BEYOND UNDECIDABLE

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ABSTRACT. The predicate complementary to the well-known Gödel's provability predicate is defined. From its recursiveness new consequences concerning the incompleteness argumentation are drawn and extended to new results of consistency, completeness and decidability with regard to Peano Arithmetic and the first order predicate calculus.

Keywords: decision problem, provability predicate, Gödel numbering.

INTRODUCTION

Of all the remarkable logical achievements of the twentieth century perhaps the most outstanding is the celebrated Gödel incompleteness argumentation of 1931 [1, 2]. In contrast to Hilbert’s program called for embodying classical mathematics in a formal system and proving that system consistent by finitary methods [4], Gödel paper showed that not even the first step could be carried out fully, any formal system suitable for the arithmetic of integers was incomplete.

The present article, in the most absolute respect for the extraordinary contribution given by Gödel to the logical inquiry, brings Gödel’s achievement into question by the definition of the refutability predicate. As it is well-known self-reference plays a crucial role in Gödel’s incompleteness argumentation and the methods of achieving self-referential statements is the so-called “diagonalization”. The refutability predicate, defined by arithmetization as a number-theoretic statement, gives rise to new consequences properly regarding Gödel’s incompleteness argumentation and the method of diagonalization. This article proposes a revision based on the logical investigation of the interactive links between provability and refutability predicates. Originally devised by Gödel in order to arithmetize metamathematical notions, Gödel numbering turns out to be the key of the problem in defining refutability with the same recursive status as provability. The inquiry comes up with a final solution for finitary methods and the related decision problem [3].

The paper is organized as follows. Firstly, in the following of this section, we introduce diagonalization and the famous incompleteness argumentation of Gödel. Section 1 presents two new primitive recursive predicates for refutability and the enucleation of some of their consequences, which represent the first main result of this paper: Gödel’s incompleteness argumentation is not a theorem in Peano Arithmetic. Section 2 shows that any formula of Peano Arithmetic is proved if and

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only if it is not refuted, and extends this result to the accomplishment of consistency and completeness for Peano Arithmetic and then to the achievement of decidability for first order predicate calculus.

**Basic Setup.** We shall assume a first order theory which adequately formalizes Peano Arithmetic (see for example the system $S$, with all the necessary assumptions, in [3] 116-175). Let us call it $P_4$. As is well known by means of the Gödel numbering, each expression in $P_4$ can refer to itself. Numerals, as usual, are defined recursively, $0$ is 0 and for any natural number $n$, $\overline{n + 1}$ is $(\overline{n})'$ (where $'$ is the Successor function). For any expression $X$ we use $\overline{X}$ to denote the corresponding Gödel number of $X$. Let us define the Gödel numbering as follows:

1. First assign different odd numbers to the primitive symbols of the language of $PA$.
2. Let $X$ be a formal expression $X_0, X_1, \ldots, X_n$, where each $X_i$, $0 \leq i \leq n$, is a primitive symbol of the language of $PA$. Then
   $$\overline{X} = p_0^{\overline{X_0}} \cdot p_1^{\overline{X_1}} \cdots p_n^{\overline{X_n}}$$
   where $p_n$ is the $n$-th prime number and $p_0 = 2$.
3. Let $X$ be composed by the formal expressions $X_0, X_1, \ldots, X_n$, then
   $$\overline{X} = p_0^{\overline{X_0}} \cdot \cdots \cdot p_n^{\overline{X_n}}.$$

For any given formula $\phi(v)$ of $PA$ we then have its Gödel number $n = \overline{\phi(v)}$. This number $n$ has a name in the language of $PA$, namely $\overline{n}$, and this name can be substituted back into $\phi(v)$. This self-reference procedure is admitted by the so-called diagonalization lemma as follows.

**Diagonalization.** For any formula $\phi$ with only the variable $v$ free there is a sentence $\delta$ such that

$$\vdash_{PA} \delta \iff \phi(\overline{\delta}).$$

The argumentation usually considered to be a proof is the following. We define the function of substitution $sb(\overline{\phi(v)} \overline{\pi}) = \overline{\phi(\overline{\pi})}$, which gives us the Gödel number of the result of replacing $v$ by the $n$-th numeral in $\phi(v)$ (see the corresponding $Sb(x^n_\pi)$ and $Sb(x^n_\chi(y))$ in [1][2]).

Let $\phi(v)$ be given and let us call $\beta(v)$ the formula $\phi(sb(v,v))$. Let $m = \overline{\beta(v)}$ and $\delta = \overline{\beta(\overline{m})}$. We shall show that $\delta$ is the sentence we were looking for. To this purpose we notice that in $PA$ they hold the following equivalences

$$\vdash \delta \iff \beta(\overline{m})$$

by definition

$$\iff \phi(sb(\overline{m}, \overline{m}))$$

by definition

$$\iff \phi(sb(\overline{\beta(v)}, \overline{m}))$$

since $m = \overline{\beta(v)}$

$$\iff \phi(\overline{\beta(m)})$$

definition of $sb$

$$\iff \phi(\overline{\delta})$$

by definition.
Gödel’s Incompleteness. We present the version of the so-called Gödel’s first incompleteness Theorem as it is given in ([3] 161-162), to which the reader can refers for the definition of the concepts which are involved.

Let \( \phi(v) \) be the formula \( \forall x \neg Pf(x, v) \), hence by diagonalization lemma we attain

\[
\vdash_{PA} \delta \iff \forall x \neg Pf(x, \overline{\delta}).
\]

Gödel’s incompleteness argumentation asserts:

(a) If \( PA \) is consistent, not \( \vdash_{PA} \delta \),

(b) If \( PA \) is \( \omega \)-consistent, not \( \vdash_{PA} \neg \delta \),

hence, if \( PA \) is \( \omega \)-consistent, \( \delta \) is an undecidable sentence of \( PA \).

The proof is as follows. Let \( q \) be the Gödel number of \( \delta \).

(a) Assume \( \vdash_{PA} \delta \). Let \( r \) be the Gödel number of a proof in \( PA \) of \( \delta \). Then \( Pf(r, q) \). Hence, \( \vdash_{PA} Pf(\overline{r}, \overline{q}) \), that is \( \vdash_{PA} Pf(\overline{r}, \overline{\delta}) \). We already have \( \vdash_{PA} \delta \iff \forall x \neg Pf(x, \overline{\delta}) \). By Biconditional Elimination, \( \vdash_{PA} \forall x \neg Pf(x, \overline{\delta}) \).

By Rule A4 (Particularization Rule), \( \vdash_{PA} \neg Pf(\overline{\delta}) \). Therefore, \( PA \) is inconsistent.

(b) Assume \( PA \) is \( \omega \)-consistent and \( \vdash_{PA} \neg \delta \). Since \( \vdash_{PA} \delta \iff \forall x \neg Pf(x, \overline{\delta}) \), Biconditional Elimination yields \( \vdash_{PA} \neg \forall x \neg Pf(x, \overline{\delta}) \) which abbreviates to (\( * \)) \( \vdash_{PA} \exists x Pf(x, \overline{\delta}) \). On the other hand, since \( PA \) is \( \omega \)-consistent, \( PA \) is consistent. But, \( \vdash_{PA} \neg \delta \). Hence, not \( \vdash_{PA} \delta \); that is, there is no proof in \( PA \) of \( \delta \). So \( Pf(n, q) \) is false for every natural number \( n \) and, therefore, \( \vdash_{PA} \neg Pf(\overline{n}, \overline{\delta}) \) for every natural number \( n \). (Remember that \( \overline{n} \) is \( \overline{\overline{x}} \)). By \( \omega \)-consistency, not \( \vdash_{PA} \exists x Pf(x, \overline{\delta}) \), contradicting (\( * \)).

1. Refutability

We are now ready to present the results with which this paper is concerned. We shall construct two new predicates by Gödel numbering. The reader can refer to the arithmetization as defined by Mendelson; the new predicates must be considered as two last relations added to the functions and relations (1-26) presented in ([2] 149-156). Let us start recalling some of the definitions involved, precisely only those we need.

\( \text{MP}(x, y, z) \): The expression with Gödel number \( z \) is a direct consequence of the expressions with Gödel numbers \( x \) and \( y \) by modus ponens,

\[
y = 2^3 \cdot x \cdot 2^{11} \cdot z \cdot 2^5 \land \text{Gd}(x) \land \text{Gd}(z).
\]

\( \text{Gen}(x, y) \): The expression with Gödel number \( y \) comes from the expression with Gödel number \( x \) by the Generalization Rule,

\[
(\exists y < y)(\text{EVbl}(v) \land y = 2^3 \cdot 2^3 \cdot 2^{13} \cdot v \cdot 2^5 \cdot x \cdot 2^5 \land \text{Gd}(x)).
\]

\( \text{Ax}(y) \): \( y \) is the Gödel number of an axiom of \( PA \):

\[
\text{Lax}(y) \lor \text{PrAx}(y).
\]

\( \text{Neg}(v) \): the Gödel number of (\( \neg \alpha \)) if \( v \) is the Gödel number of \( \alpha \):

\[
\text{Neg}(v) = 2^3 \cdot 2^9 \cdot v \cdot 2^5.
\]

\( \text{Prf}(x) \): \( x \) is the Gödel number of a proof in \( PA \):

\[
\text{We shall not reproduce entirely this long list of definitions which is already well-known (see also [4] 162-176).}
\[\exists u < x \exists v < x \exists z < x \exists w < x \ (x = 2^w \land \text{Ax}(w)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_w) \land x = u \ast 2^v \land \text{Gen}(u,v)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_z) \land \text{Fml}(u_w) \land x = u \ast 2^v \land \text{MP}(u_z, (u)_w, v)) \lor \\
[\text{Prf}(u) \land x = u \ast 2^v \land \text{Ax}(v)]].\]

For its recursiveness in other terms \(\text{Ref}(x,v)\) states \(x\) is the Gödel number of a refutation in \(\text{PA}\) of the formula with Gödel number \(v\):

\[\text{Prf}(x) \land v = (x)_{\text{lh}(x) - 1}.\]

By means of such definitions, we shall define two new predicates, \(\text{Rf}\) and \(\text{Ref}\).

\(\text{Rf}(x,v)\): \(x\) is the Gödel number of a proof in \(\text{PA}\) of the negation of the formula with Gödel number \(v\):

\[\text{Prf}(x, z) \land z = \text{Neg}(v).\]

In other terms \(\text{Rf}(x,v)\) states \(x\) is the Gödel number of a refutation in \(\text{PA}\) of the formula with Gödel number \(v\).

\(\text{Rf}\) is \textit{primitive recursive}, as the relations obtained from primitive recursive relations by means of propositional connectives are also primitive recursive (5 137). For its recursiveness \(\text{Rf}(x,v)\) is \textit{expressible} in \(\text{PA}\) by a formula \(Rf(x,v)\).

\(\text{Ref}(x)\): \(x\) is the Gödel number of a refutation in \(\text{PA}\):

\[\text{Prf}(v) \land v = \text{Neg}(x)\]

In other terms \(\text{Ref}(x)\) states \(x\) is the Gödel number of a proof in \(\text{PA}\) of its negation. \(\text{Ref}(x)\) is \textit{primitive recursive}, as the relations obtained from primitive recursive relations by means of propositional connectives are also primitive recursive. For its recursiveness \(\text{Ref}(x)\) is \textit{expressible} in \(\text{PA}\) by a formula \(\text{Ref}(x)\).

**Lemma 1.** For any natural number \(n\) and for any formula \(\alpha\) not both \(\text{Rf}(n, \neg \alpha)\) and \(\text{Prf}(n, \neg \alpha)\).

**Proof.** Let us suppose to have both \(\text{Rf}(n, \neg \alpha)\) and \(\text{Prf}(n, \neg \alpha)\). We should have then \(\text{Prf}(n) \land \neg \alpha = (n)_{\text{lh}(n)} - 1\) and \(\text{Prf}(n, z) \land z = \text{Neg}(\neg \alpha)\), i.e. \(\text{Prf}(n) \land \neg \alpha = (n)_{\text{lh}(n)} - 1\) and \(\text{Prf}(n) \land \text{Neg}(\neg \alpha) = (n)_{\text{lh}(n)} - 1\).

By the definition of \(\text{Prf}(x)\) this would mean to have

\[\exists u < n \exists v < n \exists z < n \exists w < n \ (n = 2^w \land \text{Ax}(w)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_w) \land n = u \ast 2^v \land \text{Gen}(u,v)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_z) \land \text{Fml}(u_w) \land n = u \ast 2^v \land \text{MP}(u_z, (u)_w, v)) \lor \\
[\text{Prf}(u) \land n = u \ast 2^v \land \text{Ax}(v)] \land \text{Rf}\frac{\alpha}{\neg \alpha} = (n)_{\text{lh}(n)} - 1\]

and \(\text{Neg}(\neg \alpha) = (n)_{\text{lh}(n)} - 1\).

By the four cases

1. \(n = 2^\alpha \land \text{Ax}(\alpha)\) and \(n = 2^\text{deg}(\alpha) \land \text{Ax}(\neg \alpha)\)
2. \(\text{Prf}(u) \land \text{Fml}(u_w) \land n = u \ast 2^v \land \text{Gen}(u,w, \neg \alpha)\) and \(\text{Prf}(u) \land \text{Fml}(u_w) \land n = u \ast 2^v \land \text{Gen}(u,w, \text{Neg}(\neg \alpha))\)

One can easily see that \(\text{Ref}(x,v)\) is the same as

\[\exists u < x \exists v < x \exists z < x \exists w < x \ (x = 2^v \land \text{Ax}(y) \land y = \text{Neg}(v)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_w) \land x = u \ast 2^v \land \text{Gen}(u,w, y) \land y = \text{Neg}(v)) \lor \\
[\text{Prf}(u) \land \text{Fml}(u_z) \land \text{Fml}(u_w) \land x = u \ast 2^v \land \text{MP}(u_z, (u)_w, y) \land y = \text{Neg}(v)) \lor \\
[\text{Prf}(u) \land x = u \ast 2^v \land \text{Ax}(y) \land y = \text{Neg}(v)].\]
Lemma 2. For any formula $\alpha$, and $n$ as the Gödel number of a proof in $PA$ of $\alpha$

\[
\vdash_{PA} C_{PF}(\overline{n}, \overline{\alpha}) = \overline{0} \land C_{RF}(\overline{n}, \overline{\alpha}) = \overline{1}
\]

Proof. One can easily see that the two conjuncts are true: as $n$ is the Gödel number of a proof in $PA$ of $\alpha$ $C_{PF}(\overline{n}, \overline{\alpha}) = \overline{0}$ is true. By Lemma 1 $RF(n, \overline{\alpha})$ does not hold, therefore it is true that $n$ is not the Gödel number of a refutation in $PA$ of $\alpha$. \hfill $\square$

Lemma 3. For any formula $\alpha$, and $n$ as the Gödel number of a refutation in $PA$ of $\alpha$

\[
\vdash_{PA} C_{RF}(\overline{n}, \overline{\alpha}) = \overline{0} \land C_{PF}(\overline{n}, \overline{\alpha}) = \overline{1}
\]

Proof. One can easily see that the two conjuncts are true: as $n$ is the Gödel number of a refutation in $PA$ of $\alpha$ $C_{RF}(\overline{n}, \overline{\alpha}) = \overline{0}$ is true. By Lemma 1 $PF(n, \overline{\alpha})$ does not hold, therefore it is true that $n$ is not the Gödel number of a proof in $PA$ of $\alpha$. \hfill $\square$

Lemma 4. For any formula $\alpha$

(i) not both

\[
\vdash_{PA} PF(\overline{n}, \overline{\alpha}) \vdash_{PA} RF(\overline{n}, \overline{\alpha})
\]

(ii) for $n$ as the Gödel number of a refutation in $PA$ of $\alpha$

\[
\vdash_{PA} RF(\overline{n}, \overline{\alpha}) \iff \neg PF(\overline{n}, \overline{\alpha})
\]

(iii) for $n$ as the Gödel number of a proof in $PA$ of $\alpha$

\[
\vdash_{PA} PF(\overline{n}, \overline{\alpha}) \iff \neg RF(\overline{n}, \overline{\alpha})
\]
Proof. (i) Immediately by Lemma \[1\] and the definition of being expressible which holds for both $\Phi_f(x,v)$ and $\Phi_f(x,v)$ \[2\].

(ii) Let us assume $\vdash_{PA} \Phi_f(x,v)$, then Lemma \[3\] yields $\vdash_{PA} \Phi_f(x,v)$ is true. Hence by definition $\Phi_f(x,v)$ is false, consequently $\vdash_{PA} \neg \Phi_f(x,v)$. Conversely let us assume $\vdash_{PA} \neg \Phi_f(x,v)$ then $\Phi_f(x,v)$ is true and by Lemma \[3\] we attain $\vdash_{PA} \Phi_f(x,v)$.

(iii) Let us assume $\vdash_{PA} \Phi_f(x,v)$, then Lemma \[4\] yields $\vdash_{PA} C_{\Phi_f(x,v)} = T$. Hence by definition $\Phi_f(x,v)$ is false, consequently $\vdash_{PA} \neg \Phi_f(x,v)$. Conversely let us assume $\vdash_{PA} \neg \Phi_f(x,v)$ then $\Phi_f(x,v)$ is false and by Lemma \[4\] we attain $\vdash_{PA} \Phi_f(x,v)$.

All preceding lemmas were carried out constructively, needlessly to assume consistency. We are now able to consider the consequences yielded by such lemmas to the Gödel’s argumentation.

(a') Assume $\vdash_{PA} \alpha$. Let $r$ be the Gödel number of a proof in $PA$ of $\alpha$. Then $\Phi_f(r,q)$. Hence, $\vdash_{PA} \Phi_f(x,v)$, that is $\vdash_{PA} \Phi_f(x,v)$. Hence by Lemma \[1\] (i) $\vdash_{PA} \Phi_f(x,v)$ is not admitted, which means that $r$ cannot be the Gödel number of a refutation of $\alpha$ (indeed Lemma \[2\] yields $\vdash_{PA} C_{\Phi_f(x,v)}$ = $T$). Even though we can have $\vdash_{PA} \neg \Phi_f(x,v)$, by (iii) of Lemma \[4\], then we shall have not $\vdash_{PA} \neg \Phi_f(x,v)$ by Lemma \[2\] ($\vdash_{PA} C_{\Phi_f(x,v)}$ = $T$).

(b') Assume $\vdash_{PA} \neg \alpha$. Let $r$ be the Gödel number of a proof in $PA$ of $\neg \alpha$. Then $\Phi_f(r,q)$. Hence $\vdash_{PA} \Phi_f(x,v)$ that is $\vdash_{PA} \Phi_f(x,v)$. Hence by Lemma \[1\] (i) $\vdash_{PA} \Phi_f(x,v)$ is not admitted. This means that $r$ cannot be the Gödel number of a proof of $\alpha$ (in fact, $r$ is the Gödel number of a refutation of $\alpha$, Lemma \[4\] yields $\vdash_{PA} C_{\Phi_f(x,v)}$ = $T$). Even though we can have $\vdash_{PA} \neg \Phi_f(x,v)$, by (ii) of Lemma \[4\] as well, then we shall have not $\vdash_{PA} \neg \Phi_f(x,v)$ by Lemma \[3\] ($\vdash_{PA} C_{\Phi_f(x,v)}$ = $T$).

We have thus shown that previous Lemmas prevent any accomplishment of (a) and (b) within Gödel’s argumentation \[3\]. We have then established the following theorem.

Theorem 5. By the arithmetization of the refutability predicate Gödel’s incompleteness does not hold as a theorem of $PA$.

2. CONSISTENCY, COMPLETENESS AND DECIDABILITY

A recursive predicate defines a decidable set, by reason that its characteristic function is considered to be effectively computable \[5\] 165, 249).

Let us call $T_{PA}$ the set of Gödel numbers of theorems of $PA$ and $R_{PA}$ the set of Gödel numbers of refutations of $PA$.

By the recursiveness of $\Phi_f(x,v)$, $C_{\Phi_f(x,v)} = 0$ if $v \in T_{PA}$ and $C_{\Phi_f(x,v)} = 1$ if $v \notin T_{PA}$. By the recursiveness of $\Phi_f(x,v)$, $C_{\Phi_f(x,v)} = 0$ if $v \in R_{PA}$ and $C_{\Phi_f(x,v)} = 1$ if $v \notin R_{PA}$.

We can than state the following theorem.

Theorem 6. $T_{PA}$ and $R_{PA}$ are decidable sets.

\[3\] As regard to (b), we notice that in (b'), by Lemma \[1\], $\vdash_{PA} \Phi_f(x,v)$ is not admitted for every natural number $r$ such that $\Phi_f(r,q)$ (i.e. whenever $\vdash_{PA} \neg \alpha$).
It is furthermore well-known that if we have a computable function \( f(x_1, \ldots, x_n) \) such that
\[
\begin{cases}
0 & \text{if } x_1, \ldots, x_n \in S \\
1 & \text{if } x_1, \ldots, x_n \notin S
\end{cases}
\]
(where \( S \) is a set of natural number which turns out to be decidable just by this definition), then the function \( g(x_1, \ldots, x_n) \) defined by
\[
g(x_1, \ldots, x_n) = \begin{cases}
1 & \text{if } f(x_1, \ldots, x_n) = 0 \\
0 & \text{if } f(x_1, \ldots, x_n) = 1
\end{cases}
\]
is effectively computable too. Accordingly the complement of \( S \) is decidable. One can easily see that for \( f(x_1, \ldots, x_n) \) primitive recursive, \( g(x_1, \ldots, x_n) \) is primitive recursive too. Consequently we have
\[
C_{\neg \text{prf}}(x) = \begin{cases}
1 & \text{if } C_{\text{prf}}(x) = 0 \\
0 & \text{if } C_{\text{prf}}(x) = 1
\end{cases}
\]
\[
C_{\neg \text{ref}}(x) = \begin{cases}
1 & \text{if } C_{\text{ref}}(x) = 0 \\
0 & \text{if } C_{\text{ref}}(x) = 1
\end{cases}
\]
\[
C_{\neg \text{prf}}(x, v) = \begin{cases}
1 & \text{if } C_{\text{prf}}(x, v) = 0 \\
0 & \text{if } C_{\text{prf}}(x, v) = 1
\end{cases}
\]
\[
C_{\neg \text{ref}}(x, v) = \begin{cases}
1 & \text{if } C_{\text{ref}}(x, v) = 0 \\
0 & \text{if } C_{\text{ref}}(x, v) = 1
\end{cases}
\]
where \( \neg \text{prf}, \neg \text{pf}, \neg \text{ref} \) and \( \neg \text{rf} \) are respectively complementary of \( \text{prf}, \text{pf}, \text{ref} \) and \( \text{rf} \).

Let us summarize, \( \text{prf}, \text{pf}, \text{ref} \) and \( \text{rf} \) are primitive recursive, then \( C_{\text{prf}}, C_{\text{pf}}, C_{\text{ref}} \) and \( C_{\text{rf}} \) are primitive recursive too. But \( C_{\neg \text{prf}}(x) = 1 - C_{\text{prf}}(x), C_{\neg \text{pf}}(x, v) = 1 - C_{\text{pf}}(x, v), C_{\neg \text{ref}}(x) = 1 - C_{\text{ref}}(x), \) and \( C_{\neg \text{rf}}(x, v) = 1 - C_{\text{rf}}(x, v), \) hence \( \neg \text{prf}, \neg \text{pf}, \neg \text{ref} \) and \( \neg \text{rf} \) are primitive recursive too.

We have then the following statements.

**Lemma 7.** For every \( x \)

\( \text{prf}(x) \) if and only if \( \neg \text{ref}(x) \).

*Proof.* Let us assume \( \text{prf}(x) \). \( C_{\text{prf}}(x) = 0 \). Hence \( C_{\text{prf}}(\text{neg}(x)) = 1 \), by the effective computability of \( C_{\text{prf}} \). \( \text{prf}(\text{neg}(x)) \) is false, then \( \text{ref}(x) \) is false. Accordingly, \( C_{\text{ref}}(x) = 1 \). Thus \( C_{\neg \text{ref}}(x) = 0 \) and \( \neg \text{ref}(x) \).

Conversely, let us assume \( \neg \text{ref}(x) \). Then \( C_{\neg \text{ref}}(x) = 0 \) and \( C_{\text{ref}}(x) = 1 \). If \( \text{ref}(x) \) is false by its definition \( \text{prf}(\text{neg}(x)) \) is false. Thus \( C_{\text{prf}}(\text{neg}(x)) = 1 \) and \( C_{\neg \text{prf}}(\text{neg}(x)) = 0 \). Consequently \( C_{\neg \text{prf}}(x) = 1 \), and \( C_{\text{prf}}(x) = 0 \). Hence \( \text{prf}(x) \).

\[ \square \]

If we convert to formalize “a proof in PA of \( \theta \) with \( \theta_1 \ldots \theta_r \vdash_{PA} \theta \) then we have \( \vdash_{PA} (\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \theta) \ldots)) \)” (Herbrand, 1930). Indeed Lemma 7 could be read as follows: for \( \theta_1, \ldots, \theta_r \theta \) formulas in PA \( \text{prf}((\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \theta) \ldots)) \) if and only if \( \neg \text{ref}((\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \theta) \ldots)) \).

Furthermore, by the recursiveness of \( \text{prf}(x) \), \( C_{\text{prf}}(x) = 0 \) if \( x \in T_{PA} \) and \( C_{\text{pf}}(x) = 1 \) if \( x \notin T_{PA} \). By the recursiveness of \( \text{ref}(x) \), \( C_{\text{ref}}(x) = 0 \) if \( x \in R_{PA} \) and \( C_{\text{ref}}(x) = 1 \) if \( x \notin R_{PA} \).
Lemma 8. For every \(<x, v>\)
\[
\text{Pf}(x, v) \text{ if and only if } \neg \text{Rf}(x, v).
\]

Proof. Let us assume \(\text{Pf}(x, v)\). \(C_{\text{Pf}}(x, v) = 0\). Hence \(C_{\text{Pf}}(x, \text{Neg}(v)) = 1\). Accordingly \(C_{\neg \text{Rf}}(x, v) = 0\) and \(\neg \text{Rf}(x, v)\). Conversely, let us assume \(\neg \text{Rf}(x, v)\). We have then \(\neg \text{Pf}(x, \text{Neg}(v))\) and \(C_{\neg \text{Pf}}(x, \text{Neg}(v)) = 0\). Therefore \(C_{\neg \text{Pf}}(x, v) = 1\), by the effective computability of \(C_{\neg \text{Pf}}\). Accordingly \(C_{\text{Pf}}(x, v) = 0\) and \(\text{Pf}(x, v)\).

Indeed Lemma 8 could be read as follows: for \(\theta_1, \ldots, \theta_r, \alpha\) formulas in \(\text{PA}\)
\[\text{Pf}(\neg(\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \alpha) \ldots)), \neg\alpha)\] if and only if \(\neg \text{Rf}(\neg(\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \alpha) \ldots)), \neg\alpha)\).

Lemma 9. For \(m = \neg \alpha\) and \(n = \neg(\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \alpha) \ldots))\)
\[
\begin{align*}
\text{(i)} & \quad m \in T_{\text{PA}} \text{ iff } m \notin R_{\text{PA}}, \\
\text{(ii)} & \quad n \in T_{\text{PA}} \text{ iff } n \notin R_{\text{PA}}.
\end{align*}
\]

Proof. Immediately (i) by Lemma 8, (ii) by Lemma 7.

Theorem 10. \(\text{PA}\) is consistent; that is, there is no formula \(\alpha\) such that both \(\alpha\) and \(\neg \alpha\) are theorems of \(\text{PA}\).

Proof. Let us assume \(n\) to be the Gödel number of a proof of a formula \(\alpha\) of \(\text{PA}\) and \(m\) to be the Gödel number of a proof of \(\neg \alpha\). Then \(n, m \in T_{\text{PA}}\). But since \(n\) is the Gödel number of a proof of \(\neg \alpha\) we have also \(n \in R_{\text{PA}}\), accordingly \(n\) belongs to both \(T_{\text{PA}}\) and \(R_{\text{PA}}\), which is impossible by Lemma 7.

Theorem 11. \(\text{PA}\) is complete; that is for any well formed formula \(\alpha\) of \(\text{PA}\) either \(\vdash_{\text{PA}} \alpha\) or \(\vdash_{\text{PA}} \neg \alpha\).

Proof. Let \(\alpha\) be a well formed formula of \(\text{PA}\), we can then yield by Gödel numbering \(m = \neg \alpha\). By Lemma 6 either \(m \in T_{\text{PA}}\) or \(m \in R_{\text{PA}}\). Therefore either \(\vdash_{\text{PA}} \alpha\) or \(\vdash_{\text{PA}} \neg \alpha\).

Let us call \(\text{PF}\) the full first-order predicate calculus ([5] 172). Let \(T_{\text{PF}}\) be then the set of Gödel number of theorems of \(\text{PF}\).

Theorem 12. \(T_{\text{PF}}\) is decidable.

Proof. By Gödel Completeness Theorem, a formula \(\alpha\) of \(\text{PA}\) is provable in \(\text{PA}\) if and only if \(\alpha\) is logically valid, and \(\alpha\) is provable in \(\text{PF}\) if and only if \(\alpha\) is logically valid. Hence \(\vdash_{\text{PA}} \alpha\) if and only if \(\vdash_{\text{PF}} \alpha\). Accordingly, for \(n\) as the Gödel number of a proof of \(\alpha\) in \(\text{PA}\),
\[
n \in T_{\text{PA}} \text{ iff } n \in T_{\text{PF}}.
\]

Hence, by theorem 4, \(T_{\text{PF}}\) is decidable.

Calling our attention to the diagonalization lemma we note that it holds for any formula \(\phi\) with only the variable \(v\) free. In other terms \(\phi\) can be replaced by any formula with only one free variable. Let us suppose now that a sentence \(\delta\) is a theorem of \(\text{PA}\), i.e. \(\vdash_{\text{PA}} \delta\). For \(n\) as the Gödel number of a proof in \(\text{PA}\) of \(\delta\) we have \(\vdash_{\text{PA}} Pf(\Pi, \neg \delta)\). But for \(\phi(v)\) as \(\forall x Rf(x, v)\) diagonalization lemma could have already yielded \(\vdash_{\text{PA}} \delta \iff \forall x Rf(x, \neg \delta)\) then by biconditional elimination
we have $\vdash_{PA} \forall x Rf(x, \downarrow \delta \uparrow)$. Hence, by Particularization Rule, $\vdash_{PA} Rf(\bar{n}, \downarrow \delta \uparrow)$, and therefore

$$\vdash_{PA} Pf(\bar{n}, \downarrow \delta \uparrow) \land Rf(\bar{n}, \downarrow \delta \uparrow),$$

which is false by reason of Gödel numbering itself, as proved by Lemma (1) which holds for each natural number $n$. By the tautology $(A \land B) \Rightarrow (A \iff B)$ we should then have $\vdash_{PA} Pf(\bar{n}, \downarrow \delta \uparrow) \iff Rf(\bar{n}, \downarrow \delta \uparrow)$, which openly conflicts with (iii) in Lemma (4). Finally, by lemma (5) it is always the case that for whatever formula $\delta$ and $n = \gamma(\theta_1 \Rightarrow (\theta_2 \Rightarrow \ldots (\theta_r \Rightarrow \delta) \ldots ) \uparrow$

$$\vdash_{PA} Pf(\bar{n}, \downarrow \delta \uparrow) \iff \neg Rf(\bar{n}, \downarrow \delta \uparrow).$$

We have thus established that the applicability of the diagonalization lemma to any formula $\phi$ with only the variable $v$ free leads to the assertion of a contradiction as a theorem of $PA$ and for this reason $PA$ turns out to be inconsistent. Consequently diagonalization can no longer be considered to hold as an equivalence nor replacement theorem. We shall have accordingly the following theorem.

**Theorem 13.** Diagonalization does not hold as a lemma in $PA$.

\[ \square \]

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