(m,n)-type holographic dark energy models

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Abstract

We construct $(m, n)$-type holographic dark energy models at a phenomenological level, which can be viewed as a generalization of agegraphic models with the conformal-like age as the holographic characteristic size. For some values of $(m, n)$ the holographic dark energy can automatically evolve across $\omega = -1$ into a phantom phase even without introducing an interaction between the dark energy and background matter. Our construction is also applicable to the holographic dark energy with generalized future event horizon as the characteristic size. Finally, we address the issue on the stability of our model and show that they are generally stable under the scalar perturbation.
I. INTRODUCTION

Recent cosmological observations have disclosed the current accelerated expansion of the universe driven by the exotic energy with negative pressure, which is dubbed as dark energy (DE) \[1\,–\,4\]. The dark energy scenario has attracted a great deal of attention in the last decade. Despite of many efforts in this subject, the nature of DE is the most mysterious problem in modern cosmology. The simplest candidate of dark energy is ΛCDM model, in which $\omega = -1$ is constant. Although being consistent with all observations very well, this model undergoes the fine-tuning problem and the coincidence problem \[5\,–\,6\]. After this, a lot of dynamical DE models have been proposed to solve these problems (for recent reviews we refer to \[7\,–\,8\]). As a matter of fact, for any dynamical dark energy model it contains a free parameter $\omega$ to specify, which in first principle should be derived at a statistical level, like what we have done for the ordinary matter ($\omega_m = 0$) and radiation ($\omega_r = 1/3$). Unfortunately, we know little about the microscopic property of dark energy such that the statistical mechanics on dark energy is missing. As a result, one needs to look for some principle to govern the dynamics of the state parameter of dark energy such that the evolution of the universe can be uniquely determined. Recently the most popular strategy is probably applying the holographic principle \[9\,–\,12\]. Motivated by this principle, one proposed that in a cosmological setting the total energy of system with size $L$ should not exceed the mass of a black hole with the same radius, namely

$$L^3 \Lambda^4 = L^3 \rho_{\Lambda} \leq L M_p^2,$$

(1)

While saturating this inequality by choosing the largest $L$ it gives rise to a holographic energy density

$$\rho_{\Lambda} = 3c^2 M_p^2 L^{-2},$$

(2)

where $c$ is a dimensionless constant. One usually calls the dark energy satisfying equation (2) as holographic dark energy. Now, the key issue is how to choose the holographic characteristic scale $L$. During the past years, there are many possible choices in the literature \[13\,–\,23\], in which the holographic scale $L$ can be identified with the future event horizon \[18\], the conformal age of the universe \[20\] or the Ricci scalar of the universe \[21\]. Specially the model taking the conformal age of the universe is also dubbed as new agegraphic dark energy model. Recently, Huang and Wu proposed a new holographic dark energy model with a
conformal-like age of the universe as the scale $L$ in \[22\]. This model can be consistent with the history from the inflation to the current universe.

Motivated by above progress in this paper we intend to propose a new type of holographic dark energy model at a phenomenological level which is characterized by two numbers $(m, n)$. We will demonstrate that it is quite general to construct a holographic dark energy model with an age-like scale as the holographic scale $L$. Originally people intended to propose this scale under the condition that the corresponding dark energy should be responsible for the acceleration of the current universe. As a consequence, the holographic size was previously proposed to be the future event horizon or the (conformal) age of the universe in turn in literature. In this sense, all such scales are proposed at a phenomenological level. In analogy with these conventional models, perhaps the direct physical motivation of proposing such characteristic scales in our paper is still obscure, but we have generalized the previous holographic dark energy models with significant improvements. In addition, the new introduced parameters $(m, n)$ provide us more space in theory to fit the observational data. In particular, when $(m, n)$ take some specific numbers all the agegraphic-like dark energy models previously proposed in the literature can be recovered. We will investigate the general features of $(m, n)$-type holographic dark energy models in this paper.

In addition, the stability of any dark energy model is always an important issue. Previously the relevant investigations on the stability of holographic dark energy model and (new) agegraphic dark energy models have been appeared in \[24, 25\]. In particular, the recent work in \[26\] reveals that the traditional holographic dark energy model is stable, although the perturbation of holographic dark energy is nonlocal, different from a usual fluid whose stability was defined by the sound speed square. In this paper, we will investigate the stability of our $(m, n)$-type holographic dark energy models and present an affirmative answer to this issue.

Our paper is organized as follows. In Sec. 2, we construct the $(m, n)$ type holographic dark energy model with an age-like scale as the characteristic size and discuss its general properties during the various epochs of the universe. The interaction between the dark energy and the dark (background) matter($DM$) is discussed in Sec. 3 and the coincidence problem is addressed. In Sec. 4 we briefly remark that our construction is also applicable to the models with generalized future event horizon as the holographic size in the same spirit. In Sec. 5 we shall discuss the issue of the stability of our model. The conclusions
and discussions are given in Sec. 6.

II. \((m, n)\)-TYPE HOLOGRAPHIC DARK ENERGY MODELS

We start with the standard Friedmann equations in which DE and background constituent with constant of state \(\omega_i\), are assumed to be independent, without interaction between them.

\[
3M_p^2 H^2 = \rho_\Lambda + \rho_i, \tag{3}
\]

\[
\dot{\rho}_\Lambda + 3H\rho_\Lambda(1 + \omega) = 0, \tag{4}
\]

\[
\dot{\rho}_i + 3H\rho_i(1 + \omega_i) = 0, \tag{5}
\]

where \(H = \dot{a}/a\) is the Hubble factor. In particular, \(\omega_i = 0\) for pressureless matter, whereas \(\omega_i = 1/3\) for radiation. For convenience through this paper we denote the ratio \(\rho_i/\rho_\Lambda\) by \(r\) which is related to \(\Omega_\Lambda = \rho_\Lambda/(3M_p^2 H^2)\) by \(1 + r = 1/\Omega_\Lambda\). From the Friedmann equation we easily obtain a relation between the characteristic size \(L\) and the Hubble factor \(H\) as

\[
LH = \sqrt{1 + r}. \tag{6}
\]

Furthermore, from the equations of conservation, we have

\[
r' = 3(\omega - \omega_i)r, \tag{7}
\]

where \(r' \equiv \dot{r}/H = dr/dlna\). From Eqs. (3), (4) and (5) we can work out

\[
\dot{H} = -\frac{3}{2}(1 + \frac{\omega_i r + \omega}{1 + r})H^2. \tag{8}
\]

From Eqs. (6) and (8) one can find

\[
2\frac{L'}{L} = 3(1 + \frac{\omega + \omega_i r}{1 + r}) + \frac{r'}{1 + r} = 3(1 + \omega). \tag{9}
\]

We point out that the relations derived above are general and independent of the specific form of the holographic characteristic scale. Now, we intend to construct a \((m, n)\)-type holographic dark energy model, in which the characteristic scale \(L\) is proposed to be

\[
L = \frac{1}{a^m(t)} \int_0^t a^n(t')dt', \tag{10}
\]
with \((m, n)\) being a couple of real numbers (at phenomenological level they need not be integers.). In above definition we have adopted the scale factor \(a(t_0) = 1\) for our present universe. Taking the derivative with respect to \(\ln a\) on both sides of the equation, we find

\[
\frac{L'}{L} = -m + \frac{a^{n-m}}{HL}. \tag{11}
\]

This relation together with Eq.\((9)\) leads to the equation of state for \((m, n)\)-type holographic dark energy,

\[
\omega = -1 - \frac{2}{3}m + \frac{2}{3}a^{n-m} - \frac{2}{3}m + \frac{2}{3}c\sqrt{1 + \omega}. \tag{12}
\]

In the absence of the interaction between background matter and dark energy, Eqs. \((7)\) and \((12)\) govern the evolution of \(r\) and \(\omega\). Alternatively, one can rewrite the equation of motion in terms of \(\Omega_\Lambda\) as

\[
\Omega_\Lambda' = \Omega_\Lambda(1 - \Omega_\Lambda)(3 + 3\omega_i + 2m - \frac{2\sqrt{\Omega_\Lambda a^{n-m}}}{c}). \tag{13}
\]

Next, we intend to figure out some basic constraints on the values of \((m, n)\) through the investigation on the general properties of \((m, n)\)-type holographic dark energy during the different epochs of the universe.

**A. Radiation- or Matter-dominated epoch \((a \rightarrow 0)\)**

For a radiation-dominated or matter-dominated epoch, we find the Friedmann equation Eq.\((3)\) can be approximately written as,

\[
\rho_i \propto H^2 = A^2a^{-3(1+\omega_i)}, \tag{14}
\]

where \(A\) is a constant and \(\omega_i\) is the state parameter, specifically, \(\omega_i = 1/3\) for radiation and 0 for matter. This equation implies that the scale factor evolves as \(a \propto t^{\frac{2}{3(1+\omega_i)}}\). Thus, the holographic scale \(L\) can be explicitly integrated out as

\[
L = \frac{1}{A[n + \frac{3}{2}(1 + \omega_i)]}a^{n-m+\frac{2}{3}(1+\omega_i)}. \tag{15}
\]

This solution leads to an important relation, implying that the ratio appearing in Eq.\((12)\) approaches to a constant during the radiation-dominated or matter-dominated epoch.

\[
\frac{2a^{n-m}}{3HL} = \frac{2}{3}n + 1 + \omega_i. \tag{16}
\]
As a result, one can easily find that $\Omega_\Lambda$ during that epoch evolves as

$$\Omega_\Lambda = \left( n + \frac{3}{2} + \frac{3\omega_i}{2}\right)^2 c^2 a^{2m-2n}. \quad (17)$$

It is easy to check that the above equation is consistent with Eq.(13). Moreover, the state parameter of dark energy is going to a constant, which is

$$\omega = \frac{2}{3}(n - m) + \omega_i. \quad (18)$$

Obviously, the state parameter depends on the values of $(m, n)$. We have the following remarks on the constraints on the values of $(m, n)$.

- **If** $n > m$, then $\omega > \omega_i \geq 0$. It means $r$ will increase with the expansion of the universe such that the universe could never exit from a radiation-dominated or matter-dominated epoch. Thus this case is ruled out and we will not consider it in next sections.

- **If** $n < m$, then $\omega < \omega_i$ during the Radiation- or matter-dominated epoch. In particular, when $n - m = -1$, we have $\omega = -\frac{2}{3}$ and $r \propto a^{-2}$ for $\omega_i = 0$, while $\omega = -\frac{1}{3}$ and $r \propto a^{-2}$ for $\omega_i = 1/3$. This situation recovers the new agegraphic dark energy model [20] which is $(m, n) = (0, -1)$, and the conformal age-like holographic dark energy model which is $(m, n) = (4, 3)$ [22]. In addition, we notice that in this case as $a \to 0$, the ratio $r$ goes to infinity such that there is no constraint on the value of the constant $c$.

- **If** $m = n$, then $\omega = \omega_i$ and $\rho_\Lambda \propto \rho_i$. This situation is very subtle and previously a similar discussion has been presented for the old agegraphic dark energy model which corresponds to the special case with $(m, n) = (0, 0)$ [19]. Since $\omega = \omega_i$, the ratio between the dark energy and dark matter/radiation would be a constant

$$r = \frac{1}{c^2[n + \frac{3}{2}(1 + \omega_i)]^2 - 1} > 0. \quad (19)$$

For $\omega_i = 1/3$, its positivity requires

$$c < \frac{1}{m + 2}. \quad (20)$$

$\omega = \omega_i$ implies that dark energy might intend to track the behavior of the dominated ingredient in the early stage of the universe, and thus might have the same origin.
dark matter. This potential possibility of unifying dark matter and dark energy is very interesting. However, to implement this scenario one need introduce a mechanism to make dark energy deviate from dark matter and finally the impact of dark energy must be large enough to be responsible for the acceleration of the universe at late times. We remark that in the absence of such a mechanism this scenario is hard to be realized. This difficulty might be overcome by introducing a suitable interaction between dark energy and dark matter, but here we leave this issue for further investigation in future.

B. Future with $a \gg 1$

When the interaction between dark energy and dark matter is not taken into account we only consider the case of $n < m$. When $a \gg 1$, the asymptotic behavior of the universe in future will be described by the following equations

$$\omega = -1 - \frac{2}{3}m, \quad (21)$$

$$r' = -(3 + 2m)r. \quad (22)$$

First of all, if $m \leq -1$, then $\omega > -1/3$. It means that the universe will not stay in an accelerating phase for ever. An example discussed in the previous literature is taking the particle horizon as the holographic characteristic scale, which corresponds to $m = n = -1$.

- If $m > 0$, then the holographic dark energy will behave like a phantom field.
- If $m = 0$, then the holographic dark energy will approach a cosmological constant.

The specific example is the new agegraphic dark energy model with $n = -1, m = 0$.

- If $-1 < m < 0$, the holographic dark energy can drive the universe into an accelerating phase indeed. The key point is whether this choice will be consistent with our observational data about the present universe.

C. Present days

The most strict constraints come from the observation data on our present universe. Here, we only roughly estimate the possible values for $m$ and $c$. First of all, our current universe
has an accelerating expansion, which requires that
\[ 1 + r_0 + 3\omega_0 < 0. \tag{23} \]
We find it leads to
\[ m > -1 + \frac{r_0}{2} + \frac{1}{c\sqrt{1 + r_0}}. \tag{24} \]
Since \( c \) is a positive number, if we plug the current \( r_0 \simeq \frac{1}{3} \) into this inequality, then we find a bound for \( m \), which is
\[ m > \frac{5}{6}. \tag{25} \]
Conversely, for a given \( m \), we find the constant \( c \) is constrained by
\[ c > \frac{3\sqrt{3}}{6m + 5}. \tag{26} \]
If we further require \( \omega_0 \simeq -1 \), \( c \) can be uniquely fixed by equation, which is \( c = 1/(m\sqrt{1 + r_0}) \) \( (m > 0 \text{ only}) \). In particular, when \( (m, n) = (0, -1) \) and \( (m, n) = (4, 3) \), our above estimation is in a good agreement with the results obtained by more severe constraints from observation data\textsuperscript{[27, 28]}. This implies that other types such as \( (m, n) = (1, 0), (2, 1), (3, 2) \) can also fit the data very well.

As a summary, we find the basic constraints on the \((m, n)\)-type holographic dark energy are \( n < m \) and \( m > -5/6 \).

### III. HOLOGRAPHIC DARK ENERGY WITH INTERACTION

Although the nature of both DM and DE still remains a mystery, the possibility that DE and DM can interact with each other has been widely discussed recently\textsuperscript{[29–40]}. Moreover, observational signatures on the interaction between dark ones have been investigated in the probes of the cosmic expansion history with the use of the SNIa, BAO and CMB shift data\textsuperscript{[41–43]}. The interacting dark energy has also been considered as a possible solution to the coincidence problem\textsuperscript{[29, 44–53]}. In this section, we intend to extend \((m, n)\)-type holographic dark energy models with interactions. When the interaction is taken into account, the equations of motion for \( \rho_\Lambda \) and \( \rho_i \) become
\[ \dot{\rho}_\Lambda = -3H\rho_\Lambda(1 + \omega) - Q, \tag{27} \]
\[ \dot{\rho}_i = -3H\rho_i(1 + \omega_i) + Q, \]  

where \( Q \) denotes the interacting term. From (27) and (28) we find that the interacting term has the following general form,

\[ \tilde{Q} \equiv \frac{Q}{H\rho_\Lambda} = \frac{1}{1 + r}\left[ r' - 3(\omega - \omega_i)r \right]. \]  

(29)

It is also easy to derive a general relation between \( L \) and \( \tilde{Q} \) as

\[ \tilde{Q} = r' - 2r\left( \frac{L'}{L} - \frac{3}{2} - \frac{3\omega_i}{2} \right). \]  

(30)

As we stressed in Ref. [29], four free parameters \( \omega, r, L \) and \( Q \) are not independent. Given any two of them, the dynamics of the other two will be determined. Usually, people propose the forms of \( L \) and \( Q \), and then find out the evolutions of \( \omega \) and \( r \) with observation data. Thus, after introducing the interacting term we find the equations for \( \omega \) and \( r \) in the previous section can be generalized as

\[ \omega = -1 - \frac{2}{3}m + \frac{2}{3} \frac{a^{n-m}}{c \sqrt{1 + r}} - \frac{\tilde{Q}}{3}. \]  

(31)

\[ r' = \tilde{Q}(1 + r) + 3(\omega - \omega_i)r. \]  

(32)

Obviously the interaction will change the dynamics of \( \omega \) as well as \( r \). One can alternatively write down the equation of motion for \( \Omega_\Lambda \) as

\[ \Omega_\Lambda' = \Omega_\Lambda[(3 + 3\omega_i + 2m - \frac{2a^{n-m}\sqrt{\Omega_\Lambda}}{c})(1 - \Omega_\Lambda) - \tilde{Q}\Omega_\Lambda]. \]  

(33)

It is clear that Eqs. (31) and (33) reduce to Eqs. (12) and (13) respectively in the case of \( \tilde{Q} = 0 \). Now, we turn to consider the coincidence problem with the help of interaction. We expect that the ratio \( r \) of dark matter to dark energy density varies slowly, and will finally approach to a non-zero constant at late time. For explicitness, we consider a specific form of the interaction \( \tilde{Q} = 3b^2(r + 1) \), where \( b^2 \) is a coupling constant. Its positivity is responsible for the transition from dark energy to dark matter. Repeating the calculations in the previous section, we find that the basic constraint on \( m \) and \( c \) becomes

\[ m > -\frac{5}{6} - 2b^2, \]  

(34)

\[ c > \frac{3\sqrt{3}}{6m + 5 + 12b^2}. \]  

(35)
To alleviate the coincidence problem, we are more concerned with the asymptotic value of \( r \) as \( (a \gg 1) \). Setting \( r' = 0 \) in Eq.(32), we find the ratio of dark matter to dark energy will approach to a non-zero constant, which is

\[
r_f = \frac{3b^2}{3 + 2m - 3b^2},
\]

where the value of \( r_f \) depends on \( m \) and \( b \) manifestly. This result indicates that if \( m \) is not too large, the situation that the ratio \( r \) keeps staying in a region with unit order can be easily realized, thus providing a mechanism to understand the coincidence problem.

IV. \((m, n)\) TYPE MODELS WITH A GENERALIZED FUTURE EVENT HORIZON

In this section, we would like to point out that with the same spirit our construction should be applicable to the holographic dark energy models with generalized future event horizon as the characteristic size, which has been extensively studied in literature[8]. Explicitly, we may generalize the definition of the holographic characteristic scale to

\[
L = \frac{1}{a^m(t)} \int_t^{\infty} a^n(t')dt'.
\]

Specially, when \((m, n) = (-1, -1)\) it recovers the ordinary holographic dark energy model with future event horizon. In this definition taking the derivative with respective to \( lna \) on both sides, we obtain

\[
\frac{L'}{L} = -m - \frac{a^{n-m}}{H L}.
\]

With the same algebra we may derive the equation of state as

\[
\omega = -1 - \frac{2}{3}m - \frac{2a^{n-m}\sqrt{\Omega_\Lambda}}{3c},
\]

while the equation of motion for \( \Omega_\Lambda \) reads as

\[
\Omega'_\Lambda = \Omega_\Lambda(1 - \Omega_\Lambda)(3 + 3\omega_i + 2m + \frac{2a^{n-m}\sqrt{\Omega_\Lambda}}{c}).
\]

In general case without interaction we still require that \( n \leq m \) such that the proportion of dark energy always increases with the evolution of the universe if \( m > -\frac{3}{2} \). Moreover,
under the condition of acceleration $1 + r + 3\omega < 0$, and with the use of $r_0 \simeq 1/3$ we find the number $m$ should be subject to the inequality

$$m > -\frac{5}{6} - \frac{\sqrt{3}}{2c}. \quad (41)$$

It is interesting to notice that the fate of the universe would be very different for $n = m$ and $n < m$ in future with $a \gg 1$. For $n < m$, we easily find that the asymptotic behavior of the state parameter will be depicted by the equation

$$\omega = -1 - \frac{2}{3}m, \quad (42)$$

while for $m = n$, we find its value will approach to

$$\omega = -1 - \frac{2}{3}m - \frac{2}{3c}, \quad (43)$$

which depends on the constant $c$. The latter ones of course cover the ordinary holographic dark energy model with $m = n = -1$ and $c = 1$. Based on our construction we could consider a generalized model with $m = -1$ and $n = -1 - \delta$ where $\delta$ is a small positive constant. From our above consideration it is expected that this modification will change the asymptotical behavior of the dark energy dramatically. From this point our construction here is quite different from the generalized holographic model with varying $c(z)$, which has recently been proposed in [54].

V. THE STABILITY OF $(m,n)$ TYPE HOLOGRAPHIC DARK ENERGY MODEL

In above sections we have discussed the cosmic evolution of our model. In this section we are concerned with the stability of our model. Let us start with the scalar type perturbation of the metric in flat universe, which is written as

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Phi)(dr^2 + r^2d\Omega^2), \quad (44)$$

where $\Phi$ is the Newtonian potential. Now we can use the characteristic scale without perturbation to define $r_{L0}(t)$ which is

$$L(0) \equiv L = a(t)r_{L0}(t). \quad (45)$$

When the scalar perturbation is taken into account, the characteristic scale is modified as,

$$L(\Phi) = a(t) \int_0^{r_{L}(t)} [1 - \Phi(r', t)]dr'. \quad (46)$$
We consider the 00-component of the perturbation Einstein equation in the Newtonian gauge. Making use of the variation of Friedmann equation (3) and expanding \( \Phi = \sum \Phi_l \sin \frac{l r}{r} \), we find it can be written as

\[
- \frac{\sin \frac{l r}{r}}{r} \left[ a^2 \Phi_l(t) + 3H\dot{\Phi}_l(t) + 3H^2\Phi_l(t) \right] = - \frac{3H\Phi_l(t)\dot{r}_L}{r^2_L} \left[ \sin r_L - \int_0^{r_L} \frac{\sin \frac{lr}{r'}}{r'} dr' \right].
\] (47)

Here we have used relations \( \delta L(0) = L(\Phi) - L(0) \) and \( \delta L(0) = a(t)\delta r_{L0} + r_{L0}\delta a(t) \). Moreover, it is worth to note that the above equation is independent of the specific form of the cosmological model. To investigate the stability of our model, we are mainly concerned with the asymptotic behavior of \( \dot{\Phi}_l \Phi_l \) when \( a \gg 1 \). As shown in [26, 55], there are two cases corresponding to the stability: (1) the perturbation mode is frozen when \( \dot{\Phi}_l \Phi_l \to 0 \); (2) the perturbation mode is decaying when \( \dot{\Phi}_l \Phi_l < 0 \).

### A. the stability of the model with the age-like characteristic scale

For the age-like holographic dark energy model, we define the coordinate value by Eq.(10), which is

\[
r_{L0}(t) = \frac{1}{a^{m+1}(t)} \int_0^t a^n(t')dt'.
\] (48)

Under the basic constraints \( m > n \) and \( m > -\frac{5}{6} \), we will discuss the stability of the model for two cases respectively, namely \( -\frac{5}{6} < m < 0 \) and \( m \geq 0 \).

- When \( -\frac{5}{6} < m < 0 \), employing Eq.(21) we can obtain the asymptotic behavior of dark energy density from the continuity equation [11]. It turns out that as \( a \gg 1 \), \( \rho_{\Lambda} \to 0 \) and \( L \to \infty \). For super-horizon modes, namely \( lr_{L0} \ll 1 \), we can derive the following result from Eq.(47)

\[
\dot{\Phi}_l \Phi_l = -\frac{1}{3H} \left[ \frac{l^2}{a^2} + 3H^2 \right].
\] (49)

Similarly, for sub-horizon mode \( (lr_{L0} \gg 1) \), we have

\[
\dot{\Phi}_l \Phi_l = -\frac{1}{3H} \left[ \frac{l^2}{a^2} + 3H^2 \right] + \frac{1}{L} \left[ a^n - (m + 1)c \right] \sin \frac{lr_{L0}}{r} \frac{\sin \frac{lr}{r}}{r} \to 0
\] (50)

Above two equations indicate that our model is stable when \( -\frac{5}{6} < m < 0 \).

- When \( m \geq 0 \), we notice that dark energy density \( \rho_{\Lambda} \) will approach to a constant \( (m = 0) \) or infinity\( (m > 0) \). For both cases we can derive \( r_{L0} \to 0 \) from equation [45].
As a result, for any given $l$, as $a \gg 1$, we always have $lr_{L0} \to 0$, implying that the sub-horizon modes of the perturbation are absent. Thus we just need to consider the super-horizon mode($lr_{L0} \ll 1$), which gives rise to the same result as Eq.(49).

As a summary, we conclude that when the parameters $(m, n)$ are taken values in the allowed region, we find our model is always stable under the scalar perturbations.

B. the stability of the model with the generalized event horizon as characteristic scale

We may consider the stability of the model with the generalized event horizon as characteristic scale in a parallel way. We define the coordinate $r_{L0}$ with the generalized event horizon in equation (37), which is

$$r_{L0} = \frac{1}{am+1(t)}\int_{t}^{\infty} a^n(t')dt'.$$

(51)

Under the condition $m \geq -1$, we will discuss the stability for $m > n$ and $m = n$ respectively. For $m > n$, we can further classify them into two cases.

- In the case of $-1 \leq m < 0$, dark energy density $\rho_{\Lambda}$ approaches to 0 as $a \gg 1$, which also implies the generalized event horizon $L \to \infty$. For the super-horizon modes we still have the same result as (49). For sub-horizon modes we can easily derive

$$\frac{\dot{\Phi}_l}{\Phi_l} = -\frac{1}{3H}\left[\frac{l^2}{a^2} + 3H^2\right] - \frac{1}{L}\left[a^{n-1} + (m+1)c\right]\sin lr_{L0} - \frac{\pi}{2} \to 0$$

(52)

- In the case of $m \geq 0$, $r_{L0} \to 0$ when $a \gg 1$. According to the previous analysis we can easily find that only the super-horizon modes are presented and the model is stable too.

Next, we turn to analyze the perturbation behavior for the case $m = n$. From equation(13), the future of our universe is obviously related to the parameter $c$. Here we can choose a specific value for $c$ to study the stability of our model. Without loss of generality, we set $c = 1$.

When $m \geq -1$, using equation (43) we find the perturbation modes decay as described by equation (49), which means our model is stable. In particular, for $m = n = -1$, our
discussion is consistent with the analysis presented in \cite{26}, but simpler. Our analysis above still holds for other values of parameter $c$.

Therefore, there is no instability appearing for all the perturbation modes in this sort of cosmological models.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have constructed $(m, n)$ type holographic dark energy model which can be viewed as a generalization of the ordinary holographic dark energy models appeared in literature. At phenomenological level, such a generalization provides us more space in theory to fit the observational data. In particular, for some specific values of $(m, n)$ the equation of state $\omega$ can naturally evolve cross phantom divide in the entire evolution of the universe even without introducing an interaction between dark energy and background matter. We have discussed the general features of age-like holographic dark energy models in various epochs of the universe and derived the basic constraints on the values of $(m, n)$. We have also remarked that this construction is applicable to the holographic models with generalized future event horizon as the characteristic scale.

For age-like holographic models, the case of $m = n$ is special. In this case it seems that dark energy has the same behavior as the dominant ingredient in the early epochs of the universe, implying that dark energy might be unified with dark matter, analogous to what happened in cosmological models with generalized Chaplygin gas\cite{56}. However, if DE and DM were unified at early stages, we must introduce some mechanism to make dark energy deviate from dark matter state, and eventually become dominant to be responsible for the acceleration of the universe. This might be implemented by introducing some appropriate interactions between dark energy and dark matter, and our investigation is under progress.

We have investigated the stability issue by treating holographic dark energy perturbation as global perturbation, namely the perturbation of cosmic metric. In Newtonian gauge, we have shown that when the parameters are taken values in the allowed region, our model is always stable in the dark energy dominated era.

We claim that this paper is our first step in this direction, and our focus is proposing such an original model and discussing its general features. We have found that our model satisfies all the basic requirements to be a candidate of dark energy. Further fitting with the
observational data with more quantitative precision of course is a key issue to testify our models in the next step.

Note: After we uploaded our manuscript, the observational constraint on this model appeared in [57], in which some integer parameters \((m, n)\) were analyzed in great detail. The best-fit analysis in this reference indicates that this model with \(m = n + 1\) and small \(m\) is more favored including the cases of \((m,n)=(1,0), (2,1), (3,2)\). In a word, the preliminary numerical analysis has indicated that our model can fit with the observational data very well.

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