MAGNETIC PROPERTIES OF THE SKYRMIION GAS IN TWO DIMENSIONS

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Abstract

The classical ferromagnet is analyzed in two dimensions via a mean-field treatment of the CP$^1$ model valid in the limit of low temperature and of weak magnetic field. At high skyrmion densities, we find that the magnetization begins to decrease linearly with temperature, but it then crosses over to a Curie-law at higher temperature. The comparison of these results with recent Knight-shift experiments on the quantum Hall state is fair.

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It is well known that the continuum ferromagnet supports topologically non-trivial spin textures called skyrmions in two dimension.\textsuperscript{1−3} In physical terms such skyrmions represent domain-wall loops within which the magnetization is reversed. The scale invariance characteristic of the Heisenberg model in two dimensions indicates that the corresponding skyrmion solutions have no preferred size, nor do they interact, at zero temperature. Quantum fluctuations in the form of dynamical spin-wave excitations, however, break this scale invariance, which means that skyrmions interact at non-zero temperature. Similar effects arise in the case of the two-dimensional (2D) quantum antiferromagnet, but at zero temperature as well.\textsuperscript{4}

Recent theoretical and experimental work indicates that skyrmion spin textures are tied to excess electronic charge with respect to a filled Landau level in the quantum Hall effect,\textsuperscript{5−7} at which point the groundstate is a 2D ferromagnet. Motivated by this as well as by general theoretical considerations, we shall study here the magnetic properties of a gas of skyrmions with a net topological charge density. Like it’s electronic counterpart, the total skyrmion number is necessarily quantized in integer values due to the topological nature of the 2D ferromagnet.\textsuperscript{1−3} It is also a conserved quantity, which justifies the restriction to constant skyrmion number assumed throughout. The magnetic correlations, as well as the magnetization, are calculated as a function of temperature and of external magnetic field via a meanfield analysis of the CP\textsuperscript{1} model for the classical ferromagnet.\textsuperscript{2,3,8} Quantum effects\textsuperscript{9} related to dynamical spin-wave excitations are therefore neglected. Notably, we obtain the following results in the limit of high skyrmion density: (a) the magnetic correlation length is set by the inter-skyrmion separation; (b) the paramagnetic susceptibility displays a Curie law at all temperatures due to the disordering effect of the skyrmion gas; and (c) the magnetization in fixed external field crosses over from a low-temperature linear decrease to the above Curie-law tail at high temperature. Although the latter cross-over is weak (see Fig. 1), the agreement between the theoretically predicted cross-over temperature with that observed in recent Knight-shift measurements\textsuperscript{7} on the quantum Hall effect is surprisingly good.

\textit{CP\textsuperscript{1} Model.} We now consider the thermodynamics of the classical 2D ferromagnet in the presence of a net skyrmion density. The energy functional of the corresponding
continuum Heisenberg model may be written in the CP\(^1\) form as

\[ E = 2s^2 J \int d^2 r |(\nabla - i\vec{a})z|^2, \tag{1} \]

where \( z = (z_-, z_+) \) is a complex valued doublet field constrained to unit modulus,

\[ \bar{z}z = 1, \tag{2} \]

and where the vector potential \( \vec{a} \) is tied to the \( z \) fields by

\[ \vec{a} = i(\nabla \bar{z})z. \tag{3} \]

Here \( s \) denotes the microscopic spin. The \( z \) fields are related to the normalized magnetization, \( \vec{m} \), of the Heisenberg model by

\[ \vec{m} = \bar{z}\vec{\sigma}z, \tag{4} \]

where \( \vec{\sigma} \) denotes the Pauli matrices, while the spin stiffness (1) is given by \( \rho_s = s^2 J \).

Using Eqs. (2)-(4), it can be shown that the constrained (\(|\vec{m}|^2 = 1\)) energy functional \( E = \frac{1}{2} \rho_s \int d^2 r |\nabla \vec{m}|^2 \) is equal to that of the CP\(^1\) model (1).\(^2\),\(^3\),\(^8\) Similarly, it can be shown that the skyrmion density is equal to the density of fictitious magnetic flux, \((\partial_x a_y - \partial_y a_x)/2\pi\).

To proceed further, we now write the Gibbs distribution, \( Z = \int \mathcal{D}\vec{m} \delta(|\vec{m}|^2 - 1)e^{-E/k_B T} \), in the presence of external magnetic field \( H \) along the \( z \)-axis as

\[ Z = \int \mathcal{D}z \mathcal{D}\bar{z} \mathcal{D}\vec{a} \mathcal{D}\lambda \exp \left\{ -2\beta \int d^2 r |(\nabla - i\vec{a})z|^2 + i\lambda(|z|^2 - 1) + b_Z(|z_-|^2 - |z_+|^2) \right\}, \tag{5} \]

where we define \( \beta = s^2 J/k_B T \) and \( b_Z = nE_Z/4s^2 J \), with spin density \( n \) and Zeeman energy splitting per spin \( E_Z = g\mu_B H \). Here integration over the Langrange multiplier field \( \lambda(\vec{r}) \) enforces constraint (2), while integration over the now unconstrained vector potential field \( \vec{a} \) in the Coulomb gauge, \( \nabla \cdot \vec{a} = 0 \), recovers constraint (3). We now make the basic approximations of the paper, which are (i) to enforce constraint (2) only on average over the entire 2D plane, and (ii) to neglect spatial fluctuations in the skyrmion density;\(^3\) i.e., the \( z \) fields above are integrated out first with the presumption of a homogeneous Langrange multiplier field \( \lambda(\vec{r}) = \lambda_0 \) and of a homogeneous fictitious magnetic field \( b_S = (\partial_x a_y - \partial_y a_x) \). The latter is of course equal to the net skyrmion density multiplied by
a factor of $2\pi$. The saddle-point condition $\frac{\partial}{\partial \lambda_0} \ln Z = 0$ is equivalent to the meanfield constraint $1 = \langle \tilde{z}_- z_- \rangle + \langle \tilde{z}_+ z_+ \rangle$, where

$$
\langle \tilde{z}_\pm z_\pm \rangle = (2\beta)^{-1} V^{-1} \text{tr}[-(\nabla - i\vec{a})^2 + i\lambda_0 \mp b_Z]^{-1}
$$

(6)

are the respective averages. But since the spectrum of the Hermitian operator $-(\nabla - i\vec{a})^2$ is just that of Landau levels with energies $2b_S(n + \frac{1}{2})$, each with a degeneracy per area $V$ of $b_S/2\pi$, we obtain the meanfield equation

$$
2\pi\beta = \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_S}{2b_S n + \xi^{-2} + b_Z} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_S}{2b_S n + \xi^{-2} - b_Z}
$$

(7)

for the correlation length $\xi$ set by $\xi^{-2} = i\lambda_0 + b_S$. Similarly, we obtain the meanfield expression

$$
m_z = (2\pi\beta)^{-1} \left( \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_S}{2b_S n + \xi^{-2} - b_Z} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_S}{2b_S n + \xi^{-2} + b_Z} \right)
$$

(8)

for the normalized magnetization $m_z = \langle \tilde{z}_+ z_+ \rangle - \langle \tilde{z}_- z_- \rangle$ along the $z$ direction.

Consider now the ferromagnetic regime, $b_S \to 0$, where the concentration of skyrmions is dilute with respect to the magnetic correlation length, but is not zero. The sums in Eqs. (7) and (8) may then be converted into integrals, resulting in $2\pi\beta = \ln [k_0(\xi^{-4} - b_Z^2)^{-1/4}]$ and $m_z = (2\pi\beta)^{-1} \ln [(\xi^{-2} + b_Z)/(\xi^{-2} - b_Z)]^{1/4}$ respectively. Here $k_0$ represents the momentum cut-off corresponding to the former integral. We therefore obtain the familiar result $\xi = (\xi_0^{-4} + b_Z^2)^{-1/4}$ for the correlation length, where the zero-field ferromagnetic correlation length $\xi_0 = k_0^{-1} e^{2\pi\beta}$ diverges exponentially at low temperature. The magnetization is paramagnetic in the limit of weak external field, $b_Z \to 0$, following $m_z = \chi_Z b_Z$ with $\chi_Z = (4\pi\beta)^{-1} \xi_0^2$. In general, it is given by the expression $m_z = 1 - (2\pi\beta)^{-1} \ln [k_0(\xi^{-2} + b_Z)^{-1/2}]$, which means that the normalized magnetization begins to saturate logarithmically as external field increases. The continuum limit then breaks down at $k_0 b_Z^{-1/2} \approx 1$, where $m_z$ is pathological and exceeds unity. The present results correspond to those of the quantum ferromagnet in the renormalized classical regime. The former high-field catastrophe is avoided in that case, however, where the system evolves into a quantum activated regime instead. Also, it can be easily shown that $\langle z_\pm(\vec{r}) \tilde{z}_\pm(\vec{r}') \rangle \sim \exp[-|\vec{r} - \vec{r}'|/(\xi^{-2} \mp b_Z)^{1/2}]$ at long distance $|\vec{r} - \vec{r}'|$, which by Eq.
(4) indicates that \( (\xi^2 - b_2)^{-1/2} \) gives the magnetic correlation length. Last, if \( \delta b_S(\vec{r})/2\pi \) denotes local fluctuations in the skyrmion density, while \( \delta \lambda(\vec{r}) \) denotes the corresponding fluctuations of the Langrange multiplier field, then it can be shown that the Gibbs distribution associated with such fluctuations is given by\(^3\)

\[
Z_2 = \int \mathcal{D}b_S \mathcal{D}\lambda \exp \left\{ -\frac{1}{2} \int d^2r \left[ \chi (\delta b_S)^2 + 2i\sigma (\delta b_S)(\delta \lambda) + \epsilon (\vec{\nabla}\delta \lambda)^2 \right] \right\}
\]  
(9)
in the long-wavelength limit (relative to the inter-skyrmion separation), where \( \chi \sim \xi_0^2 \), \( \sigma = 2\beta/b_S \) and \( \epsilon = \beta/b_S^2 \) (see the Appendix). Hence, the direction of steepest descent for gradients of the Lagrange multiplier field is along the real axis in the presence of a net skyrmion density. Notice also that homogeneous fluctuations \( (\vec{\nabla}\delta \lambda = 0) \) of the Langrange multiplier field are in general only marginally stable in such case. Fluctuations, \( V^{-1} (\int d^2r \delta b_S/2\pi)^2 \sim \xi_0^{-2} \), in the total skyrmion number are exponentially suppressed at low temperature, on the other hand. The latter thermally activated temperature dependence is then of course entirely consistent with the fact that \( E_S = 4\pi s^2 J \) is the energy cost of a single skyrmion.\(^1\)

Consider next the skyrmion-rich limit, \( b_S \to \infty \), where the separation between neighboring skyrmions is much less than the ferromagnetic correlation length, \( \xi_0 \). In that case the \( n = 0 \) terms corresponding to the lowest Landau level dominate the sums in Eqs. (7) and (8), which results in the meanfield equations \( 2\pi\beta = b_S \xi^2/(1 - b_Z^2 \xi^4) \) and \( m_z = \xi^2 b_Z \) respectively. Solving the former relationship explicitly, we obtain the simple expression

\[
m_z = (1 + \bar{t}^2)^{1/2} - \bar{t}
\]  
(10)
for the normalized magnetization, where

\[
\bar{t} = \frac{k_B T b_S}{E_Z \pi n}
\]  
(11)
is the reduced temperature. The magnetization therefore decreases linearly from unity at low temperature and/or high magnetic field, while it follows a Curie law, \( m_z \propto \frac{1}{2} \bar{t}^{-1} \), at high temperature and/or low magnetic field. (It is interesting to note that a similar \( H/T \) dependence is expected for the magnetization of the quantum ferromagnet in the high-temperature quantum-critical regime, yet in the absence of a net skyrmion density.\(^9\)) The effective area \( A_S(T) \) of a single skyrmion can be obtained from the comparison of Eq. (10)
at low temperature with the naive formula $m_z = 1 - 2A_S(b_S/2\pi)$ for the magnetization obtained by counting the number of reversed spins, $S = nA_S$, per skyrmion. This yields an effective area of $A_S = (k_B T/E_Z)n^{-1}$ that results from the opposition of the Zeeman energy, which tends to decrease the number of reversed spins, with the entropic part of the free energy, which tends to increase the number of reversed spins. The cross-over between these two regions naturally occurs at $\bar{t} = 1$ (see Fig. 1), where the effective size of the skyrmion is by definition on the order of the inter-skyrmion separation. Also, the correlation function for the $z$ fields has the gaussian form $|\langle z(\vec{r})\bar{z}(\vec{r'})\rangle|^2 = \exp(-\frac{1}{2}|\vec{r} - \vec{r}'|^2b_S)$ in above lowest-Landau-level approximation, which means that the magnetic correlation length is limited by the inter-skyrmion separation for all values of external magnetic field, $b_Z$, in this regime. This magnetic correlation length therefore also limits the size of a single skyrmion to the inter-skyrmion separation. Last, the long-wavelength fluctuation corrections to the present saddle-point (9) are identical to that of the skyrmion-poor regime, with the exception that $\chi = 2\beta/b_S = \sigma$ (see the Appendix). This means that fluctuations, $V^{-1}\langle(\int d^2r \delta b_S/2\pi)^2\rangle \sim \chi^{-1}$, in the total skyrmion number of the system are much larger in this regime.

Quantum Hall Effect. Let us apply the above results to the quantum Hall effect in the vicinity of unit filling, where it has been suggested that the skyrmion density $b_S/2\pi$ is equal to the excess charge density $\delta n$;\textsuperscript{5-7} i.e., the electronic filling factor, $\nu = 2\pi l_0^2 n$, is related to the skyrmion density by $|\nu - 1| = l_0^2 b_S$, where $l_0$ denotes the magnetic length in fixed magnetic field. Using these relations, we then obtain the simple expression

$$\bar{t} = 2\frac{|\nu - 1| k_B T}{\nu E_Z}$$

for the reduced temperature (11). This immediately defines a cross-over temperature between the low-temperature regime with linearly decreasing magnetization and the high-temperature Curie-law regime (see Fig. 1) given by $k_B T_* = \frac{1}{2}|1 - \nu^{-1}|-1E_Z$, which is notably independent of the spin stiffness. Recent Knight shift measurements exhibit a temperature dependence that is qualitatively similar to that predicted by Eq. (10), yet with a more pronounced cross-over.\textsuperscript{7} Such measurements were conducted in magnetic fields of $H \approx 7$ T with a Zeeman energy of $E_Z = 0.2$ meV, and at filling fractions of $\nu = 0.88$ and $\nu = 1.2$. The present theory for the skyrmion rich regime then predicts cross-over temperatures of $T_* = 8$ K and $T_* = 7$ K, respectively. The sharp cross-overs observed experimentally at roughly 9 K in both cases compare quite well with these estimates. But
do such Knight-shift measurements actually lie within the skyrmion-rich limit discussed here? Inspection of Eq. (8) for the magnetization indicates that the skyrmion density must satisfy $b_S > \max(b_Z, \xi_0^{-2})$ for this to be the case. Presuming that the momentum cut-off is set by the magnetic length, $k_0 \sim l_0^{-1}$, then the latter is equivalent to satisfying conditions (a) $b_S > b_Z$, and (b) $|1 - \nu| > e^{-E_S/k_B T}$, where $E_S = 4\pi \rho_s$ is the energy cost of creating a skyrmion in the ferromagnet. Standard estimates for the spin-stiffness $\rho_s$ of the quantum Hall state at unit filling\textsuperscript{5,9} place this energy at $E_S/k_B \sim 40$ K in a 7 T field, which means that condition (b) is easily satisfied at temperatures much less than this scale. Also, employing Eq. (11) at the cross-over ($\bar{t} = 1$) implies that condition (a) is satisfied as well, since $T_* \lesssim 10$ K falls substantially below the previous scale. Last, the present classical treatment of the ferromagnet is generally valid in the renormalized classical regime,\textsuperscript{9} $E_Z < k_B T < E_S$, which spans the majority of the experimental temperature range.\textsuperscript{7}

In conclusion, although the magnetization (10) obtained here for the classical 2D ferromagnet in the presence of a net skyrmion concentration compares favorably with recent Knight shift experiments on the quantum Hall state, it consistently overestimates the latter.\textsuperscript{7} From the theoretical side, a number of effects could be responsible for this discrepancy. First, it has been noted that fluctuations with respect to the present meanfield treatment of the classical ferromagnet are relatively large in the presence of skyrmions. More generally, it is also known that fluctuations in the skyrmion density act to confine the $z$ and $\bar{z}$ fields in strictly two dimensions.\textsuperscript{8} Such effects have not been accounted for here. Second, the inclusion of quantum mechanical spin-wave excitations will further decrease the magnetization. Last, both quantum fluctuations\textsuperscript{4} as well as the direct Coulomb interaction\textsuperscript{12} tend to crystallize the skyrmion gas. Accounting for all of the above-mentioned effects remains as a problem for the future.

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Appendix

We shall now compute the fluctuation corrections (9) to the present meanfield approximation. If we take the Landau gauge, \( \vec{a} = (0, b_S x) \), then the eigenstates of the Hermitian operator \(-(\vec{V} - i\vec{a})^2 + i\lambda_0 \) have the usual form \( \langle \vec{n}, q \rangle = \langle x' | n \rangle e^{-iqy} \), with eigenvalues \( \varepsilon_n = 2b_S n + \xi^{-2} \). Here \( |n\rangle \) denotes the eigenstates of the corresponding Harmonic oscillator while \( x' = x - b_S^{-1} q \). If we now identify the variation in the vector potential by \( \delta b = \partial_x \delta a_y - \partial_y \delta a_x \), then fluctuations correct the Gibbs distribution by a factor of

\[
\exp \left( -\frac{1}{2} V^{-1} \sum_{\vec{k}} \left\{ \left[ \Pi_{ij}^+(\vec{k}) + \Pi_{ij}^-(\vec{k}) \right] \delta a_i(\vec{k}) \delta a_j(-\vec{k}) + 2i[\Pi_{0j}^+(\vec{k}) + \Pi_{0j}^-(\vec{k})] \delta a_j(\vec{k}) \delta \lambda(-\vec{k}) + [\Pi_{00}^+(\vec{k}) + \Pi_{00}^-(\vec{k})] \delta \lambda(\vec{k}) \delta \lambda(-\vec{k}) \right\} \right), \tag{A1}
\]

where \( \Pi_{ij}^+(\vec{k}) \), \( \Pi_{0j}^+(\vec{k}) \) and \( \Pi_{00}^+(\vec{k}) \) are respectively the high-temperature limits of the Kubo formula, the Lindhard function and the hybrid Kubo-Linhard function corresponding to the \( z_\pm \) field. Also, \( \delta a_i(\vec{k}) \) and \( \delta \lambda(\vec{k}) \) denote the appropriate Fourier components. One may presume that \( \vec{k} \) be parallel to the \( x \)-axis without any loss of of generality, in which case the polarizabilities are given explicitly by

\[
\Pi_{ij}^+(\vec{k}) = V^{-1} \sum_{n,q} \frac{2\delta_{ij}}{\varepsilon_n + b_Z} - V^{-1} \sum_{n\neq n',q} \frac{\langle n, q | v_i(\vec{k}) | n', q \rangle \langle n', q | v_j(-\vec{k}) | n, q \rangle}{\varepsilon_n + \varepsilon_{n'} + b_Z}, \tag{A2}
\]

\[
\Pi_{0j}^+(\vec{k}) = V^{-1} \sum_{n\neq n',q} \frac{\langle n, q | v_j(\vec{k}) | n', q \rangle \langle n', q | e^{-ikx} | n, q \rangle}{\varepsilon_n + \varepsilon_{n'} + b_Z}, \tag{A3}
\]

\[
\Pi_{00}^+(\vec{k}) = V^{-1} \sum_{n\neq n',q} \frac{\langle n, q | e^{ikx} | n', q \rangle \langle n', q | e^{-ikx} | n, q \rangle}{\varepsilon_n + \varepsilon_{n'} + b_Z}, \tag{A4}
\]

where \( \vec{v}(\vec{k}) = 2e^{\frac{i}{2}kx}(i\vec{V} + \vec{a})e^{i\frac{k}{2}x} \) is the modified velocity operator. In general, these polarizabilities have the form \( \Pi_{00}^+(\vec{k}) = \varepsilon_\pm k^2 \), \( \Pi_{0y}^+(\vec{k}) = \sigma_\pm ik \) and \( \Pi_{yy}^+(\vec{k}) = \chi_\pm k^2 \) in the long-wavelength limit, while the remaining ones are identically zero due to gauge invariance. A direct application of (A3) and (A4) then yields \( \sigma_\pm = b_S^{-1} V^{-1} \sum_{n,q}(\varepsilon_n + b_Z)^{-1} \) and \( \varepsilon_\pm = b_S^{-}\frac{1}{2} V^{-1} \sum_{n,q}(\varepsilon_n + b_Z)^{-1} \). Employing mean-field Eq. (7), we then obtain that the net fluctuation susceptibilities (9) are given by \( \sigma = \sigma_+ + \sigma_- = 2\beta/b_S \) and \( \varepsilon = \varepsilon_+ + \varepsilon_- = \beta/b_S^2 \) in general. Similar calculations for (A2) also demonstrate ultimately
that $\chi = 2\beta/b_S = \sigma$ in the skyrmion-rich limit, $b_S \to \infty$. This identity is suggested by the present mean-field analysis, which depends crucially on the combination $\xi^{-2} = i\lambda_0 + b_S$ [see Eq. (9)].
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Figure Caption

Fig. 1. Shown is mean-field result (10) for the magnetization vs. temperature of the classical 2D ferromagnet in the limit of high skyrmion density. The pronounced Curie-law tail notably weakens the cross-over at $T_\star$, which should lie well below $4\pi \rho_s/k_B$. 
