The Second Variation for Null-Torsion Holomorphic Curves in the 6-Sphere

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Abstract

In the round 6-sphere, null-torsion holomorphic curves are fundamental examples of minimal surfaces. This class of minimal surfaces is quite rich: By a theorem of Bryant, extended by Rowland, every closed Riemann surface may be conformally embedded in the round 6-sphere as a null-torsion holomorphic curve.

In this work, we study the second variation of area for compact null-torsion holomorphic curves Σ of genus g and area 4πd, focusing on the spectrum of the Jacobi operator. We show that if g ≤ 6, then the multiplicity of the lowest eigenvalue \( \lambda_1 = -2 \) is equal to 4d. Moreover, for any genus, we show that the nullity is at least 2d + 2 - 2g. These results are likely to have implications for the deformation theory of asymptotically conical associative 3-folds in \( \mathbb{R}^7 \), as studied by Lotay.

1 Introduction

1.1 Background: Minimal Surfaces in Spheres

Let \( \Sigma^2 \) denote a closed orientable surface. In a Riemannian manifold \((M, \langle \cdot , \cdot \rangle)\), an immersed surface \( u: \Sigma^2 \to M \) is called a minimal surface if every variation \( u_t: \Sigma^2 \to M \) of \( u_0 = u \) satisfies \( \frac{d}{dt} |_{t=0} \text{Area}(u_t) = 0 \). That is, minimal surfaces are critical points of the area functional, but not necessarily global minimizers of it. The extent to which a minimal surface fails to be area-minimizing to second order can be measured by the second variation of area, which takes the form

\[
\left. \frac{d^2}{dt^2} \right|_{t=0} \text{Area}(u_t) = \int_{\Sigma} \langle \mathcal{L} \eta, \eta \rangle, \tag{1.1}
\]

where \( \eta := \frac{d}{dt} |_{t=0} u_t \) is a normal variation vector field, and where \( \mathcal{L}: \Gamma(N\Sigma) \to \Gamma(N\Sigma) \) is the Jacobi operator of the minimal surface \( u \). We will recall the standard expression for \( \mathcal{L} \) in §4.

In view of the second variation formula (1.1), it is of fundamental interest to understand the Jacobi operator of a minimal surface, which in turn motivates the study its spectrum. Indeed, recalling that \( \mathcal{L} \) is strongly elliptic [24, §I.9], it may be diagonalized with real eigenvalues

\[ \lambda_1 < \lambda_2 < \cdots < \lambda_s < 0 = \lambda_{s+1} < \lambda_{s+2} < \cdots \to \infty \]

of finite multiplicities

\[ m_1, m_2, \ldots, m_s, m_{s+1}, \ldots \]

Certain well-known invariants of minimal surfaces may be phrased in terms of this spectrum. For example, the Morse index and nullity are, respectively,

\[ \text{Ind}(u) = m_1 + \cdots + m_s \quad \text{Nullity}(u) = m_{s+1}. \]

A minimal surface is said to be stable if \( \lambda_1 \geq 0 \) and unstable if \( \lambda_1 < 0 \).
In general, computing the spectrum of $L$ is extremely difficult. There seem to be very few examples of minimal surfaces whose Jacobi spectra are known explicitly. In view of this, geometers instead seek to estimate the eigenvalues $\lambda_j$ and (sums of) multiplicities $m_j$ in terms of more computable geometric and topological quantities. Still, obtaining such bounds is a non-trivial task. Even in the classical case of (non-compact) minimal surfaces in $\mathbb{R}^3$, several outstanding open problems remain: see, for example, the excellent survey [10].

Our focus will be on compact orientable minimal surfaces (without boundary) in round spheres $M = S^n$ (of constant curvature 1) with $n \geq 3$. Pioneering work in this subject was carried out in the late 1960’s by, for example, Calabi [7], Chern [9], Simons [33], and Lawson [25]. In particular, Simons showed [33, Lemma 5.1.4] that all compact minimal surfaces in $S^n$ have $\lambda = -2$ as an eigenvalue of $L$, and hence are unstable. He also established the lower bounds

$$\text{Ind}(u) \geq n - 2 \quad \text{Nullity}(u) \geq 3(n - 2).$$

In both estimates, equality holds if and only if $u$ is the totally-geodesic $S^2$.

In the case of $n = 3$, Urbano [34] improved Simons’ bound, showing that non-totally-geodesic minimal surfaces satisfy $\text{Ind}(u) \geq 5$, with equality if and only if $u$ is the Clifford torus. This characterization of the Clifford torus was an important ingredient in Marques’ and Neves’ resolution of the Willmore conjecture [28].

In the case $n = 4$, Micallef and Wolfson [29] proved that minimal surfaces in $S^4$ of area $A$ satisfy

$$\text{Ind}(u) \geq \frac{1}{2} \left( \frac{A}{\pi} - \chi(\Sigma) \right),$$

where $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic. Recently, motivated by potential applications to the generalized Willmore conjecture in $S^n$, Kusner and Wang [23] proved that minimal surfaces of genus $g = 1$ in $S^4$ satisfy $\text{Ind}(u) \geq 6$, with equality if and only if $u$ is a Clifford torus in a totally-geodesic $S^3$.

In a different direction, if one restricts attention to the class of superminimal surfaces in $S^4$, the beautiful paper of Montiel-Urbano [30] provides remarkably precise information. They show that superminimal surfaces in $S^4$ have lowest eigenvalue $\lambda_1 = -2$ and satisfy

$$\text{Ind}(u) = m_1 = \frac{A}{\pi} - \chi(\Sigma) \quad \text{Nullity}(u) = m_2 \geq \frac{A}{\pi} + \chi(\Sigma). \quad (1.2)$$

Moreover, if $g = 0$ or $g = 1$, then equality holds in the nullity estimate. In fact, $\text{Ind}(u) \geq 10$, with equality if and only if $u$ is a (twistor deformation of a) Veronese surface. The formulas (1.2) for superminimal surfaces in $S^4$ were the primary inspiration for this work.

In even dimensions $n = 2k$, Karpukhin [21] has recently shown that a linearly full minimal surface in $S^{2k}$ of genus $g = 0$ and area $A = 4\pi d$ has index

$$\text{Ind}(u) \geq 2(k - 1)(2d - \lfloor \sqrt{8d + 1} \rfloor_{\text{odd}} + 2),$$

where $[x]_{\text{odd}}$ is the largest odd integer not exceeding $x$.

1.2 Background: Holomorphic Curves in the 6-Sphere

Among all spheres $S^n$ with $n \geq 3$, the 6-sphere is the only one that admits an almost-complex structure. In this work, we will equip $S^6$ with its standard almost-complex structure $\tilde{J}: T\tilde{S}^6 \to T\tilde{S}^6$. This almost-complex structure is compatible with the round metric, and arises from viewing
$S^6 \subset \mathbb{R}^7 = \text{Im}(\mathbb{O})$ in the imaginary octonions, as we will recall in §2.2. Having chosen $\tilde{J}$, the 6-sphere now admits a distinguished class of surfaces. That is, a holomorphic curve is a surface $u : \Sigma^2 \to S^6$ whose tangent spaces are $\tilde{J}$-invariant:

$$\tilde{J}(T_p\Sigma) = T_p\Sigma, \ \forall p \in \Sigma.$$  

It is easy to show that holomorphic curves in $S^6$ are (unstable) minimal surfaces.

In a remarkable 1982 paper, Bryant [5] studied holomorphic curves in $S^6$ by means of a “holomorphic Frenet frame,” which we discuss in §3.2. Essentially, this amounts to a decomposition of the vector bundle of $(1, 0)$-vectors along $u(\Sigma)$ into complex line subbundles

$$u^*(T^{1,0}S^6) \simeq L_T \oplus L_N \oplus L_B.$$  

(1.3)

Crucially, each of the bundles $L_T, L_N, L_B$ carries a natural holomorphic structure, though the isomorphism (1.3) generally only holds in the smooth (not holomorphic) category. By analogy with the classical case of curves in $\mathbb{R}^3$, one can extract two basic invariants: a second-order invariant (“curvature”) that is essentially the second fundamental form of the immersion, and a third-order invariant (“torsion”) that is rather more subtle. Bryant encodes the torsion as a holomorphic section

$$\Phi_{III} \in H^0(L_T^* \otimes L_N^* \otimes L_B),$$

and defines a holomorphic curve to be null-torsion if $\Phi_{III} \equiv 0$ on $\Sigma$. It is not hard to show that every holomorphic curve of genus $g = 0$ is null-torsion.

It turns out that the null-torsion condition is equivalent to the holomorphicity of the binormal Gauss map $b_u : \Sigma \to \mathbb{CP}^6$, the map sending a point $p \in \Sigma$ to its binormal real 2-plane in $T_pS^6 \subset \mathbb{R}^7$ (viewed as a complex line in $\mathbb{C}^7$). From this fact, together with the Wirtinger Theorem, it follows that the area $A$ of a null-torsion holomorphic curve is quantized. That is,

$$A = 4\pi d,$$

where $d \in \mathbb{Z}^+$ is the degree of the binormal Gauss map. Aside from the totally-geodesic 2-sphere (which has $d = 1$), all null-torsion holomorphic curves have $d \geq 6$. The moduli space of genus zero holomorphic curves in $S^6$ of a fixed degree $d \geq 6$ has been studied by Fernández [16].

In [5], Bryant derived a Weierstrass representation formula for null-torsion holomorphic curves. Using this formula, together with an algebro-geometric argument, he proved a striking result: every closed Riemann surface admits a conformal branched immersion into $S^6$ as a null-torsion holomorphic curve. This was sharpened by Rowland [31] in his 1999 Ph.D. thesis, who improved “branched immersion” to “smooth embedding.” The upshot is that, while the generic holomorphic curve is not null-torsion, the class of null-torsion curves is nevertheless extremely rich.

Since Bryant’s 1982 paper, there have been several interesting studies of holomorphic curves in the 6-sphere. For example, Sekigawa [32] classified the constant-curvature examples, Ejiri [15] classified the U(1)-invariant examples, and Hashimoto [19] obtained beautiful explicit examples of one-parameter deformations. Bolton, Vrancken, and Woodward [4] studied holomorphic curves by using harmonic sequences, and showed that every holomorphic curve in $S^6$ can only be full in a totally-geodesic $S^2, S^5$, or else the entire $S^6$. This is by no means a complete list of references; we refer the interested reader to the books of Chen [8, §19.1-19.2] and Joyce [20, §12.2] for more.

Finally, we note that the study of holomorphic curves in $S^6$ forms part of the larger study of holomorphic curves in nearly-Kähler 6-manifolds. For example, holomorphic curves in $\mathbb{CP}^3$ have been studied by Xu [35] and Aslan [2], and in $S^3 \times S^3$ by Bolton, Dioos, and Vrancken [3].
fact, holomorphic curves in nearly-Kähler 6-manifolds are precisely the links of associative cones in conical $G_2$-manifolds, and thereby serve as models for conically singular associative 3-folds. This relationship makes holomorphic curves objects of fundamental interest in $G_2$-geometry.

1.3 Main Results

In this work, we consider the Jacobi spectra of null-torsion holomorphic curves in $S^6$. Perhaps the most basic question is: What is the multiplicity $m_1$ and value $\lambda_1$ of the lowest eigenvalue of $L$?

In the early 1980’s, Ejiri [14] considered this question in the context of superminimal surfaces in $S^{2n}$, showing that $\lambda_1 = -2$. (Although his results are stated for minimal 2-spheres in $S^{2n}$, most of Ejiri’s arguments apply without change to the larger class of superminimal surfaces.) Furthermore, equipping the normal bundle with a certain holomorphic structure, which we call $\partial^\nabla$, he showed that the $\lambda_1$-eigenspace of $L$ may be identified with the space of holomorphic normal vector fields:

$$\{\eta \in \Gamma(N\Sigma) : L\eta = -2\eta\} \cong \{\text{solutions of } \partial^\nabla \xi = 0\}.$$

The Riemann-Roch Theorem then implies that

$$m_1 \geq \frac{A}{\pi} + (n - 3)\chi(\Sigma).$$

Ejiri also observed that equality holds in the case of genus $g = 0$, essentially by an application of Grothendieck’s classification of holomorphic vector bundles on $S^2 = \mathbb{CP}^1$.

Now, since null-torsion holomorphic curves in $S^6$ are, in particular, superminimal surfaces, Ejiri’s results imply that they satisfy $\lambda_1 = -2$ and

$$m_1 \geq \frac{A}{\pi}. \tag{1.4}$$

Our first result is that, in fact, equality holds for genus $g \leq 6$:

**Theorem 1.1.** Let $u : \Sigma \to S^6$ be a null-torsion holomorphic curve of genus $g$ and area $A = 4\pi d$. If $g \leq 6$ (or, more generally, if $g < \frac{1}{2}(d + 2)$), then the first multiplicity $m_1$ of the Jacobi operator is:

$$m_1 = \frac{A}{\pi} = 4d. \tag{1.4}$$

Where minimal surfaces of high genus ($g \geq 1$) and high codimension (at least 2) in round spheres are concerned, the only explicit formulas for $m_1$ that the author knows are Montiel and Urbano’s result (1.2) and Theorem 1.1 above. Our argument makes crucial use of the particular geometry of null-torsion holomorphic curves. In outline, the idea is the following. We will equip the normal bundle with a second holomorphic structure, called $\partial^D$, that arises naturally from the nearly-Kähler structure on $S^6$. Letting $S$ denote the difference tensor $S\xi := \partial^\nabla \xi - \partial^D \xi$, the Cauchy-Riemann system $\partial^\nabla \xi = 0$ is equivalent to

$$\partial^D \xi = -S\xi. \tag{1.5}$$

It turns out that the system (1.5) decouples, yielding an easy upper bound for the dimension of the solution space, which proves the theorem. Our second result is a lower bound on the nullity, valid for all genera:

**Theorem 1.2.** Let $u : \Sigma \to S^6$ be a null-torsion holomorphic curve of genus $g$ and area $A = 4\pi d$. Then the nullity of its Jacobi operator satisfies

$$\text{Nullity}(u) \geq 2d + \chi(\Sigma). \tag{1.6}$$

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Here, our argument is not original. Indeed, we closely follow the calculations in Montiel and Urbano’s study [30] of superminimal surfaces in self-dual Einstein 4-manifolds. The idea of the proof is to identify a certain subspace of
\[ \text{Null}(u) := \{ \eta \in \Gamma(N\Sigma) : L\eta = 0 \} \]
with the space of holomorphic sections of a certain line bundle whose dimension can be estimated (and for genus \( g \leq 6 \), computed explicitly) by Riemann-Roch. Note that, since we only consider a subspace of Null(\( u \)), our bound (1.6) is almost certainly not sharp. On the other hand, it appears that most of the argument extends without change to the general case of superminimal surfaces in any even-dimensional sphere \( S^{2n} \), providing an avenue for further inquiry.

1.4 Open Questions

1. Let \( u: \Sigma^2 \to S^{2n} \) be a compact orientable superminimal surface of genus \( g \). Is it always the case that Ejiri’s lower bound is satisfied:
\[ m_1 = \frac{A}{\pi} + (n - 3)\chi(\Sigma) \]
Ejiri [14] has proven this for genus \( g = 0 \), while Montiel-Urbano [30] has proven this for \( n = 2 \). Our Theorem 1.1 establishes this in the special case where \( n = 3 \), \( g \leq 6 \), and the superminimal surface is holomorphic.

2. Holomorphic curves may be studied in any nearly-Kähler 6-manifold. What can be said about the Jacobi spectrum in that generality?

3. In the 6-sphere: Can one establish a lower bound on the second eigenvalue \( \lambda_2 \)? As a first step, it would be instructive to understand the spectrum of the Boruvka sphere, the unique holomorphic curve of constant curvature \( K = \frac{1}{6} \). We show in Proposition 5.8 that the Boruvka sphere satisfies \( \lambda_2 \geq -\frac{5}{3} \). Further, Karpukhin [21, Theorem 1.7] has estimated its Morse index as \( m_1 + \cdots + m_s = \text{Ind}(u) \geq 36 \), and Ejiri’s result [14] gives \( m_1 = 24 \), implying \( m_2 + \cdots + m_s \geq 12 \), so \( \lambda_2 < 0 \).

1.5 Organization

In §2, we recall basic facts and formulas regarding minimal surfaces in \( S^6 \), holomorphic curves in \( S^6 \), and holomorphic vector bundles over Riemann surfaces. This section is largely to establish conventions and experts may wish to skip it.

In §3.1 and §3.2, we set up the moving frame for holomorphic curves in the 6-sphere. Our discussion is essentially a summary of [5, §4], though our notation is quite different. In §3.3, we take a closer look at null-torsion holomorphic curves, culminating in Proposition 3.4, which counts the holomorphic sections of \( L_N \) and \( L_B^* \), and Proposition 3.6, which justifies our tacit parenthetical claim in Theorem 1.1 that \( g \leq 6 \) implies \( g < \frac{1}{2}(d + 2) \).

Section 3.4 is at the heart of Theorem 1.1. The purpose of §3.4 is to explain how the normal bundle of a null-torsion holomorphic curve may naturally be equipped with three different holomorphic structures, which we call \( \mathcal{D}^\text{SU} \), \( \mathcal{D}^N \), and \( \mathcal{D}^D \). The operator \( \mathcal{D}^\text{SU} \) relates to the deformation theory of asymptotically conical associative 3-folds in \( \mathbb{R}^7 \), as shown by Lotay [26], while \( \mathcal{D}^N \) relates to the \((-2)\)- and 0-eigenspaces of the Jacobi operator \( L \). However, it is with respect to \( \mathcal{D}^D \) that the normal bundle splits holomorphically, which aids in the decoupling of (1.5).
In §4, we begin our study of the Jacobi operator of null-torsion holomorphic curves. Sections 4.1 and 4.2 establish the lower bound (1.4), while §4.3 proves Theorem 1.1 by analyzing (1.5). Finally, in §5.2, we reduce Theorem 1.2 to a claim (Proposition 5.3) about the image of a certain linear Cauchy-Riemann type operator, and in §5.3–§5.4 we establish Proposition 5.3.

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2 Preliminaries

In this brief section, we recall basic facts about minimal surfaces in $S^6$, holomorphic curves in $S^6$, and holomorphic vector bundles. This section primarily serves to fix notation and conventions.

2.1 Minimal Surfaces in $S^6$

Let $u: \Sigma^2 \to S^6$ be an immersed surface in the round 6-sphere of constant curvature 1. Let $\langle \cdot, \cdot \rangle$ denote the round metric on $S^6$ and let $\nabla: \Gamma(TS^6) \to \Omega^1(S^6) \otimes \Gamma(TS^6)$ denote the Levi-Civita connection. As usual, we split $u^*(T S^6) = T \Sigma \oplus N \Sigma$ into tangential and normal parts. For $X, Y \in \Gamma(T \Sigma)$ and $N \in \Gamma(N \Sigma)$, we have

$$\nabla_X Y = \nabla^\top_X Y + II(X,Y)$$
$$\nabla_X N = W_X N + \nabla^\perp_X N$$

where $\nabla^\top$ is the Levi-Civita connection on $\Sigma$, where $\nabla^\perp$ is the normal connection, where $II$ is the second fundamental form, and where $W$ is the shape operator. Recall the Weingarten equation

$$\langle W_X N, Y \rangle = -\langle II(X,Y), N \rangle.$$

The curvature tensors of $\nabla, \nabla^\top, \nabla^\perp$ will be denoted $\overline{R}, R^\top, R^\perp$, respectively. We will often use the notation $\overline{R}_{XYZ} := \overline{R}(X,Y,Z,\cdot)$, and similarly for $R^\top$ and $R^\perp$.

Suppose now that $u: \Sigma^2 \to S^6$ is a minimal surface. Let $(e_1, \ldots, e_6)$ be a local orthonormal frame with $e_1, e_2 \in T \Sigma$ and $e_3, e_4, e_5, e_6 \in N \Sigma$. We recall the Gauss equation

$$1 = K + \|II(e_1,e_1)\|^2 + \|II(e_1,e_2)\|^2$$

(2.1)

where $K$ is the Gauss curvature of $\Sigma$. We also recall the Ricci equation

$$\langle R^\top_{12} e_\alpha, e_\beta \rangle = \langle W_1(e_\beta), W_2(e_\alpha) \rangle - \langle W_1(e_\alpha), W_2(e_\beta) \rangle$$

where $3 \leq \alpha, \beta \leq 6$, and we are using the shorthand $R^\top_{12} := R^\top_{e_1,e_2}$ and $W_j := W_{e_j}$. Expressing the second fundamental form as $II(e_i,e_j) = h_{ij}^\alpha e_\alpha$, we have

$$W_1(e_\alpha) = -h_{11}^\alpha e_1 - h_{12}^\alpha e_2$$
$$W_2(e_\alpha) = -h_{12}^\alpha e_1 + h_{11}^\alpha e_2,$$
so that the Ricci equation reads
\[ \langle R^\perp_{12} e_\alpha, e_\beta \rangle = 2 \left( h^\beta_{11} h^\alpha_{12} - h^\alpha_{11} h^\beta_{12} \right). \] (2.2)

### 2.1.1 First and Second Normal Bundles

Since \( u \) is a minimal surface, its second fundamental form \( \Pi_p : \text{Sym}^2(T_p \Sigma) \to N_p \Sigma \) at \( p \in \Sigma \) is determined by \( \Pi_p(e_1, e_1) \) and \( \Pi_p(e_1, e_2) \). Therefore, the image of \( \Pi_p \), called the first normal space
\[ E_N|_p := \{ \Pi_p(X, Y) \in N_p \Sigma : X, Y \in T_p \Sigma \} = \text{span}(\Pi_p(e_1, e_1), \Pi_p(e_1, e_2)), \]
is a vector space of dimension at most 2. Letting
\[ \Sigma^o := \{ p \in \Sigma : \dim(E_N|_p) = 2 \} \]
we note that \( \Sigma^o \subset \Sigma \) is an open set, and that \( E_N := \bigcup_{p \in \Sigma^o} E_N|_p \to \Sigma^o \) is a rank 2 vector bundle, called the first normal bundle.

For \( p \in \Sigma^o \), let \( E_B|_p \) denote the second normal space, i.e., the orthogonal complement of \( E_N|_p \subset N_p \Sigma \), so that there is an orthogonal splitting
\[ N_p \Sigma = E_N|_p \oplus E_B|_p. \]
The rank 2 vector bundle \( E_B := \bigcup_{p \in \Sigma^o} E_B|_p \to \Sigma^o \) is called the second normal bundle. For a normal vector \( \eta \in N \Sigma \), we write
\[ \eta = \eta^N + \eta^B \] (2.3)
for its decomposition into first normal and second normal components. The third fundamental form \( \mathbb{III} : \text{Sym}^3(T \Sigma) \to E_B \) is defined by \( \mathbb{III}(X, Y, Z) := [\nabla_X \Pi(Y, Z)]^B \). It is a standard fact that \( \mathbb{III} \) is, in fact, symmetric in its arguments.

### 2.2 Holomorphic Curves in \( S^6 \)

Thus far, we have been regarding the round \( S^6 \) simply as a Riemannian manifold. We now equip it with extra data, namely its standard (nearly-Kähler) SU(3)-structure. To begin, let us consider the imaginary octonions \( \text{Im}(\mathbb{O}) = \mathbb{R}^7 \), equipped with the standard euclidean inner product \( g_0 \). The imaginary octonions admit a well-known cross product \( \times : \text{Im}(\mathbb{O}) \times \text{Im}(\mathbb{O}) \to \text{Im}(\mathbb{O}) \)
\[ x \times y := \frac{1}{2}(xy - yx). \]
Using the metric \( g_0 \), the cross product can be recast as a 3-form \( \phi \in \Lambda^3(\mathbb{R}^7)^* \)
\[ \phi(x, y, z) := g_0(x \times y, z) \]
called the associative 3-form. The associative 3-form is a (G2-invariant) calibration on \( \mathbb{R}^7 \), and its calibrated 3-folds are called “associative 3-folds.” That is, an associative 3-fold is an immersed submanifold \( N^3 \to \mathbb{R}^7 \) that satisfies
\[ \phi|_N = \text{vol}_N \]
where \( \text{vol}_N \) is the volume form on \( N^3 \). The study of associative 3-folds is of fundamental importance to G2-geometry [20, §12].
Returning to the round 6-sphere, let us embed $S^6 \subset \mathbb{R}^7 = \text{Im}(\mathcal{O})$ in the standard way. For each $p \in S^6$, we can use the cross product $\times$ to define a map

$$\tilde{J}_p: T_p S^6 \rightarrow T_p S^6$$

$$\tilde{J}_p(x) = p \times x.$$ 

The properties of $\times$ imply that each $(\tilde{J}_p)^2 = -\text{Id}$. The resulting bundle map $\tilde{J}: TS^6 \rightarrow TS^6$ is the standard ($G_2$-invariant) almost-complex structure on the 6-sphere. One can check that each $\tilde{J}_p: T_p S^6 \rightarrow T_p S^6$ is an isometry, and that the bilinear form on $S^6$ given by

$$\tilde{\Omega}(x,y) := \langle \tilde{J}x,y \rangle$$

is skew-symmetric and non-degenerate. In other words, the triple $(\langle \cdot, \cdot \rangle, \tilde{J}, \tilde{\Omega})$ is an almost-Hermitian (or $U(3)$-structure) on $S^6$. We emphasize that $\tilde{J}$ is not integrable, and that $\tilde{\Omega}$ is not closed.

Now, letting $\partial_r$ denote the radial vector field on $\mathbb{R}^7$, one can show that the complex 3-form $\Upsilon \in \Omega^3(S^6; \mathbb{C})$ given by

$$\Upsilon := (\partial_r \lrcorner (*\phi) + i\phi)|_{S^6}$$

is a $(3,0)$-form on $S^6$ that satisfies

$$\frac{i}{8} \Upsilon \wedge \overline{\Upsilon} = \text{vol}_{S^6}.$$ 

That is, the quadruple $(\langle \cdot, \cdot \rangle, \tilde{J}, \tilde{\Omega}, \Upsilon)$ is an SU(3)-structure on $S^6$. In fact, this SU(3)-structure satisfies the nearly-Kähler equations $d\tilde{\Omega} = 3 \text{Im} (\Upsilon)$ and $d\text{Re} (\Upsilon) = 2 \tilde{\Omega} \wedge \tilde{\Omega}$. The round 6-sphere with this SU(3)-structure is the simplest example of a strict nearly-Kähler 6-manifold.

Now, the SU(3)-structure gives rise to distinguished classes of submanifolds of the 6-sphere. In particular, an immersed surface $u: \Sigma^2 \rightarrow S^6$ is a holomorphic curve if

$$\tilde{J}(T_p \Sigma) = T_p \Sigma, \quad \forall p \in \Sigma.$$ 

Holomorphic curves are, in fact, minimal surfaces. One way to see this is to observe that holomorphic curves have extra symmetries in their second fundamental forms (see (3.10) in §3.2.2), and these symmetries imply minimality. Another way uses the following fundamental fact:

**Proposition 2.1.** Let $\Sigma^2 \subset S^6$ be an immersed surface, and let $C(\Sigma) = \{rx \in \mathbb{R}^7: r > 0, x \in \Sigma\}$ be its cone in $\mathbb{R}^7$. Then $\Sigma$ is a holomorphic curve if and only if $C(\Sigma)$ is an associative 3-fold.

So, as holomorphic curves are the links of associative cones, and since associative cones are homologically volume-minimizing, it follows that holomorphic curves are minimal surfaces.

### 2.3 Holomorphic Bundles over Riemann Surfaces

Let $E \rightarrow M$ be a complex vector bundle over a complex manifold $M$. It is well-known [22, §1.3] that a holomorphic structure on $E$ is equivalent to a $\overline{\partial}$-operator, i.e., an operator

$$\overline{\partial}: \Gamma(E) \rightarrow \Omega^{0,1}(M) \otimes \Gamma(E)$$

satisfying both the relevant Leibniz rule and $\overline{\partial}^2 = 0$. Given a complex vector bundle $E \rightarrow M$ equipped with both a connection $\nabla: \Gamma(E) \rightarrow \Omega^1(M; \mathbb{C}) \otimes \Gamma(E)$ and a holomorphic structure $\overline{\partial}$, we say that $\nabla$ and $\overline{\partial}$ are compatible if $\nabla^{0,1} = \overline{\partial}$.

Note that if $E \rightarrow \Sigma$ is a complex vector bundle over a Riemann surface, then every connection $\nabla$ on $E$ has the property that $\nabla^{0,1}$ satisfies the Leibniz rule and squares to zero. Said another way:
Proposition 2.2. Let $E \to \Sigma$ be a complex vector bundle over a Riemann surface $\Sigma$. For each connection $\nabla$, there exists a unique holomorphic structure on $E$ compatible with $\nabla$ (viz., $\mathcal{J} = \nabla^{0,1}$).

The holomorphic structure in Proposition 2.2 is often called the Koszul-Malgrange holomorphic structure for $\nabla$. However, we shall occasionally abuse terminology and refer to $\nabla$ itself as the holomorphic structure.

3 The Geometry of Holomorphic Curves in the 6-Sphere

In §3.1 and §3.2, we set up the moving frame for holomorphic curves in the 6-sphere. In §3.3, we take a closer look at the class of null-torsion holomorphic curves, the primary results being Proposition 3.4 and Proposition 3.6. In §3.4, we consider three different holomorphic structures on the normal bundle of a null-torsion holomorphic curve.

3.1 Moving Frames for $\mathbb{S}^6$

We begin by viewing $\mathbb{S}^6$ simply as an oriented Riemannian manifold (i.e., as a 6-manifold with an $\text{SO}(6)$-structure). Let $F_{\text{SO}(6)} \to \mathbb{S}^6$ denote the oriented orthonormal coframe bundle of $\mathbb{S}^6$. Let $\omega \in \Omega^1(F_{\text{SO}(6)}; \mathbb{R}^6)$ denote the tautological 1-form, and let $\psi \in \Omega^1(F_{\text{SO}(6)}; \mathfrak{so}(6))$ denote the Levi-Civita connection, so that we have

$$d\omega = -\psi \wedge \omega.$$

So, if $(e_1, \ldots, e_6)$ is a local oriented orthonormal frame on an open set $U \subset \mathbb{S}^6$, then

$$\nabla e_i = -\psi_{ij} \otimes e_j$$  \hspace{1cm} (3.1)

where we are conflating (and will continue to conflate) the 1-forms $\psi_{ij}$ on $F_{\text{SO}(6)}$ with their pullbacks $\sigma^*(\psi_{ij})$ on $U$ via the local section $\sigma: U \to F_{\text{SO}(6)}$ corresponding to $(e_1, \ldots, e_6)$.

3.1.1 The $\text{SU}(3)$-Structure

We now equip $\mathbb{S}^6$ with its standard $\text{SU}(3)$-structure $(\langle \cdot, \cdot \rangle, \tilde{J}, \tilde{\Omega}, \Upsilon)$, recalling §2.2. Let $\mathcal{P} := F_{\text{SU}(3)} \subset F_{\text{SO}(6)}$ denote the $\text{SU}(3)$-coframe bundle of $\mathbb{S}^6$. There is a natural identification $\mathcal{P} \cong \text{G}_2$, but we will not use this fact explicitly. Via the $\text{SU}(3)$-invariant splitting $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathbb{R}^6$ (orthogonal with respect to the Killing form), the restriction of the Levi-Civita connection to $\mathcal{P}$ decomposes as

$$\psi_{\mathcal{P}} = \tilde{\gamma} + T(\omega),$$  \hspace{1cm} (3.2)

where $\tilde{\gamma} \in \Omega^1(\mathcal{P}; \mathfrak{su}(3))$ is the natural $\text{SU}(3)$-connection and $T(\omega) \in \Omega^1(\mathcal{P} ; \mathbb{R}^6)$ is the intrinsic torsion of the $\text{SU}(3)$-structure. Thus, on $\mathcal{P}$, we have the first structure equations

$$d\omega = -\tilde{\gamma} \wedge \omega - T(\omega) \wedge \omega.$$

By writing the subspaces $\mathfrak{su}(3)$ and $\mathbb{R}^6$ of $\mathfrak{so}(6)$ in terms of explicit $6 \times 6$ matrices, we can express the connection matrix $\tilde{\gamma}$ in the form

$$\tilde{\gamma} = \begin{bmatrix}
0 & -\beta_{11} & \alpha_{21} & -\beta_{21} & -\alpha_{31} & -\beta_{31} \\
\beta_{11} & 0 & \beta_{21} & \alpha_{21} & \beta_{31} & -\alpha_{31} \\
-\alpha_{21} & -\beta_{21} & 0 & -\beta_{22} & \alpha_{32} & -\beta_{32} \\
\beta_{21} & -\alpha_{21} & \beta_{22} & 0 & \beta_{32} & \alpha_{32} \\
\alpha_{31} & \beta_{31} & -\alpha_{32} & -\beta_{32} & 0 & -\beta_{33} \\
\beta_{31} & \alpha_{31} & \beta_{32} & -\alpha_{32} & \beta_{33} & 0
\end{bmatrix}.$$
and calculate that the intrinsic torsion of $S^6$ is

$$T(\omega) = \frac{1}{2} \begin{bmatrix} 0 & 0 & \omega_5 & -\omega_6 & -\omega_3 & \omega_4 \\ 0 & 0 & -\omega_6 & -\omega_5 & \omega_4 & \omega_3 \\ -\omega_5 & \omega_6 & 0 & 0 & \omega_1 & -\omega_2 \\ \omega_6 & \omega_5 & 0 & 0 & -\omega_2 & -\omega_1 \\ \omega_3 & -\omega_4 & -\omega_1 & \omega_2 & 0 & 0 \\ -\omega_4 & -\omega_3 & \omega_2 & \omega_1 & 0 & 0 \end{bmatrix}.$$ 

Let $\overline{\nabla} : \Gamma(TS^6) \to \Omega^1(S^6) \otimes \Gamma(TS^6)$ denote the covariant derivative operator associated to the connection $\gamma$. If $(e_1, \ldots, e_6)$ is a local SU(3)-frame on $U \subset S^6$, we have

$$\overline{\nabla}e_i = -\tilde{\gamma}_{ij} \otimes e_j. \quad (3.3)$$

### 3.1.2 The SU(3)-Structure in Complex Notation

It will often be convenient to have complex versions of the above equations. To that end, let $\zeta \in \Omega^1(\mathcal{P}; \mathbb{C}^3)$ denote the complex tautological 1-form, where:

$$\zeta_1 = \omega_1 + i\omega_2 \quad \zeta_2 = \omega_3 + i\omega_4 \quad \zeta_3 = \omega_5 + i\omega_6$$

Let $\gamma \in \Omega^1(\mathcal{P}; \mathfrak{su}(3))$ denote the natural SU(3)-connection on $S^6$, regarded now as a complex $3 \times 3$ matrix (rather than a real $6 \times 6$ matrix). In other words:

$$(\gamma_{ij}) = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} = \begin{bmatrix} i\beta_{11} & \alpha_{21} + i\beta_{21} & -\alpha_{31} + i\beta_{31} \\ -\alpha_{21} + i\beta_{21} & i\beta_{21} & \alpha_{32} + i\beta_{32} \\ \alpha_{31} + i\beta_{31} & -\alpha_{32} + i\beta_{32} & i\beta_{33} \end{bmatrix}.$$ 

In this notation, the first and second structure equations of $S^6$ are [5], [6]

$$d\zeta_i = -\gamma_{i\ell} \wedge \zeta_\ell + \zeta_j \wedge \zeta_k$$ \quad (3.4)

$$d\gamma_{ij} = -\gamma_{ik} \wedge \gamma_{kj} + \frac{1}{2} \epsilon_{ijk} \zeta_\ell - \frac{1}{4} \delta_{ij} \zeta_\ell \wedge \zeta_\ell$$ \quad (3.5)

where $(i, j, k)$ in the first structure equation is an even permutation of $(1, 2, 3)$.

Extend both $\nabla$ and $\overline{\nabla}$ by $\mathbb{C}$-linearity to operators $\Gamma(TS^6 \otimes \mathbb{R} \mathbb{C}) \to \Gamma(TS^6 \otimes \mathbb{R} \mathbb{C}) \otimes \Omega^1(S^6; \mathbb{C})$. In terms of a local SU(3)-frame $(e_1, \ldots, e_6)$ for $TS^6$, we let

$$f_1 = \frac{1}{2} (e_1 - ie_2) \quad f_2 = \frac{1}{2} (e_3 - ie_4) \quad f_3 = \frac{1}{2} (e_5 - ie_6)$$

$$\overline{f}_1 = \frac{1}{2} (e_1 + ie_2) \quad \overline{f}_2 = \frac{1}{2} (e_3 + ie_4) \quad \overline{f}_3 = \frac{1}{2} (e_5 + ie_6).$$

Note that $(f_1, f_2, f_3)$ is a local SU(3)-frame for $T^{1,0}S^6$, while $(\overline{f}_1, \overline{f}_2, \overline{f}_3)$ is a local SU(3)-frame for $T^{0,1}S^6$. A calculation shows that

$$\nabla f_1 = \overline{\nabla} f_1 = \frac{1}{2} (\zeta_2 \otimes \overline{f}_3 - \zeta_3 \otimes \overline{f}_2)$$

$$\nabla f_2 = \overline{\nabla} f_2 = \frac{1}{2} (\zeta_3 \otimes \overline{f}_1 - \zeta_1 \otimes \overline{f}_3)$$

$$\nabla f_3 = \overline{\nabla} f_3 = \frac{1}{2} (\zeta_1 \otimes \overline{f}_2 - \zeta_2 \otimes \overline{f}_1)$$ \quad (3.6)

and

$$\overline{\nabla} f_i = \gamma_{ji} \otimes f_j$$ \quad (3.7)

where we underscore that $(\gamma_{ji}) = (\gamma_{ij})^T$. 

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3.2 Moving Frames for Holomorphic Curves in $S^6$

We now turn our attention to holomorphic curves $u: \Sigma^2 \to S^6$, always assuming for simplicity that $u$ is an (unramified) immersion. In this section, we recall Bryant’s “holomorphic Frenet frame” for $u$, which will be central to our calculations. Our discussion is essentially a self-contained summary of [5, §4], though we have changed notation in several places. Preparation of this section was aided by clarifying discussions in [18] and [27].

Before getting started, let us give a brief overview of the various complex vector bundles over $\Sigma$ that we will need. First, consider the complex rank 3 bundle $u^* (T^1,0 S^6) \to \Sigma$ of $(1,0)$-vectors along $u(\Sigma)$. Next, we let $L_T := T^1,0 \Sigma \subset u^* (T^1,0 S^6)$ and define the complex rank 2 bundle $Q_{NB} := u^* (T^1,0 S^6) / L_T$.

For $v \in u^* (T^1,0 S^6)$, we let $(v) \in Q_{NB}$ denote its projection to the quotient.

In the sequel, we will define a certain complex line subbundle $L_N \subset Q_{NB}$, from which we will set $L_B := Q_{NB} / L_N$. As above, for $(v) \in Q_{NB}$, we let $([v]) \in L_B$ denote its projection to $L_B$. In summary, we have a diagram:

$$
\begin{array}{ccc}
L_T & \longrightarrow & u^* (T^1,0 S^6) \\
\downarrow & & \downarrow (\cdot) \\
L_N & \longrightarrow & Q_{NB} \\
\downarrow & & \downarrow ([\cdot]) \\
& & L_B
\end{array}
$$

All complex vector bundles under consideration are assumed to be endowed with their obvious Hermitian metrics. As Hermitian vector bundles, we will have isomorphisms

$$u^*(T^1,0 S^6) \simeq L_T \oplus L_N \oplus L_B \tag{3.8}$$

$$Q_{NB} \simeq L_N \oplus L_B \tag{3.9}$$

We will shortly equip all of these bundles with holomorphic structures, cautioning that the isomorphisms (3.8) and (3.9) generally will not hold in the holomorphic category.

3.2.1 Holomorphic Structures

To begin, recall the (complexified) SU(3)-connection $\overline{D}$ on $T S^6 \otimes \mathbb{R} \subset \mathbb{C}$. By restriction and pull-back, we get an induced connection (still denoted $\overline{D}$) on $u^* (T^1,0 S^6) \to \Sigma$. We endow $u^* (T^1,0 S^6)$ with the Koszul-Malgrange holomorphic structure for $\overline{D}$. Since $u$ is an immersion, the complex line bundle $L_T := T^1,0 \Sigma \subset u^* (T^1,0 S^6)$ is a holomorphic line subbundle, and we equip the quotient bundle $Q_{NB} := u^* (T^1,0 S^6) / L_T$ with the induced holomorphic structure. These structures in place, we now make two frame adaptations.

3.2.2 First Adaptation

Let $(f_1, f_2, f_3)$ be an SU(3)-frame for $u^* (T^1,0 S^6)$. For our first adaptation, we consider those frames for which

$$f_1 \in L_T = T^1,0 \Sigma.$$
The set of such frames comprises a $U(2)$-subbundle $\mathcal{F}_1 \subset u^* P$ over $\Sigma$, and we will refer to such $(f_1, f_2, f_3)$ as being $U(2)$-adapted. On $\mathcal{F}_1$, we have that $\zeta_2 = \zeta_3 = 0$. Differentiating these equations and applying Cartan’s Lemma shows that there exist functions $\kappa, \mu: \mathcal{F}_1 \to \mathbb{C}$ for which

$$
\gamma_{21} = \kappa \zeta_1 \quad \quad \quad \gamma_{31} = \mu \zeta_1.
$$

Writing $\kappa = \kappa_1 + i\kappa_2$ and $\mu = \mu_1 + i\mu_2$, where $\kappa_1, \kappa_2, \mu_1, \mu_2: \mathcal{F}_1 \to \mathbb{R}$, a calculation shows that the second fundamental form may be expressed as

$$
\begin{align*}
\Pi(e_1, e_1) &= \kappa_1 e_3 + \kappa_2 e_4 + \mu_1 e_5 - \mu_2 e_6 \\
\Pi(e_1, e_2) &= -\kappa_2 e_3 + \kappa_1 e_4 + \mu_2 e_5 + \mu_1 e_6 \\
\Pi(e_2, e_2) &= -\Pi(e_1, e_1),
\end{align*}
$$

(3.10)

In particular, we observe that holomorphic curves are minimal surfaces. Equation (3.10) also shows that the functions $\kappa, \mu$ are essentially equivalent to the second fundamental form.

Using $U(2)$-adapted frames, we can understand the holomorphic structures on $L_T$ and $Q_{NB}$ more explicitly. That is, the Chern connection of $L_T$ is given by

$$
D_{LT} f_1 = \gamma_{11} \otimes f_1
$$

while the Chern connection of $Q_{NB}$ is given by

$$
\begin{align*}
D_{QNB} (f_2) &= \gamma_{22} \otimes (f_2) + \gamma_{32} \otimes (f_3) \\
D_{QNB} (f_3) &= \gamma_{23} \otimes (f_2) + \gamma_{33} \otimes (f_3).
\end{align*}
$$

We now recast the second fundamental form as a holomorphic section. In [5, Lemma 4.3], it is shown that

$$
\Phi_\Pi \in \Gamma(L_T^* \otimes L_T^* \otimes Q_{NB})
$$

where $f_1^\vee: \mathcal{F}_1 \to L_T^*$ is the dual of $f_1$. It is remarked in [5, Lemma 4.4] that $\Phi_\Pi = 0$ if and only if $u$ is the totally geodesic $S^2$. On the other hand, if $u$ is not the totally geodesic $S^2$, then the zeros of $\Phi_\Pi$ are isolated, hence finite (since $\Sigma$ is compact). To streamline further discussion, we enact the following:

**Convention:** From now on, we assume that $u$ is not totally-geodesic.

It is convenient to regard $\Phi_\Pi$ as a holomorphic section of $\text{Hom}(L_T \otimes L_T; Q_{NB})$. Thus, there is a holomorphic line subbundle, called $L_N \subset Q_{NB}$, such that

$$
\Phi_\Pi \in H^0(\text{Hom}(L_T \otimes L_T; L_N)).
$$

To be more explicit, let $F$ denote the (effective) divisor of the holomorphic section $\Phi_\Pi$, i.e.,

$$
F = \sum_{p \in \Sigma: \Phi_\Pi(p) = 0} \text{ord}_p(\Phi_\Pi) \cdot p,
$$

and let $\mathcal{O}_F \to \Sigma$ be the corresponding holomorphic line bundle. Viewing $\Phi_\Pi \in H^0((L_T \otimes L_T)^* \otimes L_N)$, it follows that

$$
L_N = \mathcal{O}_F \otimes L_T \otimes L_T.
$$

Finally, we let $L_B := Q_{NB}/L_N$ and equip $L_B$ with the induced holomorphic structure.
3.2.3 Second Adaptation

For our second adaptation, we consider the $U(2)$-adapted frames $(f_1, f_2, f_3)$ for which

$$(f_2) \in L_N.$$  

This adaptation defines a $T^2$-subbundle $\mathcal{F}_2 \subset \mathcal{F}_1 \subset u^* \mathcal{P}$ over $\Sigma$, and we refer to such frames as $T^2$-adapted. On $\mathcal{F}_2$, we have that $\gamma_{31} = 0$, so that $\mu = 0$. Differentiating $\gamma_{31} = 0$ shows that $\gamma_{32}$ is a semibasic $(1,0)$-form, and hence

$$\gamma_{32} = \tau \zeta_1$$

for some function $\tau: \mathcal{F}_2 \to \mathbb{C}$. In summary, if $(f_1, f_2, f_3)$ is a $T^2$-adapted frame, the equations (3.7) now read:

$$D \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \left( \begin{array}{ccc} \gamma_{11} & \kappa \zeta_1 & 0 \\ -\kappa \zeta_1 & \gamma_{22} & \tau \zeta_1 \\ 0 & -\tau \zeta_1 & \gamma_{33} \end{array} \right) \otimes \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \quad (3.11)$$

These are the holomorphic Frenet equations for the holomorphic curve $u: \Sigma^2 \to S^6$.

Using $T^2$-adapted frames, we can understand the holomorphic structures on $L_N$ and $L_B$ more explicitly. That is, the Chern connection of $L_N$ is given by

$$D^{L_N}(f_2) = \gamma_{22} \otimes (f_2)$$

and that of $L_B$ by

$$D^{L_B}((f_3)) = \gamma_{33} \otimes ((f_3)).$$

Now, by analogy with the familiar Frenet frame for curves in $\mathbb{R}^3$, one might be inclined to call $\tau$ the “holomorphic torsion” of $u: \Sigma^2 \to S^6$, but for the fact that $\tau$ depends on the choice of $T^2$-frame $(f_1, f_2, f_3)$. However, the “null-torsion” condition $\tau = 0$ turns out to be independent of frame. Indeed, Bryant shows [5, Lemma 4.5] that

$$\Phi = \tau \zeta_1 \otimes (f_2^\vee) \otimes ((f_3))$$

is a well-defined (frame-independent) holomorphic section. The section $\Phi$ partitions the collection of (non-totally-geodesic) holomorphic curves into three classes:

1. $\Phi = 0$ identically.
2. The zero set of $\Phi$ is finite and non-empty.
3. $\Phi$ is nowhere-vanishing.

The generic situation is (2), and relatively little is known about this case. As Bryant remarks [5, p. 225], the condition (3) is quite strong, implying a stringent relation on the line bundles $L_T, L_N, L_B$.

Holomorphic curves of type (1) are said to be null-torsion, and are the focus of this work. Note that every holomorphic curve of genus zero is null-torsion [5, Theorem 4.6] or the totally-geodesic 2-sphere. It is shown in [4] that every null-torsion holomorphic curve is linearly full in $S^6$ (i.e., is not contained in a totally-geodesic $S^5$), implying that even the simplest null-torsion curves cannot be reduced to the study of minimal Legendrians in $S^5$.

It is a remarkable fact [5, Theorem 4.10] that every compact Riemann surface admits a conformal branched immersion into $S^6$ as a null-torsion holomorphic curve. In his 1999 Ph.D. thesis [31], Rowland extended this result, showing that, in fact, every compact Riemann surface may be conformally embedded as a null-torsion holomorphic curve in $S^6$. 

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Remark. As of this writing, it is an open question whether every open Riemann surface can be conformally embedded as a null-torsion holomorphic curve in $S^6$. It seems to the author that the techniques of [1] may yield a positive solution to this problem.

3.3 Null-Torsion Holomorphic Curves

We now examine null-torsion holomorphic curves more closely. In Proposition 3.1, we give two holomorphic interpretations of the null-torsion condition. Then, in Proposition 3.3, we will see how the null-torsion condition constrains the topologies of the line bundles $L_T, L_N, L_B$. Using this, together with Riemann-Roch, we will calculate (Proposition 3.4) the number of independent holomorphic sections of $L_N$ and $L_B^*$.

3.3.1 Holomorphic Interpretations of Null-Torsion

Let $u: \Sigma^2 \to S^6$ holomorphic curve, and regard $S^6 \subset \mathbb{R}^7$ as the usual unit sphere. Its binormal Gauss map is

$$b_u: \Sigma^2 \to \text{Gr}^+_2(\mathbb{R}^7)$$

$$b_u(p) = e_5 \wedge e_6$$

where $(e_1, \ldots, e_6)$ is a $T^2$-frame at $p \in \Sigma$. One can check that $b_u$ is well-defined, independent of frame. Now, consider the map

$$\text{Gr}^+_2(\mathbb{R}^7) \to \mathbb{P}(\mathbb{C}^7) = \mathbb{CP}^6$$

$$x \wedge y \mapsto \text{span}_\mathbb{C}(x - iy)$$

where $\{x, y\}$ is orthonormal. This map is well-defined (independent of basis), injective, and its image is the complex hypersurface $\Lambda = \{[z] \in \mathbb{CP}^6: z_1^2 + \cdots + z_7^2 = 0\}$. Consequently, we may identify $\text{Gr}^+_2(\mathbb{R}^7) \simeq \Lambda \subset \mathbb{CP}^6$.

**Proposition 3.1.** Let $u: \Sigma^2 \to S^6$ holomorphic curve. The following are equivalent:

(i) $u$ is null-torsion.

(ii) The binormal Gauss map $b_u: \Sigma^2 \to \Lambda$ is holomorphic.

(iii) There is a holomorphic splitting $Q_{NB} \cong L_N \oplus L_B$.

**Proof.** The equivalence of (i) and (ii) is [5, Theorem 4.7]. For the equivalence of (i) and (iii), we simply observe that $\tau$ is essentially the second fundamental form (in the sense of complex geometry, cf. [11, Chap. V: §14] or [22, §1.6]) of the holomorphic Hermitian subbundle $L_N \subset Q_{NB}$. ♦

**Corollary 3.2.** If $u: \Sigma^2 \to S^6$ is a null-torsion holomorphic curve, then its area $A = 4\pi d$, where $d$ is the degree of the binormal Gauss map.

3.3.2 Topological Consequences

We now consider the topologies of the bundles $L_T, L_N, L_B$. Now, as Bryant points out [5, p. 224], the SU(3)-structure on $S^6$ yields a holomorphic, metric isomorphism

$$\Lambda^3(T^{1,0}S^6) \cong \mathbb{C}$$

where $\mathbb{C}$ is the trivial line bundle. Consequently, there is an isomorphism of holomorphic line bundles

$$L_T \otimes L_N \otimes L_B \cong \mathbb{C}$$
In particular, it follows that:
\[ c_1(L_T) + c_1(L_N) + c_1(L_B) = 0. \quad (3.12) \]

Here is an equivalent way to see this. Since \( \gamma \) is valued in \( \mathfrak{su}(3) \), we have
\[ \gamma_{11} + \gamma_{22} + \gamma_{33} = 0. \]
Note that \( \gamma_{11}, \gamma_{22}, \gamma_{33} \) are, respectively, the connection forms of the Chern connections on \( L_T, L_N, L_B \). From the structure equations (3.5), we may compute that their curvature \((1,1)\)-forms \( F_T, F_N, F_B \) are given by
\[
F_T = i d\gamma_{11} = (1 - 2|\kappa|^2) \text{ vol} =: K_T \text{ vol}
\]
\[
F_N = i d\gamma_{22} = \left(2|\kappa|^2 - 2|\tau|^2 - \frac{1}{2}\right) \text{ vol} =: K_N \text{ vol}
\]
\[
F_B = i d\gamma_{33} = \left(2|\tau|^2 - \frac{1}{2}\right) \text{ vol} =: K_B \text{ vol}
\]
where \( \text{ vol} = \omega_1 \wedge \omega_2 \) is the volume form on \( \Sigma \), and where \( K_T, K_N, K_B \) are defined by these equations. Note that \( K_T = K \) is simply the Gauss curvature of \( \Sigma \). Thus, we have
\[ F_T + F_N + F_B = 0, \]
and hence
\[
\frac{1}{2\pi} \int_{\Sigma} K_T \text{ vol} + \frac{1}{2\pi} \int_{\Sigma} K_N \text{ vol} + \frac{1}{2\pi} \int_{\Sigma} K_B \text{ vol} = 0,
\]
which gives (3.12) by Chern-Weil theory. In the null-torsion case, the formulas for \( K_N \) and \( K_B \) simplify, yielding:

**Proposition 3.3.** If \( u: \Sigma^2 \to \mathbb{CP}^6 \) is a null-torsion holomorphic curve of area \( A = 4\pi d \), then
\[
c_1(L_T) = \chi(\Sigma)
\]
\[
c_1(L_N) = -\chi(\Sigma) + d
\]
\[
c_1(L_B) = -d.
\]
Moreover, counting with multiplicity, there are exactly \( d - 3\chi(\Sigma) \) points \( p \in \Sigma \) at which \( \Phi_\Pi(p) = 0 \).

**Proof.** It is a standard fact that \( c_1(L_T) = c_1(T^{1,0} \Sigma) = \chi(\Sigma) \). Since \( u \) is null-torsion, we have \( \tau = 0 \), so \( K_B = -\frac{1}{2} \), so:
\[
c_1(L_B) = \frac{1}{2\pi} \int_{\Sigma} K_B \text{ vol} = -\frac{A}{4\pi}.
\]
Note that this gives another proof of the fact that null-torsion holomorphic curves have area equal to \( 4\pi \) times a positive integer. From Corollary 3.2, we know that \( A = 4\pi d \), where \( d \) is the degree of the binormal lift. So, we obtain:
\[
c_1(L_B) = -d.
\]
The last claim is simply that
\[
\sum_{p \in \Sigma: \Phi_\Pi(p) = 0} \text{ord}_p(\Phi_\Pi) = \text{deg}(F) = c_1(O_F) = c_1(L_T^+ \otimes L_T^+ \otimes L_N) = d - 3\chi(\Sigma).
\]
\[ \diamond \]
3.3.3 Holomorphic Consequences

We now invoke Riemann-Roch to understand the spaces of holomorphic sections of $L_N$ and $L_B^*$.

**Proposition 3.4.** Suppose $u: \Sigma^2 \to S^6$ is a null-torsion holomorphic curve of area $A = 4\pi d$. Then:

(a) We have:

$$h^0(L_N) = d - \frac{1}{2}\chi(\Sigma)$$

$$h^0(L_B^*) = d + \frac{1}{2}\chi(\Sigma) + h^0(L_B \otimes K_\Sigma).$$

(b) If $d > 2g - 2$, then $h^0(L_B \otimes K_\Sigma) = 0$.

**Proof.** (a) To begin, observe that

$$\deg(L_N^* \otimes K_\Sigma) = -c_1(L_N) - \chi(\Sigma) = -d < 0.$$ 

Since the line bundle $L_N^* \otimes K_\Sigma$ is negative, it has no non-trivial holomorphic sections. Therefore, by Riemann-Roch, Serre Duality, and Proposition 3.3, we obtain:

$$h^0(L_N) = h^0(L_N^* \otimes K_\Sigma) + c_1(L_N) + \frac{1}{2}\chi(\Sigma)$$

$$= 0 - \chi(\Sigma) + d + \frac{1}{2}\chi(\Sigma).$$

Similarly, we have:

$$h^0(L_B^*) = h^0(L_B \otimes K_\Sigma) + c_1(L_B^*) + \frac{1}{2}\chi(\Sigma)$$

$$= d + \frac{1}{2}\chi(\Sigma) + h^0(L_B \otimes K_\Sigma).$$

(b) Letting $\mathcal{L} := L_B \otimes K_\Sigma$, we compute

$$\deg(\mathcal{L}) = c_1(L_B) + c_1(K_\Sigma) = -d + (2g - 2).$$

Therefore, if $d > 2g - 2$, then $\deg(\mathcal{L}) < 0$, so $\mathcal{L}$ has no non-trivial holomorphic sections.

In light of Proposition 3.4(b), it is of interest to know when the hypothesis “$d > 2g - 2$” might be automatically satisfied. To that end, we recall the following facts from complex algebraic geometry. See [17, §2.3, page 253] for a proof.

**Lemma 3.5.** Let $\tilde{u}: \Sigma^2 \to \mathbb{CP}^6$ be a non-degenerate complex curve of genus $g$ and degree $d$.

(a) We have $d \geq 6$.

(b) If $d = 6$, then $\tilde{u}$ is the rational normal curve, which has genus $g = 0$.

(c) If $7 \leq d \leq 11$, then $g \leq d - 6$.

**Proposition 3.6.** Let $u: \Sigma^2 \to S^6$ be a null-torsion holomorphic curve, where $\Sigma$ is a closed surface of genus $g$ and area $A = 4\pi d$. If $g \leq 6$, then $d > 2g - 2$.

**Proof.** If $u$ is totally-geodesic, the claim is trivial, so assume otherwise. Then its binormal Gauss map $b_u: \Sigma^2 \to \Lambda \subset \mathbb{CP}^6$ is non-degenerate, so $d \geq 6$. If $d = 6$, then by Lemma 3.5(b), we have $g = 0$, fulfilling the bound $d > 2g - 2$. Thus, we may assume that $d \geq 7$ for the remainder of this proof.

If $g \leq 4$, then $2g - 2 \leq 6 < 7 \leq d$, so the bound is fulfilled in this case. Suppose now that $g = 5$. If we had $d \leq 2g - 2 = 8$, then Lemma 3.5(c) would imply that $g \leq 2$, which is absurd. Analogous reasoning holds for $g = 6$.
3.4 Structures on the Normal Bundle

Let $u: \Sigma^2 \to S^6$ be a holomorphic curve (which may or may not be null-torsion). With respect to the splitting $u^*(T S^6) = T \Sigma \oplus N \Sigma$, let $\nabla^T$, $\nabla^\perp$ and $D^\perp$, $D^\perp$ denote the tangential and normal connections for $\nabla$ and $\overline{\nabla}$. In this section, we equip the normal bundle $N \Sigma$ with various holomorphic structures and compare their properties. Since this is an important but perhaps technical point, we now provide a brief overview of this section.

Thus far, we have been focused on the $G_2$-invariant almost-complex structure $\tilde{J}$ on $S^6$. By restriction, we get a complex structure, also called $\tilde{J}$, on $N \Sigma$. Decomposing the complexification $N \Sigma \otimes_R \mathbb{C}$ into $J$-eigenbundles

$$N \Sigma \otimes_R \mathbb{C} = \tilde{N}^{1,0} \oplus \tilde{N}^{0,1}$$

we will ask how the $\mathbb{C}$-linear extensions of $\nabla^\perp$ and $D^\perp$ relate to this decomposition. The upshot is that the complex bundle $\tilde{N}^{1,0}$ can always be endowed with a holomorphic structure $\tilde{\nabla}^{SU}$ compatible with $D^\perp$. As shown by Lotay [26], this holomorphic structure plays a key role in the deformation theory of associative 3-folds in $\mathbb{R}^7$ asymptotic to the cone on $\Sigma$.

However, for our study of the second variation of area, we will need to consider a different complex structure on $N \Sigma = E_N \oplus E_B$, which we call $\hat{J}$. This alternate complex structure $\hat{J}$ will agree with $\tilde{J}$ on $E_N$, but differ on $E_B$. Again, we will decompose $N \Sigma \otimes_R \mathbb{C}$ into $\hat{J}$-eigenbundles

$$N \Sigma \otimes_R \mathbb{C} = \tilde{N}^{1,0} \oplus \tilde{N}^{0,1}$$

and consider how the $\mathbb{C}$-linear extensions of $\nabla^\perp$ and $D^\perp$ interact with this decomposition. The result is that in the null-torsion case, the complex bundle $\tilde{N}^{1,0}$ can be equipped with two different holomorphic structures: one called $\tilde{\nabla}^{\perp}$ that is compatible with $\nabla^\perp$, and another called $\tilde{\nabla}^{D^\perp}$ that is compatible with $D^\perp$. Comparing these two structures on $\tilde{N}^{1,0}$ is the idea behind Theorem 1.1.

3.4.1 The Induced Complex Structure

Let $u: \Sigma \to S^6$ be a holomorphic curve. It is easy to check that the covariant derivative operators $\nabla^\perp$ and $D^\perp$ on the normal bundle $N \Sigma$ interact with the complex structure $\tilde{J}$ as follows:

**Proposition 3.7.** The complex structure $\tilde{J}$ is not $\nabla^\perp$-parallel, but is $D^\perp$-parallel (i.e., $D^\perp \tilde{J} = 0$).

We now complexify $N \Sigma$ and extend $\tilde{J}$ by $\mathbb{C}$-linearity. In the usual way, we have a decomposition of $N \Sigma \otimes_R \mathbb{C}$ into $\tilde{J}$-eigenbundles, say

$$N \Sigma \otimes_R \mathbb{C} = \tilde{N}^{1,0} \oplus \tilde{N}^{0,1}$$

where, for example, $\tilde{N}^{1,0} = \{ \xi \in N \Sigma \otimes_R \mathbb{C}: \tilde{J} \xi = i \xi \} = \{ \frac{1}{2}(\eta - i \bar{\eta}) : \eta \in N \Sigma \}$. As complex vector bundles, there is an isomorphism $(N \Sigma, \tilde{J}) \simeq (\tilde{N}^{1,0}, i)$ via $\eta \mapsto \frac{1}{2}(\eta - i \bar{\eta})$ with inverse $\xi \mapsto \frac{1}{2}(\xi + \bar{\xi})$. There is also a well-defined isomorphism of complex vector bundles

$$\tilde{N}^{1,0}, i \to Q_{NB}$$

$$v^\perp \mapsto (v)$$

where we decompose $v \in u^*(T^{1,0} S^6)$ as $v = v^\top + v^\perp$ with tangential part $v^\top \in T^{1,0} \Sigma$ and normal part $v^\perp \in \tilde{N}^{1,0} \Sigma$. In terms of a local $T^2$-frame $(e_1, \ldots, e_6)$, the isomorphisms $(N \Sigma, \tilde{J}) \simeq (\tilde{N}^{1,0}, i) \simeq Q_{NB} \simeq L_N \oplus L_B$ are simply:

$$e_3 \mapsto f_2 \mapsto (f_2) \mapsto (f_2) \oplus 0 \quad e_5 \mapsto f_3 \mapsto (f_3) \mapsto 0 \oplus ((f_3))$$

$$e_4 \mapsto if_2 \mapsto (if_2) \mapsto (if_2) \oplus 0 \quad e_6 \mapsto if_3 \mapsto (if_3) \mapsto 0 \oplus ((if_3))$$
Now, we have already equipped $Q_{NB}$, $L_N$, and $L_B$ with holomorphic structures, and we would like to endow $\tilde{N}^{1,0}$ with a holomorphic structure as well. There is an obvious way to do this: we can use the isomorphism (3.13) to pull back the holomorphic structure on $Q_{NB}$ to $\tilde{N}^{1,0}$. Unwinding the definitions shows that this is precisely the Koszul-Malgrange structure for the $\mathbb{C}$-linear extension of $D^\perp$ on $\tilde{N}^{1,0}$.

Let us illustrate this holomorphic structure in terms of a local $T^2$-frame $(f_1, f_2, f_3)$. To begin, extend both $\nabla^\perp$ and $D^\perp$ by $\mathbb{C}$-linearity to operators

\[ \nabla^\perp, D^\perp : \Gamma(N\Sigma \otimes \mathbb{R} \mathbb{C}) \to \Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(N\Sigma \otimes \mathbb{R} \mathbb{C}). \]

From (3.6) and (3.11), we see that

\[ \nabla^\perp f_2 = \gamma_{22} \otimes f_2 + \tau \zeta_1 \otimes f_3 - \frac{1}{2} \zeta_1 \otimes \tilde{T}_3 \]
\[ D^\perp f_2 = \gamma_{22} \otimes f_2 + \tau \zeta_1 \otimes f_3 \]
\[ \nabla^\perp f_3 = \tilde{\nabla}_1 \otimes f_2 + \frac{1}{2} \zeta_1 \otimes \tilde{T}_2 + \gamma_{33} \otimes f_3 \]
\[ D^\perp f_3 = \tilde{\nabla}_1 \otimes f_2 + \gamma_{33} \otimes f_3. \]

Thus, the restriction of $\nabla^\perp$ to $\tilde{N}^{1,0}\Sigma$ does not give a well-defined connection, whereas the restriction of $D^\perp$ to $\tilde{N}^{1,0}$ does. That is, we have a covariant derivative operator

\[ D^\perp : \Gamma(\tilde{N}^{1,0}) \to \Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(\tilde{N}^{1,0}), \]

which gives $\tilde{N}^{1,0}$ the holomorphic structure described in the previous paragraph. Composing this $D^\perp$ with the projection $T\Sigma \otimes \mathbb{R} \mathbb{C} \to T^{0,1}\Sigma$ gives the corresponding $\overline{\partial}$-operator

\[ \overline{\partial}^\text{SU} : \Gamma(\tilde{N}^{1,0}) \to \Omega^{0,1}(\Sigma) \otimes \Gamma(\tilde{N}^{1,0}). \]

### 3.4.2 A Second Complex Structure

Let $u : \Sigma^2 \to \mathbb{S}^6$ be a holomorphic curve. In terms of a local $T^2$-frame $(e_1, \ldots, e_6)$, we have:

\[ \tilde{J}e_3 = e_4 \]
\[ \tilde{J}e_5 = e_6. \]

We now define a new complex structure $\tilde{J}$ on $N\Sigma$ by declaring

\[ \tilde{J}e_3 = e_4 \]
\[ \tilde{J}e_5 = -e_6. \]

The following shows how $\nabla^\perp$ and $D^\perp$ relate to $\tilde{J}$, and should be contrasted with Proposition 3.7.

**Proposition 3.8.** Let $u : \Sigma \to \mathbb{S}^6$ be a holomorphic curve. The following are equivalent:

(i) $u$ is null-torsion.

(ii) $\nabla^\perp \tilde{J} = 0$.

(iii) $D^\perp \tilde{J} = 0$.

**Proof.** Directly from (3.1), (3.2), and (3.3), one can check that

\[ \nabla^\perp(\tilde{J}e_3) - \tilde{J}(\nabla^\perp e_3) = D^\perp(\tilde{J}e_3) - \tilde{J}(D^\perp e_3) = -2\beta_{32} \otimes e_5 - 2\alpha_{32} \otimes e_6 \]
\[ \nabla^\perp(\tilde{J}e_4) - \tilde{J}(\nabla^\perp e_4) = D^\perp(\tilde{J}e_4) - \tilde{J}(D^\perp e_4) = 2\alpha_{32} \otimes e_5 - 2\beta_{32} \otimes e_6 \]
\[ \nabla^\perp(\tilde{J}e_5) - \tilde{J}(\nabla^\perp e_5) = D^\perp(\tilde{J}e_5) - \tilde{J}(D^\perp e_5) = 2\beta_{32} \otimes e_3 - 2\alpha_{32} \otimes e_4 \]
\[ \nabla^\perp(\tilde{J}e_6) - \tilde{J}(\nabla^\perp e_6) = D^\perp(\tilde{J}e_6) - \tilde{J}(D^\perp e_6) = 2\alpha_{32} \otimes e_3 + 2\beta_{32} \otimes e_4 \]

Noting that $u$ is null-torsion if and only if $\alpha_{32} = \beta_{32} = 0$ proves the claim. \(\diamondsuit\)
We now complexify \( N \Sigma \) and extend \( \tilde{J} \) by \( \mathbb{C} \)-linearity. We have a decomposition of \( N \Sigma \otimes_{\mathbb{R}} \mathbb{C} \) into \( \tilde{J} \)-eigenbundles, say
\[
N \Sigma \otimes_{\mathbb{R}} \mathbb{C} = \tilde{N}^{1,0} \oplus \tilde{N}^{0,1}
\]
where, for example, \( \tilde{N}^{1,0} = \{ \xi \in N \Sigma \otimes_{\mathbb{R}} \mathbb{C} : \tilde{J} \xi = i \xi \} = \{ \frac{1}{2}(\eta - i \tilde{J} \eta) : \eta \in N \Sigma \} \). As complex vector bundles, we have isomorphisms \((N \Sigma, \tilde{J}) \simeq (\tilde{N}^{1,0}, i) \simeq L_N \oplus L_B^*\). In terms of a local \( T^2 \)-frame, these isomorphisms are simply:
\[
e_3 \mapsto f_2 \mapsto (f_2) \oplus 0 \quad e_5 \mapsto \bar{f}_3 \mapsto 0 \oplus (\bar{f}_3) \\
e_4 \mapsto if_2 \mapsto (if_2) \oplus 0 \quad e_6 \mapsto -i\bar{f}_3 \mapsto 0 \oplus (\bar{i} \bar{f}_3).
\]

We now extend both \( \nabla^\perp \) and \( D^\perp \) by \( \mathbb{C} \)-linearity. In terms of a local \( T^2 \)-frame, we compute
\[
\begin{align*}
\nabla^\perp f_2 &= \gamma_{22} \otimes f_2 + \tau \zeta_1 \otimes f_3 - \frac{i}{2} \zeta_1 \otimes \bar{f}_3 & D^\perp f_2 &= \gamma_{22} \otimes f_2 + \tau \zeta_1 \otimes f_3 \\
\nabla^\perp \bar{f}_3 &= \frac{i}{2} \zeta_1 \otimes f_2 + \tau \zeta_1 \otimes \bar{f}_2 - \gamma_{33} \otimes \bar{f}_3 & D^\perp \bar{f}_3 &= \tau \zeta_1 \otimes \bar{f}_2 - \gamma_{33} \otimes \bar{f}_3.
\end{align*}
\]

Thus, if \( u \) null-torsion (i.e., \( \tau = 0 \)), the \( \mathbb{C} \)-linear extensions of \( \nabla^\perp \) and \( D^\perp \) both give well-defined connections on \( \tilde{N}^{1,0} \), meaning that we have covariant derivative operators
\[
\nabla^\perp, D^\perp : \Gamma(\tilde{N}^{1,0}) \to \Omega^1(\Sigma; \mathbb{C}) \otimes \Gamma(\tilde{N}^{1,0}).
\]

Composing these with the projection \( T \Sigma \otimes_{\mathbb{R}} \mathbb{C} \to T^{0,1}\Sigma \) gives the corresponding \( \bar{\partial} \)-operators:
\[
\bar{\partial}^\nabla, \bar{\partial}^D : \Gamma(\tilde{N}^{1,0}) \to \Omega^{0,1}(\Sigma) \otimes \Gamma(\tilde{N}^{1,0}).
\]

Explicitly, continuing to assume that \( u \) is null-torsion:
\[
\begin{align*}
\bar{\partial}^\nabla f_2 &= (\gamma_{22})^{0,1} \otimes f_2 & \bar{\partial}^D f_2 &= (\gamma_{22})^{0,1} \otimes f_2 \\
\bar{\partial}^\nabla \bar{f}_3 &= \frac{i}{2} \zeta_1 \otimes f_2 - (\gamma_{33})^{0,1} \otimes \bar{f}_3 & \bar{\partial}^D \bar{f}_3 &= -(\gamma_{33})^{0,1} \otimes \bar{f}_3.
\end{align*}
\]

The upshot is that if \( u \) is null-torsion, we have two different holomorphic structures to consider on the complex bundle \( \tilde{N}^{1,0} \). The corresponding spaces of holomorphic sections will be denoted
\[
H^0(\tilde{N}^{1,0}, \nabla^\perp) = \{ \xi \in \Gamma(\tilde{N}^{1,0}) : \bar{\partial}^\nabla \xi = 0 \} \quad H^0(\tilde{N}^{1,0}, D^\perp) = \{ \xi \in \Gamma(\tilde{N}^{1,0}) : \bar{\partial}^D \xi = 0 \}.
\]

As we will see, it is the bundle \((\tilde{N}^{1,0}, \nabla^\perp)\) that arises in the study of the second variation of area. On the other hand, the bundle \((\tilde{N}^{1,0}, D^\perp)\) has the desirable feature that it splits holomorphically:

**Proposition 3.9.** If \( u : \Sigma^2 \to \mathbb{S}^0 \) is a null-torsion holomorphic curve, then:

(a) The inclusion \( L_N \hookrightarrow (\tilde{N}^{1,0}, \nabla^\perp) \) is holomorphic.

(b) There is a holomorphic splitting
\[
(\tilde{N}^{1,0}, D^\perp) \cong L_N \oplus L_B^*.
\]

**Proof.** We have already seen that \( \tilde{N}^{1,0} \cong L_N \oplus L_B^* \) holds as complex vector bundles. Equations (3.16) and (3.17) show that the inclusions
\[
L_N \hookrightarrow (\tilde{N}^{1,0}, \nabla^\perp) \quad L_N \hookrightarrow (\tilde{N}^{1,0}, D^\perp) \\
L_B^* \hookrightarrow (\tilde{N}^{1,0}, D^\perp)
\]
are all holomorphic.   \( \diamond \)
To conclude this section, we use the holomorphic structure $\nabla^\perp$ on $\hat N^{1,0}$ to define a notion of “real-holomorphicity” for sections of $(N\Sigma, \hat J)$. We say that $\eta \in \Gamma(N\Sigma)$ is $(\hat J, \nabla^\perp)$-real holomorphic if $\hat \eta^{1,0} := \frac{1}{2}(\eta - i\hat J\eta)$ is $\nabla^\perp$-holomorphic. Defining the operator
\[
\hat \Theta : \Gamma(N\Sigma) \to \Omega^1(\Sigma) \otimes \Gamma(N\Sigma)
\]
where $J := \hat J|_{T\Sigma}$ is the complex structure on $T\Sigma$, it is easy to check that for $Z = \frac{1}{2}(X + iJX) \in T^{0,1}\Sigma$, we have
\[
\nabla^\perp_Z \hat \eta^{1,0} = i\left(\hat \Theta_X \eta + i\hat \Theta_{JX} \eta\right).
\]
Consequently, $\eta$ is $(\hat J, \nabla^\perp)$-real holomorphic if and only if $\hat \Theta_X \eta = 0$ for all $X \in T\Sigma$. In other words,
\[
\{\eta \in \Gamma(N\Sigma) : \hat \Theta \eta = 0\} \cong H^0(\hat N^{1,0}; \nabla^\perp)
\]
as complex vector spaces.

**Remark.** In the same way, one can speak of $(\hat J, D^\perp)$-real holomorphicity, and define a corresponding operator $\hat \Theta^D : \Gamma(N\Sigma) \to \Omega^1(\Sigma) \otimes \Gamma(N\Sigma)$. However, we will not need this concept.

### 4 The Second Variation of Area

We now begin our study of the Jacobi operator of null-torsion holomorphic curves. In §4.1, we derive a second variation formula suited to the study of null-torsion holomorphic curves in $S^0$, and in §4.2 we use this formula to give a holomorphic interpretation of the $(-2)$-eigenspace of the Jacobi operator. In §4.3, we prove Theorem 1.1.

#### 4.1 A Second Variation Formula

Let $u : \Sigma^2 \to S^{2n}$ be a minimal surface in a round $2n$-sphere of constant curvature 1, where $\Sigma$ is a compact oriented surface without boundary. Let $u_t : \Sigma \to S^{2n}$ be a variation of $F_0 = u$ with $\eta := \frac{d}{dt}|_{t=0} u_t$ a normal variation vector field. As is well-known [24, Chap I: §9], the second variation of area is given by
\[
Q(\eta) := \frac{d^2}{dt^2}\bigg|_{t=0} \text{Area}(u_t) = \int_{\Sigma} \langle \mathcal{L} \eta, \eta \rangle
\]
where the Jacobi operator $\mathcal{L} : \Gamma(N\Sigma) \to \Gamma(N\Sigma)$ is
\[
\mathcal{L} = -\Delta^\perp - B - R
\]
where, in a local orthonormal frame $(e_1, e_2)$ for $T\Sigma$, we have
\[
\Delta^\perp \eta = \nabla^\perp_{e_i} \nabla^\perp_{e_i} \eta - \nabla^\perp_{\nabla^\perp_{e_i} e_i} \eta \\
B(\eta) = \langle \Pi(e_i, e_j), \eta \rangle \Pi(e_i, e_j) \\
R(\eta) = \langle R(\eta, e_i) e_i, \eta \rangle.
\]
Note that $\Delta^\perp$ is simply the connection Laplacian for $\nabla^\perp$, the normal part of the Levi-Civita connection. The spectrum of $-\Delta^\perp$ consists of non-negative real numbers.

We now fix $\eta \in \Gamma(N\Sigma)$ and study the terms $R(\eta), \Delta^\perp \eta$, and $B(\eta)$, in that order. Let $\theta_\eta \in \Omega^1(\Sigma)$ denote the 1-form
\[
\theta_\eta(X) = \langle \nabla^\perp_X \eta, \eta \rangle.
\]
Standard arguments show (see e.g., [33], [14], [30]) that:
Proposition 4.1. Let \( u: \Sigma^2 \to \mathbb{S}^{2n} \) be an oriented minimal surface. Then
\[
\mathcal{R}(\eta) = 2\eta. \tag{4.1}
\]
and
\[
- \langle \Delta^\perp \eta, \eta \rangle = \|\nabla^\perp \eta\|^2 + \text{div}(\theta^2_\eta). \tag{4.2}
\]

We now seek a formula for the term \( \|\nabla^\perp \eta\|^2 \). For this, let \( J \) denote the complex structure on \( T\Sigma \) given by the metric and orientation, and equip \( N\Sigma \) with a complex structure \( I \) that is compatible with the metric. Associated to \( N \) on \( J \) with the metric. Associated to \( N \) on \( J \) with the metric. Associated to \( J \) with \( I \eta \), we let \( \Theta_\eta \in \Omega^1(\Sigma) \) denote the 1-form
\[
\Theta_\eta(X) = \langle \nabla^\perp_{JX} \eta, I\eta \rangle,
\]
and let \( \mathcal{D}: \Gamma(N\Sigma) \to \Gamma(N\Sigma) \otimes \Omega^1(\Sigma) \) denote the operator
\[
\mathcal{D}_X \eta = \nabla^\perp_{JX} \eta - \nabla^\perp_{X}(I\eta).
\]

We now have:

Proposition 4.2. Let \( u: \Sigma^2 \to \mathbb{S}^{2n} \) be an oriented minimal surface. Let \( I \) be any complex structure on \( N\Sigma \) that is compatible with the metric. If \( \nabla^\perp I = 0 \), then
\[
\|\nabla^\perp \eta\|^2 = \frac{1}{2} \|\mathcal{D}_\eta\|^2 - \langle R^i_{12} \eta, I\eta \rangle - \text{div}(\Theta^i_\eta). \tag{4.3}
\]

Proof. We observe that
\[
\|\mathcal{D}_\eta\|^2 = \|\nabla^\perp_{J\eta}\|^2 + \|\nabla^\perp_{\eta}(I\eta)\|^2 - 2 \left\langle \nabla^\perp_{J\eta} \eta, \nabla^\perp_{\eta}(I\eta) \right\rangle.
\]
Therefore, we have:
\[
\|\mathcal{D}_\eta\|^2 = \|\nabla^\perp_{J\eta}\|^2 + \|\nabla^\perp_{\eta}(I\eta)\|^2 - 2 \left\langle \nabla^\perp_{J\eta} \eta, \nabla^\perp_{\eta}(I\eta) \right\rangle = \|\nabla^\perp \eta\|^2 - 2 \left\langle \nabla^\perp_{\eta} \eta, \nabla^\perp_{\eta}(I\eta) \right\rangle.
\]

To evaluate the last term, we compute
\[
\left\langle \nabla^\perp_{J\eta} \eta, \nabla^\perp_{\eta}(I\eta) \right\rangle = - \left\langle \nabla^\perp_{\eta} \nabla^\perp_{J\eta} \eta, I\eta \right\rangle + e_i(\Theta_{\eta}(e_i)) = - \left\langle \nabla^\perp_{\eta} \nabla^\perp_{J\eta} \eta, \Theta_{\eta}(\nabla^\top_{\eta} e_i) \right\rangle - \delta \Theta_{\eta} = - \langle R^i_{12} \eta, I\eta \rangle - \left\langle \nabla^\perp_{[e_i,e_\eta]} \eta, I\eta \right\rangle + \Theta_{\eta}(\nabla^\top_{\eta} e_i) - \delta \Theta_{\eta} = - \langle R^i_{12} \eta, I\eta \rangle - \delta \Theta_{\eta},
\]
where \( \delta \) is the codifferential, and in the last line we used that \( \nabla^\top \) is torsion-free and commutes with \( J \). Thus, we have shown that
\[
\|\nabla^\perp \eta\|^2 = \frac{1}{2} \|\mathcal{D}_\eta\|^2 - \langle R^i_{12} \eta, I\eta \rangle - \delta \Theta_{\eta}
\]
This gives the result. \( \diamond \)
Finally, we need a formula for $B(\eta)$. For this, we specialize to the case of holomorphic curves in $S^6$ and recall the complex structure $\hat{J}$ on $N\Sigma$ defined in §3.4.2.

**Proposition 4.3.** Let $u: \Sigma^2 \to S^6$ be a holomorphic curve. Then

$$B(\eta) = \hat{J}R_{12}^\perp\eta$$

(4.4)

so that

$$\langle B(\eta), \eta \rangle = -\langle R_{12}^\perp\eta, \hat{J}\eta \rangle.$$  

(4.5)

**Proof.** Let $(e_1, \ldots, e_6)$ be a $T^2$-adapted frame. By (3.10) and the fact that $T^2$-adapted frames have $\mu = 0$, we have

$$I I(e_1, e_1) = \kappa_1 e_3 + \kappa_2 e_4$$

$$I I(e_1, e_2) = -\kappa_2 e_3 + \kappa_1 e_4.$$  

It follows that

$$B(e_3) = 2|\kappa|^2 e_3$$

$$B(e_4) = 2|\kappa|^2 e_4$$

$$B(e_5) = 0$$

$$B(e_6) = 0.$$  

On the other hand, the Ricci equation (2.2) implies that

$$R_{12}^\perp(e_3) = -2|\kappa|^2 \hat{J}e_3$$

$$R_{12}^\perp(e_4) = -2|\kappa|^2 \hat{J}e_4$$

(4.6)

Therefore, $B(\eta) = \hat{J}R_{12}^\perp\eta$, so that

$$\langle B(\eta), \eta \rangle = \langle \hat{J}R_{12}^\perp\eta, \hat{J}\eta \rangle = -\langle R_{12}^\perp\eta, \hat{J}\eta \rangle.$$  

(4.7)

We now intend to combine Propositions 4.1, 4.2, and 4.3 to arrive at a second variation formula for holomorphic curves in $S^6$. To do this, notice that Proposition 4.2 required the choice of a complex structure $I$ on $N\Sigma$ satisfying $\nabla^\perp I = 0$. In §3.4, we observed that $\hat{J}$ satisfies $\nabla^\perp \hat{J} = 0$ if and only if $u$ is null-torsion (whereas $\tilde{J}$ never has this property). Therefore, restricting to the null-torsion case and taking $I := \hat{J}$ in Proposition 4.2, we deduce:

**Corollary 4.4.** Let $u: \Sigma^2 \to S^6$ be a null-torsion holomorphic curve, where $\Sigma$ is a closed surface. For $\eta \in \Gamma(N\Sigma)$, we have:

$$Q(\eta) = \int_\Sigma \frac{1}{2} \|\hat{\Theta}\|^2 - 2\|\eta\|^2$$

(4.8)

recalling from (3.18) that $\hat{\Theta} := \nabla_{\hat{J}X}\eta - \nabla_X(\hat{J}\eta)$.

**Proof.** Using (4.1) and (4.5), we have

$$\langle L(\eta), \eta \rangle = -\langle \Delta^+\eta, \eta \rangle - \langle B(\eta), \eta \rangle - \langle R(\eta), \eta \rangle$$

$$= -\langle \Delta^+\eta, \eta \rangle + \langle R_{12}^\perp \eta, \hat{J}\eta \rangle - 2\|\eta\|^2.$$  

Next, using (4.2) and (4.3) with the choice $I := \hat{J}$, we have

$$\langle L(\eta), \eta \rangle = \frac{1}{2} \|\hat{\Theta}\|^2 - \langle R_{12}^\perp \eta, \hat{J}\eta \rangle - \text{div}(\Theta^f) + \text{div}(\theta^f) + \langle R_{12}^\perp \eta, \hat{J}\eta \rangle - 2\|\eta\|^2$$

$$= \frac{1}{2} \|\hat{\Theta}\|^2 - \text{div}(\Theta^f) + \text{div}(\theta^f) - 2\|\eta\|^2.$$  

Integrating both sides and using Stokes’ Theorem completes the proof.  

$\blacksquare$
Analogues of the second variation formula (4.8) have been observed in several other contexts. For example, a version of (4.8) was obtained by Simons [33, p. 78] for complex submanifolds of Kähler manifolds, by Ejiri [14, Lemma 3.2] for minimal 2-spheres in $S^{2n}$, by Micallef-Wolfson [29, p. 264] for minimal Lagrangians in negative Kähler-Einstein 4-manifolds, and by Montiel-Urbano [30, p. 2259] for superminimal surfaces in self-dual Einstein 4-manifolds.

4.2 The First Eigenvalue

Let $u: \Sigma^2 \rightarrow S^6$ be a null-torsion holomorphic curve. Let $\eta \in \Gamma(N\Sigma)$ be an eigenvector for $L$, so that $L\eta = \lambda \eta$ for some $\lambda \in \mathbb{R}$. Then

$$\int_{\Sigma} \frac{1}{2} \|D\eta\|^2 - 2\|\eta\|^2 = Q(\eta) = \int_{\Sigma} \lambda \|\eta\|^2.$$ 

Rearranging, we obtain

$$\int_{\Sigma} \frac{1}{2} \|D\eta\|^2 - (\lambda + 2)\|\eta\|^2 = 0.$$ 

Since $\|D\eta\|^2 \geq 0$, we deduce that $\lambda \geq -2$. Moreover, we see that $\lambda = -2$ if and only if $\eta \in \Gamma(N\Sigma)$ satisfies $D\eta = 0$, i.e., if and only if $\eta$ is $(\tilde{J}, \nabla^\perp)$-real holomorphic. Recalling (3.19), this proves:

**Proposition 4.5.** The lowest eigenvalue of the Jacobi operator satisfies $\lambda_1 \geq -2$. The multiplicity $m_1$ of the eigenvalue $\lambda = -2$ is

$$m_1 = \dim \{ \eta \in \Gamma(N\Sigma): \tilde{D}\eta = 0 \} = 2h^0(\tilde{N}^{1,0}; \nabla^\perp).$$

We now use the Riemann-Roch Theorem to derive a lower bound for $m_1$. This lower bound will show, in particular, that $m_1 \geq 4$, so that the lowest eigenvalue of $L$ is always $\lambda_1 = -2$.

**Proposition 4.6.** The multiplicity $m_1$ satisfies

$$m_1 = 4d + 2h^1(\tilde{N}^{1,0}; \nabla^\perp) \geq 4d.$$ 

**Proof.** By the Riemann-Roch Theorem for rank 2 vector bundles, the isomorphism $\tilde{N}^{1,0} \cong \Lambda_N \oplus \Lambda_B^*$ of complex vector bundles, and Proposition 3.3, we have:

$$h^0(\tilde{N}^{1,0}; \nabla^\perp) - h^1(\tilde{N}^{1,0}; \nabla^\perp) = \deg(\tilde{N}^{1,0}) + \chi(\Sigma) = c_1(\Lambda_N) - c_1(\Lambda_B) + \chi(\Sigma) = -\chi(\Sigma) + d + d + \chi(\Sigma) = 2d.$$ 

The result follows. ◊

4.3 The Multiplicity of the First Eigenvalue: Proof of Theorem 1.1

In Proposition 4.6, we obtained a lower bound for $m_1$. Our aim is to prove the following upper bound for $m_1$.

**Proposition 4.7.** Let $u: \Sigma^2 \rightarrow S^6$ be a null-torsion holomorphic curve. Then

$$h^0(\tilde{N}^{1,0}; \nabla^\perp) \leq h^0(\Lambda_N) + h^0(\Lambda_B^*).$$
Accepting this proposition on faith for a moment, we now show how it implies Theorem 1.1. 

Proof. Let $u: \Sigma^2 \to S^6$ be a null-torsion holomorphic curve satisfying $g < \frac{1}{2}(d+2)$, or equivalently, $d > 2g - 2$. Using Proposition 4.5, followed by Proposition 4.7, and finally Proposition 3.4(a) and (b), we have the upper bound 

$$m_1 = 2h^0(\hat{N}^{1,0}; \nabla) \leq 2(h^0(L_N) + h^0(L_B^*)) = 2(2d + h^0(L_B \otimes K_\Sigma)) = 4d.$$ 

Coupled with the lower bound of Proposition 4.6, we deduce that $m_1 = 4d = \frac{A}{\pi}$. $\diamond$ 

### 4.3.1 Proof of Proposition 4.7

Let $u: \Sigma^2 \to S^6$ be a null-torsion holomorphic curve. On the complex vector bundle $\hat{N}^{1,0}$, we recall the two $\overline{\partial}$-operators 

$$\overline{\partial}^\nabla, \overline{\partial}^D: \Gamma(\hat{N}^{1,0}) \to \Omega^{0,1}(\Sigma) \otimes \Gamma(\hat{N}^{1,0}).$$ 

Let $S: \Gamma(\hat{N}^{1,0}) \to \Omega^{0,1}(\Sigma) \otimes \Gamma(\hat{N}^{1,0})$ denote the difference tensor, i.e., 

$$S(\xi) := \overline{\partial}^\nabla \xi - \overline{\partial}^D \xi$$ 

for smooth sections $\xi \in \Gamma(\hat{N}^{1,0})$. Since $S$ is a tensor, it can be viewed as a pointwise operator, i.e., as a bundle map $\hat{N}^{1,0} \to \Lambda^{0,1}\Sigma \otimes \hat{N}^{1,0}$. To understand $S$ in more detail, let $(f_2, f_3)$ be a $T^2$-frame for $\hat{N}^{1,0}$ at a point $p \in \Sigma$. By (3.16) and (3.17), we have 

$$S(f_2) = 0, \quad S(f_3) = \frac{1}{2} \xi_1 \otimes f_2.$$ 

Thus, using the Hermitian vector bundle isomorphism $\hat{N}^{1,0} \simeq L_N \oplus L_B^*$, we can regard $S$ as a map 

$$S: L_N \oplus L_B^* \to \Lambda^{0,1}\Sigma \otimes L_N$$ 

such that $S|_{L_N} = 0$. 

Let $\xi \in \Gamma(\hat{N}^{1,0})$ be a smooth section. Write $\xi = \xi_2 + \xi_3$, where $\xi_2 \in \Gamma(L_N)$ and $\xi_2 \in \Gamma(L_B^*)$. The condition that $\xi$ be $\nabla^\perp$-holomorphic is equivalent to 

$$\overline{\partial}^D \xi = -S(\xi),$$ 

which (by decomposing into $L_N$ and $L_B^*$ components) is in turn is equivalent to the system 

$$\overline{\partial}^D \xi_2 = -S(\xi_3), \quad \overline{\partial}^D \xi_3 = 0.$$ 

Now, by Proposition 3.9(b), there is a holomorphic isomorphism $(\hat{N}^{1,0}, D^\perp) \cong L_N \oplus L_B^*$. Thus, the $\nabla^\perp$-holomorphicity condition can finally be rewritten as 

$$\overline{\partial}^{L_N} \xi_2 = -S(\xi_3) \quad (4.9)$$ 

$$\overline{\partial}^{L_B^*} \xi_3 = 0, \quad (4.10)$$ 

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where $\mathcal{D}^{L_N}$ and $\mathcal{D}^{L_B}$ are the respective $\mathcal{D}$-operators on $L_N$ and $L_B$.

The upshot is that the system (4.9)-(4.10) is decoupled, making it easy to count its solutions. Indeed, the solution space of (4.10) is $H^0(L_B^*)$, a $\mathbb{C}$-vector space of complex dimension $h^0(L_B^*)$. Moreover, for each $\xi_3 \in H^0(L_B)$, the set of solutions to (4.9) is either empty or an affine space of complex dimension $h^0(L_N)$. Geometrically, the set of solutions to (4.9)-(4.10) can be viewed as a bundle of complex $h^0(L_N)$-dimensional affine spaces over the set of $\xi_3 \in H^0(L_B^*)$ for which (4.9) has a solution. In conclusion, the set of $\nabla^\perp$-holomorphic sections has complex dimension at most $h^0(L_N) + h^0(L_B^*)$. This proves Proposition 4.7.

5 The Nullity: Proof of Theorem 1.2

Let $u: \Sigma^2 \to \mathbb{S}^6$ be a null-torsion holomorphic curve of area $A = 4\pi d$. The aim of this section is to prove the following lower bound on the nullity of the Jacobi operator $\mathcal{L}$ of $u$:

$$\text{Nullity}(u) \geq 2d + \chi(\Sigma).$$

The idea of the proof is to identify a subspace of $\text{Null}(u) := \{ \eta \in \Gamma(N\Sigma): \mathcal{L}\eta = 0 \}$ that is isomorphic to $H^0(K^*_\Sigma \otimes L_N)$. The dimension of $H^0(K^*_\Sigma \otimes L_N)$ will then be estimated via Riemann-Roch.

Our method in this section is not original. Indeed, our calculations are direct analogues those in Montiel and Urbano’s study [30] of superminimal surfaces in self-dual Einstein 4-manifolds. To ease notation, we enact the following conventions:

**Convention:** For the remainder of this work, we let $J$ denote the complex structure on $u^*(T\mathbb{S}^6) = T\Sigma \oplus N\Sigma$ that on $T\Sigma$ is given by the metric and orientation, and on $N\Sigma$ is given by $\tilde{J}$. Recall that as complex bundles, we have an isomorphism

$$\hat{N}^{1,0} \simeq (N\Sigma, J) = E_N \oplus E_B^*.$$

As real vector bundles, we have $N\Sigma \simeq E_N \oplus E_B$, and we will use the notation $\eta = \eta^N + \eta^B$ for the decomposition of a normal vector $\eta \in N\Sigma$ into its $E_N$ and $E_B$ components.

**Convention:** On the complex bundle $\hat{N}^{1,0}$, the only holomorphic structure we will need from now on is $\hat{N}^\mathcal{D}$. Thus, we will abbreviate “$\nabla^\perp$-holomorphic” as “holomorphic,” and abbreviate “$(\tilde{J}, \nabla^\perp)$-real holomorphic” as “real-holomorphic.” Noting that holomorphic sections of $K^*_\Sigma \otimes \hat{N}^{1,0}$ are in bijection with real-holomorphic sections of $T^*\Sigma \otimes N\Sigma$, we may identify

$$\Gamma(K^*_\Sigma \otimes L_N) \cong \{ \alpha \in \Gamma(T^*\Sigma \otimes E_N): \alpha \circ J = -J \circ \alpha \}$$

$$H^0(K^*_\Sigma \otimes L_N) \cong \{ \alpha \in \Gamma(T^*\Sigma \otimes E_N): \alpha \circ J = -J \circ \alpha \text{ and } \alpha \text{ real-holomorphic} \}.$$

These identifications will frequently be made without comment.

5.1 Preliminaries

Recall that the Levi-Civita connection $\nabla$ on $\mathbb{S}^6$ gives a tangential connection $\nabla^\perp$ on $T\Sigma$ and a normal connection $\nabla^\perp$ on $N\Sigma$. These give an induced connection $\tilde{\nabla}$ on $T^*\Sigma \otimes N\Sigma$, and an induced connection $\nabla$ on $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$. Explicitly, for $\alpha \in \Gamma(T^*\Sigma \otimes N\Sigma)$, we have

$$(\tilde{\nabla}_Y \alpha)(X) := \nabla^Y_\alpha(X) - \alpha(\nabla^X_\gamma Y), \quad (5.1)$$
and for $\beta \in \Gamma(T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma)$, we have
\[(\hat{\nabla}_Z \beta)(X, Y) := \nabla^\perp_Z (\beta(X, Y)) - \beta(\nabla^\perp_Z X, Y) - \beta(X, \nabla^\perp_Z Y). \tag{5.2}\]

We remark that if $\beta$ is a symmetric 2-tensor (i.e., $\beta(X, Y) = \beta(Y, X)$ for all $X, Y \in T\Sigma$), then $\hat{\nabla}_Z \beta$ is also a symmetric 2-tensor.

For $\alpha \in \Gamma(T^*\Sigma \otimes N\Sigma)$, we recall that its second covariant derivative $\tilde{\nabla}^2_{XY} \alpha \in \Gamma(T^*\Sigma \otimes N\Sigma)$ at $X, Y \in T\Sigma$ is given by:
\[\tilde{\nabla}^2_{XY} \alpha := (\hat{\nabla}_X \tilde{\nabla}_Y \alpha)(\cdot) = \tilde{\nabla}_X \tilde{\nabla}_Y \alpha - \tilde{\nabla}_{\tilde{\nabla}^\perp_{XY}} \alpha. \tag{5.3}\]

We also recall the Ricci identity
\[\tilde{\nabla}^2_{XY} \alpha = \tilde{\nabla}^2_{YX} \alpha + \tilde{\mathcal{R}}_{XY} \alpha \tag{5.4}\]

where $\tilde{\mathcal{R}}$ is the curvature of $\tilde{\nabla}$. A straightforward calculation shows that
\[(\tilde{\mathcal{R}}_{XY} \alpha)(Z) = R^\perp_{XY}(\alpha(Z)) - \alpha(R^\perp_{XY}(Z)). \tag{5.4}\]

We let $(\nabla^\perp)^* : \Gamma(T^*\Sigma \otimes N\Sigma) \to \Gamma(N\Sigma)$ denote the formal adjoint of $\nabla^\perp$, so that
\[\int_{\Sigma} \langle \nabla^\perp, \xi, \alpha \rangle = \int_{\Sigma} \langle \xi, (\nabla^\perp)^* \alpha \rangle \text{ for all } \xi \in \Gamma(N\Sigma), \alpha \in \Gamma(T^*\Sigma \otimes N\Sigma). \]

In terms of a local orthonormal frame $(e_1, e_2)$ on $T\Sigma$, one can compute $(\nabla^\perp)^* \alpha$ via the well-known formula:
\[\nabla^\perp)^* \alpha = -(\tilde{\nabla}_{e_i} \alpha)(e_i). \tag{5.5}\]

Similarly, we let $\tilde{\nabla}^* : \Gamma(T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma) \to \Gamma(T^*\Sigma \otimes N\Sigma)$ denote the formal adjoint of $\tilde{\nabla}$. Again, in terms of a local orthonormal frame $(e_1, e_2)$ on $T\Sigma$, one has the formula
\[\tilde{\nabla}^* \beta = -(\tilde{\nabla}_{e_i} \beta)(e_i, \cdot). \tag{5.6}\]

### 5.2 Strategy of Proof

For a fixed $\eta \in \Gamma(N\Sigma)$, we consider the section $\Psi_\eta \in \Gamma(T^*\Sigma \otimes N\Sigma)$ given by
\[\Psi_\eta(X) := \hat{\nabla}_X \eta = \nabla^\perp_{JX} \eta - J(\nabla^\perp_{\tilde{J}X} \eta). \]

The basic properties of $\Psi_\eta$ are given by the following two lemmas. Verifying Lemma 5.1 is straightforward; we will prove only Lemma 5.2.

**Lemma 5.1.** We have:
(a) $\eta$ is real-holomorphic $\iff$ $\Psi_\eta = 0$.
(b) $\Psi_\eta \circ J = -J \circ \Psi_\eta$.
(c) We have
\[(\tilde{\nabla}_{\tilde{\nabla}^\perp_\eta}(JX) = -J(\tilde{\nabla}_{\tilde{\nabla}^\perp_\eta}X). \]

**Lemma 5.2.** We have:
\[(\nabla^\perp)^* \Psi_\eta = -J(\mathcal{L}\eta + 2\eta). \]
Proof. Using (5.5) and (5.1), we calculate
\[
(\nabla^\perp)^*\Psi_\eta = -(\nabla e_i^\perp \Psi_\eta)(e_i) = -\nabla e_i^\perp (\Psi_\eta(e_i)) + \Psi_\eta(\nabla e_i^\perp)
= -\nabla e_i^\perp \nabla \eta + J\nabla e_i^\perp \eta + \nabla J(\nabla e_i^\perp) \eta - J(\nabla e_i^\perp) \eta
= -\nabla e_i^\perp \nabla \eta + \nabla J(\nabla e_i^\perp) \eta + J\Delta \eta.
\]

Now, using that \(\nabla e_i^\perp \nabla = R_{12} + \nabla_{[e_1,e_2]}\) and that \(J(\nabla e_i^\perp) = [e_1,e_2]\), we obtain
\[
(\nabla^\perp)^*\Psi_\eta = -R_{12}^\perp + J\Delta \eta
= -R_{12}^\perp + J(-\mathcal{L} \eta - 2\eta) - J(\mathcal{B} \eta)
\]
where we used that \(\Delta \eta = -\mathcal{L} \eta - \mathcal{B} \eta - 2\eta\). Finally, using (4.4), we arrive at the result. \(\diamond\)

Let \(\text{Null}(u) = \{\eta \in \Gamma(N\Sigma) : \mathcal{L} \eta = 0\}\) denote the null space of the Jacobi operator \(\mathcal{L}\). Consider the linear map
\[
G: \{\eta \in \text{Null}(u) : (\Psi_\eta)^{\mathcal{B}} = 0\} \rightarrow \Gamma(K^*_\Sigma \otimes L_N)
\eta \mapsto \hat{\Theta} \eta = \Psi_\eta.
\]
Observe that \(G\) is injective. Indeed, if \(G(\eta) = 0\), then \(\Psi_\eta = 0\), so \(\eta\) is real-holomorphic, so \(\mathcal{L} \eta = -2\eta\), but \(\mathcal{L} \eta = 0\), so \(\eta = 0\). Our main claim in this section concerns the image of \(G\):

**Proposition 5.3.** The image of \(G\) is equal to \(H^0(K^*_\Sigma \otimes L_N)\).

Accepting Proposition 5.3 on faith for a moment, we see that \(G\) gives an isomorphism
\[
\{\eta \in \text{Null}(u) : (\Psi_\eta)^{\mathcal{B}} = 0\} \cong H^0(K^*_\Sigma \otimes L_N).
\]
From this isomorphism, we now deduce Theorem 1.2:

**Proof.** Recall from Proposition 3.3 that \(c_1(L_N) = -\chi(\Sigma) + d\) and \(c_1(K^*_\Sigma) = \chi(\Sigma)\), and hence \(c_1(K^*_\Sigma \otimes L_N) = d\). By Proposition 5.3, we now estimate
\[
\text{Nullity}(u) \geq \dim_{\mathbb{R}} \{\eta \in \text{Null}(u) : (\Psi_\eta)^{\mathcal{B}} = 0\} = \dim_{\mathbb{R}}[H^0(K^*_\Sigma \otimes L_N)]
= 2h^0(K^*_\Sigma \otimes L_N)
= 2h^1(K^*_\Sigma \otimes L_N) + 2c_1(K^*_\Sigma \otimes L_N) + \chi(\Sigma)
= 2h^1(K^*_\Sigma \otimes L_N) + 2d + \chi(\Sigma),
\]
where we used Riemann-Roch in the second-to-last step. Finally, using \(h^1(K^*_\Sigma \otimes L_N) \geq 0\), we conclude the result. \(\diamond\)

**Remark.** The estimate \(h^1(K^*_\Sigma \otimes L_N) \geq 0\) can be slightly sharpened. Indeed, by Serre Duality, we have \(h^1(K^*_\Sigma \otimes L_N) = h^0(K^*_\Sigma \otimes K^*_\Sigma \otimes L^*_N)\), and we compute \(c_1(K^*_\Sigma \otimes K^*_\Sigma \otimes L^*_N) = -\chi(\Sigma) - d\). Therefore, if \(d > 2g - 2\) — which, by Proposition 3.6 holds if \(g \leq 6\) — then \(K^*_\Sigma \otimes K^*_\Sigma \otimes L^*_N\) is a negative line bundle, and hence \(h^1(K^*_\Sigma \otimes L_N) = 0\).

The remainder of this section consists of a proof of Proposition 5.3, which naturally divides into two halves. That is, in Proposition 5.7, we will show that \(\text{Im}(G) \subset H^0(K^*_\Sigma \otimes L_N)\), and in Proposition 5.9, we will show that \(H^0(K^*_\Sigma \otimes L_N) \subset \text{Im}(G)\).
5.3 Technical Lemmas

We begin by measuring the extent to which the smooth section \( \Psi_\eta \in \Gamma(T^*\Sigma \otimes N\Sigma) \) might fail to be real-holomorphic. So, for a fixed \( \eta \in \Gamma(N\Sigma) \), we consider the section \( \Omega_\eta \in \Gamma(T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma) \) given by

\[
\Omega_\eta(X, Y) := (\bar{\nabla}_X \Psi_\eta)(JY) - (\bar{\nabla}_X \Psi_\eta)(Y).
\]

We now establish the basic properties of \( \Omega_\eta \) by analogy with Lemmas 5.1 and 5.2. The analogue of Lemma 5.1 is easy:

**Lemma 5.4.** We have:

(a) \( \Psi_\eta \) is real-holomorphic \( \iff \Omega_\eta = 0 \).

(b) \( \Omega_\eta(JX,JY) = -\Omega_\eta(X,Y) \). Therefore, \( \Omega_\eta \) is an \( N\Sigma \)-valued symmetric 2-tensor on \( T\Sigma \) of trace zero.

(c) We have the identity:

\[
(\bar{\nabla}_{e_i} \Omega_\eta)(v,e_i) = (\bar{\nabla}_{e_i,v} \Psi_\eta)(Je_i) - (\bar{\nabla}_{e_i,v} \Psi_\eta)(e_i).
\]

**Proof.** (a) This is straightforward and left to the reader.

(b) Directly from the definition of \( \Omega_\eta \), we have

\[
\Omega_\eta(JX,JY) = -\Omega_\eta(X,Y).
\]

Thus, letting \((e_1,e_2)\) denote an oriented orthonormal frame on \( T\Sigma \), we have both \( \Omega_\eta(e_2,e_2) = -\Omega_\eta(e_1,e_1) \) and \( \Omega_\eta(e_1,e_2) = \Omega_\eta(e_2,e_1) \), so \( \Omega_\eta \) is an \( N\Sigma \)-valued symmetric 2-tensor of trace zero.

(c) Using (5.1) and the fact that \( \nabla^T_X(JY) = J\nabla^T_X Y \) for all \( X,Y \), the first term on the right is

\[
(\bar{\nabla}_{e_i,v} \Psi_\eta)(Je_i) = (\bar{\nabla}_{e_i} \bar{\nabla}_{Jv} \Psi_\eta)(Je_i) - (\bar{\nabla}_{\nabla^T_{e_i} Jv} \Psi_\eta)(Je_i) = \nabla^T_{e_i} ((\bar{\nabla}_{Jv} \Psi_\eta)(Je_i)) - (\bar{\nabla}_{Jv} \Psi_\eta)(J\nabla^T_{e_i} e_i) - (\bar{\nabla}_{\nabla^T_{e_i} Jv} \Psi_\eta)(Je_i),
\]

and similarly, the second term on the right is

\[
(\bar{\nabla}_{e_i,v} \Psi_\eta)(e_i) = (\bar{\nabla}_{e_i} \bar{\nabla}_{v} \Psi_\eta)(e_i) - (\bar{\nabla}_{\nabla^T_{e_i} v} \Psi_\eta)(e_i) = \nabla^T_{e_i} ((\bar{\nabla}_{v} \Psi_\eta)(e_i)) - (\bar{\nabla}_{v} \Psi_\eta)(\nabla^T_{e_i} e_i) - (\bar{\nabla}_{\nabla^T_{e_i} v} \Psi_\eta)(e_i).
\]

On the other hand, using (5.2) and the definition of \( \Omega_\eta \), the left side of the desired identity is:

\[
(\bar{\nabla}_{e_i} \Omega_\eta)(v,e_i) = \nabla^T_{e_i} ((\bar{\nabla}_{v} \Psi_\eta)(e_i)) - (\bar{\nabla}_{v} \Psi_\eta)((\bar{\nabla}_{e_i} \Psi_\eta)(e_i)) - (\bar{\nabla}_{\nabla^T_{e_i} v} \Psi_\eta)(Je_i) + (\bar{\nabla}_{\nabla^T_{e_i} v} \Psi_\eta)(e_i).
\]

Comparing terms proves the lemma. \( \diamond \)

The identities in the following lemma are straightforward to prove, but rather tedious. To streamline discussion, their verifications are deferred to the Appendix.

**Lemma 5.5.** Let \( \alpha \in \Gamma(T^*\Sigma \otimes N\Sigma) \) satisfy \( \alpha \circ J = -J \circ \alpha \). Let \((e_1,e_2)\) be a local oriented orthonormal frame on \( \Sigma \). Then for all \( v \in T\Sigma \):

\[
(\bar{\nabla}_{e_i,v}^2 \alpha)(Je_i) - (\bar{\nabla}_{e_i,v} \alpha)(e_i) = (\bar{\nabla}_{Jv,e_i} \alpha)(Je_i) - (\bar{\nabla}_{v,e_i} \alpha)(e_i) - 2\alpha(v) - (2K - 2)[\alpha(v)]B. \tag{5.7}
\]

Moreover, if \((e_1,e_2)\) is geodesic at \( p \in \Sigma \), then at the point \( p \):

\[
J(\bar{\nabla}_{v,e_i} \alpha)(v) = (\bar{\nabla}_{Jv,e_i} \alpha)(Je_i) - (\bar{\nabla}_{v,e_i} \alpha)(e_i). \tag{5.8}
\]
Using these identities, we can now give the analogue of Lemma 5.2:

**Lemma 5.6.** We have:

\[ \tilde{\nabla}^* \Omega = -\nabla_{L} + (2K - 2)(\Psi)^B. \]

**Proof.** Let \( v \in T\Sigma \). Using (5.6), followed by the symmetry \((\tilde{\nabla}_{Z} \Omega)(X,Y) = (\tilde{\nabla}_{Z} \Omega)(Y,X)\), followed by Lemma 5.4(c), followed by (5.7), we get:

\[
(\tilde{\nabla}^* \Omega)(v) = - (\tilde{\nabla}_{e_i} \Omega)(e_i, v) = -(\tilde{\nabla}_{e_i} \Omega)(v, e_i) = -[(\tilde{\nabla}^2_{e_i,Je_i} \Psi)(Je_i) - (\tilde{\nabla}^2_{e_i,v} \Psi)(e_i)] = -[(\tilde{\nabla}_{Jv,e_i} \Psi)(Je_i) - (\tilde{\nabla}_{v,e_i} \Psi)(e_i) - 2\Psi(v) - (2K - 2)[\Psi(v)]^B].
\]

Choose the local frame \((e_1, e_2)\) to be geodesic at \( p \in \Sigma \). By (5.8), at the point \( p \in \Sigma \), we have:

\[
J\Psi(\tilde{\nabla}^* \Psi)(v) = (\tilde{\nabla}^2_{Jv,e_i} \Psi)(Je_i) - (\tilde{\nabla}^2_{v,e_i} \Psi)(e_i)
\]

Using this, together with Lemma 5.2, we conclude that

\[
(\tilde{\nabla}^* \Omega)(v) = -J\Psi(\tilde{\nabla}^* \Psi)(v) + 2\Psi(v) + (2K - 2)[\Psi(v)]^B = -J\Psi - J\nabla_{\L} \eta + 2\Psi(v) + (2K - 2)[\Psi(v)]^B = -\nabla_{\L} \eta(v) + (2K - 2)[\Psi(v)]^B
\]

which is the result. \( \diamond \)

### 5.4 Proof of Proposition 5.3

We now prove that \( \text{Im}(G) \subset H^0(K_{\Sigma}^* \otimes L_N) \), which is half of Proposition 5.3. More precisely:

**Proposition 5.7.** Let \( \eta \in \Gamma(N\Sigma) \). We have

\[
\int_{\Sigma} ||\Omega\eta||^2 = 2 \int_{\Sigma} \langle \Psi_{B}, \nabla_{\L} \eta \rangle + (2 - 2K)(\Psi)^B \rangle.
\]

In particular, if \( \L \eta = 0 \) and \((\Psi_{\eta})^B = 0\), then \( \Psi_{\eta} \) is real-holomorphic.

**Proof.** First, we use the symmetries of \( \Omega_{\eta} \) given by Lemma 5.4(b) to observe that

\[
\langle (\tilde{\nabla}_{J_{e_i} \Psi_{\eta}})(Je_j), \Omega_{\eta}(e_i, e_j) \rangle = \langle (\tilde{\nabla}_{e_2} \Psi_{\eta})(e_2), \Omega_{\eta}(e_1, e_1) \rangle - \langle (\tilde{\nabla}_{e_2} \Psi_{\eta})(e_1), \Omega_{\eta}(e_1, e_2) \rangle - \langle (\tilde{\nabla}_{e_1} \Psi_{\eta})(e_2), \Omega_{\eta}(e_2, e_1) \rangle + \langle (\tilde{\nabla}_{e_1} \Psi_{\eta})(e_1), \Omega_{\eta}(e_2, e_2) \rangle = -\langle (\tilde{\nabla}_{e_2} \Psi_{\eta})(e_2), \Omega_{\eta}(e_2, e_2) \rangle - \langle (\tilde{\nabla}_{e_2} \Psi_{\eta})(e_1), \Omega_{\eta}(e_2, e_1) \rangle - \langle (\tilde{\nabla}_{e_1} \Psi_{\eta})(e_2), \Omega_{\eta}(e_1, e_2) \rangle - \langle (\tilde{\nabla}_{e_1} \Psi_{\eta})(e_1), \Omega_{\eta}(e_1, e_1) \rangle = -\langle (\tilde{\nabla}_{e_1} \Psi_{\eta})(e_j), \Omega_{\eta}(e_i, e_j) \rangle.
\]
Using this fact, we can calculate

\[ \|\Omega_\eta\|^2 = \langle \Omega_\eta(e_i, e_j), \Omega_\eta(e_i, e_j) \rangle \]

\[ = \left\langle \left( \tilde{\nabla}_e i \Psi_\eta \right)(J e_j), \Omega_\eta(e_i, e_j) \right\rangle - \left\langle \left( \tilde{\nabla}_e i \Psi_\eta \right)(e_j), \Omega_\eta(e_i, e_j) \right\rangle \]

\[ = -2 \left\langle \left( \tilde{\nabla}_e i \Psi_\eta \right)(e_j), \Omega_\eta(e_i, e_j) \right\rangle \]

and hence

\[ \int_\Sigma \|\Omega_\eta\|^2 = -2 \int_\Sigma \left\langle \left( \tilde{\nabla}_e i \Psi_\eta \right)(e_j), \Omega_\eta(e_i, e_j) \right\rangle = -2 \int_\Sigma \left\langle \Psi_\eta(e_j), (\tilde{\nabla}^* \Omega_\eta)(e_j) \right\rangle \]

\[ = 2 \int_\Sigma \left\langle \Psi_\eta, \Psi \cdot \mathcal{L}\eta + (2 - 2K)(\Psi_\eta)^B \right\rangle, \]

where we used Lemma 5.6 in the last step. Finally, note that if \( \mathcal{L}\eta = 0 \) and \( (\Psi_\eta)^B = 0 \) both hold, then \( \int_\Sigma \|\Omega_\eta\|^2 = 0 \), so that \( \Omega_\eta = 0 \), so that \( \Psi_\eta \) is real-holomorphic.

\[ \diamond \]

We now make a brief digression. In general, if \( \mathcal{L}\eta = \lambda \eta \), then Proposition 5.7 shows that:

\[ 0 \leq \frac{1}{2} \int_\Sigma \|\Omega_\eta\|^2 = \int_\Sigma \left\langle \Psi_\eta, \lambda \Psi_\eta + (2 - 2K)(\Psi_\eta)^B \right\rangle \]

\[ = \lambda \int_\Sigma \|\Psi_\eta\|^2 + \int_\Sigma (2 - 2K) \left\langle \Psi_\eta, (\Psi_\eta)^B \right\rangle \]

\[ = \lambda \int_\Sigma \|\Psi_\eta\|^2 + \int_\Sigma (2 - 2K) \| (\Psi_\eta)^B \|^2. \]

This estimate gives:

**Proposition 5.8.** Let \( u: \mathbb{S}^2 \to \mathbb{S}^6 \) be a holomorphic 2-sphere. If its Gauss curvature \( K \) satisfies \( K \geq c > 0 \), then

\[ \lambda_2 \geq -2 + 2c. \]

In particular, the Jacobi operator of the Boruvka sphere satisfies \( \lambda_2 \geq -\frac{5}{3} \).

**Proof.** Suppose \( \eta \in \Gamma(N\Sigma) \) satisfies \( \mathcal{L}\eta = \lambda \eta \) with \( \lambda > \lambda_1 = -2 \). We estimate

\[ 0 \leq \frac{1}{2} \int_\Sigma \|\Omega_\eta\|^2 = \lambda \int_\Sigma \|\Psi_\eta\|^2 + \int_\Sigma (2 - 2K) \| (\Psi_\eta)^B \|^2 \]

\[ \leq \lambda \int_\Sigma \|\Psi_\eta\|^2 + (2 - 2c) \int_\Sigma \| (\Psi_\eta)^B \|^2 \]

\[ \leq (\lambda + 2 - 2c) \int_\Sigma \|\Psi_\eta\|^2. \]

If it were the case that \( \int_\Sigma \|\Psi_\eta\|^2 = 0 \), then \( \Psi_\eta = 0 \), so \( \eta \) would be real-holomorphic and \( \lambda = -2 \), contrary to assumption. Thus, we must have \( \int_\Sigma \|\Psi_\eta\|^2 > 0 \), so \( \lambda + 2 - 2c \geq 0 \), whence the result. \[ \diamond \]

**Remark.** It is proved in \([12]\) that a holomorphic curve \( u: \mathbb{S}^2 \to \mathbb{S}^6 \) satisfying \( K \geq \frac{1}{6} \) must have either \( K \equiv \frac{1}{6} \) or \( K \equiv 1 \), hence must be either the Boruvka sphere or the totally-geodesic 2-sphere. Hence, any non-constant curvature example satisfying the hypothesis of Proposition 5.8 must have \( c \in \left(0, \frac{1}{6}\right) \). I do not know any examples of this type. We remark that it is also known \([13]\) that the pinching condition \( 0 \leq K \leq \frac{1}{6} \) implies \( K \equiv 0 \) or \( K \equiv \frac{1}{6} \). See also \([18]\) for further results.
Returning to the main discussion, we now show that \( H^0(K^*_{\Sigma} \otimes L_N) \subset \text{Im}(G) \), thereby completing the proof of Proposition 5.3, and hence of Theorem 1.2.

**Proposition 5.9.** If \( \alpha \in H^0(K^*_{\Sigma} \otimes L_N) \), then \( \alpha = \Psi_\eta \) for some \( \eta \in \text{Null}(u) \) with \( (\Psi_\eta)^B = 0 \).

**Proof.** Let \( \alpha \in H^0(K^*_{\Sigma} \otimes L_N) \), and recall the identification

\[
H^0(K^*_{\Sigma} \otimes L_N) \cong \{ \alpha \in \Gamma(T^*\Sigma \otimes E_N) : \alpha \circ J = -J \circ \alpha \text{ and } \alpha \text{ real-holomorphic} \}.
\]

Let \( \xi = \frac{1}{2} J(\nabla^\perp)^* \alpha \). Let \( (e_1, e_2) \) be a geodesic frame at \( p \in \Sigma \). Then at \( p \), we have, by (5.8) and (5.7):

\[
\Psi_\xi(v) = \frac{1}{2} J \Psi(\nabla^\perp)^* \alpha(v) = \frac{1}{2} \left[ (\nabla_{Jv,e_i}^2 \alpha)(Je_i) - (\nabla_{v,e_i}^2 \alpha)(e_i) \right]
\]

\[
= \frac{1}{2} \left[ (\nabla_{e_i, Jv}^2 \alpha)(Je_i) - (\nabla_{e_i, v}^2 \alpha)(e_i) \right] + 2\alpha(v) + (2K - 2)[\alpha(v)]^B
\]

where in the last step we used that \( [\alpha(v)]^B = 0 \). Finally, using that \( \alpha \circ J = -J \circ \alpha \) and that \( \alpha \) is real-holomorphic, we have \( (\nabla_{e_i, Jv}^2 \alpha)(Je_i) = (\nabla_{e_i, v}^2 \alpha)(e_i) \) at \( p \in \Sigma \), whence

\[
\Psi_\xi = \alpha.
\]

Now, since \( \alpha \) is real-holomorphic, it follows that \( \Psi_\xi \) is real-holomorphic, so \( \Omega_\xi = 0 \), so by Lemma 5.6, we have \( \Psi_{L_\xi} = 0 \), so that \( L_\xi \) is real-holomorphic. Therefore, \( L(L_\xi) = -2L_\xi \), so that \( L(L_\xi + 2\xi) = 0 \), and hence \( L_\xi + 2\xi = 2\eta \) for some \( \eta \in \text{Null}(u) \). Therefore,

\[
\Psi_2\eta = \Psi_{L_\xi + 2\xi} = \Psi_{L_\xi} + 2\Psi_\xi = 2\Psi_\xi = 2\alpha,
\]

whence \( \alpha = \Psi_\eta \) for an \( \eta \in \text{Null}(u) \) with \( (\Psi_\eta)^B = (\Psi_\xi)^B = 0 \). \( \Box \)

6 Appendix: Proof of Lemma 5.5

The purpose of this appendix is to prove Lemma 5.5, which we restate as Lemma 6.2. Throughout, we fix \( v \in \Gamma(T\Sigma) \), \( \eta \in \Gamma(N\Sigma) \), and a local oriented orthonormal frame \( (e_1, e_2) \) on \( \Sigma \).

**Lemma 6.1.** Let \( \alpha \in \Gamma(T^*\Sigma \otimes N\Sigma) \).

(a) We have:

\[
R^\perp_{12}(\eta) = (K - 1)J(\eta^N).
\]

(b) We have:

\[
R^\perp(e_i, Jv)\alpha(Ke_i) - R^\perp(e_i, v)\alpha(e_i) = (2K - 2)[\alpha(v)]^N
\]

\[
\alpha(R^\perp(e_i, v)e_i) - \alpha(R^\perp(e_i, Jv)e_i) = -2K\alpha(v).
\]

**Proof.** (a) Equations (4.6)-(4.7) followed by the Gauss equation (2.1) and the fact that \( J(\eta^N) = (J\eta)^N \) give

\[
R^\perp_{12}(\eta) = (K - 1)(J\eta)^N = (K - 1)J(\eta^N).
\]

(b) An easy calculation shows that

\[
R^\perp(e_i, Jv)\alpha(Ke_i) = R^\perp_{12}[\alpha(Jv)]
\]

\[
R^\perp(e_i, v)\alpha(e_i) = -R^\perp_{12}[\alpha(Jv)]
\]

\[
R^\perp(e_i, Jv)e_i = -R^\perp_{12}(Jv) = -Kv
\]

\[
R^\perp(e_i, v)e_i = -R^\perp_{12}(Jv) = -Kv
\]

The equations on the left, together with (a), give (6.1). The equations on the right give (6.2). \( \Box \)
Lemma 6.2. Let $\alpha \in \Gamma(T^*\Sigma \otimes N\Sigma)$ satisfy $\alpha \circ J = -J \circ \alpha$.

(a) We have:

\[
(\bar{\nabla}_{e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{e_i}^2 \alpha)(e_i) = (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) - 2\alpha(v) - (2K - 2)[\alpha(v)]^B.
\]

(b) If $(e_1, e_2)$ is geodesic at $p \in \Sigma$, then at the point $p$:

\[
J\Psi_{(\bar{\nabla}_+)^* \alpha}(v) = (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i).
\]

Proof. (a) Let $L$ denote the left side of the desired identity. Using the Ricci identity (5.3), followed by the formula (5.4), we have

\[
L = (\bar{\nabla}_{e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{e_i}^2 \alpha)(e_i) = (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) + (R(e_i, Jv)\alpha)(Je_i) - (R(e_i, v)\alpha)(e_i)
\]

\[
= (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) + R^2(e_i, Jv)\alpha(\alpha) - R^2(e_i, v)\alpha(e_i) + \alpha(R^2(e_i, v)\alpha(e_i) - \alpha(R^2(e_i, Jv)\alpha(\alpha))
\]

Now, using equations (6.1) and (6.2), we obtain:

\[
L = (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) + (2K - 2)[\alpha(v)]^N - 2K\alpha(v)
\]

\[
= (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) + (2K - 2)[\alpha(v)]^N - 2K\alpha(v)
\]

This proves (a).

(b) Let $(e_1, e_2)$ be a geodesic frame at $p \in \Sigma$. Then at the point $p$, we have that

\[
(\bar{\nabla}_{e_i}^2 \alpha)(Je_i) = \nabla_{e_i}^2(\alpha(\alpha)) = -J[\nabla_{e_i}^2(\alpha(\alpha))] = -J[\nabla_{e_i}^2(\alpha)(e_i)].
\]

Using this and recalling (5.5), we compute

\[
(\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},e_i}^2 \alpha)(e_i) = (\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha)(Je_i) - (\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha)(e_i)
\]

\[
= -J(\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha)(e_i) - \nabla_{\bar{v}}^2(\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha)(e_i)
\]

\[
= -J\nabla_{\bar{v}}^2[\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha](e_i) - \nabla_{\bar{v}}^2[\bar{\nabla}_{\bar{v},\bar{v}_{e_i}}^2 \alpha](e_i)
\]

\[
= J\nabla_{\bar{v}}^2((\nabla^2)^* \alpha) + \nabla_{\bar{v}}^2((\nabla^2)^* \alpha)
\]

which proves the claim. \(\Box\)

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