Stability Analysis of Superluminal Metamaterial Transmission Line with Realistic Non-Foster Negative Capacitors

Recently, it has been shown possible to go around basic dispersion energy constraints that limit bandwidth of every passive metamaterial and construct a broadband active Epsilon-Near-Zero superluminal transmission line. A basic building block of this unusual transmission line is an active ‘tank circuit’ that contains both conventional (positive) capacitor and non-Foster negative capacitor. Published theoretical studies revealed that such a ‘tank circuit’ is stable if an overall capacitance is positive. These studies assumed lossless host transmission line periodically loaded with ideal dispersionless negative capacitors. However, a possible influence of the imperfections of a realistic negative capacitor (its dispersion and loss/gain) on the stability has not been investigated so far. Here, stability analysis of realistic superluminal transmission line is performed in Laplace domain. The obtained results are in a good agreement with those obtained in recent experiments on active transmission line developed at University of Zagreb.

Key words: Non-Foster Element, Negative Capacitor, Stability, Metamaterial, Superluminal Propagation

1 INTRODUCTION

All known passive materials (or metamaterials) that have either negative (Epsilon-NeGative, Mu-NeGative) or les-than-unity (Epsilon-Near-Zero, Mu-Near-Zero) real parts of permittivity or permeability are inherently dispersive and, therefore, they suffer from narrow operating bandwidth [1]. This operating bandwidth is limited by basic energy-dispersion constrains that apply for every passive, nearly loss-free material [1]:

\[
\frac{\partial \left[ \varepsilon(\omega) \right]}{\partial \omega} > 0, \quad \frac{\partial \left[ \mu(\omega) \right]}{\partial \omega} > 0.
\]  

(1)

Here, \( \omega \) is the angular frequency while \( \varepsilon \) and \( \mu \) stand for permittivity and permeability, respectively. The constraints in (1) can be simplified to [2]:

\[
\frac{\partial \left[ \varepsilon(\omega) \right]}{\partial \omega} > 0, \quad \frac{\partial \left[ \mu(\omega) \right]}{\partial \omega} > 0.
\]  

(2)

This simplified form (2) is an equivalent of the Foster reactance theorem in circuit theory [3] that requires \( \frac{\partial X}{\partial \omega} > 0 \) and \( \frac{\partial Y}{\partial \omega} > 0 \) (\( X \) and \( Y \) being the reactance and susceptance, respectively). In the case of ENG (or MNG) and ENZ (or MNZ) materials, the constrains in (2) actually show the existence of some resonant process. This process always involves redistribution of a fraction of the energy from the electric field into the magnetic field (ENG and ENZ metamaterials) or vice versa (MNG and MNZ metamaterials). A typical example is a well-known wire medium...
Above the resonant frequency of a circuit \( \omega_0 \), there is a lack of energy stored in the electric field (it is converted into the energy stored in the magnetic field). Within this narrow band, effective relative permittivity (3) has a value between zero and one \((0 < \varepsilon_r < 1)\), indicating the ENZ behavior. From the circuit theory point of view, one may say that the dispersion in passive metamaterials occurs due to the basic difference in the frequency behavior of a capacitor and an inductor.

Very recently, it has been shown that, in some cases, it is possible to overcome basic dispersion constrains (1,2) by the use of electronic circuits that behave as negative capacitors or negative inductors (so-called non-Foster elements [5-9]). The simplest implementation (Fig. 1 b) of this basic idea involves a transmission line periodically loaded with negative capacitors. A combination of a positive capacitor \( C \) (the distributed capacitance) and the negative capacitor \( C_N \) (active non-Foster element) behaves as an active ‘tank circuit’. A role of the negative capacitor \( C_N \) is to decrease the equivalent permittivity below the free-space value achieving dispersionless ENZ behavior:

\[
\varepsilon_r(\omega) = \frac{1}{\varepsilon_0} \left( \frac{C}{C - |C_N/\Delta x|} \right).
\]

(4)

The background physics of this counter-intuitive phenomenon comes from the active nature of negative capacitor. Negative capacitor is actually a source that supplies additional current to the positive capacitor [8]. This additional current causes faster charging and therefore decreases the effective capacitance. Since the dispersion curves of a positive capacitor and a negative capacitor are exactly inverse, the resultant behavior (4) does not depend on the frequency at all. This novel principle was successfully employed in several practical realizations of active ENZ metamaterials [6-8] and achieved bandwidths varied from one octave [6,7] to more than four octaves [8]. These bandwidths are significantly better than the bandwidth of all passive metamaterials available at present. In addition, the fact that the equivalent relative permittivity is smaller than one within an extremely broad band enables the propagation with phase and group velocities higher than speed of light (superluminal propagation [8]). Numerical simulations (based on actual measurement results) in [7] clearly showed that non-Foster active ENZ metamaterial may enable construction of broadband cloaking devices. Several other applications such as dispersionless feeding networks in antenna arrays and broadband phase shifters in communication technology were also foreseen [8].

However, practical use of non-Foster ‘negative’ elements is often difficult due to inherent stability problems [2]. These circuits always include positive feedback [9] and therefore exhibit only conditional stability. Recent study [10] revealed that the stability does not depend only on the active negative elements and its parasitic effects (as it is usually believed). Actually, the stability also depends on the elements of the external passive network. Different parallel and series networks that contain non-Foster elements were analyzed in [10] and it was found that they satisfy different stability criteria. This fact is almost completely overlooked in the literature that deals with application of non-Foster elements in metamaterials and antennas. The stability criteria developed in [10] are very simple and convenient. However, they are derived using a formal mathematical approach and the physical background of instability/stability is not completely clear. In addition, the analysis in [10] deals with ideal non-dispersive non-Foster elements. The influence of the imperfections such as inherent loss/gain and a finite operating bandwidth of realistic negative capacitor, or the losses of a host transmission line, on the stability has not been investigated so far. Therefore, here we present the stability analysis of realistic superlu-
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2 ORIGIN OF INSTABILITY IN NEGATIVE CAPACITORS

Traditionally, RF and microwave engineers analyze stability in frequency domain with the help of ‘stability factors’ [3], based on the reflection coefficients at the circuit ports. However, it is very often neglected that all (external) stability factors based on the reflection coefficient assume that the Nyquist stability criterion is a \textit{priori} satisfied, i.e. that the transfer function has no poles in the right half-plane [11]. If one neglects this fact, it is possible that the analysis of the stability factor predicts stable operation even for a circuit that is inherently unstable. For instance, the modulus of the reflection coefficient of an ideal (dispersionless) negative capacitor is one, as is the modulus of an ordinary positive capacitance. Looking at this result, one could (wrongly) conclude that a pure negative capacitor is a stable device. However, a very simple and intuitive transient analysis presented in [2] clearly shows an unbounded growth of voltage across the negative capacitor, i.e. instability. This unexpected behavior comes from the fact that the electronic circuits that behave as negative capacitors or negative inductors \textit{always} apply positive feedback. This can be understood from the simple equivalent circuit of a negative capacitor shown in Fig. 2.

Here, an ordinary (positive) capacitor \((C)\) is connected in series with a dependent voltage source \(v_0(t)\) that is controlled by the input signal \(v(t)\). The amplitude of the voltage source is chosen to be twice the amplitude of the input signal \((v_0(t)=2v(t))\). This causes the net current \(i(t)\) to flow outward, which can be interpreted as the existence of an equivalent negative capacitance \((C_N)\):

\[
i(t) = C \frac{\partial}{\partial t}(v(t) - 2v(t)) = -C \frac{\partial v(t)}{\partial t} = C_N \frac{\partial v(t)}{\partial t}.
\]

The circuit in Fig. 2 can be interpreted as an ideal (voltage) amplifier with a gain equal to 2 (here, \(v(t)\) and \(v_0(t)\) are input and output signals, respectively), with a capacitor in positive feedback. Theoretically, if the input of the amplifier is left open, both, the input and output voltages will be zero and the circuit will be stable. Of course, in reality, there is always some noise present at the input of the amplifier. This noise will appear at the output, as well (however, its level will be increased due to gain of the amplifier). Output (amplified) noise will be fed back again to the input, then again amplified e.t.c. Obviously, this process resembles behavior of an ordinary oscillator and leads to the self-oscillations (every signal waveform is allowed in idealized case with an amplifier with infinite bandwidth).

In order to stabilize the circuit one should decrease the signal fed back to the input below the level that supports an onset of oscillation. For instance, this can be achieved by an additional ordinary (positive) capacitor \(C_P\) connected in parallel to the input (in parallel to the ‘negative’ capacitor). These two capacitors \((C\) and \(C_P\)) form a voltage divider described by simple equation:

\[
\frac{v_0(t)}{v(t)} = \frac{C}{C_P + C}.
\]

From (6), it can be concluded that the self-oscillation will occur if \(C_P < C\). On the other hand, the circuit will be stable if \(C_P > C\). This is actually well known analysis of a feedback factor in microwave electronics [3]. Similarly, one may connect additional positive capacitor \(C_P\) in series with negative capacitor. In this case, a simple analysis reveals that the circuit will be stable if \(C < C_P\).

The stability can also be treated in a formal way that considers a negative capacitor as a ‘black box’ (a reactance) that is connected to an external passive network [10]. It can be shown that, in order to achieve stable operation, a net capacitance of the whole circuit (containing both ‘negative’ and ‘positive’ elements that can be connected either in parallel or in a series) must be positive. Obviously, this requirement is fulfilled in superluminal ENZ transmission line in Fig. 1b \((|\Delta| < C)\). On the contrary, construction of a dispersionless ENG metamaterial unfortunately does not seem to be feasible (such a metamaterial would require existence of negative net capacitance, which is inherently unstable).

Above discussion presumes an ideal dispersionless negative capacitor with no loss (or gain). Strictly speaking, this assumption is not physical. The infinite operating bandwidth with a ‘flat’ characteristic of negative capacitance would cause an infinite growth of magnitude in the example with discharging of a capacitor in simple RC circuit [2]. Associated energy would be infinite, which certainly is not possible. In addition, the infinite operating bandwidth would support superluminal group velocity for all the frequencies from zero to infinity in [8]. This would
mean that the energy velocity is also superluminal, which (again) is not possible. The experimental results reported in [8] also showed that capacitance of realistic negative capacitor depends on the frequency (i.e., it is indeed dispersive). Finally, the results from [8] showed that realistic negative capacitor can have both loss and gain, depending on the frequency of operation (this behavior is caused by the imperfection of the amplifier). One might speculate that the inherent loss should improve stability while the inherent gain leads to instability as in the case of ordinary negative-resistance circuits and oscillators [3]. This hypothesis is analyzed in the following sections.

3 STABILITY ANALYSIS - MODEL WITH DISPERSIONLESS NEGATIVE CAPACITOR WITH LOSSES/GAIN

In order to investigate the stability of active superluminal ENZ transmission line, a simple equivalent circuit (Model I) of a differential section of a line (Fig. 3), developed from the basic idea in Fig. 1b is used. Here L and C stand for the distributed inductance and capacitance, respectively. In a first step, the negative capacitor \( C_N \) was assumed to be dispersionless. The losses of the host transmission line (together with the input resistance of a generator) are modeled by distributed resistance \( R \), while the distributed conductance \( G \) describes losses or gain (depending of the sign) an active non-Foster capacitor. The analysis is based on the transfer function \( H(s) \) of a differential segment, which (in Laplace domain) reads as:

\[
H(s) = \frac{V_o}{V_i} = \frac{1}{s^2L(C_N + C_P) + sR(C_N + C_P) + sLG + RG + 1}
\]

(7)

Here, \( V_i \) and \( V_o \) stand for the input and output voltages at the differential line segment while \( s = \sigma + j\omega \) is a complex frequency. The instability occurs if the poles of transfer function (the roots of a polynomial in the denominator in (7)) are located in the RHS of a complex plane [10]. Realistic values of the elements of the equivalent circuit in Fig. 3, taken from the experiments in [8] were used in calculation and a sample of achieved results is shown in Fig. 4 and Fig. 5. In the first example (Fig. 4) it is assumed that a line is driven by a generator with internal resistance of 50\( \Omega \) and that the conductivity \( G \) may take both positive and negative values. As expected, it was found that \(|C_P| > C_N\) is not the only condition to be fulfilled. At the same time, the requirement \( R > \frac{1}{|G|} \) should be met.

This result is consistent with the classical analysis of microwave oscillators [3] apart from (very important) fact that, even if all the resistances in the circuit (both positive and negative) are equal to zero, negative value of capacitance causes instability. In the second case (Fig. 5), it was assumed that the host line is lossless and that the circuit is driven by zero-impedance generator. In this case, it can be seen that negative conductance always causes instability, regardless of the values of \( C_N \).

In addition, the model from Fig. 3 was also used for the analysis of velocity of propagation. The exact expressions for both phase and group velocities were derived using well known basic equations:

\[
v_p = \frac{\omega}{\beta}, \quad v_g = \frac{\partial \omega}{\partial \beta}.
\]

(8)

Here, \( \beta \) stands for phase factor \((\frac{2\pi}{\lambda})\), while \( v_p \) and \( v_g \) are
phase and group velocities, respectively. The equations are too long to be included here. However, it is important to mention that they have several square roots that impose the sign ambiguities for both phase factor (\(\beta\)) and attenuation factor (\(\alpha\)). These ambiguities can be resolved if one bears in mind that the active ENZ line can support only forward waves and that the negative conductance always causes gain. Analysis of the results clearly shows that Model I (Fig. 3) correctly predicts existence of both superluminal phase and group velocities observed in [8]. Moreover, it predicts that both losses and gain of negative capacitor may alter the ‘flatness’ velocity curve. However, Model I (Fig. 3) obviously cannot predict the frequency behavior of the ENZ line because it presumes (nonphysical) dispersionless capacitor.

### 4 STABILITY ANALYSIS - MODEL WITH DISPERSIVE NEGATIVE CAPACITOR WITH LOSSES/GAIN

In the next step, it was attempted to build a more realistic model that could take the dispersion of non-Foster capacitor into account. At first, a network that comprised a simple RLC circuit and one ideal negative capacitor, dispersion of which ‘imitates’ the dispersion observed in experiments in [8], was developed. Analysis of developed network indeed correctly predicted variation of effective capacitance from negative values to positive values with frequency, consistent with experiments in [8]. However, the existence of the inductance within a model caused instability for all the frequencies, which is not consistent with measurements in [8]. Additional thorough analysis revealed that is (even theoretically) impossible to construct a passive network (complemented with an ideal negative capacitor), that should be able to ‘imitate’ behavior of realistic negative capacitor in [8]. Thus, one should go back to the basic physics and try to model realistic negative capacitor per se. Therefore, it was decided to refine a rather simple model of negative capacitor from Fig. 2. This model is nonphysical due to the assumption that the amplifier has constant gain (equal to 2) over the infinite bandwidth. We corrected this by the introduction of a frequency characteristic of the amplifier described by simple one-pole model. This more realistic circuit was used as a load of a host transmission line yielding Model II (Fig. 6).

One could analyze the stability of this model by writing down the mesh equations, deriving new transfer function and then examining its properties. We opted for a simpler approach that formally introduces effective (dispersive) capacitance (\(C_{Nef}\)) and effective (dispersive) conductance (\(G_{ef}\)) of realistic negative capacitor:

\[
C_{Nef} = C - 1 + \left(\frac{\omega}{\omega_p}\right)^2, \quad G_{ef} = -\omega \cdot C \frac{2 \cdot \left(\frac{\omega}{\omega_p}\right)}{1 + \left(\frac{\omega}{\omega_p}\right)^2}.
\]

(9)

Here, \(C\) is positive capacitor used for the inversion in negative capacitance circuit (similarly to a model in Fig. 2) and \(\omega_p\) is the angular frequency of the pole of an amplifier. Once the values of \(C_{Nef}\) and \(G_{ef}\) are calculated, they can be inserted in equation (7) and the stability can be predicted. The results of this analysis (with realistic values of parameters taken from the experiments in [8]) are given in Fig. 7 and Fig. 8. For the case in which absolute value of generated negative capacitance is lower of than a value of positive capacitance of a line segment (\(|C_N| < C_p\)), the poles are always located in LHS of a complex plane.

\[
\begin{array}{|c|c|c|c|c|}
\hline
R & G & C_P & C_N & \text{Stability} \\
\hline
0 & + & + & (-C_P, 0) & \text{stable} \\
+ & - & + & (-C_P, 0) & \text{unstable} \\
\hline
\end{array}
\]

Fig. 5: The poles loci and stability conditions for Model I (\(R = 0\Omega, L = 0.25\mu H, C_P = 70pF, C_N = -40pF, G\) is varied)
indicating stable operation (Fig. 7). Please note that the stability is achieved in spite of the fact that effective conductance $G_{ef}$ may have both negative and positive values [8]. This happens because the losses of the ENZ line ($G$), together with the positive internal resistance of the driving generator ($50\Omega$) ‘override’ generated negative conductance ($G_{ef}$) of the non-Foster capacitor.

However, if the losses (together with the positive internal resistance of the generator) do not override generated negative inductance (Fig. 8) the system becomes unstable. As in the previous case, the analysis of phase and group velocities (not shown in figures) reveal that both the losses and gain may alter the ‘flatness’ of velocity curve. However, due to physical background of Model II (Fig. 6) it is now possible to analyze the dispersion of a whole ENZ line. It was shown that one can expect (nearly) dispersionless ENZ behavior up to frequency of $\frac{\omega_p}{2\pi}$. This is in good agreement with the experiments in [8].

5 CONCLUSION

Stability of superluminal ENZ transmission line based on realistic non-Foster negative capacitors was analyzed in Laplace domain. It was shown that a simple model of dispersionless negative capacitor with losses/gain can be used for initial prediction of stability. A counterintuitive fact that an isolated negative capacitor is always unstable was explained by a simple equivalent circuit that contains an amplifier with positive feedback. More realistic version of this circuit with one pole in the gain function enables successful modeling of a realistic negative non-Foster capacitor with both gain/loss and dispersion. Stability analysis based on both models show that the net mesh capacitances and net mesh resistances must always have positive values in order to achieve stable operation. All achieved results are in a very good agreement with recent experiments performed at University of Zagreb.

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