RELATION BETWEEN ANN-CATEGORIES
AND RING CATEGORIES

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Abstract. There are different categorifications of the notion of a ring such as Ann-category due to N. T. Quang, ring category due to M. M. Kapranov and V. A. Voevodsky. The main result of this paper is to prove that every axiom in the definition of a ring category, but the axiom $x_0 = y_0$, can be deduced from the axiomatics of an Ann-category.

1. Introduction

Categories with monoidal structures $\oplus, \otimes$ (also called categories with distributivity constraints) were presented by M. L. Laplaza [3]. M. M. Kapranov and V. A. Voevodsky [2] omitted requirements of the axiomatics due to Laplaza which are related to the commutativity constraints of the operation $\otimes$. These appeared under the name ring categories.

In another approach, a monoidal category can be “smoothed” to become a category with group structure, when added the invertible objects (see Laplaza [4], Saavedra Rivano [9]). Now, if the ground category is a groupoid (i.e., each morphism is an isomorphism), then we have a group-like monoidal category (see A. Fröhlich and C. T. C. Wall [1]), or a $Gr$-category (see H. X. Sinh [11]). These categories can be classified by $H^3(\Pi, A)$. Each $Gr$-category $\mathcal{G}$ is determined by 3 invariants: The group $\Pi$ of classes of congruence objects, $\Pi$-module $A$ of automorphisms of the unit 1, and an element $\tilde{h} \in H^3(\Pi, A)$, where $\tilde{h}$ is induced by the associativity constraint of $\mathcal{G}$.

In 1987, in [6], N. T. Quang proposed a notion of an Ann-category, as a categorification of the notion of rings, when a symmetric Gr-category (also called Pic-category) is equipped with a monoidal structure $\otimes$. In [8], [7], Ann-categories and regular Ann-categories, developed from the ring extension problem, have been classified by, respectively, Mac Lane ring cohomology [5] and Shukla algebraic cohomology [10].

The aim of this paper is to clearly show the relation between these definitions of an Ann-category and a ring category.
For convenience, let us recall the definitions. Moreover, let us denote $AB$ or $A . B$ instead of $A \otimes B$.

2. Fundamental definitions

**Definition 2.1. The axiomatics of an Ann-category**

An Ann-category consists of:

i) a groupoid $\mathcal{A}$ together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

ii) a fixed object $0 \in \mathcal{A}$ together with naturality constraints $a_+, c, g, d$ such that $(\mathcal{A}, \otimes, a_+, c, (0, g, d))$ is a Pic-category.

iii) a fixed object $1 \in \mathcal{A}$ together with naturality constraints $a, l, r$ such that $(\mathcal{A}, \otimes, a, (1, l, r))$ is a monoidal $\mathcal{A}$-category.

iv) natural isomorphisms $\mathcal{L}, \mathcal{R}$:

$$
\mathcal{L}_{A,X,Y} : A \otimes (X \oplus Y) \to (A \otimes X) \oplus (A \otimes Y),
$$

$$
\mathcal{R}_{X,Y,A} : (X \oplus Y) \otimes A \to (X \otimes A) \oplus (Y \otimes A)
$$

such that the following conditions are satisfied:

(Ann-1) For each $A \in \mathcal{A}$, the pairs $(L^A, \bar{L}^A)$, $(R^A, \bar{R}^A)$ determined by relations:

$$
L^A = A \otimes -, \quad R^A = - \otimes A,
$$

$$
\bar{L}^A_{X,Y} = \mathcal{L}_{A,X,Y}, \quad \bar{R}^A_{X,Y} = \mathcal{R}_{X,Y,A}
$$

are $\oplus$-functors which are compatible with $a_+$ and $c$.

(Ann-2) For all $A, B, X, Y \in \mathcal{A}$, the following diagrams:

$$
\begin{align*}
(AB)(X \oplus Y) & \xrightarrow{a_{A,B,X \oplus Y}} A(B(X \oplus Y)) \xrightarrow{id_A \otimes L_B} A(BX \oplus BY) \\
(AB)X \oplus (AB)Y & \xrightarrow{a_{A,B,X \oplus A,B,Y}} A(BX) \oplus A(BY), \\
(X \oplus Y)(BA) & \xrightarrow{a_{X \oplus Y,B,A}} ((X \oplus Y)B)A \xrightarrow{\bar{R}^B \otimes id_A} (XB \oplus YB)A \\
X(BA) \oplus Y(BA) & \xrightarrow{a_{X,B,A \oplus Y,B,A}} (XB)A \oplus (YB)A, \\
(A(X \oplus Y))B & \xrightarrow{a_{A,X \oplus Y,B}} A((X \oplus Y)B) \xrightarrow{id_A \otimes R^B} A(XB \oplus YB) \\
(AX \oplus AY)B & \xrightarrow{\bar{R}^B} (AX)B \oplus (AY)B \xrightarrow{a_{B,A}} A(XB) \oplus A(YB),
\end{align*}
$$

where $a_{X,Y,B}$, $a_{A,B,X}$, $a_{A,B,Y}$, $a_{A,X \oplus Y,B}$, $a_{X,B,A \oplus Y,B,A}$, $a_{A,X \oplus Y,B}$ are the associativity morphisms of $\mathcal{A}$. The diagrams (1.1) and (1.1') show that $L^A$ and $R^A$ are $\oplus$-functors.

(Ann-3) For all $A, B, X, Y \in \mathcal{A}$, the following diagrams:

$$
\begin{align*}
L^A \otimes id_B & \xrightarrow{a_{A,B,X \oplus Y}} A((X \oplus Y)B) \xrightarrow{id_A \otimes R^B} A(XB \oplus YB) \\
L^A \otimes id_B & \xrightarrow{a_{A,B,X \oplus Y}} A((X \oplus Y)B) \xrightarrow{id_A \otimes R^B} A(XB \oplus YB),
\end{align*}
$$

and $\bar{L}^A \otimes id_B$ and $\bar{R}^A \otimes id_B$ are compatible with $a_+$ and $c$.
(1.3)
\[(A \oplus B)X \oplus (A \oplus B)Y \xrightarrow{L^AB} (A \oplus B)(X \oplus Y) \xrightarrow{\bar{R}^XY} A(X \oplus Y) \oplus B(X \oplus Y) \]
\[
(AX \oplus BX) \oplus (AY \oplus BY) \xrightarrow{v} (AX \oplus AY) \oplus (BX \oplus BY) \]
\[
(AX \oplus BY) \oplus (AY \oplus BX) \xrightarrow{\sim} AX \oplus BX \]
\[
(AY \oplus BX) \oplus (AX \oplus BY) \xrightarrow{\sim} AY \oplus BX \]
\[
(AX \oplus BY) \oplus (AY \oplus BX) \xrightarrow{\sim} AX \oplus BX \]
\[
(AX \oplus AY) \oplus (BX \oplus BY) \xrightarrow{\sim} AX \oplus BX \]
\[\text{commute, where } v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \rightarrow (U \oplus Z) \oplus (V \oplus T) \text{ is the unique functor built from } a_+, c, id \text{ in the monoidal symmetric category } (A, \oplus).\]

(Ann-3) For the unit object \(1 \in A\) of the operation \(\oplus\), the following diagrams commute:

\[
1(X \oplus Y) \xrightarrow{l} 1X \oplus 1Y \]
\[
(X \oplus Y)1 \xrightarrow{r} X1 \oplus Y1 \]

Remark. The commutative diagrams (1.1), (1.1') and (1.2), respectively, mean that:

\[
(a_{A,B,-}) : L^A.L^B \rightarrow L^{AB},
\]
\[
(a_{-,A,B}) : R^{AB} \rightarrow R^A.R^B,
\]
\[
(a_{A, -,B}) : L^A.R^B \rightarrow R^B.L^A
\]

are \(\oplus\)-functors. The diagram (1.3) shows that the family \((\bar{L}^X_Z)_{X,Y} = (\bar{L}_{-,X,Y})\) is an \(\oplus\)-functor between the \(\oplus\)-functors \(Z \mapsto Z(X \oplus Y)\) and \(Z \mapsto ZX \oplus ZY\), and the family \((\bar{R}^X_C)_{A,B} = (\bar{R}_{A,B,-})\) is an \(\oplus\)-functor between the functors \(C \mapsto (A \oplus B)C\) and \(C \mapsto AC \oplus BC\). The diagram (1.4) (resp. (1.4')) shows that \(l\) (resp. \(r\)) is an \(\oplus\)-functor from \(L^1\) (resp. \(R^1\)) to the unit functor of the \(\oplus\)-category \(A\).

Definition 2.2. The axiomatics of a ring category

A ring category is a category \(R\) equipped with two monoidal structures \(\oplus, \otimes\) (which include corresponding associativity morphisms \(a_{A,B,C}, a_{A,B,\otimes}\) and unit objects denoted \(0, 1\)) together with natural isomorphisms:

\[
u_{A,B} : A \oplus B \rightarrow B \oplus A, \quad v_{A,B,C} : A \otimes (B \oplus C) \rightarrow (A \otimes B) \oplus (A \otimes C),
\]
\[
w_{A,B,C} : (A \oplus B) \otimes C \rightarrow (A \otimes C) \oplus (B \otimes C),
\]
\[
x_A : A \otimes 0 \rightarrow 0, \quad y_A : 0 \otimes A \rightarrow 0.
\]

These isomorphisms are required to satisfy the following conditions.
The isomorphisms $u_{A,B}$ define on $\mathcal{R}$ a structure of a symmetric monoidal category, i.e., they form a braiding and $u_{A,B}u_{B,A} = 1$.

For any objects $A, B, C$ the following diagram commutes:

\[
\begin{array}{c}
A \otimes (B \oplus C) \xrightarrow{v_{A,B,C}} (A \otimes B) \oplus (A \otimes C) \\
A \otimes (C \oplus B) \xrightarrow{v_{A,C,R}} (A \otimes C) \oplus (A \otimes B)
\end{array}
\]

For any objects $A, B, C$ the following diagram commutes:

\[
\begin{array}{c}
(A \oplus B) \otimes (C \oplus D) \xrightarrow{w_{A,B,C,D}} AD \oplus ((B \oplus C)D) \xrightarrow{w_{A,B,C,D}} AD \oplus (BD \oplus CD) \\
((A \oplus B) \oplus C)D \xrightarrow{w_{A,B,R,C,D}} (A \oplus B)D \oplus CD \xrightarrow{w_{A,B,R,C,D}} (AD \oplus BD) \oplus CD
\end{array}
\]

For any objects $A, B, C, D$ the following diagram commutes:

\[
\begin{array}{c}
A \otimes (B \oplus C) \oplus D \xrightarrow{v_{A,B,C,D}} AB \oplus A(C \oplus D) \xrightarrow{v_{A,B,C,D}} AB \oplus (AC \oplus AD) \\
A \otimes (B \oplus C) \oplus D \xrightarrow{v_{A,B,C,D}} A(B \oplus C) \oplus AD \xrightarrow{v_{A,B,C,D}} (AB \oplus AC) \oplus AD
\end{array}
\]

For any objects $A, B, C, D$ the following diagram commutes:

\[
\begin{array}{c}
A(B(C \oplus D)) \xrightarrow{w_{A,B,C,D}} A(BC \oplus BD) \xrightarrow{w_{A,B,C,D}} A(BC) \oplus A(BD) \\
A(B(C \oplus D)) \xrightarrow{w_{A,B,C,D}} (AB)(C \oplus D) \xrightarrow{v_{A,B,C,D}} (AB)C \oplus (AB)D
\end{array}
\]
For any objects $A, B, C, D$ the following diagram commutes:

\[
\begin{array}{ccc}
((A \oplus B)C)D & \xrightarrow{w_{A,B,C,D}} & (AC \oplus BC)D & \xrightarrow{w_{AC,BC,D}} & (AC)D \oplus (BC)D \\
(A \oplus B)(CD) & \xrightarrow{w_{A,B,C,D}} & A(CD) \oplus B(CD) .
\end{array}
\]

For any objects $A, B, C, D$ the following diagram commutes:

\[
\begin{array}{ccc}
(A(B \oplus C))D & \xrightarrow{w_{A,B,C,D}} & (AB \oplus AC)D & \xrightarrow{w_{AB,AC,D}} & (AB)D \oplus (AC)D \\
A((B \oplus C)D) & \xrightarrow{w_{A,B,C,D}} & A(BD \oplus CD) & \xrightarrow{w_{A,B,D,C,D}} & A(BD) \oplus A(CD) .
\end{array}
\]

For any objects $A, B, C, D$ the diagram

\[
\begin{array}{ccc}
(A \oplus B)(C \oplus D) & \xrightarrow{w_{A,B,C,D}} & A(C \oplus D) \oplus B(C \oplus D) & \xrightarrow{w_{AC,BC,D}} & (AC \oplus AD) \oplus (BC \oplus BD) \\
& \xrightarrow{w_{A,B,C,D}} & ((AC \oplus AD) \oplus BC) \oplus BD \\
& \xrightarrow{w_{AC,BC,D}} & (AC \oplus (AD \oplus BC)) \oplus BD \\
& \xrightarrow{w_{AC,BC,D}} & ((AC \oplus BC) \oplus AD) \oplus BD \\
& \xrightarrow{w_{AC,BC,D}} & (AC \oplus (BC \oplus AD)) \oplus BD
\end{array}
\]

is commutative (the notions for arrows have been omitted, they are obvious).

The maps $x_0, y_0 : 0 \oplus 0 \to 0$ coincide.

For any objects $A, B$ the following diagram commutes:

\[
\begin{array}{ccc}
0 \oplus (A \oplus B) & \xrightarrow{w_{0,A,B}} & (0 \oplus A) \oplus (0 \oplus B) \\
0 & \xrightarrow{w_{A,B}} & 0 \oplus 0 .
\end{array}
\]
\(K12((\bullet \oplus \bullet) \otimes 0)\) For any objects \(A, B\) the following diagram commutes:

\[
\begin{array}{ccc}
A \oplus B & \overset{w_{A,B,0}}{\longrightarrow} & (A \otimes 0) \oplus (B \otimes 0) \\
\downarrow^{x_{A \oplus B}} & & \downarrow^{s_{A \otimes x_B}} \\
0 & & 0 \\
\end{array}
\]

\(K13(0 \otimes 1)\) The maps \(y_1, r_0^0 : 0 \otimes 1 \to 0\) coincide.

\(K14(1 \otimes 0)\) The maps \(x_1, l_0^0 : 1 \otimes 0 \to 0\) coincide.

\(K15(0 \otimes \bullet \otimes \bullet)\) For any objects \(A, B\) the following diagram commutes:

\[
\begin{array}{ccc}
0 \otimes (A \otimes B) & \overset{u_{0,A,B}}{\longrightarrow} & (0 \otimes A) \otimes B \\
\downarrow^{y_{A \otimes B}} & & \downarrow^{y_{A \otimes B}} \\
0 & & 0 \\
\end{array}
\]

\(K16(\bullet \otimes 0 \otimes \bullet), (\bullet \otimes \bullet \otimes 0)\) For any objects \(A, B\) the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes (0 \otimes B) & \overset{u_{A,0,B}}{\longrightarrow} & (A \otimes 0) \otimes B \\
\downarrow^{A \otimes y_B} & & \downarrow^{f_{A \otimes B}} \\
A \otimes 0 & \overset{x_A}{\longrightarrow} & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes (B \otimes 0) & \overset{u_{A,B,0}}{\longrightarrow} & (A \otimes B) \otimes 0 \\
\downarrow^{A \otimes x_B} & & \downarrow^{f_{A \otimes B}} \\
A \otimes 0 & \overset{x_A}{\longrightarrow} & 0 \\
\end{array}
\]

\(K17((0 \oplus \bullet))\) For any objects \(A, B\) the following diagram commutes:

\[
\begin{array}{ccc}
A \oplus (0 \oplus B) & \overset{u_{0,A,B}}{\longrightarrow} & (A \oplus 0) \oplus (A \oplus B) \\
\downarrow^{A \oplus y_B} & & \downarrow^{f_{A \oplus B}} \\
A \oplus B & \overset{x_{A \oplus (A \oplus B)}}{\longrightarrow} & 0 \oplus (A \oplus B) \\
\end{array}
\]
For any objects $A, B$ the diagrams
\[
\begin{align*}
(0 \oplus A) \otimes B & \xrightarrow{v_{A,B}} (0 \otimes B) \oplus (A \otimes B) \\
A \otimes B & \xrightarrow{y_{B \otimes (A \otimes B)}} 0 \oplus (A \otimes B) ,
\end{align*}
\[
\begin{align*}
A \otimes (B \oplus 0) & \xrightarrow{v_{A,B,0}} (A \otimes B) \oplus (A \otimes 0) \\
A \otimes (B \oplus 0) & \xrightarrow{y_{A \otimes B}} (A \otimes B) \oplus 0 ,
\end{align*}
\[
\begin{align*}
(A \otimes 0) \otimes B & \xrightarrow{v_{A,0,B}} (A \otimes B) \oplus (0 \otimes B) \\
(A \otimes 0) \otimes B & \xrightarrow{y_{A \otimes B}} (A \otimes B) \oplus 0 ,
\end{align*}
\[
\begin{align*}
A \otimes B & \xrightarrow{y_{A \otimes B}} (A \otimes B) \oplus 0
\end{align*}
\]
are commutative.

3. Relation between an Ann-category and a ring category

In this section, we prove that the axiomatics of a ring category, without $K10$, can be deduced from the axiomatics of an Ann-category. First, we can see that, the functor morphisms $a^\oplus, a^\otimes, u, l^\oplus, r^\oplus, v, w$, in Definiton 2.2 are, respectively, the functor morphisms $a_+, a, c, g, d, L, R$ in Definition 2.1. The isomorphisms $x_A, y_A$ coincide with the isomorphisms $\hat{L}^A, \hat{R}^A$ referred in Proposition 3.2 below.

We now prove that diagrams which commute in a ring category also hold in an Ann-category.

$K1$ obviously follows from (ii) in the definition of an Ann-category.

The commutative diagrams $K2, K3, K4, K5$ are indeed the compatibility of functor isomorphisms $(L^A, \hat{L}^A), (R^A, \hat{R}^A)$ with the constraints $a_+, c$ (the axiom Ann-1).

The diagrams $K5$ – $K9$, respectively, are indeed the ones in (Ann-2). Particularly, $K9$ is indeed the decomposition of (1.3) where the morphism $v$ is replaced by its definition diagram:

\[
\begin{align*}
(P \oplus Q) \oplus (R \oplus S) & \xrightarrow{a_+} ((P \oplus Q) \oplus R) \oplus S \xrightarrow{a_+ \oplus S} (P \oplus (Q \oplus R)) \oplus S \\
(P \oplus R) \oplus (Q \oplus S) & \xrightarrow{a_+} ((P \oplus R) \oplus Q) \oplus S \xrightarrow{a_+ \oplus S} (P \oplus (R \oplus Q)) \oplus S.
\end{align*}
\]
Proofs of $K17$, $K18$

**Lemma 3.1.** Let $P$, $P'$ be Gr-categories, $(a_+, (0, g, d)), (a'_+, (0', g', d'))$ be respective constraints, and $(F, F') : P \rightarrow P'$ be $\oplus$-functor which is compatible with $(a_+, a'_+)$. Then $(F, F')$ is compatible with the unit constraints $(0, g, d), (0', g', d')$.

First, the isomorphism $\tilde{F} : F0 \rightarrow 0'$ is determined by the composition
\[
F0 \oplus F0 \xrightarrow{\tilde{F}} F(0 \oplus 0) \xrightarrow{F(0)} F0 \leftarrow 0' \oplus F0.
\]
Since $F0$ is a regular object, there exists uniquely the isomorphism $\tilde{F} : F0 \rightarrow 0'$ such that $\tilde{F} \oplus id_{F0} = u$. Then, we may prove that $\tilde{F}$ satisfies the diagrams in the definition of the compatibility of the $\oplus$-functor $F$ with the unit constraints.

**Proposition 3.2.** In an Ann-category $A$, there exist uniquely isomorphisms
\[
\tilde{L}^A : A \otimes 0 \rightarrow 0, \quad \tilde{R}^A : 0 \otimes A \rightarrow 0,
\]
such that the following diagrams commute, i.e., $L^A$ and $R^A$ are $U$-functors respect to the operation $\oplus$.

\[
\begin{array}{cccc}
AX & \xleftarrow{\tilde{L}^A(g)} & A(0 \oplus X) \\
\downarrow{g} & & \downarrow{\tilde{L}^A} \\
0 \oplus AX & \xleftarrow{\tilde{L}^A \oplus id} & A0 \oplus AX
\end{array}
\]

(2.1)

\[
\begin{array}{cccc}
AX & \xleftarrow{\tilde{L}^A(d)} & A(X \oplus 0) \\
\downarrow{d} & & \downarrow{\tilde{L}^A} \\
AX \oplus 0 & \xleftarrow{id \oplus \tilde{L}^A} & AX \oplus A0
\end{array}
\]

(2.1')

\[
\begin{array}{cccc}
AX & \xleftarrow{\tilde{R}^A(g)} & (0 \oplus X)A \\
\downarrow{g} & & \downarrow{\tilde{R}^A} \\
0 \oplus AX & \xleftarrow{\tilde{R}^A \oplus id} & 0A \oplusXA
\end{array}
\]

(2.2)

\[
\begin{array}{cccc}
AX & \xleftarrow{\tilde{R}^A(d)} & (X \oplus 0)A \\
\downarrow{d} & & \downarrow{\tilde{R}^A} \\
AX \oplus 0 & \xleftarrow{id \oplus \tilde{R}^A} & XA \oplus 0A
\end{array}
\]

(2.2')
Proof. Since \((L^A, \bar{L}^A)\) are \(\oplus\)-functors which are compatible with the associativity constraint \(a^\oplus\) of the Picard category \((A, \oplus)\), they are also compatible with the unit constraint \((0, g, d)\) thanks to Lemma 3.1. That means there exists uniquely the isomorphism \(\hat{L}^A\) satisfying the diagrams (2.1) and (2.1'). The proof for \(\hat{R}^A\) is similar. The diagrams commute in Proposition 1 are indeed \(K_{17}, K_{18}\). \(\square\)

Proofs of \(K_{15}, K_{16}\)

Lemma 3.3. Let \((F, \hat{F}), (G, \hat{G})\) be \(\oplus\)-functors between \(\oplus\)-categories \(C, C'\) which are compatible with the constraints \((0, g, d), (0', g', d')\) and \(\hat{F}: F(0) \to 0', \hat{G}: G(0) \to 0'\) are respective isomorphisms. If \(\alpha: F \to G\) in an \(\oplus\)-morphism such that \(\alpha_0\) is an isomorphism, then the diagram

\[
\begin{array}{ccc}
F0 & \overset{\alpha_0}{\longrightarrow} & G0 \\
\downarrow F & & \downarrow G \\
0' & \overset{g'}{\longrightarrow} & \hat{G}
\end{array}
\]

commutes.

Proof. Let us consider the diagram:

\[
\begin{array}{c}
id \oplus u_0 \\
\hline
(id \oplus u_0) \\
\hline
0' \oplus F0 & \overset{\hat{F} \oplus id}{\longrightarrow} & F0 \oplus F0 & \overset{u_0 \oplus u_0}{\longrightarrow} & G0 \oplus G0 & \overset{\hat{G} \oplus id}{\longrightarrow} & 0' \oplus G0 \\
\downarrow g' & & \downarrow \hat{F} & & \downarrow \hat{G} & & \downarrow g' \\
F0 & \overset{F(g)}{\longrightarrow} & F(0 \oplus 0) & \overset{u_0 \oplus 0}{\longrightarrow} & G(0 \oplus 0) & \overset{G(g)}{\longrightarrow} & G0 \\
\downarrow u_0 & & \downarrow (V) & & \downarrow (V) & & \downarrow u_0 \\
\end{array}
\]

In this diagram, the regions (II) and (IV) commute thanks to the compatibility of \(\oplus\)-functors \((F, \hat{F}), (G, \hat{G})\) with the unit constraint; the region (III) commutes since \(u\) is a \(\oplus\)-morphism; the region (V) commutes thanks to the naturality of \(g'\). Therefore, the region (I) commutes, i.e.,

\[
\hat{G} \circ u_0 \oplus u_0 = \hat{F} \oplus u_0.
\]

Since \(F0\) is a regular object, \(\hat{G} \circ u_0 = \hat{F}\). \(\square\)

Proposition 3.4. For any objects \(X, Y \in ob A\) the following diagrams commute

\[
\begin{array}{ccc}
X \otimes (Y \otimes 0) & \overset{id \otimes L^X}{\longrightarrow} & X \otimes 0 \\
\downarrow a & & \downarrow L^X \\
(X \otimes Y) \otimes 0 & \overset{\tilde{E}^{XY}}{\longrightarrow} & 0
\end{array}
\]

(2.3)
Proof. To prove that the first diagram is commutative, let us consider the diagram:

\[ \begin{array}{ccc}
X \otimes (0 \otimes Y) & \xrightarrow{n} & (X \otimes 0) \otimes Y \\
\downarrow{id \otimes \tilde{R}^Y} & & \downarrow{\tilde{L}^X \otimes id} \\
X \otimes 0 & \xrightarrow{L} & 0 \otimes Y
\end{array} \]

where \( L = L^X \circ L^Y \). According to the axiom (1.1), \((a_{X,Y,Z})_Z\) is an \( \oplus \)-morphism from the functor \( L = L^X \circ L^Y \) to the functor \( G = L^{XY} \). Therefore, from Lemma 3.3, the region (II) commutes. The region (I) commutes thanks to the determination of \( \tilde{L} \) of the composition \( L = L^X \circ L^Y \). So the perimeter commutes.

The second diagram is proved similarly, thanks to the axiom (1.1'). To prove that the diagram (2.4) commutes, let us consider the diagram:

\[ \begin{array}{ccc}
X \otimes (0 \otimes Y) & \xrightarrow{n} & (X \otimes 0) \otimes Y \\
\downarrow{id \otimes \tilde{R}^Y} & & \downarrow{\tilde{L}^X \otimes id} \\
X \otimes 0 & \xrightarrow{\tilde{L}^X} & 0 \otimes Y
\end{array} \]

where \( H = L^X \circ R^Y \) and \( K = R^Y \circ L^X \). Then the regions (II) and (III) commute thanks to the determination of the isomorphisms \( H \) and \( K \). From the axiom (1.2), \((a_{X,Y,Z})_Z\) is an \( \oplus \)-morphism from the functor \( H \) to the functor \( K \). So from Lemma 3.3, the region (I) commutes. Therefore, the perimeter commutes. The diagrams in Proposition 3.4 are indeed \( K_{15}, K_{16} \). \( \square \)
Proof of $K_{11}$

**Proposition 3.5.** In an Ann-category, the following diagram commutes:

$$
\begin{array}{c}
0 \oplus 0 \xrightarrow{g_0 = d_0} 0 \\
\bar{R}^X \oplus \bar{R}^Y \\
(0 \otimes X) \oplus (0 \otimes Y) \xrightarrow{\bar{L}^0} 0 \otimes (X \otimes Y)
\end{array}
$$

**(2.5)**

**Proof.** Let us consider the diagram:

\[
\begin{array}{c}
(A \oplus 0)(B \oplus C) \\
\xrightarrow{d_A \otimes id} A(B \oplus C)
\end{array}
\]

\[
\begin{array}{c}
(A \oplus 0)B \oplus (A \oplus 0)C \\
\xrightarrow{(d_A \oplus id) \oplus (d_A \oplus id)} AB \oplus AC
\end{array}
\]

(V)

\[
\begin{array}{c}
(AB \oplus 0)B \oplus (AC \oplus 0)C \\
\xrightarrow{(id \oplus \bar{R}^B) \oplus (id \oplus \bar{R}^C)} (AB \oplus 0) \oplus (AC \oplus 0)
\end{array}
\]

(II)

\[
\begin{array}{c}
(AB \oplus AC) \oplus (0B \oplus 0C) \\
\xrightarrow{(id \oplus id) \oplus (\bar{R}^B \oplus \bar{R}^C)} (AB \oplus AC) \oplus (0 \oplus 0)
\end{array}
\]

(III)

\[
\begin{array}{c}
A(B \oplus C) \oplus 0(B \oplus C) \\
\xrightarrow{f_A^\prime \oplus id} A(B \oplus C) \oplus 0
\end{array}
\]

(IV)

In this diagram, the region (V) commutes thanks to the axiom I (1.3), the region (I) commutes thanks to the functorial property of $\bar{L}$; the perimeter and the region (II) commute thanks to the compatibility of the functors $\bar{R}^B \oplus \bar{R}^C$, $\bar{R}^B$, $\bar{R}^C$ with the unit constraint $(0, g, d)$; the region (III) commutes thanks to the functorial property of $v$; the region (VI) commutes thanks to the coherence for the ACU-functor $(\bar{L}^A, \hat{L}^A)$. So (IV) commutes. Note that $A(B \oplus C)$ is a regular object respect to the operation $\oplus$, so the diagram (2.5) commutes. We have $K_{11}$. \hfill $\square$

Similarly, we have $K_{12}$.

**Proofs of $K_{13}, K_{14}$**

**Proposition 3.6.** In an Ann-category, we have:

$$
\hat{L}^1 = l_0, \quad \hat{R}^1 = r_0.
$$
Proof. We will prove the first equation, the second one is proved similarly. Let us consider the diagram (2.6). In this diagram, the perimeter commutes thanks to the compatibility of $\oplus$-functor $(L^1, \hat{L}^1)$ with the unit constraint $(0, g, d)$ respect to the operation $\oplus$; the region (I) commutes thanks to the functorial property of the isomorphism $l$; the region (II) commutes thanks to the functorial property of $g$; the region (III) obviously commutes; the region (IV) commutes thanks to the axiom I(1.4). So the region (V) commutes, i.e.,

$$\hat{L}^1 \oplus id_{1.0} = l_0 \oplus id_{1.0}.$$

Since $1.0$ is a regular object respect to the operation $\oplus$, $\hat{L}^1 = l_0$.

We have $K14$.

Similarly, we have $K13$. \qed

Definition 3.1. An Ann-category $\mathcal{A}$ is strong if $\hat{L}^0 = \hat{R}^0$.

All the above results can be stated as follows.

Proposition 3.7. Each strong Ann-category is a ring category.

Remark. In our opinion, in the axiomatics of a ring category, the compatibility of the distributivity constraint with the unit constraint $(1, l, r)$ respect to the operation $\otimes$ is necessary, i.e., the diagrams of (Ann-3) should be added.

Moreover, if the symmetric monoidal structure of the operation $\oplus$ is replaced with the symmetric categorical groupoid structure, then each ring category is an Ann-category.

An open question: May the equation $\hat{L}^0 = \hat{R}^0$ be proved to be independent in an Ann-category?

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RELATION BETWEEN ANN-CATEGORIES AND RING CATEGORIES

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