BÄCKLUND TRANSFORMATIONS FOR FOURTH PAINLEVÉ HIERARCHIES

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Abstract. Bäcklund transformations (BTs) for ordinary differential equations (ODEs), and in particular for hierarchies of ODEs, are a topic of great current interest. Here we give an improved method of constructing BTs for hierarchies of ODEs. This approach is then applied to fourth Painlevé (P\text{IV}) hierarchies recently found by the same authors [Publ. Res. Inst. Math. Sci. (Kyoto) 37 327–347 (2001)]. We show how the known pattern of BTs for P\text{IV} can be extended to our P\text{IV} hierarchies. Remarkably, the BTs required to do this are precisely the Miura maps of the dispersive water wave hierarchy. We also obtain the important result that the fourth Painlevé equation has only one nontrivial fundamental BT, and not two such as is frequently stated.

1. Introduction

A classical problem, dating from the end of the nineteenth century, is that of seeking new transcendental functions defined by ordinary differential equations (ODEs). This motivated the classification of ODEs having what is today referred to as the Painlevé property, i.e., having their general solution free of movable branched singularities. In particular, it led to the discovery of the six Painlevé equations [1, 2, 3, 4], which did indeed define new transcendental functions.

The six Painlevé equations are of course second order ODEs. However the classification program embarked upon by Painlevé and co-workers foresaw, once second order ODEs had been dealt with, a classification of third order ODEs, then of fourth order ODEs, and so on. Thus Chazy [5] and Garnier [6] studied certain classes of third order ODEs, although no new transcendental was discovered at third order. Restricted classes of third order ODEs were also later considered by Exton [7] and Martynov [8, 9], unfortunately with the same result. It should be remarked that the difficulties of classification increase with the order of the equations studied; for example, at second order movable essential singularities may arise [2], whereas at third order movable natural boundaries may occur [6]. At fourth order, even the classification of dominant terms for the polynomial case was left incomplete [10].

Thus, some 20-25 years ago, the search for higher order ODEs defining new transcendental functions was in need of a new insight in order to catalyse research in...
this area. This impetus came in the form of the discovery by Ablowitz and Segur [11] of a connection between completely integrable partial differential equations (PDEs) and ODEs having the Painlevé property. This discovery not only made a remarkable connection between modern research and mathematics at the turn of the last century, but in establishing a link between integrability and the analytical properties of solutions, mirrored the prize-winning work of Kowalevski on the motion of a rigid body about a fixed point [12, 13]. It was Airault who, exploiting the fact that, for example, sitting above the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation are their respective hierarchies, first realised the next step of using higher order integrable PDEs to derive higher order ODEs with the Painlevé property. In fact she derived a whole hierarchy of ODEs, the second Painlevé ($P_{II}$) hierarchy, by similarity reduction of the KdV/mKdV hierarchies [14].

However Airault also made another important step: she obtained Bäcklund transformations (BTs) for every member of the $P_{II}$ hierarchy. A BT is a mapping between solutions of ODEs, involving naturally some identification between the parameters appearing as coefficients in the ODEs; in the case of BTs between solutions of the same ODE, this identification between parameters translates as changes in parameter values. BTs for the Painlevé equations had previously been studied in the Soviet literature; a comprehensive list of references can be found in [15], and a recent review in [16]. Today BTs are universally recognised as an important property of integrable nonlinear ODEs, and there is much interest in their derivation, especially within the context of hierarchies of ODEs. The aim of the present paper is to explore BTs for fourth Painlevé ($P_{IV}$) hierarchies.

Due therefore to the work of Ablowitz and Segur, over the last quarter century, the study of higher order analogues of the Painlevé equations, and of their properties, has been informed by knowledge of the connection with completely integrable PDEs; here we refer for example to the work of Muğan and Jrad [17, 18, 19], and Cosgrove [20]. The present authors have also exploited this connection [21, 22, 23] in the development of their own method [22] of deriving (amongst other things) hierarchies of higher order Painlevé equations together with associated underlying linear problems. Here we extend this connection still further: we find that certain features of such ODEs are directly related to the underlying structures of associated completely integrable PDEs. That is, when seeking to extend to a fourth Painlevé hierarchy [21] the pattern of BTs already known for the first member ($P_{IV}$), we find that the answer lies in the Miura transformations for the associated PDE hierarchy.

The layout of the paper is as follows. We introduce our $P_{IV}$ hierarchies in Section Two. In Section Three we give an improved method, based on the Painlevé truncation process for PDEs, of deriving auto-BTs and special integrals for hierarchies of ODEs, and as an example we apply this approach to the $P_{II}$ hierarchy of Airault. In Section Four we use this method to derive auto-BTs and special integrals for two of the $P_{IV}$ hierarchies derived in Section Two. In Section Five we identify to which BTs of $P_{IV}$ these BTs correspond. In Section Six we seek further BTs in order to extend the known pattern of BTs for $P_{IV}$ to corresponding hierarchies. Remarkably, it turns out that the BTs required to do this are precisely the known Miura maps for the associated PDE (dispersive water wave) hierarchy. In Section Seven we consider a mapping between our hierarchies which allows us to further relate the BTs derived: an important consequence of this is the result that $P_{IV}$ has only one nontrivial fundamental BT. Section Eight is devoted to conclusions.
2. Sequences of fourth Painlevé hierarchies

In our recent paper [24] we derived, along with associated linear problems, the sequence of coupled ODEs in $u = (u, v)^T$,

\begin{equation}
\mathcal{R}^n u_x + \sum_{i=0}^{n-1} c_i \mathcal{R}^i u_x + g_{n-1} \mathcal{R}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_n \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{equation}

where $c_0, \ldots, c_{n-1}$, $g_{n-1}$, $g_n$ and $g_{n+1}$, are arbitrary constants, and $\mathcal{R}$ is the recursion operator of the dispersive water wave (DWW) hierarchy [25]–[32] ($\partial_x = \partial/\partial x = d/dx$ in our ODE case (1)),

\begin{equation}
\mathcal{R} = \frac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2v + v_x \partial_x^{-1} \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}.
\end{equation}

In what follows we will consider the case which corresponds to a generalized $P_{IV}$ hierarchy, i.e. $g_{n-1} = 0$ and $g_n \neq 0$ [24]. We can then assume, using a shift on $u$, that $g_{n+1} = 0$ (note that previously we have used such a shift to set $c_{n-1} = 0$, but here we prefer to remove $g_{n+1}$). Further, using a shift on $x$, we can set $c_0 = 0$. Thus, without any loss of generality, we can assume that our generalized $P_{IV}$ hierarchy is of the form

\begin{equation}
\mathcal{R}^n u_x + \sum_{i=1}^{n-1} h_i \mathcal{R}^i u_x + g_n \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{equation}

for some constants $h_1, \ldots, h_{n-1}$ and $g_n (\neq 0)$. We note in passing that the second nontrivial member of our hierarchy ($n = 2$) is of interest for the problems that its singularity analysis presents; this was the subject of our paper [33].

The hierarchy (3) can also be written in the alternative form

\begin{equation}
B_2 \mathbf{K}_n [u] = 0,
\end{equation}

where

\begin{equation}
\mathbf{K}_n [u] = \mathbf{L}_n [u] + \sum_{i=1}^{n-1} h_i \mathbf{L}_i [u] + g_n \begin{pmatrix} 0 \\ x \end{pmatrix},
\end{equation}

$B_2$ is one of the three Hamiltonian operators of the DWW hierarchy,

\begin{equation}
B_2 = \frac{1}{2} \begin{pmatrix} 2 \partial_x & \partial_x u - \partial_x^2 \\ \partial_x v + \partial_x^2 & v \partial_x + \partial_x v \end{pmatrix},
\end{equation}

and each $\mathbf{L}_i [u]$ is the variational derivative of the Hamiltonian density corresponding to the operator $B_2$ for the $t_i$-flow of the DWW hierarchy, $u_{t_i} = \mathcal{R}^i u_x = B_2 \mathbf{L}_i [u]$.

Here we have used the fact that $\mathcal{R} = B_2 B_1^{-1}$, where

\begin{equation}
B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}
\end{equation}

is another of the Hamiltonian operators of the DWW hierarchy. We note also the recursion relation $B_1 \mathbf{L}_{i+1} [u] = B_2 \mathbf{L}_i [u]$, and that $\mathbf{L}_0 [u] = (0, 2)^T$, $\mathbf{L}_1 [u] = (v, u)^T$.

We now consider the construction of hierarchies equivalent to (4). We will also see how a reduction of order of our system (4) can be effected using the Hamiltonian structures of the DWW hierarchy. We begin by recalling the Miura maps of the
DWW hierarchy, as given by Kupershmidt [31]. The first Miura map is given by
\[ u = F[U], \]
where \( U = (U, V)^T \) and
\[ (8) \]
\[ F[U] = \left( \begin{array}{c} U \\ UV - V^2 + V_x \end{array} \right). \]
The two second Miura maps are given by \( U = \Phi[\phi] \), where \( \phi = (\phi, p)^T \) and
\[ (9) \]
\[ \Phi[\phi] = \left( \begin{array}{c} \phi + 2p \\ p \end{array} \right), \]
and \( U = \Psi[\psi] \), where \( \psi = (\psi, s)^T \) and
\[ (10) \]
\[ \Psi[\psi] = \left( \begin{array}{c} \psi - 2s \\ -s \end{array} \right). \]
That is, we have the following two sequences of Miura transformations:
\[ (11) \]
\[ \left( \begin{array}{c} u \\ v \end{array} \right) \xrightarrow{F} \left( \begin{array}{c} U \\ V \end{array} \right) \xrightarrow{\Phi} \left( \begin{array}{c} \phi \\ p \end{array} \right) \xrightarrow{\Psi} \left( \begin{array}{c} \psi \\ s \end{array} \right) \]
We now consider the first of these sequences. Using the fact that for the Miura map \( F \) we have
\[ (12) \]
\[ B_2 \bigg|_{u=F[U]} = F'[U]B (F'[U])^\dagger, \]
where \( B \) is the Hamiltonian operator of the modified DWW system,
\[ (13) \]
\[ B = \frac{1}{2} \left( \begin{array}{cc} 2\partial_x & \partial_x \\ \partial_x & 0 \end{array} \right), \]
\( F'[U] \) is the Fréchet derivative of the Miura map and \((F'[U])^\dagger\) is its adjoint, we obtain, in the same way as in the PDE case, the modified version of \( 4 \),
\[ (14) \]
\[ B (F'[U])^\dagger K_n[F[U]] = 0. \]
This we can then integrate to obtain
\[ (15) \]
\[ (F'[U])^\dagger K_n[F[U]] + (c_n, d_n)^T = 0, \]
for two arbitrary constants \( c_n \) and \( d_n \). In fact this last is equivalent to an integrated version of \( 11 \) under the BT
\[ (16) \]
\[ u - F[U] = 0, \]
\[ (17) \]
\[ (F'[U])^\dagger K_n[u] + (c_n, d_n)^T = 0. \]
It is this construction of a BT between an integrated modified hierarchy and an integrated version of our original hierarchy that lies behind the first integrals of our \( P_{IV} \) hierarchy given in [24]; this approach is described in more detail in [34]. The integrated form of \( 11 \) obtained from \( 16, 17 \) is
\[ (18) \]
\[ L_{n,x} = 2K_n + uL_n + (g_n - 2\alpha_n), \]
\[ (19) \]
\[ K_{n,x} = \frac{(K_n + \frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}g_n^2}{L_n} - vL_n, \]
where \( K_n = (K_n, L_n)^T \), and where we have set \( 2c_n + d_n = g_n - 2\alpha_n \) and \( d_n^2 = \beta_n^2 \).
In the same way, since under the composition $H = F \circ \Phi$ we have analogously to (12)
\begin{equation}
B_2 \begin{vmatrix}
\mathbf{u} = H[\phi]
\end{vmatrix} = H'[\phi]C(H'[\phi])^\dagger
\end{equation}
with
\begin{equation}
C = \frac{1}{2} \begin{pmatrix}
-2 \partial_x & \partial_x \\
\partial_x & 0
\end{pmatrix},
\end{equation}
we obtain the integrated second modified hierarchy,
\begin{equation}
(H'[\phi])^\dagger K_n[H[\phi]] + (e_n, f_n)^T = 0.
\end{equation}
It is easy to see that the constants of integration in (15) and (22) are related by $c_n = e_n$ and $d_n = f_n - 2e_n$.

For our second sequence of Miura transformations we have the composition $I = F \circ \Psi$ and, corresponding to (20),
\begin{equation}
B_2 \begin{vmatrix}
\mathbf{u} = I[\psi]
\end{vmatrix} = I'[\psi]D(I'[\psi])^\dagger
\end{equation}
with
\begin{equation}
D = \frac{1}{2} \begin{pmatrix}
-2 \partial_x & -\partial_x \\
\partial_x & 0
\end{pmatrix}.
\end{equation}
Thus we obtain the alternative integrated second modified hierarchy,
\begin{equation}
(I'[\psi])^\dagger K_n[I[\psi]] + (l_n, m_n)^T = 0,
\end{equation}
with constants of integration related to those of (15) by $c_n = l_n$ and $d_n = -m_n - 2l_n$.

The hierarchies (15)–(19), (15), (22) and (25) are all $P_{IV}$ hierarchies. In order to show this, let us consider the case $n = 1$ of these hierarchies. We have $K_1 = v$ and $L_1 = u + g_1x$, and so our system (18)–(19) reads
\begin{equation}
\begin{align*}
u_x &= 2v + u(u + g_1x) - 2\alpha_1, \\
v_x &= \frac{(v + \frac{1}{2}g_1 - \alpha_1)^2 - \frac{1}{4}\beta_1^2}{u + g_1x} - v(u + g_1x);
\end{align*}
\end{equation}
eliminating $v$ and setting $u = \pm y - g_1x$ yields the fourth Painlevé equation,
\begin{equation}
y_{xx} = \frac{1}{2} y_x^2 + \frac{3}{2} y^3 + 2g_1xy^2 + 2 \left(\frac{1}{4}g_1^2 x^2 - \alpha_1\right) y - \frac{1}{2} \beta_1^2.
\end{equation}
The system (15) reads
\begin{equation}
\begin{align*}
V_x + 2UV - V^2 + g_1xV + c_1 &= 0, \\
U_x + 2UV - U^2 - g_1(U - 2V)x + g_1 - d_1 &= 0.
\end{align*}
\end{equation}
Elimination of $V$ and setting $U = \pm y - g_1x, 2c_1 + d_1 = g_1 - 2\alpha_1$ and $d_1^2 = \beta_1^2$ yields (28). This follows immediately from the fact that in the Miura map $U = u$. However, eliminating $U$ and setting $V = \pm w$ also yields the fourth Painlevé equation,
\begin{equation}
\begin{align*}
w_{xx} &= \frac{1}{2} \frac{w_x^2}{w} + \frac{3}{2} w^3 \pm 2g_1 xw^2 + 2 \left[\frac{1}{4}g_1^2 x^2 - \frac{1}{2}(c_1 + 2d_1 - g_1)\right] w - \frac{1}{2} \beta_1^2.
\end{align*}
\end{equation}
Thus for $n = 1$, both independent variables of the first modification define versions of $P_{IV}$. We will return to the relationship between these two copies of $P_{IV}$ later.
Our first second modification (22), for \( n = 1 \), reads
\[
\phi_x - 6\phi p - 6p^2 - \phi^2 - g_1 x(\phi + 2p) + g_1 - f_1 = 0;
\]
eliminating \( \phi \) yields
\[
\frac{p_{xx}}{2} = \frac{1}{2} p^2 + \frac{3}{2} p^3 + 2g_1 x p^2 + 2 \left[ \frac{1}{4} g_1^2 x^2 - \frac{1}{2} (2f_1 - 3c_1 - g_1) \right] p - \frac{1}{2} e_1^2,
\]
i.e. the fourth Painlevé equation. Noting that in the Miura map \( V = p \), we see that this last is equivalent to (31), for the upper choice of sign, with the identification \( w = p, c_1 = e_1 \) and \( d_1 = f_1 - 2e_1 \).

Our second second modification (26), for \( n = 1 \), reads
\[
s_x + 2\psi s - 3s^2 + g_1 x s - l_1 = 0,
\]
\[
\psi_x + 6\psi s - 6s^2 - \psi^2 - g_1 x(\psi - 2s) + g_1 + m_1 = 0,
\]
equivalent to (32), (33) under \( (\phi, p, e_1, f_1) \to (\psi, -s, l_1, -m_1) \). Elimination of \( \psi \) gives
\[
s_{xx} = \frac{1}{2} s_x^2 + \frac{3}{2} s^3 - 2g_1 x s^2 + 2 \left[ \frac{1}{4} g_1^2 x^2 + \frac{1}{2} (2m_1 + 3l_1 + g_1) \right] s - \frac{1}{2} l_1^2,
\]
another version of \( P_{IV} \) equivalent to (31) for the lower choice of sign, with \( w = s, c_1 = l_1 \) and \( d_1 = -m_1 - 2l_1 \).

Thus we see that the hierarchies (13), (14), (15), (22) and (25) define sequences of \( P_{IV} \) hierarchies, as in (11). In Section Four we will derive BTs for the hierarchies (22) and (25), and show in Section Six how the known structure of BTs for \( P_{IV} \) can be replicated for \( P_{II} \) hierarchies, using the Miura transformations given above.

Before turning to the derivation of BTs and special integrals for \( P_{IV} \) hierarchies, however, we present first of all an improved method of deriving BTs for hierarchies of ODEs. As a simple but illuminating example, we apply this to the \( P_{II} \) hierarchy.

3. Bäcklund transformations for the second Painlevé hierarchy

We take the \( P_{II} \) hierarchy in the form
\[
(\partial_x + 2Y) \left( M_n[Y_x - Y^2] - \frac{1}{2} x \right) + \frac{1}{2} x - \lambda_n = 0,
\]
where \( \lambda_n \) are arbitrary parameters, and the sequence \( M_n \) satisfies the Lenard recursion relation
\[
\partial_x M_{n+1}[W] = (\partial_x^3 + 4W \partial_x + 2W_x) M_n[W],
\]
with \( M_0 = 1/2, M_1[W] = W \). In order to construct a BT for this hierarchy, we consider adapting the approach developed by Weiss for PDEs (36), and seek a “truncated Painlevé expansion”
\[
Y = \frac{\varphi_x}{\varphi} + \tilde{Y},
\]
where
\[
\tilde{Y} = \frac{1}{2} \frac{\varphi_{xx}}{\varphi_x}.
\]
For \( Y \) defined by (39), we find that
\[
Y_x - Y^2 = \tilde{Y}_x - \tilde{Y}^2 - \frac{\varphi_{xx}}{\varphi} + 2 \frac{\varphi_x}{\varphi} \tilde{Y} = \tilde{Y}_x - \tilde{Y}^2,
\]
where in order to obtain the last equality we have used (40). That is, the quantity \( Y_x - Y^2 \) is invariant under the mapping (39), (40). Thus substituting (39) into \( 38 \) yields

\[
(42) \quad \left( \frac{\partial}{\partial x} + 2\tilde{Y} - \frac{\varphi_x}{\varphi} \right) \left( M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2} x \right) + \frac{1}{2} - \lambda_n = 0,
\]

Assuming now that \( \tilde{Y} \) also satisfies the corresponding member of the \( P_{II} \) hierarchy, but now for parameter value \( \tilde{\lambda}_n \), i.e.

\[
(43) \quad \left( \frac{\partial}{\partial x} + 2\tilde{Y} \right) \left( M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2} x \right) + \frac{1}{2} - \tilde{\lambda}_n = 0,
\]

we obtain using (42) and this last,

\[
(44) \quad \frac{\varphi_x}{\varphi} = \frac{\tilde{\lambda}_n - \lambda_n}{2M_n[\tilde{Y}_x - \tilde{Y}^2] - x}.
\]

But (44) must be compatible with (40), or equivalently with the Riccati equation

\[
(45) \quad \left( \frac{\varphi_x}{\varphi} \right)_x + \left( \frac{\varphi_x}{\varphi} \right)^2 - 2\tilde{Y} \left( \frac{\varphi_x}{\varphi} \right) = 0;
\]

substituting (44) in (45) gives

\[
(46) \quad \left( \frac{\partial}{\partial x} + 2\tilde{Y} \right) \left( M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2} x \right) + \frac{1}{2}(\lambda_n - \tilde{\lambda}_n) = 0,
\]

and so comparing with (43) we see that this compatibility requires

\[
(47) \quad \lambda_n + \tilde{\lambda}_n = 1.
\]

Thus we obtain Airault’s BT (44)

\[
Y = \tilde{Y} + \frac{\tilde{\lambda}_n - \lambda_n}{x - 2M_n[\tilde{Y}_x - \tilde{Y}^2]} \quad \lambda_n + \tilde{\lambda}_n = 1.
\]

for the \( P_{II} \) hierarchy, along with the shift in parameters (47). We note that this derivation, which does not make use of the Schwarzian derivative but instead relies on the invariance of the quantity \( Y_x - Y^2 \) under the mapping (39), (40), is much simpler, and is much more widely applicable, than that presented in [37].

Special integrals of the \( P_{II} \) hierarchy are obtained by setting coefficients of different powers of \( \varphi \) in (42) to zero independently; since \( \tilde{Y}_x - \tilde{Y}^2 = Y_x - Y^2 \) we see that this gives

\[
(49) \quad M_n[Y_x - Y^2] - \frac{1}{2} x = 0,
\]

which defines solutions of (38) for \( \lambda_n = 1/2 \). We refer to [37] for further information on special integrals of the \( P_{II} \) hierarchy, and the iteration of \( P_{II} \) hierarchy Btrees.

4. Bäcklund transformations for fourth Painlevé hierarchies

We now apply the above approach to our \( P_{IV} \) hierarchy (22); since

\[
H[\phi] = \begin{pmatrix} \phi + 2p \\ \phi p + p^2 + p_x \end{pmatrix},
\]
this reads
\[
\left( \frac{1}{2} \begin{pmatrix} \phi + 2p \\ \phi p + p^2 + p_x \end{pmatrix} \begin{pmatrix} \phi + 2p \\ \phi p + p^2 + p_x \end{pmatrix} \right) K_n \left( \begin{pmatrix} e_n \\ f_n \end{pmatrix} \right) = \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).
\]

We now seek, analogously to the case of the $\text{PII}$ hierarchy above, a mapping $(\text{BT})$ between two solutions $\phi, p$ and $\tilde{\phi}, \tilde{p}$ of our $\text{PIV}$ hierarchy, of the form
\[
\phi = 2\frac{\varphi_x}{\varphi} + \tilde{\phi},
\]
\[
p = -\frac{\varphi_x}{\varphi} + \tilde{p},
\]
where
\[
\tilde{\phi} = -\frac{\varphi_{xx}}{\varphi_x}.
\]

It then follows that
\[
\phi + 2p = \tilde{\phi} + 2\tilde{p}
\]
and
\[
\phi p + p^2 + p_x = \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x - \frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x}{\varphi} \tilde{\phi} = \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x,
\]
where the last equality follows from (54). Thus we see that the quantities $\phi + 2p$ and $\phi p + p^2 + p_x$ are invariant under the mapping (52), (53), (54). Substitution of (52), (53) into (51) therefore gives
\[
\left( \begin{pmatrix} 1 \\ \varphi \end{pmatrix} - \frac{\varphi_x}{\varphi} \begin{pmatrix} \tilde{\phi} \\ \tilde{p} \end{pmatrix} \right) K_n \left( \begin{pmatrix} \tilde{\phi} + 2\tilde{p} \\ \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x \end{pmatrix} \right) + \begin{pmatrix} e_n \\ f_n \end{pmatrix} = \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).
\]

Since we assume that $\tilde{\phi}, \tilde{p}$ are solutions of a second copy of our $\text{PIV}$ hierarchy, but with parameters $\tilde{e}_n, \tilde{f}_n$, i.e.
\[
\tilde{e}_n = e_n - \hat{e}_n, \quad \tilde{f}_n = \hat{f}_n,
\]
we obtain, by elimination between (57) and this last,
\[
\tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x = \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x - \frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x}{\varphi} \tilde{\phi} = \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x,
\]
where the last equality follows from (54). Thus we see that the quantities $\phi + 2p$ and $\phi p + p^2 + p_x$ are invariant under the mapping (52), (53), (54). Substitution of (52), (53) into (51) therefore gives
\[
\begin{pmatrix} \varphi_x \\ \varphi \end{pmatrix} = L_n \left[ \begin{pmatrix} \tilde{\phi} + 2\tilde{p} \\ \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x \end{pmatrix} \right],
\]
\[
\hat{f}_n = f_n.
\]

Equation (59) must be compatible with (54), or equivalently with the Riccati equation
\[
\begin{pmatrix} \varphi_x \\ \varphi \end{pmatrix} + \begin{pmatrix} \varphi_x \\ \varphi \end{pmatrix}^2 + \tilde{\phi} \begin{pmatrix} \varphi_x \\ \varphi \end{pmatrix} = 0;
\]

substituting (59) into (61) gives
\[
(\tilde{\phi} - \partial_x) L_n \left[ \begin{pmatrix} \tilde{\phi} + 2\tilde{p} \\ \tilde{\phi} \tilde{p} + \tilde{p}^2 + \tilde{p}_x \end{pmatrix} \right] + (e_n - \hat{e}_n) = 0,
\]
and comparing this last with (58) we see that we must have $\hat{e}_n = f_n - e_n$ and so
\[
\hat{e}_n = f_n - e_n.
\]
Thus we have for our $P_{IV}$ hierarchy (51) the BT

$$
\phi = \tilde{\phi} + 2 \frac{e_n - \tilde{e}_n}{L_n \left[ \frac{\phi + 2\tilde{p}}{\phi\tilde{p} + \tilde{p}^2 + \tilde{p}_x} \right]},
$$

(64)

$$
p = \tilde{p} - \frac{e_n - \tilde{e}_n}{L_n \left[ \frac{\phi + 2\tilde{p}}{\phi\tilde{p} + \tilde{p}^2 + \tilde{p}_x} \right]},
$$

(65)

along with the shifts in parameters given by (60) and (63).

We now consider deriving BTs for the $P_{IV}$ hierarchy (25); since

$$
I[\psi] = \begin{pmatrix}
-\psi - 2s \\
-\psi s + s^2 - s_x
\end{pmatrix},
$$

(66)

this reads

$$
\begin{pmatrix}
1 \\
-2
\end{pmatrix}
\begin{pmatrix}
-\dot{s} \\
-\dot{s} + 2s + \partial_x
\end{pmatrix}
K_n
\begin{pmatrix}
\psi - 2s \\
-\psi s + s^2 - s_x
\end{pmatrix}
+ \begin{pmatrix}
l_n \\
m_n
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

(67)

Seeking a BT in the form

$$
\psi = \frac{\varphi_x}{\varphi} + \hat{\psi},
$$

(68)

$$
s = \frac{\varphi_x}{\varphi} + \hat{s},
$$

(69)

where

$$
\hat{\psi} = -\frac{\varphi_{xx}}{\varphi},
$$

(70)

and where $\hat{\psi}, \hat{s}$ are solutions of our $P_{IV}$ hierarchy for parameter values $\hat{l}_n, \hat{m}_n$.

$$
\begin{pmatrix}
1 \\
-2
\end{pmatrix}
\begin{pmatrix}
-\hat{s} \\
-\hat{s} + 2\hat{s} + \partial_x
\end{pmatrix}
K_n
\begin{pmatrix}
\psi - 2\hat{s} \\
-\psi\hat{s} + \hat{s}^2 - \hat{s}_x
\end{pmatrix}
+ \begin{pmatrix}
\hat{l}_n \\
\hat{m}_n
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

(71)

then yields

$$
\psi = \hat{\psi} + 2 \frac{l_n - \hat{l}_n}{L_n \left[ \frac{\psi - 2\hat{s}}{-\psi\hat{s} + \hat{s}^2 - \hat{s}_x} \right]},
$$

(72)

$$
s = \hat{s} + \frac{l_n - \hat{l}_n}{L_n \left[ \frac{\psi - 2\hat{s}}{-\psi\hat{s} + \hat{s}^2 - \hat{s}_x} \right]},
$$

(73)

for the shift in parameter values

$$
\hat{m}_n = m_n,
$$

(74)

$$
\hat{l}_n = -m_n - l_n.
$$

(75)

We note that the BT (72–75) follows immediately from (60), (63), (64), (65) under $(\phi, p, e_n, f_n) \rightarrow (\psi, -s, \hat{l}_n, -\hat{m}_n)$, which maps the $P_{IV}$ hierarchy (61) into the $P_{IV}$ hierarchy (77). However, this mapping does not leave the $P_{IV}$ equation in standard form, since (34) is mapped to (37), and these two equations we identify with $P_{IV}$ by setting $g_1 = 2$ and $g_1 = -2$ respectively. We return to this point later.
We now briefly consider special integrals. We see that setting coefficients of different powers of $\phi$ in (57) to zero independently gives, using the fact that $\hat{\varphi} + 2\hat{p} = \varphi + 2p$ and $\hat{\varphi} + \hat{p}^2 + \hat{p}_x = \varphi p + p^2 + p_x$, (76) \[ L_n \left[ \begin{array}{c} \phi + 2p \\ \varphi p + p^2 + p_x \end{array} \right] = 0, \] which then defines solutions of (51) provided that (77) \[ K_n \left[ \begin{array}{c} \phi + 2p \\ \varphi p + p^2 + p_x \end{array} \right] + e_n = 0 \] and (80) \[ m_n = -2l_n. \]

In the same way, at the same point in the derivation of the BT (72)–(75), setting coefficients of different powers of $\psi$ to zero independently, and using the fact that $\hat{\psi} - 2\hat{s} = \psi - 2s$ and $-\hat{\psi} + \hat{s}^2 - \hat{s}_x = -\psi s + s^2 - s_x$, gives (79) \[ L_n \left[ \begin{array}{c} \psi - 2s \\ -\psi s + s^2 - s_x \end{array} \right] = 0, \] which then defines solutions of (81) provided that (80) \[ K_n \left[ \begin{array}{c} \psi - 2s \\ -\psi s + s^2 - s_x \end{array} \right] + l_n = 0 \] and (81) \[ m_n = -2l_n. \]

Again, just as for our BTs, we have the mapping $(\phi, p, e_n, f_n) \rightarrow (\psi, -s, l_n, -m_n)$ between these special integrals. See however the discussion in the next Section.

5. Identification of Bäcklund Transformations

We now turn to the identification of the BTs obtained in the previous section. We will give to BTs for our $P_{IV}$ hierarchies the same names as are given to the case $n = 1$, i.e. to the $P_{IV}$ equation itself. First of all we fix the identification of parameters in our hierarchies with the parameters $\alpha$ and $\beta$ in $P_{IV}$ when written as (82) \[ Q_{xx} = \frac{1}{2} \frac{Q_x^2}{Q} + \frac{3}{2} Q^3 + 4xQ^2 + 2 \left( x^2 - \alpha \right) Q - \frac{1}{2} \frac{\beta^2}{Q}. \]

We take this last as the standard form of $P_{IV}$ in order to simplify the writing of parameter shifts for its BTs. We note that, since $\beta \rightarrow -\beta$ is a discrete symmetry of (82), in BTs of $P_{IV}$ we can always replace parameters corresponding to $\beta$ by $\pm \beta$.

We begin with the hierarchy (83), or (84)

(83) \[ (H'(\phi))^T K_n [H(\phi)] + (e_n, f_n)^T = 0. \]

In the case $n = 1$ this gives the system (84), (85) and, after eliminating $\phi$, equation (86). In order to identify this last with equation (82) we now set, for the entire hierarchy (84), (85), (86)

(84) \[ g_n = 2, \]
(85) \[ e_n = B_n, \]
(86) \[ f_n = \frac{1}{2} (2A_n + 3B_n + 2), \]
and similarly for the parameters $\tilde{e}_n$ and $\tilde{f}_n$ in the hierarchy (83).

We then have for the hierarchy (83) the BT (64), (65), with the corresponding shift on parameters (60), (63), i.e.

\[
\begin{align*}
A_n &= -\frac{1}{4}(2A_n - 3B_n + 6), \\
B_n &= \frac{1}{2}(2A_n + B_n + 2).
\end{align*}
\]

In the case $n = 1$ this BT reads

\[
\begin{align*}
\phi &= \tilde{\phi} + \frac{B_1 - 2A_1 - 2}{\phi + 2\tilde{p} + 2x}, \\
p &= \tilde{p} - \frac{1}{2} \frac{B_1 - 2A_1 - 2}{\phi + 2\tilde{p} + 2x} .
\end{align*}
\]

Eliminating $\tilde{\phi}$ we obtain a BT for $P_{IV}$ itself,

\[
p = \tilde{p} + \frac{(B_1 - 2A_1 - 2)\tilde{p}}{p_x - \tilde{p}^2 - 2x\tilde{p} + B_1/2 + A_1 + 1}.
\]

This BT for $P_{IV}$, along with the parameter shift (87), (88) (for $n = 1$), is often referred to as the “double dagger” ($t^\dagger$) BT (see [38, 39]). For this reason we refer to the BT (64), (65), together with the parameter shifts (87), (88), as the $t^\dagger$ BT for the $P_{IV}$ hierarchy (83).

We now turn to the hierarchy (92), or (67),

\[
(I[\psi])^I K_n[I[\psi]] + (l_n, m_n)^T = 0,
\]

In the case $n = 1$ this gives the system (35), (36) and, after eliminating $\psi$, equation (37). In order to identify equation (37) with (82) we now set, for the entire hierarchy (92),

\[
\begin{align*}
g_n &= -2, \\
l_n &= b_n, \\
m_n &= -\frac{1}{2}(2a_n + 3b_n - 2),
\end{align*}
\]

and analogously for the parameters $\tilde{l}_n$ and $\tilde{m}_n$ in the hierarchy (71).

We have for the hierarchy (92) the BT (94), (95), with the corresponding shift on parameters (93), (94), i.e.

\[
\begin{align*}
\hat{a}_n &= -\frac{1}{4}(2a_n - 3b_n - 6), \\
\hat{b}_n &= \frac{1}{2}(2a_n + b_n - 2).
\end{align*}
\]

In the case $n = 1$ this BT reads

\[
\begin{align*}
\psi &= \hat{\psi} + \frac{b_1 - 2a_1 + 2}{\psi - 2\hat{s} - 2x}, \\
s &= \hat{s} + \frac{1}{2} \frac{b_1 - 2a_1 + 2}{\psi - 2\hat{s} - 2x} ,
\end{align*}
\]

and eliminating $\hat{\psi}$ we obtain a BT for $P_{IV}$ (71) itself,

\[
s = \hat{s} + \frac{(2a_1 - b_1 - 2)\hat{s}}{\hat{s}_x + \hat{s}^2 + 2x\hat{s} - b_1/2 - a_1 + 1}.
\]
This BT for $P_{IV}$, along with the parameter shift \((96), (97)\) (for \(n = 1\)), is often referred to as the “dagger” \((\tau^\dagger)\) BT (see \([38, 39]\)); it is for this reason that we refer to the BT \((72), (73)\) together with the parameter shifts \((96), (97)\), as the $\tau^\dagger$ BT for the $P_{IV}$ hierarchy \((92)\). Here we use the letter “\(t\)” (e.g. $t^\ddagger$) for BTs related to the hierarchy \((83)\), and “\(\tau\)” (e.g. $\tau^\dagger$) for BTs related to the hierarchy \((92)\).

Finally we recall that we also have, as detailed in Section Four, special integrals for our $P_{IV}$ hierarchies. Thus we have the special integral system \((76) - (77)\), with parameters satisfying \((78)\), for the hierarchy \((83)\), where we now impose the identification \((84) - (86)\). Similarly we have the special integral system \((79) - (80)\), with parameters satisfying \((81)\), for the hierarchy \((92)\), now imposing \((93) - (95)\).

In the case \(n = 1\), with the identification \((84) - (86)\), our special integral system \((76) - (77)\) for the hierarchy \((83)\) reads

\[
\phi + 2p + 2x = 0, \\
p_x + \phi p + p^2 + B_1 = 0,
\]

for parameters satisfying \((78)\), i.e.

\[
B_1 = 2A_1 + 2.
\]

Thus we obtain the special integral of $P_{IV}$ \((81)\),

\[
p_x - p^2 - 2xp + B_1 = 0,
\]

where the parameters $A_1$ and $B_1$ of $P_{IV}$ satisfy \((103)\).

On the other hand, the special integral system \((79) - (80)\) for the hierarchy \((92)\), with the identification \((81) - (85)\), reads for \(n = 1\),

\[
\psi - 2s - 2x = 0, \\
s_x + \psi s - s^2 - b_1 = 0,
\]

with parameters satisfying \((81)\), i.e.

\[
b_1 = 2a_1 - 2.
\]

Eliminating $\psi$ then gives the special integral of $P_{IV}$ \((87)\),

\[
s_x + s^2 + 2xs - b_1 = 0,
\]

for parameters $a_1$ and $b_1$ of $P_{IV}$ satisfying \((107)\).

We note that the identifications of parameters \((81) - (86)\) and \((83) - (85)\) mean that we no longer have the simple mapping \((\phi, p, e_n, f_n) \rightarrow (\psi, -s, l_n, -m_n)\) between the hierarchies \((83)\) and \((92)\); consider for example the systems obtained for \(n = 1\), \((32), (33)\) and \((35), (36)\). Thus the BTs and special integrals obtained here are no longer equivalent under this mapping. The question of whether a mapping can be found under which they are equivalent is discussed in Section Seven.

6. Further Bäcklund Transformations for Our $P_{IV}$ Hierarchies

Thus far we have found the BTs $t^\ddagger$ and $\tau^\dagger$ for our $P_{IV}$ hierarchies. However, as is well known, for $P_{IV}$ itself, these BTs can be written as compositions of other BTs, referred to in the literature \([38, 39]\) as the “tilde” \((\tilde{t}/\tilde{\tau})\) and “hat” \((\hat{t}/\hat{\tau})\) BTs. We now show how this pattern of BTs for $P_{IV}$ can be extended to our $P_{IV}$ hierarchies.

It turns out, quite remarkably, that this can be done by considering the Miura maps between \((83)\) and \((15)\), and \((92)\) and \((15)\), as given in \((11)\). Let us begin with the Miura transformation between \((83)\) and \((16)\), as given by \((19)\). We recall that
for $n = 1$ yields equation (28). We take the lower sign in (28) and now fix the relationship between our parameters $c_n, d_n$ and $\alpha_n, \beta_n$, for the entire hierarchy (15), as

$$g_n = 2,$$

$$c_n = \frac{1}{2}(2 - 2\alpha_n - \beta_n),$$

$$d_n = \beta_n,$$

and similarly for a second copy of our hierarchy (15) in $\tilde{U}, \tilde{V}$ with parameters $\tilde{c}_n, \tilde{d}_n$, or equivalently $\tilde{\alpha}_n, \tilde{\beta}_n$.

Since we have

$$c_n = e_n,$$

$$d_n = f_n - 2e_n$$

and similarly for parameters $\tilde{c}_n, \tilde{d}_n, \tilde{e}_n,$ and $\tilde{f}_n$, we obtain the following BTs and parameter shifts:

$$\tilde{U} = \tilde{\phi} + 2\tilde{p},$$

$$\tilde{V} = \tilde{p},$$

with

$$\tilde{A}_n = -\frac{1}{4}(2 + 2\tilde{\alpha}_n - 3\tilde{\beta}_n),$$

$$\tilde{B}_n = \frac{1}{2}(2 - 2\tilde{\alpha}_n - \tilde{\beta}_n);$$

and

$$\phi = U - 2V,$$

$$p = V,$$

with

$$\alpha_n = \frac{1}{4}(2 - 2A_n - 3B_n),$$

$$\beta_n = \frac{1}{2}(2 + 2A_n - B_n).$$

For the case $n = 1$, the first of these, when written as a BT between two copies of $P_{IV}$ in $\tilde{p}, \tilde{A}_1, \tilde{B}_1$, and $\tilde{B}_1$ in $\tilde{y}, \tilde{\alpha}_1, \tilde{\beta}_1$, where $\tilde{y} = -\tilde{U} - 2x$ — reads

$$\tilde{y} = \frac{\tilde{p}_x - \tilde{p}^2 - 2x\tilde{p} + 1 - \tilde{\alpha}_1 - \tilde{\beta}_1/2}{2\tilde{p}},$$

which, together with the parameter shifts (116), (117) defines precisely the BT $\tilde{t}$.

The second of the above BTs, in the case $n = 1$, when written as a BT between (34) in $p, A_1, B_1$ and (28) in $y, \alpha_1, \beta_1$, where $y = -U - 2x$, reads

$$p = -y_x + y^2 + 2xy + 1 + A_1 - B_1/2$$

which, together with the parameter shifts (120), (121) defines precisely the BT $\hat{t}$.

We thus define the BTs (114), (115) and (118), (119) respectively, as $\tilde{t}$ and $\hat{t}$ BTs for our $P_{IV}$ hierarchies.
We now define the additional BT $S$ by $S = (\hat{t})^{-1} \circ t^i \circ (\hat{t})^{-1}$. A simple calculation gives this BT $S$ as

$$U = \hat{U},$$

$$V = \hat{V} + \frac{d_n}{L_n \left[ \hat{U} \hat{V} - \hat{V}^2 + \hat{V}x \right]},$$

with the change of parameters

$$\tilde{c}_n = c_n + d_n,$$

$$\tilde{d}_n = -d_n,$$

or equivalently

$$\tilde{\alpha}_n = \alpha_n,$$

$$\tilde{\beta}_n = -\beta_n.$$

For the case $n = 1$, for equation (28), this BT $S$ reads

$$y = \tilde{y}, \quad \tilde{\alpha}_1 = \alpha_1, \quad \tilde{\beta}_1 = -\beta_1,$$

and we recover the well known relation $t^i = \hat{t} \circ S \circ \hat{t}$ for $P_{IV}$ BTs. Our BT $S$ (124) then allows us to extend this decomposition of the BT $t^i$ as $t^i = \hat{t} \circ S \circ \hat{t}$ from $P_{IV}$ itself to our $P_{IV}$ hierarchies. This pattern of BTs, obtained here using the Miura map $U = \Phi[\phi]$ of the DWW hierarchy (which defines the BTs $\hat{t}$ and $\tilde{t}$), can be seen in Figure One. It is interesting that this Miura map, a simple linear map when considered as a mapping between $\phi$ and $U$, gives rise to BTs of our hierarchies: however, as we have seen above for $n = 1$ ($P_{IV}$), when considered as a mapping between components of our hierarchies, it is no longer a linear map.

We recall that for $n = 1$ the second component $V$ of the system (14) also defines a copy of $P_{IV}$ (31). Using the identification (84)–(86), where as usual $c_1 = e_1$ and $d_1 = f_1 - 2e_1$, we obtain that equation (31), with the upper choice of sign, is a copy of equation (34), with $w = p$ and the same parameters $A_1$ and $B_1$. Thus, in Figure One, when tracing for $n = 1$ the action of our BTs over individual components, we see that the auto-BT for (34) must be the same as the auto-BT for equation (31).

This last is as given by (125), and reads (with $V = w$, $\tilde{V} = \tilde{w}$, and eliminating $\tilde{U}$)

$$w = \tilde{w} + \frac{(B_1 - 2A_1 - 2)\tilde{w}}{\tilde{w} - \tilde{w}^2 - 2x\tilde{w} + B_1/2 + A_1 + 1},$$

with parameter shifts

$$\tilde{A}_1 = -\frac{1}{4}(2A_1 - 3B_1 + 6),$$

$$\tilde{B}_1 = \frac{1}{2}(2A_1 + B_1 + 2).$$

Thus we see that this BT is exactly the same as that for (34), i.e. (111) and (57), (58), with $n = 1$. That is, for $n = 1$, (125) is the $t^i$ BT (from $\tilde{w}$ to $w$).

From the above it also follows that the equation obtained when eliminating $\tilde{U}$ from the system (29), (30) written in terms of $U$, $\tilde{V}$, $\tilde{c}_1$, $\tilde{d}_1$, i.e.

$$\tilde{U} = -\frac{\tilde{V}_x - \tilde{V}_2^2 + 2x\tilde{V} + \tilde{c}_1}{2\tilde{V}},$$
corresponds to the $\hat{t}$ BT from (31), with upper sign, to (28), with lower sign. In the same way, the equation obtained when eliminating $V$ from the system (29), (30), (135)

\[ V = -\frac{U_x - U^2 - 2xU + 2 - d_1}{2U + 4x}, \]

corresponds to the $\hat{t}$ BT from (28), with lower sign, to (31), with upper sign. This then gives one identification of equations (134) and (135) (another is made later).

We now turn to our $\tau^{\dagger}$ BT. We recall once again the Miura maps (11) and in particular the Miura transformation between (92) and (15), as given by (10). For $n = 1$ (15) yields equation (28); we take the upper sign in (28) and change the relationship between our parameters (now labelled $\bar{c}_n, \bar{d}_n$ and $\bar{\alpha}_n, \bar{\beta}_n$, corresponding to variables $\bar{U}, \bar{V}$), to the following, again for the entire hierarchy (15):

(136) \[ g_n = -2, \]

(137) \[ \bar{c}_n = -\frac{1}{2}(2 + 2\bar{\alpha}_n + \bar{\beta}_n), \]

(138) \[ \bar{d}_n = \bar{\beta}_n, \]

and similarly for a second copy of our hierarchy (15) in $\hat{U}, \hat{V}$ with parameters $\hat{c}_n, \hat{d}_n$, or equivalently $\hat{\alpha}_n, \hat{\beta}_n$.

Since we have

(139) \[ \bar{c}_n = l_n, \]

(140) \[ \bar{d}_n = -m_n - 2l_n \]

and similarly for parameters $\hat{c}_n, \hat{d}_n, \hat{l}_n, \hat{m}_n$, we obtain the following BTs and parameter shifts:

(141) \[ \hat{U} = \hat{\psi} - 2\hat{s}, \]

(142) \[ \hat{V} = -\hat{s}, \]

with

(143) \[ \hat{a}_n = \frac{1}{4}(2 - 2\hat{\alpha}_n + 3\hat{\beta}_n), \]

(144) \[ \hat{b}_n = \frac{1}{2}(2 + 2\hat{\alpha}_n + \hat{\beta}_n); \]

and

(145) \[ \psi = \bar{U} - 2\bar{V}, \]

(146) \[ s = -\bar{V}, \]

with

(147) \[ \bar{\alpha}_n = -\frac{1}{4}(2 + 2a_n + 3b_n), \]

(148) \[ \bar{\beta}_n = -\frac{1}{2}(2 - 2a_n + b_n). \]

For the case $n = 1$, the first of these, when written as a BT between two copies of $P_{1V} - \hat{\delta}$ in $\hat{s}, \hat{a}_1, \hat{b}_1$, and (28) in $\hat{y}, \hat{\alpha}_1, \hat{\beta}_1$, where $\hat{y} = \bar{U} - 2x$ — reads

(149) \[ \hat{\psi} = -\frac{s_x + s^2 + 2x\dot{s} + 1 + \dot{\hat{\alpha}}_1 + \dot{\hat{\beta}}_1/2}{2\hat{s}}, \]
which, together with the parameter shifts (143), (144) defines the BT $\hat{\tau}$ (we recall, when comparing to the BT (123) with parameter shifts, (120), (121) for $n = 1$ — also identified as a “hat” BT, $\hat{\tau}$ — the invariance of $P_{IV}$ (82) under $\beta \to -\beta$).

The second of the above BTs, in the case $n = 1$, when written as a BT between (37) in $s$, $a_1$, $b_1$ and (28) in $\bar{y}$, $\bar{\alpha}_1$ and $\bar{\beta}_1$, where $\bar{y} = \bar{U} - 2x$, reads

\[ s = \frac{\bar{y} - \bar{y}^2 - 2x\bar{y} + 1 - a_1 + b_1/2}{2\bar{y}} \]

(150)

which, together with the parameter shifts (147), (148) defines the BT $\tilde{\tau}$ (again we recall, when comparing to (122) with parameter shifts, (116), (117) for $n = 1$ — also identified as a “tilde” BT, $\tilde{\tau}$ — the invariance of $P_{IV}$ (82) under $\beta \to -\beta$).

We thus define the BTs (141), (142) and (145), (146), with parameter shifts (143), (144) and (147), (148) respectively, as $\hat{\tau}$ and $\tilde{\tau}$ BTs for our $P_{IV}$ hierarchies.

We now define the additional BT $\sigma$ by

\[ \sigma = (\tilde{\tau})^{-1} \circ \tau^\dagger \circ (\hat{\tau})^{-1} \]

A simple calculation gives this BT $\sigma$ as

\[ \bar{U} = \hat{U}, \]

\[ \bar{V} = \hat{V} + \frac{\tilde{d}_n}{L_n} \left[ \left( \frac{\hat{U}}{\hat{V} - \beta^2 + \hat{V}_x} \right) \right], \]

(152)

with the change of parameters

\[ \hat{c}_n = \bar{c}_n + \tilde{d}_n, \]

(153)

\[ \hat{d}_n = -\tilde{d}_n, \]

(154)

or equivalently

\[ \hat{\alpha}_n = \bar{\alpha}_n, \]

(155)

\[ \hat{\beta}_n = -\bar{\beta}_n. \]

(156)

For the case $n = 1$, for equation (28), this BT $\sigma$ reads

\[ \bar{y} = \hat{y}, \quad \bar{\alpha}_1 = \hat{\alpha}_1, \quad \bar{\beta}_1 = -\hat{\beta}_1, \]

(157)

and we recover the well known relation $\tau^\dagger = \hat{\tau} \circ \sigma \circ \tilde{\tau}$ for $P_{IV}$ BTs. Our BT $\sigma$ (151) — (156) then allows us to extend this decomposition of the BT $\tau^\dagger$ as $\tau^\dagger = \hat{\tau} \circ \sigma \circ \tilde{\tau}$ from $P_{IV}$ itself to our $P_{IV}$ hierarchies. This pattern of BTs, obtained here using the Miura map $U = \Psi[\psi]$ of the DWW hierarchy, can be seen in Figure Two. Again we note that what is a simple linear map, when considered as a mapping between $\psi$ and $U$, gives rise to BTs of our hierarchies (the BTs $\hat{\tau}$ and $\tilde{\tau}$), but that, once again, and as we have seen for $n = 1$ ($P_{IV}$), when considered as a mapping between components of our hierarchies, it is no longer such a trivial mapping.

We recall that for $n = 1$ the second component $\bar{V}$ of the system (18), now written in terms of $\bar{U}$, $\bar{V}$ and with coefficients $\bar{c}_1, \bar{d}_1$, also defines a copy of $P_{IV}$ via $\bar{V} = -\bar{w}$. This copy of $P_{IV}$ is (31) in $\bar{w}$, $\bar{c}_1$ and $\bar{d}_1$, and with lower choice of sign. We thus obtain that this copy of (31), where as usual $\bar{c}_1 = l_1$ and $\bar{d}_1 = -m_1 - 2l_1$, and using the identification (88) — (95), is a copy of equation (37), with $w = s$ and the same parameters $a_1$ and $b_1$. Thus, in Figure Two, when tracing for $n = 1$ the action of our BTs over individual components, we see that the auto-BT for (37) must be the
same as the auto-BT for this copy of equation (31). This last is as given by (152), and reads (with $\tilde{V} = -\tilde{w}$, $\hat{V} = -\hat{w}$, and eliminating $\hat{U}$),

$$\tilde{w} = \hat{w} + \frac{(2a_1 - b_1 - 2)\hat{w}}{\tilde{w} + \hat{w}^2 + 2x\hat{w} - b_1/2 - a_1 + 1},$$

with parameter shifts

$$\hat{a}_1 = -\frac{1}{4}(2a_1 - 3b_1 - 6),$$

$$\hat{b}_1 = \frac{1}{2}(2a_1 + b_1 - 2).$$

Thus we see that this BT is exactly the same as that for (37), i.e. (100) and (96), (97), with $n = 1$. That is, for $n = 1$, (152) is the $\tau^\dagger$ BT (from $\hat{w}$ to $\tilde{w}$).

From the above it also follows that the equation obtained when eliminating $\hat{U}$ from the system (29), (30), written in terms of $\hat{U}, \hat{V}, \hat{c}_1, \hat{d}_1$, i.e.

$$\hat{U} = -\frac{\hat{V}_x - \hat{V}^2 + 2x\hat{V} + \hat{c}_1}{2\hat{V}},$$

corresponds to the $\tilde{\tau}$ BT from (31), with lower sign, to (28), with upper sign. In the same way, the equation obtained when eliminating $\tilde{V}$ in the system (29), (30), written in terms of $\tilde{U}, \tilde{V}, \tilde{c}_1, \tilde{d}_1$, i.e.

$$\tilde{V} = -\frac{\tilde{U}_x - \tilde{U}^2 - 2x\tilde{U} + 2 - \tilde{d}_1}{2\tilde{U} + 4x},$$

corresponds to the $\hat{\tau}$ BT from (28), with upper sign, to (31), with lower sign. Thus we have a different identification of the equations (134) and (135).

We note that the transformation induced on our original variables $u$ and $v$ by $S$ and $\sigma$ is just the identity together with $(\alpha_n, \beta_n) \rightarrow (\alpha_n, -\beta_n)$ (a discrete symmetry of the hierarchy (18), (19)).

7. A MAPPING BETWEEN OUR SEQUENCES OF $P_{IV}$ HIERARCHIES

In this section we consider a mapping between our two different sequences of fourth Painlevé hierarchies. These two different sequences are defined by the choice $g_n = 2$ or $g_n = -2$. As we noted earlier, this then means that we no longer have the mapping $(\phi, p, e_n, f_n) \rightarrow (\psi, -s, l_n, -m_n)$ between the hierarchies (22) and (25). However, we still have another mapping between these two hierarchies.

Consider the hierarchy (22), with the identifications (84)—(86), i.e.

$$(H'[\phi])^T K_n[H[\phi]] + (e_n, f_n)^T = 0,$$

where $(g_n = 2)$

$$K_n[H[\phi]] = L_n[H[\phi]] + \sum_{i=1}^{n-1} h_i L_i[H[\phi]] + 2 \left( \begin{array}{c} 0 \\ x \end{array} \right).$$

We now consider the scaling transformation $(\phi, p, x) = (\lambda^{-1}\psi, -\lambda^{-1}s, \lambda x)$. This then gives

$$L_i[H[\phi]] = \left( \begin{array}{cc} \frac{1}{\lambda^{i+1}} & 0 \\ 0 & \frac{1}{\lambda^i} \end{array} \right) L_i[H[\psi]],$$

where $i = 1, 2, \ldots, n-1$.
where in the right-hand-side of this last, derivatives are w.r.t. $\xi$. Thus

$$K_n[H[\phi]] = \left( \frac{1}{\lambda} 0 \right) \frac{1}{\lambda^n} \left[ L_n[H[\psi]] + \sum_{i=1}^{n-1} h_i \lambda^{n-i} L_i[H[\psi]] + 2 \lambda^{n+1} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right]$$

(166)

$$= \left( \frac{1}{\lambda} 0 \right) \frac{1}{\lambda^n} \left[ L_n[H[\psi]] + \sum_{i=1}^{n-1} H_i L_i[H[\psi]] - 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right],$$

where we have chosen $\lambda$ such that $\lambda^{n+1} = -1$, and have set $h_i \lambda^{n-i} = H_i$. Since also

$$H[\phi] = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \frac{1}{\lambda^n} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (H[\psi])^\dagger \left( \begin{array}{c} 1 \\ 0 \end{array} \right),$$

our hierarchy (22), i.e. (163), becomes

$$\left( \begin{array}{c} 0 \\ 0 \end{array} \right) \frac{1}{\lambda^n} \left[ L_n[H[\psi]] + \sum_{i=1}^{n-1} H_i L_i[H[\psi]] - 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right],$$

(167)

which, identifying $e_n = -l_n$ and $f_n = m_n$, is precisely the hierarchy (24), i.e.

$$\left( \begin{array}{c} 0 \\ 0 \end{array} \right) \frac{1}{\lambda^n} \left[ L_n[H[\psi]] + \sum_{i=1}^{n-1} H_i L_i[H[\psi]] - 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right] + \left( \begin{array}{c} 0 \\ 0 \end{array} \right),$$

(168)

with the identification (23) $- 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ ($g_n = -2$). Thus we have a BT from the hierarchy (24) with $g_n = -2$ to the hierarchy (24) with $g_n = 2$, given by

$$\begin{align*}
\phi, p, x &= (\lambda^{-1} \psi, -\lambda^{-1} s, \lambda \xi), \\
l_n &= -e_n, \\
m_n &= f_n, \\
H_i &= h_i \lambda^{n-i},
\end{align*}$$

(169)

where $\lambda^{n+1} = -1$. We refer to this BT as the transformation $T$. We now consider the composition $T^{-1} \circ T^\dagger \circ T$, i.e. $(\psi, \hat{s}, \hat{l}_n, \hat{m}_n) \to (\tilde{\phi}, \tilde{p}, \tilde{e}_n, \tilde{f}_n) \to (\phi, p, e_n, f_n) \to (\psi, s, l_n, m_n)$, which gives an auto-BT of the hierarchy (24). It turns out that this

$$\tau^\dagger = T^{-1} \circ T^\dagger \circ T,$$

(170)

We note that $T$ also maps the first integrals (24), (30) of the hierarchy (24), with the identifications (23) $- 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ ($g_n = -2$) to the first integrals (76), (77) of the hierarchy (24), with the identifications (24) $- 2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ ($g_n = 2$).

We now consider a corresponding scaling transformation between our hierarchy (15) for $g_n = 2$, and the same hierarchy for $g_n = -2$. This scaling transformation is induced from that between the hierarchies (24) and (26) as $(U, V, x) = (\lambda^{-1} U, \lambda^{-1} V, \lambda \xi)$. The hierarchy (15), i.e.

$$\begin{align*}
(F'U)^\dagger K_n[F[U]] + (c_n, d_n)^T &= 0, \\
K_n[F[U]] &= L_n[F[U]] + \sum_{i=1}^{n-1} h_i L_i[F[U]] + 2 \left( \begin{array}{c} 0 \\ x \end{array} \right),
\end{align*}$$

(175)

then becomes (again all derivatives are now with respect to $\xi$)

$$\begin{align*}
(F'\bar{U})^\dagger K_n[F[\bar{U}]] + (\bar{e}_n, \bar{d_n})^T &= 0,
\end{align*}$$

(177)
with

\begin{equation}
K_n[F[U]] = L_n[F[U]] + \sum_{i=1}^{n-1} H_i L_i[F[U]] - 2 \begin{pmatrix} 0 \\
\xi \end{pmatrix},
\end{equation}

and where we have identified \( c_n = -\tilde{c}_n \) and \( d_n = -\tilde{d}_n \). That is, we have the BT from \( 177, 178 \) to \( 179, 180 \).

\begin{align*}
\lambda^{-1} \tilde{U}, \lambda^{-1} \tilde{V}, \lambda \xi, \\
\tilde{c}_n = -c_n, \\
\tilde{d}_n = -d_n, \\
H_i = h_i \lambda^{n-i},
\end{align*}

with \( \lambda^{n+1} = -1 \), which, by a convenient abuse of notation, we also refer to as the transformation \( T \).

We now consider the composition \( T^{-1} \circ \tilde{t} \circ T \), i.e. \((\tilde{\psi}, \tilde{s}, \tilde{\lambda}, \tilde{m}) \rightarrow (\hat{\phi}, \hat{p}, \hat{c}_n, \hat{f}_n) \rightarrow (\tilde{U}, \tilde{V}, \tilde{c}_n, \tilde{d}_n) \rightarrow (\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n) \). This BT turns out to be precisely the BT \( \hat{\tau} \). Thus we have the relation

\begin{equation}
\hat{\tau} = T^{-1} \circ \tilde{t} \circ T.
\end{equation}

We also consider the composition \( T^{-1} \circ \hat{i} \circ T \), i.e. \((\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n) \rightarrow (U, V, c_n, d_n) \rightarrow (\phi, p, c_n, f_n) \rightarrow (\psi, s, l_n, m_n) \). This BT is \( \hat{\tau} \), and so we have

\begin{equation}
\hat{\tau} = T^{-1} \circ \hat{i} \circ T.
\end{equation}

Finally, consideration of the BT \( T^{-1} \circ S \circ T \), i.e. \((\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n) \rightarrow (\tilde{U}, \tilde{V}, \tilde{c}_n, \tilde{d}_n) \rightarrow (U, V, c_n, d_n) \rightarrow (\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n) \), leads to the conclusion

\begin{equation}
\sigma = T^{-1} \circ S \circ T.
\end{equation}

Thus we see that our transformation \( T \) is a mapping of Figure Two into Figure One, but for a different independent variable, \( x \) in Figure One being related to \( \xi \) in figure Two by \( x = \lambda \xi \) where \( \lambda^{n+1} = -1 \). Thus of course the relation \( \tau^1 = \hat{\tau} \circ \sigma \circ \hat{\tau} \) (Figure Two) is mapped into the relation \( t^1 = \tilde{t} \circ S \circ \tilde{t} \) (Figure One).

Our transformation \( T \) has some important consequences. It tells us that the pattern of BTs obtained from our second sequence of Painlevé hierarchies can be related to that obtained from our first sequence of Painlevé hierarchies. If we had only considered one sequence (e.g. the first) it might not have been obvious how to obtain a sequence (the second) having the pattern of BTs corresponding to \( \tau^1 \).

The reason why this might not have been obvious is that, for \( P_{IV} \), the BTs “tilde” and “hat” are believed to be independent. This then leads us on to another of the important consequences of our results: the BTs “tilde” and “hat” for \( P_{IV} \) are not independent, but are related by a trivial scaling of \( P_{IV} \). That is, there is only one nontrivial fundamental BT for \( P_{IV} \). This is in contrast to the claim in \( 178 \) that \( P_{IV} \) has two nontrivial fundamental BTs (“tilde” and “hat”).

Let us present our results for \( P_{IV} \) explicitly. For \( n = 1 \) we may take \( \lambda = i \) and so our transformation \( T \) from \( U \) with the identification \( 179, 180 \),

\begin{equation}
s_{e\xi} = \frac{1}{2} \frac{s^2}{s} + \frac{3}{2} s^3 + 4 \xi s^2 + 2 \left[ \xi^2 - a_1 \right] s - \frac{1}{2} b^2 \frac{s}{s},
\end{equation}

and
to (84) with the identification (85)—(86),

\[(187) \quad p_{xx} = \frac{1}{2} p_x^2 + \frac{3}{2} p^2 + 4 xp^2 + 2 [x^2 - A_1] p - \frac{1}{2} \frac{B_1^2}{p},\]

is

\[(188) \quad p = is, \quad x = i\xi, \quad a_1 = -A_1, \quad b_1 = -B_1.\]

The same transformation \(T\) provides a BT from (28) in \(\bar{y}\) and \(\xi\), with upper choice of sign and the identification (136)—(138),

\[(189) \quad \bar{y}_{\xi\xi} = \frac{1}{2} \bar{y}_x^2 + \frac{3}{2} \bar{y}^3 + 4 \xi \bar{y}^2 + 2 (\xi^2 - \bar{\alpha}_1) \bar{y} - \frac{1}{2} \frac{\bar{\beta}_1^2}{\bar{y}},\]

to (28) in \(y\) and \(x\), with lower choice of sign and the identification (109)—(111),

\[(190) \quad y_{xx} = \frac{1}{2} y_x^2 + \frac{3}{2} y^3 + 4 xy^2 + 2 (x^2 - \alpha_1) y - \frac{1}{2} \frac{\beta_1^2}{y},\]

i.e.

\[(191) \quad y = i\bar{y}, \quad x = i\xi, \quad \bar{\alpha}_1 = -\alpha_1, \quad \bar{\beta}_1 = -\beta_1.\]

In order to show explicitly that \(P_{IV}\) has only one fundamental BT it is enough to show that (184) holds, i.e. that

\[(192) \quad \tilde{\tau} = T^{-1} \circ \hat{t} \circ T.\]

Here \(\hat{t}\) is the BT (123), with parameter shifts (120) and (121), i.e.

\[(193) \quad p = -\frac{y_x + y^2 + 2xy + 1 + A_1 - B_1/2}{2y},\]

and

\[(194) \quad \alpha_1 = \frac{1}{4} (2 - 2A_1 - 3B_1),\]

\[(195) \quad \beta_1 = \frac{1}{2} (2 + 2A_1 - B_1),\]

from (140) to (187). Meanwhile, \(\tilde{\tau}\) is the BT (150), with parameter shifts (147), (148), i.e.

\[(196) \quad s = \frac{\bar{y}_x - \bar{y}^2 - 2\xi \bar{y} + 1 - a_1 + b_1/2}{2\bar{y}},\]

\[(197) \quad \bar{\alpha}_1 = -\frac{1}{4} (2 + 2a_1 + 3b_1),\]

\[(198) \quad \bar{\beta}_1 = -\frac{1}{2} (2 - 2a_1 + b_1),\]

from (189) to (186). It is easy to show that under the transformation \(T\), i.e. when (188) and (191) hold, equations (190)—(193) are mapped onto equations (196)—(199). Thus the “tilde” and “hat” BTs of \(P_{IV}\) are equivalent under a simple scaling transformation, and we see that \(P_{IV}\) has only one nontrivial fundamental auto-BT.

We note that for \(P_{IV}\) itself the transformation \(T\) can in fact be found in [40], and was also known to the authors of [38]. However these last failed to recognize that it provides a mapping between the “tilde” and “hat” BTs of \(P_{IV}\).
8. Conclusions

We have given an improved method of obtaining auto-BTs and special integrals for hierarchies of ODEs, and have used this to derive auto-BTs and special integrals for two fourth Painlevé hierarchies. We have shown how the known pattern of BTs for \( P_{IV} \) can be extended to hierarchies, observing that the BTs required to do this turn out to be precisely the Miura maps of the DWW hierarchy. Finally, we have given a mapping between our two sequences of fourth Painlevé hierarchies which allows us to relate the BTs derived for these two sequences: in particular, we have derived the result that \( P_{IV} \) has in fact only one nontrivial fundamental auto-BT.

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\[
\begin{align*}
\left( c_n \atop d_n \right) & \sim \left( \alpha_n \atop \beta_n \right) & \left( \tilde{c}_n \atop \tilde{d}_n \right) & \sim \left( \tilde{\alpha}_n \atop \tilde{\beta}_n \right) \\
\begin{pmatrix} U \\ V \end{pmatrix} & \sim s & \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} \\
\end{align*}
\]

\[t^\dagger \]

Figure One: Decomposition of the BT \(t^\dagger\) for \(P_{IV}\) hierarchies (\(g_n = 2\)).

\[
\begin{align*}
\left( \phi \atop p \right) & \sim \left( \Phi \atop \tilde{p} \right) & \left( \tilde{\phi} \atop \tilde{\Phi} \right) \\
\left( e_n \atop f_n \right) & \sim \left( A_n \atop B_n \right) & \left( \tilde{e}_n \atop \tilde{f}_n \right) & \sim \left( \tilde{A}_n \atop \tilde{B}_n \right) \\
\end{align*}
\]

\[\tau^\dagger \]

Figure Two: Decomposition of the BT \(\tau^\dagger\) for \(P_{IV}\) hierarchies (\(g_n = -2\)).