SMOOTH APPROXIMATION OF THE MODIFIED CONICAL KÄHLER-RICCI FLOW

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Abstract. We introduce the conical Kähler-Ricci flow modified by a holomorphic vector field. We construct a long-time solution of the modified conical Kähler-Ricci flow as the limit of a sequence of smooth Kähler-Ricci flows.

1. Introduction

Let $M$ be an $n$-dimensional Fano manifold with a Kähler metric $\omega_0 \in 2\pi c_1(M)$. A Kähler metric $\omega \in 2\pi c_1(M)$ is called Kähler-Einstein if it satisfies $\text{Ric}(\omega) = \omega$. For a long while, it was conjectured that the existence of Kähler-Einstein metrics is equivalent to some algebro-geometric stability in the sense of Geometric Invariant Theory (Yau-Donaldson-Tian conjecture), which was recently solved by Chen-Donaldson-Sun [CDS15] and Tian [Tia15]. Their strategy was to study the existence problem of smooth Kähler-Einstein metrics on $M$ by deforming the cone angle, i.e., study the Gromov-Hausdorff limit of conical Kähler-Einstein metrics with cone angle $2\pi \beta$ ($0 < \beta \leq 1$) along a smooth divisor $D \in | - K_M|:

$$\text{Ric}(\omega) = \beta \omega + (1 - \beta)[D]$$

when $\beta$ goes to 1, where $[D]$ is the current of integration along $D$. Although YDT conjecture has been completely settled, the existence problem of conical Kähler-Einstein metrics itself is also an interesting problem and studied extensively by many experts (cf. [LS14], [SW16]).

Now we consider more general settings: we allow $D \in | - \lambda K_M|$ ($\lambda \in \mathbb{R}_+$) to be an $\mathbb{R}$-effective divisor with simple normal crossing support and write

$$D = \sum_{i=1}^{d} \tau_i D_i$$

where $\tau_i > 0$ and $D_i$ are smooth components. We say that a Kähler current $\omega \in 2\pi c_1(M)$ is a conical Kähler metric along $(1 - \beta)D$ ($0 < \beta \leq 1$) if $\omega$ is smooth Kähler on $M \setminus D$, and asymptotically equivalent to the model conical Kähler metric near $D$: more precisely, near each point $p \in \text{Supp}(D)$ where $\text{Supp}(D)$ is cut out by the equation $\{z_1 \cdots z_r = 0\}$ ($r \leq d$) for some local holomorphic coordinates $(z^i)$, $\omega$ satisfies

$$C^{-1}\omega_{\text{model}} \leq \omega \leq C\omega_{\text{model}}$$

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for some constant $C > 0$, where
\[
\omega_{\text{model}} := \sqrt{-1} \sum_{i=1}^{r} |z^i|^{2(\beta - 1)} \tau_i dz^i \wedge d\bar{z}^i + \sqrt{-1} \sum_{i=r+1}^{n} dz^i \wedge d\bar{z}^i
\]
is the model conical Kähler metric with cone angles $2\pi (1 - (1 - \beta)\tau_i)$ along \( \{ z^i = 0 \} \).

Let \( X \) be a holomorphic vector field on \( M \) whose imaginary part \( \text{Im}(X) \) generates a torus action on the line bundles \( \mathcal{O}_M(D_i) \). Let \( H_i \) be \( \text{Im}(X) \)-invariant hermitian metrics on \( \mathcal{O}_M(D_i) \) such that the curvature of the induced hermitian metric \( H_D := \otimes_{i=1}^{d} H_i^{\tau_i} \) is \( \lambda \omega_0 \). Let \( s_i \) be the defining sections of \( \mathcal{O}_M(D_i) \) associated to \( D_i \), and set \( s_D := \otimes_{i=1}^{d} s_i^{\tau_i} \).

We define a Kähler current \( \omega^* \) as
\[
\omega^* := \omega_0 + k \sum_{i=1}^{d} \sqrt{-1} \partial \bar{\partial} |s_i|^{2(1 - (1 - \beta)\tau_i)} H_i^{\tau_i}
\]
for sufficiently small constant \( k > 0 \). Then \( \omega^* \) is a conical Kähler metric along \((1 - \beta)D\). According to [DGSW13], we say that a conical Kähler metric \( \omega \in c_1(M) \) is a conical Kähler-Ricci soliton if it satisfies
\[
\text{Ric}(\omega) = \gamma \omega + (1 - \beta)[D] + L_X \omega
\]
in the sense of distributions on \( M \), and
\[
\text{Ric}(\omega) = \gamma \omega + L_X \omega
\]
in the classical sense on \( M \setminus D \), where \( \gamma = \gamma(\lambda, \beta) := 1 - \lambda(1 - \beta) \geq 0 \) and \( L_X \omega \) is defined so that
\[
\int_M L_X \omega \wedge \zeta = - \int_M \omega \wedge L_X \zeta
\]
for any smooth \((n - 1, n - 1)\)-form \( \zeta \) on \( M \). The notion of conical Kähler-Ricci solitons is a generalization of classical Kähler-Ricci solitons (cf. [TZ00], [TZ02]) for the conical settings, and their examples in toric Fano manifolds are studied in [DGSW13] and [WZZ16].

In this paper, we introduce the following modified conical Kähler-Ricci flow (MCKRF):
\[
\begin{cases}
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \gamma \omega + (1 - \beta)[D] + L_X \omega \\
\omega|_{t=0} = \omega^*.
\end{cases}
\]
Then conical Kähler-Ricci solitons with respect to \( X \) can be viewed as the stationary points of MCKRF. We say that \( \omega = \omega(t) \ (t \in [0, \infty)) \) is a long-time solution of the above MCKRF if \( \omega(t) \) is a conical Kähler metric along \((1 - \beta)D\) for each \( t \) which satisfies the equation (1.2) in the sense of distributions on \( M \times [0, \infty) \) and can be simplified to the classical modified Kähler-Ricci flow
\[
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \gamma \omega + L_X \omega
\]
on \((M \setminus D) \times [0, \infty) \). If a long-time solution of the flow (1.2) converges to some Kähler current, it should be a conical Kähler-Ricci soliton with respect to \( X \). Thus the flow (1.2) provides a new standard method for studying the equation (1.1). In the case when \( X \equiv 0 \), Chen-Wang [CW15] established the short-time existence of the flow (1.2). Then Liu-Zhang [LZ17] and Wang [Wan16] showed the long-time

\[1\]More precisely, they dealt with the “strong” conical Kähler-Ricci flow (with some Hölder continuity assumptions for potential functions).
existence independently. On the other hand, in the general case, it seems that the flow (1.2) is considered only for $D = 0$ (cf. [TZ07], [PSSW11]).

Following the idea of [LZ17] and [Wan16], we will construct a long-time solution of (1.2) as the limit of a sequence of smooth Kähler-Ricci flows $\varphi_\epsilon$, where $\varphi_\epsilon$ ($\epsilon > 0$) is a solution of the modified twisted Kähler-Ricci flow (MTKRF) defined in Section 2. Then we show the following:

**Theorem 1.1.** Assume that $|X(\log |s_D|^2_{H_D})| < C$ on $M \setminus D$ for some constant $C > 0$. Let $\omega_{\varphi_\epsilon}$ be a long-time solution of the modified twisted Kähler-Ricci flow (2.4). Then, by passing to a subsequence $\{\epsilon_i\}$ satisfying $\epsilon_i \to 0$ as $i \to \infty$, the Kähler metric $\omega_{\varphi_{\epsilon_i}}$ converges to a solution of the modified conical Kähler-Ricci flow:

$$\begin{cases}
\frac{\partial \omega_{\varphi}}{\partial t} = -\text{Ric}(\omega_{\varphi}) + \gamma \omega_{\varphi} + (1 - \beta)[D] + L_X \omega_{\varphi} \\
\omega_{\varphi}|_{t=0} = \omega^*
\end{cases}$$

as $i \to \infty$, where $\omega_{\varphi} := \omega^* + \sqrt{-1} \partial \bar{\partial} \varphi$, and for any $t \in [0, \infty)$, the potential function $\varphi$ is Hölder continuous with respect to $\omega_0$. This convergence holds in the sense of distributions on $M \times [0, \infty)$, and in the $C^\infty_{\text{loc}}$-topology on $(M \setminus D) \times [0, \infty)$. In particular, there exists a long-time solution of the modified conical Kähler-Ricci flow.

**Remark 1.1.** (1) The assumption $|X(\log |s_D|^2_{H_D})| < C$ is a necessary condition for the existence of a conical Kähler-Ricci soliton with respect to $X$. In particular, this condition implies that $X$ is tangent to $\text{Supp}(D)$ (cf. [JLZ16, Remark 4.2]). This assumption is used only for the uniform Laplacian estimate of MTKRF (cf. Proposition 3.2).

(2) We also note that when $D$ is smooth and $\lambda \geq 1$, such a vector field $X$ automatically becomes trivial (cf. [SW16, Theorem 2.1]). This is a reason why we allow $D$ to have simple normal crossing support.

An advantage of our approach is that we do not rely on the linear theory for conical Laplacians established by Donaldson [Don12] and Chen-Wang [CW15]. At the same time, we should point out that Theorem 1.1 provides us not only the long-time existence of solutions, but also “the regularization method” to study the flow. The author expects that the conical Kähler-Ricci flow (and its regularization) method also works for the existence problem of conical Kähler-Ricci solitons. The arguments in this paper run closely in parallel to those of [LZ17] except some changes due to the modification $X$. Nevertheless, we will try to make the arguments reasonably self-contained for readers’ convenience.

The paper is organized as follows. We first review the regularization method and reduction to the Monge-Ampère flow in Section 2. Then we consider the uniform Laplacian estimate for MTKRF in Section 3. Finally, we establish the $C^\infty_{\text{loc}}$-estimate of MTKRF and give the proof of Theorem 1.1 in Section 4.

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2. Regularization and reduction to the Monge-Ampère flow

Let $\epsilon > 0$ be a small constant. As in [CPT6 Section 3.1], We define the function
\[
\chi_i(\epsilon^2 + u) := \frac{1}{1 - (1 - \beta)i} \int_0^u \frac{(\epsilon^2 + r)^{1 - (1 - \beta)i} - \epsilon^2(1 - (1 - \beta)i)}{r} dr
\]
for $i = 1, \ldots, d$ and $u \geq 0$. Then we see that the function $\chi_i(\epsilon^2 + u)$ is smooth for each $\epsilon$, and there exists uniform constants (independent of $\epsilon$) $C > 0$ and $\nu > 0$ such that for all $i$, we have
\[
0 \leq \chi_i(\epsilon^2 + u) < C
\]
provided that $u$ belongs to a bounded interval, and
\[
\omega_\epsilon \geq \nu \omega_0.
\]
We also have the convergence
\[
\chi_i(\epsilon^2 + |s_i|^2_{H_\epsilon}) \xrightarrow{\epsilon \to 0} |s_i|^2_{H_0}
\]
in the $C^\infty_{\text{loc}}$-topology on $M \setminus D_j$. Set $\chi := \sum_{i=1}^d \chi_i(\epsilon^2 + |s_i|^2_{H_\epsilon})$ and $\omega_\epsilon := \omega_0 + \sqrt{-1} \partial \bar{\partial} k \chi$. Then we have
\[
\omega_\epsilon \xrightarrow{\epsilon \to 0} \omega^*
\]
in the sense of distributions on $M$, and in the $C^\infty_{\text{loc}}$-topology on $M \setminus D$. Meanwhile, since $[D] = \lambda \omega_0 + \sum_{i=1}^d \sqrt{-1} \tau_i \partial \bar{\partial} \log |s_i|^2_{H_\epsilon}$ by the Poincaré-Lelong formula, we observe that
\[
\eta_\epsilon := \lambda \omega_0 + \sum_{i=1}^d \sqrt{-1} \tau_i \partial \bar{\partial} \log(|s_i|^2_{H_\epsilon} + \epsilon^2) \xrightarrow{\epsilon \to 0} [D],
\]
again, this convergence holds in the sense of distributions on $M$, and in the $C^\infty_{\text{loc}}$-topology on $M \setminus D$. Now We define the modified twisted Kähler-Ricci flow (MTKRF) with the twisted form $\eta_\epsilon$:
\[
\begin{cases}
\frac{\partial \omega_{\phi_\epsilon}}{\partial t} = -\text{Ric}(\omega_{\phi_\epsilon}) + \gamma \omega_{\phi_\epsilon} + (1 - \beta) \eta_\epsilon + L_X \omega_{\phi_\epsilon} \\
\omega_{\phi_\epsilon}|_{t=0} = \omega_\epsilon,
\end{cases}
\]
where $\omega_{\phi_\epsilon} := \omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \phi_\epsilon$. For an Im$(X)$-invariant Kähler metric $\omega \in 2\pi c_1(M)$, we also define an $\mathbb{R}$-valued function $\theta_X(\omega)$ by
\[
\begin{cases}
i_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X(\omega) \\
\int_M e^{\theta_X(\omega)} \omega^n = [\omega_0]^n.
\end{cases}
\]
In particular, we set $\theta_X := \theta_X(\omega_0)$. Then, from [TZ02 Proposition 1.1] and [Zhu00 Corollary 5.3] (or [BN14 Section 2.3]), we have the following:

Proposition 2.1. Let $\phi$ be a real-valued smooth function such that $\text{Im}(X)(\phi) = 0$ and $\omega_\phi := \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi \geq 0$. Then we have
\[
\begin{enumerate}
\item $\theta_X(\omega_\phi) = \theta_X + X(\phi)$,
\item $\sup_M |X(\phi)| < C$ for some constant $C$ which depends only on $\omega_0$ and $X$.
\end{enumerate}
\]
Since MTKRF preserves the initial Kähler class $[\omega_0]$, we can reduce MTKRF to the Monge-Ampère flow:

(2.6) \begin{align*}
\frac{\partial \varphi_\epsilon}{\partial t} &= \log \frac{\omega_n}{\omega_0^n} + F_0 + \gamma(k\chi + \varphi_\epsilon) + \log(\prod_{i=1}^{d}(e^2 + |s_i|_{H_i}^2))^{(1-\beta)r_i} + \theta_X(\omega_{\varphi_\epsilon}) \\
\varphi_\epsilon|_{t=0} &= c_\epsilon 0,
\end{align*}

where $c_\epsilon 0$ is a real constant such that $c_\epsilon 0 \xrightarrow{t \to 0} c_0$ and $F_0$ is the Ricci potential with respect to $\omega_0$:

(2.7) \begin{align*}
-Ric(\omega_0) + \omega_0 &= \sqrt{-1} \partial \bar{\partial} F_0 \\
\int_X e^{-F_0} \omega_0^n &= [\omega_0]^n.
\end{align*}

We often use the twisted Ricci potential $F_\epsilon$ defined by

$$F_\epsilon := F_0 + \log \left( \frac{\omega_n}{\omega_0^n} \cdot \prod_{i=1}^{d}(e^2 + |s_i|_{H_i}^2)^{(1-\beta)r_i} \right).$$

**Remark 2.1.** According to [CGP13], we see that $F_\epsilon$ is uniformly bounded.

Then the flow (2.6) can be written as

$$\begin{align*}
\frac{\partial \varphi_\epsilon}{\partial t} &= \log \frac{\omega_n}{\omega_0^n} + F_\epsilon + F_0 + \gamma(k\chi + \varphi_\epsilon) + \theta_X(\omega_{\varphi_\epsilon}) \\
\varphi_\epsilon|_{t=0} &= c_\epsilon 0.
\end{align*}$$

### 3. $C^0$-estimate, volume ratio estimate and uniform Laplacian estimate

In this section, we establish the uniform Laplacian estimate of MTKRF. First, we show the volume ratio estimate and $C^0$-estimate:

**Proposition 3.1.** Let $\varphi_\epsilon$ be the solution of (2.6). Then there exists a uniform constant $C$ (independent of $\epsilon$ and $t$) such that

$$\sup_{M \times [0,T]} |\varphi_\epsilon| \leq C^{\gamma T},$$

$$\sup_{M \times [0,T]} |\dot{\varphi}_\epsilon| \leq C e^{\gamma T}.$$ 

**Proof.** Differentiating the equation (2.6) in $t$, we have

$$\frac{d\dot{\varphi}_\epsilon}{dt} = (\Delta_{\omega_{\varphi_\epsilon}} + X)\dot{\varphi}_\epsilon + \gamma \dot{\varphi}_\epsilon.$$

By the maximum principle, we have

$$|\dot{\varphi}_\epsilon(t)| \leq |\dot{\varphi}(0)| e^{\gamma t},$$

where $\dot{\varphi}(0) = F_\epsilon + \gamma(k\chi + c_\epsilon 0) + \theta_X + X(k\chi)$. Thus, by (2.2), Proposition 2.1 and Remark 2.1, we know that $|\dot{\varphi}(0)| \leq C$ for some uniform constant $C$. Then we have

$$|\dot{\varphi}_\epsilon(t)| \leq C e^{\gamma t}.$$ 

Integrating with respect to $t$, we get

$$|\varphi_\epsilon(t)| \leq C e^{\gamma t}$$

as desired. \qed
As in the arguments in [LZ17 Proposition 3.1] and [LLZ16 Theorem 4.3], we can show the uniform Laplacian estimate for MTKRF:

**Proposition 3.2.** Let $\varphi_\epsilon$ be a solution of (2.6). Assume that there exists a uniform constant $C > 0$ such that

1. $\sup_{M \times [0, T]} |\varphi_\epsilon| < C$,
2. $\sup_{M \times [0, T]} |\dot{\varphi}_\epsilon| < C$.

Then there exists a uniform constant $A = A(\lambda, \{\tau_i\}, \beta, \omega_0, X, C)$ such that

$$A^{-1} \omega_\epsilon \leq \omega_{\varphi_\epsilon} \leq A \omega_\epsilon.$$  

**Proof.** We choose local normal coordinates $(z^i)$ with respect to $\omega_\epsilon$ where $\omega_{\varphi_\epsilon}$ is diagonal, and then reduce to local computation. Then we observe that

$$\left( \frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) \log \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} = \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \left( \Delta_{\omega_\epsilon} \left( \dot{\varphi}_\epsilon - \log \frac{\omega^n_\epsilon}{\omega^n_{\varphi_\epsilon}} + R_{\omega_\epsilon} \right) \right)$$

$$- \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \left( g^{i\bar{j}}_{\varphi_\epsilon} g_{\varphi_\epsilon, i\bar{j}} R_{\omega_\epsilon}^{\bar{i}i} \right) + \left\{ \frac{g^{ik}_{\varphi_\epsilon} \partial_k \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \partial_k \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}}{(\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon})^2} - \frac{g^{ik}_{\varphi_\epsilon} \varphi_\epsilon^t \varphi_\epsilon s t^{p}}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \right\}.$$  

The computation in [Tos15 Theorem 3.9] implies that

$$\frac{g^{ik}_{\varphi_\epsilon} \partial_k \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \partial_k \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}}{(\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon})^2} - \frac{g^{ik}_{\varphi_\epsilon} \varphi_\epsilon^t \varphi_\epsilon s t^{p}}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \leq 0.$$  

Since

$$g^{i\bar{j}}_{\varphi_\epsilon} g_{\varphi_\epsilon, i\bar{j}} R_{\omega_\epsilon}^{\bar{i}i} = \frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} R_{\omega_\epsilon}^{i\bar{i}},$$

$$n = \text{tr}_{\omega_\epsilon} \omega_0 + k \text{tr}_{\omega_\epsilon} (\sqrt{-1} \bar{\partial} \partial \chi) \geq k \Delta_{\omega_\epsilon} \chi,$$

$$\frac{\Delta_{\omega_\epsilon} \varphi_\epsilon}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} = \sum_i \varphi_{\epsilon i\bar{i}} \leq 1,$$

we have

$$\left( \frac{d}{dt} - \Delta_{\omega_{\varphi_\epsilon}} \right) \log \text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon} \leq - \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_{i,j} \frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} R_{\omega_\epsilon}^{i\bar{i} j\bar{j}}$$

$$+ \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} (F_\epsilon + \gamma (k \chi + \varphi_\epsilon) + \theta \chi (\omega_{\varphi_\epsilon})) + R_{\omega_\epsilon}$$

$$\leq - \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \sum_{i \leq j} \left( \frac{1 + \varphi_{\epsilon i\bar{i}}}{1 + \varphi_{\epsilon j\bar{j}}} + \frac{1 + \varphi_{\epsilon j\bar{j}}}{1 + \varphi_{\epsilon i\bar{i}}} - 2 \right) R_{\omega_\epsilon}^{i\bar{i} j\bar{j}}$$

$$+ \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} (\Delta_{\omega_\epsilon} F_\epsilon) + \frac{\gamma n}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} + \gamma + \frac{1}{\text{tr}_{\omega_\epsilon} \omega_{\varphi_\epsilon}} \Delta_{\omega_\epsilon} \theta \chi (\omega_{\varphi_\epsilon}).$$  

Let $C_1$ be a uniform constant such that

$$\sqrt{-1} \bar{\partial} \partial F_0 \geq -C_1 \omega_0.$$  

Then, by (2.3), we have

$$0 \leq \text{tr}_{\omega_\epsilon} (\sqrt{-1} \bar{\partial} \partial F_0 + C_1 \omega_0) \leq \nu^{-1} \text{tr}_{\omega_0} (\sqrt{-1} \bar{\partial} \partial F_0 + C_1 \omega_0) = \nu^{-1} (C_1 n + \Delta_{\omega_0} F_0).$$
Hence we have the uniform bound of $\Delta_{\omega_1} F_0$:

$$-C_1 \nu^{-1} \leq -C_1 \text{tr}_{\omega_1} \omega_0 \leq \Delta_{\omega_1} F_0 \leq \nu^{-1} (C_1 n + \Delta_{\omega_1} F_0).$$

Now we recall the arguments in [GP16, Section 2, Section 3, Section 4]. We set

$$\chi_{\rho}(\epsilon^2 + u) = \frac{1}{\rho} \int_0^u (\epsilon^2 + r)^\rho - \epsilon^2 dr$$

and define the “auxiliary function” $\Psi_{\epsilon, \rho}$ by

$$\Psi_{\epsilon, \rho} := \tilde{C} \sum_{i=1}^d \chi_{\rho}(\epsilon^2 + |s_i|^2_H),$$

where $\tilde{C} > 0$ and $\rho > 0$ are constants. Then the function $\Psi_{\epsilon, \rho}$ is uniformly bounded.

After taking suitable uniform constants $\tilde{C}$, $\rho$ and $C_2$, we have

$$- \sum_{i \geq j} \left( \frac{1 + \varphi_{eii}}{1 + \varphi_{eij}} + \frac{1 + \varphi_{ejj}}{1 + \varphi_{eii}} - 2 \right) R_{\omega_1}^{\tilde{H}} - \text{tr}_{\omega_1} \omega_1 \Delta_{\omega_1} F_1 + \Delta_{\omega_1} F_0 + C_2 \leq C_2 \sum_{i \leq j} \left( \frac{1 + \varphi_{eii}}{1 + \varphi_{eij}} + \frac{1 + \varphi_{ejj}}{1 + \varphi_{eii}} \right) + C_2 \text{tr}_{\omega_1} \omega_1 \cdot \text{tr}_{\omega_1} \omega_1 + \Delta_{\omega_1} F_0 + C_2.$$ 

Combining with the Cauchy-Schwartz inequality $n \leq \text{tr}_{\omega_1} \omega_1 \cdot \text{tr}_{\omega_1} \omega_1$, we get

$$\left( \frac{d}{dt} - \Delta_{\omega_1} \right) (\log \text{tr}_{\omega_1} \omega_1) \leq \frac{C_2}{\text{tr}_{\omega_1} \omega_1} \sum_{i \leq j} \left( \frac{1 + \varphi_{eii}}{1 + \varphi_{eij}} + \frac{1 + \varphi_{ejj}}{1 + \varphi_{eii}} \right) + \frac{C_3}{\text{tr}_{\omega_1} \omega_1} + C_2 \text{tr}_{\omega_1} \omega_1 + \frac{1}{\text{tr}_{\omega_1} \omega_1} \Delta_{\omega_1} \theta_X(\omega_1) + C_4 \leq \frac{C_2}{\text{tr}_{\omega_1} \omega_1} \left\{ \left( \sum_i \frac{1}{1 + \varphi_{eii}} \right) \left( \sum_j (1 + \varphi_{ejj}) \right) + n \right\} + \frac{C_3}{\text{tr}_{\omega_1} \omega_1} + C_2 \text{tr}_{\omega_1} \omega_1 + \frac{1}{\text{tr}_{\omega_1} \omega_1} \Delta_{\omega_1} \theta_X(\omega_1) + C_4 \leq C_5 \text{tr}_{\omega_1} \omega_1 + \frac{1}{\text{tr}_{\omega_1} \omega_1} \Delta_{\omega_1} \theta_X(\omega_1) + C_4.$$
Thus, if we set $B \in \mathbb{R}_{+}^{\rho,\theta}$, we have

$$\psi_{\varphi_{k}} \text{ is uniformly bounded since } \phi \text{ takes its maximum at } (x_0, t_0) \subset M \times [0, T].$$

According to [GP16, Section 4], we find that there exists a small uniform constant $k > 0$ such that $\omega_{k} + k' \sqrt{-1} \partial \bar{\partial} \phi_{\epsilon, \rho} \geq 0$. Thus, combining with Proposition 2.1 implies

$$|X(\varphi_{k})| \leq |X(k + \varphi_{k})| + |X(k)| \leq C_8,$$

$$|X(\psi_{\epsilon, \rho})| \leq C_9.$$
at \((x_0, t_0)\). Then we observe that
\[
\tr_{\omega, \omega_{\phi_{\epsilon}}}(x_0, t_0) \leq \frac{1}{(n-1)!}(\tr_{\omega_{\phi_{\epsilon}}, \omega_{\epsilon}})^{n-1}(x_0, t_0) \frac{\omega_{\phi_{\epsilon}}}{\omega_{\epsilon}}(x_0, t_0)
\leq \frac{C_{10}^{n-1}}{(n-1)!} \exp(\tilde{\phi}_{\epsilon} - F_{\epsilon} - \gamma(k\chi + \varphi_{\epsilon}) - \theta X(k\chi + \varphi_{\epsilon}))(x_0, t_0)
\leq C_{11}.
\]
Since \(F_{\epsilon}\) and \(\Psi_{\epsilon, \rho}\) are uniformly bounded, we find that
\[
\tr_{\omega, \omega_{\phi_{\epsilon}}} \leq C_{12}
onumber
\]
on the \(M\). Hence the flow equation (2.6) and the uniform bound of \(\varphi_{\epsilon}, \dot{\varphi}_{\epsilon}, F_{\epsilon}, X(k\chi + \varphi_{\epsilon})\) give the desired inequality (3.1) for some uniform constant \(A\). \(\Box\)

4. \(C^\infty_{\text{loc}}\)-estimate and completion of the proof of Theorem 1.1

In this section, we establish the \(C^\infty_{\text{loc}}\)-estimate of MTKRF. Let
\[
\phi_{\epsilon} := \varphi_{\epsilon} + k\chi.
\]
Then we have
\[
\omega_{\phi_{\epsilon}} := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi = \omega_{\phi_{\epsilon}}.
\]
In order to simplify the notation, we drop the explicit dependence of \(\epsilon\) and write \(\phi, \eta, \) etc. Then the equation of MTKRF can be written as
\[
\frac{\partial \omega_{\phi}}{\partial t} = -\text{Ric}(\omega_{\phi}) + \gamma \omega_{\phi} + \tilde{\eta} + LX_{\omega_{\phi}},
\]
where \(\tilde{\eta} := (1 - \beta)\eta \in (1 - \gamma)c_1(M),\) or equivalently,
\[
\frac{dg_{\phi k \bar{l}}}{dt} = -R_{\phi k \bar{l}} + \gamma g_{\phi k \bar{l}} + \tilde{\eta}_{k \bar{l}} + \nabla_{\phi k}X_{\bar{l}}.
\]
Then we can reduce the above equation to the Monge-Ampère flow:
\[
\frac{\partial \phi}{\partial t} = \log \frac{\omega_{\phi}}{\omega_0} + \gamma \phi + F + \theta X(\omega_{\phi}),
\]
where \(F\) is a twisted Ricci potential \(\sqrt{-1}\partial\bar{\partial}F = -\text{Ric}(\omega_0) + \gamma \omega_0 + \tilde{\eta}.\) Let \(\nabla_\phi\) (resp. \(\nabla_0\)) be the covariant derivative with respect to \(\omega_{\phi}\) (resp. \(\omega_0\)). We set
\[
S := |\nabla_0 g_\phi|_{\omega_{\phi}}^2 = g_{\phi i} g_{\phi j}^l g_{\phi k}^p \nabla_0 g_\phi g_{\phi q} \nabla_{\phi j} g_{\phi p \bar{l}}.
\]
If we put
\[
h^i_k := g^i_0 g_{\phi k},
\]
\[
U^k_{\bar{i}l} := (\nabla_{\phi i} h \cdot h^{-1})^k_{\bar{l}},
\]
then we have
\[
U^k_{\bar{i}l} = \Gamma^k_{\phi i l} - \Gamma^k_{0 i l},
\]
where \(\Gamma^k_{\phi i l}\) (resp. \(\Gamma^k_{0 i l}\)) is the Christoffel symbol of \(\omega_{\phi}\) (resp. \(\omega_0\)). The following proposition is an \(X\)-analogue of [LZ17, Proposition 3.3].
Proposition 4.1. Let $p \in M$ and $\phi$ be a solution of the Monge-Ampère flow $[4.3]$. We assume that there exists a constant $N > 0$ such that

$$N^{-1} \omega_0 \leq \omega_\phi \leq N \omega_0$$

on $B_r(p) \times [0, T]$, where $B_r(p)$ is a geodesic ball of radius $r > 0$ centered at $p$ with respect to $\omega_0$. Then there exists constants

$$C' = C'(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^3(B_r(p))}, \|\tilde{\eta}\|_{C^1(B_r(p))})$$

and

$$C'' = C''(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^3(B_r(p))}, \|\tilde{\eta}\|_{C^2(B_r(p))})$$

such that

$$S \leq C', \quad |\text{Rm}_\phi|^2_{\omega_\phi} \leq C''$$
on $B_{r/2}(p) \times [0, T]$. Moreover, for any $k \geq 0$ and $0 \leq \alpha < 1$, there exists constants

$$C_i^k = C_i^k(N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+1}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+2}(B_r(p))}, \|F\|_{C^0(B_r(p))}) \quad (i = 1, 2, 3)$$

such that

$$|D^k \text{Rm}_\phi|^2_{\omega_\phi} \leq C_i^1,$$

$$\|\phi\|_{C^{k+1, \alpha}} \leq C_i^2,$$

$$\|\phi\|_{C^{k+3, \alpha}} \leq C_i^3$$
on $B_{r/2}(p) \times [0, T]$.

Proof. We first establish the local version of Calabi’s $C^3$-estimates. A direct computation shows that

$$\left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S = \sum_{\alpha, q}^g m_\phi g_{\phi \beta \gamma} \bar{l}_\phi \left( g_{\phi \beta} \nabla_{\phi \gamma} \bar{\eta}_{\alpha q} - \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} U_{m \alpha}^\beta + U_{\eta \alpha}^\beta \left( g_{\phi \beta} \nabla_{\beta \gamma} \bar{\eta}_{m \alpha} - \nabla_{\beta \gamma} R_{0 \beta \alpha \eta m} \right) \right)$$

$$- \sum_{\alpha, q}^g g_{\phi \beta \gamma} l_{\phi \alpha} \left( \bar{\eta}_{\beta \gamma} + \gamma S \right)$$

$$+ \sum_{\alpha, q}^g g_{\phi \beta \gamma} l_{\phi \alpha} \left( \bar{\eta}_{\beta \gamma} X^m \cdot U_{m \alpha}^\beta + g_{\phi \beta} \bar{\eta}_{\phi \alpha} \nabla_{\phi \beta} X^\alpha \cdot U_{m \alpha}^\beta \right)$$

$$- \sum_{\alpha, q}^g g_{\phi \beta \gamma} l_{\phi \alpha} \left( \nabla_{\phi \beta} X^m \cdot U_{m \alpha}^\beta \right)$$

$$= \sum_{\alpha, q}^g g_{\phi \beta \gamma} l_{\phi \alpha} \left( \nabla_{\phi \beta} X^m \cdot U_{m \alpha}^\beta \right)$$

where (X;I)-(X;V) are additional terms arising from the holomorphic vector field $X$. Since

$$\nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} = \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} + U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m},$$

$$\nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} = \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} + U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m},$$

$$\nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} = \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} + U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m},$$

$$\nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} = \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} + U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m},$$

$$\nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} = \nabla_{\phi \beta} R_{\alpha \beta \gamma \eta m} + U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m} - U_{\beta \gamma} R_{\alpha \beta \gamma \eta m}.$$
we have
\[ g^m_{\phi} g^{\phi}_{\mu\beta} R^l_{\phi m} \left( (g^m_{\phi} \nabla^{\alpha}_{\phi} \tilde{\eta}_{\alpha} - \nabla_{\phi} R^m_{\phi j m}) U^j_{\phi \alpha} + U^m_{\phi j m} (g^k_{\phi} \nabla^{\alpha}_{\phi} \tilde{\eta}_{\alpha} - \nabla_{\phi} R^k_{\phi \alpha}) \right) \]
\[ - U^m_{\phi j m} (\tilde{\eta}_{pq} g^p_{\phi} g^r_{\phi} g^q_{\phi} g^r_{\phi} g^l_{\phi} - g^m_{\phi} \tilde{\eta}_{pq} g^l_{\phi} + g^m_{\phi} g^{\mu\beta}_{\phi} g^l_{\phi} g^l_{\phi}) - \gamma S \leq C_1 (S + 1), \]

where the constant \( C_1 \) depends only on \( N, \gamma, \omega_0 \) and \( ||\tilde{\eta}||_{C^1(B_r(p))} \). On the other hand, since

\[ \nabla_{\phi l} X^3 = \nabla_{0 l} X^3 + X^k U^3_{lk}, \]

\[ \nabla_{\phi m} \nabla_{\phi l} X^3 = \nabla_{0 m} \nabla_{0 l} X^3 - \nabla_{0 p} X^3 \cdot U^p_{ml} + \nabla_{0 l} X^3 \cdot U^3_{pm} + \nabla_{\phi m} X^k \cdot U^3_{lk} + X^k \nabla_{\phi m} U^3_{lk}, \]

in the same way as \([\text{PSSW11}] \) Section 6], we observe that

\[ |(X; I)| + |(X; II)| + |(X; V)| \leq C_2 S |\nabla_{\phi} X|_{\omega_0}, \]

\[ |(X; I)| + |(X; II)| \leq C_3 (S + 1) + S |\nabla_{\phi} X|_{\omega_0} + |X|_{\omega_0} |U|_{\omega_0} |\nabla_{\phi} U|_{\omega_0} \]

\[ \leq C_3 (S + 1) + S |\nabla_{\phi} X|_{\omega_0} + \frac{1}{2} |\nabla_{\phi} U|_{\omega_0}^2 + \frac{1}{2} |X|_{\omega_0}^2 |U|_{\omega_0}^2 \]

\[ \leq C_4 (S + 1) + \frac{1}{2} |\nabla_{\phi} U|_{\omega_0}^2 + S |\nabla_{\phi} X|_{\omega_0}, \]

where \( C_4 \) depends only on \( X, \omega_0 \) and \( N \). Thus we have

\[ \left( \frac{d}{dt} - \Delta_{\omega_0} \right) S \leq -\frac{1}{2} |\nabla_{\phi} U|_{\omega_0}^2 - |\nabla_{\phi} U|_{\omega_0}^2 + (C_2 + 1) S |\nabla_{\phi} X|_{\omega_0} + (C_1 + C_4) (S + 1). \]

On the other hand, the evolution equation of \( |X|_{\omega_0}^2 \) can be estimated as

\[ \left( \frac{d}{dt} - \Delta_{\omega_0} \right) |X|_{\omega_0}^2 = \gamma |X|_{\omega_0}^2 + (\tilde{\eta}_{ij} + \nabla_{\phi i} X^j) X^i X^j \]

\[ \leq - \frac{1}{2} |\nabla_{\phi} X|_{\omega_0}^2 + C_5. \]

Now we work in local normal coordinates \((z^i)\) with respect to \( \omega_0 \) where \( \omega_0 \) is diagonal. Since

\[ 0 \leq \text{tr} h \leq nN, \]

\[ g^i_j g^j_q g^{mk}_{\phi} \phi^q_{jk} \phi_{smq} \geq \frac{1}{N} S, \]

\[ |g^i_j \nabla_{\phi i} X^j| \leq \text{tr} h \cdot |\text{tr} \nabla_{\phi} X| \leq C_6 (S^{1/2} + 1) \leq \frac{1}{N + 1} S + C_7, \]

we observe that

\[ \left( \frac{d}{dt} - \Delta_{\omega_0} \right) \text{tr} h = \gamma \text{tr} h + g^i_j (\tilde{\eta}_{ij} + \nabla_{\phi i} X^j) - g^i_j g^m_{\phi} g^{\alpha\beta}_{\phi} R^m_{\phi j m} - g^i_j g^m_{\phi} g^{\alpha\beta}_{\phi} \phi^q_{jk} \phi_{smq} \]

\[ \leq C_8 - \frac{1}{N(N + 1)} S. \]

Let \( r \geq r_1 \geq r/2 \) and \( \kappa \) be a nonnegative smooth cut-off function that is identically equal to 1 on \( B_{r_1}(p) \) and vanishes on the outside of \( B_r(p) \). Furthermore, we assume that

\[ |\partial \kappa|_{\omega_0}, \sqrt{-1} \partial \bar{\partial} \kappa |_{\omega_0} \leq C_9, \]

We consider the function

\[ W := \kappa^2 \frac{S}{K - |X|_{\omega_0}^2} + \text{Atr} h, \]
where $K$ is a uniform constant such that \( \frac{256}{207} K \leq K - |X|_{\omega_\phi}^2 \leq K \) and $A$ is a uniform constant determined later. A direct computation shows that

\[
\left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) W = (-\Delta_{\omega_\phi} \kappa^2) \frac{S}{K - |X|_{\omega_\phi}^2} - 4\text{Re} \left( \frac{\kappa \nabla_{\omega_\phi} \kappa}{K - |X|_{\omega_\phi}^2}, \nabla_{\omega_\phi} S \right)_{\omega_\phi} - 4\text{Re} \left( \kappa \nabla_{\omega_\phi} \kappa, \frac{S \cdot \nabla_{\omega_\phi} |X|_{\omega_\phi}^2}{(K - |X|_{\omega_\phi}^2)^2} \right)_{\omega_\phi} + \frac{\kappa^2}{K - |X|_{\omega_\phi}^2} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S + \frac{\kappa^2 S}{(K - |X|_{\omega_\phi}^2)^2} \frac{d}{dt} |X|_{\omega_\phi}^2 + \frac{2\kappa^2 \text{Re}(\nabla_{\omega_\phi} |X|_{\omega_\phi}^2, \nabla_{\omega_\phi} S)_{\omega_\phi}}{(K - |X|_{\omega_\phi}^2)^2} + A \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) \text{trh.}
\]

Using (4.8), (4.9) and the facts

\[
\begin{align*}
(4.11) & \quad |\nabla_{\omega_\phi} |X|_{\omega_\phi}^2|_{\omega_\phi} \leq |X|_{\omega_\phi} |\nabla_{\omega_\phi} X|_{\omega_\phi}, \\
(4.12) & \quad |\nabla_{\omega_\phi} S|_{\omega_\phi}^2 \leq 2S(\nabla_{\omega_\phi} U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2),
\end{align*}
\]

we observe that

\[
\left| \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) \frac{S}{K - |X|_{\omega_\phi}^2} \right| \leq C_{10} S,
\]

\[
\left| \text{Re} \left( \frac{\kappa \nabla_{\omega_\phi} \kappa}{K - |X|_{\omega_\phi}^2}, \nabla_{\omega_\phi} S \right)_{\omega_\phi} \right| \leq \frac{4\sqrt{2}}{K - |X|_{\omega_\phi}^2} \kappa |\nabla_{\omega_\phi} \kappa|_{\omega_\phi} S^{1/2} (|\nabla_{\omega_\phi} U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2)^{1/2} \leq C_{11} S + \frac{\kappa^2}{4(K - |X|_{\omega_\phi}^2)} (|\nabla_{\omega_\phi} U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2),
\]

\[
\left| \text{Re} \left( \kappa \nabla_{\omega_\phi} \kappa, \frac{S \cdot \nabla_{\omega_\phi} |X|_{\omega_\phi}^2}{(K - |X|_{\omega_\phi}^2)^2} \right)_{\omega_\phi} \right| \leq C_{12} S + \frac{\kappa^2 S |\nabla_{\omega_\phi} X|_{\omega_\phi}^2}{4(K - |X|_{\omega_\phi}^2)^2}.
\]

\[
\frac{\kappa^2}{K - |X|_{\omega_\phi}^2} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S \leq - \frac{\kappa^2}{2(K - |X|_{\omega_\phi}^2)^2} (|\nabla_{\omega_\phi} U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2) + \frac{(C_2 + 1)\kappa^2 S |\nabla_{\omega_\phi} X|_{\omega_\phi}^2}{K - |X|_{\omega_\phi}^2} + \frac{\kappa^2 (C_1 + C_4)}{K - |X|_{\omega_\phi}^2} (S + 1) \leq - \frac{\kappa^2}{2(K - |X|_{\omega_\phi}^2)^2} (|\nabla_{\omega_\phi} U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2) + \frac{\kappa^2 S |\nabla_{\omega_\phi} X|_{\omega_\phi}^2}{8(K - |X|_{\omega_\phi}^2)^2} + C_{13} (S + 1),
\]

\[
\frac{\kappa^2 S}{(K - |X|_{\omega_\phi}^2)^2} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |X|_{\omega_\phi}^2 \leq - \frac{\kappa^2 S |\nabla_{\omega_\phi} X|_{\omega_\phi}^2}{2(K - |X|_{\omega_\phi}^2)^2} + C_{14} S,
\]

\[
\]
\[ \left| 2\kappa^2 \text{Re}(\nabla_\phi |X|^2_{\omega_\phi}, \nabla_\phi S)_{\omega_\phi} \right| \leq \frac{2\sqrt{2}\kappa^2}{(K - |X|_{\omega_\phi}^2)^2} |X|_{\omega_\phi} |\nabla_\phi X|_{\omega_\phi} S^{1/2} (|\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_\phi U|_{\omega_\phi}^2)^{1/2} \]

\[ \leq \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{16(K - |X|_{\omega_\phi}^2)^2} + \frac{32\kappa^2 |X|_{\omega_\phi}^2 (|\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_\phi U|_{\omega_\phi}^2)}{(K - |X|_{\omega_\phi}^2)^2} \]

\[ \leq \frac{\kappa^2 S |\nabla_\phi X|_{\omega_\phi}^2}{16(K - |X|_{\omega_\phi}^2)^2} + \frac{\kappa^2}{8(K - |X|_{\omega_\phi}^2)} (|\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_\phi U|_{\omega_\phi}^2) \]

(because \( \frac{256}{257} K < K - |X|_{\omega_\phi}^2 < K \)).

Hence, combining with (4.10), we get

\[ \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) W \leq \left( C_{10} + C_{11} + C_{13} + C_{14} - \frac{A}{N(N + 1)} \right) S + C_{13}. \]

Let \((x_0, t_0)\) be the maximum point of \(W\) on \(B_{r}((p) \times [0, T]\). If \(t_0 = 0\), then \(S\) is bounded by the initial data \(\|\phi(\cdot, 0)\|_{C^3(B_r(p))}\). Moreover, we find that \(W \equiv Atrh\) on the boundary of \(B_r(p)\) where the function \(trh\) is uniformly controlled. Then we may assume that \(t_0 > 0\) and \(x_0\) does not lie in the boundary of \(B_r(p)\). By the maximum principle, we have

\[ 0 \leq \left( C_{10} + C_{11} + C_{13} + C_{14} - \frac{A}{N(N + 1)} \right) S(x_0, t_0) + C_{13}. \]

Taking \(A := N(N + 1)(C_{10} + C_{11} + C_{13} + C_{14} + 1)\), we conclude that \(S(x_0, t_0) \leq C_{13}\). Since \(0 \leq trh \leq nN\), we have

\[ S \leq \frac{257}{256} C_{13} + AnNK \leq C_{15} \]

on \(B_{r/2}(p) \times [0, T]\), where the constant \(C_{15}\) depends only on \(N, \gamma, \omega_\phi, X, \|\phi(\cdot, 0)\|_{C^3(B_r(p))}\) and \(\|\tilde{\eta}\|_{C^1(B_r(p))}\). In particular, \(|\nabla_\phi X|_{\omega_\phi}^2\) is uniformly bounded.

Next, we establish the uniform bound of \(|\text{Rm}_\phi|_{\omega_\phi}^2\). The evolution equation of the full curvature tensor along MTKRF is

\[ \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) R_{\phi\tilde{j}i\tilde{k}} = R_{\phi\tilde{j}i\tilde{l}} R_{\phi\tilde{l}k\tilde{d}} + R_{\phi\tilde{i}k\tilde{d}} R_{\phi\tilde{j}\tilde{l}\tilde{p}} - R_{\phi\tilde{j}p\tilde{q}} R_{\phi\tilde{i}q\tilde{k}} - R_{\phi\tilde{j}p\tilde{l}} R_{\phi\tilde{i}q\tilde{k}} \]

\[ - R_{\phi\tilde{j}h\tilde{i}\tilde{k}} - \nabla_\phi \phi \nabla_\phi \nabla_\phi \phi \nabla_\phi \phi \cdot \nabla_\phi X_j - \nabla_\phi X_j \cdot R_{\phi\tilde{h}i\tilde{k}l}. \]

(4.13)

additional terms arising from \(X\)

By direct computations, we get

\[ \nabla_{\phi\tilde{j}} \nabla_{\phi\tilde{l}} \tilde{\eta}_{ij} = \nabla_{\phi\tilde{j}} \nabla_{\phi\tilde{l}} \tilde{\eta}_{ij} = U_{ij}^s \nabla_{\phi\tilde{s}} \tilde{\eta}_{ij} - \nabla_{\phi\tilde{s}} U_{ij}^s \tilde{\eta}_{sj} - \nabla_{\phi\tilde{s}} U_{ji}^s \tilde{\eta}_{ij} + U_{ij}^s \tilde{\eta}_{ij}, \]

(4.14)

\[ \nabla_{\phi\tilde{j}} U_{ij}^s = \nabla_{\phi\tilde{j}} U_{ij}^s = \delta_{ij} U_{j}^s - R_{\phi\tilde{j}k\tilde{l}} + R_{\phi\tilde{j}k\tilde{l}} \]

(4.15)

\[ \nabla_{\phi\tilde{j}} \nabla_{\phi\tilde{j}} \nabla_{\phi\tilde{j}} X_i = - \nabla_{\phi\tilde{j}} X^k \cdot R_{\phi\tilde{j}k\tilde{l}} - X^k \nabla_{\phi\tilde{j}} R_{\phi\tilde{j}k\tilde{l}} - \nabla_{\phi\tilde{j}} X^p \cdot R_{\phi\tilde{j}k\tilde{l}} + \nabla_{\phi\tilde{s}} X_i \cdot R_{\phi\tilde{s}j\tilde{l}}. \]

(4.16)
Hence, using the uniform bound of $S$, $|X|^2_{\omega_\phi}$ and $|\nabla_\phi X|^2_{\omega_\phi}$, we have
\[
\left| \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) \operatorname{Rm}_\phi \right|_{\omega_\phi} \leq C_{16}(|\operatorname{Rm}_\phi|^2_{\omega_\phi} + |\operatorname{Rm}_\phi|_{\omega_\phi} + 1) + C_{17}|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi}.
\]
Thus, by the uniform bound of $|\nabla_\phi X|^2_{\omega_\phi}$ and the equation (4.2), we obtain
\[
\left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\operatorname{Rm}_\phi|^2_{\omega_\phi} \leq C_{18}(|\operatorname{Rm}_\phi|^3_{\omega_\phi} + |\operatorname{Rm}_\phi|^2_{\omega_\phi}) + 2 \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\operatorname{Rm}_\phi|_{\omega_\phi} - |\nabla_\phi \operatorname{Rm}_\phi|^2_{\omega_\phi} - \left| \nabla_\phi \operatorname{Rm}_\phi \right|_{\omega_\phi}^2
\]
\[
(4.17) \leq C_{19}|\operatorname{Rm}_\phi|^3_{\omega_\phi} + 1 - \frac{1}{2} \left| \nabla_\phi \operatorname{Rm}_\phi \right|_{\omega_\phi}^2 - \left| \nabla_\phi \operatorname{Rm}_\phi \right|_{\omega_\phi}^2.
\]
Now we take a smaller radius $r_2$ satisfying $r_1 > r_2 > r/2$ and show that $|\operatorname{Rm}_\phi|^2_{\omega_\phi}$ is uniformly bounded on $\overline{B_{r_2}(p)}$. Let $\mu$ be a nonnegative smooth cut-off function that is identically equal to 1 on $\overline{B_{r_2}(p)}$, vanishes on the outside of $B_{r_1}(p)$ and satisfies
\[
|\nabla \mu|_{\omega_\phi}, \ |\nabla^2 \mu|_{\omega_\phi} \leq C_{20}.
\]
Let $L$ be a uniform constant satisfying $\frac{5}{12} L \leq L - S \leq L$. We consider the function
\[
G := \mu^2 \frac{|\operatorname{Rm}_\phi|^2_{\omega_\phi}}{L - S} + BS
\]
where $B$ is a uniform constant determined later. By computing, we have
\[
\left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) G = (-\Delta_{\omega_\phi} \mu^2) \frac{|\operatorname{Rm}_\phi|^2_{\omega_\phi}}{L - S} - 4\Re \left( \frac{\mu \nabla_\phi \mu}{L - S}, \nabla_\phi |\operatorname{Rm}_\phi|^2_{\omega_\phi} \right)_{\omega_\phi}
\]
\[
- 4\Re \left( \mu \nabla_\phi \mu, \frac{|\operatorname{Rm}_\phi|^2_{\omega_\phi}}{(L - S)^2} \right)_{\omega_\phi} + \mu^2 \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\operatorname{Rm}_\phi|^2_{\omega_\phi}
\]
\[
+ \frac{\mu^2 |\operatorname{Rm}_\phi|^2_{\omega_\phi}}{(L - S)^2} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S - \frac{2\mu^2 |\operatorname{Rm}_\phi|^2_{\omega_\phi}}{(L - S)^3} |\nabla_\phi S|^2_{\omega_\phi}
\]
\[
- 2\Re \left( \mu^2 \left( \frac{\nabla_\phi S}{(L - S)^2}, \nabla_\phi |\operatorname{Rm}_\phi|^2_{\omega_\phi} \right)_{\omega_\phi} \right) + B \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S.
\]
Then, by (4.8), (4.12), (4.17) and
\[
(4.18) \quad |\nabla_\phi |\operatorname{Rm}_\phi|^2_{\omega_\phi} |_{\omega_\phi} \leq |\operatorname{Rm}_\phi|_{\omega_\phi} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi})
\]
we know that
\[
\left| (-\Delta_{\omega_\phi} \mu^2) \frac{|\operatorname{Rm}_\phi|^2_{\omega_\phi}}{L - S} \right| \leq C_{21}|\operatorname{Rm}_\phi|^2_{\omega_\phi},
\]
\[
4\Re \left( \frac{\mu \nabla_\phi \mu}{L - S}, \nabla_\phi |\operatorname{Rm}_\phi|^2_{\omega_\phi} \right)_{\omega_\phi} \leq \frac{4}{L - S} |\nabla_\phi \mu|_{\omega_\phi} |\operatorname{Rm}_\phi|_{\omega_\phi} (|\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi} + |\nabla_\phi \operatorname{Rm}_\phi|_{\omega_\phi})
\]
\[
\leq C_{22}|\operatorname{Rm}_\phi|^2_{\omega_\phi} + \frac{\mu^2}{4(L - S)} (|\nabla_\phi \operatorname{Rm}_\phi|^2_{\omega_\phi} + |\nabla_\phi \operatorname{Rm}_\phi|^2_{\omega_\phi}),
\]
\[
\left| 4\text{Re} \left( \mu \nabla_\phi \mu, \frac{|\text{Rm}_\phi|^2}{|\omega_\phi|^2} \nabla_\phi S \right) \right|_{\omega_\phi} \leq \frac{4\sqrt{2} |\text{Rm}_\phi|^2}{(L - S)^2} \mu \left| \nabla_\phi \mu \right|_{\omega_\phi} S^{1/2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right)^{1/2} \\
\leq C_{23} |\text{Rm}_\phi|^2_{\omega_\phi} + \frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{4(L - S)^2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right),
\]

\[
\frac{\mu^2}{L - S} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\text{Rm}_\phi|^2_{\omega_\phi} \leq \frac{C_{19} \mu^2}{L - S} |\text{Rm}_\phi|^3_{\omega_\phi} - \frac{\mu^2}{2(L - S)} \left( |\nabla_\phi \text{Rm}_\phi|^2_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|^2_{\omega_\phi} \right) + C_{24}
\]

\[
\leq \frac{\mu^2 |\text{Rm}_\phi|^4_{\omega_\phi}}{8(L - S)^2} + C_{25} \mu^2 |\text{Rm}_\phi|^2_{\omega_\phi} - \frac{\mu^2}{2(L - S)} \left( |\nabla_\phi \text{Rm}_\phi|^2_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|^2_{\omega_\phi} \right)
\]

\[
\leq C_{26} |\text{Rm}_\phi|^2_{\omega_\phi} + \frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{8(L - S)^2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right)
\]

\[
- \frac{\mu^2}{2(L - S)} \left( |\nabla_\phi \text{Rm}_\phi|^2_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|^2_{\omega_\phi} \right) + C_{24}
\]

\[
\text{(where we used (4.15) in the last inequality)},
\]

\[
\frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{(L - S)^2} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) S \leq C_{27} |\text{Rm}_\phi|^2_{\omega_\phi} - \frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{2(L - S)^2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right),
\]

\[
2\text{Re} \left( \frac{\mu^2}{(L - S)^2} \nabla_\phi \left| \text{Rm}_\phi \right|^2_{\omega_\phi} \right) \leq \frac{2 \sqrt{2} \mu^2}{(L - S)^2} S^{1/2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right)^{1/2}.
\]

\[
\left| \text{Rm}_\phi \right|^2_{\omega_\phi} \left( |\nabla_\phi \text{Rm}_\phi|_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|_{\omega_\phi} \right) \leq \frac{64 \mu^2 S}{(L - S)^2} \left( |\nabla_\phi \text{Rm}_\phi|^2_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|^2_{\omega_\phi} \right)
\]

\[
+ \frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{16(L - S)^2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right)
\]

\[
\leq \frac{\mu^2}{8(L - S)^2} \left( |\nabla_\phi \text{Rm}_\phi|^2_{\omega_\phi} + |\nabla_{\omega_\phi} \text{Rm}_\phi|^2_{\omega_\phi} \right)
\]

\[
+ \frac{\mu^2 |\text{Rm}_\phi|^2_{\omega_\phi}}{16(L - S)^2} \left( |\nabla_\phi U|_{\omega_\phi}^2 + |\nabla_{\omega_\phi} U|_{\omega_\phi}^2 \right)
\]

\[
\text{(because } \frac{512}{513} L < L - S < L)\).
\]

As in the previous part, we may only consider an inner point \((x_0, t_0)\) which is a maximum point of \(G\) achieved on \(B_{r_1}(p) \times [0, T]\). By the maximum principle, we have

\[
0 \leq \left( C_{21} + C_{22} + C_{23} + C_{26} + C_{27} - \frac{B}{2} \right) |\text{Rm}_\phi|^2_{\omega_\phi}(x_0, t_0) + C_{28}.
\]

Now we set \(B := 2(C_{21} + C_{22} + C_{23} + C_{26} + C_{27} + 1)\). Then we obtain

\[
|\text{Rm}_\phi|^2_{\omega_\phi}(x_0, t_0) \leq C_{28}.
\]

Since \(S\) is uniformly bounded, this implies

\[
|\text{Rm}_\phi|^2_{\omega_\phi} \leq C_{29}.
\]
on $\overline{B_r^2(p)} \times [0, T]$, where $C_{29}$ depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^4(B_r^2(p))}$ and $\|\hat{\eta}\|_{C^2(B_r^2(p))}$.

Following [LZ17], we say that $\phi$ is $C^{k,\alpha}$ if its $C^{k,\alpha}$ norm can be controlled by a constant depending only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+1}(B_r^2(p))}$, $\|\hat{\eta}\|_{C^{k-1}(B_r^2(p))}$ and $\|F\|_{C^0(B_r^2(p))}$. Likewise, we say that $\phi$ is $C^{k,\alpha}$ if its $C^{k,\alpha}$ norm can be controlled by a constant depending only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+1}(B_r^2(p))}$, $\|\hat{\eta}\|_{C^{k-1}(B_r^2(p))}$ and $\|F\|_{C^0(B_r^2(p))}$. Since $|\text{Rm}_\phi|_{\omega_0}^2$ and $|\nabla X|_{\omega_0}^2$ are uniformly bounded, we know that $\phi$ is $C^{1,\alpha}$. Differentiating the equation (4.3) with respect to $z^k$, we get

$$
\frac{d}{dt} \frac{\partial \phi}{\partial z^k} = (\Delta \omega_0 + X) \frac{\partial \phi}{\partial z^k} + g_{ij} \frac{\partial g_{0ij}}{\partial z^k} - g_{ij} \frac{\partial g_{0ij}}{\partial z^k} + \frac{\partial F}{\partial z^k} + \gamma \frac{\partial \phi}{\partial z^k} + \frac{\partial \theta X}{\partial z^k} \frac{\partial \phi}{\partial z^k}.
$$

From the above Calabi’s $C^3$-estimate, we know that $\phi$ is $C^{2,\alpha}$ and then the coefficients of $\Delta \omega_0$ are $C^{0,\alpha}$. Since $F$ is the twisted Ricci potential, taking the trace with respect to $\omega_0$ yields

$$
\Delta \omega_0 F = -\text{tr}_{\omega_0} \text{Ric}(\omega_0) + \gamma + \text{tr}_{\omega_0} \hat{\eta}.
$$

Hence the $C^{1,\alpha}$-norm of $F$ on $B_r^2(p)$ only depends on $\omega_0$, $\|\hat{\eta}\|_{C^0(B_r^2(p))}$ and $\|F\|_{C^0(B_r^2(p))}$. By the standard elliptic Schauder estimates, we conclude that $\phi$ is $C^{3,\alpha}$ on $B_r^3(p) \times [0, T]$, where $r_2 > r_3 > r/2$.

Now we prove that $|\nabla X|_{\omega_0}^2 \phi \omega_0$ is uniformly bounded. First we compute the evolution equation of $U$ as

\begin{equation}
(\frac{d}{dt} - \Delta \omega_0) U_{\beta \gamma}^\beta = \nabla \omega_0 (\hat{\eta}_{\beta l} + \nabla \phi X^\beta) - \nabla \phi R_{0 \beta l} \tilde{\phi} m.
\end{equation}

Since $\hat{\eta}$, $\text{Rm}_0$ and $X$ are $t$-independent tensors, we know that

\begin{equation}
|\nabla \phi \hat{\eta}|_{\omega_0} \leq C_{30},
\end{equation}

\begin{equation}
|\nabla \phi \nabla \text{Rm}_0|_{\omega_0} + |\nabla \phi X|_{\omega_0} \leq C_{31} (1 + |\nabla \phi U|_{\omega_0}),
\end{equation}

\begin{equation}
|\nabla \phi X|_{\omega_0} \leq C_{32} (1 + |\nabla \phi U|_{\omega_0} + |\nabla \phi U|_{\omega_0}).
\end{equation}

On the other hand, by the Ricci identity, we have

\begin{equation}
(\frac{d}{dt} - \Delta \omega_0) \nabla \phi U = \nabla \phi \left( \frac{d}{dt} - \Delta \omega_0 \right) U + U * \nabla \phi (\text{Rm}_0 + \hat{\eta} + \nabla \phi X) + \text{Rm}_0 * \nabla \phi U,
\end{equation}

where $*$ means the general pairs of tensors. Thus we obtain

\begin{equation}
(\frac{d}{dt} - \Delta \omega_0) |\nabla \phi U|_{\omega_0}^2 \leq C_{33} (|\nabla \phi U|_{\omega_0}^2 + 1 + |\nabla \phi \text{Rm}_0|_{\omega_0}^2 - \frac{1}{2} |\nabla \phi \nabla \phi U|_{\omega_0}^2 - |\nabla \phi \nabla \phi U|_{\omega_0}^2).
\end{equation}

Now we set $r_3 > r'_3 > r/2$ and take a smooth cut-off function $\varphi$ such that

$$
|\partial \varphi|_{\omega_0}, |\sqrt{-1} \partial \overline{\partial} \varphi|_{\omega_0} \leq C_{34},
$$

and set

$$
I := \varphi^2 |\nabla \phi U|_{\omega_0}^2 + ES + 2 |\text{Rm}_\phi|_{\omega_0}^2,
$$

where $E$ is a uniform constant determined later. Then we see that

$$
\left( \frac{d}{dt} - \Delta_{\omega_t} \right) I \leq (-\Delta_{\omega_t}e^2)|\nabla_{\phi} U|^2_{\omega_t} - 4\text{Re}(g\nabla_{\phi} \theta, \nabla_{\phi} \nabla_{\phi} U^2_{\omega_t}) + e^2 \left( \frac{d}{dt} - \Delta_{\omega_t} \right) |\nabla_{\phi} U|^2_{\omega_t} + E \left( \frac{d}{dt} - \Delta_{\omega_t} \right) S + 2 \left( \frac{d}{dt} - \Delta_{\omega_t} \right) |\text{Rm}_{\phi}|^2_{\omega_t}.
$$

The first and second term of the RHS are estimated as

$$
|(-\Delta_{\omega_t}e^2)|\nabla_{\phi} U|^2_{\omega_t} | \leq C_{35} |\nabla_{\phi} U|^2_{\omega_t},
$$

$$
|4\text{Re}(g\nabla_{\phi} \theta, \nabla_{\phi} \nabla_{\phi} U^2_{\omega_t}) | \leq C_{36} |\nabla_{\phi} U|^2_{\omega_t} + \frac{e^2}{4} (|\nabla_{\phi} \nabla_{\phi} U|^2_{\omega_t} + |\nabla_{\phi} \nabla_{\phi} U|^2_{\omega_t}).
$$

Thus, combining with (4.8) and (4.17), we obtain

$$
\left( \frac{d}{dt} - \Delta_{\omega_t} \right) I \leq \left( C_{33} + C_{35} + C_{36} - \frac{E}{2} \right) |\nabla_{\phi} U|^2_{\omega_t} + C_{37}.
$$

Hence, if we set $E := 2(C_{33} + C_{35} + C_{36} + 1)$, the maximum principle implies the uniform bound of $|\nabla_{\phi} U|^2_{\omega_t}$ on $B_{r_t}^G(p) \times [0,T]$. Let $D$ denote the real covariant derivative with respect to $\omega_t$ (extended linearly on the space of complex tensors). Combining with the uniform bound of $|\text{Rm}_{\phi}|^2_{\omega_t}$ and (4.15), we have

$$
|DU|^2_{\omega_t} \leq C_{38}
$$
on $B_{r_t}^G(p) \times [0,T]$, where the constant $C_{38}$ depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^4(B_r(p))}$ and $\|\eta\|_{C^2(B_r(p))}$. In particular, we find that $|D^2X|^2_{\omega_t}$ is uniformly bounded. Applying $\nabla_{\phi}$ to (4.13), we see that

$$
|\nabla_{\phi} \left( \frac{d}{dt} - \Delta_{\omega_t} \right) \text{Rm}_{\phi} \right|_{\omega_t} \leq C_{39} (|\nabla_{\phi} \text{Rm}_{\phi}|_{\omega_t} + |\nabla_{\phi} \nabla_{\phi} \nabla_{\phi} \eta|_{\omega_t} + |\nabla_{\phi} \nabla_{\phi} \nabla_{\phi} X|_{\omega_t}).
$$

Applying $\nabla_{\phi}$ to (4.14) and (4.16), and using the uniform bound of $|DU|^2_{\omega_t}$, we have

$$
|\nabla_{\phi} \nabla_{\phi} \nabla_{\phi} \eta|_{\omega_t} \leq C_{40} (1 + |\nabla_{\phi} \text{Rm}_{\phi}|_{\omega_t}),
$$

$$
|\nabla_{\phi} \nabla_{\phi} \nabla_{\phi} X|_{\omega_t} \leq C_{41} (1 + |\nabla_{\phi} \text{Rm}_{\phi}|_{\omega_t} + |\nabla_{\phi}^2 \text{Rm}_{\phi}|_{\omega_t}).
$$

Combining with

$$
\left( \frac{d}{dt} - \Delta_{\omega_t} \right) \nabla_{\phi} \text{Rm}_{\phi} = \nabla_{\phi} \left( \frac{d}{dt} - \Delta_{\omega_t} \right) \text{Rm}_{\phi} + \text{Rm}_{\phi} * \nabla_{\phi} (\text{Rm}_{\phi} + \eta + \nabla_{\phi} X),
$$

we find that

$$
\left( \frac{d}{dt} - \Delta_{\omega_t} \right) |\nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_t} \leq C_{42} (|\nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_t} + 1) - \frac{1}{2} |\nabla_{\phi} \nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_t} - |\nabla_{\phi} \nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_t}.
$$

Now we take a smaller radius $r_t' > r_t > r/2$ and a smooth cut-off function $\sigma$ that is identically equal to 1 on $B_{r_t'}(p)$, vanishes on the outside of $B_{r_t'}(p)$ and satisfies

$$
|\partial \sigma|_{\omega_0}, \sqrt{-1} \partial \bar{\partial} \sigma|_{\omega_0} \leq C_{43}.
$$

We apply the maximum principle to the function $\sigma^2|\nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_0} + P|\text{Rm}_{\phi}|^2_{\omega_0}$ (where $P$ is a suitable uniform constant). Then, as in the previous argument, we find that $|\nabla_{\phi} \text{Rm}_{\phi}|^2_{\omega_0}$ is uniformly bounded on $B_{r_t'}(p) \times [0,T]$. Thus we have

$$
|\text{D}\text{Rm}_{\phi}|^2_{\omega_0} \leq C_{44}
$$
on $\overline{B_{r_3}(p)} \times [0, T]$, where $C_{44}$ depends only on $N$, $\gamma$, $\omega_0$, $X$, $\|\phi(\cdot, 0)\|_{C^5(B_r(p))}$ and $\|\tilde{\eta}\|_{C^4(B_r(p))}$.

Applying $D$ to the equation (4.2), we have

$$D\sqrt{-1}\partial\bar{\partial}\phi = DRic(\omega_\phi) + D\tilde{\eta} + D(\nabla_\phi X^\flat),$$

where $X^\flat_j := g_{\phi ij} X^i$. Taking the trace, we have

$$|\Delta_{\omega_\phi} D\phi|_{\omega_\phi} \leq |D\Delta_{\omega_\phi} \phi|_{\omega_\phi} + |DRm_\phi \ast \phi|_{\omega_\phi} + |Rm_\phi \ast D\phi|_{\omega_\phi} \leq C_{45}(|DRm_\phi|_{\omega_\phi} + |D\tilde{\eta}|_{\omega_\phi} + |D^2X|_{\omega_\phi} + |DRm_\phi|_{\omega_\phi} \phi| + |Rm_\phi|_{\omega_\phi} |D\phi|_{\omega_\phi}).$$

From the above computations and the fact that $\phi$ is $C^{1,\alpha}$, we find that $D\phi$ is $C^{1,\alpha}$, which implies that $\phi$ is $C^{2,\alpha}$. Differentiating the equation (4.3) two times and using the elliptic Schauder estimates, we have $\phi$ is $C^{1,\alpha}$ on $B_{r_4}(p) \times [0, T]$, where $r_3'' > r_4 > r/2$.

Now we establish the $C^{k,\alpha}$-estimate for $\phi$. For this, we set the following induction hypothesis:

$$(H_k) \begin{cases} |D^jRm|_{\omega_\phi}^2 \leq C_j^1 \phi \in C^{j+1,\alpha} \\ \phi \in C^{j+3,\alpha} \end{cases} \text{ on } \overline{B_{r_{j+3}}(p)} \times [0, T] \text{ for all } j = 0, 1, \ldots k,$$

where $r > r_1 > \cdots > r_{k+2} > r_{k+3} > r/2$ and the constant $C_j^1$ depends only on $N$, $\gamma$, $\omega_0$, $X$, $\|\phi(\cdot, 0)\|_{C^{j+1}(B_r(p))}$, $\|\phi\|_{C^0(B_r(p) \times [0, T])}$, $\|\tilde{\eta}\|_{C^{j+2}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$. We have already seen that this statement is established for $k = 0, 1$. Now we assume that the induction hypothesis $(H_k)$ holds for some $k \geq 1$. Since $\phi$ is $C^{k+3,\alpha}$, we observe that

$$|D^jU|_{\omega_\phi}^2 \leq C_{46} \text{ for } j = 0, 1, \ldots, k.$$

In particular, for any $t$-independent tensor $A$, we find that $|D^jA|_{\omega_\phi}^2$ is uniformly bounded for $j = 0, 1, \ldots, k + 1$. We first show the uniform bound of $|D^{k+1}U|_{\omega_\phi}^2$. Let $r, s$ ($r + s = k + 1$) are non-negative integers. Then any $(k+1)$-derivative of $U$ differs from $\nabla_\phi^s \nabla_\phi U$ by a linear combination of $D^iU \ast D^{r+s-2-i}Rm_\phi$ ($0 \leq i \leq r + s - 2$), which has been already estimated by the induction hypothesis $(H_k)$. Thus we may only consider $\nabla_\phi^s \nabla_\phi U$. Moreover, the equation (4.15) and $(H_k)$ indicate that we should only consider $\nabla_\phi^{k+1}U$. Using the Ricci identity repeatedly, we have

$$\left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) \nabla_\phi^{k+1}U = \nabla_\phi^{k+1} \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) U + \sum_{\begin{subarray}{l} p \geq 0, q \geq 1 \\ p + q = k + 1 \end{subarray}} \nabla_\phi^p U \ast \nabla_\phi^q (Rm_\phi + \tilde{\eta} + \nabla_\phi X)$$

$$\quad + \sum_{\begin{subarray}{l} p \geq 0, q \geq 1 \\ p + q = k + 1 \end{subarray}} \nabla_\phi^p Rm_\phi \ast \nabla_\phi^q U.$$

By (4.19) and $(H_k)$, we observe that

$$|\nabla_\phi^{k+1}U|_{\omega_\phi} \leq C_{47}(1 + |\nabla_\phi^{k+1}U|_{\omega_\phi} + |\nabla_\phi^{k+2}U|_{\omega_\phi}),$$

where $\nabla_\phi^{k+1}U$ and $\nabla_\phi^{k+2}U$ are non-negative integers. Therefore, $\nabla_\phi^{k+1}U$ is uniformly bounded by $C_{47}(1 + |\nabla_\phi^{k+1}U|_{\omega_\phi} + |\nabla_\phi^{k+2}U|_{\omega_\phi})$. We conclude that $\phi$ is $C^{k,\alpha}$ on $B_{r_{k+4}}(p) \times [0, T]$ for all $k \geq 0, 1$.
\[ |(\nabla^{k+1}U;\text{II})|_{\omega_\phi} \leq C_{48}(1 + |\nabla^{k+1}_{\phi}Rm_{\phi}|_{\omega_\phi} + |\nabla^{k+1}_{\phi}U|_{\omega_\phi}), \]

\[ |(\nabla^{k+1}U;\text{III})|_{\omega_\phi} \leq C_{49}(1 + |\nabla^{k+1}_{\phi}Rm_{\phi}|_{\omega_\phi}). \]

Thus the evolution equation of \(|\nabla^{k+1}_{\phi}U|_{\omega_\phi}^2\) can be estimated as

(4.23)
\[ \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\nabla^{k+1}_{\phi}U|_{\omega_\phi}^2 \leq -\frac{1}{2} |\nabla^{k+2}_{\phi}U|_{\omega_\phi}^2 - |\nabla_{\phi} \nabla^{k+1}_{\phi}U|_{\omega_\phi}^2 + C_{50} |\nabla^{k+1}_{\phi}U|_{\omega_\phi} + |\nabla^{k+1}_{\phi}Rm_{\phi}|_{\omega_\phi}^2. \]

Hence we should compute the evolution equation of \(|\nabla^{k}_{\phi}U|_{\omega_\phi}^2\) and \(|\nabla^{k}_{\phi}Rm_{\phi}|_{\omega_\phi}^2\), and add them to the above equation. It is not hard to see that

(4.24)
\[ \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\nabla^{k}_{\phi}U|_{\omega_\phi}^2 \leq C_{51} - \frac{1}{2} |\nabla^{k+1}_{\phi}U|_{\omega_\phi}^2 - |\nabla_{\phi} \nabla^{k}_{\phi}U|_{\omega_\phi}^2. \]

(4.25)
\[ \left( \frac{d}{dt} - \Delta_{\omega_\phi} \right) |\nabla^{k}_{\phi}Rm_{\phi}|_{\omega_\phi}^2 \leq C_{52} - \frac{1}{2} |\nabla^{k+1}_{\phi}Rm_{\phi}|_{\omega_\phi}^2 - |\nabla_{\phi} \nabla^{k}_{\phi}Rm_{\phi}|_{\omega_\phi}^2. \]

Actually, we can compute the first item in the same way as (4.23). For the second item, one should refer to the computation of (4.28). Hence we take a smooth cut-off function \(\varsigma\) and apply the maximum principle to the function \(\varsigma^2 |\nabla^{k+1}_{\phi}U|_{\omega_\phi}^2 + Q|\nabla^{k}_{\phi}U|_{\omega_\phi}^2 + 2|\nabla^{k}_{\phi}Rm_{\phi}|_{\omega_\phi}^2\) (for a suitable uniform constant \(Q\)) to get the uniform control of \(|\nabla^{k+1}_{\phi}U|_{\omega_\phi}^2\) in \(\bar{B}_{r'}(p) \times [0, T]\) with a smaller radius \(r_{k+3} > r'_{k+3} > r/2\). Thus we have

\[ |D^{k+1}U|_{\omega_\phi}^2 \leq C_{53} \]

on \(\bar{B}_{r_{k+3}'}(p) \times [0, T]\), where the constant \(C_{53}\) depends only on \(N, \gamma, \omega_\phi, X, \|\phi(\cdot, 0)\|_{C^{k+4}(B_{r}(p))}, \|\phi\|_{C^0(B_r(p) \times [0,T])}, \|\eta\|_{C^{k+2}(B_r(p))}\) and \(\|F\|_{C^0(B_r(p))}\). In particular, we find that \(|D^{k+2}X|_{\omega_\phi}^2\) is uniformly bounded.

Next, we establish the uniform estimate for \(|D^{k+1}Rm_{\phi}|_{\omega_\phi}^2\). As in the previous case, we may only consider the tensor of the form \(\nabla_{\phi} \nabla_{\phi} Rm_{\phi}\) for non-negative integers \(r, s\) such that \(r + s = k + 1\). Moreover, by the symmetries of \(Rm_{\phi}\), we may also assume that \(r \neq 0\).

**Case 1:** \(r, s \neq 0\).
Using the Ricci identity repeatedly, we have
\[
\left( \frac{d}{dt} - \Delta \omega \right) \nabla^s_\phi \nabla_\phi \nabla s \phi = \nabla^s_\phi \nabla_\phi \nabla s \phi \left( \frac{d}{dt} - \Delta \omega \right) \nabla s \phi
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi \nabla s_\phi \nabla^q \phi \left( \nabla s \phi + \nabla s \phi X \right)
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi \nabla s_\phi \nabla^q \phi \left( \nabla s \phi + \nabla s \phi X \right)
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi \nabla s_\phi \nabla^q \phi \left( \nabla s \phi + \nabla s \phi X \right)
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi \nabla s_\phi \nabla^q \phi \left( \nabla s \phi + \nabla s \phi X \right)
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi \nabla s_\phi \nabla^q \phi \left( \nabla s \phi + \nabla s \phi X \right)
\]
By (4.13), (4.14), (4.15), (4.16) and the uniform bound of \( |D^{k+1}U|_{\omega s}^2 \), we can estimate the first term as follows:
\[
|\left( \nabla^r \nabla^s \nabla \phi \right)\omega s| \leq C_{54}(1 + |\nabla^r \nabla^s \nabla \phi \left( \nabla \phi + \nabla \phi X \right)|_{\omega s}) + |\nabla^r \nabla^s \nabla \phi \left( \nabla \phi + \nabla \phi X \right)|_{\omega s} + |\nabla^r \nabla^s \nabla \phi \left( \nabla \phi + \nabla \phi X \right)|_{\omega s} + |\nabla^r \nabla^s \nabla \phi \left( \nabla \phi + \nabla \phi X \right)|_{\omega s}.
\]
By the used the Ricci identity and \((H_k)\).
Other terms are easier and estimated as follows:
\[
|\left( \nabla^r \nabla^s \nabla \phi \right)\omega s| + |\left( \nabla^r \nabla^s \nabla \phi \right)\omega s| \leq C_{58},
\]
\[
|\left( \nabla^r \nabla^s \nabla \phi \right)\omega s| + |\left( \nabla^r \nabla^s \nabla \phi \right)\omega s| \leq C_{59}(1 + |\nabla^r \nabla^s \nabla \phi \left( \nabla \phi + \nabla \phi X \right)|_{\omega s}).
\]
Hence we have
\[
(4.26)
\]
We can estimate the evolution equation of $|\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi}$ in a similar way to get (4.27)
\[
\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{61} - \frac{1}{2} |\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi} - |\nabla_\phi \nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi}.
\]
We take a smooth cut-off function $\tau$ that is identically equal to 1 on $B_{r_{k+3}}(p)$, vanishes on the outside of $B_{r_{k+3}}(p)$ and satisfies
\[
|\partial \tau|_{\omega_\phi}, \sqrt{-1} \partial \bar{\partial} \tau|_{\omega_\phi} \leq C_{62},
\]
where $r'_{k+3} > r''_{k+3} > r/2$. Applying the maximum principle to the function $\tau^2 |\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi} + A_1 |\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi}$ (for a suitable uniform constant $A_1$), we get
\[
|\nabla^{r-1}_\phi \nabla^2_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{63}
\]
on $B_{r_{k+3}}(p) \times [0,T]$.

**Case 2: $s = 0$.**

Using the Ricci identity repeatedly, we have
\[
\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) \nabla^{k+1}_\phi Rm_\phi = \nabla^{k+1}_\phi \left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) Rm_\phi + \sum_{p \geq 0, q \geq 1} \nabla^p_\phi Rm_\phi \ast \nabla^q_\phi (Rm_\phi + \tilde{\eta} + \nabla_\phi X)
\]
\[
+ \sum_{p \geq 0, q \geq 1} \nabla^p_\phi Rm_\phi \ast \nabla^q_\phi Rm_\phi.
\]
By (4.13), (4.14), (4.15), (4.16) and the uniform bound of $|D^{k+1}U|^2_{\omega_\phi}$, we can estimate these terms as
\[
|\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{64}(1 + |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} + |\nabla^{k+2}_\phi Rm_\phi|^2_{\omega_\phi}),
\]
\[
|\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} + |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{65}(1 + |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi}).
\]
Thus we have
(4.28)
\[
\left(\frac{d}{dt} - \Delta_{\omega_\phi}\right) |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{66}|\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} - \frac{1}{2} |\nabla^{k+2}_\phi Rm_\phi|^2_{\omega_\phi} - |\nabla_\phi \nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi}.
\]
Now we use the same cut-off function $\tau$ constructed in Case 1, and consider the function $\tau^2 |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} + A_2 |\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi}$ (for a suitable uniform constant $A_2$). Since the evolution equation of $|\nabla^{k}_\phi Rm_\phi|^2_{\omega_\phi}$ has been already estimated in (4.25), the maximum principle implies that
\[
|\nabla^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{67}
\]
on $B_{r_{k+3}}(p) \times [0,T]$. Combining with Case 1, we have
\[
|D^{k+1}_\phi Rm_\phi|^2_{\omega_\phi} \leq C_{68}
\]
on $\overline{B_{r_{k+4}}(p)} \times [0, T]$, where the constant $C_{68}$ depends only on $N, \gamma, \omega_0, X, \|\phi(\cdot, 0)\|_{C^{k+5}(B_r(p))}, \|\phi\|_{C^0(B_r(p) \times [0, T])}, \|\tilde{\eta}\|_{C^{k+3}(B_r(p))}$ and $\|F\|_{C^0(B_r(p))}$.

Applying $D^{k+1}$ to the equation (4.2) and taking the trace, we have

$$|\Delta_{\omega_\phi} D^{k+1} \dot{\phi}|_{\omega_\phi} \leq |D^{k+1} \Delta_{\omega_\phi} \dot{\phi}|_{\omega_\phi} + C_{69} \sum_{i=0}^{k+1} |D^i Rm_{\phi}|_{\omega_\phi} |D^{k+1-i} \dot{\phi}|_{\omega_\phi}$$

$$\leq C_{70} \left( |D^{k+1} Rm_{\phi}|_{\omega_\phi} + |D^{k+1} \tilde{\eta}|_{\omega_\phi} + |D^{k+2} X|_{\omega_\phi} + \sum_{i=0}^{k+1} |D^i Rm_{\phi}|_{\omega_\phi} |D^{k+1-i} \dot{\phi}|_{\omega_\phi} \right).$$

From the above estimates and $(H_k)$, we know that $|\Delta_{\omega_\phi} D^{k+1} \dot{\phi}|_{\omega_\phi}$ is uniformly bounded. Hence $D^{k+1} \dot{\phi}$ is $C^{1,\alpha}$, which implies $\dot{\phi}$ is $C^{k+2,\alpha}$. Differentiating the equation (4.3) $(k+2)$-times and applying the elliptic Schauder estimates, we find that $\phi$ is $C^{k+4,\alpha}$ on $B_{r_{k+4}}(p) \times [0, T]$ where $r_{k+4} > r_{k+4} > r/2$. Thus we have the statement $(H_{k+1})$ as desired. This completes the proof of Proposition 4.1. \hfill \Box

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1** Let $T > 0$ be a constant. By Proposition 3.1 we know that

$$\sup_{M \times [0, T]} |\varphi_\epsilon|, \sup_{M \times [0, T]} |\dot{\varphi}_\epsilon| < C(T)$$

for some constant $C(T)$ (independent of $\epsilon$). Thus Proposition 3.2 implies that

$$A(T)^{-1} \omega_t \leq \omega_{\varphi_\epsilon} \leq A(T) \omega_t$$

(4.29)

on $M$ for some constant $A(T)$ (independent of $\epsilon$). We exhaust $M \setminus D$ by a sequence of compact subsets $K$, and $[0, \infty)$ by a sequence of closed intervals $[0, T]$. From (4.29), we know that

$$N^{-1} \omega_0 \leq \omega_{\varphi_\epsilon} \leq N \omega_0$$

on $K \times [0, T]$, where the constant $N$ only depends on $K$ and $T$. Moreover, the initial data $k \chi + c_0, (1 - \beta) \eta_\epsilon, F_\epsilon$ are uniformly bounded in the $C^\infty_{\text{loc}}$-topology on $K \times [0, T]$. Thus Proposition 4.1 together with the diagonal argument implies that there exists a subsequence $\varphi_{\epsilon_t}(t)$ which converges to a function $\varphi(t)$ that is smooth on $M \setminus D$. Then, by (4.29), we also know that $\varphi_\epsilon$ is a conical Kähler metric along $(1 - \beta)D$. Now we will check that $\varphi_\epsilon$ satisfies the equation (1.2). Let $\zeta = \zeta(x, t)$ be any smooth $(n-1, n-1)$-form on $M \times [0, \infty)$ with compact support $\text{Supp}(\zeta)$. Without loss of generality, we assume that $\text{Supp}(\zeta) \subset [0, T)$. Since $F_\epsilon, \chi, \varphi_\epsilon$ are uniformly bounded on $M \times [0, T]$, for $t \in [0, T]$, dominated convergence theorem
implies that
\[
\int_M \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta = \int_M \sqrt{-1} \partial \bar{\partial} \left( \log \left( \frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} \cdot \prod_{i=1}^d \left( \epsilon^2 + |s_i|_{H_i}^2 \right)^{(1-\beta)r_i} \right) + F_0 + \gamma (\kappa \chi + \varphi_\epsilon) \right) \wedge \zeta \\
+ \int_M L_X \omega_{\varphi_\epsilon} \wedge \zeta \\
= \int_M \left( \log \left( \frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} \cdot \prod_{i=1}^d \left( \epsilon^2 + |s_i|_{H_i}^2 \right)^{(1-\beta)r_i} \right) + F_0 + \gamma (\kappa \chi + \varphi_\epsilon) \right) \wedge \sqrt{-1} \partial \bar{\partial} \zeta \\
- \int_M \omega_{\varphi_\epsilon} \wedge L_X \zeta \\
\xrightarrow{\epsilon_i \to 0} \int_M \left( \log \left( \frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} + F_0 + \gamma (\kappa \chi + \varphi_\epsilon) + \log |s_D|_{H_D}^{2(1-\beta)} \right) \right) \wedge \zeta \\
- \int_M \omega_{\varphi_\epsilon} \wedge L_X \zeta \\
= \int_M \sqrt{-1} \partial \bar{\partial} \left( \log \left( \frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n} + F_0 + \gamma (\kappa \chi + \varphi_\epsilon) + \log |s_D|_{H_D}^{2(1-\beta)} \right) \right) \wedge \zeta \\
+ \int_M L_X \omega_{\varphi_\epsilon} \wedge \zeta \\
= \int_M \left( -\text{Ric}(\omega_{\varphi_\epsilon}) + \gamma \omega_{\varphi_\epsilon} + (1-\beta)[D] + L_X \omega_{\varphi_\epsilon} \right) \wedge \zeta,
\]

On the other hand, as in the proof of [LZ17, Theorem 4.1], we have
\[
\int_M \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta = \int_M \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta.
\]

Hence, on \([0, T]\), we find that
\[
\frac{\partial}{\partial t} \int_M \omega_{\varphi_\epsilon} \wedge \zeta = \int_M \left( -\text{Ric}(\omega_{\varphi_\epsilon}) + \gamma \omega_{\varphi_\epsilon} + (1-\beta)[D] + L_X \omega_{\varphi_\epsilon} \right) \wedge \zeta \\
+ \int_M \omega_{\varphi_\epsilon} \wedge \frac{\partial \zeta}{\partial t}.
\]

Integrating the above equation on \([0, \infty)\), we get
\[
\int_{M \times [0, \infty)} \frac{\partial \omega_{\varphi_\epsilon}}{\partial t} \wedge \zeta dt = \int_0^\infty \left( \frac{\partial}{\partial t} \int_M \omega_{\varphi_\epsilon} \wedge \zeta - \int_M \omega_{\varphi_\epsilon} \wedge \frac{\partial \zeta}{\partial t} \right) dt \\
= \int_{M \times [0, \infty)} \left( -\text{Ric}(\omega_{\varphi_\epsilon}) + \gamma \omega_{\varphi_\epsilon} + (1-\beta)[D] + L_X \omega_{\varphi_\epsilon} \right) \wedge \zeta dt.
\]

Since \(\zeta\) is arbitrary, \(\omega_{\varphi_\epsilon}\) satisfies the equation (1.2) in the sense of distributions on \(M \times [0, \infty)\). Meanwhile, the equation (2.6) can be written as
\[
\frac{(\omega_0^\epsilon + \sqrt{-1} \partial \bar{\partial} \phi_\epsilon)^n}{\omega_0^n} = \exp(\phi_\epsilon - F_0 - \gamma \phi_\epsilon - \theta_X - X(\phi_\epsilon)) \cdot \prod_{i=1}^d \left( \epsilon^2 + |s_i|_{H_i}^2 \right)^{(1-\beta)r_i},
\]
where $\phi_t$, $\dot{\phi}_t$ and $X(\phi_t)$ are uniformly bounded, which implies that the $L^p$-norm of the RHS is uniformly bounded for some $p > 1$ since $\beta \in (0, 1]$. Thus the Hölder continuity of $\varphi$ with respect to $\omega_0$ is a direct consequence from Kolodziej’s work [Kol08, Theorem 2.1]. This completes the proof of Theorem 1.1. □

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