Superconducting correlations induced by charge ordering in cuprate superconductors and Fermi arc formation

E.V.L. de Mello1 and J.E. Sonier2,3

1Instituto de Física, Universidade Federal Fluminense, 24210-346 Niterói, RJ, Brazil
2Department of Physics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada.
3Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada.

We have developed a generalized electronic phase separation model of high-temperature cuprate superconductors that links the two distinct energy scales of the superconducting (SC) and pseudogap (PG) phases via a charge-density-wave (CDW) state. We show that simulated electronic-density modulations resembling the charge order (CO) modulations detected in cuprates intertwine the SC and charge orders by localizing charge and providing the energy scale for a spatially periodic SC attractive potential. Bulk superconductivity is achieved with the inclusion of Josephson coupling between nanoscale domains of intertwined fluctuating CDW and SC orders, and local SC phase fluctuations give rise to the Fermi arcs along the nodal directions of the SC gap. We demonstrate the validity of the model by reproducing the hole-doping dependence of the PG onset temperature $T^\ast$, and the SC transition temperature $T_c$ of YBa$_2$Cu$_3$O$_y$ and Bi$_{2-\delta}$Pb$_\delta$Sr$_2$La$_y$CuO$_{6+\delta}$. The results show that the periodicity of the CDW order is controlled by the PG energy scale, and the hole-doping dependence of the SC energy gap is controlled by the charge ordering free energy.

I. INTRODUCTION

Experiments using different methods have established the occurrence of short-range, incommensurate static CDW correlations in a variety of high-temperature SC cuprates. With the exception of La-based cuprates in which CDW order is accompanied by spin order, the charge order (CO) observed in different cuprate families appears to be similar. In zero magnetic field, the CDW order is essentially two-dimensional. The wave vector of the CDW order is parallel to the Cu-O bond directions along the a and b axes, and decreases in magnitude with increased charge doping. While much of the experimental data cannot distinguish between checkerboard (bidirectional) or alternating stripe (unidirectional) CO, recent resonant X-ray scattering (RXS) experiments on overdoped YBa$_2$Cu$_3$O$_y$ (Y123) and an analysis of scanning tunneling microscopy (STM) data for Bi$_2$Sr$_2$CaCu$_2$O$_8$+δ indicate that the inter-unit-cell character is one of segregated or overlapping unidirectional charge-ordered stripes. Furthermore, it has been found that the CDW order possesses a d-wave intra-unit-cell symmetry with the modulated charge primarily on the O-2p orbitals linking the Cu atoms. Since the SC order parameter also has d-wave symmetry, this local charge or bond order symmetry supports theoretical proposals that suggest the charge and SC order parameters are intimately intertwined. Some attribute the d-wave CO symmetry to quasiparticle scattering by antiferromagnetic (AF) fluctuations near a metallic quantum critical point, which also gives rise to the d-wave superconductivity. Alternatively, it has been proposed that CDW order in cuprates is a consequence of a pair-density wave (PDW) phase, in which the SC order parameter is periodically modulated in space due to the Cooper pairs having finite momentum.

The aim of our work is to establish a quantitative link between the inter-unit-cell dependence of the CO resolved by RXS and imaged in real space by STM, and the energy gaps of the PG and SC phases. Our model is based on an intrinsic propensity for mesoscale electronic phase separation below an onset temperature $T_{PS}$ that follows the hole-doping dependence of the PG temperature $T^\ast$. This picture is similar to that previously advocated by Fradkin and Kivelson. We presume the onset of fluctuating CDW order domains at $T^\ast$, where STM measurements on Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ have detected the emergence of charge stripes that extend into the overdoped regime. The short-range static CO that has been observed by X-rays at a lower temperature $T_{CO} \leq T^\ast$ is assumed to be confined to local regions where fluctuating CDW order has become pinned by disorder. Contrary to this assumption, we note that in HgBa$_2$CuO$_4$+δ (Hg1201) CDW order observed by X-ray scattering vanishes already well below optimal doping. This seems to be due to the presence of pairs of interstitial oxygens within the same unit cell specific to Hg1201. Although not captured by our model, it is also important to recognize that the pseudogap region marks the onset of an intra-unit-cell magnetic order, a true phase transition that modifies ultrasonic waves, an increase in antiferromagnetic correlations and global inversion-symmetry breaking.

Another important ingredient of our model is the experimental observation that the CDW periodicity is independent of temperature, leading us to surmise that the CDW periodicity is set by the onset of the PG at $T^\ast$. This infers that the CDW order is a consequence of the PG formation. At low doping ($p \leq 0.12$) where $T_{CO}$ decreases with decreasing doping, CDW order is potentially suppressed by a slowing down of spin fluctuations and a tendency toward static SDW order.

II. SIMULATION OF THE CHARGE-ORDERED STATE

Our approach is to first simulate spatial modulations of the electronic structure resembling experimentally resolved inter-
unit-cell CO modulations, using the time-dependent Cahn-Hilliard (CH) differential equation. Besides generating the desired CDW order, the CH approach yields the associated free-energy modulations, which we assume scales with a periodic attractive potential in the subsequent SC calculations. The starting point is the introduction of a time-dependent conserved order parameter associated with the local electronic density, $u(r, t) = [p(r, t) - p]/p$, where $p$ is the average charge density and $p(r, t)$ is the charge density at a position $r$ in the plane. The Ginzburg-Landau (GL) free energy density of the system is of the form

$$f(u) = \frac{1}{2} \varepsilon |\nabla u|^2 + V_{GL}(u,T),$$

where $V_{GL}(u,T) = -\alpha [T_{PS} - T]u^2/2 + B^2u^4/4 + ...$ is a double-well potential that characterizes the electronic phase separation below $T_{PS}$. The parameters $\alpha$ and $B$ are constants, and $\varepsilon$ controls the spatial separation of the charge-segregated patches. The CH equation obtained from the continuity equation for the local free energy current density $J = -M \nabla \mu$ (where $M$ is the charge mobility and $\mu = \partial f/\partial \mu$ is the chemical potential). Its solution is described in the Appendix I on charge ordering simulations. For each time step the CH equation is solved for $u(r, t)$, and $p(r, t)$ is obtained. We adjust the parameters of the free energy such that when the periodicity of $p(r, t)$ matches that of the experimentally observed CDW order, the calculation is stopped and the solution is taken to be the spatially-dependent static electronic density $p(r)$. Since the method described here does not generate an intra-unit-cell CO symmetry, it is applicable to systems that have SC pairing and CO symmetries other than d-wave.

Figure 1(a) shows a simulation of alternating planar domains of 90°-rotated charge stripes with an intra-domain periodicity compatible with RXS data for detwinned Y123 at $p = 0.12[\text{8}]$. Within each domain the CO wavelength is $\lambda_{CO} = 3.15a_0$, where the in-plane lattice constant is $a_0 = 3.85\text{Å}$. The CO wavelength in Y123 measured by various X-ray methods increases with increased hole doping$^{4,5,7}$. We have used such data to generate similar CO striped patterns for Y123 at $p = 0.16$ (Fig. 1(d)) and $p = 0.09$.

STM differential conductance maps for optimally-doped ($T_c = 35 \text{ K}$) and underdoped ($T_c = 32 \text{ K}$ and $T_c = 25 \text{ K}$) Bi$_{2-x}$Pb$_x$Sr$_{2-y}$La$_y$CuO$_{6+\delta}$ [(Pb, La)-Bi2201] samples$^{11,12}$ exhibit checkerboard patterns (indicative of the simultaneous presence of both CDW domains) with 6.202$a_0$, 5.102$a_0$, and 4.502$a_0$ unit cell ($a_0 = 3.83\text{Å}$) periodicity, respectively. The increase in the CO wavelength with increased hole doping agrees well with X-ray scattering and STM measurements on Bi2201 without cation substitutions$^{8}$. We have used the following formula from Ref.$^{[3]}$ to calculate the average number of holes per Cu for the (Pb, La)-Bi2201 samples in Ref.$^{11}$ $T_c/T_{c}^{max} = 1 - 250(p - 0.16)^2$, where $T_{c}^{max} = 35 \text{ K}$. This calculation yields $p = 0.126$ and $p = 0.141$ for the two underdoped samples. Figs. 2(a) and 2(d) display CO checkerboard patterns simulated by the CH equation that resemble the STM differential conductance maps for (Pb, La)-Bi2201 at $p = 0.126$ and $p = 0.16$.

The CO periodicity is manifest in the spatial dependence of the free-energy potential $V_{GL}(r)$, shown in Figs. 1(b) and...
FIG. 2. Contour plots of the electronic density $p(r)$ calculated on a square lattice of $100 \times 100$ unit cells, assuming average charge densities of $p = 0.126$ in (a) and $p = 0.16$ in (d). The charge order wavelengths are $\lambda_{CO} = 4.5a_0$ in (a) and $\lambda_{CO} = 6.2a_0$ in (d), matching the checkerboard wavelength of the STM conductance maps of underdoped ($T_c = 25$ K; $p = 0.126$) and optimally-doped (Pb, La)-Bi2201 in Refs. [13] (b), (c). Corresponding spatial dependence of the free-energy potential $V_{GL}(r)$. (c), (f) Results of calculations of the $d$-wave pairing potential $\Delta_d(r)$ displayed over a $36 \times 36$ unit cell area (in meV unit). The spatial average value of the pair potential $\langle \Delta_d(r) \rangle$ is 9.2 meV at $p = 0.126$ in (c), and 15.8 meV at $p = 0.16$ in (f).

III. SUPERCONDUCTING CALCULATIONS IN THE CHARGE-ORDERED STATE

Next we use the free-energy simulations and experimentally determined input parameters for the optimally-doped compounds to deduce the SC energy gap $\Delta_{SC}$, the pseudogap $\Delta_{PG}$, $T_c$ and $T^*$ for the underdoped samples. To derive the local SC gap we solved the Bogoliubov-deGennes (BdG) equations via self-consistent calculations based on a Hubbard Hamiltonian [Eq. [B1]]. The calculations were performed for a sub-lattice about the center of the simulated charge density maps, using periodic boundary conditions and governed by self-consistent conditions for a spatially-varying $d$-wave pairing potential $\Delta_d(r)$ and hole density $p(r)$ [Eq. [B4] and [B5]]. We find that the spatial-average $\langle \Delta_d(r) \rangle$ decreases with a reduction of $p$ (below $p = 0.16$), but increases with decreasing $\lambda_{CO}$. The latter behavior is because as the two holes are forced closer together by the narrower confining potential the binding energy of the two-body interaction increases. The results on the CuO$_2$ plane shown in Figs. 1(c), 1(f), 2(c) and 2(f) indicate that in our approach the PDW is a consequence of the CDW.

The values of $\langle V_{GL}(r) \rangle$ at optimal doping were multiplied by a scaling factor, such that the calculations (Figs. [1f] and [2f]) generate an average value of the pairing potential $\langle \Delta_d(r) \rangle$ for $p = 0.16$ that is close to the experimentally estimated value of the low-temperature SC gap $\Delta_{SC}$. To calculate $\langle \Delta_d(r) \rangle$ for the underdoped samples (Tables [I] and [II]), this same scaling factor was subsequently applied to the respective values of $\langle V_{GL}(r) \rangle$. For (Pb, La)-Bi2201, the value of $\langle V_{GL}(r) \rangle$ varies little with doping, and hence $p$ and $\lambda_{CO}$ are responsible for the hole-doping dependence of $\langle \Delta_d(r) \rangle$. Experimental estimates of the SC gap for the $p = 0.126$ and 0.141 samples are not reported, but the calculated values of $\langle \Delta_d(r) \rangle$ for the underdoped samples (Table [II]) roughly follow the trend expected if the ratio $\Delta_{SC}/k_B T_c$ is independent of $p$. In contrast to (Pb, La)-Bi2201, the doping dependence of $\lambda_{CO}$ in Y123 is weaker, and the CH simulations of charge stripes are characterized by a significant change in $\langle V_{GL}(r) \rangle$ with doping (Fig. 3). Consequently, the depth of the periodic potential plays an important role in the calculation of the doping dependence of $\langle \Delta_d(r) \rangle$ for Y123. The calculated values of $\langle \Delta_d(r) \rangle$ at $p = 0.09$ and 0.16 agree well with an empirical relation for $\Delta_{SC}(p)$ that describes a number of high-$T_c$ cuprate superconductors. The calculated result at $p = 0.12$ falls below this universal curve, which is consistent with the well-known plateau of $T_c(p)$ for Y123 near 1/8 hole doping.

An estimate of $T_c$ is obtained by self-consistently solving
superconductor in single-crystal form it is sufficient to use the tunneling proportional to the normal-state resistance immediately above $T_c$. (a,b) The temperature dependence of the average Josephson coupling energy $\langle E_j(p, T) \rangle$ (divided by Boltzmann constant $k_B$) for Y123 at $p = 0.09, 0.12,$ and 0.16 in (a), and (Pb, La)-Bi2201 at $p = 0.126, 0.141,$ and 0.16 in (b). The values of $T_c$ correspond to the intersections of the dashed straight line $\langle E_j(p, T) \rangle/k_B = T$ with the $\langle E_j(p, T) \rangle/k_B$ curves and are marked by the arrows.

The BdG equations with a temperature-dependent GL potential

$$V(T, p) = V(0, p)[1 - T/T_{PS}(p)]^2,$$

(2)

where we take $V(0, p) \approx \langle V_{GL}(r, p) \rangle$ and $T_{PS}(p)$, the onset of phase separation transition, is taken to be equan to $T^*(p)$ in the calculations. Because of the BdG approach and the above equation, the value of $\langle \Delta_d(r, T) \rangle$ decreases with increasing temperature, but the it remains finite in many regions of the system for a significant range of temperature above $T_c$. This is consistent with the body of experimental results on cuprates mentioned earlier that are suggestive of persisting SC correlations above $T_c$. Typical $\langle \Delta_d(r, T) \rangle$ plots of three Y123 compounds are shown in Fig. A2. Next we assume that bulk superconductivity is achieved via Josephson coupling between different closely spaced patches of intertwined CO and SC pairing. We assume there is SC phase coherence within the patches, and that there are many such closely spaced SC domains slightly above $T_c(p)$ forming junctions with an average tunnel y proportional to the normal-state resistance immediately above $T_c(p)$. As explained previously, for a $d$-wave superconductor in single-crystal form it is sufficient to use the following relation for the average Josephson coupling energy

$$\langle E_j(p, T) \rangle = \frac{\pi \hbar \langle \Delta_d(r, T) \rangle^4}{2e^2 R_n(p)} \tanh \left[ \frac{\langle \Delta_d(r, T) \rangle^2}{2k_B T} \right]$$

(3)

where $\langle \Delta_d(r, T) \rangle = \sum_i \langle \Delta_d(r_i, p, T) \rangle/N$ is the pairing potential. The quantity $R_n(p)$ is proportional to the normal state in-plane resistivity $\rho_{ab}(p, T_c \geq T^*(p))$. In what follows we assume $R_n(p)$ for the optimally-doped compounds, and in the case of (Pb, La)-Bi2201 we use experimental values of the in-plane resistivity ratio $\rho_{ab}(p)/\rho_{ab}(p = 0.16)$ to calculate $R_n(p)$ for the underdoped samples. Since the relationship between $T_c/T_{c2}^{max}$ and the hole concentration $p$ for (Pb, La)-Bi2201 is the same as for La-doped Bi2201, we have used available resistivity data for La-doped Bi2201 in our calculations shown in Table I. For orthorhombic Y123, we instead used experimental values of the $b$-axis resistivity ratio $\rho_b(p)/\rho_b(p = 0.16)$ from Ref. 43 to estimate $R_n(p)$ for underdoped samples.

As the temperature is lowered below $T^*$, thermal fluctuations diminish and long-range phase coherence between the individual SC domains is established when $k_B T \approx \langle E_j(p, T) \rangle$. The temperature $T$ at which this occurs defines the bulk critical temperature $T_c$. Figs. 3(a) and 3(b) show the temperature dependence of $\langle E_j(p, T) \rangle$ for both compounds at the different dopings. The intersection of the $\langle E_j(p, T) \rangle$ curves with the $k_B T$ line yields values of $T_c$ in good agreement with the actual values for Y123 and (Pb, La)-Bi2201 (Fig. 4 and Tables II and III).
TABLE I. Experimental data for $\Delta_{SC}$ and $\Delta_{PG}$ determined from universal curves that describe Y123 and a number of other high-$T_c$ cuprate superconductors (Ref. 44). An experimental value for $\Delta_{SC}$ at $p = 0.12$ is omitted, since deviations from the universal curve are expected for Y123 near 1/8 hole doping, where $T_c$ plateaus. The value of $T^* \approx 278$ K at $p = 0.09$ is estimated from a linear extrapolation of data in (Ref. 50). We tune the scaling factors explained in the text to yield the blue and green values at optimum doping. Red values are calculated with the same parameters.

|                  | $p = 0.16$ | $p = 0.12$ | $p = 0.09$ |
|------------------|------------|------------|------------|
| $\lambda_{CO}(a_0)$ | $2.49 \pm 0.16$ | $3.15 \pm 0.16$ | $3.00 \pm 0.16$ |
| $\langle V_{GL}(r) \rangle$ | $-0.156$ | $-0.110$ | $-0.105$ |
| $\rho_b(1.05T_c)(\mu\Omega \text{ cm})^{[7]}$ | $\approx 40$ | $\approx 50$ | $\approx 70$ |
| $\Delta_{SC}$ (meV) | $42 \pm 2$ | - | $24 \pm 1$ |
| $\langle \Delta_d(r) \rangle$ (meV) | $43.8$ | $25.5$ | $23.3$ |
| $T_c$ (K) | $93.4$ | $66$ | $55$ |
| $\Delta_{PG}$ | $92.0$ | $68.1$ | $55.6$ |
| $T^*$ | $76$ | $104$ | $124$ |
| $103.3 \pm 9$ | $170$ | $232$ | $278$ |
| $210 \pm 19$ | $231 \pm 21$ | |

IV. THE PSEUDOGAP

Next, we use the free-energy simulations to make a simple estimate of the PG, under the assumption that the PG appears due to the mesoscale phase separation that creates small domains of CO wavelength below $T_{PS} \approx T^*$. For Bi2201 we consider a single-particle state bound to a two-dimensional (2-D) square box of depth $\langle V_{GL}(r) \rangle$ and sides of length $\lambda_{CO}$. For Y123 we consider a single-particle state bound to a stripe-like 2-D rectangular box of depth $\langle V_{GL}(r) \rangle$, width $\lambda_{CO}$, and length equivalent to the CDW correlation length, which is much longer than $\lambda_{CO}$. We assume in both cases that the PG is proportional to the numerical solution of the corresponding 2-D Schrödinger equation for the ground state binding energy. The proportionality factor is estimated using experimental values of the pseudogap $\Delta_{PG}$ for Y123 and (Pb, La)-Bi2201 at $p = 0.16$ (Tables I and II), and the values of $\Delta_{PG}$ are calculated for the underdoped samples using the respective values of $\langle V_{GL}(r) \rangle$ and $\lambda_{CO}$. To further assess the accuracy of the results for the underdoped samples, we convert $\Delta_{PG}$ to $T^*$ using the experimental ratios $T^*/\Delta_{PG} = 170$ K/76 meV and $T^*/\Delta_{PG} = 241$ K/30 meV for optimally-doped Y123 and (Pb, La)-Bi2201, respectively. As shown in Fig. II and Table III the calculated values of $T^*$ for (Pb, La)-Bi2201 at $p = 0.126$ and 0.141 agree well with measurements of the PG onset temperature for La-doped Bi2201. Reasonable agreement is also obtained between the calculated and experimental values of $T^*$ for Y123 at $p = 0.09$ and 0.12 (Fig. II and Table III). As mentioned in the introduction we presume the onset of CO domains at $T^*$ above the long range temperature $T_{CO}(p) < T^*(p)$, a behavior that has been detected by some different experiments.

TABLE II. Experimental data for $\Delta_{SC}$ and $\Delta_{PG}$ from STM with the indicated $T_c$ values. The hole doping $p$ was determined from the $T_c$ versus $p$ relationship obtained by X-ray absorption experiments on (Pb, La)-Bi2201 and La-doped Bi2201. The in-plane resistivity $\rho_{ab}$ data area for La-doped Bi2201 and $T^*$ data are from intrinsic tunneling measurements. We tune the scaling factor explained in the text to yield the blue and green values at optimum doping. Red values are calculated with no extra parameters.

|                  | $p = 0.16$ | $p = 0.12$ | $p = 0.09$ |
|------------------|------------|------------|------------|
| $\lambda_{CO}(a_0)$ | $6.2 \pm 0.2$ | $5.1 \pm 0.2$ | $4.5 \pm 0.2$ |
| $\langle V_{GL}(r) \rangle$ | $-0.1022$ | $-0.1018$ | $-0.1021$ |
| $\rho_b(1.05T_c)(\mu\Omega \text{ cm})^{[7]}$ | $241$ | $24.8$ | $28.3$ |
| $\Delta_{SC}$ (meV) | $15$ | - | - |
| $\langle \Delta_d(r) \rangle$ (meV) | $15.8$ | $13.2$ | $9.2$ |
| $T_c$ (K) | $35$ | $32$ | $25$ |
| $35.2$ | $32.5$ | $24.7$ |
| $\Delta_{PG}$ | $30 \pm 12$ | $45 \pm 15$ | $68 \pm 20$ |
| $T^*$ | $241$ | $355$ | $446$ |
| $424$ | $327.9$ | $427.4$ |

V. COMPARISON OF CALCULATED AND EXPERIMENTAL PARAMETERS

Tables I and II contain values of experimental parameters (denoted by black text) used in the calculations for each compound, and the calculated parameters (denoted by red text). For each cuprate family the calculated values at $p = 0.16$ (which are denoted by blue and green text) were multiplied by a scaling factor to match experimental values as follows:

i) The proportionality constant between $\langle V_{GL}(r) \rangle$ and the attractive pairing potential $V$ of Eq. (B3) was adjusted to yield a calculated value of $\langle \Delta_d(r) \rangle$ that approximately equals the experimental value of the SC gap $\Delta_{SC}$ at $p = 0.16$. This proportionality constant, once determined, was subsequently used for all other values of $p$.

ii) The scaling factor between the normal resistance $R_n$ in Eq. (3) and the resistivity $\rho_b(1.05T_c)$ just above $T_c$ was adjusted until the calculated value of $T_c$ at $p = 0.16$ approximately equaled the experimental value. This same scaling factor was used for all other values of $p$.

iii) The ground state binding energy of a single-particle in a 2-D square (rectangular) box was multiplied by a proportionality factor so as to equal the PG of (Pb, La)-Bi2201 (Y123) at $p = 0.16$. Again, this same proportionality constant was used for the calculations at other dopings.

VI. FERMI ARC FORMATION

Next we show that the phase separation approach considered above is able to reproduce the ungapped portion of the Fermi surface (Fermi arcs) that is known to occur near the nodal region just above $T_c$. We start by recalling that
Figs. 1(c), 1(f), 2(c) and 2(f) show domains of SC order parameter modulations. To each domain we assign a label \( j \) and a complex SC order parameter \( \Delta_\theta(k, T) \exp(i\Phi_j) \), where 
\[
\Delta_\theta(k, T) = \Delta_\theta(T)[\cos(k_a a_0) - \cos(k_b a_0)] = \Delta_\theta(T) \cos(2\phi), \phi \text{ is the azimuthal angle measured from the}
\]
\[
-x\text{-direction in the CuO}_2 \text{ plane and } \Delta_\theta(T) \text{ is the wave function amplitude in the } j^{th} \text{ domain at temperature } T. \text{ The } d\text{-wave symmetry implies larger supercurrents flowing in the CuO}_2 \text{o plane along the antinodal directions parallel to the CuO bonds, and vanishing values along the nodal directions } \phi = \pm \pi/4 \text{ and } \pm 3\pi/4. \text{ The local intrinsic SC phase } \phi_j \text{ and the superfluid density } n_j \propto \Delta_\theta(r_j, T)^2 \text{ are canonically conjugate variables, leading to large fluctuations of the phase } \Phi_j \text{ along the nodal directions, where } n_j \text{ and } \Delta n_j \text{ vanish. This is due to the quantum uncertainty principle and we may write } \\
\Delta \Phi_j(\phi) \propto 1/\cos(2\phi) \text{ to indicate the azimuthal dependence of the phase uncertainty, which has its maximum and minimum values along the nodal and antinodal directions, respectively. Furthermore, } \Delta \Phi_j \text{ has a clear dependence on the Josephson coupling. In particular, as shown in the previous section, at } T < T_c \text{ all } \Phi_j \text{ are locked together leading to long range SC order, but at } T > T_c \text{ phase decoupling occurs because } \langle E_1(p, T) \rangle < k_B T \text{ and concomitantly } \Delta \Phi_j \text{ increases with } T \text{ up to the temperature at which } \langle E_1(p, T) \rangle \text{ vanishes. In particular, } \Delta \Phi_j \text{ increases monotonically from near zero at } T_c \text{ to very large values near } T^* \text{. Furthermore, } \Delta \Phi_j \text{ has a large anisotropy when combined with the quantum effects discussed above. The two distinct contributions are separable, such that } \Delta \Phi_j(p, T, \phi) = \Delta \Phi_j(p, T) \Delta \Phi_j(\phi). \text{ We drop the index } j \text{ because } \Delta \Phi_j(\phi) \text{ is the same for all domains, and assume } \langle E_1(p, T) \rangle \text{ is the same for all } j. \text{ These considerations imply that just above } T_c, \text{ the electrons ejected by ARPES from different domains come from regions where the SC order parameter has essentially the same } \Phi_j \text{ along the antinodal directions, and } \Delta \Phi_j(\phi) \approx 0. \text{ On the other hand, such phase coherence is lost near the nodal directions where } \Delta \Phi_j(\phi) \approx \pi \text{ is maximum. Consequently, the average SC amplitude measured by ARPES may be written as follows}:
\]

\[
\langle \Delta(d(p, T, \phi)) \rangle = \frac{\langle \cos(2\phi) \rangle}{\Delta \Phi_j(p, T, \phi)} \int_0^{\Delta \Phi_j(p, T, \phi)} \langle \Delta(p, T) \rangle \cos(\Phi) d\Phi
\]

\[
= \langle \Delta(p, T) \rangle \cos(2\phi) \frac{1}{\Delta \Phi_j(p, T, \phi)} \sin[\Delta \Phi_j(p, T, \phi)].
\]

This expression contains the two distinct contributions that weaken phase coherence, one from quantum oscillations that depends only on the azimuthal angle \( \phi \) and one from thermal oscillations that competes with the average Josephson coupling, leading to \( \Delta \Phi_j(p, T, \phi) = \Delta \Phi_j(p, T) \Delta \Phi_j(\phi) \) for all \( j \). We may take \( \Delta \Phi_j(\phi) \approx 1/\cos^2(2\phi) \), which satisfies the expected inverse cosine dependence and the square makes it symmetric and always positive around the nodal directions \( \phi = \pm \pi/4 \text{ and } \pm 3\pi/4 \). Now we can infer the functional form of \( \Delta \Phi(p, T) \) noting that for \( T < T_c \), all \( \Phi_j \) are locked together leading to long range order and \( \Delta \Phi_j \approx 0 \). On the other hand, for \( T \leq T_c \), all \( \Phi \) decouple because \( \langle E_1(p, T) \rangle < k_BT \) and \( \Delta \Phi > 0 \). Above \( T_c \), \( \langle E_1(p, T) \rangle \) decreases with increasing \( T \) and vanishes near \( T^* \). Concomitantly \( \Delta \Phi \) increases. Thus, there are three distinct temperature dependent regimes:

(i) \( T \leq T_c \): Since \( \Delta \Phi \approx 0 \), Eq. (4) is easy to solve and we obtain the “bare” expression \( \langle \Delta(d(r, T, \phi)) \rangle = \langle \Delta(d(p, T)) \rangle \cos(2\phi) \).

(ii) \( T > T_c \): \( \langle \Delta(p, T) \rangle \approx 0 \) and it is clear that there is no gap.

(iii) \( T_c < T \): Taking into account the effect of \( \langle E_1(p, T) \rangle \) we assume \( \Delta \Phi(p, T) = A[1 - \langle E_1(p, T) \rangle/k_BT_c(p)] \) where \( A \) is a constant. This expression vanishes at \( T_c \) and increases monotonically with \( p \), as expected from ARPES experiment. Thus, putting all together, \( \Delta \Phi(p, T, \phi) = [A/\cos^2(2\phi)][1 - \langle E_1(p, T) \rangle/k_BT_c(p)] \) and we obtain the value of \( A \) by comparing with the onset of the measured gapless region for a given sample. To reproduce the measured gapless regions we also assume in Eq. (4) that \( \langle \Delta(d(p, T, \phi)) \rangle \approx 0 \) whenever \( \Delta \Phi(p, T, \phi) \geq \pi \), due to destructive phase interference from electrons ejected from different domains.

Specifically we use the ARPES measurements\(^{51}\) at \( T = 102 \text{ K} \) for the Bi2212 compound with \( T_c = 92 \text{ K} \), which shows a gapless region between \( 28^0 < \phi < 62^0 \) to derive the constant \( A \). Equating \( \Delta \Phi(p \approx 0.15, T = 102\text{K}, \phi = 28^0, 62^0) = \pi \) yields \( \langle \Delta(d(p, T)) \rangle \approx 0 \) and this is possible if we take \( A = 2.84\pi \). Note that \( \phi \) is measured from the \((\pi, \pi)\) to \((0, \pi)\) direction of the Brillouin zone according to Refs. \(^{51}\) and \(^{52}\). With \( \Delta \Phi(p, T, \phi) \) determined, we may apply the derived equation to any sample. In particular, we apply this expression to the other two Bi2212 compounds measured in Ref. \(^{51}\). Some above \( T_c \) values of \( \langle E_1(p, T) \rangle \) used in the calculations are plotted in Fig.5 for illustration purpose. Accordingly we obtain for the \( T_c = 86 \text{ K} \) compound a gapless region at \( 23.8^0 < \phi < 66.2^0 \), which compares well with the experimental \( 25^0 < \phi < 65^0 \). For the underdoped \( T_c = 75 \text{ K} \) Bi2212 sample at \( T = 85 \text{ K} \), we obtain \( 36.5^0 < \phi < 53.5^0 \), which is also in good agreement with the experimental result\(^{51}\). We summarize the Fermi arc calculations for the three samples in Fig. 6 where the results of the “envelope” phase factor of \( \langle \Delta(d(p, T, \phi)) \rangle \)

\[
R = \frac{\langle \cos(2\phi) \rangle}{\Delta \Phi(p, T, \phi)} \times \sin[\Delta \Phi_j(p, T, \phi)]
\]

from Eq. (4) are in good agreement with the experiment\(^{51}\). The arrows mark the experimentally determined onset of the gapless regions for each sample, as described above.

VII. SUMMARY AND CONCLUSIONS

Our theory infers a fundamental link between the periodicity of the CDW and the PG and SC energy scales of high-temperature cuprate superconductors, and shows that within this framework it is possible to account for the onset temperatures \( T_c \) and \( T^* \) of two different cuprate families. We stress that the only quantitative assumptions made in our calculations for underdoped Y123 and (Pb, La)-Bi2201 pertain to a natural scaling factor, which we have determined by scaling calculated free energy parameters to achieve values of the PG
phase fluctuations along the nodal directions where the superfluid density vanishes. The increase of the arcs size with $p$ is reproduced because the dependence of $\langle E_J(p, T) \rangle$ on the temperature changes rapidly with doping.

Finally we address the experimental observations indicating a competition between superconductivity and CO. While X-ray experiments show a decrease of the CDW diffraction intensity and correlation length below $T_c$, these measurements seem to be detecting static charge correlations. Static CDW order competes with superconductivity by reducing the number of charge carriers available for pairing. On the other hand, our theory requires that dynamic CO is also present to induce a fluctuating hole-pair potential that scales with $V_{GL}$. While there is some evidence for CDW fluctuations from optical pump-probe and low-energy, momentum-resolved electron energy-loss spectroscopy experiments, there is currently insufficient experimental information to assess the pervasiveness or significance of fluctuating CO in cuprates.

ACKNOWLEDGMENTS

We thank S. Ono and Y. Ando for providing us with their resistivity data. We also thank Andrea Damascelli, André-Marie Tremblay and John Tranquada for informative discussions. Supported by the Brazilian agencies FAPERJ and CNPq (E.V.L.M.); the Canadian Institute for Advanced Research (CIFAR) and the Natural Sciences and Engineering Research Council of Canada (J.E.S.).

Appendix A: CHARGE ORDER SIMULATIONS

To describe the growth and development of spatial charge inhomogeneity in the CuO$_2$ planes we applied a theory of phase-ordering dynamics, whereby the system evolves through domain coarsening when quenched from a homogeneous into a broken-symmetry phase. The time-dependent CH approach provides a simple way to determine the time evolution of the CO process. The CH equation can be written in the form of the following continuity equation for the local free energy current density $J = -M \nabla^2 \left( \frac{\partial f}{\partial u} \right)$.

$$\frac{\partial u}{\partial t} = -\nabla \cdot J = -M \nabla^2 \left[ \varepsilon^2 \nabla^2 u + \frac{V_{GL}}{u} \right],$$

(A1)

where $M$ is the charge mobility that sets the phase separation time scale. The order parameter varies between $u(r, t) \sim 0$ for the homogeneous system above the phase-separation onset temperature $T_{PS}$, and $u(r, t \rightarrow \infty) = \pm 1$ for the extreme case of complete phase separation. We solved the CH equation by a stable and fast finite difference scheme with free boundary conditions. The spatial dependence of the charge density obtained by numerically solving the CH equation evolves with time $t = n \delta t$, where $n$ is the number of time steps $\delta t$. When the order parameter is conserved, as in phase separation, the charges can only exchange locally rather than over large distances. This leads to diffusive transport of the order parame-
ter. Consequently, at early times or small \( n \), we obtain charge modulations with periodicities of only a few lattice constants. Using different parameters and initial conditions we are able to reproduce the experimentally determined CO patterns in cuprates. Although these simulations are not the stable solutions of the CH equation (as is clear from the time evolution of the simulations shown in Fig. 7), the aim here is to generate periodic charge modulations with experimentally determined wavelengths that can be subsequently used to calculate the SC gap and PG in our phase- separation model. For convergence the time step \( \delta t \) and spatial step \( h \approx 1/N \) for a square lattice of \( N^2 \) sites must be such that \( \delta t \leq h^2/9 \) (Ref. 53). For the calculations here we used \( \delta t \leq h^2/10 \) and \( h = 1/100 \).

In the main paper we present detailed CO, \( T_c \) and \( T^* \) calculations for six compounds. Three of the Bi2201 and three of the Y123 families. In the next two paragraphs we give the values of some parameter used in the CH simulations.

\textbf{(Pb, La)-Bi2201:} Simulations with \( \alpha = B = 1 \), time steps of \( n = 700, 900, 1300 \) and \( \varepsilon = 0.012, 0.014, 0.0175 \) yield checkerboard CO patterns with the desired wavelengths \( \lambda_{\text{CO}} \) for (Pb, La)-Bi2201 (at \( p = 0.126, 0.141 \) and 0.16) near 4.5\( a_0 \), 5.1\( a_0 \), and 6.2\( a_0 \), respectively. The fewer time steps required to simulate the CO patterns of the underdoped samples is indicative of a reduced charge mobility, and is consistent with an increase of the normal-state resistivity. At later times (i.e., greater \( n \)) the periodic electronic structure evolves into an irregular patch-like system of segregated low- and high-charge density regions. In addition, the length scale of the system increases with the phase separated regions forming larger domains. This latter situation was considered previously\(^{45}\). Fig. 2 shows CH simulations of \( u(r, t) \) at times beyond where checkerboard CO with \( \lambda_{\text{CO}} = 4.5\ a_0 \) is observed in (Pb, La)-Bi2201 at \( p = 0.126 \).

\textbf{Y123:} Simulations with \( \alpha = 1, B = 5, \varepsilon = 0.0053, 0.0055, 0.0058 \) and time steps \( n = 35, 38, 42 \) yield charge stripe patterns with the desired wavevectors \( Q = 0.333, 0.317 \) and 0.287 r.l.u. \( \lambda_{\text{CO}} = 1/\langle Q \rangle \) estimated from (Refs. 4, 5, and 7) for Y123 at \( p = 0.09, 0.12 \) and 0.16, respectively. Note that the values of \( n \) are much shorter than needed to simulate the checkerboard CO patterns of (Pb, La)-Bi2201. Because of the fewer time steps, the simulations for Y123 are somewhat less sharp.

\section*{Appendix B: Combined Bogoliubov-deGennes (BdG) and Cahn-Hilliard (CH) Calculations}

We performed self-consistent calculations with the BdG theory (Refs. 58 and 59) for each of the CH simulated charge density maps (Figs. 11(a) and 11(d), and Figs. 2(a) and 2(d)). To calculate the SC pairing amplitude we assumed the attractive interaction potential \( V \) scales with \( V_{\text{GL}} \sim \langle \nabla V_{\text{GL}}(r) \rangle \). The SC calculations begin with the extended Hubbard Hamiltonian\(^{66,69}\). To describe the charge carriers dynamics in the CuO\(_2\) planes of the HTSC we consider this Hamiltonian in a square lattice

\[ H = - \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{\sigma} \mu_{\text{h}} n_{\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow} - \frac{V}{2} \sum_{\langle ij \rangle \sigma \sigma'} n_{i\sigma} n_{j\sigma'} , \]

where \( c_{i\sigma}^\dagger (c_{i\sigma}) \) is the usual fermionic creation (annihilation) operator at site \( i \), the spin \( \sigma \) is up \( \uparrow \) or down \( \downarrow \). \( n_{\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the number operator, and \( t_{ij} \) is the hopping between sites \( i \) and \( j \). \( U \) is the magnitude of the on-site repulsion, and \( V \) is the magnitude of the nearest-neighbor attractive interaction. \( \mu_{\text{h}} \) is the local chemical potential derived in the self-consistent process through which the local charge density is calculated by the CH equation and is maintained fixed. For (Pb, La)-Bi2201 we used nearest-neighbor hopping \( t = 0.15 \) eV, next-nearest-neighbor hopping \( t_2 = -0.27t \), and third nearest-neighbor hopping \( t_3 = 0.19t \) derived from angle-resolved photoemission spectroscopy (ARPES) dispersion relations\(^{50}\). For Y123, we used the ARPES results \( t = 0.15 \) meV, \( t_2 = -0.50t \) and \( t_3 = 0.16t \) (Ref. 61). The BdG mean-field equations are\(^{50}\)

\[ \left( \begin{array}{c} K & \Delta_d(r) \\ \Delta^*_d(r) & -K^* \end{array} \right) \left( \begin{array}{c} u_n(r) \\ v_n(r) \end{array} \right) = E_n \left( \begin{array}{c} u_n(r) \\ v_n(r) \end{array} \right) \]

with

\[ K u_n(r) = - \sum_R f_{r+r+R} u_n(r+R) + \mu(r) u_n(r) \]

\[ \Delta_d u_n(r) = \sum_R \Delta_d(r) u_n(r+R), \]

and similar equations for \( v_n(r) \), where \( r+R \) is the position of the nearest-neighbor sites, and \( \mu(r) \equiv \mu_{\text{h}} \) is the local chemical potential. These equations are solved numerically for eigenvalues \( E_n (\geq 0) \) self-consistently with the spatially-varying \( d \)-wave pairing potential\(^{58}\)

\[ \Delta_d(r) = - \frac{V}{2} \sum_n [u_n(r)]^2 v_n^*(r+R) + v_n^*(r) u_n(r+R)] \tanh \frac{E_n}{2k_B T}, \]

where \( V = V(T, p) \) was defined in Eq. \( 2 \). The results of \( \langle \Delta_d(r, T) \rangle \) are plotted in Fig. 8 for the three compounds of the Y123 system. Concomitantly, the spatially-varying hole density of charge carriers is given by

\[ p(r) = 1 - 2 \sum_n [|u_n(r)|^2 f_n + |v_n(r)|^2 (1 - f_n)], \]

where \( f_n = [\exp(E_n/k_B T + 1)^{-1} \) is the Fermi occupation function. It is important to emphasize that the spatially inhomogeneous distribution of charge generated by the CH equation for different dopings was kept fixed while the local chemical potential \( \mu(r) \) was self-consistently determined in the convergence process. This procedure incorporates the charge inhomogeneity in the calculations in a natural way.
FIG. 7. Two-dimensional CH simulations of $u(r, t)$ for $\alpha = B = 1, \varepsilon = 0.012$ and time steps (a) $n = 900$, (b) $n = 1500$, and (c) $n = 9000$. These plots are continuation of the time evolution of Fig. 2(a) with $n = 700$.

FIG. 8. Example of calculated $\langle \Delta d(p, T) \rangle$ used to obtain $T_c$: The average SC amplitudes for Y123 from BdG Eq. B4 used in the calculations of $\langle E_J(p, T) \rangle$ [see Eq. 3]. The low temperature limits of $\langle \Delta d(p, T = 0) \rangle$ are also listed in Table I.

* Corresponding author: evandro@if.uff.br

1. J. Chang, E. Blackburn, T. Holmes, N. B. Christensen, J. Larsen, J. Mesot, Ruixing Liang, D. A. Bonn, W. N. Hardy, A. Watenphul, M. V. Zimmermann, E. M. Forgan, and S. M. Hayden, “Direct observation of competition between superconductivity and charge density wave order in YBa$_2$Cu$_3$O$_{6.67}$,” Nat. Phys. 8, 871–876 (2012).

2. G. Ghiringhelli, M. Le Tacon, M. Minola, S. Blanco-Canosa, C. Mazzoli, N. B. Brookes, G. M. De Luca, A. Frano, D. G. Hawthorn, F. He, T. Loew, M. Moretti Sala, D. C. Peets, M. Salluzzo, E. Schierle, R. Sutarto, G. A. Sawatzky, E. Weschke, B. Keimer, and L. Braicovich, “Long-range incommensurate charge fluctuations in (Y, Nd)Ba$_2$Cu$_3$O$_{6+x}$,” Science 337, 821–5 (2012).

3. R. Comin, A. Frano, M. M. Yee, Y. Yoshida, H. Eisaki, E. Schierle, E. Weschke, R. Sutarto, F. He, a Soumyanarayanan, Yang He, M. Le Tacon, I. S. Elfimov, Jennifer E Hoffman, G. A. Sawatzky, B. Keimer, and A. Damascelli, “Charge order driven by Fermi-arc instability in Bi$_2$Sr$_2$−$x$La$_x$CuO$_{6+\delta}$,” Science 343, 390-2 (2014).

4. S. Blanco-Canosa, A. Frano, E. Schierle, J. Porras, T. Loew, M. Minola, M. Bluschke, E. Weschke, B. Keimer, and M. Le Tacon, “Resonant x-ray scattering study of charge-density wave correlations in YBa$_2$Cu$_3$O$_{6+x}$,” Phys. Rev. B 90, 054513 (2014).

5. M. H"ucker, N. B. Christensen, A. T. Holmes, E. Blackburn, E. M. Forgan, Ruixing Liang, D. A. Bonn, W. N. Hardy, O. Gutowski, M. V. Zimmermann, S. M. Hayden, and J. Chang, “Competing charge, spin, and superconducting orders in underdoped YBa$_2$Cu$_3$O$_y$,” Phys. Rev. B 90, 1–11 (2014).

6. Eduardo H. da Silva Neto, Riccardo Comin, Feizhou He, Ronny Sutarto, Yeping Jiang, Richard L. Greene, George A. Sawatzky, and Andrea Damascelli, “Charge ordering in the electron-doped superconductor Nd$_{2−x}$Ce$_x$CuO$_4$,” Science 347, 282–285 (2015).

7. R. Comin, R. Sutarto, E. H. da Silva Neto, L. Chauviere, R. Liang, W. N. Hardy, D. A. Bonn, F. He, G. A. Sawatzky, and A. Damascelli, “Superconductivity. Broken translational and rotational symmetry via charge stripe order in underdoped YBa$_2$Cu$_3$O$_{6+y}$,” Science 347, 1335–9 (2015).

8. R. Comin, R. Sutarto, F. He, E. H. da Silva Neto, L. Chauviere, A. Frañó, R. Liang, W. N. Hardy, D. A. Bonn, Y. Yoshida, H. Eisaki, A. J. Achkar, D. G. Hawthorn, B. Keimer, G. A. Sawatzky, and A. Damascelli, “Symmetry of charge order in cuprates,” Nat. Mater. 14, 796–800 (2015).
Subir Sachdev and Rolando La Placa, “Bond order in two-dimensional metals with antiferromagnetic exchange interactions,” *Phys. Rev. Lett.* **111**, 027202 (2013).

T. Hanaguri, C. Lupien, Y. Kohsaka, D-H Lee, M. Azuma, M. Takano, H. Takagi, and J. C. Davis, “A ‘checkerboard’ electronic crystal state in lightly hole-doped \( \text{Ca}_{2-x}\text{Na}_{x}\text{CuO}_2\text{Cl}_2 \).” *Nature* **450**, 1001–1005 (2007).

M. H. Hamidian, S. D. Edkins, Chung Koo Kim, J. C. Davis, A. P. Mackenzie, H. Eisaki, S. Uchida, M. J. Lawler, E.-A. Kim, S. Sachdev, and K. Fujita, “Atomic-scale electronic structure of the cuprate d-symmetry form factor density wave state,” *Nature* **12**, 150–156 (2016).

Subir Sachdev and Rolando La Placa, “Bond order in two-dimensional metals with antiferromagnetic exchange interactions,” *Phys. Rev. Lett.* **111**, 027202 (2013).

Efetov K. B., Meier H., and Penc P., “Pseudogap state near a quantum critical point,” *Nat. Phys.* **9**, 442–446 (2013).

Lauren E. Hayward, David G. Hawthorn, Roger G. Melko, and Subir Sachdev, “Angular fluctuations of a multi-component order describe the pseudogap of YBa\(_2\)Cu\(_3\)O\(_{6.8+\delta}\).” *Science* **343**, 1336–9 (2014).

C. Pépin, V. S. de Carvalho, T. Kloss, and X. Montiel, “Pseudogap, charge order, and pairing density wave at the hot spots in cuprate superconductors,” *Phys. Rev. B* **90**, 195207 (2014).

Eduardo Fradkin, Steven A. Kivelson, and John M. Tranquada, “Colloquium: Theory of intertwined orders in high temperature superconductors,” *Rev. Mod. Phys.* **87**, 457–482 (2015).

Yuxuan Wang, Daniel F. Agterberg, and Andrey Chubukov, “Coexistence of charge-density-wave and pair-density-wave orders in underdoped cuprates,” *Phys. Rev. Lett.* **114**, 197001 (2015).

Yuxuan Wang, Daniel F. Agterberg, and Andrey Chubukov, “Interplay between pair- and charge-density-wave orders in underdoped cuprates,” *Phys. Rev. B* **91**, 115103 (2015).

Eduardo Fradkin and Steven A. Kivelson, “High-temperature superconductivity: Ineluctable complexity,” *Nat. Phys.* **8**, 864–866 (2012).

Collin V. Parker, Pegor Aynajian, Eduardo H. da Silva Neto, Aakash Pushp, Shimppei Ono, Jinsheng Wen, Zhijun Xu, Genda Gu, and Ali Yazdani, “Fluctuating stripes at the onset of the pseudogap in the high-T\(_c\) superconductor Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Nature* **468**, 677–680 (2010).

Tabis W., B. Yu, I. Bialo, M. Bluschke, T. Kolodziej, A. Kozłowski, Y. Tang, E. Weschke, B. Vignolle, M. Hepting, H. Gretarsson, R. Sutarto, F. He, M. Le Tacon, N. Barri?i, G. Yu, and Greven M., “New insight into cuprate charge order from x-ray measurements of HgBa\(_2\)Cu\(_{4+\delta}\).” *cond-mat.supr-con/1702.03348* (2017).

B. Fauqu? Y. Sidis, V. Hinkov, S. Pailh?es, C. T. Lin, X. Chaud, and P. Bourges, “Magnetic order in the pseudogap phase of high-T\(_c\) superconductors,” *Phys. Rev. Lett.* **96**, 197001 (2006).

Li Y., Baledent V., Barcis N., Cho Y., Fauque B., Sidis Y., Yu G., Zhao X., Bourges P., and Greven M., “Unusual magnetic order in the pseudogap region of the superconductor HgBa\(_2\)Cu\(_{4+\delta}\).” *Nature* **455**, 372–375 (2008).

Shekhter Arkady, Ramshaw B. J., Liang Ruixing, Hardy W. N., Bonn D. A., Balakirev Fedor F., McDonald Ross D., Betts Jon B., Riggs Scott C., and Migliori Albert, “Bounding the pseudogap with a line of phase transitions in Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Nature* **498**, 75–77 (2013).

Chan M. K., Dorow C. J., Mangin-Thro L., Tang Y., Ge Y., Veit M. J., Yu G., Zhao X., Christianson A. D., Park J. T., Sidis Y., Steffens P., Abernathy D. L., Bourges P., and Greven M., “Commensurate antiferromagnetic excitations as a signature of the pseudogap in the tetragonal high-T\(_c\) cuprate HgBa\(_2\)Cu\(_{4+\delta}\)” *Nat. Commun.* **10**, 10819 (2016).

Zhao L., Belvin C. A., Liang R., Bonn D. A., Hardy W. N., Armitage N. P., and Hsieh D., “A global inversion-symmetry-broken phase inside the pseudogap region of YBa\(_2\)Cu\(_3\)O\(_y\).” *Nat. Phys.* **13**, 250–254 (2017).

J. Corson, R. Mallozzi, and J. Orenstein, “Vanishing of phase coherence in underdoped Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{8+\delta}\).” *Nature* **406**, 221–223 (1999).

Xu Z. A., Ong N. P., Wang Y., Kakeshita T., and Uchida S., “Vortex-like excitations and the onset of superconducting phase fluctuation in underdoped La\(_{2-x}\)Sr\(_x\)Cu\(_4\)O\(_{8}\)” *Nature* **406**, 486–488 (2000).

Kenjiro K Gomes, Abhay N Pasupathy, Aakash Pushp, Shimppei Ono, Yoichi Ando, and Ali Yazdani, “Visualizing pair formation on the atomic scale in the high-T\(_c\) superconductor Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Nature* **447**, 569–572 (2007).

Yang H.-B., Rameau J. D., Johnson P. D., Valla T., Tsvelik A., and Gu G. D., “Emergence of preformed Cooper pairs from the doped Mott insulating state in Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Nature* **456**, 7780 (2008).

A. Dubroka, M. Rössle, K. W. Kim, V. K. Malik, D. Munzar, D. N. Basov, A. A. Schafgans, S. J. Moon, C. T. Lin, D. Haug, V. Hinkov, B. Keimer, Th. Wolf, J. G. Storey, J. L. Tallon, and C. Bernhard, “Evidence of a precursor superconducting phase at temperatures as high as 180 K in RBa\(_2\)Cu\(_3\)O\(_{7-\delta}\) (R=Y,Gd,Eu) superconducting crystals from infrared spectroscopy,” *Phys. Rev. Lett.* **106**, 1–4 (2011).

Lu Li, Yuay Wang, M. J. Naughton, S. Ono, Yoichi Ando, and N. P. Ong, “Strongly nonlinear magnetization above T\(_c\) in Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Europhys. Lett.* **72**, 451 (2005).

Yauy Wang, Lu Li, M. J. Naughton, G. D. Gu, S. Uchida, and N. P. Ong, “Field-Enhanced Diamagnetism in the Pseudogap State in Underdoped Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_{6+\delta}\)” *Phys. Rev. Lett.* **106**, 1–4 (2011).
of the Cuprate Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ Superconductor in an Intense Magnetic Field," Phys. Rev. Lett. 95, 247002 (2005)

41 John W Cahn and John E Hilliard, “Free Energy of a Nonuniform System. I. Interfacial Free Energy,” J. Chem. Phys. 28, 258 (1958)

42 Ricardo Comin and Andrea Damascelli, “Resonant X-Ray Scattering Studies of Charge Order in Cuprates,” Annu. Rev. Cond. Matter Phys. 7, 369 (2016)

43 M. Schneider, R.-S. Unger, R. Mitdank, R. Müller, A. Krapf, S. Rogaschewski, H. Dwelk, C. Janowitz, and R. Manzke, “Evolution of the density of states at the Fermi level of Bi$_{2-2\delta}$Pb$_{2\delta}$Sr$_{2-x}$La$_x$CuO$_{6+\delta}$ and Bi$_2$Sr$_{2-x}$La$_x$CuO$_{6+\delta}$ cuprates with hole doping,” Phys. Rev. B 72, 014504 (2005)

44 S. Huefner, M. A. Hossain, A. Damascelli, and G. A. Sawatzky, “Two Gaps Make a High Temperature Superconductor?” Rep. Prog. Phys. 71, 062501 (2007)

45 E. V. L. de Mello and J. E. Sonier, “Charge segregation model for superconducting correlations in cuprates above $T_c$,” J. Phys.: C. M. 26, 492201 (2014)

46 S. Ono and Yoichi Ando, “Evolution of the resistivity anisotropy in Bi$_2$Sr$_{2-x}$La$_x$CuO$_{6+\delta}$ single crystals for a wide range of hole doping,” Phys. Rev. B 67, 104512 (2003)

47 Kouji Segawa and Yoichi Ando, “Transport Anomalies and the Role of Pseudogap in the 60-K Phase of YBa$_2$Cu$_3$O$_{7-\delta}$,” Phys. Rev. Lett. 86, 4907–4910 (2001)

48 A. Yurgens, D. Winkler, T. Claeson, S. Ono, and Yoichi Ando, “Intrinsic Tunneling Spectra of Bi$_2$(Sr$_{2-x}$La$_x$)CuO$_{6+\delta}$,” Phys. Rev. Lett. 90, 147005 (2003)

49 R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choinière, Francis Laliberté, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D. A. Bonn, W. N. Hardy, and Louis Taillefer, “Broken rotational symmetry in the pseudogap phase of a high-$T_c$ superconductor,” Nature 463, 519–522 (2010)

50 Sean Vig, Anshul Kogar, Vivek Mishra, Luc Venema, Melinda S. Rak, Ali A. Hossain, Peter D. Johnson, Genda D. Gu, Eduardo Fradkin, Michael R. Norman, and Peter Abbatante, “Fluctuating charge order in the optimally doped high-temperature superconductor Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$,” (2016)

51 cond-mat.supr-con/1509.04230v2

52 W. S. Lee, K. Vishik, I. M. and Tanaka, D. H. Lu, T. Sasagawa, N. Nagaosa, T. P. Devereaux, Z. Hussain, and Z-X. Shen, “Abrupt onset of a second energy gap at the superconducting transition of underdoped Bi2212,” Nature 450, 81–84 (2007)

53 Teppei Yoshida, Makoto Hashimoto, Inna M. Vishik, ZhiXun Shen, and Atsushi Fujimori, “Pseudogap, Superconducting Gap, and Fermi Arc in High-$T_c$ Cuprates Revealed by Angle-Resolved Photoemission Spectroscopy,” J. Phys. Soc. Jpn 81, 011006 (2012)

54 Keimer B., Kivelson S. A., Norman M. R., Uchida S., and Zaanen J., “From quantum matter to high-temperature superconductivity in copper oxides,” Nature 518, 179?186 (2015)

55 B. I. Spivak and S. A. Kivelson, “Negative local superfluid densities: The difference between dirty superconductors and dirty Bose liquids,” Phys. Rev. B 43, 3740–3743 (1991)

56 E. V. L. de Mello, “Disordered-based theory of pseudogap, superconducting gap, and Fermi arc of cuprates,” Europhys. Lett. 99, 37003 (2012)

57 Darius H. Torchinsky, Fahad Mahmood, Anthony T. Bollinger, Ivan Božovic, and Nuh Gedik, “Fluctuating charge-density waves in a cuprate superconductor,” Nat. Mater. 12, 387–391 (2013).

58 A. J. Bray, “Theory of phase-ordering kinetics,” Advances in Physics 43, 357–459 (1994)

59 E. V. L. de Mello and E. S. Caixeiro, “Effects of phase separation in the cuprate superconductors,” Phys. Rev. B 70, 224517 (2004)

60 M. R. Norman, M. Rand henia, H. Ding, and J. C. Campuzano, “Phenomenological models for the gap anisotropy of Bi$_2$Sr$_2$CaCu$_2$O$_{8}$ as measured by angle-resolved photoemission spectroscopy,” Phys. Rev. B 52, 615–622 (1995)

61 Matthias C. Schabel, C.-H. Park, A. Matsuiura, Z.-X. Shen, D. A. Bonn, Ruixing Liang, and W. N. Hardy, “Angle-resolved photoemission on untwinned YBa$_2$Cu$_3$O$_{6+\delta}$, I. Electronic structure and dispersion relations of surface and bulk bands,” Phys. Rev. B 57, 6090–6106 (1998)