ATTRACTION AND STABILITY OF NONLINEAR ODE'S USING CONTINUOUS PIECEWISE LINEAR APPROXIMATIONS

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Abstract. In this paper, several results concerning attraction and asymptotic stability in the large of nonlinear ordinary differential equations are presented. The main result is very simple to apply yielding a sufficient condition under which the equilibrium point (assuming a unique equilibrium) is attractive and also provides a variety of options among them the classical linearization and other existing results are special cases of the this main theorem in this paper including and extension of the well known Markus-Yamabe conjecture.

Several application examples are presented in order to analyze the advantages and drawbacks of the proposed result and to compare such results with successful existing techniques for analysis available in the literature nowadays.

1. Introduction

Stability of nonlinear continuous system of Ordinary Differential Equations (ODE system) is a very important subject either in control design or pure theoretical analysis. It is always desirable to have systematic, necessary and sufficient conditions able to predict any class of stability where Lyapunov stability is the most common definition.

Among the available techniques nowadays, we can count mainly on Lyapunov functions as the most general method whereas many particular cases including all the available analysis for linear systems and local analysis also received considerable attention, see [1] for a general reference in Lyapunov classical theory and [37], [29] for successful applications. The well known Lyapunov theorem suffers the lack of a systematic counterpart making it impossible to determine if such a Lyapunov function even exists for general cases. For instance, the example by Krasowski in [3], pp. 32, [24] and [2] shows that sometimes such a continuous Lyapunov functions does not exists.

Several tools to construct Lyapunov functions were derived in the literature, for instance, the special case in [33] provides a methodology for polynomial systems with constraints, on the other hand, the paper in [18] was the first constructive converse Lyapunov method, in fact it was proved there that if an equilibrium point possess exponential stability behavior, then a certain feasible linear programming problem yields a Lyapunov function for such a system. Later, the work in [19] improved the result proving a similar theorem for asymptotic stability instead of exponential one. Moreover, the important reference [35] shows that the analysis of Lyapunov functions using polynomial approximations is topic of interest nowadays.

Key words and phrases. Nonlinear systems, Stability, Attraction, Continuous Piecewise Linear.

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Alternative techniques to determine if a given system is stable (asymptotic or attractive) are the method conjectured by Aizerman (see [38] and [3], pp. 86, Theorem 2) and the result in [30] where \( x' \cdot f(x) \) is required to be concave. Also the Corollary 2.43 in the book [8] is an alternative to the Lyapunov’s method, in fact this result only requires to check the stability of the jacobian of a given nonlinear autonomous system at the origin (equilibrium point), the theorem also proves that if there exists a negative upper bound for all the real part of the eigenvalues of the Jacobian matrix, then there exists a region surrounding such an upper bound where the system presents exponential behavior.\(^1\)

Particular cases of ODE’s can be successfully treated specially for low dimensions; namely \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) or even discontinuous systems with linear components which can be classified under the names of Hybrid Linear, Piecewise Linear, Switching Linear, etc. For these cases, several techniques have been developed: such as, Common Quadratic Lyapunov Function, Multiple Lyapunov Functions, Poincaré Maps, Lie Algebra among many others (see for instance [21], [22], [10], [27], [26] y [28]).

Besides the interest in stability or attraction, and as it is well known, it is neither simple nor systematic to determine a region where any initial condition drives the system to an stable equilibrium point (domain of attraction). The most general analytic framework to find the domain of attraction requires to solve a Hamilton-Jacobi type partial differential equation (see [3]), in this regard the paper in [36] provides such a region by defining Lyapunov (like functions which allows to prove that the trajectories never leaves the region), whereas the classical Lyapunov functions only prove the existence of such a region without providing it. Finally notice that according the section Related Work in that paper, only two methodologies are systematic nowadays: [33] and [34].

In particular the method presented in [6] improves the early algorithm in [36] removing some computational drawbacks, however, the algorithm starts with a given Lyapunov function, otherwise, if such a function is not provided then a linearization at the origin along with a quadratic form is used to run the method, it turns out that for certain cases (called critical cases) the linearization exhibits zero or pure complex conjugate eigenvalues making the construction of the quadratic Lyapunov function more difficult with the necessity of some extra machinery [13].

The present paper presents two type of sufficient conditions, namely, a sufficient condition for asymptotic stability in nonlinear autonomous ODE’s which makes uses of a constant negative definite matrix to produce two outcomes:

- A quadratic Lyapunov function for the nonlinear ODE being analyzed.
- A subset of the domain of attraction.

The other sufficient condition is a corollary to the previous main result, in fact instead of using a constant stable matrix, it uses the jacobian of the nonlinear vector field as a function of the state-space variables, in this case the only outcome is a subset of the domain of attraction.

The results in this paper are built on the basis of the error bounds derived in [16] and [14], where given a Nonlinear autonomous ODE, then an approximation of that vector field using a Continuous Piecewise Linear (CPWL) basis is considered providing dynamic bounds for an error defined as the difference between the trajectories of the Nonlinear ODE and its CPWL approximation.

\(^1\)It turns out that the drawback is to determine such a region.
This dynamic bounds (error bounds) posses the same dynamic as the CPWL approximation ODE, in this way, stability issues for the CPWL ODE are also shared by the error bounds. Realize that once the stability for the CPWL ODE is proved, then stability for the dynamic bounds is also ensured then the trajectories of the Nonlinear ODE tends to a constant or in other words also shares stability issues.

It is worth to mention the works [12] and [11], where interval analysis it is used in order to derive a method, both for asymptotic stability and the determination of some subset of the attraction domain. In fact, in [11], an algorithm is provided to check asymptotic stability for a nonlinear ODE using successive linearization around the equilibrium at the origin. However, the stability proof is based on a quadratic Lyapunov function using the linearization aforementioned, which is not ensuring the algorithm ends in finite time for general nonlinear cases.

The work [9] is very interesting and a new result, in fact, the authors of that paper derive necessary and sufficient conditions under which a given nonlinear ODE is asymptotically stable. In this way, the results in the present paper can be compared to the results in [9] to notice that the present paper posses several results more general and also including the ones in that paper, however the tools in [9] can be utilized together with the results in the present paper, then allowing a more extensive analysis for nonlinear systems.

This paper is organized as follows: Section 2 introduces formally the problem considered; Section 2.1 proves a sufficient stability condition for CPWL ODE’s while Section 3 provides a sufficient condition for stability or attraction in a nonlinear ODE using CPWL theory. Finally, Section 4 shows some examples of application.

2. The Problem Considered

The goal of this paper is to derive sufficient conditions for attraction and asymptotic stability as well as quadratic Lyapunov functions for nonlinear ODE’s: \( \dot{x}(t) = f(x) \). The tool for such an achievement is an extension of the error bounds in [16] with the formalism in [14] and [15].

Thus, according to [16], if a given ODE \( \dot{x}(t) = f(x) \) produces a CPWL approximation, such that this approximation posses an asymptotically stable equilibrium point, then the error bounds in the mentioned reference, also posses an asymptotically stable equilibrium point, showing that the properties of stability are shared by the given ODE and its approximation.

A correction term in the development in [16] is added in this paper to take into account the possibility for the trajectories of \( \dot{x}(t) = f(x) \) and its CPWL approximation to run different simplices during some interval of time.

In this way, the scenario is as follows:

\[
\begin{align*}
\dot{x}(t) &= f(x) \quad \text{Given ODE} \\
\dot{x}_{\text{CPWL}}(t) &= A^{(k)} \cdot x_{\text{CPWL}} + B^{(k)} \quad \text{CPWL Approx.} \\
x, x_{\text{CPWL}} &\in \mathbb{R}^n
\end{align*}
\]

Recalling that the goal of the paper is to derive stability conditions of nonlinear ODE’s using its CPWL approximation in the limiting caso of the grid size tending to zero, this is only possible if the trajectories of both: nonlinear and CPWL are

\(^2\text{See }[23]\text{ for a definition of grid size.}\)
close each other. To make precise what close each other means, the concept of error bounds used in [16] and [15] will be utilized.

In this way, the formalism in [15] yields the following error bounds:

\[
\begin{align*}
\dot{x}(t) &= f(x) \quad \text{Given ODE} \\
\dot{x}_{\text{CPWL}}(t) &= A^{(k)} \cdot x_{\text{CPWL}} + B^{(k)} \quad \text{CPWL Approx.} \\
\min_{1,2} \{E^{*1}(t), E^{*2}(t)\} &\leq E^{(k)}(t) \leq \max_{1,2} \{E^{*1}(t), E^{*2}(t)\} \\
\dot{E}^{*1}(t) &= A^{(i)} \cdot E^{*1} + \xi^{ik}(t) - \lambda \\
\dot{E}^{*2}(t) &= A^{(i)} \cdot E^{*2} + \xi^{ik}(t) + \lambda \\
\end{align*}
\]

where \(\{E^{*1}, E^{*2}\} \in \mathbb{R}^{n \times 1}\), \(\lambda \in \mathbb{R}^{n \times 1}\), \(\xi^{ik}(t) = (A^{(k)} - A^{(i)}) \cdot x_{\text{CPWL}} + (B^{(k)} - B^{(i)})\) and \(E^{(k)}(t) = x(t) - x_{\text{CPWL}}(t)\) is the error vector.

Notice that extra term \(\xi^{ik}(t)\) takes into account for the possibility of the trajectories \(x(t)\) running the simplex \(i^{th}\) and \(x_{\text{CPWL}}(t)\) running the simplex \(k^{th}\).

On the other hand, to reach the goal in this paper, it is necessary to make sure that bounds in (2.1): \(\min_{1,2} \{E^{*1}(t), E^{*2}(t)\}\) and \(\max_{1,2} \{E^{*1}(t), E^{*2}(t)\}\) do not grows to infinity. Clearly, if the dynamics of such a bounds: \(\dot{E}^{*1,2}(t) = A^{(i)} \cdot E^{*1,2} + \xi^{ik}(t) \pm \lambda\) are attractive to one equilibrium point, then in each simplex the stability issues of the CPWL ODE is shared by the nonlinear one.

Since the error bounds in (2.1) can be rewritten in matrix block form:

\[
\begin{bmatrix}
\dot{E}^{*1,2}(t) \\
\dot{\xi}(t)
\end{bmatrix} = \begin{bmatrix} A^{(i)} & A^{(k)} - A^{(i)} \\
0 & A^{(k)} \end{bmatrix} \cdot \begin{bmatrix} X^{(k)} \\
A^{(k)} \end{bmatrix} + \begin{bmatrix} B^{(k)} - B^{(i)} \\
0 \end{bmatrix} + \begin{bmatrix} \pm \lambda \\
0 \end{bmatrix}
\]

The theorem in next subsection asserts that a CPWL ODE is attractive to one of its equilibrium points once all the matrices \(\bar{A}^{(k)}\) are definite negative. In this context, (2.2) will conduct to the desired goal in this paper in the view of the well known Theorem for block matrices: the eigenvalues of matrix \(\bar{A}^{(k)}\) is the junction of the eigenvalues of matrices \(\{A^{(k)}, A^{(i)}\}\).

2.1. A Sufficient Condition for Asymptotic stability in CPWL ODE’s.

In this section a theorem concerning attraction for CPWL ODE’s is recalled. In fact, the theorem was proved in [14] and it is important because it provides a tool which will be invariant when the grid size of the CPWL approximation simplicial partition tend to zero (as it will be shown in section 3).

**Theorem 1.** Given a CPWL ODE: \(\dot{x} = A^{(i)} \cdot x^{(i)} + B^{(i)}\) defined in an set \(\Omega\) which posses no equilibrium points over the frontiers. If, moreover, for each simplex \(i^{th}\), \(A^{(i)}\) is negative definite and \(\Omega\) is an invariant set, then the CPWL ODE is attractive to at least one of its equilibrium points with the initial conditions: \(x(0) \in \Omega\).
Whit this Theorem and considering the grid size of a CPWL ODE approximating a given nonlinear ODE, tending to zero (the CPWL approximation tend to the nonlinear vector field, \[23\]), it is going to be proved a theorem providing sufficient conditions for attraction and stability in a systematic way.

### 3. The Main Result: A Sufficient Condition for Stability of NonLinear ODE’s

Up to now, conditions for a CPWL ODE being asymptotically stable were derived, in particular when these ODE’s come from a CPWL approximation of a nonlinear vector field, the quality of the approximation improves as the grid size tend to zero (see \[23\]).

In order to apply this results to predict stability/attraction in Nonlinear ODE’s, two main research streams can be followed:

- **Given an ODE:** \( x(t) = f(x) \) and a CPWL approximation of it, analyzing the stability of each matrix \( A^{(i)} \) (this is the research line presented in \[22\]).

- **Given an ODE:** \( x(t) = f(x) \), utilizing the CPWL theoretical tools (like the ones in \[23\]), to derive conditions under which there exists a CPWL approximation (without explicit calculation of it), such that each matrix \( A^{(i)} \) is Hurwitz, then ensuring stability or attraction in the view of Theorem \[1\].

Following the second research line above, it is clear that the sufficient conditions in Theorem \[1\] require the following:

- To derive conditions using \( f(x) \) such that there exists a CPWL approximation for certain degree of accuracy where each matrix \( A^{(i)} \) is definite negative.

- To find an invariant set \( \Omega \) for \( \dot{x}(t) = f(x) \)

In order to provide a smooth introduction for the main result in this paper it is needed some extra machinery. In this way, the first condition above, requires the following Lemma:

**Lemma 2** (Stability Conditions for Linear Systems). Given a Linear ODE \( \dot{x}(t) = A \cdot x + B \) such that:

\[
\begin{align*}
| -F_{jk} + A_{jk} + B_j | &\leq \lambda^* \\
| B_j | &\leq \lambda^* \\
j, k = 1, 2, \ldots, n \\
F' \cdot P + P \cdot F &\leq -I \\
R &\leq \left[ P | \cdot (\lambda^* + \lambda^*) + (\lambda^* + \lambda^*)' \cdot | P | \right] \\
\lambda_R &< 1 \quad \text{The biggest eigenvalue of } R \\
| \cdot | &\text{ means the absolute value of each element}
\end{align*}
\]

Then, \( A \) is negative definite—see \[4\] for details on negative definiteness.

**Proof.** The proof is in the Appendix. \(\square\)

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5From now on stability will mean stability in the sense of Lyapunov (see \[32\], pp. 76-82 for definitions of Lyapunov stability).
It is important to remember that a positive definite matrix $P$ and symmetric such that $F' \cdot P + P \cdot F = -I$, always exists if and only if $F$ is Hurwitz (see \cite{25}, pp. 22).

On the other hand, an important corollary which will be useful in what follows is the fact that $V(x) = x' \cdot P \cdot x$ constitutes a Lyapunov function for $\dot{x}(t) = A \cdot x + B$:

**Corollary 3.** Under the hypothesis of Lemma 2, $V(x) = x' \cdot P \cdot x$ is a Lyapunov function for $\dot{x}(t) = A \cdot x + B$.

**Proof.** The proof is in the Appendix. \hfill $\Box$

Using these results for pure Linear systems and taking into account that all the development in this paper is supported by CPWL approximations, which are in essence a collection of linear systems, it is possible to prove the main result:

**Theorem 4** (Main Result). Let $\dot{x}(t) = f(x), \quad f(0) = 0, \quad f \in \mathbb{R}^n$ a Nonlinear ODE such that $f(x)$ is Lipschitz continuous in a set $\Omega$. If given three matrices $\{F, \lambda^*, \tilde{\lambda}^*\}$:

$$
\begin{align*}
F & = [F_1 \quad F_2 \quad \ldots \quad F_n] \\
\lambda^* & = [\lambda_1^* \quad \lambda_2^* \quad \ldots \quad \lambda_n^*] \\
\tilde{\lambda}^* & = [\tilde{\lambda}_1^* \quad \tilde{\lambda}_2^* \quad \ldots \quad \tilde{\lambda}_n^*] \\
F' \cdot P + P \cdot F & = -I \\
R & = \left[ P \mid (\lambda^* + \tilde{\lambda}^*) + (\lambda^* + \tilde{\lambda}^*)' \cdot | P | \right] \\
\lambda_R & < 1 \quad \text{The biggest eigenvalue of } R \\
| \cdot | & \text{ means the absolute value of each element} \\
\forall x \in \Omega & \quad \exists i \in [1, n] \quad | \frac{\partial f(x)}{\partial x} \cdot x | \leq \min_{i=1, \ldots, n} \tilde{\lambda}_i^*, \quad | \frac{\partial f(x)}{\partial x} - F | \leq (\lambda^* - \tilde{\lambda}^*)
\end{align*}
$$

where $\lambda_j$ is the maximum error between $f(x)$ and the CPWL approximation in the spirit of \cite{23} and $\hat{e}_i = [0 \quad 0 \quad \ldots \quad 1 \quad 0 \quad \ldots \hat{0}]^{i\text{th position}}$.

Then, $\dot{x}(t) = f(x)$ is attractive to the origin in $\Omega$. Moreover, $\Omega$ is a domain of attraction.

**Proof.** The proof is in the Appendix. \hfill $\Box$

This Theorem provides sufficient conditions for attraction in a region $\Omega$, so in order to extend this result to asymptotic stability, it will necessary the following Lemma:

**Lemma 5** (Lyapunov Functions). Given an ODE: $\dot{x}(t) = f(x), \quad f \in \mathbb{R}^n, \ f \text{ Lipschitz continuous with } f(0) = 0$. If given three matrices $\{F, \lambda^*, \tilde{\lambda}^*\}$:
\[
\begin{aligned}
F &= [F_1 \ F_2 \ \ldots \ F_n] \\
\lambda^* &= [\lambda_1^* \ \lambda_2^* \ \ldots \ \lambda_n^*] \\
\bar{\lambda}^* &= [\bar{\lambda}_1^* \ \bar{\lambda}_2^* \ \ldots \ \bar{\lambda}_n^*] \\
F' \cdot P + P \cdot F &= -I \\
R &= \left[ |P| \cdot (\bar{\lambda}^* + \lambda^*) + (\bar{\lambda}^* + \lambda^*)' \cdot |P| \right] \\
\lambda_R < 1 & \text{ The biggest eigenvalue of } R \\
\frac{\partial P}{\partial x} &= 0
\end{aligned}
\]

then \(V(x) = x' \cdot P \cdot x\) is a Lyapunov function for \(f(x)\) for all \(x \in \Omega\), with \(\Omega\) as follows:

\[
\Omega = \{x : |\frac{\partial f(x)}{\partial x} \cdot x| \leq \min_{i=1,\ldots,n} \bar{\lambda}_i^*, \quad |\frac{\partial f(x)}{\partial x} - F| \leq (\bar{\lambda}^* - \lambda^*)\}
\]

Proof. The proof is in the Appendix. \(\square\)

This Lemma defines a set \(\Omega\) where a Lyapunov function \(V(x) = x' \cdot P \cdot x\) is available. However, as it is well known, the set \(\Omega\) could be a subset of the truly domain of attraction. In this way, to find the complete domain of attraction, the optimization algorithm in [6] could be used, with the Lyapunov function \(V(x) = x' \cdot P \cdot x\).

Clearly, Theorem 4 and Lemma 5 prove trivially the following result:

**Theorem 6** (Sufficient Conditions for Asymptotic Stability). Under the conditions on Lemma 5, \(\dot{x}(t) = f(x)\) is asymptotically stable in the set:

\[
\Omega = \{x : |\frac{\partial f(x)}{\partial x} \cdot x| \leq \min_{i=1,\ldots,n} \bar{\lambda}_i^*, \quad |\frac{\partial f(x)}{\partial x} - F| \leq (\bar{\lambda}^* - \lambda^*)\}
\]

In this way, the first possible choice for \(F\) which always yields a non-empty region \(\Omega\) containing the equilibrium at the origin, is \(F = \frac{\partial f(x)}{\partial x}|_{x=0}\). This is not more than the classic linearization at the origin but this time, with a known validity region.

A second choice for \(F\) is: \(F = \frac{\partial f(x)}{\partial x}\), providing sufficient and systematic conditions under which \(\dot{x}(t) = f(x)\) is attractive to the origin:

**Corollary 7** (Sufficient Conditions for Attraction). Let \(x(t) = f(x), \quad f(0) = 0, \quad f \in \mathbb{R}^n\) a Nonlinear ODE such that \(f(x)\) is Lipschitz continuous in a region \(\Omega\) given by:

\[
\Omega = \{x : \lambda_R(x) < 1\}
\]

where \(R = 2(|P| \cdot |\frac{\partial f(x)}{\partial x}| \cdot x| \cdot [1 \ 1 \ldots 1] + H(x)' + H(x))\) and \(\lambda_R\) is the biggest eigenvalue of \(R\) providing \(P \cdot \frac{\partial f(x)}{\partial x} + \frac{\partial f(x)}{\partial x}' \cdot P = -I\). Then, \(x(t) = f(x)\) is attractive to the origin for all \(x \in \Omega\).

On the other hand is interesting to compare our test of stability in Theorem 4 with Theorem 7.1 in [22], pp. 115-116, where the following result is presented:
Theorem 8. Let \( z(t) \) be a piecewise \( C^1 \) trajectory of the system \( \dot{x}(t) = f(x) \) and assume that:

\[
\|f(x) - A^{(i)} \cdot x - B^{(i)} \| \leq \varepsilon^i \cdot \|x\|^2
\]

If there exists numbers \( y_i > 0 \) and symmetric matrices \( U, W \) with non-negative entries, and a symmetric matrix \( T \) such that \( P_i = F^{(i)} \cdot T \cdot F^{(i)} \) and \( \bar{P}_i = F^{(i)}' \cdot T \cdot F^{(i)} \) \( E \) \( TT \) \( F \) satisfy:

\[
\begin{cases}
-2 \cdot \varepsilon^i \cdot \gamma^{(i)} \cdot I > A^{(i)} \cdot P^{(i)} + P^{(i)} \cdot A^{(i)} + E^{(i)} \cdot U^{(i)} \cdot E^{(i)} \\
E^{(i)}' \cdot W^{(i)} \cdot E^{(i)} < P^{(i)} < \gamma^{(i)} \cdot I
\end{cases}
\]

for \( i \) in the first simplex and:

\[
\begin{cases}
-2 \cdot \varepsilon^{(i)} \cdot \gamma^i \cdot I > \bar{T}^{(i)} \cdot \bar{P}^{(i)} + \bar{P}^{(i)} \cdot \bar{T}^{(i)} + \bar{E}^{(i)} \cdot U^{(i)} \cdot \bar{E}^{(i)} \\
\bar{E}^{(i)}' \cdot \bar{W}^{(i)} \cdot \bar{E}^{(i)} < P^{(i)} < \gamma^{(i)} \cdot I
\end{cases}
\]

for \( i \) in the next simplex, then \( x(t) \) tends to zero exponentially.

Notice that in order to use this theorem and as is pointed out in [22], we need to run a CPWL approximation numerically and then extracting from that the matrices \( A^{(i)}, B^{(i)} \) to conclude stability for our given nonlinear ODE, the point is how to ensure a correct simplicial division to get precise conclusions. In that sense our result is much more stronger since we only need the evaluation of the matrix \( F \) and the Lipschitz coefficients if an estimation of the domain of attraction has to be determined, without the necessity of matrices \( A^{(i)}, B^{(i)} \).

We leave this section with an example of application of Corollary 7 and Theorem 8 utilizing the Jacobian at the origin.

Example

The well known system of Hopf in normal form is:

\[
\begin{align*}
\dot{x} &= f_1(x, y) = \alpha \cdot x - y + x \cdot (x^2 + y^2) \\
\dot{y} &= f_2(x, y) = x + \alpha \cdot y + y \cdot (x^2 + y^2)
\end{align*}
\]

Calculating the Jacobian matrix:

\[
\frac{\partial f(x)}{\partial x} = \begin{bmatrix}
\alpha + 3 \cdot x^2 + y^2 & -1 + 2 \cdot x \cdot y \\
1 + 2 \cdot x \cdot y & \alpha + x^2 + y^2
\end{bmatrix}
\]

then, the Jacobian at the origin leads:

\[
\frac{\partial f(x)}{\partial x} = \begin{bmatrix}
\alpha & -1 \\
1 & \alpha
\end{bmatrix}
\]

The eigenvalues are \( \alpha \pm \sqrt{(-1)} \), which indicates the necessity for \( \alpha < 0 \). In this way, the conditions on Theorem 8 yields:

\[
\begin{align*}
|\frac{\partial f(x)}{\partial x} - F| & \leq \lambda^*- \lambda^* \\
|\frac{\partial f(x)}{\partial x} \cdot x| & \leq \min_{i=1,2} \tilde{\lambda}^*_i
\end{align*}
\]

\[\text{More examples are given later in Section 4}\]
On the other hand, if $\lambda^*, \overline{\lambda}^*$ are such that:

$$\begin{cases}
\lambda^* = [\overline{\lambda}^* \ \lambda^*] \\
\overline{\lambda}^* = [\lambda^* \ \overline{\lambda}^*]
\end{cases}$$

where $\overline{\lambda}^*_i \in \mathbb{R}^{2 \times 1}$ and $\lambda^*_i \in \mathbb{R}^{2 \times 1}$. Then, $R$ is a matrix with rank 1, so the condition $\lambda_R < 1$, yields:

$$\begin{cases}
\lambda_R = \text{trace}(R) < 1 \\
R = \frac{1}{2} \cdot ((\overline{\lambda}^* + \lambda^*) + (\overline{\lambda}^* + \lambda^*)')
\end{cases}$$

where the operator $\text{trace}(-)$ is the sum of the diagonal elements. The condition for $\lambda_R < 1$ leads:

$$\frac{1}{2} \cdot (\overline{\lambda}^*_i + \lambda^*_i + \overline{\lambda}^*_i + \lambda^*_i) < 1$$

Finally, the set of conditions to satisfy are from equations (3.1) and (3.2):

$$\begin{cases}
|3 \cdot x^2 + y^2| \leq \overline{\lambda}^*_i - \lambda^*_i \\
|x^2 + 3 \cdot y^2| \leq \lambda^*_i - \overline{\lambda}^*_i \\
|2 \cdot x \cdot y| \leq \overline{\lambda}^*_i - \lambda^*_i \\
|\alpha \cdot x - y| + 3 \cdot x \cdot (x^2 + y^2) | \leq \min_{i=1,2} \overline{\lambda}^*_i \\
|\alpha \cdot y + x| + 3 \cdot y \cdot (x^2 + y^2) | \leq \min_{i=1,2} \lambda^*_i \\
-\frac{1}{\alpha} \cdot (\overline{\lambda}^*_i + \lambda^*_i + \overline{\lambda}^*_i + \lambda^*_i) < 1
\end{cases}$$

Noticing that the condition on $\lambda_R$ is limiting the values for $\overline{\lambda}^*_i, \lambda^*_i$, and $\overline{\lambda}^*_i, \lambda^*_i$ are not present there, then it is possible to choose any value for them. If in particular they are such that:

$$\begin{cases}
\overline{\lambda}^*_i \rightarrow +\infty \\
\lambda^*_i \rightarrow +\infty
\end{cases}$$

In this way, conditions in equation (3.3), lead:

$$\begin{cases}
|3 \cdot x^2 + y^2| \leq \overline{\lambda}^*_i - \lambda^*_i \\
|x^2 + 3 \cdot y^2| \leq \lambda^*_i - \overline{\lambda}^*_i \\
|\alpha \cdot x - y| + 3 \cdot x \cdot (x^2 + y^2) | \leq \overline{\lambda}^*_i \\
|\alpha \cdot y + x| + 3 \cdot y \cdot (x^2 + y^2) | \leq \lambda^*_i \\
-\frac{1}{\alpha} \cdot (\overline{\lambda}^*_i + \lambda^*_i + \overline{\lambda}^*_i + \lambda^*_i) < 1
\end{cases}$$

One possibility is to choose $2 \cdot \overline{\lambda}^*_i = \lambda^*_i = -\frac{8}{3}, \ 2 \cdot \overline{\lambda}^*_i = \lambda^*_i = -\frac{8}{3}$, leading a domain of attraction $x \in [-0.08, 0.08], \ y \in [-0.08, 0.08]$.

Next section is providing a set of examples to show the applicability of both: Theorem 4 and Corollary 7, together with some discussion about the accuracy when the domain of attraction obtained has to be used as an estimation to the truly global one.
4. Examples

This section is devoted to show the applicability of the main theorem (Theorem 4), as well as the its corollary. They together allow, in some cases, a precise estimation of the domain of attraction while asymptotic stability is established.

Example 1 (An example with finite domain of attraction). This first example is taken from the book [3], pp. 53:

\[
\begin{cases}
  x_1(t) = -x_1 + 2 \cdot x_1^2 \cdot x_2 \\
  x_2(t) = -x_2
\end{cases}
\]

For this system is known that the domain of attraction is given by \( x_1 \cdot x_2 < 1 \), then this is a boundary for the estimation with Theorem 4. Calculating the Jacobian:

\[
\frac{\partial f(x)}{\partial x} = \begin{bmatrix}
-1 + 4 \cdot x_1 \cdot x_2 & 2 \cdot x_1^2 \\
0 & -1
\end{bmatrix}
\]

Then, calculating at the origin:

\[
\frac{\partial f(x)}{\partial x} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

This Jacobian is clearly Hurwitz with matrix \( P \) as follows:

\[
P = \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

On the other hand, the conditions on Theorem 4 leads:

\[
\begin{cases}
|\frac{\partial f(x)}{\partial x} - F| \leq \lambda^* - \tilde{\lambda}^* \Leftrightarrow \begin{vmatrix}
4 \cdot x_1 \cdot x_2 & 2 \cdot x_1^2 \\
0 & 0
\end{vmatrix} \leq \lambda^* - \tilde{\lambda}^* \\
|\frac{\partial f(x)}{\partial x} \cdot x| \leq \min_{i=1,2} \tilde{\lambda}_i^* \Leftrightarrow \begin{vmatrix}
-x_1 + 4 \cdot x_1^2 \cdot x_2 + 2 \cdot x_1^2 \cdot x_2 \\
-x_2
\end{vmatrix} \leq \min_{i=1,2} \tilde{\lambda}_i^*
\end{cases}
\]

Analyzing the condition for \( \lambda_R \):

\[
R = \frac{1}{2} \cdot [(\lambda^* + \tilde{\lambda}^*) + (\lambda^* + \tilde{\lambda}^*)']
\]

On the other hand, if \( \lambda^*, \tilde{\lambda}^* \) are such that:

\[
\begin{cases}
\lambda^* = [\lambda_1^* \quad \lambda_2^*] \\
\tilde{\lambda}^* = [\tilde{\lambda}_1^* \quad \tilde{\lambda}_2^*]
\end{cases}
\]

where \( \lambda_1^* \in \mathbb{R}^{2 \times 1} \) and \( \tilde{\lambda}_1^* \in \mathbb{R}^{2 \times 1} \). Then, \( R \) is a matrix with rank 1, so:

\[
\lambda_R = \text{trace}(R) < 1
\]

In this way, the second condition to satisfy arises:

\[
\lambda_R < 1 \Leftrightarrow \frac{1}{2} \cdot (\lambda_{11}^* + \lambda_{22}^* + \tilde{\lambda}_{11}^* + \tilde{\lambda}_{22}^*) < 1
\]
where $\bar{\lambda} = \begin{bmatrix} \bar{\lambda}_{11} & \bar{\lambda}_{12} \\ \bar{\lambda}_{21} & \bar{\lambda}_{22} \end{bmatrix}$ and $\tilde{\lambda} = \begin{bmatrix} \tilde{\lambda}_{11} & \tilde{\lambda}_{12} \\ \tilde{\lambda}_{21} & \tilde{\lambda}_{22} \end{bmatrix}$. In summary, from equations (4.2) and (4.3):

$$\begin{cases} 
4 \cdot |x_1 \cdot x_2| \leq \bar{\lambda}_{11} - \bar{\lambda}_{11} \\
2 \cdot x_1^2 \leq \bar{\lambda}_{12} - \bar{\lambda}_{12} \\
|x_1 + 4 \cdot x_1^2 \cdot x_2 + 2 \cdot x_1^2 \cdot x_2| \leq \min_{i=1,2} \bar{\lambda}_{ii} \\
|x_2| \leq \min_{i=1,2} \bar{\lambda}_{ii} \\
\frac{1}{2} \cdot (\bar{\lambda}_{11}^* + \bar{\lambda}_{22}^* + \bar{\lambda}_{11}^* + \bar{\lambda}_{22}^*) < 1 
\end{cases}$$

If in particular, it is chosen:

$$\begin{cases} 
\tilde{\lambda}_{12} \to +\infty \\
\tilde{\lambda}_{21} \to +\infty 
\end{cases}$$

Then, the conditions in equation (4.4), yields:

$$\begin{cases} 
4 \cdot |x_1 \cdot x_2| \leq \bar{\lambda}_{11} - \bar{\lambda}_{11} \\
|x_1| \cdot |-1 + 4 \cdot x_2 + 2 \cdot x_1 \cdot x_2| \leq \bar{\lambda}_{11} \\
x_2 \leq \bar{\lambda}_{22} \\
\frac{1}{2} \cdot (\bar{\lambda}_{11}^* + \bar{\lambda}_{22}^* + \bar{\lambda}_{11}^* + \bar{\lambda}_{22}^*) < 1 
\forall x_1 \in \mathbb{R} 
\end{cases}$$

Finally choosing $\bar{\lambda}_{11}^* - \bar{\lambda}_{11}^* = 1$ and $\bar{\lambda}_{11}^* = \frac{1}{4}$, $\bar{\lambda}_{22}^* = \frac{1}{4}$, this leads:

$$\begin{cases} 
|x_1| \cdot |-1 + 4 \cdot x_2 + 2 \cdot x_1 \cdot x_2| \leq \frac{1}{4} \\
x_2 \leq \frac{1}{4} 
\end{cases}$$

The plot for this region is depicted in Figure 1. On the other hand, using Corollary 7, the domain of attraction is presented in Figure 2

**Example 2 (The Van der Pol System).** The Van der Pol system is a very well known system which serves as a benchmark in many areas of science, in particular it is stable for negative values of the parameter $\mu$:

$$\begin{align*}
\dot{x}_1(t) &= x_2 \\
\dot{x}_2(t) &= -x_1 + \mu \cdot x_2 \cdot (1 - x_1^2)
\end{align*}$$
For this example, the systematic Corollary 7 will be considered. To apply this result, it is necessary to obtain the Jacobian:

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + 2\mu \cdot x_1 \cdot x_2 & \mu \cdot (1 - x_1^2) \end{bmatrix}$$

The eigenvalues are Hurwitz in the region depicted in Figure 3. It is clear that the stable region for the eigenvalues contains the equilibrium at the origin. In this way, the zone where the eigenvalue $\lambda_R$ in Corollary 7 is less than one is plotted in Figure 4 for $\mu = -1$.

Since this system is **stiff**, the obtained stability region is smaller than the whole domain of attraction depicted in Figure 5.

Example 3 (The Markus-Yamabe conjecture, discussion). One special class of ODE’s are those with Hurwitz Jacobian matrix for all $\mathbb{R}^n$. For these systems, the intuition

---

7 in the sense that possesses two main time-scales with different magnitude producing a sudden change into the direction motion of the system—this is one reason why it is used as a benchmark for numerical ODE solver testing.
could say they are always stable. However, as it is well known since 1995 this is not true for dimensions bigger than 2 in general cases (see [31] and [7]).

Taking a look of Corollary 7 is clear that the region for stability is the intersection of two main regions:

- The region where the eigenvalues of $\frac{\partial f(x)}{\partial x}$ are Hurwitz.
- The region where $\lambda_R < 1$.

A conclusion using the theory in this paper is that the Markus-Yamabe conjecture was uncomplete, missing the second condition above.

Example 4 (An example where the test is not applicable). This example is due to Krasovskii (see [3], pp. 32, [24] and [2]). This system has an equilibrium at the origin but the jacobian is unstable for $\Re x^2$:

$$\begin{align*}
  \dot{x}_1(t) &= x_2 \\
  \dot{x}_2(t) &= -x_1 + x_2 \cdot (x_1^2 + x_2^2)^3 \cdot \sin\left(\frac{\pi}{(x_1^2 + x_2^2)}\right)^2
\end{align*}$$

Since the jacobian is unstable, it is not possible to find an stable matrix $F$ as required by Theorem 4 otherwise, in the view of Lemma 2 the conclusion would be that the jacobian is stable leading a contradiction.

Example 5 (Domain of attraction discussion). Even when the focus on this paper is to derive an analytic tool to determine stability/attraction for Nonlinear ODE’s, this example intends to discuss how precise is our prediction of the truly domain of attraction, in fact, the Normal-Hopf example in Section 3 is very conservative but it is expected that the general scenario is of acceptable conservatism. In fact, it will be shown that for many systems the prediction exhibits high precision.

A class of systems with known domain of attraction are planar systems with circular unstable limit cycles. Those dynamics can be written in polar coordinates as follows:

$$\begin{align*}
  &\dot{r}(t) = h(r) \\
  &\dot{\theta}(t) = 1
\end{align*}$$

here $h(r)$ is negative for a finite interval on $r$ containing the origin ($r = 0$), this idea is depicted in Figure 5.

Notice that in order to have $f(x = 0) = 0$, it will be needed $h(0) = 0$. Moreover for a finite circular domain of attraction, $h(\mu) = 0$ with $\mu$ a finite positive value. In this way and calculating the Jacobian:
Figure 6. Shape of $h(r)$

\[
\frac{\partial f(x)}{\partial x} = \frac{1}{r} \left[ \begin{array}{c} r \cdot h(r) + \cos(\theta)^2 \cdot (h'(r) \cdot r - h(r)) \\ r \cdot \sin(\theta) \cdot \cos(\theta) \cdot (h'(r) \cdot r - h(r)) \\ -r - \sin(\theta) \cdot \cos(\theta) \cdot (h'(r) \cdot r - h(r)) \\ r \cdot h(r) + \sin(\theta)^2 \cdot (h'(r) \cdot r - h(r)) \end{array} \right]
\]

where $h'(r) = \frac{dh(r)}{dr}$. Since the border of the domain of attraction is circular, in order to analyze the condition in Corollary 7, $\lambda R < 1$, it can be settle:

\[
r = \mu, \quad \forall \theta \in \mathbb{R}
\]

In this way, the Jacobian for $\theta = 0$:

\[
\frac{\partial f(x)}{\partial x} = \begin{bmatrix} h'(\mu) & -1 \\ 1 & 0 \end{bmatrix}
\]

then:

\[
P = \begin{bmatrix} \frac{1}{h'(\mu)} & \frac{1}{h'(\mu)} \\ \frac{1}{h'(\mu)} & -\frac{2+h'(\mu)^2}{2h'(\mu)} \end{bmatrix}
\]

Moreover:

\[
R = 2 \cdot \mu \cdot (|P| \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot |P|)
\]

Finally:

\[
\lambda R = 2 \cdot \mu \cdot \left| \frac{h'(\mu) + 2 + h'(\mu)^2}{h'(\mu)} \right|
\]

Calculating $\lambda R < 1$ it can be seen that for the interval $\mu \in [0, 0.2321]$, the estimation is perfect. Realize that this result is valid for any function $h(r)$ making valid the result for a wide range of systems.

5. Conclusions

Sufficient conditions for asymptotic stability and attraction for nonlinear ODE’s were presented. These conditions are systematic to apply, moreover the main theorem provides a region where the ODE under analysis is asymptotically stable and a quadratic Lyapunov function. Realize that such a region serves as an estimation, sometimes coincident, of the domain of attraction and the Lyapunov function can extend this region by applying methods like the one in [6].

\[\text{The rest of the cases lead equivalent results since is a circular domain.}\]
On the other hand, while Theorem 4 needs the input of a stable matrix $F$, Corollary 7 utilizes the jacobian matrix of the given nonlinear ODE. The former result provides a subset of the domain of attraction but the estimation depends strongly on the choice of that matrix $F$. In this sense, the corollary was presented to overcome this inconvenient making the method systematic.

Several examples were presented for both Theorem 4 and Corollary 7 showing the applicability of such a results, moreover Example 5 analyzed systems where the estimation of the domain of attraction is exact.

Many future directions can be depicted, in fact the eigenvalue $\lambda_R(x)$ in Corollary 7 it seems to increase from $x = 0$ until it reaches the value 1 exactly when $x$ belong to the border of the subset of the domain of attraction obtained. If this is confirmed, then this quantity $\lambda_R$ serves to determine the borders of the estimation of the domain of attraction when is equal to one.

Besides, the theory supporting all the development in this paper is based on Theorem 1 which in essence is a sufficient condition for attraction in CPWL ODE’s. Notice that the theorem is proved utilizing a finite improper integral, the finiteness is proved by requiring that all the matrices in each simplex of the CPWL under analysis being stable. This requirement could be in principle a bit conservative and should be improved allowing some of the matrices being unstable, then enlarging the attraction region provided by $\lambda_R$.

It will a topic for future research also the application of the methodologies on this paper to stabilize control systems also known as non-autonomous. In the case of control systems, one has to derive a controller but also to ensure that the system is stable at least for a finite region, in this sense, sufficient conditions for stability are the perfect scenario in order to obtain such a controllers.

6. Appendix

We present here the proof of those lemmas and theorems along the paper.

**proof of Lemma 2.** Starting from the condition:

\[ | -F + A + B | \leq \bar{\lambda}^* \]

In this way and considering a matrix $P' = [P_1, P_2, \ldots, P_n]'$, $P_j = [P_{j1}, P_{j2}, \ldots, P_{jn}]$ and multiplying both sides of equation (6.1) by $| P_{kj} |$, $j, k = 1, \ldots, n$:

\[
\begin{cases}
| P_{kj} | \cdot | -F_{jl} + A_{jl} + \bar{B}_{jl} | \leq | P_{kj} | \cdot \bar{\lambda}_{jl}^* \\
{\ast} \\
\end{cases}
\]

In matrix form:

\[ | P' | \cdot | -F + A + B | \leq | P' | \cdot \bar{\lambda}^* \]

In a similar way is also true:

\[ | ( -F + A^{(i)} + \bar{B}^{(i)})' | \cdot | P | \leq \bar{\lambda}' \cdot | P | \]

Summing equations (6.2) and (6.3):

\[ | P' | \cdot | -F + A + B | + | -F + A + B' | \cdot | P | \leq \bar{\lambda}' \cdot | P | + | P' | \cdot \bar{\lambda}^* \]
Realizing that $|P \cdot (-F + A + B) + (-F + A + B)' \cdot |P| \leq |P| \cdot (-F + A + B)^\top$ and multiplying by $|x|$, $x \in \mathbb{R}^n$ at right and left:

$$
|x| \cdot (P \cdot (-F + A + B) + (-F + A + B)' \cdot |P|) \cdot |x| \leq |x| \cdot (\lambda^* \cdot |P| + |P| \cdot \lambda^*) \cdot |x|
$$

Taking into account that $x' \cdot (\cdot) \leq |x| \cdot (\cdot) \cdot |x|$, it is possible to write:

$$
\begin{cases}
  x' \cdot (P' \cdot (-F + A + B) + (-F + A + B)' \cdot P) \cdot |x| \\
  |x| \cdot (\lambda^* \cdot |P| + |P| \cdot \lambda^*) \cdot |x|
\end{cases}
$$

Finally resorting to the property $|a + b| \geq |a| - |b|$, $\forall a, b \in \mathbb{R}$ and summing $x' \cdot (F' \cdot P + P' \cdot F) \cdot x$:

$$(6.4) \begin{align*}
  x' \cdot (P' \cdot A + A \cdot P) \cdot x & < \\
  |x| \cdot (\lambda^* \cdot |P| + |P| \cdot \lambda^*) \cdot |x| + x' \cdot (F' \cdot P + P' \cdot F) \cdot x + x' \cdot (P' \cdot B + B \cdot P) \cdot |x|
\end{align*}$$

where the property $|a + b| \geq a + b, \forall a, b \in \mathbb{R}$ was invoked. Since $F$ is stable if and only if there exists $P = P' > 0$ such that $(P' \cdot F + F' \cdot P) = -I$, with $I$ the identity matrix (see [23], pp. 22), also considering the definition of positive definiteness (see [4]), in order to guarantee that the matrix $A$ is negative definite (or stable) it is enough to require:

$$(6.5) \begin{align*}
  |x| \cdot (\lambda^* \cdot |P| + |P| \cdot \lambda^*) \cdot |x| + x' \cdot (F' \cdot P + P' \cdot F) \cdot x + x' \cdot (P' \cdot B + B \cdot P) \cdot |x| & \leq 0
\end{align*}$$

Recalling that $F' \cdot P + P' \cdot F = -I$, then equation (6.5) looks like:

$$
\begin{cases}
  |x| \cdot (\lambda^* \cdot |P| + |P| \cdot \lambda^*) \cdot |x| + x' \cdot (P' \cdot B + B \cdot P) \cdot |x| & \leq 1 \\
  x' \cdot x & > 0
\end{cases}
$$

Utilizing the condition:

$$(6.6) \begin{align*}
  |B| & \leq \lambda^*
\end{align*}$$

The requirement for stability of $A$ will be:

$$(6.7) \begin{align*}
  |x| \cdot (\lambda^* + \lambda^*) \cdot |x| + x' \cdot (P' \cdot B + B \cdot P) \cdot |x| & \leq 1
\end{align*}$$

Since this inequality should be satisfied for all $x \in \mathbb{R}^n$ in virtue of the definition of positiveness, the worst case leads:

$$(6.8) \begin{align*}
  \max_x \frac{|x| \cdot R \cdot |x|}{|x| \cdot |x|} & \leq 1
\end{align*}$$

This is called Rayleigh’s quotient and have important properties (see [4]). To finish, realize that matrices $R$ and $|P|$ are positive (in the sense that all the elements are positive), so posses positive eigenvalues with positive corresponding eigenvector (see [4], pp. 72, Theorem 18).
In this way, the condition ensuring stability for $A$ leads:

$$\lambda_R < 1$$

This completes the proof.

proof of Corollary. Under the hypothesis of Lemma and recalling the proof of this Lemma, in particular equation leads:

$$x' \cdot (P \cdot A + A' \cdot P) \cdot x < 0$$

This completes the proof.

proof of Theorem. The proof in what follows starts with the CPWL approximation bounds introduced in \[23\]:

$$|f(x) - A^{(i)} \cdot x + B^{(i)}| \leq \lambda$$

where $A^{(i)} \cdot x + B^{(i)}$ are the linear approximations for $f(x)$ in every simplex. This means that $A^{(i)} \cdot x + B^{(i)}$ is bounded by two vector fields: $\lambda + f(x)$ and $-\lambda + f(x)$, Figure is depicting this idea. Notice that the dash region in that figure is the area containing the possible CPWL vector fields for a given $\lambda$.

Since the focus is to create conditions over $f(x)$ related to \{\(\tilde{\lambda}, \tilde{\lambda}^*\}\}, we can consider the "worst case" $A^{(-\lambda)} \cdot x + B^{(-\lambda)}$ and $A^{(\lambda)} \cdot x + B^{(\lambda)}$ and show that those planes define a convex set as follows (see Figure 8):

\begin{align*}
(6.9) \quad A^{(-\lambda)} \cdot x + B^{(-\lambda)} &\leq f(x) - \lambda \leq A^{(i)} \cdot x + B^{(i)} \leq f(x) + \lambda \leq A^{(\lambda)} \cdot x + B^{(\lambda)}
\end{align*}

The key idea it is to show that the inequalities above, are satisfied for at least one point inside each simplex and centering the attention in a generic simplex with vertices: $V_i \in \mathbb{R}^n$, $i = 1, \ldots, n + 1$. In this way, if the grid size tends to zero, it is possible to prove that: $A^{(\pm \lambda)} \rightarrow A^{(i)}$, $B^{(\pm \lambda)} \rightarrow B^{(i)}$.

Then, in place of using matrices \{\(A^{(i)}, B^{(i)}\}\}, one uses the pair \{\(A^{(\pm \lambda)}, B^{(\pm \lambda)}\}\}, which can be obtained for any generic nonlinear vector field $f(x)$ without the necessity of the evaluation of any CPWL approximation which moreover has to be

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9See [39], pp. 160-165
10$V_1 \rightarrow V_2$ and $V_i \rightarrow V_j$, $i \neq j$
Figure 8. "Worst case" planes.

reduced more and more in order to consider the limiting case when the grid size tend to zero.

Once the closed-form expressions for the pairs \( \{A(\pm \lambda), B(\pm \lambda)\} \) have been obtained, it is possible to analyze attraction or stability as it going to be shown in what follows. For the sake of clarity we will only consider the analysis for \( x \geq 0 \), the case \( x \leq 0 \) shares conclusions.

\[ \mathbb{R}^1 \text{ analysis:} \] In this way, let’s calculate first these worst case planes, then based on Figure 8:

\[
\begin{align*}
P_1 &= f(V_1) + \lambda, \quad P_2 = f(V_1) - \lambda, \quad P_3 = f(V_2) + \lambda, \quad P_4 = f(V_2) - \lambda \\
f(V_1) + \lambda &= A(\lambda) \cdot V_1 + B(\lambda) \\
f(V_2) + \lambda &= A(\lambda) \cdot V_2 + B(\lambda) \\
f(V_1) - \lambda &= A^{-}(\lambda) \cdot V_1 + B^{-}(\lambda) \\
f(V_2) - \lambda &= A^{-}(\lambda) \cdot V_2 + B^{-}(\lambda)
\end{align*}
\]

In this way, the matrices \( \{A(\pm \lambda), B(\pm \lambda)\} \) are given by:

\[
\begin{align*}
A^{(\lambda)} &= \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 + f(V_1) + \lambda \\
B^{(\lambda)} &= \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 + f(V_1) + \lambda \\
A^{-}(\lambda) &= \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 + f(V_1) - \lambda \\
B^{-}(\lambda) &= \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 + f(V_1) - \lambda
\end{align*}
\]

At this point, the following conditions need to be verified:

\[
\begin{align*}
A^{-}(\lambda) \cdot x + B^{-}(\lambda) &\leq A^{(\lambda)} \cdot x + B^{(\lambda)} \\
A^{-}(\lambda) \cdot x + B^{-}(\lambda) &\leq f(x) - \lambda \\
f(x) + \lambda &\leq A^{(\lambda)} \cdot x + B^{(\lambda)} \\
\forall x &\geq 0
\end{align*}
\]

Using the notation \( V_1 - V_2 = \delta_x < 0 \) without loss of generality, the conditions above yields:
in other words:  
\[
\begin{align*}
A^{(-\lambda)} \cdot x + B^{(-\lambda)} &\leq A^{(\lambda)} \cdot x + B^{(\lambda)} \iff 0 \leq 2 \cdot \lambda \\
A^{(-\lambda)} \cdot x + B^{(-\lambda)} &\leq f(x) - \lambda \iff f(x) - f(V_1) \leq A^{(\lambda)} \cdot (x - V_1) \\
f(x) + \lambda &\leq A^{(\lambda)} \cdot x + B^{(\lambda)} \iff f(x) - f(V_1) \geq A^{(-\lambda)} \cdot (x - V_1) \\
\forall x &\geq 0
\end{align*}
\]

Considering equation (6.10), it is clear that \( A^{(\lambda)} = A^{(-\lambda)} \), then:

\[
\begin{align*}
0 &\leq 2 \cdot \lambda \\
f(x) - f(V_1) &\leq A^{(\lambda)} \cdot (x - V_1) \\
f(x) - f(V_1) &\geq A^{(\lambda)} \cdot (x - V_1) \\
\forall x &\geq 0
\end{align*}
\]

First condition is trivially satisfied taking into account that a CPWL approximation, always work with \( \lambda \geq 0 \), however, second condition means:

\[
\begin{align*}
f(x) - f(V_1) = A^{(\lambda)} \cdot (x - V_1) &\iff \frac{f(x) - f(V_1)}{(x - V_1)} = A^{(\lambda)} \\
\forall x &\geq 0
\end{align*}
\]

Considering the limiting case \( V_1 \to V_2 \), then:

\[
\begin{align*}
\lim_{V_1 \to V_2} A^{(\lambda)} = \lim_{V_1 \to V_2} = \frac{\partial f(x)}{\partial x} |_{x=V_1} \\
\forall x &\geq 0
\end{align*}
\]

In the view of the mean value theorem, this condition is always satisfied if \( f(x) \) is continuous and with first derivative continuous - see [20], pp. 148.

Once the validity of inequality (6.9) has been proved for at least one point \( x \in [V_1, V_2] \), it is missing to prove the following:

- If \( V_1 \to V_2 \), then: \( A^{(\pm \lambda)} \to A^{(i)} \), \( B^{(\pm \lambda)} \to B^{(i)} \).
- The pair \( \{A^{(\pm \lambda)}, B^{(\pm \lambda)}\} \) satisfies: \( | -F + A^{(\pm \lambda)} + B^{(\pm \lambda)} | \leq \lambda^* \), \( | B^{(\pm \lambda)} | \leq \lambda^* \).

The first item is going to be analyzed using inequality (6.9) in the vertices \( \{V_1, V_2\} \) and recalling that \( A^{(\lambda)} = A^{(-\lambda)} \):

\[
\begin{align*}
\begin{bmatrix} V_1 \ 1 \\ V_2 \ 1 \end{bmatrix} \cdot \begin{bmatrix} A^{(\lambda)} \\ B^{(-\lambda)} \end{bmatrix} &\leq \begin{bmatrix} V_1 \ 1 \\ V_2 \ 1 \end{bmatrix} \cdot \begin{bmatrix} A^{(i)} \\ B^{(i)} \end{bmatrix} \leq \begin{bmatrix} V_1 \ 1 \\ V_2 \ 1 \end{bmatrix} \cdot \begin{bmatrix} A^{(\lambda)} \\ B^{(\lambda)} \end{bmatrix}
\end{align*}
\]

using absolute values:

\[
\begin{align*}
T \cdot \left| \begin{bmatrix} A^{(\lambda)} - A^{(i)} \\ h - B^{(i)} \end{bmatrix} \right| &\leq T \cdot \left| \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \right| \iff \left| \begin{bmatrix} A^{(\lambda)} - A^{(i)} \\ h - B^{(i)} \end{bmatrix} \right| \leq |T|^{-1} \cdot \left| \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \right|
\end{align*}
\]

where \( T = \begin{bmatrix} V_1 \ 1 \\ V_2 \ 1 \end{bmatrix} \) and \( h = -A^{(\lambda)} \cdot V_1 + f(V_1) \). In this way, if \( \lambda \to 0 \), then:

\[
\left| \begin{bmatrix} A^{(\lambda)} - A^{(i)} \\ h - B^{(i)} \end{bmatrix} \right| = 0
\]

in other words: \( A^{(\lambda)} \to A^{(i)} y h \to B^{(i)} \).
The second item above, it going to be proved in what follows and states:

\[
\begin{align*}
  \left\{ \begin{array}{l}
  | -F + A^{(\pm \lambda)} + B^{(\pm \lambda)} | \leq \bar{\lambda}^* \\
  | B^{(\pm \lambda)} | \leq \bar{\lambda}^*
  \end{array} \right.
\]

For the sake of clarity in the exposition, it will be analyzed separately \(B^{(\pm \lambda)}\) and 
\(-F + A^{(\pm \lambda)} + B^{(\pm \lambda)}\).

**For \(B^{(\pm \lambda)}\):** The requirement implies to satisfy the following:

\[
\begin{align*}
  & f(V_1) + \lambda - \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 \leq \lambda_2^* \\
  & f(V_1) - \lambda - \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 \geq -\lambda_2^*
\end{align*}
\]

or, in other words:

\[
\left\{ \begin{array}{l}
  f(V_1) \leq \lambda + \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 + \lambda_2^* \\
  f(V_1) \geq \lambda - \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 - \lambda_2^*
  \end{array} \right.
\]

(6.12)

**For \(-F + A^{(\pm \lambda)} + B^{(\pm \lambda)}\):** This condition requires:

\[
\left\{ \begin{array}{l}
  -\lambda_1^* \leq -F + \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 - \lambda \\
  -F + \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 + f(V_1) - \lambda \leq \lambda_1^*
  \end{array} \right.
\]

This yields:

\[
\left\{ \begin{array}{l}
  f(V_1) \leq \lambda_1^* + F - \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) - \lambda \\
  f(V_1) \geq -\lambda_1^* + F + \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) + \lambda
  \end{array} \right.
\]

(6.13)

Finally a region where the analysis is valid, has to be obtained.

**Validity Regions in \(\mathbb{R}^1\):** Equations (6.12) and (6.13) provide a region where the analysis is valid or in other words where there exists a CPWL approximation satisfying:

\[
\left\{ \begin{array}{l}
  | -F + A^{(\pm \lambda)} + B^{(\pm \lambda)} | \leq \bar{\lambda}^* \\
  | B^{(\pm \lambda)} | \leq \bar{\lambda}^*
  \end{array} \right.
\]

In this way, the conclusions for stability and attraction valid for the CPWL vector field: \(A^{(i)} \cdot x + B^{(i)}\) are also valid for \(A^{(\pm \lambda)} \cdot x + B^{(\pm \lambda)}\) which can be immediately translated into conclusions for the nonlinear ODE.

For this reason, it is convenient to write together equations (6.12) and (6.13) to give:

\[
\left\{ \begin{array}{l}
  f(V_1) \leq \lambda + \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 + \lambda_2^* \\
  f(V_1) \geq \lambda - \left[ \frac{f(V_1) - f(V_2)}{V_1 - V_2} \right] \cdot V_1 - \lambda_2^* \\
  f(V_1) \leq \lambda_1^* + f(x = 1) - \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) - \lambda \\
  f(V_1) \geq -\lambda_1^* + f(x = 1) + \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) + \lambda
  \end{array} \right.
\]

(6.14)

The analysis will focus two main conditions:

- The bounds for \(f(V_1)\) contain equilibrium points.
• Suitable values for \( \lambda \) in order to assure that a CPWL exists for our valid region in order to validate the proof using the CPWL theory theorem developed in Section 4.1.

The first item means a positive superior bound and a negative inferior one, performing this analysis to the conditions in (6.14), it will be also obtained appropriate points \( \lambda \): 

\[
\begin{align*}
\begin{cases}
\lambda \leq \lambda_2^* + \frac{f(V_2) - f(V_1)}{V_2 - V_1} \cdot V_1 \\
\lambda \leq \lambda_2^* - \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot V_1 \\
\lambda \leq \lambda_1^* - \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) + f(x = 1) \\
\lambda \leq \lambda_1^* + \frac{f(V_1) - f(V_2)}{V_1 - V_2} \cdot (1 - V_1) - f(x = 1)
\end{cases}
\end{align*}
\]

Since \( \lambda \geq 0 \), considering the limiting case \( V_1 \to V_2 \) and naming \( V_1 = x \), this leads:

\[
\begin{align*}
\begin{cases}
x \cdot \frac{\partial f(x)}{\partial x} \geq -\lambda_2^* \\
x \cdot \frac{\partial f(x)}{\partial x} \leq \lambda_2^* \\
(1 - x) \cdot \frac{\partial f(x)}{\partial x} \leq \lambda_1^* + f(x = 1) \\
(1 - x) \cdot \frac{\partial f(x)}{\partial x} \geq -\lambda_1^* + f(x = 1)
\end{cases}
\end{align*}
\]

In summary the region where the analysis is valid can be recast as:

\[
\begin{align*}
\begin{cases}
|x \cdot \frac{\partial f(x)}{\partial x}| \leq \lambda_2^* \\
(1 - x) \cdot \frac{\partial f(x)}{\partial x} - f(x = 1) \leq \lambda_1^*
\end{cases}
\end{align*}
\]

This completes the proof for the scalar case.

**\( \mathbb{R}^n \) analysis:** The development in this section extends the previous result for \( \mathbb{R}^1 \) to vector fields in \( \mathbb{R}^n \), to do this, we consider \( 2 \cdot n \) ”worst case” planes defining a convex set as follows:

\[
\begin{align*}
\begin{cases}
A_j^{(-\lambda)} \cdot x + B_j^{(-\lambda)} \leq f_j(x) - \lambda_j \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \leq f_j(x) + \lambda_j \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \\
j = 1, 2, \ldots, n
\end{cases}
\end{align*}
\]

As for the \( \mathbb{R}^1 \) case we need to show:

\[
\begin{align*}
\begin{cases}
A_j^{(-\lambda)} \cdot x + B_j^{(-\lambda)} \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \\
A_j^{(-\lambda)} \cdot x + B_j^{(-\lambda)} \leq f_j(x) - \lambda \\
f_j(x) + \lambda \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \\
j = 1, 2, \ldots, n \\
x = [x_1, x_2, \ldots, x_n]' \\
\forall x_1 \geq 0
\end{cases}
\end{align*}
\]

(6.15)

In this way, it is first needed \( A_j^{(\pm\lambda)} \) and \( B_j^{(\pm\lambda)} \) which can be obtained using \( n + 1 \) points \( \{x_1, x_2, \ldots, x_{n+1}\} \) as follows:
The set of points \( \{y_1, y_2, \ldots, y_{n+1}\} \) is related to \( \{x_1, x_2, \ldots, x_n\} \) by:

\[
\begin{align*}
    y_j &= f_j(V_j) \pm \lambda_j \\
    x_j &= V_j \\
    \lambda &= [\lambda_1, \lambda_2, \ldots, \lambda_n]' \\
    j &= 1, \ldots, n + 1
\end{align*}
\]

where \( \{V_1, V_2, \ldots, V_{n+1}\} \) is the set of vertices containing the general simplex \( i^{th} \) considered for the analysis.

Then, it is possible to explicitly write \( A_j^{(\pm \lambda)} \) and \( B_j^{(\pm \lambda)} \) in matrix form:

\[
(6.16) \quad \begin{bmatrix} A_j^{(\pm \lambda)}' \\ B_j^{(\pm \lambda)} \end{bmatrix} = \begin{bmatrix} V'_1 & 1 \\ \vdots & \vdots \\ V'_{n+1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_j(V_1) \pm \lambda_j \\ \vdots \\ f_j(V_{n+1}) \pm \lambda_j \end{bmatrix}
\]

Next, in order to effectively calculate \( A_j^{(\pm \lambda)} \) and \( B_j^{(\pm \lambda)} \) from equation (6.16), we need the inverse of the matrix \( \begin{bmatrix} V'_1 & 1 \\ \vdots & \vdots \\ V'_{n+1} & 1 \end{bmatrix} \).

A useful way of doing this is the one presented in [17], pp. 97, where applying row operations as in the well known method of Gauss for solving linear systems of algebraic equations it is possible to transform a matrix \( M \) in \( \tilde{M} \) as follows:

\[
M = \begin{bmatrix} V'_1 & 1 \\ \vdots & \vdots \\ V'_{n+1} & 1 \end{bmatrix} \quad \leftrightarrow \quad \tilde{M} = \begin{bmatrix} V'_1 & 1 \\ \vdots & \vdots \\ V'_{n+1} & 1 \end{bmatrix}^{-1}
\]

where \( I \) is the identity matrix. Then by subtracting the first with the second row in \( M \), the second with the third and so on, we arrive to:

\[
(6.17) \quad M = \begin{bmatrix} V'_1 - V'_2 & 0 \\ V'_2 - V'_3 & 0 \\ \vdots & \vdots \\ V'_{n} - V'_{n+1} & 0 \\ V'_{n+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}
\]

On the other hand is possible to choose \( n \) vertices forming a basis (see [23] for a further reading):
\[ V_{n+1} = [V_1 \ V_2 \ \ldots \ V_n] \cdot \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} \]

then there exists \( \gamma^* \in \mathbb{R}^{n \times 1} \) such that:

\[-V_{n+1} = [V_1 - V_2 \ V_2 - V_3 \ \ldots \ V_n - V_{n+1}] \cdot \gamma^* \]

Now according the boundary configuration\(^\text{12}\) in \([23]\), pp. 58-65, equation (3.8) and more general in section (3.4.1), then it is possible to write every vertex for any simplex as follows:

\[
\begin{align*}
V_i &= m_i \cdot \delta_{x_i} \cdot \bar{e}_i + \delta_{x_i} \cdot \sum_{j=1}^{i} (-1)^{c_j} \cdot \bar{e}_j \\
i &= 1, \ldots, n
\end{align*}
\]

where \( c_j \in \{0, 1\}, m_i = 1, 2, \ldots, N_i \) with \( N_i \) the number of simplicies used in each coordinate and \( \delta_{x_i} \), their grid size. In this way:

\[
\begin{align*}
V_i - V_{i+1} &= \delta_{x_i} \cdot (\sum_{j=1}^{i} (-1)^{c_j} \cdot \bar{e}_j - \sum_{j=1}^{i} (-1)^{c_j} \cdot \bar{e}_j - (-1) \cdot \bar{e}_{i+1}) \\
i &= 1, \ldots, n
\end{align*}
\]

providing the present analysis for \( V_i > 0, i = 1, 2, \ldots, n \)-the rest of the cases are of similar consideration. This fact yields:

\[
Q = \begin{bmatrix} V_1' - V_2' \\ \vdots \\ V_n' - V_{n+1}' \end{bmatrix} = \begin{bmatrix} \delta_{x_1} & 0 & \ldots & 0 \\ 0 & \delta_{x_2} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta_{x_n} \end{bmatrix}
\]

In this way the matrix \( M \) from equation (6.17) becomes:

\[
M = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & \ldots & 0 \\ Q' & 0 & 1 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 \\ \gamma^{x'} \cdot Q' & 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & \ldots & 0 & 1 & \end{pmatrix}
\]

Performing row operations we have:

\[
M = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & \ldots & 0 \\ Q' & 0 & 1 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 \\ \gamma^{x'} \cdot Q' & 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & \ldots & 0 & 1 & \end{pmatrix}
\]

\[^{11}\) This is how the set of vertices is called in \([23]\).\]
where:

\[
I^* = \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

Realizing that \(-\gamma^* \cdot I^* = -[\gamma_1^* (-\gamma_1^* + \gamma_2^*) (-\gamma_2^* + \gamma_3^*) \ldots (-\gamma_{n-1}^* + \gamma_n^*)]\), we obtain the desired inverse:

\[
\begin{bmatrix}
V'_1 & 1 \\
\vdots \\
V'_{n+1} & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{1}{\delta_{x_1}} & 0 & \ldots & 0 & 0 \\
0 & \frac{1}{\delta_{x_2}} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\delta_{x_n}} & 0 \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & \ldots & 0 \\
\vdots \\
\vdots \\
-\gamma_1^* & -(-\gamma_1^* + \gamma_2^*) & \ldots & -(-\gamma_{n-1}^* + \gamma_n^*) & (1 + \gamma_n^*)
\end{bmatrix}
\]

Finally, \(A_j^{(\pm \lambda)}\) and \(B_j^{(\pm \lambda)}\) arise:

\[
A_j^{(\pm \lambda)}' = \begin{bmatrix}
\frac{f_j(V_1) - f_j(V_2)}{\delta_{x_1}} \\
\frac{f_j(V_2) - f_j(V_3)}{\delta_{x_2}} \\
\vdots \\
\frac{f_j(V_n) - f_j(V_{n+1})}{\delta_{x_n}}
\end{bmatrix}
\]

\[
B_j^{(\pm \lambda)} = [f_j(V_n+1) \pm \lambda_j] - \sum_{i=1}^{n} \gamma_i^* \cdot [f_j(V_i) - f_j(V_{i+1})] \quad V'_{n+1, j} = A_j^{(\pm \lambda)}
\]

where \(\gamma_i^* = \frac{V_{i+1}}{\delta_{x_i}}\) with \(V_{n+1}\) each component of the vertex \(n + 1\). Once \(A_j^{(\pm \lambda)}\) and \(B_j^{(\pm \lambda)}\) are already determined, the conditions for a convex set depicted in equation (6.15) lead:

\[
\begin{align*}
A_j^{(-\lambda)} \cdot x + B_j^{(-\lambda)} & \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \iff 0 \leq (B_j^{(\lambda)} - B_j^{(-\lambda)}) \\
& \iff 2 \cdot \lambda_j \geq 0 \\
f_j(x) - f_j(V_{n+1}) = (x - V_{n+1})' \cdot A_j^{*} \\
& \quad j = 1, 2, \ldots, n
\end{align*}
\]

As for \(\mathbb{R}^1\), considering the limiting case \(V_i \to V_{i+1}\), then:

\[
\begin{align*}
A_j^{(-\lambda)} \cdot x + B_j^{(-\lambda)} & \leq A_j^{(\lambda)} \cdot x + B_j^{(\lambda)} \iff 0 \leq (B_j^{(\lambda)} - B_j^{(-\lambda)}) \\
& \iff 2 \cdot \lambda_j \geq 0 \\
f_j(x) - f_j(V_{n+1}) = (x - V_{n+1})' \cdot \left. \frac{\partial f_j(x)}{\partial x} \right|_{x = V_{n+1}} \\
& \quad j = 1, 2, \ldots, n
\end{align*}
\]

The first condition above is always satisfied since all the CPWL approximations used along the proof provides a positive quantity \(\lambda_j\), on the other hand the last
requirement is also true for continuous vector fields $f(x)$ and with continuous first derivative in the view of the mean value theorem for vector fields in $\mathbb{R}^n$ (see [20], pp. 148).

On the other hand as for $\mathbb{R}^3$, the following it is remaining to be proved:

- If $V_1, V_2, \ldots, V_n$, then: $A^{(\pm \lambda)} \to A^{(i)}$, $B^{(\pm \lambda)} \to B^{(i)}$.
- $\{A^{(\pm \lambda)}, B^{(\pm \lambda)}\}$ satisfy: $|F_{jk} - A^{(\pm \lambda)}_{jk} - B^{(\pm \lambda)}_{jk}| \leq \tilde{\lambda}^*_{jk}$, $|B^{(i)}_j| \leq \tilde{\lambda}^*_j$, $j, k = 1, \ldots, n$.

The first item can be proved utilizing the inequality (6.15) in the $n + 1$ vertices: $V_i, \ i = 1, \ldots, n + 1$ and noticing that $A^{(\lambda)} = A^{(-\lambda)}$ (see (6.18)):

$$
\begin{bmatrix}
V'_1 & 1 \\
V'_2 & 1 \\
\vdots & \vdots \\
V'_{n+1} & 1
\end{bmatrix} \cdot \begin{bmatrix}
A^{(\lambda)} \\
B_{j}^{(-\lambda)}
\end{bmatrix} \leq \begin{bmatrix}
V'_1 & 1 \\
V'_2 & 1 \\
\vdots & \vdots \\
V'_{n+1} & 1
\end{bmatrix} \cdot \begin{bmatrix}
A^{(i)} \\
B_{j}^{(i)}
\end{bmatrix} \leq \begin{bmatrix}
\lambda_j
\end{bmatrix}.
$$

using absolute values:

$$
|T \cdot \begin{bmatrix}
A^{(\lambda)} - A^{(i)} \\
h_j - B_{j}^{(i)}
\end{bmatrix}| \leq T \cdot \begin{bmatrix}
0 \\
0 \\
\vdots \\
\lambda_j
\end{bmatrix} \Rightarrow \begin{bmatrix}
A^{(\lambda)} - A^{(i)} \\
h_j - B_{j}^{(i)}
\end{bmatrix} \leq |T^{-1}| \cdot T \cdot \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

where:

$$
T = \begin{bmatrix}
V_1 & V_2 & \ldots & V_{n+1} \\
1 & 1 & \ldots & 1
\end{bmatrix},
\quad h_j = -V'_{n+1} \cdot A^{(\lambda)}_j + f_j(V_{n+1})
$$

In this way, if $\lambda \to 0$, then:

$$
\left| \begin{bmatrix}
A^{(\lambda)} - A^{(i)} \\
h_j - B_{j}^{(i)}
\end{bmatrix} \right| = 0
$$

in other words: $A^{(\lambda)} \to A^{(i)}$ y $h \to B^{(i)}$.

The second item yields:

$$
\begin{cases}
|F_{jk} - A^{(i)}_{jk} - B^{(i)}_{jk}| \leq \tilde{\lambda}^*_{jk} \\
|B^{(i)}_j| \leq \lambda^*_j \\
j, k = 1, \ldots, n
\end{cases}
$$

As for $\mathbb{R}^3$, the analysis will be split into $B^{(\pm \lambda)}_j$ and $F_{jk} - A^{(\pm \lambda)}_{jk} - B^{(\pm \lambda)}_j$:

For $B^{(\pm \lambda)}_j$. In this case, the following should be satisfied:

$$
\begin{cases}
f_j(V_{n+1}) + \lambda_j - V'_{n+1} \cdot A^*_j \leq \tilde{\lambda}^*_j \\
f_j(V_{n+1}) - \lambda_j - V'_{n+1} \cdot A^*_j \geq -\lambda^*_j
\end{cases}
$$

Realize that $V'_{n+1} \cdot A^*_j$ play the role of $\frac{f(V_1) - f(V_2)}{\delta x}$ for the $\mathbb{R}^3$ case.
In this way:

\begin{equation}
\begin{align*}
&f_j(V_{n+1}) \leq -\lambda_j + V'_{n+1} \cdot A_j^* + \bar{\lambda}_j^* \\
&f_j(V_{n+1}) \geq \lambda_j + V'_{n+1} \cdot A_j^* - \lambda_j^*
\end{align*}
\end{equation}

For \( f_j(\bar{e}_k) - A_{jk}^{(\pm \lambda)} - B_j^{(\pm \lambda)} \). It has to be guaranteed:

\begin{equation}
\begin{align*}
&-f_j(\bar{e}_k) + A_{jk}^{(\lambda)} + B_j^{(\lambda)} \leq \bar{\lambda}_j^* \\
&-f_j(\bar{e}_k) + A_{jk}^{(-\lambda)} + B_j^{(-\lambda)} \geq -\bar{\lambda}_j^*
\end{align*}
\end{equation}

Recalling equation (6.15):

\begin{equation}
\begin{align*}
&A_{jk}^* + f_j(V_{n+1}) + \lambda_j - V'_{n+1} \cdot A_j^* - f_j(\bar{e}_k) \leq \bar{\lambda}_j^* \\
&A_{jk}^* + f_j(V_{n+1}) - \lambda_j - V'_{n+1} \cdot A_j^* - f_j(\bar{e}_k) \geq -\bar{\lambda}_j^*
\end{align*}
\end{equation}

this yields:

\begin{equation}
\begin{align*}
&f_j(V_{n+1}) \leq \bar{\lambda}_j^* + f_j(\bar{e}_k) - \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^* \\
&f_j(V_{n+1}) \geq -\bar{\lambda}_j^* + f_j(\bar{e}_k) + \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^*
\end{align*}
\end{equation}

Validity Regions in \( \mathbb{R}^n \). Comparing equations (6.19) with (6.12) and (6.20) with (6.13) for \( \mathbb{R}^n \) and \( \mathbb{R}^1 \) respectively, we immediately see that the conclusions for \( \mathbb{R}^1 \) can be readily extended to the present \( \mathbb{R}^n \) analysis to give:

\begin{equation}
\begin{align*}
&V_{(n+1)i} \geq 0 \\
&f_j(V_{n+1}) \leq -\lambda_j + V'_{n+1} \cdot A_j^* + \bar{\lambda}_j^* \\
&f_j(V_{n+1}) \geq \lambda_j + V'_{n+1} \cdot A_j^* - \bar{\lambda}_j^* \\
&f_j(V_{n+1}) \leq \bar{\lambda}_j^* + f_j(\bar{e}_k) - \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^* \\
&f_j(V_{n+1}) \geq -\bar{\lambda}_j^* + f_j(\bar{e}_k) + \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^*
\end{align*}
\end{equation}

As for \( \mathbb{R}^1 \), the present \( \mathbb{R}^n \) needs to fit the requirement of the superior bound being positive and the inferior one negative in order to contain equilibriums of \( f(x) \). In this way:

\begin{equation}
\begin{align*}
&\bar{\lambda}_j^* - \lambda_j + V'_{n+1} \cdot A_j^* \geq 0 \\
&-\bar{\lambda}_j^* + \lambda_j + V'_{n+1} \cdot A_j^* \leq 0 \\
&\bar{\lambda}_j^* + f_j(\bar{e}_k) - \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^* \geq 0 \\
&-\bar{\lambda}_j^* + f_j(\bar{e}_k) + \lambda_j - A_{jk}^* + V'_{n+1} \cdot A_j^* \leq 0
\end{align*}
\end{equation}

The appropriate values for \( \lambda_j \) arises:
\[
\left\{
\begin{array}{l}
\lambda_j \leq \bar{\lambda}_{jk} + V'_{n+1} \cdot A_j^* \\
\lambda_j \leq \bar{\lambda}_{jk} - V'_{n+1} \cdot A_j^* \\
\lambda_j \leq \bar{\lambda}_{jk} + f_j(e_k) - A^*_j + V'_{n+1} \cdot A_j^* \\
\lambda_j \leq \bar{\lambda}_{jk} - f_j(e_k) + A^*_j - V'_{n+1} \cdot A_j^* \\
j, k = 1, 2, \ldots, n
\end{array}
\right.
\]

Since \( \lambda_j \geq 0 \), writing in matrix form, considering the limiting case \( V_i \to V_{i+1} \) and naming in general \( V_{n+1} = x \), it is obtained:

\[
\begin{align*}
x' \cdot \frac{\partial f_j(x)}{\partial x} & \geq -\bar{\lambda}_j^* \\
x' \cdot \frac{\partial f_j(x)}{\partial x} & \leq \bar{\lambda}_j^* \\
(-x' + I) \cdot \frac{\partial f_j(x)}{\partial x} & \geq -\bar{\lambda}_j^* + F_k \\
(-x' + I) \cdot \frac{\partial f_j(x)}{\partial x} & \leq \bar{\lambda}_j^* + F_k \\
j = 1, 2, \ldots, n
\end{align*}
\]

where \( x' = [x', x', \ldots, x'] \) and \( F = [F_1 \ F_2 \ \ldots \ F_n] \). The conditions above can be compactly written as follows choosing the minimum bound for the first two inequalities above:

\[
\begin{align*}
| x' \cdot \frac{\partial f_j(x)}{\partial x} | & \leq \min_{j=1,2,\ldots,n} \bar{\lambda}_j^* \\
| (-x' + I) \cdot \frac{\partial f_j(x)}{\partial x} - F | & \leq \bar{\lambda}
\end{align*}
\]

(6.22)

\[
\frac{\partial f_j(x)}{\partial x} = \begin{bmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\
\frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n}
\end{bmatrix}
\]

These conditions can be further simplified as follows:

\[
\begin{align*}
| x' \cdot \frac{\partial f_j(x)}{\partial x} | & \leq \min_{j=1,2,\ldots,n} \bar{\lambda}_j^* \\
| \frac{\partial f_j(x)}{\partial x} - F' | & \leq \bar{\lambda} - \bar{\lambda}
\end{align*}
\]

(6.23)

To complete the proof, it is remarkable that the conditions obtained in this proof are independent of the coordinate system chosen.

Up to this point, it has been proved that there exists a region given by (6.23) and a precision CPWL under which all the matrices \( A^{(i)} \) are Hurwitz in each simplex, because of that and since the objective it is to prove that the aforementioned region it is also a region where the nonlinear ODE: \( \dot{x}(t)f(x) \) is attractive to its equilibrium point, it will be necessary to prove that this region is an invariant set.

By virtue of inequality (6.15), if the trajectories of: \( x(t) = A^{(i)} \cdot x + B^{(i)} \), are considered:
In order to solve the differential inequality above, the technique in \[15\] is going to be applied to give:

\[
\begin{align*}
\Omega &= \{ \lambda : \frac{\partial f(x)}{\partial x} \cdot x \leq \min_{i=1,2,...,n} \lambda_i \} \\
&= \{ \lambda : \frac{\partial f(x)}{\partial x} \cdot x \leq \min_{i=1,2,...,n} \lambda_i \} \\
&= \{ x(t) \in \Omega : \lim_{t \to \infty} (A^{(\lambda)} \cdot x + B^{(-\lambda)}) \leq A^{(\lambda)} \cdot x + B^{(\lambda)} \}
\end{align*}
\]

In this way, if \(\dot{x}(t)\) is the solution of the differential inequality, then the bounds are also contained there.

It is possible then, to assure that the trajectories given by the inequality: \(A^{(\lambda)} \cdot x + B^{(-\lambda)} \leq A^{(\lambda)} \cdot x + B^{(\lambda)} \) are contained in \(\Omega\), \(\forall t \geq 0\) if it is ensured that the bounds are also contained there.

To do this, the extremums of \(x^{1,2*}(t)\) are looked for, that means, the instants of time for which \(x^{1,2*}(t)\) reach their maximums and minimums. In order to find these maximums or minimums, the elemental calculus result for which the instant of time where the extremum occur are utilized, in this way, it is necessary to evaluate \(\dot{x}^{1,2*}(t) = 0\) and \(\dot{x}^{1,2*}(t)\) does not exist.

It is clear that the first possibility only happens if \(t \to \infty\) in the view that \(A^{(\lambda)}\) is Hurwitz, while the second one indicates \(t = 0\) due to the discontinuity introduced in considering only positive evolutions in time: \((t \in \mathbb{R}^+)\).

These reasons and considering the initial condition \(x^{1,2*}(0)\) always inside the set \(\Omega\), it will be enough to prove that: \(\lim_{t \to \infty} x^{1,2*}(t) \in \Omega\), that is:

\[
\begin{align*}
\lim_{t \to \infty} x^{1*}(t) &= -(A^{(\lambda)})^{-1} \cdot \min_{i=1,2,...,n} \lambda_i \\
\lim_{t \to \infty} x^{2*}(t) &= (A^{(\lambda)})^{-1} \cdot \min_{i=1,2,...,n} \lambda_i
\end{align*}
\]
equivalently:

\[
\begin{align*}
\lim_{t \to \infty} A^{(\lambda)} \cdot x^{1*}(t) &= -\lambda^* \leq \min_{i=1,2,...,n} \lambda_i \\
\lim_{t \to \infty} A^{(\lambda)} x^{2*}(t) &= \lambda^* \leq \min_{i=1,2,...,n} \lambda_i
\end{align*}
\]

In this way, if \(\lambda \to 0\):

\[
| \frac{\partial f(x)}{\partial x} \cdot x | \leq \min_{i=1,2,...,n} \lambda_i^*
\]

This completes the proof.
proof of Lemma 7. As it was proved in Theorem 4 each matrix $A^{(i)}$ is Hurwitz.

In this way and applying the Corollary 4.1 in [22] it is immediately seen that $V(x) = x' \cdot P \cdot x$ is a common quadratic Lyapunov function of:

$$x^{(i)}(t) = A^{(i)} \cdot x^{(i)} + B^{(i)}, \quad \forall i = 1, 2, \ldots N$$

where $N$ is the number of total simplices used for the partition of the CPWL ODE’s considered. If, on the other hand, $\frac{\partial f}{\partial x} = 0$ and using the bounds introduced in [23], it is possible to write:

$$| f(x^{(i)}) - A^{(i)} \cdot x^{(i)} - B^{(i)} | \leq \lambda, \quad \forall i = 1, 2, \ldots N$$

Multiplying both right sides by $| x^{(i)} | \cdot | P |$:

$$| x^{(i)} | \cdot | P | \cdot | f(x^{(i)}) - A^{(i)} \cdot x^{(i)} - B^{(i)} | \leq | x^{(i)} | \cdot | P | \cdot \lambda, \quad \forall i = 1, 2, \ldots N$$

Since $| x^{(i)} | \cdot (P \cdot f(x^{(i)})) - A^{(i)} \cdot x^{(i)} - B^{(i)} | \leq | x^{(i)} | \cdot | P | \cdot | f(x^{(i)}) - A^{(i)} \cdot x^{(i)} - B^{(i)} |$, it is obtained:

$$(6.24) \quad | x^{(i)} \cdot P \cdot (f(x^{(i)}) - A^{(i)} \cdot x^{(i)} - B^{(i)}) | \leq | x^{(i)} | \cdot | P | \cdot \lambda, \quad \forall i = 1, 2, \ldots N$$

With a similar procedure is also valid:

$$(6.25) \quad | (f(x^{(i)}) - A^{(i)} \cdot x^{(i)} - B^{(i)}) \cdot P \cdot x^{(i)} | \leq | x^{(i)} | \cdot | P | \cdot \lambda, \quad \forall i = 1, 2, \ldots N$$

Finally summing equations (6.24) and (6.25):

$$2 \cdot x^{(i)} \cdot P \cdot f(x^{(i)}) - 2 \cdot x^{(i)} \cdot P \cdot (A^{(i)} \cdot x^{(i)} + B^{(i)}) \leq 2 \cdot x^{(i)} \cdot | P | \cdot \lambda, \quad \forall i = 1, 2, \ldots N$$

Since $V = x^{(i)} \cdot P \cdot x^{(i)}$ is a Lyapunov function for $A^{(i)} \cdot x^{(i)} + B^{(i)}$, then

$$2 \cdot x^{(i)} \cdot P \cdot (A^{(i)} \cdot x^{(i)} + B^{(i)}) < 0$$

This is showing that:

$$2 \cdot x^{(i)} \cdot P \cdot f(x^{(i)}) - 2 \cdot x^{(i)} \cdot | P | \cdot \lambda \leq 2 \cdot x^{(i)} \cdot P \cdot (A^{(i)} \cdot x^{(i)} + B^{(i)}) < 0, \quad \forall i = 1, 2, \ldots N$$

If the limiting case $\lambda \rightarrow 0$ is considered, then in virtue of Error Bounds the Dynamics introduced in [16], it is known that the trajectories $x^{(i)} \rightarrow x(t)$, where $x(t) = f(x)$, this leads:

$$\lim_{\lambda \rightarrow 0} 2 \cdot x^{(i)} \cdot P \cdot f(x^{(i)}) - 2 \cdot x^{(i)} \cdot | P | \cdot \lambda < 0 \Leftrightarrow 2 \cdot x(t) \cdot P \cdot f(x(t)) < 0, \quad \forall i = 1, 2, \ldots N$$

This completes the proof.

proof of Corollary 4. As for the proof of Theorem 4 it will be needed a Hurwitz matrix $F$ such that:

$$\frac{\partial f(x)}{\partial x} - F \leq \tilde{\lambda}^* - \tilde{\lambda}^*.$$ In this way if $\frac{\partial f(x)}{\partial x}$ is Hurwitz for all $x \in \Omega$ and using $\tilde{\lambda}^* = \tilde{\lambda}^*$, then if $\Omega$ is as follows:
\[
\begin{align*}
\Omega &= \{ x : \lambda_R < 1 \} \\
R &= 2 \cdot ( | P | \cdot | \frac{\partial f(x)}{\partial x} \cdot x | \cdot [1 \ 1 \ldots 1] + H(x)' )
\end{align*}
\]

with \( \lambda_R \) the biggest eigenvalue of \( R \)

the ODE: \( \dot{x}(t) = f(x) \) is attractive to the origin (equilibrium point) in the view of Theorem 4.

This completes the proof.

\[\square\]

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