The inverse problem of the oscillation of a rod with a variable cross section

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Abstract. Variable cross-section rods are used in many parts and mechanisms. For example, conical rods are widely used in percussion mechanisms. The strength of such parts directly depends on the natural frequencies of vibrations. Oscillations and vibration are also used in diagnostics, for example in acoustic diagnostics of defects. The paper presents a method that allows numerically finding a variable cross section of an elastic rod from the natural frequencies of longitudinal vibrations. It is assumed that the cross-sectional area varies along the axis and is described by an exponential function of a polynomial of degree \( n \). The boundary condition on the left end is hard, on the right - elastic. It is shown that to determine \( n \) unknown coefficients of the cross section function, \( n \) natural frequencies are required.

1. Introduction
The problem of determining the natural frequencies and forms of longitudinal vibrations of rods with a variable cross-sectional area is well studied \([1, 2, 3, 4]\). This direction is relevant, because rods of various configurations are elements of many structures and machines. As a rule, such rods are modeled in the form of stepped rods \([1]\), that is, the rod consists of two or more parts with different diameters, or in the form of a conical rod \([2]\). The natural frequencies of longitudinal oscillations of a rod such that its Young’s modulus, the density, and the cross-sectional area are functions of the longitudinal coordinate are analyzed in \([3]\). For solving the corresponding problem, an integral formula is used to represent the general solution to the original Helmholtz equation with variable coefficients in terms of the general solution to the accompanying equation with constant coefficients. Frequency equations are derived in the form of rapidly converging Leibniz series for three types of boundary conditions. The results of computational experiments for various laws of rigidity change are presented in \([4]\). Intrinsic transverse vibrations of a straight rod with a rectangular cross-section with a constant height and a variable width, which varies exponentially, are considered. Using the analytical method, the values of the frequencies of natural vibrations are obtained for various functions of changing the cross section of the rod and various ways of fixing it.

In this paper, propose to describe the cross-sectional area as a function of the longitudinal coordinate. By formulation, the problem considered in this paper is the inverse, because it is not necessary to find the natural frequencies, but the law of changing the cross-sectional area by natural frequencies. Moreover, since the cross-sectional area is included in the oscillation equation, the formulation under consideration refers to coefficient inverse problems \([5]\). Inverse problem solutions are widely used in non-destructive diagnostics to detect defects. A problem...
similar in formulation was considered in [6], where a solution to the problem is given for determining the parameters of a rod crack. Similar studies are also given in [7], which describes an iterative algorithm based on the apparatus of Fredholm integral equations of the first and second kind and their discretization. The restoration of the function occurs according to the amplitude-frequency characteristics of the oscillations of the right end, and the frequencies of the oscillations should be far from the resonant ones. In the Western literature, the finite element method has been widely developed for solving inverse problems [8, 9]. In [8], the problem of identification of rod cracks is considered, where the natural frequency is used for restoration together with the natural shape. In [8], the main emphasis is on filtering input data. In [9], an identification procedure is given for determining the characteristics of a crack (location and size of a crack) based on dynamic measurements.

2. Statement of the problem

One-dimensional longitudinal vibrations of a rod with a variable cross-sectional area are described by the equation

\[ \frac{\partial}{\partial x} \left( E F(x) \frac{\partial u}{\partial x} \right) - \rho F(x) \frac{\partial^2 u}{\partial t^2} = 0, \]  

(1)

where \( u(x, t) \) is the displacement of the section with the coordinate \( x \), \( F(x) \) is the cross-sectional area at the point \( x \), the elastic modulus \( E \) and the density \( \rho \) of the rod are considered constant.

One end of the rod is fixed rigidly \( u(0) = 0 \), the border conditions at the other end may be different. From the eigenfrequencies of the longitudinal vibrations of the rod, it is required to determine the section function \( F(x) \).

3. Solution method

In the present work, it is proposed to take the cross-sectional area of the rod at point \( x \) in the form

\[ F(x) = e^{P(x)}, \]  

(2)

where \( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \), the coefficient \( a_0 \) corresponds to the value of the cross-sectional area at the reference point \( (x = 0) \) and is considered known. We will search for solution (1) in the form \( u(x, t) = y(x) \cos(\omega t) \). Then, substituting (2) in (1), taking into account that \( P'(0) = a_1 + 2a_2 x + \ldots + na_n x^{n-1} \), we obtain the Sturm-Liouville problem

\[ \begin{cases} -y'' - P'(x)y' = \lambda^2 y, \\ y(0) = 0, \quad y'(1) + Hy(1) = 0. \end{cases} \]  

(3)

The method for solving problem (3) is similar to [10], that is, we will search the general solution of equation (3) as \( y(x, \lambda) = C_1 y_1 + C_2 y_2 \). Here \( y_1, y_2 \) are linearly independent solutions of equation (3). The functions \( y_1, y_2 \) will be constructed in the form of a Maclaurin series with respect to the variables \( x \) and \( \lambda \), for which the following conditions must be satisfied

\[ \begin{align*} y_1(0, \lambda) &= 1, & y_1'(0, \lambda) &= 0, \\ y_2(0, \lambda) &= 0, & y_2'(0, \lambda) &= 1. \end{align*} \]  

(4)

Condition (4) provides linear independence due to the properties of the Wronsky determinant. Then \( y_1, y_2 \), under condition (4), are written in the form

\[ y_1(x, \lambda) = 1 - \lambda \frac{x^2}{2!} + P'(0) \cdot \lambda^2 \frac{x^3}{3!} + \left(2 \lambda^2 P''(0) - \lambda^2 \cdot P'^2(0) + \lambda^4\right) \frac{x^4}{4!} + \ldots \]
\[ y_2(x, \lambda) = x - P'(0) \frac{x^2}{2!} + \left( -P''(0) + P'^2(0) - \lambda^2 \right) \frac{x^3}{3!} + \left( -P'''(0) + 3P'(0)P''(0) - P'^3(0) + 2\lambda^2 P'(0) \right) \frac{x^4}{4!} + \ldots \]

Moreover, \( y_1, y_2 \) are entire functions with respect to \( \lambda \) for each fixed \( x \).

Substituting the general solution constructed in the form of a series into the boundary conditions (3), we obtain the characteristic equation

\[ \Delta(\lambda) = y_2'(1, \lambda) - H \cdot y_2(1, \lambda), \tag{5} \]

here \( H \) is the stiffness of the spring on the right end. If the coefficient \( H = 0 \), then we obtain free boundary conditions, if \( H = \infty \), then it is rigid, in other cases it is elastic.

Consider an example. Let the section function (2) have the form \( P(x) = -4.9 - 3x + x^2 \) and the boundary conditions at the right end are as follows \( y'(1) + 2 \cdot y(1) = 0 \). Then equation (5) takes the form \( \Delta(\lambda) = y_2'(1, \lambda) - 2 \cdot y_2(1, \lambda) \), where the linearly independent functions \( y_1, y_2 \) are constructed up to the 40th degree. From the obtained characteristic equation, it is easy to find eigenvalues \( \lambda_1 = 2.73433, \lambda_2 = 5.35295, \ldots \).

For the inverse problem, it is assumed that the eigenvalues \( \lambda_i \) are known, it is required to determine the unknown coefficients \( a_i \) of function (2). Substituting the eigenvalues in (5), we obtain a system of equations for the unknown coefficients \( a_1 \ldots a_n \). To determine \( n \) unknown coefficients of the section function, \( n \) eigenvalues are required. The complexity of the problem lies in the fact that the problem in the general case does not have a unique solution. Uniqueness is achieved in the so-called set of correctness, which is selected based on the physical properties of a particular rod. For example, the cross-sectional area of the rod cannot be less than zero, and cannot be larger than the cross-section at the attachment point. Numerical experiments have shown that different rod configurations can have very close frequency spectra.

4. Accuracy assessment

Let’s check the computational experiment on the noise of the input data. To analyze the accuracy of the method, we estimate the error in reconstructing the coefficient \( a_1 \) of the polynomial \( P(x) = -4.9 + a_1x + a_2x^2 \). Here \( a_0 = -4.9 \). As input, we take the eigenvalues given in the form \( \lambda_i = \lambda_j(1 + \gamma \psi_j) \), where \( \lambda_j \) — eigenvalue calculated with an accuracy of 9 significant digits; \( \gamma \) — noise amplitude; \( \psi_j \) — random variable with uniform distribution law defined on the segment \([-1, 1]\). Assuming \( \gamma = 10^{-n} \) (\( n = 3, \ldots, 9 \)), we study the relative error of the method presented in this paper depending on \( \gamma \). We will search for the coefficients in the interval \( a_1 = -4.4, a_2 = -2.2 \), where the solution to the system is unique \( (a_1 = -3, a_2 = 1) \). The results of a numerical experiment on the noise of the input data are shown in table 1. From the table it follows that when the error in the input data is not more than \( \gamma = 10^{-3} \), the error in reconstructing the coefficient \( a_1 \) does not exceed 3%.

5. Conclusion

Representation of a variable section in the form of function (2) allows numerically determining the law of change of the section from a finite set of eigenvalues. Another advantage of this representation is that the resulting function is always non-negative, which corresponds to the physical formulation. The accuracy of the decision depends on the choice of the number of members in the sum of the series. The inverse problem can have several solutions, however, substituting the found coefficients in equation (2) and taking into account that the coefficient \( a_0 \) is known to us, we can discard solutions that do not satisfy the physical formulation. This result suggests that for rods with different cross-section functions, the spectrum of natural frequencies of longitudinal vibrations can coincide. To unambiguously determine the parameters of such rods, it is also necessary to use a spectrum of transverse or torsional vibrations.
Table 1. The relative error of the solution, depending on the noise of the input data

| $\gamma$  | $\delta_{a}(\psi_1, \psi_2)$, % |
|-----------|----------------------------------|
| $10^{-3}$ | 0.50297 2.41989 1.46716 1.11569 0.77571 |
| $10^{-4}$ | 0.08560 0.23320 0.42030 0.06267 0.03175 |
| $10^{-5}$ | 0.01214 0.00522 0.00395 0.01118 0.00034 |
| $10^{-6}$ | 0.00354 0.00251 0.00167 0.00221 0.00203 |
| $10^{-7}$ | 0.00010 0.00015 4.69 $\cdot$ $10^{-5}$ 0.00024 0.00012 |
| $10^{-8}$ | 2.94 $\cdot$ $10^{-5}$ 7.46 $\cdot$ $10^{-6}$ 2.21 $\cdot$ $10^{-5}$ 3.38 $\cdot$ $10^{-5}$ 1.22 $\cdot$ $10^{-5}$ |
| $10^{-9}$ | 1.05 $\cdot$ $10^{-5}$ 2.1 $\cdot$ $10^{-6}$ 7.3 $\cdot$ $10^{-6}$ 7.4 $\cdot$ $10^{-6}$ 5.8 $\cdot$ $10^{-6}$ |

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