In Cohen generic extension, every countable OD set of reals belongs to the ground model

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Abstract

It is true in the Cohen generic extension of $L$, the constructible universe, that every countable ordinal-definable set of reals belongs to $L$.

Theorem 1. Let $a \in \omega^\omega$ be a Cohen-generic real over $L$. Then it is true in $L[a]$ that if $X \subseteq \omega^\omega$ is a countable OD set then $X \in L$.

One may expect such a result of any homogeneous forcing notion. For instance, Theorem 1 is true for the Solovay model (the extension of $L$ by Levy cardinal collapse up to an inaccessible cardinal [4]) — but by a different argument. One hardly can doubt that any typical homogeneous extension (Solovay-random, Sacks, Hehler, and the like) also satisfies the same result, but it’s not easy to manufacture a proof of sufficient generality.

On the contrary, non-homogeneous forcing notions may lead to models with countable OD non-empty sets of reals with no OD elements [2], and such a set can even have the form of a $\Pi^1_2$ $E_0$-equivalence class [3].

Proof. Let $C = \omega^{<\omega}$ be the Cohen forcing. First of all, it suffices to prove that (it is true in $L[a]$ that) if $X \subseteq \omega^\omega$ is a countable OD set then $X \subseteq L$. Indeed, as the Cohen forcing is homogeneous, any statement about sets in $L$, the ground model, is decided by the weakest condition.

There is a formula $\varphi(x)$ with an unspecified ordinal $\alpha_0$ as a parameter, such that $X = \{x \in \omega^\omega : \varphi(x)\}$ in $L[a]$, and then there is a condition $p_0 \in C$ such that $p_0 \subset a$ and $p_0$ $C$-forces that $\{x \in \omega^\omega : \varphi(x)\}$ is a countable set. Suppose to the contrary that $X \not\subseteq L$, so that $p_0$ also forces $\exists x (x \notin L \land \varphi(x))$.

There is a sequence $\{t_n\}_{n<\omega} \in L$ of $C$-names, such that if $b \in \omega^\omega$ is Cohen generic and $p_0 \subset b$ then it is true in $L[b]$ that $\{x \in \omega^\omega : \varphi(x)\} = \{t_n[b] : n < \omega\}$.

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where \( t[\alpha] \) is the interpretation of a \( C \)-name \( t \) by a real \( x \in \omega^\omega \). Let \( T \in L \) be the \( C \)-name for \( \{ t_n[\dot{a}] : n < \omega \} \). Thus we assume that \( p_0 \) forces

\[
T[\dot{a}] = \{ t_n[\dot{a}] : n < \omega \} = \{ x \in \omega^\omega : \varphi(x) \} \not\subseteq L,
\]

where \( \dot{a} \) is the canonical name for the \( C \)-generic real.

Let \( \dot{a}_{\text{lef}}, \dot{a}_{\text{rig}} \) be canonical \((C \times C)\)-names for the left, resp., right, of the terms of a \((C \times C)\)-generic pair of reals \( \langle a_{\text{lef}}, a_{\text{rig}} \rangle \).

**Corollary 2.** The pair \( \langle p_0, p_0 \rangle \) \((C \times C)\)-forces over \( L \) that \( T[\dot{a}_{\text{lef}}] \neq T[\dot{a}_{\text{rig}}] \).

**Proof.** \( L[a_{\text{lef}}] \cap L[a_{\text{rig}}] \cap \omega^\omega \subseteq L \) due to the mutual genericity of \( a_{\text{lef}}, a_{\text{rig}} \). \( \square \)

Now pick a regular cardinal \( \kappa > \alpha_0 \). Consider, in \( L \), a countable submodel \( M \) of \( L_\kappa \) containing \( \alpha_0 \) and all names \( t_n \) and \( T \). Let \( \pi : M \rightarrow \overline{M} \) be the Mostowski collapse onto a transitive set \( \overline{M} \).

**Corollary 3.** It is true in \( \overline{M} \) that \( \langle p_0, p_0 \rangle \) \((C \times C)\)-forces \( T[\dot{a}_{\text{lef}}] \neq T[\dot{a}_{\text{rig}}] \).

**Proof.** By the elementarity, this holds in \( M \). Further we have \( \pi(t_n) = t_n \) and \( \pi(T) = T \) because the names \( t_n \) and \( T \) belong to the transitive part of \( M \). \( \square \)

**Corollary 4.** If \( \langle a_{\text{lef}}, a_{\text{rig}} \rangle \) is a \((C \times C)\)-generic pair over \( \overline{M} \) with \( p_0 \subseteq a_{\text{lef}} \) and \( p_0 \subseteq a_{\text{rig}} \), then \( T[\dot{a}_{\text{lef}}] \neq T[\dot{a}_{\text{rig}}] \).

By the countability, there is a real \( z \in \omega^\omega \cap L \) satisfying \( z(j) = 0 \) for all \( j < \text{dom} \, p_0 \) and \( C \)-generic over \( \overline{M} \), so that \( \overline{M}[z] \) is a set in \( L \). Let \( x \in \omega^\omega \) be \( C \)-generic over \( L \), with \( p_0 \subseteq x \). Then, as \( z \in L \), the real \( y \) defined by \( y(k) = z(k) + x(k) \), \( \forall k \), is \( C \)-generic over \( L \) as well, and we have \( L[x] = L[y] \) and still \( p_0 \subseteq y \). It follows from (1) that \( T[\dot{a}_{\text{lef}}] = T[\dot{a}_{\text{rig}}] \) (an OD set of reals in \( L[x] = L[y] \)).

But on the other hand by the product forcing theorem and the choice of \( z \), the pair \( \langle x, y \rangle \) is \((C \times C)\)-generic over \( \overline{M} \), and hence \( T[\dot{a}_{\text{lef}}] \neq T[\dot{a}_{\text{rig}}] \) by Corollary 4 which is a contradiction. \( \square \) (Theorem 1)

**Remark 5.** The Solovay model \( [4] \) admits a somewhat stronger result established in \( [1] \), namely, any countable non-empty OD set of sets of reals consists of OD elements (sets of reals). We don’t know whether this is true in the Cohen generic extension \( L[a] \). \( \square \)

**Remark 6.** Is Theorem 1 true for other popular forcing notions like e.g. the random forcing? The proof above crucially employs the countability of the Cohen forcing. \( \square \)
References

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