STRONG APPROXIMATION FOR THE SUPERMARKET MODEL

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We prove three strong approximation theorems for the “supermarket” or “join the shortest queue” model—a law of large numbers, a jump process approximation and a central limit theorem. The estimates are carried through rather explicitly, and rely in part on couplings. This allows us to approximate each of the infinitely many components of the process in its own scale and to exhibit a cut-off in the set of active components which grows slowly with the number of servers.

1. Introduction. The supermarket model is a system of $N$ single-server queues. Customers arrive as a Poisson process of rate $N\lambda$, where $\lambda \in (0, 1)$. Each customer examines $d$ queues, chosen randomly from all $N$ queues, where $d \geq 2$, and joins the shortest of these $d$ queues, choosing randomly if the shortest queue is not unique. The service times of all customers are independent rate 1 exponential random variables. We will be concerned with the behavior of this model when $\lambda$ and $d$ are fixed, over a finite time interval $[0, t_0]$, as $N \to \infty$. We shall consider the case when the system starts in some well-behaved state with low server loads (in a sense to be made precise below).

This model has attracted attention because it turns out that the choice offered to customers, even if $d = 2$, dramatically reduces queue lengths (see [5, 14, 16]) and, in particular, the length of the longest queue (see [10, 11]). Given that our analysis relies on $N$ being very much larger than $d$, the model does not describe well the behavior of a real supermarket. Rather it serves as an example where a simple dynamic routing rule leads to a greatly improved performance, which is of interest in the context of communications networks.

Our results provide strong approximations for the supermarket model, and include a law of large numbers and a diffusion approximation. In arriving at these results, we have developed techniques to establish weak convergence of a sequence of Markov processes $X^N$ in infinitely many dimensions, where the jumps of $X^N$ are of order $N^{-1}$ and occur at a rate of order $N$. The classical results for fluid limits are set in a finite-dimensional context. We make essential use of the fact that the number of “active” components in $X^N$ grows only very slowly with $N$. We have used direct and quantitative methods based on exponential
martingales and strong approximation of Poisson processes by Brownian motion. These methods seem well-suited to deal with such “almost finite-dimensional” Markov processes. Earlier results for this model include laws of large numbers in [5, 6, 14, 16], quantitative concentration of measure estimates in [10] and a central limit theorem [7]. See also [9] for a preliminary version of the law of large numbers presented in the present paper (for a more general range of initial conditions).

The limiting behavior of the supermarket model as $N \to \infty$ may conveniently be described in terms of the vector $X_t = (X^k_t : k \in \mathbb{N})$, where $X^k_t$ denotes the proportion of all $N$ queues having at least $k$ customers at time $t$. The process $X = (X_t)_{t \geq 0}$ is a Markov chain. We will suppose throughout that $X_0 = x_0$ with $x_0$ nonrandom and we will suppress the dependence of $X$ on $N$ to lighten the notation. Now $X$ has the form of a density dependent Markov chain such as considered by Ethier and Kurtz in [4], Chapter 11. Thus, one might expect to be able to find a deterministic process $(x_t)_{t \geq 0}$ and a Gaussian process $(\gamma_t)_{t \geq 0}$ such that

$$X_t = x_t + O(N^{-1/2}), \quad X_t = x_t + N^{-1/2} \gamma_t + O(\log N/N).$$

However, the number of nonzero components in $X$ grows with $N$ so the standard theory does not apply.

We will see that, for small initial data, the component $X^k$ has a scale $a_k = \lambda^{1+d+\cdots+d^{k-1}}$, which, of course, decays very rapidly with $k$. Thus, the number of queues having at least $k$ customers is of order $N a_k$. We can find $m$ of order $\log \log \log N$ such that $N a_m$ is of order 1. Thus, we can exhibit a cut-off in the number of active components which grows only slowly with $N$. Below the cut-off, for $k \leq m - 1$, we prove convergence with explicit control of error probabilities for each of the $\log \log N$ active components.

We thereby obtain results of the form

$$X^k_t = x^k_t + a_k O(\sqrt{\log \log N / N a_k}),$$

$$X^k_t = x^k_t + N^{-1/2} \gamma^k_t + a_k O(\log (N a_k) / N a_k).$$

Note that each component is estimated in the correct scale, with an error depending on the number of queues active at that level. The $\log \log N$ in the first equation is a (small) price we pay for working with infinitely many components. These asymptotics will be established with a degree of uniformity in $x_0$, which thus allows a dependence of $x_0$ on $N$. The Gaussian approximation relies, as in the finite-dimensional case, on a sophisticated coupling of the compensated Poisson process with Brownian motion due to Komlós, Major and Tusnády [8].

We will give a third result, for $k \leq m - 1$, of the form

$$X^k_t = x^k_t + N^{-1/2} \tilde{\gamma}^k_t + a_k O((\log \log N / N a_k)^{3/4}).$$

Here $(\tilde{\gamma}_t)_{t \geq 0}$ is a jump process with drift which depends on $N$ but is of a simpler type than $X$ in that it is a linear function of additive Poisson noise. The
characteristics of \( \tilde{\gamma} \) are derived in a simple and canonical way from those of \( X \). Moreover, \( \tilde{\gamma} \) and \( X \) share a common filtration. The error term is larger than for the Gaussian approximation. On the other hand, the derivation is significantly simpler.

We obtain also the behavior of the queue sizes at and above the cut-off. We see a residual randomness in \( (X^m_t)_{t \geq 0} \) even for large values of \( N \). This may be approximated in terms of an \( M/M/\infty \) queue with arrival rate \( N\lambda (x^{m-1}_t)^d \) and service rate 1. Over a given finite time interval, there are no queues with lengths greater than \( m \).

Thus, we will show for the supermarket model that its infinite-dimensional character does not prevent the derivation of precise asymptotics. We expect the general approach taken here to adapt well to a number of further examples of similar character.

2. Statement of results. Let \( S_0 \) denote the set of nonincreasing sequences \( x = (x^k : k \in \mathbb{N}) \) in \([0, 1] \), where \( \mathbb{N} = \{1, 2, \ldots \} \). For \( x \in S_0 \), set \( x^0 = 1 \) and define \( \lambda^+_k(x) = \lambda((x^{k-1})^d - (x^k)^d), \lambda^-_k(x) = x^k - x^{k+1} \) and \( b_k(x) = \lambda^+_k(x) - \lambda^-_k(x) \). It is shown in [16] that, given \( x_0 \in S_0 \), there is a unique solution \( (x_t)_{t \geq 0} \) to \( x_t = b(x_t) \) in \( S_0 \). Moreover, for any other solution \( (y_t)_{t \geq 0} \) in \( S_0 \), \( x^k_t \leq y^k_t \) for all \( k \) and all \( t \geq 0 \).

Recall that \( a_k = \lambda^1 + d^2 + \ldots + d^k-1 \) for \( k \in \mathbb{N} \). Then \( a = (a_k : k \in \mathbb{N}) \) is the unique solution in \( S_0 \) to \( b(x) = 0 \) such that \( \lim_{k \to \infty} x^k = 0 \); to see this note that \( x^k_0 = \lambda(x^0)^d - \lambda(x^1)^d = a \) is independent of \( k \) for any solution \( x \).

Define \( \|x\| = \sup_k |x^k|/a_k \), set \( E = \{x \in \mathbb{R}^\mathbb{N} : \|x\| < \infty \} \) and set \( S = S_0 \cap E \). We shall work on a fixed interval \([0, t_0]\). For us, the good initial conditions will be those \( x_0 \in S \) for which \( \|x_0\| \leq \rho \) for all \( \rho \leq \infty \). Write \( S(\rho, t_0) \) for the set of such \( x_0 \). We note that, if \( x_0 \in S_0 \) with \( \|x_0\| \leq 1 \), then, by comparison with the stationary solution \( a, \|x_t\| \leq 1 \) for all \( t \), so \( x_0 \in S(1, t_0) \) for all \( t_0 \). More generally, if \( x_0 \in S \), then \( t \mapsto \|x_t\| \) is continuous (and finite) on some interval \([0, \zeta)\), so \( x_0 \in S(\rho, t_0) \) for some \( \rho < \infty \) for all \( t_0 < \zeta \). To see this, we extend \( \lambda \) and \( b \) to \( \mathbb{R}^\mathbb{N} \) by setting \( \lambda^\pm(x) = \lambda((y^{k-1})^d - (y^k)^d)^+, \lambda^\pm(x) = (y^k - y^{k+1})^+ \), where \( y^k = (x^k)^+ \); then \( b \) maps \( E \) to itself and is locally Lipschitz for the given norm [to prove this fact, carry out estimates similar to those in (5) and (6) below]; so if \( x_0 \in S \), then there is a local solution \( (x_t)_{t < \zeta} \) in \( S \), which must coincide with the global solution \( (x_t)_{t \geq 0} \) in \( S_0 \) for \( t < \zeta \).

The state-space \( I \) of the Markov chain \( X = (X_t)_{t \geq 0} \) is the set of nonincreasing sequences in \( N^{-1}[0, 1, \ldots, N] \) with finitely many nonzero terms. Thus, \( I \subseteq S \). The Lévy kernel for \( X \) is given by

\[
K(x, dy) = \sum_{k=1}^{\infty} \left[ N\lambda^+_k(x)\delta_{ek/N}(dy) + N\lambda^-_k(x)\delta_{-ek/N}(dy) \right],
\]
where $e_k$ denotes the $k$th standard basis vector. Given $m \in \mathbb{N}$, let $(\hat{X}_t^m)_{t \leq t_0}$ be a process starting from $x_0^m$ and such that $N\hat{X}_t^m$ is an $M/M/\infty$ queue, with arrival rate $N\lambda(x_0^{m-1})d$ and service rate 1. We can now state our law of large numbers.

**Theorem 2.1.** Set $m = m(N) = \inf\{k \in \mathbb{N} : Na_k \leq (\log N)^4\}$. There is a coupling of $\hat{X}_t^m$ and $X_t^m$ such that, for all $\rho \geq 1$, $t_0 > 0$ and all sequences $R(N)$ with $R(N)/\sqrt{\log \log \log N} \to \infty$, we have

\[
\sup_{x_0 \in I \cap S(\rho, t_0)} P_{x_0}(N^{3/4} | X_t^m - x_t^m | > R(N) \sqrt{a_k} \text{ for some } k \leq m - 1 \quad \text{or} \quad X_t^m \neq \hat{X}_t^m \quad \text{or} \quad X_t^{m+1} \neq 0 \text{ for some } t \leq t_0) \to 0.
\]

In particular,

\[
\sup_{k \leq m-1} \sup_{t \leq t_0} |X_t^m - x_t^m|/a_k \to 0
\]

in probability, uniformly in $x_0 \in I \cap S(\rho, t_0)$.

We know (see [16]) that if $\rho > 0$ and $\|x_0\| \leq \rho$, then $x_t^k \to a_k$ as $t \to \infty$. Thus, for $k \leq m - 1$, the proportional error in approximating $X_t^k$ by the deterministic process $x_t^k$ is small for large values of $N$.

The central limit theorem shows generically that the power $\sqrt{N}$ in Theorem 2.1 cannot be improved while the approximating process $(x_t)_{t \geq 0}$ remains deterministic. Our next result is a refined approximation which allows an improvement to $N^{3/4}$. Let $\tilde{\mu}$ be a Poisson random measure on $\mathbb{R}^N \times (0, t_0]$ with intensity $\tilde{\nu}(dy, dt) = K(x_t, dy)dt$.

For any $y$,

\[
\nabla b^k(x)y = \lambda d(x^{k-1})^{d-1}y^{k-1} - \lambda d(x^k)^{d-1}y^k - y^k + y^{k+1}.
\]

We show in Section 6 that the linear equations

\[
\tilde{\gamma}_t^k = \sqrt{N} \int_{\mathbb{R}^N \times (0, t]} y^k(\tilde{\mu} - \tilde{\nu})(dy, ds) + \int_0^t \nabla b^k(x_s)\tilde{\gamma}_s ds
\]

have a unique cadlag solution $(\tilde{\gamma}_t^k : k \in \mathbb{N}, t \leq t_0)$. Set $\tilde{X}_t = x_t + N^{-1/2}\tilde{\gamma}_t$.

**Theorem 2.2.** Define $m(N)$ as in Theorem 2.1. There is a coupling of $\tilde{X}$ and $X$, in a common filtration, such that, for all $\rho \geq 1$, $t_0 > 0$ and all sequences $\tilde{R}(N)$ with $\tilde{R}(N)/(\log \log \log N)^{3/4} \to \infty$, we have

\[
\sup_{x_0 \in I \cap S(\rho, t_0)} P_{x_0}(N^{3/4} | X_t^k - \tilde{X}_t^k | > \tilde{R}(N) a_k^{1/4} \text{ for some } k \leq m - 1, t \leq t_0) \to 0.
\]
The final result is a diffusion approximation. Let $B^+_k, B^-_k, k \in \mathbb{N}$, be independent standard Brownian motions. Set $\sigma^+_k(x) = \sqrt{\lambda^+_k(x)}$. We show in Section 6 that the linear equations

$$
\gamma^+_k(t) = \int_0^t \sigma^+_k(x_s) d B^+_k(s) - \int_0^t \sigma^-_k(x_s) d B^-_k(s) + \int_0^t \nabla b_k(x_s) \gamma_s ds,
$$

have a unique solution $(\gamma^+_k : k \in \mathbb{N}, t \leq t_0)$ with

$$
\sup_{k \in \mathbb{N}} \mathbb{E} \left( \sup_{t \leq t_0} |\gamma^+_k|^2 \right) < \infty.
$$

Set $\bar{X}_t = x_t + N^{-1/2} \gamma_t$.

**Theorem 2.3.** Define $m(N)$ as in Theorem 2.1. There is a coupling of $\bar{X}$ and $X$ such that, for all $\rho \geq 1$, $t_0 > 0$, there is a constant $\tilde{R}$, independent of $N$, such that

$$
\sup_{x_0 \in I \cap S(\rho, t_0)} \mathbb{P}_{x_0} (N|X^k_t - \bar{X}^k_t| > \tilde{R} \log(Na_k) \text{ for some } k \leq m - 1, t \leq t_0) \to 0.
$$

We remark that there are alternative versions of all three theorems in which $x_0$ is replaced by $x_0^{(m)} = (x_0^1, \ldots, x_0^{m-1}, 0, \ldots)$ and $\lambda_\pm(x)$ is replaced by $\lambda_\pm(x)^{(m)}$, so that the approximating deterministic dynamics are $(m-1)$-dimensional. The proofs are a minor modification of the proofs given below. These alternative versions would have merit in any computational implementation of the approximations since $m$ is of order only $\log \log N$.

In comparison with previous results, our theorems for the first time approximate each component of the infinite dimensional process in its own scale, while at the same time providing explicit rates of convergence. In particular, Theorem 2.1 strengthens the law of large numbers in [5, 6, 16], and Theorem 2.3 strengthens the central limit theorem in [7]. Unlike the techniques developed in [10], ours apply only on finite time intervals and do not extend to the equilibrium distribution. On the other hand, the estimates in [10] do not distinguish between the magnitudes of different components of the process.

**3. Law of large numbers.** In the first half of this section, we fix $N$ and $A, R \geq 1$ and set $m = \inf \{k \in \mathbb{N} : N a_k \leq A \}$. We will obtain, subject to certain constraints, a global estimate on the probability appearing in Theorem 2.1. In the second half we will show that this estimate becomes small as $N \to \infty$ when $A = (\log N)^4$ and when $R$ is chosen as in Theorem 2.1.

The integer $m$, which will turn out to give the maximum queue length, is of order $\log \log N / \log d$ for the values of $A$ we shall consider. To see this, fix $\rho \geq 1$ and
so that $m \in$ a Poisson random measure $\lambda d_{\text{intensity}}$. We start from $0$, starting from $0$, and assume that $\lambda 0 = 0$ and $\lambda > 0$. Consider the case $x_0 \in S(\rho, t_0)$. Then $N\lambda_m^{m+1} = N\lambda_{m+1}^{m} < N\lambda_m^{d^m} = \rho A^d / N^{d-1} \leq 1$, so $x_0^{m+1} = 0$. Set $T_1 = \inf\{t \geq 0 : X_t^{m+1} \neq 0\}$. Note that, while $X_t^{m+1} = 0$, $X_t^{m+1}$ increases at rate $N\lambda(X_t^{m-1})^d$, whereas $X_t^m + X_t^{m+1}$ increases at rate $N\lambda(X_t^{m-1})^d$ and decreases at rate $N_X^{m+1}$. We can therefore find an $M/M/\infty$ queue $(Q_t)_{t \geq 0}$, starting from $N\lambda_0^m$, without arrivals and with service rate $1$, and a Poisson random measure $\mu(dt, dx, du)$ on $(0, \infty)^3$, independent of $Q$ and of intensity $e^{-u} dt dx du$, such that, for $t \leq T_1$,

$$N_X^{m+1} = \mu \{ (s, x, u) : s \leq t < s + u, x \leq N\lambda(X_{s-}^{m})^d \}$$

and

$$N(X_t^m + X_t^{m+1}) = Q_t + \mu \{ (s, x, u) : s \leq t < s + u, x \leq N\lambda(X_{s-}^{m-1})^d \}.$$
Proof. It will suffice to show that $\mathbb{P}(T = T_i) \leq p_i$ for $i = 1, 2, 3, 4$. Recall that, for a Poisson random variable $Y$ of parameter $\nu > 0$ and for $a > 0$, we have $\mathbb{P}(Y \geq a) \leq \nu/a$. For $t < T$, we have $N\lambda(X_t^{m-1})^d \leq N\lambda(ra_m)^d \leq A^2 \nu^d/N^{d-1}$, so $X_t^{m+1}$ is dominated by a Poisson random variable $Y_1$ of parameter $p_1$, and so

$$\mathbb{P}(T = T_1) = \mathbb{P}(X_t^{m+1} = 1) \leq \mathbb{P}(Y_1 \geq 1) \leq p_1.$$ 

Since $x_0 \in S(\rho, t_0)$, we have $x_t^k \leq \rho a_k$ for all $k \in \mathbb{N}$ and all $t \leq t_0$. For $k \leq m - 1$, we have $R/\sqrt{a_k/N} \leq a_k R/\sqrt{N} \leq a_k R/\lambda$ and $a_k R / \lambda$. Hence, for $t < T$ and $k \leq m - 1$,

$$(3) \quad X_t^k \leq x_t^k + R/\sqrt{a_k/N} \leq \sigma a_k.$$ 

Then, for $t < T$,

$$N\lambda|(X_t^{m-1})^d - (x_t^{m-1})^d| \leq N\lambda d \sigma^{d-1} a_{m-1}^d |X_t^{m-1} - x_t^{m-1}|$$

$$\leq N\lambda d \sigma^{d-1} a_{m-1}^d R/(\sqrt{a_k N})$$

$$\leq A^{1-1/(2d)} d \sigma^{d-1} R/N^{(1/2)(1-1/d)},$$

where the final inequality follows from $a_m = \lambda a_{m-1}^d$. Set

$$\Delta = \mu(\{(t, x, u) : t \leq T, N\lambda(x_t^{m-1} \wedge X_t^{m-1})^d < x \leq N\lambda(x_t^{m-1} \vee X_t^{m-1})^d\}).$$

Then $\Delta$ is dominated by a Poisson random variable $Y_2$ of parameter $p_2$. Hence,

$$\mathbb{P}(T = T_2) = \mathbb{P}(\Delta = 1) \leq \mathbb{P}(Y_2 \geq 1) \leq p_2.$$ 

Note that $\lambda(x_t^{m-1})^d \leq \lambda \rho d a_{m-1}^d = \rho^d a_m$ for all $t \leq t_0$. Thus, $N\hat{X}_T^m \leq N x_0^m + Y_3$ for a Poisson random variable $Y_3$ of parameter $N\rho^d a_m t_0$ and so

$$\mathbb{P}(T = T_3) = \mathbb{P}(\hat{X}_T^m \geq r a_m) \leq \mathbb{P}(Y_3 \geq N(r - \rho) a_m) \leq p_3.$$ 

It remains to estimate $\mathbb{P}(T = T_4)$. For this, we write

$$(4) \quad X_t^k = x_0^k + M_t^k + \int_0^t b^k(X_s) \, ds$$

so that

$$X_t^k - x_t^k = M_t^k + \int_0^t (b^k(X_s) - b^k(x_s)) \, ds.$$ 

Then we use a combination of exponential martingale inequalities and Gronwall’s lemma to obtain the desired estimate. First we investigate how small $M$ will need to be to obtain the required bound on $|X_t^k - x_t^k|$ for $k \leq m - 1$. Note that

$$|\lambda_+^k(x) - \lambda_+^k(y)| \leq \lambda d(x^k \vee y^k)^d - 1 |x^k - y^k| + \lambda d(x^k \vee y^k)^{d-1} |x^k - y^k|$$

so, provided that $x^k \vee y^k \leq \sigma a_k$ and $x^k \vee y^k \leq \sigma a_{k-1}$,

$$(5) \quad |\lambda_+^k(x) - \lambda_+^k(y)| \leq d \sigma^{d-1} ((a_k/a_{k-1}) |x^k - y^k| + |x^k - y^k|).$$
Also,
\[
|\lambda_k(x) - \lambda_k(y)| \leq |x^k - y^k| + |x^{k+1} - y^{k+1}|.
\]
Hence, provided that \(x^k \vee y^k \leq \sigma a_k\) and \(x^{k-1} \vee y^{k-1} \leq \sigma a_{k-1}\), we have
\[
|b^k(x) - b^k(y)| / \sqrt{a_k} \leq L \sup_{j=k-1,k,k+1} |x^j - y^j| / \sqrt{a_j}.
\]
We note that the definitions of \(T_2\) and \(T_3\) force \(X_t^m \leq r a_m\) for all \(t < T\). Set
\[
f(t) = \sup_{k \leq m-1} \sup_{s \leq t} |X^k_s - x^k_s| / \sqrt{a_k}.
\]
Then, for \(t < T\) and \(k \leq m - 2\),
\[
|b^k(X_t) - b^k(x_t)| / \sqrt{a_k} \leq L f(t)
\]
and
\[
|b^{m-1}(X_t) - b^{m-1}(x_t)| / \sqrt{a_{m-1}} \leq L f(t) + r a_m / \sqrt{a_{m-1}}.
\]
Hence, for \(t \leq T\),
\[
f(t) \leq (M^*_t + r a_m t / \sqrt{a_{m-1}}) + L \int_0^t f(s) \, ds,
\]
where
\[
M^*_t = \sup_{k \leq m-1} \sup_{s \leq t} |M^k_s| / \sqrt{a_k}.
\]
Set \(\alpha_k = \frac{1}{2} e^{-L t_0} R \sqrt{a_k} / N\) and consider, for \(k \leq m - 1\), the stopping times \(T^k = T^k_- \land T^k_+\), where
\[
T^k_- = \inf\{t \geq 0 : \pm M^k_t > \alpha_k\}.
\]
Suppose that \(T < T^1_- \land \cdots \land T^m_-\). Then \(M^*_T \leq \frac{1}{2} e^{-L t_0} R / \sqrt{N}\). On the other hand,
\[
ra_m t_0 / \sqrt{a_{m-1}} = \lambda^{1/(2d)} ra_m^{1-1/(2d)} t_0 \leq r A^{1-1/(2d)} t_0 / N^{1-1/(2d)} \leq \frac{1}{2} e^{-L t_0} R / \sqrt{N}.
\]
So by Gronwall’s lemma,
\[
f(T) \leq e^{L t_0} (M^*_T + r a_m t_0 / \sqrt{a_{m-1}}) \leq R / \sqrt{N}.
\]
Hence,
\[
P(T = T_4) \leq \sum_{k=1}^{m-1} P(T^k \leq T)
\]
and it remains to estimate $\mathbb{P}(T^k_+ \leq T)$ for $k \leq m - 1$. For $k \in \mathbb{N}$, $x \in S$ and $\theta \in \mathbb{R}$, set
\[
\phi^k(x, \theta) = \lambda_+^k(x)h(\theta) + \lambda_-^k(x)h(-\theta),
\]
where $h(\theta) = e^\theta - 1 - \theta$. For $t < T$ and $k \leq m - 1$, we have $X^k_t \leq \sigma a_k$ so
\[
\phi^k(X^k_t, \theta) \leq \lambda \sigma d a_{k-1}^d h(\theta) + \sigma a_k h(-\theta) \leq \sigma^d a_k g(\theta),
\]
where $g(\theta) = e^\theta - 2 + e^{-\theta}$. Consider, for $\theta \geq 0$, the exponential martingale
\[
Z^k_{T^k_+} = \exp \left\{ N \theta (X^k_{T^k_+} - X^k_0) - \int_0^{T^k_+} \int_{\mathbb{R}} (e^{N \theta y^k} - 1) K(X_s, dy) ds \right\}
\]
and note that, on the event $T^k_+ \leq T$, we have
\[
Z^k_{T^k_+} \geq \exp \left\{ N \theta a_k - N \sigma^d a_k g(\theta) t_0 \right\}.
\]
By optional stopping, $\mathbb{E}(Z^k_{T^k_+}^N) \leq 1$, so
\[
\mathbb{P}(T^k_+ \leq T) \leq \exp(-N \alpha_k^2 - N \sigma^d a_k g(\theta) t_0).
\]
We choose $\alpha = \alpha_k/(2\sigma^d a_k \gamma t_0)$, where $\gamma = g(1) \leq \frac{5}{4}$. Using $R \leq t_0 \sqrt{A}$ it is straightforward to check that $\theta \leq 1$, so from Taylor’s theorem, $g(\theta) \leq \gamma \theta^2$. Hence,
\[
\mathbb{P}(T^k_+ \leq T) \leq \exp(-N \alpha_k^2/(4\sigma^d a_k \gamma t_0)) = \exp(-R^2/(16\sigma^d \gamma t_0 e^{2L_0})).
\]
The same bound applies to $\mathbb{P}(T^k_- \leq T)$. So we have shown that $\mathbb{P}(T_4 = T) \leq p_4$, as required. \qed

**Proof of Theorem 2.1.** We will determine conditions on sequences $R(N)$ and $r(N)$ so that, for $A(N) = (\log N)^4$, as $N \to \infty$, all the constraints of Proposition 3.1 are satisfied and, with an obvious notation, $p_i(N) \to 0$ for $i = 1, 2, 3, 4$. For $p_4(N) \to 0$, it suffices that $\log \log N \exp(-R^2/(20\sigma^d t_0 e^{2L_0})) \to 0$ and, hence, that $R/\sqrt{\log \log \log N} \to \infty$. For $p_3(n) \to 0$, it suffices that $r \to \infty$. For $p_2(N) \to 0$, it suffices that $A^2-1/d R^2/N^{1-1/d} \to 0$ and for $p_1(N) \to 0$, it suffices that $A r / N^{1-1/d} \to 0$. If we can also arrange that $R/\sqrt{A} \to 0$ and $r A/ (R N^{(1/2)(1-1/d)} \to 0$, then all the constraints of Proposition 3.1 will be satisfied eventually. A possible choice is to take $R(N) = N^{(1/2)(1-1/d)/(\log N)^4}$ and any sequence $R(N)$ with $R(N)/\sqrt{\log \log \log N} \to \infty$ and $R(N)/(\log N)^2 \to 0$. This proves the first part of the theorem. For the final assertion, it suffices to note that, for $k \leq m - 1$, $R(N)/\sqrt{N a_k} \leq R(N)/\sqrt{A(N)} \to 0$. \qed

We remark that the choice $R(N) = \log N$ leads to a bound of the form $p(N) \leq C N^{-(1/2)(1-1/d)}$ up to logarithmic corrections. This is the best rate of
decay of probabilities we have found. We remark also that a marginally shorter proof can be had by replacing the exponential martingale inequality by Doob’s $L^2$-inequality, at the small cost of requiring that $R(N)/\sqrt{\log \log N} \to \infty$.

4. A refinement of the fluid limit. This section leads to a proof of Theorem 2.2. The deterministic limit (for components $k \leq m - 1$) just discussed will be refined by approximating the martingale $M$ in (4) by another martingale whose characteristics are determined by the limit path, and at the same time linearizing around the limit path. The accuracy of the approximation is thereby improved from $N^{-1/2}$ to $N^{-3/4}$ at the cost of moving to an approximating process which is not deterministic but has a simple random structure, being a linear function of a Poisson random measure.

Define a measure $\tilde{\nu}$ on $\mathbb{R}^N \times (0, t_0]$ by
\[
\tilde{\nu}(dy, dt) = K(x_t, dy) dt.
\]
We will take $\tilde{\mu}$ to be a Poisson random measure with intensity $\tilde{\nu}$ coupled, in a way to be specified, with the process $X$. Define $\tilde{M} = (\tilde{M}^k_t : k \in \mathbb{N}, t \leq t_0)$ by
\[
\tilde{M}^k_t = \int_{\mathbb{R}^N \times (0, t]} y^k(\tilde{\mu} - \tilde{\nu})(dy, ds)
\]
and define $\tilde{\gamma} = (\tilde{\gamma}^k_t : k \in \mathbb{N}, t \leq t_0)$ by
\[
\tilde{\gamma}^k_t = \sqrt{N} \tilde{M}^k_t + \int_0^t \nabla b(x_s) \tilde{\gamma}^k_s ds.
\]
We show in Section 6 that we can write $\tilde{\gamma}$ as an explicit linear function of $\tilde{\mu} - \tilde{\nu}$,

\[
\tilde{\gamma}^k_t = \sqrt{N} \int_{\mathbb{R}^N \times (0, t]} \Phi_{t,s} y(\tilde{\mu} - \tilde{\nu})(dy, ds),
\]
where $(\Phi_{t,s} : s \leq t \leq t_0)$ is the $\mathbb{N} \times \mathbb{N}$ matrix-valued process given by
\[
\frac{\partial}{\partial t} \Phi_{t,s} = \nabla b(x_t) \Phi_{t,s}, \quad \Phi_{s,s} = I.
\]
Thus, $\tilde{\gamma}$ has a simpler stochastic structure than $X$. In particular, we can write the characteristic function of any finite-dimensional distribution of $\tilde{\gamma}$ in terms of $(x_t)_{t \leq t_0}$ and $(\Phi_{t,s} : s \leq t \leq t_0)$.

Recall that
\[
X_t = x_0 + M_t + \int_0^t b(X_s) ds.
\]
On the other hand, if we set $\tilde{X}_t = x_t + N^{-1/2} \tilde{\gamma}^k_t$, then
\[
\dot{\tilde{X}}_t = x_0 + \tilde{M}_t + \int_0^t b(x_s) ds + \int_0^t \nabla b(x_s)(\tilde{X}_s - x_s) ds.
\]
Set $\tilde{Y} = X - \tilde{X}$, $\tilde{D} = M - \tilde{M}$ and

$$A_t = \int_0^t (b(X_s) - b(x_s) - \nabla b(x_s)(X_s - x_s)) \, ds.$$  

Then

$$\tilde{Y}_t = \tilde{D}_t + A_t + \int_0^t \nabla b(x_s)\tilde{Y}_s \, ds. \tag{10}$$

We will obtain a good approximation if we can couple $\tilde{M}$ with $M$ to make $\tilde{D}$ small. Define kernels $K_0, K_+, K_-$ on $(0, t_0] \times E \times \mathbb{R}^N$ by

$$K_0(t, x, dy) = K(x, dy) \land K(x_t, dy),$$

$$K_\pm(t, x, dy) = (K(x, dy) - K(x_t, dy)) \pm,$$

and let $K_*(t, x, w, dx', dw')$ be the image of the measure

$$K_0(t, x, dy_0) \otimes K_+(t, x, dy_+) \otimes K_-(t, x, dy_-)$$

by the map $(x', w') = (y_0 + y_+, y_0 + y_-)$. Let $(X_t, W_t)_{t \geq 0}$ be a Markov chain, starting from $(x_0, 0)$, with time-dependent Lévy kernel $K_*$. Set

$$\tilde{\mu}_t = \sum_{\Delta W_i \neq 0} \delta_{(t, \Delta W_i)}.$$  

Then $(X_t)_{t \geq 0}$ is a Markov chain with Lévy kernel $K$ and $\tilde{\mu}$ is a Poisson random measure with intensity $\tilde{v}$. We have coupled $X$ and $W$ so that, as far as possible, they have the same jumps. Set $T_5 = \inf\{t \geq 0 : |\tilde{\gamma}_m|^2_t > r \sqrt{a_m}\}$, where $r$ is as in the previous section. Fix $\tilde{R} > 0$ and set

$$T_6 = \inf\{t \geq 0 : N^{3/4} |X^k_t - X^k_t| > \tilde{R}a^{1/4}_k \text{ for some } k \leq m - 1\}.$$  

Finally, set $\tilde{T} = T \wedge T_5 \wedge T_6$ and $\tilde{p} = \mathbb{P}(\tilde{T} < t_0)$.

**Proposition 4.1.** Assume that the conditions of Proposition 3.1 hold. Set $H = \frac{1}{d}d(d - 1)\sigma^{d-2}$ and assume, in addition, that $r A \leq N(1/2)(1-1/d)$ and

$$4HR^2t_0 \sigma^{2L_0} / N^{1/4} + 4r A t \sigma^{2L_0} / N^{1/4}(1-1/d) \leq \tilde{R} \leq 4RLt_0 \sigma^{2L_0} A^{1/4}.$$  

Then $\tilde{p} \leq p_1 + p_2 + p_3 + p_4 + p_5 + p_6$, where $p_i$ for $i = 1, 2, 3, 4$ are defined in Proposition 3.1 and

$$p_5 = 8(\rho^d + 1)t_0 \sigma^{2L_0} / r^2, \quad p_6 = 2m \exp(-\tilde{R}^2 / (20R_Lt_0 \sigma^{2L_0})).$$

**Proof.** Given Proposition 3.1, it will suffice to show that $\mathbb{P}(\tilde{T} = T_i) \leq p_i$ for $i = 5, 6$. By Proposition 6.1,

$$\mathbb{E}\left( \sup_{t \leq t_0} |\tilde{\gamma}_m|^2_t \right) \leq 8(\rho^d + 1)t_0 \sigma^{2L_0} a_m.$$
So \( \mathbb{P}(\tilde{T} = T_5) \leq p_5 \) by Chebyshev’s inequality.

We now follow an argument similar to the proof that \( \mathbb{P}(T = T_4) \leq p_4 \) in Proposition 3.1. Set

\[
\tilde{f}(t) = \sup_{k \leq m-1} \sup_{s \leq t} |\tilde{Y}^{k}_s|/a_k^{1/4},
\]

\[
\tilde{A}^*_t = \sup_{k \leq m-1} \sup_{s \leq t} |A^{k}_s|/a_k^{1/4} + \int_0^t |\tilde{Y}^{m}_s| ds/a_{m-1}^{1/4},
\]

\[
\tilde{D}^*_t = \sup_{k \leq m-1} \sup_{s \leq t} |\tilde{D}^{k}_s|/a_k^{1/4}.
\]

We recall that

\[
\nabla b^{k}(x)y = \lambda d(x^{k-1})^{d-1}y^{k-1} - \lambda d(x^{k})^{d-1}y^{k} - y^{k+1}
\]

so, provided that \( x^{k} \leq \sigma a_k \) and \( x^{k-1} \leq \sigma a_{k-1} \),

\[
|\nabla b^{k}(x)y|/a_k^{1/4} \leq L \sup_{j=k-1,k,k+1} |y_j|/a_j^{1/4}.
\]

For \( t \leq t_0 \), we have \( x^{k}_t \leq \rho a_k \leq \sigma a_k \) for all \( k \), so, for \( k \leq m-1 \),

\[
|\nabla b^{k}(x_t)\tilde{Y}_t|/a_k^{1/4} \leq L \tilde{f}(t) + \delta_{k,m-1} |\tilde{Y}^{m}_t|/a_{m-1}^{1/4}.
\]

Then, from (10), we get

\[
\tilde{f}(t) \leq \tilde{D}^*_t + \tilde{A}^*_t + L \int_0^t \tilde{f}(s) ds,
\]

so, by Gronwall’s lemma, \( \tilde{f}(t) \leq e^{Lt}(\tilde{A}^*_t + \tilde{D}^*_t) \) for all \( t \leq t_0 \). Note that, for \( k \leq m-1 \),

\[
b^{k}(y) - b^{k}(x) - \nabla b^{k}(x)(y - x)
\]

\[
= \lambda ((y^{k-1})^{d} - (x^{k-1})^{d} - d(x^{k-1})^{d-1}(y^{k-1} - x^{k-1}))
\]

\[
- \lambda ((y^{k})^{d} - (x^{k})^{d} - d(x^{k})^{d-1}(y^{k} - x^{k}))
\]

so, provided that \( x^{k}, y^{k} \leq \sigma a_k \) and \( x^{k-1}, y^{k-1} \leq \sigma a_{k-1} \),

\[
|b^{k}(y) - b^{k}(x) - \nabla b^{k}(x)(y - x)|
\]

\[
\leq H\lambda (a_k^{d-2}|y^{k-1} - x^{k-1}|^2 + a_k^{d-2}|y^{k} - x^{k}|^2).
\]

For \( t < \tilde{T} \) and \( k \leq m-1 \), we have \( |X^{k}_t - x^{k}_t| \leq R\sqrt{a_k/N} \); moreover, as we showed at (3), this implies that \( X^{k}_t \leq \sigma a_k \). Hence,

\[
|b^{k}(X_t) - b^{k}(x_t) - \nabla b^{k}(x_t)(X_t - x_t)|/a_k^{1/4}
\]

\[
\leq H\lambda R^2(a_{k-1}^{d-1} + a_k^{d-1})/(Na_k^{1/4}) \leq 2CR^2/N.
\]
Also, \(|\tilde{Y}_m^t| \leq |X_m^t - x_m^t| + N^{-1/2}|\tilde{\gamma}_m^t|\), so, for \(t < \tilde{T}\), \(|\tilde{Y}_m^t|/a_m^{1/4} \leq (ra_m + N^{-1/2}r\sqrt{a_m})/a_m^{1/4} \leq 2r(A/N)^{1-1/(4d)}\). It follows that

\[
\tilde{A}^*_T \leq 2HR^2t_0/N + 2rt_0(A/N)^{1-1/(4d)} \leq \frac{1}{2}e^{-L_{t_0}}\tilde{R}/N^{3/4}.
\]

Set \(\tilde{\alpha}_k = \frac{1}{2}e^{-L_{t_0}}\tilde{R}k^{1/4}/N^{3/4}\) and consider the stopping times \(\tilde{T}_k^\pm = \inf\{t \geq 0: \pm \tilde{D}_k^t > \tilde{\alpha}_k\}\).

Suppose that \(\tilde{T} < \tilde{T}_1 \land \cdots \land \tilde{T}_{m-1}\). Then \(\tilde{D}^*_\tilde{T} \leq \frac{1}{2}e^{-L_{t_0}}\tilde{R}/N^{3/4}\) so \(f(\tilde{T}) \leq \tilde{R}/N^{3/4}\) and \(\tilde{T} < T_6\). Hence,

\[
\mathbb{P}(\tilde{T} = T_6) \leq \sum_{k=1}^{m-1} \mathbb{P}(\tilde{T}_k \leq \tilde{T})
\]

and it remains to estimate \(\mathbb{P}(\tilde{T}_k \leq \tilde{T})\) for \(k \leq m - 1\).

For \(k \leq m - 1\), set

\[
\psi^k(t, x, \theta) = (\lambda^+_k(x) - \lambda^+_k(x_t))^+ h(\theta) + (\lambda^-_k(x) - \lambda^-_k(x_t))^+ h(-\theta)
\]

\[
+ (\lambda^+_k(x) - \lambda^+_k(x_t))^h(\theta) + (\lambda^-_k(x) - \lambda^-_k(x_t))^h(-\theta).
\]

Fix \(\theta \geq 0\) and consider, for \(k \leq m - 1\), the exponential martingale

\[
\tilde{Z}^k_t = \exp \left\{ N\theta(X^k_t - X^k_0 - W^k_t) - \int_0^t \int_{\mathbb{R}^N} (e^{N\theta y^k} - 1) K^+(s, X_s, dy) ds 
\right.
\]

\[
\left. - \int_0^t \int_{\mathbb{R}^N} (e^{-N\theta y^k} - 1) K^-(s, X_s, dy) ds \right\}
\]

\[
= \exp \left\{ N\theta \tilde{D}^k_t - N \int_0^t \psi^k(s, X_s, \theta) ds \right\}.
\]

For \(k \leq m - 2\) and \(t \leq t_0\), for \(x^k \leq \sigma a_k\) and \(x^{k-1} \leq \sigma a_{k-1}\), we can estimate as at (5), (6) and (7) to obtain

\[
\psi^k(t, x, \theta)/\sqrt{a_k} \leq g(\theta)(|\lambda^+_k(x) - \lambda^+_k(x_t)| + |\lambda^-_k(x) - \lambda^-_k(x_t)|)/\sqrt{a_k}
\]

\[
\leq Lg(\theta) \sup_{j=k-1, k, k+1} |x^j - x^j_t|/\sqrt{a_j},
\]

so, for \(t < \tilde{T}\) and \(k \leq m - 2\),

\[
\psi^k(t, X_t, \theta) \leq Lg(\theta)R\sqrt{a_k}/N.
\]

Similarly, since we assume \(R \geq 1\), \(rA \leq N^{(1/2)(1-1/d)}\) and \(Na_m \leq A\), we have, for \(t < \tilde{T}\), \(X^m_t \leq ra_m \leq R\sqrt{a_m}/N\) and we can show that (11) remains true for
$k = m - 1$. By optional stopping we have $\mathbb{E}(\tilde{Z}_k^{T_+}) \leq 1$ for all $k \leq m - 1$. But on the event $\tilde{T}_+ \leq \tilde{T}$, we have

$$\tilde{Z}_k^{T_+} \geq \exp\{N\theta \tilde{\alpha}_k - Lg(\theta) R_{t_0} \sqrt{N \tilde{\alpha}_k}\}.$$ 

We choose $\theta = 2\sqrt{N \tilde{\alpha}_k}/(5LR_{t_0} \sqrt{\tilde{\alpha}_k})$, checking that $\theta \leq 1$, so that $g(\theta) \leq 5\theta^2/4$, and deduce

$$\mathbb{P}(\tilde{T}_+ \leq \tilde{T}) \leq \exp\left\{-N^{3/2} \tilde{\alpha}_k^2/(5LR_{t_0} \sqrt{\tilde{\alpha}_k})\right\} = \exp\left\{-\tilde{R}^2/(20LR_{t_0}e^{2L_{t_0}})\right\}.$$ 

The same bound applies to $\mathbb{P}(\tilde{T}_- \leq \tilde{T})$. So we have shown that $\mathbb{P}(\tilde{T} = T_0) \leq p_6$, as required. □

**Proof of Theorem 2.2.** Choose $r(N)$ as in the proof of Theorem 2.1, so that $r(N) \to \infty$ and so $p_5(N) \to 0$. Assume $\tilde{R}(N)/(\log N)^2 \to 0$, and set $s(N) = \tilde{R}(N)/(\log \log \log N)^{3/4}$. Then $s(N) \to \infty$. Set $R(N) = s(N)(\log \log \log N)^{1/2}$ and $r(N) = N^{(1/4)(1-1/d)}/(\log N)^4$. It is straightforward to check that all the constraints in Propositions 3.1 and 4.1 are satisfied eventually. Moreover, as in the proof of Theorem 2.1, we have $p_i(N) \to 0$ for $i = 1, 2, 3, 4$. Finally, $\tilde{R}(N)^2/R \log \log \log N \to \infty$, so also $p_6(N) \to 0$, which proves the theorem. □

**5. Diffusion approximation.** In this section we prove Theorem 2.3. The method follows the lines set out in [4], Chapter 11. As we have already seen, our process $X$ has around $\log \log N$ active components, which have a wide range of scales. This will require special consideration in the implementation of the general method. We also have to deal with the fact that the variance of the diffusion approximation has degeneracies. The diffusion coefficient, obtained as the square root of the variance, then fails to be Lipschitz and some special care is needed to arrive at the desired convergence.

Let $(X^k_t: k \in \mathbb{N}, t \geq 0)$ be the supermarket process starting from $x_0$ and recall equation (4)

$$X^k_t = x^k_0 + M^k_t + \int_0^t b^k(x_s) ds.$$ 

Recall also that we set $\tilde{X}_t = x_t + N^{-1/2}\gamma_t$, where $(\gamma^k_t: k \in \mathbb{N}, t \leq t_0)$ is defined by the linear equations (2)

$$\gamma^k_t = \sqrt{N} \tilde{M}^k_t + \int_0^t \nabla b^k(x_s) \gamma_s ds$$

and

$$\sqrt{N} \tilde{M}^k_t = \int_0^t \sigma^k_+(x_s) dB^k_+(s) - \int_0^t \sigma^k_-(x_s) dB^k_-(s).$$
Set \( Y = X - \tilde{X} \) and \( D = M - \tilde{M} \). Then
\[
Y_t = D_t + A_t + \int_0^t \nabla b_k(x_s) Y_s \, ds,
\]
where \( A_t \) is defined at (9). We will obtain a good approximation if we can couple \( M \) with \( M \) to make \( D \) small.

The coupling relies on the following approximation result of [8]: there exists a constant \( c \in (0, \infty) \) and a probability space on which are defined a compensated Poisson process \( Z \) of rate 1 and a standard Brownian motion \( W \) such that, for all \( t \geq 0 \) and \( x \in \mathbb{R} \),
\[
P\left( \sup_{s \leq t} |Z(s) - W(s)| \geq c \log t + x \right) \leq ce^{-x/c}.
\]
(12)
See [13] for a recent review of developments and clarifications in connection with this result.

Given independent compensated Poisson processes \( Z^k_+, Z^k_- \), \( k \in \mathbb{N} \) of rate 1, we can construct \( X \) by the equations (4) and
\[
M^k_t = N^{-1} \left\{ Z^k_+ \left( N \int_0^t \lambda^k_+(X_s) \, ds \right) - Z^k_- \left( N \int_0^t \lambda^k_-(X_s) \, ds \right) \right\}.
\]
On the other hand, by a theorem of Knight, see, for example, [15], there exist independent Brownian motions \( W^k_+, W^k_- \), \( k \in \mathbb{N} \), such that, for all \( k \in \mathbb{N} \) and \( t \leq t_0 \),
\[
W^k_\pm \left( N \int_0^t \lambda^k_\pm(X_s) ds \right) = \sqrt{N} \int_0^t \sigma^k_\pm(\tilde{X}_s) dB^k_\pm(s).
\]
The law of \( (B^k_+, B^k_- : k \in \mathbb{N}) \), given \( (W^k_+, W^k_- : k \in \mathbb{N}) \), is given by a measurable kernel. So we may assume that these processes are defined on the same probability space as \( (Z^k_+, Z^k_- : k \in \mathbb{N}) \) and that \( (Z^k_+, W^k_+) \), \( (Z^k_-, W^k_-) \), \( k \in \mathbb{N} \), are independent copies of \( (Z, W) \).

Set \( T_7 = \inf\{ t \geq 0 : |\gamma^m_t| > r \sqrt{a_m} \} \). Fix \( \tilde{R} > 0 \) and set
\[
T_8 = \inf\{ t \geq 0 : N|X^k_t - \tilde{X}^k_t| > \tilde{R} \log(Na_k) \text{ for some } k \leq m - 1 \}.
\]
Finally, set \( \tilde{T} = T \wedge T_7 \wedge T_8 \) and \( \bar{p} = P(\tilde{T} < t_0) \).

**Proposition 5.1.** Assume that the conditions of Proposition 3.1 hold, and assume, moreover, that \( A \geq e^2 \) and \( \tilde{R} \leq \sqrt{A}/2 \). There is a constant \( C < \infty \), depending only on \( d, \lambda, \rho \) and \( t_0 \), such that, if
\[
C + C(R^2 + rA)/\log N \leq \tilde{R} \leq A/(2 \log A),
\]
then \( \bar{p} \leq p_1 + p_2 + p_3 + p_4 + p_7 + p_8 \), where \( p_1, p_2, p_3, p_4 \) are defined in Proposition 3.1 and
\[
p_7 = C/r^2, \quad p_8 = Cm(A^{-1} + (\tilde{R} \log A)^{-2}).
\]
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PROOF. Given Proposition 3.1, it will suffice to show that \( \mathbb{P}(\tilde{T} = T_i) \leq p_i \) for \( i = 7, 8 \). By Proposition 6.1,

\[
\mathbb{E}\left( \sup_{t \leq t_0} |y_t^m|^2 \right) \leq 8(\rho^d + 1)t_0e^{2L\tilde{g}}a_m.
\]

So \( \mathbb{P}(\tilde{T} = T_7) \leq p_7 \) for a suitably large \( C \) by Chebyshev’s inequality. Set

\[ f(t) = \sup_{k \leq m-1} \sup_{s \leq t} |y_s^k|/\log(Na_k), \]

\[ A_t^* = \sup_{k \leq m-1} \sup_{s \leq t} |A_s^k|/\log(Na_k) + \int_0^t |y_s^m| ds/\log(Na_{m-1}), \]

\[ D_t^* = \sup_{k \leq m-1} \sup_{s \leq t} |D_s^k|/\log(Na_k). \]

Since \( Na_{m-1} \geq A \geq e \), we have \( Na_k/\log(Na_k) \leq Na_{k-1}/\log(Na_{k-1}) \) for all \( k \leq m - 1 \). So we can use an argument from the proof of Proposition 4.1 to obtain \( \tilde{f}(t) \leq e^{Lt}(A_t^* + D_t^*) \) for all \( t \leq t_0 \).

The function \( F(s) = \log s/(N^{1-d}\lambda^d) \) is decreasing when \( s > e \) and \( A < (ANd^{-1}/\lambda)^{1/d} \), so

\[
\log(Na_{m-1})/(Na_m) = F(Na_{m-1}) \geq F((AN^{d-1}/\lambda)^{1/d}) \geq \log N/(2A).
\]

Similarly, \( \log(Na_{m-1})/\sqrt{Na_m} \geq \log N/(2\sqrt{A}) \). Hence,

\[
r(Na_m + \sqrt{Na_m})/\log(Na_{m-1}) \leq 4rA/\log N.
\]

We estimate as in the proof of Proposition 4.1 to obtain, for \( t < \tilde{T} \) and \( k \leq m - 1 \),

\[
|b^k(X_t) - b^k(x_t) - \nabla b^k(x_t)(X_t - x_t)|/\log(Na_k) \leq H\lambda R^2(\alpha_{k-1}^{d-1} + \alpha_k^{d-1})/N \log(Na_k) \leq 2HR^2/(N \log N)
\]

and

\[
|y_t^m|/\log(Na_{m-1}) \leq (ra_m + N^{-1/2}r \sqrt{a_m})/\log(Na_{m-1}) \leq 4rA/(N \log N).
\]

So \( A_t^* \leq 2t_0(HR^2 + 2rA)/(N \log N) \leq \frac{1}{2}e^{-L\tilde{g}}\tilde{R}/N \), provided \( C \) is chosen suitably large. For \( k \leq m - 1 \), set \( \alpha_k = \frac{1}{2}e^{-L\tilde{g}}\tilde{R} \log(Na_k)/N \) and consider the event \( \Omega_k = \{ \sup_{t \leq \tilde{T}} |D_t^k| > \alpha_k \} \). On \( \Omega_0 = \Omega \setminus (\Omega_1 \cup \cdots \cup \Omega_{m-1}) \), we have \( D_t^* \leq \frac{1}{2}e^{-L\tilde{g}}\tilde{R}/N \), so \( \tilde{f}(\tilde{T}) \leq \tilde{R}/N \) and \( \tilde{T} < T_8 \). Hence,

\[
\mathbb{P}(\tilde{T} = T_8) \leq \sum_{k=1}^{m-1} \mathbb{P}(\Omega_k)
\]
and it will suffice to estimate \( P(\Omega_k) \) for each \( k \leq m - 1 \). Fix \( k \leq m - 1 \). We can write \( D^k(t) = D_+(t) - D_-(t) \), where \( D_\pm(t) = D^1_\pm(t) + D^2_\pm(t) + D^3_\pm(t) \) and

\[
D^1_\pm(t) = N^{-1}(Z^k_\pm - W^k_\pm)(N \int_0^t \lambda^k_\pm(X_s) \, ds),
\]

\[
D^2_\pm(t) = N^{-1}\left\{ W^k_\pm\left(N \int_0^t \lambda^k_\pm(X_s) \, ds\right) - W^k_\pm\left(N \int_0^t \lambda^k_\pm(\bar{X}_s) \, ds\right)\right\},
\]

\[
D^3_\pm(t) = N^{-1/2} \int_0^t \{ \sigma^k_\pm(\bar{X}_s) - \sigma^k_\pm(x_s) \} \, dB^k_\pm(s).
\]

Hence, \( P(\Omega_k) \leq q^1_+ + q^1_- + q^2_+ + q^2_- + q^3_+ + q^3_- \), where, for \( j = 1, 2, 3 \),

\[
q^j_\pm = P(\Omega^j_\pm) = \{ \sup_{t \leq \bar{T}} |D^j_\pm(t)| > \alpha_k/6 \}.
\]

For \( t < \bar{T} \), we have \( \lambda^j_\pm(X_t) \leq \sigma^d a_k \), so, taking \( t = N \sigma^d a_k t_0 \) and \( x = N \alpha_k/6 - c \log t \) in (12), we obtain

\[
q^1_\pm \leq P\left( \sup_{t \leq N \sigma^d a_k t_0} |Z(t) - W(t)| > N \alpha_k/6 \right) \leq c \sigma^d t_0 N a_k e^{-N \alpha_k/(6c)} = c \sigma^d t_0 N a_k e^{-\bar{R} \log(N a_k)/(12ce^{L_0})} = c \sigma^d t_0 (N a_k)^{1-\bar{R}/(12ce^{L_0})} \leq c \sigma^d t_0 A^{1-\bar{R}/(12ce^{L_0})} \leq C/(4A),
\]

for a suitable choice of \( C \).

We turn to estimate \( q^2_\pm \). This will rely on the following continuity estimate for Brownian motion: for \( \tau, h, \delta > 0 \), setting \( n = \lfloor \tau/h \rfloor \),

\[
P\left( \sup_{s, t \leq \tau, |s-t| \leq h} |W(t) - W(s)| > \delta \right) \leq P\left( \sup_{k \in \{0, 1, \ldots, n-1\}, t \leq 2h} |W(kh + t) - W(kh)| > \delta/2 \right) \leq 2n P\left( \sup_{t \leq 2h} W(t) > \delta/2 \right) \leq (2\tau/h)e^{-\delta^2/(16h)}.
\]

For \( t < \bar{T} \), we have

\[
\bar{X}_t^k \leq x_t^k + |X_t^k - x_t^k| + |\bar{X}_t^k - X_t^k| \leq \rho a_k + R \sqrt{a_k/N} + \bar{R} \log(N a_k)/N \leq (\rho + R/\sqrt{A} + \bar{R} \log A/A) a_k \leq \sigma a_k,
\]

so

\[
N \int_0^t \lambda^k_\pm(\bar{X}_s) \, ds \leq \sigma^d t_0 N a_k.
\]
Also, using (5) and (6), for \( t < \bar{T} \),
\[
\left| N \int_0^t (\lambda^k_+(X_s) - \lambda^k_+ (\bar{X}_s)) \, ds \right| 
\leq N \sigma d^{d-1} t_0 \tilde{a}_k \{ \log(Na_{k-1})/(Na_{k-1}) + \log(Na_k)/(Na_k) \}
\leq 2d \sigma d^{d-1} t_0 \tilde{R} \log(Na_k)
\]
and
\[
\left| N \int_0^t (\lambda^-_k(X_s) - \lambda^-_k (\bar{X}_s)) \, ds \right| 
\leq \tilde{R} t_0 \{ \log(Na_k) + \log(Na_{k+1}) \mathbb{1}_{k \leq m-2} \} + r t_0 (Na_m + \sqrt{Na_m}) \delta_{k,m-1}
\leq 2t_0 \tilde{R} \log(Na_k),
\]
provided that \( C \) is sufficiently large. We take \( \tau = \sigma d t_0 N a_k \), \( h = 2d \sigma d^{d-1} t_0 \times \tilde{R} \log(Na_k) \) and
\[
\delta = N \alpha_k/6 = \tilde{R} \log(Na_k)/(12e L t_0)
\]
to obtain
\[
q_{\pm}^2 \leq \frac{\sigma N a_k}{d \tilde{R} \log(Na_k)} \exp \left\{ - \frac{\tilde{R} \log(Na_k)}{4608 d \sigma d^{d-1} t_0 e^{2L t_0}} \right\} \leq C/(4A)
\]
for a suitable choice of \( C \).

It remains to estimate \( q_{\pm}^3 \). We shall show below that there exists a constant \( C_0 \) such that, for \( t \leq t_0 \) and all \( k \in \mathbb{N} \),
\[
\mathbb{E}( |y^k_t|^2 ) \leq C_0 (x^k_t - x^k_t) \wedge (x^k_t - x^k_{t+1}).
\]
Then, for \( t < \bar{T} \),
\[
\sqrt{N} |\sigma^k_-(\bar{X}_t) - \sigma^k_-(x_t)| = \sqrt{N} \sqrt{(\bar{X}^k_t)^+ - (\bar{X}^k_{t+1})^+} - \sqrt{x^k_t - x^k_{t+1}}
= \sqrt{N} \sqrt{(\bar{X}^k_t)^+ - (\bar{X}^k_{t+1})^+} + \sqrt{x^k_t - x^k_{t+1}}
\leq (|y^k_t| + |y^k_{t+1}|)/\sqrt{x^k_t - x^k_{t+1}},
\]
so, by Doob’s \( L^2 \)-inequality,
\[
\mathbb{E} \left( \sup_{t \leq \bar{T}} |D^k_3(t)|^2 \right) \leq 4 \mathbb{E} \int_0^{t_0} N^{-1} |\sigma^k_-(\bar{X}_s) - \sigma^k_-(x_s)|^2 \, ds \leq 16 t_0 C_0/N^2.
\]
Hence,
\[
q_{\pm}^3 = \mathbb{P} \left( \sup_{t \leq \bar{T}} |D^k_3(t)| > \alpha_k/6 \right) \leq 2304 t_0 e^{2L t_0} C_0/((\tilde{R} \log(Na_k))^2 \leq C(\tilde{R} \log A)^{-2}/2
\]
for a suitable choice of $C$.

The argument for $q^3_+$ is similar. For $t < \bar{T}$,

$$
\sqrt{N}|\sigma^k_+(\tilde{X}_t) - \sigma^k_+(x_t)|\\
= \sqrt{N}\left| \left( \lambda((\tilde{X}^{k-1}_t) +) - \lambda((\tilde{X}^{k}_t) +)^- \right) - \sqrt{\lambda(x^{k-1}_t - \lambda(x^k_t))} \right|\\
\leq \lambda d \left( \left( x^{k-1}_t \vee \tilde{X}^{k-1}_t \right)^{d-1} |\gamma^{k-1}_t| + (x^k_t \vee \tilde{X}^k_t)^{d-1} |\gamma^k_t| / \sqrt{\lambda(x^{k-1}_t - \lambda(x^k_t))} \right)\\
\leq \lambda d^2 \left( \left( x^{k-1}_t \vee \tilde{X}^{k-1}_t \right)^{d-1} + (N^{-1/2} |\gamma^{k-1}_t|^{d-1}) |\gamma^{k-1}_t| / \sqrt{\lambda(x^{k-1}_t - \lambda(x^k_t))} \right) + \lambda d^2 \left( (x^k_t \vee \tilde{X}^k_t)^{d-1} + (N^{-1/2} |\gamma^k_t| |^{d-1}) |\gamma^k_t| / \sqrt{\lambda(x^{k-1}_t - \lambda(x^k_t))} \right),
$$

so, by Doob’s $L^2$-inequality,

$$
E\left( \sup_{t \leq \bar{T}} |D^3_+(t)|^2 \right) \leq 4E\int_0^{\bar{T}} N^{-1} |\sigma^k_+(\tilde{X}_s) - \sigma^k_+(x_s)|^2 ds \\
\leq 8d^2 \left( N^{-1/2} (1 + N^{-1/2} C(d)) C_0 / N^2 \right),
$$

where $C(d) = E(W(1)^{2d})$. Hence,

$$
q^3_+ = \mathbb{P}\left( \sup_{t \leq \bar{T}} |D^3_+(t)| > \alpha_k / 6 \right) \\
\leq 2152d^2 \left( N^{-1/2} (1 + N^{-1/2} C(d)) C_0 e^{2L_0} / (\bar{R} \log(Na_k))^2 \right) \\
\leq C(\bar{R} \log A)^{-2}/2
$$

for a suitable choice of $C$. On combining this with the bounds for $q^1_+$ and $q^2_+$ already found, we obtain the desired bound for $p_8$. □

**Proof of Theorem 2.3.** Set $A(N) = r(N) = (\log N)^{1/2}$ and define $\bar{m}(N) = \inf\{k \in \mathbb{N} : Na_k \leq A(N)\}$. Set $R(N) = (\log N)^{1/4}(1 \wedge t_0)/2$. It is straightforward to check that, if $C$ is the constant appearing in Proposition 5.1 and if $\bar{R} = 3C$, then all the constraints in Propositions 3.1 and 5.1 are satisfied eventually and, moreover, that $p_i(N) \to 0$ for $i = 1, 2, 3, 4, 7, 8$. Since $\bar{m}(N) \geq m(N)$, this proves the theorem. □

**6. Fluctuation variance estimates.** We have deferred from other sections the analysis of certain linear equations associated with our processes. The basic questions of existence and uniqueness in suitable spaces may be resolved by standard methods, so we review this only briefly. The more delicate result, Proposition 6.1, which is needed for the diffusion approximation, relies on the particular structure of our model.
We recall the $\mathbb{N} \times \mathbb{N}$ matrix-valued equation
\[
\frac{\partial}{\partial t} \Phi_{t,s} = \nabla b(x_t) \Phi_{t,s}, \quad \Phi_{s,s} = I,
\]
to be solved for $0 \leq s \leq t \leq t_0$. Note that, for $x_0 \in S(\rho, t_0)$ and $t \in [0, t_0]$, we have $\|\nabla b(x_t)\| \leq L$, where $\| \cdot \|$ is the operator norm corresponding to $\|x\| = \sup_k |x^k|/a_k$. Hence, it is standard that this equation has a unique continuous solution with $\|\Phi_{t,s}\| \leq e^{L(t-s)}$ for all $s, t$.

The other relevant equations may be considered as stochastic perturbations of the preceding equation. In Theorem 2.2 we used (1)
\[
\tilde{\gamma}^k_t = \sqrt{N} \tilde{M}^k_t + \int_0^t \nabla b^k(x_s) \tilde{\gamma}_s^k \, ds, \quad t \leq t_0,
\]
and in Theorem 2.3 we used (2)
\[
\gamma^k_t = \sqrt{N} \tilde{M}^k_t + \int_0^t \nabla b^k(x_s) \gamma_s^k \, ds, \quad t \leq t_0.
\]
(13)
Here
\[
\tilde{M}^k_t = \int_{\mathbb{R}^{\mathbb{N} \times (0,t]}} y^k(\tilde{\mu} - \tilde{v})(dy, ds)
\]
and
\[
\sqrt{N} \tilde{M}^k_t = \int_0^t \sigma^k_+(x_s) dB^k_+(s) - \int_0^t \sigma^k_-(x_s) dB^k_-(s).
\]
Note that
\[
N \mathbb{E}(|\tilde{M}^k_t|^2) = N \int_0^t \int_{\mathbb{R}^\mathbb{N}} (y^k)^2 K(x_s, dy) \, ds
\]
\[
= \int_0^t (\lambda^k_+(x_s) + \lambda^k_-(x_s)) \, ds = N \mathbb{E}(|M^k_t|^2)
\]
and $\lambda^k_+(x_t) + \lambda^k_-(x_t) \leq (\rho^d + 1)a_k$ for $t \leq t_0$. Hence, a standard type of iteration argument, using Doob’s $L^2$-inequality, shows that these equations have unique measurable solutions with, respectively,
\[
\mathbb{E}\left(\sup_{t \leq t_0} |\tilde{\gamma}^k_t|^2\right) \leq 8(\rho^d + 1)t_0 e^{2L^2t_0^2} a_k
\]
and
\[
\mathbb{E}\left(\sup_{t \leq t_0} |\gamma^k_t|^2\right) \leq 8(\rho^d + 1)t_0 e^{2L^2t_0^2} a_k.
\]
The details for (1) follow below; (2) may be treated in the same way. Note that, for any vector $y$ and for all $k \in \mathbb{N}$,
\[
\frac{|\nabla b^k(x_s)y^k|}{\sqrt{a_k}} \leq L \sum_{j=k-1,k,k+1} \frac{|y_j|}{\sqrt{a_j}}.
\]
Now let 
\[ \tilde{\gamma}_t^{(0)} = \sqrt{N} \tilde{M}_t , \]
and for \( n \in \mathbb{N} \),
\[ \tilde{\gamma}_t^{(n)} = \sqrt{N} \tilde{M}_t + \int_0^t \nabla b^k(x_s) \tilde{\gamma}_s^{(n-1)} \, ds . \]

Then for each \( k \),
\[
\frac{|\tilde{\gamma}_t^{(n+1),k} - \tilde{\gamma}_t^{(n),k}|^2}{a_k} \leq L^2 \left( \int_0^t \sum_{j=k-1,k,k+1} \frac{|\tilde{\gamma}_s^{(n),j} - \tilde{\gamma}_s^{(n-1),j}|}{\sqrt{a_j}} \, ds \right)^2 ,
\]
so, using Cauchy–Schwarz,
\[
\frac{|\tilde{\gamma}_t^{(n+1),k} - \tilde{\gamma}_t^{(n),k}|^2}{a_k} \leq 3L^2 \sum_{j=k-1,k,k+1} \int_0^t \frac{|\tilde{\gamma}_s^{(n),j} - \tilde{\gamma}_s^{(n-1),j}|^2}{a_j} \, ds .
\]

Then for all \( 0 \leq s \leq t \),
\[
\sup_{s \leq t} \frac{|\tilde{\gamma}_s^{(n+1),k} - \tilde{\gamma}_s^{(n),k}|^2}{a_k} \leq 3L^2 t \sum_{j=k-1,k,k+1} \int_0^t \frac{|\tilde{\gamma}_u^{(n),j} - \tilde{\gamma}_u^{(n-1),j}|^2}{a_j} \, ds .
\]

Let \( h^{(n)}(t) = \sup_k \mathbb{E}(\sup_{s \leq t} |\tilde{\gamma}_s^{(n+1),k} - \tilde{\gamma}_s^{(n),k}|^2/a_k) \); then for \( t \leq t_0 \),
\[
h^{(n)}(t) \leq 9L^2 t_0 \int_0^t h^{(n-1)}(s) \, ds .
\]

Hence,
\[
h^{(n)}(t) \leq \frac{2^n n! (3L t_0)^{2n}}{(2n)!} \tilde{M}(t_0) ,
\]
where \( \tilde{M}(t) = \sup_k \mathbb{E} \left( \sup_{s \leq t} |\tilde{M}_s^k|^2/a_k \right) \)
\[
\leq 4 \sup_k \sup_{s \leq t} \mathbb{E} \left( |\tilde{M}_s^k|^2/a_k \right) \leq 4t (\rho^d + 1) .
\]

We deduce that \( \tilde{\gamma}_t^{(n)} \) converges to a process \( \tilde{\gamma}_t \) uniformly on \([0, t_0] \), and that \( \tilde{\gamma}_t \) satisfies (1). The uniqueness part of the proof is similar. Let
\[
g(t) = \sup_k \mathbb{E} \left( \sup_{s \leq t} |\tilde{\gamma}_s^k|^2 \right)/a_k .
\]
Then it follows from the above estimates that, for $t \leq t_0$,
\[ g(t) \leq \tilde{M}(t_0) e^{gL^2 t_0^2}. \]

It may be verified by substitution that (8) gives an explicit representation of the solution of (1).

**Proposition 6.1.** The solution $(\gamma^k_t : k \in \mathbb{N}, t \leq t_0)$ to (2) satisfies
\[ \sup_{k \in \mathbb{N}} \sup_{t \leq t_0} \mathbb{E}(|\gamma^k_t|^2) / \min\{x_{t-1}^k - x_t^k, x_t^k - x_{t+1}^k\} < \infty. \]

**Proof.** Note that $\lambda^k_+(x_t) \leq d(x_{t-1}^k - x_t^k)$, so
\[ \dot{x}_t^k - x_{t+1}^k = \lambda^k_+(x_t) - \lambda^k_{+1}(x_t) - (x_t^k - x_{t+1}^k) + (x_{t+1}^k - x_t^{k+2}) \geq \lambda^k_+(x_t) - (d+1)(x_t^k - x_{t+1}^k) + (x_{t+1}^k - x_t^{k+2}). \]

Hence,
\[ \int_0^t \dot{x}_t^k(x_s) \, ds + \int_0^t (x_s^k - x_{s+1}^k) \, ds \leq e^{t(d+1)} (x_t^k - x_{t+1}^k) \]
and
\[ \int_0^t (x_s^k - x_{s+1}^k) \, ds \leq t e^{t(d+1)} (x_t^k - x_{t+1}^k). \]

Also,
\[ \int_0^t \dot{x}_t^k(x_s) \, ds \leq d \int_0^t (x_s^k - x_{s+1}^k) \, ds \leq d t e^{t(d+1)} (x_t^k - x_{t+1}^k). \]

Fix $\varepsilon > 0$ and set
\[ f(t) = \sup_{k \in \mathbb{N}} \sup_{s \leq t} \mathbb{E}(|\gamma^k_s|^2) / \delta^k_s, \]
where $\delta^k_s = \min\{x_{s-1}^k - x_s^k, x_s^k - x_{s+1}^k\} + \varepsilon$. Then $f(t_0) < \infty$. Note that
\[ \mathbb{E}\left( \left| \int_0^t \sigma^k_+(x_s) \, dB^k_+(s) - \int_0^t \sigma^k_+(x_s) \, dB^k(s) \right|^2 \right) = \int_0^t \lambda^k_+(x_s) \, ds + \int_0^t (x_s^k - x_{s+1}^k) \, ds \leq (dt + 1) e^{t(d+1)} \delta^k_t. \]

We have
\[ \nabla b^k(x) \gamma = \lambda d(x^{k-1})^{d-1} \gamma^{k-1} - (\lambda d(x^k)^{d-1} + 1) \gamma^k + \gamma^{k+1}. \]
We will make use of the following estimates:

\[
\mathbb{E}\left( \left| \int_0^t \lambda d(x_{s}^{k-1}d^{-1}y_{s}^{k-1}ds \right|^2 \right) \\
\leq \int_0^t \lambda^2 d(x_{s}^{k-1})^{2(d-1)}d^{-1}ds \int_0^t \mathbb{E}(|y_{s}^{k-1}|^2)/d^{-1}ds \\
\leq d^2 \int_0^t \lambda d(x_s^k) ds \int_0^t f(s) ds \leq (d^2 + d^3 t)e^{(d+1)t} \delta k \int_0^t f(s) ds 
\]

and

\[
\mathbb{E}\left( \left| \int_0^t \lambda d(x_{s}^k)^{d-1}y_{s}^k ds \right|^2 \right) \\
\leq \int_0^t (\lambda d(x_{s}^k)^{d-1} + 1)^2 \delta s^k ds \int_0^t \mathbb{E}(|y_{s}^k|^2)/\delta_s^k ds \\
\leq (d + 1)^2 t e^{(d+1)t} \delta k \int_0^t f(s) ds 
\]

and

\[
\mathbb{E}\left( \left| \int_0^t y_{s}^{k+1} ds \right|^2 \right) \leq \int_0^t \delta s^{k+1} ds \int_0^t \mathbb{E}(|y_{s}^{k+1}|^2)/\delta_s^{k+1} ds \leq e^{(d+1)t} \delta k \int_0^t f(s) ds. 
\]

Now, from (13), for all \( t \leq t_0 \),

\[
\mathbb{E}(y_{t}^k) \leq A \delta^k + B \delta k \int_0^t f(s) ds, 
\]

where \( A = 4(dt + 1)e^{(d+1)t} \) and \( B = 8d(d + 1)(dt + 1)e^{(d+1)t} \). So \( f(t) \leq Ae^{Bt} \) by Gronwall’s lemma. This bound does not depend on \( \varepsilon \), so the proposition follows by letting \( \varepsilon \to 0. \)

\[\square\]

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