QUASI-HEREDITARY ALGEBRAS, EXACT BOREL SUBALGEBRAS, $A_\infty$-CATEGORIES AND BOXES

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Abstract. Highest weight categories arising in Lie theory are known to be associated with finite dimensional quasi-hereditary algebras such as Schur algebras or blocks of category $O$. An analogue of the PBW theorem will be shown to hold for quasi-hereditary algebras: Up to Morita equivalence each such algebra has an exact Borel subalgebra. The category $\mathcal{F}(\Delta)$ of modules with standard (Verma, Weyl, . . . ) filtration, which is exact, but rarely abelian, will be shown to be equivalent to the category of representations of a directed box. This box is constructed as a quotient of a dg algebra associated with the $A_\infty$-structure on $\mathcal{F}(\Delta)$. Its underlying algebra is an exact Borel subalgebra.

Keywords. Highest weight category, quasi-hereditary algebra, exact Borel subalgebra, modules with standard filtrations, $A_\infty$-category, differential graded category, box.

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1. Introduction

Highest weight categories are abundant in algebraic Lie theory as well as in representation theory of finite dimensional algebras. A highest weight category with a finite number of simple objects is precisely the module category of a quasi-hereditary algebra that is unique up to Morita equivalence. A general highest weight category is made up of pieces (to be reached by a well-defined truncation process) that are finite highest weight categories. Among the most frequently studied quasi-hereditary algebras are Schur algebras of reductive algebraic
groups, blocks of the Bernstein-Gelfand-Gelfand category $O$ of semisimple complex Lie algebras - or generalisations to Kac-Moody algebras - as well as path algebras of quivers, finite dimensional algebras of global dimension two and Auslander algebras.

Highest weight categories come with important objects, the so-called standard objects, which are ‘intermediate’ between simple and projective objects. Examples of standard objects include Verma and Weyl modules. The category $\mathcal{F}(\Delta)$ of objects with standard filtration is crucial in a wide variety of contexts, including Ringel’s theory of (characteristic) tilting modules and Ringel duality, Kazhdan-Lusztig theory, relative Schur equivalences and homological questions in representation theory. In geometry, examples of standard objects come up in exceptional sequences in algebraic or symplectic geometry, where $\mathcal{F}(\Delta)$ is studied as category of ‘twisted stalks’.

The principal goal of this article is to clarify the structure of categories $\mathcal{F}(\Delta)$ of quasi-hereditary algebras in full generality. Using $A_\infty$-techniques and differential graded categories we will establish an equivalence between $\mathcal{F}(\Delta)$ and a category of representations of a box, that is, of a representation theoretic analogue of a differential graded category, satisfying strong additional properties. This makes available for the study of $\mathcal{F}(\Delta)$ the structure theory and representation theory of boxes, which is fundamental to representation theory of algebras by Drozd’s tame and wild dichotomy.

Inherent to highest weight categories and quasi-hereditary algebras is a ‘directedness’, which is reflected both in ordering conditions in the definition and in typical proofs in this area proceeding inductively along certain partial orders. Using the connection to boxes, this directedness can now be formulated precisely, as a characterisation of quasi-hereditary algebras and an equivalence of categories:

**Theorem 1.1.** A finite dimensional algebra $A$ is quasi-hereditary if and only if it is Morita equivalent to the right Burt-Butler algebra $R_B$ of a directed box $B$ if and only if it is Morita equivalent to the left Burt-Butler algebra $L_B$ of a directed box $B'$.

Moreover, the category $\mathcal{F}(\Delta)$ of $A$-modules with standard filtration is equivalent - as an exact category - to the category of representations of the box $B$.

The if part of the theorem is a rather direct consequence of the theory set up by Burt and Butler, combined with Dlab and Ringel’s ‘standardisation’ technique. The converse is more involved: Starting with a quasi-hereditary algebra we consider its category $\mathcal{F}(\Delta)$. This carries an $A_\infty$-structure, which we translate into a differential graded structure, using the concept of twisted stalks. The resulting differential graded category has additional properties that tell us that this datum actually determines a box. The directedness of the underlying algebra directly comes from the directedness of homology of standard modules. The results by Burt and Butler precisely apply to the box obtained after these translations. The category of representations of the box turns out to be precisely the category $\mathcal{F}(\Delta)$.

Turning around this result it follows that every directed box $B$ produces two quasi-hereditary algebras, which in general are different. The relation between them recovers a central symmetry within the class of quasi-hereditary algebras, Ringel duality:

**Corollary 1.2.** Let $B$ be a directed box and $L_B$ and $R_B$ its left and right Burt-Butler algebras. Then $L_B$ and $R_B$ are mutually Ringel dual quasi-hereditary algebras.

The Main Theorem 1.1 has a variety of applications obtained in this article, and potential for many more.

With respect to representation theory, it makes available for the study of $\mathcal{F}(\Delta)$ the rich supply of methods of representation theory of boxes. For instance we obtain as a consequence
(Corollary 10.6) a fundamental result by Ringel [30] asserting the existence of (relative) Auslander Reiten sequences for boxes.

A second line of applications, and in fact the original motivation for this article, is about the relation of the algebra $B$ underlying the box $\mathfrak{B}$ and the category $\mathcal{F}(\Delta)$.

In this respect, the main application of Theorem 1.1 is that it solves the following problem that has been open for about twenty years:

**Corollary 1.3.** Every quasi-hereditary algebra is Morita equivalent to a quasi-hereditary algebra $A$ (with corresponding quasi-hereditary structure) that has an exact Borel subalgebra $B$.

Here an exact Borel subalgebra, as defined in [19], satisfies properties analogous to the universal enveloping algebra of a Borel subalgebra of a semisimple complex Lie algebra, including a version of the fundamental theorem of Poincaré, Birkhoff and Witt.

The exact Borel subalgebra whose existence is claimed here, is, of course, the underlying algebra of the box whose representations form the category $\mathcal{F}(\Delta)$.

Note that there are quasi-hereditary algebras without exact Borel subalgebras, see [19], which implies that Theorem 1.1 (as well as this corollary) really fails to be true for isomorphism classes instead of Morita equivalence classes.

Existence of exact Borel subalgebras has been shown for the blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras [19], for Frobenius kernels of semisimple algebraic groups [28], for various abstractly defined classes of quasi-hereditary algebras, and more recently for certain infinite dimensional algebras [25]. Despite intensive efforts, existence for Schur algebras or for algebras of global dimension two - which, up to Morita equivalence, is a special case of our results now - could not be shown so far; more optimistic claims about Schur algebras made in [19, appendix] and in [20] rely on incomplete proofs. Various expectations raised by the early existence results and by the attempts to prove more such results, can now be explored again. Note that exact Borel subalgebras are different from and satisfy stronger properties than traditional Borel Schur algebras, as introduced by Green in [11]. Hence, in particular further investigations in the case of algebraic groups look promising.

The approach carried out here, to study categories $\mathcal{F}(\Delta)$ by means of $A_\infty$-structures and, in particular, boxes, has been outlined for the first time in the last named author’s programmatic text [27], which in detail differs substantially from the current approach by using box techniques much more heavily, while the current approach limits the use of boxes to basic and generally accessible material and instead relies on $A_\infty$- and differential graded structures. These results were presented by the last named author in lecture courses in Uppsala and in Köln. The article [27] was in turn motivated by [4], where for the first time an example of a category $\mathcal{F}(\Delta)$ has been studied using $A_\infty$-structures - or implicitly box structures - in the context of a classification of representation types.

The structure of the article is as follows. Section 2 recalls the basic notions from the representation theory of quasi-hereditary algebras, i.e. standard modules, Ringel duality and exact Borel subalgebras. In Section 3 we introduce the notion of a linear quiver, that although not necessary sometimes simplifies notation in the remainder of the article, and we fix some notation. In Section 4 we introduce the concept of an $A_\infty$-category and recall how the Yoneda algebra of a module can be regarded as an $A_\infty$-category. In Section 5 we recall the results of Keller and Lefèvre-Hasegawa that describe the category of filtered modules $\mathcal{F}(\Delta)$ as the homology of the $A_\infty$-category of twisted modules of the $A_\infty$-category
given by the Yoneda algebra. We present it in the language of [31]. Section 5 translates the results of the previous section, especially the description of the objects of the homology of the $A_\infty$-category of the twisted modules to the language of representations of quivers. Section 6 translates the results of the previous section, especially the description of the objects of the homology of the $A_\infty$-category of the twisted modules to the language of representations of quivers. Section 7 introduces the notion of a box and explains how a differential graded structure may be translated into a box. In Section 8 we prove that the dg structure constructed in Section 6 can be transferred into a box. Section 9 then recalls the theory of boxes by Burt and Butler, e.g. introduces the structure of the left and right algebra of a box and explores their connections. The final section, Section 10, then collects all the results to prove the main theorems and gives some corollaries. Especially the morphisms in the homology of the $A_\infty$-category of twisted modules are identified with morphisms between box representations via the results of Section 8. Since the notions introduced in this article are quite abstract we have added an appendix which illustrates our techniques in some examples including an example of an algebra that does not have an exact Borel subalgebra, but a Morita equivalent algebra does have such a subalgebra (see A3). In A1, the smallest non-trivial example from algebraic Lie theory is discussed: the principal block of category $O$ of $\mathfrak{sl}(2, \mathbb{C})$. Exact Borel subalgebras are, in general, not unique and boxes associated with quasi-hereditary algebras aren’t either: Example A2 exhibits two boxes with the same categories of representations; one of them is connected, while the other one is not. Example A4 provides an exact Borel subalgebra that is not associated with a box. More generally, the exact Borel subalgebras constructed here are different from those constructed before in special situations such as blocks of category $O$. The boxes produced by our approach contain stronger information than arbitrary exact Borel subalgebras.

Throughout the article we fix an algebraically closed field $\mathbb{k}$.

2. Quasi-hereditary algebras

Let $A$ be a basic finite dimensional algebra over $\mathbb{k}$ with a fixed decomposition of the unit into a sum of primitive orthogonal idempotents $1 = e_1 + \cdots + e_n$.

Unless otherwise stated our modules are left modules. By $A-\text{mod}$ we denote the category of finite dimensional left $A$-modules. For a primitive idempotent $e_i \in A$, we denote the corresponding simple module by $L(i)$, its projective cover by $P(i)$.

Furthermore we suppose that there is a partial order $\leq$ on the set of isomorphism classes of simple modules (or equivalently on the fixed set of primitive orthogonal idempotents). Let $\Delta(i)$ be the largest factor module of $P(i)$ having only composition factors $L(j)$ with $j \leq i$. The modules $\Delta(i)$ are called standard modules.

The algebra $A$ is then called quasi-hereditary (with respect to the partial order $\leq$) if $\text{End}_A(\Delta(\lambda)) \cong \mathbb{k}$ for all $\lambda$ and the kernel of the natural surjection $P(i) \twoheadrightarrow \Delta(i)$ is filtered by $\Delta(j)$ with $j > i$.

This notion can be defined dually using submodules of the indecomposable injectives, the so called costandard modules $\nabla(i)$. In the theory of quasi-hereditary algebras, the full subcategories $\mathcal{F}(\Delta)$ (respectively $\mathcal{F}(\nabla)$) of modules which can be filtered by standard (respectively costandard) modules play an important role. Ringel introduced the notion of the characteristic tilting module $T$, which is the multiplicity free module such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{add}(T)$. Here for an $A$-module $N$, we denote by $\text{add}(N)$ the full subcategory of $A-\text{mod}$, which consists of all modules isomorphic to a direct summand of $N^l$ for some $l \geq 0$. Ringel proved that $T = \bigoplus_{i=1}^n T(i)$.

We denote the Ringel dual of $A$ by $R(A) := \text{End}_A(T)^{op}$. To indicate when we are talking about $R(A)$-modules we will add a subscript $R(A)$. The functor $F(-) = \text{Hom}_A(T, -) :$
$A\text{-mod} \to R(A)\text{-mod}$ maps $T(i)$ to the indecomposable projective $R(A)$-module $P_{R(A)}(i)$. For us it will be convenient to use the characterisation of quasi-hereditary algebras given by the so called standardisation theorem by Dlab and Ringel:

**Theorem 2.1** ([8], Theorem 2). Let $\mathbb{k}$ be a field and $\mathcal{C}$ be a $\mathbb{k}$-category. A non-empty finite partially ordered set of objects $\Delta = (\Delta_1, \ldots, \Delta_n)$ of objects in $\mathcal{C}$ is called standardisable if $\text{End}(\Delta_1) \cong \mathbb{k}$, $\text{Hom}(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$ and $\text{Ext}^1(\Delta_i, \Delta_j) \neq 0$ implies $i < j$ and that those spaces are finite dimensional. Let $\Delta$ be a standardisable set. Then there is a quasi-hereditary algebra $A$, unique up to Morita equivalence, such that the subcategory $\mathcal{F}(\Delta)$ of $\mathcal{C}$ and $\mathcal{F}(\Delta_A)$ are equivalent.

We now also recall the notion of an exact Borel subalgebra introduced in [19]. As announced in the introduction we will prove that every quasi-hereditary algebra has such a subalgebra up to Morita equivalence.

**Definition 2.2.** Let $A$ be a quasi-hereditary algebra with respect to a partial ordering of the simples $\leq$. Then a subalgebra $B \subseteq A$ is called an exact Borel subalgebra provided

(B1) The algebra $B$ has the same partially ordered set of weights, i.e. isomorphism classes of simple modules, as $A$, and $(B, \leq)$ is directed, i.e. $B$ is quasi-hereditary with simple standard modules.

(B2) The tensor induction functor $A \otimes_B -$ is exact.

(B3) There is an isomorphism $A \otimes_B L_B(1) \cong \Delta_A(1)$.

3. Linear quivers, (co)diifferential tensor (co)categories

3.1. Categories over a field. A $\mathbb{k}$-bimodule $V$ is called central if the left and right $\mathbb{k}$-actions on $V$ coincide. A category $\mathcal{A}$ is called $\mathbb{k}$-category or a category over $\mathbb{k}$ if all sets $\mathcal{A}(X, Y) = \text{Hom}_A(X, Y)$ for $X, Y \in \mathcal{A}$ are central $\mathbb{k}$-bimodules and the composition of morphisms is $\mathbb{k}$-bilinear. Unless stated otherwise all categories are assumed to be $\mathbb{k}$-categories. Functors between $\mathbb{k}$-linear categories and natural transformations between them will also be assumed to be $\mathbb{k}$-linear. A $\mathbb{k}$-category $\mathcal{A}$ will be called finite dimensional provided the space $\bigoplus_{X,Y \in \mathcal{A}} \mathcal{A}(X, Y)$ is finite dimensional.

Let $S$ be a set (in many cases $S = \{1, \ldots, n\}$). The trivial $\mathbb{k}$-category on $S$ is defined to be the $\mathbb{k}$-category $\mathbb{L}_S$ with objects being the elements of $S$ and only morphisms being formal multiples of the identity morphisms, i.e. $\mathbb{L}_S(s, s') = 0$ for $s \neq s'$ and $\mathbb{L}_S(s, s) = \mathbb{k} s$. We will omit the subscript if it is clear from the context.

By $\mathbb{k} - \text{Mod}$ (respectively $\mathbb{k} - \text{mod}$) we denote the category of $\mathbb{k}$-vector spaces (respectively finite dimensional $\mathbb{k}$-vector spaces). A left $A$-module is a functor $F : \mathcal{A} \to \mathbb{k} - \text{Mod}$. $F$ is called locally finite dimensional if its image belongs to $\mathbb{k} - \text{mod}$. Right modules are $\mathcal{A}^{op}$-modules, where $\mathcal{A}^{op}$ is the opposite $\mathbb{k}$-category of $\mathcal{A}$, equivalently one can define them to be contravariant functors $\mathcal{A} \to \mathbb{k} - \text{Mod}$.

The augmentation $\pi_A : \mathcal{A} \to \mathbb{L}_A$, sending all morphisms apart from the scalar multiples of the identities to zero, where $\mathbb{L}_A$ is the subcategory of $\mathcal{A}$ given by all scalar multiples of the identity morphisms, defines for every $i \in \mathcal{A}$ a simple module $L(i) : \mathcal{A} \xrightarrow{\pi_A} \mathbb{L}_A \xrightarrow{p_i} \mathbb{k} - \text{mod}$, where $p_i$ is given by $p_i(1_j) = 0$ for $j \neq i$ and $p_i(1_i) = 1_k$. A module is called semisimple if it is the direct sum of simple modules.

If $A$ and $B$ are $\mathbb{k}$-categories, the tensor category $A \otimes B$ defined by

$$\text{Ob}(A \otimes B) := \text{Ob}(A \times B); \quad (A \otimes B)((X_1, Y_1), (X_2, Y_2)) = A(X_1, X_2) \otimes_\mathbb{k} B(Y_1, Y_2)$$

is a $\mathbb{k}$-category. An $A$-$B$-bimodule is just an $A \otimes B^{op}$-module.
3.2. Linear quivers. A \( \mathbb{k} \)-linear central bimodule over \( \mathbb{L} \), i.e. a \( \mathbb{k} \)-linear functor \( Q : \mathbb{L} \otimes \mathbb{L}^{op} \to \mathbb{k} - \text{Mod} \) is called a \( \mathbb{k} \)-quiver (or \( \mathbb{k} \)-quiver over \( S \) or linear quiver). Since the categories \( \mathbb{L}^{op} \) and \( \mathbb{L} \) are naturally isomorphic we can and will skip the \( op \). Since \( \mathbb{L} \otimes \mathbb{L} \cong \mathbb{L}_{S \times S} \) this is equivalent to giving a family of vector spaces \( Q(s, s') \) for each \( s, s' \in S \). The set \( S \) will be called objects (or points, vertices) of \( Q \) and will also be denoted by \( Q_0 \).

Morphisms between \( \mathbb{k} \)-quivers are just \( \mathbb{k} \)-linear transformations. In this way \( \mathbb{k} \)-quivers form a category \( \text{Qui}_k \).

Let \( A \) and \( B \) be \( \mathbb{k} \)-quivers over \( \mathbb{L} \). Then we define their tensor product \( A \otimes_\mathbb{L} B \) as the \( \mathbb{k} \)-quiver given by:

\[
(A \otimes_\mathbb{L} B)(i, j) := \bigoplus_{k \in \mathbb{L}} A(k, j) \otimes_\mathbb{k} B(i, k)
\]

for all \( i, j \in \mathbb{L} \).

3.3. Graded \( \mathbb{k} \)-quivers. Let \( C \) be a category. A graded module over \( C \) is defined to be a family of \( C \)-modules \( \{ M_i | i \in \mathbb{Z} \} \). A graded morphism \( f \) of degree \( d \) between two graded modules \( M = \{ M_i \} \) and \( M' = \{ M'_i \} \) is defined to be a family of \( (\mathbb{k} \text{-bilinear}) \) natural transformations \( \{ f_i : M_i \to M'_{i+d} \} \). In this way the graded \( C \)-modules form a category \( C - \text{Mod}_\mathbb{Z} \). We will denote the identity morphism by \( 1_M : M \to M \) or just by \( 1 \) if the module is clear from the context.

Because the only morphisms in \( \mathbb{L} \otimes \mathbb{L} \) are scalar multiples of the identity morphisms a graded \( \mathbb{k} \)-quiver \( A \) consists of a set of objects \( \text{Ob} A \) and a graded central \( \mathbb{k} \)-bimodule \( A(i, j) \) for each pair \( i, j \in A \).

3.4. Tensor powers and duality. Let \( A \) be a graded \( \mathbb{k} \)-quiver over \( S \) and let \( n \) be a positive integer. We will define the graded \( \mathbb{k} \)-quiver \( A^{\otimes n} \). Here we will usually skip the subscript \( \mathbb{L} \). For all \( i, j \in A \) and a vector degree \( \vec{i} \in \mathbb{Z}^n \) define the graded \( \mathbb{k} \)-bimodule by the following formulae:

\[
(A^{\otimes n}(i, j))_{\vec{i}} := \bigoplus_{i_1, \ldots, i_n \in A} A(i_{n-1}, j)_{i_n} \otimes A(i_{n-2}, i_{n-1})_{i_{n-2}} \otimes \cdots \otimes A(i, i_1)_{i_1},
\]

and

\[
(A^{\otimes n}(i, j))_i = \bigoplus_{\vec{i} \in \mathbb{Z}^n, i = |\vec{i}|} A^{\otimes n}(i, j)_{\vec{i}},
\]

where \( |\vec{i}| = i_1 + \cdots + i_n \). For \( n = 0 \) we set \( A^{\otimes 0} := \mathbb{L} \).

Note that by abuse of notation we often write \( (a_1, \ldots, a_n) \) instead of \( a_1 \otimes \cdots \otimes a_n \in A^{\otimes n} \) to make things more readable when several tensor products at different “levels” occur.

**Remark 3.1.** If \( A \) and \( B \) are linear quivers and \( n \geq 0 \), then because of the universal property of the direct sum giving a \( \mathbb{k} \)-quiver morphism \( F : A^{\otimes n} \to B \) is the same as giving the restriction of \( F \) on each of the direct summands \( A(i_{n-1}, j) \otimes A(i_{n-2}, i_{n-1}) \otimes \cdots \otimes A(i, i_1) \).

If \( A \) and \( B \) are graded \( \mathbb{k} \)-quivers over \( S \), then \( \text{Hom}_\mathbb{L}(A, B) \) is a graded vector space via:

\[
\text{Hom}_\mathbb{L}(A, B)_k = \prod_{i \in \mathbb{Z}} \text{Hom}_\mathbb{L}(A_i, B_{i+k}).
\]

Consider now graded modules \( X_i, Y_i \) and graded morphisms \( f_i : X_i \to Y_i \) for \( i = 1, \ldots, N \). We will use the Koszul-Quillen sign rule for the action of the tensor product \( f_1 \otimes \cdots \otimes f_N : X_1 \otimes \cdots \otimes X_N \to Y_1 \otimes \cdots \otimes Y_N \), i.e. for elements \( x_i \in X_i \) we have

\[
(f_1 \otimes \cdots \otimes f_N)(x_1 \otimes \cdots \otimes x_N) = (-1)^d f_1(x_1) \otimes \cdots \otimes f_N(x_N)
\]
where
\[ d = (|f_2| + \cdots + |f_N|)|x_1| + (|f_3| + \cdots + |f_N|)|x_2| + \cdots + |f_N|)|x_{N-1}|, \]
i.e. the sign indicates how often \( f_j \) and \( x_i \) are switched before arriving at the correct position. If \( M = \{ M^i \} \) is a graded \( \mathbb{k} \)-module we denote by \( sM \) the graded \( \mathbb{k} \)-module given by \( (sM)^i = M^{i+1} \) for all \( i \in \mathbb{Z} \). We call \( sM \) the **suspension** (or the **shift** of \( M \)). To distinguish elements from \( M \) and \( sM \) we will denote the image of an element \( x \in M \) in \( sM \) by \( sx \), hence \( |sx| = |x| - 1 \).

There is a duality \( \mathbb{D} \) on the category of \( \mathbb{k} \)-quivers given by \( \mathbb{D}A(i, j) = A(i, j)^* \) for all \( i, j \in \mathcal{A} \). If \( \mathbb{L} \) is finite, and \( \mathcal{A}(i, j) \) and \( \mathcal{B}(i, j) \) are finite dimensional for all \( i, j \in \mathcal{A} \), then
\[ \mathbb{D}(A \otimes B) \cong \mathbb{D}A \otimes \mathbb{D}B. \]

3.5. **Tensor (pre)category.** Define the (reduced) tensor graded \( \mathbb{k} \)-quiver \( \overline{TC} \) of a graded quiver \( \mathcal{C} \) as follows: \( \text{Ob} \overline{TC} = \text{Ob} \mathcal{C} \) and the graded \( \mathbb{k} \)-module of morphisms by
\[ \overline{TC}(i, j) = \bigoplus_{n > 0} C^\otimes n(i, j) \]
for all \( i, j \in \mathcal{C} \). The \( \mathbb{k} \)-quiver \( \overline{TC} \) can be endowed with either the structure of a precategory or of a precocategory.\(^1\) The structure of a precategory is given by \( m : \overline{TC} \otimes \overline{TC} \to \overline{TC} \) in the usual way by "concatenation":
\[ (a_1 \otimes \cdots \otimes a_n) \cdot (a'_1 \otimes \cdots \otimes a'_m) := (a_1 \otimes \cdots \otimes a_n \otimes a'_1 \otimes \cdots \otimes a'_m). \]
The restriction of the comultiplication \( \Delta : \overline{TC} \to \overline{TC} \otimes \overline{TC} \) on \( \overline{TC}(i, j) \to \bigoplus_{k \in \mathcal{C}} \overline{TC}(k, j) \otimes \overline{TC}(i, k) \) is given by
\[ \Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n). \]

In the case of multiplication we will also consider the **tensor category** \( TC = \bigoplus_{i=0}^\infty C^\otimes i \), where \( C^\otimes 0 = \mathbb{L} \) and the multiplication \( m \) is defined as above with \( \mathbb{L} \) acting as the identity morphisms. We will also denote the tensor category by \( \mathbb{L}[C] \).

A \( \mathbb{k} \)-quiver map \( d : TC \to TC \) of degree \( k \) is called a **derivation** if \( d \circ \mu = \mu \circ (1 \otimes d + d \otimes 1) \).

A \( \mathbb{k} \)-quiver morphism of degree zero \( f : \overline{TC} \to \overline{TB} \) is called a **cofunctor** provided that \( \Delta f = (f \otimes f) \Delta \). A mapping of graded quivers \( b : \overline{TC} \to \overline{TC} \) of degree \( k \) is called a **coderivation** of degree \( k \) provided that \( \Delta b = (b \otimes 1 + 1 \otimes b) \circ \Delta \).

4. **A\(_\infty\)**-CATHERGIES

4.1. **Definitions of A\(_\infty\)**-categories. There are many equivalent definitions of an \( A\(_\infty\) \)-category, which differ mostly in signs and the direction of morphisms. We restrict ourselves to the definition most useful for our purposes. This is the definition which is based on the bar-construction together with the Koszul-Quillen sign convention as in \[16\] 3.6. Usually an \( A\(_\infty\) \)-category is defined using chains of objects. Using Remark \[31\] we use the equivalent language of \( \mathbb{L} \)-modules or graded quivers (for an introduction to the subject see also \[24\] \[2\] \[23\]).

**Definition 4.1.** A graded \( \mathbb{k} \)-quiver \( \mathcal{A} \) is called a (non-unital) \( A\(_\infty\) \)-**category** provided there is a graded coderivation \( b : \overline{TS} \mathcal{A} \to \overline{TS} \mathcal{A} \) of degree 1 such that \( b^2 = 0 \).

\(^1\)Recall that precategory (respectively precocategory) means that this structure may not have identity morphisms (respectively counits).
Due to the following lemma from [16] Lemma 3.6 the condition $b^2 = 0$ can be rewritten as the following family of equalities $(A_n)$, where $n \in \mathbb{N}$, which gives another definition of $A_\infty$-category:

**Lemma 4.2.** To define a (non-unital) $A_\infty$-category is equivalent to define linear quiver mappings $b_n : (sA)^{\otimes n} \to sA$ of degree 1 such that the following relations hold:

$$A_n : \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} b_{n-k+1}(1^{\otimes j} \otimes b_k \otimes 1^{\otimes (n-j-k)}) = 0.$$  

An $A_\infty$-functor between two $A_\infty$-categories $f : A \to B$ is a cofunctor $\mathcal{T}f : \mathcal{L}sA \to \mathcal{L}sB$ which commutes with the differentials, i.e.

$$(\mathcal{T}f \otimes \mathcal{T}f) \circ \Delta = \Delta \circ \mathcal{T}f, \quad b \circ \mathcal{T}f = \mathcal{T}f \circ b.$$  

4.2. **Strictly unital and augmented $A_\infty$-algebras.** An $A_\infty$-category $\mathcal{A}$ is called **strictly unital** provided that for every $i \in \mathcal{A}$ there exists an element $s1_i \in s\mathcal{A}^{-1}(1,1)$, such that for every $sx \in s\mathcal{A}(1,j)$ holds:

$$\begin{align*}
b_2(s1_j \otimes sx) &= sx, \quad b_2(sx \otimes s1_1) = (-1)^{|x|}sx, \quad \text{and} \quad b_n(\cdots \otimes s1_1 \otimes \cdots) = 0 \quad \text{for} \quad n \neq 2.
\end{align*}$$

This may seem a bit strange but is the natural translation of the fact that $1_1$ behaves as a unit with respect to the multiplication $m_2$ (when removing the $s$).

Obviously, $\mathcal{L}$ has the structure of a strictly unital $A_\infty$-category over itself if we define $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^i = 0$ for $i \neq 0$ with a unique nonzero multiplication $b_2$. For a strictly unital $A_\infty$-category $\mathcal{A}$ the embedding $i : s\mathcal{L} \hookrightarrow s\mathcal{A}$ is an $A_\infty$-functor. The category $\mathcal{A}$ is called **augmented**, provided there exists an augmentation, i.e. an $A_\infty$-functor $\eta : \mathcal{A} \to \mathcal{L}$ such that $\eta i = \text{id}_\mathcal{L}$.

4.3. **The Yoneda algebra as an $A_\infty$-category and Kadeishvili’s theorem.** The $A_\infty$-category we have in mind is that of the Yoneda category of the set of standard modules $\text{Ext}_A^*(\Delta, \Delta)$. This can be constructed as follows. First one takes a projective resolution $P$ of $\Delta$, then $\text{Hom}_A^*(P, P)$ has a natural structure of a dg category (i.e. an $A_\infty$-category with $d_n = 0$ for $n \geq 3$) with $b_1$ induced by the differential of the projective resolution and $b_2$ by the Yoneda product. (compare e.g. [16] (3.3))

In a next step the following theorem by Kadeishvili et alii can be invoked:

**Theorem 4.3** ([15], see also [14] [32] [29] [12] [13] [26]). Let $\mathcal{A}$ be an $A_\infty$-category. Then the homology category $H^* \mathcal{A}$ (where the homology is taken with respect to $b_1$) carries the structure of an $A_\infty$-category with $b_1 = 0$ and $b_2$ induced by the $b_2$ of $\mathcal{A}$.

Since $\text{Ext}_A^*(\Delta, \Delta)$ is the homology of $\text{Hom}_A^*(P, P)$ there is an $A_\infty$-structure on this quiver. The $b_1$ on $s \text{Ext}_A^*(\Delta, \Delta)$ can be constructed inductively, see [16] (7.8, 7.9) and [24] Appendix B] for an explanation and examples.

5. **Filtered modules as twisted stalks**

In [16] (7.4) and [17] (2) categories of the form $\mathcal{F}(M)$, where $M = (M_1, \ldots, M_r)$ are modules over an associative algebra, are defined in terms of twisted objects over an $A_\infty$-category $\mathcal{A}$. Here we will recall these constructions modifying them in the spirit of [31].
5.1. The category \( \text{add} \mathcal{A} \). Given an \( A_\infty \)-category \( \mathcal{A} \) we construct what we call its **additive enlargement** \( \text{add} \mathcal{A} \). A \( \mathbb{Z} \)-graded version of it is denoted \( \Sigma \mathcal{A} \) in [31], or in a slightly different form (that can be obtained via choosing bases of the vector spaces appearing) \( \text{Mat} \mathbb{Z} \mathcal{A} \) in [16]. As usually we present it using the bar-construction:

- The set of objects in \( \text{add} \mathcal{A} \) is that of \( \mathbb{L} \)-modules.
- For \( X, Y \in \text{add} \mathcal{A} \), \((\text{add} \mathcal{A})(X, Y) \) is the graded \( \mathbb{L} \otimes \mathbb{L} \)-module given by:

\[
\text{add} \mathcal{A}(X, Y) = \text{Hom}_k(X(i), Y(j)) \otimes_k \mathcal{A}(i, j)
\]

for \( i, j \in \mathcal{A} \). Note that \( \text{Hom}_k(X(i), Y(j)) \) is assumed to have degree 0. All the grading is in \( \mathcal{A} \) and hence also the shift only acts on the second tensor factor.
- The graded multiplications \( b_n^a \) are given by:

\[
b_n^a(f_1 \otimes s_{a_1}, \ldots, f_n \otimes s_{a_n}) = f_1 \ldots f_n \otimes b_n(s_{a_1} \otimes \cdots \otimes s_{a_n}).
\]

**Lemma 5.1.** This defines on \( \text{add} \mathcal{A} \) the structure of an \( A_\infty \)-category.

**Proof.** First compute \( b_n^a(f_{j+1} \otimes s_{a_{j+1}}, \ldots, f_{j+k} \otimes s_{a_{j+k}}) = f_{j+1} \cdots f_{j+k} \otimes b_k(s_{a_{j+1}}, \ldots, s_{a_{j+k}}) \)

Now applying \( b_{n-k+1}^a \) we get:

\[
b_{n-k+1}^a(f_1 \otimes s_{a_1}, \ldots, f_j \otimes s_{a_j}, f_{j+1} \cdots f_{j+k} \otimes b_k(s_{a_{j+1}}, \ldots, s_{a_{j+k}}), f_k \otimes s_{a_{j+k+1}}, \ldots, f_n \otimes s_{a_n})
\]

\[
= (f_1 \cdots f_n) \otimes b_{n-k+1}^a(s_{a_1} \cdots s_{a_j} \otimes b_k(s_{a_{j+1}}, \ldots, s_{a_{j+k}}) \otimes s_{a_{j+k+1}} \cdots s_{a_n})
\]

Summing over the indices with the appropriate signs (from the Koszul-Quillen sign rule) yields the equality. \( \square \)

5.2. Twisted modules. A **pretwisted module** over \( \mathcal{A} \) is a pair \((X, \delta)\), where \( X \in \text{add} \mathcal{A} \) and \( s\delta \in \text{add} \mathcal{A}(X, X)_0 \)

Let \((X, \delta)\) be a pretwisted module. Suppose \( s\delta = \sum f_k \otimes s_{a_k} \), where \( f_k : X(i_k) \to X(j_k) \) and \( s_{a_k} \in \mathcal{A}(i_k, j_k) \). Then a pretwisted submodule \((X', s\delta')\) is defined by the following: A family of subspaces \( X'(i) \subseteq X(i) \) such that \( f_k(X'(i_k)) \subseteq X'(j_k) \) for all \( k \). If \( f_k' : X'(i_k) \to X'(j_k) \) denotes the restriction of \( f_k \) then define \( s\delta' = \sum (f_k' \otimes s_{a_k}) \). There is an obvious way to define the notion of a pretwisted factor module \((X/X', s\delta/s\delta')\).

**Lemma 5.2.** The notions of submodule and factor module do not depend on the choice of the presentation.

**Proof.** Since \( k \)-vector spaces form a semisimple category it is possible to write \( X = X' \oplus Y \).

Then \( \text{Hom}(X, X) \otimes s\mathcal{A} = \text{Hom}(X', X') \otimes s\mathcal{A} \oplus \text{Hom}(X', Y) \otimes s\mathcal{A} \oplus \text{Hom}(Y, X) \otimes s\mathcal{A} \) by additivity of \( \text{Hom} \) and \( \otimes \). Now if \( \sum_j g_j \otimes s\mathcal{A} \) is in that space, then the part with \( X' \to Y \) vanishes and hence one can also assume that \( g_j : X' \to X' \). It follows that the notion of a factor module also does not depend on the choice of the presentation. \( \square \)

Now a **twisted module** over \( \mathcal{A} \) is defined to be a pretwisted module \((X, \delta)\) such that:

(TM1) There exists a filtration \((0, 0) = (X_0, \delta_0) \subseteq (X_1, \delta_1) \subseteq \cdots \subseteq (X_N, \delta_N) = (X, \delta)\) of pretwisted submodules such that the quotients have zero differential, i.e. \( (X_i/X_{i-1}, s\delta_i/s\delta_{i-1}) = (X_i/X_{i-1}, 0) \). This condition will be called **triangularity** in the remainder.

(TM2) The **Maurer-Cartan equation** is satisfied, i.e.

\[
\sum_{i=1}^{\infty} b_i^a(s\delta \otimes \cdots \otimes s\delta) = 0.
\]
Lemma 5.3.

The set of objects coincides with those of \( \mathcal{A} \).

For \( X,Y \in \mathcal{A} \) we define:

\[
\begin{align*}
\text{conv}(\mathcal{A})(X,Y) := \text{Hom}_{\mathcal{A}}(X,Y). \end{align*}
\]

Now, by the theorem of Kadishvili [see Theorem 3.1], \( \mathcal{A} \) is a quiver structure on \( \text{mod}(\mathcal{A}) \).

The morphisms from \((X,\Lambda)\) to \((Y,\Lambda)\) are given by \( \text{twmod}(\mathcal{A})(X,\Lambda) \rightarrow \text{twmod}(\mathcal{A})(Y,\Lambda) \).

Note that again \( \text{Hom}_{\mathcal{A}}(X,Y) \) has degree \( 0 \). Thus because of the sign change within \( \mathcal{A} \) the shift just operates on \( \text{conv}(\mathcal{A}) \) as it does on \( \mathcal{A} \) on the right hand side.

\[
\begin{align*}
\text{conv}(\mathcal{A})(X,Y) & := \sum_{i=0}^{\infty} \text{Hom}_{\mathcal{A}}(X,Y)^{i+1} \text{Hom}_{\mathcal{A}}(X,Y)^{i}, \\
& := \sum_{i=0}^{\infty} \text{Ext}^i_{\mathcal{A}}(X,Y).
\end{align*}
\]

Theorem 5.4 [17, 57], see also \[21\] Appendix E, Theorem 3.1. Suppose \( \mathcal{A} = (\Lambda_1, \ldots, \Lambda_k) \) is a quiver structure on \( \text{twmod}(\mathcal{A}) \).

Theorem 5.3. The operations defined above define an \( \mathcal{A} \)-category structure. And Keller and Leferve-Hasegawa have proven the following theorem:

Proof. A proof can be found e.g. in [27, 6.12].
the composition
\[ \mathbb{D}s\mathcal{A}(k_n, k_0) \xrightarrow{d_n} \bigoplus_{k_1, \ldots, k_{n-1} \in \mathbb{D}s\mathcal{A}} \bigotimes_{i=0}^{n-1} \mathbb{D}s\mathcal{A}(k_{i+1}, k_i) \]
\[ \xrightarrow{[F_1 \otimes \cdots \otimes F_n]} \bigoplus_{k_1, \ldots, k_{n-1} \in \mathbb{D}s\mathcal{A}} \bigotimes_{i=0}^{n-1} \text{Hom}_k(X(k_{i+1}), X(k_i)) \]
\[ \xrightarrow{\triangleright} \text{Hom}_k(X(k_n), X(k_0)). \]

Let \( V, W \) be \( k \)-vector spaces and let \( V \) be finite dimensional. Then there exists a natural isomorphism of vector spaces
\[ M : W \otimes_k V \cong \text{Hom}_k(V^*, W) \]
where \( M(w \otimes v)(\chi) = \chi(v)w \). In particular this gives an isomorphism of graded vector spaces \( M_{X,Y} : \text{add} \mathcal{A}(X, Y) \rightarrow \text{conv} \mathcal{A}(X, Y) \).

**Theorem 6.1.** We have \( M^{\otimes n} = b_n^c M^{\otimes n} \). In particular the mappings \( b_n^c \) endow \( \text{conv} \mathcal{A} \) with the structure of an \( A_\infty \)-category such that \( M \) defines an \( A_\infty \)-isofunctor between the \( A_\infty \)-categories \( \text{add} \mathcal{A} \) and \( \text{conv} \mathcal{A} \).

**Proof.** Fix \( X_1, \ldots, X_n \in \text{add} \mathcal{A} \) and morphisms \( f_i \otimes sa_i \in \text{add} \mathcal{A}(X_i, X_{i-1}) \). We will apply the left and the right hand side of the claim to \( (f_1 \otimes sa_1) \otimes \cdots \otimes (f_n \otimes sa_n) \) and compare the results, which are elements of \( \text{Hom}_k(\mathbb{D}s\mathcal{A}, \text{Hom}_k(X_n, X_1)) \). Hence we immediately apply them to some \( \chi \in \mathbb{D}s\mathcal{A} \). For the left hand side we obtain:
\[
M^{\otimes n}(f_1 \otimes sa_1, \ldots, f_n \otimes sa_n)(\chi) = M(f_1 \cdots f_n \otimes b_n(sa_1 \otimes \cdots \otimes sa_n))(\chi) = \chi(b_n(sa_1 \otimes \cdots \otimes sa_n))f_1 \cdots f_n
\]

To calculate the right hand side we will use Sweedler notation, i.e. we write \( d_n(\chi) = \sum \chi(1) \otimes \cdots \otimes \chi(n) \). The entries of \( \overline{M}(f \otimes sa) \) and \( \overline{sa} \) have the same degrees. Thus:
\[
b_n^c M^{\otimes n}(f_1 \otimes sa_1, \ldots, f_n \otimes sa_n)(\chi) = b_n^c(M(f_1 \otimes sa_1), \ldots, M(f_n \otimes sa_n))(\chi)
\]
\[
= v_n \circ [M(f_1 \otimes sa_1) \otimes \cdots \otimes M(f_n \otimes sa_n)](\sum (-1)^{c(\overline{\chi}, \overline{sa})}M(f_1 \otimes sa_1)(\chi(1)) \otimes \cdots \otimes M(f_n \otimes sa_n)(\chi(n)))
\]
\[
= \sum (-1)^{c(\overline{\chi}, \overline{sa})}v_n(\chi(1)(sa_1)f_1 \otimes \cdots \otimes \chi(n)(sa_n)f_n)
\]
\[
= \sum (-1)^{c(\overline{\chi}, \overline{sa})}\chi(1)(sa_1)\cdots \chi(n)(sa_n)f_1 \otimes \cdots \otimes f_n
\]
\[
= d_n(\chi)(sa_1, \ldots, sa_n)f_1 \cdots f_n
\]
\[
= \chi(b_n(sa_1 \otimes \cdots \otimes sa_n))f_1 \cdots f_n
\]
The other statements follow from the bijectivity of \( M \). \( \square \)

### 6.2. Representations of \( k \)-quivers.

The following is the obvious \( k \)-analogue of the correspondence between representations of quivers and modules over the corresponding category.

**Lemma 6.2.** Let \( Q \) be a \( k \)-quiver. Let \( \mathbb{L}[Q] \) be the tensor category. Then there is a one-to-one correspondence between modules over \( \mathbb{L}[Q] \) and pairs consisting of a functor \( X_0 : \)
Then the following hold:

Proposition 6.3. A twisted convolutional module, by the following proposition.

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\[ L \to k - \text{Mod} \] (the restriction of the module to \( L \)) and an \( L \)-bimodule morphism \( X_1 : Q \to \text{Hom}_k(X_0, X_0) \) (the restriction to \( Q \), the bimodule structure of \( \text{Hom}_k(X_0, X_0) \) is given by \((i, j) \mapsto \text{Hom}_k(X_0(i), X_0(j)) \)) here: Denote by \( v_k \) the composition map as before. Then \( X \) is given on \( Q^{\otimes k} \) by

\[
X_k : Q^{\otimes k} \to \text{Hom}_k(X_0, X_0), \quad X_k = v_k \circ X^{\otimes k}
\]

via \( X = \sum_{i=1}^{\infty} X_k \) on the restriction of \( X \) to \( \bigoplus_{k=1}^{\infty} Q^{\otimes k} \).

This is for modules over \( L[Q] \). To include the relations fix a set of generators \( R \subset C \) of the ideal \( I \), such that \( C \cong L[Q]/I \). Then \( X \) as described above belongs to \( C - \text{Mod} \) iff for any \( r \in R \) we have \( X(r) = 0 \).

A representation \( X \in C - \text{Mod} \) is called rational if there exists \( N \geq 0 \) such that \( X_N = 0 \).

6.3. Convolutional presentation of twisted modules. Now let \( Q^0 = \mathbb{D} \mathbb{A}^0 \). Then a pretwisted convolutional module is a module over \( L[Q^0] \). By the previous subsection this is the same as giving a functor \( X_0 : L \to k - \text{Mod} \) and an \( L \)-bimodule isomorphism \( X_1 : Q \to \text{Hom}(X_0, X_0) \). Furthermore by the foregoing section this is the same as giving a pretwisted module. Denote the representation corresponding to the pretwisted module \((X, \delta)\) by \( R_\delta \).

The notion of a twisted module can be translated in a similar way to obtain the notion of a twisted convolutional module, by the following proposition.

Proposition 6.3. Let \((X, \delta)\) be a pretwisted module over \( A \), and \( R_{\delta} \) be the corresponding functor \( L[Q^0] \to k - \text{Mod} \). Write \( s\delta = \sum f_k \otimes sa_k \) with the \( sa_k \) being linearly independent. Then the following hold:

(i) Let \( \iota, j \in \mathbb{A}, \chi \in Q^0(\iota, j), x \in X(\iota) \). Then

\[
R_\delta(\chi)(x) = \sum_k \chi(sa_k)f_k(x).
\]

(ii) \( \delta = 0 \) if and only if \( R_\delta \) is a semisimple module.

(iii) A pretwisted module \( s\delta' \) is a pretwisted submodule of \( s\delta \) iff \( R_{\delta'} \) is a submodule of \( R_\delta \).

(iv) If \( s\delta' \subseteq s\delta \), then \( R_{\delta/s\delta'} = R_\delta/R_{\delta'} \).

(v) \( \delta \) satisfies the triangularity condition iff \( R_\delta \) is a rational module over \( L[Q^0] \).

(vi) \( \delta \) satisfies the Maurer-Cartan equation iff \( R_{\delta} \circ d = 0 \), where \( d = (d_n) \) as constructed in subsection [6.2]. In other words iff \( R_\delta \) is a \( B := L[Q^0]/(1 \text{Id} d) \)-module.

Proof. (i) The first statement just follows by recalling the definition of \( M \).

(ii) Note that a module \( R : L[Q^0] \to k - \text{Mod} \) is a semisimple module iff \( R_1 = 0 \). Now write \( s\delta = \sum f_k \otimes sa_k \). We will evaluate special "characters" \( \chi_k \) defined by \( \chi_k(a_{k'}) = \delta_{k,k'} \), the Kronecker delta. By (i) we have \( R_\delta(\chi_k) = f_k \). Hence \( R_\delta = 0 \) iff \( f_k = 0 \) for all \( k \) and hence iff \( \delta = 0 \).

(iii) Write \( s\delta(\iota, j) = \sum f_k \otimes sa_k \) and \( s\delta' = \sum f'_k \otimes sa_k \). Now by (i) for \( x \in X'(\iota) \) (suppressing the inclusions) \( R_{\delta}(\chi)(x) = \sum \chi(sa_k)f_k(x) \) and \( R_{\delta'}(\chi) = \sum \chi(sa_k)f'_k(x) \). Now for one direction note that if \( s\delta' \subseteq s\delta \), then \( f_k = f'_k \) for \( x \in X'(\iota) \), hence \( R_{\delta}(\chi)(x) = R_{\delta'}(\chi)(x) \) for \( x \in X' \) and hence \( R_{\delta'} \) is a subrepresentation of \( R_\delta \). For the reverse direction note that if \( R_{\delta'} \) is a subrepresentation of \( R_\delta \), then evaluating them on \( \chi_k \) yields \( f_k = f'_k \) on \( X' \). Hence \( s\delta' \) is a subrepresentation of \( s\delta \).

(iv) This is proved analogously.

(v) The foregoing item shows that the triangularity condition is equivalent to the existence of a composition series. Hence to rationality.
(vi) For the last claim note that because $s\delta \in \mathcal{D} sA^0$ we have that $\varepsilon((s\delta, M(f \otimes sa))) = 0$. Hence we can replace $[F_1 \otimes \cdots \otimes F_n]$ in the definition of $b^n$ by $(F_1 \otimes \cdots \otimes F_n)$. Furthermore since $M$ is an isomorphism, $\sum_n b^n_n((s\delta)^{\otimes n}) = 0$ iff

$$0 = \sum_{n=1}^{\infty} M b^n_n((s\delta)^{\otimes n}) = \sum_n b^n_n M^{\otimes n}((s\delta)^{\otimes n})$$

$$= \sum_n v_n((M(s\delta))^{\otimes n}) d_n$$

$$= \sum_n (R_{\delta})_n d_n = R_{\delta} d$$

by Theorem 6.1 definition of $b^n_n$ and the previous subsection.

□

Hence we define a **twisted convolutional module** to be a rational module over $B$, where $B$ is as defined in the foregoing proposition.

Thus there is a possibility to present the objects of the category of filtered modules in terms of a quiver with relations. Unfortunately the morphisms cannot just be module homomorphisms since the category of filtered modules is usually not abelian. The next section introduces the notion of a box which will enable us to take care also of the morphisms in this category.

### 7. Boxes and their representations

#### 7.1. Definition of box representations.
In this section we recall the notion of a box, formerly called bocs (for bimodule of coalgebra structure). For a general introduction in the context of Drozd’s tame and wild dichotomy theorem, see e.g. [9, 7, 10]. The theory of left and right algebras we are using is contained in [6] or in the nice unpublished manuscript [5]. For a more recent general introduction in a slightly different language, see e.g. [1].

**Definition 7.1.** A **box** $\mathcal{B} := (B, W, \mu, \varepsilon)$ or simply $\mathcal{B} = (B, W)$ consists of a category $B$, a $B$-$B$-bimodule $W$, a coassociative $B$-bimodule comultiplication $\mu : W \to W \otimes_B W$ and a corresponding $B$-bilinear counit $\varepsilon : W \to B$. A box not necessarily having a counit will be called a **prebox**.

Note that in this definition $W$ is not assumed to have any degree, so especially the Koszul-Quillen sign rule does not apply.

For a box $\mathcal{B}$ we can define its category of representations. Unlike most categories of representations however this category will in general not be abelian which is an advantage for our purposes.

**Definition 7.2.** The category $\mathcal{B} - \text{Mod}$ (respectively $\mathcal{B} - \text{mod}$) of **representations** (respectively **locally finite dimensional representations**) of a box $\mathcal{B}$ is defined as follows:

(i) Objects of $\mathcal{B} - \text{Mod}$ are $B$-modules (respectively locally finite dimensional $B$-modules).

(ii) Morphisms between box representations $X, Y \in \mathcal{B} - \text{Mod}$ are given by

$$\text{Hom}_B(X, Y) = \text{Hom}_{B \otimes B^{op}}(W, \text{Hom}_k(X, Y)),$$

and for morphisms $f \in \text{Hom}_B(X, Y)$ and $g \in \text{Hom}_B(Y, Z)$ their composition $gf$ in $\mathcal{B} - \text{Mod}$ is the morphism of box representations obtained by composing the following
A standard checking shows the canonical isomorphism identity morphism in \( \text{Hom} \) (see e.g. p. 93). There a morphism \( f: B \) is then defined by composing the following of the diagrams for a natural transformation.

\[
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\]

(Here, \( \nu_2 \) is the composition map defined in (6.1))

The associativity of the composition in \( \mathcal{B} - \text{Mod} \) follows from the coassociativity of \( \mu \). The identity morphism in \( \text{Hom}_{\mathcal{B}}(X, X) \) can be defined via the following composition of bimodule homomorphisms (because of the counit property):

\[
W \overset{\delta}{\to} B \overset{\delta_X}{\to} \text{Hom}_k(X, X),
\]

where \( \delta_X \) is the map which is uniquely determined by \( \delta_X(1) = 1 \) via the commutativity of the diagrams for a natural transformation).

This is a slightly altered definition of morphism compared to the traditional one (see e.g. p. 93]). There a morphism \( f : X \to Y \) is given by a homomorphism of \( B \)-modules \( f : W \otimes_B X \to Y \). The composition of two such morphisms of box representations \( f \) and \( g \) is then defined by composing the following \( B \)-module homomorphisms:

\[
gf : W \otimes_B X \overset{\mu \otimes 1_X}{\to} W \otimes_B W \overset{1_W \otimes f}{\to} W \otimes_B Y \overset{g}{\to} Z.
\]

A standard checking shows the canonical isomorphism

\[
\text{Hom}_{B \otimes_B \mathcal{B}'}(W, \text{Hom}_k(X, Y)) \cong \text{Hom}_B(W \otimes_B X, Y),
\]

and that the two definitions of composition agree via this isomorphism.

7.2. Differential graded categories and boxes. We explain here the classical transition from (certain) differential graded categories to boxes. The boxes in Section 8 will arise in this way.

**Lemma 7.3.** Let \( B \) be a category over \( S \) and let \( U_1 \) be a \( B \)-bimodule. Assume that the tensor category \( U := \bigoplus_{i=0}^\infty U_1^{\otimes i} \), where \( U_1^{\otimes 0} = B \), is endowed with a grading, such that \( |B| = 0 \) and \( |U_1| = 1 \). Suppose \( U \) is equipped with a differential \( d \) (a derivation of degree 1 that squares to 0). Denote by \( (d(B)) \) the \( B \)-bimodule generated by \( d(B) \), set \( W := U_1/(d(B)) \), and denote the canonical projection \( U_1 \to W \) by \( \pi \). Then there is a prebox \( (B, W, \mu) \), where \( \mu \) is induced by \( d_1 : U_1 \to U_1 \otimes U_1 \), i.e. \( (\pi \otimes \pi \otimes d_1 = \mu \pi \).

**Proof.** We have \( \pi \otimes (b_1, b_2) = \pi \otimes (b_1 \otimes b_2) = b_1 d(u)b_2 + b_1 d(u) b_2 + b_1 d(u) \) by definition of \( \pi \). Hence \( \mu \) is well-defined and a bimodule homomorphism. To prove coassociativity note that

\[
(\mu \otimes 1 - 1 \otimes \mu) \mu \pi = (\mu \otimes 1 \otimes 1 \otimes \mu) \pi \otimes d_1
\]

\[
= (\mu \otimes \pi \otimes \pi \otimes \mu \pi) d_1
\]

\[
= \pi \otimes ((d_1 \otimes 1 + 1 \otimes d_1) d_1)
\]

\[
= \pi \otimes d_1 d_1 |_{U_1} = 0
\]

Note that the Koszul-Quillen sign rule forces the sign change from \(-\) to \(+\) as \( U_1 \) and \( d \) are of degree 1 and hence \( (1 \otimes d_1)(u \otimes u') = -u \otimes d_1(u') \).

Now we provide a box (without pre) version of this lemma:

**Lemma 7.4.** Let \( B \) be a category over \( S \) and let \( U_1 \) be a \( B \)-bimodule. Furthermore, let \( U_1 = U_\omega \oplus \overline{U}, \) such that \( U_\omega \) is the direct sum of principal bimodules \( U_\omega = \bigoplus_{i \in \mathbb{F}} B \omega_i B \), where \( \omega_i : i \to 1 \). Suppose that the following conditions hold:

1. \( d_1 d_1 |_{U_\omega} = \omega_1 \otimes \omega_1 + r_1, \) where \( r_1 \in \overline{U} \otimes \overline{U} \),
(d2) for all \( b \in B(1, j) \) we have \( d(b) = \omega_j b - b \omega_1 + \partial b \) for some \( \partial b \in U \).

(d3) for all \( u \in U(1, j) \) we have \( d(u) = \omega_j u + u \omega_1 + \partial u \) for some \( \partial u \in U \otimes U \).

Let \( W, \mu \) be as in the foregoing lemma. Then there is a \( B \)-bimodule map \( \tilde{\varepsilon} : U_1 \to B \) given by \( \tilde{\varepsilon}(\omega_1) = 1_1 \) and \( \tilde{\varepsilon}(U) = 0 \). It induces a unique \( \varepsilon : W \to B \) with \( \varepsilon = \varepsilon \pi \) such that \( (B, W, \mu, \varepsilon) \) is a box.

Proof. By definition and (d2) we have \( \tilde{\varepsilon}(d(B)) = 0 \). Thus \( \varepsilon \) is well-defined. To see the property of the counit we compute:

\[
(\varepsilon \otimes 1)\mu \pi(u) = (\varepsilon \otimes 1)\pi^{\otimes 2} d_1(u) = (\tilde{\varepsilon} \otimes \pi)d_1(u) = (\tilde{\varepsilon} \otimes \pi)(\omega_j \otimes u + u \otimes \omega_1 + \partial u) = 1_j \otimes \pi(u)
\]

for \( u \in U(1, j) \) by (d3). and

\[
(\varepsilon \otimes 1)\mu \pi(\omega_1) = (\tilde{\varepsilon} \otimes \pi)(\omega_1 \otimes \omega_1 + r_1) = 1_1 \otimes \pi(\omega_1)
\]

by (d1). \( \square \)

An important property of boxes for applying the theory of Burt and Butler in Section 9 is that \( \overline{W} := \ker \varepsilon \) is a projective bimodule. The bimodules obtained via the methods in this subsection share this property:

**Lemma 7.5.** If \( \mathfrak{B} = (B, W) \) is a box obtained as in the foregoing lemma and assume in addition that \( U \) is a projective bimodule. Then \( \overline{W} := \ker \varepsilon \) is a projective bimodule.

Proof. Since \( \tilde{\varepsilon}(U) = 0 \), we have that \( \pi(U) \subseteq \overline{W} \). Moreover because of (d1) and (d2) we have that \( \pi \) in fact induces an isomorphism \( U \to \overline{W} \) since furthermore \( \bigoplus B\omega_1 B/(\omega_j b - b \omega_1 b \in B(1, j)) \cong B \). In particular \( \overline{W} \) is a projective bimodule. \( \square \)

### 7.3. Morphisms between box representations in terms of generators

In this subsection we are concerned with the question how to describe the morphisms in the category of modules over a box in some sense similar to representations of quivers. This makes it easier in the next section to translate the twisted modules to the box setting. First note the following universal property of projective bimodules:

**Lemma 7.6.** Let \( \varphi_k : i_k \to j_k \) be a system of generators of a projective bimodule \( \bigoplus B\varphi_k B \). Then for every bimodule \( M \) and any set of elements \( m_k : i_k \to j_k \) in \( M \) there exists a bimodule homomorphism \( f : \bigoplus B\varphi_k B \to M \) with \( f(\varphi_k) = m_k \).

Proof. This follows from the universal property of a free module. A proof can be found in [Lemma 2.6 (2)](#).

This can now be used to describe homomorphisms. Let \( X, Y \) be \( \mathfrak{B} \)-representations, where the box \( \mathfrak{B} = (B, W) \) is built from a differential graded tensor category as in the previous subsection. In addition to the assumptions there, suppose that \( U \) is a projective bimodule. Let \( I := (d(B)) \) and let \( Q_1 \) be the \( \mathbb{k} \)-quiver over \( S \) generated over \( \mathbb{k} \) by the \( \varphi_k \), a set of generators for \( U_1 \). Then

\[
\text{Hom}_{\mathfrak{B} \otimes \mathfrak{B}\text{-}\text{op}}(W, \text{Hom}_\mathbb{k}(X, Y)) = \text{Hom}_{\mathfrak{B} \otimes \mathfrak{B}\text{-}\text{op}}((\bigoplus B\varphi_k B)/I, \text{Hom}_\mathbb{k}(X, Y)) \]

\[
\begin{align*}
\downarrow & R \\
\{ R & \in \text{Hom}_{\mathfrak{B} \otimes \mathfrak{B}\text{-}\text{op}}((\bigoplus B\varphi_k B, \text{Hom}_\mathbb{k}(X, Y)) | R(I) = 0 \} \\
\downarrow & f \\
\{ f & \in \bigoplus \text{Hom}_\mathbb{k}(Q_1(i, j), \text{Hom}_\mathbb{k}(X(i), Y(j)) | R_f(I) = 0 \},
\end{align*}
\]
where \( R_f \) is given by \( R_f(b_1 \varphi_k b_2) = b_1 f(\varphi_k) b_2 \).

8. Filtered modules as box representation

Note that for the \( A_\infty \)-category \( A \) of extensions of \( \Delta \) the following properties hold:

(E1) \( A^i = 0 \) for all \( i < 0 \),

(E2) \( S \), the underlying set of \( L_\Delta \), is finite and \( A(\hat{i}, j) \) is finite dimensional for all \( i \) and \( j \),

(E3) \( A \) is strictly unital, and

(E4) \( A \) is augmented.

Denote by \( Q \) the graded \( k \)-quiver \( D_s A \). By (E1) we have that \( Q^i = 0 \) for \( i > 1 \). Now let \( T \) be the graded tensor category \( L[Q] \). The following lemma is the crucial observation for translating the category of filtered modules to the box language:

**Lemma 8.1.**

(i) \( T \) is a differential graded category with respect to \( d \) as defined above and the usual multiplication.

(ii) The ideal \( I \) generated by all \( Q^i \) with \( i < 0 \) and all \( d(Q^{-1}) \) is a differential ideal with respect to the \( d \) constructed above.

(iii) The factor \( U = T/I \) is a differential category freely generated over \( B = L[Q^0]/(L[Q^0] \cap I) \) by the \( k \)-quiver \( Q^1 \). In particular, \( U \) satisfies the conditions of Lemma 7.3 and defines a prebox \( \mathfrak{B} := (B, W, \mu) \), where \( W = U_{i_1}/(d(B)) \), and \( \mu \) is induced by \( d \).

(iv) For every \( i \in L \) denote by \( \omega_i \) the element dual to \( s_\Delta^i \). Then the augmentation decomposition \( Q = D_s L_{\Delta A} \oplus D_s \ker \eta \) satisfies the assumptions of Lemma 7.4. In particular the prebox defines a box.

(v) This box \( \mathfrak{B} \) has a projective kernel.

**Proof.**

(i) Since \( D\Delta_{T,s_A} = \mu_{T,D_s A} \) and \( 0 = \mathbb{D}(b^2) = (\mathbb{D} b)^2 \) we have that \( \mathbb{D} b \) is a differential and hence \( T \) is a differential graded category.

(ii) Since \( d \) has degree 1 this is by construction.

(iii) Since \( d \) is of degree 1 we have that \( d(I) \subseteq L[Q^0] \). Hence freeness follows.

(iv) Since \( A \) is strictly unital and augmented, the assumptions of Lemma 7.4 are satisfied.

(v) This is just Lemma 7.5 \( \square \)

Now combining all the results, the following theorem establishes the description of filtered modules that we need:

**Theorem 8.2.** Let \( A \) be a quasi-hereditary algebra with set of standard modules \( \Delta \). Then there exists a directed box \( \mathfrak{B} = (B, W) \) such that \( \text{mod } \mathfrak{B} \simeq \mathcal{F}(\Delta) \).

**Proof.** The lemma above constructs a directed box \( \mathfrak{B} \). Now using the results of Section 6 we can prove that \( R_f(I) = 0 \) is equivalent to being in \( H^0(\text{twmod} A) = Z^0(\text{twmod} A) \) since \( B^0(\text{twmod} A) = 0 \).

\[
0 = M_{b_1}^{\text{twmod} A}(sf) = M \sum_{\gamma \in \mathbb{N}^2} b_{1+|\gamma|}^{\mathbb{N}^2} (s^\delta \otimes M f \otimes s^\delta \otimes M f) \\
= \sum_{\gamma \in \mathbb{N}^2} b_{1+|\gamma|}^{\mathbb{N}^2} (s^\delta \otimes M f \otimes s^\delta \otimes M f) \\
= \sum_{\gamma \in \mathbb{N}^2} v_{1+|\gamma|}^{\mathbb{N}^2} ([M(s^\delta) \otimes M f \otimes M(s^\delta)] d_{1+|\gamma|}) \\
= \sum (R_f)_{1+|\gamma|}^{\mathbb{N}^2} (d_{1+|\gamma|}) \\
= -R_f \circ d
\]
since the degrees are fixed. For the composition the calculation is similar:

\[
Mb^w_2(sf, sg) = \sum Mb^w_{2+|\tau|}((s\delta)^{\otimes 0} \otimes sf \otimes (s\delta)^{\otimes 1} \otimes sg \otimes (s\delta)^{\otimes 2})
\]

\[
= \sum b^w_{2+|\tau|}M^{\otimes n}((s\delta)^{\otimes 0} \otimes sf \otimes (s\delta)^{\otimes 1} \otimes sg \otimes (s\delta)^{\otimes 2})
\]

\[
= \sum v_{2+|\tau|}[M(s\delta)^{\otimes 0} \otimes M(sf) \otimes M(s\delta)^{\otimes 1} \otimes M(sg) \otimes M(s\delta)^{\otimes 2}]d_{2+|\tau|}
\]

Since the sign given by \([\cdots]\) compared to the tensor product (without the Koszul-Quillen sign rule) is +1 independent of the summand we get that the two compositions coincide. \(\square\)

9. Boxes: Burt-Butler theory

In this section, we are going to review the relevant part of the results by Burt and Butler; the definition of the left and right Burt-Butler algebras (just called \(L\) and \(R\) by Burt and Butler), basic properties of these algebras and the description of the category of representations of a box as categories of induced or coinduced modules inside the module categories of the left and right algebras, respectively. We do not need any facts from the sophisticated representation theory of boxes, only the definition and some basic techniques on coalgebras and from homological algebra. All material is taken from [6]; another exposition of the same material is contained in [5].

9.1. Left and right Burt-Butler algebras. The left or right \(B\)-module \(B\) is, by definition, a left or right representation of the box \(\mathfrak{B}\) (here right representation is defined by the obvious ”dual” of left representation) and as such has endomorphism rings. These turn out to be exactly the algebras we are looking for. These have been studied by Burt and Butler [6] under the name left and right algebra of a box, respectively.

**Definition 9.1.** The **left Burt-Butler algebra** \(L_\mathfrak{B} = L\) of a given box \(\mathfrak{B} = (B, W)\) is the endomorphism ring \(\text{End}_{\mathfrak{B}^{\text{op}}}(B, B) \cong \text{Hom}_B(W_B, B_B)\) of the right module \(B\), i.e. with multiplication \(e \cdot f : W \xrightarrow{e} W \otimes W \xrightarrow{f \otimes 1} B \otimes W \simeq W \xrightarrow{\varepsilon} B\).

The **right Burt-Butler algebra** \(R_\mathfrak{B} = R\) of a given box \(\mathfrak{B}\) is the algebra \(\text{End}_\mathfrak{B}(B, B)^{\text{op}} \cong \text{Hom}_B(B_W, B_B)\) of the left module \(B\), i.e. with multiplication \(e \cdot f : W \xrightarrow{e} W \otimes W \xrightarrow{1 \otimes \varepsilon} W \otimes B \simeq W \xrightarrow{f} B\).

In both cases, \(\varepsilon\) is the identity element.

9.2. Representations of boxes as induced and coinduced modules. It is proven in [5, Proposition 1.2] that there is an \(L\)-\(R\)-bimodule structure on \(W\). This bimodule structure on \(L_W R\) defines functors \(W \otimes_R - : R\mod \rightarrow L\mod \rightarrow \text{Mod} \) and \(\text{Hom}_L(W, -) : L\mod \rightarrow R\mod \rightarrow \text{Mod}\), which will be seen to restrict to equivalences between certain subcategories.

**Definition 9.2.** The category \(\text{Ind}(B, R)\) of **induced modules** is the full subcategory of \(R\mod\) whose objects are of the form \(R_M \simeq R \otimes_B X\) for some finitely generated \(B\)-module \(X\).

The category \(\text{CoInd}(B, L)\) of **coinduced modules** is the full subcategory of \(L\mod\) whose objects are of the form \(L_M \simeq \text{Hom}_B(L, X)\) for some finitely generated \(B\)-module \(X\).

To prove that these functors are exact the following proposition is useful:

**Proposition 9.3** ([6, Proposition 2.2]). Let \(\varepsilon\) be surjective and \(\overline{W}\) be a projective bimodule which is finitely generated both as a left and a right \(B\)-module. Then there is a natural isomorphism \(R \otimes_B - \cong \text{Hom}_B(W, -)\) of functors \(B\mod \rightarrow R\mod \rightarrow \text{mod}\) and a natural
isomorphism $W \otimes_B \cong \text{Hom}_B(L, -)$ of functors $B - \text{mod} \to L - \text{mod}$. In particular these functors are exact.

**Theorem 9.4** ([6, Theorems 2.2, 2.4, 2.5]). Let $\mathcal{B} = (B, W)$ be a box. Suppose $\varepsilon$ is surjective and $W$ is a projective bimodule, finitely generated both as a left and a right $B$-module. The categories $\text{Ind}(B, R)$ and $\text{CoInd}(B, L)$ of (co-)induced modules are equivalent to the category $\mathcal{B} - \text{mod}$ of representations of the box $\mathcal{B}$.

### 9.3. Comparing extensions

For the "if" part of our main theorem the following result on comparison of extensions turns out to be useful for proving that the induced modules form a standardisable set of modules:

**Theorem 9.5** ([6, Corollary 3.5]). Suppose $\mathcal{B} = (B, W)$ is a box such that $W$ is a projective bimodule and $\varepsilon$ is surjective. Then the categories $\text{Ind}(B, R)$ and $\text{CoInd}(B, L)$ are closed under extensions and every such extension is induced by an extension of $B$. Furthermore there are maps

$$\text{Ext}^n_B(X, Y) \to \text{Ext}^n_R(R \otimes_B X, R \otimes_B Y)$$

and

$$\text{Ext}^n_B(X, Y) \to \text{Ext}^n_L(\text{Hom}_B(L, X), \text{Hom}_B(L, Y))$$

which are epimorphisms for $n = 1$ and isomorphisms for $n \geq 2$.

### 9.4. Double centraliser and Ext-injectives

A certain double centraliser property is ubiquitous in the theory of quasi-hereditary algebras, thus it is mandatory to also have it for the boxes we want to consider:

**Theorem 9.6** ([6, Proposition 2.7]). Let $\mathcal{B} = (B, W)$ be a box such that $W$ is a projective bimodule and $\varepsilon$ is surjective. Then there is a double centraliser property:

$$L \cong \text{End}_{R^{op}}(W) \quad \text{and} \quad R^{op} \cong \text{End}_L(W).$$

To prove that the left and right algebra of a box are their respective Ringel duals the following proposition will be useful:

**Theorem 9.7** ([6, Theorem 5.2]). Suppose that $\mathcal{B} = (B, W)$ is a box with projective kernel such that $\mathcal{B} - \text{mod}$ is fully additive, i.e. if $e \in \text{Hom}_B(M, M)$ is an idempotent, then there is a box representation $N$ and morphisms of box representations $\pi : M \to N$ and $\iota : N \to M$ such that $e = \iota \pi$ and $\pi \iota = \text{id}_N$. Then a module in $\text{Ind}(B, R)$ is Ext-projective iff it is projective and Ext-injective iff it belongs to $\text{add}(DW)$.

### 10. Proofs of the main results and some corollaries

In this section we will now collect all the results and prove our main theorems. We start by defining the class of directed boxes:

**Definition 10.1.** The *bigraph* of a box with projective kernel is given by the quiver of $B$ as the set of degree 0 arrows and for every copy of $Be_i \otimes e_j B$ in $W$, a dashed arrow $j \to i$. A box with projective kernel is called *directed* provided its bigraph is directed as an oriented graph.

The following theorem now is the "if" direction of our main theorem. It appeared in a slightly different language already in [3].
Theorem 10.2. Let $\mathfrak{B} = (B,W)$ be a directed box. Then the left and right algebra of $B$ are quasi-hereditary algebras.

Proof. Let $\Delta(i) := R \otimes_B L(i)$. Then according to the Ext-comparison these modules have no extensions (because the algebra $B$ has none). Now for the homomorphisms note that $\text{Hom}_R(\Delta(i), \Delta(j)) \cong \text{Hom}_B(L(i), L(j)) = \text{Hom}_B(W \otimes L(i), L(j))$. Because $W$ is a factor of a bimodule with direct summands $Bc_k \otimes e_j B$ for $k \geq 1$, $\text{Hom}_B(W \otimes L(i), L(j))$ is nonzero only if $i \leq j$ because $B$ is directed. Hence the $\Delta$ form a standard system and $R \cong R \otimes_B B$ is a projective generator. Hence $R$ is quasi-hereditary.

The proof for $L$ gives modules $\nabla(i) := \text{Hom}_R(L, L(i))$ which form a standard system, such that $\nabla(\nabla)$ contains the injective cogenerator $D(L) = \text{Hom}_B(L, DB)$. Hence $L$ is quasi-hereditary by the dual statement of the standardisation theorem.

The following part is the ”only if” direction which is the direction where all the machinery developed in this article is needed.

Theorem 10.3. Let $\Delta$ be a standard system in an abelian $k$-category and let $\mathcal{F}(\Delta)$ be the category of objects with a $\Delta$-filtration. Then there exists a directed box $\mathfrak{B} = (B,W)$ such that $\mathcal{F}(\Delta)$ is equivalent as an exact $k$-category to the category $\mathfrak{B} – \text{mod}$ of finitely generated $\mathfrak{B}$-representations.

Moreover $\mathcal{F}(\Delta)$ is equivalent as an exact $k$-category to the category of $\Delta$-filtered modules for the quasi-hereditary algebra $R_\mathfrak{B}$.

In particular, every quasi-hereditary algebra $(B, \leq)$ is Morita equivalent with the same quasi-hereditary structure to the right Burt-Butler algebra of a directed box.

Proof. We have already constructed a directed box $\mathfrak{B}$ with the wanted properties in the previous sections. By the previous theorem the right Burt-Butler algebra of that box is now a quasi-hereditary algebra and the results from the previous section by Burt and Butler guarantee that the category of filtered modules $\mathcal{F}(\Delta)$ is equivalent for both quasi-hereditary algebras. Then the Dlab-Ringel standardisation theorem tells us that the two algebras must be Morita equivalent.

Note that when starting the translation procedure upon which the above proof is based with a quasi-hereditary algebra $(A, \leq)$, the algebra $A$ gets lost already in the first step, which only keeps the category $\mathcal{F}(\Delta)$ up to categorical equivalence. Hence the resulting quasi-hereditary algebra $R_\mathfrak{B}$ is Morita equivalent, but not necessarily equal to $A$.

In the appendix we will recall and discuss an example of [19], which in the present context shows that $R_\mathfrak{B}$ need not, in general, be isomorphic to $A$ itself.

Corollary 10.4. For every quasi-hereditary algebra $A$ there is a Morita equivalent algebra $R$ which has an exact Borel subalgebra $B$.

Proof. Let $B$ be the algebra from the box $\mathfrak{B} = (B,W)$ constructed above. Obviously $B$ is directed, $B \to R, b \mapsto bc$ provides an inclusion, the tensor functor is exact by Proposition 9.3 and sends the simples to the standard modules by construction.

Theorem 10.5. Let $A$ be a quasi-hereditary algebra. Let $\mathfrak{B}$ be its box. Then $L_\mathfrak{B}$ is Morita equivalent to the Ringel dual of $A$. In particular $\mathcal{F}(\nabla_{R(A)}) \cong \mathcal{F}(\Delta_A)$ and vice versa.

Proof. Since $\mathfrak{B} – \text{mod}$ is equivalent to $\mathcal{F}(\Delta)$ it is of course fully additive. Hence $DW$ contains exactly the Ext-injective objects of $\mathfrak{B} – \text{mod} \cong \mathcal{F}(\Delta)$ as direct summands. It is well known (see e.g. [30]) that the same is true for $T$. Hence add $DW \cong \text{add} T$ via the restriction of the equivalence $\mathfrak{B} – \text{mod} \cong A – \text{mod}$, where $T$ is the characteristic tilting module of $A$. Thus
End_R(DW)^{op} \cong \text{End}_{R^{op}}(W) \cong L \text{ (by Theorem 9.6)} is Morita equivalent to the Ringel dual of } A.

The last claim follows since both are equivalent to \( \mathcal{B} - \text{mod} \) via inducing (respectively coinducing).

**Corollary 10.6** ([30 Theorem 2]). The category \( \mathcal{F}(\Delta) \) has almost split sequences.

**Proof.** This follows from the corresponding claim for the category \( \text{Ind}(B, R) \) for a box \( \mathcal{B} = (B, W) \) such that \( B \) is an Artin algebra, \( \varepsilon \) is surjective and \( \overline{W} \) is a projective bimodule which was proven in [31 (4.1)].

Also other results can now be reproven from the corresponding results for boxes, e.g. the following:

**Corollary 10.7** ([30 Theorem 5]). The characteristic tilting module is a tilting module.

**Proof.** The corresponding result is that \( W \) is a cotilting module by [31 Theorem 5.1].

In some sense a directed box of a quasi-hereditary algebra contains more information than an arbitrary exact Borel subalgebra:

**Corollary 10.8.** Let \( \mathcal{B} = (B, W) \) be a directed box, \( R_{\mathcal{B}} \) be its right algebra. Then the class of induced modules from \( B \) to \( R_{\mathcal{B}} \) is closed under extensions and there is a map

\[
\text{Ext}^n_B(X, Y) \to \text{Ext}^n_R(R \otimes_B X, R \otimes_B Y)
\]

which is an epimorphism for \( n = 1 \) and an isomorphism for \( n \geq 2 \). A similar claim holds for \( L_{\mathcal{B}} \).

**Proof.** This follows immediately from Theorem 9.5.

An example where this is not true for an exact Borel subalgebra not coming from a box is discussed in Appendix A.4.

The Ringel dual of an algebra can also be constructed as the opposite algebra of the "opposite" box:

**Proposition 10.9.** Let \( \mathcal{B} = (B, W) \) be a box. Then there is a box \( \mathcal{B}^{op} = (B^{op}, W^{op}) \) such that \( L_{\mathcal{B}} \cong R_{\mathcal{B}^{op}}^{op} \). In particular \( R(R(A)^{op})^{op} \cong A \).

**Proof.** Let \( \mathcal{B} = (B, W) \) be a box. Then \( B^{op} \) is also a category and each \( B-B \)-bimodule can also be regarded as a \( B^{op}-B^{op} \)-bimodule via the equivalence of right \( B \)- and left \( B^{op} \)-modules (and vice versa). Denote \( W \), regarded as a \( B^{op}-B^{op} \)-bimodule, as \( W^{op} \). Write \( \mu \) in Sweedler notation, i.e. \( \mu(v) = \sum v(1) \otimes v(2) \), then \( \mu^{op} \) is given by \( \mu^{op}(v) = \sum v(2) \otimes v(1) \). Hence if we compare the multiplication \( e^{op} \cdot f^{op} = f^{op} \cdot e^{op} \) in the left algebra \( \text{Hom}_B(W_B, B_B) \):

\[
W \xrightarrow{f} W \otimes_B W \xrightarrow{f \otimes 1} B \otimes_B W \xrightarrow{=} W \xrightarrow{=} B
\]

with the multiplication \( e^{op} \ast f^{op} = f^{op} \cdot e^{op} \) in the opposite algebra of the right algebra \( \text{Hom}_B(W_B, B_B)^{op} \):

\[
W^{op} \xrightarrow{\mu^{op}} W^{op} \otimes_{B^{op}} W^{op} \xrightarrow{1 \otimes f^{op}} W^{op} \otimes_{B^{op}} B^{op} \xrightarrow{=} W^{op} \xrightarrow{e^{op}} B^{op}
\]

we get the same resulting map \( v \mapsto e(f(v(1)) \otimes v(2)) = e^{op}(v(2) \ast f^{op}(v(1))) \).

Since taking the opposite algebra twice gives back the original algebra, there is an isomorphism \( R(R(A)^{op})^{op} \cong A \).
APPENDIX A. EXAMPLES

A.1. The regular block of $\mathfrak{sl}_2$. We start our series of examples by the basic algebra $A$ corresponding to a regular block of $\mathfrak{sl}_2$, that is the algebra given by the following quiver:

$$
\begin{array}{ccc}
2 & \xrightarrow{\alpha} & 1 \\
\downarrow{\beta} & & \\
1 & \xleftarrow{\alpha} & 2
\end{array}
$$

with the relation $\beta \alpha$, where we compose arrows like linear maps. It has the following projective modules (written in terms of their socle series) with standard modules consisting of the "boxed" vertices:

$$
\begin{array}{ccc}
& & \\
& 2 & \\
\downarrow{\beta} & & \\
2 & \xleftarrow{\alpha} & 1
\end{array}
$$

The $A_\infty$-Yoneda category $A$ looks as follows:

$$
\Delta(1) \xrightarrow{=} \Delta(2)
$$

where the solid arrow (of degree 1) corresponds to the extension of $\Delta(2)$ by $\Delta(1)$ given by $P(1)$ and the dashed arrow corresponds to the inclusion of $\Delta(1)$ in $\Delta(2)$. The multiplications all equal zero except for the ones with the identity morphisms that we have omitted in our picture. This is because $b_1$ is always zero since the Yoneda category is a minimal model, i.e. the homology of some algebra, and $b_{\geq 2}$ are zero if there are no paths of length $\geq 2$ in the graded quiver. Now we have to compute $\mathbb{D}sA$ according to our procedure. This is given by the following quiver (here we have included the duals of the identity morphisms).

$$
\begin{array}{ccc}
\omega_1 \xleftarrow{\alpha} \Delta(1) & \xrightarrow{=} & \Delta(2) \xrightarrow{\omega_2}
\end{array}
$$

So a Borel subalgebra is given by the path algebra of the Dynkin quiver $A_2$. To compute the right algebra of the box (which according to our results is Morita equivalent to the regular block of $\mathfrak{sl}_2$) we have to compute the opposite of the endomorphism algebra of $B$, as a representation of the box.

A representation of that box is by definition a representation of $B$ and a morphism of box representations is an assignment of linear maps making the following square commutative (were the map on the diagonal doesn’t satisfy any conditions since $\partial a = 0$):

$$
\begin{array}{ccc}
V_1 & \xrightarrow{f_{\omega_1}} & W_1 \\
V_a & \xrightarrow{f_{\phi}} & W_a \\
V_2 & \xrightarrow{f_{\omega_2}} & W_2
\end{array}
$$
So for the box representation $B$ we get the following diagram:

So the resulting opposite endomorphism algebra is 5-dimensional and hence will be isomorphic to $A$. We check this fact by computing the composition of such maps. Therefore we write two such morphisms next to each other (one with primes and one without) and since $\partial v = 0$ we just have to compute the composition corresponding to $\omega_2 v + v \omega_1$ to get the resulting map on the diagonal. Since the $\omega_i$ are grouplike, the corresponding maps are just given by the composition of the maps corresponding to the $\omega_i$:

We have omitted the solid arrows in the rightmost diagram to make it more readable. This algebra can be represented by the following matrix algebra which is isomorphic to $A^{\text{op}}$:

That the right algebra of the box is isomorphic (and not only Morita equivalent) to the original basic algebra we started with seems to be a low rank phenomenon. Even in the case of blocks of category $\mathcal{O}$ (which were proven to have Borel subalgebras in [19]) we will usually get ”bigger” algebras. This happens for example already for the singular block of $\mathfrak{sl}_3$ or for the block of the parabolic version of category $\mathcal{O}$ corresponding to the Levi subalgebra $\mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{gl}_2(\mathbb{C}) \subseteq \mathfrak{gl}_3(\mathbb{C})$. In the latter case the $A_\infty$-structure on the Ext-algebra was computed by Klamt and Stroppel in [18].

A.2. An example from the reduction algorithm. The next example gives counterexamples to some conjectures one might have, i.e. that the directed box giving a quasi-hereditary algebra is unique or that the connectedness of the biquiver given by the box implies the connectedness of the algebra. We modify the box obtained in the last example slightly, so that $a$ becomes a so-called irregular or superfluous arrow:

The next example gives counterexamples to some conjectures one might have, i.e. that the directed box giving a quasi-hereditary algebra is unique or that the connectedness of the biquiver given by the box implies the connectedness of the algebra. We modify the box obtained in the last example slightly, so that $a$ becomes a so-called irregular or superfluous arrow:
with $\partial a = v$ (and $\partial v = 0$). Now a box representation is still a representation of $B$ as before but morphisms involve the diagonal, i.e. by (d2) we must have $f_\omega V_a - W_a f_\omega + \phi = 0$.

$$
\begin{align*}
V_1 & \xrightarrow{f_\omega_1} W_1 \\
v_a & \xrightarrow{f_\omega} W_a \\
V_2 & \xrightarrow{f_\omega_2} W_2
\end{align*}
$$

For the possible endomorphisms of $B$ we thus get:

$$
\begin{align*}
\k & \to \begin{pmatrix} \lambda - r \\ -t \end{pmatrix} \\
\k^2 & \to \begin{pmatrix} r & s \\ t & u \end{pmatrix}
\end{align*}
$$

It is easy to see (the diagonal is determined by the matrix rings) that the composition of two morphisms just corresponds to the multiplication of the matrix rings. Thus the right algebra (which is again 5-dimensional) of the box is:

$$
\begin{pmatrix} \lambda & 0 & 0 \\ 0 & r & s \\ 0 & t & u \end{pmatrix}
$$

Now if we would do our process with the basic algebra corresponding to that algebra, i.e. $\k \times \k$ we would end up with the following box:

$$
\omega_1 \dashrightarrow 1 \xrightarrow{\omega_1} 2 \xrightarrow{\omega_2} 3
$$

That the two boxes have equivalent representation categories was known before under the name of "regularisation".

A.3. **An algebra not having an exact Borel subalgebra.** We have remarked that not every algebra has an exact Borel subalgebra. The following example mentioned in [19] is such an algebra. But there is an error in the argument in [19]. In contrast to our main theorem it is claimed that there is no Morita equivalent version having an exact Borel subalgebra. Here we will compute a Morita equivalent algebra having an exact Borel subalgebra. This also contradicts the result in [21] that having an exact Borel subalgebra is Morita invariant. The algebra is given by quiver and relations as follows:

$$
\begin{align*}
1 & \to 2 \to 4 \\
2 & \to 3
\end{align*}
$$
where the dotted line in the middle indicates the obvious commutativity relation. We get the following projective modules. Again we have boxed the corresponding standard modules.

The following diagram gives the $A_\infty$-Yoneda category:

Since most of the standard modules are simple, the arrows of degree 1 on the left and the dotted arrow of degree 2 just correspond to arrows in the original quiver. The solid arrow $\Delta(2) \to \Delta(4)$ corresponds to the extension given by $\text{rad } P(1)$ and the dashed arrow $\Delta(3) \to \Delta(4)$ corresponds to the inclusion.

All the $b_i$ are zero for $i \geq 4$ since there are no paths of length $\geq 4$. There is only one path of length 3, but there is no arrow (of degree 1) from $\Delta(1)$ to $\Delta(4)$, hence $b_3$ is also zero. Since $b_2$ is induced by the Yoneda product it can be calculated quite easily by choosing representatives of the arrows. We omit the calculations here.

This results in the differentials given by $d(r) = \gamma \alpha$, $\partial \beta = \varphi \gamma$ and zero in the other cases. Thus a Borel subalgebra $B$ is given by the solid arrows in this quiver and the relation $d(r) = 0$.

Now $B$ is given by the following representation where the arrows are numbered from top to bottom:
For the morphisms we will again omit the solid arrows for space reasons. The commutativities give the following endomorphisms:

\[
\begin{align*}
\k & \rightarrow \k & \k^2 & \rightarrow \k^2 \\
\mathbf{1} & \rightarrow \mathbf{1} & \mathbf{0} & \rightarrow \mathbf{0} \\
\mathbf{0} & \rightarrow \mathbf{0}
\end{align*}
\]

The composition is as follows:

\[
\begin{align*}
\k & \rightarrow \k & \k^2 & \rightarrow \k^2 \\
\mathbf{1} & \rightarrow \mathbf{1} & \mathbf{0} & \rightarrow \mathbf{0} \\
\mathbf{0} & \rightarrow \mathbf{0}
\end{align*}
\]

One checks quite straightforwardly that setting one of the parameters \(a, d, e, h\) or \(k\) to 1 and all others to 0 one gets an idempotent and that \(a\ell e = \ell, hme = m\) and \(kne = n\), where the latter corresponds to the parameter set to 1 and all others to 0. Explicit calculations show that this gives the following algebra which is Morita equivalent but not isomorphic to...
the algebra $A^{op}$:

$$
\begin{pmatrix}
    a & c & f & g & \ell \\
    0 & d & 0 & 0 & -b \\
    0 & 0 & h & i & m + b \\
    0 & 0 & j & k & n \\
    0 & 0 & 0 & 0 & e
\end{pmatrix}.
$$

We can quite easily see the Borel subalgebra $B$. If $a \in B$ then $(a \cdot \varepsilon)(\varphi) = \varepsilon(\varphi \cdot a) = \varepsilon(\varphi) \cdot a = 0$ by definition and since $\varepsilon$ is a $B$-$B$-bimodule homomorphism. This implies that $f = c$, $h = d$ and $j = \ell = m = n = 0$. The remaining parameters form an 8-dimensional algebra, which needs to be isomorphic to $B^{op}$ by the theory we developed. Thus we have the opposite of the Borel subalgebra $B^{op}$:

$$
\begin{pmatrix}
    a & c & c & g & 0 \\
    0 & d & 0 & 0 & -b \\
    0 & 0 & d & i & b \\
    0 & 0 & 0 & k & 0 \\
    0 & 0 & 0 & 0 & e
\end{pmatrix}.
$$

Alternatively one can compute by hand that $ae_1\varepsilon + de_2\varepsilon + ee_3\varepsilon + ke_4\varepsilon + \alpha e + \beta\varepsilon + \gamma\varepsilon + \delta\varepsilon + \eta\varepsilon + \zeta\varepsilon$ is a subalgebra isomorphic to $B^{op}$.

A.4. **An exact Borel subalgebra not associated with a directed box.** Note that the exact Borel subalgebras we give do not exhaust all exact Borel subalgebras. The exact Borel subalgebras given by our process have the special property that the induction functor is dense on the category of $\Delta$-filtered modules (see Theorem 9.5). An exact Borel subalgebra where this property fails was given in [21]. It is the algebra given by quiver:

$$
\begin{tikzcd}
2 \arrow{r}{\alpha} & 3 \\
1 \arrow{u}{\gamma} & 3 \arrow{u}{\delta}
\end{tikzcd}
$$

with relations $\delta\gamma = 0 = \alpha\beta$. According to the cited reference it has a Borel subalgebra given by the subquiver:

$$
\begin{tikzcd}
2 \arrow{r}{\alpha} & 3 \\
1 \arrow{u}{\gamma}
\end{tikzcd}
$$

We use our algorithm to produce an exact Borel subalgebra (of a Morita equivalent algebra) such that the induction functor is dense on $\mathcal{F}(\Delta)$. Again we start by writing down the
projective modules (with boxed standard modules):

Then the Ext-category is given by the following graded quiver (it is easy to see that the $\Delta$ have a projective resolution of length 2, hence $\text{Ext}^2$ vanishes):

$$
\begin{array}{c}
\Delta(2) \\
\alpha \\
\psi \\
\gamma \\
\varphi \\
\Delta(3) \\
\Delta(1)
\end{array}
$$

The following pushout diagram shows that $b_2(\psi, \gamma) = \gamma$ and the reason why we named $\gamma$ like that:

$$
\begin{array}{c}
\Delta(2) \\
\downarrow \\
P(1)/\Delta(3) \\
\downarrow \\
\Delta(3) \\
\downarrow \\
C \\
\downarrow \\
\Delta(1)
\end{array}
$$

where $C$ is the module

Note that $C$ also is the module which is not reached by the induction functor of the exact Borel subalgebra given above. It is also easy to see that $b_2(\alpha, \gamma) = 0$ and all other $b_i$ are zero by similar reasoning as in the other examples. Now $B$ is given by the following representation:
Now when writing the morphisms we again omit the solid arrows:

\[
\begin{array}{c}
k \rightarrow \kappa \\
/r \\/CZ
\end{array}
\]

\[
\begin{array}{c}
\kappa^2 \rightarrow \kappa^2 \\
/b \quad \ell \\
f \\
h - a \\
j \quad o
\end{array}
\]

\[
\begin{array}{c}
\kappa^4 \rightarrow \kappa^4 \\
(a \quad c \quad b \quad e) \\
0 \quad d \quad f \quad g \\
0 \quad 0 \quad h \quad i \\
0 \quad 0 \quad j \quad k
\end{array}
\]

We leave it to the reader to check that the following matrix algebra gives an algebra isomorphic to \( R^{op} \) and therefore Morita equivalent to \( A^{op} \) and that the given subalgebra gives an exact Borel subalgebra. The methods to check this are the same as in the last example:

\[
\begin{pmatrix}
ar & c & 0 & 0 \\
0 & d & e & 0 \\
0 & 0 & f & g \\
0 & 0 & 0 & h
\end{pmatrix}
\begin{pmatrix}
\ell \\
m \\
n + c \\
o
\end{pmatrix}
\preceq
\begin{pmatrix}
ar & c & 0 & 0 & 0 \\
0 & d & 0 & e & 0 \\
0 & 0 & f & 0 & 0 \\
0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & d
\end{pmatrix}
\]

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