Vibration Equations of Thick Rectangular Plates Using Mindlin Plate Theory

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Abstract: Problem statement: Rectangular steel plates are widely used in various steel structures and steel industries. For a proper design of steel plate structures and efficient use of material, the behavior, strength, buckling and post-buckling characteristics of plates should be accurately determined. Approach: Considering the significance of this matter, lateral vibration of thick rectangular plates was studied on the basis of mindlin plate theory. The exact characteristic equations for a plate which is single supported in two opposite edges are available in the literature. S-C-S-F boundary condition which covers all possible situations is selected in this study. Results: The plate frequencies were calculated for this boundary condition for a wide range of plate sizes and thicknesses. The plate mode shapes were obtained for different cases and the effect of changes in boundary conditions; size ratio and thickness on the vibration behavior of rectangular steel plates are studied. Conclusion/Recommendations: Since the results of this study is exact and without any approximation, the presented values can be used as a proper criteria to evaluate the error value of approximate methods which are used by engineers for design of steel plates. These results can provide a good gridline for efficient design and prevention of using high safety factors. Considering the wide range of steel rectangular plates, more sizes and thicknesses of plates can be studied. The behavior of plates with other boundary conditions can also be studied for future research.

Key words: Mindlin plate theory, vibration, thick plate, mode shape

INTRODUCTION

The Classic Plate Theory (CPT) provides a theoretical model of plate behavior which has some considerable advantages, which can be employed with confidence over a reasonable range of applications, but which also has significant limitations. The popularity of CPT arises from the fact that the bending behavior of a plate is expressed in terms of a single, fundamental reference quantity that is w, the lateral displacement of the middle surface. The Kirchhoff hypothesis is used in CPT that straight lines originally normal to the plate middle surface remain straight and normal during the deformation process. The consequence of using this hypothesis is that shear strain distribution through the plate thickness are uniform, but this cannot be so. To correct for this, one shear coefficient factor is introduced into the analysis and selection of these factors is of some significance.

The basic assumption of Mindlin plate theory is that a straight line originally normal to the plate middle surface is constrained to remain straight but not generally normal to the middle surface after deformations. The inclusion of shear deformation effects in Mindlin plate theory means that the two cross-sectional rotations \( \psi_x \) and \( \psi_y \) have to be considered as independent, fundamental reference quantities, in addition to w. Thus, three fundamental quantities are involved in Mindlin plate theory, against the one of CPT\(^{(1)} \).

The assumption of Mindlin plate theory implies that shear strain distribution through the plate thickness are uniform, but this cannot be so. To correct for this, one shear coefficient factor is introduced into the analysis and selection of these factors is of some significance\(^{(2)} \).

The present study is to determine the exact characteristics equations for the case of S-C-S-F. Considering the transverse shear deformation, Mindlin plate theory is used to derive the integrated equations of motion in terms of the stress resultant.
The frequency parameters which are calculated using the exact characteristic equations are obtained for this case, which can cover a wide range of plate aspect ratios $\eta$ and relative thickness ratio $\delta$. For the mentioned boundary condition S-C-S-F, Three dimensional mode shapes and their contour plots for $\eta = 2$ and $\delta = 0.1$ are shown.

**MATERIALS AND METHODS**

All the formulations provide here is for a rectangular plate of length $a$, width $b$ and uniform thickness of $h$. Such a plate is shown in Fig. 1.

The displacements along the $x_1$ and $x_2$ axes are respectively marked as $U_1$ and $U_2$ and the displacement in the direction perpendicular to plane of $x_1$ and $x_2$ is marked as $U_3$. According to Mindlin plate theory, the value of displacement components in these directions can be calculated by formulas 1:

$$
\begin{align*}
U_1 &= -x_1 \psi_1(x_1, x_2, t) \\
U_2 &= -x_3 \psi_2(x_1, x_2, t) \\
U_3 &= \psi_3(x_1, x_2, t)
\end{align*}
$$

Where $\psi_1$ and $\psi_2$ are the slope due to bending alone in the respective planes, $\psi_3$ is the transverse displacement and $t$ is the time. The strains in the form of tensor components can be derived from equation 1 and can be written as Eq. 2:

$$
\begin{align*}
\varepsilon_{11} &= -x_1 \psi_{1,1} \\
\varepsilon_{22} &= -x_2 \psi_{2,2} \\
\varepsilon_{33} &= 0 \\
\varepsilon_{12} &= -\frac{1}{2}(\psi_{1,2} + \psi_{2,1}) x_3 \\
\varepsilon_{13} &= -\frac{1}{2}(\psi_{1,3} - \psi_{3,1}) \\
\varepsilon_{23} &= -\frac{1}{2}(\psi_{2,3} - \psi_{3,2})
\end{align*}
$$

If $M_{11}$ and $M_{22}$ and $M_{12}$ are the bending and twisting moments per unit length and $Q_{1}$ and $Q_{2}$ are the shear forces per unit length, then the plate linear constitutive relationships can be expressed as Eq. 3:

$$
\begin{align*}
M_{11} &= -D(\psi_{1,2} + \nu \psi_{2,1}) \\
M_{22} &= -D(\psi_{2,2} + \nu \psi_{1,1}) \\
M_{12} &= -\frac{D}{2}(1-\nu)(\psi_{1,2} + \psi_{2,1}) \\
Q_{1} &= -k^2 G h (\psi_1 - \psi_{3,1}) \\
Q_{2} &= -k^2 G h (\psi_2 - \psi_{3,2})
\end{align*}
$$

Where $D = \frac{E h^3}{12(1-\nu^2)}$, $\nu$ as Poisson’s ratio and $E$ and $G$ as the modulus of elasticity and rigidity. The constant $k^2$ is the shear correction factor introduce to account for the non-uniformity of shear strain through the plate thickness.

The equations of motion can be derived from three dimensional equations of motion in the form of Eq. 4:

$$
\begin{align*}
\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= \rho \ddot{U}_1 \\
\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} &= \rho \ddot{U}_2 \\
\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} &= \rho \ddot{U}_3
\end{align*}
$$

where $\rho$ is mass density per unit volume. Since there is no shear force in the faces of the plate, the integration through the thickness of plate for equations 4 gives Eq. 5:

$$
\begin{align*}
M_{11,1} + M_{12,2} - Q_{1} &= \frac{1}{12} \rho h \omega^2 \psi_1 \\
M_{12,1} + M_{22,2} - Q_{2} &= \frac{1}{12} \rho h \omega^2 \psi_2 \\
Q_{1,1} + Q_{2,2} &= -\rho h \omega^2 \psi_3
\end{align*}
$$

If the coordinates are normalized to the plate planar dimensions, non-dimensional parameters can be calculated by Eq. 6:

$$
\begin{align*}
X_1 &= \frac{x_1}{a} \\
X_2 &= \frac{x_2}{b} \\
\delta &= \frac{h}{a} \\
\eta &= \frac{a}{b} \\
\beta &= \omega^2 \frac{\sqrt{\rho h}}{D}
\end{align*}
$$
where $\beta$ is frequency parameter. Equation 3 can now be written in dimensionless form as Eq. 7:

$$M_{i1} = -(\psi_{i1} + \nu \eta \psi_{21})e^{\text{aix}} = \frac{M_{i1}}{D}$$

$$M_{i2} = -(\eta \psi_{i1} + \nu \psi_{11})e^{\text{aix}} = \frac{M_{i2}}{D}$$

$$M_{i3} = -(\frac{1}{2} - \nu)(\eta \psi_{i1} + \psi_{21})e^{\text{aix}} = \frac{M_{i3}}{D}$$

$$Q_i = -(\psi_i - \psi_{i1})e^{\text{aix}} = \frac{Q_i}{K^2 G h}$$

$$Q_i = -(\psi_i - \psi_{i1})e^{\text{aix}} = \frac{Q_i}{K^2 G h}$$

In these equations, partial differentiation with respect to the normalized coordinates is represented by comma subscript. The parameters $\psi_1$, $\psi_2$ and $\psi_3$ can be given by Eq. 8:

$$\psi_i(X, X) = \psi_i(x, x, t)e^{\alpha \text{sin}}$$

$$\psi_i(X, X) = \psi_i(x, x, t)e^{\alpha \text{cos}}$$

$$\psi_i(X, X) = \psi_i(x, x, t)e^{\alpha \text{e}}$$

If the dimensionless stress resultants of Eq. 7 are substituted in Eq. 5, 9 can be derived:

$$\psi_{i1,11} + \eta \psi_{i2,21} + \frac{1}{1 + \nu} (\psi_{i1,11} + \psi_{i2,21}) - \frac{12K^2}{\delta^2}(\psi_1 - \nu \psi_{i1}) = -\frac{\beta^2 \delta^2}{6(1 - \nu)} \psi_1$$

$$\psi_{i2,11} + \eta \psi_{i2,22} + \frac{1}{1 + \nu} (\psi_{i1,12} + \psi_{i2,22}) - \frac{12K^2}{\delta^2}(\psi_2 - \nu \psi_{i2}) = -\frac{\beta^2 \delta^2}{6(1 - \nu)} \psi_2$$

$$\psi_{i3,11} + \eta \psi_{i3,22} + \frac{1}{1 + \nu} (\psi_{i1,13} + \psi_{i2,23}) = -\frac{\beta^2 \delta^2}{6(1 - \nu)} \psi_3$$

These equations can be solved if the functions $\psi_i$, $\psi_2$ and $\psi_3$ are written in the form of three dimensionless potentials $W_1$, $W_2$ and $W_3$ as Eq. 10:

$$\psi_i = (1 - \frac{2\alpha_i^2}{(1 - \nu)\alpha_i^2})W_{i1} + (1 - \frac{2\alpha_i^2}{(1 - \nu)\alpha_i^2})W_{i3} - \eta W_{i2}$$

$$\psi_2 = (1 - \frac{2\alpha_i^2}{(1 - \nu)\alpha_i^2})W_{i2} + (1 - \frac{2\alpha_i^2}{(1 - \nu)\alpha_i^2})W_{i3} - \eta W_{i1}$$

$$\psi_i = W_1 + W_2$$

The parameters $\alpha_1^2$, $\alpha_2^2$ and $\alpha_3^2$ can be calculated by Eq. 11:

$$\alpha_1^2 = \frac{\beta^2}{2} \left( \frac{\delta^2}{12} \left( \frac{1}{K^2(1 - \nu)} \right)^2 + \frac{4}{\beta^2} \right)$$

The governing equations of motion can be written as Eq. 12:

$$W_{i1} + \eta W_{i2} = -\alpha_i^2 W_i$$

$$W_{i2} + \eta W_{i3} = -\alpha_i^2 W_i$$

One set of the solutions for Eq. 12 can be Eq. 13:

$$W_i = [A_1 \sin(\lambda_i X_i) + A_2 \cos(\lambda_i X_i)] \sin(\mu_i X_i) + [B_1 \sin(\lambda_i X_i) + B_2 \cos(\lambda_i X_i)] \cos(\mu_i X_i)$$

$$W_i = [A_1 \sin(\lambda_i X_i) + A_2 \cos(\lambda_i X_i)] \sin(\mu_i X_i) + [B_1 \sin(\lambda_i X_i) + B_2 \cos(\lambda_i X_i)] \cos(\mu_i X_i)$$

$$W_i = [A_1 \sin(\lambda_i X_i) + A_2 \cos(\lambda_i X_i)] \cos(\mu_i X_i) + [B_1 \sin(\lambda_i X_i) + B_2 \cos(\lambda_i X_i)] \sin(\mu_i X_i)$$

In these equations, $A$ and $B$ are constants. $\lambda$ and $\mu$ can be found by Eq. 14:

$$\alpha_1^2 = \mu_i^2 + \eta^2 \lambda_i^2$$

$$\alpha_2^2 = \mu_i^2 - \eta^2 \lambda_i^2$$

$$\alpha_3^2 = \mu_i^2 + \eta^2 \lambda_i^2$$

It is obvious that for a simply supported edge, free edge and clamped edge Eq. 15-17 can be respectively written as:

$$\tilde{M}_{i1} = \tilde{Q}_i = 0$$

$$\tilde{M}_{i2} = \tilde{Q}_i = 0$$

$$\tilde{M}_{i3} = \tilde{Q}_i = 0$$
\[ \psi_1 = \psi_2 = \psi_4 = 0 \]  

### S-C-S-F boundary condition:  
This boundary condition is the most complicated case and covers all possible boundary conditions. 

For this case, equation 18 can be written:  

\[
\begin{align*}
\lambda_1 & = \frac{1}{\eta} \sqrt{\alpha_1^2 - m^2} \\
\lambda_2 & = \frac{1}{\eta} \sqrt{\alpha_2^2 - m^2} \\
\lambda_3 & = \frac{1}{\eta} \sqrt{\alpha_3^2 + m^2} \\
\end{align*}
\]

**RESULTS**

In this part, numerical calculations of the above equations are given to clarify the method. Poisson’s ratio is assumed to be equal to 0.3.  

The results have high accuracy and can be used for determining the accuracy of approximate methods. To illustrate the results, a typical 3D deformed mode shapes together with their corresponding deflection counter plots for plate with aspect ratio \( \eta = 2 \) and thickness ratio \( \delta = 0.1 \) are given in Fig 2.  

For different thickness to length ratios of \( \delta = 0.01, 0.05, 0.1, 0.115, 0.2 \) and aspect ratios of \( \eta = 0.4, 0.5, 2, 3, 1.15, 2, 2.5 \), the results are tabulated in Table 1. In Table 1, for every \( \delta \) and \( \eta \), the nine lowest values of frequency are displayed in ascending order.

| \( \delta \) | \( \eta = 0.4 \) | \( \eta = 0.5 \) | \( \eta = 0.7 \) | \( \eta = 0.8 \) | \( \eta = 0.9 \) | \( \eta = 1.0 \) | \( \eta = 1.2 \) | \( \eta = 1.5 \) | \( \eta = 2.0 \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |

Similarly, the results are given in detail in Table 2. In Table 2, for every \( \delta \) and \( \eta \), the nine lowest values of frequency are displayed in ascending order.

**Table 1:** First nine frequencies for rectangular thick plates with boundary condition S-C-S-F

| \( \eta \) | \( \delta \) | \( 0.4 \) | \( 0.5 \) | \( 0.7 \) | \( 0.8 \) | \( 0.9 \) | \( 1.0 \) | \( 1.2 \) | \( 1.5 \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |

Similarly, the results are given in detail in Table 2. In Table 2, for every \( \delta \) and \( \eta \), the nine lowest values of frequency are displayed in ascending order.
DISCUSSION

As it was mentioned before, the method which is used in this study is accurate and is based on the exact characteristic equations and no estimation is involved.

To assure the performance of this method, the results are compared to those of an approximate method which has acceptable accuracy Liew et al.\textsuperscript{[4]} in the case of a rectangular plate with $\delta = 0.001$. This comparison is tabulated in Table 2 for the first four frequencies. As it can be seen, the results are close which confirm the performance of the exact method. The minor differences is because of the approximations exist in the Liew non-exact method.

CONCLUSION

In this study, Mindlin plate Theory is used to investigate the free vibration of thick rectangular plates. The general characteristic equations and transversal deformations, frequencies and different mode shapes are presented for S-C-S-F boundary condition which covers all other boundary conditions. Considering the high applicability of rectangular steel thick plates and the exact results of this method, the method can be used by engineers who need the exact results for optimize plate design.

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