FINE SELMER GROUPS OF CONGRUENT $p$-ADIC GALOIS REPRESENTATIONS

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Abstract. We compare the Pontryagin duals of fine Selmer groups of two congruent $p$-adic Galois representations over admissible pro-$p$, $p$-adic Lie extensions $K_{\infty}$ of number fields $K$. We prove that in several natural settings the $\pi$-primary submodules of the Pontryagin duals are pseudo-isomorphic over the Iwasawa algebra; if the coranks of the fine Selmer groups are not equal, then we can still prove inequalities between the $\mu$-invariants. In the special case of a $\mathbb{Z}_p$-extension $K_{\infty}/K$, we also compare the Iwasawa $\lambda$-invariants of the fine Selmer groups, even in situations where the $\mu$-invariants are non-zero. Finally, we prove similar results for certain abelian non-$p$-extensions.

1. Introduction

Let $p$ be a prime, and let $F$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and uniformising element $\pi$. Suppose that $V_1$ and $V_2$ denote two $F$-representations of the absolute Galois group of a fixed number field $K$, and that $T_1 \subseteq V_1$ and $T_2 \subseteq V_2$ are two Galois invariant sublattices. We let $A_1 = V_1/T_1$ and $A_2 = V_2/T_2$ and we assume that $A_1[\pi^l]$ and $A_2[\pi^l]$ are isomorphic as Galois modules for some $l \in \mathbb{N}$. In this article, we study the Pontryagin duals of the fine Selmer groups of $A_1$ and $A_2$ over (strongly) admissible $p$-adic Lie extensions, and we compare their ranks and Iwasawa invariants.

By an admissible $p$-adic Lie extension we mean a normal extension $K_{\infty}/K$ such that only finitely many primes of $K$ ramify in $K_{\infty}$ and such that $G = \text{Gal}(K_{\infty}/K)$ is a compact, pro-$p$, $p$-adic Lie group without $p$-torsion. For any finite set $\Sigma$ of primes of $K$, an admissible $p$-adic Lie extension $K_{\infty}/K$ shall be called strongly $\Sigma$-admissible if $K_{\infty}$ contains a $\mathbb{Z}_p$-extension $L$ of $K$ such that no prime $v \in \Sigma$ and no prime of $K$ which ramifies in $K_{\infty}$ is completely split in $L$ (see also Section 2 in the literature usually only the case of the cyclotomic $\mathbb{Z}_p$-extension $L = K_{\infty}$ of $K$ is considered, see for example [Lim17a]).

The comparison of Selmer groups of congruent $p$-adic representations goes back to the seminal work of Greenberg and Vatsal (see [GV00]), who considered elliptic curves defined over $\mathbb{Q}$ with good and ordinary reduction at some odd prime $p$ (in fact the Selmer groups were studied more generally in the context of Galois representations). The main issue dealt with in the article [GV00] is the relation between algebraically and analytically (i.e. via $p$-adic $L$-functions) defined Iwasawa invariants. Roughly speaking, Greenberg and Vatsal treated the $\mu = 0$ case and only considered the cyclotomic $\mathbb{Z}_p$-extension.
Over the last years, the results in [GV00] have been generalised in many different ways and we only mention a few exemplary results. For the comparison of analytical invariants of congruent elliptic curves defined over $\mathbb{Q}$, we refer to [Hat17]; in the present article we stick to the algebraic side. Most authors have focussed on the $\mu = 0$ setting from [GV00]: if $\mu = 0$ for the Selmer group of $A_1$, then the same holds true for the Selmer group of $A_2$. Moreover, over $\mathbb{Z}_p$-extensions one can then often prove equality of $\lambda$-invariants (we refer to Section 2 for the definition of the Iwasawa invariants). Analogous results have been obtained for Selmer groups of Galois representations over the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic base field $K$ (see [HL19]) and for signed Selmer groups of Galois representations over the cyclotomic $\mathbb{Z}_p$-extension of a number field in the non-ordinary setting (see for example [Pon20, Section 3]). Moreover, there exist vast generalisations to Selmer groups attached to families of modular forms (see for example [EPW06], [Sha09] and [Bar13]).

In the present article, our main objective is the comparison of fine Selmer groups of congruent $p$-adic Galois representations over admissible $p$-adic Lie extensions in [Lim17a]. In particular, he obtained the following result: if $A_1$ and $A_2$ are attached to two $p$-adic Galois representations and $A_1[p^\infty] \cong A_2[p^\infty]$ for some sufficiently large $t$, then the $\pi$-primary submodules of the Pontryagin duals of the associated Selmer groups are pseudo-isomorphic. This comparison statement is much stronger than the previous results. We are able to prove a similar result for fine Selmer groups (see Theorem 1.1 below). Lim also studied strict Selmer groups, as introduced by Greenberg in [Gre89]. These strict Selmer groups of $p$-adic Galois representations have also been studied by Hachimori in [Hac11].

In the present article, our main objective is the comparison of fine Selmer groups of congruent $p$-adic Galois representations over admissible $p$-adic Lie extensions. These objects have previously been investigated by Lim and Sujatha in [LS18], who obtained a comparison result in the $\mu = 0$ setting under a stronger condition on the decomposition of primes in $K_\infty/K$ (see [LS18 Theorem 3.7]). Moreover, Jha studied in [Jha12] the invariance of several arithmetic properties of fine Selmer groups of modular forms in a branch of a Hida family in the $\mu = 0$ setting.

Our first main result is an analogon of the strong result of Lim in [Lim17a] for fine Selmer groups over strongly admissible $p$-adic Lie extensions, which is not restricted to the case $\mu = 0$. For an admissible $p$-adic Lie extension $K_\infty$ of $K$ and an $F$-representation $V$ of the absolute Galois group of $K$, we let $T$ denote a Galois invariant $O$-lattice in $V$ and set $A = V/T$. Let $\Sigma$ be a finite set of primes of $K$ containing all the primes above $p$ and each prime where $V$ is ramified. Then $V_{A,\Sigma}^{(K_\infty)}$ shall denote the Pontryagin dual of the $\Sigma$-fine Selmer group of $A$ over $K_\infty$ (see Section 2.2 for the precise definition).

**Theorem 1.1.** Let $A_1$ and $A_2$ be associated with two $F$-representations $V_1$ and $V_2$ of the absolute Galois group of the number field $K$. If $p = 2$, then we assume that $K$ is totally imaginary. Let $\Sigma$ be a finite set of primes of $K$ which contains the
primes above \( p \) and the sets of primes of \( K \) where either \( V_1 \) or \( V_2 \) is ramified. Let \( K_\infty/K \) be a strongly \( \Sigma \)-admissible \( p \)-adic Lie extension, and let \( G = \text{Gal}(K_\infty/K) \).

We let \( r_j = \text{rank}_{\mathcal{O}[[G]]}(Y_{A_j,K_\infty}^{(K_\infty)}) \), \( 1 \leq j \leq 2 \). Let \( l \) be the minimal integer such that \((\pi^l Y_{A_j,\Sigma}^{(K_\infty)})[\pi]\) is pseudo-null over \( \mathcal{O}[[G]] \). Then the following statements hold.

(a) If \( A_1[\pi^l] \cong A_2[\pi^l] \) as \( G_K \)-modules and \( r_2 \leq r_1 \), then \( \mu(Y_{A_1,\Sigma}^{(K_\infty)}) \leq \mu(Y_{A_2,\Sigma}^{(K_\infty)}) \).

(b) If \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \), then \( r_2 \leq r_1 \). If moreover \( r_2 = r_1 \), then

\[
\mu(Y_{A_1,\Sigma}^{(K_\infty)}) = \mu(Y_{A_2,\Sigma}^{(K_\infty)})
\]

and the modules \( Y_{A_1,\Sigma}^{(K_\infty)}/\pi^\infty \) and \( Y_{A_2,\Sigma}^{(K_\infty)}/\pi^\infty \) are pseudo-isomorphic.

(c) In particular, if \( A_1[\pi] \cong A_2[\pi] \) and \( r_2 = r_1 \), then \( \mu(Y_{A_1,\Sigma}) = 0 \) holds if and only if \( \mu(Y_{A_2,\Sigma}) = 0 \).

(d) If \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \) for some integer \( l \) such that both \((\pi^l Y_{A_j,\Sigma}^{(K_\infty)})[\pi] \), \( 1 \leq j \leq 2 \), are pseudo-null, then \( r_2 = r_1 \) and \( \mu(Y_{A_1,\Sigma}^{(K_\infty)}) = \mu(Y_{A_2,\Sigma}^{(K_\infty)}) \).

This result will be proved in Section 3.1. The basic idea of the proof is to relate the \( \pi^k \)-torsion subgroups of the fine Selmer groups of \( A_j \), \( j \in \mathbb{N} \), to certain \( \pi^k \)-fine Selmer groups (defined in Section 2) which depend only on \( A_1[\pi^k] \). In the case of admissible \( p \)-adic Lie extensions \( K_\infty/K \) which are not strongly admissible, we can derive similar results under the hypothesis that \( A_j(K_v)[\pi] = \{0\} \) for every \( v \in \Sigma \) and \( j \in \{1,2\} \) (see Theorem 3.7 below). In order to obtain this result, we use an argument which goes back to the paper of Greenberg and Vatsal (see [GV00 Proposition 2.8]), appears also in work of Mazur and Rubin (see [MR04, Lemma 3.5.3]) and has been used in, e.g. [BS10] and [Pon20]. This approach is of particular interest if one wants to treat \( \mathbb{Z}_p \)-extensions \( K_\infty \) of \( K \) in which some prime above \( p \) or a ramified prime is completely split. In the special case of \( \mathbb{Z}_p \)-extensions, and under the additional hypotheses on the \( \pi \)-torsion which have been mentioned above, we can in fact go one step further and obtain results on the \( \lambda \)-invariants, provided that the \( \mathcal{O}[[G]] \)-modules \( Y_{A_1,\Sigma}^{(K_\infty)} \) both are torsion:

**Theorem 1.2.** In the setting of Theorem 3.7, suppose that \( G \cong \mathbb{Z}_p \) and that both ranks \( r_1 \) and \( r_2 \) are zero. Then, in addition to the assertions from Theorem 3.7, the following two statements hold:

(a) If \( l \in \mathbb{N} \) is large enough such that \((\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi] = \{0\} \) and \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \), then \( \lambda(Y_{A_2,\Sigma}^{(K_\infty)}) \leq \lambda(Y_{A_1,\Sigma}^{(K_\infty)}) \).

(b) If \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \) for some \( l \) such that both \((\pi^l Y_{A_j,\Sigma}^{(K_\infty)})[\pi] = \{0\} \), \( 1 \leq j \leq 2 \), then \( \lambda(Y_{A_1,\Sigma}^{(K_\infty)}) = \lambda(Y_{A_2,\Sigma}^{(K_\infty)}) \).

We remark that we do not have to assume that the \( \mu \)-invariants vanish in Theorem 1.2.

Finally, in Section 4, we consider certain abelian non-\( p \)-extensions \( K_\infty \) of \( K \). In two different settings (inspired by the two different cases treated in Section 3), we compare the \( \mathcal{O} \)-ranks of \( Y_{A_1,\Sigma}^{(K_\infty)} \) and \( Y_{A_2,\Sigma}^{(K_\infty)} \) and derive (in-)equalities analogous to those in Theorem 1.2. We also remark that the group ring \( \mathcal{O}[[\text{Gal}(K_\infty/K)]] \) is not well-behaved in this situation and the \( \mathcal{O} \)-rank is the natural substitute for the notion of Iwasawa \( \lambda \)-invariants.
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2. Background and notation

2.1. Admissible $p$-adic Lie extensions and Iwasawa modules. We fix once and for all a rational prime $p$. Let $F$ be a finite extension of $\mathbb{Q}_p$. We will denote its ring of integers by $\mathcal{O}$ and a generator of its maximal ideal by $\pi$. Note that $\mathcal{O}/(\pi)$ is a finite field with $q = p^f$ elements, where $f$ is the inertia degree of $p$ in $F/\mathbb{Q}_p$.

For any Noetherian $\mathcal{O}$-torsion elements; for any $\pi$ are annihilated by $f$ there exists an annihilator $\pi$ is a finite field with $q$ an integer.

In this article, an admissible $p$-adic Lie extension $K_\infty$ of a number field $K$ will always be a normal extension $K_\infty/K$ such that
- $G := \text{Gal}(K_\infty/K)$ is a compact pro-$p$ $p$-adic Lie group,
- $G[p^\infty] = \{0\}$, i.e. $G$ does not contain any $p$-torsion elements, and
- the set $\mathcal{S}_{\text{ram}}(K_\infty/K)$ of primes of $K$ ramifying in $K_\infty$ is finite.

Let $\Sigma$ be a finite set of finite primes of $K$. The pro-$p$-extension $K_\infty/K$ is called strongly $\Sigma$-admissible if it is admissible and moreover contains a $\mathbb{Z}_p$-extension $L$ of $K$ such that no prime in $\Sigma \cup \mathcal{S}_{\text{ram}}(K_\infty/K)$ is completely split in $L$. In this case, we fix $L$ and denote by $H \subseteq G$ the subgroup fixing $L$. Note that any strongly $\Sigma$-admissible $p$-adic Lie extension $K_\infty/K$ is strongly $\Sigma \cup \mathcal{S}_{\text{ram}}(K_\infty/K)$-admissible.

By abuse of notation we will always assume that $\Sigma$ contains $\mathcal{S}_{\text{ram}}(K_\infty/K)$ if $K_\infty/K$ is a strongly $\Sigma$-admissible $p$-adic Lie extension.

An admissible $p$-adic Lie extension $K_\infty/K$ is called strongly admissible if it contains the cyclotomic $\mathbb{Z}_p$-extension of $K$. Since no prime of $K$ splits completely in the cyclotomic $\mathbb{Z}_p$-extension, a strongly admissible $p$-adic Lie extension is strongly $\Sigma$-admissible for every finite set $\Sigma$.

If $K_\infty/K$ is an admissible $p$-adic Lie extension, then the completed group ring $\mathcal{O}[[G]] = \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G]]$ is a Noetherian domain (see [CH01, Theorem 2.3]), and we can define the $\mathcal{O}[[G]]$-rank of a finitely generated $\mathcal{O}[[G]]$-module $X$ by

$$\text{rank}_{\mathcal{O}[[G]]}(X) = \dim_{\mathcal{F}(G)}(\mathcal{F}(G) \otimes_{\mathcal{O}[[G]]} X),$$

where $\mathcal{F}(G)$ denotes the skew field of fractions of $\mathcal{O}[[G]]$ (see [GW04, Chapter 10]).

Following Howson (see [How02, (33)]), we define the $\mu$-invariant of a finitely generated $\mathcal{O}[[G]]$-module $X$ as

$$\mu(X) = \sum_{i \geq 0} \text{rank}_{\mathcal{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty]);$$

this is a finite sum as $X$ is Noetherian.

Remark 2.1. Let $X$ be a Noetherian $\pi$-primary $\mathcal{O}[[G]]$-module. Then there exists an integer $m$ such that $\pi^m X = \{0\}$. Suppose now that $\text{rank}_{\mathcal{F}_q[[G]]}(X[\pi]) = 0$. Then there exists an annihilator $f \in \mathcal{O}[[G]] \setminus \pi \mathcal{O}[[G]]$ of $X[\pi]$. In particular,

$$\pi^{m-1} f X = \{0\} \quad \text{and} \quad \pi^{m-2} f X \subseteq X[\pi].$$

Thus, we inductively obtain that $f^m X = \{0\}$. Therefore $\text{rank}_{\mathcal{F}_q[[G]]}(X/\pi X) = 0$.

Lemma 2.2. Let $G = \text{Gal}(K_\infty/K)$ be as above, and let $X$ be a finitely generated $\mathcal{O}[[G]]$-module of rank $r$. Then

$$\text{rank}_{\mathcal{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty]) = \text{rank}_{\mathcal{F}_q[[G]]}(\pi^i X/\pi^{i+1} X) - r$$
for each $i \in \mathbb{N}$.

Proof. By [Lim17b, Proposition 4.12]$$\text{rank}_{\mathcal{O}[G]}(\pi^i X/\pi^{i+1} X) = \text{rank}_{\mathcal{O}[G]}((\pi^i X)[\pi]) + \text{rank}_{\mathcal{O}[G]}((\pi^i X)[\pi])$$and$$\text{rank}_{\mathcal{O}[G]}((\pi^i X)[\pi^\infty]/\pi^{i+1} X[\pi^\infty]) = \text{rank}_{\mathcal{O}[G]}((\pi^i X[\pi^\infty])[\pi]) + \text{rank}_{\mathcal{O}[G]}((\pi^i X[\pi^\infty])).$$Now $(\pi^i X)[\pi] = (\pi^i X[\pi^\infty])[\pi]$, $\text{rank}_{\mathcal{O}[G]}(\pi^i X) = r$ and $\text{rank}_{\mathcal{O}[G]}((\pi^i X)[\pi^\infty]) = 0$. □

The most important class of admissible $p$-adic Lie extensions are the $\mathbb{Z}_p$-extensions. A $\mathbb{Z}_p$-extension $K_\infty/K$ is a normal extension such that $G = \text{Gal}(K_\infty/K)$ is isomorphic to the additive group of $p$-adic integers. In this special case, the theory of finitely generated $\mathcal{O}[G]$-modules is very well understood: the completed group ring $\mathcal{O}[G]$ is isomorphic to the ring $\Lambda := \mathcal{O}[T]$ of formal power series in one variable. Each finitely generated torsion $\Lambda$-module $X$ is pseudo-isomorphic to an elementary $\Lambda$-module of the form

$$E_X = \bigoplus_{i=1}^s \Lambda/(\pi^{e_i}) \oplus \bigoplus_{j=1}^t \Lambda/(h_j),$$

where $h_1, \ldots, h_t \in \Lambda$ are so-called distinguished polynomials. Here pseudo-isomorphic means that there exists a $\Lambda$-module homomorphism $\varphi : X \to E_X$ with finite kernel and cokernel. One defines the (classical) Iwasawa invariants of $X$ by $\mu(X) := \sum_{i=1}^s e_i$ and $\lambda(X) := \sum_{j=1}^t \deg(h_j)$. This notation is well-defined since the classical $\mu$-invariant coincides with the invariant $\mu(X)$ given in (1) above in the special case of $\mathbb{Z}_p$-extensions:

**Lemma 2.3.** Let $X$ be a finitely generated $\Lambda$-module. Then the classical Iwasawa $\mu$-invariant is equal to

$$\sum_{i=0}^\infty \text{rank}_{\mathcal{O}[T]}((\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty])).$$

Proof. This proof is well-known (see, e.g. [Ven02, Section 3.4]), but we recall it for the convenience of the reader. Let $E$ be an elementary $\Lambda$-module that is pseudo-isomorphic to $X$. Then we can write $E = \Lambda' \oplus \bigoplus_{i=1}^s \Lambda/(\pi^{e_i}) \oplus E_\lambda$ for a torsion $\Lambda$-module $E_\lambda$ which is a finitely generated free $\mathcal{O}$-module. Therefore the classical Iwasawa invariants can be computed as

$$\mu(X) = \mu(E) = \sum_{i=0}^\infty |\{k \mid e_k \geq i + 1\}| = \sum_{i=0}^\infty \text{rank}_{\mathcal{O}[T]}((\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty]))$$

because

$$\text{rank}_{\mathcal{O}[T]}((\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty])) = \text{rank}_{\mathcal{O}[T]}((\pi^i E[\pi^\infty]/\pi^{i+1} E[\pi^\infty])) = |\{k \mid e_k \geq i + 1\}|$$

for every $i \in \mathbb{N}$. □
2.2. Fine Selmer groups. For any discrete \( \mathbb{Z}_p \)-module \( M \), we define the Pontryagin dual of \( M \) as

\[
M^\vee = \text{Hom}_{\text{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)
\]

(i.e. the set of continuous homomorphisms).

If \( K \) is a number field and \( v \) denotes any prime of \( K \), then \( K_v \) will always denote the completion of \( K \) at \( v \). We denote by \( G_K \) the Galois group \( \text{Gal}(\overline{K}/K) \), where \( \overline{K} \) denotes a fixed algebraic closure of \( K \). If \( M \) is any \( G_K \)-module, then we let \( H^i(K, M) := H^i(G_K, M) \) denote the corresponding Galois cohomology groups, \( i \in \mathbb{N} \). Moreover, if \( L/K \) is an algebraic extension, then we write \( H^i(L/K, M) = H^i(\text{Gal}(L/K), M) \) for brevity.

Now fix a number field \( K \). Let \( V \) be a finite dimensional \( F \)-representation of \( \text{Gal}(\overline{K}/K) \) for some fixed algebraic closure \( \overline{K} \) of \( K \). Let \( T \) be a Galois stable \( \mathcal{O} \)-lattice in \( V \) and write \( A = V/T \). Note that \( A \) is an \( \mathcal{O} \)-module isomorphic to \( (F/\mathcal{O})^l \) for some non-negative integer \( l = \dim(V) \). In particular, \( A = A[\pi^\infty] \) is \( \pi \)-primary, i.e. each element of the \( \mathcal{O} \)-module \( A \) is annihilated by some power of \( \pi \). By abuse of notation we will also refer to \( l \) as the dimension of \( A \).

We denote by \( S_p \) and \( S_{\text{ram}}(A) \) the set of primes of \( K \) over \( p \) and the set of primes of \( K \) where \( V \) is ramified. For any number field \( L \supseteq K \) that is unramified outside \( S_p \cup S_{\text{ram}}(A) \) we denote by \( A(L) \) the maximal submodule of \( A \) on which \( \text{Gal}(\overline{K}/L) \) acts trivially.

We mention an important and classical special case: let \( A \) be an abelian variety defined over the number field \( K \). We assume that \( F = \mathbb{Q}_p \), i.e. \( \mathcal{O} = \mathbb{Z}_p \). Let \( T = T_p(A) = \varprojlim A[p^n] \) be the Tate module of \( A \) and \( V = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \); then \( V/T \cong A[p^\infty] \). In this setting, for any field \( L \), the group \( A(L)[p^\infty] \) is the usual group of \( L \)-rational \( p \)-torsion points on \( A \). Moreover, the ramified primes correspond to the primes of \( K \) where \( A \) has bad reduction, by the criterion of Néron-Ogg-Shafarevich (see [Lan97, Theorem IV.4.1]).

For the number field \( K, \ A = V/T \) as above and a prime number \( p \), we define the (\( \pi \)-primary part of the) fine Selmer group of \( A \) over \( K \) as

\[
\text{Sel}_{0,A}(K) = \ker \left( H^1(K, A) \longrightarrow \prod_v H^1(K_v, A) \right).
\]

In our applications it will be more convenient to work with the following definition:

\[
\text{Sel}_{0,A,\Sigma}(K) = \ker \left( H^1(K_\Sigma/K, A) \longrightarrow \prod_{v \in \Sigma} H^1(K_v, A) \right)
\]

for suitable (usually finite) sets \( \Sigma \) of primes of \( K \) containing all the ramified primes of the representation \( V \) and all primes above \( p \). Let \( K_\Sigma \) be the maximal algebraic extension of \( K \) unramified outside the primes in \( \Sigma \). If \( L \subseteq K_\Sigma \) is any, non-necessarily finite, extension, then we define

\[
\text{Sel}_{0,A,\Sigma}(L) = \lim_{\lower{2pt}\hbox{$\scriptstyle K_\Sigma \subseteq L' \subseteq L$}} \text{Sel}_{0,A,\Sigma}(L'),
\]

where \( L' \) runs through all finite subextensions \( K \subseteq L' \subseteq L \). Here we note that \( K_\Sigma = L_\Sigma^{\infty} \), since \( L/K \) is unramified outside of \( \Sigma \), and therefore each \( \text{Sel}_{0,A,\Sigma}(L') \) is a subgroup of \( H^1(K_\Sigma/K, A) \).

A priori this definition depends on the choice of \( \Sigma \). But if the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \), denoted by \( K_\infty \), is contained in \( L \), then the definition becomes
independent of the set Σ by a result of Sujatha and Witte (see [SW18, Section 3]). In fact their proof depends only on the fact that none of the primes in Σ is totally split in $K_\infty/K$. Therefore the definition of the fine Selmer group does not depend on the choice of Σ if we consider strongly Σ-admissible extensions $K_\infty/K$.

Finally, we define $\pi$-fine Selmer groups, $i \in \mathbb{N}$, as

$$\text{Sel}_{0,A,\pi^i,\Sigma}(K) = \ker \left( H^1(K_\Sigma/K, A[\pi^i]) \rightarrow \prod_{v \in \Sigma} H^1(K_v, A[\pi^i]) \right),$$

where $\Sigma$ is as above. Note: these $\pi$-fine Selmer groups may depend on the choice of Σ even for algebraic extensions $L$ of $K$ which contain the cyclotomic $\mathbb{Z}_p$-extension $K_\infty$ (see [LKM16, proof of Theorem 5.1] for an example for abelian varieties).

Now let $K_\infty/K$ be an admissible $p$-adic Lie extension, and let Σ be a finite set of primes of $K$ which contains $S_{\text{ram}}(K_\infty/K) \cup S_p \cup S_{\text{ram}}(A)$ (if $p = 2$, then we assume that $K$ is totally imaginary). Then we can define fine Selmer groups of $A$ over each number field $L \subseteq K_\infty$ containing $K$. We denote the corresponding Pontryagin duals by

$$Y_{A,\Sigma}^{(L)} = \text{Sel}_{0,A,\Sigma}(L)^{\vee},$$

and we define the projective limit

$$Y_{A,\Sigma}^{(K_\infty)} = \lim_{\kappa \subseteq L \subseteq K_\infty} Y_{A,\Sigma}^{(L)}$$

with respect to the corestriction maps (where $L$ runs over the finite subextensions of $K_\infty/K$).

3. Fine Selmer groups of congruent representations

The aim of this section is to study the relation between the Iwasawa invariants of the fine Selmer groups associated with two representations $V_1$ and $V_2$ defined over the same number field $K$. The representations we consider will always satisfy a congruence condition, meaning that $A_1[\pi^i]$ and $A_2[\pi^i]$ are isomorphic as $G_K$-modules for some integer $l$ (where $A_j = V_j/T_j$ as usual). Note that this implies that the two representations have the same dimension $d$. We will always fix a set $\Sigma$ of primes in $K$ containing all ramified places for $A_1$ and $A_2$, and all places above $p$. Let $K_\infty/K$ be an admissible $p$-adic Lie extension. We consider two cases:

i) $K_\infty/K$ is strongly $\Sigma$-admissible (Section 3.1).

ii) $K_\infty/K$ is admissible and $A(K_v)[\pi] = 0$ for all $v \in \Sigma$ (Section 3.2).

Note that case ii) only becomes relevant if a prime of $\Sigma$ is completely split in $K_\infty/K$.

3.1. The generic case. In this section we prove Theorem 1.1. The main ingredient in the proof is a relation between $\text{Sel}_{0,A}(L)[\pi^i]$ and $\text{Sel}_{0,A[\pi^i]}(L)$ for any finite subextension $K \subseteq L \subseteq K_\infty$ of the $p$-adic Lie extension $K_\infty/K$.

Lemma 3.1. Let $A$ be associated with a representation of $G_K$ of dimension $d$ and let $\Sigma$ be a finite set of primes of $K$ containing $S_p$ and $S_{\text{ram}}(A)$. If $p = 2$, then we assume that $K$ is totally imaginary. Let $L/K$ be a finite extension which is contained in $K_\Sigma$. Then

$$|v_p(\text{Sel}_{0,A[\Sigma]}(L)[\pi^k]) - v_p(\text{Sel}_{0,A[\pi^i]}(L))| \leq fdk(1 + |\Sigma(L)|)$$

for each integer $k \geq 1$, where $\Sigma(L)$ denotes the set of primes of $L$ above $\Sigma$ and $f$ is the inertia degree of $p$ in $F/\mathbb{Q}_p$. 

we may conclude that
\[ (2) \]
The last equality is due to the fact that all ramified primes are contained in \( \Sigma \).
Analogously, we see that
\[ \text{exact sequence} \]
Corollary 3.3.

\[ \text{is a finite abelian group of order at most} \ p^d \text{.} \]

The surjectivity follows from the fact that \( A \) is divisible as \( O \)-module. Note further that the representation \( V \) is unramified outside \( \Sigma \). Thus, there is a well-defined action of \( \text{Gal}(K_\Sigma/L) \) on \( A \) and we can take \( K_\Sigma/L \)-cohomology of the exact sequence in order to see that the map \( h \) is surjective. Moreover,
\[ \ker(h) \cong \text{coker}(\pi^k : H^0(K_\Sigma/L, A) \rightarrow H^0(K_\Sigma/L, A)) = A(L) / \pi^k A(L). \]
The last equality is due to the fact that all ramified primes are contained in \( \Sigma \).
Analogously, we see that \( g \) is surjective and that
\[ \ker(g) \cong \bigoplus_{v \in \Sigma(L)} A(L_v) / \pi^k A(L_v). \]

We obtain the bounds \( v_p(\| \ker(h) \|) \leq dfk \) and \( v_p(\| \ker(g) \|) \leq dfk|\Sigma(L)| \). Using the exact sequence
\[ (2) \quad 0 \rightarrow \ker(s) \rightarrow \text{Sel}_0,A[\pi^k],\Sigma(L) \rightarrow \text{Sel}_0,A,\Sigma(L)[\pi^k] \rightarrow \text{coker}(s) \rightarrow 0, \]
we may conclude that \( |v_p(\| \text{Sel}_0,A[\pi^k],\Sigma(L)\|) - v_p(\| \text{Sel}_0,A,\Sigma(L)[\pi^k]\|)| \) is bounded by
\[ v_p(\| \ker(s) \|) + v_p(\| \text{coker}(s) \|) \leq v_p(\| \ker(h) \|) + v_p(\| \ker(g) \|) \leq fdk + dfk|\Sigma(L)|. \]

\[ \square \]

Corollary 3.3. Let \( A \) be associated with a representation of \( G_K \), and let \( \Sigma \) be a finite set of primes of \( K \) containing \( S_p \) and \( S_{\text{ram}}(A) \). If \( p = 2 \), then we assume that \( K \) is totally imaginary. Let \( K_\infty/K \) be a strongly \( \Sigma \)-admissible \( p \)-adic Lie extension. Then \( \text{rank}_{F_p[[C]]}(\pi^1 Y_{A,\Sigma}(K_\infty) / \pi^{i+1} Y_{A,\Sigma}(K_\infty)) \) equals
\[ \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_0,A[\pi^k],\Sigma(L)^{\vee} \big/ \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_0,A[\pi^k],\Sigma(L)^{\vee} \]
for every \( i \in \mathbb{N} \), where \( L \) runs over the finite subfields of \( K_\infty/K \).

Proof. Let \( k \in \mathbb{N} \). For every finite subextension \( L \subseteq K_\infty \) of \( K \), we consider the exact sequence
\[ 0 \rightarrow M^{(L)} \rightarrow \text{Sel}_0,A,\Sigma(L)^{\vee} \rightarrow \text{Sel}_0,A[\pi^k],\Sigma(L)^{\vee} \rightarrow N^{(L)} \rightarrow 0 \]
which is obtained from (2) by taking Pontragin duals. In particular, \( N^{(L)} \) is a finite abelian group of order at most \( p^{dfk} \), and \( M^{(L)} = \bigoplus_{v \in \Sigma(L)} G_v^{(L)} \), where each \( G_v^{(L)} \) is a finite abelian group of order at most \( p^{dkf} \).
Taking the projective limits along the $L \subseteq K_{\infty}$, we obtain an exact sequence

$$(3) \quad 0 \to M \to Y_{A,\Sigma}^{(K_{\infty})}/\pi \cdot Y_{A,\Sigma}^{(K_{\infty})} \to \lim_{K_{\subseteq L \subseteq K_{\infty}}} \text{Sel}_{0,A[\pi^i],\Sigma}(L)^{\vee} \to N \to 0,$$  

where $N$ is a finite abelian group and where $M$ is finitely generated over $O[[H]]$ because no prime $\nu \in \Sigma$ splits completely in the $\mathbb{Z}_p$-extension $K_{\infty}^H$ of $K$ which is fixed by $H \subseteq G$. In fact, replacing $K$ by a finite subextension of $K_{\infty}^H$ if necessary (this does not affect the projective limit), we may assume that actually the primes $\nu \in \Sigma$ do not split at all in $K_{\infty}^H/K$.

Letting $G := G/H \cong \mathbb{Z}_p$, the group ring $O[[\Gamma]]$ can be identified with the ring $\Lambda = O[[T]]$. Since $M$ is finitely generated over $O[[H]]$, there exists a non-constant annihilator of $M$ in $O[[G]] \cong O[[H]][[T]]$ by [CFK+05, Proposition 2.3 and Theorem 2.4]; in particular, the annihilator is not a power of $\pi$ (note: the result in [CFK+05] is formulated for the case $O = \mathbb{Z}_p$, but the proof goes through in our more general setting).

Considering now $k = i$ and $k = i + 1$, we may conclude that there exists a non-constant annihilator in $O[[G]]$ of the cokernels and kernels of both maps

$$Y_{A,\Sigma}^{(K_{\infty})}/\pi \cdot Y_{A,\Sigma}^{(K_{\infty})} \to \lim_{K_{\subseteq L \subseteq K_{\infty}}} \text{Sel}_{0,A[\pi^i],\Sigma}(L)^{\vee}$$

and

$$Y_{A,\Sigma}^{(K_{\infty})}/\pi \cdot Y_{A,\Sigma}^{(K_{\infty})} \to \lim_{K_{\subseteq L \subseteq K_{\infty}}} \text{Sel}_{0,A[\pi^{i+1}],\Sigma}(L)^{\vee}.$$

Taking quotients proves the assertion of the corollary. \qed

We need one final auxiliary

**Lemma 3.4.** Let $A_1$ and $A_2$ be associated with two representations $V_1$ and $V_2$ of $G_K$, and let $\Sigma$ be a finite set of primes of $K$ which contains $S_p \cup S_{\text{ram}}(A_1) \cup S_{\text{ram}}(A_2)$. If $p = 2$, then we assume that $K$ is totally imaginary. We assume that $A_1[\pi^i]$ and $A_2[\pi^i]$ are isomorphic as $G_K$-modules for some $i \in \mathbb{N}$, $i \geq 1$.

Then $\text{Sel}_{0,A_1[\pi^i],\Sigma}(L) \cong \text{Sel}_{0,A_2[\pi^i],\Sigma}(L)$ for every finite extension $L \subseteq K_\Sigma$ of $K$.

**Proof.** Let $\phi: A_1[\pi^i] \to A_2[\pi^i]$ be a $G_K$-module homomorphism. As $V_1$ and $V_1$ are unramified outside of $\Sigma$, the group $\text{Gal}(\overline{K}/K_{\Sigma})$ acts trivially on $A_1$ and $A_2$ and we can interpret $\phi$ as a $\text{Gal}(K_{\Sigma}/K)$-isomorphism. Then $\phi$ induces an isomorphism

$$\phi: H^1(K_{\Sigma}/L, A_1[\pi^i]) \to H^1(K_{\Sigma}/L, A_2[\pi^i])$$

of $G_K$-modules.

For any prime $\nu$ of $L$, the inclusion $G_{L,\nu} \hookrightarrow G_L$ of the local absolute Galois group at the completion $L_{\nu}$ of $L$ at $\nu$ induces an isomorphism

$$H^1(L_{\nu}, A_1[\pi^i]) \to H^1(L_{\nu}, A_2[\pi^i]).$$

The corresponding isomorphism between fine Selmer groups is now immediate. \qed

Now we turn to the proof of our first main result.
Proof of Theorem 4.4. Let $l$ be such that $(\pi^1 Y_{A_1, \Sigma}(K_\infty))[[\pi]]$ is pseudo-null. By definition of the $\mu$-invariant (see [1]) and Lemma 2.2, we have

$$\mu(Y_{A_1, \Sigma}^{(K_\infty)}) = \sum_{i=0}^{\infty} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_1, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1, \Sigma}^{(K_\infty)}) - r_1)$$

$$= \sum_{i=0}^{l-1} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_1, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1, \Sigma}^{(K_\infty)}) - r_1).$$

Now Corollary 3.3 implies that for both $j = 1$ and $j = 2$, the $F_q[[G]]$-rank of $\pi^1 Y_{A_1, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1, \Sigma}^{(K_\infty)}$ equals

$$\text{rank}_{F_q[[G]]}(\lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_1[\pi^{i+1}], \Sigma}(L)^Y / \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_1[\pi^i], \Sigma}(L)^Y).$$

Using that $A_1[\pi^i] \cong A_2[\pi^i]$, Lemma [3.4] implies that $\text{Sel}_{0, A_1[\pi^i], \Sigma}(L) \cong \text{Sel}_{0, A_2[\pi^i], \Sigma}(L)$ for every $i \leq l$. By [4], we may conclude that

$$\mu(Y_{A_1, \Sigma}^{(K_\infty)}) = \sum_{i=0}^{l-1} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_1, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1, \Sigma}^{(K_\infty)}) - r_1)$$

$$= \sum_{i=0}^{l-1} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_2, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2, \Sigma}^{(K_\infty)}) - r_1)$$

$$\leq \sum_{i=0}^{l-1} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_2, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2, \Sigma}^{(K_\infty)}) - r_2) = \mu(Y_{A_2, \Sigma}^{(K_\infty)}).$$

Here we used the hypothesis $r_2 \leq r_1$, i.e. $-r_1 \leq -r_2$.

Now we prove assertion (b). In the following, we abbreviate $\text{rank}_{F_q[[G]]}$ to $r$. If $A_1[\pi^{i+1}] \cong A_2[\pi^{i+1}]$ then

$$r(\pi^i Y_{A_1, \Sigma}(K_\infty)/\pi^{i+1} Y_{A_1, \Sigma}(K_\infty)) = r(\lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_1[\pi^{i+1}], \Sigma}(L)^Y / \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_1[\pi^i], \Sigma}(L)^Y)$$

$$= r(\lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_2[\pi^{i+1}], \Sigma}(L)^Y / \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_2[\pi^i], \Sigma}(L)^Y)$$

$$= r(\pi^i Y_{A_2, \Sigma}(K_\infty)/\pi^{i+1} Y_{A_2, \Sigma}(K_\infty)).$$

Using [Lim17b] Proposition 4.12 and the definition of $l$, we obtain

$$r(\pi^i Y_{A_1, \Sigma}(K_\infty)/\pi^{i+1} Y_{A_1, \Sigma}(K_\infty)) = r((\pi^i Y_{A_1, \Sigma}(K_\infty))[\pi]) + \text{rank}_{Z_q[[G]]}(\pi^i Y_{A_1, \Sigma}(K_\infty))$$

$$= 0 + r_1,$n
and similarly

$$r(\pi^i Y_{A_2, \Sigma}(K_\infty)/\pi^{i+1} Y_{A_2, \Sigma}(K_\infty)) \geq r_2.$$n

This proves the first claim of (b).

If $r_2 = r_1$, then it follows from the above that $\text{rank}_{F_q[[G]]}((\pi^i Y_{A_2, \Sigma}(K_\infty))[\pi]) = 0$. In view of Remark 2.2, this implies that

$$\mu(Y_{A_2, \Sigma}^{(K_\infty)}) = \sum_{i=0}^{l-1} (\text{rank}_{F_q[[G]]}(\pi^i Y_{A_2, \Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2, \Sigma}^{(K_\infty)}) - r_2) = \mu(Y_{A_1, \Sigma}^{(K_\infty)}),$$
Let now \( w \) be a place in \( L \) above a prime \( v \in \Sigma \). Using analogous arguments we can derive from the hypothesis \( A(K_v)[\pi] = \{0\} \) that
\[
H^1(L_w, A[\pi^i]) \cong H^1(L_w, A)[\pi^i].
\]

\( \square \)
Corollary 3.6. Let $A$ be as above, let $K_{\infty}/K$ be an admissible $p$-adic Lie extension, and let $\Sigma$ be a finite set of primes of $K$ containing $S_{\text{ram}}(K_{\infty}/K) \cup S_p \cup S_{\text{ram}}(A)$. If $p = 2$, then we assume that $K$ is totally imaginary. Assume that $A(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$. Then

$$\frac{Y^{(K_{\infty})}_{A,\Sigma}}{\pi^iY^{(K_{\infty})}_{A,\Sigma}} \cong \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A[\pi^i], \Sigma}(L)^{\vee}$$

for every $i \in \mathbb{N}, i \geq 1$, where the projective limit is taken over the finite normal subextensions $L$ of $K_{\infty}/K$.

Proof. In view of Lemma 3.5 we have isomorphisms

$$\text{Sel}_{0, A, \Sigma}(L)[\pi^i] \cong \text{Sel}_{0, A[\pi^i], \Sigma}(L)$$

for each $L$. By duality $\pi^iY^{(L)}_{A, \Sigma}$ is precisely the group acting trivial on $\text{Sel}_{0, A, \Sigma}(L)[\pi^i]$. Therefore

$$\frac{Y^{(K_{\infty})}_{A,\Sigma}}{\pi^iY^{(K_{\infty})}_{A,\Sigma}} \cong (\text{Sel}_{0, A, \Sigma}(L)[\pi^i])^{\vee} \cong \text{Sel}_{0, A[\pi^i], \Sigma}(L)^{\vee}.$$  

The result now follows since

$$\frac{Y^{(K_{\infty})}_{A,\Sigma}}{\pi^iY^{(K_{\infty})}_{A,\Sigma}} \cong \lim_{K \subseteq L \subseteq K_{\infty}} \frac{Y^{(L)}_{A,\Sigma}}{\pi^iY^{(L)}_{A,\Sigma}}.$$  

We can now prove an analogous of Theorem 1.3.

Theorem 3.7. Let $A_1$ and $A_2$ be associated with two representations $V_1$ and $V_2$ of $G_K$. Let $K_{\infty}/K$ be an admissible $p$-adic Lie extension, and let $G = \text{Gal}(K_{\infty}/K)$. Let $\Sigma$ be a finite set of primes of $K$ which contains $S_{\text{ram}}(K_{\infty}/K), S_p$ and the sets of primes of $K$ where either $V_1$ or $V_2$ is ramified. If $p = 2$, then we assume that $K$ is totally imaginary.

Suppose that $A_v(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$ and $j \in \{1, 2\}$. We let $r_j = \text{rank}_{G[[G]]}(Y^{(K_{\infty})}_{A_j, \Sigma}), 1 \leq j \leq 2$. Let $l$ be minimal such that $(\pi^iY^{(K_{\infty})}_{A_1, \Sigma})[\pi]$ is pseudonull.

Then the statements from Theorem 1.3 hold.

Proof. Suppose that $A_1[\pi^i] \cong A_2[\pi^i]$ as $G_K$-modules. Then

$$\pi^{i}Y^{(K_{\infty})}_{A_1, \Sigma} / \pi^{i+1}Y^{(K_{\infty})}_{A_1, \Sigma} \cong \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_1[\pi^{i+1}], \Sigma}(L)^{\vee} / \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_1[\pi^i], \Sigma}(L)^{\vee}$$

$$\cong \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_2[\pi^{i+1}], \Sigma}(L)^{\vee} / \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_2[\pi^i], \Sigma}(L)^{\vee}$$

$$\cong \pi^{i}Y^{(K_{\infty})}_{A_2, \Sigma} / \pi^{i+1}Y^{(K_{\infty})}_{A_2, \Sigma}$$

for every $i < l$, by Corollary 3.6 and Lemma 3.4. It follows in particular that both $F_q[[G]]$-modules have the same rank. Therefore we can proceed as in the proof of Theorem 1.3.

Proof of Theorem 1.3. The hypothesis in (a) implies that $\pi^iY^{(K_{\infty})}_{A_1, \Sigma}$ is $\mathcal{O}$-free. Let $E$ be an elementary $\Lambda$-module pseudo-isomorphic to $\pi^iY^{(K_{\infty})}_{A_1, \Sigma}$. Since $\pi^iY^{(K_{\infty})}_{A_1, \Sigma}$ is $\mathcal{O}$-free, the maximal finite submodule of $Y^{(K_{\infty})}_{A_1, \Sigma}$ is annihilated by $\pi^i$. For any finitely generated $\mathcal{O}$-module $M$ we denote by $\text{rank}_q(M)$ the dimension of $M/\pi M$ as $F_q$-vector space. Then $\text{rank}_q(\pi^iY^{(K_{\infty})}_{A_1, \Sigma}) = \text{rank}_q(E)$ by the argument given in
the proof of [Kle17, Proposition 3.4(i)]: as \( \pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})} \) is \( \mathcal{O} \)-free, we have an injection \( \varphi: \pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})} \to E \) with finite cokernel. Moreover, since multiplication by \( \pi \) is injective on \( E \), the quotients \( E/\text{im}(\varphi) \) and \( \pi E/\text{im}(\varphi) \) are isomorphic, proving that

\[
\text{rank}_q(\pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}) = \text{rank}_q(\text{im}(\varphi))
\]
equals rank \( q \)

Therefore

\[
|\pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}/\pi^{l+1} \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}| = q^\text{rank}_q(\pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}) = q^\text{rank}_q(\text{im}(\varphi)) = q^{\lambda(E)} = q^{\lambda(\mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})})}.
\]

On the other hand, the maximal finite \( \Lambda \)-submodule of \( \mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})} \) need not be annihilated by \( \pi^l \); therefore

\[
|\pi^l \mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})}/\pi^{l+1} \mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})}| \geq q^{\lambda(\mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})})}.
\]

If both \( \pi^1 \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})} \) and \( \pi^l \mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})} \) are \( \mathcal{O} \)-free, then we can exchange the roles of \( A_1 \) and \( A_2 \) and obtain equality of \( \lambda \)-invariants. \( \Box \)

4. NON \( p \)-EXTENSIONS

In this final section we study the growth of fine Selmer groups of congruent Galois representations over abelian algebraic extensions of \( K \) which are the compositum of finite \( r \)-extensions for suitable primes \( r \neq p \). If \( p = 2 \), then we always assume that \( K \) is totally imaginary. Similarly as in Sections 3.1 and 3.2, we distinguish between two different settings, starting with one resembling the case which has been studied in Section 3.2.

**Theorem 4.1.** Let \( p \) be a fixed prime, let \( A_1 \) and \( A_2 \) be associated with two representations of \( G_K \), and let \( K_{\infty}/K \) be an abelian algebraic extension. Let \( \mathcal{P} \) be the set of primes \( r \) such that \( K_{\infty}/K \) contains a finite subextension of degree \( r \) over \( K \). Let \( \Sigma \) be a finite set of primes of \( K \) which contains \( S_p \), \( S_{\text{ram}}(K_{\infty}/K) \) and \( S_{\text{ram}}(A_j) \), \( j = 1, 2 \).

We assume that \( \dim(A_1) = \dim(A_2) = d \), that \( r \geq q^d \) for each \( r \in \mathcal{P} \), and that \( A_j(K_v)[\pi] = \{0\} \) for \( j = 1, 2 \) and every \( v \in \Sigma \). Then the following statements hold:

(a) If \( A_1[\pi] \cong A_2[\pi] \) as \( G_K \)-modules, then \( \mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})} \) is a finitely generated \( \mathcal{O} \)-module if and only if \( \mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})} \) is finitely generated over \( \mathcal{O} \).

(b) Suppose that both \( \mathcal{Y}_{A_j, \Sigma}^{(K_{\infty})} \), \( j = 1, 2 \), are finitely generated over \( \mathcal{O} \). Let \( \ell \in \mathbb{N} \) be large enough such that \( (\pi^\ell \mathcal{Y}_{A_j, \Sigma}^{(K_{\infty})})[\pi] = \{0\} \). If \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \) as \( G_K \)-modules, then \( \text{rank}_\mathcal{O}(\mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})}) \leq \text{rank}_\mathcal{O}(\mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}) \).

(c) In the setting of (b), suppose that \( A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}] \) for some \( l \) such that both \( (\pi^l \mathcal{Y}_{A_j, \Sigma}^{(K_{\infty})})[\pi] \) are trivial. Then \( \text{rank}_\mathcal{O}(\mathcal{Y}_{A_2, \Sigma}^{(K_{\infty})}) = \text{rank}_\mathcal{O}(\mathcal{Y}_{A_1, \Sigma}^{(K_{\infty})}) \).

**Proof.** The proof is analogous to the proofs of Theorems 3.7 and 1.2. Instead of Lemma 3.5, we apply the following

**Lemma 4.2.** Let \( A \) be associated with a \( G_K \)-representation of dimension \( d \), let \( \Sigma \) be a finite set of primes of \( K \) containing \( S_p \cup S_{\text{ram}}(A) \). If \( p = 2 \), then we assume that \( K \) is totally imaginary. Assume that \( A(K_v)[\pi] = \{0\} \) for every \( v \in \Sigma \).
Let $L \subseteq K_\Sigma$ be a finite normal extension of $K$ such that each prime number $r$ dividing $[L : K]$ satisfies $r \geq q^d$. Then

$$\text{Sel}_{0, A, \Sigma}(L)[\pi^r] \cong \text{Sel}_{0, A, \Sigma}(L)$$

for each $i \in \mathbb{N}$, $i \geq 1$.

**Proof.** By assumption $A(K)[\pi] = \{0\}$. We mimic the proof of [NSW08, Corollary (1.6.13)] and show that also $H^0(L, A[\pi]) = A(L)[\pi] = \{0\}$. Let $r$ be the smallest prime number dividing $[L : K]$. Since $A(L)[\pi] \setminus A(K)[\pi]$ is the disjoint union of $\text{Gal}(L/K)$-orbits with more than one element, the cardinality of each such orbit is divisible by some prime $r' \geq r$. Thus if there exists at least one orbit containing more than one element, then

$$|A(L)[\pi]| \geq |A(K)[\pi]| + r' \geq |A(K)[\pi]| + r.$$  

On the other hand, $|A(L)[\pi]| \leq q^d$. Since $r \geq q^d$ by assumption, we obtain that such a non-trivial orbit cannot exist. Therefore $A(L)[\pi] = \{0\}$. Now we can proceed as in the proof of Lemma 3.3. \hfill \square

Using this lemma, we can derive a variant of Corollary 3.6 for all finite normal subextensions of $K_\infty$. Then we can proceed as in the proofs of Theorems 1.1 and 1.2. \hfill \square

Now we turn to the second result for non-$p$-extensions. As in Theorem 4.1, we let $P = P(K_\infty)$ be the set of prime numbers $r$ such that $K_\infty$ contains an extension of $K$ of degree $r$.

**Theorem 4.3.** Let $p$ be a fixed prime, let $A_1$ and $A_2$ be associated with two $G_K$-representations, and let $K_\Sigma/K$ be an abelian algebraic extension such that $p \not\in P(K_\infty)$. Let $\Sigma$ be a finite set of primes of $K$ which contains $S_p$, $S_{\text{ram}}(K_\infty/K)$ and $S_{\text{ram}}(A_j)$, $j = 1, 2$.

We assume that each prime $v \in \Sigma$ is finitely split in $K_\infty/K$.

(a) If $A_1[\pi] \cong A_2[\pi]$ as $G_K$-modules, then $Y_{A_1, \Sigma}(K_\infty)$ is a finitely generated $\mathcal{O}$-module if and only if $Y_{A_2, \Sigma}(K_\infty)$ is finitely generated over $\mathcal{O}$.

Suppose now that for each $j \in \{1, 2\}$ and every $w \in \Sigma(K_\infty)$, the group $A_j(K_\infty, w)[\pi^\infty]$ is finite. Then also the following statements hold:

(b) Suppose that both $Y_{A_j, \Sigma}(K_\infty)$, $j = 1, 2$, are finitely generated over $\mathcal{O}$. Let $l \in \mathbb{N}$ be large enough such that $(\pi^l Y_{A_j, \Sigma}(K_\infty))[\pi] = \{0\}$ and $\pi^l A_1(K_\infty, w)[\pi^\infty] = \{0\}$ for every $w \in \Sigma(K_\infty)$. If $A_1[\pi^l+1] \cong A_2[\pi^l+1]$ as $G_K$-modules, then

$$\text{rank}_\mathcal{O}(Y_{A_2, \Sigma}(K_\infty)) \leq \text{rank}_\mathcal{O}(Y_{A_1, \Sigma}(K_\infty)).$$

(c) In the setting of (b), if $A_1[\pi^l+1] \cong A_2[\pi^l+1]$ for some $l$ such that both $(\pi^l Y_{A_j, \Sigma}(K_\infty))[\pi]$ and all the groups $\pi^l A_j(K_\infty, w)[\pi^\infty]$, $j \in \{1, 2\}$, are trivial, then

$$\text{rank}_\mathcal{O}(Y_{A_2, \Sigma}(K_\infty)) = \text{rank}_\mathcal{O}(Y_{A_1, \Sigma}(K_\infty)).$$

**Proof.** We first note that $Y_{A_j, \Sigma}(K_\infty)$ is finitely generated over $\mathcal{O}$ if and only if the quotient $Y_{A_j, \Sigma}(K_\infty)/\pi Y_{A_j, \Sigma}(K_\infty)$ is finite. The exact sequence 3 from the proof of Corollary 33 implies that the kernels and cokernels of the maps

$$Y_{A_j, \Sigma}(K_\infty)/\pi Y_{A_j, \Sigma}(K_\infty) \longrightarrow \lim_{K \subseteq L \subseteq K_\infty} \text{Sel}_{0, A_j, \Sigma}(L)$$


are finite for both $j = 1$ and $j = 2$ (here we use the hypothesis that each $v \in \Sigma$ is finitely split in $K_{\infty}/K$). The assertion (a) therefore follows from arguments similar to those used in the proof of Theorem 1.2.

More generally, we have exact sequences

$$0 \to M_k \to \frac{Y_{A_j}(K_{\infty})}{\pi^K Y_{A_j}(K_{\infty})} \to \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_j[\pi^K], \Sigma}(L)^{\vee} \to N_k \to 0$$

for finite abelian groups $M_k$ and $N_k$, $k \in \mathbb{N}$. Moreover, from the proof of Lemma 6.1, we obtain exact sequences

$$(5) \quad 0 \to N_k \to A_j(K_{\infty})[\pi^K] \to \frac{\pi^K A_j(K_{\infty})[\pi^K]}{\pi^K A_j(K_{\infty})[\pi^K]} \to C_k \to 0,$$

$$0 \to C_k \to \bigoplus_{w \in \Sigma(K_{\infty})} A_j(K_{\infty, w})[\pi^K] \to \frac{\pi^K A_j(K_{\infty, w})[\pi^K]}{\pi^K A_j(K_{\infty, w})[\pi^K]} \to M_k \to 0$$

for every $k \in \mathbb{N}$ and $j = 1, 2$ (note that $M_k \cong M_k^{\vee}$ and $N_k \cong N_k^{\vee}$). Therefore $|\pi^K Y_{A_j[\pi^K], \Sigma}(L)^{\vee}|$ differs from

$$\lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_j[\pi^K+1], \Sigma}(L)^{\vee} \bigg/ \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_j[\pi^K], \Sigma}(L)^{\vee}$$

by a factor $\frac{|M_k||N_k+1|}{|N_k||M_k+1|}$ which is smaller than or equal to 1 because (5) implies that $\frac{|N_k+1|}{|N_k||M_k+1|}$ can be strictly less than or equal to 1 because (5) implies that $\frac{|N_k+1|}{|N_k||M_k+1|}$. In fact, for $k = l$ and $j = 1$, this factor is 1 by our hypotheses. Therefore

$$|\pi^K Y_{A_1[\pi^K], \Sigma}(L)^{\vee}| = \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_2[\pi^{l+1}], \Sigma}(L)^{\vee} \bigg/ \lim_{K \subseteq L \subseteq K_{\infty}} \text{Sel}_{0, A_2[\pi^l], \Sigma}(L)^{\vee}$$

because $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$. Note that the factor $\frac{|M_k||N_k+1|}{|N_k||M_k+1|}$ can be strictly smaller than 1 for $A_2$. This happens if $\pi^K$ does not annihilate the $\pi$-primary subgroups of the $A_2(K_{\infty, w})$. We have thus shown that

$$\text{rank}_q(\pi^K Y_{A_2[\pi^K], \Sigma}) \leq \text{rank}_q(\pi^K Y_{A_1[\pi^K], \Sigma})$$

where $\text{rank}_q$ is defined as in the proof of Theorem 1.2. The assertion (b) follows since

$$|\pi^K Y_{A_1, \Sigma}(K_{\infty})| = q^{\text{rank}_q(Y_{A_1[\pi^K], \Sigma})}$$

because $(\pi^K Y_{A_1[\pi^K], \Sigma})[\pi] = \{0\}$ by assumption; the $O$-rank of $Y_{A_2[\pi^K] \Sigma}$ can be strictly smaller than the corresponding $q$-rank, as in the proof of Theorem 1.2.

Finally, (c) follows by reverting the roles of $A_1$ and $A_2$ in the previous proof. □

Remark 4.4. Going through the proof of the theorem, one sees that actually the finiteness of $A_2(K_{\infty, w})[\pi^K]$ is needed only for at least one $w \in \Sigma(K_{\infty})$. Moreover, if one assumes that $A(K_{\infty})[\pi] = \{0\}$, then one can drop completely the condition that the group $A_2(K_{\infty, w})[\pi^K]$ is finite for every $w \in \Sigma(K_{\infty})$ in point (b) of the above theorem.

Remark 4.5. In order to give some evidence for the finiteness assumptions in the last two parts of Theorem 4.3, we mention some known results in the special setting of abelian varieties. In the following, we let $A$ be an abelian variety defined over the number field $K$, and we consider $O = \mathbb{Z}_p$.

Actually the following conditions are sufficient for ensuring finite torsion groups, i.e. not only finite $p$-torsion for some fixed prime $p$. 


(i) If $K_\infty/K$ is a finite extension, then the torsion subgroup of $A(K_\infty,w)$ is finite for each prime $w$ by the theorem of Mattuck (see [Mat55]).

(ii) If $A$ has potentially good and ordinary reduction at some prime $q$, then the torsion subgroup of $A(K_\infty,w)$ is finite for each $w$ if $K_\infty$ is a finite extension of the cyclotomic $\mathbb{Z}_q$-extension of $K$ (see [Ima75]).

(iii) For global fields, more is known: let $\Omega$ be the field obtained from $K$ by adjoining all roots of unity in some fixed algebraic closure of $K$ (i.e., $\Omega$ contains the cyclotomic $\mathbb{Z}_q$-extensions for all primes $q$). Then it follows from results of Ribet (see [KLS1] Appendix, Theorem 1]) that the torsion group of $\Omega(A)$ is finite.

We conclude by mentioning a special setting, namely of an elliptic curve $A = E$ defined over $K$, in which $Y_{A,\Sigma}^{(K_\infty)}$ is known to be finitely generated over $\mathcal{O} = \mathbb{Z}_p$ for an infinite non-$p$-extension $K_\infty$ of $K$.

Example 4.6. Let $N$ be an imaginary quadratic number field, and let $E$ be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_N$ of $N$. Let $q > 3$ be a prime of good reduction which splits in $N$, $q\mathcal{O}_N = q\mathcal{O}_N$. Let $K$ be an abelian extension of $N$ which is tamely ramified at $q$, and let $K_\infty = K \cdot N(E[q^\infty])$.

Now suppose that $p \neq q$ is a prime number which is co-prime with $6[K:N]$. We assume that $p$ splits in $N/Q$, does not ramify in $K/N$ and that $E$ has good reduction at the primes of $N$ above $p$. Let $\Sigma$ be a finite set of primes of $K$ which contains $S_{\text{ram}}(K_\infty/K)$, $S_p$ and $S_{\text{ram}}(E)$. If $E(K)[p] = \{0\}$ and $E(K_v)[p] = \{0\}$ for every $v \in \Sigma$, then $\text{rank}_{\mathbb{Z}_p}(Y_{E_\Sigma}^{(K_\infty)})$ is finite for every elliptic curve $E'$ which is defined over $K$ and satisfies $S_{\text{ram}}(E') \subseteq \Sigma$ and $E'[p] \cong E[p]$ as $G_K$-modules.

Indeed, by [Lam15] Theorem 1.2, the hypotheses of the corollary imply that in fact $X_{E_\Sigma}^{(K_\infty)}$ is finitely generated over $\mathbb{Z}_p$. Now we can apply Theorem 4.1.

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