Rough bi-Heyting algebra and its applications on Rough bi-intuitionistic logic

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Abstract

A Rough semiring \((T, \Delta, \nabla)\) is considered to describe a special distributive Rough semiring known as a Rough bi-Heyting algebra. A bi-Heyting algebra is an extension of boolean algebra and it is accomplished by weaker notion of complements namely pseudocomplement \(\ast\), dual pseudocomplement \(\ast\), relative pseudocomplement \(\rightarrow\) and dual relative pseudocomplement \(\leftarrow\). In this paper, it is proved that the elements of the Rough semiring \((T, \Delta, \nabla)\) are accomplished with the pseudocomplement, relative pseudocomplement along with their duals. The definition of pseudocomplement leads to the concept of Brouwerian Rough semiring structure \((T, \Delta, \nabla, \ast, RS(\emptyset), RS(U))\) on the Rough semiring \((T, \Delta, \nabla)\). Also it is proved \((T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U))\) is a Rough bi-Heyting algebra. The concepts are illustrated with the examples. As an application, this Rough bi-Heyting algebra is used to model Rough bi-intuitionistic logic. The syntax is defined and three types of semantics for Rough bi-intuitionistic logic are defined and validated.

Keywords: Stone algebra, Bi-Heyting algebra, Bi-intuitionistic logic, Formal semantic, Kripke semantic, Algebraic semantic.

1 Introduction

Rough set theory was introduced by (Pawlak 1982) to process the incomplete information with the concept of lower and upper approximation in the approximation space \((U, R)\). Many research work has been carried out in rough set theory in connection with distributive lattices. (Praba et al 2015) in their work proved that the distributive lattice \(T\) with the binary operations Praba \(\Delta\) and Praba \(\nabla\) is a Rough semiring. In this work, we define the Rough bi-Heyting algebra taking the Rough semiring \((T, \Delta, \nabla)\) as our underlying structure. The idea behind defining Heyting algebra is from the Brouwer’s student, Arend Heyting in 1930. Arend developed the methods and concepts used to solve the problems arising in intuitionistic logic with the help of Heyting algebra. Whereas the bi-Heyting algebra was introduced to formalize bi-intuitionistic logic which is found in (Rauszer 1977) and (Rauszer 1974). Heyting algebra which results from the study of pseudocomplement and relative pseudocomplement contains the hierarchy of algebraic structures between boolean algebras and lattices. The concept of extending relative pseudocomplements to posets that are characterized through binary operation is pointed out by (Chajda 2020) and (Chajda 2018). The construction of Stone algebra for any distributive lattice is given in (Adam 2015) and the work is first initiated by (Raymond Balbes 1973). A Heyting algebra together with its properties of dual pseudocomplement and dual relative pseudocomplement can be studied in (Sankappanavar 1985).

The idea of defining hierarchy of increasing semantics in bi-intuitionistic logic from the least Kripke semantic to the general algebraic semantic is found in (Holliday and Guram 2019). Whereas the results on linear Kripke semantic and its characterization using logic have been found in (Arnold Beckmann 2017) and (Arnold Beckmann 2007). The implementation of logic-algebraic connection is seen in (Pagliani 1994), (Pagliani 1997) and which is later proved to form a regular double Stone algebra (Stephen Comer 1993). The considered Rough semiring \((T, \Delta, \nabla)\) establishes the relation between bi-intuitionistic logic and bi-Heyting algebra. This new approach in defining Rough bi-Heyting algebra is accomplished with the help
of (Praba and Mohan 2013), (Praba et al 2015) and (Manimaran et al 2017). In this paper, this idea of defining bi-Heyting algebra and its applications in bi-intuitionistic logic are extended to the semiring on Rough sets defined on a given approximation space $I = (U, R)$.

This paper is organized into the following sections. Preliminary definitions of a Rough semiring $(T, \Delta, \nabla)$ and bi-Heyting algebra are provided in Section 2. The existence of Rough bi-Heyting algebra on a Rough semiring $(T, \Delta, \nabla)$ is established in Section 3. Imposing the logical notions of bi-intuitionistic logic on Rough bi-Heyting algebra is discussed in Section 4. This paper’s conclusion and future work are presented in Section 5.

2 Preliminaries

In this section, the formal definitions of Rough set and bi-Heyting algebra are given.

2.1 Rough set

A structure $I = (U, R)$ where $U$ is a non-empty finite set of objects, called universe and $R$ is an arbitrary equivalence relation on $U$, is called an approximation space. The partition induced by the relation $R$ consists of equivalence class denoted $[x]_R$, is a subset of $U$ containing $X$. The lower and upper approximation defined for any $X \subseteq U$ is

$R_-(X) = \{x \in U \mid [x]_R \subseteq X\}$

$R^-(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$

Definition 1. (Pawlak 1982) If $X$ is an arbitrary subset of $U$, then the Rough set $RS(X)$ is an ordered pair $(R_-(X), R^-(X))$. The set of Rough sets is defined as $T = \{RS(X) \mid X \subseteq U\}$.

Definition 2. (Praba and Mohan 2013) Let $X, Y \subseteq U$. The Praba join of $X$ and $Y$ is denoted by $X \Delta Y$ and defined as $X \Delta Y = X \cup Y$ if $IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$. If $IW(X \cup Y) > IW(X) + IW(Y) - IW(X \cap Y)$, then the equivalence classes obtained by the union of $X$ and $Y$ are identified. The elements of that class belonging to $Y$ are deleted and the new set is named $Y$. Now, we obtain $X \Delta Y$. This process is repeated until

$IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$

Definition 3. (Praba and Mohan 2013) If $X, Y \subseteq U$, then an element $x \in U$ is called pivot element, if $[x]_R \not\subseteq X \cap Y$, but $[x]_R \cap X \neq \emptyset$ and $[x]_R \cap Y \neq \emptyset$.

Definition 4. (Praba and Mohan 2013) If $X, Y \subseteq U$, then the set of pivot elements of $X$ and $Y$ is called the pivot set of $X$ and $Y$ and is denoted by $P_{X \cap Y}$.

Definition 5. (Praba and Mohan 2013) Let $X, Y \subseteq U$. The Praba meet of $X$ and $Y$ is denoted $X \nabla Y$ and defined as $X \nabla Y = \{x \in U \mid [x]_R \subseteq X \cap Y\} \cup P_{X \cap Y}$. Here each pivot element in $P_{X \cap Y}$ is the representative of that particular class.

Theorem 1. (Praba and Mohan 2013) For any two sets $X, Y$ in $U$,

1. $RS(X \Delta Y)$ is the least upper bound of $RS(X)$ and $RS(Y)$

2. $RS(X \nabla Y)$ is the greatest lower bound of $RS(X)$ and $RS(Y)$

Theorem 2. (Praba et al 2015) For any given approximation space $I = (U, R)$, $(T, \Delta, \nabla)$ is a semiring called Rough semiring.

2.2 Bi-Heyting Igebra

Definition 6. Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. The pseudocomplement of the element $x \in L$ is the greatest element $x^*$ such that $x \wedge x^* = 0$.

Definition 7. A Stone algebra $(L, \vee, \wedge, ^*, 0, 1)$ is a pseudocomplemented distributive lattice in which $a^* \vee a^{**} = 1$ for all $a \in L$. 

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Definition 8. Let \((L, \lor, \land, 0, 1)\) be a bounded distributive lattice and let \(x, y \in L\). If \(\sup \{z \in L \mid x \land z \leq y\}\) exists, then it is said to be the relative pseudocomplement of \(x\) with respect to \(y\) and denoted by \(x \leftarrow y\).

Definition 9. A Heyting algebra \((L, \lor, \land, \to, 0, 1)\) is a distributive lattice with the least element such that a relative pseudocomplement exists for every pair of elements.

Definition 10. Let \((L, \lor, \land, 0, 1)\) be a bounded distributive lattice. The dual pseudocomplement of the element \(x \in L\) is the least element \(x^+\) such that \(x \lor x^+ = 1\).

Definition 11. A dual Stone algebra \((L, \lor, \land, ^+, 0, 1)\) is a dual pseudocomplemented distributive lattice in which \(a^+ \land a^{++} = 0\) for all \(a \in L\).

Definition 12. Let \((L, \lor, \land, 0, 1)\) be a bounded distributive lattice and let \(x, y \in L\). If \(\inf \{z \in L \mid z \lor y \geq x\}\) exists, then it is said to be the dual relative pseudocomplement of \(x\) with respect to \(y\) and denoted by \(x \leftarrow y\).

Definition 13. A dual Heyting algebra \((L, \lor, \land, ^-, 0, 1)\) is a distributive lattice with the greatest element such that a dual relative pseudocomplement exists for every pair of elements.

Definition 14. A bi-Heyting algebra \((L, \lor, \land, ^-, 0, 1)\) is a distributive lattice with a Heyting algebra and a dual Heyting algebra.

3 Rough bi-Heyting algebra

Throughout this section, we employ \(I = (U, R)\) is an approximation space and \((T, \Delta, \nabla)\) is a Rough semiring (Praba et al 2015).

Definition 15. In a Rough semiring \((T, \Delta, \nabla)\), the element \(RS(X) \in T\) is said to have pseudocomplement if there exists a greatest element \(RS(X)^* \in T\), disjoint from \(RS(X)\) such that \(RS(X) \nabla RS(X)^* = RS(X \nabla X)^* = RS(\emptyset)\). The pseudocomplement \(RS(X)^*\) is defined as

\[ RS(X)^* = \max_{RS(Y)} \{RS(Y) \in T \mid RS(X) \nabla RS(Y) = RS(X \nabla Y) = RS(\emptyset)\} \]

Definition 16. A pseudocomplemented Rough semiring \((T, \Delta, \nabla, ^*, RS(\emptyset), RS(U))\) is a distributive Rough semiring \((T, \Delta, \nabla)\) with the least element \(RS(\emptyset)\) such that every element in \(T\) has a pseudocomplement.

Note that the pseudocomplemented Rough semiring \((T, \Delta, \nabla, ^*, RS(\emptyset), RS(U))\) has a greatest element (say) \(RS(\emptyset)^* = RS(U)\). So that \(RS(\emptyset)^* \nabla RS(\emptyset)^* = RS(\emptyset)^* \nabla RS(U) = RS(\emptyset \nabla U) = RS(\emptyset)\). This pseudocomplemented Rough semiring \((T, \Delta, \nabla, ^*, RS(\emptyset), RS(U))\) is also called Brouwerian Rough semiring.

Theorem 3. The pseudocomplement of \(RS(X)\) is unique.

Proof. For any \(RS(X) \in T\), the pseudocomplement of \(RS(X)\) is defined by

\[ RS(X)^* = \max_{RS(Y)} \{RS(Y) \in T \mid RS(X) \nabla RS(Y) = RS(X \nabla Y) = RS(\emptyset)\} \]

Now claim \(RS(X)^* = RS(E - P^{-}(X))\) to prove \(RS(X) \nabla RS(E - P^{-}(X)) = RS(\emptyset)\). Let \(E\) be the set of equivalence classes defined on \(U\). Consider, \(RS(X) \nabla RS(E - P^{-}(X)) = RS(X \nabla E - P^{-}(X))\). Consider \(X \nabla (E - P^{-}(X)) = \{x \in U \mid [x]_R \subseteq X \cap E - P^{-}(X)\} \cup P_{Y \subseteq E - P^{-}(X)} = A \cup B\) (say) where \(A = \{x \in U \mid [x]_R \subseteq X \cap E - P^{-}(X)\}\) and \(B = P_{Y \subseteq E - P^{-}(X)}\). Since \(X \cap E - P^{-}(X) = \emptyset\), the set \(A = \emptyset\) and there is no pivot element which is common to \(X\) and \(E - P^{-}(X)\). Hence the set \(B = \emptyset\).

\[ : RS(X \nabla E - P^{-}(X)) = RS(\emptyset) \]

To Prove \(RS(X)^*\) is maximum

Let \(RS(X_1)^*\) be the maximum such that \(RS(X) \nabla RS(X_1)^* = RS(X \nabla X_1^*) = RS(\emptyset)\). To prove \(RS(X)^*\) is maximum, it is enough to prove \(RS(X)^* = RS(X_1)^*\). It is clear that

\[ RS(X)^* \subseteq RS(X_1)^* \]

(1)

Now to prove \(RS(X_1)^* \subseteq RS(X)^*\). Particularly, to prove \((P_-(X_1)^*), P^{-}(X_1)^*) \subseteq (P_-(X)^*), P^{-}(X)^*)\). So claim \(P_-(X_1)^* \subseteq P_-(X)^*\).

Let \(x \in P_-(X_1)^*\) then \([x]_R \subseteq X_1^*\) but \([x]_R \cap X = \emptyset\)
⇒ \{x\}_R \subseteq X^* (\text{since } X^*_1 \subseteq X^*)
⇒ x \in P_-(X^*)
∴ P_-(X^*_1) \subseteq P_-(X^*)

Now claim $P^-(X^*_1) \subseteq P^-(X^*)$. Let $y \in P^-(X^*_1)$ then $[y]_R \cap X^*_1 \neq \emptyset$. If $[y]_R \cap X \neq \emptyset$ then $y \in P_X \cup X^*_1$ is not possible since $RS(X \cup X^*_1) = RS(\emptyset)$.

\[
\begin{align*}
\Rightarrow [y]_R \cap X &= \emptyset \\
\Rightarrow [y]_R \cap X^* &\neq \emptyset \\
\Rightarrow y &\in P^-(X^*)
\end{align*}
\]
∴ $P^-(X^*_1) \subseteq P^-(X^*)$
∴ $(P_-(X^*_1), P^-(X^*)) \subseteq (P_-(X^*), P^-(X^*))$

$RS(X^*_1) \subseteq RS(X^*)$  \hspace{1cm} (2)

From (1) and (2), $RS(X)^* = RS(X^*_1)^*$

Hence the pseudocomplement $RS(X)^*$ exists and it is maximum. By the definition, if the pseudocomplement exists then it is unique. Therefore, the pseudocomplement of $RS(X)$ is unique.  \hspace{1cm} \Box

**Example 1.** (Praba et al 2015) Consider an approximation space $I = (U, R)$ where $U = \{x_1, x_2, x_3, x_4\}$ be the universe and $R$ is an equivalence relation defined on $U$.

Let $X = \{x_1, x_2, x_3, x_4\} \subseteq U$, then the equivalence classes induced by the relation $R$ is given by $X_1 = [x_1]_R = \{x_1, x_3\}$ and $X_2 = [x_2]_R = \{x_2, x_4\}$. Then the collection of Rough sets obtained is, $T = \{RS(\emptyset), RS(\{x_1\}), RS(\{x_2\}), RS(X_1), RS(X_2), RS(\{x_1\} \cup \{x_2\}), RS(X_1 \cup \{x_2\}), RS(\{x_1\} \cup X_2), RS(X_1 \cup X_2), RS(U)\}$

Table 1 defines the pseudocomplement of the Rough sets in $T$.

| $RS(X)$         | $RS(X)^*$       |
|-----------------|-----------------|
| $RS(\emptyset)$ | $RS(U)$         |
| $RS(\{x_1\})$  | $RS(X_1)$       |
| $RS(\{x_2\})$  | $RS(X_2)$       |
| $RS(\{x_1\} \cup \{x_2\})$, $RS(X_1 \cup \{x_2\})$, $RS(\{x_1\} \cup X_2)$, $RS(X_1 \cup X_2)$ | $RS(\emptyset)$ |

Table 1:

**Example 2.** (Praba and Mohan 2013) Consider an approximation space $I = (U, R)$ where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the universe and $R$ is an equivalence relation defined on $U$.

Let $X = \{x_1, x_3, x_5, x_6\} \subseteq U$, then the equivalence classes induced by the relation $R$ is given by $X_1 = [x_1]_R = \{x_1, x_3\}$, $X_2 = [x_2]_R = \{x_2, x_4\}$ and $X_3 = [x_5]_R = \{x_5\}$. Then the collection of Rough sets obtained are, $T = \{RS(\emptyset), RS(\{x_1\}), RS(\{x_2\}), RS(X_1), RS(X_2), RS(X_3), RS(\{x_1\} \cup \{x_2\}), RS(\{x_1\} \cup X_2), RS(\{x_1\} \cup X_3), RS(\{x_2\} \cup X_3), RS(\{x_2\} \cup X_3), RS(X_1 \cup X_2), RS(X_1 \cup X_3), RS(X_1 \cup X_3), RS(X_1 \cup X_3), RS(X_1 \cup X_3), RS(X_1 \cup X_3), RS(X_1 \cup X_3), RS(U)\}$.

The pseudocomplement of the Rough sets in $T$ are defined in Table 2.

| $RS(X)$         | $RS(X)^*$       |
|-----------------|-----------------|
| $RS(\emptyset)$ | $RS(U)$         |
| $RS(\{x_1\})$, $RS(X_1)$ | $RS(X_2 \cup X_3)$ |
| $RS(\{x_2\})$, $RS(X_2)$ | $RS(X_1 \cup X_3)$ |
| $RS(X_1)$       | $RS(X_1 \cup X_3)$ |
| $RS(\{x_1\} \cup \{x_2\})$, $RS(X_1 \cup \{x_2\})$, $RS(\{x_1\} \cup X_2)$, $RS(X_1 \cup X_2)$ | $RS(\emptyset)$ |
| $RS(\{x_1\} \cup X_2)$, $RS(X_1 \cup X_3)$ | $RS(X_1 \cup X_3)$ |
| $RS(\{x_2\} \cup X_3)$, $RS(X_2 \cup X_3)$ | $RS(X_2)$ |
| $RS(\{x_2\} \cup X_3)$, $RS(X_2 \cup X_3)$ | $RS(X_1)$ |
| $RS(\{x_1\} \cup \{x_2\} \cup X_3)$, $RS(X_1 \cup \{x_2\} \cup X_3)$, $RS(\{x_1\} \cup X_2 \cup X_3)$, $RS(X_1 \cup X_2 \cup X_3)$ | $RS(\emptyset)$ |

Table 2:

**Example 3.** The following example indicates that $RS(X)^* \Delta RS(X)^* = RS(U)$ is not always true.

From Example-2, let $E = \{X_1, X_2, X_3\}$ be the set of equivalence classes obtained by the relation $R$ on $U$. Consider $RS(\{x_1\} \cup \{x_2\}) \in T$ and its pseudocomplement is given by $RS(\{x_1\} \cup \{x_2\})^* = \{x_1, x_2, x_3\}$.
Proof. Consider, satisfy the condition of relative pseudocomplement such that
\[ RS(E - P^-(\{x_1\} \cup \{x_2\})) = RS(E - X_1 \cup X_2) = RS(X_3) \]
So that \( RS(\{x_1\} \cup \{x_2\}) \Delta RS(\{x_1\} \cup \{x_2\})^* = RS(\{x_1\} \cup \{x_2\}) \Delta RS(X_3) = RS(\{x_1\} \cup \{x_2\} \cup X_3) \neq RS(U) \).

Similarly for \( RS(X_2) \in T \) and its pseudocomplement \( RS(X_2)^* = RS(E - P^-(X_2)) = RS(X_1 \cup X_3) \) such that \( RS(X_2) \Delta RS(X_2)^* = RS(X_2) \Delta RS(X_1 \cup X_3) = RS(U) \).

**Remark 1.** When \( X \) is the union of one (or) more equivalence classes then \( RS(X) \Delta RS(X)^* = RS(U) \).

**Definition 17.** A pseudocomplemented Rough semiring \( (T, \Delta, \nabla,^*, RS(\emptyset), RS(U)) \) is called a Rough Stone algebra if \( RS(X)^* \Delta RS(X)^* = RS(U) \) for all \( RS(X) \in T \).

**Lemma 1.** The pseudocomplemented Rough semiring \( (T, \Delta, \nabla,^*, RS(\emptyset), RS(U)) \) is a Rough Stone algebra.

**Proof.** To prove \( T \) is a Rough Stone algebra, it is enough to show that for any \( RS(X) \in T \),
\[ RS(X)^* \Delta RS(X)^* = RS(U) \tag{3} \]
The pseudocomplement of any Rough set \( RS(X) \in T \) is defined to be
\[ RS(X)^* = RS(E - P^-(X)) \]
and so,
\[ RS(X)^* = RS(E - P^-(X))^* \]
\[ = RS(E - P^-(E - P^-(X))) \text{ (since } RS(X)^* = RS(E - P^-(X)) \text{)} \]
\[ = RS(E - (E - P^-(X))) = RS(P^-(X)) \]
Hence \( RS(X)^* \Delta RS(X)^* = RS(E - P^-(X)) \Delta RS(P^-(X)) = RS(E - P^-(X)) \Delta P^-(X) = RS(E) = RS(U) \).

**Example 4.** (Praba et al 2015) (Praba and Mohan 2013) The following example shows that the Rough Stone algebra condition is satisfied by the elements of \( T \).

From Example-1, let \( E = \{X_1, X_2\} \) be the set of equivalence classes obtained by the relation \( R \) on \( U \). Consider, \( RS(\{x_2\})^* = RS(E - P^-(\{x_2\})) = RS(E - X_2) = RS(X_1) \) and \( RS(\{x_2\})^* = RS(P^-(\{x_2\})) = RS(X_2) \). Therefore, \( RS(\{x_2\})^* \Delta RS(\{x_2\})^* = RS(X_1) \Delta RS(X_2) = RS(U) \).

From Example-2, let \( E = \{X_1, X_2, X_3\} \) be the set of equivalence classes in which \( RS(\{x_1\} \cup X_3) = RS(E - P^-(\{x_1\} \cup X_3)) = RS(E - X_1 \cup X_3) = RS(X_2) \) and \( RS(\{x_1\} \cup X_3)^* = RS(P^-(\{x_1\} \cup X_3)) = RS(X_1 \cup X_3) \). Therefore, \( RS(\{x_1\} \cup X_3)^* \Delta RS(\{x_1\} \cup X_3)^* = RS(X_2) \Delta RS(X_1 \cup X_3) = RS(U) \). Thus, any Rough set in \( T \) satisfies the condition of Rough Stone algebra.

**Remark 2.** From Example-4, the defined \( RS(X)^* \) and \( RS(X)^* \) also satisfies that \( RS(X)^* \Delta RS(X)^* = RS(\emptyset) \).

**Definition 18.** A element \( RS(Y \rightarrow Z) \) is a relative pseudocomplement of \( RS(Y) \) with respect to \( RS(Z) \) such that
\[ RS(Y) \rightarrow RS(Z) = \sup_{RS(W)} \{ RS(W) \in T \mid RS(Y) \nabla RS(W) \leq RS(Z) \} \]

**Remark 3.** The pseudocomplement of \( RS(X) \) is also represented in terms of relative pseudocomplement as \( RS(X)^* = RS(X) \rightarrow RS(\emptyset) \).

**Theorem 4.** The relative pseudocomplement of \( RS(Y) \) with respect to \( RS(Z) \) is unique.

**Proof.** Consider,
\[ RS(Y \rightarrow Z) = RS((Y \nabla Z) \Delta (E - P^-(Y - Y \nabla Z))) \]
satisfy the condition of relative pseudocomplement such that \( RS(Y) \nabla RS(Y \rightarrow Z) \leq RS(Z) \)
\[ \iff RS(Y \nabla (Y \rightarrow Z)) \leq RS(Z) \]
\[ \Rightarrow RS(Y \nabla ((Y \nabla Z) \Delta (E - P^-(Y - Y \nabla Z))) \leq RS(Z) \]
\[ \iff RS(Y \neg ((Y \neg Z) \Delta E - P^-(Y - Y \neg Z))) \leq RS(Z) \] (4)

The relative pseudocomplement condition is verified by solving (4) in the following cases.

**Case 1**
Assume \( Y \neg Z = \emptyset \). Then from (4)
\[ RS(\emptyset) \leq RS(Z) \]

**Case 2**
Assume \( Y \neg Z \neq \emptyset \). Then from (4), consider
\[ RS(Y \neg ((Y \neg Z) \Delta E - P^-(Y - Y \neg Z))) = RS(Y \neg (Y \neg Z) \Delta E - P^-(Y - Y \neg Z)) \leq RS(Y \neg Z) \leq RS(Z) \]
\[ : RS(Y \neg ((Y \neg Z) \Delta E - P^-(Y - Y \neg Z))) \leq RS(Z) \]

Now to prove \( RS(Y \rightarrow Z) \) is maximum. Suppose if there exists \( RS(K) \in T \) satisfying the condition in (4) such that
\[ RS(K \neg Y) \leq RS(Z) \] (5)

It is enough to prove \( RS(K) \leq RS(Y \rightarrow Z) \). Let \( x \in K \) and if \( x \in Y \neg Z \) then \( x \in Y \rightarrow Z \)

Therefore, \( K \subseteq Y \rightarrow Z \)
\[ RS(K) \leq RS(Y \rightarrow Z) \]

On the other hand, let \( x \in K \) and \( x \notin Y \neg Z \). Suppose if \( x \in Y \), then \( x \in K \neg Y \)
\[ \Rightarrow x \in Z \text{ (by (5))} \]
\[ \Rightarrow x \in Y \neg Z \]

which is a contradiction.

Therefore, \( x \notin Y \neg Z \)
\[ \Rightarrow x \notin Y \]
\[ \Rightarrow x \notin P^-(Y - Y \neg Z) \]
\[ \Rightarrow x \in E - P^-(Y - Y \neg Z) \]
\[ \Rightarrow x \in Y \rightarrow Z \]

Therefore, \( K \subseteq Y \rightarrow Z \) and
\[ RS(K) \leq RS(Y \rightarrow Z) \]

Thus \( RS(Y \rightarrow Z) \) is the maximum element in \( T \) satisfying the condition of relative pseudocomplement. Hence the pseudocomplement of \( RS(Y) \) relative to \( RS(Z) \) exists and it is unique. \( \square \)

**Example 5.** (Praba et al 2015)(Praba and Mohan 2013) Consider an approximation space \( I = (U, R) \) as in Example-1 and Example-2, then the Table-3 and Table-4.1 & 4.2, shows the relative pseudocomplement between the elements of \( T \)

| \( \rightarrow \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) | \( RS(\emptyset) \) | \( RS(x_1) \) | \( RS(x_2) \) | \( RS(x_1 \cup x_2) \) | \( RS(x_1 \cup \{x_2\}) \) | \( RS(x_1 \cup \{x_1\}) \) | \( RS(x_2 \cup \{x_1\}) \) |

Table 3:

**Definition 19.** A Rough Heyting algebra (or) Rough pseudo-boolean algebra \((T, \Delta, \nabla, \rightarrow, RS(\emptyset), RS(U))\) is a distributive Rough semiring \((T, \Delta, \nabla)\) with the least element \( RS(\emptyset) \) in which the relative pseudocomplement is defined for every pair of elements in \( T \).
Theorem 5. For any given approximation space \( I = (U, R) \), the Rough semiring \((T, \Delta, \forall)\) is a Rough Heyting algebra.

**Proof.** \((T, \Delta, \forall)\) be a distributive Rough semiring (Praba et al 2015) and by Theorem-4, the relative pseudocomplement exists for every pair of elements in \( T \). Therefore, \((T, \Delta, \forall, \rightarrow, RS(\emptyset), RS(U))\) is a Rough Heyting algebra. \(\blacksquare\)

**Definition 20.** Let \((T, \Delta, \forall)\) be a Rough semiring. The dual pseudocomplement of an element \( RS(X) \) in \( T \), is the least element \( RS(X)^{\perp} \) such that \( RS(X)^{\perp}AR(X)^{\perp} = RS(U) \) and it is defined as

\[
RS(X)^{\perp} = \min_{RS(Y) \in T} \{ RS(Y) \mid RS(X)\Delta RS(Y) = RS(X\forall Y) = RS(U) \}
\]

**Theorem 6.** The dual pseudocomplement of \( RS(X) \) is unique.

**Proof.** For every \( RS(X) \in T \), the dual pseudocomplement of \( RS(X)^{\perp} \) is defined as,

\[
RS(X)^{\perp} = \min_{RS(Y) \in T/RS(X)} \{ RS(Y) \mid RS(X)\Delta RS(Y) = RS(X\forall Y) = RS(U) \}
\]

Let \( E \) be the set of equivalence classes defined on \( U \). Now claim \( RS(X)^{\perp} = RS(E - P_{-}(X)) \), to prove \( RS(X)\Delta RS(X)^{\perp} = RS(U) \). Consider,

\[
RS(X)\Delta RS(E - P_{-}(X)) = RS(X\Delta E - P_{-}(X))
\]

Here,

\[
X \forall E - P_{-}(X) = X \cap E - P_{-}(X) \text{ if}
\]

\[
IW(X \cup E - P_{-}(X)) = IW(X) + IW(E - P_{-}(X)) - IW(X \cap E - P_{-}(X))
\]

\[
\therefore RS(X\Delta E - P_{-}(X)) = RS(X \cup E - P_{-}(X)) = RS(U)
\]

**To Prove** \( RS(X)^{\perp} \) is minimum

Choose another minimal dual pseudocomplement \( RS(X_{1})^{\perp} \) such that \( RS(X)\Delta RS(X_{1})^{\perp} = RS(X\Delta X_{1}^{\perp}) =
Now to prove, \( RS(X)^+ \) is minimum, it is enough to show that \( RS(X)^+ = RS(X_1)^+ \). It is clear that

\[
RS(X_1)^+ \subseteq RS(X)^+
\]

(6)

Now to prove, \( RS(X)^+ \subseteq RS(X_1)^+ \) (i.e) To prove \((P_-(X^+), P^-(X^+)) \subseteq (P_-(X_1^+), P^-(X_1^+))\)

Claim: \( P_-(X^+) \subseteq P_-(X_1^+) \)

Let \( x \in P_-(X^+) \) then \([x]_R \subseteq X^+\)

\[
\Rightarrow [x]_R \subseteq X_1^+ \quad \text{(since } X^+ \subseteq X_1^+) \\
\Rightarrow x \in P_-(X_1^+)
\]

\[
\therefore P_-(X^+) \subseteq P_-(X_1^+)
\]

Now claim, \( P^-(X^+) \subseteq P^-(X_1^+) \)

Let \( y \in P^-(X^+) \) then \([y]_R \cap X^+ \neq \emptyset \) (since \( RS(X \Delta X^+) = RS(U) \)). If \([y]_R \cap X = \emptyset \) then \([y]_R \cap X_1^+ \neq \emptyset \) (since \( RS(X \Delta X^+) = RS(U) \))

\[
\Rightarrow y \in P^-(X_1^+)
\]

\[
\therefore P^-(X^+) \subseteq P^-(X_1^+)
\]

If \([y]_R \cap X \neq \emptyset \) then \( y \in P_{X \cap X^+} \) (since \( RS(X \Delta X^+) = RS(U) \))

\[
\Rightarrow [y]_R \cap X_1^+ \neq \emptyset \\
\Rightarrow y \in P^-(X_1^+)
\]

\[
\therefore P^-(X^+) \subseteq P^-(X_1^+)
\]

\[
RS(X)^+ \subseteq RS(X_1)^+ 
\]

(7)

From (6) and (7), it is obtained that \( RS(X)^+ = RS(X_1)^+ \). Therefore \( RS(X)^+ \) is minimum and it exists for any \( RS(X) \in T \). By the definition, if the dual pseudocomplement exists then it is unique. Hence the dual pseudocomplement of \( RS(X) \) is unique.

**Example 6.** (Praba et al 2015) (Praba and Mohan 2013) For an approximation space \( I = (U, R) \) where \( U \) is the non-empty finite set as in Example-1 and Example-2, the dual pseudocomplement of the elements of \( T \) are given in Table 6 and Table 7

| \( RS(X) \) | \( RS(X)^+ \) |
|---|---|
| \( RS(\emptyset) \), \( RS(\{x_1\}) \), \( RS(\{x_2\}) \), \( RS(\{x_1\} \cup \{x_2\}) \) | \( RS(U) \) |
| \( RS(X_1) \), \( RS(X_1 \cup \{x_2\}) \) | \( RS(X_1) \) |
| \( RS(X_2) \), \( RS(\{x_1\} \cup X_2) \) | \( RS(\emptyset) \) |
| \( RS(X_1 \cup X_2) \) | \( RS(\emptyset) \) |

*Table 6:*

| \( RS(X) \) | \( RS(X)^+ \) |
|---|---|
| \( RS(\emptyset) \), \( RS(\{x_1\}) \), \( RS(\{x_2\}) \), \( RS(\{x_1\} \cup \{x_2\}) \) | \( RS(U) \) |
| \( RS(X_1) \), \( RS(X_1 \cup \{x_2\}) \) | \( RS(X_2 \cup X_3) \) |
| \( RS(X_2) \), \( RS(\{x_1\} \cup X_2) \) | \( RS(X_1 \cup X_2) \) |
| \( RS(X_1) \), \( RS(\{x_1\} \cup x_3) \), \( RS(\{x_2\} \cup x_3) \), \( RS(\{x_1\} \cup \{x_2\} \cup X_3) \), \( RS(\{x_1\} \cup \{x_2\} \cup \{x_3\}) \) | \( RS(X_1 \cup X_2) \) |
| \( RS(X_1 \cup X_2) \) | \( RS(X_3) \) |
| \( RS(X_1 \cup X_3) \), \( RS(X_1 \cup \{x_2\} \cup X_3) \) | \( RS(X_2) \) |
| \( RS(X_2 \cup X_3) \), \( RS(\{x_1\} \cup X_2 \cup X_3) \) | \( RS(X_1) \) |
| \( RS(X_1 \cup X_2 \cup X_3) \) | \( RS(\emptyset) \) |

*Table 7:*

**Example 7.** The following example shows that \( RS(X) \lor RS(X)^+ = RS(\emptyset) \) is not always true.

From Example-2, let \( E = \{X_1, X_2, X_3\} \) be the set of equivalence classes. Consider \( RS(\{x_1\} \cup \{x_2\} \cup X_3) \in T \) whose dual pseudocomplement is given by \( RS(\{x_1\} \cup \{x_2\} \cup X_3)^+ = RS(E - X_3) = RS(X_1 \cup X_2) \)
So that \( RS(\{x_1\} \cup \{x_2\} \cup X_3) \nabla RS(\{x_1\} \cup \{x_2\} \cup X_3)^+ = RS(\{x_1\} \cup \{x_2\} \cup X_3) \nabla RS(X_1 \cup X_2) = RS(\{x_1\} \cup \{x_2\}) \).

Similarly for \( RS(X_2 \cup X_3) \in T \) and its dual pseudocomplement \( RS(X_2 \cup X_3)^+ = RS(E - X_2 \cup X_3) = RS(X_1) \) such that \( RS(X_2 \cup X_3) \nabla RS(X_2 \cup X_3)^+ = RS(X_2 \cup X_3) \nabla RS(X_1) = RS(\emptyset) \).

But this condition is satisfied when \( X \) is the union of one (or) more equivalence classes.

**Definition 21.** A dual Rough Stone algebra is a dual pseudocomplemented Rough semiring \( (T, \Delta, \nabla, ^+, RS(\emptyset), RS(U)) \) satisfying \( RS(X)^+ \nabla RS(X)^{++} = RS(\emptyset) \) for all \( RS(X) \in T \).

**Lemma 2.** The dual pseudocomplemented Rough semiring \( (T, \Delta, \nabla, ^+, RS(\emptyset), RS(U)) \) is a dual Rough Stone algebra.

**Proof.** Proof is straight forward. \( \square \)

**Example 8.** (Praba et al 2015)(Praba and Mohan 2013) The following example describe the elements of \( T \), satisfying the dual Rough Stone algebra condition.

From Example-1, let \( E = \{X_1, X_2\} \) be the set of equivalence classes. Consider, \( RS(X_1 \cup \{x_2\})^+ = RS(E - X_1) = RS(X_2) \) and \( RS(X_1 \cup \{x_2\})^{++} = RS(X_1) \). Therefore, \( RS(X_1 \cup \{x_2\})^+ \nabla RS(X_1 \cup \{x_2\})^{++} = RS(X_2) \nabla RS(X_1) = RS(\emptyset) \).

From Example-2, let \( E = \{X_1, X_2, X_3\} \) be the set of equivalence classes in which \( RS(\{x_1\} \cup X_2 \cup X_3)^+ = RS(E - X_2 \cup X_3) = RS(X_1) \) and \( RS(\{x_1\} \cup X_2 \cup X_3)^{++} = RS(X_2 \cup X_3) \). Therefore, \( RS(\{x_1\} \cup X_2 \cup X_3)^+ \nabla RS(\{x_1\} \cup X_2 \cup X_3)^{++} = RS(X_1) \nabla RS(X_2 \cup X_3) = RS(\emptyset) \). Therefore, the dual Rough Stone algebra condition is satisfied by the elements of \( T \).

**Definition 22.** A double Rough Stone algebra \( (T, \Delta, \nabla, ^*, ^+, RS(\emptyset), RS(U)) \) is a Rough Stone algebra and dual Rough Stone algebra with the unary operation of pseudocomplementation and dual pseudocomplementation.

**Definition 23.** A element \( RS(Y \leftarrow Z) \) is the dual relative pseudocomplement of \( RS(Y) \) with respect to \( RS(Z) \) such that

\[
RS(Y) \leftarrow RS(Z) = \inf_{RS(V)} \{ RS(V) \in T \mid RS(Z) \Delta RS(V) \geq RS(Y) \}
\]

**Remark 4.** The dual pseudocomplement of \( RS(X) \in T \) is expressed in terms of dual relative pseudocomplement and it is defined by \( RS(X)^+ = RS(U) \leftarrow RS(X) \).

**Theorem 7.** The dual relative pseudocomplement of \( RS(Y) \) with respect to \( RS(Z) \) is unique.

**Proof.** Consider,

\[
RS(Y \leftarrow Z) = RS(Y \nabla P^- (Y - Y \nabla Z))
\]
satisfy the condition that \( RS(Z) \Delta RS(Y \leftarrow Z) \geq RS(Y) \)

\[
RS(Z) \Delta RS(Y \nabla P^- (Y - Y \nabla Z)) \geq RS(Y)
\] (8)

**Case 1**
Assume \( Y \nabla Z = \emptyset \). Then (8) becomes

\[
RS(Z) \Delta RS(Y \nabla P^- (Y)) = RS(Z \Delta Y \nabla P^- (Y)) = RS(\{Z \Delta Y\} \nabla (Z \Delta P^- (Y))) \geq RS(Y \nabla Y) = RS(Y)
\]
\[
\therefore RS(Z) \Delta RS(Y \leftarrow Z) \geq RS(Y)
\]

**Case 2**
Assume \( Y \nabla Z \neq \emptyset \). Then from (8)

\[
RS(Z) \Delta RS(Y \nabla P^- (Y - Y \nabla Z)) = RS(\{Z \Delta Y\} \nabla (Z \Delta P^- (Y - Y \nabla Z)))
\]
It is clear that, $RS(Z \Delta Y) \geq RS(Y)$
Claim: $RS(Z \Delta P^-(Y - Y \nabla Z)) \geq RS(Y \Delta Z)$
It is clear that $RS(Z) \leq RS(Z \Delta P^-(Y - Y \nabla Z))$ and it remains to show that $RS(Y) \leq RS(Z \Delta P^-(Y - Y \nabla Z))$. Let $x \in Y$ then either $x \notin Y \nabla Z$ or $x \in Y \nabla Z$.
If $x \notin Y \nabla Z$ then $x \in Y - Y \nabla Z$

$$\Rightarrow x \in P^-(Y - Y \nabla Z)$$
$$\Rightarrow x \in Z \Delta P^-(Y - Y \nabla Z)$$
$$\therefore Y \subseteq Z \Delta P^-(Y - Y \nabla Z)$$
$$\therefore RS(Y) \leq RS(Z \Delta P^-(Y - Y \nabla Z))$$

If $x \in Y \nabla Z$ then $x \notin Y - Y \nabla Z$

$$\Rightarrow x \notin P^-(Y - Y \nabla Z)$$
$$\Rightarrow x \in Z \Delta P^-(Y - Y \nabla Z) \text{ (since } x \in Y \nabla Z)$$
$$\therefore Y \subseteq Z \Delta P^-(Y - Y \nabla Z)$$
$$\therefore RS(Y) \leq RS(Z \Delta P^-(Y - Y \nabla Z))$$
$$\therefore RS(Z \Delta P^-(Y - Y \nabla Z)) \geq RS(Y \Delta Z)$$

Then (8) becomes

$$RS(Z) \Delta RS(Y \nabla P^-(Y - Y \nabla Z)) = RS((Z \Delta Y) \nabla (Z \Delta P^-(Y - Y \nabla Z)) \geq RS(Z \Delta Y)$$
$$\geq RS(Y)$$
$$\therefore RS(Z) \Delta RS(Y \leftarrow Z) \geq RS(Y)$$

Now to prove $RS(Y \leftarrow Z)$ is minimum. If there exists $RS(K) \in T$ satisfying (8) such that

$$RS(K \Delta Z) \geq RS(Y) \tag{9}$$

It is to prove now $RS(Y \leftarrow Z) \leq RS(K)$. Let $x \in Y \leftarrow Z$

$$\Rightarrow x \in Y \nabla P^-(Y - Y \nabla Z)$$
$$\Rightarrow x \in Y \text{ and } x \notin Z$$
$$\Rightarrow x \in K \Delta Z \text{ (by (9))}$$

But $x \notin Z$

$$\Rightarrow x \in K$$
$$Y \leftarrow Z \subseteq K$$
$$RS(Y \leftarrow Z) \leq RS(K)$$

Therefore $RS(Y \leftarrow Z)$ is the minimum element in $T$ for $RS(Y)$ relative to $RS(Z)$ exists and it is unique.

**Example 9.** (Praba et al 2015) (Praba and Mohan 2013) For a given approximation space $I = (U, R)$ as in Example-1 and Example-2, the Table 8 and Table 9.1.9.2 shows the dual relative pseudocomplement between the elements of $T$

| → | RS(Ø) | RS({x1}) | RS({x2}) | RS({x1, x2}) | RS(x1) | RS(x2) | RS({x1} ∪ {x2}) | RS(x1) ∪ {x2} | RS(U) |
|---|---|---|---|---|---|---|---|---|---|
| RS(Ø) | RS(Ø) | RS(Ø) | RS(Ø) | RS(Ø) | RS(Ø) | RS(Ø) | RS(Ø) | RS(U) |
| RS({x1}) | RS({x1}) | RS(Ø) | RS({x1}) | RS(Ø) | RS({x1}) | RS(Ø) | RS({x1}) | RS(U) |
| RS({x2}) | RS({x2}) | RS({x2}) | RS(Ø) | RS({x2}) | RS(Ø) | RS({x2}) | RS(U) |
| RS(X1) | RS(X1) | RS(X1) | RS(X1) | RS(X1) | RS(X1) | RS(X1) | RS(U) |
| RS(X2) | RS(X2) | RS(X2) | RS(X2) | RS(X2) | RS(X2) | RS(X2) | RS(U) |
| RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(X1 ∪ X2) | RS(U) |

Table 8:
| ~ | RS(∅) | RS(∪∅) | RS(∪∅) | RS(X∪Y) | RS(Y∪X) | RS(X∪Y) | RS(Y∪X) | RS((X∪Y)∪X) | RS((Y∪X)∪X) | RS(Y∪(X∪Y)) | RS((X∪Y)∪(Y∪X)) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| RS(∅) | RS(∅) | RS(∅) | RS(∅) | RS(∅) | RS(X∪Y) | RS(Y∪X) | RS(X∪Y) | RS((X∪Y)∪X) | RS((Y∪X)∪X) | RS(Y∪(X∪Y)) | RS((X∪Y)∪(Y∪X)) |
| RS(∪∅) | RS(∅) | RS(∅) | RS(∅) | RS(∅) | RS(X∪Y) | RS(Y∪X) | RS(X∪Y) | RS((X∪Y)∪X) | RS((Y∪X)∪X) | RS(Y∪(X∪Y)) | RS((X∪Y)∪(Y∪X)) |
| RS(∪∅) | RS(∅) | RS(∅) | RS(∅) | RS(∅) | RS(X∪Y) | RS(Y∪X) | RS(X∪Y) | RS((X∪Y)∪X) | RS((Y∪X)∪X) | RS(Y∪(X∪Y)) | RS((X∪Y)∪(Y∪X)) |
| RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) |
| RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) | RS(Y∪X) |
| RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) | RS(X∪Y) |
| RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) | RS((X∪Y)∪X) |
| RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) | RS((Y∪X)∪X) |
| RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) | RS(Y∪(X∪Y)) |
| RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) | RS((X∪Y)∪(Y∪X)) |

Table 9.1:

Table 9.2:

**Definition 24.** A dual Rough Heyting algebra \( (T, \Delta, \nabla, \rightarrow, RS(\emptyset), RS(U)) \) (or) dual Rough pseudo-boolean algebra is a distributive Rough semiring \( (T, \Delta, \nabla) \) with the greatest element \( RS(U) \) in which the dual relative pseudocomplement is defined for every pair of elements in \( T \).

**Theorem 8.** For the given approximation space \( I = (U, R) \), the Rough semiring \( (T, \Delta, \nabla) \) is a dual Rough Heyting algebra.

Proof. \((T, \Delta, \nabla)\) be a Rough semiring (Praba et al 2015) and from Theorem-7, the dual relative pseudocomplement exists for every pair of elements in \( T \). Hence \((T, \Delta, \nabla, \rightarrow, RS(\emptyset), RS(U))\) is a dual Rough Heyting algebra.

**Definition 25.** A Rough bi-Heyting algebra \( (T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U)) \) is a Rough semiring \( (T, \Delta, \nabla) \) with a Rough Heyting algebra and a dual Rough Heyting algebra.

**Theorem 9.** For the given approximation space \( I = (U, R) \) and a Rough semiring \( (T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U)) \) is a Rough bi-Heyting algebra.

Proof. The proof is straight forward.

**Remark 5.** Let \((T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U))\) be a Rough bi-Heyting algebra. Then the following properties

- \( RS(X) \rightarrow RS(Y) = RS(U) \) if and only if \( RS(X) \leq RS(Y) \)
- \( RS(X) \rightarrow RS(U) = RS(U) \)
- \( RS(\emptyset) \rightarrow RS(X) = RS(U) \)
- \( RS(U) \rightarrow RS(X) = RS(X) \)
- \( RS(X) \leftarrow RS(Y) = RS(\emptyset) \) if and only if \( RS(X) \leq RS(Y) \)
- \( RS(X) \leftarrow RS(U) = RS(\emptyset) \)
• \( RS(X) \leftarrow RS(\emptyset) = RS(X) \)
• \( RS(\emptyset) \leftarrow RS(X) = RS(\emptyset) \)

holds for all \( RS(X), RS(Y) \in T \).

4 Rough bi-intuitionistic logic on Rough bi-Heyting algebra

Bi-intuitionistic logic was introduced by (Rauszer 1974) through the axiomatic calculus, which is an conventional extension of intuitionistic propositional logic. This bi-intuitionistic logic has a binary operation \( \leftarrow \), dual to the intuitionistic logic and dual pseudocomplement \( + \) definable from \( \leftarrow \). This section aims to introduce the Rough Kripke semantic and Rough algebraic semantic for the Rough bi-intuitionistic logic. In literature, bi-Heyting algebras are used as models for bi-intuitionistic reasoning. Arend Heyting had previously established intuitionistic logic to formalize Brouwer’s intuitionism. As a result, intuitionistic logic’s ideas were derived from classical logic. The difference in their interpretation is that in intuitionistic logic, the law of excluded middle, which is admissible in classical logic, is not allowed. The reasoning behind rough and vague information is based on bi-intuitionistic logic in this context.

Consider an approximation space \( I = (U, R) \), in which \( U \) is a non-empty finite set of objects and \( R \) is an equivalence relation on \( U \). The equivalence classes \( \{X_1, X_2, ... X_n\} \) are obtained using \( R \). In this case, the cardinality of the \( m \) equivalence classes \( \{X_1, X_2, ... X_m\} \) is more than one, but the cardinality of the remaining \( n - m \) equivalence classes \( \{X_{m+1}, X_{m+2}, ... X_n\} \) is equal to one. Let us suppose that each object in \( U \) symbolizes a logical statement (or) a proposition, with a truth-value of either true (or) false, indicated by 1 (or) 0 correspondingly, to introduce the logical concepts in the provided approximation space. The mathematical assertions provided by Rough bi-intuitionistic logic represent propositional variables (or) algebraic models built using the connectives Praba \( \Delta \), Praba \( \nabla \), \( \rightarrow \) and \( \leftarrow \). The logic in obtaining the formula \( RS(\Phi) \) with finite constructions by using a propositional variable is known as Rough bi-intuitionistic propositional calculus(RBi-IPC). This is extended to Rough bi-intuitionistic predicate calculus(RBi-IQC) with the universal quantifiers \( \forall \) (for all) and existential quantifier \( \exists \) (there exists). Thus Rough bi-Heyting algebra is taken as a fundamental structure to model the algebraic models of Rough bi-intuitionistic logic.

Definition 26. Let \( \{X_1, X_2, ... X_n\} \) be the collection of equivalence classes. If \( X_i = \{x_{i_1}, x_{i_2}, ... x_{i_i}\} \), then the truth value of an equivalence class \( X_i \) is defined by,

\[
Tr(X_i) = Tr(x_{i_1}).Tr(x_{i_2}) \ldots Tr(x_{i_i}) \text{, where } x_{i_1}, x_{i_2}, ... x_{i_i} \in X_i \text{ for } i = 1, 2, ... n
\]

where \( \{x_{i_1}, x_{i_2} ... x_{i_i}\} \) denotes the logical statement belonging to the equivalence class \( X_i \) taking the truth value 0 (or) 1.

Definition 27. For any \( RS(X) \in T \), the truth value of \( RS(X) \) is defined by

\[
Tr(RS(X)) = (Tr(R_{\leftarrow}(X)), Tr(R_{\rightarrow}(X)))
\]

where \( Tr(R_{\leftarrow}(X)) = max\{Tr(X_k) | X_k \in R_{\leftarrow}(X)\} \) and \( Tr(R_{\rightarrow}(X)) = max\{Tr(X_l) | X_l \in R_{\rightarrow}(X)\} \)

Note that the possible pair of truth values obtained for any element in \( T \) are \((0,0),(0,1)\) and \((1,1)\) by the assumption of truth value 0 (or) 1 to the logical statement in \( U \).

4.1 Syntax of Rough bi-intuitionistic logic

Let us begin by defining a finite set of atomic propositions in \( U \), where the elements of \( U \) are called propositional variables. For any logical statement in \( U \), the statement formula is obtained in such a way that every pair of elements in \( T \) are connected using the set of bi-intuitionistic connectives \( \{\Delta, \nabla, \rightarrow, \leftarrow\} \). The formulas of Rough bi-intuitionistic logic is inductively defined as,

1. Every propositional variable \( \{x_1, x_2, ... x_r\} \) is a formula (or) a well-formed formula.
2. If \( RS(\Phi) \) and \( RS(\Psi) \) are well-formed formula, then \( RS(\Phi)^+, RS(\Phi)^+, RS(\Phi)\Delta RS(\Psi), RS(\Phi)\nabla RS(\Psi), RS(\Phi) \rightarrow RS(\Psi) \) and \( RS(\Phi) \leftarrow RS(\Psi) \) are also well-formed formulas.
A formula $RS(\Phi)$ formed using the connectives of bi-intuitionistic logic is said to be provable if $RS(\Phi)$ is true in all the worlds (elements in) $T$ and is denoted by $\vDash RS(\Phi)$.

**Definition 28.** For any $RS(X) \in T$, the statement formula in $T$ is defined as,

The formula for Rough IPC is,

$$RS(\Phi), RS(\Psi) := RS(\{x_i\}) | RS(\emptyset) | RS(U) | RS(\Phi)\Delta RS(\Psi) | RS(\Phi)\nabla RS(\Psi) | RS(\Phi)^* | RS(\Phi) \rightarrow RS(\Psi)$$

The formula for dual Rough IPC is,

$$RS(\Phi), RS(\Psi) := RS(\{x_i\}) | RS(\emptyset) | RS(U) | RS(\Phi)\Delta RS(\Psi) | RS(\Phi)\nabla RS(\Psi) | RS(\Phi)^+ | RS(\Phi) \leftarrow RS(\Psi)$$

Therefore, the Rough bi-IPC is defined as

$$RS(\Phi), RS(\Psi) := RS(\{x_i\}) | RS(\emptyset) | RS(U) | RS(\Phi)\Delta RS(\Psi) | RS(\Phi)\nabla RS(\Psi) | RS(\Phi)^* | RS(\Phi)^+ | RS(\Phi) \rightarrow RS(\Psi) | RS(\Phi) \leftarrow RS(\Psi)$$

**Definition 29.** Let $RBHey(T) = (T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$ be a Rough bi-Heyting algebra. The truth value function $TrV$ is a function from the set of atomic propositions in $U$ to the set $\{0, 1\}$.

In other words, this truth value function $TrV$ extends from sending propositional formulas to each element in $T$ using the connectives to the set $\{0, 1\}$.

**Definition 30.** A model $\mathfrak{M}$ is a structure $\mathfrak{M} = (RBHey(T), TrV)$, where $RBHey(T)$ is a Rough bi-Heyting algebra and $TrV$ is a truth value function.

**Definition 31.** For a model $\mathfrak{M} = (RBHey(T), TrV)$, where $T$ is the set of possible worlds. The relation $\mathfrak{M}, RS(X) \vDash RS(\Phi)$ is defined such that the well-formed formula $RS(\Phi)$ is true in the world $RS(X)$ for a model $\mathfrak{M}$. The propositions in the world $RS(X)$ is true when it belongs to it and false when it doesn’t.

**Definition 32.** A formula $RS(\Phi)$ is said to be possibly satisfiable, if there exists a model $\mathfrak{M} = (RBHey(T), TrV)$ and $RS(X) \in T$ such that the truth value obtained for $RS(\Phi)$ is $(0, 1)$.

**Definition 33.** A formula $RS(\Phi)$ is said to be satisfiable, if for every model $\mathfrak{M} = (RBHey(T), TrV)$ and for every $RS(X) \in T$ such that the truth value obtained for $RS(\Phi)$ is $(1, 1)$.

**Remark 6.** A formula $RS(\Phi)$ is not satisfiable in the model $\mathfrak{M} = (RBHey(T), TrV)$ if the truth value obtained is $(0, 0)$.

### 4.2 Semantics for Rough bi-intuitionistic logic

The most common approach in Rough bi-intuitionistic propositional logic is the semantic approach. The truth-value of the statement formula is determined by the truth-value of the formula formed by the connectives in classical logic. In Rough bi-intuitionistic logic, the propositions considered are assigned a truth value to the statement, either true if 1 or false if 0. The value taken by every proposition is not always absolute, but it is assigned by the truth value function $TrV$. The truth-value of the formula $RS(\Phi)$ is calculated by first solving rules of inference using Rough bi-Heyting operators. Then the truth-value of the conclusion is calculated in the fixed world in $T$. The proposition is semantically true if its value is 1. Otherwise, it is semantically false.

#### 4.2.1 Formal Semantic

Formal semantic is used as a mathematical tool in the study of logic. One can understand the logical connectives in formal semantics in terms of satisfaction relation. A formal semantic for classical propositional calculus can be defined.

**Definition 34.** For the truth value function $TrV$ assigns a value 0 (or) 1 to each propositional variable in $U$. The satisfaction relation for a model $\mathfrak{M} = (RBHey(T), TrV)$ and $RS(X) \in T$ is defined as follows,

1. $\mathfrak{M}, RS(X) \vDash RS(U)$ and $\mathfrak{M}, RS(X) \nvdash RS(\emptyset)$

2. $\mathfrak{M}, RS(X) \vDash RS(\{x_i\})$ iff $TrV(RS(\{x_i\})) = (0, 1)$ for all $i$. 
3. $\mathfrak{M}, RS(X) \models RS(\Phi) \Delta RS(\Psi)$  if $\mathfrak{M}, RS(X) \models RS(\Phi)$ or $\mathfrak{M}, RS(X) \nvdash RS(\Psi)$
4. $\mathfrak{M}, RS(X) \models RS(\Phi) \nabla RS(\Psi)$  if $\mathfrak{M}, RS(X) \models RS(\Phi)$ and $\mathfrak{M}, RS(X) \nvdash RS(\Psi)$
5. $\mathfrak{M}, RS(X) \models RS(\Phi)^*$  if $\mathfrak{M}, RS(X) \nvdash RS(\Phi)$
6. $\mathfrak{M}, RS(X) \models RS(\Phi)^+$  if $\mathfrak{M}, RS(X) \nvdash RS(\Phi)$
7. $\mathfrak{M}, RS(X) \models RS(\Phi) \rightarrow RS(\Psi)$  if $\mathfrak{M}, RS(X) \nvdash RS(\Phi)$ or $\mathfrak{M}, RS(X) \nvdash RS(\Psi)$
8. $\mathfrak{M}, RS(X) \models RS(\Phi)^* \iff RS(\Psi)$  if $\mathfrak{M}, RS(X) \models RS(\Phi)$ and $\mathfrak{M}, RS(X) \nvdash RS(\Psi)$

Illustration 1: Let us consider the set of atomic propositions $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ for the model $\mathfrak{M} = (RHHey(T), TrV)$ and any world $RS(X_1 \cup \{x_2\} \cup X_3) \in T$. Then the following model validates whether it is a formal semantic for the well-formed formula $RS(\Phi)$ and $RS(\Psi)$.

Consider,

$x_1 =$ Bad habits are mortal
$x_2 =$ All humans are mortal
$x_3 =$ Therefore bad habits are mortal to humans
$x_4 =$ All mortal things have fragments
$x_5 =$ Therefore humans are a collective form of fragments
$x_6 =$ All fragment things are made of a molecule

Conclusion: Therefore, humans are made of molecules

For the given world $RS(X_1 \cup \{x_2\} \cup X_3) \in T$ the value taken by the propositions in $U$ are $\{1, 1, 1, 0, 1, 0\}$. The equivalence classes $X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_1, x_6\}$ and $X_3 = \{x_5\}$ are obtained using the equivalence relation $R$ on $U$. So the truth value of $X_1, X_2, X_3$ are found to be $1, 0$ and $1$. Let $RS(\Phi)$ and $RS(\Psi)$ be the well-formed formulas defined by

$RS(\Phi) = RS(X_1 \cup X_3) \rightarrow RS(\{x_1\}) = RS(X_1 \cup X_3)$ and $RS(\Psi) = RS(X_1 \cup \{x_2\} \cup X_3) \rightarrow RS(\{x_2\})$

| $RS(\Phi)^*$ | $RS(\Phi)^+$ | $RS(\Phi) \rightarrow RS(\Psi)$ | $RS(\Phi)^* \rightarrow RS(\Psi)$ | $RS(\Phi)^* \nabla RS(\Psi)$ | $RS(\Phi)^* \Delta RS(\Psi)$ |
|----------------|----------------|-------------------------------|---------------------------------|--------------------------------|--------------------------------|
| $RS(\Phi)^*$    | $RS(\Phi)^+$   | $RS(\Phi) \rightarrow RS(\Psi)$ | $RS(\Phi)^* \rightarrow RS(\Psi)$ | $RS(\Phi)^* \nabla RS(\Psi)$ | $RS(\Phi)^* \Delta RS(\Psi)$ |

Table 11: Connectives of well-formed formulas

From Table 11, the satisfaction relation for the formal semantic will be verified

1. $TrV(RS(\Phi)) = TrV(RS(X_1 \cup X_3)) = (1, 1)$ and $TrV(RS(\Psi)) = TrV(RS(\{x_2\})) = (0, 0)$
   $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \models RS(\Phi)$ and $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Psi)$
2. $RS(\Phi) \Delta RS(\Psi) = RS(X_1 \cup X_3) \Delta RS(\{x_2\}) = RS(X_1 \cup \{x_2\} \cup X_3)$
   $TrV(RS(\Phi) \Delta RS(\Psi)) = TrV(RS(X_1 \cup \{x_2\} \cup X_3)) = (1, 1)$
   $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Phi) \Delta RS(\Psi)$ with $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \models RS(\Phi)$ or $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Psi)$
3. $RS(\Phi) \nabla RS(\Psi) = RS(X_1 \cup X_3) \nabla RS(\{x_2\}) = RS(\emptyset)$
   $TrV(RS(\Phi) \nabla RS(\Psi)) = TrV(RS(\emptyset)) = (0, 0)$
   $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Phi) \nabla RS(\Psi)$
4. $RS(\Phi) \rightarrow RS(\Psi) = RS(X_1 \cup X_3) \rightarrow RS(\{x_2\}) = RS(X_2)$
   $TrV(RS(\Phi) \rightarrow RS(\Psi)) = TrV(RS(X_2)) = (0, 0)$
   $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Phi) \rightarrow RS(\Psi)$
5. $RS(\Phi) \leftarrow RS(\Psi) = RS(X_1 \cup X_3) \leftarrow RS(\{x_2\}) = RS(X_1 \cup X_3)$
   $TrV(RS(\Phi) \leftarrow RS(\Psi)) = TrV(RS(X_1 \cup X_3)) = (1, 1)$
   $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Phi) \leftarrow RS(\Psi)$ if $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \models RS(\Phi)$ and $\mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \nvdash RS(\Psi)$
6. \( RS(\Phi)^* = RS(X_2) \) and \( RS(\Psi)^* = RS(X_1 \cup X_3) \)

\[ TrV(RS(\Phi)^*) = TrV(RS(X_2)) = (0, 0) \text{ and } TrV(RS(\Psi)^*) = TrV(RS(X_1 \cup X_3)) = (1, 1) \]

\( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \neq RS(\Phi)^* \) and \( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) = RS(\Psi)^* \) if \( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \neq RS(\Psi) \)

7. \( RS(\Phi)^+ = RS(X_2) \) and \( RS(\Psi)^+ = RS(U) \)

\[ TrV(RS(\Phi)^+) = TrV(RS(X_2)) = (0, 0) \text{ and } TrV(RS(\Psi)^+) = TrV(RS(U)) = (1, 1) \]

\( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \neq RS(\Phi)^+ \) and \( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) = RS(\Psi)^+ \) if \( \forall \mathfrak{M}, RS(X_1 \cup \{x_2\} \cup X_3) \neq RS(\Psi) \)

**Interpretation:**

In this, the truth value of the proposition in \( U \) is either 0 or 1. In the considered world \( RS(X_1 \cup \{x_2\} \cup X_3) \) in \( T \), the truth values of well-formed formulas and associated connectives are calculated. This verifies the formal semantics with the help of satisfiability relation.

### 4.3 Standard Semantic

Standard semantic is the collection of formal semantic for the bi-intuitionistic propositional logic and its extension. Standard semantic is extended from formal semantic by modifying the semantics of Rough bi-Heyting algebraic operators. Here the hierarchy of different semantics for the Rough bi-intuitionistic propositional system are presented. The two types of standard semantics to be discussed are Rough Kripke semantic and Rough algebraic semantic which provides the way to compute the formulas with the operations on Rough bi-Heyting algebra.

#### 4.3.1 Rough Kripke semantic

A Kripke semantic is a widely used semantic in the Bi-IPC and Bi-IQC. A Rough Kripke semantic is a structure \( \mathfrak{M} = (RBHey(T), TrV) \), where \( RBHey(T) \) is a Rough bi-Heyting algebra with \( T \) is the non-empty set of worlds and \( TrV \) is the truth value function. Here each \( RS(X) \in T \) has a set of propositions that are true in it, when it belongs to \( RS(X) \), and a set of propositions that are false in it when it doesn’t belong.

**Definition 35.** A Rough Kripke model is a structure \( \mathfrak{M} = (RBHey(T), TrV) \) with

1. \( RBHey(T) \) is a Rough bi-Heyting algebra, and
2. \( TrV \) is a truth value function.

**Definition 36.** For any \( RS(X) \in T \), the Rough upset of \( RS(X) \) is a subset of \( T \) defined by

\( \text{Rough upset } (RS(X)) = \{ RS(Y) \in T \mid RS(X) \leq RS(Y) \} \)

**Definition 37.** For any \( RS(X) \in T \), the Rough downset of \( RS(X) \) is a subset of \( T \) defined as

\( \text{Rough downset } (RS(X)) = \{ RS(Y) \in T \mid RS(Y) \leq RS(X) \} \)

**Definition 38.** Let \( \mathfrak{M} = (RBHey(T), TrV) \) be a Rough Kripke model, let \( x_i \in U \) and \( RS(\Phi), RS(\Psi) \) be the well-formed formulas. For the given model, the satisfiability at \( RS(X) \) is defined as,

1. \( \mathfrak{M}, RS(X) \models RS(U) \) and \( \mathfrak{M}, RS(X) \nmid RS(\emptyset) \)
2. \( \mathfrak{M}, RS(X) \models RS(\{x_i\}) \) if \( TrV(RS(\{x_i\})) = (0, 1) \) for all \( i \).
3. \( \mathfrak{M}, RS(X) \models RS(\Phi) \triangleq RS(\Psi) \) if \( \mathfrak{M}, RS(X) \models RS(\Phi) \) or \( \mathfrak{M}, RS(X) \nmid RS(\Psi) \)
4. \( \mathfrak{M}, RS(X) \models RS(\Phi) \lor RS(\Psi) \) if \( \mathfrak{M}, RS(X) \models RS(\Phi) \) and \( \mathfrak{M}, RS(X) \nmid RS(\Psi) \)
5. \( \mathfrak{M}, RS(X) \models RS(\Phi) \rightarrow RS(\Psi) \) if for all \( RS(Y) \in \text{Rough upset } (RS(X)) : \mathfrak{M}, RS(Y) \nmid RS(\Phi) \) or \( \mathfrak{M}, RS(Y) \nmid RS(\Psi) \)
6. \(M, RS(X) \models RS(\Phi) \leftrightarrow RS(\Psi)\) iff for some \(RS(Y) \in \text{Rough downset } (RS(X)) : M, RS(Y) \models RS(\Phi)\) and \(M, RS(Y) \not\models RS(\Psi)\)

7. \(M, RS(X) \models RS(\Phi)^+\) iff for all \(RS(Y) \in \text{Rough upse}\ (RS(X)) : M, RS(Y) \not\models RS(\Phi)\)

8. \(M, RS(X) \models RS(\Phi)^-\) iff for some \(RS(Y) \in \text{Rough downset } (RS(X)) : M, RS(Y) \not\models RS(\Phi)\)

Illustration 2: For the given set of atomic propositions \(U = \{x_1, x_2, x_3, x_4, x_5, x_6\}\) and for the model \(M = (RBHey(T), TrV)\) and any \(RS(\{x_1\} \cup X_3) \in T\), the following well-formed formula \(RS(\Phi)\) and \(RS(\Psi)\) validate whether it is a Rough Kripke-structure.

Consider the set of propositions as in Illustration 1, the equivalence classes on \(U\) are \(X_1, X_2\) and \(X_3\) and their corresponding truth values are 0,0 and 1 are obtained from the world \(RS(\{x_1\} \cup X_3)\) in \(T\). The formulas \(RS(\Phi)\) and \(RS(\Psi)\) are defined as

\[RS(\Phi) = ([RS(\{x_1\} \cup \{x_2\}) \rightarrow RS(X_2)] \rightarrow [RS(\{x_1 \cup X_3\) \leftarrow RS(X_3)]) = RS(X_1)\) and \(RS(\Psi) = RS(X_2 \cup X_3)\)

\[\forall \mathcal{X} \mathcal{Y} (RS(\mathcal{X}) \leftrightarrow RS(\mathcal{Y})\)

| \(RS(\Phi)^+\) | \(RS(\Phi)^-\) | \(RS(\Phi)^{+}\) | \(RS(\Phi)^{-}\) |
|-----------------|-----------------|-----------------|-----------------|
| RS(X_1 \cup X_3) | RS(X_2) \cup X_3 | RS(X_2 \cup X_3) | RS(X_1) \cup X_3 |

| RS(\Phi) \rightarrow RS(\Psi) \rightarrow RS(\Phi)| RS(\Phi) \rightarrow RS(\Psi) \rightarrow RS(\Phi) |
|-----------------|-----------------|
| RS(X_2) \cup X_3 | RS(X_1) \cup X_3 |

Table 12: Connectives of well-formed formulas

\[RS(\Phi) \rightarrow RS(\Psi) \rightarrow RS(\Phi)\] in the Rough upse\(RS(\{x_1\} \cup X_3)\)

Table 13: Calculation for the satisfiability of \(RS(\Phi) \rightarrow RS(\Psi)\) in the Rough upse\(RS(\{x_1\} \cup X_3)\)

| RS(X_1 \cup X_3) \rightarrow RS(X_1) | RS(X_2) \cup X_3 | RS(X_1) \cup X_3 |
|-----------------|-----------------|-----------------|
| (1,1) | (1,1) | (1,1) |

| RS(X_1 \cup X_3) \rightarrow RS(X_1) | RS(X_2) \cup X_3 | RS(X_1) \cup X_3 |
|-----------------|-----------------|-----------------|
| (1,1) | (1,1) | (1,1) |

| RS(X_1 \cup X_3) \rightarrow RS(X_1) | RS(X_2) \cup X_3 | RS(X_1) \cup X_3 |
|-----------------|-----------------|-----------------|
| (0,0) | (0,0) | (0,0) |

| RS(X_1 \cup X_3) \rightarrow RS(X_1) | RS(X_2) \cup X_3 | RS(X_1) \cup X_3 |
|-----------------|-----------------|-----------------|
| (0,0) | (0,0) | (0,0) |

Table 14: Calculation for the satisfiability of \(RS(\Phi) \leftarrow RS(\Psi)\) in the Rough downse\(RS(\{x_1\} \cup X_3)\)

From Table 12,13, and 14, the satisfaction relation for the Rough kripke-structure semantic will be verified.

1. \(TrV(RS(\Phi)) = TrV(RS(X_1)) = (0,0)\) and \(TrV(RS(\Psi)) = TrV(RS(\{x_2\} \cup X_3)) = (1,1)\)

\[M, RS(\{x_1\} \cup X_3) \not\models RS(\Phi)\) and \(M, RS(\{x_1\} \cup X_3) \models RS(\Psi)\)

2. \(RS(\Phi) \Delta RS(\Psi) = RS(X_1 \cup \{x_2\} \cup X_3)\)

\[TrV(RS(X_1 \cup \{x_2\} \cup X_3)) = (1,1)\]

\[M, RS(\{x_1\} \cup X_3) \models RS(\Phi) \Delta RS(\Psi)\) iff \(M, RS(\{x_1\} \cup X_3) \models RS(\Phi)\) or \(M, RS(\{x_1\} \cup X_3) \models RS(\Psi)\)

3. \(RS(\Phi) \nabla RS(\Psi) = RS(\emptyset)\)

\[TrV(RS(\emptyset)) = (0,0)\]

\[M, RS(\{x_1\} \cup X_3) \not\models RS(\Phi) \nabla RS(\Psi)\)

4. \(RS(\Phi) \rightarrow RS(\Psi) = RS(X_2 \cup X_3)\)

\[TrV(RS(X_2 \cup X_3)) = (1,1)\]

\[M, RS(\{x_1\} \cup X_3) \not\models RS(\Phi) \rightarrow RS(\Psi)\) iff for all \(RS(Y) \in \text{Rough upse}\ (RS(\{x_1\} \cup X_3)) : M, RS(Y) \not\models RS(\Phi)\) (or) \(M, RS(Y) \models RS(\Psi)\)

Hence \(M, RS(\{x_1\} \cup X_3) \not\models RS(\Phi) \rightarrow RS(\Psi)\) iff for all \(RS(Y) \in \text{Rough upse}\ (RS(\{x_1\} \cup X_3))\) with \(M, RS(Y) \models RS(\Psi)\)

5. \(RS(\Phi) \leftarrow RS(\Psi) = RS(X_1)\)

\[TrV(RS(X_1)) = (0,0)\]
\[ M, RS(\{x_1 \} \cup X_3) \models RS(\Phi) \iff RS(\Psi) \]
iff for some \( RS(Y) \in \text{Rough downset} \left( RS(\{x_1 \} \cup X_3) \right) : M, RS(Y) \not\models RS(\Phi) \]
Hence \( M, RS(\{x_1 \} \cup X_3) \not\models RS(\Phi) \iff RS(\Psi) \]

6. \( RS(\Phi)^+ = RS(X_2 \cup X_3) \) and \( RS(\Psi)^+ = RS(X_1) \)
\[ Tr_V(RS(X_2 \cup X_3)) = (1, 1) \quad \text{and} \quad Tr_V(RS(X_1)) = (0, 0) \]
\[ M, RS(\{x_1 \} \cup X_3) \models RS(\Phi)^+ \iff \text{for all } RS(Y) \in \text{Rough upset} \left( RS(\{x_1 \} \cup X_3) \right) : M, RS(Y) \not\models RS(\Phi) \]
and \( M, RS(\{x_1 \} \cup X_3) \not\models RS(\Phi)^+ \iff \text{for all } RS(Y) \in \text{Rough upset} \left( RS(\{x_1 \} \cup X_3) \right) : M, RS(Y) \not\models RS(\Phi) \)
Hence, \( M, RS(\{x_1 \} \cup X_3) \not\models RS(\Phi)^+ \iff \text{for } RS(Y) = \{ RS(\{x_1 \} \cup X_3), RS(\{x_1 \} \cup \{x_2 \} \cup X_3), RS(\{x_1 \} \cup X_3) \} \)
we have \( M, RS(\{x_1 \} \cup X_3) \not\models RS(\Phi) \) and \( M, RS(Y) \not\models RS(\Psi)^+ \)

7. \( RS(\Phi)^+ = RS(X_2 \cup X_3) \) and \( RS(\Psi)^+ = RS(X_1 \cup \{x_2 \} \cup \{x_6 \}) \)
\[ Tr_V(RS(X_2 \cup X_3)) = (1, 1) \quad \text{and} \quad Tr_V(RS(X_1 \cup \{x_2 \} \cup \{x_6 \})) = (0, 0) \]
\[ M, RS(\{x_1 \} \cup X_3) \models RS(\Phi)^+ \quad \text{and} \quad M, RS(\{x_1 \} \cup X_3) \models RS(\Psi)^+ \quad \text{iff} \quad \text{for some } RS(Y) \in \text{Rough downset} \left( RS(\{x_1 \} \cup X_3) \right) : M, RS(Y) \not\models RS(\Phi) \] and \( M, RS(Y) \not\models RS(\Psi) \)
Hence \( M, RS(\{x_1 \} \cup X_3) \models RS(\Phi)^+ \quad \text{iff} \quad RS(Y) = \{ RS(\emptyset), RS(\{x_1 \}), RS(X_3), RS(\{x_1 \} \cup X_3) \} \)
with \( M, RS(Y) \not\models RS(\Phi) \) and \( M, RS(Y) \not\models RS(\Psi)^+ \)

**Interpretation:**
The world considered for computing the truth values of well-formed formulas and their connectives is \( RS(\{x_1 \} \cup X_3) \). Between well-formed formulas, the operations Praba \( \Delta \) and Praba \( \nabla \) are the same as in formal semantics. However, the Rough bi-Heyting algebra connectives \( *, \dagger, \rightarrow, \leftarrow \) are verified in the Rough Kripke model using Rough upset and Rough downset of \( RS(\{x_1 \} \cup X_3) \).

### 4.3.2 Rough algebraic semantic

Rough algebraic semantic aims at algebraic interpretation of the bi-intuitionistic logic using Rough bi-Heyting algebra. The algebraic models of Rough bi-intuitionistic logic with the set of four binary operations \( \{\Delta, \nabla, \rightarrow, \leftarrow\} \) along with the minimal element \( RS(\emptyset) \) and maximal element \( RS(U) \) is a Rough bi-Heyting algebra \( RBHey(T) = (T, \Delta, \nabla, \rightarrow, \leftarrow, RS(\emptyset), RS(U)) \) satisfy the following conditions

1. \( RS(X) \rightarrow RS(X) = RS(U) \)
\[ RS(X) \leftarrow RS(X) = RS(\emptyset) \]
2. \( RS(X) \nabla (RS(X) \rightarrow RS(Y)) = RS(X) \nabla RS(Y) \)
\[ RS(X) \Delta (RS(X) \leftarrow RS(Y)) = RS(X) \]
3. \( (RS(X) \rightarrow RS(Y)) \nabla RS(Y) = RS(Y) \)
\[ (RS(X) \leftarrow RS(Y)) \Delta RS(Y) = RS(Y) \Delta RS(Y) \]
4. \( RS(X) \rightarrow (RS(Y) \nabla RS(Z)) = (RS(X) \rightarrow RS(Y)) \nabla (RS(X) \rightarrow RS(Z)) \)
\[ (RS(X) \Delta RS(Y)) \leftarrow RS(Z) = (RS(X) \leftarrow RS(Z)) \Delta (RS(Y) \leftarrow RS(Z)) \]

**Example 10.** Let \( U = \{x_1, x_2, x_3, x_4, x_5, x_6\} \) be the set of atomic propositions. For the Rough sets \( RS(X) = RS(X_1 \cup \{x_2\}) \), \( RS(Y) = RS(X_2 \cup X_3) \) and \( RS(Z) = RS(\{x_1\} \cup \{x_2\} \cup X_3) \) in \( T \), the following axioms validate whether the conditions of Rough algebraic semantics are satisfied.

For the given \( RS(X), RS(Y) \) and \( RS(Z) \) in \( T \),

1. Since by the properties of Rough bi-Heyting algebra, \( RS(X) \rightarrow RS(Y) = RS(U) \) and \( RS(X) \leftarrow RS(Y) = RS(\emptyset) \) iff \( RS(X) \subseteq RS(Y) \). Hence,
\[ RS(X_1 \cup \{x_2\}) \rightarrow RS(X_1 \cup \{x_2\}) = RS(U) \quad \text{and} \quad RS(X_1 \cup \{x_2\}) \leftarrow RS(X_1 \cup \{x_2\}) = RS(\emptyset) \]
2. \( RS(X_1 \cup \{x_2\}) \nabla (RS(X_1 \cup \{x_2\}) \rightarrow RS(X_2 \cup X_3)) = RS(X_1 \cup \{x_2\}) \nabla RS(X_2 \cup X_3) \)
L.H.S: \( RS(X_1 \cup \{x_2\}) \nabla (RS(X_1 \cup \{x_2\}) \rightarrow RS(X_2 \cup X_3)) = RS(X_1 \cup \{x_2\}) \nabla RS(X_2 \cup X_3) = RS(\{x_2\}) \)
R.H.S: \( RS(X_1 \cup \{x_2\}) \nabla RS(X_2 \cup X_3) = RS(\{x_2\}) \)
Hence \( RS(X) \nabla (RS(X) \rightarrow RS(Y)) = RS(X) \nabla RS(Y) \)
In this study, a Rough bi-Heyting algebra for the Rough semiring \((T, \Delta, \nabla)\) is obtained. As a result, for the Rough bi-Heyting algebra \((T, \Delta, \nabla, \rightarrow, \leftarrow, RS(0), RS(U))\), a Rough bi-intuitionistic logic exists, which helps in modeling various semantics. In future work, the focus is on developing this approach to describe varying algebraic structures.

5 Conclusion

In this study, a Rough bi-Heyting algebra for the Rough semiring \((T, \Delta, \nabla)\) is obtained. As a result, for the Rough bi-Heyting algebra \((T, \Delta, \nabla, \rightarrow, \leftarrow, RS(0), RS(U))\), a Rough bi-intuitionistic logic exists, which helps in modeling various semantics. In future work, the focus is on developing this approach to describe varying algebraic structures.

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