Resonating mean-field theoretical approach to two-gap superconductivity with high-$T_c$∗

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Abstract

In the recent paper (referred to as I), the resonating mean-field theory (Res-MFT) has been applied and shown to effectively describe the so-called two-gap superconductivity (SC). In I, a state with large quantum fluctuations has been approximated by superposition of two MF wave functions composed of Hartree-Bogoliubov (HB) wave functions with different correlation structures. Particularly in I, using a suitable chemical potential, at $T=0$ the two-gap SC in MgB$_2$ has been well described by the Res-HBT. Furthermore the Res-HB ground state generated with HB wave functions has almost explained value of the ground-state correlation energy in all the correlation regimes including an intermediate coupling regime. In the present paper we will apply the Res-HBT to the two-gap SC with high critical temperature $T_c$. We will aim at constructing a theoretical foundation for phenomenological theories of the two-gap SC at $T=0$ and finite temperature. In the single-gap case we will find a new formula leading to a higher $T_c$ than the usual HB's.

Keywords: Res-MF theory; BCS model; Two-gap superconductivity

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1 Introduction

A topical two-gap superconductivity (SC) with critical temperature $T_c = 39$K has been recently discovered in MgB$_2$ [2]. It may be expected to open a new area in the vigorous pursuit by the radical spirit of the resonating mean-field theories (Res-MFTs) [3, 4] to develop a theoretical framework appropriate to explore the problem of high $T_c$ in superconductors. In particular, fermion systems with large quantum fluctuations show serious difficulties in many-body problems at finite temperature. To approach such problems, Fukutome has developed the resonating Hartree-Fock theory (Res-HFT) [3] and Fukutome and one of the present authors (S.N.) have extended it directly to the resonating Hartree-Bogoliubov theory (Res-HBT) to include pair correlations [4, 5]. In the recent paper (referred to as I) [6], the Res-HBT has been applied and demonstrated to effectively describe one of an exciting topic of current interest, the two-gap SC. An appearance of Thermal Gap Equation in the Res-HBT is a manifestation of the analogy of the Res-HBT with the usual BCS and HBT [7, 8].

Before the discovery of high-$T_c$ superconductor, much effort had been devoted to raising $T_c$ of the usual BCS superconductor in the weak coupling regime [7, 8, 9] and to obtaining the Eliashberg’s critical temperature in the strong coupling regime [10, 11, 12]. The $T_c$ for MgB$_2$ is 39K, which is close to or even higher than the upper theoretical value predicted by the BCS theory [13]. Even if SC in MgB$_2$ is phonon-mediated, a model beyond the simple BCS model or the Eliashberg model is required. The existence of two energy-gaps in MgB$_2$ at $T = 0$ has been predicted phenomenologically by Kortus et al. [14] and Liu et al. [15] employing a BCS-like weak-coupling theory, using the effective $\sigma$ and $\pi$ two-band model. They have obtained $\Delta_\sigma = 7.4$ [meV] and $\Delta_\pi = 2.4$ [meV]. On the other hand, employing the Eliashberg’s strong-coupling theory [10], Choi et al. [16] have obtained $\Delta_\sigma = 6.8$ [meV] and $\Delta_\pi = 1.8$ [meV]. Beyond such theoretical great successes, the Res-HB ground state generated with HB wave functions which are equivalent to the coherent state representations (CS reps) [17], is expected to almost explain the value of the ground-state correlation energy in all the correlation regimes including an intermediate coupling regime.

To go beyond such phenomenologies, we develop a tentative temperature-dependent Res-HBT. A temperature dependent variation is made to satisfy diagonalization conditions for thermal Res-Fock-Bogoliubov (Res-FB) operators along a way different from the usual thermal-BCS theory [18, 19, 20]. We derive formulas for determining $T_c$ and the behaviour of the gap near $T = 0$ and $T_c$. In the particular case of an equal magnitude of gaps but with two different phases, we find new formulas boosting up $T_c$ to a higher value than the usual HB’s value. Finally we stress to attempt a projection-method approach to a rigorous thermal Res-HBT. A partition function in an $SO(2N)$ ($N$: Number of fermion states) group can be calculated using the projection method proposed in the Res-MFTs [3, 4].

In Section 2, we give a diagonalization condition for thermal Res-FB operators which is the generalization of the condition to the Res-HBT’s proposed by the Ozaki’s method [21]. The diagonalized thermal Res-HB density matrix is expressed in the form of a Fermi-Dirac distribution with Res-HB eigenvalues. These lead to a self-consistent thermal Res-HB gap equation, from which we get formulas to determine $T_c$ and the behaviour of gaps near $T = 0$ and $T_c$. Finally in Section 3, we give a summary and further perspectives. In Appendices, we give a resonating mean-field free energy, variation of the resonating mean-field free energy, diagonalization of thermal HB density matrix and necessary integral formulas to calculate new formulas for the gap at intermediate temperature.
2 Thermal resonating HB equation

The Res-HB eigenvalue equation $\mathcal{F} \cdot u_p = \epsilon_p u_p$ in I can be extended to the thermal Res-HB eigenvalue equation. We obtain the thermal Res-HB equation in Appendix A. The Res-HBT implies that every HB eigenfunction in an HB resonating state has its own orbital-energy. We can derive the thermal Res-HB eigenvalue equation which is given in Appendix B. The thermal HB density matrix is determined as $W_{rrp}[\mathcal{F}_p] = [1+\exp(\beta\mathcal{F}_p)]^{-1}$ which is proved also in Appendix B. Using a Bogoliubov transformation $g_1p$ and $g_2p$ in I and (B.9) and (B.10), $W_{11p}[\mathcal{F}_p]$ and $W_{22p}[\mathcal{F}_p]$ in I are diagonalized as follows:

$$\widetilde{W}_{rp} = g^\dagger_{rp} W_{rrp}[\mathcal{F}_p] g_{rp} = \begin{bmatrix} \tilde{w}_{rp} & 0 \\ 0 & 1 - \tilde{w}_{rp} \end{bmatrix}, \quad (r = 1, 2),$$

where

$$\tilde{w}_{rp} = \frac{1}{1 + e^{\beta \epsilon_{rp}}}, \quad 1 - \tilde{w}_{rp} = \frac{1}{1 + e^{-\beta \epsilon_{rp}}}, \quad \left( \beta = \frac{1}{k_B T} \right)$$

which are the generalizations of the Ozaki's results [21] to the Res-MFT. By making the Bogoliubov transformation $g_{rp}$, eigenvalues $\tilde{\epsilon}_{rp}$ are obtained by diagonalization of the thermal Res-FB operators $\mathcal{F}_p$ with additional terms $(H[W_{rrp}] - E) |\psi_{rp}\rangle^2$ ($r = 1, 2$) (B.10). The thermal HB interstate density matrix in the whole Res-HB subspace is given as the direct sum:

$$W_p[\mathcal{F}_p] = g_p \widetilde{W}_p g^\dagger_p = \bigoplus_{r=1}^2 W_{rrp}[\mathcal{F}_p], \quad W_{rr}[\mathcal{F}_p] = g_{rp} \widetilde{W}_{rp} g^\dagger_{rp}. \quad (2.3)$$

Suppose a tilde thermal Res-HB density operator $\tilde{W}_{1p}$ for equal-gaps to be

$$\tilde{W}_{1p} = \begin{bmatrix} \tilde{W}^\dagger_{1p} \cdot I_2 & 0 \\ 0 & \tilde{W}^\dagger_{1p} \cdot I_2 \end{bmatrix}, \quad \tilde{W}^\dagger_{1p} = \begin{bmatrix} \tilde{w}^\dagger_{1p} & 0 \\ 0 & (1 - \tilde{w}^\dagger_{1p}) \cdot I_2 \end{bmatrix}. \quad (2.4)$$

Here $I_2$ is the two-dimensional unit matrix. Performing the unitary transformation by $\tilde{g}_{1p}^\dagger$, we obtain the following thermal Res-HB density matrix $\tilde{W}^\dagger_{1p}$.

$$W^\dagger_{1p} = \tilde{g}_{1p} \tilde{W}_{1p} \tilde{g}_{1p}^\dagger = \tilde{g}_{1p} \begin{bmatrix} \tilde{w}^\dagger_{1p} \cdot I_2 & 0 \\ 0 & (1 - \tilde{w}^\dagger_{1p}) \cdot I_2 \end{bmatrix} \tilde{g}_{1p}^\dagger = \begin{bmatrix} \frac{1}{2} \{ 1 - \cos \tilde{\theta}_{1p} \left( 2 \tilde{w}^\dagger_{1p} \right) \} \cdot I_2 & (\{ + \}) \times \frac{1}{2} \sin \tilde{\theta}_{1p} e^{-i\tilde{\phi}} \left( 1 - 2 \tilde{w}^\dagger_{1p} \right) \cdot I_2 \\ (\{ - \}) \times \frac{1}{2} \sin \tilde{\theta}_{1p} e^{i\tilde{\phi}} \left( 1 - 2 \tilde{w}^\dagger_{1p} \right) \cdot I_2 & \frac{1}{2} \{ 1 + \cos \tilde{\theta}_{1p} \left( 2 \tilde{w}^\dagger_{1p} \right) \} \cdot I_2 \end{bmatrix}. \quad (2.5)$$

The $\tilde{W}^\dagger_{2p}$ has the same form as (2.5). The Res-FB operator $\mathcal{F}^\dagger_{1(2)p}$ for spin-up state with upper and lower signs, corresponding to Case I (4.5) and Case II (4.6) in I, is expressed as

$$\mathcal{F}^\dagger_{1(2)p} = \begin{bmatrix} \mathcal{F}^\dagger_{+\epsilon_p} \cdot I_2 & \{ + \} \times \mathcal{F}^\dagger_{\Delta p} \cdot I_2 \\ \{ + \} \times \mathcal{F}^\dagger_{\Delta p} \cdot I_2 & -\mathcal{F}^\dagger_{-\epsilon_p} \cdot I_2 \end{bmatrix}. \quad (2.6)$$

$$\mathcal{F}^\dagger_{+\epsilon_p} = \frac{1}{2} \left\{ \epsilon_p + 2(H[W] - E_{g\ell}^{\text{Res}}) \frac{\sin^2 \theta_p}{\cos \theta_p} \left( \frac{\cos^2 \theta_p}{2} \right) + \frac{\Delta^2}{\epsilon_p} \cdot [\det z_{12}]^{\frac{1}{2}} \right\} \cdot \frac{1}{1 + [\det z_{12}]^{\frac{1}{2}}}. \quad (2.7)$$
\[ \mathcal{F}_{\Delta p}^{\pm} = \mathcal{F}_\Delta^{\mp} = -\frac{1}{2} \Delta \left\{ N(0)V \cdot \text{arcsinh} \left( \frac{1}{x} \right) \right\} \pm \left( \frac{1}{1 \pm \text{det} z_{12}} \right)^{\frac{1}{2}}, \quad (x = \frac{\Delta}{\hbar \omega_D}) \]  

(2.8)

for Case I (upper sign) and Case II (lower sign) where

\[ \text{det} z_{12} = \exp \left\{ -2N(0)\hbar \omega_D \left\{ \ln(1 + x^2) + 2x \cdot \text{arctan} \left( \frac{1}{x} \right) \right\} \right\}. \]  

(2.9)

At finite temperature, using the formulas (2.1) and (2.2) we require correspondence relations \( \cos \theta_{\rho T} \Rightarrow \cos \theta_{1p} \) and \( \sin \theta_{\rho T} \Rightarrow \sin \theta_{1p} \) given through

\[
\begin{align*}
\cos \theta_p &= \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta_T^2}} = \frac{\mathcal{F}_{\epsilon_p T}^{\uparrow} + \mathcal{F}_{-\epsilon_p T}^{\uparrow}}{2 \varepsilon_{1p}} \left( 1 - 2 \bar{w}_{1p}^\uparrow \right), \\
\sin \theta_p &= \frac{\Delta_T}{\sqrt{\varepsilon_p^2 + \Delta_T^2}} = -\frac{\mathcal{F}_{\Delta_T}^{\uparrow}}{\varepsilon_{1p}} \left( 1 - 2 \bar{w}_{1p}^\uparrow \right),
\end{align*}
\]  

(2.10)

Notice the multiplication factor \( 1 - 2 \bar{w}_{1p}^{\uparrow(i)} \). The \( \bar{\varepsilon}_{1p} (= \bar{\varepsilon}_p) \) is the quasi-particle (QP) energy:

\( \bar{\varepsilon}_p = \sqrt{\left( \mathcal{F}_{\epsilon_p T}^{\uparrow} + \mathcal{F}_{-\epsilon_p T}^{\uparrow} \right)^2 / 4 + \mathcal{F}_{\Delta_T}^{\uparrow^2}} \). Equations in L.H.S. of (2.10) are unified into one equation in R.H.S. It is easily shown that equation (2.10) plays the role of self-consistency condition at \( T = 0 \). Dividing numerator and denominator, respectively by \( (\varepsilon_p^2 + \Delta_T^2)^{3/2} \), equation in R.H.S. of (2.10) is rewritten as

\[ \frac{\varepsilon_p^2}{(\varepsilon_p^2 + \Delta_T^2)^{3/2}} \left( \frac{2 \mathcal{F}_{\Delta_T}^{\uparrow}}{\Delta_T} \right)(1 - 2 \bar{w}_{p}^{\uparrow}) = \bar{w}_{1p}^{\uparrow}. \]  

(2.11)

Now we demand a new condition for thermal gap equation

\[ \sum_p \left\{ \frac{\varepsilon_p}{(\varepsilon_p^2 + \Delta_T^2)^{3/2}} \frac{1}{2} \left( \mathcal{F}_{\epsilon_p T}^{\uparrow} + \mathcal{F}_{-\epsilon_p T}^{\uparrow} \right) - \frac{\varepsilon_p^2}{(\varepsilon_p^2 + \Delta_T^2)^{3/2}} \left( -\frac{\mathcal{F}_{\Delta_T}^{\uparrow}}{\Delta_T} \right) \right\} \left( 1 - 2 w_p^{\uparrow} \right) = 0, \]  

(2.12)

which leads to

\[ \left\{ 1 - N(0)V \cdot \text{arcsinh} \left( \frac{1}{x_T} \right) \right\} \sum_p A_p \equiv \left[ \text{det} z_{12} \right]^{\frac{1}{2}} \sum_p A_p \]  

\[ + \bar{E}_{\text{Res}(\pm)} \hbar \omega_D \sum_p B_p + \Delta_{T'}^2 \cdot \left[ \text{det} z_{12} \right]^{\frac{1}{2}} \sum_p C_p = 0, \]  

(2.13)

Here we calculate the term \( (H[W] - E_{\text{Res}}^{\text{Res}}) \) in (2.7) using the solutions for the Res-HB CI equation obtained in I and denote the result as \( \bar{E}_{\text{Res}(\pm)} \). We also give the following definitions for \( \sum_p A_p, \sum_p B_p \) and \( \sum_p C_p \):

\[ \sum_p A_p, \sum_p B_p, \sum_p C_p \equiv \sum_p \left[ \frac{\varepsilon_p^2}{(\varepsilon_p^2 + \Delta_T^2)^{3/2}}, \frac{1}{\varepsilon_p^2 + \Delta_T^2}, \frac{1}{(\varepsilon_p^2 + \Delta_T^2)^{3/2}} \right] \left( 1 - 2 \bar{w}_{p}^{\uparrow} \right). \]  

(2.14)

Rearranging (2.13), it is cast to
where

\[ \frac{1}{N(0) V} = \arcsinh \left( \frac{1}{x_T} \right) \left[ 1 \pm 2N(0) \arcsinh \left( \frac{1}{x_T} \right) \right] \left[ \Delta_T \cdot \Delta_T \sum_p B_p \right] \frac{\rho_p}{\rho_p + \Delta_T} \cdot \left[ \det z_{12} \right]^{\frac{3}{2}}. \]

which reduces to the Res-HB gap equation (4.10) in I as \( T \rightarrow 0 \). Using a variable \( \varepsilon = \xi \Delta_T \) instead of \( \varepsilon \), the summations \( \sum_p A_p, \sum_p B_p \) and \( \sum_p C_p \) near \( T = 0 \) can be computed to be

\[ \frac{\sum_p A_p}{2N(0)} = \int_0^{\frac{\pi}{2}} \frac{d\xi}{(\xi - 1)^2} = \arcsinh \left( \frac{1}{x_T} \right) - \frac{1}{\sqrt{1 + x_T^2}}. \]

\[ \frac{\Delta_T \sum_p B_p}{N(0)} = 2 \arctan \left( \frac{1}{x_T} \right) - \frac{\Delta_T^2 \sum_p C_p}{2N(0) \Delta_T} = \frac{1}{\sqrt{1 + x_T^2}}. \]

Substituting (2.16) into (2.15) and near \( T = 0 \) approximating as

\[ \arctan \left( \frac{1}{x_T} \right) \approx \frac{\pi}{2} - x_T, \quad \frac{1}{\sqrt{1 + x_T^2}} \approx 1 - x_0 x_T, \quad (0 < x_0 \ll 1) \]

then, near \( T = 0 \) we have the gaps for Case I (4.5) in I as

\[ \Delta^I_T = \Delta_0 \left[ 1 - \frac{\left[ \det z_{12} \right]^{\frac{3}{2}}}{N(0) V} \right] \left[ 1 + \arcsinh \left( \frac{\hbar \omega_D}{\Delta_0} \right) \right]^{-1} \]

\[ \times \left[ 1 - \frac{\left[ \det z_{12} \right]^{\frac{3}{2}}}{N(0) V} + \pi N(0) \Delta_0 \left[ \det z_{12} \right]^{\frac{3}{2}} \left[ 1 + \arcsinh \left( \frac{\hbar \omega_D}{\Delta_0} \right) \right]^{-1} \right], \]

and Case II (4.6) in I with the aid of \( [\det z_{12}]^{\frac{3}{2}} \approx 1 - 2\pi N(0) \hbar \omega_D x_0 (0 < x_0 \ll 1) \) easily derived from Taylor expansion of (2.9), as

\[ \Delta^II_T = \left\{ \hbar \omega_D - \frac{\Delta_0 \hbar \omega_D}{\pi - 1 - \frac{\pi}{2} \cdot N(0) V - (1 - N(0) V) \cdot [\det z_{12}]^{-\frac{3}{2}} + \Delta_0 \arcsinh \left( \frac{\hbar \omega_D}{\Delta_0} \right) \right\}. \]

In the opposite limit \( T \rightarrow T^I_c \) (\( T_c \) for Case I) the gap becomes very small, \( [\det z_{12}]^{\frac{3}{2}} \rightarrow 1 \), then \( F^I_{\Delta \gamma} \rightarrow -\Delta N(0) V \arcsinh (\hbar \omega_D / \Delta) / 4 \) and \( (F^I_{\Delta \gamma} + F^I_{-\Delta \gamma}) / 2 \rightarrow \varepsilon / 4 \) if we use the last relation in (5.9) and the second one in (5.16) in I. We have an approximate QP energy \( \tilde{\varepsilon}^I_p \approx \sqrt{\varepsilon^2 + \left\{ -\Delta N(0) V \arcsinh (\hbar \omega_D / \Delta) \right\}^2 / 4} \). This is because the two HB WFs have different correlation structures \( \psi_2 = \pi \) and \( \psi_1 = 0 \). In such a case, returning to the original form of the BCS gap equation but with the modified QP energy \( \tilde{\varepsilon}^I_p \), the thermal gap equation is expressed as

\[ 1 = V / 2 \sum_p \left( 1 - 2 \tilde{\varepsilon}^I_p / \tilde{\varepsilon}^I_p \right) \]

and leads to the integral form

\[ 1 = \frac{V}{2} \sum_p \frac{4}{\varepsilon_p} \tanh \left( \frac{\varepsilon_p}{8k_B T} \right) = 4N(0)V \int_0^{\hbar \omega_D} d\varepsilon \frac{1}{\varepsilon} \tanh \left( \frac{\varepsilon}{8k_B T} \right). \]
Introduce a dimensionless variable \( y_1 = \varepsilon / 8 k_B T^4 \) and its upper-value \( y_1^{\text{up}} \equiv \hbar \omega_D / 8 k_B T_c^4 \). Integrating R.H.S. of (2.20) by parts, it is approximated as follows:

\[
\frac{1}{4 N(0)V} \simeq \ln y_1^{\text{up}} - \int_0^{\infty} dy \ln \sech^2 y = \ln y_1^{\text{up}} + \ln \left( \frac{4 e^\gamma}{\pi} \right)
\]

\[
= \ln \left( \frac{e^\gamma \hbar \omega_D}{2 \pi k_B T_c^4} \right) \equiv \ln \left( \frac{\theta_D}{T_c^4} \right), \quad \left( \theta_D \equiv \frac{\hbar \omega_D}{k_B} : \text{Debye temperature} \right)
\]

(2.21)

where we have used the formula in the textbook [22]. Number \( \gamma \) is the Euler’s constant \( (\gamma \simeq 0.5772) \) and \( e^\gamma \simeq 1.781 \). Finally a small rearrangement yields

\[
T_c^4 = 0.283 \theta_D e^{-\frac{1}{4 N(0)V}},
\]

(2.22)

which should be compared with the Eliashberg’s formula [23] and the usual HB’s one for \( T_c \)

\[
T_c = 1.130 \theta_D e^{-\frac{1}{N(0)V}}.
\]

(2.23)

The new formula (2.22) gives a high critical temperature, e.g., \( T_c^4 = 72.87 \)K for \( N(0)V = 0.25 \) and \( \theta_D = 700 \)K. This \( T_c^4 \) is in contrast to \( T_c \) obtained by the usual HB’s (2.23), i.e., \( T_c = 14.49 \)K for the same values of \( N(0)V \) and \( \theta_D \).

We are now in a stage to discuss the behaviour of the gap near \( T_c \). In the above the modified QP energy \( \tilde{\varepsilon} = \sqrt{\varepsilon^2 + \left[ -\Delta N(0)V \text{arcsinh} (\hbar \omega_D / \Delta) \right]^2} / 4 \) plays a crucial role to boost the \( T_c \) in (2.22) comparing with the numerical result in (2.23). Notice the existence of the numerical factor 1/4. Now let us consider \( \Delta_T^1 \) near \( T_c^4 \). Using this form of QP energy, the gap equation is roughly rewritten as

\[
\frac{1}{4 N(0)V} = \frac{1}{4} \int_0^{\hbar \omega_D} \frac{d \varepsilon}{\varepsilon} \frac{1}{\varepsilon} \tanh \left( \frac{\varepsilon}{2 k_B T} \right) \simeq \int_0^{\hbar \omega_D} \frac{d \varepsilon}{\varepsilon} \frac{1}{\varepsilon} \tanh \left( \frac{\varepsilon}{8 k_B T} \right)
\]

\[
- \left\{ \Delta_T N(0)V \cdot \text{arcsinh} \left( \frac{\hbar \omega_D}{\Delta_T} \right) \right\}^2 \int_0^{\hbar \omega_D} d \varepsilon \left( \frac{1}{\varepsilon^3} \tanh \left( \frac{\varepsilon}{8 k_B T} \right) - \frac{1}{\varepsilon^2} \frac{1}{8 k_B T} \sech^2 \left( \frac{\varepsilon}{8 k_B T} \right) \right),
\]

(2.24)

from which we obtain

\[
\frac{1}{4 N(0)V} = \ln \left( \frac{\hbar \omega_D}{k_B T_c^4} \right) - \frac{7}{8 \pi^2 \zeta(3)} \left( \frac{2 \pi}{e^\gamma} \right)^2 \left( \frac{\hbar \omega_D}{k_B T_c^4} \right)^2 \left\{ N(0)V x_T \cdot \text{arcsinh} \left( \frac{1}{x_T} \right) \right\}^2
\]

\[
\simeq \ln \left( \frac{\hbar \omega_D}{k_B T_c^4} \right) + \frac{\bar{T}_c - T}{T_c^4} - \frac{7}{8 \pi^2 \zeta(3)} \left( \frac{2 \pi}{e^\gamma} \right)^2 \left( 1 - \frac{\bar{T}_c - T}{T_c^4} \right)^2
\]

\[
\times \left( \frac{\hbar \omega_D}{k_B T_c^4} \right)^2 \left\{ N(0)V x_T \cdot \text{arcsinh} \left( \frac{1}{x_T} \right) \right\}^2,
\]

(2.25)

where \( \hbar \omega_D / k_B T_c^4 \equiv e^\gamma / 2 \pi \cdot \hbar \omega_D / k_B T_c \). For details see Appendix C. Using \( \text{arcsinh} (1/ x_T) \simeq \ln (2 / x_T) \simeq -(x_T - 2) / 2 + \cdots (2 / x_T > 1 / 2) \), (2.21) and (2.25), we get \( \Delta_T^1 \) near \( T_c^4 \) as

\[
\Delta_T^1 \simeq 2 \pi \sqrt{\frac{2}{7 \zeta(3)} N(0)V \left( 1 - \frac{T_c^4 - T}{T_c^4} \right)} \sqrt{\frac{T_c^4 - T}{T_c^4}}.
\]

(2.26)

The \( \sqrt{T_c^4 - T} \) dependence of \( \Delta_T^1 \) is more complicated than the usual one [19, 20].
For Case II, \((\cal{F}^\dagger_{+\varepsilon_pT} + \cal{F}^\dagger_{-\varepsilon_pT}), \cal{F}^\dagger_T\) and \(\varepsilon_p^\dagger\) become infinite simultaneously in
the limit \(\Delta_T \to 0\) \((x_T \to 0)\) due to the existence of \(1 - [\det z_{12}]^{1/2}\) in denominator.
Then mathematical handling for such a problem is too difficult and therefore we can not easily get a
formula for \(T_{c}^{\text{II}}\) in an analytical way as we did in Case I. Denote \(T_{c}\) for Case II as \(T_{c}^{\text{II}}\). At
\(T \approx T_{c}^{\text{II}}\), \(\Delta_{T}^{\text{II}}\) almost vanishes and \(1 - [\det z_{12}]^{1/2}/2\pi N(0)\ho\omega_{D}x_T\). Using \(\cal{F}^\dagger_T\) in (2.8), we
reach to the following asymptotic forms: \(\cal{F}^\dagger_T \to - \left((4\pi N(0))^{-1}N(0)\arcsinh(\ho\omega_{D}/\Delta_T)\right)\)
and \((\cal{F}^\dagger_{+\varepsilon_pT} + \cal{F}^\dagger_{-\varepsilon_pT})/2 \to \left((4\pi N(0))^{-1}\varepsilon_p/\Delta_T\right)\). The QP energy \(\varepsilon_p^\dagger\) is approximately calculated to be \(\varepsilon_p^\dagger \approx (4\pi N(0))^{-1}\varepsilon_p/\Delta_T\) \((0 < \Delta_T \ll 1)\). Here we discard
the contribution from \(\cal{F}^\dagger_{\Delta_T}\), comparing with the one from \((\cal{F}^\dagger_{+\varepsilon_pT} + \cal{F}^\dagger_{-\varepsilon_pT})/2\). As was done
previously, returning again to the original form of the BCS gap equation but with another
modified QP energy \(\varepsilon_p^\dagger\), the thermal gap equation is obtained as
\(1 = V/2 \sum_{\rho} \left(1 - 2\bar{w}_p^{\dagger}/\varepsilon_p^\dagger\right)\), which also leads to the integral form

\[
1 = \frac{V}{2} \sum_{\rho} \frac{4\pi N(0)\Delta_T^{\text{II}}}{\varepsilon_p} \tanh\left(\frac{\varepsilon_p}{2k_B T^{\text{II}} \cdot 4\pi N(0)\Delta_T^{\text{II}}}\right)
= N(0)V \int_0^{\ho\omega_{D}} d\varepsilon \frac{4\pi N(0)\Delta_T^{\text{II}}}{\varepsilon} \tanh\left(\frac{\varepsilon}{2k_B T^{\text{II}} \cdot 4\pi N(0)\Delta_T^{\text{II}}}\right). \tag{2.27}
\]

Introduce a dimensionless variable \(y_T^{\text{II}} \equiv \varepsilon / \left(2k_B T \cdot 4\pi N(0)\Delta_T^{\text{II}}\right)\) and its upper-value \(y_T^{\text{II}}_{\text{u}} \equiv \ho\omega_{D} / \left(2k_B T^{\text{II}} \cdot 4\pi N(0)\Delta_T^{\text{II}}\right)\). Integrating the last equation in (2.27) by parts, it is approximately calculated as

\[
\frac{1}{4N(0)V} \frac{1}{\pi N(0)\Delta_T^{\text{II}}} \approx \ln y_T^{\text{II}} + \ln\left(\frac{4e^\gamma}{\pi}\right),
= \ln\left(\frac{1}{\pi N(0)\Delta_T^{\text{II}}} \frac{\ho\omega_{D}}{2\pi k_B T^{\text{II}}}\right) = \ln\left(\frac{1}{\pi N(0)\Delta_T^{\text{II}}} \frac{\theta_D}{T^{\text{II}}}\right), \tag{2.28}
\]

which reads

\[
\Delta_T^{\text{II}} = \frac{\theta_D}{\pi N(0)T^{\text{II}}} \exp\left\{- \frac{1}{4N(0)V \pi N(0)\Delta_T^{\text{II}}}\right\} \approx \frac{\theta_D}{\pi N(0)T^{\text{II}}} \left(1 - \frac{1}{4N(0)V \pi N(0)\Delta_T^{\text{II}}}\right), \tag{2.29}
\]

from which we obtain an equation to determine \(\Delta_T^{\text{II}}\) very near \(T_{c}\) as

\[
\Delta_T^{\text{II}} - \frac{1}{\pi N(0)} \frac{\theta_D}{T^{\text{II}}} \Delta_T^{\text{II}} + \frac{1}{\pi N(0)} \frac{\theta_D}{T^{\text{II}}} \frac{1}{4N(0)V \pi N(0)\Delta_T^{\text{II}}} = 0. \tag{2.30}
\]

Then we have a solution for \(\Delta_T^{\text{II}}\) as

\[
\Delta_T^{\text{II}} = \frac{1}{\pi N(0)} \frac{\theta_D}{T^{\text{II}}} \frac{1}{4N(0)V \pi N(0)}, \tag{2.31}
\]

in which at \(\Delta_T^{\text{II}} = T_{c}^{\text{II}}\), the \(\Delta_T^{\text{II}}\) vanishes. Then finally we can determine the critical temperature \(T_{c}^{\text{II}}\) for Case II as

\[
T_{c}^{\text{II}} = \frac{2e^\gamma}{\pi} \theta_D N(0)V = 1.334\theta_D N(0)V. \tag{2.32}
\]
The simple formula (2.32) gives a high critical temperature, e.g., $T^H_c = 198K$ for $N(0)V = 0.25$ and $\theta_D = 700K$. Finally $\Delta^H_T$ near $T^H_c$ can be approximately obtained as

$$\Delta^H_T \approx -\frac{e^\gamma}{2\pi \pi N(0)} \frac{1}{T^H_c} \frac{\theta_D}{T^H_c} \left( T - T^H_c \right)$$

(2.33)

which is linearly dependent on $T - T^H_c$. It is very interesting that we could find such a dependence of $\Delta^H_T$, comparing with the usual dependence $\sqrt{T - T^H_c}$ of $\Delta^H_T$.

In intermediate temperature region the modified QP energy $\bar{\epsilon}$ is approximated as $\bar{\epsilon} = \sqrt{\epsilon^2 + \Delta^2_T} \left\{ 2 \left(1 \pm \left| \det z_{12} \right| \frac{z}{T} \right)^2 \right\}^{-1}$. When $\epsilon \gg \Delta_T$, each term $\Sigma_p A_p$, $\Sigma_p B_p$ and $\Sigma_p C_p$ in (2.14) is approximately computed as

$$\frac{\Sigma_p A_p}{2N(0)} = \int_0^{\hbar \omega_B/2\pi} d\epsilon \frac{\epsilon^2}{(\epsilon^2 + \Delta^2_T)^{3/2}} \tanh \left( \frac{\bar{\epsilon}^{(\pm)}}{2k_BT} \right) = \ln \left( \frac{4e^\gamma}{\pi y_T^{(\pm)}} \right) - \frac{21}{2\pi^2} \zeta(3) \left( y_T^{(\pm)} \bar{x}_T \right)^2$$

$$\frac{\Delta_T \Sigma_p B_p}{2N(0)} = 0, \quad \frac{\Delta^2_T \Sigma_p C_p}{2N(0)} = \int_0^{\hbar \omega_B/2\pi} d\epsilon \frac{1}{(\epsilon^2 + \Delta^2_T)^{3/2}} \tanh \left( \frac{\bar{\epsilon}^{(\pm)}}{2k_BT} \right) = \frac{7}{\pi^2} \zeta(3) \left( y_T^{(\pm)} \bar{x}_T \right)^2$$

(2.34)

whose details are presented in Appendix C. Taking only a leading term, $A_p$ and $C_p$ terms in (2.34) are approximated to be

$$\frac{\Sigma_p A_p}{2N(0)} \approx \ln \left( \frac{4e^\gamma}{\pi y_T^{(\pm)}} \right) - \frac{21\zeta(3)}{2\pi^2} \left( y_T^{(\pm)} \bar{x}_T \right)^2 = \ln \left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right) - \alpha_T^{(\pm)}$$

$$\frac{\Delta^2_T \Sigma_p C_p}{2N(0)} \approx \frac{7\zeta(3)}{\pi^2} \left( y_T^{(\pm)} \bar{x}_T \right)^2 = \frac{2}{3} \alpha_T^{(\pm)}$$

(2.35)

$$\alpha_T^{(\pm)} \equiv \frac{21\zeta(3)}{2\pi^2} \left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right)^2 \left( \frac{\Delta_T}{k_BT} \right)^2 = \Delta_T^{H} \equiv \Delta_T N(0) \text{Var} \sinh \left( \frac{\hbar \omega_B}{\Delta_T} \right)$$

Substituting these results into (2.15), we have

$$\left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right)^2 x_T^{(\pm)} \left( N(0) \text{Var} \sinh \left( \frac{1}{x_T} \right) \right)^2 \left( N(0) \text{Var} \sinh \left( \frac{1}{x_T} \right) - \left( 1 - \left| \det z_{12} \right| \right) \right)$$

$$- \frac{2\pi^2}{21\zeta(3)} \ln \left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right) \left( \frac{T}{\theta_D} \right)^2 N(0) \text{Var} \sinh \left( \frac{1}{x_T} \right)$$

$$+ \frac{2\pi^2}{21\zeta(3)} \left( 1 - \left| \det z_{12} \right| \right) \ln \left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right) \left( \frac{T}{\theta_D} \right)^2$$

$$+ \frac{2}{3} \left( \frac{e^\gamma}{\pi} \frac{1}{1 \pm \left| \det z_{12} \right| T} \right)^2 x_T^{(\pm)} \left[ \text{det } z_{12} \right] = 0$$

(2.36)
to be solved analytically for a given $T$, which is rewritten as

$$
\left(\frac{\partial}{\partial T} \right)^2 \frac{1}{1 \pm \frac{1}{2} \left( \frac{V}{\theta_D} \right)^2} \left\{ N(0)V \arcsinh \left( \frac{1}{x_T} \right) \right\}^2
- \frac{2\pi^2}{21\zeta(3)} \ln \left( \frac{\partial}{\partial T} \right) \left( \frac{T}{\theta_D} \right)^2 \left\{ N(0)V \arcsinh \left( \frac{1}{x_T} \right) - \left( \frac{T}{\theta_D} \right)^2 \right\}
= \frac{2}{3} \left( \frac{e}{\pi} \right)^2 \left( \frac{1}{1 \pm \frac{1}{2} \left( \frac{V}{\theta_D} \right)^2} \right) \left\{ N(0)V \arcsinh \left( \frac{1}{x_T} \right) \right\}^2
$$

(2.37)

from which we obtain an equation to determine $\Delta_T$ for a given $T$ as

$$
N(0)V \frac{\partial}{\partial T} \frac{1}{1 \pm \frac{1}{2} \left( \frac{V}{\theta_D} \right)^2} \arcsinh \left( \frac{1}{x_T} \right)
= - \frac{2\pi^2}{21\zeta(3)} \ln \left( \frac{\partial}{\partial T} \right) \left( \frac{T}{\theta_D} \right)^2 \left\{ N(0)V \arcsinh \left( \frac{1}{x_T} \right) \right\}^2
$$

(2.38)

This is classified into the following two cases:

Case I: $[\det z_{12}]^\frac{1}{T} \approx 0.3$

$$
0.436 N(0) V x_T \arcsinh \left( \frac{1}{x_T} \right) = - \sqrt{0.782} \ln(0.436) - \ln \left( \frac{T}{\theta_D} \right) \cdot \left( \frac{T}{\theta_D} \right),
$$

(2.39)

from which, using the approximate relation $\arcsinh(1/x_T) \approx (x_T - 2)/2$, finally we have a solution for $x_T$ ($N(0)V = 0.25$ and $\theta_D = 700K$) as

$$
x_T = 1 - \frac{1}{\sqrt{0.436 \times 0.25}} \sqrt{0.436 \times 0.25 - 2 \sqrt{0.782} \ln(0.436) - \ln \left( \frac{T}{700} \right) \cdot \left( \frac{T}{700} \right)},
$$

(2.40)

Case II: $1 - [\det z_{12}]^\frac{1}{T} \approx 2 \pi N(0)\hbar \omega_D x_T$ and $\ln \left( \frac{1}{x_T} \right) \approx \arcsinh \left( \frac{1}{x_T} \right)$

$$
(0.283)^2 \left\{ N(0)V \right\}^2 \arcsinh^2 \left( \frac{1}{x_T} \right) - 0.782 \left[ \pi N(0)\hbar \omega_D \right]^2 \left( \frac{T}{\theta_D} \right)^2 \arcsinh \left( \frac{1}{x_T} \right)
- 0.782 \left[ \pi N(0)\hbar \omega_D \right]^2 \left[ \ln(0.283) - \ln \left( 2 \pi N(0)\hbar \omega_D \right) - \ln \left( \frac{T}{\theta_D} \right) \cdot \left( \frac{T}{\theta_D} \right) \right]^2 = 0,
$$

(2.41)

whose solution is easily obtained as

$$
\arcsinh \left( \frac{1}{x_T} \right) = (0.283)^2 \left\{ N(0)V \right\}^2 \arcsinh \left( \frac{1}{x_T} \right) + (0.283)^2 \left\{ N(0)V \right\}^2 \left( \frac{T}{\theta_D} \right)^2 \arcsinh \left( \frac{1}{x_T} \right)
- \left\{ (0.283)^2 \left[ \pi N(0)\hbar \omega_D \right]^2 \left( \frac{T}{\theta_D} \right)^2 + (0.283)^2 \left\{ N(0)V \right\}^2 \cdot 0.782 \times \left[ \ln(0.283) - \ln \left( 2 \pi N(0)\hbar \omega_D \right) - \ln \left( \frac{T}{\theta_D} \right) \cdot \left( \frac{T}{\theta_D} \right) \right] \right\}^\frac{1}{2},
$$

(2.42)
from which, using again the approximate relation \( \text{arcsinh}(1/x_T) \approx (x_T - 2)/2 \), finally we have a solution for \( x_T \) \( (N(0)V = 0.25, \; N(0)\omega_D = 0.01 \) and \( \theta_D = 700\text{K} ) \) as

\[
x_T = 2 + 2 \left( 0.283 \right)^2 \left( 0.25 \right)^{-2} \{ 3.14 \times 0.01 \} \left( \frac{T}{700} \right) \left[ 0.391 \{ 3.14 \times 0.01 \} \left( \frac{T}{700} \right) \right.

\[
- \left\{ (0.391)^2 \{ 3.14 \times 0.01 \}^2 \left( \frac{T}{700} \right)^2 + (0.283)^2 (0.25)^2 \times 0.782 \right. \\

\times \left\{ \ln(0.283) - \ln \{ 2 \times 3.14 \times 0.01 \} - \ln \left( \frac{T}{700} \right) \right\} \right]^{\frac{1}{2}}.
\]

(2.43)

It is very interesting to investigate behaviour of the temperature dependence of the gap \( \Delta_T \) for Case I and Case II.
3 Summary and further perspectives

We have concentrated on derivation of thermal gap equations within the framework of Res-HBA. From the Res-FB operators $\mathcal{F}_1$ and $\mathcal{F}_2$ with equal-gaps, we have found the diagonalization conditions for them, which are essentially of the same form as the former one. It leads to the self-consistent thermal Res-HB gap equation and makes possible to derive the new formulas to determine $T_c$ and the gaps near $T = 0$ and $T_c$. The formula for Case I gives a high $T_c^1$, e.g., $T_c^1 = 72.87$K for $N(0)V = 0.25$ and Debye temperature $\theta_D = 700$K. This is in contrast with $T_c$ of the usual HB formula giving $T_c = 14.49$K for the same values of $N(0)V$ and $\theta_D$. The formula for Case II gives also a high $T_c^\Pi$, e.g., $T_c^\Pi = 198$K for the same values of $N(0)V$ and $\theta_D$. The temperature dependence of the gap near $T = 0$ and $T_c$ becomes more complicated than that of the usual HB and Abrikosov descriptions [19, 20]. At intermediate temperature, we have got the solutions of $\Delta_T$ for Case I and Case II. We have taken $[\det z_{12}]^{1/2} \approx 0.3$ (Case I) and $N(0)\hbar\omega_0 = 0.01$ (Case II) to get real solutions. We have got, however, $x_T = 0.65$ (Case I) and 1.05 (Case II) for $T = 70$K which are a little bit large compared with the real solutions. Improvement of such results should be made. Further, it would have been better to draw numerical aspects of the temperature dependence in all the correlation regimes. This is possible in the near future.

For unequal two-gaps, it is also possible to realize the above-mentioned diagonalization condition for Res-FB operators $\mathcal{F}_{rp}(r = 1, 2)$. Transforming by a unitary matrix $\hat{g}_{rp}, \mathcal{F}_{rp}$ is easily diagonalized. Noticing the same correspondence as the one in (2.10), $\cos \theta_{rp} \rightarrow \cos \theta_{rp}$ and $\sin \theta_{rp} \rightarrow \sin \theta_{rp}$, we assume each diagonalization condition (2.11) holds even in this case. Then we obtain coupled equations through a function of $\Delta_{1T}$ and $\Delta_{2T}$ expressed as

$$1 = \frac{\varepsilon_p^2}{(\varepsilon_p^2 + \Delta_{1T}^2)^{3/2}} \left( \frac{2\mathcal{F}_{r,\Delta}}{\Delta_{1T}} \right) \left( 1 - 2\hat{w}_{rp}^\dag \right), \quad \hat{w}_{rp}^\dag = \frac{1}{1 + e^{\beta \varepsilon_{rp}}}$$

which reduces to equation in R.H.S. of (2.10) if $\Delta_{1T} = \Delta_{2T}$. The quantities $\mathcal{F}_{r,\Delta}$ and $\mathcal{F}_{r,\pm \delta}$ are given by the equations similar to (5.9) in I but with more complicated forms of $\Delta_{1T}$ and $\Delta_{2T}$. For the time being, as was done in the previous section we here also use the function $(\varepsilon_p^2 + \Delta_{1T}^2)^{3/2}$ by which we divide numerator and denominator, respectively, in (3.1). After equating the numerator to the denominator and using the relation $1 - 2\hat{w}_{rp}^\dag = \tanh(\hat{\varepsilon}_{rp}/2k_BT)$, we sum up over $p$, namely integrate both sides of the equation over $\varepsilon$, to achieve the optimized conditions. We obtain coupled thermal Res-HB gap equations and reach our ultimate goal of computing temperature-dependent two-gaps. Along such a strategy and method, we will make a numerical analysis to demonstrate the behaviour of temperature-dependent two-gaps.

To solve such a problem, we must provide a rigorous thermal Res-HBA. We have an expression for partition function in an $SO(2N)$ CS rep $|g\rangle$ [17], $\text{Tr}(e^{-\beta H}) = 2^{N-1}\int \langle g | e^{-\beta H} | g \rangle \, dg$ ($dg$ is the group integration on group $SO(2N)$). Following Fukutome [3], introducing the projection operator $P$ to the Res-HB subspace, the partition function in the Res-HB subspace is computed as $\text{Tr}(P e^{-\beta H})$. This kind of trace formula is calculated within the Res-HB subspace by using the Laplace transform of $e^{-\beta H}$ and the projection method which leads to an infinite matrix continued fraction. A thermal variation of the Res-HB free energy is carried out after the inverse Laplace transform of matrix elements of $e^{-\beta H}$. This is made parallel to the usual thermal BCS theory, which will be given in a separate paper.
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A Resonating mean-field free energy $F[Z]_{\text{Res}}$

Suppose a matrix $Z$ to be a usual FB operator. Introduce a quadratic HB Hamiltonian and the usual HB free energy [21]

$$
H[Z] = \frac{1}{2}[c, c^\dagger] Z \begin{bmatrix} c & c^\dagger \end{bmatrix}, \quad Z^\dagger = Z,
$$

$$
F[Z] = \text{Tr}(\tilde{W}[Z]H) + \frac{1}{\beta} \text{Tr}\{\tilde{W}[Z] \ln(\tilde{W}[Z])\}, \quad \tilde{W}[Z] = \frac{e^{-\beta H[Z]}}{\text{Tr}(e^{-\beta H[Z]})},
$$

which leads to

$$
\begin{aligned}
\langle H \rangle_Z &= \frac{\text{Tr}(e^{-\beta H[Z]}H)}{\text{Tr}(e^{-\beta H[Z]})}, \\
\langle H[Z] \rangle_Z &= \frac{\text{Tr}(e^{-\beta H[Z]}H[Z])}{\text{Tr}(e^{-\beta H[Z]})}.
\end{aligned}
$$

We have another form for this free energy, i.e., a well-known formula expressed in terms of a usual HB density matrix $W[Z]$ as

$$
F[Z] = \langle H \rangle_Z + \frac{1}{2\beta} \text{Tr}\{W[Z] \ln(W[Z]) + (1_{2N} - W[Z]) \ln(1_{2N} - W[Z])\},
$$

$$
W[Z] \equiv \begin{bmatrix} R[Z] & K[Z] \\ -K[Z]^* & 1_N - R[Z]^* \end{bmatrix},
$$

and the trace formulas for the pair operators

$$
\text{Tr}\{\tilde{W}[Z] (E^\beta_{\alpha} + \frac{1}{2} \delta_{\alpha\alpha})\} = R_{\alpha\beta}[Z], \quad \text{Tr}\{\tilde{W}[Z] E_{\beta\alpha}\} = K_{\alpha\beta}[Z].
$$

We give $Z$ by a direct sum of $\mathcal{F}_r$, $Z = \bigoplus_{r=1}^n \mathcal{F}_r$, and assume each $\mathcal{F}_r (r=1, \ldots, n)$ is a Res-FB operator. Then, instead of the above $H[Z]$ in (A.1), we introduce a quadratic Res-HB Hamiltonian

$$
H[Z]_{\text{Res}} \equiv \frac{1}{2} [c^\dagger, c, \ldots, c^\dagger, c, \ldots, c^\dagger, c]
\begin{bmatrix} \mathcal{F}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{F}_n \end{bmatrix}
\begin{bmatrix} c \\ c^\dagger \\ \vdots \\ c^\dagger \\ \vdots \\ c^\dagger \end{bmatrix}.
$$

Along the same way as the one in (A.1), the Res-HB free energy can also be defined.

Now let us introduce a projection operator $P$ ($P^2 = P = P^\dagger$) to the Res-HB subspace, $P|\Psi\rangle = |\Psi_{\text{Res}}\rangle$ as $P \equiv \sum_{r,s=1}^n |g_r\rangle(S^{-1})_{rs} \langle g_s|$, where the $|g_r\rangle$'s are HB wave functions and $S = (S_{rs}) = [\det z_{rs}]^{1/2}$ is an $n \times n$ matrix composed of the overlap integrals and $S^\dagger = S$. Using the projection operator $P$, we give the Res-HB free energy in the form

$$
F[Z]_{\text{Res}} = \text{Tr}(\tilde{W}[Z]_{\text{Res}} H) + \frac{1}{\beta} \text{Tr}\{\tilde{W}[Z]_{\text{Res}} \ln(\tilde{W}[Z]_{\text{Res}})\}, \quad \tilde{W}[Z]_{\text{Res}} \equiv \frac{P e^{-\beta H[Z]_{\text{Res}}}}{\text{Tr}(P e^{-\beta H[Z]_{\text{Res}}})},
$$

in which, by making Taylor expansion of $\ln P = \ln\{1 - (1 - P)\}$ and using $P^2 = P$, we have

$$
\frac{1}{\beta} \text{Tr}\{\tilde{W}[Z]_{\text{Res}} \ln(\tilde{W}[Z]_{\text{Res}})\} = -\langle H[Z]_{\text{Res}} \rangle_{Z;\text{Res}} - \frac{1}{\beta} \ln\text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}}).
$$
A natural extension of the HB free energy to the Res-HB free energy is easily made. Then, the Res-HB free energy is given as
\[ F[Z]_{\text{Res}} = \langle H - H[Z]_{\text{Res}} \rangle_{Z;\text{Res}} - \frac{1}{\beta} \ln \text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}}), \]
(A.8)

\[ \langle H[Z]_{\text{Res}} \rangle_{Z;\text{Res}} \equiv \frac{\text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}} H[Z]_{\text{Res}})}{\text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}})}, \quad \langle H \rangle_{Z;\text{Res}} \equiv \frac{\text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}} H)}{\text{Tr}(Pe^{-\beta H[Z]_{\text{Res}}})}. \]
(A.9)

Consider the whole Res-HB subspace \(|\Psi^{\text{Res}(k)}\rangle = \sum_{i=1}^{n} c_i^{(k)} |g_i\rangle, \ (k = 1, \cdots, n)\) in which the Res-state with index \(k = 1\) and the Res-states with indices \(k = 2, \cdots, n\) stand for the Res-ground one and the Res-excited ones, respectively. In order to determine the thermal \(|g_r\rangle\)'s and thermal mixing coefficients \(c_i^{(k)}\)'s by the variational method, we use a temperature dependent Lagrangian with the Lagrange multiplier term \(E\) to secure the normalization condition \(\langle \Psi^{\text{Res}(k)}|\Psi^{\text{Res}(k)}\rangle = 1\):
\[ L[Z]_{\text{Res}}^{\text{HB}} = \sum_{r,s=1}^{n} \left\{ H[\mathcal{W}_{rs}[Z]_{\text{Res}}] - E^{(k)} \right\} \cdot [\det z_{rs}]^{\frac{1}{2}} c_r^{(k)*} c_s^{(k)}. \]
(A.10)

The variation of (A.10) is made in a quite parallel manner to the one in the previous works [3, 4]. We omit detailed derivations for such parallel cases. From the variation of \(L[Z]_{\text{Res}}^{\text{HB}}\) with respect to \(c_r^{(k)*}\) for any \(k\), we get a thermal Res-HB CI equation to determine \(c_s^{(k)}\)
\[ \sum_{r,s=1}^{n} \left\{ H[\mathcal{W}_{rs}[Z]_{\text{Res}}] - E^{(k)} \right\} \cdot [\det z_{rs}]^{\frac{1}{2}} c_r^{(k)*} c_s^{(k)} = 0. \]
(A.11)

We define \(D_{rs}\) and \(\bar{D}_{rs}\) as \(D_{rs} \equiv u_s z_{rs}^{-1} u_r^\dagger\) and \(\bar{D}_{rs} \equiv \delta u_s z_{rs}^{-1} u_r^\dagger\), respectively, where \(z_{rs} \equiv u_r^* u_s\) and \(u_r^T \equiv [b_r^T, a_r^T]\) (\(a_r\) and \(b_r\): r-th HB amplitudes). The variations of thermal HB interstate density matrix \(W[Z]_{\text{Res}}\) and thermal overlap integral \([\det z]^{1/2}\) are given by
\[ \delta W_{rs}[Z]_{\text{Res}} = D_{rs}(12N - W_{rs}[Z]_{\text{Res}}) + (12N - W_{rs}[Z]_{\text{Res}}) \bar{D}_{rs}, \]
\[ \delta [\det z_{rs}]^{\frac{1}{2}} = \frac{1}{2} \text{Tr}(D_{rs} + \bar{D}_{rs}) \cdot [\det z_{rs}]^{\frac{1}{2}}. \]
(A.12)

We obtain also the variation of Hamiltonian matrix element \(H[\mathcal{W}_{rs}[Z]_{\text{Res}}]\) as
\[ \delta H[\mathcal{W}_{rs}[Z]_{\text{Res}}] = \frac{1}{2} \text{Tr}\{\mathcal{F}[\mathcal{W}_{rs}[Z]_{\text{Res}}]\delta W_{rs}[Z]_{\text{Res}}\}, \]
\[ \mathcal{F}[\mathcal{W}_{rs}[Z]_{\text{Res}}] = \begin{bmatrix} F_{rs}[Z]_{\text{Res}} & D_{rs}[Z]_{\text{Res}} \\ -D_{rs}^*[Z]_{\text{Res}} & -F_{rs}^*[Z]_{\text{Res}} \end{bmatrix}, \]
(A.13)

Following I and the Res-HF theory [3], writing \(L[Z]_{\text{Res}}^{\text{HB}} = \sum_{k=1}^{n} \sum_{r,s=1}^{n} L_{rs}^{\text{HB}(k)}[Z]_{\text{Res}} c_r^{(k)*} c_s^{(k)}\) and \(L_{rs}^{\text{HB}(k)}[Z]_{\text{Res}} = \{H[\mathcal{W}_{rs}[Z]_{\text{Res}}] - E^{(k)}\} \cdot [\det z_{rs}]^{1/2}\), from the variation of \(L_{\text{Res}}^{\text{HB}}\) for any \(k\), we obtain a thermal Res-HB equation to determine the thermal mean field wave function \(u_r\) as
\[ \sum_{k=1}^{n} \sum_{s=1}^{n} K_{rs}^{(k)}[Z]_{\text{Res}} c_r^{(k)*} = 0, \]
\[ K_{rs}^{(k)}[Z]_{\text{Res}} = \{(12N - W_{rs}[Z]_{\text{Res}}) \mathcal{F}[\mathcal{W}_{rs}[Z]_{\text{Res}}] + H[\mathcal{W}_{rs}[Z]_{\text{Res}}] - E^{(k)}\} \cdot W_{rs}[Z]_{\text{Res}} \cdot [\det z_{rs}]^{\frac{1}{2}}. \]
(A.14)

From (A.14) we can derive a thermal Res-HB eigenvalue equation. See next Appendix.
B Variation of resonating mean-field free energy $F[Z]_{\text{Res}}$

The thermal Res-HB coupled eigenvalue equations is expressed as follows:

\[
\begin{align*}
[F_r[Z]_{\text{Res}}u_r]_i = \epsilon_r u_{ri}, \quad \epsilon_r & = \epsilon_s - \sum_{k=1}^n \left\{ H[W_{rr}[Z]_{\text{Res}}] - E^{(k)} \right\} |c_r^{(k)}|^2, \\
F_r[Z]_{\text{Res}} &= F_r[Z]_{\text{Res}}^\dagger = F[W_{rr}[Z]_{\text{Res}} \sum_{k=1}^n |c_r^{(k)}|^2 \\
&\quad + \sum_{k=1}^n \sum_{s=1}^n \left\{ \mathcal{K}_{rs}^{(k)} [Z]_{\text{Res}} c_r^{(k)*} c_s + \mathcal{K}_{rs}^{(k)} [Z]_{\text{Res}} c_r^{(k)} c_s^{(k)*} \right\} .
\end{align*}
\]

We call the $2N \times 2N$ matrix $F_r[Z]_{\text{Res}}$ the thermal Res-FB operator. From now let us denote $W_{rs}[Z]_{\text{Res}}, F_r[Z]_{\text{Res}}$ and $\mathcal{K}_{rs}^{(k)}[Z]_{\text{Res}}$ simply as $W_{rs}, F_r$ and $\mathcal{K}_{rs}^{(k)}$, respectively. First, due to idempotent-like product properties $W_{rs}W_{rs}=W_{rs}$ and $W_{sr}W_{sr}=W_{sr}$, we have important relations $\mathcal{K}_{rs}^{(k)}W_{rs}=\mathcal{K}_{rs}^{(k)}$ and $\mathcal{K}_{rs}^{(k)}W_{sr}=\left\{ H[W_{sr}]-E^{(k)} \right\} W_{sr}[\det z_{sr}]^{1/2}$. Next multiplication of the second equation in (B.1) by $W_{rr}$ from the right yields

\[
F_r W_{rr} = F[W_{rr}] W_{rr} \sum_{k=1}^n |c_r^{(k)}|^2 \\
= F[W_{rr}] W_{rr} \sum_{k=1}^n |c_r^{(k)}|^2 - \sum_{k=1}^n \mathcal{K}_{rs}^{(k)} |c_r^{(k)}|^2 - \sum_{k=1}^n H[W_{rr}] - E^{(k)} W_{rr} |c_r^{(k)}|^2 \\
+ \sum_{k=1}^n \sum_{s=1}^n \left\{ \mathcal{K}_{rs}^{(k)} c_r^{(k)*} c_s + \mathcal{K}_{rs}^{(k)} c_r^{(k)} c_s^{(k)*} \right\} .
\]

Using (A.11), the second term in the last line of R. H. S. of (B.2) is vanished. Substituting to $\mathcal{K}_{rs}^{(k)}$ its explicit form obtained from (A.14), thus, $F_r W_{rr}$ is cast into

\[
F_r W_{rr} = F[W_{rr}] W_{rr} \sum_{k=1}^n |c_r^{(k)}|^2 \\
- \sum_{k=1}^n \left\{ (1_{2N} - W_{rr}) F[W_{rr}] + 2 \left\{ H[W_{rr}] - E^{(k)} \right\} \right\} W_{rr} |c_r^{(k)}|^2 + \sum_{k=1}^n \sum_{s=1}^n \mathcal{K}_{rs}^{(k)} c_r^{(k)*} c_s
\]

and taking hermitian conjugate of both sides of (B.3), we have $F_r W_{rr} = W_{rr} F_r$. This means the two hermitian matrices $F_r$ and $W_{rr}$ have common eigenvectors to diagonalize them so that it leads to (B.1). Therefore, (B.1) and the relation $F_r W_{rr} = W_{rr} F_r$ are equivalent. Due to the idempotency relation $W_{rs}^2=W_{rs}$, $F_r W_{rr} = W_{rr} F_r$ is equivalent to $F_r W_{rr} - W_{rr} F_r W_{rr} = 0$. Further multiplying (B.3) by $W_{rr}$ from the left and using the explicit form of $\mathcal{K}_{rs}^{(k)}$, we obtain

\[
W_{rr} F_r W_{rr} = W_{rr}^2 F[W_{rr}] W_{rr} \sum_{k=1}^n |c_r^{(k)}|^2 - \sum_{k=1}^n 2 \left\{ H[W_{rr}] - E^{(k)} \right\} W_{rr}^2 |c_r^{(k)}|^2 \\
+ \sum_{k=1}^n \sum_{s=1}^n W_{rr} \left\{ (1_{2N} - W_{rr}) F[W_{rr}] + H[W_{rr}] - E^{(k)} \right\} W_{rs} [\det z_{rs}]^{1/2} c_r^{(k)*} c_s
\]

Subtracting (B.4) from (B.3), it is easy to derive an equivalence relation

\[
\sum_{k=1}^n \sum_{s=1}^n \mathcal{K}_{rs}^{(k)} c_r^{(k)*} c_s = F_r W_{rr} - W_{rr} F_r W_{rr}.
\]

Thus, the equivalence of (B.1) with (A.14) has been proved. The above equivalent relation (B.5) also makes a crucial role in the variation of the Res-HB free energy.
Let us introduce the following Res-HB free energy $F[Z]_{\text{Res}}^{\text{HB}}$ which is quite similar to (A.3) but involves the HB interstate density matrix instead of the usual HB density matrix:

$$
F[Z]_{\text{Res}}^{\text{HB}(1)} = \sum_{k=1}^{n} \sum_{r,s=1}^{n} \left\{ H[W_{rs}] - E^{(k)} \right\} \cdot \left[ \det z_{rs} \right] \sum_{s} c^{(k)*}_{r} c^{(k)}_{s},
$$

$$
F[Z]_{\text{Res}}^{\text{HB}(2)} = \frac{1}{2} \sum_{r,s=1}^{n} \left\{ \text{Tr} \left\{ W_{rs} \ln W_{rs} + (1_{2N} - W_{rs}) \ln (1_{2N} - W_{rs}) \right\} \right\},
$$

$$
F[Z]_{\text{Res}}^{\text{HB}} = F[Z]_{\text{Res}}^{\text{HB}(1)} + F[Z]_{\text{Res}}^{\text{HB}(2)}.
$$

We are now in a stage to make a variation of the Res-HB free energy. Using the variational formulas (A.12), the thermal Res-HB equation (A.14), the equivalence relation (B.5) and the commutability $[F_{r}, W_{rr}] = 0$, it is made as follows:

$$
\delta F[Z]_{\text{Res}}^{\text{HB}(1)} = \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \sum_{k=1}^{n} \sum_{s=1}^{n} \{ K_{rs}^{(k)} c^{(k)*}_{r} c^{(k)}_{s} \} \cdot \delta u_{r} \right\}
+ \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \delta u_{r} u_{r}^{\dagger} \sum_{k=1}^{n} \sum_{s=1}^{n} K_{rs}^{(k)*} c^{(k)*}_{r} c^{(k)}_{s} \right\}
= \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( F_{r} W_{rr} - W_{rr} F_{r} W_{rr} \right) u_{r} \delta u_{r}^{\dagger} \right\}
+ \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} W_{rr} \right) \delta W_{rr} \right\}
= \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} \right) \delta W_{rr} \right\}
+ \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} W_{rr} \right) \delta W_{rr} \right\}
= \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} \right) \delta W_{rr} \right\}
+ \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} \right) \delta W_{rr} \right\}
$$

(B.7)

$$
\delta F[Z]_{\text{Res}}^{\text{HB}(2)} = \frac{1}{2} \sum_{r=1}^{n} \text{Tr} \left\{ \ln \{ W_{rr} - W_{rr} F_{r} \} \right\} \delta W_{rr}
+ \sum_{r=1}^{n} \frac{1}{2} \text{Tr} \left\{ \left( W_{rr} - W_{rr} F_{r} \right) \delta W_{rr} \right\}
$$

(B.8)

the last of which has no contribution since $(1_{2N} - W_{rs})W_{rs} = 0$. Then, the variational equation $\delta F[Z]_{\text{Res}}^{\text{HB}} = \delta F[Z]_{\text{Res}}^{\text{HB}(1)} + \delta F[Z]_{\text{Res}}^{\text{HB}(2)} = 0$ leads to $\ln \{ W_{rr} - W_{rr} F_{r} \} = -\beta F_{r}$, in which we have used the variational relations $\delta W_{rr} = u_{r} \delta u_{r}^{\dagger} + \delta u_{r} u_{r}^{\dagger} + \delta u_{r}^{\dagger} u_{r}$. Thus we obtain $W_{rr} = \exp \{ -\beta F_{r} \}$ from which we have $W_{rr} = \exp \{ -\beta F_{r} \}$. Finally we can reach the r-th thermal HB density matrix $W_{rr}[Z]_{\text{Res}}$ expressed in terms of the r-th thermal Res-FB operator $F_{r}[Z]_{\text{Res}}$ as $W_{rr}[Z]_{\text{Res}} = \frac{1}{1_{2N} + \exp \{ \beta F_{r}[Z]_{\text{Res}} \}}$ where we have used the relation $[F_{r}, W_{rr}] = 0$. By using a Bogoliubov transformation $g_{r}$, $W_{rr}[Z]_{\text{Res}}$ is diagonalized as follows:

$$
W_{r} = r^{\dagger} W_{rr}[Z]_{\text{Res}} g_{r} = \begin{pmatrix} \tilde{w}_{ri} & 0 \\ 0 & 1_{N} - \tilde{w}_{ri} \end{pmatrix}, \quad \tilde{w}_{ri} = \frac{1}{1 + \exp \{ -\beta \epsilon_{ri} \}} (r=1, \ldots, n),
$$

(B.9)

$$
W_{r} = r^{\dagger} W_{rr}[Z]_{\text{Res}} g_{r} = \begin{pmatrix} \tilde{w}_{ri} & 0 \\ 0 & 1_{N} - \tilde{w}_{ri} \end{pmatrix}, \quad \tilde{w}_{ri} = \frac{1}{1 + \exp \{ -\beta \epsilon_{ri} \}} (i=1, \ldots, N).
$$

The diagonalization of the r-th thermal Res-FB operator $F_{r}[Z]_{\text{Res}}$ by the same Bogoliubov transformation $g_{r}$ leads us to the eigenvalue $\epsilon_{ri}$. To this eigenvalue by adding a term $\sum_{k=1}^{n} \left\{ H[W_{rr}[Z]_{\text{Res}} - E^{(k)}] \right\} |c^{(k)}_{r}|^{2} \cdot 1_{2N}$, the usual HB type of the eigenvalue $\epsilon_{ri}$ is realized. Using (B.9) we can derive the inverse transformation of (B.9) in the form

$$
g_{r}^{\dagger} W_{rr}[F_{r}] g_{r} = \frac{1}{1_{2N} + \exp \{ \beta (F_{r}[Z]_{\text{Res}} + \sum_{k=1}^{n} \left\{ H[W_{rr}[Z]_{\text{Res}} - E^{(k)}] \right\} |c^{(k)}_{r}|^{2} \cdot 1_{2N}) \}} g_{r}
$$

(B.10)
C Calculations of $\Sigma_p A_p$, $\Sigma_p B_p$ and $\Sigma_p C_p$ at intermediate temperature

First we give a integral formula

$$\int_0^\infty dy \left\{ \frac{1}{y^3} \tanh y - \frac{1}{y^2} \sech^2 y \right\} = \frac{7}{\pi^2} \zeta(3), \quad (C.1)$$

which can be derived by using the famous mathematical formulas [22]

$$\frac{1}{y} \tanh y = 8 \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 \pi^2 + 4y^2}, \quad \frac{1}{y^3} \tanh y - \frac{1}{y^2} \sech^2 y = 64 \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 \pi^2 + 4y^2}^2. \quad (C.2)$$

Adopting a new integral variable $y = (2m-1)\pi/2 \cdot \tan \theta$, an integral of the second formula in (C.2) is easily carried out for $h\omega_D \gg 1$ as

$$64 \int_0^{\pi/2} dy \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 \pi^2 + 4y^2}^2 = 32 \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3 \pi^3} \int_0^\pi d\theta \frac{1}{1+\tan^2 \theta} = \frac{7}{\pi^2} \zeta(3), \quad (C.3)$$

where we have used $\sum_{m=1}^{\infty} (2m-1)^{-3} = (7/8) \cdot \zeta(3)$, and $\zeta(3) = \pi^3/25.79436$ [22].

Next the modified QP energy $\tilde{\varepsilon}$ is approximated as $\tilde{\varepsilon}^{(\pm)} = \sqrt{\varepsilon^2 + \Delta_T^2} \left\{ 2 \left[ 1 \pm \det z_{12} \right] \right\}^{-1}$. Let us introduce a new variable $y$ by $\varepsilon = 4 \left( 1 \pm \det z_{12} \right) k_B T y$ and quantities $\tilde{x}_T = \Delta_T / h\omega_D$ and $y_T^{(\pm)} = \sqrt{\varepsilon^2 + \Delta_T^2} \left\{ 4 \left( 1 \pm \det z_{12} \right) \right\}^{-1} h\omega_D / k_B T$ where $\Delta_T \equiv \Delta_T N(0) V \arcsinh(h\omega_D / \Delta_T)$.

If $\varepsilon \gg \Delta_T$, $\Sigma_p A_p$, $\Sigma_p B_p$ and $\Sigma_p C_p$ in (2.14) are recast to the following integrals up to $\Delta_T$:

$$\frac{\Sigma_p A_p}{2N(0)} \simeq \int_0^{h\omega_D} dy \int_0^{\pi/2} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta_T^2} \sech^2 y} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) - \int_0^{h\omega_D} dy \int_0^{\pi/2} d\varepsilon \frac{\Delta_T^2}{\varepsilon^2 + \Delta_T^2} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right), \quad (C.4)$$

$$\frac{\tilde{\Delta}_T \Sigma_p B_p}{N(0)} \simeq \int_0^{h\omega_D} dy \int_0^{\Delta_T} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) - \int_0^{h\omega_D} dy \int_0^{\Delta_T} \frac{\tilde{\Delta}_T^2}{2 \varepsilon \Delta_T} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) = 0, \quad (C.5)$$

$$\frac{\tilde{\Delta}_T^2 \Sigma_p C_p}{2N(0)} \simeq \int_0^{h\omega_D} dy \int_0^{\Delta_T} \frac{\tilde{\Delta}_T^2}{\varepsilon^2 + \Delta_T^2} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) - \int_0^{h\omega_D} dy \int_0^{\Delta_T} \frac{\tilde{\Delta}_T^2}{\varepsilon^2 + \Delta_T^2} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) \equiv \left( y_T^{(\pm)} \tilde{x}_T \right)^2 \int_0^{\pi/2} dy \int_0^{\pi/2} d\varepsilon \frac{1}{y^2} \tanh \left( \frac{\tilde{\varepsilon}^{(\pm)}}{2k_B T} \right) \left( y_T^{(\pm)} \tilde{x}_T \right)^2. \quad (C.6)$$

To get a finite value of $\Sigma_p A_p$, expanding (C.4) around $y_T^{(\pm)} \tilde{x}_T$, (C.4) is boldly approximated as

$$\frac{\Sigma_p A_p}{2N(0)} \simeq \int_0^{y_T^{(\pm)}} dy \frac{1}{y} \tanh y - \frac{3}{2} \left( y_T^{(\pm)} \tilde{x}_T \right)^2 \int_0^{y_T^{(\pm)}} dy \left\{ \frac{1}{y^3} \tanh y - \frac{1}{y^2} \sech^2 y \right\}. \quad (C.7)$$

In a similar way we also get a roughly approximated integral form for (C.6) as

$$\frac{\tilde{\Delta}_T^2 \Sigma_p C_p}{2N(0)} \simeq \left( y_T^{(\pm)} \tilde{x}_T \right)^2 \int_0^{y_T^{(\pm)}} dy \left\{ \frac{1}{y^3} \tanh y - \frac{1}{y^2} \sech^2 y \right\}. \quad (C.8)$$

Integrations of (C.7) and (C.8) are easily made by using the integration formula (C.1) if we take the upper-value $y_T^{(\pm)}$ to be infinite.
References

[1] S. Nishiyama, J. da Providência and H. Ohnishi, in Proceedings of the 26th International Workshop on Condensed Matter Theories, ed. M. de Llano, C. Fiolhais and J. da Providência, (Nova Science Publishers, Inc., USA), Condensed Matter Theories Vol. 18 (2003), 169-178; in Proceedings of the 29th International Workshop on Condensed Matter Theories, ed. H. Akai, A. Hosaka, H. Toki and F. Bary Malik, (Nova Science Publishers, Inc., USA), ibid. Vol. 21 (2007), 13-25

[2] J. Nagamatsu, N. Nakagawa, T. Muranaka, Y. Zenitani and J. Akimitsu, Nature 410 (2001), 63-64

[3] H. Fukutome, Prog. Theor. Phys. 80 (1988), 417-432

[4] S. Nishiyama and H. Fukutome, Prog. Theor. Phys. 85 (1991), 1211-1222

[5] S. Nishiyama and H. Fukutome, J. Phys. G: Nucl. Part. Phys. 18 (1992), 317-328; H. Ohnishi, J. da Providência and S. Nishiyama, in Proceedings of the 29th International Workshop on Condensed Matter Theories, ed. H. Akai, A. Hosaka, H. Toki and F. Bary Malik, (Nova Science Publishers, Inc., USA), Condensed Matter Theories Vol. 21 (2007), 395-406

[6] S. Nishiyama, J. da Providência, C. Providência and H. Ohnishi, Adv. Studies Theor. Phys. 4 (2010), 283-303.

[7] J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys. Rev. 108 (1957), 1175-1204

[8] N.N. Bogoliubov, Usp. Fiz. Nauk 67 (1959), 549-580 [Soviet Phys. Uspekhi 67 (1959), 236-254]

[9] N.N. Bogoliubov, V.V. Tolmachev, D.V. Shirkov, A New Method in the Theory of Superconductivity (Consultants Bureau, New York, 1959)

[10] G.M. Eliashberg, Zh. Exp. Teor. Fiz. 38 (1960), 966-976 [Soviet Phys. JETP 11 (1960), 696 -702]

[11] J.R. Schrieffer, Theory of Superconductivity (W.A. Benjamine, New York, 1964)

[12] The Problem of High Temperature Superconductivity, edited by V.L. Ginzburg and D.A. Kirzhnits (Consultant Bureau, New York, 1982)

[13] R.D. Parks (ed.), Superconductivity, 2 Vols. (Marcel Dekker, New York, 1969)

[14] Jens Kortus, I.I. Mazin, K.D. Belashchenko, V.P. Antropov and L.L. Boyer, Phys. Rev. Lett. 86 (2001), 4656-4659

[15] Amy Y. Liu, I.I. Mazin and Jens Kortus, Phys. Rev. Lett. 87 (2001), 87005-1-87005-4

[16] H.J. Choi, D. Roundy, H. Sun, M.L. Cohen and S.G. Louie, Phys. Rev. B66 (2002), 020513-1(R)-020513-4(R); Nature 418 (2002), 758-760
[17] A.M. Perelomov, Commun. Math. Phys. 26 (1972), 222-236; Usp. Fiz. Nauk 123 (1977), 23-55 [Sov. Phys. Usp. 20 (1977), 703-720]

[18] I.M. Khalatnikov and A.A. Abrikosov, Adv. Phys. 8 (1959), 45-86

[19] A.A. Abrikosov, L.P. Gor’kov and I. Ye. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics*, (Pergamon, Oxford, 1965)

[20] A.A. Abrikosov, *Fundamentals of the Theory of Metals*, (North-Holland, Amsterdam, 1988)

[21] M. Ozaki, J. Math. Phys. 26 (1985), 1514-1520.

[22] I.S. Gradshteyn and I.M. Ryzhik, A. Jeffrey, Editor, *Tables of Integrals, Series and Products*, Fifth Edition, Academic Press, London, 1994.

[23] G.M. Eliashberg, Zh. Exp. Teor. Fiz. 39 (1960), 1437-1441 [Soviet Phys. JETP, 12 (1961), 1000-1002]