FEEDBACK VERTEX SET ON CHORDAL BIPARTITE GRAPHS

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Abstract. Let $G = (A, B, E)$ be a bipartite graph with color classes $A$ and $B$. The graph $G$ is chordal bipartite if $G$ has no induced cycle of length more than four. Let $G = (V, E)$ be a graph. A feedback vertex set $F$ is a set of vertices $F \subseteq V$ such that $G - F$ is a forest. The feedback vertex set problem asks for a feedback vertex set of minimal cardinality. We show that the feedback vertex set problem can be solved in polynomial time on chordal bipartite graphs.

1 Introduction

The feedback vertex set problem asks for a minimum set of vertices that meets all cycles of a graph. The feedback vertex set problem is a benchmark problem for fixed-parameter algorithms, exact algorithms, approximation algorithms, and for algorithms on special graph classes [16]. In this paper we show that the feedback vertex set problem can be solved in polynomial time for bipartite graphs without chordless cycles of length more than four.

The feedback vertex set problem can be solved in time $O(1.7548^n)$ [17]. If a graph $G$ has a feedback vertex set with at most $k$ vertices then so does any minor of $G$. It thus follows from the graph minor theorem that the problem is fixed-parameter tractable [39]. It is possible to reduce $k$-feedback vertex set to a quadratic kernel in polynomial time [42]. At the moment, the best algorithm for $k$-feedback vertex set seems to run in time $O(3.83^k n^2)$ [10].

The feedback vertex set problem is NP-complete [18] and remains so on bipartite graphs and on planar graphs. Recently, it was shown that the problem remains NP-complete on tree convex bipartite graphs [43]. There is a factor two approximation algorithm [4].

Note that the problem can be formulated in monadic second-order logic without quantification over subsets of edges and it follows that the problem can be solved in $O(n^3)$ time for graphs of bounded treewidth or rankwidth [11]. The problem can also be solved in polynomial time on, e.g., interval graphs, chordal graphs, permutation graphs, cocomparability graphs, convex bipartite graphs, and AT-free graphs [31,33,34,35]. For information on various graph classes we refer to [7,20,37,41].

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2 Preliminaries

If $A$ and $B$ are sets then we use $A + B$ and $A - B$ to denote $A \cup B$ and $A \setminus B$ respectively. For a set $A$ and an element $x$ we also write $A + x$ and $A - x$ instead of $A + \{x\}$ and $A - \{x\}$.

A graph is a pair $G = (V, E)$ where $V$ is a finite, nonempty set and where $E$ is a set of two-element subsets of $V$. We call the elements of $V$ the vertices or points of the graph. We denote the elements of $E$ as $(x, y)$ where $x$ and $y$ are vertices. We call the elements of $E$ the edges of the graph. If $e = (x, y)$ is an edge of a graph then we call $x$ and $y$ the endpoints of $e$ and we say that $x$ and $y$ are adjacent. The neighborhood of a vertex $x$ is the set of vertices $y$ such that $(x, y) \in E$. We denote this neighborhood by $N(x)$. The closed neighborhood of a vertex $x$ is defined as $N[x] = N(x) + x$. The degree of a vertex $x$ is the cardinality of $N(x)$. Let $W \subseteq V$ and let $W \neq \emptyset$. The graph $G[W]$ induced by $W$ has $W$ as its set of vertices and it has those edges of $E$ that have both endpoints in $W$. If $W \subset V$, $W \neq V$, then we write $G - W$ for the graph induced by $V \setminus W$. If $W$ consists of a single vertex $x$ then we write $G - x$ instead of $G - \{x\}$. We usually denote the number of vertices of a graph by $n$ and the number of edges of a graph by $m$.

A path is a graph of which the vertices can be linearly ordered such that the pairs of consecutive vertices form the set of edges of the graph. We call the first and last vertex in this ordering the terminal vertices of the path. We denote a path with $n$ vertices by $P_n$. To denote a specific ordering of the vertices we use the notation $[x_1, \ldots, x_n]$. Let $G$ be a graph. Two vertices $x$ and $y$ of $G$ are connected by a path if $G$ has an induced subgraph which is a path with $x$ and $y$ as terminals. Being connected by a path is an equivalence relation on the set of vertices of the graph. The equivalence classes are called the components of $G$. A cycle consists of a path with at least three vertices with one additional edge that connects the two terminals of the path. We denote a cycle with $n$ vertices by $C_n$. The length of a path or a cycle is its number of edges.

A clique in a graph is a nonempty subset of vertices such that every pair in it is adjacent. An independent set in a graph is a nonempty subset of vertices with no edges between them. A graph $G = (V, E)$ is bipartite if there is a partition $\{A, B\}$ of the set of vertices $V$ in two independent sets. One part of the partition may be empty. We use the notation $G = (A, B, E)$ to denote a bipartite graph with independent sets $A$ and $B$ and we call $A$ and $B$ the color classes of $G$. A bipartite graph $G = (A, B, E)$ is complete bipartite if any two vertices in different color classes are adjacent.

**Definition 1.** A bipartite graph $G = (A, B, E)$ is chordal bipartite if it has no induced cycle of length more than four.

Chordal bipartite graphs were introduced in [19]. They are characterized by the property of having a ‘perfect edge without vertex elimination ordering.’

**Definition 2.** Let $G = (A, B, E)$ be a bipartite graph. An edge $e = (x, y)$ is bisimplicial if $N(x) \cup N(y)$ induces a complete bipartite subgraph in $G$. 

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Let $G = (A, B, E)$ be a bipartite graph and let $(e_1, \ldots, e_m)$ be an ordering of the edges of $G$. Define $G_0 = G$, $E_0 = E$ and for $i = 1, \ldots, m$ define $G_i = (A, B, E_i)$ as the graph with $E_i = E_{i-1} - \{e_i\}$. Thus $G_i$ is obtained from $G_{i-1}$ by removing the edge $e_i$ but not its endvertices. The ordering $(e_1, \ldots, e_m)$ is a perfect edge without vertex elimination ordering if $e_i$ is bisimplicial in $G_{i-1}$ for $i \in \{1, \ldots, m\}$.

**Theorem 1 ([7, 26])**. A bipartite graph is chordal bipartite if and only if it has a perfect edge without vertex elimination ordering.

If $G$ is chordal bipartite then any bisimplicial edge can start a perfect edge without vertex elimination ordering. Thus Theorem 1 provides a greedy recognition algorithm for chordal bipartite graphs [7, 26, 27].

We also need the following characterization of chordal bipartite graphs.

A graph is chordal if it has no induced cycle of length more than three [21]. A famous theorem of Dirac shows that a graph is chordal if and only if every minimal separator is a clique [14] (see the next section for the definition of a minimal separator). A vertex is simplicial if its neighborhood induces a clique. A graph is chordal if and only if every induced subgraph has a simplicial vertex [14]. Thus the graph has a ‘perfect elimination ordering’ of the vertices by removing simplicial vertices from the graph one by one.

Let $k \geq 3$. A $k$-sun is a chordal graph with $2k$ vertices, partitioned into two ordered sets $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$. The graph induced by $C$ is a clique and, for $i \in \{1, \ldots, n\}$, $s_i$ is adjacent to $c_i$ and $c_{i+1}$ where $c_{n+1} = c_1$. A graph is strongly chordal if it is chordal and has no induced sun. The structure of strongly chordal graphs was first analyzed by Farber at about the same time during which the class of chordal bipartite graphs, as ‘totally balanced matrices,’ was investigated by various other authors. It turns out that the two classes of graphs have, essentially the same structure.

Consider a bipartite graph $G = (A, B, E)$. Let $G_A$ be the chordal graph obtained from $G$ by adding an edge between every pair of vertices in $A$.

**Theorem 2 ([12])**. A bipartite graph $G = (A, B, E)$ is chordal bipartite if and only if $G_A$ is strongly chordal.

Strongly chordal graphs have a special perfect elimination ordering of its vertices.

**Definition 3.** Let $G$ be a graph. A vertex $x$ is simple if the vertices of $N[x]$ can be ordered $x_1, \ldots, x_t$ such that, for $i = 1, \ldots, t - 1$, $N[x_i] \subseteq N[x_{i+1}]$.

**Theorem 3 ([2, 3, 9, 15, 29, 32])**. A graph is strongly chordal if and only if every induced subgraph has a simple vertex.

Let $G = (A, B, E)$ be chordal bipartite. It is easy to see that if $B \neq \emptyset$, then $B$ has a vertex that is simple in $G_A$ (see, e.g., [22, 25]).
The ordering of the vertices of a strongly chordal graph, obtained by repeatedly removing a simple vertex is called a ‘simple elimination ordering.’

More information on the structure of chordal bipartite graphs can be found in the books [6,7,20,37,41] and in various papers, e.g., [3]. We refer to the literature for many different kinds of characterizations. Chordal bipartite graphs can be recognized in \( O(n^2) \) time [27,40]. Several NP-complete problems can be solved in polynomial time on chordal bipartite graphs. Others, such as the domination problem, remain NP-complete [13,38].

**Definition 4.** Let \( G = (V,E) \) be a graph. A set \( F \subseteq V \) is a feedback vertex set of \( G \) if \( G - F \) has no induced cycle.

The feedback vertex set problem asks for a feedback vertex set of minimal cardinality. In this paper we show that the feedback vertex set problem can be solved in polynomial time on chordal bipartite graphs.

### 3 Separators in chordal bipartite graphs

In this section we analyze the structure of chordal bipartite by means of minimal separators.

**Definition 5.** Let \( G = (V,E) \) be a graph. A set \( S \subseteq V \) is a separator of \( G \) if \( G - S \) has at least two components.

If every vertex of a separator \( S \) has a neighbor in a component \( C \) of \( G - S \) then we say that \( C \) is close to \( S \). A separator \( S \) is minimal if \( G - S \) has two components that are close to \( S \).

**Remark 1.** A similar concept was introduced in [19]. In this paper the authors prove that a graph is chordal bipartite if and only if every ‘minimal edge separator’ is complete bipartite. Here, a minimal edge separator is a minimal separator which separates two edges with nonadjacent endpoints into distinct components.

The following lemma generalizes the result of [19].

**Lemma 1.** Let \( G = (A,B,E) \) be chordal bipartite. Let \( S \) be a minimal separator of \( G \). Then \( G[S] \) is complete bipartite.

**Proof.** Assume that \( S \) has two nonadjacent vertices \( x \in A \) and \( y \in B \). Since \( S \) is minimal there are two components \( C_1 \) and \( C_2 \) in \( G - S \) that are close to \( S \). For \( i \in \{1,2\} \) let \( p_i \) be a neighbor of \( x \) in \( C_i \) and let \( q_i \) be a neighbor of \( y \) in \( C_i \). Then \( p_i \in B \) and \( q_i \in A \) since \( G \) is bipartite. Since \( G[C_i] \) is connected there exist a path \( P_i \) in \( G[C_i] \) with terminals \( p_i \) and \( q_i \). We may choose \( p_i \) and \( q_i \) such that \( x \) and \( y \) have no other neighbors than \( p_i \) and \( q_i \) in \( P_i \). Since \( C_1 \) and \( C_2 \) are components of \( G - S \) no vertex of \( P_1 \) is adjacent to any vertex of \( P_2 \). It follows that \( P_1 + P_2 + \{x,y\} \) induces a cycle of length at least 6 which is a contradiction. This proves the lemma. \( \square \)
Lemma 2. Let $G = (A, B, E)$ be chordal bipartite. Let $S$ be a minimal separator of $G$ and let $C$ be a component of $G - S$ that is close to $S$. If $S \cap A \neq \emptyset$ then there exists a vertex $x$ in $C$ with

$$N(x) \cap S = S \cap A.$$ 

Proof. Let $C' \neq C$ be another component of $G - S$ that is close to $S$. Consider the subgraph induced by $S + C_1 + C_2$. For simplicity, denote this subgraph also by $G$. Assume that $S \cap A \neq \emptyset$. Then $E \neq \emptyset$ otherwise $S$ is not a minimal separator.

If $C$ contains only one vertex $x$ then $x$ is adjacent to all of $S$. Then $S \cap B = \emptyset$ and $N(x) \cap S = S \cap A$ because $G$ is bipartite and $C$ is close to $S$. Assume that $C$ contains at least two vertices.

Since $G$ is chordal bipartite it has a bisimplicial edge $e = (p, q)$. Assume that $p \in A$ and $q \in B$. We consider the following cases.

Assume that $p$ and $q$ are both in $S$. Because $C$ and $C'$ are close to $S$ the neighbor in $C$ and $q$ has a neighbor in $C'$. This is a contradiction since no vertex of $C$ is adjacent to any vertex of $C'$ and thus $e$ is not bisimplicial.

Assume that $p \in S$ and $q \in C'$ is close to $S$ has a neighbor in $C'$. This contradicts the assumption that $e$ is bisimplicial.

Assume that $p \in S$ and $q \in C'$. By the previous argument $C' = \{q\}$. Also $S \cap B = \emptyset$ since every vertex of $S$ has a neighbor in $C'$. Let $x$ be any neighbor of $p$ in $C$. Since $(p, q)$ is bisimplicial, and $N(q) = S$, $x$ is adjacent to all vertices of $S$ as well.

Assume that $p$ and $q$ are both in $C'$. Let $G'$ be the graph obtained from $G$ by removing the edge $e$ but not its endpoints. Then $G'$ is chordal bipartite. If the component $C'$ remains connected after removal of $e$ we can use induction since $S$ is a minimal separator of $G'$, $C$ is a component of $G' - S$ that is close to $S$ and $G'$ has fewer edges than $G$.

Assume that the removal of $e$ disconnects $C'$. Then one of $p$ and $q$ has only neighbors in $S$. First assume that $p$ has only neighbors in $S$. Let $D$ be the component of $G' - S$ that contains $q$. If every vertex of $N(p) \cap S$ has a neighbor in $D$ then $D$ is close to $S$ in $G'$ and we can apply induction as above. Assume that some vertex $s \in S$ has $N(s) \cap C' = \{p\}$. If $q$ has a neighbor $q'$ in $D$ then $s$ is adjacent to $q'$ which is a contradiction. Thus $C' = \{p, q\}$, $N(p) \cap S = S \cap B$, and $N(q) \cap S = S \cap A$. Now $S \cap A$ is a minimal separator of $G'$ with close components $C + (S \cap B) + \{p\}$ and $\{q\}$. Let

$$G'' = G' - ((S \cap B) \cup \{p\}).$$

Then $S \cap A$ is a minimal separator in $G''$ with close components $C$ and $\{q\}$. By induction $C$ has a vertex adjacent to all vertices of $S \cap A$.

Now assume that $q$ has only neighbors in $S \cap A$ in $G'$. Let $D$ be the component of $G' - S$ that contains $p$. Assume there exists a vertex $s \in S$ with $N(s) \cap C' = \{q\}$. If $p$ has a neighbor $p' \in D$ then $s$ is adjacent to $p'$ which is a contradiction. Thus in this case $C' = \{p, q\}$ and we obtain the result as above. Otherwise, every vertex of $N(q) \cap S$ has a neighbor in $D$. Then $S$ is a minimal separator in $G'$ and we can use induction.
Assume that \( p \) and \( q \) are both in \( C \). If the removal of the edge \( e = (p, q) \) does not disconnect \( C \) then \( S \) is a minimal separator in the graph \( G' \), obtained by removing \( e \), and \( C' \) and \( C \) are close to \( S \) in \( G' \). Otherwise, it follows as in the case where \( p \) and \( q \) are both in \( C' \) that either we can apply induction on a component \( D \subseteq C \) of \( G' - S \) or,

\[
C = \{p, q\} \quad \text{and} \quad N(p) \cap S = S \cap B \quad \text{and} \quad N(q) \cap S = S \cap A.
\]

This proves the lemma. \( \square \)

**Lemma 3.** Let \( G = (A, B, E) \) be chordal bipartite. Let \( S \) be a minimal separator and let \( C \) be a component of \( G - S \) that is close to \( S \). If \( C \) has an edge then it has an edge which is bisimplicial in \( G \).

**Proof.** First notice that the claim holds true when \( C \) has only two vertices; in that case, by Lemma 1, \( S + C \) induces a complete bipartite graph and the edge in \( C \) is bisimplicial. If \( S = \emptyset \) then \( C \) is a component of \( G \) and the claim follows since \( G[C] \) has a bisimplicial edge and this edge is also bisimplicial in \( G \).

Let \( C' \) be another component of \( G - S \) that is close to \( S \). Consider the subgraph induced by \( S + C + C' \). It suffices to prove the claim for this induced subgraph. For simplicity we call this induced subgraph also \( G \). Let \( \{p, q\} \) be a bisimplicial edge in \( G \) with \( p \in A \) and \( q \in B \). We consider the following cases.

Assume that \( p \in S \) and that \( q \in C \). The vertex \( q \) has a neighbor \( x \) in \( C \) since \( |C| > 1 \) and \( G[C] \) is connected. Since \( S \) is minimal, \( p \) has a neighbor \( y \in C' \). This is a contradiction since \( x \) and \( y \) are not adjacent.

Assume that \( p \in S \) and that \( q \in S \). The vertex \( p \) has a neighbor \( x \in C \) and the vertex \( q \) has a neighbor \( y \in C' \) since \( C \) and \( C' \) are close to \( S \). This is a contradiction since \( x \) and \( y \) are not adjacent.

Assume that \( p \in S \) and that \( q \in C' \). If \( q \) has a neighbor \( y \in C' \) then we derive a contradiction as above. Thus \( C' = \{q\} \) and \( N(q) = S \) and \( S \cap B = \emptyset \). Let \( \Omega = N(p) \cap C \). Then every vertex of \( \Omega \) is adjacent to every vertex of \( S \) since \( \{p, q\} \) is bisimplicial. Let \( C_1, \ldots, C_t \) be the components of \( G[C] - \Omega \). Assume that a component, say \( C_1 \) has at least two vertices. Let \( S' \subseteq S + \Omega \) be the set of vertices with a neighbor in \( C_1 \). We claim that \( S' \) is a minimal separator. First notice that \( C_1 \) is a component of \( G - S' \) and that every vertex of \( S' \) has a neighbor in \( C_1 \) by construction. Also, \( p \) and \( q \) are not in \( S' \) since they have no neighbors in \( C_1 \). Let \( C'' \) be the component of \( G - S' \) that contains the edge \( \{p, q\} \). Then every vertex of \( S' \) has a neighbor in \( C'' \) since it is adjacent to \( p \) or to \( q \). We can now use induction on the number of vertices in the component \( C \) and conclude that \( C_1 \) has an edge which is bisimplicial in \( G \).

Assume that every component of \( G[C] - \Omega \) has only one vertex. Notice that \( C - \Omega \) has only vertices in \( A \) since \( G[C] \) is connected and \( \Omega \subseteq N(p) \subseteq B \) since \( G \) is bipartite and \( p \in A \). Consider the vertices of \( C \cap A \) and \( \Omega \). Note that \( C \cap A \neq \emptyset \) since \( |C| > 1 \) and \( G[C] \) is connected and bipartite. The graph induced by \( (C \cap A) + \Omega \) has a bisimplicial edge. This edge is also bisimplicial in \( G \) since every vertex of \( \Omega \) is adjacent to every vertex of \( S \).

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Assume that \( p \in C' \) and that \( q \in C' \). If the removal of the edge \((p, q)\) from the graph leaves \( C' \) connected then the claim follows by induction on the number of edges in \( G \). Otherwise, after removal of the edge \((p, q)\), one of \( p \) and \( q \) has only neighbors in \( S \). Say \( p \) has only neighbors in \( S \). Let \( G' \) be the graph obtained from \( G \) by removing the edge \((p, q)\) and let \( D \) be the component of \( G' - S \) that contains \( q \). If every vertex of \( S \) has a neighbor in \( D \) then \( S \) is a minimal separator in \( G' \) with close components \( C \) and \( D \). In that case we can proceed by induction on the number of edges in \( G \). Assume some vertex \( s \in S \) has no neighbors in \( D \). Then \( s \) is adjacent to \( p \) since \( C' \) is close to \( S \). If \( q \) has a neighbor in \( D \) then we arrive at a contradiction since \( s \) is adjacent to this neighbor. We may now conclude that \( C' = (p, q) \).

The graph \( G' \) is chordal bipartite. Thus it has a bisimplicial edge \((a, b)\) with \( a \in A \) and \( b \in B \). Assume that \( a = p \). Then \( b \in S \). Let \( \Gamma \subseteq C \) be the set of neighbors of \( b \) in \( C \). Then every vertex of \( \Gamma \) is adjacent to every vertex of \( S \cap B \). Since every vertex of \( S \cap B \) is adjacent to \( a \) and \((a, b)\) is bisimplicial. Consider the components \( O_1, \ldots, O_t \) of \( G[C] - \Gamma \). Assume that \( |O_1| > 1 \). Let \( S' \subseteq S + \Gamma \) be the subset that has neighbors in \( O_1 \). Then \( S' \) is a minimal separator in \( G \). Since \( |O_1| < |C| \) we can use induction and conclude that \( G(O_1) \) has an edge which is bisimplicial in \( G \). Assume that all components \( O_i \) have only one vertex. Then \( C - \Gamma \) has only vertices in \( B \) since \( C \) is connected and \( \Gamma \subseteq A \). Also, \( C - \Gamma \neq \emptyset \) since \( C \) is connected and \( |C| > 1 \). Consider the subgraph \( G'' \) of \( G \) induced by \( C + (S \cap A) + q \). Notice that \( G'' \) is connected and that \( S \cap A \) is a minimal separator in \( G'' \) with close components \( C \) and \( \{q\} \). By induction on the number of vertices in \( G \) we may conclude that \( C \) has an edge which is bisimplicial in \( G'' \). This edge is also bisimplicial in \( G \) which follows from the fact that every vertex of \( \Gamma \) is adjacent to every vertex of \( S \cap B \).

Assume that \( a \neq p \) and that \( b \neq q \). Assume that \( a \in S \) and \( b \in S \). Then \( a \) has a neighbor in \( C \) and \( b \) is adjacent to \( p \), which is a contradiction. Assume that \( a \in S \) and that \( b \in C \). Since \( |C| > 1 \) and \( C \) is connected, \( b \) has a neighbor \( x \in C \). Then we obtain a contradiction since \( a \) is adjacent to \( q \) and \((x, q) \notin E \).

We conclude that \( a \in C \) and that \( b \in C \). Then \((a, b)\) is a bisimplicial edge in the graph \( G \) since \( a \) and \( b \) have only neighbors in \( C + S \).

This proves the lemma. \( \square \)

**Corollary 1.** Let \( G = (A, B, E) \) be chordal bipartite and let \( S \) be a minimal separator of \( G \). Let \( C \) be a component of \( G - S \) that is close to \( S \). Assume that \( S \cap A \neq \emptyset \) and that \( S \cap B \neq \emptyset \). Then there exist adjacent vertices \( x \) and \( y \) in \( C \) such that

\[
N(x) \cap S = S \cap A \quad \text{and} \quad N(y) \cap S = S \cap B.
\]

**Corollary 2.** Let \( G \) be chordal bipartite. The number of minimal separators in \( G \) is \( O(n + m) \).

**Remark 2.** A somewhat weaker upperbound for the number of minimal separators in a chordal bipartite graph, i.e., \( O(n + \binom{m}{2}) \), was obtained in [20] (see also [7]). Note that there exists an algorithm with polynomial delay that lists all the minimal separators of an arbitrary graph [23].
4 Maximal chordal bipartite graphs

Let \( k \) be a natural number. A k-tree is a chordal graph, defined recursively as follows \[5\]. A k-tree on \( k+1 \) vertices is a clique. Given a k-tree \( G \) with \( n \) vertices, one can obtain a k-tree with \( n+1 \) vertices by introducing a new vertex and by making that adjacent to a clique with \( k \) vertices in \( G \).

Note that k-trees have a decomposition tree of the following form. It is a rooted tree with points colored from a set of \( k+1 \) colors. The first \( k+1 \) vertices closest to the root form a simple path and all the points on this path have different colors. The rest of the tree branches arbitrarily and there is no restriction on the coloring of the points.

Note that we can assign a unique \((k+1)\)-clique to each point in the tree as follows. For the points that are in the simple path attached to the root it is a \((k+1)\)-clique on the points of the path. For any other point \( x \), the clique is the same as that of its parent, except that the point that has the same color as \( x \) is replaced by \( x \).

It follows that a decomposition tree provides a proper coloring of the vertices of a k-tree with \( k+1 \) colors, that is, no two adjacent vertices receive the same color. By the recursive definition of a k-tree, this coloring is unique up to a permutation of the colors.

**Lemma 4.** Let \( G = (V, E) \) be a k-tree. Let \( T \) be a decomposition tree for \( G \). Then for any subset \( S \subseteq \{1, \ldots, k+1\} \) the graph induced by the vertices with a color in \( S \) is an \((|S|−1)\)-tree.

**Proof.** Consider a vertex \( x \) with color \( i \). Consider the path \( P \) from \( x \) to the root. Then, for any \( j \neq i \), the vertex \( x \) is adjacent to that vertex with color \( j \neq i \) on \( P \) that is furthest from the root.

Let \( S \) be a subset of the colors. Construct a \((|S|−1)\)-tree as follows. The root-clique is the subset of the root-clique of \( T \) with colors in \( S \). For any vertex with a color in \( S \), make it adjacent to the vertex on its path to the root in \( T \) that is furthest from the root and that has a color in \( S \). \( \square \)

**Remark 3.** Note that in a decomposition tree of a sun-free k-tree, for every node the children are colored with at most two different colors. This holds true for the decomposition tree induced by any subset of the colors.

The following lemma is basically the same as Theorem [2]. We include it because it eases the description of the following decomposition of chordal bipartite graphs.

**Lemma 5 ([15],[24],[32]).** Let \( G = (A, B, E) \) be chordal bipartite. Let \( G^*_B \) be the graph obtained from \( G \) by making a clique of every neighborhood of a vertex in \( A \). Then \( G^*_B \) is strongly chordal.
Proof. We write $N^+(x)$ for the neighborhood of a vertex $x$ in $G_B^*$. Consider a vertex $x \in A$ which is simple in $G_B$. We claim that $x$ is simple in $G_B^*$. Assume this is not the case. Then there exist two vertices $p$ and $q$ in $N(x)$ that have ‘private neighbors’ $p'$ and $q'$ in $G_B^*$. That is,

$$p' \in N^+(p) - N^+(q) \quad \text{and} \quad q' \in N^+(q) - N^+(p).$$

Assume that $p' \in A$ and that $q' \in B$. Since $q'$ is adjacent to $q$ in $G_B^*$, there exists a vertex $q'' \in A$ which is adjacent to $q$ and $q'$. Note that $q'' \neq p'$ since $p'$ is not adjacent to $q$. Also, $q''$ is not adjacent to $p$ since $q'$ is not adjacent to $p$. Then $x$ is not simple in $G_B$. The case where $p'$ and $q'$ are both in $B$ is similar.

This proves that $G_B^*$ has a simple elimination ordering of the vertices in $A$. Let $x \in A$ be simple in $G_B^*$. If some vertex $y \in N(x)$ has no other neighbors in $A$ than $x$, then it is simple in $G_B^* - x$. Thus we obtain an augmented simple elimination ordering of all the vertices in $G_B^*$.

This proves the lemma. \qed

From [3] we have the following embedding theorem. An easier proof of this is described in [32].

**Theorem 4 ([3,32]).** Let $G = (A, B, E)$ be chordal bipartite and let $\omega + 1$ be the maximal cardinality of a clique in the graph $G_B^* - A$. There exists a sequence $T_0, \ldots , T_\omega$ such that, for $i = 0, \ldots, \omega$ the following holds.

1. $T_0$ is a spanning tree on the vertices of $B$ such that for each vertex $x \in A$ with $N(x) \neq \emptyset$, the set $N(x)$ induces a subtree of $T_0$;
2. For $i = 1 \geq 1$, $T_i = (B, E_i)$ is a strongly chordal $i$-tree and it spans the vertices of $B$;
3. For $i \geq 1$, each $(i+1)$-clique in $T_i$ is the union of two $i$-cliques in $T_{i-1}$;
4. For each vertex $x \in A$ with $N(x) \neq \emptyset$, the set $N(x)$ appears as a maximal clique in one of the $T_i$’s.

Consider the 0/1-adjacency matrix $Q$ with the rows indexed by the vertices in $A$ and the columns indexed by the vertices in $B$. Note that there may be multiple copies of identical rows in $Q$; the model does not reflect this fact. For any two rows in $Q$ we may add the intersection if it is not present already; the new matrix is the incidence matrix of a chordal bipartite graph. Possibly some rows in $Q$ correspond to $k$-cliques that are contained only in one $(k+1)$-clique (see, e.g., [23,36]). We can extend the $k$-trees with additional cliques such that each $k$-tree spans all vertices of $B$.

We describe the data structure that we use in the next section. Consider a strongly chordal $(k-1)$-tree $T_{k-1}$. Consider the maximal clique-tree $T$ for $T_{k-1}$, obtained as described at the start of this section. The $k$-tree $T_k$ is obtained by a procedure which can be described as follows [32]. Consider the linegraph $L$ of $T$; thus $L$ is a claw-free blockgraph. Let $T'$ be a spanning tree of $L$. Each vertex of $T'$ is a $(k+1)$-clique, which is the union of the two $k$-cliques that are the endpoints of the corresponding line in $T$. 

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Let \( H \) be a \( k \)-tree on \( n \) vertices. Then all maximal cliques in \( H \) have \( k + 1 \) vertices and there are \( n - k \) of them. This follows by a simple induction from the recursive definition of \( k \)-trees. It follows from Theorem 4 that a chordal bipartite graph \( G = (A, B, E) \) can be embedded into a maximal chordal bipartite graph \( G' = (A', B, E) \) with

\[
|A'|^* = \sum_{k=0}^{||B||} (||B|| - k) = \sum_{k=0}^{||B||} k = \left( \frac{||B|| + 1}{2} \right),
\]

where \( |A'|^* \) does not take into account multiple copies of vertices in \( A \) with the same neighborhood [32]. Here we assume that the matrix \( A \) has no row that contains only zeros (otherwise 1 should be added to the formula above).

Let \( G = (A, B, E) \) be chordal bipartite. A maximal embedding of \( G \) is a chordal bipartite graph \( G' \) obtained as described above. A chordal bipartite graph is maximal if its bipartite adjacency matrix has a maximal number of different rows.

5 Feedback vertex set on chordal bipartite graphs

\[\text{Lemma 6.} \ Let \ G = (A, B, E) \ be \ complete \ bipartite. \ Let \ F \ be \ a \ feedback \ vertex \ set \ of \ G. \ Then \]

\[|A - F| \leq 1 \ or \ |B - F| \leq 1.\]

\[\text{Proof.} \ If \ A \ and \ B \ both \ have \ two \ vertices \ that \ are \ not \ in \ F \ then \ G - F \ has \ an \ induced \ 4\text{-cycle.} \]

\[\text{Definition 6.} \ Let \ G = (A, B, E) \ be \ a \ bipartite \ graph. \ A \ hyperedge \ is \ the \ neighborhood \ of \ a \ vertex \ in \ A.\]

Note that different vertices in \( A \) may define the same hyperedge.

\[\text{Lemma 7.} \ Let \ G = (A, B, E) \ be \ chordal \ bipartite \ and \ let \ R \ be \ a \ hyperedge. \ Assume \ that \ there \ exist \ hyperedges \ A \ and \ B \ such \ that \]

\[(i) \ |A \cap R| = |B \cap R| = 1, \ and \]
\[(ii) \ A \cap R \neq B \cap R.\]

Then \( A \cap B = \emptyset. \)

\[\text{Proof.} \ To \ prove \ this \ we \ use \ the \ following \ characterization \ of \ chordal \ bipartite \ graphs \ (see, \ e.g., \ [11, \ Proposition \ 3]). \ Let \ G = (A, B, E) \ be \ a \ bipartite \ graph. \ Then \ G \ is \ chordal \ bipartite \ if \ and \ only \ if \ for \ any \ three \ vertices \ x, y \ and \ z \ in \ A \ at \ least \ one \ of \ N(x), N(y) \ and \ N(z) \ contains \ the \ intersection \ of \ the \ other \ two. \]

This characterization proves the lemma. \]

\[\text{Theorem 5.} \ There \ exists \ a \ polynomial-time \ algorithm \ that \ solves \ the \ feedback \ vertex \ set \ problem \ on \ chordal \ bipartite \ graphs.\]
Proof. Let $G = (A, B, E)$ be chordal bipartite. We describe the algorithm for an embedding of $G$ into a maximal chordal bipartite graph $M = (A', B, E)$. Each hyperedge, defined as the neighborhood of a vertex $x$ in $A'$, has a multiplicity, which is either zero if $x \notin A$ or else it is the number of vertices $x'$ with $N(x') = N(x)$.

We decompose $M_B$ as follows. First consider the hyperedges with a maximal number of vertices. Say these hyperedges have $k + 1$ vertices. The subgraph of $M_B - A$ is a $k$-tree $H_k$ and the maximal cliques are the hyperedges with $k + 1$ vertices. Decompose $H_k$ as described in Section 4.

Next consider the hyperedges with $k$ vertices. These define a $(k - 1)$-tree. For each maximal clique $C$ in $H_k$ consider the vertices $V_C$ in the subtree rooted at $C$. Decompose the subgraph of $M_B - A$ induced by $V_C$ into a $(k - 1)$-tree, rooted at $C$.

Continue this decomposition using the $i$-trees for $i = k, \ldots, 1$.

Let $C$ be a hyperedge and let $k + 1 = |C|$. The subtree $G_C$ of $C$ is the collection of hyperedges contained in the graph induced by the vertices in the subtree of the $k$-tree rooted at $C$ (including $C$). Note that some of these hyperedges may have cardinality larger than $k + 1$. Let $A_C$ denote the vertices $x$ in $A$ such that $N(x)$ is a hyperedge in the subtree rooted at $C$.

Consider a hyperedge $C = N(x)$ for some $x \in A$. We consider the following cases for the shape of a maximal forest in the graph induced by the vertices in the subtree rooted at $C$ and prove that the maximal cardinality can be determined in polynomial time.

First assume that a maximum forest $T$ contains the vertex $x$ and no other copies of $x$. If a subtree of $T$ that contains $x$ has vertices outside $C$ then it contains a vertex $y \in A_C$ such that $|N(y) \cap C| = 1$. Consider the vertices $b \in C$ and let $Q_b$ be the set of vertices $y \in A_C$ with $N(y) \cap C = \{b\}$. Consider a maximal subtrees of $C$ with a root $C'$, such that for all vertices $z \in A_{C'}$ the following holds:

$$|N(z) \cap C| \leq 1 \quad \text{and} \quad (|N(z) \cap C| = 1 \Rightarrow N(z) \cap C = \{b\}).$$

Find the maximal forest in the subtree rooted at $C'$ by a table look-up. By Lemma 7, we may add up the cardinalities of all these forests to find a maximal forest rooted at $x$.

Assume that the multiplicity of $x$ is more than one. Consider a forest that contains more than one copy of $x$. Then it contains at most one vertex in $C$. Let $b \in C$ and assume that the forest contains the vertex $b$. Let $C'$ be the root of the maximal subtree as defined above. By table look-up we may find the maximum forest in the subtree rooted at $C'$.

The vertices in $C - b$ are in the feedback vertex set. To find the maximum forest in the subgraph induced by the vertices in the subtree rooted at $C$ that are not in the subtree rooted at $C'$ we proceed as follows. Remove the vertices of $C - b$. If we remove the corresponding columns from the maximal bipartite
adjacency matrix, the new matrix is, obviously, totally balanced and the intersection of any two rows is a row in the reduced matrix. It follows that we can use the same data structure, except that the multiplicities of some of the hyperedges change. If a hyperedge contains a vertex that is in the feedback vertex set, then the multiplicity of that hyperedge becomes zero. Consider a hyperedge that does not contain any vertices in the feedback vertex set. Increase the multiplicity by the number of vertices \( x \in A \) for which the reduced neighborhood is exactly that hyperedge. Some vertices \( x \in A \) may have their neighborhood contained in the feedback vertex set. That is, the row of \( x \) in the reduced matrix becomes zero. These vertices become isolated and can be added to any maximal forest. Rerun the dynamic programming algorithm for the subtrees rooted at \( C \) with the new multiplicities, except for the subtree rooted at \( C' \). Add up the cardinalities of these maximum forests and maximize over the choices of \( b \in C \).

For hyperedges in the subtree of \( C \) we update the maximum forest in the same manner. Let \( Q \) be a hyperedge in the subtree rooted at \( C \) and let \( q \in A \) be such that \( N(q) = Q \). Some hyperedges in the subtree of \( C \) that are not in the subtree rooted at \( Q \), intersect \( Q \) in one vertex. Update the maximal cardinality of a maximal forest that contains \( q \) as described above.

Consider the case where a maximum forest does not contain \( x \). Decrease the multiplicity of \( C \) by one and update the values for all hyperedges in the subtree of \( C \) as described above. Let \( |C| = k + 1 \). Then \( C \) contains exactly two hyperedges \( C_1 \) and \( C_2 \) of cardinality \( k \). Let \( C'_1 \) be a hyperedge of maximal cardinality in the subtree rooted at \( C'_1 \) and consider the decomposition tree rooted at \( C'_1 \) induced by the hyperedges contained in the subtree of \( C \).

Note that the number of calls to each hyperedge from one of its ancestors is bounded by a fixed polynomial. This proves that the algorithm terminates in a polynomial number of steps.

Lemma 8. Let \( G = (A, B, E) \) be a bipartite graph and let \( x \) and \( y \) be two vertices in \( A \). Assume that \( N(x) \subseteq N(y) \). Assume that there exists a feedback vertex set \( F \) in \( G \) with \( x \in F \) and \( y \notin F \). Let
\[
F' = (F - x) + y.
\]

Then \( F' \) is also a feedback vertex set.

Proof. Assume not. Let \( C \) be an induced cycle in \( G - F' \). Then \( x \in C \) otherwise \( C \) is an induced cycle in \( G - F \). Let
\[
C' = (C - x) + y.
\]
Then \( C' \) is a cycle in \( G - F \) and obviously, \( C' \) contains an induced cycle in \( G - F \). This is a contradiction. \( \Box \)

Let \( G = (A, B, E) \) be a chordal bipartite graph and let \( x \in A \) be a simple vertex in \( G_B \). Let \( [x_1, \ldots, x_\ell] \) be an ordering of \( N_G(x) \) such that
\[
\text{for } i = 1, \ldots, \ell - 1, \quad N_G(x_i) \subseteq N_G(x_{i+1}).
\]
It follows from Lemma [8] that there is a minimum feedback vertex set $F$ such that, for some threshold $t \in \{0, \ldots, \ell\}$,

$$
\forall 1 \leq i \leq \ell \quad x_i \in F \quad \text{if and only if} \quad i \leq t.
$$

6 Concluding remarks

Define the chordality of a graph $G$ as the length of a longest induced cycle in $G$. We are not aware of any class of graphs of bounded chordality on which the feedback vertex set problem is NP-complete.

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