COMBINATORIAL CONSTRUCTION OF GELFAND-TSETLIN MODULES FOR $\mathfrak{gl}_n$

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Abstract. We propose a new effective method of constructing explicitly Gelfand-Tsetlin modules for $\mathfrak{gl}_n$. We obtain a large family of irreducible modules (conjecturally all) that have a basis consisting of Gelfand-Tsetlin tableaux, the action of the Lie algebra is given by the Gelfand-Tsetlin formulas and with all Gelfand-Tsetlin multiplicities equal 1. As an application of our construction we prove necessary and sufficient condition for the Gelfand and Graev’s continuation construction to define a module which was conjectured by Lemire and Patera.

1. Introduction

A classical paper of Gelfand and Tsetlin [9] describes a basis of irreducible finite dimensional modules over the Lie algebra $\mathfrak{gl}_n$. This is one of the most remarkable results of the representation theory of Lie algebras which triggered a strong interest and initiated a development of the theory of Gelfand-Tsetlin modules in [3], [10], [11], [16], [17], [18], [20], among others. Gelfand-Tsetlin representations are related to Gelfand-Tsetlin integrable systems studied by Guillemin and Sternberg [12], Kostant and Wallach [13], [14], Colarusso and Evens [1], [2]. Each tableau in the basis of a finite dimensional representation is an eigenvector of the Gelfand-Tsetlin subalgebra $\Gamma$, certain maximal commutative subalgebra of the universal enveloping algebra of $\mathfrak{gl}_n$. Hence, any such tableau corresponds to a maximal ideal of $\Gamma$. Gelfand-Tsetlin theory had a successful development for infinite dimensional representations in [15], [4] where it was shows that irreducible Gelfand-Tsetlin modules are parametrized up to some finiteness by the maximal ideals of $\Gamma$. The significance of the class of Gelfand-Tsetlin modules is in the fact that they form the largest subcategory of $\mathfrak{gl}_n$-modules (in particular weight modules with respect to a fixed Cartan subalgebra) where there is some understanding of irreducible modules. The main remaining problem is how to construct explicitly these modules.

There were essentially two main approaches to generalize Gelfand-Tsetlin basis and construct explicitly new irreducible modules. One, starting from [3], was aiming to construct generic Gelfand-Tsetlin modules, that is those having a basis consisting of tableaux with no integer differences between the entries of the same row. These modules were constructed in [5]. Next step was to consider 1-singular case when there is just one pair in only one row with integer difference. A break through was a recent paper [6] where such irreducible modules were explicitly constructed, further generalization obtained in [7].
A different approach in constructing new irreducible modules is due to Gelfand and Graev [8] who presented a systematic study of formal analytic continuations of both the labelling and the algebra structure of finite dimensional representations of $\mathfrak{gl}_n$. Imposing certain conditions on entries of a tableau Gelfand and Graev described new infinite dimensional irreducible modules with a basis consisting of tableaux and the algebra action given by the classical Gelfand-Tsetlin formulas. We will call this condition \textit{GG-condition}. However, Lemire and Patera [15] showed that some of these analytic continuations, in fact, are not representations. They conjectured a necessary and sufficient condition, here called the \textit{LP-condition}, for Gelfand-Graev continuation to define a module and proved it for $\mathfrak{gl}_3$ and $\mathfrak{gl}_4$ (partial cases).

Our first result establishes this conjecture.

\textbf{Theorem I.} Gelfand-Graev continuation defines a $\mathfrak{gl}_n$-module if and only if it satisfies the LP condition.

We propose a new combinatorial method of constructing irreducible Gelfand-Tsetlin modules by continuation from tableau satisfying more general conditions which we call, following the tradition, \textit{FRZ-condition}. Any tableau satisfying the LP-condition also satisfies the FRZ-condition but the latter is much more general. Each tableau $L$ satisfying the FRZ-condition defines the Gelfand-Tsetlin character $\chi_L$ and leads to the Gelfand-Tsetlin module $V(L)$ with character $\chi_L$. A tableau $L$ is \textit{critical} if it has equal entries in one or more rows different from the top row. Otherwise, tableau is noncritical. Our second main result is

\textbf{Theorem II.} Let $L$ be a tableau satisfying the FRZ-condition. There exists a unique irreducible Gelfand-Tsetlin $\mathfrak{gl}_n$-module $V(L)$ with character $\chi_L$ having the following properties:

- $V(L)$ has a basis consisting of noncritical tableaux with standard action of the generators of $\mathfrak{gl}_n$.
- All Gelfand-Tsetlin multiplicities of $V(L)$ equal 1.

We will call a class of Gelfand-Tsetlin $\mathfrak{gl}_n$-modules from Theorem II \textit{admissible} modules. Hence, we have a combinatorial way to explicitly construct a vast number of new irreducible Gelfand-Tsetlin modules. It is interesting to know the place of admissible modules in the category of all Gelfand-Tsetlin modules. We make the following conjecture

\textbf{Conjecture 1.} If $V$ is irreducible Gelfand-Tsetlin $\mathfrak{gl}_n$-module with Gelfand-Tsetlin multiplicities 1 which has a basis consisting of noncritical tableaux with standard action of the generators of $\mathfrak{gl}_n$ then $V \simeq V(L)$ for some tableau $L$ satisfying the FRZ-condition.

We prove Conjecture 1 in the case $n \leq 4$. 
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2. Notation and conventions

Throughout the paper we fix an integer $n \geq 2$. The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers $m$ such that $m \geq a$. Similarly, we define $\mathbb{Z}_{<a}$, etc. By $\mathfrak{gl}_n$ we denote the general linear Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ - the standard basis of $\mathfrak{gl}_n$ of elementary matrices. We fix the standard Cartan subalgebra $\mathfrak{h}$, the standard triangular decomposition and the corresponding basis of simple root s of $\mathfrak{gl}_n$. The weights of $\mathfrak{gl}_n$ will be written as $n$-tuples $(\lambda_1, ..., \lambda_n)$.

The general linear Lie algebra $\mathfrak{gl}(n)$ is isomorphic to the Lie algebra generated by the elements $e_i, f_i (1 \leq i \leq n-1)$ and $h$ with the following relations:

\begin{enumerate}
  \item $[h, h'] = 0$ \quad $h, h' \in \mathfrak{h}$
  \item $[e_i, f_j] = \delta_{ij} h_i$
  \item $[h, e_i] = \alpha_i (h) e_i \quad [h, f_j] = -\alpha_j (h_i) f_j$
  \item $[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0 \quad (|i - j| = 1)$
  \item $[e_i, e_j] = [f_i, f_j] = 0 \quad (|i - j| > 1)$
\end{enumerate}

For a Lie algebra $\mathfrak{a}$ by $U(\mathfrak{a})$ we denote the universal enveloping algebra of $\mathfrak{a}$. Throughout the paper $U = U(\mathfrak{gl}_n)$. For a commutative ring $R$, by $\text{Specm } R$ we denote the set of maximal ideals of $R$.

We will write the vectors in $\mathbb{C}^{n(n+1)/2}$ in the following form:

$$L = (l_{ij}) = (l_{n1}, ..., l_{nn}, l_{n-1,n-1}, ..., l_{21,22})$$

For $1 \leq j \leq i \leq n$, $\delta^{ij} \in \mathbb{Z}^{n(n+1)/2}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{kt}$ are zero.

For $i > 0$ by $S_i$ we denote the $i$th symmetric group. Throughout the paper we set $G := S_n \times \cdots \times S_1$.

3. Gelfand-Tsetlin Theorem and Gelfand-Tsetlin modules

In 1950, I Gelfand and M. Tsetlin give an explicit realization of all irreducible finite dimensional modules for $\mathfrak{gl}_n$. Let us remember the construction.

**Definition 3.1.** For a vector $L = (l_{ij})$ in $\mathbb{C}^{n(n+1)/2}$, by $T(L)$ we will denote the following array with entries $\{l_{ij} \mid 1 \leq j \leq i \leq n\}$

\[
\begin{array}{cccccc}
  l_{n1} & l_{n2} & \cdots & l_{n,n-1} & l_{nn} \\
  l_{n-1,1} & \cdots & l_{n-1,n-1} \\
  \cdots & \cdots & \cdots \\
  l_{21} & l_{22} \\
  l_{11}
\end{array}
\]


Such an array will be called a Gelfand-Tsetlin tableau of height \( n \). A Gelfand-Tsetlin tableau of height \( n \) is called standard if \( l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0} \) and \( l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{> 0} \) for all \( 1 \leq i \leq k \leq n - 1 \).

**Theorem 3.2** ([3]). Let \( L(\lambda) \) be the finite dimensional irreducible module over \( \mathfrak{gl}_n \) of highest weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \). Then there exist a basis of \( L(\lambda) \) consisting of all standard tableaux \( T(L) \) with fixed top row \( l_{0j} = \lambda_j - j + 1 \). Moreover, the action of the generators of \( \mathfrak{gl}_n \) on \( L(\lambda) \) is given by the Gelfand-Tsetlin formulas:

\[
E_{k,k+1}(T(L)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1} (l_{ki} - l_{k+1,j})}{\prod_{j \neq i}^{k+1} (l_{ki} - l_{kj})} \right) T(L + \delta^{ki}),
\]

\[
E_{k+1,k}(T(L)) = \sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k-1} (l_{ki} - l_{k-1,j})}{\prod_{j \neq i}^{k-1} (l_{ki} - l_{kj})} \right) T(L - \delta^{ki}),
\]

\[
E_{kk}(T(L)) = \left( k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L),
\]

if the new tableau \( T(L \pm \delta^{ki}) \) is not standard, then the corresponding summand of \( E_{k,k+1}(T(L)) \) or \( E_{k+1,k}(T(L)) \) is zero by definition.

For \( m \leq n \) let \( \mathfrak{gl}_m \) be the Lie subalgebra of \( \mathfrak{gl}_n \) spanned by \( \{ E_{ij} \mid i, j = 1, \ldots, m \} \). We have the following chain

\[ \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n. \]

It induces the chain \( U_1 \subset U_2 \subset \ldots \subset U_n \) for the universal enveloping algebras \( U_m = U(\mathfrak{gl}_m), 1 \leq m \leq n \). Let \( Z_m \) be the center of \( U_m \). The subalgebra \( \Gamma \) of \( U = U(\mathfrak{gl}_n) \) generated by \( \{ Z_m \mid m = 1, \ldots, n \} \) is the Gelfand-Tsetlin subalgebra of \( U \) [6].

We define now our main objects.

**Definition 3.3.** A finitely generated \( U \)-module \( M \) is called a Gelfand-Tsetlin module (with respect to \( \Gamma \)) if

\[
M = \bigoplus_{m \in \text{Spec} \Gamma} M(m),
\]

where

\[ M(m) = \{ v \in M \mid m^k v = 0 \text{ for some } k \geq 0 \}. \]

The Gelfand-Tsetlin support of \( M \) is the set \( \text{Supp}_{GT}(M) := \{ m \in \text{Spec} \Gamma \mid M(m) \neq 0 \} \). The Gelfand-Tsetlin multiplicity of \( m \) in \( M \) is the dimension of \( M(m) \).

The category of Gelfand-Tsetlin modules is closed under the operations of taking submodules (cf. [3], [6]), quotients and direct sums.

One can choose the following generators of \( \Gamma \): \( \{ e_{mk} \mid 1 \leq k \leq m \leq n \} \), where

\[
c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m \}^k} E_{i_1,i_2} E_{i_2,i_3} \ldots E_{i_k,i_1}.
\]

Let \( \Lambda \) be the polynomial algebra in the variables \( \{ \lambda_{ij} \mid 1 \leq j \leq i \leq n \} \). The action of the symmetric group \( S_i \) on \( \{ \lambda_{ij} \mid 1 \leq j \leq i \} \) induces the action of \( G \) on
Λ. There is a natural embedding \( i : \Gamma \rightarrow \Lambda \) given by \( i(c_{mk}) = \gamma_{mk}(\lambda) \) where

\[
\gamma_{mk}(\lambda) = \sum_{i=1}^{m} (\lambda_{mi} + m - 1)^{k} \prod_{j \neq i} \left( 1 - \frac{1}{\lambda_{mi} - \lambda_{mj}} \right).
\]

Hence, \( \Gamma \) can be identified with \( \text{Specm} \Lambda \).

**Remark 3.4.** In what follows, we will identify the set \( \text{Specm} \Lambda \) of maximal ideals of \( \Lambda \) with the set \( C_{n(n+1)/2} \). There are finitely many maximal ideals of \( \Lambda \) whose image under the surjective map \( \pi : \text{Specm} \Lambda \rightarrow \text{Specm} \Gamma \) is a fixed maximal ideal of \( \Gamma \). These maximal ideals of \( \Lambda \) are obtained from each by permutations of the group \( G \).

**Remark 3.5.** It was shown in \([20]\) that the action of the generators \( c_{rs} \) of \( \Gamma \) on any basis tableau in irreducible finite dimensional module is given by

\[
c_{rs}(T(L)) = \gamma_{rs}(l) T(L),
\]

where the polynomials \( \gamma_{rs}(L) \) are defined in \([9]\). In particular, any irreducible finite dimensional module is a Gelfand-Tsetlin module and the Gelfand-Tsetlin subalgebra is diagonal in the tableaux basis described in Theorem 3.2.

Our goal is to construct explicitly new families of irreducible Gelfand-Tsetlin modules of \( \mathfrak{gl}_n \). Our approach involves constructing of certain admissible sets of relations analogous to relations that define standard tableaux. For each such set of relations we define an infinite family of nonisomorphic Gelfand-Tsetlin modules.

### 4. Admissible relations

For convenience we denote \( l_{ij} \geq l_{st} \) (respectively \( l_{ij} > l_{st} \)) if \( l_{ij} - l_{st} \in \mathbb{Z}_{\geq 0} \) (respectively \( \mathbb{Z}_{>0} \)).

Set \( \mathcal{W} := \{(i, j) \mid 1 \leq j \leq i \leq n\} \). For pairs \((i, j), (s, t)\) from \( \mathcal{W} \) we write \((i, j) \geq (s, t)\) (respectively \((i, j) > (s, t)\)) and we say that \( T(L) \) satisfies such relation if \( l_{ij} \geq l_{st} \) (respectively \( l_{ij} > l_{st} \)). Set

\[
\mathcal{S} := \{(i + 1, j) \geq (i, j) > (i + 1, j + 1) \mid 1 \leq j \leq i \leq n - 1\}.
\]

A tableau \( T(L) \) is standard if and only if \( T(L) \) satisfies all the relations in \( \mathcal{S} \).

Set

\[
\mathcal{R} := \{(i, j) \geq (i - 1, j')\}, \quad \{(i - 1, j') > (i, j) \mid 1 \leq j \leq i \leq n, 1 \leq j' \leq i - 1\}
\]

and \( \mathcal{R}^0 = \{(n, i) \geq (n, j) \mid 1 \leq i \neq j \leq n\} \). We say that \( T(L) \) satisfies the relation \((n, i) \geq (n, j)\) if \( l_{ni} \geq l_{nj} \). From now on we will consider sets of relations on \( \mathcal{W} \) which are subsets of \( \mathcal{R} \cup \mathcal{R}^0 \).

**Definition 4.1.** Let \( \mathcal{C} \) be any subset of \( \mathcal{R} \cup \mathcal{R}^0 \).

1. We say that a Gelfand-Tsetlin tableau \( T(L) \) satisfies \( \mathcal{C} \) if \( T(L) \) satisfies all the relations in \( \mathcal{C} \).
2. Let \( T(L) \) be a Gelfand-Tsetlin tableau satisfying \( \mathcal{C} \).
   (i) Denote by \( \mathcal{B}_C(T(L)) \) the set of all tableaux \( T(L + z) \) satisfying \( \mathcal{C} \), \( z \in \mathbb{Z}_{\geq 0}^{n(n+1)/2} \), and by \( \mathcal{V}_C(T(L)) \) the complex vector space spanned by \( \mathcal{B}_C(T(L)) \).
   (ii) We call a tableau \( T(L) \) noncritical if \( l_{ki} \neq l_{kj} \) for all \( 1 \leq i < j \leq k \leq n - 1 \), and critical otherwise.
   (iii) We say that \( T(L) \) is a \( \mathcal{C} \)-realization if \( T(L) \) satisfies \( \mathcal{C} \) and for any \( T(R) \in \mathcal{B}_C(T(L)) \), \( T(R) \) is noncritical.
Our goal is to determine for which $\mathcal{C}$ and $T(L)$ one can define an action of $\mathfrak{gl}_n$ on $V_{\mathcal{C}}(T(L))$ by the Gelfand-Tsetlin formulas. If $T(L)$ is critical then the Gelfand-Tsetlin formulas are not defined on $T(L)$ and, hence, $V_{\mathcal{C}}(T(L))$ should not contain any critical tableau.

**Definition 4.2.** Let $\mathcal{C}$ be a subset of $\mathcal{R} \cup \mathcal{R}^0$. We call $\mathcal{C}$ admissible if for any $\mathcal{C}$-realization $T(L)$, the vector space $V_{\mathcal{C}}(T(L))$ has a structure of a $\mathfrak{gl}_n$-module, endowed with the action of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.

We are going to define algorithmically sets of admissible relations starting from any subset of $\mathcal{R}$.

Let $\mathcal{C}$ be a subset of $\mathcal{R}$, denote by $\mathfrak{W}(\mathcal{C})$ the set of all $(i,j)$ in $\mathfrak{W}$ such that $(i,j) \geq (\text{respectively } >, \leq, <)(r,s) \in \mathcal{C}$ for some $(r,s)$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two subsets of $\mathcal{C}$. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are disconnected if $\mathfrak{W}(\mathcal{C}_1) \cap \mathfrak{W}(\mathcal{C}_2) = \emptyset$, otherwise $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected. A subset $\mathcal{C} \subseteq \mathcal{R}$ is called decomposable if it can be decomposed into the union of two disconnected subsets of $\mathcal{R}$, otherwise $\mathcal{C}$ is called indecomposable.

Note that any subset of $\mathcal{R}$ is a union of disconnected indecomposable sets. The decomposition is unique.

### 4.1. Noncritical sets of relations. As we saw above an admissible set of relations does not allow critical tableau $T(L)$ as its realization. noncritical case. So as the first step in constructing of admissible relations we define noncritical sets.

We start with the following

**Proposition 4.3.** Let $\mathfrak{W}$ be any subset of $\mathfrak{W}$ and $\mathcal{C}$ any subset of $\mathcal{R}$ such that $\mathfrak{W}(\mathcal{C}) \subseteq \mathfrak{W}$. If there is a tableau $T(L)$ satisfying $\mathcal{C}$ such that $l_{ki} = l_{kj}$, $1 \leq k \leq n - 1$, $i \neq j$, for some $(k,i),(k,j) \in \mathfrak{W}$, then for any $T(R)$ satisfying $\mathcal{C}$ there exists $z \in \mathbb{Z}^{\mathfrak{W}(\mathcal{C})}$ with $z_{rs} = 0$ if $(r,s) \notin \mathfrak{W}$ such that $T(Q) = T(R + z)$ has $q_{ki} = q_{kj}$ and:

1. $\min\{r_{st} \mid (s,t) \in \mathfrak{W}\} \leq \min\{q_{st} \mid (s,t) \in \mathfrak{W}\}$.
2. $\max\{q_{st} \mid (s,t) \in \mathfrak{W}\} \leq \max\{r_{st} \mid (s,t) \in \mathfrak{W}\}$.

**Proof.** We prove the statement by induction on $\#\mathfrak{W}$. When $\#\mathfrak{W} = 2,3$ it is easy to verify. Assume the statement is true for $\#\mathfrak{W} < m$ and consider a tableau $T(L)$ satisfying $\mathcal{C}$ such that $l_{ki} = l_{kj}$, $1 \leq k \leq n - 1$, $i \neq j$, for some $(k,i),(k,j) \in \mathfrak{W}$.

(a) Suppose $l_{ki} = l_{kj} = \max\{l_{st} \mid (s,t) \in \mathfrak{W}\}$. Let $\mathfrak{W}_1 = \{(s,t) \in \mathfrak{W} \mid l_{st} = l_{ki}\}$. Let $\mathfrak{W}_1 \subseteq \mathfrak{W}$ and $T(Q) = T(R + z)$ be a tableau such that $q_{st} = \min\{r_{st}\}$ for any $s \leq n - 1$. Then $T(Q)$ satisfies the conditions in the statement. Suppose now that $\mathfrak{W}_1 \not\subseteq \mathfrak{W}$. Let $\mathfrak{W}_2 = \{(s,t) \mid l_{st} = \min\{l_{st'} \mid (s',t') \in \mathfrak{W}\}\}$ and let $C_1$ be the subset of $\mathcal{C}$ obtained by removing relations that involve $\mathfrak{W}_2$. Then $C_1 \subseteq \mathfrak{W} \setminus \mathfrak{W}_2$. By assumption there exists $T(R + z)$ satisfying $\mathcal{C}_1$. Let $T(Q)$ be the tableau such that $q_{st} = (r + z)_{rs}$ if $(r,s) \notin \mathfrak{W}_2$ and $q_{st} = \min\{r_{st} \mid (s,t) \in \mathfrak{W}_2\}$ if $(s,t) \in \mathfrak{W}_2$. Since $z_{st} = 0$ for any $(s,t) \in \mathfrak{W}_2$ one has $\min\{q_{st} \mid (s,t) \in \mathfrak{W}_2\} \leq \min\{q_{st} \mid (s,t) \in \mathfrak{W} \setminus \mathfrak{W}_2\}$. Then $T(R + z)$ satisfies $\mathcal{C}$ and (i), (ii) are satisfied.

(b) Suppose $l_{ki} = l_{kj} = \min\{l_{st} \mid (s,t) \in \mathfrak{W}\}$. This case is similar to (a).

(c) Suppose $\min\{l_{st} \mid (s,t) \in \mathfrak{W}\} < l_{ki} = l_{kj} < \min\{l_{st} \mid (s,t) \in \mathfrak{W}\}$. This is similar to the case $\mathfrak{W}_1 \not\subseteq \mathfrak{W}$ in (a).
Corollary 4.4. Let $\mathcal{C}$ be any subset of $\mathcal{R}$. If there is a critical tableau $T(L)$ satisfying $\mathcal{C}$ such that $l_{ki} = l_{kj}$, for some $(k, i), (k, j) \in \mathfrak{M}(\mathcal{C})$, $1 \leq k \leq n - 1$, $i \neq j$, then for any $T(R)$ satisfying $\mathcal{C}$ there exist $z \in \mathbb{Z}^{n_{m-1}}$ such that $T(R + z) = (q_{rs})$ is critical with $q_{ki} = q_{kj}$.

Proof. Let $\mathfrak{V} = \mathfrak{B}(\mathcal{C})$ as in Proposition 4.3. The statement can be proved by induction on $\# \mathfrak{V}$.

This suggests the following definition.

Definition 4.5. Let $\mathcal{C}$ be an indecomposable set. We call $\mathcal{C}$ noncritical if for any $T(L)$ satisfying $\mathcal{C}$ one has $l_{ki} \neq l_{kj}$, $1 \leq k \leq n - 1$, $i \neq j$, $(k, i), (k, j) \in \mathfrak{B}(\mathcal{C})$. An arbitrary $\mathcal{C}$ is called noncritical if every disconnected indecomposable subset of $\mathcal{C}$ is noncritical, otherwise $\mathcal{C}$ is critical.

Remark 4.6.
(i) A noncritical set $\mathcal{C}$ has infinitely many tableaux realizations.
(ii) $\mathcal{S}$ is noncritical. Any standard tableau is an $\mathcal{S}$-realization.
(iii) $\emptyset$ is noncritical. Any generic tableau is an $\emptyset$-realization.
(iv) $\mathcal{C} = \{(2, 1) \geq (1, 1), (2, 2) \geq (1, 1)\}$ is critical since there exist a tableau $T(L)$ satisfying $\mathcal{C}$ with $l_{21} = l_{22}$.

Remark 4.7. Suppose $\mathcal{C}$ is critical. Then for any $T(L)$ satisfying $\mathcal{C}$ there exists a critical tableau in $\mathcal{B}_C(T(L))$ by Corollary 4.4. Suppose now $\mathcal{C}$ is noncritical. If $T(L)$ satisfies $\mathcal{C}$ and $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if $(k, i)$ and $(k, j)$ in the same indecomposable subset of $\mathcal{C}$, then every tableau in $\mathcal{B}_C(T(L))$ is noncritical.

From now on we only consider noncritical sets of relations.

4.2. Extended relations. Let $\mathcal{C}$ be a noncritical subset of $\mathcal{R}$, $T(L)$ a $\mathcal{C}$-realization. Let $\mathcal{C}^* = \mathcal{C} \cup \mathcal{C}^0$ where $\mathcal{C}^0$ is an arbitrary subset of $\mathcal{R}^0$. Then $\mathcal{C}^*$ can be decomposed into the union of disconnected indecomposable subsets of $\mathcal{R} \cup \mathcal{R}^0$, i.e. $\mathcal{C}^* = \mathcal{C}_1^* \cup \cdots \cup \mathcal{C}_m^*$. Proposition 4.3 and Corollary 4.4 hold for $\mathcal{C}^*$.

We call $\mathcal{C}^*$ noncritical if $\mathcal{C}^*$ satisfies the following conditions:

(i) $\mathcal{C}^*$ is noncritical and $\mathcal{C}^0$ contains no cycles.
(ii) There exists a tableau satisfying $\mathcal{C}^*$.
(iii) $\mathcal{C}_i^* \cap \mathcal{C}_j^* \cap \mathcal{C}^0$ defines an order in $\mathfrak{B}(\mathcal{C}_i^*) \cap \mathfrak{B}(\mathcal{C}_j^*)$ for each $1 \leq i \leq m$.

Remark 4.8. All tableaux in $\mathcal{B}_C(T(L))$ have the same top row. For any $\mathcal{C}$-realization $T(L)$ there exists $\mathcal{C}^*$ such that $\mathcal{B}_C(T(L)) = \mathcal{B}_{C^*}(T(L))$. Observe that $l_{ki} - l_{kj}$ can be an integer for $(k, i)$ and $(k, j)$ in different indecomposable subset of $\mathcal{C}$. Clearly, we can always choose $\mathcal{C}^*$ such that $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if $(k, i)$ and $(k, j)$ are in the same $\mathfrak{B}(\mathcal{C}_s^*)$, $1 \leq s \leq m$.

Definition 4.9. Let $\mathcal{C}^*$ be a union of disconnected indecomposable sets, $\mathcal{C}^* = \mathcal{C}_1^* \cup \cdots \cup \mathcal{C}_m^*$, and $T(L)$ a Gelfand-Tsetlin tableau satisfying $\mathcal{C}^*$. We call $T(L)$ a $\mathcal{C}^*$-realization if $T(L)$ satisfies $\mathcal{C}^*$, and $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if $(k, i)$ and $(k, j)$ are in the same $\mathfrak{B}(\mathcal{C}_s^*)$, $1 \leq s \leq m$.

From now on by $\mathcal{C}$ we will always mean $\mathcal{C}^*$ and by $V_C(T(L))$ we will mean $V_{\mathcal{C}^*}(T(L))$, where $T(L)$ is a $\mathcal{C}^*$-realization.
4.3. Reduced sets. Our next step towards constructing the admissible sets of relations is to define reduced sets.

We have

**Proposition 4.10.** Let $\mathcal{C}$ be an indecomposable noncritical set. The relations in $\mathcal{C}$ define an order on the set $\{(k, j) \mid (k, j) \in \mathcal{V}(\mathcal{C})\}$ for any $1 \leq k \leq n - 1$: $(k, j_1) > (k, j_2) > \cdots > (k, j_r)$.

**Proof.** Suppose there exist two tableaux $T(L)$ and $T(R)$ satisfying $C$ with $l_{ki} > l_{kj}$ and $r_{ki} < r_{kj}$. For any positive integers $s$ and $t$, the tableau $T(Q) = T(sL + tR)$ with entries $q_{ij} = sl_{ij} + tr_{ij}$, satisfies $C$. In particular, for $s = r_{kj} - r_{ki}$ and $t = l_{ki} - l_{kj}$ one has $q_{ki} = q_{kj}$ which is a contradiction. The order in top row is given by $C$. Thus all tableaux satisfying $C$ has the same order on $\{(k, j) \mid (k, j) \in \mathcal{V}(\mathcal{C})\}$. □

Note that any standard tableau $T(L)$ satisfies $l_{k1} > \cdots > l_{kk}$ for any $k$.

The following statement can be easily proved by the induction on $\#(\mathcal{V}(\mathcal{C}))$.

**Proposition 4.11.** Let $\mathcal{C}$ be any noncritical set and $m$ be any positive integer. There exists a tableau $T(L)$ satisfying $C$ such that $|l_{ij} - l_{st}| \geq m$ for any two pairs $(i, j), (s, t) \in \mathcal{V}(\mathcal{C})$.

**Definition 4.12.** Let $\mathcal{C}_1, \mathcal{C}_2$ be noncritical sets of relations. We say that $\mathcal{C}_1$ implies $\mathcal{C}_2$ if any tableau that satisfies $\mathcal{C}_1$ also satisfies $\mathcal{C}_2$. We say that $\mathcal{C}_1$ is equivalent to $\mathcal{C}_2$ if $\mathcal{C}_1$ implies $\mathcal{C}_2$ and $\mathcal{C}_2$ implies $\mathcal{C}_1$.

**Definition 4.13.** Let $\mathcal{C}$ be any noncritical set of relations. We call $\mathcal{C}$ reduced, if for every $(k, j) \in \mathcal{V}(\mathcal{C})$ the following conditions are satisfied:

(i) There exist at most one $i$ such that $(k, j) > (k + 1, i) \in \mathcal{C}$, $1 \leq k \leq n$.
(ii) There exist at most one $i$ such that $(k + 1, i) \geq (k, j) \in \mathcal{C}$, $1 \leq k \leq n$.
(iii) There exist at most one $i$ such that $(k, j) \geq (k - 1, i) \in \mathcal{C}$, $1 \leq k \leq n$.
(iv) There exist at most one $i$ such that $(k - 1, i) > (k, j) \in \mathcal{C}$, $1 \leq k \leq n$.

We will show now that distinct reduced sets of relations and nonequivalent.

**Proposition 4.14.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be reduced sets. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent if and only if $\mathcal{C}_1 = \mathcal{C}_2$.

**Proof.** Assume $\mathcal{C}_1$ and $\mathcal{C}_2$ to be equivalent. Then $\mathcal{V}(\mathcal{C}_1) = \mathcal{V}(\mathcal{C}_2)$. Let $m = \#(\mathcal{V}(\mathcal{C}_1)) = \#(\mathcal{V}(\mathcal{C}_2))$. The statement can be proved by induction on $m$. When $m < 3$ it is easy to verify. By Proposition 4.11 there is a tableau $T(L)$ that satisfies $C_1$ and $|l_{ij} - l_{st}| \geq 2$ for any $(i, j), (s, t) \in \mathcal{V}(C_1)$. Suppose $l_{ki} = \max\{l_{st} \mid (s, t) \in \mathcal{V}(C_1)\}$. Let $C'_1$ be the set of all relations in $\mathcal{C}_1$ that involve $(k, i)$, $(t = 1, 2)$. Then $C_1 \setminus C'_1$ and $C_2 \setminus C'_2$ are equivalent and $\#(\mathcal{V}(C_1 \setminus C'_1)) < \#(\mathcal{V}(C_1))$. One has that $\mathcal{C}_1 \setminus C'_1 = \mathcal{C}_2 \setminus C'_2$. If $C'_1 \neq C'_2$, then there exists $T(L + r\delta_{kt})$ that satisfies only one of $C'_1$ or $C'_2$. It contradicts to the equivalence of $C_1$ and $C_2$. Thus $C_1 = C_2$.

We can prove now

**Theorem 4.15.** Any noncritical set of relations is equivalent to a unique reduced set of relations.

**Proof.** It is sufficient to prove that any indecomposable set is equivalent to a reduced set. Let $\mathcal{C}$ be indecomposable and $(k + 1, i) \geq (k, j)$, $(k + 1, i') \geq (k, j) \in \mathcal{C}$. By Proposition 4.10 and definition of $\mathcal{C}$, $(k + 1, i)$ and $(k + 1, i')$ are ordered. Suppose first $(k + 1, i) > (k + 1, i')$. If we show that any tableau $T(R)$ satisfying $\mathcal{C} \setminus \{(k + 1, i) \geq (k, j) \}$
\[(k, j)\] satisfies also \(r_{k+1, i} \geq r_{k, j}\), then \(C \setminus \{(k + 1, i) \geq (k, j)\}\) implies \(C\). Assume we have a tableau \(T(R)\) satisfying \(C \setminus \{(k + 1, i) \geq (k, j)\}\) with \(r_{k+1, i} < r_{k, j}\). Then \(r_{k+1, j} < r_{k+1, i}\). Take a tableau \(T(L)\) satisfying \(C\). Hence its entries satisfy \(l_{k, j} \leq l_{k+1, i} < l_{k+1, i}'\). For any positive integers \(s\) and \(t\), \(T(Q) = T(sL + tR)\) satisfies \(C \setminus \{(k + 1, i) \geq (k, j)\}\). In particular, if \(s = r_{k+1, i} - r_{k+1, i}\) and \(t = l_{k+1, i} - l_{k+1, i}'\) then \(q_{k+1, i} = q_{k+1, i}'\). Since \(T(Q)\) satisfies the noncritical set \(C\) we obtain a contradiction. Thus we have \(r_{k+1, i} \geq r_{k, j}\). If \((k + 1, i') > (k + 1, i)\) then using the same arguments one has that \(C \setminus \{(k + 1, i') \geq (k, j)\}\) implies \(C\).

All other cases can be proved similarly. Thus any noncritical set is equivalent to a reduced one. The uniqueness follows from Proposition 4.14. \(\square\)

Recall that \(G\) denotes the group \(S_n \times \cdots \times S_1\).

**Definition 4.16.** For any \(\sigma = (\sigma[n], \sigma[n-1], \ldots, \sigma[2], \sigma[1]) \in G\) and \(C \subseteq R\) denote by \(\sigma(C)\) the set of relations:

\[\{(i, \sigma[i])(j)\} \geq (\text{resp. }>(r, \sigma[r])(s)) \mid (i, j) \geq (\text{resp. }>(r, s) \in C\}.

If \(C\) is any noncritical reduced subset of \(R \cup R^0\) and \(V_C(T(L))\) is a \(\mathfrak{g}_n\)-module then \(V_C(T(L)) \leq V_{\sigma(C)}(T(\sigma L))\). So it is sufficient to consider the noncritical sets that satisfy the following condition: \((k, i) \geq (k, j), 1 \leq k \leq n - 1\) and \((n, i) \geq (n, j)\) only if \(i < j\).

**4.4 Cross elimination.**

**Definition 4.17.** Let \(C\) be an indecomposable noncritical subset of \(R \cup R^0\). A subset of \(C\) of the form \(\{(k, i) \geq (k + 1, t), (k + 1, s) < (k, j)\}\) with \(i < j\) and \(s < t\) will be called a cross. We will identify the cross \(\{(k, i) \geq (k + 1, t), (k + 1, s) < (k, j)\}\) with the 5-tuple \([k, i, s, j, t]\).

Let \(C\) be an indecomposable noncritical subset of \(R \cup R^0, [k, i, s, j, t]\) a cross in \(C\) and \(T(L)\) a Gelfand-Tsetlin tableau satisfying \(C\) such that \(V_C(T(L))\) is a module. Set \(C_1 = C \setminus [k, i, s, j, t]\) and define \(C_2\) as follows:

(i) If \(l_{ki} > l_{k+1, s}\) and \(l_{k+1, s} \geq l_{k, j} > l_{k+1, t}\)
\[C_2 := C_1 \cup \{(k, i) > (k + 1, s) \geq (k, j) > (k + 1, t)\} \]

(ii) If \(l_{ki} > l_{k+1, s}\) and \(l_{k+1, t} \geq l_{k, j}\)
\[C_2 := C_1 \cup \{(k, i) > (k + 1, s) \geq (k, j) \geq (k + 1, t)\} \]

(iii) If \(l_{k+1, s} \geq l_{k, i} \geq l_{k, j} > l_{k+1, t}\)
\[C_2 := C_1 \cup \{(k + 1, s) \geq (k, i), (k, j) > (k + 1, t)\} \]

(iv) If \(l_{k+1, s} \geq l_{k, i} > l_{k+1, t}\) and \(l_{k+1, t} \geq l_{k, j}\)
\[C_2 := C_1 \cup \{(k + 1, s) \geq (k, i) > (k + 1, t) \geq (k, j)\} \]

We have \(B_{C_2}(T(L)) \subseteq B_C(T(L))\), \(V_{C_2}(T(L))\) is a module, and \(T(L)\) is contained in \(V_{C_2}(T(L))\). If \(C_2\) is not reduced one can remove some relations by Theorem 4.15.

Assume that \(C\) is subset of \(R \cup R^0\) that has a cross. Define a lexicographical order \(\preceq\) on the set of crosses, choose the minimal cross \([k, i, s, j, t]\) and define \(C_1\) as above. Then \([k, i, s, j, t]\) \(\preceq [k_1, t_1, s_1, j_1, t_1]\) for any cross \([k_1, t_1, s_1, j_1, t_1]\) in \(C_1\) and \(T(L)\) is contained in \(V_{C_2}(T(L))\). Repeating this procedure we obtain after finitely many steps that \(T(L)\) is contained in \(V_C(T(L))\) where \(\tilde{C}\) does not have any cross.

Let \(C\) be an indecomposable set. We say that \(C\) is pre-admissible if it satisfies the following conditions:
Lemma 4.19. Let $F$ be an admissible set. It is sufficient to consider only pre-admissible sets.

4.5. $\mathfrak{F}$ set.

Proposition 4.18. Let $C$ be an indecomposable pre-admissible set. If $T(L)$ is a $C$-realization such that $l_{ki} - l_{kj} = 1$, then one of the following cases hold.

(i) $\{ (k, i) > (k, j) \} \subseteq C$.

(ii) $\{ (k, i) > (k, j) \} \subseteq C$.

(iii) $(n - 1, i) > (n, s) \in C$ and $(n, t) \geq (n - 1, j) \in C$, for some $s \leq t$.

Proof. We note that there exists a relation $(k, i) > (k, j) \in C$ such that $l_{ki} = l_{kj}$ or $(k, i) > (k, j) \in C$ such that $l_{ki} = l_{kj}$. Indeed, otherwise $T(L + \delta_{ki})$ satisfies $C$ which contradicts $C$ being noncritical. Similalry we have $(k, j) \in C$ such that $l_{kj} = l_{ki}$ or $(k, j) \in C$ with $l_{kj} = l_{ki}$.

(a) Suppose $(k, i) > (k + 1, s) \in C$ and there is no $(k, i) > (k - 1, s') \in C$.

If $r_{k+1,s} > r_{kj}$ then $r_{ki} < r_{kj}$ by Proposition 4.10 and $T(R - (r_{ki} - r_{kj})a_{ki})$ satisfies $C$ which is a contradiction. Hence $C$ implies $(k + 1, s) > (k, j)$ and $C \cup \{ (k, i) > (k, j) \}$ is equivalent to $C$.

Let $k \leq n - 2$. If $(k + 1, s) \geq (k, j) \in C$ then $C \cup \{ (k + 1, s) > (k, j) \} \subseteq C$. If $\{ (k + 1, s) > (k, j) \}$ implies $(k + 1, s) > (k, j)$. By the uniqueness of the reduced set one has that $(k + 1, s) > (k, j) \in C$. Similarly, if $(k + 1, t) \geq (k, j) \in C$ then $(k + 1, s) \geq (k, j) \in C$. If $k = n - 1$ then there exists $(k + 1, t) > (k, j) \in C$.

(b) Suppose $(k, i) > (k + 1, s) \in C$ and there is no relation $(k, i) > (k + 1, s') \in C$.

(c) Suppose $\{ (k, i) > (k + 1, s), (k, i) > (k - 1, t) \} \subseteq C$. Then $(k + 1, s') \geq (k, j) \in C$ or $(k - 1, t) > (k, j) \in C$. The statement follows since there is no cross in $C$.

Now we introduce our main sets of relations which lead to admissible sets. Denote by $\mathfrak{F}$ the set of all indecomposable $C$ satisfying the following condition: for every adjoining pair $(k, i)$ and $(k, j)$, $1 \leq k \leq n - 1$ one of the following holds

\[
\{ (k, i) > (k + 1, s), (k, i) > (k - 1, t) > (k, j) \} \subseteq C, \\
\{ (k, i) > (k + 1, s), (k + 1, t) > (k, j) \} \subseteq C, s < t.
\]

The following lemma is easy to verify.

Lemma 4.19. Let $C \in \mathfrak{F}$, $T(L)$ a $C$-realization. Then for any $T(R) \in B_{C}(T(L))$, if $r_{ki} - r_{kj} = 1$ then $\# \{ r_{k+1,i'}, r_{k-1,j'} \mid r_{k+1,i'} = r_{kj}, r_{k-1,j'} = r_{ki} \} \geq 2$.

Set

\[
e_{ki}(L) = \begin{cases} 0, & \text{if } T(L) \notin B(T(L)), \\ \prod_{k+1}^{k}(l_{ki} - l_{kj}), & \text{if } T(L) \in B(T(L)). \end{cases}
\]
One has Lemma 4.20.

Proof. \(\Phi(L, z)\) is given for any \(t\) by

\[
\begin{cases}
0, & \text{if } T(L) \notin \mathcal{B}(T(L)) \\
\prod_{i=1}^{k} (k_i - l_{i+1}), & \text{if } T(L) \in \mathcal{B}(T(L))
\end{cases}
\]

(16) \(h_k(L) = \left\{ \begin{array}{ll}
2 \sum_{i=1}^{k} l_{k_i} - 2 \sum_{i=1}^{k-1} l_{k_i} - \sum_{i=1}^{k+1} l_{k+1,i} - 1, & \text{if } T(L) \notin \mathcal{B}(T(L)) \\
0, & \text{if } T(L) \in \mathcal{B}(T(L))
\end{array} \right.\)

(17) \(\Phi(L, z_1, \ldots, z_m) = \left\{ \begin{array}{ll}
1, & \text{if } T(L + z_1 + \ldots + z_t) \in \mathcal{B}(T(L)) \text{ for any } t \\
0, & \text{otherwise.}
\end{array} \right.\)

One has \(h_k(L + z_1 + \ldots + z_m) = f_k(L + z_1 + \ldots + z_m) = h_k(L + z_1 + \ldots + z_m) = 0\) if \(\Phi(L, z_1, \ldots, z_m) = 0\).

We will denote by \(T(v)\) the tableau with variable entries \(v_{ij}\).

**Lemma 4.20.** Let \(C \in \mathcal{F}\), \(T(L)\) any tableau satisfying \(C\).

(i) If \(T(L + \delta^{k_j}) \notin \mathcal{B}(T(L))\) and \(l_{k,i} - l_{k,j} \neq 1\) for any \(i\), then

\[
\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k_j}) = 0.
\]

(ii) If \(T(L - \delta^{k_j}) \notin \mathcal{B}(T(L))\) and \(l_{k,j} - l_{k,i} \neq 1\) for any \(i\), then

\[
\lim_{v \rightarrow l} f_k(v)e_k(v - \delta^{k_j}) = 0.
\]

(iii) If \(l_{k,i} - l_{k,j} = 1\), then \(T(L + \delta^{k,j}), T(L - \delta^{k,i}) \notin \mathcal{B}(T(L))\), and

\[
\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k,j}) - f_k(v)e_k(v - \delta^{k,i}) = 0.
\]

**Proof.** Since \(T(L + \delta^{k_j}) \notin \mathcal{B}(T(L))\), we have \(\{(k + 1, s) \geq (k, j)\} \subseteq C\) or \(\{(k - 1, t) > (k, j)\} \subseteq C\). Suppose \(\{(k + 1, s) \geq (k, j)\} \subseteq C\) and \(T(L + \delta^{k_j}) \notin \mathcal{B}(T(L))\). Then \(l_{k+1,s} = l_{k,j}\) and by direct computation one has \(\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k_j}) = 0\). Suppose \(T(L + \delta^{k_j})\) does not satisfy the relation \(l_{k,1-t} - l_{k,j} \in \mathbb{Z}_{>0}\). Then we have \(l_{k,1-t} = l_{k,j} + 1\) and \(\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k_j}) = 0\). Thus one has \(\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k_j}) = 0\).

The proof of (ii) is similar to (i).

It is clear that \(T(L - \delta^{k_j})\), \(T(L + \delta^{k,j+1}) \notin \mathcal{B}(T(L))\) if \(l_{k,j} - l_{k,j+1} = 1\). By Lemma 4.19 \#\(\{l_{k+1,i'}, l_{k-1,j'} \mid l_{k+1,i'} = l_{k,j}, l_{k-1,j'} = l_{k1}\} \geq 2\). By direct computation one has

\[
\lim_{v \rightarrow l} e_k(v)f_k(v + \delta^{k,j+1}) - f_k(v)e_k(v - \delta^{k,j}) = 0.
\]

The following lemma is easy to verify.

**Lemma 4.21.** Let \(C \in \mathcal{F}\), \(z^{(1)}, z^{(2)} \in \mathbb{Z}^{n(m-1)}\). Denote \(I_1 = \{ (i, j) \mid z^{(1)}_{ij} \neq 0 \}\), \(I_2 = \{ (i, j) \mid z^{(2)}_{ij} \neq 0 \}\). If \(I_1 \cap I_2 = \emptyset\) and for any \((i_1, j_1) \in I_1\), \((i_2, j_2) \in I_2\) there is no relation between \((i_1, j_1)\) and \((i_2, j_2)\), then \(T(R + z^{(1)} + z^{(2)}) \neq 0\) if and only if \(T(R + z^{(1)}) \neq 0\) and \(T(R + z^{(2)}) \neq 0\).
4.6. **Necessary and sufficient conditions of admissibility.** Now we are ready to describe admissible sets of relations.

**Theorem 4.22.** $C$ is admissible if and only if $C$ is union of disconnected sets from $\mathcal{G}$.

**Proof.** Let $C$ be a union of disconnected sets from $\mathcal{G}$. We will show that $C$ is admissible. It is sufficient to consider the case when $C$ is union of two disconnected subsets from $\mathcal{G}$. Suppose $C = C_1 \cup C_2$.

Let $T(L)$ be any $C$-realization. In order to prove that $V_C(T(L))$ is a module one needs to show that for any $T(R) \in B_C(T(L))$

(i) $[h_i, h_j]T(R) = 0$ for $h_i, h_j \in b$

(ii) $[e_i, f_j]T(R) = \delta_{ij}h_i T(R)$,

(iii) $[h, c_i]T(R) = \alpha_i(h)c_i T(R)$, $[h, f_j]T(R) = -\alpha_j(h)f_j T(R)$,

(iv) $[e_i, [e_i, e_j]]T(R) = [f_i, [f_i, f_j]]T(R) = 0$ for $|i - j| = 1$,

(v) $[e_i, e_j]T(R) = [f_i, f_j]T(R) = 0$ for $|i - j| > 1$.

First we show that $[e_i, [e_i, e_j]]T(R) = 0$ for $|i - j| = 1$.

\begin{equation}
[e_i, [e_i, e_j]]T(R) = \sum_{r,s,t} \Phi(R, \delta^{jr}, \delta^{is}) e_{jr}(R)e_{is}(R + \delta^{jr} + \delta^{is}) T(R + \delta^{jr} + \delta^{is} + \delta^{it})
\end{equation}

Now we consider the coefficients by nonzero tableaux $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$.

(i) Let $s = t$.

(a) Suppose there is no relation between $(i, s)$ and $(j, r)$. By Lemma 4.21 $\Phi(R, \delta^{jr}, \delta^{is}) = \Phi(R, \delta^{is}, \delta^{jr}) = 1$. Then the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is the limit of the coefficient of $T(v + \delta^{jr} + 2\delta^{is})$ when $v \rightarrow R$ (here $T(v)$ again is a tableau with variable entries). Thus the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is zero.

(b) Suppose there exists a relation between $(i, s)$ and $(j, r)$. Without loss of generality we assume that this relation is $C' = \{(i, s) \geq (j, r)\}$. Let $T(v')$ be the tableau with $v'_s = l_{v'_{jr}} = 1_{v_{jr}}$ if $(s', t') = (i, s)$ or $(j, r), v$ and variable entries otherwise. Then $T(v')$ is a $C'$-realization and $V_{C'}(T(v'))$ is a module, that is for arbitrary generic (no integral relations) values of free variables in $v'$. Let $z^{(1)}, z^{(2)} \in \{\delta^{jt}, \delta^{is}\}$. Then $\Phi(R, z^{(1)}, z^{(2)}) = \Phi(v, z^{(1)}, z^{(2)})$ where $z^{(1)} = z^{(2)}$ only if $z^{(1)} = z^{(2)} = \delta^{is}$. Therefore the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is the limit of the coefficient of $T(v + \delta^{jr} + 2\delta^{is})$ when $v \rightarrow R$, hence, it is zero.

(ii) Suppose $s \neq t$. Then there is no relation between $(i, s)$ and $(i, t)$.

(a) Suppose there is no relation between $(j, r)$ and $(i, s)$ or between $(j, r)$ and $(i, t)$. Then the value of function $\Phi$ that appears along with $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ is $1$ by Lemma 4.21. Thus the coefficient of $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ is zero similarly to (a) in (i).
(b) Suppose there is a relation between \((j, r)\) and one of \(\{(i, s), (i, t)\}\). Similarly to (b) in (i), one has that the coefficient of \(T(R + \delta^{jr} + \delta^{is} + \delta^{it})\) is zero.

(c) Suppose there exist relations between \((j, r)\) and both \(\{(i, s), (i, t)\}\). In this case \((j, r), (i, s), (i, t)\) are in the same indecomposable set. If \(r_{is} - r_{it} = 1\) then \(r_{jr} = r_{it}\) and there exists \(r'\) such that \(\{(i, s) \geq (i-1, r') > (i, t)\}\) \(\subseteq C\) and \(r_{i-1, r'} = r_{is}\). It contradicts with \(T(R + \delta^{jr} + \delta^{is} + \delta^{it})\) nonzero. Therefore \(r_{is} - r_{it} > 1\). Then \(r_{is} - r_{jr} > 1\) or \(r_{jr} - r_{it} > 1\). Without loss of generality we assume that \(r_{jr} - r_{it} > 1\). Let \(C' = \{(i, s) \geq (j, r)\}\) and \(T(v')\) the tableau with \(v'_{s't'} = l_{s't'}\) if \((s', t') = (i, s)\) or \((j, r)\) and variable entries otherwise. Then \(T(v')\) is a \(C'\)-realization and \(V_{C'}(T(v'))\) is a module. Let \(z(1), z(2) \in \{\delta^{jr}, \delta^{is}, \delta^{it}\}\). One has that \(\Phi(R, z^{(1)}, z^{(2)}) = \Phi(v, z^{(1)}, z^{(2)})\) whenever \(z^{(1)} \neq z^{(2)}\). Therefore the coefficient of \(T(R + \delta^{jr} + 2\delta^{is})\) is the limit of the coefficient of \(T(v + \delta^{jr} + 2\delta^{is})\) when \(v \to R\), which is zero.

The proof of (i) (iii) (iv) and \([f_i, [f_i, f_j]]T(R) = 0\) \(\langle i, j \rangle = 1\), is similar.

In the following we show that \([e_i, f_j]T(R) = \delta_{ij}h_iT(R)\). We have

\[
[e_i, f_j]T(R) = \sum_{r=1}^{j} \sum_{s=1}^{i} \Phi(R, -\delta^{jr})f_jr(R)e_{is}(R + \delta^{is})T(R - \delta^{jr} + \delta^{is}) \\
- \sum_{r=1}^{j} \sum_{s=1}^{i} \Phi(R, \delta^{is})e_{is}(R)f_jr(R + \delta^{is})T(R - \delta^{jr} + \delta^{is}).
\]

(19)

Now we consider the coefficients of nonzero tableaux \(T(L - \delta^{jr} + \delta^{is})\). If \((i, r) \neq (j, s)\) then the coefficient of \(T(L - \delta^{jr} + \delta^{is})\) is zero similarly to the above case and, hence, \([e_i, f_j]T(R) = 0\) if \(i \neq j\).

Suppose \(i = j = k\). The coefficient of \(T(R - \delta^{jr} + \delta^{is})\) is zero if \(r \neq s\).

By Corollary 4.20, the coefficient of \(T(R)\) is

\[
\lim_{v \to R} \left( \sum_{r=1}^{k} \sum_{s=1}^{k} f_{kr}(v)e_{ks}(v + \delta^{kr}) - \sum_{r=1}^{k} \sum_{s=1}^{k} e_{ks}(v)f_{kr}(v + \delta^{ks}) \right) \\
= \lim_{v \to R} h_k(v) = h_k(R).
\]

Hence \([e_i, f_j]T(R) = \delta_{ij}h_iT(R)\) and \(C\) is admissible.

Conversely, assume that \(C\) is admissible. We will show that \(C\) is a union of disconnected sets from \(F\). Suppose first that \(C\) is indecomposable. If it does not satisfy condition (13), then one can choose a tableau that satisfies \(C\) and \(l_{ki} - l_{kj} = 1\) and \(l_{ks} - l_{kt} \neq 1\) if \((s, t) \neq (i, j)\). Thus \([e_k, f_k]T(L) = h_kT(L)\) and \(V_C(T(L))\) is a module. Suppose \(C = C_1 \cup \cdots \cup C_m\). Without loss of generality we may assume that \(C_1\) does not satisfy condition (13). Then one can choose a tableau that satisfies \(C\) and \(l_{ki} - l_{kj} = 1\) for some \((k, i), (k, j) \in \Omega(C_1)\) and \(l_{ks} - l_{kt} \neq 1\) if \((s, t) \neq (i, j)\). We see that \([e_k, f_k]T(L) \neq h_kT(L)\) and \(V_C(T(L))\) is a module. \(\square\)

4.7. Constructing admissible sets. Theorem 4.22 provides a combinatorial description of admissible sets of relations leading to a construction of new Gelfand-Tsetlin modules. Now we describe an effective method of constructing of admissible subsets of \(R\) which we call relations removal method (RR-method for short).

Let \(C\) be any admissible subset of \(R\) and \(T(L)\) a \(C\)-realization. Fix \((k, i) \in \Omega(C)\) and suppose that \(T(L + m\delta^{ki})\) is a \(C\)-realization for infinitely many choices of \(m \in \mathbb{Z}\). Denote by \(\tilde{C}_k\) the set of relations obtained from \(C\) by removing all relations that
involves \((k, i)\). We say that \(\tilde{\mathcal{C}} \subseteq \mathcal{C}\) is obtained from \(\mathcal{C}\) by the RR-method if it is obtained by a sequence of such removing of relations for different indices.

**Example 4.23.** If \(x - y \in \mathbb{Z}_{\geq 0}\) denote

\[
\begin{array}{c}
    x \\
    \downarrow \\
    y
\end{array}
\]

If \(x - y \in \mathbb{Z}_{> 0}\) denote

\[
\begin{array}{c}
    x \\
    \downarrow \\
    y
\end{array}
\]

Then all relations satisfied by a standard tableau of height 3 can be expressed as in diagram (i). If we fix \(a, b, x, y, z\) in the diagram, \(c\) has infinitely many choices to satisfy the relations. So we obtain the diagram (ii) after remove the relation between \(y\) and \(c\). If we fix \(a, b, x, z\) in the diagram (ii), \(y\) has infinitely many choices. So we obtain the diagram (iii) after remove the relations between \(y\) and \(b\), and \(y\) and \(z\). Finally, if \(b, x, z\) are fixed in the diagram (iii) then \(a\) has infinitely many choices and we can remove the relation between \(a\) and \(x\) obtaining the diagram (iv).

The following diagrams show all possible ways to remove the relations. In these cases we are assuming that the only arrows from the position \((i, j)\) are the ones shown in the diagram.
Theorem 4.24. Let $C_1$ be any admissible subset of $\mathcal{R} \cup \mathcal{R}^0$. If $C_2$ is obtained from $C_1$ by the RR-method then $C_2$ is admissible.

Proof. Suppose $C_2$ is obtained from $C_1$ by removing the relations involving $(i,j)$. Let $T(L)$ be a $C_2$-realization. To show that $V_{C_2}(T(L))$ is a module it is sufficient to prove that for any $T(L+z) \in \mathcal{B}_{C_2}(T(L))$, $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ and any relation $g_1 = g_2$ in $U(\mathfrak{g}_{l_n})$ we have $g_1 T(L+z) = g_2 T(L+z)$. Consider a tableau $T(L'+z+m\delta^{ij})$ with sufficiently large integer $m$ that satisfies relations involving $(i,j)$, where $l'_{st} = l_{st}$ if $(s,t) \neq (i,j)$. Then $T(L'+z+m\delta^{ij})$ is a $C_1$-realization and $V_{C_1}(T(L'+z+m\delta^{ij}))$ is a module. We have

$$g_1 T(L'+z+m\delta^{ij}) = \sum_{w_1 \in A_1} g_{1w_1} (L'+z+m\delta^{ij}) T(L'+z+w_1),$$

$$g_2 T(L'+z+m\delta^{ij}) = \sum_{w_2 \in A_2} g_{2w_2} (L'+z+m\delta^{ij}) T(L'+z+w_2),$$

where $A_1$ and $A_2$ are sets such that $T(L'+z+w_1)$ and $T(L'+z+w_2)$ are nonzero in the respective formulas. Since $m$ is large one has that $T(L+z+w_1) = 0$ in $V_{C_2}(T(L))$ if and only if $T(L'+z+w_1) = 0$ in $V_{C_1}(T(L'))$. Similarly, $T(L+z+w_2) = 0$ in $V_{C_2}(T(L))$ if and only if $T(L'+z+w_2) = 0$ in $V_{C_1}(T(L'))$. Thus,

$$g_1 T(L+z) = \sum_{w_1 \in A_1} g_{1w_1} (L+z) T(L+z+w_1),$$

$$g_2 T(L+z) = \sum_{w_2 \in A_2} g_{2w_2} (L+z) T(L+z+w_2).$$

Since $V_{C_1}(T(L'+m\delta^{ij}+z))$ is a module for infinitely many values of $m$ and $g_{iw_1}(L+z)$, $i = 1, 2$ are rational functions in one variable, we conclude that $A_1 = A_2$ and $g_{1w_1}(L+z) = g_{2w_2}(L+z)$ for all $w$ and $V_{C_2}(T(L))$ is a module.

Since $\mathcal{S} \cup \mathcal{S}^0$ is admissible we immediately have from Theorem 4.24.

Corollary 4.25. If $C$ is a subset of $\mathcal{S}$ obtained by the RR-method then $C$ is admissible.

Denote by $\mathfrak{F}_1$ the set of all indecomposable subsets of $\mathcal{S}$ obtained by the RR-method. Then any $C \in \mathfrak{F}_1$ is admissible. Let $\mathfrak{F}_2$ the collection of $\{\sigma(C) \mid C \in \mathfrak{F}_1, \sigma \in G\}$.

Corollary 4.26. Any $C \in \mathfrak{F}_2$ is admissible.

Proof. The Gelfand-Tsetlin formulas are invariant under the action of elements of the group $G$.

Denote by $\mathfrak{F}_3$ the union of disconnected sets in $\mathfrak{F}_2$.

Corollary 4.27. Any set $C$ in $\mathfrak{F}_3$ is admissible.
Proof. It follows from Corollary 4.26 and Theorem 4.22.

**Proposition 4.28.** Set \( k \leq n-1 \). The following sets of relations are not admissible.

\[
\begin{array}{ccc}
l_{k+1,j} & l_{k,j+1} & l_{k,j} \\
l_{k,j-1} & l_{k-1,j} & l_{k,j}
\end{array}
\]

(i) \quad (ii)

**Proof.** Suppose \( C \) is the set of relations of the diagram (i). If \( k \leq n-2 \) then consider any \( C \)-realization \( T(L) \). If \( k = n-1 \) then let \( T(L) \) be any \( C \)-realization such that \( l_{nj} - l_{ni} \neq 0 \) for all \( i \neq j \). Let \( T(R) \) be a tableau in \( V_C(T(L)) \) such that \( r_{kj,l} = r_{k+1,j-1} - 1, r_{kj} = r_{k+1,j} \). By direct computation one has that \( e_k f_k T(R) - f_k e_k T(R) \neq h_k T(R) \). Thus \( V_C(T(L)) \) is not a module. Similarly, suppose \( C_2 \) is the set of relations of the diagram (ii). If \( T(L) \) is any \( C \)-realization then \( V_{C_2}(T(L)) \) is not a module.

**Example 4.29.** Let \( C \) be the set of relations defined by the following diagram.

\[
\begin{array}{ccc}
(3,1) & (3,2) \\
(2,1) & (2,2) \\
(1,1)
\end{array}
\]

If \( T(L) \) is a tableau satisfying \( C \) then by Theorem 4.22 and by Propositions 4.28 one has:

(i) \( V_C(T(L)) \) is a module if \( l_{31} \geq l_{32} + 1 \).

(ii) \( V_C(T(L)) \) is not a module if \( l_{31} < l_{32} + 1 \).

**Remark 4.30.** Let \( C_1 \) be the set of relations of the diagram (i), \( C_2 \) the set of relations of the diagram (ii). Both \( C_1 \) and \( C_2 \) are noncritical. Proposition 4.28 shows that these sets are not admissible. For any \( C_2 \)-realization \( T(L) \), \( V_{C_2}(T(L)) \) is not a module. However, there exist a \( C_1 \)-realization \( T(L) \) such that \( V_{C_1}(T(L)) \) is a module. Indeed, let \( n = 3 \) and fix any complex numbers \( a, b \). The following tableau is \( C_1 \)-realization and \( V_{C_1}(T(L)) \) is a module.

\[
T(L):= \begin{bmatrix} 3 & 3 & a \\ 4 & 2 \\ b \end{bmatrix}
\]

The reason is that \( T(L) \) is \( C \)-realization for another admissible relation, namely \( C = \{(3,2) \geq (2,2), (2,1) > (3,1)\} \).

5. Admissible Gelfand-Tsetlin modules

From now on we will assume that \( C \) is an admissible subset of \( R \cup R^0 \) and consider a \( \mathfrak{gl}_n \)-module \( V_C(T(L)) \). It is endowed with the action of \( \mathfrak{gl}_n \) by the Gelfand-Tsetlin formulas.
We will analyze the action of the Gelfand-Tsetlin subalgebra $\Gamma$ on modules $V_C(T(L))$. First we show that the action of $\Gamma$ is preserved by the RR-method. namely, we have

**Lemma 5.1.** Let $C_1$ and $C_2$ be admissible sets such that $C_2$ is obtained from $C_1$ by the RR-method. Then $\Gamma$ acts on $V_{C_1}(T(L))$ by (10) for any $C_1$-realization $T(L)$ if and only the it acts on $V_{C_2}(T(L))$ by (10) for any $C_2$-realization $T(L)$.

**Proof.** Suppose $C_2$ is obtained from $C_1$ by removing the relations involving $(i,j)$ and the action of $\Gamma$ on $V_{C_2}(T(L))$ is given by (10). Let $T(L') = T(L+z)$, $z \in \mathbb{Z}^{(n+1)}$, be a tableau satisfying $C_2$ such that $z_{rs} = 0$ if $(r,s) \neq (i,j)$ and with negative $z_{ij}$ with sufficiently large $|z_{ij}|$. Then $T(L')$ is a $C_2$-realization. Then for all possible $m$ and $k$ the equality $c_{mk}(T(L')) = \gamma_{mk}(l')T(L')$ holds for infinitely many choices of $z_{ij}$. Since $|z_{ij}|$ is sufficiently large we have that $T(L+z') = 0$ in $V_{C_1}(T(L))$ if and only if $T(L'+z') = 0$ in $V_{C_2}(T(L))$ for suitable $z' \in \mathbb{Z}^{(n+1)}$ with $|z'_{ij}| \leq n$. Considering $\gamma_{mk}(l')$ as a rational function in $z_{ij}$ we conclude that $c_{mk}(T(L)) = \gamma_{mk}(l')T(L)$. This action extends to the whole $V_{C_1}(T(L))$. The converse statement can be proved similarly. \qed

We have

**Theorem 5.2.** For any admissible $C$ the module $V_C(T(L))$ is a Gelfand-Tsetlin module with diagonalisable action of the generators of the Gelfand-Tsetlin subalgebra given by the formula (10).

**Proof.** If $C = \emptyset$ then $V_C(T(L))$ is generic Gelfand-Tsetlin module. The action of $\Gamma$ on $V_C(T(L))$ is given by (10) by Lemma 5.1 since $\emptyset$ is obtained from $S$ by finitely many steps. Again, if $C$ is an arbitrary admissible set then applying the RR-method we get $\emptyset$ after finitely many steps. Lemma 5.1 implies the statement. \qed

We call $V_C(T(L))$ admissible Gelfand-Tsetlin module associated with the admissible set of relations $C$. Note that $V_C(T(L))$ is infinite dimensional if $C \neq S \cup S^0$. We have immediately from the construction

**Proposition 5.3.** Admissible Gelfand-Tsetlin module $V_C(T(L))$ is irreducible if and only if $C$ is the maximal set of relations satisfied by $T(L)$.

**Definition 5.4.** We will say that a tableau $T(L)$ satisfies the FRZ-condition if it is a $C$-realization of some admissible set $C \subset R \cup R^0$ and it is not a $C'$-realization for any larger admissible set $C'$.

Therefore, each admissible set $C$ defines infinitely many tableaux satisfying the FRZ-condition, each of which gives rise to an irreducible admissible Gelfand-Tsetlin module. These irreducible modules form infinitely many isomorphism classes.

**Theorem 5.5.** For any $m \in \text{Specm} \Gamma$ from the Gelfand-Tsetlin support of $V_C(T(L))$, the Gelfand-Tsetlin multiplicity of $m$ is one.

**Proof.** The action of $\Gamma$ is given by the formulas (10), and hence determined by the values of symmetric polynomials on the entries of the rows of the tableaux. Given two Gelfand-Tsetlin tableaux $T(L)$ and $T(R)$, we have $c_{rs}(T(L)) = c_{rs}(T(R))$ for any $1 \leq s \leq r \leq n$ if and only if $L = \sigma(R)$ for some $\sigma \in G$. In particular, $T(R) \in \mathcal{B}_C(T(L))$ and $L = \sigma(R)$ for some $\sigma \neq id$ imply $T(Q) \in \mathcal{B}_C(T(L))$ with $q_{ki} = q_{kj}$ for some $1 \leq j \neq i \leq n-1$. \qed
Theorem 4.22 Proposition 5.3 and Theorem 5.5 imply the statements of the Theorem II. The uniqueness of module $V(L)$ follows from its explicit construction.

We consider particular cases of generic and highest weight modules. We use Theorem 4.24 to show that generic Gelfand-Tsetlin modules are admissible and obtain their explicit construction.

Corollary 5.6. Let $T(L)$ be a generic Gelfand-Tsetlin tableau of height $n$. Denote by $B(T(L))$ the set of all Gelfand-Tsetlin tableaux $T(R)$ satisfying $r_{nj} = l_{nj}$, $r_{ij} - r_{ij} - l_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n - 1$. Then $V(T(L)) = \text{span} B(T(L))$ has a structure of a $\mathfrak{gl}_n$-module with the action of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.

Proof. Applying the RR-method to $S$, after finitely many steps we can remove all the relations in $S$. Theorem 4.24 implies the statement. □

The following proposition gives a family of highest weight modules that can be realized as $V_C(T(L))$ for some admissible set of relations $C$.

Proposition 5.7. Set $\lambda = (\lambda_1, \ldots, \lambda_n)$. The irreducible highest weight module $L(\lambda)$ is admissible Gelfand-Tsetlin module if $\lambda_i - \lambda_j \not\in \mathbb{Z}$ or $\lambda_i - \lambda_j > i - j$ for any $1 \leq i < j \leq n - 1$.

Proof. Let $i_{11}, i_{12}, \ldots, i_{1n_1}, i_{21}, i_{22}, \ldots, i_{2n_2}, \ldots, i_{m1}, i_{m2}, \ldots, i_{mn_m}$ be a rearrangement of $\{1, 2, \ldots, n\}$ such that $\lambda_{i_{rs}} - \lambda_{i_{rt}} > i_{rs} - i_{rt}$ for any $1 \leq s < t < n_r$ and $\lambda_{i_{rs}} - \lambda_{i_{rt}} \not\in \mathbb{Z}$ for any $r \neq j$.

Let $C = \{(k + 1, i_{rs}) \geq (k, i_{rs}) \geq (k + 1, i_{rs+1}) \mid 1 \leq k \leq n - 1\}$. If $T(L)$ is the Gelfand-Tsetlin tableau such that $l_{ni} = \lambda_{ni} - i + 1$ for $1 \leq i \leq n$ and $l_{ki} = l_{ni}$ for $1 \leq i \leq n - 1$, then $T(L)$ is a $C$-realization and $V_C(T(L))$ is isomorphic to the irreducible highest weight module $L(\lambda)$. □

6. Tableaux Gelfand-Tsetlin modules

In this section we discuss the place of admissible Gelfand-Tsetlin modules among the all Gelfand-Tsetlin modules having a realization by Gelfand-Tsetlin formulas.

We start with the following natural question: if $C$ is not an admissible set, is there a tableau $T(L)$ satisfying $C$ and such that $V_C(T(L))$ is a $\mathfrak{gl}_n$ module? The answer is positive. Here is an example.

Example 6.1. Let $C$ be the set of the following relations:

$(4, 1) \geq (4, 2) \geq (4, 3) \geq (4, 4)$

$(4, 2) \geq (3, 1), (3, 3) > (4, 3)$

$(k + 1, i) \geq (k, i), (k, i) > (k + 1, i + 1), 1 \leq k \leq 2$.

Let $T(L)$ be the following tableau

\[
\begin{array}{cccc}
4 & 3 & 0 & -1 \\
3 & 2 & 1 & \\
3 & 2 & \\
3 & & & \\
\end{array}
\]

Then $V_C(T(L))$ is a module but the action of $\Gamma$ is not given by $(10)$.\]
This example suggests the following definition.

**Definition 6.2.** We say that a Gelfand-Tsetlin $\mathfrak{gl}_n$-module $V$ is a tableaux module if it satisfies the following conditions:

(i) $V$ has a basis consisting of noncritical tableaux.

(ii) The action of $\mathfrak{gl}_n$ on $V$ is given by the Gelfand-Tsetlin formulas.

(iii) All Gelfand-Tsetlin multiplicities of $V$ are 1.

(iv) The action of $\Gamma$ on $V$ is given by the formula \([10]\).

Hence, the module $V_C(T(L))$ from the example above is not a tableaux module. As we showed in the previous section any admissible Gelfand-Tsetlin module is a tableaux module. We believe that the converse also holds, namely we have

**Conjecture 6.3.** If $V$ is a tableaux Gelfand-Tsetlin $\mathfrak{gl}_n$-module then $V$ is isomorphic to some $V_C(T(L))$ with admissible $C$.

Note that this conjecture together with \([16], \text{Lemma 1}\) imply Conjecture 1 stated in the introduction. In the rest of this section we justify the conjecture and show it in some particular cases.

**Lemma 6.4.** Let $\mathcal{C}$ be an admissible set, $T(L)$ a tableau satisfying $\mathcal{C}$. If there is no tableau $T(R \pm \delta^{n-1}, r)$ satisfying $\mathcal{C}$ for any $1 \leq r \leq n - 1$ where $r_{ij} = l_{ij}$ for $1 \leq j \leq i$, $n - 1 \leq i \leq n$, then $\mathcal{C} = S$ and $l_{ni} - l_{n,i+1} = 1$ for $1 \leq i \leq n - 1$.

**Proof.** Suppose the entries in the $(n-1)$-th row are contained in $m$ disconnected subsets of $\mathcal{C}$. Let $\mathcal{C}_1$ be a subset of $\mathcal{C}$. Assume $(n-1,j), 1 \leq j \leq j_1$ are contained in $\mathcal{C}_1$. By condition \([13]\), there are at least $j_1 - 1$ entries of the $n$-th row in $\mathfrak{W}(\mathcal{C}_1)$. Moreover, if $(n,s) \geq (n-1,1)$ (respectively $(n-1,j_1) > (n,s)$) is not in $\mathfrak{W}(\mathcal{C}_1)$ then $T(R \pm \delta^{n-1}, 1)$ (respectively $T(R \pm \delta^{n-1}, j_1)$) satisfies $\mathcal{C}$ which is contradiction. Hence, $\mathfrak{W}(\mathcal{C}_1)$ contains at least $j_1 + 1$ elements of the $n$-th row. Thus, all the entries of the $(n-1)$-th row are contained in the same disconnected subset of $\mathcal{C}$ and the relations between the $(n-1)$-th row and the $n$-th row are as follows:

$$
\{(n,i) \geq (n-1,i) > (n,i+1) \mid 1 \leq i \leq n-1\}.
$$

Therefore $\mathcal{C} = S$ by \([13]\). \(\square\)

**Proposition 6.5.** Let $n = 4$, $\mathcal{C}$ a non-admissible set, $T(L)$ a tableau satisfying $\mathcal{C}$. If $\mathcal{C}$ is the maximal relation set satisfied by $T(L)$, $V_C(T(L))$ is a $\mathfrak{gl}_n$-module and the action of $\mathfrak{gl}_n$ is given by the Gelfand-Tsetlin formula \([4]\), then $V_C(T(L))$ is not admissible, that is the action of $\Gamma$ is not given by \([10]\).

**Proof.** Suppose $k$ is the minimal such that there exist $(k,i)$, $(k,j)$ that do not satisfy the condition \([13]\). Then for any fixed top row there exists a tableau $T(L)$ such that $l_{ki} - l_{kj} = 1$ and $\#\{l_{k+1,i'}, l_{k-1,j'} \mid l_{k+1,i'} = l_{kj}, l_{k-1,j'} = l_{ki}\} = 1$. Now we consider $\mathfrak{gl}_{k+1}$-module generated by $T(L)$. We have

$$
[e_k, f_k]T(R) = \sum_{r,s=1}^{k} \Phi(L, -\delta^{kr})f_{kr}(l)e_{ks}(l + \delta^{kr})T(L - \delta^{kr} + \delta^{ks})
$$

\begin{equation}
- \sum_{r,s=1}^{k} \Phi(L, \delta^{ks})e_{ks}(l)f_{kr}(l + \delta^{ks})T(L - \delta^{kr} + \delta^{ks}).
\end{equation}
The same formula holds for the tableau \( T(v) \) with distinct variable entries (considered as a generic tableau). Then the coefficient of \( T(L) \) in (20) is the same as the coefficient of \( T(L) \) in \( h_k T(L) \) if and only if

\[
\lim_{v \to l} \left( \sum_{\phi(L, -v) = 0} f_k(v) e_k(v + \delta v) - \sum_{\phi(L, v) = 0} e_k(v) f_k(v + \delta v) \right) = 0.
\]

By Lemma 4.20 (i) and (ii) one has

\[
\lim_{v \to l} \left( \sum_{\phi(L, -v) = 0} f_k(v) e_k(v + \delta v) - \sum_{\phi(L, v) = 0} e_k(v) f_k(v + \delta v) \right) = \lim_{v \to l} \left( \sum f_k(v) e_k(v + \delta v) - e_k(v) f_k(v + \delta v) \right),
\]

where the sum runs over all pairs \((i, j)\) such that \( l_{ki} - l_{kj} = 1 \) and \((k, i), (k, j)\) do not satisfy condition (13). If there is only one such pair \((i, j)\) then by direct computation we obtain \( \lim_{v \to l} f_k(v) e_k(v + \delta v) - e_k(v) f_k(v + \delta v) \neq 0 \). If there are two such pairs then there exists \( T(Q) = T(R \pm \delta k-1,s) \) satisfying \( C \) where \( r_{ij} = l_{ij} \) for \( 1 \leq j \leq i, k - 1 \leq i \leq k + 1 \). For every such pair \((k, i), (k, j)\),

\[
\lim_{v \to l} f_k(v) e_k(v + \delta v) - e_k(v) f_k(v + \delta v) \]

can be written as \((k-1,s-a)b\), where \( a = l_{kj} \). Since \( V_c(T(L)) \) is a module, one has

\[
(l_{k-1,s} - a_1) b_1 + (l_{k-1,s} - a_2) b_2 = 0 \quad \text{and} \quad (l_{k-1,s} + 1 - a_1) b_1 + (l_{k-1,s} + 1 - a_2) b_2 = 0.
\]

Then \( b_1 = b_2 = 0, a_1 b_1 + a_2 b_2 = 0 \) and \( a_1 \neq a_2 \). Thus \( b_1 = b_2 = 0 \) which is a contradiction. Hence, there is no tableau \( T(Q) = T(R \pm \delta k-1,s) \) satisfying \( C \) where \( r_{ij} = l_{ij} \) for \( 1 \leq j \leq i, k - 1 \leq i \leq k + 1 \). By Lemma 6.4 \( U(\mathfrak{g}_{k+1})T(L) \) is one dimensional with unique tableau \( T(L) \). Let \( T(L') \) be a tableau with \( l_{st}^{t_{st}} = l_{st} \) for \( 1 \leq t \leq s \leq k, l_{k+1,t} = l_{k,t} \) for \( 1 \leq t \leq k \) and \( l_{k+1,k+1} = l_{k,k} + 1 \). One has \( c_{k+1,t}T(L') = g_{k+1,l}(L')T(L), t = 1, \ldots, k + 1 \). We see that the action of \( \Gamma \) is different from formula (10). Thus \( V_c(T(L')) \) is not an admissible module. \( \square \)

**Remark 6.6.**

(i) Let \( n = 3 \), \( C \) a maximal non admissible set of relations, satisfied by \( T(L) \). Then \( V_c(T(L)) \) is not a \( \mathfrak{g}_n \) module.

(ii) \( V = \text{span}T(L) = (3,0|2) \) is a one dimensional tableau module. Its basis is not equal to any set \( B_C(T(L)) \).

(iii) Let \( n \leq 4 \), \( V \) a tableau \( \mathfrak{g}_n \)-module without condition (iv). If \( V \) is not isomorphic to any \( V_c(T(L)) \) then \( V \) is not admissible.

We immediately have from Proposition 6.5

**Corollary 6.7.** Let \( n \leq 4 \), \( V \) a tableau Gelfand-Tsetlin \( \mathfrak{g}_n \)-module. Then \( V \) is isomorphic to \( V_c(T(L)) \) for some admissible set of relations \( C \) and some tableau \( T(L) \) satisfying \( C \).

### 7. Gelfand-Graev continuation

In this section we prove necessary and sufficient condition for the Gelfand and Graev’s continuation.

For sake of convenience we will use our notation to describe Gelfand and Graev’s continuations. In [8] the standard labelling of tableaux given in [9] is slightly
modified and the action of generating elements of the Lie algebra is given on this new basis. To each \( k = 1, 2, \ldots, n - 1 \) we assign a pair of integers \( \{i_k, i'_k\} \) where \( i_k \in \{0, 1, \ldots, k\} \), \( i'_k \in \{1, 2, \ldots, k + 1\} \), and \( i_k < i'_k \). For each such set of indices one defines a Hilbert space \( H\{i_k, i'_k\} \) having an orthonormal basis labeled by the set of all possible tableaux of integers where the top row is fixed and the other components satisfy the following set of inequalities:

\[
\begin{align*}
(22) & \quad l_{n,i} > l_{n,i+1} & 1 \leq i \leq n - 1 \\
(23) & \quad l_{k,j} < l_{k,j+1} & j < k \leq n \\
(24) & \quad l_{k+1,j-1} \geq l_{kj} > l_{k+1,j} & j \leq i_k \\
(25) & \quad l_{k+1,j} \geq l_{kj} > l_{k+1,j+1} & i_k \leq j \leq i'_k \\
(26) & \quad l_{k+1,j+1} \geq l_{kj} > l_{k+1,j+2} & j \geq i'_k.
\end{align*}
\]

By the **GG-condition** we will call the following set of relations:

\[
\begin{align*}
\{(n,i) > (n,i+1)|1 \leq i \leq n-1\} \\
\cup\{(k+1,j-1) \geq (k,j) > (k+1,j) & | j \leq i_k\} \\
\cup\{(k+1,j) \geq (k,j) > (k+1,j+1) & | i_k \leq j \leq i'_k\} \\
\cup\{(k+1,j+1) \geq (k,j) > (k+1,j+2) & | i'_k \leq j\}.
\end{align*}
\]

Lemire and Patera [15] gave counterexamples and showed that for certain sets of indices the Gelfand-Tsetlin formulas do not define a representation of the algebra \( \mathfrak{gl}_n \). In fact, they claimed (though without proof) that a necessary condition to have a representation of \( \mathfrak{gl}_n \) on \( H\{i_k, i'_k\} \) is that for each \( k = 2, 3, \ldots, n - 1 \) one has

\[
i_{k-1}, i'_{k-1} \in \{0, i_k, i'_k - 1, k\}.
\]

The GG-condition together with this restriction we will call the **LP-condition**. In [13] it was given an example of a tableau for \( \mathfrak{gl}_3 \) which does not satisfy the LP-condition and does not generate a module. This is not sufficient to conclude that \( H\{i_k, i'_k\} \) is not a module if it contains a basis tableau which does not satisfies the LP-condition, since \( H\{i_k, i'_k\} \) not only depends on the choice of pairs \( i_k, i'_k \) but also on the top row of the tableau. Lemire and Patera showed that the LP-condition is sufficient to have a module structure on \( H\{i_k, i'_k\} \) for \( \mathfrak{gl}_3 \) and in some cases for \( \mathfrak{gl}_4 \).

The following is clear

**Lemma 7.1.** Fix \( i_k, i'_k \). If \( r_{ij} \geq k, 1 \leq j \leq i \) satisfy the GG-condition then there exists a tableau in \( H\{i_k, i'_k\} \) that satisfies the GG-condition and \( i_{ij} = r_{ij} \) \( i \geq k \)

\[
1 \leq j \leq i.
\]

**Proof of Theorem I.** Note that the relation set given by the GG-condition is a noncritical set. Suppose that LP-condition is satisfied by a tableau \( T(L) \). Consider pairs \( i_k, i'_k \) when \( k \) ranges from \( n - 1 \) to \( 1 \). For each \( k \), if the choice of \( i_{k-1}, i'_{k-1} \) satisfies the LP-condition then every pair \( (k,j), (k,j+1) \) satisfies the condition [13]. Thus \( T(L) \) satisfies the admissible set of relations \( C, V_C(T(L)) \simeq H\{i_k, i'_k\} \) is a module.

Now we show that the LP-condition is necessary. Let \( k \) be the maximal such that \( \{i_{k-1}, i'_{k-1}\} \not\subset \{0, i_k, i'_k - 1, k\} \).
(i) Assume that one of \( \{i_{k-1}, i'_{k-1}\} \) is not in \( \{0, i_k, i'_k - 1, k\} \). Without loss of generality we assume that \( i_{k-1} \notin \{0, i_k, i'_k - 1, k\} \). By Lemma 7.1 for any fixed top row there exists a tableau that satisfies the GG-condition in \( H(i_k, i'_k) \) and \( l_{k, i_k-1} - l_{k, i_{k-1}} = 1, l_{k-1,j} \neq l_{k,i_k} \) for any \( j \). The pair \( (l_{k, i_k-1} - l_{k, i_{k-1}}, l_{k, i_{k-1}}) \) is the only pair in the \( k \)-th row such that \( l_{k, i_k-1} - l_{k, i_{k-1}} = 1 \)

\[
\#\{ (l_{k+1,i'}, l_{k-1,j'}) \mid l_{k+1,i'} = l_{k, i_{k-1}} - 1, l_{k-1,j'} = l_{k, i_{k-1}} \} = 1.
\]

By direct computation one has

\[
\lim_{v \to t} \left( \sum_{\Phi(L, -\delta^{iv})=0} f_{kr}(v) e_{kr}(v + \delta^{iv}) - \sum_{\Phi(L, \delta^{iv})=0} e_{kr}(v) f_{kr}(v + \delta^{kr}) \right) = \lim_{v \to t} \left( f_{kr}(v_{k, i_k - 1}(v) e_{kr}(v_{k, i_k - 1}(v + \delta^{i, i_{k-1}})) - e_{kr}(v_{k, i_k - 1}(v)) f_{kr}(v_{k, i_k - 1}(v + \delta^{k, i_{k-1}})) \right)
\]

which is nonzero. Thus \( H(i_k, i'_k) \) is not a module.

(ii) Suppose both of \( \{i_{k-1}, i'_{k-1}\} \) are not in \( \{0, i_k, i'_k - 1, k\} \). By Lemma 7.1 for any fixed top row there exists a tableau that satisfies the GG-condition in \( H(i_k, i'_k) \) and \( l_{k, i_k-1} - l_{k, i_{k-1}} = 1, l_{k, i'_k - 1} - l_{k, i'_{k-1} + 2} = 1 \) and \( l_{k-1,j} \neq l_{k, i_{k-1}}, l_{k-1,j} \neq l_{k, i'_{k-1} + 2} \) for any \( j \). By Lemma 7.20 (i) and (ii), one has

\[
\lim_{v \to t} \left( \sum_{\Phi(L, -\delta^{iv})=0} f_{kr}(v) e_{kr}(v + \delta^{iv}) - \sum_{\Phi(L, \delta^{iv})=0} e_{kr}(v) f_{kr}(v + \delta^{kr}) \right) = \lim_{v \to t} \left( \sum_{r=i_{k-1} + 1, i'_{k-1} + 2} f_{kr}(v) e_{kr}(v + \delta^{iv}) - \sum_{r=i_{k-1}, i'_{k-1} + 1} e_{kr}(v) f_{kr}(v + \delta^{kr}) \right).
\]

By Lemma 7.21 \( l_{k-1,l_k} \) has at least 2 choices if we fix all other \( l_{st}, 1 \leq t \leq s, k-1 \leq s \leq n \). Applying same argument as in the proof of Proposition 6.5 one can show that it is impossible to have zero limit for all these tableaux. Thus \( H(i_k, i'_k) \) is not a module.

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