Dynamic resource allocation problems are ubiquitous, arising in inventory management, order fulfillment, online advertising, and other applications. We initially focus on one of the simplest models of online resource allocation: the multisecretary problem. In the multisecretary problem, a decision maker sequentially hires up to $B$ out of $T$ candidates, and candidate ability values are drawn i.i.d. from a distribution $F$ on $[0, 1]$. First, we investigate fundamental limits on performance as a function of the value distribution under consideration. We quantify performance in terms of regret, defined as the additive loss relative to the best performance achievable in hindsight. We present a novel fundamental regret lower bound scaling of $\Omega(T^{\frac{3}{2}} \cdot \frac{1}{\beta})$ for distributions with gaps in their support, with $\beta$ quantifying the mass accumulation of types (values) around these gaps. This lower bound contrasts with the constant and logarithmic regret guarantees shown to be achievable in prior work, under specific assumptions on the value distribution. Second, we introduce a novel algorithmic principle, Conservativeness with respect to Gaps (CwG), which yields near-optimal performance with regret scaling of $\tilde{O}(T^{\frac{3}{2}} \cdot \frac{1}{\beta})$ for any distribution in a class parameterized by the mass accumulation parameter $\beta$. We then turn to operationalizing the CwG principle across dynamic resource allocation problems. We study a general and practical algorithm, Repeatedly Act using Multiple Simulations (RAMS), which simulates possible futures to estimate a hindsight-based approximation of the value-to-go function. We establish that this algorithm inherits theoretical performance guarantees of algorithms tailored to the distribution of resource requests, including our CwG-based algorithm, and find that it outperforms them in numerical experiments.

Key words: revenue management, online matching, simulation-based algorithms, regret analysis

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1. Introduction

Online resource allocation provides a comprehensive framework for scenarios that involve allocating finite resources to requests arriving over time, with the objective of maximizing the overall reward. This model encompasses several well-studied problems, such as the multisecretary problem \cite{Arlotto2019, Bray2022}, network revenue management \cite{Talluri2006, Vera2021, Bumpensanti2020}, and order fulfillment \cite{Jasin2015}.

Prior work mainly explores these problems under one of two distributional assumptions on the request types: (i) atomic distributions supported on a few points \cite{Vera2021, Bumpensanti2020} and (ii) non-atomic distributions with contiguous support \cite{Lueker1998, Bray2022}. Under these cases, impressive constant and logarithmic regret guarantees have been established, where regret is defined as the expected difference between the total reward under the optimal hindsight policy and the total reward gathered under an online policy.

However, for many important applications, neither of these two assumptions adequately capture reality. For instance, consider the order fulfillment problem encountered by e-commerce platforms like Amazon or Walmart. This is an online matching problem with spatially distributed demand (different zip codes or counties) with product inventory housed in various warehouses scattered across a geographic area. The fulfillment team aims to minimize cumulative shipping costs by dynamically matching each demand to a warehouse which has the item available. Warehouses have limited inventory, and decisions must be made in real-time. This problem can be framed within the online resource allocation problem paradigm. Yet, the aforementioned assumptions made in the prior literature do not capture key features of this setting: (i) the number of demand locations (types) is large (for instance, there are over 40,000 zip codes in the United States), and (ii) these demand locations are spatially clustered with gaps (regions with no demand), a natural characteristic of geographical landscapes such as rivers, mountains, deserts, etc. Hence, atomic distributions with a low number of types or non-atomic distributions with contiguous support fail to capture the salient features of such a problem. Aside from modeling concerns, the near-optimal
algorithms developed for each of the two classes of distributions mentioned above are tailored to that particular class of distributions.

The above motivation leads us to the following research questions: (i) What (request type) distribution features drive achievable performance, and how does regret scale as a function of the underlying distribution? (ii) What algorithmic principles allow one to achieve optimal regret scaling? (iii) Is there a unifying near-optimal algorithm that is agnostic to the underlying distribution’s features?

To isolate and examine key performance drivers, we will initially focus on one of the simplest online resource allocation problems: the multisecretary problem, which is a special case of both the network revenue management problem as well as the online matching (order fulfillment) problem (we refer to Appendix EC.7 for a more extensive discussion on the latter connection). In the multisecretary problem, a decision-maker (DM) with a budget to hire $B$ secretaries is presented with a series of $T$ independent values representing candidate abilities. The DM must make irrevocable “accept” (i.e., hire) or “reject” decisions on the fly, aiming to maximize the (expected) sum of the chosen candidates’ abilities.

We make three main contributions. The first two are in the context of the multisecretary problem: fundamental lower bounds on regret, and an algorithmic principle to achieve the optimal regret scaling. Our third contribution is a unifying and practical algorithm for achieving near optimal regret performance in general resource allocation problems. We now elaborate on these contributions.

(i) Drivers of regret: In the context of the multisecretary problem, we identify a novel fundamental driver of regret which is characterized by a parameter $\beta$, which quantifies the mass accumulation of types around gaps (interval with zero probability mass). Using this parameter $\beta$ we characterize a broad class of distributions with gaps, which we refer to as $(\beta, \varepsilon_0, \delta)$-clustered distributions (cf. Definition 1). The class of $(\beta, \varepsilon_0, \delta)$-clustered distributions is a superset of the class of discrete distributions [Arlotto and Gurvich 2019], and the class of non-atomic distributions with continuous support over $[0, 1]$ and density uniformly bounded away from zero.
We establish a universal lower bound (for any policy) on the growth rate of the regret as a function of the parameter $\beta$ which quantifies how mass accumulates around gaps. In particular, we establish that any policy must incur $\Omega(T^{1/2 - \frac{1}{2(1+\beta)}})$ regret in the worst-case (cf. Theorem 1) for a $(\beta, \varepsilon_0, \delta)$-clustered distribution. This is in stark contrast to prior results which prove regret scaling of $\Theta(1)$ (Arlotto and Gurvich 2019) for the case of distributions with a few discrete types and $\Theta(\log T)$ (Bray 2022) for a special class of non-atomic distributions.

We also show that our lower bound on the regret scaling is achievable up to polylogarithmic factors. To the best of our knowledge, ours is the first result of its kind; notably the regret scaling we establish is polynomial in $T$ for $\beta > 0$ and an entire spectrum of regret scalings are possible. As $\beta$ increases, so does the exponent $\frac{1}{2} - \frac{1}{2(1+\beta)}$ (from 0 to 1/2), characterizing the “hardness” of the problem instance.

(ii) Algorithmic Principle: It turns out the workhorse certainty equivalent (CE) policy is insufficient to deal with general type distributions which have gaps in the support, already in the case of the multisecretary problem. For such distributions, we introduce a new algorithmic principle we call Conservativeness with respect to gaps (CwG); which makes a crucial modification to the CE policy. The idea is that if at any time the CE threshold is close to the boundary of a gap, CwG instead uses the gap as the acceptance threshold to avoid incurring large regret in the future. We establish that this enables the policy to mitigate the risk of incurring large regret (in the event that the threshold for the hindsight optimal falls on the opposite side of that gap). We use this principle to design a near-optimal algorithm, dubbed CwG, for the $(\beta, \varepsilon_0, \delta)$-clustered distributions. Its worst-case regret scales as $\tilde{O}(T^{1/2 - \frac{1}{2(1+\beta)}})$, matching the scaling of the lower bound in $T$ up to polylogarithmic terms (cf. Theorem 2).

For the case of a few discrete types, our algorithm recovers bounded regret, as in Arlotto and Gurvich (2019) (cf. Corollary 2). For the special class of non-atomic distributions with density bounded away from zero, CwG is identical to CE since there are no gaps and we recover the logarithmic regret scaling result of Lueker (1998) and Bray (2022) (cf. Corollary 1).
(iii) **Unifying Algorithm:** Returning to general resource allocation problems, we propose a versatile algorithm called **Repeatedly Act using Multiple Simulations (RAMS)**, which offers a practical and data-driven approach to resource allocation. At each \( t \), RAMS simulates multiple future demand scenarios. Each possible allocation decision at \( t \) results in different cumulative rewards in hindsight, in each demand scenario. RAMS greedily selects the allocation decision which maximizes the average over scenarios of the cumulative reward in hindsight. Unlike previous algorithms, RAMS does not require to be tuned to specific distribution features, and by its design can organically leverage the data-driven simulations of the future which are typically available in practical applications. In terms of performance, we establish a meta result (Theorem 3) that shows that RAMS is guaranteed to inherit the regret performance guarantee of any algorithm satisfying certain conditions (specified in Theorem 3). This result, in conjunction with Theorem 2, implies that RAMS is near-optimal for the multisecretary problem and naturally incorporates the *conservativeness with respect to gaps* principle. Furthermore, our meta theorem, together with existing results on other algorithms in the literature, tells us that RAMS is near-optimal in a variety of settings for NRM and Order Fulfillment problems.

### 1.1. Related Literature

The classical secretary problem was introduced by Cayley (1875) and Moser (1956). The multisecretary variant of the above problem was initially studied by Kleywegt and Papastavrou (1998) and Kleinberg (2005). Recently, Arlotto and Gurvich (2019) showed that, when the distribution of types is discrete, regret is bounded uniformly for all values of the number of candidates \( T \) and the hiring budget \( B \), where the constant may scale with the reciprocal of the minimum probability mass on any type. In order to prove this result, they devise an adaptive policy called the Budget-Ratio (BR) policy where they compare the ratio of the remaining budget to the remaining number of candidates to interview and make the hire/reject decision by comparing the budget ratio to some fixed thresholds. This regret guarantee, in conjunction with a lower bound on regret from Kleinberg (2005) yields a tight understanding of the class of distributions supported on a few discrete types.
Note that the classical secretary problem and its generalization considered in \cite{Kleinberg2005} do not assume the knowledge of the reward distribution. However, following the work of \cite{Arlotto2019}, the variant of multisecretary with distributional knowledge has also been referred to as the multisecretary problem and we will also employ this terminology.

At the other extreme, for a continuum of types, \cite{Lueker1998, Bray2022} show that instead of the regret being uniformly bounded, the best possible scaling for a certain class of non-atomic distributions with contiguous support is $\Theta(\log T)$ (\cite{Bray2022} shows that this is true for the more general network revenue management problem as well). In the context of the multisecretary problem, they devise a simple threshold policy based on the budget ratio to achieve this regret scaling. However, the class of non-atomic distributions considered in these papers requires the probability density function to be bounded away from zero. In a parallel line of inquiry, the set of distributions examined by Blumrosen and Holenstein (2008) bears close resemblance to our own. Yet, there are marked differences in the settings and results. Specifically, Blumrosen and Holenstein (2008) concentrate on the auction setting involving a single item and restrict their study to continuous distributions.

The multisecretary problem is a special case of a broader class of network revenue management (NRM) problems, or more broadly dynamic resource constrained reward collection problems; see \cite{Balseiro2023} for a recent survey and unified modeling framework for this class of problems. There is a wide variety of applications in auction theory (\cite{Kleinberg2005}), online resource allocation (\cite{Kleywegt1998, Talluri2006}), order fulfillment (\cite{Jasin2015}), among others. Note that this literature typically assumes a small number of types.

\cite{Vera2021, Vera2021} generalized the arguments in \cite{Arlotto2019} to a broader class of online packing and online matching problems and proved a uniform regret guarantee across all values of capacity $B$ and time horizon $T$. They developed a technique called \textit{compensated coupling} and used it to prove a constant regret guarantee without requiring any non-degeneracy assumptions. \cite{Bumpensanti2020} also proved constant regret guarantees for a class of NRM problems, however their algorithm and proof techniques differ from those of
Vera and Banerjee (2021), Vera et al. (2021). While all these papers impressively establish constant regret bounds, all of them assume a few discrete types, and their regret bounds scale polynomially in the number of types. However in many practical systems, the number of types is, in fact, large.

Simulation-based algorithms have been studied in the network revenue management literature (Talluri and Van Ryzin 1999, Kunnumkal et al. 2012), albeit without any regret guarantees. The idea in these papers is to solve multiple stochastic optimization problems with different realizations instead of a single fluid relaxation and average the shadow prices of the different optimization problems and implement a bid-price control. Recently, Freund and Banerjee (2019) and Sinclair et al. (2022) have used related ideas to develop algorithms for online bin packing with a few types.

Another line of research connected to our work is on prophet inequalities, in particular $k$-unit prophet inequalities ($k$ corresponds to the budget $B$ described earlier). The $k$-unit prophet inequality problem, originally studied in Hajiaghayi et al. (2007), analyzes the competitive ratio which is defined as the ratio of the expected performance of an algorithm to the expected performance of the hindsight optimal in the worst case over the reward distributions, where the focus is on deriving tight guarantees in terms of $k$. The seminal work of Alaei (2014) proved a guarantee of $1 - 1/\sqrt{k + 3}$ on the competitive ratio and since then this result has been improved upon by Chawla et al. (2020) and Jiang et al. (2022b). One key distinction between this stream and our work is that we consider i.i.d values from a known distribution, which allows to prove stronger guarantees on the regret. The competitive ratio results above would imply a regret scaling of $\Theta(\sqrt{T})$, whereas we show that if the distribution is known and i.i.d, it is possible to do better even under the worst-case when the budget $B$ scales linearly in $T$ (cf. Theorem 2).

Organization of the paper. Section 2 describes the model. In Section 3, we describe a general family of distributions, dubbed $(\beta, \varepsilon_0, \delta)$-clustered distributions, and provide novel fundamental limits on regret scaling. In Section 4 we state our key conservativeness with respect to gaps (CwG) algorithmic principle and provide near-optimal regret scaling for $(\beta, \varepsilon_0, \delta)$-clustered distributions in the context of the multisecretary problem. In Section 5 we discuss our unifying algorithm RAMS. We conclude in Section 6. Due to space constraints, all proofs have been relegated to the appendix.
2. Model

We consider a dynamic resource allocation problem with a known finite time horizon \( T \). There are \( d \) resources and the decision maker is endowed with an initial budget vector \( B \in \mathbb{R}^d \) for the resources. At each time \( t = 1, 2, \ldots, T \), a request \( \theta_t \) is drawn independently from a type set \( \Theta \) via some distribution \( F \) which is known to the decision maker. Upon observing a request \( \theta_t \), the decision maker takes an action \( a_t \in \mathcal{A}(B_t, \theta_t) \) where \( \mathcal{A}(B_t, \theta_t) \) is the set of feasible actions at time \( t \) which depends on the remaining budget \( B_t \) and the request \( \theta_t \). Let \( \mathcal{A} \triangleq \bigcup_{B \geq 0} \bigcup_{\theta \in \Theta} \mathcal{A}(B, \theta) \) denote the set of all possible actions. Upon taking an action \( a_t \), the decision maker collects a reward \( r_t \) which depends on the request \( \theta_t \) and the action \( a_t \). We denote by \( r : \Theta \times \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \) the reward function. Taking an action consumes resources and the amount of resource consumed depends on the request \( \theta_t \) and is denoted by a consumption function \( c : \Theta \times \mathcal{A} \rightarrow \mathbb{R}^d \) where \( c(\theta, a) \) is the amount of \( k \)-th resource consumed when the request is \( \theta \) and action is \( a \). Given a request \( \theta_t \) and action \( a_t \), the remaining budget is updated as per \( B_{t+1} = B_t - c(\theta_t, a_t) \); the action \( a_t \) is required to be such that each coordinate of \( B_{t+1} \) is non-negative. We assume that there is a null action \( a_0 \in \mathcal{A} \) which consumes no resources and generates no reward, i.e., \( r(\theta, a_0) = 0 \) for all \( \theta \in \Theta \) and \( c(\theta, a_0) = 0_{d \times 1} \) for all \( \theta \in \Theta \). Further, we will assume that \( |r(\theta, a)| \leq 1 \) and \( \|c(\theta, a)\|_\infty \leq 1 \) for all \( \theta \in \Theta \) and \( a \in \mathcal{A} \).

A policy is said to be an online (non-anticipating) policy if the decision on the \( t \)-th request is based only on the request \( \theta_t \) at time \( t \), the past requests, \( \{\theta_j\}_{j=1}^{t-1} \) and the history of the actions \( \{a_j\}_{j=1}^{t-1} \) up to the time \( t \). Let \( U_1, U_2, \ldots, U_T \) be a sequence of random variables that are independent and uniformly distributed over \([0, 1]\) and independent of the requests \( \theta_1, \theta_2, \ldots, \theta_T \). (The \( U \)s will allow us to accommodate randomized policies.) Define the filtration \( \mathcal{F}_t = \sigma(\theta_1, U_1, \theta_2, U_2, \ldots, \theta_t, U_t) \) for all \( t \in [T] \). A feasible online policy \( \pi \) is a sequence of \( \{\mathcal{F}_t : t \in [T]\} \)-measurable random variables \( \{a^\pi_1, a^\pi_2, \ldots, a^\pi_T\} \) such that \( \sum_{t=1}^T c(\theta_t, a^\pi_t) \leq B \) almost surely. We define the set of feasible online policies as \( \Pi(B, T) \). For any feasible and online policy \( \pi \in \Pi(B, T) \), define \( R^\pi_t = \sum_{k=1}^t r(\theta_k, a^\pi_k), \forall t \in [T] \) to be the accumulated reward up to time \( t \). The total expected reward under a policy \( \pi \in \Pi(B, T) \) is given by \( V^\pi_1(B, T) = \mathbb{E}[R^\pi_T] = \mathbb{E} \left[ \sum_{t=1}^T r(\theta_t, a^\pi_t) \right] \). Fix \( T \in \mathbb{N} \) and \( B \in \mathbb{R}_{d \geq 0} \), the objective is to maximize the total expected reward given by \( V^\pi_1(B, T) = \sup_{\pi \in \Pi(B, T)} V^\pi_1(B, T) \).
Next we consider the hindsight (hs), full-information version of the problem in which the requests \( \theta_{\geq 1} = \{ \theta_1, \theta_2, \ldots, \theta_T \} \) are known apriori. In the hindsight setting, the problem essentially reduces to solving \( V_1^{hs}(B, T; \theta_{\geq 1}) = \max_{a} \{ \sum_{t=1}^{T} r(\theta_t, a_t) : (a_1, a_2, \ldots, a_T) \in |A|^T \text{ and } \sum_{t=1}^{T} c(\theta_t, a_t) \leq B \} \) and the total expected value by the hindsight optimal problem is given as \( V_1^{hs}(B, T) = \mathbb{E}[V_1^{hs}(B, T; \theta_{\geq 1})] \).

It trivially follows that \( V_1^{hs}(B, T) \geq V_1^{\pi}(B, T), \forall \pi \in \Pi(B, T) \) and \( \forall B \in [T] \). To measure the performance of a feasible online policy \( \pi \in \Pi(B, T) \), we consider the hindsight problem as a benchmark and define the (expected) regret of the policy \( \pi \) as the difference between the expected value of the hindsight problem and the expected value attained by the policy \( \pi \) i.e., \( \text{Regret}(B, T; \pi) \triangleq V_1^{hs}(B, T) - V_1^{\pi}(B, T) \).

We also define the (minimum achievable, expected) regret as the difference between the expected value of the hindsight problem and the expected value under the optimal online policy \( \pi^* \in \Pi(B, T) \).

\[
\text{Regret}(B, T) = \inf_{\pi \in \Pi(B, T)} \text{Regret}(B, T; \pi) = V_1^{hs}(B, T) - V_1^{\pi*}(B, T).
\]

In what follows, we will focus on characterizing the growth rate of \( \text{Regret}(B, T) \) as a function of \( T \) and the characteristics of the distribution of types. Next we discuss the three important classes of online resource allocation problems.

**Network Revenue Management.** In this problem each request \( \theta = (r_\theta, c_\theta) \) is presented with a single reward \( r_\theta \geq 0 \) and a consumption vector \( c_\theta \in \mathbb{R}^d \). We have that \( A = \{a_0 = \text{reject}, a_1 = \text{accept}\} \).

The reward and consumption functions are given as

\[
r(\theta, \text{reject}) = 0, \quad c(\theta, \text{reject}) = \mathbf{0}_{d \times 1}
\]

\[
r(\theta, \text{accept}) = r_\theta, \quad c(\theta, \text{accept}) = c_\theta.
\]

**Online Matching (Order Fulfillment).** In this problem each request \( \theta = r_\theta \) is presented with a vector of rewards \( r_\theta \in \mathbb{R}^d \). Each request wants to consume at most one unit of any single resource.

The action set is \( A = \{a_0, a_1, \ldots, a_d\} \) where \( a_k \) denotes that the request is matched to resource \( k \)
with \( a_0 \) being the null action denoting that the request is rejected. The reward and consumption functions are given as

\[
\begin{align*}
  r(\theta, a_0) &= 0, \\
  c(\theta, a_0) &= 0_{d \times 1} \\
  r(\theta, a_k) &= r_{\theta,k}, \\
  c(\theta, a_k) &= e_k, 
\end{align*}
\]

where \( r_{\theta,k} \) denotes the \( k \)-th coordinate of \( r_{\theta} \) and \( e_k \) is a \( d \)-dimensional vector with the \( k \)-th coordinate being one and all other coordinates being zero.

*Multisecretary Problem.* For the case of one resource \((d = 1)\), network revenue management and online matching are equivalent problems and this special case is referred to as the multisecretary problem. We have that \( c(\theta, \text{accept}) = 1 \) for all \( \theta \in \Theta \). In the context of the multisecretary problem, the request type (equivalently, reward) will be referred to as the candidate ability.

### 3. Fundamental Limits on Achievable Performance

To delve deeper into the intrinsic drivers of performance, we initially focus on the multisecretary problem – a cornerstone model in online resource allocation. Clearly, any lower bound established for the multisecretary problem directly translates into a lower bound for a broader range of online resource allocation problems like NRM and online matching. We now define two classes of distributions under which the multisecretary problem has been previously studied.

**Assumption 1 (Small Number of Types).** The type (reward) distribution \( F \) is supported on a finite set and the rewards are assumed to be in the interval \([0, 1]\).

**Remark 1.** Many prior works refer to this as the “finite types setting”, and establish constant regret guarantees (see, e.g., [Arlotto and Gurvich 2019], [Bumpensanti and Wang 2020], [Vera and Banerjee 2021]). However, these guarantees scale linearly with the number of types. Hence, they are most relevant when the size of discrete types set is small. To emphasize this aspect, we use the phrases “small number of types” or “small discrete set” or “few types” to describe this setting.

**Assumption 2 (Infinitely Many Types with density bounded away from zero).** The type (reward) distribution \( F \) is supported on an infinite set and \( F \) admits a density \( f \) which is
bounded from below and above, i.e., there exist $0 < \nu \leq \bar{\nu} < \infty$ such that $\nu \leq f(\theta) \leq \bar{\nu}$ for all $\theta \in \Theta$.

The rewards are assumed to be in the interval $[0, 1]$.

To interpolate between these two classes of distributions, we will introduce a general class of distributions which will capture the distributions with a few types and infinitely many types with bounded density as special cases.

3.1. General Class of Distributions For the multisecretary Problem

We will anchor our analysis around a general family of distributions which allow for gaps in the type space and can capture as special cases discrete distributions as well as the non-atomic distributions with density uniformly bounded away from zero. We call this family $(\beta, \epsilon_0, \delta)$-clustered distributions. For any $q \in [0, 1]$, we define $F^{-1}(q) \triangleq \inf\{v : F(v) \geq q\}$.

**Definition 1 (Valid) (\(\beta, \epsilon_0, \delta\)-clustered distributions).** Fix $\beta \in [0, \infty)$, $\epsilon_0 \in (0, 1]$ and $\delta \in [0, 1]$. A distribution $F$ is said to be $(\beta, \epsilon_0, \delta)$-clustered if there exists $n \in \mathbb{N} \cup \{0\}$ and gap quantiles $q^*_0 = 0 < q^*_1 < \cdots < q^*_n < q^*_{n+1} = 1$ such that we have

(a) (Generalized cluster “density” requirement) $\forall i \in [n + 1], \forall q, \tilde{q} \in (q^*_{i-1}, q^*_i]$, we have that $|F^{-1}(q) - F^{-1}(\tilde{q})| \leq C|q - \tilde{q}|^{\frac{1}{1+\epsilon_0}} + \delta$ for some constant $C < \infty$.

(b) (Cluster size requirement) $q^*_i - q^*_{i-1} \geq \epsilon_0, \forall i \in [n + 1]$.

Let $\mathcal{F}_{\beta, \epsilon_0, \delta}$ denote the class of $(\beta, \epsilon_0, \delta)$-clustered distributions. This class includes a wide variety of distributions. An important sub-class is the one with $\delta = 0$, which we denote by $\mathcal{F}_{\beta, \epsilon_0}$. We refer to distributions in this subclass as $(\beta, \epsilon_0)$-clustered.

Define $H_i \triangleq [F^{-1}((q^*_i)^+), F^{-1}(q^*_i)]$, for all $i \in [n + 1]$, where $F^{-1}(q^+) \triangleq \lim_{\epsilon \rightarrow 0^+} F^{-1}(q + \epsilon)$. We will refer to the $(H_i)$’s as mass clusters or just clusters. We will use the term gaps to refer to the complementary intervals $G_i \triangleq (F^{-1}(q^*_i), F^{-1}((q^*_i)^+))$ for $i \in [n]$, and the intervals at the extremes $G_0 = [0, F^{-1}(0^+)), G_{n+1} = (F^{-1}(1), 1]$, since they contain no probability mass. The requirement (a) can be thought of as a within-cluster “density” requirement, which becomes weaker as $\beta$ increases; we can think of $\beta$ as quantifying the within-cluster mass density (with a decreasing relationship).
When $\delta = 0$, this requirement corresponds to $F^{-1}$ being $(1/(\beta + 1))$-Hölder continuous on the mass clusters. Requirement (b) is a cluster size requirement, $\varepsilon_0$ being the minimum cluster size; this requirement becomes more stringent as $\varepsilon_0$ increases. The parameter $\delta$ provides us with additional flexibility in modelling our distributions. One such practically relevant class of distributions is the one with a large number of discrete types, which can be modelled using the parameter $\delta$ (cf. Example [4]). In general, there is some flexibility on how the distributions are modelled, more specifically how the types are aggregated into clusters, and this is associated with a tradeoff between $\delta$ and $\varepsilon_0$ (and potentially $\beta$). Please refer to Appendix [EC.8] for more details.

Next we present some examples of $(\beta, \varepsilon_0, \delta)$-clustered distributions including discrete distributions, as well the uniform distribution, along with the appropriate choices of gap quantiles.

**Example 1 (Discrete Distributions).** Consider a discrete distribution (as studied in [Arlotto and Gurvich 2019]). Let the support be $\{\theta_1, \theta_2, \ldots, \theta_n\}$ with probability masses $\{p_1, p_2, \ldots, p_n\}$. Assume that $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n \leq 1$. We make use of the natural choice of gap quantiles $q_i^{\star} = \sum_{j=1}^{i} p_j$ for all $i \in [n-1]$, leading to gaps $G_0 = [0,a_1), G_i = (a_i, a_{i+1}) \forall i \in [m-1], G_n = (a_n, 1]$ and clusters $H_i = \{a_i\} \forall i \in [n]$. Now for $q, \tilde{q} \in (q_{i-1}^{\star}, q_i^{\star}] = Q_i$, we have that $|F^{-1}(q) - F^{-1}(\tilde{q})| = 0 \leq |q - \tilde{q}|$, i.e., the cluster density requirement is satisfied for $\beta = 0$ and $\delta = 0$. Defining $\varepsilon_0 \triangleq \min\{p_1, p_2, \ldots, p_n\}$ the cluster size requirement is satisfied. Therefore the discrete distribution belongs to the class of $(0, \varepsilon_0)$-clustered distributions where $\varepsilon_0$ is the minimum probability mass in the support.

**Example 2 (Non-atomic Distributions with Contiguous Support).** Consider the non-atomic distributions with pdf $f$ considered in [Bray (2022)] (Assumption 2). Assume that there exists $\alpha_0 > 0$ such that $f(x) \geq \alpha_0, \forall x \in [0,1]$. (The uniform distribution over $[0,1]$ is a special case of these distributions with $f(x) = 1$ for all $x \in [0,1]$.) Such distributions are $(\beta = 0, \varepsilon_0 = 1, \delta = 0)$-clustered distributions with $n = 0$ gaps, i.e., $F^{-1}$ is $1$-Hölder continuous over the interval $(0,1]$ with the constant $C = 1/\alpha_0$. The gap quantiles are only the trivial ones $q_0^{\star} = 0$ and $q_n^{\star} = 1$. There is a single mass cluster $H_1 = [0,1]$ with mass 1, which clearly satisfies the cluster density requirement with $\beta = 0, \varepsilon_0 = 1$ and $\delta = 0$. 
Example 3 (A class of bimodal distributions). An example of a \((\beta, \varepsilon_0)\)-clustered distribution with \(n = 1\) gap (with gap quantile \(q_1^* = 1/2\)), for general \(\beta \geq 0\) and \(\varepsilon_0 = 1/2\), which we will make use of to prove our lower bound results is presented below:

\[
F_\beta(x) = \begin{cases} 
-2 \cdot 4^\beta \cdot \left(\frac{1}{4} - x\right)^{\beta+1} + \frac{1}{2} & 0 \leq x \leq \frac{1}{4} \\
\frac{1}{2} & \frac{1}{4} \leq x \leq \frac{3}{4} \\
2 \cdot 4^\beta \cdot \left(x - \frac{3}{4}\right)^{\beta+1} + \frac{1}{2} & \frac{3}{4} \leq x \leq 1 
\end{cases}
\]

\[
F^{-1}_\beta(q) = \begin{cases} 
\frac{1-(1-2q)^{\frac{1}{\beta+1}}}{4}, & 0 \leq q \leq \frac{1}{2} \\
\left[\frac{1}{4}, \frac{3}{4}\right] & q = \frac{1}{2} \\
\frac{(2q-1)^{\frac{1}{\beta+1}}+3}{4}, & \frac{1}{2} < q \leq 1
\end{cases}
\]

It is easy to see that \(F^{-1}_\beta\) in (1) is a \((\beta, 1/2)\)-clustered distribution, with one gap \(G_1 = (1/4, 3/4)\) and clusters \(H_1 = [0, 1/4]\) and \(H_2 = [3/4, 1]\). Refer to Figure 1 for a plot of the density \(f_\beta\) and the CDF \(F_\beta\) of the \((\beta, 1/2)\)-clustered distribution defined in (1).

![Figure 1](image.png)

**Figure 1** (L) PDF \(f_\beta\) for \(\beta = 0, 1, 2\). Notice the gap from 1/4 to 3/4, (R) CDF \(F_\beta\) for \(\beta = 0, 1, 2\).

Observe that \((\beta, \varepsilon_0)\)-clustered distributions already allow us to capture not only the previously studied distributions such as distributions with few types and continuous distributions (with density bounded below), but also a mixture of atomic and non-atomic distributions with gaps. As mentioned previously, the parameter \(\delta\) provides us with additional flexibility to model distributions with a large number of discrete types, which may be of practical relevance. One such example is that of many small discrete types which we present below.
Example 4 (Many Small Discrete Types). Fix a small $\delta > 0$ and consider a discrete distribution with many small types supported on the points $S = \{0, \delta, 2\delta, \ldots, 1/4\} \cup \{3/4, 3/4 + \delta, 3/4 + 2\delta, \ldots, 1\}$ with probability mass $2\delta$ on each of the points in $S$. This constitutes a setting with many small discrete types since there are a large number of atomic types (separated by small empty intervals) and the probability mass of each type is small, i.e., it is proportional to $\delta$. This instance of many small discrete types captures the salient feature of the order fulfillment problem that there are a large number of demand types (e.g., zipcodes) with each demand type having small probability mass and these demand types are spatially clustered with possibly large gaps between different clusters of demand types. As $\delta \to 0$, we recover the bimodal uniform distribution $F_0$ in the limit. One can similarly consider similar many-small-discrete-type analogs for other $(\beta, \varepsilon_0)$-clustered distributions. Note that the many small types need not be uniformly spaced. We require that the maximum distance between the discretized grid points be at most $\delta$. In such discretizations, we have some flexibility in choosing which empty intervals to classify as “gaps”. In the case of many small types, if the size of the empty intervals (due to discretization) is at most $\delta$ then we may consider the entire clump of these many small types as belonging to one cluster (say, $H_i$) and hence one quantile interval ($Q_i$).

3.2. Fundamental Lower bound on Performance

In this section we present a novel driver of regret scaling: the shape of the candidate ability (or value) distribution around gaps which is characterized by the parameter $\beta \in [0, \infty)$ and show that for $\beta > 0$, polynomial regret scaling is unavoidable. To focus on the scaling with parameter $\beta$, we fix $\delta = 0$ and $\varepsilon_0 = 1/2$.

Theorem 1 (universal lower bound). Fix $\delta = 0, \varepsilon_0 = 1/2$ and consider any $\beta \in [0, \infty)$. Then there exists a candidate-ability distribution $F \in \mathcal{F}_{\beta, \varepsilon_0}$, a time horizon $T_0 < \infty$, a universal constant $c > 0$ such that, for all $T \geq T_0$ and for any online policy $\pi \in \Pi(B, T)$, we have that

$$\sup_{B \in [T]} \text{Regret}(B, T; \pi) \geq \left(\frac{c}{1 + \beta}\right)T^\frac{1}{2 - \frac{1}{\beta(\beta + 1)}} \mathbb{1}\{\beta > 0\} + c \log T \mathbb{1}\{\beta = 0\}.$$
This theorem provides an impossibility result: it says that for any fixed $\beta \in [0, \infty)$, there exists a distribution for which no online policy can achieve a better regret scaling than the one presented in Theorem 1. This lower bound also highlights that the fundamental limits of the regret scaling are governed by the parameter $\beta$ which characterizes the curvature of the distribution around the gap boundaries. We observe that as $\beta \to \infty$, the scaling of regret approaches $\sqrt{T}$; i.e., no matter the online policy, it will suffer regret nearly as large as that of a simple non-adaptive policy. Hence $\beta$ can be seen as characterizing the “hardness” of an instance. The parameter $\beta$ has a physical interpretation as well. It captures how mass accumulates in the type space. For some intuition, consider the $F_{\beta}$ distribution described in (1) and consider the gap boundary at 3/4. As we move from the boundary point 3/4 to a distance $\delta$ into the adjacent cluster, i.e., to $3/4 + \delta$, the probability mass accrued grows as $C\delta^{\beta+1}$ for some universal constant $C > 0$. Alternately, to accrue a probability mass of $\varepsilon$, we need to move a distance $C\varepsilon^{1/(\beta+1)}$ from the boundary 3/4 into the adjacent cluster. Therefore as $\beta$ increases, the distance one needs to travel to collect a probability mass of $\varepsilon$ also increases and this property is what makes the instances harder as $\beta$ increases.

For $\beta = 0$, our lower bound follows from Bray (2022). To establish our bound for $\beta > 0$, we consider the distributions $F_{\beta}$ defined in (1). At a high level, we consider two events of $\Omega(1)$ probability – one is a perturbation of the other – under one event (denoted as $H$), there are more than the expected number of arrivals with values at least 3/4 (“high” types) and hence the hindsight threshold is (slightly) more than 3/4, and on the other event (denoted as $L$), there are fewer than expected number of arrivals with value at least 3/4 and hence the hindsight threshold is (slightly) less than 1/4. While the hindsight optimal policy does well on both the events, the optimal online policy can only do well on one or the other but not in both. We show that any online algorithm must make at least $\Omega(\sqrt{T})$ mistakes on at least one of the two events, and leveraging how the mass accumulates over space (characterized by Definition 1), one may show that the cost of each of these mistakes $\Omega(T^{-1/(\beta+1)})$. Combining the two gives us that the cumulative regret scales as $\Omega(T^{1/2 - 1/(2(\beta+1))})$. We elaborate on this in the formal proof in Appendix EC.1.
4. Algorithmic Design Principles for Near Optimal Performance

Having established a spectrum of fundamental performance boundaries, it is natural to inquire if it is possible to achieve these limits, and if so, what algorithms are capable of attaining these fundamental limits. A prevalent algorithmic principle in the network revenue management literature is the Certainty Equivalent (CE) heuristic. This approach solves a deterministic approximation of a stochastic optimization problem by substituting random variables with their expected values. Given its widespread use, the CE heuristic emerges as a natural initial candidate for analysis and characterization of achievable performance. In this section, we will focus on the CE heuristic for non-atomic distributions to avoid any tie-breaking issues which are present for atomic distributions. For the multisecretary problem, the CE heuristic is defined as follows: at each time $t$ (before the arrival of request $\theta_t$), given a remaining budget $B_t$ and remaining number of time steps $T - t + 1$, we compute the budget ratio $B_t/(T - t + 1)$ and accept the request $\theta_t$ if and only if $r(\theta_t, \text{accept}) \geq F^{-1}(1 - B_t/(T - t + 1))$. Note that the CE heuristic employs an adaptive threshold at each time $t$.

4.1. Failure of the CE policy under many types with gaps

Indeed, in the case of non-atomic distributions with density uniformly bounded away from zero, Lueker (1998) and Bray (2022) showed that CE achieves $O(\log T)$ regret, and that this is the best scaling achievable. However, it turns out that as soon as one introduces a gap in these non-atomic distributions (as in Example 3), the performance of CE degrades significantly. This phenomenon is documented in the proposition below.

**Proposition 1 (Failure of CE).** Fix any $\eta \in (0, 1)$ and $\varepsilon \in (0, 1/2]$. Suppose the candidate-ability distribution $F$ is any non-atomic distribution that has a gap of length at least $\eta$, i.e., $\exists c \in (0, 1 - \eta)$ such that $F(c) = F((c + \eta)^-) \leq \varepsilon$, and such that there is mass at least $\varepsilon$ on each side of the gap, i.e., $\min\{F(c), 1 - F(c)\} \geq \varepsilon$. Then for the CE policy, there exists $T_0 \equiv T_0(\varepsilon) < \infty$, a constant $c \equiv c(\eta, \varepsilon) > 0$ and $B \in [T]$ such that $\text{Regret}(B, T; \text{CE}) \geq c \sqrt{T}$ for all $T \geq T_0$.

The regret of the CE policy increases dramatically if there is a gap in the types, even when one maintains the uniform distribution of types (or any other distribution) outside of the gap.
As a matter of fact, the regret scaling is as large as that of a non-adaptive policy. The result in Proposition 1 is analogous to the results for few types in the literature. The main driver of \( \Omega(\sqrt{T}) \) regret scaling for both the many types with gaps and finite types settings is degeneracy, i.e., situations where the dual variables corresponding to the initial fluid model LP are not unique. This issue is well documented in the setting with finitely many types [Bumpensanti and Wang 2020, Vera and Banerjee 2021], but also manifests in the case of non-atomic distributions with gaps. As such the proof of Proposition 1 follows from the proof of the analogous result for finitely many types in [Bumpensanti and Wang 2020, Proposition 2].

4.2. Conservativeness with respect to gaps

We observed that the CE policy breaks down for distributions with many types and “gaps” (intervals) of absent types; it suffers \( \Omega(\sqrt{T}) \) regret, as large as that of a non-adaptive algorithm. We identified that the main driver for the \( \Omega(\sqrt{T}) \) regret of the CE policy is the presence of gaps. To solve this issue, we introduce a new algorithmic principle which we call “conservativeness with respect to gaps” (CwG), and use it to provably achieve near optimal regret scalings for the \((\beta, \varepsilon_0, \delta)\)-clustered distributions which allow for gaps. The idea of CwG is that if there is a risk that the acceptance threshold based on CE will move across a given gap in the future, then CwG uses that gap as the acceptance threshold instead of using the CE-based threshold. Based on the CwG principle, we devise a new policy with the same name, which we present in Algorithm 1.

The algorithm operates in two phases. For simplicity, assume that \( T \geq \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil \). We begin by describing the first phase. For the first \( \tilde{T} \triangleq T - \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil \) steps the algorithm uses the CwG principle, where if the re-solving threshold \( p_t^{CE} \) is close to a gap, we modify it by instead using the quantile corresponding to the boundary of the gap as our acceptance threshold \( p_{CwG}^t \). It remains to clarify how the quantile threshold \( p_{CwG}^t \) translates to an accept/reject decision for the arrival at \( t \). After observing the type \( \theta_t \), we form the set of corresponding quantiles \( I_t \). If \( I_t \) is a singleton (this is the case if \( \theta_t \) does not lie at an atom of \( F \)) then we have that its unique element \( X_t = F(\theta_t) \). If \( \theta_t \) lies at an atom of \( F \), the set \( I_t \) is a corresponding interval (recall Example 1). If
Algorithm 1: Conservativeness with respect to Gaps (CwG)

Input: Time Horizon $T$, Hiring Budget $B$, $(\beta, \varepsilon_0, \delta)$-clustered dist. $F$ with gaps $G_i = (a_i, b_i)$.

Initialize: $B_1 = B, q_i^* = F(a_i) = F(b_i)$, $\forall i \in [n], \tilde{T} = \max\{0, T - \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil\}$

for $t = 1$ to $\tilde{T}$ do

\[ p_t^{CE} = 1 - \frac{B_t}{T-t+1} \]

\[ S_t = \left\{ i : p_t^{CE} \in B \left( q_i^*, \sqrt{\frac{2\log(T-t+1)}{T-t+1}} \right) \right\} \]

if $S_t = \emptyset$ then

\[ p_t^{CwG} = p_t^{CE} \]

else

\[ j_t^* = \arg\min_{i \in S_t} |p_t^{CE} - q_i^*| \]

\[ p_t^{CwG} = q_{j_t}^* \]

end

Observe a candidate of ability $\theta_t$ and form the set $I_t = \{q \in [0,1] : F^{-1}(q) = \theta_t\}$

Let $X_t$ be a uniform sample from the set $I_t$

if $X_t \geq p_t^{CwG}$ and $B_t > 0$ then

Hire the candidate and $B_{t+1} \leftarrow B_t - 1$

else

Reject the candidate and $B_{t+1} \leftarrow B_t$

end

end

Define $p_{T+1}^{CE} = 1 - \frac{B_{T+1}}{T-T}$

for $t = \tilde{T} + 1$ to $T$ do

Observe a candidate of ability $\theta_t$ and form the set $I_t = \{q \in [0,1] : F^{-1}(q) = \theta_t\}$

Let $X_t$ be a uniformly random sample from the set $I_t$

if $X_t \geq p_{T+1}^{CE}$ and $B_t > 0$ then

Hire the candidate and $B_{t+1} \leftarrow B_t - 1$

else

Reject the candidate and $B_{t+1} \leftarrow B_t$

end

end
If \( p_t^{\text{CwG}} \notin \mathcal{I}_t \) then the hire/reject decision is unambiguous. The only case of ambiguity is \( p_t^{\text{CwG}} \in \mathcal{I}_t \). To handle this case, we make use of randomization to break ties by drawing \( X_t \) uniformly from the interval \( \mathcal{I}_t \), and hiring the candidate only if the \( X_t \) is weakly greater than \( p_t^{\text{CwG}} \).

We now describe the second phase of the algorithm. In the final \( \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil \) time steps, the radius \( \sqrt{2 \log \tau/\tau} \) (where \( \tau = T - t + 1 \) is the number of remaining time steps) by which we measure the closeness of CE threshold \( p_t^{\text{CE}} \) and the gap quantiles \( \{q^*_i\}_{i=1}^n \) becomes too large, i.e. \( \sqrt{2 \log \tau/\tau} > \varepsilon_0/2 \). This results in more than one gap quantiles being in the \( \sqrt{2 \log \tau/\tau} \) neighborhood of \( p_t^{\text{CE}} \) which in turn makes the choice of \( p_t^{\text{CwG}} \) ambiguous and further complicates the regret analysis. In order to avoid this ambiguity and simplify the analysis, we employ a static allocation policy in the second phase: we solve for the certainty equivalent threshold \( p_{\tilde{T} + 1}^{\text{CE}} \) at time \( \tilde{T} + 1 \), and use that threshold for the remaining \( \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil \) time steps.

### 4.2.1. Performance Analysis

**Theorem 2.** For any \( \beta \in [0, \infty), \varepsilon_0 \in (0, 1] \) and \( \delta \in (0, 1] \), suppose the candidate-ability distribution \( F \) with associated gaps is \((\beta, \varepsilon_0, \delta)\)-clustered. Then for all \( T \in \mathbb{N} \) and for all \( B \in \mathbb{N} \), there exists a universal constant \( C < \infty \) such that the regret of the CwG policy is upper bounded as

\[
\text{Regret}(B, T; \text{CwG}) \leq C(1 + 1/\beta)(\log T)^{1/2 + \frac{1}{2\gamma + 1}} T^{1/2 - \frac{1}{2\gamma + 1}} \cdot 1\{\beta > 0\} + C(\log T)^2 1\{\beta = 0\} + C\delta \sqrt{T \log T} + C \sqrt{\log(1/\varepsilon_0)/\varepsilon_0}.
\]

(2)

**Discussion of Theorem 2** The regret upper bound can be decomposed as shown in (2), where each of the terms has a different driver. The terms in (♠) are driven by the shape of the reward distribution around gaps and is characterized by the parameter \( \beta \in [0, \infty) \). Comparing the term (♠) to the lower bound in Theorem 1, we note that the scaling of the upper bound matches the scaling of the lower bound in \( T \) up to a polylogarithmic factor and hence the proposed CwG policy is near-optimal. In the case of the CE policy, we had identified that the main driver of its worst case regret of \( \Theta(\sqrt{T}) \) was the presence of gaps in the distribution of candidate abilities. Theorem 2 tells...
us that one can overcome the difficulty introduced by gaps in the distribution by using the CwG principle that we devised. The term in (♦) is driven by the parameter \( \delta \) which allows us to model distributions with many small discrete types (cf. Example [1]). We will typically assume that \( \delta \) is small and may scale as \( o(1/\sqrt{T}) \). Note that for the extreme cases of a few types (cf. Example [1]) or continuous distributions (cf. Example [2]), we have that \( \delta = 0 \) and hence the term in (♦) disappears. The term in (♥) is driven by the minimum probability mass \( \epsilon_0 \) and is typically assumed to be a constant in \((0,1]\). The contribution of (♥) is attributable to the regret accrued due to the static allocation rule employed in Algorithm [1] in the last \([64\log(1/\epsilon_0)/\epsilon_0^2]\). In terms of scaling of (♥), it matches up to polylogarithmic factors the lower bound on regret scaling of \( \Omega(1/\epsilon_0) \) presented in Lemma 1 of Arlotto and Gurvich (2019).

**Corollary 1.** Suppose the candidate-ability distribution is \( F_0 \) where \( F_0 \) is as defined in [1] with \( \beta = 0 \). Then we have that for all \( T \in \mathbb{N} \) and for all \( B \in [T] \) the regret of our CwG policy is upper bounded as \( \text{Regret}(B,T;\text{CwG}) \leq C(\log T)^2 \) for the universal constant \( C < \infty \) in Theorem [2].

**Discussion of Corollary [1]**. This corollary follows immediately from Theorem [2] by setting \( \beta = 0 \) and \( \delta = 0 \). The distribution \( F_0 = \text{Uniform}([0, 1/4] \cup [3/4, 1]) \) is a natural variant of the uniform distribution with a gap. Corollary [1] shows that regret of CwG scales as \( O((\log T)^2) \) for the distribution \( F_0 \). This is a significant improvement on the \( \Omega(\sqrt{T}) \) regret scaling of the CE policy for the same distribution \( F_0 \), and the regret of the CwG policy for the \( F_0 \) distribution is only a \( \log T \) factor larger than the regret for the uniform distribution. The key takeaway from Corollary [1] in conjunction with Proposition [1] is that the presence of gaps is not a fundamental driver of the achievable regret performance, and one can overcome the difficulty posed by gaps by using the CwG principle.

**Corollary 2 (Constant Regret for discrete distributions)**. Suppose the candidate-ability distribution is \( F \) where \( F \) is a discrete distribution as described in Example [7]. Then, for all \( T \in \mathbb{N} \) and for all \( B \in [T] \), we have \( \text{Regret}(B,T;\text{CwG}) \leq C\sqrt{\log(1/\epsilon_0)/\epsilon_0} \) for a universal constant \( C < \infty \).
Remark 2. The discrete distribution considered in Example 1 belongs to the class of \((0, \varepsilon_0)\)-clustered distributions and hence from Corollary 1 it follows that \(\text{Regret}(B, T) = \mathcal{O}((\log T)^2)\). However, recall from Example 1 that for discrete distributions we have that \(\forall i \in [n+1], \forall q, \tilde{q} \in Q_i, |F^{-1}(q) - F^{-1}(\tilde{q})| = 0\), and this distinguishes discrete distributions from general \((0, \varepsilon_0)\)-clustered distributions. This distinction allows us to obtain stronger regret guarantees than the one implied by Corollary 1 and recover the result of [Arlotto and Gurvich (2019)]. The proof of Corollary 2 follows by modifying the analysis leading to Theorem 2. The modifications enable us to eliminate the \(C(\log T)^2\) term in the regret bound in Theorem 2. We defer the details to Appendix EC.3.

Corollary 3 (Regret for non-atomic distribution with contiguous support). For any \(\beta \in [0, \infty), \varepsilon_0 = 1, \text{ and } \delta = 0\), suppose the candidate-ability distribution \(F\) is \((\beta, \varepsilon_0 = 1, \delta = 0)\)-clustered (\(F\) has no non-trivial gaps). Then for all \(T \in \mathbb{N}\) and for all \(B \in [T]\), there exists a universal constant \(C < \infty\) such that the regret of our CwG policy is

\[
\text{Regret}(B, T; \text{CwG}) \leq C \left(1 + \frac{1}{\beta}\right) T^{\frac{1}{2} - \beta (1+\delta)} \mathbb{1}\{\beta > 0\} + C \log T \mathbb{1}\{\beta = 0\}
\]

Discussion of Corollary 3. This corollary follows immediately from Theorem 2 by setting \(\varepsilon_0 = 1\) and \(\delta = 0\), except for some polylogarithmic factors. The class of \((\beta, \varepsilon_0 = 1, \delta = 0)\)-clustered distributions allows for the pdf \(f\) to be zero at some points. An example of such a distribution is given by \(\tilde{F}_\beta(x) = \left(0.5 - 2^\beta (0.5 - x)^{\beta + 1}\right) \mathbb{1}\{x \leq 0.5\} + \left(0.5 + 2^\beta (x - 0.5)^{\beta + 1}\right) \mathbb{1}\{x > 0.5\}\) where the pdf \(f\) is zero at \(x = 0.5\). Since there are no non-trivial gaps for the distribution \(\tilde{F}_\beta\), we choose to treat the whole interval \([0, 1]\) as a single cluster and hence have \(\varepsilon_0 = 1\). It can be easily verified that \(\tilde{F}_\beta\) satisfies the “cluster density requirement” in Definition 1 with \(\delta = 0\). Note that the distribution \(\tilde{F}_\beta\) is not admissible under the assumptions of Bray (2022) for \(B = T/2\) and \(\beta > 0\). Since there are no gaps of positive length in \((\beta, 1)\)-clustered distributions, the CwG policy boils down to the CE policy. If the probability density function \(f\) is bounded below by a constant, we have \(\beta = 0\) and we recover the \(\mathcal{O}(\log T)\) scaling in Bray (2022). If \(f\) is zero at some points, then the regret scaling is determined by \(\beta\) which quantifies how the mass accumulates around types where \(f\) is zero. This result, in conjunction with Theorem 1, proves that the CE policy is near-optimal in the absence of non-trivial gaps.
4.3. Achieving Conservativeness with respect to Gaps via a Simulation-based Policy

In Algorithm 1 if the re-solving threshold $p_{t}^{CE}$ at time $t = T − \tau + 1$ was within $\sqrt{2\log\tau/\tau}$ of a gap, we modified it as by instead using the quantile corresponding to the boundary of the gap as our acceptance threshold. An alternative to this method is a simulation-based approach, which we’ll outline next, followed by a full treatment in the next section.

Consider the bimodal uniform distribution, described by (1) with $\beta = 0$. Assume the CE threshold at time $t$, denoted as $p_{t}^{CE}$, is $1/2 - \epsilon$, where $\epsilon$ is sufficiently small ($\epsilon < \sqrt{2\log\tau/\tau}$, where $\tau = T − t + 1$). Under Algorithm 1 the CwG quantile threshold is set to $p_{t}^{CwG} = 1/2$. Consequently, only abilities with values of at least $3/4$ will be accepted at time $t$. This is illustrated in Figure 2a where the threshold shifts from $F^{-1}(p_{t}^{CE}) = 1/4 − 2\epsilon$ (in red) to $F^{-1}(p_{t}^{CwG}) = 1/4$ (in blue).

On the other hand, consider the following simulation-based approach: simulate multiple future demand scenarios. For the $i$-th simulated scenario, let $\theta_{[B_{t}]}^{(i)}$ denote the value of the $B_{t}$-th largest candidate ability on the simulated sample path $\hat{\theta}_{\geq t+1}^{(i)}$, where $B_{t}$ is the remaining budget. The candidate with ability $\theta_{t}$ is accepted if $\theta_{t} \geq K_{t}^{−1} \sum_{k=1}^{K_{t}} \theta_{[B_{t}]}^{(i)}$ where $K_{t}$ is the number of scenarios. Figure 2b illustrates this simulation-based approach using three simulated demand scenarios, with the $B_{t}$-th largest value in each of the demand scenarios (denoted as $\{\theta_{[B_{t}]}^{(i)}\}_{i=1}^{3}$) being depicted as the dashed green lines. The average of these values (depicted as a solid green line in Figure 2b) falls within the gap interval $(1/4, 3/4)$, resulting in only abilities of at least $3/4$ being accepted.
at time $t$. The simulation-based approach yields the same action as the carefully crafted CwG policy (Algorithm [1]). Interestingly, as we will later explore in Section 5, this simulation-based approach inherits the regret guarantee of the CwG policy (cf. Corollary EC.1), and outperforms the CwG policy in numerical experiments (cf. Figure 3b). It is worth noting that the $\{\theta_{[B_t]}^{(i)}\}_{i=1}^3$ values represent shadow prices for the single resource (the hiring budget) under the three different demand scenarios. In the simulation-based approach, the candidate ability $\theta_t$ is accepted if its reward $\theta_t$ exceeds the approximated average shadow price, obtained by averaging the shadow prices over multiple demand scenarios, i.e., $\sum_{i=1}^3 \theta_{[B_t]}^{(i)}/3$. Importantly, as we present next, this simulation-based idea is not limited to the multisecretary problem but can be applied more broadly to dynamic resource allocations, such as network revenue management and online matching, and notably inherits performance guarantees which hold for any algorithm satisfying certain conditions in these settings.

5. Unifying Algorithm: Repeatedly Act using Multiple Simulations

In this section, we will lift the idea of using simulations to drive decisions from the multisecretary setting to the broader class of NRM and online matching problems. We dub the resulting natural and versatile simulation-based algorithm Repeatedly Act using Multiple Simulations (RAMS). Prior to formally presenting RAMS, we will establish some notations. Let $V_h^{hs}(B_t; \theta_{\geq t})$ denote the hindsight optimal value for a given tail sequence of requests $\theta_{\geq t} \triangleq \{\theta_t, \ldots, \theta_T\}$ and remaining budget $B_t$,

$$V_h^{hs}(B_t; \theta_{\geq t}) \equiv \max_{a \in \mathcal{A}^{T-t+1}} \sum_{k=t}^T r(\theta_k, a_k) \text{ s.t. } \sum_{k=t}^T c(\theta_k, a_k) \leq B_t. \quad (3)$$

Furthermore, it is natural to define $V_{T+1}^{hs}(B_{T+1}, \emptyset) \equiv 0, \forall B_{T+1}$. We will assume access to a simulator $\mathcal{S}$ which takes as input a history $\mathcal{H}$ of request arrivals and random seed $U$ and produces a simulated demand scenario. Here a demand scenario is a tail sequence of requests $\theta_{\geq t+1}$; we remark that the order of requests in a tail sequence will not matter to RAMS, since it will perform a hindsight-based calculation. Note that the assumption of access to a simulator is a weaker and more practical assumption than knowledge of the distribution $F$. This permits RAMS to be
a data-driven algorithm where distributional knowledge $F$ is replaced by a high fidelity simulator based on historical data. Additionally, while most of our previous discussion was focused on a stationary setting with i.i.d requests, RAMS could be applicable in non-stationary settings where the request types may have some form of temporal correlations, corresponding to the reality of many applications. This is due to the fact that RAMS is completely agnostic to the underlying type distribution.

### 5.1. Algorithmic Description

The basic idea behind RAMS is as follows: given the remaining budget $B_t$ at time $t$, upon observing a request $\theta_t$, we simulate $K_t$ sample paths of the future denoted as $\{\tilde{\theta}^{(i)}_{\geq t+1}\}_{i=1}^{K_t}$. On each of these simulated sample paths $\tilde{\theta}^{(i)}_{\geq t+1}$, we compute the maximum achievable cumulative reward in hindsight under each possible action $a \in \mathcal{A}(B_t, \theta_t)$ at time $t$, denoted by $Q_{t}^{bs}(B_t; a; \tilde{\theta}^{(i)}_{\geq t})$ where $\tilde{\theta}^{(i)}_{\geq t} \equiv \{\theta_t\} \cup \tilde{\theta}^{(i)}_{\geq t+1}$. For each action $a \in \mathcal{A}(B_t, \theta_t)$ we average over the $K_t$ simulated sample paths, and choose the action which maximizes the average cumulative reward, i.e.,

$$\arg\max_{a \in \mathcal{A}(B_t, \theta_t)} K_t^{-1} \sum_{i=1}^{K_t} Q_{t}^{bs}(B_t; a; \tilde{\theta}^{(i)}_{\geq t+1}).$$

We formally describe RAMS in Algorithm 2.

For a feasible online policy $\pi$, given a state $B_t$ and an action $a$ which is feasible in that state $a \in \mathcal{A}(B_t, \theta_t) \subseteq \mathcal{A}$, define the following $Q$-function

$$Q_{t}^{*}(B_t; a; \theta_t) = r(\theta_t, a) + \mathbb{E} \left[ \sum_{k=t+1}^{T} r(\theta_k, a_k^\pi) \right], \quad Q_{t}^{*}(B_t; a; \theta_t) = \max_{\pi \in \Pi(B_t-c(\theta_t, a), T-t+1)} Q_{t}^{*}(B_t; a; \theta_t).$$

The action under the optimal online policy is $\arg\max_{a \in \mathcal{A}} Q_{t}^{*}(B_t; a; \theta_t)$, however computing this dynamic programming solution may be infeasible in general. Instead RAMS utilizes the “hindsight-based” approximation to the $Q$-function, estimated from simulated futures,

$$K_t^{-1} \sum_{i=1}^{K_t} Q_{t}^{bs}(B_t; a; \tilde{\theta}^{(i)}_{\geq t+1}) \approx \mathbb{E}_{\theta_{\geq t+1}} [Q_{t}^{bs}(B_t; a; \theta_{\geq t})]$$

as a proxy to make allocation decisions. Note that $\mathbb{E}_{\theta_{\geq t+1}} [Q_{t}^{bs}(B_t; a; \theta_{\geq t})] \geq Q_{t}^{*}(B_t; a; \theta_t)$ and from (4), we have that $Q_{t}^{bs}(B_t; a; \tilde{\theta}^{(i)}_{\geq t+1}) = V_{t+1}^{bs}(B_t - c(\theta_t, a); \tilde{\theta}^{(i)}_{\geq t+1}) + r(\theta_t, a)$. Next we define marginal compensation for a given action $a$ at time $t$ [Vera and Banerjee 2021]. Intuitively speaking, marginal compensation is the minimum payment one must make to an agent who knows the future to persuade that agent to take action $a$ at time $t$ on a realized sample path.
**Algorithm 2: Repeatedly Act using Multiple Simulations (RAMS)**

**Input:** Time Horizon $T$, Budget $B \in \mathbb{R}_+^d$, simulator $S$, Sequence of number of simulated sample paths $\{K_t\}_{t=1}^T$

**Initialize:** $B_1 = B$, $\mathcal{H} = \emptyset$

for $t = 1$ to $T$ do

Observe the request $\theta_t$

$\mathcal{H} \leftarrow \mathcal{H} \cup \{\theta_t\}$

Make $K_t$ conditionally independent calls to the simulator $S$ with history $\mathcal{H}$ and random seed $U \sim \text{Unif}([0, 1])$ (denote the $K_t$ simulated sample paths of requests as $\{\tilde{\theta}^{(i)}_{\geq t+1}\}_{i=1}^K$.)

for $i = 1$ to $K_t$ do

for $a \in A(B_t, \theta_t)$ do

$$Q_{hs}^t \left(B_t, a; \tilde{\theta}^{(i)}_{\geq t}\right) = r(\theta_t, a) + \left\{ \max_{(a_k)_{k>t}} \sum_{k>t} r(\tilde{\theta}^{(i)}_{k}, a_k) \text{ s.t. } \sum_{k>t} c(\tilde{\theta}^{(i)}_{k}, a_k) \leq B_t - c(\theta_t, a) \right\} \tag{4}$$

end

end

Take the action $a_t = \arg \max_{a \in A(B_t, \theta_t)} K_t^{-1} \sum_{i=1}^{K_t} Q_{hs}^t \left(B_t, a; \tilde{\theta}^{(i)}_{\geq t}\right)$

$B_{t+1} \leftarrow B_t - c(\theta_t, a_t)$

end

**Definition 2 (Marginal Compensation).** Given budget $B_t \geq 0$ and tail sequence of requests $\theta_{\geq t}$ for some $t \in [T]$, for any action $a \in A(B_t, \theta_t)$, we define

$$\partial \mathcal{R}_t(B_t, a; \theta_{\geq t}) \triangleq V_{hs}^t(B_t; \theta_{\geq t}) - \left[ V_{hs}^t \left( B_t - c(\theta_t, a); \theta_{\geq t+1} \right) + r(\theta_t, a) \right] \tag{5}$$

$$\partial \mathcal{R}_t(B_t, a) \triangleq \mathbb{E}_{\theta_{\geq t}} \left[ \partial \mathcal{R}_t(B_t, a; \theta_{\geq t}) | B_t \right]. \tag{6}$$

We refer to $\partial \mathcal{R}_t(B_t, a; \theta_{\geq t})$ as marginal compensation and $\partial \mathcal{R}_t(B_t, a)$ as the expected marginal compensation. A key fact from [Vera and Banerjee 2021, Lemma 1] is that the expected regret of a policy can be decomposed as the sum of the expected marginal compensations for the actions taken by the policy, as formalized below.
Lemma 1. For all $T \in [N]$ and budget $B \in [T]$, consider any online policy $\pi \in \Pi(B,T)$ and let $B_t^\pi$ denote the remaining budget at time $t$ under policy $\pi$. Then we have that

$$\text{Regret}(B,T;\pi) = \sum_{t=1}^{T} E_{B_t^\pi} \left[ \partial R_t(B_t^\pi, a_t^\pi) \right].$$  (7)

Lemma 2 (RAMS is equivalent to minimizing expected marginal compensation). Given a budget $B_t$, request $\theta_t$ and a collection of simulated sample paths $\{\tilde{\theta}_{t+1}^{(i)}\}_{i=1}^{K_t}$, RAMS takes an action $a_t \in A(B_t, \theta_t)$ at time $t$ which minimizes the simulation-based estimate of expected marginal compensation, i.e. $a_t = \arg \min_{a \in A(B_t, \theta_t)} K_t^{-1} \sum_{i=1}^{K_t} \partial R_t(B_t, a; \tilde{\theta}_{t+1}^{(i)})$ where $\tilde{\theta}_{t+1}^{(i)} = \{\theta_t\} \cup \tilde{\theta}_{t+1}^{(i)}$.

Lemma 2 follows immediately from (4) and (5) and provides an alternate description of RAMS.

5.2. Performance Analysis: Meta Theorem for RAMS

Since the expected regret of the policy is the sum of the expected marginal compensations (Lemma 1), and RAMS performs a simulation-based minimization of the expected marginal compensation (Lemma 2), it follows that RAMS provides the “best achievable” regret performance (in a certain sense). This reasoning is formalized in the following meta theorem.

Theorem 3 (Meta Performance of RAMS). Consider an online resource allocation problem with horizon $T$, number of resources $d$, initial budget $B \in \mathbb{R}^d$, a finite action set $A$ and request distribution $F$ as defined in Section 2. Assume the following

(i) There exists an algorithm $\text{ALG}$ for the online resource allocation problem such that the expected marginal compensation is uniformly bounded at each $1 \leq t \leq T$ as per $\sup_{B_t \geq 0} \partial R_t(B_t, a_t^{\text{ALG}}) \leq \Delta_t(\text{ALG})$ where $B_t$ is the remaining budget at time $t$ and $a_t^{\text{ALG}}$ is the action under $\text{ALG}$.

(ii) There exists a constant $C \equiv C(F) < \infty$ such that the marginal compensation in a time step is uniformly bounded by $C$, i.e., $\sup_{B_t, a, \theta \geq t} \partial R_t(B_t, a; \theta_{\geq t}) \leq C$ for all $t \geq 1$.

Let $K_t$ denote the number of simulated sample paths drawn at time $t$. Then for any $\eta > 2$, there exists a constant $C \equiv C(\eta, |A|, C(F)) < \infty$, such that

$$\text{Regret}(B,T;\text{RAMS}) \leq \sum_{t=1}^{T} \Delta_t(\text{ALG}) + C \sum_{t=1}^{T} K_t^{-\frac{1}{\eta}}.$$
Discussion of Theorem 3. Note that while the theorem has been stated for the i.i.d setting, Theorem 3 can also apply to non-stationary settings with some form of temporal correlations. Theorem 3 states that the regret of RAMS can be broken down into two components: $\Delta_t(\text{ALG})$ and $K_t^{-\frac{1}{\eta}}$. The former term $\Delta_t(\text{ALG})$ follows from the assumed uniform (over the states) upper bound on the expected compensation $\partial R_t(B_t, a_t^{\text{ALG}})$ under algorithm ALG, while the latter term $K_t^{-\frac{1}{\eta}}$ is due to the finite number of simulated sample paths. Theorem 3 states that RAMS inherits – up to sampling error – the best (uniform) regret guarantee which holds for any algorithm. Our numerical observations show that RAMS outperforms regret-optimal algorithms tailored for specific distributions or problem contexts, without the need for tuning (see Section 5.4). Notably, neither RAMS nor the meta theorem (Theorem 3) require prior knowledge of these optimized algorithms. As long as there exist algorithms that satisfy assumption (i) and that (ii) holds, RAMS achieves the same regret scaling.

We highlight that there exist algorithms developed in this and prior work for different problem settings which satisfy assumption (i) (cf. Corollaries EC.1-EC.3). Coming to assumption (ii), in the context of network revenue management problem, this assumption holds under mild conditions, as captured in the following claim.

Claim 1. In the context of the NRM problem, for any request type $\theta = (r_{\theta}, c_{\theta}) \in \Theta$, assume that the consumption vector $c_{\theta}$ is bounded i.e., $\nu \leq \|c_{\theta}\|_{\infty} \leq \bar{\nu}$ for $0 < \nu \leq \bar{\nu} < \infty$. Then we have that $\sup_{B_t, a; \theta \in \Theta} \partial R_t(B_t, a; \theta) \leq d r_{\max} \bar{\nu} / \nu \triangleq C(F)$ where $d$ is the number of resources and $r_{\max} \equiv \max_{\theta \in \Theta} r_{\theta} \leq 1$ (by assumption).

Note that the sufficient condition in Claim 1 permits many (or infinitely many) consumption types, in contrast to the typical assumption in the prior literature of a small number of consumption types (with some notable exceptions Lueker 1998, Arlotto and Xie 2020, Li and Ye 2022, Bray 2022).

Combining Theorem 3 with analyses of specific algorithms, we can show that RAMS achieves the same regret scaling as that of the CwG algorithm (Algorithm 1) for the class of $(\beta, \varepsilon_0, \delta)$–clustered distributions (Corollary EC.1). Zooming out from the multisecretary problem, we consider the
more general network revenue management and online matching problems. We show that under the
assumption of a small number of discrete types, RAMS achieves bounded regret scaling for both the
network revenue management (Corollary \textbf{EC.2(a)}) and online matching (Corollary \textbf{EC.3}). Under
infinitely many types and some structural assumptions, RAMS achieves logarithmic (Corollary
\textbf{EC.2(b)}) and log-squared regret (Corollary \textbf{EC.2(c)}) scaling for the general NRM problem in line
with state of the art algorithms presented in Bray (2022) and Jiang et al. (2022a) respectively.
Detailed assumptions and corollaries are presented in Appendix \textbf{EC.5} due to space constraints.

5.3. Connection of RAMS to prior work

Due to the equivalence of RAMS to minimizing the expected compensation at each time period (cf.
Lemma \textbf{2}), RAMS follows the “Bayes Selector” principle developed in Vera and Banerjee (2021).
However, the focus of Vera and Banerjee (2021) is on settings with a few types and hence their
algorithm has been tailored for such settings, whereas RAMS is a very general algorithm which
does not require any knowledge of the underlying assumptions on the type space.

In the context of network revenue management, RAMS is a refined version of the dual
averaging policy proposed in Talluri and Van Ryzin (1999), where dual prices are computed
for multiple demand scenarios and the allocation decisions are made by averaging these dual
prices over the different scenarios. Under RAMS, given a remaining budget $B_t$, a request $\theta_t$
is accepted if

$$K_t^{-1} \sum_{i=1}^{K_t} Q_i(B_t, \text{accept}; \tilde{\theta}_i^{(t)}) \geq K_t^{-1} \sum_{i=1}^{K_t} Q_i(B_t, \text{reject}; \tilde{\theta}_i^{(t)}).$$

Assume that there exists a dual vector $\mu(B_t; \tilde{\theta}_{2^t+1})$ for (3) with tail sequence $\tilde{\theta}_{2^t+1}$ such that first order
approximation of $V_{t+1}(B_t; \tilde{\theta}_{2^t+1})$ is good, i.e., $V_{t+1}(B_t; \tilde{\theta}_{2^t+1}) - V_{t+1}(B_t - c(\theta_t, \text{accept}); \tilde{\theta}_{2^t+1}) \approx
\mu(B_t; \tilde{\theta}_{2^t+1})^\top c(\theta_t, \text{accept})$. Then, using (3), (4) and the fact that $r(\theta_t, \text{reject}) = 0$ and $c(\theta_t, \text{reject}) = 0$, under RAMS, the request $\theta_t$ is accepted if

$$r(\theta_t, \text{accept}) \geq \frac{1}{K_t} \sum_{i=1}^{K_t} \left(V_{t+1}(B_t; \tilde{\theta}^{(i)}_{2^t+1}) - V_{t+1}(B_t - c(\theta_t, \text{accept}); \tilde{\theta}^{(i)}_{2^t+1})\right)$$

$$\approx \frac{1}{K_t} \sum_{i=1}^{K_t} \mu(B_t; \tilde{\theta}^{(i)}_{2^t+1})^\top c(\theta_t, \text{accept}) = \left(\frac{1}{K_t} \sum_{i=1}^{K_t} \mu(B_t; \tilde{\theta}^{(i)}_{2^t+1})\right)^\top c(\theta_t, \text{accept}).$$

average dual price for bid price control.
Therefore, assuming that the first order approximation is good, RAMS will accept the request \( \theta_t \) if the reward exceeds the sum of the average dual prices for the resources it consumes, and this resembles the bid price control policy [Talluri and Van Ryzin 1998]. Thus we see that dual averaging is, in fact, an approximate version of RAMS for settings in which individual actions have a “small” impact, and our theoretical backing for RAMS (Theorem 3) provides new justification for why dual averaging should work well in such settings. Dual averaging is very practical and requires only a small adaptation of dual-based dynamic resource allocation systems based on model predictive control, which are typical in the industry, e.g., in supply chain optimization. Specifically, it only requires the construction of multiple demand scenarios. The hindsight problem for each scenario can be solved in parallel (using the existing MPC solver as is) and then a simple dual averaging layer can be inserted before the decision making layer.

RAMS can be viewed as the manifestation in our setting of the so-called Multi Forecast–Model Predictive Control (MF-MPC) policy which appears in the control literature, e.g., see Shen and Boyd (2021) and citations therein. In MF-MPC, one constructs multiple plausible forecasts of the future, termed scenarios, and constructs a different plan for each of the possible scenarios, while imposing the constraint that the plans must agree on the present action to be chosen. This process is repeated each time an action is to be chosen. The connection with MF-MPC further reveals an illuminating interpretation for RAMS: Suppose all uncertainty about the future will be resolved right after the current action is chosen. What current action is optimal in this proxy problem? This is the action chosen by RAMS at each time; after all, by definition, RAMS solves the Bellman equation for this proxy problem. This interpretation throws light on the approximation underlying RAMS, and may help us –in future work– to understand how well RAMS (or, more generally, any compensation-based approach) can approximate the optimal MDP solution in a given setting.

5.4. Numerical Simulations

We perform numerical experiments under different assumptions and for different problem classes. For the multisecretary problem, we study the performance of the CwG algorithm for different
distributions (Figure 3a), compare the performance of CE, CwG and RAMS for the bimodal uniform distribution $F_0$ (Figure 3b), and study the impact of $\beta > 0$ (Figure 3c). In addition, we consider the general network revenue management problem with a few types and two resources and compare the performance of previous algorithms with that of RAMS (Figure 3d). In each of the settings that we consider, we vary the time horizon $T$, and consider a budget of $B = T/2 \times 1_{d \times 1}$ where $d$ is the number of resources. We note that this starting budget leads to the worst-case regret scaling for the instances with gaps which we consider. Overall, our simulation results confirm our theoretical predictions, including the importance of the conservativeness with respect to gaps principle, and demonstrate superior numerical performance of the RAMS algorithm.

**Figure 3a.** We numerically study the regret scaling of the CwG policy as a function of the time horizon $T$ for different distributions. The distributions we consider are: (i) bimodal uniform distribution $F_0 = \text{Uniform}([0,1/4] \cup [3/4,1])$, (ii) the uniform distribution over $[0,1]$ and (iii) a discrete distribution over a few types $\{0.25,0.5,0.75\}$ and the probability mass being $1/3$ for each of the points. We numerically evaluate the average regret for different number of candidates $T$ (with the budget varying as $B = T/2$) and fit a curve (as shown in the dashed lines) to observe the regret scaling. For each of the three distributions considered, we empirically observe that the regret scaling is consistent with our theoretical guarantees as implied by Corollary 1 (log squared regret) for the bimodal distribution, Corollary 3 (logarithmic regret) for the uniform distribution and Corollary 2 (bounded regret) for the discrete distribution with few types.

**Figure 3b.** We numerically study the average regret scaling of the CE, CwG and RAMS policy for the bimodal uniform distribution $F_0 = \text{Unif}([0,1/4] \cup [3/4,1])$ with gap in the interval $[1/4,3/4]$. We fit a curve (as shown in dashed lines) to observe the regret scaling. For each of the three policies considered, we empirically observe that the regret scaling is consistent with our theoretical guarantees as implied by Proposition 1 for the CE policy, Corollary 1 for the CwG policy and Corollary EC.1 for the RAMS policy. While both CwG and RAMS have the same regret scaling, we observe that RAMS has superior numerical performance over CwG since RAMS is designed to
minimize the compensation and hence the regret, whereas CwG is designed to optimize only the scaling of the compensation (and hence the regret scaling).

**Figure 3c** To assess the influence of the parameter \( \beta \), we examine the performance of the CE (equivalently CwG) algorithm on the gapless version of the \( F_\beta \) distribution, as described in [1] for \( \beta \in \{0, 1, 2, 3\} \). From Theorem [1] and Corollary [3] we know that CE has the optimal regret scaling.
We fit a curve (shown in dashed lines) to the empirical average regret for different values of time horizon $T$ and observe that the regret for $\beta \in \{1, 2, 3\}$ scales polynomially in the time horizon with the exponent given by $\frac{1}{2} - \frac{1}{2(1+\beta)}$ and this is consistent with our guarantees in Corollary 3.

**Figure 3d.** We consider an NRM problem with two resources and six types. The types $\theta = (r_\theta, c_\theta)$ are given as $\xi_1 = (1.0, [1, 0]), \xi_2 = (0.6, [1, 0]), \xi_3 = (1, [0, 1]), \xi_4 = (0.5, [0, 1]), \xi_5 = (0.9, [1, 1]), \xi_6 = (0.8, [1, 1])$. The requests arrive i.i.d with $P(\theta_t = \xi_j) = 0.2, \forall j \in \{1, 2, 3, 4\}$ and $P(\xi_t = \xi_j) = 0.1, \forall j \in \{5, 6\}$. We compare the performance of RAMS against two near optimal algorithms - Infrequent Resolving with Thresholding (IRT) [Bumpensanti and Wang 2020] and Bayes Selector (BS) [Vera and Banerjee 2021]. We observe that for all the three algorithms that we consider, the regret increases initially but converges to a constant for sufficiently large $T$. We observe that amongst all the three algorithms considered, RAMS either matches or improves upon the algorithms.

### 6. Conclusion

In this work, we considered dynamic resource allocation problems and investigated the impact of distributional assumptions on algorithmic performance. By focusing on the multisecretary problem, we gained valuable insights into the fundamental drivers and limits of algorithmic regret performance. We identified a novel driver of regret, characterized by the parameter $\beta$, which measures the concentration of types around gaps. We introduced the Conservativeness with respect to Gaps (CwG) principle, and used it to develop an innovative algorithmic approach that mitigates the limitations of the widely used certainty-equivalent (CE) policy. The CwG principle, along with its associated CwG algorithm, achieves near-optimal regret scaling of $\tilde{O}(T^{\frac{1}{2} - \frac{1}{2(1+\beta)}})$ for a broad class of distributions with gaps parameterized by $\beta$. Furthermore, we analyzed the natural Repeatedly Act using Multiple Simulations (RAMS) algorithm, which offers a general-purpose solution for online resource allocation problems (not just the multisecretary problem), which is applicable to any distribution of requests. RAMS is practical and data-driven, relying on simulated future demand scenarios to drive decision making. Heuristically speaking, RAMS is equivalent to a bid price control policy where the bid prices are computed by averaging the shadow prices of the hindsight optimal
problem for multiple scenarios. This requires a minor adaptation of existing dual-based systems which is an industry default.

Recently, there has been a growing interest in studying online resource allocation problems in the presence of horizon uncertainty [Besbes and Sauré 2014, Balseiro et al. 2022, Bai et al. 2023, Aouad and Ma 2022]. Specifically, Bai et al. (2023) demonstrate that by leveraging an alternative fluid benchmark, it is possible to achieve a sublinear regret scaling of $O(\sqrt{T})$, through the use of a static policy. Nevertheless, a naïve implementation of the RAMS approach yields regret (relative to the alternative fluid benchmark of Bai et al. (2023)) that scales linearly. Whether RAMS can be adapted to attain sublinear regret remains unknown. We leave the exploration of this intriguing question, as well as other related queries surrounding the development of near-optimal algorithms under horizon uncertainty, for future endeavors.

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Appendix

The Appendix is organized as follows. Appendix EC.1 provides the proof of the universal lower bound in Theorem 1. Appendix EC.2 provides the proof of Theorem 2. Appendix EC.3 and Appendix EC.4 provides the proof of Corollaries 2 and 3 respectively. Appendix EC.5 discusses how RAMS is able to recover both the new and prior regret guarantees in the context of the multisecretary problem and the more general network revenue management and online matching problems. Appendix EC.6 provides the proof of Theorem 3 and the corollaries EC.1, EC.2 and EC.3. Appendix EC.7 provides a discussion of the connections between the order fulfillment problem and the multisecretary problem. Appendix EC.8 provides some details on the different possible clusterings for \((\beta, \varepsilon_0, \delta)\)-clustered distributions.

EC.1. Proof of Theorem 1

First we will consider the case of \(\beta = 0\). For the uniform distribution over \([0, 1]\), we have that \(\beta = 0\) and from Proposition 4 of Bray (2022), Theorem 1 follows for \(\beta = 0\). Therefore our focus will on the case of \(\beta > 0\). Fix \(\beta > 0\) and fix a number \(g \geq 0\). In the context of Example 3, we have that \(g = \frac{1}{2}\). Consider a distribution supported on the set \(S \triangleq [0, \ell] \cup [u, 1]\) where \(\ell \triangleq \frac{1}{2} - \frac{g}{2}\) and \(u \triangleq \frac{1}{2} + \frac{g}{2}\). For \(g = 1/2\), we have that \(\ell = \frac{1}{4}\) and \(u = \frac{3}{4}\). For a fixed \(\beta > 0\) and \(g, \ell, u\) as defined above, consider the following candidate ability distribution \(F_{\beta, \ell, u}\),

\[
F_{\beta, \ell, u}(x) = \begin{cases} 
-\frac{(\ell-x)^{1+\beta}}{2^{1+\beta}} + \frac{1}{2}, & 0 \leq x \leq \ell \\
\frac{1}{2}, & \ell \leq x \leq u \\
\frac{(x-u)^{1+\beta}}{2^{1+\beta}} + \frac{1}{2}, & u \leq x \leq 1 
\end{cases} \tag{EC.1}
\]

For \(g > 0\), we can easily verify that \(F_{\beta, \ell, u}\) is a \((\beta, \varepsilon_0 = \frac{1}{2})\)-clustered distribution and for \(g = 0\), \(F_{\beta, \ell, u}\) is a \((\beta, \varepsilon_0 = 1)\)-clustered distribution. Next, we will fix the time horizon \(T > 0\) and set the budget \(B \triangleq \lfloor \frac{1}{2} T \rfloor\). Define \(\Delta_\beta \triangleq T^{-\frac{1}{2(1+\beta)}} (1-u) = T^{-\frac{1}{2(1+\beta)}} \ell\). Define \(c_0 \triangleq \left(\frac{19}{128}\right)^{1+\beta} > 1\). Define the following quantities:

\[
\alpha_0 \triangleq c_0 - 1 > 0, \quad \tilde{\Delta}_\beta \triangleq \alpha_0 \Delta_\beta, \quad \ell_1 \triangleq \ell - \Delta_\beta, \quad \ell_2 \triangleq \ell - c_0 \Delta_\beta, \quad u_1 \triangleq u + \Delta_\beta, \quad u_2 \triangleq u + c_0 \Delta_\beta. \tag{EC.2}
\]
Now we will partition the set $S \triangleq [0, \ell] \cup [u, 1]$ into the following sets (refer to Figure EC.1):

$$\mathcal{I}_L = [0, \ell_2), \mathcal{I}_{M_1} = [\ell_2, \ell_1), \mathcal{I}_{M_2} = [\ell_1, \ell], \mathcal{I}_{M_3} = [u, u_1), \mathcal{I}_{M_4} = [u_1, u_2), \mathcal{I}_H = [u_2, 1]$$

Further define the sets $\mathcal{I}_{M_e} \triangleq \mathcal{I}_{M_2} \cup \mathcal{I}_{M_3}$ and $\mathcal{I}_{M_p} \triangleq \mathcal{I}_{M_1} \cup \mathcal{I}_{M_4}$.

![Figure EC.1](image-url)

Partition of the set $S = [0, \ell] \cup [u, 1]$ into disjoint set $\mathcal{I}_L = [0, \ell_2), \mathcal{I}_{M_1} = [\ell_2, \ell_1), \mathcal{I}_{M_2} = [\ell_1, \ell], \mathcal{I}_{M_3} = [u, u_1), \mathcal{I}_{M_4} = [u_1, u_2), \mathcal{I}_H = [u_2, 1]$, where $\ell_1 \triangleq \ell - \Delta_\beta, \ell_2 \triangleq \ell - c_0 \Delta_\beta, u_1 \triangleq u + \Delta_\beta, u_2 \triangleq u + c_0 \Delta_\beta$ and $\Delta_\beta \triangleq (c_0 - 1) \Delta_\beta$.

Let $\theta_{\geq 1}$ denote the sequence of candidate abilities and define $N(\mathcal{A}, t_1, t_2)$ denote the number of candidate abilities in the set $\mathcal{A}$ that arrive in the time interval $[t_1, t_2]$. Formally, the random variable $N(\mathcal{A}, t_1, t_2)$ is defined as

$$N(\mathcal{A}, t_1, t_2) \triangleq \sum_{k=t_1}^{t_2} 1 \{ \theta_k \in \mathcal{A} \}, \quad \forall \mathcal{A} \subseteq S, t_1, t_2 \in \{1, 2, \ldots, T\} \quad \text{(EC.3)}$$

Let $\mu_{t_1}^{t_2}(A) \triangleq \mathbb{E}[N(\mathcal{I}_H, t_1, t_2)]$ denote the mean of the random variable $N(\mathcal{I}_H, t_1, t_2)$. Next we define the following set of events:

$$\mathcal{H}_1 \triangleq \left\{ \frac{T}{4} - \frac{\sqrt{T}}{2} \leq N(\mathcal{I}_H, 1, B) \leq \frac{T}{4} \right\} \quad \text{(EC.4)}$$

$$\mathcal{H}_2 \triangleq \left\{ \frac{T}{4} - 4\sqrt{T} \leq N(\mathcal{I}_H, B + 1, T) \leq \frac{T}{4} - 3\sqrt{T} \right\} \quad \text{(EC.5)}$$

$$\mathcal{H}_2^\prime \triangleq \left\{ \frac{T}{4} + \frac{\sqrt{T}}{2} \leq N(\mathcal{I}_H, B + 1, T) \leq \frac{T}{4} + \frac{3\sqrt{T}}{2} \right\} \quad \text{(EC.6)}$$

$$\mathcal{C}_1 \triangleq \left\{ \frac{\sqrt{T}}{4} \leq N(\mathcal{I}_{M_e}, 1, B) \leq \sqrt{T} \right\} \quad \text{(EC.7)}$$

$$\mathcal{C}_2 \triangleq \left\{ \frac{\sqrt{T}}{4} \leq N(\mathcal{I}_{M_e}, B + 1, T) \leq \sqrt{T} \right\} \quad \text{(EC.8)}$$

$$\mathcal{P}_1 \triangleq \left\{ \frac{\sqrt{T}}{256} \leq N(\mathcal{I}_{M_p}, 1, B) \leq \frac{\sqrt{T}}{64} \right\} \quad \text{(EC.9)}$$

$$\mathcal{P}_2 \triangleq \left\{ \frac{\sqrt{T}}{256} \leq N(\mathcal{I}_{M_p}, B + 1, T) \leq \frac{\sqrt{T}}{64} \right\} \quad \text{(EC.10)}$$

Further we define the events $\mathcal{H} \triangleq \mathcal{H}_1 \cap \mathcal{H}_2, \mathcal{H}^\prime \triangleq \mathcal{H}_1 \cap \mathcal{H}_2, \mathcal{C} \triangleq \mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{P} \triangleq \mathcal{P}_1 \cap \mathcal{P}_2$. 
Discussion of the Hindsight Optimal. Conditional on the event $\tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P}$, we have that the total number of arrivals in the set $\mathcal{I}_{H}$ is more than the budget $B$, i.e., $N(\mathcal{I}_{H}, 1, T) \geq \frac{1}{2}T \geq B$ and hence the hindsight optimal must reject all the arrivals in the set $\mathcal{I}_{L} \cup \mathcal{I}_{M_{p}} \cup \mathcal{I}_{M_{c}}$ and possibly some arrivals in the set $\mathcal{I}_{H}$. However, conditional on the event $\mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$, we have that total number of arrivals in the set $\mathcal{I}_{H} \cup \mathcal{I}_{M_{c}} \cup \mathcal{I}_{M_{p}}$ is less than the budget $B$, i.e., $N(\mathcal{I}_{H}, 1, T) + N(\mathcal{I}_{M_{c}}, 1, T) + N(\mathcal{I}_{M_{p}}, 1, T) \leq \frac{1}{2}T - \frac{31}{32}\sqrt{T} = B$ for sufficiently large $T$ and hence the hindsight optimal must accept all the arrivals in the set $\mathcal{I}_{H} \cup \mathcal{I}_{M_{p}} \cup \mathcal{I}_{M_{c}}$ and possibly some arrivals in the set $\mathcal{I}_{L}$.

Let $N_{\text{DP}}^{DP}(A, t_{1}, t_{2})$ denote the number of accepted candidates by the DP (optimal dynamic programming policy) with ability in the set $A$ and they arrive in the time interval $[t_{1}, t_{2}]$ which we formally define as:

$$N_{\text{DP}}^{DP}(A, t_{1}, t_{2}) \triangleq \sum_{k=t_{1}}^{t_{2}} \mathbb{1}\{\theta_{k} \in A, \pi_{DP}^{t_{1}}(k) = \text{accept}\}, \quad \forall A \subseteq \mathcal{S}, t_{1}, t_{2} \in \{1, 2, \ldots, T\} \quad \text{(EC.11)}$$

Define the event $\mathcal{E}$ which says that under the optimal online policy, the number of accepted candidates up till time $B (= \lfloor \frac{1}{2}T \rfloor)$ is at least one eighth of the number of arrivals in set $\mathcal{I}_{M_{c}}$ up till time $B$, i.e.,

$$\mathcal{E} \triangleq \left\{ N_{\text{DP}}^{DP}(\mathcal{I}_{M_{c}}, 1, B) \geq \frac{N(\mathcal{I}_{M_{c}}, 1, B)}{8} \right\} \quad \text{(EC.12)}$$

Proof Strategy. Our proof will proceed by considering the following events: (a) $\mathcal{E} \cap \tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P}$ and (b) $\mathcal{E}^{c} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$. In case (a), from the discussion about the hindsight optimal policy, the hindsight optimal policy will reject all the arrivals in the set $\mathcal{I}_{M_{c}}$ but DP will accept at least $\frac{1}{32}\sqrt{T}$ arrivals in set $\mathcal{I}_{M_{c}}$ in the time interval $[1, B]$. This will result in the DP incorrectly rejecting at least $\frac{1}{32}\sqrt{T}$ arrivals in interval $\mathcal{I}_{H}$ and the cost of each of these mistakes is at least $\alpha_{0}\Delta_{F}$. In case (b), from the discussion about the hindsight optimal policy, the hindsight optimal policy will accept all the arrivals in the interval $\mathcal{I}_{M_{c}}$ but the DP accepts at most $\frac{1}{8}\sqrt{T}$ arrivals in the interval $\mathcal{I}_{M_{c}}$. This implies that at least $\frac{1}{8}\sqrt{T}$ arrivals in the interval $\mathcal{I}_{M_{c}}$ are incorrectly rejected. This will result in the DP incorrectly accepting at least $\frac{1}{8}\sqrt{T}$ arrivals in the interval $\mathcal{I}_{L}$ and the cost of each of these mistakes is again at least $\alpha_{0}\Delta_{F}$. Informally speaking, we can lower bound the expected regret as

$$\text{Regret}(B, T; \text{DP}) \geq c \left( \mathbb{P} \left( \mathcal{E} \cap \tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P} \right) + \mathbb{P} \left( \mathcal{E}^{c} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \right) \right) \left( \# \text{ of mistakes} \times \text{cost/mistake} \right)$$
Assuming we can show that $\Pr(\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}) + \Pr(\mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}) \geq \gamma > 0$ for some $\gamma \in (0,1)$, we have that the number of mistakes is $\Omega(\sqrt{T})$ and the cost of each mistake is $\Omega(T^{-\frac{1}{1+\beta}})$. Combining all this will provide the lower bound guarantee as desired for $\beta > 0$.

Consider the random variable $\Lambda(B,T;\text{DP})$

$$\Lambda(B,T;\text{DP}) = \sum_{t=1}^{T} \theta_i a_{t}^{n} - \sum_{t=1}^{T} \theta_i a_{t}^{\text{DP}}$$

(EC.13)

Next we will formalize our proof strategy using the following two lemmas.

**Lemma EC.1.** Consider the event $\mathcal{E} \cap \tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P}$, then we have that

$$\mathbb{E} \left[ \Lambda(B,T;\text{DP}) | \mathcal{E} \cap \tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P} \right] \geq \frac{\alpha_0 T}{32} T^{\frac{1}{2}} - \frac{\gamma}{1+\beta},$$

where $\alpha_0 = (\frac{128}{129})^{\frac{1}{1+\beta}} - 1$ defined in (EC.2) and $\ell = \frac{1}{2} - \frac{\gamma}{2}$.

**Lemma EC.2.** Consider the event $\mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$, then we have that

$$\mathbb{E} \left[ \Lambda(B,T;\text{DP}) | \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \right] \geq \frac{\alpha_0}{8} T^{\frac{1}{2}} - \frac{\gamma}{1+\beta},$$

where $\alpha_0 = (\frac{128}{129})^{\frac{1}{1+\beta}} - 1$ defined in (EC.2).

We defer the proofs of Lemmas EC.1 and EC.2 to Sections EC.1.1 and EC.1.2 respectively. Finally, we have that

$$\text{Regret}(B,T;\text{DP}) \overset{(a)}{=} \mathbb{E} \left[ \Lambda(B,T;\text{DP}) \right],$$

$$\geq \mathbb{E} \left[ \Lambda(B,T;\text{DP}) | \mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \right] \Pr(\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P})$$

$$+ \mathbb{E} \left[ \Lambda(B,T;\text{DP}) | \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \right] \Pr(\mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}),$$

$$\geq \frac{\alpha_0}{32} T^{\frac{1}{2}} - \frac{\gamma}{1+\beta}\left( \Pr(\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}) + \Pr(\mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}) \right),$$

(EC.14)

where (a) follows from the definition of (expected) regret, (b) follows from total law of expectations, (c) follows from Lemmas EC.1 and EC.2.

Observe that $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2$, $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$, $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$ and moreover $\mathcal{C}_1 \perp \mathcal{C}_2$, $\mathcal{P}_1 \perp \mathcal{P}_2$ and $\mathcal{H}_1 \perp \mathcal{H}_2$, $\tilde{\mathcal{H}}_2$ since the events $\mathcal{H}_1, \mathcal{C}_1, \mathcal{P}_1$ only depend on the arrivals in the time interval...
\([1, B]\) i.e. \(\{\theta_k\}_{k=1}^B\) whereas the events \(\mathcal{H}_2, \tilde{\mathcal{H}}_2, \mathcal{C}_2\) and \(\mathcal{P}_2\) only depend on the arrivals in the time interval \([B + 1, T]\) i.e. \(\{\theta_k\}_{k=B+1}^T\) and the arrivals by assumption are \(i.i.d.\) Additionally, the events \(\mathcal{E}, \mathcal{E}^c\) also only depend on the arrivals in the interval \([1, B]\) and hence are independent of \(\mathcal{H}_2, \tilde{\mathcal{H}}_2, \mathcal{C}_2\) and \(\mathcal{P}_2\). Therefore, we have that

\[
P(\mathcal{E} \cap \tilde{\mathcal{H}} \cap \mathcal{C} \cap \mathcal{P}) + P(\mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P})
\]

\[= P(\mathcal{E} \cap \mathcal{H}_1 \cap \tilde{\mathcal{H}}_2 \cap C_1 \cap C_2 \cap \mathcal{P}_1 \cap \mathcal{P}_2) + P(\mathcal{E}^c \cap \mathcal{H}_1 \cap \tilde{\mathcal{H}}_2 \cap C_1 \cap C_2 \cap \mathcal{P}_1 \cap \mathcal{P}_2)\]

\[\geq \min \left\{ P(\tilde{\mathcal{H}}_2 \cap C_2 \cap \mathcal{P}_2), P(\mathcal{H}_2 \cap C_2 \cap \mathcal{P}_2) \right\} \cdot (P(\mathcal{E} \cap \mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1) + P(\mathcal{E}^c \cap \mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1)),\]

\[= \min \left\{ P(\tilde{\mathcal{H}}_2 \cap C_2 \cap \mathcal{P}_2), P(\mathcal{H}_2 \cap C_2 \cap \mathcal{P}_2) \right\} P(\mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1), \quad \text{(EC.15)}\]

where (a) follows from the definition of \(\mathcal{H}, \tilde{\mathcal{H}}, \mathcal{C}\) and \(\mathcal{P}\), (b) follows from the fact that \(\mathcal{E} \cap \mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1 \perp \tilde{\mathcal{H}}_2 \cap C_2 \cap \mathcal{P}_2\) and \(\mathcal{E}^c \cap \mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1 \perp \mathcal{H}_2 \cap \mathcal{C}_2 \cap \mathcal{P}_2\) using the arguments presented previously, (c) follows trivially, (d) follows from the law of total probability.

Now it suffices to show to that there exists a constant \(\alpha > 0\) independent of \(T\) such that for all \(T\) sufficiently large, we have that \(P(\tilde{\mathcal{H}}_2 \cap C_2 \cap \mathcal{P}_2), P(\mathcal{H}_2 \cap C_2 \cap \mathcal{P}_2), P(\mathcal{H}_1 \cap C_1 \cap \mathcal{P}_1) \geq \alpha\). Using a CLT argument, one can easily see that \(P(\mathcal{H}_1), P(\tilde{\mathcal{H}}_2), P(\mathcal{H}_2) \geq \alpha' > 0\) and \(P(C_1), P(C_2), P(P_1), P(P_2) \geq \alpha'\) for \(\alpha'' \neq \alpha'\). However the events \(\mathcal{H}_1, C_1, P_1\) (similarly \(\mathcal{H}_2, C_2, P_2\) and \(\tilde{\mathcal{H}}_2, C_2, P_2\)) are correlated and hence proving \(P(\tilde{\mathcal{H}}_2 \cap C_2 \cap P_2), P(\mathcal{H}_2 \cap C_2 \cap P_2), P(\mathcal{H}_1 \cap C_1 \cap P_1) \geq \alpha\) requires a conditioning argument which we will illustrate now. We will argue this the event \(\mathcal{H}_1 \cap C_1 \cap P_1\) and the exact same argument works for the events \(\tilde{\mathcal{H}}_2 \cap C_2 \cap P_2\) and \(\mathcal{H}_2 \cap C_2 \cap P_2\). We have that

\[
P(\mathcal{H}_1 \cap C_1 \cap P_1) \overset{(a)}{=} P(C_1 \cap P_1) P(\mathcal{H}_1 | C_1 \cap P_1),
\]

\[\overset{(b)}{=} P(\mathcal{H}_1) - P(\mathcal{H}_1 | (C_1 \cap P_1)^c) P((C_1 \cap P_1)^c),
\]

\[\overset{(c)}{=} P(\mathcal{H}_1) - (P(C_1^c) + P(P_1^c)),
\]

where (a) follows from the definition of conditional probability, (b) follows from the law of total probability i.e. \(P(\mathcal{H}_1) = P(\mathcal{H}_1 | C_1 \cap P_1) P(C_1 \cap P_1) + P(\mathcal{H}_1 | (C_1 \cap P_1)^c) P((C_1 \cap P_1)^c)\) and (c) follows.
from the fact that \( P(H_1 | (C_1 \cap P_1)^c) \leq P((C_1 \cap P_1)^c) \leq P(C_1^c) + P(P_1^c) \) where the first inequality follows from the fact that \( P(H_1 | (C_1 \cap P_1)^c) \leq 1 \) and the second inequality follows from the union bound. Using the exact same arguments we have that

\[
P(H_2 \cap C_2 \cap P_2) = P(H_2 | (C_1 \cap P_1)^c) \leq P(C_1^c) + P(P_1^c),
\]

Next we present a few lemmas which would imply that

\[
P(H_1 \cap C_1 \cap P_1) \geq 0.001 \quad \text{and} \quad P(H_2 \cap C_2 \cap P_2) \geq 0.001.
\]

**Lemma EC.3.** There exists \( T_0 < \infty \) such that for all \( T \geq T_0 \), we have that \( P(H_1), P(H_2), P(H_2) \geq 0.003 \)

**Lemma EC.4.** There exists \( T_0 < \infty \) such that for all \( T \geq T_0 \), we have that \( P(C_1^c), P(C_2^c), P(P_1^c), P(P_2^c) \leq 0.001 \). \( \square \)

We defer the proofs of Lemma EC.3 and EC.4 to Appendix EC.1.3 and EC.1.4 respectively. Using Lemmas EC.3 and EC.4 and (EC.15), we have that \( P(E \cap \tilde{H} \cap C \cap P) + P(E^c \cap \tilde{H} \cap C \cap P) \geq 10^{-6} \), combined with (EC.14) concludes the proof.

**EC.1.1. Proof of Lemma EC.1**

Recall the definition of the random variable \( \Lambda(B, T; \text{DP}) = \sum_{k=1}^T \theta_k \pi_{hs}^k - \sum_{k=1}^T \theta_k \). For a sequence of candidate ability arrivals \( \theta \geq 1 \), we can define the following random set of indices \( J_{hs} \) and \( J_{DP} \) as

\[
J_{hs}(A) \triangleq \{ k : \theta_k \in A \text{ and } \pi_{hs}^k = 1 \}, \quad J_{DP}(A) \triangleq \{ k : \theta_k \in A \text{ and } \pi_{DP}^k = 1 \}, \quad \forall A \subseteq S \quad (\text{EC.16})
\]

Notice that we can equivalently write the sum of values chosen under the hindsight optimal and the DP policy as

\[
\begin{align*}
\sum_{i=1}^T \theta_i \pi_{hs}^i &= \sum_{k \in J_{hs}(I_L)} \theta_k + \sum_{k \in J_{hs}(I_{M_L})} \theta_k + \sum_{k \in J_{hs}(I_{M_C})} \theta_k + \sum_{k \in J_{hs}(I_H)} \theta_k \\
\sum_{i=1}^T \theta_i \pi_{DP}^i &= \sum_{k \in J_{DP}(I_L)} \theta_k + \sum_{k \in J_{DP}(I_{M_L})} \theta_k + \sum_{k \in J_{DP}(I_{M_C})} \theta_k + \sum_{k \in J_{DP}(I_H)} \theta_k
\end{align*}
\]

(EC.17) \quad (EC.18)
Now conditional on the event $\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$, we have that $\mathcal{J}^{hs}(I_L) = \mathcal{J}^{hs}(I_{M_p}) = \mathcal{J}^{hs}(I_{M_c}) = \emptyset$ and $|\mathcal{J}^{hs}(I_H)| = B$ and we have that $|\mathcal{J}^{dp}(I_{M_c})| \geq \frac{1}{32} \sqrt{T}$. This follows from the fact under the event $\mathcal{E}$, the DP accepts at least $\frac{1}{8} N(I_{M_c}, 1, B)$ and from the event $\mathcal{C}_1$, it follows that $N(I_{M_c}, 1, B) \geq \frac{1}{4} \sqrt{T}$.

Using this we have that

$$\sum_{i=1}^{T} \theta_i \pi^{hs}_i = \sum_{k \in \mathcal{J}^{hs}(I_H) \setminus \mathcal{J}^{dp}(I_H)} \theta_k - \left( \sum_{k \in \mathcal{J}^{dp}(I_L)} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_p})} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_c})} \theta_k \right),$$  \hspace{1cm} (EC.19)

Since any online policy can select at most $B$ candidates and the offline policy will select the top $B$ candidates, we have that conditional on the event $\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$,

$$|\mathcal{J}^{dp}(I_H)| + |\mathcal{J}^{hs}(I_H) \setminus \mathcal{J}^{dp}(I_H)| \geq |\mathcal{J}^{dp}(I_H)| + |\mathcal{J}^{dp}(I_{M_c})| + |\mathcal{J}^{dp}(I_{M_p})| + |\mathcal{J}^{dp}(I_L)|$$  \hspace{1cm} (EC.20)

Conditional on the event $\mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}$, we have that

$$\mathbb{E} \left[ \Pi(B, T; \text{DP}) | \mathcal{E} \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \right]$$

\hspace{1cm} \overset{(a)}{=} \sum_{k \in \mathcal{J}^{hs}(I_H) \setminus \mathcal{J}^{dp}(I_H)} \theta_k - \left( \sum_{k \in \mathcal{J}^{dp}(I_L)} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_p})} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_c})} \theta_k \right),$$

\hspace{1cm} \overset{(b)}{\geq} |\mathcal{J}^{hs}(I_H) \setminus \mathcal{J}^{dp}(I_H)| (u + c_0 \Delta_\beta) - \left( \sum_{k \in \mathcal{J}^{dp}(I_L)} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_p})} \theta_k + \sum_{k \in \mathcal{J}^{dp}(I_{M_c})} \theta_k \right),$$

\hspace{1cm} \overset{(c)}{\geq} |\mathcal{J}^{hs}(I_H) \setminus \mathcal{J}^{dp}(I_H)| (u + c_0 \Delta_\beta) - \left( |\mathcal{J}^{dp}(I_L)| \ell + |\mathcal{J}^{dp}(I_{M_p})| (u + c_0 \Delta_\beta) + |\mathcal{J}^{dp}(I_{M_c})| (u + \Delta_\beta) \right),$$

\hspace{1cm} \overset{(d)}{\geq} |\mathcal{J}^{dp}(I_L)| \left[ (u + c_0 \Delta_\beta) - \ell \right] + |\mathcal{J}^{dp}(I_{M_p})| \left[ (u + c_0 \Delta_\beta) - (u + c_0 \Delta_\beta) \right]$$

\hspace{1cm} + |\mathcal{J}^{dp}(I_{M_c})| \left[ (u + c_0 \Delta_\beta) - (u + \Delta_\beta) \right],$$

\hspace{1cm} \overset{(e)}{=} |\mathcal{J}^{dp}(I_L)| \left[ (u + c_0 \Delta_\beta) - \ell \right] + |\mathcal{J}^{dp}(I_{M_c})| \alpha_0 \Delta_\beta,$$

\hspace{1cm} \overset{(f)}{\geq} \frac{\alpha_0 \ell}{32} \left( \frac{1}{2} - \frac{1}{2} \right),$$

where (a) follows from $\text{(EC.18)}$ and $\text{(EC.19)}$, (b) follows from the fact that $\sum_{k \in S} a_k \geq |S| \min_{k \in S} \{a_k\}$ and by construction, for all the arrivals in the set $I_H$, $\theta_k \geq u + c_0 \Delta_\beta$, (c) follows similar to (b), (d) follows from $\text{(EC.20)}$, (e) follows from the definition of $\alpha_0$ in $\text{(EC.2)}$, (f) follows from the fact that $|\mathcal{J}^{dp}(I_{M_c})| \geq \frac{1}{32} \sqrt{T}$ due to the event $\mathcal{E}$ and from the fact that $|\mathcal{J}^{dp}(I_L)| \geq 0$. $\square$
EC.1.2. Proof of Lemma EC.2

Recall the definitions of set of indices \( \mathcal{J}^{hs} \) and \( \mathcal{J}^{DP} \) from (EC.16) and decomposition of the sum of values chosen under hindsight optimal and the DP policy as given in (EC.17) and (EC.18). Recall from the discussion of the hindsight optimal that under the event \( \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \), the hindsight optimal will accept all the candidates with abilities in the set \( \mathcal{I}_H, \mathcal{I}_{M_p}, \mathcal{I}_{M_c} \) and possibly some candidates in the set \( \mathcal{I}_L \). Conditional on the event \( \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \), we have that,

\[
B \overset{\text{(a)}}{=} |\mathcal{J}^{hs}(\mathcal{I}_H)| + |\mathcal{J}^{hs}(\mathcal{I}_{M_p})| + |\mathcal{J}^{hs}(\mathcal{I}_{M_c})| + |\mathcal{J}^{hs}(\mathcal{I}_L)| \\
\overset{\text{(b)}}{=} |\mathcal{J}^{hs}(\mathcal{I}_H) \setminus \mathcal{J}^{DP}(\mathcal{I}_H)| + |\mathcal{J}^{DP}(\mathcal{I}_H)| + |\mathcal{J}^{hs}(\mathcal{I}_{M_p}) \setminus \mathcal{J}^{DP}(\mathcal{I}_{M_p})| + |\mathcal{J}^{DP}(\mathcal{I}_{M_p})| \\
+ |\mathcal{J}^{hs}(\mathcal{I}_{M_c}) \setminus \mathcal{J}^{DP}(\mathcal{I}_{M_c})| + |\mathcal{J}^{DP}(\mathcal{I}_{M_c})| + |\mathcal{J}^{hs}(\mathcal{I}_L)|, \\
\overset{\text{(c)}}{=} |\mathcal{J}^{DP}(\mathcal{I}_H)| + |\mathcal{J}^{DP}(\mathcal{I}_{M_p})| + |\mathcal{J}^{DP}(\mathcal{I}_{M_c})| + |\mathcal{J}^{DP}(\mathcal{I}_L)|, \\
\overset{\text{(d)}}{=} |\mathcal{J}^{DP}(\mathcal{I}_H)| + |\mathcal{J}^{DP}(\mathcal{I}_{M_p})| + |\mathcal{J}^{DP}(\mathcal{I}_{M_c})| + |\mathcal{J}^{DP}(\mathcal{I}_L)| + |\mathcal{J}^{hs}(\mathcal{I}_L)|,
\]

where (a) follows from the fact the hindsight optimal will accept exactly \( B \) candidates, (b) follows from the fact that for countable set \( A, B \) such that \( B \subseteq A \), we have that \( |A| = |A \setminus B| + |B| \), (c) follows from the fact that any online policy will accept at most \( B \) candidates, (d) follows for the same reason as (b). This implies the following inequality,

\[
|\mathcal{J}^{hs}(\mathcal{I}_H) \setminus \mathcal{J}^{DP}(\mathcal{I}_H)| + |\mathcal{J}^{hs}(\mathcal{I}_{M_p}) \setminus \mathcal{J}^{DP}(\mathcal{I}_{M_p})| + |\mathcal{J}^{hs}(\mathcal{I}_{M_c}) \setminus \mathcal{J}^{DP}(\mathcal{I}_{M_c})| \geq |\mathcal{J}^{DP}(\mathcal{I}_L) \setminus \mathcal{J}^{hs}(\mathcal{I}_L)| \quad \text{(EC.21)}
\]

Conditional on the event \( \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \), we have that

\[
\mathbb{E}[\Lambda(B, T; DP) | \mathcal{E}^c \cap \mathcal{H} \cap \mathcal{C} \cap \mathcal{P}] \\
\overset{\text{(a)}}{=} \left( \sum_{A \in \{I_H, I_{M_p}, I_{M_c}\}} \sum_{k \in \mathcal{J}^{hs}(A) \setminus \mathcal{J}^{DP}(A)} \theta_k + \sum_{A \in \{I_H, I_{M_p}, I_{M_c}\}} \sum_{k \in \mathcal{J}^{DP}(A)} \theta_k + \sum_{k \in \mathcal{J}^{hs}(I_L)} \theta_k \right) \\
- \left( \sum_{A \in \{I_H, I_{M_p}, I_{M_c}\}} \sum_{k \in \mathcal{J}^{DP}(A)} \theta_k + \sum_{k \in \mathcal{J}^{DP}(I_L) \setminus \mathcal{J}^{hs}(I_L)} \theta_k + \sum_{k \in \mathcal{J}^{hs}(I_L)} \theta_k \right), \\
\overset{\text{(b)}}{=} \sum_{A \in \{I_H, I_{M_p}, I_{M_c}\}} \sum_{k \in \mathcal{J}^{hs}(A) \setminus \mathcal{J}^{DP}(A)} \theta_k - \sum_{k \in \mathcal{J}^{DP}(I_L) \setminus \mathcal{J}^{hs}(I_L)} \theta_k.
\]
\[
\begin{align*}
(c) & \geq |J^\text{hs}(I_H) \setminus J^\text{DP}(I_H)| (u+c_0 \Delta_\beta) + |J^\text{hs}(I_{M_p}) \setminus J^\text{DP}(I_{M_p})| (\ell-c_0 \Delta_\beta) \\
& \quad + |J^\text{hs}(I_{M_c}) \setminus J^\text{DP}(I_{M_c})| (\ell-\Delta_\beta) - |J^\text{DP}(I_L) \setminus J^\text{hs}(I_L)| (\ell-c_0 \Delta_\beta) \\
& \quad \overset{d}{\geq} |J^\text{hs}(I_H) \setminus J^\text{DP}(I_H)| [(u+c_0 \Delta_\beta) - (\ell-c_0 \Delta_\beta)] + |J^\text{hs}(I_{M_c}) \setminus J^\text{DP}(I_{M_c})| [(\ell-\Delta_\beta) - (\ell-c_0 \Delta_\beta)] \\
& \quad + |J^\text{hs}(I_{M_p}) \setminus J^\text{DP}(I_{M_p})| [(\ell-c_0 \Delta_\beta) - (\ell-c_0 \Delta_\beta)] \\
& \quad \overset{(e)}{=} |J^\text{hs}(I_H) \setminus J^\text{DP}(I_H)| [(u+c_0 \Delta_\beta) - (\ell-c_0 \Delta_\beta)] + |J^\text{hs}(I_{M_c}) \setminus J^\text{DP}(I_{M_c})| \alpha_0 \Delta_\beta, \\
& \quad \overset{(f)}{\geq} \frac{\alpha_0 \ell}{8} T^{1/2 - \frac{1}{2pi}},
\end{align*}
\]

where (a) follows from (EC.17) and (EC.18), (b) follows trivially, (c) follows from the fact \( \theta_k \{ \theta_k \in \mathcal{I}_H \} \geq u+c_0 \Delta_\beta, \theta_k \{ \theta_k \in \mathcal{I}_{M_p} \} \geq \ell-c_0 \Delta_\beta, \theta_k \{ \theta_k \in \mathcal{I}_{M_c} \} \geq \ell-\Delta_\beta \) and \( \theta_k \{ \theta_k \in \mathcal{I}_L \} \leq \ell-c_0 \Delta_\beta \), (d) follows from (EC.21), (e) follows from the definition of \( \alpha_0 = c_0 - 1 \), (f) follows from the fact that 

\[
|J^\text{hs}(I_{M_c}) \setminus J^\text{DP}(I_{M_c})| \geq \frac{1}{8} \sqrt{T}
\]

which is due to fact that under the event \( \mathcal{H} \cap \mathcal{C} \cap \mathcal{P} \), the hindsight optimal will accept all the arrivals in the set \( \mathcal{I}_{M_c} \) however under the event \( \mathcal{E}^c \) will accept at most \( \frac{1}{8} \sqrt{T} \) arrivals in the first \( B \) time steps and this will result in incorrectly rejecting at least 

\[
\frac{1}{2} \sqrt{T} - \frac{1}{8} \sqrt{T} = \frac{3}{8} \sqrt{T}
\]

arrivals in the set \( \mathcal{I}_{M_c} \). \( \square \)

**EC.1.3. Proof of Lemma [EC.3]**

Recall the definition of the events \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \tilde{\mathcal{H}}_2 \) as defined in (EC.4), (EC.5) and (EC.6) respectively. We have that \( N(I_H, 1, B) \sim \text{Bin}(B, p_{I_H}) \), where \( p_{I_H} = \frac{1}{2} - \frac{129}{512} T^{-\frac{1}{2}} \). Therefore we have that 

\[
\mu^B_1(I_H) = \mathbb{E}[N(I_H, 1, B)] \approx \frac{T}{4} - \frac{129}{512} \sqrt{T}
\]

since \( B \approx T/2 \). Therefore we can write the event \( \mathcal{H}_1 \) as

\[
\mathcal{H}_1 = \left\{ \mu^B_1(I_H) - \frac{127}{512} \sqrt{T} \leq N(I_H, 1, B) \leq \mu^B_1(I_H) + \frac{129}{512} \sqrt{T} \right\}
\]

Therefore we have that

\[
\mathbb{P}(\mathcal{H}_1) \overset{(a)}{=} \mathbb{P} \left( \mu^B_1(I_H) - \frac{127}{512} \sqrt{T} \leq N(I_H, 1, B) \leq \mu^B_1(I_H) + \frac{129}{512} \sqrt{T} \right)
\]

\[
\overset{(b)}{=} \mathbb{P} \left( -\frac{127}{512} \sqrt{T} \leq N(I_H, 1, B) - \mu^B_1(I_H) \leq \frac{129}{512} \sqrt{T} \right)
\]

\[
\overset{(c)}{=} \mathbb{P} \left( -\frac{127}{256} \frac{1}{\sqrt{p_{I_H}(1-p_{I_H})}} \leq \frac{N(I_H, 1, B) - \mu^B_1(I_H)}{\sqrt{Bp_{I_H}(1-p_{I_H})}} \leq \frac{129}{256} \frac{1}{\sqrt{p_{I_H}(1-p_{I_H})}} \right)
\]
\[ (d) \geq \mathbb{P} \left( 0 \leq \frac{N(I_H, 1, B) - \mu_1^B(I_H)}{\sqrt{B}p_I(1 - p_I)} \leq \frac{258}{256} \right) \]

\[ (e) \leq \mathbb{P} \left( \frac{N(I_H, 1, B) - \mu_1^B(I_H)}{\sqrt{B}p_I(1 - p_I)} \leq \frac{258}{256} \right) - \mathbb{P} \left( \frac{N(I_H, 1, B) - \mu_1^B(I_H)}{\sqrt{B}p_I(1 - p_I)} \leq 0 \right) \]

\[ (f) \geq \Phi \left( \frac{258}{256} \right) - \Phi(0) - \frac{c}{\sqrt{T}} \]

\[ (g) \geq 0.34 - \frac{c}{\sqrt{T}} \]

where (a) follows from definition of event \( H_1 \), (b,c) follows trivially, (d) follows the fact that

\[ \left\{ 0 \leq \frac{N(I_H, 1, B) - \mu_1^B(I_H)}{\sqrt{B}p_I(1 - p_I)} \leq \frac{258}{256} \right\} \subseteq \left\{ - \frac{127}{256} \frac{1}{p_I(1 - p_I)} \leq \frac{N(I_H, 1, B) - \mu_1^B(I_H)}{\sqrt{B}p_I(1 - p_I)} \leq \frac{129}{256} \frac{1}{p_I(1 - p_I)} \right\} \]

since \( p_I(1 - p_I) \leq 1/4 \), (e) follows trivially, (f) follows from Berry Esseen Theorem and (g) follows trivially. Now there exists a \( T_0 < \infty \) such that for all \( T \geq T_0 \), we have that \( \mathbb{P}(H_1) \geq 0.003. \) An analogous proof follows for \( H_2 \) and \( \tilde{H}_2 \) as well, we omit it to avoid repetition.

\[ \Box \]

**EC.1.4. Proof of Lemma \text{EC.4}**

Recall the definition of events \( C_1, C_2, P_1 \) and \( P_2 \) as defined in (EC.7), (EC.8), (EC.9) and (EC.10) respectively. We have that \( N(I_{M_c}, 1, B) \sim \text{Bin}(B, p_{I_{M_c}}) \) where \( p_{I_{M_c}} = T^{-\frac{1}{2}} \). Therefore we have that \( \mu_1^B(I_{M_c}) = \mathbb{E}[N(I_{M_c}, 1, B)] \approx \sqrt{T}/2 \) since \( B \approx T/2 \). Now the event \( C_1 \) can be written as

\[ C_1 = \left\{ \frac{\mu_1^B(I_{M_c})}{2} \leq N(I_{M_c}, 1, B) \leq 2\mu_1^B(I_{M_c}) \right\} \]

Therefore we have that

\[ \mathbb{P}(C_1) \overset{(a)}{=} \mathbb{P} \left( \{ N(I_{M_c}, 1, B) \geq 2\mu_1^B(I_{M_c}) \} \cup \{ N(I_{M_c}, 1, B) \leq \mu_1^B(I_{M_c})/2 \} \right) \]

\[ \overset{(b)}{\leq} \mathbb{P} \left( \{ N(I_{M_c}, 1, B) \geq 2\mu_1^B(I_{M_c}) \} \right) + \mathbb{P} \left( \{ N(I_{M_c}, 1, B) \leq \mu_1^B(I_{M_c})/2 \} \right) \]

\[ \overset{(c)}{\leq} cT^{-\frac{1}{2}} \]

where (a) follows from definition of \( C_1 \), (b) follows from union bound, (c) follows from Berry Esseen theorem as applied before. From this it follows that there exists \( T_0 < \infty \) such that \( \mathbb{P}(C_1) \leq 0.001 \) for all \( T \geq T_0 \). An analogous proof follows for \( C_2, P_1 \) and \( P_2 \), we omit it to avoid repetition. \( \Box \)
**EC.2. Details and Analysis of CwG Policy**

In this section we will provide some more details about the CwG algorithm (Algorithm 1) and also provide the proof of Theorem 2. In Section EC.2.1 we provide a discussion about the phase structure of Algorithm 1. In Section EC.2.2 we define the concept of hindsight-to-go (HTG) which will aid our analysis. In Section EC.2.3 we provide a proof outline for Theorem 2. In Section EC.2.4 we will provide some helper lemmas to formalize our analysis with their proofs deferred to Section EC.2.6. In Section EC.2.5 we provide the formal proof of Theorem 2.

**EC.2.1. Phase Structure of Algorithm 1**

The phase structure of the CwG policy has been devised to simplify the analysis of the CwG policy. The key idea of the CwG policy is that if the CE threshold $p_t^{CE}$ at time $t$ is within a ball of radius $\Delta_t \triangleq \sqrt{2 \log \tau / \tau}$ (where $\tau = T - t + 1$ is the number of remaining time steps) of a gap quantile $q^*_i$, then the CwG threshold is set to the gap quantile $q^*_i$ itself. As $t$ increases, so does the size of the radius and hence eventually there will be more than one gap quantiles in this ball. If there are more than one gap quantiles in $\Delta_t$, we need a tie-breaking rule to decide which gap quantile the CwG threshold is assigned to. This tie-breaking rule further complicates an already involved analysis and hence to simplify the technical analysis, we define the CwG algorithm by dividing it into two phases.

In the first phase, it suffices to ensure that there will always be at most one gap quantile in the $\Delta_t$-neighbourhood of $p_t^{CE}$ for any value of $p_t^{CE}$ and there is no need for a tie-breaking rule. One way to ensure this, is to find $t^*$ such that for all $t \leq t^*$, we have that $\sqrt{2 \log \tau / \tau} \leq \varepsilon_0 / 2$. Note that irrespective of the value of $p_t^{CE}$, there is at most one gap quantile in the $\Delta_t$ neighborhood of $p_t^{CE}$. Further, note that for $t \leq T - 2$, $\sqrt{2 \log \tau / \tau}$ is increasing in $t$ and hence it suffices to verify that $\sqrt{2 \log \tau^* / \tau^*} \leq \varepsilon_0 / 2$ for $\tau^* = \lceil 64 \log(1/\varepsilon_0) / \varepsilon_0^2 \rceil$. Given that we are guaranteed to have at most one gap quantile in $\Delta_t$-neighbourhood of $p_t^{CE}$, our analysis is great simplified.

The second phase is of length $\lceil 64 \log(1/\varepsilon_0) / \varepsilon_0^2 \rceil$ and we use a static allocation rule in the second phase. The contribution to regret because of the static policy is at most $C \sqrt{\log(1/\varepsilon_0) / \varepsilon_0}$ for some universal constant $C < \infty$. 
EC.2.2. Hindsight To Go (HTG) and HTG Threshold

Let \( q^\theta_{\geq_2}(n) \) denote the \( n \)-th largest value quantile in \( q^\theta_{\geq_2} \) for an integer \( n \in \mathbb{N} \). Define the following quantile values \( q^l_t = q^\theta_{\geq_2}(B_t + 1) \) and \( q^u_t = q^\theta_{\geq_2}(B_t) \) and denote their corresponding values by \( l_t = F^{-1}(q^l_t), u_t = F^{-1}(q^u_t) \), where \( B_t \) is the remaining budget at time \( t \). Note that since the principle of compensated coupling is to persuade the hindsight policy to take the same action as the online policy using sufficient compensations, the hindsight policy at time \( t \) may look different from the hindsight policy initially and being adapted to the budget which evolves according to the online policy. To distinguish between the two, at any time \( t \), we will instead refer to the hindsight policy as the Hindsight To Go (HTG) policy, which due to coupling follows the same actions as the online policy up till time \( t - 1 \) and then from time \( t \) onwards takes the optimal hindsight decision with arrivals in \( \omega_{\geq_2} \) given the remaining budget \( B_t \). Given the CwG quantile threshold \( p^\text{CwG}_t \), we define \( p^\text{HTG}_t = \arg \max_{x \in [q^l_t, q^u_t]} |p^\text{CwG}_t - x| \) when \( B_t > 0 \), otherwise \( p^\text{HTG}_t = 1 \). The reason to adopt this particular \( p^\text{CwG}_t \) dependent definition of \( p^\text{HTG}_t \) is that the compensation needed at time \( t \) will now be bounded above by the separation between the CwG threshold and the HTG threshold in value space.

EC.2.3. Proof Outline

We first provide a proof outline. Recall \( \tilde{T} = T - \left\lceil 64 \log(1/\varepsilon_0) \right\rceil / \varepsilon_0^2 \) in Algorithm 1 (the CwG policy). The algorithm operates in two phases, the first phase includes time steps \( t \) such that \( 1 \leq t \leq \tilde{T} \) while the second phase consists of the remaining time steps \( t \) such that \( \tilde{T} + 1 \leq t \leq T \).

Analysis of First Phase. The analysis of the first phase makes use of the regret decomposition given in Lemma 1. To bound the expected compensation term \( \mathbb{E}_{\tau^\theta} [\partial R_t(B_t^\pi, a_t^\pi)] \) in Lemma 1 for \( \pi = \text{CwG} \), we will analyse two thresholds: the CwG quantile threshold denoted as \( p^\text{CwG}_t \) and Hindsight To Go (HTG) quantile threshold \( p^\text{HTG}_t \). Note that given a tail sequence \( \theta_{\geq_2} \) and the remaining budget \( B_t \), the Hindsight To Go threshold is set such that on the sample path \( \theta_{\geq_2} \), the top \( B_t \) candidates are chosen. We bound the expected compensation at time \( t \) for \( t \in [1, \tilde{T}] \) and we do so by dividing the analysis into two events: (a) \( E_t = \{1 - B_t / \tau > 4\sqrt{\log \tau / \tau}\} \) and (b) \( E_t^c = \{1 - B_t / \tau \leq 4\sqrt{\log \tau / \tau}\} \)
where \( \tau = T - t + 1 \). At any time either of the two events arises and we bound the expected compensation conditional on each of the two events. The analysis for both the events utilizes the same recipe. We show that with high probability the difference between \( \text{CwG} \) quantile threshold \( p^\text{CwG}_t \) and the \( \text{HTG} \) quantile threshold \( p^\text{HTG}_t \) is bounded above by \( C \sqrt{\log \tau / \tau} \) (Lemma EC.7). As a result of this, we establish that with high probability the two thresholds \( p^\text{CwG}_t \) and \( p^\text{HTG}_t \) belong to the same cluster (Lemma EC.8). Now compensation is need at time \( t \) only if there is a candidate ability arrival \( \theta_t \) such that its quantile \( F(\theta_t) \) lies between the two thresholds \( p^\text{CwG}_t \) and \( p^\text{HTG}_t \) and the amount of compensation is bounded by \( |F^{-1}(p^\text{CwG}_t) - F^{-1}(p^\text{HTG}_t)| \) (Lemma EC.9). Using Lemmas EC.7 EC.8 EC.9 and definition of the \((\beta, \varepsilon_0, \delta)\)-clustered distribution, we show that the expected compensation at time \( t \) is bounded as follow.

**Lemma EC.5.** There is a universal constant \( C < \infty \) such that the following occurs. For any \( \beta \in [0, \infty) \), \( \varepsilon_0 \in (0, 1] \), and \( \delta \in (0, 1] \), suppose the candidate-ability distribution \( F \) with associated gaps is \((\beta, \varepsilon_0, \delta)\)-clustered. Then for \( t \in \{1, 2, \ldots , \tilde{T} \} \), for the \( \text{CwG} \) policy we have that the expected compensation at time \( t \) is bounded above as

\[
\sup_{B_t \geq 0} \partial \mathcal{R}_t (B_t, a^\text{CwG}_t) \leq C \left( (\log \tau / \tau)^{\frac{1}{2} + \frac{1}{\pi (1 + \beta)} + \delta \sqrt{\log \tau / \tau}} \right),
\]

where \( \tau = T - t + 1 \). Note that the above implies that

\[
\mathbb{E}_{B_t} [\partial \mathcal{R}_t (B_t, a^\text{CwG}_t)] \leq C \left( (\log \tau / \tau)^{\frac{1}{2} + \frac{1}{\pi (1 + \beta)} + \delta \sqrt{\log \tau / \tau}} \right).
\]

Using Lemma 1 and Lemma EC.5, the cumulative regret accrued up till time \( \tilde{T} \) is upper bounded by \( C \left( (\log T)^{\frac{1}{2} + \frac{1}{\pi (1 + \beta)}} T \frac{1}{2 - \frac{1}{\pi (1 + \beta)}} \mathbb{1} \{ \beta > 0 \} + \log^2 T \mathbb{1} \{ \beta = 0 \} + \delta \sqrt{T \log T} \right) \).

**Analysis of Second Phase.** Recall that the \( \text{CwG} \) policy (Algorithm 2) in the last \( 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \) time steps, makes use of the static allocation policy where we solve for the CE quantile threshold \( p^\text{CE}_T \) and thereafter use the time invariant quantile threshold \( p^\text{CE}_T \). Using a well known fact in the network revenue management literature, we know that the regret accrued under a static allocation policy is upper bounded as \( C \sqrt{\text{horizon length}} \) for some universal constant \( C < \infty \). Since the \( \text{CwG} \) policy (Algorithm 1) employs the static allocation policy for the last \( \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil \) time steps, the regret accrued over the last \( T - \tilde{T} \) time steps is upper bounded as \( C \sqrt{\log(1/\varepsilon_0)/\varepsilon_0} \). Adding up the regret over the two phases results in the regret scaling in Theorem 2.
EC.2.4. Preliminaries and Helper Lemmas

We introduce some helper lemmas which we will use to prove the regret bound. We defer the proof of these lemmas to Appendix EC.2.6. Let $t$ denote the current time step and $\tau = T - t + 1$ denote the remaining number of time steps. Assume that $T \geq \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil$ and define $\tilde{T} = T - \lfloor 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rfloor$. Define the following events for $t \leq \tilde{T}$:

\begin{equation}
A_{1,t} = \{|p_{t}^{CE} - p_{t}^{HTG}| \leq \sqrt{2 \log \tau/\tau}\}, \tag{EC.22}
\end{equation}

\begin{equation}
A_{2,t} = \{||p_{t}^{CwG} - p_{t}^{HTG}| \leq 3 \sqrt{\log \tau/\tau}\}, \tag{EC.23}
\end{equation}

\begin{equation}
A_{3,t} = \bigcup_{i=1}^{n+1} \{p_{t}^{CwG} \in \bar{Q}_{i}, p_{t}^{HTG} \in \bar{Q}_{i}\}, \tag{EC.24}
\end{equation}

where $\bar{Q}_{i} = [q_{i}^{l}, q_{i}^{u}]$ and $n$ denotes the number of gaps. The interpretation of $A_{3,t}$ is that the CwG policy threshold and the HTG policy threshold belong (weakly) to the same mass cluster. The following lemmas show that these three events are very likely to occur for $t \leq \tilde{T}$:

**Lemma EC.6.** Consider the event $A_{1,t}$ defined in (EC.22). We have that $\mathbb{P}(A_{1,t}^c) \leq 2/\tau^4$.

**Lemma EC.7.** Consider the event $A_{2,t}$ defined in (EC.23). We have that $\mathbb{P}(A_{2,t}^c) \leq 2/\tau^4$.

**Lemma EC.8.** Consider the event $A_{3,t}$ defined in (EC.24). We have that $\mathbb{P}(A_{3,t}^c) \leq 2n(n+1)/\tau^4$, where $n$ is the number of gaps.

Let $q_{t}^{\theta} = F(\theta_t)$ be the quantile of the candidate ability $\theta_t$ at time $t$. If $p_t^{CwG} < q_{t}^{l}$ then we have that $p_t^{HTG} = q_{t}^{u}$ and compensation is needed only if $q_{t}^{\theta} \in [p_t^{CwG}, p_t^{HTG}]$. If $p_t^{CwG} > q_{t}^{u}$ then we have that $p_t^{HTG} = q_{t}^{l}$ and compensation is needed only if $q_{t}^{\theta} \in [p_t^{HTG}, p_t^{CwG}]$. If $p_t^{CwG} \in (q_{t}^{l}, q_{t}^{u})$, then no compensation is required.

**Lemma EC.9.** Let $q_{t}^{\theta} = F(\theta_t)$ denote the quantile corresponding to $\theta_t$. Compensation needs to be provided only if $q_{t}^{\theta} \in (\min\{p_t^{CwG}, p_t^{HTG}\}, \max\{p_t^{CwG}, p_t^{HTG}\})$; let $\partial R_t(B_t, a_t^{CwG})$ denote the compensation. Then we have that $\partial R_t(B_t, a_t^{CwG}) \leq \max\{F^{-1}(p_t^{CwG}) - F^{-1}(p_t^{HTG}), F^{-1}(p_t^{HTG}) - F^{-1}(p_t^{CwG})\}$. 
EC.2.5. Formal Proof of Theorem 2

Proof of Theorem 2 Define $\hat{T} = T - \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil$ and define $\tau_0 = \lceil 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rceil$. Consider some time $t \leq \hat{T}$ and let $\tau = T - t + 1$ denote the remaining time. Recall that $p_t^{\text{CwG}} \in \mathcal{F}_t$ and $p_t^{\text{HTG}}$ depends on the candidate abilities $\theta_{\geq t}$ but only via the $B_t$-th largest quantile $q_t^u$ and $B_t+1$-th largest quantile $q_t^l$. To facilitate our analysis, we employ the so-called principle of deferred decisions, and only reveal $q_t^u$ and $q_t^l$ (in addition to the history up to time $t$ i.e. $\mathcal{F}_t$), which uniquely determines $p_t^{\text{HTG}}$. Define the event $\mathcal{L}_t \triangleq \{p_t^{\text{CwG}} \leq q_t^l\}$ and $\mathcal{H}_t \triangleq \{p_t^{\text{CwG}} \geq q_t^u\}$. For the rest of the proof, we will condition on the event $\mathcal{L}_t$ and prove an upper bound on the expected compensation $\partial \mathcal{R}_t(B_t, a_t)$ (conditional on $\mathcal{L}_t$). A similar bound can be analogously shown under the event $\mathcal{H}_t$ and we omit the details to avoid repetition. Let $q_t^\theta$ denote the quantile corresponding to the candidate ability $\theta_t$. Now compensation is needed only if $q_t^\theta \in [p_t^{\text{CwG}}, q_t^u]$. Let $\mathcal{C}_t$ denote the event that compensation is needed i.e., the action under the CwG threshold is different from the action under the HTG threshold. Given $q_t^l$ and $q_t^u$, we know that the $\tau$ periods to go include a random subset of $B_t$ quantiles located above $q_t^u$ (these quantiles are i.i.d uniform in $[q_t^u, 1]$) and the remaining $\tau - B_t$ quantiles are below $q_t^l$ (these quantiles are i.i.d uniform in $[0, q_t^l]$). If $p_t^{\text{CwG}} \in (q_t^l, q_t^u)$, no compensation is needed. Compensation is needed only if $q_t^\theta \in [p_t^{\text{CwG}}, q_t^l]$ and this event occurs if (a) the realized quantile $q_t^\theta = q_t^l$ or (b) $q_t^\theta \in [p_t^{\text{CwG}}, q_t^l]$ is one of the $\tau - B_t - 1$ lower quantiles. The probability of case (a) is $1/\tau$ and probability of (b) is $(\tau - B_t - 1)(q_t^l - p_t^{\text{CwG}})/q_t^l \tau$. Combining the two we have that

$$
P(\mathcal{C}_t|\mathcal{F}_t, q_t^l, q_t^u, \mathcal{L}_t) = \frac{1}{\tau} p_t^{\text{CwG}} \leq q_t^l + \frac{(\tau - B_t - 1)(q_t^l - p_t^{\text{CwG}})}{q_t^l \tau} \quad \text{(EC.25)}$$

where $\tau = T - t + 1$ and $(x)_+ = \max\{x, 0\}$. Using Lemma [EC.9] and (EC.25), we have the following bound the expected compensation

$$
\mathbb{E} \left[ \partial \mathcal{R}_t(B_t^{\text{CwG}}, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{F}_t, q_t^l, q_t^u, \mathcal{L}_t \right] \leq \frac{|F^{-1}(q_t^u) - F^{-1}((p_t^{\text{CwG}})^+)|}{\tau} + \frac{(q_t^l - p_t^{\text{CwG}})|F^{-1}(q_t^u) - F^{-1}((p_t^{\text{CwG}})^+)|}{q_t^l \tau} \quad \text{(EC.26)}
$$
Next we need to bound the ratio \((\tau - B_t - 1)/(q^l_t/\tau)\) and at any time \(t \leq T - \tau_0\), exactly of the following complementary events occurs: (a) \(\mathcal{E}_t = \{1 - B_t/\tau > 4\sqrt{\log \tau/\tau}\}\) and (b) \(\mathcal{E}^c_t = \{1 - B_t/\tau \leq 4\sqrt{\log \tau/\tau}\}\). Recall the event \(A_{3,t}\) defined in [EC.24] which states that

\[
A_{3,t} = \{q^l_t, q^u_t, p^{CWG}_t \text{ are quantiles belonging (weakly) to the same cluster}\}
\]

Next we will establish an upper bound on (EC.26) for each of the events \(\mathcal{E}_t\) and \(\mathcal{E}^c_t\).

**Case (a):** \(\mathcal{E}_t = \{1 - B_t/\tau > 4\sqrt{\log \tau/\tau}\}\). Define the following events:

\[
A_{4,t} \triangleq \{q^l_t \geq (1/2)(1 - B_t/\tau)\},
\]

\[
A_{5,t} \triangleq A_{3,t} \cap A_{4,t}.
\]

Under the event \(A_{3,t}\), from Definition 1 (a) it follows that 

\[
|F^{-1}(q^u_t) - F^{-1}(p^{CWG}_t)| \leq |q^u_t - p^{CWG}_t|^{1/\tau + \delta} + \delta.
\]

Now, on the event \(A_{4,t}\), we have that \((\tau - B_t - 1)/(\tau q^l_t) \leq 2\). We have that

\[
\mathbb{P}(A_{4,t}^c | B_t, \mathcal{E}_t) = \mathbb{P}(q^l_t < (1/2)(1 - B_t/\tau)) \leq \mathbb{P}(\text{Binomial}(\tau, (1/2)(1 - B_t/\tau)^-) \geq \tau - B_t - 1)
\]

\[
\leq \exp(-\Omega(\tau - B_t)) \leq C/(\tau - B_t)^4 \leq C/\tau^2.
\]

where the last inequality follows from the case assumption that \(\tau - B_t \geq 4\sqrt{\tau \log \tau}\) and the inequality is true for some appropriately defined constant \(C < \infty\). It follows that

\[
\mathbb{P}(A_{4,t}^c | \mathcal{E}_t) \leq C/\tau^2.
\]  

(EC.27)

Using (EC.26), and the definitions of the events \(A_{3,t}\) and \(A_{4,t}\), we have that

\[
\mathbb{E}\left[\partial R_{t}(B_t, a^{CWG}_t, \theta_{\geq t}) | \mathcal{F}_t, q^l_t, q^u_t, \mathcal{L}_t, \mathcal{E}_t\right]
\]

\[
\leq 1_{A_{3,t}} \cdot \left[|q^u_t - p^{CWG}_t|^{1/\tau + \delta}/\tau + 21_{A_{3,t}}1_{A_{4,t}} \left[|q^u_t - p^{CWG}_t|^{1+1/\tau + \delta} + |q^u_t - p^{CWG}_t|\delta\right] + 1_{A_{3,t}} + 1_{A_{4,t}}\right],
\]

(EC.28)

\[
\leq |q^u_t - p^{CWG}_t|^{1/\tau + \delta}/\tau + 2|q^u_t - p^{CWG}_t|^{1+1/\tau + \delta} + 2|q^u_t - p^{CWG}_t|\delta + 1_{A_{4,t}} + 1_{A_{5,t}}.
\]

(EC.29)
where the first inequality follows from \( q_t^u - p_t^{\text{CwG}} \leq q_t^u - p_t^{\text{CwG}} \), and the second inequality follows from the fact that \( 1_{A_{3,1}} \leq 1 \). Using the definition of the event \( A_{2,t} \) in (EC.23) and Lemma [EC.7], we have that for all \( \alpha \in (0, 2] \), we have

\[
\mathbb{E} \left[ |q_t^u - p_t^{\text{CwG}}|^{\alpha} \right] \leq \mathbb{E} \left[ 1_{A_{2,t}} |q_t^u - p_t^{\text{CwG}}|^{\alpha} + 1_{A_{2,t}} \right] \leq 3^\alpha (\log \tau/\tau)^{\alpha/2} + 2/\tau^4 \leq C (\log \tau/\tau)^{\alpha/2}. \quad (\text{EC.30})
\]

Taking expectations on both sides of (EC.29), we obtain that

\[
\mathbb{E}[\partial \mathcal{R}_t(B_t, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{E}_t, \mathcal{L}_t] \\
\leq \left( \mathbb{E} \left[ |q_t^u - p_t^{\text{CwG}}|^{1/\tau} + \delta \right] /\tau + 2\mathbb{E} \left[ |q_t^u - p_t^{\text{CwG}}|^{1+1/\tau^2} \right] + 2\mathbb{E} \left[ |q_t^u - p_t^{\text{CwG}}| \right] \mathbb{P}(A_{3,1}|\mathcal{E}_t) + \mathbb{P}(A_{3,1}|\mathcal{E}_t), \right.
\]  

\[
\left. \leq 6 (\log \tau/\tau)^{1/\tau} \mathcal{L}_t /\tau + \delta /\tau + 36 (\log \tau/\tau)^{1+1/\tau^2} + 6\delta \sqrt{\log \tau/\tau} + \mathbb{P}(A_{3,1}|\mathcal{E}_t) + C/\tau^2, \quad (\text{EC.31})
\]

where inequality (i) follows from the taking expectation on both sides, and inequality (ii) follows from using (EC.30) for the first, third and fourth summands, and the sixth summand follows from (EC.27).

Case (b): \( \mathcal{E}_t^c = \{1 - B_t/\tau \leq 4\sqrt{\log \tau/\tau}\} \). The event \( 1 - B_t/\tau \leq 4\sqrt{\log \tau/\tau} \) implies that \((\tau - B_t)/\tau \leq 4\sqrt{\log \tau/\tau} \), and obviously we have \((q_t^u - p_t^{\text{CwG}})/q_t^u \leq 1 \). Therefore the second term in the RHS of (EC.26) is bounded above as

\[
|q_t^u - p_t^{\text{CwG}}| |F^{-1}(q_t^u) - F^{-1}((p_t^{\text{CwG}})^+)| /\tau - B_t \leq 4\sqrt{\log \tau/\tau} \leq F^{-1}(q_t^u) - F^{-1}((p_t^{\text{CwG}})^+) | q_t^u - p_t^{\text{CwG}} |^{1+1/\tau^2} + 4\delta \sqrt{\log \tau/\tau} + 1_{A_{3,1}}.
\]

Therefore we can upper bound \( \mathbb{E}[\partial \mathcal{R}_t(B_t, \mu_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{F}_t, q_t^u, q_t^u, \mathcal{E}_t, \mathcal{L}_t] \) as

\[
\mathbb{E}[\partial \mathcal{R}_t(B_t, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{F}_t, q_t^u, \mathcal{E}_t, \mathcal{L}_t] \\
\leq \mathbb{E}[\partial \mathcal{R}_t(B_t, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{F}_t, q_t^u, \mathcal{E}_t, \mathcal{L}_t] \\
\leq 4\sqrt{\log \tau/\tau} | q_t^u - p_t^{\text{CwG}} |^{1+1/\tau^2} + 4\delta \sqrt{\log \tau/\tau} + 1_{A_{3,1}}.
\]

Taking expectations on both sides we get that

\[
\mathbb{E}[\partial \mathcal{R}_t(B_t, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{E}_t, \mathcal{L}_t] \\
\leq \mathbb{E}[\partial \mathcal{R}_t(B_t, a_t^{\text{CwG}}, \theta_{\geq t})|\mathcal{E}_t, \mathcal{L}_t] \\
\leq 6 (\log \tau/\tau)^{1+1/\tau^2} /\tau + \delta /\tau + 24 (\log \tau/\tau)^{1+1/\tau^2} + 4\delta \sqrt{\log \tau/\tau} + \mathbb{P}(A_{3,1}|\mathcal{E}_t), \quad (\text{EC.32})
\]
where the second inequality follows from the fact that the first and the second term are bounded by (EC.30). This completes for event \( E_c \). From Lemma EC.8, it follows that \( \mathbb{P}(A_{\leq t}^c) \leq 2n(n+1)/\tau^4 \).

\[
\sum_{t=1}^{\bar{T}} \mathbb{P}(A_{\leq t}^c) \leq \sum_{t=\tau_0}^{\bar{T}} 2n(n+1)/\tau^4 \leq n(n+1) \varepsilon_0^6 (\ast) \leq 1
\]

where \((\ast)\) follows since if there are \( n \) gaps, there are \( n+1 \) clusters, and hence \( \varepsilon_0 \leq 1/(n+1) \).

Combining (EC.31) and (EC.32), for a constant \( C < \infty \) we have that

\[
E \left[ \partial R_t(B_t, a_t^{cW}, \theta_{\geq t}) \mid L_t \right] \leq C \left( \frac{1}{\beta} \right) \left( \frac{1}{\beta+1} \right) + C\delta \sqrt{\log \tau / \tau} + \mathbb{P}(A_{\leq t}^c) \quad (EC.33)
\]

An identical bound holds for the regret contribution from the event \( H_t \) where \( H_t = \{ P_t^{cW} \geq q_u \} \), by a symmetric argument. As a result, we can bound the expected total regret at time \( t \) as per

\[
E \left[ \partial R_t(B_t, a_t^{cW}, \theta_{\geq t}) \mid L_t \right] \leq C \left( \log \tau / \tau \right) \frac{1}{\beta+1} + C\delta \sqrt{\log \tau / \tau} + \mathbb{P}(A_{\leq t}^c)
\]

Note that the RHS for (EC.34) does not depend on the remaining budget \( B_t \) and hence we have a uniform bound on the expected compensation given below.

\[
\sup_{B_t \geq 0} \partial R_t(B_t, a_t^{cW}) \leq C \left( \frac{1}{\beta} \right) \left( \frac{1}{\beta+1} \right) + C\delta \sqrt{\log \tau / \tau} + \mathbb{P}(A_{\leq t}^c) \quad (EC.35)
\]

This further implies that

\[
E_{B_t} \left[ \partial R_t(B_t, a_t^{cW}) \right] \leq C \left( \frac{1}{\beta} \right) \left( \frac{1}{\beta+1} \right) + C\delta \sqrt{\log \tau / \tau} + \mathbb{P}(A_{\leq t}^c) \quad (EC.36)
\]

Now summing this bound from \( t = 1 \) to \( t = \tilde{T} \), we have, using (EC.33) that for a constant \( C < \infty \), we have that

\[
\sum_{t=1}^{\tilde{T}} E_{B_t} \left[ \partial R_t(B_t, a_t^{cW}) \right] \leq C \left[ (1 + 1/\beta)(\log T) \frac{1}{\beta+1} \frac{1}{\beta+1} T \frac{1}{\beta+1} \right] \cdot 1 \{ \beta > 0 \} + (\log T)^2 1 \{ \beta = 0 \}
\]

\[
+ C\delta \sqrt{T \log T}.
\]

Finally, consider time steps \( t \) such that \( \tilde{T} + 1 \leq t \leq T \). In the last \( 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \) time steps, we make use of the static allocation policy and as noted before the regret accrued during the static allocation policy is upper bounded by \( C \sqrt{\tau_0} = C \sqrt{\log(1/\varepsilon_0)/\varepsilon_0} \) for some universal constant \( C < \infty \). Combining the two parts completes the proof. \( \square \)
EC.2.6. Proof of Helper Lemmas

**Proof of Lemma EC.6.** Let us assume that \( p_t^{CE} \geq p_t^{HTG} + \sqrt{2 \log \tau / \tau} \). Now conditional on \( B_t \) and given the knowledge of \( p_t^{HTG} \), we know that there are \( B_t \) arrivals with quantile larger than \( p_t^{HTG} \) and \( \tau - B_t \) arrivals with quantiles less than \( p_t^{HTG} \). Let \( X_t \triangleq \text{Ber}(\tau, (p_t^{CE} - \sqrt{2 \log \tau / \tau})^+) \) with \( \mathbb{E}[X_t|B_t] = (\tau - B_t - \sqrt{2 \tau \log \tau})^+ \). Then we have that

\[
\mathbb{P}(p_t^{CE} \geq p_t^{HTG} + \sqrt{2 \log \tau / \tau}|B_t) \leq \mathbb{P}(X_t \geq \tau - B_t|B_t) \leq \mathbb{P}(X_t - \mathbb{E}[X_t|B_t] \geq \sqrt{2 \log \tau / \tau}|B_t) \leq 1/\tau^4
\]

where the last inequality follows from the Hoeffding inequality. It follows that \( \mathbb{P}(p_t^{CE} \geq p_t^{HTG} + \sqrt{2 \log \tau / \tau}) \leq 1/\tau^4 \). Analogously, we can show the same for the case of \( p_t^{HTG} \geq p_t^{CE} + \sqrt{2 \log \tau / \tau} \). \( \square \)

**Proof of Lemma EC.7.** We have that \( |p_t^{CwG} - p_t^{HTG}| = |p_t^{CwG} - p_t^{CE} + p_t^{CE} - p_t^{HTG}| \leq |p_t^{CwG} - p_t^{CE}| + |p_t^{CE} - p_t^{HTG}| \). By the definition of the algorithm we have that \( |p_t^{CwG} - p_t^{CE}| \leq \sqrt{2 \log \tau / \tau} \). Now conditional on \( B_t \), we have that event \( A_{1,t} \) implies the event \( A_{2,t} \) and hence we have that \( A_{2,t} \) implies \( A_{1,t} \) which implies that \( \mathbb{P}(A_{2,t}|B_t) \leq \mathbb{P}(A_{1,t}|B_t) \). Using the proof of Lemma EC.6, we have that \( \mathbb{P}(A_{2,t}|B_t) \leq 2/\tau^4 \) and the claim of the lemma follows. \( \square \)

**Proof of Lemma EC.8.** We have that \( A_{3,t} = \bigcup_{i,j:Q_i \cap Q_j \neq \emptyset} \{p_t^{CwG} \in Q_i, p_t^{HTG} \in Q_j\} \) where \( A^0 \) denotes the interior of the set \( A \). Consider the event \( \{p_t^{CwG} \in Q_i, p_t^{HTG} \in Q_j\} \) such that \( Q_i \cap Q_j = \emptyset \).

From the definition of \( p_t^{CwG} \) in Algorithm 1 and the fact that \( Q_i \cap Q_j = \emptyset \) implies that \( |p_t^{CE} - p_t^{HTG}| \geq \sqrt{2 \log \tau / \tau} \). Using Lemma EC.6 and the union bound completes the proof. \( \square \)

**Proof of Lemma EC.9.** Assume that \( p_t^{CwG} \leq q_t^u \), then according to the definition of \( p_t^{HTG} \), we have that \( p_t^{HTG} = q_t^u \). Compensation is provided only if \( q_t^u \in [p_t^{CwG}, p_t^{HTG}] \). Suppose that is the case, then we have that \( F^{-1}((p_t^{CwG})^+) \leq \theta_t \leq F^{-1}(p_t^{CwG}) = F^{-1}(q_t^u) = u_t \). The CwG policy would accept the candidate with ability \( \theta_t \) since \( \theta_t \geq F^{-1}((p_t^{CwG})^+) \) where as the HTG would want to reject the candidate with ability \( \theta_t \), because in the future it knows that it can select a candidate with ability at least \( u_t \geq \theta_t \). Hence to persuade the HTG, we need to compensate it \( u_t - \theta_t = F^{-1}(p_t^{HTG}) - \theta_t \) and maximum compensation can hence be \( F^{-1}(p_t^{HTG}) - F^{-1}((p_t^{CwG})^+) \). An analogous analysis can be done for the case when \( p_t^{CwG} \geq q_t^u \) which follows similarly. \( \square \)
EC.3. Proof of Corollary 2

Proof of Corollary 2. The discrete distribution considered is a \((\beta = 0, \varepsilon_0, \delta = 0)\)-clustered distribution for \(\varepsilon_0 = \min_{1 \leq i \leq m} \{f_i\}\). As done for the general case above, our analysis for the case of discrete distributions as considered in the Example 1 also follows in two parts. The regret accrued during the second part due to the static allocation policy is upper bounded by \(C \sqrt{\log(1/\varepsilon_0)/\varepsilon_0}\) for some universal constant \(C < \infty\). Next we will consider the first part. The argument for the first part will mirror the analysis presented in the proof of Theorem 2 except for one important improvement we make for this special case. Consider the regret contribution of sample paths satisfying \(\mathcal{L}_t \triangleq \{p_t^{\text{CWG}} \leq q^t_l\}\) as we did previously. (Again, there is a analogous analysis for the symmetric event \(\mathcal{H}_t \triangleq \{p_t^{\text{CWG}} > q^t_u\}\), which we omit to avoid repetition.) The only but important distinction in the case of discrete distributions is that on the event \(A_{3,t}\), which is that \(q^t_l, q^t_u\) and \(p_t^{\text{CWG}}\) are quantiles belonging to the same cluster, the compensation is given as \(F^{-1}(q^t_u) - F^{-1}(p_t^{\text{CWG}})\), however for discrete distributions, we have that \(F^{-1}(q^t_u) - F^{-1}(p_t^{\text{CWG}}) = 0\). Previously, in the general case, we had upper bounded \(F^{-1}(q^t_u) - F^{-1}(p_t^{\text{CWG}})\) by \(|q^t_u - p_t^{\text{CWG}}|^{/(1+\beta)} + \delta\) using Definition 1. Because \(F^{-1}(q^t_u) - F^{-1}(p_t^{\text{CWG}}) = 0\) on the event \(A_{3,t}\), from (EC.26), we have that

\[
E \left[ \partial R_t(B_t, a^{\text{CWG}}_t, \theta_{\geq t}) \mid F_t, q^t_l, q^t_u, \mathcal{L}_t, \mathcal{E}_t \right] \leq 1_{A_{3,t}} + 1_{A_{4,t}},
\]

because \(\sup_{B_t, a_t, \theta_{\geq t}} \partial R_t(B_t, a_t, \theta_{\geq t}) \leq 1\). Taking expectations on both sides, we have that

\[
E \left[ \partial R_t(B_t, a^{\text{CWG}}_t, \theta_{\geq t}) \mid \mathcal{L}_t \right] \leq \mathbb{P}(A_{3,t}^c) + \mathbb{P}(A_{4,t}^c) \leq \mathbb{P}(A_{3,t}^c) + C/\tau^2,
\]

Summing this upper bound from \(\tau = \tau_0\) to \(\tau = T\), we get that the summation is upper bounded by a universal constant \(C < \infty\) using (EC.33). Combining the regret accrued in the two parts, we get the required result.

□

EC.4. Proof of Corollary 3

Proof of Corollary 3. Since by assumption, there are no gaps in the distribution, we have that \(p_t^{\text{CWG}} = p_t^{\text{CE}}\) for all \(t\). Our analysis will follow along the same lines as the analysis for Theorem 2 with \(\delta = 0\). From (EC.26), we have that

\[
E \left[ \partial R_t(B_t, a^{\text{CWG}}_t, \theta_{\geq t}) \mid F_t, q^t_l, q^t_u, \mathcal{L}_t \right] \leq \frac{|F^{-1}(q^t_u) - F^{-1}(p_t^{\text{CWG}})|}{\tau}
\]
This is where our proof departs from the proof of Theorem 2. The fact that the CwG policy boils down to the CE policy when there are no non-trivial gaps simplifies the analysis to a great extent. Instead of considering two cases to bound the ratio $(\tau - B_t - 1)/(q_i^u)$, we can bound it much simply. From the definition of $p_i^{CE}$, we have that $p_i^{CE} = 1 - B_t/\tau$, which implies that $(\tau - B_t - 1)/(q_i^u)$ ≤ $p_i^{CE} \tau/(q_i^u \tau)$ ≤ 1 since $p_i^{CwG} = p_i^{CE}$ and we are considering the sample paths on which $p_i^{CwG} \leq q_i^u$. Since $F$ is a $(\beta, \varepsilon_0 = 1, \delta = 0)$-clustered distribution, we have that

$$\mathbb{E} \left[ \partial \mathcal{R}_t(B_t, a_i^{CwG}, \theta_{\geq t}) | \mathcal{F}_t, q_i^u, q_i^l, \mathcal{L}_t \right] \leq \frac{|q_i^u - p_i^{CwG}|^{1+\frac{1}{1+\beta}}}{\tau} + |q_i^u - p_i^{CwG}|^{1+\frac{1}{1+\beta}}$$

Taking expectations, we have that

$$\mathbb{E} \left[ \partial \mathcal{R}_t(B_t, a_i^{CwG}, \theta_{\geq t}) | \mathcal{L}_t \right] \leq \mathbb{E} \left[ |q_i^u - p_i^{CwG}|^{1+\frac{1}{1+\beta}} \right] / \tau + \mathbb{E} \left[ |q_i^u - p_i^{CwG}|^{1+\frac{1}{1+\beta}} \right] \leq C \left( \tau^{-\frac{1}{\alpha+1}} + \tau^{-\frac{1}{\alpha+1} - \frac{1}{2}} \right),$$

where the second inequality follows from the fact that $\mathbb{E} [|q_i^u - p_i^{CwG}|^\alpha] \leq C(T-t+1)^{-\alpha/2}$ for any $\alpha \in (0, 2]$ and the fact that $1/(1+\beta) \in (0, 1]$ and $1+1/(1+\beta) \in (0, 2]$. Recall that $\mathbb{E} [\partial \mathcal{R}_t(B_t, a_i^{CwG}, \theta_{\geq t})] \leq 2\mathbb{E} [\partial \mathcal{R}_t(B_t, a_i^{CwG}, \theta_{\geq t}) | \mathcal{L}_t]$]. Summing this over $T$ time steps gives us the regret scaling in Corollary 3. To complete the proof, we will prove that for all $t \leq T-1$ and $\alpha \in (0, 2]$, we have that $\mathbb{E} [|q_i^u - p_i^{CwG}|^\alpha] \leq C(T-t+1)^{-\alpha/2}$. This inequality follows from the Hoeffding inequality as shown below.

$$\mathbb{E} \left[ |q_i^u - p_i^{CwG}|^\alpha \right] = \int_0^\infty \mathbb{P} (|q_i^u - p_i^{CwG}|^\alpha \geq x) \, dx \leq 2 \int_0^\infty \exp \left( -2\tau x^{2/\alpha} \right) \, dx = 2^{-\alpha/2} \alpha \Gamma(1+2/\alpha) \tau^{-\alpha/2},$$

This completes the proof of Corollary 3. □

**EC.5. Recovering existing regret guarantees for RAMS**

In this section, we provide corollaries which show that RAMS attains near-optimal regret scaling for a variety of online resource allocation problems under different assumptions. As a first application of Theorem 3, we consider the multisecretary problem. For analytical simplicity, we consider a
minor variant of RAMS where for the first $\bar{T} = T - \lfloor 64 \log(1/\varepsilon_0) / \varepsilon_0^2 \rfloor$ time steps, we implement RAMS as stated in Algorithm 2 and in the final $\lceil 64 \log(1/\varepsilon_0) / \varepsilon_0^2 \rceil$ time steps, we implement a static threshold policy as done in the case of Algorithm 1. This minor variant of RAMS inherits the guarantees established in Theorem 2.

**Corollary EC.1 (β-dependent regret for multisecretary).** Consider the multisecretary problem with the candidate ability distribution $F$ being $(\beta, \varepsilon_0, \delta)$-clustered for some fixed $\beta \in [0, \infty), \varepsilon_0 \in (0, 1]$ and $\delta \in [0, 1]$. Fix the parameter $\eta > 2$ in Theorem 3. Assume that the number of sample paths drawn at time $t$ is sufficiently large, specifically $K_t \geq (T - t + 1)^{\eta + \nu}$ for some $\nu > 0$. Then there exists a constant $C \equiv C(F, \eta, \nu) < \infty$, such that for all $T \in \mathbb{N}$ and $B \in \mathbb{N}$, the regret for RAMS is bounded above as

$$\text{Regret}(B, T; \text{RAMS}) \leq C(1 + 1/\beta)(\log T)^{1/\beta} T^{1/\beta} - (1/\beta) \cdot 1 \{\beta > 0\} + C(\log T)^2 1 \{\beta = 0\}$$

$$+ C\delta \sqrt{T \log T} + C\sqrt{\log(1/\varepsilon_0)/\varepsilon_0}.$$

Next we zoom out from the multisecretary problem and consider the more general network revenue management and online matching problems. We present four assumptions under which these problems have been studied. These assumptions are stated in the notation introduced in this paper.

**Assumption EC.1 (Small number of types for NRM).** The type distribution $F$ is supported on a discrete set $\{(r_1, c_1), (r_2, c_2), \ldots, (r_n, c_n)\}$ with $c_{\theta,k} \in \{0, 1\}$ for all $\theta \in \{1, \ldots, n\}, k \in \{1, \ldots, d\}$.

**Assumption EC.2 (Infinitely many types for NRM with density bounded below).** The consumption random vector $c_{\theta}$ is bounded i.e. $\nu \leq \|c_{\theta}\|_{\infty} \leq \bar{\nu}$ for $0 < \nu \leq \bar{\nu} < \infty$ for all $\theta \in \Theta$. Conditional on the consumption vector $c_{\theta}$, the reward distribution $F_{\theta}$ is assumed to be $(\beta = 0, \varepsilon_0 = 1)$-clustered with reward random variable $r_{\theta}$ being bounded in $[0, 1]$.

**Assumption EC.3 (Infinitely many types for NRM).** The consumption random vector $c_{\theta}$ is supported on a small discrete set $\{c_1, \ldots, c_n\}$ with $c_{\theta,k} \in \{0, 1\}$ for all $\theta \in \{1, \ldots, n\}$ and $k \in \ldots$
Conditional on the consumption vector $c_\theta$, the reward distribution $F_\theta$ is assumed to be $(\beta = 0, \varepsilon_0)$-clustered distribution with $\varepsilon_0 \in (0, 1]$ and the reward random variable $r_\theta$ being bounded in $[0, 1]$.

**Assumption EC.4 (Small number of types for Online Matching).** The type distribution $F$ is supported on a discrete set of reward vectors $\{r_1, \ldots, r_n\}$ where $r_\theta \in [0, 1]^d$ for all $\theta \in \{1, \ldots, n\}$.

**Discussion of the assumptions.** Recall Assumptions 1 (a few discrete types) and 2 (continuous types) for the multisecretary problem. Assumptions EC.1 and EC.4 are a natural generalization of Assumption 1 in the context of network revenue management and online matching respectively and is often a standard assumption in this literature (Vera and Banerjee 2021, Bumpensanti and Wang 2020, Jasin and Kumar 2012). Assumptions EC.2 and EC.3 are a generalization of Assumption 2 for the NRM problem with multiple resources. Assumption EC.2 resembles the assumption studied in Bray (2022), however Assumption EC.2 is stronger than the one in Bray (2022) in the sense that Bray (2022) allows for arbitrarily small consumption vectors (i.e., $\nu = 0$) while Assumption EC.2 assumes that consumption vectors are bounded below. Additionally Bray (2022) allows for unbounded rewards while Assumption EC.2 assumes that the rewards are bounded in the interval $[0, 1]$. Note that we study a stronger version of the assumptions in Bray (2022) for the sake of technical simplicity and conjecture that RAMS will achieve the same logarithmic regret scaling under the assumptions studied in Bray (2022). The key similarity between Assumption EC.2 and the assumption studied in Bray (2022) is that both assumptions imply that the fluid problem is non-degenerate which enables the logarithmic regret scaling. Assumption EC.3 while being similar to Assumption EC.2 allows for degeneracy in the fluid problem and was recently studied by Jiang et al. (2022a). There are two key distinctions between Assumptions EC.2 and EC.3 (i) Assumption EC.3 only permits a few consumption types and (ii) Assumption EC.3 allows for gaps in the (conditional) reward distributions which in turn permits degeneracy in the fluid problem.

Theorem 3 tells us that RAMS inherits the regret guarantees previously established for other algorithms, under Assumptions EC.1-EC.4. This is formalized in the following corollaries. Note
that in all our regret guarantees provided below, the only scaling parameters are the time horizon $T$ and the budget $B$ and all other parameters are considered constant. Moreover, we emphasis that the distribution $F$ is initially fixed and its parameters do not scale with the scaling parameter $T$ and $B$. Therefore, the minimum probability parameter $\varepsilon_0$ for the distributions considered in Assumptions EC.2 and EC.3 is also fixed and subsumed in the constants presented below.

**Corollary EC.2 (Regret for NRM).** Consider the network revenue management problem with request distribution $F$. Fix the parameter $\eta > 2$ in Theorem 3. Assume that the number of sample paths drawn at time $t$ is large enough as per $K_t \geq (T-t+1)^{\eta+\nu}$ for some $\nu > 0$. Then there exists a constant $C \equiv C(F,\eta,\nu) < \infty$, such that for all $T \in \mathbb{N}$ and $B \in \mathbb{R}^d$, we have that

(a) (Constant Regret with few types) If $F$ satisfies Assumption EC.1, $\text{Regret}(B,T;\text{RAMS}) \leq C$.

(b) (Logarithmic Regret) If $F$ satisfies Assumption EC.2, $\text{Regret}(B,T;\text{RAMS}) \leq C \log T$.

(c) (Log-Squared Regret) If $F$ satisfies Assumption EC.3, $\text{Regret}(B,T;\text{RAMS}) \leq C \log^2 T$.

**Corollary EC.3 (Constant Regret for Online Matching).** Consider the online matching setting with request distribution $F$ satisfying Assumption EC.4. Fix the parameter $\eta > 2$ in Theorem 3. Assume that the number of sample paths drawn at time $t$ is large enough as per $K_t \geq (T-t+1)^{\eta+\nu}$ for some $\nu > 0$. Then there exists a constant $C \equiv C(F,\eta,\nu) < \infty$ such that for all $T \in \mathbb{N}$ and $B \in \mathbb{R}^d$, the regret for RAMS is bounded above as $\text{Regret}(B,T;\text{RAMS}) \leq C$.

**EC.6. Proofs Related to RAMS**

**EC.6.1. Proof of Claim 1**

*Proof of Claim 1.* Given any budget $B_t \geq 0$ and any sample path $\theta_{t+1}$, if the hindsight to go (HTG) policy decides to accept the request $\theta_t$, we can make it reject the request $\theta_t$ by paying a maximum compensation of $r_{max}$. On the flip side, the hindsight to go policy can extract at most $r_{max}\nu/\nu$ in the future for every resource $\theta_t$ makes use of, hence if the hindsight to go (HTG) policy wants to reject $\theta_t$, we can make it accept the request $\theta_t$ by paying a compensation of $dr_{max}\nu/\nu$ since the request $\theta_t$ can make use of at most $d$ resources. □
Using (5), for a simulated sample path \( \theta_{\geq t}^{(i)} \triangleq \{ \theta_t, \theta_{t+1}^{(i)} \} \), we have that
\[
\partial \mathcal{R}_t(B_t, a, \theta_{\geq t}^{(i)}) = V_{t+1}^{hs}(B_t; \theta_{\geq t}^{(i)}) - V_{t}^{hs}(B_t - c(\theta_t, a); \theta_{\geq t}^{(i)} + r(\theta_t, a))
\]
\[
= V_{t}^{hs}(B_t; \theta_{\geq t}^{(i)}) - Q_{t}^{hs}(B_t, a; \theta_{\geq t}^{(i)}).
\]

Note that the term \( V_{t}^{hs}(B_t; \theta_{\geq t}^{(i)}) \) does not depend on the action \( a \in \mathcal{A}(B_t, \theta_t) \) and hence we have
\[
\arg\max_{a \in \mathcal{A}(B_t, \theta_t)} K_t^{-1} \sum_{i=1}^{K_t} Q_{t}^{hs}(B_t, a; \tilde{\theta}_{\geq t}^{(i)}) = \arg\min_{a \in \mathcal{A}(B_t, \theta_t)} K_t^{-1} \sum_{i=1}^{K_t} \partial \mathcal{R}_t(B_t, a; \tilde{\theta}_{\geq t}^{(i)}),
\]
\[
(\text{EC.37})
\]
i.e., RAMS takes an action \( a \in \mathcal{A}(B_t, \theta_t) \) which minimizes the simulation-based estimate of the expected marginal compensation.

**EC.6.3. Proof of Theorem 3**

Given a budget \( B_t \) and a request \( \theta_t \) at time \( t \), from Algorithm 2, it follows that the action under the RAMS policy is given by:
\[
a_t^{\text{RAMS}} = \arg\max_{a \in \mathcal{A}} \frac{1}{K_t} \sum_{i=1}^{K_t} Q_{t}^{hs}(B_t, a; \tilde{\theta}_{\geq t}^{(i)}),
\]
\[
(\text{EC.38})
\]
where \( K_t \) denotes the number of simulated sample paths used at time \( t \), \( Q_{t}^{hs}(B_t, a; \theta_{\geq t}^{(i)}) \) is defined in (4), \( \tilde{\theta}_{\geq t}^{(i)} \triangleq \{ \theta_t, \tilde{\theta}_{t+1}^{(i)}, \tilde{\theta}_{t+2}^{(i)}, \ldots, \tilde{\theta}_T^{(i)} \} \) and \( \{ \tilde{\theta}_{t+1}^{(i)}, \tilde{\theta}_{t+2}^{(i)}, \ldots, \tilde{\theta}_T^{(i)} \} \) denote the \( i \)-th sequence of simulated sample paths. From Lemma 2, it follows that the action under the RAMS policy can be equivalently written as
\[
a_t^{\text{RAMS}} = \arg\min_{a \in \mathcal{A}} \frac{1}{K_t} \sum_{i=1}^{K_t} \partial \mathcal{R}_t(B_t, a; \tilde{\theta}_{\geq t}^{(i)}),
\]
\[
(\text{EC.39})
\]
where \( \partial \mathcal{R}_t(B_t, a; \tilde{\theta}_{\geq t}^{(i)}) = \max_{a \in \mathcal{A}} Q_{t}^{hs}(B_t, a; \theta_{\geq t}^{(i)}) - Q_{t}^{hs}(B_t, a; \tilde{\theta}_{\geq t}^{(i)}). \)

From the regret decomposition lemma of Vera and Banerjee (2021), it follows that
\[
\text{Regret}(B, T; \text{RAMS}) = \sum_{t=1}^{T} \mathbb{E}_{\mu_{\text{RAMS}}} \left[ \partial \mathcal{R}_t(B_t^{\text{RAMS}}, a_t^{\text{RAMS}}) \right].
\]
\[
(\text{EC.40})
\]
We note that $E_{B_t} [\partial R_t (B_t^{\text{RAMS}}, a_t^{\text{RAMS}})] \leq \sup_{B_t \geq 0} \partial R_t (B_t, a_t^{\text{RAMS}})$ for all $t \in \{1, 2, \ldots, T\}$. Now to prove to Theorem 3 it suffices for us to show that

$$\sup_{B_t \geq 0} \partial R_t (B_t, a_t^{\text{RAMS}}) \leq \Delta_t (\text{ALG}) + C(\eta, |A|, C) K_t^{-\frac{1}{\eta}},$$

where $\Delta_t (\text{ALG})$ is the uniform upper bound assumed in condition (i) at time $t$ under ALG. We will begin by upper bounding the quantity $\partial R_t (B_t, a_t^{\text{RAMS}})$. For some fixed parameter $\eta > 2$, conditional on the budget $B_t$, and request $\theta_t$, define the following “good” event $G_t$

$$G_t = \cap_{a \in A} \{ \left[ K_t^{-1} \sum_{i=1}^{K_t} \partial R_t (B_t, a; \theta^{(i)}_{\geq t}) - E [\partial R_t (B_t, a; \theta_{> t})] \right] \leq K_t^{-\frac{1}{\eta}} \}.$$  

(31)

Using the definition of $\partial R_t (B_t, a_t^{\text{RAMS}})$ and the tower property we have,

$$\partial R_t (B_t, a_t^{\text{RAMS}}) = E [\partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) | B_t] = E \left[ E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) | \theta_t, B_t \right] | B_t \right].$$  

(32)

Now we further write the inner (conditional) expectation $E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) | \theta_t, B_t \right]$ as

$$E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) | \theta_t, B_t \right] = E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) 1_{G_t} | \theta_t, B_t \right] + E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) 1_{\neg G_t} | \theta_t, B_t \right].$$

Now we have two terms (♣) and (♠) to bound. We begin by bounding the term (♣). We have that

$$E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) 1_{G_t} | \theta_t, B_t \right] \leq \left( K_t^{-1} \sum_{i=1}^{K_t} \partial R_t (B_t, a_t^{\text{RAMS}}; \theta^{(i)}_{\geq t}) + K_t^{-\frac{1}{\eta}} \right) 1_{G_t},$$

(33)

$$\leq \left( K_t^{-1} \sum_{i=1}^{K_t} \partial R_t (B_t, a_t^{\text{ALG}}; \theta^{(i)}_{\geq t}) + K_t^{-\frac{1}{\eta}} \right) 1_{G_t},$$

(34)

$$\leq \left( E \left[ \partial R_t (B_t, a_t^{\text{ALG}}; \theta_{> t}) | \theta_t, B_t \right] + 2K_t^{-\frac{1}{\eta}} \right) 1_{G_t},$$

(35)

where (a) follows from definition of event $G_t$ applied to the action $a_t^{\text{RAMS}}$, (b) follows from the fact that RAMS takes the action according to (EC.39), (c) follows from definition of event $G_t$ applied to the action $a_t^{\text{ALG}}$ and (d) follows from that fact that $1_{G_t} \leq 1$. Using this it follows that

$$\text{(♣)} = E \left[ \partial R_t (B_t, a_t^{\text{RAMS}}; \theta_{> t}) 1_{G_t} | \theta_t, B_t \right] \leq E \left[ \partial R_t (B_t, a_t^{\text{ALG}}; \theta_{> t}) | \theta_t, B_t \right] + 2K_t^{-\frac{1}{\eta}}.$$
Next we bound the term \((\bigstar)\). Define \(Y_i(a) \triangleq \partial \mathcal{R}_t(B_t, a; \theta_{\geq t}^i)\). From Assumption (ii) in Theorem 3 we have that \(\partial \mathcal{R}_t(B_t, a; \theta_{\geq t}) \leq C\) almost surely for all \(B_t \geq 0, a \in \mathcal{A}\) and \(\theta_{\geq t}\). Therefore we have that \(\{Y_i(a)\}_{i=1}^{K_t}\) are i.i.d random variables with \(|Y_i(a)| \leq C\) almost surely. Hence we have that

\[
(\bigstar) \leq CE[\mathcal{G}_t^\ast | \theta_t, B_t] = C\mathbb{P}(\mathcal{G}_t^\ast | \theta_t, B_t) \leq C(\eta, |\mathcal{A}|, C)K_t^{-\frac{1}{\eta}}
\]

where the last inequality follows from union bound and Hoeffding’s inequality as described below.

\[
\mathbb{P}(\mathcal{G}_t^\ast | \theta_t, B_t) \leq \sum_{a \in \mathcal{A}} \mathbb{P}\left(\left| K_t^{-1} \sum_{i=1}^{K_t} Y_i(a) - \mathbb{E}[Y_i(a)| \theta_t, B_t] \right| > K_t^{-\frac{1}{\eta}} \right),
\]

\[
\lessapprox 2|\mathcal{A}| \exp\left(-\frac{2K_t^{-2}}{C^2t}\right),
\]

\[
\lessapprox 2|\mathcal{A}| \exp\left(\frac{2K_t^{-2}}{\eta^2} / C^2\right),
\]

\[
\lessapprox C(\eta, |\mathcal{A}|, C)K_t^{-\frac{1}{\eta}}
\]

where (a) follows from union bound, (b) follows from Hoeffding’s inequality, (c) follows trivially, (d) follows for some appropriate constant \(C(\eta, |\mathcal{A}|, C)\) since \(\exp(-x) \leq C(p)x^{-p}\) for some \(p > 0\).

Given the bound on \((\bigstar)\) and \((\bigstar)\), we have that

\[
\mathbb{E}[\partial \mathcal{R}_t(B_t, a_{t, \text{RAMS}}^R; \theta_{\geq t}) | \theta_t, B_t] \leq \mathbb{E}[\partial \mathcal{R}_t(B_t, a_{t, \text{ALG}}^R; \theta_{\geq t}) | \theta_t, B_t] + C(\eta, |\mathcal{A}|, C)K_t^{-\frac{1}{\eta}}.
\]

Using (EC.42), we have that

\[
\partial \mathcal{R}_t(B_t, a_{t, \text{RAMS}}^R) \leq \partial \mathcal{R}_t(B_t, a_{t, \text{ALG}}^R) + C(\eta, |\mathcal{A}|, C)K_t^{-\frac{1}{\eta}}.
\]

Taking a supremum over the budget \(B_t \geq 0\), we have that

\[
\sup_{B_t \geq 0} \partial \mathcal{R}_t(B_t, a_{t, \text{RAMS}}^R) \leq \sup_{B_t \geq 0} \partial \mathcal{R}_t(B_t, a_{t, \text{ALG}}^R) + C(\eta, |\mathcal{A}|, C(F))K_t^{-\frac{1}{\eta}} \leq \Delta_t(\text{ALG}) + C(\eta, |\mathcal{A}|, C(F))K_t^{-\frac{1}{\eta}},
\]

where the last inequality follows from Assumption (i). This completes the proof. \(\square\)
EC.6.4. Proof of Corollaries [EC.1] [EC.2] and [EC.3]

From Theorem 3, recall that the regret upper bound for RAMS (or minor variants of RAMS) can be decomposed as a sum of the following two terms

\[
\text{Regret}(B, T; \text{RAMS}) \leq \sum_{t=1}^{T} \Delta_t(\text{ALG}) + C \sum_{t=1}^{T} K_t^{-\frac{1}{\eta}}
\]

where the constant \( C < \infty \) is a function of the parameter \( \eta \), size of the action set \(|A|\) and the distribution \( F \). Note that (\( \triangledown \)) is common across different problem settings and assumptions while (\( \diamondsuit \)) needs to be dealt with separately. For each Corollary [EC.1] [EC.2] and [EC.3], we have that

\[
K_t \geq (T - t + 1)^{\eta} - 1 \eta + \nu
\]

where \( \eta > 2 \) is a fixed parameter from Theorem 3 and \( \nu > 0 \) is a chosen parameter. This implies that

\[
K_t^{-\frac{1}{\eta}} \leq (T - t + 1)^{-1 - \frac{\nu}{\eta}} \leq \int_{1}^{T} x^{-1 - \frac{\nu}{\eta}} dx \leq C(\eta, |A|, F, \nu) \text{ since } \nu/\eta > 0.
\]

Since this is common across all the corollaries, we have that the contribution to regret due to the number of simulated sample paths \( K_t \) is a constant (depending on \( \eta, \nu, F \) and \(|A|\)). The only thing remaining to bound is (\( \diamondsuit \)) under different assumptions and problem settings.

**Proof of Corollary [EC.1]**. From Lemma [EC.5], it follows that for \( t \leq \tilde{T} = T - \lfloor 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rfloor \), we have that \( \sup_{B_t \geq 0, a_t \in A} \partial R_t(B_t, a_t; \theta_t) \leq C \left( (\log \tau/\tau) \frac{1}{2} + \frac{1}{2(1 + \beta)} \right) \) which implies that for \( t \leq \tilde{T}, \Delta_t(\text{CwG}) = C \left( (\log \tau/\tau) \frac{1}{2} + \frac{1}{2(1 + \beta)} \right) \). Summing \( \Delta_t(\text{CwG}) \) from \( t = 1 \) to \( t = \tilde{T} \), implies that the regret contribution is at most \( C(\log T)^{\frac{1}{2} + \frac{1}{2(1 + \beta)}} T^{\frac{1}{2}} \frac{1}{(1 + \beta)} \{(\beta > 0) + \log^2 T \{\beta = 0\} + \delta \sqrt{T \log(T)} \}. \) Since we are considering a variant of RAMS which employs a static allocation policy (same as the one deployed in Algorithm 1) for the last \( \lfloor 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rfloor \), the regret accrued over the last \( \lfloor 64 \log(1/\varepsilon_0)/\varepsilon_0^2 \rfloor \) time steps is upper bounded as \( C \sqrt{\log(1/\varepsilon_0)/\varepsilon_0} \). Adding up all the contributions (including due to (\( \triangledown \))), we attain the same regret scaling as in Theorem 2. \( \square \)

**Proof of Corollary [EC.2]**. For each of the Assumptions [EC.1] [EC.2] and [EC.3] we have that \( \sup_{B_t \geq 0, a_t \in A, \theta_t \geq 1} \partial R_t(B, a_t; \theta_t) \leq d \) for all \( t \in \{1, \ldots, T\} \) since the offline will need a compensation of atmost \( r_{\max} = 1 \) per resource in the future for accepting or rejecting the request \( \theta_t \). Since there are \( d \) fixed resources, the compensation is atmost \( d \). Now under different assumptions, we have different algorithms with different values for \( \Delta_t(\text{ALG}) \).
(a) **Under Assumption EC.1.** From (9) in [Vera and Banerjee (2021)](VeraBanerjee2021), we have that for the Bayes Selector algorithm described in Algorithm 2 of [Vera and Banerjee (2021)](VeraBanerjee2021),

\[ \sup_{B_t \geq 0} \partial R_t(B_t, \mu^\text{BayesSelector}_t) \leq \frac{d}{\exp(-c\tau)} := \Delta_t(\text{BayesSelector}) \] for \( t \leq T - T_0 \) where \( \tau = T - t + 1 \), and \( c, T_0 \) are constants which depend only on the distribution \( F \). Using the fact that in the last constant \( T_0 \), the regret accrued is atmost \( dT_0 \) and \( \int_1^T \frac{d}{\exp(-c\tau)}d\tau \leq C \), we have that the total regret accrued by RAMS under Assumption EC.1 is at most a constant \( C \) which depends on the parameters \( \eta > 2, \nu > 0 \), number of resources \( d \) and the distribution \( F \).

(b) **Under Assumption EC.2.** From Lemma 5, 8, 9 and 10 of [Jiang et al. (2022a)](Jiang2022a), for the Bid Price algorithm described in Algorithm 3 of [Jiang et al. (2022a)](Jiang2022a), we have that

\[ \sup_{B_t \geq 0} \partial R_t(B_t, \mu^\text{BidPrice}_t) \leq C/\tau := \Delta_t(\text{BidPrice}) \] for \( t \leq T - T_0 \) where \( \tau = T - t + 1 \) and \( C, T_0 \) are constants which depend only on the distribution \( F \). Using the fact that in the last constant \( T_0 \), the regret accrued is atmost \( dT_0 \) and \( \int_1^T C/\tau d\tau \leq C \log T \), we have that the total regret accrued by RAMS under Assumption EC.2 is at most \( C \log T \) where the constant depends on the parameters \( \eta > 2, \nu > 0 \), number of resources \( d \) and the distribution \( F \).

(c) **Under Assumption EC.3.** From Theorem 1 of [Jiang et al. (2022a)](Jiang2022a), for the Boundary Attracted algorithm described in Algorithm 2 of [Jiang et al. (2022a)](Jiang2022a), we have that

\[ \sup_{B_t \geq 0} \partial R_t(B_t, \mu^\text{BoundaryAttracted}_t) \leq C \log \tau/\tau := \Delta_t(\text{BoundaryAttracted}) \] for \( t \leq T - T_0 \) where \( \tau = T - t + 1 \) and \( C, T_0 \) are constants which depend only on the distribution \( F \). Using the fact that in the last constant \( T_0 \), the regret accrued is atmost \( dT_0 \) and \( \int_1^T C \log \tau/\tau d\tau \leq C \log^2 T \), we have that the total regret accrued by RAMS under Assumption EC.3 is at most \( C \log^2 T \) where the constant which depends on the parameters \( \eta > 2, \nu > 0 \), number of resources \( d \) and the distribution \( F \).

This concludes the proof for all three cases. \( \square \)

**Proof of Corollary EC.3.** The proof follows analogously to the proof of Corollary EC.2 under Assumption EC.1. \( \square \)
EC.7. Relating the order fulfillment problem to the multisecretary problem

EC.7.1. Motivating Example

Let’s explore the following example to illuminate our point: Consider two Amazon fulfillment centers, located respectively in Salt Lake City, Utah, and Sacramento, California, as presented in Figure EC.2. Both states have a total of over two thousand zip codes, which are spatially clustered and represent distinct demand locations.

The United States sees an estimated total demand volume of around eighty million Amazon packages delivered weekly. Without a precise state-wise breakdown of these deliveries, we can reasonably assume that combined deliveries in California and Utah amount to no more than five million each week. Based on these figures, we calculate a total demand volume \( T \) of \( 5 \times 10^6 \), and a total number of demand locations or types \( D \) of \( 2 \times 10^3 \).

Assuming uniform demand across these locations, our model suggests that at any given time \( t \), the probability of receiving a demand request from type \( j \) is \( D^{-1} \approx T^{-\frac{1}{2}} \), which scales with the total demand volume. This differs from settings studied previously, which considered atomic distributions with a few types and implicitly assumed that the probability of receiving a demand request at a given time was independent of the total demand volume - an assumption inconsistent with the example we have described.

Alternatively, we could consider the setting where infinitely many types exist over a contiguous support. However, this approximation falls short in the presence of natural geographical features like the Sierra Nevada desert which creates gaps, as depicted in Figure EC.2. Neither of these previously explored models satisfactorily fit this stylized order fulfillment problem. Instead, what we encounter is a scenario characterized by many types with gaps. Demand request at time \( t \) is independent of the total demand volume and hence does not align well with the aforementioned example. On the other extreme, one could consider the setting with infinitely many types over a contiguous support but clearly such an approximation is wanting in the presence of gaps introduced by natural geographical features like the desert in Nevada as shown in Figure EC.2. Therefore neither of the previously studied models are a good fit for this stylized order fulfillment problem. What we have are essentially many types with gaps.
EC.7.2. Stylized model of order fulfillment

Inspired by our example illustrated above, we consider a stylized order fulfillment problem with the demand locations being spatially distributed over the unit square $[0,1]^2$ and two fulfillment centers (FCs) denoted as FCA and FCB with a total inventory in the two warehouses being $T$. The initial inventory in FCA and FCB is denoted as $I^A_t$ and $I^B_t$ respectively. Now at each time $t$, a request $\xi_t$ arrives given by the coordinates $(x_t, y_t) \in [0,1]^2$ which is drawn from some spatial demand distribution $Q$ with measure $\mu_Q$. Given the inventory levels $I^A_t$ and $I^B_t$ at time $t$, the order fulfillment problem is to decide which fulfillment center to serve the request $\xi_t$ from. The goal is to minimize the total matching distance between the requests and the fulfillment center from which they are served. It is easy to see that this problem can be easily translated into the multisecretary problem. We will illustrate this correspond via an example as shown in Figure EC.3.

In the stylized example illustrated in Figure EC.3, we assume that the demand locations are uniformly distributed in regions $R_1$ and $R_3$ with no demand in region $R_2$. The two fulfillment
centers FCA and FCB are located at (0, 0) and (1, 1) respectively. Let $d_A((x, y)) = |x| + |y|$ and $d_B((x, y)) = |1 - x| + |1 - y|$ denote the (Manhattan) distance from the fulfillment centers FCA and FCB respectively. The hindsight optimal problem for the order fulfillment problem is the following integer program.

$$\min_{z_t} \sum_{t=1}^{T} d_A((x_t, y_t)) z_t + d_B((x_t, y_t))(1 - z_t)$$  \hspace{1cm} (EC.43)

s.t. \hspace{1cm} \sum_{t=1}^{T} z_t = I_A^1 \\
\hspace{1cm} z_t \in \{0, 1\}, \hspace{0.5cm} \forall t
$$

The objective in (EC.43) can be equivalently written as $\sum_{t=1}^{T} (d_A((x_t, y_t)) - d_B((x_t, y_t))) z_t + d_B((x_t, y_t))$ and hence we can cast the minimization problem into the following maximization problem.

$$\max_{z_t} \sum_{t=1}^{T} (d_B((x_t, y_t)) - d_A((x_t, y_t))) z_t$$  \hspace{1cm} (EC.44)

s.t. \hspace{1cm} \sum_{t=1}^{T} z_t = I_A^1 \\
\hspace{1cm} z_t \in \{0, 1\}, \hspace{0.5cm} \forall t
$$

Figure EC.3  Stylized example for order fulfillment with no demand from region $\mathcal{R}_2$
The optimization problem in (EC.44) is the multisecretary problem with reward \(\hat{r}((x_t, y_t)) = d_B((x_t, y_t)) - d_A((x_t, y_t))\) we consider in this work with appropriate scaling. Observe that \(\hat{r}((x_t, y_t)) \in [-2, 2]\), therefore we can scale the reward and define the types as \(\theta_t = (\hat{r}((x_t, y_t)) + 2)/4 \in [0, 1]\). We can translate the spatial demand distribution \(Q\) into the distribution over the types \(\theta_t\) as follows.

\[
\mathbb{P}(\theta_t \leq z) = \mathbb{P}(\hat{r}((x, y)) \leq 4z - 2)
\]

\[
= \mathbb{P}(d_B((x, y)) - d_A((x, y)) \leq 4z - 2)
\]

\[
= \mathbb{P}((1 - x + 1 - y) - (x + y) \leq 4z - 2)
\]

\[
= \mathbb{P}(x + y \geq 2 - 2z)
\]

Using the fact that demand locations are uniformly distribution in regions \(R_1\) and \(R_3\), we have that

\[
\mathbb{P}(\theta_t \leq z) = \begin{cases} 
\frac{25}{8}z^2 & z \in [0, \frac{2}{5}], \\
\frac{1}{2} & z \in [\frac{2}{5}, \frac{3}{5}], \\
1 - \frac{25}{8}(1 - z)^2 & z \in [\frac{3}{5}, 1].
\end{cases}
\]

Note that this is a \((\beta = 0, \varepsilon_0 = \frac{1}{2})\)-clustered distribution with a gap interval \([\frac{2}{5}, \frac{3}{5}]\). Note that the gap in the demand location for the order fulfillment translates into a gap in the type distribution for the multisecretary problem. Moreover for simplicity we assume that the demand locations in regions \(R_1\) and \(R_3\) are distributed over a contiguous support but we can further discretize these regions into many small types which are clustered close to each other. This captures the more realistic setting where various zipcodes are spatially close to each other. In the context of the multisecretary problem, this is captured via the \((\beta, \varepsilon_0, \delta)\)-clustered distribution (recall Definition 1).

**EC.8. Some details on the \((\beta, \varepsilon_0, \delta)\)-clustered distributions**

As mentioned in Section 3.1, there is some flexibility is how we may model a distribution or define clusters. Additionally, the parameter \(\delta\) allows us to model distributions with many small types. In
this section, we will discuss how the same distribution can have different characterizations due to different clustering and choice of parameters $\beta, \varepsilon_0$ and $\delta$. We will discuss this using two examples. For each of the examples, we will discuss two different possible clusterings and their impact on the regret guarantees. The two examples we will consider will be atomic distributions and let $F$ be the continuous limit of those atomic distributions described below.

$$F(x) = \begin{cases} 
-8(1/4 - x)^2 + 1/2, & 0 \leq x \leq 1/4 \\
1/2, & 1/4 \leq x \leq 3/4 \\
8(x - 3/4)^2 + 1/2, & 3/4 \leq x \leq 1.
\end{cases}$$

Note that it can be easily verified that the distribution $G$ above is a $(\beta = 1, \varepsilon_0 = 1/2, \delta = 0)$-clustered distribution. Let us consider two other atomic distributions with probability mass functions denoted as $p_1$ and $p_2$ respectively and defined using the parameters $\eta_1$ and $\eta_2$ as follows,

$$p_1(1/4 - k\eta_1) = p_1(3/4 + k\eta_1) = 16\eta_1^2/(4\eta_1 + 1), \forall k \in \{0, 1, \ldots, 1/4\eta_1\}$$

$$p_2(1/4 - k\eta_2) = p_2(3/4 + k\eta_2) = 16\eta_2^2/(4\eta_2 + 1), \forall k \in \{0, 1, \ldots, 1/4\eta_2\}$$

For $\eta_1 = 1/24$, we get the distribution with probability mass function $p_1$ is supported on twelve points and is an example of distribution with a few types (refer to center figure in Figure EC.4) with the minimum probability mass being 1/42. For $\eta_2 = 1/2400$, we get the distribution with probability mass function $p_2$ is supported on twelve hundred points and can be considered an example of many small points since the number of types are large (1200) and each type has a small probability mass (at most $2 \times 10^{-3}$) (refer to the right figure in Figure EC.4). Note that as $\eta_1, \eta_2 \to 0$, we have that $p_1, p_2 \to f$.

There are two natural ways that the distribution $p_1$ can be modelled as a $(\beta, \varepsilon_0, \delta)$-clustered distribution. First way is as a distribution with a few types (as modelled in Example 1), where we have twelve mass clusters $H = \bigcup_{k=0}^{5} \{k/24, (19+k)/24\}$ corresponding to the twelve points on which the distribution is supported with eleven gap intervals $G = (\bigcup_{k=0}^{4} (k/24, (k+1)/24) \cup ((19+k)/24, (20+k)/24)) \cup (5/24, 19/24)$. We can easily verify that the
distribution \( p_1(x) \) satisfies the conditions in Definition 1 with \( \beta = 0, \varepsilon_0 = \min_{x: p_1(x) > 0} p_1(x) = 1/42 \) and \( \delta = 0 \). The second way to model this distribution is by having only two mass clusters \( H_1 = [0, 1/4] \) and \( H_2 = [3/4, 1] \) with one gap interval \( G = (1/4, 3/4) \). By considering only two clusters, we have that \( \varepsilon_0 = 1/2 \) since the total probability mass in both the clustered is 1/2 each. It is easy to see that for any choice of \( \beta \in [0, \infty) \), to satisfy condition (a) in Definition 1 we must choose \( \delta = \eta_1 > 0 \). While both ways are valid in terms of modelling the distribution, the theoretical guarantees implied by the two different characterizations of the same distribution lead to two different regret scalings. Under the first way where \( p_1 \) is modelled as a \((\beta = 0, \varepsilon_0 = 1/42, \delta = 0)\)–clustered, we get constant regret scaling, while under the second way where \( p_1 \) is modelled as a \((\beta = 0, \varepsilon_0 = 1/2, \delta = 1/24)\)–clustered, we get that the regret will scale as \( \tilde{O}(\sqrt{T}) \). Note that these regret scalings not only follow from the bounds in Theorem 2 but also due to the fact that CwG algorithm in Algorithm 1 will operate differently under the two different characterizations of the same distribution \( p_1 \) since the gaps are defined differently under the two different characterizations.

![Figure EC.4](image)

**Figure EC.4** (Left) PDF \( f_\beta \) of distribution \( F \) (Center) a few types (Right) many small types

Coming to the distribution \( p_2 \), again there are two ways that the distribution \( p_2 \) can be modelled as a \((\beta, \varepsilon_0, \delta)\)–clustered distribution. Since strictly speaking, \( p_2 \) is an atomic distribution albeit with many types, we can model is similar to how we modelled an atomic distribution with a few types. Building on that, we would have that 1200 mass clusters \( H = \bigcup_{k=0}^{598} \{k/2400, (1801 + k)/2400\} \) with 1199 gap intervals \( G = (\bigcup_{k=0}^{598} (k/2400, (k + 1)/2400) \cup ((1801 + k)/2400, (1802 + k)/2400)) \cup (599/2400, 1801/2400) \). We can easily verify that the distribution \( p_2(x) \) satisfies the conditions in
Definition 1 with $\beta = 0, \varepsilon_0 = \min\{x : p_2(x) > 0\} p_2(x) = 1/360600$ and $\delta = 0$. The second way to model this distribution is having only two mass clusters $H_1 = [0, 1/4]$ and $H_2 = [3/4, 1]$ with one gap interval $G = (1/4, 3/4)$. By considering only two clusters, we have that $\varepsilon_0 = 1/2$. It is easy to verify that for $\beta = 1$ and $\delta = \eta_2$, we satisfy the condition (a) in Definition 1. Note that under the first way, we have that $\varepsilon_0$ is very small and for most reasonable and practical values of the time horizon $T$, we may have that $1/\varepsilon_0 \sim T$ and hence the theoretical guarantees implied by Theorem 2 may be vacuous. On the other hand, in the second characterization as $(\beta = 1, \varepsilon_0 = 1/2, \delta = \eta_2)$-clustered distribution, we have that $\delta \sim 1/\sqrt{T}$ for reasonable values of $T$ and implied regret scaling is $\tilde{O}(T^{1/4})$ (sublinear regret). Note that the CwG algorithm (Algorithm 1) operates differently under the two different characterizations of the same distribution $p_2$. 