Identical particle and lepton mass effects in the decay

\[ H \rightarrow Z^*(\rightarrow \tau^+\tau^-) + Z^*(\rightarrow \tau^+\tau^-) \]

S. Groote, L. Kaldamäe and M. Naeem

Füüsika Instituut, Tartu Ülikool, W. Ostwaldi 1, 50411 Tartu, Estonia

Abstract

We consider identical particle and lepton mass effects in the cascade decay \( H \rightarrow Z^*(\rightarrow \tau^+\tau^-) + Z^*(\rightarrow \tau^+\tau^-) \). Since the scale of the problem is set by the off-shellness \( p_a^2 \) and \( p_b^2 \) of the respective gauge bosons in the limits \( (4m_{\tau}^2 \leq p_a^2, p_b^2 \leq (m_H - 2m_{\tau})^2) \) and not by \( m_H^2 \), lepton mass effects are nonnegligible in particular close to the threshold of the off-shell decays. We calculate the rates and single angle decay distributions and compare them with the corresponding rates and single angle decay distributions for the nonidentical particle decays \( H \rightarrow Z^*(\rightarrow e^+e^-) + Z^*(\rightarrow \mu^+\mu^-) \) involving negligible lepton masses.
1 Introduction

The decay channel $H \rightarrow ZZ^{(*)} \rightarrow 4\ell$ was one of the main decay channels for the observation of the Standard Model Higgs boson in 2012 by the ATLAS and CMS collaborations [1]. Detailed studies of this decay channel have been performed in a couple of PhD theses [3, 4, 5, 6, 7, 8, 9] and in a few partially unpublished reports [10, 11, 12]. Most authors who have studied the decay $H \rightarrow \ell\ell\ell\ell$ of the Higgs boson into four leptons have shied away from a detailed investigation of identical particle effects in the decay distributions of $H \rightarrow Z^*Z^* \rightarrow \ell^+\ell^−\ell^+\ell^−$. A first appraisal of the importance of identical particle effects [13, 14, 15] can be obtained from a comparison of the branching ratios of a 125 GeV Higgs decaying into nonidentical and identical lepton pairs in Ref. [16]. The branching ratios listed in Ref. [16] are $B(H \rightarrow e\mu\mu) = 5.93 \cdot 10^{-5}$ and $B(H \rightarrow eeee) = 3.27 \cdot 10^{-5}$. The approximate factor of two between the two rates reflects (i) the statistical factor of $1/4$ and (ii) the doubling of noninterference contributions in the identical particle case. The small deviation of the rate ratio from the exact value of 2 must be assigned to the contributions of the two interference terms. Judging from the numbers calculated in Ref. [16] one concludes that the interference contributions add constructively and are approximately 10% in size.

There is a multitude of Feynman diagrams that contribute to $H \rightarrow \ell\ell\ell\ell$ for massive leptons as for $\ell = \tau^\pm$. We divide these into the three classes I, II and III. The first class contains the Feynman diagrams that contribute to $H \rightarrow \ell\ell\ell\ell$ also in the zero lepton mass case. When $m \neq 0$ one has in addition contributions proportional to $g_{H\ell\ell}$ (class II). Finally, the class III contributions comprise loop-induced higher order contributions such as $H \rightarrow \gamma^*\gamma^* \rightarrow \ell\ell\ell\ell$. In Fig. 1 we present an exemplary diagram for each of the classes.

The paper is organised as follows. In Sec. 2 we concentrate on the discussion of class I contributions where we attempt to clarify several issues concerning identical particle effects in the decay $H \rightarrow \ell\ell\ell\ell$, including narrow width effects. In Sec. 3 we further analyse lepton mass effects in these decays and provide numerical results for the single angle decay
The four-body decay \( H \rightarrow Z^*(\rightarrow \tau^+\tau^-) + Z^*(\rightarrow \tau^+\tau^-) \) differs very much from e.g. the decay \( H \rightarrow Z(\rightarrow \mu^+\mu^-) + Z^*(\rightarrow \tau^+\tau^-) \) in that one has to take into account interference effects resulting from the fact that one has two pairs of identical particles in the former decay. According to the two diagrams in Fig. 2 one has the two amplitudes

\[
\begin{align*}
M_A &= M \left( \tau^+(p_1), \tau^-(p_3), \tau^+(p_2), \tau^-(p_4) \right) \\
M_B &= M \left( \tau^+(p_1), \tau^-(p_4), \tau^+(p_2), \tau^- (p_3) \right)
\end{align*}
\]

where \( M_B \) is obtained from \( M_A \) by exchanging the two \( \tau^+ \) leptons. The two additional configurations where the \( \tau^- \) leptons are exchanged or where both \( \tau^+ \) and \( \tau^- \) leptons are exchanged simultaneously are topologically equivalent to the above two diagrams (I) and (II) and...
Figure 2: Feynman diagrams A and B contributing to $H \to Z^* (\tau^+ \tau^-) + Z^* (\to \tau^+ \tau^-)$ should therefore be discarded. When squaring the amplitudes one obtains

$$|M_A + M_B|^2 = |M_A|^2 + 2 \text{Re}(M_A M_B^*) + |M_B|^2.$$  \hspace{1cm} (2)

Let us add a few general remarks.

It is clear that the rate contributions of $|M_A|^2$ and $|M_B|^2$ are identical to each other since their mutual contributions are obtained by the exchange $p_3 \leftrightarrow p_4$ which also leaves the measure of the phase space integration invariant. In the case when one neglects the interference contribution $2 \text{Re}(M_A M_B^*)$ one therefore obtains the relation $\Gamma(H \to \tau^+ \tau^- \tau^+ \tau^-) = 1/2 \cdot \Gamma(H \to \tau^+ \tau^- \mu^+ \mu^-)$ where the statistical factor $1/4$ has been taken into account in the identical fermion case. The nondiagonal interference contribution proportional to $2 \text{Re}(M_A M_B^*)$ corresponds to the absorptive part of a fermionic one-loop diagram compared to the fermionic two-loop diagram of the diagonal contribution, as can be inferred from the calculation of $|M_A + M_B|^2$ illustrated in Fig. (B). One must therefore be careful to include an extra minus sign in the nondiagonal contribution.

In the following we shall separately calculate the rate for the two non-interference contributions $|M_A|^2 + |M_B|^2$ and the interference contributions $2 \text{Re}(M_A M_B^*)$. Henceforth we shall refer to the non-interference contribution as the diagonal contribution and the interference contribution as the nondiagonal contribution.
Figure 3: Contributions (a) $|M_A|^2 = |M_B|^2$ and (b) $\text{Re}(M_AM_B^*) = \text{Re}(M_BM_A^*)$ from $|M_A + M_B|^2$. These contributions can be obtained as absorptive parts of fermionic two- and one-loop diagrams via cuts of the diagrams (c) and (d), respectively.
2.1 The diagonal noninterference contribution

The contributions of the diagonal contributions \(|M_A|^2 = |M_B|^2\) to the rate are not difficult to evaluate since they can be seen to factorize as described in some detail in Ref. [17]. Using the results of Ref. [17] the differential rate corresponding to the contribution of \(|M_A|^2 + |M_B|^2 = 2|M_A|^2\) can be written in the form (including the identical particle factor of 1/4)

\[
\frac{d\Gamma^{AA}}{dp_a^2 dp_b^2}(p_a^2, p_b^2) = \frac{1}{4} \cdot \frac{2\alpha^3}{9\pi^2 m_H^2} |\vec{p}_{Z\ast}(p_a^2, p_b^2)| |\vec{p}_{\ell}(p_a)| |\vec{p}_{\ell}(p_b)| \frac{m_Z^2}{256 \sin^6 \theta_W \cos^6 \theta_W} \\
\times \frac{1}{(p_a^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} (p_b^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \frac{1}{p_a^2 p_b^2} \\
\times \left\{ v_{\ell}^2 (1 + 2 \frac{m_a^2}{p_a^2}) P_{1\mu\nu}(p_a) + a_\ell^2 \left( (1 - 4 \frac{m_a^2}{p_a^2}) P_{1\mu\nu}(p_a) - 3 a_\ell^2 \cdot 2 \frac{m_a^2}{p_a^2} F_S^2(p_a^2) P_{0\mu\nu}(p_a) \right) \right\} \\
\times \left\{ v_{\ell}^2 (1 + 2 \frac{m_b^2}{p_b^2}) P_{1\mu\nu}(p_b) + a_\ell^2 \left( (1 - 4 \frac{m_b^2}{p_b^2}) P_{1\mu\nu}(p_b) - 3 a_\ell^2 \cdot 2 \frac{m_b^2}{p_b^2} F_S^2(p_b^2) P_{0\mu\nu}(p_b) \right) \right\} \quad (3)
\]

where \(v_{\ell} = -1 + 4 \sin^2 \theta_W\) and \(a_\ell = -1\) are the vector and axial vector couplings of the Z boson. \(\alpha = e^2/(4\pi)\) is the fine structure constant for which we use the value \(\alpha(m_H) \approx 1/120\). For the kinematics we have used \(p_a = (E_a; \vec{p}_{Z\ast})\) and \(p_b = (E_b; -\vec{p}_{Z\ast})\) with

\[
E_a = m_H^2 + p_a^2 - p_b^2, \quad E_b = m_H^2 - p_a^2 + p_b^2, \quad |\vec{p}_{Z\ast}| = \frac{1}{2m_H} \sqrt{\lambda(m_H^2, p_a^2, p_b^2)}, \quad (4)
\]

and \(p_a p_b = \frac{1}{2}(m_H^2 - p_a^2 - p_b^2)\), where \(\lambda(a, b, c) := a^2 + b^2 + c^2 - 2ab - 2ac - 2bc\) is the Källén function. The remaining phase space factors \(|\vec{p}_1| = |\vec{p}_3| = |\vec{p}_\ell(p_a)|\) and \(|\vec{p}_2| = |\vec{p}_4| = |\vec{p}_\ell(p_b)|\) are calculated in the respective rest frames of the decaying vector bosons and read

\[
|\vec{p}_\ell(p_a)| = \frac{1}{2} \sqrt{p_a^2 - 4m^2} =: \frac{1}{2} \sqrt{p_a^2} v_a, \quad |\vec{p}_\ell(p_b)| = \frac{1}{2} \sqrt{p_b^2 - 4m^2} =: \frac{1}{2} \sqrt{p_b^2} v_b. \quad (5)
\]

Note the appearance of spin-1 and spin-0 projectors given by

\[
P_{1\mu\nu}(p_a) = g^{\mu\nu} - \frac{p_a^\mu p_a^\nu}{p_a^2}, \quad P_{0\mu\nu}(p_a) = \frac{p_a^\mu p_a^\nu}{p_a^2} \quad (6)
\]

and similarly \(P_{1\mu\nu}(p_b)\) and \(P_{0\mu\nu}(p_b)\). The factor \(F_S(p^2) = 1 - p^2/m_Z^2\) multiplying the scalar contribution in Eq. (3) is a result of having used the unitary gauge for the gauge boson
propagator (see Ref. [17]). The spin-0 piece of the unitary gauge boson propagators carries the helicity flip factors $m^2/p_a^2$ and $m^2/p_b^2$ which we need to take into account when discussing lepton mass effects in the decay $H \to \ell\ell\ell\ell$. The relevant Lorentz contractions in Eq. (3) can be calculated to be

$$P_{1}^{\mu\nu}(p_a)P_{1}^{\mu\nu}(p_b) := \rho_{TT}(p_a^2, p_b^2) = 2 + \frac{(p_ap_b)^2}{p_a^2p_b^2} = 3 + \frac{m_H^2|\vec{p}\cdot Z^*|^2}{p_a^2p_b^2}, \quad (7)$$

$$P_{1}^{\mu\nu}(p_a)P_{0}^{\mu\nu}(p_b) := \rho_{TS}(p_a^2, p_b^2) = 1 - \frac{(p_ap_b)^2}{p_a^2p_b^2} = -\frac{m_H^2|\vec{p}\cdot Z^*|^2}{p_a^2p_b^2}, \quad (8)$$

$$P_{0}^{\mu\nu}(p_a)P_{0}^{\mu\nu}(p_b) := \rho_{SS}(p_a^2, p_b^2) = \frac{(p_ap_b)^2}{p_a^2p_b^2} = 1 + \frac{m_H^2|\vec{p}\cdot Z^*|^2}{p_a^2p_b^2}. \quad (9)$$

Following the standard convention we label the components of the $1 \otimes 1$ spin–spin density matrix elements in Eq. (7) by labels $T$ (transverse) and $S$ (scalar). The scalar–scalar contribution $\rho_{SS}$ in Eq. (3) appears multiplied with the product $m^2/p_a^2 \cdot m^2/p_b^2$ of helicity flip factors and can be neglected for all practical purposes.

The contractions (7), (8) and (9) stand for spin–spin density matrix elements which determine the angular coefficients of the angular decay distributions of the subsequent decays $Z^* \to \ell\ell$ [17]. The contractions have been written in two different forms. The first equations are suitable for a discussion of the large recoil region where $p_a^2$ and $p_b^2$ are small. The second equations are suitable for the low recoil region where $|\vec{p}\cdot Z^*|$ is small. In the large recoil region the dominant contributions can be seen to be given by $\rho_{TT} \approx \rho_{TS} \approx \rho_{SS}$, whereas on has $\rho_{SS} \approx 3\rho_{TT}$ and $\rho_{TS} \approx 0$ in the low recoil region.

The rate is obtained from Eq. (3) by $p_a^2$ and $p_b^2$ integration according to

$$\Gamma^{AA} = \int_{4m^2}^{(m_H-2m)^2} dp_a^2 \int_{4m^2}^{(m_H-\sqrt{p_a^2})^2} dp_b^2 \frac{d\Gamma^{AA}}{dp_a^2 dp_b^2}(p_a^2, p_b^2). \quad (10)$$

The integrations in Eq. (10) can be performed numerically by using MATHEMATICA. The results are given in Table I where we list the integrated rates for the three cases $\ell = e, \mu, \tau$. For the mass of the Higgs boson we use the central value $m_H = 125.09 \pm 0.24$ GeV [18].
2.2 The nondiagonal interference contribution

The rate calculation corresponding to the interference contribution $\text{Re}(M_A M_B^\dagger)$ is considerably more difficult. One reason is that the integrand does not factorize into $p_a$- and $p_b$-side contributions as in the diagonal case (see Eq. (3)). As a result the requisite angular integrations can no longer be done analytically as was possible in the diagonal case.

We begin by setting up the five-dimensional phase space. As in Ref. [20, 21] (see also Ref. [22]) we choose the following five phase space variables: the two invariant masses $p_a^2 = (p_1 + p_3)^2$ and $p_b^2 = (p_2 + p_4)^2$, the two polar angles $\theta_a$ and $\theta_b$, and the azimuthal angle $\phi$ (see Fig. 4). For the twice differential rate corresponding to the nondiagonal interference contribution $2 \text{Re}(M_A M_B^\dagger)$ one obtains (including the identical particle factor of 1/4)

$$\frac{d\Gamma^{AB}}{dp_a^2 dp_b^2(p_a^2, p_b^2)} = \frac{\alpha^3}{4} \frac{m_H^2}{16 \pi^3 m_W^2} |\vec{p}^2_{Z^*}(p_a^2, p_b^2)| \frac{|\vec{p}_\ell(p_a)| |\vec{p}_\ell(p_b)|}{\sqrt{p_a^2} \sqrt{p_b^2}} \frac{m_Z^2}{256 \sin^6 \theta_W \cos^6 \theta_W}$$

$$\times \int d\cos \theta_a d\cos \theta_b d\phi \left( v_\ell^4 N_0 + v_\ell^2 a_\ell^2 N_2 + a_\ell^4 N_4 \right)$$

$$\times \frac{D_a D_b D_c D_d + m_Z^2 \Gamma_Z^2 (-D_a D_b + D_a D_c + D_a D_d + D_b D_c + D_b D_d - D_c D_d) + m_Z^4 \Gamma_Z^4}{(D_a^2 + m_Z^2 \Gamma_Z^2)(D_b^2 + m_Z^2 \Gamma_Z^2)(D_c^2 + m_Z^2 \Gamma_Z^2)(D_d^2 + m_Z^2 \Gamma_Z^2)},$$

where the numerator factors $N_0$, $N_2$, and $N_4$ are too long to be presented here.
The pole factors \( D_i \) for \( i = a, b, c, d \) in Eq. (11) read

\[
D_a = (p_1 + p_3)^2 - m_Z^2 = p_a^2 - m_Z^2, \\
D_b = (p_2 + p_4)^2 - m_Z^2 = p_b^2 - m_Z^2, \\
D_c = (p_1 + p_4)^2 - m_Z^2 = p_c^2 - m_Z^2 = 2m^2 - m_Z^2 \\
\quad + \frac{1}{4} \left( (m_H^2 - p_a^2 - p_b^2)(1 - \cos \theta_a \cos \theta_b v_a v_b) - 2\sqrt{p_a^2 p_b^2} \cos \phi \sin \theta_a \sin \theta_b v_a v_b \\
\quad + \sqrt{\lambda(m_H^2, p_a^2, p_b^2)}(\cos \theta_a v_a - \cos \theta_b v_b) \right), \\
D_d = (p_2 + p_3)^2 - m_Z^2 = p_d^2 - m_Z^2 = 2m^2 - m_Z^2 \\
\quad + \frac{1}{4} \left( (m_H^2 - p_a^2 - p_b^2)(1 - \cos \theta_a \cos \theta_b v_a v_b) - 2\sqrt{p_a^2 p_b^2} \cos \phi \sin \theta_a \sin \theta_b v_a v_b \\
\quad - \sqrt{\lambda(m_H^2, p_a^2, p_b^2)}(\cos \theta_a v_a - \cos \theta_b v_b) \right),
\]

where \( D_c \) and \( D_d \) depend on the angles which is the second reason for not being able to obtain an analytical result. The cosines of the angles are given by

\[
\cos \theta_a = \frac{\vec{p}_a \cdot \vec{p}_1}{|\vec{p}_a||\vec{p}_1|}, \quad \cos \theta_b = \frac{\vec{p}_b \cdot \vec{p}_2}{|\vec{p}_b||\vec{p}_2|}, \quad \cos \phi = \frac{\vec{p}_1^\perp \cdot \vec{p}_2^\perp}{|\vec{p}_1^\perp| |\vec{p}_2^\perp|}
\]

where \( \vec{p}_i^\perp \) is the component of \( \vec{p}_i \) perpendicular to \( \vec{p}_a \) (or \( \vec{p}_b \), respectively), given by

\[
\vec{p}_1^\perp = \vec{p}_1 - \frac{(\vec{p}_1 \cdot \vec{p}_a)\vec{p}_a}{|\vec{p}_a|^2}, \quad \vec{p}_2^\perp = \vec{p}_2 - \frac{(\vec{p}_2 \cdot \vec{p}_b)\vec{p}_b}{|\vec{p}_b|^2}.
\]

in the rest frame of the decaying \( Z \) bosons.

Note that due to the crossing of momenta, for the nondiagonal interference contribution it is necessary to switch to a more general notation involving a set of four invariant masses \( p_a^2, p_b^2, p_c^2 \) and \( p_d^2 \) instead of the initial first two of this set on which they depend via the angles. The input for this calculation and the kinematics necessary for it is transferred to the numerical integration routine VEGAS [23]. The kinematics is expressed in terms of four-vectors in the rest frames of the decaying \( Z \) bosons with polar angles \( \theta_a \) and \( \theta_b \), boosted to the rest frame of the Higgs boson via the rapidities

\[
\lambda_a = \arctanh \left( \sqrt{\lambda(m_H^2, p_a^2, p_b^2)} \right), \quad \lambda_b = -\arctanh \left( \sqrt{\lambda(m_H^2, p_a^2, p_b^2)} \right),
\]

9
and turned around the z axis through $\phi$, resulting in

\[
p_{1/3} = \frac{1}{2} \sqrt{p_a^2} \left( \cosh \lambda_a \pm v_a \cos \theta_a \sinh \lambda_a; \pm v_a \sin \theta_a \cos \phi, \pm v_a \sin \theta_a \sin \phi, \sinh \lambda_a \pm v_a \cos \theta_a \cosh \lambda_a \right),
\]

\[
p_{2/4} = \frac{1}{2} \sqrt{p_b^2} \left( \cosh \lambda_b \pm v_b \cos \theta_b \sinh \lambda_b; \pm v_b \sin \theta_b, 0, \sinh \lambda_b \pm v_b \cos \theta_b \cosh \lambda_b \right). \tag{16}
\]

\subsection{2.3 The phase space for $p_a^2$ and $p_b^2$}

The phase space domain in $(p_a^2, p_b^2)$ to be integrated over can be investigated by looking at the phase space. As pointed out in Ref. [24], the boundary of the phase space domain can be found by demanding that the measure is real, i.e. all the radicands are positive. Claiming that

\[
1 - \frac{4m^2}{p_a^2} \geq 0, \quad 1 - \frac{4m^2}{p_b^2} \geq 0, \quad \lambda(m_H^2, p_a^2, p_b^2) \geq 0, \tag{17}
\]

the first two inequalities can be resolved to $p_a^2, p_b^2 \leq 4m^2$. For the last one we replace $p_a^2$ and $p_b^2$ by the squared invariant masses $m_a^2$ and $m_b^2$ and obtain

\[
(m_H^2 - (m_a + m_b)^2)(m_H^2 - (m_a - m_b)^2) > 0 \tag{18}
\]

The only “physical” restriction is given by $m_a + m_b \leq m_H$ which is also very intuitive. Therefore, the phase space domain is given by the intersection of $m_a \geq 2m$, $m_b \geq 2m$ and $m_a + m_b \leq m_H$ as shown in Fig. 5. In terms of $p_a^2$ and $p_b^2$ the integrations limits are given by

\[
4m^2 \leq p_a^2 \leq (m_H - 2m)^2, \quad 4m^2 \leq p_b^2 \leq (m_H - \sqrt{p_a^2})^2. \tag{19}
\]

In case of the narrow width approximation (NWA), the phase space domain is restricted to the two line segments along $p_a^2 = m_Z^2$ and $p_b^2 = m_Z^2$ and indicated in Fig. 6 in dark blue, cutting the light blue phase space domain. If for instance the first $Z$ boson is on shell, $m_a^2 = m_Z^2$, the phase space is the vertical line segment limited by $4m^2 \leq p_b^2 \leq (m_H - m_Z)^2$. 

10
Figure 5: Phase space domain for the invariant masses $m_a$ and $m_b$ in dependence on the Higgs boson mass $m_H$, as restricted by the three straight lines $m_a = 2m$, $m_b = 2m$ and $m_a + m_b = m_H$ and painted in light blue. The localizations of the two $Z$ poles at $m_a = m_Z$ and $m_b = m_Z$ are indicated by two line segments in dark blue cutting the phase space domain. Note that the diagram is used for illustrative reasons only and not fixed to the physical masses $m = m_\tau$, $m_Z$ and $m_H$. 
As suggested by Ježabek and Kühn \cite{25, 26}, singularities at $p_a^2 = m_Z^2$ and $p_b^2 = m_Z^2$ (and eventually singularities occurring in connection with $p_c^2$ and $p_d^2$) are defended by adding a Breit–Wigner contribution to the denominator factor, for instance

$$D_a := (p_a^2 - m_Z^2) \to (p_a^2 - m_Z^2) + im_Z\Gamma_Z =: D_a + i\varepsilon.$$  \hfill (20)

One obtains

$$|M_A|^2 = \frac{N^{AA}}{(D_a + i\varepsilon)(D_b + i\varepsilon)(D_a - i\varepsilon)(D_b - i\varepsilon)} = \frac{N^{AA}}{(D_a^2 + \varepsilon^2)(D_b^2 + \varepsilon^2)}.$$  \hfill (21)

The nondiagonal interference contribution is more complicated. Taking the numerator to be symbolically $N^{AB} = N^{AB}_r + iN^{AB}_i$, the product

$$M_A M_B^* = \frac{N^{AB}_r + iN^{AB}_i}{(D_a + i\varepsilon)(D_b + i\varepsilon)(D_c - i\varepsilon)(D_d - i\varepsilon)},$$  \hfill (22)

leads to

$$\text{Re}(M_A M_B^*) = \frac{\text{Re} \left((D_a - i\varepsilon)(D_b - i\varepsilon)(D_c + i\varepsilon)(D_d + i\varepsilon)(N^{AB}_r + iN^{AB}_i)\right)}{(D_a^2 + \varepsilon^2)(D_b^2 + \varepsilon^2)(D_c^2 + \varepsilon^2)(D_d^2 + \varepsilon^2)}.$$  \hfill (23)

The numerical results for the nondiagonal contribution $\Gamma^{AB}$ obtained by VEGAS is shown in Table 1 for the three leptons $\ell = e, \mu, \tau$.

Table 1: Diagonal and nondiagonal contributions $\Gamma^{AA}$ and $\Gamma^{AB}$ in units of $10^{-7}$ GeV to the decay $H \to Z^*(\to \ell^+\ell^-) + Z^*(\to \ell^+\ell^-)$ for the three leptons $\ell = e, \mu, \tau$

| $\ell$ | $\Gamma^{AA}$ | $\Gamma^{AB}$ |
|---|---|---|
| $\ell = e$ | 2.420(1) | 0.2484(1) |
| $\ell = \mu$ | 2.420(1) | 0.2484(1) |
| $\ell = \tau$ | 2.332(1) | 0.2432(1) |
Table 2: Dependence of the rate ratios of diagonal and nondiagonal contributions $\Gamma^{AA}$ and $\Gamma^{AB}$ on the $Z$ width for the decay $H \rightarrow Z^* (\rightarrow \tau^+ \tau^-) + Z^* (\rightarrow \tau^+ \tau^-)$. The fourth column gives the rate ratio in absolute values and the fifth column in units of $\Gamma_Z/m_Z$.

| $\Gamma_Z$ [GeV] | $\Gamma^{AA}$ [GeV] | $\Gamma^{AB}$ [GeV] | $\Gamma^{AB}/\Gamma^{AA}$ | $\Gamma^{AB}/\Gamma^{AA}$ $\Gamma_Z/m_Z$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| 2.4952            | 2.420 $\cdot 10^{-7}$ | 2.484(1) $\cdot 10^{-8}$ | 10.2% | 3.75     |
| 1.0               | 5.419(5) $\cdot 10^{-7}$ | 2.546(2) $\cdot 10^{-8}$ | 4.7% | 4.28     |
| 0.5               | 1.0350(2) $\cdot 10^{-6}$ | 2.569(3) $\cdot 10^{-8}$ | 2.5% | 4.53     |
| 0.2               | 2.518(7) $\cdot 10^{-6}$ | 2.580(4) $\cdot 10^{-8}$ | 1.0% | 4.67     |
| 0.1               | 4.97(2) $\cdot 10^{-6}$ | 2.569(6) $\cdot 10^{-8}$ | 0.5% | 4.94     |
| 0.05              | 9.56(6) $\cdot 10^{-6}$ | 2.585(8) $\cdot 10^{-8}$ | 0.27% | 4.93     |

2.4 Narrow width approximation

We have mentioned before that the nondiagonal interference contribution $\Gamma^{AB} \sim \text{Re}(M_A M^*_B)$ to the rate $\Gamma = \Gamma^{AA} + \Gamma^{AB}$ is suppressed relative to the diagonal non-interference contribution $\Gamma^{AA} \sim |M_A|^2$. Technically this comes about by the fact that there is a phase space momentum mismatch between the peaking regions of diagram A and diagram B. This mismatch becomes larger as the width $\Gamma_Z$ becomes smaller. The net result is that the phase space integration of the nondiagonal contribution tends to a constant value independent of the width $\Gamma_Z$. This is illustrated in Table 2 where we list the numerical values of the rate ratios of the nondiagonal and diagonal contributions for different values of the $Z$ width.

The entries of Table 2 indicate that the width dependence of the rate ratio tends to $\Gamma^{AB}/\Gamma^{AA} \sim \Gamma_Z/m_Z$ as $\Gamma_Z \rightarrow 0$. The limiting behavior of the $\Gamma_Z$ dependence of this rate ratio can in fact be analyzed quantitatively with the help of the $\delta$ distribution representa-
\[
\lim_{\Gamma_Z \to 0} \frac{1}{(p_a^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} = \frac{\pi}{m_Z \Gamma_Z} \delta(p_a^2 - m_Z^2). \tag{24}
\]

In order to extract the narrow width dependence of the rate it is sufficient to analyze the peaking region close to e.g. \(p_a^2 = m_Z^2\). We first discuss the diagonal contribution to the rate. Close to \(p_a^2 = m_Z^2\) the integral corresponding to the diagonal contribution can be cast into the form

\[
\lim_{\Gamma_Z \to 0} \int dp_a^2 \frac{F(p_a^2)}{(p_a^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} = \frac{1}{m_Z \Gamma_Z} \int dp_a^2 \frac{m_Z \Gamma_Z F(p_a^2)}{(p_a^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} = \frac{\pi}{m_Z \Gamma_Z} \delta(p_a^2 - m_Z^2) \tag{25}
\]

where the function \(F(p_a^2)\) is regular at \(p_a^2 = m_Z^2\).

The nondiagonal contribution corresponding to Eq. (11) is less singular for \(p_a^2 \to m_Z^2\). One only has a single Breit–Wigner pole denominator that determines the functional behavior close to \(p_a^2 = m_Z^2\) which we write as

\[
\frac{1}{(p_a^2 - m_Z^2) - im_Z \Gamma_Z} = \frac{(p_a^2 - m_Z^2) + im_Z \Gamma_Z}{(p_a^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \tag{26}
\]

for the contribution of e.g. \(M_A M_B^*\). The imaginary part is cancelled by the conjugate contribution \(M_A^* M_B\). The structure of the \(p_a^2\) integral is now given by

\[
\lim_{\Gamma_Z \to 0} \int dp_a^2 \frac{(p_a^2 - m_Z^2) F_1(p_a^2)}{(p_a^2 - m_Z^2)^2} = \frac{\pi}{m_Z \Gamma_Z} \int dp_a^2 \delta(p_a^2 - m_Z^2)(p_a^2 - m_Z^2) F_1(p_a^2) = 0. \tag{27}
\]

The phase space integration over the single pole region \(p_a^2 \sim m_Z^2\) thus tends to zero as \(\Gamma_Z \to 0\). The integration over the remaining phase space region results in a constant value independent of the \(Z\) boson width.

The upshot of our analysis is that one has \(\lim_{\Gamma_Z \to 0} \Gamma_A^A \sim m_Z/\Gamma_Z\) for the diagonal contribution whereas the nondiagonal contribution tends to a constant value, i.e. \(\Gamma_A^B/\Gamma_A^A \sim \Gamma_Z/m_Z\) as also indicated in Table 2. The limiting proportionality factor linking the two quantities can be read off Table 2 and is given by \(\Gamma_A^B/\Gamma_A^A \approx 2.74 \Gamma_Z/m_Z\).

The investigation in this subsection was prompted by an misleading statement in the literature that the nondiagonal interference contribution is suppressed at \(O(\Gamma_Z^2/m_Z^2)\) [27].
3 Single angle decay distributions

With two different pairs of particles discernible in experiments as in e.g. Ref. [17] we would be able to distinguish between the two channels, reconstruct the momenta \( p_a \) and \( p_b \), and measure the relative angles \( \theta_a \) and \( \theta_b \). However, with two pairs of identical particles this is not possible any more. A distinction can of course be made between positively and negatively charged leptons, and the angles \( \theta^a := \theta_{13}, \theta^b := \theta_{24}, \theta^c := \theta_{14} \) and \( \theta^d := \theta_{23} \) between all couples of differently charged leptons can be measured. In addition, the momenta \( p_{ij} = p_i + p_j \) for the partial channels of these couples can be reconstructed, the ladder being the momenta \( p_a, p_b, p_c \) and \( p_d \) already introduced before. If the \( Z \) bosons would be visible in the experiment, one could measure differential decay rates \( d\Gamma/dp_i^2 d\cos \theta^i \) (\( i = a, b, c, d \)). However, not knowing the actual channel, the observable accessible in experiment is the mean value

\[
\frac{d\Gamma}{dp^2 d\cos \theta} = \frac{1}{4} \left( \frac{d\Gamma}{dp_a^2 d\cos \theta^a} + \frac{d\Gamma}{dp_b^2 d\cos \theta^b} + \frac{d\Gamma}{dp_c^2 d\cos \theta^c} + \frac{d\Gamma}{dp_d^2 d\cos \theta^d} \right).
\]

(28)

\( p^2 \) stands for the measured momentum square of the reconstructed \( Z \) while \( \theta \) is the opening angle of the \( \tau \) pair for which the \( Z \) boson is reconstructed. The four separate parts in Eq. (28) are referred to as \( a, b, c \) and \( d \) channel contributions. By using

\[
\frac{df}{dy} \bigg|_{y=g(x)} = \int \frac{df}{dy} \delta(y - g(x)) \, dy = \int \frac{df}{dx} \delta(g(x) - y) \, dx,
\]

(29)

the differential decay rates can be calculated to obtain

\[
\frac{d\Gamma}{dp_i^2 d\cos \theta^i} = \int \frac{d\Gamma}{dp_a^2 dp_b^2 d\cos \theta_a d\cos \theta_b d\phi} \delta \left( p_i^2 (p_a^2, p_b^2, \theta_a, \theta_b, \phi) - p_i^2 \right) \times \delta \left( \cos \theta^i (p_a^2, p_b^2, \theta_a, \theta_b, \phi) - \cos \theta^i \right) \, dp_a^2 dp_b^2 d\cos \theta_a d\cos \theta_b d\phi,
\]

(30)

where \( p_i^2 (p_a^2, p_b^2, \theta_a, \theta_b, \phi) \) and \( \cos \theta^i (p_a^2, p_b^2, \theta_a, \theta_b, \phi) \) are the explicit expressions for the squared channel momenta and cosines in terms of the kinematic quantities \( p_a^2, p_b^2, \theta_a, \theta_b \) and \( \phi \). The kinematical details are left to Appendix A. In Fig. [6] the results of the VEGAS...
integration is shown for different squares energies $p^2 = 50, 100, 200$ and $500 \text{GeV}^2$. For small $p^2$ the function is peaked close to the upper limit $\cos \theta = +1$. A detailed analysis shows that the first peak at lower values of $\cos \theta$ is proliferated by the diagonal noninterference contribution while the second peak close to the threshold already present in the diagonal noninterference contribution is enhanced by the nondiagonal interference contribution.

4 Class II contributions

As emphasized in the Introduction, the class I contributions considered up to this point and given by the cascade process $H \rightarrow Z^*(\rightarrow \tau^+\tau^-) + Z^*(\rightarrow \tau^+\tau^-)$ are not the only contributions to $H \rightarrow \ell\ell\ell\ell$. Instead, because of the finite tau lepton mass, there are also class II diagrams which were mentioned (though, in a different context) in Ref. [28]. Together with class III two-loop diagrams we obtain sixteen diagrams (with multiple boson choices) shown in Fig. 7 (outgoing lepton momenta from top to bottom are $p_1$, $p_2$, $p_3$ and $p_4$). Though, concentrating only on the one-loop class I and II contributions, the situation is not as hopeless as it might look at the first sight. The diagrams of the second and third line in Fig. 7 can be combined to give an effective contribution. As the momentum carried by the intermediate boson line in diagrams (2a) and (3a) is given by $p_2 + p_4 = p_b$, there are similarities in the kinematics with diagram (1a). Denoting the combined diagrams of the second and third line by $J_i$ and of the first line as before by $M_i$ ($i = a, b, c, d$ – however, containing also intermediate Higgs bosons), one obtains four $4 \times 4$ matrices $J_{ij} = J_i J_j^\dagger$, $K_{ij} = J_i M_j^\dagger$, $L_{ij} = M_i J_j^\dagger$ and $M_{ij} = M_i M_j^\dagger$.

4.1 The contributions $J_{ij}$, $K_{ij}$, $L_{ij}$ and $M_{ij}$

The different contributions to the absolute square of the matrix element are given in Tab. 3 where rows and columns stand for the first and second index, respectively. Calculating the sum over the different squared decay channels $aa$, $ab$, … divided by the symmetry factor.
Figure 6: Angular dependence of $d\Gamma/(dp^2d\cos\theta)$ for different squared energies $p^2$
Figure 6: (cont.) Angular dependence of $d\Gamma/(dp^2d\cos\theta)$ for different squared energies $p^2$
Figure 7: Diagrams contributing to $H \rightarrow \tau^+ \tau^+ \tau^- \tau^-$
4, the contribution $M + (L + K) + J$ reads

$$(5.89(2) - 4.62(7) \cdot 10^{-3} + 0.362(2)) \cdot 10^{-8} \text{GeV} = 6.25(2) \cdot 10^{-8} \text{GeV}.$$ 

If instead of $m = m_\tau = 1.77682 \text{GeV}$ we choose $m = m_\mu = 113.429 \text{MeV}$ or $m = m_e = 0.511 \text{MeV}$, the contributions of $J_{ij}$, $K_{ij}$ and $L_{ij}$ to the cross section decrease rapidly. $J_{ij}$ is dominated by the diagonal elements $i = j$, leading to $1.08 \cdot 10^{-10} \text{GeV}$ for $m = m_\mu$ and $1.4 \cdot 10^{-15} \text{GeV}$ for $m_e$. The mixed contributions $K_{ij}$ and $L_{ij}$ are not dominated by the diagonal elements. These contributions combined are given by $-1.96 \cdot 10^{-13} \text{GeV}$ ($m = m_\mu$) and $-3.9 \cdot 10^{-18} \text{GeV}$ ($m = m_e$). Therefore, in all these cases the contribution of $M_{ij}$ dominates (cf. Tab. 4). For $m = m_\mu$ one obtains

$$(6.12(2) - 1.96(5) \cdot 10^{-5} + 0.0108(2)) \cdot 10^{-8} \text{GeV} = 6.13(2) \cdot 10^{-8} \text{GeV},$$

and for $m = m_e$ one obtains

$$(6.12(2) - 3.9(1) \cdot 10^{-10} + 1.4(1) \cdot 10^{-7}) \cdot 10^{-8} \text{GeV} = 6.12(2) \cdot 10^{-8} \text{GeV}.$$ 

Therefore, even though the phase space the main contribution $M_{ij}$ increases for decreasing lepton mass, the total cross section decreases due to the additional diagrams.

## 5 Conclusions

In this paper we have studied the decay channel $H \to Z^* (\to \ell^+ \ell^-) + Z^* (\to \ell^+ \ell^-)$ into identical leptons in detail. We have dealt with the peculiarities of identical particle effects and worked on the dependence on the lepton mass. We have found that for increasing lepton mass the decay rates decrease for class-I contributions while class-II contributions invert this trend. We have shown that nondiagonal class-I interference contributions are suppressed compared to diagonal class-I noninterference contributions by about a factor of 10. Lepton mass dependent class-II contributions correct the result by 6% but can be safely
Table 3: Contributions of $J_{ij}$, $K_{ij}$, $L_{ij}$ and $M_{ij}$ to the total rate for $m = m_\tau$. The values are given in units of $10^{-7}$ GeV.

|    | a          | b          | c          | d          |
|----|------------|------------|------------|------------|
| $J_{ij}$ | +0.1440(6) | +0.00927(4) | +0.00928(4) | +0.000837(3) |
| $K_{ij}$ | -0.0009(1) | -0.0006(1) | -0.0006(1) | -0.0009(1)  |
| $L_{ij}$ | -0.0009(1) | -0.0007(1) | -0.0007(1) | -0.0009(1)  |
| $M_{ij}$ | +2.357(9)  | +0.2446(8) | +0.2446(8) | +2.357(9)   |
Table 4: Contributions of $M_{ij}$ to the total rate for different lepton masses $m = m_e, m_\mu, m_\tau$.

The values are given in units of $10^{-7}$ GeV.

| $m = m_e$ | $a$     | $b$     | $c$     | $d$     |
|-----------|---------|---------|---------|---------|
| $a$       | 2.447(9)| 0.2501(8)| 0.2501(8) | 2.447(9) |
| $b$       | 0.2501(8) | 2.46(2) | 2.46(2) | 0.2501(8) |
| $c$       | 0.2501(8) | 2.46(2) | 2.46(2) | 0.2501(8) |
| $d$       | 2.447(9) | 0.2501(8) | 0.2501(8) | 2.447(9) |

| $m = m_\mu$ | $a$     | $b$     | $c$     | $d$     |
|-------------|---------|---------|---------|---------|
| $a$         | 2.446(9) | 0.2501(8) | 0.2501(8) | 2.446(9) |
| $b$         | 0.2501(8) | 2.46(2) | 2.46(2) | 0.2501(8) |
| $c$         | 0.2501(8) | 2.46(2) | 2.46(2) | 0.2501(8) |
| $d$         | 2.446(9) | 0.2501(8) | 0.2501(8) | 2.446(9) |

| $m = m_\tau$ | $a$     | $b$     | $c$     | $d$     |
|--------------|---------|---------|---------|---------|
| $a$          | 2.357(9) | 0.2446(8) | 0.2446(8) | 2.357(9) |
| $b$          | 0.2446(8) | 2.37(2) | 2.37(2) | 0.2446(8) |
| $c$          | 0.2446(8) | 2.37(2) | 2.37(2) | 0.2446(8) |
| $d$          | 2.357(9) | 0.2446(8) | 0.2446(8) | 2.357(9) |
neglected for the lighter leptons. Mixed contributions between class-I and class-II processes can be neglected in all cases. We have dwelled on the narrow width approximation and we have shown that in the limit of a vanishing vector boson width $\Gamma_Z$ the nondiagonal interference contribution $\Gamma^{AB}$ stays constant while the diagonal noninterference contribution $\Gamma^{AA}$ grows, leading to the approximate narrow width limit $\Gamma^{AB}/\Gamma^{AA} \to 5(\Gamma_Z/m_Z)$. As a possible observable for future experiments we worked on single angle decay distributions and identified the contributions of the diagonal and nondiagonal terms to the two separate peaks of the distribution, indicating a clear assignment of the lepton momenta to the intermediate virtual $Z$ bosons or a mixture of those, respectively.

**Acknowledgments**

This work was supported by the Estonian Institutional Research Support under grant No. IUT2-27, by the Estonian Science Foundation under grant No. 8769, and by the European Regional Development Fund under Grant No. TK133. We would like to thank A. Denner, M. Rauch and J. Wang for useful discussions. S.G. acknowledges the support by the Mainz Institute of Theoretical Physics (MITP) and by the centers of excellence PRISMA and PRISMA+. This paper is finished in greatful remembrance on our deceased colleague Jürgen G. Körner who initiated this research.

**A Kinematics of the single angle decay distributions**

In this Appendix we deal in detail on the kinematics of the single angle decay distributions. Of particular interest is the treatment of the delta distribution in Eq. (30) and the construction of the corresponding integration limits for VEGAS. One has

\[
p_a^2 = (p_1 + p_3)^2 = p_a^2, \quad p_b^2 = (p_2 + p_4)^2 = p_b^2, \]

\[
p_c^2 = (p_1 + p_4)^2 = 2m^2 + \frac{1}{2} p_a p_b \left( \cosh(\lambda_a - \lambda_b) + v_a \cos \theta_a \sinh(\lambda_a - \lambda_b) + \right.
\]
of the momentum three-vectors, 

\[ p_a^2 = (p_2 + p_3)^2 = 2m^2 + \frac{1}{2} \sqrt{p_a^2 p_b^2} \left( \cosh(\lambda_a - \lambda_b) - v_a \cos \theta_a \sinh(\lambda_a - \lambda_b) + \right. \\
\left. - v_b \cos \theta_b \sinh(\lambda_a - \lambda_b) + v_a v_b \cos \theta_a \cos \theta_b \cosh(\lambda_a - \lambda_b) + v_a v_b \sin \theta_a \sin \theta_b \cos \phi \right) \]

(A1)

where \( v_a, v_b, \lambda_a \) and \( \lambda_b \) are given in terms of \( m^2, p_a^2, p_b^2 \) and \( m_H^2 \). For the function \( \cos \theta^i(p_a^2, p_b^2, \theta_a, \theta_b, \phi) \) we have to take into account the scalar products and normalizations of the momentum three-vectors,

\[ \cos \theta_{ij} = \frac{\vec{p}_i \cdot \vec{p}_j}{||\vec{p}_i|| ||\vec{p}_j||}, \]

(A2)

where for instance \( \cos \theta_{13} = \cos \theta_{13} \) can be constructed with the help of

\[ |\vec{p}_1|^2 = -(1 + \cos^2 \theta_a \sinh^2 \lambda_a)m^2 + \]

\[ + \frac{p_a^2}{4} \left( \cosh^2 \lambda_a + \cos^2 \theta_a \sinh^2 \lambda_a + 2v_a \cos \theta_a \sinh \lambda_a \cosh \lambda_a \right), \]

\[ |\vec{p}_3|^2 = -(1 + \cos^2 \theta_a \sinh^2 \lambda_a)m^2 + \]

\[ + \frac{p_a^2}{4} \left( \cosh^2 \lambda_a + \cos^2 \theta_a \sinh^2 \lambda_a - 2v_a \cos \theta_a \sinh \lambda_a \cosh \lambda_a \right), \]

\[ \vec{p}_1 \cdot \vec{p}_3 = (1 + \cos^2 \theta_a \sinh^2 \lambda_a)m^2 - \frac{p_a^2}{4}(1 - \sin^2 \theta_a \sinh^2 \lambda_a). \]

(A3)

One can use

\[ \delta \left( \frac{\vec{p}_1 \cdot \vec{p}_3}{||\vec{p}_1|| ||\vec{p}_3||} - \cos \theta_{13} \right) = |\vec{p}_1||\vec{p}_3|\delta(\vec{p}_1 \cdot \vec{p}_3 - |\vec{p}_1||\vec{p}_3| \cos \theta_{13}) \]

(A4)

in order to get rid of the ratio in the argument of the delta distribution, and the square root hidden in the absolute values of the three-vectors can be removed by using

\[ \delta \left( (\vec{p}_1 \cdot \vec{p}_3)^2 - |\vec{p}_1|^2 |\vec{p}_3|^2 \cos^2 \theta_{13} \right) = \frac{1}{2|\vec{p}_1||\vec{p}_3||\vec{p}_1 \cdot \vec{p}_3|} \]

\[ \times \left( \delta \left( \frac{\vec{p}_1 \cdot \vec{p}_3}{||\vec{p}_1|| ||\vec{p}_3||} - \cos \theta_{13} \right) + \delta \left( \frac{\vec{p}_1 \cdot \vec{p}_3}{||\vec{p}_1|| ||\vec{p}_3||} + \cos \theta_{13} \right) \right). \]

(A5)

The duplication of zeros corresponds to a multiplicity of solutions for the zeroth of the argument of the delta distribution. Solving \( (\vec{p}_1 \cdot \vec{p}_3)^2 - |\vec{p}_1|^2 |\vec{p}_3|^2 \cos^2 \theta_{13} = 0 \) for \( \cos \theta_a \) leads
In Fig. 8 we compare the original function \( \cos \theta_{13}(\cos \theta_a) \) (left panel) with the four branches \( a, b, c \) and \( d \) for the inverted function \( \cos \theta_a(\cos \theta_{13}) \) (right panel). Obviously, only half of the branches have to be taken, leading to two solutions (zeros)

\[
c_a = \frac{\pm \left( p_a^2 \cosh^2 \lambda_a + 4m^2 \right) \sin^2 \theta_{13} - 2p_a^2 \pm \sqrt{4p_a^2 \cos^2 \theta_{13}(p_a^2 - 4m^2 \cosh^2 \lambda_a \sin^2 \theta_{13})}}{\left( p_a^2 - 4m^2 \right) \sin^2 \lambda_a \sin^2 \theta_{13}}
\]

(A6)

which can be written symbolically as

\[
\begin{align*}
    c_a^{(a)} &= \sqrt{\frac{N_a + R_a}{D_a}}, \\
    c_a^{(b)} &= \sqrt{\frac{N_a - R_a}{D_a}}, \\
    c_a^{(c)} &= -\sqrt{\frac{N_a + R_a}{D_a}}, \\
    c_a^{(d)} &= -\sqrt{\frac{N_a - R_a}{D_a}}.
\end{align*}
\]

(A7)

In the equations, we can write again in the compact form

\[
c_a^+ = \left\{ \begin{array}{ll}
    c_a^{(a)} & \text{for } \cos \theta_{13} \leq 0, \\
    c_a^{(b)} & \text{for } \cos \theta_{13} \geq 0,
\end{array} \right.
\]

\[
c_a^- = \left\{ \begin{array}{ll}
    c_a^{(c)} & \text{for } \cos \theta_{13} \leq 0, \\
    c_a^{(d)} & \text{for } \cos \theta_{13} \geq 0,
\end{array} \right.
\]

(A8)

which can be written again in the compact form

\[
c_a^+ = \pm \sqrt{\left( p_a^2 \cosh^2 \lambda_a + 4m^2 \right) \sin^2 \theta_{13} - 2p_a^2 - 2 \cos \theta_{13} \sqrt{p_a^2(p_a^2 - 4m^2 \cosh^2 \lambda_a \sin^2 \theta_{13})}}
\]

\[
\left( p_a^2 - 4m^2 \right) \sinh^2 \lambda_a \sin^2 \theta_{13}
\]

(A9)
Employing the known rule $\delta(f(x)) = \sum \delta(x-x_i)/|f'(x_i)|$ where the sum runs over the zeros $x_i$ of $f(x)$, each term in the sum of delta distributions will lead to the replacement of $\cos \theta_a$ by the corresponding zero $c_a^{\pm}$. After a long calculation involving a lot of simplifications by hand (but checked afterwards numerically), we obtain

$$\delta \left( \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1||\vec{p}_3|} - \cos_{13} \right) = \frac{1}{C_a} \left( \delta(\cos \theta_a - c_a^+) + \delta(\cos \theta_a - c_a^-) \right),$$

(A10)

where

$$C_a := v_a \sinh \lambda_a \frac{\sqrt{r_a n_a}}{2 t_a^{3/2}} ,$$

$$r_a := \left( p_a^2 \sin^2 \lambda_a + 4 m^2 (1 - \cosh^2 \lambda_a \cos^2 \theta_{13}) \right) \sin^2 \theta_{13} - q_a ,$$

$$n_a := \left( p_a^2 - 4 m^2 \cosh^2 \lambda_a (2 - \cos^2 \theta_{13}) \right) \sin^2 \theta_{13} + q_a ,$$

$$t_a := \frac{q_a}{\sin^2 \theta_{13}} - 4 m^2 \cosh^2 \lambda_a \sin^2 \theta_{13} ,$$

$$q_a := \left( \sqrt{p_a^2 + \cos \theta_{13}} \sqrt{p_a^2 - 4 m^2 \cosh^2 \lambda_a \sin^2 \theta_{13}} \right)^2 ,$$

$$d_a := (p_a^2 - 4 m^2) \sin^2 \lambda_a \sin^2 \theta_{13} .$$

(A11)

It is easy to see that $c_a^{\pm} = \pm \sqrt{r_a/d_a}$. A corresponding explicit relation is possible between $\theta_{24}$ and $c_b$, but an analytic resolution fails for $\theta_{14}$ and $\theta_{23}$.

### A.1 Constraints for the angle $\theta_{13}$

In performing the calculation with VEGAS, the calculation experiences a couple of branching points, i.e. phase space values have to be excluded where either the radicant

$$p_a^2 - 4 m^2 \cosh^2 \lambda_a \sin^2 \theta_{13}$$

(A12)

in $q_a$ or the radicant $r_a$ of the final result for $c_a^{\pm}$ becomes negative. The restriction to the former one imposes the minimum condition

$$| \cos \theta_{13} | \geq \sqrt{1 - \frac{p_a^2}{4 m^2 \cosh^2 \lambda_a}} ,$$

(A13)

26
while the restriction to the latter one leads to the maximum condition

$$\cos \theta_{13} \leq \frac{p_a^2(\sinh^2 \lambda_a - 1) + 4m^2}{p_a^2(\sinh^2 \lambda_a + 1) + 4m^2}. \quad (A14)$$

Both constraints are displayed in Fig. 9, leaving a narrow region where $\cos \theta_{13}$ is defined. In Fig. [11] we cut these surfaces of constraint at $p_a^2 = 100 \text{ GeV}^2$. The region is narrow in particular for small $p_a^2$, explaining the pile-up of the peak close to the upper boundary shown in Fig. [6]

Note that the dependence of the opening angle $\theta^a = \theta_{13}$ between the two leptons on the angle $\theta_a$ of the leptons with respect to the momentum direction of the $Z$ boson shown in the left panel of Fig. [8] can be understood physically: If $\theta_a = 0$ or $\theta_a = \pi$, i.e. the
two $\tau$ leptons are emitted in the boost direction, the boosted momenta remain in this direction. The maximal effect is found for $\theta_a = \pi/2$ where the two $\tau$ leptons are emitted perpendicularly to the $Z$ momentum direction. In this case the two momenta are boosted to the front. However, one has to be careful about joining the branches. If channel $a$ is quite “light” (i.e. $p_a^2$ small), the boost is close to the light cone and the $\tau$ lepton emitted in back direction will be boosted to the front as well. This is shown in the left panel of Fig. 10 for decreasing values $\sqrt{p_a^2} = 20$ GeV, 14 GeV and 10 GeV. In case of $\sqrt{p_a^2} = 10$ GeV as in the right panel of Fig. 10 one cannot combine two branches to obtain a function but remains with four distinct branches which have to be considered for $\cos \theta_{13} > 0$. The situation flips for $\sinh \lambda_a < v_a \cosh \lambda_a$ or

$$p_a^2 < m \frac{m_h^2 - p_b^2}{m_h - m} \quad \Leftrightarrow \quad p_b^2 < m_h^2 - \frac{m_h - m}{m} p_a^2 \quad (A15)$$

which is a pure mass effect. The flip point can be found in Fig. 11 as the apex of the inverted parabola. This means that values with negative values of $\cos \theta_{13}$ below this threshold have
Figure 11: Constraints on $\cos \theta_{13}$ in dependence on $\sqrt{p^2_b}$ for $p^2_a = 100 \text{ GeV}^2$

to be omitted. Instead, one has to take into account all four branches, as being obvious from the right panel of Fig. 10, leading to the modified result

\[
\delta \left( \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1||\vec{p}_3|} - \cos_{13} \right) = \Theta(\cos \theta_{13}) \times \\
\times \left[ \frac{1}{2C_a} \left( \delta(\cos \theta_a - c_{a}^{(a)}) + \delta(\cos \theta_a - c_{a}^{(c)}) \right) + \frac{1}{2C_a} \left( \delta(\cos \theta_a - c_{a}^{(b)}) + \delta(\cos \theta_a - c_{a}^{(d)}) \right) \right],
\]

where (for $\cos \theta_{13} > 0$ which is expressed by the Heaviside step function) one has $c_{a}^{(a)} = c_{a}^{+}$, $c_{a}^{(c)} = c_{a}^{-}$ and $c_{a}^{(b)} = \bar{c}_{a}^{+}$, $c_{a}^{(d)} = \bar{c}_{a}^{-}$ with $\bar{c}_{a}^{\pm} = \pm \sqrt{\bar{r}_a/d_a}$, $\bar{C}_a$ and $\bar{r}_a$ calculated by replacing $q_a$ by

\[
\bar{q}_a := \left( \sqrt{p^2_a - \cos \theta_{13}} \sqrt{p^2_a - 4m^2 \cosh^2 \lambda_a \sin^2 \theta_{13}} \right)^2,
\]

i.e. by formally replacing $\cos \theta_{13} \rightarrow -\cos \theta_{13}$. However, this second integration region can again be omitted, because for $p^2_a < 4m^2 \cosh^2 \lambda_a$ and $\cos \theta_{13} < 0$ the expression $\bar{t}_a$ is negative, as

\[
p^2_a - 4m^2 \cosh^2 \lambda_a (1 - \cos^2 \theta_{13}) < 4m^2 \cosh^2 \lambda_a \cos^2 \theta_{13},
\]

\footnote{Otherwise, for small values of $\sqrt{p^2_a}$ one would obtain a small cusp close to $\cos \theta_{13} = -1$.}
\[ \sqrt{p_a^2 - \cos \theta_{13}} \sqrt{p_a^2 - 4m^2 \cosh^2 \lambda_a (1 - \cos^2 \theta_{13})} < 2m \cosh \lambda_a \sin^2 \theta_{13}, \]

\[ \bar{q}_a = \left( \sqrt{p_a^2 - \cos \theta_{13}} \sqrt{p_a^2 - 4m^2 \cosh^2 \lambda_a (1 - \cos^2 \theta_{13})} \right)^2 < 4m^2 \cosh^2 \theta_{13} \sin^4 \theta_{13} \quad (A18) \]

and, therefore,

\[ \bar{t}_a = \frac{q_a}{\sin^2 \theta_{13}} - 4m^2 \cosh^2 \lambda_a \sin^2 \theta_{13} < 0. \quad (A19) \]

Drawing a straight line from the apex in Fig 11 to the bottom of the diagram and skipping the region left to this line, one ends up with the integration region that is purely functional.

### A.2 Constraints for the angle \( \theta_{14} \)

Though the method explained before does not work for \( \theta_c = \theta_{14} \) and \( \theta_d = \theta_{23} \), there is a possibility to solve the delta distributions also in these cases. For this note that the momentum square equations for \( p_c^2 \) and \( p_d^2 \) in Eqs. (A1) can be solved for \( \cos \phi \) almost trivially, inserted into the angular cosine equations, and solved analytically for \( \cos \theta_a \) (or \( \cos \theta_b \)). As the procedure is quite similar for the \( c \) and \( d \) channels, the results are written down for the \( c \) channel only. Solving for \( \cos \phi \), one obtains the solution \( \cos \phi_0 \) with

\[ v_a v_b \sin \theta_a \sin \theta_b \cos \phi_0 = \frac{2(p_c^2 - 2m^2)}{\sqrt{p_a^2 p_b^2}} + \]

\[ - (1 + v_a v_b \cos \theta_a \cos \theta_b) \cosh(\lambda_a - \lambda_b) - (v_a \cos \theta_a + v_b \cos \theta_b) \sin(\lambda_a - \lambda_b) \quad (A20) \]

and, therefore,

\[ \delta \left( p_c^2(p_a^2, p_b^2, \theta_a, \theta_b, \phi) - p_c^2 \right) = \frac{2(p_c^2 p_b^2)^{-1/2}}{v_a v_b |\sin \theta_a \sin \theta_b \sin \phi|} \delta(\phi_0 - \phi). \quad (A21) \]

There seems to be a risk that \( \theta_a, \theta_b, v_a \) or \( v_b \) might vanish. Kinematically, this is the case if one of the decay planes is no longer spanned up and, therefore, the relative angle \( \phi \) is not given. If the expressions to which we insert \( \cos \phi \) also contain these factors, one can assign an arbitrary value to \( \phi \), but if this is not the case, these points have to be skipped artificially for the numerical integration by VEGAS. A hint for the first case is that if
we insert \( \cos \phi \) into the second equation, these factors cancel out. Solving for \( \cos \theta_b \) one obtains

\[
\cos \theta_b = \frac{N_b^c \pm c_b R_b^c}{D_b^c} =: c_b^\pm
\]

(note the absence of a general square root) with

\[
D_b^c = \sqrt{p_a^2 p_b^2 v_b^2 \sinh^2 \lambda_b \left[ c_a^2 - c_b^2 \right]},
\]

\[
N_b^c = \sqrt{p_a^2 p_b^2 v_b \sinh \lambda_b c_a (2p_c^2 - 4m^2) - p_a^2 p_b^2 v_b \sinh \lambda_b \cosh \lambda_b \left[ c_a^2 - c_b^2 \right]},
\]

\[
R_b^c = \sqrt{p_a^2 p_b^2 v_b \sinh \lambda_b \sqrt{(2p_c^2 - 4m^2)^2 - 4m^2 p_a^2 (c_a^2 - c_b^2)}},
\]

where we have used the abbreviation \( c_\theta := \cos \theta_1 \sqrt{s_a^2 + v_a^2 \sin^2 \theta_a} \) as well as \( c_a := \cosh \lambda_a + v_a \cos \theta_a \sinh \lambda_a \) and \( s_a := \sinh \lambda_a + v_a \cos \theta_a \cosh \lambda_a \) in order to obtain a compact form. Also in this case the calculation of \( |f'(x_i)| \) and, therefore, \( C_c \) was successful. Analysing the pole structure as before, it can be seen that only \( c_b^+ \) can be considered as a relevant zero. One obtains

\[
\delta \left( \frac{\vec{p}_1 \cdot \vec{p}_4}{|\vec{p}_1||\vec{p}_4|} - \cos \theta_{14} \right) = \frac{1}{C_c} \delta(c_b^+ - \cos \theta_b),
\]

where

\[
C_c = \frac{\sqrt{p_a^2 p_b^2 v_b \sinh \lambda_b (c_a^2 - c_b^2)^2 n_c}}{\sqrt{s_a^2 + v_a^2 \sin^2 \theta_a d_c^3}}
\]

with

\[
n_c := c_a \left( (2p_c^2 - 4m^2)^2 - 4m^2 p_a^2 (c_a^2 - c_b^2) \right) + \]
\[
+ c_\theta (2p_c^2 - 4m^2) \sqrt{(2p_c^2 - 4m^2)^2 - 4m^2 p_a^2 (c_a^2 - c_b^2)},
\]

\[
d_c := c_\theta (2p_c^2 - 4m^2) + c_a \sqrt{(2p_c^2 - 4m^2)^2 - 4m^2 p_a^2 (c_a^2 - c_b^2)}.
\]

The function \( C_c(\cos \theta^c) \) shown in Fig. 12 for different energies \( \sqrt{p_c^2} \) is quite similar to the function \( C_a(\cos \theta^a) \). Therefore, we expect similar angular distributions.

The results obtained so far lead to the constraints we are looking for. The first condition
is

$$(2p_c^2 - 4m^2)^2 - 4m^2 p_a^2 (c_a^2 - c_b^2) \geq 0 \iff \cos^2 \theta_{14} \geq \frac{1}{s_a^2 + v_a^2 \sin^2 \theta_a} \left( c_a^2 - \frac{(2p_c^2 - 4m^2)^2}{4m^2 p_a^2} \right).$$

(A29)

On the other hand, the condition $c_b^+ \leq 1$ leads to $\cos^2 \theta_{14} \leq c_a^2 / (s_a^2 + v_a^2 \cos^2 \theta_a)$ which is satisfied trivially, and

$$\cos^2 \theta_{14} \leq \frac{((2p_c^2 - 4m^2) - \sqrt{p_a^2 p_b^2 (s_a^2 + v_a^2 \sin^2 \theta_a) s_b^2})^2}{p_a^2 p_b^2 (s_a^2 + v_a^2 \sin^2 \theta_a) s_b^2}$$

(A30)

with $s_b := \sinh \lambda_b + v_b \cosh \lambda_b$ and $c_b := \cosh \lambda_b + v_b \sinh \lambda_b$. This is a quite weak condition, working only close to $\cos \theta_a = -1$ (anticollinear lepton with momentum $p_1$).

References

[1] G. Aad et al. [ATLAS], “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” Phys. Lett. B 716 (2012),
[2] S. Chatrchyan et al. [CMS], “Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC,” Phys. Lett. B 716 (2012), 30–61

[3] A. Rinkevičius, “An Observation of a Higgs Boson in the H to ZZ to Four Leptons Decay Channel and the Studies of Its Spin-Parity Properties,” PhD thesis, University of Florida, 2014

[4] M. A. Snowball, “Observation of a Higgs Boson in the H to ZZ to 4L Decay Channel and using Z to 4L Decays as a Calibration Tool in Studies of the Higgs Boson Properties,” PhD thesis, University of Florida, 2014

[5] F. M. Garay, “Studies of the Higgs boson using the $H \rightarrow ZZ \rightarrow 4l$ decay channel with the ATLAS detector at the LHC,” PhD thesis, University of Edinburgh, 2016

[6] L. Finco, “Measurement of the ZZ cross-section in the fully leptonic decay channel in association with jets, and studies on vector boson scattering with the CMS Experiment,” PhD thesis, Università degli Studi di Turino, 2016

[7] G. J. Cree, “Direct Measurement of the Higgs Boson Mass, Natural Width, and Cross Section Times Branching Ratio to Four Leptons Using a Per-Event Lineshape in the Higgs to ZZ to Four Lepton Decay Channel with the ATLAS Detector,” PhD thesis, Carleton University, 2017

[8] M. McKay, “Measurement of Inclusive Fiducial and Differential Cross Sections in the $H \rightarrow ZZ* \rightarrow 4l$ Decay Channel in p-p Collisions at 13 TeV with the Full ATLAS Run 2 Dataset,” PhD thesis, Southern Methodist University, 2020

[9] A. Gaile, “Measurements of Higgs boson properties in the Golden Channel and development of MTD DCS and CS and DSS in CMS experiment,” PhD thesis, Riga Technical University, to be defended
[10] P. Avery et al., “Precision studies of the Higgs boson decay channel $H \to ZZ \to 4\ell$ with MEKD,” Phys. Rev. D 87 (2013) no.5, 055006

[11] K. Nikolopoulos [ATLAS], “Search for the Standard Model Higgs boson in the $H \to ZZ^* \to 4\ell$ decay channel with the ATLAS detector,” PoS ICHEP2012 (2013), 057

[12] CMS Collaboration [CMS], “Vector Boson Scattering prospective studies in the $ZZ$ fully leptonic decay channel for the High-Luminosity and High-Energy LHC upgrades,” Report No. CMS-PAS-FTR-18-014

[13] G. Ranft and J. Ranft, “Identical Particle Effect in Azimuthal Correlations at Small Rapidity Separation,” Phys. Lett. B 53 (1974), 188–190

[14] W. M. De Muynck, “Distinguishable and Indistinguishable Particle Descriptions of Systems of Identical Particles,” Int. J. Theor. Phys. 14 (1975), 327–346

[15] A. N. Schellekens and W. L. van Neerven, “Identical Particle Effects in Quark Quark Lepton Pair Production,” Report No. THEF-NYM-80-7

[16] A. Denner, S. Heinemeyer, I. Puljak, D. Rebuzzi and M. Spira, “Standard Model Higgs-Boson Branching Ratios with Uncertainties,” Eur. Phys. J. C 71 (2011), 1753

[17] S. Berge, S. Groote, J. G. Körner and L. Kaldamäe, “Lepton-mass effects in the decays $H \to ZZ^* \to \ell^+\ell^-\tau^+\tau^-$ and $H \to WW^* \to \ell\nu\tau\nu$,” Phys. Rev. D 92 (2015) no.3, 033001

[18] G. Aad et al. [ATLAS and CMS Collaborations], “Combined Measurement of the Higgs Boson Mass in $pp$ Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments,” Phys. Rev. Lett. 114 (2015), 191803

[19] N. Christ, “Possible CP Violation in $K^\pm \to \pi^\pm\pi^0\gamma$,” Phys. Rev. 159 (1967), 1292
[20] N. Cabibbo and A. Maksymowicz, “Angular Correlations in Ke-4 Decays and Determination of Low-Energy pi-pi Phase Shifts,” Phys. Rev. 137 (1965), B438 [Erratum-ibid. 168 (1968), 1926]

[21] A. Pais and S. B. Treiman, “Pion Phase-Shift Information from Kl-4 Decays,” Phys. Rev. 168 (1968), 1858.

[22] L. Cappiello, O. Cata, G. D’Ambrosio and D.N. Gao, “K⁺ → π⁺π⁰e⁺e⁻: a novel short-distance probe,” Eur. Phys. J. C72 (2012), 1872 [Erratum-ibid. C72 (2012), 2208]

[23] G. P. Lepage, “Adaptive multidimensional integration: VEGAS enhanced,” J. Comput. Phys. 439 (2021), 110386

[24] L. Kaldamäe and S. Groote, “Virtual and real processes, the Källén function, and the relation to dilogarithms,” J. Phys. G 42 (2015) no.8, 085003

[25] M. Ježabek and J.H. Kühn, “QCD Corrections to Semileptonic Decays of Heavy Quarks,” Nucl. Phys. B314 (1989), 1

[26] M. Ježabek and J.H. Kühn, “The Top width: Theoretical update,” Phys. Rev. D48 (1993), 1910 [Erratum-ibid. D49 (1994), 4970]

[27] B. A. Kniehl, “The Higgs Boson Decay H→Z gg,” Phys. Lett. B 244 (1990), 537

[28] A. Ali, J.G. Körner, Z. Kunszt, J. Willrodt, G. Kramer, G. Schierholz and E. Pietarinen, “Four Jet Production in e⁺e⁻ Annihilation,” Phys. Lett. B 82 (1979), 285

[29] M. Böhm, A. Denner and H. Joos, “Gauge Theories of the Strong and Electroweak Interactions,” 3. Ed., B.G. Teubner, Stuttgart, 2001