A LOWER BOUND OF THE $L^2$ NORM ERROR ESTIMATE FOR THE ADINI ELEMENT OF THE BIHARMONIC EQUATION

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Abstract. This paper is devoted to the $L^2$ norm error estimate of the Adini element for the biharmonic equation. Surprisingly, a lower bound is established which proves that the $L^2$ norm convergence rate can not be higher than that in the energy norm. This proves the conjecture of [Lascaux and Lesaint, Some nonconforming finite elements for the plate bending problem, RAIRO Anal. Numer. 9 (1975), pp. 9–53.] that the convergence rates in both $L^2$ and $H^1$ norms can not be higher than that in the energy norm for this element.

1. Introduction

For the numerical analysis of finite element methods for fourth order elliptic problems, one unsolved fundamental problem is the $L^2$ norm error estimates [1, 3, 14]. In a recent paper [7], we analyzed several mostly popular lower order elements: the Powell-Sabin $C^1 − P_2$ macro element [12], the nonconforming Morley element [11, 14, 16, 17], the $C^1−Q_2$ macro element [6], the nonconforming rectangle Morley element [15], and the nonconforming incomplete biquadratic element [13, 18]. In particular, we proved that the best $L^2$ norm error estimates for these elements were at most of second order and could not be two order higher than that in the energy norm.

The Adini element [1, 3, 8, 14] is one of the earliest finite elements, dating back over 50 years. It is a nonconforming finite element for the biharmonic equation on rectangular meshes. The shape function space contains the complete cubic space and two additional monomials on each rectangle. In 1975, Lascaux and Lesaint analyzed this element and showed that the consistency error was of second order for uniform meshes, the same as the approximation error, and thus obtained a second order convergence rate [8]; see also [9, 10] and [4, 19]. This in
particular implies at least a second order $H^1$ norm convergence rate. Lascaux and Lesaint also conjectured that it did not seem possible to improve this estimate; see [8, Remark 4.5]. However, they did not provide a rigorous proof or justification for this remark.

The purpose of this paper is to analyze the $L^2$ norm error estimate for the Adini element [1, 3, 8, 14]. There are two main ingredients for the analysis. One is a refined property of the canonical interpolation operator, which is proved by a new expansion method. The other is an identity for $(-f, e)_{L^2(\Omega)}$, where $f$ is the right-hand side function and $e$ is the error. Such an identity separates the dominant term from the other higher order terms, which is the key to use the aforementioned refined property of the interpolation operator. Based on these factors, a lower bound of the $L^2$ norm error estimate is surprisingly established which proves that the best $L^2$ norm error estimate is at most of order $O(h^2)$. Thus, by the usual Poincare inequality, this indicates that the best $H^1$ norm error estimate is also at most of order $O(h^2)$. This gives a rigorous proof of the conjecture from [8] that the convergence rates in both $L^2$ and $H^1$ norms can not be higher than that in the energy norm.

The paper is organized as follows. In the following section, we present the Adini element and define the canonical interpolation operator. In Section 3, based on a refined property of the canonical interpolation operator and an identity for $(-f, e)_{L^2(\Omega)}$, we prove the main result that the $L^2$ norm error estimate has a lower bound which indicates that the convergence rates in both $L^2$ and $H^1$ norms are at most of order $O(h^2)$. In Section 4, we analyze the refined property of the canonical interpolation operator. In Section 5, we establish the identity for $(-f, e)_{L^2(\Omega)}$. In Section 6, we end this paper by the conclusion and some comments.

2. The Adini Element Method

We consider the model fourth order elliptic problem: Given $f \in L^2(\Omega)$ find $w \in W := H^2_0(\Omega)$, such that

$$(2.1) \quad a(w, v) := (\nabla^2 w, \nabla^2 v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \text{ for any } v \in W.$$ 

where $\nabla^2 w$ denotes the Hessian matrix of the function $w$.

To consider the discretization of (2.1) by the Adini element, let $T_h$ be a uniform regular rectangular triangulation of the domain $\Omega \subset \mathbb{R}^2$ in the two dimensions. Given $K \in T_h$, let $(x_c, y_c)$ be the center of $K$, the horizontal length $2h_{x,K}$, the vertical length $2h_{y,K}$, which define the
meshsize $h := \max_{K \in \mathcal{T}_h} \max(h_{x,K}, h_{y,K})$ and affine mapping:

\begin{align}
\xi := \frac{x - x_c}{h_{x,K}}, \quad \eta := \frac{y - y_c}{h_{y,K}} \quad \text{for any } (x, y) \in K.
\end{align}

Since $\mathcal{T}_h$ is a uniform mesh, we define $h_x := h_{x,K}$ and $h_y := h_{y,K}$ for any $K$. On element $K$, the shape function space of the Adini element reads

\begin{align}
Q_{Ad}(K) := P_3(K) + \text{span}\{x^3, y^3\},
\end{align}

here and throughout this paper, $P_\ell(K)$ denotes the space of polynomials of degree $\leq \ell$ over $K$. The Adini element space $W_h$ is then defined by

$W_h := \{v \in L^2(\Omega) : v|_K \in Q_{Ad}(K) \text{ for each } K \in \mathcal{T}_h, v \text{ and } \nabla v \text{ is continuous at the internal nodes, and vanishes at the boundary nodes on } \partial \Omega\}$.

The finite element approximation of Problem (2.1) reads: Find $w_h \in W_h$, such that

\begin{align}
a_h(w_h, v_h) := (\nabla^2 w_h, \nabla^2 v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \text{for any } v_h \in W_h,
\end{align}

where the operator $\nabla^2_h$ is the discrete counterpart of $\nabla^2$, which is defined element by element since the discrete space $W_h$ is nonconforming.

Given $K \in \mathcal{T}_h$, define the canonical interpolation operator $\Pi_K : H^3(K) \to Q_{Ad}(K)$ by, for any $v \in H^3(K)$,

\begin{align}
(\Pi_K v)(P) = v(P) \quad \text{and} \quad \nabla(\Pi_K v)(P) = \nabla v(P)
\end{align}

for any vertex $P$ of $K$. The interpolation $\Pi_K$ has the following estimates:

\begin{align}
|v - \Pi_K v|_{H^\ell(K)} \leq Ch^{4-\ell}|v|_{H^4(K)}, \quad \ell = 1, 2, 3, 4,
\end{align}

provided that $v \in H^4(K)$. Herein and throughout this paper, $C$ denotes a generic positive constant which is independent of the meshsize and may be different at different places. Then the global version $\Pi_h$ of the interpolation operator $\Pi_K$ is defined as

\begin{align}
\Pi_h|_K = \Pi_K \text{ for any } K \in \mathcal{T}_h.
\end{align}

3. A LOWER BOUND OF THE $L^2$ NORM ERROR ESTIMATE

This section proves the main result of this paper, namely, a lower bound of the $L^2$ norm error estimate. The main ingredients are a lower bound of $a_h(w - \Pi_h w, \Pi_h w)$ in Lemma 4.2 and an identity for $(-f, w - w_h)_{L^2(\Omega)}$ in Lemma 5.1.

For the analysis, we list two results from [8] and [9, 10].
Lemma 3.1. Let \( w \in H^2_0(\Omega) \cap H^4(\Omega) \) be the solution of problem (2.1). It holds that
\[
|a_h(w, v_h) - (f, v_h)| \leq Ch^2|w|_{H^4(\Omega)}\|\nabla^2_h v_h\|_{L^2(\Omega)} \text{ for any } v_h \in W_h.
\]

Lemma 3.2. Let \( w \) and \( w_h \) be solutions of problems (2.1) and (2.4), respectively. Suppose that \( w \in H^2_0(\Omega) \cap H^4(\Omega) \). Then,
\[
\|\nabla^2_h (w - w_h)\|_{L^2(\Omega)} \leq Ch^2|w|_{H^4(\Omega)}.
\]

Theorem 3.3. Let \( w \in H^2_0(\Omega) \cap H^4(\Omega) \) and \( w_h \) be solutions of problems (2.1) and (2.4), respectively. Then, there exists a positive constant \( \alpha \) which is independent of the meshsize such that
\[
\alpha h^2 \leq \|w - w_h\|_{L^2(\Omega)},
\]
provided that \( \|f\|_{L^2(\Omega)} \neq 0 \) and that the meshsize is small enough.

Proof. The main ingredients for the proof are Lemma 5.1 and Lemma 4.2. Indeed, it follows from Lemma 5.1 that
\[
(-f, w - w_h)_{L^2(\Omega)} = a_h(w, \Pi_h w - w_h) - (f, \Pi_h w - w_h)_{L^2(\Omega)}
\]
\[
+ a_h(w - \Pi_h w, w - \Pi_h w) + a_h(w - \Pi_h w, w - \Pi_h w) + 2(f, \Pi_h w - w)_{L^2(\Omega)} + 2a_h(w - \Pi_h w, \Pi_h w).
\]
The first two terms on the right-hand side of (3.4) can be bounded by Lemmas 3.1 and 3.2, and the estimates of (2.6), which leads to
\[
|a_h(w, \Pi_h w - w_h) - (f, \Pi_h w - w_h)_{L^2(\Omega)}| \leq Ch^2|w|_{H^4(\Omega)}\|\nabla^2_h (\Pi_h w - w_h)\|_{L^2(\Omega)}
\]
\[
\leq Ch^2(\|\nabla^2_h (\Pi_h w - w)\|_{L^2(\Omega)} + \|\nabla^2_h (w - w_h)\|_{L^2(\Omega)})
\]
\[
\leq Ch^4|w|_{H^4(\Omega)}^2.
\]
The estimates of the third and fifth terms on the right-hand side of (3.4) follow immediately from (2.6), which gives
\[
|a_h(w - \Pi_h w, w - \Pi_h w) + 2(f, \Pi_h w - w)_{L^2(\Omega)}| \leq Ch^4(\|w\|_{H^4(\Omega)} + \|f\|_{L^2(\Omega)})\|w\|_{H^4(\Omega)}.
\]
From the Cauchy-Schwarz inequality, the triangle inequality, Lemma 3.2, and (2.6) it follows that
\[
|a_h(w - \Pi_h w, w_h - \Pi_h w)| \leq Ch^4|w|_{H^4(\Omega)}^2.
\]
The last term on the right hand-side of (3.4) has already been analyzed in Lemma 4.2 which reads
\[
\beta h^2 \leq (\nabla^2_h (w - \Pi_h w), \nabla^2_h \Pi_h w)_{L^2(\Omega)}.
\]
for some positive constant $\beta$. A combination of these estimates states
\[ \delta h^2 \leq \langle -f, w - w_h \rangle_{L^2(\Omega)}, \]
for some positive constant $\delta$ which is independent of the meshsize provided that the meshsize is small enough. This plus the definition of the $L^2$ norm of $w - w_h$ prove
\[ \| w - w_h \|_{L^2(\Omega)} = \sup_{0 \neq d \in L^2(\Omega)} \frac{\langle d, w - w_h \rangle_{L^2(\Omega)}}{\|d\|_{L^2(\Omega)}} \geq \frac{\langle -f, w - w_h \rangle_{L^2(\Omega)}}{\| -f \|_{L^2(\Omega)}} \geq \frac{\delta}{\| f \|_{L^2(\Omega)}} h^2. \]
Setting $\alpha = \delta/\| f \|_{L^2(\Omega)}$ completes the proof. \hfill \Box

**Remark 3.4.** By the Poincare inequality, it follows that
\[ \alpha h^2 \leq \| \nabla (w - w_h) \|_{L^2(\Omega)}. \]

## 4. A Refined Property of the Interpolation Operator $\Pi_h$

This section establishes a lower bound of $a_h(w - \Pi_h w, \Pi_h w)$. To this end, given any element $K$, we follow [5, 7] to define $P_K v \in P_4(K)$ by
\begin{equation}
\int_K \nabla^\ell P_K v dx dy = \int_K \nabla^\ell v dx dy, \quad \ell = 0, 1, 2, 3, 4,
\end{equation}
for any $v \in H^4(K)$. Here and throughout this paper, $\nabla^\ell v$ denotes the $\ell$-th order tensor of all $\ell$-th order derivatives of $v$, for instance, $\ell = 1$ the gradient, and $\ell = 2$ the Hessian matrix, and that $\nabla^\ell_h$ are the piecewise counterparts of $\nabla^\ell$ defined element by element. Note that the operator $P_K$ is well-posed. It follows from the definition of $P_K$ in (4.1) that
\begin{equation}
\nabla^4 P_K v = \Pi_0,K \nabla^4 v
\end{equation}
with $\Pi_0,K$ the $L^2$ constant projection operator over $K$. Then the global version $\Pi_0$ of the interpolation operator $\Pi_0,K$ is defined as
\begin{equation}
\Pi_0|_K = \Pi_0,K \text{ for any } K \in \mathcal{T}_h.
\end{equation}

**Lemma 4.1.** For any $u \in P_4(K)$ and $v \in Q_{Ad}(K)$, there holds that
\begin{equation}
\langle \nabla^2(u - \Pi_K u), \nabla^2 v \rangle_{L^2(K)} = -\frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 u}{\partial x^2 \partial y^2} \frac{\partial^2 v}{\partial x^2} dx dy - \frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 u}{\partial x^2 \partial y^2} \frac{\partial^2 v}{\partial y^2} dx dy.
\end{equation}
Proof. Let $\xi$ and $\eta$ be defined as in (2.2). It follows from the definition of $Q_{Ad}(K)$ that
\[
\frac{\partial^2 v}{\partial x^2} = a_0 + a_1 \xi + a_2 \eta + a_3 \xi \eta,
\]
\[
\frac{\partial^2 v}{\partial y^2} = b_0 + b_1 \xi + b_2 \eta + b_3 \xi \eta,
\]
\[
\frac{\partial^2 v}{\partial x \partial y} = c_0 + c_1 \xi + c_2 \eta + c_3 \xi^2 + c_4 \eta^2,
\]
for some interpolation parameters $a_i, b_i, i = 0, \ldots, 3,$ and $c_i, i = 0, \ldots, 4.$ Since $u \in P_4(K),$ we have
\[
u = u_1 + \frac{h_{x,K}^4}{4!} \frac{\partial^4 u}{\partial x^4} \xi^4 + \frac{h_{y,K}^4}{4!} \frac{\partial^4 u}{\partial y^4} \eta^4 + \frac{h_{x,K}^4}{4} \frac{\partial^4 u}{\partial x \partial y^2} \xi^2 \eta^2,
\]
where $u_1 \in Q_{Ad}(K).$ Note that $\Pi_K u_1 = u_1,$ and
\[
\Pi_K \xi^4 = 2\xi^2 - 1, \quad \Pi_K \eta^4 = 2\eta^2 - 1, \quad \text{and} \quad \Pi_K \xi^2 \eta^2 = \xi^2 + \eta^2 - 1.
\]
This implies
\[
u - \Pi_K \nu = \frac{h_{x,K}^4}{4!} \frac{\partial^4 u}{\partial x^4} (\xi^2 - 1)^2 + \frac{h_{y,K}^4}{4!} \frac{\partial^4 u}{\partial y^4} (\eta^2 - 1)^2
\]
\[
+ \frac{h_{x,K}^4 h_{y,K}^4}{4} \frac{\partial^4 u}{\partial x^2 \partial y^2} (\xi^2 - 1)(\eta^2 - 1).
\]
Therefore
\[
\frac{\partial^2 (\nu - \Pi_K \nu)}{\partial x^2} = \frac{h_{x,K}^4}{4!} \frac{\partial^4 u}{\partial x^4} (12\xi^2 - 4) + \frac{h_{y,K}^4}{4} \frac{\partial^4 u}{\partial x \partial y^2} (\eta^2 - 1),
\]
\[
\frac{\partial^2 (\nu - \Pi_K \nu)}{\partial y^2} = \frac{h_{y,K}^4}{4!} \frac{\partial^4 u}{\partial y^4} (12\eta^2 - 4) + \frac{h_{x,K}^4}{4} \frac{\partial^4 u}{\partial x^2 \partial y^2} (\xi^2 - 1),
\]
\[
\frac{\partial^2 (\nu - \Pi_K \nu)}{\partial x \partial y} = h_{x,K} h_{y,K} \frac{\partial^4 u}{\partial x \partial y^2} \xi \eta.
\]
Since
\[
\int_K (12\xi^2 - 4)(a_0 + a_1 \xi + a_2 \eta + a_3 \xi \eta)dxdy = 0,
\]
and
\[
\int_K (\eta^2 - 1)(a_1 \xi + a_2 \eta + a_3 \xi \eta)dxdy = 0,
\]
a combination of (4.5) and (4.6) plus some elementary calculation yield
\[
\int_K \frac{\partial^2 (\nu - \Pi_K \nu)}{\partial x^2} \frac{\partial^2 v}{\partial x^2} dxdy = -\frac{h_{y,K}^4}{3} \int_K \frac{\partial^4 u}{\partial x^4} \frac{\partial^2 v}{\partial x^2} dxdy.
\]
A similar argument proves

\[
\begin{align*}
&\int_K \frac{\partial^2 (u - \Pi_K u)}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \, dx \, dy = -\frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 u}{\partial x^2 \partial y^2} \frac{\partial^2 v}{\partial y^2} \, dx \, dy, \\
&\int_K \frac{\partial^2 (u - \Pi_K u)}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \, dx \, dy = 0,
\end{align*}
\]

(4.8)

which completes the proof. \(\square\)

The above lemma can be used to prove the following crucial lower bound.

**Lemma 4.2.** Suppose that \(w \in H^2_0(\Omega) \cap H^4(\Omega)\) be the solution of Problem (2.1). Then,

\[
\beta h^2 \leq (\nabla^2_h (w - \Pi_h w), \nabla^2_h \Pi_h w)_{L^2(\Omega)},
\]

(4.9)

for some positive constant \(\beta\) which is independent of the meshsize \(h\) provided that \(\|f\|_{L^2(\Omega)} \neq 0\) and that the meshsize is small enough.

**Proof.** Given \(K \in T_h\), let the interpolation operator \(P_K\) be defined as in (4.1), which leads to the following decomposition

\[
\begin{align*}
&\quad (\nabla^2_h (w - \Pi_h w), \nabla^2_h \Pi_h w)_{L^2(\Omega)} \\
&= \sum_{K \in T_h} (\nabla^2(P_K w - \Pi_K P_K w), \nabla^2 P_K w)_{L^2(K)} \\
&\quad + \sum_{K \in T_h} (\nabla^2(I - \Pi_K)(I - P_K)w, \nabla^2 \Pi_K w)_{L^2(K)} \\
&= I_1 + I_2.
\end{align*}
\]

(4.10)

Let \(u = P_K w\) and \(v = \Pi_k w\) in Lemma 4.1. The first term \(I_1\) on the right-hand side of (4.10) reads

\[
I_1 = -\sum_{K \in T_h} \frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 P_K w}{\partial x^2 \partial y^2} \frac{\partial^2 \Pi_K w}{\partial y^2} \, dx \, dy \\
- \sum_{K \in T_h} \frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 P_K w}{\partial x^2 \partial y^2} \frac{\partial^2 \Pi_K w}{\partial x^2} \, dx \, dy,
\]
which can be rewritten as

\[
I_1 = - \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial x^2} dxdy - \sum_{K \in \mathcal{T}_h} \frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial y^2} dxdy
\]

\[
+ \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^4 (I - P_K)w}{\partial x^2 \partial y^2} \left( \frac{h_{y,K}^2}{3} \frac{\partial^2 \Pi_K w}{\partial x^2} + \frac{h_{x,K}^2}{3} \frac{\partial^2 \Pi_K w}{\partial y^2} \right) dxdy
\]

\[
+ \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \left( \frac{h_{x,K}^2}{3} \frac{\partial^2 (I - \Pi_K)w}{\partial x^2} + \frac{h_{y,K}^2}{3} \frac{\partial^2 (I - \Pi_K)w}{\partial y^2} \right) dxdy.
\]

By the commuting property of \((4.2)\),

\[
\frac{\partial^4 (I - P_K)w}{\partial x^2 \partial y^2} = (I - \Pi_{0,K}) \frac{\partial^4 w}{\partial x^2 \partial y^2}.
\]

Note that

\[
\|
\frac{\partial^2 \Pi_K w}{\partial y^2}\|_{L^2(K)} + \|
\frac{\partial^2 \Pi_K w}{\partial x^2}\|_{L^2(K)} \leq C \|w\|_{H^3(K)}.
\]

This, the error estimates of \((2.6)\), yield

\[
I_1 = - \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial x^2} dxdy
\]

\[
- \sum_{K \in \mathcal{T}_h} \frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial y^2} dxdy
\]

\[
+ O(h^2) \|(I - \Pi_0)\nabla^4 w\|_{L^2(\Omega)} \|w\|_{H^3(\Omega)}.
\]

Since the mesh is uniform, an elementwise integration by parts yields

\[
- \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial x^2} dxdy
\]

\[
= \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \left( \frac{\partial^3 w}{\partial x^2 \partial y} \right)^2 dxdy - h_y^2 \int_{\Gamma_y} \frac{\partial^3 w}{\partial x^2 \partial y} \frac{\partial^2 w}{\partial x \partial \nu_2} d\Gamma,
\]

where \(\Gamma_y\) is the boundary of \(\Omega\) that parallels to the x-axis, and \(\nu_2\) is the second component of the unit normal vector \(\nu = (\nu_1, \nu_2)\) of the boundary. Since \(\frac{\partial w}{\partial y} = 0\) on \(\Gamma_y\), \(\frac{\partial^3 w}{\partial x^2 \partial y} = 0\) on \(\Gamma_y\). Hence,

\[
- \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial x^2} dxdy = \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \int_K \left( \frac{\partial^3 w}{\partial x^2 \partial y} \right)^2 dxdy
\]
A similar procedure shows

$$- \sum_{K \in \mathcal{T}_h} \frac{h_{x,K}^2}{3} \int_K \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial y^2} \, dx dy = \sum_{K \in \mathcal{T}_h} \frac{h_{x,K}^2}{3} \int_K \left( \frac{\partial^3 w}{\partial x \partial y^2} \right) \, dx dy.$$ 

Therefore

$$I_1 = \sum_{K \in \mathcal{T}_h} \frac{h_{y,K}^2}{3} \left\| \frac{\partial^3 w}{\partial x^2 \partial y} \right\|^2_{L^2(K)} + \sum_{K \in \mathcal{T}_h} \frac{h_{x,K}^2}{3} \left\| \frac{\partial^3 w}{\partial x \partial y^2} \right\|^2_{L^2(\Omega)}$$

$$+ \mathcal{O}(h^2) \left\| (I - \Pi_0) \nabla^4 w \right\|_{L^2(\Omega)} \left\| w \right\|_{H^3(\Omega)}.$$

The second term $I_2$ on the right-hand side of (4.10) can be estimated by the error estimates of (2.6) and the commuting property of (4.2), which reads

$$|I_2| \leq C h^2 \left\| (I - \Pi_0) \nabla^4 w \right\|_{L^2(\Omega)} \left\| w \right\|_{H^3(\Omega)}.$$

Since the piecewise constant functions are dense in the space $L^2(\Omega)$, 

$$\left\| (I - \Pi_0) \nabla^4 w \right\|_{L^2(\Omega)} \to 0 \text{ when } h \to 0.$$ 

Since $\left\| f \right\|_{L^2(\Omega)} \neq 0$ implies that $\left\| \frac{\partial^2 w}{\partial x \partial y} \right\|_{H^1(\Omega)} \neq 0$ (see more details in the following remark), a combination of (4.10)-(4.12) proves the desired result.

**Remark 4.3.** For the rectangular domain $\Omega$ under consideration, the condition $\left\| \frac{\partial^2 w}{\partial x \partial y} \right\|_{H^1(\Omega)} \neq 0$ holds provided that $\left\| f \right\|_{L^2(\Omega)} \neq 0$. In fact, if $\left\| \frac{\partial^2 w}{\partial x \partial y} \right\|_{H^1(\Omega)} = 0$, $w$ is of the form

$$w = c_0 xy + h(x) + g(y),$$

for some function $h(x)$ with respect to $x$, and $g(y)$ with respect to $y$. Then, the boundary condition concludes that both $h(x)$ and $g(y)$ are constant. Hence the boundary condition indicates $w \equiv 0$, which contradicts with $w \neq 0$.

**Remark 4.4.** The expansion (4.4) was analyzed in [4, 9, 19]. Herein we give a new and much simpler proof. Moreover, compared with the regularity $H^5$ needed therein, the analysis herein only needs the regularity $H^4$.

**Remark 4.5.** The idea herein can be directly extended to the eigenvalue problem investigated in [4, 19], which improves and simplifies the analysis therein and proves that the discrete eigenvalue produced by the Adini element is smaller than the exact one provided that the mesh-size is sufficiently small. In addition, such a generalization weakens the regularity condition from $u \in H^5(\Omega)$ to $u \in H^4(\Omega)$ where $u$ is the eigenfunction.
5. An identity of \((-f, w - w_h)\)

This section establishes the identity of \((-f, w - w_h)\) which is one man ingredient for the proof of Theorem 3.3.

**Lemma 5.1.** Let \(w\) and \(w_h\) be solutions of problems (2.1) and (2.4), respectively. Then,

\[
(-f, w - w_h)_{L^2(\Omega)} = a_h(w, \Pi_h w - w_h) - (f, \Pi_h w - w_h)_{L^2(\Omega)}
\]

\[+ a_h(w - \Pi_h w, w - \Pi_h w) + a_h(w - \Pi_h w, w_h - \Pi_h w)
\]

\[+ 2(f, \Pi_h w - w)_{L^2(\Omega)} + 2a_h(w - \Pi_h w, \Pi_h w).
\]

**(5.1)**

**Proof.** We start with the following decomposition

\[
(-f, w - w_h)_{L^2(\Omega)} = \sum (f, w - w_h) + a_h(w, w - w_h) - a_h(w, w - w_h)
\]

\[= (-f, w - \Pi_h w)_{L^2(\Omega)} + (-f, \Pi_h w - w_h)_{L^2(\Omega)}
\]

\[+ a_h(w - \Pi_h w, w - \Pi_h w) + a_h(w, w - \Pi_h w) - a_h(w, w - w_h).
\]

**(5.2)**

The last two terms on the right-hand side of (5.2) allow for a further decomposition:

\[
a_h(w - \Pi_h w, w - \Pi_h w) - a_h(w, w - w_h)
\]

\[= a_h(w - \Pi_h w, w - w_h) + a_h(\Pi_h w, w - \Pi_h w)
\]

\[+ a_h(w - \Pi_h w, w - w_h) - a_h(\Pi_h w, w - w_h)
\]

\[= a_h(w - \Pi_h w, w - \Pi_h w) + a_h(\Pi_h w, w - \Pi_h w)
\]

\[+ a_h(\Pi_h w, w - w_h).
\]

**(5.3)**

It follows from the discrete problem (2.4) and the continuous problem (2.1) that the last term on the right-hand side of (5.3) can be divided as

\[
- a_h(\Pi_h w, w - w_h)
\]

\[= (f, \Pi_h w - w)_{L^2(\Omega)} - a_h(w, \Pi_h w - w)
\]

\[= (f, \Pi_h w - w)_{L^2(\Omega)} - a_h(w - \Pi_h w, \Pi_h w - w)
\]

\[+ a_h(\Pi_h w, \Pi_h w - w).
\]

**(5.4)**

A summary of (5.2)-(5.4) completes the proof. \(\square\)

**Remark 5.2.** The importance of the identity of (5.1) lies in that such a decomposition separates the dominant term \(2a_h(w - \Pi_h w, \Pi_h w)\) from the other higher order terms, which is the key to employ Lemma 4.2.
6. The conclusion and comments

This paper presents the analysis of the $L^2$ norm error estimate of the Adini element. It is proved that the best $L^2$ norm error estimate is at most of order $O(h^2)$ which can not be improved in general. This result in fact indicates that the nonconforming Adini element space can not contain any conforming space with an appropriate approximate property. This will cause further difficulty for the a posteriori error analysis. In fact, the reliable and efficient a posteriori error estimate for this element is still missing in the literature, see [2] for more details.

References

[1] S. C. Brenner, L. R. Scott, The mathematical theory of finite element methods, Springer-Verlag, 1996.
[2] C. Carstensen, D. Gallistl, and J. Hu, A posteriori error estimates for nonconforming finite element methods for fourth-order problems on rectangles, Numer. Math. DOI 10.1007/s00211-012-0513-5.
[3] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, 1978; reprinted as SIAM Classics in Applied Mathematics, 2002.
[4] J. Hu and Y. Q. Huang, The correction operator for the canonical interpolation operator of the Adini element and the lower bounds of eigenvalues, Science China Mathematics, 55(2012), pp. 187–196.
[5] J. Hu, Y. Q. Huang, and Q. Lin, The lower bounds for eigenvalues of elliptic operators by nonconforming finite element methods, [arXiv:1112.1145v1[math.NA]] 6 Dec 2011.
[6] J. Hu, Y. Q. Huang and S. Y. Zhang, The lowest order differentiable finite element on rectangular grids, SIAM J. Numer. Anal., 49(2011), pp. 1350–1368.
[7] J. Hu and Z. C. Shi, The best $L^2$ norm error estimate of the lower order finite element methods for the fourth order problem, J. Comput. Math., 30(2012), pp. 449–460.
[8] P. Lascaux and P. Lesaint, Some nonconforming finite elements for the plate bending problem, RAIRO Anal. Numer., 9 (1975), pp. 9–53.
[9] P. Luo and Q. Lin, High accuracy analysis of the Adini’s nonconforming element, Computing, 68(2002), pp. 65–79.
[10] S. P. Mao and S. C. Chen, Accuracy analysis of Adinis non-conforming plate element on anisotropic meshes, Commun. Numer. Meth. Engng., 22(2006), pp. 433–440.
[11] L. S. D. Morley, The triangular equilibrium element in the solutions of plate bending problem, Aero. Quart., 19(1968), pp. 149–169.
[12] M. J. D. Powell and M. A. Sabin, Piecewise quadratic approximations on triangles, ACM Transactions on Mathematical Software, 3-4 (1977), pp. 316–325.
[13] Z. C. Shi, On the convergence of the incomplete biquadratic nonconforming plate element, Math. Numer. Sinica, 8(1986), pp. 53–62.
[14] Z. C. Shi and M. Wang, The finite element method (in Chinese), Science Press, Beijing, 2010.
[15] M. Wang, Z. C. Shi and J. C. Xu, Some n-rectangle nonconforming elements for fourth order elliptic equations, J. Comp. Math. 25(2007), pp. 408–420.
[16] M. Wang and J. C. Xu, The Morley element for fourth order elliptic equations in any dimensions, Numer. Math., 103(2006), pp. 155–169.
[17] M. Wang and J. C. Xu, Minimal finite element spaces for 2m-th order partial differential equations in $\mathbb{R}^n$, Math. Comp., 82(2012), pp. 25–43.
[18] M. Q. Wu, The incomplete biquadratic nonconforming plate element (in Chinese), Journal of Suzhou University, 1(1983), pp. 20–29.
[19] Y. D. Yang, A posteriori error estimates in Adini finite element for eigenvalue problems, J. Comp. Math., 18(2000), pp. 413–418.

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