Brane world unification of quark and lepton masses and its implication for the masses of the neutrinos

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A TeV-scale scenario is constructed in an attempt to understand the relationship between quark and lepton masses. This scenario combines a model of early (TeV) unification of quarks and leptons with the physics of extra large dimensions. It demonstrates a relationship between quark and lepton mass scales at rather “low” (TeV) energies which will be dubbed as early quark-lepton mass unification. It also predicts that the masses of the neutrinos are naturally light and Dirac. There is an interesting correlation between neutrino masses and those of the unconventional charged fermions which are present in the early unification model. If these unconventional fermions were to lie between 200 GeV and 300 GeV, the Dirac neutrino mass scale is predicted to be between \(\sim 0.07\mathrm{eV}\) and \(\sim 1\mathrm{eV}\).

I. INTRODUCTION

Are quark and lepton masses related? This question has been addressed almost thirty years ago in a famous paper by [1]. In [1], the equality of the \(\tau\) lepton and bottom quark masses at the GUT scale \(M_{GUT} \sim 10^{15} - 10^{16}\) GeV gives rise to, for the particular case of \(SU(5)\) considered in [1], the equality of the \(\tau\) lepton and bottom quark masses at \(M_{GUT}\). After renomalization-group (RG) evolution down to low energies, a remarkable “prediction” for the \(b\) quark mass was made, although the complete story was significantly more complicated. Despite the enormous popularity of GUT, questions started to arise as to whether or not there are actually structures instead of simply a “desert” between the electroweak scale and \(M_{GUT}\). If so, how would quark and lepton masses be related if they were to have early unification?

The hope that new physics is lurking somewhere in the TeV region has given rise in the past decade or two to a flurry of activities which resulted in a rich diversity of topics with a variety of motivations. A common thread in all of these activities is the prediction of new particles of one kind or another. It goes without saying that discoveries of these new particles will vindicate all the efforts put into it. The present paper will rely on two of such scenarios with a special emphasis put into the relationship between quark and lepton masses, including the issues of neutrino mass: Is it Dirac or Majorana? Why is it so small?

Two TeV scenarios which form the focus of this paper are the following: (1) Early petite unification of quarks and leptons and its use in the attempt to explain the smallness of neutrino masses and the hierarchy of quark masses.

In [3], the Standard Model (SM) with three independent couplings is merged into a group \(G_S \otimes G_W\) with two independent couplings at some scale which is supposed to be in the TeV region. The choice of the Pati-Salam \(SU(4)_{PS}\) [13] for \(G_S\) was used. This scenario allowed us to compute \(\sin^2\theta_W(M_Z^2)\) and to use it to constraint the choices of \(G_W\). The preferred choice of \(G_W\) was the gauge group \(SU(4)_{PS} \otimes SU(2)^4\) with an early unification scale of several hundreds of TeVs. Recent precise measurements of \(\sin^2\theta_W(M_Z^2)\) coupled with a surge of interest in TeV scale physics have prompted [4] to reexamine the petite unification idea. There it was shown that the petite unification scale is lowered considerably, to less than \(10\mathrm{TeV}\), due to the increase of \(\sin^2\theta_W(M_Z^2)\) as compared with its value of twenty three years ago. This has the effect of practically ruling out \(SU(4)_{PS} \otimes SU(2)^4\) due to severe problems with the decay rate for \(K_L \rightarrow \mu e\) among other things. Two favorite models emerged: \(SU(4)_{PS} \otimes SU(2)^3\) and \(SU(4)_{PS} \otimes SU(3)^2\), both of which nicely and naturally avoid the \(K_L \rightarrow \mu e\) problem due principally to the existence of new types of fermions. A detailed analysis of \(SU(4)_{PS} \otimes SU(2)^3\) was performed by [5], including a two-loop renormalization group (RG) analysis and a discussion of the physics of the new unconventional fermions. Early unification in this model takes place at a mass scale \(M = \mathcal{O}(1 - 2 \mathrm{TeV})\).

On another front, Ref. [6] has constructed a model which made use of the mechanism of wave function overlap along an extra compact dimension to “explain” the smallness of Dirac neutrino masses. An \(SU(2)_R\) symmetry was assumed and was subsequently spontaneously broken, giving rise to a phenomenon in which one member of the right-handed doublet has a narrow wave function, while the other member acquires a broad wave function (both localized at the same point along the extra dimension). The overlap of the wave function of the left-handed doublet with the wave functions of the right-
handed fields gives rise to the splitting between the effective four-dimensional Yukawa couplings (and eventually between the masses) of neutral and charged leptons or of the up and down quarks. This splitting can be large or small depending on the separation $d$ (along the extra dimension) between the wave functions for left-handed and right-handed fields as demonstrated in [9]. It was further noticed that there is a deep connection between the separation $d_1$ for the lepton sector and the separation $d_q$ for the quark sector, giving rise to a relationship between quark and lepton masses, a common feature in Grand Unified Theories (GUT).

At this point, one might ask about the distinction between the present scenario and a possible attempt to incorporate a GUT scenario for the masses in the context of Large Extra Dimensions. First, it is fair to say that, in order to achieve Grand Unification above the compactification scale, some rather strong assumption has to be made about the behaviour of the running couplings, namely a power-law running. Because of this dynamical assumption, the running masses used in extrapolating the values at the GUT scale to low energies will also suffer from large uncertainties. This is very unlike the logarithmic behaviour used in [10]. In our case, quark-lepton unification is achieved at a scale comparable to the compactification scale and the predictions made there can be extrapolated down to the $Z$-mass using familiar renormalization group techniques. In fact, since the quark-lepton unification scale is an order of magnitude or so larger than the $Z$-mass, there will not be much “running”.

The plan of the paper will be as follows. First, we present a brief review of the essential elements that go into the wave function overlap scenario in extra dimensions. We then briefly review the ideas of early quark-lepton unification with a special emphasis on the group structure and fermion representations. We then show how one can connect these two ideas to relate the overall mass scales in the mass matrices of the quark sector to those of the lepton sectors. We finish with a numerical illustration of those results along with their physical implications, including neutrino masses. We will present predictions for Dirac neutrino masses. Whether or not Majorana neutrino masses are needed is a question which depends on the predicted values for Dirac neutrino masses. We will show a correlation between the masses of the neutrinos and those of the unconventional fermions which are present in the early unification model. If the latter fermions are required to have a mass between the electroweak scale and approximately 1 TeV, it is shown that the Dirac neutrino masses are too small for the see-saw mechanism [14] to provide the bulk of neutrino masses if, as it is natural to assume, the Majorana scale is of the order of the early unifications scale. It is also shown that if the mass of the unconventional fermions is taken to lie between 200 GeV and 300 GeV, the range of the Dirac neutrino mass is found to be between $\sim 0.07 \, eV$ and $\sim 1 \, eV$. In fact, there is a recent interest in the possibility that the neutrino mass might be either mostly or pure Dirac and there are questions about the popular see-saw mechanism itself [15].

II. EXTRA DIMENSION, EARLY QUARK-LEPTON UNIFICATION AND MASS RELATIONSHIP

Two TeV-scale scenarios are briefly summarized below with the purpose of exposing their common threads and ultimately combining them in order to obtain an understanding of the possible relationship between quark and lepton masses and the smallness of the neutrino masses.

A. Effective Yukawa couplings in models with extra dimensions

In its simplest version, an effective Yukawa coupling (which would be proportional to the mass of the fermion) is defined, in four dimensions, as proportional to the size of the wave function overlap between left-handed and right-handed fermions along a compact fifth (spatial) dimension [11]. Among the many applications of this idea, one can cite for example the attempts to give an explanation for the smallness of the neutrino masses [8, 9]. One can either arbitrarily choose the locations, along the extra dimension, of the localized wave functions for the left-handed and right-handed neutrinos in such a way that the overlap is tiny, or one can try to build a model in which the tiny overlap comes out more or less naturally as Ref. [9] had done. In [9], the size of the neutrino overlap came out small as compared with the size of the charged lepton overlap. A brief review of how this happens as described in Ref. [9] will be given below. The main point of these works is that the four-dimensional effective Yukawa couplings can be small even if the fundamental (four-dimensional) Yukawa coupling is of order unity.

Let us start with one extra spatial dimension $y$ compactified on an orbifold $S_1/Z_2$ and having a length $L$. Let us, as an example, take a lepton $SU(2)_L$ doublet, $L^{(L)}(x, y)$, and another lepton $SU(2)_R$ doublet, $L^{(R)}(x, y)$, where the superscripts refer to the groups respectively. Since a fermion in five dimensions is a Dirac fermion, it will have both chiralities (left and right-handed) under four dimensions i.e. $\psi = (\psi_L + \psi_R)$, where $\psi_{L,R} = P_{L,R}\psi$, with $P_{L,R} = (1 \mp \gamma_5)/2$ being the usual four-dimensional chiral projection operator. The notations that were used in [9] and here will be as follows. For the $SU(2)_L$ doublet, we use $L^{(L)}(x, y) = (l^{(L)}_L + i^{(L)}_R)$, while for the $SU(2)_R$ doublet, we use $L^{(R)}(x, y) = (l^{(R)}_L + i^{(R)}_R)$. One can choose the $Z_2$ parity for these fields such that the only zero modes are $l^{0,(L)}_L(x, y) = l_L(x)\xi_L(y)$ and $l^{0,(R)}_R(x, y) = l_R(x)\xi_R(y)$. With the introduction of the appropriate background scalar fields, these zero modes can be localized at some points along $y$. The effective Yukawa coupling (which
would determine the mass of the fermion) is then proportional to the overlap between $\xi_L(y)$ and $\xi_R(y)$. (Let us recall that $\xi_L(y)$ and $\xi_R(y)$ are doublets.) The main focus of this scenario is the construction of a model for the $SU(2)_R$ doublet $\xi_R(y)$. This construction will be repeated below but a few words might be illuminating here. The wave functions for the up and down members of $\xi_R(y)$, although localized at the same point along $y$, have very different shapes: one which is wide and the other which is narrow. It is this disparity in shapes of the “right-handed” wave functions that, when overlapping with the common “left-handed” wave function, gives rise to the hierarchy in mass among up and down members of the doublet.

For the sake of clarity, a review of the model of this is warranted here. Since the main object is the construction of $\xi_R(y)$, one will concentrate on $L(R)$. A summary of the main results of this can now be given. First, the localization of $\xi_L(y)$ can be achieved by a coupling of $L(R)$ to a background scalar field which develops a kink solution along $y$. Two background scalar fields are needed in this scenario: a singlet field $\phi_S$ whose kink solution localizes the wave functions of both members of $\xi_R(y)$ at the same location while keeping their shapes identical, and a triplet $\Phi_T = (\phi_T^1, \phi_T^2, \phi_T^3)$ whose kink solution is responsible for drastically changing the shapes of the two wave functions while keeping the localization points the same. (As mentioned in this, these background scalars are chosen to be odd under $Z_2$ so that they do not have zero modes.)

Below is how it works.

The minimum energy solutions used in this are as follows

$$\langle \Phi_T \rangle = \langle \phi_T^3 \rangle T_3 / 2 = \begin{pmatrix} h_T(y) & 0 \\ 0 & -h_T(y) \end{pmatrix}, \quad (1)$$

and

$$\langle \phi_S \rangle = h_S(y), \quad (2)$$

where generically $h(y) = \nu \tanh(\mu y)$, with $\mu = \sqrt{\lambda/2\nu}$ being typically the “thickness” of the domain wall. Coupled with the Yukawa coupling $L_{Y2} = f_S^{(1)} \bar{L}^{(R)} \Phi_T L^{(R)} + f_T^{(1)} \bar{L}^{(R)} \phi_S L^{(R)}$, one obtains the following equations for $\xi_R(y)$:

$$\partial_y \xi_R^{\nu}(y) + (f_S^{(1)} h_S(y) + f_T^{(1)} h_T(y)) \xi_R^{\nu}(y) = 0, \quad (3)$$

$$\partial_y \xi_R^{e}(y) + (f_S^{(1)} h_S(y) - f_T^{(1)} h_T(y)) \xi_R^{e}(y) = 0, \quad (4)$$

The solutions for the up and down members of $\xi_R(y)$ (which will have the superscripts $\nu$ and $e$ respectively) are given as

$$\xi_R^{\nu,e}(y) = k_{\nu,e} \exp\left(-\int_0^y dy' \left(f_S^{(1)} h_S(y') \pm f_T^{(1)} h_T(y')\right)\right), \quad (5)$$

where $k_{\nu,e}$ are normalization factors. The immediate implication of Eq. (5) can be seen as follows. Using $h(y) = v \tanh(\mu y)$ in Eq. (5) one obtains

$$\xi_R^{\nu,e}(y) = k_{\nu,e} e^{-(C_S \ln(\cosh(\mu y)) \pm C_T \ln(\cosh(\mu_T y)))}, \quad (6)$$

where $C_S, T = f_S, T / (\lambda S, T / 2)^{1/2}$. If the parameters of the two scalar potentials are such that $C_S \ln(\cosh(\mu y)) \approx C_T \ln(\cosh(\mu_T y))$, one can immediately see that $\xi_R^{\nu}(y)$ is narrow while $\xi_R^{e}(y)$ is broad. When these wave functions overlap with the left-handed wave function (common for both $\nu$ and $e$), one can observe a large disparity between the two effective Yukawa couplings. A crucial quantity which enters this hierarchy in Yukawa couplings is the separation between the left-handed wave function and the right-handed wave functions (localized at the same point along $y$) and which was denoted by $\Delta y^{(i)}$ in this.

The model described above has been espoused in Ref. as a mechanism for naturally small Dirac neutrino masses. Furthermore, using the same wave function profiles for the right-handed quarks, an interesting connection between quark and lepton mass hierarchies was noticed in this. Basically, it was a connection between $\Delta y^{(i)}$ and $\Delta y^{(g)}$. Possible symmetry reasons for this connection were left open in this. It is the purpose of this paper to elucidate the relationship between quark and lepton mass hierarchies by considering explicitly a model of TeV-scale quark-lepton unification. To set the stage for that discussion, a brief summary of the early unification model is presented below.

### B. Early TeV-scale quark-lepton unification

The model that was presented in this and discussed in detail in this is based on the gauge group

$$G_{PUT} = SU(4)_PS \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H. \quad (7)$$

This group is characterized by two independent gauge couplings: $g_S$ for $SU(4)_PS$ and $g_W$ for $SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$. Where a permutation symmetry is assumed among the three $SU(2)$’s. $G_{PUT}$ is assumed to be broken down to the Standard Model in two steps, namely

$$G_{PUT} \rightarrow G_1 \rightarrow G_2 \rightarrow SU(3)_c \otimes U(1)_{EM}, \quad (8)$$

where

$$G_1 = SU(3)_c \otimes U(1)_S \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H, \quad (9)$$

and

$$G_2 = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y. \quad (10)$$

In this scheme, quarks and leptons, which are generic terms for color triplets and color singlets respectively, are grouped into quartets of $SU(4)_PS$. The scale of such a
quark-lepton unification is denoted by $M$ as seen above. In contrast with GUT where such a unification occurs close to the Planck scale, it has been shown in \([\square]\), and particularly in \([\square, \triangle]\), that $M \leq 2$ TeV. That such a low scale of unification can be achieved is a distinctive feature of this model. A sketch of the arguments is presented below.

At this point, it is worth noticing that, if the scale(s) of extra dimensions is comparable with the Petite Unification scale, physics which are related to the breaking of Petite Unification can be extrapolated to “low energies” with little, if any, uncertainties coming from physics beyond the compactification scale. This is in contrast with a typical GUT scenario embedded in large extra dimensions since its scale which would normally lie above the compactification scale. As a consequence, there are large uncertainties associated with the extrapolation of “GUT” physics down to the Z-mass for example.

The main idea of Petite Unification has to do with the assumption that the SM, with three independent couplings: $g_s, g_2$ and $g_1$, is merged into the PUT group $G_S \otimes G_W$ which is characterized by two independent couplings: $g_S$ and $g_W$. As a result, one can compute $\sin^2 \theta_W(M_Z^2)$ as a function of the PUT unification scale $M$ as shown in \([\square, \triangle]\). The highly precise value of $\sin^2 \theta_W(M_Z^2) = 0.23113(15)$, along with the requirement that $M \leq 10$ TeV, allows us to severely restrict the choices of $G_W$, resulting in the preferred model mentioned at the beginning of this section. (Two other models were also found: $SU(4)_PS \otimes SU(2)^4$ and $SU(4)_PS \otimes SU(3)^2$, with the former being, in some sense, ruled out due to severe problems with the process $K_L \rightarrow \mu e$ unless some exotic mechanisms are invoked, for example an embedding of the model into five dimensions with the gauge symmetry breaking accomplished by orbifold boundary conditions \([\square, \triangle]\).)

Two crucial elements in the computation of $\sin^2 \theta_W(M_Z^2)$ are the group theoretical factor $\sin^2 \theta_W$ (= 1/3 for $G_W = SU(2)^3$) and the factor $C_S$ which appears in the expression $Q = T_{3L} + T_Y = Q_W + C_S T_{15}$, where $Q_W$ is the “weak” charge corresponding to the group $G_W$, and $T_{15}$ is the unbroken diagonal generator of the $SU(4)_PS$. The value of $C_S$ depends on how quarks and leptons transform under $SU(4)_PS \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$. For instance, $C_S = \sqrt{2/3}$ if fermions transform as $(4, 2, 1, 1)$ for example, while $C_S = \sqrt{8/3}$ if they transform as $(4, 2, 1, 2)$ or $(4, 1, 2, 2)$. This is shown in details in \([\square, \triangle]\). Since $\sin^2 \theta_W(M_Z^2) = (1/3)(1 - 0.067 C_S^2 - \log \text{terms})$ (see \([\square]\)) and coupled with the requirement that $M \leq 10$ TeV (which makes for little “running” between $M$ and the electroweak scale, and hence small log terms), it was found \([\square]\) that the only acceptable fermion representations are the ones for which $C_S = \sqrt{8/3}$. Using this value for $C_S$ \([\square, \triangle]\), a detailed computation of $\sin^2 \theta_W(M_Z^2)$, up to two loops \([\square]3\), determines the Petite Unification scale to be less than 2 TeV. This fermion content will be the one that will be used in this paper.

For the sake of clarity, an explicit description of the fermions of the model is presented below.

Under $SU(4)_PS \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$, the fermions transform as

$$\Psi_L = (4, 2, 1, 2)_L = \left( \begin{array}{c} (d^c(1/3), \bar{U}(4/3)) \\ (u^c(-2/3), \bar{D}(1/3)) \end{array} \right) \left( \begin{array}{c} \tilde{l}_u(-1), \nu(0) \\ \tilde{l}_d(-2), l(-1) \end{array} \right)_L \quad (11)$$

$$\Psi_R = (4, 2, 1, 2)_R = \left( \begin{array}{c} (d^c(1/3), \bar{U}(4/3)) \\ (u^c(-2/3), \bar{D}(1/3)) \end{array} \right) \left( \begin{array}{c} \tilde{l}_u(-1), \nu(0) \\ \tilde{l}_d(-2), l(-1) \end{array} \right)_R \quad (12)$$

As one can see, this model contains, besides conventionally charged fermions, unconventional fermions with charges up to 4/3 for the quarks and down to $-2$ for the leptons.

To understand the notations in Eqs. \(11, 12\), one notices the following conventions.

- \(SU(2)_{L,R}\) doublets:

$$\left( \begin{array}{c} d^c(1/3) \\ u^c(-2/3) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \nu(0) \\ l(-1) \end{array} \right)_{L,R},$$

$$\left( \begin{array}{c} \bar{U}(4/3) \\ \bar{D}(1/3) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \tilde{l}_u(-1) \\ \tilde{l}_d(-2) \end{array} \right)_{L,R} \quad (13)$$

- \(SU(2)_H\) doublets:

$$\left( \begin{array}{c} \bar{U}(4/3) \\ d^c(1/3) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \nu(0) \\ \tilde{l}_u(-1) \end{array} \right)_{L,R},$$

$$\left( \begin{array}{c} \bar{D}(1/3) \\ u^c(-2/3) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \tilde{l}(-1) \\ \tilde{l}_d(-2) \end{array} \right)_{L,R} \quad (14)$$

- \(SU(4)_{PS}\) quartets:

$$\left( \begin{array}{c} \bar{d}^c(1/3) \\ \tilde{l}_u(-1) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \bar{D}(1/3) \\ l(-1) \end{array} \right)_{L,R},$$

$$\left( \begin{array}{c} \bar{U}(4/3) \\ \nu(0) \end{array} \right)_{L,R}, \quad \left( \begin{array}{c} \bar{u}^c(-2/3) \\ \tilde{l}_d(-2) \end{array} \right)_{L,R} \quad (15)$$

Note that due the particular nature of the fermion representation in this model, it is

$$\left( \begin{array}{c} d^c(1/3) \\ u^c(-2/3) \end{array} \right)_{L,R} = i\tau_2 \left( \begin{array}{c} u(2/3) \\ d(-1/3) \end{array} \right)_{L,R} \quad (16)$$

which appears instead of the more familiar-looking $(u(2/3), d(-1/3))$. \(\quad \end{proof}\)
As emphasized in [4, 5], a nice feature of this model is the absence of tree-level flavor-changing neutral currents (FCNC) because the $SU(2)_{L}$ and $SU(4)_{PS}$ transitions only connect conventional to unconventional fermions as can be seen above. A process such as $K_{L} \rightarrow \mu e$ can only occur at one loop and can easily be made to obey the experimental upper bound.

Since the natural scales of the scenarios described in Sections II A and II B are both in the TeV range, it is worthwhile to see if a “marriage” of some sorts can be made between these two scenarios and if, as a result, some light can be shed concerning the relationship between quark and lepton masses and the smallness of the neutrino masses.

C. Connection between the scales of quark and lepton masses

If “quarks” and “leptons” (in the generic sense as discussed above) can be unified at the TeV scale, there is a good possibility that whatever gives rise to their masses will also determine the relationship between their mass scales. We will show below that such a possibility does exist within the framework of the two scenarios described above.

The basic model used in this paper is $SU(4)_{PS} \otimes SU(2)_{L} \otimes SU(2)_{R} \otimes SU(2)_{H}$. As stated above, this group spontaneously breaks down to $SU(2)_{L} \otimes U(1)_{Y}$ and then to $U(1)_{em}$ at the scales $M \sim \text{few TeV's}$ and $v \sim 250 \text{ GeV}$ respectively. Upon embedding this model in five dimensions, with the fifth dimension $y$ compactified on an $S_{1}/Z_{2}$ orbifold, it is shown below that the following features occur: 1) The breaking of $SU(4)_{PS}$ splits the positions, along $y$, of wave functions of the zero modes of “quarks” and “leptons”; 2) The breaking of $SU(2)_{R}$ gives rise to two vastly different profiles for the wave functions of the “right-handed” zero modes; 3) Since a $SU(2)_{L}$ doublet groups together a conventional quark (or lepton) with an unconventional one, the breaking of $SU(2)_{H}$ splits the locations, along $y$, of the wave functions of the conventional fermions relative to those of the unconventional ones; 4) And finally, the breaking of $SU(2)_{L} \otimes U(1)_{Y}$ provides a mass scale for all the fermions. As we have explained in the previous section, a crucial quantity that appears in the hierarchy of masses among the up and down members of an $SU(2)_{L}$ doublet is the separation along $y$ between the wave function of the left-handed doublet and that of the right-handed fields, namely $\Delta y_{(l,o)}$. We will show below that points # 1, 2 and 3 help establish a relationship between $\Delta y_{(l)}$ and $\Delta y_{(q)}$.

In the construction of the model, one important point we would like to stress is the following. The model $SU(4)_{PS} \otimes SU(2)_{L} \otimes SU(2)_{R} \otimes SU(2)_{H}$ contains unconventional quarks and leptons which were assumed to be heavy enough to escape detection. The fate of these fermions were well described in Ref. 3. For the purpose of this paper, we will simply require that all unconventional fermions are heavy. This will be one constraint which will be used below.

1. $SU(4)_{PS} \otimes SU(2)_{L} \otimes SU(2)_{R} \otimes SU(2)_{H}$ in five dimensions

The first step one would like to do is to embed $SU(4)_{PS} \otimes SU(2)_{L} \otimes SU(2)_{R} \otimes SU(2)_{H}$ in five dimensions. Let us first denote, in five dimensions, the fermions presented in Section II B by

$$\Psi_{L}(x, y) = (4, 2, 1, 2),$$

$$\Psi_{R}(x, y) = (4, 1, 2, 2).$$

Let us recall that, in five dimensions, these are four-component Dirac fields, i.e. they have both left- and right-handed components. The superscripts $\{L\}$ and $\{R\}$ are used for two reasons: (a) to denote the transformation of the left-handed doublet and that of the right-handed doublet groups together a conventional and then to $SU(2)_{R}$; and (b) to show that the surviving zero modes are related to these fields. By choosing the appropriate $Z_{2}$ parity for these fields, the zero modes of $\Psi_{L}(x, y)$ and $\Psi_{R}(x, y)$ are

$$\Psi_{L}(x, y) = \psi_{L}(x) \xi_{L}(y),$$

$$\Psi_{R}(x, y) = \psi_{R}(x) \xi_{R}(y),$$

respectively.

We wish to localize $\xi_{L}(y)$ and $\xi_{R}(y)$ along $y$. This is accomplished by coupling these fields to some background scalar fields. To see the group representations of these scalar fields, we consider the following bilinears:

$$\Psi_{L}(x, y)\Psi_{L}^{(L)}(x, y) = (1 + 15, 1 + 3, 1, 1 + 3),$$

$$\Psi_{R}(x, y)\Psi_{R}^{(R)}(x, y) = (1 + 15, 1 + 3, 1 + 3).$$

From Eq. 20, one can see that some possible scalar fields which can couple to these fermions would transform like $(1, 1, 1, 1), (15, 1, 1, 1), (15, 1, 1, 3), (15, 1, 3, 1), \text{etc.}$. We will show step-by-step below how these scalar fields help establish the link between $\Delta y_{(l)}$ and $\Delta y_{(q)}$. We will successively invoke one scalar at a time and show how it modifies the behaviour of the fermion zero modes.

As we shall see, the scalar fields which are needed for our scenario are the following:

$$\Phi_{S_{1}, S_{2}} = (1, 1, 1, 1)$$

$$\Sigma = (15, 1, 1, 1),$$

$$\Phi_{R} = (1, 1, 3, 1),$$

$$\Phi_{H} = (15, 1, 1, 3).$$
2. The role of the singlet scalar fields $\Phi_{S_1, S_2} = (1, 1, 1, 1)$

The Yukawa coupling of this field with the fermions takes the form

$$\mathcal{L}_{Y1} = f_S (\bar{\Psi}^L \psi^L + \bar{\psi}^R \Psi^R) \Phi_{S_1},$$

(23)

with $f_S > 0$. We assume a kink solution for $\Phi_{S_1}$,

$$\langle \Phi_{S_1} \rangle = h_S(y),$$

(24)

which localizes all fermions at the same point $y = 0$ along $y$. In fact, the equation governing the zero modes, at this stage, is

In order to be more general, we will allow the fermions to be localized, still at this stage, at some common arbitrary point which might be different from the origin. The most economical scenario is one in which the “left-handed” fermions are localized at that other point. This can be accomplished by the following coupling:

$$\mathcal{L}_{Y1p} = -f_{S'} \Phi_{S_2} \bar{\psi}^L \psi^L,$$

(25)

where $f_{S'} > 0$ and the negative sign in front of it is an arbitrary choice. Assuming

$$\langle \Phi_{S_2} \rangle = \delta,$$

(26)

we obtain the following equations for the zero modes:

$$\partial_y \xi_L(y) + \{f_S h_S(y) - f_{S'} \delta\} \xi_L(y) = 0.$$

(27a)

$$\partial_y \xi_R(y) + \{f_S h_S(y)\} \xi_R(y) = 0.$$

(27b)

We will present below two scenarios.

- Scenario I:

$$\delta = 0.$$

(28)

- Scenario II:

$$\delta \neq 0.$$

(29)

At this stage, Scenario I implies that all (left and right-handed) fermions are localized at the origin. Scenario II implies that the right-handed ones are localized at the origin while the left-handed ones are localized at a common point away from the origin. As we shall see in the last section, it will be Scenario II with $\delta \neq 0$ that is favored phenomenologically.

It is clear that this is not the end of the story because the effective Yukawa couplings to an $SU(2)_L$-doublet Higgs field, which depend on the overlap of the left and right wave functions, would be universal for all fermions, a clearly undesirable feature. We therefore have to split the various wave functions along $y$. To do this, one has to invoke scalars which transform non-trivially under $SU(4)_P \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$. This is what we will proceed to do next.

3. The roles of $\Sigma = (15, 1, 1, 1)$ and $\Phi_H = (15, 1, 1, 3)$

From the group representations of $\Sigma = (15, 1, 1, 1)$ and $\Phi_H = (15, 1, 1, 3)$, one can see that, in principle, both $\Psi^L$ and $\Psi^R$ can couple to $\Sigma$ and $\Phi_H$. However, for reasons of economy, we shall see that it is sufficient to couple $\Psi^L$ to $\Phi_H$.

We now concentrate on $\xi_L(y)$. As it is mentioned above, one has to differentiate the unconventional fermions from the conventional ones as well as the “quarks” from the “leptons”. Let us remind ourselves that the conventional and unconventional fermions are grouped into $SU(2)_H$ doublets as shown in Eq. (14).

To differentiate the aforementioned fermions, we need to break both $SU(4)_P$ and $SU(2)_H$. This is accomplished by the use of $\Phi_H = (15, 1, 1, 3)$ and of $\Sigma = (15, 1, 1, 1)$. The Yukawa interaction between $\Psi^L$, $\Phi_H$ and $\Sigma = (15, 1, 1, 1)$ is given by

$$\mathcal{L}_{Y2} = \bar{\Psi}^L (f_H \Phi_H + f_S \Sigma) \psi^L,$$

(30)

with $f_H, f_S > 0$.

We will assume a vacuum expectation value for $\Phi_H$ and $\Sigma$ as follows

$$\langle \Sigma \rangle = \sigma \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right),$$

(31a)

$$\langle \Phi_H \rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right) \otimes \left(\begin{array}{c} v_H \\ 0 \\ 0 \\ -v_H \end{array}\right),$$

(31b)

where the first matrix on the right-hand side of Eq. (31) refers to the direction $T_{15}$ of $SU(4)_P$ whereas the second matrix in the second equation refers to the direction $T_3$ in $SU(2)_H$, all of which refer to $(15, 1, 1, 3)$. Here $\sigma$ and $v_H$ are constants.

When Eq. (31) is combined with Eq. (28), the equation for the left-handed zero modes is now given by

$$\partial_y \xi_L(y) + \{f_S h_S(y) - f_{S'} \delta + f_H \langle \Phi_H \rangle + f_S \langle \Sigma \rangle\} \xi_L(y) = 0.$$

(32)

We now make an important assumption:

$$f_H v_H = f_S \sigma.$$

(33)

This assumption has a far-reaching consequence: All unconventional quarks and leptons will have large wave function overlaps resulting in “large” mass scales for those sectors as we shall see below.

Making use explicitly of Eq. (31) and the assumption $\mathcal{L}_{Y1p}$, one can now rewrite $\mathcal{L}_{Y1p}$ in terms various $SU(2)_L$ doublets as follows

$$\partial_y \xi'^Q_L(y) + \{f_S h_S(y) + 2 f_H v_H - f_{S'} \delta\} \xi'^Q_L(y) = 0,$$

(34a)
\[ \partial_y \zeta^Q_L(y) + \{f_s \, h_S(y) - f_{s'} \, \delta\} \zeta^Q_L(y) = 0, \quad (34b) \]

\[ \partial_y \zeta^L_L(y) + \{f_s \, h_S(y) - 6 f_H \, v_H - f_{s'} \, \delta\} \zeta^L_L(y) = 0, \quad (34c) \]

\[ \partial_y \zeta^L_R(y) + \{f_s \, h_S(y) - f_{s'} \, \delta\} \zeta^L_R(y) = 0, \quad (34d) \]

where the superscripts \(Q, \bar{Q}, L, \bar{L}\) refer to the normal quark, unconventional quark, normal lepton, and unconventional lepton \(SU(2)_L\) doublets as shown in Eq. (4). The above equations show the splitting between conventional and unconventional fermions as well as between quarks and leptons. As we shall show below, this splitting is crucial to the success of this model.

From Eqs. (34), it is also easy to see that

\[ \zeta^Q_L(y) = \zeta^L_R(y). \quad (35) \]

4. The role of \(\Phi_R = (1, 1, 3, 1)\)

We have encountered in Section II A the \(SU(2)_R\) triplet scalar field whose kink solution, when combined with a singlet kink, gives rise to very different profiles for the wave functions of the up and down members of an \(SU(2)_R\) fermion doublet. In the present context, the triplet scalar field is now \(\Phi_R = (1, 1, 3, 1)\). Its coupling to \(\Psi(R)\) can be written as

\[ \mathcal{L}_{Y3} = f_R \bar{\Psi}^{(R)} \Phi_R \Psi^{(R)}, \quad (36) \]

with \(f_R > 0\). Notice that here we use \(f_R\) instead of the notation \(f_T\) used in Section II A in order to be consistent with the notation used for \(\Phi_R\).

The minimum energy solution for \(\Phi_R\) can be written as

\[ \langle \Phi_R \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} h_T(y) & 0 \\ 0 & -h_T(y) \end{pmatrix}, \quad (37) \]

where we have assumed that there is a kink solution for \(\Phi_R\).

When one combines Eq. (38) with Eqs. (36, 37), the equation for the zero modes looks as follows.

\[ \partial_y \xi_R(y) + \{f_s \, h_S(y) + f_R \, (\Phi_R)\} \xi_R(y) = 0. \quad (38) \]

Let us define the following effective kinks:

\[ h^{\text{sym}}(y) = f_s \, h_S(y) + f_R \, h_T(y), \quad (39a) \]

\[ h^{\text{asym}}(y) = f_s \, h_S(y) - f_R \, h_T(y). \quad (39b) \]

Eq. (38) then takes the following forms:

\[ \partial_y \xi^Q_{R \uparrow}(y) + h^{\text{sym}}(y) \xi^Q_{R \uparrow}(y) = 0, \quad (40a) \]

\[ \partial_y \xi^Q_{R \downarrow}(y) + h^{\text{asym}}(y) \xi^Q_{R \downarrow}(y) = 0, \quad (40b) \]

\[ \partial_y \xi^{L \uparrow}_R(y) + h^{\text{sym}}(y) \xi^{L \uparrow}_R(y) = 0, \quad (40c) \]

\[ \partial_y \xi^{L \downarrow}_R(y) + h^{\text{asym}}(y) \xi^{L \downarrow}_R(y) = 0. \quad (40d) \]

In Eqs. (40) and according to the particle content given in Eqs. (4), “\(Q, \uparrow\)” refers to \(d^c(1/3)\) and \(\tilde{U}(4/3)\). Likewise, “\(Q, \downarrow\)” refers to \(u^-(-2/3)\) and \(\tilde{D}(1/3)\). Similarly, “\(L, \uparrow\)” refers to \(\nu(0)\) and \(l_u(-1)\) while “\(L, \downarrow\)” refers to \(l(-1)\) and \(\tilde{l}_d(-2)\). It is also clear, from Eqs. (40), that

\[ \xi^Q_{R \uparrow}(y) = \xi^{L \uparrow}_R(y) = \xi^{\uparrow}, \quad (41a) \]

\[ \xi^Q_{R \downarrow}(y) = \xi^{L \downarrow}_R(y) = \xi^{\downarrow}. \quad (41b) \]

A quick look at Eq. (39) reveals that \(u_R(2/3)\) and \(\tilde{D}_R(1/3)\) as well as \(l_R(-1)\) and \(\tilde{l}_d, R(-2)\) have “broad” wave functions while \(d_R(-1/3)\) and \(\tilde{U}_R(1/3)\) and \(\nu_R(0)\) and \(\tilde{l}_u, R(-1)\) have “narrow” wave functions. These features will be shown explicitly below.

From Eqs. (40), one can easily see that all right-handed wave functions are localized at the origin. They have, however, different profiles, a situation which is very similar to the scenario which is summarized in Section II A. It is this difference in profiles, when combined with the different locations of left-handed wave functions, which gives rise to the disparity in mass scales.

We turn next our attention to the separations along \(y\) between left-handed and right-handed fermions which are crucial, along with the different profiles, in determining the mass scale of each sector.

5. Wave function localizations in a linear approximation

To see heuristically how Eqs. (40) help split the locations of the “quarks” and “leptons” along the extra dimension, let us make a linear approximation to the kink solutions \(h_S(y)\) and \(h_T(y)\), namely

\[ h_S(y) \approx 2 \mu^2_S y. \quad (42a) \]

\[ h_T(y) \approx 2 \mu^2_T y. \quad (42b) \]
Let us recall that, with the linear approximation, a wave function behaves like a Gaussian $\xi(y) \propto \exp(-\mu^2 y^2)$. It then follows that that the overlap between two functions separated by a distance $\Delta y$ along $y$ goes like $\exp(-\mu^2 (\Delta y)^2)$ (where for heuristic purpose $\mu^2$ are taken to be the same for both wave functions which in general is not the case). The effective Yukawa couplings in four dimensions are proportional to the overlaps between “right-handed” and “left-handed” fermions and it can be seen that they can be “large” or “small” depending on whether or not $(\Delta y)^2 \gg \mu^2$ or $(\Delta y)^2 \ll \mu^2$. What we will set out to derive in our model is the relationship between $\Delta y^{(l)}$ and $\Delta y^{(d)}$.

With the approximation $\exp(-\mu^2 y^2)$, let us apply it to Eqs. $\text{(42)}$, $\text{(43a)}$, $\text{(43b)}$, $\text{(44)}$, $\text{(45)}$, $\text{(46)}$, $\text{(47a)}$, $\text{(47b)}$.

From Eqs. $\text{(46)}$, the locations of the Up-members and the Down-members of an $SU(2)_L$ doublet, for both conventional and unconventional quarks, are found to be at the origin. We shall denote that common point by $y_R = 0$. (44)

For the left-handed zero modes, their locations will depend on $f_{S\prime}$ and $f_H v_H$. For convenience, let us define the following quantity:

\[ r = \frac{f_{S\prime}}{2 f_H v_H}. \] (45)

We will assume that $r < 1$. From Eqs. $\text{(46)}$, the locations of the $SU(2)_L$ doublets for conventional and unconventional quarks are

\[ y_{Q_L} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 - r), \] (46a)

\[ y_{Q_R} = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r, \] (46b)

while the locations of the “left-handed” doublets are

\[ y_{l_R} = \left(3 \frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 + \frac{r}{3}), \] (47a)

\[ y_{l_L} = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r. \] (47b)

One important comment is in order at this point. From the above equations $\text{(46)}$, $\text{(47)}$, as well as $\text{(48)}$, it is clear that there are five independent parameters: $f_S \mu_S^2$, $f_{S\prime} \delta$, $f_R \mu_R^2$, $f_2 \sigma$, and $f_H v_H$, although Eq. $\text{(49)}$ reduces to four independent parameters. What this implies is that, out of eight locations, there are three (or four) predictions. In principle, we would then obtain three (or four) predictions for mass scales once the other four (or three) are fixed by the choices of the aforementioned parameters. We shall come back to this point below.

In order to make sense out of the above locations, a few remarks are in order here. The effective Yukawa coupling, in four dimensions, which governs the fermion mass scale depends on the overlap between left-handed and right-handed fermions. This overlap depends on the separation between the two fermions as well as on the shapes of the fermion wave functions. As mentioned above, the spread of the wave functions is crucial in our scenario. This spread is roughly proportional to $1/\mu$. From Eqs. $\text{(43a)}$, one can deduce that, for the right-handed wave functions, $l_R(-1)$ $l_{d,R}(-2)$, $u_R(2/3)$ and $\tilde{D}_R(1/3)$ have broad wave functions since $\mu_{\text{sym}}^2 = f_S \mu_S^2 - f_R \mu_R^2$. On the other hand, $d_R(-1/3)$, $\tilde{U}_R(4/3)$, $\nu_R(0)$ and $\tilde{l}_{u,R}(-1)$ have narrow wave functions since $\mu_{\text{asym}}^2 = f_S \mu_S^2 + f_R \mu_R^2$. All the left-handed wave functions, on the other hand, have a spread of the order of $1/\sqrt{f_S \mu_S^2}$. How do these facts translate into the disparities in mass scales? To answer this question, one has to look at the separations between the left-handed and right-handed wave functions.

From Eqs. $\text{(48)}$, $\text{(49)}$, one can readily derive the following left-right separations:

- For the “quarks”:

\[ |\Delta y_U| \equiv |y_R - y_{Q_L}| = \left|\left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 - r)\right|, \] (48a)

\[ |\Delta y_D| \equiv |y_R - y_{Q_R}| = \left|\left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 - r)\right|, \] (48b)

\[ |\Delta y_G| \equiv |y_R - y_{Q_L}| = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r, \] (48c)

\[ |\Delta y_{\tilde{D}}| \equiv |y_R - y_{Q_L}| = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r. \] (48d)

- For the “leptons”:

\[ |\Delta y_{\nu}| \equiv |y_R - y_{l_L}| = 3 \left|\left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 + \frac{r}{3})\right|, \] (49a)

\[ |\Delta y_{l_{(-1)}}| \equiv |y_R - y_{l_L}| = 3 \left|\left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) (1 + \frac{r}{3})\right|, \] (49b)

\[ |\Delta y_{\tilde{l}_{(-1)}}| \equiv |y_R - y_{l_L}| = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r, \] (49c)

\[ |\Delta y_{\tilde{l}_{(-2)}}| \equiv |y_R - y_{l_L}| = \frac{f_{S\prime}}{2 f_S \mu_S^2} = \left(\frac{f_H}{f_S}\right) \left(\frac{v_H}{\mu_S}\right) r. \] (49d)
Comparing Eqs. (19) with Eqs. (15), we arrive at the following important relationship between conventional quarks and leptons:

\[ |\Delta y_{\text{Lepton}}| = 3 \left( \frac{1 + \frac{r}{1 - r}}{1 - r} \right) |\Delta y_{\text{Quark}}|, \]  

(50)

where \( r < 1 \). For Scenario I, one would have \( r = 0 \) and one would simply have \( |\Delta y_{\text{Lepton}}| = 3|\Delta y_{\text{Quark}}| \). For Scenario II, where \( r \neq 0 \), one obtains the above relationship. What (50) implies is the following important fact: \( |\Delta y_{\text{Lepton}}| \geq 3|\Delta y_{\text{Quark}}| \). This means that the scales of the lepton sector are generally a bit smaller than those of the quark counterpart.

It is also useful to derive a relationship between the separations of unconventional and conventional fermions. From Eqs. (16, 17, 18), it is straightforward to derive the following relationship between the common left-right separation of the unconventional quarks and leptons and that of the conventional quarks:

\[ |\Delta y_{\text{Unconventional}}| = \left( \frac{r}{1 - r} \right) |\Delta y_{\text{Quark}}|, \]  

(51)

Another useful form for (51) can be obtained by using Eq. (50), namely

\[ |\Delta y_{\text{Unconventional}}| = \left( \frac{|\Delta y_{\text{Lepton}}| - 3|\Delta y_{\text{Quark}}|}{4} \right) |\Delta y_{\text{Quark}}|. \]  

(52)

From (50) and (51), one notices that, in Scenario I where \( r = 0 \), one obtains the simple relations: \( |\Delta y_{\text{Lepton}}| = 3|\Delta y_{\text{Quark}}| \) and \( |\Delta y_{\text{Unconventional}}| = 0 \).

The above relationship is important for the following reasons. First, it implies that the left-handed wave function for the leptons is situated much further (by a factor of three) away from the right-handed ones than is the case for the quarks. It then means that the wave function overlaps which determine the effective four-dimensional Yukawa couplings would, in principle, be much smaller for the lepton sector than for its quark counterpart, resulting in a large disparity in mass scales between the two sectors. This is actually what happens in reality. The details of that disparity, in our scenario, will also depend on the difference in the wave function profiles. This will be discussed in the next section.

Since \( |\Delta y| = \frac{f_R \delta}{2 f_S v_2^2} \) for both unconventional quarks and leptons, this implies that unconventional quarks and leptons have comparable mass scales which can be “large”. How large this might be is the subject of the next section.

To summarize, the model contains four independent parameters: \( f_S \mu_2^2 \), \( f_R \mu_1^2 \), \( f_R v_H \), and \( r \). From these, we have two independent wave function profiles for the right-handed zero modes, one independent separation \( |\Delta y_{\text{Quark}}| \), and one parameter \( r \). Once \( r \) and \( |\Delta y_{\text{Quark}}| \) are specified, all other separations can be computed.

We now present some numerical illustrations of the above results. Our strategy will be as follows. First, we write down the coupling between the left-handed fermions, right-handed fermions and a Higgs field whose VEV gives rise to fermion masses. Next, we (arbitrarily) fix the two right-handed wave function profiles. We then choose \( |\Delta y_{\text{Quark}}| \) so that the mass scales of the conventional Up and Down quark sectors come out correctly. With the same wave function profiles, we next choose \( r \) so that, upon the use of Eq. (50), the mass scale of the charged lepton sector comes out correctly. We will show below that, as a result, we obtain predictions for the mass scale of the Dirac neutrino sector as well as those of the unconventional quarks and leptons. An alternative way to fix the parameters is to choose one of the right-handed wave functions, for example the one that belongs to the charged leptons and the Up quark sector, fix \( r \) so that the mass scales come out correctly, fix the second right-handed wave function function so that the Down quark sector comes out correctly. Once this is done, the above predictions will come out the same.

III. COMPUTATION OF MASS SCALES AND IMPLICATIONS

One can now use the results of Ref. [1] and Section II A to estimate mass scales of the normal quark and lepton sectors as well as those of the unconventional fermions. Since the scope of this paper is the construction of a model showing a relationship between mass scales of “quarks” and “lepton” sectors, we shall ignore issues such as fermion mixings in the mass matrices. Higher dimensional models have been built to tackle quark mass hierarchies, mixing angles and CP phase (see e.g. [11] and [12]). We will therefore concentrate on the overall mass scales that appear in various mass matrices.

A. SM Fermion-Higgs coupling

By “SM Fermion-Higgs coupling”, we mean that the Higgs field that couples with left-handed and right-handed fermions transforms non-trivially under \( SU(2)_L \). Since \( \Xi^{(L)} = (4, 2, 1, 2) \), \( \Xi^{(R)} = (4, 1, 2, 2) \) and \( \Psi^{(L)} \Psi^{(R)} = (1 + 15, 2, 2, 1 + 3) \), an appropriate Higgs field (the simplest choice) could be the following field:

\[ H = (1, 2, 2, 1). \]  

(53)

The Yukawa coupling with this field can be written as

\[ \mathcal{L}_Y = k_1 \bar{\Psi}^{(L)} H \Psi^{(R)} + k_2 \bar{\Psi}^{(L)} \tilde{H} \Psi^{(R)} + H.c., \]  

(54)

where \( \tilde{H} = \tau_2 H^* \tau_2 \) and where, in principle, \( k_2 \) can be different from \( k_1 \). Assuming the extra dimension to be compactified on an orbifold \( S_1 \times S_2 \) and an even \( Z_2 \) parity for \( H \), it follows that \( H \) can have a zero mode. This zero mode can be written as \( H^0(x, y) = K \phi(x) \) where \( \phi(x) \) is a 4-dimensional Higgs field with dimension \( M \) (mass) and \( K \), a constant, has a dimension \( \sqrt{M} \), since
$H$ has a dimension $M^{3/2}$ in five dimensions. Notice that $k_{1,2}$ have a dimension $M^{-1/2}$. We define the following dimensionless couplings:

$$g_{Y1,2} = k_{1,2} K.$$

(55)

The VEV of $\phi$ is assumed to be

$$\langle \phi \rangle = \left( \frac{v_1}{\sqrt{2}} 0 \frac{v_2}{\sqrt{2}} \right).$$

(56)

Eqs. (54) and (56) provide mass scales which appear in mass matrices as follows

$$M_{U,D,\nu,\bar{l},D,\bar{l},\bar{d}} = \Lambda_{U,D,\nu,\bar{l},D,\bar{l},\bar{d}},$$

(57)

where $\Lambda_{U,D,\nu,\bar{l},D,\bar{l},\bar{d}}$ are the mass scales in question and $M_{U,D,\nu,\bar{l},D,\bar{l},\bar{d}}$ are matrices whose elements will depend on models of fermion masses (e.g. [11]). The subscripts are self-explanatory.

Using the fermion representations (15) and Eqs. (54, 56), the mass scales that appear in Eq. (53) now take the following forms:

$$\Lambda_U = g_{U1} v_2 / \sqrt{2} + g_{U2} v_1 / \sqrt{2},$$

(58a)

$$\Lambda_D = g_{D1} v_1 / \sqrt{2} + g_{D2} v_2 / \sqrt{2},$$

(58b)

$$\Lambda_{\nu} = g_{\nu1} v_1 / \sqrt{2} + g_{\nu2} v_2 / \sqrt{2},$$

(58c)

$$\Lambda_{\bar{l}} = g_{\bar{l}1} v_2 / \sqrt{2} + g_{\bar{l}2} v_1 / \sqrt{2},$$

(58d)

$$\Lambda_{\bar{d}} = g_{\bar{d}1} v_2 / \sqrt{2} + g_{\bar{d}2} v_1 / \sqrt{2},$$

(58e)

$$\Lambda_{\bar{l}} = g_{\bar{l}1} v_1 / \sqrt{2} + g_{\bar{l}2} v_2 / \sqrt{2},$$

(58f)

where

$$g_{U1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{down}}(y),$$

(59a)

$$g_{D1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{up}}(y),$$

(59b)

$$g_{\nu1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{up}}(y),$$

(59c)

$$g_{\bar{l}1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{down}}(y),$$

(59d)

$$g_{\bar{d}1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{down}}(y),$$

(59e)

$$g_{\bar{l}1,2} = g_{Y1,2} \int_0^L dy \xi_Q^L(y) \xi_{R}^{\text{up}}(y).$$

(59f)

The way $v_1$ and $v_2$ appear in Eqs. (58) should be clearly understood that it has to do with the fermion content as shown in Eqs. (15), and that is the reason why the first two equations appear in the forms shown above.

There are two possibilities concerning Eqs. (58).

- $g_{Y1} = g_{Y2}$:

This is a rather economical option, in terms of reducing the number of parameters. From Eqs. (58) and (59), it is easy to see that, if $g_{Y1} = g_{Y2}$, the ratios of scales will simply be $\Lambda_{Y} = \Lambda_{Y}$, as demonstrated in the next section where we will show how to obtain these masses.

- $g_{Y1} \neq g_{Y2}$:

In the following, we will choose $\Lambda_U$ and $\Lambda_D$ as two independent inputs. From them, we can extract $\Lambda_{\nu}$, $\Lambda_{\bar{l}}$, $\Lambda_{\bar{d}}$, and $\Lambda_{\bar{l}}$. Once the parameter $r$ is chosen, all other mass scales can be predicted.
• $g_{Y1} \neq g_{Y2}$:
  For the case when $g_{Y1} \neq g_{Y2}$, one can still obtain the following ratios which depend only on ratios of wave function overlaps:

$$
\frac{\Lambda_U}{\Lambda_D} = \frac{\int_0^L dy \xi_L^f(y) \xi_R^{down}(y)}{\int_0^L dy \xi_Q^f(y) \xi_R^{down}(y)}, \quad (61a)
$$

$$
\frac{\Lambda_U}{\Lambda_D} = \frac{\int_0^L dy \xi_L^f(y) \xi_R^{up}(y)}{\int_0^L dy \xi_Q^f(y) \xi_R^{up}(y)}, \quad (61b)
$$

$$
\frac{\Lambda_U}{\Lambda_D} = \frac{\int_0^L dy \xi_L^f(y) \xi_R^{up}(y)}{\int_0^L dy \xi_Q^f(y) \xi_R^{down}(y)}, \quad (61c)
$$

$$
\frac{\Lambda_U}{\Lambda_D} = \frac{\int_0^L dy \xi_L^f(y) \xi_R^{down}(y)}{\int_0^L dy \xi_Q^f(y) \xi_R^{down}(y)}, \quad (61d)
$$

What are the implications of the above two cases? First, one chooses the two parameters $f_S \mu_{S}^2$ and $f_R \mu_{R}^2$, so that $\xi_R^{up}(y)$ and $\xi_R^{down}(y)$ are fixed. Next, we choose $\Lambda_U$ and $\Lambda_D$ as two independent mass scales. In the first case where $g_{Y1} = g_{Y2}$, these two scales are used to extract $|\Delta y_{Quark}|$. One can then choose the parameter $r$ so that the scale $\Lambda_U$- is fixed. Once this is done, all other scales- neutrinos, unconventional fermions- can be predicted. For the second case where $g_{Y1} \neq g_{Y2}$, one has to both choose $|\Delta y_{Quark}|$ and $r$ in order to fix the first ratio in (61a). All other ratios in (61) can then be predicted.

B. Some numerical examples

In this section, we will present some numerical illustrations of the above ideas. A more comprehensive numerical analysis will be presented elsewhere. We find a surprising correlation between the Dirac neutrino masses and those of the unconventional fermions. As we shall see below, by requiring the unconventional fermions to be heavier than the top quark but at the same time NOT too much heavier than the electroweak scale e.g. $< 650 GeV$, it is found that the largest Dirac neutrino mass is bounded from below by a value roughly of the order of $0.1 eV$ and from above by a value of the order of $30 eV$. From this result alone, it is hard to see how one can incorporate Majorana neutrinos in this scenario since the Dirac neutrinos alone are naturally light. Actually, the detailed numerical analysis of (61) shows that, in order to keep the early unification scale below $2 TeV$, the masses of the unconventional fermions are constrained to be less than $300 TeV$ which, as we shall see below, sets the following bound for the heaviest Dirac neutrino: $0.1 eV < m_{\nu_{\text{heaviest}}} < 1 eV$. Or one can turn things around by using some of our knowledge, however uncertain, about neutrino masses to set bounds on the masses of the unconventional fermions. For example, if we set the largest neutrino mass to lie between $0.1 eV$ and $1 eV$ assuming it is of the Dirac type, the unconventional fermions are constrained to have a mass between the top quark mass and $300 GeV$.

Our numerical strategy is as follows. 1) For a given pair of $f_S h_S(y)$ and $f_R h_T(y)$, we use the ratio $\Lambda_D/\Lambda_U$ to find $|\Delta y_{Quark}|$. Actually, it is the difference between $f_S h_S(y)$ and $f_R h_T(y)$ that is important. 2) We then choose $r$ so that $|\Delta y_{Lepton}|$, which is given in terms of $|\Delta y_{Quark}|$ via Eq. (61), gives the correct ratio $\Lambda_U/\Lambda_D$. 3) After Steps 1 and 2 have been carried out, one can make predictions for the mass scales of the Dirac neutrino sector as well as those for the unconventional fermions, using Eqs. (60). After this is done, one can then decide whether or not a Majorana neutrino mass term is needed, depending on the value of the Dirac mass.

To be general, we start out with $r \neq 0$.

For the zero mode right-handed wave functions, we use expressions similar to those found in Eq. (61), namely

$$
\xi_R^{up,down}(y) = k_{up,down} e^{-\left(C_S \ln(\cosh(\mu_S y)) + C_R \ln(\cosh(\mu_R y))\right)}
$$

where $k_{up,down}$ are normalization factors and $C_{S,R} = f_{S,R}/(\lambda_{S,T}^2/2)^{1/2}$ are factors which involve the Yukawa couplings as well as the self-couplings in the scalar potentials. (Let us recall generically that $h(y) = v \tanh(\mu y)$, with $v = \sqrt{\chi/2}$.) The wave functions for the left-handed zero modes are given by

$$
\xi_L^f(y) = k_L e^{-\left(C_S \ln(\cosh(\mu_S(y-y_i)))\right)},
$$

where $k_L$ is a normalization factor, $i = Q, L, \bar{Q}$ and $y_i$ are given by (15b) and (17).

Using (62) and (63), we can now illustrate the results presented in the previous section with a numerical example. To translate ratios of mass scales into ratios of actual mass eigenvalues, an assumption has to be made concerning the mass matrices themselves. Explicitly, the relationship between the mass scales $\Lambda$ and the mass matrices $\mathcal{M}$ can be written as

$$
\mathcal{M} = \Lambda \tilde{\mathcal{M}},
$$

where $\tilde{\mathcal{M}}$ is a dimensionless matrix whose form depends on a particular model. An exhaustive general analysis of different types of mass matrices is beyond the scope of this paper. For the purpose of illustration in this paper, we will make the following reasonable assumptions concerning the relationship between the mass scale that appears as a common factor in the mass matrix and the largest eigenvalue, namely

$$
\frac{m_3}{3} \leq \Lambda_U \leq m_t, \quad (65a)
$$

$$
\frac{m_3}{3} \leq \Lambda_D \leq m_b, \quad (65b)
$$
\[
\frac{m_t}{3} \leq \Lambda_{l-} \leq m_r,
\]  
(65c)

where \(m_t\), \(m_b\), and \(m_r\) are the largest eigenvalues of the Up-quark, Down-quark, and charged lepton mass matrices respectively. The lower bounds in Eqs. (65) refer to a pure democratic mass matrix \[15\] where, apart from the common scale factor, all elements are unity and the largest eigenvalue is three times the scale factor. Such a pure democratic mass matrix is unrealistic but it is included here for completeness. The upper bound refers to a class of hierarchical mass matrices where the largest eigenvalue is approximately the scale factor itself \[15\]. In between these two bounds, there exists models e.g. \[11, 12\] which are almost but not quite of the pure phase mass matrix type \[19\]. We will assume below that the Up-quark, Down-quark and charged lepton sectors have mass matrices with “similar” behaviour, only in the sense that the relationship between the scale factors and the largest mass eigenvalues is assumed to be the same.

For the purpose of illustration, we will use, for the largest eigenvalues, \(m_t\), \(m_b\) and \(m_r\) evaluated at \(M_Z\), and neglect any “running” between \(M_Z\) and the early unification scale taken to be comparable to the scale of compactification. We take

\[
m_t(M_Z) = 181 \text{ GeV}; \quad m_b(M_Z) = 3 \text{ GeV}; \quad m_r(M_Z) = 1.747 \text{ GeV}.
\]  
(66)

With the above numbers and with the remarks made above concerning the relationship between the scale factors and the largest mass eigenvalues, we can write

\[
\frac{\Lambda_D}{\Lambda_U} \approx \frac{m_b(M_Z)}{m_t(M_Z)} \approx 0.0166,
\]  
(67a)

\[
\frac{\Lambda_{l-}}{\Lambda_U} \approx \frac{m_r(M_Z)}{m_t(M_Z)} \approx 0.00965.
\]  
(67b)

The ratios \(\frac{\Lambda_D}{\Lambda_U}\) and \(\frac{\Lambda_{l-}}{\Lambda_U}\) are now used to predict the mass scales \(\Lambda\) for the neutrino sector as well as for the unconventional fermion sectors.

As we have mentioned earlier, the mass scales of the neutrinos are correlated with those of the unconventional fermions. This will be shown in the six examples below for the more general case of \(r \neq 0\). For comparison, we will also show a result for \(r = 0\).

First we would like to remind ourselves that it is the difference between \(C_S \ln(\cosh(\mu S y))\) and \(C_R \ln(\cosh(\mu R y))\) in the wave functions that is important. In consequence, we will set \(C_S = C_R = 1\), choose \(\mu_S = 1\) (in some unit), and vary \(\mu_R\).

In order to present the results in a transparent way, we prefer to write expressions such as \(\exp(-\ln(\cosh(y)))\) instead of the equivalent expression \(1/\cosh(y)\).

- \(\xi_R^{up}(y) = \frac{1}{\sqrt{1.53}} \exp\{ -\ln(\cosh(y)) - \ln(\cosh(0.7 y)) \} \); \(\xi_R^{down}(y) = \frac{1}{\sqrt{1.53}} \exp\{ -\ln(\cosh(y) + 7.815) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y)) + \ln(\cosh(0.7 y)) \} \);
- \(\xi_L^{down}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y) + 7.53) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y) - 24.715) \} \); \(\xi_L^{down}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y - 0.531)) \} \).

The predictions are:

\[
\frac{\Lambda_\nu}{\Lambda_{l-}} = 0.278 \times 10^{-6}, \]  
(68a)

\[
\frac{\Lambda_D}{\Lambda_U} = \frac{\Lambda_{l-}}{\Lambda_U} = 7.3, \]  
(68b)

\[
\frac{\Lambda_D}{\Lambda_U} = \frac{\Lambda_{l}}{\Lambda_U} = 7.93. \]  
(68c)

- \(\xi_R^{up}(y) = \frac{1}{\sqrt{1.53}} \exp\{ -\ln(\cosh(y)) - \ln(\cosh(0.7 y)) \} \);
- \(\xi_R^{down}(y) = \frac{1}{\sqrt{1.53}} \exp\{ -\ln(\cosh(y) + 7.815) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y)) + \ln(\cosh(0.7 y)) \} \);
- \(\xi_L^{down}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y) + 7.53) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y) - 24.715) \} \); \(\xi_L^{down}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y - 0.531)) \} \).

The predictions are:

\[
\frac{\Lambda_\nu}{\Lambda_{l-}} = 0.5 \times 10^{-7}, \]  
(69a)

\[
\frac{\Lambda_D}{\Lambda_U} = \frac{\Lambda_{l-}}{\Lambda_U} = 5.46, \]  
(69b)

\[
\frac{\Lambda_D}{\Lambda_U} = \frac{\Lambda_{l}}{\Lambda_U} = 5.82. \]  
(69c)

- \(\xi_R^{up}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y)) - \ln(\cosh(0.7 y)) \} \);
- \(\xi_R^{down}(y) = \frac{1}{\sqrt{1.332}} \exp\{ -\ln(\cosh(y) + 7.815) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y)) + \ln(\cosh(0.7 y)) \} \);
- \(\xi_L^{down}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y) + 7.53) \} \);
- \(\xi_L^{up}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y) - 24.715) \} \); \(\xi_L^{down}(y) = \frac{1}{\sqrt{2}} \exp\{ -\ln(\cosh(y - 0.531)) \} \).

The predictions are:

\[
\frac{\Lambda_\nu}{\Lambda_{l-}} = 0.13 \times 10^{-7}. \]  
(70a)
The predictions are:

\[ \frac{\Lambda_U}{\Lambda_l} = 0.381 \times 10^{-9}, \]  

\[ \frac{\Lambda_D}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 2.834, \]  

\[ \frac{\Lambda_{\tilde{D}}}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 1.858. \]

The predictions are:

\[ \frac{\Lambda_U}{\Lambda_l} = 0.381 \times 10^{-9}, \]  

\[ \frac{\Lambda_{\tilde{D}}}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 2.834, \]  

\[ \frac{\Lambda_{\tilde{U}}}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 1.39. \]

\[ e_{R_{\tilde{U}}}^{up}(y) = \frac{1}{\sqrt{1.383}} \exp\{-\ln(cosh(y)) - \ln(cosh(0.79 y))\}; \]
\[ e_{R_{\tilde{D}}}^{down}(y) = \frac{1}{\sqrt{1.383}} \exp\{-\ln(cosh(y)) + \ln(cosh(0.79 y))\}; \]
\[ e_{L_{\tilde{U}}}^{up}(y) = \frac{1}{\sqrt{1.383}} \exp\{-\ln(cosh(y)) - \ln(cosh(0.79 y))\}; \]
\[ e_{L_{\tilde{D}}}^{down}(y) = \frac{1}{\sqrt{1.383}} \exp\{-\ln(cosh(y)) + \ln(cosh(0.79 y))\}; \]
\[ e_{L_{\tilde{D}}}^{up}(y) = \frac{1}{\sqrt{2}} \exp\{-\ln(cosh(y) - 7.07)\}; \]
\[ e_{L_{\tilde{D}}}^{down}(y) = \frac{1}{\sqrt{2}} \exp\{-\ln(cosh(y) - 29.17)\}; \]
\[ e_{L_{\tilde{D}}}^{up}(y) = \frac{1}{\sqrt{2}} \exp\{-\ln(cosh(y) - 1.99)\}. \]

The implications of the above results are given for the upper and lower bounds in [68] in Tables I and II. We use the values in Eq. (66) for the estimates given in these tables. The predictions coming from [68] are listed in the second, third, and fourth columns.

We end this section by briefly showing that the case \( r = 0 \) which gives the interesting relations \( |\Delta y_{\text{Lepton}}| = 3|\Delta y_{\text{Quark}}| \) and \( |\Delta y_{\text{Unconventional}}| = 0 \) and which means that one can also predict the charged lepton mass scale in terms of the one for the quarks, is, unfortunately, not good. For example, taking the quark wave functions used in [72] and using the above relations, one obtains a prediction for the mass scale of the charged lepton sector to be approximately 11 GeV. This, by itself, rules out the case \( r = 0 \). Incidentally, the neutrino mass scale comes out to be \( \sim 2.5 \) keV and those of the unconventional fermions come out to be \( \sim 590-700 \) GeV, although

| \( \lambda_U \) | \( \lambda_{\tilde{l}} \) | \( \lambda_{\tilde{D}} \) |
|---|---|---|
| Eq. 68 | 486 eV | 1321 GeV | 1435 GeV |
| Eq. 68 | 87 eV | 988 GeV | 1053 GeV |
| Eq. 68 | 23 eV | 802 GeV | 791 GeV |
| Eq. 68 | 0.67 eV | 513 GeV | 336 GeV |
| Eq. 68 | 0.23 eV | 456 GeV | 252 GeV |
| Eq. 68 | 0.065 eV | 406 GeV | 181 GeV |

| \( \lambda_{\tilde{l}} \) | \( \lambda_{\tilde{D}} \) |
|---|---|
| Eq. 68 | 162 eV | 440 GeV |
| Eq. 68 | 29 eV | 329 GeV |
| Eq. 68 | 7.7 eV | 267 GeV |
| Eq. 68 | 0.22 eV | 171 GeV |
| Eq. 68 | 0.077 eV | 152 GeV |
| Eq. 68 | 0.022 eV | 135 GeV |

The predictions are:

\[ \frac{\Lambda_U}{\Lambda_l} = 0.371 \times 10^{-9}, \]
\[ \frac{\Lambda_{\tilde{D}}}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 2.246, \]
\[ \frac{\Lambda_{\tilde{D}}}{\Lambda_U} = \frac{\Lambda_{\tilde{l}}}{\Lambda_U} = 0.998. \]
these numbers are irrelevant since the prediction for the charged lepton sector is already wrong.

We now discuss the implications of Tables I and II for the more general case $r \neq 0$.

C. Implications of Tables I and II

To obtain a better understanding of the numerical results presented in Tables I and II, we will assume that the mass matrices for the unconventional fermions are such that, for each sector, the fermions are approximately degenerate. That is because, if it were not the case, a mass splitting similar to the normal fermions (quarks and charged leptons) would render at least one fermion for each sector to be lighter than the top quark. It goes without saying that none has been seen so far. We will therefore assume that the masses of the unconventional fermions are approximately equal to the mass scales $\Lambda$'s.

- Table I:
The numerical results given in this table are for the case where the mass matrices of the normal quarks and leptons are highly hierarchical as we had mentioned earlier.

One obvious remark that one can make by looking at Table I is the following. There is a clear relationship between the masses of the unconventional fermions and those of the neutrinos: As the unconventional masses increase so do neutrino masses. However, if we restrict the masses of the unconventional fermions to be less than one TeV, one notices that the Dirac mass of the neutrinos cannot exceed a few hundreds eVs. In scenarios such as this one, one might expect that Majorana masses, if they exist, would typically be also of the order of TeVs. The see-saw mass for the light neutrino would then be roughly at most of the order of a few hundred eV$^2$/(1 TeV) $\approx 10^{-8}$ eV. As a result, the bulk of the neutrino mass, in this scenario, is Dirac. Furthermore, if the unconventional fermions are not too heavy, say lighter than 500 GeV, nor too light, i.e. heavier than the top quark, the neutrino mass scales vary between 1 eV and approximately 0.07 eV. From Table I, one can tentatively conclude that if the unconventional fermions are heavy, i.e. with masses ranging from 300 GeV to 500 GeV, the neutrino mass scales will range between 0.2 eV and 0.7 eV. This would imply that one might have a situation in which neutrinos are nearly degenerate in order to satisfy the oscillation data. If, on the other hand, the unconventional fermions were to be lighter, i.e. with masses ranging from 181 GeV to 400 GeV, one could have a scenario in which the neutrino mass matrix is hierarchical.

- Table II:
This is the extreme case of democratic mass matrices for the normal fermions. If the masses of the unconventional fermions were to lie between 181 GeV and 400 GeV, the neutrino mass scales would be of the order of a few eVs or more. In light of cosmological constraints as well as of oscillation data, this particular case might even be ruled out. It is amusing to note that this scenario of extreme democratic mass matrices for the normal fermions does not work but itself, regardless of the neutrino sector, because it cannot reproduce the correct mass spectrum and the CKM matrix.

- Intermediate cases:
In between the above two bounds, there are models, e.g. (11, 12), which deviate somewhat from a pure democratic mass matrix model but which can fit fairly well the mass spectrum as well as the CKM matrix. In this model, the mass scales are roughly half the value of the largest mass eigenvalues. In rescaling Table I by a factor of 1/2, one notices that, in order for the lightest unconventional fermion to be heavier than the top quark, the smallest neutrino mass scale is around 0.2 – 0.4 eV. This implies that neutrinos are nearly degenerate.

Although a more extensive investigation of the above questions for various scenarios of mass matrices is warranted-a subject of a next paper- a preliminary conclusion can be drawn from the above results. From the consideration of the lower bound on the lightest unconventional fermion, it appears that neutrinos in our scenario are more likely to be near-degenerate with mass lying around a few tenths of an eV. This would imply that mixing angles as deduced from various oscillation data mainly come from the charged lepton sector.

IV. SUMMARY

We have presented, in this paper, a model of quark-lepton mass unification which “marries” two TeV-scale scenarios: Early Unification of Quarks and Leptons \cite{3, 4, 5} and Large Extra Dimensions \cite{7, 8, 9} (as applied to neutrino masses), has been presented. Explicitly, the early unification model $G_{PUT} = SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$ is embedded in 4+1 dimensions with the extra spatial dimension, $y$, being compactified on an orbifold $S_1/Z_2$. Chiral zero modes are localized along the extra spatial dimension by kinks that come from two background scalar fields, one of which transforms non-trivially under $G_{PUT}$. Additional scalars are used to break $G_{PUT}$ down to the SM. The model contains the following features.

- The breaking of $SU(4)_{PS}$ splits the positions, along $y$, of wave functions of the zero modes of “quarks” and “leptons”.

The breaking of $SU(2)_R$ gives rise to two vastly different profiles for the wave functions of the “right-handed” zero modes.

Since a $SU(2)_H$ doublet groups together a conventional quark (or lepton) with an unconventional one, the breaking of $SU(2)_H$ splits the locations, along $y$, of the wave functions of the conventional fermions relative to those of the unconventional ones.

The breaking of $SU(2)_L \otimes U(1)_Y$ provides a mass scale for all the fermions.

The size of the effective Yukawa couplings, which depends on the overlap between right- and left-handed wave functions, is characterized by the separation along the extra dimension $y$ between these two wave functions, $\Delta y$. In this model, we were able to show that there is a relationship between the quark separation, $\Delta y^{(q)}$, and the lepton separation $\Delta y^{(l)}$, and also that of the unconventional fermions. This is due to the early unification scenario discussed in [4, 5] and again in this paper. It translates into relationships between mass scales that appeared in fermion mass matrices and are valid in the TeV range. This is what is referred to as early quark-lepton mass unification in our scenario. A summary of its ramifications is listed below.

A feature of the $SU(4)_P \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$ model is the existence of “quarks” and “leptons” with unconventional electric charges. As described in the model, these unconventional fermions acquire masses from the same sources as conventional fermions. Because of the existence of relationships between mass scales of different sectors, a consequence of early quark-lepton unification, we have found a strong correlation between the Dirac masses of the neutrinos and those of the unconventional fermions: The neutrino Dirac masses increase the heavier the unconventional fermions become. For example, by requiring that the masses of these unconventional fermions lie between the top mass and $1\,\text{TeV}$ (the lower bound is experimentally obvious while the upper bound refers more to the wish of not having a strong Yukawa coupling regime), we found that the mass scales of the neutrino sector range from approximately $0.07\,\text{eV}$ to roughly $80\,\text{eV}$, as can be seen in Tables I and II. The Dirac masses of the neutrinos are naturally small in this scenario. Any Majorana contribution to the total mass through the see-saw mechanism would have to be negligible.

From the above arguments, our scenario accommodates naturally light Dirac neutrinos without having to use the see-saw mechanism. Since there is a correlation between neutrino masses and those of the unconventionally charged fermions in the $SU(4)_P \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_H$ model, a light neutrino of mass less than $1\,\text{eV}$ (as a consequence of data seems to “indicate” [20]) will imply that the unconventionally charged fermions are not too heavy (see Tables I and II) and there might be a chance to observe them, if they exist, at future colliders [21].

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Conference on the Seesaw Mechanism and the Neutrino Mass, Paris, France, 10-11 Jun 2004, [hep-ph/0411194].

[16] Z. Chako, L. J. Hall and M. Perelstein, JHEP 0301, 001 (2003).

[17] See e.g. P. Kaus and S. Meshkov, Phys. Rev. D 42, 1863 (1990), and references therein.

[18] There is a long history on this subject. Below is an incomplete set of references. H. Fritzsch, Phys. Lett. B 73, 317 (1978); P. Ramond, R. G. Roberts and G. G. Ross, Nucl. Phys. B 406, 19 (1993) [arXiv:hep-ph/9303320]; D. s. Du and Z. z. Xing, Phys. Rev. D 48, 2349 (1993); L. J. Hall and A. Rasin, Phys. Lett. B 315, 164 (1993) [arXiv:hep-ph/9303303]; H. Fritzsch and D. Holtmannspotter, Phys. Lett. B 338, 290 (1994) [arXiv:hep-ph/9406241]; P. S. Gill and M. Gupta, J. Phys. G 21, T (1995). P. S. Gill and M. Gupta, Phys. Rev. D 56, 3143 (1997) [arXiv:hep-ph/9707443]; H. Lehmann, C. Newton and T. T. Wu, Phys. Lett. B 384, 249 (1996); Z. z. Xing, J. Phys. G 23, 1563 (1997) [arXiv:hep-ph/9609204]; K. Kang and S. K. Kang, Phys. Rev. D 56, 1511 (1997) [arXiv:hep-ph/9704253]; T. Kobayashi and Z. z. Xing, Mod. Phys. Lett. A 12, 561 (1997) [arXiv:hep-ph/9609486]; T. Kobayashi and Z. z. Xing, Int. J. Mod. Phys. A 13, 2201 (1998) [arXiv:hep-ph/971232]; J. L. Chkareuli and C. D. Froggatt, Phys. Lett. B 450, 158 (1999) [arXiv:hep-ph/9812499]; J. L. Chkareuli, C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B 626, 307 (2002) [arXiv:hep-ph/0109156]; A. Mondragon and E. Rodriguez-Jauregui, “The breaking of the flavor permutation symmetry: Mass textures and the Phys. Rev. D 59, 093009 (1999) [arXiv:hep-ph/9902240]; H. Nishiura, K. Matsuda and T. Fukuyama, Phys. Rev. D 60, 013006 (1999) [arXiv:hep-ph/9902385]; G. C. Branco, D. Emmanuel-Costa and R. Gonzalez Felipe, Phys. Lett. B 477, 147 (2000) [arXiv:hep-ph/9911418]; R. Rosenefeld and J. L. Rosner, “Hierarchy and anarchy in quark mass matrices, or can hierarchy tolerate Phys. Lett. B 516, 408 (2001) [arXiv:hep-ph/0106335]; H. Fritzsch and Z. z. Xing, Phys. Lett. B 555, 63 (2003) [arXiv:hep-ph/0212195].

[19] G. C. Branco, J. I. Silva-Marcos and M. N. Rebelo, Phys. Lett. B 237, 446 (1990); G. C. Branco and J. I. Silva-Marcos, Phys. Lett. B 359, 166 (1995); G. C. Branco, D. Emmanuel-Costa and J. I. Silva-Marcos, Phys. Rev. D 56, 107 (1997).

[20] For the latest, see e.g. S. Hannestad, [hep-ph/0412181]; J. A. Aguilar-Saavedra, G. C. Branco and F. R. Joaquim, Phys. Rev. D 69, 073004 (2004) [arXiv:hep-ph/0310305]; V. Barger, D. Marfatia and K. Whisnant, Int. J. Mod. Phys. E 12, 569 (2003) [arXiv:hep-ph/0308123].

[21] See e.g. P. Frampton, P. Q. Hung, M. Sher, Phys. Rept. 330, 263 (2000) [arXiv:hep-ph/9903387].