Ordinary differential equations in algebras of generalized functions

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Abstract

A local existence and uniqueness theorem for ODEs in the special algebra of generalized functions is established, as well as versions including parameters and dependence on initial values in the generalized sense. Finally, a Frobenius theorem is proved. In all these results, composition of generalized functions is based on the notion of c-boundedness.

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1 Introduction

At the time of their introduction in the 1980s ([2], [3]), algebras of generalized functions in the Colombeau setting were primarily intended as a tool for treating nonlinear (partial) differential equations in the presence of singularities. Since then, many types of differential equations have been studied in the Colombeau setting (see [16], together with the references given therein, and the first part of [9] for a variety of examples). Nevertheless, the authors of [10] feel compelled to declare some 15 years later that “a refined theory of local solutions of ODEs is not yet fully developed” (p. 80). In fact, this state of affairs has not changed much since then. It is the purpose of this article to lay the foundations for such a theory, with composition of generalized functions based on the concept of c-boundedness.

As the basic object of study one may view the differential equation \( \dot{u}(t) = F(t, u(t)) \) with initial condition \( u(t_0) = \bar{x}_0 \). Since \( u(t) \) gets plugged into the second slot of \( F \) it is evident that one has to adopt a suitable concept of composition of generalized functions in order to give meaning to the right-hand side of the ODE, keeping in mind that in general, the composition of generalized functions is not defined.

One way of handling the composition \( u \circ v \) of generalized functions \( u, v \) is to assume the left member \( u \) to be tempered (see [10, Subsection 1.2.3] for a...
definition). In this setting, a number of results on ODEs have been established, including a global existence and uniqueness theorem ([12, Theorem 3.1], [10, Theorem 1.5.2]). A more recent concept of composing generalized functions goes back to Aragona and Biagoni (cf. [1]): Here, the right member \( v \) is assumed to be compactly bounded (c-bounded) into the domain of \( u \) (see Section 2 for details); then the composition \( u \circ v \) is defined as a generalized function. It is this latter approach we will adopt in this article. It seems to be suited better to local questions; moreover, the concept of c-boundedness permits an intrinsic generalization to smooth manifolds ([10, Subsection 3.2.4], contrary to that of tempered generalized distributions.

In a number of contributions, the notion of c-boundedness has already been taken as the basis for the treatment of generalized ODEs. The first instance, dating back to [15], served as a tool for an application to a problem in general relativity, see [10, Lemma 5.3.1] and the improved version in [6, Lemma 4.2]. Theorem 3.1 of [14]—where a theory of singular ordinary differential equations on differentiable manifolds is developed—provides a global existence and uniqueness result for autonomous ODEs on \( \mathbb{R}^n \). Theorem 1.9 in [11] establishes existence of a solution assuming an \( L^1 \)-bound (as a function of \( t \), uniformly on \( \mathbb{R}^n \) with respect to the second slot) on the representatives of \( F \). Finally, the study of the Hamilton-Jacobi equation in the framework of generalized functions in [7] led to some local existence and uniqueness results for ODEs, in a setting adapted to this particular problem. We will discuss one of these Theorems in more detail in Section 3.

A special feature of the existence and uniqueness results 3.1 and 3.8 in Section 3 consists in their capacity to simultaneously allow generalized values both for \( \tilde{t}_0 \) and \( \tilde{x}_0 \) in the initial conditions, and to have, nevertheless, the domain of existence of the local solution equal to the one in the classical case.

The results of this article may be viewed as extending and refining the material of Chapter 5 of [4]. Section 2 makes available the necessary technical prerequisites. Local existence and uniqueness results for ODEs in the c-bounded setting are the focus of Section 3: Following the basic theorem handling the initial value problem mentioned above, two more statements are established covering ODEs with parameters and \( \mathcal{G} \)-dependence of the solution on initial values, respectively. Section 4, finally, presents a generalized version of the theorem of Frobenius, also in the c-bounded setting.

## 2 Notation and preliminaries

For subsets \( A, B \) of a topological space \( X \), we write \( A \subset B \) if \( A \) is a compact subset of the interior \( B^o \) of \( B \). By \( B_r(x) \) we denote the open ball with centre \( x \) and radius \( r > 0 \). We will make free use of the exponential law and the argument swap (flip), i.e. for functions \( f : X \times Y \to Z \) we will write \( f(x)(y) = f(x,y) = f^\text{fl}(y,x) = f^\text{fl}(y)(x) \).

Generally, the special Colombeau algebra can be constructed with real-valued or with complex-valued functions. For the purposes of the present article we consider the real version only. Concerning fundamentals of (special) Colombeau algebras, we follow [10, Subsection 1.2].

In particular, for defining the special Colombeau algebra \( \mathcal{G}(U) \) on a given
is a well-defined generalized function in \(G\) with representatives 

2.2. Proposition. Let 

\[ (u_ε)_ε \in \mathcal{E}(U) | \forall K \subset U \forall α \in \mathbb{N}_0^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^α u_ε(x)| = O(ε^{-N}) \text{ as } ε \to 0, \]

\[ \mathcal{N}(U) := \{(u_ε)_ε \in \mathcal{E}(U) | \forall K \subset U \forall α \in \mathbb{N}_0^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^α u_ε(x)| = O(ε^m) \text{ as } ε \to 0. \]

Elements of \(\mathcal{E}_M(U)\) and \(\mathcal{N}(U)\) are called moderate and negligible functions, respectively. By [10, Theorem 1.2.3], \((u_ε)_ε\) is already an element of \(\mathcal{N}(U)\) if the above conditions are satisfied for \(α = 0\). \(\mathcal{E}_M(U)\) is a subalgebra of \(\mathcal{E}(U)\), \(\mathcal{N}(U)\) is an ideal in \(\mathcal{E}_M(U)\). The special Colombeau algebra on \(U\) is defined as 

\[ \mathcal{G}(U) := \mathcal{E}_M(U)/\mathcal{N}(U). \]

The class of a moderate net \((u_ε)_ε\) in this quotient space will be denoted by \[\{(u_ε)_ε\}].\] A generalized function on some open subset \(U\) of \(\mathbb{R}^n\) with values in \(\mathbb{R}^m\) is given as an \(m\)-tuple \((u_1, \ldots, u_m)\) \(\in \mathcal{G}(U)^m\) of generalized functions \(u_j \in \mathcal{G}(U)\) where \(j = 1, \ldots, m\).

\(U \to \mathcal{G}(U)\) is a fine sheaf of differential algebras on \(\mathbb{R}^n\).

The composition \(v \circ u\) of two arbitrary generalized functions is not defined, not even if \(v\) is defined on the whole of \(\mathbb{R}^m\) (i.e., if \(u \in \mathcal{G}(U)^m\) and \(v \in \mathcal{G}(\mathbb{R}^m)^l\)). A convenient condition for \(v \circ u\) to be defined is to require \(u\) to be "compactly bounded" (c-bounded) into the domain of \(v\). Since there is a certain inconsistency in [10] concerning the precise description of c-boundedness (see [5, Section 2] for details) we include the explicit definition of this important property below.

For a full discussion, see again [5, Section 2].

2.1. Definition. Let \(U\) and \(V\) be open subsets of \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively.

1. An element \((u_ε)_ε\) of \(\mathcal{E}_M(U)^m\) is called c-bounded from \(U\) into \(V\) if the following conditions are satisfied:

   i) There exists \(ε_0 \in (0, 1]\), such that \(u_ε(U) \subseteq V\) for all \(ε \leq ε_0\).

   ii) For every \(K \subset U\) there exist \(L \subset V\) and \(ε_0 \in (0, 1]\) such that \(u_ε(K) \subseteq L\) for all \(ε \leq ε_0\).

The collection of c-bounded elements of \(\mathcal{E}_M(U)^m\) is denoted by \(\mathcal{E}_M[U,V]\).

2. An element \(u\) of \(\mathcal{G}(U)^m\) is called c-bounded from \(U\) into \(V\) if it has a representative which is c-bounded from \(U\) into \(V\). The space of all c-bounded generalized functions from \(U\) into \(V\) will be denoted by \(\mathcal{G}[U,V]\).

2.2. Proposition. Let \(u \in \mathcal{G}(U)^m\) be c-bounded into \(V\) and let \(v \in \mathcal{G}(V)^l\), with representatives \((u_ε)_ε\) and \((v_ε)_ε\), respectively. Then the composition

\[ v \circ u := [(v_ε \circ u_ε)_ε] \]

is a well-defined generalized function in \(\mathcal{G}(U)^l\).
Generalized functions can be composed with smooth classical functions provided they do not grow “too fast”: The space of slowly increasing smooth functions is given by

$$\mathcal{O}_M(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) | \forall \alpha \in \mathbb{N}_0^n \exists N \in \mathbb{N}_0 \exists C > 0 : |\partial^\alpha f(x)| \leq C(1 + |x|)^N \forall x \in \mathbb{R}^n \}.$$

2.3. Proposition. If \( u = [(u_\varepsilon)_{\varepsilon}] \in \mathcal{G}(U)^m \) and \( v \in \mathcal{O}_M(\mathbb{R}^m) \), then

$$v \circ u := [(v \circ u_\varepsilon)_{\varepsilon}]$$

is a well-defined generalized function in \( \mathcal{G}(U) \).

We call \( \mathcal{R} := \mathcal{E}_M/N \) the ring of generalized numbers, where

$$\mathcal{E}_M := \{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} | \exists N \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \},$$

$$\mathcal{N} := \{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} | \forall m \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \to 0 \}.$$

For \( u := [(u_\varepsilon)_{\varepsilon}] \in \mathcal{G}(U) \) and \( x_0 \in U \), the point value of \( u \) at \( x_0 \) is defined as the class of \((u_\varepsilon(x_0))_{\varepsilon}\) in \( \mathcal{R} \).

On

$$U_M := \{(x_\varepsilon)_\varepsilon \in U^{(0,1)} | \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \iff \forall m \in \mathbb{N} : |x_\varepsilon - y_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by \( \bar{U} := U_M/\sim \) the set of generalized points. For \( U = \mathbb{R} \) we have \( \bar{\mathbb{R}} = \mathcal{R} \). Thus, we have the canonical identification \( \mathbb{R}^n = \bar{\mathbb{R}}^n = \mathbb{R}^n \).

The set of compactly supported points is

$$\bar{U}_c := \{ \bar{x} = [(x_\varepsilon)_{\varepsilon}] \in \bar{U} | \exists K \subset \subset U \exists \varepsilon_0 \in (0,1] \forall \varepsilon \leq \varepsilon_0 : x_\varepsilon \in K \}.$$ 

Obviously, for \( u \in \mathcal{G}(U) \) and \( \bar{x} \in \bar{U}_c \), \( u(\bar{x}) \) is a generalized number, the generalized point value of \( u \) at \( \bar{x} \).

A point \( \bar{x} \in \bar{U}_c \) is called near-standard if there exists \( x \in U \) such that \( x_\varepsilon \to x \) as \( \varepsilon \to 0 \) for one (thus, for every) representative \((x_\varepsilon)_{\varepsilon}\) of \( x \). In this case we write \( \bar{x} \approx x \).

Two generalized functions are equal in the Colombeau algebra if and only if their generalized point values coincide at all compactly supported points ([10, Theorem 1.2.46]). By [13], it is even sufficient to check the values at all near-standard points. We will need a slightly refined result which is easy to prove using the techniques of [10, Theorem 1.2.46] and [13]:

2.4. Proposition. Let \( u \in \mathcal{G}(U \times V) \). Then

$$u = 0 \text{ in } \mathcal{G}(U \times V) \iff u(\cdot, \bar{y}) = 0 \text{ in } \mathcal{G}(U) \text{ for all near-standard points } \bar{y} \in \bar{V}_c.$$
3 Local existence and uniqueness results for ODEs

In the first theorem of this section we give sufficient conditions to guarantee a (unique) solution of the local initial value problem

\[ \dot{u}(t) = F(t, u(t)), \quad u(\tilde{t}_0) = \tilde{x}_0, \]  

where \( I \) is an open interval in \( \mathbb{R} \), \( U \) an open subset of \( \mathbb{R}^n \), \( F \in \mathcal{G}(I \times U)^n \), \( \tilde{t}_0 \in \tilde{I}_c \) and \( \tilde{x}_0 \in \tilde{U}_c \). A generalized function \( u \in \mathcal{G}[J, U] \) (where \( J \) is some open subinterval of \( I \)) is called a (local) solution of (1) on \( J \) around \( \tilde{t}_0 \in \tilde{I}_c \) with initial value \( \tilde{x}_0 \) if the differential equation in (1) is satisfied in \( \mathcal{G}(J)^n \) and the initial condition in (1) is satisfied in the set \( \tilde{U} \) of generalized points.

Reflecting our decision to employ the concept of c-boundedness to ensure the existence of compositions, a solution on some subinterval \( J \) of \( I \) will be a c-bounded generalized function from \( J \) into \( U \) satisfying (1). Due to the c-boundedness of \( u \) the requirement for \( \tilde{x}_0 \) to be compactly supported in fact does not constitute a restriction.

Theorem 3.1 generalizes Theorem 5.2 of [4] insofar as the domain of existence of the local solution precisely equals the one in the classical case whereas the solution in [4] is only defined on a strictly smaller interval. Moreover, the present version establishes uniqueness with respect to the largest sensible target space (i.e., \( U \)), as opposed to the more restricted statement in [4].

3.1. Theorem. Let \( I \) be an open subinterval of \( \mathbb{R} \), \( U \) an open subset of \( \mathbb{R}^n \), \( \tilde{t}_0 \) a near-standard point in \( \tilde{I}_c \) with \( \tilde{t}_0 \approx t_0 \in I \), \( \tilde{x}_0 \in \tilde{U}_c \) and \( F \in \mathcal{G}(I \times U)^n \).

Let \( \alpha \) be chosen such that \( [t_0 - \alpha, t_0 + \alpha] \subset I \). Let \( (\tilde{x}_0\epsilon)_{\epsilon} \) be a representative of \( \tilde{x}_0 \) and \( L \subset \subset U \), \( \tilde{\epsilon}_0 \in (0,1] \) such that \( \tilde{x}_0\epsilon \in L \) for all \( \epsilon \leq \tilde{\epsilon}_0 \). With \( \beta > 0 \) satisfying \( L_{\beta} := L + B_{\beta}(0) \subset \subset U \) set

\[ Q := [t_0 - \alpha, t_0 + \alpha] \times L_{\beta} \quad (\subset \subset I \times U). \]

Assume that \( F \) has a representative \( (F_{\epsilon})_{\epsilon} \) satisfying

\[ \sup_{(t,x) \in Q} |F_{\epsilon}(t,x)| \leq a \quad (\epsilon \leq \tilde{\epsilon}_0) \]  

for some constant \( a > 0 \). Then the following holds:

(i) The initial value problem

\[ \dot{u}(t) = F(t, u(t)), \quad u(\tilde{t}_0) = \tilde{x}_0, \]  

has a solution \( u \in \mathcal{G}[J, W] \) where \( J = (t_0 - h, t_0 + h) \) with \( h = \min(\alpha, \frac{\tilde{\epsilon}_0}{a}) \) and \( W = L + B_{2\beta}(0) \).

(ii) Every solution of (3) in \( \mathcal{G}[J, U] \) is already an element of \( \mathcal{G}[J, W] \).

(iii) The solution of (3) is unique in \( \mathcal{G}[J, U] \) if, in addition to (2),

\[ \sup_{(t,x) \in J \times W} |\partial_{2}F_{\epsilon}(t,x)| = O(|\log \epsilon|) \]  

holds.
Proof. Throughout the proof, it suffices to consider only values of \( \varepsilon \) not exceeding \( \varepsilon_0 \). Moreover, we can assume without loss of generality that

\[
|\tilde{t}_0 - t_0| \leq \frac{h}{4} \quad \text{holds for all } \varepsilon \leq \varepsilon_0. \tag{5}
\]

(i) In a first step we fix \( \varepsilon \) and solve the (classical) initial value problem

\[
\dot{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t)), \quad u_\varepsilon(t_0) = \tilde{x}_0, \tag{6}
\]
on a suitable subinterval of \([t_0 - h, t_0 + h]\). To this end, set

\[
\delta_\varepsilon := \sup \{ |\tilde{t}_0 - t_0| \mid 0 < \varepsilon' \leq \varepsilon \} \quad \text{and} \quad J_\varepsilon := [t_0 - h + \delta_\varepsilon, t_0 + h - \delta_\varepsilon],
\]

both for \( \varepsilon \leq \varepsilon_0 \); note that \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). By this choice, we have \( J_\varepsilon \subseteq [t_0 - h, t_0 + h] \). Indeed, from \( t \in J_\varepsilon \) we infer \( |t - t_0| \leq |t - t_0| + |t_0 - \tilde{t}_0| \leq h - \delta_\varepsilon + \delta_\varepsilon \). The solution \( u_\varepsilon \) of (6) now is obtained as the fixed point of the operator \( T_\varepsilon : X_\varepsilon \to X_\varepsilon \) defined by

\[
(T_\varepsilon f)(t) := \tilde{x}_0 + \int_{t_0}^t F_\varepsilon(s, f(s)) \, ds \quad (t \in J_\varepsilon)
\]

where \( X_\varepsilon := \{ f : J_\varepsilon \to L_\beta \mid f \text{ is continuous} \} \) becomes a complete metric space when being equipped with the metric \( d(f, g) := \|f - g\|_\infty = \sup_{t \in J_\varepsilon} |f(t) - g(t)| \). That \( T_\varepsilon \) in fact maps \( X_\varepsilon \) into \( X_\varepsilon \) is immediate from

\[
| (T_\varepsilon f)(t) - \tilde{x}_0 | \leq \left| \int_{t_0}^t | F_\varepsilon(s, f(s)) | \, ds \right| \leq a \cdot |t - t_0| \tag{7}
\]

by noting that \( a \cdot |t - t_0| \leq ah \leq \beta \) for \( t \in J_\varepsilon \).

Now the existence of a fixed point of \( T_\varepsilon \) (hence, of a solution of (6)) follows from Weissinger’s fixed point theorem ([17, §1], [8, I.1.6 (A5)]) by the following argument: A variant of [10, Lemma 3.2.47] referring only to the second slot (see [4, Remark 3.12] for an explicit variant theorem) yields a positive constant \( \gamma \) (depending on \( \varepsilon \)) such that \( |F_\varepsilon(t, x) - F_\varepsilon(t, y)| \leq \gamma |x - y| \) for all \( (t, x), (t, y) \in Q \). From this we derive, by induction, \( |(T_\varepsilon^k f)(t) - (T_\varepsilon^k g)(t)| \leq \frac{\gamma}{k!}(t - \tilde{t}_0)^k \|f - g\|_\infty \) for \( t \in [\tilde{t}_0, t_0 + h - \delta_\varepsilon] \) and \( k \in \mathbb{N}_0 \). The case of \( t \in [t_0 - h + \delta_\varepsilon, \tilde{t}_0] \) being similar, we finally arrive at \( \|T_\varepsilon^k f - T_\varepsilon^k g\|_\infty \leq \frac{(h\varepsilon')^k}{k!} \|f - g\|_\infty \) which, due to \( \sum_{k=0}^{\infty} (h\varepsilon')^k = e^{h\varepsilon'} < \infty \), suffices for an appeal to Weissinger’s theorem. We obtain a solution \( u_\varepsilon \) of (6) on \( J_\varepsilon \) taking values in \( L_\beta \). Moreover, \( u_\varepsilon(t) \in W := L + B_\beta(0) \) for \( t \in J_\varepsilon \) by (7).

If \( \delta_\varepsilon = 0 \) (i.e., if \( t_0 \) is standard) then \( u_\varepsilon \) is defined on \([t_0 - h, t_0 + h]\) and we set \( \tilde{u}_\varepsilon := u_\varepsilon \); by (7), \( \tilde{u}_\varepsilon(J) \subseteq W \). In the case \( \delta_\varepsilon > 0 \), Lemma 3.3 provides \( \tilde{u}_\varepsilon \in C^\infty([t_0 - h, t_0 + h], W) \) being equal to \( u_\varepsilon \) on \( J_\varepsilon \). We now obtain, by differentiating \( \tilde{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t)) \), an estimate of the form

\[
|\tilde{u}_\varepsilon(t)| \leq |\partial_1 F_\varepsilon(t, u_\varepsilon(t))| + |\partial_2 F_\varepsilon(t, u_\varepsilon(t))| \cdot |\tilde{u}_\varepsilon(t)| \leq C \varepsilon^{-N}
\]
with constants $C > 0$ and $N \in \mathbb{N}$ not depending on $\varepsilon$. The estimates for the higher-order derivatives of $u_\varepsilon$ are now obtained inductively by differentiating the equation for $\tilde{u}_\varepsilon$.

Concerning $C$-boundedness of $(\tilde{u}_\varepsilon)_\varepsilon$ from $J$ into $W$ let $J^1 := [t_0 - h', t_0 + h']$ with $\frac{4}{3} < h' < h$. For $\varepsilon$ small enough as to satisfy $2\delta \varepsilon \leq h - h'$, we have $J^1 \subseteq \tilde{J}_\varepsilon$.

(7) now yields $\tilde{u}_\varepsilon(J^1) = u_\varepsilon(J^1) \subseteq \tilde{L} + \tilde{B}_{\tilde{u}(h', \delta \varepsilon)} \subseteq \tilde{L} + B_{\beta}(0)$.

Now that we have shown that the net $(\tilde{u}_\varepsilon)_\varepsilon$ represents a member of $\mathcal{G}[J, W]$ ($\subseteq \mathcal{G}[J, U]$), it follows from the result established for fixed $\varepsilon$ that the class of $(\tilde{u}_\varepsilon)_\varepsilon$ is a solution of (3) on $J$ in the sense specified at the beginning of this section:

Due to the fact that equality in Colombeau spaces involves null estimates only on compact subsets of the domain, it indeed suffices that every $\tilde{u}_\varepsilon$ satisfies the (classical) equation on $\tilde{J}_\varepsilon$, taking into account $\delta \varepsilon \to 0$.

(ii) Assume that $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[J, U]$ satisfies $\dot{v}(t) = F(t, v(t))$ and $v(\tilde{t}_0) = \tilde{x}_0$. With $t_0 = \tilde{x}_0$ and $F_\varepsilon$ as in part (i) we have $v_\varepsilon(t_0) = \tilde{x}_0 + \eta_\varepsilon$ and $\dot{v}_\varepsilon(t) = F_\varepsilon(t, v_\varepsilon(t)) + n_\varepsilon(t)$ for some $(\eta_\varepsilon)_\varepsilon \in \mathcal{N}^n$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}(\mathcal{N})^n$, respectively.

In order to show that $v \in \mathcal{G}[J, W]$ with $W = L + B_{\beta}(0)$ we again choose $J^1 = [t_0 - h', t_0 + h'] \subseteq J$ with $\frac{4}{3} < h' < h$. Setting $\delta := \frac{1}{3}(h - h')$, we select $\varepsilon_1$ such that $\varepsilon_1$, the three conditions $|\eta_\varepsilon| < \frac{4}{3}, \int_{\tilde{J}_\varepsilon} |\eta_\varepsilon| ds < \frac{4}{3}$ and $a|\delta \varepsilon| < \frac{4}{3}$ are satisfied. Now for $\varepsilon \leq \varepsilon_1$, we claim that $|v_\varepsilon(t) - \tilde{x}_0| \leq \frac{2}{3}(h + h')$ holds for all $t \in J^1_\varepsilon := [t_0, t_0 + h']$ if $|v_\varepsilon(t) - \tilde{x}_0| < \frac{2}{3}(h + h')$ for all $t \in J^1$, then we are done. Otherwise, choose $t^\ast$ minimal in $J^1_\varepsilon$ with $|v_\varepsilon(t^\ast) - \tilde{x}_0| = \frac{2}{3}(h + h')$.

We demonstrate that, in fact, $t^\ast = t_0 + h'$. From the estimate

$$
a \frac{1}{2} (h + h') = |v_\varepsilon(t^\ast) - \tilde{x}_0| \leq |\eta_\varepsilon| + \int_{t_0 \varepsilon}^{t^\ast} |\eta_\varepsilon| ds + \int_{t_0 \varepsilon}^{t^\ast} |F_\varepsilon(t, v_\varepsilon(t))| ds \\
\leq \frac{\delta}{3} + \frac{\delta}{3} + a|\delta \varepsilon| + a(t^\ast - t_0) \\
\leq \frac{a}{2} (h - h') + a(t^\ast - t_0)
$$

it readily follows that $t^\ast \geq t_0 + h'$, and thus $t^\ast = t_0 + h'$. Since, by a similar argument, $|v_\varepsilon(t) - \tilde{x}_0| \leq \frac{2}{3}(h + h')$ holds also for all $t \in J^1_\varepsilon := [t_0 - h', t_0 + h']$, we finally arrive at

$$v_\varepsilon(J^1) \subseteq \tilde{L} + \tilde{B}_{\tilde{u}(h, \delta \varepsilon)}(0) \subseteq \tilde{L} + B_{\beta}(0) = W.$$

This proves that $v$ is $C$-bounded from $J$ into $W$.

(iii) Let $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[J, U]$ be another solution and $(n_\varepsilon)_\varepsilon \in \mathcal{N}^n$, $(\eta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathcal{N})^n$ as above. By (ii), $v \in \mathcal{G}[J, W]$. As before let $J^1 := [t_0 - h', t_0 + h']$ (with $\frac{4}{3} < h' < h$) be a compact subinterval of $J$. Since both $(\eta_\varepsilon)_\varepsilon$ and $(n_\varepsilon)_\varepsilon$ are $C$-bounded from $J$ into $W$, there exists a compact subset $K$ of $W$ such that $u_\varepsilon(J^1) \subseteq K$ and $v_\varepsilon(J^1) \subseteq K$ for $\varepsilon$ sufficiently small. Moreover, we can assume $\delta \varepsilon < h - h'$. Applying the second-slot version of [10, Lemma 3.2.47] to the function $F_\varepsilon$ and some (fixed) compact set $K'$ with $K \subset K' \subset \subset W = L + B_{\beta}(0)$ yields a constant $C'$ (only depending on $K'$) such that

$$|F_\varepsilon(t, x) - F_\varepsilon(t, y)| \leq C' \sup_{(s, z) \in J \times K'} (|F_\varepsilon(s, z)| + |\partial_2 F_\varepsilon(s, z)|) \cdot |x - y| \\
\leq C' (a + C_1 |\log \varepsilon|) \cdot |x - y|.$$
holds for all \( t \in J^1 \) and all \( x, y \in K \) (note that \( J^1 \times K' \subseteq J \times W \subseteq Q \)) where \( C_1 > 0 \) is the constant provided by (4). Therefore, for \( t \in J^1 \) it follows that 

\[
|v_\varepsilon(t) - u_\varepsilon(t)| \leq \left| \tilde{y}_{0\varepsilon} - \tilde{x}_{0\varepsilon} \right| + \left| \int_{t_{0\varepsilon}}^t \left( |F_\varepsilon(s, v_\varepsilon(s)) - F_\varepsilon(s, u_\varepsilon(s))| + |n_\varepsilon(s)| \right) \, ds \right|
\]

\[
\leq |\tilde{v}_{0\varepsilon}| + \left| \int_{t_{0\varepsilon}}^t |n_\varepsilon(s)| \, ds \right| + C'(a + C_1 |\log \varepsilon|) \cdot \left| \int_{t_{0\varepsilon}}^t |v_\varepsilon(s) - u_\varepsilon(s)| \, ds \right|
\]

\[
\leq C_2 \varepsilon^m + (C_3 + C_4 |\log \varepsilon|) \cdot \left| \int_{t_{0\varepsilon}}^t |v_\varepsilon(s) - u_\varepsilon(s)| \, ds \right|
\]

for suitable constants \( C_2, C_3, C_4 > 0 \) and arbitrary \( m \in \mathbb{N} \). By Gronwall’s Lemma, we obtain

\[
\sup_{t \in J^1} |v_\varepsilon(t) - u_\varepsilon(t)| \leq C_2 \varepsilon^m \cdot e^{(C_3 + C_4 |\log \varepsilon|) |f|_{1,t_{0\varepsilon}} 1 \, ds|} \leq C_0 \varepsilon^{m-h} C_4
\]

for some constant \( C_0 > 0 \) (note that \( |\tilde{y}_{0\varepsilon} - t_0| \leq h' + \delta_\varepsilon < h \)). This concludes the proof of the theorem.

3.2. Remark. (i) The proof of Theorem 3.1 establishes the following statement on the level of representatives: For any given representatives \( (\tilde{t}_{0\varepsilon})_\varepsilon \) of \( \tilde{t}_0 \) \( (\tilde{t}_{0\varepsilon} \to t_0 \in I) \), \( (\tilde{x}_{0\varepsilon})_\varepsilon \) of \( \tilde{x}_0 \in U_\varepsilon \) and \( (F_\varepsilon)_\varepsilon \) of \( F \in \mathcal{G}(I \times U)^n \) satisfying (2) the following holds: If \( \alpha, L, \varepsilon_0 \) and \( \beta \) are chosen as in Theorem 3.1 (including condition (5) as to \( \varepsilon_0 \)), then \( u \) has a representative \( (\tilde{u}_\varepsilon)_\varepsilon \) that on every compact subinterval of \( J \) satisfies the classical initial value problem (6) for \( \varepsilon \) sufficiently small.

(ii) If \( \tilde{t}_0 \) is standard, i.e. (without loss of generality) \( \tilde{t}_{0\varepsilon} = t_0 \in I \) for all \( \varepsilon \), then \( \delta_\varepsilon = 0 \) and every \( u_\varepsilon \) exists (as a solution of (6)) even on \([t_0 - h, t_0 + h]\).

(iii) If \( \tilde{x}_0 \) is standard, i.e. (without loss of generality) \( \tilde{x}_{0\varepsilon} = x_0 \in U \) for all \( \varepsilon \), then \( L := \{x_0\} \) yields \( L_\beta = B_\beta(x_0) \) as in the classical case.

3.3. Lemma. (i) Let \( a < a_1 < a_2 < b_2 < b_1 < b \) and \( L \) be a (non-empty) open subset of \( \mathbb{R}^n \). Then for \( f \in C^\infty([a, b], U) \) being given, there exists \( \tilde{f} \in C^\infty([a, b], U) \) with \( \tilde{f} = f \) on some open neighbourhood of \([a_2, b_2]\).

(ii) For any given positive \( \delta \), the function \( \tilde{f} \) can be chosen such as to satisfy 

\[
\tilde{f}([a, b]) \subseteq f([a_1, b_1]) \cup B_\delta(f(a_1)) \cup B_\delta(f(b_1)).
\]

Proof. (i) Choose \( \delta > 0 \) as to satisfy \( B_\delta(f(a_1)) \cup B_\delta(f(a_2)) \subseteq U \). Choose \( \eta > 0 \) such that \( f(t) \in B_\delta(f(a_1)) \) holds for \( t \in [a_1, a_1 + 2\eta] \) and \( f(t) \in B_\delta(f(b_1)) \) holds for \( t \in [b_1 - 2\eta, b_1] \); without loss of generality we may assume \( \eta < \frac{1}{2} \) \( \min(a_2 - a_1, b_1 - b_2) \).

Now let \( \psi \) be a smooth function with \( 0 \leq \psi \leq 1 \) such that \( \psi = 1 \) on \([a_1 + 2\eta, b_1 - 2\eta]\) and \( \psi = 0 \) outside \([a_1 + \eta, b_1 - \eta]\). Then \( f \) defined on \([a, b]\) by

\[
\tilde{f}(t) := \begin{cases} 
  f(a_1) & t \in [a, a_1 + \eta] \\
  f(t) \psi(t) + f(a_1)(1 - \psi(t)) & t \in [a_1, a_2] \\
  f(t) & t \in [a_1 + 2\eta, b_1 - 2\eta] \\
  f(t) \psi(t) + f(b_1)(1 - \psi(t)) & t \in [b_2, b_1] \\
  f(b_1) & t \in [b_1 - \eta, b] 
\end{cases}
\]
satisfies all requirements since each of the five defining terms is smooth and on overlaps the two relevant terms give rise to the same values.

(ii) is clear from the proof of (i).

Theorem 3.1 is distinguished from the related result [7, Theorem 4.5] by the following features: The existence statement (i) of Theorem 3.1 does not require logarithmic control of derivatives of \( F \) which, by contrast, is assumed in [7]; the domain interval of the solution in Theorem 3.1 equals the classical (open) one given by \( (t_0 - h, t_0 + h) \) with \( h = \min(\alpha, \frac{\beta}{2}) \) while in [7] one has to take \( h < \min(\alpha, \frac{\beta}{2}) \); finally, the boundedness assumption on \( F \) in [7] refers to the whole open domain of \( F \) whereas in Theorem 3.1 it suffices to have boundedness of \( F \) on the (compact) subset \( Q \). Generally, all existence and uniqueness results for ODEs in [7] are tailored for applications of the method of characteristics to the generalized Hamilton-Jacobi problem; hence the setting of [7] always includes initial conditions as parameters, necessitating the logarithmic growth condition even for existence results (compare Theorem 3.8 below).

The following three examples illustrate the significance of the boundedness assumption on \( F \) by displaying increasing obstacles against obtaining a generalized solution from the classical ones obtained for fixed \( \varepsilon \), in the absence of condition (2).

3.4. Example. Let \( F \in \mathcal{G}(\mathbb{R} \times \mathbb{R}) \) be given by the representative \( F_\varepsilon(t, x) := \frac{1}{2}(2 - \frac{1}{1+e^2}) \), and let \( t_0 = 0 \) and \( x_0 = 0 \). Then \( F \) fails to satisfy condition (2) on any neighbourhood of \( (t_0, x_0) \). Nevertheless, there exists a unique global solution for every \( \varepsilon \). Integrating \( \dot{x}(t) = F_\varepsilon(t, x) \) yields \( \frac{x}{2} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} x) = \frac{1}{2} t \). Setting \( f(x) = \frac{x}{2} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} x) \), we obtain \( u_\varepsilon(t) := f^{-1}(\frac{1}{2} t) \) as the solution of the classical initial value problem. By Proposition 2.3, \( (u_\varepsilon)_\varepsilon \in \mathcal{E}_A(\mathbb{R}) \). However, \( (u_\varepsilon)_\varepsilon \) is not \( c \)-bounded. Hence, \( u_\varepsilon \) solves the differential equation for every \( \varepsilon \) but on any interval around 0, the generalized function \( (u_\varepsilon)_\varepsilon \) is not a solution of the initial value problem in the setting of the \( c \)-bounded theory of ODEs since the composition \( F(t, u(t)) \) exists only componentwise on the level of representatives, yet not in the sense of Proposition 2.2.

3.5. Example. Let \( F \in \mathcal{G}(\mathbb{R} \times \mathbb{R}) \) be given by the representative \( F_\varepsilon(t, x) := \frac{2}{\varepsilon} \), and let \( t_0 = 0 \) and \( x_0 = 1 \). Again, \( F \) does not satisfy condition (2) on any neighbourhood of \( (t_0, x_0) \). For each \( \varepsilon \), there exists a unique (even global) solution \( u_\varepsilon(t) = e^{\frac{2}{\varepsilon}} \). However, \( (u_\varepsilon)_\varepsilon \) is not moderate on any neighbourhood of 0.

3.6. Example. Let \( F \in \mathcal{G}(\mathbb{R} \times (\mathbb{R} \setminus \{-1\})) \) be defined by the representative \( F_\varepsilon(t, x) := -\frac{t}{x+1} \cdot g(\varepsilon) \) where \( g : (0, 1] \rightarrow \mathbb{R} \) is a smooth map satisfying \( g(\varepsilon) \rightarrow \infty \) for \( \varepsilon \rightarrow 0 \). Let \( t_0 = 0 \) and \( x_0 = 0 \). Then \( F \) violates condition (2) on any neighbourhood of \( (t_0, x_0) \). For every \( \varepsilon \) we obtain (unique) solutions \( u_\varepsilon(t) = \sqrt{1 - g(\varepsilon)T^2} - 1 \) that are defined, at most, on the open interval \( (-1/\sqrt{g(\varepsilon)}, 1/\sqrt{g(\varepsilon)}) \). Hence, there is not even a common domain.

In this example, \( F \) failing to satisfy condition (2) leads to shrinking of the solutions’ domains as \( \varepsilon \rightarrow 0 \). Note that this result is not a consequence of the rate of growth of \( |F_\varepsilon(t, x)| \) on any compact set; rather, it only matters that \( |F_\varepsilon(t, x)| \) does increase infinitely (as \( \varepsilon \rightarrow 0 \)).

Theorem 3.1 can handle jumps as the following example shows.
3.7. Example. Let $I$ be an open interval in $\mathbb{R}$ and $U$ an open subset of $\mathbb{R}^n$. Consider the initial value problem

$$
\dot{u}(t) = f(t, u(t)) \cdot (\iota H)(t) + g(t, u(t)), \quad u(t_0) = x_0,
$$

where $f, g \in C^\infty(I \times U, \mathbb{R}^n)$, $t_0 \in I$, $x_0 \in U$, and where $\iota H$ denotes the embedding of the Heaviside function $H$ into the Colombeau algebra. If $\rho$ is a mollifier (i.e., a Schwartz function on $\mathbb{R}$ satisfying $\int \rho(x) \, dx = 1$ and $\int x^\alpha \rho(x) \, dx = 0$ for all $\alpha \geq 1$), then a representative $(\iota H)_\varepsilon$ of $\iota H$ is given by $H\varepsilon(t) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon} \rho \left( \frac{t-x}{\varepsilon} \right) \, dx$.

Fix some $\alpha > 0$ such that $[t_0 - \alpha, t_0 + \alpha] \subseteq I$ and choose an open subset $W$ of $U$ with $x_0 \in W \subseteq \overline{W} \subset U$. A short computation shows that $|H\varepsilon(t)| \leq \|\rho\|_{L^1(\mathbb{R})}$ for all $t$. Thus, $|f(t, x) \cdot H\varepsilon(t) + g(t, x)| \leq a_1 \|\rho\|_{L^1(\mathbb{R})} + a_2 =: a$ on $[t_0 - \alpha, t_0 + \alpha] \times \overline{W}$ for some constants $a_1, a_2 > 0$. Hence, by Theorem 3.1, the initial value problem (8) possesses a solution $u$ in $\mathcal{G}[J, W]$ where $J := (t_0 - h, t_0 + h)$ and $h = \min \left( \alpha, \frac{\text{dist}(\alpha, \partial W)}{a} \right)$. Since the initial value problem also satisfies (4), the solution is unique in $\mathcal{G}[J, U]$. 

Next, we turn our attention to generalized ODEs including parameters. In view of our goal to establish a Frobenius theorem in the present setting, we want the solution to be $G$-dependent on the parameter.

3.8. Theorem. Let $I$ be an open subinterval of $\mathbb{R}$, $U$ an open subset of $\mathbb{R}^n$, $P$ an open subset of $\mathbb{R}^n$, $t_0$ a near-standard point in $\mathcal{I}_\varepsilon$ with $t_0 \approx t_0 \in \mathcal{I}_\varepsilon$ and $F \in \mathcal{G}(I \times U \times P)$. 

Let $I_\varepsilon$ be chosen such that $[t_0 - \alpha, t_0 + \alpha] \subset I$. Let $(\tilde{x}_0 \varepsilon) \varepsilon$ be a representative of $\tilde{x}_0$ and $L \subset U$, $\tilde{x}_0 \in (0, 1]$ such that $\tilde{x}_0 \varepsilon \in L$ for all $\varepsilon \leq \varepsilon_0$. With $\beta > 0$ satisfying $L_\beta := L + B_\beta(0) \subset U$ set

$$
Q := [t_0 - \alpha, t_0 + \alpha] \times L_\beta \quad (\subset \subset I \times U).
$$

Assume that $F$ has a representative $(F_\varepsilon) \varepsilon$ satisfying

$$
\sup_{(t, x, p) \in Q \times P} |F_\varepsilon(t, x, p)| \leq a \quad (\varepsilon \leq \varepsilon_0)
$$

for some constant $a > 0$ and that for all compact subsets $K$ of $P$

$$
\sup_{(t, x, p) \in Q \times K} |\partial_2 F_\varepsilon(t, x, p)| = O(|\log \varepsilon|).
$$

Then the following holds: There exists $u \in \mathcal{G}[P \times J, W]$ with $J := [t_0 - h, t_0 + h]$, $h = \min \left( \alpha, \frac{\varepsilon_0}{2} \right)$ and $W = L + B_\beta(0)$ such that for all $\tilde{p} \in \tilde{P}_\varepsilon$ the map $u(\tilde{p}, \cdot) \in \mathcal{G}[J, W]$ is a solution of the initial value problem

$$
\dot{u}(t) = F(t, u(t), \tilde{p}), \quad u(t_0) = \tilde{x}_0.
$$

The solution $u$ is unique in $\mathcal{G}[P \times J, U]$. 

Proof. Existence: Let $(\tilde{t}_0 \varepsilon) \varepsilon$ be a representative of $\tilde{t}_0$. Proceeding as in the proof of Theorem 3.1, we set $\delta_\varepsilon := \sup\{[\tilde{t}_0 \varepsilon' - t_0] \mid 0 < \varepsilon' \leq \varepsilon\}$ and $J_\varepsilon := [t_0 - h + \delta_\varepsilon, t_0 + h - \delta_\varepsilon]$. For every $p \in P$ there exists a net of (classical) solutions $u_\varepsilon(p, \cdot) : J_\varepsilon \rightarrow L_\beta$ of the initial value problem

$$
\dot{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t), p), \quad u_\varepsilon(\tilde{t}_0 \varepsilon) = \tilde{x}_0 \varepsilon \quad (\varepsilon \leq \varepsilon_0),
$$

(11)
satisfying \( u_\varepsilon(p,J^c_\varepsilon) \subseteq W \). By the classical Existence and Uniqueness Theorem for ODEs with parameter, the mappings \((p,t) \mapsto u_\varepsilon(p,t)\) are \(C^\infty\). Lemma 3.3 provides \( \tilde{u}_\varepsilon \in C^\infty(P \times [t_0 - h, t_0 + h], W) \) being equal to \( u_\varepsilon \) on \( \tilde{J}_\varepsilon := [t_0 - h + 2\delta_\varepsilon, t_0 + h - 2\delta_\varepsilon] \).

In order to show that \((\tilde{u}_\varepsilon)_\varepsilon\) is moderate on \( J \) it again suffices to establish the corresponding estimates for \((u_\varepsilon)_\varepsilon\). C-boundedness of \((u_\varepsilon)_\varepsilon\) is shown as in the proof of Theorem 3.1.

The moderateness of \((u_\varepsilon)_\varepsilon\) will be shown in three steps: First we consider derivatives with respect to \( t \), then only derivatives with respect to \( p \) and, finally, mixed derivatives.

The \( \mathcal{E}_M \)-estimates for \( u_\varepsilon(p,t) \), \( \partial_2 u_\varepsilon(p,t) \) and all its derivatives with respect to \( t \) are obtained in the same way as in the proof of Theorem 3.1.

Next, we consider the derivatives with respect to \( p \). Differentiating the integral equation corresponding to the initial value problem (on the level of representatives) with respect to \( p \) yields

\[
\partial_1 u_\varepsilon(p,t) = \int_{t_0}^{t} \left( \partial_2 F_\varepsilon(s, u_\varepsilon(p,s), p) \cdot \partial_1 u_\varepsilon(p,s) + \partial_3 F_\varepsilon(s, u_\varepsilon(p,s), p) \right) ds. \tag{12}
\]

Let \( K_1 \times K_2 \subseteq P \times J \) and \((p,t) \in K_1 \times K_2 \). By \( u_\varepsilon(K_1 \times K_2) \subseteq L_\beta \subseteq U \) and (10), we obtain

\[
|\partial_1 u_\varepsilon(p,t)| \leq h C_1 \varepsilon^{-N_1} + \left| \int_{t_0}^{t} C_2|\log \varepsilon| \cdot |\partial_1 u_\varepsilon(p,s)| \, ds \right|
\]

for constants \( C_1, C_2 > 0 \) and some fixed \( N \in \mathbb{N} \). By Gronwall’s Lemma, it follows that

\[
|\partial_1 u_\varepsilon(p,t)| \leq h C_1 \varepsilon^{-N_1} \cdot e^{|\int_{t_0}^{t} C_2|\log \varepsilon| \, ds|} \leq (h C_1) \varepsilon^{-(N_1 + h C_2)}.
\]

Differentiating (12) \( i - 1 \) times with respect to \( p \) \((i \in \mathbb{N})\) gives an integral formula for \( \partial_i^1 u_\varepsilon(p,t) \). Observe that in this formula \( \partial_i^1 u_\varepsilon(p,t) \) itself appears on the right-hand side only once, namely with \( \partial_2 F_\varepsilon(s, u_\varepsilon(p,s), p) \) as coefficient, and that the remaining terms contain only \( \partial_i^1 \)-derivatives of \( u_\varepsilon \) of order less than \( i \). Thus, we may estimate the higher-order derivatives with respect to \( p \) inductively by differentiating equation (12) and applying Gronwall’s Lemma.

Finally, it remains to handle the case of mixed derivatives. For arbitrary \( i \in \mathbb{N} \) we have

\[
\partial_i^1 \partial_2^1 u_\varepsilon(p,t) = \frac{\partial^i}{\partial p^i} \frac{\partial}{\partial t} \left( \tilde{x}_0 + \int_{t_0}^{t} F_\varepsilon(s, u_\varepsilon(p,s), p) \, ds \right) = \frac{\partial^i}{\partial p^i} F_\varepsilon(t, u_\varepsilon(p,t), p).
\] \tag{13}

By carrying out the \( i \)-fold differentiation on the right-hand side of equation (13), we obtain a polynomial expression in \( \partial_i^1 F_\varepsilon(t, u_\varepsilon(p,t), p) \), \( \partial_i^2 F_\varepsilon(t, u_\varepsilon(p,t), p) \) and \( \partial_i^k u_\varepsilon(p,t) \) for \( 1 \leq k \leq i \) all of which satisfy the \( \mathcal{E}_M \)-estimates. The estimates for \( \partial_i^1 \partial_2^1 u_\varepsilon(p,t) \) with \( j \geq 2 \) are now obtained inductively by differentiating equation (13) with respect to \( t \).

**Uniqueness:** By Proposition 2.4, it suffices to show that for every near-standard point \( \tilde{p} \in \tilde{P}_\varepsilon \) the solution \( u(\tilde{p},.) \) is unique in \( \mathcal{G}[J,U] \). For a fixed near-standard point \( \tilde{p} = [(\tilde{p}_z)_\varepsilon]_\varepsilon \in \tilde{P}_\varepsilon \), condition (10) implies the condition for uniqueness (4) in Theorem 3.1 with respect to \( G_\varepsilon(t,x) := (F_\varepsilon(\cdot,\cdot,\tilde{p}_z))_\varepsilon \), yielding uniqueness of \( u(\tilde{p},.) \) in \( \mathcal{G}[J,U] \).
3.9. Remark. Similarly to Remark 3.2 (i), a corresponding statement on the level of representatives can be extracted from the proof of the preceding theorem. Also (ii) and (iii) of Remark 3.2 apply.

Requiring also \( \tilde{x}_0 \) in the initial condition in Theorem 3.8 to be near-standard, we even can prove \( G \)-dependence of the solution on the initial values.

3.10. Theorem. Let \( I \) be an open subinterval of \( \mathbb{R} \), \( U \) an open subset of \( \mathbb{R}^n \), \( P \) an open subset of \( \mathbb{R}^3 \), \( t_0 \) a near-standard point in \( \tilde{I} \), with \( t_0 \approx t_0 \in I \), \( \tilde{x}_0 \) a near-standard point in \( \tilde{U} \), with \( \tilde{x}_0 \approx x_0 \in U \) and \( F \in \mathcal{G}(I \times U \times P)^n \).

With \( \alpha > 0 \) and \( \beta > 0 \) satisfying \( [t_0 - \alpha, t_0 + \alpha] \subset \subset I \) and \( \overline{B}_\beta(x_0) \subset \subset U \), respectively, set

\[
Q := [t_0 - \alpha, t_0 + \alpha] \times \overline{B}_\beta(x_0) \quad \subset \subset I \times U.
\]

Assume that \( F \) has a representative \((F_\varepsilon)_\varepsilon\) satisfying

\[
\sup_{(t,x,p) \in Q \times P} |F_\varepsilon(t,x,p)| \leq a \quad (\varepsilon \leq \varepsilon_0) \quad (14)
\]

for some constant \( a > 0 \) and \( \varepsilon_0 \in (0, 1] \) and that for all compact subsets \( K \) of \( P \)

\[
\sup_{(t,x,p) \in Q \times K} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log \varepsilon|). \quad (15)
\]

Then the following holds: For fixed \( h \in \left(0, \min \left(\alpha, \frac{\alpha}{4}\right)\right) \) there exist open neighbourhoods \( J_1 \) of \( t_0 \) in \( J := (t_0 - h, t_0 + h) \) and \( U_1 \) of \( x_0 \) in \( U \) and a generalized function \( u \in \mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)] \) with \( \gamma \in (0, \beta) \) and \( \beta - \gamma > 0 \) sufficiently small, such that for all \((\tilde{t}, \tilde{x}_1, \tilde{p}) \in \tilde{J}_1 \times \tilde{U}_1 \times \tilde{P} \) the map \( u(\tilde{t}, \tilde{x}_1, \tilde{p}, \ldots) \in \mathcal{G}[J, B_\gamma(x_0)] \) is a solution of the initial value problem

\[
\dot{u}(t) = F(t, u(t), \tilde{p}), \quad u(\tilde{t}_1) = \tilde{x}_1. \quad (16)
\]

The solution \( u \) is unique in \( \mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)] \).

Proof. Existence: The basic strategy of the proof is to consider \((\tilde{t}_0, \tilde{x}_0)\) as part of the parameter and apply Theorem 3.8. However, we will have to cope with some technicalities.

Let \((\tilde{t}_0)_\varepsilon\) and \((\tilde{x}_0)_\varepsilon\) be representatives of \( \tilde{t}_0 \) and \( \tilde{x}_0 \), respectively. From now on, we always let \( \varepsilon \leq \varepsilon_0 \). Let \( \lambda \in (0, 1) \) and set

\[
\tilde{I} := (-\lambda \alpha, \lambda \alpha), \quad I := (t_0 - (1 - \lambda)\alpha, t_0 + (1 - \lambda)\alpha).
\]

Choose \( \mu \in (0, \frac{\alpha}{4}) \), set \( \gamma := \beta - 2\mu \) and define

\[
\tilde{U} := B_{\gamma + \mu}(0), \quad U_1 := B_{\mu}(x_0).
\]

Then \( \tilde{I} + I_1 = (t_0 - \alpha, t_0 + \alpha) \subseteq \tilde{I} \) and \( \tilde{U} + U_1 = B_\beta(x_0) \subseteq U \). Hence, we may define \( G_\varepsilon : \tilde{I} \times \tilde{U} \times (I_1 \times U_1 \times P) \to \mathbb{R}^n \) by

\[
G_\varepsilon(t, x, (t_1, x_1, p)) := F_\varepsilon(t + t_1, x + x_1, p).
\]

Obviously, \((G_\varepsilon)_\varepsilon\) is moderate and, therefore, \( G := [(G_\varepsilon)_\varepsilon] \) is in \( \mathcal{G}(\tilde{I} \times \tilde{U} \times (I_1 \times U_1 \times P))^n \). Now let \( \delta \in (0, \lambda \alpha) \) and \( \eta \in (0, \gamma - \mu) \). By assumptions (14)
and \((15)\), we obtain \(|G(\varepsilon x, (t_1, x_1, p))| \leq a\) for all \((t, x, (t_1, x_1, p)) \in B\varepsilon(0) \times B\varepsilon(0) \times (I_1 \times U_1 \times P)\) and \(|\partial G(\varepsilon x, (t_1, x_1, p))| = O(|\log \varepsilon|)\) for all \(K \subset I_1 \times U_1 \times P\) and \((t, x, (t_1, x_1, p)) \in B\varepsilon(0) \times B\varepsilon(0) \times K\). By Theorem 3.8, there exists \(v \in G[(I_1 \times U_1 \times P) \times \hat{J}, B\varepsilon(0)]\) with \(\hat{J} := (-\hat{h}, \hat{h})\) and \(\hat{h} = \min(\delta, 2)\) such that for all \((\tilde{t}_1, \tilde{x}_1, \tilde{p}) \in \tilde{I}_1 \times \tilde{U}_1 \times \tilde{P}\), the map \(v(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \cdot) \in G[\hat{J}, B\varepsilon(0)]\) is a solution of the initial value problem

\[
\dot{v}(t) = G(t, v(t), (\tilde{t}_1, \tilde{x}_1, \tilde{p})), \quad v(0) = 0.
\]

The solution \(v\) is unique in \(G[(I_1 \times U_1 \times P) \times \hat{J}, \hat{U}]\).

By Remark 3.9, there exists a representative \((v\varepsilon)_\varepsilon\) of \(v\) that satisfies the classical initial value problem for all \((t_1, x_1, p) \in I_1 \times U_1 \times P\) and \(\varepsilon\) sufficiently small. Let \(\varepsilon \in [\varepsilon_1, 1]\), \(\hat{h} := a(h)\) and \(\varepsilon_1 := \min((1 - \varepsilon)\hat{h}, (1 - \lambda)\varepsilon_0)\). Set \(\tilde{J} := (t_0 - h, t_0 + h)\) and \(\tilde{J}_1 := (t_0 - h_1, t_0 + h_1)\). Then \(J_1 \subseteq J \subseteq \tilde{J}\). We now define \(u\varepsilon : J_1 \times U_1 \times P \times J \to \mathbb{R}^n\) by

\[
u\varepsilon(t_1, x_1, p, t) := v\varepsilon(t_1, x_1, p, t - t_1) + x_1.
\]

The map \(u\varepsilon\) is well-defined since \(J_1 \subseteq I_1\) and

\[
|t - t_1| \leq |t - t_0| + |t_0 - t_1| \leq h + h_1 \leq \varepsilon a(h) + (1 - \varepsilon)\hat{h} = \hat{h}.
\]

The moderateness of \((u\varepsilon)_\varepsilon\) is an immediate consequence of the moderateness of \((v\varepsilon)_\varepsilon\). By (18) and since \(x_1 - x_0 \in B\varepsilon(0)\) for all \(x_1 \in U_1\), it follows that

\[
u\varepsilon(J_1 \times U_1 \times P \times \tilde{J}) \subseteq v\varepsilon(I_1 \times U_1 \times P \times \tilde{J}) + x_1 \subseteq B\varepsilon(0) + x_1 \subseteq B\varepsilon(x_0) - x_0 + x_1 \subseteq B\varepsilon(x_0) + B\varepsilon(0) = B\varepsilon(x_0),
\]

i.e., \(u := [(u\varepsilon)_\varepsilon]\) is an element of \(G[J_1 \times U_1 \times P \times J, B\varepsilon(x_0)]\). Furthermore, the function \(u\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \cdot, \cdot)\) satisfies

\[
\frac{\partial}{\partial t} u\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t) = \frac{\partial}{\partial t} (v\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t - \tilde{t}_1) + \tilde{x}_1) = G\varepsilon(t, v\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t - \tilde{t}_1), (\tilde{t}_1, \tilde{x}_1, \tilde{p}, \cdot)) = F\varepsilon(t, v\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t - \tilde{t}_1) + \tilde{x}_1, \tilde{p}) = F\varepsilon(t, u\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t), \tilde{p})
\]

and

\[
u\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \tilde{t}_1) = v\varepsilon(\tilde{t}_1, \tilde{x}_1, \tilde{p}, 0) + \tilde{x}_1 = \tilde{x}_1
\]

for all \((\tilde{t}_1, \tilde{x}_1, \tilde{p}) = ((\tilde{t}_1)_\varepsilon, ((\tilde{x}_1)_\varepsilon), (\tilde{p})_\varepsilon) \in \tilde{I}_1 \times \tilde{U}_1 \times \tilde{P}\) and \(t \in J\). Thus, \(u(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \cdot)\) is a solution of the initial value problem (16).

Note that for any \(h \in (0, \min(\alpha, \frac{2}{\lambda})\) the constants \(\lambda, \mu, \delta, \eta, \hat{h}\) and \(\varepsilon\) can be chosen within their required bounds such that all the necessary inequalities in the construction of \((u\varepsilon)_\varepsilon\) are satisfied.

**Uniqueness:** By Proposition 2.4, it suffices to show that for every near-standard point \((\tilde{t}_1, \tilde{x}_1, \tilde{p}) = ((\tilde{t}_1)_\varepsilon, ((\tilde{x}_1)_\varepsilon), (\tilde{p})_\varepsilon) \in \tilde{I}_1 \times \tilde{U}_1 \times \tilde{P}\) the solution \(u(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \cdot)\) is unique in \(G[J, B\varepsilon(x_0)]\); Let \((\tilde{t}_1, \tilde{x}_1) \to (t_1, x_1) \in J_1 \times U_1\) for \(\varepsilon \to 0\). Assume that \(w(t_1, \tilde{x}_1, \tilde{p}) \in G[J, B\varepsilon(x_0)]\) is another solution of (16). For brevity’s sake we simply write \(w = w(t_1, \tilde{x}_1, \tilde{p})\) and \(w(t_1, \tilde{x}_1, \tilde{p})\), respectively.
We will show that \( w|_{(t_0-r,t_0+r)} = u|_{(t_0-r,t_0+r)} \) holds for any \( r \in (0,h) \). Since \( G \) is a sheaf, the equality of \( w \) and \( u \) then also holds on \( J \).

Now, let \( r \in (0,h) \) and set \( \rho := \frac{1}{2} (h-r) \). Define \( \tilde{w} : B_{r+p}(t_0-t_1) \to B_{\gamma+\mu}(0) \) by \( \tilde{w}(t) := w(t+\tilde{t}_1) - \tilde{x}_1 \). From \( \tilde{t}_1e \to t_1 \) as \( \varepsilon \to 0 \) it follows that \( \tilde{w} \) is well-defined. Then, by the choice of \( \rho \) and Proposition 2.2, \( \tilde{w} \in \mathcal{G}[B_{r+p}(t_0-t_1), B_{\gamma+\mu}(0)] \). Moreover, \( \tilde{w} \) is a solution of the initial value problem (17). Since \( B_{r+p}(t_0-t_1) \subseteq J \) and solutions of (17) are unique in \( \mathcal{G}[J, B_{\gamma+\mu}(0)] \), it follows that \( \tilde{w} = \bar{w}(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \ldots)|_{B_{r+p}(t_0-t_1)} \). Noting that
\[
w(t) = \bar{w}(\tilde{t_1}) + \tilde{x}_1 = v(\tilde{t}_1, \tilde{x}_1, \tilde{p}, t-\tilde{t}_1) + \tilde{x}_1 = u(t),
\]
we finally arrive at \( w|_{(t_0-r,t_0+r)} = u|_{(t_0-r,t_0+r)} \). \( \square \)

3.11. Remark. Concerning representatives, a remark analogous to 3.9 also applies to Theorem 3.10.

4 A Frobenius theorem in generalized functions

In this section, we will use the following notation: By \( \mathbb{R}^{m \times n} \) we denote the space \( \mathbb{R}^m \), viewed as the space of \((m \times n)\)-matrices over \( \mathbb{R} \). A similar convention applies to \( \mathbb{R}^{m \times n} \) and \( \mathcal{G}(U)^{m \times n} \). For any \( u \in \mathcal{G}(U)^m \) the derivative \( Du \) can be regarded as an element of \( \mathcal{G}(U)^{m \times n} \).

Now we are ready to prove a generalized version of the Frobenius Theorem.

4.1. Theorem. Let \( U \) be an open subset of \( \mathbb{R}^n \), \( V \) an open subset of \( \mathbb{R}^n \) and \( F \in \mathcal{G}(U \times V)^{m \times n} \). Let \( \alpha > 0 \) be chosen such that \( B_\alpha(x_0) \subset U \). Let \( (\bar{y}_0, \varepsilon) \) be a representative of \( \bar{y}_0 \) and \( L \subset \subset V, \varepsilon_0 \in (0,1] \) such that \( \bar{y}_{0 \varepsilon} \in L \) for all \( \varepsilon \leq \varepsilon_0 \).

With \( \beta > 0 \) satisfying \( L_\beta := L + B_\beta(0) \subset \subset V \) set
\[
Q := B_\alpha(x_0) \times L_\beta \quad (\subset \subset U \times V).
\]
Assume that \( F \) has a representative \( (F_\varepsilon) \) satisfying
\[
\sup_{(x,y) \in Q} |F_\varepsilon(x,y)| \leq a \quad (\varepsilon \leq \varepsilon_0) \quad (19)
\]
for some constant \( a > 0 \) and
\[
\sup_{(x,y) \in Q} |\partial_2 F_\varepsilon(x,y)| = O(|\log \varepsilon|). \quad (20)
\]

Then the following are equivalent:

(A) For all \( (\bar{x}_0, \bar{y}_0) \in \bar{U}_\varepsilon \times \bar{V}_\varepsilon \) with \( \bar{x}_0 \approx x_0 \in U \) the initial value problem
\[
Du(x) = F(x, u(x)), \quad u(\bar{x}_0) = \bar{y}_0 \quad (21)
\]
has a unique solution \( u(\bar{x}_0, \bar{y}_0) \in \mathcal{G}[U(\bar{x}_0, \bar{y}_0), W] \), where \( U(\bar{x}_0, \bar{y}_0) \) is an open neighbourhood of \( \bar{x}_0 \) in \( U \) and \( W = L + B_\beta(0) \).

(B) The integrability condition is satisfied, i.e., the mapping
\[
(x, y, v_1, v_2) \mapsto DF(x, y)(v_1, F(x, y)(v_1))(v_2) \quad (22)
\]
is symmetric in \( v_1, v_2 \in \mathbb{R}^n \) as a generalized function in \( \mathcal{G}(U \times V \times \mathbb{R}^n \times \mathbb{R}^n)^m \).
Proof. We follow the line of argument of the classical proof based on the ODE theorem with parameters.

(A) ⇒ (B): By Proposition 2.4, we only have to check the integrability condition (22) for all near-standard points \( \tilde{v}_1, \tilde{v}_2 \in \mathbb{R}_c^n \) and \( (\tilde{x}, \tilde{y}) \in \tilde{U}_c \times \tilde{V}_c \). By (A), there exists a solution \( u \) of the initial value problem \( Du(x) = F(x, u(x)), u(\tilde{x}) = \tilde{y} \). Writing \( Du \) as \( D = F \circ (\text{id}, u) \), we obtain

\[
D^2u(\tilde{x})(\tilde{v}_1, \tilde{v}_2) = (D^2u(\tilde{x})(\tilde{v}_1))(\tilde{v}_2) = (D(F \circ (\text{id}, u))(\tilde{x}))(\tilde{v}_1))(\tilde{v}_2)
\]

for all near-standard points \( \tilde{v}_1, \tilde{v}_2 \in \mathbb{R}_c^n \). The last expression is symmetric in \( \tilde{v}_1 \) and \( \tilde{v}_2 \) since, by Schwarz’s Theorem, \( D^2u(\tilde{x}) \) has this property.

(B) ⇒ (A): Let \( \tilde{x}_0 = [(\tilde{x}_{0c})_c] \) be a near-standard point in \( \tilde{U}_c \) with \( \tilde{x}_0 \approx x_0 \) and let \( \tilde{y}_0 \in \tilde{V}_c \).

Existence: Choose \( \delta \in (0, \alpha) \) and set \( \gamma := \alpha - \delta \). We can assume without loss of generality that \( \tilde{x}_{0c} \in B_{\delta}(x_0) \) for all \( \varepsilon \leq \varepsilon_0 \). Then, for \( t \in (-\gamma, \gamma) \) and \( v \in B_1(0) \subseteq \mathbb{R}^n \), we have \( \tilde{x}_{0c} + tv \in B_\alpha(x_0) \subseteq U \) and, thus, the function

\[
G_\varepsilon : \left(-\gamma, \gamma\right) \times V \times B_1(0) \rightarrow \mathbb{R}^m
\]

with parameter \( v \in B_1(0) \). Then the conditions of Theorem 3.8 are satisfied, i.e.,

\[
|G_\varepsilon(t, y, v)| \leq a \quad \text{and} \quad \partial_2G_\varepsilon(t, y, v) = O(\|\log \varepsilon\|)
\]

for all \( (t, y, v) \in \bar{B}_\eta(0) \times L_\beta \times B_1(0) \) with \( \eta \in (0, \gamma) \) fixed. From Theorem 3.8, it follows that there exists a generalized function \( f \in \mathcal{G}[B_1(0) \times J, W] \) with \( J := [-h, h], h := \min(\eta, \frac{\varepsilon_0}{4}) \) and \( W := L + B_\beta(0) \) such that \( f(v, \cdot) \) is a solution of (23) for all \( v \in B_1(0) \). Fix some \( r \in (0, h) \) and \( \lambda \in (0, 1) \) and set

\[
U(\tilde{x}_0, \tilde{y}_0) := B_{\lambda r}(x_0).
\]

Assuming without loss of generality that \( |x_0 - \tilde{x}_{0c}| < (1 - \lambda)r \) for all \( \varepsilon \leq \varepsilon_0 \), the function \( u_\varepsilon(\tilde{x}_0, \tilde{y}_0) : U(\tilde{x}_0, \tilde{y}_0) \rightarrow W \) given by

\[
u_\varepsilon(\tilde{x}_0, \tilde{y}_0)(x) := f_\varepsilon \left( \frac{1}{r}(x - \tilde{x}_{0c}), r \right)
\]

is well-defined. By Proposition 2.2, \( u(\tilde{x}_0, \tilde{y}_0) := [(u_\varepsilon(\tilde{x}_0, \tilde{y}_0))] \in \mathcal{G}[U(\tilde{x}_0, \tilde{y}_0), W] \).

From now on, we will denote \( u(\tilde{x}_0, \tilde{y}_0) \) simply by \( u \).

The fact that \( u \) is indeed a solution of (21) follows from

\[
\partial_1f(v, t)(w) = F(x_0 + tv, f(v, t))(tw) \quad \text{in} \quad \mathcal{G}((-h, h) \times B_1(0) \times \mathbb{R}^n)^m.
\]
Assuming this to be true for the moment, we have

\[
Du(x)(\tilde{w}) = \left( \frac{\partial}{\partial x} f \left( \frac{x - \tilde{x}_0}{r}, r \right) \right)(\tilde{w}) = \partial_1 f \left( \frac{x - \tilde{x}_0}{r}, r \right) \left( \frac{1}{r} \tilde{w} \right)
\]

\[
= F(x, u(x))(\tilde{w})
\]

for all \( \tilde{w} \in \mathbb{R}^n \). Applying Proposition 2.4 to the above equation, we obtain

\[
Du(x) = F(x, u(x)) \text{ in } \mathcal{G}[U(\tilde{x}_0, \tilde{y}_0), W].
\]

Moreover, we observe that \( f(0, .) \) is the (in \( \mathcal{G}[-h, h], W] \) constant function \( t \mapsto \tilde{y}_0 \), and hence we obtain \( u(\tilde{x}_0) = f(\tilde{x}_0 - \tilde{x}_0, r) = \tilde{y}_0 \). Thus, \( u \) is indeed a solution of the initial value problem (21).

To complete the proof of existence, it remains to show (24): Consider the net \( (k_{\varepsilon}) \) given by

\[
k_{\varepsilon}(t, v, w) := \partial_1 f_\varepsilon(v, t)(w) - F\varepsilon(\tilde{x}_{\varepsilon} + tw, f_\varepsilon(v, t))(tw).
\]

Note that, by Proposition 2.2, \( k := [(k_{\varepsilon})_\varepsilon] \) is a well-defined generalized function in \( \mathcal{G}[-h, h] \times B_1(0) \times \mathbb{R}^m \). Let \( \tilde{v} \in B_1(0) \) and \( \tilde{w} \in \mathbb{R}^n \). Differentiating \( k(t, \tilde{v}, \tilde{w}) \) with respect to \( t \), using the fact that \( f(\tilde{v}, .) \) is a solution of (23) and setting \( \tilde{z} = (\tilde{x}_0 + t\tilde{v}, f(\tilde{v}, t)) \), we obtain

\[
\dot{k}(t, \tilde{v}, \tilde{w}) = \partial_1 F(\tilde{z})(t\tilde{w}, \tilde{v}) + \partial_2 F(\tilde{z})(\partial_1 f(\tilde{v}, t)(\tilde{w}), \tilde{v}) - D F(\tilde{z})(\tilde{v}, F(\tilde{z})(\tilde{v}))(t\tilde{w}).
\]

Applying the integrability condition (B) to the last term on the right-hand side, we arrive at

\[
\dot{k}(t, \tilde{v}, \tilde{w}) = \left( \partial_2 F(\tilde{x}_0 + t\tilde{v}, f(\tilde{v}, t))^{\tilde{B}}(\tilde{v}) \right) \cdot k(t, \tilde{v}, \tilde{w}). \tag{25}
\]

Moreover, observe that \( k(0, \tilde{v}, \tilde{w}) = 0 \) in \( \mathbb{R}^m \). Hence, \( k(\tilde{v}, \tilde{w}) \) is a solution of a linear initial value problem. Setting \( A_\varepsilon(t) := \partial_2 F(\tilde{x}_0 + t\tilde{v}, f(\tilde{v}, t))^{\tilde{B}}(\tilde{v}) \), it follows from (20) that

\[
\sup_{t \in (-h, h)} |A_\varepsilon(t)| = O(|\log \varepsilon |).
\]

By a Gronwall argument similar to the one in the uniqueness proof of Theorem 3.1 we infer that \( k(\tilde{v}, ., \tilde{w}) = 0 \) is the only solution of (25). By Proposition 2.4, we conclude that \( k = 0 \) in \( \mathcal{G}[-h, h] \times B_1(0) \times \mathbb{R}^m \), thereby establishing the claim.

Uniqueness: Let \( \tilde{u} \in \mathcal{G}[B_{\lambda r}(x_0), W] \) be another solution of (21). We will show that \( \tilde{u}|_{B_{\lambda r}(x_0)} = u|_{B_{\lambda r}(x_0)} \) for all \( s < \lambda r \). Since \( \mathcal{G} \) is a sheaf, the equality then also holds on \( B_{\lambda r}(x_0) = U(\tilde{x}_0, \tilde{y}_0) \).

Let \( s \in (0, \lambda r) \) and let \( \tilde{v} = [(\tilde{v}_c)]_c \in B_1(0) \). Setting \( \sigma := \frac{1}{r} (\lambda r - s) \), we define \( g(\tilde{v}, .) := (s - 2\sigma, s + 2\sigma) \rightarrow W \) by \( g(\tilde{v}, t) := \tilde{u}(\tilde{x}_0 + t\tilde{v}) \). From \( \tilde{x}_{\varepsilon} \rightarrow x_0 \) as \( \varepsilon \rightarrow 0 \) it follows that \( g(\tilde{v}, .) \) is well-defined. Then, by the choice of \( s \) and by Proposition 2.2, \( g(\tilde{v}, .) \in \mathcal{G}[(-s - 2\sigma, s + 2\sigma), W] \). Moreover, \( g(\tilde{v}, .) \) is a solution of (23) for \( v = \tilde{v} \). Since \((-s - 2\sigma, s + 2\sigma) \subseteq J \) and solutions of (23) are unique in \( \mathcal{G}[J, W] \), it follows that \( g(\tilde{v}, .) = f(\tilde{v}, .) \) for all \( \tilde{v} \in B_1(0) \).

By Proposition 2.4, \( g : (v, t) \rightarrow g(v, t) \) is equal to \( f \) on \( (-s - 2\sigma, s + 2\sigma) \). Observe that for \( c_1, c_2 > 0 \) the generalized functions \( (v, t) \rightarrow f \left( \frac{1}{c_1} v, c_1 t \right) \) and
\((v, t) \mapsto f\left(\frac{1}{c_2}v, c_2t\right)\) are equal on the intersection of their domains. Hence, we obtain

\[
\bar{u}(x) = g\left(\frac{1}{s + \sigma}(x - \tilde{x}_0)\right)(s + \sigma) = f\left(\frac{1}{s + \sigma}(x - \tilde{x}_0)\right)(s + \sigma)
\]

\[
= f\left(\frac{1}{r}(x - \tilde{x}_0)\right)(r) = u(x),
\]

thereby establishing the claim.

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