Maxwell’s Conjecture of the Demon creating a Temperature Difference is False

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Abstract

We argue that Maxwell’s demon is incapable of creating a nonzero temperature difference. Hence, it does not destroy equilibrium and the second law is never at risk, contrary to the claim by Maxwell and accepted by many. It is therefore remarkable that despite this, the demon paradox has been a valuable source of new ideas. We use two independent arguments, one using classical equilibrium thermodynamics by extending Brillouin’s approach, and the other one using equilibrium statistical mechanics and the central limit theorem.

Maxwell’s demon, which has been puzzling scientists since 1867, stands between two neighboring chambers $\Sigma_1$ and $\Sigma_2$ (having fixed and identical volumes) sharing a thermally insulating and impenetrable wall with a hole, and having an ideal gas containing $N$ particles. The demon $D$ opens or closes the small hole at will to select faster particles to go from $\Sigma_2$ into $\Sigma_1$ and slower particles from $\Sigma_1$ into $\Sigma_2$: Maxwell does not describe how $D$ knows their energies. The chambers and $D$ form an isolated system $\Sigma$ of volume $V$. The wall, the hole, and $D$ act like an inert piston in a cylinder of gas is treated. Thus, we will focus only the ideal gas, whose entropy we denote by $S$. Maxwell considered $\Sigma$ to be initially in equilibrium (EQ) having a temperature $T$ before $D$ intervenes. He conjectured as self-evident that after a while $D$ creates a nonzero temperature difference $\Delta T = T_1 - T_2 > 0$ between the temperatures $T_1$ and $T_2$ of $\Sigma_1$ and $\Sigma_2$, respectively, without any expenditure of work.

The paradox has generated a tremendous amount of debate and some confusion among the best minds of our time since its inception. The endeavor has been a constant source of major conceptual advances and some challenges in theoretical physics and in the philosophy of science that has resulted in the demon problem to undergo many modifications including those requiring an open $\Sigma$.

The main source of confusion in these developments has been the concept of any work done by the demon as it performs “measurements” of energies, and has required the concepts of feedback mechanism, information (including mutual information) entropy, Landaufer’s principle, minimum dissipation, erasure, etc. The most common response is that $D$ somehow manages to create enough entropy to salvage the second law but it is just as common for not everyone to agree to the actual manner in which this happens. Maxwell’s conjecture $\Delta T > 0$ seems very natural and almost self-evident because of the separation of more energetic (fast, $\epsilon_f > 3T/2$) and less energetic (slow, $\epsilon_s < 3T/2$) particles, and is universally accepted without ever being questioned; here $\epsilon_{mp} = 3T/2$ is the most probable energy per particle.

This is surprising as equilibrium thermodynamic and statistical mechanics tells us that for a macroscopic system, only the most probable state matters; the improbable states are almost irrelevant. In view of this, we have decided to carefully investigate the conjecture. A negative answer will make the above modifications, though interesting and mentally challenging in their own rights, not physically or conceptually relevant for solving the demon paradox. Our interest is not to continue the above controversies as they do not pertain to the verification of Maxwell’s conjecture in an isolated system. We will also ask if there is a need for any measurement and erasure, and if the loss of equilibrium is mere fluctuation prior to reset.

We will closely follow Brillouin, who seems to be the...
first to treat particle flows carefully assuming $\Delta T > 0$, but the main interest was to determine the entropy generated by observation through light. He does not account for how often particles pass through the open door and how much average energies they transport during $\Delta t$. This is unfortunate as slow particles will go through more often than the fast particles so it is possible for them to transport more energy than the fast ones; see Eq. (3) below. This issue has never been discussed for the demon paradox so far. This is where we deviate from Brillouin.

We show by two independent methods that D cannot create a nonzero $\Delta T$ and destroy EQ. Thus, D remains in EQ with the gas and there is no need for resetting D. The two approaches are (i) classical equilibrium thermodynamics with no fluctuations and (ii) equilibrium statistical mechanics involving fluctuations. We say nothing about an interacting $\Sigma$, which we will discuss elsewhere.

The average number of particles $dN_x$ per unit volume with the (dimensional) energy $x = \epsilon / T$ due to translation motion in the interval $dx$ is given by [21–23] $dN_x = (2N/V)\sqrt{\pi x^3}e^{-x^2}dx$ with $\int_0^\infty dN_x = N/V$. The average energy carried by these particles is $x^2dN_x$. It will be useful to suppress unnecessary constant terms and introduce the following two unnormalized probability distribution functions (pdfs)

$$f(x) = \sqrt{\pi x}e^{-x^2}, \eta(x) = xf(x) = x^{3/2}e^{-x^2}. \quad (1)$$

Here, $f(x)$ is the pdf for (the random variable) $x$, and $\eta(x)$ for the energy carried by these particles. They are shown by the blue and red curves in Fig. 1. The areas under curves are $\Gamma(3/2)$ and $\Gamma(5/2)$, respectively, with the ratio $\Gamma(5/2)/\Gamma(3/2)$ giving the average energy $\bar{x} = 3/2$. Each function has a single maximum: $f_{\text{max}} \approx 0.4289$ at $x = x_m = 1/2$ shown by the location of the blue vertical line, and $\eta_{\text{max}} \approx 0.4099$ at $x = \bar{x} = (7/2)/3$ shown by the location of the red vertical line; here, $x_m$ is the most probable value, the mode of $f(x)$ and is different from the mean $\bar{x} = x_{\text{mp}} / T$. [It should be remarked that $f(x)$ can also be interpreted as the average energy per degree of freedom of the particle in the interval $dx$ so that the location of the blue line denotes the average energy $\bar{x}_{\text{mp}} = 1/2$ per degree of freedom over $(0, \infty)$.

We now follow slow ($x_s$) and fast ($x_f$) particles. We assume that D has very sharp faculty but not so sharp that he can determine the particle energy with infinite precision. We introduce a very small but nonzero positive quantity $\delta \sim 10^{-14}-10^{-11}$ to be the limit of D’s precision [24] so that all particles $x_{\text{mp}}$, lie in a window $\delta \bar{x} = \bar{x} - \delta \leq x \leq \bar{x} + \delta$. Particles $x_s$ with $x < \bar{x} - \delta$ and $x_f$ with $x > \bar{x} + \delta$ occupy, respectively, the solid light blue and the solid dark red portions of the $x$-axis in Fig. 1. D only manipulates these particles but not the $x_{\text{mp}}$ in the window $\delta \bar{x}$ on the $x$-axis. The dotted blue-red horizontal blue line at the height $f(\bar{x}) \approx 0.2733$ cuts the blue curve at $x_1 \approx 0.0895$ and $x = \bar{x}$, and the red curve at $x_2 \approx 0.6492$ and $x_3 \approx 2.8835$. We see that for $x \geq x_1$, $f(x_s) \geq f(x_f)$. Thus, slow particles appear more often than the fast particles for $x \geq x_1$. From $\eta(x)$ over $(x_2, x_3)$, we can easily conclude that $\Delta \eta = \eta(x_2) - \eta(x_1)$ can have any sign, a fact that will be important in thermodynamic consideration.

**Thermodynamics**: For simplicity, we will use the term “body” and denoted by $\Sigma_b$ to refer to any one of $\Sigma, \Sigma_1,$ and $\Sigma_2$ as their thermodynamic discussion is very similar. The body $\Sigma_b$ is described by the observables $(E_i, N_i)$; its volume is kept fixed. As $\Sigma_1$ and $\Sigma_2$ together require four observables, we also need the same number to specify $\Sigma$. Using $E_1$ and $E_2$, we introduce combinations $E = E_1 + E_2$ and $E' = E_1 - E_2$; similarly, we introduce $N = N_1 + N_2$ and $N' = N_1 - N_2$. They now uniquely specify $\Sigma$. As $E$ and $N$ are fixed, only $E' = 2E_1 - E$ and $N' = 2N_1 - N$ will change. Let us consider the changes $dE' = 2dE_1$ and $dN' = 2dN_1$, which are caused by transfer of fast and slow particles so that

$$dE_1 = dE_1 - dE_2, dN_1 = dN_1 - dN_2. \quad (2)$$

It should be noted that $dE_i$ and $dE_2$, or $dN_1$ and $dN_2$ do not represent two independent variations as their differences are single variations $dE_1$ and $dN_1$, respectively. Let us consider their significance. From $S(E, N, E', N') = S(E_1, N_1) + S(E_2, N_2)$, we have

$$dS = (\beta_1 - \beta_2)dE_1 + (\mu_2 \beta_2 - \mu_1 \beta_1)dN_1 \quad (3)$$

in $\Sigma$, where $\beta_1 = 1/T_1 = \partial S_1/\partial E_1$ and $\beta_2 = 1/T_2 = \partial S_2/\partial E_2$, and $\mu_1 = -\partial S_1/\partial N_1$ and $\mu_2 = -\partial S_2/\partial N_2$ of $\Sigma_1$ and $\Sigma_2$. For the second law to be valid, each term on the right must be nonnegative; otherwise not. In equilibrium, they both vanish identically.

Brillouin [8] also uses $\bar{x}$ as the cutoff for slow and fast particles as we have done. However, he merely follows Maxwell’s conjecture that after a certain time $\Delta t$, D creates nonzero $\Delta T << T$. He then proceeds with one fast particle moving from $\Sigma_2$ into $\Sigma_1$, and one slow particle from $\Sigma_1$ into $\Sigma_2$, and finds $\Delta Q = 3(e_1 + e_2)/2 > 0$ without accounting for their probabilities; here, we are using the notation of Brillouin. As fast particles have smaller probabilities than the slow particles, see $f(x)$ in Fig. 1, it is possible that D selects two slow particles for a fast particle so that the net energy transfer is

$$\Delta Q = 3(-1 + e_1 + 2e_2)/2 < 0, \quad (4)$$

which results in $\Delta S_i > 0$. This is different from $\Delta Q > 0$ above and resulting in $\Delta S_i < 0$ [8]. Eqs. (13) and (14)]. However, it is not correct to interpret $\Delta S_i > 0$ as supporting the second law and $\Delta S_i < 0$ as violating the second law as $\Delta S_i$ is evaluated using a few (one or two) particles.

We see from the above discussion that it is not just the energies but also probabilities of the particles play important roles in determining the energy transfer. Thus, we need to consider $\Delta \eta(x)$, which is the analog of the quantity $\Delta Q$ of Brillouin for two particles. Consider the solid black horizontal line near the peak of $\eta(x)$ that meets the red curve at a slow and a fast particles having the same
energy to give $\Delta \eta = 0$. By taking the black line at different angles will give us $\Delta \eta$ (or $\Delta Q$) of both signs. Thus, there is no one particular sign of $\Delta \eta$. This casts doubts on the claim by Maxwell and supported by Brillouin’s calculation that $\Sigma_1$ has a higher temperature than $\Sigma_2$ after many particle transfers during $\Delta \tau$. Indeed, it is just as possible to have the opposite situation as seen from Eq. (4). In this case, the flow of two slow particles into $\Sigma_2$ raises its energy over $\Sigma_1$ by more than the increase in the energy of $\Sigma_1$ by one fast particle over the same duration of time.

Indeed, it is quite possible that during $\Delta \tau$, the fluctuating signs of $\Delta \eta$ yield a zero average so there is no energy difference between $\Sigma_1$ into $\Sigma_2$. We first recognize that the concept of temperature, heat flow, entropy, entropy generations, etc. are macroscopic concepts so D must let a very large number of particles through over $\Delta \tau$ and average over them to give

$$dE_1 = a_1dT_1 + a_2d\mu_1, dN_1 = b_1dT_1 + b_2d\mu_1, \quad (5)$$

by treating $E_1$ and $N_1$ equivalently as functions of $T_1$ and $\mu_1$ of $\Sigma_1$; the coefficients are also functions of $T_1$ and $\mu_1$. For $\Sigma_2$, we have similar relations with the same EQ coefficients but with $dT_2 = -dT_1$ and $d\mu_2 = -d\mu_1$.

As the presence of $x_{mp}$ is never affected, $dE_1$ and $dN_1$ must be expressed as in Eq. (2). Using Eq. (3), we finally obtain

$$ds = [\Delta b_1 - \Delta (\mu_1)b_1] dT_1 + [\Delta b_2 - \Delta (\mu_2)b_2] d\mu_1, \quad (6)$$

where $\Delta b = b_1 - b_2$ and $\Delta (\mu_1)b = \mu_1b_1 - \mu_2b_2$. We now apply this equation to the initial EQ state of $\Sigma_b$, just when D begins to sort out particles. As $\Delta b = 0$ and $\Delta (\mu_1)b = 0$ in the EQ state, we see that $dS_b = 0$ so D does not affect $S, S_1,$ and $S_2$. This means that the EQ is not destroyed and $T_b = T, \mu_b = \mu$ during $\Delta \tau$. As $dT_1 = 0$ and $d\mu_1 = 0$, we see that even $dE_1 = 0$ and $dN_1 = 0$ so

$$dE_t = dE_s, dN_t = dN_s \quad (7)$$

over $\Delta \tau$. It is now clear that the situation does not change at all no matter how long D operates the hole. We have thus falsified Maxwell’s conjecture. We emphasize that the above argument does not require using the second law as we did not exploit the nonnegativity of the two terms in Eq. (5).

**Statistical Mechanics:** We now give the statistical mechanical argument, which is independent of the above thermodynamic argument in that we consider fluctuations that were neglected above. As D needs to average over many particles, $f(x)$ and $\eta(x)$ of single particles are not useful. We need to consider the entire gas, which requires considering the sum $X = \sum_{i=1}^{N} x_i$ of the independent and identically distributed (the same mean $\mathcal{F} = 3/2$ and the standard deviation $\sigma = \sqrt{3/2}$) random variables $x_i$ of the $i$th particle. It follows from the central limit theorem ($N >> 1$) [21] that $x \approx X/N$, which is relevant for single particles, has a normal distribution $\mathcal{N}(\mathcal{F}, \sigma^2/\sqrt{N})$ with mean $\mathcal{F}$, where it also has its peak, and the standard deviation $\sigma/\sqrt{N}$. Thus, $f(x)$ will approach the central limit $\bar{f}(x)$ of a normal distribution:

$$\bar{f}(x) = \frac{\sqrt{\mathcal{F}}}{2} \mathcal{N}(\mathcal{F}, \sigma^2/\sqrt{N}) = \sqrt{\frac{N}{8 \sigma}} \exp\left[-\frac{N}{2} \left( \frac{x - \mathcal{F}}{\sigma} \right)^2 \right];$$

the prefactor $A_f = \sqrt{\mathcal{F}/2}$ is inserted to ensure $\bar{f}(x)$ has the same area as $f(x)$. Similarly, $\eta(x)$ will approach the central limit $\bar{\eta}(x) = x \ f(x)$ with its peak at $\mathcal{F}$ as expected. Oncoming particles towards the hole during $\Delta \tau$ are any of $N$ particles on average, so their pdf is $f(x)$ and not $f(x)$. The maximum of $\bar{f}(x)$ gives the most probable state for the thermodynamic consideration, which as discussed below is very sharp with relative fluctuations $\approx 1/\sqrt{N}$ that are almost vanishingly small due to the presence of $N$ in the exponent and are neglected in thermodynamic consideration above [23, see discussion on pages 5 and 28]. As is well known, the mode, median and the mean for $f(x)$ are all equal to $\mathcal{F}$. A gas particle in equilibrium thermodynamics is governed not by the broad maximum of $f(x)$ in Fig. 4 but by the sharp maximum of $\bar{f}(x)$. This is consistent with Boltzmann’s observation that the average energy is determined by the most probable energy $3T/2$. The entropy $S(E)$ of the gas is also a function of the most probable energy $E = 3NT/2$ with the inverse temperature given by the thermodynamic derivative $1/T = \partial S(E)/\partial E$. Thus, even $S(E)$ is determined by the peak at $\mathcal{F}$.

The discussion of $\bar{f}(x)$ above is very general and applies to any $\Sigma_b$. Thus, we have three functions $\bar{f}_b(x) = f(x), \bar{f}_1(x), \bar{f}_2(x)$, for $\bar{f}_1(x)$, Eq. (4). Let the hole initially be open so that all bodies have the same temperature. Consequently, all functions have their peaks at $\mathcal{F}$. Each body has all kinds of particles: $x_{mp}, x_s, x_t$. For any nonzero $\delta$, we have $\alpha = (\delta/\sigma)^2/2 > 0$

$$\bar{f}_b(x) \quad or \quad \bar{f}_b(x_t) \lesssim \sqrt{Ne^{-\mathcal{F}a}} \approx 0. \quad (9a)$$

Let us consider the case $N = 10^{24}$ and $\alpha = 10^{-22}$ so that

$$\sqrt{Ne^{-\mathcal{F}a}} = 10^{12}e^{-100} \approx 3.758 \times 10^{-12}, \quad (10)$$

a fantastically small number to justify the above approximation. Thus, $x_s$ and $x_t$ particles have extremely low probabilities and make no difference in determining the temperature, which is determined by $x_{mp}$ alone.

The averaging over many particles is done using $\bar{f}_b(x)$ as noted above. As a rule, D never allows $x_{mp}$ to be exchanged. This means that the peak in $\bar{f}_b(x)$ remains in the same place at $\mathcal{F}$ even after D begins to manipulate particle flows. Indeed, the next to the most probable particles $x_{mp}$ are fast particles with energies in the range $(\mathcal{F} + \delta < x_i' < \mathcal{F} + 2\delta)$ and slow particles with energies in the range $(\mathcal{F} - 2\delta < x_i' < \mathcal{F} - \delta)$, respectively. As we see from Eqs. (9a,10), the pdf’s for $x_s'$ or $x_t'$, although themselves equal, are relative to that of $x_{mp}$ given by the ratio $\bar{f}_b(x_s')/\bar{f}_b(x_{mp}) \approx e^{-\mathcal{F}a} \approx 10^{-44}$. This means...
that D will observe about $10^{44}$ most probable particles before opening the hole to let a $x'_f$ or $x'_b$ particle through. As the temperature and chemical potentials of $\Sigma_b$ are almost surely determined by $x_{mp}$, and as D has not affected $x_{mp}$ and the peak $f_b(x_{mp})$, they are almost surely given by $T$ and $\mu$. In other words, any rare transfer of slow and fast particles cannot affect the temperature and chemical potential in $\Sigma_b$, so $E_b$ and $N_b$ are also not affected. This immediately falsifies Maxwell’s conjecture that seems to be universally accepted as was also concluded thermodynamically in Eq. (7). The statistical argument now clarifies the significance of Eq. (7) by rare events so it is independent of the thermodynamic argument, and provides another support for the falsification of Maxwell’s conjecture for the simple reason that the thermodynamic argument does not depend on the choice of $\delta$. As the entropy $S_b$ of $\Sigma_b$ is also determined by the most probable distribution, it has the same value as before D begins to operate. This is again in accordance with the previous conclusion that $\Delta S = 0$. We have also arrived at the same conclusion earlier [23] using a nonequilibrium approach.

By focussing on $x_s$ and $x_f$ particles, Maxwell had hoped to create energy imbalance between the chambers. But to suggest this improbable energy imbalance results in a temperature imbalance does not work as the temperature is a macroscopic concept defined more precisely by the derivative ($\partial E/\partial S$)$_b$ or by $f_b(x_{mp})$. It is well known from the fluctuation theory [23] that fluctuations in $S_b$ such as due to $x'_f$ or $x'_b$ particle transfer occur at the same temperature $T$ of $\Sigma_b$. The gas in each chamber even after including these fluctuations will have the original $T$ describing $f_b(x_{mp})$. Thus, no perpetual motion machine of the second kind can be formed by using equilibrium fluctuations in a system. As $\Delta S = 0$, there is no need to bring in the concept of information as there is no second law violation to resolve. Whether we need it for Szilard’s engine is a separate issue since there we deal with a system in contact with a heat bath [9, 11]. Our discussion here is for an isolated system for which entropy reduction is a fundamental problem as Maxwell has observed. We discuss the engine in a separate publication [26].

As $f_b(x)$ is symmetric about $x_{mp}$, $x_s$ and $x_f$ have the same probability distribution. Only if we start with the two chambers at different temperatures will be obtain $dE_1 \neq 0$, see Eq. [5], as was also discussed by Feynman [4]. But this does not happen with the Maxwell’s demon, where we start with an equilibrium state. In our discussion, we have found no need to worry about the mechanism used by D to observe particles and of intelligence as was the case with Brillouin [8]. As the initial equilibrium of $\Sigma$ is not destroyed, D always remains in EQ with the two chambers so its entropy cannot change, which justifies treating it as thermodynamically inert. As the operations performed by D does not alter its initial state, there is no resetting required.

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