On the Graver basis of block-structured integer programming

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Abstract

We consider the 4-block $n$-fold integer programming (IP), in which the constraint matrix consists of $n$ copies of small matrices $A$, $B$, $D$ and one copy of $C$ in a specific block structure. We prove that, the $\ell_\infty$-norm of the Graver basis elements of 4-block $n$-fold IP is upper bounded by $O_{FPT}(n^{s_c})$ where $s_c$ is the number of rows of matrix $C$ and $O_{FPT}$ hides a multiplicative factor that is only dependent on the parameters of the small matrices $A, B, C, D$ (i.e., the number of rows and columns, and the largest absolute value among the entries). This improves upon the existing upper bound of $O_{FPT}(n^{2s_c})$ [13]. We provide a matching lower bounded of $\Omega(n^{s_c})$, which even holds for an arbitrary non-zero integral element in the kernel space. We then consider a special case of 4-block $n$-fold in which $C$ is a zero matrix (called 3-block $n$-fold IP). We show that, surprisingly, 3-block $n$-fold IP admits a Hilbert basis whose $\ell_\infty$-norm is bounded by $O_{FPT}(1)$, despite the fact that the $\ell_\infty$-norm of its Graver basis elements is still $\Omega(n)$. Finally, we provide upper bounds on the $\ell_\infty$-norm of Graver basis elements for 3-block $n$-fold IP. Based on these upper bounds, we establish algorithms for 3-block $n$-fold IP and provide improved algorithms for 4-block $n$-fold IP.

Keywords: Integer Programming; Graver basis; Fixed parameter tractable

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1 Introduction

An integer program (IP) can be written as

$$\min \{ w \cdot x : Hx = b, l \leq x \leq u, x \in \mathbb{Z}^m \}$$

where all the numbers (i.e., the coordinates of $H, w, b, l, u$) are integers. We call $H$ as the constraint matrix. Integer programming is a strong mathematical tool for modeling various optimization problems, based on which many parameterized and approximation algorithms have been developed. In general, integer programming is NP-hard \cite{3}. Lenstra \cite{24} showed a polynomial time algorithm when the number of variables is fixed, which was improved later by Kannan \cite{20}. A somewhat complementary algorithm, which runs in pseudo-polynomial time when the number of constraints (the number of rows of $H$, excluding $l \leq x \leq u$ in IP (1)) is fixed was provided by Papadimitriou \cite{26}. Very recently, Eisenbrand and Weismantel \cite{9} gave an important improvement on the running time by utilizing Steinitz Lemma \cite{12}. Subsequent improvement and lower bounds were obtained by Jansen and Rohwedder \cite{19}.

Despite the research into IPs with fixed number of variables or constraints, there is also a strong interest in the research of IPs where the number of variables and constraints are part of the input, but with the constraint matrix $H$ having a specific structure. One of the most prominent examples is the class of IPs with $H$ being a totally unimodular matrix, which was further extended recently by Artmann et al. \cite{1}. Another important example is the so-called 4-block $n$-fold integer programming, which has received increasing attention in recent years \cite{13,8,23}. We focus on such block-structured integer programming in this paper.

1.1 Problem definition.

We define 4-block $n$-fold IP as follows. A constraint matrix $H$ is called a 4-block $n$-fold matrix, if it consists of small matrices $A, B, C$ and $D$ and can be written as follows:

$$H = \begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(n)} = \begin{pmatrix} C & D & D & \cdots & D \\ B & A & 0 & 0 \\ B & 0 & A & 0 \\ \vdots & \ddots & \ddots & \ddots \\ B & 0 & 0 & A \end{pmatrix}$$

Here $A, B, C, D$ are $s_i \times t_j$ matrices, $i = A, B, C, D$, respectively, and the big matrix $H$ consists of $n$ copies of $A, B, D$ and one copy of $C$. Notice that by plugging $A, B, C, D$ into the above blocked structure we require that $s_C = s_D, s_A = s_B, t_B = t_C$ and $t_A = t_D$. Let $\Delta$ be the largest absolute value among all the entries of $A, B, C, D$. Given $H$, we will be focusing on the following IP throughout this paper

$$(IP)_{n,b,l,u,w} : \min \{ w \cdot x : Hx = b, l \leq x \leq u, x \in \mathbb{Z}^{nt_B + nt_A} \}. \quad (1)$$

Removing $B$ and $C$ from $H$, the remaining matrix is called an $n$-fold matrix. Removing $C$ and $D$ from $H$, the remaining matrix is called a two-stage stochastic matrix. Throughout this paper, we denote by $E$ and $F$ these two matrices, i.e.,

$$E := \begin{pmatrix} D & D & \cdots & D \\ A & 0 & 0 \\ 0 & A & 0 \\ \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & A \end{pmatrix} \quad F := \begin{pmatrix} B & A & 0 & 0 \\ B & 0 & A & 0 \\ \vdots & \ddots & \ddots \\ B & 0 & 0 & A \end{pmatrix}$$
Replacing $H$ with $E$ or $F$ in IP (1), the resulted ILP is called $n$-fold IP or two-stage stochastic IP, respectively.

Specifically, if $C = 0$ in $H$, we denote the matrix by $H_0$. Replacing $H$ with $H_0$, we define the resulted IP (1) as a 3-block $n$-fold IP.

As is observed before (see, e.g., [14]), the constraint $Hx = b$ can be replaced with $Hx \leq b$ or $Hx \geq b$ (or even part of inequalities being “$\geq$” and part of them being “$\leq$”). Inequalities can be transformed into equalities by introducing $O(n)$ additional variables and modify $A$ and $D$ into $\tilde{A}$ and $\tilde{D}$ (the technique is the same as that in Section 2, Feasibility and Optimality). Therefore, IP (1) with the constraint $Hx \leq b$ ($Hx \geq b$) or $H_0x \leq b$ ($H_0x \geq b$) are equivalent to 4-block $n$-fold IP or 3-block $n$-fold IP, respectively, and we also call them 3-block or 4-block $n$-fold IP.

1.2 Motivation

4-block $n$-fold integer programming is an important research topic that has received increasing attention in recent years. Although a 4-block $n$-fold IP has a restricted structure, it is still general enough to be capable of modeling a variety of fundamental combinatorial optimization problems. For example, its special case, $n$-fold integer programming, can be used to model various problems in scheduling [21, 18], computational social choice and stringology [22]. The two-stage stochastic version of these combinatorial problems, as well as various other stochastic problems with second order dominance relations can be modeled using 4-block $n$-fold IP [10, 13].

From a theoretical point of view, it is crucial to understand to what extend an IP with a special structure can be solved efficiently. Hemmecke and Schultz [16] showed that two-stage stochastic IP can be solved in $f_{sto}(s_A, s_B, t_A, t_B, \Delta)n^3L$ time for some computable function $f_{sto}$ (where $L$ is the length of the input). In 2013, Hemmecke, Onn and Romanchuk [14] showed that $n$-fold IP can be solved in $f_{n.f}(s_A, s_D, t_A, t_D, \Delta)n^3L$ for some computable function $f_{n.f}$. Very recently, improved algorithms with a better running time have been developed for two-stage stochastic IP [23] and $n$-fold IP [8, 23]. Adopting the concept of fixed parameter tractability (FPT) (see, e.g., the book [7] as a nice introduction), we take $s_i, t_i$ ($i = A, B, C, D$) and $\Delta$ as parameters, and write $O_{FPT}$ to hide a computable function that is only dependent on the parameters. Then the above results indicate that two-stage stochastic IP and $n$-fold IP both admit algorithms of running time $O_{FPT}(n^{O(1)})L$ and are thus both in FPT. In contrast, the best known algorithm for 4-block $n$-fold IP has a running time of $\min\{O_{FPT}(n^{2k}t_A+3)L),O_{FPT}(n^{k(A,B)}t_A+3)L)\}$ [13], where $k(A, B)$ is some parameter that is dependent on $s_A, s_B, t_A, t_B, \Delta_{A,B}$ (where $\Delta_{A,B}$ is the largest absolute value among all entries of $A, B$). As the existence of $k(A, B)$ follows from a saturation result in commutative algebra, even a rough estimation of $k(A, B)$ (say, singly or doubly exponential) is not clear so far. Given the recent progress in two-stage stochastic IP and $n$-fold IP [8, 23], it becomes a very natural question whether an improved algorithm can be designed for 4-block $n$-fold IP. In particular, is 4-block $n$-fold IP in FPT?

Towards an algorithmic improvement, it is crucial to understand the Graver basis of 4-block $n$-fold IP. Indeed, all the algorithms so far for 4-block $n$-fold IP as well as its two special cases (namely two-stage stochastic IP and $n$-fold IP) rely on the same augmentation framework, as we will provide details in Section 2. Such an augmentation framework applies to an arbitrary IP. The reason that we can have a better algorithm for 4-block $n$-fold IP and its special cases, rather than one that is exponential in the number of variables or constraints, is that its Graver basis has a nice structure. In particular, the $\ell_\infty$-norm of two-stage stochastic IP and $n$-fold IP are both bounded by $O_{FPT}(1)$, whereas they admit FPT algorithms using the augmentation framework. In contrast, the $\ell_\infty$-norm (or 1-norm) of 4-block $n$-fold IP is only bounded by $\min\{O_{FPT}(n^{2k}t_A),O_{FPT}(n^{k(A,B)}t_A)\}$ [13]. If an $O_{FPT}(1)$ upper bound can be established for 4-block $n$-fold IP, then an FPT algorithm follows. This motivates us to study the Graver basis of 4-block $n$-fold IP and its special cases.
1.3 Our Contribution

Firstly, we show that the $\ell_\infty$-norm of Graver basis elements for 4-block $n$-fold IP is upper bounded by $O_{FPT}(n^c)$ (Theorem 2), improving the existing upper bound of $O_{FPT}(n^{2c})$ [13]. We also establish the first explicit lower bound of $\Omega(n^c)$ (Theorem 4). It is thus tight up to an FPT factor. Indeed, our lower bound even shows that for some $H$, any non-zero integral element of $\{x: Hx = 0\}$ has an $\ell_\infty$-norm at least $\Omega(n^c)$. Therefore, even an algorithm that augments via other basis instead of Graver basis may have to deal with an augmentation step that is unbounded (by $O_{FPT}(1)$).

Secondly, we study a special case of 4-block $n$-fold IP, namely 3-block $n$-fold IP where $C = 0$. We show that, unlike 4-block $n$-fold IP, 3-block $n$-fold IP admits a Hilbert basis whose $\ell_\infty$-norm is bounded by $O_{FPT}(1)$ (Theorem 5). Unfortunately, the $\ell_\infty$-norm of Graver basis elements of 3-block $n$-fold IP is at least $\Omega(n)$. We complement our results by establishing an upper bound of $\min\{O_{FPT}(n^c), O_{FPT}(n^\frac{5}{2}) + 1\}$ (Theorem 2 and Theorem 6). The upper bound of $O_{FPT}(n^\frac{5}{2}) + 1$, which is singly exponential in $t_A$, is much more involved compared with the other upper bound. This seems to coincide with the existing results for 4-block $n$-fold IP [13], where an upper bound that depends on $A, B$ (instead of $C, D$) is more complicated. Our proof relies on a completely new approach, which first establishes a specific decomposition and then modify it into a sign-compatible decomposition through merging summands. This may be of separate interest for deriving upper bounds on the norms of Graver basis for other problems, particularly for deriving an upper bound on the $\ell_\infty$-norm of Graver basis for 4-block $n$-fold IP which has an explicit dependency on $s_A, s_B, t_A, t_B$ in the exponent of $n$.

Thirdly, combining our upper bounds on the $\ell_\infty$-norm of Graver basis elements and the new algorithmic progress in $n$-fold IP [8, 23], we establish an algorithm of running time $O_{FPT}(n^{5/2} + 3)\log n$ for 4-block $n$-fold IP and an algorithm of running time $\min\{O_{FPT}(n^{5/2} + 3)\log^3 n), O_{FPT}(n^{5/2} + 1)\log^3 n)\}$ for 3-block $n$-fold IP.

2 Preliminary

Notations. Any $(t_B + nt_A)$-dimensional vector $x$ can be written into $n + 1$ “bricks” such that $x = (x^0, x^1, \cdots, x^n)$ where $x^0$ is $t_B$-dimensional and each $x^i$, $1 \leq i \leq n$, is $t_A$ dimensional. We call $x^i$ as the $i$th brick for $0 \leq i \leq n$. We write $0_{s \times t}$ for an $s \times t$ matrix consisting of 0, and $I_t$ for an $t \times t$ identity matrix. For a vector or a matrix, we write $|| \cdot ||_\infty$ to denote the maximal absolute value of its coordinates (elements). For two column vectors $x, y$ of the same dimension, we write $x \cdot y$ for its inner product.

Throughout this paper, we write $O_{FPT}(1)$ to represent a parameter that is only dependent on $\Delta, s_A, s_B, s_C, s_D, t_A, t_B, t_C, t_D$ where $\Delta$ is the maximal absolute value among all the entries of $A, B, C, D$, that is, $O_{FPT}(1)$ is only dependent on the small matrices $A, B, C, D$ and is independent of $n$. For any computable function $f(x)$, we write $O_{FPT}(f)$ to represent a computable function $f'(x)$ such that $|f'(x)| \leq O_{FPT}(1) \cdot |f(x)|$, and $\Omega_{FPT}(f)$ to represent a function $f''(x)$ such that $|f''(x)| \geq \Omega(1) \cdot |f(x)|$.

Two vectors $x$ and $y$ are called sign-compatible if $x_i \cdot y_i \geq 0$ holds for every pair of coordinates $(x_i, y_i)$. Furthermore, we call a summation $\sum_i x_i$ a sign-compatible summation if for every $i, j$ the summands $x_i$ and $x_j$ are sign-compatible.

We provide a brief introduction to the notions needed for solving a general integer programming. We refer the readers to a nice book [6] for details.

Graver basis. Consider the general integer linear programming in the standard form:

$$
\min \{w \cdot x : Hx = b, l \leq x \leq u, x \in \mathbb{Z}^m\}
$$

(2)

We define Graver basis, which was introduced in [11] by Graver. We define a partial order $\subseteq$ in $\mathbb{R}^m$ in the
following way:

For any \( x, y \in \mathbb{R}^m \), \( x \sqsubseteq y \) if and only if for every \( 1 \leq i \leq n \), \( |x_i| \leq |y_i| \) and \( x_i \cdot y_i \geq 0 \).

Given any subset \( X \subseteq \mathbb{R}^n \), we say \( x \) is an \( \sqsubseteq \)-minimal element of \( X \) if \( x \in X \) and there does not exist \( y \in X \), \( y \neq x \) such that \( y \sqsubseteq x \). It is known that every subset of \( \mathbb{Z}^m \) has finitely many \( \sqsubseteq \)-minimal elements.

**Definition 1.** The Graver basis of an integer \( m' \times m \) matrix \( H \) is the finite set \( G(H) \subseteq \mathbb{Z}^n \) which consists of all the \( \sqsubseteq \)-minimal elements of \( \text{ker}_{\mathbb{Z}}(H) = \{ x \in \mathbb{Z}^m | Hx = 0, x \neq 0 \} \).

Any \( x \in \text{ker}_{\mathbb{Z}}(H) \), \( x \neq 0 \) can be written as \( x = \sum_{i} \alpha_i g_i(H) \), where \( \alpha_i \in \mathbb{Z}_+ \), \( g_i(H) \in G(H) \) and \( g_i(H) \sqsubseteq x \).

**Augmentation algorithms for IP and Graver-best oracle.** There is a general framework for solving an integer programming by utilizing Graver basis, which is implemented in a series of recent papers (see, e.g., \[4, 14, 18, 22\]). A very recent paper by Koutecký, Levin and Onn \[23\] gives a nice explanation on this framework. In the following we briefly recap their explanation. We define a Graver-best augmentation procedure as follows. Given an arbitrary feasible solution \( x_0 \) for IP \( \Omega \), for any \( g \in G(H) \) and \( \rho \in \mathbb{Z}_+ \) we say \( (g, \rho) \) is a Graver augmentation pair if \( w(x_0 + \rho g) < wx_0 \) and \( l \leq x_0 + \rho g \leq u \), i.e., \( x_0 + \rho g \) is a feasible solution with a strictly better objective value. We say \( h \in \mathbb{Z}^m \) is a Graver-best augmentation step if it holds that \( x_0 + h \) is feasible and \( w(x_0 + h) \leq w(x_0 + \rho g) \) for any Graver augmentation pair \( (g, \rho) \). Given a feasible solution \( x_0 \) for IP \( \Omega \), a Graver-best augmentation procedure works iteratively as follows:

(i) If no Graver-best augmentation step exists, return \( x_0 \) is optimal;

(ii) If there exists some Graver-best augmentation step \( h \), set \( x_0 \leftarrow x_0 + h \) and go to step (i).

We define a Graver-best oracle as such that given an input of IP \( \Omega \) that consists of an integer matrix \( H \), integer vectors \( w, b, l, u \) and a feasible solution \( x \), it returns a Graver-best step \( h \) for \( x \).

The following theorem is due to \[23\], which generalizes the result in \[25\].

**Theorem 1.** \[23\] Given a Graver best oracle and an initial feasible solution for IP \( \Omega \). IP \( \Omega \) can be solved by a strongly polynomial oracle algorithm.

**Approximate Graver-best oracle.** In general, finding a Graver-best augmentation step is difficult. However, if some additional information on the Graver basis is known, e.g., if the Graver basis element of \( G(H) \) has an \( \ell_\infty \)-norm bounded by some value \( \xi \), then we are able to restrict our attention to the following:

\[
\min \{ w \cdot \rho x : Hx = 0, l \leq x_0 + \rho x \leq u, \rho \in \mathbb{Z}_+, x \in \mathbb{Z}^m, ||x||_\infty \leq \xi \} \tag{3}
\]

An algorithm for IP \( \Omega \) serves as a Graver-best oracle. It has been observed in \[8\], very recently, that we do not really need to solve IP \( \Omega \) optimally. Indeed, it suffices to find out an \( \mathcal{O}(1) \)-approximation solution for IP \( \Omega \), which, in turn, gives us an approximate Graver-best oracle. Why an approximate Graver-best oracle suffices? Let \( \rho^* \) and \( g^* \) be such that \( w \cdot \rho^* g^* \) is the minimal among all the pairs \( (g, \rho) \in G(H) \times \mathbb{Z}_+ \) and \( l \leq x_0 + \rho g \leq u \). It has been observed before (see, e.g., \[15, 24\]) that \( |w \cdot \rho^* g^*| \geq 1/\Omega_{FPT}(n) \cdot w \cdot (x^* - x_0) \) where \( x^* \) is the optimal solution. Therefore, an optimal solution to IP \( \Omega \) allows us to reduce the gap between \( x_0 \) and \( x^* \) by a multiplicative factor of \( 1 - 1/\Omega_{FPT}(n) \), implying that \( \mathcal{O}(n \log |w \cdot x^* - w \cdot x_0|) \) augmentation steps suffice to reach \( x^* \). It is easy to see that instead of an optimal solution to IP \( \Omega \), any \( \mathcal{O}(1) \)-approximation solution also allows us to reduce the gap by a factor of \( 1 - 1/\Omega(n) \). This observation allows us to restrict the value of \( \rho^* \)'s to be the form of \( 2^k \) for \( k \in \mathbb{Z}_{\geq 0} \). Given an explicitly upper and lower bound, we know that \( \rho \leq \max \{ ||u - x_0||_\infty, ||l - x_0||_\infty \} \). If, however, no explicit upper or lower bound is known for some variable, we can use some proximity result from the linear programming relaxation \[5\] or simply use a standard upper bound of \( (n\Delta)^{\mathcal{O}(n)} \) where \( \Delta = ||H||_\infty \) (which is also used in the Lenstra’s algorithm \[24\]).
Therefore, we can restrict that \( \rho \) only takes \( O(n \log(n\Delta)) \) distinct values of the form \( 2^0, 2^1, 2^2, \ldots \). For each fixed value \( \rho_0 = 2^k \), we solve the following IP:

\[
\min \{ w \cdot x : Hx = 0, 1 \leq x_0 + \rho_0 x \leq u, x \in \mathbb{Z}^m, ||x||_{\infty} \leq \xi \} \tag{4}
\]

It is clear that an optimal or \( O(1) \)-approximation solution for IP \( \text{(4)} \) suffices for us to derive an optimal solution for IP \( \text{(2)} \).

**Feasibility and Optimality.** Finding a feasible solution of \( \left( \begin{array}{c} C \\ D \\ \hline B \\ A \end{array} \right) \overset{(n)}{=} b, 1 \leq x \leq u \) is equivalent to finding an optimal solution of an augmented IP with the same 4-block structure but has a trivial initial feasible solution. We briefly describe this procedure as follows (this is also useful in our analysis).

Let \( D = (D, I_d, 0_{t_d \times t_d}) \) and \( \bar{A} = (A, 0_{t_d \times t_d}, I_{t_d}) \). Let \( \bar{y} = (\bar{y}^1, \bar{y}^1, \bar{y}^2, \ldots, \bar{y}^m, \bar{y}^m) \) with \( \bar{y}^i \) and \( \bar{y}^i \) being an \( s_A \)- and \( s_D \)-dimensional vectors, respectively. Let \( \bar{x} = \bar{x}^1, \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m, \bar{x}^m \). Now it is easy to see that if we take \( x^i_0 = 0, y^i_0 = b^i, \bar{y}^i_0 = 0 \) for \( 2 \leq i \leq n, y^i_0 = b^i \) for \( 1 \leq i \leq n \), then \( x^i_0 + y^i_0 \) is a feasible solution to

\[
\left( \begin{array}{c} C \\ D \\ \hline B \\ A \end{array} \right) \bar{x} = b, 1 \leq \bar{x} \leq u
\]

If we minimize an objective function of \( ||y||_1 \) for the above augmented IP, its optimal solution with the objective value of 0 implies a feasible solution to the original IP. Although \( ||y||_1 \) is not linear, we can use the standard technique to make it linear, i.e., we can write \( y = y_+ - y_- \) and add the constraint \( y_+, y_- \geq 0 \). It is easy to verify that such a modification does not destroy the 4-block \( n \)-fold structure. Also notice that this approach only involves modifying \( A \) and \( D \), and therefore applies if \( B = C = 0 \).

**Fitness theorems for \( n \)-fold and two-stage stochastic matrices.**

Consider an \( n \)-fold matrix \( E \) that consists of \( A \) and \( D \) (i.e., \( B = C = 0 \) in a 4-block \( n \)-fold matrix). It is shown that, the \( || \cdot ||_1 \)-norm of any Graver basis element of \( E \) is \( O_{FPT}(1) \). More precisely, we have the following lemma.

**Lemma 1.** \( \text{[24][27]} \) Let \( E \) be an \( n \)-fold matrix. There exists some integer \( \kappa = f_{nj}(s_A, s_D, t_A, t_D, \Delta) \) for some computable function \( f_{nj} \) and

\[
M(A) = \{ h \in \mathbb{Z}^m | h \text{ is the sum of at most } \kappa \text{ elements of } \mathcal{G}(A_2) \},
\]

such that for any \( g = (g^1, g^2, \ldots, g^m) \in \mathcal{G}(E) \) we have \( \sum_{i \in I} g^i \in M(A) \) for any \( I \subseteq \{1, 2, \ldots, n\} \).

**Lemma 2.** \( \text{[2][23]} \) Let \( F \) be a two-stage stochastic matrix, \( g_\infty(H) = \max_{g \in \mathcal{G}(H)} ||g||_{\infty} \) and \( a = \max\{ ||A||_{\infty}, ||B||_{\infty} \} \). Then \( g_\infty(H) \leq f_{st}(s_A, t_A, s_B, t_B, \Delta) \) for some computable function \( f_{st} \).

It is remarkable that the above lemma actually holds for a more general class of matrices called multi-stage stochastic matrices.

**Steinitz lemma** Steinitz lemma has been utilized in several recent papers \( \text{[8][9]} \) to establish a better algorithm for integer programming. We will also utilize it in this paper.

**Lemma 3.** \( \text{[2][22]} \) Let an arbitrary norm be given in \( \mathbb{R}^\kappa \) and assume that \( ||x_i|| \leq \zeta \) for \( 1 \leq i \leq m \) and \( \sum_{i=1}^m x_i = x \). Then there exists a permutation \( \pi \) such that for all positive integers \( \ell \leq m \),

\[
|| \sum_{i=1}^\ell x_{\pi(i)} - \frac{\ell - \kappa}{m} x || \leq \kappa \zeta.
\]

### 3 4-block \( n \)-fold integer programming

In this section we consider IP \( \text{(1)} \) for arbitrary \( H \).
3.1 Upper bound on the $\ell_\infty$-norm of Graver basis

The goal of this subsection is to prove the following theorem.

**Theorem 2.** For any 4-block $n$-fold matrix $H$ and $g(H) \in G(H)$, $\|g(H)\|_\infty \leq O_{FPT}(n^\kappa)$.

**Proof.** Let $g \in G(H)$. As $F \cdot g = 0$, there exist $\alpha_j \in \mathbb{Z}_+$, $g_j(F) \in G(F)$ and $g_j(F) \subseteq g$ such that

$$g = \sum_{j=1}^m \alpha_j g_j(F).$$

Furthermore, $\|g_j(F)\|_\infty = O_{FPT}(1)$ according to Lemma 3.

Let $h_j = C \cdot g_j(F) + \sum_{i=1}^n D g_j(F)$, which is an $s_C$-dimensional vector such that $\|h_j\|_\infty = O_{FPT}(n)$. As $Hg = 0$, it follows that

$$\sum_{j=1}^m \alpha_j h_j = \underbrace{h_1 + h_1 + \cdots + h_1}_{\alpha_1} + \underbrace{h_2 + h_2 + \cdots + h_2}_{\alpha_2} + \cdots + \underbrace{h_m + h_m + \cdots + h_m}_{\alpha_m} = 0,$$

i.e., the sequence of $h_j$'s sum up to 0. According to Lemma 3, there exists a permutation of the sequence such that

$$\|\sum_{i=1}^\ell z_i\|_\infty \leq s_C \cdot O_{FPT}(n) = O_{FPT}(n), \quad \forall \ell \leq m'.$$

where $m' = \sum_{j=1}^m \alpha_j$ and $z_1, z_2, \ldots, z_{m'}$ is a permutation of the sequence $h_1, h_1, \ldots, h_2, h_2, \ldots, h_m, h_m, \ldots, h_m$. Let $\tau = O_{FPT}(n)$ be the upper bound on $\|\sum_{i=1}^\ell z_i\|_\infty$, then we know that $\sum_{i=1}^{\ell_1} z_i \in \{-\tau, -\tau + 1, \ldots, \tau\}^\kappa$. Consequently, if $m' > (2\tau + 1)^\kappa + 1$, there exists $\ell_1 < \ell_2$ such that $\sum_{i=1}^{\ell_1} z_i = \sum_{i=1}^{\ell_2} z_i$, i.e., $\sum_{i=1}^{\ell_2 - \ell_1} z_i = 0$. Recall that every $z_i$ corresponds to some $h_j$. Suppose $\sum_{i=1}^{\ell_2 - \ell_1} z_i = \sum_{j=1}^m \alpha'_j h_j$ for $\alpha'_j \leq \alpha_j$, then by the definition of $h_j$ it follows that

$$C \left( \sum_{j=1}^m \alpha'_j g_j^0(F) \right) + \sum_{i=1}^{n} D \left( \sum_{j=1}^m \alpha'_j g_j^i(F) \right) = 0.$$

Hence, $H \sum_{j=1}^m \alpha'_j g_j^i(F) = 0$. That is, if $m' = \sum_{j=1}^m \alpha_j > (2\tau + 1)^\kappa + 1$, then there exists some $g' = \alpha'_j g_j(F)$ such that $Hg' = 0$ and $g' \subseteq g$, contradicting the fact that $g \in G(H)$. Thus, $\sum_{j=1}^m \alpha_j \leq (2\tau + 1)^\kappa + 1$, implying that $\|g\|_\infty = O_{FPT}(n^\kappa)$.

**Remark.** The idea of the proof above seems to only work for the parameter $s_C$. An explicit upper bound that depends on $A, B$ is far from clear (albeit that we know an upper bound of $O_{FPT}(n^\kappa(A, B))$ with some unknown function $k(A, B)$). In the following section we will provide an upper bound that is singly exponential in $t_A$ for the special case when $C = 0$ using a completely different approach.

Theorem 2 implies the following, whose proof is exactly the same as Theorem 7.

**Theorem 3.** There exists an algorithm for 4-block $n$-fold IP that runs in $O_{FPT}(n^{\kappa t_B + 3}) \log n$ time.

3.2 Lower bound on the $\ell_\infty$-norm of Graver basis

We prove an even stronger result which gives a lower bound for any element in $ker_{\mathbb{Z}^{n+nt_A}}(H) = \{x \in \mathbb{Z}^{n+nt_A} : Hx = 0, x \neq 0\}$.
**Theorem 4.** There exists a 4-block n-fold matrix H such that \( s_C = s_D = t - 1 \), \( t_C = t_D = t \), \( s_A = s_B = t_A = t_B = t \), and for any \( y \in \text{ker}_{\mathbb{Z}B \times \mathbb{A}}(H) = \{ x : Hx = 0, x \neq 0 \} \), \( \| y \|_\infty = \Omega(n^{-1}) \).

**Proof.** We let \( A = I_{t \times t} \), \( B = -I_{t \times t} \). We define \((t - 1) \times t\) matrices \( D \) and \( C \) such that

\[
D = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}, \quad C = \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}
\]

Consider any \( y \in \text{ker}_{\mathbb{Z}^{(n+1)t}}(x : Hx = 0, x \neq 0) \). According to \( Ay^0 - By^i = 0 \), we know that \( y^0 = y^i \) for every \( 1 \leq i \leq n \). According to \( Cy^0 + \sum_{k=1}^{n} Dy^i = 0 \), we have \( (C + nD)y^0 = 0 \), i.e.,

\[
\begin{pmatrix}
(n-1) & -n & 0 & \cdots & 0 & 0 \\
0 & n-1 & -n & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & n-1 & -n
\end{pmatrix} \cdot y = 0
\]

Let \( y^0 = (y_1, y_2, \cdots, y_t) \), the following is true:

\[
(n-1)y_i = ny_{i+1}, \quad 1 \leq i \leq t - 1
\]  

(5)

It is easy to see that as long as \( y \neq 0 \), we have \( y^0 \neq 0 \) and consequently \( y_i \neq 0 \) for every \( 1 \leq i \leq t \). Furthermore, Eq (5) indicates that either \( y_i > 0 \) for all \( i \), or \( y_i < 0 \) for all \( i \). Suppose \( y_i > 0 \) (the other case can be proved in the same way). According to \((n-1)y_{i-1} = ny_i, y_i-1 \) is divisible by \( n \). Let \( y_{i-1} = ez_{i-1} \) for some \( z_{i-1} \in \mathbb{Z} \neq 0 \). According to \((n-1)y_{i-2} = ny_{i-1} = n^2z_{i-2} \), we know that \( y_{i-2} \) is divisible by \( n^2 \). Let \( y_{i-2} = n^2z_{i-2} \) and we plug it into \((n-1)y_{i-3} = ny_{i-2} \). In general, suppose we have shown that \( y_{i-k} = n^kz_{i-k} \) for all \( k \leq k_0 \). Now for \( k = k_0 + 1 \), we have \((n-1)y_{i-k_0-1} = ny_{i-k_0} = n^{k_0+1}z_{i-k_0} \), then \( y_{i-k_0-1} \) is divisible by \( n^{k_0+1} \). Hence, we conclude that \( y_1 \) is divisible by \( n^{-1} \), i.e., \( \| y \|_\infty = \Omega(n^{-1}) \) and Theorem 4 is proved.

### 4 3-block n-fold integer programming

In this section we consider IP (1) where \( H = H_0, \) i.e., \( C = 0. \) The goal of this section is to show the following three main results: 1). There exists a Hilbert basis for \( \text{ker}_{\mathbb{Z}^{(n+1)t}}(H_0) = \{ x \in \mathbb{Z}^{(n+1)t} : H_0x = 0, x \neq 0 \} \) such that the \( \ell_\infty \)-norm of every basis element is bounded by \( \mathcal{O}_{\text{FPT}}(1) \). This gives a sharp contrast to Theorem 4 since when \( C \neq 0 \), any non-zero element of \( \text{ker}_{\mathbb{Z}^{(n+1)t}}(H) \) may have an \( \ell_\infty \)-norm at least \( \Omega(n^k) \). 2). Any Graver basis element of \( G(H_0) \) has an \( \ell_\infty \)-norm bounded by \( \min \{ \mathcal{O}_{\text{FPT}}(n^k), \mathcal{O}_{\text{FPT}}(n^{k+1}) \} \). We also complement this upper bound by establishing a lower bound of \( \Omega(n) \). 3). There exists an algorithm of running time \( \min \{ \mathcal{O}_{\text{FPT}}(n^{k+1}) \log^5 n, \mathcal{O}_{\text{FPT}}(n^{k+1}) \log^3 n \} \) for 3-block n-fold IP by utilizing our bound on the \( \ell_\infty \)-norm of the Graver basis and the general framework from [23] (see Section 2) Augmentation algorithms for IP and Graver-best oracle).

#### 4.1 Decomposition with bounded \( \ell_\infty \)-norm

The goal of this subsection is to prove the following theorem.

**Theorem 5.** There exists some \( \xi = \mathcal{O}_{\text{FPT}}(1) \) such that for any \( g \in \mathbb{Z}^{(n+1)t} \) satisfying \( H_0g = 0 \), there exist a finite sequence of vectors \( e_1, e_2, \cdots \) such that \( e_h \in \mathbb{Z}^{(n+1)t} \), \( H_0e_h = 0, \| e_h \|_\infty \leq \xi, e_0^0 \subseteq g^0 \) and \( g = \sum_h e_h. \)
Recall that $x^0$ always refer to the first $t_B$ coordinates of a $(t_B + nt_A)$-dimensional vector $x$. Note that $e_i$‘s do not necessarily lie in the same orthant.

**Proof.** Since $H_0g = 0$, we know that $F \cdot g = 0$. Therefore, there exist $\alpha_j \in \mathbb{Z}_+, g_j(F) \subseteq g$ such that

$$g = \sum_j \alpha_j g_j(F),$$

where $g_j(F) \in G(F)$. Consider each $g_j(F)$. As $F$ is a two-stage stochastic matrix, by Lemma 2 it holds for every $j$ that $\|g_j(F)\|_\infty = O_FPT(1)$. Note that each $g_j(F)$ can be written into $n + 1$ bricks such that $g_j(F) = \left( g_j^0(F), g_j^1(F), \ldots, g_j^n(F) \right)$ where $g_j^0(F)$ is a $t_B$-dimensional vector, and $g_j^i(F)$ is a $t_A$-dimensional vector for every $1 \leq i \leq n$. It is obvious that $\|g_j^i(F)\|_\infty = O_FPT(1)$ for every $0 \leq i \leq n$, and it holds that

$$B g_j^0(F) + A g_j^i(F) = 0, \quad \forall 1 \leq i \leq n.$$

We first prove the following claim.

**Claim 1.** For every $g_j(F)$ and $1 \leq \ell \leq |G(A)|$, there exist some $v_j^\ell$, $\alpha_{j,\ell}^i \in \mathbb{Z}_{\geq 0}$ such that

- $g_j^\ell(F) - v_j^\ell = \sum_{\ell'=1}^{\ell} \alpha_{j,\ell'}^i g_{\ell'}(A), \quad \forall 1 \leq i \leq n$.
- For every $1 \leq \ell \leq |G(A)|$, either $|\{i : \alpha_{j,\ell}^i > 0\}| = 0$, or $|\{i : \alpha_{j,\ell}^i > 0\}| \geq n/2$.
- Let $\alpha_{\max} = 2 \max_h \|g_h(F)\|_\infty = O_FPT(1)$. Then $\max_{i,\ell} |\alpha_{j,\ell}^i| \leq \alpha_{\max}$.
- $\|v_j^\ell\|_\infty = O_FPT(1)$.

**Proof of Claim 7.** Consider an arbitrary $v_j$ such that $\left( \begin{array}{c} g_j^0(F) \\ \vdots \\ g_j^{n}(F) \end{array} \right) \in G([B,A])$. We have $A(g_j^i(F) - v_j) = 0$ for every $1 \leq i \leq n$, hence there exist $\tilde{\alpha}_{j,\ell}^i \in \mathbb{Z}_+$, $g_{\ell}(A) \in G(A)$ and $g_{\ell}(A) \subseteq g_j^i(F) - v_j$ such that

$$g_j^i(F) - v_j = \sum_{\ell} \tilde{\alpha}_{j,\ell}^i g_{\ell}(A), \quad \forall 1 \leq i \leq n.$$

Note that $\| \left( \begin{array}{c} g_j^0(F) \\ \vdots \\ g_j^{n}(F) \end{array} \right) \|_\infty \leq \max_h \|g_h(F)\|_\infty = \alpha_{\max}/2$, consequently $\|g_j^i(F) - v_j\|_\infty \leq \alpha_{\max}$, and $\tilde{\alpha}_{j,\ell}^i \leq \alpha_{\max}$.

Consider the cardinality of the set $\{i : \alpha_{j,\ell}^i > 0\}$. If $1 \leq |\{i : \alpha_{j,\ell}^i > 0\}| \leq |n/2|$, we say $\ell$ is *unbalanced* for $g_j(F)$. Let $\tilde{\alpha}_{j,\max} = \max_{1 \leq i \leq n} \tilde{\alpha}_{j,\ell}^i$ and $UB_j$ be the set of all unbalanced indices $\ell$, we define

$$v_j^\ell := v_j + \sum_{\ell \in UB_j} \tilde{\alpha}_{j,\max} g_{\ell}(A),$$

then

$$g_j^i(F) - v_j^\ell = \sum_{\ell \notin UB_j} \tilde{\alpha}_{j,\ell}^i g_{\ell}(A) + \sum_{\ell \in UB_j} (\tilde{\alpha}_{j,\max} - \tilde{\alpha}_{j,\ell}^i) \cdot (-g_{\ell}(A)), \quad \forall 1 \leq i \leq n.$$

Note that $-g_{\ell}(A) \in G(A)$. For all the $g_{\ell}(A)$’s in $G(A)$ that do not appear in the above equation, their coefficients are 0. Furthermore, we have $|\alpha_{j,\ell}^i| \leq \alpha_{\max}$ and $|\tilde{\alpha}_{j,\ell}^i| \leq \alpha_{\max}$ for all $i, \ell$. As $\|v_j\|_\infty = O_FPT(1)$, $\|g_{\ell}(A)\|_\infty = O_FPT(1)$, we know that $\|v_j^\ell\|_\infty = O_FPT(1)$. Thus, the claim is proved. \qed
We call \((g_1^0(F), v_1^0, v_2^0, \ldots, v_y^0)\) as a canonical vector (of \(g_j(F)\)). Since \(||v_j^0||_\infty = O_{\text{FPT}}(1)\) and \(||g_0^0(F)||_\infty = O_{\text{FPT}}(1)\), there are at most \(\tau = O_{\text{FPT}}(1)\) different kinds of canonical vectors. This means, there may be different \(g_k(F)'s\) with the same canonical vector. We list all the \(\tau\) possible canonical vectors and let \(r_j := (u_j^0, v_j^0, v_j^1, \ldots, v_j^y)\) be the \(j\)-th one. Let \(CA_j\) be the set of indices of all \(g_k(F)'s\) whose canonical vector is \(r_j\), then we have

\[
g = \sum_{j=1}^\tau \left( \sum_{k \in CA_j} \alpha_k \right) r_j + \sum_{j=1}^\tau \sum_{k \in CA_j} \alpha_k (g_k(F) - r_j).
\] (6)

We say an \(n\)-dimensional vector \(\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n) \in \mathbb{Z}_{\geq 0}^n\) is balanced, if \(\alpha = 0\), or \(||\alpha||_\infty \leq \alpha_{\text{max}} = O_{\text{FPT}}(1)\) and \(||\{i : \alpha^i > 0\}\| \geq n/2\). Then the following observation is true.

**Observation 1.** For any nonzero balanced vector \(\alpha\) it holds that \(||\alpha||_1 \geq n/2 \cdot \alpha'/\alpha_{\text{max}}\) for every \(1 \leq i \leq n\).

Using the concept of a balanced vector, Claim [1] indicates that if \(r_j\) is a canonical vector of \(g_k(F)\), then \(g_k(F) - v_j^y = \sum_{i=1}^n \alpha_{j,\ell} g_i(A)\) such that the vector \((\alpha_{j,\ell}^1, \alpha_{j,\ell}^2, \ldots, \alpha_{j,\ell}^n)\) is a balanced vector.

The nice thing about balanced vectors is that we can have the following claim, which will be used several times later.

**Claim 2.** Let \(y_1, y_2, \ldots, y_k\) be a sequence of balanced vectors in \(\mathbb{Z}_{\geq 0}^n\) such that \(||\sum_{h=1}^k y_h||_1 \leq n\Lambda\) where \(\Lambda = O_{\text{FPT}}(1)\), then \(||\sum_{h=1}^k y_h||_\infty \leq 2\alpha_{\text{max}}\Lambda = O_{\text{FPT}}(1)\).

**Proof of Claim 2** We prove by contradiction. Suppose on the contrary that \(||\sum_{h=1}^k y_h||_\infty > 2\alpha_{\text{max}}\Lambda\), then there exists some \(i'\) such that \(\sum_{h=1}^k y_{h,i'} > 2\alpha_{\text{max}}\Lambda\). Since \(y_h's\) are balanced vectors, according to Observation [1] we have

\[
||\sum_{h=1}^k y_h||_1 \geq n \cdot \frac{\sum_{h=1}^k y_{h,i'}}{2\alpha_{\text{max}}} > n\Lambda,
\]

which contradicts the fact that \(||\sum_{h=1}^k y_h||_1 \leq n\Lambda\). Hence, the claim is true.

Since \(r_j\) is a canonical vector of \(g_k(F)\), by Claim [1] there exist balanced vectors \(\beta_{k,\ell}\) such that Eq (6) can be rewritten as

\[
g' = \sum_{j=1}^\tau \left( \sum_{k \in CA_j} \alpha_k \right) v_j^y + \sum_{j=1}^\tau \sum_{k \in CA_j} \alpha_k \left( \sum_{\ell=1}^{\lfloor G(A) \rfloor} \beta_{k,\ell} g_\ell(A) \right), \quad \forall 1 \leq i \leq n,
\]

or equivalently,

\[
g' = \sum_{j=1}^\tau \alpha_j' v_j^y + \sum_{\ell=1}^{\lfloor G(A) \rfloor} \beta_\ell' g_\ell(A), \quad \forall 1 \leq i \leq n,
\] (7)

where \(\alpha_j' = \sum_{k \in CA_j} \alpha_k\) and each vector \(\beta_\ell' = (\beta_{1,\ell}^1, \ldots, \beta_{n,\ell}^n)\) is the summation of some balanced vectors.

As \([0, D, D, \ldots, D]g = 0\), we have

\[
\sum_{j=1}^\tau n\alpha_j' D v_j^y + \sum_{\ell=1}^{\lfloor G(A) \rfloor} \left( \sum_{i=1}^n \beta_{\ell,i}^i \right) g_\ell(A) = 0.
\] (8)

Note that \(|G(A)| = O_{\text{FPT}}(1)\), the equation above can be rewritten as

\[
[DV_1^1, DV_2^2, \ldots, DV_r^y, g_1(A), g_2(A), \ldots, g_{\lfloor G(A) \rfloor}(A)] \cdot (n\alpha_1', n\alpha_2', \ldots, n\alpha_z', \sum_{i=1}^n \beta_{1,i}^1, \ldots, \sum_{i=1}^n \beta_{i,\ell}^i) = 0.
\] (9)
Let \( V = [Dv^*_1, Dv^*_2, \ldots, Dv^*_r; g_1(A), g_2(A), \ldots, g_{|G(A)|}(A)] \), there exist \( \lambda_k \in \mathbb{Z}_+ \) and \( g_k(V) \in G(V) \), such that

\[
(n\alpha_1', n\alpha_2', \ldots, n\alpha_r', \sum_{i=1}^n \beta_i^1, \ldots, \sum_{i=1}^n \beta_i^{|G(A)|}) = \sum_k \lambda_k g_k(V).
\]

Note that since \( \alpha_i', \beta_i' \geq 0 \), we can restrict that every \( g_j(V) \in \mathbb{Z}_{k=0}^{\lambda_k|G(A)|} \).

There are two possibilities regarding the values of \( \lambda_k \).

**Case 1.** \( \lambda_k < n \) for every \( k \). In this case we prove that \( ||g||_\infty = O_{FPT}(1) \) and Theorem 5 follows directly. Note that \( V \) is a matrix of \( O_{FPT}(1) \) size with \( ||V||_\infty = O_{FPT}(1), \) hence \( ||g_k(V)||_\infty = O_{FPT}(1) \) and \( |G(V)| = O_{FPT}(1) \). Therefore, \( n\alpha_j' < n|G(V)| \cdot \max_k ||g_k(V)||_\infty = O_{FPT}(n) \), implying that \( \alpha_j' = O_{FPT}(1) \). Consider the vector \( \beta = (\beta^1, \ldots, \beta^n) \) where \( \beta^i = \sum_i \beta^i_i \). As \( \beta^i \geq 0 \),

\[
||\beta||_1 = (\sum_{i=1}^n \beta^i_1, \ldots, \sum_{i=1}^n \beta^i_{|G(A)|})_1 \leq \sum_k \lambda_k ||g_k(V)||_1 = O_{FPT}(n).
\]

Recall that \( \beta^i \) is the summation of balanced vectors, whereas \( \beta = \sum_i \beta^i \) is also the summation of balanced vectors. Using Claim 2 \( ||\beta||_\infty = O_{FPT}(1) \).

Combining the fact that \( ||V^*_j||_\infty = O_{FPT}(1) \) and \( ||g_j(A)||_\infty = O_{FPT}(1) \), we conclude that \( ||g^0||_\infty = O_{FPT}(1) \) for \( 1 \leq i \leq n \). Meanwhile, as \( g^0 = \sum_j \alpha_j u^*_j \), we have \( ||g^0||_\infty = O_{FPT}(1) \). Hence, \( ||g||_\infty = O_{FPT}(1) \).

**Case 2.** \( \lambda_k \geq n \) for some \( k \). For ease of description, we take the viewpoint of a packing problem. We view each canonical vector \( r^*_j \) and \( g_j(A) \) as an item, whereas there are \( \tau + |G(A)| \) different kinds of items. There are \( n+1 \) different bins. Bin 0 can only be used to pack items \( r^*_j, 1 \leq j \leq \tau \), and bin \( i (1 \leq i \leq n) \) can only be used to pack items \( g_j(A), 1 \leq \ell \leq |G(A)| \). Currently there are \( \alpha_j' \) copies of item \( r^*_j \) in bin 0, and \( \beta_j^i \) copies of item \( g_j(A) \) in bin \( i \). This is called a packing profile. Now we want to split this packing profile into several “sub-profiles”, i.e., we want to determine integers \( \mu^j, \sigma^j_\ell \in \mathbb{Z}_{\geq 0} \) such that the followings are true:

(i) \( \mu^j, \sigma^j_\ell = O_{FPT}(1) \) and \( \mu^j + \sigma^j_\ell > 0 \).

(ii) \( \sum_h \mu^j_h = \alpha_j', \sum_h \sigma^j_\ell_h = \beta^i_\ell \).

(iii) \( [Dv^*_1, Dv^*_2, \ldots, Dv^*_r; g_1(A), g_2(A), \ldots, g_{|G(A)|}(A)] \cdot (n\mu^1, n\mu^2, \ldots, n\mu^r, \sum_{i=1}^n \sigma^1_\ell, \ldots, \sum_{i=1}^n \sigma^r_\ell) = 0 \) for every \( h \).

A packing with \( \mu^j_h \) copies of \( r^*_j \) in bin 0 and \( \sigma^j_\ell \) copies of \( g_j(A) \) in bin \( i \) is called a sub-profile. Any sub-profile corresponds to a \( (t_A + nt_B) \)-dimensional vector \( e_h = (e^0_h, e^1_h, \ldots, e^n_h) \) where

\[
e^0_h = \sum_{j=1}^{\tau} \mu^j_h u^*_j,
\]

\[
e^i_h = \sum_{j=1}^{\tau} \mu^j_h v^*_j + \sum_{i=1}^{\sum_{t_i=1}^{|G(A)|}} \sigma^j_\ell \ g_i(A), \quad \forall 1 \leq i \leq n
\]

If all the three conditions on sub-profiles hold, then we know that \( ||e_h||_\infty = O_{FPT}(1) \), \( g = \sum_h e_h \) and \( H_0 e_h = 0 \) (to see why \( H_0 e_h = 0 \) holds, simply observe that \( F r^*_j = 0 \) and condition (iii) implies that \( [0, D, D, \ldots, D]e_h = 0 \)), and furthermore, there are at most \( \sum_j \alpha_j' + \sum_i \beta_i^j \) sub-profiles, which is finite. Hence, \( g = \sum e_h \) and the theorem is proved.

We will construct \( e_h \)'s iteratively. Once \( e_h \) is constructed, we continue our decomposition procedure on \( g - \sum_{k=1}^h e_k \).
Suppose we have constructed $e_1$ to $e_{h_0-1}$ where conditions (i) and (iii) are satisfied for each $e_h$, $\alpha_j' = \sum_{h=1}^{h_0-1} \mu^h_j \geq 0$, $\beta_j' = \sum_{h=1}^{h_0-1} \sigma^h_j \geq 0$ and furthermore, each vector $\tilde{\beta}_\ell = (\tilde{\beta}_1^\ell, \cdots, \tilde{\beta}_n^\ell)$ where $\tilde{\beta}_j^\ell = \beta_j^\ell - \sum_{h=1}^{h_0-1} \sigma^h_j$ can be expressed as a summation of all but one balanced vectors, more precisely, there exist balanced vectors $\phi_{\ell,k} \in \mathbb{Z}_{\geq 0}^n$, $1 \leq k \leq k_{\text{max}}$ such that

$$\tilde{\beta}_\ell = \sum_{k=1}^{k_{\text{max}}-1} \phi_{\ell,k} + \phi_{\ell,k_{\text{max}}}$$

where $\phi_{\ell,k_{\text{max}}} \equiv \phi_{\ell,k_{\text{max}}}$.

We show how to construct $e_{h_0}$. Let $\alpha_j' = \alpha_j' - \sum_{h=1}^{h_0-1} \mu^h_j$. According to condition (iii) of each $e_h$, we know that

$$[Dv_1^\tau, Dv_2^\tau, \cdots, Dv_n^\tau, g_1(A), g_2(A), \cdots, g_{|G(A)|}(A)] \cdot (n\alpha_1', n\alpha_2', \cdots, n\alpha_n', \sum_{i=1}^n \tilde{\beta}_1^i, \cdots, \sum_{i=1}^n \tilde{\beta}_n^i) = 0$$

Consequently, there exist $\lambda_k' \in \mathbb{Z}_{\geq 0}$ and $g_k \in \mathbb{Z}^{n+|G(A)|+|G(V)|}$ such that

$$(n\alpha_1', n\alpha_2', \cdots, n\alpha_n', \sum_{i=1}^n \tilde{\beta}_1^i, \cdots, \sum_{i=1}^n \tilde{\beta}_n^i) = \sum_k \lambda_k' g_k(V).$$

There are two possibilities.

**Case 2.1** If there exists some $\lambda_k' \geq n$, we consider the vector $ng_k(V)$. Let $ng_k(V) = (n\zeta_1, n\zeta_2, \cdots, n\zeta_{\tau+|G(A)|})$. We set $\mu_j^{h_0} = \zeta_j = \mathcal{O}_{FPT}(1)$ for $1 \leq j \leq \tau$. We set the values of $\sigma_{\ell}^{j,h_0}$ such that $\sum_{i=1}^n \sigma_{\ell}^{j,h_0} = n\zeta_{\tau+\ell}$. Consequently, condition (iii) is satisfied for $e_{h_0}$. Now it suffices to set the values of each $\sigma_{\ell}^{j,h_0}$ such that they are bounded by $\mathcal{O}_{FPT}(1)$. Equivalently, this means out of the $\tilde{\beta}_j$ copies of $g_k(A)$, our goal is to take $\sigma_{\ell}^{j,h_0}$ copies such that in total we take $n\zeta_{\tau+\ell}$ copies and $\sigma_{\ell}^{j,h_0} = \mathcal{O}_{FPT}(1)$. We achieve this in a simple greedy way. Let $k^*$ be the index such that

$$\sum_{k=k^*+1}^{k_{\text{max}}-1} ||\phi_{\ell,k}||_1 + ||\phi_{\ell,k_{\text{max}}}||_1 < n\zeta_{\tau+\ell} \leq \sum_{k=k^*}^{k_{\text{max}}-1} ||\phi_{\ell,k}||_1 + ||\phi_{\ell,k_{\text{max}}}||_1$$

Let $\tilde{\phi}_{\ell,k^*} \equiv \phi_{\ell,k^*}$ be an arbitrary vector such that

$$||\tilde{\phi}_{\ell,k^*}||_1 + \sum_{k=k^*+1}^{k_{\text{max}}-1} ||\phi_{\ell,k}||_1 + ||\phi_{\ell,k_{\text{max}}}||_1 = n\zeta_{\tau+\ell}.$$  

We set $\sigma_{\ell}^{j,h_0} = \tilde{\phi}_{\ell,k^*} + \sum_{k=k^*+1}^{k_{\text{max}}-1} \phi_{\ell,k} + \phi_{\ell,k_{\text{max}}}$. It is obvious that in total we have taken $n\zeta_{\tau+\ell}$ copies of $g_k(A)$.

Now it remains to show that $||\sigma_{\ell}^{j,h_0}||_\infty = ||\tilde{\phi}_{\ell,k^*} + \sum_{k=k^*+1}^{k_{\text{max}}-1} \phi_{\ell,k} + \phi_{\ell,k_{\text{max}}}||_\infty = \mathcal{O}_{FPT}(1)$. To see this, notice that each $\phi_{\ell,k}$ is a balanced vector, hence

$$||\phi_{\ell,k^*}||_1 + \sum_{k=k^*+1}^{k_{\text{max}}-1} ||\phi_{\ell,k}||_1 + ||\phi_{\ell,k_{\text{max}}}||_1 \leq n\zeta_{\tau+\ell} + 2n\alpha_{\text{max}} = \mathcal{O}_{FPT}(n).$$

According to Claim 2 $||\tilde{\phi}_{\ell,k^*} + \sum_{k=k^*+1}^{k_{\text{max}}-1} \phi_{\ell,k} + \phi_{\ell,k_{\text{max}}}||_\infty = \mathcal{O}_{FPT}(1)$. Consequently, $||\sigma_{\ell}^{j,h_0}||_\infty = \mathcal{O}_{FPT}(1)$.

Also notice that after we take $\sigma_{\ell}^{j,h_0}$ copies of $g_k(A)$,

$$\tilde{\beta}_\ell - \sigma_{\ell}^{j,h_0} = \sum_{k=1}^{k_{\text{max}}-1} \phi_{\ell,k} + (\phi_{\ell,k^*} - \tilde{\phi}_{\ell,k^*}).$$

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which is still the summation of all but one balanced vector. Hence we can continue to decompose \( g - \sum_{h=1}^{h_0} e_h \).

**Case 2.2** \( \lambda_k^j < n \) for every \( k \). We claim that \( \| g - \sum_{h=1}^{h_0-1} e_h \|_\infty = O_{FPT}(1) \). If this claim is true, then \( g = \sum_{h=1}^{h_0-1} e_h + (g - \sum_{h=1}^{h_0-1} e_h) \), and Theorem 5 is proved. To show the claim, we use a similar argument as that of case 1. First, \( n\bar{\alpha}' \leq (\sum_k \lambda_k) \cdot \max_k \| g_k(V) \|_\infty = O_{FPT}(n) \), hence \( \bar{\alpha}' = O_{FPT}(1) \). Second, we consider the \( n \)-dimensional vector \( \beta = \sum_{\ell=1}^{[G(A)]} \beta_{\ell} \). Let \( \tilde{\beta}'_{\ell} = \sum_{k=1}^{k_{\text{max}}} \phi_{\ell,k} \) and \( \beta' = \sum_{\ell=1}^{[G(A)]} \beta'_{\ell} \). Given that \( \phi_{\ell,k_{\text{max}}} \leq \phi_{\ell,k_{\text{max}}} \) and \( \phi_{\ell,k_{\text{max}}} \) is a balanced vector, \( \| \beta'_{\ell} \|_1 \leq \| \beta'_{\ell} \|_1 + n\alpha_{\text{max}} \cdot \| G(A) \| \leq \sum_k \lambda_k \cdot \max_k \| g_k(V) \|_1 + n\alpha_{\text{max}} \cdot \| G(A) \| = O_{FPT}(n) \).

Note that \( \beta' \) is the summation of balanced vectors. According to Claim 2, \( \| \beta' \|_\infty = O_{FPT}(1) \), consequently \( \| \beta \|_\infty \leq \| \beta' \|_\infty = O_{FPT}(1) \). Combining the fact that \( \| u'_1 \|_\infty = O_{FPT}(1) \), \( \| v'_j \|_\infty = O_{FPT}(1) \) and \( \| g_{\ell}(A) \|_\infty = O_{FPT}(1) \), we conclude that \( \| g - \sum_{h=1}^{h_0-1} e_h \|_\infty = O_{FPT}(1) \).}

Theorem 5 indicates that, there exists a Hilbert basis for \( 3 \)-block \( n \)-fold IP with FPT-bounded \( \ell_\infty \)-norms. The following lemma provides a slightly more compact form of decomposition, which will also be utilized later.

**Lemma 4.** There exist a set of \( q' = O_{FPT}(1) \) vectors \( S = \{ \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_q \} \) of such that for any \( y \in \ker_{Z_{[2^{2^n} + \eta]}_A}(H_0) \), there exist \( \alpha_h, \beta_k \in Z_{\geq 0} \) and at most \( 2nt_A - 1 \) vectors \( d_\ell = (0, g_\ell(E)) \) where \( g_\ell(E) \in G(E) \) such that \( \alpha_h = 0 \) if \( \tilde{e}_h^0 \nsubseteq y^0, \| \tilde{e}_h \|_\infty \leq \xi = O_{FPT}(1) \), and

\[
y = \sum_{h=1}^{q'} \alpha_h \tilde{e}_h + \sum_{\ell} \beta_\ell d_\ell. \tag{10}
\]

Furthermore, \( d_\ell \)'s lie in the same orthant and the set \( S \) can be computed in \( O_{FPT}(n^3 L) \) time where \( L \) is the length of the encoding.

**Proof.** According to Theorem 5 there exist \( e_1, e_2, \ldots, e_k \) with \( \| e_h \|_\infty \leq \xi, e_h^0 \nsubseteq y^0 \) such that \( y = \sum_{h=1}^{k} e_h \). Let \( u_1, u_2, \ldots, u_\eta \) be all the \( k \)-dimensional vectors whose \( \ell_\infty \)-norm is bounded by \( \xi \), then \( \eta = O(\xi^k) = O_{FPT}(1) \). For any \( u_j \), we pick an arbitrary \( \tilde{e}_j \) such that \( H_0 \tilde{e}_j = 0 \) and \( \tilde{e}_j^0 = u_j \). Note that such a \( \tilde{e}_j \) can be found out by solving an \( n \)-fold IP, which can be done in \( O_{FPT}(n^3 L) \) time [14]. Among \( e_1 \) to \( e_k \), we define \( K_j = \{ e_h : e_h^0 = u_j, 1 \leq h \leq k \} \). We have

\[
y = \sum_{j=1}^{\eta} \tilde{e}_j \cdot |K_j| + \sum_{j=1}^{\eta} \sum_{e_h \in K_j} (e_h - \tilde{e}_j)
\]

Since \( H_0 e_h = 0 \), it is easy to see that \( \sum_{j=1}^{\eta} \sum_{e_h \in K_j} (e_h - \tilde{e}_j) = (0, e') \) where \( e' \) is a feasible solution to \( Ex = 0 \) where \( E \) is an \( n \)-fold matrix. According to [14], there exists at most \( 2nt_A - 1 \) vectors \( g_\ell(E) \in G(E) \), \( g_\ell(E) \subseteq e' \) and \( \beta_\ell \in Z_+ \) such that \( e' = \sum_\ell \beta_\ell g_\ell(E) \). Define \( d_\ell = (0, g_\ell(E)) \), we have

\[
\sum_{j=1}^{\eta} \sum_{e_h \in K_j} (e_h - \tilde{e}_j) = \sum_{\ell} \beta_\ell d_\ell.
\]

Thus, the lemma is proved.

It is remarkable that we can further restrict that the \( \tilde{e}_h \)'s also lie in the same orthant (see Lemma 7).
4.2 A sign-compatible decomposition

We have shown in the previous subsection that any vector of $\ker_{\mathbb{Z}^{n+m}}(H_0) = \{ x \in \mathbb{Z}^{n+m} : H_0 x = 0, x \neq 0 \}$ can be expressed as a conic combination of vectors in $\ker_{\mathbb{Z}^{n+m}}(H_0)$ whose $\ell_{\infty}$-norm is bounded by $O_{FPT}(1)$. However, it is not clear how such a result can be utilized directly for an augmentation algorithm. The current augmentation algorithms for 4-block $n$-fold IP as well as other related IPs all rely on Graver basis. This is due to the fact that if there exists a feasible or optimal augmentation vector that can be decomposed into a conic combination of Graver basis elements which lie in the same orthant, then any of these Graver basis elements itself also provides a feasible augmentation. This fact is, unfortunately, no longer true if we take a conic combination of some Hilbert basis elements that do not necessarily lie in the same orthant. Towards the algorithm for 3-block $n$-fold IP, we resort to Graver basis. We show the following upper bounds on the $\ell_{\infty}$-norm of the Graver basis for 3-block $n$-fold IP.

**Theorem 6.** For any 3-block $n$-fold matrix $H_0$ and $g(H_0) \in \mathcal{G}(H_0)$, $||g(H_0)||_{\infty} \leq O_{FPT}(n^{1.2} + 1)$.

In [13], Hemmecke, Köppe and Weismantel provide two upper bounds on the $\ell_{\infty}$-norm of Graver basis element for a general 4-block $n$-fold IP, which is $O_{FPT}(n^{2.6})$ and $O_{FPT}(n^{k(A,B)})$ where $k(A,B)$ is some unknown function that is dependent on $s_A, s_B, t_A, t_B, ||A||_{\infty}, ||B||_{\infty}$. Indeed, as the existence of such a $k(A,B)$ follows from some algebraic argument, even a rough estimation of $k(A,B)$ is not clear, despite that it is highly unlikely for $k(A,B)$ to be some polynomial of the parameters. In this paper, we have established, in Theorem 2 that there exists a tight bound of $O_{FPT}(n^{1.2})$ which depends on $s_c$. An explicit upper bound that depends on $A,B$, however, is still unclear. Theorem 6 provides such an upper bound for the special case where $C$ is a zero matrix.

Following the line of arguments in previous papers [2, 13, 15, 17], it seems very difficult to get rid of the exponential dependency of $k(A,B)$. To prove the Theorem 6, we use a completely different approach.

We give a brief overview of the proof of Theorem 6. The basic idea is to show that if $||g(H_0)||_{\infty}$ is too large, then we are able to find some $z \subseteq g(H_0)$ and $H_0 z = 0$, contradicting the fact that $g(H_0)$ is a Graver basis element. Suppose $y = g(H_0)$ and $||y||_{\infty}$ is very large. The most crucial idea is that we do not search directly for $z \subseteq y$, but rather search for $z \subseteq \tilde{y}$ where $\tilde{y}$ is called a “centralization” of $y$, and then prove that such a $z$ also satisfy that $z \subseteq y$. Roughly speaking, we will divide the $n$ bricks of $y$, i.e., $y^i$ for $i = 1, 2, \ldots, n$, into $\sigma = O_{FPT}(1)$ groups $N_1, N_2, \ldots, N_\sigma$ such that for any $k \in N_j$, $\tilde{y}^k \approx \frac{1}{N_j} \sum_{i \in N_j} y^i$. Why do we need to take such a detour in the proof? The problem is that by directly arguing on $y$ we run into a bound that is similar as [13]. Therefore, we use a completely different approach – we adopt the decomposition of Theorem 5 and then modify such a decomposition into a sign-compatible one by “merging” summands. Towards this, we first prove a merging lemma (Lemma 5) which states that given a summation of a sequence of vectors, we can always divide them into disjoint subsets such that the summation of vectors in each subset becomes sign-compatible. The merging lemma can turn an arbitrary decomposition into a sign-compatible one, albeit the fact that the cardinality of each subset is exponential in the dimension of the vectors (which means the $\ell_{\infty}$-norm of the summands will explode by a factor that is exponential in the dimension). Consequently, if we directly merge the $O_{FPT}(n)$-dimensional vectors in the decomposition of Theorem 5 we get a very weak bound. To handle this, we consider an alternative sum $\tilde{y}$, which is derived by averaging multiple bricks of $y$ as we mentioned above. By altering the decomposition of $y$, we get a decomposition of $\tilde{y}$ such that the following is true: all the $n+1$ bricks of each vector-summand can be divided into $O_{FPT}(1)$ subsets where in each subset the bricks are identical. This indicates that, although we are summing up $O_{FPT}(n)$-dimensional vectors to $\tilde{y}$, it is essentially the same as summing up $O_{FPT}(1)$-dimensional vectors. Such a transformation comes at a cost – summands summing up to $\tilde{y}$ do not have $O_{FPT}(1)$-bounded $\ell_{\infty}$-norms, indeed, each vector-summand consists of $n$ bricks whose $\ell_{\infty}$-norm is $O_{FPT}(1)$, and at most 1 brick (which is a $t_A$-dimensional vector) whose $\ell_{\infty}$-norm is $O_{FPT}(n)$. Applying our merging lemma, we derive a sign-compatible decomposition of $\tilde{y}$ where the summands have an $\ell_{\infty}$-norm bounded by $O_{FPT}(n^{1.2} + 1)$. It remains
to show that, at least one vector-summand $z$ in the sign-compatible decomposition of $\vec{y}$ also satisfies that $z \sqsubseteq y$. This goes back to the definition of $\vec{y}$. We are averaging bricks of $y$, but which bricks shall we average? Each brick is a $t_A$-dimensional vector and we consider each coordinate. We set up a threshold $\Gamma$. If the absolute value of a coordinate is larger than $\Gamma$, we say it is (positive or negative) large. Otherwise it is small. Therefore, each brick can be characterized by identifying its coordinates being positive large, negative large or small (which is defined as the quantity type of a brick). We only average the bricks of the same quantity type. By doing so, we can ensure that $\vec{y}'$ is roughly sign-compatible with $y'$ – if the $j$-th coordinate of $y'$ is positive or negative large, then this coordinate of $\vec{y}'$ is also positive or negative. Hence, any $z \sqsubseteq \vec{y}'$ is almost sign-compatible with $y$ – indeed, if we can ensure additionally that the $j$-th coordinate of $z'$ is 0 as long as the $j$-th coordinate of $y'$ is small, then we can conclude that $z \sqsubseteq y$. This “if” can be proved, and we get Theorem \[6\]

### 4.2.1 A merging lemma

**Lemma 5.** Let $x_1, x_2, \cdots, x_m$ be a sequence of integers such that $x = \sum_{i=1}^m x_i$, and $|x_i| \leq \zeta$. Then the $m$ integers can be partitioned into $m'$ subsets $T_1, T_2, \cdots, T_{m'}$ satisfying that: $\bigcup_{j=1}^{m'} T_j = \{1, 2, \cdots, m\}$, and for every $1 \leq j \leq m'$ it holds that $\sum_{i \in T_j} x_i \equiv x, |T_j| \leq 6\zeta + 2$.

**Proof.** Without loss of generality we assume that $x \geq 0$ (otherwise we argue on $-x_i$'s). If $m \leq 6\zeta + 2$ the lemma is trivial. Otherwise we apply Steinitz lemma (Lemma \[3\]) to the integral sequence $x_1, x_2, \cdots, x_m$ and there exists a permutation $\pi$ such that for all $1 \leq \ell \leq m$ it holds that

$$|\sum_{i=1}^{\ell} x_{\pi(i)} - \ell - 1 \over m | x| \leq \zeta.$$ 

Now we consider the first $3\zeta + 2$ numbers $x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(3\zeta + 2)}$. There are two possibilities regarding $(3\zeta + 1)x/m$.

If $(3\zeta + 1)x/m > \zeta$, then since $-\zeta \leq \sum_{i=1}^{3\zeta + 2} x_{\pi(i)} - (3\zeta + 1)x/m \leq \zeta$, we know that $\sum_{i=1}^{3\zeta + 2} x_{\pi(i)} \geq 0$, and is consequently sign-compatible with $x$. Further notice that the summation of the remaining integers satisfies that $\sum_{i=m-3\zeta + 1}^{m} x_{\pi(i)} \geq x - (3\zeta + 1)x/m - \zeta$. Given that $m \geq 6\zeta + 2$, $x - (3\zeta + 1)x/m \geq (3\zeta + 1)x/m > \zeta$, the summation of the remaining integers is still positive.

Otherwise $(3\zeta + 1)x/m \leq \zeta$, and consequently $0 \leq (\ell - 1)x/m \leq \zeta$ for any $1 \leq \ell \leq 3\zeta + 2$. This implies that the values of the $3\zeta + 2$ numbers $\sum_{i=1}^{\ell} x_{\pi(i)}$ where $1 \leq \ell \leq 3\zeta + 2$ must lie in the set $\{-\zeta, -\zeta + 1, \cdots, 2\zeta\}$, i.e., there must exist two distinct integers $\ell_1 < \ell_2$ and $\ell_1, \ell_2 \leq 3\zeta + 2$ such that $\sum_{i=1}^{\ell_1} x_{\pi(i)} = \sum_{i=1}^{\ell_2} x_{\pi(i)}$. Consequently, $\sum_{i=1}^{\ell_2 - \ell_1} x_{\pi(i)} = 0$. Further notice that by taking out the sequence of integers $x_{\ell_1 + 1}, \cdots, x_{\ell_2}$, the summation is of the remaining integers is still $x \geq 0$.

Hence, as long as $m \geq 6\zeta + 2$, we can always select at most $3\zeta + 2$ integers whose summation is non-negative, and if we delete them, the summation of the remaining integers is still non-negative. Hence, we can carry on our argument on the remaining integers, and the lemma is proved.

We can extend the above lemma to higher dimensions using the same basic idea but a much more involved analysis.

In the following we write $O^*(x^k)$ to represent a function that is bounded by $(cx)^k$ for some constant $c$.

**Lemma 6.** Let $x_1, x_2, \cdots, x_m$ be a sequence of vectors in $\mathbb{Z}^k$ such that $x = \sum_{i=1}^m x_i$, and $||x_i||_{\infty} \leq \zeta$. Then the $m$ vectors can be partitioned into $m'$ subsets $T_1, T_2, \cdots, T_{m'}$ satisfying that: $\bigcup_{j=1}^{m'} T_j = \{1, 2, \cdots, m\}$, and for every $1 \leq j \leq m'$ it holds that $\sum_{i \in T_j} x_i \equiv x, |T_j| = O^*(\zeta^k)$.

**Proof.** Again we assume without loss of generality that $x \geq 0$ (if some of the coordinates are negative, then we take the negation of every $x_i$ and $x$ at this coordinate). For $x = (x^1, x^2, \cdots, x^k)$, we further assume that
\( x^1 \leq x^2 \leq \cdots \leq x^\kappa \) (Notice that here \( x^j \in \mathbb{Z} \)). By Steinitz lemma (Lemma 3), there exists a permutation \( \pi \) such that for all \( 1 \leq \ell \leq m \) it holds that

\[
\| \sum_{i=1}^{\ell} x_{\pi(i)} - \frac{\ell - \kappa}{m} x \|_\infty \leq \zeta.
\]

For simplicity, we reorder all the vectors such that \( x_{\pi(i)} \) is at the \( i \)-th location, i.e., we assume that the given vectors \( x_i \) satisfy that

\[
\| \sum_{i=1}^{\ell} x_i - \frac{\ell - \kappa}{m} x \|_\infty \leq \zeta.
\]  

(11)

Our goal is to show the following claim:

**Claim 3.** There always exists a subset \( T \) such that, \( |T| = O^*(\zeta^\zeta) \), \( \sum_{i \in T} x_i \subseteq x \) and \( x - \sum_{i \in T} x_i \geq 0 \).

If the claim is true, we can iteratively apply it to cut the whole sequence of vectors into subsets \( T_1, T_2, \ldots, T_m \) and Lemma 4 is proved.

To prove Claim 3, we need the following two claims. For simplicity, we say a subset \( T \) is conformal if \( \sum_{i \in T} x_i \subseteq x \) and \( x - \sum_{i \in T} x_i \geq 0 \).

**Claim 4.** For any \( 1 \leq j \leq \kappa \), if there exists some \( \mu_j \) such that \( \frac{\mu_j}{m} x^j > 2\zeta \geq \frac{\mu_{j-1}}{m^j} x^j \) and \((\mu_j - 1) x^j / x^j > \kappa + (6\zeta + 1) \zeta \), then there exists a subset \( T \) such that \(|T| \leq 3(6\zeta + 1) \zeta \mu_j + \kappa \) and \( T \) is conformal, i.e., \( \sum_{i \in T} x_i \subseteq x \) and \( x - \sum_{i \in T} x_i \geq 0 \).

Proof of Claim 4: If \( m \leq 3(6\zeta + 1) \zeta \mu_j + \kappa \), then we take \( T = \{1, 2, \ldots, m\} \). In the following we assume that \( m > 3(6\zeta + 1) \zeta \mu_j + \kappa \). Recall that \( x^1 \leq x^2 \leq \cdots \leq x^\kappa \), whereas \( \frac{\mu_j}{m} x^h > 2\zeta \) for any \( h \geq j \). Consider an arbitrary subsequence of vectors whose length is \( \mu_j \), say, \( x_{i_0}, x_{i_0+1}, \ldots, x_{i_0+\mu_j-1} \). By Eq (3), we have

\[
\| \sum_{i=1}^{i_{0-1}} x_i - \frac{i_0 - 1 - \kappa}{m} x \|_\infty \leq \zeta, \quad \text{and} \quad \| \sum_{i=1}^{i_{0+\mu-1}} x_i - \frac{i_0 + \mu - 1 - \kappa}{m} x \|_\infty \leq \zeta.
\]

(12)

Consequently, for any \( h \geq j \), it follows that

\[
\sum_{i=1}^{i_{0-1}} x_i^h \leq \frac{i_0 - 1 - \kappa}{m} x^h + \zeta, \quad \text{and} \quad \sum_{i=1}^{i_{0+\mu-1}} x_i^h \geq \frac{i_0 + \mu - 1 - \kappa}{m} x^h - \zeta.
\]

Thus,

\[
\sum_{i=i_0}^{i_{0+\mu-1}} x_i^h \geq \frac{\mu_j}{m} x^h - 2\zeta \geq \frac{\mu_j}{m} x^h - 2\zeta > 0, \quad \forall h \geq j
\]

(13)

This means, the summation of any adjacent \( \mu \geq \mu_j \) vectors satisfies that the sum is positive on every \( h \)-th coordinate for \( h \geq j \).

Meanwhile, by Eq (12) we have

\[
\sum_{i=1}^{i_{0-1}} x_i^h \geq \frac{i_0 - 1 - \kappa}{m} x^h - \zeta, \quad \text{and} \quad \sum_{i=1}^{i_{0+\mu-1}} x_i^h \leq \frac{i_0 + \mu - 1 - \kappa}{m} x^h + \zeta.
\]

Thus,

\[
\sum_{i=i_0}^{i_{0+\mu-1}} x_i^h \leq \frac{\mu_j}{m} x^h + 2\zeta, \quad \forall h \geq j.
\]

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Meanwhile, we have
\[
\sum_{i=1}^{m} x_i^h \geq \frac{m - \kappa}{m} x^h - \zeta \geq \frac{\mu}{m} \cdot \frac{m - \kappa}{\mu} - \zeta, \quad \forall h \geq j.
\]
Thus,
\[
\sum_{i=1}^{m} x_i^h - \sum_{i=0}^{\ell_0 - 1} x_i^h \geq \frac{\mu}{m} \cdot \left( \frac{m - \kappa}{\mu} - 1 \right) - 3\zeta, \quad \forall h \geq j.
\] (14)

Recall that \( \frac{\mu}{m} x^h > 2\zeta \), as long as \( m - \kappa \geq 3\mu \), we know that \( \sum_{i=1}^{m} x_i^h - \sum_{i=0}^{\ell_0 - 1} x_i^h > 0 \) for all \( h \geq j \).

Next we consider the \( h \)-th coordinate for \( h < j \). Recall that \( \frac{\mu - 1}{m} x^j \leq 2\zeta \). As \( (\mu - 1) \frac{x^j}{x^h} > \kappa + (6\zeta + 1)^{j-1} \mu_j \), it follows directly that
\[
\kappa + (6\zeta + 1)^{j-1} \mu_j x^h \leq 2\zeta, \quad \forall h \leq j - 1.
\]

Now we consider the \( 1 + (6\zeta + 1)^{j-1} \) vectors \( \sum_{i=1}^{\ell} x_i \) for \( \ell \in \mathcal{L}_j \) where \( \mathcal{L}_j = \{ \kappa, \kappa + \mu_j, \kappa + 2\mu_j, \ldots, \kappa + (6\zeta + 1)^{j-1} \mu_j \} \). For any \( \ell \in \mathcal{L}_j \), it is clear that
\[
|\sum_{i=1}^{\ell} x_i| \leq \frac{\ell - \kappa}{m} x^h + 2\zeta \leq 3\zeta, \quad \forall h \leq j - 1
\]
that is, \( \sum_{i=1}^{\ell} x_i \in \{-3\zeta, -3\zeta + 1, \ldots, 3\zeta \} \) for all \( \ell \in \mathcal{L}_j \) and \( h \leq j - 1 \). Hence, if we consider the projection of \( \sum_{i=1}^{\ell} x_i \) onto its first \( j - 1 \) coordinates, this projection lies within \( \{-3\zeta, -3\zeta + 1, \ldots, 3\zeta \}^{j-1} \), implying that there must exist \( \ell_1 < \ell_2 \) such that \( \sum_{i=1}^{\ell_1} x_i \) and \( \sum_{i=1}^{\ell_2} x_i \) have the same projection. Hence, the first \( j - 1 \) coordinates of \( \sum_{i=1}^{\ell_2} x_i \) are all 0. Furthermore, we observe the followings: 1) \( \ell_2 - \ell_1 > \mu_j \), whereas for \( h \geq j \), the \( h \)-th coordinate of \( \sum_{i=1}^{\ell} x_i \) is positive according to Eq (13). 2) \( \ell_2 - \ell_1 \leq (6\zeta + 1)^{j-1} \mu_j \) and \( m \geq 3(6\zeta + 1)^{j-1} \mu_j + \kappa \), whereas for \( h \geq j \), the \( h \)-th coordinate of \( \sum_{i=1}^{\ell} x_i - \sum_{i=1}^{\ell_2} x_i \) is also positive according to Eq (14). Hence, \( \sum_{i=1}^{\ell_2} x_i \in \mathcal{L} \) and \( \sum_{i=1}^{m} x_i - \sum_{i=1}^{\ell_2} x_i \geq 0 \), i.e., by taking \( T = \{ \ell_1 + 1, \ell_1 + 2, \ldots, \ell_2 \} \), Claim 3 holds true.

Now we come to the proof of Claim 3.

Proof of Claim 3. We prove the claim by induction on the following hypothesis.

Statement: For \( 1 \leq j \leq \kappa \), either there exists some \( T \) which is conformal (i.e., \( \sum_{i \in T} x_i \subseteq \mathcal{X} \) and \( \mathcal{X} = \sum_{i \in T} x_i \geq 0 \)) and \( |T| = O^*(\zeta^{(\kappa-j+1)\kappa}) \), or there exists some \( \mu_j = O^*(\zeta^{(\kappa-j+1)\kappa}) \) such that \( \frac{\mu_j}{m} x^j > 2\zeta \geq \frac{\mu_j - 1}{m} x^j \).

We first prove the statement for \( j = k \). Taking \( \mu_k = (6\zeta + 1)^{k-1} + \kappa = O^*(\zeta^k) \). There are two possibilities. If \( \frac{\mu_k}{m} x^k \leq 2\zeta \), then \( \frac{\mu_k}{m} x^j \leq 2\zeta \) for all \( 1 \leq j \leq k \). Consequently, for \( \ell \in \mathcal{L} = \{ \kappa, \kappa + 1, \ldots, \mu_k \} \), we have
\[
||\sum_{i=1}^{\ell} x_i||_m \leq \frac{\mu_k}{m} x^k + \zeta \leq 3\zeta, \quad \forall i \in \mathcal{L}
\]
i.e., \( \sum_{i=1}^{\ell} x_i \in \{-3\zeta, -3\zeta + 1, \ldots, 3\zeta \} \). Since \( |\mathcal{L}| = (6\zeta + 1)^{k+1} + 1 \), there exist \( \ell_1 < \ell_2 \) and \( \ell_1, \ell_2 \in \mathcal{L} \) such that \( \sum_{i=1}^{\ell_1} x_i = \sum_{i=1}^{\ell_2} x_i \), i.e., \( \sum_{i=1}^{\ell_1} x_i = 0 \). Taking \( T = \{ \ell_1 + 1, \ldots, \ell_2 \} \), the statement is true.

Otherwise, \( \frac{\mu_k}{m} x^k > 2\zeta \). Then there exists some \( \mu_k \leq \mu_k' = O^*(\zeta^k) \) such that \( \frac{\mu_k}{m} x^k > 2\zeta \geq \frac{\mu_k - 1}{m} x^k \). That is, the statement is also true.

Suppose the statement (hypothesis) holds for \( j \geq j_0 \), we prove it for \( j = j_0 - 1 \). According to the statement, either there exists some \( T \) satisfying Claim 3 with \( |T| = O^*(\zeta^{(\kappa-j_0+1)\kappa}) \), or there exists some
\[ \mu_{j_0} = O^*(\zeta^{(\kappa-j_0+1)\kappa}) \text{ such that } \frac{\mu_{j_0}}{\mu} x^{j_0} > 2 \zeta \geq \frac{\mu_{j_0}-1}{\mu} x^{j_0}. \] If the former case is true, then obviously the same \( T \) satisfies that \( |T| \leq O^*(\zeta^{(\kappa-j_0+2)\kappa}) \), implying that the statement is true for \( j = j_0 - 1 \). Hence, from now on we assume the latter case is true. According to Claim 4 if \( (\mu_{j_0} - 1) x^{j_0} > \kappa + (6 \zeta + 1) \mu_{j_0} \), then there exists a subset \( T \) which is conformal and \( |T| \leq (6 \zeta + 1) \mu_{j_0} \mu_{j_0} \). Plugging in \( \mu_{j_0} = O^*(\zeta^{(\kappa-j_0+1)\kappa}) \), we have \( |T| = O^*(\zeta^{(\kappa-j_0+2)\kappa}) \), that is, \( (\mu_{j_0} - 1) x^{j_0} > \kappa + (6 \zeta + 1) \mu_{j_0} \). Thus, in the following we assume that \( (\mu_{j_0} - 1) x^{j_0} \leq \kappa + (6 \zeta + 1) \mu_{j_0} \). Notice that \( x^j / m \leq \zeta \) (as \( ||x_j||_{\infty} \leq \zeta \)). According to \( \frac{\mu_{j_0}}{m} x^{j_0} > 2 \zeta \), we know \( \mu_{j_0} \geq 2 \), whereas
\[
\frac{x^{j_0}}{\mu_{j_0} - 1} \leq \frac{\kappa + (6 \zeta + 1) \mu_{j_0}}{\mu_{j_0} - 1},
\]
and consequently
\[
\frac{\mu_{j_0}}{\mu_{j_0} - 1} \cdot \frac{\kappa + (6 \zeta + 1) \mu_{j_0}}{m} x^{j_0} \leq 2 \zeta.
\]
Since \( \mu_{j_0} = O^*(\zeta^{(\kappa-j_0+1)\kappa}) \), we can conclude that \( \frac{\mu_{j_0}}{\mu_{j_0} - 1} \cdot \kappa + (6 \zeta + 1) \mu_{j_0} = O^*(\zeta^{(\kappa-j_0+2)\kappa}) \), hence, there exists some \( \mu_{j_0-1} = O^*(\zeta^{(\kappa-j_0+2)\kappa}) \) such that \( \frac{\mu_{j_0}-1}{m} x^{j_0-1} \geq \frac{\mu_{j_0}-1}{m} x^{j_0-1} \). Thus, the statement holds for \( j = j_0 - 1 \).

Now we have proved the statement for all \( 1 \leq j \leq \kappa \). Taking \( j = 1 \), either there exists some subset \( T \) which is conformal and \( |T| = O^*(\zeta^{\kappa^2}) \), and Claim 3 is proved. Or there exists some \( \mu_1 = O^*(\zeta^{\kappa^2}) \) such that \( \frac{\mu_{j_0}}{m} x^1 > 2 \zeta \). As \( x^1 \leq x^j \) for all \( j \leq \kappa \), it holds that \( \frac{\mu_{j_0}}{m} x^1 > 2 \zeta \). There are two possibilities. If \( m \leq 2 \mu_1 + \kappa = O^*(\zeta^{\kappa^2}) \), we simply take \( T = \{1, 2, \cdots, m\} \). Otherwise, we have
\[
|| \sum_{i=1}^{m+1} x_i - \frac{\mu_1}{m} x ||_{\infty} \leq \zeta, \quad \text{and} \quad || \sum_{i=1}^{m} x_i - \frac{m - \kappa}{m} x ||_{\infty} \leq \zeta.
\]
Consequently, for any \( 1 \leq j \leq \kappa \), it follows that
\[
0 \leq \frac{\mu_1}{m} x^j - \zeta \leq \frac{\mu_1}{m} x^j \leq \frac{\mu_1}{m} x^1 + \zeta, \quad \text{and} \quad \sum_{i=1}^m x^j_i \geq \frac{m - \kappa}{m} x^j - \zeta \geq 2 \frac{\mu_1}{m} x^j - \zeta > \frac{\mu_1}{m} x^j + \zeta.
\]
Hence, taking \( T = \{1, 2, \cdots, \mu_1 + \kappa\} \) we have that \( \sum_{i \in T} x_i \equiv x \) and \( \sum_{i=1}^m x_i - \sum_{i \in T} x_i \geq 0 \), and \( |T| = O^*(\zeta^{\kappa^2}) \), i.e., Claim 3 is proved.

Iteratively applying Claim 3 Lemma 6 is proved.

**Remark.** It is notable that a weaker version of Lemma 6 can also be proved, in a much simpler way, by iteratively applying Lemma 3 to each individual dimension. However, by doing so we get an upper bound of \( O^*(\zeta^{2\kappa}) \) on \( |T_j| \)'s, which is much worse.

### 4.2.2 A decomposition in two or thants

Towards the proof of Theorem 6 we need the following lemma which gives an “almost” sign-compatible decomposition.

**Lemma 7.** For any \( y \in \text{ker} \mathbb{Z}_m \cdot \mu \) (H0), there exist \( q = O_{FPP}(1) \) vectors \( \epsilon_{h}, \alpha_0, \beta_\ell \in \mathbb{Z}_+ \), and at most \( 2m - 1 \) vectors \( d_\ell = (0, g_\ell(E)) \) where \( g_\ell(E) \in \mathbb{G}(E) \) such that \( \epsilon_{h} \subseteq y \), \( ||\epsilon_{h}||_{\infty} \leq \zeta' = O_{FPP}(1) \), \( y = \sum_{h=1}^q \alpha_0 \epsilon_{h} + \sum_\ell \beta_\ell d_\ell \), and moreover, all the \( \epsilon_{h} \)'s lie in the same orthant, and all the \( d_\ell \)'s lie in the same orthant.
Note that \( e_h \) and \( d_\ell \) do not necessarily lie in the same orthant. The \( \xi \) in this lemma is the same as that in Theorem 5.

Towards the proof, we first show a simpler result, which will also be utilized later.

Proof of Lemma 7. Continuing the proof of Lemma 4, we consider the \( \bar{e}_j \)'s where \( K_j \neq \emptyset \). If they are all sign-compatible, the lemma is proved. Otherwise we try to apply Lemma 6 to the sequence of vectors that consists of \( |K_j| \) copies of \( \bar{e}_j \). Note that we cannot directly apply the lemma as \( \bar{e}_j \)'s have very high dimensions. However, if we consider the bricks \( \bar{e}_j \), since \( \| \bar{e}_j \|_\infty \leq \xi \), there are at most \( \xi^{O(t_A)} \) different kinds of bricks. Consequently, if we consider the vectors that consists of the \( \eta \) bricks \( \{ \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_\eta \} \), there are at most \( \theta = \xi^{O(\eta n)} = O_{FPT}(1) \) different kinds of vectors for \( 1 \leq i \leq n \). We let these vectors be \( \phi_1, \phi_2, \ldots, \phi_\theta \). Now we consider the “reduced” vectors \( Rd(\bar{e}_j) \) that only consists of distinct vectors. More precisely, we define the set of indices \( I_{n_j} = \{ i : (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_\eta) = \phi_j \} \). For each \( I_{n_j} \), we pick an arbitrary \( i_j \in I_{n_j} \) and define a \( (t_B + \theta t_A) \)-dimensional vector \( Rd(e_h) = (\bar{e}^0_h, \bar{e}^1_h, \bar{e}^2_h, \ldots, \bar{e}^\theta_h) \). Note that \( \bar{e}_h \) is simply a vector that copies the bricks of \( Rd(\bar{e}_h) \) for multiple times. Now we consider the sequence that consists of \( |K_j| \) copies of \( Rd(\bar{e}_j) \). Applying Lemma 6, we can divide these vectors into disjoint subsets \( S_1, S_2, \ldots, S_m \) such that the summation of vectors in each subset is sign-compatible, and each subset has cardinality bounded by \( \xi^{O(t_B+\theta t_A)} \). Consequently, \( \sum_{h \in S_j} \bar{e}_h \)'s are also sign-compatible. Let \( e_j = \sum_{h \in S_j} \bar{e}_h \), we have

\[
y = \sum_{j=1}^{m} e_j + \sum_{\ell} \beta_\ell d_\ell.
\]

It remains to show there are at most \( O_{FPT}(1) \) different kinds of \( e_j \)'s. To see this, consider the reduced vector \( e_j' = \sum_{i \in S_j} \bar{e}_i \) and notice that \( e_j \) is duplicating the bricks of \( e_j' \) at locations indicated by \( I_{n_j} \). Hence, it suffices to show that there are at most \( O_{FPT}(1) \) different kinds of \( e_j' \)'s. Note that \( \| e_j' \|_\infty \leq \xi \cdot \xi^{O(t_B+\theta t_A)} \), and it is of \( (t_B + \theta t_A) \)-dimensional. Hence, there are only \( O_{FPT}(1) \) different kinds of \( e_j' \)'s, and the lemma is proved.

Our next goal is to further make \( e_h \)'s and \( d_\ell \)'s sign-compatible. Given \( y \) and a decomposition satisfying Lemma 7 we call \( e_h \)'s as the principle vectors and \( d_\ell \)'s as the add-ons. The basic idea is to merge principle vectors and add-ons such that they become sign-compatible, and we will mainly use Lemma 6 to achieve this. However, there is a problem in applying Lemma 6 directly as the dimension is too high. Again we try to utilize the idea in the proof of Lemma 7 note that principle vectors are good in the sense that they can be reduced to lower dimensional vectors such that they are duplicating the bricks of lower dimensional vectors in fixed locations. While add-ons do not have such a nice structure, they are “sparse” according to Lemma 7 that is, only an \( O_{FPT}(1) \) number of their bricks are non-zero. This will allow us to achieve the desired merging.

4.2.3 Defining types of bricks

Prior to our merging process, let \( \Gamma \) be some positive integer to be determined later. We will eventually set its value within \( O_{FPT}(1) \), but for ease of analysis on its value at the end, we will first treat it as an unbounded parameter and write \( O_{FPT}(\Gamma) \) in the following.

We first define a quantity type. For every \( t_A \)-dimensional vector \( x = (x_1, x_2, \ldots, x_{t_A}) \), we compare every coordinate \( x_j \) with \( \Gamma \). If \( x_j \leq \Gamma \), we say the \( j \)-th coordinate of \( x \) is small. Otherwise, we say it is large. A large coordinate may be positive or negative, hence each coordinate of \( x \) can be of three kinds: small, positive large and negative large. The quantity type of each \( x \) is defined as a \( t_B \)-dimensional vector which stores the kind of each \( x_j \). It is obvious there are at most \( 3^{t_A} \) different quantity types.

Next, we define a principle type for every \( y^j \). Note that \( \| e_j \|_\infty \leq \xi^j \). For each \( 1 \leq i \leq n \), we define the vector \( (e^i_1, e^i_2, \ldots, e^i_q) \) as the principle type of \( y^j \). There are at most \( (\xi^j)^{O(t_A)} = O_{FPT}(1) \) different principle types.
Consider the bricks of \( y \). \( y^t \)'s with the same quantity type and principle type are called to have the same type. There are at most \( 6^\lambda \cdot (\xi^t)_{O(d \alpha)} = O_{FPT}(1) \) different types. We pick an arbitrary brick, say, brick 1 as a specific brick and let \( N_1 = \{1\} \). For the remaining bricks (brick 2 to brick \( n \)), we divide them into \( \sigma - 1 = O_{FPT}(1) \) sets such that bricks in the same set have the same type, i.e., we let \( N_2, \ldots, N_\sigma \) be the set of indices of the bricks that have the same type, and let \( n_j = |N_j| \). For simplicity, we reorder the bricks of \( y \) such that \( N_j = \{i_{j-1} + 1, i_{j-1} + 2, \ldots, i_{j-1} + n_j\} \) where \( i_{j-1} = n_1 + n_2 + \cdots + n_{j-1} \).

### 4.2.4 Centralization

According to Lemma 1 every \( d_h^j \), as well as \( \sum_h d_h^j \), is the summation of at most \( O_{FPT}(1) \) elements of \( G(A) \). Let \( v_1, v_2, \ldots, v_\lambda \) be all the non-zero \( t_A \)-dimensional vectors that \( d_h^j \) can take. For simplicity, we allow \( d_h^j \)'s to be the same and rewrite Eq (10) as

\[
y = \sum_{h=1}^{q} \alpha_h e_h + \sum_h d_h.
\]

Note that in the above summation we simply add each \( d_h \) separately by \( \beta_h \) times.

For ease of description, let us now take a scheduling point of view. We assume there are \( n \) machines. The \( t_B \)-dimensional load of machine \( i \) is defined by \( y^i \). This load is contributed by two parts, \( \sum_{h=1}^{q} \alpha_h e_h^i \) and \( \sum_h d_h^i \). For every \( i \in N_j \), the first part \( \sum_{h=1}^{q} \alpha_h e_h^i \) is the same, while the second part might be different. We can view each \( v_k \) as a \( t_A \)-dimensional job. Obviously there are only \( \lambda = O_{FPT}(1) \) different kinds of jobs, albeit that each job may have multiple identical copies. Let \( \psi(j, k) \) be the total number of copies of job \( k \) on machines in \( N_j \).

We define a vector \( y_f \) such that \( y_f^0 = y^0 \), \( y_f^1 = y^1 \) and

\[
y_f^k = \frac{1}{n_j} \cdot \sum_{i \in N_j} y^i = \sum_{h=1}^{q} \alpha_h e_h^i + \frac{1}{n_j} \cdot \sum_{i \in N_j} \sum_h d_h^i, \quad \forall k \in N_j, 2 \leq j \leq \sigma
\]  

(15)

Ideally, we would like to argue on \( y_f \). However, \( y_f \) may be fractional. Therefore, in the following we define an integral vector \( \tilde{y} \approx y_f \) and call it the centralization of \( y \).

We give the precise definition of \( \tilde{y} \) as follows. Let \( \psi(j, k) \) be the number of copies of job \( v_k \) on machines in \( N_j \). We (almost) evenly distribute these jobs among machines such that every machine gets \( \lfloor \psi(j, k)/n_j \rfloor \) or \( \lceil \psi(j, k)/n_j \rceil \) copies. To make it unique, we further restrict that machines with smaller indices in \( N_j \) always have the same or more number of copies. By doing so, we construct a new vector \( \tilde{y} \). Note that \( \tilde{y} \) consists of the same number of jobs as \( y \), only that jobs are distributed among machines in a different (more even) way.

As we re-distribute \( v_k \)'s such that for every machine in \( N_j \), the number of copies of each \( v_k \) differs by at most 1, the following lemma is straightforward.

**Lemma 8.**

\[
||\tilde{y}^j - y_f^j||_\infty \leq \sum_{k=1}^{\lambda} ||v_k||_\infty.
\]

### 4.2.5 Decomposition of \( \tilde{y} \)

We create new \( (t_B + nt_A) \)-dimensional vectors in the following way. For simplicity, we define \( \psi^\lambda(j, k) = \lfloor \psi(j, k)/n_j \rfloor \) and \( \psi^\varphi(j, k) = \psi(j, k) - n_j \psi^\lambda(j, k) \), i.e., they are the quotient and residue of \( \psi(j, k) \) divided
by \( n_j \), respectively. For every \( 2 \leq j \leq \sigma \), we create \( \psi^q(j,k) \) copies of a vector \( \mathbf{md}(j,k) \) and one copy of \( \overline{\mathbf{md}}(j,k) \) such that

\[
\mathbf{md}^i(j,k) = \begin{cases} 
\mathbf{v}_k, & i \in N_j \\
-n_j \mathbf{v}_k, & i = 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\overline{\mathbf{md}}^i(j,k) = \begin{cases} 
\mathbf{v}_k, & 1 \leq i \leq \psi^q(j,k) \\
-\psi^q(j,k) \cdot \mathbf{v}_k, & i = 1 \\
0, & \text{otherwise}
\end{cases}
\]

Using the above notations, it is clear that for any \( i \geq 2 \), \( \tilde{y}^i \) consists of \( \psi^q(j,k) \) copies of \( \mathbf{md}^i(j,k) \) and one copy of \( \overline{\mathbf{md}}(j,k) \), i.e., we have the following:

\[
\tilde{y}^i = \sum_{h=1}^q \alpha_h \mathbf{e}_h^i + \sum_{j=1}^\sigma \sum_{k=1}^\lambda \left( \psi^q(j,k) \cdot \mathbf{md}^i(j,k) + \overline{\mathbf{md}}^i(j,k) \right), \quad \forall i \geq 2 \tag{16}
\]

The above equation is also true for \( i = 0 \) as \( \mathbf{md}^0(j,k) = \overline{\mathbf{md}}^0(j,k) = 0 \) for every \( 1 \leq j \leq \sigma \), \( 1 \leq k \leq \lambda \).

Furthermore, we have the following observations which follow directly from the definitions of \( \mathbf{md}(j,k) \) and \( \overline{\mathbf{md}}(j,k) \).

**Observation 2.** \( H_0 \cdot \mathbf{md}(j,k) = 0 \) and \( H_0 \cdot \overline{\mathbf{md}}(j,k) = 0 \) for all \( 1 \leq j \leq \sigma \) and \( 1 \leq k \leq \lambda \).

**Observation 3.** For any \( i = 0 \) or \( 2 \leq i \leq n \), \( \mathbf{md}(j,k) = \mathcal{O}_{\text{FPT}}(1) \), \( \overline{\mathbf{md}}(j,k) = \mathcal{O}_{\text{FPT}}(1) \); For \( i = 1 \), \( \mathbf{md}^1(j,k) = \mathcal{O}_{\text{FPT}}(n) \), \( \overline{\mathbf{md}}^1(j,k) = \mathcal{O}_{\text{FPT}}(n) \).

It is clear that Eq (16) does not necessarily hold for \( i = 1 \). Let us consider

\[
\eta = \tilde{y}^1 - \sum_{h=1}^q \alpha_h \mathbf{e}_h^1 - \sum_{j=1}^\sigma \sum_{k=1}^\lambda \left( \psi^q(j,k) \cdot \mathbf{md}^1(j,k) + \overline{\mathbf{md}}^1(j,k) \right).
\]

We have the following lemma.

**Lemma 9.** \( D \eta = 0 \).

**Proof.** Note that \( H_0 \mathbf{d}_i = 0 \) for each \( \mathbf{d}_i \), whereas \( D \sum_{h=1}^n \mathbf{d}_h^i = 0 \). Since \( \tilde{y} \) is constructed from \( y \) by re-distributing the bricks \( \mathbf{d}_h^i \) (i.e., by shifting it from brick \( i \) to brick \( i' \)), it holds that

\[
D \sum_{i=1}^n (\tilde{y}^i - \sum_{h=1}^q \mathbf{e}_h^i) = 0.
\]

Plugging in Eq (16), we have

\[
0 = D\tilde{y}^1 + D \sum_{i=2}^n \tilde{y}^i - D \sum_{i=1}^n \sum_{h=1}^q \mathbf{e}_h^i
\]

\[
= D\tilde{y}^1 + D \sum_{i=2}^n \left( \sum_{h=1}^q \alpha_h \mathbf{e}_h^i + \sum_{j=1}^\sigma \sum_{k=1}^\lambda \left( \psi^q(j,k) \cdot \mathbf{md}^1(j,k) + \overline{\mathbf{md}}^1(j,k) \right) \right) - D \sum_{i=1}^n \sum_{h=1}^q \mathbf{e}_h^i
\]

\[
= D\tilde{y}^1 - D \sum_{h=1}^q \mathbf{e}_h^1 + D \sum_{j=1}^\sigma \sum_{k=1}^\lambda \left( -\psi^q(j,k) \cdot \mathbf{md}^1(j,k) - \overline{\mathbf{md}}^1(j,k) \right) = D\eta.
\]

Here the third equation makes use of the fact that \( \mathbf{md}^1(j,k) = -\sum_{i=2}^n \mathbf{md}^i(j,k) \) and \( \overline{\mathbf{md}}^1(j,k) = -\sum_{i=2}^n \overline{\mathbf{md}}^i(j,k) \).

\hfill \Box

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Recall that by definition $y^1 - \sum_{h=1}^q e^1_h$ is a weighted sum of $v_k$’s, and so is $\sum_{j=1}^\lambda \sum_{k=1}^\lambda \left( \psi^j(j,k) \cdot \text{md}^1(j,k) + \overline{\text{md}}^1(j,k) \right)$. Hence, $\eta$ is also a weighted sum of $v_k$’s, and we let $\eta = \sum_k \gamma_k v_k$ where $\gamma_k \in \mathbb{Z}$ for $1 \leq k \leq \lambda$. According to Lemma 9, we have that

$$\sum_{k=1}^\lambda \gamma_k \cdot Dv_k = 0.$$ 

Equivalently, the above equation can be written as

$$(\gamma_1, \gamma_2, \ldots, \gamma_\lambda) \cdot [Dv_1, Dv_2, \ldots, Dv_\lambda] = 0.$$ 

Consequently, if we define the matrix $DV = [Dv_1, Dv_2, \ldots, Dv_\lambda]$, which is an $O_{FPT}(1) \times O_{FPT}(1)$ matrix, then there exist $\gamma'_k \in \mathbb{Z}_+$ and $g_k(DV) \in G(DV)$, $g_k(DV) \subseteq (\gamma_1, \gamma_2, \ldots, \gamma_k)$ such that

$$(\gamma_1, \gamma_2, \ldots, \gamma_\lambda) = \sum_{h=1}^\omega g_k(DV),$$ 

where $\omega \leq |G(DV)| = O_{FPT}(1)$. Consequently, we have

$$\eta = \sum_{h=1}^\omega \gamma'_h \left( \sum_{k=1}^\lambda g^k_h(DV) \cdot v_k \right),$$ 

Note that here each $g^k_h(DV) \in \mathbb{Z}$ is the $k$-th coordinate of $g_k(DV)$. Furthermore, $||g_k(DV)||_\infty = O_{FPT}(1)$.

We define new vectors $od(h)$ such that

$$od^i(h) = \begin{cases} \sum_{k=1}^\lambda g^k_h(DV)v_k, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$ 

Recall that $A v_k = 0$ and $\sum_{k=1}^\lambda g^k_h(DV) \cdot Dv_k = 0$, we have the following observation.

Observation 4. $||od(h)||_\infty = O_{FPT}(1)$ and $H_0 \cdot od(h) = 0$ for all $1 \leq h \leq \omega$.

Now we derive the following decomposition of $\tilde{y}$:

$$\tilde{y} = \sum_{h=1}^q \alpha_h e_h + \sum_{j=1}^\sigma \sum_{k=1}^\lambda \left( \psi^j(j,k) \cdot \text{md}^1(j,k) + \overline{\text{md}}^1(j,k) \right) + \sum_{h=1}^\omega \gamma'_h \cdot od(h),$$

(17)

4.2.6 A sign-compatible decomposition of $\tilde{y}$

We will find a sign-compatible decomposition of $\tilde{y}$ in this subsection, and show in the next subsection that at least one element of the decomposition lie in the same orthant of $\tilde{y}$.

Recall Eq (17). We observe that all the vectors involved have a nice structure in the sense that they can be divided into $O_{FPT}(1)$ segments where every segment consists of identical bricks. More precisely, for every $1 \leq j \leq \sigma$, let $\pi_j$ be the permutation of $\{1, 2, \ldots, k\}$ such that the $\lambda$ residues can be ordered as $\psi^j(j, \pi_j(1)) \leq \psi^j(j, \pi_j(2)) \leq \cdots \leq \psi^j(j, \pi_j(\lambda))$. Additionally, we define $\psi^j(j, \pi_j(0)) = t_{j-1} + 1$ for $j \geq 2$, $\psi^j(j, \pi_j(\lambda + 1)) = t_j - 1$ and $\psi^j(1, \pi_j(0)) = 2$ (as machine 1 is special and should be excluded). We can divide the $n + 1$ bricks of a $((t_B + n_{\text{A}})$-dimensional vector into $2 + 2(\lambda + 1) \sigma$ groups as follows:

- Group 0 consists of brick 0 (the first $t_B$ dimensions), which is 0 for all the $\text{md}(j, k)$ and $\overline{\text{md}}(j, k)$.
- Group 1 consists of only machine (brick) 1.
For $1 \leq j \leq \sigma$ and $1 \leq k \leq \lambda + 1$, Group $k + 1 + (j - 1)(\lambda + 1)$ consists of brick $\psi^r(j, \pi(k - 1)) + 1$ to brick $\psi^r(j, \pi(k))$.

Hence, each vector is divided into $2 + 2(\lambda + 1)\sigma$ segments where each segment contains its bricks within one group. See the following figure as an illustration of the grouping. Here circles of different colors represent different $v_k$'s. Note that if we take a “snapshot” of any vector ($e_h$ or $md(j, k)$) on the bricks within a group (see the bricks among two adjacent red lines in the figure), we see that all of these bricks are identical (for otherwise some of the residues shall lie within the indices of these bricks, which contradicts the grouping). More precisely, we have the following.

**Observation 5.** Let $Gr_\ell$ be the indices of bricks in Group $\ell$, then for every $i_1, i_2 \in Gr_\ell$, we have $e_{i_1}^h = e_{i_2}^h$ and $md^{i_1}(j, k) = md^{i_2}(j, k)$ for $1 \leq h \leq q$, $1 \leq j \leq \sigma$, $1 \leq k \leq \lambda$.

Furthermore, notice that $Gr_\ell$'s is a further sub-division of $N_1, N_2, \ldots, N_\sigma$, hence we have the following observation.

**Observation 6.** For any $i_1, i_2 \in Gr_\ell$, $y^{i_1}$ and $y^{i_2}$ have the same type.

Now we are able to define reduced vectors. For $z = e_h$ or $md(j, k)$ or $md(j, k)$ or $od(h)$ or $\tilde{y}$, we define $Rd(z)$ as a $(t_B + t_A + 2(\lambda + 1)\sigma t_A)$-dimensional vector that consists of $2 + 2(\lambda + 1)\sigma$ bricks where the $\ell$-th brick $Rd^\ell(z)$ equals any brick of $z$ in the group $Gr_\ell$ (as they are the same by Observation 5). Furthermore, Eq (17) implies the following:

$$Rd(\tilde{y}) = \sum_{i=1}^q \alpha_i Rd(e_h) + \sum_{j=1}^\lambda \sum_{k=1}^\sigma (\psi^d(j, k) \cdot Rd(md(j, k)) + Rd(md(j, k))) + \sum_{h=1}^\omega \gamma_h' \cdot Rd(od(h)). \quad (18)$$

If we want to make the rightside of Eq (17) into a sign-compatible summation, it suffices to make the above Eq (18) into a sign-compatible summation, and this is achievable by utilizing Lemma 6. To derive a good bound, we will apply Lemma 6 twice in a separate way.

By Observation 3 we have the following.
Consequently, \( \text{Rd} = \sum_i \text{Rd}(z_i) \).

We define \( \text{Rd}(x)[\overline{1}] \) as the projection of the vector \( \text{Rd}(x) \) onto the subspace by excluding \( \text{Rd}^1(x) \). Hence, we have

\[
\text{Rd}(\overline{y})[\overline{1}] = \sum_i \text{Rd}(z_i)[\overline{1}].
\]

According to Observation 7, we have \( ||\text{Rd}(z_i)[\overline{1}]||_\infty \leq O_{\text{FPT}}(1) \), whereas by Lemma 6, we can find disjoint subsets \( T_1, T_2, \ldots, T_m \) such that \( |T_j| = O_{\text{FPT}}(1) \) and \( \sum_{i \in T_j} \text{Rd}(z_i)[\overline{1}] \) is equal to \( \text{Rd}(\overline{y})[\overline{1}] \) and \( \text{Rd}(\overline{y})[\overline{1}] = \sum_j (\sum_{i \in T_j} \text{Rd}(z_i)[\overline{1}]) \).

Now we consider \( \text{Rd}^1(x) \)'s. By Eq (20) we have

\[
\text{Rd}^1(\overline{y}) = \sum_{j=1}^m \sum_{i \in T_j} \text{Rd}^1(z_i).
\]

Given that \( \text{Rd}^1(\text{md}(j,k)), \text{Rd}^1(\overline{\text{md}}(j,k)) = O_{\text{FPT}}(n) \), \( \text{Rd}(\overline{e}_h) = O_{\text{FPT}}(1) \), and \( |T_j| = O_{\text{FPT}}(1) \), we can conclude that \( ||\sum_{i \in T_j} \text{Rd}^1(z_i)||_\infty = O_{\text{FPT}}(n) \). Applying Lemma 6, we can further find \( m'' \) disjoint sets \( T'_1, T'_2, \ldots, T'_m \subseteq \{1, 2, \ldots, m'\} \) such that \( |T'_h| = O_{\text{FPT}}(n^{\frac{1}{2}}) \), \( \bigcup_{h=1}^{m''} = \{1, 2, \ldots, m'\} \) and \( \sum_{j \in T'_h} \sum_{i \in T_j} \text{Rd}^1(z_i) \subseteq \text{Rd}^1(\overline{y}) \). Hence, Eq (19) can be rewritten as:

\[
\overline{y} = \sum_{h=1}^{m''} \left( \sum_{j \in T'_h} \sum_{i \in T_j} z_i \right),
\]

where for every \( h \) it holds that \( \sum_{j \in T'_h} \sum_{i \in T_j} z_i \subseteq \overline{y} \), \( ||\sum_{j \in T'_h} \sum_{i \in T_j} z_i ||_\infty = O_{\text{FPT}}(n^{\frac{1}{2}}) \), i.e., the following lemma is true.

**Lemma 10.** Let \( H_0 \overline{y} = 0 \) and \( \overline{y} \) be the centralization of \( y \), then there exist \( z_h \)'s such that \( H_0 z_h = 0 \), \( z_h \subseteq \overline{y} \), \( ||z_h||_\infty = O_{\text{FPT}}(n^{\frac{1}{2}}) \) and \( \overline{y} = \sum_{h=1}^{m''} z_h \). Furthermore, the \( n+1 \) bricks of each \( z_h \) can be divided into \( 2 + 2(\lambda + 1)\sigma = O_{\text{FPT}}(1) \) groups such that for any \( i_1, i_2 \in Gr_t \), \( z_{h_1}^{i_1} = z_{h_2}^{i_2} \), and \( y^{i_1}, y^{i_2} \) have the same type.

4.2.7 A sign-compatible decomposition of \( y \)

Let \( \Gamma = \sum_{k=1}^{2} ||y_k||_\infty = O_{\text{FPT}}(1) \). Let \( z_h \)'s be the same as that in Lemma 10. The goal of this subsection is to prove the following lemma.

**Lemma 11.** If \( m'' > 2 \Gamma \cdot t_A \cdot (2 + 2(\lambda + 1)\sigma) \), then there exists some \( h_0 \) such that \( z_{h_0} \subseteq y \).

Towards the proof, we need the following observation and lemma. For an arbitrary \( (t_B + n\lambda) \)-dimensional vector \( z \), we define \( y^j[i] \) the \( j \)-th coordinate of the brick \( z^j \). Recall the definition of \( y_f \). As the average is taken among bricks of the same type, we have the following observation.
Corollary 2. If \( y^i[j] \) is large, then \(|y^i[j]| > \Gamma\). Otherwise, \(|y^i[j]| \leq \Gamma\).

By Lemma \[8\] we have the following corollary.

**Corollary 1.**
- If \( y^i[j] \) is positive large, then \( \tilde{y}^i[j] > 0 \).
- If \( y^i[j] \) is negative large, then \( \tilde{y}^i[j] < 0 \).
- If \( y^i[j] \) is small, then \(|\tilde{y}^i[j]| \leq 2\Gamma\).

Using the above corollary, we have the following lemma, which implies directly Lemma \[11\]

**Lemma 12.** If \( m'' > 2\Gamma t_A \cdot (2 + 2(\lambda + 1)\sigma) \), then there exists some \( z_{h_0} \) such that

- If \( y^i[j] \) is positive large, then \( z_{h_0}^i[j] \geq 0 \).
- If \( y^i[j] \) is negative large, then \( z_{h_0}^i[j] \leq 0 \).
- If \( y^i[j] \) is small, then \( z_{h_0}^i[j] = 0 \).

**Proof.** First, by Lemma \[10\] we have \( z_h \subseteq \tilde{y} \) for every \( 1 \leq h \leq m'' \). If \( y^i[j] \) is positive large, by Corollary \[1\] we have \( \tilde{y}^i[j] > 0 \), then \( z_{h_0}^i[j] \geq 0 \). Similarly if \( y^i[j] \) is negative large we have \( z_{h_0}^i[j] \leq 0 \). It remains to consider small coordinates. Consider the following set:

\[
Z_s = \{h : \exists 1 \leq i \leq n, 1 \leq j \leq t_A \text{ such that } y^i[j] \text{ is small and } z_{h_0}^i[j] \neq 0\}.
\]

We claim that, \(|Z_s| \leq 2\Gamma t_A \cdot (2 + 2(\lambda + 1)\sigma)\). Suppose on the contrary that this claim is not true, then \( Z_s \) contains more than \( 2\Gamma t_A \cdot (2 + 2(\lambda + 1)\sigma) \) elements, and consequently there exists some \( 1 \leq \ell_0 \leq 2 + 2(\lambda + 1)\sigma \) such that \(|Z_s \cap Gr_{t_0}| > 2\Gamma t_A\). As \( 1 \leq j \leq t_A \), there exists some \( j_0 \) such that

\[
|\{h : y^i[j_0] \text{ is small and } z_{h_0}^i[j_0] \neq 0, i \in Gr_{t_0}\}| > 2\Gamma.
\]

Note that for all \( i \in Gr_{t_0} \), \( z_{h_0}^i[j_0] \) takes the same value, hence, for an arbitrary \( i_0 \in Gr_{t_0} \) we have that

\[
|\{h : y^i_0[j_0] \text{ is small and } z_{h_0}^i[j_0] \neq 0\}| > 2\Gamma.
\]

Let \( Z_s[j_0] = \{h : y^i_0[j_0] \text{ is small and } z_{h_0}^i[j_0] \neq 0\} \). According to Corollary \[1\] \(|y^i_0[j_0]| \leq 2\Gamma\). Meanwhile the fact that \( z_h \subseteq \tilde{y} \) implies that either \( z_{h_0}^i[j_0] > 0 \) for all \( h \in Z_s[j_0] \), or \( z_{h_0}^i[j_0] < 0 \) for all \( h \in Z_s[j_0] \). In either case, we conclude that \(|\sum_{h \in Z_s[j_0]} z_{h_0}^i[j_0]| > 2\Gamma\). As \( \tilde{y} = \sum_h z_h \) is a sign-compatible decomposition, we have \(|\tilde{y}^i_0[j_0]| > 2\Gamma \), which is a contradiction. Hence, \(|Z_s| \leq 2\Gamma t_A \cdot (2 + 2(\lambda + 1)\sigma)\). Thus, if \( m'' > 2\Gamma t_A \cdot (2 + 2(\lambda + 1)\sigma)\), there must exist some \( h_0 \) such that \( z_{h_0}^i[j] = 0 \) for all \( i, j \) where \( y^i[j] \) is small. \[\square\]

It is clear that the \( z_{h_0} \) in Lemma \[12\] satisfies that \( z_{h_0} \subseteq \tilde{y} \), whereas Lemma \[11\] is proved.

Recall that \(|z_h|_\infty = O_{FPT}(n^2)\), then there exists some function \( f(A, B, C, D) \) that only depends on the small matrices \( A, B, C, D \) (or more precisely, the parameters \( \Delta, s_A, s_B, s_C, s_D, t_A, t_B, t_C, t_D \)) such that \(|z_h|_\infty = f(A, B, C, D) \cdot n^2\). Consequently, the following corollary follows directly from Lemma \[11\]

**Corollary 2.** If \(|y|_1 > 2\Gamma \cdot t_A \cdot (2 + 2(\lambda + 1)\sigma) \cdot (t_B + nt_A) \cdot f(A, B, C, D) \cdot n^2 + 2\Gamma \cdot (t_B + nt_A)\), then there exists some \( z_{h_0} \) such that \(|z_{h_0}|_\infty \leq f(A, B, C, D) n^2 \) and \( z_{h_0} \subseteq \tilde{y} \).
Theorem 7.\footnote{\cite{23,8}} Given Theorem 6, the following theorem follows by combining the idea from \cite{13} and a recent progress

\[\sum_{i \in N_f} ||y^i||_1 \geq \sum_{i \in N_f} ||y^i||_1 - \Gamma \cdot n_f \cdot t_A\]

According to Lemma \ref{lem:graver}, we know that \(||\vec{y} - y^i||_\infty \leq \Gamma\), hence

\[\sum_{i \in N_f} ||y^i||_1 \geq \sum_{i \in N_f} ||y^i||_1 - \Gamma \cdot n_f \cdot t_A \geq \sum_{i \in N_f} ||y^i||_1 - 2\Gamma \cdot n_f \cdot t_A, \quad \forall 1 \leq i \leq n\]

Recall that \(\vec{y}^0 = y^0\), hence,

\[||\vec{y}||_1 \geq ||y||_1 - 2\Gamma \cdot (t_B + nA).\]

If \(||y||_1 > 2\Gamma \cdot t_A \cdot (2 + 2(\lambda + 1) \sigma) \cdot (t_B + nA) \cdot f(A, B, C, D) n^2 + 2\Gamma \cdot (t_B + nA),\) then

\[||y||_1 > 2\Gamma \cdot t_A \cdot (2 + 2(\lambda + 1) \sigma) \cdot (t_B + nA) \cdot f(A, B, C, D) n^2 + 2\Gamma \cdot (t_B + nA) = \Omega_{FPT}(n^2 + 1),\]

then \(y\) is not a Graver basis element. Hence, Theorem 6 is true.

4.3 Running time of the augmentation algorithm for 3-block \(n\)-fold IP

Given Theorem 6, the following theorem follows by combining the idea from \cite{13} and a recent progress in \cite{23,8}.

Theorem 7.\footnote{\cite{23,8}} There exists an algorithm for 3-block \(n\)-fold IP that runs in \(\min\{O_{FPT}(n^{2(n+3)}) \log^3 n, O_{FPT}(n^{(2n+1)/3} \log^3 n)\}\) time.

Proof. According to Section 2 (Approximate Graver-best oracle), it suffices for us to solve the following IP for each fixed value \(\rho_0 = 2^0, 2^1, 2^2, \ldots:\)

\[\min \{w \cdot x : H_0 x = 0, 1 \leq x_0 + \rho_0 x \leq u, x \in \mathbb{Z}^m, ||x||_\infty \leq \min \{O_{FPT}(n^5), O_{FPT}(n^4 + 1)\}\}\]

Let \(x_*\) be the optimal solution. Given that \(||x_*||_\infty \leq O_{FPT}(n^2 + 1),\) we can guess \(x^0_*\) and there are \(O_{FPT}(n^{(2n+1)/3})\) different possibilities. For each guess, say, \(x^0_* = v\), we solve the following problem:

\[\min \{w \cdot x : H_0 x = 0, 1 \leq x_0 + \rho_0 x \leq u, x \in \mathbb{Z}^m, x^0_0 = v\}\]

By fixing \(x^0_*\), the above problem becomes exactly an \(n\)-fold IP, which can be solved efficiently in \(O_{FPT}(n^3 \log^3 n)\) time \cite{8}. Notice that \(\rho_0\) may take \(O_{FPT}(n \log n)\) distinct values, the overall running time is \(\min \{O_{FPT}(n^{2(n+3)}) \log^3 n, O_{FPT}(n^{(2n+1)/3} \log^3 n)\}\).\qed
4.4 Lower bound on the $\ell_\infty$-norm of Graver basis

Given Theorem 5 it seems that we may expect the Graver basis of 3-block $n$-fold IP can be bounded by $O_{\text{FPT}}(1)$. Unfortunately, the following theorem indicates that this is impossible.

**Theorem 8.** There exists a 3-block $n$-fold matrix $H_0$ such that for some $g \in G(H_0)$, $\|g\|_\infty = \Omega(n)$.

**Proof.** Let $B = 1$, which is a $1 \times 1$ identity matrix. Let $A = (1, -1)$, $D = (1, 0)$. Consider the vector $y = (y^0, y^1, \cdots, y^n)$ with $y^0 = 1$, $y^1 = (n-1, n)$ and $y^i = (-1, 0)$ for every $2 \leq i \leq n$. It is easy to verify that $y^0 + Ay^i = 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^n Dy^i = 0$. In the following we show that $y$ is a Graver basis element. As $H_0y = 0$, there exist $\alpha_j \in \mathbb{Z}_+$, $g_j(H_0) \in G(H_0)$, $g_j(H_0) \subseteq y$ such that $y = \sum_j \alpha_j g_j(H_0)$. Among all of these $g_j(H_0)$’s, there exists some $j$ such that $g_j(H_0) \neq 0$. Given that $y^0 = 1$, it holds that $g_j^0(H_0) = 1$. Let $g_j(H_0) = (1, x^1, x^2, \cdots, x^n)$. The fact that $g_j(H_0) \subseteq y$ implies that $x^i = (x^i_1, 0)$ for $2 \leq i \leq n$. As $1 + Ax^i = 0$, $x^i_1 = -1$ for $2 \leq i \leq n$, and consequently $x^i_1 = n - 1$ according to $\sum_{i=1}^n Dx^i = 0$. Hence, $g_j(H_0) = y$, and Theorem 8 is proved. 

5 Conclusion

We consider 4-block $n$-fold IP and its important special case 3-block $n$-fold IP, both generalizing the well-known two-stage stochastic IP and $n$-fold IP. We show that, 3-block $n$-fold IP admits a Hilbert basis whose $\ell_\infty$-norm is bounded in $O_{\text{FPT}}(1)$, while any non-zero integral element in the kernel space of 4-block $n$-fold IP may have an $\ell_\infty$-norm at least $\Omega(n^c)$). We provide a matching upper bound on the $\ell_\infty$-norm of the Graver basis for 4-block $n$-fold IP, which gives an exponential improvement upon the best known result. We also establish an upper bound of $\min\{O_{\text{FPT}}(n^c), O_{\text{FPT}}(n^{3/2} + 1)\}$ on the $\ell_\infty$-norm of the Graver basis for 3-block $n$-fold IP.

It remains as an important open problem whether 4-block $n$-fold IP, or even its special case 3-block $n$-fold IP, is in FPT. Our results indicate that, using the current augmentation framework, it is unlikely to derive an FPT algorithm. Another important open problem is whether the $\ell_\infty$-norm of the Graver basis elements of 3-block $n$-fold IP is bounded by $n^{O(1)}$, which is independent of the parameters.

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