Use Residual Correction Method and Monotone Iterative Technique to Calculate the Upper and Lower Approximate Solutions of Singularly Perturbed Non-linear Boundary Value Problems

Chi-Chang Wang\(^1\) and Cha’o-Kuang Chen\(^2\)

\(^1\)Department of Mechanical and Aeronautical Engineering, Feng Chia University, Taichung, Taiwan, R.O.C
\(^2\)Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan, R.O.C

*Corresponding Author / E-mail: chicwan@fcu.edu.tw, TEL: 886-4-24517250-3512

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This paper seeks to use the proposed residual correction method in coordination with the monotone iterative technique to obtain upper and lower approximate solutions of singularly perturbed non-linear boundary value problems. First, the monotonicity of a non-linear differential equation is reinforced using the monotone iterative technique, then the cubic-spline method is applied to discretize and convert the differential equation into the mathematical programming problems of an inequation, and finally based on the residual correction concept, complex constraint solution problems are transformed into simpler questions of equational iteration. As verified by the four examples given in this paper, the method proposed hereof can be utilized to fast obtain the upper and lower solutions of questions of this kind, and to easily identify the error range between mean approximate solutions and exact solutions.

1. Introduction

Let us begin by considering the boundary problems of a singularly perturbed problem; its equation is given as follows:

\[
\epsilon^2 u'' = f(x)u' + g(x)u + N(x, u, u') + h(x), \quad x \in (a, b)
\]

\[
u(a) = \alpha, \quad u(b) = \beta
\]

Among the above equation, \(\epsilon\) and \(N(x, u, u')\) are a small parameter and a non linear term respectively, with reference to the existence and uniqueness of the solutions of such problem, please see illustrations in Bender and Orszag [1]; while the related study and applications can be referred to in [2]-[6]. Usually in the process of obtaining upper and lower solutions of a differential equation, the maximum principle of a differential equation must be relied on to establish the monotonic relation of the residual function of the differential equation with its equational solutions. To find out the monotonic relation of such problem, the Equations (1) and (2) is reformulated into the following residual equations:

\[
R(x, u) = \epsilon^2 u'' - f(x)u' - g(x)u - N(x, u, u') - h(x)
\]

\[
R(x = a, u) = \alpha - u(a)
\]

NOMENCLATURE

| Symbol  | Description                                                                 |
|---------|-----------------------------------------------------------------------------|
| \(E_p(x)\) | maximum possible error of mean approximate solutions, \((\bar{u} - \bar{u})/2\) |
| \(E_r(x)\) | actual error of mean approximate solutions, \(\pi - \text{analytical solution}\) |
| \(h\) | grid point interval, \(x_i - x_{i-1}\)                                    |
| \(N\) | number of grid points                                                      |
| \(R(x)\) | residual of a differential equation                                          |
| \(R_i\) | residual correction value on a calculation grid point                       |
| \(u\) | function                                                                    |
| ^m | iteration times for residual correction                                     |
| ^{-} | mean approximate solutions, \((\bar{u} + \bar{u})/2\)                       |
| ^{\sim} | approximate solutions                                                      |
| ^{\wedge} | initially assumed function or value obtained from the previous iteration    |
| \(\cap, \cup\) | upper and lower approximate solutions                                       |
| ^i | serial number of calculation grid points                                     |

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Where function R is called the residual function of the differential equation in \( x \in (a, b) \) or boundary, or abbreviated as residual. Given that function \( u(x) \) is the exact solution that makes the residuals of Equations (3)-(5) all satisfy zero, and further assume that approximate functions \( \tilde{u}(x) \) and \( \bar{u}(x) \) have definitions in \( x \in (a, b) \) and continuous derivatives till second order. Based on the maximum principle of a second-order monotonic differential equation (Protten [7] and Wang and Hu [13]), if

\[
\frac{\partial R}{\partial u} = -g(x) = \frac{\partial N(x,u,u')}{\partial u} \leq 0
\]  
(6)

then when the following conditions are satisfied

\[
R(x, \tilde{u}) \geq R(x,u) = 0 \geq R(x,\bar{u}), \ x \in (a,b)
\]  
(7)

\[
R(x = a, \tilde{u}) \geq R(x = a,u) = 0 \geq R(x = a,\bar{u})
\]  
(8)

\[
R(x = b, \tilde{u}) \geq R(x = b,u) = 0 \geq R_g(x = b,\bar{u})
\]  
(9)

The inequality given below will also be established accordingly, namely:

\[
\tilde{u}(x) \leq u(x) \leq \bar{u}(x)
\]  
(10)

In Inequation (10), the approximate solutions \( \tilde{u}(x) \) and \( \bar{u}(x) \) are called the lower and upper solutions of the exact solution \( u(x) \) respectively. And a differential equation that possesses such relation is said to be of monotonicity.

Actually when solving a singularly perturbed non-linear problem, Inequation (6) is not always be satisfied. Therefore, this paper presents a “monotone iterative” concept to reinforce the monotonicity of a differential equation, allowing the maximum principle of differential equations to be applied to a wider range of singularly perturbed non-linear problems. For this purpose, a monotonicity correction parameter \( \lambda \) is first added into Equation (1) to replace the original equation with the following equation:

\[
e^\lambda u' = f(x,u') + g(x) - \lambda \tilde{u}' - h(x)
\]  
(11)

then after putting Equation (12) into Equation (11) and making rearrangements, the following equation is generated:

\[
R(x,\tilde{u}) = e^\lambda \tilde{u}' - f(x)\tilde{u}' - g(x)\tilde{u}' - \tilde{N}(x,\tilde{u},\tilde{u}') - h(x)
\]  
(13)

\[
e^\lambda (\bar{u} - \tilde{u}) = e^\lambda \bar{u}' - f(x)\bar{u}' - g(x)\bar{u}' - \tilde{N}(x,\bar{u},\bar{u}') - h(x)
\]  
(14)

then based on Inequation (6), when

\[
g(x) + \lambda^2 \geq 0
\]  
(15)

Equation (14) satisfies Inequation (6), indicating that Equation (13) which is rewritten from Equation (3) using the monotone iterative technique has ensured the possible existence of its monotonicity with addition of \( \lambda^2 \). It should be worth noticing that, when Equation (13) is used to obtain the lower solution of the exact solution, besides \( R(x, \tilde{u}) \) must be greater than or equal to zero, the both sums of the last two terms in Equation (14) must be greater than or equal to zero in \( x \in (a,b) \), so as to ensure that the sum of all terms in the right of equal sign in Equation (14) is positive. The reason for this condition is that: when the sum of all terms in the right of equal sign in Equation (14) is greater than or equal to zero, this implies that \( \lambda \tilde{u}(x) \) is smaller than or equal to the exact solution “zero”, yet Expression (12) indicates that only a negative \( \lambda \tilde{u}(x) \) can make the obtained approximate solution \( \tilde{u}(x) \) be smaller than or equal to the exact solution \( u(x) \). And similarly, this is also applicable in obtaining the upper solution. So following summing-up of all constraint conditions, besides the monotonicity that is required to satisfy Expression (6) when calculating the solutions, the following shall also be satisfied in the process of obtaining the lower solution:

\[
R(x,\bar{u}) = e^\lambda \bar{u}' - f(x)\bar{u}' - g(x)\bar{u}' - \bar{N}(x,\bar{u},\bar{u}') - h(x) \geq 0
\]  
(16)

\[
\lambda^2 \geq 0, \ x \in (a,b)
\]  
(17)

\[
R(x = a) = \alpha - u(a) \geq 0
\]  
(18)

\[
R(x = b) = \beta - u(b) \geq 0
\]  
(19)


\[
\lambda \tilde{u}(x) = \text{Max}(\tilde{u}(x)) \text{ may be found and } \tilde{u}(x) \leq u(x) \text{ may be ensured. Similarly, in order to obtain the upper solution, the following}
\]  

conditions, besides Expression (6), shall also be satisfied

\[ R(x, \tilde{u}) = e^x \tilde{u}' - f(x)\tilde{u}' - g(x)\tilde{u}, \quad x \in (a, b) \]  \hspace{1cm} (20)

\[- \tilde{N}(x, \tilde{u}, \tilde{u}'') - h(x) - \lambda^2 (\tilde{u}' - \bar{u}) \leq 0 \]

\[ 2\lambda^2 (u - \tilde{u}) + (\tilde{N} - N) \leq 0, \quad x \in (a, b) \]  \hspace{1cm} (21)

\[ R_b(x = a) = u - a(u(a) \leq 0 \]

\[ R_b(x = b) = \beta - u(b) \leq 0 \]  \hspace{1cm} (22)

\[ R_{m}^{m+1} = R_{m}^{m} - \text{Max}(R(x)), \quad x_{i-1} \leq x \leq x_{i+1} \]  \hspace{1cm} (25)

\[ R_{m}^{m+1} = R_{m}^{m} - \text{Min}(R(x)), \quad x_{i-1} \leq x \leq x_{i+1} \]  \hspace{1cm} (26)

Similarly, an approximate solution \( \tilde{u}(x) = \text{Min}(\tilde{u}(x)) \) which is the optimal upper solution may also be obtained, and \( \tilde{u}(x) \geq u(x) \) may be ensured.

2. Mathematical Backgrounds

2.1 Residual Correction Method

Though the gain of the upper and lower approximate solutions is useful in analyzing accuracy or credibility of a solution, given the considerable complexity and difficulty often in deriving the optimal solutions of mathematical programming problems under such constraints, such theory has thus been under theoretical study in the long run but has not been applied practically in solving complex problems. To the author’s knowledge, there were only attempts made by Lin and Chen [8] and Chang and Lee [9] recently to obtain solutions of mathematical programming problems under such constraint conditions, such theory has thus been under theoretical study in the long run but has not been applied practically in solving complex problems. To the author’s knowledge, there were only attempts made by Lin and Chen [8] and Chang and Lee [9] recently to.
These formulae are recurrence relations, and can be expressed in matrix form as

\[
\begin{bmatrix}
 b_0 & c_0 & \cdots & \cdots & m_0 \\
 a_1 & b_1 & c_1 & \cdots & m_1 \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 \cdots & \cdots & \cdots & a_{N-1} & b_{N-1} & c_{N-1} & m_{N-1} \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_N & b_N & c_N & m_N \\
\end{bmatrix}
\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_i \\ \vdots \\ x_{N-1} \end{bmatrix}
= \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_i \\ \vdots \\ d_{N-1} \end{bmatrix}
\]

(34)

where B0, C0, D0, AN, BN and DN can be acquired based on boundary conditions. Since Formula (34) consists of N+1 equations and takes the form of a tridiagonal matrix, then Thomas Algorithm can be used to calculate the second-order differential value of the differential equation swiftly. This paper seeks to use Formula (28) to obtain the second derivative \( M_i^{n+1} \) of a second partial differential equation; then trace back to extract the first differential derivative \( u_i^{n+1} \) and the functional value \( u_i \) using the basic relation of cubic spline function. In addition, the value of the function at any point can be expressed as follow:

\[
u_i(x) = M_i \frac{(x_j - x)^3}{6h_i} + M_i \frac{(x - x_{i-1})^3}{6h_i} + u_{i-1} \frac{M_{i-1}b_i^2}{6h_i} s_i - x \]

\[
u_i(x) = M_i \frac{(x - x_{i-1})^3}{6h_i}
\]

\[
u_i = \begin{cases} u_{i-1} \frac{M_{i-1}b_i^2}{6h_i} s_i - x \quad & x \in [x_{i-1}, x_i] \\ u_i \frac{M_i b_i^2}{6h_i} x - x_{i-1} \quad & \end{cases}
\]

(35)

3. Result and Discussion

In order to verify the correctness of the method proposed in this paper, the following four examples of singularly perturbed non-linear problems are given for validation purpose.

Example 1. Kadalbajoo and Patidar [3]

\[ \varepsilon^2 u'' + u = 1, 0 < x < 1 \]

(46)
\[ u(0) = 1, \quad u(1) = 1 \quad (47) \]

In order to derive the upper and lower approximate solutions, we adopt the technique proposed in the paper to rewrite Equation (46) into a monotone iteration equation as follows:

\[
R(x, \tilde{u}) = \varepsilon^2 \tilde{u}'''' + \tilde{u}'' - \lambda^2 (\tilde{u} - \hat{u}) = \varepsilon^2 \tilde{u}'''' + 2\tilde{u}' - \lambda^2 (u - \hat{u}) \quad (48)
\]

As the above-mentioned equation satisfies Formula (6) when \( \lambda = 0 \), then when the following inequations are satisfied

\[
R(x, \tilde{u}) = \varepsilon^2 \tilde{u}'''' + \tilde{u}'' \
R(x = 0, \tilde{u}) = 1 - \tilde{u}(0) \geq 0
R(x = 1, \tilde{u}) = 1 - \tilde{u}(1) \geq 0
\]

an optimal lower approximate solution \( \tilde{u}(x) = \text{Max}(\tilde{u}(x)) \) will be found and vice versa, when

\[
R(x, \tilde{u}) = \varepsilon^2 \tilde{u}'''' + \tilde{u}'' \
R(x = 0, \tilde{u}) = 1 - \tilde{u}(0) \leq 0
R(x = 1, \tilde{u}) = 1 - \tilde{u}(1) \leq 0
\]

an optimal upper approximate solution \( \hat{u}(x) = \text{Min}(\hat{u}(x)) \) can also be obtained. As described previously on cubic spline, we discretize Expressions (49)-(54) and use residual correction method to obtain solutions and residual values. The distribution of these solutions and the residual values before and after correction is shown as in Table 1 and Fig. 1. As Fig. 1 indicates, for a differential equation without residual correction, its residual value only satisfies the distribution of zero at grid points (i.e. \( R_i = 0 \)), but it is not ensured that the residual values in subintervals of all grid points are greater/smaller than or equal to zero. Thus, as shown in Fig. 1, because the residual value is smaller than or equal to zero near the point \( x=0.5 \) but is greater than or equal to zero at both sides of the point \( x=0.5 \), the relationship between the approximate solutions obtained without residual correction and the exact solutions cannot be determined at this point. Additionally, regardless of the original residual value distribution, it can be seen that following residual correction as described in this text, the residual value is corrected to satisfy the residual value distribution required for Formula (49) in which the residual value is greater than or equal to zero or for Formula (52) where the residual value is smaller than or equal to zero. Therefore, the relation between the obtained approximate solutions and the exact solutions is as shown in Table 1 where the obtained upper approximate solution \( \tilde{u}(x) \) is always greater than the obtained lower solution \( \hat{u}(x) \), and the both solutions tend increasingly to both sides of the exact solution as numbers of grid points increase. That is to say:

\[
u(1/2) = 1 - x + 2\varepsilon^2 \left[ \log 1 + e^{(2x - 1)/\varepsilon^2} \right] - \log \left( 1 + e^{-1/\varepsilon^2} \right) = 0.5173286795139999
\]

This suggests that the residual correction method presented in this paper can be used to correctly obtain the upper and lower approximate solutions. Secondly, the upper and lower approximate solutions obtained with such method can it easy to analyze the maximum error of approximate solutions. In other words, if the final approximate solution \( \pi(x) \) is a mean value between the upper solution and lower solution, it can still be convinced that the maximum error of the approximate solutions \( \pi(x) \) at any point must be smaller than the maximum possible error, even though the exact solution is not identified

\[
E_p(x) = \frac{\hat{u}(x) - \tilde{u}(x)}{2}, \quad 0 < x < 1
\]

As Table 1 indicates, the maximum possible error gained in this paper is roughly the same as the actual maximum error obtained in Kadalbajoo and Patidar [3]. However, it is worth noting that the actual maximum error obtained in Kadalbajoo and Patidar [3] is a value of difference between the numerical solution and the exact solution, while the maximum possible error \( E.p(x) \) mentioned in this paper is a maximum possible error of mean approximate solutions under conditions of unidentified exact solution. If the exact solution is known, the maximum actual error \( E.p(x) \) resulting in this paper will be outlined as in the last column of Table 1, in which one can see that the error magnitude is far smaller than the maximum possible error obtained in this paper. This is still the case for the fewest grid points (e.g. \( N=16 \) or 32), owing to residual value correction which leads to the residual values' symmetrical distribution on both sides of zero, as shown in Fig. 1, thus the obtained upper and lower approximate solutions are characterized by basic symmetry on both sides of the exact solution. So the mean approximate solution derived from the upper and lower average values will be in extreme proximity to the exact solution, allowing the method proposed in this paper to have the characteristics of obtaining optimal approximate solutions with few grid points.

Fig. 2 shows the numerical solutions of this example, while Table 2 and Fig. 3 indicates the error range analysis of approximate solutions of this example as the parameter \( e^2 \) increases when the number of grid points is 256. As shown in Table 2 and Fig. 3, the maximum possible error (i.e. \( \hat{u} - \tilde{u} \)) and actual maximum error increases with the parameter \( e^2 \). Because there is an inflexion point
at \( x=0.5 \), as demonstrated in Fig. 2, the error increase at this point is particularly apparent, as shown in Fig. 3. But in any way, the actual maximum error is still far smaller than the maximum possible error.

Thus ensuring that the term \( (e^u - e^{\hat{u}}) \) in Equation (59) is greater than or equal to zero, or smaller than or equal to zero. So in this example, \( \lambda \) is not necessary to secure the monotonicity. Accordingly, when the following formula is satisfied

\[
R(x, \hat{u}) = \epsilon^2 \left( \frac{d^2}{dx^2} + 2 \frac{d}{dx} \right) u - \lambda^2 (u - \hat{u}) \\
= \epsilon^2 \left( \frac{d^2}{dx^2} + 2 \frac{d}{dx} \right) u - \lambda^2 (u - \hat{u}) \geq 0, \quad 0 < x < 1
\]  

(62)

an optimal lower approximate solution \( \hat{u}(x) = Max(\hat{u}(x)) \) is available. Reversely when

\[
R(x, \hat{u}) = \epsilon^2 \left( \frac{d^2}{dx^2} + 2 \frac{d}{dx} \right) u - \lambda^2 (u - \hat{u}) \leq 0, \quad 0 < x < 1
\]  

(63)

an optimal upper approximate solution \( \hat{u}(x) = Min(\hat{u}(x)) \) can also be secured.

Example 3. Kadalbajoo and Patidar [3]

\[
\epsilon^2 \left( \frac{d^2}{dx^2} + u \right) + u^2 = 0, \quad 0 < x < 1
\]  

(64)

\[
u(0) = 1, \quad u(1) = 1
\]  

(65)

its residual equation after monotone iteration is

\[
R(x, \hat{u}) = \epsilon^2 \left( \frac{d^2}{dx^2} + (2+1)\hat{u} + \hat{u} - \lambda^2 (u - \hat{u}) \right) \\
= \epsilon^2 \left( \frac{d^2}{dx^2} + (2+1)\hat{u} - \lambda^2 (u - \hat{u}) \right) \geq 0
\]  

(66)

To secure the equational monotonicity, the value of \( \lambda \) can be selected, on basis of observations of approximate solutions, among values big enough to ensure \( \lambda^2 + u \geq 0 \).

Example 4. Chang and Howes [2] and Kadalbajoo and Patidar [3]

\[
\epsilon^2 \left( \frac{d^2}{dx^2} + u \right) + u^2 = 0, \quad 0 < x < 1
\]  

(67)

\[
u(0) = 2, \quad u(1) = 1
\]  

(68)

its residual equation after monotone iteration is

\[
R(x, \hat{u}) = \epsilon^2 \left( \frac{d^2}{dx^2} + \hat{u} + \hat{u} - \lambda^2 (u - \hat{u}) \right) \\
= \epsilon^2 \left( \frac{d^2}{dx^2} + \hat{u} - \lambda^2 (u - \hat{u}) \right) \geq 0
\]  

(69)

Similarly, to enable the formula to be of monotonicity, the value of \( \lambda \) can be obtained, on basis of observations of approximate solutions, from values that are big enough to ensure \( \lambda^2 - \hat{u} \geq 0 \).

However in this example, the first-order differential value of its solutions is always smaller than or equal to zero, as shown in Fig. 2. So we can directly let the value of \( \lambda^2 \) be zero.

Fig. 4-6 and Table 2 shows the error analysis of the approximate
solutions extracted in Examples 2-4 varies with the parameter $\varepsilon^2$.

In Fig. 4-6, the value of difference between the upper solution and lower solutions is always positive, which indicates that the monotonic iteration method proposed in this paper can be applied to strengthen the monotonicity of a differential equation, thus ensuring the obtained upper and lower approximate solutions are greater or smaller than the exact solution in all intervals of $x$. Secondly, although the maximum possible errors of approximate solutions increase as $\varepsilon^2$ decreases, the method mentioned in this paper still can be effective in defining the maximum possible errors of solutions under conditions of unknown exact solutions. Besides, if based on the double mesh principle (Dollan et al. [12]), the approximate solutions are considered as the exact solutions, the exact maximum error obtained in this paper is far smaller than the estimated maximum possible error, as displayed in Table 2. This indicates that the obtained upper and lower approximate solutions can not only define the error range of approximate solutions, but also the mean approximate solution derived from the upper and lower approximate solutions is characterized by more accuracy.

Fig. 2 Numerical solutions obtained in Examples 1-4

Fig. 3 Difference between upper and lower approximate solutions with different values of $\varepsilon^2$ ($N=256$) in Example 1

Fig. 4 Difference between upper and lower approximate solutions with different values of $\varepsilon^2$ ($N=256$) in Example 2

Fig. 5 Difference between upper and lower approximate solutions with different values of $\varepsilon^2$ ($N=256$) in Example 3

Fig. 6 Difference between upper and lower approximate solutions with different values of $\varepsilon^2$ ($N=256$) in Example 4

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Table 1. Approximate solutions obtained in Example 1 at x=0.5 when $\varepsilon^2=1/40$ and comparison of its maximum possible errors with exact maximum errors in x-axis.

| $N$ | lower solutions $\tilde{u}$ | average solutions $\bar{u} = (\tilde{u} + \tilde{u})/2$ | upper solutions $\tilde{u}$ | $Max. E_p(x)$ | $Max. E_r^a(x)$ | Max. error [3]$b$ |
|-----|-----------------------------|----------------------------------|-----------------------------|--------------|--------------|----------------|
| 16  | 0.5051643                   | 0.51529852                       | 0.5283724                   | 1.1E-02      | -5.6E-04     | ---            |
| 32  | 0.5147307                   | 0.51753660                       | 0.5204921                   | 2.9E-03      | 2.8E-04      | ---            |
| 64  | 0.5163818                   | 0.51747349                       | 0.5181557                   | 9.0E-04      | -6.0E-05     | 1.4E-03        |
| 128 | 0.5171602                   | 0.51737515                       | 0.5175010                   | 1.7E-04      | 2.0E-06      | 3.6E-04        |
| 256 | 0.5172893                   | 0.51734243                       | 0.5173758                   | 4.3E-05      | 3.9E-06      | 9.0E-05        |
| 512 | 0.5173193                   | 0.51733382                       | 0.5173398                   | 1.0E-05      | 9.0E-07      | 2.3E-05        |
| 1024| 0.5173201                   | 0.51732517                       | 0.5173301                   | 5.0E-06      | -3.5E-06     | 5.6E-06        |

$^a$ $E_r^a(x) = \bar{u}(x) - \text{analytical solution}$

$^b$ numerical solution – analytical solution

Table 2. Analysis and comparison of maximum possible errors and exact maximum errors of approximate solutions obtained in Examples 1-4 at different values of $\varepsilon^2$.

| Examples (grind number) | $\varepsilon^2$ | Present | Max. error [3]$b$ |
|-------------------------|-----------------|---------|------------------|
|                         | $Max. E_p(x)$   | $Max. E_r^a(x)$ |
| Ex. 1 ($N=256$)         | 1/10            | 1.3E-05 | 2.5E-06           | 2.3E-05        |
|                         | 1/20            | 2.1E-05 | 3.3E-06           | 4.5E-05        |
|                         | 1/40            | 4.3E-05 | 3.9E-06           | 9.0E-05        |
|                         | 1/80            | 8.5E-05 | 2.9E-06           | 1.8E-04        |
|                         | 1/160           | 2.2E-04 | 8.6E-05           | 3.5E-04        |
|                         | 1/320           | 3.6E-04 | 3.3E-05           | 6.2E-04        |
|                         | 1/640           | 7.4E-04 | 2.6E-06           | 9.5E-04        |
| Ex. 2 ($N=256$)         | 1/10            | 9.9E-05 | 2.8E-05           | 2.1E-04        |
|                         | 1/20            | 3.9E-04 | 1.1E-05           | 9.7E-04        |
|                         | 1/40            | 1.6E-04 | 4.5E-05           | 4.2E-03        |
|                         | 1/80            | 7.1E-03 | 1.4E-04           | 1.8E-02        |
|                         | 1/160           | 4.0E-02 | 6.6E-04           | 8.7E-02        |
| Ex. 3 ($N=256$)         | 1/10            | 7.4E-05 | 1.8E-05           | 3.4E-04        |
|                         | 1/20            | 3.0E-04 | 7.5E-05           | 1.8E-03        |
|                         | 1/40            | 1.2E-03 | 3.0E-04           | 8.7E-03        |
|                         | 1/80            | 5.4E-03 | 9.2E-04           | 4.1E-02        |
| Ex. 4 ($N=16$)          | 1/2             | 2.4E-05 | -7.2E-06          | ---             |
|                         | 1/4             | 3.6E-05 | 2.2E-06           | 2.7E-03        |
|                         | 1/8             | 2.6E-04 | 4.2E-05           | 1.8E-02        |
|                         | 1/16            | 1.0E-03 | 1.5E-05           | ---             |

$^a$ $E_r^a(x) = \bar{u}(x) - \text{analytical solution}$

$^b$ numerical solution – analytical solution
4. Conclusions

As verified by the four examples in this article, the monotone iteration technique and residual correction method proposed in this paper have the following characteristics:

a. Capable of strengthening the monotonicity of a differential equation;
b. Fewest times required for residual correction and relatively rapid in solving inequational constraint mathematical programming problems;
c. Effective to obtain the upper and lower approximate solutions of singularly perturbed problems correctly;
d. Easy in determining the maximum possible error range when the exact solution is unknown;
e. Symmetric of difference between the upper/lower approximate solutions and the exact solutions, thus leading to improved accuracy of the mean approximate solutions.

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