\[ d^3 = 0, \quad d^2 = 0\] 

**Differential calculi on certain non-commutative (super) spaces.**

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**Abstract:** In this paper, we construct a covariant differential calculus on a quantum plane with two-parametric quantum group as a symmetry group. The two cases \( d^2 = 0 \) and \( d^3 = 0 \) are completely established.

We also construct differential calculi \( n = 2 \) and \( n = 3 \) nilpotent on super quantum spaces with one and two-parametric symmetry quantum supergroup.

**Keywords:** non-commutative (super) plane, non-commutative differential calculi \( d^3 = 0, d^2 = 0 \).

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1 Introduction:

A non-commutative quantum (super) space \[1, 2\] is an unital, associative algebra with a quantum (super) group as a symmetry group. These objects \[3, 4\] have enriched the arena of mathematics and mathematical physics: they appear in the context of theory of knots and braids \[5\], as well as in the study of Yang-Baxter equations \[6\]. Quantum (super) groups are deformations of the enveloping (super) algebra of classical Lie groups in the sense that one recovers the classical (super) commutator when the deformation parameters go to some particular values. Usually the generators of a quantum (super) group are assumed to commute with the non-commuting coordinates of the corresponding (super) plane. As a consequence, the quantum (super) plane admits a quantum group as a symmetry group with only one parameter: \((GL_q(1/1)) GL_q(2)\) \[7\]. More generally, one can obtain a multiparametric quantum (super) group, if one relaxes this property (commutation between space coordinates and group generators), namely, \(GL_{p,q}(1/1)\) and \(GL_{p,q}(2)\) \[8\] respectively in the two dimensional quantum superplane and quantum plane cases.

Many authors \[8, 10, 11, 12, 13, 14, 15, 16\] have also studied differential calculus with nilpotency \(n = 2\) on (super) spaces with one or two-parameter (super) group as symmetry groups. An adequate way leading to generalization of this ordinary differential calculus arises from the graded differential algebra \[17, 18, 19, 20, 21, 22\]. The latter involves a complex parameter that satisfies some conditions allowing to obtain a consistent generalized differential calculus. The most important property of this calculus is that the operator “\(d\)” satisfies \(d^n = 0 / d^l \neq 0, 1 \leq l \leq n-1\) and it contains as a consequence, not only first differentials \(dx^i, i = 1...m\), but involves also higher order differentials \(d^j x^i, j = 1...n - 1\).

In this paper, we construct covariant differential calculus \(d^3 = 0\) on certain quantum (super) spaces with one or two-parametric quantum group as a symmetry groups. We will show that our differential calculus is covariant under the algebra with a quantum group structure. The complex \(j\), which appears in the Leibniz rule, is a third-root of unity and will be an interesting and non trivial aspect of the differential calculus that we will introduce.

This paper is organized as follows:

In section 2 we start by recalling the two-parameter quantum group acting on a two-dimensional quantum plane. We also establish \(n = 2\) and \(n = 3\) covariant differential calculi on this space following R. Coquereaux approach \[14, 15\]. It will be noticed that some modifications have been brought up to this approach in order to adapt it to the two-parameter quantum group symmetry and the \(n = 3\) differential calculus. In section 3, the same method will be applied to construct
the $n = 2$, $n = 3$ covariant differential calculus on $1 + 1$-dimensional superspace with one parameter quantum supergroup as a supersymmetry group. In section 4 we generalize the results of section 3 by taking the two-parameter quantum group acting covariantly on the superspace.

2 Differential calculus on a two-parametric quantum plane.

2.1 Preliminaries

The two dimensional quantum plane is an associative algebra generated by two non-commuting coordinates $x$ and $y$ satisfying the relation:

$$xy = qyx, \quad q \neq 0, 1 \quad (q \in C). \quad (1)$$

In order to have a two-parameter quantum group $GL_{p,q}^p(2)$ as a symmetry group of such a space, one must assume that the coordinates do not commute in general with elements defining this group.

Indeed, for a generic element $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_{p,q}^p(2)$, the relations between the matrix entries and the coordinates are assumed to be:

$$
\begin{align*}
xa &= q_{11} ax \quad &ya &= q_{21} ay \\
xb &= q_{12} bx \quad &yb &= q_{22} by \\
xc &= q_{13} cx \quad &yc &= q_{23} cy \\
d &= q_{14} dx \quad &yd &= q_{24} dy.
\end{align*}
\quad (2)
$$

The coordinates $x$ and $y$ transform under $T$ and $^tT$ (transposed matrix) as:

$$
\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
$$

$$
\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{^tT} \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

The requirement that the transformed coordinates obey a similar relation as eq(1) (not necessarily with the same deformation parameter $q$) i.e.

$$
\begin{align*}
x'y' &= \bar{q} y' x', \quad \bar{q} \in C \\
x''y'' &= \bar{q} y'' x'', \quad \bar{q} \in C.
\end{align*}
\quad (3)
\quad (4)
$$
and taking account of the defining relations of \( GL_{p,q'}(2) \) \[13\],

\[
\begin{align*}
ab &= p \, ba \\
ac &= q' \, ca \\
bd &= q' \, db \\
pbc &= q' \, cb \\
ad - da &= (p - \frac{1}{q'}) \, bc,
\end{align*}
\]

for some non zero \( p, q' \) with \( pq' \neq -1 \), implicates further constraints on the involved parameters:

\[
\bar{q} = \bar{q}
\]

and

\[
\begin{align*}
q_{11} &= 1 & q_{21} &= qq^{-1}k \\
q_{12} &= \bar{q}p^{-1} & q_{22} &= \bar{q}p^{-1}[q - (p - q^{-1})k] \\
q_{13} &= \bar{q}q'^{-1} & q_{23} &= \bar{q}q'^{-1}[\bar{q} - (p - q'^{-1})k] \\
q_{14} &= \bar{q}q'^{-1}k & q_{24} &= \bar{q}q'^{-1}p^{-1}[\bar{q} - (p - q'^{-1})k].
\end{align*}
\]

One can check that the matrix \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is indeed an element of the quantum group \( GL_{p,q'}(2) \) and is consistent with Hopf algebra structures \[9\]. For supplementary properties and results concerning the quantum group \( GL_{p,q'}(2) \) see for example \[13\].

It is clear that many quantum planes could be associated to this two-parameter quantum group, depending on choices of the \( q_{ij} \)'s. In the following, we shall confine our selves to the case \( \bar{q} = q \), which corresponds to the standard definition of the quantum plane.

### 2.2 Differential calculus with nilpotency \( n = 2 \) \( (d^2 = 0) \).

Our aim in this section, is to construct a differential calculus on the previously defined quantum plane. We proceed using the same approach as the one adopted in \[14\] \[15\] \[20\].

We start by defining the exterior differential "\( d \)" which satisfies the usual properties, namely:

i/ Linearity

\( d(\alpha + \beta) = d(\alpha) + d(\beta) \)

ii/ Nilpotency

\( d^2 = 0 \).

iii/ Leibniz rule

\( d(uv) = d(u)v + (-1)^n ud(v) \),
where \( u \in \Omega^n \). \( \Omega^n \) is the space of forms with degree \( n \),

\[
d : \Omega^n \to \Omega^{n+1}.
\]

\( \Omega^0 \) is the algebra of functions defined on the quantum plane. We have also:

\[
d(x) = dx, \quad d(y) = dy \quad \text{and} \quad d(1) = 0.
\]

From (2), we deduce:

\[
\begin{align*}
(dx)a &= q_{11} a(dx) \\
(dy)a &= q_{21} a(dy) \\
(dx)b &= q_{12} b(dx) \\
(dy)b &= q_{22} b(dy) \\
(dx)c &= q_{13} c(dx) \\
(dy)c &= q_{23} c(dy) \\
(dx)d &= q_{14} d(dx) \\
(dy)d &= q_{24} d(dy).
\end{align*}
\]

One can write a priori \( xdx, xdy, ydx \), and \( ydy \) in terms of \( (dx)x, (dy)x, (dx)y \) and \( (dy)y \), by mean of 16 unknown coefficients [14]. Imposing the covariance of the obtained relations under \( GL_{p,q'}(2) \), and differentiating eq(1), permit to fix 15 of the 16 unknown coefficients. The associativity of the expression \( (xdx)dy = x(dx dy) \) enables us to fix the last unknown parameter.

We notice that in the usual case \( GL_q(2) \) this approach yields directly the desired differential calculus. However, when \( GL_{p,q'}(2) \) is a symmetry group, we obtain additional conditions on the parameters \( k \) and \( q' \):

\[
q' = q, \quad k = \frac{q'}{p}.
\]

Then eq(6) becomes:

\[
q_{11} = q_{13} = 1, \quad q_{12} = q_{14} = q_{21} = q_{23} = q^{-1}, \quad q_{22} = q_{24} = q^2 p^{-2}.
\]

So, the covariant differential calculus is given by:

\[
\begin{align*}
xdx &= \frac{1}{pq} dx x \\
xy &= \frac{1}{pq} dy y \\
xy &= (\frac{1}{pq} - 1) dy x + \frac{1}{q} dx y
\end{align*}
\]

and the differential algebra is \( \Omega_{q,p}^{x,y} = \{ x, y, dx, dy \} \).

It is remarkable that the differential calculus on the quantum plane with \( GL_q(2) \), as a symmetry group [11][12][14], can be obtained from the two-parameter one eq(11) in the \( p \to q \) limit.

5
As in the ordinary case, the differential operator $d$ can be realized by

$$d := dx \partial x + dy \partial y.$$ 

Based on this realization one can construct a gauge field theory on the two-parameter quantum plane. This should be achieved formally as in [20].

The nilpotent differential calculus can be extended to higher orders, as there is no reason to constrain this one to $n = 2$ nilpotency [17, 18, 19, 20, 23, 24, 25, 26].

In the following section, we generalize the differential calculus on the quantum plane with one-parameter symmetry group [20] to the two-parameter one, this is done by extending the $n = 2$ differential calculus obtained here to $n = 3$ case.

### 2.3 Differential calculus with nilpotency $n = 3$ ($d^3 = 0$).

Let us introduce the differential operator ”$d$” that satisfies the following conditions:

1. Linearity
2. Nilpotency
   $$d^3 = 0, \quad d^2 \neq 0.$$
3. Leibniz rule
   $$d(uv) = (du)v + (j)^n ud(v).$$

where $j$ is the cubic root of unity: $j = e^{\frac{2\pi i}{3}}, 1 + j + j^2 = 0$. $u$ is an element of $\Omega^n$, the space of forms with degree $n$. It is a subspace of the differential algebra $\hat{\Omega}^n_{x,y} = \{x, y, dx, dy, d^2x, d^2y\}$. The new objects $d^2x$ and $d^2y$ which appear are defined by:

$$d(dx) = d^2(x) = d^2x, \quad d(dy) = d^2(y) = d^2y,$$

these are ”forms” with degree two.

In order to ensure the covariance of the differential calculus under the two-parameter symmetry group $GL_{p,q}(2)$, we proceed as in the previous section. However, instead of the last step where we have used the associativity property, we shall use the independence between the two different 2-forms $z d^2z'$ and $dz dz'$, where $z, z' = x, y$. Below, we will discuss how to recover this property.

The same constraints on $q'$ and $k eq(9)$ are recovered, thus the $q_{ij}$’s are the same as in eq(10). The covariant differential calculus is then given by:

$$
\begin{align*}
x dx &= j^2 dx x \\
y dy &= j^2 dy y \\
x dy &= -\frac{jq}{1 + qp} dy x + \frac{j^2 qp - 1}{1 + qp} dx y \\
y dx &= \frac{j^2 - qp}{1 + qp} dy x - \frac{jq}{1 + qp} dx y
\end{align*}
$$
\[ x \, d^2 x = j^2 \, d^2 x \quad x \, d^2 y = -\frac{j q}{1 + q p} \, d^2 y \, x + \frac{j^2 q p - 1}{1 + q p} \, d^2 x \, y \]
\[ y \, d^2 y = j^2 \, d^2 y \quad y \, d^2 x = \frac{j^2 - q p}{1 + q p} \, d^2 y \, x - \frac{j p}{1 + q p} \, d^2 x \, y \]  \quad \text{(12)}
\[ d x \, d^2 x = j \, d^2 x \, d x \quad d x \, d^2 y = -\frac{q}{1 + q p} \, d^2 y \, d x + \frac{j q p - j^2}{1 + q p} \, d^2 x \, d y \]
\[ d y \, d^2 y = j \, d^2 y \, d y \quad d y \, d^2 x = \frac{j - j^2 q p}{1 + q p} \, d^2 y \, d x - \frac{p}{1 + q p} \, d^2 x \, d y \]
\[ d x \, d y = q \, d y \, d x \quad d^2 x \, d^2 y = q \, d^2 y \, d^2 x. \]

Moreover, a realization of "\( d \)" in terms of partial derivatives:
\[ d = d x \partial_x + d y \partial_y \]  \quad \text{(13)}

permits us to have \( (d x)^3 = (d y)^3 = 0 \) \[20\].

We note that the differential algebra \( \tilde{\Omega}_{x,y}^{q,p} \), defined above, is not associative. One can check this statement by first assuming that this property (associativity) is preserved, then deriving some inconsistent relations. Especially, one expects, due to this assumption, the two expressions \( (x \, d x) \, d y \) and \( x \, (d x \, d y) \) to be equal. However, using (12) and successively moving the parenthesis, one obtains two expressions which are manifestly not equal, unless \( p q = j^2 \).

Thus, the differential algebra \( \tilde{\Omega}_{x,y}^{q,p} \) is associative only when \( p q = j^2 \), otherwise it is not.

Another associative 3-nilpotent differential algebra, for \( p q = j \), can be constructed basing on the method already mentioned in section (2.2), with a proper substitution of the differential operator \( d^2 = 0 \) with the one \( d^3 = 0 \), \( d^2 \neq 0 \). It follows from this method that the commutation relations between the coordinates and their first order differentials are given (by the first ones) in (11). The first, second and third differentiations of these relations give rise to the remaining commutation relations between \( x, y, d x, d y, d^2 x \) and \( d^2 y \).

The results of \[20\] (i.e., differential calculus on a reduced quantum plane respectively with \( q^3 = 1 \) and \( q^N = 1 \)) can be recovered as limiting cases of the one obtained here (12); this is done by taking the adequate limit \( p \rightarrow q \) (respectively with \( q^3 = 1 \) and \( q^N = 1 \)).

It is also remarkable that the case \( n = 3 \) differential calculus was applied to introduce interesting "Higher order gauge theories" \([18, 19, 20]\). Indeed, an interesting manner to do this (in the present case) is to pursue the same steps of \[20\].

Another important question arises at this step is how to adapt the techniques
applied in subsections (2.2) and (2.3) to the quantum superplane. This will be developed in the next section.

3 Differential calculus on a one-parameter quantum superplane.

3.1 $n = 2$ Differential calculus.

The 1+1 dimensional quantum superspace, in Manin's approach \cite{2,7}, is an algebra generated by a bosonic and a fermionic coordinate satisfying the relations:

$$x\theta = q \theta x, \quad q \neq 0, 1$$
$$\theta^2 = 0.$$  

(14)  

(15)

In analogy with the quantum plane, a symmetry supergroup of this space is $GL_q(1/1)$, and a generic element of this supergroup is a supermatrix: $T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$, where $a, d$ are bosonic elements commuting with $x$ and $\theta$ while $\beta, \gamma$ are fermionic elements commuting with $x$, anticommuting with $\theta$ and obeying the following relations:

$$a\beta = q \beta a \quad d\beta = q \beta d$$
$$a\gamma = q \gamma a \quad d\gamma = q \gamma d$$
$$\beta\gamma + \gamma\beta = 0 \quad \beta^2 = \gamma^2 = 0$$

(16)

These relations can also be obtained by imposing the invariance of eqs(14,15) under $T$ and $^{st}T = \begin{pmatrix} a & -\gamma \\ \beta & d \end{pmatrix}$ (supertranspose).

Many authors studied the differential calculus on this superspace \cite{8,10,28}. Here we construct the differential calculus based on the same technique adopted by R. Couquereaux \cite{14} which is used in the previous section, with however, some modifications to adapt it to this superspace. We introduce an exterior differential operator "$d$" satisfying the properties:

i/ Linearity

$$d(\lambda u) = (-1)^{\hat{\lambda}} \lambda d(u),$$

(17)

where the parity $\hat{\lambda} = 0, 1$ respectively, if $\lambda$ is a bosonic or a fermionic element.

ii/ Nilpotency

$$d^2 = 0.$$
iii/Leibniz rule

\[ d(uv) = (du)v + (-1)^{\hat{u}}(-1)^{\deg u}u(dv), \quad (18) \]

where \( \hat{u} \) is the parity of \( u \) and \( \deg u \) is the degree of the differential form \( u \).

Note that consistency requires that \( d\theta \) commutes with \( a, d, \beta, \gamma \) and \( dx \) commutes with \( a, d \) and anticommutes with \( \beta, \gamma \).

The same method applied in section 2.2 yields:

\[ \begin{align*}
xdx & = q^{-2} \, dx \, x \\
\theta \, d\theta & = d\theta \, \theta \\
dx \, d\theta & = q^{-1} \, d\theta \, dx \\
(d\theta)^2 & = 0,
\end{align*} \quad (19) \]

and the associative differential algebra is denoted \( \Omega^q_{x,\theta} = \{x, \theta, dx, d\theta\} \).

As in section 2.3, one can apply the same method to generalize the differential calculus on the superspace to higher orders \( (d^3 = 0) \). This is the aim of the next section.

3.2 Differential calculus on superspace with nilpotency

\( n = 3 \) \( (d^3 = 0) \).

We proceed as in section 2.3, in order to construct the \( n = 3 \) covariant differential calculus on superspace. We introduce a differential operator "\( d \" satisfying the usual requirements, namely: linearity is the same as in eq(17), the nilpotency will be changed to \( n = 3 \) \( (d^3 = 0) \) and the Leibniz rule, eq(18) becomes:

\[ d(uv) = (du)v + (-1)^{\hat{u}}(j)^{\deg u}u(dv). \quad (20) \]

The resulting differential algebra \( \tilde{\Omega}^q_{x,\theta} \) is generated by, \( x, \theta, dx, d\theta, d^2x \) and \( d^2\theta \) satisfying:

\[ \begin{align*}
xdx & = j^2 \, dx \, x \\
xd\theta & = -\frac{jq}{1+q^2} \, d\theta \, x + \frac{j^2 q^2 - 1}{1+q^2} \, dx \, \theta \\
\theta \, d\theta & = d\theta \, \theta \\
dx \, d\theta & = -q \, d\theta \, dx \\
(d\theta)^2 & = 0 \\
x \, d^2x & = j^2 \, d^2x \, x \\
x \, d^2\theta & = -\frac{jq}{1+q^2} \, d^2\theta \, x + \frac{j^2 q^2 - 1}{1+q^2} \, d^2x \, \theta \\
\theta \, d^2\theta & = -d^2\theta \, \theta \\
\theta \, d^2x & = \frac{j^2 - q^2}{1+q^2} \, d^2x \, x - \frac{jq}{1+q^2} \, d^2x \, \theta \\
\theta \, d^2\theta & = -d^2\theta \, \theta
\end{align*} \quad (21) \]
\[ dx \, d^2x = j \, d^2x \, dx \quad \quad \quad dx \, d^2\theta = \frac{q}{1 + q^2} \, d^2\theta \, dx + \frac{jq^2 - j^2}{1 + q^2} \, d^2x \, d\theta \]
\[ d\theta \, d^2\theta = j^2 \, d^2\theta \, d\theta \quad \quad \quad d\theta \, d^2x = \frac{j^2q^2 - j}{1 + q^2} \, d^2\theta \, dx - \frac{q}{1 + q^2} \, d^2x \, d\theta \]
\[ d^2x \, d^2\theta = q \, d^2\theta \, d^2x \quad \quad \quad (d^2\theta)^2 = 0. \]

Let us point out that the differential algebra \( \tilde{\Omega}_{x,\theta}^q \) is not associative, unless \( q = j \). In the case \( q \neq j \), one can recover this property by following the same steps mentioned at the end of section 2.3 with the adequate modifications.

4 Differential calculus on a two-parameter quantum superplane.

4.1 Differential calculus with nilpotency \( n = 2 \), \((d^2 = 0)\).

In this section, we generalize the results of section 3, in the sense that we choose a two-parametric quantum supergroup \( GL_{p,q'}(1/1) \) as a symmetry group for the superplane eqs(14, 15). This group will be introduced using the same method as in section 2 \[8, 9\].

The entries of a matrix element \( T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \) of \( GL_{p,q'}(1/1) \) satisfy the following non trivial relations:

\[
\begin{align*}
  a\beta &= p \beta a & d\beta &= p \beta d \\
  a\gamma &= q' \gamma a & d\gamma &= q' \gamma d \\
  p \beta \gamma + q' \gamma \beta &= 0 & \beta^2 &= \gamma^2 = 0 \\
  ad - da &= (q'^{-1} - p) \beta \gamma.
\end{align*}
\]

As it is done in section 2 this superspace is covariant under \( T \) and \( ^sT \) (super-transpose), and the analogous of eq(2) are:

\[
\begin{align*}
  xa &= k \, a \, x & \theta a &= qq^{-1}\, p^{-1} \, k \, a \, \theta \\
  xb &= \bar{q}p^{-1} \, k \, b \, x & \theta b &= -qq^{-2}.q'^{-1}p^{-2} \, k \, b \, \theta \\
  xc &= \bar{q}q'^{-1} \, k \, c \, x & \theta c &= -qq^{-2}q'^{-2}p^{-1} \, k \, c \, \theta \\
  xd &= \bar{q}^2q'^{-1}p^{-1}k \, d \, x & \theta d &= qq^3q'^{-2}p^{-2} \, k \, d \, \theta.
\end{align*}
\]

We are interested in establishing a covariant differential calculus on this superspace in the case \( \bar{q} = \bar{q}' = q \). To achieve this construction, for \( n = 2 \), we introduce a differential operator \( "d" \) satisfying the same properties as in section 3.2 (Linearity eq(17), nilpotency and Leibniz rule eq(18)). The associative differential algebra
\( \Omega_{x,\theta}^{p,q} = \{ x, \theta, dx, d\theta \} \) is generated by the following relations:

\[
\begin{align*}
    x \, dx &= (qp)^{-1} \, dx \\
    \theta \, d\theta &= d\theta \, \theta \\
    dx \, d\theta &= p^{-1} \, d\theta \, dx
\end{align*}
\]

\( x \, d\theta = p^{-1} \, d\theta \, x \)

\( \theta \, dx = \frac{q (1 - (qp)^{-1})}{1 + qp} \, d\theta \, x - q^{-1} \, dx \, \theta \) \hspace{1cm} (24)

\( (dx)^2 = 0. \)

We have used \( q' = q \) and \( k = \frac{q}{p} \), which, as in eq(9), are consequences of the requirement of the covariance of \( \Omega_{x,\theta}^{p,q} \) under \( GL_{p,q}(1/1) \).

As expected, in the limit \( p \to q \), we recover \( \Omega_{x,\theta}^q \) and relations (19).

4.2 Differential calculus with nilpotency \( n = 3 \) \((d^3 = 0)\).

The technique used in sections (2.3) and (3.3), allows us to construct the \( n = 3 \) differential algebra \( \tilde{\Omega}_{x,\theta}^{p,q} = \{ x, \theta, dx, d\theta, d^2 x, d^2 \theta \} \):

\[
\begin{align*}
    x \, dx &= j^2 \, dx \, x \\
    \theta \, d\theta &= d\theta \, \theta \\
    dx \, d\theta &= -q \, d\theta \, dx
\end{align*}
\]

\( x \, d\theta = -\frac{jq}{1 + qp} \, d\theta \, x + \frac{j^2 qp - 1}{1 + qp} \, dx \, \theta \)

\( \theta \, dx = \frac{qp - j^2}{1 + qp} \, d\theta \, x + \frac{jp}{1 + qp} \, dx \, \theta \)

\( (d\theta)^2 = 0 \)

\( x \, d^2 x = j^2 \, d^2 x \, x \\
\theta \, d^2 \theta = -d^2 \theta \, \theta \\
dx \, d^2 x = j \, d^2 x \, dx \)

\( x \, d^2 \theta = -\frac{jq}{1 + qp} \, d^2 \theta \, x + \frac{j^2 qp - 1}{1 + qp} \, d^2 x \, \theta \) \hspace{1cm} (25)

\( \theta \, d^2 x = \frac{j^2 - qp}{1 + qp} \, d^2 \theta \, x - \frac{jp}{1 + qp} \, d^2 x \, \theta \)

\( dx \, d^2 \theta = \frac{q}{1 + qp} \, d^2 \theta \, dx + \frac{jqp - j^2}{1 + qp} \, d^2 x \, d\theta \)

\( d\theta \, d^2 \theta = j^2 \, d^2 \theta \, d\theta \\
dx \, d^2 \theta = \frac{q}{1 + qp} \, d^2 \theta \, dx - \frac{p}{1 + qp} \, d^2 x \, d\theta \)

\( d^2 x \, d^2 \theta = q \, d^2 \theta \, d^2 x \)

\((d^2 \theta)^2 = 0.\)

The same limit as in section 2.3, namely \( p \to q \), yields \( \tilde{\Omega}_{x,\theta}^q \). The differential algebra \( \tilde{\Omega}_{x,\theta}^{p,q} \) is not associative. In order to restore this property we proceed as mentioned at the end of sections 2.3 and 3.2.

One physical application of the differential calculi (sections 3 and 4) is to construct a supersymmetric gauge field theory on the quantum superplane (with one or two parameter quantum supergroup as symmetry groups; the latter will be a generalization of the former). However, this is not straightforward, since one should firstly start by defining a supersymmetric covariant derivative.
5 Conclusion:

In this paper, we have constructed differential calculi on certain quantum (super) spaces. Namely, the $n = 2$ and $n = 3$ nilpotent differential calculi on the quantum plane with two parametric quantum group ($GL_{p,q}(2)$) as a symmetry group was obtained. We have also considered two cases of quantum superplanes related to the one and two-parametric quantum supergroups $GL_q(1|1)$ and $GL_{p,q}(1|1)$, as symmetry groups, respectively. The related $n = 2$ and $n = 3$ differential calculi were also established.

In general, the differential calculus can be applied to formulate gauge field theories [30, 31, 32]. As a consequence, the results obtained here permit us to construct gauge theories on the corresponding non-commutative spaces [29]. Indeed, for the quantum space (section 2), this can be done using the same techniques of [20], where the symmetry group is a one-parameter.

The non-commutative supersymmetric case (sections 3 and 4) will be treated in the same fashion, with however, more care since it is essential first, to define a covariant supersymmetric derivative [33, 34].

We note that the differential calculus was also applied to derive a corresponding quantum oscillator, where the latter is seen as a representation of the former [35]. It will be interesting to achieve this with the differential calculus in section 2, as the resulting quantum oscillator will be two-parameter dependent.

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