On the Nakano vanishing theorem

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Abstract

In this note, we state various generalisations of the Nakano vanishing theorem under weak positivity assumptions, and compare them with the known results.

1 Introduction

In this note, we give the following generalized version of the Nakano vanishing theorem.

Theorem 1. Let $X$ be an $n$-dimensional projective manifold and $L$ a nef holomorphic line bundle over $X$. Then we have

$$H^p(X, \Omega^q_X \otimes L) = 0$$

for any $p + q > n + \max(\dim(B_+(L)), 0)$. Here $B_+(L)$ denotes the augmented base locus (or non-ample locus) of $L$. When $B_+(L) = \emptyset$, we define by convention that its dimension is $-1$.

Here we recall the definition of $B_+(L)$. Given an ample line bundle $A$ over $X$, the augmented base locus is defined by

$$B_+(L) := \bigcap_{m > 0} \text{Bs}(mL - A)$$

where Bs means the base locus of a line bundle.

We recall classically (cf. [BBP]) that $B_+(L) = \emptyset$ if and only if $L$ is ample and $B_+(L) \neq X$ if and only if $L$ is big. Thus we have the Nakano vanishing theorem in the case that $B_+(L) = \emptyset$.

We observe that in this example $B_+(L) = E$ where $E$ is the exceptional divisor.

Now let $l := \dim(B_+(L))$. When $l = -1$, the theorem is true by the Nakano vanishing theorem. When $l \leq 0$, we show that in fact $L$ is ample. In this case, there exists some $m > 0$ and $s_0, \cdots, s_k \in H^0(X, mL - A)$ such that

$$\text{Bs}(s_0, \cdots, s_k) = \{x_0, \cdots, x_l\}.$$  

These sections induce a singular metric $h_0$ on $mL - A$ with analytic singularity at the discrete points $\{x_0, \cdots, x_l\}$. Its curvature is a closed positive $(1,1)$-current which is smooth outside $\{x_0, \cdots, x_l\}$. By [Dem92] Lemma 6.3 $mL - A$ is nef. Hence $L$ is ample.

Now let $l > 0$ and suppose by induction that the theorem has been verified for $\dim(B_+(L)) \leq l-1$. We recall the concepts involved in the theorem of Nakamay on base loci [Nak].

Definition 1. Given a nef and big divisor $L$ on $X$, the null locus Null$(L)$ of $L$ is the union of all positive dimensional subvarieties $V \subset X$ with

$$(L^{\dim V} \cdot V) = 0.$$
We observe that for any smooth divisor $D$ of $X$ and such a line bundle, 
$$\text{Null}(L|_D) \subset \text{Null}(L).$$

**Theorem 2.** (Nakamaye). If $L$ is an arbitrary nef and big divisor on $X$, then 
$$B_+(L) = \text{Null}(L).$$

Fix $A_2$ a very ample divisor on $X$. By Bertini theorem with a general choice we can assume that $D \in |A_2|$ is smooth. Since $A_2$ is very ample we can assume that $D \cap B_+(L) \subset B_+(L)$. More precisely, for a general choice of $D$, no $l-$dimensional component of $B_+(L)$ is contained in $D$. Since $L$ is nef and big, we have by Nakamaye theorem $\text{Null}(L) = B_+(L)$. By the definition of $\text{Null}(L)$ we have 
$$(L^{n-1} \cdot D) > 0.$$ 

In other words, $L|_D$ is big. Using another time the Nakamaye theorem, we find that 
$$B_+(L|_D) = \text{Null}(L|_D) \subset \text{Null}(L) \cap D \subset B_+(L).$$

In particular, $\dim B_+(L|_D) \leq \dim B_+(L) - 1$.

Recall the following elementary lemma (3.24) in [SS].

**Lemma 1.** Let $L$ be a holomorphic line bundle over $X$, let $D$ be a smooth hyper-surface in $X$, and let $p,q \geq 0$ be fixed. If 
$$(a)H^p(X, \Omega^q_X \otimes [D] \otimes L) = 0,$$
$$(b)H^{p-1}(D, \Omega^{q-1}_D \otimes L|_D) = 0,$$
$$(c)H^{p-1}(D, \Omega^{q}_D \otimes ([D] \otimes L)|_D) = 0,$$

then we have 
$$H^p(X, \Omega^q_X \otimes L) = 0.$$ 

Since $[D] \otimes L$ is ample ($L$ is nef), the hypotheses (a) (c) of the lemma is verified by the Nakano vanishing theorem. Since 
$$(p-1) + (q-1) > \dim D + l - 1,$$
the condition (b) is satisfied by the inductive hypothesis.

This finishes the proof.

**Remark 1.** It would be interesting to know whether the theorem is still valid without assuming $L$ to be nef. Here principally, we use the nef condition in two places: in the Nakamaye theorem and in the fact that the sum of an ample divisor and a nef divisor is ample.

Here, following some ideas of Demailly, we give the following more general version of the Nakano vanishing theorem.

**Theorem 3.** Let $X$ be a $n$-dimensional projective manifold, $L$ a holomorphic line bundle and $A$ an ample line bundle over $X$. Assume that for sufficiently large $m \in \mathbb{N}$ and general hyper-surfaces in the linear system $H_1, \cdots, H_k \in |mA|$, the restriction $L|_{H_1 \cap \cdots \cap H_k}$ is ample. Then for $p+q > n$, we have 
$$H^q(X, \Omega^p_X \otimes L) = 0.$$ 

**Proof.** By duality, it is equivalent to show that for $p+q < n-k$, we have 
$$H^q(X, \Omega^p_X \otimes L^{-1}) = 0.$$ 

Since the hyper-surface $H_i$ is supposed to be general, we can assume that any intersection of type $H_1 \cap \cdots \cap H_l$ is smooth for any $l$ and of dimension $n - l$ for any $l \leq k$.

For $m$ big enough such that $mA + L$ is ample, hence by Nakano vanishing theorem we have the vanishing $p + q < n - k$ 
$$H^q(X, \Omega^p_X \otimes L^{-1} \otimes \mathcal{O}(-H_1)) = 0.$$
From the short exact sequence

\[ 0 \to \Omega_X^p \otimes L^{-1} \otimes \mathcal{O}(-H_1) \to \Omega_X^p \otimes L^{-1} \to (\Omega_X^p \otimes L^{-1})|_{H_1} \to 0 \]

we know that to prove the desired vanishing it is enough to show that for \( p + q < n - k \)

\[ H^q(X, (\Omega_X^p \otimes L^{-1})|_{H_1}) = 0. \]

From the short exact sequence

\[ 0 \to TH_1 \to TX|_{H_1} \to \mathcal{O}(H_1)|_{H_1} \to 0 \]

we have the exact sequence (using the fact that \( \mathcal{O}(H_1) \) is of rank one)

\[ 0 \to \mathcal{O}(-H_1)|_{H_1} \otimes \Omega_H^{-1} \otimes \mathcal{O}(H_1)|_{H_1} \to \Omega_H \to \Omega_H^p \to 0. \]

We take the tensor product with \( L^{-1}|_{H_1} \) and the long exact sequence associated to the corresponding short exact sequence. By the Nakano vanishing theorem, we find

\[ H^i(H_1, \Omega_H^1 \otimes (L^{-1} \otimes \mathcal{O}(-H_1))|_{H_1}) = 0 \]

for any \( i + j < n - 1 \). It is enough to prove that

\[ H^q(H_1, \Omega_H^p \otimes L^{-1}|_{H_1}) = 0 \]

for \( p + q < n - k \). But this is true by the Nakano vanishing theorem and our assumption. \( \square \)

**Remark 2.** By the proof of the theorem, it is enough to take \( m \) so large that \( mA + L \) is ample, and \( H_i \in |mA| \) so that \( H_1 \cap \cdots \cap H_i \) is smooth and of dimension \( n - l \) for any \( l \leq k \), and \( L|_{H_1 \cap \cdots \cap H_k} \) is ample.

As pointed out by A. Höring, it is interesting to compare this result to the following theorem 2 of [Kur13]:

Let \( X \) be a smooth projective variety, \( L \) a divisor, \( A \) a very ample divisor on \( X \). If \( L|_{E_1 \cap \cdots \cap E_k} \) is big and nef for a general choice of \( E_1, \cdots, E_k \), then \( H^i(X, \mathcal{O}_X(K_X + L)) = 0 \) for \( i > k \).

**Remark 3.** Our first theorem is a special case of this general version. Since \( L \) is nef, it is nef on the complete intersection of the hyper-surfaces \( H_1, \ldots, H_l \) where \( l := \dim(B_+(L)) \). On the other hand, for such general hyper-surfaces, we can assume that the intersection \( B_+(L) \cap H_1 \cap \cdots \cap H_l \) is finite points. By the definition of stable base locus, \( L|_{H_1 \cap \cdots \cap H_l} \) is ample outside these finite points. Hence in fact, \( L|_{H_1 \cap \cdots \cap H_l} \) is ample.

The \( k \)-ampleness condition defined by Sommese [Som] is also a sufficient condition for the condition stated in Theorem 3.30. We start by recalling the definition.

**Definition 2.** A holomorphic line bundle \( L \) on a compact complex manifold \( X \) is said to be \( k \)-ample \((0 \leq k \leq n - 1)\) if there exists a positive integer \( N \) such that \( NL \) spans at each point of \( X \) and the Kodaira morphism associated to \( NL \) has at most \( k \)-dimensional fibres.

Changing \( N \) in the definition by a possible large multiple of \( N \) we can assume that the Kodaira morphism associated to \( NL \) is the Iitaka fibration. Denote \( \Phi : X \to Z \) the fibration where \( Z \) is a projective variety. Denote \( A_{z,j} \) \((z \in Z, j \in \mathbb{N})\) the irreducible components of the fibre of \( z \) (i.e. \( \Phi^{-1}(z) \)). By a general choice of \( H_1 \), we can assume that for any \( z, j \) the hyper-surface \( H_1 \) intersecting \( A_{z,j} \) defines a divisor of \( A_{z,j} \) by the lemma stated below. Similarly, with a general
choice of $H_1, \cdots, H_k$ we can assume that for any $z, j$ $H_1 \cap \cdots \cap H_k \cap A_{z,j}$ is a finite set, by the assumption that $\dim A_{z,j} \leq k$. In other words, the restriction of the Kodaira morphism

$$\Phi : H_1 \cap \cdots \cap H_k \to Z$$

is a finite morphism. Since $L|_{H_1 \cap \cdots \cap H_k}$ is pull back of $\mathcal{O}(1)$ via $\Phi$, $L|_{H_1 \cap \cdots \cap H_k}$ is ample on $H_1 \cap \cdots \cap H_k$. (Recall that the pull back of an ample line bundle under a finite morphism is ample.)

**Lemma 2.** Let $\Phi : X \to Z$ be the fibration such that all the fibers have dimension $\leq k$. Assume $X$ is projective. Then there exists $H \subset X$ a general very ample divisor such that the restriction $\Phi_H : H \to Z$ of $\Phi$ on $H$ has all fibers of dimension $\leq (k-1)$.

**Proof.** Denote $A_{z,j}$ ($z \in Z, j \in \mathbb{N}$) the irreducible components of the fibre of $z$ (i.e. $\Phi^{-1}(z)$). It is equivalent to demand the restriction to each $A_{z,j}$ of the defining section $\sigma$ of $H$ is non trivial. Let $A$ be an ample divisor on $X$. Denote $V_{z,j}$ the linear subspace of $H^0(X, mA)$ such that $\sigma|_{A_{z,j}} \equiv 0$. We want to choose $\sigma$ such that $\sigma \in H^0(X, mA) \setminus \bigcup_{z,j} V_{z,j}$. Notice that the family $A_{z,j}$ parameetrized by $z, j$ forms a bounded family in the Hilbert scheme of $X$. A sufficient condition to find $\sigma$ as above is that for $m$ large enough

$$\dim Z + \dim V_{z,j} < h^0(X, mA).$$

Without loss of generality, we can assume that $A$ is very ample on $X$. Hence, by boundedness, we have for $m$ large enough independent of $z, j$ a surjective restriction morphism

$$H^0(X, mA) \to H^0(A_{z,j}, mA).$$

As $V_{z,j}$ is the kernel of this morphism, it is enough to take $m$ so large that

$$\dim Z < h^0(A_{z,j}, mA).$$

For $A_{z,j}$ with positive dimension, the regular part of $A_{z,j}$ is a smooth submanifold of $X$. Since $A$ is very ample, it generates 1-jets of the regular part of $A_{z,j}$ at any point. Hence $H^0(A_{z,j}, NA)$ generates any $m$-fold symmetric product of 1-jets of $A_{z,j}$ at some regular point. In other words,

$$h^0(A_{z,j}, mA) > \left( \frac{m}{\dim A_{z,j}} \right) \geq m.$$

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