Optimal Prefix Codes with Fewer Distinct Codeword Lengths are Easier to Construct

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Abstract. A new method for constructing minimum-redundancy prefix codes is described. This method does not explicitly build a Huffman tree; instead it uses a property of optimal codes to find the codeword length of each weight. The running time of the algorithm is shown to be $O(nk)$, which is asymptotically faster than Huffman’s algorithm when $k = o(\log n)$, where $n$ is the number of weights and $k$ is the number of distinct codeword lengths. We also sketch a matching lower bound of $\Omega(nk)$ for any such construction algorithm, indicating that our algorithm is asymptotically optimal in terms of $n$ and $k$.

1 Introduction

Minimum-redundancy coding plays an important role in data compression applications [14]. Minimum-redundancy prefix codes give the best possible compression of a finite text when we use one static code for each symbol of the alphabet. This encoding is extensively used in various fields of computer science, such as picture compression, data transmission, etc. Therefore, the methods used for calculating sets of minimum-redundancy prefix codes that correspond to sets of input symbol weights are of great interest [3, 9, 7, 10].

The minimum-redundancy prefix code problem is to determine, for a given list $W = [w_1, \ldots, w_n]$ of $n$ positive symbol weights, a list $L = [l_1, \ldots, l_n]$ of $n$ corresponding integer codeword lengths such that $\sum_{i=1}^{n} 2^{-l_i} = 1$, and $\sum_{i=1}^{n} w_i l_i$ is minimized. (Throughout the paper, when Kraft inequality, $\sum_{i=1}^{n} 2^{-l_i} \leq 1$, is satisfied, it is satisfied with an equality.) Once we have the codeword lengths corresponding to a given list of weights, constructing a corresponding prefix code can be easily done in linear time using standard techniques.

Finding a minimum-redundancy code for $W = [w_1, \ldots, w_n]$ is equivalent to finding a binary tree with minimum-weight external path length $\sum_{i=1}^{n} w(x_i) l(x_i)$ among all binary trees with leaves $x_1, \ldots, x_n$, where $w(x_i) = w_i$ and $l(x_i) = l_i$ is the depth of $x_i$ in the corresponding tree. Hence, if we define a leaf as a weighted node, the minimum-redundancy prefix code problem can be defined as the problem of constructing an optimal binary tree for a given list of leaves.

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Based on a greedy approach, Huffman’s algorithm [6] constructs specific optimal trees, which are referred to as Huffman trees. The Huffman algorithm starts with a list $\mathcal{H}$ containing $n$ leaves whose values correspond to the given $n$ weights.

In the general step, the algorithm selects the two nodes with the smallest values in the current list of nodes $\mathcal{H}$ and removes them from the list. Next, the removed nodes become children of a new internal node, which is inserted in $\mathcal{H}$. To this internal node is assigned a value that is equal to the sum of the values of its children. The general step repeats until there is only one node in $\mathcal{H}$, the root of the Huffman tree. The internal nodes of a Huffman tree are thereby assigned values throughout the algorithm; The value of an internal node is the sum of the weights of the leaves of its subtree. The Huffman algorithm requires $O(n \log n)$ time and linear space. Van Leeuwen [12] showed that the time complexity of Huffman’s algorithm can be reduced to $O(n)$ if the input list is already sorted.

A distribution-sensitive algorithm is an algorithm whose running time relies on how the distribution of the input affects the output [8, 11]. For example, a related such algorithm is that of Moffat and Turpin [10]; where they show how to construct a minimum-redundancy prefix code on a sorted-by-weight alphabet of $n$ symbols and $r$ distinct symbol weights in $O(r + r \log(n/r))$ time. The algorithms proposed in [7] are, in a sense, also distribution-sensitive, since their additional space complexities depend on the maximum codeword length of the output code; The B-LazyHuff algorithm [7] runs in $O(n)$ time and requires $O(l)$ extra storage to construct a minimum-redundancy prefix code on a sorted-by-weight alphabet, where $l$ is the maximum codeword length.

Throughout the paper, we interchangeably use the terms leaves and weights. Unless otherwise stated, we assume that the input weights are unsorted. Unless explicitly stated, when mentioning a node of a tree we mean that it is either a leaf or an internal node. We number the levels of the tree bottom up starting from level 0, i.e. the root will have the highest level number $l$, its children will be at level $l - 1$, and the leaves furthest from the root will be at level 0 (this may be different from the standard level numbering). A weight at level $j$ is then given a codeword of length $l - j$.

In this paper, we give an asymptotically optimal distribution-sensitive algorithm for constructing minimum-redundancy prefix codes which runs in $O(nk)$. We use the symbol $k$ to represent the number of different codeword lengths, i.e. $k$ is the number of levels that have leaves in the corresponding optimal binary tree. We also sketch a matching lower bound of $\Omega(nk)$ for any such construction algorithm. If the sequence of weights is sorted, our algorithm uses $O(\log^{2k-1} n^3)$ comparisons, which is sub-linear if the value of $k$ is small.

The paper is organized as follows: In Section 2, we give a property of optimal trees corresponding to prefix codes, on which our construction algorithm relies. In Section 3, we give the basic algorithm and prove its correctness. We show in Section 4 how to implement the algorithm to ensure the distribution-sensitive behavior; the bound on the running time we achieve in this section is exponential with respect to $k$. In Section 5, we improve our algorithm to achieve the $O(nk)$ bound. We conclude the paper and sketch the lower bound proof in Section 6.
2 The Exclusion property

Consider a binary tree $T$ that corresponds to a list of $n$ weights $[w_1, \ldots, w_n]$ and has the following properties:

1. The $n$ leaves of $T$ correspond to the given $n$ weights.
2. The value of a node equals the sum of the weights of the leaves of its subtree.
3. For every level of $T$, let $y_1, y_2, \ldots$ be the nodes of that level in non-decreasing order with respect to their values, then $y_{2i-1}$ and $y_{2i}$ are siblings for all $i \geq 1$.

We define the exclusion property [1, 2] for $T$ as follows: $T$ has the exclusion property if and only if the values of the nodes at a level are not smaller than the values of the nodes at the lower levels.

Lemma 1. [1] Given a prefix code whose corresponding tree $T$ has the aforementioned properties, the given prefix code is optimal and $T$ is a Huffman tree if and only if $T$ has the exclusion property.

Proof. First, assume that $T$ does not have the exclusion property. It follows that there exist two nodes $y$ and $y'$ at levels $\eta$ and $\eta'$ such that $\eta < \eta'$ and $\text{value}(y) > \text{value}(y')$. Swapping the subtree of $y$ with the subtree of $y'$ results in another tree with a smaller external path length and a different list of levels, implying that the given prefix code is not optimal.

Next, assume that $T$ has the exclusion property. Let $[x_1, \ldots, x_n]$ be the list of leaves of $T$, with $w(x_i) \leq w(x_{i+1})$. We prove by induction on the number of leaves $n$ that $T$ is an optimal binary tree that corresponds to an optimal prefix code. The base case follows trivially when $n = 2$. As a result of the exclusion property, the two leaves $x_1, x_2$ must be at the lowest level of $T$. Also, Property 3 of $T$ implies that these two leaves are siblings. Alternatively, there is an optimal binary tree with leaves $[x_1, \ldots, x_n]$, where the two leaves $x_1, x_2$ are siblings; a fact that is used to prove the correctness of Huffman’s algorithm [6]. Remove $x_1, x_2$ from $T$, replace their parent with a leaf $x_1 + x_2$ whose weight equals $w(x_1) + w(x_2)$, and let $T'$ be the resulting tree. Since $T'$ has the exclusion property, it follows using induction that $T'$ is an optimal tree with respect to its leaves $[x_1 + x_2, x_3, \ldots, x_n]$. Hence, $T$ is an optimal tree and corresponds to an optimal prefix code. Property 3 ensures that every two consecutive nodes in the non-decreasing order of values are siblings, which is precisely the way Huffman’s algorithm constructs the codes. It follows that the tree $T$ is a Huffman tree.

The sibling property was introduced by Gallager [5] (see also [13]). The sibling property states that if the nodes of a tree that corresponds to a prefix code are numbered in a non-decreasing order by their values, then this tree is a Huffman tree if and only if every two consecutive nodes (except the root) in this ordering are siblings. In fact, the sibling property is equivalent to Property 3 combined with the exclusion property. This equivalence can be directly proved, indicating the equivalence of a tree $T$ that has the exclusion property to a Huffman tree.

In general, building a tree $T$ that has the exclusion property by evaluating all the internal nodes of $T$ requires $\Omega(n \log n)$. This follows from the fact that once
we have built $T$ the sorted order of the input weights will be known, a problem that requires $\Omega(n \log n)$ in the algebraic decision-tree model. It is crucial to mention that we do not have to explicitly construct $T$ in order to find optimal codeword lengths. Instead, we only need to find the values of some of, and not all, the internal nodes to maintain the exclusion property.

3 The basic construction method

Given a list of weights, we build a corresponding optimal tree bottom up. Starting with the lowest level (level 0), a weight is momentarily assigned to a level as long as its value is less than the sum of the two nodes with the smallest values at that level. The Kraft inequality is enforced by making sure that the number of nodes at every level is even, and that the number of nodes at the highest level containing leaves is a power of two. This will result in some weights changing their initially assigned levels when moved upwards.

3.1 Example

For the sake of illustration, consider a list with thirty weights: ten weights have the value 2, ten have the value 3, five the value 5, and five the value 9.

To construct the optimal codes, we start by finding the smallest two weights in the list: these will have the values 2, 2. We now identify all the weights in the list with value less than 4, the sum of these two smallest weights. All these weights, ten weights of value 2 and ten of value 3, will be momentarily placed at level 0. The number of nodes at this level is even, so we move to the next upper level (level 1). We identify the smallest two nodes at level 1, amongst the two smallest internal nodes resulting from combining nodes of level 0 and the two smallest weights among those remaining in the list; these will be the two internal nodes 4, 4 whose sum is 8. All the remaining weights with value less than 8 are placed at level 1. This level now contains an odd number of nodes: ten internal nodes and five weights of value 5. To make this number even, we move the node with the largest value to the, still empty, next upper level (level 2). The node to be moved, in this case, is an internal node with value 6. Moving an internal node one level up implies moving the weights in its subtree one level up. So, the subtree consisting of the two weights of value 3 is moved one level up. At the end of this stage, level 0 contains ten weights of value 2 and eight weights of value 3; level 1 contains two weights of value 3 and five weights of value 5. For level 2, the smallest two internal nodes have values 6, 8 and the smallest weight in the list has value 9. This means that all the five remaining weights in the list will go to level 2. Since we are done with all the weights, we only need to enforce the condition that the number of nodes at level 3 is a power of two. Level 2 now contains eight internal nodes and five weights, for a total of thirteen nodes. All we need to do is to move the three nodes with the largest values, from level 2, one level up. The largest three nodes at level 2 are the three internal nodes of values 10, 12 and 12. So, we move eight weights of value 3 and two weights of
value 5 one level up. As a result, the number of nodes at level 3 will be 8. The root will then be at level 6.

The final distribution of weights will be: ten weights of value 2 at level 0; ten weights of value 3 and three weights of value 5 at level 1; and the remaining weights, two of value 5 and five of value 9, at level 2. The corresponding codeword lengths are 6, 5 and 4 respectively.

3.2 The algorithm

The idea of the algorithm should be clear. We construct an optimal tree by maintaining the exclusion property for all the levels. Once the weights are placed at the levels in such a way that the exclusion property is satisfied, the property will be satisfied for the internal nodes. Adjusting the number of nodes at each level will not affect the exclusion property, since we are always moving the largest nodes one level up to a still empty level. A formal description follows. (Note that the main ideas of our basic algorithm described in this subsection are pretty similar to those of the Lazy-Traversal algorithm described in [7].)

1. Let \( W \) be the list of symbol weights (not necessarily sorted). The smallest two weights are found, removed from \( W \), and placed at the lowest level 0; Their sum \( S \) is computed. The list \( W \) is scanned and all weights less than \( S \) are removed and placed at level 0. If the number of leaves at level 0 is odd, the leaf with the largest weight among these leaves is moved to level 1.

2. In the general iteration, after moving weights from \( W \) to level \( \eta \), determine the weights from \( W \) that will go to level \( \eta + 1 \) as follows. Find the smallest two internal nodes at level \( \eta + 1 \), and the smallest two weights remaining in \( W \). Find the smallest two values amongst these four, and let their sum be \( S \). Scan \( W \) for all weights less than \( S \), and move them to level \( \eta + 1 \). If the number of nodes at level \( \eta + 1 \) is odd, move the subtree of the node with the largest value among these nodes to level \( \eta + 2 \).

3. When \( W \) is exhausted, let \( m \) be the number of nodes at the highest level that has leaves. Move the \( 2^{\lceil \log_2 m \rceil} - m \) subtrees of the nodes with the largest values, from such level, one level up.

3.3 Proof of correctness

To guarantee its optimality following Lemma 1, we need to show that both the Kraft inequality and the exclusion property hold for the constructed tree.

By construction, the number of nodes at every level of the tree is even. At Step 3 of the algorithm, if \( m \) is a power of 2, no subtrees are moved up and Kraft inequality holds. Otherwise, we move \( 2^{\lceil \log_2 m \rceil} - m \) nodes to the upper level, leaving \( 2m - 2^{\lceil \log_2 m \rceil} \) nodes at this level other than those of the subtrees that have just been moved one level up. Now, the number of nodes at the next upper level is \( m - 2^{\lceil \log_2 m \rceil} - 1 \) internal nodes resulting from combining pairs of nodes at this level, plus the \( 2^{\lceil \log_2 m \rceil} - m \) nodes that we have just moved. This sums up to \( 2^{\lceil \log_2 m \rceil} - 1 \) nodes, that is a power of 2, and Kraft inequality holds.
Throughout the algorithm, we maintain the exclusion property by making sure that the sum of the two nodes with the smallest values is larger than all the values of the nodes at this level. When we move a subtree one level up, the root of this subtree is the node with the largest value at its level. Hence, all the nodes of this subtree at a certain level will have the largest values among the nodes of this level. Moving these nodes one level up will not alter the exclusion property. We conclude that the resulting tree has the exclusion property.

4 The detailed construction method

Up to this point, we have not shown how to evaluate the internal nodes needed by our basic algorithm, and how to search within the list $W$ to decide which weights are at which levels. The basic intuition behind the novelty of our approach is that it does not require evaluating all the internal nodes of the tree corresponding to the prefix code, and would thus surpass the $\Theta(n \log n)$ bound for several cases, a fact that will be asserted in the analysis. We show next how to implement the basic algorithm in a distribution-sensitive behavior.

4.1 An illustrating example

The basic idea is clarified through an example with $3n/2 + 2$ weights ($n$ is a power of 2). Assume that the resulting optimal tree will turn out to have $k = 3$: $n$ leaves at level 0, $n/2$ at level 1, and two at level $\log_2 n$. Note that the $3n/2$ leaves at levels 0 and 1 combine to produce two internal nodes at level $\log_2 n$.

In such case, we show how to apply our algorithm such that the optimal codeword lengths will be produced in linear time. Determining the weights at level 0 can be easily done by finding the smallest two weights and scanning through the list of weights. Determining the weights at level 1 can also be easily done after finding the smallest two internal nodes at level 1, resulting from the smallest four weights from level 0, and scanning through the list of the remaining weights. A more involved task that we need to do next is to evaluate the smallest node $y$ of the two internal nodes at level $\log_2 n$, which amounts to identifying the smallest $n/2$ nodes amongst the nodes at level 1. In order to be able to achieve this in linear time, we need to do it without having to evaluate all $n/2$ internal nodes resulting from the pairwise combinations of the $n$ weights at level 0. We show that this can be done through a simple pruning procedure. The nodes at level 1 consist of two sets: one set has $n/2$ leaves whose weights are known and thus their median $M$ can be found in linear time [4], and another set containing $n/2$ internal nodes which are not known but whose median $M'$ can still be computed in linear time, by simply finding the two middle weights of the $n$ leaves at level 0 and adding them. Assuming without loss of generality that $M > M'$, then the larger half of the $n/2$ weights at level 1 can be safely discarded as not contributing to $y$, and the smaller half of the $n$ weights at level 0 are guaranteed to contribute to $y$. The above step is repeated recursively on
a problem half the size. This results in a procedure satisfying the recurrence
\[ T(n) = T(n/2) + O(n) , \]
and hence \( T(n) = O(n) \).

If the list of weights is already sorted, no comparisons are required to find
\( M \) or \( M' \). The total number of comparisons needed will satisfy the recurrence
\[ C_s(n) = C_s(n/2) + O(1) , \]
and hence \( C_s(n) = O(\log n) \).

### 4.2 The algorithm

Let \( \eta_1 = 0 < \eta_2 < \ldots \eta_j \) be the levels that have already been assigned weights
at some step of our algorithm (other levels only have internal nodes), \( n_i \) be the
count of the leaves so far assigned to level \( \eta_i \), and \( N_j = \sum_{i=1}^{j} n_i \).

At this iteration, we are looking forward to find the next upper level \( \eta_{j+1} \)
that will be assigned weights by our algorithm. We use the fact that the weights
that have already been assigned to these \( j \) levels are the only weights that may
contribute to the values of the internal nodes below and up to level \( \eta_{j+1} \).

Consider the internal node \( \aleph_j \) at level \( \eta_j \), where the sum of the counts of
the weights contributing to level-\( \eta_j \) internal nodes whose values are smaller (or
larger) than that of \( \aleph_j \) is at most \( N_j / 2 \). We call \( \aleph_j \) the splitting node of the
internal nodes at level \( \eta_j \). In other words, if we define the multiplicity of a node
to be the number of leaves in its subtree, then \( \aleph_j \) is the weighted-by-multiplicity
median within the sorted-by-value sequence of the internal nodes at level \( \eta_j \).

Analogously, consider a node \( \aleph'_j \) (not necessarily an internal node) at level \( \eta_j \),
where the sum of the counts of the weights contributing to level-\( \eta_j \) internal nodes
whose values are smaller (or larger) than that of \( \aleph'_j \) plus the count of level-\( \eta_j \)
leaves whose values are smaller (or larger) than that of \( \aleph'_j \) is at most \( N_j / 2 \). We
call \( \aleph'_j \) the splitting node of all the nodes at level \( \eta_j \). Informally, \( \aleph_j \) is an internal
node that splits the weights below level \( \eta_j \) in two groups having almost equal
counts, and \( \aleph'_j \) is the node that splits the weights below and up to level \( \eta_j \) in
two groups having almost equal counts.

**Finding the splitting node.** Consider the following pruning procedure which
finds the splitting node \( \aleph'_j \) of all the nodes at level \( \eta_j \) by utilizing another procedure
that identifies the splitting nodes of the internal nodes at the same level.
We find the leaf \( M \) with the median weight among the list of the \( n_j \) weights
already assigned to level \( \eta_j \) (partition the \( n_j \) list into two sublists around \( M \)),
and recursively evaluate the splitting node \( M' \) of the internal nodes at level \( \eta_j \)
using the list of the \( N_{j-1} \) weights of the lower levels (partition the \( N_{j-1} \) list into
two sublists around \( M' \)). Comparing the values of \( M \) and \( M' \), assume without
loss of generality that \( M > M' \). We conclude that the weights that are larger
than \( M \) must be larger than \( \aleph'_j \), and the internal nodes whose values are smaller
than \( M' \) must be smaller than \( \aleph'_j \). The two corresponding sublists are accordingly discarded, and a new median \( M \) and a new splitting node \( M' \) are found
for the remaining two sublists. The pruning procedure continues until only the
node \( \aleph'_j \) remains. As a byproduct, we also know which weights contribute to the
nodes at level \( \eta_j \) whose values are smaller (or larger) than that of \( \aleph'_j \).
Now, consider the problem of finding the splitting node $\aleph_{j+1}$ of the internal nodes at level $\eta_{j+1}$. Observe that $\aleph'_j$ is a descendant of $\aleph_{j+1}$, so we start by recursively finding the node $\aleph'_j$. Let $\alpha$ be the count of the nodes whose values are smaller than $\aleph'_j$ at level $\eta_j$. Knowing that exactly $\lambda = 2^{\eta_{j+1} - \eta_j}$ nodes from level $\eta_j$ contribute to every internal node at level $\eta_{j+1}$, we conclude that the largest $\beta = \alpha - \lambda \cdot \lfloor \alpha / \lambda \rfloor$ nodes among these $\alpha$ nodes, as well as the smallest $\lambda - \beta - 1$ nodes among those whose values are larger than $\aleph'_j$, are the remaining nodes contributing to $\aleph_{j+1}$. We proceed by finding such nodes, a procedure that requires recursively evaluating more splitting nodes at level $\eta_j$, in a way that will be illustrated in the next subsection.

To summarize, the splitting node $\aleph_{j+1}$ of level $\eta_{j+1}$ is evaluated as follows. The aforementioned pruning procedure is applied to split the weights already assigned to the lower $j$ levels to three groups; those contributing to $\aleph'_j$, those contributing to the nodes of level $\eta_j$ that are smaller than $\aleph'_j$, and those contributing to the nodes of level $\eta_j$ that are larger than $\aleph'_j$. The weights contributing to $\aleph_{j+1}$ are: the weights of the first group, the weights among the second group contributing to the largest $\beta$ nodes smaller than $\aleph'_j$, and the weights among the third group contributing to the smallest $\lambda - \beta - 1$ nodes larger than $\aleph'_j$.

Let $T(N_j, j)$ be the time required to find $\aleph_{j+1}$. The total amount of work, in all the recursive calls, required to find the medians among the $n_j$ weights assigned to level $\eta_j$ is $O(n_j)$. During the pruning procedure to evaluate $\aleph'_j$, the time for the $i$-th recursive call to find a splitting node at level $\eta_j$ is $T(N_{j-1}/2^{i-1}, j - 1)$. The pruning procedure, therefore, requires up to $\sum_{i \geq 1} T(N_{j-1}/2^{i-1}, j - 1) + O(n_j)$ time. To find the $i$-th smallest (or largest) nodes among each group which constitutes at most $N_{j-1}/2$ of the leaves, several calls to evaluate splitting nodes are also initiated. The time for the $i$-th such recursive call is $T(N_{j-1}/2^i, j - 1)$, for a total of $\sum_{i \geq 1} T(N_{j-1}/2^i, j - 1) + O(n_j)$ time for each of the two groups (see next subsection). Summing up the bounds, the next relations follow:

\[
T(N_1, 1) = O(n_1),
\]
\[
T(N_j, j) \leq \sum_{i \geq 1} T(N_{j-1}/2^{i-1}, j - 1) + 2 \sum_{i \geq 1} T(N_{j-1}/2^i, j - 1) + O(n_j).
\]

Substitute with $T(a, b) \leq c \cdot 4^b a$, for $a < N_j$, $b < j$, and some big enough constant $c$. Then,

\[
T(N_j, j) \leq c \cdot 4^{j-1} N_{j-1} \left( \sum_{i \geq 1} 1/2^{i-1} + 2 \sum_{i \geq 1} 1/2^i \right) + O(n_j),
\]
\[
< c \cdot 4^j N_{j-1} + c \cdot n_j.
\]

Using the fact that $N_j = N_{j-1} + n_j$, then

\[
T(N_j, j) = O(4^j N_j).
\]
Finding the \( t \)-th smallest (or largest) node. Consider the node \( \mathbb{3}_j \) at level \( \eta_j \), which has the \( t \)-th smallest (or largest) value among the nodes at level \( \eta_j \). The following recursive procedure is used to evaluate \( \mathbb{3}_j \).

As for the case of finding the splitting node, we find the leaf with the median weight \( M \) among the list of the \( n_j \) weights already assigned to level \( \eta_j \), and evaluate the splitting node \( M' \) of the internal nodes at level \( \eta_j \) (applying the above recursive procedure) using the list of the \( N_j-1 \) leaves of the lower levels. Comparing \( M \) to \( M' \), we can discard one of the four sublists - the two sublists of \( n_j \) leaves and the two sublists of \( N_j-1 \) leaves - as not contributing to \( \mathbb{3}_j \). Repeating this pruning procedure, we identify the weights that contribute to \( \mathbb{3}_j \) and hence evaluate \( \mathbb{3}_j \). As a byproduct, we also know which weights contribute to the nodes at level \( \eta_j \) whose values are smaller (or larger) than that of \( \mathbb{3}_j \).

Let \( T'(N_j, j) \) be the time required by the above procedure. Then,

\[
T'(N_j, j) \leq \sum_{i \geq 1} T(N_{j-1}/2^{i-1}, j - 1) + O(n_j) = O(4^j N_j).
\]

Finding \( \eta_{j+1} \) (the next level that will be assigned weights). We start by finding the minimum weight \( w \) among the weights remaining in \( W \) at this point of the algorithm, and use this weight to search within the nodes at level \( \eta_j \) in a manner similar to binary search. The basic idea is to find the maximum number of the smallest-valued nodes at level \( \eta_j \), such that the sum of their values is less than \( w \). We find the splitting node \( N'_j \) at level \( \eta_j \), and evaluate the sum of the weights contributing to the nodes at that level whose values are smaller than that of \( N'_j \). Comparing this sum with \( w \), we decide which sublists of the \( N_j \) leaves to proceed to find its splitting node. At the end of this searching procedure, we would have identified the weights contributing to the \( \gamma \) smallest nodes at level \( \eta_j \), such that the sum of their values is less than \( w \) and \( \gamma \) is maximum (\( \gamma \geq 2 \)). We conclude by setting \( \eta_{j+1} \) to be equal to \( \eta_j + \lceil \log_2 \gamma \rceil \).

To prove the correctness of this procedure, consider any level \( \eta_j \) such that \( \eta_j < \eta < \eta_j + \lceil \log_2 \gamma \rceil \). The values of the two smallest internal nodes at level \( \eta \) are contributed to by at most \( 2^{\eta-\eta_j+1} \leq 2^{\lceil \log_2 \gamma \rceil} \leq \gamma \) nodes from level \( \eta_j \). Hence, the sum of these two values is less than \( w \). For the exclusion property to hold, no weights are assigned to any of these levels. On the contrary, the values of the two smallest internal nodes at level \( \eta_j + \lceil \log_2 \gamma \rceil \) are contributed to by more than \( \gamma \) nodes from level \( \eta_j \), and hence their sum is more than \( w \). For the exclusion property to hold, at least this weight \( w \) is to be assigned to this level.

The time required by this procedure is the \( O(n) \) time to find the weight \( w \) among the weights remaining in \( W \), plus the time for the calls to find the splitting nodes. Let \( T''(N_j, j) \) be the time required by this procedure. Then,

\[
T''(N_j, j) \leq \sum_{i \geq 1} T(N_j/2^{i-1}, j) + O(n) = O(4^j N_j + n).
\]

Maintaining Kraft inequality. After deciding the value of \( \eta_{j+1} \), we need to maintain Kraft inequality in order to produce a binary tree corresponding to the
optimal prefix code. This is accomplished by moving the subtrees of the \( \nu \) nodes with the largest values from level \( \eta_j \) one level up. Let \( \mu \) be the number of nodes currently at level \( \eta_j \) and let \( \lambda = 2^{\eta_j+1-\eta_j} \), then the number of the nodes to be moved up is \( \nu = \lambda \cdot \lceil \mu / \lambda \rceil - \mu \). Note that when \( \eta_{j+1} - \eta_j = 1 \) (as in the case of our basic algorithm), then \( \nu \) equals one if \( \mu \) is odd and zero otherwise.

To establish the correctness of this procedure, we need to show that both the Kraft inequality and the exclusion property hold. For a realizable construction, the number of nodes at level \( \eta_j \) has to be even, and if \( \eta_{j+1} - \eta_j = 1 \), the number of nodes at level \( \eta_j + 1 \) has to divide \( \lambda / 2 \). If \( \mu \) divides \( \lambda \), no subtrees are moved to level \( \eta_j + 1 \) and Kraft inequality holds. If \( \mu \) does not divide \( \lambda \), then \( \lambda \cdot \lceil \mu / \lambda \rceil - \mu \) nodes are moved to level \( \eta_j + 1 \), leaving \( 2\mu - \lambda \cdot \lceil \mu / \lambda \rceil \) nodes at level \( \eta_j \) other than those of the subtrees that have just been moved one level up. Now, the number of nodes at level \( \eta_j + 1 \) is \( \mu - \lambda \cdot \lceil \mu / \lambda \rceil / 2 \) internal nodes resulting from the nodes of level \( \eta_j \), plus the \( \lambda \cdot \lceil \mu / \lambda \rceil - \mu \) nodes that we have just moved. This sums up to \( \lambda \cdot \lceil \mu / \lambda \rceil / 2 \) nodes, which divides \( \lambda / 2 \), and Kraft inequality holds. The exclusion property holds following the same argument given in Section 3.3.

The running time of this procedure is the time needed to find the weights contributing to the \( \nu \) nodes with the largest values at level \( \eta_j \), which is \( O(4^j N_j) \).

Summary of the algorithm.

1. The smallest two weights are found, moved from \( W \) to the lowest level \( \eta_1 = 0 \), and their sum \( S \) is computed. The rest of \( W \) is searched for weights less than \( S \), which are moved to level 0 as well.
2. In the general iteration of the algorithm, after assigning weights to \( j \) levels, perform the following steps:
   (a) Find \( \eta_{j+1} \) (the next level that will be assigned weights).
   (b) Maintain the Kraft inequality at level \( \eta_j \) (by moving the \( \nu \) subtrees with the largest values from this level one level up).
   (c) Find the values of the smallest two internal nodes at level \( \eta_{j+1} \), and the smallest two weights from those remaining in \( W \). Find the two nodes with the smallest values among these four, and let their sum be \( S \).
   (d) Search the rest of \( W \), and move the weights less than \( S \) to level \( \eta_{j+1} \).
3. When \( W \) is exhausted, maintain Kraft inequality at the highest level that has been assigned weights.

4.3 Complexity analysis

Using the bounds deduced for the described steps of the algorithm, we conclude that the time required by the general iteration is \( O(4^j N_j + n) \).

To complete the analysis, we need to show the effect of maintaining the Kraft inequality on the complexity of the algorithm. Consider the scenario when, as a result of moving subtrees one level up, all the weights at a level move up to the next level that already had other weights. As a result, the number of levels that contain leaves decreases. It is possible that within a single iteration the number
of such levels decreases to half its value. If this happens for several iterations, the amount of work done by the algorithm would have been significantly large compared to, $k$, the actual number of distinct codeword lengths. Fortunately, this scenario will not happen quite often. In the next lemma, we bound the number of iterations performed by the algorithm by $2k$. We also show that at any step of the algorithm the number of levels that are assigned weights, and hence the number of iterations performed, is at most twice the number of the distinct optimal codeword lengths for the weights that have been assigned so far.

**Lemma 2.** Consider the set of weights that will have the $\tau$-th largest optimal codeword length at the end of the algorithm. During the execution of the algorithm, these weights will be assigned to at most two consecutive (with respect to the levels that contain leaves) levels, with level numbers, at most, $2\tau - 1$ and $2\tau$. Hence, the number of iterations performed by the algorithm is at most $2k$.

**Proof.** Consider a set of weights that will turn out to have the same codeword length. During the execution of the algorithm, assume that some of these weights are assigned to three levels. Let $\eta_j < \eta_{j+1} < \eta_{j+2}$ be such levels. Since we are maintaining the exclusion property throughout the algorithm and since $\eta_j + 1 < \eta_{j+2}$, there will exist some internal nodes at level $\eta_j + 1$ whose values are strictly smaller than the values of the weights at level $\eta_{j+2}$ (some may have the same value as the smallest weight at level $\eta_{j+2}$). The only way for such weights to catch each other at the same tree level would be as a result of moving subtrees up to maintain the Kraft inequality. Suppose that, at some point of the algorithm, the weights that are currently at level $\eta_j$ are moved up to catch the weights at level $\eta_{j+2}$. It follows that the internal nodes that are currently at level $\eta_j + 1$ will accordingly move to the next upper level of the moved weights. As a result, the exclusion property will not hold; a fact that contradicts the behavior of our algorithm. It follows that these weights will never be at the same tree level.

We prove the second part of the lemma by induction. The base case follows easily for $\tau = 1$. Assume that the argument is true for $\tau - 1$. By induction, the levels of the weights that will have the $(\tau - 1)$-th largest optimal codeword length will be assigned to the at most $2\tau - 3$ and $2\tau - 2$ levels. From the exclusion property, it follows that the weights that have the $\tau$-th largest optimal codeword length must be at the next upper levels. Using the first part of the lemma, the number of such levels is at most two. It follows that these weights are assigned to the, at most, $2\tau - 1$ and $2\tau$ levels among those assigned weights.

Hence, the weights with the $\tau$-th largest optimal codeword length will be assigned within $2\tau$ iterations. Since the number of distinct codeword lengths is $k$, the number of iterations performed by the algorithm is at most $2k$. $\square$

Using Lemma 2, the time required by our algorithm to assign the set of weights whose optimal codeword length is the $j$-th largest, among all distinct lengths, is $O(4^j n) = O(16^j n)$. Summing for all such lengths, the total time required by our algorithm is $\sum_{j=1}^{k} O(16^j n) = O(16^k n)$.

Consider the case when the list of weights $W$ is already sorted. The following theorem follows using similar recursive relations to those in this section.
Theorem 1. If the list of weights is presorted, constructing minimum-redundancy prefix codes can be done using $O(\log^{2k-1}n^3)$ comparisons.

Corollary 1. For $k < c \cdot \log n / \log \log n$, and any constant $c < 0.5$, the above algorithm requires $o(n)$ comparisons if the list of weights is presorted.

5 The improved algorithm

The drawback of the algorithm we described in the previous section is that it uses many recursive median-finding calls. We perform the following enhancement to the algorithm, if we start with an unsorted list of weights. The basic idea we use here is to incrementally process the assigned weights throughout the algorithm by partitioning them into unsorted blocks, such that the weights of one block are smaller or equal to the smallest weight of the succeeding block. The time bound required by the recursive calls improves when handling these shorter blocks.

The invariant we maintain is that during the execution of the general iteration of the algorithm, after assigning weights to $j$ levels, the weights that have already been assigned to a level $\eta_j$, $j' \leq j$, are partitioned into blocks each of size at most $\max\{n_j/4^j, 1\}$, such that the weights of each block are smaller or equal to the smallest weight of the succeeding block. To accomplish this invariant, once we assign weights to a level, the weights of each block among those already assigned to all the lower levels are partitioned into four almost equal blocks, by finding the weights at the three quartiles and partitioning around these weights. Using Lemma 2, the number of iterations performed by the algorithm is at most $2k$. The amount of work required for this partitioning is $O(n)$ per iterations, for a total of an extra $O(nk)$ time for the partitioning procedure.

For $j - j' \geq \log_4 N_j'$, all the weights assigned to level $\eta_j'$ and the lower levels are already sorted as a result of the partitioning procedure. We maintain the invariant that the internal nodes of all these levels are evaluated, by performing the following incremental evaluation procedure once the above condition is satisfied. The internal nodes at level $\eta_{j'-1}$ must have been evaluated in a previous iteration, since the above condition must have been satisfied for level $\eta_{j'-1}$. What we need to do is to merge the sorted sequence of the weights assigned to level $\eta_{j'-1}$ with the sorted sequence of the internal nodes of level $\eta_{j'-1}$ and evaluate the corresponding internal nodes at level $\eta_{j'}$. This extra work can be done in a total of $O(n)$ time. As a result, finding the value of a node (the splitting node or the $t$-th smallest node) within any of these $j'$ levels can now be done in constant time, as indicated within the recursive relations below.

The basic step for all our procedures is to find the median weight among the weights already assigned to a level $\eta_j$. This step can now be done faster. To find such median weight, we can identify the block that has such median (the middle block) in constant time, then we find the required weight in $O(\max\{n_j/4^j, 1\})$, which is the size of the block at this level. Let $G_j(N_j', j')$ be the time performed by the improved algorithm at and below level $j'$ while assigning the weights at level $j$, where $j' \leq j$. The following recursive relations follow:
\[ G_j(N_1, 1) = O(\max\{n_1/4^{j-1}, 1\}), \]
\[ G_j(N_{j'}, j') = O(1) \quad \text{if } j - j' \geq \log_4 N_{j'}, \]
\[ \leq \sum_{i \geq 1} G_j(N_{j'-1}/2^{i-1}, j' - 1) + 2 \sum_{i \geq 1} G_j(N_{j'-1}/2^{i}, j' - 1) \]
\[ + O(\max\{n_{j'}/4^{j'-1}, 1\}) \quad \text{otherwise.} \]

Substitute with \( G_j(a, b) \leq c \cdot \max\{a/4^{i-b}, 1\} \), for \( a < N_{j'}, \ b < j' \), and some big enough constant \( c \). Then,

\[ G_j(N_{j'}, j') \leq c \cdot \max\{N_{j'-1}/4^{j'-j'+1}, (\sum_{i \geq 1} 1/2^{i-1} + 2 \sum_{i \geq 1} 1/2^i) + n_{j'}/4^{j'-j'}, 1\} \]
\[ < c \cdot \max\{(N_{j'-1} + n_{j'})/4^{j'-j'}, 1\}. \]

Since \( N_{j'} = N_{j'-1} + n_{j'} \), it follows that

\[ G_j(N_{j'}, j') = O(\max\{N_{j'}/4^{j'-j'}, 1\}). \]

The work done to assign the weights at level \( j \) is therefore

\[ G_j(N_j, j) = O(N_j) = O(n). \]

Since the number of iterations performed by the algorithm is at most \( 2k \), by Lemma 2. Summing up for these iterations, the running time for performing the recursive calls is \( O(nk) \). The next main theorem follows.

**Theorem 2.** Constructing minimum-redundancy prefix codes can be done in \( O(nk) \) time.

### 6 Conclusion

We gave a distribution-sensitive algorithm for constructing minimum-redundancy prefix codes, whose running time is \( O(nk) \). For small values of \( k \), this algorithm asymptotically improves over other known algorithms that require \( O(n \log n) \); it is quite interesting to know that the construction of optimal codes can be done in linear time when \( k \) turns out to be a constant. For small values of \( k \), if the sequence of weights is already sorted, the number of comparisons performed by our algorithm is asymptotically better than other known algorithms that require \( O(n) \) comparisons. For such sorted sequences, the number of comparisons required is poly-logarithmic when \( k \) is a constant.

We have shown in [1] that the verification of a given prefix code for optimality requires \( \Omega(n \log n) \) in the algebraic decision-tree model. Such lower bound was illustrated through an example of a prefix code that has \( k = \Theta(\log n) \) distinct
codeword lengths. Since the construction is harder than the verification, it follows that constructing the solution for the codes of this example requires $\Omega(n \log n)$. This implies a lower bound of $\Omega(nk)$ for constructing optimal prefix codes, for otherwise we could have been able to construct the optimal codes for the example in [1] in $o(n \log n)$. This implies that our algorithm is asymptotically optimal.

One remaining question is if it is possible or not to make the algorithm faster in practice by avoiding so many recursive calls to a median-finding algorithm.

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