Kernel Ridgeless Regression is Inconsistent in Low Dimensions
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Abstract. We show that for a large class of shift-invariant kernels, the kernel interpolation estimator is inconsistent in fixed dimensions, even with bandwidth adaptive to the training set.

1. Introduction. Recent empirical evidence has shown that certain algorithms, contrary to classical learning theory, can interpolate noisy data (achieve zero training error) while also generalizing well out of sample (low test error) [2,10,14]. We have also seen this phenomenon rigorously analyzed in theory for parametric methods such as linear regression, and random feature regression [1,3,7,9], as well as non-parametric methods such as kernel regression with singular kernels [4–6].

Many theoretical results demonstrating this phenomenon of “benign overfitting” (using the terminology of [1]) assume a high-dimensional regime where the data dimension $d$ grows with the sample size $n$, i.e., $d = \omega_n(1)$. However, it remains unclear whether this phenomenon is common when the data dimension is fixed.

A natural setting to examine this question is in the context of kernel machines, where the kernel interpolation estimator (also called kernel ridgeless regression) corresponds to the minimum norm interpolation in a reproducing kernel Hilbert space (RKHS). This is an important practical and theoretical setting, which also provides independent control on the data dimension $d$.

Indeed kernel interpolation has been studied in this light. The authors of [8] showed benign overfitting holds in high dimensions$^1$. On the other hand, recent work of Rakhlin and Zhai [11] showed that for the Laplace kernel, the kernel interpolation scheme is inconsistent in low dimensions even with an adaptive bandwidth.

In this paper, we generalize the result of [11] to a broad class of shift-invariant kernels satisfying mild spectral assumptions. Important examples of these kernels include the Laplace, Gaussian, and Dirichlet kernels.

Our counterexample uses a simple data model of the grid on the unit circle for $d = 1$, and, by extension, a multidimensional torus (the product of unit circles) when $d > 1$. For the sake of clarity, we outline the $d = 1$ case in the main body of the paper, and generalize to $d > 1$ in the Appendix (see Section C of the Appendix).

2. Problem setup.

Notation. We denote functions by lowercase letters $a$, sequences by uppercase letters $A$, vectors by lowercase bold letters $\mathbf{a}$, matrices by uppercase bold letters $\mathbf{A}$. $\mu$, $\mu_n$, $\mu_N$ are probability measures. Sequences are indexed using square-brackets, $A[k]$ where $k \in \mathbb{Z}$. For a vectors, functions, sequences, $\langle \mathbf{a}, \mathbf{b} \rangle$, $\langle a, b \rangle$, $\langle A, B \rangle$ denote their Euclidean, $L^2_\mu$, $\ell^2(\mathbb{Z})$ inner

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$^1$We refer to data dimension $d = \omega_n(1)$ as high dimensional and $d = O_n(1)$ as low dimensional.
products respectively, while $\|a\|, \|a\|_1, \|A\|_1$ denote corresponding induced norms, whereas $\|a\|_1, \|a\|_1, \|A\|_1$ denote their respective 1-norms. Like the $L^1$ norm, other norms or inner products will be pointed out explicitly. For a nonnegative integer $N$, we denote the set $\{0, 1, \ldots, N-1\}$ by $[N]$. We use $j$ to denote $\sqrt{-1}$, and an overline, $\overline{a}$, to denote complex conjugation. For asymptotic notation $O_Q(\cdot), o_Q(\cdot), o_Q(\cdot), \omega_Q(\cdot)$, have their usual meaning with $Q \to \infty$. In the absence of a subscript, the limit is with respect to sample size $n$.

We use $N \in \mathbb{N}$ as a resolution hyperparameter. For a sequence $G \in \ell^1(\mathbb{Z})$, and a fixed $N$, we define an $N$-hop subsequence $G_\ell \in \ell^1(\mathbb{Z})$ to be the sequence

\begin{equation}
G_\ell = \{G[mN + \ell]\}_{m \in \mathbb{Z}} \text{ defined for } \ell \in \{0, 1, \ldots, N-1\}.
\end{equation}

**Nonparametric regression.** We consider a supervised learning problem where we have $n$ labeled samples $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y} \subseteq \mathbb{R}^d \times \mathbb{R}$, with labels generated as,

\[ y_i = f^*(x_i) + \xi_i, \quad \{x_i\}_{i=1}^n \sim \mu_n, \quad \xi_i \sim \mathbb{P}_\xi, \]

for some unknown target function $f^*$. Here, we assume $\mu_n$ is a distribution on $\mathcal{X}$ that weakly converges to $\mu$ as $n \to \infty$. The noise distribution $\mathbb{P}_\xi$ is centered with a finite variance $\sigma^2 > 0$.

The estimation task is to propose an estimator $\hat{f}_n: \mathcal{X} \to \mathbb{R}$, where $\mathbf{y} = (y_i) \in \mathbb{R}^n$ is the vector of all labels. An estimator’s performance (or generalization error) is measured in terms of its mean squared error,

\[ \text{MSE}(\hat{f}_n, f^*) := \left\| \hat{f}_n - f^* \right\|_{L_2}^2 = \int_{\mathcal{X}} \left( \hat{f}_n(x) - f^*(x) \right)^2 \, d\mu(x). \]

**Consistency for a function.** We say that an estimator $\hat{f}_n$ is consistent for a target function $f^*$ if,

\[ \lim_{n \to \infty} \text{MSE}(\hat{f}_n, f^*) \to 0, \quad \text{in probability} \]

and inconsistent for that function otherwise. Note that we add the qualifier, in probability, because in general $\hat{f}_n$ is a random variable due to the randomness in $\xi = (\xi_i)$, and consequently so is $\text{MSE}(\hat{f}_n, f^*)$. While consistency usually is defined for a statement of the form $\hat{f}_n \xrightarrow{p} f^*$, the continuous mapping theorem [13, Thm. 2.3] tells us that these two statements are equivalent due to the continuity of the square function. Importantly, a sufficient condition for $\hat{f}_n$ to be inconsistent for $f^*$ is that,

\[ \lim_{n \to \infty} \mathbb{E}_\xi \text{MSE}(\hat{f}_n, f^*) > 0. \]

Our proof strategy is to establish such a condition.

**Kernel interpolation.** For an RKHS $\mathcal{H}$, the kernel interpolation estimator is given by,

\begin{equation}
\hat{f}_n = \operatorname{argmin}_{f \in \mathcal{H}} \left\| f \right\|_H \quad \text{subject to } f(x_i) = y_i \quad \text{for } i = \{1, 2, \ldots, n\}.
\end{equation}

Every RKHS is in one-to-one correspondence with a positive definite kernel function $K: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. Define the kernel matrix $K = (k(x_i, x_j))$ of pairwise evaluations of the kernel on
the training data. Due to the representer theorem [12], the solution to (2.2) lies in the span of $n$ basis functions $K(x_i, x)$ and can be written as

$$\hat{f}_n(x) = \sum_{i=1}^{n} \hat{\alpha}_i K(x_i, x), \quad \hat{\alpha} = (\hat{\alpha}_i) \in \mathbb{R}^n$$

where $y \in \mathbb{R}^n$ is the vector of all labels. Here $K$ is invertible because the kernel is positive definite, otherwise interpolation is not always possible. The (Riesz) representer of a given kernel $K$ at a datum $x_*$ is an element of $\mathcal{H}$, denoted by $K(x_*, \cdot) : \mathcal{X} \rightarrow \mathbb{R}$. It is the evaluation functional of $x_* \in \mathcal{X}$, i.e., $\langle f, K(x_*, \cdot) \rangle_{\mathcal{H}} = f(x_*)$ for all $f \in \mathcal{H}$. The basis functions above are thus the representers of the training data $\{x_1, x_2, \ldots, x_n\}$.

We define the restriction operator, and its adjoint, the extension operator, defined below,

$$R_n : \mathcal{H} \rightarrow \mathbb{R}^n \quad R_n f := (f(x_i)) \in \mathbb{R}^n, \quad \forall f \in \mathcal{H}$$

$$R^*_n : \mathbb{R}^n \rightarrow \mathcal{H} \quad R^*_n \alpha := \sum_{i=1}^{n} \alpha_i K(x_i, \cdot) \in \mathcal{H}, \quad \forall \alpha = (\alpha_i) \in \mathbb{R}^n$$

that evaluates the function on the data. Here, since $L^2_{\mu_n} \cong \mathbb{R}^n$ are isometric, we are abusing notation in favour of simpler expressions. This gives us the following equations

$$y = R_n f^* + \xi, \quad \text{and} \quad \hat{f}_n = R^*_n K^{-1} y.$$

For an RKHS we have two data dependent operators, the integral operator and the empirical operator, respectively given by,

$$\mathcal{I}_K f(x) = \int_{\mathcal{X}} K(x, x') f(x') \, d\mu(x'), \quad (2.3)$$

$$\mathcal{I}_n^\mathcal{X} f(x) = \int_{\mathcal{X}} K(x, x') f(x') \, d\mu_n(x'). \quad (2.4)$$

Eigenfunctions of $\mathcal{I}_K$ that form a countable orthonormal basis of $L^2_\mu$ can be used to provide an alternate representation for the $\mathcal{H}$-norm via the identity,

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{k \in \mathbb{Z}} \frac{\langle f, \varphi_k \rangle}{\sigma_k} \frac{\langle g, \varphi_k \rangle}{\sigma_k} \sigma_k$$

where $\langle \sigma_k, \varphi_k \rangle$ is an eigen-pair, i.e., $\mathcal{I}_K \varphi_k = \sigma_k \cdot \varphi_k$, with $\sigma_k \in \mathbb{R}_+$ and $\varphi_k \in L^2_\mu$.

**Fourier analysis:** We recall some useful quantities from Fourier analysis to be used later.

**Definition 2.1 (Fourier basis).** Let $\phi_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$, which satisfy

$$\langle \phi_k, \phi_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-t)t} \, dt = 1_{\{k=t\}}$$

An important tool in our analysis is the Fourier series representation of functions $\mathcal{X} \mapsto \mathbb{R}$. In general, any functions $\mathbb{R} \rightarrow \mathbb{R}$ periodic with period $2\pi$, adhere to such a representation.
Definition 2.2 (Fourier Series). For $f \in L^1_{[-\pi, \pi)}$, let $F$ be the Fourier series indexed by $k \in \mathbb{Z}$,
\[
f(t) = \sum_{k \in \mathbb{Z}} F[k] \phi_k(t) = \sum_{k \in \mathbb{Z}} F[k] e^{jkt}, \quad \forall t \in [-\pi, \pi)
\]
\[
F[k] = \langle f, \phi_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jkt} \, dt, \quad \forall k \in \mathbb{Z}
\]

Definition 2.3 (DFT Matrix). The normalized discrete Fourier transform (DFT) matrix is
\[
U = [u_0 \cdots u_{N-1}], \quad u_\ell = \frac{1}{\sqrt{N}} [1 \ e^{-j\frac{2\pi}{N} \ell} \ldots e^{-j\frac{2\pi}{N} (N-1) \ell}]^T, \quad \ell \in [N].
\]
Notice that $UU^H = U^H U = I$, where we use $^H$ to denote the conjugate transpose (hermitian) of a matrix.

Proposition 2.4 (Parseval). For a continuous $f : [-\pi, \pi) \to \mathbb{R}$ with Fourier series $F$,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, dt = \sum_{k \in \mathbb{Z}} |F[k]|^2.
\]

3. Model. We now describe a setting and demonstrate that kernel interpolation is inconsistent.

Data distribution: Grid on the unit circle. We describe the case of $d = 1$ and focus on $\mathcal{X} = [-\pi, \pi)$, viewed as the unit circle. An extension to $d > 1$ is deferred to Appendix C where we consider $[-\pi, \pi)^d$. We consider the continuous distribution $\mu := \text{Uniform}([-\pi, \pi))$ and discrete distributions indexed by a resolution hyperparameter $N \in \mathbb{N}$, given by
\[
\mu_N(x) := \frac{1}{N} \sum_{i=0}^{N-1} \delta(x - x_i) \quad x_i := \frac{2\pi}{N} i - \pi, \quad x \in [-\pi, \pi)
\]
i = 0, \ldots, N - 1, i.e., the uniformly spaced grid on the unit circle $[-\pi, \pi)$, where $\delta(\cdot)$ is the Dirac delta function, for which
\[
\int_{\mathbb{R}} f(x) \delta(x - y) \, dx = f(y), \quad \forall y \in \mathbb{R}
\]
We call $N$ the resolution parameter of the grid on $[-\pi, \pi)$, and assume $N$ is even for simplicity. Observe that $\mu_N$ weakly converges to $\mu$. For $d = 1$, the total number of samples $n$ equals the resolution $N$.

For $d > 1$, we consider $\mathcal{X} = [-\pi, \pi)^d$, the product of $d$ unit circles, and the respective grids, along each dimension. Thus $N$ is the number of samples per dimension, whereby the total number of samples $n = N^d$.

Shift-invariant kernels. We consider (periodic) kernels parameterized by a positive bandwidth parameter $M$,
\[
K(x, x') = g \left( M (x - x' \mod [-\pi, \pi]) \right), \quad x, x' \in \mathcal{X}
\]
\footnote{In machine learning, the bandwidth is often denoted by $1/M$.}
for some even function \( g : \mathbb{R} \to \mathbb{R} \). where we denote,

\[
\theta \mod [-\pi, \pi) = ((\theta + \pi) \mod 2\pi) - \pi \in [-\pi, \pi)
\]

We denote the RKHS corresponding to \( K \) by \( \mathcal{H} \). For ease of notation, when \( M = 1 \), we write the kernel \( K_0 \). We refer to this as the base kernel and the base space \( \mathcal{H}_0 \).

Define \( G_0, G : \mathbb{Z} \to \mathbb{C} \) as the following Fourier series, i.e.

\[
\begin{align*}
G[\theta \mod [-\pi, \pi)] &= \sum_{k \in \mathbb{Z}} G[k] \exp(jk\theta) \\
G[k] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(M\theta) \exp(-jk\theta) \, d\theta \\
(3.2a) \\
G[\theta \mod [-\pi, \pi)] &= \sum_{k \in \mathbb{Z}} G_0[k] \exp(jk\theta) \\
G_0[k] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \exp(-jk\theta) \, d\theta \\
(3.2b)
\end{align*}
\]

While usually the bandwidth scales the input, we note our analysis also holds for different mechanisms, e.g., the Dirichlet kernel (see Appendix C.2). For positive definite kernels, \( g \) is an even function whereby we have,

\[
G[k] = G[-k] \geq 0 \quad \forall k \in \mathbb{Z}
\]

Hence the above complex exponentials in equation (3.2a) can be written as cosine series, although we use complex exponentials for ease of exposition.

**Proposition 3.1.** \( u_\ell \) and \( \overline{u}_\ell \) are eigenvectors of \( K = (K(x_i, x_j)) \in \mathbb{R}^{N \times N} \), with eigenvalue \( \lambda_\ell = N \|G_\ell\|_1 \), i.e., \( K u_\ell = \lambda_\ell u_\ell \) and \( K \overline{u}_\ell = \lambda_\ell \overline{u}_\ell \). Furthermore,

\[
K = \sum_{\ell=0}^{N-1} \lambda_\ell \overline{u}_\ell u_\ell^H, \quad K^{-1} = \sum_{\ell=0}^{N-1} \frac{1}{\lambda_\ell} \overline{u}_\ell u_\ell^H, \quad K^{-2} = \sum_{\ell=0}^{N-1} \frac{1}{\lambda_\ell^2} \overline{u}_\ell u_\ell^H.
\]

**Proposition 3.2.** For any \( M > 0 \), the Fourier basis are eigenfunctions of the kernel integral operator \( T_K \) with eigenvalues \( G \), i.e., we have,

\[
T_K \phi_k = G[k] \cdot \phi_k
\]

The proofs to these propositions are provided in Appendix B.

For \( \mu_N \), we define the restriction operator \( R_N \), and its adjoint, the extension operator,

\[
\begin{align*}
R_N : \mathcal{H} \to \mathbb{R}^N \\
R_N f &= \left( f \left( \frac{2\pi}{N} (i-1) - \pi \right) \right)_i \in \mathbb{R}^N \\
R_N^* : \mathbb{R}^N \to \mathcal{H} \\
R_N^* \alpha := \sum_{i=0}^{N-1} \alpha_i K(x_i, \cdot) \in \mathcal{H}
\end{align*}
\]

Here we are using the fact that \( L^2_{\mu_N} \cong \mathbb{R}^N \). We also use the notation

\[
\langle \alpha, K(X_N, \cdot) \rangle_N := \sum_{i=0}^{N-1} \alpha_i K(x_i, \cdot)
\]
to keep expressions simple. With this notation, the labels and the kernel interpolator can be written as

\[ \hat{f}_N = R_N^* K^{-1} y = \langle K^{-1} y, K(X_N, \cdot) \rangle_N \] (3.3)

Definition 3.3 (Span of Riesz Representers). Functions in the range of \( R_N^* \), and of \( \mathcal{T}^N_K \), are in the span of the representers \( \{ K(x_i, \cdot) \}_{i=1}^N \).

Target function. We assume the target function lies in the base RKHS \( H_0 \), i.e., \( H \) with \( M = 1 \), and has a norm \( \| f^* \|_{H_0} = O_{M,N}(1) \). As the target function is defined on the unit circle, it admits a Fourier series,

\[ f^* = \sum_{k \in \mathbb{Z}} V[k] \phi_k. \] (3.4)

Let \( P_X \) be the \( L^2_\mu \)-projection operator onto the span of the representers, i.e.,

\[ P_X f := \arg\min_{h \in H} \left\{ \| f - h \| : h = \sum_{i=1}^N \alpha_i K(x_i, \cdot) \text{ for some } (\alpha_i) \in \mathbb{R}^N \right\} \] (3.5a)

\[ \alpha^* := (\alpha^*_i) \text{ such that } P_X f^* = \sum_{i=1}^N \alpha^*_i K(x_i, \cdot) \] (3.5b)

\[ f^*_\perp := f^* - P_X f^*, \] (3.5c)

where \( f^*_\perp \) is orthogonal to all functions in \( \text{Span} \{ K(x_i, \cdot) \} \). An immediate identity using the evaluation operator \( R_N \) is,

\[ R_N P_X f^* = K \alpha^* \quad \text{and} \quad R_N^* \alpha^* = P_X f^* \]

We can decompose the target function as

\[ f^* = P_X f^* + f^*_\perp = \sum_{i=0}^{N-1} \alpha^*_i K(x_i, \cdot) + f^*_\perp = \langle \alpha^*, K(X_N, \cdot) \rangle_N + f^*_\perp \]

Using this, the vector of labels, and the kernel interpolation estimator can be written as,

\[ y = R_N f^* + \xi = R_N P_X f^* + R_N f^*_\perp + \xi = K \alpha^* + R_N f^*_\perp + \xi \] (3.6)

\[ \hat{f}_N = R_N^* K^{-1} y = P_X f^* + \langle K^{-1} R_N f^*_\perp, K(X_N, \cdot) \rangle_N + \langle K^{-1} \xi, K(X_N, \cdot) \rangle_N \] (3.7)

where we have used the expression from Equation (3.3).

4. Main result: Inconsistency of kernel interpolation. Our main result holds under certain assumptions on the shift-invariant kernels. Below, we assume \( M', i, i^* \) are all non-negative integers. Recall that \( G \) is the Fourier series of the kernel function, see Equation (3.2a).
Assumption 1 (Scale). \[ \sum_{k \in \mathbb{Z}} |G[k]| < \infty. \]

Assumption 2 (Spectral Tail). For all \( k \in \mathbb{Z}_{\geq 0} \), there exists a constant \( C_1 > 0 \) such that,
\[
|G[M'k + i]| \leq \frac{C_1|G[i]|}{1 + k^2}.
\]
holds for all \( M' \geq M \) and for all \( i \leq M' \), except \( o_M(M') \) many.

Assumption 3 (Spectral Head). There exist constants \( C_2, C_3 \in \mathbb{R}^+ \) and \( i^* \in \mathbb{Z}_{\geq 0} \) such that for \( M \geq C_2 \), we have that for all \( M' < M \), \( |G[i^*]| \leq C_3|G[i^* + M']| \) and \( |G_0[i^*]| > 0 \).

Remark 1. Note that if the coefficients \( G[i] \) are bounded from above and below by a monotonically decreasing function with sufficiently fast decay, e.g., for some constants \( c, C > 0 \), we have \( cf(k) \leq G[k] \leq Cf(k) \), where \( f(k) = e^{-k^2} \), or \( f(k) = \frac{1}{1+k^2} \), then these conditions are satisfied.

The proof of the following Proposition is provided in Appendix B.

Proposition 4.1. The Gaussian \( g(t) = \exp(-t^2) \), Laplace \( g(t) = \exp(-|t|) \), and Dirichlet \( g(t) = \frac{\sin((M+1/2)t)}{\sin(t/2)} \) kernels satisfy Assumptions 1-3.

We present the main results in the following theorems. Recall that \( \mathcal{H}_0 \) is the base RKHS.

Theorem 4.2 (Inconsistency for all Bandwidths). For any shift-invariant kernel satisfying Assumptions 1-3, there exists a function with constant \( \mathcal{H}_0 \)-norm for which kernel interpolation will be inconsistent for any bandwidth, even adaptive to the data set.

Theorem 4.3 (Inconsistency for all Functions). For any shift-invariant kernel satisfying Assumptions 1-2, with a bandwidth \( M \leq N \), kernel interpolation will be inconsistent for all target functions that can be expressed as convergent Fourier series. In particular, kernel interpolation with any fixed bandwidth will be inconsistent for all such functions.

To prove these results we apply Fourier analysis to compute an exact expression for the MSE of kernel interpolation. We decompose the MSE for a target function into three components - (i) an approximation error, measuring how close the target function is to the span of the representers, (ii) a noiseless estimation error, measuring the error in the absence of noise, and (iii) a noisy estimation error, measuring the average error if the target function is 0.

We then apply Parseval’s Theorem, which relates these errors terms to the Fourier series of the target function, and of the kernel. Proving that the MSE is bounded away from 0 will rely on our assumptions on the tail and the head of the kernel spectrum.

5. Decomposition of mean squared error. We now derive an exact expression for the MSE as a sum of three error terms: the approximation error, the noise-free estimation error, and the noisy estimation error. This useful expression will allow us to prove the main theorems of the previous section.

Recall the definition of \( f^*_\perp, \alpha^*, P_X f^* \) from equation (3.5). We have the following lemma.

Lemma 5.1 (MSE Decomposition). For any square integrable \( f^* \), the kernel interpolation
estimator satisfies,

\[
\mathbb{E}_\xi \text{MSE} (\hat{f}_N, f^*) = \left\| f^* - P_X f^* \right\|^2 + \left\| \langle \mathbf{K}^{-1} R_N \{ f^* - P_X f^* \} , K(X_N, \cdot) \rangle_N \right\|^2
\]

Proof. Since \( P_X f^* - \hat{f}_N \in \text{Span} \{ K(x_i, \cdot) \} \), the Pythagorean theorem for the triangle \( \{ f^*, P_X f^*, \hat{f}_N \} \), yields,

\[
\text{MSE} (\hat{f}_N, f^*) = \left\| f^* - \hat{f}_N \right\|^2 = \left\| f^* - P_X f^* \right\|^2 + \left\| P_X f^* - \hat{f}_N \right\|^2
\]

Notice that the estimation error above is random, since \( \hat{f}_N \) is random due to the randomness in \( \xi \). Using equation (3.7), we can further decompose the average estimation error into two error terms,

\[
\mathbb{E}_\xi \| P_X f^* - \hat{f}_N \|^2 = \mathbb{E}_\xi \left( \| \mathbf{K}^{-1} R_N f^* \perp, K(X_N, \cdot) \rangle_N + \langle \mathbf{K}^{-1} \xi, K(X_N, \cdot) \rangle_N \right)^2
\]

where the cross term cancels out since the noise is 0-mean. This concludes the proof.

Lemma 5.2. For a target function \( f^* = \sum_{k \in \mathbb{Z}} V[k] \phi_k \), we have

(a) Approximation error: \( \mathcal{E}^{\text{apx}} := \| f^* - P_X f^* \|^2 = \sum_{i=0}^{N-1} \| V_i \|^2 - \| G_i \|^2 = \sum_{i=0}^{N-1} \mathcal{E}^{\text{apx}}_i \)

(b) Noise-free estimation error:

\( \mathcal{E}^{\text{free}} := \| \mathbf{K}^{-1} R_N f^* \perp, K(X_N, \cdot) \rangle_N \|^2 = \sum_{i=0}^{N-1} \frac{1}{N} \left( \frac{\| V_i \|_1}{\| G_i \|_1} - \frac{\langle G_i, V_i \rangle}{\| G_i \|^2} \right)^2 \| G_i \|^2 = \sum_{i=0}^{N-1} \mathcal{E}^{\text{free}}_i \)

(c) Averaged noisy estimation error:

\( \mathcal{E}^{\text{noise}} := \mathbb{E}_\xi \| \mathbb{K}^{-1} \xi, K(X_N, \cdot) \rangle_N \|^2 = \sum_{i=0}^{N-1} \frac{\sigma^2}{N} \left( \frac{\| G_i \|_1}{\| G_i \|^2} \right)^2 = \sum_{i=0}^{N-1} \mathcal{E}^{\text{noise}}_i \)
Together, this yields that the MSE for the function \( f^* \) is,

\[
(5.2) \quad \mathbb{E}_\xi \text{MSE} \left( \hat{f}_N, f^* \right) = \sum_{i=0}^{N-1} \xi_i = \sum_{i=0}^{N-1} \epsilon_i^{\text{apx}} + \epsilon_i^{\text{free}} + \epsilon_i^{\text{noise}}
\]

\[
(5.3) \quad \xi_i := \|V_i\|^2 - \frac{(G_i, V_i)^2}{\|G_i\|^2} + \frac{1}{N} \left( \|V_i\|_1 - \frac{(G_i, V_i)}{\|G_i\|^2} \right)^2 \|G_i\|^2 + \frac{\sigma^2}{N} \left( \|G_i\| \right)^2
\]

Appendices A.1, A.2, and A.3 provide proofs for Lemma 5.2 (a), (b), and (c) respectively.

6. Proof of main result. We are now ready to prove the main results. The proof strategy is to obtain an \( \Omega(1) \) lower bound on \( \mathbb{E}_\xi \text{MSE} \left( \hat{f}_N, f^* \right) \). Equation (5.2) expressed this quantity as a sum of \( N \) non-negative quantities. We show that at least \( \Omega(N) \) of these quantities, \( \epsilon_i \), are \( \Omega(1/N) \).

Proof of Theorem 4.2. When \( M > N \), we show that the approximation error is large for a cosine function in the base RKHS \( \mathcal{H}_0 \). On the other hand, when \( M \leq N \) we show that the averaged noisy estimation error has a constant lower bound.

Case 1, \( M > N \). In this case we show the approximation error is bounded away from 0. Since \( M > N \), by Assumption 3, there exists a constant integer \( i^* \), such that \( |G[i^*]| \leq C_3 |G[N+i^*]| \). Now let \( f^* \) be the (real-valued) function with Fourier coefficients \( V[i^*] = V[-i^*] = \frac{1}{\sqrt{2}} \), and \( V[k] = 0 \) for \( |k| \neq i^* \). Using this, we can lower bound the approximation error as,

\[
\epsilon_{i^*}^{\text{apx}} \geq \frac{2V[i^*]^2}{2\pi} \left( 1 - \frac{|G[i^*]|^2}{\sum_{m \in \mathbb{Z}} |G[mN+i^*]|^2} \right) \geq \frac{1}{2\pi} \left( 1 - \frac{|G[i^*]|^2}{|G[N+i^*]|^2 + |G[i^*]|^2} \right) \geq \frac{1}{2\pi(1+C_3)}
\]

The fact that \( G[i^*] > 0 \) from the Assumption 3, also allows us to conclude that

\[
\|f^*\|_{\mathcal{H}_0} = \frac{|V[k]|^2}{G[i^*]} < \infty.
\]

Case 2, \( M \leq N \). In this case we show that the noisy estimation error is bounded away from 0. Define, \( \Delta_i := \|G_i\|_1 - |G[i]| = \sum_{m \neq 0} |G[mN+i]| \geq 0 \). Assumption 2 says that for all but \( o(N) \) terms \( i \in [N] \), we have,

\[
(6.1) \quad \Delta_i = \sum_{m \neq 0} |G[mN+i]| \leq C_1 |G[i]| \sum_{m \neq 0} \frac{1}{1+m^2} \leq 4C_1 |G[i]|.
\]

For such an \( i \), we can lower bound the noisy estimation error term \( \epsilon_i^{\text{noise}} \) as,

\[
\epsilon_i^{\text{noise}} = \sigma^2 \frac{\|G_i\|^2}{\|G_i\|^2} = \sigma^2 \frac{\|G_i\|^2}{N} \left( |G[i]| + \Delta_i \right)^2 \geq \sigma^2 \frac{\|G[i]\|^2}{N} \frac{2|G[i]|^2 + 2|\Delta_i|^2}{2|\Delta_i|^2} \geq \frac{\sigma^2}{2N(1+4C_1)}
\]

\[
\epsilon_i^{\text{noise}} = \Omega(\sigma^2)
\]

where the last line holds since there are \( \Omega(N) \) such \( i \in [N] \) for which equation (6.1) holds.
Proof of Theorem 4.3. Theorem 4.3 follows from the proof of Theorem 4.2. We showed that for $M \leq N$ (Case 2 above), the noisy estimation error $\mathcal{E}^{\text{noise}}$ satisfies $\mathcal{E}^{\text{noise}} = \Omega(\sigma^2)$. Since $\mathcal{E}^{\text{noise}}$ is independent of the target function $f^*$, the statement of Theorem 4.3 follows.

7. Inconsistency of kernel interpolation for $d > 1$. We can perform an analysis similar to the one outlined above for the case of $d > 1$. To generalize the main results, we also generalize Assumptions 1-3 for the kernel to Assumptions 4-6 for dimensions greater than one. Under these assumptions, we show the main results hold.

Theorem 7.1 (Inconsistency for all Bandwidths). For any shift-invariant kernel satisfying Assumptions 4-6, there exists a function with constant $\mathcal{H}_0$-norm for which kernel interpolation will be inconsistent for any bandwidth, even adaptive to the data set.

Theorem 7.2 (Inconsistency for all Functions). For any shift-invariant kernel satisfying Assumptions 4-5, with a bandwidth $M \leq N$, kernel interpolation will be inconsistent for all target functions that can be expressed as convergent Fourier series. In particular, kernel interpolation with any fixed bandwidth will be inconsistent for all such functions.

Further, these results hold for the Gaussian, Laplace, and Dirichlet kernels.

Proposition 7.3. The Gaussian $g(\|x\|) = \exp(-\|x\|^2)$, Laplace $g(x) = \exp(-\|x\|)$, and Dirichlet $g(x) = (2\pi)^{-d} \sum_{m \in \{-M, \ldots, M\}^d} e^{-j\langle m, x \rangle}$ kernels satisfy Assumptions 4-6.

8. Conclusion. Kernel machines are widely used in machine learning. We show that ridgeless kernel regression is inconsistent in fixed dimension even with adaptive bandwidth. This provides a generalization of the main result in [11] (where it was shown for the Laplace kernel) to a broad class of radial kernels including Gaussian and Dirichlet kernels.

The Blessing of Dimensionality. When the dimensionality scales with the number of samples, kernel ridgeless regression can generalize [8]. Our results provide additional evidence that high dimensions can dissipate the error due to noise. In particular, under our assumptions on the kernel, for the expression of the noisy estimation error (Lemma C.3(c)), the constants decay exponentially with dimension. This was also observed in [11] for the Laplace kernel.

Further, to counteract the error due to noise, the bandwidth should be much larger than the data resolution in each dimension. However, when the dimensions scale with the number of samples, the resolution $N = n^{1/d}$ in each dimension is approximately constant, and therefore the bandwidth does not need to increase. As increasing the bandwidth in general will worsen the approximation error, the constancy of the bandwidth is a form of the blessing of dimensionality.

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Appendices.

Appendix A. Decomposition of MSE: Proof of Lemma 5.2. Appendices A.1, A.2, and A.3 provide proofs for Lemma 5.2 (a), (b), and (c) respectively.

A.1. Approximation error: Proof of Lemma 5.2(a). The proof proceeds by applying the Pythagorean theorem to the triangle \( \{0, f^*, P_X f^*\} \) in \( L^2_\mu \). The following lemma gives exact expressions for projection of the target function and its norm.

**Lemma A.1 (Projection).** For \( f^* = \sum_{k \in \mathbb{Z}} V[k]\phi_k \)

\[
P_X f^* = \sum_{\ell=0}^{N-1} \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|^2} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN+\ell}, \quad \text{and} \quad \|P_X f^*\|^2 = \sum_{\ell=0}^{N-1} \frac{\langle G_\ell, V_\ell \rangle^2}{\|G_\ell\|^2},
\]

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We get,

\[ \|f^* - P_X f^*\|^2 = \|f^*\|^2 - \|P_X f^*\|^2 = \|V\|^2 - \sum_{\ell=0}^{N-1} \frac{\langle G_{\ell}, V_\ell \rangle^2}{\|G_{\ell}\|^2}. \]

**Proof of Lemma A.1.** Note that Lemma B.5 shows that \( \left\{ \frac{\psi_{\ell}}{\|\psi_{\ell}\|} \right\}_{\ell=0}^{N-1} \) is an orthonormal basis for \( \text{Span}\{K(x_{\ell}, \cdot)\} \). Consequently, we have

\[ P_X f^* = \sum_{\ell=0}^{N-1} \left( \frac{f^*, \psi_{\ell}}{\|\psi_{\ell}\|} \right) \psi_{\ell} \quad \text{and} \quad \|P_X f^*\|^2 = \sum_{\ell=0}^{N-1} \left( \frac{f^*, \psi_{\ell}}{\|\psi_{\ell}\|} \right)^2 \]

We compute these projections below. For \( \ell \in [N] \),

\[ \sqrt{\|G_{\ell}\|_1} \left( f^*, \psi_{\ell} \right) = \left( \sum_{k \in \mathbb{Z}} V[k] \phi_{k, \ell} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN+\ell} \right) = \sum_{m, k \in \mathbb{Z}} G[mN + \ell] V[k] \mathbb{1}_{\{k=mN+\ell\}} = \langle G_{\ell}, V_\ell \rangle \]

Thus, we get that,

\[ \left( \frac{f^*, \psi_{\ell}}{\|\psi_{\ell}\|} \right) \frac{\psi_{\ell}}{\|\psi_{\ell}\|} = \frac{\langle G_{\ell}, V_\ell \rangle}{\|G_{\ell}\|^2} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN+\ell} \]

The claims follow immediately.

**A.2. Noise-free estimation error: Proof of Lemma 5.2(b).** Let \( E \) be the fourier series of \( \langle K^{-1} R_N \{ f^* - P_X f^* \}, K(X_N, \cdot) \rangle \). From Lemma B.3 we have,

\[ E[k] = \sqrt{N} R_N \{ f^* - P_X f^* \}^\top K^{-1} u_{k \mod N} \cdot G[k] \]

By Parseval’s theorem (Proposition 2.4), we conclude,

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \langle K^{-1} R_N \{ f^* - P_X f^* \}, K(X_N, t) \rangle \right)^2 dt = \sum_{k \in \mathbb{Z}} |E[k]|^2 \]

First, we show that

\[ R_N \{ f^* - P_X f^* \}^\top K^{-1} u_{\ell} = \left( \frac{\|V_{\ell}\|_1}{\|G_{\ell}\|_1} - \frac{\langle G_{\ell}, V_{\ell} \rangle}{\|G_{\ell}\|^2} \right) \]

From Proposition 3.1 we have \( K^{-1} u_{\ell} = u_{\ell} \cdot \frac{1}{\|G_{\ell}\|_1} \). Thus by Lemma A.1, we can write \( P_X f^* \) on the data as

\[ P_X f^*(x_i) = \sum_{\ell=0}^{N-1} \frac{\langle G_{\ell}, V_{\ell} \rangle}{\|G_{\ell}\|^2} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN+\ell}(x_i) = \sum_{\ell=0}^{N-1} \frac{\langle G_{\ell}, V_{\ell} \rangle}{\|G_{\ell}\|^2} \|G_{\ell}\|_1 \overline{u_{\ell}} \]

\[ f^*(x_i) = \sum_{k \in \mathbb{Z}} V[k] \phi_k(x_i) = \sum_{\ell=0}^{N-1} \|V_{\ell}\|_1 \overline{u_{\ell}} \]

\[ \|f^* - P_X f^*\|^2 = \|f^*\|^2 - \|P_X f^*\|^2 = \|V\|^2 - \sum_{\ell=0}^{N-1} \frac{\langle G_{\ell}, V_{\ell} \rangle^2}{\|G_{\ell}\|^2}. \]
We thus have
\[ R_N \{ f^* - P_X f^* \}^\top K^{-1} u_\ell = \sum_{i,\ell=0}^{N-1} \left( \frac{\| V_{\ell,i} \|_1 - \langle G_{\ell,i}, V_{\ell,i} \rangle}{\| G_{\ell,i} \|_1^2} \right) \frac{\pi \langle e_i, u_\ell \rangle}{N \| G_{\ell,i} \|_1} \]

This gives,
\[ \sum_{k \in \mathbb{Z}} |E[k]|^2 = \frac{1}{N} \sum_{\ell=0}^{N-1} \left( \frac{\| V_{\ell} \|_1}{\| G_{\ell} \|_1} - \frac{\langle V_{\ell}, G_{\ell} \rangle}{\| G_{\ell} \|_1^2} \right)^2 \| G_{\ell} \|_1^2. \]

**A.3. Noisy estimation error: Proof of Lemma 5.2(c).** We derive this by an application of Parseval’s theorem. Define the Fourier series,
\[ \langle K^{-1} \xi, K(X_N, t) \rangle_N = \sum_{k \in \mathbb{Z}} \tilde{E}[k] e^{jkt} \]
By Proposition 2.4 (Parseval’s theorem), we have,
\[ \mathbb{E}_\xi \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \langle K^{-1} \xi, K(X_N, t) \rangle_N \right|^2 dt \right] = \sum_{k \in \mathbb{Z}} \mathbb{E}_\xi \left| \tilde{E}[k] \right|^2 = \sum_{i=0}^{N-1} \sum_{m \in \mathbb{Z}} \mathbb{E}_\xi \left| \tilde{E}[mN+i] \right|^2 \]
\[ \overset{(a)}{=} \sum_{i=0}^{N-1} \sum_{m \in \mathbb{Z}} \| G[mN+i] \|^2 \mathbb{E}_\xi \left| \xi^\top K^{-1} u_i \right|^2 \cdot N \overset{(b)}{=} \sigma^2 \sum_{i=0}^{N-1} \| G_i \|^2 \left( u_i^\top K^{-2} u_i \right) \cdot N \]
\[ \overset{(c)}{=} \sigma^2 \sum_{i=0}^{N-1} \| G_i \|^2 \frac{1}{N^2 \| G_i \|_1^2} N = \sigma^2 \sum_{i=0}^{N-1} \| G_i \|^2 \frac{1}{N \| G_i \|_1^2} N \]
where we have used Lemma B.3 in (a), and Lemma B.1 in (b), and Proposition 3.1 in (c).

**Appendix B. Miscellaneous results.**

**Proof of Proposition 4.1 (d=1): Gaussian and Laplace kernels satisfy the assumptions.** The proofs for \( d > 1 \) (including the Dirichlet kernel \( d \geq 1 \) proofs) are provided in Appendix C.2.

**Gaussian kernel.** In Lemma B.2 we show that when \( g(t) = \exp(-M^2t^2) \) we have \( G[k] = G[0] e^{-k^2/4M^2} \). Thus \( \sum_{k \in \mathbb{Z}} \exp(-k^2/4M^2) < \infty \), proving that Assumption 1 is satisfied.

Next, for all \( i \in \mathbb{Z}^+ \) and \( M' \geq M \),
\[ \frac{G[M'k+i]}{G[i]} = \exp(-(kM' + i)^2/4M'^2 + i^2/4M^2) \leq \exp(-k^2M'^2/M^2) \leq e^{-k^2} \leq \frac{1}{1+k^2} \]
This shows Assumption 2 holds. For \( k = 1 \), and \( M' \leq M \) in the equation above we have,
\[ \frac{G[M'+i]}{G[i]} = \exp(-(M'+i)^2/4M'^2 + i^2/4M^2) \geq e^{-1} \exp(iM'/2M^2) = \Omega_M(1), \]
which shows Assumption 3.
Laplace kernel. For the Laplace kernel, \( G[k] = \frac{M(1 - (-1)^k e^{-\pi M})}{M^2 + k^2} \). Hence, \( \sum_{k \in \mathbb{Z}} |G[k]| \leq \frac{M}{M^2 + k^2} < \infty \), proving that Assumption 1 is satisfied.

Now, for \( M' \geq M \), define \( a = M/M' \leq 1 \) and \( b = i/M' \leq 1 \) and consider,
\[
\frac{G[M'k + i]}{G[i]} \leq \frac{M' + i^2}{M' + i^2} = \frac{a^2 + b^2}{a^2 + b^2} \leq \frac{a^2 + b^2}{1 + k^2}.
\]
The last inequality holds if \( (2 - a^2 - b^2)k^2 + (a^2 + b^2) \geq 0 \) which is always true since \( a, b \leq 1 \). This proves assumption 2. Similarly, for \( M' \leq M \), and \( k = 1 \), define \( a = \frac{M'}{M} \leq 1 \) and \( b = \frac{i}{M} \leq 1 \),
\[
\frac{G[M' + i]}{G[i]} \geq \frac{M' + i^2}{M' + i^2} = \frac{1 + a^2}{1 + (b + a)^2} \geq \frac{1}{3}
\]
hence Assumption 3 holds.

Intermediate Lemmas.

Lemma B.1. Let \( \xi \) be a random vector with \( \mathbb{E} \xi \xi^\top = \sigma^2 I \), then for \( u \in \mathbb{C}^N \), \( \mathbb{E}_\xi |\xi \xi^\top K^{-1} u|^2 = \sigma^2 \cdot u^\top K^{-2} u \).

Proof. \( \mathbb{E}_\xi |\xi \xi^\top K^{-1} u|^2 = \mathbb{E}_\xi u^\top K^{-1} \xi \xi^\top K^{-1} u = \sigma^2 u^\top K^{-2} u \).

Lemma B.2. For the Gaussian kernel, there is a constant \( c > 0 \) such that,
\[
G[k] = G[0]e^{-k^2/4M^2}
\]

Proof. We begin by noting the following identity,
\[
(e^{-f(t)} g(t))' = e^{-f(t)} g'(t) - e^{-f(t)} f'(t) g(t)
\]
Using the cosine series definition of \( G[k] \), consider the following quantity,
\[
k \cdot G[k] + 2M^2 \frac{\partial}{\partial k} G[k] = \int_{-\pi}^{\pi} e^{-M^2 t^2} \cos(kt) k \, dt - e^{-M^2 t^2} 2M^2 \sin(kt) t \, dt
\]
\[
= e^{-M^2 t^2} \sin(kt) \bigg|_{t=-\pi}^{\pi} = 0
\]
where we have used \( f(t) = M^2 t^2 \) and \( g(t) = \sin(kt) \) in equation (B.1). Solving this differential equation yields that \( G[k] = G[0]e^{-k^2/4M^2} \), for all \( k \in \mathbb{R} \).

Lemma B.3. For \( \beta \in \mathbb{C}^N \), let \( B \) be the Fourier series of the function \( \langle \beta, K(X_N, \cdot) \rangle \). Then,
\[
B[k] = \sqrt{N} \beta^\top u_k \text{mod } N \cdot G[k], \quad k \in \mathbb{Z}
\]

Proof. Use the Fourier series definition to get,
\[
\sum_{i=1}^{N} \beta_i \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x_i, t) \, dt = \sum_{i=1}^{N} \beta_i \frac{1}{2\pi} \int_{-\pi}^{\pi} g(M(x_i - t) \text{mod } [-\pi, \pi]) \, dt
\]
\[
= \sum_{i=1}^{N} \beta_i \frac{1}{2\pi} \int_{-\pi}^{\pi} g(M\tau) e^{-jk\tau} e^{-jkx_i} d\tau = \sqrt{N} \beta^\top u_k \text{mod } N G[k]
\]
since \( e^{-jkx_i} = e^{-j\frac{2\pi}{N} ki} = e^{-j\frac{2\pi}{N} (k \text{ mod } N)i} = \sqrt{N} u_{k \text{ mod } N} \). This concludes the proof.
**Eigenfunctions of \( T_K, T_K^N \) and eigenvectors of \( K \).**

**Proof of Proposition 3.1.** It suffices to show the eigenvector equation for the unnormalized version of \( u_\ell \). We start by noting that

\[
K_{im'} = g(M(x_{m'} - x_i)) = \sum_{m \in \mathbb{Z}} G[m] e^{jm(x_{m'} - x_i)} = \sum_{m \in \mathbb{Z}} G[m] e^{jm \frac{2\pi}{N} (m' - i)}.
\]

Using this, we have

\[
(Ku_\ell)_i = \sum_{m'=0}^{N-1} K_{im'} e^{-j \frac{2\pi}{N} m' \ell} = \sum_{m'} \sum_{m \in \mathbb{Z}} G[m] e^{jm \frac{2\pi}{N} (m' - i)} e^{-j \frac{2\pi}{N} m' \ell}
\]

\[
= \sum_{m \in \mathbb{Z}} G[m] e^{-j \frac{2\pi}{N} m i} \sum_{m'=0}^{N-1} e^{j \frac{2\pi}{N} (m - \ell)m'} = N \sum_{m \in \mathbb{Z}} G[mN + \ell] e^{-j \frac{2\pi}{N} (mN + \ell)i}
\]

\[
= N e^{-j \frac{2\pi}{N} \ell i} \sum_{m \in \mathbb{Z}} G[mN + \ell] e^{-j \frac{2\pi}{N} mN} = e^{-j \frac{2\pi}{N} \ell i} N \sum_{m \in \mathbb{Z}} G[mN + \ell] = e^{-j \frac{2\pi}{N} \ell i} \cdot N \|G_\ell\|_1
\]

This proves \( Ku_\ell = N \|G_\ell\|_1 u_\ell \). The rest follows from standard results on linear algebra. □

**Proof of Proposition 3.2.** Observe that

\[
T_K \{\phi_k\}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x, x') \phi_k(x') dx' = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(M(x' - x \mod [-\pi, \pi])) e^{j k x'} dx'
\]

\[
= e^{j k x} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} g(Mu) e^{-j k u} du = G[\ell] \phi(x)
\]

This proves the claim.

The following lemma relates the eigenfunctions of the empirical covariance operator defined in equation (2.4) to the eigenvectors of the kernel matrix.

**Lemma B.4 (Eigenfunctions of \( T_K^N \)).** Let \( (\lambda, \psi) \) be an eigenvalue-eigenfunction pair of \( T_K^N \). Assume \( K \) is invertible. Then for \( \lambda > 0 \), a unit-norm eigenfunction \( \psi \) satisfies,

\[
(B.3) \quad \psi = \sum_{i=1}^{n} \frac{e_i}{\sqrt{n\lambda}} K(x_i, \cdot),
\]

where \( e = (e_i) \in \mathbb{C}^n \) is a unit-norm eigenvector of \( K \) satisfying, \( K e = n\lambda e \).

We apply the above lemma to the setting described in Section 3. The proof is provided in Appendix D.4 in the supplementary materials.

**Lemma B.5 (Eigenfunctions of \( T_K^N \)).** The eigenfunctions for \( T_K^N \) are,

\[
\psi_\ell = \frac{1}{\sqrt{\|G_\ell\|_1}} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN + \ell}, \quad \ell \in [N].
\]

They satisfy, \( T_K^N \psi_\ell = \|G_\ell\|_1 \psi_\ell \), and their norms satisfy \( \|\psi_\ell\|_\mathcal{H} = 1 \), and \( \|\psi_\ell\| = \frac{1}{\sqrt{\|G_\ell\|_1}} \|G_\ell\|_1 \). Furthermore, \( \psi_\ell \) are orthogonal in \( L^2 \), i.e., \( \langle \psi_\ell, \psi_k \rangle = 0 \) for \( k \neq \ell \).
Proof of Lemma B.5. By Lemma B.4, we have

$$
\psi_\ell = \left \langle \frac{u_\ell}{\sqrt{N \|G_\ell\|_1}}, K(X_N, \cdot) \right \rangle_N = \left \langle \frac{u_\ell}{\sqrt{N \|G_\ell\|_1}}, K(X_N, \cdot) \right \rangle_N
$$

Then using Lemma B.3 we have a Fourier series expansion of the form

$$
\psi_\ell = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{\|G_\ell\|_1}} \frac{u_k \mod N G[k]}{\sqrt{\|G_\ell\|_1}} \sum_{k \in \mathbb{Z}} \frac{u_k \mod N G[k] \phi_k}{\sqrt{\|G_\ell\|_1}} = \frac{1}{\sqrt{\|G_\ell\|_1}} \sum_{m \in \mathbb{Z}} G[mN + \ell] \phi_{mN + \ell}
$$

To see the orthogonality, suppose $k, \ell \in \{0, 1, \ldots, N - 1\}$ and $k \neq \ell$. Then

$$
\langle \psi_\ell, \psi_k \rangle = \frac{1}{\sqrt{\|G_\ell\|_1} \sqrt{\|G_k\|_1}} \sum_{m, m' \in \mathbb{Z}} G[mN + \ell] G[m'N + k] \langle \phi_{mN + \ell}, \phi_{m'N + k} \rangle = 0
$$

For $L^2$ norm, substitute $k = \ell$ above to get,

$$
\langle \psi_\ell, \psi_\ell \rangle = \frac{1}{\|G_\ell\|_1} \sum_{m, m' \in \mathbb{Z}} G[mN + \ell] G[m'N + \ell] \langle \phi_{mN + \ell}, \phi_{m'N + \ell} \rangle = \frac{1}{\|G_\ell\|_1} \|G_\ell\|^2
$$

This proves the claim.

Appendix C. Extending results to higher dimensions. Proofs missing from this section are provided in the supplementary materials.

Notation. In this section $d \geq 1$ and $n = N^d$. By $[N]^d$ we denote the $d$-fold Cartesian product of $[N] := \{0, 1, \ldots, N - 1\}$. For vectors $p, q$, we write $p \leq q$ to indicate a coordinate-wise inequality, i.e., for all coordinates $i$, we have $p_i \leq q_i$. Similarly, $p \not\leq q$ indicates $p \leq q$ is violated, i.e., there exists a coordinate $i$ for which $p_i > q_i$. We similarly define $p \geq q$ and $p \not\geq q$. We also denote $0$ and $1$ to be the vectors of all 0’s and all 1’s respectively in a dimension compatible with the expression. For a scalar $C$, the expression $p \leq C$ means $p \leq C \cdot 1$, and similarly $p \geq C, p \not\leq C, p \not\geq C$.

We consider sequences indexed by $\mathbb{Z}^d$, and for such sequences we extend the definition of $N-$hop subsequences from equation (2.1) in the following manner. For a fixed $N \in \mathbb{N}$, and a sequence $G \in l^1(\mathbb{Z}^d)$, and for $\ell \in [N]^d$, let

$$
G_\ell \in l^1(\mathbb{Z}^d) \quad G_\ell[m] = G[mN + \ell], \quad \forall m \in \mathbb{Z}^d
$$

be the $N$-hop subsequence with entries given as above. For $k \in \mathbb{Z}^d$, and $x \in \mathbb{R}^d$

$$
k \mod N := (k_1 \ (\text{mod } N), k_2 \ (\text{mod } N), \ldots, k_d \ (\text{mod } N)) \in [N]^d
$$

$$
x \mod [-\pi, \pi) := (x_1 \ (\text{mod } [-\pi, \pi)), x_2 \ (\text{mod } [-\pi, \pi)), \ldots, x_d \ (\text{mod } [-\pi, \pi))) \in [-\pi, \pi)^d,
$$

where we remind the reader of notation from equation (3.1). We denote by $[-\pi, \pi)^d$ the Cartesian product of $d$ unit circles $[-\pi, \pi)$ along each dimension. We refer to this as the unit torus.
Definition C.1 (Fourier basis). For $k \in \mathbb{Z}^d$ and $x \in [-\pi, \pi]^d$, define $\phi_k(x) := \exp(j<k, x>) = \prod_{i=1}^{d} \exp(jk_i x_i)$. This basis satisfies $\langle \phi_k, \phi_\ell \rangle = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \exp(j<k, x>) \, dx = \mathbf{1}_{\{k = \ell\}}$.

A target function defined on the unit torus admits a Fourier series,

$$f^* = \sum_{k \in \mathbb{Z}^d} V[k] \phi_k \quad V[k] = \langle f^*, \phi_k \rangle.$$  \hspace{1cm} (C.1)

Definition C.2 (DFT Matrix $d > 1$). The normalized DFT matrix in $d > 1$ is

$$U_d = [u_0 \ldots u_{(N-1)1}] \in \mathbb{C}^{N^d \times N^d}, \quad u_{\ell, p} := N^{-d/2} \exp\left(-j\frac{2\pi}{N} \langle \ell, p \rangle\right), \quad \ell, p \in [N]^d$$

Data distribution. For $d > 1$, the continuous distribution is $\mu = \text{Uniform}([-\pi, \pi]^d)$ and the discrete distribution over $x \in [-\pi, \pi]^d$, with $n = N^d$ samples, to be

$$\mu_n(x) := \frac{1}{N^d} \sum_{\ell \in [N]^d} \delta(x - x_\ell), \quad (x_\ell)_i = \frac{2\pi}{N} \ell_i - \pi, \quad \forall \ell \in [N]^d, \text{ and } \forall i \in [N].$$

The $n = N^d$ samples $\{x_\ell\}$ are indexed by elements of $[N]^d$, where $x_\ell \in [-\pi, \pi)^d$ and has coordinates given by the expression above. Note again that $\mu_n$ weakly converges to $\mu$. In the rest of this section we use

$$\langle \alpha, K(X_n, \cdot) \rangle_n := \sum_{p \in [N]^d} \alpha_p K(x_p, \cdot)$$

to keep the notation simple.

We start with a result analogous to Lemma 5.1, when $d > 1$.

Lemma C.3 (Decomposition of MSE for $d > 1$). For a target function $f^* = \sum_{k \in \mathbb{Z}^d} V[k] \phi_k$,

(a) Approximation error: $\mathcal{E}^\text{apx} := \|f^* - P_X f^*\|^2 = \sum_{p \in [N]^d} \|V_p\|^2 - \frac{\langle G_p, V_p \rangle^2}{\|G_p\|^2} = \sum_{p \in [N]^d} \mathcal{E}^\text{apx}_p$

(b) Noise-free estimation error:

$$\mathcal{E}^\text{free} := \|\langle K^{-1} R_n \{f^* - P_X f^*\}, K(X_n, \cdot) \rangle_n \|^2 = \sum_{p \in [N]^d} \frac{\|G_p\|^2}{N^d} \left(\frac{\|V_p\|_1}{\|G_p\|_1} - \frac{\langle G_p, V_p \rangle}{\|G_p\|^2}\right)^2$$

$$= \sum_{p \in [N]^d} \mathcal{E}^\text{free}_p$$

(c) Averaged noisy estimation error:

$$\mathcal{E}^\text{noise} := \mathbb{E}_\xi \|\langle K^{-1} \xi, K(X_n, \cdot) \rangle_n \|^2 = \sum_{p \in [N]^d} \frac{\sigma^2}{N^d} \frac{\|G_p\|^2}{\|G_p\|_1} = \sum_{p \in [N]^d} \mathcal{E}^\text{noise}_p$$
Together, this yields that the MSE for this function is,

(C.2) \[ \mathbb{E}_\xi \text{MSE} \left( \hat{f}_N, f^* \right) = \sum_{p \in [N]^d} \varepsilon_p = \sum_{p \in [N]^d} \varepsilon_p^{\text{apx}} + \varepsilon^\text{free}_p + \varepsilon^\text{noise}_p \]

(C.3) \[ \varepsilon_p := \|V_p\|^2 - \frac{\langle G_p, V_p \rangle^2}{\|G_p\|^2} + \frac{\|G_p\|^2}{N^d} \left( \frac{\|V_p\|}{\|G_p\|} - \frac{\langle G_p, V_p \rangle}{\|G_p\|^2} \right)^2 + \frac{\sigma^2}{N^d} \left( \frac{\|G_p\|}{\|G_p\|_1} \right)^2 \]

The derivation for \( d > 1 \) is similar to the \( d = 1 \) case. Appendices D.1, D.2, and D.3 in the supplementary materials provide proofs for Lemma C.3 (a), (b), and (c) respectively.

Assumption 4 (Scale). \( \sum_{k \in \mathbb{Z}^d} |G[k]| < \infty. \)

Assumption 5 (Spectral Tail). For all \( k \in \mathbb{Z}^d \), there exists a dimension-dependent constant \( C_{1,d} > 0 \) such that,

(C.4) \[ |G[kM' + p]| \leq C_{1,d}|G[p]| \prod_{i=1}^d \frac{1}{1 + k_i^2} \]

holds for all \( M' \geq M \) and for all \( 0 \leq p \leq M' \), except \( o_{M'}((M')^d) \) many.

Assumption 6 (Spectral Head). There exist dimension-dependent constants \( C_{2,d}, C_{3,d} \in \mathbb{R}_+ \), \( p^* \in \mathbb{Z}_{\geq 0} \), and \( 0 \leq m^* \leq 1 \) with \( m^* \neq 0 \), such that for \( M \geq 2 \), we have that for all \( M' \leq M \), \( |G[p^*]| \leq C_{3,d}|G[p^* + M'm^*]| \) and \( |G_0[p^*]| > 0 \).

C.1. Proof of Theorem 7.1. When \( M \leq N \) we show that the averaged noisy estimation error is large. On the other hand, when \( M > N \), we show that the approximation error is large for a cosine function in the base RKHS \( \mathcal{H}_0 \).

Case 1, \( M > N \). In this case we show the approximation error is bounded away from 0. Since \( M > N \), by Assumption 6, there exists a vector \( p^* \) of constant integers, and an \( 0 \leq m^* \leq 1 \) with \( m^* \neq 0 \), such that \( |G[p^*]| \leq C_{3,d}|G[m^*N + p^*]| \). Now let \( f^* \) be the (real-valued) function with Fourier coefficients \( V[p^*] = V[-p^*] = \sqrt{\frac{(2^d)^{d-1}}{2}} \), and \( V[k] = 0 \) for \( |k| \neq i^* \). Using this, we can lower bound the approximation error as,

\[ \varepsilon_p^{\text{apx}} \geq \frac{2V[p^*]^2}{(2\pi)^d} \left( 1 - \frac{|G[p^*]|^2}{\sum_{m \in \mathbb{Z}^d} |G[mN + p^*]|^2} \right) \geq \frac{1}{2\pi} \left( 1 - \frac{|G[p^*]|^2}{|G[m^*N + p^*]|^2 + |G[p^*]|^2} \right) \geq \frac{1}{2\pi(1 + C_{3,d})} \]

Case 2, \( M \leq N \). In this case we show that the noisy estimation error is bounded away from 0. Define, \( \Delta_p := \|G[p]\|_1 - |G[p]| = \sum_{m \neq 0} |G[mN + p]| \geq 0 \). Assumption 5 says that for all but \( o(N^d) \) terms \( p \in [N]^d \), we have,

(C.5) \[ \Delta_p = \sum_{m \neq 0} |G[mN + p]| \leq C_{1,d}|G[p]| \sum_{m \neq 0} \prod_{i=1}^d \frac{1}{1 + m_i^2} \leq 4^d C_{1,d} |G[i]| \]
For such an $p$, we can lower bound the noisy estimation error term $\mathcal{E}_p^{\text{noise}}$ as,

$$\mathcal{E}_p^{\text{noise}} = \frac{\sigma^2}{N^d} \left\| \frac{|G_p|}{\|p\|_1^2} \right\|^2 \geq \frac{\sigma^2}{N^d} \left( \frac{|G_p|^2}{|\Delta_p|^2} \right) \geq \frac{\sigma^2}{2N^d \|G_p\|^2 + 2 |\Delta_p|^2} \geq \frac{\sigma^2}{2N^d (1 + 4^d C_1, d)}$$

since there are $\Omega(N^d)$ such $p \in [N]^d$ for which equation (C.5) holds.

**Proof of Theorem 7.2.** As in the $d = 1$ case, Theorem 7.2 follows from the proof of Theorem 7.1. We showed that for $M \leq N$ (Case 2 above), the noisy estimation error $\mathcal{E}^{\text{noise}}$ satisfies $\mathcal{E}^{\text{noise}} = \Omega(\sigma^2)$. Since $\mathcal{E}^{\text{noise}}$ is independent of the target function $f^*$, the statement of Theorem 4.3 follows.

**C.2. Special cases of kernels for $d > 1$.** For a kernel $g : \mathbb{R}^d \to \mathbb{R}$, define the function $g_1 : \mathbb{R} \to \mathbb{R}$ as $g_1(x) = g((x, 0, \ldots, 0))$. Let $G^1[k]$, for $k \in \mathbb{Z}$, be the sequence of Fourier series coefficients for $g_1$ on the unit circle. Note that this is exactly $G[k]$ for the one-dimensional version of a kernel $g$.

**Proof of Lemma 7.3:** Gaussian, Laplace, and Dirichlet satisfy the assumptions.

**Dirichlet kernel.** For the Dirichlet kernel, $G[k] = 1\{ |k| \leq M \}$. Therefore, $\sum_{k \in \mathbb{Z}^d} |G[k]| = (2M + 1)^d < \infty$, proving Assumption 4.

To prove Assumption 5, let $M' \geq M$, $p \geq 0$. Suppose $k \neq 0$. Suppose $p \not\in M$. Then $G[p] = 0$, which means $G[kM' + p] = 0$, satisfying equation (C.4). Otherwise, suppose $p \leq M$. Therefore, $G[p] = 1$, and, $G[kM' + p] \leq 1$, satisfying equation (C.4). Suppose instead $k = 0$. Then $G[p + kM'] = G[p]$, satisfying equation (C.4). So, Assumption 5 holds. Finally, let $p = 0$, and $k = (1, 0, \ldots, 0)$. Then, for all $M' \leq M$, $G[kM' + p] = G[p] = 1$. Hence, Assumption 6 holds.

**Gaussian kernel.** For the Gaussian kernel, by Lemma D.2 and Lemma B.2, we have

$$G[k] = G[0] \exp \left( -\frac{\|k\|^2}{4M^2} \right)$$

Thus $\sum_{k \in \mathbb{Z}^d} \exp \left( -\frac{\|k\|^2}{M^2} \right) < \infty$, proving that Assumption 4 is satisfied. Next, for all $p \in (\mathbb{Z}^+)^d$ and $M' \geq M$,

$$\frac{G[M'k + p]}{G[p]} = \exp \left( -\frac{\|kM' + p\|^2}{4M^2} + \frac{\|p\|^2}{4M^2} \right) \leq \exp \left( -\frac{\|k\|^2}{2M^2} \right) \leq \prod_{i=1}^d \frac{1}{1 + k_i^2}$$

This shows Assumption 5 holds. Now for $k = (1, 0, \ldots, 0)$, $p = 0$, and $M' \leq M$ in the equation above we have,

$$\frac{G[kM' + p]}{G[p]} = \exp \left( -(M')^2/4M^2 \right) \geq \exp (-1/4) = \Omega_M(1),$$

which shows Assumption 6.
**Laplace kernel.** By Lemma D.3, for a constant $c_{k,d} < O(1)$

$$G[k] \leq c_{k,d} \prod_{i=1}^{d} \frac{M(1 - (-1)^{k_i} \exp(-\pi M))}{M^2 + k_i^2}$$

Hence $\sum_{k \in \mathbb{Z}^d} |G[k]| \leq \left(\sum_{k \in \mathbb{Z}} \frac{M_{c_{k,d}}}{M^2 + k_i^2}\right)^d < \infty$, proving that Assumption 4 is satisfied. Now, for $M' \geq M$, $|p| \leq M'$ and $k \in \mathbb{Z}$, recall,

$$\frac{G^1[M'k + p]}{G^1[p]} \lesssim \frac{2}{1 + k^2}.$$  

Therefore, for $k \in \mathbb{Z}^d$ and $p \in S_{M'}$, and a constant $c_{k,d}$, by Lemma D.3,

$$\frac{G[M'k + p]}{G[p]} \leq c_{k,d}^d \prod_{i=1}^{d} \frac{G^1[M'k_i + p_i]}{G^1[p_i]} \lesssim (2c_{k,d})^d \prod_{i=1}^{d} \frac{1}{1 + k_i^2}.$$  

This proves Assumption 5. Now recall that for $M' \leq M$, $\frac{G^1[M']}{G^1[0]} \geq \frac{1}{3}$. Then, for $k = (1, 0, \ldots, 0) \in \mathbb{Z}^d$, $\frac{G[kM']}{G[k]} = \frac{G^1[k,M']}{G^1[0]} \geq \frac{1}{3}$ hence Assumption 6 holds.

**Lemma C.4.** For $\beta = (\beta_p) \in \mathbb{C}^{N^d}$, the Fourier series of the function $\langle \beta, K(X_n, \cdot) \rangle_n$ is,

$$B[k] = N^{d/2} \beta^\top u_k \mod N \cdot G[k], \quad k \in \mathbb{Z}^d$$

The proof for this lemma is provided in Appendix D.4 in the supplementary materials.
Appendix D. Supplementary Materials: Proofs to technical lemmas.

Proposition D.1 (Parseval’s theorem in high dimensions). For a function \( f : [-\pi, \pi]^d \to \mathbb{R} \) with Fourier series coefficients \( F[k] \) for \( k \in \mathbb{Z}^d \), we have

\[
\sum_{k \in \mathbb{Z}^d} |F[k]|^2 = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} |f(t)|^2 dt.
\]

Lemma D.2. For the kernels of the form \( g(x) = \exp(-\|x\|_p^p) \) (e.g. the Gaussian kernel), the multidimensional Fourier series coefficients decomposes into a product of the 1-dimensional Fourier series coefficients. In particular, for \( k \in \mathbb{Z}^d \),

\[
G[k] = \prod_{i=1}^d G^1[k_i]
\]

Proof of Lemma D.2. The LHS corresponds to \( g(M(x \mod [-\pi, \pi]^d)) \) which equals,

\[
\prod_{i=1}^d g(M(x_i \mod N)) = \prod_{i=1}^d \left( \sum_{k_i \in \mathbb{Z}} G^1[k_i] \exp(j k_i x_i) \right) = \sum_{k \in \mathbb{Z}^d} \prod_{i=1}^d G^1[k_i] \exp(j \langle k, x \rangle) .
\]

Lemma D.3. For the Laplace kernel, \( g(t) = \exp(-\|t\|M^d) \), we have for a constant \( c_{k,d} > 0 \),

\[
\prod_{i=1}^d G^1[k_i] \leq G[k] \leq c_{k,d} \prod_{i=1}^d G^1[k_i]
\]

Proof of Lemma D.3. We can write the Fourier coefficient \( G[k] \) for \( k \in \mathbb{Z}^d \) as,

\[
G[k] = \int_{[-\pi, \pi]^d} \exp(-\|x\|M^d) \exp(j \langle k, x \rangle) \ dx
\]

Both inequalities are a manifestation of the equivalence between the 1- and 2- norms,

\[
d^{-\frac{1}{2}} \|x\|_1 \leq \|x\| \leq \|x\|_1
\]

The first claim then follows as,

\[
\int_{[-\pi, \pi]^d} \exp(-\|x\|M^d) \exp(j \langle k, x \rangle) \ dx \geq \int_{[-\pi, \pi]^d} \exp(-\|x\|_1 M^d) \exp(j \langle k, x \rangle) \ dx
\]

\[
= \prod_{i \in [N]} \left( \int_{[-\pi, \pi]} \exp(-|x_i| M) \exp(j k_i x_i) \ dx_i \right) = \prod_{i=1}^d G^1[k_i]
\]

For the second claim, note that,

\[
\int_{[-\pi, \pi]^d} \exp(-\|x\|M^d) \exp(j \langle k, x \rangle) \ dx \leq \int_{[-\pi, \pi]^d} \exp(-\|x\|_1 M^d d^{-1/2}) \ dx
\]

\[
= \prod_{i \in [N]} \left( \int_{[-\pi, \pi]} \exp(-|x_i| M d^{-1/2d}) \exp(j k_i x_i) \ dx_i \right)
\]
Further, for constants $c_{k,d} > 0$,

$$
\int_{-\pi}^{\pi} \exp(-|x_1|Md^{-1/(2d)}) \exp(jk_1x_1) = \frac{Md^{-1/(2d)}(1 - (-1)^k \exp(-\pi Md^{-1/(2d)}))}{M^2d^{-1/d} + k^2}
\leq \frac{c_{k,d}M}{M^2d^{-1/d} + k^2} \leq c_{k,d}G^1[k_1]
$$

\textit{Eigenfunctions of } \mathcal{T}_K, \mathcal{T}_K^\circ \textit{ and eigenvectors of } \mathbf{K} \textit{ for } d > 1. \text{ The proofs of the following two statements are provided in Appendix D.4.}

**Proposition D.4.** Proposition 3.1 holds with $u_\ell$ from Definition C.2 and $\mathbf{K} = (K(x_p, x_{p'})) \in \mathbb{R}^{N_d \times N_d}$, with eigenvalue $\lambda_\ell = N^d \|G_\ell\|_1$, i.e., $\mathbf{K}u_\ell = \lambda_\ell u_\ell$.

**Lemma D.5 (Eigenfunctions of } \mathcal{T}_K^{N,d}). \textit{The eigenfunctions for the empirical operator } \mathcal{T}_K^{N,d} \textit{ are,}

$$
\psi_\ell = \frac{1}{\sqrt{\|G_\ell\|_1}} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN + \ell}, \quad \ell \in [N]^d.
$$

They satisfy, $\mathcal{T}_K^{N,d} \{\psi_\ell\} = \|G_\ell\|_1 \psi_\ell$, and their norms satisfy $\|\psi_\ell\|_H = 1$, as well as $\|\psi_\ell\| = \frac{1}{\sqrt{\|G_\ell\|_1}} \|G_\ell\|$.

Furthermore, $\psi_\ell$ are orthogonal in $L^2_H$, i.e., $\langle \psi_\ell, \psi_\ell' \rangle = 0$ for $\ell \neq \ell'$.

**D.1. Approximation error: Proof of Lemma C.3(a).** Once again, the proof proceeds by applying the Pythagorean theorem to the triangle $\{0, f^*, P_Xf^*\}$ in $L^2$. The following lemma gives exact expressions for projection of the target function and its norm.

**Lemma D.6 (Projection).** For $f^* = \sum_{k \in \mathbb{Z}^d} V[k] \phi_k$,

$$
P_X f^* = \sum_{\ell \in [N]^d} \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|^2} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN + \ell}, \quad \text{and} \quad \|P_X f^*\|^2 = \sum_{\ell \in [N]^d} \frac{\|G_\ell\|^2}{\|G_\ell\|^2} \langle G_\ell, V_\ell \rangle^2
$$

We get,

$$
\|f^* - P_X f^*\|^2 = \|f^*\|^2 - \|P_X f^*\|^2 = \|V\|^2 - \sum_{\ell \in [N]^d} \frac{\|G_\ell\|^2}{\|G_\ell\|^2} \langle G_\ell, V_\ell \rangle^2
$$

**Proof of Lemma D.6.** Note that Lemma D.5 shows that $\{\psi_\ell/\|\psi_\ell\|\}_{\ell \in [N]^d}$ is an orthonormal basis for $\text{Span}\{K(x_\ell, \cdot)\}$. Consequently, we have

$$
P_X f^* = \sum_{\ell \in [N]^d} \langle f^*, \psi_\ell/\|\psi_\ell\| \rangle \psi_\ell/\|\psi_\ell\|, \quad \text{and} \quad \|P_X f^*\|^2 = \sum_{\ell \in [N]^d} \langle f^*, \psi_\ell/\|\psi_\ell\| \rangle^2
$$
We compute these projections. For \( \ell \in [N]^d \),
\[
\langle f^*, \psi_\ell \rangle = \frac{1}{\sqrt{\|G_\ell\|_1}} \left( \sum_{k \in \mathbb{Z}^d} V[k] \phi_k, \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN+\ell} \right)
\]
\[
= \frac{1}{\sqrt{\|G_\ell\|_1}} \sum_{m,k \in \mathbb{Z}^d} G[mN + \ell] V[k] \langle \phi_k, \phi_{mN+\ell} \rangle
\]
\[
= \frac{1}{\sqrt{\|G_\ell\|_1}} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] V[mN + \ell] = \frac{\langle G_\ell, V_\ell \rangle}{\sqrt{\|G_\ell\|_1}}
\]
Thus, we get that,
\[
\langle f^*, \psi_\ell \rangle \psi_\ell = \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|^2} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN+\ell}
\]
The claims follow immediately.

**D.2. Noise-free estimation error: Proof of Lemma C.3(b).** Let \( F \) be the fourier series of \( \langle K^{-1} R_n \{ f^* - P_X f^* \}^\top K(X_n, \cdot) \rangle_n \). From Lemma C.4 we have,
\[
F[k] = \sqrt{N^d} R_n \{ f^* - P_X f^* \}^\top K^{-1} u_k \mod N^d \cdot G[k]
\]
By Parseval’s theorem (Proposition D.1), we conclude,
\[
\frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left( \langle K^{-1} R_n \{ f^* - P_X f^* \}^\top K(X_n, t) \rangle_n \right)^2 dt = \sum_{k \in \mathbb{Z}^d} |F[k]|^2
\]
We will show that
\[
R_n \{ f^* - P_X f^* \}^\top K^{-1} u_\ell = \left( \frac{\|V_\ell\|_1}{\|G_\ell\|_1} - \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|_1^2} \right)
\]
We have \( K^{-1} u_\ell = u_\ell \cdot \frac{1}{N^d \|G_\ell\|_1} \). By Lemma A.1, we can write \( P_X f^* \) on the data as
\[
P_X f^*(x_p) = \sum_{\ell \in [N]^d} \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|^2} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN+\ell}(x_p) = \sum_{\ell \in [N]^d} \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|^2} \|G_\ell\|_1 \frac{u_{\ell p}}{\|G_\ell\|_1}
\]
\[
f^*(x_p) = \sum_{k \in \mathbb{Z}^d} V[k] \phi_k(x_p) = \sum_{\ell \in [N]^d} \|V_\ell\|_1 \frac{u_{\ell p}}{\|G_\ell\|_1}
\]
We thus have
\[
R_n \{ f^* - P_X f^* \}^\top K^{-1} u_\ell = \sum_{p, \ell \in [N]^d} \left( \frac{\|V_\ell\|_1}{\|G_\ell\|_1} - \frac{\langle G_\ell, V_\ell \rangle}{\|G_\ell\|_1^2} \right) \frac{u_{\ell p}}{N^d \|G_\ell\|_1^2}
\]
This gives,
\[
\sum_{k \in \mathbb{Z}^d} |F[k]|^2 = \frac{1}{N^d} \sum_{\ell \in [N]^d} \left( \frac{\|V_\ell\|_1}{\|G_\ell\|_1} - \frac{\langle V_\ell, G_\ell \rangle}{\|G_\ell\|_1^2} \right)^2 \|G_\ell\|^2.
\]
D.3. Noisy estimation error: Proof of Lemma C.3(c). Similar to $d = 1$, we derive this by an application of Parseval’s theorem.

Define the Fourier series,

$$\langle K^{-1} \xi, K(X_n, t) \rangle_n = \sum_{k \in \mathbb{Z}^d} E[k] \exp(j \langle k, t \rangle)$$

By Proposition D.1 (Parseval’s theorem), we have,

$$E[\xi] \left( \frac{1}{2\pi} \int_{[-\pi, \pi]^d} |\langle K^{-1} \xi, K(X_n, t) \rangle_n|^2 dt \right) = \sum_{k \in \mathbb{Z}^d} E[k] |E[k]|^2 = \sum_{p \in [N]^d} \sum_{m \in \mathbb{Z}^d} E[|mN + p|]^2$$

$$= \sigma^2 \sum_{p \in [N]^d} \|G_p\|^2 \frac{1}{N^{2d} \|G_p\|^2 N^d} = \sigma^2 \sum_{p \in [N]^d} \frac{\|G_p\|^2}{N^d \|G_p\|^2}$$

where we have used Lemma C.4 in the second, and Lemma B.1 in the third, and Proposition D.4 in the last line.

D.4. Additional Proofs.

Proof of Lemma B.4. We will first show that $\psi$ can be written as a linear combination of the $n$ representers $\{K(x_i, \cdot)\}$.

(D.2) \[ \psi = \sum_{i=0}^{n-1} \beta_i K(x_i, \cdot) \]

Let $\psi \in H$ be an eigenfunction of $T^n_K$ with eigenvalue $\lambda$. Then by definition of $T^n_K$ we have,

(D.3) \[ \lambda \psi = T^n_K(\psi) = \frac{1}{n} \sum_{i=1}^{n} \langle K(x_i, \cdot), \psi \rangle_H K(x_i, \cdot) = \frac{1}{n} \sum_{i=1}^{n} \psi(x_i) K(x_i, \cdot) \]

where the last equality holds due to the reproducing property of the kernel. Define $\beta_i = \frac{\psi(x_i)}{n\lambda}$ to show (D.2). Next, rewriting the equation for an eigenfunction $\psi$, expressed as (D.2), we get

(D.4) \[ T^n_K \left( \sum_{i=1}^{n} \beta_i K(x_i, \cdot) \right) = \lambda \sum_{i=1}^{n} \beta_i K(x_i, \cdot). \]

By definition of $T^n_K$ however we get,

(D.5) \[ T^n_K \left( \sum_{i=0}^{n} \beta_i K(x_i, \cdot) \right) = \frac{1}{n} \sum_{i,j=1}^{n} \beta_i \langle K(x_j, \cdot), K(x_i, \cdot) \rangle_H K(x_j, \cdot) = \frac{1}{n} \sum_{j=1}^{n} (K\beta_j) K(x_j, \cdot) \]

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Evaluating functions on the RHS of equations (D.2) and (D.5) at \( x_\ell \) yields,

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i K(x_i, x_j) K(x_j, x_i) = \lambda \sum_{i=1}^{n} \beta_i K(x_i, x_i) \quad \text{for all } \ell \in \{0, 1, \ldots, n - 1\}
\]

Compactly these \( n \) equations can be written as:

\[
K^2 \beta = n\lambda K \beta \implies K \beta = n\lambda \beta
\]

since \( K \) is inverible. Thus \( \beta \) is a scaled eigenvector of \( K \). It remains to determine the scale of \( \beta \) that defines \( \psi \).

Now, the norm of \( \psi \) can be simplified as

\[
\|\psi\|^2_\mathcal{H} = \left\langle \sum_{i=1}^{n} \beta_i K(x_i, \cdot), \sum_{j=1}^{n} \beta_j K(x_j, \cdot) \right\rangle_\mathcal{H} = \sum_{i,j=1}^{n} \beta_i \overline{\beta_j} \langle K(x_i, \cdot), K(x_j, \cdot) \rangle_\mathcal{H}
\]

\[
= \beta^T K \beta = n\lambda \|\beta\|^2.
\]

Since \( \psi \) is unit norm, we have \( \|\beta\| = \frac{1}{\sqrt{n\lambda}} \). This concludes the proof.

\textit{Proof of Lemma C.4.} Use the Fourier series definition to get,

\[
\sum_{p \in [N]d} \beta_p \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} g(M(t - x_p \mod [-\pi, \pi])) \exp(-j \langle k, t \rangle) \, dt
\]

\[
= \sum_{p \in [N]d} \beta_p \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} g(M \tau) \exp(-j \langle k, \tau \rangle) \exp(-j \langle k, x_p \rangle) \, d\tau = N^{d/2} \beta^T u_k \mod N G[k]
\]

This concludes the proof.

\textit{Proof of Proposition D.4.} It suffices to show the eigenvector equation for the unnormalized version of \( u_\ell \). We start by noting that

\[
K_{p,m'} = g(M(x_{m'} - x_p)) = \sum_{m \in \mathbb{Z}^d} G[m] e^{j \langle m, x_{m'} - x_p \rangle} = \sum_{m \in \mathbb{Z}^d} G[m] e^{j \frac{2\pi}{N} \langle m, m' - p \rangle}.
\]

Using this, we have

\[
(Ku_\ell)_q = \sum_{p \in [N]d} K_{q,p} u_\ell_p = \sum_{m' \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} G[m'] \exp \left( j \frac{2\pi}{N} \langle m', p - q \rangle \right) \exp \left( -j \frac{2\pi}{N} \langle p, \ell \rangle \right)
\]

\[
= \sum_{m' \in \mathbb{Z}^d} G[m'] e^{-j \frac{2\pi}{N} \langle q, p \rangle} \sum_{m \in \mathbb{Z}^d} e^{j \frac{2\pi}{N} \langle m' - \ell, p \rangle} = N^d \sum_{m \in \mathbb{Z}^d} G[mN + \ell] e^{-j \frac{2\pi}{N} \langle mN + \ell, q \rangle}
\]

\[
= e^{-j \frac{2\pi}{N} \langle \ell, q \rangle} N^d \sum_{m \in \mathbb{Z}^d} G[mN + \ell] = e^{-j \frac{2\pi}{N} \langle \ell, q \rangle} N^d \lambda_\ell
\]

This proves \( Ku_\ell = N^d \|G_\ell\|_1 u_\ell \). The rest follows from standard results on linear algebra.
Proof of Lemma D.5. By Lemma B.4, we have

$$\psi_\ell = \left\langle \frac{u_\ell}{\sqrt{N^d \|G_\ell\|_1}}, K(X_n, \cdot) \right\rangle_n = \left\langle \frac{u_\ell}{\sqrt{N^d \|G_\ell\|_1}}, K(X_n, \cdot) \right\rangle_n$$

Then using Lemma C.4 we have a Fourier series expansion of the form

$$\psi_\ell = \sum_{k \in \mathbb{Z}^d} \sqrt{N^d} \frac{u_\ell^H}{\sqrt{N^d \|G_\ell\|_1}} u_k \mod N G[k] \phi_k = \frac{1}{\|G_\ell\|_1} \sum_{k \in \mathbb{Z}^d} u_\ell^H u_k \mod N G[k] \phi_k$$

$$= \frac{1}{\|G_\ell\|_1} \sum_{m \in \mathbb{Z}^d} G[mN + \ell] \phi_{mN+\ell}$$

To see the orthogonality, suppose $k, \ell \in [N]^d$ and $k \neq \ell$. Then

$$\langle \psi_\ell, \psi_k \rangle = \frac{1}{\|G_\ell\|_1 \|G_k\|_1} \sum_{m, m' \in \mathbb{Z}^d} G[mN + \ell] G[m'N + k] \langle \phi_{mN+\ell}, \phi_{m'N+k} \rangle = 0$$

For $L^2$ norm, substitute $k = \ell$ above to get,

$$\langle \psi_\ell, \psi_\ell \rangle = \frac{1}{\|G_\ell\|_1} \sum_{m, m' \in \mathbb{Z}^d} G[mN + \ell] G[m'N + \ell] \langle \phi_{mN+\ell}, \phi_{m'N+\ell} \rangle = \frac{1}{\|G_\ell\|_1} \|G_\ell\|^2$$

This proves the claim.