Junction conditions at spacetime singularities

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A classical model for the extension of singular spacetime geometries across their singularities is presented. The regularization introduced by this model is based on the following observation. Among the geometries that satisfy Einstein’s field equations there is a class of geometries, with certain singularities, where the components of the metric density and their partial derivatives remain finite in the limit where the singularity is approached. Here we exploit this regular behavior of the metric density and elevate its status to that of a fundamental variable – from which the metric is constructed. We express Einstein’s field equations as a set of equations for the metric density, and postulate junction conditions that the metric density satisfies at singularities. Using this model we extend certain geometries across their singularities. The following examples are discussed: radiation dominated Friedmann-Robertson-Walker Universe, Schwarzschild black hole, Reissner-Nordström black hole, and certain Kasner solutions. For all of the above mentioned examples we obtain a unique extension of the geometry beyond the singularity.

I. INTRODUCTION

General relativity (GR) is in excellent agreement with all experimental and observational tests of gravity. However, when a GR solution develops a singularity it is sometimes impossible to use Einstein’s field equations to extend the geometry beyond the hypersurface where the singularity resides. The standard lore is that this incompleteness should be resolved by a quantum gravity theory that may become important when the curvature length scale is of the order of Planck length. It is possible that quantum-gravity phenomena can smooth out singularities and thereby resolve the nonextendibility of singular GR solutions. Alternatively, it is possible that the correct description of gravity in the vicinity of singularities requires first an extension or a modification of classical GR (e.g. classical string theory introduces modifications to GR \cite{1}). In this case, quantization of gravity can take place only after our classical understanding of gravity has been modified \cite{2}. Therefore, there is a good motivation to extend or modify GR so that it would not breakdown at singularities. Moreover, quantum theories can inherit singularities that appear in their corresponding classical theories (e.g. a charged point particle gives rise to an infinite electrostatic energy in classical electrodynamics, a related ultraviolet divergent behavior also appears in quantum electrodynamics). Therefore, handling spacetime singularities at the classical level may provide a useful preliminary step towards a more regular quantum gravity analysis of singularities. Furthermore, this classical analysis could shed light on the extension of spacetime geometries beyond their singularities.

In this manuscript we introduce a classical model for gravity that retains its predictive power for a class of spacetime singularities. In the construction of the model we make sure that the following requirements are satisfied: First, to ensure agreement with observational constraints, we demand that except for singularities the model would coincide with GR. Second, we demand that the model would be able to provide predictions for the extension of certain spacetime geometries beyond their singularities.

Below we use our model to extend certain GR solutions across their singularities. The following examples are studied: Friedmann-Robertson-Walker Universe (FRW), Schwarzschild black hole, charged Reissner-Nordström black hole (RN), and Kasner Universe. For other classical approaches to singularities see Penrose \cite{3} and Tod \cite{4}.

The regularization that we introduce is based on the following observation. There is a class of spacetime singularities where the components of the metric density (in certain coordinates) attain finite values in the limit where the singularity is approached. As an example let us consider a Schwarzschild black hole. In the Schwarzschild coordinates the metric reads $g^{\text{Sch}}_{\mu\nu} = \text{diag}(-b/r, r/b, r^2, r^2 \sin^2 \theta)$, where $b = r - 2M$, and $M$ denotes the mass of the black hole. In the limit $r \rightarrow 0$, the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ diverges thereby indicating that there is a spacetime singularity at $r = 0$. Here we focus attention on the contravariant metric density $g^{\alpha\beta}_{\text{Sch}}$ which is equal to the product $\sqrt{-\det(g^{\text{Sch}}_{\mu\nu})} g^{\alpha\beta}_{\text{Sch}}$. In the Schwarzschild coordinates, it reads $g^{\alpha\beta}_{\text{Sch}} = \text{diag}(-b^{-1}r^3 \sin \theta, rb \sin \theta, \sin \theta, \sin \theta^{-1})$. Notice that in the limit of interest the components of $g^{\alpha\beta}_{\text{Sch}}$ and their partial derivatives remain finite, though the complete matrix $g^{\alpha\beta}_{\text{Sch}}$ becomes degenerate. In this paper we exploit this non-divergent behavior of the metric density and elevate its status to that of a fundamental variable, while the metric becomes a constructed variable, which is constructed from the metric density. We express Einstein’s field equations as a set of equations for the metric density and postulate junction conditions at the singularities. We use this framework to extend certain geometries across their singularities. At the singularities we find that the components of the metric density

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are continuous and differentiable even though the geometry remains ill-defined. In fact our model does not remove any physical singularity. On the contrary, as in GR we find that the Schwarzschild geometry becomes singular as \( r \to 0 \). However, in our framework the spacetime geometry becomes a constructed entity, which is constructed from a non-divergent fundamental quantity — the metric density. Since the metric is no longer a fundamental dynamical variable, its singular behavior at a spacetime singularity does not obstruct the continuation of the solution beyond the singularity.

This paper is organized as follows: in Sec. II the classical model is presented, in Sec. III we discuss few examples in detail, and Sec. IV provides conclusions.

II. THE CLASSICAL MODEL

Let us begin by defining the metric density \( g^{\mu\nu} \) to be a four-dimensional symmetric contravariant tensor density \( \in \mathbb{R} \) of weight -1. In the standard GR formulation the metric is represented by a non-degenerate matrix. Recall, however, that the metric density may become degenerate in the limit where a spacetime singularity is approached. For example, the Schwarzschild metric density \( g_{\text{Sch}}^{\alpha\beta} \) becomes a degenerate matrix in the limit \( r \to 0 \). Exploiting the non divergent behavior of \( g_{\text{Sch}}^{\alpha\beta} \), our first step is to extend the solution \( g_{\text{Sch}}^{\alpha\beta} \) in a continuous manner, and allow the metric density to attain its degenerate configuration in the limit.

Here the covariant metric density \( g_{\alpha\beta} \) is defined to be the adjoint matrix (also called the adjugate matrix) of \( g^{\alpha\beta} \)

\[
\begin{align*}
g_{\alpha\beta} & \equiv \text{adj}(g^{\alpha\beta}) ,
\end{align*}
\]

This means that \( g_{\alpha\beta} \) is equal to the transpose of the cofactor matrix of \( g^{\alpha\beta} \). From definition (6) it follows that \( g_{\alpha\beta} \) transforms as a covariant tensor density of weight \(-1\) under a general coordinate transformation \( \mathbf{g} \), and furthermore Eq. (6) guarantees that \( g_{\alpha\beta} \) remains well defined even if \( g^{\alpha\beta} \) becomes degenerate. For example, in the Schwarzschild singularity both \( g_{\alpha\beta} \) and \( g_{\text{Sch}}^{\alpha\beta} \) as well as their partial derivatives (up to all orders), all have a well defined limit as \( r \to 0 \). By contrast, the singular behavior of the Kretschmann scalar at the Schwarzschild singularity implies that the metric must exhibit some kind of singular behavior at this limit.

Notice that "observables" (3,4,5) that are constructed directly from the metric-density remain well defined even if domain \( D \) contains a singularity where \( g^{\mu\nu} \) takes the form of a nondivergent degenerating matrix. By contrast, a curve in \( D \) that passes through such a singularity may have an ill defined length. From Eq. (6) we find that

\[
\begin{align*}
\left(g^{\mu\nu}\right)_{\alpha\beta} & = g_{\alpha\beta}.
\end{align*}
\]

In analogy with the metric, one can use the metric density to construct evolution equations for other fields. Equations of motion that depend directly on the metric density may retain their predictive power at singularities where the metric density becomes degenerate. For example, in a fixed RN background the scalar wave equation \( \left(g^{\alpha\beta}\partial_{,\alpha}\right)_{,\beta} = 0 \) determines the transmission of scalar waves through the RN singularity [7].

1 We ignore coordinate singularities, e.g. at \( \theta = 0 \), since these singularities can be removed by a coordinate transformation.

2 More precisely, since the diverging Kretschmann scalar is equal to a sum of products of terms that depend on the metric (including the metric, its inverse, and their derivatives up to the second) it follows that in all possible coordinates at least one of these metric dependent terms must diverge at the singularity.
Next we define the covariant metric to be
\[ g_{\alpha\beta} \equiv -(-g)^{-1/2} \mathcal{g}_{\alpha\beta}. \] (8)

It follows from Eqs. (10,11,12,13) that for \( g \neq 0 \) we have
\[ g = g; \; g^{\alpha\beta}g_{\beta\gamma} = \delta_{\gamma}^{\alpha}, \] (9)
\[ g_{\alpha\beta} = -\sqrt{-g}g_{\alpha\beta}; \; g^{\alpha\beta} = \sqrt{-g}g_{\alpha\beta}. \] (10)

Here \( g \equiv \det(g_{\alpha\beta}) \).

We now construct the equations of motion for the metric density. For this purpose we cast Einstein’s field equations in a densitized format. Using the standard Landau-Lifshitz formulation \( \mathcal{S} \) and units where \( G = c = 1 \) we obtain
\[ (g^{\mu\nu} \mathcal{g}_{\alpha\beta} - g^{\alpha\mu} \mathcal{g}_{\nu\beta})_{\alpha\beta} = 16\pi \mathcal{T}^{\mu\nu}_{\text{total}}. \] (11)

Here \( \mathcal{T}^{\mu\nu}_{\text{total}} = \mathcal{T}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{LL} \), \( \mathcal{T}^{\mu\nu} = (-g)T^{\mu\nu} \), where \( T^{\mu\nu} \) denotes the energy-momentum tensor depending on the matter fields and the metric density via Eqs. (10,11), and
\[ t^{\mu\nu}_{LL} = \frac{1}{16\pi} \left[ 2g^{\mu\rho} \mathcal{g}_{\alpha\beta}^{\lambda} \omega + \frac{1}{2} g^{\alpha\nu} \mathcal{g}_{\lambda\beta}^{\rho} \omega \sigma + \frac{1}{16\pi} g^{\alpha\rho} \mathcal{g}_{\beta\lambda}^{\sigma} \omega + 2g^{\sigma\rho} \mathcal{g}_{\lambda\beta}^{\nu} \omega \right]. \] (12)

Here \( t^{\mu\nu}_{LL} = (-g)t^{\mu\nu}_{LL} \), where \( t^{\mu\nu}_{LL} \) denotes the Landau-Lifshitz pseudo-tensor \( \mathcal{S} \). For \( g \neq 0 \) we may substitute Eqs. (11) into the equations of motion (11) and recover the standard Einstein’s field equations depending on the metric. In this case, Eq. (11) implies that \( \mathcal{T}^{\mu\nu}_{\text{total}} \) satisfies the Landau-Lifshitz energy-momentum conservation law \[ \partial_\nu \mathcal{T}^{\mu\nu}_{\text{total}} = 0. \] (12)

Let us consider a vacuum singularity, e.g., the Schwarzschild singularity, where \( g^{\mu\nu} \) is finite but degenerate. Notice that some of the terms in \( t^{\mu\nu}_{LL} \) have negative powers of \( g \). These terms become ambiguous when the metric density becomes degenerate. This does not mean that \( t^{\mu\nu}_{LL} \) must diverge at singularity, e.g., in the Schwarzschild coordinates \( t^{\mu\nu}_{LL} \) has a well defined limit as the singularity is approached. Still the ambiguity in \( t^{\mu\nu}_{LL} \) may give rise to difficulties in the extension of the solution across the singularity by using Eq. (11) alone. To overcome these difficulties we shall postulate new junction conditions at the singularity.

Our goal is to find junction conditions that can be combined with the equations of motion (11) in a manner that would provide a unique extension of the solution across the hypersurface where the singularity resides. We would like to exploit the fact that the components of the metric density and their derivatives may remain finite as the singularity is approached, and so it is possible to demand that these components would have some degree of smoothness at the singularity. It is tempting to demand the components of the metric density would be analytical functions of the coordinates. In fact requiring analyticity of the metric is a standard method to continue the Kerr geometry through its ring singularity (see e.g. [10]). However, in some respect demanding analyticity of the metric density is a too strong requirements. An analytical continuation completely determines the extension of the solution, so it is no longer necessary to solve the equations of motion. This however contradicts our goal of keeping the standard physical picture where the solution is evolved via differential equations, and the junction conditions merely determines how the solution is extended across the hypersurface with the singularity.

A weaker condition than analyticity is smoothness (i.e. \( C^\infty \)). Indeed an assumption of a smooth metric is often found in standard GR theorems. For example, in a rigorous initial value formulation one assumes smooth initial data \( \mathcal{I} \). In analogy with this standard GR assumption, we postulate that the components of metric density be smooth functions of the coordinates. More precisely, we focus attention to metric density configurations that satisfy Eq. (11), not necessarily in vacuum, where \( g^{\alpha\beta} \) is smooth away from a genuine singularity of the geometry, and its components and all of their partial derivatives have a well defined limit as the singularity is approached. We assume that \( g \) may be zero at most on a hypersurface, but not in an open set (i.e. not in the bulk), and demand that at a singular hypersurface, where \( g = 0 \), the components of \( g^{\mu\nu} \) remain smooth. We then use these conditions together with Eq. (11) to extend the solution across the singularity. Let us now discuss few examples in detail.

III. EXAMPLES

A. Friedmann-Robertson-Walker

First we consider a flat FRW Universe filled with a perfect fluid. In the standard GR formulation this solution is nonextendible since its geometry becomes singular at the big bang. By virtue of the underlying symmetry we express the metric density as \( g^{\alpha\beta} = A(\eta)\eta^{\alpha\beta} \), where \( g^{\mu\nu} \) denotes the Minkowski metric, and the scale factor is given by \( a = \sqrt{A} \), where \( A \geq 0 \). In GR we fix the signature of the metric to be, say \((- + + +)\). The matter source term reads \( \mathcal{T}^{\mu\nu} = \text{diag}(\rho, \rho, \rho, \rho) \), where \( p = A^{-3}\rho \) and \( \rho = A^{-3}\rho \) are the proper pressure and energy density, respectively. For \( \rho \neq 0 \) we substitute the expressions for \( \mathcal{T}^{\mu\nu} \) and \( g^{\mu\nu} \) into Eq. (11) and obtain
\[ 3A^2 = 32\pi \rho \] (13)
\[ 3A^2 - 4\pi A = 32\pi \rho. \] (14)
Here an overdot denotes differentiation with respect to conformal time \( \eta \). We assume an equation of state of the form \( p = \omega \rho \), where \( \omega \) is a constant. Substituting this relation into Eqs. (13,14) gives \( \rho \propto A^{3(1-\omega)/2} \). We
substitute $\tilde{\rho}(A)$ into Eq. (13) and solve for $A$. Recall that we seek a solution where $g^{\alpha\beta}$ and all of its partial derivatives have a well defined limit as singularity is approached. For the particular coordinates in use, this condition is satisfied for a radiation dominated Universe characterized by $w = 1/3$. Demanding that $g^{\alpha\beta}$ be smooth at the singularity gives

$$A \propto \eta^2.$$  

(15)

This solution describes a bounce at the big bang.

Notice that the matter source term $\Sigma^\mu\nu$ is continuous at the singularity, but the complete source term $\Sigma^\mu\nu_{\text{total}}$ has an ambiguity at $A = 0$, since $\Sigma^\mu\nu_{\text{ext}}$ contains the combination $A/A$, which is ambiguous for $A = 0$. Imposing continuity, we define $\Sigma^\mu\nu_{\text{ext}}(0)$ to be the limit of $\Sigma^\mu\nu_{\text{ext}}(\eta)$ as $\eta \to 0$. This gives, for all values of $\eta$,

$$16\pi \Sigma^\mu\nu_{\text{ext}} = -\tilde{A}^2 \text{diag}(3/2, 7/2, 7/2, 7/2).$$

Substituting this expression into the right hand side of Eq. (11) shows that solution (15) satisfies the equation of motion (11) at the singularity. Moreover, evaluating the divergence of $\Sigma^\mu\nu_{\text{total}}$ shows that the energy-momentum conservation law (12) is satisfied at the singularity. By contrast, it is hard to make sense of the covariant conservation law $\nabla\alpha T^{\beta\gamma} = 0$ at the big bang, since both $\nabla\alpha$ and $T^{\beta\gamma}$ diverge at the singularity.

The above solution shows that the FRW singularity is described by a field configuration where both $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are smooth, but the metric as defined in Eqs. (15) can not be constructed. This suggests the interpretation that a degenerate metric density describes a pre-metric configuration which is more primitive than a geometry.

### B. Black holes

Next we study a spherically symmetric charged black hole characterized by a mass $M$ and a charge $Q$. Substituting the RN solution into Eq. (10) we find that for $r > 0$ the non-zero components of the metric density are given by

$$\bar{\theta}_{\text{RN}}^\mu = -r^2 \sin \theta \quad g_{\text{RN}}^{\theta\theta} = \sin \theta$$

(16)

$$g_{\text{RN}}^{r\theta} = \sqrt{\theta_{\text{RN}}^r r + Q^2} \sin \theta, \quad g_{\text{RN}}^{\phi\phi} = (\sin \theta)^{-1},$$

and $\sqrt{\theta_{\text{RN}}^r r} = r^2 \sin \theta$. We focus on the domain $r < r_-$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$, and $M > |Q|$. Notice that in the limit $r \to 0$ the RN metric density becomes a non-divergent degenerate matrix. The $r = 0$ surface corresponds to a genuine singularity since the scalar $R_{\mu\nu}R^{\mu\nu}$ diverges in the limit. We now extend the solution across this singularity to the domain $r < 0$. For this purpose, we first have to solve Maxwell equations in vacuum. To overcome the singularity in the electromagnetic field we cast Maxwell equations in a densitized format, and obtain

$$\bar{\hat{s}}^{\mu\nu} = 0.$$  

(17)

FIG. 1: Penrose-Carter diagram of the extended RN space-time.

Here $\bar{\hat{s}}^{\mu\nu} = \sqrt{-\hat{g}} F^{\mu\nu}$, where $F^{\mu\nu}$ is the electromagnetic field tensor. For $r > 0$ the non-zero components of the RN solution $\bar{\hat{s}}^{\mu\nu}_{\text{RN}}$ read

$$\bar{\hat{s}}^{rr}_{\text{RN}} = -\bar{\hat{s}}^{tt}_{\text{RN}} = Q \sin \theta.$$  

(18)

Notice that Eq. (17) has no singularity at $r = 0$, and solution (18) satisfies the Maxwell equations (17) in the entire domain $-\infty < r < r_-$. We now solve Eq. (11) in $r < 0$. Exploiting the underlying symmetry we substitute $g^{\alpha\beta} = \text{diag}(-C^4(r) \sin \theta/D(r), D(r) \sin \theta, \sin \theta, (\sin \theta)^{-1})$ into Eq. (11), where we introduced the combination $C^4/D$ in $g^{\mu\nu}$ to slightly simplify the equations. Demanding that $g^{rr}, g^{tt}$ be smooth at $r = 0$, and using Eqs. (16) determines $C(r)$ and $D(r)$ uniquely. The complete solution for $-\infty < r < r_-$ is given by Eq. (16). This expression for the extended geometry is also obtained for the special case of the Schwarzschild solution where $Q = 0$. Figure 1 shows the Penrose-Carter diagram of the RN extension for $Q \neq 0$. We should mention that this extension had been previously conjectured in Ref. [7]. Notice that the components of this extended RN metric density are analytical in the domain of interest, and in particular they are well defined at the $r = 0$ singularity. However, the metric as defined in Eqs. (18) can not be constructed at the singularity. Similar to the FRW example, we define $\Sigma^\mu\nu_{\text{total}}(0)$ to be the limit of $\Sigma^\mu\nu_{\text{total}}(r)$ as $r \to 0$. With this definition, the equation of motion (11) and the conservation law (12) are satisfied at the singularity.

The geometry of a rotating Kerr (and Kerr-Newmann) black hole can be analytically extended through its ring singularity using the standard GR formalism (see e.g. [10]), and therefore a regularization of the singularity is not essential in this case. Nevertheless, it interesting to note that in the Boyer-Lindquist coordinates the components of the metric density of the Kerr (and Kerr-Newmann) solution are analytical in the neighborhood of the singularity. Here again the metric density takes the form of a degenerate non-diverging matrix at the singularity.
C. Kasner

The Kasner solution is a vacuum solution which is flat and homogeneous but anisotropic. In the Kasner coordinates \((t, x, y, z)\) the metric density of the Kasner solution is given by

\[
\mathbf{g}^{\alpha\beta}_{\text{Kas}} = \text{diag}(-t, t^{1-2p_1}, t^{1-2p_2}, t^{1-2p_3}).
\]

(19)

Here the three parameters \(p_i\), satisfy

\[
p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.
\]

(20)

The Kasner geometry is regular for \(t > 0\) but has a singularity at \(t = 0\). To be able to use our model and extend the geometry to the domain \(t < 0\), we should check that the components of the metric density, and their partial derivatives, have a well defined limit as the singularity is approached. Here however there is a difficulty. Notice that the constraints \([20]\) imply that at least one of the \(p_i\) parameters is greater than \(1/2\), and so at least one of the components of \(\mathbf{g}^{\alpha\beta}_{\text{Kas}}\) must diverge at the singularity. To resolve this difficulty we should seek a coordinate transformation that removes the singularity from \(\mathbf{g}^{\alpha\beta}_{\text{Kas}}\). Below we provide such a transformation for the case where the \(p_i\) parameters are given by three different rational numbers.

Let us consider first the domain \(t > 0\) and introduce the coordinate transformation \(t = e^{q_s}t\), where \(s\) is a positive integer whose value is specified below, and \(\epsilon = 1\). In the new coordinates \((\eta, x, y, z)\) the metric density reads

\[
\mathbf{g}^{\alpha'\beta'}_{\text{Kas}} = \text{diag}(-s^{-1}\eta, s\eta^{q_1}, s\eta^{q_2}, s\eta^{q_3}).
\]

(21)

Here \(q_i = 2s(1 - p_i) - 1\). Let us denote the combinations \((1 - p_i)\) with \(m_i,n_i\), where \(m_i,n_i\) are positive integers. Setting \(s = n_1n_2n_3\) implies that the parameters \(q_1,q_2,q_3\) are given by positive odd integers. Notice that the components of the Kasner metric density \(\mathbf{g}^{\alpha\beta}_{\text{Kas}}\) and their partial derivatives, both have a well defined limit as \(\eta \to 0\) as desired.

We are now ready to use our model and extend the solution to \(\eta < 0\). Notice that solution \([21]\) satisfies Eq. \([11]\) in \(\eta < 0\), though here the signature is \((+,−,−,−)\). Imposing our junction conditions by demanding that the components of the metric density be smooth at the \(\eta = 0\) singularity, ensures that the parameters \(s\) and \(q_1,q_2,q_3\) have the same values for \(\eta < 0\) and \(\eta > 0\). This uniquely determines the continuation of the geometry across the singularity. It is now possible to transform back to the original coordinates (For the case where \(s\) is an even number, we set \(\epsilon = -1\) for \(\eta < 0\), so that negative values of \(\eta\) correspond to negative values of \(t\)).

IV. CONCLUSIONS

Singularities mark the breakdown of the laws of general relativity. There is a widely accepted view that a viable quantum theory of gravity should have a mechanism that removes singularities, for example by some quantum-gravity phenomena that smooth them out. In this paper we have presented a classical model that illustrates an alternative point of view. We showed that it is possible to keep the singularities in the solution, so that they become a legitimate part of the physical model.

The construction of our model was based on the observation that the metric density encodes the same amount of information as the metric, but unlike the metric, its components and their partial derivatives may remain finite at a class of spacetime singularities. Our strategy was to exploit this property, and replace the metric with the metric density as the fundamental variable of gravity. By retaining the dynamical evolution of GR, and supplementing it with junction conditions we showed that it is possible to extend certain singular geometries across their singularities.

It would be interesting to see if our model could be used to extend singular geometries in \(d\) dimensions. Curiously, however, not all the observables can be adjusted to accommodate a \(d\) dimensional metric density. In particular notice that \(g_{\alpha\beta}\) transforms as a tensor density of weight \(-(d-3)\), and so the inner product \((u^\alpha, v^\beta)\) defined by Eq. \([5]\) is a scalar only if \(d = 4\).

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[6] This property follows from the transformation law for \(g^{\alpha\beta}\) together with the following properties of the adjoint \(adj(AB) = adj(A)adj(B)\), where \(A, B\) are matrices, and \(adj(C) = C^{-1}\) det(C) where \(C\) is an invertible matrix (notice that the coordinate transformation matrix \(\frac{\partial x'^\alpha}{\partial x^\alpha}\) is invertible).
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