Dilaton gravity black holes with regular interior

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Abstract

In a one parameter family of dilaton gravity theories which allow the coupling of the dilaton to gravity and to a $U(1)$ gauge field to differ, we have found the existence of everywhere regular spacetimes describing black holes hiding expanding universes inside their horizon.

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1. Introduction

Two-dimensional dilaton gravity theories have become very popular toy models for investigating issues related to black hole evaporation and quantum gravity in general ([1, 2]). One of the most discussed is the Callan-Giddings-Harvey-Strominger (CGHS) model defined by the two-dimensional action

$$S^{(2)}_1 = \int d^2x \sqrt{-g^{(2)}} e^{-2\phi} [R^{(2)} + 4(\nabla \phi)^2 + 4\lambda^2]. \quad (1.1)$$

This action is supposed to describe the s-wave perturbations along the infinite deep throat of an extremal four dimensional black hole, solution of the “string originated” low energy action

$$S^{(4)}_1 = \frac{1}{2\pi} \int d^4x \sqrt{-g^{(4)}} e^{-2\phi} [R^{(4)} + 4(\nabla \phi)^2 - \frac{1}{2} F^2] \quad (1.2)$$

where $F_{\mu\nu}$ is a $U(1)$ gauge field. These magnetically charged extremal black holes are essentially nonsingular. The singularity is confined to the bottom of an infinite throat along which the geometry is effectively two dimensional, being the radius of the transverse two spheres nearly constant (see for instance [1] and [4]). A standard Kaluza-Klein reduction along this infinite throat leads from the four dimensional action $S^{(4)}_1$ to the CGHS action $S^{(2)}_1$.

The virtue of the CGHS model is its exact solvability. Moreover, starting from this it was also possible to construct exactly solvable models which include the backreaction on the geometry of quantized fields (massless scalars) [5]. The process of black hole formation and evaporation in two dimensions should then correspond to the scattering of low-energy quanta off (quantum mechanically) stable extremal four dimensional black holes. Among these two dimensional semiclassical models, one of the most popular is due to Russo, Susskind and Thorlacius, the so called RST model [6].

In [7] it was found a simple generalization of this latter which is still exactly integrable. The classical part of the corresponding action is

$$S^{(2)}_n = \int d^2x \sqrt{-g^{(2)}} [e^{-\frac{2\phi}{n}} (R^{(2)} + \frac{4}{n} (\nabla \phi)^2) + 4\lambda^2 e^{-2\phi}] \quad (1.3)$$

$n$ is a parameter which we suppose here positive definite. Note that for $n = 1$ one obtains the CGHS action $S^{(2)}_1$. In this paper we shall find the four dimensional “ancestor” of the 2D theory described by $S^{(2)}_n$ and see how the value of the parameter $n$ deeply influences the inner structure of the black hole solutions.
2. Spherically symmetric black holes

The theory we shall consider in this paper is given by the following one parameter 4D action

\[ S^{(4)}_n = \frac{1}{2\pi} \int d^4x \sqrt{-g} \{ e^{-2\phi} [R^{(4)} + (6 - \frac{2}{n^2})(\nabla \phi)^2] - \frac{1}{2} e^{-\frac{2\phi}{n}} F^2 \} \],

(2.1)

which allows a different coupling of the dilaton with gravity and the \( U(1) \) gauge field. This action is related to \( S^{(4)}_1 \) by a rescaling of the dilaton \( \phi \to \frac{\phi}{n} \) and by a conformal transformation \( g_{\mu\nu} \to e^{2\alpha\phi} g_{\mu\nu} \) with \( \alpha = 1 - \frac{1}{n} \). The equations of motion which follow from the action \( S^{(4)}_n \) are

\[ R^{(4)}_{\mu\nu} - \frac{1}{2} R^{(4)} g_{\mu\nu} + 2(1 - \frac{1}{n^2})\nabla_\mu \phi \nabla_\nu \phi + 2\nabla_\mu \nabla_\nu \phi - e^{2(\frac{n-1}{n})\phi} F_{\mu\gamma} F^{\gamma}_{\nu} + \]

\[ + g_{\mu\nu}[(1 + \frac{1}{n^2})(\nabla \phi)^2 - 2\nabla^2 \phi + \frac{1}{4} e^{2(\frac{n-1}{n})\phi} F^2] = 0, \]

\[ R^{(4)} - (6 - \frac{2}{n^2})(\nabla \phi)^2 + (6 - \frac{2}{n^2})\nabla^2 \phi - \frac{1}{2n} F^2 e^{2(\frac{n-1}{n})\phi} = 0, \]

\[ \nabla_\mu (e^{-\frac{2\phi}{n}} F^{\mu\nu}) = 0, \]

(2.2) (2.3) (2.4)

which are obtained by varying \( S^{(4)}_n \) with respect to the metric, the dilaton and the Maxwell field respectively.

Spherically symmetric magnetically charged black hole solutions of these equations take the form

\[ ds^2 = e^{2\phi_0} (1 - r_0/r)^{1-n} [\frac{(1 - r_H/r)}{(1 - r_0/r)} dt^2 + \frac{dr^2}{(1 - r_H/r)(1 - r_0/r)} + r^2 d\Omega^2], \]

\[ e^{-\frac{2\phi}{n}} = e^{-\frac{2\phi_0}{n}} (1 - r_0/r), \]

\[ F_{\theta\varphi} = Q \sin \theta. \]

(2.5) (2.6) (2.7)

Here \( \phi_0 \) is an arbitrary constant corresponding to the value of the dilaton at infinity,

\[ r_H = 2M \]

(2.8)

is the location of the event horizon and

\[ r_0 = \frac{Q^2}{2M} e^{-\frac{2\phi_0}{n}}. \]

(2.9)

The surface \( r = r_0 \) represents, for \( n = 1 \), a singularity; for other values of \( n \) we postpone relevant discussion to section III.

\( Q \) represents the magnetic charge and \( M \) is related to the gravitational mass of the hole \( m_g = (r_H - nr_0)/2 \). They are chosen so that \( r_0 \leq r_H \) and \( nr_0 < r_H \) (this last
inequality ensures the gravitational mass to be positive). Finally, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the metric on the unit two-sphere.

Extremal black holes are defined by $r_0 = r_H$. In the case $n = 1$ these are well known \[4\] to be completely regular, the singularity being pushed off to infinite proper distance. The geometry along this “throat” is effectively two-dimensional: the radius of the transverse two-sphere is approximatively constant $r \simeq r_0 = 2M$.

We shall consider the general case ($n$ arbitrary), but solutions close to extremality. In view of the conformal relation between the actions $S_n^{(4)}$ and $S_1^{(4)}$, and, therefore, between the corresponding solutions of the field equations, we parametrize the four dimensional metric as

$$ds^2 = g_{ab}^{(2)} dx^a dx^b + \tilde{R}^2 d\Omega^2,$$  \hspace{1cm} (2.10)

where $g_{ab}^{(2)} (a, b = 1, 2)$ is the radial part of the metric and

$$\tilde{R} = \exp[(\frac{n-1}{n})\phi + \frac{\phi_0}{n}]r$$  \hspace{1cm} (2.11)

with $r \simeq r_0$ (down the throat).

Inserting the decomposition eq. (2.10) in the action $S_n^{(4)}$ of eq. (2.1) and performing a standard Kaluza-Klein reduction along the throat one obtains, after integration over the angular variables,

$$S_n^{(2)} = \int d^2x \sqrt{-g^{(2)}}[e^{-\frac{2\phi}{n}}(\tilde{R}^{(2)} + \frac{4}{n}(\nabla\phi)^2) + 4\lambda^2 e^{-2\phi}],$$  \hspace{1cm} (2.12)

where we have defined

$$4\lambda^2 \equiv \frac{2e^{-\frac{2\phi_0}{n}}}{r_0^2} - \frac{Q^2 e^{-\frac{4\phi_0}{n}}}{r_0^4}. $$  \hspace{1cm} (2.13)

The action $S_n^{(2)}$ of eq. (2.12) coincides exactly with the action of the two-dimensional theory introduced in ref. \[7\] (see our eq. (1.3)). The correspondence between the two-dimensional black hole solutions of the action $S_n^{(2)}$ and the near extremal four dimensional black hole solutions of $S_n^{(4)}$ can be easily established by introducing the expansion parameter $\epsilon$, which measures the deviation from extremality \[8\]

$$\epsilon = \frac{r_H}{r_0} - 1.$$  \hspace{1cm} (2.14)

It is also useful to define a new spatial coordinate $\sigma$ by

$$e^{2\lambda\sigma} = 2\lambda r.$$  \hspace{1cm} (2.15)
In the region $r - r_0 \ll r_0$, the solutions eqs. (2.3), (2.6) can be well approximated, in the limit $\epsilon \ll 1$, by

$$ds^2 = e^{2(1-n)\lambda \sigma}[-(1 - \epsilon e^{-2\lambda \sigma})dt^2 + \frac{d\sigma^2}{(1 - \epsilon e^{-2\lambda \sigma})} + \frac{1}{4\lambda^2}d\Omega^2], \quad (2.16)$$

$$\phi = -n\lambda \sigma \quad (2.17)$$

where we have set, as in ref. [8], $r_0 \equiv \frac{1}{2\lambda}$ for convenience.

Provided one identifies the parameter $\epsilon$ with the two-dimensional ADM mass $M$ via the relation

$$\epsilon = \frac{M}{\lambda} \quad (2.18)$$

the radial part of the metric eq. (2.14) coincides with the black hole solutions of the action $S_n^{(2)}$ (see ref. [7]).

Furthermore, as shown in Appendix A, also the axially symmetric dilaton Ernst like solutions, describing quasi extremal accelerating black holes, become, in the point particle limit, almost spherically symmetric as the horizon is approached, of the form given by eqs. (2.5), (2.6) and (2.7). On the basis of the previous discussion we can therefore conclude that the two-dimensional action $S_n^{(2)}$ describes the “low energy excitations” along the throat of extremal black holes, which can be either exactly spherically symmetric or (in the point particle limit) even accelerated by an external field.

3. Spacetime structure

Let us now analyse the spacetime structure of the spherically symmetric black hole solutions of $S_n^{(4)}$ given in the previous section. These solutions are labelled by the parameter $n$. This dependence, as we shall see, introduces profound qualitative changes in the resulting picture of the spacetime with respect to the case $n = 1$ which are worth to be emphasized.

We rewrite for convenience the line element of eq. (2.3) in the form

$$ds^2 = e^{2\phi_0} \left( \frac{r}{r + r_0} \right)^{1-n}[-(1 - \frac{\epsilon r_0}{r})dt^2 + \frac{(\frac{r}{r_0} + 1)^2}{(1 - \frac{\epsilon r_0}{r})(\frac{r}{r_0})^2}dr^2 + (r + r_0)^2d\Omega^2], \quad (3.1)$$

where we have simply shifted $r$ by a constant quantity $r_0$ and $\epsilon$ is given as in eq. (2.14).

The dilaton, on the other hand, reads

$$e^{\frac{2\phi}{\pi}} = e^{\frac{2\phi_0}{\pi}} \left( \frac{r}{r + r_0} \right). \quad (3.2)$$

The range of $r$ is now $0 < r < \infty$ and the event horizon is located at $r = \epsilon r_0$. 

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The \( n \) dependence of this family of solutions is essentially contained in the conformal factor \((\frac{r}{r+r_0})^{1-n}\), which becomes trivial when \( n = 1 \).

One should furthermore note that at infinity \((r \to \infty)\), irrespectively of the value of \( n \), the metric becomes asymptotically minkowskian and the dilaton approaches a constant value \( \phi_0 \).

The Ricci scalar for the metric eq. (3.1) is

\[
R^{(4)} = e^{-2\phi_0} r^{-2}(r + r_0)^{-n-3}[rr_0(\frac{1}{2} + 3\epsilon n + 3n - \frac{3}{2}n^2) + \frac{r_0^3}{2}(3n^2 - 1)].
\]

We shall first consider extremal black holes, namely \( \epsilon = 0 \), for which the horizon disappears and the Killing vector \( \frac{\partial}{\partial t} \) is everywhere timelike.

In the limit \( r \to 0 \) from eq. (3.3) we obtain that \( R^{(4)} \sim r^{n-1} \).

We can then infer that for \( n \geq 1 \) the surface \( r = 0 \) is regular and the all spacetime as well. One can show that this surface lies at infinite proper distance and for \( n > 1 \) timelike radial geodesics never go close to it as they bounce back to infinity \((r \to \infty)\) at turning points. Note also that for \( n = 1 \) \( R^{(4)} \to \text{const.} \), whereas for \( n > 1 \) \( R^{(4)} \to 0 \). The difference between the two cases is well represented in Figs. I. For \( n = 1 \), see Fig. Ia, we have the usual bottomless hole with constant transverse radius \((r \simeq r_0)\). For \( n > 1 \), on the other hand, the area of the two-sphere, being proportional to \( r^{1-n} \), increases indefinitely as \( r \to 0 \) (see Fig. Ib). Note however that the gravitational mass \( m_g \) measured by an asymptotic observer at large \( r \) is negative semidefinite for \( n \geq 1 \).

Finally, for \( n < 1 \) \( r = 0 \) is a singular surface. The singularity is lightlike and lies at a finite geodesic distance. It is caused by the two spheres crushing to zero area as \( r \to 0 \) (see Fig. II).

Let us consider now non extremal \((\epsilon \neq 0)\) black holes. These have a regular event horizon located at \( r = r_H = \epsilon r_0 \) which separates the space-time in an exterior (asymptotically flat) region \( r > r_H \) where the Killing vector \( \frac{\partial}{\partial t} \) is timelike and an interior region \( r < r_H \) where \( \frac{\partial}{\partial t} \) is spacelike. Note that within this “Kantowski-Sacks” region \( r \) is a timelike variable.

From eq. (3.3) the scalar curvature behaves in the limit \( r \to 0 \) as \( R^{(4)} \to r^{n-2} \).

For \( n < 2 \), \( r = 0 \) is a spacelike singularity located at finite geodesic distance. The causal structure of the space-time is the same as for the familiar Schwarzschild solution. Note however that the three-dimensional spatial curvature \( \delta R \) of the surfaces \( r = \text{const.} \) behaves

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1 By the way, in the axisymmetric case this is no longer true. The metric considered in appendix A at infinity becomes conformal to the dilaton Melvin universe [9].

2 This, like all the other results here presented, are confirmed by a careful analysis of the other curvature invariants \( R^{(4)} \) and \( R^{(4)} \).

3 Test particles are supposed to be free falling along the geodesics of the metric (3.1).
as $3 R \to r^{n-1}$. Therefore for $n < 1$ we have the usual cigar like singularity: the area of the two spheres crushes to zero (like in Fig. II). For $1 \leq n < 2$ the spatial sections have regular curvature, but the spacetime is still singular for $r = 0$, the divergence in $(4) R$ coming from the extrinsic curvature.

Finally, for $n \geq 2$ the inner region is everywhere regular and infinitely wide (like in Fig. Ib). A static observer does not notice anything peculiar about these black holes: they are characterized by the mass and charge and as usual their attractive gravitational force increases the more the observer is close to the horizon. For a courageous observer willing to investigate the inner structure of these bodies the picture appears more exciting. The event horizon represents as usual the no return surface, but as he crosses it he finds himself immersed in a new expanding universe which does not look like a Friedmann-Robertson-Walker universe: it is still homogeneous but not isotropic. Black holes of this kind represent wormholes to other universes, the door being as small as the horizon area.

4. Conclusions

In this paper we have put on physical ground the 2D action $S_n^{(2)}$ introduced in ref. [7]. This action describes perturbations along the throat of extreme 4D dilaton black holes arising as solutions of the action $S_n^{(4)}$. The presence of the parameter $n$ in this latter allows the coupling of the dilaton to gravity and to the Maxwell field to differ.

For a range of values of $n$ we have found solutions describing everywhere regular asymptotically flat spacetimes. The black holes in this case hide inside their horizons infinite expanding homogeneous universes of the Kantowski-Sacks type.

Caution is however required to take this conclusion as granted. The inspection of eq. (3.2) reveals that the dilaton diverges to $+\infty$ as $r \to 0$. So this region corresponds to strong coupling and quantum effects are likely expected to alter, even significantly, the nice classical picture here derived.

More general dilaton gravity models resulting in singularity free solutions are discussed in ref. [10].

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Appendix A. Axially symmetric solutions

As it has been shown in ref. [11] the action $S_1^{(4)}$ admits axially symmetric solutions describing magnetically charged black holes accelerated by a background magnetic field. These are extensions including the dilaton field of the well known Ernst solution [12]. In view of the conformal relation between the action $S_1^{(4)}$ and our action $S_n^{(4)}$, the axially symmetric Ernst like solutions of this latter can be simply written as

$$ds^2 = e^{2\phi_0} \Lambda(x, y)^{1-n} \left( F(y) \right)^{-n} \left\{ F(x)(G(y)dt^2 - \frac{dy^2}{G(y)}) + F(y) \left( \frac{dx^2}{G(x)} + \frac{G(x)}{\Lambda^2(x, y)} d\varphi^2 \right) \right\} ,$$

(A.1)

$$e^{-\frac{2\phi}{n}} = e^{-\frac{2\phi_0}{n}} \Lambda(x, y) \frac{F(y)}{F(x)},$$

(A.2)

$$A_{\varphi} = -\frac{e^{\phi_0}}{B \Lambda(x, y)} [1 + Bqx] + k$$

(A.3)

where

$$\Lambda(x, y) = \left( 1 + Bqx \right)^2 + \frac{B^2}{2A^2(x-y)^2} G(x) F(x),$$

(A.4)

$$F(\xi) = (1 + r_- A \xi), \quad G(\xi) = (1 - \xi^2 - r_+ A \xi^3).$$

(A.5)

For a detailed description of these solutions we refer to refs. [11], [1].

The metric contains essentially four parameters $r_+, r_-, A$ and $B$. $r_+$ and $r_-$ are related to the ADM mass and total charge respectively, whereas $A$ and $B$ to the acceleration of the hole and to the strength of the external magnetic field. Following [11] the parameters $r_+$ and $A$ are chosen such that $r_+ A < 2/(3\sqrt{3})$, so that the function $G(\xi)$ has three real distinct roots. Call these roots $\xi_2, \xi_3$ and $\xi_4$ and further define $\xi_1 \equiv -\frac{1}{r_- A}$. In the parameter region where $\xi_1 \leq \xi_2 < \xi_3 < \xi_4$ the surface $y = \xi_2$ represents the black hole horizon and $y = \xi_3$ the acceleration horizon.

$x$ and $\varphi$ are angular coordinates, $\xi_3 \leq x \leq \xi_4$ and $0 \leq \varphi \leq \frac{4\pi \Lambda(\xi_3)}{G'(\xi_3)}$. This last restriction comes from the requirement that the metric has no nodal singularity, i.e.

$$G'(\xi_3) \Lambda(\xi_4) = -G'(\xi_4) \Lambda(\xi_3).$$

(A.6)

The purpose to exploit in detail the form of these solutions is because one can show that along the throat of nearly extremal accelerating black holes (extremality is here defined as $\xi_1 = \xi_2$) the axially symmetric metric eq. (A.1), the dilaton eq. (A.2) and the Maxwell field (A.3) approach, as $y \to \xi_2$, the spherically symmetric simple form of eqs. (2.5), (2.6) and (2.7) provided that $r_+ A \ll 1$. In the case of exact extremality this last restriction on the parameters is not required.
To this end, let us introduce the quantity
\[ \epsilon \equiv \frac{\xi_1 - \xi_2}{\xi_2} \]  
which measures deviation from extremality. We suppose \( \epsilon \ll 1 \). One can then write from eqs. (A.5) in the limit \( y \to \xi_2 \)
\[ F(\xi) = r_- A(\xi - \xi_2(1 + \epsilon)), \]  
\[ G(\xi) = -(r_+ A)(\xi - \xi_2(\xi - \xi_3)(\xi - \xi_4)). \]  
Moreover, using the regularity condition eq. (A.6) and (A.8), (A.9) we get
\[ 1 + 2Bq\xi_2 + B^2q^2(\xi_2(\xi_3 + \xi_4) - \xi_3\xi_4) = 0. \]
Using this last result we can rewrite \( \Lambda \) as
\[ \Lambda(x, y) = \alpha(x - \xi_2) - B^2q^2(x - \xi_3)(x - \xi_4)\frac{(-\xi_2\epsilon)}{x - \xi_2}, \]
where we have defined
\[ \alpha = 2Bq + B^2q^2(\xi_3 + \xi_4). \]
As in ref. [9] we can now define the new coordinates \( r, t', \theta \) and \( \tilde{\phi} \)
\[ y = \frac{\xi_2}{r}, \]
\[ t = \frac{t'}{\sqrt{r_+ r_-(-\alpha\xi_2)(\xi_4 - \xi_2)(\xi_3 - \xi_2)}}, \]
\[ x = \frac{1}{2}[\xi_3 + \xi_4 - \frac{\xi_4 - \xi_3 + (\xi_4 + \xi_3 - 2\xi_2)\cos\theta}{\cos\theta + (\xi_4 + \xi_3 - 2\xi_2)/(\xi_4 - \xi_3)]}, \]
\[ \varphi = \frac{2\Lambda(\xi_3)}{G'(\xi_3)}\tilde{\varphi}, \]
where
\[ \hat{r}_+ = \frac{1}{A} \sqrt{\frac{r_-(-\alpha\xi_2)}{r_+(\xi_4 - \xi_2)(\xi_3 - \xi_2)}}. \]
At this point the metric eq. (A.1), the dilaton eq. (A.2) and the Maxwell field (A.3) in terms of the new variables read
\[ ds^2 = e^{2\phi_0}(-\xi_2\alpha)^{-n}[1 + g(x)\frac{\epsilon}{\xi_2(1 - \frac{\epsilon}{\xi_2})}]^{1-n}. \]
\[
\{ (1 + \epsilon)^{-n}(1 + \frac{\epsilon}{1 - \frac{r}{\xi}})^{(1+n)}(1 - \frac{\hat{r}^+}{r})(1 - \frac{\hat{r}^+}{r}(1 - \epsilon))^{-n}\} [-dt'^2 + \frac{dr^2}{(1 - \frac{r}{r^+})^2}] + \\
(1 + \epsilon)^{1-n}(1 + \frac{\epsilon}{1 - \frac{r}{\xi}})^n(\hat{r}_+^2)[1 - \frac{\hat{r}^+}{r}(1 - \epsilon)]^{1-n}[d\theta^2 + \sin^2 \theta d\phi^2(1 - \frac{2g(x)}{\alpha} \epsilon \frac{1}{\xi_2 (1 - \frac{r}{\xi_2})^2})],
\]

\[e^{-\frac{2\phi}{\alpha}} = e^{-\frac{2\phi_0}{\alpha}}(-\xi_2 \alpha)[1 + \frac{g(x)}{\alpha} \frac{\epsilon}{\xi_2 (1 - \frac{r}{\xi_2})^2}] (1 + \epsilon) \frac{1 + \frac{\hat{r}^+}{r}(1 - \epsilon)}{(1 + \frac{\hat{r}^+}{r}(1 - \epsilon)), \quad (A.19)\]

\[A_{\phi} = \left[-\frac{e^{\phi_0} (1 + B q x)}{B \alpha (x - \xi_2)[1 + \frac{g(x)}{\alpha} \frac{\epsilon}{\xi_2 (1 - \xi_2)^2}] + k] \frac{2\Lambda(\xi_3)}{G'(\xi_3)} \right] \quad (A.20)\]

where \(g(x) = B^2 q^2 (x - \xi_3)(x - \xi_4)\) and \(x\) is considered as a function of \(\theta\) by eq. \(A.15\). The result is straightforward in the exact extreme case \((\epsilon = 0)\) as all \(x\) dependent parts in eqs. \((A.18)-(A.20)\) drop off \([9]\). In the quasi extreme case \((\epsilon \ll 1)\) let us consider the point particle limit (or equivalently the limit of small acceleration) by defining

\[\tau = r_+ A.\quad (A.21)\]

In the limit \(\tau \ll 1\) it is easy to see that the \(x\) dependent parts of the metric eq. \((A.18)\) are of order \(O(\tau \epsilon)\), this because \(\xi_2 \sim \frac{1}{\tau}\). To order \(O(\epsilon)\) the metric reads

\[ds^2 = e^{+2\phi_0 (-\xi_2 \alpha)^{-n}}(1 + \epsilon) \{(1 - \frac{\hat{r}^+}{r})(1 - \frac{\hat{r}^+}{r}(1 - \epsilon))^{-n}\} [-dt'^2 + \frac{dr^2}{(1 - \frac{r}{r^+})^2}] + \hat{r}_+^2[1 - \frac{\hat{r}^+}{r}(1 - \epsilon)]^{1-n}d\Omega^2\]

which defining \(\hat{r}_- \equiv \hat{r}_+(1 - \epsilon)\) becomes

\[ds^2 = e^{2\phi_0 (-\xi_2 \alpha)^{-n}}(1 + \epsilon)[1 - \frac{\hat{r}^-}{\hat{r}}]^{1-n}\{-\frac{(1 - \frac{\hat{r}^-}{\hat{r}})}{(1 - \frac{\hat{r}^+}{\hat{r}})} dt'^2 + \frac{dr^2}{(1 - \frac{\hat{r}^+}{\hat{r}})(1 - \frac{\hat{r}^-}{\hat{r}})} + \hat{r}^2_+d\Omega^2\}.\]

Similarly, for the dilaton and Maxwell field we find

\[e^{-\frac{2\phi}{\alpha}} = e^{-\frac{2\phi_0}{\alpha}}(-\xi_2 \alpha)[1 - \frac{\hat{r}^-}{\hat{r}}],\quad (A.24)\]

\[A_{\phi} = \hat{q}(1 - \cos \theta)\quad (A.25)\]

where \(\hat{q} \equiv \frac{\hat{r}^+_+ e^{\phi_0} (-\alpha \xi_2)^{-1/2}}{\sqrt{2}}\). The quasi extremal \((\epsilon \ll 1)\) point particle \((\tau \ll 1)\) axially symmetric solutions therefore become, as the horizon is approached, spherically symmetric of the simple form in eqs. \((2.3)\), \((2.6)\) and \((2.7)\), with the obvious replacement of \(r_0\) and \(r_H\) with \(\hat{r}_-\) and \(\hat{r}_+\) respectively.
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Figure caption:

Fig. I: qualitative picture of
a) spatial section of the $n = 1$ extreme ($\epsilon = 0$) black hole
b) spatial section of the $n > 1$ extreme black hole

Fig. II: spatial section of the $n < 1$ black hole
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