Analyzing Stable Solutions of $3 \times 3$ and $3 \times 4$ van der Pol Oscillators 
Coupled as a Torus Shape

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Abstract  Synchronization phenomena are observed in many situations and places, and are used for various purposes. Many synchronization phenomena in coupled van der Pol oscillators, which are electronic circuits, have been analyzed and reported. We previously discovered and reported a wavelike propagating phase state between adjacent oscillators. However, the analysis of many synchronization phenomena is not yet sufficient and is a very important task. In this paper, we analyze synchronization states on torus shapes in which the number of oscillators per column is three and the number of oscillators per row is three or four. Furthermore, theoretical results are compared with simulation results.

Keywords: coupled oscillators, synchronization, torus shape

1. Introduction

Synchronization is an important natural phenomenon because living creatures, communication systems, planet systems, and etc. could not exist without synchronization. For example, we can observe synchronization phenomena as the simultaneous firing of many pacemaker cells in our hearts and the simultaneous flashing of fireflies in Southeast Asia. Furthermore, synchronization phenomena are used in industrial products such as communication systems. Therefore, the analysis of synchronization phenomena is very important, and many studies have been reported[1]-[4].

Synchronization phenomena also can be observed on coupled van der Pol oscillators, which are electronic circuits. Synchronization phenomena on van der Pol oscillators coupled as a ladder, a ring, and a two-dimensional lattice (2D lattice) have been theoretically analyzed and reported[5]-[7]. We discovered a wave motion on arrays of many van der Pol oscillators coupled as a ladder, a ring, and a 2D lattice[8],[9]. This wave motion can also be observed on a cross constructed from four coupled ladders[10]. This motion is called a phase-inversion wave. A phase-inversion wave propagates while changing its phase state between adjacent oscillators from in-phase synchronization to anti-phase synchronization, or from anti-phase synchronization to in-phase synchronization. When many oscillators are coupled as a ladder and are synchronizing in-phase, we can observe the generation of two phase-inversion waves from around a center oscillator of the ladder, and each phase-inversion wave propagates in a different direction[8]. Furthermore, the oscillators between two phase-inversion waves exhibit anti-phase synchronization. However, when phase-inversion waves do not exist, we cannot observe anti-phase synchronization on the ladder. Anti-phase synchronization can be observed on a ring with an even number of oscillators. Thus, the theoretical analysis of synchronization phenomena on the ring is required to analyze the phase-inversion waves on the ladder, and the theoretical analysis of a coupled oscillator system with a torus shape, which is a 2D lattice without an edge, is required to analyze special phenomena such as phase-inversion waves on a 2D lattice. Therefore, various synchronization phenomena including special phenomena such as phase-inversion waves are observed on coupled oscillators, and the theoretical analysis of synchronization states in steady states on oscillators coupled in a torus shape is important for understanding various synchronization phenomena.

In this paper, we theoretically analyze synchronization phenomena on a torus of $3 \times 3$ oscillators, which is the smallest arrangement of coupled oscillators in a torus shape, and we increase the number of oscillators of each row to four and theoretically analyze the observable synchronization phenomena. These phe-
nominal on a 3×4 torus are compared with those on a 3×3 torus, and we investigate whether the observable synchronization phenomena in each column and row depend on the numbers of oscillators in each column and row.

Furthermore, torus-shaped circuits are simulated by the fourth-order Runge-Kutta method, and theoretical analysis results are compared with simulation results.

2. Circuit Model

3×3 or 3×4 van der Pol oscillators are coupled by inductors to form torus shapes (see Figs. 1–3).

A nonlinear negative resistor in each van der Pol oscillator is expressed as follows (see Fig. 1).

\[
f(v_{mn}) = -g_1 v_{mn} + g_3 v_{3mn}^3 \]

The circuit equation of van der Pol oscillator OSC
t11
in Figs. 2 and 3 is expressed as

\[
\frac{C dv_{11}}{dt} + \frac{1}{L} \int v_{11} dt - g_1 v_{11} + g_3 v_{311}^3 \\
- \frac{1}{L_0} \int (v_a - v_{11}) dt - \frac{1}{L_0} \int (v_{31} - v_{11}) dt \\
+ \frac{1}{L_0} \int (v_{11} - v_{12}) dt + \frac{1}{L_0} \int (v_{11} - v_{21}) dt = 0
\]

(3×3 oscs.: a = 13, 3×4 oscs.: a = 14)

Stable solutions are derived by normalizing Eq. (2) and using the averaging method. The derived stable solutions are compared with the simulation results obtained using the fourth-order Runge-Kutta method.

3. Theoretical Analysis

Equation (3) is obtained by differentiating both sides of Eq. (2).

\[
\frac{C d^2v_{11}}{dt^2} + \left(\frac{1}{L} + \frac{4}{L_0}\right) v_{11} - g_1 \frac{dv_{11}}{dt} + 3g_3 v_{11}^2 \frac{dv_{11}}{dt} \\
- \frac{1}{L_0} \left(v_{12} + v_a + v_{21} + v_{31}\right) = 0
\]

(3×3 oscs.: a = 13, 3×4 oscs.: a = 14)

\[
\begin{align*}
\tau &= \sqrt{\frac{1}{2CL} + \frac{1}{CL_0}} \\
v_{mn} &= \sqrt{\frac{g_1}{3g_3}} x_{mn}
\end{align*}
\]

Equation (5) is obtained by normalizing Eq. (3) with Eq. (4).

\[
2L + L_0 \frac{g_1}{2LL_0} \sqrt{\frac{1}{3g_3}} \frac{d^2x_{11}}{dt^2} + \frac{g_1}{3g_3} \left(\frac{4L + L_0}{LL_0}\right) x_{11} \\
- g_1 \frac{g_1}{3g_3} \sqrt{\frac{2L + L_0}{2CLL_0}} \frac{dx_{11}}{dt} + 3g_3 v_{11}^2 \frac{2L + L_0}{2CLL_0} \frac{dx_{11}}{dt} \\
- \frac{1}{L_0} \sqrt{\frac{g_1}{3g_3}} \left(x_{12} + x_a + x_{21} + x_{31}\right) = 0
\]

(3×3 oscs.: a = 13, 3×4 oscs.: a = 14)

Multiplying both sides of Eq. (5) by \(\frac{2LL_0}{2L + L_0} \sqrt{\frac{3g_3}{g_1}}\) gives
\[ \frac{d^2 x_{11}}{d \tau^2} + 2 \left( \frac{4L + L_0}{2L + L_0} \right) x_{11} - g_1 \sqrt{\frac{2LL_0}{C(L_0 + 2L)}} \frac{dx_{11}}{d \tau} + g_1 \frac{dx_{11}}{d \tau} \left( \frac{2LL_0}{C(L_0 + 2L)} \right) \]

\[ - \frac{2L}{2L + L_0} (x_{12} + x_a + x_{21} + x_{31}) = 0 \]

\[ (3 \times 3 \text{ oscs.}: a = 13, \quad 3 \times 4 \text{ oscs.}: a = 14) \]

Equation (6) is normalized using Eq. (7) to derive this paper, nonlinearity is assumed to be very weak \( \alpha \) where 

\[ \alpha \cdot \]

which is approximated to Eq. (18), because the nonlinearity is assumed to be very weak and \( \varepsilon \) is assumed to be very small. Therefore, these circuits can be analyzed by the averaging method.

**3.1 Torus of 3×3 oscillators**

When 3×3 oscillators are coupled, Eq. (9) is obtained from the nine simultaneous differential equations obtained by normalizing the circuit equations of all oscillators in Fig. 2 by the same method.

\[ \begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} & \dot{x}_{13} \\ \dot{x}_{21} & \dot{x}_{22} & \dot{x}_{23} \\ \dot{x}_{31} & \dot{x}_{32} & \dot{x}_{33} \end{bmatrix} + \begin{bmatrix} 1 + \alpha & -\alpha & -\alpha \\ -\alpha & 1 + \alpha & -\alpha \\ -\alpha & -\alpha & 1 + \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \]

\[ + \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1 + \alpha & -\alpha & -\alpha \\ -\alpha & 1 + \alpha & -\alpha \\ -\alpha & -\alpha & 1 + \alpha \end{bmatrix} \]

\[ = \frac{\varepsilon}{3} \begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} & \dot{x}_{13} \\ \dot{x}_{21} & \dot{x}_{22} & \dot{x}_{23} \\ \dot{x}_{31} & \dot{x}_{32} & \dot{x}_{33} \end{bmatrix} \]

\[ = \frac{\varepsilon}{3} \begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} & \dot{x}_{13} \\ \dot{x}_{21} & \dot{x}_{22} & \dot{x}_{23} \\ \dot{x}_{31} & \dot{x}_{32} & \dot{x}_{33} \end{bmatrix} \]

where \( \varepsilon = \frac{d}{d \tau} \).

This equation is rewritten as 

\[ \ddot{x} + Bx + xB = \varepsilon \dot{x} - \frac{1}{3} \varepsilon \dot{x} \]

where 

\[ \dot{x} = \begin{bmatrix} 1 + \alpha & -\alpha & -\alpha \\ -\alpha & 1 + \alpha & -\alpha \\ -\alpha & -\alpha & 1 + \alpha \end{bmatrix} \]

\[ \dot{y} = \begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} & \dot{x}_{13} \\ \dot{x}_{21} & \dot{x}_{22} & \dot{x}_{23} \\ \dot{x}_{31} & \dot{x}_{32} & \dot{x}_{33} \end{bmatrix} \]

Table 1 shows the three eigenvalues (\( \lambda_1 - \lambda_3 \)) and three eigenvectors (\( \vec{E}_1 - \vec{E}_3 \)) of matrix \( B \). \( k_1, k_2, \) and \( k_3 \) are set as 1, and the three eigenvectors are combined as matrix \( P \).

\[ P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

The inverse matrix of \( P \) is calculated as 

\[ P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \]

\[ x = PyP^{-1} \]

Then, Eq. (13) is substituted into Eq. (10) to give 

\[ P \dot{y} + BPyP^{-1} + PyP^{-1}B = \varepsilon P \dot{y} - \frac{1}{3} \varepsilon \dot{z} \]

Multiplying by matrix \( P \) from the right side and multiplying by matrix \( P^{-1} \) from left side give 

\[ \ddot{y} + Dy + yD = \varepsilon \dot{y} - \frac{1}{3} \varepsilon \dot{P}^{-1} \dot{P} \]

where matrix \( D \) is the diagonal matrix 

\[ D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \]

\[ \dot{x}^2_{mn} \text{, which denotes each element of matrix } \dot{x}, \text{ is represented as } 3x^2_{mn} \dot{x}_{mn}. \] If the matrix, with elements \( \dot{x}^2_{mn} \dot{x}_{ij} \) is expressed as \( f \), then \( \dot{x} = 3f \), and \( \frac{1}{3} \varepsilon \dot{P}^{-1} \dot{P} \) is expressed as \( \dot{P}^{-1} f \) \( P \).

Then, Eq. (15) is rewritten as 

\[ \ddot{y} + Dy + yD = \varepsilon (\dot{y} - \dot{P}^{-1} f P) \]

which is approximated to Eq. (18), because the nonlinearity is very weak (0 < \( \varepsilon < 1 \)).

\[ \ddot{y} + Dy + yD = 0 \]

The Laplace transforme of Eq. (18) is 

\[ Y_{mn} = \frac{\dot{y}_{mn}(0)}{s^2 + \lambda_m + \lambda_n} + \frac{\dot{y}_{mn}(0)}{s^2 + \lambda_m + \lambda_n} \]

\[ \lambda_m + \lambda_n \text{ is set as } \omega^2_{mn}, \text{ and Eq. (20) is derived by the inverse Laplace transform of Eq. (19), } \]

\[ y_{mn} = y_{mn}(0) \cos(\omega_{mn} \tau) + \dot{y}_{mn}(0) \sin(\omega_{mn} \tau) \]

where \( \dot{y}_{mn}(0) \) is rewritten as \( \dot{y}_{mn}(0) \).
Each value of $\omega_{mn}^2$ is shown in Table 2. Equation (20) is deformed and differentiated with respect to $\tau$ to give
\begin{align*}
\frac{dy_{mn}}{d\tau} = \rho_{mn} \omega_{mn} \cos(\omega_{mn} \tau + \theta_{mn}) \\
\frac{d^2y_{mn}}{d\tau^2} = \rho_{mn} \omega_{mn} \cos(\omega_{mn} \tau + \theta_{mn}) \\
\end{align*}
where
\[ \theta_{mn} = \tan^{-1} \frac{y_{mn}(0)}{x_{mn}(0)} \quad \text{and} \quad \rho_{mn} = \sqrt{x_{mn}^2(0) + y_{mn}^2(0)} \]
Amplitude $\rho_{mn}$ and phase angle $\theta_{mn}$ are assumed to be time-varying variables.
\begin{align*}
\begin{cases}
    y_{mn} = \rho_{mn}(\tau) \sin(\varphi_{mn}(\tau)) \\
    \frac{dy_{mn}}{d\tau} = \rho_{mn}(\tau) \omega_{mn} \cos(\varphi_{mn}(\tau))
\end{cases}
\end{align*}
The derivative of Eq. (24) with respect to $\tau$ is
\begin{align*}
\frac{d^2y_{mn}}{d\tau^2} &= \frac{d\rho_{mn}(\tau)}{d\tau} \omega_{mn} \cos(\varphi_{mn}(\tau)) \\
&\quad - \rho_{mn}(\tau) \omega_{mn} \frac{d\theta_{mn}(\tau)}{d\tau} \sin(\varphi_{mn}(\tau))
\end{align*}
Substituting Eqs. (23)–(25) into Eq. (17) gives
\begin{align*}
\frac{d\rho_{mn}(\tau)}{d\tau} \omega_{mn} \cos(\varphi_{mn}(\tau)) \\
&\quad - \rho_{mn}(\tau) \omega_{mn} \frac{d\theta_{mn}(\tau)}{d\tau} \sin(\varphi_{mn}(\tau))
\end{align*}
Next, $\rho_{mn}(\tau) \omega_{mn} \cos(\varphi_{mn}(\tau)) - (P^{-1} f P)_{mn}$ on the right side of Eq. (26) is replaced with $f_{mn}(y_{mn}, \dot{y}_{mn})$ to give
\begin{align*}
\frac{d\rho_{mn}(\tau)}{d\tau} \omega_{mn} \cos(\varphi_{mn}(\tau)) \\
&\quad - \rho_{mn}(\tau) \omega_{mn} \frac{d\theta_{mn}(\tau)}{d\tau} \sin(\varphi_{mn}(\tau))
\end{align*}
Equation (23) is differentiated with respect to $\tau$ to give an equation equivalent to Eq. (24), and Eq. (28) is derived.

![Table 2: Values of $\omega_{mn}^2 = \lambda_m + \lambda_n$ (see Fig. 2)]

![Equation (28) derived from Eq. (27).](image)

3.2 Torus of $3 \times 4$ oscillators

The matrices of Eq. (31) are obtained from the $3 \times 4 = 12$ simultaneous differential equations obtained by normalizing the circuit equations of all oscillators in Fig. 3 by the same method as in Sect. 3.1.

![Matrix $B$](image)

Table 3 shows the 12 eigenvalues ($\lambda_{11}$–$\lambda_{34}$) and 12 eigenvectors ($E_{11} - E_{34}$) of matrix $B$. $k_1 - k_{12}$ are set as 1, the 12 eigenvectors are combined as matrix $P$ in Eq. (34), and the inverse matrix of $P$ is calculated (Eq. (35)).

![Inverse matrix of $P$](image)
The Laplace transform of Eq. (41) is

\[
Y_{mn} = \frac{s y_{mn}(0)}{s^2 + \lambda_{mn}} + \frac{\dot{y}_{mn}(0)}{s^2 + \lambda_{mn}}
\]  

(42)

\[B = \begin{bmatrix}
2 + 2\alpha & -\alpha & 0 & -\alpha & -\alpha & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 \\
-\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 & 0 & 0 & -\alpha & 0 & 0 \\
0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 & 0 & -\alpha & 0 & 0 \\
-\alpha & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 & 0 & 0 & -\alpha \\
0 & -\alpha & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 & 0 \\
0 & 0 & -\alpha & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & 0 & -\alpha \\
0 & 0 & 0 & -\alpha & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 2 + 2\alpha & -\alpha & 0 & 0 & 0 \\
\end{bmatrix} \]  

(32)

\[
\dot{\mathbf{x}} = \mathbf{P}\dot{\mathbf{y}} + \mathbf{y}
\]

(36)

\[
P^{-1} = \frac{1}{12} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\
0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 & -2 & 0 & 2 & 2 \\
-2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 & 2 \\
0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 4 & 0 & 4 \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 4 & 0 & 4 & 0 \\
0 & 2 & 0 & -2 & 0 & 4 & 0 & 2 & 0 & -2 & 0 & 2 \\
2 & 0 & -2 & 0 & 4 & 0 & 2 & 0 & -2 & 0 & 2 & 2 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 2 & -2 \\
1 & -1 & 1 & -1 & -2 & 2 & -2 & 2 & 1 & -1 & -1 & -1 \\
\end{bmatrix}
\]

(35)

\[
D = \begin{bmatrix}
\lambda_{11} & 0 \\
0 & \ddots \\
0 & 0 & \lambda_{33}
\end{bmatrix}
\]

(39)

\[
\mathbf{x} \text{ is rewritten using matrix } \mathbf{P} \text{ and arbitrary matrix } \mathbf{y} \text{ as }
\]

\[
\mathbf{x} = \mathbf{P}\mathbf{y}
\]

Equation (36) is substituted into Eq. (33) to give

\[
P\ddot{\mathbf{y}} + \mathbf{B}\dot{\mathbf{y}} = \epsilon \mathbf{P}\dot{\mathbf{y}} - \frac{1}{3}\epsilon\dot{\mathbf{z}}
\]

(37)

Then, Eq. (38) is obtained by multiplying Eq. (37) by matrix \(P^{-1}\) from the left side.

\[
\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} = \epsilon\ddot{\mathbf{y}} - \frac{1}{3}\epsilon P^{-1}\dot{\mathbf{z}}
\]

(38)

Here, matrix \(D\) is the diagonal matrix.

\[
\lambda_{mn} \text{ is set as } \omega_{mn}^2, \text{ and Eq. (43) is derived by the inverse Laplace transform of Eq. (42).}
\]

\[
y_m = y_{mn}(0)\cos(\omega_{mn}\tau) + \dot{y}_{mn}(0)\sin(\omega_{mn}\tau)
\]

(43)

where \(\dot{y}_{mn}(0)\) is rewritten as \(\dot{y}_{mn}(0)\). Each value of \(\omega_{mn}^2\) is shown in Table 4. Equation (43) is deformed and differentiated with respect to \(\tau\) to give

\[
\begin{cases}
y_{mn} = \rho_{mn}\sin(\omega_{mn}\tau + \theta_{mn}) \\
\frac{dy_{mn}}{d\tau} = \rho_{mn}\omega_{mn}\cos(\omega_{mn}\tau + \theta_{mn})
\end{cases}
\]

(44)

where

\[
\theta_{mn} = \tan^{-1}\left(\frac{\dot{y}_{mn}(0)}{y_{mn}(0)}\right) \text{ and } \rho_{mn} = \sqrt{y_{mn}(0)^2 + \dot{y}_{mn}(0)^2}
\]

Amplitude \(\rho_{mn}\) and phase angle \(\theta_{mn}\) are assumed to be time-varying variables.

\[
\begin{cases}
y_{mn} = \rho_{mn}(\tau)\sin(\omega_{mn}\tau + \theta_{mn}(\tau)) \\
\frac{dy_{mn}}{d\tau} = \rho_{mn}(\tau)\omega_{mn}\cos(\omega_{mn}\tau + \theta_{mn}(\tau))
\end{cases}
\]

(45)

\[
\omega_{mn}\tau + \theta_{mn}(\tau) \text{ is represented as } \varphi_{mn}(\tau), \text{ and Eq. (45) is rewritten as}
\]

\[
y_{mn} = \rho_{mn}(\tau)\sin(\varphi_{mn}(\tau))
\]

(46)

\[
\frac{dy_{mn}}{d\tau} = \rho_{mn}(\tau)\omega_{mn}\cos(\varphi_{mn}(\tau))
\]

(47)

The derivative of Eq. (47) with respect to \(\tau\) is

\[
\frac{d^2y_{mn}}{d\tau^2} = \frac{dp_{mn}(\tau)}{d\tau}\omega_{mn}\cos(\varphi_{mn}(\tau)) - \rho_{mn}(\tau)\omega_{mn}\frac{d\theta_{mn}(\tau)}{d\tau}\sin(\varphi_{mn}(\tau))
\]

(48)

Substituting Eqs. (46)–(48) into Eq. (40) gives

\[
\frac{dp_{mn}(\tau)}{d\tau}\omega_{mn}\cos(\varphi_{mn}(\tau)) - \rho_{mn}(\tau)\omega_{mn}\frac{d\theta_{mn}(\tau)}{d\tau}\sin(\varphi_{mn}(\tau))
\]

(49)
Table 3  Eigenvalues and eigenvectors of matrix $B$ (torus of 3×3 oscillators)

| Eigenvalues | Eigenvectors |
|-------------|--------------|
| $\lambda_{11} = 2 - 2\alpha$ | $E_{11}^T = k_1(1,1,1,1,1,1,1,1,1,1)$ |
| $\lambda_{12} = 2 + \alpha$ | $E_{2}^T = k_2(-1,1,1,1,1,1,1,1,1,1)$ |
| $\lambda_{13} = 2 + \alpha$ | $E_{3}^T = k_3(-1,1,1,1,1,1,1,1,1,1)$ |
| $\lambda_{14} = 2 + 2\alpha$ | $E_{4}^T = k_4(-1,1,1,1,1,1,1,1,1,1)$ |
| $\lambda_{21} = 2$ | $E_{5}^T = k_5(0,-1,0,1,0,-1,0,1,0,-1,0)$ |
| $\lambda_{22} = 2$ | $E_{6}^T = k_6(-1,0,1,0,-1,0,1,0,-1,0,1)$ |
| $\lambda_{23} = 2 + 3\alpha$ | $E_{7}^T = k_7(0,1,0,-1,0,0,0,0,-1,0,1)$ |
| $\lambda_{24} = 2 + 3\alpha$ | $E_{8}^T = k_8(1,0,-1,0,0,0,0,-1,0,1,0)$ |
| $\lambda_{31} = 2 + 3\alpha$ | $E_{9}^T = k_9(0,1,0,-1,0,1,0,0,0,0,0)$ |
| $\lambda_{32} = 2 + 3\alpha$ | $E_{10}^T = k_{10}(1,0,-1,0,-1,0,1,0,0,0,0)$ |
| $\lambda_{33} = 2 + 5\alpha$ | $E_{11}^T = k_{11}(1,-1,1,-1,0,0,0,0,-1,1,1)$ |
| $\lambda_{34} = 2 + 5\alpha$ | $E_{12}^T = k_{12}(1,-1,1,-1,1,-1,1,0,0,0,0)$ |

Table 4  Values of $\omega_{mn}^2 = \lambda_{mn}$ (see Fig. 3)

| $m, n$ | $\lambda_{mn} = \omega_{mn}^2$ | $m, n$ | $\lambda_{mn} = \omega_{mn}^2$ |
|--------|-------------------------------|--------|-------------------------------|
| 1, 1   | $\lambda_{11} = 2 - 2\alpha$ | 2.3    | $\lambda_{23} = 2 + 3\alpha$  |
| 1, 2   | $\lambda_{12} = 2 + \alpha$  | 2.4    | $\lambda_{24} = 2 + 3\alpha$  |
| 1, 3   | $\lambda_{13} = 2 + \alpha$  | 3.1    | $\lambda_{31} = 2 + 3\alpha$  |
| 1, 4   | $\lambda_{14} = 2 + 2\alpha$ | 3.2    | $\lambda_{32} = 2 + 3\alpha$  |
| 2, 1   | $\lambda_{21} = 2$           | 3.3    | $\lambda_{33} = 2 + 5\alpha$  |
| 2, 2   | $\lambda_{22} = 2$           | 3.4    | $\lambda_{34} = 2 + 5\alpha$  |

Next, $\rho_{mn}(\tau)\omega_{mn}\cos(\varphi_{mn}(\tau)) - (\mathbf{P}^{-1}\mathbf{f})_{mn}$ on the right side of Eq. (49) is replaced with $f_{mn}(y_{mn}, \dot{y}_{mn})$ to give

$$
\frac{d\rho_{mn}(\tau)}{d\tau} = \omega_{mn}\cos(\varphi_{mn}(\tau)) - \rho_{mn}(\tau)\omega_{mn}\frac{d\theta_{mn}(\tau)}{d\tau}\sin(\varphi_{mn}(\tau))
$$

(50)

Equation (46) is differentiated with respect to $\tau$ to give an equation equivalent to Eq. (47), and Eq. (51) is derived.

$$
\frac{d\rho_{mn}(\tau)}{d\tau}\sin(\varphi_{mn}(\tau)) + \rho_{mn}(\tau)\frac{d\theta_{mn}(\tau)}{d\tau}\cos(\varphi_{mn}(\tau)) = 0
$$

(51)

Equations (52) and (53) are derived by substituting Eq. (51) into Eq. (50).

$$
\frac{d\rho_{mn}(\tau)}{d\tau} = \frac{\varepsilon\cos(\varphi_{mn}(\tau))f_{mn}(y_{mn}, \dot{y}_{mn})}{\omega_{mn}}
$$

(52)

$$
\frac{d\theta_{mn}(\tau)}{d\tau} = -\frac{\varepsilon\sin(\varphi_{mn}(\tau))f_{mn}(y_{mn}, \dot{y}_{mn})}{\rho_{mn}(\tau)\omega_{mn}}
$$

(53)

3.3 Both toruses

When $0 < \varepsilon < 1$, Eqs. (29), (30), (52), and (53) are approximately zero, so the averaging method can be applied. In other words, $\frac{d\rho_{mn}(\tau)}{d\tau}$ and $\frac{d\theta_{mn}(\tau)}{d\tau}$ in Eqs. (29), (30), (52), and (53) can be approximated to values averaged over infinite time.

$$
\frac{d\rho_{mn}(\tau)}{d\tau} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \varepsilon\cos(\varphi_{mn}(\tau))f_{mn}(y_{mn}, \dot{y}_{mn}) \right) d\tau
$$

(54)

$$
\frac{d\theta_{mn}(\tau)}{d\tau} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \varepsilon\sin(\varphi_{mn}(\tau))f_{mn}(y_{mn}, \dot{y}_{mn}) \right) d\tau
$$

(55)

Table 5 shows calculation results of Eq. (54) for a torus of 3×3 oscillators. Table 6 shows the results obtained when a torus of 3×4 oscillators is calculated. All the calculation results of Eq. (55) are zero, so $\theta_{mn}$ does not change with time and depends only on initial values.

4. Stability

4.1 Torus of 3×3 oscillators

The 9×9 Jacobian matrix $J_1$ (see Eq. (56)) is derived by replacing $\frac{d\rho_{mn}(\tau)}{d\tau}$ in Table 5 with $y_{mn}(\tau)$.

$$
J_1 = \begin{bmatrix}
    \frac{d\rho_{11}(\tau)}{d\theta_{11}(\tau)} & \frac{d\rho_{11}(\tau)}{d\theta_{12}(\tau)} & \cdots & \frac{d\rho_{11}(\tau)}{d\theta_{19}(\tau)} \\
    \frac{d\rho_{12}(\tau)}{d\theta_{11}(\tau)} & \frac{d\rho_{12}(\tau)}{d\theta_{12}(\tau)} & \cdots & \frac{d\rho_{12}(\tau)}{d\theta_{19}(\tau)} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{d\rho_{91}(\tau)}{d\theta_{11}(\tau)} & \frac{d\rho_{92}(\tau)}{d\theta_{12}(\tau)} & \cdots & \frac{d\rho_{99}(\tau)}{d\theta_{19}(\tau)}
\end{bmatrix}
$$

(56)

We calculate the values of $\rho_{mn}$ for which all nine equations in Table 5 are zero. These $\rho_{mn}$ values are
Then, Eq. (57) is substituted in Eq. (13) to derive

4.2 Torus of 3×4 oscillators

We obtain the stable solutions of the torus of 3×4 oscillators by using the same method as for the torus of 3×3 oscillators. We derive the 12 equations in Table 6 from the 12×12 Jacobian matrix $J_2$. Table 8 shows three patterns that are stable solutions, which are calculated by using the 12 equations in Table 6.

<Pattern 2>

The values are substituted into Eq. (44) to obtain

$$y_{11} = 2 \sin(\omega_{11} \tau + \theta_{11}) \quad \text{(other } y_{mn} = 0)$$

Then, Eq. (59) is substituted into Eq. (36) to derive

$$x_{mn} = 2 \sin(\omega_{11} \tau + \theta_{11})$$

(m=1, 2, or 3, and n=1, 2, 3, or 4) (60)

We can see from Eq. (60) that all oscillators are synchronized in-phase in this stable solution. The phase states of each oscillator in this stable solution are shown in Fig. 5.

<Pattern 3>

The following stable solution is obtained.

$$x_{mn} = (-1)^{n} \times 2 \sin(\omega_{14} \tau + \theta_{14})$$

(m=1, 2, or 3, and n=1, 2, 3, or 4) (61)

It can be seen from Eq. (61) that the three oscillators in each column are synchronized in-phase, and the four oscillators in each row are synchronized anti-phase. The phase states of each oscillator in this stable solution are shown in Fig. 6.

| Pattern | Value of $\rho$ |
|---------|-----------------|
| 1       | $\rho_{11} = 6$, other $\rho_{mn} = 0$ |

Table 7 All stable solutions (torus of 3×3 oscillators)

| $m$ | $n$ | $x_{11}$ | $x_{12}$ | $x_{13}$ | $x_{21}$ | $x_{22}$ | $x_{23}$ | $x_{31}$ | $x_{32}$ | $x_{33}$ |
|-----|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 1   | 1   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 1   | 2   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 1   | 3   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 2   | 1   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 2   | 2   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 2   | 3   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 3   | 1   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 3   | 2   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |
| 3   | 3   | $\frac{\pi}{3}$ ($27 - 2\rho_{11} - 9\rho_{12} - 2\rho_{13} - 3\rho_{21} - 9\rho_{22} - 2\rho_{23} - 2\rho_{31} - 9\rho_{32} - 2\rho_{33}$) |

Table 5 Calculation results of $\frac{d^2 \rho_{mn}(\tau)}{d\tau^2}$ (torus of 3×3 oscillators)

Fig. 4 Phase states of each oscillator of Pattern 1 (torus of 3×3 oscillators)

Fig. 5 Phase states of each oscillator of Pattern 2 (torus of 3×4 oscillators)
The stable solution is

\[
\begin{bmatrix}
    x_{11} \\
    x_{12} \\
    x_{13} \\
    x_{14} \\
    x_{21} \\
    x_{22} \\
    x_{23} \\
    x_{24} \\
    x_{31} \\
    x_{32} \\
    x_{33} \\
    x_{34}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    -2 \\
    0 \\
    2 \\
    0 \\
    -2 \\
    0 \\
    2 \\
    0 \\
    -2 \\
    0 \\
    2
\end{bmatrix} \sin(\omega_{21} \tau + \theta_{21}) + \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix} \sin(\omega_{22} \tau + \theta_{22})
\]

(62)

In each column, the three oscillators are synchronized in-phase. The phase states between the first and third columns are synchronized anti-phase, and the phase states between the second and fourth columns are synchronized anti-phase. However, there is no constraint condition between the first and third columns or between the second and fourth columns. The phase states of each oscillator in this stable solution are shown in Fig. 7.

5. Comparison between Theoretical and Simulation Results

We compare theoretical results with simulation results. The simulation results are shown in Figs. 8–13. In Figs. 8–12, each attractor of each oscillator is shown in each square graph, and the two voltages of adjacent oscillators are summed and shown as a function of time in rectangular graphs. When the phase state between adjacent oscillators is in-phase synchronization, the amplitude is large, and when anti-phase synchronization is observed, the amplitude is zero.
5.1 Torus of $3 \times 3$ oscillators

Simulation result of Pattern 1 ($\alpha = 0.5$ and $\varepsilon = 0.05$) (see Fig. 8)

We can see from Fig. 8 that all oscillators are in-phase synchronizations. The simulation result is the same as the result of the theoretical analysis.

5.2 Torus of $3 \times 4$ oscillators

Simulation result of Pattern 2 ($\alpha = 0.5$ and $\varepsilon = 0.05$) (see Fig. 9)

We can see from Fig. 9 that all oscillators are in-phase synchronizations. This simulation result is the same as the result of the theoretical analysis.

Simulation result of Pattern 3 ($\alpha = 0.5$ and $\varepsilon = 0.05$) (see Fig. 10)

We can see from Fig. 10 that the phase states between adjacent oscillators are in-phase synchronization in each column and anti-phase synchronization in each row. This simulation result is the same as the result of the theoretical analysis.

Simulation result of Pattern 4 ($\alpha = 0.5$ and $\varepsilon = 0.05$) (see Figs. 11 and 12)

The initial phase differences between the first and second columns, the second and third columns, and the third and fourth columns are set as approximately $\frac{5\pi}{18}$, $\frac{13\pi}{18}$, and $\frac{5\pi}{18}$, respectively. The simulation result is shown in Fig. 11 and the phase differences in Fig. 11 are shown in Fig. 13. The phase differences between the first and second columns, and the third and fourth columns are approximately $50^\circ$, indicating stability. The phase differences between the second and third columns, and the fourth and first columns are approximately $130^\circ$, indicating stability. Furthermore, all phase differences between adjacent oscillators in each column are zero.

Thus, the initial phase differences are maintained in this simulation result.

Next, the initial phase differences are changed from those in the previous result. We set the initial phase differences between the first and second columns, and the third and fourth columns as approximately zero. The initial phase difference between the second and third columns is approximately $\pi$. The simulation result obtained with these initial values is shown in Fig. 12. The initial phase differences are maintained in this simulation result.

Therefore, the simulation and theoretical results are the same.
In this paper, synchronization phenomena were theoretically analyzed on systems of coupled van der Pol oscillators in torus shapes when the number of oscillators in each column was three and the number of oscillators in each row was three or four. In the case of $3 \times 3$ oscillators, only in-phase synchronization were obtained in each column and row. On the other hand, in the case of $3 \times 4$ oscillators, in-phase synchronization, anti-phase synchronization, and special anti-phase synchronization (Pattern 4) were observed in each row, but only in-phase synchronization was observed in each column. We compared the theoretical and simulation results of four stable synchronization states, and confirmed that the same phenomena occurred in both results.

For these toruses, we conclude that the observable synchronization phenomena in the row or column direction depend on the number of oscillators in the row or column direction, respectively.

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References

[1] Y. Kuramoto: Self-entertainment of a population of coupled non-linear oscillators, International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics 39, pp. 420-422, Jan. 1975.
[2] A. N. Burkitt and G. M. Clark: Analysis of integrate-and-fire neurons: synchronization of synaptic input and spike output, Neural Computation, Vol. 11, No. 4, pp. 871-901, 1999.
[3] J.-S. Li, I. Dasanayake and J. Ruths: Control and synchronization of neuron ensembles, IEEE Trans. Automatic Control, Vol. 58, No. 8, pp. 1919-1930, 2013.
[4] Y. Lin, F. Li and M. Ma: Synchronization of bio-nanomachines based on molecular diffusion, IEEE Sensors Journal, Vol. 16, No. 16, pp. 7267-7277, 2016.
[5] T. Endo and S. Mori: Mode analysis of a multimode ladder oscillator, IEEE Trans. Circuits and Systems, Vol. CAS-23, No. 2, pp. 100-113, Feb. 1976.
[6] T. Endo and S. Mori: Mode analysis of a two-dimensional low-pass multimode oscillator, IEEE Trans. Circuits and Systems, Vol. CAS-23, No. 9, pp. 517-530, Sep. 1976.
[7] T. Endo and S. Mori: Mode analysis of a ring of a large number of mutually coupled van der Pol oscillators, IEEE Trans. Circuits and Systems, Vol. CAS-25, No. 1, pp. 7-18, Jan. 1978.
[8] M. Yamauchi, M. Wada, Y. Nishio and A. Ushida: Wave propagation phenomena of phase states in oscillators coupled by inductors as a ladder, IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences, Vol. E82-A, No. 11, pp. 2592-2598, Nov. 1999.
[9] Y. Todani, M. Yamauchi and Y. Nishio: Phase-inversion waves in simultaneously existing two synchronization modes of 2D oscillator networks, Proceedings of International Symposium on Nonlinear Theory and its Applications (NOLTA 2014), pp. 361-364, Sep. 2014.
[10] M. Tanaka, M. Yamauchi and Y. Nishio: Phase-inversion waves propagating in an in-phase synchronization on oscillators coupled as a cross, IEEE Trans. Circuits and Systems, Vol. CAS-66, No. 12, pp. 4807-4816, Dec. 2019.

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