HYPERCYCLIC COMPOSITION OPERATORS ON THE LITTLE BLOCH SPACE $B_0$ AND THE BESOV SPACES $B_p$

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ABSTRACT. Let $S(\mathbb{D})$ be the collection of all holomorphic self-maps on $\mathbb{D}$ of the complex plane $\mathbb{C}$, and $C_\varphi$ the composition operator induced by $\varphi \in S(\mathbb{D})$. We obtain that there are no hypercyclic composition operators on the little Bloch space $B_0$ and the Besov space $B_p$.

1. Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc in the complex plane $\mathbb{C}$ and $S(\mathbb{D})$ be the collection of all holomorphic self-maps on $\mathbb{D}$. We denote $dA(z) = dx dy$ the Lebesgue area measure on $\mathbb{C}$. For the composition operator $C_\varphi$ induced by $\varphi \in S(\mathbb{D})$ is defined as

$$C_\varphi f(z) = f \circ \varphi(z), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}. $$

The one-to-one holomorphic functions that map $\mathbb{D}$ onto itself, called the Möbius transformations, and denoted by $\mathcal{M}$ (also $\text{Aut}(\mathbb{D})$), have the form $\lambda \varphi_a$, where $|\lambda| = 1$ and $\varphi_a$ is the basic conformal automorphism defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D},$$

for $a \in \mathbb{D}$. The following identities are easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2}$$

and

$$(1 - |z|^2)|\varphi_a'(z)| = 1 - |\varphi_a(z)|^2. \quad (1.1)$$

A linear space $X$ of analytic functions on the open unit disk $\mathbb{D}$ is said to be Möbius-invariant, if $f \circ S \in X$ for all $f \in X$ and all $S \in \mathcal{M}$ and $X$ has a seminorm $\| \cdot \|_X$ such that $\| f \circ S \|_X = \| f \|_X$ for each $f \in X$ and each $S \in \mathcal{M}$.

The well-known Möbius-invariant function space—the Besov spaces $B_p$ ($1 < p < \infty$) are defined as following

$$B_p = \{ f \in H(\mathbb{D}) : \| f \|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty \},$$

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Indeed, by (1.1) and the above identity also holds for \( S \) where
\[
\| f \|_{B_p} = \inf \left\{ \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z) : f(z) = \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z), z \in \mathbb{D} \right\}.
\]

Although the measure \( \lambda \) is not a finite measure on \( \mathbb{D} \), it is a \( \text{M"obius-invariant} \).
Indeed, by (1.1)
\[
d\lambda(\varphi_a(z)) = \frac{|\varphi_a'(z)|^2}{(1 - |\varphi_a(z)|^2)^2} d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2} = d\lambda(z).
\]

Hence we have the following change-of-variable formula
\[
\int_{\mathbb{D}} f \circ \varphi_a(z) d\lambda(z) = \int_{\mathbb{D}} f(u) d\lambda(u),
\]
for every positive measurable function \( f \) on \( \mathbb{D} \). From which it is easily seen that
\[
\| f \circ \varphi_a \|_{B_p} = \| f \|_{B_p}, \tag{1.2}
\]
and the above identity also holds for \( S = \lambda \varphi_a \in \mathcal{M} \) with \( |\lambda| = 1 \). Thus
\[
\text{if } f \in B_p \text{ then } f \circ S \in B_p \text{ for all } S \in \mathcal{M}.
\]
That is, \( B_p (1 < p < \infty) \) are \( \text{M"obius-invariant} \) spaces.

For \( p = 1 \), the Besov space \( B_1 \) consists of the analytic functions \( f \) on \( \mathbb{D} \) that admit the representation
\[
f(z) = \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z), \quad z \in \mathbb{D},
\]
where \( \{a_n\} \in l^1 \) and \( \lambda_n \in \mathbb{D} \) for \( n \in \mathbb{N} \). The norm in \( B_1 \) is defined as
\[
\| f \|_{B_1} = \inf \left\{ \sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z), z \in \mathbb{D} \right\}.
\]

It is evident that \( B_1 \) is the \( \text{M"obius invariant} \) subset of the bounded analytic functions space \( H^\infty \). On the other hand, \( B_1 \) has the following definition,
\[
B_1 = \{ f \in H(\mathbb{D}) : \| f \|_{B_1} = \int_{\mathbb{D}} |f''(z)| dA(z) < \infty \},
\]
even though the above semi-norm is not \( \text{M"obius-invariant} \), the Besov space \( B_1 \) is the minimal \( \text{M"obius-invariant} \) space (see, e.g. [1, 2]).

It is well-known that \( B_p (1 < p < \infty) \) are Banach spaces endowed with the norm denoted by \( \| f \|_p \),
\[
\| f \|_p^p = |f(0)|^p + \| f \|_{B_p}^p,
\]
Another \( \text{M"obius-invariant} \) space of analytic functions on \( \mathbb{D} \) is the Bloch space \( \mathcal{B} \),
\[
\mathcal{B} = \{ f \in H(\mathbb{D}) : \| f \|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \}.
\]

By (1.1) it follows that
\[
\| f \circ \varphi_a \|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)(f \circ \varphi_a)'(z)
\]
\[
= \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(\varphi_a(z))| \varphi_a'(z)
\]
\[
= \sup_{z \in \mathbb{D}} (1 - \varphi_a(z)^2)|f'(\varphi_a(z))|
\]
\[
= \| f \|_{\mathcal{B}}, \tag{1.3}
\]
for all $a \in \mathbb{D}$, and the above identities also hold for all $S \in \mathcal{M}$. That is, if $f \in \mathcal{B}$ then $f \circ S \in \mathcal{B}$ for all $S \in \mathcal{M}$.

Hence $\mathcal{B}$ is a Möbius-invariant space.

The little Bloch space $\mathcal{B}_0$ consists of all $f \in \mathcal{B}$ such that

$$
\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.
$$

Replacing "sup" by " lim " in (1.3), we get that $f \circ \varphi_a \in \mathcal{B}_0$ for every $f \in \mathcal{B}_0$ and $a \in \mathbb{D}$. Similarly, $\mathcal{B}_0$ is also a Möbius-invariant space. Both the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ are Banach spaces under the norm

$$
\|f\|_{\text{Bloch}} = |f(0)| + \|f\|_{\mathcal{B}}.
$$

The above Möbius-invariant spaces have the relationship $\mathcal{B}_1 \subset \mathcal{B}_p \subset \mathcal{B}_q \subset \mathcal{B}$ for each $1 < p < q < \infty$ (see, e.g. [25, Lemma 1.1]). Moreover, $\mathcal{B}_1$ is a subset of the little Bloch space $\mathcal{B}_0$ (see [28]) and the Bloch space $\mathcal{B}$ is maximal among all Möbius-invariant Banach spaces of analytic functions on $\mathbb{D}$ (see, e.g. [22]). The Besov space $\mathcal{B}_2$ is often referred to as the Dirichlet space $\mathcal{D}$, which is a Hilbert space with inner product

$$
\langle f, g \rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}dA(z) / \pi.
$$

The problem of boundedness and compactness of $C_{\varphi}$ has been studied in many function spaces, we refer the readers to the books [8, 25, 27, 29]. In the recent time, the papers [4, 5, 9, 10, 12, 20] play important parts in the theory of the hypercyclicity of composition operators $C_{\varphi}$ acting on analytic function spaces.

In the following, we introduce some definitions in dynamic systems. Let $L(X)$ denote the space of all linear continuous operators on a separable infinite dimensional Banach space $X$. For a positive integer $n$, the $n$-th iterate of $T \in L(X)$ denoted by $T_n$, is the function obtained by composing $T$ with itself $n$ times.

A continuous linear operator $T \in L(X)$ is called hypercyclic provided there is some $f \in X$ such that the orbit

$$
\text{Orb}(T, f) = \{T^n f : n = 0, 1, \cdots \}
$$

is dense in $X$. Such a vector $f$ is said to be a hypercyclic vector for $T$. Therefore, if a Banach space $X$ admits a hypercyclic operator, $X$ must be separable and infinite dimensional.

Since the polynomials are dense in the little Bloch space $\mathcal{B}_0$ (see, e.g. [27, Proposition 3.10]) and the polynomials are dense in $\mathcal{B}_p$ ($1 \leq p < \infty$) (see, e.g. [27, Proposition 6.2]), thus the little Bloch space $\mathcal{B}_0$ and $\mathcal{B}_p$ ($1 \leq p < \infty$) are separable infinite dimensional Banach spaces. This is why we investigate the composition operators on $\mathcal{B}_0$ and $\mathcal{B}_p$ ($1 \leq p < \infty$). For motivation, examples and background about linear dynamics we refer the reader to the books [6] by Bayart and Matheron, [9] by Grosse-Erdmann and Manguillot, and articles by Godefroy and Shapiro [11]. The papers [16]-[18] investigate other aspects of the hypercyclic property.

This paper is inspired by the result [9, Theorem 1.8]: "No linear fractional composition operator is hypercyclic on the Dirichlet space $\mathcal{D}$." We refer the readers to the paper [26], which contains the proof. Now in this paper, we want to characterize the hypercyclicity of composition operator $C_{\varphi}$ acting on $\mathcal{B}_0$ and $\mathcal{B}_p$ ($1 \leq p < \infty$). We will show that "No linear fractional composition operator
is hypercyclic on the little Bloch space $B_0$, $B_1$ and $B_p$ \((2 \leq p < \infty)\).” Since the Besov space $B_p$ is the Dirichlet space $D$, hence we generalize the above result to some extent. The paper is organized as follows: some lemmas are listed in section 2 and the main results are given in section 3 and section 4.

Throughout the remainder of this paper, $C$ will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Auxiliary Results

A linear fractional transformation is a mapping of the form

$$\varphi(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. We will write $\text{LFT}(\mathbb{D})$ to refer to the set of all such maps, which are self-maps of the unit disk $\mathbb{D}$. Those maps that take $\mathbb{D}$ onto itself are precisely the members of $\text{Aut}(\mathbb{D})$, so that $\text{Aut}(\mathbb{D}) \subset \text{LFT}(\mathbb{D}) \subset S(\mathbb{D})$.

We classify those maps according to their fixed point behaviour, see [23, p. 5]:

(a) Parabolic members of $\text{LFT}(\mathbb{D})$ have their fixed point on $\partial \mathbb{D}$.

(b) Hyperbolic members of $\text{LFT}(\mathbb{D})$ must have an attractive fixed point in $\overline{\mathbb{D}}$, with the other fixed point outside $\mathbb{D}$, and lying on $\partial \mathbb{D}$ if and only if the map is an automorphism of $\mathbb{D}$.

(c) Loxodromic and elliptic members of $\varphi \in \text{LFT}(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside $\mathbb{D}$. The elliptic ones are precisely the automorphisms in $\text{LFT}(\mathbb{D})$ with this fixed point configuration.

The following two lemmas are well-known, so we omit the details.

**Lemma 2.1.** [27, p. 82 (3.5)] For each $f \in B$, we have

$$|f(z)| \leq \|f\|_{\text{Bloch}} \log \frac{2}{1 - |z|^2}.$$

**Lemma 2.2.** [28, Theorem 8] For every $f \in B_p$ with $1 < p < \infty$, we have

$$|f(z)| \leq C\|f\|_{B_p} \left(\log \frac{2}{1 - |z|^2}\right)^{1-1/p}.$$  

**Remark 2.3.** From Lemma 2.1 and Lemma 2.2, we obtain that the norm convergence in $B_0$ (respectively, $B_p$ \((1 < p < \infty)\)) implies pointwise convergence.

**Lemma 2.4.** Let $\varphi \in S(\mathbb{D})$ with an interior fixed point on $\mathbb{D}$. Suppose that $C_\varphi$ is bounded on $B_0$ (respectively, $B_p$ \((1 < p < \infty)\)). Then the operator $C_\varphi$ is not hypercyclic on $B_0$ (respectively, $B_p$ \((1 < p < \infty)\)).

**Proof.** We prove for the little Bloch space $B_0$. Let $a \in \mathbb{D}$ be the fixed point of $\varphi$. Suppose that $f \in B_0$ is hypercyclic for $C_\varphi$ and for each $g \in B_0$ there exists a sequence $\{n_k\}$ such that $C_\varphi^{n_k} f$ tends to $g$ in $B_0$, that is,

$$\|C_\varphi^{n_k} f - g\|_{\text{Bloch}} \to 0 \text{ as } k \to \infty.$$

By Remark 2.3 and $f(\varphi^{n_k}(a)) = f(a)$ for every $k \in \mathbb{N}$, it follows that

$$g(a) = \lim_{k \to \infty} (C_\varphi^{n_k} f)(a) = \lim_{k \to \infty} (C_\varphi^{n_k} f)(a) = \lim_{k \to \infty} f(\varphi^{n_k}(a)) = f(a),$$

that is not the case for every $g \in B_0$. Thus the operator $C_\varphi$ is not hypercyclic on $B_0$. The proof for the Besov spaces $B_p$ is similar, so we omit the details. This ends the proof. \qed
Remark 2.5. In the following, we need only consider \( \varphi \) is parabolic or hyperbolic case.

The following lemma is a necessary condition of the hypercyclic composition operator \( C_\varphi \) on \( B_0 \) and \( B_p \).

**Lemma 2.6.** Suppose that \( \varphi \in S(\mathbb{D}) \) and the bounded composition operator \( C_\varphi \) is hypercyclic on \( B_0 \) (respectively, \( B_p \), \( 1 < p < \infty \)), then the compositional symbol \( \varphi \) is univalent.

**Proof.** We prove for the space \( B_0 \). Suppose that \( \varphi(z_1) = \varphi(z_2) \) for some \( z_1, z_2 \in \mathbb{D} \) with \( z_1 \neq z_2 \). We pick \( g \in B_0 \) such that \( g(z_1) \neq g(z_2) \), and let \( f \in B_0 \) be a hypercyclic vector for \( C_\varphi \). By Lemma 2.1, for each \( n \in \mathbb{N} \),

\[
|C_\varphi^n f - g|(z) \leq \|C_\varphi^n f - g\|_\text{Bloch} \log \frac{2}{1 - |z|^2}.
\]

So for \( \epsilon := |g(z_1) - g(z_2)| > 0 \), we choose \( n \in \mathbb{N} \) be large enough so that

\[
|C_\varphi^n f - g|(z) < \frac{\epsilon}{4} \text{ for } z = z_1, z_2.
\]

On the other hand, since \( \varphi(z_1) = \varphi(z_2) \), it follows that

\[
\epsilon = |g(z_1) - g(z_2)| \leq |g(z_1) - C_\varphi^n f(z_1)| + |C_\varphi^n f(z_1) - g(z_2)|
= |g(z_1) - C_\varphi^n f(z_1)| + |f(\varphi^n(z_1)) - g(z_2)|
= |g(z_1) - C_\varphi^n f(z_1)| + |f(\varphi^n(z_2)) - g(z_2)|
= |g(z_1) - C_\varphi^n f(z_1)| + |C_\varphi^n f(z_2) - g(z_2)|
< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\]

we get a contraction. So \( \varphi \) must be univalent. The proof for the \( B_p \), \( 1 < p < \infty \) is similar. This completes the proof. \( \square \)

**Lemma 2.7.** [19] Schwarz-Pick Lemma] For \( \varphi \in S(\mathbb{D}) \), we have

\[
\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| \leq 1
\]

for all \( z \in \mathbb{D} \).

**Lemma 2.8.** [13] Theorem 6.7] Let \( T \) be an operator on a complex Fréchet space \( X \). If \( x \in X \) is such that \( \{ \lambda T^n x, \lambda \in \mathbb{C}, |\lambda| = 1, \text{ and } n \in \mathbb{N}_0 \} \) is dense in \( X \), then \( \text{orb}(x, TX) \) is dense in \( X \) for each \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \).

In particular, for any \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \), \( T \) and \( \lambda T \) have the same hypercyclic vectors, that is,

\[
HC(T) = HC(\lambda T).
\]

3. Hypercyclicity on the little Bloch space \( B_0 \)

For \( \varphi \in S(\mathbb{D}) \) and by the Schwarz-Pick lemma (Lemma 2.7), we obtain that

\[
\|C_\varphi f\|_{\text{Bloch}} = |f(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(\varphi(z))\varphi'(z)|
\leq |f(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |\varphi(z)|^2)|f'(\varphi(z))|
\leq |f(\varphi(0))| + \|f\|_B < \infty,
\]
then the composition operator $C_\varphi$ is always a bounded operator from $B$ into $B$. Moreover, if $\varphi \in B_0$, then $C_\varphi$ maps $B_0$ into $B_0$. In this section, we always assume $C_\varphi$ is bounded on $B_0$.

**Case I.** $\varphi \in \text{Aut}(\mathbb{D})$.

**Theorem 3.1.** Suppose $\varphi \in \text{Aut}(\mathbb{D})$ and the composition operator $C_\varphi$ is bounded on $B_0$. Then $C_\varphi$ is not hypercyclic on $B_0$.

**Proof.** Suppose that there is an $f \in B_0$ such that the set $\{C_k^\varphi f : k \in \mathbb{N} \cup \{0\}\}$ is dense in $B_0$. By (1.3) it follows that

$$\|C_k^\varphi f\|_{\text{Bloch}} = |f(\varphi^k(0))| + \|f \circ \varphi^k\|_B = |f(\varphi^k(0))| + \|f\|_B.$$  

For $f_1(z) = z \in B_0$, there exists a subsequence $\{\varphi^{k_j}\}_j$ such that

$$\|C_{\varphi^{k_j}} f - f_1\|_{\text{Bloch}} \to 0 \text{ as } j \to \infty. \quad (3.1)$$

From which and Remark 2.3, it is clear that for $z = 0$,

$$f(\varphi^{k_j}(0)) \to 0 \text{ as } j \to \infty. \quad (3.2)$$

By (3.1) and (3.2), we have

$$\|f\|_{\text{Bloch}} = |f(0)| + \|f\|_B \to |f(0)| + 1. \quad (3.3)$$

On the other hand, there exists another sequence $\{\varphi^{k_j}\}_j$ such that

$$\|C_{\varphi^{k_j}} f - f_2\|_{\text{Bloch}} \to 0 \text{ as } j \to \infty.$$  

Similarly, $f \circ \varphi^{k_j}(0) \to 0$ as $j \to \infty$. Besides

$$\|f\|_{\text{Bloch}} = |f(0)| + \|f\|_B \to |f(0)| + \|f\|_B + |f \circ \varphi^{k_j}(0)| \to |f(0)| + \|f\|_B + |f \circ \varphi^{k_j}(0)| \to |f(0)| + \frac{2\sqrt{3}}{9}. \quad (3.4)$$

Combining (3.3) and (3.4), we get a contraction. Therefore, the composition operator $C_\varphi$ is not hypercyclic on $B_0$. This completes the proof. \hfill \Box

**Case II.** $\varphi \notin \text{Aut}(\mathbb{D})$. For this case, we only consider $\varphi$ with no interior fixed point in $\mathbb{D}$ by Remark 2.5.

**Theorem 3.2.** Suppose $\varphi \notin \text{Aut}(\mathbb{D})$ and $\varphi$ has no interior fixed point in $\mathbb{D}$. Further assume that the composition operator $C_\varphi$ is bounded on $B_0$, then $C_\varphi$ is still not hypercyclic on $B_0$.

**Proof.** Suppose that $C_\varphi$ is hypercyclic on $B_0$ and $f \in B_0$ is a hypercyclic vector for $C_\varphi$. Hence for every $g \in B_0$, there exists $\{\varphi^k\}_k$ satisfying

$$\|C_{\varphi^k} f - g\|_{\text{Bloch}} \to 0 \text{ as } k \to \infty.$$
In particular, we choose \( g(z) = nz \) for a fixed \( n \in \mathbb{N} \), then there exists a subsequence \( \{\varphi_{kj}\} \) such that

\[
\|C_{\varphi_{kj}} f - nz\|_\text{Bloch} \to 0 \quad \text{as} \quad j \to \infty.
\]

From which and Remark 2.3, it follows that

\[
f(\varphi_{kj}(0)) \to 0 \quad \text{as} \quad j \to \infty. \tag{3.5}
\]

By Schwarz-Pick Lemma (Lemma 2.7) and (3.5), we get

\[
\|C_{\varphi_{kj}} f\|_\text{Bloch} = |f(\varphi_{kj}(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(\varphi_{kj}(z))| (\varphi_{kj})'(z)
\]

\[
\leq |f(\varphi_{kj}(0))| + \sup_{z \in \mathbb{D}} (1 - |\varphi_{kj}(z)|^2) |f'(\varphi_{kj}(z))| (1 - |\varphi_{kj}(z)|^2) |f'(\varphi_{kj}(z))| 
\]

\[
= |f(\varphi_{kj}(0))| + \sup_{z \in \mathbb{D}} (1 - |\varphi_{kj}(z)|^2) |f'(\varphi_{kj}(z))| 
\]

\[
\leq |f(\varphi_{kj}(0))| + \|f\|_B 
\]

\[
\to \|f\|_B = \|f\|_\text{Bloch} - |f(0)| < \infty, \quad j \to \infty. \tag{3.6}
\]

At the same time, since

\[
\|nz\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)n = n \to \infty \quad \text{as} \quad n \to \infty.
\]

Thus

\[
\|C_{\varphi_{kj}} f\|_\text{Bloch} \to \infty, \quad j \to \infty. \tag{3.7}
\]

Combining (3.6) and (3.7), we get a contraction. Thus \( C_{\varphi} \) is not hypercyclic on \( B_0 \).

This completes the proof. \( \Box \)

From Lemma 2.8, we obtain that \( HC(C_{\varphi}) = HC(\lambda C_{\varphi}) \) for \( |\lambda| = 1 \), hence the following corollary holds.

**Corollary 3.3.** For every \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and \( \varphi \in S(\mathbb{D}) \). Suppose that the composition operator \( C_{\varphi} \) is bounded in \( B_0 \), then the operator \( \lambda C_{\varphi} \) is not hypercyclic on \( B_0 \).

Since the Besov space \( B_1 \subset B_0 \), we obtain the following corollary,

**Corollary 3.4.** For every \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and \( \varphi \in S(\mathbb{D}) \). Suppose that the composition operator \( C_{\varphi} \) is bounded in \( B_1 \), then the operator \( \lambda C_{\varphi} \) is not hypercyclic on \( B_1 \).

4. Hypercyclicity on \( B_p \)

This section is similar to section 3, we include the brief proof for the convenience of the readers.

**Case I.** \( \varphi \in \text{Aut}(\mathbb{D}) \).

**Theorem 4.1.** Suppose \( \varphi \in \text{Aut}(\mathbb{D}) \) and the composition operator \( C_{\varphi} \) is bounded on \( B_p \) \((1 < p < \infty)\), then \( C_{\varphi} \) is not hypercyclic on \( B_p \).
Suppose that there is an \( f \in B_p \) such that \( \{ C^k \varphi f : k \in \mathbb{N} \cup \{0\} \} \) is dense in \( B_p \). By (1.2), it follows that
\[
\| C^{k_j} \varphi f \|_p = |f(\varphi^{k_j}(0))|^p + \| f \circ \varphi^{k_j} \|_{B_p}^p
\]
\[
= |f(\varphi^{k_j}(0))|^p + \| f \|_{B_p}^p.
\]
For \( f_1(z) = z \in B_p \), there exists a subsequence \( \{ \varphi^{k_j} \} \) such that
\[
\| C^{k_j} \varphi f - f_1 \|_p \to 0 \quad \text{as} \quad j \to \infty.
\]
From which and Remark 2.3, it is clear that for \( z = 0 \),
\[
f(\varphi^{k_j}(0)) \to 0 \quad \text{as} \quad j \to \infty.
\]
By the above three formulas, we have
\[
\| f \|_p^p = |f(0)|^p + \| f \|_{B_p}^p
\]
\[
= |f(0)|^p + |C^{k_j} \varphi f|_p^p - |f(\varphi^{k_j}(0))|^p
\]
\[
\to |f(0)|^p + \| f_1 \|_p^p, \quad j \to \infty
\]
\[
= |f(0)|^p + \frac{1}{2(p-1)}. \quad (4.1)
\]
On the other hand, there exists another sequence \( \{ \varphi^{k_j} \} \) such that
\[
\| C^{k_j} \varphi f - f_2 \|_p \to 0 \quad \text{as} \quad j \to \infty.
\]
Similarly, \( f \circ \varphi^{k_j}(0) \to 0 \) as \( j \to \infty \). Using the Beta function
\[
B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx, \quad \text{for} \; p > 0, \; q > 0,
\]
it follows that
\[
\| f \|_p^p = |f(0)|^p + \| f \|_{B_p}^p
\]
\[
= |f(0)|^p + |C^{k_j} \varphi f|_p^p - |f \circ \varphi^{k_j}(0)|^p
\]
\[
\to |f(0)|^p + \| f_2 \|_{B_p}^p, \quad j \to \infty
\]
\[
= |f(0)|^p + 2p^{-1}B(p, 2(p-1) + 1, p-1). \quad (4.2)
\]
From (4.1) and (4.2), we get a contraction. Thus the composition operator \( C \varphi \) is not hypercyclic on \( B_p \) \((1 < p < \infty)\). This completes the proof. \( \square \)

**Case II.** \( \varphi \notin Aut(\mathbb{D}) \), we only consider the case \( p = 2 \geq 0 \).

**Theorem 4.2.** Suppose \( \varphi \notin Aut(\mathbb{D}) \) and \( \varphi \) has no interior fixed point in \( \mathbb{D} \). Further assume that the composition operator \( C \varphi \) is bounded on \( B_p \) \((2 \leq p < \infty)\), then \( C \varphi \) is still not hypercyclic on \( B_p \).

**Proof.** Suppose that \( C \varphi \) is hypercyclic on \( B_p \) and \( f \in B_p \) is a hypercyclic vector for \( C \varphi \). Then for each \( g \in B_p \), there exists \( \{ \varphi^k \}_k \) satisfying
\[
\| C^{k_j} \varphi f - g \|_p \to 0 \quad \text{as} \quad k \to \infty.
\]
In particular, we choose \( g(z) = nz^2 \) for a fixed \( n \in \mathbb{N} \), then there exists a subsequence \( \{ \varphi^{k_j} \}_j \) such that
\[
\| C^{k_j} \varphi f - nz^2 \|_p \to 0 \quad \text{as} \quad j \to \infty.
\]
From which and Remark 2.3, it follows that
\[ f(\varphi^{k_j}(0)) \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \]  
(4.3)
For \( p \geq 2 \), by Schwarz-Pick lemma (Lemma 2.7) and Lemma 2.6, we get that
\[
\|C_{\varphi^{k_j}}f\|^p_p = |f(\varphi^{k_j}(0))|^p + \|f \circ \varphi^{k_j}\|^p_{B_p} \\
= |f(\varphi^{k_j}(0))|^p + \int_{D} |(f \circ \varphi^{k_j})'(z)|^p(1 - |z|^2)^{p-2}dA(z) \\
= |f(\varphi^{k_j}(0))|^p + \int_{D} |f'(\varphi^{k_j}(z))(\varphi^{k_j})'(z)|^p(1 - |z|^2)^{p-2}dA(z) \\
= |f(\varphi^{k_j}(0))|^p + \int_{D} \frac{(1 - |\varphi^{k_j}(z)|^2)^2}{(1 - |\varphi^{k_j}(z)|^2)^2} |f'(\varphi^{k_j}(z))(\varphi^{k_j})'(z)|^p(1 - |\varphi^{k_j}(z)|^2)^{p-2}dA(z) \\
\leq |f(\varphi^{k_j}(0))|^p + \int_{D} \frac{(1 - |\varphi^{k_j}(z)|^2)^2}{(1 - |\varphi^{k_j}(z)|^2)^2} |f'(\varphi^{k_j}(z))(\varphi^{k_j})'(z)|^p(1 - |\varphi^{k_j}(z)|^2)^{p-2}dA(z) \quad (\varphi \text{ is univalent}) \\
= |f(\varphi^{k_j}(0))|^p + \int_{D} |f'(w)|^p(1 - |w|^2)^{p-2}dA(w) \\
\leq |f(\varphi^{k_j}(0))|^p + \int_{D} |f'(w)|^p(1 - |w|^2)^{p-2}dA(w) \\
= |f(\varphi^{k_j}(0))|^p + \|f\|^p_{B_p} \\
\rightarrow \|f\|^p_{B_p} = \|f\|^p_p - |f(0)|^p < \infty, \quad j \rightarrow \infty. \tag{4.4}
\]
At the same time, since
\[ \|nz^2\|^p_p = n^p2^{p-1}B\left(\frac{p}{2} + 1, p - 1\right) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty. \]
Thus
\[ \|C_{\varphi^{k_j}}f\|^p_p \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty. \tag{4.5} \]
From the above, comparing (2.14) with (2.15), we get a contraction. Thus \( C_{\varphi} \) is not hypercyclic on \( B_p \) (\( 2 \leq p < \infty \)). This completes the proof. \( \square \)

**Corollary 4.3.** For any \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and \( \varphi \in S(\mathbb{D}) \). Suppose that the composition operator \( C_{\varphi} \) is bounded in \( B_p \) (\( 2 \leq p < \infty \)), then the operator \( \lambda C_{\varphi} \) is not hypercyclic on \( B_p \).

**Open question:** Is the composition operator \( C_{\varphi} \) hypercyclic on the space \( B_p \) (\( 1 < p < 2 \)) with \( \varphi \notin \text{Aut}(\mathbb{D}) \)?

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