Quantum-critical conductivity of marginal Fermi-liquids

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(Dated: July 26, 2022)

Abstract

We present exact results for the electrical and thermal conductivity at low temperatures and frequencies in the quantum-critical region for fermions on a lattice scattering with the collective fluctuations of the quantum xy model. The model is applicable to the fluctuations of the loop-current order in cuprates as well as to a class of quasi-two dimensional heavy-fermion and other metallic antiferromagnets, and proposed recently also for the possible loop-current order in Moiré twisted bi-layer graphene, and for WSe$_2$. All these metals have a linear in temperature electrical resistivity in the quantum-critical region of their phase diagrams, often termed "Planckian" resistivity. The solution of the integral equation for the vertex in the Kubo equation for transport shows that all vertex renormalizations except due to Aslamazov-Larkin processes are absent. The latter appear as an Umklapp scattering matrix, which is shown to give only a temperature independent multiplicative factor for electrical conductivity which is non-zero in the pure limit only if the Fermi-surface is large enough. We also show that the vertex renormalization and the logarithmic enhancement of the marginal Fermi-liquid specific heat do not appear in the thermal conductivity. The results for transport properties are derived for any Fermi-surface on any two-dimensional lattice. As an example, the numerical factor in the linear in $T$ resistivity is explicitly calculated for large enough circular Fermi-surfaces on a square lattice.
I. INTRODUCTION

The discovery of normal state properties in cuprates which violate the quasi-particle concept of Landau Fermi-liquids have been intensely discussed in the last three decades [1]. The most prominently discussed [2] of these properties is the resistivity which has a linear in temperature dependence from the lowest temperature investigated by suppressing superconductivity often up to temperatures of $O(10^3)\,K$. Optical conductivity, Raman scattering, nuclear relaxation rates, tunneling conductance, etc. were similarly discovered to have anomalous frequency and temperature dependence. Soon thereafter, similar properties were discovered in heavy fermion compounds, for reviews see [3, 4], and in the normal state of high temperature Fe-based compounds, for reviews see [5, 6], and more recently in twisted bi-layer graphene (TBG) [7, 8] and twisted bi-layer WSe$_2$ [9] (TBWSe).

All the anomalous properties in all these metals in their critical regime follow from the marginal Fermi-liquid (MFL) hypothesis [10] that there must exist fluctuations whose absorptive part is a scale invariant function $F(\omega/T)$, equivalently a function of $1/\tau$ where $\tau$ is the imaginary time rather than $1/\tau^2$ as in a Fermi-liquid, and only weakly dependent on momentum. Some exactly soluble models of purely mathematical interest such as the SY model [11] and the related SYK-like models [12] arrive at the same result. Several predictions of this hypothesis have been experimentally verified. Especially important for this paper is the prediction, verified with Angle resolved photoemission (ARPES) [13–16] and angular dependence of magneto-resistance [17] that the imaginary part of the single-particle self-energy is proportional to $\max(\omega, T)$ and only weakly dependent on momentum both along and normal to the Fermi-surface. Thus the transport scattering rate for electrical and thermal conductivity have the same frequency and temperature dependence as the single-particle scattering rate. In this paper this issue and the ratio between the two scattering rates is investigated precisely.

MFL posits a singularity at $T = 0$, i.e. a quantum-critical point [18]. It was suggested that there must be a phase transition ending at a quantum-critical point as a function of doping [19] in cuprates. Its physical nature as a loop-current order, odd in time-reversal and inversion was predicted [20, 21] using a microscopic model taking into account the charge transfer nature of the cuprates [22]. The model for the quantum-critical fluctuations of the order parameter is the quantum xy-model coupled to fermions (QXY-F) [23]. The applicability of the quantum xy model to antiferromagnetic quantum-critical points in heavy fermions and Fe-based compounds has been
shown\(^{[24]}\), as also for the TBG (E. Berg and C.M. Varma - unpublished 2022) and TBWSe (Liang Fu - private communication 2022). The fluctuations of this model have been derived analytically\(^{[25]}\) as well as through quantum-Monte-carlo calculations\(^{[26, 27]}\) and are functions of \(\omega/T\) as in the MFL hypothesis but the momentum dependence have interesting differences. Direct evidence of the fluctuations of the model over the entire momentum region is found by neutron scattering in heavy-fermions and in an Fe based antiferromagnets\(^{[28-30]}\).

A variety of experiments are now consistent\(^{[31-41]}\) with the predicted broken symmetry in cuprates, while the broken symmetry relevant to the quantum-fluctuations in the heavy-fermions and other metallic antiferromagnets is of-course obvious. Further experiments are required to ascertain predicted broken symmetry\(^{[42, 43]}\) in TBG and TBWSe. The recent verification of the prediction of a specific heat \(\propto T \ln T\) close to the fermion density where the loop-current order is extrapolated to turn on as \(T \rightarrow 0\), gives direct evidence for a quantum-critical point in cuprates\(^{[44, 45]}\). The same singularity in the specific heat has earlier been observed at the antiferromagnetic criticality in heavy-fermion compounds\(^{[3]}\) and in Fe-based compounds\(^{[6]}\), which show a linear in \(T\) resistivity. An additional recent feature in all these compounds\(^{[46, 47]}\) as well as in TBG\(^{[48]}\) and TWSe\(^{[9]}\) is that the resistivity is linear also in an applied magnetic field \(|H|\) with magnitude such that the magneto-resistance at \(\mu_B|H| = k_B T\) is similar to the zero field resistance at \(T\). The theory for this phenomena and quantitative comparison with experiments has also been recently given\(^{[49]}\) based on the theory of the quantum xy model.

Calculations of transport properties at finite temperatures is a hard problem. Even in the problem of transport with electron-phonon interactions, in which although everything essential was understood long ago by Peierls (for a historical review see\(^{[50]}\)), an exact solution at low temperatures including Umklapp scattering has not been possible, although a detailed formalism was developed by Holstein\(^{[51]}\). For the Hubbard model in two dimensions, precise results including Umklapp scattering have been found only to second order in the interaction parameter by Mae-bashi and Fukuyama\(^{[52, 53]}\).

We are able to present an exact low temperature theory for electrical and thermal conductivity in the problem of transport with scattering of fermions by the fluctuations calculated for the QXY-F. That this is possible is due only to the simplicity and unusual nature of the correlation function for this model\(^{[23, 25, 54]}\). The simplicity comes from the fact that the spectra of fluctuations is a product of a function of frequency and of momentum. Moreover, the spatial correlation length normalized to the lattice constant is exponentially smaller than the temporal correlation length.
normalized to the short-time cut-off. These results are available from a precise quantum-Monte-
carlo calculation on a lattice including the renormalization of the critical fluctuations through
coupling to fermions \[26, 27\] as well as a renormalization group calculation \[25\].

As argued by Peierls \[50\], the electrical conductivity in a pure metal without Umklapp scatter-
ing is infinite at all temperatures. This is true both for a continuum model for a metal or a lattice
model in which the current is proportional not to the momentum but the group-velocity of particles
near the Fermi surface. For simple kinematic reasons Umklapp scattering is usually ineffective for
fluctuations close to \( q = 0 \) because of scarcity of low energy excitations of the order of \( k_B T \) at
such \( q \). It leads to extra powers of temperature in transport relaxation rates compared to single-
particle relaxation rates (for example \( T^5 \) in resistivity and \( T^3 \) in single-particle relaxation rates in
electron-phonon scattering). In the cuprates and most likely in the other quantum-critical prob-
lems with linear in \( T \) resistivity, the momentum relaxation rate which determines the resistivity
has the same temperature dependence as the single-particle relaxation rates. It is then necessary
for adequate phase space in scattering that there also be low energy excitations at large as well
as small \( q \). This is one of the rather unique conditions met in the fluctuations spectra derived for
the quantum-xy model coupled to fermions summarized in Appendix A by Eq. \( 70 \). We show
that Umklapp scattering in this case gives only a geometry dependent but temperature independent
numerical factor. Interestingly, we find that Umklapp scattering is not required for relaxing energy
current which determines the thermal conductivity. We give reasons based on symmetry why this
is so.

We should mention that we use the phrase "Planckian" only to mean that the transport scattering
rate is proportional to \( k_B T / \hbar \), with no implication that the constant of proportionality is 1, for
which we find no basis either in experiments or in theory.

In a significant recent development \[55\] Else and Senthil (ES) have shown that the observed
proportionality of the resistivity \( \rho(T) \) to the temperature \( T \), or more generally of the conduc-
tivity \( \sigma(\omega, T) \) scaling as \( \frac{1}{T} F(\omega / T) \) as in the marginal Fermi-liquid hypothesis, implies for two-
dimensional models in the pure limit for which momentum is the only conserved quantity at \( T = 0 \),
that the scattering is from critical fluctuations of a vector order parameter which transforms as cur-
rent, i.e. it is odd in time-reversal and in inversion. These are the symmetries of the loop-current
order as well as of a variety of models which can be described by the quantum xy model. As ex-
plained by ES such a conservation strictly holds only for circular Fermi-surfaces, while for a more
general Fermi-surface there are an infinite number of conservation laws at \( T = 0 \). In this paper,
we also discuss conservation laws in the pure limit for a general Fermi-surface and for \( T \neq 0 \). An infinite number of conservation laws exist even at \( T \neq 0 \) in the pure limit if there are regions on the Fermi-surface in which Umklapp scattering is kinematically not allowed. In that case, the resistivity even at finite temperatures is zero. We show that for a circular Fermi-surface on a square lattice \( \frac{k_F a}{\pi} \) must be larger than about 0.552 for their to be finite resistivity.

The order of presentation in this paper is as follows: We begin with the well-known Kubo formula for transport properties in Sec. II. The previously obtained results for the propagator of the fluctuations for the quantum-xy model coupled to fermions are briefly summarized in Appendix A. We use these to specify the self-energies of the fermions and the vertices in the Kubo equations. In sub-section II-C, the Kubo equations are re-cast in terms of an equation for the velocity distribution functions \( \Phi \) which makes it easy to introduce the memory matrix \( M \) for calculations of transport. We separate the three distinct physical contributions to \( M \), due to self-energy of fermions, and two distinct types of vertices to the collective modes. This part is quite general. We then use the properties of the quantum-xy model in Sec. II-D to show that the first two yield identity and the last is simply the vertex renormalization for Umklapp scattering. Conservation laws for small enough Fermi-surfaces and their consequence are discussed in Sec. II-E. Some important cancellations using Ward identities show that the logarithmically renormalized mass of MFL never appears in the electrical conductivity or thermal conductivity. At low temperatures the transport properties can be evaluated exactly for any shape of Fermi-surface and lattice. Finally as an example, the coefficient of the linear in \( T \) resistivity and \( T \) independent thermal conductivity are given for circular Fermi-surface of various sizes in a square lattice in terms of the coupling constant to the collective fluctuations. Some details of the calculations and summary of old results are given in three appendices.

II. CONDUCTIVITY

The Kubo formula expresses the conductivity \( \sigma(\omega, T) \) in terms of the retarded current-current correlation function for zero momentum transfer \( \chi_{J,J}(\omega, T) \)

\[
\sigma(\omega, T) = e^2 \frac{\chi_{J,J}(\omega, T) - \chi_{J,J}(0, T)}{i\omega}.
\]

\( \chi(\omega, T) \) is the analytic continuation of

\[
\chi_{J,J}(i\omega_n) = -\frac{2}{\beta V} \sum_n v_{px} G(p, i\epsilon_n + i\omega_n) G(p, i\epsilon_n) \Lambda(p, i\epsilon_n; i\omega_n).
\]

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FIG. 1. Diagrammatic representation of the Kubo equation for the conductivity. The external field coupling to fermion charge times their velocity is shown as a dotted line. The lines are exact single-particle Green’s functions \( G(p, \omega) \). \( \Lambda \) is the renormalized current vertex.

as shown in Fig. 1 (\( e, \beta, \) and \( V \) are the elementary charge, the inverse temperature, and the volume of the system, respectively.) \( G(p, i\epsilon_n) \) is the renormalized single-particle Green’s function, \( v_{px} \) is the \( x \) component of the bare velocity and \( \Lambda \) is the renormalized vertex coupling to external electric field.

The formula for conductivity can be written in the following transparent and familiar form (re-derived below),

\[
\sigma(\omega, T) = \frac{2e^2}{V} \sum_p \left( -\frac{\partial f(\epsilon^*_p)}{\partial \epsilon^*_p} \right) v^*_p v_{px} \Phi(p, \omega).
\] (3)

Here \( f(\epsilon) = 1/(e^{\beta \epsilon} + 1) \), \( v^*_p \) is the velocity renormalized due to interactions, and \( \Phi(p, \omega) \) is the velocity distribution function renormalized exactly for the effect of interactions. The latter is the appropriate Boltzmann distribution function.

A similar Kubo formula gives the results for the thermal conductivity \( \kappa \). We will present results of evaluation of \( \kappa \) as well as \( \sigma \).

The interesting and hard part for a calculation of the conductivity is the calculation of the external field (EF) - electron vertex \( \Lambda \). We structure our calculation applying the Baym–Kadanoff conserving scheme \([51, 56, 57]\) to the loop-current order and an extension to collective fluctuations of the Memory matrix formalism used by Maebashi and Fukuyama \([52, 53]\). The inter-relationship of the Memory matrix method to the more familiar many body techniques \([58, 59]\), used for example to derive the Landau-Boltzmann transport equations, is given below.
FIG. 2. Diagrammatic representation of (a) the renormalized collective mode propagator. In our case, the exact collective mode propagator is already available from quantum Monte-carlo and renormalization group calculations. (b) the renormalized fermion propagator and (c) the irreducible vertex.

A. Self-energies and Vertices

Let $D(q, i\nu)$ be the propagator of the collective modes with which the fermions scatter with a coupling function to them $g(p - p')$. The imaginary parts of the retarded self-energies $\Pi_R(q, \nu)$ for the collective modes and $\Sigma_R(p, \varepsilon)$ for the fermions are represented diagrammatically by Fig. 2 and are given by,

$$\text{Im}\Pi_R(q) = -2\pi^2 \sinh \frac{\nu}{2T} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} |g(q)|^2 A(p)A(p - q)\text{sech}\frac{\varepsilon}{2T}\text{sech}\frac{\varepsilon - \nu}{2T},$$

(4)

$$\text{Im}\Sigma_R(p) = \pi \cosh \frac{\varepsilon}{2T} \int \frac{d^{d+1}p'}{(2\pi)^{d+1}} |g(p - p')|^2 \text{Im}D_R(p - p')A(p')\text{cosech}\frac{\varepsilon - \varepsilon'}{2T}\text{sech}\frac{\varepsilon'}{2T},$$

(5)

where $A(p)$ is the single-particle spectral function,

$$A(p) = -\frac{1}{\pi}\text{Im}G_R(p).$$

(6)

Here we have used the $(d + 1)$-momentum notation $q = (q, \nu), p = (p, \varepsilon)$, and $p' = (p, \varepsilon')$ for brevity, and taken the limit of infinite volume so that $(1/V) \sum_p (2\pi)^{-d} \int dp$. The Monte-carlo calculations of the critical fluctuations of the quantum xy model on a lattice with coupling to fermions provides the collective fluctuation propagator $D$ including its self-energy through coupling to fermions and Umklapp scattering due to the square lattice. Therefore,
FIG. 3. Diagrammatic representation for the integral equation for the external Field vertices in calculation of conductivity. This is a sum of three parts shown successively in the three lines. First is the vertex coupling to the renormalized fermion propagators, second and third are the vertex coupling to the collective modes. The second line gives what may be called "Maki-Thompson" diagrams and the third corresponds to the "Aslamazov-Larkin" diagrams.

we shall not need $D_0$ in Fig. 2(a) explicitly but the fact that $D$ is a functional of the fermion propagator is important to note for calculating the vertex, as shown below. The availability of the renormalized propagator of the fluctuations for the wiggly lines in Fig. 2 so that they need not be re-calculated in the evaluation of the Fermion-vertices below is crucial for the developments below.

The irreducible vertex $I$, represented in Fig. 2(c) (using which one can calculate the vertex $\Lambda$ as well as the fermion self-energy) is given by the functional derivative $I = \delta \Sigma / \delta G$ \[56, 57\]. Because the exact Green’s function $D$ is a functional of $G$, the irreducible vertex $I$ includes contributions not only from what might be called the Maki-Tompson (MT) type diagram (the second line in the diagrams shown in Fig. 3) but also from the two Aslamazov-Larkin (AL) type diagrams (the third line in the digram), thorough the functional derivative of $D$ with respect to $G$. We will find that the MT type diagrams do not contribute for the present problem but the AL diagrams are essential for providing the Umklapp factor. Also the first line of Fig. 3 gives no correction from the self-energy of the propagator and is therefore just the bare velocity. This as well as the absence of the MT diagram was noted earlier \[60\] but the contribution of the AL diagrams was not noted.

We will need the analytic continuation of the EF electron vertex part given by an integral
equation represented in Fig. 3. This is accomplished following the classical paper by Êliashberg [61]. By taking $i\omega_m \rightarrow \omega + i0^+$ with $\omega = 0$ after the analytic continuation of $\Lambda(p, i\epsilon_n; i\omega_m)$ to the region 2 shown by Fig. 4(a) on the complex energy plane of $i\epsilon_n = z$, we obtain

$$\Lambda_2(p) = \tilde{v}_x(p) + \cosh \frac{\varepsilon}{2T} \int \frac{d^{d+1}p'}{(2\pi)^{d+1}} \mathcal{I}(p, p') \frac{\varepsilon'}{2T} G_R(p') G_A(p') \Lambda_2(p').$$

(7)

$\Lambda_2$ alone will enter in subsequent calculations. Its definition follows Êliashberg: The first term on the right come from the diagrams including $G_R G_R$ and $G_A G_A$ and they satisfies the Ward identity or the continuity equation,

$$\tilde{v}_x(p) = v_{px} + \frac{\partial \text{Re} \Sigma_R(p)}{\partial p_x}. \quad (8)$$

The second term comes from the particle-hole sections with $G_R G_A$ and $\mathcal{I}(p, p')$ is given by

$$\mathcal{I}(p, p') = \frac{1}{2i} \left[ I_{22}^H(p, p') - I_{22}^H(p, p') \right] \text{cosech} \frac{\varepsilon - \varepsilon'}{2T} + \frac{1}{2i} \left[ I_{22}^M(p, p') - I_{22}^M(p, p') \right] \text{cosech} \frac{\varepsilon + \varepsilon'}{2T}, \quad (9)$$

where $I_{22}^H$, $I_{22}^M$, and $I_{22}^\mathcal{H}$ are the analytic continuations of the irreducible vertex $I(p, i\epsilon_n, p', i\epsilon_n'; i\omega_m)$ to the regions shown by Fig. 4(b) with $i\epsilon_n = z$ and $i\epsilon_n' = z'$. Note that the conservation of charge leads to a Ward–Takahashi identity that relates the imaginary part of the fermion self-energy to the irreducible vertex as

$$2\text{Im} \Sigma_R(p) \text{sech} \frac{\varepsilon}{2T} = -\int \frac{d^{d+1}p'}{(2\pi)^{d+1}} \mathcal{I}(p, p') A(p') \frac{\varepsilon'}{2T}. \quad (10)$$

By use of $\tilde{v}_x(p)$ and $\Lambda_2(p)$, the Kubo formula, Eq. (1), can be written as

$$\sigma(T) = 2e^2 \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left( -\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \tilde{v}_x(p) G_R(p) G_A(p) \Lambda_2(p), \quad (11)$$

where $-\partial f(\varepsilon)/\partial \varepsilon = (1/4T) \text{sech}^2(\varepsilon/2T)$.

We introduce the electron velocity distribution function $\Phi(p)$, which occurs in Boltzmann equation for transport. We now conveniently re-write the Kubo formula, Eq. (1) in terms of $\Phi(p)$ instead of $\Lambda_2(p)$ as promised earlier in Eq. (3),

$$\Phi(p) = -\frac{\Lambda_2(p)}{2\text{Im} \Sigma_R(p)}. \quad (12)$$

Noting that

$$G_R(p) G_A(p) = \frac{\text{Im} G_R(p)}{\text{Im} \Sigma_R(p)}. \quad (13)$$
we can rewrite Eqs. (7) and (11). As a result, the electrical conductivity is given by

$$\sigma(T) = 2e^2 \int \frac{d^{d+1}p}{(2\pi)^d} \left( -\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \tilde{v}_x(p) A(p) \Phi(p), \quad (14)$$

where $\Phi(p)$ satisfies

$$\tilde{v}_x(p) \sech \frac{\varepsilon}{2T} = -2\text{Im} \Sigma_R(p) \Phi(p) \sech \frac{\varepsilon}{2T} - \int \frac{d^{d+1}p'}{(2\pi)^d} \mathcal{I}(p, p') A(p') \Phi(p') \sech \frac{\varepsilon'}{2T}$$

$$= \int \frac{d^{d+1}p'}{(2\pi)^d} \mathcal{I}(p, p') A(p') [\Phi(p) - \Phi(p')] \sech \frac{\varepsilon'}{2T}. \quad (15)$$

We have used Eq. (10) for the second equality and $\mathcal{I}(p, p')$ is the contributions from the three processes shown in Fig. 2(c) for which Eq. (7) leads to

$$\mathcal{I}(p, p') = -|g(p-p')|^2 \text{Im} D_R(p-p') \text{cosech} \frac{\varepsilon - \varepsilon'}{2T}$$

$$+ 2\pi^2 \int \frac{d^{d+1}p_1}{(2\pi)^{d+1}} |g(p-p_1)|^4 D_R(p-p_1) D_A(p-p_1)$$

$$\times \left( A(p_1) A(p_1 - p + p') \sech \frac{\varepsilon_1 - \varepsilon + \varepsilon'}{2T} \sech \frac{\varepsilon_1 + \varepsilon' - \varepsilon_1}{2T} - A(p_1) A(p + p' - p_1) \sech \frac{\varepsilon_1 + \varepsilon' - \varepsilon_1}{2T} \right). \quad (16)$$

The first and second terms on the right comes from the MT and AL type diagrams, respectively. Noting that

$$D_R(q) D_A(q) = \frac{\text{Im} D_R(q)}{\text{Im} \Pi_R(q)}. \quad (17)$$
we can rewrite Eq. (16) as

\[
\mathcal{I}(p, p') = \pi W(p-p') \int \frac{d^{d+1}p_1}{(2\pi)^d} A(p_1) A(p_1-p+p') \sech \frac{\varepsilon_1}{2T} \sech \frac{\varepsilon_1-\varepsilon+\varepsilon'}{2T} \\
+ \pi \int \frac{d^{d+1}p_1}{(2\pi)^d} W(p-p_1) \left( A(p_1) A(p_1-p+p') \sech \frac{\varepsilon_1}{2T} \sech \frac{\varepsilon_1-\varepsilon+\varepsilon'}{2T} \\
- A(p_1) A(p+p'-p_1) \sech \frac{\varepsilon_1}{2T} \sech \frac{\varepsilon+\varepsilon'-\varepsilon_1}{2T} \right),
\]

(18)

where

\[
W(q) = \frac{|g(q)|^4 \text{Im} D_R(q)}{\text{Im} \Pi_R(q)}.
\]

(19)

Then Eq. (15) can also be written as

\[
\tilde{v}_x(p) \sech \frac{\varepsilon}{2T} = \pi \int \frac{d^{d+1}p'}{(2\pi)^d} \frac{d^{d+1}p_1}{(2\pi)^d} W(p-p') A(p_1) A(p-p'+p_1) A(p') \\
\times \left[ \Phi(p) - \Phi(p') + \Phi(p_1) - \Phi(p-p'+p_1) \right] \\
\times \sech \frac{\varepsilon_1}{2T} \sech \frac{\varepsilon-\varepsilon'+\varepsilon_1}{2T} \sech \frac{\varepsilon'}{2T}.
\]

(20)

Assuming Landau’s quasiparticles, we can show that Eqs. (3) and (14) are equivalent, and Eq. (20) corresponds to the Boltzmann equation for the problem of transport for fermions coupled to collective fluctuations in the limit of vanishing external frequency and wave vector. This is accomplished in Appendix B, but here we proceed without such an assumption.

B. Memory matrix formalism

Define the inner product in momentum space for a given dimensionless energy variable \( t = \varepsilon/T \) as

\[
\langle \tilde{v}_x(t) | \Phi(t) \rangle = \frac{1}{N(Tt)} \int \frac{d\mathbf{p}}{(2\pi)^d} \tilde{v}_x(p, Tt) A(p, Tt) \Phi(p, Tt),
\]

(21)

where \( N(\varepsilon) = (2\pi)^{-d} \int A(p, \varepsilon) d\mathbf{p} \) is the density of states of fermions per spin. We can then write the dc conductivity given by the Kubo formula, Eq. (1) as

\[
\sigma(T) = \frac{e^2}{2} \int_{-\infty}^{\infty} dt \langle \tilde{v}_x(t) | \Phi(t) \rangle N(Tt) \sech^2 \frac{t}{2}.
\]

(22)

Let \( \hat{M}''(t, t') \) be the imaginary part of the memory matrix \( \hat{M} \) acting on this inner product space,

\[
\hat{M}''(t, t') \Phi_x(t') = \frac{1}{N(Tt')} \int \frac{d\mathbf{p'}}{(2\pi)^d} \int \frac{d\mathbf{p}}{(2\pi)^d} \hat{M}''(p, t; p', t') A(p', Tt') \Phi(p', Tt')
\]

(23)
with

\[ M''(p, Tt, p', Tt') = -2\text{Im} \Sigma_R(p, Tt) \frac{N(Tt)}{A(p, Tt)} (2\pi)^d \delta^d(p - p') \delta(t - t') \]

\[ -T \sqrt{N(Tt)} \mathcal{I}(p, Tt, p', Tt') \sqrt{N(Tt')} \]

Then Eq. (15) can be written in the following compact form:

\[ |\tilde{\nu}_x(t)| \sqrt{N(Tt)} \text{sech} \frac{t}{2} = \int_{-\infty}^{\infty} dt' \tilde{M}''(t, t') |\Phi(t')| \sqrt{N(Tt')} \text{sech} \frac{t'}{2}. \]  

Defining the inverse matrix through \( \int_{-\infty}^{\infty} \tilde{M}''(t, t_1)^{-1} \tilde{M}''(t_1, t') dt_1 = \hat{1} \delta(t - t') \), where \( \hat{1} \) is the unit matrix in the inner product space, we obtain

\[ \sigma(T) = \frac{e^2}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle \tilde{v}_x(t) | \tilde{M}''(t, t')^{-1} | \tilde{v}_x(t') \rangle \sqrt{N(Tt)} \text{sech} \frac{t}{2} \sqrt{N(Tt')} \text{sech} \frac{t'}{2}. \] \( \text{(26)} \)

This is a rigorous result of the dc conductivity in the memory matrix formalism at all temperatures.

In parallel with the above result, the dynamical conductivity \( \sigma(\omega, T) \) is written in terms of the memory matrix \( \tilde{M}(t, t'; \omega) = \tilde{M}'(t, t'; \omega) + i \tilde{M}''(t, t') \) for low frequencies as

\[ \sigma(\omega, T) = \frac{i e^2}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle \tilde{v}_x(t) | [\omega \hat{1} \delta(t - t') + \tilde{M}(t, t'; \omega)]^{-1} | \tilde{v}_x(t') \rangle \times \sqrt{N(Tt)} \text{sech} \frac{t}{2} \sqrt{N(Tt')} \text{sech} \frac{t'}{2}, \] \( \text{(27)} \)

where the real part \( \tilde{M}'(t, t'; \omega) \) is a matrix acting on the inner product space satisfying a similar equation as Eq. (23) and

\[ M'(p, Tt, p', Tt'; \omega) = -\omega \left[ \frac{\partial \text{Re} \Sigma_R(p, Tt)}{T \partial t} \frac{N(Tt)}{A(p, Tt)} (2\pi)^d \delta^d(p - p') \delta(t - t') + \frac{1}{4} \sqrt{N(Tt)} \text{sech} \frac{t}{2} \Gamma^k(p, Tt, p', Tt') \sqrt{N(Tt')} \text{sech} \frac{t'}{2} \right]. \] \( \text{(28)} \)

\( \Gamma^k(p, p') \) is the full vertex obtainable from \( \Lambda \) in the \( k \to 0 \) before \( \omega \to 0 \) limit. Eq. (27) is a rigorous form of \( \sigma(\omega, T) \) for low frequencies.

C. Low-temperature electrical and thermal conductivities

At low temperature, we can replace all the single-particle spectral functions \( A(p, Tt) \) in Eq. (26) by \( A(p, 0) \). For a metal with a Fermi surface satisfying Luttinger’s theorem, the latter is given by

\[ A(p, 0) = \delta(\varepsilon_p), \] \( \text{(29)} \)
where $\tilde{\epsilon}_p = \epsilon_p + \text{Re} \Sigma_R(p,0)$ with $\epsilon_p$ giving the non-interacting energy dispersion relative to the chemical potential, and $\tilde{v}_x(p, T t)$ can also be replaced by $\tilde{v}_x(p, 0) = \partial \tilde{\epsilon}_p / \partial p_x \equiv \tilde{v}_{px}$. By Eq. (26), the low-temperature electrical conductivity is given by

$$\sigma = \frac{e^2}{2} N(0) \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle \tilde{v}_x | \hat{M}''(t, t')^{-1} | \tilde{v}_x \rangle \text{sech} \frac{t}{2} \text{sech} \frac{t'}{2}. \quad (30)$$

This result holds regardless of Fermi or non-Fermi liquids, and the leading temperature dependence of the conductivity originates from the energy and temperature dependence of the critical fluctuations in the memory matrix.

Consider the case where Im$\Sigma_R(p, T t)$ is linear in temperature. Details will be studied in the subsequent sub-sections. In this case, from Eq. (10), we see that $\mathcal{I}(p, T t, p', T t')$ is independent of temperature. Then, replacing $N(T t)$ with $N(0)$ in Eq. (24), $\hat{M}''(t, t')$ is proportional to $T$. Let us call $u_p$ a conserved quantity at low temperatures if the following equation, which is of the same form as Eq. (10), holds for $u_p$ to linear order in $T$:

$$2 \text{Im} \Sigma_R(p, T t) u_p \text{sech} \frac{t}{2} = -T \int_{-\infty}^{\infty} dt' \int \frac{dp'}{(2\pi)^d} \mathcal{I}(p, T t, p', T t') \delta(\tilde{\epsilon}_p) u_p \text{sech} \frac{t'}{2}. \quad (31)$$

For example, $u_p = p_x$ corresponds to the conservation of crystal momentum in the absence of Umklapp scattering. In the memory matrix formalism, Eq. (31) can be written simply as

$$\int_{-\infty}^{\infty} dt' \hat{M}''(t, t') | u \rangle \text{sech} \frac{t'}{2} = 0. \quad (32)$$

Hence $\hat{M}''(t, t')$ has zero eigenvalue for each conserved quantity $u_p$ at low temperatures. If $u_p$ has an overlap with $\tilde{v}_{px}$ on the Fermi surface, the coefficient of linear in $T$ resistivity vanishes. This is true for small Fermi surfaces where Umklapp is ineffective. However, as will be demonstrated in Sec. II-E, for large Fermi surfaces where Umklapp scattering occurs frequently, there is no such conserved quantity and a nonzero linear in $T$ resistivity results.

For specific calculations, it is useful to introduce the Fermi surface harmonics $\psi_L(p)$ which are orthonormalized as $\langle \psi_L | \psi_{L'} \rangle = (2\pi)^{-d} \int dp \psi_L(p) \psi_{L'}(p) \delta(\tilde{\epsilon}_p) / N(0) = \delta_{L,L'}$ and complete $\sum_L | \psi_L \rangle \langle \psi_L | = \hat{1}$. Since the bare vertex is an odd function of momentum, only the harmonics satisfying $\psi_L(-p) = -\psi_L(p)$ can form a basis and $\psi_L(p)$ with $L = 1$ is chosen to be proportional to $\tilde{v}_{px}$, i.e., $\psi_1(p) = \tilde{v}_{px} / \langle \tilde{v}_{px}^2 \rangle^{1/2}$ where $\langle \tilde{v}_{px}^2 \rangle = \langle \tilde{v}_x^2 \rangle$ is the average of $\tilde{v}_x^2$ on the Fermi surface. Then we get

$$\sigma = \frac{e^2}{2} N(0) \sum_L \sum_{L'} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle \tilde{v}_x | \psi_L \rangle \langle \psi_L | \hat{M}''(t, t')^{-1} | \psi_{L'} \rangle \langle \psi_{L'} | \tilde{v}_x \rangle \text{sech} \frac{t}{2} \text{sech} \frac{t'}{2}$$

$$= 2e^2 N(0) \sum_L \sum_{L'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[\hat{M}''(t, t')^{-1}]_{11}}{4 \cosh(t/2) \cosh(t'/2)} dt dt'. \quad (33)$$
From Eqs. (23) and (24), the matrix elements of \( \hat{M}'(t, t') \) when the Fermi surface harmonics are used as the basis are given by

\[
[\hat{M}'(t, t')]_{LL'} \equiv \langle \psi_L | \hat{M}'(t, t') | \psi_{L'} \rangle = -\frac{2}{N(0)} \int \frac{dp}{(2\pi)^d} \text{Im} \Sigma_R(p, Tt) \psi_L(p) \psi_{L'}(p) \delta(\tilde{\epsilon}_p) - \frac{T}{N(0)} \int \frac{dp}{(2\pi)^d} \int \frac{dp'}{(2\pi)^d} \psi_L(p) \delta(\tilde{\epsilon}_p) \Im \Sigma_L(p, Tt, p', Tt') \delta(\tilde{\epsilon}_p') \psi_{L'}(p').
\] (34)

The first term on the right comes from the self-energy correction. The second term comes from the vertex corrections includes the contributions from MT and AL type diagrams. From Eq. (16), the former is given by

\[
[\hat{M}'_{MT}(t, t')]_{LL'} = \frac{T}{2N(0)} \int \frac{dp}{(2\pi)^d} \int \frac{dp'}{(2\pi)^d} \psi_L(p) \delta(\tilde{\epsilon}_p) \delta(\tilde{\epsilon}_p') \psi_{L'}(p') \times |g(p - p')|^2 \text{Im} D_R(p - p', Tt - Tt') \text{sech} \frac{t - t'}{2}. \] (35)

From Eq. (18), the latter is given by

\[
[\hat{M}'_{AL}(t, t')]_{LL'} = \frac{T}{2N(0)} \int \frac{dp}{(2\pi)^d} \int \frac{dp'}{(2\pi)^d} \psi_L(p) \delta(\tilde{\epsilon}_p) \delta(\tilde{\epsilon}_p') \times \int_{-\infty}^{\infty} dx |g(p - p')|^2 \text{Im} D_R(p - p', Tt - Tt') \text{sech} \frac{t - t'}{2} \times \frac{1}{x} \text{sech} \frac{t + x}{2} + \text{sech} \frac{t - x}{2} \\
\times \frac{1}{x} \text{sech} \frac{t' + x}{2} + \text{sech} \frac{t' - x}{2} \\
\times \int \frac{dkdk'}{(2\pi)^d} \Delta^d(p - p' + k - k') \delta(\tilde{\epsilon}_k) \delta(\tilde{\epsilon}_{k'}) \psi_{L'}(k') \\
\times \left[ \int \frac{dk_1dk_1'}{(2\pi)^d} \Delta^d(p - p' + k_1 - k_1') \delta(\tilde{\epsilon}_{k_1}) \delta(\tilde{\epsilon}_{k_1'}) \right]^{-1}, \] (36)

where \( \Delta^d(p - p' - q) \equiv \sum_G \delta^d(p - p' - q - G) \) is a \( d \)-dimensional delta function extended to the lattice with \( G \) being the reciprocal lattice vectors including a zero vector, and then normal and Umklapp scatterings are described by \( G = 0 \) and \( G \neq 0 \), respectively.

In a similar manner, we obtain the thermal conductivity \( \kappa \) at low temperature as

\[
\kappa = 2TN(0)\langle v_x^2 \rangle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{M}'(t, t')^{-1}|_{11} tt'}{4 \cosh(t/2) \cosh(t'/2)} dt dt'. \] (37)

Note that from Eq. (36), \( \hat{M}'_{AL}(t, t') = \hat{M}'_{AL}(t, -t') = \hat{M}'_{AL}(-t, t') \), while the bare vertex for \( \kappa \) is an odd function of the energy variable. Thus the AL vertex corrections do not contribute to \( \kappa \).
D. Planckian dissipation for criticality of the quantum xy-model coupled to fermions

We use the critical fluctuations and their coupling to fermions summarized in Appendix A, assuming they are valid for fluctuations of any vector field obeying $U(1)$ symmetry in the quantum-critical region and coupled to fermions as in the model solved in [23, 25–27]. The self-energy of the fermions is then effectively momentum independent for all calculations so that with the coupling function specified in Appendix A,

$$|g(q)|^2 \text{Im} D_R(q, \nu) = -\bar{g}^2 \tanh \left( \frac{\nu}{2T} \right).$$

We apply the memory matrix formalism described in the previous section to the case of Eq. (38).

Substituting Eq. (38) into Eq. (5), the full energy and temperature dependences of the imaginary part of the self-energy is given by

$$\text{Im} \Sigma(p, \varepsilon) = -\bar{g}^2 N(0) \varepsilon \coth \frac{\varepsilon}{2T}.$$ (39)

Since the imaginary part of the self energy is independent of momentum, the self-energy correction $\hat{M}_{SE}''(t, x)$ is proportional to the unit matrix $\hat{1}$ whose matrix element is $\delta_{L,L'}$

$$\hat{M}_{SE}''(t, x) = 2\bar{g}^2 N(0) T f(t) \delta(t-x) \hat{1},$$ (40)

where

$$f(t) = t \coth(t/2).$$ (41)

Because of the momentum-independent fluctuations and $\psi_L(-p) = -\psi_L(p)$, Eq. (35) shows that the MT vertex corrections are zero,

$$\hat{M}_{MT}''(t, x) = \hat{0}.$$ (42)

Substituting Eq. (38) into Eq. (36) we obtain the AL vertex correction as follows:

$$\hat{M}_{AL}''(t, x) = -\bar{g}^2 N(0) T [F(t, x) + F(t, -x)] \hat{B},$$ (43)

where

$$F(t, x) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{\tanh(y/2)}{y \cosh((y + t)/2) \cosh((y + x)/2)} dy,$$ (44)
and the matrix elements of \( \hat{B} \) are given by
\[
B_{LL'} = \frac{2}{N(0)^2} \int \frac{dp dp'}{(2\pi)^2} \psi_L(p) \delta(\hat{\epsilon}_p) \delta(\hat{\epsilon}_{p'}) \\
\times \int \frac{dk dk'}{(2\pi)^d} \Delta^d(p - p' + k - k') \delta(\hat{\epsilon}_k) \delta(\hat{\epsilon}_{k'}) \psi_L(k') \\
\times \left[ \int \frac{dk_1 dk'_1}{(2\pi)^d} \Delta^d(p - p' + k_1 - k'_1) \delta(\hat{\epsilon}_{k_1}) \delta(\hat{\epsilon}_{k'_1}) \right]^{-1}.
\]

Then the effect of the vertex corrections is described by the matrix \( \hat{B} \). Gauge invariance relates the vertex corrections to the self-energy. This relation corresponds to the identity
\[
\int_{-\infty}^{\infty} \frac{F(t, x)}{\cosh(x/2)} \, dx = \frac{f(t)}{\cosh(t/2)}.
\]

Now, following the exact solution of the Boltzmann equation by Jensen, Smith, and Wilkins \[62\], we consider the eigenvalue equation
\[
f(t) \varphi_{n, \pm}(t) = \lambda_{n, \pm} \int_{-\infty}^{\infty} dx F(t, x) \varphi_{n, \pm}(x),
\]
where \( \varphi_{n, +}(t) \) and \( \varphi_{n, -}(t) \) are even and odd functions of \( t \), respectively, \( \varphi_{n, \pm}(t) = \pm \varphi_{n, \pm}(\pm t) \), and they are normalized according to
\[
\int_{-\infty}^{\infty} dt f(t) \varphi_{n, \pm}(t) \varphi_{n, \pm}(t) = \delta_{n, m}.
\]

All the eigenvalues are greater than or equal to 1. From Eq. (46), the eigenfunction with the smallest eigenvalue \( \lambda_{1, +} = 1 \) is given by
\[
\varphi_{1, +}(t) = \frac{1}{\pi \cosh(t/2)}.
\]

Using the eigenvalue \( \lambda_{n, \pm} \geq 1 \) and the eigenfunctions \( \varphi_{n, \pm}(t) \), we can write Eqs. (40) and (43) as
\[
\hat{M}''_{SE}(t, x) = 2g^2 N(0) T f(t) f(x) \sum_n \sum_{s=\pm} \varphi_{n, s}(t) \varphi_{n, s}(x) \hat{1} - \hat{B},
\]
\[
\hat{M}''_{AL}(t, x) = -2g^2 N(0) T f(t) f(x) \sum_n \frac{\varphi_{n, +}(t) \varphi_{n, +}(x)}{\lambda_{n, +}} \hat{B}.
\]

Therefore we obtain the imaginary part of the memory matrix, \( \hat{M}''(t, x) = \hat{M}''_{SE}(t, x) + \hat{M}''_{AL}(t, x) \), for the loop-current-order fluctuations
\[
\hat{M}''(t, x) = 2g^2 N(0) T f(t) f(x) \sum_n \left( \frac{\varphi_{n, +}(t) \varphi_{n, +}(x)}{\lambda_{n, +}} (\lambda_{n, +} \hat{1} - \hat{B}) + \sum_n \varphi_{n, -}(t) \varphi_{n, -}(x) \hat{1} \right).
\]
The inverse matrix can be easily obtained as
\[ \hat{M}''(t, x)^{-1} = \frac{1}{2\bar{g}^2 N(0) T} \sum_n \left( \varphi_{n,+}(t) \varphi_{n,+}(x) \lambda_{n,+} \left( \lambda_{n,+} \hat{1} - \hat{B} \right)^{-1} + \sum_n \varphi_{n,-}(t) \varphi_{n,-}(x) \hat{1} \right). \]  

(53)

Since \( \hat{M}''(t, x)^{-1} \propto 1/T \), we see that the loop-current-order fluctuations give rise to the Planckian dissipation where the transport relaxation time is proportional to \( \hbar/k_B T \) (we have re-inserted the Planck constant \( \hbar \) and the Boltzmann constant \( k_B \)).

Substituting Eq. (53) into Eqs. (33) and (37), we obtain.
\[ \sigma = \frac{e^2 \langle v_x^2 \rangle}{\bar{g}^2 T} \sum_n \left( \int_{-\infty}^{\infty} \frac{\varphi_{n,+}(t)}{2 \cosh(t/2)} dt \right)^2 \left[ \frac{\lambda_{n,+}}{\lambda_{n,+} \hat{1} - \hat{B}} \right]_{11}, \]  

(54)

\[ \kappa = \frac{\langle v_x^2 \rangle}{\bar{g}^2} \sum_n \left( \int_{-\infty}^{\infty} \frac{t \varphi_{n,-}(t)}{2 \cosh(t/2)} dt \right)^2. \]  

(55)

Note that the effect of the vertex corrections \( \hat{B} \) vanishes for the thermal conductivity \( \kappa \) because the bare vertex \( \propto t \) for \( \kappa \) is orthogonal to the even eigenfunctions \( \varphi_{n,+}(t) \), while the electrical conductivity has the vertex corrections to the self-energy contribution,
\[ \sigma_{SE} = \frac{e^2 \langle v_x^2 \rangle}{\bar{g}^2 T} \sum_n \left( \int_{-\infty}^{\infty} \frac{\varphi_{n,+}(t)}{2 \cosh(t/2)} dt \right)^2. \]  

(56)

Using the completeness
\[ f(t) \sum_n \sum_{s=\pm} \varphi_{n,s}(t) \varphi_{n,s}(x) = \delta(t-x), \]  

(57)

\( \sigma_{SE} \) and the thermal conductivity can be calculated as
\[ \sigma_{SE} = \frac{e^2 \langle v_x^2 \rangle}{\bar{g}^2 T} \int_{-\infty}^{\infty} \frac{\tanh(t/2) \cosh^2(t/2) f(t)}{4t} dt = \frac{7\zeta(3)}{\pi^2} \frac{e^2 \langle v_x^2 \rangle}{\bar{g}^2 T}, \]  

(58)

\[ \kappa = \frac{\langle v_x^2 \rangle}{\bar{g}^2} \int_{-\infty}^{\infty} \frac{t \tanh(t/2)}{4 \cosh^2(t/2)} dt = \frac{\langle v_x^2 \rangle}{\bar{g}^2}. \]  

(59)

Due to the Planckian dissipation, the thermal conductivity is a constant independent of temperature. The Lorentz number \( L \) for the loop-current-order fluctuations can be described by
\[ L \equiv \frac{\kappa}{\sigma T} = \frac{\kappa}{\sigma_{SE} T} \frac{\rho}{\rho_{SE}} = \frac{6}{7\zeta(3)} \frac{\rho}{\rho_{SE}} L_0, \]  

(60)

where \( \rho = 1/\sigma, \rho_{SE} = 1/\sigma_{SE} \), and \( L_0 = (\pi^2/3) (k_B/e)^2 \) is the Lorentz number of normal metals. Since \( \frac{6}{7\zeta(3)} = 0.713 \), the value of \( L/L_0 \) is about 70% of the value of \( \rho/\rho_{SE} \).
By using Eq. (56), we can write Eq. (54) as

\[
\frac{\sigma}{\sigma_{SE}} = 1 + \frac{2\pi^2}{7\zeta(3)} \sum_{n=1}^{\infty} \left( \int_{-\infty}^{\infty} \frac{\varphi_{n,+}(t)}{2 \cosh(t/2)} dt \right)^2 \left[ \frac{\lambda_{n,+}}{(\lambda_{n,+} - 1)\hat{1} + \hat{C}} - \hat{1} \right]_{11}, \tag{61}
\]

where

\[\hat{C} \equiv \hat{1} - \hat{B}.\tag{62}\]

If we keep only the \(n = 1\) term in the sum over \(n\) in Eq. (61), we obtain the inequality

\[
\frac{\sigma}{\sigma_{SE}} > 1 - \frac{8}{7\zeta(3)} + \frac{8}{7\zeta(3)}[\hat{C}^{-1}]_{11}. \tag{63}\]

By replacing all eigenvalues \(\lambda_{n,+}\) with the minimum eigenvalue \(\lambda_{1,+} = 1\), on the other hand, we obtain the inequality

\[
\frac{\sigma}{\sigma_{SE}} < [\hat{C}^{-1}]_{11}. \tag{64}\]

Therefore the lower and upper bounds of \(\rho/\rho_{SE}\) are given by

\[
\left( \frac{\rho}{\rho_{SE}} \right)_{L.B.} = \frac{1}{[\hat{C}^{-1}]_{11}}, \tag{65}
\]
\[
\left( \frac{\rho}{\rho_{SE}} \right)_{U.B.} = \frac{7\zeta(3)/8}{[\hat{C}^{-1}]_{11} + 7\zeta(3)/8 - 1}. \tag{66}
\]

Since \(7\zeta(3)/8 = 1.0518\), however, these bounds are almost equal, so that we can take the lower bound for evaluating the electrical resistivity. The results for its explicit evaluation for the case of a circular Fermi-surface in a square lattice are given in the next sub-section.

The vertex corrections \(\hat{B}\) due to Umklapp scattering can never be ignored for \(\sigma\) because in that case \([\hat{C}^{-1}]_{11}\) diverges due to the momentum conservation. Then the coefficient of the \(T\) linear term in the resistivity vanishes. So Umklapp scattering is essential for the linear in \(T\) resistivity. However, for the case of the critical fluctuations of the Qxy-F model, it provides no temperature dependent factors. This is crucial to be in accord with experiments on single-particle self-energy and the specific heat.

To understand physically the absence of Umklapp corrections in \(\kappa\), consider the single-particle decay of thermally excited fermions in the following process: A particle with a momentum \(p_1\) greater than the Fermi momentum \(k_F\) interacts with a particle inside the Fermi sphere with a momentum \(p_2\), \((|p_2| < k_F)\) and decays into two particles with momenta \(p_3\) and \(p_4\) which are both outside the Fermi sphere. Since momentum is conserved in this process \((p_1 + p_2 - p_3 - p_4 = 0)\),
the charge current of the whole system does not decay due to the feedback effect of momentum coming back from other particles. Energy is also conserved in this process: If $\epsilon$ is the energy measured from the chemical potential, $\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 = 0$. However, since $\epsilon_1, \epsilon_3,$ and $\epsilon_4$ are positive while $\epsilon_2$ is negative, the product of $\epsilon$ and momentum is not conserved. For this reason, the energy current decays even without Umklapps for the fermions.

These results can be put in context of the arguments due to Peierls, quoted for example in [63]. Peierls has argued that Umklapp scattering is essential for finite thermal conductivity from phonons in the pure limit, but not for finite electronic thermal conductivity when that is defined, as is customary and done here, making sure that the electronic conductivity induced by thermal gradient is absent. We have substantiated this argument. One should note however that the fermion self-energy $\Sigma$ on a lattice includes the effect of Umklapp scattering so that while one can say that the vertex renormalization is absent for the electronic thermal conductivity, it does not necessarily imply that the Umklapp scattering is absent.

It should be noted that there is no $\log(T)$ correction due to mass renormalization for the electrical conductivity. There is also no $\log(T)$ enhancement of the thermal conductivity which appears in the critical specific heat due to the marginal Fermi-liquid single-particle quasi-particle renormalization.

Recently, the breakdown of the Wiedemann-Franz law in strongly correlated electron systems and Weyl semimetals has been studied based on the Boltzmann equation [64, 65]. We note that the disappearance of the quasi-particle renormalization in the thermal conductivity is generally true for other problems such as the heavy-Fermi-liquids, in which the self-energy is nearly momentum independent.

E. Results for Umklapp factor on a square lattice with a circular Fermi-surface

Let us call $\rho_{SE}$ the resistivity calculated with self-energy alone, i.e. by putting $\hat{B} = 0$ and the actual resistivity by $\rho$. The evaluation of $\hat{C} = \hat{1} - \hat{B}$ is quite non-trivial even for a circular fermi-surface and given in Appendix C. As shown in the previous sub-section, $\rho/\rho_{SE}$ is given by the inverse of the 11 components of the inverse matrix of $\hat{C}$. The infinite-dimensional matrix $\hat{C}$ can be approximated by a square matrix of $L_{\text{max}} \times L_{\text{max}}$ and the limit of $L_{\text{max}} \to \infty$ gives the correct value. We present the results for numerical evaluation for the quantity $\rho/\rho_{SE}$ for various $L_{\text{max}} (= 1, 2, 4, 8, 16, 32)$ and for a circular Fermi-surface in Fig.5.
FIG. 5. Reduction factor of the resistivity due to the vertex corrections, i.e. the Umklapp factor, for a circular Fermi-surface of radius $k_F$ in a square lattice of unit-cell $a$, calculated as a function of $k_Fa/\pi$ ($1/2 < k_Fa/\pi < 1/\sqrt{2}$) for different number of $L_{\text{max}}$. The inset shows the $L_{\text{max}}$ dependence of the Umklapp factor for $k_Fa/\pi = 2 - \sqrt{2}$.

We first turn to the kinematics for various ratios $k_Fa/\pi$, which underlies the calculations. This is explained in Fig. 6. For $k_Fa/\pi < 0.5$, where the diameter of the Fermi surface is less than half of the reciprocal lattice vector, the resistivity is zero because there is no Umklapp scattering. When $0.5 < k_Fa/\pi < 4/((\sqrt{2} + \sqrt{34}) \approx 0.552$, the momentum involved in Umklapp scattering is restricted to the four "hot" portions of the Fermi surface shown in red, and only normal scattering occurs in the "cold" portions shown in blue in Fig. 6(a). The entire Fermi-surface becomes "hot" for $k_Fa/\pi \approx 0.552$ as shown in Fig. 6(b). We find in Fig. 5 that the resistivity is zero if there is any part of the Fermi-surface which is "cold". This is an interesting effect arising from the fact that $\sigma$ is proportional to $[C^{-1}]_{11}$ in Eq. (61), and $\hat{C}$ has obviously zero eigenvalues. As mentioned in Appendix C, we can write $\hat{C} = \hat{C}_N + \hat{C}_U$ in general, where $\hat{C}_N$ and $\hat{C}_U$ are the contributions from normal and Umklapp scatterings, respectively. $\hat{C}_U$ is not a zero matrix in the presence of the
FIG. 6. Kinematics of Umklapp scattering for $p - p' + k - k' = G$ with $p' = k' = -k$ and $G = (2\pi/a, 0)$. Two Brillouin zones are considered and it is demonstrated that for $k_F a/\pi < 0.5$, there is no Umklapp scattering. (a) There is a "hot" part of the Fermi-surface near the corners where Umklapp is allowed but the major part of the Fermi-surface is "cold" for $0.5 < k_F a/\pi < 4/((\sqrt{2} + \sqrt{34}) \approx 0.552$. (b) For larger $k_F a$, the entire Fermi-surface is hot and only then is their finite resistivity.

"hot" part, but it has an infinite number of zero eigenvalues in the presence of the "cold" part. This is because the "cold" regions always short-circuit the "hot" regions, or more precisely because we can choose $u_p$ as the conserved quantity in Eq. (31), which has a value only at points in the "cold" regions and is always zero in the "hot" regions. In the present two dimensions, $\hat{C}_N$ is a zero matrix, i.e., $\hat{C} = \hat{C}_U$, so this choice of $u_p$ does not produce any extra contribution from normal scattering. As a result, there are an infinite number of conserved quantities for a two-dimensional Fermi surface with the "cold" part. This anomaly disappears when impurity scattering is taken into account, because then no region is "cold" to begin with.

In Fig. 5, we find that the convergence with respect to $L_{max}$ is slow. This is especially so around $k_F a/\pi = 2 - \sqrt{2} \approx 0.586$, where $\rho/\rho_{SE}$ has a local minimum. But we have investigated
FIG. 7. Umklapp factor of the resistivity for a circular Fermi surface on a square lattice with $k_F a / \pi > 1/\sqrt{2}$, where higher Umklapp vectors such as $(2\pi/a, 2\pi/a)$ become players. The inset shows a circular Fermi surface of radius $k_F \approx 0.874 \pi/a$ centered at the M point, which corresponds to about 20% hole doping. Around this value of $k_F$, the convergence with respect to $L_{\text{max}}$ is fast and the Umklapp factor is nearly constant at about 0.78.

this region more exhaustively with increased $L_{\text{max}}$ to assure that the value of the normalized resistivity there is about 0.2 but saturating at about 0.5 at large $k_F a$. If the Fermi surface is large enough, so that higher Umklapp vectors $(2\pi/a, 2\pi/a)$ become players and the true resistivity $\rho$ then becomes closer to $\rho_{\text{SE}}$. For the about 20% hole doped cuprates near quantum-critical doping, the Fermi-surface is large. We have also calculated $\rho/\rho_{\text{SE}}$ for $k_F a / \pi > 1/\sqrt{2}$, where there is Umklapp scattering also for the next reciprocal vectors. As seen in Fig. 7, the convergence with respect to $L_{\text{max}}$ is fast and the Umklapp factor is nearly constant at about 0.78 around $k_F a / \pi = 2\sqrt{3}/5\pi \approx 0.874$, which corresponds to the radius of an oblate Fermi surface centered at the M point of the 20% hole doped cuprates when approximated by a circle. We would therefore expect $0.5 < \rho/\rho_{\text{SE}} < 1$. This is in line with the estimates given in Ref. [2] for the relative magnitude of
the imaginary part of the self-energy in the diagonal direction, the coefficient of the $T \ln T$ specific heat and the resistivity scattering rate estimated from experiments.

In this context, the result derived by Maebashi and Fukuyama \cite{52, 53} is worth recalling. For transport theory in dimension $d \to \infty$ or for infinite number of neighbors, as in Dynamical mean-field calculations, the Umklapp factor is identity. This should be equally true for the SYK-type models \cite{12} in which $N$-sites are coupled to each other with $N \to \infty$.

III. CONCLUSIONS

We have solved the integral equation for the vertex coupling to external fields in the Kubo expression for electrical and thermal conductivities, using scattering of fermions from the known propagator for the quantum fluctuations of the xy model coupled to fermions. We have derived exact results at low temperatures for transport properties. This has been possible only because of the simplicity and the unusual nature of the scale-invariant spectra of fluctuations. The simplicity is due to the fact that the fluctuations are due to two orthogonal topological objects, one fluctuating only in space and the other only in time, and the fact that fermions couple only to gradients in space or time of the phase variable in the xy model. This model has direct applicability to a variety of physical problems of experimental interest which give experimental results with linear in $T$ and linear in $H$ resistivity, temperature independent thermal conductivity and $T \ln(\omega_c/T)$ specific heat, with coefficients which are simply related to each other. This is also the only example of a physically applicable model we know for which the Kubo equation for transport has been solved. The general theory presented should be of considerable technical interest although in the specific problem solved it provides corrections only by a factor of about $3/4$ for Fermi-surfaces of interest in electrical resistivity and none for thermal conductivity to the simpler calculations done earlier.

We hope that this work, through its exact results and their quantitative agreement with experiments, provides an unambiguous understanding of the universality of the ”strange metal” anomalies near quantum-critical points discovered in experiments in a variety of different materials with quite different order parameters. These include the cuprates with loop-current order without breaking translation symmetry, planar antiferromagnets and incommensurate Ising antiferromagnets, TBG if it indeed has loop-current order breaking translation symmetry, as well as TBWS. We have emphasized that the reason for the universality near quantum-criticality is that they are all described by the quantum xy-model coupled to fermions. It follows also that the
superconductivity in them is promoted by scattering of fermions from the same fluctuations which give the normal state strange metal anomalies.

Acknowledgements: HM is grateful to M. Ogata and H. Matsuura for fruitful discussions. CMV wishes to acknowledge several discussions and email exchanges with Dominic Else and Senthil Todadri which were useful in understanding their work. The work of HM was partly supported by Grants-in-Aid for Scientific Research from the Japan Society for the Promotion of Science (Grant Nos. JP21K03426, JP18K03482, and JP18H01162) and JST-Mirai Program, Japan (Grant No. JPMJMI19A1). CMV wishes to thank James Analytis, Robert Birgeneau and Joel Moore for arranging for him to be a "re-called Professor" at the Physics department of University of California, Berkeley, where part of this work was done and to Aspen Center for Physics where part of this work was done last summer. Aspen Center for Physics is partially supported by the National Science Foundation of USA through grant PHY-1607611.
IV. APPENDICES

A. Summary of the critical fluctuations of quantum xy model coupled to fermions

The quantum xy model in 2+1 dimensions is defined in terms of the action $S_{qxy}$ of a rotor of fixed length and angle $\theta(x, \tau)$ at a point $x$ and imaginary time $\tau$, which is periodic in the inverse temperature $(0, \beta)$. It is given by the following action:

$$S_{qxy} = -K_0 \sum_{\langle x, x' \rangle} \int_0^\beta d\tau \cos(\theta_{x,\tau} - \theta_{x',\tau}) + \frac{1}{2E_c} \sum_x \int_0^\beta d\tau \left( \frac{d\theta_x}{d\tau} \right)^2.$$  \hspace{1cm} (67)

The variable $\theta$ refers to different quantities in different physical situations. It refers to the direction of the anapole vector in the loop-current order in cuprates [21], to the in-plane antiferromagnetic order-vector in some heavy-fermion transformed on a bi-partite lattice, to a model coupled ferromagnetically in the plane [24], or to the phase of an incommensurate antiferromagnetic Ising order in another heavy fermion [24]. If the lattice anisotropy is more than 4-fold, it is irrelevant in the quantum model and is ignored. Recently, it has been suggested, (E. Berg and C.M. Varma, Unpublished - 2022) the loop-current order proposed in Moiré twisted bi-layer graphene [42, 43], (which is remarkably similar to an order presented for Graphene due to nearest neighbor interactions [66]), and WSe$_2$ (Liang Fu - private communication 2022) also fall in the quantum-xy class.

The model of Eq. (67) has only Lorentz-invariant fluctuations at long wave-lengths, which are inadequate to address questions in problems with fermions. It must be coupled to fermions to give interesting results. $\theta(x, \tau)$ is not gauge invariant and cannot couple to any fermion variable. $\nabla \theta(x, \tau)$ is proportional to a collective mode current, and if $\theta$ refers to the direction of a vector which is time-reversal and inversion-odd, it can couple to the fermion current. The conjugate variable in the quantum-rotor problem, i.e. the angular momentum $\frac{d\theta(x, \tau)}{d\tau}$ or in a Hamiltonian formulation $L_z(x, \tau)$ similarly couples to the local fermion angular momentum. After integrating over the fermions (which can be done formally in terms of the fermion current-current correlation function), both of these give contributions of the same functional form to the Lagrangian. This form is of the Caldeira-Leggett form, and is

$$S_{c-f} = \frac{\alpha}{4\pi^2} \sum_{\langle x, x' \rangle} \int d\tau d\tau' \frac{\pi^2}{\beta^2} \left[ (\theta_{x,\tau} - \theta_{x',\tau}) - (\theta_{x,\tau'} - \theta_{x',\tau'}) \right]^2 \sin^2 \left( \frac{\pi|\tau - \tau'|}{\beta} \right).$$  \hspace{1cm} (68)
This looks complicated but it looks much simpler when expressed as a function of real frequency \( \omega \) and momentum \( q \):
\[
\alpha \frac{4}{\pi^2} |\omega| q^2 |\theta(q, \omega)|^2.
\]
\( \alpha \) is the conductivity of the fermions in the limit \( q \to 0, \omega \to 0, T \to 0 \) made dimensionless in terms of \( e^2/hc \). One could also couple to \( \cos(\theta(x, \tau)) \).
This has also been investigated and is found to be irrelevant when (68) is present.

The quantum xy model coupled to fermion which has been investigated by quantum-Monte-carlo exhaustively is given by
\[
S_{qxy-F} \equiv S_{qxy} + S_{c-f}. \tag{69}
\]
As a function of \( K_0, E_c \) and \( \alpha \) has three different lines of transitions. We will be interested in the problem under discussion where the quantum-critical point fluctuations are at the transition in which \( \theta(x, \tau) \) orders both in space and time. The most important quantity to calculate, which is directly relevant to the calculation of the properties of fermions is the correlation function of the fluctuations
\[
D(r, \tau) \equiv < e^{-i\theta(r, \tau)} e^{i\theta(0, 0)} > \propto < L_z(r, \tau) L_z(0, 0) > \tag{70}
\]
The proportionality of the fluctuations of the angular momentum variable to those of \( e^{i\theta} \) has been shown \[67\]. The result for the fluctuations are very accurately given by \[26, 27, 68\]
\[
D(r, \tau) = D_0 \frac{T^2}{\tau} e^{-(\tau/\xi_r)^{1/2}} \ln(r/a) e^{-r/\xi_r}. \tag{71}
\]
\( D_0 \) provides the magnitude of the fluctuations which is given by the square magnitude of the orbital current moments per unit-cell. \( \tau_c^{-1} \) is the ultraviolet energy cut-off and \( a \) is the lattice constant. Further the spatial correlation \( \xi_r \) is negligible for all practical purposes compared to the temporal correlation length:
\[
\xi_r/a = \ln(\xi_r/\tau_c). \tag{72}
\]
The most remarkable feature of (71) is that they are of product form in time and space. This arises because of the nature of the two kinds of orthogonal topological excitations responsible for the correlation functions noted in \[23, 54\] one of which propagates only in space and the other only in time.

An example of the accuracy of the extensive calculations \[26, 27\] from which the above conclusions are drawn is given by Fig. 8.

After making the Villain transformation, the model is harmonic for the spin-wave like fluctuations of the \( \theta \) so that they can be integrated out \[23, 25, 54\]. This leaves two kinds of topological
FIG. 8. This figure gives an example of the correlation function $D(x, \tau)$ (called $G$ in the original paper), calculated in [27]. All results are for fixed $K_0$ and $E_c$ and varying $\alpha/4\pi^2$. Top left at small $\tau$ $D$ is shown as a function of $x$, while top-right shows it at small $x$ as a function of $\tau$. With such fixed values, there is a critical point for $\alpha/(4\pi^2) = 0.014 \pm 0.0002$. These curves can be collapsed and shown to be of a scaling form near the critical form from which the correlation length in time $\xi_\tau$ and in space $\xi_x$ can be deduced as shown at bottom left. In bottom right, the relation between $\xi_\tau$ and $\xi_x$ is shown to be $\xi_x \sim \xi_\tau^{1/2}$. The minimum value of $z$ with which these results can be fit is 6 but $z \to \infty$, i.e. $\xi_x \sim \log \xi_\tau$ fits very well.

fluctuations, vortices interacting logarithmically in space and local in time, and another species - warps, which are jumps in imaginary time of $\theta(x, \tau)$, which interact locally in space but logarithmically in time, together with another term involving warps alone in which the interaction is Lorentz - invariant. Direct evidence for warps and vortices with such properties is observed from the quantum-monte-carlo calculations.

Remarkably, the transformed problem in terms of two kinds of topological excitations is as well soluble as the Kosterlitz solution [69, 70] of the classical xy model in first order RG. The first
order RG solution reproduces essentially all but not all features of the Monte-Carlo solution. A remarkable result from Monte-carlo is that at the critical point with varying dimensionless ratios \(K_0E_c, \alpha\), not only are the correlation functions of the same form but the amplitude \(D_0\) is also independent of them. This is not reproduced by the RG, nor is the fact that the correlation decays as \(e^{-(\tau/\xi)^{1/2}}\) and not as \(e^{-(\tau/\xi)}\).

The correlation function calculated by RG can be continued to real frequency and momentum only numerically for finite correlation lengths. The results can be well fitted to the form given by the analytical calculations:

\[
D(\mathbf{q}, \omega) = D_0 \left( \ln \frac{\omega_c}{\max(\omega, \pi T)} - i \tanh \frac{\omega}{\sqrt{(2T)^2 + \xi_{\tau}^{-2}}} \right) \xi_{\tau}^{-2} q^2 + \xi_{\tau}^{-2}. \tag{73}
\]

The relation in Eq. (72) makes the problem effectively spatially local. Moreover the coupling to the fermions of the fluctuations comes from the coupling of their current to the fermion current and of their angular momentum to the fermion angular momentum [71, 72]. Both of these lead to a vertex coupling to fermions

\[
g(\mathbf{q}) = g_0 q^2. \tag{74}
\]

where \(q\) is the momentum transfer. This is of the same form as used in the calculation of the coupling of fluctuations to fermions, which were eliminated to give (68). Eqs. (73) and (74) have been used to calculate the fermion self-energy. The wiggly lines in all the figures in the text refers to (73).

### B. Landau-Boltzmann transport coefficients

For Landau Fermi liquids or marginal Fermi-liquids, the single-particle spectral function is approximated by

\[
A(p, \varepsilon) \approx a_p \delta(\varepsilon - \varepsilon^*_p). \tag{75}
\]

Even though in marginal Fermi-liquids the quasi-particle weight \(a_p\) is non-zero only due to a logarithmic factor, a Fermi-surface is well defined so that for low temperatures we can use Eq. (75) without introducing errors, where all the effects of \(a_p\) are canceled out in the theory such that \(A(p, 0) = a_p \delta(\varepsilon^*_p) = \delta(\varepsilon_p)\). (It is important to note that this cancellation occurs also for the marginal Fermi-liquid, where \(a_p(\varepsilon^*_p) \rightarrow 0\), as \(1/\ln(\varepsilon^*_p)\).) Hence, the low-temperature conductivity can be obtained without using the Fermi liquid assumption, Eq. (75), and the result given
by Eq. (33) does not include any quasi-particle weight. However, Landau theory gives a fam-
iliar physical picture of quasiparticles carrying electric and thermal currents, so here we derive the
Landau-Boltzmann transport equations for a model of fermions on a lattice scattering with collec-
tive fluctuations.

Substituting Eq. (75) into Eq. (14), we get

\[
\sigma(T) = 2e^2 \int \frac{dp}{(2\pi)^d} \left( -\frac{\partial f(\epsilon_p^*)}{\partial \epsilon_p^*} \right) v_{px}^* \Phi(p),
\]

(76)

where \( v_{px}^* = a_p \bar{v}_x(p, \epsilon_p^*) \) and \( \Phi(p) = \Phi(p, \epsilon_p^*) \). This is equivalent to Eq. (3). Then Eq. (20) is
given by

\[
v_{px}^* \text{sech} \frac{\epsilon_p^*}{2T} = \frac{1}{2} \int \frac{dp' dk dk'}{(2\pi)^2d} S(p, k; p', k') \text{sech} \frac{\epsilon_{p'}^*}{2T} \text{sech} \frac{\epsilon_k^*}{2T} \text{sech} \frac{\epsilon_{k'}^*}{2T}
\]

\[
\times [\Phi(p) - \Phi(p') + \Phi(k) - \Phi(k')],
\]

(77)

where

\[
S(p, k; p', k') = 2\pi W(p - p', \epsilon_p^* - \epsilon_{p'}^*) a_p a_{p'} a_k a_{k'}
\]

\[
\times \Delta^d(p - p' + k - k') \delta(\epsilon_p^* - \epsilon_{p'}^* + \epsilon_k^* - \epsilon_{k'}^*).
\]

(78)

Noting that

\[
f(\epsilon) \tilde{f}(\epsilon') f(\epsilon_1) \tilde{f}(\epsilon - \epsilon' + \epsilon_1) = \frac{1}{16} \text{sech} \frac{\epsilon}{2T} \text{sech} \frac{\epsilon'}{2T} \text{sech} \frac{\epsilon_1}{2T} \text{sech} \frac{\epsilon - \epsilon' + \epsilon_1}{2T},
\]

(79)

where \( \tilde{f}(\epsilon) = 1 - f(\epsilon) \), we can write Eq. (77) in the familiar form of the Landau-Boltzmann
transport equation as

\[
v_{px}^* \left( -\frac{\partial f(\epsilon_p^*)}{\partial \epsilon_p^*} \right) = \frac{2}{T} \int \frac{dp' dk dk'}{(2\pi)^2d} S(p, k; p', k') \tilde{f}(\epsilon_p^*) f(\epsilon_{p'}^*) f(\epsilon_k^*) \tilde{f}(\epsilon_{k'}^*)
\]

\[
\times [\Phi(p) - \Phi(p') + \Phi(k) - \Phi(k')].
\]

(80)

It should be noted, however, that for fermions scattering with collective fluctuations, \( W \) in \( S \) is
given by Eq. (19).

Let us write Eq. (77) in terms of the imaginary part of the quasiparticle’s memory matrix \( M^* \)
which we may define through:

\[
v_{px}^* \text{sech} \frac{\epsilon_p^*}{2T} = \int \frac{dp'}{(2\pi)^d} M^{*''}(p, p') \Phi(p') \text{sech} \frac{\epsilon_{p'}^*}{2T}
\]

(81)

Then we obtain the dc conductivity in terms of the inverse matrix \( [M^{*''}]^{-1} \) using the definition
(81),

\[
\sigma(T) = \frac{e^2}{2T} \int \frac{dp dp'}{(2\pi)^{2d}} v_{px}^* \text{sech} \frac{\epsilon_p^*}{2T} [M^{*''}]^{-1}(p, p') v_{p'x}^* \text{sech} \frac{\epsilon_{p'}^*}{2T}.
\]

(82)
Similarly, we can obtain the thermal conductivity \( \kappa \) and the Seebeck coefficient \( S \):

\[
\kappa = \frac{1}{T} \left( L_{22} - \frac{L_{12}L_{21}}{L_{11}} \right),
\]

\[
S = \frac{L_{12}}{TL_{11}},
\]

where the transport coefficients \( L_{ij} \) are given by

\[
L_{11} = \frac{e^2}{2T} \int \frac{dp dp'}{(2\pi)^2} v_{px}^* \frac{\epsilon_p^*}{2T} [M^*]^{-1}(p, p') v_{px}^* \frac{\epsilon_{p'}}{2T},
\]

\[
L_{12} = L_{21} = -\frac{e}{2T} \int \frac{dp dp'}{(2\pi)^2} v_{px}^* \frac{\epsilon_p}{2T} [M^*]^{-1}(p, p') v_{px}^* \frac{\epsilon_{p'}}{2T},
\]

\[
L_{22} = \frac{1}{2T} \int \frac{dp dp'}{(2\pi)^2} v_{px}^* \frac{\epsilon_p}{2T} [M^*]^{-1}(p, p') v_{px}^* \frac{\epsilon_{p'}}{2T}.
\]

C. Umklapp vertex for a circular Fermi-surface

From Eq. (45) with use of \( \psi_L(-p) = -\psi_L(p) \), the matrix elements of \( \hat{C} = \hat{1} - \hat{B} \) are obtained as

\[
C_{LL'} = \frac{1}{N(0)^2} \int \frac{dp dp'}{(2\pi)^4} \psi_L(p) \delta(\epsilon_p) \delta(\epsilon_{p'})
\times \int \frac{dk dk'}{(2\pi)^2} \Delta^d(p - p' + k - k') \delta(\epsilon_k) \delta(\epsilon_{k'})
\times \left[ \psi_{L'}(p) - \psi_{L'}(p') + \psi_{L'}(k) - \psi_{L'}(k') \right]
\times \left[ \frac{1}{(2\pi)^2} \Delta_{i}(p - p' + k_i - k_i') \delta(\epsilon_{k_i}) \delta(\epsilon_{k_i'}) \right]^{-1}.
\]

Since \( \Delta^d(p - p' + k - k') = \sum_{G} \delta^d(p - p' + k - k' - G) \), the matrix can be separated as \( \hat{C} = \hat{C}_N + \hat{C}_U \), where \( \hat{C}_N \) and \( \hat{C}_U \) are the contributions from normal (\( G = 0 \)) and Umklapp (\( G \neq 0 \)) scatterings, respectively. \( \hat{C}_N \) has a zero eigenvalue corresponding to conservation of crystal momentum; in three dimensions it is generally a nonzero matrix. However, two dimensions are special. Consider, for example, the circular Fermi surface shown in Fig. 9(a). For the given \( p \) and \( p' \), there are only two possible processes for which \( p, p', k, \) and \( k' \) are on the Fermi surface and \( p + k = p' + k' \); one process is described by \( k = -p \) and \( k' = -p' \), the other by \( k = p' \) and \( k' = p \). Since \( \psi_{L'}(p) - \psi_{L'}(p') + \psi_{L'}(k) - \psi_{L'}(k') = 0 \) in Eq. (88) for the both processes, \( \hat{C}_N \) is a zero matrix. As discussed in Ref. [53], this result holds broadly for noncircular Fermi surfaces on a two-dimensional lattice, where \( \hat{C} = \hat{C}_U \).

Let us consider in more detail \( \hat{C} \) for a two-dimensional circular Fermi surface with a Fermi wavenumber \( k_F \). As shown in Fig. 9 for the given \( p, p' \), and \( G \) (we include normal processes by
FIG. 9. Normal and Umklapp scattering processes for a two-dimensional circular Fermi surface. For the given \( p, p', \) and \( G \), there are only two processes that satisfy \( p - p' + k - k' = G \), indicated by \( \pm \). These processes are related by \( k_\pm' = -k_\mp \). (a) For normal scattering (\( G = 0 \)), \( k_+ = -p, k_+ = -p', k_- = p' \) and \( k_- = p \), so that \( \psi_L(p) - \psi_L(p') + \psi_L(k_\pm) - \psi_L(k_\mp) \) in Eq. (88) is identically zero. (b) For Umklapp scattering (\( G \neq 0 \)), \( \psi_L(k_+) - \psi_L(k'_+) = \psi_L(k_-) - \psi_L(k'_-) \) but \( \psi_L(p) - \psi_L(p') + \psi_L(k_+) - \psi_L(k'_+) \) becomes nonzero.

For \( G = 0 \), the two possible sets of solutions satisfying \( p - p' + k - k' = G \) for \( p, p', k, \) and \( k' \) on the Fermi surface is given by \( (k, k') = (k_\pm, k'_\pm) \). These solutions are explicitly given by

\[
k'_\pm, x = -k_\mp, x = \frac{q_x}{2} \pm \frac{q_y}{2} \sqrt{\frac{4k_F^2}{q^2} - 1}, \tag{89}
\]

\[
k'_\pm, y = -k_\mp, y = \frac{q_y}{2} \pm \frac{q_x}{2} \sqrt{\frac{4k_F^2}{q^2} - 1}, \tag{90}
\]

where \( q = p - p' - G \). Let \( \theta \) and \( \theta' \) be the angles of \( p \) and \( p' \), respectively. Then, depending on the reciprocal vector \( G = (G_x, G_y) \), the angles \( \alpha_\pm \) of \( k'_\pm \), which are functions of \( \theta \) and \( \theta' \), is
obtained through

\[
\cos \alpha_{\pm}(\theta, \theta'; G) = \frac{1}{2} \left( \cos \theta - \cos \theta' - \frac{G_x}{k_F} \right) \\
\pm \frac{1}{2} \left( \sin \theta - \sin \theta' - \frac{G_y}{k_F} \right) \frac{\sqrt{1 - X^2(\theta, \theta'; G)}}{X(\theta, \theta'; G)},
\]

(91)

\[
\sin \alpha_{\pm}(\theta, \theta'; G) = \frac{1}{2} \left( \sin \theta - \sin \theta' - \frac{G_y}{k_F} \right) \\
\pm \frac{1}{2} \left( \cos \theta - \cos \theta' - \frac{G_x}{k_F} \right) \frac{\sqrt{1 - X^2(\theta, \theta'; G)}}{X(\theta, \theta'; G)},
\]

(92)

where

\[
X(\theta, \theta'; G) = \frac{1}{2} \sqrt{\left( \cos \theta - \cos \theta' - \frac{G_x}{k_F} \right)^2 + \left( \sin \theta - \sin \theta' - \frac{G_y}{k_F} \right)^2}.
\]

(93)

The average over the Fermi surface is given by

\[
\frac{1}{N(0)} \int \frac{d\mathbf{p}}{(2\pi)^2} \delta(\tilde{\epsilon}_p) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi},
\]

(94)

where \(\tilde{\epsilon}_p = (p^2 - k_{F}^2)/2m\) and \(N(0) = m/2\pi\), but the effect of the mass \(m\) is canceled in \(\hat{C}\). The Fermi surface harmonics \(\psi_L(p)\) are given by

\[
\psi_L(p) = \sqrt{2} \cos(2L - 1)\theta,
\]

(95)

where the \(L = 1\) Fermi surface harmonics is proportional to the \(x\) component of the velocity or the momentum, \(\psi_1(p) = \sqrt{2} \cos \theta = \sqrt{2} p_x/k_F\). Therefore we obtain the matrix elements of \(\hat{C}\) for the two-dimensional circular Fermi surface as

\[
C_{LL'} = 2 \sum_{G \neq 0} \int_{-\pi}^{\pi} \frac{d\theta d\theta'}{(2\pi)^2} w(\theta, \theta'; G) \cos\{(2L - 1)\theta\} \\
\times \left[ \cos\{(2L' - 1)\theta\} - \cos\{(2L' - 1)\theta'\} \\
- \cos\{(2L' - 1)\alpha_+(\theta, \theta'; G)\} - \cos\{(2L' - 1)\alpha_-(\theta, \theta'; G)\} \right].
\]

(96)

Here \(w(\theta, \theta'; G)\) is a weight function satisfying \(\sum_{G} w(\theta, \theta'; G) = 1\),

\[
w(\theta, \theta'; G) = \frac{\Theta(1 - X^2(\theta, \theta'; G))}{X(\theta, \theta'; G) \sqrt{1 - X^2(\theta, \theta'; G)}} \\
\times \left( \sum_{G'} \frac{\Theta(1 - X^2(\theta, \theta'; G'))}{X(\theta, \theta'; G') \sqrt{1 - X^2(\theta, \theta'; G')}} \right)^{-1},
\]

(97)

where \(\Theta(x)\) is the Heaviside step function. Note that \(G = 0\) is excluded from the summation for the reciprocal lattice vector \(G\) in Eq. (96), corresponding to the two-dimensional speciality \(\hat{C} = \hat{C}_U\) mentioned above. However, the sum over \(G'\) in Eq. (97) includes \(G' = 0\).
If \( k_F a < \pi/2 \), \( w(\theta, \theta'; G) \) vanishes for \( G \neq 0 \) [note that for example, \( G_x/k_F > 4 \) for \( G = (2\pi/a, 0) \) and the step function in Eq. (97) is zero]. Then, by Eq. (96), \( \hat{C} \) gets equal to a zero matrix, \( \hat{C} = \hat{0} \). Hence, the coefficient of the \( T \) linear term in the resistivity vanishes in the absence of Umklapp scattering.

The numerical evaluation of \( C_{LL'} \) for various choices of \( L_{max} \) has been carried out only for a circular Fermi-surface of various radii on a square. The results for the resistivity factor from Umklapp are presented in Sec. II-E.

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