Monopole and quadrupole polarization effects on the $\alpha$-particle description of $^8$Be

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Abstract

We investigate the effect of monopole and quadrupole modes on the elastic $\alpha - \alpha$ resonance structure of $^8$Be. To this end we make a fully microscopic coupled channels calculation with three coupled channels, using the Algebraic Model. The continuum spectrum and wave functions are analyzed in terms of the individual channels to understand the nature of the resonances. It is shown that both monopole and quadrupole modes have a non-negligible effect on the resonances in the $\alpha - \alpha$ continuum.

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I. INTRODUCTION

$^8$Be is known to be a strongly clustered nucleus that appears through relatively short-lived resonances just above the $\alpha - \alpha$ scattering threshold. A low-lying rotational band is experimentally apparent and suggests a strong deformation of the 8-particle resonance system. A systematic survey of the full spectrum of $^8$Be, including a review of available theoretical and experimental results, has been made in the work of Ajzenberg-Selove [1] and is now available in a revised version [2]. Bacher et al. [3] have reported partial phase shifts in the $\alpha - \alpha$ collisions for even states up to $L = 6$ in the range up to 35 MeV excitation energy. Together with [4], these experiments reveal no resonance states with measurable widths above 25 up to 50 MeV in the elastic $\alpha - \alpha$ channel. Arena et al. [5] point out the need of including inelastic channels such as $^4$He($^4$He,d)$^6$Li, $^4$He($^4$He,n)$^7$Be, and $^4$He($^4$He,p)$^7$Li if one wants to find states of high excitation energy. These authors report the possible existence of highly excited $^8$Be levels for $L = 6$ and $L = 12$ at about 41 MeV and $E_x = 43$, and around 50 MeV for $L = 2$ up to $L = 10$.

Because of the strong experimental evidence for predominance of cluster structure in $^8$Be, many theoretical approaches based on cluster structures have been considered. Microscopic cluster models are known to provide valuable information about the structure of light nuclei [6, 7] and in particular of $^8$Be. The Resonating Group Method (RGM) has often been used [8, 9, 10, 11]. The low-lying rotational structure of $^8$Be, both in position and width, is reproduced by elastic $\alpha - \alpha$ scattering calculations with effective interactions. The Coulomb interaction plays an important role in the correct position and width of these states, in particular for $L = 0$ groundstate of $^8$Be [12, 13].

Collective $A$-particle deformation models for light nuclei have been under discussion for several decades. A meaningful classification scheme has been derived for such models through the irreducible representations of the non-compact Sp(2, R) group. A good description of the low-energy spectra of light nuclei [14], and more specifically of the rotational structure of $^8$Be [15] has been obtained within these models. It has also been demonstrated that the quadrupole Sp(2,R) model of $^8$Be and the $\alpha$-particle description have an important overlap, and thus are complementary in the description of this nucleus [16], a conclusion confirmed in [17, 18, 19]. It seems therefore appropriate to study the coexistence and competition between collectivity and clustering in light nuclei through a combined approach. Filippov
et al. [20], [21], [22] have already investigated cluster-monopole and cluster-quadrupole descriptions of \(^8\)Be.

The deformation aspect in \(^8\)Be has been studied within a cluster approach by introducing a quadrupolar polarization of the \(\alpha\)-particles [23], or a monopolar distortion of the \(\alpha\)-particles [24].

In this work we propose a model in which the \(\alpha\)-particle description and the collective (8-particle) quadrupole and monopole modes are coupled. We consider a scattering approach within an energy range in which the \(\alpha\) cluster channel is open, and both collective channels are closed, thus limiting ourselves to the elastic \(\alpha - \alpha\) cluster decay. The effects of the collective channels will then only be apparent in the compound system during resonance lifetimes.

We implement our coupled channels approach within the Modified J-Matrix Method (MJM) [25], [26] also known as the Algebraic Model [27], [28]. It determines an approximate solution of the Schrödinger equation in terms of square-integrable bases, and maps both scattering or bound-state boundary conditions from configuration to the space of basis expansion coefficients. As such it allows for a simultaneous treatment of open and closed channels. The MJM is an extension of the J-Matrix Method (JM) [29] using an oscillator basis. The MJM allows one to treat long-range interactions, including the Coulomb potential, in that basis. It provides convergence in terms of number of basis functions with reasonable basis sizes, which is important because calculation of Hamiltonian matrix elements is the bulk of the computational load of the method.

The multi-channel approach of this work allows for a clear-cut analysis of the elastic \(\alpha - \alpha\) phase shifts and corresponding wave functions in terms of the contribution from the individual channels, leading to a physical interpretation of the resonances. It indicates the importance of collective degrees of freedom in the compound system.

Our approach is most suited to handle two-body Gaussian interactions, mainly because of the oscillator expansion in the MJM. We consider two-body potentials such as the one proposed in [30] and determined within a Hartree-Fock approximation, and the ones from [31], [32] determined within an RGM approach, to calculate the \(^8\)Be spectrum, and check the validity of our conclusions.

The paper is organized as follows. In section II we elaborate on the combined cluster-collective model description for \(^8\)Be and formulate the multi-channel MJM scattering ap-
II. A COUPLED-CHANNELS CLUSTER-COLLECTIVE MODIFIED J-MATRIX APPROACH

The Modified J-Matrix Method [25], [26], also referred to as the Algebraic Version of the RGM [28], has become a well-tested approach for nuclear structure calculations involving multi-channel cluster and/or collective descriptions for light nuclei. The application of the MJM is based on an expansion in terms of oscillator basis states in the respective collective coordinates (intercluster distance, monopole radius, quadrupole deformation, ...). We refer to the papers of Vasilevsky et al. [33], [34], [35], [36] for detailed properties of the individual channel wave functions, and the multi-channel formulation of the MJM [37] with non-orthogonal bases.

The model considered here for $^8$Be consists of a wave function containing three structure components distinguished by a specific collective coordinate. These three components represent the $\alpha - \alpha$ cluster, the Sp(2,R) quadrupole and the Sp(2,R) monopole modes

$$\Psi = \Psi^C + \Psi^Q + \Psi^M. \quad (1)$$

The structure of a single cluster is described by a wave function $\Psi_i (\alpha_i)$

$$\Psi_i (\alpha_i) = \Psi_i \left( q_1^{(i)}, q_2^{(i)}, q_3^{(i)} \right), \quad (i = 1, 2) \quad (2)$$
centered around its centre of mass $R_i$.

A two-cluster wave function can then be written as

$$\Psi^C (q_1, ..., q_7) = A \left[ \Psi_1 (\alpha_1) \Psi_2 (\alpha_2) \Psi_R (r) \right], \quad (3)$$

where $A$ stands for the antisymmetrization operator over all 8 particles, and $\Psi_R (r)$ represents the relative motion of both clusters, $r$ being the corresponding Jacobi coordinate.

To limit the computational complexity of the problem, the cluster wave functions are frozen, and constructed as Slater determinants of harmonic oscillator ($0s$)-states, corresponding to the groundstate shell-model configuration of the cluster. The $\Psi_R (r)$ wave function
for the relative motion will be represented by an expansion in terms of an oscillator basis in $r$. As the cluster states are frozen and built of $(0s)$-orbitals, the quantum numbers reduce to those of the inter-cluster wave function only. The set of quantum numbers is unambiguously defined and is obtained from the reduction of the symmetry group $U(3) \supset O(3)$ of the one-dimensional oscillator. This reduction provides the quantum numbers $n$ for the radial excitation, and $L, M$ for the angular momentum of the two-cluster system. The two-cluster wave function can be decomposed as

$$
\Psi_{LM}^C = A[\psi_1(\alpha_1) \psi_2(\alpha_2) \psi_{LM}(r)]
= \sum_n c_n^L A[\psi_1(\alpha_1) \psi_2(\alpha_2) \phi_{nLM}(r)] = \sum_n c_n^L \psi_{nLM}^C,
$$

where the $\phi_{nLM}(r)$ are the three-dimensional harmonic oscillator states. The oscillator parameter $b$ is the same for both the individual particle $(0s)$ states, and the expansion for the relative motion.

The oscillator decomposition of collective $Sp(2,R)$ quadrupole and monopole components are most easily introduced through the standard step operators

$$
\Psi_{LM}^Q = \sum_n c_n^Q N_{LM}^{(Q)} P_{LM} [A_{Q}^+]^n \Phi_0 = \sum_n c_n^Q \psi_{nLM}^Q;
$$

$$
\Psi_{LM}^M = \sum_n c_n^{LM} N_{LM}^{(M)} [A_{M}^+]^n P_{LM}\Phi_0 = \sum_n c_n^{LM} \psi_{nLM}^M,
$$

where $\Phi_0$ is a $0\hbar\omega$ shell-model vacuum state. For $^8$Be this is a Slater determinant with a $(0s)^4(0p)^4 = (000)(001)^4$ configuration (in Cartesian $(n_x, n_y, n_z)$ oscillator notation) and it has SU(3) $(\lambda, \mu) = (40)$ classification. The $P_{LM}$ stands for the traditionally angular momentum projection operator and $N_{LM}^{(M)}, N_{LM}^{(Q)}$ are norm factors. The (translationally invariant) step operators $A_{\mu\nu}^+$ in a Cartesian notation $(\mu, \nu = x, y, z)$ are written, in terms of the standard harmonic oscillator creation operators $a_\mu^+(i)$ for particle $i$:

$$
A_{\mu\nu}^+ = \sum_{i=1}^A a_\mu^+(i)a_\nu^+(i) - \frac{1}{A} \sum_{i,j=1}^A a_\mu^+(i)a_\nu^+(j)
$$

so that

$$
A_{Q}^+ = A_{zz}^+;
A_{M}^+ = A_{xx}^+ + A_{yy}^+ + A_{zz}^+.
$$
The projection after excitation for the quadrupole mode in (3) is necessary because the step operator $A^+_Q$ contains both $L = 0$ and $L = 2$ components. In the monopole case, the projection operator commutes with the step operators. A single projection of the $0\hbar\omega$ state, which contains $L = 0, 2$ and 4 components suffices. Thus the monopole mode contributes only to those $L$-subspaces. Contrary to the cluster and quadrupole modes, this limits the monopole Hilbert space to the latter angular momenta. We will therefore limit ourselves to $L = 0, 2$ and 4 for all modes throughout this work.

We use the same oscillator parameter $b$ for $\Psi^Q_{LM}$ and $\Psi^M_{LM}$ functions as we do for the cluster wave function $\Psi^C_{LM}$. This makes the calculation of overlap and Hamiltonian matrix elements significantly easier.

One has of course orthogonality with respect to the quantum numbers $(n, L, M)$, but not with respect to the channel label:

$$\langle \psi^\tau_{nLM} | \psi^\nu_{n'L'M'} \rangle \sim \delta_{nn'}\delta_{LL'}\delta_{MM'},$$

(9)

where $\tau$ and $\nu$ stand for the cluster $(C)$, monopole $(M)$ and quadrupole $(Q)$channels. In particular it is well known [16] that with a common choice of overall $b$ the $n = 0$ basis states of all modes are identical:

$$\psi^C_{0LM} = \psi^Q_{0LM} = \psi^M_{0LM}.$$  

(10)

This means that care should be taken in the interpretation of the results when attributing an effect to some channel or other.

The calculation of overlap and Hamiltonian matrix elements is most easily performed by considering Gaussian-type generating functions for the three oscillator expansions, which are (up to $L$-projection)

$$\psi^C(R) = \mathcal{A}[\Psi_1(A_1) \Psi_2(A_2) \phi(r|R)]$$

$$= \mathcal{A} \left[ \Psi_1(A_1) \Psi_2(A_2) \exp \left\{ -\frac{1}{2b^2}r^2 + \frac{\sqrt{2}}{b^2} (Rr) - \frac{1}{2b^2}R^2 \right\} \right]$$

$$= \mathcal{A} \left[ \Psi_1(A_1) \Psi_2(A_2) \exp \left\{ (R \cdot A^+_C) \right\} \exp \left\{ -\frac{1}{2b^2}r^2 \right\} \right];$$

$$\psi^Q(\epsilon) = \exp{\epsilon A^+_Q} \Phi_0;$$

$$\psi^M(\nu) = \exp{\nu A^+_M} \Phi_0,$$

(11)

where we introduced the step operator $A^+_C = a^+(r)$ for the oscillator decomposition of the cluster wave function.
Because of the Gaussian nature of the generating functions, matrix elements for the overlaps and Gaussian two-body operators can be calculated in a straightforward way. From these the matrix elements in the discrete \((n,L,M)\) basis can then be obtained by e.g. recurrence techniques \cite{28}, \cite{37}.

By substituting the expansions \(4, 5, 6\) as an ansatz for the solution \(11\) in the Schrödinger equation, the latter reduces to an infinite matrix equation to be solved for the coefficients \(c_n^C, c_n^Q\) and \(c_n^M\):

\[
\sum_{\tau'} \sum_m \left\langle nL, \tau \left| \hat{H} - E \right| mL, \tau' \right\rangle c_{mL}^{\tau'} = 0.
\]

We solve this equation by considering the Modified J-Matrix approach, which was formulated in terms of an oscillator decomposition of the trial solution, and provides fast convergence in a finite subset of the model subspace. In the J-Matrix approach the boundary conditions (for scattering as well as bound-state solutions) are translated from coordinate into the space of basis expansion coefficients, and asymptotic solutions can be obtained from a three-term recurrence relation for the expansion coefficients for high \(n\). Considering the coefficients to equal the asymptotic values from a given \(n = N\) on, and imposing a matching condition between the interaction region and asymptotic regions, only a reasonably sized matrix equation remains to be solved. In the Modified J-Matrix approach asymptotic contributions for the potential behavior, in particular for the Coulomb term, are taken into account in a semi-classical way through a modified recurrence equation. This was shown to reduce the size \(N\) of the remaining matrix equation drastically.

For the scattering boundary condition the asymptotic regular \((c_n^{(+)}\) and irregular \((c_n^{(-)}\) solutions are obtained so that the asymptotic expansion coefficients of the solution can be written as

\[
c_{nL}^{as} \rightarrow c_{nL}^{(-)}(kR_{n,l}) - Sc_{nL}^{(+)}(kR_{n,l}), \quad n \rightarrow \infty,
\]

where \(S\) stands for the \(S\)-matrix and reflects the matching condition. It is to be determined by solving the remaining matrix equation. \(k = \sqrt{2mE/\hbar^2}\) is the momentum corresponding to energy \(E\), and \(R_{n,l}\) are the oscillator turning points for the channel under consideration. In a single-channel approach \(S\) is a scalar quantity related to the phase-shift only.

For the bound-state boundary condition only the exponentially decaying solution can be retained, and its expansion coefficients are

\[
c_{nL}^{as} \rightarrow \exp(-\kappa R_{n,l})/\sqrt{R_{n,l}}, \quad n \rightarrow \infty
\]
with \( \kappa = \sqrt{2m|E|/\hbar^2} \). In the equations (13) and (14) the energy \( E \) is determined with respect to threshold of the corresponding channels.

In our approach we have three different channels, so a multi-channel MJM formulation is necessary. We take the same form as in [35] and write

\[
c^{(0)\tau}_{nL} = c^{(0)\tau}_{nL} + \delta_{\nu \tau} c^{(-)\tau}_{nL} - S_{\nu \tau} c^{(+)}_{nL} \tag{15}
\]

with \( \nu \) the entrance channel, and \( \tau \) any other coupled channel. Substitution of (15) in the matrix form of the Schrödinger equation leads to

\[
\sum_{\tau'} \sum_{m<N} \langle nL, \tau | \hat{H} - E | mL, \tau' \rangle c^{(0)\tau'}_{mL} - \sum_{\tau'} S_{\nu \tau'} \left[ \beta_0^{(+)} \delta_{n,0} \delta_{\nu \tau} + V^{(+)}_{n} \tau \tau' \right] = -\beta_0^{(-)} \delta_{n,0} \delta_{\nu \tau} - V^{(-)\nu}_{n} \tau \tag{16}
\]

and

\[
V^{(\pm)\tau \tau'}_{n} = \sum_{m=0}^{\infty} \langle nL, \tau | \hat{V} | mL, \tau' \rangle c^{(\pm)\tau'}_{m}, \tag{17}
\]

where \( \hat{V} \) stands for the two-body interaction and \( \beta_0 \) accounts for the traditional regularization of the irregular asymptotic solution (see for instance [28] or [35]).

This system of equations should then be solved for the residual coefficients \( c^{(0)\tau}_{nL} \) and the \( S \)-matrix elements \( S_{\tau \tau'} \). We consider in equations (16, 17) a near-interaction region with \( n < N \) and a far-interaction region with \( n \geq N \). The choice of \( N \) is such that one can expect the residual expansion coefficients \( \{ c^{(0)\tau}_{nL} \} \) to be negligibly small in the far-interaction region. The total number of equations for a given entrance channel \( \nu \) then equals to \( N_{ch} (N + 1) \), and solving the set of equations by traditional numerical linear algebra leads to the \( N_{ch} N \) residual coefficients \( \{ c^{(0)\tau}_{nL} ; \tau = C, Q, M ; n = 0..N - 1 \} \) and \( N_{ch} S \)-matrix elements \( \{ S_{\nu \tau} ; \tau = C, Q, M \} \). The set of equations has to be solved for all \( N_{ch} \) entrance channels. A final parameter of the calculation concerns the summation in (17). Because the potential matrix elements decrease rapidly when \( |n - m| \) gets large, we can truncate this sum at some \( M > N \).

In this paper we limit ourselves to the situation in which only the cluster channel is open, so that \( \nu = C \), because this is the dominant channel for \(^8\text{Be}\) in our model description. The boundary conditions (13) are appropriate for an open channel, and are therefore used for the cluster channel; conditions (14) are appropriate for a closed channel and are thus applied.
to the monopole and quadrupole channels. This choice limits the energy range between the monopole threshold at 0 MeV (all 8 particles infinitely apart) and the cluster threshold. The quadrupole threshold is even higher, because of the forced polarization condition.

III. NUMERICAL APPLICATION AND RESULTS

In a microscopic calculation the choice of the effective nucleon-nucleon (NN) interaction remains a crucial point. We limit ourselves to effective NN interactions of Gaussian form, which lead to a straightforward evaluation of matrix elements in the cluster-collective model space. One well-known example is Volkov [30] force, which was essentially determined and used within a Hartree-Fock context. This force binds both the deuteron triplet as well as the dinucleon singlet. Gaussian forces that discriminate between the deuteron triplet and dinucleon triplet are the Minnesota [31] and the Hasegawa-Nagata [38], [39] potentials. A modified version of the latter was proposed in [32]. These interactions were considered and tested in $^\alpha - N$ and $^\alpha - ^\alpha$ RGM scattering calculations.

In the current calculations we consider the Volkov (V1), Modified Hasegawa-Nagata (MHN) and Minnesota (Mi) forces. We include the Coulomb interaction which is necessary to produce the $L = 0$ ground-state as a narrow resonance just above the $^\alpha - ^\alpha$ threshold. The parameters are chosen to reproduce both the ground state energy and size of $^4$He. The Majorana exchange part accounts for nuclear matter properties. It does not influence the ground state energy and size of $^4$He, but affects the deformation in $p$-shell nuclei significantly [31].

To obtain the phase-shifts for elastic $^\alpha - ^\alpha$ scattering we solve (16), and determine resonance positions and widths in the usual numerical way through

$$\frac{d^2\delta_l}{dE^2} = 0 \implies E_r, \quad \Gamma = 2 \left( \frac{d\delta_l}{dE} \right)_{E_r}^{-1}. \quad (18)$$

We fix the common oscillator parameter $b$ for the C, Q and M expansion bases for each of the potentials so as to optimize an acceptable $^\alpha - ^\alpha$ threshold. These values can be found in Table I. We have also slightly modified the Majorana parameters of the Minnesota and modified Hasegawa-Nagata potentials to reproduce comparable values for the lowest $L = 0, 2$ and 4 resonances forming the ground state rotational band in $^8$Be; the values for the modified Majorana parameters are also shown in Table I. Also shown in the table are
TABLE I: Parameters used for Minnesota (Mi), Volkov (V1) and modified Hasegawa-Nagata (MHN) potentials and corresponding cluster and quadrupole threshold energies with respect to the monopole break-up energy.

| Parameter                              | Mi  | V1  | MHN |
|----------------------------------------|-----|-----|-----|
| Original Majorana parameter           | 0.52| 0.6 | 0.39|
| Modified Majorana parameter           | 0.57| 0.6 | 0.43|
| Oscillator parameter $b$ (fm)          | 1.28| 1.37| 1.32|
| $E_{g.s.}$ of $^4$He, MeV              | -24.69| -27.09| -29.01|
| $E_{th}(C)$, MeV                       | -49.37| -54.17| -58.02|
| $E_{th}(M)$, MeV                       | 0.0 | 0.0 | 0.0 |
| $E_{th}(Q)$, MeV                       | 39.22| 23.16| 25.28|

the respective threshold energies for the three channels, which amounts to twice the binding energy of the $\alpha$-particle for the cluster channel, zero energy (all 8 particles apart) for the monopole channel, and a positive value for the quadrupole channel (all 8 particles apart under the quadrupole deformation restriction).

In Figure 1 we show the elastic $\alpha$-$\alpha$ scattering $L = 0, 2$ and $4$ phase shifts within the energy region between the $\alpha$-$\alpha$ and monopole thresholds as obtained from the full MJM calculations, including experimental data from [3] and [4]. The parameters of the calculation were chosen for convergent phase shifts and, with $N = 50$ in (16) for the near interaction region, and $M = 100$ for truncating the sum in (17) for all channels. The common states in the different channels for $n = 0$ have been taken into account properly, and forbidden Pauli states are explicitly removed from the calculation.

One immediately recognizes from Figure 1 the low-lying rotational structure of $^8$Be. A rich resonance structure beyond 30 MeV above the cluster threshold appears through the coupling with the collective channels. This is made clear in Figure 2 where the pure cluster phase shifts are compared to those of the fully coupled ones for V1 and $L = 0$. The effects are qualitatively well reproduced by all potentials considered, implying that the effect is a genuine one, independent of specific choice of NN-interaction. Table II lists the resonance parameters obtained from the phase shifts in Figure 1 by applying (18).
TABLE II: Resonance energies and widths in the coupled channel calculation for Minnesota (Mi), Volkov (V1) and modified Hasegawa-Nagata (MHN) potentials. Resonance energies are in MeV, widths in keV.

IV. ANALYSIS

The results of the previous section indicate an important effect of the quadrupole and/or monopole deformation on the elastic \( \alpha - \alpha \) continuum, appearing below the thresholds of the collective modes as relatively narrow resonances above 30 MeV.

In order to analyze these results we examine the multi-channel wave function for channel
content, by calculating channel weights:

\[ W_{\nu,L} = \sum_n (c_{nL}^\nu)^2 \]  

(19)

Figure 3 shows the channel weights for each of the three channels (\(\alpha - \alpha\), monopole and quadrupole) as a function of energy for V1 and \(L = 0\). The figure includes the resonance positions for reference. This picture indicates the strong polarization effects at the resonance energies, emphasizing that the preferred configurations for \(^8\)Be in the compound system at these resonance energies are dominated by the monopole and/or quadrupole modes. Even more, the collective resonances suggest a decoupled picture in the sense that each resonance is either essentially of a quadrupole or monopole nature. It should be kept in mind however that the channels are not orthogonal to each other, and that this fact blurs a clear-cut comparison.

In order to further analyze the results in terms of the collective modes we suggest to consider the orthogonal complement of the quadrupole and monopole bases with respect to the \(\alpha\)-channel. This effectively removes the non-orthogonality coupling between the \(\alpha\)-channel and the collective ones. We denote the resulting subspaces by \(\tilde{Q}\), \(\tilde{M}\) and \(\tilde{Q}\tilde{M}\), when the coupled orthogonal complement is used. The transformed bases are easily obtained through Schmidt orthogonalization as

\[ \psi_{\tilde{M}}^{nL} = \left[ \psi_{nL}^M - \left( \psi_{nL}^M | \psi_{nL}^C \right) \psi_{nL}^C \right] / \sqrt{1 - \left( \psi_{nL}^M | \psi_{nL}^C \right)^2}; \]  

\[ \psi_{\tilde{Q}}^{nL} = \left[ \psi_{nL}^Q - \left( \psi_{nL}^Q | \psi_{nL}^C \right) \psi_{nL}^C \right] / \sqrt{1 - \left( \psi_{nL}^Q | \psi_{nL}^C \right)^2}. \]  

(20)

A further orthogonalization between the \(\tilde{Q}\) and \(\tilde{M}\) subspaces is not appropriate, as we will only be interested in either quadrupole or monopole contributions for the characterization of resonance states.

The multi-channel Hilbert subspace is now separated into two non-overlapping parts, the open \(\alpha\)-channel and the closed orthogonal collective components. This is similar to the Feshbach method [40] of projecting out the states of the “external” decay channel (the cluster one) at continuum energy \(E\), and constructing the effective Hamiltonian in the “internal” space of many body collective states. We calculate the bound eigenstates \(E_{iL}^{\tilde{M}\tilde{Q}}\) in the orthogonal to the cluster channel subspace and compare these to the resonance energies.
from the fully coupled scattering calculation. The results for all potentials are summarized in Table III. It is immediately apparent that the eigenenergies of the coupled orthogonal subspace correspond almost exactly to the resonance energies, except for the “ground-state” one which can be completely attributed to the Coulomb barrier in the \( \alpha - \alpha \) configuration. In Figure 4, we show for \( V_1 \) and \( L = 0 \) the content of the energy wave function in terms of the eigenstates of the orthogonal complement \( (O_{L,j} = \langle \psi^E_L | \psi^MQ_j \rangle^2) \). It confirms the one-to-one correspondence of the resonances to the orthogonal complement eigenstates.

A final aspect to be studied is the specific polarization nature of the resonances suggested by Figure 4. To this end we calculate the spectrum in both the \( \tilde{Q} \) and \( \tilde{M} \) spaces separately. The combination of these uncoupled spectra is comparable to the coupled \( \tilde{Q} \) and \( \tilde{M} \) diagonalization, indicating a limited dynamical coupling between both subspaces. We indicate this in Table III by showing for each eigenstate of the coupled orthogonal complement the content \( (\langle \psi^M_{iL} | \psi^MQ_{jL} \rangle^2 \text{ or } \langle \psi^M_{iL} | \psi^Q_{jL} \rangle^2) \) of the most prominent uncoupled eigenstate. This confirms in most cases an almost pure polarization mode for each coupled eigenstate, and thus corresponding resonance. These results are seen to be qualitatively identical for all three potentials considered in this work, and for all \( L = 0, 2 \) and 4 values.

From Table III one notices that the energy of the resonance states lies above the corresponding eigenenergy of the collective orthogonal complement. This confirms the results of ref. [41] where it was shown that the coupling between orthogonal open and closed channels transforms bound state from the closed channel into a resonance with an energy above the bound state one.

Table III also indicates a rotational behavior for the quadrupole related resonances, closely following a \( L(L+1) \) energy spacing, whereas the monopole related resonances remain roughly at the same energy.

V. CONCLUSIONS

We have presented the results for the elastic \( \alpha - \alpha \) spectrum for \(^8\text{Be}\) obtained from a calculation in which the \( \alpha \)-cluster configuration is coupled to the collective Sp(2,R) quadrupole and monopole modes. The energy range considered is between the \( \alpha \)-channel and the monopole (full 8 particle decay) thresholds. In order to accommodate both the open \( \alpha \)-channel and the closed collective channels we considered a multi-channel version of the Modified J-Matrix
The results indicate, apart from the well-known low-lying rotational band attributed to Coulomb repulsion in the \( \alpha - \alpha \) description, a rich spectrum of relatively narrow resonances above 30 MeV. We have shown that the resonances are connected to the eigenstates of the collective subspace, orthogonalized to the open \( \alpha \)-channel. More specifically we have shown that the resonances are essentially of quadrupole or monopole nature, and thus exhibit a specific polarization of the nucleus. This indicates that both the quadrupole and monopole eigenmodes remain mainly uncoupled as it was shown earlier [14] for bound sd-nuclei. It shows that both collective symmetries are important in the compound 8-particle system at specific energies in \( \alpha - \alpha \) scattering.

### TABLE III: Comparison of the resonance energies \( E_{iL}^{MQ} \) in the orthogonal complement to the cluster mode, denoted by \( \tilde{M}_Q \). All energies are in MeV. "Mode" stands for the content of the most prominent orthogonal monopole \( \psi_{jL}^M \) or quadrupole \( \psi_{jL}^M \) eigenstates in particular eigenstate \( \psi_{iL}^{MQ} \).

|  | \( E_r \) | \( E_{iL}^{MQ} \) | Mode | \( E_r \) | \( E_{iL}^{MQ} \) | Mode | \( E_r \) | \( E_{iL}^{MQ} \) | Mode |
|---|---|---|---|---|---|---|---|---|---|
| \( 0^+_2 \) | 34.78 | 32.02 | 92% \( \tilde{Q}_1 \) | 30.51 | 29.90 | 95% \( \tilde{Q}_1 \) | 34.13 | 33.78 | 97% \( \tilde{Q}_1 \) |
| \( 0^+_3 \) | 44.45 | 44.19 | 75% \( \tilde{Q}_2 \) | 41.72 | 41.52 | 90% \( \tilde{Q}_2 \) | 46.37 | 46.04 | 89% \( \tilde{Q}_2 \) |
| \( 0^+_4 \) | 48.41 | 48.31 | 56% \( \tilde{M}_1 \) | 45.92 | 45.77 | 43% \( \tilde{M}_1 \) | 51.71 | 51.42 | 66% \( \tilde{M}_1 \) |
| \( 0^+_5 \) | x | x | x | 21.06 | 51.02 | 72% \( \tilde{Q}_3 \) | 56.33 | 56.08 | 63% \( \tilde{Q}_3 \) |
| \( 0^+_6 \) | x | x | x | 53.41 | 53.33 | 63% \( \tilde{M}_2 \) | x | x | x |
| \( 2^+_1 \) | 35.12 | 34.33 | 90% \( \tilde{Q}_1 \) | 32.59 | 31.96 | 94% \( \tilde{Q}_1 \) | 36.41 | 36.08 | 96% \( \tilde{Q}_1 \) |
| \( 2^+_2 \) | 46.51 | 46.21 | 66% \( \tilde{Q}_2 \) | 43.48 | 43.27 | 93% \( \tilde{Q}_2 \) | 48.23 | 47.90 | 97% \( \tilde{Q}_2 \) |
| \( 2^+_4 \) | 47.17 | 47.17 | 58% \( \tilde{M}_1 \) | 44.48 | 44.48 | 66% \( \tilde{M}_1 \) | 50.63 | 49.69 | 71% \( \tilde{M}_1 \) |
| \( 2^+_5 \) | x | x | x | 52.84 | 52.39 | 50% \( \tilde{Q}_3 \) | 57.63 | 57.40 | 41% \( \tilde{Q}_3 \) |
| \( 4^+_2 \) | 40.89 | 39.89 | 80% \( \tilde{Q}_1 \) | 37.75 | 36.94 | 88% \( \tilde{Q}_1 \) | 42.12 | 41.85 | 93% \( \tilde{Q}_1 \) |
| \( 4^+_3 \) | 47.26 | 46.95 | 62% \( \tilde{M}_1 \) | 44.17 | 43.84 | 71% \( \tilde{M}_1 \) | 49.52 | 49.03 | 78% \( \tilde{M}_1 \) |
| \( 4^+_4 \) | x | x | x | 48.01 | 47.59 | 87% \( \tilde{Q}_2 \) | 52.95 | 52.64 | 91% \( \tilde{Q}_2 \) |
| \( 4^+_5 \) | x | x | x | 53.68 | 53.60 | 86% \( \tilde{M}_2 \) | x | x | x |
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FIG. 1: Phase shifts obtained in the multi-channel MJM approach with Minnesota (solid), Volkov (dotted) or MHN (dashed) forces for $L = 0$ (a), $L = 2$ (b) and $L = 4$ (c). Triangles indicate experimental data from [3], [4]. $E_{\alpha\alpha}$ is the c.m. energy with respect to the cluster threshold.
FIG. 2: $L = 0$ phase shifts for the Volkov potential in the one-channel cluster (dotted) and three-channel cluster-collective (solid line) approach. $E_{\alpha-\alpha}$ is the c.m. energy with respect to the cluster threshold.
FIG. 3: $L = 0$ cluster (a), monopole (b) and quadrupole (c) channel weights obtained with the Volkov force. $E_{\alpha-\alpha}$ is the c.m. energy with respect to the cluster threshold.
FIG. 4: Overlap of $L = 0$ energy wave function with the 1$^{\text{st}}$ (a), 2$^{\text{nd}}$ (b), 3$^{\text{rd}}$ (c), 4$^{\text{th}}$ (d) and 5$^{\text{th}}$ (e) eigenstates of the orthogonal complement to the cluster mode, calculated with the Volkov force. $E_{\alpha-\alpha}$ is the c.m. energy with respect to the cluster threshold.