GLOBAL EXISTENCE OF STRONG SOLUTIONS FOR
2-DIMENSIONAL NAVIER-STOKES EQUATIONS ON EXTERIOR
DOMAINS WITH GROWING DATA AT INFINITY

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Abstract. It is proved the existence of a unique, global strong solution to the
two-dimensional Navier-Stokes initial-value problem in exterior domains in the
case where the velocity field tends, at large spatial distance, to a prescribed
velocity field that is allowed to grow linearly.

1. Introduction. In this paper we investigate the flow of an incompressible, vis-
cous fluid in exterior domains $\Omega \subset \mathbb{R}^2$ and for data growing linearly at infinity. The
equations governing this flow are given by

$$
\begin{align*}
\partial_t U &- \Delta U + U \cdot \nabla U + \nabla P = F, & \text{in } \Omega \times (0,T), \\
\text{div } U &= 0, & \text{in } \Omega \times (0,T), \\
U(x,0) &= U_0(x), & \text{in } \Omega, \\
U(x,t) &= 0, & \text{on } \partial \Omega \times (0,T), \\
\lim_{|x| \to \infty} \left[ U(x,t) + m(x) \right] &= 0, & \text{in } [0,T],
\end{align*}
$$

(1)

where $m = \mathfrak{M} \cdot x$, with $\mathfrak{M}$ a $2 \times 2$ traceless matrix.

Problems of this kind arise in a number of fundamental questions of fluid me-
chanics. Of particular significance are the cases

$$
\begin{align*}
\mathfrak{M} &= m_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{M} = \begin{pmatrix} 0 & m_2 \\ m_3 & 0 \end{pmatrix},
\end{align*}
$$

where $m_1 \in \mathbb{R}, i = 1, 2, 3$, with $m_1$ and at least one of $m_2, m_3$ being non-
zero.

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Notice that the set of equations in (1) can be equivalently rewritten in the form

\[
\begin{cases}
& u_t - \Delta u + u \cdot \nabla u + M \cdot \nabla u + u \cdot \nabla M + \nabla p = f, \quad \text{in } \Omega \times (0, T), \\
& \text{div } u = 0, \quad \text{in } \Omega \times (0, T), \\
& u(x, 0) = u_0(x), \quad \text{in } \Omega, \\
& u(x, t) = 0, \quad \text{on } \partial \Omega \times (0, T), \\
& \lim_{|x| \to \infty} u(x, t) = 0, \quad \text{in } [0, T],
\end{cases}
\]

with \( M \in W^{1,\infty}_{\text{loc}}(\bar{\Omega}) \) satisfying certain conditions explained in detail in Section 3.

The investigation of problems of this kind started with the corresponding stationary situation on \( \Omega = \mathbb{R}^3 \). More precisely, in his celebrated paper [11] Leray posed the question of whether one could prove blowup of the usual Navier-Stokes equations

\[
\begin{cases}
& v_t - \Delta v + v \cdot \nabla v + \nabla \pi = 0, \quad x \in \mathbb{R}^3, t > 0, \\
& \text{div } v = 0, \quad x \in \mathbb{R}^3, t > 0,
\end{cases}
\]

by construction backward self-similar solutions. In fact, let \( M = \text{diag } (\lambda_1, \lambda_2, \lambda_3) \) with \( \lambda_i \in \mathbb{R} \) for \( i = 1, 2, 3 \) and \( \text{tr } M = 0 \). Then stationary solutions to (3) of the form \( v(x) = u(x) + Mx \) and \( \pi(x) = p(x) - \frac{1}{2} |Mx|^2 \) satisfy

\[
\begin{cases}
& -\Delta u + u \cdot \nabla u + Mx \cdot \nabla u + Mu + \nabla p = 0, \quad x \in \mathbb{R}^3, \\
& \text{div } u = 0, \quad x \in \mathbb{R}^3.
\end{cases}
\]

If \( \text{tr } M > 0 \), then equation (4) relates to backward self-similar solutions of (3) with linear strain. In the particular case where \( \lambda_1 = \lambda_2 = \lambda_3 > 0 \), equation (4) is often called ‘Leray’s equation’. It was proved by Nečas, Růžička and Šverák in [13] that in this case any weak solution to (4) lying in \( L^3(\mathbb{R}^3) \) has to be trivial, implying that, in a suitable class, one cannot construct a blowing-up solution to (3) by this idea of Leray. Later, Tsai [15] was able to remove the decay assumption originally made in [13] on \( u \). The corresponding problem for the Euler equation was investigated by Chae in [1].

The case \( \text{tr } M = 0 \) and \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \) is also of special interest. In this case, equation (4) has an explicit solution which is called the Burger vortex.

First results on the analysis of the instationary problem (2) on \( \Omega = \mathbb{R}^3 \) for \( M = R \), where \( R \) denotes the rotation matrix, are due to Hishida [9]. For extensions of this result to arbitrary matrices with vanishing trace or to more general type of functions as well as to the \( L^p \)-setting, we refer to the work of Hieber, Sawada and Rhandi, see e.g. [7], [8]. Their method makes use of the theory of Ornstein-Uhlenbeck operators and yields local solutions. It was recently proved by Fang, Han and Zhang in [3] that even in the case of density dependent fluids the problem is globally wellposed provided the underlying domain coincides with \( \Omega = \mathbb{R}^2 \). Their proof is based on the Bony decomposition technique from harmonic analysis.

Starting from this situation in this paper we give a further contribution to the global in time resolution of problem (2) in exterior domains \( \Omega \subset \mathbb{R}^2 \) with boundary regularity of class 2. Our accomplishment is twofold. On the one hand, we prove existence (and uniqueness) of solutions possessing regularity in space and time. On the other hand, we allow for the more general case where the spatial region of flow is a (smooth enough) exterior domain \( \Omega \). This assumption is more appropriate as the one of \( \Omega = \mathbb{R}^2 \), from the physical viewpoint, when describing the stagnation and shear flow past a body that we previously recalled.
In order to achieve these goals we employ an approach different and much simpler than techniques from harmonic analysis, the latter being, in fact, known to work only for the Cauchy problem (that is, when $\Omega$ is the whole space/plane), and, in addition, to furnish very little information about the time regularity: see [7, 3]. Our approach, instead, relies on the use of the classical Galerkin’s method in conjunction with the technique of “invading domains”, on a suitable family of approximating problems (see Section 3), and allows us to obtain more general and complete results of the type specified above.

More specifically, our first main result, Theorem 3.1, states that for any $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$ (see the next section for the notation), equation (2) admits a unique, global strong solution. By “strong” we mean that the velocity field possesses up to the second spatial derivatives and first time-derivatives for a.a. $t > 0$, while the associated pressure gradient exists for a.a. $t > 0$. Moreover, it is shown in Theorem 4.1 that this solution becomes even more regular provided the data possess further regularity.

We conclude this introductory section with the following remarks. In the first place, we wish to emphasize that our approach would be able to produce solutions with the above degree of regularity also in the case when $\Omega$ is a (sufficiently smooth) exterior domain of $\mathbb{R}^3$. However, the corresponding existence results would only hold locally in time, due to the lack of a “good” energy estimate that might ensure the finiteness of the enstrophy, that is to say, the $L^2$-norm of the gradient of the velocity field over the space-time region $\Omega \times (0, T)$, arbitrary $T > 0$. This circumstance then leads to our second related remark, namely, that it is just the lack of control on the total enstrophy the reason why, to date, the three-dimensional steady state counterpart of our problem is not known to have a solution, even for small data.

2. Preliminary results.

In this section we fix the notation and prove a first preliminary result. To this end, let $\Omega \subset \mathbb{R}^2$ be an open set. As usual, the space $L^2_\sigma(\Omega)$ is defined as the completion of the class of smooth vector fields having zero divergence and compact support in $\Omega$ with respect to the $\|\cdot\|_2$-norm. Then $L^2(\Omega) = L^2_\sigma(\Omega) \oplus G(\Omega)$, where $G(\Omega) = \{ h \in L^2(\Omega) : h = \nabla \phi, \text{ for some } \phi \in L^1_{\text{loc}}(\Omega) \text{ with } \nabla \phi \in L^2(\Omega) \}$.

We denote by $P$ the Helmholtz projection mapping $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$. In the following, $\Omega \subset \mathbb{R}^2$ will always denote an exterior domain in $\mathbb{R}^2$, i.e. the complement of a compact set in $\mathbb{R}^2$, with boundary $\partial \Omega$ of class 2. The Stokes operator $A_S$ in $L^2_\sigma(\Omega)$ is then defined by

$$A_S u := P \Delta u, \quad \text{with} \quad D(A_S) := H^2(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega).$$

We start with the following preliminary result. For $\Omega \subset \mathbb{R}^2$ as defined above, consider the initial boundary value problem

$$\begin{cases}
  u_t + u \cdot \nabla u + a \cdot \nabla u + u \cdot A &= \Delta u - \nabla p + f, & \text{in } \Omega \times (0, T), \\
  \text{div } u &= 0, & \text{in } \Omega \times (0, T), \\
  u(x, 0) &= u_0(x), & \text{in } \Omega, \\
  u(x, t) &= 0, & \text{on } \partial \Omega \times (0, T), \\
  \lim_{|x| \to \infty} u(x, t) &= 0, & \text{in } (0, T),
\end{cases} \quad (5)$$

where $a : \Omega \mapsto \mathbb{R}^2$ and $A : \Omega \mapsto \mathbb{R}^{2 \times 2}$ are vector and second order tensor functions, respectively, with bounded support in $\Omega$, satisfying

$$a, \text{ div } a, A \in L^\infty(\Omega).$$
Lemma 2.1. Let \( f \in L^2(0,T;L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \cap L^2_\sigma(\Omega) \). Then, given \( T > 0 \) there exists a unique solution \((u,p)\) to (5) satisfying

\[
u \in C([0,T];L^2(\Omega) \cap H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \quad u_t \in L^2(0,T;L^2(\Omega)) \quad \text{and} \quad p \in L^2(0,T;L^2(\Omega)).
\]

Moreover, suppose \( u_0 \in D(A_S) \) and \( f_t \in L^2(0,T;L^2(\Omega)) \). Then

\[
u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),
\]

and the following estimates hold

\[
\|u_t(t)\|_2^2 + \int_0^T \|\nabla u_t(t)\|_2^2 \, ds \leq N^2 + c \left( (T + D_0^2 e^{CT}) N^2 + D_0^2 e^{CT} + D_1^2 \right),
\]

where \( c = c(\Omega,\kappa) \), for any \( \kappa \geq \max\{||\text{div} \, a||_\infty, ||A||_\infty\} \), and

\[
D_0^2 := \|u_0\|_2^2 + \int_0^T \|f(t)\|_2^2 \, dt \quad \text{and} \quad D_1^2 := \int_0^T \|f(t)\|_2^2 \, dt
\]

\[
N := \|F(u_0 \cdot \nabla u_0 + a \cdot \nabla u_0 + A \cdot u_0 - \Delta u_0 - f)\|_2.
\]

Finally, if besides the stated assumptions, \( f \in L^\infty(0,T;L^2(\Omega)) \) then

\[
u \in L^\infty(0,T;H^2(\Omega)), \quad \nabla p \in L^\infty(0,T;L^2(\Omega)),
\]

and if, in addition \( \partial \Omega \) is of class 3 and \( f \in L^2(0,T;H^1(\Omega)) \), then

\[
u \in L^2(0,T;H^1(\Omega)), \quad D^2 p \in L^2(0,T;L^2(\Omega)).
\]

Proof. The system of equations (5) differs from the analogous Navier-Stokes system by lower order linear terms with time-independent coefficients of bounded support. Consequently, uniqueness in the class (6) can be inferred by a routine argument (see, e.g., [10, Chapter 6, Theorem 1]), and its proof will be omitted. As for existence, we observe that it is enough to establish a priori estimates in the specified functional class. In fact, the claimed result will then follow by the standard Galerkin’s method in combination with the “invading domains” technique [6]. For the application we have in mind, the only issue that must be handled carefully is the dependence of these estimates on the coefficients \( a \) and \( A \). We start our proof by multiplying both sides of the first line of (5) by \( u_t \), integrating by parts and obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 = \frac{1}{2} \left( \text{div} \, a, |u|^2(t) \right) - (A \cdot u(t), u(t)) + (f(t), u(t)).
\]

Since, also by Cauchy-Schwarz inequality,

\[
\left( \text{div} \, a, |u|^2(t) \right) - (A \cdot u(t), u(t)) \leq ||\text{div} \, a||_\infty \|u(t)\|_2^2 + ||A||_\infty \|u\|_2^2,
\]

\[
(f(t), w) \leq \frac{1}{2} \left( \|f(t)\|_2^2 + \|w\|_2^2 \right), \quad w \in L^2(\Omega),
\]

from (10) we deduce

\[
\frac{d}{dt} \|u(t)\|_2^2 + 2 \|\nabla u(t)\|_2^2 \leq \beta \|u(t)\|_2^2 + \|f(t)\|_2^2,
\]

with \( \beta := 1 + ||\text{div} \, a||_\infty + 2 ||A||_\infty \). Gronwall’s lemma furnishes

\[
\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \, ds \leq \mathcal{D}_0 \exp \beta t, \quad t \in [0,T].
\]
We next multiply (5) by $A_S u = \mathbb{P} \Delta u$ for $u \in D(A_S)$ and integrate over $\Omega$. We thus obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_2^2 + \|\mathbb{P} \Delta u(t)\|_2^2 = (u(t) \cdot \nabla u(t), \mathbb{P} \Delta u(t)) + (a \cdot \nabla u(t), \mathbb{P} \Delta u(t)) \\
+ (A \cdot u(t), \mathbb{P} \Delta u(t)) - (f(t), \mathbb{P} \Delta u(t)).
\]  
(13)

Arguing as in (11), we show
\[
(a \cdot \nabla u(t), \mathbb{P} \Delta u(t)) + (A \cdot u(t), \mathbb{P} \Delta u(t)) \\
\leq \|a\|_\infty \|\nabla u(t)\|_2^2 + \|A\|_\infty \|u(t)\|_2^2 + \frac{1}{2} \|\mathbb{P} \Delta u(t)\|_2^2,
\]  
(14)

We next recall from [5], [2] and [6] the following three inequalities
\[
\|w\|_4^2 \leq c \|w\|_2 \|\nabla w\|_2, \quad w \in H_0^1(\Omega),
\]  
(15)
\[
\|\nabla w\|_4^2 \leq c \|\nabla w\|_2 \|D^2 w\|_2, \quad w \in H^2(\Omega),
\]  
(16)
\[
\|D^2 w\|_2 \leq c (\|A_S w\|_2 + \|\nabla w\|_2), \quad w \in D(A_S),
\]  
(17)

where the constant $c$ depends only on $\Omega$. By Cauchy-Schwarz inequality,
\[
(u(t) \cdot \nabla u(t), \mathbb{P} \Delta u(t)) \leq 2 \|u(t) \cdot \nabla u(t)\|_2^2 + \frac{1}{8} \|\mathbb{P} \Delta u(t)\|_2^2.
\]

On the other hand, by Hölder inequality and (15)–(16), we deduce
\[
\|u(t) \cdot \nabla u(t)\|_2^2 \leq c \|u(t)\|_2^2 \|\nabla u(t)\|_2^2 \|D^2 u(t)\|_2^2,
\]  
(18)

so that we obtain
\[
(u(t) \cdot \nabla u(t), \mathbb{P} \Delta u(t)) \leq c \|u(t)\|_2^2 \|\nabla u(t)\|_2^2 \|D^2 u(t)\|_2^2 + \frac{1}{8} \|\mathbb{P} \Delta u(t)\|_2^2.
\]  
(19)

From the latter, (17) and Cauchy inequality we find
\[
(u(t) \cdot \nabla u(t), \mathbb{P} \Delta u(t)) \leq c \|u(t)\|_2^2 (\|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^2) + \frac{1}{4} \|\mathbb{P} \Delta u(t)\|_2^2.
\]  
(20)

Combining (12)–(14), and (20), we infer
\[
\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{3}{2} \|\mathbb{P} \Delta u(t)\|_2^2 \leq c \left[\|\nabla u(t)\|_2^2 + \exp(\beta t)(1 + \|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^2) + \|f(t)\|_2^2\right],
\]
where $c = c(\Omega, \|a\|_\infty, \|A\|_\infty, \mathcal{D}_0)$. By Gronwall’s lemma in the latter relation in conjunction with (12) and (17), we deduce that
\[
\|\nabla u(t)\|_2^2 + \int_0^t \|D^2 u(s)\|_2^2 ds \leq F(\Omega, \|a\|_\infty, \|A\|_\infty, \|\nabla u_0\|_2, \mathcal{D}_0, t), \quad t \in [0, T],
\]  
(21)

where $F$ is continuous in $t$. We now multiply both sides of (5) by $u_t$ and integrate by parts to see
\[
\frac{1}{2} \frac{d}{dt} \|u_t(t)\|_2^2 + \|u_t(t)\|_2^2 = -(u(t) \cdot \nabla u(t), u_t(t)) \\
- (a \cdot \nabla u, u_t(t)) - (A \cdot u_t(t), u_t(t)) + (f(t), u_t(t)).
\]  
(22)
By the Cauchy-Schwarz inequality and (19), it is not hard to see that (22) leads us to
\[
\frac{d}{dt} \| \nabla u(t) \|_2^2 + \| u_t(t) \|_2^2 \leq c \left( \| u(t) \|_2^2 \| \nabla u(t) \|_2^2 \| D^2 u(t) \|_2 + \| u(t) \|_2^2 \right) + \| f(t) \|_2^2,
\]
with \( c = c(\Omega, \| a \|_\infty, \| A \|_\infty) \). Integrating both sides of this differential inequality between 0 and \( t \), and taking into account (21), we deduce
\[
\int_0^t \| u_s(s) \|_2^2 ds \leq G(\Omega, \| a \|_\infty, \| A \|_\infty, \| \nabla u_0 \|_2, D_0, t), \quad t \in [0, T], \tag{23}
\]
where \( G \) is a continuous function in \( t \). Employing estimates (12), (21), and (23) in conjunction with Galerkin’s method and the “invading domains” technique [10, 6], under the assumption \( u_0 \in H^1_0(\Omega) \cap L^2(\Omega) \), it becomes a routine task to show the existence of a solution to (5) with \( u \) satisfying
\[
\begin{align*}
  u &\in C([0, T]; L^2(\Omega) \cap H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
  u_t &\in L^2(0, T; L^2(\Omega)).
\end{align*} \tag{24}
\]
Furthermore, from (5)\(_1\), we also deduce \( \nabla p \in L^2(0, T; L^2(\Omega)) \), for some \( p \in L^2(0, T; L^2(\Omega)) \). We shall next show that this solution is, in fact, more regular, provided \( u_0 \) and \( \Omega \) are more regular as well. To this end, we begin to operate with the time derivative on both sides of (5)\(_1\) and multiply the resulting equation by \( u_t \). We thus get
\[
\frac{1}{2} \frac{d}{dt} \| u_t(t) \|_2^2 + \| \nabla u_t(t) \|_2^2 = -(u_t(t) \cdot \nabla u(t), u_t(t)) \\
+ \frac{1}{2} (\text{div} a, u_t^2(t)) - (A \cdot u(t), u_t(t)) + (f_t(t), u_t(t)). \tag{25}
\]
By the Cauchy-Schwarz inequality,
\[
-(A \cdot u(t), u_t(t)) \leq \frac{1}{2} \| A \|_\infty \| u(t) \|_2^2 + \frac{1}{2} \| u_t(t) \|_2^2,
\]
while, by Hölder inequality, (15), and Cauchy inequality we deduce
\[
-(u_t(t) \cdot \nabla u(t), u_t(t)) \leq \| u_t(t) \|_2^2 \| \nabla u(t) \|_2 \leq c \| u_t(t) \|_2 \| \nabla u(t) \|_2 \| \nabla u(t) \|_2 \\
\leq \frac{1}{2} \| u_t(t) \|_2^2 \| \nabla u(t) \|_2^2 + \frac{1}{2} \| \nabla u(t) \|_2^2. \tag{27}
\]
Collecting (25)–(27), and taking into account that
\[
(\text{div} a, u_t^2(t)) \leq \| \text{div} a \|_\infty \| u_t(t) \|_2^2,
\]
produces
\[
\frac{d}{dt} \| u_t(t) \|_2^2 + \| \nabla u_t(t) \|_2^2 \leq c \left[ (1 + \| \nabla u(t) \|_2^2) \| u_t(t) \|_2^2 + \| u(t) \|_2^2 + \| f_t(t) \|_2^2 \right], \tag{28}
\]
where \( c = c(\Omega, \kappa) \). We next use Gronwall’s lemma in (28) to show
\[
\| u_t(t) \|_2^2 \leq \left[ \| u_t(0) \|_2^2 + c \int_0^t \left( \| f_t(s) \|_2^2 + \| u(s) \|_2^2 \right) ds \right] \exp \left\{ c(t + \int_0^t \| \nabla u(s) \|_2^2 ds) \right\} \tag{29}
\]
from which, with the help of (12) and (5), the inequality (7) follows under the further condition that \( u_0 \in D(A_S) \). We thus get,

\[ u_t \in L^\infty(0, T; L^2(\Omega)) , \]

while from from (28) and (7) we establish (7) along with the following:

\[ u \in L^2(0, T; H^1(\Omega)) . \]

In order to show the further property for \( u \) and \( p \) listed in (9), we rewrite (5) as follows, for almost all \( t \in [0, T] \),

\[
\begin{align*}
\Delta u(t) - \nabla p(t) &= F, & \text{in } \Omega \times \{t\}, \\
\text{div } u(t) &= 0, & \text{in } \Omega \times \{t\}, \\
u(t) &= 0, & \text{on } \partial \Omega \times \{t\},
\end{align*}
\]  

with

\[ F := u_t + u \cdot \nabla u + a \cdot \nabla u + A \cdot u - f . \]  

From the established regularity properties of \( u \) it follows that \( F \in L^\infty(0, T; L^2(\Omega)) \) provided \( f \in L^\infty(0, T; L^2(\Omega)) \) and \( F \in L^2(0, T; H^1(\Omega)) \) provided \( f \in L^2(0, T; H^1(\Omega)) \), so that the remaining properties of \( u \) and \( p \) are a consequence of (30)–(31), and of classical results on the exterior steady-state Stokes problem; see [5].

3. Existence of global strong solutions. Objective of this section is to show existence and uniqueness of a global strong solution to the initial-boundary value problem

\[
\begin{align*}
\partial_t U + U \cdot \nabla U &= \Delta U - \nabla P + F, & \text{in } \Omega \times (0, T), \\
\text{div } U &= 0, & \text{in } \Omega \times (0, T), \\
U(x, 0) &= U_0(x), & \text{in } \Omega, \\
U(x, t) &= 0, & \text{on } \partial \Omega \times (0, T), \\
\lim_{|x| \to \infty} [U(x, t) + m(x)] &= 0, & \text{in } [0, T],
\end{align*}
\]  

where \( m = \mathcal{M} \cdot x \), with \( \mathcal{M} \) a \( 2 \times 2 \) traceless matrix. We begin to notice that the set of equations in (32) can be put in the following equivalent form

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + M \cdot \nabla u + u \cdot \nabla M &= \Delta u - \nabla p + f , & \text{in } \Omega \times (0, T), \\
\text{div } u &= 0, & \text{in } \Omega \times (0, T), \\
u(x, 0) &= u_0(x), & \text{in } \Omega, \\
u(x, t) &= 0, & \text{on } \partial \Omega \times (0, T), \\
\lim_{|x| \to \infty} u(x, t) &= 0 , & \text{in } [0, T],
\end{align*}
\]  

where \( M \in W^{1, \infty}_{\text{loc}}(\Omega) \) satisfies the conditions

\[
\begin{align*}
(i) \quad & \text{div } M = 0 \quad \text{in } \Omega; \\
(ii) \quad & M = 0 \quad \text{at } \partial \Omega; \\
(iii) \quad & \sup_{x \in \Omega} |M(x)|(1 + |x|)^{-1} < \infty; \\
(iv) \quad & \nabla M \in L^\infty(\Omega) .
\end{align*}
\]  

In fact, let \( \zeta = \zeta(x) \), \( x \in \mathbb{R}^2 \), be a smooth function of bounded support that is 1 in a neighborhood of \( \partial \Omega \). Set

\[
\psi = \frac{1}{2} (\mathcal{M}_{12} x_2^2 - \mathcal{M}_{21} x_1^2) + \mathcal{M}_{11} x_1 x_2
\]

where \( \mathcal{M}_{ij}, i, j = 1, 2 \), are the entries of \( \mathcal{M} \), and define the vector field

\[
w(x) := (\partial_2 (\zeta \psi), -\partial_1 (\zeta \psi)) .
\]
Using also the fact that \( M_{11} = -M_{22} \), we at once deduce (i) \( w = M \cdot x \), \( x \in \partial \Omega \), (ii) \( \text{div} w = 0 \) in \( \Omega \), and (iii) \( w \in C^0(\mathbb{R}^2) \). Therefore, if we now put
\[
 u(x, t) := U(x, t) - w(x) + M \cdot x
\]
\[
p := P - \frac{1}{2}(M_{11} + M_{21}M_{21})(x_1^2 + x_2^2)
\]
\[
M(x) := w(x) - M \cdot x
\]
\[
f := F - w \cdot \nabla w + \Delta w,
\]
it is immediately checked that (32) goes into (33), with \( M \) satisfying (34).

The following theorem provides our first result on the existence of global strong solutions.

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^2 \) be an exterior domain with boundary of class 2 and let \( T > 0 \). Assume that \( f \in L^2(0, T; L^2(\Omega)) \) and \( u_0 \in L^2_\sigma(\Omega) \). Then there exists exactly one solution \((u, \nabla p)\) to (33)–(34) satisfying
\[
u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \nabla u \in C((0, T]; L^2(\Omega)),
\]
\[
t^\frac{1}{2}D^2u \in L^2(0, T; L^2(\Omega)), \quad t^\frac{1}{2}u_t \in L^2(0, T; L^2_{\text{loc}}(\Omega)),
\]
\[
t^\frac{1}{2}p \in L^2(0, T; L^2_{\text{loc}}(\Omega)), \quad t^\frac{1}{2}\nabla p \in L^2(0, T; L^2(\Omega)).
\]
The above assertion means of course that \((u, p)\) is satisfying (33)\(_{1,2}\) a.e. in \( \Omega \times (0, T) \), that \( u \) attains (33)\(_3\) in the sense of \( L^2 \), (33)\(_4\) in the sense of traces and (33)\(_5\) uniformly pointwise for a.a. \( t \in [0, T] \).

**Proof.** The first step of our proof consists of replacing (32) with a family of suitably “approximating” problems. To this end, let \( \psi = \psi(r) \) be a smooth, non-negative real function such that \( \psi(r) = 1 \) for \( r \leq 1 \), and \( \psi(r) = 0 \) for \( r \geq 2 \), and set
\[
\psi_R(x) = \psi\left(\frac{|x|}{R}\right), \quad x \in \mathbb{R}^2,
\]
where \( R \in \mathbb{N} \) is chosen large enough \( (R > R_0, \text{say}) \) in order to ensure that \( \psi_R(\sigma) = 1 \) for every \( \sigma \in \partial \Omega \). It is plain that \( \psi_R = 1 \) for \( |x| \leq R \), and \( \psi_R(x) = 0 \) for \( |x| \geq 2R \), and that for all \( |\alpha| \geq 1 \), \( \text{supp}(D^\alpha \psi_R) \subseteq \{ x \in \mathbb{R}^2 : R \leq |x| \leq 2R \} \). Moreover, there exists a suitable constant \( c_\psi > 0 \), independent of \( R \), such that
\[
|D^\alpha \psi_R(x)| \leq \frac{c_\psi}{|x|^m}, \quad |\alpha| = m \geq 0.
\]
(35)

For \( R > R_0 \), set
\[
M_R(x) := \psi_R(x) M(x), \quad x \in \Omega.
\]
Since \( \text{div} M(x) = 0 \), it follows from (35) and the assumptions on \( M \) that
\[
|\text{div} M_R(x)| = |M(x) \cdot \nabla \psi_R(x)| \leq c_{M, \psi}, \quad x \in \Omega,
\]
(36)
for some \( c_{M, \psi} > 0 \). Consider now the following family of problems “approximating” (32)
\[
\begin{cases}
  u_t + u \cdot \nabla u + M_R \cdot \nabla u + u \cdot \nabla M_R = \Delta u - \nabla p + f, & \text{in } \Omega \times (0, T), \\
  u = 0, & \text{in } \Omega \times (0, T), \\
  u(x, 0) = u_0(x), & \text{in } \Omega, \\
  u(x, t) = 0, & \text{in } \partial \Omega \times (0, T), \\
  \lim_{|x| \to \infty} u(x, t) = 0, & \text{in } [0, T],
\end{cases}
\]
(37)
where \( \{u_0(x)\}_{R \in \mathbb{N}} \subset L^2(\Omega) \cap H^1(\Omega) \) converges to \( u_0 \) in \( L^2 \). Clearly, \( a := M_R \) and \( A := \nabla M_R \) obey the assumptions of Lemma 2.1, so that, by that lemma,
we deduce the existence of a unique solution \((u_R, \nabla p_R)\) to (37) belonging to the class defined in (6). Our objective in the following is to show that in the limit \(R \to \infty\), this solution converges to a solution to (33)–(34) satisfying the properties stated in the theorem. In what follows, and unless otherwise stated, we shall omit for simplicity the subscript “\(R\)” for \(u\) and \(p\). Moreover, by \(C\), we denote generic constants independent of \(R\). Multiplying both sides of (37) by \(u\) and integrating by parts, we readily show that

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 = \frac{1}{2} (\text{div} M_R, u^2(t)) - (u(t) \cdot \nabla M_R, u(t)) + (f(t), u(t)) .
\]

In view of (34), (35), and (36) we show

\[
\frac{1}{2} \text{div} M_R, u^2(t)) - (u(t) \cdot \nabla M_R, u(t)) \leq C \|u(t)\|_2^2 .
\]

Replacing the latter inequality in the previous equation, we find

\[
\frac{d}{dt} \|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 \leq C \|u(t)\|_2^2 + \|f(t)\|_2^2 ,
\]

so that, by Gronwall’s lemma, and observing that that \(\|u_{0R}\|_2^2 \leq 2\|u_0\|_2^2\), we infer

\[
\|u_R(t)\|_2^2 + 2 \int_0^t \|\nabla u_R(s)\|_2^2 ds \leq \left(2\|u_0\|_2^2 + \int_0^T \|f(s)\|_2^2 ds\right) \exp\{CT\}, \quad t \in [0, T] ,
\]

where we have temporarily restored the original notation. We next multiply through both sides of (37) by \(\mathbb{P}\Delta u\) and integrate by parts to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_2^2 + \|\mathbb{P}\Delta u(t)\|_2^2 = (u(t) \cdot \nabla u(t), \mathbb{P}\Delta u(t)) + (M_R \cdot \nabla u(t), \mathbb{P}\Delta u(t)) + (u(t) \cdot \nabla M_R, \mathbb{P}\Delta u(t)) - (f(t), \mathbb{P}\Delta u(t)) .
\]

Arguing as in (19)–(20), we obtain

\[
(u(t) \cdot \nabla u(t), \mathbb{P}\Delta u(t)) \leq C \|u(t)\|_2^2 \|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \frac{1}{4} \|\mathbb{P}\Delta u(t)\|_2^2 ,
\]

while, using (34) and Cauchy-Schwarz inequality, we show

\[
(u(t) \cdot \nabla M_R, \mathbb{P}\Delta u(t)) \leq C \|u(t)\|_2^2 + \frac{1}{8} \|\mathbb{P}\Delta u(t)\|_2^2 ,
\]

\[
-(f(t), \mathbb{P}\Delta u(t)) \leq C \|f(t)\|_2^2 + \frac{1}{8} \|\mathbb{P}\Delta u(t)\|_2^2 .
\]

The investigation of the second term on the right-hand side of (39) needs more work. Recall first that, by definition, there exists a function \(\chi \in L^2_{\text{loc}}(\Omega)\), with \(\nabla \chi \in L^2(\Omega)\) such that \(\mathbb{P}\Delta u = \Delta u - \nabla \chi\), and

\[
\|\nabla \chi\|_2 \leq C \|\Delta u\|_2 .
\]

We thus have

\[
(M_R \cdot \nabla u(t), \mathbb{P}\Delta u(t)) = (M_R \cdot \nabla u(t), \Delta u(t)) - (M_R \cdot \nabla u(t), \nabla \chi(t)) := \mathcal{T}_1 + \mathcal{T}_2 .
\]

Recalling that \(\psi_R \equiv 1\) and \(M \equiv 0\) on \(\partial \Omega\) and integrating by parts yields

\[
\mathcal{T}_1 = \int_{\Omega} \psi_R M_i \partial_i u_j \partial_k^2 u_j dx = - \int_{\Omega} \partial_k [\psi_R M_i \partial_i u_j] \partial_k u_j dx
\]

\[
= - \int_{\Omega} \partial_k \psi_R M_i \partial_i u_j \partial_k u_j dx - \int_{\Omega} \psi_R \partial_k M_i \partial_i u_j \partial_k u_j dx - \int_{\Omega} \psi_R M_i \partial_k^2 u_j \partial_k u_j dx .
\]
Therefore, using (34) in conjunction with (35), and observing that
\[ \partial_{ij} u_j(s) \partial_k u_j(s) = \frac{1}{2} \partial_j |\nabla u|^2, \quad i = 1, 2, \] (45)
from the preceding relations we deduce
\[ \mathcal{F}_1 \leq C \|\nabla u(t)\|^2_2 - \frac{1}{2} (M_R, \nabla (\nabla u(t))^2) = C \|\nabla u(t)\|^2_2 + \frac{1}{2} (\text{div } M_R, (\nabla u(t))^2), \]
which, in view of (36), allows us to conclude
\[ \mathcal{F}_1 \leq C \|\nabla u(t)\|^2_2. \] (46)
We next observe that
\[ \mathcal{F}_2 = -(M_R \cdot \nabla u(t) - u \cdot \nabla M_R, \nabla \chi(t)) + (u(t) \cdot \nabla M_R, \nabla \chi(t)) := \mathcal{F}_2^{(1)} + \mathcal{F}_2^{(2)}. \] (47)
By (34), (42), Cauchy-Schwarz inequality, and (17) we obtain
\[ \mathcal{F}_2^{(2)} \leq C \|u(t)\|^2_1, \quad \mathcal{F}_2^{(2)} \leq \frac{1}{8} \|\Delta u(t)\|^2_2. \] (48)
Moreover, since
\[ \text{div} [M_R \cdot \nabla u(t) - u \cdot \nabla M_R] = u \cdot \nabla (\text{div } M_R), \]
by a double integration by parts we show
\[ \mathcal{F}_2^{(1)} = -(\text{div } M_R, u(t) \cdot \nabla \chi(t)), \]
which, in turn, again by (36), (42), Cauchy-Schwarz inequality, and (17) allows us to conclude
\[ \mathcal{F}_2^{(1)} \leq C \|u(t)\|^2_2 + \frac{1}{8} \|\Delta u(t)\|^2_2. \] (49)
Combining (39)–(41) with (43)–(50), we infer
\[ \frac{d}{dt} \|\nabla u(t)\|^2_2 + \frac{1}{2} \|\Delta u(t)\|^2_2 \leq C [\|u(t)\|^2 + 2 (\|\nabla u(t)\|^2_2 + \|\nabla u(t)\|^2_2) + \|\nabla u(t)\|^2_2 + \|f(t)\|^2_2]. \] (50)
Setting
\[ Y(t) := t \|\nabla u(t)\|^2_2, \]
\[ G(t) := \|\nabla u(t)\|^2_2 + C (\|f(t)\|^2_2 + \|u(t)\|^2_2), \]
\[ H(t) := C (\|u(t)\|^2_2 \|\nabla u(t)\|^2_2 + \|\nabla u(t)\|^2_2 + 1), \]
from (51) we derive, in particular,
\[ Y'(t) \leq G(t) + H(t) Y(t), \quad Y(0) = 0, \]
which, in turn, by Gronwall’s lemma furnishes the following bound for $Y(t)$
\[ Y(t) \leq \int_0^T G(s) ds \exp \left\{ \int_0^T H(s) ds \right\}, \quad t \in [0, T]. \]
Recalling (52) and using (38) from this relation we obtain (restoring the original notation)
\[ t^2 \|\nabla u_R(t)\|^2_2 \leq K, \]
where in (53) and in the rest of the proof $K$ denotes a generic positive constant depending only on $\Omega, T, \|u_0\|_2$ and the norm of $f$ in $L^2(0; T; L^2(\Omega))$. We now multiply (51) by $t$, integrate the resulting relation over $[0, T]$ and use (53) to show
\[ \int_0^T t \|\Delta u_R(t)\|^2_2 dt \leq K. \]
Combining this latter with (53) and (17), we thus conclude
\[ \int_0^T t \| D^2 u_R(t) \|^2_{L^2} dt \leq K. \]  
(54)

Our next objective is to prove an estimate for the pressure. To this end, we observe that from (37) and (49) it follows that \( p \) satisfies the following Neumann problem (in the sense of distribution) for almost all \( t \in [0, T] \)
\[ \Delta p = \text{div} \left( f - u \cdot \nabla u - 2u \cdot \nabla M_R - u \text{div} M_R + \Delta u \right) := \text{div} F \quad \text{in } \Omega \times \{ t \}, \]

\[ \frac{\partial p}{\partial n} = (\Delta u + f) \cdot n \equiv F \cdot n \quad \text{at } \partial \Omega \times \{ t \}, \]
where \( n \) denotes the unit outer normal to \( \partial \Omega \). Then, from classical results, we know that
\[ \| \nabla p(t) \|_2 \leq C \| F(t) \|_2, \]
that is, on account of (32) and (36),
\[ \| \nabla p(t) \|_2^2 \leq C (\| f(t) \|_2^2 + \| u(t) \cdot \nabla u(t) \|_2^2 + \| u(t) \|_{L^2(\Omega)}^2). \]

Multiplying both sides of this relation by \( t \) and using (18), (38), (53) yields
\[ t \| \nabla p(t) \|_2^2 \leq C \left[ t \| f(t) \|_2^2 + \| u(t) \cdot \nabla u(t) \|_2^2 (t^{\frac{1}{2}} \| \nabla u(t) \|_2^2) (t^{\frac{1}{2}} \| D^2 u(t) \|_2^2) + t \| u(t) \|_{L^2(\Omega)}^2 \right] \]
\[ \leq C \left[ t \| f(t) \|_2^2 + K \| \nabla u(t) \|_2^2 (t^{\frac{1}{2}} \| D^2 u(t) \|_2^2) + t \| u(t) \|_{L^2(\Omega)}^2 \right] \]
\[ \leq C \left[ t \| f(t) \|_2^2 + K \| \nabla u(t) \|_2^2 + t \| D^2 u(t) \|_2^2 + t \| u(t) \|_{L^2(\Omega)}^2 \right], \]
from which, with the help of (38), (54), and Poincaré’s inequality (here \( p \) is normalised by the condition \( \int_{\Omega_t} p(x,t)dx = 0, \) a.a. \( t \in [0, T] \)), for a fixed \( b > 0 \) we conclude
\[ \int_0^T t \| p_R(t) \|_{L^2(\Omega')} dt \leq K', \quad \int_0^T t \| \nabla p_R(t) \|_2^2 dt \leq K, \]  
(55)

where \( \Omega' \subset \Omega \) is any bounded open set, and \( K' \) depends also on \( \Omega' \). Finally, from (37) (18), (38), (53)–(55), we also deduce
\[ \int_0^T t \| u_R(t) \|_{L^2(\Omega')} dt \leq K, \]  
(56)

where \( \Omega' \subset \Omega \) is an arbitrary bounded open set. The estimates (38), (53)–(55) imply the existence of a sequence \( \{ u_{R_i}, p_{R_i} \}_{R_i \in \mathbb{N}} \) converging in the natural weak and weak-* topologies suggested by (38), (53)–(55) to \( (u, p) \) satisfying
\[ u \in L^\infty(0, T; H(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega)), \quad t^{\frac{1}{2}} \nabla u \in L^\infty(0, T; L^2(\Omega)), \]
\[ t^{\frac{1}{2}} D^2 u \in L^2(0, T; L^2(\Omega)), \quad t^{\frac{1}{2}} u_t \in L^2(0, T; L^{2,\text{loc}}(\Omega)) , \]
\[ t^{\frac{1}{2}} p \in L^2(0, T; L^{2,\text{loc}}(\Omega)), \quad t^{\frac{1}{2}} \nabla p \in L^2(0, T; L^2(\Omega)). \]  
(57)

Moreover, by (38), (56) and Aubin–Lions Lemma, we have \( u_{R_i} \to u \) in the norm of \( L^2(\eta; T; L^2(\Omega')) \), for any \( \eta > 0 \) and any bounded open set \( \Omega' \subset \Omega \). By standard arguments, e.g., [4, 10], we show that \( (u, p) \) satisfies (33) on a.e. \( (x, t) \in \Omega \times (0, T) \), and that \( u \) attains the initial condition (33) in the sense of \( L^2 \)-convergence. It is also clear that \( u \) obeys (33) in the trace sense. Finally, taking into account that \( u(t) \in H^2(\Omega) \) for a.e. \( t \in [0, T] \), condition (33) is satisfied uniformly, for a.e. \( t \in [0, T] \); see [5, Theorem II.9.1].

In order to conclude the existence part, it remains to show the claimed continuity properties in time of \( u \) and \( \nabla u \). From (57) we find \( u \in L^\infty(0, T; L^2(\Omega)) \) with \( t^{\frac{1}{2}} u_t \in L^2(0, T; L^{2,\text{loc}}(\Omega)) \) from which it follows \( u \in C_w([0, T]; L^2(\Omega)) \). We shall next
show that the function \( t \in (0, T] \to \|u(t)\|_2 \) is continuous, thus proving the desired property. To this end, we multiply both sides of (33) by \( \psi_R u \) and integrate by parts over \( \Omega \times (s, t) \), \( s > 0 \). Taking into account the regularity properties (57), this procedure leads to

\[
\frac{1}{2} \left( \| \sqrt{\psi_R} u(t) \|_2^2 - \| \sqrt{\psi_R} u(s) \|_2^2 \right) = - \int_s^t \left[ \frac{1}{2} (\psi_R u(\tau), \nabla u^2(\tau)) 
- \frac{1}{2} (\text{div} (\psi_R M), u^2(\tau)) + (\psi_R u(\tau) \cdot \nabla M, u(\tau)) 
- (\psi_R \Delta u(\tau), u(\tau)) - (\psi_R \nabla p(\tau), u(\tau)) - (\psi_R f(\tau), u(\tau)) \right] d\tau .
\]

We now take the limit \( R \to \infty \) in (58). Using (36) and bearing in mind (57), we show

\[
\|u(t)\|_2^2 - \|u(s)\|_2^2 = -2 \int_s^t \left[ (u(\tau) \cdot \nabla M, u(\tau)) - (\Delta u(\tau), u(\tau)) - (\psi_R f(\tau), u(\tau)) \right] d\tau ,
\]

which, in turn, again by the properties (57) proves the desired continuity.

We shall next show continuity for \( \nabla u \). From (57) we obtain, in particular, for all \( \eta > 0 \)

\[
\psi_R u(t) \in L^2(\Omega \times (0, T]; L^2(\Omega)) , \quad u_t \in L^2(\eta, T; L^2(\Omega)) ,
\]

so that, by a classical interpolation theorem [12, Théorème 2.1], we deduce \( u \in C((0, T]; H^1(\Omega')) \), for each bounded open set \( \Omega' \subset \Omega \).

Since \( \nabla u \in L^\infty((\eta, T; L^2(\Omega))) \), we conclude \( \nabla u \in C_w((0, T]; L^2(\Omega)) \).

As a consequence, to prove the claimed continuity property for \( \nabla u \), it suffices to show, as before, that the map \( t \in (0, T] \to \|\nabla u(t)\|_2 \) is continuous. To this end, we multiply (32) by \( \psi_R \Delta u \) and integrate by parts over \( \Omega \times (s, t) \), \( s > 0 \). We thus show

\[
\frac{1}{2} \left( \| \sqrt{\psi_R} \nabla u(t) \|_2^2 - \| \sqrt{\psi_R} \nabla u(s) \|_2^2 \right) = \int_s^t \left[ (\nabla \psi_R \nabla u(\tau), u(\tau)) 
- (\psi_R u(\tau) \cdot \nabla u(\tau), \Delta u(\tau)) - (\psi_R M \cdot \nabla u(\tau), \Delta u(\tau)) 
- (\psi_R u(\tau) \cdot \nabla M, \Delta u(\tau)) - \| \sqrt{\psi_R} \Delta u(\tau) \|_2^2 
+ (\psi_R \nabla p(\tau), \Delta u(\tau)) - (\psi_R f(\tau), \Delta u(\tau)) \right] d\tau .
\]

Arguing as in (44)–(45),

\[
(\psi_R u \cdot \nabla M, \Delta u) = - \int_\Omega \partial_k \psi_R M_i \partial_i u_j \partial_k u_j \, dx - \int_\Omega \psi_R \partial_k M_i \partial_i u_j \partial_k u_j \, dx 
+ \int_\Omega \text{div} (\psi_R M) |\nabla u|^2 \, dx .
\]

Furthermore, from (32), (18), and (57) it easily follows that

\[
u_t(x, t) = - M \cdot \nabla u(x, t) + G(x, t) , \quad G \in L^2(\eta, T; L^2(\Omega)) , \quad \text{arbitrary } \eta > 0 . \quad (61)
\]

Therefore, taking into account (60)–(61), the properties of \( \psi_R \), and, again, (18), and (57) we may pass to the limit \( R \to \infty \) in (59) to infer

\[
\|\nabla u(t)\|_2^2 - \|\nabla u(s)\|_2^2 = -2 \int_s^t \left[ (u(\tau) \cdot \nabla u(\tau), \Delta u(\tau)) - \int_\Omega \partial_k M_i \partial_i u_j (\tau) \partial_k u_j (\tau) \, dx 
+ (u(\tau) \cdot \nabla M, \Delta u(\tau)) + \| \Delta u(\tau) \|_2^2 - (\nabla p(\tau), \Delta u(\tau)) + (f(\tau), \Delta u(\tau)) \right] d\tau ,
\]
from which, with the help of (57), the desired continuity follows. The existence proof is thus completed.

In order to prove uniqueness, let \((u, p), (u_1, p_1)\) be two solutions to (33)–(34) satisfying the properties stated in the theorem and corresponding to the same \(f\) and \(u_0\). Setting \(v = u_1 - u, \phi = p_1 - p\), we find
\[
v_t + v \cdot \nabla v + v \cdot \nabla u + u \cdot \nabla v + M \cdot \nabla v + v \cdot \nabla M = \Delta v - \nabla \phi, \quad \text{in} \ \Omega \times (0, T),
\]
\[
\text{div} v = 0, \quad \text{in} \ \Omega \times (0, T),
\]
\[
v(x, 0) = 0, \quad \text{in} \ \Omega,
\]
\[
v(x, t) = 0, \quad \text{on} \ \partial \Omega \times (0, T)
\]
\[
\lim_{|x| \to \infty} v(x, t) = 0, \quad \text{in} \ [0, T].
\]
(62)

By repeating a by now familiar procedure, we multiply both sides of (62) by \(\psi_R v\) and integrate by parts over \(\Omega \times [s, t], 0 < s < t < T\). Then
\[
\|\sqrt{\psi_R} v(t)\|^2_2 - \|\sqrt{\psi_R} v(s)\|^2_2 = 2 \int_s^t \left[ \frac{1}{2} (\nabla \psi_R \cdot (v(\tau) + u(\tau)), v^2(\tau)) + \frac{1}{2} (\text{div} (\psi_R M), v^2(\tau)) + (\psi_R v(\tau) \cdot \nabla u(\tau), v(\tau)) - (\psi_R \Delta v(\tau), v(\tau)) + (\psi_R \nabla \phi(\tau), v(\tau)) \right] d\tau.
\]
(63)

Passing to the limit \(R \to \infty\) in (63), and employing the fact that both solutions are in the class (57), it is not hard to show that (63) furnishes
\[
\|v(t)\|^2_2 - \|v(s)\|^2_2 = 2 \int_s^t \left[ (v(\tau) \cdot \nabla M, v(\tau)) + (v(\tau) \cdot \nabla u(\tau), v(\tau)) - \|\nabla v(\tau)\|^2_2 \right] d\tau.
\]
(64)

However, as it is well known \([4, \S 4]\) and easy to establish by means of (15),
\[
\int_s^t (v(\tau) \cdot \nabla u(\tau), v(\tau)) d\tau \leq c \int_s^t \|u(\tau)\|_{1,2}^2 \|v(\tau)\|^2_2 d\tau + \int_s^t \|\nabla v(\tau)\|^2_2 d\tau.
\]
As a result, recalling that \(\nabla M \in L^\infty(\Omega)\) from the latter inequality and (64) we deduce
\[
\|v(t)\|^2_2 - \|v(s)\|^2_2 \leq c \int_s^t \left[ (1 + \|u(\tau)\|_{1,2}^2) \|v(\tau)\|^2_2 \right] d\tau.
\]
(65)

We now pass to the limit \(s \to 0\) in this inequality, use the fact that \(v(0) \equiv 0\) along with Gronwall’s lemma and conclude \(v \equiv 0\) in \(L^2(\Omega \times [0, T])\). The proof of Theorem 3.1 is complete.

4. More regular solutions. It is the aim of this section to show that the solution to (33) determined in Theorem 3.1 is more regular provided this is true for \(u_0\) and \(f\). More specifically, we have the following result.

**Theorem 4.1.** Let \(\Omega\) and \(f\) be as in Theorem 3.1 and suppose that \(u_0 \in L^2(\Omega) \cap H^1_0(\Omega)\). Let \(T > 0\). Then, the solution \((u, \nabla p)\) to equation (33) given in Theorem 3.1 satisfies
\[
\begin{align*}
&u \in C(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad u_t \in L^2(0, T; L^2_{\text{loc}}(\Omega)), \\
p &\in L^2(0, T; L^2_{\text{loc}}(\Omega)), \quad \nabla p \in L^2(0, T; L^2(\Omega)).
\end{align*}
\]
Moreover, if
\[ f_t \in L^2(0,T;L^2(\Omega)), \quad u_0 \in D(A_S), \quad M \cdot \nabla u_0 \in L^2(\Omega), \]
then
\[ u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \]  
Moreover, if
\[ f \in L^\infty(0,T;L^2_{\text{loc}}(\Omega)) \cap L^2(0,T;H^1_{\text{loc}}(\Omega)), \]
then
\[ u \in L^\infty(0,T;H^2_{\text{loc}}(\Omega)) \cap L^2(0,T;H^3_{\text{loc}}(\Omega)), \]
\[ \nabla p \in L^\infty(0,T;L^2_{\text{loc}}(\Omega)) \cap L^2(0,T;H^1_{\text{loc}}(\Omega)). \]

Proof. The starting point is again the “approximating” problem (37), for which the existence of a solution in the functional class stated in the theorem is ensured by Lemma 2.1, for each \( R \in \mathbb{N}, \ R > R_0 \). In view of the assumed regularity on \( u_0 \), this time we may apply Gronwall’s lemma to (50) without using the “weight” \( t \).

The starting point is again the “approximating” problem (37), for which the starting point is again the “approximating” problem (37), for which the proof of Theorem 3.1, one easily shows that the solution \((u,p)\) has the same meaning as in the proof of Theorem 3.1. As a consequence, using the assumptions on \( u \) and the properties of \((u,p)\), we show the better estimates
\[ \|\nabla u_R(t)\|_2 + \int_0^T \|D^2 u_R(\tau)\|_2^2 d\tau \leq K, \quad t \in [0,T], \]
where \( K \) has the same meaning as in the proof of Theorem 3.1. As a consequence, by the proof of Theorem 3.1, one easily shows that the solution \((u,p)\) will be in the class (57), but, this time, without the weight \( t^2 \). Also, the continuity of \( t \mapsto \nabla u(t) \) in the \( L^2 \) norm can be proved up to the time \( t = 0 \). We leave the details to the reader.

Next we observe that, under the further assumption (66), again by Lemma 2.1, \( u_{R_t} \) satisfies the estimate (see (7))
\[ \|u_{R_t}(t)\|_2^2 \leq \left[ N^2 + c \left( D^2 + D_0^2 e^t \right) \right] \exp \left\{ c \left( t + D_0^2 e^t \right) \right\}, \quad t \in [0,T], \]
where \( D_0, D_1 \) are defined in (8), \( c = c(\Omega, \kappa) \), with \( \kappa \) any number such that
\[ \kappa \geq \max \left\{ \|\text{div} (M_R)\|_\infty, \|\nabla M_R\|_\infty \right\}, \]
and
\[ N := \|P(u_0 \cdot \nabla u_0 + \psi_{R_t} M \cdot \nabla u_0 + \nabla (\psi_{R_t} M) \cdot u_0 - \Delta u_0 - f)\|_2. \]
In view of (34) and (36), the constant \( c \) can be made independent of \( R \). Furthermore, using the assumptions on \( u_0 \) and the properties of \( M \) and \( \psi_{R_t} \), we show
\[ N \leq c \left( \|u_0\|_{2,2} + \|M \cdot \nabla u_0\|_2 \right) \]
with \( c = c(\Omega, M) \). As a result, we conclude
\[ \|u_{R_t}(t)\|_2 \leq K, \quad t \in [0,T]. \]
The latter, in turn, combined with (7) allows us to conclude
\[ \|u_{R_t}(t)\|_2 + \int_0^T \|\nabla u_{R_t}(\tau)\|_2^2 d\tau \leq K, \quad t \in [0,T], \]
which ensures that the limit solution \( u \) is in the class (67). Finally, the remaining properties of \((u,p)\) are obtained exactly as in the last part of the proof of Lemma 2.1, namely, by rewriting (33) as a Stokes problem of the type (30) where now
\[ F := u_t + u \cdot \nabla u + M \cdot \nabla u + \nabla M \cdot u - f. \]
From the already established regularity properties of \( u \) and (68) it follows that
\[ F \in L^\infty(0,T;L^2_{\text{loc}}(\Omega)) \cap F \in L^2(0,T;H^1_{\text{loc}}(\Omega)), \]
so that the further regularity properties of $u$ are a consequence of classical local results for the (steady-state) Stokes problem [5]. The theorem is completely proved.

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