ON MODULES $M$ WITH $\tau(M) \cong \nu \Omega^{d+2}(M)$ FOR ISOLATED SINGULARITIES OF KRULL DIMENSION $d$

RENÉ MARCZINZIK

Abstract. A classical formula for the Auslander-Reiten translate $\tau$ says that $\tau(M) \cong \nu \Omega^{2}(M)$ for every indecomposable module $M$ of a selfinjective Artin algebra. We generalise this by showing that for a $2d$-periodic isolated singularity $A$ of Krull dimension $d$, we have for the Auslander-Reiten translate of an indecomposable non-projective Cohen-Macaulay $A$-module $M$, $\tau(M) \cong \nu \Omega^{d+2}(M)$ if and only if $\text{Ext}^{d+1}_A(M,A) = \text{Ext}^{d+2}_A(M,A) = 0$. We give several applications for Artin algebras.

Introduction

A classical result in Auslander-Reiten theory for Artin algebras is that for every indecomposable non-projective module $X$ over a selfinjective Artin algebra one has $\tau(X) \cong \nu \Omega^{2}(X)$, see for example proposition 3.7. in [ARS]. This isomorphism has several important applications for the Auslander-Reiten theory of selfinjective algebras, we refer for example to [DP] and [KZ].

In section 2 of [RZ] it was recently noted that for a general Artin algebra $A$, every indecomposable non-projective module $X$ satisfies $\tau(X) \cong \nu \Omega^{2}(X)$ in case $\text{Ext}^{1}_A(X,A) = \text{Ext}^{2}_A(X,A) = 0$. As a main result of this article, we sharpen their result by showing that also the converse is true.

Theorem. Let $A$ be an Artin algebra with an indecomposable non-projective module $M$. Then the following are equivalent:

1. $\text{Ext}^{1}_A(X,A) = \text{Ext}^{2}_A(X,A) = 0$.
2. $\text{Tr}(X)$ is reflexive.
3. $\tau(X) \cong \nu \Omega^{2}(X)$.

We actually prove a much more general result for isolated singularities and obtain the previous theorem as a special case by setting $d = 0$. We refer to the next section for the relevant definitions.

Theorem. Let $A$ be a $2d$−periodic isolated singularity of Krull dimension $d$ and $X \in \text{CM}(A)$ indecomposable non-projective. Then the following are equivalent:

1. $\text{Ext}^{d+1}_A(X,A) = \text{Ext}^{d+2}_A(X,A) = 0$.
2. $\text{Tr}_d(X)$ is reflexive.
3. $\tau(X) \cong \nu \Omega^{d+2}(X)$.

In the final section we give several applications of our main result for Artin algebras. I thank Jeremy Rickard for allowing me to use his proof in 1.2.

1. A characterisation of $\tau(M) \cong \nu \Omega^{d+2}(M)$

We assume that $A$ is always a (not necessarily commutative) noetherian ring and modules are finitely generated right modules unless otherwise stated. We write $(-)^*$ short for the functor $\text{Hom}_A(-,A)$. Recall that a module $X$ is called torsionless in case the natural evaluation map $\text{ev}_M : M \to M^{**}$ is injective and it is called reflexive in case $\text{ev}_M$ is an isomorphism. For a general full subcategory $C$ of $\text{mod} - A$, we denote by $\overline{C}$ the subcategory $C$ modulo projective objects (called the stable category of $C$) and by $\overline{C}$ we denote the subcategory $C$ modulo injective objects (called the costable category of $C$). We now define the Auslander-Bridger transpose $\text{Tr}(X)$ of a module $X$: Let

$P_1 \xrightarrow{g} P_0 \to X \to 0$
Theorem 1.1. Let $A$ be a noetherian ring. Then $\text{Tr}$ induces a duality $\text{mod} - A \rightarrow \text{mod} - A^{\text{op}}$ with $\text{Tr} \circ \text{Tr} \cong \text{id}_{\text{mod} - A}$.

Proof. See proposition 2.6. in [AB].

We will need the following equivalent characterisations of reflexive modules. The proof that (2) implies (1) is due to Jeremy Rickard.

Proposition 1.2. Let $A$ be a noetherian algebra and $M$ an $A$-module. The following are equivalent:

1. $M$ is reflexive.
2. $M \cong M^{**}$.
3. $\text{Ext}^1_A(\text{Tr}(M), A) = \text{Ext}^2_A(\text{Tr}(M), A) = 0$.

Proof. The equivalence between (1) and (3) is well known, see formula (0.1) in the introduction of [AB]. (1) implies (2) by definition. We now show that (2) implies (1): Assume $M \cong M^{**}$. Then $M$ is a dual and for any dual the canonical evaluation map $M \rightarrow M^{**}$ is a split monomorphism, see for example Proposition 1.1.9. of [C] (note that the book [C] is about commutative rings but the same proof applies to non-commutative rings). Thus $M$ is a direct summand of $M^{**} = M$ and therefore $M \cong M \oplus N$ for some non-zero module $N$ in case $M$ is not reflexive. We show that this leads to a contradiction and thus $M$ has to be reflexive. In case $N \neq 0$, there is a strictly increasing chain of submodules of $M$ as follows: $N < N^2 < N^3 < \ldots$, which contradicts that $M$ is finitely generated and thus noetherian.

We now give the relevant definitions on isolated singularities, where we closely follow section 3 of [Iya]. We refer to [Iya] for more information and examples. Let $R$ be a commutative noetherian complete local ring of Krull dimension $d$ and assume furthermore that $R$ is a Gorenstein ring. Recall that an $R$-module $X$ is called Cohen-Macaulay in case $X = 0$ or $\dim(X) = d = \text{depth}(X)$.

We always assume that an $R$-algebra $A$ is module-finite and semiperfect so that projective covers exist. We define the full subcategory $\text{CM}(A) := \{X \in \text{mod} - A | X$ is Cohen-Macaulay as an $R$-module $\}$. We call $A$ an $R$-order in case $A \in \text{CM}(A)$. We call an $R$-order $A$ an isolated singularity in case the Hom-spaces modulo projectives $\text{Hom}(X, Y)$ have finite length as $R$-modules for all $X, Y \in \text{CM}(A)$. For example for $d = 0$ the isolated singularities are exactly the Artin algebras. We assume in the following that $A$ is always an isolated singularity of Krull dimension $d$. We have a duality $D_d := \text{Hom}_R(-, R) : \text{CM}(A) \rightarrow \text{CM}(A^{\text{op}})$ and use this to define the Nakayama functors as $\nu := D_d(-)^* \circ \nu^{-1} := (-)^* \circ D_d$. The following theorem motivates the definition of the Auslander-Reiten translates for isolated singularities:

Theorem 1.3. Let $A$ be an isolated singularity of Krull dimension $d$. Then there is a duality $\Omega^d \text{Tr} : \text{CM}(A) \rightarrow \text{CM}(A^{\text{op}})$ with $(\Omega^d \text{Tr})^2 \cong \text{id}_{\text{CM}(A)}$.

Proof. See for example [Iya], theorem 3.4.

We set $\text{Tr}_d := \Omega^d \text{Tr}$ and define the Auslander-Reiten translate $\tau$ of an isolated singularity of Krull dimension $d$ as $\tau := D_d \text{Tr}_d$ and the inverse Auslander-Reiten translate $\tau^{-1}$ as $\tau^{-1} := \text{Tr}_d D_d$. In this text we are interested in a homological characterisation for indecomposable non-projective modules $M \in \text{CM}(A)$ with $\tau(M) \cong \nu \Omega^{d+2}(M)$ for isolated singularities.

Lemma 1.4. Let $A$ be an isolated singularity. Let $X$ be an indecomposable non-projective $A$-module. Then $X^* \cong \Omega^2 \text{Tr}(X)$.

Proof. Let $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ be a minimal projective presentation of $X$ and apply the functor $(-)^*$ to it. Since for a projective module $P$ also $P^*$ is projective, we obtain by the definition of the Auslander-Bridger transpose the following exact sequence, where $(P_0)^*$ and $(P_1)^*$ are projective and give a minimal projective presentation of $\text{Tr}(X)$:

$$X^* \rightarrow (P_0)^* \rightarrow (P_1)^* \rightarrow \text{Tr}(X) \rightarrow 0.$$ 

Thus $X^* \cong \Omega^2 \text{Tr}(X)$, since we assumed that $X$ as no projective summands.
Corollary 1.5. Let $A$ be an isolated singularity with an indecomposable non-projective module $X$. Then $X$ is reflexive if and only if $\Omega^2 Tr \Omega^2 Tr(X) \cong X$.

Proof. By Lemma 1.6, $X$ is reflexive if and only if $X \cong X^{**}$. Assume first that $X$ is indecomposable non-projective and reflexive. Note that $X^*$ can not be projective, or else also $X^{**} \cong X$ would be projective, which contradicts our assumptions on $X$.

By Lemma 1.6, we have $X^* \cong \Omega^2 Tr(X)$ and then $X^{**} \cong \Omega^2 Tr \Omega^2 Tr(X)$. Now assume that $X$ is indecomposable non-projective with $\Omega^2 Tr \Omega^2 Tr(X) \cong X$. Then also $\Omega^2 (Tr(X))$ is non-projective and $\Omega^2 Tr \Omega^2 Tr(X) \cong X^{**}$ forces $X \cong X^{**}$ and thus $X$ is reflexive.

We call an isolated singularity $A$ of Krull dimension $d$ $l$-periodic in case the functor $\Omega^l : \text{CM}(A) \rightarrow \text{CM}(A)$ is isomorphic to the identity functor.

Lemma 1.6. Let $A$ be a $2d$-periodic isolated singularity with Krull dimension $d$ and $X \in \text{CM}(A)$ indecomposable non-projective. Then $\text{Tr}(\Omega^d(X)) \cong \Omega^d(\text{Tr}(X))$ in the stable category.

Proof. We use that $\Omega^d Tr \Omega^d Tr \cong id$ in the stable category by Lemma 1.3. We have for every $X \in \text{CM}(A)$: $\Omega^{2d}(X) \cong X$. This is equivalent to $\Omega^d Tr \Omega^d Tr(X) \cong X$. Note that when for $Y \in \text{mod-A}$: $\Omega^d Tr(Y) = Z \in \text{CM}(A)^{op}$, then also $Y = \text{Tr} \Omega^d(Z) \in \text{CM}(A)$. This shows that $Y = \text{Tr} \Omega^d(X) \in \text{CM}(A)$.

We now apply the functor $\Omega^d Tr$ to $\Omega^d Tr \Omega^d Tr(X) \cong X$ and use that $(\Omega^d Tr)^2 \cong id$ to obtain:

$$(\Omega^d Tr \Omega^d Tr) \text{Tr}(\Omega^d(X)) \cong \Omega^d Tr(X)$$

which is gives

$$\text{Tr}(\Omega^d(X)) \cong \Omega^d Tr(X).$$

Theorem 1.7. Let $A$ be a $2d$-periodic isolated singularity of Krull dimension $d$ and $X \in \text{CM}(A)$ indecomposable non-projective. Then the following are equivalent:

1. $\text{Ext}^{d+1}_A(X,A) = \text{Ext}^{d+2}_A(X,A) = 0$.
2. $\text{Tr}_d(X)$ is reflexive.
3. $\tau(X) \cong \nu \Omega^{2d+2}(X)$.

Proof. We have

$$\text{Ext}^{d+1}_A(X,A) = \text{Ext}^{d+2}_A(X,A) = 0 \iff$$

$$\text{Ext}^1_A(\Omega^d(X),A) = \text{Ext}^2_A(\Omega^d(X),A) = 0 \iff$$

$$\text{Ext}^1_A(\text{Tr}(\text{Tr}(\Omega^d(X))),A) = \text{Ext}^2_A(\text{Tr}(\Omega^d(X)),A) = 0 \iff$$

$$\text{Ext}^1_A(\text{Tr}(\text{Tr}(\text{Tr}(\Omega^d(X)))),A) = \text{Ext}^2_A(\text{Tr}(\text{Tr}(\Omega^d(X))),A) = 0 \iff$$

$$\text{Tr}_d(X)$$

is reflexive, by our characterisation (3) of reflexive modules in Lemma 1.2. Here we used that $\text{Ext}^i_A(\text{Tr}(\text{Tr}(\Omega^d(X))),A) \cong \text{Ext}^i_A(\text{Tr}(\Omega^d(\text{Tr}(X))),A)$ for $i = 1,2$ in the stable category since we have $\text{Tr}(\Omega^d(X)) \cong \Omega^d(\text{Tr}(X))$ in the stable category by the previous lemma. This gives the equivalence between (1) and (2).

Now assume that $\tau(X) \cong \nu \Omega^{2d+2}(X)$. Applying the duality $D_d$ on both sides and using that $(-)^* \cong \Omega^2 \text{Tr}(-)$, by Lemma 1.3 this is equivalent to $\text{Tr}_d(X) \cong \Omega^2 \text{Tr} \Omega^{2d+2}(X)$. We simplify this using our assumption that $\text{Tr}(\Omega^d(X)) \cong \Omega^d(\text{Tr}(X))$ and $(-)^* \cong \Omega^2 \text{Tr} \Omega^2 \text{Tr}$ in the stable category:

$$\Omega^2 \text{Tr} \Omega^{2d+2}(X) \cong \Omega^2 \text{Tr} \Omega^2 \text{Tr} \Omega^d(X)$$

$$\cong (\text{Tr} \Omega^d(X))^* \cong (\Omega^d \text{Tr}(X))^* \cong (\text{Tr}(X))^*. $$

Thus the isomorphism $\tau(X) \cong \nu \Omega^{2d+2}(X)$ is equivalent to $\text{Tr}_d(X) \cong (\text{Tr}_d(X))^*$, which is equivalent to $\text{Tr}_d(X)$ being reflexive by (2) of Lemma 1.2.
Example 1.8. Let $d \geq 1$. Let $S = K[[x_0, \ldots, x_d]]$ the formal power series ring in $(d + 1)$-variables and $f \in (x_0, \ldots, x_d)$. Set $A = S/(f)$ and assume that $A$ is an isolated singularity. In this case $A$ is local (and thus semiperfect) and has Krull dimension $d$. $A$ is called an isolated hypersurface singularity. We refer to [LW] and [Y] for more information on such algebras and several classification results. According to theorem 4.9. of [LW], we have $Ω^2 \cong id$ in $CM(A)$ and thus $A$ is 2d-periodic and our main result 1.7 applies to such algebras to give $τ(M) \cong ν(M)$ in case $d$ is even and $τ(M) \cong νΩ^2(M)$ in case $d$ is odd.

2. Applications for Artin algebras

We restrict in this section to Artin algebras (that is to the case of Krull dimension $d = 0$). We assume that the reader is familiar with the basic representation theory of Artin algebras and refer for example to [ARS]. We denote the duality of an Artin algebra by $D$. We call a module $M$ $τ$-perfect in case $τ(M) \cong νΩ^2(M)$ and $τ^{-1}$-perfect in case $τ^{-1}(M) \cong ν^{-1}Ω^{-2}(M)$. $Per(τ(A))$ is defined as the full subcategory of $τ$-perfect modules and $Per^{-1}(A)$ is defined as the full subcategory of $τ^{-1}$-perfect modules. We denote by Ref($A$) the full subcategory of reflexive modules. Furthermore, we call a module $N$ coreflexive in case $D(N)$ is reflexive and we define the subcategory Coref($A$) as the full subcategory of coreflexive $A$-modules. We state the case $d = 0$ of our main result here explicitly due to its importance:

Theorem 2.1. Let $A$ be an Artin algebra with an indecomposable non-projective module $X$. Then the following are equivalent:

1. $Ext^1_A(X, A) = Ext^2_A(X, A) = 0$.
2. $Tr(X)$ is reflexive.
3. $X$ is $τ$-perfect, that is $τ(X) \cong νΩ^2(X)$.

We also state the dual of the previous theorem:

Theorem 2.2. Let $A$ be an Artin algebra with an indecomposable non-injective module $X$. Then the following are equivalent:

1. $Ext^1_A(D(A), X) = Ext^2_A(D(A), X) = 0$.
2. $Tr(D(X))$ is reflexive.
3. $X$ is $τ^{-1}$-perfect, that is $τ^{-1}(X) \cong ν^{-1}Ω^{-2}(X)$.

In selfinjective Artin algebras it is in fact true that for every module $M$ with $τ(M) \cong νΩ^2(M)$, we even have $τ(M) \cong νΩ^2(M) \cong Ω^2ν(M)$. But this fails in general as we show in the next example:

Example 2.3. Let $A$ be the Nakayama algebra, given by quiver and relations, with Kupisch series $[2, 2, 2, 1]$ with simple modules numbered from 0 to 3. Then the simple module $S_0$ satisfies $τ(S_0) \cong νΩ^2(S_0) \cong S_1$, but $Ω^2ν(S_0) \cong 0$.

The next proposition shows that the classification of $τ^{-1}$-perfect modules is essentially equivalent to the classification of reflexive modules for general Artin algebras.

Proposition 2.4. Let $A$ be an Artin algebra.

1. $τ$ induces an equivalence of categories between $Per(τ(A))$ and $Coref(A)$.
2. $τ^{-1}$ induces an equivalence of categories between $Per^{-1}(A)$ and $Ref(A)$.

Proof. We only prove (1), since the proof of (2) is dual. $τ$ is an equivalence between $mod_A$ and $mod_A$. Thus when we restricted to the subcategory $Per(τ(A))$, $τ$ induces an equivalence to the image $τ(Per(τ(A)))$. By [2.1] $X \in Per(τ(A))$ implies that $Tr(X)$ is reflexive and thus $τ(X) = DTr(X)$ is coreflexive. Thus the image is contained in $Coref(A)$. Now $τ : Per(τ(A)) \to Coref(A)$ is also dense since when $Y \in Coref(A)$ is given, we have that $τ^{-1}(Y)$ is $τ$-perfect. This is true since by [2.1] $τ^{-1}(Y)$ is $τ$-perfect if and only if $Tr(τ^{-1}(Y))$ is reflexive, which is equivalent to $DTr(τ^{-1}(Y)) = τ(τ^{-1}(Y)) = Y$ being coreflexive.

We can also obtain a new characterisation of selfinjective Artin algebras from our main theorem:

Corollary 2.5. Let $A$ be an Artin algebra. Then the following are equivalent:

1. $A$ is selfinjective.
2. Every simple module is $τ$-perfect.
Proof. Since \( \tau \cong \nu \Omega^2 \) in a selfinjective Artin algebra, it is clear that (1) implies (2). Now assume that (2) holds. Then we have \( \text{Ext}^i_A(S, A) = 0 \) for every simple modules \( S \) by \([2.1]\). But by corollary 2.5.4. of [Ben], we have \( \text{Hom}_A(S, \Omega^{-1}(A)) \cong \text{Ext}^i_A(S, A) = 0 \) for every simple module \( S \). Thus the socle of \( \Omega^{-1}(A) \) is zero and thus also \( \Omega^{-1}(A) \) is zero. This means that \( A \) is isomorphic to its injective envelope. But this shows that every projective module is injective and thus \( A \) is selfinjective.

We give two applications of this result to algebras with dominant dimension at least 2 and algebras that are 2-Iwanaga-Gorenstein. Recall that the dominant dimension of a module \( M \) with minimal injective coresolution \( (I_i) \) is defined as the smallest number \( k \) such that \( I_k \) is not projective. The dominant dimension of an algebra is by definition the dominant dimension of the regular module. It is well known that an algebra \( A \) has dominant dimension at least one if and only if there is a minimal faithful projective-injective left \( A \)-module of the form \( Af \) for some idempotent \( f \). Furthermore, such an algebra has dominant dimension at least two if and only if \( A \cong \text{End}_{fA}(Af) \) in addition.

**Corollary 2.6.** Let \( A \) be an algebra with a minimal faithful projective-injective left module \( Af \) with dominant dimension at least two. There is an equivalence of categories

\[
\text{mod} - fAf/(\text{add}(Af)) \to \text{Per}_{\tau^{-1}}(A).
\]

**Proof.** Since \( A \) has dominant dimension at least two and by a result of Morita in \([1]\) theorem 3.3., an \( A \)-module \( X \) has dominant dimension at least two if and only if \( X \) is reflexive. Now the full subcategory of modules with dominant dimension at least two \( \text{Dom}_2 \) is equivalent to \( \text{mod} - fAf \) and the projective modules are sent to the modules in \( \text{add} - Af \) (this is a special case of lemma 3.1. in \([AP\]) . This means that there is an equivalence \( \text{Ref}(A) = \text{Dom}_2 \cong \text{mod} - fAf/(\text{add}(Af)) \). But by \([2.4]\) \( \text{Ref}(A) \) is equivalent to \( \text{Per}_{\tau^{-1}}(A) \), which finishes the proof.

**Example 2.7.** Let \( K \) be an infinite field and \( V \) an \( n \)-dimensional vector space. Let \( A \) be the Schur algebra \( S(n, r) \) for \( n \geq r \). Then \( A \) has dominant dimension at least two and \( fAf \) is isomorphic to the symmetric group algebra \( KS_r \) and \( Af \) to \( V^{\otimes n} \), see for example theorem 1.2. in \([KSX]\). The previous corollary gives us in this case an equivalence between the module category of the algebra of the symmetric group modulo the subcategory \( \text{add}(V^{\otimes n}) \) to the cosubcategory of \( \tau^{-1} \)-perfect modules over the Schur algebra.

Recall that an Artin algebra \( A \) is called \( n \)-Iwanaga-Gorenstein in case the left and right injective dimensions of the regular modules \( A \) are equal to \( n \) for a natural number \( n \). An \( A \)-module \( X \) over an \( n \)-Iwanga-Gorenstein algebra is called Gorenstein projective in case \( \text{Ext}^i_A(X, A) = 0 \) for all \( i \geq 1 \). All projective modules are Gorenstein projective by definition and one of the main goals of Gorenstein homological algebra is to find a classification of all Gorenstein projective modules for a given Artin algebra. In the next corollary we give a new characterisation of Gorenstein projective modules for 2-Iwanga-Gorenstein algebras. Note that this is a large class of algebras that contain for example all cluster-tilted algebras by \([KR]\) and all algebras considered by \([GLS]\).

**Corollary 2.8.** Let \( A \) be a 2-Iwanga-Gorenstein algebra. Then an indecomposable module \( X \) is Gorenstein projective if and only if \( X \) is \( \tau \)-perfect, that is \( \tau(X) \cong \nu \Omega^2(X) \).

**Proof.** Since \( A \) has injective dimension at most 2, we have that for an arbitrary module \( X \) \( \text{Ext}^i_A(X, A) = 0 \) for all \( i \geq 1 \) if and only if \( \text{Ext}^i_A(X, A) = \text{Ext}^i_A(X, A) = 0 \). But this condition is by \([2.1]\) equivalent to \( X \) being \( \tau \)-perfect.

**Example 2.9.** Let \( A \) be the Nakayama algebra with Kupisch series \([3,3,4]\) and primitive idempotents \( e_0, e_1 \) and \( e_2 \). \( A \) is 2-Iwanga-Gorenstein. The indecomposable non-projective Gorenstein projective \( A \)-modules are \( e_0A/e_0J^1 \) and \( e_1A/e_1J^2 \).

**References**

[AB] Auslander, M.; Bridger, M.: Stable Module Theory Memoirs of the American Mathematical Society, No. 94 American Mathematical Society, Providence, R.I. 1969 146 pp.
[APT] Auslander, M.; Platzeck, M. I.; Todorov, G.: Homological theory of idempotent Ideals, Transactions of the American Mathematical Society, Volume 332, Number 2, August 1992.

[ARS] Auslander, M.; Reiten, I.; Smalo, S.: Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, 36, Cambridge University Press, Cambridge, 1997. xiv+425 pp.

[Ben] Benson, D. J.: Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge Studies in Advanced Mathematics, Volume 30, Cambridge University Press, 1991.

[C] Christensen, L. W.: Gorenstein dimensions, Lecture Notes in Mathematics, 1747, Berlin: Springer. viii, 204 p. (2000).

[DP] Diveris, K.; Purin, M.: Vanishing of self-extensions over symmetric algebras, Journal of Pure and Applied Algebra 218 (2014) 962-971.

[GLS] Geiss, C.; Leclerc, B.; Schroer, J.: Quivers with relations for symmetrizable Cartan matrices I: Foundations, Inventiones Mathematicae 209 (2017), 61-158.

[Iya] Iyama, O.: Auslander-Reiten theory revisited, Trends in representation theory of algebras and related topics. Proceedings of the 12th international conference on representations of algebras and workshop (ICRA XII), EMS Series of Congress Reports, 349-397 (2008).

[KR] Keller, B.; Reiten, I.: Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, Advances in Mathematics Volume 211, Issue 1, 1 May 2007, Pages 123-151.

[KZ] Kerner, O.; Zacharia, D.: Auslander-Reiten theory for modules of finite complexity over self-injective algebras, Bull. London Math. Soc. 43 (2011) 44-56.

[KSX] Koenig, S.; Slungard, I. H.; Xi, C.: Double Centralizer Properties, Dominant Dimension, and Tilting Modules. Journal of Algebra Volume 240, Issue 1, 1 June 2001, Pages 393-412.

[LW] Leuschke, G. J.; Wiegand, R.: Cohen-Macaulay Representations. Mathematical Surveys and Monographs, Volume 181, 2012.

[M] Morita, K.: Duality in QF-3 rings. Mathematische Zeitschrift 108(1969), 237-252.

[RZ] Ringel, C. M.; Zhang, P.: Gorenstein-projective and semi-Gorenstein-projective modules. https://arxiv.org/abs/1808.01809.

[Y] Yoshino, Y.: Maximal Cohen-Macaulay Modules over Cohen-Macaulay Rings. London Mathematical Society Lecture Note Series Book 146, (1990).

Institute of algebra and number theory, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: marcziire@mathematik.uni-stuttgart.de