Strong-coupling expansions for chiral models of electroweak symmetry breaking

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Abstract

We consider chiral $U(N) \times U(N)$ models with fermions in the limit of infinitely large local bare Yukawa coupling. When the scalar field is subject to non-linear constraint, phase transitions in these models are seen to be identical to those in the corresponding purely bosonic ones. Relaxed the non-linear constraint, we compute the seventh-order strong-coupling series for the susceptibility in these models and analyze them numerically for the $U(2) \times U(2)$ case. We find that in four dimensions the approach to the phase transition follows to a good accuracy the mean-field critical behavior, indicating the absence of non-trivial fixed points at strong coupling and being consistent with the first-order nature of the transition. In three dimensions, the strongly-coupled bosonic $U(2) \times U(2)$ model (without gauge fields) has a first-order transition strong enough to accommodate electroweak baryogenesis only for a narrow region of the bare parameter space.

* Alfred P. Sloan Foundation Fellow; DOE Outstanding Junior Investigator
Phase transitions in chiral sigma models are of interest both in three dimensions where they serve as models of the chiral phase transition in QCD \[1\], and in four dimensions where the nature of the phase transition determines viability of certain models of dynamical electroweak symmetry breaking \[2, 3, 4, 5\]. In the latter case, the analysis should include fermions with large Yukawa couplings because they can influence the nature of the transition. In ref.\[4\], it was argued, based on an approximate method of dealing with strongly coupled fermions, that a large Yukawa interaction can increase an otherwise unacceptably small hierarchy between the symmetry breaking scale and the cutoff. Ref.\[5\] considered the case of moderate \((O(1))\) bare Yukawa coupling and found no substantial increase in the hierarchy for that case.

In this note we consider \(U(N) \times U(N)\) chiral models with fermions in the limit of infinite bare Yukawa coupling. This limit should give us some insight into the behavior of systems with finite but large (much larger than one) bare Yukawa coupling \(y\). For models where the order parameter is a single scalar field, it is known that the phase diagram in the limit \(y \to \infty\) depends on the choice of a lattice form for the Yukawa interaction \[6\]. This is natural when there are phase transitions that involve both ferromagnetic and antiferromagnetic order, because in that case the order parameter varies on the scale of lattice spacing and that variation cannot be removed by a redefinition of the field. Here we are interested in a paramagnetic-ferromagnetic transition, so we should choose a lattice action that has it.

We use the local form of Yukawa interaction

\[
S_Y = y \sum_i (\bar{\chi}_i^a \phi_i \psi_i^a + \bar{\psi}_i^a \phi_i^\dagger \chi_i^a)
\]  

(1)

where \(i\) labels sites of a four-dimensional lattice with unit spacing; \(\psi_i^a\) (\(\bar{\psi}_i^a\)) and \(\chi_i^a\) (\(\bar{\chi}_i^a\)) are staggered fermions, each has one spin component per site but forms a fundamental (conjugated fundamental) representation of the respective \(U(N)\) and in addition comes in \(N_c\) "colors", \(a = 1, \ldots, N_c\). This corresponds to having \(8N_c\) Dirac fermions in a continuum theory. Just as in the case of a single scalar \[6\], in the limit \(y \to \infty\) the fermion fields can be integrated away leaving a local correction to the
potential for $\phi$. In our case, this changes scalar potential $V(\phi)$ into

$$\tilde{V}(\phi) = V(\phi) - N_c \text{Tr} \ln(\phi_i^\dagger \phi_i).$$

(2)

This result holds when fermions have a gauge interaction.

In the non-linear limit of the model, when the order parameter is subject to the constraint $\phi_i^\dagger \phi_i \propto 1$, the correction to the potential is a constant. We see that bosonic correlators of a non-linear model with $y \to \infty$ coincide with those of the corresponding model without fermions. Hence, the phase transitions in the two models are identical. If the phase transition in the model without fermions is of the first order, as suggested by the previous work, so is the phase transition in the model with fermions in the limit of infinite Yukawa coupling, and with exactly the same strength. (This applies also to models with an hermitean order parameter considered in ref.\[8\].)

Even when the non-linear constraint is relaxed, there can be no second-order or weakly first-order ferromagnetic-paramagnetic transition in a $y \to \infty$ model if there is none in its bosonic counterpart. This is a simple consequence of universality, since the logarithmic interaction in (2) is a local one. So, for the purpose of searching for non-trivial fixed points and weakly first-order transitions, it is in principle sufficient to study the purely bosonic model, without the logarithm, although we have also done independent analysis of the effective theory with the potential $\tilde{V}$.

We have computed the strong-coupling expansion for the effective bosonic theory to the seventh order and done numerical analysis of the series for the $U(2) \times U(2)$ case. Our results are consistent with the absence of non-trivial fixed points at strong coupling both in the case with and without fermions. Assuming that the phase transition is of the first order, we find that it is a relatively weak one for most of the bare parameter space. We also considered the $U(2) \times U(2)$ model without fermions in three dimensions and found that except for a narrow region of bare parameters near the line beyond which the potential becomes unbounded from below, the phase transition (without gauge fields) is too weakly first-order to provide deviations from equilibrium required for electroweak baryogenesis \[7\].
Thus, we consider the linear $U(N) \times U(N)$ model for which the bosonic action $S$ is

$$\begin{align*}
S &= - \sum_{\langle ij \rangle} \text{Tr} (\phi_i^\dagger \phi_j) + \text{h.c.} + \sum_i V(\phi_i) \\
V(\phi_i) &= -\mu^2 \text{Tr}(\phi_i^\dagger \phi_i) + \lambda_1 [\text{Tr}(\phi_i^\dagger \phi_i)]^2 + \lambda_2 \text{Tr}(\phi_i^\dagger \phi_i)^2 ;
\end{align*}$$

$\phi$ is now an arbitrary complex $2 \times 2$ matrix. We consider only the case $\lambda_2 > 0$. The bare action (3) is then bounded from below if $N\lambda_1 + \lambda_2 > 0$. The non-linear limit is recovered if we let $\mu^2$, $\lambda_2$, and $N\lambda_1 + \lambda_2$ to infinity with $\mu^2/(N\lambda_1 + \lambda_2)$ fixed.

The strong coupling expansion for (3) is the expansion in powers of the first term in (3). Coefficients in that expansion are functions of the bare parameters $\mu^2$, $\lambda_1$ and $\lambda_2$ and can be expressed through linear combinations of products of invariant ordinary (non-functional) integrals such as

$$\begin{align*}
I_1 &= Z^{-1} \int \text{Tr}(\phi^\dagger \phi) \exp[-\tilde{V}(\phi)] d\phi , \\
I_2 &= Z^{-1} \int \text{Tr}(\phi^\dagger \phi)^2 \exp[-\tilde{V}(\phi)] d\phi , \\
I_3 &= Z^{-1} \int [\text{Tr}(\phi^\dagger \phi)]^2 \exp[-\tilde{V}(\phi)] d\phi , \\
I_4 &= Z^{-1} \int \text{Tr}(\phi^\dagger \phi)^3 \exp[-\tilde{V}(\phi)] d\phi \quad \text{etc.}
\end{align*}$$

where $\tilde{V}$ is given in (2) and $Z = \int \exp[-\tilde{V}(\phi)] d\phi$.

The invariant integrals $I_n$ of (5), as functions of the bare parameters, have the form

$$I_n = (2\lambda_s)^{-p/2} F_n(\lambda_2/\lambda_s, \mu^2/\sqrt{2\lambda_s})$$

where

$$\lambda_s = \lambda_1 + \lambda_2$$

and $p$ is the total power of $\phi$ and $\phi^\dagger$ in the traces in the integrand: $p = 1$ for $I_1$, $p = 2$ for $I_2$ and $I_3$, $p = 3$ for $I_4$ etc. So, for fixed $\lambda_2/\lambda_s$ and $\mu^2/\sqrt{2\lambda_s}$, the strong coupling series is a series in powers of $\beta = (2\lambda_s)^{-1/2}$.

\footnote{In statistical mechanics, such expansions are called high-temperature expansions \cite{footnote}. We chose not to use that terminology to avoid confusion with field theories at finite temperature.}
We have computed the seventh-order series for zero-momentum susceptibility
\[ \chi = \sum_i \langle \text{Tr} \phi_0 \phi_i^\dagger \rangle \] (7)
for three- and four-dimensional lattices in terms of the invariant integrals for a general
\( U(N) \times U(N) \) model with infinite bare Yukawa coupling. Calculation of the integrals
and analysis of the series were done for the \( U(2) \times U(2) \) case. The series were obtained
by the recursion method of ref. [10]. The complete series will be presented elsewhere;
here we limit ourselves to the results of the analysis.‡

Let us consider first the purely bosonic model (\( N_c = 0 \)) on the four-dimensional
simple hypercubic lattice. The values of bare parameters for which Shen [3] finds
evidence for a first-order transition at large bare couplings correspond in our notation§
to \( \lambda_2/\lambda_s = 1 \) (\( \lambda_1 = 0 \)) and \( \mu^2/\sqrt{2\lambda_s} = 2.41775 \). With these parameters, the series for
the susceptibility for the \( U(2) \times U(2) \) model is
\[ \chi/\chi_0 = 1 + 9.98596\beta + 90.66614\beta^2 + 822.72732\beta^3 + 7347.40485\beta^4 \\
+ 65668.03299\beta^5 + 583436.92981\beta^6 + 5186731.98528\beta^7 + ... \] (8)
where \( \beta = (2\lambda_s)^{-1/2} \) and \( \chi_0 \) is the zeroth-order susceptibility which has been factored
out.

The series were analyzed using Zinn-Justin’s method [15, 16]. Denote the coeffi-
† Comparison of our series to known limiting cases has found one discrepancy. The \( N = 2 \)
case with \( \lambda_2 = 0 \) includes the \( O(8) \) model. For that model on the fcc (face-centered cubic) lattice
we obtain the 7th order coefficient of 14492289.1770 while Table 8.5 of ref. [11] lists 14490203.7349.
Other coefficients through the 7th order for that model on the fcc and bcc (body-centered cubic)
lattices agree. Our computer program generating the series has also reproduced, through the 7th
order, the known results for models with fewer components of order parameter: the classical \( XY \)
model [12] and the general one-component model [13] on various three-dimensional lattices, and the
Ising model on simple and face-centered hypercubic four-dimensional lattices [14]. This gives us
confidence that our program generates series correctly.
§ Our \( \mu^2 \) is \( -m^2 \) of ref. [3] minus the number of nearest neighbors; our \( \lambda_1 \) and \( \lambda_2 \) are four times
those of ref. [3].
cients in (8) as \( a_n \), so that
\[
\chi / \chi_0 = \sum_{n=0}^{\infty} a_n \beta^n .
\] (9)

In Zinn-Justin’s method one forms the ratios
\[
s_n = - \left( \ln \frac{a_n a_{n-2}}{a_{n-1}^2} \right)^{-1} .
\] (10)

Then, estimates of the susceptibility exponent \( \gamma \) are obtained as
\[
\gamma_n = 1 + 2 \frac{s_n + s_{n-1}}{(s_n - s_{n-1})^2} ,
\] (11)
and estimates of the critical ”temperature” as
\[
\beta_{c,n}^{-1} = \left( \frac{a_n}{a_{n-2}} \right)^{1/2} \exp \left[ \frac{s_n + s_{n-1}}{(s_n - s_{n-1})s_n} \right] .
\] (12)

For the series (8) we obtain
\[
\{ \gamma_3, ..., \gamma_7 \} = \{ 1.0011, 1.0013, 0.9986, 0.9989, 0.9991 \} ,
\] (13)
\[
\{ \beta_{c,3}^{-1}, ..., \beta_{c,7}^{-1} \} = \{ 9.072, 9.158, 8.940, 8.952, 8.892 \} .
\] (14)

We observe that the estimates for \( \gamma \) are very close to 1 and are remarkably stable for such a short series.

The values of couplings \( \lambda_1 \) and \( \lambda_2 \) to which a given estimate refers are determined a posteriori, through the critical value \( \beta_c \). Our estimates for \( \beta_c \) and the critical exponent become less stable as we try to move in the region of larger \( \beta_c \) or, equivalently, smaller \( \lambda_s \). As a result, we could not probe the weak coupling region \( \lambda_s \lesssim 1 \). For the purely bosonic theory, the usual perturbation is applicable for small \( \lambda \), so this is not a very significant limitation. In the strong-coupling region \( \lambda_s \gtrsim 1 \), except for a narrow region near the line \( 2\lambda_1 + \lambda_2 = 0 \), stable estimates were obtained and they were consistent with \( \gamma = 1 \). We interpret these results as follows.

At a second-order phase transition, the correlation length grows to infinity and susceptibility is singular, \( \chi \sim (\beta_c - \beta)^{-\gamma} \). However, even for a first-order transition, as it is approached, the correlation length and the susceptibility may grow somewhat before the transition takes place. \( \gamma = 1 \) is the mean-field value and it implies, in the
renormalization group language, that in our case the approach to the phase transition is controlled by the trivial fixed point \( \lambda_1 = \lambda_2 = 0 \). This fixed point is known to be infrared unstable by perturbation theory [1] — the signal of a first-order transition.

A seventh-order series probes distances of about seven lattice spacings. If there were an infrared fixed point with \( \gamma \neq 1 \) at strong coupling, we could detect it even with such a short series by taking the bare couplings close to the fixed point. Because non-trivial critical behavior was not found for any \( \lambda_1, \lambda_2 \) with large \( \lambda_s \), we can claim the absence of strongly-coupled infrared fixed points, except for the unlikely case that such a fixed point has \( \gamma \) very close to 1.

If we assume that the phase transition is, in fact, of the first order, then for the \( U(2) \times U(2) \) model it is a relatively weak one for most of the bare parameter space. Indeed, we have seen that the susceptibility follows the mean-field, second-order, behavior up to rather large scales. Equivalently, the correlation length grows to the size of at least several lattice spacings. It is also instructive to compare the estimates (14) for the critical "temperature" \( \sqrt{2\lambda_s} \) of what our series sees as a second-order transition with the value for which ref.[3] finds a first-order transition. In our normalization, that value is \( \sqrt{2\lambda_s^f.o.} = 2\sqrt{20} = 8.94427 \), rather close to the numbers in (14). The fact that mistaking the first-order transition for a second-order one makes only a small error in the critical "temperature" confirms that the transition is only weakly first-order.

The correlation length for electroweak interactions is the scale of physics responsible for the symmetry breaking and cannot be much above, say, 2 TeV. Therefore, an hierarchy between the correlation length and the cutoff by a factor of 10 or so should be sufficient for consistency of models of electroweak symmetry breaking in which the cutoff scale is not much above 20 TeV.

Near the line \( 2\lambda_1 + \lambda_2 = 0 \), beyond which the potential becomes unbounded from below, the transition becomes more strongly first-order. We will describe the change in the behavior of the series in that region while discussing the three-dimensional case, for which this change may have an application to electroweak baryogenesis.
In three dimensions, our results indicate that for small $\lambda_2$ and large $\lambda_1$, the approach to the phase transition, at the length scale probed by the 7th order series, is controlled by the non-trivial $O(8)$ fixed point, but for most of the bare couplings plane, by the trivial point $\lambda_1 = \lambda_2 = 0$. Assuming that the transition is of the first order, we find that it remains a weak one, except for the vicinity of $2\lambda_1 + \lambda_2 = 0$. For the simple cubic lattice and $\lambda_2/\lambda_s = 0.5$, $\mu^2/\sqrt{2\lambda_s} = -5$, corresponding to the values of fig.3 of ref.\[17\], we obtain

$$\{\gamma_3, ..., \gamma_7\} = \{1.0005, 1.0005, 1.0009, 1.0007, 0.9997\}, \quad (15)$$
$$\{\beta_{\gamma,3}^{-1}, ..., \beta_{\gamma,7}^{-1}\} = \{1.023, 1.030, 1.011, 1.008, 1.020\} \quad (16)$$

in good agreement with the mean field value $\gamma = 1$ and the "temperature" of the first order transition $\sqrt{2\lambda_s} \approx 1$ deduced from fig.3 of ref.\[17\]. Similar results were obtained for stronger couplings. This is in accord with the results of a numerical simulation of the non-linear $U(2) \times U(2)$ model in three dimensions \[18\], which has found that to observe the first-order phase transition one has to go to lattices as large as $16^3$.

Near the line $2\lambda_1 + \lambda_2 = 0$, or $\lambda_2/\lambda_s = 2$, the behavior of the series changes. Compare the following estimates for the critical exponent obtained for the simple cubic lattice, $\mu^2/\sqrt{2\lambda_s} = -2$, and different values of $\lambda_2/\lambda_s$: for $\lambda_2/\lambda_s = 1.55, 1.6, 1.65$, respectively,

$$\{\gamma_3, ..., \gamma_7\} = \{0.9996, 0.9996, 0.9990, 0.9992, 0.9991\},$$
$$\{\gamma_3, ..., \gamma_7\} = \{0.9992, 0.9993, 0.9988, 0.998 + 1.45 \times 10^{-7}i, 3.16 + 0.886i\},$$
$$\{\gamma_3, ..., \gamma_7\} = \{2.91 + 0.218i, 5.09, -1.47 + 1.34i, -31.5 - 533.i, 8.38 + 2.55i\}.$$

Imaginary values reflect the appearance of negative coefficients in the series which starts to display irregular behavior; the prescription for determining the signs of imaginary parts was chosen arbitrarily. We see that as we approach the line $\lambda_2/\lambda_s = 2$, the irregular behavior of the series sets off in lower order. In some systems, the presence of negative coefficients in a series signals an unphysical singularity close to
\( \beta = 0 \), which can be mapped further away by an appropriate change of the expansion variable \([16]\). In our case, there is a physical reason for such behavior — the decrease in the correlation length as the phase transition becomes more strongly first-order.

For the electroweak phase transition, the cutoff in the effective three-dimensional theory is of order of temperature. To prevent the washout of baryon asymmetry, the expectation value of the order parameter after the transition should be at least of order of temperature \([7]\) and, at strong coupling, so should be the inverse correlation lengths in both phases. So, we find that for the strongly-coupled \( U(2) \times U(2) \) model without gauge fields, the condition for electroweak baryogenesis is realized only in a narrow region near the line \( 2\lambda_1 + \lambda_2 = 0 \).

The phase transition becomes more strongly first-order for larger \( U(N) \times U(N) \) groups \([18, 19]\). However, from the results of ref. [18] we were unable to conclude whether for \( N = 3 \) it is strong enough to preserve the baryon asymmetry for general values of the couplings.

Finally, let us turn to the \( y \to \infty \) limit of the four-dimensional \( U(2) \times U(2) \) model with fermions. Though universality implies that there can be no infrared fixed points or weak first-order transitions other than those of the purely bosonic model, we have done an independent series analysis. The results are indeed similar to those in the bosonic case. Stable estimates of \( \gamma \) were obtained for different \( N_c \) for most of the strong-coupling region, \( \lambda_s \gtrsim 1 \), and they were consistent with \( \gamma = 1 \). The phase transition becomes more strongly first-order in the vicinity of the line \( 2\lambda_1 + \lambda_2 = 0 \). Similar results were obtained for the face-centered hypercubic lattice (with naive fermions), although Zinn-Justin’s method often did not lead to stable estimates of \( \gamma \) in that case.

To summarize, expanding in the inverse bare Yukawa constant \( y \) gives the effective bosonic theory \([2]\) for the \( y \to \infty \) limit of chiral models of electroweak symmetry breaking that contain heavy fermions. The non-linear limit of that effective theory coincides with that of the model without fermions. For the linear \( U(2) \times U(2) \) model in four dimensions, the analysis of the seventh-order strong coupling series for the
susceptibility is consistent with the absence of any non-trivial fixed points at large bare self-coupling, both in the case without fermions and in the effective theory of the $y \to \infty$ limit. Assuming that the phase transition in the strongly-coupled $U(2) \times U(2)$ model is of the first order, it is a relatively weak one, both in four and (in the case without fermions) three dimensions, except for a region near the line beyond which the bare potential is unstable.

We are grateful to T. Clark and S. Love for getting us interested in the subject and many helpful discussions. S.K. was supported in part by Alfred P. Sloan Foundation and in part by the U.S. Department of Energy under grant DE-AC02-76ER01428 (Task B). R.S. was supported in part by a grant from Purdue Research Foundation.

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