The recurrence function of a random Sturmian word.

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Abstract
This paper describes the probabilistic behaviour of a random Sturmian word. It performs the probabilistic analysis of the recurrence function which can be viewed as a waiting time to discover all the factors of length \( n \) of the Sturmian word. This parameter is central to combinatorics of words. Having fixed a possible length \( n \) for the factors, we let \( \alpha \) to be drawn uniformly from the unit interval \([0,1]\), thus defining a random Sturmian word of slope \( \alpha \). Thus the waiting time for these factors becomes a random variable, for which we study the limit distribution and the limit density.

1 Introduction

Recurrent and Sturmian words. The recurrence function measures the “complexity” of an infinite word and describes the possible occurrences of finite factors inside it together with the maximal gaps between successive occurrences. This recurrence function is thus widely studied, notably in the case of Sturmian words. Sturmian words are central in combinatorics of words, as they are in a precise sense the simplest infinite words which are not eventually periodic. With each Sturmian word, associated with an irrational number \( \alpha \) which are close to the right end of the interval \([0,1]\), one has

\[
S(\alpha,n) := \frac{R(\alpha,n) + 1}{n} = 1 + \frac{q_k(\alpha)}{n},
\]

where \( n \rightarrow S(\alpha,n) \) is now a sequence of random variables.

Proposition 1.1. The following holds for the recurrence quotient defined in (1.2):

(i) For any irrational real \( \alpha \), one has

\[
\liminf_{n \rightarrow \infty} S(\alpha,n) \leq 3.
\]

(ii) For almost any irrational \( \alpha \), and any \( c > 0 \) one has

\[
\limsup_{n \rightarrow \infty} \frac{S(\alpha,n)}{\log n} = +\infty, \quad \limsup_{n \rightarrow \infty} \frac{S(\alpha,n)}{(\log n)^{1+c}} = 0
\]

This result also shows that the quotient recurrence is “small” for integers \( n \) which are close to the right end of the interval \([q_{k-1}(\alpha), q_k(\alpha)]\), whereas it is “large” when \( n \) is close to \( q_{k-1}(\alpha) \) (see Figure 1).

Two different probabilistic settings. Here, we adopt a probabilistic approach, and consider a random Sturmian word, associated with a random irrational slope \( \alpha \) of the unit interval. There are now two possibilities:

(i) fix the integer \( n \) (corresponding to the length of the factors, which will further tend to \( \infty \)); now the index \( k \) of the interval \([q_{k-1}(\alpha), q_k(\alpha)]\) which contains \( n \) is a random variable \( k = k(\alpha,n) \). This model may be called the model “with a large fixed \( n \)”. The sequence \( n \rightarrow S(\alpha,n) \) is now a sequence of random variables.

\[\{q_{k-1}(\alpha), q_k(\alpha)\} \text{ which contains } n. \] More precisely, for any \( n \in [q_{k-1}(\alpha), q_k(\alpha)] \), one has

\[
R(\alpha,n) = n - 1 + q_k(\alpha) + q_{k-1}(\alpha).
\]
We here deal with the recurrence quotient within the model (with large fixed \( n \)) previously studied in \([1]\). We return to our model “with a large fixed \( n \)” in \([6]\). and we return to it in Section 5.1.

We exhibit a limit for their distribution, and prove that there exists a limit density, as \( n \to \infty \). We also study the conditional expectation of the recurrence quotient, when we exclude the possibility for \( n \) to be too close of the left end of the interval \([q_{k-1}(\alpha), q_k(\alpha)]\). And we exhibit a class of events, for which the order of this conditional mean value is exactly of order \( \log n \). This can be viewed as a probabilistic extension of the Morse and Hedlund result (compare with Proposition 1.1). Our proofs use elementary methods: they are based on a precise comparison between an integral and its Riemann sum ; however, the integral is improper (but convergent) and the Riemann sum is constrained by a coprimality condition, what we call a “coprime Riemann sum”.

We also introduce a general family of functions, called continuant-functions or \(Q\)-functions, which are defined via the sequence of continuants \( k \mapsto q_k(\alpha) \). The recurrence quotient is an instance of such a function, but the other “geometric” parameters of interest provide other natural examples of such a notion. And the framework of the paper is well-adapted to the study of a general \(Q\) function.

**Plan of the paper.** Section 2 gives a precise definition of the parameters under study, introduces the class of \(Q\)-functions and states our three results: limit distributions in Theorem 2.1, limit densities in Theorem 2.2 and conditional expectations in Theorem 2.3. Section 3 is devoted to the proof of the first two results, whereas Section 4 focuses on the study of conditional expectations. Section 5 compares the results obtained in the two models, the present model (with large fixed \( n \)), and the model (with large fixed \( k \)) previously studied in \([6]\).

**2 General framework and main results.**

The section first makes precise the notions that were informally defined in the introduction, notably Sturmian words and recurrence. Then, it introduces parameters which describe the geometry of the “continuant intervals” or the position of the integer \( n \) inside the continuant interval. Section 2.3 defines the class of \(Q\) functions that provides a convenient framework for our study. Then, we state Theorems 2.1, 2.2 in Sections 2.5 and 2.6, for general \(Q\)-functions. We return to our specific parameters of interest, notably the recurrence function in Section 2.7, with two figures (Figures 2 and 3). Section 2.8 concludes with conditional expectations.

**2.1 More on Sturmian words and recurrence function.** We consider a finite set \(A\) of symbols, called...
alphabet. Let \( u = (u_n)_{n \in \mathbb{N}} \) be an infinite word in \( \mathcal{A}^N \). A finite word \( w \) of length \( n \) is a factor of \( u \) if there exists an index \( m \) for which \( w = u_m \ldots u_{m+n-1} \). Let \( \mathcal{L}_u(n) \) stand for the set of factors of length \( n \) of \( u \). Two functions describe the set \( \mathcal{L}_u(n) \) inside the word \( u \), namely the complexity and the recurrence function.

**Complexity.** The (factor) complexity function of the infinite word \( u \) is defined as the sequence \( n \mapsto p_u(n) := |\mathcal{L}_u(n)| \). The eventually periodic words are the simplest ones, in terms of the complexity function, and satisfy \( p_u(n) \leq n \) for some \( n \).

The simplest words that are not eventually periodic satisfy the equality \( p_u(n) = n + 1 \) for each \( n \geq 0 \). Such words do exist, they are called Sturmian words. Moreover, Morse and Hedlund provided a powerful arithmetic description of Sturmian words (see also [2] for more on Sturmian words).

**Proposition 2.1.** [Morse and Hedlund] \( \exists \) Associate with a pair \((\alpha,b)\in[0,1]^2\) the two infinite words \( \mathfrak{S}(\alpha,b) \) and \( \overline{\mathfrak{S}}(\alpha,b) \) whose \( n \)-th symbols are respectively

\[
\begin{align*}
\mathfrak{u}_n &= [\alpha(n+1) + b] - [\alpha n + b], \\
\overline{u}_n &= [\alpha(n+1) + b] - [\alpha n + b].
\end{align*}
\]

Then a word \( u \in \{0,1\}^N \) is Sturmian if and only if it equals \( \mathfrak{S}(\alpha,b) \) or \( \overline{\mathfrak{S}}(\alpha,b) \) for a pair \((\alpha,b)\) formed with an irrational \( \alpha \in ]0,1[ \) and a real \( b \in [0,1[ \).

**Recurrence.** It is also important to study where finite factors occur inside the infinite word \( u \). An infinite word \( u \in \mathcal{A}^N \) is uniformly recurrent if every factor of \( u \) appears infinitely often and with bounded gaps. More precisely, denote by \( w_u(q,n) \) the minimal number of symbols \( u_q \) with \( k \geq q \) which have to be inspected for discovering the whole set \( \mathcal{L}_u(n) \) from the index \( q \). Then, the integer \( w_u(q,n) \) is a sort of “waiting time” and \( u \) is uniformly recurrent if each set \( \{w_u(q,n) \mid q \in \mathbb{N} \} \) is bounded, and the recurrence function \( n \mapsto R_u(n) \) is defined as

\[ R_u(n) := \text{max}\{ w_u(q,n) \mid q \in \mathbb{N} \}. \]

We then recover the usual definition: Any factor of length \( R_u(n) \) of \( u \) contains all the factors of length \( n \) of \( u \), and the length \( R_u(n) \) is the smallest integer which satisfies this property. The inequality \( R_u(n) \geq p_u(n) + n - 1 \) thus holds.

Any Sturmian word is uniformly recurrent. Its recurrence function only depends on the slope \( \alpha \) and is thus denoted by \( n \mapsto R(\alpha,n) \). As we already said, it only depends on \( \alpha \) via the sequence of its continuants \( k \mapsto q_k(\alpha) \), and satisfies \( [1,1] \).

**2.2 Position parameters.** Besides the recurrence quotient, there are also three other parameters \( \nu, \mu, \rho \) which describe the geometry of the interval \([q_{k-1}(\alpha), q_k(\alpha)] \) which contains \( n \) (this is the case for \( \rho \)) or the position of \( n \) inside this interval (the case for \( \mu \) and \( \nu \))

\[
\begin{align*}
(2.3) & \quad \rho(\alpha,n) = \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \\
(2.4) & \quad \mu(\alpha,n) := \frac{n - q_{k-1}(\alpha)}{q_k(\alpha) - q_{k-1}(\alpha)}, \quad \nu(\alpha,n) = \frac{n}{q_k(\alpha)}. \\
(2.5) & \quad S(\alpha,n) = 1 + \frac{1 + \rho(\alpha,n)}{\nu(\alpha,n)}.
\end{align*}
\]

As \( \nu(\alpha,n) \) belongs to the interval \([\rho(\alpha,n),1[, \) the following bound holds

\[
(2.6) \quad 2 + \rho(\alpha,n) \leq S(\alpha,n) \leq 2 + \frac{1}{\rho(\alpha,n)}
\]

(the lower bound holds for \( n \) close to \( q_k(\alpha) \) whereas the upper bound is attained for \( n = q_{k-1}(\alpha) \)).

The ratio \( \rho(\alpha,n) \) belongs to \( [0,1[, \) and the Borel-Bernstein Theorem proves that \( \lim \inf_{n \to \infty} \rho(\alpha,n) = 0 \) for almost any irrational \( \alpha \). This is the main step for proving Proposition [1,1].

**2.3 \( Q \)-functions.** More generally, we are interested in functions whose definition strongly depends on the partition defined by the continuants, and consider the functions \( (\alpha,n) \mapsto \Lambda(\alpha,n) \) that are associated with some function \( f \) and are written as

\[
(2.7) \quad \Lambda(\alpha,n) = f \left( \frac{q_{k-1}(\alpha)}{n}, \frac{q_k(\alpha)}{n} \right),
\]

as soon as \( n \in [q_{k-1}(\alpha), q_k(\alpha)] \).

In the following, we restrict ourselves to a function \( f \) that satisfies the following three properties

(i) it is written as the non trivial quotient of two linear functions

\[
(2.8) \quad f(x,y) = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2};
\]

(ii) it is defined on the unbounded rectangle

\[ \mathcal{R} := \{(x,y) \mid 0 < x \leq 1 < y\}, \]

(iii) it is non negative on \( \mathcal{R} \).
A function \( \Lambda \) which is written as in (2.7) in terms of such a function \( f \) is called a continuant-function, or a \( Q \)-function.

Our four parameters of interest, namely the recurrence quotient, the ratio \( \rho \) and the two parameters which describe the position of integer \( n \) with respect to the interval \( [q_k-1(\alpha), q_k(\alpha)] \) are \( Q \)-functions, associated to the following functions

\[
  f_S(x, y) = 1 + x + y,
\]

\[
f_{\rho}(x, y) = \frac{x}{y}, \quad f_{\mu}(x, y) = \frac{1 - x}{y - x}, \quad f_{\nu}(x, y) = \frac{1}{y}.
\]

### 2.4 Probabilistic setting.

We recall the present setting, already described in the introduction. We consider a fixed integer \( n \), and a random real \( \alpha \) in the unit interval \([0, 1] \). The sequence \( \Lambda_n(\alpha) := \Lambda(\alpha, n) \) is now a sequence of random variables. We are interested in the limit distribution of the sequence when \( n \to \infty \). Does there exist a limit distribution? a limit density?

### 2.5 General results - distributions.

In the distribution study, we associate with a real \( \lambda \geq 0 \) the subdomain of \( \mathcal{R} \),

\[
  \Delta_f(\lambda) := \{(x, y) \mid 0 \leq x \leq 1 \leq y; \ f(x, y) \leq \lambda \}
\]

(which is a convex domain due to the particular form of the function \( f \)), and associate the integral

\[
  I_f(\lambda) = \iint_{\Delta_f(\lambda)} \omega(x, y)dx\,dy = I[\omega, \Delta_f(\lambda)],
\]

which involves the function \( \omega \) defined on \( \mathcal{R} \) as

\[
  \omega(x, y) = \frac{1}{y(x + y)},
\]

whose integral on \( \mathcal{R} \) satisfies \( I(\omega, \mathcal{R}) = \pi^2/12 \). The associated density

\[
  \psi(x, y) = \frac{12}{\pi^2} \frac{1}{y(x + y)}
\]

plays a fundamental role in the sequel, as our originally discrete distribution smooths out (converges weakly) to the distribution associated with the density \( \psi \), as the following result shows:

**Theorem 2.1.** Consider a \( Q \)-function associated with a function \( f \). Then the sequence \( n \to \Lambda_n(\alpha) \) as \( n \to \infty \) admits a limit distribution, and the sequence

\[
  F_n(\lambda) := \mathbb{P}[\Lambda_n \leq \lambda] = \frac{12}{\pi^2} I_f(\lambda) + O\left(\frac{1}{n}\right),
\]

involves the integral \( I_f(\lambda) \) defined in (2.10). Moreover, the constant does not depend on the pair \( (f, \lambda) \).

### 2.6 General results - densities.

For the densities, we deal with boundary curves \( \{(x, y) \mid f(x, y) = \lambda\} \) and their intersection with \( \mathcal{R} \). We prove the following:

**Theorem 2.2.** Consider a \( Q \)-function associated with a function \( f \) which is written as in (2.8). Then,

(a) The function \( \lambda \to I_f(\lambda) \) and its derivative \( J_f \) exist for any \( \lambda \). The derivative \( J'_f \) exists except perhaps on a finite set, which contains the point \( b_1/b_2 \) and two possible other values \( \lambda_0 \) and \( \lambda_1 \). The following holds:

(i) At each of the points \( \lambda = \lambda_i \), the function \( J_f \) admits a left and a right derivative, each of them being finite.

(ii) When the determinant \( r(a, b) := a_1 b_2 - a_2 b_1 \) is zero, the derivative \( J'_f \) exists at \( \lambda = b_1/b_2 \).
and the constants in the recurrence quotient arise when \( \nu \) or \( \mu \) are small. In particular, the event \( \{\nu_n \geq \epsilon(n)\} \) gathers the reals \( \alpha \) for which the integer \( n \) is not too close to the left end of the interval \( [q_{k-1}(\alpha), q_k(\alpha)] \), and, at the same time, the length of the interval \( [q_{k-1}(\alpha), q_k(\alpha)] \) is of the same order as the right end \( q_k(\alpha) \). We then consider a sequence \( \epsilon(n) \to 0 \), and condition with one of the events

\[
\{\rho_n \geq \epsilon(n)\}, \quad \{\nu_n \geq \epsilon(n)\}, \quad \{\mu_n \geq \epsilon(n)\}.
\]

Theorem 2.3. Consider a parameter \( \Gamma \in \{\rho, \mu, \nu\} \) defined in (2.3) and (2.4). Then the conditional expectation of the recurrence quotient \( S_n \) with respect to the event \( \{\Gamma_n \geq \epsilon(n)\} \) satisfies

\[
\mathbb{E}[S_n | \Gamma_n \geq \frac{1}{n}] = \frac{12}{\pi^2} \log n + O(1).
\]

This result exhibits a sequence of events, for which the integer is not too close to the left ends of interval \( [q_{k-1}(\alpha), q_k(\alpha)] \). When we are sure not to be too close to this left end, the quotient of the recurrence quotient is (on average) of order \( \log n \). This can be viewed as a counterpart of Proposition 1.1. We return to this study at the end of Section 4.
3 Proofs of Theorems 2.1 and 2.2

We first introduce the main objects of interest: continued fraction expansions and coprime Riemann sums. Then, we prove the existence of limit distribution and limit densities for a general Q function. The proof of Theorem 2.1 has three main steps, described in Sections 3.3, 3.4, and 3.5, and we conclude the proof of Theorem 2.1 in Section 3.6. Section 3.6 is devoted to the proof of Theorem 2.2.

3.1 Continued fractions, fundamental intervals and continuants. (See here [14] for more details). The continued fraction of an irrational number \( \alpha \) of the unit interval \([0,1]\) is

\[
\alpha = \frac{1}{m_1 + \frac{1}{m_2 + \frac{1}{\ldots + \frac{1}{m_k + \frac{1}{\ldots}}}}}.
\]

Truncated at depth \( k \), it gives rise to a rational number \( \frac{p_k}{q_k} \) associated with a coprime integer pair \((p_k, q_k)\). The numerator \( p_k = p_k(\alpha) \) and the denominator \( q_k = q_k(\alpha) \) are uniquely defined by the irrational number \( \alpha \).

All the irrational numbers \( \alpha \) which begin with the same sequence \( m = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k \) belong to an interval, called a fundamental interval of depth \( k \) and denoted by \( I_k(m) \). As the irrational numbers of \( I_k(m) \) have the same convergents of order \( \ell \leq k \), we denote their numerator and denominator by \( p_\ell(m), q_\ell(m) \). The ends of the interval \( I_k(m) \) are

\[
\begin{align*}
p_k(m) & = p_k(\alpha), \\
q_k(m) & = q_k(\alpha).
\end{align*}
\]

As the equality \( |p_k(m)q_{k-1}(m) - p_{k-1}(m)q_k(m)| = 1 \) holds, the length of the fundamental interval involves the function \( \omega \) defined in (2.11) under the form

\[
|I_k(m)| = \omega(q_{k-1}(m), q_k(m)).
\]

This explains why the function \( \omega \) defined in (2.11) and the associated density \( \psi \) are ubiquitous in the study of the Q-functions.

3.2 Distributions. Strategy of the proof. There are two main steps in the proofs of Theorem 2.1.

(i) Discrete step. We express in Proposition 3.1 the distribution of a Q function in terms of a variant of a Riemann sum, that is called in the following a “coprime” Riemann sum. This type of “constrained” Riemann sum was already considered in [6].

(ii) Continuous step. We compare the “coprime” Riemann sum to the associated integral. We begin by the comparison of the “plain” Riemann sum to the integral in Proposition 2.2, then, we take into account the coprimality condition in Proposition 3.3. We extend here the results of [6] which are only proven for finite domains.

3.3 Distributions and Riemann sums. We begin with the alternative expression of a Q-function \( \Lambda \) associated with \( f \), (already defined in (2.7)), which is written with the Iverson bracket under the form

\[
\Lambda(\alpha, n) = \sum_{k \geq 0} f \left( \frac{q_{k-1}(\alpha)}{n}, \frac{q_k(\alpha)}{n} \right) \left[ n \in [q_{k-1}(\alpha), q_k(\alpha)] \right].
\]

The distribution of a Q-function associated with \( f \) is

\[
P[\Lambda_n \leq \lambda] = \int_0^1 d\alpha \sum_{k \geq 0} \left[ \left( \frac{q_{k-1}(\alpha)}{n}, \frac{q_k(\alpha)}{n} \right) \in \Delta_f(\lambda) \right].
\]

For each \( k \), the family of fundamental intervals \( I_k(m) \) defines a pseudo-partition when \( m \) goes through \( \mathbb{N}^k \), and, for any \( \alpha \in I_k(m) \), the equality \( q_k(\alpha) = q_k(m) \) holds. We deduce

\[
P[\Lambda_n \leq \lambda] = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{N}^k} |I_k(m)| \left[ \left( \frac{q_{k-1}(m)}{n}, \frac{q_k(m)}{n} \right) \in \Delta_f(\lambda) \right].
\]

Then, with the expression of the length \(|I_k(m)|\) in terms of the function \( \omega \) given in (3.15) and the fact that \( \omega \) is homogeneous of degree -2, we obtain

\[
|I_k(m)| = \frac{1}{n^\kappa} \omega \left( \frac{q_{k-1}(m)}{n}, \frac{q_k(m)}{n} \right).
\]

Now, as we go through all the sequences \( m \in \mathbb{N}^* \), the coprime pairs \((q_{k-1}(m), q_k(m))\) give rise to all the coprime pairs \((a, b)\). Moreover, each coprime pair \((a, b)\), except the pair \((1,1)\), appears exactly twice, due to the existence of two continued fraction expansions, the proper one (in which the last digits strictly greater than

\[\text{The Iverson bracket is a Boolean function defined by } [P] = 1 \text{ as soon as Property } P \text{ is true}\].
1), and the improper one (in which the last digit is equal to 1). Then, the equality holds
\[ \mathbb{P}[\Lambda_n \leq \lambda] = \frac{2}{n^2} \sum_{(a,b) \in \mathbb{Z}^2} \omega\left(\frac{a}{n}, \frac{b}{n}\right) \left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Delta_f(\lambda)\right]. \]

The right member is the Riemann sum of the function \(2\omega\) on the domain \(\Delta_f(\lambda)\) with step \(1/n\), with an extra condition of coprimality. More generally, for a function \(g\) integrable on a subset \(\Omega\), we are led to the following two Riemann sums with step \(1/n\): the first one \(R_n(g,\Omega)\) is the usual one,
\[ R_n(g,\Omega) = \frac{1}{n^2} \sum_{(a,b) \in \mathbb{Z}^2} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega\right], \]
whereas the second one \(\tilde{R}_n(g,\Omega)\) takes into account the coprimality of \((a,b)\), and is called the "coprime" Riemann sum,
\[ \tilde{R}_n(g,\Omega) = \frac{1}{n^2} \sum_{(a,b) \in \mathbb{Z}^2 \text{ coprime}} g\left(\frac{a}{n}, \frac{b}{n}\right) \left[\left(\frac{a}{n}, \frac{b}{n}\right) \in \Omega\right]. \]

We summarize:

**Proposition 3.1.** Consider a \(Q\)-function \(\Lambda\) associated with a function \(f\). Then the distribution \(F_n(\lambda) := \mathbb{P}[\Lambda_n \leq \lambda]\) is expressed with a coprime Riemann sum,
\[ F_n(\lambda) = \tilde{R}_n\left(2\omega, \Delta_f(\lambda)\right), \tag{3.16} \]
which involves the density \(\omega\) defined in \((2.11)\) and the domain \(\Delta_f(\lambda)\) defined in \((2.9)\).

The previous result extends if we replace \(\Delta_f(\lambda)\) by any other domain \(\Omega \subset \mathcal{R}\). In particular, in Section 4, we will deal with two \(Q\)-functions \(\Lambda\) and \(\Gamma\) associated respectively to \(f\) and \(g\), together with the domain
\[ \Delta_{f,g}(\lambda, \epsilon) := \{(x,y) \in \mathcal{R} \mid f(x,y) \geq \lambda, g(x,y) \geq \epsilon\}, \tag{3.17} \]
and use the equality
\[ \mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \tilde{R}_n\left(2\omega, \Delta_{f,g}(\lambda, \epsilon)\right). \tag{3.18} \]

### 3.4 Usual Riemann sums and integrals.
We first deal with the usual Riemann sum, and compare it to its associated integral \(I(g,\Omega)\). This is a classical proof, but we consider improper integrals and we wish to have precise error terms.

We now deal (only within this subsection) with \( \Omega := [0, 1] \times (0, \infty) \), consider a subset \(\Omega \subset \mathcal{S}\) and associate with it the family of subsets
\[ \Omega(k) := \Omega \cap ([0, 1] \times [k, k+1]), \] for any \(k \geq 1\), which form a pseudo-partition of \(\Omega\). We also consider a positive function \(g\) defined on \(\Omega\) of class \(C^1\), bounded on any bounded subset on \(\Omega\) for which the following two finite bounds:
\[ C_g(\Omega, k) := \sup\{g(x,y) \mid (x,y) \in \Omega(k)\}, \]
\[ D_g(\Omega, k) := \sup \left\{ \left| \frac{\partial g}{\partial y}(x,y) \right| \mid (x,y) \in \Omega(k) \right\}, \]
define sequences whose associated series are convergent,
\[ \sum_{k \geq 0} C_g(\Omega, k) < \infty, \quad \sum_{k \geq 0} D_g(\Omega, k) < \infty. \]

Their sums are denoted by \(C_g(\Omega)\) and \(D_g(\Omega)\), and we denote by \(M_g(\Omega)\) their maximum.

Such a function \(g\) is called strongly decreasing on \(\Omega\) with bound \(M_g(\Omega)\). Such a function is integrable on \(\Omega\) and the inequality \(I(g,\Omega) \leq M_g(\Omega)\) holds.

**Proposition 3.2.** Consider the domain \(\mathcal{S}\) defined in \((3.19)\) and a function \(g\) which is strongly decreasing on a convex \(\Omega \subset \mathcal{S}\) with bound \(M_g(\Omega)\). Then, the Riemann sum of the function \(g\) on \(\Omega\) compares to the integral,
\[ |R_n(g,\Omega) - I(g,\Omega)| \leq \frac{5}{n} M_g(\Omega). \tag{3.20} \]

**Proof.** We will prove the estimate, for each \(k \geq 0\),
\[ |R_n(g,\Omega(k)) - I(g,\Omega(k))| \leq \frac{4}{n} \left( C_g(\Omega, k) + D_g(\Omega, k) \right). \]

This will entail the result by taking the sum over \(k \geq 0\).

We consider the elementary squares of side \(1/n\), namely
\[ R_{a,b} = \left[ \frac{a}{n}, \frac{a+1}{n} \right] \times \left[ \frac{b}{n}, \frac{b+1}{n} \right], \]
and we concentrate on those which meet \(\Omega(k)\). There are two cases for such rectangles \(R_{a,b}\), namely
\[ (i) \quad R_{a,b} \subset \Omega(k), \quad \text{or} \quad (ii) \quad R_{a,b} \cap \Omega(k)^c \neq \emptyset. \]

In the first case \((i)\), the definition of the bound \(D_g\) entails the estimate
\[ \left| \frac{1}{n^2} g\left(\frac{a}{n}, \frac{b}{n}\right) - I(g, R_{a,b}) \right| \]
\[ \leq \frac{4}{n} \left( C_g(\Omega, k) + D_g(\Omega, k) \right). \]

\[ \text{By convention, we consider that } C_g(\Omega, k) \text{ and } D_g(\Omega, k) \text{ are 0 if the set } \Omega(k) \text{ is empty.} \]
\[
\leq I \left( \left| g \left( \frac{a}{n}, \frac{b}{n} \right) - g \right|, R_{a,b} \right) \leq \frac{1}{n^2} D_g(\Omega, k). \]

As the number of such squares is at most \( n^2 \), the contribution from case (i) is at most \( (1/n) D_g(\Omega, k) \).

In the second case (ii), the positivity of \( g \) and the definition of the bound \( C_g \) entails the estimate
\[
\left| \frac{1}{n^2} g \left( \frac{a}{n}, \frac{b}{n} \right) - I(g, \Omega \cap R_{a,b}) \right| \leq \frac{1}{n^2} C_g(\Omega, k).
\]

But, the convexity of \( \Omega \) entails that there are at most \( 4n \) such squares, and the contribution of the second case is at most \( (4/n) C_g(\Omega, k) \). The constant 4 is explained in the Annex.

### 3.5 Coprime Riemann sums and integrals

The following result is an extension of the results obtained in [3], that are only proven for finite domains.

**Proposition 3.3.** Consider a positive function \( g \) defined on \( \mathcal{R} \), homogeneous of degree \( -\beta \) there with \( \beta > 1 \). Such a function is strictly decreasing on \( \mathcal{R} \). Consider also a convex subset \( \Omega \subset \mathcal{R} \). Then, the coprime Riemann sum of the function \( g \) on \( \Omega \) compares to the integral of \( g \) on \( \Omega \), namely
\[
\left| \tilde{R}_n(g, \Omega) - \frac{6}{\pi^2} I(g, \Omega) \right| \leq \frac{1}{n} \left( 1 + 5\zeta(\beta) \right) M_g(\mathcal{R}).
\]

**Proof.** To filter the cases in which \( \gcd(a, b) > 1 \), we use the Mobius function \( \mu \) which performs “inclusion-exclusion”. The Mobius function \( \mu : \mathbb{N} \to \{-1, 0, +1\} \) satisfies
\[
\sum_{d | n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}
\]

We consider the restricted “coprime” Riemann sum, where the sum is taken over the pairs \((a, b)\) with \( \gcd(a, b) = 1 \), namely
\[
n^2 \tilde{R}_n(g, \Omega) = \sum_{(a, b) \in \mathbb{Z}^2, \gcd(a, b) = 1} g \left( \frac{a}{n}, \frac{b}{n} \right) \left[ \left( \frac{a}{n}, \frac{b}{n} \right) \in \Omega \right].
\]

We then “insert” the \( \mu \)-function inside this restricted “coprime” Riemann sum,
\[
n^2 \tilde{R}_n(g, \Omega) = \sum_{(a, b) \in \mathbb{Z}^2} g \left( \frac{a}{n}, \frac{b}{n} \right) \left[ \left( \frac{a}{n}, \frac{b}{n} \right) \in \Omega \right] \left( \sum_{d | \gcd(a, b)} \mu(d) \right).
\]

As the point \((a/n, b/n)\) belongs to \( \mathcal{R} \) with \( a > 0 \), the inequality \( \gcd(a, b) \leq n \) holds. Then, interverting the summations entails the equality
\[
n^2 \tilde{R}_n(g, \Omega) = \sum_{d \leq n} \mu(d) \sum_{(a, b) \in \mathbb{Z}^2} g \left( \frac{ad}{n}, \frac{bd}{n} \right) \left[ \left( \frac{ad}{n}, \frac{bd}{n} \right) \in \Omega \right].
\]

Finally, the following equality holds
\[
\tilde{R}_n(g, \Omega) = \sum_{d \leq n} \mu(d) R_n(g_d, \Omega_d),
\]
and involves the function \( g_d \) and the subset \( \Omega_d \) defined as
\[
g_d(x, y) = g(dx, dy), \quad \Omega_d = \frac{1}{d} \Omega.
\]

As the inclusion \( \Omega_d \subset S \) holds, we now apply the previous Proposition 3.2 to each (plain) Riemann sum \( R_n(g_d, \Omega_d) \) and obtain
\[
\left| R_n(g_d, \Omega_d) - I(g_d, \Omega_d) \right| \leq \frac{5}{n} M_{g_d}(\Omega_d).
\]

We now use three properties. We first remark the equality
\[
I(g_d, \Omega_d) = \frac{1}{d^2} I(g, \Omega),
\]
due to the change of variables \((x', y') = (dx, dy)\). Second, the series of general term \( \mu(d)/d^2 \) is convergent, and, with the Mobius inversion, its sum equal \( 1/\zeta(2) \) and
\[
\left| \sum_{d \leq n} \mu(d) \frac{6}{\pi^2} - 1 \right| \leq \frac{1}{n}.
\]

Third, we relate the bound \( M_{g_d}(\Omega_d) \) to its analogous. As \( g \) is homogeneous of degree \(-\beta\), its derivative is homogeneous of degree \((-\beta - 1)\) and the two relations
\[
g_d(x, y) = g(dx, dy) = \frac{1}{d^\beta} g(x, y),
\]
\[
\frac{\partial g_d}{\partial y}(x, y) = \frac{d}{dy} \frac{\partial g}{\partial y}(dx, dy) = \frac{1}{d^\beta} \frac{\partial g}{\partial y}(x, y),
\]
hold for \((x, y) \in \mathcal{R} \). As \( g \) and its derivative are 0 outside \( \mathcal{R} \), the same holds for \( g_d \) and its derivative, and
\[
M_{g_d}(\Omega_d) = \frac{1}{d^\beta} M_g(\Omega_d \cap \mathcal{R}) \leq \frac{1}{d^\beta} M_g(\mathcal{R}).
\]

Then, as \( \beta > 1 \), one has
\[
\sum_{d \leq n} M_{g_d}(\Omega_d) \leq \zeta(\beta) M_g(\mathcal{R}).
\]

With the three previous properties, together with Eq. (3.23), we obtain the final result.
3.6 Distributions. Proof of Theorem 2.1. Theorem 2.1 is a particular case of the previous Proposition 3.3 when it applies to \( \omega \) and \( \Delta_f(\lambda) \) defined in (2.11) and (2.9). The function \( \omega \) is homogeneous of degree -2 and the domain \( \Delta_f(\lambda) \) is convex, as it is the intersection of the unbounded rectangle \( R \) with the halfplane \( \{ f(x, y) \leq \lambda \} \). Applying Proposition 3.3 then proves Theorem 2.1.

3.7 Proof of Theorem 2.2. Assertion (a) is proven in the Annex. We prove now Assertion (b). We let
\[
F_n(\lambda) := \mathbb{P}[\Lambda_n \leq \lambda], \quad F_\infty(\lambda) = \frac{12}{\pi^2} I_f(\lambda).
\]
We know that the derivative \( J_f(\lambda) \) of \( \lambda \mapsto I_f(\lambda) \) exists. This is the same for the function \( F_\infty \) and we wish to estimate the difference
\[
\left| \frac{F_n(\lambda + \epsilon(n)) - F_n(\lambda)}{\epsilon(n)} - F'_\infty(\lambda) \right|.
\]
We begin with the triangle inequality
\[
(3.24) \quad \left| \frac{F_n(\lambda + \epsilon(n)) - F_n(\lambda)}{\epsilon(n)} - F'_\infty(\lambda) \right| \\
\leq \left| \frac{F_n(\lambda + \epsilon(n)) - F_\infty(\lambda + \epsilon(n))}{\epsilon(n)} \right| + \left| \frac{F_\infty(\lambda) - F_n(\lambda)}{\epsilon(n)} \right| \\
+ \left| \frac{F_\infty(\lambda + \epsilon(n)) - F_\infty(\lambda)}{\epsilon(n)} - F'_\infty(\lambda) \right|.
\]
With the special form of function \( f \), the domain \( \Delta_f(\lambda) \) is convex, and Theorem 2.1 provides the estimates
\[
|F_n(\lambda) - F_\infty(\lambda)| = O(1/n), \\
|F_n(\lambda + \epsilon(n)) - F_\infty(\lambda + \epsilon(n))| = O(1/n),
\]
where the constant in the \( O \)-terms does not depend on \( \lambda \) and \( \epsilon(n) \). Then, the first two terms in Inequality (3.24) are \( O(1/(n\epsilon(n))) \) which tends to 0 because \( n\epsilon(n) \to \infty \). For the last term in (3.24), we use Taylor expansion of order 2 of the function \( F_\infty \) together Assertion (a).

4 Conditional expectations. Proof of Thm 2.3

We now focus on conditional expectations. Our final purpose is to prove Theorem 2.3 which is devoted to the recurrence quotient. However, we begin by a more general study and we obtain in Section 4.3 a general result on conditional expectations (Theorem 4.4). We then apply it in Section 4.4 to the particular case of the recurrence quotient, and this provides Theorem 4.3 which can be viewed itself as an extension of Theorem 2.3.

4.1 Limit expectation of bounded \( Q \) functions. Thus far, we dealt with distributions of \( Q \)-functions. Now, we consider expected values of a \( Q \)-function, and use the equality
\[
\mathbb{E}[\Lambda_n] = \int_0^\infty \mathbb{P}[\Lambda_n \geq \lambda] d\lambda,
\]
valid when \( \Lambda \geq 0 \), as in our case. We consider here the case of a \( Q \) function \( \Lambda \) associated with a bounded function \( f \) (which is the case when \( b_2 \) is not zero). It is then possible to interchange the the limit and the integral and use Theorem 2.1.

When reversing the order of integration, we first integrate with respect to \( \lambda \), and we are led to the integral
\[
(4.25) \quad \mathbb{E}_\psi[f] := \frac{6}{\pi^2} I(f : 2\omega, R)
\]
which is exactly the expectation \( \mathbb{E}_\psi[f] \) of the function \( f \) on the rectangle \( R \) with respect to the density \( \psi := (12/\pi^2) \omega \). We thus obtain the following result which provides an extension of Theorem 2.1.

**Theorem 4.1.** Consider a \( Q \)-function \( \Lambda \) associated with a function \( f \) bounded by \( B_f \). Then the sequence \( n \mapsto \Lambda_n \) admits a limit expected value as \( n \to \infty \) equal to the expectation \( \mathbb{E}_\psi[f] \) of the function \( f \) on the rectangle \( R \) with respect to the density \( \psi := (12/\pi^2) \omega \), and
\[
(4.26) \quad \mathbb{E}[\Lambda_n] = \mathbb{E}_\psi[f] + B_f \left( \frac{1}{n} \right),
\]
where the constant in the \( O \)-term does not depend on \( f \) and \( \lambda \).

4.2 Case of the recurrence quotient. The function \( f \) associated with the recurrence quotient \( S(\alpha,n) \) is \( f_S(x,y) = 1 + x + y \). It is unbounded on \( \mathcal{R} \), and the function \( f_S \) is not integrable with respect to \( \psi \). In fact, by the argument of Proposition 3.1 the expected value can be worked out to be
\[
\mathbb{E}[S_n] = \hat{R}_n(2\omega f_S, \mathcal{R}),
\]
and here \( \hat{R}_n(2\omega f_S, \mathcal{R}) \) is infinite for each \( n \).

This is why we consider the conditional expectations for the sequence \( S_n \) with respect to an event \( \Gamma_n \geq \epsilon(n) \) associated with another \( Q \)-function \( \Gamma \), namely
\[
\mathbb{E}[S_n | \Gamma_n \geq \epsilon(n)].
\]
We will choose in the sequel the \( Q \)-function \( \Gamma \) from the set \( \{ \mu, \nu, \rho \} \) and a positive sequence \( \epsilon(n) \) tending to 0 not all too quickly.
4.3 General conditional expectations. We consider more general conditional expectations,

$$\mathbb{E}[\Lambda_n | \Gamma_n \geq \epsilon] \quad (\epsilon > 0)$$

when $\Gamma$ is a $\mathcal{Q}$-function associated with a function $g$ which tends to 0 for $y \to \infty$. (This means that the pair $(b_1, b_2)$ in (2.8) satisfies $b_1/b_2 = 0$). The subset

$$\{(x, y) \in \mathcal{R} \mid g(x, y) \geq \epsilon\}$$

is bounded for $\epsilon > 0$, and we denote, for $\epsilon > 0$,

$$B_{f|g}(\epsilon) := \sup\{f(x, y) \mid g(x, y) \geq \epsilon\} < \infty.$$ 

In this case, the expectation of $f$ with respect to $\psi$ conditioned to the event $[g \geq \epsilon]$ is well defined, and denoted as

$$\mathbb{E}_\psi[f | g \geq \epsilon].$$

The following holds, and its proof (similar to the proof of Theorem 2.1) is in the Annex.

**Theorem 4.2.** Consider two $\mathcal{Q}$-functions $\Lambda$ and $\Gamma$ with respective associated functions $f$ and $g$. Assume that $g$ tends to 0 for $y \to \infty$. Then the conditional expectation of $\Lambda_n$ with respect to the event $[\Gamma_n \geq \epsilon]$ satisfies

$$\mathbb{E}[\Lambda_n | \Gamma_n \geq \epsilon] \cdot \mathbb{P}[\Gamma_n \geq \epsilon] = \mathbb{E}_\psi[f | g \geq \epsilon] \cdot \mathbb{P}_\psi[g \geq \epsilon]$$

$$+ B_{f|g}(\epsilon) O\left(\frac{1}{n}\right)$$

where the constant in the $O$-term does not depend on either $f$, $g$ or $\epsilon$.

4.4 Return to the conditional expectation of the recurrence quotient. Proof of Theorem 2.3

We will prove here a stronger version of Theorem 2.3, where the remainder terms are more precise.

**Theorem 4.3.** Consider a parameter $\Gamma \in \{\rho, \mu, \nu\}$ defined in (2.2) and (2.4), and a sequence $n \to \epsilon(n)$ which tends to 0 with $\epsilon(n) = \Omega(1/n)$. Then the conditional expectation of the recurrence quotient $S_n$ with respect to the event $[\Gamma_n \geq \epsilon(n)]$ satisfies

$$\mathbb{E}[S_n | \Gamma_n \geq \epsilon(n)] = \frac{12}{\pi^2} |\log \epsilon(n)| + C(\Gamma)$$

$$+ O\left(\frac{1}{\epsilon(n)n} + \epsilon(n)|\log \epsilon(n)|^2\right).$$

Moreover, the constants $C(\Gamma)$ satisfy

$$C(\nu) = +1, \quad C(\mu) = 0, \quad C(\rho) = +1.$$

Proof. The proof is an application of Theorem 4.2. First, a direct computation with Theorem 2.1 shows that if $\Gamma$ is one of the $\mathcal{Q}$-functions $\rho$, $\mu$ or $\nu$, the following estimates hold

$$\mathbb{P}[\Gamma_n \geq \epsilon(n)] = 1 + O(\epsilon(n) + 1/n).$$

Along with the bounds and the integrals provided in Figure 5 (in the Annex), this implies the result. \(\blacksquare\)

Now, Theorem 2.3 is an immediate application of Theorem 4.3 by taking $\epsilon(n) = 1/n$.

5 Comparison between the two models.

We now compare the two models, the present model (with a large fixed $n$) and the model which was previously studied in [1], namely, the model with a large fixed $k$. We first recall our result of [1], we then study the number of continuants $q_k(\alpha)$ in an interval of the form $[n, cn]$ for $c > 1$.

5.1 Results in the previous model. When the integer $n$ of the interval $[q_{k-1}(\alpha), q_k(\alpha)]$ is at a position $\mu$ there, the recurrence quotient admits the expression

$$S_k(\mu)(\alpha) = f_{\mu}\left(\frac{q_{k-1}(\alpha)}{q_k(\alpha)}\right)$$

which involves the function

$$f_{\mu}(x) := 1 + \frac{1 + x}{x + \mu(1-x)}.$$ 

The main idea of the study “with a fixed $k$” relates the recurrence quotient and the $k$-th iterate of the Euclidean transfer operator $H$ via the equality

$$\mathbb{E}[S_k^{(\mu)}] = H^k\left[x \mapsto \frac{f_{\mu}(x)}{1+x}\right](0).$$

The operator $H$ admits nice dominant spectral properties, and, notably, the celebrated Gauss density

$$\rho(x) = \frac{1}{\log 2} \frac{1}{1+x}$$

as its fixed density. This leads to the estimate

$$\lim_{k \to \infty} \mathbb{E}[S_k^{(\mu)}] = 1 + \frac{1}{\log 2} \int_0^1 \frac{1}{t + \mu(1-t)} dt.$$

More precisely, we have shown the following in [1]: for the sequence $\mu_k = \tau^k$, with $\tau \in [\varphi^2, 1]$, (where $\varphi$ is the inverse of the Golden ratio), the following holds

$$\mathbb{E}[S_k^{(\tau^k)}] \sim \frac{1}{2 \log k} |\log \tau| \quad (k \to \infty).$$
5.2 Relation between the two models. We now wish to relate the two (asymptotic) models: the present model “with fixed large \( n \)” and the previous model “with fixed large \( k \)?” Of course, these two models should be close if the behaviour of the sequence \( k \mapsto q_k(\alpha) \) does not depend too strongly on \( \alpha \), and we know that it is not the case. However, the behaviour of the sequence \( k \mapsto \log q_k(\alpha) \) is much more regular, as it is well known (see for instance [5]) that

\begin{equation}
\lim_{k \to \infty} \frac{1}{k} \log q_k(\alpha) = L = \frac{\pi^2}{12 \log 2} \quad \text{for almost all } \alpha .
\end{equation}

Consider first the present model “with \( n \) fixed”, and a sequence \( \ell \mapsto n(\ell) = \tau^\ell \). Then Theorem 2.3 is written as

\begin{equation}
\mathbb{E}[S_{n(\ell)} \mid \mu_{n(\ell)} \geq \tau^{-\ell}] \sim \left[ \frac{12}{\pi^2 \log \tau} \right] \ell ,
\end{equation}

Furthermore, as \( n(\ell) \) belongs to the interval \([q_{k-1}(\alpha), q_k(\alpha)]\), the existence of the limit for the quotient \( q_k(\alpha)/k \), that holds for almost any \( \alpha \), and is recalled in (5.33) entails the relation between the index \( \ell \) and the index \( k := k(\alpha, n(\ell)) \), that holds for almost any \( \alpha \), namely

\begin{equation}
\log n(\ell) = \ell \log \tau \sim \frac{\pi^2}{12 \log 2} k(\alpha, n(\ell)) .
\end{equation}

Now, we deal with the model “with \( k \) fixed”, and we consider that the index \( k(\alpha, n(\ell)) \) satisfies (5.35) everywhere. Then, the application of the result in the model “with \( k \) fixed”, described in (5.32), should entail

\begin{equation}
\mathbb{E}[S_{k}^{(\tau^\ell)}] \sim \left[ \frac{1}{\log 2 \log \tau} \right] k \sim \left[ \frac{12}{\pi^2 \log^2 \tau} \right] \ell .
\end{equation}

Remark that the conditional events are not the same in the two equations (5.34) and (5.36):

- in (5.34), the event is \( \{ \alpha \mid \mu(\alpha, n(\ell)) \geq \tau^{-\ell} \} \),
- in (5.36), the event is \( \{ \alpha \mid \mu(\alpha, n(\ell)) \sim \tau^{-\ell} \} \).

This (heuristic) comparison exhibits in both cases a linear growth with respect to \( \ell \). However, the events of interest are not the same, and we have considered that the index \( k(\alpha, n(\ell)) \) satisfies (5.35) everywhere.

5.3 Number of continuants in an interval. There is also an interesting connection between the two models, that counts the number of terms of the sequence \( k \mapsto q_k(\alpha) \) belongs to the interval \([n, cn]\), for some fixed \( c > 1 \). We thus study the function

\( (\alpha, n) \mapsto T(\alpha, n) := \sum_{k \geq 0} \| q_k(\alpha) \in [n, cn] \| . \)

Proposition 5.1. Consider the Lévy constant \( \kappa := \exp \left( \pi^2/(12 \log 2) \right) \). Then the mean number of continuants in the interval \([n, cn]\) tends to 1 as \( n \to \infty \).

Proof. Even if \( T \) is not a \( \mathcal{Q} \)-function, its expectation \( \mathbb{E}[T_n] \) is expressed as a Riemann sum of the function \( 2\omega \), in a domain \( T_c \). However, the domain \( T_c \) is not a subset of the rectangle \( R \). We have indeed

\begin{equation}
\mathbb{E}[T_n] = \int_0^1 T(\alpha, n) d\alpha = \int_0^1 d\alpha \sum_k \| q_k(\alpha) \in [n, cn] \|
\end{equation}

\begin{align}
&= \sum_k \sum_{m \in \mathbb{N}^k} \| q_k(\alpha) \in [n, cn] \| \omega(\frac{q_k(\alpha)}{n}) \\
&= \frac{1}{n^2} \sum_k \sum_{m \in \mathbb{N}^k} \omega(\frac{q_k(\alpha)}{n}) \\
&= 2 \sum_{(a, b) \in \mathbb{Z}^2} \omega(\frac{a}{n}, \frac{b}{n}) \\
&= R_n(2\omega, T_c), \text{ with } T_c = \{(x, y) \mid x \leq y, 1 \leq y \leq c \} .
\end{align}

Even if \( T_c \) is not a subset of \( R \), Proposition 3.3 applies, and the coprime Riemann series admits a limit equal to the integral

\begin{equation}
\frac{6}{\pi^2} I(2\omega, T_c) = \frac{12 \log c}{\pi^2 \log 2} .
\end{equation}

6 Conclusions
Beginninging from the question “what does the recurrence function of a random Sturmian words look like?”, we define and work within a model that is natural at least from an algorithmic standpoint: pick a large integer \( n \) and let the slope of the word be drawn at random from \([0, 1]\). We are led to the notion of the so-called \( \mathcal{Q} \)-functions: functions that, given \( n \) and a slope \( \alpha \), place \( n \) within the sequence of continuants \( k \mapsto q_k(\alpha) \) of \( \alpha \), namely consider the index \( k := k(\alpha, n(\ell)) \), and then return a value depending only on the two ratios \((1/n)q_k-1(\alpha)\) and \((1/n)q_k(\alpha)\). The recurrence quotient of Sturmian words defines such a \( \mathcal{Q} \) function, via a Theorem of Morse and Hedlund, where \( n \) is the length of the factors and \( \alpha \) the slope of the word.

Then, we study the distribution of a general \( \mathcal{Q} \)-function. It defines in fact a sequence of distributions, and we prove that the limit distribution and the limit densities exist. They all involve, as a sort of reference density, the density \( \psi \) defined in (2.12), which plays a similar role to that of the Gauss density (defined in (5.30)) when one studies functions that depend on the ratio \( q_{k-1}(\alpha)/q_k(\alpha) \), and appears in our study [1].
Our results apply in particular to the recurrence quotient of Sturmian words; we exhibit the limit distribution (and the limit density) of such a quotient. We wish to compare this probabilistic study to the results of Morse and Hedlund, which exhibit extreme behaviours, attained when \( n \) is close to the left border \( q_{k-1}(\alpha) \) of the interval \([q_{k-1}(\alpha), q_k(\alpha)]\) containing the integer \( n \). That is why we also consider conditional expectations, our conditional events are related to the various parameters which describe the position of the integer \( n \) inside \([q_{k-1}(\alpha), q_k(\alpha)]\). We then compare this “constrained probabilistic” behaviours to the extreme behaviours, in a precise manner.

We had previously performed a similar study in [1] under another probabilistic model, where it is rather the index \( k \) of the interval \([q_{k-1}(\alpha), q_k(\alpha)]\) the integer \( n \) belongs to that is fixed. Then for \( k \to \infty \), we exhibited limit distribution and limit densities all of which involve, as a sort of reference density, the Gauss density. The two models are clearly different, but the two types of results show certain similarities.

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7 Annex

7.1 Proof of the constant 4 in Proposition 3.2.

To see where the constant 4 comes from, we first replace \( \Omega(k) \) by a closed convex polygon \( C_n \subset \Omega \) without affecting the bound : in each square \( R_{a,b} \) of the second case, pick a point in \( \Omega(k) \) and then take the convex hull. If \( \Omega(k) \) is a closed convex polygon, we go through the border in clockwise order and look at the grid rectangles we encounter as explained in Figure 4.

Figure 4: The convex domain \( \Omega \), and, in blue, a convex polytope \( P \). These two convex sets have the same grid squares that intersect both themselves and their complement. We traverse the polygon clockwise from the lowest vertex. Each time we intersect a horizontal line we move \( \pm 1 \) square horizontally in the grid, similarly for the vertical lines, and diagonals. Being the polygon convex, once we stop moving upwards vertically (at most \( n \) steps), we can only move downwards (at most \( n \) steps) when moving vertically. A similar observation for the horizontal case tells us that there can be at most \( 2n \) horizontal steps.

7.2 Proof of Theorem 2.2. (a). Assertion (a) of Theorem 2.2 describes the main properties of the first two derivatives of the function \( \lambda \mapsto I_f(\lambda) \). These properties are closely related to the geometry of the figure formed with the rectangle \( R \) together with the set of lines \( F \) containing a given point \((x_0, y_0)\).

The set of lines \( F \). In the set \( F \) of lines, defined as

\[
\mathcal{F} := \{ f(x, y) = \lambda \mid \lambda \in \mathbb{R} \},
\]

the equation of the line \( f(x, y) = \lambda \) is written in terms of coefficients described in (7.8) as

\[
(7.37) \quad (a_1 x + b_1 y + c_1) - \lambda (a_2 x + b_2 y + c_2) = 0.
\]

The case where the two vectors \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) are colinear is excluded, as in this case \( f(x, y) \) is constant. The case where \( b_1 = b_2 = 0 \) is also excluded as we wish that \( f \) depend on \( y \). Then, there is at most one vertical line in \( F \).

There are two cases for the set \( F \) defined in (7.37).
(i) the case when the determinant \( r(a, b) := a_1 b_2 - a_2 b_1 \)

is zero and in this case the determinant \( r(a, c) := a_1 c_2 - a_2 c_1 \) is not zero. The set \( \mathcal{F} \) is formed with parallel lines of slope \(-a_1/b_1\). This is for instance the case of the recurrence quotient with slope \(-1\) or the case of \( \nu \) with slope 0.

(ii) the case when the determinant \( r(a, b) := a_1 b_2 - a_2 b_1 \)

is not zero. In this case, we can choose \( r(a, b) = 1 \) due to the homogeneity of the problem. Then, the set \( \mathcal{F} \) is formed with all the lines which contain the point \((x_0, y_0)\) uniquely defined by the relations

\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} = \begin{pmatrix}
  -c_1 \\
  -c_2
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} = \begin{pmatrix}
  r(b, c) \\
  r(a, c)
\end{pmatrix}.
\]

The point \((x_0, y_0)\) is called the basic point of \( \mathcal{F} \). Remark that case (i) can be seen as the limit of the case (ii) when \((x_0, y_0)\) tends to \(\infty\) in the direction \(a_1/b_1\). The basic points attached to our parameters \(\rho, \mu\) are \((0, 0)\) for \(\rho\) and \((1, 1)\) for \(\mu\).

In the set \( \mathcal{F} \) of basic point \((x_0, y_0)\), the value of \(\lambda\) and the inverse \(1/\tau\) of the slope \(\tau\) of the line \(f(x, y) = \lambda\), are related via linear fractional transformations with determinant equal to 1, namely

\[
\lambda = F(\tau) = \frac{a_1 \tau + b_1}{a_2 \tau + b_2} \quad \tau = G(\lambda) = \frac{b_2 \lambda - b_1}{a_2 \lambda - b_2}.
\]

In the set \( \mathcal{F} \) of basic point \((x_0, y_0)\), the parametrization of the line \(f(x, y) = \lambda\) of slope \(1/\tau\) is thus

\[
x = x_0 + \tau(y - y_0), \quad \tau = G(\lambda).
\]

Expressions of \( I_f \) and its derivative. Consider a function \( f \) as in (2.3); denote by \( \delta_f(\tau) \) the segment (possibly empty or unbounded) which is the intersection of the line \( f(x, y) = \lambda = F(\tau) \) of slope \(1/\tau\) with the rectangle \( \mathcal{R} \). Now, the function \( f \) is fixed, the point \((x_0, y_0)\) is fixed, and all the indices which involve \( f \) are removed. There is an open interval \( \mathcal{D} \) which gathers the values of \(\tau\) for which the segment \(\delta(\tau)\) is not empty, and we denote by \( A(\tau), B(\tau) \) the ordinates of the two ends of the segment \(\delta(\tau)\).

As soon as the line \( f(x, y) = F(\tau) \) is not horizontal, we consider the natural parametrization \( h_\tau \) of the line \(\delta(\tau)\), namely a map \( h_\tau : [A(\tau), B(\tau)] \to \delta(\tau) \) which associates to \( \tau \) the point

\[
h_\tau(\tau) = h(\tau, y) = (x_0 + \tau(y - y_0), y)
\]

of the line \(\delta(\tau)\). The map \( \tau \mapsto h_\tau \) is of class \(C^\infty\) on \( \mathcal{D} \). Using the change of variables \((\theta, y) \mapsto (h(\theta, y), y)\), and its Jacobian \(|(\partial h)/(\partial \theta))(y, \theta)| = |y - y_0|\), the integral

\[
L(\tau) := L_f(\tau) := I_f(F(\tau)) = I_f \circ F(\tau) \quad \text{is written as}
\]

\[
L(\tau) = \int_{-\infty}^{\tau} d\theta \int_{A(\theta)}^{B(\theta)} Q(\theta, y) dy,
\]

with \( Q(\theta, y) = \omega(x_0 + \theta(y - y_0), y)|y - y_0|\).

(We have used the fact that \( F \) is increasing). Then the derivative of \( L \) admits the expression

\[
L'(\tau) = \int_{A(\tau)}^{B(\tau)} Q(\tau, y)dy
\]

The function \( L' \) is itself differentiable on the set \( \mathcal{D} \), except perhaps on a finite set (as we will see now) and involves the previous functions under the form

\[
L''(\tau) = \int_{A(\tau)}^{B(\tau)} \frac{\partial Q}{\partial \tau}(\tau, y)dy
\]

(7.39)

\[
+ B'(\tau)Q(\tau,B(\tau)) - A'(\tau)Q(\tau,A(\tau)),
\]

with \( \frac{\partial Q}{\partial \tau}(\tau, y) = \frac{\partial \omega}{\partial x}(x_0 + \tau(y - y_0), y)|y - y_0|^2 \).

We prefer to deal with the function \( L \), as it is easier to “see the geometry”. We will return to the function \( I \) and its two derivatives with the relations

\[
I'(\lambda) = \frac{L'(\tau)}{F'(\tau)}, \quad I''(\lambda)F'(\tau)^2 = L''(\tau) - L'(\tau)\frac{F''(\tau)}{F'(\tau)},
\]

and use the special form of \( F \) defined in (7.37).

The role of the corners. The values of \(\tau\) in \( \mathcal{D} \) for which \( I' \) is a priori not differentiable are those for which the line of slope \(1/\tau\) is vertical or meets one of the two “corners” of \( \mathcal{R} \), namely the slope \(1/\tau_0\) for which it meets the point \((0, 1)\), and the slope \(1/\tau_1\) for which it meets the point \((0, 1)\).

There are now two different geometric cases: the generic case (G) or the exceptional case (E), described as follows:

(G) If the point \((x_0, y_0)\) does not belong to the line \( y = 1 \), there are exactly two lines in \( \mathcal{F} \), each of them containing one corner of \( \mathcal{R} \), associated with two distinct values \(\tau_0\) and \(\tau_1\).

(E) If the point \((x_0, y_0)\) belongs to the line \( y = 1 \), there is only one value \(\tau_0 = \tau_1 = \infty\).

Finally, there are at most three values of \(\tau\) in the set \([0, \tau_0, \tau_1]\) where \( I' \) is possibly not differentiable. But, \( I' \) possesses at each finite \(\tau\) a left and a right derivative, each of them being finite. This is thus the same for the
**Parameter $\Gamma$** | **Bound for $S$** | **$\mathbb{E}_\psi[f_S|f_T \geq \epsilon(n)] P_\psi[f_T \geq \epsilon(n)]$**
---|---|---
$\rho$ | $S \leq 2 + 1/\rho \implies B_{f_S|f_T}(\epsilon) = O(1/\epsilon)$ | $A[\log(\epsilon(n)) + 1 - A\epsilon(n)|\log(\epsilon(n))|]$ 
$\mu$ | $S \leq 1 + 1/\mu \implies B_{f_S|f_T}(\epsilon) = O(1/\epsilon)$ | $A[\log(\epsilon(n)) + \frac{A}{1-\epsilon(n)}\epsilon(n)|\log(\epsilon(n))|]$ 
$\nu$ | $S \leq 1 + 2/\nu \implies B_{f_S|f_T}(\epsilon) = O(1/\epsilon)$ | $A[\log(\epsilon(n)) + 1$ 

Figure 5: In the second column, the bounds for $S$ for each parameter $\Gamma \in \{\rho, \mu, \nu\}$. In the third column, the values of the product $\mathbb{E}_\psi[f_S|f_T \geq \epsilon(n)] P_\psi[f_T \geq \epsilon(n)]$ needed to apply Theorem 4.2. The constant $A$ is $12/\pi^2$.

The bounded convex subset already described in (3.17)

$$\Delta_{f,g}(\lambda, \epsilon) := \{(x, y) \in \mathcal{R} \mid f(x, y) \geq \lambda, g(x, y) \geq \epsilon\}.$$ 

We have remarked in Section 3 that a slight extension of Proposition 3.1 entails the equality

$$\mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \hat{R}_n (2\omega, \Delta_{f,g}(\lambda, \epsilon)) .$$

Moreover, with the convexity of the domain $\Delta_{f,g}(\lambda, \epsilon) \subset \mathcal{R}$, Proposition 3.3 applies, yielding

$$\mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] = \frac{12}{\pi^2} I[\omega, \Delta_{f,g}(\lambda, \epsilon)] + O \left( \frac{1}{n} \right) .$$

Now we integrate on $\lambda$, noticing that we need only integrate from 0 to $B_{f,g}(\epsilon)$

$$\int_0^\infty \mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] d\lambda = \frac{12}{\pi^2} \int_0^\infty I[\omega, \Delta_{f,g}(\lambda, \epsilon)] d\lambda + B_{f,g}(\epsilon) O \left( \frac{1}{n} \right) .$$

We are led to the integral of $\omega$ on the domain of $\mathbb{R}^3$ defined by

$$\{(x, y, \lambda) \in \mathcal{R} \times \mathbb{R}_{\geq 0} \mid f(x, y) \geq \lambda, g(x, y) \geq \epsilon\}.$$ 

We interchange the summation, and we first integrate with respect to $\lambda$ (which provides the term $f(x, y)$), and obtain

$$\int_0^\infty I[\omega, \Delta_{f,g}(\lambda, \epsilon)] d\lambda = \int_{(x, y) \in \mathcal{R}, s(x, y) \geq \epsilon} \omega(x, y) \cdot f(x, y) dxdy ,$$

$$= \frac{\pi^2}{12} \mathbb{E}_\psi[f[g \geq \epsilon] \cdot P_\psi[g \geq \epsilon] .$$

7.3 **Proof of Theorem 4.2**

**Proof.** The conditional expectation is a ratio; the denominator is $\mathbb{P}[\Gamma_n \geq \epsilon]$ whereas the numerator

$$\int_0^\infty \mathbb{P}[\Lambda_n \geq \lambda, \Gamma_n \geq \epsilon] d\lambda .$$

Associate with the pair $(\Lambda, \Gamma)$ its function pair $(f, g)$ and, for any pair $(\lambda, \epsilon)$ of positive real numbers, consider

\[
\tau \left( B'(\tau) Q(\tau, B(\tau)) - A'(\tau) Q(\tau, A(\tau)) \right) \rightarrow 1
\]