Portfolio choice under cumulative prospect theory: sensitivity analysis and an empirical study

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Abstract
A sensitivity analysis of the impact of cumulative prospect theory (CPT) parameters on a Mean/Risk efficient frontier is performed through a simulation procedure, assuming a Multivariate Variance Gamma distribution for log-returns. The optimal investment problem for an agent with CPT preferences is then investigated empirically, by considering different parameters’ combinations for the CPT utility function. Three different portfolios, one hedge fund and two equity portfolios are considered in this study, where the Modified Herfindahl index is used as a measure of portfolio diversification, while the Omega ratio and the Information ratio are used as measures of performance.

Keywords Cumulative prospect theory · Non-convex optimization · Robustness and sensitivity analysis · Hedge funds

1 Introduction

The expected utility theory (EUT) has been the dominant theory for making decisions under risk for decades and it is widely employed by both academics and practitioners. Nonetheless, this decision paradigm has been criticized for not being always consistent with the agents’ observed behavior. There is substantial empirical and experimental evidence that human behavior deviates from the key assumptions of expected utility. The best known counter-example to the EUT which exploits the certainty effect was

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introduced by the French economist Allais (1953). In response to this criticism several alternatives have been proposed, most notably the dual theory (Yaari 1987), the rank-dependent utility theory (Quiggin 1993) and the Cumulative Prospect Theory (Kahneman and Tversky 1979; Tversky and Kahneman 1992). However, Kahneman and Tversky (1979), in particular, demonstrated empirically that in a variety of decision problems preferences systematically violated the axioms of expected utility theory. In their experiments, the authors found that investors are more sensitive to losses than gains and behave differently on gains and losses: investors can be risk averse in gains but risk seeking in losses. To capture investors’ behavior in Kahneman and Tversky (1979) and Tversky and Kahneman (1992) Kahneman and Tversky developed, respectively, the Prospect Theory (PT) and the Cumulative Prospect Theory (CPT). Both theories have two constituent parts: (i) a loss-averse utility which has also a point of non-differentiability; (ii) a non linear transformation of the scale of probabilities; by means of this transformation low probabilities are overweighted and high probabilities are underweighted.

Although this theory was published many years ago, few theoretical and empirical works have been done [see among others (Mayer and Hens 2013; Giorgi et al. 2007; Hens and Mayer 2014)] and Barberis (2013). An analytical treatment of a single-period portfolio choice model with a CPT agent has been proposed, for example, in Bernard and Ghossoub (2010); He and Zhou (2011); Pirvu and Kwak (2014). Bernard and Ghossoub (2010) and He and Zhou (2011) deal with a portfolio with one risky and one riskless asset. In Bernard and Ghossoub (2010) the authors show that the optimal holding, when it exists, is related to a generalized Omega measure\(^2\) and that a CPT investor is very sensitive to the skewness of stock excess returns. He and Zhou (2011) carry out an analytical study of the CPT model and derive explicit solutions for some special cases. Moreover, they investigate the sensitivity of the optimal risky allocation with respect to the model’s parameters. In Pirvu and Kwak (2014) an optimal portfolio problem for risky assets with multivariate Gh skewed t distribution is considered.

The aforementioned papers are mainly of theoretical nature, rather than empirical, since they look for the explicit formulation of the optimal portfolio under restrictive assumptions on the number of assets and on the distribution of their returns. However,

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1 One of the main assumptions of EU theory is the independence axiom, which states that the intrinsic value that an individual places on any particular outcome in a gamble will not be influenced by the other possible outcomes (either within that gamble or within other gambles to which the gamble is being compared), or by the size of the probability of the outcome occurring. The axiom requires that, when comparing gambles, all common outcomes that have the same probability of occurring will be viewed by the individual as irrelevant. In a famous criticism of EU, Maurice Allais argued that under certain conditions individuals will systematically violate independence. Allais proposition is known as the Allais paradox (or the common consequence effect), and has been empirically supported in subsequent analysis [see, among the others (Morrison 1967; MacCrimmon and Larsson 1979; Kahneman and Tversky 1979; Camerer and Kunreuther 1989)].

2 Omega Ratio is defined by:

$$\Omega = \frac{\mathbb{E}(R_P - \tau)^+}{\mathbb{E}(\tau - R_P)^+},$$

where \(R_P\) denotes the return of the portfolio and \(\tau\) is a specified threshold. \(\Omega\) ratio is very sensitive to \(\tau\), which can be different from investor to investor. In the empirical analysis \(\tau\) is set equal to 0. For a given threshold, a higher ratio indicates that the portfolio provides more expected gains than expected losses and so it would be preferred by an investor.
applying the Cumulative) Prospect Theory to finance is quite challenging from a computational point of view. The prospect theory objective function is non-concave and non-smooth. In this paper, the non-concavity, which arises from both the \textit{S-shaped} utility function and the non trivial probability distortions have been dealt with by adopting a global search optimization algorithm. Furthermore, to overcome the non-differentiability of the CPT function, a spline-smoothing technique has been employed (as in Giorgi et al. 2007).

An important work considering the CPT portfolio selection problem from an empirical viewpoint is Hens and Mayer (2014). This paper differs from Hens and Mayer (2014) and contributes to the existing empirical literature in the following dimensions:

– It focuses on the mean-risk analysis of the investment decision problem under the so called \textit{certainty equivalent} associated to the CPT objective function, defined as:

\begin{equation}
\rho_{CPT}(X) := -(V^{CPT})^{-1}(E[V^{CPT}(X)]). \tag{1}
\end{equation}

Risk measures of type (1) have been introduced and studied in details from an axiomatic point of view in Müller (2007) and, later, in Mastrogiacomo and Gianin (2015). They provide alternative criteria for investors’ decision making under risk. Instead of minimizing the portfolio variance for a given level of portfolio return \( \mu_T \), one can minimize \( \rho_{CPT}(X) \). Within this paradigm a combination of assets is referred to as efficient when, for a given value of \( \mu_T \), it carries the lowest level of risk, which is represented by the certainty equivalent of the portfolio. Analogously to the mean-variance analysis we call efficient frontier the set of all points \((\rho_{CPT}(X), E[X])\) in the plane corresponding to all efficient portfolios \(X\). This paper assesses the direct impact of the CPT parameters on the mean-\(\rho_{CPT}(X)\) efficient frontiers. The analysis is performed through a simulation procedure where assets’ returns are generated from a multivariate Variance Gamma distribution. To the best of our knowledge the direct impact of the CPT parameters on the mean-\(\rho_{CPT}(X)\) efficient frontier has not yet been considered in the literature.

– It investigates the effect of the CPT parameters on portfolio performance and diversification. To perform this analysis two different investment universes are considered: one based on hedge funds and one on the US equity market.

– To understand portfolio allocation, psychological realism is enhanced, e.g. by allowing for the possibility of less than fully rational thinking. In the empirical part portfolios with different characteristics are considered. The first is a hedge fund portfolio. It is well known that hedge fund returns are characterized by high skewness and fat tails. In the literature on hedge funds portfolio allocation, the expected utility is sometimes approximated through a fourth order Taylor expansion [see among others (Jondeau and Rockinger 2006; Hitaj et al. 2012; Martellini and Ziemann 2010; Jondeau et al. 2007; Mantegna and Stanley 2000; Xiong and Idzorek 2011; Hitaj and Zambruno 2018; Patton 2004)]. Therefore the Taylor expansion of the expected utility does not consider the whole distribution of assets’ returns, while the CPT model does. In order to validate the results, two equity portfolios are also considered, whose components are selected from the \textit{S&P 500} index. For the sake of comparison, the global minimum variance (GMV) and an optimal
mean-variance (MV) portfolio are also constructed. In order to assess the magnitude of potential gains (losses) that can be realized by an investor when using a CPT portfolio model instead of a GMV or a MV model, the out-of-sample performance measures such as Omega Ratio and Information Ratio are analysed.

All numerical tests are carried out without any a priori assumption concerning the returns distribution. The results clearly indicate that the CPT parameters play an important role on the diversification of a portfolio: agents with a high loss aversion and a high risk aversion choose more diversified portfolios, while agents with a high risk preference in losses exhibit less diversified portfolios. We also notice that CPT portfolios are characterized by higher (lower) out-of-sample performances compared to Mean-Variance and Global Minimum-Variance portfolios when assets’ returns are not normally distributed. For the sake of completeness, it is also tested whether, relative to GMV portfolios the Information Ratios are statistically different from zero, based on the p values (see Goodwin 1998). In case of non normally distributed assets’ returns, relying on the t-statistic test, we obtain that the potential Information Ratio increases of a CPT portfolio, with GMV as benchmark, is statistically different from zero in most of the cases, with a significance level of 15%, while when assets’ returns are approximately normally distributed the potential loss in terms of Information Ratios is mostly not statistically significant.

The paper is organized as follows: Sect. 2 briefly recalls the CPT portfolio model. Section 3 explains the sensitivity analysis performed and presents the results obtained from the simulated and real data, and Sect. 4 draws some conclusions.

2 Portfolio optimization under cumulative prospect theory

In this section a brief review of the theory behind the CPT and an explanation of the computational procedure adopted to determine the CPT value function are reported.

2.1 The main components of cumulative prospective theory

This section, recalls the framework of Tversky and Kahneman (1992) and explains the main components of the . The three key elements of the decision making process of a CPT-investor are: (1) the agent is primarily concerned with the deviation of her final wealth from a reference level τ, whereas the Expected-Utility maximizing investor is interested only on the final value of her wealth. (2) She reacts differently to gains and losses. Risk aversion is generally assumed in economic analysis of decision under uncertainty. However, risk-seeking choices against losses are consistently observed. Risk seeking is prevalent when people must choose between a sure loss and a substantial probability of a larger loss. Moreover investors are distinctively more sensitive to losses than to gains (the latter behavior is called loss aversion). (3) Investors do not evaluate random outcomes using the physical probabilities but base their decision upon distorted probabilities. This distortion explains the investors’ tendency to overweight small probabilities corresponding to extreme losses or gains. We will now

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3 In case of mean-variance portfolio the considered risk aversion parameter is one.
formalize these three components explaining the decision process of the CPT-investor and introduce the notation that will be adopted throughout the analysis.

Let $x$ denote the return of the portfolio and by $\tau$ the reference level of wealth at the end of the period. Define the deviation $D$ from the reference level by $D = x - \tau$. Notice that $D < 0$ corresponds to losses, while $D \geq 0$ corresponds to gains. In the empirical analysis we work with assets' returns and consider $\tau = 0$. Let us denote by $F_D$ the cumulative distribution function of $D$ and by $S_D$ the survival distribution function, i.e. the function $S_D(x) := 1 - F_D(x)$. The second component of the decision of a CPT-investor is the value function $v(D)$, which is defined as:

$$v(D) = \begin{cases} 
D^\alpha, & D \geq 0 \\
-\lambda(-D)^\beta, & D < 0
\end{cases}$$

(2)

where $\alpha, \beta \in (0, 1)$ and $\alpha \leq \beta, \lambda \geq 1$. $\lambda$ is related to the so called loss aversion, which is defined as the behavioral phenomena that makes losses loom larger than gains. The parameters $\alpha$ and $\beta$ are related to the curvature of the value function on the positive and negative domains, respectively and have a financial interpretation in terms of, respectively, risk aversion in gains and risk preference in losses.

The third element of the decision making of a CPT-investor lies in the systematic distortion of the physical probability measure. The probability distortion process may be slightly different for losses (negative deviations $D$) and for gains (positive deviations $D$).

The probability weighting function for gains, denoted by $T^+ : [0, 1] \rightarrow [0, 1]$, is defined as:

$$T^+(p) = \frac{p^\delta}{(p^\delta + (1 - p)^\delta)^{\frac{1}{\delta}}}$$

for $\delta \in (0, 1]$; analogously, the probability weighting function for losses, denoted by $T^- : [0, 1] \rightarrow [0, 1]$, is given by:

$$T^-(p) = \frac{p^\gamma}{(p^\gamma + (1 - p)^\gamma)^{\frac{1}{\gamma}}}$$

for $\gamma \in (0, 1]$. We notice that $T^+, T^-$ are differentiable with $T^\pm(0) = 0, T^\pm(1) = 1$. One can also verify that, assuming $\gamma > 0.28$ and $\delta > 0.28$ ensures that $T^\pm$ are increasing (see Barberis and Huang 2008). The value function of the CPT-investor is defined as

$$V^{CPT}(D) = V^+(D) - V^-(D)$$

where

$$V^+(D) = \int_{0}^{+\infty} v(w)d(-T^+(S_D(w)))$$

(3)

$$V^-(D) = \int_{-\infty}^{0} v(w)dT^-(F_D(w)).$$
We notice that \( V^{CPT} \) is a sum of two Choquet integrals; in (Barberis and Huang (2008), pag. 7) it is stated that it is well-defined when \( \alpha, \beta < 2 \min(\delta, \gamma) \).

### 2.2 Problem formulation

A portfolio optimization model based on CPT objective function maximization (also considered in Hens and Mayer 2014; Barberis and Huang 2008) can be formulated as follows:

\[
\max_{w \in \mathcal{W}} V^{CPT}(w \cdot X),
\]

where \( X = (X_1, \ldots, X_n) \) is a random vector of assets’ returns, \( V^{CPT} \) is the CPT investor value function and \( \mathcal{W} \) is the set of all admissible portfolio weights. We assume that \( \mathcal{W} \) is given by:

\[
\mathcal{W} := \left\{ w \in \mathbb{R}^n : \sum_{i=1}^n w_i = 1, \ w_i \geq 0, \ \text{for} \ i = 1, \ldots, n \right\},
\]

where \( w_i \) stands for the fraction of wealth invested in asset \( i \). In the empirical analysis short sales are excluded. Due to the non-differentiability and non-concavity of the components of prospect theory, analytic solutions for CPT-based asset allocation problems can only be expected in very simple cases. Closed-form expressions for portfolio selection problems in presence of investors behaving according to CPT have been considered, for example, in the case of normally, elliptical or skew-normally distributed returns (see, respectively, Barberis and Huang 2008; Pirvu and Schulze 2012; Bernard and Ghossoub 2010). Yet, many asset allocation problems involve non-normally and non-elliptically distributed returns since even at the level of global indices, returns of stocks, bonds and commodities typically have fat tails and are skewed. In what follows, a numerical optimization approach is adopted for solving a portfolio selection problem (4) with several assets. The considered algorithm avoids distributional assumptions on the assets’ returns and leads to a numerical study based on an actual market data.

#### 2.2.1 The numerical optimization approach

Suppose that \( X_{(k)}^{(k)}_{k=1,\ldots,N} \) is a random sample for the assets’ return vector \( X \). For a given portfolio \( w \), the portfolio returns of the random sample are \( (w \cdot X_{(k)}^{(k)})_{k=1,\ldots,N} \). In order to evaluate numerically the CPT value function corresponding to \( w \), we proceed as follows. First, the vector of all possible portfolio returns is sorted in an increasing order and denoted by \( (w \cdot \xi^{(1)}, \ldots, w \cdot \xi^{(N)}) \). We have:

\[
w \cdot \xi^{(1)} \leq \cdots \leq w \cdot \xi^{(n)} \leq 0 \leq w \cdot \xi^{(n+1)} \leq \cdots \leq w \cdot \xi^{(N)}.
\]

Let \( (\tilde{p}_k)_{k=1,\ldots,N} \) be the corresponding vector of probabilities. Here \( n \) divides the portfolio returns in two groups: the first one is represented by \( w \cdot \xi^{(1)}, \ldots, w \cdot \xi^{(n)} \) and
corresponds to losses; the second one is represented by $w \cdot \xi^{(n+1)}, \ldots, w \cdot \xi^{(N)}$ and corresponds to gains. In fact, if $v$ is the piecewise power value function defined in (2), then $v(w \cdot \xi^{(k)}) \leq 0$ (respectively, $v(w \cdot \xi^{(k)}) \geq 0$) for $w \cdot \xi^{(k)} \leq 0$ (respectively, $w \cdot \xi^{(k)} \geq 0$).

Second, let us define the probability distortion functions (see Tversky and Kahneman 1992; Barberis 2013 for more details):

$$
\pi_N^+(w \cdot \xi^{(N)}) = T^+(1 - (\bar{p}_1 + \cdots + \bar{p}_{N-1}))
$$
$$
= T^+(p_N)
$$

$$
\pi_k^+(w \cdot \xi^{(k)}) = T^+(1 - (\bar{p}_1 + \cdots + \bar{p}_{k-1})) - T^+(1 - (\bar{p}_1 + \cdots + \bar{p}_k))
$$
$$
= T^+(\bar{p}_N + \cdots + \bar{p}_k) - T^+(\bar{p}_N + \cdots + \bar{p}_{k+1}), \ n < k \leq N - 1
$$

and

$$
\pi_1^-(w \cdot \xi^{(1)}) = T^-(\bar{p}_1)
$$

$$
\pi_k^-(w \cdot \xi^{(k)}) = T^-(\bar{p}_1 + \cdots + \bar{p}_k) - T^-(\bar{p}_1 + \cdots + \bar{p}_{k-1}) \ 2 \leq k \leq n.
$$

The objective function is then:

$$
V_{CPT}(w \cdot X) = V^+(w \cdot X) - V^-(w \cdot X)
$$

$$
= \sum_{k=n+1}^{N} v(w \cdot \xi^{(k)})\pi_k^+(w \cdot \xi^{(k)}) - \sum_{k=1}^{n} v\left( w \cdot \xi^{(k)} \right) \pi_k^-(w \cdot \xi^{(k)}).
$$

The optimal portfolio, hence, is computed through an approximation procedure which involves the integrals introduced in (3). It corresponds to the average of the $N$ values of $v(X_w^{(k)})$ and takes into account the distorted probability functions $T^+$ and $T^-$. 

**Remark 1** We stress that the procedure described above will allow to solve the CPT selection problem numerically. In fact, the CPT optimal portfolio will be determined by solving the following maximization problem numerically:

$$
\max_{w \in W} \max_{w \in W} V_{CPT}(w \cdot X) = \max_{w \in W} \left( \sum_{k=n+1}^{N} u(w \cdot \xi^{(k)})\pi_k^+(w \cdot \xi^{(k)}) \right.
$$

$$
- \sum_{k=1}^{n} u\left( w \cdot \xi^{(k)} \right) \pi_k^-(w \cdot \xi^{(k)}) \left. \right).
$$

The optimization problem (7) turns out to be quite difficult from a numerical viewpoint. Since the value function is convex for losses and concave for gains, the objective function is no longer quasi-concave and the first order condition may only describe local optima. Moreover, the objective function is not differentiable at the reference point. Consequently, the problem belongs to the class of non-smooth optimization problems. Therefore standard optimization algorithms, such as the gradient based
method, cannot be used. In order to illustrate that the problem (4) is not convex, in Fig. 1 the Kahneman and Tversky CPT value function for a portfolio of three hedge fund indexes is plotted: Absolute Return Index, Distressed Restructuring Index and Merger Arbitrage Index, which have been downloaded from HFR Database. The objective function is evaluated on daily log-returns from 2010 to 2014 on a grid of $40 \times 40$ portfolio weights $w_1, w_2 \in [-1, 1]$, while $w_3$ is determined by the budget constraint. Notice that when plotting the associated surface, we allow for short selling. This choice, here, is done only for illustrative purpose. In the rest of the paper we assume no short selling. The presence of multiple local minima and the non-smoothness of the surfaces indicates that a simple gradient-based method cannot be used to solve the problem.

We enforce differentiability at the reference point $\tau$ using a cubic spline of the form $p(X_\tau) = aX_\tau^3 + bX_\tau^2 + cX_\tau + d$ in a neighborhood $[-\eta, \eta]$ ($\eta > 0$) of the reference point $\tau = 0$. The coefficients of the polynomial are computed in such a way that the following equalities hold:

\[
\begin{align*}
    p(-\eta) &= v(-\eta) \\
    p(\eta) &= v(\eta) \\
    p'(-\eta) &= v'(-\eta) \\
    p'(\eta) &= v'(\eta).
\end{align*}
\]

The above conditions ensure that at the extremal points of the interval $[-\eta, \eta]$ the value function $v$ and the polynomial $p$ take the same value and first derivative. Therefore the original problem (4) is replaced by the following one:

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4 Here $X_\tau$ indicates the portfolio return. The use of a $B$-spline interpolation to deal with non-smoothness and non differentiability is not a new idea in finance [see (Potaptchik et al. 2008; Grüne and Semmler 2004) and the references therein].
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$$\max_{w \in \mathcal{W}} V^{CPT, \eta}(w \cdot X),$$  \hspace{1cm} (8)

where

$$V^{CPT, \eta}(X_p) := \begin{cases} p(X_p) & \text{if } X_p \in [-\eta, \eta] \\ v(X_p) & \text{otherwise.} \end{cases}$$

In the empirical analysis $\eta = 10^{-4}$ is considered. The smaller $\eta$ is, the better the approximation is.

The non-convexity of problem (8) is overcome by adopting the Global Search Optimization Algorithm\(^5\) of MATLAB. To determine good starting points for the Global Search algorithm, each axis is split in 10 equal parts\(^6\) and then all the permutations among the $n$ axis, whose components sum to 1 are considered. These points are candidates for the initial point of the Global Search algorithm. We evaluate the objective function in each of these points and select as starting points 10 of them that give the objective function the highest values. We ran the Global Search algorithm in parallel for the 10 starting points. The solution is the one that maximizes the objective function in (8).

3 Sensitivity analysis of the CPT parameters on the efficient frontiers

Central to portfolio theory is the premise that investment decisions are made to achieve an optimal risk/return tradeoff under a budget constraint. Such trade-off is usually determined through the specification of an appropriate measure of risk and then through the selection of those combinations of assets that are the most efficient, in the sense of providing the lowest level of risk for a desired level of expected return. The pioneering work in portfolio selection by Markowitz and Selection (1959) and Roy (1952) is based solely on the mean and variance of returns.

In this section the optimal risk/return portfolios are analysed, basing on the following measure of risk:

$$\rho^{CPT}(X) := -(V^{CPT})^{-1}(\mathbb{E}[V^{CPT}(X)]).$$  \hspace{1cm} (9)

The above definition (9) identifies the exposure of a position $X$ with the certainty equivalent of an expected utility maximizer having utility function $V^{CPT}$. We stress that, for a given utility function $v$, the quantity $v^{-1}(\mathbb{E}[v(X)])$ is the so called certainty equivalent of an expected utility maximizer. This is also known under the name of a quasi-linear mean, and has a long history (see Muliere and Parmigiani 1993 and references therein).

CPT certainty equivalent optimization is an alternative formulation of the investor’s decision problem: instead of minimizing the variance for a given level of

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\(^5\) For more information on the Global Search algorithm see. http://it.mathworks.com/help/gads/how-globalsearch-and-multistart-work.html.

\(^6\) The finer the grid is, the more precise the solution is, and the longer the computation time will be.
portfolio return $\mu_T$, we minimize $\rho_{CPT, \eta}$ ($\rho_{CPT, \eta}$ for short). To be more precise, due to the computational complexity of the CPT objective function, the following optimization problem is solved numerically:

$$\min_{w \in W} \rho_{CPT, \eta}(X \cdot w),$$

where $\mu_T$ denotes the investor’s desired level of return. No short positions are allowed and portfolio weights should sum up to 1. X defines the space of all possible portfolios. The solution of problem (10) is denoted by $X_{\mu_T}$.

In the application below the set of efficient portfolios with respect to the introduced measure of risk is found by recursively searching for the minimum risk portfolio as the expected return $\mu_T$ increases.

The impact of the CPT parameters on the mean-$\rho_{CPT, \eta}$ efficient frontiers is analyzed and compared with the mean-variance approach. For this purpose, a Monte Carlo simulation procedure is considered. Asset returns are simulated according to the Multivariate Variance Gamma distribution.\(^7\) The simulated portfolios are composed of 4 assets and have different combinations for skewness and kurtosis (normally distributed and not normally distributed).

### 3.1 Effect of CPT parameters on efficient frontiers

In this subsection the results for the efficient frontier associated with the portfolio problem (10) are presented. These results refer to investors with a degree of loss aversion, risk aversion in gains, risk preference in losses and distortions related to those estimated by Tversky and Kahneman (1992). The median exponent of the value function is 0.6 for the risk aversion and 0.88 for risk preference in losses, in accord with the principle of diminishing sensitivity: the impact of a change diminishes with the distance from the reference point. The median $\lambda$ was 2.25, indicating pronounced loss aversion. The median values of $\gamma$ and $\delta$, respectively, are 0.61 and 0.69. These values identifies investors which overweight low probabilities and underweight moderate and high probabilities (for both positive and negative prospects). Moreover, under these values, people are relatively insensitive to probabilities in the middle of the range.

Figure 2 below presents the results of five different efficient portfolios in the mean $-\rho_{CPT, \eta}$ space; returns are modelled by a symmetric MVG distribution with moderate level of kurtosis. The efficient frontiers of risky assets which yield the lowest $\rho_{CPT, \eta}$ risk for fixed expected return are obtained through the optimization procedure described in Sect. 2.2. Panel 2a displays mean-$\rho_{CPT, \eta}$ efficient frontiers for subjects with a different degree of loss aversion ($\lambda$), but the same level of risk aversion for gains ($\alpha$), risk preference for losses ($\beta$) and probability distortions ($\delta, \gamma$) in accordance with

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\(^7\) Given the first four moments of each asset and the corresponding correlation matrix, we implement an algorithm able to estimate the parameters of the MVG distribution (see Hitaj and Mercuri 2013a, b). We then simulate a sample of $10^4$ observations for the log-returns by means of the MVG r.v. with the desired parameters. The reason for using the MVG to simulate assets’ returns is related to the fact that this distribution can capture some of the stylized facts of assets’ returns (see Hitaj and Mercuri 2013b and the references therein).
the parameters estimated in Tversky and Kahneman (1992). The mean-variance efficient frontier (black dashed line) in the $\rho_{CPT,\eta}$ scale is reported as well. Panel 2a shows that for a given level of mean, a higher level of loss aversion implies a higher degree of risk. This behaviour becomes clear if we recall that the parameter $\lambda$ refers to people’s tendency to strongly prefer avoiding losses than acquiring gains (Kahneman and Tversky 1979). Consequently, the greater the value of the loss aversion is, the greater the loss of satisfaction in front of the same decrease in wealth is. This implies that, as $\lambda$ increases, the CPT value function of a risky position decreases and, consequently, the corresponding $\rho_{CPT,\eta}$ increases. Panel 2b refers to investors with the same level of loss aversion, risk preference in losses and weighting probability functions, but a different degree of risk aversion in gains $\alpha$. Thus $\lambda$, $\beta$, $\delta$, $\gamma$ are fixed and $\alpha$ changes. In this case, for a given level of mean, higher values of $\alpha$ determine higher degrees of risk only up to a certain level, after which the situation may change.

Panel 2c considers the mean-$\rho_{CPT,\eta}$ efficient frontiers for $\lambda$, $\alpha$, $\delta$, $\gamma$ fixed and $\beta$ variable. Here, we observe that for a given expected return, a more risk-lover (in losses) agent (hence with $\beta$ close to 0) should consider his portfolio less risky than an agent

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8 Loss aversion implies, for example, that one who loses one hundred dollars will lose more satisfaction than that gained from one hundred dollars windfall.

9 The greater $\beta \in (0, 1)$, the lower the attitude to risk in losses.
with greater $\beta$. Panel 2d (respectively, panel 2e) indicates the behaviour of the efficient frontiers when the only parameter that changes is $\gamma$ (respectively, $\delta$); hence, here we can see how the probability weighting functions $T^-$ (resp. $T^+$) affect the efficient frontiers. We emphasize that the situation in the two graphics is different. In particular, in Panel 2d (where $\gamma$ changes), the efficient frontiers move to the right as $\gamma$ increases; the opposite happens in Panel (2e). To interpret this trend, we recall firstly that, as $\gamma$ increases, the weighting probability function for losses tends to the identity function (i.e. no distortion of probability occurs with respect to losses). On the contrary, as $\gamma$ decreases, the probabilities of extreme losses are over-weighted, while the probabilities of returns closed to the reference point $\tau$ or below it are under-weighted. It is reasonable that the impact of extreme losses on a CPT value function depends on how overweighted their probabilities are. Hence we would expect that, as $\gamma$ increases, the impact of extreme losses decreases. Consequently, the same portfolio is perceived as less risky by a CPT investor with higher $\gamma$. This reasoning is in accordance with the experimental evidence. Similarly, it should be intuitive that when $\delta$ decreases, the extreme gains are over-weighted and hence the expectation of the portfolio is overestimated. This implies that the riskiness of the portfolio is underestimated in comparison with the absence of distortion. The lower $\delta$ is, the smaller the risk associated with the portfolio is.

### 3.2 Empirical analysis

In dealing with real market data, different CPT portfolios are constructed by considering different values for the parameters of the CPT value function. In particular, $\alpha = [0.2, 0.4, 0.5, 0.6, 0.8, 0.9]$, $\beta = [0.2, 0.4, 0.5, 0.6, 0.8, 0.9]$, $\lambda = [1, 1.5, 2.25]$, $\delta = [0.6, 0.75, 0.9]$ and $\gamma = [0.6, 0.75, 0.9]$. We construct the CPT portfolios by considering all the permutations between these values of parameters, taking into account the constraints on the CPT parameters explained in Sect. 2. In total 567 CPT portfolios are considered. The effect of the CPT parameters on the optimal allocation strategies and the comparison with other traditional portfolios is analysed in this section. As a robustness check, three portfolios with different characteristics are considered. The first portfolio is composed by 12 hedge funds indexes and it is representative of non-normally distributed portfolios. In fact, hedge funds are usually characterized by an asymmetric and leptokurtic behavior. The second and third investment universes are made of 14 equities selected from the S&P 500 index. One collects the 14 equities with the lowest Jarque–Bera test value; we refer to this portfolio as equity portfolio 1. The other is also composed by 14 equities but now selected randomly and it is indicated as equity portfolio 2. These portfolios have been introduced for comparative purposes and to test the effects of a normality assumption.

To evaluate portfolio strategies, several criteria exists. In this paper we focus on the diversification of portfolio weights (see Sect. 3.2.1 below for more details) and on risk-return performance measures (see Sect. 3.2.2). Both these aspects are investigated through a rolling window (RW) type methodology. The rolling window methodology

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10 The data set is taken from [www.hedgefundresearch.com](http://www.hedgefundresearch.com).

11 The data set for the second and third portfolios is downloaded from Bloomberg.
is characterized by an in-sample period of length \( n \) and an out-of-sample period of length \( h \). Results related to diversification and risk-performance measure are obtained, respectively, by means of in-sample analysis and out-of-sample analysis: the data of the first in-sample window of width \( n \), from \( t = 1 \) to \( t = n \), are used to estimate the optimal weights \(^{12}\) and to quantify the diversification of the portfolio in the first window. These weights are kept constant in the out-of-sample period, from \( t = n + 1 \) to \( t = n + h \), where the out-of-sample performance is calculated. Next, the window is rolled \( h \) steps forward, using data from \( t = h + 1 \) to \( n + h \), while discarding the first \( h \) observations. The weights of the relevant portfolio are updated by solving the optimal allocation problem in the new subsample and the performance is re-estimated using data from \( t = n + h + 1 \) to \( n + 2h \). And so forth. In this paper we use a rolling window strategy with an in-sample-period of length 250 (or 175) days \(^{13}\) and an out-of-sample period of length 5 days (or 20 days). \(^{14}\)

The three data sets adopted in this case study consist of daily log-returns over the January 2008–November 2014 period and therefore include a sample size of 1737 observations for each portfolio. Tables 1, 3 and 5 present summary statistics and Jarque–Bera test values, respectively, for the hedge fund portfolio, equity portfolio 1 and equity portfolio 2.

From Table 1 we observe that the skewness of each hedge fund index (except Merger Arbitrage that has a skewness of 2.205) ranges from \(-4.135\) to \(-0.180\), indicating that the empirical distributions of daily hedge fund returns are skewed to the left. The kurtosis are well above 3 (the highest value is 75.771 and the lowest value is around 5.4) for all components, indicating the presence of fat tails and thus a deviation from the normal distribution. This is also confirmed by the JB test, which rejects the normality assumption at 5% significance level for all the hedge fund indexes under investigation. Table 3 presents the annual statistics for equity portfolio 1. Although the null hypothesis of normality over all the period under analysis 2008–2014 is rejected, it can be shown that for subsamples of length 1 year (or 6 months) the majority of the components are normally distributed. \(^{15}\) We also note that the skewness and kurtosis values are lower than those of the hedge fund portfolio. We thus consider this portfolio as approximately (close to) normal. Table 5 shows the statistics for equity portfolio 2. It is evident that the null hypothesis of normality distribution, using the Jarque–Bera test, is rejected for all the components. In particular, the null hypothesis of normality is rejected for almost all the components in the majority of sub-periods with length of 1 year (or 6 months). Summarizing, Tables 1, 3 and 5 show that the portfolios under analysis have different characteristics; two are portfolios whose components deviates from the normal distribution, and one has approximately normally distributed returns in the in-sample periods.

\(^{12}\) The optimal portfolio is obtained by solving problem (8) for a CPT investor [or by solving for the GMV or the MV portfolio considering as feasible region (5)].

\(^{13}\) We refer to the in-sample period also as 1 year (or 6 months).

\(^{14}\) We refer to the out-of-sample period also as 1 week (or 1 month).

\(^{15}\) More details about the results concerning the JB test for sub-samples are omitted for space constraints, but they are available upon request for the interested reader.
Table 1 Annual descriptive statistics of the 12 hedge funds indexes under observation

| Index                                           | Mean  | STD   | Skewness | Kurtosis | p value | JB-test |
|-------------------------------------------------|-------|-------|----------|----------|---------|---------|
| 'HFRX Absolute Return Index'                    | -0.023| 0.027 | -0.928   | 9.732    | 0.001   | 3529.16 |
| 'HFRX ED: Distressed Restructuring Index'       | -0.051| 0.045 | -3.010   | 40.621   | 0.001   | 105,057.98 |
| 'HFRX ED: Merger Arbitrage Index'               | 0.030 | 0.049 | 2.205    | 75.771   | 0.001   | 384,679.20 |
| 'HFRX EH: Equity Market Neutral Index'          | -0.010| 0.040 | -0.180   | 5.892    | 0.001   | 614.66  |
| 'HFRX Equal Weighted Strategies Index'          | -0.010| 0.030 | -1.763   | 19.522   | 0.001   | 20,656.46 |
| 'HFRX Equity Hedge Index'                       | -0.020| 0.069 | -0.886   | 9.219    | 0.001   | 3026.45 |
| 'HFRX Event Driven Index'                       | 0.003 | 0.050 | -1.306   | 16.662   | 0.001   | 14,002.92 |
| 'HFRX Global Hedge Fund Index'                  | -0.012| 0.039 | -1.339   | 14.107   | 0.001   | 9447.82 |
| 'HFRX Macro/CTA Index'                          | -0.013| 0.057 | -0.338   | 5.417    | 0.001   | 455.69  |
| 'HFRX Market Directional Index'                 | -0.007| 0.064 | -1.619   | 19.560   | 0.001   | 20,607.25 |
| 'HFRX Relative Value Arbitrage Index'           | -0.009| 0.049 | -1.871   | 38.123   | 0.001   | 90,299.09 |
| 'HFRX RV: FI-Convertible Arbitrage Index'      | -0.055| 0.081 | -4.135   | 41.606   | 0.001   | 112,817.85 |

The period under analysis spans January 2008 to November 2014, consisting in 1737 daily observations. For each hedge fund, we report annual mean, annual standard deviation, skewness, kurtosis, JB-Test.
3.2.1 Diversification

The portfolio diversification is evaluated using the modified Herfindahl Index, which is the measure of portfolio concentration defined as:

\[ H_I = \frac{\sum_{i=1}^{n} w_i^2 - \frac{1}{n}}{1 - \frac{1}{n}}. \]

\( H_I \) takes value 0 for the equally-weighted portfolio (that is portfolios with \( w_i = \frac{1}{n} \) for all \( i = 1, \ldots, n \)); it takes value 1 when all is invested in one asset (highest concentrated portfolio).

Figure 3 reports the average \( H_I \) for optimally selected hedge fund portfolios considering several values of the risk aversion, loss aversion and risk preference parameters in case of the rolling window methodology with in-sample period of 1 year and moving step \( h \) equal to 1 week. We also report the average \( H_I \) of the GMV and MV portfolios. Every point in the figure (with the exception of the two points describing the GMV and the MV diversification) is identified by a data structure including the loss aversion \( \lambda \), the risk aversion \( \alpha \), the risk preference in losses \( \beta \) and the average \( H_I \). The figure is divided into 6 sectors, which correspond to different values of \( \alpha \). The horizontal axis reports the value of \( \beta \) and the value of \( \lambda \) is indicated near each point. The average \( H_I \) is displayed on the vertical axis. The CPT parameters \( \gamma, \delta \) are fixed, respectively to 0.6 and 0.9 while \( \alpha, \beta \) and \( \lambda \) vary. As expected, the GMV model exhibits the most diversified portfolio while the MV model is among the most concentrated ones. In fact, for this portfolio the average \( H_I \) is greater than 0.9.\(^{16}\) It can be observed that for given risk aversion \( \alpha \) and risk preference in losses \( \beta \), the average \( H_I \) increases with \( \lambda \). This means that investors with higher loss aversion prefer less diversified portfolios. When \( \beta \) increases and all the other parameters are kept constant, the portfolio diversification decreases. In addition, when \( \alpha \) increases and the other parameters remain constant the portfolio diversification increases.

Moreover we observe that the effects of the CPT parameters on portfolio diversification are the same also for the other two portfolios under analysis.\(^{17}\) This leads us to conclude that the effect of CPT parameter values is independent from the portfolio’s characteristics.

We stress that analogous conclusions hold when considering different values of \( \gamma \) and \( \delta \) and also a different rolling window size.

3.2.2 Risk-return performance

Financial analysts and individual investors need effective tools to choose the best investment opportunity among available financial products. Risk-adjusted perfor-

\(^{16}\) The fact the MV portfolio is concentrated is due to the value of the risk aversion, which is fixed at 1. The level of risk aversion has an important impact on the portfolio (see Hitaj and Zambruno (2016) where a detailed analysis has been performed in case of a CARA utility function). Considering different levels of risk aversion goes beyond the objective of this paper.

\(^{17}\) For space constraints the results obtained for the two equity portfolios are not reported, but are available upon author request.
Performance measures are used to compare the absolute returns or the relative returns (i.e. excess returns) to the risk taken to earn this return. In this paper, we examine the performance of the three datasets using the Information Ratio (IR) and the Omega Ratio ($\Omega$) and we then compare such quantities with the performances of the GMV and MV portfolios.

Loosely speaking, the Information Ratio is defined as the average excess return per unit of volatility in excess return:

$$IR = \frac{\bar{R}_P - \bar{R}_{ref}}{\sigma(R_P - \bar{R}_{ref})}.$$  

Here $\bar{R}_{ref}$ is the average return of a specified reference portfolio (the GMV portfolio, in this work). This performance measure allows us to check that the risk taken by the manager in deviating from the reference portfolio is sufficiently rewarded. Therefore a higher and positive IR is preferred. In order to understand if the out-of-sample IR gains are statistically significant different from zero, the $t$-statistic proposed in Goodwin (1998) is used.

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18 Of course each portfolio manager has his own reference portfolio. The reason for choosing the GMV is that, when dealing with $\mu/\sigma$ efficient frontier, it exhibits the minimum risk ($\sigma$).
Fig. 4 Hedge fund portfolio: out-of-sample monthly $IR_{GMV}$ for different levels of $\alpha$, $\beta$ and $\lambda$. Rolling-window 1 year—1 week. The point labelled MV indicates the result obtained from the Mean Variance model and the one labeled GMV indicates the result obtained using the Global Minimum Variance. The figure is divided in 6 parts, one for each level of $\alpha$. On the x-axis the value of $\beta$ is indicated; each point in the figure is associated with a number that indicates the value of $\lambda$. On the y-axis the value of the average out-of-sample Information Ratio (with respect to the GMV portfolio) for a particular combination of parameters is reported.

Omega Ratio is defined as:

$$\Omega = \frac{\mathbb{E}(R_P - \tau)^+}{\mathbb{E}(\tau - R_P)^+},$$

where $R_P$ denotes the return of the portfolio and $\tau$ is a specified threshold. For a given threshold, a higher ratio indicates that the portfolio provides more expected gains than expected losses and so would be preferred by an investor. The Omega Ratio results are reported only for practical illustration and we do not validate them on statistical basis since this measure introduces uncertainty on the out-of-sample results as it depends on the reference point.

Figures 4 and 5 report the hedge fund portfolio results, respectively, for the IR and $\Omega$, fixing $\gamma = 0.6$, $\delta = 0.9$ in the CPT value function, in case of the rolling window ‘1 year (250 days)—1 week (5 days)’. These figures are structured in the same way as Fig. 3 despite the fact that on the y-axis different quantities are reported. From Fig. 4 we notice that the IR of the CPT portfolios is almost always positive, meaning that

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19 We point out that $\Omega$ ratio is very sensitive to $\tau$ which can be different from investor to investor. In this analysis $\tau$ is set equal to 0.

20 Rolling window ‘1 year—1 week’ means that the in-sample period is 1 year and the out-of-sample period is 1 week.
the CPT portfolios are almost always better than GMV portfolio. The same conclusion can be derived for $\Omega$, since this quantity, for CPT portfolios, is higher than that of the GMV and MV ones (see Fig. 5). Moreover we observe that the Information Ratio is always positive independently from the rolling-window length and values of $\gamma$ and $\delta$.

In general better out-of-sample results in terms of Information Ratio are obtained in case of rolling-window 1 year—1 week. In order to understand whether the gain in terms of Information Ratio of a CPT portfolio with respect to the Global Minimum Variance is statistically significant different from zero we run a $t$-statistic as proposed in Goodwin (1998). In Table 2 are reported the percentages, over 567 CPT portfolios, when the gain in terms of Information Ratio is statistically significant different from zero, for the hedge fund portfolio, considering different significance levels and different rolling window strategies. From these results we can conclude that at a significant level of 15% for more than a half of the CPT portfolios (over the 567 under consideration) we have an IR statistically significant different from 0 for almost all the rolling-window strategies.

Comparing the portfolios in terms of $\Omega$—see Fig. 5, where the in-sample period is long (1 year) and the out-of-sample period is short (1 week)—we can conclude that

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21 For space constraints the results obtained in case of the other three rolling window strategies are not reported, but can be obtained from the authors upon request.
In Sects. A.1 and A.2 the out-of-sample results for the two equity portfolios are reported. We observe that when assets’ returns are near to the normal distribution (equity portfolio 1) only occasionally CPT portfolios perform better than the GMV or the MV one. This result is valid with respect to both the Information Ratio and the Omega Ratio as reported, respectively, in Figs. 6 and 7 in “Appendix 1”. This evidence suggests that when assets’ returns are almost normally distributed a more complicated model (such as the CPT) may lead to a loss in terms of both (out-of-sample) Information Ratio and Omega Ratio. In order to validate if the loss obtained in terms of Information Ratio, is statistically significant, the $t$-statistic proposed in Goodwin (1998) is used. Statistics on the Information Ratio associated with the equity portfolio 1, for different significance levels and different rolling window strategies, are reported in Table 4 in “Appendix 1”. From these results it is evident that at a confidence level of 85% there are less than 0.5% of the overall CPT portfolios with a statistically significant loss. Therefore, the conclusion drawn from the analysis is that the model based on the CPT value function may lead to losses, but these losses are not statistically significant, according to the Information Ratio.

Out-of-sample Information and Omega Ratios for the equity portfolio 2 are reported, respectively, in Figs. 8 and 9 in “Appendix 2”. As analysed previously, the components of this portfolio are not normally distributed and it is not surprising that the conclusions that can be drawn for this portfolio are the same as those of the hedge fund portfolio.

We can summarize the evidences by saying that in case of portfolios whose assets’ returns are far from being normal, more sophisticated CPT-based models lead to higher out-of-sample performance compared to that obtained by GMV or MV portfolios. The gains obtained in this case are statistically significant from zero. While when assets’ returns are nearly normal, sticking to the GMV portfolio leads almost always to higher out-of-sample performance. Nevertheless, the loss obtained from using a CPT-based approach is not statistically significant.

### 4 Conclusions

In this paper the impact of the CPT parameters on the mean-$\rho_{CPT,\eta}$ efficient frontier is analysed. For this purpose, a simulation procedure for asset returns is implemented. From the collected results, it is clear that each parameter influences the behavior of the

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**Table 2** Hedge fund portfolio: percentage when gain in terms of Information Ratio of the CPT portfolios, having as benchmark the Global Minimum Variance portfolio, is statistically significant different from 0, for the 4 rolling-window strategies considered and different levels of significance

| Sign. level | Hedge fund portfolio |
|-------------|----------------------|
|             | 1%  | 5%  | 10% | 15% |
| RW 250-5    | 0.4 | 3.4 | 19.0| 64.2|
| RW 175-5    | 0.0 | 7.8 | 19.9| 46.5|
| RW 175-20   | 2.8 | 19.2| 30.7| 53.9|
| RW 250-20   | 0.0 | 1.4 | 12.3| 54.2|
efficient frontier approximately in the same way, independently from the degree of portfolio diversification. Moreover, the characteristics of the CPT portfolios using actual market data have been also investigated. To this purpose a portfolio composed of hedge fund indices and two equity portfolios are considered. Different CPT portfolios are constructed (corresponding to different choices of the CPT objective function parameters, 567 in total) and compared with well known portfolio strategies, such as the Global Minimum Variance (GMV) and a Mean-Variance (MV) optimal portfolio.

Empirically, we observe that CPT portfolios are more diversified than the MV, but less diversified than the GMV one. Moreover, it is evident that CPT parameters play an important role on the portfolio diversification: increasing $\lambda$ (or $\alpha$) and keeping all the other parameters fixed, leads to a more diversified portfolio. If $\beta$ increases (while all the other parameters remain constant) the diversification decreases. These results are valid independently from the characteristics of the returns distribution.

The out-of-sample analysis suggests that the CPT model leads to a higher Omega ratio compared with the MV or GMV when the distribution of assets’ returns is far from the normal one. When calculating the Information ratio of CPT portfolios (567 in total), having as benchmark the Global Minimum Variance, in all tested strategies, the CPT portfolios have a positive IR when assets’ returns are not normally distributed: CPT portfolios are thus performing consistently better than the benchmark (GMV). In order to analyse if these gains are statistically different from zero a $t$-statistic test (see Goodwin 1998) has been considered. We conclude that at a significance level of 15% more than half of the CPT portfolios have an IR which is statistically significant different from zero, for almost all the rolling-window strategies, in case of assets’ returns whose distribution is far away from the normal one. While when assets’ returns distribution is near to the normal one, the majority of the CPT portfolios considered lead to a loss compared to GMV and MV. Nevertheless in the majority of the cases this loss is not statistically different from zero.

We believe these results are due to the fact that CPT is considering all the distribution of assets’ returns, while MV (GMV) relies only on the first two moments of the distribution. When assets’ returns are normally distributed it is well known, in the literature, that a quadratic utility function (or the GMV) will be a good candidate for an investor. When however assets’ returns are not normally distributed, a model based on realistic assumptions consistent with stylized market evidencies will lead to better performance than GMV (or MV). In the literature, in order to overcome the limits of the MV (or the GMV), models that consider higher moments have been proposed. In this paper a different approach (the CPT utility function) has been used. The results obtained are consistent with those obtained in the EUT when higher moments are introduced in the portfolio allocation problem [see for instance Hitaj and Zambruno (2016)]. It is of great interest a comparative analysis between optimal CPT portfolios and optimal strategies based on returns’ higher moments. This topic is left for future research.

This paper sheds some light on CPT portfolios’ characteristics and their performance relative to GMV and MV portfolio, in particular in the domain of risky portfolios, based on stocks or hedge funds.
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A Appendix

In the following the results obtained for the two equity portfolios under analysis are reported.

A.1 Equity portfolio 1 results

See Tables 3, 4 and Figs. 6, 7.

Table 3  Annual descriptive statistics for the equity portfolio 1

| Bloomberg ticker | Annual Mean | Annual SD | Skewness | Kurtosis | p value | JB-test |
|------------------|-------------|-----------|----------|----------|---------|---------|
| AMAT UW          | 0.045       | 0.360     | -0.152   | 7.129    | 0.001   | 1243.31 |
| LRCX UW          | 0.093       | 0.408     | -0.076   | 5.375    | 0.001   | 410.97  |
| ATI UN           | -0.135      | 0.565     | -0.273   | 7.330    | 0.001   | 1382.04 |
| CSX UN           | 0.127       | 0.361     | -0.104   | 6.794    | 0.001   | 1047.47 |
| BMY UN           | 0.117       | 0.249     | 0.134    | 7.430    | 0.001   | 1428.64 |
| NSC UN           | 0.110       | 0.330     | -0.361   | 7.352    | 0.001   | 1411.89 |
| EMC UN           | 0.075       | 0.322     | 0.098    | 7.817    | 0.001   | 1685.79 |
| KSS UN           | 0.036       | 0.345     | -0.032   | 7.597    | 0.001   | 1533.49 |
| HON UN           | 0.070       | 0.300     | -0.112   | 7.072    | 0.001   | 1206.26 |
| ETN UN           | 0.052       | 0.340     | -0.234   | 6.519    | 0.001   | 914.36  |
| TJX UN           | 0.221       | 0.284     | -0.058   | 7.779    | 0.001   | 1657.74 |
| PHM UN           | 0.105       | 0.582     | 0.130    | 6.578    | 0.001   | 933.58  |
| PAYX UW          | 0.043       | 0.242     | 0.127    | 7.447    | 0.001   | 1439.45 |
| PH UN            | 0.084       | 0.349     | -0.018   | 6.392    | 0.001   | 834.80  |

These components are the S&P 500 equities with the lowest Jarque–Bera (JB) test value. The period under analysis spans January 2008 to November 2014, consisting in 1737 daily observations. For each equity, we report annual mean, annual standard deviation, skewness, kurtosis, JB-test.
Table 4  

| Sign. level | Equity portfolio 1 |
|-------------|--------------------|
|             | 1%     | 5%     | 10%    | 15%    |
| RW 250-5    | 0.00   | 0.18   | 0.18   | 0.35   |
| RW 175-5    | 0.00   | 0.00   | 0.00   | 0.00   |
| RW 175-20   | 0.00   | 0.00   | 0.00   | 0.00   |
| RW 250-20   | 0.00   | 0.00   | 0.18   | 0.18   |

These results clearly indicate that the loss that we may have in terms of out-of-sample performance in using the CPT instead of the GMV is non statistically significant from zero, in the case assets’ returns are near to the normal distribution.

Fig. 6  

Equity portfolio 1 in case of rolling-window 1 year (250 days)—1 week (5 days). The figure is divided in 6 sectors, one for each level of $\alpha$. On the $x$-axis the value of $\beta$ is indicated; each point in the figure is associated with a number that indicates the value of $\lambda$. On the $y$-axis the value of the average out-of-sample Information Ratio (with respect to the GMV portfolio) for a particular combination of parameters is reported. It is evident that the Information Ratio in the majority of the cases under consideration is less than zero meaning that, in an out-of-sample perspective, the CPT portfolios lead to a loss with respect to the GMV one.
Fig. 7 Equity portfolio 1: Rolling-window 1 year—1 week. The figure is divided in 6 parts, one for each level of \( \alpha \). On the x-axis the value of \( \beta \) is indicated; each point in the figure is associated with a number that indicates the value of \( \lambda \). On the y-axis the value of the average out-of-sample monthly Omega Ratio for different levels of \( \alpha \), \( \beta \) and \( \lambda \) is reported. The point labelled MV indicates the result obtained from the Mean Variance model and the one labeled GMV indicates the result obtained using the Global Minimum Variance.

A.2 Equity portfolio 2 results

See Tables 5, 6 and Figs. 8, 9.

Table 5 Annual descriptive statistics for the components of equity portfolio 2

| Bloomberg ticker | 01/2008-11/2014 Annual Mean | Annual SD | Skewness | Kurtosis | p value | JB-test |
|------------------|----------------------------|-----------|----------|----------|---------|---------|
| TSS UN           | 0.029                      | 0.289     | −0.734   | 12.659   | 0.001   | 6924.87 |
| LUV UN           | 0.177                      | 0.373     | −0.506   | 9.138    | 0.001   | 2807.55 |
| ROST UW          | 0.284                      | 0.301     | 0.337    | 7.931    | 0.001   | 1797.04 |
| CTSH UW          | 0.174                      | 0.403     | −0.428   | 12.335   | 0.001   | 6374.66 |
| VZ UN            | 0.031                      | 0.237     | 0.316    | 11.407   | 0.001   | 5155.73 |
| ESV UN           | −0.085                     | 0.435     | −0.518   | 11.494   | 0.001   | 5311.23 |
| WY UN            | 0.026                      | 0.385     | −0.424   | 8.423    | 0.001   | 2185.37 |
| AIV UN           | 0.061                      | 0.557     | −0.518   | 14.999   | 0.001   | 10,521.26 |
| MSFT UW          | 0.046                      | 0.297     | 0.137    | 12.207   | 0.001   | 6154.25 |
| PXD UN           | 0.155                      | 0.508     | −0.575   | 9.122    | 0.001   | 2815.19 |
| FMC UN           | 0.099                      | 0.406     | −0.664   | 18.799   | 0.001   | 18,234.78 |
| HCBK UW          | −0.062                     | 0.363     | −0.058   | 9.943    | 0.001   | 3497.70 |
These components are selected randomly from S&P 500 investment universe. The period under analysis spans January 2008 to November 2014, consisting in 1737 daily observations. For each equity, we report annual mean, annual standard deviation, skewness, kurtosis, JB-Test. These components are selected randomly from the components of S&P 500 index.

Table 6

| Sign. level | Equity portfolio 2 |
|-------------|-------------------|
|             | 1%    | 5%    | 10%   | 15%   |
| RW 250-5    | 1.23  | 25.22 | 60.32 | 79.01 |
| RW 175-5    | 0.00  | 4.23  | 18.05 | 51.69 |
| RW 175-20   | 0.00  | 2.65  | 26.93 | 56.92 |
| RW 250-20   | 0.00  | 3.88  | 24.87 | 56.74 |

At a significance level of 15% in the majority of the cases the gain in the out-of-sample performance, in terms of Information Ratio, is statistically significant different from zero, when assets’ returns are far away from the normal distribution.

Fig. 8

Equity portfolio 2: rolling-window 1 year (250 days)—1 week (5 days). The figure is divided in 6 parts, one for each level of $\alpha$. On the $x$-axis the value of $\beta$ is indicated; each point in the figure is associated with a number that indicates the value of $\lambda$. On the $y$-axis the value of the average out-of-sample Information Ratio (with respect to the GMV portfolio) for a particular combination of parameters is reported. Having always an Information Ratio greater than zero means that, in an out-of-sample perspective, the CPT model leads to a gain with respect to the GMV.
Fig. 9  Equity portfolio 2: rolling-window 1 year—1 week. The figure is divided in 6 parts, one for each level of $\alpha$. On the x-axis the value of $\beta$ is indicated; each point in the figure is associated with a number that indicates the value of $\lambda$. On the y-axis the value of the average out-of-sample monthly Omega Ratio for different levels of $\alpha$, $\beta$ and $\lambda$ is reported. The point labeled $MV$ indicates the result obtained from the Mean Variance model and the one labeled $GMV$ indicates the result obtained using the Global Minimum Variance. It is evident that in an out-of-sample the majority of the CPT portfolios lead to gains, in terms of Omega Ratio, with respect to the GMV and to the MV ones.

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