Parity proofs of the Kochen-Specker theorem based on the Lie algebra E8

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Abstract The 240 root vectors of the Lie algebra E8 lead to a system of 120 rays in a real 8-dimensional Hilbert space that contains a large number of parity proofs of the Kochen-Specker theorem. After introducing the rays in a triacontagonal representation due to Coxeter, we present their Kochen-Specker diagram in the form of a “basis table” showing all 2025 bases (i.e., sets of eight mutually orthogonal rays) formed by the rays. Only a few of the bases are actually listed, but simple rules are given, based on the symmetries of E8, for obtaining all the other bases from the ones shown. The basis table is an object of great interest because all the parity proofs of E8 can be exhibited as subsets of it. We show how the triacontagonal representation of E8 facilitates the identification of substructures that are more easily searched for their parity proofs. We have found hundreds of different types of parity proofs, ranging from 9 bases (or contexts) at the low end to 35 bases at the high end, involving projectors of various ranks and multiplicities. After giving an overview of the proofs we found, we present a few concrete examples of the proofs that illustrate both their generic features as well as some of their more unusual properties. In particular, we present a proof involving 34 rays and 9 bases that appears to provide the most compact proof of the KS theorem found to date in 8 dimensions.

1 Introduction

The exceptional Lie algebra E8 plays a role in a number of physical theories such as supergravity and heterotic string theory[1]. Here we show that its system of root vectors can be used to exhibit a large number of “parity proofs” of the Kochen-Specker (KS) theorem[2] ruling out the existence

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of noncontextual hidden variables theories as viable alternatives to quantum mechanics. The fact that the root vectors of E8 could be made to serve this end was pointed out, in different ways, by Lisonek et al.\cite{3} and by Runge and van Oystaeyen\cite{4}. However the proof in \cite{4} is unrelated to parity proofs, while \cite{3}, though proving that $2^{1940}$ parity proofs exist, does not list even a single example of such a proof. The purpose of this paper is to supplement the observations in \cite{3} and \cite{4} by (a) presenting a general framework (namely, the “basis table”) within which the parity proofs can be exhibited, (b) showing how the symmetries of E8 can be exploited to simplify the search for its parity proofs, (c) providing an overview of the parity proofs found by our search and, finally, (d) presenting a few concrete examples of the proofs in order to convey some feeling for their variety and intricacy.

Parity proofs of the KS theorem are appealing because they take no more than simple counting to verify. What a parity proof is, and how it accomplishes its goals, are matters that will be explained later in this paper.

We make a few remarks about E8, to shed light on the way it is used in this paper. For us E8 is simply a collection of 240 vectors (namely, its roots) in a real 8-dimensional Euclidean space. These vectors define the vertices of a semiregular polytope discovered by Thorold Gosset\cite{5} and sometimes described by the symbol $4_{21}$. The vectors have eight real coordinates that can be chosen in a variety of ways. A particularly judicious choice, for our purposes, is the “triacontagonal representation” of Coxeter and Shepard\cite{6}. In this representation the coordinates of the vectors are chosen in such a way that if only the first two coordinates are retained (which amounts to projecting the vectors orthogonally from eight dimensions down to two), the tips of the vectors lie at the vertices of eight regular triacontagons lying on concentric circles. Such a projection is shown in Fig.10 of \cite{6}. The eight rings of thirty points are easily picked out in the figure, while the dense network of lines connecting pairs of points are projections of the 6720 edges of Gosset’s polytope.

Two slight modifications of this figure will convert it into the Kochen-Specker diagram of E8: the first is that only one member of each pair of diametrically opposite vertices should be retained, since one is concerned with rays rather than vectors in a proof of the KS theorem; and the second is that the line segments corresponding to the edges of Gosset’s polytope should be replaced by new segments that connect only pairs of vertices that correspond to orthogonal rays. Rather than construct such a diagram, we later present a “basis table” that conveys essentially the same information in a more useful form. The basis table is simply a listing of all the bases (i.e., sets of eight mutually orthogonal rays) in the system. Because there are 2025 bases, and this is too large a number to display explicitly, we list just a few bases and give a simple set of rules (based on the symmetries of E8) for generating all the bases from the ones shown. It should be pointed out that the entire basis table can in fact be generated from any one of its elements by applying products of all
possible powers of three basic symmetry operations of E\textsubscript{8}. The basis table is of central importance in this paper because all the parity proofs of E\textsubscript{8} can be exhibited as subsets of it.

We say a few words about the broader area in which this work is set, to provide some perspective. After their initial discovery in two-qubit systems\textsuperscript{[8,9,10,11,12]}, parity proofs were discovered in three-\textsuperscript{[13,14]} and higher-qubit\textsuperscript{[15,16]} systems, in systems of rays derived from the four-dimensional regular polytopes\textsuperscript{[17,18,19]} and the root systems of exceptional Lie algebras\textsuperscript{[20]} and, very recently, in a remarkably compact six-dimensional system of complex rays\textsuperscript{[21]} in connection with which an experiment has also been reported\textsuperscript{[22]}. Aside from revealing the many guises in which quantum contextuality can arise in spaces of different dimensionality, parity proofs are interesting because they have a variety of applications: they can be used to derive state-independent inequalities for ruling out noncontextuality\textsuperscript{[23,25,26,28]} and Bell inequalities for identifying fully nonlocal correlations\textsuperscript{[29]}; they have applications to quantum games\textsuperscript{[30]}, quantum zero-error communication\textsuperscript{[31]}, quantum error correction\textsuperscript{[32,33]} and the design of relational databases\textsuperscript{[34]}; they can be used to witness the dimension of quantum systems\textsuperscript{[35]}; and they underlie surprising phenomena such as the quantum pigeonhole effect\textsuperscript{[36,37,38]}. Although the KS theorem is theoretically compelling, it has been argued\textsuperscript{[39,40,41]} that the finite precision of real measurements nullifies practical attempts at verifying it. There is some debate about this matter\textsuperscript{[42]}, but it should be mentioned that methods of establishing contextuality that are not open to this objection have been proposed\textsuperscript{[43,44,45,46]}. Spekkens\textsuperscript{[47]} has recently expanded upon the conditions that must be satisfied by realistic experiments that claim to rule out noncontextual ontological models. It has been argued in\textsuperscript{[48]} that contextuality is the source of the speedup in many quantum information protocols. This brief survey is far from complete, but it serves to show that the KS theorem and quantum contextuality are at the heart of many current research efforts.

The plan of this paper is as follows. Section\textsuperscript{[2]} introduces the triacontagonal representation of the 120 rays derived from the root vectors of E\textsubscript{8}, and shows how their symmetries can be exploited to give an efficient construction of their basis table. Section\textsuperscript{[3]} points out some interesting substructures within E\textsubscript{8} that have been shown in the past to give rise to proofs of the KS theorem. Section\textsuperscript{[4]} reviews the notion of a parity proof and shows how the triacontagonal representation of E\textsubscript{8} facilitates the identification of subsets of its bases, of distinct symmetry types, that each house a large number of parity proofs. The significance of these smaller subsets is that they are far more easily searched for parity proofs than the entire system. After giving an overview of the parity proofs we found among the different subsets of bases, we present a few examples of the parity proofs that illustrate their important features. Section\textsuperscript{[5]} concludes with a discussion of our results.
2 The E8 system: rays and bases

The 240 root vectors of E8 come in 120 pairs, with the members of each pair being the negatives of each other. Choosing just one member from each pair yields the 120 rays associated with E8. Each ray has eight real coordinates, which may be chosen in a variety of ways. We use a coordinatization due to Richter\[7\], which differs from the one introduced earlier by Coxeter and Shepard\[6\]. Let \( \omega = \exp(\frac{i\pi}{30}) \) and let \( \tau = \frac{1 + \sqrt{5}}{2} \) be the golden ratio. Define \( a, b, c \) and \( d \) as the positive numbers satisfying the equations

\[
2a^2 = 1 + 3^{-1/2}5^{-1/4}\tau^{-3/2}, \quad 2b^2 = 1 + 3^{-1/2}5^{-1/4}\tau^{-3/2}, \quad 2c^2 = 1 - 3^{-1/2}5^{-1/4}\tau^{-3/2}.
\]

For any integer \( n \), let \( c_n = \omega^n + \omega^{-n} = 2\cos(\frac{2\pi n}{30}) \) and define the quantities

\[
r_1 = a/c, \quad r_2 = b/c, \quad r_3 = c/c, \quad r_4 = d/c, \quad r_5 = a/c, \quad r_6 = b/c, \quad r_7 = c/c, \quad r_8 = d/c.
\]

The 120 rays \( |i\rangle \) \((i = 1, \cdots, 120)\) are then defined as

\[
\begin{align*}
|n + 1\rangle &= (r_1\omega^{2n}, r_4\omega^{22n}, r_6\omega^{14n+1}, r_7\omega^{26n+1}) \quad \text{for} \ 0 \leq n \leq 14 \\
|n + 16\rangle &= (r_4\omega^{2n}, -r_1\omega^{22n}, r_7\omega^{14n+1}, -r_6\omega^{26n+1}) \quad \text{for} \ 0 \leq n \leq 14 \\
|n + 23\rangle &= (r_7\omega^{2n}, -r_6\omega^{19+22n}, -r_1\omega^{24+14n}, r_4\omega^{18+26n}) \quad \text{for} \ 8 \leq n \leq 14 \\
|n + 38\rangle &= (r_4\omega^{2n}, -r_7\omega^{19+22n}, -r_6\omega^{24+14n}, r_1\omega^{18+26n}) \quad \text{for} \ 8 \leq n \leq 7 \\
|n + 53\rangle &= (r_6\omega^{2n}, r_7\omega^{19+22n}, r_4\omega^{24+14n}, r_1\omega^{18+26n}) \quad \text{for} \ 8 \leq n \leq 7 \\
|n + 61\rangle &= (r_8\omega^{2n}, -r_5\omega^{22n}, -r_3\omega^{14n+1}, r_2\omega^{26n+1}) \quad \text{for} \ 0 \leq n \leq 14 \\
|n + 76\rangle &= (r_5\omega^{2n}, r_8\omega^{22n}, -r_2\omega^{14n+1}, -r_3\omega^{26n+1}) \quad \text{for} \ 0 \leq n \leq 14 \\
|n + 83\rangle &= (r_2\omega^{2n}, r_3\omega^{19+22n}, -r_5\omega^{24+14n}, -r_8\omega^{18+26n}) \quad \text{for} \ 8 \leq n \leq 14 \\
|n + 98\rangle &= (r_3\omega^{2n}, r_2\omega^{19+22n}, r_5\omega^{24+14n}, -r_8\omega^{18+26n}) \quad \text{for} \ 8 \leq n \leq 7 \\
|n + 113\rangle &= (r_3\omega^{2n}, -r_2\omega^{19+22n}, r_5\omega^{24+14n}, -r_8\omega^{18+26n}) \quad \text{for} \ 0 \leq n \leq 7,
\end{align*}
\]

with each ray being an 8-component column vector whose (real) components are given by the real and imaginary parts of the four complex numbers listed for \( |i\rangle \). We will use \( \langle i| \) to denote the 8-component row vector that is the transpose of \( |i\rangle \).

Let us denote by the letters \( \text{A}, \cdots, \text{H} \) each consecutive set of 15 rays (thus \( \text{A} \) denotes rays 1–15, \( \text{B} \) rays 16–30, etc.). If we add to each group of 15 rays all their negatives, we get groups of 30 vectors whose first two coordinates define the vertices of regular triacontagons in the plane, with the triacontagons corresponding to the eight letter groups being concentric to one another. This

\[1\] For example, the components of the column vector \( |3\rangle \), in the proper order, are \( r_1 \cos \left( \frac{2\pi}{15} \right), r_1 \sin \left( \frac{2\pi}{15} \right), r_4 \cos \left( \frac{4\pi}{15} \right), r_4 \sin \left( \frac{4\pi}{15} \right), r_6 \cos \left( \frac{6\pi}{15} \right), r_6 \sin \left( \frac{6\pi}{15} \right), r_7 \cos \left( \frac{7\pi}{15} \right), r_7 \sin \left( \frac{7\pi}{15} \right) \).
is just the triacontagonal representation of the roots of E8 (or of the vertices of Gosset’s polytope $4_{21}$) mentioned in the introduction. Although the coordinates we have introduced for the rays are identical to those of Richter, our numbering of the rays is a bit different from his (in essence, we have swapped some of his triacontagons and rotated some of them relative to the others for convenience in the presentation of some of our results).

A straightforward calculation shows that each of the 120 rays is orthogonal to 63 others and that the rays form 2025 bases. Each ray occurs in 135 bases and its only companions in these bases are the 63 other rays it is orthogonal to. We will denote this system of rays and bases by the symbol $120 \cdot 135 \cdot 2025$, with the subscript on the left indicating the multiplicity of each of the rays (i.e., the number of bases it occurs in) and that on the right the number of rays in each basis. The product of the numbers in the left half of the symbol equals the product on the right, as it should. The basis table of E8 (i.e., the complete list of all its bases) is saturated, by which we mean that all the orthogonalities between its rays are represented in it. Because of this, the basis table is completely equivalent to the Kochen-Specker diagram of its rays. However it has the great advantage over the Kochen-Specker diagram that it is easy to interpret and work with.

Later we will encounter other systems of rays and bases having a high degree of symmetry, and the notation we have introduced above is easily modified to deal with such cases. For example, a system of 45 rays that forms 15 bases, with 30 of the rays being of multiplicity 2 and the other 15 of multiplicity 4 can be represented by the symbol $30 \cdot 2 \cdot 15 \cdot 4 \cdot 15$ (again the sum of the products of each number on the left with its subscript equals the product of the number and its subscript on the right). The parity proofs we will present later, which are subsets of the basis table, can also be described by symbols of this kind.

We now present the basis table of E8. Figure 1 shows 15 bases that contain all 120 rays once each. The entire basis table can be derived from these 15 bases by permuting the rays in them in the manner we now describe. Let $V$ be the permutation of order 9 with the cycle decomposition $V = (1 \ 5 \ 9 \ 13 \ 53 \ 40 \ 82 \ 105 \ 11)(2 \ 91 \ 90 \ 55 \ 28 \ 42 \ 54 \ 119 \ 49)(3 \ 47 \ 38 \ 66 \ 31 \ 30 \ 41 \ 103 \ 12)(4 \ 10 \ 51 \ 89 \ 117 \ 106 \ 87 \ 27 \ 36)(6 \ 93 \ 97 \ 101 \ 86 \ 71 \ 48 \ 69 \ 113)(7 \ 14 \ 79 \ 67 \ 33 \ 29 \ 64 \ 32 \ 100)(8 \ 95 \ 99 \ 45 \ 44 \ 92 \ 112 \ 63 \ 78)(15 \ 104 \ 34 \ 46 \ 109 \ 77 \ 118 \ 107 \ 85)(16 \ 120 \ 98 \ 60 \ 61 \ 75 \ 18 \ 35 \ 68)(17 \ 73 \ 20 \ 24 \ 59 \ 96 \ 58 \ 57 \ 94)(19 \ 23 \ 76 \ 52 \ 84 \ 56 \ 21 \ 25 \ 37)(22 \ 74 \ 116 \ 108 \ 115 \ 72 \ 62 \ 50 \ 70)(65 \ 80 \ 88 \ 102 \ 83 \ 110 \ 114 \ 81 \ 111)(26 \ 43 \ 39)$ and $W$ the permutation of order 15 with the cycle decomposition $W = (1 \cdots 15)(16 \cdots 30)(31 \cdots 45)(46 \cdots 60)(61 \cdots 75)(76 \cdots 90)(91 \cdots 105)(106 \cdots 120)$, where the dots signify all the integers between the two extremes. Let each basis in

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2 This is a graph whose vertices correspond to the rays and whose edges join vertices corresponding to orthogonal rays.

3 By the cycle $(1 \ 5 \ 9 \ 13 \ \cdots \ 11)$, we mean the permutation in which 1 goes to 5, 5 to 9, 9 to 13 ... and 11 to 1.
Figure 1 be assigned the label \((0,0,l)\), where the first two labels are fixed and the third varies, in integer steps, from 0 to 14. Then any other member of the basis table, which is assigned the label \((n,m,l)\) with \(0 \leq l, n \leq 14\) and \(0 \leq m \leq 8\), can be generated by applying suitable powers of \(W\) and \(V\) to one of the bases in Fig.1, as described by the equation \((n,m,l) = W^n V^m (0,0,l)\).

The number of bases that can be generated in this way is \(15 \cdot 9 \cdot 15 = 2025\), which is the entire basis table.

| Basis label | Basis               |
|-------------|---------------------|
| \((0,0,0)\) | 1 7 62 66 70 73 107 111 |
| \((0,0,1)\) | 29 115 33 11 74 61 5 52   |
| \((0,0,2)\) | 64 34 85 102 19 88 101 94 |
| \((0,0,3)\) | 58 78 92 42 47 110 112 116 |
| \((0,0,4)\) | 55 117 80 51 96 106 41 108 |
| \((0,0,5)\) | 27 81 84 82 21 59 114 14   |
| \((0,0,6)\) | 10 23 38 103 37 56 53 113   |
| \((0,0,7)\) | 2 72 44 104 95 32 48 25     |
| \((0,0,8)\) | 17 89 76 54 26 119 77       |
| \((0,0,9)\) | 87 109 4 16 22 86 79 49     |
| \((0,0,10)\)| 100 31 120 45 60 57 13 30   |
| \((0,0,11)\)| 91 90 43 15 75 20 105 63   |
| \((0,0,12)\)| 99 71 36 28 93 39 6 50     |
| \((0,0,13)\)| 24 46 67 35 118 12 3 69    |
| \((0,0,14)\)| 83 8 40 97 98 18 65 68     |

Fig. 1 Fifteen bases of the E8 system, involving the rays 1-120 once each. The first column shows the three-index label of each basis.

In Figure 2 we show the 9 blocks of bases obtained by applying all powers of \(V\) to the block of Figure 1. The remaining blocks of bases can be obtained by applying powers of \(W\) to these nine blocks. Since an application of \(W\) amounts, for the most part, to increasing the ray numbers by one, these other blocks are easily written down.

The construction we have given of the basis table can be compressed even further by introducing a permutation of order 15, which we will term \(U\), whose cycle decomposition is given by the eight columns of numbers obtained by aligning the bases in Fig.1 and reading down them vertically\(^4\). Then all the bases can be generated from the first basis of Fig.1 by applying powers of the permutations \(U, V\) and \(W\) to it, as described by the equation

\[
(n, m, l) = W^n V^m U^l (0,0,0) ,
\]

with \(0 \leq l \leq 14, 0 \leq m \leq 8\) and \(0 \leq n \leq 14\). This procedure works even if an arbitrary basis is substituted for \((0,0,0)\) as the seed basis. The three-index

\(^4\) To be explicit, \(U = (1\ 29\ 64 \cdots 83)(7\ 115\ 34 \cdots 8) \cdots (111\ 52\ 94 \cdots 68)\), where there are 8 cycles and each consists of 15 numbers.
Fig. 2 The nine blocks of bases obtained by applying all powers of $V$, from 0 to 8, to the block of Figure 1. The powers increase as one goes from left to right and up to down. Applying $V$ to the block at the bottom right gives back the block at the top left, which is just the block of Figure 1.

The label $(n, m, l)$ serves as a convenient shorthand for the basis if one wishes to avoid listing all its rays.
Instead of describing $U$, $V$, and $W$ by the permutations they perform on the rays, one can represent them by the $8 \times 8$ orthogonal matrices

\[
U = \begin{pmatrix}
-39|1\rangle\langle11| + 29|7\rangle\langle33| + 62|66\rangle\langle62| + 74|70\rangle\langle70| - 5|107\rangle\langle107| - 52|111\rangle\langle111|
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
5|1\rangle\langle14| + 1|62\rangle\langle31| - 50|62\rangle\langle70| - 31|62\rangle\langle70| - 22|62\rangle\langle70| - 20|62\rangle\langle70| - 85|62\rangle\langle70| + 65|111\rangle\langle111|
\end{pmatrix},
\]

and

\[
W = \begin{pmatrix}
2|1\rangle\langle8| + 63|62\rangle\langle62| + 67|66\rangle\langle66| + 71|70\rangle\langle70| + 74|70\rangle\langle70| + 108|70\rangle\langle70| + 112|111\rangle\langle111|
\end{pmatrix},
\]

as one can easily check by applying them to the column vectors representing the rays and verifying that they produce the desired permutations.

We can construct all the symmetries of $E_8$ by looking at the mappings of a fixed basis onto all the bases of the system. Let $x_1x_2\cdots x_8$ and $y_1y_2\cdots y_8$ be two bases and let $y'_1y'_2\cdots y'_8$ be some permutation of the numbers in the latter. Consider the orthogonal transformation

\[
T = (-1)^{n_1}|y'_1\rangle\langle x_1| + (-1)^{n_2}|y'_2\rangle\langle x_2| + \cdots + (-1)^{n_8}|y'_8\rangle\langle x_8|,
\]

where $n_i \in (0, 1)$ for $1 \leq i \leq 8$. Keeping the basis $x_1x_2\cdots x_8$ fixed and letting $y_1y_2\cdots y_8$ vary, the number of transformations of the form (5) that one can construct is the number of possibilities for the variable basis ($2025$) times the number of permutations of the variable basis labels ($8!$) times the number of possibilities for the signs of the terms ($2^8$). However, an investigation shows that only $1/30$ of the $8!$ permutations of the basis labels lead to symmetries of $E_8$, so the total number of its symmetries is $2025 \cdot 8! \cdot 2^8 = 696729600$, which equals the number of $192 \cdot 10!$ given by Coxeter [5] as the order of the symmetry group of Gosset’s polytope $4_{21}$.

### 3 Substructures within the E8 system

The $E_8$ system contains a number of interesting substructures that yield proofs of the KS theorem. These substructures have all been studied in the past, and we discuss each of them briefly.

Two interesting substructures are the Lie algebras $E_7$ and $E_6$, whose roots/rays can be exhibited as subsets of the roots/rays of $E_8$. The rays of $E_7$ are simply the 63 rays orthogonal to any ray of $E_8$; these rays lie in a 7-dimensional space, and if one adjoins to them all their negatives, one gets the 126 roots of $E_7$. The 63 rays of $E_7$ form 135 bases, with each ray occurring in 15 bases. This system can be described by the symbol $63_{15}-135_7$, and it is saturated. The basis table of $E_7$ is easily extracted from that of $E_8$ by picking out the 135 bases involving a particular ray and then dropping that ray from these bases. This construction shows that the symmetry group of $E_7$ is a subgroup of index 240 of that of $E_8$; thus its order is $192 \cdot 10!/240$, which agrees
with the figure of $8 \cdot 9!$ given by Coxeter for the order of the symmetry group of the associated 7-dimensional polytope $3_21$. The rays of E7 can be used to give a proof of the KS theorem. Because the system is saturated, the proof requires showing that it is impossible to assign noncontextual 0/1 values to the rays in such a way that each of the 135 bases has just a single ray assigned the value 1 in it, and this is easily done using a “proof-tree” argument. An alternative proof of the KS theorem based on the rays of E7 has been given by Ruuge\cite{20}.

If one picks the 36 rays orthogonal to any two nonorthogonal rays of E8, one gets the rays of the E6 system, and the roots of E6 are these 36 rays along with their negatives. The rays of E6 do not form even a single basis (i.e., a set of six mutually orthogonal rays) and so do not yield a proof of the KS theorem. This was pointed out by Ruuge\cite{20}, who discussed how two rotated copies of E6 could be superposed to give a system of rays that yields a proof of the KS theorem.

Another interesting subsystem of E8 is what we will term a Kernaghan-Peres (KP) set\cite{13}. Such a set consists of 40 rays that form 25 bases, with each ray occurring in five bases, so that its symbol is $40_5-25_8$. Kernaghan and Peres\cite{13} constructed such a set as the simultaneous eigenstates of five sets of mutually commuting observables of a system of three qubits. The caption to Fig.3 explains how KP sets can be extracted from the bases of E8 and the figure gives an example of a set constructed using this procedure. The caption to Fig.4 gives a simple procedure for obtaining all the parity proofs in a KP set\cite{14} and the figure gives one example of each of the three types of parity proofs contained in the KP set of Fig.3 (see the beginning of Sec. 4 for an explanation of the notion of a parity proof). Finally, Fig.5 gives an example of a “pseudo” KP set that closely resembles a KP set but is not one, and in fact yields no proofs of the KS theorem.

\footnote{The argument is a reductio ad absurdum: one assumes that a noncontextual value assignment exists and then shows that it leads to a contradiction. Since E7 has a symmetry group that is transitive on its rays, one can begin, without loss of generality, by assigning an arbitrary ray the value 1. This forces all rays orthogonal to that ray to have the value 0, and one then finds that a basis with three rays having the value 0 appears. Assigning one of the remaining rays in this basis the value 1 forces a basis with five rays having the value 0 to appear. However assigning either of the remaining rays in this basis the value 1 forces a basis with all its rays assigned the value 0 to appear, which is not allowed. To avoid this conflict, one must proceed backwards along the chain and make alternative choices for the rays assigned the value 1 at every earlier step of the argument and see if any of these alternative possibilities leads to a valid value assignment. One then finds that all the alternatives lead to a situation in which at least one basis has all its rays assigned the value 0, showing that a valid value assignment does not exist and proving the KS theorem. The “proof-tree” leading to this contradiction has eight branches, with each branch leading to a contradiction at the fourth step.}
Fig. 3 A Kernaghan-Peres (KP) set can be extracted from the bases of E8 by choosing any five ("seed") bases from one of the blocks in Fig. 2 (or a block obtained from one of these blocks by applying a power of $W$ to it) and supplementing them by 20 bases, chosen from the full set of 2025, that each have four rays from one seed basis and four rays from another. The 20 added bases always come in 10 complementary pairs, with the members of each pair originating in the same pair of seed bases and having no rays in common. In order that this construction gives rise to a KP set, it is necessary that each ray occurs once with 17 other rays and thrice with 6 other rays in the five bases in which it occurs. Shown above is a KP set constructed according to this procedure, with its five seed bases shown in the first column and its 10 pairs of complementary bases in the second and third columns; the seed bases were picked from the block in the top left corner of Fig. 2 and each pair of complementary bases is shown on a line. It can be checked that this set of rays and bases has the symbol $40_{15}^{5}$ and that each of its rays has the pattern of companions stated earlier.

Fig. 4 Parity proofs can be extracted from any KP set by picking one member from each of the 10 pairs of complementary bases and supplementing them with the needed seed bases. There are three types of parity proofs that can be constructed in this way, and they involve the addition of one, three or five seed bases. These proofs have the symbols $28_{2}^{2}8_{1}^{4}$, $24_{2}^{2}14_{1}^{4}13_{8}$ and $20_{2}20_{1}^{4}15_{8}$, and one example of each, constructed from the KP set of Fig. 3, is shown in the three columns above (with the seed bases always shown at the end). The 10 bases in the first step of the construction can be picked in $2^{10} = 1024$ ways, and all of them can be extended into valid parity proofs.

4 Parity proofs in the E8 system

We will say that a set of projectors in a Hilbert space of even dimension furnishes a parity proof of the KS theorem if the projectors form an odd number of bases in such a way that each projector occurs in an even number of the bases (a basis is any set of mutually orthogonal projectors that sums to the
identity, and we will allow for the possibility that the projectors are not all of the same rank). Such a set of projectors proves the KS theorem because it is impossible to assign noncontextual 0/1 values to them in such a way that the sum of the values assigned to the projectors in any basis is always 1. Because an even-odd conflict makes this assignment impossible, we refer to this type of proof as a parity proof. In [3] it was pointed out that the E8 system has $2^{1940}$ parity proofs in it, but no examples of such proofs were given. In this section we would like to describe a straightforward method we used to discover a large number of these proofs, and then present a few examples of them. These proofs are far more numerous and varied than those in the KP sets we know to be contained in the E8 system.

We will discuss only critical parity proofs, where by a critical proof we mean one that ceases to provide a proof of the KS theorem if even a single basis is dropped from it. We restrict ourselves to critical proofs to avoid redundancy, since many noncritical proofs can often be reduced to the same critical proof. We present just a few of the more striking critical proofs we found from among the staggeringly large number that exist.

In Sec. 2 we introduced the letters A to H for each consecutive set of 15 rays of E8. These letters can be used to attach an 8-letter label to each basis. For example, the basis 1 7 62 66 70 73 107 111 would have the label AAEDEEHH. Viewed in terms of their labels (which specify how the rays of a basis are distributed over the triacontagons of E8), the bases fall into 33 families with distinct triacontagon profiles. We made the important discovery that if one looks at only the bases of a particular family, the parity proofs housed by them


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*Fig. 5* The construction that gave rise to the KP set of Fig. 3 also leads to many “pseudo” KP sets, like the one above. Although a $40_5-25_8$ set, it is not a KP set because each of its rays occurs four times with a particular companion in five bases, which never happens in a KP set. Further, this “pseudo” KP set does not yield a proof of the KS theorem because it is possible to make valid noncontextual 0/1 assignments to its rays in many ways; one such assignment consists of assigning 1’s to the rays 1, 85, 17, 93 and 68 and 0’s to all the others.

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6 Dropping a single basis from a parity proof leaves an even number of bases, which can never provide a parity proof of the KS theorem. However the reduced system may not admit a valid noncontextual value assignment to its rays, and so provide a proof of the KS theorem. If this happens, the original parity proof would not be deemed critical.
could be unearthed with relatively little effort. In the following subsections we discuss the parity proofs we found in a few of the families.

4.1 Type 1 Bases and their parity proofs

Let us term bases with the profile AAEEEEHH, BBFFFFGG, CCEEGGGG or DDFHHHHH as Type 1 bases. The 15 bases with each of these profiles give a parity proof that can be characterized by the symbol $15_4^{30_2-15_8}$. Figure 6 shows the proof given by the bases with profile AAEEEEHH (the proofs given by the other three profiles are very similar).

The Type 1 bases give rise to two other types of parity proofs if bases of different profiles can be combined. These proofs are characterized by the symbols $15_4^{45_2-35_8}$ and $45_4^{50_2-35_8}$ and there are 12 versions of each (all structurally identical, but involving different rays). The properties of the three different types of parity proofs made up only of Type 1 bases are summarized in the first row of Figure 7.

4.2 Type 2 Bases and their parity proofs

We will term bases with the profile AABBEEFF or CCDDGGHH as Type 2 bases. There are 30 bases with each of these profiles, and therefore 60 Type 2 bases in all. These bases contain just the two types of parity proofs shown in the second row of Figure 7.

Figure 8 shows two $36_2-9_8$ proofs of this class that seem very similar at first sight, but are subtly different from one another. While both proofs involve 36 rays that occur two times each over 9 bases, the proof on the left always has the following pairs of rays occur together over the bases: (1,66), (7,62), (16,86), (22,87), (9,84), (19,64), (21,76), (14,89), (17,82), (2,72), (11,61), (27,77), (29,74), (26,81), (4,79), (12,67), (24,69) and (6,71). Thus each of these pairs of rays can be regarded as defining a two-dimensional subspace, or a rank-2 projector, and the proof can be reinterpreted as involving 18 rank-2 projectors that each occur twice over nine bases; this situation can be captured...
in the symbol $18_2^2-9_4$, where the superscript on the left indicates the rank of the projectors and the subscript their multiplicity, and the subscript on the right that each of the 9 bases is made up of four rank-2 projectors. For the proof on the right, 18 of the rays can be paired into the rank-2 projectors $(1,16)$, $(62,87)$, $(75,80)$, $(70,90)$, $(82,72)$, $(65,85)$, $(67,77)$, $(11,26)$ and $(6,21)$, while the remaining 18 rays are associated with rank-1 projectors; thus the symbol of this proof can be written as $9_2^{18_2^2-9_4}$, with the superscripts and subscripts on the left having the same meaning as before and the subscript on the right indicating that each basis is made up of six projectors (four of rank-1 and two of rank-2).

4.3 Type 3 Bases and their parity proofs

Type 3 bases are those with the profile EEEFFGGHH. There are 45 such bases involving 60 rays, and they form a 60$^6$-45$^8$ system. Despite their small number, the bases of this system are a fecund lot and give rise to 20 different types of parity proofs, each of which can come in hundreds to thousands of versions. The symbols of all the possible proofs are shown in the third row of Figure 7. When the different versions of each of the proofs are taken into account, the total number of distinct proofs is 700,326. Figure 9 shows a
Fig. 8 Two parity proofs made up exclusively of Type 2 bases with the profile AABBEEFF or CCDDGGHH. The proof on the left involves only rank-2 projectors and is characterized by the symbol 18\textsuperscript{2} \cdot 9, while the proof on the right involves a mixture of rank-2 and rank-1 projectors and is characterized by the symbol 9\textsuperscript{2} \cdot 18\textsuperscript{2} \cdot 9\textsuperscript{2} (see text for explanation). If one ignores the distinction between rank-1 and rank-2 projectors and focuses only on the rays, both proofs can be described by the common symbol 36\textsuperscript{2} \cdot 9\textsuperscript{2} (indicating that there are 36 rays that each occur twice over the nine bases).

Fig. 9 A 6\textsubscript{6} \cdot 12\textsubscript{4} \cdot 42\textsubscript{2} \cdot 21\textsubscript{4} parity proof made up of 21 Type 3 bases with the profile EEF-FGGHH. The rays of multiplicity 6 are 66, 81, 94, 98, 118, 119, those of multiplicity 4 are 63, 65, 67, 69, 77, 78, 84, 85, 95, 97, those of multiplicity 2 are 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, and all the other rays have multiplicity 2.

6\textsubscript{6} \cdot 12\textsubscript{4} \cdot 42\textsubscript{2} \cdot 21\textsubscript{4} proof of this class involving rays of multiplicity 6, 4 and 2.

4.4 Type 4 Bases and their parity proofs

Type 4 bases have the profile AABBCCDD. There are 75 such bases involving 60 rays, and they form a 60\textsubscript{10} \cdot 75\textsubscript{8} system. We have found over 400 different types of parity proofs in this system, with each coming in anywhere from scores to thousands of versions. We show just nine of these proofs in the last row of Figure 7. There are no critical proofs with more than 27 bases in this class. The number of proofs in this class greatly exceeds those in the previous three classes combined. Figure 10 shows a 36\textsuperscript{2} \cdot 9\textsuperscript{2} proof of this class and Fig. 11 shows a rather unusual proof consisting entirely of rank-2 projectors.

We end by presenting a proof of this class, in Fig. 12, that involves 34 rays (2 of multiplicity four and 32 of multiplicity two) that occur over 9 bases. This proof is more economical than the best proofs found earlier in 8 dimensions, which involve 36 rays occurring an even number of times over 11 bases\textsuperscript{13} or 81 bases\textsuperscript{49} or 9 bases\textsuperscript{14}. As explained in the caption to Fig. 12, this proof can also be interpreted as involving 26 rank-1 projectors and 4 rank-2 projec-
Fig. 10 A $36_2$-9s parity proof made up of 9 Type 4 bases with the profile AABBCCDD.
Twenty of the rays can be grouped into the rank-2 projectors $(1,19)$, $(7,25)$, $(16,40)$, $(10,46)$, $(5,20)$, $(12,51)$, $(11,26)$, $(22,31)$, $(24,45)$ and $(4,55)$, while the remaining rays define 16 rank-1 projectors. Also, eight of the bases involve 6 projectors (two of rank-2 and four of rank-1) while the remaining basis involves four rank-2 projectors. Thus a more descriptive symbol for this proof is $10^2_{2}16^1_{2}8_{4}14$.

Fig. 11 A parity proof made up of 11 Type 4 bases with the profile AABBCCDD. It consists of 44 rays that each occur twice over the 11 bases, so its symbol is $44_2$-11s. However a more careful examination shows that the rays can be paired into the 22 rank-2 projectors $(1,19)$, $(2,44)$, $(4,16)$, $(5,32)$, $(7,34)$, $(25,52)$, $(33,48)$, $(17,59)$, $(14,20)$, $(15,30)$, $(11,29)$, $(37,58)$, $(38,47)$, $(8,41)$, $(28,46)$, $(23,56)$, $(13,31)$, $(45,60)$, $(22,49)$, $(10,43)$, $(20,53)$ and $(42,57)$ that each occur twice over the bases. Thus a more descriptive symbol for this proof would be $22^2_{2}$-11s, with the subscript of 4 on the right indicating that there are four rank-2 projectors in each basis.

5 Discussion

We pointed out at the end of Sec. 3 that the bases of E8 have 33 different triacontagon profiles. Our survey of parity proofs in Sec. 4 covered just four of these profiles, so it is clear that we have left the vast majority of the proofs untouched. The basis table of E8 presented in this paper serves a convenient template for displaying all the proofs in this gargantuan system.

It is interesting that the triacontagonal representations of both the 600-cell and Gosset’s polytope lead to some of the simplest parity proofs contained in them. In the case of the 600-cell, the vertices project into four triacontagons, with two of the triacontagons uniting to yield a parity proof of 15 bases and the other two triacontagons yielding the complementary proof (i.e., one involving
Fig. 12 A 32_24_98 parity proof made up of 9 Type 4 bases with the profile AABBCCDD, with rays 25 and 10 being of multiplicity four and all the others of multiplicity two. However the pairs of rays (1,19), (4,55), (5,20) and (16,40) always occur together over the bases, with each pair occurring twice. Interpreting these pairs as rank-2 projectors and the remaining 26 rays as rank-1 projectors allows us to attach the more descriptive symbol 4^2_2^2 2^1 4 24 1^1 8^1_8 to this proof, where the subscripts in the second half of the symbol indicate that there are eight bases of 7 projectors (with 6 being of rank-1 and 1 of rank-2) and one basis of 8 projectors (with all being of rank-1).

all the rays not present in the earlier proof). In the case of Gosset’s polytope, the vertices project into eight triacontagons, and one can construct a parity proof (actually four different proofs) by picking out 15 bases that each span all the triacontagons in the same way. The great virtue of the triacontagonal representation for Gosset’s polytope (or E8) is, of course, that it allows the bases to be organized into smaller families that are more easily searched for their parity proofs. Although we have unearthed only a tiny fraction of the parity proofs present in E8, their variety and intricacy seems to exceed that in any of the other systems we have studied to date. This is doubtless due to the large basis table of E8 (at 2025 bases, a record) and its huge symmetry group (of over 10^8 elements).

A comment should be made about the experimental measurements needed to realize the bases of E8, on which all the parity proofs of this paper depend. It might be asked if the projectors corresponding to some of the bases can be realized as simultaneous eigenstates of commuting three-qubit observables that are tensor products of Pauli operators of the individual qubits. While this is true of some of the bases, such as the ones we have identified as the Kernaghan-Peres sets, it is not true of the bases in general. The simplest way of generating an arbitrary basis from the computational basis is by following Eq. (1) and applying a product of the appropriate powers of the three unitary operators $U, V$ and $W$. Designing efficient quantum gates for these operators is an interesting problem that we will not take up here. However it seems worth pursuing because a recent experiment\[50\] has successfully generated several KS sets in a three-qubit system and holds out the possibility of eventually generating the more complex sorts of KS sets considered here.

It was pointed out in \[24\] that any KS proof can be converted into an inequality that is satisfied by any noncontextual hidden variables theory but violated in measurements carried out on an arbitrary quantum state. It might
be asked what the extent of the violation is for the parity proofs discussed in this paper. The answer to this question has already been given in an earlier work of ours [18]. We showed there that for any basis-critical parity proof (i.e., one which fails if even a single basis is omitted from it), the upper bound of the inequality for any noncontextual hidden variable theory is $B - 2$ (where $B$ is the number of bases in the proof) whereas quantum mechanics predicts the value of $B$. This gap of 2 between the values predicted by hidden variable theories and quantum mechanics is a universal feature of all basis critical parity proofs. Thus the present proofs do not offer any particular advantage, from this point of view, over the many similar proofs [12, 14, 16, 18, 19] we found earlier.

Gosset’s polytope is the real representative of a complex polytope known as Witting’s polytope [6]. Coxeter [51] has carried out a systematic study of a large number of complex polytopes. It is possible that the ray systems derived from some of them might yield new proofs of the KS theorem. Whether this is true, and of what use it might be, are matters that remain to be explored.

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