(G\) simple, and then classify invariant Einstein metrics on all previously unexplored three-locally-symmetric spaces for \(G\) simple.

The paper is organized as follows. In Section 2, we give the correspondence between the classification of three-locally-symmetric spaces \(G/H\) and
that of certain involution pairs of $G$. In Section 3, the classification of three-
locally-symmetric spaces $G/H$ is given for $G$ simple based on the theory on
involutions of compact Lie groups. We list them in Table 1 in Theorem 3.15.
Furthermore we prove the isotropy summands are pairwise nonisomorphic
for the new three-locally-symmetric spaces. It makes the method of [25]
valid to give the classification of invariant Einstein metrics on these spaces
in Section 4.

**Remark 1.1.** The minute that we uploaded this paper on www.arXiv.org,
we received an Email from Prof. Nikonorov with the paper [29] on three-
locally-symmetric spaces which were called generalized Wallach spaces. The
classification of three-locally-symmetric spaces for $G$ simple was obtained in
[29] based on the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces [7, 20, 23]. We
note that the classification by Nikonorov is more general since it includes
the case of non-simple $G$.

### 2. Three-locally-symmetric spaces $G/H$ and
involution pairs of $G$

Assume that $G/H$ is a three-locally-symmetric space. Then $g = h \oplus p_1 \oplus
p_2 \oplus p_3$, and it is easy to see that the Lie brackets satisfy

\[(2.1) \quad [h, p_i] \subset p_i, \quad [p_i, p_i] \subset h, \quad [p_i, p_j] \subset p_k\]

for any $i \in \{1, 2, 3\}$ and $\{i, j, k\} = \{1, 2, 3\}$. Define a linear map $\theta_1$ on $g$ by

\[
\theta_1|_{h \oplus p_1} = id, \quad \theta_1|_{p_2 \oplus p_3} = -id.
\]

By the equation (2.1), $\theta_1[X, Y] = [\theta_1(X), \theta_1(Y)]$ for any $X, Y \in g$. It follows
that $\theta_1$ is an automorphism, and then an involution of $g$. Similarly, define
another linear map $\theta_2$ on $g$ by

\[
\theta_2|_{h \oplus p_2} = id, \quad \theta_2|_{p_1 \oplus p_3} = -id,
\]

which is also an involution of $g$. Moreover we have $\theta_1 \theta_2 = \theta_2 \theta_1$, $h = \{X \in \mathfrak{g}|\theta_1(X) = X, \theta_2(X) = X\}$, $p_1 = \{X \in \mathfrak{g}|\theta_1(X) = X, \theta_2(X) = -X\}$, $p_2 = \{X \in \mathfrak{g}|\theta_1(X) = -X, \theta_2(X) = X\}$, and $p_3 = \{X \in \mathfrak{g}|\theta_1(X) = -X, \theta_2(X) = -X\}$.

On the other hand, let $G$ be a compact semisimple connected Lie group
with the Lie algebra $\mathfrak{g}$ and $\rho, \varphi$ be involutions of $\mathfrak{g}$ satisfying $\rho \varphi = \varphi \rho$. Then
we have a decomposition
\[ g = h \oplus p_1 \oplus p_2 \oplus p_3, \]
corresponding to \( \rho, \varphi \), where
\[ h = \{ X \in g | \rho(X) = X, \varphi(X) = X \}, \]
\[ p_1 = \{ X \in g | \rho(X) = X, \varphi(X) = -X \}, \]
\[ p_2 = \{ X \in g | \rho(X) = -X, \varphi(X) = X \}, \]
and
\[ p_3 = \{ X \in g | \rho(X) = -X, \varphi(X) = -X \}. \]
It is easy to check that
\[ [h, p_i] \subset p_i, \quad [p_i, p_i] \subset h, \quad [p_i, p_j] \subset p_k \]
for any \( i \in \{1, 2, 3\} \) and \( \{i, j, k\} = \{1, 2, 3\} \). Let \( H \) denote the connected Lie subgroup of \( G \) with the Lie algebra \( h \). If every \( p_i \) for \( i \in \{1, 2, 3\} \) is an irreducible \( \text{ad} h \)-module, then \( G/H \) is a three-locally-symmetric space.

In summary, there is a one-to-one correspondence between the set of three-locally-symmetric spaces and the set of commutative involution pairs of \( g \) such that every \( p_i \) for \( i \in \{1, 2, 3\} \) is an irreducible \( \text{ad} h \)-module.

### 3. The classification of three-locally-symmetric spaces

The section is to give the classification of three-locally-symmetric spaces for a compact simple Lie group. By the discussion in Section 2, it turns to the classification of certain commutative involution pairs.

Let \( G \) be a compact simple connected Lie group with the Lie algebra \( g \) and \( (\theta, \tau) \) be an involution pair of \( G \) with \( \theta \tau = \tau \theta \). Then for \( \theta \), we have a decomposition,
\[ g = k + m, \]
where \( k = \{ X \in g | \theta(X) = X \} \) and \( m = \{ X \in g | \theta(X) = -X \} \). Since \( \theta \tau = \tau \theta \), we know \( \tau(X) \in k \) for any \( X \in k \), which implies that \( \tau|_k \) is an involution of \( k \). Roughly to say, we can classify commutative involution pairs of \( g \) by studying the extension of an involution of \( k \) to \( g \). But an important problem is when an involution of \( k \) can be extended to an involution of \( g \).

Cartan and Gantmacher made great attributions on the classification of involutions on compact Lie groups. The theory on the extension of involutions of \( k \) to \( g \) can be found in [8], which is different in the method from that in [37]. There are also some related discussion in [13–15]. The following are the theories without proof.

Let \( t_1 \) be a Cartan subalgebra of \( k \) and let \( t \) be a Cartan subalgebra of \( g \) containing \( t_1 \).
Theorem 3.1 (Gantmacher Theorem). With the above notations, \( \theta \) is conjugate with \( \theta_0 e^{adH} \) under \( \text{Aut} \mathfrak{g} \), where \( H \in \mathfrak{t}_1 \) and \( \theta_0 \) is an involution which keeps the Dynkin diagram invariant.

Let \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) be a fundamental system of \( \mathfrak{t} \) and \( \phi = \sum_{i=1}^{n} m_i \alpha_i \) be the maximal root respectively. Let \( \alpha'_i = \frac{1}{2}(\alpha_i + \theta_0(\alpha_i)) \). Then \( \Pi' = \{ \alpha'_1, \ldots, \alpha'_n \} \) consisting different elements in \( \{ \alpha'_1, \ldots, \alpha'_n \} \) is a fundamental system of \( \mathfrak{g}_0 \), where \( \mathfrak{g}_0 = \{ X \in \mathfrak{g} | \theta_0(X) = X \} \). Denote by \( \phi' = \sum_{i=1}^{l} m'_i \alpha'_i \) the maximal root of \( \mathfrak{g}_0 \) respectively. Furthermore we have

Theorem 3.2 ([37]). If \( H \neq 0 \), then for some \( i \), we can take \( H \) satisfying

\[
\alpha'_i = \alpha_i; \quad \langle H, \alpha'_i \rangle = \pi \sqrt{-1}; \quad \langle H, \alpha'_j \rangle = 0, \forall j \neq i.
\]

Here \( m'_i = 1 \) or \( m'_i = 2 \).

Moreover, \( \mathfrak{k} \) is described as follows.

Theorem 3.3 ([37]). Let the notations be as above. Assume that \( \alpha_i \) satisfies the identity (3.1).

1) If \( \theta_0 = Id \) and \( m_i = 1 \), then \( \Pi - \{ \alpha_i \} \) is a fundamental system of \( \mathfrak{k} \), and \( \phi \) and \( -\alpha_i \) are the highest weights of \( \text{ad}_{m_i} \mathfrak{k} \) corresponding to the fundamental system.

2) If \( \theta_0 = Id \) and \( m_i = 2 \), then \( \Pi - \{ \alpha_i \} \cup \{ -\phi \} \) is a fundamental system of \( \mathfrak{k} \), and \( -\alpha_i \) is the highest weight of \( \text{ad}_{m_i} \mathfrak{k} \) corresponding to the fundamental system.

3) If \( \theta_0 \neq Id \), then \( \Pi' - \{ \alpha'_i \} \cup \{ \beta_0 \} \) is a fundamental system of \( \mathfrak{k} \), and \( -\alpha_i \) is the highest weight of \( \text{ad}_{m_i} \mathfrak{k} \) corresponding to the fundamental system.

Remark 3.4. In Theorem 3.3, the dimension of \( C(\mathfrak{t}) \), i.e. the center of \( \mathfrak{t} \), is 1 for case (1); 0 for cases (2) and (3), \( \beta_0 \) in case (3) is the highest weight of \( \text{ad}_{m_i} \mathfrak{k} \) for \( \theta = \theta_0 \) corresponding to \( \Pi' \).

Now for any involution \( \tau^\mathfrak{t} \) of \( \mathfrak{t} \), we can write \( \tau^\mathfrak{t} = \tau^\mathfrak{t}_0 e^{adH^\mathfrak{t}} \), where \( \tau^\mathfrak{t}_0 \) is an involution on \( \mathfrak{t} \) which keeps the Dynkin diagram of \( \mathfrak{t} \) invariant, \( H^\mathfrak{t} \in \mathfrak{t}_1 \) and \( \tau^\mathfrak{t}_0(H^\mathfrak{t}) = H^\mathfrak{t} \). Since \( e^{adH^\mathfrak{t}} \) is an inner-automorphism, naturally we can extend \( e^{adH^\mathfrak{t}} \) to an automorphism of \( \mathfrak{g} \). Moreover,

Theorem 3.5 ([37]). The involution \( \tau^\mathfrak{t}_0 \) can be extended to an automorphism of \( \mathfrak{g} \) if and only if \( \tau^\mathfrak{t}_0 \) keeps the weight system of \( \text{ad}_{m_i} \mathfrak{k} \) invariant.
If $C(\mathfrak{k}) \neq 0$, then $\dim C(\mathfrak{k}) = 1$. Thus $\tau_0^\mathfrak{k}(Z) = Z$ or $\tau_0^\mathfrak{k}(Z) = -Z$ for any $Z \in C(\mathfrak{k})$.

**Theorem 3.6 ([37]):** Assume that $C(\mathfrak{k}) \neq 0$ and $\tau_0^\mathfrak{k}(Z) = Z$ for any $Z \in C(\mathfrak{k})$. If $\tau^\mathfrak{k}$ can be extended to an automorphism of $\mathfrak{g}$, then $\tau^\mathfrak{k}$ can be extended to an involution of $\mathfrak{g}$.

For the other cases, we have the following theorems.

**Theorem 3.7 ([37]):** Assume that $C(\mathfrak{k}) = 0$, or $C(\mathfrak{k}) \neq 0$ but $\tau_0^\mathfrak{k}(Z) = -Z$ for any $Z \in C(\mathfrak{k})$. If $\tau$ is an automorphism of $\mathfrak{g}$ extending an involution $\tau^\mathfrak{k}$ of $\mathfrak{k}$, then $\tau^2 = \text{Id}$ or $\tau^2 = \theta$. Furthermore, the following conditions are equivalent:

1) There exists an automorphism $\tau$ of $\mathfrak{g}$ extending $\tau^\mathfrak{k}$ which is an involution.

2) Every automorphism $\tau$ of $\mathfrak{g}$ extending $\tau^\mathfrak{k}$ is an involution.

Then it is enough to determine when the automorphism extending $\tau^\mathfrak{k}$ is an involution.

**Theorem 3.8 ([37]):** Let $\tau_0$ be the automorphism of $\mathfrak{g}$ extending the involution $\tau_0^\mathfrak{k}$ on $\mathfrak{k}$. Then $\tau_0^2 = \text{Id}$ except $\mathfrak{g} = A_n$ and $n$ is even. For $e^{\text{ad}H^\mathfrak{k}}$, we have:

1) If $\theta_0 \neq \text{Id}$, then the natural extension of $e^{\text{ad}H^\mathfrak{k}}$ is an involution.

2) Assume that $\theta_0 = \text{Id}$. Let $\alpha'_1, \ldots, \alpha'_k$ be the roots satisfying $\langle \alpha'_j, H \rangle \neq 0$. Then the natural extension of $e^{\text{ad}H^\mathfrak{k}}$ is an involution if and only if $\sum_{j=i}^{i_k} m'_j$ is even.

In particular, for every case in Theorem 3.3,

**Theorem 3.9 ([8, 37]):** If $\tau$ is an involution of $\mathfrak{g}$ extending an involution $\tau^\mathfrak{k}$ on $\mathfrak{k}$, then every extension of $\tau^\mathfrak{k}$ is an involution of $\mathfrak{k}$, which is equivalent with $\tau$ or $\tau\theta$.

Up to now, we can obtain the classification of commutative involution pairs by that of $(\theta, \tau)$ based on the above theory. By Theorem 3.9, for an involution $\tau^\mathfrak{k}$ on $\mathfrak{k}$ which can be extended to an involution $\tau$ of $\mathfrak{g}$, we have two involution pairs $(\theta, \tau)$ and $(\theta, \theta\tau)$ which determine the same three-locally-symmetric space. So, without loss of generality, we denote by $\tau$ the
natural extension of $\tau^t$. Let $\mathfrak{h} = \{ X \in \mathfrak{t} | \tau^t(X) = X \}$, $\mathfrak{p}_1 = \{ X \in \mathfrak{t} | \tau(X) = -X \}$, $\mathfrak{p}_2 = \{ X \in \mathfrak{m} | \tau(X) = X \}$, and $\mathfrak{p}_3 = \{ X \in \mathfrak{m} | \tau(X) = -X \}$. We shall pick up certain pairs $(\theta, \tau^t)$ by the following steps:

**Step 1:** we obtain the classification of the pairs $(\theta, \tau^t)$ satisfying that the extension $\tau$ of $\tau^t$ is an involution and $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are irreducible as ad-modules.

**Step 2:** among the pairs given by Step 1, we remain only one if several pairs determine the same three-locally-symmetric space.

Here we don’t list all the pairs satisfying Steps 1 and 2 but give some examples and remarks.

**Example 3.10.** Let $G = A_l(l \geq 1)$ and the Dynkin diagram with the maximal root is

![Dynkin Diagram for $A_l$]

We have the following cases:

1) $l = 1$, $\theta = e^{adH}$, where $\langle H, \alpha_1 \rangle = \pi \sqrt{-1}$; $\tau^t|_{C(t)} = -Id$.

2) $l \geq 3$ is odd, $\theta = e^{adH}$, where $\langle H, \alpha_{\frac{l+1}{2}} \rangle = \pi \sqrt{-1}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{ \alpha_i \}$; $\tau^t(\alpha_k) = \alpha_{l+1-k}$ for $k \neq \frac{l+1}{2}$, and $\tau^t|_{C(t)} = Id$.

3) $l \geq 2$, $\theta = e^{adH}$, where $\langle H, \alpha_i \rangle = \pi \sqrt{-1}$ for some $2 \leq i \leq \frac{l+1}{2}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{ \alpha_i \}$; $\tau^t = e^{adH_1}$, where $\langle H_1, \alpha_j \rangle = \pi \sqrt{-1}$ for some $\alpha_j \in \Pi - \{ \alpha_i \}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{ \alpha_i, \alpha_j \}$. Furthermore, we may require that $1 \leq i \leq \frac{l+1}{2}$ and $2i \leq j \leq \frac{l+i+1}{2}$ for Step 2.

**Example 3.11.** Let $G = B_l(l \geq 2)$ and the Dynkin diagram with the maximal root is

![Dynkin Diagram for $B_l$]

We have the following cases:

1) $\theta = e^{adH}$, where $\langle H, \alpha_i \rangle = \pi \sqrt{-1}$ for some $2 \leq i \leq l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{ \alpha_i \}$; $\tau^t = e^{adH_1}$, where $\langle H_1, \alpha_j \rangle = \pi \sqrt{-1}$ for some
Remark 3.13. The case (1) of Example 3.11 is valid for $2 < i \leq l$ and $\frac{i+1}{2} \leq j \leq i-1$. Moreover, we may assume that $2 < i \leq l$ and $\frac{i}{2} \leq j \leq i-1$.

2) $\theta = e^{adH}$, where $\langle H, \alpha_i \rangle = \pi \sqrt{-1}$ for some $2 \leq i \leq l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^t(\alpha_1) = -\phi$, $\tau^t(-\phi) = \alpha_1$, and $\tau^t(\alpha_k) = \alpha_k$ for any $\alpha_k \in \Pi - \{\alpha_1, \alpha_i\}$. Moreover, we may assume that $\frac{l+1}{2} \leq i \leq l$.

3) $\theta = e^{adH}$, where $\langle H, \alpha_i \rangle = \pi \sqrt{-1}$ for some $2 \leq i \leq l$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$; $\tau^t = \tau^t_0 e^{adH_1}$, where $\tau^t_0(\alpha_1) = -\phi$, $\tau^t_0(-\phi) = \alpha_1$, and $\tau^t_0(\alpha_k) = \alpha_k$ for any $\alpha_k \in \Pi - \{\alpha_1, \alpha_i\}$, $\langle H_1, \alpha_j \rangle = \pi \sqrt{-1}$ for some $2 \leq j < i$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i, \alpha_j\} \cup \{-\phi\}$. Furthermore, we may require that $[\frac{2l+3}{3}] \leq i \leq l$ and $[\frac{i+2}{2}] \leq j \leq 2i - l$.

Remark 3.12. Let $G = A_l$ where $l \geq 1$ is odd. Take the involution $\theta = e^{adH}$, where $\langle H, \alpha_{i+1} \rangle = \pi \sqrt{-1}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_i\}$. Then $\tau^t$ defined by $\tau^t(\alpha_k) = \alpha_{i+1+k}$ and $\tau^t(\alpha_{i+1+k}) = \alpha_k$ for any $1 \leq k \leq \frac{l-1}{2}$ is an involution on $\mathfrak{k}$. By the above theory, if $\tau^t$ can be extended to an involution of $\mathfrak{g}$, we obtain $\tau^t(-\alpha_{i+1}) = \phi$. It is equivalent to $\tau^t|_{C(\mathfrak{k})} = -Id$. If $l \geq 3$, $\mathfrak{p}_1$ is reducible. If $l = 1$, we get the case (1) of Example 3.10.

Remark 3.13. The case (1) of Example 3.11 is valid for $i = 1$ and $j$, which determines the same three-locally-symmetric space with $i' = j$ and $j' = j - 1$. If we take $\theta$ as that of the case (1) of Example 3.11 for $i = 1$, then $\tau^t|_{C(\mathfrak{k})} = -Id$ can be extended to an involution of $\mathfrak{g}$. This pair determines the same three-locally-symmetric space as the case (1) of Example 3.11 for $i = l$. Every involution $\tau^t$ in cases (2) and (3) of Example 3.11 is an outer-automorphism on $\mathfrak{k}$, which can be extended to an inner-automorphism of $\mathfrak{g}$.

Example 3.14. Let the Dynkin diagram with the maximal root of $G = C_l(l \geq 3)$ be

\[
\begin{array}{cccccccc}
-\phi & \alpha_1 & \alpha_{i-1} & \alpha_i & \alpha_{i+1} & \alpha_{i+2} & \alpha_l \\
\end{array}
\]

Consider the following involution pair $(\theta, \tau)$ of $G = C_l$. Here $\theta = e^{adH}$ is an involution of $\mathfrak{g}$, where $\langle H, \alpha_l \rangle = \pi \sqrt{-1}$ and others zero. Then $\mathfrak{k}$ is the direct sum of $A_{l-1}$ and the center of one-dimension. Given an involution $\tau^t = e^{adH_1}$ of $\mathfrak{k}$, where $\langle H_1, \alpha_j \rangle = \pi \sqrt{-1}$ for some $j \in \{1, 2, \ldots, l - 1\}$ and $\langle H, \alpha_k \rangle = 0$ for any $\alpha_k \in \Pi - \{\alpha_j, \alpha_l\}$. Then the extension of $\tau^t$ is an involution of $\mathfrak{g}$ and

1) $\mathfrak{p}_1$ is an irreducible ad$\mathfrak{g}$-module.
2) By a result in [8], the extension of \( \tau^k \) is \( \tau \) or \( \tau \theta \). Here \( \tau \) is also denoted by \( e^{ad H_1} \), where \( \langle H_1, \alpha_j \rangle = \pi \sqrt{-1} \) and others zero.

We can prove that \( p_2 \) is a reducible \( ad_H \)-module. In fact, the Dynkin diagram of \( \mathfrak{h} + p_2 \), i.e. the set of fixed points of \( \tau \), is

\[
\begin{array}{cccc}
-\phi & \alpha_1 & \alpha_{j-1} & \alpha_{j+1} \\
\circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

Then by the definition of \( \theta \), the Dynkin diagram of \( \mathfrak{h} \) is

\[
\begin{array}{cccc}
\alpha_1 & \alpha_j & \alpha_{l-1} \\
\circ & \circ & \circ \\
\end{array}
\]

and on \( \mathfrak{h} + p_2 \), \( \langle H, \alpha_1 \rangle = \pi \sqrt{-1} \), \( \langle H, -\phi \rangle = \pi \sqrt{-1} \), and others zero. That is, \( p_2 \) is reducible.

By the similar discussions case by case, we classify three-locally-symmetric spaces as follows.

**Theorem 3.15.** The classification of three-locally-symmetric spaces \( G/H \) for a connected, compact and simple Lie group \( G \) is given in Table 1. In Table 1, \( A_1 = B_1 = C_1 \), \( B_2 = C_2 \), \( A_3 = D_3 \), \( D_1 = T \) and \( A_0 = B_0 = C_0 = D_0 = \{e\} \).

**Remark 3.16.** The well-known examples of three-locally-symmetric spaces are the following:

1) \( SU(2) = SU(2)/\{e\} \),
2) \( SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3)) \),
3) \( SO(l + m + n)/SO(l) \times SO(m) \times SO(n) \),
4) \( SO(2n)/U(1) \times U(n - 1) \),
5) \( Sp(l + m + n)/Sp(l) \times Sp(m) \times Sp(m) \),
6) \( E_6/U(1) \times U(1) \times Spin(8) \),
7) \( F_4/Spin(8) \).

The first one is the three-locally-symmetric space of type \( A-I \) in Table 1 of Theorem 3.15, the second one is of type \( A-III \), the third one corresponds to types \( B-I \), \( B-II \), \( B-III \), \( D-I \), \( D-II \), \( D-III \), and \( D-IV \), the fourth one is of type \( D-V \), the fifth one is of type \( C-I \), the sixth one is of type \( E_6-I \), and the seventh one is of type \( F_4-I \). For the above cases, \( p_1 \), \( p_2 \) and \( p_3 \) have been
proved to be irreducible, and pairwise nonisomorphic with respect to the adjoint action of the Lie algebra \( \mathfrak{h} \) on \( \mathfrak{p} \) except \( SO(n + 2)/SO(n) \), which is of type \( B-II \) for \( i = l \) and \( D-IV \).

**Example 3.17.** ([14]) Let \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \) be a fundamental system of \( F_4 \) such that the Dynkin diagram of \( F_4 \) is

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\end{array}
\]

Consider the involution \( \theta = e^{\text{ad}H} \) defined by

\[
\langle H, \alpha_4 \rangle = \pi \sqrt{-1}; \quad \langle H, \alpha_j \rangle = 0, \forall j \neq 4.
\]

Let \( \phi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \). Then the Dynkin diagram of \( \mathfrak{f} \) is

| Type | \( G \) | \( H \) | Type | \( G \) | \( H \) |
|------|-------|-------|------|-------|-------|
| \( A-I \) | \( A_1 \) | \{e\} | \( A-II \) | \( A_l \) | \( T \times A_{l-1}^{-\frac{1}{2}} \) \( l \geq 3 \) is odd |
| \( A-III \) | \( A_l \) | \( T^2 \times A_{l-1} \times A_{j-i-1} \times A_{i-j} \) \( 1 \leq i \leq \left\lfloor \frac{l+1}{3} \right\rfloor \) \( 2i \leq j \leq \left\lfloor \frac{l+i+1}{3} \right\rfloor \) | \( B-I \) | \( B_{l-i} \times D_j \times D_{i-j} \) \( 2 < i \leq l \) \( \frac{i}{2} \leq j \leq i-1 \) |
| \( B-II \) | \( B_l \) | \( B_{l-i} \times B_{l-j} \) \( \frac{l+1}{2} \leq i \leq l \) | \( B-III \) | \( B_{l-i} \times B_{l-j} \times B_{l-i-j} \) \( \left\lfloor \frac{2l+3}{3} \right\rfloor \leq i \leq l \) \( \left\lfloor \frac{i+2}{2} \right\rfloor \leq j \leq 2i-l \) |
| \( C-I \) | \( C_l \) | \( C_i \times C_{j-i} \times C_{l-j} \) \( 1 \leq i \leq \left\lfloor \frac{l}{3} \right\rfloor \) \( 2i \leq j \leq \left\lfloor \frac{l+i}{3} \right\rfloor \) | \( D-I \) | \( D_l \times D_{l-i} \times D_{l-j} \) \( 1 \leq i \leq \left\lfloor \frac{l}{3} \right\rfloor \) \( 2i \leq j \leq \left\lfloor \frac{l+i}{3} \right\rfloor \) |
| \( D-II \) | \( D_l \) | \( B_{l-i} \times D_{l-i} \) \( 1 \leq i \leq l-2 \) | \( D-III \) | \( D_l \times B_{l-i} \times B_{l-j} \times B_{l-j-1} \) \( 1 \leq i \leq l-2 \) \( i \leq j \leq \left\lfloor \frac{l+i-1}{2} \right\rfloor \) |
| \( D-IV \) | \( D_l \) | \( D_{l-1} \) | \( D-V \) | \( D_l \times T^2 \times A_{l-2} \) |
| \( E_6-I \) | \( E_6 \) | \( T^2 \times D_4 \) | \( E_6-II \) | \( E_6 \times A_1 \times A_1 \times A_3 \) |
| \( E_6-III \) | \( E_6 \times A_1 \times C_3 \) | \( E_7-I \) | \( E_7 \times A_1 \times A_1 \times D_4 \) |
| \( E_7-II \) | \( E_7 \times A_1 \times A_5 \) | \( E_7-III \) | \( E_7 \times D_4 \) |
| \( E_8-I \) | \( E_8 \times A_1 \times A_1 \times D_6 \) | \( E_8-II \) | \( E_8 \times D_4 \times D_4 \) |
| \( F_4-I \) | \( F_4 \times D_4 \) | \( F_4-II \) | \( F_4 \times A_1 \times A_1 \times C_2 \) |

**Table 1:** Classification of three-locally-symmetric spaces.
The involution $\tau$ is the extension of $\tau^\ell$, where $\tau^\ell = e^{\text{ad}H_1}$ satisfies

$$\langle H_1, \alpha_3 \rangle = \pi \sqrt{-1}; \quad \langle H, \alpha_1 \rangle = \langle H_1, \alpha_1 \rangle = \langle H_1, \alpha_2 \rangle = \langle H_1, -\phi \rangle = 0.$$ 

Then $\phi_1 = -(\alpha_2 + 2\alpha_3 + 2\alpha_4)$ be the maximal root of $\ell$. Then the Dynkin diagram of $\mathfrak{h}$ is

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By Theorem 3.3, \( p_3 \) is the irreducible representation of \( \mathfrak{h} \) with the highest weight \( -\alpha'_3 \). For the fundamental system \( \{ -\phi, \alpha_1, \alpha_2, -\phi \} \), \( p_3 \) is the irreducible representation of \( \mathfrak{h}_1 \) with the highest weight \( -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4) \), which is a fundamental dominant weight corresponding to \( -\phi \). The discussion for \( p_2 \) is similar.

Hence, for the fundamental system \( \{ -\phi, \alpha_1, \alpha_2, -\phi \} \) of \( \mathfrak{h} \), we conclude that the highest weights of \( p_1, p_2 \) and \( p_3 \) as \( \text{ad} \mathfrak{h} \) modules are fundamental dominant weights of \( \mathfrak{h} \) corresponding to \( \alpha_2, -\phi_1 \) and \( -\phi \) respectively, which are pairwise nonisomorphic.

By Theorem 3.15, we obtain in Table 2 the dimensions of \( p_1, p_2 \) and \( p_3 \) for the following cases. Clearly \( p_1, p_2 \) and \( p_3 \) are pairwise nonisomorphic for types \( A\text{-II}, E_6\text{-III} \) and \( E_7\text{-II} \). We can prove the same result for the other cases similar to Example 3.17.

In summary, we have the following theorem.

**Theorem 3.18.** Let \( G/H \) be a three-locally-symmetric space in Theorem 3.15 with the decomposition \( \mathfrak{p} = p_1 \oplus p_2 \oplus p_3 \). Then \( p_1, p_2, p_3 \) are pairwise nonisomorphic with respect to the adjoint action of the Lie algebra \( \mathfrak{h} \) on \( \mathfrak{p} \) except types \( B\text{-II} \) for \( i = l \) and \( D\text{-IV} \).

### 4. Einstein metrics on three-locally-symmetric spaces

There are some studies on the geometry of three-locally-symmetric spaces. In particular, a lot of studies on invariant Einstein metrics have been done for some three-locally-symmetric spaces independently. For example,
(1) The flag manifolds $SU(3)/T_{max}, Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$, $F_4/Spin(8)$ known as Wallach spaces admit invariant Riemannian metrics of positive section curvature ([35]). The invariant Einstein metrics on the first space are classified in [16], on the other two spaces in [32]. In any case, there are exactly four invariant Einstein metrics up to proportionality.

(2) The invariant Einstein metrics on the Kähler C-spaces $SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))$, $SO(2n)/U(1) \times U(n - 1)$, $E_6/U(1) \times U(1) \times Spin(8)$ are classified in [22]. Every space admits four invariant Einstein metrics up to proportionality. Another approach to $SU(n_1 + n_2 + n_3)/S(U(n_1) \times U(n_2) \times U(n_3))$ is given in [5].

(3) The Lie group $SU(2)$ considered as $SU(2)/\{e\}$ admits only one left-invariant Einstein metric which is a metric of constant curvature [9].

(4) It is proved in [28] that every three-locally-symmetric space admits at least on invariant Einstein metric. Furthermore, it is proved in [25] that $Sp(l + m + n)/Sp(l) \times Sp(m) \times Sp(n)$ admits exactly four invariant Einstein metrics up to proportionality and that $SO(l + m + n)/SO(l) \times SO(m) \times SO(n)$ admits one, two, three or four invariant Einstein metrics up to proportionality. In particular, it is demonstrated in [21] that $SO(n + 2)/SO(n)$ admits just one Einstein metric up to isometry and homothety for $n \geq 3$, the space $SO(4)/SO(2)$ has two such metrics from the classification theorem for five dimensional homogeneous compact Einstein manifolds [3].

In summary, invariant Einstein metrics on three-locally-symmetric spaces are studied for types A-I, A-III, B-I, B-II, B-III, C-I, D-I, D-II, D-III, D-IV, D-V, $E_6$-I and $F_4$-I in Theorem 3.15. The following is to classify invariant Einstein metrics on every three-locally-symmetric space in Theorem 3.15 except the above cases.

By Theorem 3.18, in the decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are pairwise nonisomorphic with respect to the adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{p}$. Then we can give the classification of invariant Einstein metrics on these spaces following the theory in [25, 28, 36].

Let $d_i$ denote the dimension of $\mathfrak{p}_i$, and let $\{e^i_j\}$ be an orthonormal basis in $\mathfrak{p}_i$ with respect to $\langle \cdot, \cdot \rangle = -B(\cdot, \cdot)$, where $i = 1, 2, 3$ and $1 \leq j \leq d_i = \dim \mathfrak{p}_i$. Let $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum_{\alpha, \beta, \gamma} \langle [e^\alpha_i, e^\beta_j], e^\gamma_k \rangle^2$, where $\alpha, \beta, \gamma$ range from 1 to $d_1, d_2, d_3$ respectively. Then $\begin{bmatrix} k \\ ij \end{bmatrix}$ are symmetric in all three indices and $\begin{bmatrix} k \\ ij \end{bmatrix} = 0$ if two indices coincide. Let $c_i$ be the Casimir constant of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{p}_i$. If $\{e^j_0\}_{1 \leq j \leq \dim \mathfrak{h}}$ is an orthonormal basis in $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$ and $e$ is an arbitrary unit vector in $\mathfrak{p}_i$, then $c_i = \sum_j \langle [e^j_0, e], [e^j_0, e] \rangle$. For
three-locally-symmetric spaces, by [25, 36],

\[(4.1) \quad 2A = \begin{bmatrix} k \\ ij \\ ik \end{bmatrix} + \begin{bmatrix} j \\ ik \end{bmatrix} = d_i(1 - 2c_i).\]

Let \(\rho\) be an invariant metric on \(G/H\). We identify it with the corresponding ad\(\mathfrak{h}\)-invariant \((\cdot, \cdot)\) on \(\mathfrak{p}\). Since \(\mathfrak{p}_i\) are irreducible and pairwise nonisomorphic, we have

\[(\cdot, \cdot) = x_1(\cdot, \cdot)|_{\mathfrak{p}_1} \oplus x_2(\cdot, \cdot)|_{\mathfrak{p}_2} \oplus x_3(\cdot, \cdot)|_{\mathfrak{p}_3}\]

for some positive real numbers \(x_i\). The Ricci curvature \(\text{Ric}(\cdot, \cdot)\) of the metric \((\cdot, \cdot)\) is also ad\(\mathfrak{h}\)-invariant. It is easy to see \(\text{Ric}(\cdot, \cdot)|_{\mathfrak{p}_i} = r_i(\cdot, \cdot)|_{\mathfrak{p}_i}\) for some real numbers \(r_i\). As that given in [28], we have the following formula

\[r_i = \frac{1}{2x_i} + \frac{A}{2d_i} \left( \frac{x_i}{x_jx_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_i x_k} \right).\]

Here \(\{i, j, k\} = \{1, 2, 3\}\). Put \(a_i = \frac{A}{d_i}\). Then we have

\[
\begin{cases}
  r_1 = \frac{1}{2x_1} + \frac{a_1}{2} \left( \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right) \\
  r_2 = \frac{1}{2x_2} + \frac{a_2}{2} \left( \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right) \\
  r_3 = \frac{1}{2x_3} + \frac{a_3}{2} \left( \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} \right)
\end{cases}
\]

Now the invariant metric \((\cdot, \cdot)\) is Einstein if and only if \(r_1 = r_2 = r_3\). If \(a_i = a_j\) for \(i \neq j\), then the equations \(r_i = r_j\) and \(r_i = r_k\) for \(k \neq i, j\) become

\[
\begin{cases}
  (x_j - x_i)(x_k - 2a_i(x_i + x_j)) = 0, \\
  x_j(x_k - x_i) + (a_i + a_k)(x_i^2 - x_k^2) + (a_k - a_i)x_j^2 = 0
\end{cases}
\]

If \(x_j = x_i\), then the second equation is

\[(4.2) \quad (1 - 2a_k)x_i^2 - x_ix_k + (a_i + a_k)x_k^2 = 0.\]

If \(a_k = 1/2\), we have only one family of proportional Einstein metrics. Otherwise, \(1 - 2a_k > 0\), hence, all real solutions of the equation \((4.2)\) are positive. Then there exist one family of proportional Einstein metrics for \(\Delta_1 = 0\), two families for \(\Delta_1 > 0\), and none for \(\Delta_1 < 0\). Here \(\Delta_1\) is the discriminant.
of (4.2). If \( x_k = 2a_i(x_i + x_j) \), then the second equation is

\[
(a_i + a_k)(1 - 4a_i^2)x_i^2 - (1 - 2a_i + 8a_i^2(a_i + a_k))x_i x_j + (a_i + a_k)(1 - 4a_i^2)x_j^2 = 0.
\]

If \( a_i = 1/2 \) then the equation (4.3) has no solution. Otherwise, \( 1 - 4a_i^2 > 0 \), hence, all real roots of the equation (4.3) are positive. Then there exist one family of proportional Einstein metrics for the discriminant \( \Delta_2 = 0 \), two families for \( \Delta_2 > 0 \), and none for \( \Delta_2 < 0 \). Here \( \Delta_2 \) is the discriminant of (4.3).

In particular, if \( a_1 = a_2 = a_3 \), then we have the following theorem.

**Theorem 4.1 ([25] Theorem 3).** If \( G/H \) is a three-locally-symmetric space in Theorem 3.15 satisfying \( a_1 = a_2 = a_3 \), then, for \( a_1 \notin \{ \frac{1}{2}, \frac{1}{4} \} \), \( G/H \) admits exactly four nonproportional invariant Einstein metrics. The parameters \( \{x_1, x_2, x_3\} \) has the form \((t, t, t)\), \((1 - 2a_1)t, 2a_1t, 2a_1t\), \((2a_1t, 1 - 2a_1)t, 2a_1t\), or \((2a_1t, 2a_1t, 1 - 2a_1)\). For \( a_1 = \frac{1}{2} \) and \( a_1 = \frac{1}{4} \), every invariant Einstein metric is proportional to the standard metric.

The following is the method for calculating \( c_i \) given in [25]. In detail, \( \mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{p}_i \) is a subalgebra of \( \mathfrak{g} \). Let \( K_i \) be the connect Lie subgroup in \( G \) with the Lie algebra \( \mathfrak{k}_i \). In this case, the homogeneous spaces \( K_i/H \) and \( G/K_i \) are locally symmetric [9]. If \( K_i \) does not act almost effectively on \( M = K/H \), consider its subgroup acting on \( M = K_i/H = \tilde{K}_i/\tilde{H} \) almost effectively, here \( \tilde{H} \) denotes the corresponding isotropy group. The pair of algebras \((\tilde{\mathfrak{k}}_i, \tilde{\mathfrak{h}})\) is irreducible symmetric [9]. If \( \tilde{\mathfrak{k}}_i \) is simple, then its Killing form \( B_{\tilde{\mathfrak{k}}_i} \) is proportional to the restriction of the Killing form of \( \mathfrak{g} \) to \( \tilde{\mathfrak{k}}_i \), i.e. \( B_{\tilde{\mathfrak{k}}_i} = \gamma_i B|_{K_i} \). By Lemma 1 in [25], \( c_i = \gamma_i/2 \). It follows that

\[
a_i = \frac{A}{d_i} = \frac{1 - \gamma_i}{2}.
\]

From the above formulae and results in [25, 28, 36], we can classify invariant Einstein metrics on three-locally-symmetric spaces case by case.

4.1. Invariant Einstein metrics on the three-locally-symmetric space of type \( A-II \)

For this case, \( \mathfrak{h} \oplus \mathfrak{p}_2 = C_k \), where \( l = 2k - 1 \) for \( k \geq 2 \). By the table for \( \gamma_i \geq \frac{1}{2} \) given in [17],

\[
\gamma_2 = \frac{k + 1}{2k}.
\]
In fact, there is a method to compute every $\gamma_i$ in [17]. Here for three-locally-symmetric spaces, from (4.1) and the dimensions in Table 2, we calculate directly

$$\gamma_1 = \frac{1}{2}, \quad \gamma_3 = \frac{k - 1}{2k}.$$ 

It follows that

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{k - 1}{4k}, \quad a_3 = \frac{k + 1}{4k}.$$ 

Let $x_1 = 1$. The equations $r_1 = r_2 = r_3$ are equivalent to

$$\begin{cases} x_2^2 - (2k + 1)x_3^2 + 4kx_2x_3 - 4kx_2 + 2k + 1 = 0, \\ x_2^2 - x_3^2 + 2x_3 - 2x_2 + \frac{1}{k} = 0. \end{cases}$$

Excluding the summand containing $x_2^2$ form the first equation, we obtain

$$x_2(4kx_3 - 4k + 2) = 2kx_3^2 + 2x_3 + \frac{1}{k} - 2k - 1.$$ 

For this case, $4kx_3 - 4k + 2 \neq 0$. Expressing $x_2$ by $x_3$ from (4.4) and inserting it into second one, we obtain

$$12k^4x_3^4 - (48k^4 - 8k^3)x_3^3 + (72k^4 - 36k^3 - 4k^2)x_3^2 - (48k^4 - 48k^3 + 4k^2 + 4k)x_3 + 12k^4 - 20k^3 + 7k^2 + 2k - 1 = 0.$$ 

Denote by $U_0(x_3)$ the left side of the above equation. Similar to [25], define Sturm’s series by

$$U_1(x_3) = 48k^4x_3^3 - (144k^4 - 24k^3)x_3^2 + (144k^4 - 72k^3 - 8k^2)x_3 - 48k^4 + 48k^3 - 4k^2 - 4k,$$

$$U_2(x_3) = (6k^3 + 3k^2)x_3^2 - 12k^3 - 2k^2 - \frac{8}{3}k)x_3 + 6k^3 - 4k^2 - \frac{7}{6}k + \frac{5}{6},$$

$$U_3(x_3) = \frac{32k}{27(2k + 1)^2}[(108k^4 - 18k^3 - 12k^2 + 4k)x_3 - 108k^4 + 36k^3 + 27k^2 - 7k - 1],$$

$$U_4(x_3) = -\frac{27(4k^2 - 1)^2(27k^4 - 9k^2 + 1)}{4(-54k^3 + 9k^2 + 6k - 2)^2}.$$ 

It is easy to see that

$$U_0(0) > 0, U_1(0) < 0, U_2(0) > 0, U_3(0) < 0, U_4(0) < 0;$$

$$U_0(\infty) > 0, U_1(\infty) > 0, U_2(\infty) > 0, U_3(\infty) > 0, U_4(\infty) < 0.$$
Denote by \( Z(0) \) the number of the sign changes in the series \( U_0(0), U_1(0), U_2(0), U_3(0), U_4(0) \) (neglecting zeros) and \( Z(\infty) \) the number of sign changes in \( U_0(\infty), U_1(\infty), U_2(\infty), U_3(\infty), U_4(\infty) \), where \( U_i(\infty) \) denotes the leading coefficient of \( U_i(x_3) \) which defines the sign of \( U_i(x_3) \) as \( x_i \to \infty \). By Sturm’s theorem [34], the number of real roots of (4.1) on \((0, \infty)\) is equal to \( Z(0) - Z(\infty) = 2 \) since \( U_0(0) \neq 0 \). By the discussion in [25], the homogeneous manifold of type \( A-II \) in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality.

### 4.2. Invariant Einstein metrics on the three-locally-symmetric space of type \( E_6-II \)

For this case, \( h \oplus p_1 = A_1 \oplus A_5 \) and \( p_1 \subset A_5 \). By the tables in [17], \( \gamma_1 = \frac{1}{2} \).

From (4.1) and the dimensions in Table 2, \( \gamma_2 = \frac{1}{2} \) and \( \gamma_3 = \frac{2}{3} \). It follows that

\[
    a_1 = a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{6}.
\]

If \( x_1 = x_2 \), then the equation (4.2) is

\[
    \frac{2}{3} x_1^2 - x_1 x_3 + \frac{5}{12} x_3^2 = 0.
\]

The discriminant \( 1 - \frac{40}{36} < 0 \), which implies that the equation has no solution. If \( x_3 = \frac{1}{2} (x_1 + x_2) \), then the equation (4.2) equals with

\[
    15 x_1^2 - 34 x_1 x_2 + 15 x_2^2 = 0.
\]

The discriminant \( 34^2 - 30^2 = 16^2 > 0 \), which implies that the equation has two solutions.

That is, the homogeneous manifold of type \( E_6-II \) in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters \((x_1, x_2, x_3)\) have the form \((\frac{5}{3} t, t, \frac{4}{3} t)\), or \((\frac{3}{5} t, t, \frac{4}{5} t)\), where \( t > 0 \).

### 4.3. Invariant Einstein metrics on the three-locally-symmetric space of type \( E_6-III \)

For this case, \( h \oplus p_1 = A_1 \oplus A_5 \) and \( p_1 \subset A_5 \). By the tables in [17], \( \gamma_1 = \frac{1}{2} \).

From (4.1) and the dimensions in Table 2, \( \gamma_2 = \frac{3}{4} \) and \( \gamma_3 = \frac{5}{12} \). It follows
that

\[ a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{8}, \quad a_3 = \frac{7}{24}. \]

Let \( x_1 = 1 \). The equations \( r_1 = r_2 = r_3 \) are equivalent to

\[
\begin{cases}
    x_2^2 - 13x_3^2 + 24x_2x_3 - 24x_2 + 13 = 0, \\
    5x_2^2 - 5x_3^2 + 12x_3 - 12x_2 + 2 = 0.
\end{cases}
\]

Excluding the summand containing \( x_2^2 \) form the first equation, we obtain

\begin{equation}
    x_2(40x_3 - 36) = 20x_3^2 + 4x_3 - 21.
\end{equation}

For this case, \( 40x_3 - 36 \neq 0 \). Expressing \( x_2 \) by \( x_3 \) from (4.5) and inserting it into second one, we obtain

\[ 1200x_3^4 - 4960x_3^3 + 7048x_3^2 - 4152x_3 + 855 = 0, \]

which has two real solutions \( x_3 \approx 1.8845 \) or \( x_3 \approx 0.4838 \). From (4.5), \( x_2 \approx 1.4618 \) or \( x_2 \approx 0.8640 \).

That is, the homogeneous manifold of type \( E_6 \)-III in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters \((x_1, x_2, x_3) \approx (t, 1.4618t, 1.8845t)\) or \((x_1, x_2, x_3) \approx (t, 0.8640t, 0.4838t)\), where \( t > 0 \).

4.4. Invariant Einstein metrics on the three-locally-symmetric space of type \( E_7 \)-I

For this case, \( \mathfrak{h} \oplus \mathfrak{p}_1 = A_1 \oplus D_6 \) and \( \mathfrak{p}_1 \subset D_6 \). By the tables in [17], \( \gamma_1 = \frac{5}{9} \).

From (4.1) and the dimensions in Table 2, \( \gamma_2 = \gamma_3 = \frac{5}{9} \). It follows that

\[ a_1 = a_2 = a_3 = \frac{2}{9}. \]

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type \( E_7 \)-I in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters \((x_1, x_2, x_3)\) have the form \((t, t, t), \left(\frac{2}{9}t, \frac{4}{9}t, \frac{4}{9}t\right), \left(\frac{4}{9}t, \frac{5}{9}t, \frac{4}{9}t\right), \) or \((\frac{4}{9}t, \frac{4}{9}t, \frac{5}{9}t)\), where \( t > 0 \).
4.5. Invariant Einstein metrics on the three-locally-symmetric space of type $E_7$-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_2 = A_1 \oplus D_6$ and $\mathfrak{p}_2 \subset D_6$. By the tables in [17], $\gamma_2 = \frac{5}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_1 = \frac{4}{9}$ and $\gamma_3 = \frac{2}{3}$. It follows that

$$a_1 = \frac{5}{18}, \quad a_2 = \frac{2}{9}, \quad a_3 = \frac{1}{6}.$$

Let $x_1 = 1$. The equations $r_1 = r_2 = r_3$ are equivalent to

$$\begin{cases} x_2^2 + 4x_3^2 - 9x_2x_3 + 9x_2 - 4 = 0, \\ 7x_2^2 - 7x_3^2 + 18x_3 - 18x_2 - 1 = 0. \end{cases}$$

Excluding the summand containing $x_2^2$ form the first equation, we obtain

(4.6) $$x_2(63x_3 - 81) = 35x_3^2 - 18x_3 - 27.$$ For this case, $63x_3 - 81 \neq 0$. Expressing $x_2$ by $x_3$ from (4.6) and inserting it into second one, we obtain

$$2744x_3^4 - 13482x_3^3 + 24732x_2^2 - 19926x_3 + 5832 = 0,$$

which has two real solutions $x_3 \approx 1.5535$ or $x_3 \approx 0.7302$. From (4.6), $x_2 \approx 1.7489$ or $x_2 \approx 0.6139$.

That is, the homogeneous manifold of type $E_7$-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3) \approx (t, 1.7489t, 1.5535t)$ or $(x_1, x_2, x_3) \approx (t, 0.6139t, 0.7302t)$, where $t > 0$.

4.6. Invariant Einstein metrics on the three-locally-symmetric space of type $E_7$-III

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = A_7$. It is the same as that of type $E_7$-II. That is, $\gamma_1 = \frac{4}{9}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{4}{9}$. It follows that

$$a_1 = a_2 = a_3 = \frac{5}{18}.$$

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type $E_7$-III in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3)$ have the form $(t, t, t), (\frac{4}{9}t, \frac{5}{9}t, \frac{5}{9}t), (\frac{5}{9}t, \frac{4}{9}t, \frac{5}{9}t), \text{ or } (\frac{5}{9}t, \frac{5}{9}t, \frac{4}{9}t)$, where $t > 0$. 
4.7. Invariant Einstein metrics on the three-locally-symmetric space of type $E_8$-I

For this case, $h \oplus p_2 = A_1 \oplus E_7$ and $p_2 \subset E_7$. By the tables in [17], $\gamma_2 = \frac{3}{5}$. From (4.1) and the dimensions in Table 2, $\gamma_1 = \frac{7}{15}$ and $\gamma_3 = \frac{3}{5}$. It follows that

$$a_1 = \frac{4}{15}, \quad a_2 = a_3 = \frac{1}{5}.$$  

If $x_2 = x_3$, then the equation (4.2) is

$$\frac{7}{15} x_2^2 - x_1 x_2 + \frac{7}{15} x_1^2 = 0.$$  

The discriminant $1 - (\frac{14}{15})^2 > 0$, which implies that the equation has two solutions. If $x_1 = \frac{2}{5}(x_2 + x_3)$, then the equation (4.2) equals with

$$147x_2^2 - 281x_2x_3 + 147x_3^2 = 0.$$  

The discriminant $281^2 - 294^2 < 0$, which implies that the equation has no solution.

That is, the homogeneous manifold of type $E_8$-I in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3)$ have the form $(qt, t, t)$, where $t > 0$ and $q$ is the root of the equation $7x^2 - 15x + 7 = 0$.

4.8. Invariant Einstein metrics on the three-locally-symmetric space of type $E_8$-II

For this case, $h \oplus p_1 = D_8$. It is the same as that of type $E_8$-I. That is, $\gamma_1 = \frac{7}{15}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{7}{15}$. It follows that

$$a_1 = a_2 = a_3 = \frac{4}{15}.$$  

By Theorem 4.1, i.e. Theorem 3 in [25], the homogeneous manifold of type $E_8$-II in Theorem 3.15 admits exactly four invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3)$ have the form $(t, t, t)$, $(\frac{7}{15} t, \frac{8}{15} t, \frac{8}{15} t)$, $(\frac{8}{15} t, \frac{7}{15} t, \frac{8}{15} t)$, or $(\frac{8}{15} t, \frac{8}{15} t, \frac{7}{15} t)$, where $t > 0$. 

4.9. Invariant Einstein metrics on the three-locally-symmetric space of type $F_4$-II

For this case, $\mathfrak{h} \oplus \mathfrak{p}_1 = D_4$. By the tables in [17], $\gamma_1 = \frac{7}{5}$. From (4.1) and the dimensions in Table 2, $\gamma_2 = \gamma_3 = \frac{4}{5}$. It follows that

$$a_1 = \frac{1}{9}, \quad a_2 = a_3 = \frac{5}{18}.$$  

If $x_2 = x_3$, then the equation (4.2) is

$$\frac{7}{9} x_2^2 - x_1 x_2 + \frac{7}{18} x_1^2 = 0.$$  

The discriminant $1 - \frac{98}{81} < 0$, which implies that the equation has no solution. There exists none Einstein metrics. If $x_1 = \frac{5}{9} (x_2 + x_3)$, then the equation (4.2) equals with

$$196x_2^2 - 499x_2x_3 + 196x_3^2 = 0.$$  

The discriminant $499^2 - 392^2 > 0$, which implies that the equation has two solutions.

That is, the homogeneous manifold of type $F_4$-II in Theorem 3.15 admits exactly two invariant Einstein metrics up to proportionality. The parameters $(x_1, x_2, x_3)$ have the form $(\frac{5}{9}(q + 1)t, qt, t)$, where $t > 0$ and $q$ is the root of the equation $196x_2^2 - 499x + 196 = 0$.

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