ON THE SZÜSZ’S SOLUTION TO
GAUSS’ PROBLEM

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Abstract

The present paper deals with Gauss’ problem on continued fractions. We present a new proof of a theorem which Szüsz applied in order to solve this problem. To be noted, that we obtain the value 0.759... for q, which has been optimized by Szüsz in his 1961 paper "Über einen Kusminschen Satz", where the value 0.485 is obtained for q. In our proof, we make use of an important property of the Perron-Frobenius operator of τ under γ, where τ is the continued fraction transformation, and γ is the Gauss’ measure.

Keywords: continued fractions, Gauss-Kuzmin problem

1 INTRODUCTION

Let ξ ∈ [0, 1), and let

\[ ξ = \frac{1}{d_1 + \frac{1}{d_2 + \ldots + \frac{1}{d_n + \ldots}}} = [0; d_1, d_2, \ldots, d_n, \ldots] \]

be the regular continued fraction expansion of ξ. On October 25, 1800, Gauss wrote in his diary that (in modern notation):

\[ \lim_{n \to \infty} \lambda(\{ξ ∈ [0, 1); τ^n(ξ) ≤ z\}) = \frac{\log(z + 1)}{\log 2}, 0 ≤ z ≤ 1, \quad (1) \]

where λ is the Lebesgue measure and τ : [0, 1) → [0, 1) is the continued fraction transformation defined by

\[ τ(ξ) := \frac{1}{ξ - \left[ \frac{1}{ξ} \right]}, ξ \neq 0; τ(0) := 0, \quad (2) \]
where \([\cdot]\) denotes the floor (entire) function. Latter on, in a letter dated January 30, 1812, Gauss asked Laplace to provide an estimate of the error term \(r_n(z)\), defined by
\[
r_n(z) := \lambda(\tau^{-n}([0, z])) - \frac{\log(z + 1)}{\log 2}, n \geq 1.
\]
Gauss’ proof has never been found. The first who prove (1) and at the same time answered to Gauss question was Kuzmin. In 1928, Kuzmin [3] showed that
\[
r_n(z) = O(q^{\sqrt{n}}),
\]
with \(q \in (0, 1)\), uniformly in \(z\). Independently, Paul Lévy showed one year later that
\[
r_n(z) = O(q^n),
\]
with \(q = 0.7\ldots\), uniformly in \(z\). From that moment onwards, a great number of such Gauss-Kuzmin theorems followed. To mention a few: F. Schweiger (1968), P. Wirsing [6] (1974 - which determined that the optimal value of \(q\) is equal to 0.303663002), K.I. Babenko (1978), and more recently M. Iosifescu (1992).

2 THE GAUSS-KUZMIN TYPE EQUATION

An essential ingredient in any proof of whichever Gauss-Kuzmin theorem is the following observation. Let \(\xi \in [0, 1)\setminus\mathbb{Q}\) and put
\[
\tau_k := \tau^k(\xi), k \geq 0,
\]
where \(\tau : [0, 1) \to [0, 1)\) is the continued fraction transformation defined in (2). From (2) it follows that
\[
0 \leq \tau_{n+1} \leq x \Leftrightarrow \tau_n \in \bigcup_{i \in \mathbb{N}_+} \left[ \frac{1}{x + i}, \frac{1}{i} \right).
\]
Thus, if we put
\[
F_n(x) := \lambda(\{\xi \in [0, 1); \tau^n(\xi) \leq x\}), n \geq 0,
\]
then
\[
F_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \left( F_n \left( \frac{1}{i} \right) - F_n \left( \frac{1}{x + i} \right) \right), n \geq 0,
\]
relation called the Gauss-Kuzmin type equation.

3 IMPORTANT RESULT

Let \(B(I)\) the Banach space of all bounded measurable functions \(f : I \to \mathbb{C}\), \(I := [0, 1]\).
Proposition. If \( f \in B(I) \) is non-decreasing, then \( Uf \) is non-increasing, where \( U \) is the Perron-Frobenius operator of \( \tau \) under \( \gamma \), with \( \gamma \) the Gauss’ measure which is defined on \( B_{[0,1]} \) - Borel \( \sigma \)-algebra of sets on \([0,1]\), by

\[
\gamma(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}, A \in B_{[0,1]}.
\]

Proof. Let \( f \) be a non-decreasing function. Thus, if \( x < y \), then \( f(x) \leq f(y) \).

We evaluate the difference \( Uf(y) - Uf(x) \). We have, \( Uf(y) = \sum_{i \in \mathbb{N}^+} P_i(y) f \left( \frac{1}{x+i} \right) \)

and \( Uf(x) = \sum_{i \in \mathbb{N}^+} P_i(x) f \left( \frac{1}{x+i} \right) \), where \( P_i(x) = \frac{x+1}{(x+i)(x+i+1)} \). Thus,

\[
Uf(y) - Uf(x) = S_1 + S_2,
\]

where

\[
S_1 = \sum_{i \in \mathbb{N}^+} P_i(y) \left( f \left( \frac{1}{x+i} \right) - f \left( \frac{1}{x+i+1} \right) \right) - \sum_{i \in \mathbb{N}^+} (P_i(y) - P_i(x)) f \left( \frac{1}{x+i} \right) f \left( \frac{1}{x+1} \right).
\]

Since \( f \) is non-decreasing, and \( \frac{1}{x+i+1} > \frac{1}{x+i} \), then \( f \left( \frac{1}{x+i+1} \right) \geq f \left( \frac{1}{x+i} \right) \).

Thus, \( S_1 \leq 0 \). We will show that \( S_2 \geq 0 \) too. We have that \( \sum_{i \in \mathbb{N}^+} P_i(u) = 1 \), \( u \in I \), and therefore we obtain:

\[
S_2 = -\sum_{i \in \mathbb{N}^+} (P_i(y) - P_i(x)) f \left( \frac{1}{x+i+1} \right) - \frac{1}{x+1} \sum_{i \in \mathbb{N}^+} (P_i(y) - P_i(x)) f \left( \frac{1}{x+1} \right) f \left( \frac{1}{x+2} \right) \]

\[
= -\sum_{i \in \mathbb{N}^+} \left( f \left( \frac{1}{x+1} \right) - f \left( \frac{1}{x+i} \right) \right) \left( P_i(y) - P_i(x) \right).
\]

Now, it is easy to show that the function \( P_1 \) is decreasing, while the functions \( P_i, i \geq 2 \), are all increasing. Also,

\[
f \left( \frac{1}{x+1} \right) - f \left( \frac{1}{x+i} \right) \geq f \left( \frac{1}{x+i+1} \right) - f \left( \frac{1}{x+2} \right) \geq 0, i \geq 2.
\]

Therefore,

\[
S_2 = -\sum_{i \geq 2} \left( f \left( \frac{1}{x+1} \right) - f \left( \frac{1}{x+i} \right) \right) \left( P_i(y) - P_i(x) \right)
\]

\[
\leq -\left( f \left( \frac{1}{x+1} \right) - f \left( \frac{1}{x+2} \right) \right) \sum_{i \geq 2} \left( P_i(y) - P_i(x) \right) \leq 0.
\]

Thus, \( Uf(y) - Uf(x) \leq 0 \).

4 THE GAUSS-KUZMIN THEOREM

We will give a simple proof that

\[
F_n(x) = \frac{\log(x+1)}{\log 2} + O(q^n),
\]
where $0 < q < 1$ or, to be exactly, $q = 0.7594\ldots$. In fact, we will proof the following:

**Theorem.** Let $f_0(x)$ be any twice differentiable function defined on $[0, 1]$ with $f_0(0) = 0$ and $f_0(1) = 1$. Let the sequence of functions $f_1(x), f_2(x), \ldots$ be defined by the recursion formula

$$f_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \left( f_n \left( \frac{1}{i} \right) - f_n \left( \frac{1}{x+i} \right) \right).$$

Then

$$f_n(x) = \frac{\log(x+1)}{\log 2} + O(q^n),$$

where $0 < q < 1$ or, to be exactly, $q = 0.7594\ldots$.

It is clear that for $f_0(x) = x = F_0(x)$, this theorem will establish Gauss’ claim and provide an answer to his problem.

**Proof.** Instead of studying $f_n(x)$ directly, we look at the derivative:

$$f'_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \frac{1}{(x+i)^2} f_n' \left( \frac{1}{x+i} \right). \quad (4)$$

Let us introduce another sequence of functions $g_0, g_1, \ldots$ defined by

$$g_n(x) = (x+1)f'_n(x).$$

Then the recursion formula (4) is transformed into

$$\frac{g_{n+1}(x)}{x+1} = \sum_{i \in \mathbb{N}_+} \frac{1}{(x+i)^2} \frac{g_n \left( \frac{1}{x+i} \right)}{x+i + 1} = \sum_{i \in \mathbb{N}_+} \frac{1}{(x+i)(x+i+1)} g_n \left( \frac{1}{x+i} \right) \Rightarrow$$

$$\Rightarrow g_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \frac{x+1}{(x+i)(x+i+1)} g_n \left( \frac{1}{x+i} \right) = \sum_{i \in \mathbb{N}_+} P_i(x) g_n \left( \frac{1}{x+i} \right) = U g_n,$$

where $P_i(x) = \frac{x+1}{(x+i)(x+i+1)}, i \in \mathbb{N}_+$, and $U$ is the Perron-Frobenius operator of $\tau$ under $\gamma$.

If we can show that $g_n(x) = \frac{1}{\log 2} + O(q^n)$, then an integration will establish the theorem for $f_n(x)$, because integrating $\frac{1}{x+i}$ will give $\log(x+1)$ term together with a bounded expression on a bounded interval times the $O(q^n)$ term, which will remain $O(q^n)$. To demonstrate that $g_n(x)$ has this desired form, it suffices to establish that $g'_n(x) = O(q^n)$, as the $\frac{1}{\log 2}$ constant in $g_n(x)$ will follow from the normalization requirement that $f_0(0) = 0$ and $f_0(1) = 1$.

We have:

$$P_i(x) = \frac{x+1}{(x+i)(x+i+1)} = \frac{i}{x+i+1} - \frac{i-1}{x+i},$$

where $0 < q < 1$ or, to be exactly, $q = 0.7594\ldots$. In fact, we will proof the following:

**Theorem.** Let $f_0(x)$ be any twice differentiable function defined on $[0, 1]$ with $f_0(0) = 0$ and $f_0(1) = 1$. Let the sequence of functions $f_1(x), f_2(x), \ldots$ be defined by the recursion formula

$$f_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \left( f_n \left( \frac{1}{i} \right) - f_n \left( \frac{1}{x+i} \right) \right).$$

Then

$$f_n(x) = \frac{\log(x+1)}{\log 2} + O(q^n),$$

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$$\Rightarrow g_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \frac{x+1}{(x+i)(x+i+1)} g_n \left( \frac{1}{x+i} \right) = \sum_{i \in \mathbb{N}_+} P_i(x) g_n \left( \frac{1}{x+i} \right) = U g_n,$$

where $P_i(x) = \frac{x+1}{(x+i)(x+i+1)}, i \in \mathbb{N}_+$, and $U$ is the Perron-Frobenius operator of $\tau$ under $\gamma$.

If we can show that $g_n(x) = \frac{1}{\log 2} + O(q^n)$, then an integration will establish the theorem for $f_n(x)$, because integrating $\frac{1}{x+i}$ will give $\log(x+1)$ term together with a bounded expression on a bounded interval times the $O(q^n)$ term, which will remain $O(q^n)$. To demonstrate that $g_n(x)$ has this desired form, it suffices to establish that $g'_n(x) = O(q^n)$, as the $\frac{1}{\log 2}$ constant in $g_n(x)$ will follow from the normalization requirement that $f_0(0) = 0$ and $f_0(1) = 1$.

We have:

$$P_i(x) = \frac{x+1}{(x+i)(x+i+1)} = \frac{i}{x+i+1} - \frac{i-1}{x+i},$$
\[ g_{n+1}(x) = \sum_{i \in \mathbb{N}_+} \left( \frac{i}{x+i+1} - \frac{i-1}{x+i} \right) g_n \left( \frac{1}{x+i} \right) \Leftrightarrow \]

\[ g'_{n+1}(x) = -\sum_{i \in \mathbb{N}_+} \frac{i}{(x+i+1)^2} \left( g_n \left( \frac{1}{x+i} \right) - g_n \left( \frac{1}{x+i+1} \right) \right) - \sum_{i \in \mathbb{N}_+} \frac{P_i(x)}{(x+i)^2} g'_n \left( \frac{1}{x+i} \right). \]

By applying the mean value theorem of calculus to the difference

\[ g_n \left( \frac{1}{x+i} \right) - g_n \left( \frac{1}{x+i+1} \right), \]

we obtain

\[ g_n \left( \frac{1}{x+i} \right) - g_n \left( \frac{1}{x+i+1} \right) = \left( \frac{1}{x+i} - \frac{1}{x+i+1} \right) g'_n \left( \frac{1}{x+\theta_i} \right), \]

where \( 1 < \theta_i < i \).

Thus, from (5), we have:

\[ g'_{n+1}(x) = -\sum_{i \in \mathbb{N}_+} \frac{i}{(x+i)(x+i+1)^3} g'_n \left( \frac{1}{x+\theta_i} \right) - \sum_{i \in \mathbb{N}_+} \frac{P_i(x)}{(x+i)^2} g'_n \left( \frac{1}{x+i} \right). \]

Let \( M_n \) be the maximum of \( |g'_n(x)| \) on \([0, 1] \), i.e. \( M_n = \max_{x \in [0, 1]} |g'_n(x)| \).

Then, from (6), we have that:

\[ M_{n+1} \leq M_n \max_{x \in [0, 1]} \left( \sum_{i \in \mathbb{N}_+} \frac{i}{(x+i)(x+i+1)^3} + \sum_{i \in \mathbb{N}_+} \frac{P_i(x)}{(x+i)^2} \right). \]

We now must calculate the maximum value of the sums in this expression. To this end, define function \( h \) by

\[ h(x) = \sum_{i \in \mathbb{N}_+} \frac{P_i(x)}{(x+i)^2}, x \in [0, 1]. \]

Note that for \( \varphi(x) = x^2, x \in [0, 1], \) we have \( h(x) = U \varphi(x). \) Since \( \varphi \) is increasing, and using the proposition from Section 3, we have that \( h \) is decreasing. Hence, \( h(x) \leq h(0) \), and since \( h(0) = \sum_{i \in \mathbb{N}_+} \frac{P_i(0)}{i^2} = \sum_{i \in \mathbb{N}_+} \frac{1}{i^3(i+1)}. \) Therefore, (7)
become:

\[
M_{n+1} \leq M_n \sum_{i \in \mathbb{N}_+} \left( \frac{1}{i(i+1)^3} + \frac{1}{i^3(i+1)^3} \right)
\]

\[
= M_n \sum_{i \in \mathbb{N}_+} \left( \frac{1}{i^3} - \frac{1}{i^2} + \frac{1}{i} - \frac{1}{i+1} + \frac{1}{(i+1)^3} \right)
\]

\[
= M_n (\zeta(3) - \zeta(2) + 1 + \zeta(3) - 1)
\]

\[
= M_n (2\zeta(3) - \zeta(2)),
\]

where \(\zeta(n)\) denotes the Riemann zeta function. Hence, \(2\zeta(3) - \zeta(2) = 0.7594798 \ldots\). Thus, \(M_{n+1} < M_n q\), where \(q = 0.7594798 \ldots\), and \(q_{n+1}^n (x) = O(q^{n+1})\), which proves the theorem.

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