Uncertainty Principles and Differential Operators on the Weighted Bergman Space

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Abstract
We classify self-adjoint first-order differential operators on weighted Bergman spaces on the unit disc and answer questions related to uncertainty principles for such operators. Our main tools are the discrete series representations of SU(1, 1). This approach has the promise to generalize to other bounded symmetric domains.

1 Introduction
In this paper we address several questions about weighted Bergman spaces on the unit disc from the perspective of representation theory. We rederive an uncertainty principle by Soltani [6] and give a partial answer to an open question posed by Yong and Zhu [10]. The question is if there are self-adjoint operators on the Bergman space for which the commutator is a non-zero multiple of the identity. We show that this cannot happen if the operators are first-order differential operators. Also, we give a classification of first-order differential operators on the weighted Bergman space. Such a classification has already been carried out for unweighted Bergman space in [7, 8]. Moreover, we show that they arise from the derived representation of the discrete series representations. The representation theoretic approach we propose seems to generalize to bounded symmetric domains. Lastly, we set up an isomorphism between Bergman spaces with shifted weights in the Hilbert space case, and conjecture that this extends to the whole family of Bergman spaces.

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2 Weighted Bergman Space

Let \( H(D) \) be the space of all holomorphic functions on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( dz \) denote the Euclidean measure on \( \mathbb{C} \) and define the weighted measure \( d\nu_\xi (z) := \frac{(\xi + 1)}{\pi} (1 - |z|^2) \xi \ d\nu \). For \( \xi > -1 \), this is a probability measure and therefore the weighted Bergman space defined by

\[
A_p^\xi := \left\{ f \in H(D) \left| \left\| f \right\|_{A_p^\xi} := \left( \int_D |f(z)|^p \ d\nu_\xi (z) \right)^{1/p} < \infty \right. \right\}
\]

is non-trivial for \( \xi > -1 \). The weighted Bergman space is a reproducing kernel Banach space for \( 1 \leq p < \infty \), and the reproducing kernel given by \( K_\xi (z, w) = \frac{1}{(1-zw)^{\xi + 2}} \) satisfies \( f(z) = \int_D f(w) K_\xi (z, w) \ d\nu_\xi (w) \) for \( f \in A_p^\xi \).

In this paper we will mostly focus on the Hilbert space \( A_2^\xi \) with inner product

\[
\langle f, g \rangle_\xi := \int_D f(z) \overline{g(z)} \ d\nu_\xi (z),
\]

which has an orthonormal basis \( e_n(z) := \sqrt{\frac{(\xi + 2)n}{n!}} z^n \). We also denote the norm on this space by \( \| \cdot \|_\xi \).

We define additional Hilbert spaces of analytic functions over \( \mathbb{D} \)

\[
A_{\xi, n}^2 := \left\{ f \in H(D) \left| \left\| f \right\|_{A_{\xi, n}^2} := \left| f(0) \right|^2 + \int_D |f^{(n)}(z)|^2 \ d\nu_\xi (z) < \infty \right. \right\}
\]

for \( n \in \mathbb{N} \). These spaces will later be identified as domains of certain differential operators. Notice that the spaces \( A_{\xi, n}^2 \) satisfies the containment \( \cdots \subseteq A_{\xi, n+1}^2 \subseteq A_{\xi, n}^2 \subseteq \cdots \subseteq A_{\xi, 1}^2 \subseteq A_2^\xi \) (see [6]).

3 Uncertainty Principles and Lie Groups

Let \( H \) be a Hilbert space with inner product denoted \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). For an operator \( A : H \rightarrow H \) the domain will be denoted \( \mathcal{D}(A) \). Assuming that \( A \) is densely defined and closable, let \( \overline{A} \) denote the closure of \( A \). For self- or skew-adjoint operators \( A, B : H \rightarrow H \), the following uncertainty principle can easily be derived

\[
\left| \langle [A, B]u, u \rangle \right| \leq 2 \| (A + a)u \| \| (B + b)u \| \text{ for all } u \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}([A, B])
\]

where \( a, b \in \mathbb{R} \). In general it is not true that this uncertainty principle extends to

\[
\left| \langle [A, B]u, u \rangle \right| \leq 2 \| (A + a)u \| \| (B + b)u \| \text{ for all } u \in \mathcal{D}(A) \cap \mathcal{D}(B) \cap \mathcal{D}([A, B]),
\]
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since \([A, B]\) may not even be densely defined. However, when the operators involved arise from representations of Lie groups as described below, the uncertainty principle can be extended.

Recall that \((\pi, H)\) is a representation of a group \(G\) if it satisfies

a) \(\pi(x) : H \rightarrow H\) is linear,

b) \(\pi(e) = I\),

c) \(\pi(xy) = \pi(x)\pi(y)\)

for \(x, y \in G\). The representation is called unitary if every \(\pi(x)\) is unitary.

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\) and exponential mapping \(\exp : \mathfrak{g} \rightarrow G\) (we will also denote \(\exp(X)\) by \(e^X\)). The Lie bracket on \(\mathfrak{g}\) will be denoted \([\cdot, \cdot]\).

If \((\pi, H)\) is a representation of the Lie group \(G\), then the space of smooth vectors \(H^\infty_\pi\) is the collection of vectors \(v \in H\) for which \(x \mapsto \pi(x)v\) is smooth from \(G\) to \(H\). It is well-known that \(H^\infty_\pi\) is dense in \(H\). Therefore an \(X \in \mathfrak{g}\) defines a densely defined differential operator \(\pi(X)\) by

\[
\pi(X)v = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(tX))v
\]

with domain \(H^\infty_\pi\). If \(\pi\) is unitary, the operator \(\pi(X)\) is skew-symmetric and thus closable, and its closure, denoted \(\overline{\pi}(X)\), is skew-adjoint. The domain of \(\overline{\pi}(X)\) is exactly the collection of vectors for which (3.1) converges in \(H\). It turns out that for \(X, Y \in \mathfrak{g}\) the operator \([\overline{\pi}(X), \overline{\pi}(Y)]\) is closable and its closure is \(\overline{\pi}([X, Y])\). By [3] the following uncertainty principle holds.

**Theorem 1** Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\), and let \((H, \pi)\) be a unitary representation of \(G\). Suppose that \(X, Y \in \mathfrak{g}\). Then for \(x, y \in \mathbb{R}\),

\[
|\langle \overline{\pi}([X, Y])u, u \rangle| \leq 2\|\overline{\pi}(X) + xu\|\|\overline{\pi}(Y) + yu\|
\]

where \(u \in D(\overline{\pi}(X)) \cap D(\overline{\pi}(Y)) \cap D(\overline{\pi}([X, Y]))\).

**4 Uncertainty Principles on Weighted Bergman Spaces**

In the following we will investigate properties of operators arising from the discrete series representations of the Lie group \(SU(1, 1)\) of \(2 \times 2\) complex matrices with determinant 1 which leave the form \(z_1 \bar{w}_1 - z_2 \bar{w}_2\) invariant:

\[
SU(1, 1) := \left\{ \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.
\]

The corresponding Lie algebra is

\[
su(1, 1) := \left\{ \begin{bmatrix} ia & b \\ \overline{b} & -ia \end{bmatrix} : a \in \mathbb{R}, b \in \mathbb{C} \right\}.
\]
For an element of SU(1, 1),

\[ x = \begin{bmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{bmatrix} \in \text{SU}(1, 1), \]

consider the discrete series representation \( \pi_\xi(x) : A^2_\xi \rightarrow A^2_\xi \) given by

\[ \pi_\xi(x)f(z) = \frac{1}{(-\beta z + \alpha)^{\xi+2}} f\left( \frac{\alpha z - \beta}{-\beta z + \alpha} \right). \]

It is known that \((\pi_\xi, A^2_\xi)\) is a unitary representation of \(\text{SU}(1, 1)\) when \(\xi > -1\) is an integer. Moreover, it defines a unitary representation of the universal covering group of \(\text{SU}(1, 1)\) for all \(\xi > -1\). In this paper we do not need to make a distinction between these groups, since their Lie algebras are the same and the exponential mapping is a local diffeomorphism.

Let us define the following elements \(X, Y, Z \in \text{su}(1, 1)\) by

\[ X := \begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix}, \quad Y := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z := \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \]

which are a basis for \(\text{su}(1, 1)\). It follows that

\[ \begin{align*}
\pi_\xi(X)f(z) &= -2izf'(z) - (\xi + 2)if(z), \\
\pi_\xi(Y)f(z) &= z^2 f'(z) - f'(z) + (\xi + 2)z f(z), \\
\pi_\xi(Z)f(z) &= if'(z) - 2izf'(z) + iz^2 f'(z) + (\xi + 2)iz f(z) - (\xi + 2)if(z).
\end{align*} \tag{4.1} \]

In the following, we show that an uncertainty principle on \(A^2_\xi\) presented in [6] can be obtained from the discrete series representations. Define \(W \in \text{su}(1, 1)\) to be

\[ W := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y - X. \]

We see that that \(\pi_\xi(W) = \pi_\xi(Z) - \pi_\xi(X)\), and since \([W, Y] = -2X\), then

\[ \pi_\xi([W, Y])f(z) = -2\pi_\xi(X)f(z) = 4iz f'(z) + (\xi + 2)2if(z). \tag{4.2} \]

**Lemma 2** The domain of \(\mathcal{F}_\xi(X)\) is \(A^2_{\xi,1}\).

**Proof** We will adopt the notation that \((\xi + 2)_k = \frac{\Gamma(k+\xi+2)}{\Gamma(\xi+2)}\). From [6], we have for \(f \in A^2_\xi\) and \(g \in A^2_{\xi,n}\) that

\[ \|f\|_\xi^2 = \sum_{k=0}^{\infty} |a_k|^2 \frac{k!}{(\xi + 2)_k} \quad \text{and} \quad \|g\|_{\xi,n}^2 = |b_0|^2 + \sum_{k=1}^{\infty} |b_k|^2 \frac{k^{2n}k!}{(\xi + 2)_k}. \]
where \( a_k, b_k \in \mathbb{C} \). Now, let \( f \in \mathcal{D}(\pi_\xi(\mathcal{X})) \). Since \( f \) is holomorphic,

\[
\pi_\xi(\mathcal{X}) f(z) = -2iz \frac{d}{dz} \sum_{k=0}^\infty a_k z^k - (\xi + 2)i \sum_{k=0}^\infty a_k z^k
\]

\[
= \sum_{k=0}^\infty -2ika_k z^k - \sum_{k=0}^\infty (\xi + 2)i a_k z^k = \sum_{k=0}^\infty -i(2k + \xi + 2)a_k z^k
\]

for \( a_k \in \mathbb{C} \). Thus,

\[
\|\pi_\xi(\mathcal{X}) f\|^2_{\xi} = \sum_{k=0}^\infty (2k + \xi + 2)^2 |a_k|^2 \frac{k!}{(\xi + 2)_k}. \tag{4.3}
\]

For any \( k \geq 0 \) we have that

\[
(2k + \xi + 2)^2 = 4k^2 + 8k\xi + 8k + \xi^2 + 4\xi + 4
\]

\[
\leq 4k^2 + 8k^2\xi + 8k^2 + 4k^2\xi^2 + 8k^\xi + 4k^2 = 4(\xi + 2)^2 k^2.
\]

Also, for any \( k \geq 1 \), it follows that

\[
k^2 \leq 4k^2 + 4k(\xi + 2) + \xi^2 + 4(\xi + 1) = (2k + \xi + 2)^2.
\]

Letting \( c := (\xi + 2)^2 |a_0|^2 \) (the \( k = 0 \) term from \( \|\pi_\xi(\mathcal{X})\|^2_{\pi_\xi} \)), we have that

\[
\left( |a_0|^2 + \sum_{k=1}^\infty |a_k|^2 \frac{k^2 k!}{(\xi + 2)_k} \right) + (c - |a_0|^2)
\]

\[
\leq \sum_{k=0}^\infty |a_k|^2 \frac{(2k + \xi + 2)^2 k!}{(\xi + 2)_k}
\]

\[
\leq 4(\xi + 2)^2 \left( |a_0|^2 + \sum_{k=1}^\infty |a_k|^2 \frac{k^2 k!}{(\xi + 2)_k} \right)
\]

which gives the inequality

\[
\|f\|^2_{\xi,1} + (c - |a_0|^2) \leq \|\pi_\xi(\mathcal{X}) f\|^2_{\xi} \leq 4(\xi + 2)^2 \|f\|^2_{\xi,1}.
\]

We conclude that \( \mathcal{D}(\pi_\xi(\mathcal{X})) = \mathcal{A}^2_{\xi,1} \). \( \square \)

From [6], the operators \( \frac{d}{dz} \) and \( z^2 \frac{d}{dz} + (\xi + 2)z \) have domain \( \mathcal{A}^2_{\xi,1} \). Since \( \pi_\xi(\mathcal{Y}) \) and \( \pi_\xi(\mathcal{W}) \) are linear combinations of these operators, then their domains contain \( \mathcal{A}^2_{\xi,1} \), i.e., \( \mathcal{D}(\pi_\xi(\mathcal{Y})), \mathcal{D}(\pi_\xi(\mathcal{W})) \supseteq \mathcal{A}^2_{\xi,1} \).*
Theorem 3 For \( f \in \mathcal{A}_{\xi}^{2} \) and \( w, y \in \mathbb{R} \),
\[
(\xi + 2)\|f\|_{\mathcal{A}_{\xi}^{2}}^{2} + 2(\langle zf', f \rangle)_{\xi} \leq \left\|(1 + z^2)f' + ((\xi + 2)z + iw)f\right\|_{\xi}.
\]

Note that this uncertainty principle is identical to (2.5) in [6].

**Proof** We have from Lemma 2 that \( \mathcal{D}(\pi_{\xi}(\mathcal{W}, \mathcal{Y})) = \mathcal{A}_{\xi}^{2} \). Then, \( \mathcal{D}(\pi_{\xi}(\mathcal{W})) \cap \mathcal{D}(\pi_{\xi}(\mathcal{Z})) = \mathcal{A}_{\xi}^{2} \). Apply Theorem 1 to obtain the inequality. \( \square \)

There are, of course, many uncertainty principles for Bergman spaces, but we only present this one here. One may ask, if there is a distinguished uncertainty principle. For example, Zhu posed the question of whether there exists an uncertainty principle on \( \mathcal{A}_{\xi}^{2} \) such that the commutator is a multiple of the identity operator [10]. The next result shows that it is not possible to find such an uncertainty principle from the representation \( (\pi_{\xi}, \mathcal{A}_{\xi}^{2}) \). As it turns out (see the next section) this also shows that such an uncertainty principle cannot be found for self-adjoint first-order differential operators on the Bergman space. The reason is that any first-order differential operator is of the form \( i\pi_{\xi}(X) + d \) for \( X \in \mathfrak{su}(1, 1) \) and \( d \in \mathbb{R} \).

**Theorem 4** Given \( (\pi_{\xi}, \mathcal{A}_{\xi}^{2}) \), there are no \( A, B \in \mathfrak{su}(1, 1) \) such that \( \pi_{\xi}([A, B]) = \eta \) for \( \eta \in \mathbb{C}\setminus\{0\} \).

**Proof** Since \( \mathfrak{g} = \mathfrak{su}(1, 1) \) is simple (so \( [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \)) and \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) form a basis, it is enough to show that \( \pi_{\xi}(\sigma\mathcal{X} + \tau\mathcal{Y} + \lambda\mathcal{Z}) \) is never a non-zero multiple of the identity for \( \sigma, \tau, \lambda \in \mathbb{R} \). Notice
\[
\pi_{\xi}(\sigma\mathcal{X} + \tau\mathcal{Y} + \lambda\mathcal{Z}) = \sigma\pi_{\xi}(\mathcal{X}) + \tau\pi_{\xi}(\mathcal{Y}) + \lambda\pi_{\xi}(\mathcal{Z})
\]
\[
= \left[ (\tau + i\lambda)z^2 + (-2\sigma - i2\lambda)z + (-\tau + i\lambda) \right] \frac{d}{dz}
\]
\[
+ \left[ (\xi + 2)(\tau + i\lambda)z + (\xi + 2)i(-\sigma - \lambda) \right]. \quad (4.4)
\]

To obtain a multiple of the identity, the coefficient \( * \) must be 0, which means \( \sigma + \lambda = 0 \). However, that would mean \( ** \) is also 0, which tells us that we cannot obtain a non-zero multiple of the identity. \( \square \)

### 5 First-Order Differential Operators on Weighted Bergman Spaces

In this section, we characterize self-adjoint extensions of first-order symmetric differential operators on \( \mathcal{A}_{\xi}^{2} \). Many of the results found here are already in [1], but
the connection to representation theory is new. In particular we show that all first-order symmetric differential operators come from the discrete series representations of SU($1, 1$). Using this characterization, we also provide a partial answer to a question about uncertainty principles raised by Yong and Zhu [10]. Some of the operator theoretic results in this section have also been proven in more generality in [1], but we wish to highlight the connection with representation theory and include the details here.

Let $P(D)$ be the space of polynomials on $D$. Naturally, $P(D) \subseteq A_2^2$. We define first-order differential operators be of the form $f \frac{d}{dz} + g$ where $f, g \in A_2^2$. Denote $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ for $a_k, b_k \in \mathbb{C}$. Let $L := f \frac{d}{dz} + g$ where $\mathcal{D}(L) = P(D)$. We first justify that the domain makes sense.

**Lemma 5** If $p$ is a polynomial, then $Lp$ is in $A_2^2$ and therefore the operator $L$ is densely defined.

**Proof** We will proceed by induction. When $N = 0$, notice that $La_0 = a_0 g$, and since $g \in A_2^2$, then $La_0 \in A_2^2$. Assume $L \sum_{k=0}^{M} c_k z^k \in A_2^2$ for some $M$. Since $L$ is a linear operator,

$$
L \sum_{k=0}^{M+1} c_k z^k = L \left( \sum_{k=0}^{M} c_k z^k + c_{M+1} z^{M+1} \right) = L \sum_{k=0}^{M} c_k z^k + c_{M+1} L z^{M+1}.
$$

Since $|z| < 1$, it follows that

$$
\|L z^{M+1}\|_2 = \left\| \sum_{k=0}^{\infty} (M+1)a_k z^{k+M} + \sum_{k=0}^{\infty} b_k z^{k+M+1} \right\|_2
\leq (M+1) \left\| \sum_{k=0}^{\infty} a_k z^{k+M} \right\|_2 + \left\| \sum_{k=0}^{\infty} b_k z^{k+M+1} \right\|_2
\leq \|g\|_2 < \infty.
$$

Since $L z^{M+1} \in A_2^2$, then $L \sum_{k=0}^{M+1} c_k z^k \in A_2^2$. Thus, $L$ is densely defined.

Define the operator $T$ by $Th = fh' + gh$ and let $\mathcal{D}(T)$ be the set of all $h \in A_2^2$ for which $fh' + gh \in A_2^2$. Notice that $\mathcal{D}(L) \subseteq \mathcal{D}(T) \subseteq A_2^2$.

**Lemma 6** The operator $T$ is closed.

The proof follows the argument for Theorem 1.1 in [7] or Theorem 3.9 in [1], but we include it here for completeness. The lemma implies that the operator $L$ is closable.

**Proof** Assume that $(x_n, y_n)$ is in the graph of $T$ for all $n$ and that the sequence converges to $(x, y) \in A_2^2 \times A_2^2$. Then $x_n$ converges to $x$ in $A_2^2$ and therefore, by Lemma 2.4 in [11], $x_n(z)$ and $x_n'(z)$ converge uniformly on compact sets to $x(z)$ and $x'(z)$ respectively.
This implies that \( y_n(z) = Tx_n(z) = f(z)x_n'(z) + g(z)x_n(z) \) converges uniformly on compact sets to \( f(z)x'(z) + g(z)x(z) = Tx(z) \). Since \( y_n \) converges to \( y \) in \( A_\xi^2 \) we get that \( Tx = y \) which implies that the graph of \( T \) is closed. \( \square \)

Define the space
\[
H^\infty_\xi = \left\{ f(z) \in H(\mathbb{D}) \left| \sum_{k=0}^{\infty} |a_k|^2 \xi^k = \xi(\xi + 2) + 2k \right. ^m < \infty, m \in \mathbb{N} \right\}. \tag{5.1}
\]
This is the space of smooth vectors for the representation \( \pi_\xi \) [4] and more generally [2]. Notice that if \( f \in H^\infty_\xi \), then \( f' \) is also in \( H^\infty_\xi \), since
\[
\| f' \|_\xi^2 = \sum_{k=1}^{\infty} |ka_k| \xi^{k-1} \| z \|^2_\xi
\]
and \( \| z^{k-1} \|^2_\xi = \frac{k+\xi+1}{k} \| z \|^2_\xi \) (choose \( m = 1 \) in (5.1) and compare norms).

**Lemma 7** If \( f \) and \( g \) are polynomials, then \( S = f(z) \frac{d}{dz} + g(z) \) with domain \( \mathcal{D}(S) = H^\infty_\xi \) is densely defined. Moreover, \( S \subseteq T \) and is therefore closable. Lastly, \( \overline{L} = \overline{S} \).

Notice that the space of smooth vectors is related (but not equal) to the spaces introduced in Section 3.1 of [1].

**Proof** We only need to prove the last statement, which requires that the graph \( G(L) \) is dense in the graph \( G(\overline{S}) = \overline{G(S)} \). Since multiplication by \( g(z) \) is a bounded operator on \( A_\xi^2 \), it is enough to work with \( g = 0 \). Let \( f(z) = \sum_{k=0}^{m} b_k z^k \). If \( (h, \overline{Sh}) \) is in \( \overline{G(S)} \), then there is a vector \( \tilde{h} \in H^\infty_\xi \) such that \( \| h - \tilde{h} \|_\xi \) and \( \| \overline{Sh} - \overline{S\tilde{h}} \|_\xi \) are both small. Let \( \tilde{h}(z) = \sum_{k=0}^{\infty} a_k z^k \) with coefficient satisfying (5.1). Define the polynomial \( \tilde{h}_n(z) = \sum_{k=0}^{n} a_k z^k \). Then \( \overline{S\tilde{h}} - L\tilde{h}_n = S(\tilde{h} - \tilde{h}_n) \) has norm
\[
\| \overline{S\tilde{h}} - L\tilde{h}_n \|_\xi \leq \sum_{\ell=0}^{m} |b_\ell| \left( \sum_{k=n+1}^{\infty} |ka_k|^2 \| z^{k+\ell-1} \|^2_\xi \right)^{1/2} \tag{5.2}
\]
Since \( \| z^{k-1} \|^2_\xi = \frac{\xi+1+k}{k} \| z^k \|^2_\xi \leq (\xi + 2) \| z \|^2_\xi \) and \( \frac{\| z^{k+\ell} \|^2_\xi}{\| z^k \|^2_\xi} = \frac{(k+\ell)! \Gamma(\xi+2)}{\Gamma(k+\ell+\xi+2)} \) converges to 1 as \( k \to \infty \), by the ratio test and (5.1), the finite sum in (5.2) converges to 0 as \( n \to \infty \). This shows that \( \| h - \tilde{h}_n \|_\xi \) and \( \| \overline{Sh} - L\tilde{h}_n \|_\xi \) are both small when \( n \) is large enough, and therefore \( G(L) \) is dense in \( G(\overline{S}) \). \( \square \)

**Theorem 8** The operator \( L = f \frac{d}{dz} + g \) with domain \( \mathcal{D}(L) = P(\mathbb{D}) \) is symmetric if and only if \( f = a_0 + a_1 z + a_0 z^2 \) and \( g = b_0 + (\xi + 2)a_0 z \), where \( a_1, b_0 \in \mathbb{R} \).

The first part of the proof is similar to [7], but is included here for completeness.
**Proof** Since any polynomial is a finite linear combination of the $e_n$’s, it is enough to work with the basis elements.

Assume $L$ is symmetric. For any $z \in \mathbb{D}$,

$$L e_n(z) = \sum_{k=0}^{\infty} n a_{k-n} \sqrt{\frac{(\xi + 2)_n}{n!}} z^{k+n} + \sum_{k=0}^{\infty} b_k \sqrt{\frac{(\xi + 2)_n}{n!}} z^k$$

$$= \sum_{k=0}^{\infty} n a_{k-n+1} \sqrt{\frac{(\xi + 2)_n}{n!}} z^{k+1} + \sum_{k=0}^{\infty} b_k \sqrt{\frac{(\xi + 2)_n}{n!}} z^k$$

$$= \sum_{k=0}^{\infty} n a_{k-n+1} \sqrt{\frac{(\xi + 2)_n k!}{(\xi + 2)_k n!}} e_k(z) + \sum_{k=0}^{\infty} b_k \sqrt{\frac{(\xi + 2)_n k!}{(\xi + 2)_k n!}} e_k(z)$$

$$= \sum_{k=0}^{\infty} \sqrt{\frac{(\xi + 2)_n k!}{(\xi + 2)_k n!}} (n a_{k-n+1} + b_k) e_k(z).$$

Note that $e_k(z)$ is a multiple of $z^k$ and $\sqrt{\frac{k!}{(\xi + 2)_k} \leq 1}$. By the series comparison test using the results of Lemma 2.1 in [7], the series $L e_n(z)$ converges, so

$$L e_n = \sum_{k=0}^{\infty} \sqrt{\frac{(\xi + 2)_n k!}{(\xi + 2)_k n!}} (n a_{k-n+1} + b_k) e_k.$$

Now, by orthonormality of the $\{e_k\}_{k \in \mathbb{N}_0}$, we have that

$$\langle L e_n, e_m \rangle_\xi = \sqrt{\frac{(\xi + 2)_n m!}{(\xi + 2)_m n!}} (n a_{m-n} + b_{m-n}),$$

and similarly for $\langle e_n, L e_m \rangle_\xi$,

$$\langle e_n, L e_m \rangle_\xi = \sqrt{\frac{(\xi + 2)_n m!}{(\xi + 2)_m n!}} (m a_{n-m} + b_{n-m}).$$

Notice that $L$ is symmetric if and only if

$$\sqrt{\frac{(\xi + 2)_n m!}{(\xi + 2)_m n!}} (n a_{m-n} + b_{m-n}) = \sqrt{\frac{(\xi + 2)_m n!}{(\xi + 2)_m n!}} (m a_{n-m} + b_{n-m})$$

(5.3)

for $n, m \in \mathbb{N}_0$. By properties of the Gamma function,

$$(\xi + 2)_1 = \frac{\Gamma(\xi + 2 + 1)}{\Gamma(\xi + 2)} = \frac{(\xi + 2)\Gamma(\xi + 2)}{\Gamma(\xi + 2)} = (\xi + 2),$$
so we have in particular that

\[ g = Le_0 = \sum_{k=0}^{\infty} \sqrt{\frac{k!}{(\xi + 2)_k}} b_k e_k = \sum_{k=0}^{\infty} \sqrt{\frac{(\xi + 2)_k}{k!}} (k\bar{a}_{1-k} + \bar{b}_{-k})e_k \]

\[ = \bar{b}_0 + \sqrt{(\xi + 2)_1\bar{a}_0}e_1 = \bar{b}_0 + (\xi + 2)\bar{a}_0 z = \bar{b}_0 + (\xi + 2)\bar{a}_0 z. \]

Using (5.3) for \( m = 0 \) and \( n = 0 \), we have \( b_0 = \bar{b}_0 \in \mathbb{R} \). Then, we derive that

\[ f = \frac{1}{\sqrt{\xi + 2}} Le_1 - gz \]

\[ = \left( \frac{1}{\sqrt{\xi + 2}} \sum_{k=0}^{\infty} \sqrt{\frac{(\xi + 2)_k}{(\xi + 2)_k}} (a_k + b_{k-1})e_k \right) - (\bar{b}_0 + (\xi + 2)\bar{a}_0 z)z \]

\[ = \left( \frac{1}{\xi + 2} \sum_{k=0}^{\infty} \sqrt{\frac{(\xi + 2)_k}{k!}} (ka_{2-k} + b_{1-k})e_k \right) - (\bar{b}_0 z + (\xi + 2)\bar{a}_0 z^2) \]

\[ = \left[ \frac{b_1}{\xi + 2} + (\bar{a}_1 + \bar{b}_0)z + \frac{(\xi + 2)_2}{\xi + 2} \bar{a}_0 z^2 \right] - (\bar{b}_0 z + (\xi + 2)\bar{a}_0 z^2) \]

\[ = \frac{b_1}{\xi + 2} + \bar{a}_1 z + \frac{(\xi + 2)_2}{\xi + 2} \bar{a}_0 z^2 = \frac{b_1}{\xi + 2} + \bar{a}_1 z + \bar{a}_0 z^2. \]

Using (5.3) for \( m = 0 \) and \( n = 1 \),

\[ \sqrt{(\xi + 2)a_0} = \frac{b_1}{\sqrt{\xi + 2}} \iff a_0 = \frac{b_1}{\xi + 2}, \]

and for \( m = 1 \) and \( n = 1 \), we get that \( a_1 = \bar{a}_1 \in \mathbb{R} \).

Now assume that \( f, g \) are of the described form, then \( L = (a_0 + a_1 z + (\bar{a}_0 z^2)) \frac{d}{dz} + (b_0 + (\xi + 2)\bar{a}_0 z) \) for \( a_1, b_0 \in \mathbb{R} \). Adopting the convention that \( e_{-k} = 0 \) for \( k \in \mathbb{N} \), we have

\[ Le_n = [(a_0 + a_1 z + (\bar{a}_0 z^2)) \frac{d}{dz} + (b_0 + (\xi + 2)\bar{a}_0 z)](\xi + 2)_n z^n \]

\[ = [na_0 z^{n-1} + (na_1 + b_0)z^n + (n + \xi + 2)\bar{a}_0 z^{n+1}] \frac{(\xi + 2)_n}{n!} z^n \]

\[ = na_0 \sqrt{(n + \xi + 1) e_{n-1} + (na_1 + b_0) e_n + (n + \xi + 2)\bar{a}_0 \sqrt{(n + 1)(\xi + 2)_n}} e_{n+1} \]

\[ = a_0 \sqrt{n(n + \xi + 1) e_{n-1} + (na_1 + b_0) e_n + \bar{a}_0 \sqrt{(n + \xi + 2)(n + 1)} e_{n+1}}. \]

It follows by orthogonality that

\[ \langle Le_n, e_{n-1} \rangle_{\xi} = a_0 \sqrt{n(n + \xi + 1)} = \langle e_n, Le_{n-1} \rangle_{\xi}, \]
\[ \langle L e_n, e_n \rangle_\xi = na_1 + b_0 = \langle e_n, L e_n \rangle_\xi \]
\[ \langle L e_n, e_{n+1} \rangle_\xi = a_0 \sqrt{(n + \xi + 2)(n + 1)} = \langle e_n, L e_{n+1} \rangle_\xi , \]
and that \( \langle L e_n, e_m \rangle_\xi = 0 = \langle e_n, L e_m \rangle_\xi \) when \( m \) is not \( n - 1, n \) or \( n + 1 \), so \( L \) is symmetric. \( \Box \)

One of our main results in this section is the following classification of self-adjoint first-order differential operators in terms of the discrete series representations.

**Theorem 9** Given \( a, b \in \mathbb{R} \) and \( c \in \mathbb{C} \), define \( L = (cz^2 + az + \overline{c}) \frac{d}{dz} + ((\xi + 2)cz + b) \) with \( \mathcal{D}(L) = P(\mathbb{D}) \). Then \( L \) is symmetric and equal to the restriction of an operator of the form \( S = i \pi_\xi (X) + d \) with \( \mathcal{D}(S) = H^\infty_\xi \) to for \( d \) real and \( X \in \mathfrak{su}(1, 1) \). Moreover, \( \overline{L} = i \pi_\xi (X) + d \) is self-adjoint.

**Proof** Theorem 8 gives that \( L \) is symmetric. Let \( \sigma, \tau, \lambda \in \mathbb{R} \). For any \( X \in \mathfrak{su}(1, 1) \), we can express \( i \pi_\xi (X) \) as

\[
i(\sigma \pi_\xi (X) + \tau \pi_\xi (\mathfrak{h}^g) + \lambda \pi_\xi (\mathfrak{h}^f)) = \left[ (-\lambda + i \tau)z^2 + (2\sigma + 2\lambda)z + (-\lambda - i \tau) \right] \frac{d}{dz} + \left[ (\xi + 2)(-\lambda + i \tau)z + (\xi + 2)(\sigma + \lambda) \right].
\]

which can be simplified to

\[
[(-\lambda + i \tau)z^2 + (2\sigma + 2\lambda)z + (-\lambda - i \tau)] \frac{d}{dz} + [(\xi + 2)(-\lambda + i \tau)z + 2(\sigma + \lambda)].
\]

This is equivalent to saying

\[
i \pi_\xi (X) = (cz^2 + az + \overline{c}) \frac{d}{dz} + ((\xi + 2)cz + a)
\]

for \( a \in \mathbb{R} \) and \( c \in \mathbb{C} \). Notice that

\[
L = (cz^2 + az + \overline{c}) \frac{d}{dz} + ((\xi + 2)cz + b)
\]

\[
(cz^2 + az + \overline{c}) \frac{d}{dz} + \left( (\xi + 2)cz + \frac{(\xi + 2)a}{2} \right) + \left( b - \frac{(\xi + 2)a}{2} \right) = d
\]

We know that \( \overline{L} = \overline{S} \) from Lemma 7. That \( \overline{S} = i \pi (X) + d \) is self-adjoint has been shown in [5]. \( \Box \)

By a similar argument to Villone [7, Theorem 1.2], this implies that \( \overline{L} = T \).
Remark 10  This results shows that the polynomials are a core for $\pi_\xi(X)$ for every $X \in su(1, 1)$ for the discrete series representation. This seems to be a new result, although it is likely not surprising to experts.

It seems that one should be able to characterize self-adjoint differential operators of any order, studied in [7], via the extension of $\pi_\xi$ to the universal enveloping algebra of $su(1, 1)$.

Theorem 9 shows that any self-adjoint first-order differential operator on $A^2_\xi$ is generated by $\pi_\xi$ and the identity. This allows us to strengthen our answer to Zhu’s question in [10].

Theorem 11 There are no first-order self-adjoint differential operators $A, B$ on $A^2_\xi$ such that $[A, B] = \eta$ for $\eta \in \mathbb{C}\setminus\{0\}$.

Proof  Let $A, B$ be first-order self-adjoint differential operators on $A^2_\xi$. We know that $A = \pi_\xi(X)$ and $B = \pi_\xi(Y)$ for some $X, Y \in su(1, 1)$. Then

$$[A, B] = [\pi_\xi(X), \pi_\xi(Y)] \subseteq \pi_\xi([X, Y]).$$

Apply the fact that $[A, B] = \pi_\xi([X, Y])$ which can never be a non-zero multiple of the identity by Theorem 4.

6 Relation Between Bergman Spaces with Shifted Weights

In this section, we show that the weighted Bergman space, $A^2_\xi$, can be related to a shifted weighted Bergman space.

Theorem 12 The operator $\tilde{L} := z \frac{d}{dz} + c$ with $c$ not being 0 or a negative integer extends to an isomorphism between $A^2_\xi$ and $A^2_{\xi+2}$.

Proof  The argument for linearity is clear. Let $f, g \in A^2_\xi$. Denote $f(z) := \sum_{k=0}^{\infty} a_k z^k$, and let $g(z) := \sum_{k=0}^{\infty} b_k z^k$. Then

$$\tilde{L} f(z) = \tilde{L} g(z) \iff \sum_{k=0}^{\infty} (k+c)a_k z^k = \sum_{k=0}^{\infty} (k+c)b_k z^k \iff a_k = b_k$$

for every $k \in \mathbb{N}_0$, so $\tilde{L}$ is injective.

Let $f \in A^2_{\xi+2}$ and denote $f(z) := \sum_{k=0}^{\infty} a_k z^k$, and let $g(z) := \sum_{k=0}^{\infty} \frac{a_k}{k+c} z^k$. Notice that $\sup_{k \in \mathbb{N}_0} \frac{(k+c)^2}{(k+\xi+3)(k+\xi+2)} \to 1$ as $k \to \infty$, and is always positive if $c \notin \{-3, -2, -1, 0\}$, so there is $D > 0$, such that $\frac{1}{D} \leq \frac{(k+c)^2}{(k+\xi+3)(k+\xi+2)} \leq D$ for any $k \in \mathbb{N}_0$. Then,

$$\|g\|_{\xi}^2 = \sum_{k=0}^{\infty} \frac{1}{(k+c)^2} |a_k|^2 \|z^k\|_{\xi}^2.$$
\[
\begin{align*}
\leq & \frac{D}{(\xi + 3)(\xi + 2)} \sum_{k=0}^{\infty} \frac{(\xi + 3)(\xi + 2)}{(k + \xi + 3)(k + \xi + 2)} |a_k|^2 \|z^k\|_\xi^2 \\
= & \frac{D}{(\xi + 3)(\xi + 2)} \sum_{k=0}^{\infty} |a_k|^2 \|z^k\|_\xi^2 = \frac{D}{(\xi + 3)(\xi + 2)} \|f\|_\xi^{2} \lt \infty,
\end{align*}
\]
so \(g \in A_\xi^2\). Since \(\widetilde{L}g(z) = f(z)\) for every \(z \in \mathbb{D}\), then \(\widetilde{L}\) is surjective. For any \(f \in A_\xi^2\) denoted \(f(z) := \sum_{k=0}^{\infty} a_k z^k\), we have

\[
\begin{align*}
\frac{(\xi + 3)(\xi + 2)}{D} \sum_{k=0}^{\infty} |a_k|^2 \|z^k\|_\xi^2 &= (\xi + 3)(\xi + 2) \sum_{k=0}^{\infty} \frac{1}{D} |a_k|^2 \|z^k\|_\xi^2 \\
&\leq (\xi + 3)(\xi + 2) \sum_{k=0}^{\infty} \frac{(k + c)^2}{(k + \xi + 3)(k + \xi + 2)} |a_k|^2 \|z^k\|_\xi^2 \\
&\leq (\xi + 3)(\xi + 2) \sum_{k=0}^{\infty} D|a_k|^2 \|z^k\|_\xi^2 = D(\xi + 3)(\xi + 2) \sum_{k=0}^{\infty} |a_k|^2 \|z^k\|_\xi^2,
\end{align*}
\]
so \(\widetilde{L}\) is continuous. \(\square\)

**Remark 13** If \(\tilde{L} = \frac{2}{\xi+2} \frac{d}{dz} + 1\) and \(K^w_\xi(z) = K_\xi(z, w)\) is the reproducing kernel for \(A_\xi^2\), then \(\widetilde{L}K^w_\xi(z) = K^w_{\xi+1}(z) = K_{\xi+1}(z, w)\). Such differential operators have been considered in [9]. Also, Theorem 12 shows that \(D(\pi_\xi(X)) = A_\xi^2 = A_{\xi+1}^2\) as sets when \(\xi > 1\).

Zhu in [11, p. 50] shows that \(\mathcal{R} := z \frac{d}{dz}\) is a surjective map from \(A_\xi\) to \(A_{\xi+2}\). Theorem 12 shows that \(\mathcal{R}\) (surjective) plus particular \(c\)'s (injective) is an isomorphism between the spaces. This is not necessarily true for any \(c \in \mathbb{R}\). We now state a generalization we believe is possible.

**Remark 14** We conjecture that Theorem 12 can be extended to show that \(\widetilde{L}\) extends to an isomorphism between \(A_\xi^p\) and \(A_{\xi+2}^p\). This would involve a study of multipliers between the two spaces. This also opens up the question of which differential operators \(q_1 \frac{d}{dz} + q_2\) extend to isomorphisms between Bergman spaces if \(q_1, q_2\) are polynomials.

**Author Contributions** All authors contributed equally.

**Declarations**

**Conflict of interest** The authors declare no competing interests.
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