New Sasaki–Einstein 5-manifolds

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Abstract
By estimating the $\delta$-invariants of certain log del Pezzo surfaces, we prove that closed simply connected 5-manifolds $2(S^2 \times S^3) \# nM_2$ allow Sasaki-Einstein structures, where $M_2$ is the closed simply connected 5-manifold with $H_2(M_2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $nM_2$ is the $n$-fold connected sum of $M_2$, and $2(S^2 \times S^3)$ is the twofold connected sum of $S^2 \times S^3$.

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1 | INTRODUCTION

A Riemannian manifold $(M, g)$ is called Sasakian if the cone metric $r^2 g + dr^2$ defines a Kähler metric on $M \times \mathbb{R}^+$. If the metric $g$ satisfies the Einstein condition, that is, $\text{Ric}_g = \lambda g$ for some constant $\lambda$, then the metric $g$ is called Einstein. A significant number of closed simply connected Sasaki–Einstein manifolds, in particular 5-manifolds, have been discovered based on the method that was introduced by Kobayashi \cite{21} and developed by Boyer, Galicki, and Kollár \cite{8, 9, 23}. The upshot of their method is briefly presented in \cite{24} as follows. A quasi-regular Sasakian structure on a manifold $L$ can be written as the unit circle subbundle of a holomorphic Seifert $\mathbb{C}^*$-bundle over a complex algebraic orbifold $(S, \Delta)$, where $\Delta = \sum (1 - \frac{1}{m_i})D_i$, functions of $m_i$ are positive...
integers, and functions of $D_i$ are distinct irreducible divisors. A simply connected Sasakian manifold $L$ is Einstein if and only if $-(K_S + \Delta)$ is ample, the first Chern class of $c_1(L/S)$ is a rational multiple of $-(K_S + \Delta)$, and there is an orbifold Kähler–Einstein metric on the orbifold $(S, \Delta)$.

Links of quasi-homogeneous hypersurface singularities are Seifert circle bundles over the corresponding projective hypersurfaces in weighted projective spaces. For a brief explanation, we consider a quasi-smooth hypersurface $X$ defined by a quasi-homogeneous polynomial $F(z_0, z_1, \ldots, z_n)$ in variables $z_0, \ldots, z_n$ with weights $w(z_i) = a_i$ in a weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$. Denote by $\deg_w(F)$ the degree of $F(z_0, z_1, \ldots, z_n)$ with respect to the weights $w(z_i) = a_i$. The equation $F(z_0, z_1, \ldots, z_n) = 0$ also defines a hypersurface $\tilde{X}$ in $\mathbb{C}^{n+1}$ that is smooth outside the origin. The link of $X$ is a smooth compact manifold of real dimension $2n - 1$ defined by the intersection

$$L_X = S^{2n+1} \cap \tilde{X},$$

where $S^{2n+1}$ is the unit sphere centered at the origin in $\mathbb{C}^{n+1}$. Note that it is simply connected if $n \geq 3$ [29, Theorem 5.2].

Suppose that $m := \gcd(a_1, \ldots, a_n) > 1$ and $\gcd(a_0, a_1, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_n) = 1$ for each $i = 1, \ldots, n$. Set $b_0 = a_0$ and $b_i = \frac{a_i}{m}$ for $i = 1, \ldots, n$. Then, the weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is not well formed, while the weighted projective space $\mathbb{P}(b_0, b_1, \ldots, b_n)$ is well formed (see [19, Definition 5.1]). There is a quasi-homogeneous polynomial $G(x_0, \ldots, x_n)$ in variables $x_0, \ldots, x_n$ with weights $w(x_i) = b_i$ such that $F(z_0, z_1, \ldots, z_n) = G(z_0^m, z_1, \ldots, z_n)$. The equation $G(x_0, \ldots, x_n) = 0$ defines a quasi-smooth hypersurface $Y$ in $\mathbb{P}(b_0, b_1, \ldots, b_n)$. We suppose that $\deg_w(F) - \sum a_i < 0$ and $Y$ is well formed in $\mathbb{P}(b_0, b_1, \ldots, b_n)$ (see [19, Definition 6.9]). Denote by $D$ the divisor on $Y$ cut by $x_0 = 0$. We may consider the log pair $(Y, \frac{m-1}{m}D)$ as a Fano orbifold. The method by Kobayashi has evolved into the following assertion through the works of Boyer, Galicki, and Kollár.

**Theorem 1.1** [8, Theorem 2.1; 21, Theorem 5]. If $(Y, \frac{m-1}{m}D)$ allows an orbifold Kähler–Einstein metric, then there is a Sasaki–Einstein metric on the link $L_X$ of $X$.

Closed simply connected 5-manifolds are completely classified by Barden and Smale [3], [32]. In particular, Smale has classified all the closed simply connected spin 5-manifolds [32], which are called Smale 5-manifolds. For a positive integer $m$, up to diffeomorphisms, there is a unique closed simply connected spin 5-manifold $M_m$ with $H_2(M_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Furthermore, a closed simply connected spin 5-manifold $M$ is of the form

$$M = kM_\infty \# M_{m_1} \# \ldots \# M_{m_r},$$

where $kM_\infty$ is the $k$-fold connected sum of $S^2 \times S^3$ for a non-negative integer $k$ and $m_i$ is a positive integer greater than 1 with $m_i$ dividing $m_{i+1}$.

Many efforts have been made to classify all the closed simply connected Sasaki–Einstein 5-manifolds. To be precise, for each Smale 5-manifold (every Sasaki–Einstein manifolds are spin), we want to determine whether it has a quasi-regular Sasaki–Einstein structure or not. Such efforts and their results are summarized in [20]. Toward complete classification, three conjectures were proposed in [20]. One of them is as follows.

**Conjecture 1.2.** For each integer $k \leq 8$ and $n \geq 2$, the Smale 5-manifold $kM_\infty \# nM_2$ admits a Sasaki–Einstein metric.
The conjecture has been verified for $k = 0, 1$ so far [23], [31]. Also, $2M_\infty \# nM_2$ is proven to allow a Sasakian metric of positive Ricci curvature [10, Theorem B]. In this article, we prove the conjecture for $k = 2$.

**Main Theorem.** For every positive integer $n$, the Smale manifold $2M_\infty \# nM_2$ allows a Sasakian–Einstein metric.

**Proof.** Due to Theorem 1.1, it is enough to show that there are Kähler–Einstein del Pezzo orbifold surfaces whose links are $2M_\infty \# nM_2$. Such del Pezzo orbifold surfaces are provided by Theorem 4.1 in Section 4. □

## 2 KÄHLER–EINSTEIN METRIC AND K-STABILITY

The theory on Kähler–Einstein metrics and K-stability of Fano varieties and the theory on valuative criterions for K-stability have developed dramatically for the last ten years.

In 2016 Fujita and Odaka introduced a new invariant of a Fano variety, so-called $\delta$-invariant, which has evolved into a criterion for K-stability through the work of Blum and Jonsson. The $\delta$-invariant measures how singular the average divisors of sections that form bases for plurianticanonical linear systems are, using their log canonical thresholds.

Let $X$ be a projective $\mathbb{Q}$-factorial normal variety and $\Omega$ be a $\mathbb{Q}$-divisor on $X$ such that the log pair $(X, \Omega)$ has at worst Kawamata log terminal singularities. We suppose that $(X, \Omega)$ is a log $\mathbb{Q}$-Fano variety, that is, the divisor $-(K_X + \Omega)$ is ample.

**Definition 2.1.** Let $m$ be a positive integer such that $|-m(K_X + \Omega)|$ is non-empty. Set $\ell_m = h^0(X, \mathcal{O}_X(-m(K_X + \Omega)))$. For a section $s$ in $H^0(X, \mathcal{O}_X(-m(K_X + \Omega)))$, we denote the effective divisor of the section $s$ by $D(s)$. If $\ell_m$ sections $s_1, \ldots, s_{\ell_m}$ form a basis of the space $H^0(X, \mathcal{O}_X(-m(K_X + \Omega)))$, then the anticanonical $\mathbb{Q}$-divisor

$$D := \frac{1}{\ell_m} \sum_{i=1}^{\ell_m} \frac{1}{m} D(s_i)$$

is said to be of $m$-basis type. We set

$$\delta_m(X, \Omega) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the log pair } (X, \Omega + \lambda D) \text{ is log canonical for every effective } \mathbb{Q}\text{-divisor } D \text{ of } m\text{-basis type} \right\}.$$

The $\delta$-invariant of $(X, \Omega)$ is defined by the number

$$\delta(X, \Omega) = \limsup_m \delta_m(X, \Omega).$$

To study the $\delta$-invariant from local viewpoints, we set

$$\delta_{Z,m}(X, \Omega) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the log pair } (X, \Omega + \lambda D) \text{ is log canonical along } Z \text{ for every effective } \mathbb{Q}\text{-divisor } D \text{ of } m\text{-basis type} \right\}.$$
for a closed subvariety $Z$ of $X$. The local $\delta$-invariant of $(X, \Omega)$ along $Z$ is defined by the number

$$\delta_Z(X, \Omega) = \limsup_m \delta_{Z,m}(X, \Omega).$$

Using the $\delta$-invariant, Blum–Jonsson ([7]) and Fujita–Odaka ([18]) set up a criterion for K-(semi)stability in an algebro-geometric way. Due to the result [28, Theorem 1.5], the criterion reads as follows.

**Theorem 2.2.** A log $\mathbb{Q}$-Fano variety $(X, \Omega)$ is K-stable (respectively, K-semistable) if and only if $\delta(X, \Omega) > 1$ (respectively, $\geq 1$).

The bridge between K-polystability and existence of Kähler–Einstein metrics has been completely established for log Fano pairs [4, 5, 13–16, 26–28, 33, 34].

**Theorem 2.3.** A Fano orbifold $(X, \Omega)$ is K-polystable if and only if it allows an orbifold Kähler–Einstein metric.

Since K-stability implies K-polystability by definition, a K-stable Fano orbifold admits an orbifold Kähler-Einstein metric.

### 3 TOOLS FOR $\delta$-INARIANT

Let $S$ be a surface with at most cyclic quotient singularities and $D$ an effective $\mathbb{Q}$-divisor on the surface $S$. Also let $p$ be a point of $S$.

**Lemma 3.1.** Suppose that $p$ is a smooth point of $S$. If the log pair $(S, D)$ is not log canonical at $p$, then $\text{mult}_p(D) > 1$.

**Proof.** See [25, Proposition 9.5.13], for instance. \(\square\)

Let $C$ be an integral curve on $S$ that passes through the point $p$. Suppose that $C$ is not contained in the support of the divisor $D$. If $p$ is a smooth point of the surface $S$ and the log pair $(S, D)$ is not log canonical at $p$, then it follows from Lemma 3.1 that $D \cdot C > 1$.

**Lemma 3.2.** Suppose that $p$ is a cyclic quotient singularity of type $\frac{1}{n}(a, b)$, where $a$ and $b$ are coprime positive integers that are also coprime to $n$. If the log pair $(S, D)$ is not log canonical at $p$ and $C$ is not contained in the support of the divisor $D$, then

$$D \cdot C > \frac{1}{n}.$$

**Proof.** This follows from [22, Proposition 3.16], Lemma 3.1, and [11, Lemma 2.2]. \(\square\)

In general, the curve $C$ may be contained in the support of the divisor $D$. In this case, we write

$$D = rC + \Omega,$$
where \( r \) is a positive rational number and \( \Omega \) is an effective \( \mathbb{Q} \)-divisor on \( S \) whose support does not contain the curve \( C \). We suppose that \((S, C)\) is purely log terminal around \( p \).

**Lemma 3.3.** Suppose that \( r \leq 1 \) and the log pair \((S, D)\) is not log canonical at \( p \).

1. If \( p \) is a smooth point, then
   \[ C \cdot \Omega \geq (C \cdot \Omega)_p > 1, \]
   where \((C \cdot \Omega)_p\) is the local intersection number of \( C \) and \( \Omega \) at \( p \).

2. If \( p \) is a cyclic quotient singularity of type \( \frac{1}{n}(a, b) \), then
   \[ C \cdot \Omega > \frac{1}{n}. \]

**Proof.** See [11, Lemma 2.5]. \(\square\)

We now let \((S, \Omega)\) be a log del Pezzo surface that admits only Kawamata log terminal singularities. It follows from [6, Corollary 1.3.2] that a log del Pezzo surface is a Mori dream space. Let \( C \) be a prime divisor over \( S \) and let \( \pi : \tilde{S} \to S \) be a birational morphism such that \( C \) is a divisor on \( \tilde{S} \).

We first set
\[
S_{S, \Omega}(C) = \frac{1}{(-(K_S + \Omega))^2} \int_0^\infty \text{vol} (\pi^*(-(K_S + \Omega)) - tC) dt.
\] (3.4)

Taken the definition of the basis-type divisors in Definition 2.1 into consideration, it is natural to expect that a divisor of \( m \)-basis type cannot carry a prime divisor with big multiplicity for a sufficiently large \( m \). Indeed, the following bound is originally given in [18, Lemma 2.2].

**Lemma 3.5.** For a given real number \( \epsilon > 0 \), there is an integer \( \mu \) such that whenever \( m > \mu \), we have
\[
\text{ord}_C(\pi^*(D)) \leq S_{S, \Omega}(C) + \epsilon
\]
for every ample \( \mathbb{Q} \)-divisor \( D \) of \( m \)-basis type with respect to \((S, \Omega)\).

**Proof.** See [12, Theorem 2.9]. \(\square\)

The following describe how to estimate \( \delta(S, \Omega) \) from local viewpoints, which are developed in [1] and simplified in [2] and [17].

Suppose that \((S, \Omega + C)\) is purely log terminal. Let \( \tau \) be the supremum of the positive real numbers \( t \) such that \(-(K_S + \Omega) - tC\) is big. For a real number \( u \in (0, \tau) \), we write the Zariski decomposition of \(-(K_S + \Omega) - uC\) as
\[
-(K_S + \Omega) - uC \equiv P(u) + N(u),
\]
where \( P(u) \) and \( N(u) \) are the positive and the negative parts, respectively. Let \( p \) be a point on \( C \). Define

\[
h(u) = (P(u) \cdot C) \cdot \text{ord}_p \left( N(u) \big|_C \right) + \int_0^\infty \text{vol} \left( P(u) \big|_C - vp \right) dv, \tag{3.6}
\]

and then put

\[
S(W^C_{\star, \star}; p) = \frac{2}{(-(K_S + \Omega))^2} \int_0^\tau h(u) du. \tag{3.7}
\]

Recall that we have the following adjunction formula:

\[
(K_S + \Omega + C) \big|_C = K_C + \Delta,
\]

where \( \Delta \) is the different for \( K_S + \Omega + C \). Then, the log discrepancy of the log pair \((C, \Delta)\) along the divisor \( p \) is

\[
A_{C, \Delta}(p) = 1 - \text{ord}_p(\Delta).
\]

If \( p \) is a quotient singular point of type \( \frac{1}{n}(a, b) \), then

\[
A_{C, \Delta}(p) = \frac{1}{n} - (\Omega \cdot C)_p. \tag{3.8}
\]

**Theorem 3.9.** The local \( \delta \)-invariant of \((S, \Omega)\) at the point \( p \) satisfies the inequality

\[
\delta_p(S, \Omega) \geq \min \left\{ \frac{1}{S_{S, \Omega}(C)}, \frac{A_{C, \Delta}(p)}{S(W^C_{\star, \star}; p)} \right\}.
\]

**Proof.** This immediately follows from [17, Theorem 4.8 (2) and Corollary 4.9] because the point \( p \) is the only prime divisor over the curve \( C \) with the center at \( p \) in [17, Definition 3.11]. \( \square \)

### 4 SASAKI–EINSTEIN 5-MANIFOLDS \(2M_\infty \# nM_2\)

For each integer \( n \geq 2 \), let \( \hat{S}_n \) be a quasi-smooth hypersurface of degree \( 4(2n + 1) \) in \( \mathbb{P}(2, 4n, 4n + 1) \). This hypersurface appears in [10] to give a Sasakian metric of positive Ricci curvature to \( 2M_\infty \# nM_2 \). By using appropriate coordinate changes, we may assume that the surface is defined by

\[
w^2x + yz(x - y^n) + zx\hat{A}_{4n+2}(x, y) + x\hat{A}_{8n+2}(x, y) = 0,
\]

where \( x, y, z, w \) are quasi-homogeneous coordinates with \( \text{wt}(x) = 2, \text{wt}(y) = 4, \text{wt}(z) = 4n, \text{wt}(w) = 4n + 1 \), and \( \hat{A}_{4n+2}(x, y), \hat{A}_{8n+2}(x, y) \) are quasi-homogeneous polynomials of degrees \( 4n + 2 \) and \( 8n + 2 \), respectively, in \( x, y \).
We use the same notation $x, y, z, w$ for homogeneous coordinates of the weighted projective space $\mathbb{P}(1, 2, 2n, 4n + 1)$ with $wt(x) = 1$, $wt(y) = 2$, $wt(z) = 2n$, and $wt(w) = 4n + 1$. Let $S_n$ be the quasi-smooth hypersurface of degree $2(2n + 1)$ in $\mathbb{P}(1, 2, 2n, 4n + 1)$ defined by

$$wx + yz(z - y^n) + xA_{2n+1}(x, y) + xA_{4n+1}(x, y) = 0,$$

where $A_{2n+1}(x, y)$ and $A_{4n+1}(x, y)$ are the quasi-homogeneous polynomials of degrees $2n + 1$ and $4n + 1$ defined by $\hat{A}_{4n+2}(x, y)$ and $\hat{A}_{8n+2}(x, y)$, respectively, with weights $wt(x) = 1$ and $wt(y) = 2$. We denote by $W$ the irreducible divisor on $S_n$ cut by $w = 0$. As an orbifold, $S_n$ can be regarded as the log del Pezzo surface $(S_n, \frac{1}{2}W)$.

We consider the link of $S_n$. It follows from [30, Corollary] that the link of $S_n$ has the second Betti number 2. The curve $W$ is isomorphic to a smooth curve of degree $2n + 1$ in $\mathbb{P}(1, 1, n)$, and hence, its genus is $n$. It then follows from [23, Theorem 5.7] that the torsion part of the second homology group of the link is $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\otimes n}$. Consequently, the link of $S_n$ is diffeomorphic to $2M_\infty \# nM_2$ by [32, Theorem].

Therefore, Theorem 1.1 implies that the following statement guarantees existence of a Sasaki–Einstein metric on $2M_\infty \# nM_2$. In other words, Main Theorem immediately follows from the theorem below.

**Theorem 4.1.** For $n \geq 2$, $(S_n, \frac{1}{2}W)$ allows an orbifold Kähler–Einstein metric.

We remark here that $2M_\infty \# M_2$, $2M_\infty \# 3M_2$, and $2M_\infty \# 4M_2$ are already shown to admit Sasaki–Einstein metrics [10, Theorem A; 20, Corollaries 2.5 and 2.7].

## 5 PROOF OF THEOREM 4.1

Due to Theorems 2.2 and 2.3, in order to prove Theorem 4.1, it is enough to show that

$$\delta \left( S_n, \frac{1}{2}W \right) > 1.$$

In this section, we achieve this inequality by verifying

$$\delta_p \left( S_n, \frac{1}{2}W \right) \geq \frac{20n + 5}{20n + 4}$$

for each point $p$ in $S_n$.

Since the hypersurface $S_n$ is quasi-smooth, it has only singularities inherited from the singularities of the ambient space $\mathbb{P}(1, 2, 2n, 4n + 1)$. The surface passes through exactly four distinct singular points $O_w = [0 : 0 : 0 : 1]$, $O_z = [0 : 0 : 1 : 0]$, $O_0 = [0 : 1 : 0 : 0]$, $O_1 = [0 : 1 : 1 : 0]$ of $\mathbb{P}(1, 2, 2n, 4n + 1)$. The point $O_w = [0 : 0 : 0 : 1]$, where $\mathbb{P}(1, 2, 2n, 4n + 1)$ has a quotient singularity of type $\frac{1}{4n+1}(1, 2n)$, yields a cyclic quotient singularity of type $\frac{1}{4n+1}(1, n)$ on $S_n$. Similarly, the surface $S_n$ also gains cyclic quotient singular points at $O_z = [0 : 0 : 1 : 0]$, $O_0 = [0 : 1 : 0 : 0]$, and $O_1 = [0 : 1 : 1 : 0]$ of types $\frac{1}{2n}(1, 1), \frac{1}{2}(1, 1),$ and $\frac{1}{2}(1, 1)$, respectively.

Denote by $C_x$ the divisor on $S_n$ cut by $x = 0$. The divisor $C_x$ consists of three components. To be precise,

$$C_x = L_{xy} + R_0 + R_1,$$
where $L_{xy}$ is defined by $x = y = 0$, $R_0$ by $x = z = 0$, and $R_1$ by $x = z - y^n = 0$. Each pair of these three curves meet only at $O_w$.

Their intersection numbers on $S_n$ are as follows:

\[
L^2_{xy} = -\frac{4n - 1}{2n(4n + 1)}, \quad R_0^2 = R_1^2 = -\frac{2n + 1}{2(4n + 1)},
\]

\[
L_{xy} \cdot R_0 = L_{xy} \cdot R_1 = \frac{1}{4n + 1}, \quad R_0 \cdot R_1 = \frac{n}{4n + 1}.
\] (5.1)

The divisor $-(K_{S_n} + \frac{1}{2} W)$ is equivalent to $\frac{3}{2} C_x$ and its self-intersection number is

\[
\left( K_{S_n} + \frac{1}{2} W \right)^2 = \frac{9(2n + 1)}{8n(4n + 1)}.
\]

We also obtain

\[
-(K_{S_n} + \frac{1}{2} W) \cdot L_{xy} = \frac{3}{2}(L_{xy} + R_0 + R_1) \cdot L_{xy} = \frac{3}{4(4n + 1)},
\]

\[
-(K_{S_n} + \frac{1}{2} W) \cdot R_i = \frac{3}{2}(L_{xy} + R_0 + R_1) \cdot R_i = \frac{3}{4(4n + 1)},
\]

where $i = 1, 2$, from (5.1).

The irreducible curves $L_{xy}, R_0, R_1$ belong to the boundary of the pseudoeffective cone of $S_n$ since they are of negative self-intersection. For $t > \frac{3}{2}$, the divisor

\[
\frac{3}{2} C_x - tL_{xy} = \left( \frac{3}{2} - t \right)L_{xy} + \frac{3}{2} R_0 + \frac{3}{2} R_1
\]

is not pseudoeffective. Set

\[
P_L(t) := \begin{cases} 
\frac{3}{2} C_x - tL_{xy} & \text{for } 0 \leq t \leq \frac{3}{4}, \\
\frac{3}{2} C_x - tL_{xy} - \frac{4t - 3}{2}(R_0 + R_1) & \text{for } \frac{3}{4} \leq t \leq \frac{3}{2},
\end{cases}
\]

\[
N_L(t) := \begin{cases} 
0 & \text{for } 0 \leq t \leq \frac{3}{4}, \\
\frac{4t - 3}{2}(R_0 + R_1) & \text{for } \frac{3}{4} \leq t \leq \frac{3}{2}.
\end{cases}
\] (5.2)
For each \( i = 0, 1 \),
\[
\left( \frac{3}{2} C_x - tL_{xy} \right) \cdot R_i = \frac{3 - 4t}{4(4n + 1)}.
\]

This implies that the divisor \( P_L(t) \) is nef for \( 0 \leq t \leq \frac{3}{4} \). Furthermore, for \( \frac{3}{4} \leq t \leq \frac{3}{2} \), \( P_L(t) \) is a nef divisor with \( P_L(t) \cdot R_0 = P_L(t) \cdot R_1 = 0 \) and \( N_L(t) \) is negative definite. Consequently, the Zariski decomposition of \( \frac{3}{2} C_x - tL_{xy} \) is given by
\[
\frac{3}{2} C_x - tL_{xy} \equiv P_L(t) + N_L(t)
\]
for \( 0 \leq t \leq \frac{3}{2} \). We then see that the volume of \( \frac{3}{2} C_x - tL_{xy} \) is
\[
\text{vol} \left( \frac{3}{2} C_x - tL_{xy} \right) = \begin{cases} 
- \frac{4(4n - 1)t^2 + 12t - 9(2n + 1)}{8n(4n + 1)} & \text{for } 0 \leq t \leq \frac{3}{4}, \\
\frac{(3 - 2t)^2}{8n} & \text{for } \frac{3}{4} \leq t \leq \frac{3}{2}, \\
0 & \text{for } t \geq \frac{3}{2},
\end{cases}
\]
and the function in (3.4) is given by
\[
S_{S_n, \frac{1}{2}W}(L_{xy}) = \frac{3n + 1}{2(2n + 1)}.
\]

Moreover, note that
\[
\text{ord}_p \left( N_L(t) \big|_{L_{xy}} \right) = 0,
\]
for \( p \in L_{xy} \setminus \{O_w\} \). Then, the function in (3.6) is given by
\[
h_{p,L}(t) = \int_0^\infty \text{vol} \left( P_L(t) \big|_{L_{xy}} - vp \right) dv
= \frac{1}{2} \left( P_L(t) \cdot L_{xy} \right)^2
= \begin{cases} 
\frac{1}{32n^2(4n + 1)^2} (2(4n - 1)t + 3)^2 & \text{for } 0 \leq t \leq \frac{3}{4}, \\
\frac{1}{32n^2} (-2t + 3)^2 & \text{for } \frac{3}{4} \leq t \leq \frac{3}{2},
\end{cases}
\]
and the value in (3.7) for $p \in L_{xy} \setminus \{O_w\}$ is given by

$$S(W^c_*; p) = \frac{16n(4n+1)}{9(2n+1)} \int_0^{\frac{3}{2}} h_{p,t}(t) dt$$

$$= \frac{4n + 1}{18n(2n+1)} \left\{ \int_0^{\frac{3}{2}} \frac{1}{(4n+1)^2} (2(4n-1)t + 3)^2 dt + \int_{\frac{3}{2}}^3 (3 - 2t)^2 dt \right\}$$

$$= \frac{4n^2 + 3n + 1}{4n(2n+1)(4n+1)}.$$  

(5.5)

We now use indices $i, j$ such that $\{i, j\} = \{0, 1\}$. For $t > \frac{3}{2}$, the divisor

$$\frac{3}{2} C_x - t R_i = \frac{3}{2} L_{xy} + \left( \frac{3}{2} - t \right) R_i + \frac{3}{2} R_j$$

is not pseudoeffective. Put

$$P_{R_i}(t) := \begin{cases} \frac{3}{2} C_x - t R_i & \text{for } 0 \leq t \leq \frac{3}{4n}, \\ \frac{3}{2} C_x - t R_i - \frac{4nt - 3}{2(2n-1)} (L_{xy} + R_j) & \text{for } \frac{3}{4n} \leq t \leq \frac{3}{2}, \end{cases}$$

(5.6)

$$N_{R_i}(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{3}{4n}, \\ \frac{4nt - 3}{2(2n-1)} (L_{xy} + R_j) & \text{for } \frac{3}{4n} \leq t \leq \frac{3}{2}. \end{cases}$$

We have

$$\left( \frac{3}{2} C_x - t R_i \right) \cdot L_{xy} = \frac{3 - 4nt}{4n(4n+1)}, \quad \left( \frac{3}{2} C_x - t R_i \right) \cdot R_j = \frac{3 - 4nt}{4(4n+1)},$$

and hence we see that $P_{R_i}(t)$ is nef for $0 \leq t \leq \frac{3}{4n}$. The divisor $P_{R_i}(t)$ is a nef divisor with $P_{R_i}(t) \cdot L_{xy} = P_{R_i}(t) \cdot R_j = 0$ and $N_{R_i}(t)$ is negative definite for $\frac{3}{4n} \leq t \leq \frac{3}{2}$. Therefore, the Zariski decomposition of $\frac{3}{2} C_x - t R_i$ is given by (5.6).

Consequently, the volume is given by

\[
\text{vol} \left( \frac{3}{2} C_x - t R_i \right) = \begin{cases} \frac{-4n(2n+1)t^2 + 12nt - 9(2n+1)}{8n(4n+1)} & \text{for } 0 \leq t \leq \frac{3}{4n}, \\ \frac{(3 - 2t)^2}{8(2n-1)} & \text{for } \frac{3}{4n} \leq t \leq \frac{3}{2}, \\ 0 & \text{for } t \geq \frac{3}{2}, \end{cases}
\]

(5.7)

and the function in (3.4) is given by

$$S_{S_{\frac{1}{2}} W}(R_i) = \frac{4n^2 + 3n + 1}{4n(2n+1)}.$$  

(5.8)
For a point $p \in R_i \setminus \{O_w\}$, note that

$$\text{ord}_p \left( N_{R_i}(t) |_{R_i} \right) = 0$$

on $0 \leq t \leq \frac{3}{2}$. Thus, for $p \in R_i \setminus \{O_w\}$ the function in (3.6) is given by

$$h_{p,R_i}(t) = \int_0^\infty \text{vol} \left( P_{R_i}(t) |_{R_i} - v p \right) dv$$

$$= \frac{1}{2} (P_{R_i}(t) \cdot R_i)^2$$

$$= \begin{cases} \frac{1}{2(4n+1)^2} \left( \left( \frac{1}{2} + n \right) t + \frac{3}{4} \right)^2 & \text{for } 0 \leq t \leq \frac{3}{4n} \\ \frac{1}{32(2n-1)^2} (-2t + 3)^2 & \text{for } \frac{3}{4n} \leq t \leq \frac{3}{2} \end{cases},$$

and the value in (3.7) is given by

$$S(W_{R_i}^{R_i}; p) = \frac{16(4n+1)}{9(2n+1)} \int_0^{\frac{3}{4}} h_{p,R_i}(t) dt$$

$$= \frac{16n(4n+1)}{9(2n+1)} \left\{ \int_0^{\frac{3}{4n}} \frac{1}{(2(4n+1))^2} \left( \left( \frac{1}{2} + n \right) t + \frac{3}{4} \right)^2 dt \\ + \int_{\frac{3}{4n}}^{\frac{3}{2}} \frac{1}{32(2n-1)^2} (-2t + 3)^2 dt \right\}$$

$$= \frac{8n^2 + 7n + 1}{8n(2n+1)(4n+1)}.$$

Let $C_\gamma$ be the curve on $S_n$ cut by $y = \gamma x^2$ for a constant $\gamma$. It consists of two irreducible curves. One is $L_{xy}$ and the other is the curve $R$ defined by

$$y - \gamma x^2 = w + \gamma x z (z - \gamma^n x^{2n}) + z A_{2n+1}(x, \gamma x^2) + A_{4n+1}(x, \gamma x^2) = 0.$$

Their intersection numbers are as follows:

$$R^2 = \frac{1}{2n}, \quad L_{xy} \cdot R = \frac{1}{2n}.$$

From these intersection numbers we obtain

$$-\left( K_{S_n} + \frac{1}{2} W \right) \cdot R = \frac{3}{4} C_\gamma \cdot R = \frac{3}{4} (L_{xy} + R) \cdot R = \frac{3}{4n}.$$

Also we see that $W$ and $R$ meets at $O_z$ with local intersection number $\frac{1}{2n}$ and

$$W \cdot R = 2 + \frac{1}{2n}.$$
Besides the singular point $O_2$, the curve $R$ meets $W$ either transversally at two distinct smooth points or tangentially at a single smooth point with local intersection number 2.

Since $L_{xy}$ is of negative self-intersection, $\frac{3}{4}C_γ - tR$ is not pseudoeffective for $t > \frac{3}{4}$. Put

$$P_R(t) := \begin{cases} 
\frac{3}{4}C_γ - tR & \text{for } 0 \leq t \leq \frac{3}{2(4n + 1)}, \\
\frac{3}{4}C_γ - tR - \frac{2(4n + 1)t - 3}{2(4n - 1)}L_{xy} & \text{for } \frac{3}{2(4n + 1)} \leq t \leq \frac{3}{4},
\end{cases}$$

(5.10)

and

$$N_R(t) := \begin{cases} 
0 & \text{for } 0 \leq t \leq \frac{3}{2(4n + 1)}, \\
\frac{2(4n + 1)t - 3}{2(4n - 1)}L_{xy} & \text{for } \frac{3}{2(4n + 1)} \leq t \leq \frac{3}{4},
\end{cases}$$

Then, we have

$$P_R(t) \cdot L_{xy} = \begin{cases} 
\frac{1}{2n} \left( \frac{3}{2(4n + 1)} - t \right) & \text{for } 0 \leq t \leq \frac{3}{2(4n + 1)}, \\
0 & \text{for } \frac{3}{2(4n + 1)} \leq t \leq \frac{3}{4},
\end{cases}$$

and hence we see that $P_R(t)$ is nef for $0 \leq t \leq \frac{3}{4}$. Consequently, the Zariski decomposition of $\frac{3}{4}C_γ - tR$ is given by (5.10) for $0 \leq t \leq \frac{3}{4}$. Thus, the volume is given by

$$\text{vol} \left( \frac{3}{2}C_γ - tR \right) = \begin{cases} 
\frac{4(4n + 1)t^2 - 12(4n + 1)t + 9(2n + 1)}{8n(4n + 1)} & \text{for } 0 \leq t \leq \frac{3}{2(4n + 1)}, \\
\frac{(3 - 4t)^2}{4(4n - 1)} & \text{for } \frac{3}{2(4n + 1)} \leq t \leq \frac{3}{4}, \\
0 & \text{for } t \geq \frac{3}{4},
\end{cases}$$

(5.11)

and the value in (3.4) is given by

$$S_{S_n, \frac{1}{2}W}(R) = \frac{4n^2 + 3n + 1}{2(2n + 1)(4n + 1)}.$$  

(5.12)

We now consider an effective $\mathbb{Q}$-divisor $D$ numerically equivalent to $-(K_{S_n} + \frac{1}{2}W)$. We may write

$$D = aW + bL_{xy} + b_0R_0 + b_1R_1 + \Delta,$$

(5.13)

where $a, b, b_i$ are non-negative rational numbers and $\Delta$ is an effective $\mathbb{Q}$-divisor whose support contains none of the curves $W, L_{xy}, R_0, R_1$. Also we may write

$$D = aW + bL_{xy} + cR + \Omega,$$

(5.14)

where $\Omega$ is an effective $\mathbb{Q}$-divisor whose support contains none of $W, L_{xy}$, and $R$.  


Lemma 5.15. For a sufficiently large integer $m$, suppose that $D$ is of $m$-basis type with respect to the log del Pezzo surface $(S_n, \frac{1}{2}W)$. Then

$$a < \frac{1}{8n}; \quad b < \frac{3n + 2}{2(2n + 1)}; \quad b_0, b_1 < \frac{8n^2 + 6n + 3}{8n(2n + 1)}; \quad c < \frac{3}{10}.$$  

Proof. The first inequality immediately follows from

$$\frac{1}{D^2} \int_0^\infty \text{vol}(D - tW)dt = \int_0^{\frac{3}{2(4n+1)}} \left(1 - \frac{2(4n+1)}{3}t\right)^2 dt = \frac{1}{2(4n+1)} < \frac{1}{8n}$$

via Lemma 3.5.

Also, it follows from (5.4), (5.8), (5.12), and Lemma 3.5 that

$$b \leq S_{S_n, \frac{1}{2}W}(L_{xy}) + \varepsilon = \frac{3n + 1}{2(2n + 1)} + \varepsilon < \frac{3n + 2}{2(2n + 1)},$$

$$b_i \leq S_{S_n, \frac{1}{2}W}(R_i) + \varepsilon = \frac{4n^2 + 3n + 1}{4n(2n + 1)} + \varepsilon < \frac{8n^2 + 6n + 3}{8n(2n + 1)},$$

$$c \leq S_{S_n, \frac{1}{2}W}(R) + \varepsilon = \frac{4n^2 + 3n + 1}{2(2n + 1)(4n + 1)} + \varepsilon < \frac{3}{10},$$

where $\varepsilon$ is a sufficiently small positive rational number. \hfill \Box

From now on, we put $\lambda = \frac{20n+5}{20n+4}$.

Theorem 5.16. For a smooth point $p$,

$$\delta_p(S_n, \frac{1}{2}W) \geq \lambda.$$  

Proof. With a sufficiently large positive integer $m$, let $D$ be a $Q$-divisor of $m$-basis type with respect to the log del Pezzo surface $(S_n, \frac{1}{2}W)$. It is enough to show that the log pair

$$(S_n, \frac{1}{2}W + \lambda D)$$

is log canonical on the smooth locus of $S_n$.

Suppose that the log pair is not log canonical at a smooth point $p$.

We write the divisor $D$ as in (5.13), that is,

$$D = aW + bL_{xy} + b_0R_0 + b_1R_1 + \Delta,$$

where $a, b$, and $b_i$ are non-negative rational numbers and $\Delta$ is an effective $Q$-divisor whose support contains none of the curves $W, L_{xy}, R_0, R_1$. Lemma 5.15 shows that

$$b < \frac{4}{5}; \quad b_0, b_1 < \frac{3}{5}.$$  

Suppose that the point $p$ lies on $L_{xy}$. Since $\lambda b \leq 1$ and $p \notin W \cup R_0 \cup R_1$, the log pair

$$(S_n, L_{xy} + \lambda \Delta)$$
is not log canonical at $p$. We then obtain a contradiction from Lemma 3.3 and the inequality

$$\Delta \cdot L_{xy} = (D - aW - bL_{xy} - b_0R_0 - b_1R_1) \cdot L_{xy} \leq (D - bL_{xy}) \cdot L_{xy} = \frac{3 + 2b(4n - 1)}{4n(4n + 1)} < \frac{1}{\lambda}.$$ 

We now suppose that the point $p$ lies on $R_i$. Since $\lambda b_i \leq 1$ and $p \not\in L_{xy} \cup W \cup R_j$, where $i \neq j$, the log pair

$$(S_n, R_i + \lambda \Delta)$$

is not log canonical at $p$. This also yields an absurd inequality

$$\Delta \cdot R_i = (D - aW - bL_{xy} - b_0R_0 - b_1R_1) \cdot R_i \leq (D \cdot R_i - R_i^2) = \frac{4n + 5}{4(4n + 1)} < \frac{1}{\lambda}.$$ 

Therefore, the point $p$ must be located outside the curves $C_x$.

Let $C$ be a curve in the pencil $|\mathcal{O}_{S_n}(2)|$ that passes through the point $p$. Since the curve $C$ is cut by $y = \gamma x^2$ for some constant $\gamma$, it consists of two irreducible curves $L_{xy}$ and $R$. As in (5.14), we now may write

$$D = aW + bL_{xy} + cR + \Omega,$$

where $\Omega$ is an effective $\mathbb{Q}$-divisor whose support contains none of $W, L_{xy}, R$. Lemmas 5.15 implies that

$$a < \frac{1}{8n}; \quad b < \frac{4}{5}; \quad c < \frac{3}{10}.$$ 

The log pair

$$(S_n, \left(\frac{1}{2} + \lambda a\right)W + \lambda bL_{xy} + \lambda cR + \lambda \Omega)$$

is not log canonical at $p$.

Suppose that $p \not\in W$. Since $\lambda c \leq 1$, the log pair

$$(S_n, R + \lambda \Omega)$$

is not log canonical at $p$ either. Lemma 3.3 then implies an absurd inequality

$$\frac{1}{\lambda} < \Omega \cdot R = (D - aW - bL_{xy} - cR) \cdot R \leq D \cdot R = \frac{3}{4n}.$$ 

This means that the point $p$ must belong to $W$. Then the log pair

$$(S_n, \left(\frac{1}{2} + \lambda a\right)W + R + \lambda \Omega)$$

is not log canonical at $p$.

The curve $R$ meets $W$ at $p$ either transversally or tangentially. When they meet at $p$ tangentially, their local intersection number at $p$ is 2.
We first consider the case when the curve $R$ meets $W$ at $p$ transversally. In this case, we can easily obtain a contradiction,

$$1 < \left( \left( \frac{1}{2} + \lambda a \right) W + \lambda \Omega \right) \cdot R$$

$$\leq \left( \frac{1}{2} + \lambda a \right) + \lambda \Omega \cdot R$$

$$= \left( \frac{1}{2} + \lambda a \right) + \lambda(D - aW - bL_{xy} - cR) \cdot R$$

$$\leq \frac{1}{2} + \lambda D \cdot R = \frac{1}{2} + \lambda \frac{3}{4n} < 1$$

from Lemma 3.3. Therefore, the curve $R$ meets $W$ at $p$ with local intersection number 2. Note that

$$\operatorname{mult}_p(\Omega) \leq \Omega \cdot R \leq \frac{3}{4n}.$$

Let $\phi : \tilde{S}_n \to S_n$ be the blowup at $p$ and $E$ be its exceptional curve. Then

$$\phi^* \left( K_{\tilde{S}_n} + \left( \frac{1}{2} + \lambda a \right) W + \lambda bL_{xy} + \lambda cR + \lambda \Omega \right)$$

$$= K_{\tilde{S}_n} + \left( \frac{1}{2} + \lambda a \right) \tilde{W} + \lambda b\tilde{L}_{xy} + \lambda c\tilde{R} + \lambda \tilde{\Omega} + dE,$$

where $\tilde{W}$, $\tilde{L}_{xy}$, $\tilde{R}$, and $\tilde{\Omega}$ are the proper transforms of $W$, $L_{xy}$, $R$, and $\Omega$, respectively. Here $d = \lambda(a + c) + \lambda \operatorname{mult}_p(\Omega) - \frac{1}{2}$. Since $d \leq 1$ and $\lambda \operatorname{mult}_p(\Omega) \leq 1$, the log pair

$$\left( \tilde{S}_n, \left( \frac{1}{2} + \lambda a \right) \tilde{W} + \lambda b\tilde{L}_{xy} + \lambda c\tilde{R} + \lambda \tilde{\Omega} + dE \right)$$

is not log canonical at the point $q$ where $E$, $\tilde{R}$, and $\tilde{W}$ meet. Let $\psi : \tilde{S}_n \to \tilde{S}_n$ be the blowup at the point $q$ and let $F$ be the exceptional curve of $\psi$. Denote the proper transforms of $\tilde{W}$, $\tilde{L}_{xy}$, $\tilde{R}$, $\tilde{\Omega}$, and $E$ by $\tilde{W}$, $\tilde{L}_{xy}$, $\tilde{R}$, $\tilde{\Omega}$, and $\tilde{E}$, respectively. Then

$$\psi^* \left( K_{\tilde{S}_n} + \left( \frac{1}{2} + \lambda a \right) \tilde{W} + \lambda b\tilde{L}_{xy} + \lambda c\tilde{R} + \lambda \tilde{\Omega} + dE \right)$$

$$= K_{\tilde{S}_n} + \left( \frac{1}{2} + \lambda a \right) \tilde{W} + \lambda b\tilde{L}_{xy} + \lambda c\tilde{R} + \lambda \tilde{\Omega} + d\tilde{E} + eF,$$

where $e = \lambda(a + c) + d + \lambda \operatorname{mult}_q(\tilde{\Omega}) - \frac{1}{2}$.

Since

$$e = 2\lambda(a + c) + \lambda \left( \operatorname{mult}_p(\Omega) + \operatorname{mult}_q(\tilde{\Omega}) \right) - 1 \leq 2\lambda(a + c + \operatorname{mult}_p(\Omega)) - 1 \leq 1,$$

the log pair

$$\left( \tilde{S}_n, \left( \frac{1}{2} + \lambda a \right) \tilde{W} + \lambda b\tilde{L}_{xy} + \lambda c\tilde{R} + \lambda \tilde{\Omega} + d\tilde{E} + F \right)$$
is not log canonical at a point on $F$. Note that the curves $W$, $R$, and $E$ meet $F$ transversally at distinct points. However, the inequalities

$$\lambda \Omega \cdot F = \lambda \text{mult}_q(\Omega) \leq \lambda \text{mult}_p(\Omega) \leq \frac{3\lambda}{4n} < 1,$$

$$\left(\frac{1}{2} + \lambda a\right) + \lambda \Omega \cdot F \leq \left(\frac{1}{2} + \lambda a\right) + \frac{3\lambda}{4n} < 1,$$

$$\lambda c + \lambda \Omega \cdot F \leq \lambda c + \frac{3\lambda}{4n} < 1,$$

$$d + \lambda \Omega \cdot F = \lambda (a + c) + 2\lambda \text{mult}_p(\Omega) - \frac{1}{2} < 1$$

imply that the log pair above is log canonical along the curve $F$ by Lemma 3.3. This is a contradiction. Consequently, the log pair $(S_n, \frac{1}{2}W + \lambda D)$ must be log canonical in the smooth locus of $S_n$.

**Remark 5.17.** Another way to verify Theorem 5.16 is to apply a general version of Theorem 3.9 as in the proof of Theorem 5.18 below. We, however, use a direct method that is a bit simpler and demonstrate an instructive and basic approach to estimations of $\delta$-invariants.

**Theorem 5.18.** For singular points $p = O_2$, $O_0$, and $O_1$,

$$\delta_p\left(S_n, \frac{1}{2}W\right) \geq \lambda.$$  

**Proof.** For the singularity $O_2$, we obtain the log discrepancy

$$A_{L_{xy}L}(O_2) = 1 - \text{ord}_{O_2}(\Delta_L) = \frac{1}{4n}$$

from the adjunction formula

$$\left(K_{S_n} + \frac{1}{2}W + L_{xy}\right)_{L_{xy}} = K_{L_{xy}} + \Delta_L.$$

Then, (5.4), (5.5), and Theorem 3.9 imply that

$$\delta_{O_2}\left(S_n, \frac{1}{2}W\right) \geq \min \left\{ \frac{2(2n+1)}{3n+1}, \frac{(2n+1)(4n+1)}{4n^2+3n+1} \right\} > \lambda.$$

Similarly, for the singularity $O_1$, we obtain the log discrepancies

$$A_{R_i\Lambda_i}(O_1) = 1 - \text{ord}_{O_1}(\Lambda_i) = \frac{1}{4}$$

from the adjunction formula

$$\left(K_{S_n} + \frac{1}{2}W + R_i\right)_{R_i} = K_{R_i} + \Lambda_i.$$
Theorem 3.9 with (5.8) and (5.9) then yields
\[ \delta_{O_i} \left( S_n, \frac{1}{2} W \right) \geq \min \left\{ \frac{4n(2n+1)}{4n^2 + 3n + 1}, \frac{2n(2n+1)(4n+1)}{8n^2 + 7n + 1} \right\} > \lambda. \]

Theorem 5.19. For the singular point \( O_w \),
\[ \delta_{O_w} \left( S_n, \frac{1}{2} W \right) \geq \lambda. \]

Proof. Let \( D \) be a \( \mathbb{Q} \)-divisor of \( m \)-basis type with respect to the log del Pezzo surface \( (S_n, \frac{1}{2} W) \) for a sufficiently large positive integer \( m \). It is enough to show that the log pair
\[ \left( S_n, \frac{1}{2} W + \lambda D \right) \]
is log canonical at \( O_w \). Since the point \( O_w \) is away from the curve \( W \), we will prove that \( (S_n, \lambda D) \) is log canonical at \( O_w \).

Suppose that \( (S_n, \lambda D) \) is not log canonical at \( O_w \). As (5.13), we write
\[ D = b L_{xy} + b_0 R_0 + b_1 R_1 + \Lambda, \]
where \( b \) and \( b_i \) are non-negative rational numbers and \( \Lambda \) is an effective \( \mathbb{Q} \)-divisor whose support contains none of the curves \( L_{xy}, R_0, R_1 \). By Lemma 5.15,
\[ b < \frac{3n + 2}{2(2n + 1)}; \quad b_0, b_1 < \frac{8n^2 + 6n + 3}{8n(2n + 1)}. \]

Let \( \phi: \hat{S}_n \to S_n \) be the weighted blowup at \( O_w \) with weights \((1, n) \) and \( F \) be its exceptional curve. Then
\[ K_{\hat{S}_n} = \phi^*(K_{S_n}) - \frac{3n}{4n + 1} F. \]
Denote the proper transforms of \( L_{xy}, R_0, R_1 \), and \( \Lambda \) by \( \hat{L}_{xy}, \hat{R}_0, \hat{R}_1 \), and \( \hat{\Lambda} \), respectively.

The exceptional curve \( F \) contains one singular point of \( \hat{S}_n \), where \( F \) and \( \hat{L}_{xy} \) intersect. It is a cyclic quotient singularity of type \( \frac{1}{n}(-1, 1) \).

We have
\[ \hat{L}_{xy} = \phi^*(L_{xy}) - \frac{1}{4n + 1} F, \quad \hat{R}_i = \phi^*(R_i) - \frac{n}{4n + 1} F, \quad \hat{\Lambda} = \phi^*(\Lambda) - \frac{\mu}{4n + 1} F, \]
where \( \mu \) is a non-negative rational number, and hence
\[ K_{\hat{S}_n} + \lambda (b \hat{L}_{xy} + b_0 \hat{R}_0 + b_1 \hat{R}_1 + \hat{\Lambda}) + \left( \frac{3n}{4n + 1} + \lambda \theta \right) F = \phi^*(K_{S_n} + \lambda D), \]
where
\[ \theta = \frac{b + n(b_0 + b_1) + \mu}{4n + 1}. \]
Since \( F^2 = -\frac{4n+1}{n} \), we obtain
\[
\hat{L}_{xy}^2 = -\frac{1}{2n}, \quad \hat{R}_0^2 = \hat{R}_1^2 = -\frac{1}{2}, \quad \hat{L}_{xy} \cdot \hat{R}_0 = \hat{L}_{xy} \cdot \hat{R}_1 = \hat{R}_0 \cdot \hat{R}_1 = 0,
\]
\[
\hat{L}_{xy} \cdot F = \frac{1}{n}, \quad \hat{R}_0 \cdot F = \hat{R}_1 \cdot F = 1.
\]

For the estimation of \( \theta \), we first compute the volume of \( \phi^*(D) - tF \). Since \( \hat{L}_{xy}, \hat{R}_0, \hat{R}_1 \) are of negative self-intersection, and
\[
\phi^*(D) - tF \equiv \phi^* \left( \frac{3}{2} C_x \right) - tF = \frac{3}{2} (\hat{L}_{xy} + \hat{R}_0 + \hat{R}_1) + \left( \frac{3(2n+1)}{2(4n+1)} - t \right) F,
\]
for \( t > \frac{6n+3}{8n+2} \), the divisor \( \phi^*(D) - tF \) is not pseudoeffective. Put
\[
P_F(t) = \begin{cases} 
\frac{3}{2} (\hat{L}_{xy} + \hat{R}_0 + \hat{R}_1) + \left( \frac{3(2n+1)}{2(4n+1)} - t \right) F & \text{for} \ 0 \leq t \leq \frac{3}{4(4n+1)} , \\
\left( \frac{3(2n+1)}{2(4n+1)} - t \right) (2\hat{L}_{xy} + 2\hat{R}_0 + 2\hat{R}_1 + F) & \text{for} \ \frac{3}{4(4n+1)} \leq t \leq \frac{3(2n+1)}{2(4n+1)} ,
\end{cases}
\]
\[
N_F(t) = \begin{cases} 
0 & \text{for} \ 0 \leq t \leq \frac{3}{4(4n+1)} , \\
\left( 2t - \frac{3}{2(4n+1)} \right) (\hat{L}_{xy} + \hat{R}_0 + \hat{R}_1) & \text{for} \ \frac{3}{4(4n+1)} \leq t \leq \frac{3(2n+1)}{2(4n+1)} .
\end{cases}
\]

For \( 0 \leq t \leq \frac{3}{4(4n+1)} \),
\[
P_F(t) \cdot \hat{L}_{xy} = \frac{1}{n} \left( \frac{3}{4(4n+1)} - t \right), \quad P_F(t) \cdot \hat{R}_0 = P_F(t) \cdot \hat{R}_1 = \frac{3}{4(4n+1)} - t.
\]

For \( \frac{3}{4(4n+1)} \leq t \leq \frac{3(2n+1)}{2(4n+1)} \),
\[
P_F(t) \cdot \hat{L}_{xy} = P_F(t) \cdot \hat{R}_0 = P_F(t) \cdot \hat{R}_1 = 0.
\]

Therefore, the divisor \( P_F(t) \) is nef. The Zariski decomposition of \( \phi^*(D) - tF \) is given by
\[
P_F(t) + N_F(t).
\]

Thus, the volume is given by
\[
\text{vol}(\phi^*(D) - tF) = \begin{cases} 
\frac{9(2n+1)}{8n(4n+1)} - \frac{4n+1}{n} t^2 & \text{for} \ 0 \leq t \leq \frac{3}{4(4n+1)} , \\
\frac{1}{n} \left( \frac{3(2n+1)}{2(4n+1)} - t \right)^2 & \text{for} \ \frac{3}{4(4n+1)} \leq t \leq \frac{3(2n+1)}{2(4n+1)} .
\end{cases}
\]
so the value in (3.4) is given by

$$S_{S, \frac{1}{2}w}(F) = \frac{1}{D^2} \int_0^\infty \text{vol}(\phi^*(D) - tF)dt = \frac{4n + 3}{4(4n + 1)}.$$ 

Thus, it follows from Lemma 3.5 that for a sufficiently small positive real number $\varepsilon$

$$\frac{b + n(b_0 + b_1) + \mu}{4n + 1} = \theta < \frac{4n + 3}{4(4n + 1)} + \varepsilon. \tag{5.20}$$

It implies that

$$\frac{3n}{4n + 1} + \lambda \theta < 1.$$ 

Therefore, the log pair

$$(S_n, \lambda bL_{xy} + \lambda b_0R_0 + \lambda b_1R_1 + \lambda \Lambda + F)$$

is not log canonical at some point $q$ on $F$.

We first suppose that $q \in F \setminus \hat{L}_{xy} \cup \hat{R}_0 \cup \hat{R}_1$. Then the log pair $(\hat{S}, \lambda \Lambda + F)$ is not log canonical at the point $q$. Lemma 3.3 then implies

$$\frac{1}{\lambda} < \Lambda \cdot F = \frac{\mu}{n}.$$ 

However, if $b_i \leq b_j$, then the inequality

$$0 \leq \hat{\Lambda} \cdot \hat{R}_i = \frac{3}{4(4n + 1)} - \frac{b}{4n + 1} + \frac{b_j(2n + 1)}{2(4n + 1)} - \frac{b_jn}{4n + 1} - \frac{\mu}{4n + 1}$$

yields the opposite inequality

$$\mu \leq \frac{3}{4} - b - b_jn + \frac{b_j(2n + 1)}{2} \leq \frac{3}{4} + \frac{b_j}{2} \leq \frac{3}{4} + \frac{8n^2 + 6n + 3}{16n(2n + 1)} < \frac{n}{\lambda}.$$ 

Similarly, if $b_j < b_i$, then the inequality $0 \leq \hat{\Lambda} \cdot \hat{R}_j$ produces a contradiction. Therefore, the point $q$ must be one of the intersection points $F \cap \hat{R}_0, F \cap \hat{R}_1, \text{and } F \cap \hat{L}_{xy}$.

We first consider the case when $q$ is the intersection point of $F$ and $\hat{R}_i$. Then the log pair $(\hat{S}, \lambda(\hat{\Lambda} + b_i\hat{R}_i) + F)$ is not log canonical at $q$. We then have

$$\frac{1}{\lambda} < (\hat{\Lambda} + b_i\hat{R}_i) \cdot F = \frac{\mu}{n} + b_i.$$ 

From (5.20) we obtain

$$\frac{1}{\lambda} + b_j < \frac{b}{n} + (b_0 + b_1) + \frac{\mu}{n} < \frac{4n + 3}{4n} + \varepsilon.$$ 

On the other hand, from

$$0 \leq \hat{\Lambda} \cdot \hat{R}_j = \frac{3}{4(4n + 1)} - \frac{b}{4n + 1} + \frac{b_j(2n + 1)}{2(4n + 1)} - \frac{b_jn}{4n + 1} - \frac{\mu}{4n + 1}, \tag{5.21}$$
we obtain
\[
\frac{\mu}{n} + b_i \leq \frac{3}{4n} - \frac{b}{n} + \frac{b_j(2n + 1)}{2n}.
\]

Then
\[
\frac{1}{\lambda} = \frac{20n + 4}{20n + 5} < \frac{\mu}{n} + b_i \leq \frac{3}{4n} + \frac{b_j(2n + 1)}{2n},
\]

and hence
\[
\frac{80n^2 - 44n - 15}{10(2n + 1)(4n + 1)} < b_j.
\]

This yields a contradictory inequality
\[
\frac{1}{\lambda} + b_j > \frac{20n + 4}{20n + 5} + \frac{80n^2 - 44n - 15}{10(2n + 1)(4n + 1)} = \frac{40n - 7}{20n + 10}.
\]

Consequently, \( q \) must be the intersection point of \( F \) and \( \hat{L}_{xy} \), which is a singular point of type \( \frac{1}{n}(-1, 1) \).

Then the log pair \((\hat{S}, \lambda(\hat{L} + b\hat{L}_{xy}) + F)\) is not log canonical at \( q \). We then obtain
\[
\frac{1}{n\lambda} < (\hat{L} + b\hat{L}_{xy}) \cdot F = \frac{\mu}{n} + \frac{b}{n}
\]

from Lemma 3.3. Meanwhile, if \( b_j \leq b_i \), we use (5.21) to obtain
\[
4(b + \mu) - 3 \leq b_0 + b_1.
\]

Together with (5.20) this implies that
\[
\frac{b}{4n + 1} + \frac{n}{4n + 1}(4(b + \mu) - 3) + \frac{\mu}{4n + 1} = (b + \mu) - \frac{3n}{4n + 1} < \frac{4n + 3}{4(4n + 1)} + \epsilon.
\]

Thus
\[
\frac{20n + 4}{20n + 5} = \frac{1}{\lambda} < b + \mu \leq \frac{16n + 3}{16n + 4} + \epsilon.
\]

This is absurd.

Therefore, we may conclude that the log pair \((S_n, \frac{1}{2}W + \lambda D)\) is log canonical at \( O_w \). \(\square\)

**Proof of Theorem 4.1.** Theorems 5.16, 5.18, and 5.19 immediately imply
\[
\delta(S_n, \frac{1}{2}W) > 1.
\]

Then Theorems 2.2 and 2.3 complete the proof. \(\square\)

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