ENVELOPING ALGEBRAS OF RESTRICTED LIE
SUPERALGEBRAS SATISFYING NON-MATRIX
POLYNOMIAL IDENTITIES

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Abstract. Let \( L \) be a restricted Lie superalgebra with its
enveloping algebra \( u(L) \) over a field \( F \) of characteristic \( p > 2 \). A
polynomial identity is called non-matrix if it is not satisfied by
the algebra of \( 2 \times 2 \) matrices over \( F \). We characterize \( L \) when
\( u(L) \) satisfies a non-matrix polynomial identity. In particular, we
characterize \( L \) when \( u(L) \) is Lie solvable, Lie nilpotent, or Lie
super-nilpotent.

1. Introduction

A variety of associative algebras over a field \( F \) is called non-matrix
if it does not contain \( M_2(F) \), the algebra of \( 2 \times 2 \) matrices over \( F \). A
polynomial identity (PI) is called non-matrix if \( M_2(F) \) does not satisfy
this identity. Latyshev in his attempt to solve the Specht problem
proved that any non-matrix variety generated by a finitely generated
algebra over a field of characteristic zero is finitely based \[L80]\. The
complete solution of the Specht problem in the case of characteristic
zero is given by Kemer \[K91, K81]\.

Although several counterexamples are found for the Specht problem
in the positive characteristic \[AK]\, the development in this area has
lead to some interesting results. Kemer has investigated the relation be-
tween PI-algebras and nil algebras. Amitsur \[Am]\ had already proved
that the Jacobson radical of a relatively-free algebra of countable rank
is nil. Restricting to non-matrix varieties, Kemer \[K96]\ proved that
the Jacobson radical of a relatively-free algebra of a non-matrix variety
over a field of positive characteristic is nil of bounded index. Recently
these varieties have been further studied in \[MPR]\.

Enveloping algebras satisfying polynomial identities were first con-
sidered by Latyshev \[L63]\ by proving that the universal enveloping
algebra of a Lie algebra \( L \) over a field of characteristic zero satisfies a

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PI if and only if $L$ is abelian. Latyshev’s result was extended to positive characteristic by Bahturin [B74]. Passman [P] and Petrogradsky [P91] considered the analogous problem for restricted Lie algebras and their envelops.

Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra with the bracket $(, )$. We denote the enveloping algebra of $L$ by $u(L)$. All algebras in this paper are over a field $\mathbb{F}$ of characteristic $p > 2$ unless otherwise stated. In case $p = 3$ we add the axiom $((y, y), y) = 0$, for every $y \in L_1$. This identity is necessary to embed $L$ in $u(L)$. Restricted Lie superalgebras whose enveloping algebras satisfy a polynomial identity have been characterized by Petrogradsky [P92]. The purpose of this paper is to characterize restricted Lie superalgebras whose enveloping algebras satisfy a non-matrix PI. Our results unify the results of Riley, Shalev, and Wilson in [RS, RW] where they characterize restricted Lie algebras whose enveloping algebras satisfy a non-matrix PI. Our first main result is as follows.

**Theorem 1.1.** The following statements are equivalent:

1. $u(L)$ satisfies a non-matrix PI.
2. The commutator ideal of $u(L)$ is nil of bounded index.
3. $u(L)$ satisfies a PI, $(L_0, L_0)$ is $p$-nilpotent, and there exists an $L_0$-module $M$ of codimension at most 1 in $L_1$ such that $(M, L_1)$ is $p$-nilpotent and $(L_1, L_0) \subseteq M$.

We prove Theorem 1.1 in Section 3. The classification of finite dimensional Clifford algebras is used in the proof. In the course of proving Theorem 1.1 it was of interest for us to know whether a variant of Cayley-Hamilton Theorem holds for matrices over the Grassmann algebra. Note that the Grassmann algebra satisfies the identity $[x, y, z] = 0$. So in general we ask the following:

**Problem 1.2.** Let $G$ be a nilpotent (solvable) Lie algebra over a field of positive characteristic. Does a variant of Cayley-Hamilton Theorem hold for $M_n(G)$?

Szigeti [Sz] proved that if $H$ is a nilpotent Lie ring of any characteristic and $T \in M_n(H)$, then there exists a polynomial $f(t) \in H[t]$ such that the left substitution of $T$ in $f(t)$ is zero. The degree of $f$ is $n^m$ where $m$ is the nilpotency class of $H$. However the leading coefficient of $f$ is factorial in $n$, so this result is essentially not useful in positive characteristic.

Every $\mathbb{Z}_2$-graded associative algebra $A = A_0 \oplus A_1$ over $\mathbb{F}$ can be given the structure of a restricted Lie superalgebra via $(x, y) = xy - (-1)^{ij}yx$ for every $x \in A_i$ and $y \in A_j$. We denote the usual Lie bracket of $A$
by \([u,v] = uv - vu\). We emphasize that the terms Lie nilpotent or Lie solvable are used with respect to the usual Lie bracket \([,]\) whereas Lie super-nilpotent refers to the super-bracket \((,\)).

The variety of Lie solvable algebras includes Lie nilpotent and Lie super-nilpotent algebras. We characterize \(L\) when \(u(L)\) is Lie solvable in the following theorem.

**Theorem 1.3.** Let \(L\) be a restricted Lie superalgebra. Then \(u(L)\) is Lie solvable if and only if \((L,L)\) is finite-dimensional, \((L_0, L_0)\) is \(p\)-nilpotent, and there exists a subspace \(M \subseteq L_1\) of codimension at most 1 such that \((M, L_1)\) is \(p\)-nilpotent and \((L_1, L_0) \subseteq M\).

Proof of Theorem 1.3 is given in Section 4. Furthermore, restricted Lie superalgebras whose enveloping algebras are Lie super-nilpotent or Lie nilpotent are characterized in Theorem 4.8 and Theorem 5.5, respectively. Stewart [St] proved that if \(H\) is a nilpotent ideal of a Lie algebra \(L\) such that \(L/H'\) is nilpotent then \(L\) is nilpotent. In contrast we prove that if \(I\) a nilpotent two-sided ideal of an associative algebra \(R\) such that \(R/I^2\) is Lie nilpotent then \(R\) is Lie nilpotent, see Proposition 5.4.

### 2. Preliminaries

Let \(A = A_0 \oplus A_1\) be a vector space decomposition of a non-associative algebra. We say that this is a \(Z_2\)-grading of \(A\) if \(A_i A_j \subseteq A_{i+j}\), for every \(i, j \in Z_2\) with the understanding that the addition \(i + j\) is mod 2. The components \(A_0\) and \(A_1\) are called even and odd parts of \(A\), respectively. Note that \(A_0\) is a subalgebra of \(A\). One can associate a Lie superbracket to \(A\) by defining \((x, y) = xy - (-1)^{ij}yx\) for every \(x \in A_i\) and \(y \in A_j\).

If \(A\) is associative, then for any \(x \in A_i, y \in A_j\) and \(z \in A\) the following identities hold:

1. \((x, y) = -(−1)^{ij}(y, x),\)
2. \((x, (y, z)) = ((x, y), z) + (−1)^{ij}(y, (x, z)).\)

The above identities are the defining relations of Lie superalgebras. Furthermore, \(A\) can be viewed as a Lie algebra by the usual Lie bracket \([u, v] = uv - vu\). Let \(B\) and \(C\) be subspaces of \(A\). We denote by \([B, C]\) the subspace spanned by all commutators \([b, c]\), where \(b \in B\) and \(c \in C\). Then \([B, n C] = [[B, n−1 C], C],\) for every \(n \geq 2\). If \(B\) is a nilpotent subalgebra of \(A\), the nilpotence index of \(B\) is the least integer \(k\) such that \(B^{k+1} = 0\). The lower Lie central series of \(A\) is defined by setting \(\gamma_1(A) = A\) and \(\gamma_n(A) = [\gamma_{n-1}(A), A]\), for every \(n \geq 2\). The Lie derived series of \(A\) is defined by setting \(\delta_0(A) = A\).
and $\delta_n(A) = [\delta_{n-1}(A), \delta_{n-1}(A)]$, for every $n \geq 1$. Recall that $A$ is called Lie nilpotent if $\gamma_n(A) = 0$, for some $n$, and $A$ is called Lie solvable if $\delta_m(A) = 0$, for some $m$. The nilpotence class of $A$ is the least integer $c$ such that $\gamma_{c+1}(A) = 0$. The derived length of $A$ is the least integer $d$ such that $\delta_d(A) = 0$. Long commutators are left tapped, that is $[x, y, x] = [[x, y], z]$, and $[x, y]$ denotes the commutator $[x, y, \ldots, y]$, where $y$ occurs $k$ times. Analogous definitions hold for the super-bracket of $A$. We denote by $\gamma_n^s(A)$ the $n$-th term of the Lie super-central series of $A$.

If $L$ is a Lie superalgebra we denote the bracket of $L$ by $(, )$. The adjoint representation of $L$ is given by $\text{ad} : L \to L$, $\text{ad}(x(y)) = (y, x)$, for all $x, y \in L$. The notion of restricted Lie superalgebras can be easily formulated as follows:

**Definition 2.1.** A Lie superalgebra $L = L_0 \oplus L_1$ is called restricted, if there is a $p$th power map $L_0 \to L_0$, denoted as $[p]$, satisfying

(a) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$, for all $x \in L_0$ and $\alpha \in \mathbb{F}$,

(b) $(y, x^{[p]}) = (y, x)$, for all $x \in L_0$ and $y \in L$,

(c) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, for all $x, y \in L_0$ where $s_i$ is the coefficient of $\lambda^{i-1}$ in $(\text{ad} (\lambda x + y))^{p-1}(x)$.

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra. For example, every $\mathbb{Z}_2$-graded associative algebra inherits a restricted Lie superalgebra structure.

Let $L$ be a restricted Lie superalgebra. We denote the enveloping algebra of $L$ by $u(L)$. The augmentation ideal $\omega(L)$ is the ideal of $u(L)$ generated by $L$. The analogue of the PBW Theorem is as follows. We refer to [B92] for basic background.

**Theorem 2.2.** Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra and let $\mathcal{B}$ be a totally ordered basis for $L$ consisting of $\mathbb{Z}_2$-homogeneous elements. Then $u(L)$ has a basis consisting of PBW monomials, that is, monomials of the form $x_1^{a_1} \ldots x_s^{a_s}$ where $x_1 < \cdots < x_s$ in $\mathcal{B}$, $0 \leq a_i < p$ whenever $x_i \in L_0$, and $0 \leq a_i \leq 1$ whenever $x_i \in L_1$.

The fact that $u(L)$ is $\mathbb{Z}_2$-graded can be explained as follows. For every Lie superalgebra $L = L_0 \oplus L_1$ we can associate a linear map $\sigma : L \to L$ such that $\sigma^2 = 1$. Indeed, $\sigma(x) = x$, for all $x \in L_0$ and $\sigma(y) = -y$, for all $y \in L_1$. The converse of this is also true, that is every linear map of $L$ of order 2 induces a $\mathbb{Z}_2$-grading on $L$. Now suppose that $L = L_0 \oplus L_1$ is a Lie superalgebra and let $\sigma$ be the corresponding
linear map. Then $\sigma$ extends to an automorphism of $A = u(L)$ of order 2. Now, we can write $A = A_0 \oplus A_1$, where

$$A_0 = \langle u \in A \mid \sigma(u) = u \rangle_Z, \quad A_1 = \langle u \in A \mid \sigma(u) = -u \rangle_Z.$$ 

Since $\sigma$ is an automorphism of $A$, the parity of a PBW monomial $x_1^{a_1} \ldots x_s^{a_s}$ is equal to the parity of the number of odd $x_i$ with exponent 1.

Since $L$ embeds into $u(L)$ we identify $x[p]$ with $x^p$, for every $x \in L_0$. Furthermore, if $x \in L_0$ and $y \in L$ then the bracket $(x, y)$ in $L$ is the same as the bracket $[x, y]$ in $u(L)$. By an ideal of $L$ we always mean a restricted ideal, that is $I$ is an ideal of $L$ if $(I, L) \subseteq I$ and $I_0$ is closed under the $p$-map. Let $H$ be a subalgebra of $L$. We denote by $H'$ the commutator subalgebra of $H$, that is $H' = (H, H)$. For a subset $X \subseteq L$, we denote by $\langle X \rangle_p$ the restricted ideal of $L$ generated by $X$. Also, $\langle X \rangle_Z$ denotes the subspace spanned by $X$. An element $x \in L_0$ is called $p$-nilpotent if there exists some non-negative integer $t$ such that $x^{p^t} = 0$. Also, recall that $X$ is said to be $p$-nil if every element $x \in X$ is $p$-nilpotent and $X$ is $p$-nilpotent if there exists a positive integer $k$ such that $x^{p^k} = 0$, for every $x \in X$.

We shall call $L$ a nilpotent $L_0$-module if $(L, n L) = 0$, for some $n$. Note that Engel’s Theorem holds for Lie superalgebras, see [Sch], for example.

**Theorem 2.3** (Engel’s Theorem). Let $L$ be a finite-dimensional Lie superalgebra such that $ad x$ is nilpotent, for every homogeneous element $x \in L$. Then $L$ is nilpotent.

The proof of the following lemma follows from Engel’s Theorem and the fact that $(ad x)^2 = \frac{1}{2}ad (x, x)$, for every $x \in L_1$.

**Lemma 2.4.** Let $L = L_0 \oplus L_1$ be a finite-dimensional restricted Lie superalgebra. If $L_0$ is $p$-nilpotent then $L$ is nilpotent.

**Lemma 2.5.** Let $L$ be a restricted Lie superalgebra. Then $\omega(L)$ is associative nilpotent if and only if $L$ is finite-dimensional and $L_0$ is $p$-nilpotent.

**Proof.** The if part follows from the PBW Theorem. We prove the converse by induction on $\dim L$. Since $L$ is nilpotent, by Lemma 2.4 there exists a non-zero element $z$ in the center $Z(L)$ of $L$. Since $Z(L)$ is homogeneous we may assume that either $z \in L_0$ or $z \in L_1$. If $z \in L_1$ then $z^2 = (z, z) = 0$. If $z \in L_0$ then we can replace $z$ with its $p$-powers so that $z^{p^t} = 0$. So in either case $z^p = 0$ in $u(L)$. Now consider $H = L/<z>_p$. Then by induction hypothesis $\omega(H)$ is nilpotent.
This means that \( \omega^m(L) \subseteq (z_p)_p u(L) \), for some \( m \). It then follows that \( \omega^{mp}(L) \subseteq (z^p)_p u(L) = 0 \), as required. \( \square \)

We shall need the following result of Petrogradsky.

**Theorem 2.6** ([P92]). Let \( L = L_0 \oplus L_1 \) be a restricted Lie superalgebra. Then \( u(L) \) satisfies a PI if and only if there exist homogeneous restricted ideals \( M \subseteq N \subseteq L \) such that

1. \( \dim L/N \) and \( \dim M \) are both finite.
2. \( N' \subseteq M, M' = 0 \).
3. The restricted Lie subalgebra \( M_0 \) is \( p \)-nilpotent.

The Grassmann (or exterior) algebra \( G \) on a vector space \( V \) is defined by the relation: \( v^2 = 0 \), for all \( v \in V \). This relation implies that \( uv = -vu \), for all \( u, v \in V \). Note that \( G \) satisfies \( [x, y, z] = 0 \). Since the identity \( (x + y)^p = x^p + y^p \) holds in any Lie nilpotent algebra of class at most \( p \), it follows that \( G \) is nil of index \( p \).

Let \( V \) be an \( F \)-vector space and \( Q : V \to F \) a quadratic form, that is \( Q \) satisfies the following conditions:

1. \( Q(\alpha x) = \alpha^2 Q(x), x \in V, \alpha \in F \).
2. \( B(x, y) = Q(x + y) - Q(x) - Q(y) \) is bilinear.

Let \( T(V) = F \oplus V \oplus V \otimes V \oplus \cdots \) be the tensor algebra on \( V \). The associated Clifford algebra to \( V \) and \( Q \) is defined by \( C(V, Q) = T(V)/\langle x \otimes x - Q(x)1 \rangle \). It is known that if \( Q \) is non-degenerate and \( \dim V = n \) is even then \( C(V, Q) \) is a central simple algebra of dimension \( 2^n \), see [J] for example.

### 3. General non-matrix PI

Let \( R \) be an algebra satisfying a non-matrix polynomial identity \( f \). It follows from the structure theory of PI-algebras that \([R, R]R\) is nil. Indeed, by Posner’s theorem, if a prime ring satisfies \( f \), it must be commutative. Since every semiprime ring is a subdirect sum of prime rings, if a semiprime ring satisfies \( f \), it must be commutative. But \( R/N \) is semiprime, where \( N \) is the nil radical of \( R \) (the sum of all nil two-sided ideals). So, \( R/N \) must be commutative. Thus, \([R, R]R\) is nil. However, a stronger result holds using a theorem of Kemer.

**Theorem 3.1** ([K96]). The Jacobson radical of a relatively-free algebra of a non-matrix variety over a field of positive characteristic is nil of bounded index.

**Corollary 3.2.** If \( R \) satisfies a non-matrix PI over a field of positive characteristic, then \([R, R]R\) is nil of bounded index.
Proof. By Theorem 3.1, the radical $J$ of the relatively-free algebra $U$ of the variety defined by $R$ is nil of bounded index. We note that $U/J$ is commutative since it is semisimple satisfying a non-matrix PI. Thus, $[U, U]U \subseteq J$, implying that $[U, U]U$ is nil of bounded index. Thus $[R, R]R$ is nil of bounded index.

**Proposition 3.3.** If $u(L)$ satisfies a non-matrix PI then there exists an $L_0$-module $M$ of codimension at most 1 in $L_1$ such that $(M, L_1)$ is $p$-nilpotent and $(L_1, L_0) \subseteq M$.

Proof. Let $M$ be the set of all $y \in L_1$ such that $(y, y)$ is $p$-nilpotent. By Theorem 2.6, there exists a homogeneous ideal $N$ of $L$ of finite codimension such that $(N_1, N_1)$ is $p$-nilpotent. In particular, $N_1 \subseteq M$. Since $[R, R]R$ is nil of bounded index, by Corollary 3.2, it follows that $(M, L_1)$ is $p$-nilpotent. Note that $(L_1, L_0) \subseteq M$. It is enough to show that the codimension of $M$ in $L_1$ is at most 1. Suppose to the contrary. Let $K = \langle (L_0, L_0) + M \rangle_p$. Without loss of generality we can replace $L$ with $L/M$. So, $(a, a)$ is not $p$-nilpotent, for every $a \in L_1$. Let $a$ and $b$ be linearly independent elements of $L_1$ and set $x = (a, a)$, $y = (b, b)$, and $z = (a, b)$. We replace $L$ with its subalgebra $H = L_0 \oplus \langle a, b \rangle_p$. Note that $(a, b)$ is not $p$-nilpotent. Let $X$ be the subset of $R$ consisting of all $x^i y^j z^k$, where $i, j, k$ are non-negative integers. Note that $X$ is a multiplicative set, $1 \in X$ and $0 \notin X$. Now we consider the localization $R_X$ of $R$ with respect to $X$. Consider the subalgebra $S$ of $R_X$ generated by all $r/s, r, s \in X$. Let $m$ be a maximal ideal of $S$ and set $F = S/m$. Let $V = H_1 \otimes_{\mathbb{F}} F$. We denote by $\bar{r}$ the cost representative of an element $r \in S$. Consider the $F$-bilinear map $\phi : V \times V \to F$ induced by $\phi(u, v) = (u, v)$, for every $u, v \in H_1$. Then the Clifford algebra $C$ over $F$ defined by $\phi$ is the $F$-subalgebra of $R_F = R_X \otimes_{\mathbb{F}} F$ generated by $a, b$. By the classification of Clifford algebras, $C$ is a central simple algebra. We also claim that $R_F$ satisfies a non-matrix PI. Note that $(H_0, H_1) = 0$. We observe that the commutator ideal $[R_F, R_F]R_F$ of $R_F$ is generated by $[a, b] \otimes 1, [a, b]a \otimes 1,$ and $[a, b]b \otimes 1$. Since each of these generators are nil, by Corollary 3.2, it follows that $[R_F, R_F]R_F$ is nilpotent. Thus, $R_F$ is nilpotent-by-commutative which is a non-matrix PI. Thus, $C$ satisfies a non-matrix PI and being central simple, $C$ must be commutative. So, $ab = ba$. But then $2ab = (a, b) \in L_0$ which contradicts the PBW Theorem because $a$ and $b$ are linearly independent. This contradiction implies that $L_1$ is 1-dimensional. Hence, $M$ has codimension at most 1 in $L_1$. □

**Lemma 3.4.** Let $L$ be a restricted Lie superalgebra and $N$ a homogeneous abelian ideal of $L$ of finite codimension. Let $I$ be an ideal of
that is stable under the adjoint action of $L$ on $N$. If $I$ is nil of bounded index then so is $Iu(L)$.

Proof. Let $R = u(L)$. Let $x_1, \ldots, x_n \in L_0$ and $y_1, \ldots, y_m \in L_1$ such that

$$L = N + \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle_F.$$ 

By the PBW Theorem $R$ has a basis consisting of the monomials of the form

$$w x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_m^{\beta_m},$$

where the $w$'s are PBW monomials lying in $u(N)$. Let $D = u(N)$. Note that $R$ is a left $D$-module of finite rank $r = p^m 2^m$. Thus, under the left regular representation of $R$, we can embed $R$ into $M_r(D)$. Since $I$ is stable under the adjoint action of $L$ on $N$, we have $RI = IR = RI'R$. We observe that $RI$ embeds in $M_r(D)$. Therefore, it suffices to show that $M_r(I)$ is nil of bounded index. Note that $u(N)$ is the tensor product of a commutative algebra with the Grassmann algebra. Thus, $u(N)$ satisfies the identity $[x, y, z] = 0$. Hence, $I$ satisfies $[x, y, z] = 0$, too. Recall Levitzki’s Theorem and Shirshov’s Height Theorem stating that every $t$-generated PI algebra which is nil of bounded index is (associative) nilpotent of a bound given as a function of $s, t, and d$, where $d$ is the degree of the polynomial identity, see [Lev] and [Sh]. So, if $S$ is any $r^2$-generated subalgebra of $I$ then there exists a constant $k$ such that $S^k = 0$. Now, let $T \in M_r(I)$ and denote by $S$ the subalgebra of $I$ generated by all entries of $T$. So, $T^i \in M_s(S^i)$, for every $i$. Since $S^k = 0$, we get $T^k = 0$. Since $k$ is independent of $T$, it follows that $M_r(I)$ is nil of bounded index, as required. \[\square\]

Proof of Theorem 1.1. (1) $\iff$ (2): follows from Corollary 3.2
(1) $\Rightarrow$ (3): follows from Proposition 3.3 and Corollary 3.2
(3) $\Rightarrow$ (2): we shall use the fact that the class of nil algebras of bounded index is closed under extensions. Let $L_1 = M + \mathbb{F}z$, where $(M, L_1)$ is $p$-nilpotent and $(L_1, L_0) \subseteq M$. Since $u(L)$ satisfies a PI, by Theorem 2.6, there exists a homogenous restricted ideal $N$ of $L$ such that $N'$ is finite-dimensional and $(N')_0$ is $p$-nilpotent. Note that $(N, N)$ is nilpotent, by Lemma 2.4. Thus, $\omega(N')$ is associative nilpotent, by Lemma 2.5. Hence, $N'u(L)$ is associative nilpotent. Consider the natural map $u(L) \rightarrow u(L/N')$. It suffices to consider $L/N'$. In other words, we may assume that $N$ is abelian. Let $B$ be the restricted ideal of $L$ generated by $(N, L)$. We observe that $I = Bu(N)$ is nil of bounded index. Furthermore, by Lemma 3.4, $Iu(L) = Bu(L)$ is also a nil ideal of $R$ of bounded index. But $Iu(L)$ is the kernel of the homomorphism
Thus, we can replace $L$ with $L/B$. In other words, we can assume that $N$ is central in $L$. It follows that $L'$ is finite-dimensional. Note that $N_1 u(N)$ is nil of index $p$. Since $N_1$ is a central ideal of $L$, $N_1 R$ is nil of bounded index, by Lemma 3.4. Thus, we may assume that $N_1 = 0$. It follows that $L$ is a restricted Lie algebra for which $L'$ finite-dimensional and $p$-nilpotent. Thus, the associative ideal of $R$ generated by $L'$ is associative nilpotent, by Lemma 2.5. So, we may assume that $L' = 0$. But then $R$ is a commutative algebra and the assertion holds.

\section{Lie Solvable}

\begin{proposition}
If $u(L)$ is Lie solvable then $\dim (L, L)$ is finite.
\end{proposition}

The following result is proved by Zalesskii and Smirnov \cite{ZS} and independently by Sharma and Srivastava \cite{SS}.

\begin{theorem}
Let $R$ be a Lie solvable ring of derived length $t \geq 2$. The two-sided ideal $\mathcal{J}$ of $R$ generated by $[[R, R], [R, R], R]$ is associative nilpotent of index bounded by a function of $t$.
\end{theorem}

Throughout this section, we denote $u(L)$ by $R$ and $\mathcal{J}$ is used to denote the associative ideal of $u(L)$ generated by $[[R, R], [R, R], R]$.

The proof of Proposition 4.1 breaks down to several parts. First we prove that $(L_0, L_0)$ is finite-dimensional. In this case, it is enough to assume that $L$ is a restricted Lie algebra. We remark that this assertion is proved in \cite{RS}, however we can offer a new shorter proof. Let us first recall the following result of Neumann originating from Group theory, see \cite{N, B87}.

\begin{theorem}
Let $\phi : U \times V \rightarrow W$ be a bilinear map, where $U, V, W$ are vector spaces over $\mathbb{F}$. Suppose that there exist constants $m, \ell$ such that for each $u \in U$ the codimension of its annihilator in $V$ is bounded by $m$, and for each $v \in V$ the codimension of its annihilator in $U$ is bounded by $\ell$. Then $\dim \langle \phi(U, V) \rangle_{\mathbb{F}} \leq m\ell$.
\end{theorem}

\begin{lemma}
Let $K$ be a restricted Lie algebra. If $u(K)$ is Lie solvable then $K'$ is finite-dimensional.
\end{lemma}

\begin{proof}
Clearly $K$ is solvable.

Step 1. We can assume $\delta_2(K) = 0$. We use induction on the derived length $s$ of $K$. If $K' = 0$, there is nothing to prove. So, assume
that \( s \geq 2 \). Let \( H = K/\langle \delta_{s-1}(K) \rangle_p \). Since \( u(H) \) is Lie solvable, we can assume by induction hypothesis that \( [H, H] \) is finite-dimensional. Since \( [K, K] \) is \( p \)-nilpotent, by Corollary 3.2 it is enough to prove that \( \delta_{s-1}(K) \) is finite-dimensional. Without loss of generality we can replace \( K \) with \( \langle \delta_{s-2}(K) \rangle_p \). So, \( \delta_2(K) = 0 \).

**Step 2.** We can assume \( \gamma_3(K) = 0 \). We prove that \( \gamma_3(K) \) is finite-dimensional. Then we replace \( K \) with \( K/\langle \gamma_3(K) \rangle_p \). We apply Theorem 4.3 to the function \( \varphi : K \times K' \to \gamma_3(K) \) given by \( \varphi(x, y) = [x, y] \), for every \( x \in K \) and \( y \in K' \). It is enough to prove that \( \dim [x, K'] \) and \( \dim [y, K] \) are bounded for every \( x \in K \) and \( y \in K' \). Fix \( x \in K \). For every \( y \in K' \), we write \( y' = [y, x] \). Since \( K' \) is abelian, we have

\[
y_1x, y_2x] + [y_1y_2x, x] = 2y_1'y_2x \in [R, R],
\]

for every \( y_1, y_2 \in K' \). Since \( [yx, x] = y'x \in [R, R] \) and \( \text{char}(\mathbb{F}) \neq 2 \), we have

\[
y_1'y_2x, y_1'x] = y_1'^2y_2x \in \delta_2(R),
\]

Thus,

\[
y_1'^2y_2x, y_2] = -y_1'^2y_2x \in \mathcal{J},
\]

which implies that \( \dim [x, K'] \) is bounded since \( \mathcal{J} \) is associative nilpotent. Now let \( y \in K' \) and set \( \hat{x} = [x, y] \), for every \( x \in K \). We also define \( x_1 \circ x_2 = x_1\hat{x}_2 + \hat{x}_1x_2 \), for every \( x_1, x_2 \in K \). Thus \( x_1 \circ x_2 = [x_1x_2, y] \in [R, R] \) and \( [x_1x_2y, y] = (x_1 \circ x_2)y \in [R, R] \). Since \( K \) is abelian, we have

\[
[x_1 \circ x_2, (x_1 \circ x_2)y] = 2(x_1 \circ x_2)\hat{x}_1\hat{x}_2 \in \delta_2(R).
\]

Hence,

\[
[(x_1 \circ x_2)\hat{x}_1\hat{x}_2, y] = 2\hat{x}_1^2\hat{x}_2^2 \in \mathcal{J},
\]

which implies that \( \dim [y, K] \) is bounded since \( \mathcal{J} \) is associative nilpotent.

**Step 3.** \( K' \) is finite-dimensional. We apply Theorem 4.3 to the function \( \varphi : K \times K \to K' \) given by \( \varphi(x, y) = [x, y] \), for every \( x, y \in K \). Thus, it suffices to show that \( \dim [x, K] \) is bounded for every \( x \in K \). This is similar to Step 1 taking into account that \( \gamma_3(K) = 0 \). Thus, \( K' \) is finite-dimensional, as required. \( \square \)

By Theorem 2.6, there exists a homogenous restricted ideal \( N \) of \( L \) such that \( N' \) is finite-dimensional. In order to prove that \( (L, L) \) is finite-dimensional it suffices to replace \( L \) with \( L/N' \). In the next two lemmas, we assume that \( L \) has an abelian ideal \( N \) of finite codimension.
Lemma 4.5. If \( u(L) \) is Lie solvable then \((L_1, L_1)\) is finite-dimensional.

Proof. We apply Theorem 4.3 to the function \( \varphi : L_1 \times L_1 \to (L_1, L_1) \)
given by \( \varphi(y, z) = (y, z) \), for \( y, z \in L_1 \). Since \( N \) is abelian and \( \dim L/N \)
is finite, it is enough to prove that \( \dim (y, N_1) \) is finite, for every \( y \in L_1 \).
Fix \( y \in L_1 \) and let \( z' = (z, y) \), for every \( z \in L \). Note that \([z_1 z_2, z_3] = z_1 (z_2, z_3) - (z_1, z_3) z_2 \), for every \( z_1, z_2, z_3 \in L_1 \).
Since \( N \) is abelian, we have \([y_1, y_2] = 2y_1 y_2 \in [R, R] \) and \([y_1, y] = y_1' - 2y y_1 \in [R, R] \), for every
\( y_1, y_2 \in N_1 \). Since \( \text{char} (\mathbb{F}) \neq 2 \), we get 
\[
[y_1' - 2y y_1, y_1 y_2] = 2[y_1 y_2, y] y_1 = 2y_1' y_1 y_2 \in \delta_2(R).
\]
Thus, \([y_1 y_2, y, y_1 y_2] = y_1 y_1 y_1 y_2 \in \mathcal{J} \). We deduce that \( \dim (y, N_1) \) is
finite as \( \mathcal{J} \) is nilpotent by Theorem 4.2.

In order to finish the proof of Proposition 4.1, it remains to prove the following:

Lemma 4.6. If \( u(L) \) is Lie solvable then \((L_1, L_0)\) is finite-dimensional.

Proof. Consider the function \( \varphi : L_0 \times L_1 \to (L_0, L_1) \) given by \( \varphi(x, y) = (x, y) \), for every \( x \in L_0 \) and \( y \in L_1 \). By Theorem 4.3 and using the
fact that \( N' = 0 \) and \( \dim L/N \) is finite, it is enough to prove that \( \dim (N_0, y) \) and \( \dim (N_1, x) \) are finite, for every \( x \in L_0 \) and \( y \in L_1 \).
Since \( N \) is abelian, we have
\[
[[x_1, x_3 y], [x_2, y], y] = 2[x_1, y] [x_2, y] [x_3, y] \in \mathcal{J},
\]
\[
[[x, y_1], [x, x y_2], y_3] = 2[x, y_1] [x, y_2] [x, y_3] \in \mathcal{J},
\]
for every \( x_1, x_2, x_3 \in N_0 \) and \( y_1, y_2, y_3 \in N_1 \). Since, by Theorem 4.2
\( \mathcal{J} \) is nilpotent, the assertion follows.

We next show that if \( R \) is Lie super-nilpotent then \( R \) is in fact Lie solvable.

Lemma 4.7. Let \( R = R_0 \oplus R_1 \) be any superalgebra. If \( \gamma^{*}_{c+1}(R) = 0 \)
then \( \delta_{c}(R) \subseteq R_0 \). Consequently, \( R \) is Lie solvable.

Proof. Observe that \( \delta_{1}(R) = [R, R] \subseteq R_0 + (R_1, R_0) \). Hence,
\[
\delta_{2}(R) \subseteq R_0 + (R_1, R_0, R_0),
\]
Continuing this way yields \( \delta_{m}(R) \subseteq R_0 + (R_{1,m} R_0), \) for every \( m \geq 1 \).
So, \( \delta_{c}(R) \subseteq R_0 \).

Theorem 4.8. Let \( L = L_0 \oplus L_1 \) be a restricted Lie superalgebra. Then
\( u(L) \) is Lie super-nilpotent if and only if \( L \) is nilpotent, \( L' \) is finite-dimensional, \((L_0, L_0)\) is \( p \)-nilpotent, and there exists an \( L_0 \)-module \( M \)
of codimension at most 1 in \( L_1 \) such that \((M, L_1)\) is \( p \)-nilpotent and
\((L_1, L_0) \subseteq M \).
Proof. Suppose that $R = u(L)$ is Lie super-nilpotent. Then, by Lemma 4.7, $R$ is Lie solvable. So, the if part follows from Corollary 3.2 and Propositions 3.3 and 4.1. We prove the converse. Let $z \in L_1$ such that $L_1 = M + \mathbb{F}z$ and set $H = L_0 + M$.

Step 1. We may assume $(H, H) = 0$. Since $H$ is nilpotent, there exists a non-zero element $x \in (H, H) \cap Z(H)$, where $Z(H)$ is the center of $H$. Note that $(x, z) \in (H, H) \cap Z(H)$. Since $L$ is nilpotent, we can replace $x$ with $(x, kz)$, for some $k$, so that $x \in Z(L)$. Since $(L, L)$ is graded, we can assume either $x = x_0$ or $x = x_1$. If $x \in L_0$ then either $x \in (L_0, L_0)$ or $x \in (M, L_1)$. In either case, $x$ is $p$-nilpotent. Let $s$ be the least integer such that $x^{p^s+1} = 0$. Then we replace $x$ with $x^{p^s}$.

Thus, $x^p = 0$. On the other hand, if $x \in L_1$ then $(x, x) = 0$. Thus, in either case $x^p = 0$. Now $I = \langle x \rangle_p$ is a restricted ideal of $L$ and we consider $L/I$. Suppose that $u(L/I) = R/IR$ is Lie super-nilpotent. This means that there exists an integer $m$ such that $(R, mR) \subseteq IR$. Since $(I, R) = 0$, we get

$$(IR, (p-1)m R) \subseteq I^p R = 0.$$ 

Hence, $(R, pm R) = 0$. So it is enough to prove that $u(L/I)$ is Lie super-nilpotent. Hence, by induction on $\dim (H, H)$, we can assume $(H, H) = 0$.

Step 2. We may assume $(H, z) = 0$. Let $y$ be a homogeneous element in $H$ such that $x = (y, z) \neq 0$. Since $x \in H$, we can replace $x$ with $(x, i z)$, for some $i$, so that $(x, z) = 0$. So, $x \in Z(L) \cap (L, L)$. Now we can use a similar argument as in Step 1, to show that we can replace $L$ with $L/\langle x \rangle_y$. So, by induction on $\dim (H, z)$, we can assume $(H, z) = 0$.

By Steps 1 and 2, we can assume $(H, L) = 0$. It is now easy to verify that $\gamma^3_3(R) = 0$, as required. □

Proof of Theorem 1.3. The if part follows from Corollary 3.2 and Propositions 3.3 and 4.1. We prove the converse. Let $K$ be the ideal of $L$ generated by $(L, L_0)$. Since $K_0$ is $p$-nilpotent and $(L, L)$ finite-dimensional, it follows by Lemma 2.5 that $\omega(K)$ is associative nilpotent. Thus, $Ku(L)$ is associative nilpotent. So, it is enough to prove that $u(L/K) = u(L)/Ku(L)$ is Lie solvable. Hence, we can assume $L_0$ is central in $L$. In a similar manner, we can assume $(M, L) = 0$. Thus, $L$ is nilpotent and it follows from Theorem 4.8 that $u(L)$ is Lie super-nilpotent. Thus, by Lemma 4.7, $u(L)$ is Lie solvable, as required. □
5. Lie nilpotence

In this section we characterize $L$ when $u(L)$ is Lie nilpotent.

**Lemma 5.1.** If $u(L)$ is Lie nilpotent then either $(L_1, L_1)$ is $p$-nilpotent or $\dim L_1 \leq 1$ and $(L_1, L_0) = 0$.

**Proof.** Suppose that $(L_1, L_1)$ is not $p$-nilpotent. By Proposition 3.3, there exists a subspace $M$ of $L_1$ of codimension at most 1 such that $(M, L_1)$ is $p$-nilpotent. Let $L_1 = M + \mathbb{F}z$. Then $(z, z)$ is not $p$-nilpotent. Let $y \in M$. So, $(z, z)$ and $(y, z)$ are linearly independent. Now for every $n \geq 1$, we have

$$[y, n z] = (-2)^n z^n y + \sum_{i+j=n} \alpha_{i,j} z^i (y, z),$$

where in the sum above $i, j$ are positive integers and $\alpha_{i,j} \in \mathbb{F}$. Let $m$ such that $[y, 2m z] = 0$. Note that $(z^2)^m y$ is a PBW monomial of degree $m + 1$ and the rest of the PBW monomials in Equation (1) have degree strictly less than $m + 1$. Thus, $(z^2)^m y = 0$. Since $(z, z)$ is not nilpotent, we deduce that $y = 0$. Thus, $M = 0$ and $L_1$ is 1-dimensional. Let $x \in L_0$. Then $[z, x] = \alpha z$, for some $\alpha \in \mathbb{F}$. But $[z, n x] = \alpha^n z = 0$, for some $n$. So, $\alpha = 0$. Thus, $(L_1, L_0) = 0$, as required. □

**Lemma 5.2.** Let $L$ be a Lie superalgebra and $W$ a finite-dimensional homogeneous subspace of $L$. If $\dim (L, L)$ is finite, then there exists a homogeneous subspace $V$ of $L$ of finite codimension such that $(W, V) = 0$.

**Proof.** Let $w_1, \ldots, w_n$ be a homogeneous basis for $W$. Since $\dim (w_1, L)$ is finite, there exists a homogeneous subspace $V_1$ of $L$ of finite codimension such that $(w_1, V_1) = 0$. Now we replace $L$ with $V_1$. So, there exists a homogeneous subspace $V_2$ of $V_1$ of finite codimension such that $(w_2, V_2) = 0$. Continuing this way, we get a sequence of subspaces

$$L = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n,$$

so that $\dim V_i / V_{i+1}$ is finite and $(w_i, V_i) = 0$, for every $i \geq 1$. Clearly, $V_n$ has the desired property. □

**Lemma 5.3.** Let $R$ be an associative algebra and $A, B, C$ some subspaces of $R$. Then

$$[AB, n C] \subseteq \sum_{i=0}^{n} [A, i C][B, n-i C]$$

**Proof.** Induction on $n$. □
Stewart [St] proved that if \( H \) is a nilpotent ideal of a Lie algebra \( L \) such that \( L/H' \) is nilpotent then \( L \) is nilpotent. We prove the following for the Lie nilpotence of associative algebras.

**Proposition 5.4.** Let \( I \) be a two-sided ideal of an associative algebra \( R \). If \( I \) is associative nilpotent of index \( c \) and \( R/I^2 \) is Lie nilpotent of class \( d \), then \( \gamma_{\mu(c,d)}(R) = 0 \), where \( \mu(c, d) = 2cd - c - d + 2 \).

**Proof.** Induction on \( c \). If \( c = 1 \), the result is obvious. So, suppose that \( c \geq 2 \). Now, \( I_k = I/I^{k+1} \) is an ideal of \( R_k = R/I^{k+1} \). Note that \( R_k/I_k \) is Lie nilpotent and the nilpotence index of \( I_k \) is at most \( c - 1 \), for every integer \( 1 \leq k \leq c - 1 \). So, we may assume \( \gamma_{\mu(k,d)}(R) \leq I^{k+1} \), for every \( 1 \leq k \leq c - 1 \). By Lemma 5.3, we have

\[
\gamma_{\mu(c,d)}(R) \leq [I^2, 2cd - 2d - c + 1 \ R]
\leq \sum_i [I, R][I, 2cd - 2d - c + 1 - i \ R],
\]

where in the sum above \( i \) is in the range \( 0 \leq i \leq 2cd - 2d - c + 1 \). Now we claim that every term in the sum is zero. Indeed, if \( i \leq d - 1 \), then

\[
[I, R][I, 2cd - 2d - c + 1 - i \ R] \leq I[R, 2cd - 2d - c + 1 - i \ R] \leq I\gamma_{\mu(c-1,d)}(R) \leq I^{c+1} = 0.
\]

On the other hand, if \( i \geq d \) then there exists an integer \( j \geq 2 \) such that \( i \) belongs to the interval

\[
2(j - 1)d - d - (j - 1) + 1 \leq i < 2jd - d - j + 1.
\]

Since \( i \leq 2cd - 2d - c + 1 \), we have \( j \leq c \). If \( j \leq c - 1 \) then, by induction hypothesis, we have

\[
[I, R][I, 2cd - 2d - c + 1 - i \ R] \leq \gamma_{\mu(j-1,d)}(R)[I, 2(c-j)d - d - (c-j)+1 \ R]
\leq I^j \gamma_{\mu(c-j,d)}(R)
\leq I^j I^{c-j+1} = I^{c+1} = 0.
\]

If \( j = c \) then,

\[
[I, R][I, 2cd - 2d - c + 1 - i \ R] \leq \gamma_{\mu(c-1,d)}(R)I
\leq I^c I = I^{c+1} = 0.
\]

The claim is now proved. Thus, \( \gamma_{\mu(c,d)}(R) = 0 \), as required. \( \square \)

**Theorem 5.5.** Let \( L = L_0 \oplus L_1 \) be a restricted Lie superalgebra. Then \( u(L) \) is Lie nilpotent if and only if \( L' \) is finite dimensional, \( L \) is a nilpotent \( L_0 \)-module, \((L_0, L_0)\) is p-nilpotent, and either \((L_1, L_1)\) is p-nilpotent or \( \dim L_1 \leq 1 \) and \((L_1, L_0) = 0 \).
Proof. Proposition 4.1, Lemma 5.1, and Corollary 3.2 are combined to give the proof of the if part. We prove the converse. Suppose first that \( \dim L_1 \leq 1 \) and \( \langle L_0, L_0 \rangle = 0 \). Since \( L_0 \) is nilpotent, by induction on \( \dim \langle L_0, L_0 \rangle \), we can assume \( \langle L_0, L_0 \rangle = 0 \). It follows that \( u(L) \) is Lie nilpotent of class at most two. Now consider the case where \( \langle L_1, L_1 \rangle = 0 \) is \( p \)-nilpotent. Let \( R = u(L) \).

**Step 1.** We may assume \( L_1 \) has a subspace \( P \) of finite codimension such that \( \langle L_1 + \langle L_0, L_0 \rangle, P \rangle = 0 \). By Lemma 5.2, there exists a subspace \( K \) of \( L_1 \) of finite codimension such that \( \langle (L, L), K \rangle = 0 \). It follows that \( \langle (K, K), L \rangle = 0 \). Let \( N = \langle (K, K) \rangle_{p} \subseteq \langle L_0 \rangle \) and set \( I = NR \). We claim it is enough to prove that \( u(L/N) \) is Lie nilpotent. Indeed, suppose \( n^k(u(L/N)) = 0 \). This means that \( n^k(R) \subseteq NR \). Thus, \( n^k(R) \subseteq [NR, R] = N[R, R] \), since \( N \) is central in \( R \). Hence,

\[
\gamma_{nk}(R) \subseteq \omega^k(N)R.
\]

Since \( \omega(N) \) is associative nilpotent, by Lemma 2.1, it follows that \( R \) is Lie nilpotent. So, we replace \( L \) with \( L/N \). In other words, we can assume that \( \langle K, K \rangle = 0 \). Let \( V \) be a finite-dimensional subspace of \( L_1 \) such that \( L_1 = V + K \). By Lemma 5.2, there exists a subspace \( P \) of \( K \) of finite codimension such that \( \langle V, P \rangle = 0 \). Thus, \( \langle L_1 + \langle L_0, L_0 \rangle, P \rangle = 0 \). Let \( W \) be a finite-dimensional subspace of \( L_1 \) such that \( L_1 = W + P \). We may assume that \( \langle L_1, L_0 \rangle \subseteq W \).

**Step 2.** We may assume \( \langle L_1, L_1 \rangle = \langle L_0, L_0 \rangle = 0 \). Consider the ideal \( N = \langle W + \langle L_0, L_0 \rangle \rangle_{p} \) of \( L \). By Lemma 2.1, \( \omega(N) \) is associative nilpotent. Thus, \( NR \), the associative ideal of \( R \) generated by \( N \), is nilpotent. Furthermore, \( (NR)^2 = \omega^2(N)R \). Now, by Proposition 4.1, it is enough to prove that \( R = R/\omega^2(N)R \) is Lie nilpotent. Since \( \langle N', R \rangle \subseteq \omega^2(N)R \), it is enough to prove that \( R/\langle N', R \rangle \) is Lie nilpotent. So, we replace \( L \) with \( L/\langle N' \rangle_{p} \). Thus, Note that \( \langle W, W \rangle = \langle W, \langle L_0, L_0 \rangle \rangle = 0 \). It follows by Step 1 that \( \langle L_1, L_1 \rangle = 0 \). Since \( L_0 \) is nilpotent, a simple induction on \( \dim \langle L_0, L_0 \rangle \) and using Step 1 enables us to assume that \( \langle L_0, L_0 \rangle = 0 \).

**Step 3.** \( R \) is Lie nilpotent. Note that the subalgebra of \( R \) generated by \( L_1 \) is the Grassmann algebra \( G \) (we assume \( 1 \in G \)). Let \( A \) be the subalgebra of \( R \) generated by \( L_0 \). Note that \( U(L) = GA \). For every \( k \geq 1 \), let

\[
S_k = \sum [L_1, \cdots, L_0],
\]
where in the sum above the $k_i$ are all positive integers and $\sum_{i=1}^{r} k_i \geq k$. Note that $S_k R = RS_k$, for every $k \geq 1$. We claim that $\gamma_{2k+1}(R) \subseteq S_k R$, for every $k \geq 1$. Since $(L_1, L_0)$ is finite-dimensional and $(L_n, L_0) = 0$, for some $n$, it follows that $S_k = 0$, for some $k$. So, it is enough to prove the claim. We proceed by induction on $k$. Since $L'_0 = 0$, we have

$$[R, R] = [G A, G A] \subseteq [G A, G A] + [G A, G A] \subseteq S_1 R + [G, G] A.$$

Since $[G, G] = 0$, we get

$$\gamma_3(R) \subseteq S_1 R + [[G, G] A, G A].$$

Now suppose that $\gamma_{2k+1}(R) \subseteq S_k R$. We have

$$[S_k G A, G A] = G[S_k G A, A] + [S_k G A, G A] \subseteq S_{k+1} R + [S_k G A, G A] = S_{k+1} R.$$

Thus,

$$\gamma_{2k+3}(R) \subseteq S_{k+1} R + [[S_k G, G] A, G A] \subseteq S_{k+1} R + G[[S_k G, G] A, A] + [[S_k G, G] A, G A] \subseteq S_{k+1} R + [S_k G, G] A + [[S_k G, G], G A] \subseteq S_{k+1} R + [[G, G], G A] = S_{k+1} R.$$

This finishes the induction step. The proof is complete. \(\square\)

REFERENCES

[AK] E.V. Aladova, A.N. Krasilnikov, Polynomial identities in nil-algebras, Trans. Amer. Math. Soc. 361 (2009), no. 11, 5629–5646.

[Am] S.A. Amitsur, A generalization of Hilbert’s Nullstellensatz, Proc. Amer. Math. Soc. 8 (1957), 649–656.

[B92] Y. Bahturin, A. Mikhalev, V.M. Petrogradsky, M. Zaicev, Infinite Dimensional Lie Superalgebras, (Walter de Gruyter, Berlin, 1992).

[B87] Y. Bahturin, Identical relations in Lie algebras, (VNU Science Press, Utrecht, 1987.)

[B74] Y. Bahturin, Identities in the universal envelopes of Lie algebras, J. Austral. Math. Soc. 18 (1974), 10–21.

[Br] A. Braun, The nilpotency of the radical in a finitely generated P.I. ring, J. Algebra 89 (1984), 375–396.

[J] N. Jacobson, Basic Algebra II, second ed., (W.H. Freeman and Company, New York, 1989).

[K96] A.R. Kemer, PI-algebras and nil algebras of bounded index, Trends in Ring Theory, CMS Conf. Proc., Miskolc, (1996), (Amer. Math. Soc., Providence, 1998), 22, 59–69.
[K91] A.R. Kemer, *Ideal of Identities of Associative Algebras*, (Amer. Math. Soc., Providence, RI, 1991), Vol. 87.

[K81] A.R. Kemer, Nonmatrix varieties, *Algebra and Logic* **19** (1981), 157–178.

[L80] V.N. Latyshev, Nonmatrix varieties of associative algebras, *Mat. Zametki* **27** (1980), no. 1, 147–156.

[L77] V.N. Latyshev, The complexity of nonmatrix varieties of associative algebras, I, II. *Algebra i Logika* **16** (1977), no. 2, 149–183, 184–199, 249–250.

[L63] V.N. Latyshev, Two remarks on PI-algebras, *Sibirsk. Mat. Zh.* **4** (1963), 1120–1121.

[Lev] J. Levitzki, On a problem of A. Kurosch, *Bull. Amer. Math. Soc.* **52** (1946), 1033–1035.

[MPR] S.P. Mishchenko, V.M. Petrogradsky, A. Regev, Characterization of nonmatrix varieties of associative algebras, *Israel J. Math.* to appear.

[N] P.M. Neumann, An improved bound for BFC p-groups. *J. Austral. Math. Soc.* **11** (1970), 19-27.

[P] D.S. Passman, Enveloping algebras satisfying a polynomial identity, *J. Algebra*. **134**(2) (1990), 469–490.

[P92] V.M. Petrogradski, Identities in the enveloping algebras for modular Lie superalgebras, *J. Algebra* **145** (1992), no. 1, 1–21.

[P91] V.M. Petrogradsky, The existence of an identity in a restricted envelope, *Mat. Zametki* **49**(1) (1991), 84–93.

[RW] D.M. Riley, M.C. Wilson, Group algebras and enveloping algebras with nonmatrix and semigroup identities, *Comm. Algebra* **27** (7) (1999), 3545–3556.

[R97] D.M. Riley, PI-algebras generated by nilpotent elements of bounded index, *J. Algebra* **192** (1997), no. 1, 1–13.

[RS] D.M. Riley, A. Shalev, The Lie structure of enveloping algebras, *J. Algebra* **162** (1993), 46–61.

[Sch] M. Scheunert, The theory of Lie superalgebras, *Lecture Notes in Math.* **716** (1979).

[SS] R.K. Sharma and J.B. Srivastava, Lie solvable rings, *Proc. Amer. Math. Soc.* **94** (1985), 1–8.

[Sh] A.I. Shirshov, On rings with identity relations, *Mat. Sb.* 43 (85) (1957), 277–283.

[St] I. Stewart, Infinite-dimensional Lie algebras in the spirit of infinite group theory, *Compositio Mathematica* **22** (1970), 313–331.

[Sz] J. Szigeti, New determinants and the Cayley-Hamilton theorem for matrices over Lie nilpotent rings, *Proc. Amer. Math. Soc.* **125** (8) (1997), 2245–2254.

[ZS] E. Zalesskii, M.B. Smirnov, Associative rings satisfying the identity of Lie solvability (in Russian), *Vestsi Akad. Navuk. BSSR Ser. Fiz.-Mat. Navuk* **2** (1982), 15–20.

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