Abstract

A tandem duplication denotes the process of inserting a copy of a segment of DNA adjacent to its original position. More formally, a tandem duplication can be thought of as an operation that converts a string \( S = AXB \) into a string \( T = AXXB \), and is denoted by \( S \Rightarrow T \). As they appear to be involved in genetic disorders, tandem duplications are widely studied in computational biology. Also, tandem duplication mechanisms have been recently studied in different contexts, from formal languages, to information theory, to error-correcting codes for DNA storage systems.

The problem of determining the complexity of computing the tandem duplication distance between two given strings was proposed by [Leupold et al., 2004] and, very recently, it was shown to be NP-hard for the case of unbounded alphabets [Lafond et al., 2019].

In this paper, we significantly improve this result and show that the tandem duplication distance problem is NP-hard already for the case of strings over an alphabet of size \( \leq 5 \). We also consider the existence problem: given strings \( S \) and \( T \) over the same alphabet, decide whether there exists a sequence of duplications converting \( S \) into \( T \). A polynomial time algorithm that solves this (existence) problem was only known for the case of the binary alphabet. We focus on a special class of strings—here referred to as purely alternating—that generalize the special structure of binary strings to larger alphabets. We show that for the case of purely alternating strings from an alphabet of size \( \leq 5 \), the existence problem can be solved in linear time.

2012 ACM Subject Classification Theory of computation → Design and analysis of algorithms; Theory of computation → Problems, reductions and completeness; Mathematics of computing → Combinatorics on words

Keywords and phrases tandem duplication, tandem repeats, duplication distance, NP-hardness, purely alternating strings
1 Introduction

Since the draft sequence of the human genome was published, it has been known that a very large part of it consists of repeated substrings \[10\]. Among the different types of possible repetitions, one talks about a tandem repeat when a pattern of one or more nucleotides is repeated and the repetitions are adjacent to each other. For instance, in the word CTACTAGTCA, the substring CTACTA is a tandem repeat. As tandem repeats appear to be correlated to several genetic disorders \[4, 17, 18\], the study of tandem duplication mechanisms has attracted the interest of different communities also outside the specific area of computational biology \[15, 16, 1, 3, 7, 2\].

Problem definition. Formally, a tandem duplication (TD) (of length \(h\))—later simply referred to as a duplication—is an operation on a string \(S\) that copies a substring \(X\) (of length \(h\)) of \(S\) and inserts the copy after the occurrence of \(X\) in \(S\). In other words, a TD transforms \(S = AXB\) into \(AXXB\). Given another string \(T\), we write \(S \Rightarrow T\) if there exist strings \(A, B, X\) such that \(S = AXB\) and \(T = AXXB\), e.g., \(1213451 \Rightarrow 1213413451\). More generally, we write \(S \Rightarrow_k T\) if there exist some \(k\) such that \(S \Rightarrow_k T\).

The tandem duplication distance between two strings \(S\) and \(T\), indicated with \(\text{dist}_{TD}(S, T)\), is the minimum value of \(k\) satisfying \(S \Rightarrow_k T\). If \(S \Rightarrow_k T\) does not hold, then \(\text{dist}_{TD}(S, T) = \infty\).

For example, \(\text{dist}_{TD}(0121, 0101211) = 2\), since: (i) we can apply the following sequence of duplications \(0121 \Rightarrow 010121 \Rightarrow 0101211\); (ii) and it is easy to verify that no single duplication can turn \(0121\) into \(0101211\).

We can now define the following two natural problems about the possibility of converting a string \(S\) into a string \(T\) by only using tandem duplications:

TANDEM DUPLICATION DESCENDENCE (TD-EXIST)
Input: Two strings \(S\) and \(T\) over the same alphabet \(\Sigma\).
Question: Is \(\text{dist}_{TD}(S, T) < \infty\)？

TANDEM DUPLICATION DISTANCE (TD-DIST)
Input: Two strings \(S\) and \(T\) over the same alphabet \(\Sigma\) and an integer \(k\).
Question: Is \(\text{dist}_{TD}(S, T) \leq k\)？

Determining the complexity of computing the tandem duplication distance between two given strings (the TD-Dist problem) was posed in \(\[15\]\) and, only very recently, it was shown to be NP-hard in the case of unbounded alphabets \[9\]. Here, we significantly improve this result by showing that the tandem duplication distance problem is NP-hard for the case of bounded alphabets of size \(\geq 5\).

For both the result of \(\[9\]\) and ours, it is assumed that the strings \(S\) and \(T\) satisfy \(S \Rightarrow_k T\). In general, the complexity of deciding if a string \(S\) can be turned into a string \(T\) by a sequence of tandem duplications (the TD-Exist problem) is still an open problem for alphabets of size \(> 2\). In the second part of the paper we also consider the existence problem (TD-Exist) focusing on a special class of strings—which we call purely alternating (see Section 2 for the definition)—that generalize the special structure of binary strings to larger alphabets. We show that a linear time algorithm for the TD-Exist problem exists for every alphabet of size \(\leq 5\) if the strings are purely alternating. In a final section we also discuss the limit of the approach used here for larger alphabets \(|\Sigma| > 5\).

Related Work. The first papers explicitly dealing with tandem duplication mechanisms are probably in the area of formal languages \[15, 14, 5, 12, 13\]. In particular, in \[14, 15\],
unbounded duplication language and the $k$-bounded duplication languages are defined as the set of words generated via tandem duplications (of constant length, in the $k$-bounded variant), and the focus is on decidability issues and classification of these languages in the Chomsky hierarchy. In the same line of research, more recently, Jain et al. [6] proved that $k$-bounded duplication languages are regular for $k \leq 3$. Problems of duplication distance were recently considered in an information theoretic perspective in [1], where the authors studied, for the binary alphabet case, extremal questions regarding the number of tandem duplications required to generate a binary word starting from its unique seed (the duplication free word from which it can be generated). In the same paper, the authors also considered approximate duplication operations. In [3] Farnoud et al. began the study of the average information content of a $k$-bounded duplication language, as the capacity of a duplication system. Later, Jain et al. [6] introduced the notion of expressiveness to measure a language’s capability to generate words that contain certain desired substrings. A complete characterization of fully expressive $k$-bounded duplication languages was provided by Jain et al. for all alphabet sizes and all $k$. Motivated by problems arising from DNA storage applications, Jain et al. [7] proposed the study of codes that correct tandem duplications to improve the reliability of data storage, and gave optimal construction for the case where tandem duplication length is at most two. In [2], Chee et al. investigated algorithms associated with these codes and particularly, they focused on the question of confusability, i.e., whether given words $x$ and $y$ there are two sequences of tandem duplications such that the resulting words $x'$ and $y'$ are equal. They show that even for small duplication lengths, the solutions to this question are nontrivial, and exact solutions are provided for the case of tandem duplications of size at most three.

2 Preliminary notions

We borrow some of the terminology from [1][9]. For a string $S$ we write $\Sigma(S)$ to denote the alphabet of the string, namely the set of characters that occur in $S$. If $|\Sigma(S)| = q$ we say that the string is $q$-ary. A substring of $S$ is a contiguous sequence of characters within $S$. A prefix (resp. suffix) is a substring that occurs at the beginning (resp. end) of $S$. A subsequence of $S$ is a string that can be obtained by successively deleting zero or more characters from $S$. If a string $S'$ is a subsequence of the string $S$ we write $S' \subseteq S$.

A square string is a string of the form $XX$, i.e. a concatenation of two identical substrings. A string is square-free if it doesn’t contain any substring which is a square string. Given a string $S$, a contraction is the reverse of a tandem duplication. That is, it takes a square string $XX$ contained in $S$ and deletes one of the two copies of $X$. We write $T \Rightarrow S$ if there exist strings $A$, $B$, $X$ such that $T = AXXB$ and $S = AXB$. We also define $T \Rightarrow_1 S$ and $T \Rightarrow_2 S$ for contractions analogously as for TDs (note that $T \Rightarrow_1 S$ if and only if $S \Rightarrow_1 T$ and $T \Rightarrow_2 S$ if and only if $S \Rightarrow_2 T$).

For two strings $A$ and $B$, if $A \Rightarrow_1 B$, we say that $A$ is an ancestor of $B$ and $B$ is a descendant of $A$. An ancestor $A$ of $B$ is a root of $B$ if it is square-free.

A run in a string is a maximal substring consisting of one or more copies of a single symbol. Given a string $S$ containing $k$ runs, the run-length encoding of $S$, denoted $RLE(S)$, is a sequence $s^1_1 s^1_2 \ldots s^1_k$ such that each $s^i_l$ indicates the $i$-th run of $S$, consisting of the symbol $s_i$ repeated $l_i$ times. We write $|RLE(S)|$ to indicate the number of runs contained in $S$, in this case $|RLE(S)| = k$. For example, given the string $S = 111001222$ we have $RLE(S) = 1^30^11^22^2$ and $|RLE(S)| = 4$.

A $q$-ary string $S$ is purely alternating if, after choosing an order for the symbols contained
in the alphabet of \( S \) and relabelling \( \Sigma(S) = \{0, ..., q - 1\} \), the first run has the symbol 0 and each run containing the \( i \)-th symbol is followed by a run containing the \((i + 1) \mod q\)-th symbol\(^3\). For example, the string 001220112 is purely alternating, but the string 01202 is not. Note that all binary strings (that up to relabelling we assume to start with 0) are purely alternating.

Given a \( q \)-ary purely alternating string \( S \), a group of \( S \) is a substring \( X \) of \( S \) containing exactly \( q \) runs of \( S \).

## 3 The Hardness of \( \text{TD-Dist} \) for alphabets of size 5

In this section we show that given two strings \( S \) and \( T \), over the same alphabet \( \Sigma \) and such that \( S \Rightarrow_{\ast} T \), finding the minimum number of duplications required to transform \( S \) into \( T \) is NP-hard when \( |\Sigma| = 5 \). Sometimes during the proof, it’s useful to imagine a sequence of duplications \( S \Rightarrow_{k} T \) as their respective sequence of contractions \( T \Rightarrow_{k} S \).

During the following proofs, we will need to create strings \( S \) and \( S' \) such that \( \text{dist}_{\text{TD}}(S, S') = \ell \), for a specific desired value \( \ell \). To do so, we begin with the following definition, recalling that \( S' \subseteq S \) means that \( S' \) is a subsequence of \( S \).

\[ ▶ \textbf{Definition 1.} \ A \text{ string } S \text{ is almost square-free if there exists a square-free string } S_{SF} \text{ such that } S_{SF} \subseteq S \subseteq S_{SF}' , \text{ where } S_{SF}' \text{ is the string obtained from } S_{SF} \text{ by duplicating every single character}. \]

For example, the string 01120022 is almost square-free, while the strings 01122201 and 0012212 are not.

\[ ▶ \textbf{Lemma 2.} \ Let Z be an almost square free string. Then, the only contractions possible on Z are of size 1, i.e., those that remove one of two consecutive equal characters. \]

\[ \textbf{Proof.} \ \text{We argue by contradiction. Let } Z_{SF} \text{ be the square free string such that } Z_{SF} \subseteq Z \subseteq Z_{SF}' , \text{ where } Z_{SF}' \text{ is the string obtained from } Z_{SF} \text{ by duplicating every single character.} \]

Suppose that a contraction of size \( \geq 1 \) is possible on \( Z \). Hence \( Z = \text{ADDDB} \) with \( |D| > 1 \). Let us indicate by \( x^R \) each character in \( DD \) that is a copy of an original character \( x \) of \( Z_{SF} \) and has been added when producing \( Z_{SF}' \). Let us also indicate by \( D^{(1)} \) and \( D^{(2)} \) the two copies of \( D \) in \( DD \). We have two cases according to whether the last character of \( D^{(1)} \) and the first character of \( D^{(2)} \) are copies or not. We will show that in either case we reach a contradiction.

\[ \text{Case 1. } D^{(1)} = wx \text{ and } D^{(2)} = x^Rw'. \text{ Let } z \text{ denote the last character of } w. \text{ Note that } z \neq x \text{ for otherwise we would have a repetition in } Z_{SF}. \text{ The same argument shows that the first character, say } y, \text{ of } w' \text{ must be different from } x. \text{ Then, we must have } D^{(1)} = xyw\hat{x} \text{ and } D^{(2)} = x^Ryw\hat{x}. \text{ If we now remove from } D^{(1)}D^{(2)} \text{ all the copied characters (which were not originally in } Z_{SF} \text{) we get that } xyw\hat{x}y\hat{w}x \text{ is a substring of } Z_{SF}, \text{ where } \hat{w} \text{ is the version of } w \text{ without duplicates. We can easily see that we got a square contradicting the square free hypothesis on } Z_{SF}. \]

\[ \text{Case 2. } D^{(1)} = wx \text{ and } D^{(2)} = yw', \text{ for some } x \neq y. \text{ Hence, we must have } D^{(1)} = y\hat{w}x \text{ and } D^{(2)} = y\hat{w}x, \text{ for some word } \hat{w}. \text{ If we remove, as in the previous case, all the duplicates from } D^{(1)} \text{ and } D^{(2)}, \text{ we get that } y\hat{w}xw\hat{x} \text{ is a substring of } Z_{SF}, \text{ for some word } \hat{w} \text{ that does not start with } y \text{ and does not end with } x. \text{ This is again a contradiction since } y\hat{w}xw\hat{x} \text{ is a square, hence it cannot be a substring of } Z_{SF}. \]

\[ \]
Lemma 3. Let $S_{SF}$ be a square-free string and $S_{SF}'$ be the string obtained from $S_{SF}$ by duplicating every single character. If $S$ is an almost square-free string such that $S_{SF} \subseteq S \subseteq S_{SF}'$, then $\text{dist}_{TD}(S_{SF}, S) = |S| - |S_{SF}|$.

Proof. Note that if we contract $S$ into $S_{SF}$ by deleting single characters one by one, then we need $|S| - |S_{SF}|$ contractions. Let $\ell = |S| - |S_{SF}|$.

To see that this is also a lower bound, let us consider an arbitrary sequence of contractions $S_{SF} \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_{\ell - 1} \rightarrow S_{SF}$. It is not hard to see that for each $i = 1, \ldots, \ell - 1$ the string $S_i$ is almost square free and $S_{SF} \subseteq S_i \subseteq S_{SF}'$. Then, by the previous lemma, it follows that each contraction removes a single character, hence $\ell' \geq \ell$, which concludes the proof.

Note that as a particular case of the previous lemma, we have that $\text{dist}(S_{SF}, S_{SF}') = |S_{SF}|$.

The Cost-Effective Subgraph problem. In order to show the hardness of TD-Dist, we reduce from the Cost-Effective Subgraph problem described below, whose hardness was shown in [9].

Cost-Effective Subgraph Problem (CESG)

Input: A graph $G = (V, E)$, an integer cost $c \in \mathbb{N}_{>0}$, and an integer cost bound $k$.

Question: Is there a set of vertices $X \subseteq V$ such that, denoting by $E(X)$ the edges inside of $X$, i.e., $E(X) = \{uv \in E : u, v \in X\}$, we have $\text{cost}(X) = c \cdot (|E(G)| - |E(X)|) + |X| \cdot |E(X)| \leq k$?

Theorem 4. [9] The Cost-Effective Subgraph problem is NP-hard.

To get some intuition on where the hardness of this problem lies, note that edges "outside" of $X$ have cost $c$ and edges "inside" have costs $|X|$. Therefore, the problem is to find some balance between the size of $X$ and the number of edges it induces.

In the reduction to TD-Dist, paying for the edges is mapped to having to operate many contractions. In particular, the structure of the input strings $S$ and $T$ will be such that an initial set of contractions $X$ must be chosen. Each substring will have two ways to be contracted: one of cost $c$ and the other of cost $|X|$.

The idea closely follows the reduction in [9] for the unbounded alphabet case. In that case, the string $S$ and in particular its component substrings (the gadgets used to reduce from the graph problems) are made of distinct characters. The string $T$ is basically composed of the same substrings after duplicating each single character, i.e., going from a substring $X$ to $X'$ or to some $X''$, such that $X \subseteq X' \subseteq X''$. This is in particular useful to control the cost of the available and necessary contractions. The key idea we employ for making the reduction work in the case of an alphabet of size 5 is to substitute substrings of a square free ternary string. These substrings are appropriately chosen, in order to avoid that when they are put together in the input strings $S$ and $T$ some squares or unwanted duplications are produced. The only contractions allowed will be those satisfying the desiderata of the original proof of [9]. For controlling the number of available and necessary contractions, we rely on the property of (almost) square free strings recorded in Lemma 3. However, we also need to overcome some additional non trivial technical hurdles, that make the hard part of the reduction (equivalence in the direction from the string problem to the graph problem) more involved than the original proof in [9].

3.1 A reduction from CESG to TD-DIST on alphabets of size 5

In this section we will prove the following theorem.
Theorem 5. The TD-Dist problem is NP-complete over alphabets of size 5.

To see that the problem is in NP, note that, if \( S \Rightarrow T \), then \( \text{dist}_{TD}(S, T) \leq |T| \) because each contraction from \( T \) to \( S \) removes at least one character. Thus a sequence of contractions (equivalently, duplications) has polynomial size and can also be clearly verified in polynomial time.

As anticipated in the previous section, for the hardness proof, we reduce from the Cost-Effective Subgraph problem. Let \( (G, c, r) \) be an instance of Cost-Effective Subgraph, letting \( n := |V(G)| \) and \( m := |E(G)| \). Recall that \( c \) is the “outsider edge” cost and the question is whether there is a subset \( X \subseteq V(G) \) such that \( c(m - |E(X)|) + |X||E(X)| \leq r \).

We denote \( V(G) = \{v_1, \ldots, v_n\} \) and \( E(G) = \{e_1, \ldots, e_m\} \). The ordering of vertices and edges is arbitrary but fixed for the rest of the proof. In the construction of the gadget used in the proof, it will be useful to have the possibility to refer to an edge \( e_i \) also as \( e_i = e_{i + lm} \) for every integer \( l \geq 0 \). More precisely, when talking about edge \( e_k \) with an index \( k > m \), we mean the edge \( e_{((k-1) \mod m)+1} \).

The structure of the reduction. We show how to construct \( S \) and \( T \). We fix two large integer parameters \( d, p \in \mathbb{N} \) with \( p \) being a multiple of \( m \). Precisely, it is enough to set \( d = m + 1 \), \( p = m(n + m)^{10} \), which guarantees also that these values are polynomial in the size of the CESG instance.

In order to present the structure of the input strings \( S \) and \( T \) for TD-Dist, we need to prepare some smaller strings which will constitute the building blocks for \( S \) and \( T \). These building blocks are obtained by iteratively “slicing off” suffixes from a large square-free string, denoted by \( O \): the first building block will be a suffix of \( O \); the second will be a suffix of the (square free) string obtained after removing the first building block from \( O \); the third building block will be a suffix of the (square free) string obtained after removing the first and second building block from the suffix of \( O \); etc.

For concreteness, let us define \( O \) to be a ternary square-free string of length at least \( p^{10} \) over the alphabet \( \{0, 1, 2\} \). Note that creating \( O \) takes polynomial time using a square-free morphism, for example Leech’s morphism \( 0 \Rightarrow 1, 1 \Rightarrow 2, 2 \Rightarrow 20 \). Let us now proceed to define the building blocks. Refer to Fig. 1 for a pictorial description of this process.

![Figure 1](image.png)

For each \( i \in [n] \), starting from \( n \) and going down to 1, we define the string \( X_i \) to be the suffix of length \( d \) of \( O \) after removing the suffix \( X_{i+1}X_{i+2} \ldots X_n \). We also let \( X_i^d \) be the almost square-free string obtained from \( X_i \) by duplicating each single character, and such that (by Lemma 3) \( \text{dist}_{TD}(X_i, X_i^d) = d \).

Let \( O^{-X} \) be the string obtained from \( O \) after removing the suffix \( X_1X_2 \ldots X_n \).

We define \( B_0 \) as the minimum suffix of \( O^{-X} \) such that \( |B_0| \geq \max(2d - 2, dn + 2d - 1) \) and that doesn’t start with the character 0. We then define \( B_0^d \) as the minimum suffix of...
the string obtained by removing the suffix $B_{0'}$ from $O^{-X}$ such that $|B_{0'}| > |B_0|$ and that doesn’t start with 0. We also define the strings $B_{0'}^*$ and $B_{0'}^*$ to be almost square-free and:

\[ \text{dist}_{TD}(B_{0'}, B_{0'}) = dc + 2d - 2 \] (1)

\[ \text{dist}_{TD}(B_{0'}, B_{0'}^*) = dn + 2d - 1 \] (2)

To create $B_{0'}$ and $B_{0'}^*$, we simply duplicate in $B_{0'}$ and $B_{0'}^*$ the appropriate number of single characters.

Let $O^{-X-B_{0'}-B_{0'}}$ the string obtained by removing from $O$ the suffix $B_{0'}B_{0'}X_1\ldots X_n$.

For $i=1, \ldots 2p$ we define the string $B_i$ as the minimum suffix of the string obtained from $O^{-X-B_{0'}-B_{0'}}$ after removing the suffix $B_{i-1}B_{i-2} \ldots B_1$ and such that:

- $|B_i| > |B_{i-1}|$ (for $i = 1$, we require $|B_1| > |B_{0'}|$);
- if $i \neq 2p$ then $B_i$ does not start with 0;
- if $i = 2p$ then $B_i$ starts with 0.

Notice that the length of each string $B_{0'}, B_{0'}$, $B_1$, $B_2$, $\ldots$, $B_{2p}$ is incremental, all the strings are ternary square-free and the only $B_j$ string that starts with the character 0 is $B_{2p}$.

We now use these strings as building blocks to create larger strings. We will use two additional characters, denoted $L$ and $\$$, to separate the components of our strings and to control the type of repeats that are created, equivalently, the contractions that will be possible.

For each $q \in [2p]$, we define:

\[ B_q = B_q^0B_{q-1}^0 \ldots B_2^0B_1^0B_{0'}^0B_{0'}^0B_{0'}^0 \] (3)

\[ B_q^1 = B_q^1B_{q-1}^1 \ldots B_2^1B_1^1B_{0'}^1B_{0'}^1B_{0'}^1 \] (4)

Note that $B_q$ is square free since it is equivalent to a suffix of $O$ in which we have inserted characters not in the alphabet of $O$. Also, we have that $B_q^1$, $B_q^0$, $B_q^0$ are almost square free and by Lemmas 3, 2 we also have that:

- $\text{dist}_{TD}(B_q, B_q^0) = \text{dist}_{TD}(B_{0'}, B_{0'}^0) = dc + 2d - 2$
- $\text{dist}_{TD}(B_q, B_q^1) = \text{dist}_{TD}(B_{0'}, B_{0'}^1) = dn + 2d - 1$
- $\text{dist}_{TD}(B_q^1, B_q^0) = \text{dist}_{TD}(B_{0'}, B_{0'}^0) = dc + 2d - 2$
- $\text{dist}_{TD}(B_q^0, B_q^0) = \text{dist}_{TD}(B_{0'}, B_{0'}^1) = dn + 2d - 1$

We then define the strings:

\[ X = X_1X_2\ldots X_n \]

\[ X^d = X_1^d X_2^d \ldots X_n^d \] (5)

and use these strings to build larger blocks which will be employed to construct the gadget representing the edges of the graph $G$. For each edge $e_i = v_a v_b$, where $i \in [p]$ and where $v_a$ and $v_b$ are the incident vertices, we let:

\[ X_{e_i} = X_1^d \ldots X_{a-1}^d X_a X_{a+1}^d \ldots X_{b-1}^d X_b X_{b+1}^d \ldots X_n^d \]

Thus in $X_{e_i}$, all $X_k$ substrings are “doubled”, i.e., turned into $X_k^d$, except $X_a$ and $X_b$, those whose indices coincide with those of the incident vertices.

For $i \in [p]$, define:

\[ E_i = B_i^0X_{e_i}LB_{2p}^1X_L \]

which we call edge gadget.
With the above building blocks, we are finally ready to define the strings $S$ and $T$:

$$S = B_{2p}X_1L = B_{2p}S_2B_{2p-1}S \ldots \cdot B_2S_1B_1S_BB_0S_BX_1X_2 \ldots X_nL$$

$$T = B_{2p}^{0}X^{cd}LB_{2p}^{1}X_1L'E_1E_2 \ldots E_p$$

$$= B_{2p}^{0}X^{cd}LB_{2p}^{1}X_1L'B_1^{01}X_{e_1}LB_{2p}^{1}X_1L'B_2^{01}X_1L \ldots [B_{2p}^{01}X_{e_p}LB_{2p}^{1}X_1L]$$

where, in the second line, the brackets $[]$ are only added for the sake of highlighting the substring which will be significant in the following arguments, but they are not actual characters of $T$.

Observe that

- the string $S$ is square-free, since it coincides with a suffix of the square-free string $O$ in which new characters $S$ and $L$ have been added, which are not from the alphabet of $O$.
- for every $a \in \{0, 1, 01\}$, every string $B_{p}^{a}X^{d}L$ is almost square-free, since it can be obtained by starting from a suffix of the square-free string $S$ and doubling a subset of its characters.

**Proof of Theorem 5.** The result will be consequence of showing that $G$ has a subgraph $W$ of cost at most $r$ if and only if $T$ can be contracted to $S$ using at most $p/m \cdot d(r + nm) + 4cdn$ contractions.

In the remaining part of this section we provide the proof of the only if ($\Rightarrow$ direction), and defer the more involved ($\Leftarrow$ direction) to the appendix.

($\Rightarrow$ direction). Suppose that $G$ has a subgraph $W$ of cost at most $r$. Thus $c(m - |E(W)|) + |W||E(W)| \leq r$. We will show a sequence of contractions to convert $T$ into $S$.

**Contracting edge gadgets ‘out of’ $W$.** First, for each edge $e_i$ with at least one incident vertex not in $W$ we will use $dn + dc$ contractions to remove the gadget substring $B_{e_i}$ from $T$. These contractions will act on the part of $T$ constituting the substring $B_{2p}^{0}X_1L'E_i = B_{2p}^{1}X_{e_i}LB_{2p}^{1}X_1L$, where, again, we added brackets only to indicate the occurrence of $E_i$ that will be removed. By $(1)-(4)$, we can first contract $B_{2p}^{0}$ to $B_{2p}^{1}$ using $dc + 2d - 2$ contractions, then contract $X_{e_i}$ to $X$ using $d(n - 2)$ contractions. The result is the $B_{2p}^{1}X_1L'B_{2p}^{1}X_1L$ substring, which becomes $B_{2p}^{1}X_1L$ using two contractions (see below). This sums up to $dc + 2d - 2 + dn - 2d + 2 = dc + dn$ moves. Summarizing, the sequence of contractions works as follows (here the brackets surround the $E_i$ substring and what remains of it)

$$B_{2p}^{1}X_{e_i}LB_{2p}^{1}X_1L \Rightarrow B_{2p}^{1}X_1L'[B_{2p}^{1}X_{e_i}LB_{2p}^{1}X_1L] \quad (dc + 2d - 2 \text{ contractions})$$

$$\Rightarrow B_{2p}^{1}X_1L'[B_{2p}^{1}X_1L] \quad (d(n - 2) \text{ contractions})$$

$$= B_{2p}S_{2p-1}S \ldots \cdot B_{1+1}S_{1+1}S_{1}X_1L'B_{2p}^{1}X_1LB_{2p}^{1}X_1L$$

$$\Rightarrow B_{2p}S_{2p-1}S \ldots \cdot B_{1+1}S_{1+1}S_{1}X_1L'[B_{2p}^{1}X_1L] \quad (1 \text{ contraction})$$

$$= B_{2p}^{1}X_1L[B_{2p}^{1}X_1L] \quad (1 \text{ contraction})$$

It is to be noted that after these contractions, each remaining $E_i$ gadget substring is still preceded by $B_{2p}^{1}X_1L$. This guarantees that we can repeat the same procedure to remove the gadget $E_i$ for each $e_i$ incident to at least one vertex non in $W$ (including the $E_i$ gadgets for which $i > m$). Overall, the procedure is repeated on $p/m \cdot (m - |E(W)|)$ gadgets.

After this, the only $E_i$ gadget substrings still present in the resulting string $T'$, are those for which $e_i$ has both incident vertices in $W$. Let $E(W)$ denote the set of edges with both incident vertices in $W$ and $1 \leq i_1 < i_2 < \ldots < i_\ell \leq p$ be the indices of these edges. Notice that $\ell$ counts the number of all $\frac{p}{m}$ copies of the edges in $E(W)$, i.e., $\ell = \frac{p}{m} |E(W)|$. Then we have

$$T' = B_{2p}^{0}X^{cd}LB_{2p}^{1}X_1L'B_{1+1}^{01}X_{e_{i_1}}LB_{2p}^{1}X_1L'B_{1+2}^{01}X_{e_{i_2}}LB_{2p}^{1}X_1L \ldots [B_{2p}^{01}X_{e_{i_\ell}}LB_{2p}^{1}X_1L]$$
Contracting edge gadgets *inside* $W$.

First, we use contractions to convert the leftmost substring $X^d$ into the substring $X_W$, where, for each $v_i \in W$, instead of the string $X^d_i$ we have $X_i$, i.e., if $W = \{v_1, \ldots, v_{|W|}\}$ then

$$X_W = X^d_1 \ldots X^d_{i-1} X_i X^d_{i+1} \ldots X^d_{|W|} X^d_{|W|+1} \ldots X^d_n.$$

Contracting the (only) occurrence of $X^d$ substring in $T'$ into $X_W$ requires $d|W|$ contractions.

Let $T''$ denote the resulting string.

$$T'' = B^0_{2p} X_W L B^1_{2p} \mathcal{X} [E^0_1 X_{e_1} L B^1_{2p} \mathcal{X} L] [E^0_1 X_{e_2} L B^1_{2p} \mathcal{X} L] \ldots [E^0_1 X_{e_i} L B^1_{2p} \mathcal{X} L],$$

where, for each edge $e_i = (v_j, v_k)$ ($t = 1, \ldots, \ell$) with both incident vertices in $W$, the substring $X_W$ contains the substrings $X_j$ and $X_k$ (instead of the $X^d_j$ and $X^d_k$, originally in $X^d$).

Let us now focus on the prefix of $T''$ up to the end of the edge gadget $E_{i_1}$:

$$B^0_{2p} X_W L B^1_{2p} \mathcal{X} [E_{i_1}] [E_{i_2}] \ldots [E_{i_\ell}]$$

We first contract $B^0_{2p}$ to $B^1_i$, and contract $X_{e_i}$ to $X_W$. In the resulting substring $B^0_{2p} X_W L [E^0_i X_W L B^1_{2p} \mathcal{X} L]$, one contraction suffices to remove the second half. This is summarized as follows, where for conciseness, we use $[E_{i_\ell}] = [B^0_{0i} X_{e_i} L B^1_{2p} \mathcal{X} L]$:

$$T'' = B^0_{2p} X_W L B^1_{2p} \mathcal{X} L [E_{i_1}] [E_{i_2}] \ldots [E_{i_\ell}]$$
$$= B^0_{2p} X_W L B^1_{2p} \mathcal{X} L [B^0_{i_1} X_{e_1} L B^1_{2p} \mathcal{X} L] [E_{i_2}] \ldots [E_{i_\ell}]$$
$$\Rightarrow B^0_{2p} X_W L B^1_{2p} \mathcal{X} L [B^0_{1i} X_{e_i} L B^1_{2p} \mathcal{X} L] [E_{i_2}] \ldots [E_{i_\ell}]$$
$$= B^0_{2p} X_W L B^1_{2p} \mathcal{X} L [E_{i_2}] \ldots [E_{i_\ell}]$$

Since the structure of the final string (10) is the same as the initial one (6), and for each $j = 2, \ldots, \ell$ we have $e_i \in E(W)$, we can repeat the same sequence of contractions to remove each $E_{i_j}$.

The resulting string is then $B^0_{2p} X_W L B^1_{2p} \mathcal{X} L$. We can now contract $X_W$ to $X$ using $d(n - |V|) \leq dn$ contractions and then $B^0_{2p}$ and $B^1_{2p}$ to $B_2p$, by using $dc + 2d + 2 + dn + 2d - 1 = d(c + n + 4) - 3$ additional contractions. Finally, with one more contraction of the second half of the string we obtain $S$. Therefore we have that the total the number of contractions is upper bounded by:

$$\frac{p}{m} \cdot (m - |E(W)|) \cdot (dc + dn) + \frac{p}{m} \cdot |E(W)| \cdot (dn + d|W|) + dn + d(c + n + 4) - 3$$
$$\leq \frac{p}{m} \cdot (m - |E(W)|) \cdot (dc + dn) + \frac{p}{m} \cdot |E(W)| \cdot (dn + d|W|) + 4cdn$$

as desired.

4 Polynomial time computable distance for purely alternating strings

In this section, we investigate the existence of polynomial time algorithms to decide whether a purely alternating string $S$ can be transformed into another purely alternating string $T$ through a series of duplications, i.e., if $S \Rightarrow T$. 
Definition 6. Let $S$ and $T$ be two strings with run-length encodings $\text{RLE}(S) = 0^{l_1}1^{l_2} \ldots s_{n}^{l_n}$ and $\text{RLE}(T) = 0^{l'_1}1^{l'_2} \ldots t_{m}^{l_m}$. We say that the run $s_i^{l_i}$ matches the run $t_j^{l_j}$ if $s_i = t_j$ and $l_i \leq l_j$. We also say that the string $S$ matches the string $T$ if $n = m$ and for each $i = 1, \ldots, n$ we have that $s_i^{l_i}$ matches $t_j^{l_j}$. If $S$ matches $T$ we write $S \preceq T$.

Note that the existence of a string $S'$ that matches $T$ and that satisfies $S \Rightarrow S'$, implies that $S \Rightarrow T$: we can convert $S'$ into $T$ by duplications on single letters.

Definition 7. Given two $q$-ary strings $S$ and $T$, we say that the operation $S = AXB \Rightarrow AXXB = T$ is a normal duplication if one of the following conditions holds: (i) $X$ is a $q$-ary string with exactly $q$ runs and $\text{RLE}(X) = x_1^{l_1}x_2^{l_2} \ldots x_{n-1}^{l_{n-1}}x_n^{l_n}$; (ii) $X$ is a unary string with exactly one run and $\text{RLE}(X) = x_1^{l_1}$.

Case (i) (resp. (ii)) we say that the duplication is normal of type 1 (resp. normal of type 2).

We write $S \Rightarrow_{\text{N}} T$ if there exists a normal duplication converting $S$ into $T$. More generally, we write $S \Rightarrow_{\text{k}} T$ if there exist $S_1, \ldots, S_k$ such that $S \Rightarrow_{\text{N}} S_1 \Rightarrow_{\text{N}} \cdots \Rightarrow_{\text{N}} S_k \Rightarrow_{\text{N}} T$. We also write $S \Rightarrow_{\text{N}}^* T$ if there exists some $k$ such that $S \Rightarrow_{\text{N}}^* T$.

In perfect analogy with the definition of contractions given in section 2, we define normal contractions: $T \Rightarrow_{\text{N}}^* S$ and $T \Rightarrow_{\text{N}}^* S$ by $T \Rightarrow_{\text{N}}^* S$ if and only if $S \Rightarrow_{\text{N}}^* T$.

Intuitively, normal duplications are effective in converting a string $S$ into a string $S'$ that matches a string $T$ because they keep the resulting string purely alternating and create new runs that are as small as possible; these runs allow the string $S'$ to match many strings.

We proceed to characterizing pairs of strings $S$ and $T$ such that $S \Rightarrow_{\text{N}}^* T$. Because of the space limitation, the proofs of the following technical lemmas are deferred to the appendix.

Lemma 8. Fix $2 \leq q \leq 5$. Let $S$ and $T$ be purely alternating strings over the same $q$-ary alphabet $\Sigma = \{0, 1, 2, \ldots, q-1\}$. Let $\text{RLE}(S) = 0^{l_1}1^{l_2} \ldots s_{n}^{l_n}$ and $\text{RLE}(T) = 0^{l'_1}1^{l'_2} \ldots t_{m}^{l_m}$ be the run-length encodings of the strings. Then, $S \Rightarrow_{\text{N}}^* T$ if and only if there exists a function $f : \{1, \ldots, n-q+2\} \Rightarrow \{1, \ldots, m-q+2\}$ such that:

1. $f(1) = 1$ and $f(n-q+2) = m-q+2$
2. $f(i) = j \Rightarrow s_i = t_j$ and for each $u = 0, \ldots, q-2$ we have that $l_{i+u} \leq l'_{j+u}$
3. $f(i) = j$ and $f(i') = j'$ and $i < i' \Rightarrow j < j'$
4. if $q = 5$ and $f(i) = j$ and $f(i+1) = j' \neq j+1 \Rightarrow$ there exists a substring $M$ in $T$ starting in a position $p$ such that $j \leq p \leq j'$ with the form $M = s_{i+3}^{l_{i+3}}s_{i+4}^{l_{i+4}} \ldots s_{i+q}^{l_{i+q}}s_{i+q+1}^{l_{i+q+1}}$ such that for each $u = 0, 1, \ldots, q-3$ it holds that $l_{i+3+u} \leq l'_{p+u}$ and $l_{i+1} \leq l'_{p+q-2}$.

The next step in our analysis consists in showing that for alphabets of size $\leq 5$ normal duplications are as powerful as all possible duplications. We use the following claim (whose proof is deferred to the appendix).

Claim 9. Let $S$ and $T$ be $q$-ary purely alternating strings such that there exists a series of duplications $S \Rightarrow_{\text{N}}^* T$. Then for each duplication, if we call the duplicated substring $X$, we have that $|\text{RLE}(X)| \mod q \leq 1$.

Lemma 10. Let $S$ and $T$ be purely alternating strings over the same alphabet $\Sigma$ of size $\leq 5$. Then $S \Rightarrow_{\text{N}}^* T$ if and only if $S \Rightarrow_{\text{N}}^* T$.

Proof. Exploiting Claim 9 we can show that any duplication in a string over an alphabet of size $\leq 5$ can be simulated by normal duplications. We limit ourselves to show here the complete argument for for the case of alphabet of size 5 where the use of all properties in Lemma 8 is more explicit. The remaining cases are deferred to the appendix.
Claim. Let $S$ and $T$ be a quinary purely alternating strings over the alphabet $\Sigma = \{0,1,2,3,4\}$. If there exists a duplication $S \Rightarrow T$, then we can create a series of normal duplications $S \Rightarrow \cdots \Rightarrow T'$ such that $T'$ matches $T$.

Proof. Let $S = AXB \Rightarrow AXXB = T$ be the original duplication. Like before, we create a string that matches $XX$ starting from $X$ through normal duplications, depending on how many runs are contained in $X$. By Claim 9 the only possible cases are $|RLE(X)| \mod 5 \in \{0,1\}$

Case 1. $|RLE(X)| = 1$. It means that the only effect of the original duplication is to extend one of the runs of $S$. For this reason we know that $S$ already matches $T$.

Case 2. $|RLE(X)| \mod 5 = 0$, we suppose that the string $X$ starts with a 0 (rotate the characters if it starts with any other symbol), so the run-length encoding of $X$ is in the form

$RLE(X) = 0^l_11^l_22^l_30^l_4\ldots1^n_{l-1}2^n_{l}$.

If $|RLE(X)| = 10$, we use the following sequence of normal duplications to match $XX$:

$XX = 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

Here we see a problem: for $X'$ to match $XX$, we must have $l_4 \leq l_9$. If we are in the case that $l_4 > l_9$, we can apply a different series of normal duplications:

$XX = 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

$\Rightarrow 0^l_11^l_22^l_34^l_50^l_61^l_72^l_83^l_94^{1l}_{10}$

In this case $X'$ matches $XX$ if $l_4 \geq l_9$, therefore for $|RLE(X)| = 10$ we can construct the series of normal duplications in both cases.

Finally, let us assume that $X$ contains $5r$ runs for some $r > 2$. Then, in order to produce a string through normal duplications that matches $XX$, it suffices to execute the duplications explained before, then continue with $|RLE(X)|/5 - 2$ normal duplications containing the four adjacent runs of length 1 plus another adjacent run. This pushes to the right the original runs of $X$ remaining, together with $3^l_{14}$ (in the first case) or $3^l_{19}$ (in the second case). We can see that $3^l_{14}$ in $X'$ will be in the same position as $3^l_{19}$ in $XX$ and vice versa. $\diamond$

*Proof.* The algorithm computes the run length encoding of $S$ and $T$ and then decides about the existence of the function $f$ satisfying the properties of Lemma 8. By Lemma 10 we have

---

**Theorem 11.** Let $\Sigma$ be an alphabet of size $\leq 5$. There exists a algorithm that for every pair of purely alternating strings $S$ and $T$ over $\Sigma$ can decide in linear time whether $S \Rightarrow T$.

**Proof.** The algorithm computes the run length encoding of $S$ and $T$ and then decides about the existence of the function $f$ satisfying the properties of Lemma 8. By Lemma 10 we have
that \( S \Rightarrow T \) if and only if \( S \Rightarrow^* T \). By Lemma \( \text{[8]} \) this latter condition holds if and only if there exists a function \( f \) satisfying the conditions 1-4 in Lemma \( \text{[8]} \).

Therefore, to prove the claim it is enough to show that the existence of such a function \( f \) can be decided in linear time. This is easily attained by employing the following greedy approach (see, e.g., Algorithm \( \text{[1]} \) in appendix): once the values of \( f(1) = 1, \ldots, f(i-1) = j \) have been fixed, sets the assignment \( f(i) = j' \) to the smallest \( j' \) such that \( l_i + u \leq l_{j'+u} \) for each \( u = 0, \ldots, k-1 \) and if this condition does not hold for \( j' = j + 1 \), then \( j' \) is the smallest integer \( j' > j \) that guarantees the existence of a \( j < p < j' \) satisfying condition 4 in Lemma \( \text{[8]} \). The correctness of this approach can be easily shown by a standard exchange argument (see, e.g., \( \text{[8]} \)) and it is deferred to the appendix for the sake of the space limitations. The resulting algorithm takes \( O(|S| + |T|) \) time, since it only needs to scan a constant number of times each component of the run length encoding of \( T \) and \( S \).

4.1 A final remark on senary strings and some open problems

It turns out that the technique we used to prove Lemmas \( \text{[8]} \), \( \text{[10]} \) is not generalizable to the case of larger alphabets: For purely alternating strings with \( |\Sigma(S)| = 6 \), we can’t always simulate a general duplication with normal duplications. For example, take a duplication \( AXB \Rightarrow AXXB \) where \( X \) is the string: \( X = 0^2, 1^1, 2^2, 3^1, 4^2, 5^1, 0^1, 1^2, 2^3, 4^1, 5^2 \).

By exhaustive computer search, we were able to check all possible normal duplications: since each normal duplication adds six runs, it suffices to check series of two normal duplications: No pair of normal duplications can to convert the string \( X \) into a string \( T' \preceq XX \). In the proof of Lemma \( \text{[10]} \) for alphabets of size 5, one hurdle to overcome was the relationship between the length of two runs of \( XX \), namely \( l_4 \) and \( l_9 \). But according to the direction of the inequality involving these quantities there was the possibility to choose alternative first normal duplications. In the case of senary strings, there appear to be more inequalities to consider and the failure of the exhaustive search might be explained by the existence of duplications where some inequalities simultaneously hold that would impose incompatible choices of normal duplications. We don’t know whether this has an implication on the polynomial time solvability of the problem \( \text{TD-Exist} \) already in this simple special case, and we leave it as a first step for future research. More generally, the main algorithmic problems that are left open by our results are the complexity of \( \text{TS-Dist} \) for binary alphabets (more generally, whether our hardness result can be extended to smaller alphabets) and the complexity of \( \text{TD-Exist} \) for arbitrary ternary alphabets. Also, on the basis of the hardness result, approximation algorithms for the distance problem is another interesting direction for future research.

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Appendix

A Proof of Theorem 5, \((\Leftarrow)\) direction

Assume that there exists a sequence of \(N \leq p/m \cdot d(r + nm) + 4cdn\) contractions \(T \mapsto T(1) \mapsto T(2) \mapsto \cdots \mapsto T(N) = S\), where \(T(l)\) denotes the string obtained from \(T\) after the first \(l\) contractions have been performed (for definiteness, we let \(T(0) = T\)).

For the sake of the analysis, it is convenient to think that for each string \(T(l)\) encountered in this process, the characters of \(T(l)\) are spread over \(|T|\) positions, with some positions left blank if \(|T(l)| < |T|\).

In particular, when a contraction \(T(l) = ADDB \mapsto ADB = T(l + 1)\) is performed, we imagine that the positions occupied in \(T(l)\) by the characters of the second copy of \(D\) are simply left empty in the \(T(l + 1)\). Therefore, every character of \(T(l)\) is uniquely identified with the (same) character of \(T\) that occupies the same position. If the position of a character (or a substring) \(\gamma\) of \(T\) is empty in \(T(l)\) we say that \(\gamma\) has been removed. Conversely, if a character or a substring \(\gamma\) of \(T(l)\) occupies some of the positions of a substring \(\gamma'\) of \(T\) we say that \(\gamma\) belongs to \(\gamma'\). This way we can keep track of the the evolution of each single character of \(T\) through the intermediate step of the sequence of contractions.

A block of \(T(l)\) is a maximal substring \(P\) of \(T(l)\) satisfying the following properties: the last character of \(P\) is \(L\); and this is the only occurrence of \(L\) in \(P\). Hence, the first character of \(P\) is either preceded by \(L\) or it is the first character of \(T(l)\).

\(\triangleright\) Claim 12. Each \(E_i\) substring must be removed in \(T(\alpha)\).

Proof. Let \(BXL\) be the first, leftmost block \(B^0_{2p}X\) of \(T\). We show that for each run \(r\) contained in \(BXL\), at least a character contained in \(r\) is not removed in \(T(N)\). Assume that \(l\) is the smallest integer for which in \(T(l)\) an entire run of \(BXL\) is removed. Let \(D\) be the string that was contracted from \(T(l - 1)\) to \(T(l)\) (so that \(T(l - 1)\) contained \(DD\) as a substring, and the second \(D\) substring gets removed from \(T(l - 1)\)). The first character removed must be in \(BXL\), otherwise we can’t remove any of its runs. If \(D\) does not contain \(L\), then \(DD\) is entirely contained in \(BXL\), but this is not possible since \(BXL\) is almost square-free, so at most we could have that \(DD = aa\), where \(a\) is a single character. If the right \(D\) contains the \(L\) character, then also the left \(D\) must contain \(L\), but this is not possible by construction. We can see that the number of runs contained in \(BXL\) is the same as the number of runs contained in \(S\). Therefore, all the characters of \(S\) belong to \(BXL\), implying that every character in the string \(T\) that doesn’t belong to \(BXL\) is removed in \(T(N)\), in particular all the characters of every \(E_i\) substring.

Notice that in \(T\), \(E_i = B^0_{3i}X_{e_i}L\) has two blocks. We write \(E'_i = B^0_{3i}X_{e_i}L\) to denote the first block of \(E_i\). We let \(E'_i(l)\) be the substring of \(T(l)\) formed by all the characters that belong to \(E'_i\). Noting that \(E'_i(l)\) is any possibly empty subsequence of \(E'_i\).

For any \(a \in \{0, 1, 01\}\) and \(j \in [2p]\), a block \(BXL\) is called a \(B^j_{3p}X\)-block if \(B_jX\) is a subsequence of \(BXL\) and \(BXL\) is a subsequence of \(B^j_{3p}X\). In other words, \(BXL\) has the same runs of \(B^j_{3p}X\) in the same order, but some of its duplicated characters may have been contracted into a single character. A \(B^j_{3p}X\)-cluster is a string obtained by concatenating an arbitrary number of \(B^j_{2p}X\)-blocks. We write \((B^j_{3p}X)^*\) to denote a possibly empty \(B^j_{3p}X\)-cluster.
Claim 13. For any \( l \), \( T(l) \) has the form:

\[
BXL(B_{2p}^0XL)^*E_{i_1}(l)(B_{2p}^1XL)^*E_{i_2}(l)(B_{2p}^2XL)^* \ldots E_{i_h}(l)(B_{2p}^1XL)^*
\]

where:
- \( BXL \) is a \( B_{2p}^0XL \)-block
- \( i_1 < i_2 < \cdots < i_h \)
- each \((B_{2p}^1XL)^* \) is a \( B_{2p}^1XL \)-cluster
- for each \( j \in \{i_1, \ldots, i_h\} \), \( E_{i_j}(l) \) is a \( B_{2p}^1XL \)-block.

**Proof.** The statement is true for \( l = 0 \) by construction. Assume the claim is false and, like before, let \( l \) be the smallest integer for which \( T(l) \) is a counter-example to the claim. Thus we may assume that \( T(l - 1) \) has the same form as in the claim statement. Let \( D \) be the string that was contracted from \( T(l - 1) \) to \( T(l) \). If \( D \) does not contain a \( L \) character, then \( DD \) is entirely contained in a single block. But each block is almost square-free, so \( DD = aa \) for some character \( a \). We can see that a contraction like this can’t remove an entire run of the block, so each \( B_{2p}^0XL \)-block will remain a \( B_{2p}^0XL \)-block. Assume instead that the last character of \( D \) is \( L \). Then \( DD = D'LDL \) for some string \( D' \), and removing the second \( D'L \) half only removes entire blocks of \( T(l - 1) \). As this block cannot be the first block \( BXL \) and since each \( E_{i_j}(l - 1) \) is itself a block, this preserves the form of the claim.

Therefore we may assume that the last character of \( D \) is not \( L \), but that \( D \) has at least one \( L \) character. It’s easy to see that the second condition is preserved. For the other conditions, we have four cases to consider depending on which blocks the leftmost character and rightmost character removed belongs to, i.e., where the right half of \( DD \) starts and ends in \( T(l - 1) \).

1. The leftmost character removed belongs to \( BXL \). We are assuming that \( D \) contains the character \( L \), so the right \( D \) contains the \( L \) at the end of \( BXL \). This is the only \( L \) contained in \( BXL \) thus the left \( D \) can’t contain one, a contradiction.

2. The leftmost character removed belongs to a \( E_{i_j}(l - 1) \) substring for some \( j \in [h] \) and the rightmost character removed belongs to a \( B_{2p}^1XL \)-cluster. We show that a duplication of this type is impossible. Considering that \( D \) contains a \( L \), the right half of \( DD \) must contain the \( L \) of \( E_{i_j}(l - 1) \). If the leftmost character removed is the first character of \( E_{i_j}(l - 1) \), then the left \( D \) ends with a \( L \). But we are assuming that \( D \) doesn’t end with \( L \), hence we know that the first character removed is not the first character of \( E_{i_j}(l - 1) \). Furthermore, since the right \( D \) contains a \( L \), also the left \( D \) must contain an \( L \). We know that the first character of the right \( D \) isn’t the first character of \( E_{i_j}(l - 1) \), therefore the left \( D \) contains at least a character after the \( L \). Let’s call \( D' \) (resp. \( D'' \)) the maximum suffix of the right (resp. left) \( D \) that doesn’t contain \( L \). We can then visualize the contraction as follows:

\[
T(l - 1) = \underbrace{E_{i_j}(l - 1)(B_{2p}^1XL)^*}_{D} \underbrace{(B_{2p}^1XL)^*}_{D'} \underbrace{E_{i_{j+1}}(l - 1)(B_{2p}^1XL)^*}_{D''} \ldots
\]

For \( DD \) to be a valid contraction, we must have that \( D' = D'' \), but this is impossible since no \( E_{i_j}(l - 1) \) string starts with 0, while \( B_{2p}^1 \) starts with 0 by construction, hence a contradiction.

3. The leftmost character removed belongs to a \( E_{i_{j+k}}(l - 1) \) substring for some \( j \in [h] \) and the rightmost character removed belongs to \( E_{i_{j+k}}(l - 1) \) for some integer \( k \) such that
1 ≤ k ≤ h − j. Assume that k > 1, then we have that LE_{i,j+1}(l − 1) is a substring of the right D. This implies that also the left D must contain this substring, but this is impossible because the number of runs of each block E′_i(l − 1) is different and each B_{2p}\mathcal{X}L-block has more runs than E_{i,j+1}(l − 1) by construction. Therefore we know that the rightmost character removed belongs to E′_{i,j+1}(l − 1). To summarize, so far we showed that if the leftmost character removed belongs to E′_i(l − 1) and all the previous assumptions hold, the contraction looks like this:

\[
T(l − 1) = \ldots LE_{i,j}(l − 1)(B_{2p}\mathcal{X}L)^* E_{i,j+1}(l − 1) \ldots
\]

We now show that the leftmost character of DD belongs either to the B_{2p}\mathcal{X}L-cluster exactly before E′_i(l − 1), or to the first block. Assume that the leftmost character of DD belongs to E′_{i,j−1}(l − 1) (we know that it can’t belong to any E′_{i,j−k}(l − 1) with k > 1). Then if the leftmost character removed is the first $ of E′_i(l − 1) = B_{i,j}$B_{i,j−1}$B_{i,j−2}$… or any character further left, we know that the right D must contain the substring $B_{i,j}$,$ and therefore also the left D must contain it. This is impossible, since every substring B_j has a different number of runs, E′_{i,j−1}(l − 1) = B_{i,j−1}$B_{i,j−2}$… is a block, so it is preceded by a Ł, and every E′_{i,j−k}(l − 1) with k > 1 doesn’t contain $B_{i,j−1}$ as a substring by construction. If the leftmost character removed is more to the right than the first $ of E′_i(l − 1), then LB_{i,j}$ is a substring of the left D, therefore also the right D must contain this substring. This means that there exists a string E′_{i,j+k}(l − 1) = B_{i,j+k}$B_{i,j+k−1}$… with k > 0 that contains B_j$ as a prefix, but this is impossible by construction. Therefore we showed that the leftmost character of DD belongs either to the B_{2p}\mathcal{X}L-cluster exactly before E′_i(l − 1), or to the first block.

We now prove that the result of a contraction like this is the same as removing the E′_i(l − 1) string (possibly along with some B_{2p}\mathcal{X}L-blocks). As before, we call D′ (resp. D′′) the maximum suffix of the right (resp. left) D that doesn’t contain Ł. The contraction now can be imagined as:

\[
T(l − 1) = \ldots (B_{2p}\mathcal{X}L)(B_{2p}\mathcal{X}L)^* E_{i,j}(l − 1)(B_{2p}\mathcal{X}L)^* E_{i,j+1}(l − 1) \ldots
\]

Notice that the leftmost character removed in E′_i(l − 1) = B_{i,j}$B_{i,j−1}$B_{i,j−2}$… must be more to the left than the first $ of E′_i(l − 1) for an argument similar to the one used before. We know that D′ = D′′, otherwise DD couldn’t be a valid contraction. Then if we imagine deleting the right D, we can see that the form of the resulting string T(l) is equal to T(l − 1) without E′_i(l − 1) (and some B_{2p}\mathcal{X}L-blocks). This implies that the form of the claim is preserved.

4. The leftmost character removed belongs to a B_{2p}\mathcal{X}L-cluster. Assume that the rightmost character removed belongs to a different (B_{2p}\mathcal{X}L)^*-cluster. In this case, for some j > 0, the right D must contain LE′_j(l − 1) as a substring, but as shown before this is impossible. Assume instead that both the leftmost and the rightmost characters removed belong to the same B_{2p}\mathcal{X}L-cluster. We know that the leftmost character removed isn’t the first of a block. Let’s call D′ the maximum suffix of the right D that doesn’t contain Ł and D′′ the minimum prefix of the right D that contains Ł. We can see that the contraction must have the following structure:
Notice that removing the right $D$, the resulting string is still a $B_{2p}^1\mathcal{XL}$-cluster because $D'D'' = B_{2p}^1\mathcal{XL}$, hence the structure of the claim is preserved.

If the rightmost character removed doesn’t belong to any $B_{2p}^1\mathcal{XL}$-cluster, it must belong to $E_j'(l - 1)$ for some $j > 0$. We know that $E_j'(l - 1)$ must be the block immediately after the cluster. Let $D'$ (resp. $D''$) be the maximum suffix of the right (resp. left) $D$ that doesn’t contain $L$. The contraction must have the following form:

$$T(l - 1) = \ldots \underbrace{(B_{2p}^1 \mathcal{XL})}_{D} (B_{2p}^1 \mathcal{XL})^{*} \underbrace{(B_{2p}^1 \mathcal{XL})}_{D} (B_{2p}^1 \mathcal{XL})^{*} \underbrace{(B_{2p}^1 \mathcal{XL})}_{D} \ldots$$

For $DD$ to be a contraction, we must have that $D' = D''$. We notice that $D'$ can’t begin with the character 0, while $D''$ must begin with a 0 by construction. Hence $D' \neq D''$, a contradiction.

We covered the cases in which the leftmost character removed belongs to the first block $BX\mathcal{L}$, a substring $E_j'(l - 1)$ for some $l > 0$ or a $B_{2p}^1\mathcal{XL}$-cluster, hence we exhausted every possibility and we conclude the claim.

\[\blacktriangleright\textbf{Claim 14.}\quad \text{For any } l, \text{ if the contraction } C_l \text{ from } T(l) \text{ to } T(l + 1) \text{ removes characters from the substring } $B_{2p} B_{2p}^* \mathcal{X} \text{ belonging to some block } E_j'(l) \text{, then } C_l \text{ removes all the characters contained in } $B_{2p} B_{2p}^* \mathcal{X} \text{ and cannot affect any other substring of the type } $B_{2p} B_{2p}^* \mathcal{X} \text{ belonging to a different block } E_j'(l) \text{ of } T(l).\]

\textbf{Proof.}\ Suppose that the claim is false. Let’s first prove the first part: suppose that there exists a contraction $DD$ on $T(l)$ for some integer $l \geq 0$ that removes some but not all of the characters of the subsequence $B_{2p} B_{2p}^* \mathcal{X}$ that belongs to $E_j'(l)$ for some $j > 0$. Note that by Claim 13, we know that the string $T(l)$ has the form described there. Like before, $D$ must contain at least one $L$ character, otherwise we would have that $D = aa$ for some character $a$. Thus, the left $D$ must contain the substring $LB_{2p} \mathcal{X}$ that belongs to $E_j'(l)$, but the right $D$ cannot contain this substring since it is unique in $T(l)$. Therefore there cannot exist a contraction that removes some but not all of the characters of $B_{2p} B_{2p}^* \mathcal{X}$.

Now let’s prove the second part of the Claim: let $DD$ be a contraction on $T(l)$ such that its right $D$ affects the substring $B_{2p} B_{2p}^* \mathcal{X}$ belonging to some block $E_{i_k}'(l)$ and $DD$ also affects the substring $B_{2p} B_{2p}^* \mathcal{X}$ belonging to another block $E_{i_j}'(l)$, for some integers $0 < j < k$. We refer to the subsequence of the type $B_{2p} B_{2p}^* \mathcal{X}$ that belongs to $E_{i_j}'(l)$ (resp. $E_{i_k}'(l)$) with $B_{2p} \mathcal{X}$ that belongs to the block $E_{i_k}'(l)$ (the first part of the block) then the substring $LB_{2p} \mathcal{X}$ would be entirely contained in the left $D$ or in the right $D$, but this substring is unique. We therefore know that the first character removed belongs to the substring $B_{2p} \mathcal{X}$. The right $D$ must contain the symbol $L$, but cannot contain the first $\mathcal{X}$ of the block $E_{i_k}'(l)$. We also know that the maximum suffix of the left $D$ that doesn’t contain $L$ doesn’t begin with 0, therefore the last character removed must belong to $E_{i_k+1}'$. Therefore, the duplication looks like this:
The minimum prefix of the right $D$ that contains two $\$$ characters is $D'BB_{n-1}^+$, where $D'$ is a string that doesn’t contain the characters $\$ and $\&$. Thus, the left $D$ must have the same prefix, but remember that the strings $B_{01}$, $B_{00}$, $B_1$, $B_2$, ..., $B_{2p}$ contain an incremental number of runs, so this is only possible if the first character of $DD'$ belongs to a $\B_{2p}\$-subproblem.

Therefore, we have that $DD'$ affects only $B_k$ but not $B_i$, hence a contradiction and we conclude the proof of the claim.

Notice that $T$ has one occurrence of the $X^d = X_1^d \ldots X_n^d$ substring. We will therefore refer to the $X^d$ string of $T$ without ambiguity. For $i \in [n]$, let $X_i(l)$ denote the substring of $T(l)$ formed by all the characters that belong to the $X_i^d$ substring of $X^d$. We will say that $X_i$ is activated in $T(l)$ if $X_i(l) = X_i$. Intuitively speaking, $X_i$ is activated in $T(l)$ if it has undergone $d$ contractions to turn it from $X_i^d$ into $X_i$.

\begin{itemize}
\item \textbf{Claim 15.} Let $i \in [p]$, and suppose that the substring $\$B_{01}\$B_{00}$ of $X^d$ that belongs to $E'_i$ is not removed in $T(l-1)$ but is removed in $T(l)$. Let $t$ be the number of $X_i$’s that were activated in $T(l-1)$. Suppose that $a_i$ and $a_i^*$ are the end points of edge $e_i$.

Then the number of contractions that have affected the substring $\$B_{01}\$B_{00}\$ of $X^d$ at $E'_i$ is at least $dc + dn - 1$ if $X_{a_i}$ or $X_{a_i^*}$ is not activated in $T(l-1)$, or at least $\min\{dt + dn, dc + dn - 1\}$ if $X_{a_i}$ and $X_{a_i^*}$ are both activated in $T(l-1)$.

\textbf{Proof.} By \cite{cite} in $T(l-1)$, $E'_i(l-1)$ belongs to a $\B_{01}\$-block. Let’s call $E_i(l-1)$ (resp. $E_i$) the substring $\$B_{01}\$B_{00}$ of $X^d$ that belongs to $E'_i(l-1)$ (resp. $E'_i$). As $E_i(l-1)$ gets removed completely after the $t$-th contraction of some substring $DD'$, it follows that $D$ must contain a substring that is equal to $E_i(l-1)$. The right $D$ of the $DD'$ square certainly contains the $B_i(l-1)$ substring that gets removed, but consider the copy of $B_i(l-1)$ in the first $D$ of the $DD'$ square. That is, we can represent the contraction as

\[
T' = \underbrace{D_1B_i(l-1)D_2}_{D} \underbrace{B_1B_i(l-1)D_2}_{D} \overbrace{T''}^{B_i(n-1)}
\]

where $D = D_1B_i(l-1)D_2$ and $B_i(l-1)$ is a substring equal to $B_i(l-1)$. By Claim \cite{cite} we know that there are only two blocks that can contain $B_i(l-1)$: either it is $BXL$, which is the $\B_{01}\$-block at the start of $T(l-1)$, or it is a $\B_{2p}\$-block from the cluster preceding $E_i(l-1)$. We analyze these two cases, which will prove the two cases of the claim.

Suppose that $B_i(l-1)$ is located in the first block $BXL$ of $T(l-1)$. Note that since $X_{a_i}$ contains $X_{a_i}$ and $X_{a_i^*}$ in their contracted form (as opposed to $X_{a_i}^d$ or $X_{a_i^*}^d$), $X_{a_i}$ and $X_{a_i^*}$ must be activated in $T(l-1)$ for the $DD'$ contraction to be possible. Moreover, for $B_i(l-1)$ to be equal to a substring of $BXL$, every other $X_j$ with $j \neq i, i_2$ that is activated must be contracted in $B_i(l-1)$ (i.e., $B_i$ contains $X_j^d$, but must contain $X_j$ in $B_i(l-1)$). This requires at least $d(l-2)$ contractions. Moreover, $B$ contains the $B_{01}$ substring whereas $B_i$ contains $B_{01}$. There must have been at least $dn + 2d - 1$ affecting the $\B_{01}^{\max}$ substring of $B_i$. Counting the contractions removing $B_i(l-1)$, this implies the existence of $d(l-2) + dn + 2d - 1 + 1 = dn + dt$ contractions affecting $B_i$.

If instead $B_i(l-1)$ was located in a $\B_{2p}\$-block, call this block $P$, then it suffices to note that $P$ contains $B_{01}$ as a substring whereas $B_i$ contains $B_{01}^p$. Counting the contraction that removes $B_i(l-1)$, it follows that at least $dc + 2d - 1$ contractions must have affected $B_i$.\hfill\begin{comment}\end{comment}
We have shown that $E^i_j$ can be removed from $T(l)$: either by a contraction acting also on the $BXL$ subarray at the start of $T(l - 1)$, or by a contraction that uses a block from a $BjX$-$XL$-cluster.

We say that the $E^i_j$'s removed by the former type of contraction are of Type 1, and otherwise we say that it is of Type 2.

If all Type 1 $E^i_j$'s were removed while having the same set of activated $X^i_j$'s, it would be easier to account for the contribution of the contractions in the first block and the contractions in the edge gadgets. Since this is not necessarily the case, we exploit the fact that in the sequence of edge gadgets there are many sequences of consecutive edge gadgets within which the Type 1 are removed with the same set of activated $X^i_j$'s. We show that considering only these subsequences of contractions suffices for guaranteeing the bound.

Fix $k \in [p]$, and let $\text{act}(E^i_j)$ denote the set of activated $X^i_j$'s in the $T(l - 1)$ where $E^i_j$ gets removed (i.e. $E^i_j$ is not removed from $T(l - 1)$ but is removed from $T(l)$). Let us partition $[p]$ into intervals of integers $P_a = [1 + am \ldots m + am]$, where $a \in \{0, \ldots, p/m - 1\}$. We say that interval $P_a$ is homogeneous if, for each $i, j \in P_a$ such that $E^i_j$ and $E^i_j$ are of Type 1, $\text{act}(E^i_j) = \text{act}(E^i_j)$. In other words, $P_a$ is homogeneous if all the Type1 $E^i_j$ substrings corresponding to those in $P_a$ are removed with the same set of activated $X^i_j$'s.

The following claim was proved in [9]. We include the proof here for the sake of self-containment.

\begin{claim} [9] There are at least $p/m - 2n$ homogeneous intervals. \end{claim}

\begin{proof} Observe that once an $X^i_j$ is activated, it remains so for the rest of the contraction sequence. Since there are $n$ of the $X^i_j$'s, there are only $n + 1$ possible values for $\text{act}(E^i_j)$ (counting the case when none of them are activated). There are $p/m$ intervals, and it follows that at most $n + 1 \leq 2n$ of them are not homogeneous, which gives the desired result. \end{proof}

We can now complete the proof, closely following the argument in [9]. For each edge gadget, let $\text{cost}(E^i_j)$ be the overall number of contractions used that act on $E^i_j$. Let $P_{a_1}, \ldots, P_{a_h}$ be the set of homogeneous intervals, where, by the previous claim, we know that $h \geq p/m - 2n$. Let $P_a = \arg\min_{j \in [h]} \sum_{i \in P_a} \text{cost}(E^i_j)$.

By Claim [9] no two $E^i_j$'s share their cost. Then the total number of contractions, which is at least the total number of contractions acting on edge gadgets in the homogeneous intervals, $P_{a_1}, \ldots, P_{a_h}$, is lower bounded as:

$$N \geq \sum_{a \in \{0, 1, \ldots, \frac{p}{m} - 1\}} \sum_{i \in P_a} \sum_{j \in [a]} \text{cost}(E^i_j) \geq \sum_{j \in [a]} \sum_{i \in P_{a_j}} \text{cost}(E^i_j) \geq \left( \frac{p}{m} - 2n \right) \sum_{i \in P_a} \text{cost}(E^i_j), \quad (11)$$

where in the last inequality we used the minimality of $P_a$. Notice that even if we are ignoring the cost of the contractions involved in non-homogeneous intervals, but this does not affect our argument aiming at showing a lower bound on the total number of contractions.

Assume that there is at least one $i \in P_a$ such that $E^i_j$ is of Type 1. Then by Claim 4, $\text{cost}(E^i_j)$ is either at least $\min\{dc + dn - 1, dt + dn\}$ where $t = |\text{act}(E^i_j)|$, or $\text{cost}(E^i_j)$ is at least $dc + dn - 1$. If $dt + dn \geq dc + dn - 1$, we may assume that $E^i_j$ is of Type 2 since removing $E^i_j$ using Type 2 contractions will not increase its cost. We will therefore assume that if there is at least one $E^i_j$ of Type 1 in $P_a$, then $dt + dn < dc + dn - 1$ and thus $\text{cost}(E^i_j) \geq dt + dn$. 

\begin{proof} [\text{Proof.}] Observe that once an $X^i_j$ is activated, it remains so for the rest of the contraction sequence. Since there are $n$ of the $X^i_j$'s, there are only $n + 1$ possible values for $\text{act}(E^i_j)$ (counting the case when none of them are activated). There are $p/m$ intervals, and it follows that at most $n + 1 \leq 2n$ of them are not homogeneous, which gives the desired result. \end{proof}
Now, choose any $i$ in $P_3$ such that $E'_i$ is of Type 1, and let $W$ be the set of vertices of $G$ corresponding to those in $act(E'_i)$. That is, $e_j \in W$ if and only if $X_j$ is activated when $E'_i$ gets removed. If there does not exist an $E'_i$ of Type 1 to choose, then define $W = \emptyset$. Denote $|W| = t$ and $|E(W)| = s$. Then, we have

$$\sum_{i \in P_3} \text{cost}(E'_i) \geq (m-s)(dc + dn - 1) + s(dt + dn).$$

(12)

For any $E'_i$ where $i \in P_4$, by Claim 14, either $e_i$ is not in $W$ and $\text{cost}(E'_i) \geq dc + dn - 1$, or $e_i$ is in $W$ and $\text{cost}(E'_i) \geq dt + dn$. Notice that it is crucial here that $P_4$ is homogeneous as this guarantees that every Type 1 $E'_i$ uses the same value of $t$ in the cost $dt + dn$.

The desired result will be direct consequence of the following final claim

\[\triangleright\] Claim 17. $W$ is a subgraph of $G$ of cost $c(m - s) + ts \leq r$.

\textbf{Proof.} Assume by contradiction that $c(m - s) + ts > r$. Since all the quantities are integral, we have $c(m - s) + ts \geq r + 1$. We will show that this contradicts the standing assumption on the number of contractions. From (11) and (12) it follows that the total number of contractions can be lower bounded as follows:

$$N \geq \left(\frac{p}{m} - 2n\right)[(m-s)(dc+dn-1)+s(dt+dn)]$$

$$= \left(\frac{p}{m} - 2n\right) \cdot d \cdot [c(m-s)+st+nm] + \left(\frac{p}{m} - 2n\right)(s-m)$$

$$\geq \left(\frac{p}{m} - 2n\right) \cdot d \cdot [r+1+nm] + \left(\frac{p}{m} - 2n\right)(s-m)$$

$$= \left(\frac{p}{m} - 2n\right) \cdot d \cdot [r+nm] + \left(\frac{p}{m} - 2n\right)(d+s-m)$$

$$= \frac{p}{m} \cdot d \cdot (r+nm) - 2dn(r+nm) + \left(\frac{p}{m} - 2n\right)(d+s-m)$$

$$\geq \frac{p}{m} \cdot d \cdot (r+nm) + 4cdn,$$

where the last inequality holds for the large values $d = m+1$ and $p = m(n+m)^{10}$ fixed for the reduction.

Since assuming $r < c(m - s) + ts$ we reached a contradiction to the hypothesis on $N$, it must indeed hold that $r \geq c(m - s) + ts = \text{cost}(W)$ as desired. This concludes the proof of the claim and of the theorem.

\textbf{B} The proof of Lemma 8

\textbf{Lemma 8}. Fix $q \in \{2, 3, 4, 5\}$. Let $S$ and $T$ be purely alternating strings over the same $q$-ary alphabet $\Sigma = \{0, 1, 2, \ldots, q-1\}$. Let RLE$(S) = 0^{s_1}1^{s_2} \ldots s_n^q$ and RLE$(T) = 0^{t_1}1^{t_2} \ldots t_m^q$ be the run-length encodings of the strings. Then, $S \Rightarrow_N^n T$ if and only if there exists a function $f : \{1, \ldots, n-q+2\} \rightarrow \{1, \ldots, m-q+2\}$ such that:

1. $f(1) = 1$ and $f(n-q+2) = m-q+2$
2. $f(i) = j \implies s_i = t_j$ and for each $u = 0, \ldots, q-2$ we have that $l_{i+u} \leq t_{j+u}$
3. $f(i) = j$ and $f(i') = j'$ and $i < i' \implies j < j'$
4. if $q = 5$, and $f(i) = j$ and $f(i + 1) = j' \neq j + 1 \implies$ there exists a substring $M$ in $T$ starting in a position $p$ such that $j \leq p \leq j'$ with the form $M = s_{p+3}^{p+3}, s_{p+4}^{p+4}, \ldots, s_{p+q}^{p+q}, s_{p+q+1}^{p+q+2}$ such that for each $u = 0, 1, \ldots, q-3$ it holds that $l_{i+3+u} \leq t_{p+u}$ and $l_{i+1} \leq t_{p+q+2}$. 

\textbf{Lemma 8}. Fix $q \in \{2, 3, 4, 5\}$. Let $S$ and $T$ be purely alternating strings over the same $q$-ary alphabet $\Sigma = \{0, 1, 2, \ldots, q-1\}$. Let RLE$(S) = 0^{s_1}1^{s_2} \ldots s_n^q$ and RLE$(T) = 0^{t_1}1^{t_2} \ldots t_m^q$ be the run-length encodings of the strings. Then, $S \Rightarrow_N^n T$ if and only if there exists a function $f : \{1, \ldots, n-q+2\} \rightarrow \{1, \ldots, m-q+2\}$ such that:

1. $f(1) = 1$ and $f(n-q+2) = m-q+2$
2. $f(i) = j \implies s_i = t_j$ and for each $u = 0, \ldots, q-2$ we have that $l_{i+u} \leq t_{j+u}$
3. $f(i) = j$ and $f(i') = j'$ and $i < i' \implies j < j'$
4. if $q = 5$, and $f(i) = j$ and $f(i + 1) = j' \neq j + 1 \implies$ there exists a substring $M$ in $T$ starting in a position $p$ such that $j \leq p \leq j'$ with the form $M = s_{p+3}^{p+3}, s_{p+4}^{p+4}, \ldots, s_{p+q}^{p+q}, s_{p+q+1}^{p+q+2}$ such that for each $u = 0, 1, \ldots, q-3$ it holds that $l_{i+3+u} \leq t_{p+u}$ and $l_{i+1} \leq t_{p+q+2}$.
Proof. (sketch) We first prove the “only if” direction of the statement. We argue by induction on the number of normal duplications. Let \( \ell \) be such number, i.e., \( S \Rightarrow^N S_1 \Rightarrow^N S_2 \Rightarrow^N \cdots \Rightarrow^N S_{\ell-1} \Rightarrow^N T \). Let \( f_0(i) = i \) for each \( i = 1, \ldots, n - q + 2 \).

Clearly, if \( \ell = 0 \), then \( S = T \) and \( f_0 \) satisfies properties 1-4.

Assume now that \( \ell > 0 \) and that the claim is true for \( \ell - 1 \), i.e., there is a function \( f_{\ell-1} \) associated to \( S \Rightarrow^N \ell-1 \) \( S_{\ell-1} \) that satisfies properties 1-4.

Case 1. If the duplication \( S_{\ell-1} \Rightarrow^N T \) is of type 2 then it is easy to see that \( f_\ell = f_{\ell-1} \) satisfies the claim.

Case 2. If the duplication \( S_{\ell-1} \Rightarrow^N T \) is of type 1, let \( j^* \) be the first run of \( \text{RLE}(S_{\ell-1}) \) involved in this normal duplication. Then, the function

\[
 f_\ell(i) = \begin{cases} 
 f_{\ell-1}(i) & f_{\ell-1}(i) \leq j^* \\
 f_{\ell-1}(i) + q & f_{\ell-1}(i) > j^*
 \end{cases}
\]

satisfies properties 1-3.

For property 4., in the case \( q = 5 \), we split the analysis into two sub-cases, according to whether the gap between \( f_\ell(i) \) and \( f_\ell(i+1) \) is created by the last duplication or not. Let \( i^* \) be such that \( f_\ell(i^*) = j^* \).

Sub-case (i). \( f_\ell(i+1) = f_\ell(i) + 1 \) and also \( f_{\ell-1}(i+1) \neq f_{\ell-1}(i) + 1 \). Then there is a substring \( M \) in \( S_{\ell-1} \) starting in a position \( f_{\ell-1}(i) \leq p \leq f_{\ell-1}(i+1) \) satisfying the property. Since this substring involves \( q - 1 \) runs of \( S_{\ell-1} \) it is not eliminated by the duplication \( S_{\ell-1} \Rightarrow^N T \) which can only shift it to the right (by exactly \( q \) positions, which, by definition of \( f_\ell \), only happens if \( f_\ell(i+1) = f_{\ell-1}(i+1) + q \)). Hence \( M \) is also a substring of \( T \) and it appears in a position \( p' \) between \( f_{\ell}(i) \) and \( f_{\ell}(i+1) \).

Sub-case (iii). \( i = i^* \) and \( f_{\ell-1}(i+1) = f_{\ell-1}(i) + 1 \). By definition, we have \( f_{\ell}(i+1) = f_{\ell}(i) + q + 1 \neq f_{\ell}(i) + 1 \). In this case, it is not hard to verify that with \( p = f_{\ell}(i^*) + 3 \) the property is also satisfied.

We now show the (if) direction.

Assume there exists a function \( f \) satisfying properties 1-4. Because of properties 1-3, we have that there exists \( \ell_1 \leq \ell_2 \), and \( 0 \leq r < q \) such that \( n = \ell_1 \times q + r \) and \( m = \ell_2 \times q + r \). Without loss of generality, we can assume \( r = 0 \). Let \( \Delta = \Delta(S,T) = |\text{RLE}(T)|/n - |\text{RLE}(S)|/n = \ell_2 - \ell_1 \). We argue by induction on \( \Delta \).

For \( \Delta = 0 \) it must be \( f(i) = i \), for each \( i \), hence, there is a sequence of normal duplications of type 2 that satisfies the claim.

Assume now \( \Delta > 0 \) and that (induction hypothesis) the claim holds true for any pair of strings \( S' \) and \( T' \) for which \( \text{RLE}(S') = \ell'_1 \times q, \text{RLE}(T') = \ell'_2 \times q \) and \( \Delta' = \Delta(S',T') = \ell'_2 - \ell'_1 \leq \Delta - 1 \).

Let \( i \) be the smallest integer such that \( f(i) = j \) and \( f(i+1) = j' > j + 1 \).—Such an \( i \) must exist, otherwise, we are in the case \( \Delta = 0 \).—Hence, the fact that the strings are purely alternating, together with \( f \) satisfying property 2., implies that \( f(i+1) = (j+1) + u \times q \) for some \( u \geq 1 \).

Consider \( S \) obtained from \( S \) by a normal duplication that starts on the \( i \)th run of \( S \). Then

\[
 \text{RLE}(\hat{S}) = \ldots, s_{i+1}^{l_1} s_{i+2}^{l_1} \ldots, s_{i+1}^{l_1} s_{i+2}^{l_1} \ldots s_{i+q-1}^{l_1} s_{i+q}^{l_1} \ldots, s_{i+q-1}^{l_1} s_{i+q+1}^{l_1} \ldots s_{i+q-1}^{l_1} \ldots,
\]

where the number underneath each component specifies its index in \( \text{RLE}(\hat{S}) \).
Note that, by property 2., because of \( f(i) = j \), we have that \( l_i \leq l'_j, l_{i+1} \leq l'_{j+1}, l_{i+2} \leq l'_{j+2}, \ldots, l'_{i+q-2} \leq l'_{j+q-2} \). Hence, letting \( \tilde{l}_i \) be the length of the \( k \)th run of \( S \), we also have \( \tilde{l}_i \leq l'_j, \tilde{l}_{i+1} \leq l'_{j+1}, \tilde{l}_{i+2} \leq l'_{j+2}, \ldots, \tilde{l}_{i+q-2} \leq l'_{j+q-2} \). Moreover we have \( 1 = \tilde{l}_{i+q-1} \leq l'_{j+q-1} \), and \( 1 = \tilde{l}_{i+q} \leq l'_{j+q} \).

Finally, we have \( 1 = \tilde{l}_{i+q-1} \leq l'_{f(i+1)-2} \), and \( 1 = \tilde{l}_{i+q} \leq l'_{f(i+1)-2}. \) And, by the hypothesis on \( f(i+1) \) it holds that

\[
\tilde{l}_{i+q+u} = \tilde{l}_{i+u} \leq l'_{f(i+1)+u}, \text{ for } u = 1, \ldots, q - 1. \tag{13}
\]

Therefore, for \( q < 5 \), we have that the function

\[
\tilde{f}(k) = \begin{cases} 
  f(k) & k \leq i \\
  f(i) + u & 1 \leq u \leq \min\{2, q - 2\}, k = i + u, \\
  f(i + 1) - 2 & k = i + q - 1 \\
  f(i + 1) - 1 & k = i + q \\
  f(k - q) & k \geq i + q + 1,
\end{cases}
\]

satisfies properties 1 - 3 for \( \tilde{S} \) and \( T \). Since \( \Delta(\tilde{S}, T) < \Delta \), by induction hypothesis there is a sequence of normal duplications \( \tilde{S} \Rightarrow T \), that combined with the above normal duplication \( S \Rightarrow \tilde{S} \) proves the claim.

Let us now consider the case \( q = 5 \). Let \( p \) be the index satisfying 4. Clearly, it holds that \( p = j + 3 + v \times q \), for some \( 0 \leq v \leq u - 1 \). By property 4, we have

\[
\tilde{l}_{i+3+u} \leq l_{i+3+u} \leq l'_{p+u} \text{ for } u = 0, 1, \ldots, q - 3 \text{ and } \tilde{l}_{i+q+1} = \tilde{l}_{i+1} \leq l'_{p+q-2}. \tag{14}
\]

We distinguish three subcases according to whether: (i) \( p = j + 3 \); (ii) \( p = j' - 3 \); (iii) \( j + 3 < p < j' - 3 \). In each case, our argument will be to show that there is a sequence of normal duplication that can convert \( S \) into a string \( \tilde{S} \) with more groups of runs—hence, such that \( \Delta(\tilde{S}, T) < \Delta \)—for which there exists a function \( \tilde{f} \) satisfying properties 1-4. (with respect to \( \tilde{S} \) and \( T \) and such that \( \tilde{f}(k) = f \) for \( k \leq i + 1 \). This will imply, by induction hypothesis that \( \tilde{S} \Rightarrow ^{N} \tilde{T} \), hence proving our claim.

For the sake of conciseness, in the following argument we will denote by \( \lceil a \rceil^4 \) a normal duplication on a string \( A \) that acts on the runs of \( A \) from the \( a \)th one to the \( (a + 3) \)th one. Recall that \( i \) is the smallest integer such that \( f(i) = j \) and \( f(i + 1) = j' > j + 1 \).

Subcase 1. \( p = j + 3 \). Consider the duplication \( \lceil i \rceil^5 \) and indicate with \( \tilde{S}^{(i)} \) the resulting string.

If \( f(i + 1) = i + q + 1 \) we have that \( \tilde{S}^{(i)} \) satisfies our desiderata, i.e., indicating with \( \tilde{f}^{(i)}_j \), the length of the \( j \)th run of \( \tilde{S}^{(i)} \) we have \( \tilde{f}^{(i)}_k \leq l_k^i \) for each \( k = 1, \ldots f(i + 1) \). Hence, the function

\[
\tilde{f}(k) = \begin{cases} 
  k & k \leq f(i + 1) \\
  f(k - i - 1) & k > i + 1.
\end{cases}
\tag{15}
\]

satisfies 1-4. w.r.t. \( \tilde{S}^{(i)} \) and \( T \) and the desired result follows by induction.

If \( f(i + 1) = i + 1 + u \times q \) for some \( u > 1 \), we use first a duplication \( \lceil i + 3 \rceil^{\tilde{S}^{(i)}} \) and then repeatedly for \( u = 3, 4, \ldots, u \) (always on the newly obtained string \( \tilde{S}^{(u)} \)) the duplication \( \lceil i + u \rceil^{\tilde{S}^{(u)}} \). The resulting string \( \tilde{S}^{(u)} \) has a long sequence of runs of size 1 and at each iteration the sequence of runs \( \tilde{s}_{i+1}^{(u)}, \tilde{s}_{i+2}^{(u)}, \tilde{s}_{i+2}^{(u+2)}, \ldots, \tilde{s}_{i+2^u}^{(u)} \) gets shifted to the right until the run \( \tilde{s}_{i+1}^{(u)} \) becomes the \( f(i + 1) \)th run in \( \tilde{S}^{(u)} \) and we have that the function in \( \tilde{f}^{(u)} \) satisfies 1-4.
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w.r.t. $\hat{S}^{(u)}$ and $T$ and the desired result follows by induction. The process can be visualized as follows (where we are assuming $s_i = 0$):

\[
S = \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots
\]

\[
\Rightarrow \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots = \hat{S}^{(1)}
\]

\[
\Rightarrow \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots = \hat{S}^{(2)},
\]

where $M$ denotes a sequence of runs that satisfies the property 4.

**Subcase 2.** $p = j' - 3$. Again we perform first the duplication $[i]^S$ and indicate with $\hat{S}^{(1)}$ the resulting string. We then perform duplication $[i+2]^S$. The process can be visualized as follows (where we are assuming $s_i = 0$):

\[
\hat{S}^{(1)} \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots
\]

\[
\Rightarrow \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots = \hat{S}^{(2)},
\]

where $M$ denotes a sequence of runs that satisfies the property 4. in $\hat{S}^{(2)}$.

Like before, we continue with $(j' - j)/5 - 2$ normal duplications producing the sequence of strings $\hat{S}^{(2)} \Rightarrow \ldots \Rightarrow N \hat{S}^{(u)}$ where for each $w = 2, \ldots, u - 1$, the duplication is $[i + 3]^S(u)$. As in the previous case, we have that the function in \[15\] satisfies 1.-4. w.r.t. the resulting string $\hat{S}^{(u)}$ and $T$ and the desired result follows by induction.

**Subcase 3.** $j + 3 < p < j' - 3$. As before, we perform first the duplication $[i]^S$ and indicate with $\hat{S}^{(1)}$ the resulting string. We then perform duplication $[i+2]^S$, and on the resulting string, denote by $\hat{S}^{(2)}$, we use duplication $[i+2]^S$. The process can be visualized as follows (where we are assuming $s_i = 0$):

\[
\hat{S}^{(1)} \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots
\]

\[
\Rightarrow \ldots 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} 0^i 1^{i+1} 2^{i+2} 3^{i+3} 4^{i+4} \ldots = \hat{S}^{(2)}
\]

where, in each intermediate string, we have indicated with $M$ a sequence of runs that satisfies the property 4.

We can now apply $(j' - j)/5 - 3$ normal duplications, choosing the four runs with length one to the left or the right of $M$. In this way we can position the substring $M$ to match the one in $T$ that is guaranteed by the fourth property and also shift the run $s_{i+1}$ to become the $f(i+1)$th run of the resulting string $\hat{S}^{(u)}$. As before, this implies that the function in \[15\] satisfies 1.-4. w.r.t. the resulting string $\hat{S}^{(u)}$ and $T$ and the desired result follows by induction.
C The proof of Claim 9 and Lemma 10: the case $|\Sigma| \in \{2, 3, 4\}$

Claim 9. Let $S$ and $T$ be $q$-ary purely alternating strings such that there exists a series of duplications $S \Rightarrow T$. Then for each duplication, if we call the duplicated substring $X$, we have that $|RLE(X)| \mod q \leq 1$.

Proof. Assume that in the series of duplications $S \Rightarrow T$ there exists a duplication $S' = AXB \Rightarrow AXXB = T'$ with $|RLE(X)| \mod q > 1$. In this case, the string $XX$ contains in the middle a substring $ab$, in which $a$ and $b$ are not consecutive characters in $S$. This happens because $a$ corresponds to the last character of $X$ and $b$ is the first character of $X$ and $|RLE(X)| \mod q > 1$, implying that $a \neq (b - 1) \mod q$. For this reason $T'$ can’t be purely alternating, but since $T$ is, there must exist some other duplication that eliminates the adjacency $ab$. This is impossible, since no duplication eliminates adjacencies, but can only create new ones.

Binary strings ($|\Sigma| = 2$)

Lemma 18. Let $S$ and $T$ be binary (purely alternating) strings. If there exists a duplication $S \Rightarrow T$, then we can create a series of normal duplications such that $S \Rightarrow^N T$.

Proof. Let $S = AXB \Rightarrow AXXB = T$ be the original duplication, which we assume not to be normal. We show how we can have a string $T \preceq XX$ that matches $XX$ and such that $X \Rightarrow^N T$. The process consists in creating the necessary number of pairs of consecutive runs $0^i$ $1^i$ of size 1.

$$X = 0^i_0 1^i_1 0^i_0 1^i_1 \ldots 0^i_{l-1} 1^i_l$$

$$\Rightarrow^N 0^i_0 1^i_1 0^i_0 1^i_1 \ldots 0^i_{l-1} 1^i_l$$

$$\Rightarrow^N 0^i_0 1^i_1 1^i_0 1^i_1 0^i_1 1^i_2 0^i_3 1^i_4 \ldots 0^i_{l-1} 1^i_l$$

$$\Rightarrow^N :$$

$$\Rightarrow^N 0^i_0 1^i_1 1^i_0 1^i_1 0^i_1 1^i_2 0^i_3 1^i_4 \ldots 0^i_{l-1} 1^i_l = T \preceq XX.$$ 

Since $T \preceq XX$ implies $T \Rightarrow^N XX$ (by type 2 normal duplications), we have $X \Rightarrow^N T \Rightarrow^N XX$, which complete the proof of the claim.

Ternary strings

Lemma 19. Let $S$ and $T$ be ternary purely alternating strings. If there exists a duplication $S \Rightarrow T$, then we can create a series of normal duplications $S \Rightarrow^N \cdots \Rightarrow^N T$.

Proof. Let $S = AXB \Rightarrow AXXB = T$ be the original duplication. We create a string that matches $XX$ starting from $X$ through normal duplications, depending on how many runs are contained in $X$. By Claim 9 the only possible cases are $|RLE(X)| \mod 3 \in \{0, 1\}$

- If $|RLE(X)| = 1$ it means that the only effect of the original duplication is to extend one of the runs of $S$. For this reason we know that $S$ already matches $T$.

- If $|RLE(X)| \mod 3 = 0$, we suppose that the string $X$ starts with a 0 (rotate the characters if it starts with any other symbol), so the run-length encoding of $X$ is in the form $RLE(X) = 0^i_0 1^i_1 2^i_0 0^i_1 \ldots 1^i_{l-1} 2^i_l$. If $|RLE(X)|$ is equal to 3 we can just execute a normal duplication on the same runs. If $|RLE(X)| = 6$, we can execute the following sequence of normal duplications and observe that the resulting string matches $XX$: 

$$X = 0^i_0 1^i_1 2^i_0 0^i_1 \ldots 1^i_{l-1} 2^i_l$$

$$\Rightarrow^N :$$

$$\Rightarrow^N 0^i_0 1^i_1 2^i_0 0^i_1 \ldots 1^i_{l-1} 2^i_l 0^i_3 1^i_4 \ldots 0^i_{l-1} 1^i_l = T \preceq XX.$$ 

Since $T \preceq XX$ implies $T \Rightarrow^N XX$ (by type 2 normal duplications), we have $X \Rightarrow^N T \Rightarrow^N XX$, which complete the proof of the claim.
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\[ X = 0^{l_i} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_6} \]
\[ \Rightarrow N 0^{l_i} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_6} \]
\[ \Rightarrow N 0^{l_i} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_6} \]
\[ \Rightarrow N 0^{l_i} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_6} \]
\[ \Rightarrow N 0^{l_i} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_6} \]

since the last string matches \( XX \) there is a series of normal duplications (of type 2) that converts it into \( XX \).

We can see that using a procedure similar to this one, starting from every string \( X \) we can produce another string through normal duplications that matches \( XX \). We need to execute the duplications explained before and continue with \( |RLE(X)|/3 - 2 \) normal duplications on the runs \( 1^{l_2} 2^{l_1} \). In this way we create a sequence of runs with length 1, pushing the original runs of \( X \) to the right to be matched with their corresponding copies in \( XX \).

- If \( |RLE(X)| \mod 3 = 1 \) and we are not in the first case, then the effect of this duplication is the same as the effect of a duplication with \( |RLE(X)| \mod 3 = 0 \) followed by the extension of a run. We can therefore treat this case with the same technique we used for the previous one.

Quaternary strings

\textbf{Lemma 20.} Let \( S \) and \( T \) be quaternary purely alternating strings. If there exists a duplication \( S \Rightarrow \Rightarrow T \), then we can create a series of normal duplications \( S \Rightarrow \Rightarrow \Rightarrow T' \) such that \( T' \) matches \( T \).

\textbf{Proof.} Let \( S = AXB \Rightarrow AXBX = T \) be the original duplication. Like before, we create a string that matches \( XX \) starting from \( X \) through normal duplications, depending on how many runs are contained in \( X \). By Claim \( 9 \), the only possible cases are \( |RLE(X)| \mod 4 \in \{ 0, 1 \} \)

- If \( |RLE(X)| \mod 4 = 1 \) it means that the only effect of the original duplication is to extend one of the runs of \( S \), or that the effect can be simulated with the technique used for the case \( |RLE(X)| \mod 4 = 0 \).

- If \( |RLE(X)| \mod 4 = 0 \), suppose that the run-length encoding of \( X \) is in the form
  \[ RLE(X) = 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} \ldots 3^{n-1} 4^{l_n} \]
  If \( |RLE(X)| \mod 4 = 0 \) we can just execute a normal duplication on the same runs. If \( |RLE(X)| = 8 \), we can execute the following sequence of normal duplications and observe how the resulting string matches \( XX \):

\[ X = 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]
\[ \Rightarrow 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]
\[ \Rightarrow 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]
\[ \Rightarrow 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]
\[ X = 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]
\[ XX = 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} 4^{l_5} 1^{l_6} 2^{l_7} 3^{l_8} \]

Like before, we can see that using a procedure similar to this one, starting from every string \( X \) we can produce another string through normal duplications that matches \( XX \).

We need to execute the duplications explained before and continue with \( |RLE(X)|/4 - 2 \) normal duplications on the runs \( 0^{l_1} 1^{l_2} 2^{l_3} 3^{l_4} \). In this way we create a sequence of runs with length 1, pushing the original runs of \( X \) to the right to be matched with their corresponding copies in \( XX \).
D The Algorithm in the proof of Theorem 11

Here we show the pseudocode of an algorithm that can be used to check the existence of a mapping from $\text{RLE}(S)$ to $\text{RLE}(T)$ that guarantees that $S \Rightarrow^*_N T$.

Algorithm 1

Input: Two $q$-ary purely alternating strings $S$ and $T$ ($q \in \{2, 3, 4, 5\}$)
Output: yes if and only if there exists a function $f$ satisfying 1-4 in Lemma 8
Create $\text{RLE}(S) = s_1^1, s_2^2, \ldots, s_n^n$ and $\text{RLE}(T) = t_1^1, t_2^2, \ldots, t_m^m$;
if $s_1 \neq t_1$ or $s_n \neq t_m$ or there is $u \in [q-1]$ s.t. $l_u > l'_u$ or $l_{n-u+1} > l'_{m-u+1}$ then
\[ \text{return no;} \]
else
\[ f(1) = 1, f(n-q+2) = m-q+2; \]
\[ i = 2, j = 2; \]
while $i \leq n-q+2$ and $j \leq m-q+2$ do
\[ \text{while } l_{i+q-2} < l'_{j+q-2} \text{ and } i \leq n-q+2 \text{ do} \]
\[ f(i) = j; i = i+1; j = j+1; \]
if $i \leq n-q+2$ then
\[ p = f(i-1) + 3; \]
while there is $u \in \{0, 1, \ldots, q-3\}$ s.t. $l_{(i-1)+3+u} > l'_{p+u}$ or $l_i > l'_{p+q-2}$ do
\[ p = p + q; \]
if $p > m-q+3$ then
\[ \text{return no;} \]
\[ j = p - 2 + q; \]
while there is $u \in \{0, 1, \ldots, q-2\}$ s.t. $l_{i+u} > l'_{j+u}$ and $j < m-q+2$ do
\[ j = j + q; \]
if $j \leq m-q+2$ then
\[ f(i) = j; i = i+1; j = j+1 \]
else
\[ \text{return no;} \]
if $i = n-q+1$ and $j = m-q+1$ then
\[ \text{return yes;} \]
else
\[ \text{return no;} \]