Gravity in a warped 6D world with an extra 2D sphere

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Corrections to Newton’s inverse law have been so far considered, but not clear in warped higher dimensional worlds, because of complexity of the Einstein equation. Here we give a model of a warped 6D world with an extra 2D sphere. We take a general energy-momentum tensor, which does not depend on a special choice of bulk matter fields. The 6D Einstein equation reduces to the spheroidal differential equation, which can be easily solved. The gravitational potential in our 4D universe is calculated to be composed of infinite series of massive Yukawa potentials coming from the KK mode, together with Newton’s inverse law. The series of Yukawa type potentials converges well to behave as $1/r^3$ near $r = 0$.

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I. INTRODUCTION

There is a long history in the search of corrections to Newton’s inverse law in gravity. Especially, gravity in higher dimensional worlds has been drawn many interests, because the gravitational potential is expected to reflect sharply on number of dimensions. This kind of work, however, turns out to be extremely hard, because of complexity of the Einstein equation in higher dimensions. So, this problem of corrections to Newton’s inverse law has been still not clear, especially in warped higher dimensional worlds.

In this paper we concentrate on gravity in the 6D world with an extra 2D sphere. Here, the extra 2D sphere preserves spherically symmetric, whereas the 4D world carries metrics with a warp factor $\phi(\theta)$. The compact 2D extra space is expected to induce the gravitational potentials of Yukawa type coming from KK modes, which are corrections to Newton’s inverse law. The Yukawa potential appears in several theoretical models, such as unification theories that predict new fundamental interactions with a massive gauge boson, a massive Brans-Dicke scalar, a light dilaton and so on [1]. Randall-Sundrum summed up the series of Yukawa potentials in the warped 5D world with branes [2], Salvio also has discussed the same potential in the 6D supergravity [3, 4].

We now proceed with the background line element, which is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + a^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $g_{\mu\nu} = \delta(\theta)\eta_{\mu\nu}$, $x^\mu$ are coordinates for our 4D spacetime, while $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$ are coordinates for the extra 2D spherical surface with a constant radius $a$.

One of the most characteristic things is that the warp factor can be fixed almost completely from the positive energy condition of the 6D energy-momentum tensor (EMT) of bulk matter fields. One of possible results is given by

$$\phi(\theta) = \epsilon \exp (a \sin^2 \theta),$$

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where \( 4\alpha \leq 1 \) and \( \Lambda a^2 \leq -2 \). Here \( \epsilon \) is an arbitrary constant and \( \Lambda \) the 6D cosmological constant. In this approach we take a general EMT, which does not depend on a special choice of bulk matter fields.

We can easily see that the energy of any particle running along the geodesic line above is given by

\[
E = \left( \frac{d\theta}{d\tau} \right)^2 - \frac{A^2}{\phi^2(\theta)},
\]

where \( A \) is a constant. The potential has a maximum value \( -A^2/(\epsilon e^\alpha)^2 \) at \( \theta = \pi/2 \) and the lowest value \( -A^2/\epsilon^2 \) at \( \theta = 0, \pi \). As for massless particles we can see that they can extend into the 6D world.

For the most general perturbation around the background \( g_{IJ} \) we calculate the gravitational potential in our 4-dimensional universe. In this formulation we would like to point out that the traceless energy-momentum tensor is quite naturally appeared in our 6-dimensional model. Furthermore, we should stress that the complicated Einstein equations reduce to the simple differential equations for the spheroidal functions, which can be easily solved.

The gravitational potential in our 4-dimensional universe is calculated to be composed of infinite series of massive Yukawa potentials coming from the KK mode, together with Newtonian’s inverse law. The series of Yukawa type potentials can be summed up for the large \( n \), say \( n > n_0 \) to give \( \epsilon^2/(a r^3) \) near \( r = 0 \).

In Sec. II we give the warp factor and discuss gravitational wave equations in the 6D world. In Sec. III eigenfunctions of differential operator of the Einstein equation are obtained. In Sec. IV the gravitational potential is calculated by means of the Green functional method. The final section is devoted to concluding remarks. We prepare the Appendix A for fixing the warp factor from the requirement of the positive energy condition for the 6D EMT.

### II. GRAVITATIONAL WAVE EQUATION

The Einstein equation with the 6D metric \( g_{IJ} \) is given by

\[
R_{IJ} - \frac{1}{2}g_{IJ}R + \Lambda g_{IJ} = \kappa T_{IJ},
\]

where \( \kappa = 8\pi G_6 \), \( R_{IJ} = R_{IKJ}^\kappa \), and we follow notations in Wald’s book [5]. This equation can be rewritten as

\[
R_{IJ} - \frac{1}{2}g_{IJ}\Lambda = \kappa \left( T_{IJ} - \frac{1}{4}g_{IJ}T \right),
\]

with \( T = g^{IJ}T_{IJ} \). The factor \( 1/4 \) is characteristic in the 6D model.

We look for background solutions of Eq. (2.2) for the warp factor \( \phi(\theta) \) with the ansatz for EMT of bulk matter fields [6–9]

\[
T_{\mu\nu} = -g_{\mu\nu}f_1(\theta),
T_{55} = -g_{55}f_2(\theta),
T_{66} = -g_{66}f_3(\theta).
\]

All other elements vanish. Relevant quantities with the metric (1.2) are inserted into the \((\mu\nu)\) component of Eq. (2.2) to give

\[
3\frac{\phi''}{\phi} + \cot\theta \frac{\phi''}{\phi} + \frac{\Lambda a^2}{2} = -\frac{1}{4}\kappa a^2 (f_2 + f_3).
\]
In the same way, we get
\[ 4 \phi'' - 1 + \frac{1}{2} \Lambda a^2 = -\frac{1}{4} \kappa a^2 (4f_1 - 3f_2 + f_3), \]  
(2.5)
\[ 4 \cot \theta \phi' - 1 + \frac{1}{2} \Lambda a^2 = -\frac{1}{4} \kappa a^2 (4f_1 + f_2 - 3f_3), \]  
(2.6)
for the (55) and (66) component respectively. From Eqs. (2.4)-(2.6) we obtain

\[ f_1 = -\frac{1}{\kappa a^2} \left[ 3 \phi'' + \phi \phi'' + \cot \theta \phi' \right] - 1 + \Lambda a^2, \]  
(2.7)
\[ f_2 = -\frac{1}{\kappa a^2} \left[ 6 \phi'' + 4 \cot \theta \phi' \right] + \Lambda a^2, \]
\[ f_3 = -\frac{1}{\kappa a^2} \left[ 6 \phi'' + 4 \phi \phi'' \right] + \Lambda a^2. \]

The left-hand side quantities \( f_i \)'s correspond to EMT \( T_{iJ} \)'s, whereas the right-hand sides are those of Ricci tensors \( R_{iJ} \)'s in Eq. (2.1). Note that the Ricci tensor \( R_{iJ} \) is always divergence free, i.e., \( \Delta R_{iJ} = 0 \) because of the Bianchi identity, where \( \Delta \) is the covariant derivative. Hence also so is the EMT, i.e., \( \Delta T_{iJ} = 0 \) for any function \( \phi \).

Now we would like to fix the functional form of \( \phi \). We use the positive energy condition of the 6D EMT, in a region \( 0 \leq \theta \leq \pi \), that is,
\[ u_i T^{IJ} u_J = f_1 + u_5 u_5 (f_1 - f_2) + u_6 u_6 (f_1 - f_3) \geq 0, \]  
(2.8)
where \( u_I \) is a unit time-like vector in 6-dimensions. We assume that the warp factor is given by \( \phi = \epsilon \exp (\alpha \sin^2 \theta) \), where \( \epsilon \) and \( \alpha \) are arbitrary parameters. In order to fix them, \( \phi \) is substituted into inequalities, \( f_1 \geq 0, f_1 - f_2 \geq 0, \) and \( f_1 - f_3 \geq 0 \). Then we have
\[ \phi(\theta) = \epsilon e^{(\alpha \sin^2 \theta)}, \]  
(2.9)
with \( 4\alpha \leq 1 \) and \( \Lambda a^2 \leq -2 \). [see Appendix A]

Let us now consider the most general perturbation \( g_{IJ}^{(1)} = h_{IJ} \) around the background metric \( g_{IJ}^{(0)} \). We follow the technique previously obtained [10]. The line element is
\[ ds^2 = g_{IJ} dx^I dx^J = (g_{IJ}^{(0)} + g_{IJ}^{(1)}) dx^I dx^J, \]  
(2.10)
where
\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} = \phi^2(\theta) \eta_{\mu\nu} + h_{\mu\nu}, \quad g_{\mu5} = g_{\mu5}^{(1)} = h_{\mu5}, \quad g_{\mu6} = g_{\mu6}^{(1)} = h_{\mu6}, \]
\[ g_{55} = g_{55}^{(0)} + g_{55}^{(1)} = a^2 + h_{55}, \quad g_{66} = g_{66}^{(0)} + g_{66}^{(1)} = a^2 \sin^2 \theta + h_{66}, \quad g_{56} = g_{56}^{(1)} = h_{56}. \]

We put here the gauge conditions
\[ \partial_K h^K = 0, \quad \partial_K h_K^5 = 0, \quad h_{56} = 0. \]  
(2.12)

We now expand \( R_{iJ}, T_{iJ} \) in order \( h_{IJ} \) as
\[ R_{iJ} = R_{iJ}^{(0)} + R_{iJ}^{(1)}, \quad T_{iJ} = T_{iJ}^{(0)} + T_{iJ}^{(1)}. \]  
(2.13)
where

\[ T^{(0)\mu}_\nu = -g^{(0)\mu}_\nu f_1(\theta) = -\phi^2(\theta)\eta_{\mu\nu} f_1(\theta) , \quad (2.14) \]

\[ T^{(0)}_{55} = -g^{(0)}_{55} f_2(\theta) = -a^2 f_2(\theta) , \quad (2.15) \]

\[ T^{(0)}_{66} = -g^{(0)}_{66} f_3(\theta) = -a^2 \sin^2 \theta f_3(\theta) . \]

We shall restrict our attention to the most interesting case of a static particle with mass \( m_0 \) located at \( \vec{x} = \vec{0}, \theta = 0 \). In this case EMT is given by

\[ T^{(1)}_{\mu\nu} = \tau_{\mu\nu}(x)\delta(\theta) = m_0\delta^{(0)}_{\mu\nu}\delta^{(3)}(\vec{x})\delta(\theta) , \quad (2.15) \]

\[ T^{(1)}_{IJ} = 0 , (I, J \neq 0) \]

\( \tau_{\mu\nu} \) being in order \( h_{\mu\nu} \).

The first-order equation in Eq. \( 2.22 \) is given by

\[ R^{(1)}_{IJ} = \frac{1}{2} g_{IJ}^{(1)} \Lambda + \frac{1}{4} \kappa g_{IJ}^{(1)} T^{(0)} + \frac{1}{4} \kappa g_{IJ}^{(0)} T^{(1)} = \kappa \left[ T^{(1)}_{IJ} - \frac{1}{4} g_{IJ}^{(0)} T^{(1)} \right] \equiv \kappa \Sigma_{IJ} . \quad (2.16) \]

where \( T = g^{IJ} T_{IJ} = T^{(0)} + T^{(1)} + \tilde{T}^{(1)} \) with

\[ T^{(0)} = g^{(0)IJ} T^{(0)}_{IJ} = -4f_1(\theta) - f_2(\theta) - f_3(\theta) , \]

\[ T^{(1)} = g^{(0)IJ} T^{(1)}_{IJ} = \tau_{\chi}(x)\delta(\theta) , \]

\[ \tilde{T}^{(1)} = g^{(1)IJ} T^{(1)}_{IJ} = -h^\lambda_\chi f_1(\theta) - h^5_\chi f_2(\theta) - h^6_\chi f_3(\theta) . \]

The source terms \( \Sigma_{IJ} \) are explicitly given by

\[ \Sigma_{\mu\nu}(x, \theta) = \left[ \tau_{\mu\nu}(x) - \frac{1}{4} g^{(0)}_{\mu\nu} \tau(x) \right] \delta(\theta) = \left( \delta^{(0)}_{\mu \lambda} \delta^{(0)}_{\nu \sigma} + \frac{1}{4} \eta_{\mu\nu} \right) m_0 \delta^{(3)}(\vec{x}) \delta(\theta) , \quad (2.17) \]

\[ \Sigma_{55}(x, \theta) = -\frac{1}{4} a^2 \tau(x) \delta(\theta) = \frac{1}{4} \left( \frac{a}{\epsilon} \right)^2 m_0 \delta^{(3)}(\vec{x}) \delta(\theta) , \]

\[ \Sigma_{36}(x, \theta) = -\frac{1}{4} a^2 \sin^2 \theta \tau(x) \delta(\theta) = 0 , \]

where \( \tau(x) \equiv \tau_{\chi}(x) \). We solve Eq. \( 2.16 \) under such approximation that the source term \( \Sigma_{55} \) is negligible compared with \( \Sigma_{\mu\nu} \), that is, \( |\Sigma_{55}| \ll |\Sigma_{\mu\nu}| \). This will be realized by the assumption \( (a/\epsilon) \ll 1 \). We can then put as \( \Sigma_{55} \approx 0 \).

In a previous work \[10\] we found special solutions of Eq. \( 2.16 \):

\[ h^\lambda_\chi = h^5_\chi = h^6_\chi = 0 , \quad (2.18) \]

\[ \partial^K h^K_6 = 0 , \quad (2.19) \]

\[ h_{\mu\mu} = h_{5\mu} = 0 . \quad (2.20) \]

Thanks to these results \( 2.18, 2.19 \) and \( 2.20 \) together with gauge conditions \( 2.12 \), the \( (\mu\nu) \) components of Eq. \( 2.16 \) reduces to

\[ -\frac{1}{2\phi^2} \Box h_{\mu\nu} - \frac{1}{2a^2 \sin^2 \theta} \partial^2 \phi h_{\mu\nu} \]

\[ -\frac{1}{2a^2} \left( \partial^2 \phi h_{\mu\nu} + \frac{\cos \theta}{\sin \theta} \partial_{\theta} h_{\mu\nu} - 8 \left( \frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} + \frac{\cos \theta \phi'}{\sin \theta} \phi' \right) h_{\mu\nu} + (2 - 2\Lambda a^2) h_{\mu\nu} \right) \]

\[ = \kappa [\tau_{\mu\nu}(x) - \frac{1}{4} g^{(0)}_{\mu\nu} \tau(x)] \delta(\theta) , \quad (2.21) \]
where $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$.

Eq. (2.21) can be solved by means of the Green function, which is given by eigenfunctions of the differential operator $L$ in the left-hand side. Separating the four-dimensional mass term by

$$h_{\mu\nu}(x, \theta, \varphi) = h_{\mu\nu}(0) \exp(ik_{\mu} x^{\mu}) f(\theta) k(\varphi), \quad k_{\mu} k^{\mu} = -M^2,$$

with $h_{\lambda}(0) = 0$, and substituting $\phi = \epsilon \exp(\alpha \sin^2 \theta)$ into Eq. (2.21), we get, for $0 \leq \theta \leq \pi$,

$$-\frac{1}{2a^2} \left[ \partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + Q(\theta) + \frac{1}{\sin^2 \theta} \partial^2_{\varphi} \right] h_{\mu\nu} \equiv Lh_{\mu\nu} = 0.$$

where

$$Q(\theta) = \lambda - c^2 z^2, \quad z = \cos \theta, \quad x = 1 - \Lambda a^2,$$

$$\lambda = 16\alpha + 2x + \frac{M^2 a^2}{c^2} (1 - 2\alpha),$$

$$c^2 = 2\alpha (24 - \frac{M^2 a^2}{c^2}),$$

for a small $\alpha$. Here we have used an approximation that $\phi = \epsilon \exp(\alpha \sin^2 \theta) \simeq \epsilon (1 + \alpha \sin^2 \theta)$. Eq. (2.23) becomes a simply separable equation. Hence we get $(\partial_\theta^2 + m^2) k(\varphi) = 0$, where $m$ should take integral values, because $k(\varphi)$ is a $2\pi$-periodic function. Putting the boundary condition $k(0) = 0$, we get $k(\phi) = A \sin(m\varphi)$. The equation for $f(\theta)$ is, therefore, given by

$$\left[ \partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + Q_1(\theta) \right] f(\theta) = 0.$$

where

$$Q_1(\theta) = \lambda - c^2 z^2 - \frac{m^2}{1 - z^2},$$

III. EIGENFUNCTIONS

The equation (2.24) is known as the spheroidal wave equation. The regular solution is given by in terms of associated Legendre functions apart from normalization factors

$$f_{mn}(c, z) = \sum_l d_{lmn} P_{m+l}^n(z),$$

with eigenvalues

$$\lambda_{mn} = \sum_{k=0}^{\infty} l_{2k} c^{2k},$$

where

$$l_0 = n(n + 1),$$

$$l_2 = \frac{1}{2} \frac{1}{1 - \frac{(2m - 1)(2m + 1)}{(2n - 1)(2n + 3)}}.$$
So, in the following we set $m=0$ in Eqs. (3.1), (3.2) and (3.4).

From the equation

$$\lambda_n = l_0 + l_2 c^2 + l_4 c^4 + \cdots = 16\alpha + 2x + \frac{M^2 a^2}{c^2} (1 - 2\alpha)$$  (3.5)

we have a mass formula

$$\frac{M^2 a^2}{c^2} \simeq \frac{l_0 - 2x + 16\alpha(3l_2 - 1)}{1 - 2\alpha(1 - l_2)}. \quad (3.6)$$

The 0-mass is obtained when the denominator is zero. The approximate formula for $\alpha \simeq 0$ is

$$\frac{M^2 a^2}{c^2} \simeq l_0 - 2x = n(n + 1) - 6. \quad (3.7)$$

These masses correspond to KK masses.

Normalization of Eq. (3.1) with $m = 0$ is given by [11]

$$\int_{-1}^{1} dz f_n^2(c, z) = \frac{2}{2n + 1}. \quad (3.8)$$

Hence the normalization factor is given by $N = \sqrt{(2n + 1)/2}$.

To sum up, the mass-eigenfunctions of Eq. (3.1) are defined by

$$h_n(z) = \sqrt{\frac{2n + 1}{2}} f_n(c, z) = \sqrt{\frac{2n + 1}{2}} \sum_i d_{nl} P_l(z) \quad (3.9)$$

Finite mass eigenvalues are approximately given by

$$M_n = \frac{\epsilon \sqrt{n(n + 1) - 6}}{a} \quad (3.10)$$

From the formula $\sum_l d_{nl} = 1$ [11], we see $h_n(c, z = 1) = \sqrt{(2n + 1)/2}$.

**IV. GREEN FUNCTION**

The Einstein equation (2.23) for $h_{\mu\nu}$, neglecting $\varphi$-dependence, is given by

$$L h_{\mu\nu} = \kappa \Sigma_{\mu\nu}(x, \theta), \quad (4.1)$$

where

$$L = -\frac{1}{2\varphi^2(\theta)} \partial_\lambda \partial^\lambda - \frac{1}{2a^2} [\partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + Q(\theta)] \quad (4.2)$$

$$Q(\theta) = (\lambda - c^2 \cos^2 \theta)|_{M=0}, \quad (4.3)$$

$$\Sigma_{\mu\nu}(x, \theta) = (\tau_{\mu\nu}(x) - \frac{1}{4} g_{\mu\nu}^{(0)} \tau(x)) \delta(\theta). \quad (4.4)$$
Here $\Sigma_{\mu\nu}(x, \theta)$ is the traceless energy-momentum tensor, consistent with the traceless condition $h^\mu_\mu = 0$. We shall restrict our attention to the most interesting case of a static particle with mass $m_0$ located at $\vec{x} = \theta = 0$. In this case the energy-momentum tensor is given by $\tau_{00}(x) = m_0 \delta(\vec{x})$ and $\text{others} = 0$. Then we have $\Sigma_{00}(X) = (3/4)m_0 \delta(\vec{x}) \delta(\theta)$.

The solution $h_{\mu\nu}(X)$ can be derived by means of the Green function as

$$h_{\mu\nu}(x) = \int d^5x' \ G_R(X, X') \kappa \Sigma_{\mu\nu}(X'), \quad X = (x, \theta),$$

The Green function is defined by

$$LG_R(X, X') = \delta^5(X - X'), \quad (4.5)$$

where

$$G_R(X, X') = L^{-1} \sum_n h_n(X) h^\dagger_n(X')$$

$$= (-2) \sum_n \int \frac{d^4p}{(2\pi)^4} \frac{e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}}{\sigma(\vec{p}) \delta(p^2 - M_n^2)} h_n(\theta) h^\dagger_n(\theta').$$

Here $h_n(\theta)$ are mass eigenfunctions of the differential operator $L$.

Following Tanaka et al. [12] we define the stationary Green function by

$$G_R(\vec{x}, \theta; \vec{x}', \theta') = \int_{-\infty}^{\infty} dt' G_R(X, X')$$

$$= 2 \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}}{\sigma(\vec{p}^2) \delta(p^2 + M_n^2)} h_n(\theta) h^\dagger_n(\theta')$$

$$= 2\phi^2(\theta) \frac{1}{4\pi r} \left[ h_2(\theta) h^\dagger_2(\theta) + \sum_{n \geq 3} e^{-M_n r} h_n(\theta) h^\dagger_n(\theta') \right],$$

where $r = |\vec{x} - \vec{x}'|$. The gravitational potential between masses $m_0$ and $m_1$ separated by $r$ is given, by putting $\theta = \theta' = \vec{x} = \vec{x}' = 0$, as follows:

$$V(r) = -\frac{1}{2} m_1 h_{00}(r) = -\frac{3}{8} m_0 m_1 G_R(\vec{x}, 0; 0, 0)$$

$$= -6\pi G_N m_0 m_1 \phi^2(0) \frac{1}{4\pi r} \left[ h_2(0) h^\dagger_2(0) + \sum_{n \geq 3} e^{-M_n r} h_n(0) h^\dagger_n(0) \right],$$

According to Eqs. (3.7) and (3.8), we have

$$V(r) = -G_N \frac{m_0 m_1}{r} \left( 1 + \sum_{n=3}^{\infty} \alpha_n e^{-M_n r} \right),$$

with

$$G_N = \frac{15}{4} e^2 G_6,$$

$$\alpha_n = \frac{2n + 1}{5}, \quad \frac{M_n a}{\epsilon} \simeq \sqrt{n(n + 1) - 6} \quad (n \geq 3).$$
V. CONCLUDING REMARKS

In the warped 6D world with an extra 2D sphere we have succeeded to calculate the gravitational potential, which is given by Eq. (4.9) with (4.10). We have taken a general EMT, which does not depend on a special choice of bulk matter fields. We have fixed the warp factor to be \( \phi = \epsilon \exp(\alpha \sin^2 \theta) \) with \( 1/4 \geq \alpha > 0 \) and \( x = 1 - \Lambda a^2 \geq 3 \) from the positive energy condition of EMT. Actually we have made of an approximation that \( \alpha \) is so small enough \( 1 >> \alpha \). The 6D Einstein equation has reduced to the spheroidal differential equation, which can be easily solved.

The infinite series of Yukawa type potentials coming from KK-modes can be summed up for the large \( n \), say \( n > n_0 \). Namely, the gravitational potential reduces to

\[
V(r) = -G_N \frac{m_0 m_1}{r} \left( 1 + \sum_{n=3}^{n_0} \alpha_n e^{-M_n r} + f(r) \right),
\]

where

\[
f(r) = \frac{2}{5} x^{n_0} \left[ \frac{n_0}{1-x} + \frac{x}{(1-x)^2} \right],
\]

with \( x = e^{-\epsilon r/a} \). The function \( f(r) \) behaves as \( (\epsilon)^2/(a^2 r^3) \) near \( r = 0 \). Two latter correction terms to Newton’s \( 1/r \) law is remarkable in the warped 6D world with the small extra 2D sphere, to be experimentally checked.

The 4D Newton constant \( G_N \) is given by \( G_N = \epsilon^2 G_6 \). The smallness of \( G_N \), therefore, may reflect that of \( \epsilon \). If we choose \( \epsilon = 10^{-19} \) against \( G_6 = 1 \text{GeV}^{-2} \), then we get the present value of \( G_N = 10^{-38} \text{GeV}^{-2} \).

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Appendix A: The positive energy condition of EMT

The warp factor \( \phi(\theta) \) is expanded, around \( \theta = 0 \), into the Taylor series, \( \phi(\theta) = \phi_0 + \phi_1 \theta + \phi_2 \theta^2 + \phi_3 \theta^3 + \cdots \). This form is substituted into \( f_i(\theta) \) to yield, dropping the factor \( 1/(\kappa a^2) \).

\[
f_1 = \frac{3\phi_1 \phi_0 + \theta(12\phi_2 \phi_0 + 6\phi_1^2 + (\Lambda a^2 - 1)\phi_0^2)}{\theta \phi(\theta)^2} \geq 0, \quad (A1)
\]

\[
f_1 - f_2 = -\frac{\phi_1 \phi_0 + \theta(-4\phi_2 \phi_0 + 4\phi_1^2 + \phi_0^2)}{\theta \phi(\theta)^2} \geq 0, \quad (A2)
\]

\[
f_1 - f_3 = -\frac{3\phi_1 \phi_0 + \theta(-4\phi_2 \phi_0 + \phi_0^2)}{\theta \phi(\theta)^2} \geq 0, \quad (A3)
\]

In a limit \( \theta \to 0 \), we have consistent results \( \phi_1 = 0, \phi_0/4 \geq \phi_2 > 0, (1 - \Lambda a^2)\phi_0/12 \geq \phi_2 > 0 \). Hence we get \( \phi(\theta) = \phi_0 + \phi_2 \theta^2 + \cdots \), where \( \phi_0/4 \geq \phi_2 > 0, 1 - \Lambda a^2 = x \geq 3 \).

This form suggests us generally to put an ansatz for \( \phi(\theta) \)

\[
\phi(\theta) = \epsilon \exp(\alpha \sin^2 \theta), \quad (A4)
\]
where
\[
\frac{1}{4} \geq \alpha > 0, \quad \text{and} \quad x \geq 3,
\]  
(A5)

This form is substituted into inequalities, \( f_1 \geq 0, f_1 - f_2 \geq 0 \) and \( f_1 - f_3 \geq 0 \) to yield
\[
f_1 - f_3 = -16\alpha^2 s^4 + 2s^2\alpha(1 + 8\alpha) + 1 - 4\alpha.
\]  
(A6)

Here \( s = \sin \theta \). If \( 0 \leq 4\alpha \leq 1 \), this equation is always positive in a reason \( 0 \leq s^2 \leq 1 \). This can be seen by calculating roots of \( f_1 - f_3 = 0 \) for \( s^2 \).
\[
f_1 - f_2 = 10\alpha s^2 + 1 - 4\alpha,
\]  
(A7)

This is nonnegative when \( 0 \leq 4\alpha \leq 1 \). Finally, \( f_1 \) is given by
\[
f_1 = 24\alpha^2 s^4 + s^2\alpha(18 - 24\alpha) + x - 12\alpha,
\]  
(A8)

where \( x = 1 - \Lambda a^2 \). Eq. (A8) is positive if \( 4\alpha \leq 1 \) and \( x \geq 3 \).

In conclusion, the warp factor \( \phi = \epsilon \exp(\alpha \sin^2 \theta) \) makes EMT positive when \( 4\alpha \leq 1 \) and \( x \geq 3 \), (This means \( \Lambda a^2 \leq -2 \)).

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