Noncritical $\mathfrak{osp}(1|2, \mathbb{R})$ $\mathcal{M}$-theory matrix model with an arbitrary time dependent cosmological constant

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**Abstract:** Dimensional reduction of the $D = 2$ minimal super Yang-Mills to the $D = 1$ matrix quantum mechanics is shown to double the number of dynamical supersymmetries, from $\mathcal{N} = 1$ to $\mathcal{N} = 2$. We analyze the most general supersymmetric deformations of the latter, in order to construct the noncritical 3D $\mathcal{M}$-theory matrix model on generic supersymmetric backgrounds. It amounts to adding quadratic and linear potentials with arbitrary time dependent coefficients, namely, a cosmological ‘constant,’ $\Lambda(t)$, and an electric flux background, $\rho(t)$, respectively. The resulting matrix model enjoys, irrespective of $\Lambda(t)$ and $\rho(t)$, two dynamical supersymmetries which further reveal three hidden $\mathfrak{so}(1,2)$ symmetries. All together they form the supersymmetry algebra, $\mathfrak{osp}(1|2, \mathbb{R})$. Each $\mathfrak{so}(1,2)$ multiplet in the Hilbert space visualizes a dynamics constrained on either Euclidean or Minkowskian $dS_2/AdS_2$ space, depending on its Casimir. In particular, all the unitary multiplets have the Euclidean $dS_2/AdS_2$ geometry. We argue that the matrix model provides holographic duals to the 2D superstring theories on various backgrounds having the spacetime signature Minkowskian if $\Lambda(t) > 0$, or Euclidean if $\Lambda(t) < 0$. In particular, we conjecture that the choice of $\Lambda$ being negative constant corresponds to the $\mathcal{N} = 2$ super Liouville theory.

**Keywords:** $\mathcal{M}$-theory, AdS/CFT, supersymmetry.
1. Introduction

String or $\mathcal{M}$-theory dress all the known supersymmetric gauge theories with the insightful geometrical pictures by the notion of holography or $AdS$/CFT correspondence \[1, 2\]. In particular, the symmetry group of a gauge theory is identified as the isometry of the corresponding higher dimensional string/$\mathcal{M}$-theory background. Conversely, different string theories - bosonic or supersymmetric, critical or noncritical - on various backgrounds are expected to have holographic dual gauge theories.

In the present paper, we attempt to address the dual of the noncritical 2D superstring, which is a yet unresolved problem despite of some progress \[3, 4, 5, 6, 7, 8, 9\], from the $\mathcal{M}$-theory point of view \[10, 11, 12\]. The spacetime dimension two is singular in the sense that the holographic dual of 2D superstring theory, whatever its concrete form is, should share many common features with the corresponding noncritical $\mathcal{M}$-theory matrix model.

As is well known, superstring lives in 2, 3, 4, 6 and 10 dimensions, while the supermembrane exits in dimensions one higher, i.e. 3, 4, 5, 7 and 11, since only in those spacetime
dimensions the relevant Fierz identities hold. Although the pioneering work on super p-branes \[13\] excludes the possibility of the space-filling p-branes i.e. p-branes propagating in \((p + 1)\)-dimensional target spacetime, supermembrane does exit in three dimensions, since the Fierz identity for the supermembrane works manifestly, from \(\gamma^{012} = 1\),

\[
(d\bar{\theta}\gamma_{\mu}d\theta)(d\bar{\theta}\gamma^{\mu}\nu d\theta) = \epsilon^{\mu\nu\lambda}(d\bar{\theta}\gamma_{\mu}d\theta)(d\bar{\theta}\gamma_{\lambda}d\theta) = 0, \tag{1.1}
\]

where \(d\theta\) is a bosonic spinor. The matrix regularization \[14, 15\] of the supermembrane prescribes the replacement of the Poisson bracket appearing in the light cone gauged membrane action by a matrix commutator. For 3D supermembrane action, it leads to a supersymmetric and gauged version of a one matrix model, where the local gauge symmetry originates from the area preserving diffeomorphism for the Poisson bracket, and the appearance of only one matrix is due to the light cone gauge, i.e. \(3 - 2 = 1\).

The resulting \(N \times N\) matrix model, at least for the ‘flat’ 3D background, can be also obtained by the dimensional reduction of the 2D minimal super Yang-Mills\(^1\) to \(D = 1\), and it is supposed to describe exactly the D0-brane dynamics of the discrete light cone momentum sector, \(p_- = N/R\), in \(\mathcal{M}\)-theory compactified on a light-like circle, \(x^- \sim x^- + 2\pi R\), as initially proposed by Banks, Fischler, Shenker and Susskind for the critical \(\mathcal{M}\)-theory \[17, 18\]. As for the D0-branes, the local gauge symmetry is required to reflect the identical nature of the \(N\) D-particles \[19\].

Also for the noncritical 2D superstring, almost by definition, its holographic dual should be one dimensional, supersymmetric and gauged theories. In the presence of RR electric field, \(F\), the low energy effective action of 2D string theory typically reads, neglecting the massless tachyon and putting \(\alpha' \equiv 1\) \[20, 21, 22\],

\[
S_{2D} = \int d^2x\sqrt{-g}\left[ e^{-2\Phi}\left(8 + R + 4(\nabla\Phi)^2\right) - \frac{1}{2}F^2\right], \tag{1.2}
\]

where \(-\frac{1}{2}F^2\) plays the role of the negative cosmological constant, and the solutions are characterized by the \(AdS_2\)-like geometries.\(^2\) Indeed, switching off the dilaton completely we have the \(AdS_2\) solution, while turning on \(\Phi\), one has static extremal black hole-like solutions \[23, 24, 25\]. In the asymptotic region the latter becomes the usual linear dilaton vacuum, and in the “near-horizon” region it approaches to \(AdS_2\) with the dilaton reaching the critical value, \(\Phi_c = -\ln(\frac{1}{4}F)\).

\(^1\)Recently all the minimal noncritical super Yang-Mills (except \(D = 3\)) have been identified in the noncritical superstring theories \[14\].

\(^2\)In two dimensions the geometries of \(AdS_2\) and \(dS_2\) coincide, and we will distinguish them by the sign of the ‘cosmological constant’. Also it is to be reminded that

\[
k_0^2 - k_1^2 - k_2^2 = R^2 > 0 \quad : \text{Euclidean } AdS_2/dS_2 \quad (\text{hyperboloid of two sheets}),
\]

\[
k_0^2 - k_1^2 - k_2^2 = -R^2 < 0 \quad : \text{(Minkowsian) } AdS_2/dS_2 \quad (\text{hyperboloid of one sheet}).
\]
However, the effective action, (1.2), can not be thoroughly trusted due to the $\alpha'$ corrections as well as the tachyon tadpoles. Necessarily one has to work on the full sigma model (e.g. [25]) with the difficulty of dealing with background fluxes. Hence to find the exact nontrivial superstring background is not an easy task. And also for the $\mathcal{M}$-theory, the matrix regularization of the supermembrane action is not always straightforward for generic nontrivial backgrounds.

In this work, we take supersymmetry itself as the principal guideline to tackle the problem of constructing the noncritical $3D \mathcal{M}$-theory matrix model on generic supersymmetric backgrounds. Namely after the dimensional reduction of the $D = 2$ super Yang-Mills to the $D = 1$ matrix quantum mechanics, we analyze all the possible deformations of the latter without breaking any supersymmetry. We show that the most general supersymmetric deformations simply amount to adding quadratic and linear potentials with arbitrary time dependent coefficients, namely, a cosmological ‘constant,’ $\Lambda(t)$, and an electric flux background, $\rho(t)$, respectively. The latter couples to the $\mathfrak{u}(1)$ sector or the “center of mass” only, while the $\mathfrak{su}(N)$ sector and the $\mathfrak{u}(1)$ sector are completely decoupled. Remarkably we find that, irrespective of $\Lambda(t)$ and $\rho(t)$, the resulting matrix model always enjoys two dynamical supersymmetries, not just one as in the $2D$ minimal super Yang-Mills. Namely after the dimensional reduction, the number of supersymmetry is doubled, from $\mathcal{N} = 1$ to $\mathcal{N} = 2$. Furthermore, again for arbitrary $\Lambda(t)$ and $\rho(t)$, these two supersymmetries reveal three hidden nontrivial bosonic symmetries. All together the five symmetries form the super Lie algebra, $\mathfrak{osp}(1|2, \mathbb{R})$, where the even part corresponds to $\mathfrak{so}(1, 2)$ i.e. the isometry of the Euclidean or Minkowskian $dS_2/AdS_2$. We introduce a projection map from the phase space to a three dimensional ‘$\mathfrak{so}(1, 2)$ hyperspace’ associated with the bosonic symmetries. The dynamics therein is always constrained on a two dimensional rigid surface, Euclidean $dS_2/AdS_2$ or Minkowskian $dS_2/AdS_2$, depending on the sign of the $\mathfrak{so}(1, 2)$ Casimir for each multiplet in the Hilbert space. The richness of the matrix model comes from the arbitrariness of the time dependent coefficients, $\Lambda(t)$, $\rho(t)$, and the vast amount of supermultiplets existing in the Hilbert space each of which has its own two dimensional geometries.

The organization of the present paper is as follows. In section 2, we analyze the most general supersymmetric deformations of the matrix model having the $2D$ super Yang-Mills origin. We discuss its symmetries, Hamiltonian dynamics and the BPS configurations. We also comment on the relation to the matrix cosmology. Section 3 is devoted to the detailed analysis on the underlying supersymmetry algebra, $\mathfrak{osp}(1|2, \mathbb{R})$, both from the kinematical and dynamical point of view. In particular, we show that all the unitary multiplets correspond to the Euclidean $dS_2/AdS_2$ geometry, rather than the Minkowskian one. The last section, Sec.4 includes our conclusion that the matrix model with different choices of $\Lambda(t)$ and $\rho(t)$ can provide holographic duals to various $2D$ superstring or superconformal theories.
2. Noncritical $\mathfrak{osp}(1|2, \mathbb{R})$ $\mathcal{M}$-theory matrix model

2.1 Derivation of the matrix model and supersymmetry enhancement

In two dimensional Minkowskian spacetime the fermion satisfies the Majorana-Weyl condition, resulting in only one component real spinor. After the dimensional reduction to $D = 1$, the 2$D$ super Yang-Mills leads to the following supersymmetric matrix model, which can be also obtained by the matrix regularization of the 3$D$ supermembrane action in the light cone gauge,

$$\mathcal{L} = \text{tr} \left[ \frac{1}{2} D_t XD_t X + i \frac{i}{2} \psi D_t \psi + X \psi \psi \right],$$

(2.1)

where $X, \psi$ are respectively bosonic or fermionic $N \times N$ Hermitian matrices. With a gauge potential, $A_0 = A_0^\dagger$, the covariant time derivative reads, in our convention,

$$D_t = \partial_t - i [A_0, \ ].$$

(2.2)

Bosons, $X, A_0$, have the mass dimension 1, while the fermion, $\psi$, has the mass dimension $\frac{3}{2}$, so that the Lagrangian has the mass dimension, 4.

The supersymmetry transformation, $\delta_{\text{YM}}$, descending from the 2$D$ super Yang-Mills theory is, with a constant supersymmetry parameter, $\varepsilon$,

$$\delta_{\text{YM}} A_0 = \delta_{\text{YM}} X = i \psi \varepsilon, \quad \delta_{\text{YM}} \psi = D_t X \varepsilon.$$

(2.3)

Now we look for the generalizations of the above Lagrangian as well as the supersymmetry transformations. First of all, we note from

$$\text{tr} \left[ i \frac{1}{2} \psi D_t \psi + X \psi \psi \right] = \text{tr} \left[ i \frac{1}{2} \psi \partial_t \psi + (X - A_0) \psi \psi \right],$$

(2.4)

that in order to cancel the cubic terms of $\psi$ in any possible supersymmetry variation which will transform the bosons, $(X - A_0)$ to the fermion, it is inevitable to impose \(^3\)

$$\delta A_0 = \delta X.$$

(2.5)

Hence, introducing a time dependent function, $f(t)$, we let the generalized supersymmetry transformation be

$$\delta A_0 = \delta X = i f(t) \psi \varepsilon, \quad \delta \psi = \left( f(t) D_t X + \Delta \right) \varepsilon,$$

(2.6)

where $\Delta$ is a bosonic quantity having the mass dimension 2, and its explicit form is to be determined shortly. After some straightforward manipulation, we obtain

$$\delta \mathcal{L} = \text{tr} \left[ i \psi \varepsilon \left( D_t (f \Delta X + \Delta) - f \Delta X + i [X, \Delta] \right) \right] + \partial_t \mathcal{K},$$

(2.7)

\(^3\)Essentially this rigidity corresponds to the Fierz identity, $\text{tr} \left( \bar{\psi} \gamma^\mu [\bar{\varepsilon} \gamma_\mu \psi, \psi] \right) = 0$, relevant to the existence of the minimal super Yang-Mills in 2, 3, 4, 6, 10 dimensions.
where the total derivative term is given by

\[ K = \text{tr}(D_t X \delta X - i \frac{1}{2} \psi \delta \psi) . \tag{2.8} \]

Of course, the simplest case where \( f(t) = 1 \) and \( \Delta = 0 \) reduces to the supersymmetry of the original 2D super Yang-Mills, (2.3). For the generic cases, we are obliged to set

\[ \Delta = -\dot{f} X - \kappa 1 , \tag{2.9} \]

and obtain the following supersymmetry invariance,

\[ \delta \left[ \mathcal{L} + \text{tr} \left( \frac{1}{2} (\ddot{f}/f) X^2 + (\dot{\kappa}/f) X \right) \right] = \partial_t K . \tag{2.10} \]

This essentially leads to a novel supersymmetric matrix model with two arbitrary time dependent functions, \( \kappa(t) \), \( f(t) \), as spelled out in Eq.(2.12).

For given functions, \( \Lambda(t) \) and \( \rho(t) \), there exist two sets of ‘solutions’ given by \( f(t) \), \( \kappa(t) \) to satisfy the following second order differential equations,

\[ \Lambda = \ddot{f}/f \quad \rho = \dot{\kappa}/f . \tag{2.11} \]

Thus, surprisingly, there are two dynamical supersymmetries in the matrix model, even for the case,\(^{5}\) \( \Lambda = \rho = 0 \). This will further reveal three nontrivial bosonic symmetries as we discuss in the next subsection.

Rather than taking (2.9) one might attempt to close the supersymmetry invariance by adding other terms to \( \mathcal{L} \). However, since there exits only one component spinor, there can not be any mass term for the fermion\(^{6}\) as \( \text{tr}(\psi \psi) = 0 \). Thus, as long as we restrict on the ‘non-derivative corrections’, the above generalization is the most generic one.

\(^{4}\)The integral constant for \( \kappa(t) \) corresponds to the kinematical supersymmetry.

\(^{5}\)This kind of supersymmetry enhancement after the dimensional reduction can be also noticed elsewhere. For example, in the earlier works [26, 27], we derived the effective worldvolume gauge theories for the longitudinal D5 and D2 branes on the maximally supersymmetric 11D pp-wave background. After the dimensional reductions to \( D = 1 \), both of them lead to a matrix quantum mechanics which is equivalent to the BMN \( \mathcal{M} \)-theory matrix model [28] up to field redefinitions. The formers have only four dynamical supersymmetries, while the BMN model has 32 supersymmetries, 16 dynamical and 16 kinematical. The physical reason for the enhancement is that the D-branes which the higher dimensional gauge theories describe preserve only the fraction of the full \( \mathcal{M} \)-theory supersymmetries, \( \frac{4}{32} \). The same reasoning also holds for the present \( \mathfrak{o-sp}(1|2, \mathbb{R}) \) \( \mathcal{M} \)-theory matrix model having three supersymmetries, two dynamical and one kinematical. As we see shortly, all the BPS states preserve only one supersymmetry, breaking the other two. Hence, the minimal 2D super Yang-Mills can be interpreted as the worldvolume action of the longitudinal M2-brane which preserves only one supersymmetry. However, it remains somewhat mysterious that the total number of supersymmetries is three, a rather unusual odd number.

\(^{6}\)This is a special feature only present in the matrix quantum mechanics of the 2D super Yang-Mills origin. In fact, in the higher dimensional cases one needs to add the fermion mass term for the supersymmetry invariance as in the BMN matrix model [28] or [29].
2.2 Noncritical $osp(1|2, \mathcal{R})$ $\mathcal{M}$-theory matrix model: Final form

With an arbitrary time dependent ‘cosmological constant’, $\Lambda(t)$, having the mass dimension two and an arbitrary time dependent ‘electric flux background’, $\rho(t)$, having the mass dimension three, the generic form of the noncritical 3D $\mathcal{M}$-theory matrix model reads

$$L_{osp(1|2, \mathcal{R})} = \text{tr} \left[ \frac{i}{2} (D_t X)^2 + i \frac{i}{2} \psi D_t \psi + X \psi \psi + \frac{1}{2} \Lambda(t) X^2 + \rho(t) X \right].$$  (2.12)

The Lagrangian corresponds to the most general supersymmetric deformations of the $\mathcal{N} = 2'$ matrix quantum mechanics of the 2D super Yang-Mills origin. The matrix model is to describe the noncritical 3D supermembrane in a controllable manner through the matrix regularization, and our claim is further that it also provides holographic duals to 2D superstring theories, as discussed in the last section.

The matrix model is equipped with the standard local gauge symmetry,

$$X \rightarrow gXg^{-1}, \quad \psi \rightarrow g\psi g^{-1}, \quad A_0 \rightarrow gA_0g^{-1} - i \partial_t gg^{-1}, \quad g \in U(N),$$  (2.13)

and enjoys two dynamical supersymmetries,

$$\delta_{\pm} A_0 = \delta_{\pm} X = i f_{\pm}(t) \psi \varepsilon_{\pm}, \quad \delta_{\pm} \psi = \left( f_{\pm}(t) D_t X - \dot{f}_{\pm}(t) X - \kappa_{\pm}(t) 1 \right) \varepsilon_{\pm},$$  (2.14)

where $\varepsilon_+$, $\varepsilon_-$, are two real supersymmetry parameters, and $f_{\pm}(t)$, $\kappa_{\pm}(t)$ are the two different solutions of the second order differential equations,

$$\dot{f}_{\pm}(t) = f_{\pm}(t) \Lambda(t), \quad \kappa_{\pm}(t) := \int_{t_0}^{t} dt' \rho(t') f_{\pm}(t').$$  (2.15)

The above two dynamical supersymmetries further reveal three hidden nontrivial bosonic symmetries,\footnote{It is worth to note that the three bosonic symmetries are still valid in the bosonic matrix model obtained after putting $\psi \equiv 0$,}

$$\delta_{++} A_0 = \delta_{++} X = f_+ \left( f_+ D_t X - \dot{f}_+ X - \kappa_+ 1 \right), \quad \delta_{++} \psi = 0,$$

$$\delta_{--} A_0 = \delta_{--} X = f_- \left( f_- D_t X - \dot{f}_- X - \kappa_- 1 \right), \quad \delta_{--} \psi = 0,$$

$$\delta_{(+-)} A_0 = \delta_{(+-)} X = 2 f_+ f_- D_t X - \left( f_+ f_- + f_- f_+ \right) X - \left( f_+ \kappa_- + f_- \kappa_+ \right) 1, \quad \delta_{(+-)} \psi = 0.$$  (2.16)

As we see shortly, all the five symmetries form the $osp(1|2, \mathcal{R})$ superalgebra, while the three bosonic symmetries correspond to its even part, $sp(2, \mathcal{R}) \equiv so(1, 2) \equiv sl(2, \mathcal{R})$.

$\delta_{++} A_0$ is the $so(1, 2)$ superalgebra, and the other two $so(2, \mathcal{R})$ superalgebras are the $so(1, 2)$ superalgebra.
Apart from the dynamical supersymmetries, there is the usual kinematical supersymmetry, corresponding to the integral constant of $\kappa(t)$,

$$\delta A_0 = \delta X = 0, \quad \delta\psi = \varepsilon 1. \quad (2.17)$$

Note that $\rho(t)$ couples to the $u(1)$ sector or the “center of mass” only, and that the $su(N)$ sector and the $u(1)$ sector are completely decoupled. Moreover, when $\Lambda \neq 0$, a field redefinition such as $X \to X - (\rho/\Lambda)1$ reveals an (infinite) chain of duality given by

$$\rho(t) \longrightarrow \rho_{\text{new}}(t) := \frac{d^2}{dt^2} \left( \frac{\rho(t)}{\Lambda(t)} \right). \quad (2.18)$$

### 2.3 Hamiltonian and the Dirac bracket

The Euler-Lagrangian equations read

$$D_t D_t X - \psi\psi - \Lambda(t)X - \rho(t)1 = 0, \quad D_t \psi + i [X, \psi] = 0, \quad (2.19)$$

$$[D_t X, X] + i\psi\psi = 0 \quad : \text{Gauss constraint}.$$ 

Up to the Gauss constraint or the first-class constraint, the cubic vertex term vanishes, $\text{tr}(X\psi\psi) \simeq 0$, so that the Hamiltonian becomes simply a harmonic oscillator type, being free of the fermion,

$$H = \text{tr} \left[ \frac{1}{2}P^2 - \frac{1}{2}\Lambda(t)X^2 - \rho(t)X \right], \quad P := D_t X. \quad (2.20)$$

In fact, for any gauge invariant object,

$$\mathcal{F} = \text{tr} \left[ F(X, P, \psi, t) \right], \quad (2.21)$$

the Euler-Lagrangian equations, (2.19), imply

$$\frac{d\mathcal{F}}{dt} = \text{tr} \left[ P \frac{\partial}{\partial X} + \left( \Lambda(t)X + \rho(t)1 \right) \frac{\partial}{\partial P} \right] \mathcal{F} - i\text{tr} \left( [X, F] \right) + \frac{\partial\mathcal{F}}{\partial t}$$

$$= [\mathcal{F}, H]_{D.B.} + \frac{\partial\mathcal{F}}{\partial t}. \quad (2.22)$$

The Dirac bracket for our matrix model is given by, after taking care of the primary second-class constraint for the fermion [30, 31],

$$[\mathcal{F}, \mathcal{G}]_{D.B.} = \frac{\partial\mathcal{F}}{\partial X^a_b} \frac{\partial\mathcal{G}}{\partial P^b_a} - \frac{\partial\mathcal{F}}{\partial P^a_b} \frac{\partial\mathcal{G}}{\partial X^b_a} + i(-1)^{\#\mathcal{F}} \frac{\partial\mathcal{F}}{\partial \psi^a_b} \frac{\partial\mathcal{G}}{\partial \psi^b_a}, \quad (2.23)$$

where $a, b$ are the $N \times N$ matrix indices, while $\#\mathcal{F} = 0$ or 1, depending on the spin statistics of $\mathcal{F}$, i.e. 0 for the boson and 1 for the fermion.
Due to the five symmetries of the action, there are five conserved quantities given by the Noether charges. For their explicit expressions we refer (3.29) and (3.30).

Since the $\text{su}(N)$ and $\text{u}(1)$ sectors are completely decoupled, it is convenient to introduce the trace over either the $\text{su}(N)$ or $\text{u}(1)$ sector only,

$$
\begin{align*}
\text{tr}_{\text{su}(N)} \left[ F(X, P, \psi) \right] &:= \text{tr} \left[ F \left( X - N^{-1} \text{tr}(X), \ P - N^{-1} \text{tr}(P), \ \psi - N^{-1} \text{tr}(\psi) \right) \right], \\
\text{tr}_{\text{u}(1)} \left[ F(X, P, \psi) \right] &:= \text{tr} \left[ F \left( N^{-1} \text{tr}(X), \ N^{-1} \text{tr}(P), \ N^{-1} \text{tr}(\psi) \right) \right].
\end{align*}
$$

Accordingly the quadratic Hamiltonian decomposes into the two distinct pieces,

$$
H = H_{\text{su}(N)} + H_{\text{u}(1)},
$$

where

$$
H_{\text{su}(N)} = \text{tr}_{\text{su}(N)} \left[ \frac{1}{2} P^2 - \frac{1}{2} \Lambda(t) X^2 \right], \quad H_{\text{u}(1)} = \text{tr}_{\text{u}(1)} \left[ \frac{1}{2} P^2 - \frac{1}{2} \Lambda(t) X^2 - \rho(t) X \right].
$$

### 2.4 BPS states and the cosmological principle

From the supersymmetry transformations of the fermion, the BPS equations are

$$
\frac{d}{dt} X(t) = D_t X(t) = \dot{f}_\pm(t) X(t) + \kappa_\pm(t) 1,
$$

so that the generic BPS configurations decompose into the traceless and $\text{u}(1)$ parts,

$$
X(t) = f_+(t) X + h_+(t) 1 \quad \text{or} \quad X(t) = f_-(t) X + h_-(t) 1,
$$

where $X$ is an arbitrary traceless constant matrix, and $h_\pm(t)$ are the solutions of the first order differential equation, $\dot{f}_\pm h_\pm = \dot{f}_\pm h_\pm + \kappa_\pm$, corresponding to the center of mass position, $N^{-1} \text{tr} X(t) = h_\pm(t)$.  

Since $f_+(t) \neq f_-(t)$, the BPS state preserves only one supersymmetry out of three supersymmetries (two dynamical and one kinematical). It is interesting to note that for arbitrary time dependent functions, say $f_+(t)$ and $h_+(t)$, there exits a supersymmetric matrix model where $X(t) = f_+(t) X + h_+(t) 1$ corresponds to a BPS state, and furthermore there exits always its “twin” BPS state given by $' + \rightarrow - '$.

Utilizing the gauge symmetry (2.13), one can diagonalize $X$ in order to show the positions of the $N$ D-particles in the BPS sector,

$$
X(t) = \text{diag} \left( x_1(t), x_2(t), \cdots, x_N(t) \right) = f_\pm(t) \text{diag} \left( x_1, x_2, \cdots, x_N \right) + h_\pm(t) 1.
$$

A remarkable fact is that all D-particles have precisely the same relative movement, same position, same velocity, same acceleration, etc. up to the constant scaling factors which
entirely depend on their initial positions or so called the co-moving coordinates. This matches precisely with the "homothetic ansatz" adopted in the cosmology literature in order to incorporate the cosmological principle \[32, 33\]. In fact, the second order differential equation, \( \ddot{f} \pm f \dot{\Lambda}, \) (2.15) can be identified as the Raychaudhuri’s equation in cosmology, where \( \Lambda \) is indeed the time dependent cosmological "constant". Also, in the matrix approach to the cosmology \[33, 34, 35\], it is natural to associate \( \Lambda \) as the non-relativistic cosmological constant, and associate \( \Lambda > 0 \) and \( \Lambda < 0 \) with the de-Sitter and Anti-de-Sitter space respectively accounting the repulsive and attractive potential. Thus, although the geometries of \( dS_2 \) and \( AdS_2 \) coincide, we distinguish them by the sign of \( \Lambda \), throughout the paper.

3. \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra

After the standard quantization, \([\mathcal{F}, \mathcal{G}]_{D, B} \rightarrow -i[\mathcal{F}, \mathcal{G}],\) the present \( \mathfrak{osp}(1|2, \mathbb{R}) \) matrix model leads to the following ‘Heisenberg ⊕ Clifford’ algebra,

\[
\left[ X^a_{\ b}, P^c_{\ d} \right] = i \delta^a_d \delta^c_b, \quad \{ \psi^a_{\ b}, \psi^c_{\ d} \} = \delta^a_d \delta^c_b. \tag{3.1}
\]

In Sec. 3.1, utilizing the above algebra alone, especially from the \( \mathfrak{su}(N) \) sector only, we construct explicitly the generators of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra.\(^8\) The number of odd generators is two, and this is consistent with the fact that there are two dynamical supersymmetries in the matrix model, rather than one. Sec. 3.2 is devoted to the analysis on the unitary irreducible representations of the superalgebra, \( \mathfrak{osp}(1|2, \mathbb{R}) \). Further analysis on the superalgebra from the dynamical point of view is given in Sec. 3.3.

3.1 \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra - kinematical point of view

There are five real generators in \( \mathfrak{osp}(1|2, \mathbb{R}) \) which we take as

\[
Q_P := \text{tr}_{\mathfrak{su}(N)}(\psi P), \quad Q_X := \text{tr}_{\mathfrak{su}(N)}(\psi X), \tag{3.2}
\]

and

\[
K_0 := \frac{1}{2}\text{tr}_{\mathfrak{su}(N)}(P^2 + X^2), \quad K_1 := \frac{1}{2}\text{tr}_{\mathfrak{su}(N)}(P^2 - X^2), \quad K_2 := \frac{1}{2}\text{tr}_{\mathfrak{su}(N)}(XP + PX). \tag{3.3}
\]

Alternatively we may construct the generators out of the full \( \mathfrak{u}(N) \) matrices, including the \( \mathfrak{u}(1) \) part and using the ordinary trace, i.e. \( \text{tr}_{\mathfrak{su}(N)} \rightarrow \text{tr}. \) All the results below will remain identical, but the resulting \( \mathfrak{so}(1, 2) \) Casimir will not be a conserved time independent operator, which is not what we want. In order to account for the kinematical supersymmetry,

\(^8\)For the construction of other various algebras, see [33].
we may also include one more odd generator, \( Q \), however this commutes with any generator above in the \( \text{su}(N) \) sector.

All the super-commutator relations\(^9\) of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra follow simply from the ‘Heisenberg ⊕ Clifford’ algebra, (3.1),

\[
Q_P^2 = \frac{1}{2}(K_0 + K_1), \quad Q_X^2 = \frac{1}{2}(K_0 - K_1), \quad \{Q_P, Q_X\} = K_2,
\]

\[
[K_0, Q_P] = +iQ_X, \quad [K_0, Q_X] = -iQ_P, \quad [K_1, Q_P] = -iQ_X,
\]

\[
[K_1, Q_X] = -iQ_P, \quad [K_2, Q_P] = +iQ_P, \quad [K_2, Q_X] = -iQ_X,
\]

\[
[K_1, K_2] = -2iK_0, \quad [K_0, K_1] = +2iK_2, \quad [K_2, K_0] = +2iK_1.
\]

The Casimir of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra reads

\[
C_{\mathfrak{osp}(1|2, \mathbb{R})} = C_{\mathfrak{so}(1,2)} + i [Q_P, Q_X], \quad \left[ C_{\mathfrak{osp}(1|2, \mathbb{R})}, \text{anything} \right] = 0, \quad (3.5)
\]

where the \( \mathfrak{so}(1, 2) \) Casimir is given by

\[
C_{\mathfrak{so}(1,2)} = K_0^2 - K_1^2 - K_2^2
\]

\[
= \frac{1}{4} \left\{ \text{tr}_{\text{su}(N)}(P^2), \text{tr}_{\text{su}(N)}(X^2) \right\} - \frac{1}{4} \left[ \text{tr}_{\text{su}(N)}(XP + PX) \right]^2 \quad (3.6)
\]

\[
= 2 \left\{ Q_X^2, Q_P^2 \right\} - \{Q_X, Q_P\}^2.
\]

The root structure of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra can be identified by complexifying the generators as\(^10\)

\[
Q_+ := Q_P + iQ_X, \quad Q_- := Q_P - iQ_X = Q_+^\dagger, \quad (3.7)
\]

\[
K_+ := K_1 + iK_2, \quad K_- := K_1 - iK_2 = K_+^\dagger.
\]

The Cartan subalgebra has only one element, \( K_0 \), and all others are either raising, \( Q_+, K_+ \),

\(^9\)Although the \( \mathfrak{osp}(1|2, \mathbb{R}) \) super-commutator relations above are direct consequences of the ‘Heisenberg ⊕ Clifford’ algebra, the way to express the generators in terms of \( X, P \) and \( \psi \) is not unique. In fact, \( \mathfrak{so}(1, 2) \) algebra was identified thirty years ago \(^{[37]}\) using a ‘non-polynomial’ basis in the conformal matrix model having the inverse square potential, and based on the observation, Strominger proposed that the conformal matrix model is dual to 2D type IIA string theory on \( \text{AdS}_2 \) \(^{[22]}\) (see also \(^{[38, 39]}\)).

\(^{10}\)For further analysis by us on the root structures of super Lie algebras, see \(^{[40, 41]}\).
or lowering, $Q_-, K_-$, operators to satisfy

$$Q_+^2 = K_+,$$
$$Q_-^2 = K_-,$$
$$\{Q_-, Q_+\} = 2K_0,$$
$$[K_0, Q_+] = +Q_+,$$
$$[K_+, Q_+] = 0,$$
$$[K_-, Q_+] = +2Q_-,$$
$$[K_0, Q_-] = -Q_-,$$
$$[K_+, Q_-] = -2Q_+,$$
$$[K_-, Q_-] = 0,$$
$$[K_0, K_+] = +2K_+,$$
$$[K_0, K_-] = -2K_-,$$
$$[K_-, K_+] = 4K_0. \tag{3.8}$$

In the Cartan basis, the Casimir operators, (3.5), (3.6), read

$$C_{\text{osp}(1|2, \mathbb{R})} = C_{\text{so}(1, 2)} + \frac{1}{2} \{Q_-, Q_+\}, \quad C_{\text{so}(1, 2)} = K_0^2 - \frac{1}{2} \{K_-, K_+\}. \tag{3.9}$$

The $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra can be represented by $(2 + 1) \times (2 + 1)$ real supermatrices, $M$, satisfying

$$M^T \mathcal{J} + \mathcal{J} M = 0, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

so that its generic form reads, with the even and odd real Grassmann entries, $x, \theta$,

$$M = \begin{pmatrix} x_2 & -x_+ & \theta_1 \\ x_- & -x_2 & \theta_2 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}. \tag{3.11}$$

Note that the $2 \times 2$ bosonic part corresponds to $\mathfrak{sp}(2, \mathbb{R}) \equiv \mathfrak{so}(1, 2) \equiv \mathfrak{sl}(2, \mathbb{R})$, as it corresponds to $x_\mu \gamma^\mu$, where $x_\pm = x_0 \pm x_1$, and $\gamma^\mu$ is the $\mathfrak{so}(1, 2)$ gamma matrix,

$$\gamma^0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.12}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \quad \eta = \text{diag}(-++).$$

In fact, with the notion of a $(1 + 2)$-dimensional two component Majorana spinor and its ‘charge conjugate’,

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Q_P \\ Q_X \end{pmatrix}, \quad \tilde{Q} := Q^T \gamma^0 = (Q_X - Q_P), \tag{3.13}$$

the $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra, (3.4), can be rewritten in a compact form,

$$\{Q, \tilde{Q}\} = \gamma^\mu K_\mu, \quad [K_\mu, Q] = i\gamma_\mu Q, \quad [K_\mu, K_\nu] = 2i\epsilon_{\mu\nu\lambda}K^\lambda, \tag{3.14}$$

where $\epsilon_{\mu\nu\lambda}$ is the usual three form with $\epsilon_{012} \equiv 1$. Note also, from $\tilde{Q} Q = [Q_X, Q_P]$, that the $\mathfrak{osp}(1|2, \mathbb{R})$ Casimir operator, $C_{\text{osp}(1|2, \mathbb{R})}$, (3.5) is indeed manifestly $\text{SL}(2, \mathbb{R})$ invariant.
In a similar fashion to above, one can also equip the bosonic operators with the SL(2, R) covariant structure,

\[ \mathcal{V} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{pmatrix} := \begin{pmatrix} P \\ X \end{pmatrix}, \quad K_\mu = \frac{1}{2} \mathcal{V}_\mu \gamma^\nu \mathcal{V}_\nu. \tag{3.15} \]

### 3.2 Unitary irreducible representations of \( \mathfrak{osp}(1|2, \mathbb{R}) \)

In order to analyze the unitary irreducible representations or unitary supermultiplets of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra spanned by the five real generators, (3.2), (3.3), one needs to take \( K_0 \) as the ‘good’ quantum number operator to diagonalize it. Different choice of the good quantum number operator, e.g. \( K_2 \), is not compatible with the unitarity, as it would lead to the raising and lowering operators with the pure imaginary unit, such as \( [K_2, (K_1 \pm K_0)] = \pm 2i(K_1 \pm K_0) \).

Any \( \mathfrak{osp}(1|2, \mathbb{R}) \) supermultiplet decomposes into \( \mathfrak{so}(1,2) \) multiplets. We first review briefly the general properties of the latter or the unitary irreducible representations of \( \mathfrak{so}(1,2) \).

From (3.9) and the commutator relations, we get

\[ C_{\mathfrak{so}(1,2)} + 1 + K_{\pm} K_+ = (K_0 \mp 1)^2. \tag{3.16} \]

From the Hermitian conjugacy property, \( K_+ = K_+^\dagger \), the third term on the left hand side, \( K_\pm K_+ \), is positive semi-definite, while the possible minimum value of the right hand side for the states in a unitary irreducible representation may lie

\[ 0 \leq \min[(K_0 \mp 1)^2] \leq 1, \tag{3.17} \]

if the raising or lowering operators act nontrivially ever. But, this is impossible when \( C_{\mathfrak{so}(1,2)} > 0 \). In this case, the unitary representation is infinite dimensional and characterized by the existence of either the lowest weight state obeying

\[ K_- |l,l\rangle = 0, \quad K_0 |l,l\rangle = l |l,l\rangle, \quad C_{\mathfrak{so}(1,2)} = l(l-2) > 0, \quad l > 2, \tag{3.18} \]

or the highest weight state obeying

\[ K_+ |h, h\rangle = 0, \quad K_0 |h, h\rangle = h |h, h\rangle, \quad C_{\mathfrak{so}(1,2)} = h(h+2) > 0, \quad h < -2. \tag{3.19} \]

When \( C_{\mathfrak{so}(1,2)} = 0 \), there exists only one trivial state, \( |0,0\rangle \), satisfying

\[ K_\pm |0,0\rangle = 0, \quad K_0 |0,0\rangle = 0. \tag{3.20} \]

When \( -1 \leq C_{\mathfrak{so}(1,2)} < 0 \), the representation is called the ‘continuous principal series’. It is infinite dimensional, and the lowest or highest weight state may or may not exit. If there

\[ ^{11} \text{For further analysis see e.g. [42].} \]
is a lowest or highest weight state, then its good quantum number is \( +1 \pm \sqrt{\mathcal{C}_{\mathfrak{so}(1,2)} + 1} \) or \( -1 \pm \sqrt{\mathcal{C}_{\mathfrak{so}(1,2)} + 1} \), respectively but not simultaneously. When \( \mathcal{C}_{\mathfrak{so}(1,2)} < -1 \), there must be neither lowest nor highest weight state, and the representation is called the ‘continuous supplementary series’.

As for the present \( \mathfrak{osp}(1|2, \mathbb{R}) \) matrix model, \( K_0 \) is positive definite for the unitary multiplets as
\[
K_0 = \frac{1}{2} \text{tr}_{\mathfrak{su}(N)} (P^2 + X^2) = \text{tr}_{\mathfrak{su}(N)} (A^\dagger A) + \frac{1}{2} (N^2 - 1) \geq \frac{1}{2} (N^2 - 1),
\]
where
\[
A := \frac{1}{\sqrt{2}} (P - iX), \quad [A^a_b, A^{\dagger c}_d] = \delta^a_d \delta^c_b.
\]
There exists always a lowest weight state in any \( \mathfrak{so}(1,2) \) multiplet, and from (3.16), the \( \mathfrak{so}(1,2) \) Casimir is bounded below
\[
\mathcal{C}_{\mathfrak{so}(1,2)} \geq \frac{1}{4} (N^2 - 1)(N^2 - 5) \quad \text{for} \quad N \geq 3,
\]
\[
\mathcal{C}_{\mathfrak{so}(1,2)} > 0 \quad \text{or} \quad \mathcal{C}_{\mathfrak{so}(1,2)} = -\frac{3}{4} \quad \text{for} \quad N = 2.
\]

Now as for the \( \mathfrak{osp}(1|2, \mathbb{R}) \) unitary supermultiplet, we first note that the odd roots, \( Q \), shift the ‘good’ quantum number by one unit, half of what \( K \) do. Hence the odd roots move one \( \mathfrak{so}(1,2) \) multiplet to another inside a \( \mathfrak{osp}(1|2, \mathbb{R}) \) supermultiplet, but at most once due to \( Q^2 = K \). Similar to (3.16), we also have
\[
\mathcal{C}_{\mathfrak{osp}(1|2, \mathbb{R})} + K_+ K_- \pm Q_\pm = K_0 (K_0 + 1).
\]

After all, utilizing all the facts above, we conclude that any unitary irreducible representation of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra satisfying the positiveness, (3.21), is infinite dimensional and characterized by the existence of the super-lowest weight state obeying
\[
Q_- |l_s, l_s\rangle = 0, \quad K_0 |l_s, l_s\rangle = l_s |l_s, l_s\rangle, \quad \mathcal{C}_{\mathfrak{osp}(1|2, \mathbb{R})} = l_s (l_s - 1), \quad l_s \geq \frac{1}{2} (N^2 - 1).
\]

Furthermore, the \( \mathfrak{osp}(1|2, \mathbb{R}) \) unitary supermultiplet always decomposes into two \( \mathfrak{so}(1,2) \) multiplets whose lowest weight states are given by
\[
|l_s, l_s\rangle \quad \text{and} \quad |l_s + 1, l_s + 1\rangle = \frac{1}{\sqrt{2l_s}} Q_+ |l_s, l_s\rangle.
\]

---

\[12\] In fact, from (3.6), expressing \( \mathcal{C}_{\mathfrak{so}(1,2)} \) in terms of the odd generator, the trace of \( \mathcal{C}_{\mathfrak{so}(1,2)} \) also ‘formally’ shows the positiveness,
\[
\text{tr} \mathcal{C}_{\mathfrak{so}(1,2)} \approx \text{tr} (-[Q_X, Q_P]^2) \geq 0.
\]

The subtlety is due to the infinite sum over the infinite dimensional \( \mathfrak{so}(1,2) \) multiplet.
3.3 $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra - dynamical point of view

The Noether charges corresponding to the two dynamical supersymmetries, \((2.14)\), decompose into the $\mathfrak{su}(N)$ and $\mathfrak{u}(1)$ parts,

\[
\text{tr}(i\psi \delta_{\pm} \psi) = if_{\pm}(t) \left( Q_{\mathfrak{su}(N)}^\pm + Q_{\mathfrak{u}(1)}^\pm \right) \varepsilon_{\pm},
\]

\[
Q_{\mathfrak{su}(N)}^\pm := \text{tr}_{\mathfrak{su}(N)} \left[ \psi \left( P - g_{\pm}(t)X \right) \right],
\]

\[
Q_{\mathfrak{u}(1)}^\pm := \text{tr}_{\mathfrak{u}(1)} \left[ \psi \left( P - g_{\pm}X - (\kappa_{\pm}/f_{\pm}) L \right) \right].
\]

(3.26)

where we put

\[
g_{\pm}(t) := \frac{\dot{f}_{\pm}(t)}{f_{\pm}(t)}, \quad \dot{g}_{\pm} + g_{\pm}^2 = \Lambda(t).
\]

(3.27)

Because the Hamiltonian as well as the above two supercharges in the $\mathfrak{su}(N)$ sector can be expressed in terms of the previous “kinematical” basis, $Q_X, Q_P, K_0, K_1, K_2$, \((3.2), (3.3)\), the underlying supersymmetry algebra must correspond to $\mathfrak{osp}(1|2, \mathbb{R})$, no matter what the dynamics is. However, the use of the above supercharges, $Q_{\mathfrak{su}(N)}\pm$, will not lead to simple expressions for the superalgebra. For example, from the conservation of the Noether charge and Eq.\((2.22)\), the commutator relation between the Hamiltonian and the supercharge reads in a less economic manner,

\[
[H, Q_{\mathfrak{su}(N)}^\pm] = ig_{\pm} Q_{\mathfrak{su}(N)}^\pm + i \frac{\dot{g}_{\pm}}{g_+ - g_-} \left( Q_{\mathfrak{su}(N)}^+ - Q_{\mathfrak{su}(N)}^- \right).
\]

(3.28)

Henceforth, in order to analyze the underlying $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra in a simple fashion but still to keep track of the dynamical properties, we slightly modify the basis of the odd generators and keep the Hamiltonian explicitly as a $\mathfrak{so}(1, 2)$ generator. Note that the change of basis requires the time dependent coefficients due to $\Lambda(t)$, as $Q_{\mathfrak{su}(N)}^\pm = Q_P - g_{\pm}(t)Q_X$. Hence, only with specific time dependent coefficients we can write down the time independent conserved quantities, as one can expect from \((2.22)\). All together there are five conserved “true” Noether charges corresponding to the five symmetries, \((2.14), (2.16)\). Namely we have the two fermionic conserved Noether charges for the two dynamical supersymmetries,

\[
f_{\pm} Q_{\mathfrak{su}(N)}^\pm = f_{\pm} Q_P - \dot{f}_{\pm} Q_X,
\]

(3.29)

and three bosonic conserved Noether charges for the $\mathfrak{so}(1, 2)$ symmetries \((2.16)\),

\[
(f_{\pm} Q_{\mathfrak{su}(N)}^\pm)^2 = \frac{1}{2} \left( f_{\pm}^2 + \dot{f}_{\pm}^2 \right) K_0 + \frac{1}{2} \left( f_{\pm}^2 - \dot{f}_{\pm}^2 \right) K_1 - f_{\pm} \dot{f}_{\pm} K_2,
\]

\[
\{ f_{+} Q_{\mathfrak{su}(N)}^+, f_{-} Q_{\mathfrak{su}(N)}^- \} = \left( f_{+} f_{-} - \dot{f}_{+} \dot{f}_{-} \right) K_0 + \left( f_{+} f_{-} - \dot{f}_{+} \dot{f}_{-} \right) K_1 - \left( f_{+} \dot{f}_{-} + f_{-} \dot{f}_{+} \right) K_2.
\]

(3.30)
Apart from the above five Noether charges, both of the \( \mathfrak{osp}(1|2, \mathbb{R}) \) and \( \mathfrak{so}(1, 2) \) Casimir operators, \( C_{\mathfrak{osp}(1|2, \mathbb{R})} \) \((3.5)\) and \( C_{\mathfrak{so}(1, 2)} \) \((3.6)\), are also conserved time independent quantities, since they do not include any explicit time dependency and they commute with the Hamiltonian, for sure.

\[\text{3.3.1 \ \mathfrak{osp}(1|2, \mathbb{R}) superalgebra when } \Lambda(t) \neq 0\]

In a similar fashion to the standard harmonic oscillator analysis, we first set a pair of operators,

\[A_\pm(t) := \frac{P \pm \sqrt{\Lambda(t)} X}{\sqrt{2}},\]  

and define a pair of even generators in \( \mathfrak{osp}(1|2, \mathbb{R}) \) by

\[J_\pm(t) := \text{tr}_{\text{su}(N)} \left[ A_\pm(t)^2 \right],\]  

as well as a pair of odd generators,

\[Q_\pm(t) := \text{tr}_{\text{su}(N)} \left[ \psi A_\pm(t) \right].\]

Note that \( Q_\pm \) coincide with the actual supercharges, \( Q_\pm \) \((3.26)\), provided that \( \Lambda(t) \) is constant.

The Hamiltonian for the \( \text{su}(N) \) sector is then

\[H_{\text{su}(N)} = \frac{1}{2} \text{tr}_{\text{su}(N)} \left[ A_+(t) A_-(t) + A_-(t) A_+(t) \right] = \text{tr}_{\text{su}(N)} \left[ A_\pm(t) A_\mp(t) \right] = \frac{1}{2} i \sqrt{\Lambda(t)} (N^2 - 1),\]  

and from the quantization relation,

\[\left[ A_-(t)^a_b, A_+(t)^c_d \right] = -i \sqrt{\Lambda(t)} \delta^a_d \delta^c_b,\]  

we obtain such as

\[H_{\text{su}(N)}, A_\pm = \mp i \sqrt{\Lambda(t)} A_\pm, \quad [J_-, A_+] = -2i \sqrt{\Lambda(t)} A_-, \quad [J_+, A_-] = +2i \sqrt{\Lambda(t)} A_+,\]  

where we set

\[\hat{A}_\pm := A_\pm - N^{-1} \text{tr} (A_\pm) 1.\]  

Now, the \( \mathfrak{osp}(1|2, \mathbb{R}) \) superalgebra reads in terms of \( Q_\pm, J_\pm, H_{\text{su}(N)} \),

\[Q_+^2 = \frac{1}{2} J_+, \quad Q_-^2 = \frac{1}{2} J_-, \quad \{ Q_+, Q_- \} = H_{\text{su}(N)};\]

\[\left[ J_-, Q_+ \right] = -2i \sqrt{\Lambda(t)} Q_-, \quad \left[ J_+, Q_- \right] = +2i \sqrt{\Lambda(t)} Q_+, \quad \left[ H_{\text{su}(N)}, Q_\pm \right] = \mp i \sqrt{\Lambda(t)} Q_\pm;\]

\[\left[ J_-, J_+ \right] = -4i \sqrt{\Lambda(t)} H_{\text{su}(N)}, \quad \left[ H_{\text{su}(N)}, J_\pm \right] = \mp 2i \sqrt{\Lambda(t)} J_\pm, \quad \left[ J_\pm, Q_\pm \right] = 0.\]  

\[\text{13In fact, when } \Lambda(t) \text{ is constant, } A_\pm \text{ correspond to the generators of } W_\infty \text{ algebra } \mathfrak{R}.\]
Especially, the $\mathfrak{so}(1, 2)$ Casimir operator, (3.41), can be reexpressed as

$$C_{\mathfrak{so}(1, 2)} = K_0^2 - K_1^2 - K_2^2 = \frac{1}{\Lambda(t)} \left[ \frac{1}{2} \{ J_+, J_- \} - H_{su(N)} \right] . \tag{3.39}$$

From $J_\pm^\dagger = J_\mp$ for $\Lambda > 0$ and $J_+^\dagger = J_-$ for $\Lambda < 0$, there exists a SO(1, 2) rotation which transforms $H_{su(N)}$ to $K_1$ if $\Lambda > 0$ or $K_0$ if $\Lambda < 0$.

3.3.2 $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra when $\Lambda(t) = 0$

If $\Lambda = 0$, $A_+ \equiv A_-$, and the above super-commutator relations (3.38) do not faithfully represent the super Lie algebra, $\mathfrak{osp}(1|2, \mathbb{R})$. In order to do so, one needs to define the generators differently. When $\Lambda = 0$ we have $f_+ = 1$, $f_- = t$, and the corresponding two supercharges, (3.26), in the $su(N)$ sector are

$$Q_{\Lambda=0}^+ = \text{tr}_{su(N)}[\psi P] , \quad Q_{\Lambda=0}^- = \text{tr}_{su(N)}[\psi(P - t^{-1} X)] , \tag{3.40}$$

while the Hamiltonian is given by

$$H_{su(N)} = \frac{1}{2} \text{tr}_{su(N)}(P^2) = \frac{1}{2} (K_0 + K_1) . \tag{3.41}$$

Rather than $Q_{\Lambda=0}^\pm$, we adopt the kinematical odd generators, (3.2),

$$Q_P = \text{tr}_{su(N)}(\psi P) = Q_{\Lambda=0}^+ , \quad Q_X = \text{tr}_{su(N)}(\psi X) = t \left( Q_{\Lambda=0}^+ - Q_{\Lambda=0}^- \right) , \tag{3.42}$$

and write the $\mathfrak{osp}(1|2, \mathbb{R})$ superalgebra in terms of the real basis,

$$Q_P^2 = H_{su(N)} , \quad Q_X^2 = V_{su(N)} := \frac{1}{2} \text{tr}_{su(N)}(X^2) , \quad \{ Q_P, Q_X \} = K_2 ,$$

$$[H_{su(N)}, Q_X] = -iQ_P , \quad [H_{su(N)}, Q_P] = 0 , \quad [V_{su(N)}, Q_P] = 0 , \quad [V_{su(N)}, Q_X] = 0 ,$$

$$[V_{su(N)}, Q_P] = +iQ_X , \quad [K_2, Q_X] = -iQ_X , \quad [K_2, Q_P] = +iQ_P ,$$

$$[H_{su(N)}, V_{su(N)}] = -iK_2 , \quad [K_2, H_{su(N)}] = +2iH_{su(N)} , \quad [K_2, V_{su(N)}] = -2iV_{su(N)} . \tag{3.43}$$

In particular, the $\mathfrak{so}(1, 2)$ Casimir operator, (3.6), reads

$$C_{\mathfrak{so}(1, 2)} = K_0^2 - K_1^2 - K_2^2 = 2 \{ H_{su(N)} , V_{su(N)} \} - K_2^2 . \tag{3.44}$$
4. Discussion and conclusion

We have derived a $\mathcal{N} = 2$ supersymmetric matrix model, (2.12), with quadratic and linear potentials whose coefficients are arbitrary time dependent ‘cosmological constant’, $\Lambda(t)$, and ‘electric flux background’, $\rho(t)$. The matrix model corresponds to the most general supersymmetric deformations of the matrix quantum mechanics having the 2D super Yang-Mills origin. We have shown that, for arbitrary $\Lambda(t)$ and $\rho(t)$, the matrix model enjoys two dynamical supersymmetries, $Q_1, Q_2$, and three bosonic symmetries, $K_0, K_1, K_2$, which amount to the superalgebra, $\mathfrak{osp}(1|2, \mathbb{R})$, (3.14),

$$\{ Q, \bar{Q} \} = \gamma^\mu K_\mu, \quad [ K_\mu, Q ] = i \gamma_\mu Q, \quad [ K_\mu, K_\nu ] = 2i \epsilon_{\mu\nu\lambda} K^\lambda. \quad (4.1)$$

If the matrix model had only one supersymmetry as in the 2D minimal super Yang-Mills, the $\mathfrak{osp}(1|2, \mathbb{R})$ structure would be absent.

The matrix model is to describe the noncritical 3D $\mathcal{M}$-theory on generic supersymmetric backgrounds in a controllable manner through the matrix regularization, and our claim is further that, with the arbitrariness of $\Lambda(t)$ and $\rho(t)$, it also provides holographic duals to various two dimensional superstring theories, as we argue below.

4.1 Normalizable and non-normalizable wave functions

At the quantum level, the wave function satisfies

$$\text{tr} \left[ \frac{1}{2} P^2 - \frac{1}{2} \Lambda(t) X^2 - \rho(t) X \right] |\Psi(t)\rangle = i \frac{\partial}{\partial t} |\Psi(t)\rangle. \quad (4.2)$$

As is well known, when $\Lambda(t)$ is negative, the potential is bounded below and the normalizable or unitary wave functions have the discrete spectrum. Namely, from (3.36), the lowering operator, $\hat{A}_-$, lowers the eigenvalue of $H_{su(N)}$ by the unit $\sqrt{|\Lambda|}$. However, since $H_{su(N)}$ should be positive definite for the unitary representations, there must be a ground state which is annihilated by $\hat{A}_-$,

$$\hat{A}_- |0\rangle = 0, \quad |0\rangle = e^{-\frac{1}{2} \sqrt{|\Lambda|} u X^2} |P \equiv 0\rangle. \quad (4.3)$$

All other excited states are then constructed by acting the raising operator, $\hat{A}_+$, to the ground state. Due to the Gauss constraint, one needs to restrict on the gauge singlets, which can be simply done by taking the ‘trace’ of the $u(N)$ indices in all the possible ways [44]. The quantum states in the Hilbert space then form the unitary irreducible representations of the superalgebra, $\mathfrak{osp}(1|2, \mathbb{R})$, and in particular, their $\mathfrak{so}(1, 2)$ Casimir is positive definite for $N \geq 3$, (3.22). The energy spectrum is discretized by the unit $\sqrt{|\Lambda(t)|}$, and the zero point vacuum energy is, from (3.34), $\frac{1}{2} \sqrt{|\Lambda(t)|} (N^2 - 1)$. The vacuum has the degeneracy, $2^{(N^2/2)}$, due to the fermions. The non-vanishing zero point energy refers to the existing two other bosonic charges in the superalgebra apart from the Hamiltonian.
On the other hand, when $\Lambda(t)$ is positive, the wave functions can not be normalizable, as the raising and lowering operators shift the energy spectrum by the imaginary unit, $i\sqrt{\Lambda}$, while $H_{su(N)}$ should have real eigenvalues for the normalizable states. Physically, this amounts to the fact that the matrix model describes the Fermi sea (see e.g. [45, 35]).

Therefore, in order to have a unifying description for arbitrary $\Lambda(t)$, the full Hilbert space of the $M$-theory matrix model should include not only the normalizable states but also the non-normalizable states, allowing both the unitary and the non-unitary representations of $osp(1|2, \mathbb{R})$. The former is relevant only to the case $\Lambda(t) < 0$. Without the concern about the normalizability, the following Schrödinger equation has always solutions for arbitrary energy, $E(t)$,

$$-\text{tr} \left[ \frac{1}{2} \left( \frac{\partial}{\partial X} \right)^2 + \frac{1}{2} \Lambda(t) X^2 + \rho(t) X \right] \Psi(X, t) = E(t) \Psi(X, t). \quad (4.4)$$

In particular, the momentum operator $P = -i \frac{\partial}{\partial X}$ is no longer necessarily real. Again the raising and lowering operators, $\hat{A}_\pm$, generate new solutions with the shifted energy, $E(t) \pm i\sqrt{\Lambda(t)}$.

The reason to consider the phase space over the complex planes rather than the real lines is manifest in the path integral formalism, since when $\Lambda(t) > 0$, the local minima of the Hamiltonian are located on the genuine complex planes rather than the real lines so that one should take $P$ Hermitian and $X$ anti-Hermitian, or vice versa.

All the BPS states are non-normalizable or non-unitary. From its defining property,

$$Q_{\text{BPS}} |\text{BPS}\rangle = 0, \quad Q_{\text{BPS}} = \text{tr} \left[ \psi \left( f P - \dot{f} X - \kappa 1 \right) \right] = Q_{\text{BPS}}^\dagger, \quad (4.5)$$

and the positive definite property of $Q_{\text{BPS}}^2 = \frac{1}{2} \text{tr} \left( f P - \dot{f} X - \kappa 1 \right)^2$ for the unitary states, if the BPS state were normalizable, it would mean $f P |\text{BPS}\rangle = (\dot{f} X + \kappa 1) |\text{BPS}\rangle$. But this contradicts with the quantization, $[X, P] \neq 0$.

### 4.2 Projection to the ‘$so(1, 2)$ hyperspace’

We consider a projection map from the full phase space to the ‘$so(1, 2)$ hyperspace’ given by the “coordinates”, $K_0, K_1, K_2, \{3,3\}$. The induced dynamics therein is subject to

$$\dot{K}_\mu = i [H_{su(N)}, K_\mu] = 2 \epsilon_{\mu \nu \lambda} T^\nu K^\lambda, \quad T^\nu := \left( \frac{1}{2} (1 - \Lambda), \frac{1}{2} (1 + \Lambda), 0 \right), \quad (4.6)$$

$$T^\mu K_\mu = H_{su(N)}, \quad T^\mu T_\mu = \Lambda.$$ 

Naturally, as seen from $\{3,12\}$, the $so(1, 2)$ hyperspace is equipped with the $so(1, 2)$ metric, $\eta = \text{diag}(-+++)$. The $so(1, 2)$ Casimir, $C_{so(1,2)}$, $\{3,6\}$ highlights the geometrical picture,

$$C_{so(1,2)} = K_0^2 - K_1^2 - K_2^2. \quad (4.7)$$
$C_{so(1,2)}$ is a conserved time independent operator, since it does not include any explicit time dependency and it commutes with the Hamiltonian, just like the $osp(1|2, \mathbb{R})$ Casimir. Classically, this can be also seen, from (4.6), as $K^\mu \dot{K}_\mu = 0$. Therefore, we observe that for each $so(1,2)$ multiplet in the Hilbert space, the corresponding $so(1,2)$ hyperspace dynamics is constrained on a two dimensional rigid surface such that

$$
\begin{align*}
\text{Euclidean } dS_2/AdS_2 & \quad \text{if } C_{so(1,2)} > 0, \\
\text{Minkowskian } dS_2/AdS_2 & \quad \text{if } C_{so(1,2)} < 0, \\
\text{Null cone} & \quad \text{if } C_{so(1,2)} = 0.
\end{align*}
$$

Surely the specific value of the Casimir for each multiplet is to be superselected just like any boundary condition in quantum field theories. This also fits into the 3D $M$-theory picture, to include or provide holographic dual descriptions to all the superstring theories. The richness of the $osp(1|2, \mathbb{R}) M$-theory matrix model originates from the arbitrariness of the cosmological constant, $\Lambda(t)$ and the electric flux background, $\rho(t)$ as well as the vast amount of existing $so(1,2)$ multiplets in the Hilbert space each of which has its own two dimensional geometry.

However, if we restrict on the unitary irreducible representations, i.e. the normalizable sector relevant to the case $\Lambda(t) < 0$, we have the bound for the Casimir, (3.22),

$$
C_{so(1,2)} \geq \frac{1}{4} (N^2 - 1) (N^2 - 5).
$$

Thus, the corresponding geometry is always Euclidean $dS_2/AdS_2$ if $N \geq 3$. As for the non-normalizable or non-unitary sector, the above bound does not hold.

The bound can be also understood classically as

$$
C_{so(1,2)} = \text{tr}_{su(N)} \left( P^2 \right) \text{tr}_{su(N)} \left( X^2 \right) - \left[ \text{tr}_{su(N)} \left( PX \right) \right]^2
$$

$$
= \text{tr}_{su(N)} \left( P^2 \right) \text{tr}_{su(N)} \left[ X - \frac{\text{tr}_{su(N)} \left( PX \right)}{\text{tr}_{su(N)} \left( P^2 \right)} P \right]^2.
$$

This is positive semi-definite if both $X$ and $P$ are Hermitian, as is the case for the expectation values of the unitary states. Otherwise, of course, not. Especially, when $P$ is Hermitian and $X$ is anti-Hermitian or vice versa, as in the path integral formalism for $\Lambda > 0$, $C_{so(1,2)}$ is negative semi-definite, implying the Minkowskian $dS_2/AdS_2$ geometry. From (3.27), among the on-shell configurations, only the BPS configurations saturate the bound, $C_{so(1,2)} = 0$.

From (3.39), (1.6), a ‘dispersion relation’ follows

$$
\dot{K}_\mu K^\mu = 4 \left( H_{su(N)}^2 - \Lambda K_\mu K^\mu \right) = 4 \left( H_{su(N)}^2 + \Lambda(t) C_{so(1,2)} \right) = 2 \left\{ J_+ (t) , J_- (t) \right\},
$$

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which shows that the ‘mass’ is conserved if Λ(t) is constant. Furthermore, we have the positive semi-definite bound both for the unitary states,

\[
\begin{align*}
\text{For } & \Lambda(t) = 0, \quad \dot{K}_\mu \dot{K}^\mu = 4H(t)^2 \geq 0, \\
\text{For } & \Lambda(t) > 0, \quad \dot{K}_\mu \dot{K}^\mu = 8 \left\{ Q_+ Q^+_+ , Q_- Q^+_+ \right\} \geq 0, \\
\text{For } & \Lambda(t) < 0, \quad \dot{K}_\mu \dot{K}^\mu = 2 \left\{ J_+, J^+_+ \right\} \geq 0,
\end{align*}
\]

(4.12)

and also for the non-unitary states for which Λ > 0 and \( P, iX \) being Hermitian or anti-Hermitian,

\[
\dot{K}_\mu \dot{K}^\mu = \left[ \text{tr}_{su(N)}(P^2 + \Lambda X^2) \right]^2 - \Lambda \left[ \text{tr}_{su(N)}(PX + XP) \right]^2 \geq 0.
\]

(4.13)

The equality holds only for the trivial case, \( X = P = 0 \). Thus, the velocity vector, \( \dot{K}_\mu \) is always space-like, which is natural for the Euclidean geometry of the unitary states. But in the Minkowskian space, as for the non-unitary states with \( \Lambda > 0 \), it implies the superluminal behavior, i.e. “tachyon”. As shown above, all the BPS configurations correspond to the null geometry, and hence not tachyonic.

To summarize, the normalizable or the unitary sector in the Hilbert space relevant to the case \( \Lambda < 0 \) is characterized by the Euclidean \( dS_2/AdS_2 \) geometry, while the non-normalizable or the non-unitary sector relevant to the case \( \Lambda > 0 \) has the Minkowskian geometry. All the BPS states always correspond to the null geometry, i.e. \( \mathcal{C}_{\mathfrak{so}(1,2)} = 0 \). When \( \Lambda(t) > 0 \) and \( \mathcal{C}_{\mathfrak{so}(1,2)} < 0 \), i.e. the Minkowskian de-Sitter geometry, from (4.12), the particles in the \( \mathfrak{so}(1,2) \) hyperspace are tachyonic and can not be supersymmetric.

### 4.3 Holographic dual to 2D superstring

Various matrix models with potentials having a single maximum have been proposed as dual candidates of 2D string theories on \( AdS_2 \) or \( AdS_2 \)-type backgrounds with the rolling tachyon or the linear dilaton [46, 47, 48, 49, 20, 22, 53]. The continuum or so called the double scaling limit in the matrix models zoom in on the maximum of the potential, effectively leaving a single upside down harmonic potential [50, 51, 52], precisely the same feature as our \( \mathfrak{osp}(1|2, \mathbb{R}) \) matrix model shares when \( \Lambda > 0 \). Furthermore, the Hermitian matrix itself is supposed to represent the non-Abelian open string tachyon [46], and this is manifest in our dispersion relation, (4.12), for the case of \( \Lambda > 0 \) and \( \mathcal{C}_{\mathfrak{so}(1,2)} < 0 \). Thus, we conclude that when \( \Lambda(t) \) is positive, the \( \mathfrak{osp}(1|2, \mathbb{R}) \) \( \mathcal{M} \)-theory matrix model provides holographic duals to the two dimensional Minkowskian superstring theories. The relevant sector in the matrix model Hilbert space is then the non-normalizable or non-unitary one satisfying \( \mathcal{C}_{\mathfrak{so}(1,2)} < 0 \). From (4.12), the choice of the decreasing \( \Lambda(t) \), like \( \Lambda(t) = e^{-t/t_0} \), seems appropriate for the description of the tachyon condensation [53] or the D-brane decay [46]. Further investigation is to be required.
On the other hand, when $\rho = 0$, for the constant positive $\Lambda$ the generic BPS configurations (2.28) are given by the hyperbolic functions,

$$X(t) = \cosh \left( \sqrt{\Lambda} t \right) X(0) + \frac{\kappa}{\sqrt{\Lambda}} \sinh \left( \sqrt{\Lambda} t \right),$$  \hspace{1cm} (4.14)

while for the constant negative $\Lambda$ they are the usual harmonic oscillators,

$$X(t) = \cos \left( \sqrt{|\Lambda|} t \right) X(0) + \frac{\kappa}{\sqrt{|\Lambda|}} \sin \left( \sqrt{|\Lambda|} t \right).$$  \hspace{1cm} (4.15)

The latter is also consistent with the Euclidean 2D superstring theory or the $\mathcal{N} = 2$ super Liouville theory results [53, 54, 55, 56, 57, 58]. The classical shape of the so called FZZT brane (falling Euclidean D0-brane), which is given as time dependent boundary state, precisely matches with (4.13). Thus, we expect that when $\Lambda(t)$ is negative, the $\mathfrak{osp}(1|2, \mathbb{R})$ M-theory matrix model provides holographic dual description of 2D Euclidean superstring theories or superconformal theories. In particular, if $\Lambda$ is negative constant, it corresponds to the $\mathcal{N} = 2$ super Liouville theory, with the relation to the ‘Liouville background charge’, $Q_{\text{Liouville}} = 2|\Lambda|$.

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