On weak annihilators and nilpotent associated primes of skew PBW extensions

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ABSTRACT
We investigate the notions of weak annihilator and nilpotent associated prime defined by Ouyang and Birkenmeier (Weak annihilator over extension rings, 2012) in the setting of skew PBW extensions. We extend several results formulated in the literature concerning annihilators and associated primes of commutative rings and skew polynomial rings to a more general setting of algebras not considered before. We exemplify our results with families of algebras appearing in the theory of enveloping algebras, noncommutative algebraic geometry, and theoretical physics. Finally, we present some ideas for future research.

1. Introduction
Throughout the paper, every ring is associative (not necessarily commutative) with identity unless otherwise stated. For a ring $R$, $P(R)$, $N(R)$, and $N^*(R)$ denote the prime radical of $R$, the set of all nilpotent elements of $R$, and the upper radical $N^*(R)$ of $R$ (i.e., the sum of all its nil ideals of $R$), respectively. It is well-known that $N^*(R) \subseteq N(R)$, and if the equality $N^*(R) = N(R)$ holds, Marks [59] called $R$ an NI ring. Due to Birkenmeier et al. [15], $R$ is called 2-primal if $P(R) = N(R)$ (Sun [83] used the term weakly symmetric for these rings). Notice that every reduced ring (a ring that has no non-zero nilpotent elements) is 2-primal. The importance of 2-primal rings is that they can be considered as a generalization of commutative rings and reduced rings. For more details about 2-primal rings, see Birkenmeier et al. [17] and Marks [60].

For a subset $B$ of a ring $R$, the sets $r_R(B) = \{ r \in R \mid Br = 0 \}$ and $l_R(B) = \{ r \in R \mid rB = 0 \}$ represent the right and left annihilator of $B$ in $R$, respectively. Properties of annihilators of subsets in rings have been investigated by several authors [9, 21, 22, 47, 88]. For example, Kaplansky [47] introduced the Baer rings as those rings for which the right (left) annihilator of every nonempty subset of the ring is generated by an idempotent element. This concept has its roots in functional analysis, having close links to $C^*$-algebras and von Neumann algebras. Closely related to Baer rings are the quasi-Baer rings. A ring $R$ is said to be quasi-Baer if the left annihilator of an ideal of $R$ is generated by an idempotent. Now, $R$ is called right (left) p.p. (principally projective) or left Rickart ring if the right (left) annihilator of each element of $R$ is generated by an idempotent (equivalently, any principal left ideal of the ring is projective). Birkenmeier [16] defined a ring $R$ to be right (left) principally quasi-Baer (or simply right (left) p.q.-Baer) ring if the right annihilator of each principal right (left) ideal of $R$ is generated (as a right ideal) by an idempotent. Families of rings satisfying Baer, quasi-Baer, p.p. and p.q.-Baer conditions have been presented in the literature. For instance, Armendariz [4] established that if $R$ is a reduced ring, then the commutative polynomial ring $R[x]$ is a Baer ring if and only if $R$ is a Baer ring [4, Theorem B].

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Dedicated to the memory of Professor V. A. Artamonov.
Birkenmeier et al. [16] showed that the quasi-Baer condition is preserved by many polynomial extensions and proved that a ring $R$ is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer. For more details about these ring-theoretical properties, commutative and noncommutative examples, see [16, 34, 35, 39, 75], and references therein, and the excellent treatment by Birkenmeier et al. [17].

As a generalization of right and left annihilators, Ouyang and Birkenmeier [65] introduced the notion of weak annihilator of a subset in a ring, and investigated its properties over Ore extensions (also known as skew polynomial rings) $R[\sigma; \delta]$ defined by Ore [69]. More exactly, for a ring $R$ and $B$ a subset of $R$, they defined $\text{ann}_B R = \{ r \in R \mid br \in B \}$ for all $b \in B$, which is called the weak annihilator of $B$ in $R$ [65, Definition 2.1]. The theory developed by them makes use of the notion of compatible ring in the sense of Annin [1] (see also Hashemi and Moussavi [35]). Briefly, if $R$ is a ring, $\sigma$ is an endomorphism of $R$, and $\delta$ is a $\sigma$-derivation of $R$, then (i) $R$ is said to be $\sigma$-compatible if for each $a, b \in R$, $ab = 0$ if and only if $a \sigma(b) = 0$ (necessarily, the endomorphism $\sigma$ is injective). (ii) $R$ is called $\delta$-compatible if for each $a, b \in R$, $ab = 0$ implies $a \delta(b) = 0$. (iii) If $R$ is both $\sigma$-compatible and $\delta$-compatible, then $R$ is called $(\sigma, \delta)$-compatible. For $R$ a $(\sigma, \delta)$-compatible 2-primal ring, Ouyang and Birkenmeier [65] proved different results concerning (principal) ideals generated by nilpotent elements [65, Theorems 2.1, 2.2, 2.3, 2.4, and 2.5].

With the aim of investigating if the results obtained by Ouyang and Birkenmeier hold in noncommutative polynomial extensions more general than Ore extensions, in this paper we consider the class of skew Poincaré-Birkhoff-Witt extensions introduced by Gallego and Lezama [25] which include families of noncommutative rings appearing in ring theory and algebraic geometry (see Section 2 for a description of the generality of these objects with respect to other families of noncommutative rings). Ring-theoretical, homological and geometrical properties of these objects have been investigated by some people (e.g. [5, 24, 33, 38, 54, 82, 84]).

The paper is organized as follows. In Section 2, we recall some definitions and results about skew PBW extensions and compatible rings for finite families of endomorphisms and derivations of rings. Section 3 contains the original results of the article concerning weak annihilator ideals of skew PBW extensions (Theorems 3.4, 3.5, 3.7, 3.8, 3.10, 3.11, and 3.13). Next, in Section 4, we characterize the nilpotent associated prime ideals of skew PBW extensions (Theorems 4.3 and 4.4). As expected, our results generalize those above corresponding skew polynomial extensions presented by Ouyang et al. [65, 67]. It is worth mentioning that this work is a sequel of the study of ideals of skew PBW extensions that has been realized by different authors (e.g. [33, 55, 57, 64, 73]). In this way, the results formulated in this paper about associated primes extend or contribute to those presented by Annin [1], Bhat [14], Brewer and Heinzer [18], Faith [22], Leroy and Matczuk [53], and Niño et al. [63]. Finally, Section 5 illustrates the results established in Sections 3 and 4 with several noncommutative algebras that cannot be expressed as Ore extensions.

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the classical numerical systems. We assume the set of natural numbers including zero. The symbol $k$ denotes a field and $k^* := k \setminus \{0\}$.

2. Skew Poincaré-Birkhoff-Witt extensions

Skew PBW extensions were defined by Gallego and Lezama [25] with the aim of generalizing Poincaré-Birkhoff-Witt extensions introduced by Bell and Goodearl [10] and Ore extensions of injective type defined by Ore [69]. Over the years, several authors have shown that skew PBW extensions also generalize families of noncommutative algebras such as 3-dimensional skew polynomial algebras introduced by Bell and Smith [11], diffusion algebras defined by Isaev et al. [40], ambiskew polynomial rings introduced by Jordan [41–43], solvable polynomial rings introduced by Kandri-Rody and Weispfenning [46], almost normalizing extensions defined by McConnell and Robson [61], skew bi-quadratic algebras recently introduced by Bavula [8], and others (see [24] for more details). Relations between skew PBW extensions and other noncommutative algebras having PBW bases can be found in [24, 27, 81], and references therein.
Definition 2.1. [25, Definition 1] Let $R$ and $A$ be rings. We say that $A$ is a skew PBW extension (also known as $\sigma$-PBW extension) over $R$, which is denoted by $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$, if the following conditions hold:

(i) $R$ is a subring of $A$ sharing the same multiplicative identity element.

(ii) there exist elements $x_1, \ldots, x_n \in A$ such that $A$ is a left free $R$-module with basis given by $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$, where $x_1^0 \cdots x_n^0 := 1$.

(iii) For each $1 \leq i \leq n$, there exists an element $c_i \in \mathbb{R} \setminus \{0\}$ such that $x_i r - c_i x_i r \in R$.

(iv) For any elements $1 \leq i, j \leq n$, there exists $d_{ij} \in R \setminus \{0\}$ such that $x_i x_j - d_{ij} x_j x_i + c_{ij} r = r_{ij} + \sum_{k=1}^{\infty} r_{k}^{(ij)} x_k$.

From Definition 2.1 it follows that every non-zero element $f$ of a skew PBW extension can be uniquely expressed as $f = a_0 x_0 + a_1 x_1 + \cdots + a_m x_m$, with $a_i \in R$ and $x_i \in \text{Mon}(A)$, for $0 \leq i \leq m$ ($x_0 := 1$) [25, Remark 2].

Thinking about Ore extensions [69], if $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a skew PBW extension over $R$, then for every $1 \leq i \leq n$ there exists an injective endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$ [25, Proposition 3]. When necessary, we will write $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$.

Definition 2.2. Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension over $R$.

(i) [25, Definition 4] $A$ is called quasi-commutative if the conditions (iii) and (iv) in Definition 2.1 are replaced by the following: (iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{ir} \in R \setminus \{0\}$ such that $x_i r = c_{ir} x_i r$.

(ii) [25, Definition 4] $A$ is called bijective if $\sigma_i$ is bijective for each $1 \leq i \leq n$, and $d_{ij}$ is invertible, for any $1 \leq i, j \leq n$.

(iii) [55, Definition 2.3] $A$ is called of endomorphism type if $\delta_i = 0$, for every $i$. In addition, if every $\sigma_i$ is bijective, then $A$ is said to be a skew PBW extension of automorphism type.

Some relations between skew polynomial rings [69], PBW extensions [10] and skew PBW extensions are presented below.

Remark 2.3. (i) If $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an iterated Ore extension such that $\sigma_i$ is injective for $1 \leq i \leq n$; $\delta_i(r_i) \in R$, for $r_i \in R$ and $1 \leq i \leq n$; $\sigma_i(x_i) = c_i x_i + d_i$, for $i < j$, $c_i, d_i \in R$, where $c_i$ has a left inverse; $\delta_i(x_i) \in R + R x_1 + \cdots + R x_n$, for $i < j$, then $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \cong R(R)\langle x_1, \ldots, x_n \rangle$ [56, Example 5(3)].

(ii) Skew polynomial rings are not included in PBW extensions. For instance, the quantum plane defined as the quotient $k[x, y]/(xy - qyx | q \in \mathbb{k}\otimes k)$ is an Ore extension of injective type given by $k[y][x; \sigma]$, where $\sigma(y) = qy$, but it is clear that this cannot be expressed as a PBW extension over $k$ or $k[y]$.

(iii) Skew PBW extensions are not contained in skew polynomial rings. For example, the universal enveloping algebra $U(g)$ of a Lie algebra $g$ is a PBW extension [10, Section 5], and hence a skew PBW extension, but in general it cannot be expressed as an Ore extension.

(iv) If $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a quasi-commutative skew PBW extension over a ring $R$, then $A$ is isomorphic to an iterated Ore extension of endomorphism type (i.e., that is, $R[x; \sigma]$ with $\sigma$ surjective) [56, Theorem 2.3].

(v) Skew PBW extensions of endomorphism type are more general than iterated Ore extensions of endomorphism type [38, Remark 3.4].

Following [25, Section 3], if $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a skew PBW extension over $R$, then we consider the following notation:
(i) For the families $\Sigma$ and $\Delta$ in Definition 2.1, from now on, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ belongs to $\mathbb{N}^n$, we will write $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\sigma^{-\alpha} := \sigma_n^{\alpha_n} \circ \cdots \circ \sigma_1^{\alpha_1}$, and $\delta^\alpha := \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) Let $\preceq$ be a total order defined on $\text{Mon}(A)$. If $x^\alpha \preceq x^\beta$ but $x^\alpha \neq x^\beta$, we will write $x^\alpha > x^\beta$. As we saw above, if $f$ is a non-zero element of $A$, then $f$ can be expressed uniquely as $f = a_0 X_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R$, $X_m > \cdots > X_1$, and $X_0 := 1$. Eventually, we use expressions as $f = a_0 Y_0 + a_1 Y_1 + \cdots + a_m Y_m$, with $a_i \in R$, and $Y_m > \cdots > Y_1$. We define $\text{lm}(f) := X_m$, the leading monomial of $f$; $\text{lc}(f) := a_m$, the leading coefficient of $f$; $\text{lt}(f) := a_m X_m$, the leading term of $f$; $\text{exp}(f) := \text{exp}(X_m)$, the order of $f$. Notice that $\deg(f) := \max(\deg(X_i))_{i=1}^m$. If $f = 0$, then $\text{lm}(0) := 0$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$. Finally, we also consider $X > 0$ for any $X \in \text{Mon}(A)$, and hence we extend $\succeq$ to $\text{Mon}(A) \cup \{0\}$.

From [25, Definition 11], if $\succeq$ is a total order on $\text{Mon}(A)$, then we say that $\succeq$ is a monomial order on $\text{Mon}(A)$ if the following conditions hold:

- For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in \text{Mon}(A)$, the relation $x^\beta \succeq x^\alpha$ implies $\text{lm}(x^\gamma x^\alpha x^\lambda) \succeq \text{lm}(x^\gamma x^\alpha x^\lambda)$ (i.e., the total order is compatible with multiplication).
- $x^\alpha \succeq 1$, for every $x^\alpha \in \text{Mon}(A)$.
- $\succeq$ is degree compatible, i.e., $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

Monomial orders are also called admissible orders. The condition (iii) of the previous definition is needed in the proof of the fact that every monomial order on $\text{Mon}(A)$ is a well order, that is, there are not increasing decreasing chains in $\text{Mon}(A)$ [25, Proposition 12].

(iii) For a skew PBW extension $A = \sigma(R) (x_1, \ldots, x_n)$ over a ring $R$ and $B$ a subset of $R$, $BA$ denotes the set $\{ f = \sum_{i=0}^{\infty} b_i x_i \mid b_i \in B, \text{ for all } i \}$.

Propositions 2.4 and 2.5 are very useful in the calculations presented in Sections 3 and 4.

**Proposition 2.4.** [25, Theorem 7] If $A$ is a polynomial ring with coefficients in $R$ with respect to the set of indeterminates $\{x_1, \ldots, x_n\}$, then $A = \sigma(R) (x_1, \ldots, x_n)$ is a skew PBW extension over $R$ if and only if the following conditions hold:

1. For each $x^\alpha \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If $r$ is left invertible, so is $r_\alpha$.
2. For each $x^\alpha, x^\beta \in \text{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

**Proposition 2.5.** [73, Proposition 2.7 and Remark 2.8] Let $A = \sigma(R) (x_1, \ldots, x_n)$ be a skew PBW extension over $R$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $r$ is an element of $R$, then

$$
x^\alpha r = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left( \sum_{j=1}^{\alpha_n} x_n^{-j} \delta_n j^{-1}(\sigma_n^{-1}(r)) j^{-1} \right) x_n^{\alpha_n}
$$

\[+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left( \sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{-j} \delta_{n-1} j^{-1}(\sigma_{n-1}^{-1}(\sigma_n^{-1}(r))) j^{-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}
\]

\[+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left( \sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{-j} \delta_{n-2} j^{-1}(\sigma_{n-2}^{-1}(\sigma_n^{-1}(r))) j^{-1} \right) x_{n-2}^{\alpha_{n-2}} x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}
\]

\[+ \cdots + x_1^{\alpha_1} \left( \sum_{j=1}^{\alpha_2} x_2^{-j} \delta_{2} j^{-1}(\sigma_2^{-1}(\sigma_3^{-1}(\cdots(\sigma_n^{-1}(r)))) j^{-1} \right) x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}
\]

\[+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \sigma_j^0 := \text{id}_R \text{ for } 1 \leq j \leq n.
\]
If \( a_i, b_j \in R \) and \( X_i := x_1^{a_{i1}} \cdots x_n^{a_{in}}, \ Y_j := b_1 y_1 \cdots b_j y_m \), when we compute every summand of \( a_iX_i b_jY_j \) we obtain products of the coefficient \( a_i \) with several evaluations of \( b_j \) in \( \sigma 's \) and \( \delta 's \) depending on the coordinates of \( \alpha_i \). This assertion follows from the expression:

\[
\begin{align*}
a_iX_i b_jY_j &= a_i \sigma^{a_i}(b_j)x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m} \\
&\quad + a_i x_1^{a_{i1}} p_{a_1} \sigma_{a_1}^{a_1}(\cdots (\sigma_{a_{in}}(b_j))) x_2^{a_{i2}} \cdots x_n^{a_{in}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m} \\
&\quad + a_i x_1^{a_{i1}} x_2^{a_{i2}} p_{a_1} \sigma_{a_1}^{a_1}(\cdots (\sigma_{a_{in}}(b_j))) x_3^{a_{i3}} \cdots x_n^{a_{in}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m} \\
&\quad \vdots \\
&\quad + a_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_{i(n-2)}^{a_{i(n-2)}} p_{a_{i(n-1)}} \sigma_{a_{i(n-1)}}^{a_{i(n-1)}}(b_j) x_{i(n-1)}^{a_{i(n-1)}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m} \\
&\quad + a_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_{i(n-1)}^{a_{i(n-1)}} p_{a_{i(n)}} x_{i(n)}^{a_{i(n)}} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m},
\end{align*}
\]

where the polynomials \( p's \) are given by Proposition 2.4.

### 2.1. \((\Sigma, \Delta)\)-compatible and weak \((\Sigma, \Delta)\)-compatible rings

As we said in the introduction, following Annin [1] (see also Hashemi and Moussavi [35]), for \( \sigma \) an endomorphism of a ring \( R \) and \( \delta \) a \( \sigma \)-derivation, a ring \( R \) is said to be \( \sigma \)-compatible if for every \( a, b \in R \), we have \( ab = 0 \) if and only if \( a\sigma(b) = 0 \) (necessarily, \( \sigma \) is injective). \( R \) is called \( \delta \)-compatible if for each \( a, b \in R \), \( ab = 0 \) implies \( a\delta(b) = 0 \). If \( R \) is both \( \sigma \)-compatible and \( \delta \)-compatible, then \( R \) is said to be \((\sigma, \delta)\)-compatible. From [34, Lemma 3.3] we know that \( (\sigma, \delta) \)-compatible rings generalize the \( \sigma \)-rigid rings (an endomorphism \( \sigma \) of a ring \( R \) is called rigid if \( \sigma(a) = a \) implies \( a = 0 \), where \( a \in R \); \( R \) is said to be rigid if there exists a rigid endomorphism \( \sigma \) of \( R \) introduced by Krempa [49]). Thinking in the context of skew PBW extensions, Hashemi et al. [32] and Reyes and Suárez [75] introduced independently the \((\Sigma, \Delta)\)-compatible rings as a natural generalization of \((\sigma, \delta)\)-compatible rings. Examples, ring and module theoretic properties of these structures can be found in [30, 33, 57, 72, 78].

Since the notion of \((\sigma, \delta)\)-compatible ring was used by Ouyang and Birkenmeier [65] in their treatment of weak annihilators of Ore extensions, it is to be expected that we have to consider the notion of \((\Sigma, \Delta)\)-compatible ring to characterize this type of annihilators in the setting of skew PBW extensions. In this way, next we consider a finite family of endomorphisms \( \Sigma \) and a finite family \( \Delta \) of \( \sigma \)-derivations of a ring \( R \) in a similar way to the established in Definition 2.1.

**Definition 2.6** ([32, Definition 3.1]; [75, Definition 3.2]). \( R \) is said to be \( \Sigma \)-compatible if for each \( a, b \in R \), \( a\sigma^\alpha(b) = 0 \) if and only if \( ab = 0 \), where \( \alpha \in \mathbb{N}^n \). \( R \) is said to be \( \Delta \)-compatible if for each \( a, b \in R \), it follows that \( ab = 0 \) implies \( a\delta(b) = 0 \), where \( \beta \in \mathbb{N}^n \). If \( R \) is both \( \Sigma \)-compatible and \( \Delta \)-compatible, then \( R \) is called \( (\Sigma, \Delta) \)-compatible.

The following proposition is the natural generalization of [35, Lemma 2.1].

**Proposition 2.7** ([32, Lemma 3.3]; [75, Proposition 3.8]). If \( R \) is a \((\Sigma, \Delta)\)-compatible ring, then for every elements \( a, b \) belonging to \( R \), we have the following assertions:

1. If \( ab = 0 \), then \( a\sigma^\theta(b) = a\sigma^\theta(a)b = 0 \), where \( \theta \in \mathbb{N}^n \).
2. If \( a\sigma^\beta(b) = 0 \), for some \( \beta \in \mathbb{N}^n \), then \( ab = 0 \).
3. If \( ab = 0 \), then \( a\delta^\theta(b) = a\delta^\theta(a)\sigma^\delta(b) = 0 \), where \( \theta, \beta \in \mathbb{N}^n \).

**Example 2.8.** Let \( \mathbb{F}_4 = \{0, 1, a, a^2 \} \) be the field of four elements. Consider the ring of polynomials \( \mathbb{F}_4[z] \) and let \( R = \mathbb{F}_4[z]/(z^2) \). For simplicity, we identify the elements of \( \mathbb{F}_4[z] \) with their images in \( R \). Let \( \Sigma = \{\sigma_{ij}\} \) be the family of endomorphisms of \( R \) defined by \( \sigma_{ij}(a) = a^i \) and \( \sigma_{ij}(z) = a^jz \), for \( 1 \leq i \leq 2 \) and \( 0 \leq j \leq 2 \). Consider the skew PBW extension \( A = \sigma(R)\langle x_1, x_1, x_1, x_2, x_2, x_{2,1}, x_{2,2} \rangle \), where the indeterminates satisfy the commutation relations \( x_i x_j = x_j x_i \), for all \( 1 \leq i, j \leq 2 \) and \( 0 \leq i, j \leq 2 \).
On the other hand, for \( a'z \in R \), we have that \( x_{ij}a'z = \sigma_{ij}(a'z)x_{ij} = (a')_i^j a'z x_{ij} = a'^{i+j} z x_{ij} \), where \( a'^{i+j} \in \mathbb{F}_2 \), \( 1 \leq i, j \leq 2 \) and \( 1 \leq r, j \leq 2 \). This example can be extended to any finite field \( \mathbb{F}_p \) with \( p \) a prime number. It is straightforward to show that \( R \) is \( \Sigma \)-compatible, and thus \( A \) is a skew PBW extension over a \( \Sigma \)-compatible ring \( R \).

**Example 2.9.** Let \( k[t] \) be the commutative polynomial ring over the field \( k \) in the indeterminate \( t \). We consider the identity endomorphism \( \sigma \) of \( k[t] \) and \( \delta(t) := 1 \). Having in mind that \( k[t] \) is reduced, we can show that \( k[t] \) is a \((\sigma, \delta)\)-compatible ring. Let \( R \) be the ring defined as

\[
R = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in k[t] \right\}.
\]

The idea is to extend the usual derivation. In this way, the endomorphism \( \sigma \) of \( k[t] \) is extended to the endomorphism \( \overline{\sigma} : R \to R \) by defining \( \overline{\sigma}((a_{ij})) = (\sigma(a_{ij})) \) for every \( 1 \leq i, j \leq 2 \), and the \( \sigma \)-derivation \( \delta \) of \( k[t] \) is also extended to \( \overline{\delta} : R \to R \) by considering \( \overline{\delta}((a_{ij})) = (\delta(a_{ij})) \) for every \( 1 \leq i, j \leq 2 \). Then the Ore extension \( R[x; \overline{\sigma}, \overline{\delta}] \) is a skew PBW extension over \( R \) which is \((\overline{\sigma}, \overline{\delta})\)-compatible.

Reyes and Suárez [76] defined the concept of weak \((\Sigma, \Delta)\)-compatible ring as a generalization of \((\Sigma, \Delta)\)-compatible rings and weak \((\sigma, \delta)\)-compatible rings introduced by Ouyang and Liu [66]. For the next definition, consider again a finite family of endomorphisms \( \Sigma \) and a finite family \( \Delta \) of \( \Sigma \)-derivations of a ring \( R \).

**Definition 2.10.** [76, Definition 4.1] \( R \) is said to be weak \( \Sigma \)-compatible if for each \( a, b \in R \), \( a a^\alpha (b) \in N(R) \) if and only if \( a b \in N(R) \), where \( \alpha \in \mathbb{N}^n \). \( R \) is said to be weak \( \Delta \)-compatible if for each \( a, b \in R \), \( a b \in N(R) \) implies \( a^\delta \beta (b) \in N(R) \), where \( \beta \in \mathbb{N}^n \). If \( R \) is both weak \( \Sigma \)-compatible and weak \( \Delta \)-compatible, then \( R \) is called weak \((\Sigma, \Delta)\)-compatible.

It is clear that the following result extends Proposition 2.7.

**Proposition 2.11.** [76, Proposition 4.2] If \( R \) is a weak \((\Sigma, \Delta)\)-compatible ring, then the following assertions hold:

1. If \( ab \in N(R) \), then \( a a^\alpha (b), \sigma^\alpha (a)b \in N(R) \), for all \( \alpha, \beta \in \mathbb{N}^n \).
2. If \( \sigma^\alpha (a)b \in N(R) \), for some element \( \alpha \in \mathbb{N}^n \), then \( ab \in N(R) \).
3. If \( a^\alpha (b) \in N(R) \), for some element \( \beta \in \mathbb{N}^n \), then \( ab \in N(R) \).
4. If \( ab \in N(R) \), then \( \sigma^\alpha (a)\delta^\beta (b), \delta^\beta (a)\sigma^\alpha (b) \in N(R) \), for \( \alpha, \beta \in \mathbb{N}^n \).

The next example shows that there exists a weak \((\Sigma, \Delta)\)-compatible ring which is not \((\Sigma, \Delta)\)-compatible.

**Example 2.12.** Let \( R \) be a ring and \( M \) an \((R, R)\)-bimodule. The trivial extension of \( R \) by \( M \) is defined as the ring \( T(R, M) := R \oplus M \) with the usual addition and the multiplication defined as \((r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + m_1 r_2)\), for all \( r_1, r_2 \in R \) and \( m_1, m_2 \in M \). Notice that \( T(R, M) \) is isomorphic to the matrix ring (with the usual matrix operations) of the form \((\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})\), where \( r \in R \) and \( m \in M \). In particular, we call \( S_2(\mathbb{Z}) \) the ring of matrices isomorphic to \( T(\mathbb{Z}, \mathbb{Z}) \).

Consider the ring \( S_2(\mathbb{Z}) \) given by

\[
S_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.
\]

Let \( \sigma_1 = \text{id}_{S_2(\mathbb{Z})} \) be the identity endomorphism of \( S_2(\mathbb{Z}) \), and let \( \sigma_2 \) and \( \sigma_3 \) be the two endomorphisms of \( S_2(\mathbb{Z}) \) defined by
respectively. Notice that the set of nilpotent elements of \( S_2(\mathbb{Z}) \) is given by

\[
N(S_2(\mathbb{Z})) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}.
\]

It is straightforward to see that \( S_2(\mathbb{Z}) \) is weak \( \Sigma \)-compatible where \( \Sigma = \{\sigma_1, \sigma_2, \sigma_3\} \), but \( S_2(\mathbb{Z}) \) is not \( \sigma_3 \)-compatible. For example, if we take the matrices \( C, D \in S_2(\mathbb{Z}) \) given by

\[
C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

then it is clear that \( C\sigma_3(D) = 0 \) but \( CD = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0 \), whence \( S_2(\mathbb{Z}) \) is not \( \Sigma \)-compatible.

Ouyang and Liu [66, Lemma 2.13] characterized the nilpotent elements in an Ore extension over a weak \((\sigma, \delta)\)-compatible and NI ring. Since skew PBW extensions generalize Ore extensions of injective type, Reyes and Suárez [76] extended this result as the following proposition shows. The NI property for skew PBW extensions was recently studied by Suárez et al. [82].

**Proposition 2.13.** [76, Theorem 4.6] If \( A = \sigma(R)(x_1, \ldots, x_n) \) is a skew PBW extension over a weak \((\Sigma, \Delta)\)-compatible and NI ring \( R \), then \( f = a_0 + a_1X_1 + \cdots + a_mX_m \in N(A) \) if and only if \( a_i \in N(R) \), for every \( i \).

Notice that Proposition 2.13 also generalizes the results presented by Ouyang and Birkenmeier for Ore extensions [65, Lemma 2.6], and Ouyang and Liu for differential polynomials rings [67, Lemma 2.12]. Furthermore, since weak \((\Sigma, \Delta)\)-compatible rings contain strictly \((\Sigma, \Delta)\)-compatible rings, we have immediately the following corollary for skew PBW extensions over \((\Sigma, \Delta)\)-compatible rings.

**Corollary 2.14.** If \( A = \sigma(R)(x_1, \ldots, x_n) \) is a skew PBW extension over a \((\Sigma, \Delta)\)-compatible and NI ring \( R \), then the following statements hold:

1. \( N(A) \) is an ideal of \( A \) and \( N(A) = N(R)\langle x_1, \ldots, x_n \rangle \).
2. \( f = a_0 + a_1X_1 + \cdots + a_mX_m \in N(A) \) if and only if \( a_i \in N(R) \), for every \( i = 0, \ldots, m \).

**Corollary 2.14** generalizes the corresponding theorem for Ore extensions of injective type over \((\sigma, \delta)\)-compatible and 2-primal rings [65, Corollary 2.2], and the result for differential polynomials rings over \( \delta \)-compatible reversible rings [67, Corollary 2.13].

**Proposition 2.13** and **Corollary 2.14** are key for the other results of the paper. Precisely, the following example shows that the NI condition is not superfluous.

**Example 2.15.** Let \( M_2(\mathbb{Z}_4) \) be the ring of matrices of size \( 2 \times 2 \) over the ring \( \mathbb{Z}_4 \), and consider the Ore extension \( M_2(\mathbb{Z}_4)[x; \sigma] \) where \( \sigma \) is the endomorphism of \( M_2(\mathbb{Z}_4) \) defined as \( \sigma \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \).

It is clear that \( M_2(\mathbb{Z}_4) \) is a \( \sigma \)-compatible ring. On the other hand, an important fact about \( M_2(\mathbb{Z}_4) \) is that the sum of nilpotent elements of the ring is not always a nilpotent element, i.e., it is not a NI ring. Indeed, we have \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in N(M_2(\mathbb{Z}_4)) \), but \( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin N(M_2(\mathbb{Z}_4)) \), which shows that \( M_2(\mathbb{Z}_4) \) is not NI.

Additionally, for \( f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_4)[x; \sigma] \), we have \( f(x) \) is an element of \( N(M_2(\mathbb{Z}_4))[x; \sigma] \) but

\[
f(x)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^2 x^2 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \neq 0.
\]

In this way, \( f(x) \notin N(M_2(\mathbb{Z}_4))[x; \sigma] \) and so \( N(M_2(\mathbb{Z}_4))[x; \sigma]) \neq N(M_2(\mathbb{Z}_4))[x; \sigma] \).
Example 2.16. Let $M_2(\mathbb{Z}_2)$ be the ring of matrices of size $2 \times 2$ over the field $\mathbb{Z}_2$, and the Ore extension $M_2(\mathbb{Z}_2)[x; \sigma]$ where $\sigma$ is the identity endomorphism of $M_2(\mathbb{Z}_2)$. Clearly, $M_2(\mathbb{Z}_2)$ is a $\sigma$-compatible ring which is not NI. Indeed, we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in N(M_2(\mathbb{Z}_2))$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin N(M_2(\mathbb{Z}_2))$, concluding that $M_2(\mathbb{Z}_2)$ is not NI. Again, for the element $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)[x; \sigma]$, we obtain $f(x) \in N(M_2(\mathbb{Z}_2))[x; \sigma]$ but

$$f(x)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}x + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}x \neq 0.$$ 

In this way, $f(x) \notin N(M_2(\mathbb{Z}_2))[x; \sigma]$ and so $N(M_2(\mathbb{Z}_2))[x; \sigma] \neq N(M_2(\mathbb{Z}_2))[x; \sigma]$.

Ouyang et al. [68] introduced the notion of skew $\pi$-Armendariz ring. Briefly, if $R$ is a ring with an endomorphism $\sigma$ and a $\sigma$-derivation $\delta$, then $R$ is called skew $\pi$-Armendariz ring if for polynomials $f(x) = \sum_{i=0}^l a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j$ in $\mathbb{R}[x; \sigma, \delta]$, $f(x)g(x) \in N(R)$ implies that $a_ib_j \in N(R)$, for each $0 \leq i \leq l$ and $0 \leq j \leq m$. Skew $\pi$-Armendariz rings are more than skew Armendariz rings when the ring of coefficients is $(\sigma, \delta)$-compatible [68, Theorem 2.6], and also extend $\sigma$-Armendariz rings defined by Hong et al. [37] considering $\delta$ as the zero derivation. For a detailed description of Armendariz rings in the commutative and noncommutative context, including skew PBW extensions, see [4, 17, 39].

Ouyang and Liu [66] showed that if $R$ is a weak $(\sigma, \delta)$-compatible and NI ring, then $R$ is skew $\pi$-Armendariz ring [66, Corollary 2.15]. Reyes [71] formulated the analogue of skew $\pi$-Armendariz ring in the setting of skew PBW extensions. For a skew PBW extension $A = \sigma(R) \langle x_1, \ldots, x_n \rangle$ over a ring $R$, we say that $R$ is skew $\Pi$-Armendariz ring if for elements $f = \sum_{i=0}^l a_i x_i$ and $g = \sum_{j=0}^m b_j Y_j$ belong to $A$, $fg \in N(A)$ implies $a_ib_j \in N(R)$, for each $0 \leq i \leq l$ and $0 \leq j \leq m$. Reyes [71, Theorem 3.10] showed that if $R$ is reversible (following Cohn [20, p. 641], $R$ is called reversible if $ab = 0$ implies $ba = 0$, where $a, b \in R$) and $(\Sigma, \Delta)$-compatible, then $R$ is skew $\Pi$-Armendariz ring. Related to this result, the following proposition generalizes [67, Lemma 2.14] and [65, Corollary 2.3].

Proposition 2.17. [76, Theorem 4.7] Let $A = \sigma(R) \langle x_1, \ldots, x_n \rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $f = a_0 + a_1 X_1 + \cdots + a_m X_m \in N(A)$ and $g = b_0 + b_1 Y_1 + \cdots + b_t Y_t$ are elements of $A$, then $fg \in N(A)$ if and only if $a_ib_j \in N(R)$, for all $i, j$.

3. Weak Annihilator Ideals

In this section, we study the weak notion of annihilator introduced by Ouyang and Birkenmeier [65] in the setting of skew PBW extensions.

Definition 3.1. [65, Definition 2.1] Let $R$ be a ring. For $X \subseteq R$, it is defined $N_R(X) = \{a \in R \mid xa \in N(R) \text{ for all } x \in X\}$, which is called the weak annihilator of $X$ in $R$. If $X$ is a singleton, say $X = \{r\}$, we use $N_R(r)$ to denote $N_R(\{r\})$.

Notice that for $X \subseteq R$, the sets given by $\{a \in R \mid xa \in N(R) \text{ for all } x \in X\}$ and $\{b \in R \mid bx \in N(R) \text{ for all } x \in X\}$ coincide. Moreover, $l_R(X)$, $r_R(X) \subseteq N_R(X)$, which is called the weak annihilator of $X$ in $R$. If $R$ is reduced, then $r_R(X) = l_R(X) = N_R(X)$. In addition, if $N(R)$ is an ideal of $R$, then $N_R(X)$ is also an ideal of $R$ [65, p. 346].

Example 3.2. [65, Example 2.1] Let $T_2(\mathbb{Z})$ be the $2 \times 2$ upper triangular matrix ring over the ring of integers $\mathbb{Z}$ and $X = \{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in \mathbb{Z}\} \subseteq T_2(\mathbb{Z})$. Then $r_{T_2(\mathbb{Z})}(X) = 0$ and $N_{T_2(\mathbb{Z})}(X) = \{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in \mathbb{Z}\}$, whence $r_{T_2(\mathbb{Z})}(X) \neq N_{T_2(\mathbb{Z})}(X)$. This shows that a weak annihilator is not an immediate generalization of an annihilator.

We recall some properties of weak annihilators.
Proposition 3.3. [65, Proposition 2.1] If $X$ and $Y$ are subsets of a ring $R$, then we have the following:

1. $X \subseteq Y$ implies $N_R(Y) \subseteq N_R(X)$.
2. $X \subseteq N_R(N_R(X))$.
3. $N_R(X) = N_R(N_R(N_R(X)))$.

Proof. First, let us show that $R$. Iffor each subset $X \subseteq R$, let $N\text{Ann}_R(R) := \{N_R(U) \mid U \subseteq R\}$ and $N\text{Ann}_A(A) := \{N_A(V) \mid V \subseteq A\}$. For $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$, we denote by $\{a_0, \ldots, a_m\}$ or $C_f$ the set of the coefficients of $f$, and for a subset $V \subseteq A$, $CV = \bigcup_{f \in U} C_f$.

The following theorem establishes a bijective correspondence between weak annihilators of $A$ and weak annihilators of $R$. This result generalizes [77, Theorem 3.21] which was formulated for classical annihilators.

Theorem 3.4. If $A = (\sigma(R) \left<x_1, \ldots, x_n\right>$ is a skew PBW extension over a $(\Sigma, \Delta)$-compatible and NI ring $R$, then the correspondence $\varphi : \text{NAnn}_R(R) \to \text{NAnn}_A(A)$, given by $\varphi(N_R(U)) = N_R(U)A$, for every $N_R(U) \in \text{NAnn}_R(R)$, is bijective.

Proof. First, let us show that $\varphi$ is well defined, i.e., $N_R(U) = N_R(U)A$, for every nonempty subset $U$ of $R$. Consider $f = a_0 + a_1X_1 + \cdots + a_iX_i \in N_R(U)A$ and $r \in U$. Since $a_i \in N_R(U)$, for all $0 \leq i \leq l$, this implies that $ra_i \in N(R)$, for each $i$. Thus, by Corollary 2.14, we get $rf = ra_0 + ra_1X_1 + \cdots + ra_iX_i \in N(R)A = N(A)$. Hence, $N_R(U)A \subseteq N_A(U)$.

Now, let us prove that $N_A(U) \subseteq N_R(U)A$. If $f = a_0 + a_1X_1 + \cdots + a_iX_i \in N_A(U)$, then $rf = ra_0 + ra_1X_1 + \cdots + ra_iX_i \in N(R)A$, for every element $r$ of $U$. This means that $ra_i \in N(R)$, for all $0 \leq i \leq l$, and so $a_i \in N_R(U)$, for each $i$, whence $f \in N_R(U)A$. Therefore, $N_A(U) = N_R(U)A$.

Let $U, V$ be two subsets of $R$ such that $\varphi(N_R(U)) = \varphi(N_R(V))$. By definition of $\varphi$, $N_A(U) = N_A(V)$. In particular, we obtain $N_R(U) = N_R(V)$, that is, $\varphi$ is injective.

Finally, let us show that $\varphi$ is surjective. Let $N_A(V) \in \text{NAnn}_A(A)$ and consider $g = b_0 + b_1Y_1 + \cdots + b_sY_s \in N_A(V)$, for a subset $V$ of $A$. Then $fg \in N(A)$, for all $f = a_0 + a_1X_1 + \cdots + a_iX_i \in V$, and by Proposition 2.17, $a_ib_j \in N(R)$, for each $i, j$. Thus, $b_j \in N_R(C_V)$, for all $0 \leq j \leq s$, whence $g \in N_R(C_V)A$, and so $N_A(V) \subseteq N_R(C_V)A$. Since $N_R(C_V)A \subseteq N_A(V)$, we obtain that $N_A(V) = N_R(C_V)A = \varphi(N_R(C_V))$, i.e., $\varphi$ is surjective. □

The following theorem generalizes [65, Theorem 2.1].

Theorem 3.5. Let $A = (\sigma(R) \left<x_1, \ldots, x_n\right>$ be a skew PBW extension over a $(\Sigma, \Delta)$-compatible and NI ring $R$. If for each subset $X \not\subseteq N(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element, then for each subset $U \not\subseteq N(A)$, $N_A(U)$ is generated as an ideal by a nilpotent element.

Proof. Let $U$ be a subset of $A$ with $U \not\subseteq N(A)$. By Corollary 2.14, $CU \not\subseteq N(R)$. There exists $c \in N(R)$ such that $N_R(C_U) = cU$. Let us show that $N_A(U) = cA$. Let $f = a_0 + a_1X_1 + \cdots + a_iX_i \in U$ and $g = b_0 + b_1Y_1 + \cdots + b_sY_s \in A$. The idea is to show that $fg \in N(A)$. Using that $cg = cb_0 + cb_1Y_1 + \cdots + cb_sY_s$, we get

$$fcg = \sum_{k=0}^{s+1} \left( \sum_{i+j=k} a_iX_icb_jY_j \right) = \sum_{k=0}^{s+1} \left( \sum_{i+j=k} a_i\sigma^{a_i}(cb_j)X_iY_j + a_ip_{a_i,cb_j}Y_j \right).$$

Since $a_icb_j \in N(R)$, for every $0 \leq i \leq l$, $0 \leq j \leq s$, [33, Proposition 3.3] this implies that $a_i\sigma^{a_i}(cb_j) \in N(R)$. Notice that $a_i\sigma^{a_i}(\delta^{\beta}(cb_j))$ and $a_i\delta^{\beta}(\sigma^{a_i}(cb_j))$ are elements of $N(R)$, for every $\alpha, \beta \in \mathbb{N}^n$. In addition, by Proposition 2.5, the polynomial $p_{a_i,cb_j}$ involves elements obtained evaluating $\sigma$'s and $\delta$'s
Theorem 3.8. Let $A$ be a skew PBW extension over a $(\Sigma, \Delta)$-compatible and NI ring $R$. For each principal right ideal $pR \not\subseteq N(R)$, $N_R(pR)$ is generated as an ideal by a nilpotent element, then for each principal right ideal $fA \not\subseteq N(A)$, $N_A(fA)$ is generated as an ideal by a nilpotent element.
**Proof.** Let \( f = a_0 + a_1X_1 + \cdots + a_lX_l \in A \) with \( fA \nsubseteq N(A) \). If \( a_iR \subseteq N(R) \), for all \( 0 \leq i \leq l \), by Corollary 2.14, we have \( fA \subseteq N(A) \), a contradiction. In this way, there exists \( 0 \leq i \leq l \) such that \( a_iR \nsubseteq N(R) \). Thus, there exists \( c \in N(R) \) such that \( N_R(a_iR) = cR \). Let us show that \( N_A(fA) = cA \). Consider \( g = b_0 + b_1Y_1 + \cdots + b_lY_l \) and \( h = c_0 + c_1Z_1 + \cdots + c_lZ_l \) elements of \( A \). By Theorem 3.5, \( gh \in N(R)A = N(A) \) and by Corollary 2.14, for every \( f \in A \), \( fgh \in N(A) \). Therefore, we obtain \( cR \subseteq N_A(fA) \), that is, \( cA \subseteq N_A(fA) \). Let \( p = p_0 + p_1Y_1 + \cdots + p_sY_s \in N_A(fA) \). Then \( fAp \subseteq N(A) \), for every \( f = a_0 + a_1X_1 + \cdots + a_lX_l \in A \). In particular, \( fAp \subseteq N(A) \). Let \( r \in R \). We get \( rp = rp_0 + rp_1Y_1 + \cdots + rp_sY_s \), which implies

\[
frp = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_iX_irp_jY_j \right) = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_i\sigma^{a_i}(rp_j)X_iY_j + a_{p0}rp_jY_j \right) \in N(R)A.
\]

Therefore, \( a_i\sigma^{a_i}(rp_j) \in N(R) \) and so \( a_irp_j \in N(R) \), for every \( i, j \). In particular, we obtain \( p_j \in N_R(a_iR) = cR \), and thus there exists \( r_j \in R \) such that \( p_j = cr_j \). Hence, \( p = p_0 + p_1Y_1 + \cdots + p_sY_s \in c(r_0 + r_1Y_1 + \cdots + r_sY_s) \in cA \). Therefore, we conclude that \( p \in cA \) and so \( N_A(fA) \subseteq cA \).

**Example 3.9.** (i) Once again, let \( A \) be the skew PBW extension over the ring \( \frac{\mathbb{F}[x_1]}{(x_1^2)} \) considered in Example 2.8. Let \( p = \frac{x_1+1}{x_1} \) such that \( p \frac{\mathbb{F}[x_1]}{(x_1^2)} \nsubseteq N \left( \frac{\mathbb{F}[x_1]}{(x_1^2)} \right) \). Notice that \( N_{\frac{\mathbb{F}[x_1]}{(x_1^2)}} \left( \frac{\mathbb{F}[x_1]}{(x_1^2)} \right) = z \frac{\mathbb{F}[x_1]}{(x_1^2)} \), where \( z \in N \left( \frac{\mathbb{F}[x_1]}{(x_1^2)} \right) \). By Theorem 3.8 \( N_A(fA) \) is generated by a nilpotent element for any principal right ideal \( fA \) with \( fA \nsubseteq N(A) \) and \( A = \sigma(R) \{ x_1, x_1, x_1, x_1, x_2, x_2, x_2, x_2, x_2 \} \).

(ii) In Example 2.9, let \( p \in R \) such that \( pR \nsubseteq N(R) \), where \( p = \begin{pmatrix} p(t) & q(t) \\ 0 & q(t) \end{pmatrix} \), for some \( p(t), q(t) \in k[t] \) and \( p(t) \neq 0 \). It is straightforward to see that

\[
N_R(pR) = \left\{ \begin{pmatrix} 0 & q(t) \\ 0 & 0 \end{pmatrix} \mid q(t) \in k[t] \right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R,
\]

with \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R) \). By Theorem 3.8, \( N_{R[x, \sigma, \delta]}(fR[x; \sigma, \delta]) \) is generated by a nilpotent element for any principal right ideal \( fR[x; \sigma, \delta] \) where \( fR[x; \sigma, \delta] \nsubseteq N(R[x; \sigma, \delta]) \) with \( f \in R[x; \sigma, \delta] \).

**Theorem 3.10** generalizes [65, Theorem 2.4].

**Theorem 3.10.** If \( A = \sigma(R) \{ x_1, \ldots, x_n \} \) is a skew PBW extension of endomorphism type over a \( \Sigma \)-compatible and NI ring \( R \), then the following statements are equivalent:

1. For each principal right ideal \( pR \nsubseteq N(R) \), \( N_R(pR) \) is generated as an ideal by a nilpotent element.
2. For each principal right ideal \( fA \nsubseteq N(A) \), \( N_A(fA) \) is generated as an ideal by a nilpotent element.

**Proof.** By Theorem 3.8, it suffices to show (2) \( \Rightarrow \) (1). The proof is very similar to the reasoning above and follows the same ideas. \( \square \)

The next result extends [65, Theorem 2.5].

**Theorem 3.11.** Let \( A = \sigma(R) \{ x_1, \ldots, x_n \} \) be a skew PBW extension over a \( (\Sigma, \Delta) \)-compatible and NI ring \( R \). If for each \( p \notin N(R) \), \( N_R(p) \) is generated as an ideal by a nilpotent element, then for each \( f \notin N(A) \), \( N_A(f) \) is generated as an ideal by a nilpotent element.

**Proof.** Consider \( f = a_0 + a_1X_1 + \cdots + a_lX_l \in A \) with \( f \notin N(A) \), and let us see that \( N_A(f) \) is generated as an ideal by a nilpotent element. If \( a_iR \subseteq N(R) \), for all \( 0 \leq i \leq l \), Corollary 2.14 implies that \( f \in N(A) \),
which is a contradiction. Hence, there exist $0 \leq i \leq l$ such that $a_i \notin N(R)$, and so there is $c \in N(R)$ such that $N_R(a_i) = cR$. The idea is to show that $N_A(f) = cA$. If $h = c_0 + c_1Z_1 + \cdots + c_iZ_i \in A$, by Proposition 3.5, we have $fch \in N(R)A = N(A)$. Therefore, we obtain that $ch \in N_A(f)$. On the other hand, let $p = p_0 + p_1Y_1 + \cdots + p_sY_s \in N_A(f)$. For $f = a_0 + a_1X_1 + \cdots + a_lX_l \in A$, we have $fp \in N(A)$. Thus, we obtain that

$$fp = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_iX_ip_jY_j \right) = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_i\sigma^{a_i}(p_j)X_iY_j + a_ip_{a_i}(p_j)Y_j \right) \in N(R)A.$$

Therefore, $a_i\sigma^{a_i}(p_j) \in N(R)$ and so $a_ip_j \in N(R)$ for every $i, j$. In particular, we obtain that $p_j \in N_R(a_i) = cR$, and thus there exists $r_j \in R$ such that $p_j = cr_j$, whence $p = p_0 + p_1Y_1 + \cdots + p_sY_s = c(r_0 + r_1Y_1 + \cdots + r_sY_s) \in cA$, and so $p \in cA$. \hfill \Box

Example 3.12. (i) If $A$ is a skew PBW extension over $\mathbb{F}[x] \langle z^2 \rangle$, let $p \in \mathbb{F}[x] \langle z^2 \rangle$ such that $p \notin N\left( \mathbb{F}[x] \langle z^2 \rangle \right)$. Notice that $N_{\mathbb{F}[x] \langle z^2 \rangle}(p) = zN_{\mathbb{F}[x] \langle z^2 \rangle}(p)$, where $z \in N\left( \mathbb{F}[x] \langle z^2 \rangle \right)$. By using Theorem 3.11 we conclude that the ideal $N_A(f)$ is generated by a nilpotent element for any element $f$ where $f \notin N(A)$ and $A = \sigma(R) \langle x_{1,0}, x_{1,1}, x_{2,2}, x_{2,0}, x_{2,1}, x_{2,2} \rangle$.

(ii) In Example 2.9, let $p \in R$ such that $p \notin N(R)$, where $p = \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix}$, for some $p(t), q(t) \in \mathbb{k}[t]$ and $p(t) \neq 0$. It is easy to see that

$$N_R(p) = \left\{ \begin{pmatrix} 0 & q(t) \\ 0 & 0 \end{pmatrix} \mid q(t) \in \mathbb{k}[t] \right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R,$$

where $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$. By using Theorem 3.11 we obtain that the ideal $N_{R[\mathbb{k}[\sigma, \sigma^2]]}(f)$ is generated by a nilpotent element of $R[x; \sigma, \sigma^2]$, for any element $f \in R[x; \sigma, \sigma^2]$ where $f \notin N(R[\mathbb{k}[\sigma, \sigma^2]])$.

For a skew PBW extension of endomorphism type over a ring $R$, the following theorem characterizes extensions for which every weak annihilator of an element is generated by a nilpotent element. This result generalizes [65, Theorem 2.6].

Theorem 3.13. If $A = \sigma(R) \langle x_1, \ldots, x_n \rangle$ is a skew PBW extension of endomorphism type over a $\Sigma$-compatible and NI ring $R$, then the following statements are equivalent:

1. For each $p \notin N(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element.
2. For each $f \notin N(A)$, $N_A(f)$ is generated as an ideal by a nilpotent element.

Proof. By Theorem 3.11, it suffices to show (ii) $\Rightarrow$ (i). Let $p \in R$ with $p \notin N(R)$, whence $p \notin N(A)$. Thus, there exists $f = a_0 + a_1X_1 + \cdots + a_lX_l \in N(A)$ such that $N_A(p) = fA$. Notice that $f = a_0 + a_1X_1 + \cdots + a_lX_l \in N(A)$ and so $a_i \in N(R)$, for all $0 \leq i \leq l$ by Corollary 2.14. We may assume that $a_0 \neq 0$. Now, we show that $N_R(p) = a_0R$. Since $a_0 \in N(R)$ and $N(R)$ is an ideal of $R$, we obtain $p_{a_0}R \subseteq N(R)$, that is, $a_0R \subseteq N_R(p)$. If $m \in N_R(p)$, then, we get that $m \in N_A(p)$. Thus, there exists $g = b_0 + b_1Y_1 + \cdots + b_sY_s \in A$ such that

$$m = fg = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_iX_ib_jY_j \right) = \sum_{k=0}^{s+l} \left( \sum_{i+j=k} a_i\sigma^{a_i}(b_j)X_iY_j \right).$$
Hence, we have $m = a_0 b_0 \in a_0 R$, and thus $N_R(p) \subseteq a_0 R$. Therefore, we conclude that $N_R(p) = a_0 R$ where $a_0 \in N(R)$. □

4. Nilpotent associated prime ideals

It is well-known that associated prime ideals are an important tool in areas such as commutative algebra and algebraic geometry concerning the theory of primary decomposition of ideals. Briefly, for a ring $R$ and a right $R$-module $N_R$, the right annihilator of $N_R$ is denoted by $r_R(N_R) = \{ r \in R : N_Rr = 0 \}$, and $N_R$ is called prime if $N_R \neq 0$ and $r_R(N_R) = r_R(N'_R)$, for every non-zero submodule $N'_R \subseteq N_R$ [1, Definition 1.1]. If $M_R$ is a right $R$-module, an ideal $P$ of $R$ is called an associated prime of $M_R$ if there exists a prime submodule $N_R \subseteq M_R$ such that $P = r_R(N_R)$. The set of associated primes of $M_R$ is denoted by $\text{Ass}(M_R)$ [1, Definition 1.2]. One of the most important results on these ideals was proved by Brewer [1] extended the result above to the noncommutative setting of Ore extensions (Leroy and Matczuk [53] also studied these ideals by defining a better condition than the compatibility), while Niño et al. [63] formulated this result for skew PBW extensions.

Since Ouyang and Birkenmeier [65] introduced the notion of nilpotent associated prime as a generalization of associated prime, and characterize these ideals of the Ore extension $R[x; \sigma, \delta]$ in terms of the nilpotent associated primes of $R$, a natural task is to investigate this kind of ideals in the context of skew PBW extensions with the aim of generalizing the results formulated in [65]. This is the purpose of this section.

We start by recalling some definitions presented by Ouyang and Birkenmeier [65].

**Definition 4.1.** Let $R$ be a ring.

(i) [65, Definition 3.1] Let $I$ be a right ideal of a non-zero ring $R$. $I$ is called a right quasi-prime ideal if $I \nsubseteq N(R)$ and $N_R(I) = N_R(I')$, for every right ideal $I' \subseteq I$ and $I' \nsubseteq N(R)$.

(ii) [65, Definition 3.2] Let $N(R)$ be an ideal of a ring $R$. An ideal $P$ of $R$ is called a nilpotent associated prime of $R$ if there exists a right quasi-prime ideal $I$ such that $P = N_R(I)$. The set of nilpotent associated primes of $R$ is denoted by $\text{NAss}(R)$.

An important concept in the characterization of nilpotent associated prime ideals is the notion of good polynomial. These polynomials were first used by Shock [80] with the aim of proving that the uniform dimensions of a ring $R$ and the polynomial ring $R[x]$ are equal (Annin [1] also considered such polynomials in his study of associated prime ideals of Ore extensions). A polynomial $f(x) \in R[x]$ is called good polynomial if it has the property that the right annihilators of the coefficients of $f(x)$ are equal. Shock proved that, given any non-zero polynomial $f(x) \in R[x]$, there exists $r \in R$ such that $f(x)r$ is good [80, Proposition 2.2]. Ouyang and Birkenmeier [65, Definition 3.3] introduced the notions of nilpotent degree and nilpotent good polynomial to study the nilpotent associated prime ideals of Ore extensions. Motivated by their ideas, we present the following definitions in the setting of skew PBW extensions.

**Definition 4.2.** Let $A = \sigma(R)(x_1, \ldots, x_n)$ be a skew PBW extension over a ring $R$, and consider an element $f = r_0 + r_1 X_1 + \cdots + r_k X_k + \cdots + r_1 X_1 \neq N(R)A$, where $X_1 > X_{i-1} > \cdots > X_k > X_{k-1} > \cdots > X_1$, $\text{lm}(f) = x^\alpha$ and $\text{lc}(f) = r_1$.

(i) If $r_k \notin N(R)$ and $r_i \in N(R)$, for all $i > k$, then we say that the nilpotent degree of $f$ is $k$, which is denoted as $\text{Ndeg}(f)$. If $f \in N(R)A$, then we define $\text{Ndeg}(f) = -1$.

(ii) Suppose that the nilpotent degree of $f$ is $k$. If $N_R(r_k) \subseteq N_R(r_i)$, for all $i \leq k$, then we say that $f$ is a nilpotent good polynomial.
The following result generalizes [65, Lemma 3.1] and [67, Lemma 2.16].

**Theorem 4.3.** Let \( A = \sigma(R)(x_1, \ldots, x_n) \) be a skew PBW extension over a \((\Sigma, \Delta)\)-compatible and NI ring \( R \). For any \( f = r_0 + r_1 X_1 + \cdots + r_l X_l \in N(R)A \), there exists \( r \in R \) such that \( fr \) is a nilpotent good polynomial.

**Proof.** Let us assume that the result is false and suppose that \( f = r_0 + r_1 X_1 + \cdots + r_k X_k + \cdots + r_l X_l \notin N(A) \) is a counterexample of minimal nilpotent degree \( \text{Ndeg}(f) = k \), that is, \( fr \) is not a nilpotent good polynomial. In particular, if \( r = 1 \), we have that \( f \) is not a nilpotent good polynomial. Hence, there exists \( i < k \) such that \( N_R(r_i) \not\subseteq N_r(r_i) \). Thus, we can find \( b \in R \) such that \( b \in N_R(r_i) \) and \( b \notin N_R(r_i) \), whence \( r_i b \notin N(R) \) and \( r_k b \in N(R) \).

Notice that the degree \( k \) coefficient of \( fb \) is \( r_k \sigma^\alpha_k(b) + \sum_{i=k+1}^l r_i p_{\alpha_i,b} \). Now, \( r_k \sigma^\alpha_k(b) \in N(R) \) due to the \((\Sigma, \Delta)\)-compatibility of \( R \). On the other hand, we have \( \text{Ndeg}(f) = k \), whence \( r_i \in N(R) \), for all \( i > k \). This means that \( r_i p_{\alpha_i,b} \in N(R) \), for all \( i > k \) and so \( \sum_{i=k+1}^l r_i p_{\alpha_i,b} \in N(R) \) since \( N(R) \) is an ideal of \( R \). Therefore, \( fb \) has nilpotent degree at most \( k - 1 \). Since \( r_i b \notin N(R) \), we have \( fb \notin N(R)A \). By the minimality of \( k \), there exists \( c \in R \) with \( fbc \) a nilpotent good polynomial. However, this contradicts the fact that \( f \) is a counterexample since \( bc \in R \) and \( f(bc) \) is a nilpotent good polynomial. This concludes the proof. \( \square \)

**Theorem 4.4** characterizes the nilpotent associated primes ideals of a skew PBW extension over a \((\Sigma, \Delta)\)-compatible and NI ring. This result extends different assertions in the literature about associated prime ideals (e.g., [1, Theorem 2.1], [63, Theorem 3.1.2], [65, Theorem 3.1], and [67, Theorem 3.1 and Corollary 3.2]).

**Theorem 4.4.** If \( A = \sigma(R)(x_1, \ldots, x_n) \) is a skew PBW extension over a \((\Sigma, \Delta)\)-compatible and NI ring \( R \), then

\[
\text{NAss}(A) = \{ PA \mid P \in \text{NAss}(R) \}.
\]

**Proof.** With the aim of establishing the desired equality, we prove the two inclusions. Let \( P \in \text{NAss}(R) \). By **Definition 4.1** (ii), there exists a right ideal \( I \not\subseteq N(R) \) with \( I \) a right quasi-prime ideal of \( R \) and \( P = N_R(I) \). Let us show that \( PA = N_A(IA) \) and also that \( IA \) is a quasi-prime ideal. We show first that \( PA = N_A(I) \). Let \( i = a_0 + a_1 X_1 + \cdots + a_m X_m \in IA \) and let \( f = b_0 + b_1 Y_1 + \cdots + b_l Y_l \in PA \). Then,

\[
if = \sum_{k=0}^{m+l} \left( \sum_{i+j=k} a_i X_i b_j Y_j \right) = \sum_{k=0}^{m+l} \left( \sum_{i+j=k} a_i \sigma^\alpha_i(b_j) X_i Y_j + a_i p_{\alpha_i,b} Y_j \right).
\]

Since \( a_i b_j \in N(R) \) for every \( i, j \), from [33, Proposition 3.3] it follows that \( a_i \sigma^\alpha_i(b_j) \in N(R) \). Furthermore, the polynomial \( p_{\alpha_i,b} \) involves elements obtained while evaluating \( \sigma \)'s and \( \delta \)'s (depending on the coordinates of \( a_i \)) in the element \( b_j \) by **Proposition 2.5**. This means that \( a_i p_{\alpha_i,b} \in N(R) \) for every \( i, j \) and thus, we obtain \( if \in N(R)A = N(A) \). Hence, \( PA \subseteq N_A(I) \).

Conversely, let \( f = b_0 + b_1 Y_1 + \cdots + b_m Y_m \in N_A(I) \), then \( if \in N(A) = N(R)A \), for all \( i = a_0 + a_1 X_1 + \cdots + a_m X_m \in IA \), whence \( a_i \sigma^\alpha_i(b_j) \in N(R) \), which implies that \( a_i b_j \in N(R) \) [33, Proposition 3.3]. Since \( a_i \in I \), we have \( b_j \in N_R(I) = P \). Therefore, we get \( f \in PA \) and thus \( N_A(I) \subseteq PA \). Hence, we conclude that \( PA = N_A(I) \).

Since the ideal \( I \) is a right quasi-prime ideal, we have \( I \not\subseteq N(R) \), which implies that \( IA \not\subseteq N(A) \). Let us show that for any right ideal \( U \) of \( R \), if \( U \not\subseteq N(A) \) and \( U \subseteq IA \), then \( N_A(U) = N_A(I) \). Let us see first that \( N_A(I) \subseteq N_A(U) \). If \( f = a_0 + a_1 X_1 + \cdots + a_m X_m \in N_A(I) \), this means that \( if \in N(A) \), for all \( i \in IA \). In particular, since \( U \subseteq IA \), then \( if \in N(A) \), for all \( i \in U \), whence \( f \in N_A(U) \). Hence, we have \( N_A(I) \subseteq N_A(U) \).
Conversely, let $C_U \subseteq R$ consisting of all coefficients of elements of $U$. Let us first consider $P'$ the right ideal of $R$ generated by $C_U$. Since $U \not\subseteq N(A) = N(R)A$, this means that $C_U \not\subseteq N(R)$, and hence $P' \subseteq I$ and $P' \not\subseteq N(R)$. Thus, as $I$ is a right quasi-prime ideal this implies that $N_R(P') = N_R(I) = P$. Now, if $f = a_0 + a_1X_1 + \cdots + a_nX_n \in N_A(U)$ and $u = u_0 + u_1Y_1 + \cdots + u_lY_l \in U$, then $uf \in N(A)$, whence $u_i\sigma^\beta(j)(a_j) \in N(R)$ and therefore $u_ia_j \in N(R)$, for all $0 \leq i \leq t$, $0 \leq j \leq m$. Since $R$ is a NI ring, $N(R)$ is an ideal, $u_ia_j \in N(R)$ implies $a_j \in N(R)$ and thus $u_iRa_j \in N(R)$ gives that $(u_iRa_j)^2 \in N(R)$. Therefore,

$$a_j \in N_R(P') = N_R(I) = P, \quad \text{for all} \quad 0 \leq j \leq m.$$ 

Let $i = b_0 + b_1Z_1 + \cdots + b_rZ_r \in IA$. We have $b_ma_j \in N(R)$, and so $b_m\sigma^\alpha(\delta^j(a_j))$ and $a_m\delta^j(\sigma^\alpha(a_j))$ are elements of $N(R)$, for every $\alpha, \beta \in \mathbb{N}$. Therefore, we get that if $j \in N(R)A = N(A)$, which implies that $f \in N_A(IA)$. Hence, we conclude $N_A(U) \subseteq N_A(IA)$. Thus, we have proved that $PA = N_A(IA)$ and also that $IA$ is a quasi-prime ideal.

Let $I \in NAss(A)$. By Definition 4.1 (ii), there exists a right ideal $J \not\subseteq N(A)$ with $J$ a right quasi-prime ideal of $A$ and $I = N_A(J)$. Let $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin N(A) = N(R)A$ and $m \in J$. Since $J \not\subseteq N(A)$, we may assume that $m$ is nilpotent and $\text{Ndeg}(m) = k$, by Theorem 4.3. We consider $I_0 = mA$ the principal right ideal of $A$ generated by $m$. Since $m \notin N(A) = N(R)A$ this implies that $I_0 = mA \not\subseteq N(R)A = N(A)$, whence $N_A(I) = N_A(I_0) = N_A(mA) = I$ because $I$ is a quasi-prime ideal. Now, we consider the right ideal $mkR$, and let us denote $U = N_R(mkR)$.

Let us prove first that $I = UA$. Let $g = b_0 + b_1Y_1 + \cdots + b_Y \in UA$. Since $b_j \in U$, then $m_Rb_j \in N(R)$ for all $0 \leq i \leq l$. Furthermore, $m$ is nilpotent good polynomial and $\text{Ndeg}(m) = k$, hence $m_Rb_j \in N(R)$, for all $0 \leq i \leq k$, and $0 \leq j \leq l$. On the other hand, for all $i > k$, $m_i \in N(R)$. Thus, we get $m_Rb_j \in N(R)$, for all $0 \leq i \leq n$ and $0 \leq j \leq l$. Now, for any element $h \in A$ where $h = h_0 + h_1Z_1 + \cdots + h_pZ_p$, we have that $m_Rh \in N(R)$, for all $0 \leq i \leq n$, $0 \leq d \leq p$, and $0 \leq j \leq l$. Therefore, by $(\Sigma, \Delta)$-compatibility of $R$, we obtain $m_Ig \in N(A)$. This implies that $g \in N_A(mA) = I$, and so $UA \subseteq I$. Conversely, let $g = b_0 + b_1Y_1 + \cdots + b_Y \in I$. Since $m_Rg \in N(A)$, it follows that $m_Rb_j \in N(R)$, for all $0 \leq i \leq n$, and $0 \leq j \leq l$. Thus, we have that $b_j \in N_R(mkR)$, for all $0 \leq j \leq l$, and so $g \in UA$. Hence, we conclude $I \subseteq UA$ which implies that $I = UA$.

Now, let $mkR$ be the principal right ideal of $A$ generated by $mk$. The idea is to show that $mkR$ is a quasi-prime ideal. Since $mk \notin N(R)$, we have $mk_R \not\subseteq N(R)$. The idea is to show that for any right ideal $Q \subseteq mkR$ with $Q \not\subseteq N(R)$, $Q$ does not contain $N_R(Q) = N_R(mkR)$. Assume that a right ideal $Q \subseteq mkR$, and $Q \not\subseteq N(R)$. Then $N_R(mkR) \not\subseteq Q$ by Proposition 3.3. Now, we show that $N_R(Q) \subseteq N_R(mkR)$. Let $W$ be the following set $W = \{mr \mid r \in Q\}$, and let $WA$ be the right ideal of $A$ generated by $W$.

First, notice that $WA \subseteq mA$. Since $Q \not\subseteq N(R)$, there exists $a \in R$ such that $m_Ra \in Q$ and $m_Ra \notin N(R)$. If $mk(m_Ra) \in N(R)$, then we have $m_Ra \in N(R)$ which contradicts the fact that $m_Ra \notin N(R)$. Thus $mk(m_Ra) \notin N(R)$, and hence $m_Ra \notin N(A)$, by Proposition 2.17. This implies that $WA \not\subseteq N(A)$. Since $I$ is a quasi-prime ideal, we obtain $N_A(WA) = N_A(mA) = I$.

Suppose $q \in N_R(Q)$. Then $r \in N(R)$, for each $r \in Q$. Now, for any $mr \in WA$ where $f = a_0 + a_1Y_1 + \cdots + a_Y \in A$, the term of $mr$ is $m_Ra_1Y_1$. The idea is to show that $m_Ra_1Y_1 \in N(R)$. Since $r \in N(R)$ and $N(R)$ is an ideal, it follows that $r \in N(R) \Rightarrow qr \in N(R) \Rightarrow ra_1q \in N(R) \Rightarrow ra_1q \in N(R)$.

If $ra_1q \in N(R)$, then $m_Ra_1q \in N(R)$. Thus, due to the $(\Sigma, \Delta)$-compatibility of $R$, we have that $m_Ra_1Y_1q \in N(R)A$ which implies that $m_Rf \in N(R)A = N(A)$. Hence, for any $\sum mr_i \in WA$ it follows that $\sum (m_Rf_i)q \in N(A)$. Therefore, $q \in N_A(WA) = I = UA$, and so $q \in U = N_R(mkR)$. Thus, $N_R(Q) \subseteq N_R(mkR)$, and this implies that $N_R(Q) = N_R(mkR)$. Hence, we conclude that $mkR$ is quasi-prime ideal.

Example 4.5. Since Example 2.9 satisfies the conditions of Theorem 4.4, it follows that $NAss(R[x; \overline{\alpha}, \overline{\delta}]) = NAss(R)[x; \overline{\alpha}, \overline{\delta}]$. On the other hand, the right ideals of $R$ are given by
The relevance of the results presented in the paper is appreciated when we apply them to algebraic

\[ \begin{align*}
I_1 &= \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} R, & I_2 &= \begin{pmatrix} 0 & 0 \\
0 & q(t) \end{pmatrix} R, & I_3 &= \begin{pmatrix} 0 & p(t) \\
0 & 0 \end{pmatrix} R, \\
I_4 &= \begin{pmatrix} 0 & p(t) \\
0 & q(t) \end{pmatrix} R, & I_5 &= \begin{pmatrix} q(t) & p(t) \\
0 & 0 \end{pmatrix} R, & I_6 &= \begin{pmatrix} q(t) & p(t) \\
0 & s(t) \end{pmatrix} R,
\end{align*} \]

for \( p(t), q(t) \in \mathbb{k}[t] \). Additionally, we can observe that the only quasi-prime ideals of \( R \) are \( I_2, I_4, I_5, \) and \( I_6 \). Furthermore, \( N_R(I_2) = N_R(I_4) = N_R(I_5) = N_R(I_6) = I_3 \), where \( I_3 = N(R) \). Thus, it follows that \( \text{NAss}(R) = \{N(R)\} = \{I_3\} \). Therefore, we conclude that \( \text{NAss}(R) = \{N(R)\} = \{I_3\} \).

The compatibility condition is not superfluous in Theorem 4.4. The next example show that if \( R \) is not \( (\Sigma, \Delta) \)-compatible, then this result 4.4 can be failed.

**Example 4.6.** [31, p. 3796] Let \( R = \mathbb{Z}_2[t]/\langle t^2 \rangle \), where \( \mathbb{Z}_2[t] \) is the polynomial ring over the field \( \mathbb{Z}_2 \) of two elements, and \( \langle t^2 \rangle \) is the ideal of \( \mathbb{Z}_2[t] \) generated by \( t^2 \). Consider the derivation \( \delta \) over \( R \) defined by \( \delta(T) = 1 \) where \( T = t + \langle t^2 \rangle \), and let \( R[x; \delta] = \mathbb{Z}_2[t]/\langle t^2 \rangle | x; \delta \) be the corresponding Ore extension. Since \( t^2 = 0 \) but \( \delta(t) \neq 0 \), the \( \delta \)-compatibility condition fails here. If we set \( e_{11} = tx, e_{12} = t, e_{21} = tx^2 + x \) and \( e_{22} = 1 + tx \) in \( R[x; \delta] \), then they from a system of matrix units in \( R[x; \delta] \). The centralizer of these matrix units in \( R[x; \delta] \) is \( \mathbb{Z}_2[x^2] \). Therefore \( R[x; \delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y] \), where \( M_2(\mathbb{Z}_2)[y] \) is the polynomial ring over \( M_2(\mathbb{Z}_2) \). So, \( \langle T \rangle \) is a nilpotent associated prime ideal of \( R \), but \( \langle T \rangle \) is not a nilpotent associated prime ideal of \( R[x; \delta] \).

**5. Examples**

The relevance of the results presented in the paper is appreciated when we apply them to algebraic

structures more general than those considered by Ouyang et al. [65, 67], i.e., noncommutative rings which cannot be expressed as Ore extensions. In this section, we consider several and remarkable families of rings that have been studied in the literature which are subfamilies of skew PBW extensions and cannot be expressed as Ore extensions. Of course, the list of examples is not exhaustive.

**Definition 5.1 ([11]; [79, Definition C4.3]).** A 3-dimensional algebra \( A \) is a \( \mathbb{k} \)-algebra generated by the indeterminates \( x, y, z \) subject to the relations \( yz - \alpha xy = \lambda, zx - \beta xz = \mu, \) and \( xy - \gamma yx = \nu \), where \( \lambda, \mu, \nu \in \mathbb{k} \) and \( \alpha, \beta, \gamma \in \mathbb{k}^* \). \( A \) is called a 3-dimensional skew polynomial \( \mathbb{k} \)-algebra if the set \( \{x^i y^j z^k \mid i, j, k \geq 0\} \) forms a \( \mathbb{k} \)-basis of the algebra.

Up to isomorphism, there are fifteen 3-dimensional skew polynomial \( \mathbb{k} \)-algebras [79, Theorem C4.3.1]. Different authors have studied ring-theoretical and geometrical properties of these algebras (e.g., [43, 70, 74]).

It follows from Definition 5.1 that every 3-dimensional skew polynomial algebra is a skew PBW extension over the field \( \mathbb{k} \). For these algebras, Theorem 4.3 guarantees the existence of nilpotent good polynomials and Theorem 4.4 characterizes the nilpotent associated prime ideals. In particular, since \( \mathbb{k} \) is a field (and hence reduced), the only nilpotent associated prime ideal is the zero ideal.

**Example 5.2.** Following Havliček et al. [36, p. 79], the \( \mathbb{C} \)-algebra \( U_q'(\mathfrak{s}\mathfrak{o}_3) \) is generated by the indeterminates \( I_1, I_2, \) and \( I_3 \), subject to the relations given by

\[ I_2 I_1 - q I_1 I_2 = -q^{\frac{1}{2}} I_3, \quad I_3 I_1 - q^{-1} I_1 I_3 = q^{-\frac{1}{2}} I_2, \quad I_3 I_2 - q I_2 I_3 = -q^{\frac{1}{2}} I_1, \]

where \( q \) is a non-zero element of \( \mathbb{C} \). It is straightforward to show that \( U_q'(\mathfrak{s}\mathfrak{o}_3) \) cannot be expressed as an iterated Ore extension. However, this algebra is a skew PBW extension over \( \mathbb{C} \), i.e., \( U_q'(\mathfrak{s}\mathfrak{o}_3) \cong \sigma(\mathbb{C})(I_1, I_2, I_3) \) [24, Example 1.3.3]. On the other hand, notice that \( \mathbb{C} \) is a \( (\Sigma, \Delta) \)-compatible NI ring with the identity endomorphism of \( \mathbb{C} \) and the trivial \( \sigma \)-derivation. Thus, since that \( r_C(X) = N_C(X) = \)
For $AW(3)$, Theorem 4.3 guarantees the existence of nilpotent good polynomials, and Theorem 4.4 describes the nilpotent associated prime ideals. In particular, since that for $AW$ the only nilpotent associated prime ideal is the zero ideal. All the theorems mentioned above are valid for the reduced ring with the identity endomorphism of $AW$.

Example 5.3. Zhedanov [89, Section 1] introduced the Askey-Wilson algebra $AW(3)$ as the algebra generated by three operators $K_0, K_1, \text{and } K_2$, that satisfy the commutation relations $[K_0, K_1]_\omega = K_2, [K_2, K_0]_\omega = BK_0 + C_1K_1 + D_1, \text{and } [K_1, K_2]_\omega = BK_1 + C_0K_0 + D_0$, where $B, C_0, C_1, D_0, D_1, \text{are the structure constants of the algebra, which Zhedanov assumes are real, and the } q\text{-com mutator } [\cdot, \cdot]_\omega \text{ is given by } [\cdot, \cdot]_\omega := e^{\omega}\cdot \cdot - e^{-\omega} \cdot \cdot \cdot \cdot, \text{where } \omega \text{ is an arbitrary real parameter. Notice that in the limit } \omega \to 0, \text{the algebra } AW(3) \text{ becomes an ordinary Lie algebra with three generators } (D_0 \text{ and } D_1 \text{ are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as } e^{\omega}K_0K_1 - e^{-\omega}K_1K_0 = K_2, e^{\omega}K_2K_0 - e^{-\omega}K_0K_2 = BK_0 + C_1K_1 + D_1, \text{and } e^{\omega}K_1K_2 - e^{-\omega}K_2K_1 = BK_1 + C_0K_0 + D_0. \text{According to these relations that define the algebra, it is clear that } AW(3) \text{ cannot be expressed as an iterated Ore extension. Nevertheless, using techniques such as those presented in [24, Theorem 1.3.1], it can be shown that } AW(3) \text{ is a skew PBW extension of endomorphism type, that is, } AW(3) \cong \sigma(\mathbb{R})(K_0, K_1, K_2). \text{Since that } \mathbb{R} \text{ is reduced, } r_\mathbb{R}(X) = N_\mathbb{R}(X) = [0], \text{for all } X \nsubseteq N(\mathbb{R}). \text{Hence, the characterization of the weak annihilators of } N_{AW(3)}(U), N_{AW(3)}(f \cdot AW(3)), \text{and } N_{AW(3)}(f), \text{for certain subsets, principal right ideals, and elements of algebra } AW(3), \text{follows from Theorems 3.5, 3.8, and 3.11, respectively. Furthermore, Theorems 3.7, 3.10, and 3.13 are also valid for the description of the weak annihilators mentioned above. For } AW(3), \text{Theorem 4.3 guarantees the existence of nilpotent good polynomials, and Theorem 4.4 describes the nilpotent associated prime ideals. In particular, since that } \mathbb{R} \text{ is a field (hence reduced), the only nilpotent associated prime ideal is the zero ideal. All the theorems mentioned above are valid for } AW(3) \text{ if we change } \mathbb{R} \text{ to any reduced ring } R \text{ by considering } e^{\omega} \text{ as a non-zero element of } R \text{ and } e^{-\omega} \text{ as its multiplicative inverse.}

Example 5.4. As is well-known, algebras whose generators satisfy quadratic relations such as Clifford algebras, Weyl-Heisenberg algebras, and Sklyanin algebras, play an important role in analysis and mathematical physics. Motivated by these facts, Golovashkin and Maximov [26] considered the algebras $Q(a, b, c)$, with two generators $x$ and $y$, generated by the quadratic relations $yx = ax^2 + bxy + c^2y^2,$ where the coefficients $a, b, \text{and } c \text{ belong to an arbitrary field } k \text{ of characteristic zero. They studied conditions on these elements under which such an algebra has a PBW basis of the form } \{x^m y^n \mid m, n \in \mathbb{N}\}.

In [26, Section 3], Golovashkin and Maximov presented a necessary and sufficient condition for the existence of a PBW basis as above in the case $ac + b \neq 0$, while in [26, Section 5], for the case $ac + b = 0$, they proved that if $b \neq 0, -1$, then $Q(a, b, c)$ has a PBW basis, and if $b = -1$, then the elements $\{x^m y^n \mid m, n \in \mathbb{N}\}$ are linearly independent but do not form a PBW basis of $Q(a, b, c)$.

One can check that if $a, b, \text{and } c$ are not zero simultaneously, then $Q(a, b, c)$ is neither an Ore extension of $K, K[x] \text{ nor of } K[y]$. On the other hand, if $b \neq 0 \text{ and } c = 0$, then one can check that $Q(a, b, c)$ is a skew PBW extension over $K[x], \text{i.e., } Q(a, b, c) \cong \sigma(K[x])(y). \text{Note that } K[x] \text{ is a } (\Sigma, \Delta)\text{-compatible reduced ring with the identity endomorphism of } K(x) \text{ and the trivial } \sigma\text{-derivation. Since } K[x] \text{ is a reduced ring, we get } r_{K[x]}(X) = N_{K[x]}(X) = [0], \text{for all } X \nsubseteq N(K[x]). \text{Hence, the characterization of the weak annihilators of } N_{Q(a,b,c)}(U), N_{Q(a,b,c)}(f \cdot Q(a,b,c)), \text{and } N_{Q(a,b,c)}(f), \text{for some subsets, principal right ideals, and elements of algebra } Q(a,b,c), \text{follows from Theorems 3.5, 3.8, and 3.11, respectively. Also, Theorems 3.7, 3.10, and 3.13 are also valid for the description of these weak annihilators. For the algebra } Q(a,b,c), \text{the existence of nilpotent good polynomials and the characterization of the nilpotent associated prime ideals follow from Theorems 4.3 and 4.4, respectively. Notice that if } a = 0, \text{ then one
can check that \(Q(a, b, c)\) is a skew PBW extension over \(k[y]\), i.e., \(Q(a, b, c) \cong (k[y])/(\sigma (k[y])(x))\) and all the previous results are satisfied too. Similarly, if we change \(k\) to a reduced ring \(R\), all results are still valid for \(Q(a, b, c)\).

**Example 5.5.** [19, Section 25.2] The basis of the algebra \(g = so(5, \mathbb{C})\) consists of the elements \(J_{\alpha \beta} = -J_{\beta \alpha}\), \(\alpha, \beta = 1, 2, 3, 4, 5\) satisfying the commutation relations \([J_{\alpha \beta}, J_{\mu \nu}] = \delta_{\beta \mu}J_{\alpha \nu} + \delta_{\alpha \nu}J_{\beta \mu} - \delta_{\beta \nu}J_{\alpha \mu} - \delta_{\alpha \mu}J_{\beta \nu}\).

Having in mind the classical PBW theorem for the universal enveloping algebra \(U(so(5, \mathbb{C}))\) of \(so(5, \mathbb{C})\), and since \(U(so(5, \mathbb{C}))\) is a PBW extension of \(\mathbb{C}\) [10, Section 5], then \(U(so(5, \mathbb{C}))\) is a skew PBW extension over \(\mathbb{C}\), i.e., \(U(so(5, \mathbb{C})) \cong \sigma(\mathbb{C})(J_{\alpha \beta} | 1 \leq \alpha \leq \beta \leq 5)\). The weak annihilator of certain subsets, principal right ideals, and elements of algebra \(U(so(5, \mathbb{C}))\) are characterized by Theorems 3.5, 3.8, and 3.11, respectively. Furthermore, Theorems 3.7, 3.10, and 3.13 hold since \(U(so(5, \mathbb{C}))\) is a skew PBW extension of endomorphism type. Theorem 4.3 asserts the existence of nilpotent good polynomials and Theorem 4.4 describes the nilpotent associated prime ideals of \(U(so(5, \mathbb{C}))\).

**Example 5.6.** With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [28] and Greenberg and Messiah [29] introduced by means of generators and relations the parafermionic and parabosonic algebras. For the completeness of the paper, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [45]. Let \([\square, \triangle] := \square \triangle - \triangle \square\) and \([\square, \square] := \square \square + \triangle \triangle\).

Consider the \(k\)-vector space \(V_F\) freely generated by the elements \(f_i^+, f_j^-\), with \(i, j = 1, \ldots, n\). If \(T(V_F)\) is the tensor algebra of \(V_F\) and \(I_F\) is the two-sided ideal \(I_F\) generated by the elements \([f_i^\xi, f_j^\eta]\), \(f_i^\xi f_j^\eta - \frac{1}{2}(\xi - \eta)^2 \delta_{ij} f_i^\xi + \frac{1}{2}(\xi - \eta)^2 \delta_{ij} f_j^\eta\), for all values of \(\xi, \eta, \varepsilon = \pm 1\), and \(i, j, k = 1, \ldots, n\), then the parafermionic algebra in \(2n\) generators \(P_F^{(n)}(n\ parafermions)\) is the quotient algebra of \(T(V_F)\) with the ideal \(I_F\), that is,

\[
P_F^{(n)} = \frac{T(V_F)}{\langle \{f_i^\xi, f_j^\eta| f_i^\xi f_j^\eta - \frac{1}{2}(\xi - \eta)^2 \delta_{ij} f_i^\xi + \frac{1}{2}(\xi - \eta)^2 \delta_{ij} f_j^\eta | \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \ldots, n \rangle}.
\]

It is well-known (e.g., [45, Section 18.2]) that a parafermionic algebra \(P_F^{(n)}(n\ parafermions)\) is isomorphic to the universal enveloping algebra of the simple complex Lie algebra \(so(2n + 1)\) (according to the classification of the simple complex Lie algebras, e.g., Kac [44]), i.e., \(P_F^{(n)} \cong U(so(2n + 1))\). On the other hand, \(P_F^{(n)}\) is a skew PBW extension over \(k\), that is, \(P_F^{(n)} \cong \sigma(k)(f_i^\xi, f_j^\eta)\), for values \(\xi, \eta = \pm 1\), and \(1 \leq i, j \leq n\). Since \(k\) is a field (hence reduced), then \(\mathfrak{n}_k(X) = \mathfrak{N}_k(X) = \{0\}\), for all \(X \not\subseteq N(k)\). This implies that the characterization of the weak annihilators of \(N_F^{(n)}(U)\), \(N_F^{(n)}(f P_F^{(n)}),\) and \(N_F^{(n)}(f)\), for some subsets, principal right ideals, and elements of algebra \(P_F^{(n)}\), follow from Theorems 3.5, 3.8, and 3.11, respectively. Furthermore, Theorems 3.7, 3.10, and 3.13 are also valid for the description of these weak annihilators. We recall that these subsets, principal right ideals and elements, are not contained in the set of nilpotent elements of the algebra \(P_F^{(n)}\). For the parafermionic algebra, Theorem 4.3 guarantees the existence of nilpotent good polynomials and Theorem 4.4 describes the nilpotent associated prime ideals.

Similarly, if \(V_B\) denotes the \(k\)-vector space freely generated by the elements \(b_i^+, b_j^-\), \(i, j = 1, \ldots, n\), \(T(V_B)\) is the tensor algebra of \(V_B\), and \(I_B\) is the two-sided ideal of \(T(V_B)\) generated by the elements \([b_i^\xi, b_j^\eta]\), \(b_i^\xi b_j^- - (\varepsilon - \eta)\delta_{ij} b_i^\xi - (\varepsilon - \xi)\delta_{ij} b_j^\eta\), for all values of \(\xi, \eta, \varepsilon = \pm 1\), and \(i, j, k = 1, \ldots, n\), then the parabosonic algebra \(P_B^{(n)}(n\ parabosons)\) is defined as the quotient algebra \(P_B^{(n)} / I_B\), that is,

\[
P_B^{(n)} = \frac{T(V_B)}{\langle \{b_i^\xi, b_j^\eta| b_i^\xi b_j^\eta - (\varepsilon - \eta)\delta_{ij} b_i^\xi - (\varepsilon - \xi)\delta_{ij} b_j^\eta | \xi, \eta, \varepsilon = \pm 1, i, j = 1, \ldots, n \rangle}.
\]
It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$, that is, $P_B^{(n)} \cong U(B(0, n))$. For more details about parafermionic and parabosonic algebras, see [45, Proposition 18.2] and references therein.

Similar to the case of parafermionic algebra, the parabosonic algebra is a skew PBW extension over $k$, that is, $P_B^{(n)} \cong \sigma(k)(b_{ij}^0, b_{ij}^1)$ for values $\xi, \eta = \pm 1$, and $1 \leq i, j \leq n$. In this case, the results mentioned for the parafermionic algebras hold for parabosonic algebras.

**Example 5.7.** Other algebraic structures that illustrate the results obtained in the paper concerns examples of generalized Weyl algebras, down-up algebras, and ambiskew polynomial rings. For the completeness of the paper, we briefly present the definitions and some relations between these algebras (see [41–43] for a detailed description).

Given an automorphism $\sigma$ and a central element $a$ of a ring $R$, Bavula [6] defined the generalized Weyl algebra $R(\sigma, a)$ as the ring extension of $R$ generated by the indeterminates $X^-$ and $X^+$ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(sl_2)$, and ambiskew polynomial rings (see below the definition of these objects). Generalized Weyl algebras have been extensively studied in the literature by various authors (see [6–8, 42], and references therein).

The down-up algebras $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, were defined by Benkart and Roby [12, 13] as generalizations of algebras generated by a pair of operators, precisely, the “down” and “up” operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered sets $P$. Let us see the details.

Consider a partially ordered set $P$ and let $\mathbb{C}P$ be the complex vector space with basis $P$. If for an element $p$ of $P$, the sets $\{x \in P \mid x > p\}$ and $\{x \in P \mid x < p\}$ are finite, then we can define the “down” operator $d$ and the “down” operator $u$ in $\text{End}_\mathbb{C}(\mathbb{C}P)$ as $u(p) = \sum_{x < p} x$ and $d(p) = \sum_{x > p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define $d$ and $u$). For any $\alpha, \beta, \gamma \in \mathbb{C}$, the down-up algebra is the $\mathbb{C}$-algebra generated by $d$ and $u$ subject to the relations $d^2 u = \alpha u d + \beta u^2 + \gamma d$ and $d u^2 = \alpha u^2 + \beta u^2 d + \gamma u$. A partially ordered set $P$ is called $(q, r)$-differential if there exist $q, r \in \mathbb{C}$ such that the down and up operators for $P$ satisfy both relations, and $\alpha = q(q + 1), \beta = -q^3,$ and $\gamma = r$. From [13], we know that for $0 \neq \lambda \in \mathbb{C}, A(\alpha, \beta, \gamma) \cong A(\alpha, \beta, \lambda \gamma)$. This means that when $\gamma \neq 0$, no problem if we assume $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [12, Section 1].

Remarkable examples of down-up algebras include the enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$ and some of its deformations introduced by Witten [85] and Woronowicz [87]. Related to the theoretical properties of these algebras, Kirksman et al. [48] proved that a down-up algebra $A(\alpha, \beta, \gamma)$ is Noetherian if and only if $\beta$ is non-zero. As a matter of fact, they showed that $A(\alpha, \beta, \gamma)$ is a generalized Weyl algebra in the sense of Bavula [6], and that $A(\alpha, \beta, \gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [85, 86] introduced a 7-parameter deformation of the universal enveloping algebra $U(sl_2)$. By definition, Witten’s deformation is a unital associative algebra over a field $k$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\xi = (\xi_1, \ldots, \xi_7)$ of elements of $k$. This algebra, denoted by $W(\xi)$, is generated by the indeterminates $x, y, z$ subject to the defining relations $x z - \xi_1 x z = \xi_2 x, z y - \xi_3 y z = \xi_4$, and $y x - \xi_5 x y = \xi_6 z^2 + \xi_7 z$. From [12, Section 2], we know that a Witten’s deformation algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5 \xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4,$$  \tag{5.1}$$
is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha, \beta, \gamma)$ with not both $\alpha$ and $\beta$ equal to 0 is isomorphic to a Witten deformation algebra $W(\xi)$ whose parameters satisfy (5.1).

Since algebras $W(\xi)$ are filtered, Le Bruyn [51, 52] studied the algebras $W(\xi)$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be conformal $sl_2$ algebras that are generated by the indeterminates $x, y, z$ over a field $k$ subject to
the relations given by \( zx - axz = x, \) \( zy - ayz = y, \) and \( yx - cxy = bz^2 + z. \) In the case \( c \neq 0 \) and \( b = 0, \) the conformal \( \mathfrak{sl}_2 \) algebra with these three defining relations is isomorphic to the down-up algebra \( A(\alpha, \beta, \gamma) \) with \( \alpha = c^{-1}(1 + ac), \) \( \beta = -ac^{-1} \) and \( \gamma = -c^{-1}. \) Notice that if \( c = b = 0 \) and \( a \neq 0, \) then the conformal \( \mathfrak{sl}_2 \) algebra is isomorphic to the down-up algebra \( A(\alpha, \beta, \gamma) \) with \( \alpha = a^{-1}, \beta = 0, \) and \( \gamma = -a^{-1}. \) As one can check, conformal \( \mathfrak{sl}_2 \) algebras are not Ore extensions but skew PBW extensions over \( k[z]. \)

Of interest for the examples in the paper, Kulkarni \([50]\) showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal \( \mathfrak{sl}_2 \) algebra or to an iterated Ore extension (double Ore extension). More exactly, following \([50, \text{Theorem 3.0.3}]\) if \( \xi_1 \xi_3 \xi_5 \xi_4 \neq 0 \) or \( \xi_1 \xi_5 \xi_3 \xi_2 \neq 0, \) then \( W(\xi) \) is isomorphic to one of the following algebras: (i) a conformal \( \mathfrak{sl}_2 \) algebra with generators \( x, y, z \) and relations given above or (ii) an iterated Ore extension.

Considering the results presented in this paper, we can assert that weak annihilators of subsets, principal right ideals, and elements of conformal \( \mathfrak{sl}_2 \) algebras are characterized by Theorems 3.5, 3.8, and 3.11, respectively. About the theory of nilpotent associated prime ideals, Theorem 4.3 establishes the existence of nilpotent good polynomials, and Theorem 4.4 describes the nilpotent associated prime ideals of the algebra.

Finally, Jordan \([42]\) introduced a certain class of iterated Ore extensions called ambiskew polynomial rings (these structures have been studied by Jordan at various levels of generality \([41]\)) that contains different examples of noncommutative algebras. As one can check, these polynomial rings are examples of skew PBW extensions, so the results presented above can be illustrated with these algebras.

**Example 5.8.** Bavula \([8]\) defined the skew bi-quadratic algebras with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis. From their definition, it is clear that these algebras are strictly contained in skew PBW extensions. Notice that if the ring of coefficients is a \((\Sigma, \Delta)\)-compatible NI ring, then the characterization of the weak annihilators for subsets, principal right ideals, and elements of the algebra, follow from Theorems 3.5, 3.8, and 3.11. If, in addition, the ring of coefficients is a field, Theorems 3.7, 3.10, and 3.13 are also valid for the description of all of them. For the bi-quadratic algebras, Theorem 4.3 guarantees the existence of nilpotent good polynomials, and Theorem 4.4 describes its nilpotent associated prime ideals.

**6. Future work**

Having in mind that some authors considered the notion of compatibility (and related ring-theoretical properties) for modules over skew PBW extensions \([58, 63, 72]\), a first and natural task is to characterize nilpotent associated prime ideals over these objects. Related with this, since Macdonald \([62]\) introduced a dual theory to primary decomposition called secondary representation whose key objects are known as attached primes, which were considered by Annin \([2, 3]\) with the aim of studying the behavior of the attached prime ideals of inverse polynomial modules over skew polynomial rings of automorphism type \( R[x; \sigma] \), a second task is to determine the attached primes of polynomial modules over skew PBW extensions with the aim of generalizing Annin's results.

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**References**

1. Annin, S. (2004). Associated primes over Ore extension rings. *J. Algebra Appl.* 3(2):193–205.
2. Annin, S. (2008). Attached primes over noncommutative rings. *J. Pure Appl. Algebra* 212(3):510–521.
3. Annin, S. (2011). Attached primes under skew polynomial extensions. *J. Algebra Appl.* 10(3):537–547.
Bell, A. D., Goodearl, K. R. (1988). Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions. *Pacific J. Math.* 131(1):13–37.

Bell, A. D., Smith, S. P. (1990). Some 3-dimensional skew polynomial rings. University of Wisconsin, Milwaukee, preprint.

Benkart, G. (1998). Down-up algebras. *J. Algebra* 209(1):305–344.

Bhat, V. K. (2010). Associated prime ideals of weak σ-rigid rings and their extensions. *Algebra Discrete Math.* 10(1):8–17.

Birkenmeier, G. F., Heatherly, H. E., Lee, E. K. (1993). Completely prime ideals and associated radicals. In: Jain, S. K., Rizvi, S. T., eds. *Ring Theory*. Granville, OH, 1992. Singapore and River Edge: World Scientific, pp. 102–129.

Birkenmeier, G. F., Kim, J. Y., Park, J. K. (2001). Principally quasi-Baer rings. *Commun. Algebra* 29(2):639–660.

Birkenmeier, G. F., Park, J. K., Rizvi, S. T. (2013). *Extensions of Rings and Modules*. New York: Birkhäuser.

Breuer, J., Heinzner, W. (1974). Associated primes of principal ideals. *Duke Math. J.* 41(1):1–7.

Burdič, Č., Navrátil, O. (2009). Decomposition of the enveloping algebra so(5). In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A., eds. *Generalized Lie Theory in Mathematics, Physics and Beyond*. Berlin, Heidelberg: Springer, pp. 297–302.

Cohn, P. M. (1999). Reversible rings. *Bull. Lond. Math. Soc.* 31(6):641–648.

Faith, C. (1989). Rings with zero intersection property on annihilators: ZIP rings. *Publ. Math.* 33(2):329–338.

Faith, C. (1991). Annihilator ideals and associated primes and Kasch-McCoy commutative rings. *Commun. Algebra* 19(7):1867–1892.

Faith, C. (2000). Associated primes in commutative polynomial rings. *Commun. Algebra* 28(8):3983–3986.

Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., Venegas, H. (2020). *σ*-compatible skew PBW extensions. *J. Algebraic Syst.* 7(1):1–29.

Green, H. S. (1953). A generalized method of field quantization. *Phys. Rev.* 90(2):270.

Greenberg, O. W., Messiah, A. M. L. (1965). Selection rules for parafields and the absence of para particles in nature. *Phys. Rev.* 138(5B):B1155–B1167.

Havlíček, M., Klimyk, A. U., Pošta, S. (2000). Central elements of the algebras $U(\mathfrak{so}_m)$ and $U(\mathfrak{iso}_m)$. *Czech. J. Phys.* 50(1):79–84.

Hong, C. Y., Kwak, T. K., Rizvi, S. T. (2006). Extensions of generalized Armendariz rings. *Algebra Colloq.* 13(2):253–266.

Higuera, S., Reyes, A. (2022). A survey on the fusible property of skew PBW extensions. *J. Algebraic Syst.* 10(1):1–29.

Hong, C. Y., Kim, N. K., Kwak, T. K. (2000). Ore extensions of Baer and p.p.-rings. *J. Pure Appl. Algebra* 151(3):215–226.

Isaev, A. P., Pyatov, P. N., Rittenberg, V. (2001). Diffusion algebras. *J. Phys. A.* 34(29):5815–5834.
[41] Jordan, D. A. (1995). Finite-dimensional simple modules over certain iterated skew polynomial rings. *J. Pure Appl. Algebra* 98(1):45–55.

[42] Jordan, D. A. (2000). Down-up algebras and Ambiskew polynomial rings. *J. Algebra* 228(1):311–346.

[43] Jordan, D. A., Wells, I. (1996). Invariants for automorphisms of certain iterated skew polynomial rings. *Proc. Edinb. Math. Soc.* (2) 39(3):461–472.

[44] Kac, V. G. (1977). Lie superalgebras. *Adv. Math.* 26(1):8–96.

[45] Kanakoglou, K., Daskaloyannis, C. (2009). Bosonisation and parastatistics. In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A., eds. *Generalized Lie Theory in Mathematics, Physics and Beyond*. Berlin, Heidelberg: Springer, pp. 207–218.

[46] Kandri-Rody, A., Weispfenning, V. (1990). Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.* 9(1):1–26.

[47] Krempa, J. (1996). Some examples of reduced rings. *Algebra Colloq.* 3(4):289–300.

[48] Lezama, O., Acosta, J. P., Reyes, A. (2015). Prime ideals of skew PBW extensions. *Rev. Un. Mat. Argentina* 56(2):39–55.

[49] Lezama, O., Reyes, A. (2014). Some homological properties of skew PBW extensions. *Commun. Algebra* 42(3):1200–1230.

[50] Lezama, O., Reyes, A. (2020). Minimal prime ideals of skew PBW extensions over 2-primal compatible rings. *Rev. Colombiana Mat.* 54(1):39–63.

[51] Louzari, M., Reyes, A. (2020). Minimal prime ideals of skew PBW extensions over 2-primal compatible rings. *Rev. Colombiana Mat.* 54(1):39–63.

[52] Louzari, M., Reyes, A. (2020). Generalized rigid modules and their polynomial extensions. In: Siles Molina, M., El Kaoutit, L., Louzari, M., Ben Yakoub, L., Benslimane, M., eds. *Associative and Non-Associative Algebras and Applications*. MAMAA 2018. Springer, Proceeding in Mathematics & Statistics, Vol. 311. Cham: Springer, pp. 147–158.

[53] Louzari, M., Reyes, A. (2020). Generalized rigid modules and their polynomial extensions. *Commun. Algebra* 48(2):866–878.

[54] Louzari, M., Reyes, A. (2020). Generalized rigid modules and their polynomial extensions. In: Siles Molina, M., El Kaoutit, L., Louzari, M., Ben Yakoub, L., Benslimane, M., eds. *Associative and Non-Associative Algebras and Applications*. MAMAA 2018. Springer, Proceeding in Mathematics & Statistics, Vol. 311. Cham: Springer, pp. 147–158.

[55] Lezama, O., Reyes, A. (2014). Some homological properties of skew PBW extensions. *Commun. Algebra* 42(3):1200–1230.

[56] Lezama, O., Reyes, A. (2019). Armendariz modules over skew PBW extensions. *Commun. Algebra* 47(3):1248–1270.

[57] Ouyang, L., Birkenmeier, G. F. (2012). Weak annihilator over extension rings. *Bull. Malays. Math. Sci. Soc.* 35(2):345–347.

[58] Ouyang, L., Liu, J. (2011). On weak $(\alpha, \delta)$-compatible rings. *Int. J. Algebra* 5(26):1283–1296.

[59] Ouyang, L., Liu, J. (2012). Weak associated primes over differential polynomial rings. *Rocky Mountain J. Math.* 42(5):1583–1600.

[60] Ouyang, L., Liu, J., Xiang, Y. (2013). Ore extensions of skew $\pi$-Armendariz rings. *Bull. Iranian Math. Soc.* 39(2):355–368.

[61] Ore, O. (1933). Theory of non-commutative polynomials. *Ann. Math.* (2) 34(3):480–508.

[62] Redman, I. T. (1999). The homogenization of the three dimensional skew polynomial algebras of type I. *Commun. Algebra* 27(11):5587–5602.

[63] Reyes, A. (2018). $\sigma$-PBW Extensions of Skew $\Pi$-Armendariz rings. *Far East J. Math. Sci. (FJMS)* 103(2):401–428.

[64] Reyes, A. (2019). Armendariz modules over skew PBW extensions. *Commun. Algebra* 47(3):1248–1270.

[65] Reyes, A., Rodríguez, C. (2021). The McCoy condition on Skew Poincaré-Birkhoff-Witt extensions. *Commun. Math. Stat.* 9(1):1–21.

[66] Reyes, A., Suárez, H. (2018). A notion of compatibility for Armendariz and Baer properties over skew PBW extensions. *Rev. Un. Mat. Argentina* 59(1):157–178.

[67] Reyes, A., Suárez, H. (2020). Skew Poincaré-Birkhoff-Witt extensions over weak compatible rings. *J. Algebra Appl.* 19(12):2050225.

[68] Reyes, A., Suárez, H. (2021). Radicals and Köthe’s conjecture for Skew PBW extensions. *Commun. Math. Stat.* 9(2):119–138.
[78] Reyes, A., Suárez, H. (2021). Skew PBW extensions over symmetric rings. *Algebra Discrete Math.* 32(1):76–102.

[79] Rosenberg, A. (1995). *Non-commutative Algebraic Geometry and Representations of Quantized Algebras.* Mathematics and Its Applications, Vol. 330. Dordrecht: Springer.

[80] Shock, R. C. (1972). Polynomial rings over finite dimensional rings. *Pacific J. Math.* 42(1):251–257.

[81] Seiler, W. M. (2010). *Involution. The Formal Theory of Differential Equations and Its Applications in Computer Algebra.* Algorithms and Computation in Mathematics, Vol. 24. Berlin: Springer.

[82] Suárez, H., Chacón, A., Reyes, A. (2022). On NI and NJ skew PBW extensions. *Commun. Algebra* 50(8):3261–3275.

[83] Sun, S. H. (1991). Noncommutative rings in which every prime ideal is contained in a unique maximal ideal. *J. Pure Appl. Algebra* 76(2):179–192.

[84] Tumwesigye, A. B., Richter, J., Silvestrov, S. (2020). Centralizers in PBW extensions. In: Silvestrov, S., Malyarenko, A., Rancic, M., eds. *Algebraic Structures and Applications. SPAS 2017.* Springer, Proceedings in Mathematics & Statistics, Vol. 317, Springer, pp. 469–490.

[85] Witten, E. (1990). Gauge theories, vertex models, and quantum groups. *Nuclear Phys. B* 330(2–3):285–346.

[86] Witten, E. (1991). Quantization of Chern-Simons Gauge theory with complex gauge group. *Commun. Math. Phys.* 137(1):29–66.

[87] Woronowicz, S. L. (1987). Twisted SU(2)-group, an example of a non-commutative differential calculus. *Publ. Res. Inst. Math. Sci.* 23:117–181.

[88] Zelmanowitz, J. M. (1976). The finite intersection property on annihilator right ideals. *Proc. Amer. Math. Soc.* 57(2):213–216.

[89] Zhedanov, A. S. (1991). “Hidden symmetry” of Askey–Wilson polynomials. *Theoret. Math. Phys.* 89(2):1146–1157.