Positive maps, doubly stochastic matrices and new family of spectral conditions

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Abstract. We provide partial classification of positive maps in matrix algebras which is based on a family of spectral conditions. This construction recovers many well known examples of positive maps. We displayed interesting connection between positive maps and doubly stochastic matrices.

1. Introduction
One of the most important problems of quantum information theory [1] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e. whether it is separable or entangled. For low-dimensional systems there exists simple necessary and sufficient condition for separability. The celebrated Peres-Horodecki criterion [2, 3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transpose is positive. Unfortunately, for higher-dimensional systems there is no single universal separability condition.

It turns out that the above problem may be reformulated in terms of positive linear maps in operator algebras: a state $\rho$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable iff $(\text{id} \otimes \Phi)\rho$ is positive for any positive map $\Phi$ which sends positive operators on $\mathcal{H}_2$ into positive operators on $\mathcal{H}_1$. Therefore, a classification of positive maps between operator algebras $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ is of primary importance. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [4]-[12] (for the recent review paper see [13]).

In the present paper we perform partial classification of positive linear maps which is based on spectral conditions. Actually, the presented method enables one to construct maps with a desired degree of positivity — so called $k$-positive maps with $k = 1, 2, \ldots, d = \min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\}$. Completely positive (CP) maps correspond to $d$-positive maps, i.e. maps with the highest degree of positivity. These maps are fully classified due to Stinespring’s theorem [16]. Now, any positive map which is not CP can be written as $\Phi = \Phi_+ - \Phi_-$, with $\Phi_{\pm}$ being CP maps. However, there is no general method to recognize the positivity of $\Phi$ from $\Phi_+ - \Phi_-$. We show that suitable spectral conditions satisfied by the pair $(\Phi_+, \Phi_-)$ guarantee the $k$-positivity of $\Phi_+ - \Phi_-$. There is an interesting connection between positive maps in matrix algebras and doubly stochastic matrices. We provide several examples of maps and the corresponding families of doubly stochastic matrices. It was argued that such matrices may occur in the very basic description of quantum phenomena [14] (for the interesting discussion see [15]). It is hoped that the connection between positive maps and doubly stochastic matrices may shed a new light into
the problem of positive maps. This part of our paper is inspired very much by the beautiful lecture by James Louck (see this volume).

2. Quantum entanglement, Schmidt number and all that

Consider a Hilbert space being the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, with $\dim \mathcal{H}_i = d_i < \infty$. It is clear that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is isomorphic to the space of linear operators from $L(\mathcal{H}_1, \mathcal{H}_2)$, i.e. each $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ corresponds to some linear operator $F : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Indeed, taking orthonormal basis $\{e_i\} (i = 1, \ldots, d_1)$ in $\mathcal{H}_1$ and $\{f_\alpha\} (\alpha = 1, \ldots, d_2)$ in $\mathcal{H}_2$, one has

$$\psi = \sum_{i=1}^{d_1} \sum_{\alpha=1}^{d_2} \psi_{i\alpha} e_i \otimes f_\alpha ,$$

where $\psi_{i\alpha}$ are complex coefficients. Defining

$$Fe_i := \sum_{\alpha=1}^{d_2} \psi_{i\alpha} f_\alpha ,$$

one finds

$$\psi = \sum_{i=1}^{d_1} e_i \otimes Fe_i ,$$

which establishes the above-mentioned isomorphism. Let us note that the normalization condition

$$\langle \psi | \psi \rangle = \sum_{i=1}^{d_1} \sum_{\alpha=1}^{d_2} |\psi_{i\alpha}|^2 = 1 ,$$

gives rise to

$$\text{tr} FF^* = 1 ,$$

for the corresponding operator $F$. Denote by

$$s_1 \geq s_2 \geq \ldots \geq s_d , \quad d := \min\{d_1, d_2\} ,$$

the singular values of $F$. One introduces the Schmidt rank of $\psi$:

$$\text{SR}(\psi) := \text{rank} F ,$$

that is, the number of non-vanishing singular values of $F$. It is well known that $\psi$ is separable if and only if $\text{SR}(\psi) = 1$. Otherwise it is entangled.

Now, the vector space $L(\mathcal{H}_1, \mathcal{H}_2)$ may be equipped with the following family of $k$-norms:

$$||F||_k^2 := \sum_{i=1}^{k} s_i^2 ,$$

where $s_i$ denote the singular values of $F$, and $1 \leq k \leq d$. Note, that for $k = 1$ one reproduces the operator norm

$$||F||_1^2 := s_1^2 = ||F||^2 ,$$

whereas for $k = d$ one reproduces the Hilbert-Schmidt norm

$$||F||_d^2 := \sum_{i=1}^{d} s_i^2 = \text{Tr} F^* F = ||F||_{\text{HS}}^2 .$$
Moreover, it is clear that
$$||F||_1 \leq \ldots \leq ||F||_d .$$
(9)

Note, that if $F$ corresponds to a normalized vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, then $||F||_d \equiv 1$ from the normalization condition $\langle \psi | \psi \rangle = 1$. Interestingly, if $\psi$ is separable, then
$$||F||_1 = \ldots = ||F||_d = 1 .$$
(10)

If $\psi$ is maximally entangled, i.e. $s_1^2 = \ldots = s_d^2 = 1/d$, then
$$||F||^2_k = \frac{k}{d} ,$$
(11)
for $k = 1, \ldots, d$.

3. Positive maps in matrix algebras

Let $M_d$ denote a set of $d \times d$ complex matrices. A linear map
$$\Phi : M_{d_1} \longrightarrow M_{d_2} ,$$
(12)
is called positive if it maps semi-positive definite matrix $X \in M_{d_1}$ into semi-positive definite matrix $\Phi(X) \in M_{d_2}$. It is called unital if
$$\Phi(\mathbb{I}_{d_1}) = \mathbb{I}_{d_2} ,$$
(13)
where $\mathbb{I}_d$ is a unit matrix in $M_d$. It is called trace preserving if
$$\text{Tr} \Phi(X) = \text{Tr} X ,$$
(14)
for all $X \in M_{d_1}$. Note that trace preserving positive maps map quantum states of $d_1$–level system into quantum states of $d_2$–level system. It is clear that a map (12) is positive if it satisfies
$$\langle \psi | \Phi(|\phi\rangle \langle \phi|) | \psi \rangle \geq 0 ,$$
(15)
for each $\phi \in \mathbb{C}^{d_1}$ and each $\psi \in \mathbb{C}^{d_2}$. Unfortunately, the above condition is very difficult to check. Let us recall that a similar condition for a positivity of a matrix $X \in M_d$
$$\langle \psi | X | \psi \rangle \geq 0 ,$$
(16)
for all $\psi \in \mathbb{C}^d$ reduces to the spectral problem for $X$: all eigenvalues of $X$ have to be nonnegative. It is no longer true for (15). This property does not reduce to any spectral condition and it makes the problem of positive maps very nontrivial.

Let $\text{id}_k : M_k \longrightarrow M_k$ denote an identity map, i.e. $\text{id}_k(X) = X$ for any $X \in M_k$. A positive map (12) is called $k$-positive if
$$\text{id}_k \otimes \Phi : M_k \otimes M_{d_1} \longrightarrow M_k \otimes M_{d_2} ,$$
(17)
is positive. Note, that $M_k \otimes M_d$ may be considered as an algebra of $k \times k$ matrices with entries from $M_d$. Finally, $\Phi$ is called completely positive (CP) if it is $k$-positive for $k = 1, 2, \ldots$. One shows that $\Phi$ is CP iff it is $d$-positive, where $d = \min\{d_1, d_2\}$. Denoting by $\mathcal{P}_k$ a convex cone of $k$-positive maps one arrives at the following chain of cones
$$\text{positive maps} = \mathcal{P}_1 \supset \mathcal{P}_2 \supset \ldots \supset \mathcal{P}_d = \text{CP maps} .$$
(18)
Now, CP maps are fully characterized due the celebrated results of Stinespring, Kraus and Choi [16, 17] any CP map (12) may be represented as follows

$$\Phi(X) = \sum_{\alpha} V_{\alpha} X V_{\alpha}^* ,$$  \hspace{1cm} (19)$$

where the so-called Kraus operators $V_{\alpha}$ are linear operators $V_{\alpha}: \mathbb{C}^d_1 \rightarrow \mathbb{C}^d_2$. It turns out that CP property reduces to a simple spectral condition. Introducing so called Choi-Jamiołkowski matrix [6, 19]

$$\hat{\Phi} := \sum_{i,j=1}^{d_1} |e_i\rangle \langle e_j| \otimes \Phi(|e_i\rangle \langle e_j|) ,$$  \hspace{1cm} (20)$$

where $\{e_1, \ldots, e_{d_1}\}$ defines an orthonormal basis in $\mathbb{C}^{d_1}$, one shows that $\Phi$ is CP if and only if the corresponding matrix $\hat{\Phi}$ is semi-positive definite. It is no longer true for $k$-positive maps with $k < d$. Now, the condition for $k$-positivity of $\Phi$ may be rewritten as the following condition for the corresponding matrix $\hat{\Phi}$

$$\langle \psi| \hat{\Phi} |\psi\rangle \geq 0 ,$$  \hspace{1cm} (21)$$

for all $\psi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ such that $\text{SN}(\psi) \leq k$. For $k = 1$ it gives therefore

$$\langle \psi_1 \otimes \psi_2| \hat{\Phi} |\psi_1 \otimes \psi_2\rangle \geq 0 ,$$  \hspace{1cm} (22)$$

for $\psi_j \in \mathbb{C}^{d_j}$. We stress that the above condition can not be reduced for any spectral condition for the matrix $\hat{\Phi}$.

4. Positive maps: examples

Let us illustrate our definitions with a few simple examples.

**Example 1** Transposition in $M_d$:

$$\Phi(X) = X^T .$$  \hspace{1cm} (23)$$

The map is trivially positive since transposition does not change the spectrum of the Hermitian operator. Moreover, it is trace preserving and unital. For example for $d = 2$ the corresponding $\hat{\Phi}$ matrix reads as follows

$$\hat{\Phi} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix} ,$$  \hspace{1cm} (24)$$

where dots represent zeros. Its spectrum is given by $\{-1, 1, 1, 1\}$ and hence it is not CP. It is well known that transposition is not CP for any $d$ (actually, it is even not 2-positive).

**Example 2** A map in $M_d$ known as the reduction map

$$\Phi(X) = \mathbb{I}_d \text{Tr} X - X .$$  \hspace{1cm} (25)$$

It is easy to check for positivity: indeed for any rank-1 projector $P = |\psi\rangle \langle \psi|$ one gets

$$\Phi(P) = \mathbb{I}_d - P = P^\perp ,$$  \hspace{1cm} (26)$$

where $P^\perp$ defines a projector onto a hyperplane orthogonal to $\psi$. Hence, $\Phi(P)$ being a projector is semi-positive definite. Again, for $d = 2$ it is easy to find the corresponding $\hat{\Phi}$ matrix

$$\hat{\Phi} = \begin{pmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} ,$$  \hspace{1cm} (27)$$
with the spectrum given by \{-1, 1, 1, 1\}. It shows that reduction map is positive but not CP. Interestingly, one may show that composing reduction with transposition one obtains a CP map.

**Example 3** A family of Choi maps in \(M_d\)

\[
\Phi_\mu(X) = I_d \text{Tr} X - \mu X , \quad \mu \leq 1 .
\]  

(28)

Note, that for \(\mu = 1\) it reproduces the reduction map (25). Now, it is well known [18] that if

\[
\frac{1}{k + 1} < \mu \leq \frac{1}{k} ,
\]

then \(\Phi_\mu\) is \(k\)-positive but not \((k + 1)\)-positive. Hence, if \(\mu \leq 1/d\), then \(\Phi_\mu\) is CP. Again, for \(d = 2\) it is easy to find the corresponding \(\hat{\Phi}_\mu\) matrix

\[
\hat{\Phi}_\mu = \begin{pmatrix} 1 - \mu & \cdot & \cdot & -\mu \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -\mu & \cdot & \cdot & 1 - \mu \end{pmatrix},
\]

(30)

with the spectrum given by \{\(-2\mu, 1, 1, 1\)\}. Clearly, for \(\mu \leq 1/2\) the map is CP.

5. **Positive maps vs. doubly stochastic matrices**

Linear positive maps generalize a notion of a nonnegative matrix, i.e. a matrix \(X\) with nonnegative entries \(X_{ij} \geq 0\). Such matrices play important role in classical stochastic dynamics. Consider two stochastic systems described by discrete probability distributions \(p = (p_1, \ldots, p_d)\) and \(q = (q_1, \ldots, q_d)\). Now, a transition \(p \rightarrow q\) is described by a linear map \(p = Sq\), where \(S\) is a real \(d_2 \times d_1\) matrix such that \(S_{ij} \geq 0\), and \(\sum_i S_{ij} = 1\). If \(d_1 = d_2 = d\) and

\[
S_{ij} \geq 0 , \quad \sum_{i=1}^d S_{ij} = 1 , \quad \sum_{j=1}^d S_{ij} = 1 ,
\]

(31)

then \(S\) is called doubly stochastic (or bistochastic). A doubly stochastic matrix has one extra property – it keeps equilibrium distribution \(p_0 = (1/d, \ldots, 1/d)\) invariant.

Let us observe that each trace preserving, unital positive map \(\Phi : M_d \rightarrow M_d\) gives rise to a family of doubly stochastic matrices. Indeed, take two orthonormal basis \{\(e_1, \ldots, e_d\)\} and \{\(f_1, \ldots, f_d\)\} in \(\mathbb{C}^d\) and define

\[
S_{ij} := \langle e_i | \Phi(|f_j\rangle\langle f_j|)|e_i \rangle .
\]

(32)

By the very definition of positivity of \(\Phi\) one has \(S_{ij} \geq 0\). Now

\[
\sum_{i=1}^d S_{ij} = \sum_{i=1}^d \langle e_i | \Phi(|f_j\rangle\langle f_j|)|e_i \rangle = \text{Tr} \Phi(|f_j\rangle\langle f_j|) ,
\]

(33)

and using \(\text{Tr} \Phi(X) = \text{Tr} X\) one gets \(\text{Tr} \Phi(|f_j\rangle\langle f_j|) = 1\). Moreover, one finds

\[
\sum_{j=1}^d S_{ij} = \langle e_i | \sum_{j=1}^d \Phi(|f_j\rangle\langle f_j|)|e_i \rangle .
\]

(34)

Taking into account that \(\sum_{j=1}^d \Phi(|f_j\rangle\langle f_j|) = \Phi(\sum_{j=1}^d |f_j\rangle\langle f_j|) = \Phi(I_d)\) and the fact that \(\Phi\) is unital, i.e. \(\Phi(I_d) = I_d\), one obtains \(\langle e_i | \sum_{j=1}^d \Phi(|f_j\rangle\langle f_j|)|e_i \rangle = 1\). It shows that \(S_{ij}\) defined in (32) is doubly stochastic.
Example 4 Consider reduction map (25) in $M_2$. Let us take the standard orthonormal basis in $\mathbb{C}^2$ — $\{e_1, e_2\}$ — and consider

$$S_{ij} := \langle e_i | \Phi(\langle e_j | e_j \rangle) | e_i \rangle.$$  (35)

One easily finds

$$S_{11} = S_{22} = 0, \quad S_{12} = S_{21} = 1,$$

which defines a permutation (and hence doubly stochastic) matrix.

Example 5 Consider Choi map (28) in $M_2$. One easily finds

$$S = (1 - \mu) \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + \mu \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix},$$  (36)

that is, Birkhoff decomposition of $S$ into convex combination of permutation matrices.

Example 6 Consider the following family of maps in $M_3$:

$$\Phi_{abc} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} =$$

$$= \frac{1}{a + b + c} \begin{pmatrix} aX_{11} + bX_{22} + cX_{33} & -X_{12} & -X_{13} \\ -X_{21} & aX_{22} + bX_{33} + cX_{11} & -X_{23} \\ -X_{31} & -X_{32} & aX_{33} + bX_{11} + cX_{22} \end{pmatrix},$$  (37)

with $a, b, c \geq 0$. Now, $\Phi_{abc}$ is positive if and only if the following conditions are satisfied

(i) $a \geq 0$,
(ii) $a + b + c \geq 2$,
(iii) if $a \leq 1$, then $bc \geq (1 - a)^2$.

Moreover, $\Phi_{abc}$ is CP iff $a \geq 2$. Note, that $\Phi_{abc}$ is unital and trace preserving. The corresponding doubly stochastic matrix $S$ reads as follows

$$S = \frac{1}{a + b + c} \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix},$$  (38)

that is

$$S = a' \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + b' \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + c' \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}.$$  (39)

Hence, one finds again the Birkhoff decomposition of $S$ into convex combination of permutation matrices with

$$a' = \frac{a}{a + b + c}, \quad b' = \frac{b}{a + b + c}, \quad c' = \frac{c}{a + b + c}.$$  (40)

Interestingly, $S$ is not only doubly stochastic but circulant as well.

Let us recall that a class of $d \times d$ doubly stochastic matrices defines a convex polytope in $\mathbb{R}^{d^2}$ known as the Birkhoff polytope $B_d$. The principal fact about doubly stochastic matrices is the Birkhoff theorem [22] which states that the set $B_d$ of doubly stochastic matrices of order $d$ is the convex hull of the set of permutation matrices (of order $d$), and furthermore that the vertices
(extreme points) of $B_d$ are precisely the permutation matrices. It means that any $S \in B_d$ may be represented as

$$S = \sum_{\pi} a_{\pi} P_{\pi}, \quad (41)$$

where $\pi$ enumerates permutations of order $d$ and $P_{\pi}$ is a $d \times d$ permutation matrix, i.e.

$$(P_{\pi})_{ij} = \delta_{i,\pi(j)}. \quad (42)$$

Coefficients $a_{\pi}$ satisfy: $a_{\pi} \geq 0$ and $\sum_{\pi} a_{\pi} = 1$. It should be stressed that not all permutation matrices are linearly independent. It turns out [22] that among $d!$ elements there are exactly $b_d = \left(2^{d-1} + 1\right)$ linearly independent.

Interestingly, any $S \in B_d$ gives rise to the unital, trace preserving CP map in $M_d$. Indeed, using (41) let us define $\Phi : M_d \rightarrow M_d$ by

$$\Phi(X) = \sum_{\pi} a_{\pi} P_{\pi} X P_{\pi}^*, \quad (43)$$

which defines Kraus representation of $\Phi$ with the corresponding Kraus operators $V_{\pi} = \sqrt{a_{\pi}} P_{\pi}$. Note that (43) is unital and trace preserving. Indeed, one has

$$\Phi(I_d) = \sum_{\pi} a_{\pi} P_{\pi} P_{\pi}^* = I_d, \quad (44)$$

where we used the fact that $P_{\pi}$ defines an orthogonal matrix. Moreover,

$$\text{Tr} \Phi(X) = \sum_{\pi} a_{\pi} \text{Tr} (P_{\pi} X P_{\pi}^*) = \sum_{\pi} a_{\pi} \text{Tr} X = \text{Tr} X. \quad (45)$$

The above properties are equivalent to

$$\sum_{\pi} V_{\pi} V_{\pi}^* = \sum_{\pi} V_{\pi}^* V_{\pi} = I_d.$$

It shows that any classical stochastic operation defined by the Birkhoff decomposition (41) may be generalized to the corresponding quantum operation defined by a CP map (43).

### 6. Spectral class of positive maps

Consider a vector space of linear operators $\mathcal{L}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$. It is $D$-dimensional ($D = d_1 d_2$) Hilbert space equipped with the Hilbert-Schmidt scalar product

$$(F, G)_{HS} := \text{Tr} F^* G. \quad (46)$$

Let $\{V_1, \ldots, V_D\}$ denotes an orthonormal (with respect to (46)) basis in $\mathcal{L}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$. Let us define two CP maps

$$\Phi_-(X) := \sum_{\alpha=1}^{L} \lambda_{\alpha} V_{\alpha} X V_{\alpha}^*, \quad (47)$$

and

$$\Phi_+(X) := \sum_{\alpha=L+1}^{D} \lambda_{\alpha} V_{\alpha} X V_{\alpha}^*, \quad (48)$$

with $1 \leq L < D$, such that

$$\lambda_{\alpha} \geq 0, \quad \alpha = 1, \ldots, L, \quad (49)$$
and
\[ \lambda_\alpha > 0 \ , \ \alpha = L + 1, \ldots, D \ , \]  
(50)
It implies that the corresponding matrices \( \hat{\Phi}_\pm \) satisfy
\[ \hat{\Phi}_- \geq 0 \ , \ \hat{\Phi}_+ > 0 \ . \]  
(51)
Finally, let us define
\[ \Phi := \Phi_+ - \Phi_- . \]  
(52)
Our main result consists in the following

**Theorem 1**

Let \( \sum_{\alpha=1}^{L} ||V_\alpha||^2_k < 1 \). If the following spectral conditions are satisfied
\[ \lambda_\alpha \geq \mu_k \ , \ \alpha = L + 1, \ldots, D \ , \]  
(53)
where
\[ \mu_\ell := \frac{\sum_{\alpha=1}^{L} \lambda_\alpha ||V_\alpha||^2_k}{1 - \sum_{\alpha=1}^{L} ||V_\alpha||^2_k} \ , \]  
(54)
then \( \Phi \) is \( k \)-positive. If moreover \( \sum_{\alpha=1}^{L} ||V_\alpha||^2_{k+1} < 1 \) and
\[ \mu_{k+1} > \lambda_\alpha \ , \ \alpha = L + 1, \ldots, D \ , \]  
(55)
then \( \Phi \) being \( k \)-positive is not \((k+1)\)-positive.

For the proof we refer to [20].

7. **Spectral class – examples**

Surprisingly this simple construction recovers many well known examples of positive maps.

**Example 7** Transposition in \( M_2 \). One finds
\[ \Phi_-(X) = \lambda_1 V_1 X V_1^* , \]  
(56)
and
\[ \Phi_+(X) = \sum_{\alpha=2}^{4} \lambda_\alpha V_\alpha X V_\alpha^* , \]  
(57)
where \( \lambda_1 = \ldots = \lambda_4 = 1 \), and
\[ V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ V_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \ V_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]  
(58)
It is easy to check that
\[ (V_\alpha, V_\beta)_{\text{HS}} = \delta_{\alpha\beta} . \]  
(59)
Moreover
\[ ||V_1||^2_1 = \frac{1}{2} \ , \ ||V_4||^2_2 = 1 . \]  
(60)
One easily finds \( \mu_1 = 1 \) and hence condition (53) is trivially satisfied \( \lambda_\alpha \geq \mu_1 \) for \( \alpha = 2, 3, 4 \).
Example 8 Reduction map in $M_d$

$$\Phi(X) = \mathbb{I}_d \text{Tr} X - X.$$  \hfill (61)

One finds

$$\Phi_-(X) = \lambda_1 V_1 X V_1^*,$$  \hfill (62)

and

$$\Phi_+(X) = \sum_{\alpha=2}^{D} \lambda_\alpha V_\alpha X V_\alpha^*,$$  \hfill (63)

where $\lambda_1 = d - 1$ and $\lambda_2 = \ldots = \lambda_D = 1$. Moreover

$$V_1 = \frac{1}{\sqrt{d}} \mathbb{I}_d,$$  \hfill (64)

and the remaining $V_\alpha$ are arbitrary $d \times d$ matrices such that

$$(V_\alpha, V_\beta)_{HS} = \delta_{\alpha\beta},$$  \hfill (65)

and $\text{Tr} V_\alpha = 0$ (for $\alpha = 2, \ldots, D$). Again, one finds $\mu_1 = 1$ and hence condition (53) is trivially satisfied $\lambda_\alpha \geq \mu_1$ for $\alpha = 2, \ldots, D = d^2$. Now, since $V_1$ corresponds to the maximally entangled state one has $1 - ||V_1||_2^2 = (d - 2)/d < 1$. Hence, condition (55)

$$\mu_2 = \frac{2d^2 - 2}{d^2 - 1} > \lambda_\alpha, \quad \alpha = 2, \ldots, D,$$  \hfill (66)

implies that $\Phi$ is not a 2-positive.

For other examples of positive maps belonging to our spectral class see [20, 21].

8. Conclusions

We provide partial classification of positive linear maps based on family of spectral conditions. It is shown that our class does contain well known examples of positive maps. We displayed interesting connection between positive (unital and trace preserving) maps and doubly stochastic matrices. It turns out [20] that our scheme may be easily generalized to the multipartite setting. The subject deserves further studies in order to investigate further properties of positive maps from the spectral class.

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