Fibrations of genus two on complex surfaces

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1 Introduction

A singular fibration on a complex surface $M$ consists of a non-constant holomorphic map $\mathcal{P}$ from $M$ to a compact Riemann surface $S$. The genus of the fibration is the genus of a “generic” fiber (cf. Section 2 for further details). Up to birational transformations (i.e. compositions of blow-ups and blow-downs) every algebraic surface $M$ is given as a singular fibration, possibly, in more than one way.

Recall that $M$ is called ruled (resp. elliptic) if it carries a singular fibration of genus 0 (resp. 1). The structure of ruled surfaces was known to the Italian School of Algebraic Geometry whereas Kodaira has provided a similar picture for elliptic surfaces. Although these works are widely quoted in the literature, the same question concerning genus 2 fibrations is a much less known topic. A very detailed classification of the structure of the singular fibers was obtained by Namikawa et Ueno [N-U]. This paper is itself based on a series of papers by Ueno et al. [O], [U-1], [U-2] and their references, concerning the description of conjugacy classes of $\text{Sp}(2,\mathbb{Z})$ and their fixed point sets on Siegel upper half plane of degree 2 (cf. below). The main result of [N-U] is a list of “discrete” invariants associated to a singular fiber, such as the monodromy, the degree and the “modulos point”, allowing to recognize the fiber unambiguously.

Yet most of these papers are not widely known by experts in complex geometry/topology and the applications of their results are not numerous when compared to the case of elliptic surface. For higher genus fibrations, there is no systematic treatment available as far as we know.

Compared to [N-U] this paper is devoted to a slightly different question. Suppose for example that a singular fiber is fixed. Our purpose is to present models for a fibered neighborhood of this fiber. If you think of a singular fiber as the limiting object of a degenerating family of curves, our purposes is to quantify the way in which the family in question is degenerating. This is expected to be more useful for topologists than the classification of the singular fibers alone. Indeed, our models allows for example to determine the topology of the manifold on a neighborhood of the fiber. As a consequence the topology of the ambient manifold becomes codified in a finite number of these models, corresponding to the existing singular fibers, complemented by a regular fibration. Computation of classical invariants become then very straightforward.

To state our main result, the structure of the irreducible components, along with their intersections and self-intersections, of a singular fiber is going to be called its combinatorial data.

**Theorem 1.1** Let $\mathcal{P}_1 : M_1 \to D$ and $\mathcal{P}_2 : M_2 \to D$ be two fibrations of genus 2 over the disk $D \subset \mathbb{P}^1$ whose unique singular fibers are those sitting over $0 \in D$. Suppose that the combinatorial data of the singular fiber $\mathcal{P}_1^{-1}(0)$ is the same as $\mathcal{P}_2^{-1}(0)$, then there exists a $C^\infty$-diffeomorphism that conjugates the two fibrations. Furthermore, this diffeomorphism is transversely holomorphic.

The main consequence of the preceding theorem is that the only mechanism through which a tubular neighborhood of $\mathcal{P}_1^{-1}(0)$ may not be holomorphically equivalent to a tubular neighborhood of $\mathcal{P}_2^{-1}(0)$
consists of having complex structures on the fibers of $P_1$ that do not match the corresponding structures on the fibers of $P_2$. In particular, the variation of the complex structures on the fibers of $P_1$, $P_2$ plays a crucial role in this study. Also, in this sense, the statement above is clearly sharp. Some additional details concerning these issues will be mentioned at the end of Section 4.

Naturally it should be pointed out that a difficulty associated to the applications of the above mentioned results lies in their extension. This applies also to Theorem 1.1. Similarly the number of possible “local models” increases rapidly with the genus: for genus 2 it is much higher than for genus 1. Nonetheless it seems conceivable that in many potential applications, we shall deal with specific fibrations whose nature of the singular fibers form a smaller set of possibilities. It would therefore be desirable to have a method to work out the structure of these fibers in a more direct way, without passing through the long list of all possible models. The present paper is also somehow devoted to this question. In fact, our aim is to introduce a systematic way to “describe” the neighborhood of a singular fiber out of rather basic information such as its combinatorial data. This is, indeed, the most important upshot of this paper: if the combinatorial data of a singular fiber is known, then the fibration is totally determined on a neighborhood of this fiber. We mention that this is essentially true regardless of the genus of the fibration. Indeed, whereas the proof of Theorem 1.1 carried out in Section 4 is detailed only for genus 2 fibrations, the reader will notice that most of the argument is applicable to fibrations of arbitrary genus. In this sense, our work becomes to a good extent insensitive to the genus of the fibration.

To be more precise and better explain the contents of this paper, let us consider the classification of genus 2 fibrations in more detail. First note that the combinatorial data of the singular fibers of these fibrations can easily be described as shown by Ogg in [O] (cf. also Section 2.2). This however does not provide much insight into the structure of the fibration around the singular fibers. Ultimately, the contribution of this paper may be viewed as a way to recover the structure of this fibration “immediately” provided that the structure of the singular fiber is known. In particular, the statement of Theorem 1.1 should promptly be reduced to Ogg’s classification which is briefly recalled in Section 2.2. We believe that these results have their own interest, but they also to provide a more “compact” view of the preceding classification of these fibrations which is spread in a few papers.

Concerning the classification of genus 2 fibrations obtained by Ueno and its collaborators, their technique consists of considering the associated Jacobian fibration over the punctured disc. The method is essentially based on the observation that the “singular fiber” is a fixed point for the corresponding monodromy map. The classification problem is then translated into the problem of classifying conjugacy classes of $Sp(2, \mathbb{Z})$ and their fixed point sets on Siegel upper half plane of degree 2. The latter problem being tackled in [U-1], [U-2], [U-3]. Whereas this technique is very interesting in itself, it is not immediately adapted to certain geometric applications involving for example the study of the limit of a degenerating family of genus 2 curves. For questions having similar geometric nature, we believe that our method is more directly accessible. Curiously, by reverting the “arrow” in the equivalence between classes of conjugacy of $Sp(2, \mathbb{Z})$ and the classification of genus 2 fibrations, our methods also allow for new proofs of the previous results. It should be mentioned that our results also allow for explicit formulas for the Canonical Line Bundle of genus 2 fibrations analogue to the well-known formula in the case of elliptic surfaces. Similarly another application of our result consists of a formula for the fundamental group of our surface analogous to the formula due to Kodaira, Moishezon and Dolgachev for the fundamental group of elliptic surfaces (see [F], page 189). method of Jacobian fibrations was also employed in [X] to study some global question concerning genus 2 fibrations, in particular the relations among their numerical invariants. It would be interesting to see what can be said about these surfaces by using the models for the neighborhoods of the singular fibers.

Finally in what follows we shall not be concerned with the problem of “realization” of the models
for the singular fibers. In fact, from the perspective of Theorem 1.1, the existence of all the “possible” models is an easy consequence of Winters theorem in [W]. More global questions related to the possibility of coexistence for the possible models in a same fibration and to the variation of the complex structure of the fibers, which is itself intimately related to the possibility of strengthen Theorem 1.1 to produce a holomorphic conjugacy, are to be discussed somewhere else. In particular, we add that the method of Jacobian fibrations was also employed in [X] to study some global question concerning genus 2 fibrations, in particular the relations among their numerical invariants. It would be interesting to see what can be said about these surfaces by using the models for the neighborhoods of the singular fibers. Suitable fibered sums of the fibrations given in Section 3 might be good candidates to provide interesting examples of surfaces.

Let us close this Introduction by giving the organization of this paper. The main theme of this work may be summarized as follows: the language and some fundamental results used in the dynamical study of singular holomorphic foliations lends itself well to analyse the structure of singular fibrations. The central part of our results is obtained through these methods and it is expressed in this dynamical language. In doing so, we believe the general procedure for dealing with fibrations of arbitrary genus will further be clarified. To help us to attain this objective, we made some effort to write a rather self-contained paper. In particular we included a brief review of Ogg’s paper [O]. Similarly some definitions and results well-known to experts in the theory of singularities of holomorphic foliations are carefully stated. Of course this has made the paper a few pages longer, however this “background” material is needed to make the contents of the article promptly accessible to most readers.

Here is the description of the contents of the paper by sections. Section 2 consists of background material. Since most of this work is more naturally expressed in the language of holomorphic foliations/vector fields, in the first part of Section 2, we review the general notions related to their singularities and associated indices. In the second part of this section we give a brief and self-contained account of Ogg’s paper [O]. Section 3 provides some explicit examples of genus 2 fibrations having singular fibers exhibiting non-linear singularities (in our language this assertion is translated into the fact that the singularity of the singular fiber is not the transverse intersection of two smooth locally irreducible components). In standard literature of algebraic geometry these fibers correspond to non-reduced ones. The mentioned examples also show that these fibers are organized as dual fibers of certain fibrations possessing only linear singularities. In this way the regular part of the fibrations naturally interpolates between the two singular fibers in question.

Section 4 contains most of the bulk of our methods. It is this section that Theorem 1.1 is proved showing that the structure of the singular fiber determines the structure of the fibration on its neighborhood. The proof of this theorem follows standard techniques of foliation theory. It relies on the study of the holonomy of the singular fiber and in its Abelian character which is established with the help of the classical picture of the flower associated to the local dynamics of a germ of diffeomorphism of \((\mathbb{C}, 0)\) tangent to the identity. This section contains an appendix, which is named Section 5, where some specific results about the singular points of a singular fiber are proved. The reason for this appendix is that it seemed to us to exist a small gap in the literature concerning these results in the way we need them here.

Finally in Section 6 it is another appendix where the monodromy of all possible singular fibers are computed through standard geometric techniques. Some remarks involving certain monodromy maps and special tesselations of the plane and of the hyperbolic disc are also made. It is to be noted that the full list containing all the resulting models of singular fibers, along with their monodromy maps etc, was not included in this paper since this would amount to duplicating most of the papers [O] and [N-U].

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of the complex structure of fibers that are implicitly alluded to in the “comments and questions” added at the end of Section 4. Also we would like to thank G. Tian and S. Sharif for the interest showed in this paper.

2 Fibrations, Singularities and Indices

2.1 Singularities and indices

A singular fibration on a compact complex surface $M$ is a non-constant holomorphic map $\mathcal{P}$ from $M$ to a compact Riemann surface $S$. Given one such map $\mathcal{P}$, the finite set $\{p_1, \ldots , p_s\} \subset S$ consisting of the critical values of $\mathcal{P}$ is such that $\mathcal{P}$ defines a regular fibration of $M \setminus \bigcup_{i=1}^{s} \mathcal{P}^{-1}(p_i)$ over $S \setminus \{p_1, \ldots , p_s\}$. The genus of $\mathcal{P}$ is simply the genus of the fiber of the mentioned (regular) fibration or, equivalently, the genus of a “generic” fiber of $\mathcal{P}$. The preimages $\mathcal{P}^{-1}(p_i)$ of the critical values of $\mathcal{P}$ are called singular fibers.

Given a critical value $p_i \in S$ and a neighborhood $V_i \subseteq S$ of $p_i$, the set $\mathcal{P}^{-1}(V_i)$ is going to be called a tubular neighborhood of $\mathcal{P}^{-1}(p_i)$. To a large extent, describing the structure of a singular fibration $\mathcal{P} : M \rightarrow S$ is tantamount to provide models for “small” tubular neighborhoods of its singular fibers. In fact, one passes from the set $\bigcup_{i=1}^{s} \mathcal{P}^{-1}(V_i)$ to the whole surface $M$ by filling the complement $M \setminus \bigcup_{i=1}^{s} \mathcal{P}^{-1}(p_i)$ in with a regular fibration. Naturally, by “models” we mean “normal forms” for the restriction of the fibration to $\mathcal{P}^{-1}(V_i)$ and not only the geometry of the open set $\mathcal{P}^{-1}(V_i) \subseteq M$ or, in other words, “normal forms” for the singular foliation induced on $\mathcal{P}^{-1}(V_i)$ by the restriction of the fibration. In view of what precedes, the purpose of this work is to present explicit models for the tubular neighborhood of a singular fiber in a fibration $\mathcal{P} : M \rightarrow S$ whose genus is 2.

Since this work deals with tubular neighborhoods as above, throughout this paper we place ourselves in the following setting: let $D \subset \mathbb{C}$ be the unit disc and suppose that $\mathcal{P}$ is a proper holomorphic map from an open complex surface $M$ to $D$ which satisfies the conditions below:

1. $\mathcal{P}$ defines a regular fibration of $M \setminus \mathcal{P}^{-1}(0)$ over $D \setminus \{0\}$ with fibers of genus 2.
2. $\mathcal{P}^{-1}(0)$ is a connected singular fiber.

There is a particularly simple way to obtaining meromorphic vector fields tangent to the fibers of $\mathcal{P}$. For this suppose that we are given a meromorphic section $\eta$ of the Canonical Line Bundle $K_M$ of $M$. We then define a meromorphic vector field $X$ on $M$ by means of the equation

$$\eta_p(X, \ldots ) = D_p \mathcal{P}$$

whenever both sides are defined and where $D_p \mathcal{P}$ stands for the differential of $\mathcal{P}$ at $p \in M$. Clearly $X$ is a meromorphic vector field tangent to the level sets of $\mathcal{P}$. Hence the singular foliation $\mathcal{F}$ induced by the local orbits of $X$ (and referred to as the foliation associated to $X$) is nothing but the foliation induced by the level sets of $\mathcal{P}$. In particular, $\mathcal{F}$ admits $\mathcal{P} : M \rightarrow D \subset \mathbb{C}$ as a non-constant holomorphic first integral. We also note that the singularities of $\mathcal{F}$ are exactly the singular points of the singular fiber $\mathcal{P}^{-1}(0)$.

In the sequel $\mathcal{F}$ will stand for a singular holomorphic foliation defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. A separatrix for $\mathcal{F}$ is an analytic curve passing by the origin and invariant by the foliation. Suppose that $\mathcal{F}$ possesses a smooth separatrix $\mathcal{S}$. Modulo changing coordinates, we can suppose that $\mathcal{S}$ coincides with the axis $\{y = 0\}$. In these coordinates, there is a holomorphic vector field $Y$, with isolated singularities and tangent to $\mathcal{F}$, having the form

$$Y = F(x,y) \partial / \partial x + G(x,y) \partial / \partial y$$

(2)
where, in addition, $G$ is divisible by $y$. Following [C-S] we define the index of $S$ with respect to $F$ (at $(0,0) \in \mathbb{C}^2$) by letting

$$\text{Ind}_{(0,0)}(F, S) = \text{Res}_{x=0} \frac{\partial}{\partial y} \left( \frac{G}{F} \right) (x,0) dx .$$

Note, in particular, that $\text{Ind}_{(0,0)}(F, S) = 0$ if $(0,0)$ is a regular point of $F$. As it is easy to see the definition above does not depend on the choices made. The index can be interpreted as measuring an “infinitesimal” self-intersection of $S$. This interpretation is materialized by the Camacho-Sad formula [C-S] as follows. Assume that $S$ is represented by a global compact Riemann surface $D$ embedded in a complex surface $M$ and invariant by the foliation $F$ (implicitly we assume now that $F$ is defined on a neighborhood of $D \subset M$). Denoting by $p_1, \ldots, p_s$ the singularities of $F$ lying in $D$, we can consider, for each $i = 1, \ldots, s$, the index $\text{Ind}_{p_i}(F, D)$. If $D \cdot D$ stands for the self-intersection of $D$ then one has

$$\sum_{i=1}^s \text{Ind}_{p_i}(F, D) = D \cdot D . \quad (3)$$

In the sense that it measures an “infinitesimal” self-intersection, the index behaves naturally with respect to blow-ups. Recall that, if, as above, $S$ actually represents a globally defined curve $D$, then the self-intersection of $D$ falls by one unity if it is blown-up at a regular point. The index recovers this behavior: if $\tilde{F}$ denotes the blow-up of $F$ at the origin, then the proper transform $\tilde{S}$ of $S$ is a smooth separatrix for some singularity of $\tilde{F}$. The index of $\tilde{S}$ w.r.t. $\tilde{F}$ equals exactly the index of $S$ w.r.t. $F$ minus $1$. This allows us to define the index for an irreducible singular separatrix as well. This goes as follows. If $D$ now represents a singular curve and $p \in D$ is a singular point, then the self-intersection of the blow-up $\tilde{D}$ of $D$ at $p$ and the self-intersection of the initial curve $D$ are related by Kodaira’s conductor formula. We then use the analogous formula to define the index of an irreducible, possibly singular, separatrix for $F$. Namely, if $S$ is a separatrix and $\tilde{F}$ (resp. $\tilde{S}$) stands for the blow-up of $F$ (resp. the proper transform of $S$), then one has

$$\text{Ind}_{(0,0)}(F, S) = \text{Ind}_q(\tilde{F}, \tilde{S}) + m[\pi^{-1}(0)] \cdot \tilde{S} , \quad (4)$$

where $q = \pi^{-1}(0) \cap \tilde{S}$ and where $m$ is the multiplicity of $\pi^{-1}(0)$ as component of $\pi^*(S)$. Finally the intersection product $[\pi^{-1}(0)] \cdot \tilde{S}$ should be regarded as the usual intersection multiplicity between the curves in question. Since the proper transform of $S$ under a sequence of blow-up transformations will eventually become smooth, the above formula allows us to unequivocally define all these indices. It is rather easy to check that the resulting index does not depend neither on the number nor on the sequence of blow-up transformations used to turn $S$ into a smooth separatrix.

We can now close this section with the classification of singularities of foliations that appears as singular points of fibrations of genus 2. Let then $F$ be a singular holomorphic foliation defined about $(0,0) \in \mathbb{C}^2$. The simplest possible singularity that $F$ may have is of course a linear singularities. In this paper, a singularity of a foliation $F$ will be called linear if there are local coordinates $(x, y)$ where the foliation is locally given as the orbits of the vector field

$$Z = mx \partial / \partial x - ny \partial / \partial y ,$$

$m, n \in \mathbb{N}^*$ Suppose now that $F$ is realized as singularity of a fibration having genus equal to 2. Then, apart from linear singularities and modulo blow-ups, the classification carried out in [Re] states that $F$ possesses a (local) holomorphic normal form belonging to the list below. These foliations are defined in terms of tangent vector fields. Note that the subscripts used to identify each model is a reference to the corresponding first integral of the foliation/vector field.
1. $Z_{1,1,1} = x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y$. The corresponding separatizes and their associated indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{x = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{x = y\}) = -2$. The (primitive) first integral of $Z_{1,1,1}$ being $xy(x-y)$.

2. $Z_{1,1-2} = (2y-x^2)\partial/\partial x + 2xy\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y = 0\}) = -2$. The (primitive) first integral of $Z_{1,1-2}$ being $y(x^2 - y)$.

3. $Z_{2-3} = 3y^2\partial/\partial x + 2x\partial/\partial y$. The separatrix and its index are $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y^3 = 0\}) = 0$. The (primitive) first integral of $Z_{2-3}$ being $x^2 - y^3$.

4. $Z_{2,2,1} = x(3y-2x)\partial/\partial x + y(3x-2y)\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{x = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = -3/2$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x = y\}) = -4$. The (primitive) first integral of $Z_{2,2,1}$ being $y^3(x^2 - y)$.

5. $Z_{3,1,1} = x(2y-x)\partial/\partial x + y(4x-3y)\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{x = 0\}) = -2/3$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{x = y\}) = -4$. The (primitive) first integral of $Z_{3,1,1}$ being $x^3y(x^2 - y)$.

6. $Z_{4,2-1} = (5y-4x^2)\partial/\partial x + 2xy\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = -1/2$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y = 0\}) = -8$. The (primitive) first integral of $Z_{4,2-1}$ being $y^3(x^2 - y)$.

7. $Z_{2-5} = 5y^4\partial/\partial x + 2x\partial/\partial y$. The separatrix and its index is $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y^5 = 0\}) = 0$. The (primitive) first integral of $Z_{2-5}$ being $x^2 - y^5$.

8. $Z_{2,(2,3)^3} = (5y-2x^2)\partial/\partial x + 6xy\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = -3$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y = 0\}) = -4/3$. The (primitive) first integral of $Z_{2,(2,3)^3}$ being $y^3(x^2 - y)^3$.

9. $Z_{3,2-1} = (4y-3x^2)\partial/\partial x + 2xy\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = -2/3$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y = 0\}) = -6$. The (primitive) first integral of $Z_{3,2-1}$ being $y^3(x^2 - y)$.

10. $Z_{1,2-3} = (4y^2 - x^2)\partial/\partial x + 2xy\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = -2$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y^3 = 0\}) = -2$. The (primitive) first integral of $Z_{1,2-3}$ being $y(x^2 - y^3)$.

11. $Z_{1,1,2-1} = (2xy - x^3)\partial/\partial x + (3x^2 - y)\partial/\partial y$. The separatizes along with their indices are $\text{Ind}_{(0,0)}(\mathcal{F}, \{y = 0\}) = \text{Ind}_{(0,0)}(\mathcal{F}, \{x^2 - y = 0\}) = -3$ and $\text{Ind}_{(0,0)}(\mathcal{F}, \{x = 0\}) = -2$. The (primitive) first integral of $Z_{1,1,2-1}$ being $xy(x^2 - y)$.

**Remark 2.1** The above list deserves some comments especially since this paper was “publicized” as a rather self-contained one. In [Re] this list is obtained as a corollary of a result related to a more type of vector fields. As a matter of fact, if we take into account our previous knowledge of Ogg’s list, in particular the knowledge of the analytic set formed by the separatizes of the foliation in question, the normal forms above can easily be deduced from the method discussed in Section 5, the Appendix devoted to the structure of singularities of fibrations, to which we refer to further details. Note also that the normal forms considered above involve not only the structure of the set formed by the separatizes but also the existence of certain multiplicities associated with its irreducible components (cf. for example model 1 and model 4). This complementary information is also provided by our method in Section 5.
2.2 Dinkyn Diagrams - The structure of the singular fiber

Kodaira [K] has shown that the only possible singular fibers for an elliptic fibration are either a rational curve with a node or a cusp (the so-called pinched torus), or a sum of rational curves of self-intersection $-2$ of one of the following seven types:

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1 1 1
1
1
1 1
1
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Figure 1: Kodaira’s classification of singularities of a pencil of elliptic curves

Kodaira’s argument is elementary in nature. It was extended by Ogg, [O], and independently by Iitaka, to fibrations of genus 2. For the convenience of the reader, and also because it fits the purpose of this paper of presenting a systematic treatment of questions relative to singular fibrations, we are going to summarize Ogg’s classification here. To begin with, let $\mathcal{P} : M \to S$ denote again a fibration consisting of curves of genus $g$, with connected fibers. Throughout this section, the fibers of $\mathcal{P}$ are supposed to be minimal in the sense that they do not contain a irreducible component consisting of a rational curve with self-intersection $-1$. Note that this assumption does not force $M$ to be minimal as well, in other words $M$ may contain $-1$ rational curves. Yet, this suffices to allow us to treat fibrations and pencils on the same footing.

Note that every two fibers are equivalent as divisors, i.e. they belong to the same homology class whether or not they are singular. Hence, the self-intersection of any fiber $L$ is $L \cdot L = L^2 = 0$. Moreover, if $D$ belongs to the group generated by the irreducible components of a fiber $L$, we still have $L \cdot D = 0$. Next, we recall that the arithmetic genus of a fiber $L$ is defined by

$$g(L) = 1 + \frac{1}{2}(L^2 + L \cdot K),$$

where $K$ stands for the canonical divisor of $M$. In particular, if $L = nD$, then the condition of $g$ being an integer forces $n = 1$ in the case of $g = 2$. So, if a fiber has components of a single type, it can only contain one single component. On the other hand, if a fiber $L$ is of the form $L = nD + \Gamma$, $D$ and $\Gamma$ distinct, then $0 = L \cdot \Gamma = nD \cdot \Gamma + \Gamma^2$. In turn, this equation implies that $\Gamma^2 < 0$. Finally, if $L = \sum_i n_i \Gamma_i$, where all $\Gamma_i$’s are distinct components, we have $\Gamma_i \cdot K \geq 0$, and $2g - 2 = \sum_i n_i \Gamma_i \cdot K$.

Now, by exploiting the fact that the arithmetic genus $g(\Gamma_i)$ of every irreducible component $\Gamma_i$ must be a non-negative integer, we conclude that $\Gamma_i^2$ is odd exactly when $\Gamma_i$ is odd. Also, $\Gamma_i^2 \geq -2 - \Gamma_i \cdot K$. 
Summarizing, setting the genus $g$ of the whole fiber $L$ equal to 2, the above relations yield the following five possibilities for the structure of a singular fiber:

Type A: $\Gamma_i \cdot K = 1 \quad \Gamma_i^2 = -1 \quad g = 1$;
Type B: $\Gamma_i \cdot K = 1 \quad \Gamma_i^2 = -3 \quad g = 0$;
Type C: $\Gamma_i \cdot K = 2 \quad \Gamma_i^2 = -2 \quad g = 0$;
Type D: $\Gamma_i \cdot K = 2 \quad \Gamma_i^2 = -4 \quad g = 1$;
Type E: $\Gamma_i \cdot K = 0 \quad \Gamma_i^2 = -2 \quad g = 0$;

From the relation $\sum_i n_i (\Gamma_i \cdot K) = 2g - 2 = 2$, we see that a reducible fiber of a genus 2 fibration has only one out of five choices:

i) It has a component of type $C$, and all other fibers are of type $E$;
ii) It has a component of type $D$, and all other fibers are of type $E$;
iii) It has a component of type $A$ with multiplicity 2, and all other fibers are of type $E$;
iv) It has a component of type $A + B$, and all other fibers are of type $E$;
v) It has a component of type $B$ with multiplicity 2, and all other fibers are of type $E$;

The strategy from now on is to study separately each of the five cases above. The possible intersection numbers of different components will give rise to combinatorial relations that will determine the shape of the corresponding singular fibers.

Let us start with the case i) which is the simplest one. In other words, $L$ contains a component of type $C$ with multiplicity 1, and all its other components are of type $E$. Here we first note that $\Gamma \cdot (L - \Gamma) = 2$, so that $\Gamma$ intersects the rest of the fiber twice. Hence, by the argument of Kodaira, it can only be one of the seven types of cycles (of self-intersection $-2$) described in [K], where one of the components of multiplicity 1 is replaced by $\Gamma$. This concludes the classification of case i).

The reader should note that, in what follows, we shall keep the numbering used by Ogg in [O] when referring to singular fibers. In particular, the example above is said to be of “Type 1” in [O]).

![Figure 2: Type 1 singularities](image)

Let us continue by analyzing the case ii). Therefore $L$ is supposed to contain a component $\Gamma$ of type $D$. Then, $\Gamma \cdot (L - \Gamma) = 4$, ie, $\Gamma$ intersects the rest of the fiber four times more.

Certainly, one possibility for $L$ is to have $\Gamma$ joining two “Kodaira components” (Type 2), which will be excluded henceforth.
Assume that $\Gamma_1$ (of type E) intersects $\Gamma$. We claim that $\Gamma_1 \cdot \Gamma \leq 2$, for $0 = L \cdot \Gamma_1 = \Gamma_1^2 + \Gamma_1 \cdot \Gamma + \Gamma_1 \cdot \Gamma_2$, and $L$ is connected (so $\Gamma_1 \cdot \Gamma_2 \geq 0$). However, if $\Gamma_1 \cdot \Gamma = 2$, then either $\Gamma_1$ intersects $\Gamma$ in two points, or the must have a double contact. If $\Gamma_1$ appears with multiplicity 1, then the fiber ends at $\Gamma_1$, and we must be in the cases already counted as Types 1 and 2. If $\Gamma_1$ has multiplicity 2, then $\Gamma_1$ must meet $2\Gamma_2$, and so on, giving rise to Type 3. Therefore we may assume that $\Gamma$ (of Type D) intersects all the other components in at most 1 point.

Consider now the case $\Gamma_1 \cdot \Gamma = 1$. Let $m$ be the multiplicity of $\Gamma_1$ in $L$, $m \in \{1, 2, 3, 4\}$. Then, $\Gamma_1$ meets the rest of the fiber in $\Gamma_1 \cdot (L - m\Gamma_1) = 2m - 1$ more times. Under these conditions, we have the following algorithm to determine the possible models, i.e., the possible multiplicities of the irreducible components:

**Step 0:** Set $\ell^0 = m$, $\ell^{-1} = 1$ = (number of times that $\Gamma_1$ meets $\Gamma$).

**Step 1:** Write $2\ell^0 - 1$ (the number of times $\Gamma_1$ intersects the rest of the fiber) as a sum of positive integers

$$2\ell_1^0 + \cdots + \ell_{k_1}^0,$$

such that $2\ell_j^0 \geq \ell^0$ for all $j \in \{1, \cdots, k_1\}$.

If for all $j \in \{1, \cdots, k_1\}$ we have that $2\ell_j^0 = \ell^0$, then the fiber is complete, and we can stop here. Otherwise, there exists at least one $\ell_{k_1, i_1}$ such that $2\ell_{k_1, i_1}^0 - \ell^0 > 0$.

**Step 2:** For all $\ell_{k_1, i_j}$ from Step 2, we will write

$$2\ell_{k_1, i_j}^0 - \ell^0 = \ell_{k_1, i_j}^2 + \cdots + \ell_{k_2, i_j}^2,$$

such that $2\ell_{p, i_j}^2 \geq \ell_{k_1, i_j}^0$, for all $p \in \{1, \cdots, k_2\}$.

We stop if equality holds for all $p$. Otherwise, we proceed inductively to

**Step s:** We write, for all $\ell_{k_{s-1}, i_j}^{s-1}$ from Step s - 1,

$$2\ell_{k_{s-1}, i_j}^{s-1} - \ell_{k_{s-2}, i_j}^{s-2} = \ell_{k,s, i_j}^{s} + \cdots + \ell_{k_{s-1}, i_j}^{s},$$

such that $2\ell_{p, i_j}^{s} \geq \ell_{k_{s-1}, i_j}^{s-1}$, for all $p \in \{1, \cdots, k_s\}$.

There are three possibilities for this algorithm. Either it ends in finite steps, generating explicitly the Types 3 until 11 in [Q]; or it generates an obvious impossibility at finite time, case that should be discarded; or finally, as it is clear, for $s$ large enough, the only possible decomposition in integers is the trivial one. This would represent the fiber with an infinite number of components of type $E$, with multiplicities $m, 2m - 1, 2(2m - 1) - m$, ..., also impossible.

In order to illustrate the method above, let us study, for example, the case when $m = 4$.

We can write $8 - 1 = 7$ as 7, 3 + 4, 2 + 5, or 2 + 2 + 3. The latter three, if we keep running the algorithm, will generate the following three models:

For the trivial decomposition 7 = 7, $\Gamma_1$ will meet the fiber 14 - 4 = 10 more times. We can write 10 as 5 + 5, 4 + 6, or 10. The former ones lead to impossibilities. Hence, $\Gamma_1$ meets $10\Gamma_2$, $\Gamma_2$ also of type $E$.

Writing $20 - 7 = 13$ as 5 + 8, 6 + 7, or 13 will clearly lead to Type 7, an impossibility, or an infinite chain.

The cases for the other values of $m$ are completely analogous. This completes the case where the singular fiber $L$ is of type ii).
Figure 3: If a fiber contains a component of type $D$ meeting a component of multiplicity 4

In order to describe the fibers that only contains components of Types A, B and E, we can develop algorithms of the same fashion as the one above, simply by following the same reasoning. We just need to make further considerations with respect to the number of different components that a fixed one meets in the singular fiber. Explicitly describing those algorithms, though, would make the notation unnecessarily heavy, and we prefer to omit it. For the complete list of possible examples, see [O].

3 Some explicit dual examples arising from non-linear singularities

In this section, we study some of the normal forms of vector fields listed at the end of Section 2. We shall constructively obtain examples of dual fibers by turning the pencils obviously associated to the mentioned normal forms into actual fibrations by means of successive blow-ups.

**Example A.** We start with the potential given by $F(x, y) = x^5 y^4 (x - y)$. It is singular along the lines $x = 0$, $y = 0$, $z = 0$ and blows up at the line at infinity (to be denoted by $\Delta$).

Let $P_1$, $P_2$ and $P_3$ be the intersection of those lines, as in Figure 4.

![Figure 4: Singular Sets of the vector fields](image)

On a neighborhood of $P_3$, we write $x = \frac{1}{u}$, $y = \frac{u}{w}$. The potential becomes $F(u, v) = \frac{u^4 (1 - v)}{v^n}$, and our
singularity looks like $\frac{v^4}{u^3}$ near $u = v = 0$.

A sequence of blow-ups of the singularities is pictured below. Each of the numbers on the components indicates its self-intersection.

![Figure 5: Blow up of first singularity](image)

Analogously, we can study the singularity at the point $P_2$, obtaining the model below.

![Figure 6: Resolution of second singularity](image)

Finally, the singularity at $P_1$ yields

![image]

Gathering all the information about the fiber, we obtain the following dual fibers, which are classified as Type 20 (since it has a component of self-intersection 3) and a pinched torus in [O].

Note that in Figure 7, the components with self-intersection $-1$ were collapsed. Since we are primarily interested in classifying minimal models we shall proceed in this way every time we reach a $-1$-component.

**Example B.** The associated potential is $F(x, y) = x^5y^2(x - y)^3$. The process of blowing up singularities
Figure 7: Type 20 and a pinched torus

is analogous to the one carried out in the discussion of Model N6. The dual fibers obtained at the end are Type 7 and Type 16 of Ogg’s list:

Figure 8: Types 7 and 16

Example C. The potential is $F(x, y) = x^4 y^3(x - y)$, and its associated dual fibers are Type 25 and a pinched torus, with quartic tangency.

Example D. The examples discussed above are non-ramified in the sense of Section 2. Let us now describe the remaining ramified examples. Let us begin by considering the potential $F(x, y) = x^2 y^2(x - y)$. The dual fibers associated to the corresponding fibration are of Type 44 and Type 8:

Example E. The potential for this model is $F(x, y) = x^3 y(x - y)$. The dual fibers associated to those models are of Type 21 and Type 36:
Figure 9: Types 25 and a pinched torus

Figure 10: Types 44 and 8

Figure 11: Types 21 and 36
4 Canonical Models for Hyperelliptic Fibrations

The goal of this section is to prove Theorem 1.1 which can be thought of as a local uniformization theorem for fibrations of genus 2. The strategy of the proof relies on the study of the singularities of each model in Ogg’s list. In particular we shall check that the eigenvalues of each of these singularities are uniquely determined by the combinatorial data of the singular fiber itself.

Concerning the index of a separatrix with respect to a singular foliation \( F \), as defined in Section 2, there is a special case that are going to be rather useful in what follows. Suppose for a moment that \( F \) is a local singular foliation defined about \( (0,0) \in \mathbb{C}^2 \). Suppose, in addition, that at the origin \( F \) has eigenvalues \( \lambda_1, \lambda_2 \neq 0 \) with quotient belonging to \( \mathbb{C} \setminus \mathbb{R}_+ \). By definition this means that \( F \) can be represented by a local holomorphic vector field whose linear part at the origin has \( \lambda_1, \lambda_2 \) as eigenvalues.

It is well-known that, in this case, there are local coordinates \((x, y)\) where \( F \) is represented by a vector field of the form

\[
\lambda_1 x (1 + \text{h.o.t.}) \frac{\partial}{\partial x} + \lambda_2 y (1 + \text{h.o.t.}) \frac{\partial}{\partial y}.
\]

In particular \( F \) possesses exactly two separatrices given in the above coordinates by the axes \( \{x = 0\} \) and \( \{y = 0\} \). A direct inspection shows that

\[
\text{Ind}_{(0,0)} (F, \{y = 0\}) = \frac{\lambda_2}{\lambda_1} \quad \text{and} \quad \text{Ind}_{(0,0)} (F, \{x = 0\}) = \frac{\lambda_1}{\lambda_2}.
\]

Hence

\[
\text{Ind}_{(0,0)} (F, \{y = 0\}) = \frac{1}{\text{Ind}_{(0,0)} (F, \{x = 0\})}.
\]

It is particularly easy to detect when a singular point of a fibration \( F \) possesses non-zero eigenvalues. This is due to the proposition below.

**Proposition 4.1** Let \( \mathcal{P} \) be a singular fibration on a complex surface \( M \) having \( \mathcal{P}^{-1}(0) \) as a singular fiber. A singular point \( p \) of \( \mathcal{P}^{-1}(0) \) possesses non-zero eigenvalues as a singularity of \( \mathcal{P} \) if and only if \( p \) is a nodal singularity of \( \mathcal{P}^{-1}(0) \). Furthermore, if \( p \) has non-zero eigenvalues, then it is automatically a linear singularity of \( \mathcal{P} \). In fact, there are local coordinates \((x, y)\) about \( p \) in which \( \mathcal{P} \) coincides with the foliation given by

\[
m x \partial / \partial x - n y \partial / \partial y.
\]

with \( m, n \in \mathbb{N}^* \).

In the case of fibrations having genus 2, this proposition is a corollary of the classification of all possible singularities carried out in [Re] and recalled in Section 2. The Proposition 4.1 is however valid for arbitrary fibrations. Whereas the contents of Proposition 4.1 are probably well-known, it is slightly stronger than claiming the existence of coordinates in which \( \mathcal{P}^{-1}(0) \) is given by \( x^m y^n = 0 \). This issue and its generalizations to more complicated singularities seemed to us to deserve slightly more accurate discussion which appears in Section 5 (Appendix 1).

It follows from the preceding that the structure of a fibration around a nodal singularity of a singular fiber is totally determined by its eigenvalues. Based on this remark, we are going to prove the following stronger statement.

**Proposition 4.2** Let \( \mathcal{P} : M \to D \subset \mathbb{C} \) denote a fibration of genus 2 having \( \mathcal{P}^{-1}(0) \) as singular fiber. Then the Dynkin diagram of \( \mathcal{P}^{-1}(0) \) analytically determines the structure of the fibration \( \mathcal{P} \) on a neighborhood of every singular point.
The proposition above can also be stated as follows. Suppose that \( P_1 : M \to D \subset \mathbb{C} \) (resp. \( P_2 : M \to D \subset \mathbb{C} \)) are fibrations of genus 2 whose singular fibers \( P_1^{-1}(0), P_2^{-1}(0) \) have isomorphic Dynkin diagrams. Then, if \( p_1 \in P_1^{-1}(0), p_2 \in P_2^{-1}(0) \) are corresponding singularities in this Dynkin diagram, then there is a local holomorphic diffeomorphism from \( p_1 \) to \( p_2 \) sending fibers of \( P_1 \) to fibers of \( P_2 \).

**Proof of Proposition 4.2** Let us start with the case of fibers having only nodal singularities. As already seen, on a neighborhood of a nodal singularity, the fibration \( P \) is given by a linear vector field \( mx\partial/\partial x - ny\partial/\partial y \), \( m, n \in \mathbb{N}^* \). In particular the indices of each axis \( \{x = 0\}, \{y = 0\} \) with respect to \( P \) verify Equation 5. All we have to do is to check that the eigenvalues of each singularity of \( P^{-1}(0) \) is then determined by its position in the corresponding Dynkin diagram.

The easiest way to deal with this question is to check each model with linear singularities individually. To avoid unnecessary repetition, we shall explain explicitly some examples that contain the general procedure.

To illustrate our method, let us consider the model of singular fiber noted **Model 4** in [O], cf. Figure 12.

![Figure 12: Description of the eigenvalues for Model 4](image)

Let us begin by the components that only intersect the rest of the fiber once.

\[
\text{Ind}_{p_i}(\mathcal{F}, S_i) = S_1 \cdot S_i = -4 = \frac{\lambda_{2,1}}{\lambda_{1,1}} \quad \text{and} \quad \text{Ind}_{p_i}(\mathcal{F}, S_i) = S_i \cdot S_i = -2 = \frac{\lambda_{2,i}}{\lambda_{1,i}},
\]

for \( i = 2, 3, 6 \).

Note that, by changing variables (or re-scaling coordinates), we can always take one of the eigenvalues of this type of components (say, for example, \( \lambda_{1,i}, i = 1, 2, 3, 6 \)) to be equal to 1. Hence, we determine that \( \lambda_{2,1} = -4 \), and \( \lambda_{2,i} = -2 \), for \( i = 2, 3, 6 \).

Also,

\[
\text{Ind}_{p_5}(\mathcal{F}, S_5) + \text{Ind}_{p_6}(\mathcal{F}, S_5) = \frac{1}{\lambda_{2,6}} + \frac{\lambda_{2,5}}{\lambda_{1,5}} = -2
\]

\[
\sum_{j=1}^{4} \text{Ind}_{p_j}(\mathcal{F}, S_7) = \sum_{j=1}^{4} \frac{\lambda_{2,j}}{\lambda_{1,j}} = -2.
\]

The first equation determines \( \text{Ind}_{p_5}(\mathcal{F}, S_5) = -\frac{3}{2} \), and the last determines \( \text{Ind}_{p_4}(\mathcal{F}, S_7) = -\frac{3}{4} \).
Note that we would still have an extra relation given by the separatrix $S_4$, which is redundant. This phenomenon will occur in all examples where the fiber model corresponds to a graph with trivial fundamental group. It is due to the trivial fact that, in such graphs, the number of vertices is strictly smaller than the number of edges.

Still considering only linear singularities, there is another possible case in which some of the components of the singular fiber form a loop. As a prototype for these cases, let us consider Model 10 in [O], cf. Figure 13.

![Figure 13: Description of the eigenvalues for Model 10](image)

Following the same reasoning as above, and writing $\text{Ind}_{p_i}(\mathcal{F}, S_j) = I_{i,j}$, we see that

$$I_{1,1} + I_{4,1} = -\frac{4}{3}; \quad I_{1,2} + I_{2,2} = -2; \quad I_{2,3} + I_{3,3} = -4; \quad \text{and} \quad I_{3,4} + I_{4,4} = -2.$$ 

This $4 \times 8$ system has a rank 4 space of solutions, which will give us uniqueness when we impose the conditions that, if two edges $S_i$ and $S_j$ intersect, then for any $p_q \in S_i \cap S_j$, $I_{q,i}I_{q,j} = 1$.

Once again, this example reflects the general idea: whenever we have a loop in the graph representing the fiber, we will necessarily have that the number $N$ of edges will equal the number of vertices. Camacho-Sad Index formula will provide a $N \times 2N$ linear system of rank $N$, that together with the observation on the previous paragraph, imply the uniqueness of the indices of the singularities for each model.

The study of the singularities of each of the models in [O] that have linear singularities will coincide with one of the two cases depicted above, and therefore is going to be omitted.

It remains to discuss the cases corresponding to the singular fibers exhibiting non-linear singularities. Again by virtue of Proposition 4.1, non-linear singularities are detected as the singular points of $\mathcal{P}^{-1}(0)$ that are not of nodal type. Since these singularities were classified in [Re], we have the full list of possible normal forms given in Section 2. It can directly be checked that each possible normal form is totally identified by the nature of the singularity of $\mathcal{P}^{-1}(0)$ (as a singularity of a curve) along with the corresponding indices. For example when discussing the Example B in Section 3, we have found a singularity consisting of two smooth separatrices with quadratic tangency and having indices $-3$ and $-4/3$. There is only one possible normal form whose separatrices satisfy all these conditions, namely they are smooth curves with quadratic tangency and the mentioned indices. It follows that whenever one has a singularity of $\mathcal{P}^{-1}(0)$ verifying these conditions, the local structure of $\mathcal{P}$ must be given by the foliation associated to the vector field $Z_{2,(2-3)^3}$ (item 7 in the list of ramified singularities at the end of Section 2).
Summarizing to deal with the cases where there are non-linear singularities we proceed as follows. By considering the linear singularities appearing in the model in question, we repeat the preceding analysis to conclude that the indices of the separatrices of the non-linear non-linear singularity are uniquely determined. By the above observation, these indices together with the form of the singularity formed by the separatrices themselves characterize unequivocally the local normal form of $\mathcal{P}$. This concludes the proof of the proposition.

What precedes provides a complete description of the structure of the fibration $\mathcal{P}$ on a neighborhood of a singular point of $\mathcal{P}^{-1}(0)$. Thus, by now, we have understood the “local pieces” that can be used to build $\mathcal{P}$. Our next task will consist of working out the possible assembling of these pieces so as to obtain a better global understanding of $\mathcal{P}$. As far as this section is concerned, our global understanding is represented by Theorem [2.1].

The key notion that will lead us to the proof of this theorem is the holonomy associated to $\mathcal{P}^{-1}(0)$. To explain this notion, let us fix an irreducible component $D$ of $\mathcal{P}^{-1}(0)$. Then $L = D \setminus \text{Sing}(\mathcal{P}^{-1}(0))$ can be viewed as a regular leaf of the foliation induced $\mathcal{P}$. Let then $\Sigma$ denote a local transverse section to $L$ at a point $x$. In particular we note that every fiber of $\mathcal{P}$ can cut $\Sigma$ at a uniformly bounded number of points. Identifying $x$, $\Sigma$ with a neighborhood of $0 \in \mathbb{C}$, the holonomy of $L$ provides us a representation

$$\rho_D : \pi_1(L) \rightarrow \text{Diff}(\mathbb{C}, 0)$$

where $\pi_1(L)$ stands for the fundamental group of $L$.

**Lemma 4.3** The image $\rho_D(\pi_1(L)) \subset \text{Diff}(\mathbb{C}, 0)$ is a finite Abelian group. In suitable coordinates it is generated by a rational rotation.

**Proof.** To show that $\rho_D(\pi_1(L))$ is Abelian, let us suppose for a contradiction that it is not the case. Hence there are elements $h_1, h_2 \in \rho_D(\pi_1(L))$ such that $h = h_1 \circ h_2 \circ h_1^{-1} \circ h_2^{-1}$ is not reduced to the identity. However the resulting local diffeomorphisms $h \neq \text{Id}$ obviously satisfies $h'(0) = 1$. The local dynamics of this type of diffeomorphism is known as the “flower” and it possesses infinite orbits accumulating at $0 \in \mathbb{C}$. This means that nearby fibers of $\mathcal{P}$ would accumulate on $\mathcal{P}^{-1}(0)$ what is impossible.

Let now $h \neq \text{Id}$ be an arbitrary element of $\rho_D(\pi_1(L))$. Clearly we must have $|h'(0)| = 1$ since otherwise $h$ or $h^{-1}$ would have orbits accumulating at $0 \in \mathbb{C}$ what is impossible. Hence we can set $h(z) = e^{2\pi i \theta} z + \cdots$. If $\theta$ is not rational then we would still have fiber of $\mathcal{P}$ intersecting $\Sigma$ a number arbitrarily large of times. As already seen, this is again impossible. Thus we finally conclude that $h(z) = e^{2\pi i p/q} z + \cdots$. In particular $h^q(z) = z + \cdots$. In view of the preceding argument, it follows that $h^q = \text{Id}$.

The preceding shows that $\rho_D(\pi_1(L))$ is a finitely generated Abelian group all of whose elements are of finite order. It follows then that $\rho_D(\pi_1(L))$ is finite and hence conjugate to a group of rational rotations. Clearly the latter group must be cyclic and this completes the proof of the lemma.

We are now ready to start the proof of Theorem [2.1]. Consider fibrations $\mathcal{P}_1 : M_1 \rightarrow \mathbb{D}$ and $\mathcal{P}_2 : M_2 \rightarrow \mathbb{D}$ as in the statement. To simplify the discussion, we drop the assumption of having minimal singular fibers i.e. $\mathcal{P}_1^{-1}(0)$, $\mathcal{P}_2^{-1}(0)$ are allowed to contain rational curves with self-intersection equals to $-1$. The advantage of doing so is that all the singularities of $\mathcal{P}_1$, $\mathcal{P}_2$ can be supposed to be linear. Let us start with the most common case:

**Proposition 4.4** Let $\mathcal{P}_1$, $\mathcal{P}_2$ be as above and suppose that all the irreducible components of $\mathcal{P}_1^{-1}(0)$, $\mathcal{P}_2^{-1}(0)$ are rational curves. Then the statement of Theorem [2.1] holds.
Proof. Let $D^1$ be an irreducible component of $\mathcal{P}_1^{-1}(0)$ and denote by $D^2$ the corresponding component of $\mathcal{P}_2^{-1}(0)$. Denote by $p^1_i, \ldots, p^r_i$ the singularities of $\mathcal{P}$ lying in $D^1$. Each $p^1_i$ is a linear singularity whose local holonomy is conjugate to a rational rotation. Fixed a base point $x_0 \in D^1 \setminus \{p^1_1, \ldots, p^1_r\}$, the fundamental group of $D^1 \setminus \{p^1_1, \ldots, p^1_r\}$ is generated by loops $c_1, \ldots, c_k$ based at $x_0$ and each of them encircling a single singularity $p^1_1, \ldots, p^1_r$. In particular $c_i$ is freely homotopic to a small circle about $p_i$ characterizing the local holonomy of this singularity. The only non-trivial relation verified by the generators $c_1, \ldots, c_k$ is $c_1 \cdot c_2 \cdots c_k = \text{id}$. Fixed a local transverse section $\Sigma$ at $x_0$, the holonomy of the regular leaf of $\mathcal{P}$ given precisely by $D^1 \setminus \{p^1_1, \ldots, p^1_r\}$ can be identified to a homomorphism

$$\rho : \pi_1(D^1 \setminus \{p^1_1, \ldots, p^1_r\}) \longrightarrow \text{Diff}(\mathbb{C}, 0).$$

Recalling that $\rho(c_i) = h_i$ is conjugate to a rational rotation, we can denote its order by $n_i \in \mathbb{N}^*$. Thus the image $\rho(\pi_1(D^1 \setminus \{p^1_1, \ldots, p^1_r\})) \subset \text{Diff}(\mathbb{C}, 0)$ is generated by local diffeomorphisms $h_1, \ldots, h_r$ satisfying the following relations:

$$h_1^{n_1} = \cdots = h_r^{n_r} = h_1 \circ \cdots \circ h_r = \text{id}. \quad (9)$$

The argument used in the proof of Lemma 4.3 still implies that $\rho(\pi_1(D^1 \setminus \{p^1_1, \ldots, p^1_r\}))$ is Abelian. Thus it is also finite and generated by a single rational rotation whose order is totally determined by the orders $n_1, n_2, \ldots, n_r$. In turn this means that $\rho(\pi_1(D^1 \setminus \{p^1_1, \ldots, p^1_r\}))$ is generated by a rational rotation whose order is determined explicitly by the eigenvalues associated to the singularities $p^1_1, \ldots, p^1_r$.

The proof of the proposition now goes as follows. We consider the corresponding components $D^1, D^2$ along with the singularities $\{p^1_1, \ldots, p^1_r\} \subset D^1$ (resp. $\{p^2_1, \ldots, p^2_r\} \subset D^2$). According to Proposition 4.2 for every $i = 1, \ldots, r$ the eigenvalues of $\mathcal{P}_1$ at $p^1_i$ coincide with those of $\mathcal{P}_2$ at $p^2_i$. The preceding then ensures us that the holonomy of the leaf $D^1 \setminus \{p^1_1, \ldots, p^1_r\}$ w.r.t. $\mathcal{P}_1$ is analytically conjugate to the holonomy of $D^2 \setminus \{p^2_1, \ldots, p^2_r\}$ w.r.t. $\mathcal{P}_2$ since they are both conjugate to the same rational rotation. With this information in hand, we can proceed to construct a conjugacy between $\mathcal{P}_1$ and $\mathcal{P}_2$ on a neighborhood of $D^1, D^2$ as follows. For each $i = 1, \ldots, r$ and $j = 1, 2$, let $W_i^j$ denote a small neighborhood of $p^j_i$. In particular $D^1 \setminus \bigcup_{i=1}^r W_i^1$ (resp. $D^2 \setminus \bigcup_{i=1}^r W_i^2$) is a compact part of the leaf $D^1 \setminus \{p^1_1, \ldots, p^1_r\}$ (resp. $D^2 \setminus \{p^2_1, \ldots, p^2_r\}$). The holomorphic conjugacy between the corresponding holonomies can then be lifted to a (transversely holomorphic) $C^\infty$-conjugacy between $\mathcal{P}_1, \mathcal{P}_2$ on neighborhoods of $D^1 \setminus \bigcup_{i=1}^r W_i^1$, $D^2 \setminus \{p^2_1, \ldots, p^2_r\}$. The linear character of the singularities $p^j_i$ along with their saddle-like behavior assured by the sign of the eigenvalues makes it easy to check that this conjugacy can be extended to the neighborhoods $W_i^j$.

A simple repetition of the preceding argument, combined with an induction argument, shows that the conjugacy constructed above can be extended from the component $D^1 \subset \mathcal{P}_1^{-1}(0)$ (resp. $D^2 \subset \mathcal{P}_2^{-1}(0)$) to a subsequent component of $\mathcal{P}_1^{-1}(0)$ (resp. $\mathcal{P}_2^{-1}(0)$). For further details, the reader can check Remark 4.5 below. The mentioned induction procedure eventually leads to the desired conjugacy defined on neighborhoods of $\mathcal{P}_1^{-1}(0), \mathcal{P}_2^{-1}(0)$. The proposition is proved.

Remark 4.5 The induction procedure mentioned above deserves some further comments. Let $D^1 = D^1_1$ be a rational curve contained in $\mathcal{P}_1^{-1}(0)$. The holonomy of $D^1$ w.r.t. $\mathcal{P}_1$ as defined above may differ from a similar notion of holonomy that takes into account the entire singular fiber $\mathcal{P}_1^{-1}(0)$. To explain this difference, let call tha above defined holonomy of the holonomy generated at $D^1_1$. Next consider another rational curve $D^1_2$ contained in $\mathcal{P}_1^{-1}(0)$ and intersecting $D^1_1$ at a linear singularity $p$. Denote by $m, -n$ the eigenvalues of $\mathcal{P}_1$ at $p$ so that the local holonomy $h^1_1$ of $D^1_1$ (resp. $h^1_2$ of $D^1_2$) is conjugate to a rotation of order $m$ (resp. $n$). The singularity $p$ can allow part of the holonomy of $D^1_1$ to be “transmitted” to $D^1_2$ (and vice-versa). Indeed, the effect of the singularity $p$ is to change a rotation of order $m$ into a rotation of order $n$. Thus, if the holonomy group generated at $D^1_1$ has order exactly
m then there is no transmission from $D_1^1$ to $D_2^1$. However if the group in question has order strictly larger than $m$, then those transformations that are not in the subgroup generated by $h^1_1$ will induce non-trivial transformations on $D_2^1$. These new transformations need not be contained in the holonomy group generated at $D_2^1$. Finally, the group of transformations induced at an irreducible component $D_1^1$ by all these transformations will be called the total holonomy of $D_1^1$ w.r.t. $P_1$.

The argument employed in the proof of Proposition 4.4 actually shows that every two corresponding irreducible components $D^1_1 \in P_1^{-1}(0)$ and $D^2_2 \in P_2^{-1}(0)$ have conjugate total holonomy. The meaning of the induction procedure mentioned at the end of this proof should now be clearer.

**Remark 4.6** The total holonomy group of a component $D$ of a singular fiber $P^{-1}(0)$ has a natural relation of the *multiplicity* of the component in question. Recall that this multiplicity is defined in terms of the divisors naturally associated to $P^{-1}(0)$ which should belong to the same cohomology class as a regular fiber. In terms of total holonomy, this multiplicity is nothing but the order of the “total holonomy group of $D$ w.r.t. $P$”. In fact, these numbers are nothing but the number of intersections of a nearby fiber with a local transverse section $\Sigma$ through a point of $D$.

To close the section we shall push forward the method employed in the preceding proposition to set up the proof of Theorem 1.1.

---

**Proof of Theorem 1.1** In view of the preceding, we can restrict our attention to singular fibers $P^{-1}_1(0), P^{-1}_2(0)$ containing an elliptic curve among their irreducible components. Consider corresponding elliptic components $E^1 \in P^{-1}_1(0)$ and $E^2 \in P^{-1}_2(0)$. We shall use analogous definitions of holonomy generated at $E^1$ and of total holonomy at $E^2$, cf. Remark multiplicity1. Let then $p^1_1, \ldots, p^1_r$ be the singularities of $P_1$ lying in $E^1$. To each of these singularities it is associated to a local holonomy $h^1_1, \ldots, h^1_r$. Besides each $h^1_i$ has finite order $n^1_i$. Again in view of Propostion 4.2 there are corresponding singularities $p^2_1, \ldots, p^2_r$ in $E^2$ that are locally conjugate to $p^1_1, \ldots, p^1_r$ in $E^1$.

Since $E^1$ is not simply connected, the holonomy generated at $E^1$ is given by the free group in the generators $h^1_1, \ldots, h^1_r$ and $f^1_1, f^1_2$ where $f^1_1, f^1_2$ to the generators of the fundamental group of $E^1$. Analogously definitions apply to $E^2$ and $f^2_1, f^2_2$. The additional difficulty involved in the present case is that we have to show that $\Gamma_1 = \langle h^1_1, \ldots, h^1_r, f^1_1, f^1_2 \rangle$ and $\Gamma_2 = \langle h^2_1, \ldots, h^2_r, f^2_1, f^2_2 \rangle$ are still conjugate. In particular we have to settle a correspondence between $f^1_1, f^1_2$ and $f^2_1, f^2_2$ which are, in principle, arbitrary.

To establish the existence of a conjugacy between $\Gamma_1, \Gamma_2$ we first observe that these groups are still Abelian. Then they also have finite order due, for example, to Remark 4.6. Thus we only need to show that $f^1_1$ (resp. $f^2_1$) has the same order of $f^2_2$ (resp. $f^2_2$) up to relabeling these generators. Actually the following stronger claim holds.

**Claim:** All the local diffeomorphisms $f^1_1, f^1_2, f^2_1, f^2_2$ coincide with the identity.

**Proof of the claim.** We just have to check the models of singular fibers possessing an elliptic curve as component. These are models labeled “Type 1”, “Type 12”, “Type 13” and “Type 14” in Ogg’s classification [O]. Except for “Type 1”, the multiplicity of these elliptic curves were seen to be equal to 1. Thus the total holonomy groups of these components are all trivial and the claim results, cf. Remark 4.6.

As to “Type 1” we simply note that these singular fibers were obtained by choosing one of the seven Kodaira’s models (for elliptic fibrations) and then replace one of its component having multiplicity one by an elliptic curve. Therefore the elliptic curve still has multiplicity one and the preceding argument applies to it. The claim is proved and so is Theorem 1.1. 

\[\square\]
Some comments regarding the complex structure of the regular fibers: Here are some informal questions and facts about fibrations of genus 2 that are naturally raised by Theorem 1.1. We hope to pursue these questions further elsewhere and, in particular, to apply them to the study of global aspects of genus 2 fibrations as those discussed in [X] and, more recently, in [C-P].

Consider fibrations $P_1: M_1 \to D$ and $P_2: M_2 \to D$ as in the statement of Theorem 1.1. It is natural to wonder when it is possible to have a holomorphic conjugacy between these fibrations. Clearly, a necessary condition for the existence of such a conjugacy it a correspondence between regular fibers of $\cap_1$ and $P_2$ as Riemann surfaces (modulo a local change of coordinates in the disc $D$). Conversely, it is not hard to check that this condition is also sufficient. Therefore, to decide about the existence of holomorphic conjugacies we have the following cases of particular interest.

Case 1: Suppose that the complex structure of the regular fiber of $\cap_1$ (resp. $P_2$) does not vary. In this case the only further “holomorphic” invariant of these fibrations is the complex structure of the fibers. We ignore however if all the moduli space of a genus 2 curve can be realized as a fibration with singular fiber having fixed type (the answer seems to be not but it would be of interest to work out the sharp conditions).

In favor of the interest of this Case 1, we point out that for several types of singular fiber it can be proved that the complex structure of the neighbors fibers is constant. Thus in the presence of those fiber, we are then reduced to the existence of a single “holomorphic” invariant as above.

Case 2: If the complex structure of the fibers of $\cap_1$ (resp. $P_2$) is allowed to vary. Although it is known this does not happen in a global genus 2 fibration on a compact surface, this might be possible on the semi-local context of a fibration over the disc. Naturally this phenomenon does happen in specific cases of higher genus fibrations. Here the necessary invariant is rather more subtle as it must take into account the whole family of fibers. Yet reasonably standard deformation techniques seems to have a saying on the “dimension” of these invariants for a fixed type of singular fiber.

5 Appendix 1: Linear and degenerate singularities in fibrations

The purpose of this appendix/section is primarily to provide a proof of Proposition 5.1 below that generalizes to higher genus the contents of Proposition 4.1. However, apart from linear singularities, the method presented below is also effective for degenerate singularities as it will be explained in the sequel.

Proposition 5.1. Let $\mathcal{P}$ be a proper holomorphic mapping onto the disc defining a fibration having $\mathcal{P}^{-1}(0)$ as singular fiber. Denoting by $\mathcal{F}$ the singular foliation induced by the level curves of $\mathcal{P}$, a singularity $p \in \mathcal{P}^{-1}(0)$ of $\mathcal{F}$ possesses non-zero eigenvalues if and only if $p$ is a nodal singularity of $\mathcal{P}^{-1}(0)$. Furthermore, in this case, there are local coordinates $(x, y)$ around $p$ in which $\mathcal{F}$ is given by the vector field $mx\partial/\partial x - ny\partial/\partial y$, for some $m, n \in \mathbb{N}^*$. 

Before starting the proof of Proposition 5.1, it is convenient to make a few additional comments. Recall that $\mathcal{F}$ possesses a non-constant holomorphic first integral on a neighborhood of $p$. Still denoting by $\mathcal{P}$ this first integral, it promptly follows the existence of local coordinates $(x, y)$ in which $\mathcal{P}$ becomes $x^ny^mg(x, y)$ where $m, n \in \mathbb{N}^*$ and where $g$ is a holomorphic function verifying $g(0, 0) \neq 0$. The fact that $g$ can actually be chosen to be constant is however less obvious. This is, indeed, equivalent to the existence of a local conjugacy between the linear foliation given by $mx\partial/\partial x - ny\partial/\partial y$ (i.e. defined by the first integral $x^ny^m$) and the foliation defined by the first integral $x^ny^mg(x, y)$. The more general
problem of recognizing the local model of the singularity out of the structure of the set formed by its separatrizes is still less clear when the set in question is more degenerate, for example given by an irreducible cusp such as \( x^2 + y^3 = 0 \). By the way, these statements are false for foliations if they need not have a holomorphic first integral. Another difficulty has to do with the “multiplicities” of the irreducible components. To explain this, consider for example the model \( Z_{2,2.1} = x(3y - 2x)\partial/\partial x + y(3x - 2y)\partial/\partial y \) appearing in the list given at the end of paragraph 2.1. A simple consideration of the set formed by the separatrizes would hint us to consider only the first integral \( xy(x - y) \) and the model in question would be missed for its first integral is \( x^2y^2(x - y) \). In view of the preceding, we decided to include a rather detailed exposition of a “dynamical” method that has the advantage of “automatically” taking care of these difficulties.

**Proof.** We shall give a very detailed proof which also applies to much more general situations. First choose and fix, once and for all, a holomorphic vector field \( Z \) defined on a neighborhood of \( p \). Besides \( Z \) is tangent to \( \mathcal{F} \) and it has isolated singularities. With this choice made, the eigenvalues of \( \mathcal{F} \) at \( p \) (resp. the order of \( \mathcal{F} \) at \( p \)) are, by definition, the eigenvalues of the linear part of \( Z \) at \( p \) (resp. the order of \( Z \) at \( p \)). The eigenvalues are therefore defined only up to a multiplicative constant.

Let us first suppose that \( \mathcal{F} \) has one eigenvalue \( \lambda_1 \) different from zero. We claim that the second eigenvalue \( \lambda_2 \) is also different from zero. To check the claim, suppose that \( \lambda_2 = 0 \). The singularity of \( \mathcal{F} \) at \( p \) is then called a *saddle-node* and it is known since Dulac that there are coordinates \((x_1, y_1)\) where \( Z \) becomes

\[
Z = [x(1 + \lambda y^p) + R(x_1, y_1)]\partial/\partial x_1 + y^{p+1}\partial/\partial y_1
\]

with \( p \geq 1 \). In particular, the axis \( \{y_1 = 0\} \) is invariant under \( \mathcal{F} \). An elementary computation of its local holonomy \( h \) shows that it has the form \( h(z) = z + z^{p+1} + \cdots \). The local dynamics of \( h \) was already mentioned (it is the “flower”) and, in particular, \( h \) has orbits non-trivially accumulating on \( 0 \in \mathbb{C} \). Therefore every holomorphic function that is constant over the orbits of \( h \) must be constant everywhere. This gives us a contradiction since \( P \) is constant over the leaves of \( \mathcal{F} \) (i.e. \( P \) is a first integral of \( \mathcal{F} \)) and hence also over the orbits of \( h \). The resulting contradiction implies that \( \lambda_2 \neq 0 \).

It is now obvious that the linear part of \( Z \) must admit a first integral itself. It then follows that this linear part is given by \( mx_2\partial/\partial x_2 - ny_2\partial/\partial y_2 \) in suitable coordinates. Besides we can actually now choose \((x_2, y_2)\) so that \( Z = mx_2(1 + \text{h.o.t.})\partial/\partial x_2 - ny_2(1 + \text{h.o.t.})\partial/\partial y_2 \). To complete the proof in this first case, it remains only to check that the foliation \( \mathcal{F} \) associated to \( Z \) is, in fact, conjugate to the linear foliation given by \( mx_2\partial/\partial x_2 - ny_2\partial/\partial y_2 \). According to a well-known conjugacy lemma due to Mattei and Moussu [M-M], this happens if only if the local holonomy of the axis \( \{y_2 = 0\} \) w.r.t. \( \mathcal{F} \) has finite order. Again it is clear that \( h'(0) = e^{2\pi i n/m} \) so that \( h^m \) has the form \( h^m(z) = z + \cdots \). Now, unless \( h^m(z) = z \), the local dynamics of \( h^m \) is again the “flower”. As already seen, this is incompatible with the existence of a non-trivial first integral. Thus we must have \( h^m(z) = z \) and the proof of the proposition is over in this first case.

To establish Proposition 5.1, it is now sufficient to show that the eigenvalues of \( Z \) cannot vanish simultaneously. Therefore we suppose for a contradiction that this is the case. We then use Seidenberg’s theorem [S] asserting the existence of a finite sequence of blow-up maps

\[
\mathcal{F} = \mathcal{F}_0 \overset{\pi_1}{\leftarrow} \mathcal{F}_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_k}{\leftarrow} \mathcal{F}_k
\]

such that \( \mathcal{F}_k \) has only singularities with at least one eigenvalue different from zero. Applied to our case, it follows from the preceding that \( \mathcal{F}_k \) has eigenvalues of the form \( m, -n, m, n \in \mathbb{N} \), at all its singularities. Next we have:

**Claim 1.** The exceptional divisor \((\pi_1 \circ \cdots \circ \pi_k)^{-1}(p)\) is fully invariant by \( \mathcal{F}_k \).
Proof of Claim 1. If $C$ is an irreducible component of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(p)$ that is not invariant by $\mathcal{F}_k$, then the leaves $L$ of $\mathcal{F}_k$ that are transverse to $C$ are projected onto separatrizes of $\mathcal{F}$ at $p$ since $\pi_1 \circ \cdots \circ \pi_k$ is a proper mapping. However it is obvious that the separatrizes of $\mathcal{F}$ at $p$ are exactly given by the components of $\mathcal{P}^{-1}(0)$ passing through $p$. Therefore there are only finite many of these. The resulting contradiction proves the claim.

Let $q$ be the center of the last punctual blow-up $\pi_k$ leading to $\mathcal{F}_k$. By choosing the Seidenberg resolution minimal, we can suppose that both eigenvalues of $\mathcal{F}_{k-1}$ vanish at $q$. Now we have:

Claim 2. The order of $\mathcal{F}_{k-1}$ at $q$ is at least 2.

Proof of Claim 2. Let $Z_{k-1}$ be a holomorphic vector field with isolated singularities defined on a neighborhood of $q$ and tangent to $\mathcal{F}_{k-1}$. The statement of the claim is equivalent to saying that $Z_{k-1}$ has trivial linear part at $q$. To check this, we just need to show that the linear part of $Z_{k-1}$ at $q$ cannot be (non-trivially) nilpotent. In turn, this amounts to the following straightforward verifications:

- If (the linear part of) $Z_{k-1}$ is nilpotent at $q$ and $\mathcal{F}_{k-1}$ admits a smooth separatrix at $q$, then the blow-up of $\mathcal{F}_{k-1}$ at $q$ leads to a foliation having a single singularity on the added component of the exceptional divisor which happens to have order 2.

- If (the linear part of) $Z_{k-1}$ is nilpotent at $q$ but $\mathcal{F}_{k-1}$ does not admit a smooth separatrix at $q$, then the mentioned blow-up still has a unique singularity on the added component of the exceptional divisor. However this singularity is still nilpotent but now admitting a smooth separatrix (namely the one induced by the component in question).

The completes the proof of the claim.

To finish the argument, let us choose coordinates $(x, y)$ about $q$ so that $q = (0, 0)$ and $\mathcal{F}_{k-1}$ may be viewed as being defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. By assumption, the order of $\mathcal{F}_{k-1}$ at $(0, 0) \in \mathbb{C}^2$ is at least 2. Furthermore the blow-up of $\mathcal{F}_{k-1}$ leads to a foliation $\mathcal{F}_k$ having singularities $p_1, \ldots, p_r$ over the exceptional divisor $\pi^{-1}(0)$. For every $i = 1, \ldots, r$, the eigenvalues of $\mathcal{F}_k$ at $p_i$ are $m_i, -n_i$ ($m_i, n_i \in \mathbb{N}^*$). In particular, at every $p_i$, $\mathcal{F}_k$ possesses exactly two separatrizes. These separatrizes are smooth and one of them coincides with the exceptional divisor $\pi^{-1}(0)$. The other separatrix will be denoted by $S_i$ and it is transverse to $\pi^{-1}(0)$.

Claim 3. One has $r \geq 3$.

Proof of Claim 3. It is a trivial calculation to check that, with the preceding assumption, if the blown-up foliation $\mathcal{F}_k$ has strictly less than 3 singularities on $\pi^{-1}(0)$ then the eigenvalues of $\mathcal{F}_{k-1}$ at $(0, 0)$ are already different from zero.

We are finally ready to complete the proof. Note that we have not yet used the assumption that $p$ is a nodal singularity of $\mathcal{P}^{-1}(0)$. This translates into saying that the separatrices of $\mathcal{F}$ at $p$ consists of two smooth curves with normal crossings. To obtain the desired contradiction, consider the separatrices $S_1, \ldots, S_r$ mentioned above. If none of these separatrices is contained in a component of the total exceptional divisor $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(p)$, then their projections yield $r \geq 3$ irreducible components for the set of separatrices of $\mathcal{F}$ at $p$. Since this is impossible, there is at least one of these separatrices, say $S_1$, that is contained in the exceptional divisor. Reversing the Seidenberg’s resolution, we can suppose that at some stage $k'$ there is a singularity $q'$ satisfying the following conditions:

- The set of separatrices of $\mathcal{F}_{k'}$ at $q'$ contains at least $r \geq 3$ components which are the corresponding projections of $S_1, \ldots, S_r$ (which are still denoted by $S_1, \ldots, S_r$).
• $S_1$ is contained in the component of the exceptional divisor introduced in the stage of Seidenberg’s resolution going from $k' - 1$ to $k'$.

Therefore, at the singularity $q'$, $F_{k'}$ has at least $r - 1 \geq 2$ components that are not contained in the component of the exceptional divisor in question. If none of them were again contained in the total exceptional divisor, then their projections would yield at least two components of the set of separatrices of $F$ at $p$ which are tangent to each other. Since this is again impossible, it follows that again one of these separatrices, say $S_2$ is contained in the total exceptional divisor. Repeating the argument above, it follows the existence of a singularity $q''$ occurring for the foliation $F_{k''}$ which has $S_2, \ldots, S_r$ as separatrices. Besides $S_2$ is contained in the component of the exceptional divisor added at the passage $k'' - 1$ to $k''$. The remaining components $S_3, \ldots, S_r$ are all tangent to $S_1$. Thus, at the stage $k'' - 1$ they project onto singular components of the set of separatrices of the corresponding singularity. Note that there is at least one such singular separatrix since $r \geq 3$. Being singular, these separatrices cannot be contained in the exceptional divisor. They therefore produce singular components for the set of separatrices of $F$ at $p$. This is the final contradiction proving Proposition 5.1.

Let us close this appendix by showing how the discussion above can be adapted to handle more degenerate set of separatrices. Consider for example the case of a set of separatrices given by $y(x^2 - y)$, i.e. a quadratic tangency between two smooth germs of curves. This goes as follows. After blowing up the singularity in question, we obtain a single singularity $p$ over the exceptional divisor $C_1 = \pi^{-1}(0)$. This singularity has exactly three separatrices namely, the exceptional divisor $C_1$ itself and the proper transforms of $y = 0$ and $y = x^2$. After blowing-up $p_1$, we shall obtain 3 singularities $q_1, q_2, q_3$ over the new exceptional divisor $C_2$ corresponding to the intersection of $C_2$ with the proper transforms of the separatrices at $p$. According to our previous proposition, all these three singularities are linear. Since the proper transform of $C_1$ has self-intersection equal to $-2$, the index formula shows that the eigenvalues at the corresponding singularity are $1, -2$. Similarly the eigenvalues of the other two singularities are $1, -4$ as it follows from the knowledge of the self-intersection of the irreducible components of the singular fiber combined with the index formula. Because the holonomy group of the leaf consisting of $C_2 \setminus \{q_1, q_2, q_3\}$ must be Abelian, as already explained, we conclude that this group is cyclic generated by a rotation of order four. We can now show that the only possible model for this singularity is precisely given by the first integral $y(x^2 - y)$ by simply repeating the construction of a holomorphic conjugacy as discussed in Section 4. We leave to the reader to adapt this argument to the remaining cases, further details can be found in [Re].

6 Appendix 2: Monodromy of fibrations

In this last section we are going to show how to effectively compute the monodromy map of a singular fiber of a genus 2 fibration. Similarly to our previous results, most of the discussion applies equally well to fibrations of arbitrary genus. The computation will be based on the standard method of stable reduction as it will be explained later. Nonetheless, at the end of the section a different method of computation will be sketched. This second method is related to special tesselations of the complex plane or of the hyperbolic disc.

Since most of our discussion can be carried over for fibrations of higher genus, let us begin the discussion by recalling the notion of **multiple fiber**. Let then $P : M \to D$ be a fibration as before and consider the standard coordinate $z$ on $D$. Suppose first that $P^{-1}(0)$ is irreducible. In this case, its multiplicity is defined as the vanishing order of the pull-back $P^*z$ over $P^{-1}(0)$. In general, let $P^{-1}(0) = \sum C_i$ be the decomposition of $P^{-1}(0)$ into irreducible components and denote by $k_i$ the vanishing order of $P^*z$ over $C_i$. The multiplicity $k$ of $P^{-1}(0)$ is now defined as the greatest common
divisor of the \( k_i \)'s. On the other hand, it is clear that \( k.\mathcal{P}^{-1}(0) \) is homologous, as divisor, to a generic fiber of \( \mathcal{P} \). In particular, we have

\[
\mathcal{P}^{-1}(0) \cdot \mathcal{P}^{-1}(0) = 0 \quad \text{and} \quad \mathcal{P}^{-1}(0) \cdot K_M = \frac{1}{k} \mathcal{P}^{-1}(\lambda) \cdot K_M
\]

for a generic \( \lambda \in D \). By adjunction, it follows that \( \mathcal{P}^{-1}(\lambda) \cdot K_M \) equals \( 2(g-1) \). Thanks to the integrality of the virtual genus, we then conclude that the multiplicity \( k \) of the singular fiber must divide \( g - 1 \). In particular, all fibers of a genus 2 fibration are simple. To make the connection between these notions of multiplicity and the point of view of foliations used in this paper, it suffices to check the contents of Remark 4.6.

A very nice reference for the background material used in the sequel is [BHPV], Chapter 3. The singular fiber \( \mathcal{P}^{-1}(0) \) is said to be stable if it verifies the following conditions: a) \( \mathcal{P}^{-1}(0) \) contains no \(-1\)-curve; b) \( \mathcal{P}^{-1}(0) \) is reduced; c) the singularities of \( \mathcal{P}^{-1}(0) \) are linear with eigenvalues 1, \(-1\). In particular, when dealing with genus 2 fibrations, Ogg’s list immediately yields all possible stable fibers. First we have the cases of a regular (genus 2) fiber, of two elliptic curves with a normal crossing (or a single curve with one double point), a rational curve with two double points or combinations of these. Another typical example is the case of two elliptic curves (of self-intersection \(-1\)) joined by a string of rational curves with self-intersection \(-2\). Naturally each elliptic curve can also be replaced by a (singular) curve as above whose genus happens to be 1 (for example a rational curve with one double point). For our purposes, the loop constituted by \( k \)-rational curves of self-intersections \(-2\) counts as an elliptic curve. All possible stable models are obtained by assembling these pieces. Thus, we have for example the fibers below:

\[
\begin{align*}
B &:egin{array}{c}
\hline
k_1 - 1 \\
\hline
k_2 - 1 \\
\hline
k_3 - 1 \\
\hline
\end{array} \\
D &:egin{array}{c}
\hline
k_1 - 1 \\
\hline
\end{array} \\
C &:egin{array}{c}
k - 1 \\
\hline
\end{array} \\
\end{align*}
\]

The main advantage of having a stable fiber concerns the computation of the corresponding monodromy. Naturally the fact that the singularities of \( \mathcal{P}^{-1}(0) \) have eigenvalues 1, \(-1\) is equivalent to saying that these singularities are of “Morse type” ie. locally given by the first integral \( x^2 + y^2 \). The effect of each of these singularities on the monodromy map is then explicitly quantified by the classical Picard-Lefschetz formula \( T(a) = a - \sum_i (a, e_i) e_i \), where \( a \in H_1(\mathcal{P}^{-1}(0), \mathbb{Z}) \) and where \( T \) stands for the corresponding monodromy map. For example, for the fibers appearing in Figure 6 the monodromy
maps are respectively given by the matrices

\[
\begin{bmatrix}
1 & 0 & k_2 + k_3 & -k_3 \\
0 & 1 & -k_3 & k_1 + k_3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & k \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & 0 & k_2 & 0 \\
0 & 1 & 0 & k_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

We also point out that, in slightly loose terms, the effect on the monodromy of a loop consisting of \(k\) rational curves of self-intersections equal to \(-2\) is given by the matrix

\[
\begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix}.
\]

To compute the monodromy map associated to a general (non stable) fiber, we can use the “stable reduction” method. This goes as follows. Fix a singular fiber \(\mathcal{P}^{-1}(0)\) and consider \(k \in \mathbb{N}^*\) such that the multiplicity \(k_i\) of every irreducible component of \(\mathcal{P}^{-1}(0)\) divides \(k\). In practice we choose \(k\) as the least common multiple of these multiplicities. If multiple fibers are allowed, then the mentioned multiplicities are obtained accordingly, cf. Remark \[4.6\]. Let us define the \(k\)-root fibration of \(\mathcal{P}\). For this let \(D \subset \mathbb{C}\) be the unit disc and denote by \(\delta_k : D \to D\) the mapping \(z \mapsto z^k\). The fiber product \(D \times_D M\) is defined as the analytic subvariety of \(D \times M\) consisting of those points \((z, p)\) satisfying the condition \(z^k = \mathcal{P}(p)\).

Denote by \(\overline{D \times_D M}\) the minimal resolution of the normalization of \(D \times_D M\). Naturally \(\overline{D \times_D M}\) comes equipped with a projection \(\overline{\mathcal{P}} : \overline{D \times_D M} \to D\) realizing \(\overline{D \times_D M}\) as a fibration having the same genus as \(\mathcal{P}\). Now we have:

**Proposition 6.1** \(\overline{\mathcal{P}} : \overline{D \times_D M} \to D\) has a stable fiber over \(0 \in D \subset \mathbb{C}\).

**Proof.** It is very well-known. Let us just summarize some important points of it. Note that it is irrelevant for our purposes whether or not \(\mathcal{P}^{-1}(0)\) is minimal. Therefore, modulo performing finitely many blow-ups, we can assume that all the singularities of \(\mathcal{P}^{-1}(0)\) are linear. Hence they are locally given by the first integral \(x^n y^m\). Next we only need to work out the effect of the above procedure at the points of the singular fiber. If we consider a regular point of \(\mathcal{P}^{-1}(0)\), then the foliation becomes simply \(y^{k_i}\) for some \(k_i\) as above. The definition of the fiber product locally becomes \(z^k = y^{k_i}\). Since \(k_i\) divides \(k\) it follows that the fiber \(\mathcal{P}^{-1}\) splits into \(k_1\) regular surfaces.

Now consider a singular point \(x^n y^m\) with g.c.d. \((m, n) = 1\). Let \(d\) be the quotient between the order total holonomy of the component corresponding to \(y = 0\) and \(m\). Finally set \(r = k / mnd\). The fiber product decomposes into \(d\) copies of the domain of definition of the function \(z = \sqrt[n d]{x^n y^m}\). After normalization, these copies are separated and the resolution of the singularities leads to a string of rational curves whose singular points possess \(-1, -1\) as eigenvalues. \(\square\)

To complete the above argument, let us consider a string as above. The rational curves lying at the extremities of the string will be denoted \(C_1\) and \(C_l\). The index formula says that the remaining rational curves have self-intersection equal to \(-2\). Similarly, unless, \(C_1, C_l\) intersect other components of the singular fiber (not belonging to the mentioned string), their self-intersection is \(-1\). Hence they can be collapsed. Continuing the procedure, the whole string eventually will disappear. Another remark is that a string as above, may have non-trivial total holonomy, as it happens for example for the model \(I_b^* = D_{1+b}\) of Kodaira’s table for elliptic surfaces. In this case, two or more of these strings can be glued together so as to form a loop of rational curves with self-intersection \(-2\).

It is easy to recognize the stable model of an arbitrary fiber. For example consider the model depicted in figure[14](where the components are rational curves with self-intersection \(-2\) unless otherwise indicated).
Figure 14: A singular fiber and its stable model

The central curve containing 4 singularities has multiplicity 2, cf. Remark 4.6 and can be viewed as an elliptic orbifold (cf. below). The square-root of this fiber is therefore stable and consisting of the loop of rational curves joined to an elliptic curve by a string of rational curves, figure 14. The monodromy of the initial fiber is therefore the square-root of the monodromy of its stable model, with suitable orientations. Since the latter is easily computable, we obtain that the monodromy of the initial fiber can be represented by the matrix

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & k \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
A^2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & k \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Let us finish by sketching, without proof, a more direct method to compute these monodromy maps. It is straightforward to turn our description into precise statements. We leave out of our discussion components constituted by a loop of rational curves whose effect on the monodromy was already mentioned. The general principle is that rational curves with two singular points can be ignored for most of purposes. A similar argument applies to higher genus components with trivial holonomy. In the genus 2 case, the only interesting case that remain consists of rational curves containing 3 or more singular points. We shall only discuss their contribution to the total monodromy map of the fibration.

Thus let \( C \) be a rational curve containing singularities \( p_1, \ldots, p_r \). To each singularity \( p_i \) we associate its multiplicity \( n_i \) namely, the order of the local holonomy of \( C \) around \( p_i \). In this sense \( C \) can be viewed as an orbifold denoted by \( \hat{C} \). To define the Euler characteristic \( \text{eu}(\hat{C}) \) of \( \hat{C} \), we denote by \( k \) the order of the holonomy group of \( C \) namely the least common multiple of all the \( n_i \)'s. The eigenvalues of the singularity \( p_i \) can then be written as \( k_i, k \). Now \( \text{eu}(\hat{C}) \) is defined in accordance with Hurwitz formula by

\[
\text{eu}(\hat{C}) = k(2 - r) + \sum_{i=1}^{r} \text{g.c.d.}(k_i, k).
\]

The orbifold \( \hat{C} \) will be called elliptic (resp. hyperbolic) if \( \text{eu}(\hat{C}) \) vanishes (resp. \( \text{eu}(\hat{C}) < 0 \)).

It is immediate to check that there are four types of elliptic orbifolds namely: \( \hat{C}_{2222} \) (\( r = 4 \) with \( n_1 = n_2 = n_3 = n_4 = 2 \)), \( \hat{C}_{333} \) (\( r = 3, n_1 = n_2 = n_3 = 3 \)), \( \hat{C}_{244} \) (\( r=3, n_1 = 2, n_2 = n_3 = 4 \)) and \( \hat{C}_{236} \) (\( r = 3, n_1 = 2, n_2 = 3, n_3 = 6 \)). Being elliptic these orbifolds already appear in the context of elliptic fibrations. Their monodromy map in an elliptic fibration also accounts for their “contribution” to the monodromy map in a higher genus fibration. Let us first describe in these cases how these monodromy maps can be recovered. To begin with, consider the orbifold \( \hat{C}_{333} \). Its holonomy group is well-known, it coincides with the group generated by reflections on the edges of an euclidean equilateral
The degree 3 ramified covering of $\hat{C}_{333}$ is an elliptic curve obtained as follows. Consider a regular (euclidean) hexagon with edges labelled by the symbols $A, B, C, A^{-1}, B^{-1}, C^{-1}$. The standard “cut-and-paste” procedure to obtain an elliptic curve from this hexagon consists of cutting it along the diagonal $X = A + B$ and paste it back along $A$. The usual picture of the rectangle has edges $X = A + B$ and $Y = C + B$. On the other hand, the group of covering automorphisms of the hexagon is generated by a rotation $\sigma$ of angle $2\pi/3$ about its center. It therefore yields the identifications $A \mapsto C$, $B \mapsto A^{-1}$ and $C \mapsto B^{-1}$.

The action of $\sigma$ on the homology of the elliptic curve is nothing but the monodromy of the orbifold. To express it in the basis $X, Y$ we simply note that $\sigma(X) = C + A^{-1} = Y - X$ whereas $\sigma(Y) = -X$. The resulting matrix is therefore

\[
\begin{pmatrix}
-1 & 1 \\
-1 & 0
\end{pmatrix}.
\]

The remaining elliptic cases are similar. For the orbifold $\hat{C}_{244}$ we consider reflections on the edges of the triangle with angles $\pi/2, \pi/4, \pi/4$ while the triangle corresponding to $\hat{C}_{236}$ has angles $\pi/2, \pi/3, \pi/6$. Finally the orbifold $\hat{C}_{2222}$ is generated by reflections on the edges of a square. Note that these procedure not only give the monodromy of the singular fiber but it also characterizes the complex structure (geometry) of the neighbor regular fibers.

Let us now consider the case of hyperbolic orbifolds appearing in a singular fiber of a fibration having genus 2. Incidentally all these orbifolds have Euler characteristic equal to 2. There are nine types of orbifolds as it can be seen by checking on Ogg’s list. Three of them have 4 singularities and the other six orbifolds have 3 singularities thus arising from triangular groups. Consider for example the orbifold whose singularities $p_1, p_2, p_3$ have eigenvalues respectively equal to 2, 5, 4, 5 and 4, 5. The group of symmetry for the corresponding ramified covering of this orbifold should be searched on the Hyperbolic disc $D$ what is, of course, reminiscent from the fact that the orbifold is hyperbolic (and the ramified covering in question is genus 2 surface). It follows immediately that this group generated by the reflection on the edges of a (equilateral) hyperbolic triangle all of whose angles equal $\pi/5$. Analogously to the elliptic case discussed above, the corresponding ramified covering is a genus 2 curve obtained from a decagon generated by the rotations about a vertex of the above mentioned hyperbolic triangle. The generator of the monodromy group being precisely the rotation of order $2\pi/5$ about this vertex. The monodromy can now easily be worked out in detail. In particular the geometry of the fiber is also clarified. As to the monodromy, we can simply note that it is a “primitive” $5^{th}$-root of the identity matrix. This leads to the representative

\[
\begin{pmatrix}
0 & -1 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

The other cases can similarly be treated.

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\[1\] here and in the sequel we make a standard abuse of notation. Note that reflection on a straight line reverses the orientation of the plan, thus by the group “generated by reflections on the edges of a polygon” it is actually meant the index 2 subgroup of the former consisting of those elements that preserve orientation.
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