A CALCULUS OF INCONSISTENCY I: SENTENTIAL LOGIC

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Abstract. We describe a graph-theoretic syntax for self-referential formulas as well as a four-valued logic to include contradictory and independent formulas. We then explore the degree to which generalized truth tables can be realized in our theory, and go on to describe a model theory for sentential calculus, wherein models are allowed to include contradictions (such as the “Liar”) and formulas that result from them as an integral part of their structure. This sets the groundwork for a sequel in which we construct models of set theory that include contradictions.

1. Introduction

Any logic that permits self-reference opens itself to paradox, as illustrated by the Liar’s paradox, a simple form of which states “This statement is false.” In their study of the subject, Barwise and Etchemendy [BE] address this and similar paradoxes by introducing a “non-Russellian” logic that includes a hierarchy of discourse in which self-referential statements, such as the Liar, are either true or false.

If we wish to include self-referential statements without abandoning the Russellian system of logic, we are faced with contradictions. Classically, one has the high school proof that any contradiction implies that all statements are contradictions, thereby contaminating the whole universe with contradiction. Consider the simplest variant of this argument: “If $p$ is true then $p \lor q$ is true for any formula $q$, but then if $p$ is also false, it follows that $q$ must be true.” In our analysis of this argument, if $p$ is a contradiction and $q$ is false, then $p \lor q$ is also a contradiction, and our rules for computing truth will not permit us to conclude anything more about $q$. In this way we avoid contaminating the entire model.

The approach we shall use to compute truth is the traditional hierarchical approach: the truth of formulas that are not subformulas of axioms can only be computed from “below”; that is, from a knowledge of the truth of subformulas. They can only be inferred from above if they are subformulas of axioms, or through arguments (such as rules of inference) outside the model.

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With regard to rules of inference, we shall see that all the classical rules of inference still apply, but that few of them are tautologies (in particular, modus ponens is not a tautology).

Our models of logic will be based on graphs analogous to those used by Axcel to model set theory with antifoundation in [A]. In this paper we restrict our syntax to sentential logic; richer forms of syntax such as that in Smullyan [S] will be developed in a future paper. We shall also find that the most natural setting in which to include contradictions is through the use of a four-valued system of logic, where the two additional values are \( L \) ("is a lie") for contradictions, and \( V \) ("is vacuous") for independent statements.

In an unpublished manuscript, Linton [L] outlines a four-valued system of logic whose four truth values are somewhat reminiscent of ours, although he does not interpret them as we do here, nor does he develop a calculus of their use or an application to self-referential statements.

In §2, we describe the basic calculus of our four-valued system of logic and observe that all formulas in classical propositional logic can be identified with special formulas in our sense. In §3, we describe (Proposition 3.3 and Theorem 3.9) exactly what kinds of truth tables can be realized by formulas. In §4 we outline our model theory for sentential calculus, describe how to determine truth in a model, and give models of sentential calculus that extend the classical one; one in which the Liar is a contradiction, and another in which the Liar is false.

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2. Preliminaries

Axcel [A] constructed anti-foundational models of set theory using pointed graphs with cycles to model membership hierarchies in sets. In the same vein, the formulas in our version of sentential logic will be pointed graphs with cycles. If \( \Gamma \) is a set, recall that a \( \Gamma \)-graph consists of a finite set \( Z \) of \textit{directed edges} together with functions \( S : Z \to \Gamma \) and \( T : Z \to \Gamma \) called \textit{source} and \textit{target} maps respectively. We refer to the sources and targets of the arrows as \textit{nodes}. If \( z \) is a directed edge, then the node \( T(z) \) is called a \textit{child} of \( S(z) \), while \( S(z) \) is called a \textit{parent} of \( T(z) \).

A \textit{subgraph} of a \( \Gamma \)-graph \( G \) is a \( \Gamma \)-graph \( H \) each of whose nodes is a node in \( G \) such that \( H \) is closed under children in \( G \), and if \( z \) is an edge in \( G \) between nodes in \( H \), then \( z \) is in \( H \). A \textit{family} is a subgraph consisting of a node and all its children. A \textit{path} through the graph is a finite sequence of nodes \( \langle n_0, \ldots, n_k \rangle \), each connected to the next by a directed edge. If such a path exists then \( n_k \) is a \textit{descendant} of \( n_0 \). A \textit{pointed} \( \Gamma \)-graph is a pair \( (G, n^*) \) where \( G \) is a \( \Gamma \)-graph and \( n^* \) is a distinguished node in \( G \).
The nodes in our graphs will be propositional letters and labelled k-ary boolean operators: pairs \((p, \gamma)\) where \(p\) is a label (in some alphabet set) and \(\gamma : \{t, f\}^k \to \{t, f\}\) for some \(k \geq 0\). (We can think of propositional letters as labelled \(-\infty\)-ary boolean operators so that all nodes are labelled operators.)

**Definitions 2.1.** Let \(\Gamma\) be the collection of all labelled boolean operators and propositional letters. A formula is a triple \(\Phi = (G, n^*, F)\) where:

1. \((G, n^*)\) is a pointed \(\Gamma\)-graph such that nodes with \(k\) children are \(k\)-ary boolean operators for \(k > 0\) and nodes with no children are either constant boolean operators or propositional letters.
2. \(F\) is a set of nodes of \(G\) which are said to be free.

In order that the interpretation of a graph not be ambiguous, we assume that the set of children of each node is ordered. A pointed subgraph of a formula \(\Phi\) is called a subformula if its free nodes are free in \(\Phi\). Its distinguished node need not coincide with that of \(\Phi\).

**Remark 2.2.** In formulas of classical sentential logic, the propositional letters can be thought of as free variables. We have generalized this idea by designating arbitrary nodes in a formula as free.

Note that each propositional letter can occur only once in a formula, whereas the same boolean operator can occur multiple times with different labels. Following are two simple examples of formulas. In displaying a formula, we omit the labels on the boolean operators, we arrange the arrows originating at each boolean operator from left to right to reflect the order of their children (when order is important, as in the case of non-symmetric operators), we place a "\(*\)" next to the distinguished node, and we show free nodes in double circles:

```
      a
     ↬*
    ↓  ↓
 p  q

 a b
    ↬*∨
   ↓  ⬇
 p
```

**Remarks 2.3.**

1. Formula \(a\) is the classical formula \(p \to q\).
2. Cycles in the underlying graph permit us to encode self-referential formulas. For instance, \(b\) above can be read as "\(p \lor b\)".
3. We can encode formulas in classical sentential logic (such as \(a\) above) using trees as the underlying graphs (see below).
4. Although we disallow duplicate copies of propositional letters in our graphs, classical formulas mentioning a propositional letter more than once can obviously still be realized by identifying the corresponding nodes.
Barwise and Etchemendy construct a syntax for self-referential statements in [BE]. Following is a list of some of their examples showing their notation and our equivalent representation of these statements as formulas.

**Examples 2.4.**

1. **The Liar ("This proposition is false."):**
   \[ \phi = [Fa \ \phi] \]
   \[\neg * \]

2. **The Strengthened Liar ("The Liar is false."):**
   \[ \phi = [Fa \ \phi] \]
   \[ \psi = [Fa \ \phi] \]
   \[\neg * \neg \]

3. **Liar Cycle of Length Three ("The next proposition is true. The next proposition is true. The first proposition is false."):**
   \[ \phi_1 = [Tr \ \phi_2] \]
   \[ \phi_2 = [Tr \ \psi] \]
   \[ \psi = [Fa \ \phi_1] \]
   \[\neg \]

4. **The Contingent Liar ("Max has the three of clubs and this proposition is false."):**
   \[ \phi = [Max \ H \ 3C] \land [Fa \ \phi] \]
   \[\neg \]

5. **Contingent Liar Cycle ("Max has the three of clubs. The next proposition is true. At least one of the first two propositions is false."):**
   \[ \phi_1 = [Max \ H \ 3C] \]
   \[ \phi_2 = [Tr \ \psi] \]
   \[ \psi = [Fa \ \phi_1] \lor [Fa \ \phi_2] \]
   \[\neg \]

Note that the propositional letter \( p \) corresponds to the proposition \([Max \ H \ 3C] \).
Again, \( p \) corresponds to \([Max \ H \ 3C]\).

(6) Löb’s Paradox (“If this proposition is true, then Max has the three of clubs.”):
\[
\phi = [Fa \ \phi] \lor [Max \ H \ 3C]
\]
(Note that the node \( p \) is not free.)

(7) Gupta’s Puzzle (“Max has the three of clubs. The last two propositions are true. At least one of the last two propositions is false. Claire has the three of clubs. At most one of the first three propositions is true.”):
\[
\begin{align*}
\phi_1 &= [Max \ H \ 3C] \\
\phi_2 &= [Tr \ \psi_1] \land [Tr \ \psi_2] \\
\phi_3 &= \phi_2 \\
\psi_1 &= [Claire \ H \ 3C] \\
\psi_2 &= ([Fa \ \phi_1] \land [Fa \ \phi_2]) \lor ([Fa \ \phi_1] \land [Fa \ \phi_3]) \lor ([Fa \ \phi_2] \land [Fa \ \phi_3])
\end{align*}
\]

\( p \) corresponds to the proposition \([Max \ H \ 3C]\) and \( q \) to the proposition \([Claire \ H \ 3C]\). \( \Phi \) is the boolean operator that says “at most one of these three is true.” The location of the star will change depending upon which truth value we wish to compute (see below).

**Definitions 2.5.** An **evaluation** of a formula \( \Phi \) is a function \( e \) from the set of free nodes to the set \( \{ T, F, V, L \} \). A node that is mapped to \( T \) or \( F \) is said to be **bound as true** or **false** respectively. A **proposition** is a pair \( (\Phi, e) \) where \( \Phi \) is a formula and \( e \) is an evaluation.

If \( (\Phi, e) \) is a proposition with underlying set of nodes \( N \), then a **hypothesis** on \( (\Phi, e) \) is a function \( H : N \rightarrow \{ t, f \} \). (Note that \( H \) is not required to take nodes bound as true (respectively false) to \( t \) (respectively \( f \)). However, we shall see shortly that it will rapidly evolve to one that does.)

If \( H \) is a hypothesis on \( (\Phi, e) \) such that \( H(n^*) = t \) (respectively \( f \)), then we say that the formula or proposition is **assumed true** (respectively **assumed false**).

To describe rules for changing the evaluations on unbound nodes, we need some notation. Let \( c \) be a node with children \( \{ c_1, \ldots, c_n \} \) (so that \( c \) is an \( n \)-ary boolean operator — note that one of the \( c_i \) may be \( c \) itself) and denote by \( \rho = (\epsilon c_1, \ldots, \epsilon c_n; \epsilon c) \) a row in the truth table of \( c \). Thus, each \( \epsilon \) is either \( t \) or \( f \), and \( \epsilon c \) is the **output** value of \( c \) determined by the **inputs** \( \epsilon c_i \). Note that
permitting loops in the graphs has the effect that rows of the truth table may include truth values for the same node in two slots: as input and output. If \( C \) is a nonempty collection of children of \( c \), denote by \( \rho|C \) the sub-tuple \((\epsilon_{c_i}; \epsilon_c)\) of \( \rho \) with inputs indexed on the children in \( C \). If \( H \) is a hypothesis on a proposition whose underlying graph includes \( c \) and its family, denote by \( H|C \) the corresponding tuple \((\mu_{c_i}; \mu_c)\) of \( ts \) and \( fs \) determined by \( H \).

**Definition 2.6.** Let \( H, K \) be hypotheses on a proposition \((\Phi, e)\). We say that \( K \) is an elementary consequence of \( H \), and write \( H \vdash K \), if \( K \) is obtained from \( H \) by changing its value on a single node \( c \) in one of the following ways:

1. If \( c \) is bound as \( T \) and \( H(c) = f \), then \( K(c) = \neg H(c) \).
2. If \( c \) is bound as \( F \) and \( H(c) = t \), then \( K(c) = \neg H(c) \).
3. If \( c \) is bound as \( L \), then \( K(c) = \neg H(c) \).
4. If \( c \) is not bound and is a parent (possibly of itself), let \( C \) be a nonempty subset of its children, and let \( \rho \) be a row of the truth table of \( c \) such that \( H|C \neq \rho|C \), but does agree if we change only the output coordinate \( \mu_c \) of \( H|C \) to its negation. Then \( K(c) = \neg H(c) \).
5. If \( c \) is not bound and is a child (possibly of itself), let \( d \) be a parent, let \( C \) be a subset of the children of \( d \) containing \( c \), and let \( \rho \) be a row of the truth table of \( c \) such that \( H|C \neq \rho|C \), but does agree if we change only the input coordinate \( \mu_c \) of \( H|C \) to its negation. Then \( K(c) = \neg H(c) \).

Note that no elementary consequence can change the value of any node bound as \( V \). We say that \( K \) is a consequence of \( H \), or that the assumption \( H \) leads to the conclusion \( K \), and write \( H \Rightarrow K \), iff \( K \) is obtained from \( H \) by following a finite sequence of elementary consequences.

**Examples 2.7.**

Note that one of the following must hold for any proposition \((\Phi, e)\):

(\( \mathbf{T} \)) Every assumption that \((\Phi, e)\) is false leads to the conclusion that it is true, and not every assumption that \((\Phi, e)\) is true leads to the conclusion that it is false.
(F) Every assumption that \((\Phi, e)\) is true leads to the conclusion that it is false, and not every assumption that \((\Phi, e)\) is false leads to the conclusion that it is true.

(V) Not every assumption that \((\Phi, e)\) is true leads to the conclusion that it is false, and not every assumption that \((\Phi, e)\) is false leads to the conclusion that it is true.

(L) Every assumption that \((\Phi, e)\) is false leads to the conclusion that it is true, and every assumption that \((\Phi, e)\) is true leads to the conclusion that it is false.

Remarks 2.8.

(1) In the cases that not every assumption that \((\Phi, e)\) is true (respectively false) leads to the conclusion that it is false (respectively true), we will sometimes say that \((\Phi, e)\) can get stuck in true (respectively false).

(2) It follows from the definitions that every node bound as \(T\) becomes \(t\) under an elementary consequence, and its value is subsequently fixed as \(t\). Similarly, nodes bound as \(F\) turn \(f\) and remain so.

The above observation leads us to a four-valued system of logic:

Definition 2.9. A proposition \((\Phi, e)\) has a truth value \(P \in \{T, F, V, L\}\) depending on which of the above possibilities holds.

Remarks 2.10.

(1) \(T\) stands for “true”, \(F\) for “false”, \(V\) for “vacuous”, and \(L\) for “lie.”

(2) The Liar (“This statement is false.”)

\[
\begin{array}{c}
\circ \circ \\
\ast \\
\end{array}
\]

has truth value \(L\). On the other hand, the Vacuous Affirmation (“This statement is true.”):

\[
\begin{array}{c}
\circ \\
\ast \\
\end{array}
\]

has truth value \(V\).

(3) Let’s compute the truth values of the other formulas in Example 2.4. Note that they agree with the heuristic reasoning found in Barwise and Etchemendy [BE]. The strengthened liar has truth value \(L\), as does any liar cycle. The contingent liar and contingent liar cycle have truth value \(F\) when the propositional letter is bound as \(F\), and truth value \(L\) when the propositional letter is bound as \(T\). L"ob’s Paradox has truth value \(T\) (Note that in this example no nodes are free and we are computing the truth value of the propositional letter \(p\)). In Gupta’s puzzle, under the assumption that Claire has the three of clubs and Max does not we bind \(p\) as \(F\) and \(q\) as \(T\). By starring each node in succession we see that the nodes corresponding
Definition 2.11. A formula is **well-grounded** if no node is a descendant of itself and every node is a descendant of the distinguished node. It is **strongly well-grounded** if, in addition, the free nodes are precisely the propositional letters.

Note that there is a natural bijection $\Psi : W \to P$ from the collection $W$ of strongly well-grounded formulas to the collection $P$ of formulas in sentential calculus. If $e$ is an evaluation of $\Phi \in W$ with the propositional letters bound as $T$ or $F$, then $\Psi(\Phi)$ has a truth value coinciding with the truth value of $(\Phi, e)$ under the assignment of values to the propositional letters under $\Psi$ (note that the truth value under these circumstances is always $T$ or $F$). We can therefore identify the strongly well-grounded formulas with the corresponding formulas of sentential calculus.

Example 2.12. $p \leftrightarrow \neg p$ can be identified with the following well-grounded formula:

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3. Logical Equivalence and Classification of Propositions

Definition 3.1. A formula $\Phi$ is **completely connected** if each boolean operator in $\Phi$ has every node (including itself) as a child.

Given a formula $\Phi$ with $k$ nodes, we can define a completely connected formula $\Phi'$ by replacing each boolean operator $\gamma : \{t, f\}^q \to \{t, f\}$ with a $k$-ary boolean operator $\gamma' : \{t, f\}^k \to \{t, f\}$ obtained by composing $\gamma$ with the evident projection $\pi : \{t, f\}^k \to \{t, f\}^q$. Note that $\Phi$ and $\Phi'$ are indistinguishable in the sense that corresponding hypotheses on $\Phi$ and $\Phi'$ have corresponding elementary consequences.

Definitions 3.2. The **truth table** of a formula with $k$ free nodes is the function $h : \{T, F, L, V\}^k \to \{T, F, L, V\}$ obtained by computing the truth value of the starred node resulting from each evaluation of the free nodes. (Implicit here is an ordering of the free nodes, since we are thinking of $\{T, F, L, V\}^k$ as the collection of all evaluations of the free nodes.) If a formula contains no free nodes then its truth table consists of its (single) truth value. The **restricted truth table** of a formula is the restriction of its truth table to $\{T, F\}^k$.

Examples 3.3.

(1) The truth table of the Liar is $L$. 

(2) If $\Phi$ is any formula with $k$ free nodes in which the distinguished node $n^*$ is free, then its truth table is the projection $\pi : \{T, F, L, V\}^k \to \{T, F, L, V\}$ onto the coordinate associated with $n^*$.

(3) Following is a representation of the truth table $\{T, F, L, V\}^2 \to \{T, F, L, V\}$ of the formula $p \land q$:

| T | F | L | V |
|---|---|---|---|
| T | T | F | L |
| F | F | F | F |
| L | L | F | L |
| V | V | F | V |

Definitions 3.4. Two formulas with the same free nodes are logically equivalent if they have the same truth table. They are weakly logically equivalent if they have the same restricted truth table. A (strong) tautology is a formula whose truth table has constant value $T$ while a weak tautology is a formula whose restricted truth table has constant value $T$.

The above definitions beg the following questions:

1. Are weakly logically equivalent formulas logically equivalent?
2. Are all truth tables realizable? That is, is every function $h : \{T, F, L, V\}^k \to \{T, F, L, V\}$ the truth table of some formula with $k$ free nodes?
3. Are all restricted truth tables realizable?
4. If $\Psi$ is a subformula of $\Phi$, and we replace $\Psi$ by any equivalent sub-formula $\Psi'$, is the resulting formula logically equivalent to $\Phi$?

The easiest question to answer is the first. The formulas $p \lor \neg p$ and $p \leftrightarrow p$ are weakly logically equivalent but not logically equivalent; the first returns $L$ when $p$ is bound as $L$, while the second returns $T$.

We shall answer the second question negatively below, and in the process describe which truth tables are realizable. To do so involves first answering the third question affirmatively.

**Proposition 3.5.** All restricted truth tables are realizable. That is, every function $h : \{T, F\}^k \to \{T, F, L, V\}$ is the restricted truth table of some formula $\Phi$.

**Proof.** The desired formula $\Phi$ will be completely connected with $k$ free nodes $n_1, \ldots, n_k$ that are propositional letters and a single node $n^*$ which is not free. The distinguished node $n^*$ is a (labelled) $(k + 1)$-ary boolean operator.
\[\varphi(\epsilon_1, \ldots, \epsilon_k, \epsilon) = \begin{cases} 
  t & \text{if } h(\epsilon_1, \ldots, \epsilon_k) = T \\
  f & \text{if } h(\epsilon_1, \ldots, \epsilon_k) = F \\
  e & \text{if } h(\epsilon_1, \ldots, \epsilon_k) = V \\
  \neg e & \text{if } h(\epsilon_1, \ldots, \epsilon_k) = L
\end{cases} \]

Here the last coordinate corresponds to the starred node. It is then easy to check that \(\Phi\) behaves as desired. \(\square\)

Next, we turn to the question of which truth tables can be realized. First, partially order the four truth values as follows:

**Definition 3.6.** \(\prec\) is the partial ordering defined on \(\{T, F, L, V\}\) by the following diagram:

\[V \prec F, T \prec L\]

Thus, formulas with lesser truth values are more prone to getting stuck.

If \((\epsilon_1, \ldots, \epsilon_k) \in \{T, F, L, V\}^k\), then let \(L\) (resp. \(V\)) be the set of indices \(i\) for which \(\epsilon_i = L\) (resp. \(V\)). First, the definition of truth leads to the following observation:

**Proposition 3.7.** If \(\Phi\) is a formula with \(k\) free nodes and restricted truth table \(h:\{T, F\}^k \to \{T, F, L, V\}\) and if \((\epsilon_i) \in \{T, F, L, V\}^k\), then

\[h((\epsilon_i)) \geq \inf_{\mu_i \in \{T, F\}, \epsilon_i \in V} \sup_{\mu_i \in \{T, F\}, \epsilon_i \in L} \{h(\mu_1, \ldots, \mu_k) \mid \mu_i = \epsilon_i \text{ if } \epsilon_i \in \{T, F\}\}\]

**Proof.** From the definition of truth, replacing one or more bound \(T\) or \(F\) nodes by bound \(L\) nodes replaces the truth value of \(\Phi\) by values at least as large as the supremum of the original values. Subsequently replacing one or more of the remaining nodes by \(V\) nodes results in exactly the infimum of the current values. \(\square\)

**Remarks 3.8.**

1. Note that reversing sup and inf in the above proposition gives a lower bound, and hence a weaker result.

2. The construction in Proposition 3.5 produces formulas whose values on elements of \(\{T, F, L, V\}^k\) can be computed by replacing the inequality above by an equality. Thus, it must be shown that we can introduce additional appropriate elementary consequences in the construction there without affecting the values on \(\{T, F\}^k\).

**Theorem 3.9.** Every instance of the inequality in Proposition 3.7 can be realized.

**Proof.** Suppose \(h\) satisfies the inequality in Proposition 3.7, let \(g\) be the corresponding restricted truth table, and let \(\Phi\) be the realization of \(g\) constructed in Proposition 3.5. It suffices to show that we can adjust \(\Phi\), without
altering its restricted truth table, in such a way that the starred node can change in a prescribed way (from \( t \) to \( f \), from \( f \) to \( t \) or both) when any specified collection \( S \) of free nodes is bound as \( L \). Denote this desired elementary consequence (on the starred node) by \( c_S \).

We accomplish this by expanding \( \Phi \) as follows. First, add a new free node \( n_S \) for every nonempty subset \( S \) of nodes in the original collection of free nodes. Next, construct a new formula \( \Psi \) whose restricted truth table agrees with that of \( \Phi \) on all evaluations in which all the new nodes are set to \( f \), but causes the desired change \( c_S \) in the starred node precisely when \( n_S \) is \( t \) and \( n_{S'} = f \) for all \( S' \neq S \).

Next, we turn each new free node in \( \Psi \) into a “lie detector” as follows: for each \( S \), write \( S = \{n_1, \ldots, n_r\} \) and replace \( n_S \) by

\[
[n_1 \land \cdots \land n_r] \land \neg [n_1 \lor \cdots \lor n_r]
\]

Thus, the new nodes are no longer free. Call the resulting formula \( \Psi' \). If the original free nodes are bound as \( T \) or \( F \), and if \( H \) is any hypothesis on \( \Phi \) that causes it to become stuck, then that same hypothesis on \( \Psi' \) with each \( n_S \) set to false, and the conjunctions and disjunctions in \( \Theta \) set to the appropriate values, is still stuck. Therefore, \( \Phi \) and \( \Psi' \) have the same restricted truth table. On the other hand, if any subset \( S \) of free nodes is bound as \( L \), then \( n_S \)—and only \( n_S \)—can turn to \( t \) in any configuration in which the old circuit was stuck. But since turning \( n_S \) on has the desired effect, we are done. \( \Box \)

Example 3.10. It is instructive to consider the special case \( k = 1 \). Call a formula with one free node that is also a propositional letter a gate. More specifically, if \( P_1, P_2, P_3 \) and \( P_4 \) are truth values, then a \( P_1P_2P_3P_4 \)-gate is a gate with truth table \( T \mapsto P_1, F \mapsto P_2, L \mapsto P_3, V \mapsto P_4 \). For instance, negation can be viewed as an \( FTLV \)-gate. \[ \square \] has shown that there are exactly 25 gates out of a possible 256, and has constructed simple formulas of each type. One non-existent gate is an \( FTVL \)-gate.

To end this section, we turn to the question of replacing subformulas by equivalent ones. Since it is possible for a formula to have edges pointing to different nodes in a subformula, we shall restrict the kinds of subformulas we can swap.

Definition 3.11. If \( \Phi \) and \( \Psi \) are any two formulas with no nodes in common, and \( p \) is a propositional letter in \( \Phi \), then the formula obtained by replacing the node \( p \) by the formula \( \Psi \) is called the substitution of \( \Psi \) into \( \Phi \) at node \( p \), and is written as \( s(\Psi, \Phi, p) \)

Proposition 3.12. Let \( \Phi, \Psi_1, \Psi_2 \) be formulas such that \( \Phi \) has no nodes in common with \( \Psi_1 \) or \( \Psi_2 \), and \( \Psi_1 \) and \( \Psi_2 \) are logically equivalent. Let \( p \) be a propositional letter in \( \Phi \). Then \( \Phi_1 = s(\Psi_1, \Phi, p) \) and \( \Phi_2 = s(\Psi_2, \Phi, p) \) are logically equivalent.
Proof. Let \( e \) be an evaluation of \( \Phi_1 \) and suppose that the proposition \((\Phi_1, e)\) has truth value \( P \). That \((\Phi_2, e)\) also has truth value \( P \) follows easily from Definition 2.6. We show one case as an illustration: Suppose that \( \Psi_1 \) has truth value \( L \) under the current evaluation (thus \( \Psi_2 \) also has truth value \( L \)), and suppose that \((\Phi_1, e)\) can get stuck in \( T \). Let \( H_1 \) be a hypothesis on \((\Phi_1, e)\) such that \( H_1(n^*) = t \) and there is no consequence \( H_1' \vdash H_1 \) with \( H_1'(n^*) = f \). Define the hypothesis \( H_2 \) on \((\Phi_2, e)\) by \( H_2(x) = H_1(x) \) for all nodes \( x \neq p \) in \( \Phi \) and let \( H_2(x) \) be arbitrary for \( x \) in \( \Psi_2 \). It is then easy to see that there is no \( H_2' \vdash H_2 \) with \( H_2'(n^*) = f \). The other cases are just as easy. \( \square \)

4. Models of Sentential Logic

It may seem most natural to generalize the notion of a model (see, for example, [CK]) by defining a model to be an assignment of truth values to the propositional letters. However, we wish to create models which admit inconsistencies of an arbitrary nature, for instance models in which a propositional letter and its negation are both true. We would also like to create models in which natural inconsistencies such as the Liar can become true or false. The following definition serves our needs in this paper, and will also serve as the basis for our mathematical model theory in the sequel [WW].

Definition 4.1. A model \( M \) of sentential logic is a tuple \( (M, A, B, C, D) \) where \( M \) is a nonempty set of propositional letters, and \( A, B, C \) and \( D \) are disjoint sets of formulas without free nodes.

Remarks 4.2.

(1) The formulas in \( A \) are called true axioms. Similarly, the formulas in \( B, C \) and \( D \) are called false axioms, contradictory axioms and independent axioms, respectively. When determining truth, each axiom will be bound to have the truth value determined by which of these four sets it is a member (see below).

(2) Since the nodes in our formulas are labelled boolean operators, it follows that models are possible in which different instances (distinguished by their labels) of the same formula are assigned different truth values as axioms. For instance, some Liars could be true and others false.

Definition 4.3. Let \( M \) be a model. A formula \( \Phi \) in \( M \) is a formula with no free nodes whose propositional letters are all in \( M \).

We now show how to compute the truth value of a formula \( \Phi \) in \( M \).

Definition 4.4. A subaxiom of an axiom \( A \) in \( M \) is a proper subformula \( B \) of \( A \).

Definition 4.5. The truth values of formulas in \( M \) are determined as follows:
(1) If \( \Phi \) is an axiom, then its truth value is determined by its membership in \( A, B, C \) or \( D \) as above.

(2) If \( \Phi \) is a subaxiom but not an axiom, its truth value \( R(\Phi) \) will be defined as a limit of a nondecreasing sequence of truth values \( R_n(\Phi) \), as follows: For any axiom \( A \) of which \( \Phi \) is a subaxiom let \( A^* \) be \( A \) with the star moved to the node for \( \Phi \), bind all axioms by their truth values under (1) that occur as subformulas of \( A \), and compute the truth value of \( A^* \). Denote this truth value by \( R_A \). Then take
\[
R_1(\Phi) = \sup \{ R_A \mid \Phi \text{ a subformula of } A \}.
\]
To obtain \( R_n(\Phi) \) from \( R_{n-1}(\Phi) \), proceed as in the definition of \( R_1 \), but also bind all subaxioms other than \( \Phi \) and occurring as subformulas of \( A^* \) by their truth value under \( R_{n-1} \). Let \( R(\Phi) = \sup_{n<\omega} R_n(\Phi) \).

(3) If \( \Phi \) is not an axiom or subaxiom, bind all its subformulas that are axioms or subaxioms by their truth values under (1) and (2), and then compute the value of the starred node as usual.

Remarks 4.6. (1) Technically, in computing truth we are using formulas identical to the ones we are interested in, but with certain nodes free so that we can bind them.

(2) Note that we do not permit formulas that are not subaxioms to inherit truth values from formulas containing them as we do in Definition 4.5(2). If we did then the presence of a single \( L \) formula would have the consequence that every formula that is not a subaxiom would have truth value \( L \). Indeed, if \( \Phi \) is \( L \) and not a subaxiom, and \( \Psi \) is any other formula that cannot get stuck in \( F \), then \( \Phi \land \Psi \) is seen to be \( L \). But binding this formula as \( L \) leads to \( \Psi \) having truth value \( L \). If \( \Psi \) can get stuck in \( F \) then a similar argument works using a disjunction instead of a conjunction. By contrast Definition 4.5 will allow us to admit contradictions and contain them.

We write \( M \models \Phi \), \( M \models F \Phi \), \( M \models L \Phi \) or \( M \models V \Phi \) if \( \Phi \) takes on truth value \( T, F, L \) or \( V \) respectively. \( M \) is complete if there are no well-grounded formulas \( \Phi \) in \( M \) such that \( M \models V \Phi \) (in particular, \( D \) contains no well-grounded formulas).

Example 4.7. Let \( M = (M, A, B, C, D) \) where \( M = \{ p \} \), \( A = \{ p, \neg [= p] \} \) and \( B = C = D = \emptyset \). Then the subaxiom \([= p]\) has truth value \( L \) in \( M \) as the following diagram shows.

We now relate our models to classical models of sentential calculus.

Definitions 4.8. Let \( \Phi \) be any formula in the model \( M \). Say that \( \Phi \) is generically inconsistent if its truth value in \( M \) differs from its truth value obtained by binding only the propositional letters that occur as axioms by their truth values in \( M \).
Also, call a well-grounded formula $\Phi$ in the model $\mathcal{M}$ \textbf{special in} $\mathcal{M}$ if each node in $\Phi$ is either a conjunction, disjunction, negation, equals, or a propositional letter, and such that no subformulas of $\Phi$ of the form “$\equiv \Psi$” are axioms. Further, we require that, if $\Psi$ is a subformula of $\Phi$ that is a subaxiom of some axiom $A$, then the formula “$\equiv \Psi$” is also a subaxiom of $A$ (so that the “$\equiv$” node is bound to at least the same value as its target when we evaluate truth). In particular, the starred node of any axiom in $\Phi$ must point to only “$\equiv$” nodes.

\textbf{Lemma 4.9.} Let $\mathcal{M}$ be a model all of whose axioms are either propositional letters assumed true or false, or other well-grounded formulas assumed true. Furthermore, assume that every propositional letter in $\mathcal{M}$ is an axiom. If there is a generically inconsistent special formula $\Phi$ in $\mathcal{M}$, then $\Phi$ has an $L$ subformula.

\textit{Proof.} Note that, since each propositional letter in $\mathcal{M}$ is assigned a unique truth value, $\mathcal{M}$ determines an evaluation $e$ of every formula whose propositional letters are free and in $\mathcal{M}$. Let $\Phi$ be as in the hypothesis and assume that $\Phi$ has no $L$ subformulas. It follows that the truth value of any subaxiom which is a subformula of $\Phi$ can be computed by choosing any axiom of which it is a subformula, binding all axioms, and then computing the truth value of the subaxiom’s distinguished node as usual.

Let $\Psi$ be a minimal subformula of $\Phi$ which is generically inconsistent under the evaluation $e$ determined by $\mathcal{M}$. Then $\Psi$ must be an axiom or subaxiom by minimality. If the distinguished node $n^*$ of $\Psi$ is an “$\equiv$” node, then the definition of a special formula implies that it either has the same truth value as its target, contradicting minimality, or is $L$, contrary to assumption. Thus $n^*$ is not an “$\equiv$” node.

Since $\Psi$ must be an axiom or a subaxiom, the special property of $\Phi$ now guarantees that $n^*$ points to only “$\equiv$” nodes, each of which is the distinguished node of a subaxiom so that all of these “$\equiv$” nodes are bound (when we evaluate truth). Since $\Psi$ is generically inconsistent, binding only the propositional letters will lead any hypothesis to one in which $n^*$ has the opposite truth value. Further the chain of elementary consequences can be arranged to affect $n^*$ only in the last step (because $\Psi$ is well-grounded).

If $n^*$ and the propositional letters are bound, then the same chain of elementary consequences except for the last is still possible. Since proper subformulas of $\Psi$ are generically consistent, all nodes except $n^*$ end up in their values as subaxioms. Therefore, binding $n^*$ to its given truth value will result in an elementary consequence in which the truth value of one of its children $c$ changes to the opposite truth value of its associated subaxiom $C$ (recall that $n^*$ must be a conjunction, disjunction or negation; see remark below).

One has $C < \Psi < A$ for some axiom $A$ (where $<$ indicates subformula). A lower bound of the truth value of $C$ is obtained by binding all axioms that occur in $A$ and then computing the truth value of the node $c$. However,
any hypothesis on \( A \) will lead to one in which \( n^* \) is assigned its given truth value. Subsequently, as we have seen, a further sequence of elementary consequences will change \( c \) to the opposite truth value. The definition of truth value of subformulas now tells us that \( C \) must be \( L \) in the model, contrary to assumption.

\[ \square \]

Remark 4.10. The definition of special need not be so restrictive as to include only conjunctions, disjunctions, negations and equals. However, we need to avoid operators which either are, or can behave like, constant boolean operators. For example the operator \( \leftrightarrow \) can behave as a constant unary operator as in \( p \leftrightarrow p \). If this formula is hypothesized as false, then there is no elementary consequence that changes the value of its only child \( p \).

If \( T \) is a set of well-grounded formulas, let \( T' \) be obtained from \( T \) by writing each formula in disjunctive normal form, and then replacing each non-starred node \( n \) in each formula by \( "= n" \). Call this expanded disjunctive normal form. If \( T' \) is now the set of axioms in a model \( M \), then each axiom is special. It follows that any classical theory \( T \) is equivalent (in the sense that it has the same models) to one with a special set of axioms.

Theorem 4.11. Let \( T \) be a theory and \( M \) a model in the classical sense. Form the model \( M(T) \) by letting \( A \) consist of the formulas of \( T \) written in expanded disjunctive normal form, together with the true propositional letters in \( M \), and \( B \) the false propositional letters in \( M \). Then \( M \) is a model of \( T \) in the classical sense iff there are no well-grounded \( L \) formulas in \( M(T) \). When this is the case, the well-grounded formulas in \( M(T) \) have the same truth values as the corresponding formulas in \( M \).

Proof. First observe that if \( T' \) is the theory \( T \), but with all axioms written in expanded disjunctive normal form, then \( M \) is a model of \( T' \) iff \( M \) is a model of \( T' \). Therefore we assume without loss of generality that the formulas in \( T \) are in expanded disjunctive normal form.

If \( M \) is a model of \( T \), then it is clear that all well-grounded formulas in \( M(T) \) get stuck in their appropriate truth values (\( T \) or \( F \)), and hence there are no well-grounded \( L \) formulas. Conversely, if \( M \) is not a model of \( T \), then there exists a well-grounded formula \( \Phi \) such that \( T \vdash \Phi \) but \( \Phi \) is false in the model \( M \). This implies that at least one axiom in \( T \) is generically inconsistent as a formula in \( M(T) \). Since each axiom is special, the lemma implies that this axiom has an \( L \) subformula.

In \[BE\] Barwise and Etchemendy describe a class of models which include false liars. It is therefore natural to ask if there is an interesting class of models in our sense in which the Liar is false. Uninteresting models can be constructed by simply binding every formula with an arbitrary truth value. In an interesting model, truth values must, for the most part, be computed according to Definition \[4.5\].
Definition 4.12. If $\Phi_1, \ldots, \Phi_n$ are (not necessarily well-grounded) formulas and $\Psi$ is a well-grounded formula with $n$ propositional letters, let $\Psi'$ be obtained from $\Psi$ by replacing its propositional letters with the $\Phi_i$. Then we say that the pair $(\Psi', (\Phi_i))$ is relatively well-grounded.

Proposition 4.13. If $T$ is any consistent theory with a classical model $M$, then it has a model $N(T)$ in our sense in which all well-grounded formulas have the same truth values as the corresponding formulas in $M$. Further, there are no $L$ formulas in $N(T)$, the Liar is false in $N(T)$, and no relatively well-grounded formula is an axiom or subaxiom. (Thus, for instance, the negation of the Liar is true).

Proof. We construct the model $N(T)$ as the $\omega$-limit of an inductively defined sequence $M_i$. Define $M_0 = M(T)$, as in Theorem 4.11, and assume that $M_i = (A_i, B_i, C_i, D_i)$, $(i < n)$ have been defined, with $A_i = A_{i+1}$, $B_i \subseteq B_{i+1}$, and $C_i = D_i = \emptyset$, and such that all well-grounded formulas have the same truth values as the corresponding formulas in $M$. To define $M_{i+1}$, take $A_{i+1} = A_i$, $C_{i+1} = D_{i+1} = \emptyset$, and

$$B_{i+1} = B_i \cup \{ \Phi \mid \Phi \text{ a minimal } L \text{ formula in } M_i \}$$

Since we are decreasing some truth values in passing from $M_i$ to $M_{i+1}$, it follows that no new $L$ formulas are introduced. Further, since by assumption, no well-grounded formula can be $L$, their truth values are not affected. Also, if a relatively well-grounded formula is $L$ in $M_i$, it cannot be a minimal $L$ formula, and so it will remain unbound when we evaluate truth. Finally, it is not hard to see that there are no $L$ formulas in the limit. \qed

Remarks 4.14.

(1) We cannot eliminate all $L$ formulas in just one step. For instance, the following $L$ formula remains $L$ in $M_1$ because only the bottom left node is bound as $F$ in $M_1$.

(2) We cannot avoid creating $V$ statements in general. The following formula has truth value $T$ until the Liar subformula (rightmost node) is bound as $F$, whereupon the formula becomes $V$. (Note also that we can replace the Liar on the right by any $L$ formula.)

(3) We cannot expect to eliminate $V$ statements by binding them as $T$ or $F$ without (re-)introducing $L$ statements. In the following pair of formulas, $\Phi$ stands for any $V$ subformula. Binding $\Phi$ as false causes the first formula to change from $T$ to $L$. On the other hand, binding $\Phi$ as true causes the second formula to change from $T$ to $L$. Note
that re-binding these new $L$ statements will lead again to further $V$ statements (parenthetical comment in (2)).

(4) Barwise and Etchemendy [BE] seem to avoid $V$ statements in their models of Austinian logic by limiting the amount of information that is talked about in their “situations.” Our models, on the other hand, include all formulas.

We end this section with a brief discussion of rules of inference in the hope of stimulating further research.

Since an instance of modus ponens may be bound as $F$, $V$, or $L$ in a model, we cannot expect a general set of rules of inference to hold in every model of sentential logic. However, models in which all the axioms are propositional letters and bound as either $T$ or $F$ are interesting in this regard, since a large class of rules of inference hold “from without” in such models.

**Definition 4.15.** Say a model is simple if each axiom is a propositional letter assumed true or false, and every propositional letter in $M$ is an axiom.

As usual, say that modus ponens holds externally if $M \models \Phi$ and $M \models [\Phi \rightarrow \Psi]$ implies $M \models \Psi$, and similarly for the other common rules of inference from classical sentential logic. If $\Phi$ is a (weak or strong) tautology (see Definition 3.4), write $R(\Phi)$ for the rule of inference that says $M \models [\Phi \rightarrow \Psi]$. If $\Phi$ is a (weak or strong) tautology, write $R(\Phi)$ for the rule of inference that says $M \models \Phi$.

**Proposition 4.16.** If $M$ is a simple model, then the following rules of inference hold: modus ponens, modus tollens, contrapositive, chain rule, disjunctive inference, De Morgan, simplification, conjunction, and disjunctive syllogism. Further, the rule $R(\Phi)$ holds iff $\Phi$ is a strong tautology.

**Proof.** The last statement is a consequence of our rules for computing truth. In the case of modus ponens, suppose that $M \models F \Psi$, then $\Phi \rightarrow \Psi$ has truth value $F$. If $M \models L \Psi$, then $\Phi \rightarrow \Psi$ has truth value $L$. Finally, if $M \models V \Psi$, then $\Phi \rightarrow \Psi$ has truth value $V$. We therefore rule these possibilities out by assumption. The remaining rules listed can be checked one-by-one. 

Note that the modus ponens formula $[p \land (p \rightarrow q)] \rightarrow q$ is a (strictly) weak tautology, and so cannot hold “internally” in any classical model. Further, not every weak tautological implication leads to an external rule of inference, as illustrated by the weak tautological implication $(q \rightarrow (p \lor \neg p))$. Indeed, if $M \models q$, then $M \not\models (p \lor \neg p)$ if $p$ is $L$, since then $M \models L (p \lor \neg p)$.

In general, a good set of rules of inference and a good theory of argument should include rules that predict values other than $T$. 

\[ \neg (* \lor V \Phi) \rightarrow (\neg \Phi \lor \Phi) \]
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