Uniform Stabilization of 3D Navier–Stokes Equations in Low Regularity Besov Spaces with Finite Dimensional, Tangential-Like Boundary, Localized Feedback Controllers

IRENA LASIECKA, BUDDHlKA PRIYASAD & ROBERTO TRIGGANI

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Abstract

The present paper provides a solution in the affirmative to a recognized open problem in the theory of uniform stabilization of 3-dimensional Navier–Stokes equations in the vicinity of an unstable equilibrium solution, by means of a ‘minimal’ and ‘least’ invasive feedback strategy which consists of a control pair \(\{v, u\}\) (Lasiecka and Triggiani in Nonlinear Anal 121:424–446, 2015). Here \(v\) is a tangential boundary feedback control, acting on an arbitrary small part \(\tilde{\Gamma}\) of the boundary \(\Gamma\); \(u\) is a localized, interior feedback control, acting tangentially on an arbitrarily small subset \(\omega\) of the interior supported by \(\tilde{\Gamma}\). The ideal strategy of taking \(u = 0\) on \(\omega\) is not sufficient. A question left open in the literature is: can such feedback control \(v\) of the pair \(\{v, u\}\) be asserted to be finite dimensional also in dimension \(d = 3\)? We here give an affirmative answer to this question, thus establishing an optimal result. To achieve the desired finite dimensionality of the feedback tangential boundary control \(v\), it is here then necessary to abandon the Hilbert-Sobolev functional setting of past literature and replace it with a “right” Besov space setting of lower regularity. These spaces are ‘close’ to \(L^3(\Omega)\) for \(d = 3\). This functional setting is significant. It is in line with recent well-posedness results in the full space of the non-controlled N–S equations (Escauriaza et al. in Math Subj Classif 35K:76D, 1991; Rusin and Sverak in Minimal initial data for potential Navier–Stokes singularities. arXiv:0911.0500; Jia and Šverák in SIAM J Math Anal 45(3):1448–1459, 2013; Gallagher et al. in Math Ann 355(4):1527–1559, 2013). A double key feature of such Besov spaces with tight indices is that they do not recognize compatibility conditions while having a sufficiently high topological level to handle the 3d-nonlinearity in the analysis of well-posedness and uniform stabilization. The proof is constructive and is “optimal” also regarding the “minimal” number of tangential boundary feedback controllers needed. The new setting requires the solution of novel technical and conceptual issues. These include establishing maximal regularity up to \(T = \infty\) in the required suitably identified
“right” Besov setting for the overall closed-loop linearized problem with tangential feedback control applied on the boundary. This result is also a new contribution to the area of maximal regularity as the operator to which it applies incorporates a boundary feedback control term rather than homogeneous boundary conditions. It escapes direct use of perturbation theory. Finally, the very ability to stabilize even the finite dimensional unstable projected system is linked to a Unique Continuation Property of a suitably over-determined (adjoint) Oseen eigenproblem, which requires the presence of the interior tangential-like control $u$ acting on $\omega$.

1. Introduction

1.1. Controlled Dynamic Navier–Stokes Equations

Let $\Omega$ be an open connected bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma = \partial \Omega$. Unless otherwise stated, $\Gamma$ will be assumed of class $C^2$ throughout the paper. Some results will require less/different boundary assumptions, as noted. For purposes of illustration, let $\omega$ be at first an arbitrary collar (layer) of the boundary $\Gamma$ in the interior of $\Omega$, $\omega \subset \Omega$ (Fig. 1). For each point $\xi \in \omega$, we consider the (sufficiently smooth) curve ($d = 2$) or surface ($d = 3$) $\Gamma_\xi$, which is the parallel translation of the boundary $\Gamma$, passing through $\xi \in \omega$ and lying in $\omega$. Let $\tau(\xi)$ be a unit tangent vector to the oriented curve $\Gamma_\xi$ at $\xi$, if $d = 2$; and let $\tau(\xi) = [\tau_1(\xi), \tau_2(\xi)]$ be an orthonormal system of oriented tangent vectors lying on the tangent plane to the surface $\Gamma_\xi$ at $\xi$, if $d = 3$, and obtained as isothermal parametrization via a 1-1 conformal mapping of a suitable open set in $\mathbb{R}^2$ with canonical basis $e_1 = \{1, 0\}, e_2 = \{0, 1\}$. See [52, Appendix] for details and references. We shall in particular focus on and study the case where $\omega$ is a localized collar based on an arbitrarily small, connected portion $\tilde{\Gamma}$ of the boundary $\Gamma$ (Fig. 2). Let $m$ denote the characteristic function of the collar set $\omega : m \equiv 1$ in $\omega$, $m \equiv 0$ in $\Omega/\omega$.

We consider the following Navier–Stokes equations perturbed by a force $f$ and subject to the action of a pair $\{v, u\}$ of controls (to be described below):

\begin{align*}
  y_t(t, x) - v_0 \Delta y(t, x) + (y \cdot \nabla) y + \nabla \pi(t, x) - (m(x)u) \tau &= f(x) \quad \text{in } Q \quad (1.1a) \\
  \text{div } y &= 0 \quad \text{in } Q \quad (1.1b)
\end{align*}

Fig. 1. Internal collar $\omega$ of full boundary $\Gamma$
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\[ y = v \quad \text{on } \Sigma \]  
\[ y(0, x) = y_0(x) \quad \text{in } \Omega. \]  

Here \( Q = (0, \infty) \times \Omega, \Sigma = (0, \infty) \times \Gamma \) and the constant \( \nu_o > 0 \) is the viscosity coefficient. In (1.1c), \( v \) is a \( d \)-dimensional tangential boundary control \( \nu \cdot v \equiv 0 \) on \( \Gamma \), possibly supported on an arbitrarily small connected part \( \tilde{\Gamma} \) of the boundary, where \( \nu \) is the unit outward normal to \( \Gamma \). Instead, \( u \) is a scalar \( (d = 2) \) or a two dimensional vector \( u = [u^1, u^2] \) \( (d = 3) \) interior “tangential” control acting in the ‘tangential direction’ \( \tau \) (that is, parallel to the boundary) in the small boundary layer \( \omega: (mu)\tau \), where for \( d = 3 \) (Fig. 2),

\[ (mu)\tau = [(mu^1)\tau_1 + (mu^2)\tau_2] \quad \text{for short}, \quad m \equiv 1 \quad \text{on } \omega; \quad m \equiv 0 \quad \text{in } \Omega \setminus \omega; \]  

see [52, Appendix]. The scalar function \( \pi \) is the unknown pressure.

**Notation:** As already done in the literature, for the sake of simplicity, we shall adopt the same notation for function spaces of scalar functions and function spaces of vector valued functions. Thus, for instance, for the vector valued \( (d\)-valued) velocity field \( y \) or external force \( f \), we shall simply write say \( y, f \in L^q(\Omega) \) rather than \( y, f \in (L^q(\Omega))^d \) or \( y, f \in L^q(\Omega) \). This choice is unlikely to generate confusion. The initial condition \( y_0 \) and the body force \( f \in L^q(\Omega) \) are given. The scalar function \( \pi \) is the unknown pressure.

### 1.2. Stationary Navier–Stokes Equations

The following result represents our basic starting point:

**Theorem 1.1.** Consider the following steady-state Navier–Stokes equations in \( \Omega \):

\[-v_o \Delta y_e + (y_e \cdot \nabla) y_e + \nabla \pi_e = f \quad \text{in } \Omega \]  
\[ \text{div} \, y_e = 0 \quad \text{in } \Omega \]  
\[ y_e = 0 \quad \text{on } \Gamma. \]  

Let \( 1 < q < \infty \). For any \( f \in L^q(\Omega) \) there exits a solution (not necessarily unique)

\( (y_e, \pi_e) \in (W^{2,q}(\Omega) \cap W^{1,d}_0(\Omega)) \times W^{1,q}(\Omega)/\mathbb{R}. \)
For the Hilbert case \( q = 2 \), see [19, Thm 7.3 p 59]. For the general case \( 1 < q < \infty \), see [4, Thm 5.iii p 58].

**Remark 1.1.** It is well-known [49,56,75] that the stationary solution is unique when “the data is small enough, or the viscosity is large enough” [75, p 157; Chapt 2] that is, if the ratio \( \|f\|/\nu_o^2 \) is smaller than some constant that depends only on \( \Omega \) [33, p 121]. When non-uniqueness occurs, the stationary solutions depend on a finite number of parameters [33, Theorem 2.1, p 121] asymptotically, in the time dependent case.

**Remark 1.2.** The case where \( f(x) \) in (1.1a) is replaced by \( \nabla g(x) \) is noted in the literature as arising in certain physical situations, where \( f \) is a conservative vector field. In this case, a solution of the stationary problem (1.2) is \( y_e \equiv 0, \pi_e = g \). The analysis of this relevant case will be discussed in the Orientation, Case 3a) in Section 1.5; in Remark 2.2 at the end of Section 2; and in Problem #3, Appendix C.

### 1.3. The Stabilization Problem

**1.3.1. Its Physical and Mathematical Importance.** With reference to the viscous Navier–Stokes fluid in a bounded region of the two- or three-dimensional space under the action of a given time-independent driving mechanism such as a body force, an attractive physical description of the feedback stabilization problem in fluid dynamics has been given in the report of Referee #1 of paper [53]. We quote from the review (see also [53, below Eq (1.4)]):

It is experimentally observed, and analytically and numerically validated, that if the magnitude of the driving mechanism, call it \( |\mathcal{D}| \), is below a certain critical value, \( C \), then the corresponding flow of the liquid is time-independent as well, and it is also unique and stable. However, if \( |\mathcal{D}| > C \), then another motion (not necessarily of steady nature) appears and is stable. Eventually, when \( |\mathcal{D}| \) becomes very large, the corresponding motion is of chaotic nature, and turbulence sets in. The stabilization problem consists in avoiding the occurrence of the above process by forcing the flow to keep its original steady-state regime though a suitable feedback control.

In other words, for large Reynolds numbers \( 1/\nu_0 \), the steady state solution \( y_e \) becomes unstable in a quantitative sense to be made more precise below in (1.3), and may cause turbulence: it is therefore important to be able to suppress turbulence asymptotically in time by selecting a suitable feedback control action. As to turbulence theory, one of its main features is the so called phenomenon of energy cascade, dating back to Kolmogorov, whereby the average energy at any given scale is governed by three elements: the input from the driving force; the inertial effects that transfer energy toward lower scales; and dissipation due to viscosity. A recent contribution aimed at a better understanding of the mathematical mechanisms due to turbulence is [23]. Our goal in this paper is to suppress turbulence—potentially caused by an external force—asymptotically in time.
Assumption of instability. Let \( d = 2, 3 \). We label “unstable” the uncontrolled \((u \equiv 0, v \equiv 0)\) system (1.1) to mean that the corresponding Oseen operator \( A_q \) in (1.11), which depends on \( y_e \) via (1.10), has a finite number, say \( N \), of not necessarily distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N \) on the complex half plane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq 0 \} \) which we then order according to their real parts, so that

\[
\ldots \leq \text{Re} \lambda_{N+1} < 0 \leq \text{Re} \lambda_N \leq \ldots \leq \text{Re} \lambda_1,
\]

each \( \lambda_i, i = 1, \ldots, N \) being an unstable eigenvalue repeated according to its geometric multiplicity \( \ell_i \). We shall next indicate with \( M(\leq N) \) the number of distinct unstable eigenvalues of \( A_q \). Thus, the corresponding uncontrolled linearized \( w \)-problem (1.28) \((v = 0, u = 0)\) is described by a strongly continuous analytic semigroup generated by such Oseen operator, Appendix A, which moreover is unstable. We seek to counteract such instability by devising a suitable feedback control strategy \( \{v, u\} \) [acting on the state \( y(t) \) only], that not only produces the corresponding linearized \( w \)-problem to be (globally) uniformly stable, but—in addition—forces the overall nonlinear problem (1.1) to be uniformly stable in the vicinity of such ‘unstable’ equilibrium solution \( y_e \). Henceforth, unless otherwise stated, we consider the pair \( \{\tilde{\Gamma}, \omega\} \), where \( \tilde{\Gamma} \) is an arbitrarily small connected portion of the boundary \( \Gamma = \partial \Omega \), and \( \omega \) is an arbitrarily small interior subset of \( \Omega \), supported by \( \tilde{\Gamma} \) as in Fig 2. Moreover, following [53], the pair \( \{v, u\} \) of stabilizing feedback controls will be acting on \( \tilde{\Gamma}, \omega \). This means that: the boundary feedback control \( v \) will be taken as acting tangentially (no normal component) on the arbitrary small connected portion \( \tilde{\Gamma} \) of the boundary \( \Gamma = \partial \Omega \); the interior control \( u \) will be taken as acting tangential-like (parallel to the boundary) on the arbitrary small interior subset \( \omega \) of \( \Omega \), supported by \( \tilde{\Gamma} \) (Fig. 2).

**1.3.2. Main Features of Present Solution.** The following features of the stabilization problem solved in this paper—both at the level of state-of-the-art of the literature as well as new contributions—need to be stressed at the outset (a more extensive explanation is postponed to Section 1.4.4, following the main Theorem A):

1. **Insufficiency of the pair \( \{v, u \equiv 0\} \) on \( \{\tilde{\Gamma}, \omega\} \).** Use of the sole tangential boundary feedback control \( v \) as localized on \( \tilde{\Gamma} \) (that is, \( u \equiv 0 \) on \( \omega \)) is not sufficient as a stabilizing control, regardless of whether \( v \) is finite or infinite dimensional. Indeed only such \( v \) is not capable even to uniformly stabilize (with arbitrary decay rate) the finite dimensional unstable projected system \( w_N \) in (3.8), as the algebraic controllability condition (4.11b) with \( u \equiv 0 \) would fail. Refer to the Unique Continuation Property of Lemma 4.3 and paragraph below it. This is due to the counterexample in [26], to be discussed in the Orientation Section 1.5 and in Problem #1 of Appendix C.

2. **Minimal extra condition to be added to \( v \).** Thus, the addition of an interior, tangential-like feedback control \( u \), as localized on the companion domain \( \omega \) is a minimal extra condition, with which to supplement such \( v \).

**Remark 1.3.** To put the above point (2) in proper perspective, we recall that uniform stabilization of the Navier–Stokes equations \( d = 2, 3 \), by means of a localized
feedback interior control (with no constraint on being tangential-like, if supported on a patch of the boundary) was solved in [11] in a Hilbert space setting. The recent solution [50] in the Besov setting of the present paper \( d = 2, 3 \), improves [11] in both results and methods; for instance, on the test for the finite dimensional controllability condition and the nonlinear analysis. In particular, in [50] the minimal number of required feedback controls is equal to the largest geometric multiplicity of the unstable eigenvalues, rather than the algebraic multiplicity as in [11].

(3) Main contribution: finite dimensionality of \( v \). Regarding the described pair \( \{v, u\} \) on \( \{\tilde{F}, \omega\} \) present state-of-the-art has succeeded [52,53] in establishing local exponential stabilization (asymptotic turbulence suppression) near an unstable equilibrium solution \( y_e \) by means of a localized finite-dimensional tangential feedback boundary control \( v \) of the pair \( \{v, u\} \) on \( \{\tilde{F}, \omega\} \), in the Hilbert setting in two cases: (i) when the dimension \( d = 2 \), (ii) when the dimension \( d = 3 \) but the initial condition \( y_0 \) in (1.1d) is compactly supported. In the general \( d = 3 \) case, handling of the non-linearity of the N–S problem forces a Hilbert space setting with a high-topology \( H^{1/2+\varepsilon}(\Omega) \), \( \varepsilon > 0 \), whereby the compatibility conditions kick in. These then cannot allow the boundary stabilizing feedback control \( v \) to be finite dimensional in general for \( d = 3 \). In the case \( d = 3 \), the obstruction due to the compatibility conditions in the Hilbert setting of all the past literature was recognized also by other authors. In [62], the compatibility condition between the initial state and the feedback controller at \( t = 0 \) is achieved by choosing a time-varying control operator, rather than a static control operator as in the present paper. Same in [5]: this is the dynamic controller of the title of these authors’ paper. Moreover, in both these works, the boundary control has tangential as well as normal components. In contrast, in our present paper, we take of course the feedback boundary control \( v = F(\cdot) \) on \( \tilde{F} \) to be tangential with \( F \) a static operator. The main goal and contribution of the present work is to remove the deficiency noted in (ii) on the localized tangential stabilizing feedback boundary control \( v \) of the pair \( \{v, u\} \) on \( \{\tilde{F}, \omega\} \) in the case \( d = 3 \), and thus obtain constructively local uniform feedback stabilization of (1.1) near an unstable equilibrium solution \( y_e \), by means of a control pair \( \{v, u\} \) on \( \{\tilde{F}, \omega\} \) with a stabilizing tangential, boundary localized static feedback control operator \( v = F(\cdot) \) in (1.1d). Here, the feedback operator \( F \) is in feedback form as given by (1.23). The key point is that \( F \) is also finite dimensional in the case \( d = 3 \). This solves the affirmative an established and recognized open problem in this area.

(4) Strategy: from the Hilbert setting of the literature to a new tight, low regularity Besov space setting. To this end, we need therefore to go beyond the Hilbert setting of the literature and thus achieve local uniform feedback stabilization near an equilibrium solution \( y_e \) in the case \( d = 3 \) in a function space enjoying the following two features: on the one hand, it must have a topological level high enough as to accommodate the N–S nonlinearity for \( d = 3 \); and on the other hand, it must be of low regularity as to not recognize boundary conditions, in order not to be subject to compatibility conditions. Thus, the present paper will provide a feedback stabilization pair \( \{v, u\} \), in (1.1c) and in (1.1a)
respectively, both finite-dimensional also in the case \( d = 3 \) (in the case of \( u \), this is already known [11,52]) and spectral based, this time however within the “right”, tight Besov-setting, that fulfills the above two requirements. In particular, well-posedness and local exponential stability for the velocity field \( y \) near an unstable equilibrium solution \( y_e \) will be achieved for \( d = 3 \) in the topology of the Besov space \( \tilde{B}^{2,2/p}_{q,p}(\Omega) \) in (1.15b) (‘close’ to \( L^3(\Omega) \) for \( d = 3 \)) which does not recognize compatibility conditions. See Remark 1.4.

### 1.4. Main Results

Before stating the main results, we need to introduce the necessary mathematical setting.

#### 1.4.1. Preliminaries: Helmholtz Decomposition

A first difficulty one faces in extending the local exponential feedback stabilization result near an equilibrium solution \( y_e \) with tangential control pair \( \{v, u\} \) of the original problem (1.1) from the Hilbert-space setting in [11,12,53] to the \( L^q \)/Besov setting is the question of the existence of a Helmholtz (Leray) projection for the domain \( \Omega \subset \mathbb{R}^d \). More precisely: Given an open set \( \Omega \subset \mathbb{R}^d \), the Helmholtz decomposition answers the question as to whether \( L^q(\Omega) \) can be decomposed into a direct sum of the solenoidal vector space \( L^q_\sigma(\Omega) \) and the space \( G^q(\Omega) \) of gradient fields. Here,

\[
L^q_\sigma(\Omega) = \{y \in C_0^\infty(\Omega) : \text{div } y = 0 \text{ in } \Omega\}^p
\]

\[
= \{g \in L^q(\Omega) : \text{div } g = 0; \ g \cdot \nu = 0 \text{ on } \partial \Omega\},
\]

for any locally Lipschitz domain \( \Omega \subset \mathbb{R}^d, d \geq 2 \) [Ga.1, p119]

\[
G^q(\Omega) = \{y \in L^q(\Omega) : y = \nabla p, \ p \in W^{1,q}_{loc}(\Omega) \text{ where } 1 \leq q < \infty\}. \tag{1.4}
\]

Both of these are closed subspaces of \( L^q \).

**Definition 1.1.** Let \( 1 < q < \infty \) and \( \Omega \subset \mathbb{R}^n \) be an open set. We say that the Helmholtz decomposition for \( L^q(\Omega) \) exists whenever \( L^q(\Omega) \) can be decomposed into the direct sum (orthogonal for \( d = 2 \))

\[
L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega). \tag{1.5}
\]

The unique linear, bounded and idempotent (that is \( P^2_q = P_q \)) projection operator \( P_q : L^q(\Omega) \longrightarrow L^q_\sigma(\Omega) \) having \( L^q_\sigma(\Omega) \) as its range and \( G^q(\Omega) \) as its null space is called the Helmholtz projection.

Here below we collect a subset of known results about Helmholtz decomposition. We refer to [39, Section 2.2], in particular for the comprehensive Theorem 2.2.5 in this reference, which collects domains for which the Helmholtz decomposition is known to exist. These include the following cases:
(i) any open set \( \Omega \subset \mathbb{R}^d \) for \( q = 2 \), that is with respect to the space \( L^2(\Omega) \); more precisely, for \( q = 2 \), we obtain the well-known orthogonal decomposition (in the standard notation, where \( \nu = \text{unit outward normal vector on } \Gamma \)) [19, Prop 1.9, p 8]

\[
L^2(\Omega) = H \oplus H^\perp
\]

\[
H = \{ \phi \in L^2(\Omega) : \text{div } \phi \equiv 0 \text{ in } \Omega; \; \phi \cdot \nu \equiv 0 \text{ on } \Gamma \}
\]

\[
H^\perp = \{ \psi \in L^2(\Omega) : \psi = \nabla h, \; h \in H^1(\Omega) \};
\]

(ii) a bounded \( C^1 \)-domain in \( \mathbb{R}^d \) [27], \( 1 < q < \infty \), or [34, Theorem 1.1 p 107, Theorem 1.2 p 114] for \( C^2 \)-boundary;

(iii) a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \) \( (d = 3) \) and for \( \frac{3}{2} - \epsilon < q < 3 + \epsilon \) sharp range [27];

(iv) a bounded convex domain \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), \( 1 < q < \infty \) [27].

On the other hand, on the negative side, it is known that there exist domains \( \Omega \subset \mathbb{R}^d \) such that the Helmholtz decomposition does not hold for some \( q \neq 2 \) [58].

Assumption (H-D) Henceforth in this paper, we assume that the bounded domain \( \Omega \subset \mathbb{R}^d \) under consideration admits a Helmholtz decomposition for the values of \( q \), \( 1 < q < \infty \), here considered at first, for the linearized problem (1.28) below. This is the case for domains of class \( C^2 \), as assumed. The final results Theorems A of Section 1.4.4 for the non-linear problem (1.1) will require \( q > 3 \), see (7.18), in the case of interest \( d = \text{dim } \Omega = 3 \).

Let \( \Omega \subset \mathbb{R}^d \) be an open set and let \( 1 < q < \infty \). The Helmholtz decomposition exists for \( L^q(\Omega) \) if and only if it exists for \( L^{q'}(\Omega) \), and we have: (adjoint of \( P_q \))

\[
P_q^* = P_{q'} \quad \text{(in particular } P_2 \text{ is orthogonal)}, \quad \text{where } P_q \text{ is viewed as a bounded operator } L^q(\Omega) \rightarrow L^{q'}(\Omega), \quad P_q^* = P_{q'} \quad \text{as a bounded operator } L^{q'}(\Omega) \rightarrow L^q(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1. \]

See [39, Prop 2.2.2 p6], [27,34, Ex. 16 p115],[50]. Through out the paper we shall use freely that [50, Appendix A]

\[
(L^q_0(\Omega))' = L^{q'}_0(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1.
\]

1.4.2. Preliminaries: The Stokes and Oseen Operators. First, for \( 1 < q < \infty \) fixed, the Stokes operator \( A_q \) in \( L^q_0(\Omega) \) with Dirichlet boundary conditions is defined by [36, p 1404], [39, p 1]

\[
A_qz = -P_q \Delta z, \quad \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_0(\Omega).
\]

The operator \( A_q \) has a compact inverse \( A_q^{-1} \) on \( L^q_0(\Omega) \), hence \( A_q \) has a compact resolvent on \( L^q_0(\Omega) \). Next, we introduce the first order Oseen perturbation \( L_e \) as given by

\[
L_e(z) = (y_e \cdot \nabla)z + (z \cdot \nabla)y_e.
\]
Accordingly we define the first order operator $A_{o,q}$, via (1.8) and (1.9), as

$$A_{o,q}z = P_q L_e(z) = P_q [(y_e \cdot \nabla)z + (z \cdot \nabla)y_e],$$

$$\mathcal{D}(A_{o,q}) = \mathcal{D}(A_1^{1/2}) = W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega) \subset L_\sigma^q(\Omega),$$  \hspace{1cm} (1.10)

where $A_1^{1/2}$ is defined in (A.6) below in Appendix A. Thus, $A_{o,q}A_1^{-1/2}$ is a bounded operator on $L_\sigma^q(\Omega)$, and thus $A_q$ is bounded on $\mathcal{D}(A_1^{1/2})$. This leads to the definition of the Oseen operator

$$A_q = -(\nu A_q + A_{o,q}), \quad \mathcal{D}(A_q) = \mathcal{D}(A_1) \subset L_\sigma^q(\Omega),$$  \hspace{1cm} (1.11)

also with compact resolvent.

### 1.4.3. Preliminaries: Definition of Besov Spaces $B^s_{q,p}$ on Domains of Class $C^1$ as Real Interpolation of Sobolev Spaces.

Let $m$ be a positive integer, $m \in \mathbb{N}$, $0 < s < m$, $1 \leq q < \infty$, $1 \leq p < \infty$, then we define [36, p 1398], [65]

$$B^s_{q,p}(\Omega) = (L^q(\Omega), W^{m,q}(\Omega))_{s \frac{m}{m-p}}$$  \hspace{1cm} (1.12)

[82, p. xx]. This definition does not depend on $m \in \mathbb{N}$. This clearly gives

$$W^{m,q}(\Omega) \subset B^s_{q,p}(\Omega) \subset L^q(\Omega) \quad \text{and} \quad \|y\|_{L^q(\Omega)} \leq C \|y\|_{B^s_{q,p}(\Omega)}.$$  \hspace{1cm} (1.13)

We shall be particularly interested in the following special real interpolation space of $L^q$ and $W^{2,q}$ spaces ($m = 2, s = 2 - \frac{2}{p}$):

$$B^{2-\frac{2}{p}}_{q,p}(\Omega) = (L^q(\Omega), W^{2,q}(\Omega))_{1-\frac{1}{p},p}.$$  \hspace{1cm} (1.14)

Our interest in (1.14) is due to the following characterization [3, Theorem 3.4], [36, p1399]: if $A_q$ denotes the Stokes operator introduced in (1.8), then

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{ g \in B^{2-\frac{2}{p}}_{q,p}(\Omega) : \text{div } g = 0, \ g|_{\Gamma} = 0 \right\}$$

if $\frac{1}{q} < 2 - \frac{2}{p} < 2$  \hspace{1cm} (1.15a)

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{ g \in B^{2-\frac{2}{p}}_{q,p}(\Omega) : \text{div } g = 0, \ g \cdot n|_{\Gamma} = 0 \right\}$$

$$\equiv \tilde{B}^{2-\frac{2}{p}}_{q,p}(\Omega)$$

if $0 < 2 - \frac{2}{p} < \frac{1}{q}$; or $1 < p < \frac{2q}{2q - 1}$.  \hspace{1cm} (1.15b)

**Remark 1.4.** The intended goal of the present paper is to obtain the sought-after feedback stabilization result in a function space is of sufficiently low regularity as to not recognize boundary conditions of the I.C. while, on the other hand, it has a sufficiently high topological level to account for the 3d-nonlinearity in the analysis of well-posedness and uniform stabilization. Thus, we need to avoid the
case in (1.15a), as this implies a Dirichlet homogeneous B.C. Instead, we need to fit into the case (1.15b), where the conditions $\text{div } g \equiv 0$ and $g \cdot v|_\Gamma = 0$ are just features of the underlying space $L^d_\rho(\Omega)$, see (1.4). We shall then impose the condition $2 - \frac{2}{p} < \frac{1}{q}$. Moreover, for the linearized feedback abstract $w$-problem (5.3) below of the translated non-linear feedback $z$-problem (1.29), the final well-posedness and global uniform stabilization result, Theorems 5.1 and 5.4 hold in general for $2 \leq q < \infty$. However, in the analysis of well-posedness and stabilization of the nonlinear N–S translated feedback $z$-problem (1.29), we shall need to impose a constraint $q > d = \dim \Omega$, in particular $q > 3$, see Equation (7.18), to obtain the embedding $W^{1,q} \hookrightarrow L^\infty(\Omega)$ in case of interest $d = 3$. In conclusion, via (1.15b), the range of $p$ is $1 < p < 6/5$ for $d = 3$; and $1 < p < 4/3$ for $d = 2$. In such setting, the compatibility conditions on the boundary of the initial conditions are not recognized. This feature is precisely our key objective within the stabilization problem and removes the shortcoming of the prior Hilbert space setting noted below point (ii) above. For $d = 3$, the space is (1.15b) in ‘close’ to $L^3(\Omega)$. To appreciate this relationship, we note that $L^3(\mathbb{R}^3)$ is the space of minimal initial data for potential Navier–Stokes singularities for the $3 - d$ uncontrolled Navier–Stokes equations on the full space [25, 35, 63]. See Remark 1.6 below.

1.4.4. Main Contributions of the Present Paper For $\dim \Omega = d = 2, 3$, local-in-space (semigroup) well-posedness on the space of maximal regularity $\mathcal{X}^{\infty}_{p,q,\sigma}(\mathbb{A}_{F,q}) \hookrightarrow C([0, \infty); \tilde{B}^{2-2/p}_q(\Omega))$ of the N–S dynamics (1.1) as well as local exponential uniform stabilization near $y_e$ on the space $\tilde{B}^{2-2/p}_q(\Omega)$, $q > d$, $1 < p < 2q/2q-1$ with finite dimensional feedback control pair $\{v, u\}$ on $\{\tilde{F}, \omega\}$.

In line with Section 1.3.2, the present main contributions are obtained by means of the feedback control pair $\{v, u\}$ acting on $\{\tilde{F}, \omega\}$, see Fig. 2, with $v$ (as well as $u$) finite dimensional also for $d = 3$. This is the content of the next main Theorem A.

Main Theorem A. (On problem (1.1)). Let $\Omega$, $\dim \Omega = d = 2, 3$, be a bounded domain of class $C^2$, thus satisfying the Helmholtz decomposition assumption of Definition (1.1). Let $\tilde{\Gamma}$ be an arbitrary small, open connected subset of $\Gamma = \partial \Omega$, of positive measure, supporting the corresponding arbitrary small interior collar $\omega$ (Fig. 2). With reference to the N–S dynamics (1.1), consider a given equilibrium solution $y_e$ of problem (1.2), as guaranteed by Theorem 1.1. Assume $y_e$ is unstable, that is, the eigenvalues of the corresponding Oseen operator $\mathbb{A}_q$ satisfy condition (1.3). Let $q > d$, $1 < p < 2q/2q-1$. Thus $1 < p < 6/5$ for $d = 3$ and $1 < p < 4/3$ for $d = 2$. For $\rho > 0$, define the ball $\mathcal{V}_\rho$ in $\tilde{B}^{2-2/p}_q(\Omega)$ by

$$\mathcal{V}_\rho = \left\{ y_0 \in \tilde{B}^{2-2/p}_q(\Omega) : \|y_0 - y_e\|_{\tilde{B}^{2-2/p}_q(\Omega)} < \rho \right\}, \quad \rho > 0. \quad (1.16)$$

There exists $\rho_0 > 0$ sufficiently small, such that, if $0 < \rho < \rho_0$, for each initial condition $y_0 \in \mathcal{V}_\rho$ there exist a tangential boundary feedback controller $v$ and a tangential-like interior feedback controller $u$, defined respectively by
through bounded operators $F \in \mathcal{L}(L^q_0(\Omega), L^q(\widetilde{\Gamma}))$, and $\widetilde{G} \in \mathcal{L}(L^q_0(\Omega))$, defined explicitly in (1.23), (1.24) below, both finite-dimensional, with $v$ supported on $\tilde{\Gamma}$ and tangential along $\tilde{\Gamma}$, and $u$ with tangential-like internal action $(mu)\tau = (mu^1)\tau_1 + (mu^2)\tau_2$ for $d = 3$, supported on a collar $\omega$ of $\tilde{\Gamma}$, such that the corresponding closed loop system (1.1) due to the action of such pair $\{v, u\}$

\[
\begin{aligned}
&v = F(y - y_e), \text{ supported on } \tilde{\Gamma}, \quad v \cdot v|_{\Gamma} = 0; \quad u = \widetilde{G}(y - y_e) \text{ supported on } \omega \\
&\quad (1.17)
\end{aligned}
\]

has the following two properties:

(a) the feedback system (1.18) is well-posed on the space of maximal regularity, see (6.8), of the operator $\hat{A}_{F,q}$:

\[
X_{p,q,\sigma}^\infty(\hat{A}_{F,q}) = L^p(0, \infty; D(\hat{A}_{F,q})) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega))
\]

\[
\subset X_{p,q}^\infty = L^p(0, \infty; W^{2,q}(\Omega)) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega))
\]

\[
X_{p,q,\sigma}^\infty(\hat{A}_{F,q}) \hookrightarrow C\left([0, \infty); \tilde{B}_{q,p}^{-2/2} \Omega\right)
\]

\[
D(\hat{A}_{F,q}) \equiv \{\phi \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega) : \phi|_{\Gamma} = F\phi\},
\]

that is, as a nonlinear s.c. semigroup on $C\left([0, \infty); \tilde{B}_{q,p}^{-2/2} \Omega\right)$. Here $\hat{A}_{F,q}$ defined in (5.4) or (6.2), is a-fortiori the generator of a strongly continuous analytic uniformly stable semigroup on either $L^q_\sigma(\Omega)$ or $\tilde{B}_{q,p}^{-2/2} \Omega$, describing the linearized $w$-problem in (5.3) in feedback form, see Theorem 5.4.

(b) such closed loop system (1.18) is exponentially stable on $\tilde{B}_{q,p}^{-2/2} \Omega$: by taking, if necessary, $\rho_0 > 0$ further smaller, there exists a constant $\tilde{\gamma}$, $0 < \tilde{\gamma} < |Re \lambda_{N+1}|$, and a constant $C_{\tilde{\gamma}} \geq 1$, depending on $q$, such that

\[
\|y(t) - y_e\|_{\tilde{B}_{q,p}^{-2/2} \Omega} \leq C_{\tilde{\gamma}}e^{-\tilde{\gamma}t} \|y_0 - y_e\|_{\tilde{B}_{q,p}^{-2/2} \Omega}, \quad t \geq 0, \quad y_0 \in \mathcal{V}_\rho.
\]

The bounded finite-dimensional feedback operators $F \in \mathcal{L}(L^q_0(\Omega), L^q(\tilde{\Gamma}))$ and $\widetilde{G} \in \mathcal{L}(L^q_0(\Omega))$ have the following form:

\[
F(y - y_e) = \sum_{k=1}^K \left(\mathcal{P}_N(y - y_e), p_k\right)_{w^q_N} f_k, \quad \text{supported on } \tilde{\Gamma}
\]

\[
(1.23)
\]
Here, $P_N$ is the projector, given explicitly in (3.3a), of $L^q_\sigma(\Omega)$ onto $W^q_{\,N}$, $L^q_\sigma(\Omega) = W^q_{\,N} \oplus W^q_{\,N}$. $W^q_{\,N} = P_N L^q_\sigma(\Omega)$ being the finite dimensional subspace of $L^q_\sigma(\Omega)$, spanned by the generalized eigenvectors corresponding to the unstable eigenvalues in (1.3) of the Oseen operator. Here, $\langle \cdot , \cdot \rangle_{W^q_{\,N}}$ denotes the duality pairing between $W^q_{\,N} \in L^q_\sigma(\Omega)$ and $(W^q_{\,N})^* \in (L^q_\sigma(\Omega))'$ = $L^q_\sigma(\Omega)'$, by (1.7): $\langle h_1, h_2 \rangle_{W^q_{\,N}} = \int_{\Omega} h_1 h_2 \, d\Omega$. The vectors $p_k, q_k$ in $(W^q_{\,N})^* \subset L^q_\sigma(\Omega)$ as well as the boundary vectors $f_k$ are constructed explicitly in Section 4, in the proof of Theorem 4.1. In particular (see Appendix B, in particular (B.5))

$$f_k \in \mathcal{F} = \text{span} \left\{ \frac{\partial \varphi^*_{ij}}{\partial v} : i = 1, \ldots, M; \ j = 1, \ldots, \ell_i \right\} \in W^{2-1/q,q}(\Gamma), \ q \geq 2,$$

(1.25)

$1/q + 1/q' = 1$, where $\varphi^*_{ij} \in W^{3,q}(\Omega)$, see (B.5) in Appendix B, are the eigenfunctions of the adjoint of the Oseen operator, see (4.8a), corresponding to the $M$ distinct unstable eigenvalue $\lambda_i$, with geometric multiplicity $\ell_i$. Finally, $K = \sup \{ \ell_i ; i = 1, \ldots, M \}$. □

The key, new feature of the above Theorem A is that the localized tangential boundary feedback control $v$ (the static operator $F$ supported on $\tilde{\Gamma}$) can be chosen to be finite dimensional also for $d = 3$, as documented by (1.23). This was a recognized open problem in the literature, even for a boundary feedback control, let alone a tangential one, see Section 1.3.2. To obtain such finite dimensionality, it was critical to abandon the prior Hilbert-Sobolev setting of the literature and descend to a lower level regularity setting of the Besov space in (1.15b) where the compatibility conditions are not recognized, Remark 1.4.

**Remark 1.5.** (On the constants $\gamma_0 \approx |\text{Re}\lambda_{N+1}|$, $\gamma'$ of decay rates) With reference to the instability assumption (1.3), throughout the paper we let $\gamma_0$ be a preassigned constant, just below $|\text{Re}\lambda_{N+1}| : 0 < \gamma_0 = |\text{Re}\lambda_{N+1}| - \varepsilon$, $\varepsilon > 0$ fixed and arbitrarily small. Next in Theorem 4.1, we let $\gamma_1 > 0$ be an arbitrarily large constant, in particular $\gamma_1 > \gamma_0$. Then, the conclusion of Theorem 4.1, Equations (4.4) and (4.5), gives the global exponential decay $e^{-\gamma_1 t}$ for the finite dimensional feedback $w_N$-dynamics (4.6) in, respectively, the $L^q_\sigma(\Omega)$- and $\tilde{B}^{2-2/p}_{q,p}(\Omega)$-norms. Next, precisely because $\gamma_1 > \gamma_0$, Theorem 5.1, Equations (5.17) and (5.19) (where full proof is in [53]) yields the global exponential decay $e^{-\gamma_0 t}$ for the feedback linearized $w$-dynamics (5.3), equivalently the s.c. analytic semigroup generated by the operator $\tilde{A}_{F,q}$ on either $L^q_\sigma(\Omega)$ or $\tilde{B}^{2-2/p}_{q,p}(\Omega)$. Finally, the proof of Theorem B given in Section 8 yields the local exponential decay $e^{-\gamma_f t}$ in (1.30) for $z(t) = y(t) - y_e$ in feedback form (1.29) for all I.C. sufficiently close to zero for the variable $z$, respectively to $y_e$ for the variable $y$ in (1.18), (1.23), (1.24). Remark 8.1 supports
the intuitive expectation that “the larger the constant $\gamma_0 \approx |Re\lambda_{N+1}|$, the larger $\tilde{y}$”. In conclusion, the desired exponential decay rate is dictated by $\gamma_0 \approx |Re\lambda_{N+1}|$. See also (C.10) in Appendix C.

Remark 1.6. (Distinction of the space $L^3$) In the case of the uncontrolled N–S equations defined on the full space $\mathbb{R}^3$, extensive research efforts have recently lead to the discovery that the space $L^3(\mathbb{R}^3)$ has the following property for the issue of well-posedness. Assuming that some divergence free initial data in $L^3(\mathbb{R}^3)$ would produce finite time singularity, then there exists a so-called minimal blow-up initial data in $L^3(\mathbb{R}^3)$ [35, 41]. More precisely, let $y$ now be a solution of the N–S equations (1.1a, 1.1b, 1.1c, 1.1d) with $m \equiv 0$, $f \equiv 0$, as defined on the whole space $\mathbb{R}^3$. For any divergence free I.C. $y_0 \in L^3(\mathbb{R}^3)$, denote by $T_{\max}(y_0)$ the maximal time of existence of the mild solution starting from $y_0$. Define

$$\rho_{\max} = \sup \{ \rho : T_{\max}(y_0) = \infty \text{ for every divergence free } y_0 \in L^3(\mathbb{R}^3), \text{ with } \|y_0\|_{L^3(\mathbb{R}^3)} < \rho \}.$$ 

The following result holds [41, Theorem 4.1, p 14]: Suppose that $\rho_{\max} < \infty$. Then there exists some $y_0 \in L^3(\mathbb{R}^3)$, $\|y_0\|_{L^3(\mathbb{R}^3)} = \rho_{\max}$, whose $T_{\max}(y_0) < \infty$, that is the corresponding solution blows up in finite time. Of the numerous works that followed the pioneering work of [55] along this line of research, we quote in addition [25, 63, 66, 67]. Frequency localized regularity criteria for the N–S in $\mathbb{R}^d$, $d = 3$, related also to [25] are given in [16, p 127].

I.5. Orientation

Given the complexity of the problem, whose solution proceeds through various phases, we insert the present encompassing orientation at the very outset of our treatment, even though its full content can be documented and understood only after considerable further reading of the present paper. One may wish to refer back to it as reading proceeds.

The Stabilization Feedback Control Paradigm: purely boundary control action versus arbitrarily short portion of the boundary.

Case 1: Tangential Boundary Feedback Control Action on an Arbitrarily Small Portion of the Boundary. First, ideally, one would like to establish uniform stabilization of the above problem (1.1) by use of only the boundary feedback control $v$ (thus, with localized interior, tangential-like control $u \equiv 0$), subject to two additional desirable features (regardless, at this stage, of its finite dimensionality):

(i) the boundary control $v$ is applied only on an (arbitrarily) small portion $\tilde{\Gamma}$ of the boundary $\Gamma$;

(ii) such $v$ acts only tangentially along $\tilde{\Gamma}$, so that the normal component is not needed (a sort of minimal control action). Tangential actuation is attractive and technologically feasible: it is described as implementable in the engineering community, by means of jets of air [7, 15, 43, p1696].

Is such idealized purely boundary, tangential feedback control $v$ acting only on a portion $\tilde{\Gamma}$ of the boundary $\Gamma$ a possible stabilizing control? The answer is in
the negative. As is known since the studies of boundary feedback stabilization of classical parabolic equations with Dirichlet boundary traces in the feedback loop as acting on the Neumann boundary conditions [54], a critical potential obstruction arises already at the level of the finite-dimensional analysis: more precisely, at the level of enforcing feedback stabilization with large decay rate of the (assumed unstable) finite-dimensional projected $w_N$—system (3.8a, 3.8b) of the linearized $w$—problem (1.28) (with $u \equiv 0$). To achieve this requirement, one needs to verify the algebraic Kalman (or Hautus) rank conditions, corresponding to the unstable eigenvalues $\lambda_i$ in (1.3) with geometric multiplicity $\ell_i$ of the linearized Oseen operator; actually, equivalently, of its adjoint. In the present case, these turn out to be: rank $W_i = \ell_i$, see the matrix $W_i$ in (4.9), with entries restricted only on $\tilde{\Gamma}$, for each distinct unstable eigenvalue $\lambda_i, i = 1, \ldots, M$ in (1.3). Such entries involve the normal derivatives $\{\partial_n \phi_{ij}^*, j = 1, \ldots, \ell_i\}$ on $\tilde{\Gamma}$ of the eigenvectors $\{\phi_{ij}^*, j = 1, \ldots, \ell_i\}$ of the adjoint $A_q^*$ of the Oseen operator $A_q$ in (1.11). See (4.8a). In turn, such algebraic controllability conditions are equivalent to the unique continuation property of the Oseen eigenproblem (C.1a, C.1b, C.1a) \iff (C.2) of Appendix C. Actually, and equivalently, of its adjoint eigenproblem. Such unique continuation property with over-determined conditions $\phi |_{\tilde{\Gamma}} \equiv 0, \partial_n \phi|_{\tilde{\Gamma}} \equiv 0$ only on a portion $\tilde{\Gamma}$ of the boundary $\Gamma$ as in (C.1c) is false. In fact, as collected in Appendix C, reference [26] provides a simple counterexample to such unique continuation property even for the Stokes problem ($y_e \equiv 0$) on the 2-dimensional half-space $\{(x, y) : x \in \mathbb{R}^+, y \in \mathbb{R}\}$, with over-determination on the infinite boundary $\{x = 0\}$. Such counterexample on the half-space can then be transformed via partition of unity in a counterexample of the unique continuation property on a bounded domain $\Omega$ with over-determination on any sub-portion of its boundary $\partial \Omega$. Thus, stabilization (with large decay rate) of the (assumed unstable) finite dimensional projected $w_N$—system (3.8a, 3.8b)—hence of the linearized $w$—problem (1.28)—by means only of the boundary feedback control $v$ active only on the small portion $\tilde{\Gamma}$ of the boundary $\Gamma$ is not possible [and thus with localized interior, tangent-like control $u \equiv 0$ on $\omega$]. This is in contrast with purely parabolic (heat-type) stabilization, where the required unique continuation result is available [79,80].

Case 2: the necessity of complementing the localized tangential boundary feedback control $v$ with a corresponding localized interior tangential-like control $u$. See Fig. 2. If one insists on a boundary control action $v$ active only on a portion of the boundary $\tilde{\Gamma}$, one then needs an extra condition. A weakest extra condition is to complement such $v$ with a localized, interior, tangential-like control $u$, acting on an arbitrarily small patch $\omega$, supported by $\tilde{\Gamma}$ (as it was introduced in [52,53]). This is a sort of minimal extra requirement for keeping $v$ acting only on $\tilde{\Gamma}$. The role of this additional localized interior, tangential-like control $u$ is to guarantee that the corresponding unique continuation property (4.12) \iff (4.14) of Lemma 4.3, augmented this time with the interior condition $\phi^* \cdot \tau \equiv 0$ on $\omega$ in (4.12c), now holds true. This is equivalent to the UCP of Problem #2: (C.6a,b,c) \iff (C.7) in Appendix C. In short: the unique continuation property of Problem #1 (C.1a, C.1b, C.1c) \iff (C.2) without the extra condition $\phi \cdot \tau \equiv 0$ on $\omega$, is false, and this is then "corrected" by falling into the unique continuation of Problem #2, (C.6a,b,c) \iff (C.7) augmented with the interior condition $\phi \cdot \tau \equiv 0$ on $\omega$, which is true.
Technically, \( \text{rank} \left[ -v_0 W_i | U_i \right] = \ell_i \) as in (4.11b) is true; while \( \text{rank} \left[ -v_0 W_i \right] = \ell_i \) is false. Consequently the correspondingly augmented controllability matrix in (4.11b) satisfies the required Kalman rank conditions. In this sense, therefore, the results of the present paper (ultimately, Theorem B yielding tangential null-feedback stabilization in the vicinity of the unstable origin, of the translated \( z \)-problem (1.27)) are optimal, also in terms of the smallness of the required control action for \( v \) and \( u \). Moreover, \( v \) in feedback form is shown here for the first time to be finite-dimensional also in the case \( d = 3 \). This is the key new contribution of the present work (finite dimensionality of the internal tangential-like feedback control \( u \) is not an issue, see [53]). Recall also Remark 1.3.

Case 3: tangential boundary control \( v \) on the whole boundary \( \Gamma \). If, on the other hand, one insists on only exercising tangential boundary feedback control action \( v \)—and thus dispensing altogether with the localized, interior, tangential-like control \( u \)—then such boundary feedback control action \( v \) will have to be applied, as a first preliminary attempt, to the entire boundary \( \Gamma \). Would then be possible to establish uniform stabilization with only a feedback control \( v \) acting tangentially on the entire boundary \( \Gamma \) (regardless of its finite dimensionality)? The answer is Yes, as long as the corresponding unique continuation property (UCP) with over-determination on the whole boundary \( \Gamma \) holds true: by duality Problem #3 (C.8a, C.8b, C.8c) \( \implies \) (C.9) in Appendix C. The proof of such version of uniform stabilization with only feedback control \( v \) (\( u \equiv 0 \)) acting tangentially on the whole boundary \( \Gamma \) and being finite dimensional is unchanged, subject only to invoking said UCP for the corresponding unstable equilibrium solution \( y_e \). Thus, the obstruction is again the validity of the unique continuation property of the Oseen eigenproblem (corresponding to the unstable distinct eigenvalues \( \lambda_i, i = 1, \ldots, M \), in (1.3), with—this time—over-determination \( \varphi|_{\Gamma} \equiv 0, \partial_\nu \varphi|_{\Gamma} \equiv 0 \) on the entire boundary \( \Gamma \): that is Problem #3, implication (C.8) \( \implies \) (C.9) in Appendix C. Is such UCP always true? Only partial results are presently known.

a) Such required unique continuation property is true in dimension \( d = 2, 3 \), if the equilibrium solution \( y_e = 0 \) (Stokes eigenproblem) or, more generally, \( y_e \) is in a sufficiently small ball of the origin in the \( W^{1,\infty} \)-norm. Several very different proofs are given in [79,80]: As noted in Remarks 1.2 and 2.2, the case \( y_e = 0 \) is actually physically quite important as it occurs for instance when the forcing function \( f \) in (1.1a) or (1.2a) is a conservative vector field \( f = \nabla g \) (say an electrostatic field): in which case a solution of problem (1.2a,b,c) is \( y_e = 0 \) and \( \pi = g \), modulo constant. Moreover, the “good” equilibrium solutions (which yield the required unique continuation property with over-determination on the entire boundary \( \Gamma \)) form an open set in, say, the \( W^{1,\infty} \) space topology: if \( y_e \) is “good”, then there is a full ball in the \( W^{1,\infty} \)-topology that contains “good” \( y_e \) [79,80].

What is the implication, in any, of the validity of the corresponding UCP in the case \( y_e = 0 \) on the problem of the present paper?

**Enhancement of decay rate:** See Remark 2.2. Of course, with \( y_e = 0 \), the corresponding Stokes problem (which now replaces the general Oseen problem) is already uniformly stable, with, say a decay rate \(-|\text{Re}(\lambda_i)|\) where \( \text{Re} \lambda_i < 0 \) for the Stokes operator \(-A_q\) in (1.8). A most valuable variation of the problem
under investigation, whose solution is contained in the treatment of the present paper, is as follows: enhance the stability of the linearized (uniformly stable) uncontrolled w-problem (1.28a, 1.28b, 1.28c, 1.28d) (with \( u \equiv 0, v \equiv 0 \)) from the given natural margin \(-|\text{Re}(\lambda_1)|\) to an arbitrarily preassigned decay rate \(-k^2\), by means of only a tangential boundary finite dimensional feedback control \( v \) of the same form as the operator \( F \) in (1.23) as applied this time to the entire boundary \( \Gamma \). To this end, it suffices to apply the procedure of the present paper (with \( u \equiv 0 \)) to a finite dimensional projected space spanned by the eigenvectors of the Stokes operator corresponding to its finitely many eigenvalues \( \lambda_i \) with \( |\text{Re}(\lambda_i)| \leq k^2 \). This, in turn, will provide stability enhancement of the non-linear problem (1.1) in the vicinity of \( y_e = 0 \) (or small \( y_e \)), with only \( v = F(y - y_e) \) on all of \( \Gamma (u \equiv 0) \).

b) In the two dimensional case, \( d = 2 \), there is a genericity result [10] about the validity of the required unique continuation problem (C.8) \( \Rightarrow \) (C.9) in Appendix C with over-determination on the whole boundary.

1.6. Comparison with the Literature

To put the present paper in the context of the literature, we first summarize its main contributions, expanding on the Orientation of Section 1.5.

Contributions of the present paper.

1. The stabilizing control strategy of the present paper is an optimal result for the feedback uniform stabilization of the N–S dynamics (1.1) for \( d = 3 \). It consists of a pair \( \{v, u\} \) of finite dimensional feedback controls requiring a “minimum” control action or support \( \{\tilde{\Gamma}, \omega\} \) and minimal number \( K \): a localized \( K \)-dimensional tangential boundary feedback control \( v \) in (1.1d) acting on an arbitrarily small open connected portion \( \tilde{\Gamma} \) of the boundary \( \Gamma \), \( v \cdot \nu = 0 \) on \( \tilde{\Gamma} \), implemented as \( v = F(\cdot) \), with \( F \) a static operator, and a localized interior \( K \)-dimensional feedback control \( u \) in (1.1a) acting tangential-like (parallel to the boundary) on an arbitrarily small interior patch \( \omega \) supported by \( \tilde{\Gamma} \) (Fig. 2). The number \( K \) is \( K = \sup\{\ell_i : i = 1, \ldots, M\} \), the maximal geometric multiplicity of the distinct unstable eigenvalues of the Oseen problem in (1.3).

As documented in the Orientation of Section 1.5, the interior tangential-like controller \( u \) cannot be dispensed with, if one insists in controlling from an arbitrarily small portion \( \tilde{\Gamma} \) of the boundary. This is due to the counter-example in [26] to the unique continuation property of the over-determined Oseen eigenproblem in (C.1a, C.1b, C.1c), (C.2) of Appendix C, leading to the implication noted in (C.5). Thus: minimal support \( \{\tilde{\Gamma}, \omega\} \), minimal number \( K \), no normal component for \( v \) and \( u \).

2. The main contribution of the present paper over the literature is in asserting that the tangential boundary feedback control \( v \) is finite dimensional also for \( d = 3 \) in full generality, in fact \( K \)-dimensional, through a constructive algorithm. This is an affirmative solution to a recognized open problem. To achieve this desired goal, it was necessary to abandon the Hilbert-Sobolev setting of all prior literature on this problem and employ, for the first time, save for [50], the lower
regularity setting of the Besov space framework $\tilde{B}^{2-2/p}_{q,p}(\Omega)$ in (1.15b) with tight indices, $1 < p < 6/5$, $q > 3$ for $d = 3$, ‘close’ to $L^3(\Omega)$, which does not recognize compatibility conditions as explained in Remark 1.4, while being of sufficiently high topological level as to handle the $3d$, N–S nonlinearity.

3. We have noted in 1 the positive feature in that the finite dimensionality $K$ of the feedback stabilizing controllers $v$ in (1.1c) and $u$ in (1.1a) is equal to the max of the geometric multiplicity—not the algebraic multiplicity as in [8,9,11–14]—of the distinct unstable eigenvalues of the Oseen operator. This is due to the proof, given originally in [52], for checking the controllability condition (4.11b) of the finite dimensional projected system $w_N$ in (3.8). Not only does this proof rest on the max geometric rather than the max algebraic multiplicity of the unstable eigenvalues, but it also much simplifies the somewhat complicated and unnecessary Gram–Schmidt orthogonalization process of [8,11] by employing direct, explicit, sharp tests.

4. Finally, the present work offers a much more attractive and preferable proof over past literature of the ultimate non-linear result: the well-posedness and uniform stabilization of the original (modulo translation) non-linear $z$-problem (7.1), given in Sections 7 and 8. This new proof now rests on the fundamental preliminary property of maximal $L^p$ regularity of the linearized boundary feedback problem (5.3) or (6.2a), or generator $A_{F,q}$ up to $T = \infty$, as stated in Theorem 6.1. Such maximal regularity-based proof up to $T = \infty$ is much cleaner and effective over the original proof for the non-linear boundary stabilization result as given in [12]; and even more so over the approximation argument of the nonlinear operator $N_q$ in (2.17) given in the case of localized feedback control given in [11], [8, Chapter 4]. For maximal regularity literature, see [24,47,48,83,84], following the original contributions [17,22] as well as the classical contributions of [69,70,73] for Navier–Stokes equations, and later [21] for systems of incompressible flows, and [20] for the heat equations on exterior domains. In reference [21], the functional setting is Besov-space-based with either exterior domains or bounded domains. Besov spaces are first introduced on $\mathbb{R}^n$ [21, p 10] and next defined on bounded domains by restriction, [21, p 18], as an alternative to definition (1.12).

The origin of the studies on the uniform stabilization problem of Navier–Stokes equations. The problem of boundary feedback stabilization of unstable linear classical parabolic equations was investigated extensively in the period, say 1974–1983, see [54,76–78]. The study of uniform stabilization of Navier–Stokes equations apparently initiated with the pioneering work of Fursikov [29–31], first in $2d$, next in $3d$. However, this work used open-loop boundary controls not closed loop feedback controls. The nature and dimensionality of the obtained boundary controllers (whether finite or infinite dimensional, whether tangential or otherwise) was not an issue covered by the method of these papers. Fursikov’s work was soon followed by paper [11] which tackled and solved, instead, the (preliminary) problem of uniform stabilization of the Navier–Stokes equations, $d = 2, 3$, by means of a localized interior finite dimensional control. This was implemented as a high-gain, Riccati-based feedback control. All these studies—and the subsequent ones such as
[32] till the present work, some of which are noted below—were carried out in a Hilbert-Sobolev-settings [An improvement over [11] in both content of results and effectiveness of proofs with spectral based, explicit interior localized controllers on the same interior uniform stabilization problem is contained in the authors’ paper [50]. For the first time, its analysis is carried out in the same lower regularity Besov setting of the present boundary stabilization study].

**Tangential Boundary feedback stabilization.** Paper [11] on uniform stabilization by localized interior feedback control opened then the way to a first analysis of the tangential boundary stabilization problem in [12] via a high gain, Riccati-based boundary control, followed by an axiomatic approach, still Riccati-based, in [13], both low and high gain, as well as a complementary, spectral-based approach in [14]. These works required some spectral assumptions of the Oseen eigenvalue problem, equivalent to a unique continuation property for a corresponding over-determined Oseen eigenproblem. See Appendix C. It was only in [52,53] that uniform stabilization with a localized feedback control pair \((v, u)\), as described above, was resolved in an “optimal” way regarding the minimal amount of their support \(\tilde{\Gamma}, \omega\). Moreover, this setting of [52,53] had the advantage of not requiring any property or assumptions on the distinct unstable eigenvalues of the Oseen operator in (1.3), as it was the case in prior literature, since the required corresponding unique continuation property can be shown in this context to hold true (Lemma 4.3), [equivalently, by duality, Problem #2: (C.6a, b, c) \(\Rightarrow (C.7)\)] due to an extra condition \(\varphi^* \cdot \tau \equiv 0\) in (4.12c); or \(\varphi \cdot \tau \equiv 0\) in \(\omega\), in (C.6c), dictated by the employment of the interior localized tangential-like control \(u\). As noted in Section 1.3.2, in reference [53] the issue of finite dimensionality of the tangential boundary feedback controller component was resolved positively only for \(d = 2\) and for \(d = 3\) only in the case of Initial Conditions being compactly supported. The general case for \(d = 3\) was left open. It is resolved here in the affirmative.

**Oblique boundary stabilization; dynamic boundary feedback.** We have already noted references [5,62] which use for \(d = 3\), oblique (rather than tangential) boundary feedbacks, which moreover are dynamic rather than static. Finally, reference [9] investigates stabilization with an oblique boundary control—that is one with an additional normal component. The normal component however is not expressed in feedback form. In addition, two strong assumptions K1 and K.2 are made. The first is the simplifying assumption that the distinct unstable eigenvalues of the Oseen operators be semisimple (geometric = algebraic multiplicity). The second assumes that all \(N\) unstable eigenvalues have dual eigenvectors whose normal derivatives are linearly independent as \(L^2\) functions on the whole boundary \(\Gamma\). This is much stronger than the conditions (already given in [12]) that require the much weaker property that for each distinct unstable eigenvalue \(\lambda_i\) with geometric multiplicity \(\ell_i\), only the traces \(\partial_{\nu}\varphi_{ij}^*\) as in (C.5) of Appendix C be linearly independent, \(i = 1, 2, \ldots, M; j = 1, \ldots, \ell_i\). In both [8, 9], the number of controls equals the max algebraic multiplicity of the unstable eigenvalues of the Oseen operator, see [8, Eq (3.19)].
1.7. Beginning of the Proof of Theorem A: Translated Nonlinear Navier–Stokes z-Problem and Corresponding Linearized w-Problem. Reduction to Zero Equilibrium

We return to Theorem 1.1 which provides an equilibrium pair \( \{ y_e, \pi_e \} \). Then, as in [12, 52] we translate by \( \{ y_e, \pi_e \} \) the original N–S problem (1.1). Thus we introduce new variables

\[
z = y - y_e, \quad \chi = \pi - \pi_e
\]

and obtain the translated problem in \( \{ z, \chi \} \)

\[
\begin{cases}
z_t - v_0 \Delta z + L_e(z) + (z \cdot \nabla)z + \nabla \chi - (m(x)u)\tau = 0 & \text{in } Q \\
\text{div } z = 0 & \text{in } Q \\
z = v & \text{on } \Sigma \\
z(0, x) = z_0(x) = y_0(x) - y_e(x) & \text{on } \Omega
\end{cases}
\]

(1.27a) (1.27b) (1.27c) (1.27d)

where \( v \cdot \nu = 0 \) on \( \Sigma \) and the first order Oseen perturbation \( L_e \) is given by \( L_e(z) = (y_e \cdot \nabla)z + (z \cdot \nabla)y_e \) as defined in (1.9). We shall accordingly first study the local null feedback stabilization of the \( z \)-problem (1.27), that is, feedback stabilization in a neighborhood of the origin.

Our strategy will be to select constructively feedback control operators \( v = F(z) \) and \( u = \tilde{G}(z) \), with \( v \) tangential \( v \cdot \nu = 0 \) on \( \Gamma \) and supported only on \( \tilde{\Gamma} \), and \( u \) tangential-like supported only on \( \omega \), and both \( F \) and \( \tilde{G} \) bounded and finite dimensional also for \( d = 3 \). For \( d = 2 \) this was achieved in the Hilbert space setting [53]. To this end, it will be critical to show global uniform stabilization of the following linearization of the non-linear \( z \)-problem (1.27) near the equilibrium solution \( y_e \)

\[
\begin{cases}
w_t - v_0 \Delta w + L_e(w) + \nabla \chi - (m(x)u)\tau = 0 & \text{in } Q \\
\text{div } w = 0 & \text{in } Q \\
w = v & \text{on } \Sigma \\
w(0, x) = w_0(x) & \text{on } \Omega
\end{cases}
\]

(1.28a) (1.28b) (1.28c) (1.28d)

\( v \cdot \nu = 0 \) on \( \Sigma \). The above main Theorem A for problem (1.1) is an immediate corollary of the following main Theorem B for the translated non-linear \( z \)-problem (1.27).

**Main Theorem B.** (On problem (1.27)) Under the same assumptions and in the same notation of Theorem A, in particular, \( q > 3, 1 < p < 6/5 \) for \( \dim \Omega = 3 \), and with the same boundary vectors \( f_k \) in (1.25) and interior vectors \( p_k, q_k, u_k \) as below (1.24), consider the following feedback version of the translated non-linear
z-problem (1.27), corresponding to the abstract version (7.1):

\[
\begin{aligned}
zt - v_o \Delta z + L_e(z) + (z \cdot \nabla)z + \nabla \chi & \quad (1.29a) \\
- \left( m \left( \sum_{k=1}^{K} \langle P_N z, q_k \rangle_{W^q_N} u_k \right) \right) \cdot \tau & = 0 \quad \text{in } Q \quad (1.29b) \\
\text{div } z & = 0 \quad \text{in } Q \quad (1.29c) \\
z = \sum_{k=1}^{K} \langle P_N z, p_k \rangle_{W^q_N} f_k & \quad \text{on } \Sigma \quad (1.29d) \\
z(0, x) = z_0(x) = y_0(x) - y_e(x) & \quad \text{on } \Omega \quad (1.29e)
\end{aligned}
\]

There exists a positive constant \( r_1 > 0 \) (identified in (7.26), (8.18) such that if

\[
\| z_0 \|_{\tilde{B}_{q,p}^{2-2/p} (\Omega)} \leq r_1,
\]

then, the following local well-posedness and uniform feedback stabilization results hold true:

(i) the feedback problem (1.29) admits a unique (fixed point nonlinear semigroup) solution \( z \) in the space \( C \left( [0, \infty); \tilde{B}_{q,p}^{2-2/p} (\Omega) \right) \), where \( X^\infty_{p,q,\sigma} (\tilde{H}_{p,q}) \hookrightarrow C \left( [0, \infty); \tilde{B}_{q,p}^{2-2/p} (\Omega) \right) \) is the space of maximal regularity in (1.19) and (1.20).

(ii) Moreover, if the constant \( r_1 > 0 \) in (1.30) is sufficiently small as in (8.18), then there exists a constant \( \tilde{\gamma} > 0 \) and a corresponding constant \( C_{\tilde{\gamma}} \geq 1 \), (depending on \( q \)) such that the guaranteed solution \( z \) satisfies the exponential decay

\[
\| z(t) \|_{\tilde{B}_{q,p}^{2-2/p} (\Omega)} \leq C_{\tilde{\gamma}} e^{-\tilde{\gamma} t} \| z_0 \|_{\tilde{B}_{q,p}^{2-2/p} (\Omega)}, \quad t \geq 0.
\]

Remark 8.1 at the end of Section 8 supports qualitatively the intuitive expectation that “the larger the global decay rate \( \gamma_0 \approx |Re \lambda_{N+1}|, \gamma_0 > 0 \) in (5.17) of Theorem 5.4 of the linearized \( w \)-problem (1.11) in feedback form as in (5.3), the larger the local decay rate \( \tilde{\gamma} \) in (1.31).

The proof of the well-posedness part in \( X^\infty_{p,q,\sigma}, \) as well as non-linear semigroup well-posedness on \( \tilde{B}_{q,p}^{2-2/p} (\Omega) \) of Theorem B is given (in its concluding arguments) in Section 7, while the exponential decay (1.31) is established (in its concluding arguments) in Section 8. Recalling \( z = y - y_e, \chi = \pi - \pi_e \) from (1.9), we see at once that Theorem B implies Theorem A.

**Orientation on the semigroup formalism.**

We offer the following considerations to explain the semigroup formalism of the present paper for the benefit of communities outside control theory:

(i) Start with system (1.27) and take initially the B.C. \( z = 0 \) on \( \Sigma \) (instead of (1.27c)). This is plainly a homogeneous B.C.
Next, take the inhomogeneous B.C. \( z = v(t, x) \) on \( \Sigma \), where \( v \) is a function, say an open loop control function, to be selected so that the corresponding \( z \)-solution achieves a desired goal.

Third, in order to achieve such desired goal (in our case, local stabilization near an unstable equilibrium solution \( y_e \)), require that \( v \) be expressed in closed loop feedback form as in (1.29d), rewritten now as

\[
z = \sum_{k=1}^{K} \langle P_N z, p_k \rangle_{W_N^u} f_k = 0.
\]

Such B.C. may then be considered to be homogeneous for the \( z \)-system in feedback form.

In order to use the semigroup formalism, one needs to transfer the boundary feedback term \( v \) in closed loop form into the equation through ‘a Dirichlet map’ (as defined by problem (2.1) and thus one then needs to extend the original dynamic operator, which originally had homogeneous B.C., as done in (2.19).

2. Abstract Models for the Non-linear \( z \)-Problem (1.27) and the Linearized \( w \)-Problem (1.28) in the \( L^q \)-Setting

We shall next provide abstract models for the translated non-linear \( z \)-problem (1.27) and its corresponding linearized \( w \)-problem (1.28) in the \( L^q \)-setting. This will be the counterpart (extensions) of these introduced in [12] and used in [14, 53]. The \( L^q \)-setting will require a wealth of non-trivial additional results: from the well-posedness and regularity from the boundary of the stationary Oseen problem (that is, the definition of the Dirichlet map \( \mathcal{D} \) with range in \( L^q(\Omega) \) : \( g \rightarrow \mathcal{D} g = \psi \) in (2.1) below) to the definition of the adjoint \( \mathcal{A}_q^* = \mathcal{A}_q^* = A_q^* \) for short, (in the \( L^q \) sense) of the Oseen operator \( \mathcal{A}_q \) in (1.11), to the critical meaning of \( \mathcal{D}^* \mathcal{A}_q^* \). These results will be provided below. They will be the perfect counterpart of those obtained in [12], in the Hilbert setting. For the original idea of Dirichlet map we refer to [51].

2.1. Well-Posedness in the \( L^q \)-Setting of the Non-homogeneous Stationary Oseen Problem: The Dirichlet Map \( \mathcal{D} : \text{Boundary} \rightarrow \text{Interior} \)

Recalling the first order operator \( L_e(\psi) = (\psi \cdot \nabla) y_e + (y_e \cdot \nabla) \psi \) from (1.9) and introducing the differential expression \( \mathcal{A} \psi = -\nu_0 \Delta \psi + L_e(\psi) \), we consider the stationary, boundary non-homogeneous Oseen problem on \( \Omega \):

\[
\begin{align*}
\mathcal{A} \psi + \nabla \pi^* &= -\nu_0 \Delta \psi + L_e(\psi) + \nabla \pi^* = 0 & \text{on } \Omega \\
\text{div } \psi &= 0 \text{ in } \Omega; & \psi = g \text{ on } \Gamma, & \text{g} \cdot \nu = 0 \text{ on } \Gamma
\end{align*}
\]

(2.1a) (2.1b)
Remark 2.1. Postponing regularity issues to the second part of the present subsection, our purpose here is to introduce the Dirichlet map $g \rightarrow \psi$, from the boundary datum to the interior solution of the above Oseen problem, following [12, Chapter 3]. As noted and discussed in [12, Ch 3, Orientation at p. 21; Appendix A.2, pp 99–102], problem (2.1) may not define a unique solution $\psi$; that is, the operator $g \rightarrow \psi$ may have a nontrivial (finite dimensional) null space. To overcome this, one replaces in (2.1) the differential expression $A\psi = -\nu_o \Delta \psi + Le(\psi)$ with its translation $k + A$, for a positive constant $k$, sufficiently large as to obtain a unique solution $\psi$. As seen in the subsequent results below, we can take $k = 0$ whenever the Stokes operator is perturbed only by a first order operator such as $A\psi = -\nu_o \Delta \psi + (a \cdot \nabla)\psi$, $\text{div } a = 0$, with $a$ sufficiently regular. Moreover, as documented in [12] in the Hilbert setting $q = 2$ and restated below in the general $L^q$-setting, the expression $D^*(k)A^*(k)$ does not depend on the translation parameter $k$. Thus, at the end, also in name of simplicity of notation, we are here justified to admit henceforth that problem (2.1) (with $k = 0$) defines a unique solution $\psi$. We shall then denote the ‘Dirichlet’ map $g \rightarrow \psi$ by $D : Dg = \psi$ in the notation of (2.1).

The following two regularity results of the Oseen equation below are critical for our subsequent development. The are the perfect counterpart of the results given in [12] in the Hilbert setting. To state properly the conclusion of uniqueness, they will refer to the Oseen equation with only a first order term, such as

$$\begin{cases} -\nu_o \Delta \psi + (a \cdot \nabla)\psi + \nabla \pi^* = 0 \text{ in } \Omega \\ \text{div } \psi = 0 \text{ in } \Omega \\ \psi = g \text{ on } \Gamma \end{cases}$$

(2.2a, 2.2b, 2.2c)

Theorem 2.1. [4, Thm 15, p 37, where a more general result is given] Let

$a \in L^3(\Omega), \text{ div } a \equiv 0; \ g \in W^{1-\frac{1}{q},q}(\Gamma), \ 2 < q < \infty, \ g \cdot \nu = 0 \text{ on } \Gamma.$

(2.3)

Then problem (2.2) has a unique solution $(\psi, \pi^*) \in W^{1,q}(\Omega) \times L^q(\Omega)/\mathbb{R}$ continuously; there is a constant $C > 0$ such that

$$\|\psi\|_{W^{1,q}(\Omega)} + \|\pi^*\|_{L^q(\Omega)/\mathbb{R}} \leq C \left(1 + \|a\|_{L^3(\Omega)}^2\right) \|g\|_{W^{1-\frac{1}{q},q}(\Gamma)}.$$  

(2.4)

Theorem 2.2. [4, Thm 2, p 6, where a more general result is given] Let

$a \in L^3(\Omega), \text{ div } a \equiv 0; \ g \in W^{-\frac{1}{q},q}(\Gamma), \ \frac{3}{2} < q < \infty, \ g \cdot \nu = 0 \text{ on } \Gamma.$

(2.5)

Then problem (2.2) has a unique solution

$(\psi, \pi^*) \in L^q(\Omega) \times W^{-1,q}(\Omega)/\mathbb{R}$

(2.6)

continuously; there is a constant $C_a > 0$ (explicitly depending on the norm of $\|a\|_{L^3(\Omega)}$) such that

$$\|\psi\|_{L^q(\Omega)} + \|\pi^*\|_{W^{-1,q}(\Omega)/\mathbb{R}} \leq C_a \|g\|_{W^{-\frac{1}{q},q}(\Gamma)}.$$  

(2.7)
We note that, in Theorem 2.2, we have also $\Delta \psi \in \left( Y_{r', p'}(\Omega) \right)' = \text{dual of } Y_{r, p}(\Omega) = \{ \varphi \in W^{1, r}_0(\Omega), \text{div } \varphi \in W^{1, q}_0(\Omega) \}, \ 1 < r, q < \infty$, but we shall not need this result [4, p 6].

Returning to our Oseen problem (2.1) of interest, we have $y_e \in W^{2, q}(\Omega) \cap W^{1, q}_0(\Omega)$ from Theorem 1.1, hence the embedding $W^{2, q}(\Omega) \hookrightarrow C(\overline{\Omega})$ holds true for $d = 3, \ q > \frac{3}{2}$ [44, p 79], [1, p 97]. Thus, we can apply Theorems 2.1 and 2.2 to problem (2.1) and obtain the following results, where, with $\psi = Dg$, the range of $D$ is in $L^q_\sigma(\Omega)$, since $\text{div}(Dg) \equiv 0$ in $\Omega$, $(Dg) \cdot v|_\Gamma = g \cdot v|_\Gamma = 0$, see (1.4):

$$g \in W^{(1-\frac{1}{q})/(1-\theta), \theta \cdot \frac{1}{q} \cdot q}(\Gamma) \quad g \cdot v = 0 \text{ on } \Gamma; \ 0 < \theta < 1 \quad \longrightarrow \quad Dg = \psi \in W^{(1-\theta), q}(\Omega) \cap L^q_\sigma(\Omega) \quad (2.8)$$

so that, as $\left( 1 - \frac{1}{q} \right)(1-\theta) - \frac{\theta}{q} = 0$ for $\theta = 1 - \frac{1}{q}$, we also obtain

$$g \in U_q \equiv \{ \tilde{g} \in L^q(\Gamma), \ \tilde{g} \cdot v = 0 \text{ on } \Gamma \} \quad \longrightarrow \quad Dg \in W^{(1-q)/\theta, q}(\Omega) \cap L^q_\sigma(\Omega) \quad (2.9)$$

all continuously. This property will be further complemented by additional information in (2.64) below. In the Hilbert space setting, $q = 2$, we re-obtain the regularity results, that were derived in [12, Theorem A.2.2 p 102], where we recall (1.6a, 1.6b, 1.6c)

$$g \in H^s(\Gamma), \ -\frac{1}{2} \leq s \leq \frac{1}{2} \quad g \cdot v = 0 \text{ on } \Gamma \quad \longrightarrow \quad Dg = \psi \in H^{s+\frac{1}{2}}(\Omega) \cap H. \quad (2.10)$$

2.2. Abstract Model for the Non-linear Translated $z$-Problem (1.27).

We re-write Equation (1.27a) as $z_t + \mathbb{A} z + (z \cdot \nabla) z + \nabla \chi - (mu) \tau = 0$ recalling the differential expression $\mathbb{A}$ defined above (2.1a), and next subtract $\mathbb{A} \psi = \mathbb{A} Dg = -\nabla \pi^*$ from (2.1a), where presently $g = v$ on $\Gamma, v \cdot v = 0$ on $\Gamma$. We obtain

$$z_t + \mathbb{A} (z - Dv) + (z \cdot \nabla) z + \nabla (\chi - \pi^*) - (m(x)u) \tau = 0 \quad \text{in } Q \quad (2.11)$$

Next we apply to (2.11) the Helmholtz projector $P_q$, and obtain [notice that $P_q \psi = z_t$, since $z_t \in L^q_\sigma(\Omega)$ [div $z_t \equiv 0, \ z_t \cdot v = v_t \cdot v = 0$ on $\Gamma$] since $P_q \nabla (\chi - \pi^*) \equiv 0$]

$$z_t + P_q \mathbb{A} (z - Dv) + P_q (z \cdot \nabla z) - P_q ((m(x)u) \tau) \equiv 0, \quad (2.12)$$

where, via (2.1a), (1.9),

$$P_q \mathbb{A} f = -v_o P_q \Delta f + P_q \left[ (y_e \cdot \nabla) f + (f \cdot \nabla) y_e \right]. \quad (2.13)$$

For $1 < q < \infty$ fixed, we recall the Stokes operator $A_q$ in $L^q_\sigma(\Omega)$, the perturbation operator $A_{o, q}$ and the Oseen operator $A_q$, from (1.8), (1.10), (1.11), respectively, to get

$$A_q z = -P_q \Delta z, \ \mathcal{D}(A_q) = W^{2, q}(\Omega) \cap W^{1, q}_0(\Omega) \cap L^q_\sigma(\Omega). \quad (2.14)$$
\[ A_{o,q}z = P_q L_e(z) = P_q [(y_e \cdot \nabla)z + (z \cdot \nabla)y_e], \quad D(A_{o,q}) = D(A_q^{1/2}) \]
\[ = W_0^{1,q}(\Omega) \cap L_q^q(\Omega) \subset L_q^q(\Omega). \quad (2.15) \]
\[ A_q = -(v_o A_q + A_{o,q}), \quad D(A_q) = D(A_q) \subset L_q^q(\Omega). \quad (2.16) \]

Finally, we define the projection of the nonlinear portion of (1.27a) as
\[ N_q(z) = P_q[(z \cdot \nabla)z] \quad (2.17) \]
Thus, after using (2.14)–(2.17) in (2.12), the N–S translated problem (2.12) can be rewritten as the following abstract equation on \( L_q^q(\Omega) \), \( 1 < q < \infty \):
\[ z_t - A_q(z - Dv) + N_qz - P_q[(mu)\tau] = 0, \text{ on } L_q^q(\Omega), \quad v \cdot v = 0 \text{ on } \Gamma \quad (2.18) \]
in factor form on \( L_q^q(\Omega) \). Next, extending the original Oseen operator \( L_q^q(\Omega) \supset D(A_q) \rightarrow L_q^q(\Omega) \) to \( A_{e,q} : L_q^q(\Omega) \rightarrow \left[ D(A_q^*) \right]' \), by isomorphism, we arrive at the definitive abstract model
\[ \{ z_t - A_{e,q}z + N_qz + A_{e,q}Dv - P_q[(mu)\tau] = 0 \text{ on } \left[ D(A_q^*) \right]' \}
\[ z(x, 0) = z_0(x) = y_0(x) - y_e \text{ in } L_q^q(\Omega) \] in additive form, on \( \left[ D(A_q^*) \right]' \).

2.3. Abstract Model of the Linearized \( w \)-Problem (1.28) of the Translated \( z \)-Model (1.27)

Still for \( 1 < q < \infty \), the abstract model (in additive form) of the linearized \( w \)-problem in (1.28) is obtained from (2.19) by dropping the nonlinear term
\[ \{ w_t - A_{e,q}w + A_{e,q}Dv - P_q[(mu)\tau] = 0 \text{ on } \left[ D(A_q^*) \right]' \}
\[ w(x, 0) = w_0(x) = y_0(x) - y_e \text{ in } L_q^q(\Omega). \] (2.20)

2.4. The Adjoint Operators \( D^*, (A_q)^* = A_q^* \) and \( (A_{p,q})^* = A_{o,q}^* \), \( (A_q)^* = A_q^* = -(v_o A_q^* + A_{o,q}^*), \) \( 1 < q < \infty \)

(i) Regarding the Helmholtz projection \( P_q \) and its adjoint \( P_q^* \), we recall from the statement above (1.7) that \( P_q \in \mathcal{L}(L_q^q(\Omega)) \), while \( P_q^* = P_q \in \mathcal{L}(L_q^q(\Omega)), \) \( 1/q + 1/q = 1 \).
(ii) Define as in (2.9)
\[ U_q = \{ g \in L_q^q(\Gamma) : g \cdot v = 0 \text{ on } \Gamma \}. \quad (2.21) \]
We have seen in (2.9) that
\[ D : U_q = \{ g \in L_q^q(\Gamma) : g \cdot v = 0 \text{ on } \Gamma \} \rightarrow W^{1/4,q}(\Omega) \cap L_q^q(\Omega), \] (2.22)
so that the dual \( D^* \) satisfies
\[
D^* : W^{-1/q, q'} \rightarrow L^q(\Gamma). \tag{2.23}
\]

(iii) The adjoint \( A_q^* : L^q_{\sigma} (\Omega) \supset \mathcal{D}(A_q^*) \rightarrow L^q_{\sigma} (\Omega), \ 1/q + 1/q' = 1 \) of the Stokes operator \( A_q \) in (2.14)
\[
\{ A_q f_1, f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ f_1, A_q^* f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} , \quad f_1 \in L^q_{\sigma}, \ f_2 \in L^q_{\sigma}'. \tag{2.24}
\]
(duality pairing \( L^q_{\sigma} \rightarrow L^q_{\sigma}' \) is
\[
A_q^* f_2 = -P_{q'} \Delta f_2 , \mathcal{D}(A_q^*) = W^{2, q'}(\Omega) \cap W^1_{0, q'}(\Omega) \cap L^q_{\sigma}'(\Omega). \tag{2.25}
\]

**Proof.** For \( f_1 \in \mathcal{D}(A_q) \subset L^q_{\sigma} (\Omega) \subset L^q(\Omega), \) so that \( A_q f_1 \in L^q_{\sigma} (\Omega) \) and \( f_2 \in \mathcal{D}(A_q^*) \subset L^q_{\sigma} (\Omega) \subset L^q(\Omega) \) so that \( A_q^* f_2 \in L^q_{\sigma}'(\Omega), \) and \( P_{q'} f_2 = f_2, \) we compute from (2.14): with \( P_{q'} = P_q' \) by the statement above (1.7)
\[
- \{ A_q f_1, f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ P_q \Delta f_1, f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ \Delta f_1, P_q^* f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ \Delta f_1, f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = (2.27)
\]
\[
= \int_{\mathcal{D}} f_1 \Delta f_2 \ d\Omega + \int_{\mathcal{D}} \frac{\partial f_1}{\partial \nu} f_2 \ d\Gamma - \int_{\mathcal{D}} f_1 \frac{\partial f_2}{\partial \nu} \ d\Gamma
\]
\[
= \{ f_1, \Delta f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ P_q f_1, \Delta f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = \{ f_1, \Delta f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} = - \{ f_1, A_q^* f_2 \}_{L^q_{\sigma}, L^q_{\sigma}'} \tag{2.28}
\]
since \( f_1 \in W^1_{0, q'}(\Omega) \) by (2.14); and so \( f_1 \big|_\Gamma = 0; \) and since \( f_2 \in W^1_{0, q'}(\Omega) \) by (2.25) and so \( f_2 \big|_\Gamma = 0. \) Moreover, \( P_q f_1 = f_1, \) since \( f_1 \in L^q_{\sigma}(\Omega) \). Equation (2.28) proves (2.25). \( \square \)

(iv) Similarly from \( A_{o,q} = P_q L_e : \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W^1_{0, q'}(\Omega) \cap L^q_{\sigma}(\Omega) \rightarrow L^q_{\sigma}(\Omega), \) in (2.15), we obtain
\[
(A_{o,q})^* = A_{o,q}^* \ (\text{for short}) = P_{q'} (L_e)^* : W^{-1, q'}(\Omega) \rightarrow L^q_{\sigma}(\Omega) \tag{2.29}
\]
where the expression of \((L_e)^*\), which is not needed, is given in [12, p 55], [52, below (54)], [29].

(v) As a consequence of (ii), (iii) we have \((A_q)^* = A_{q}^* = -(v_0 A_q^* + A_{o,q}^*) \), \( D(A_q^*) = D(A_q^*). \)

2.5. The Operator \( D^* A_q^* \)

**Theorem 2.3.** Let \( 1 < q < \infty. \) Let \( v \in \mathcal{D}(A_q^*) = \mathcal{D}(A_q^*) = W^{2, q'}(\Omega) \cap W^1_{0, q'}(\Omega) \cap L^q_{\sigma}(\Omega) \), by (2.25), \( \frac{1}{q} + \frac{1}{q'} = 1 \) so that \( \frac{\partial v}{\partial \nu} \big|_\Gamma \in W^{1-1/q', q'}(\Gamma) \subset L^q(\Gamma). \) Let \( g \in L^q(\Gamma), g \cdot v = 0 \) on \( \Gamma. \) Then
\[
\{ D^* A_q^* v, g \}_{L^q(\Gamma), L^q(\Gamma)} = v_0 \left[ \frac{\partial v}{\partial \nu}, g \right]_{L^q(\Gamma), L^q(\Gamma)}, \tag{2.30}
\]
where \( q > 3 \) for \( d = 3; \) and \( q > 2 \) for \( d = 2. \)
Proof. We shall first prove (2.30) with \( g \in W^{1-1/q,q}(\Gamma) \), \( g \cdot v = 0 \) on \( \Gamma \); and then extend the validity of (2.30) to \( g \in L^q(\Gamma) \), \( g \cdot v = 0 \) on \( \Gamma \) by density. By (2.16), \( A_q = -(v_o A_q + A_{o,q}) \). Accordingly, we consider \( D^* A_q^* \) in Step 1 and \( D^* A_{o,q}^* \) in Step 2.

Step 1: Let \( v \in \mathcal{D}(A_q^*) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}(\Omega) \), so that \( A_q^* v \in L^{q'}(\Omega) \), and let initially \( g \in W^{1-1/q,q}(\Gamma) \subset L^q(\Gamma) \), \( g \cdot v = 0 \) on \( \Gamma \), so that \( Dg \in W^{1,q}(\Omega) \cap L^q(\Omega) \) by (2.8) with \( \theta = 0 \). Our first step is to show

\[
-D^* A_q^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \int_{\Gamma} v \Delta(Dg) \, d\Gamma + \int_{\Gamma} \frac{\partial v}{\partial \nu} g \, d\Gamma,
\]

(2.31)

where the integral term under \( \Omega \) is well-defined as a duality pairing with \( v \in W^{1,q'}_0(\Omega) \) and \( \Delta(Dg) \in W^{-1,q}(\Omega) \); while the integral term under \( \Gamma \) is well-defined as a duality pairing with \( \frac{\partial v}{\partial \nu} \) in \( L^q'(\Gamma) \) and \( g \in L^q(\Gamma) \).

In fact, we compute—and the computations in (2.32) through (2.34) below actually work even for \( g \in W^{1-1/q,q}(\Gamma) \) so that \( Dg \in L^q(\Omega) \) by (2.8) with \( \theta = 1 \), and hence \( P_q Dg = Dg \)

\[
-D^* A_q^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = -\langle A_q^* v, Dg \rangle_{W^{1,q'}_0(\Omega), W^{q'}(\Omega)}
\]

(by (2.25))

\[
= \langle P_q^* \Delta v, Dg \rangle_{L^{q'}(\Omega), L^q(\Gamma)} = \langle \Delta v, P_q^* Dg \rangle_{L^{q'}(\Omega), L^q(\Gamma)}
\]

(2.33)

\[
= \langle \Delta v, Dg \rangle_{L^q(\Omega), L^q(\Gamma)}
\]

(2.34)

where in going from (2.33) to (2.34) we have recalled \( P_q^* = P_q \) by the statement above (1.7). Next, we apply Green’s theorem in (2.34) and get

\[
-D^* A_q^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \int_{\Omega} \Delta v Dg \, d\Omega - \int_{\Omega} v \Delta(Dg) \, d\Omega
\]

\[
+ \int_{\Gamma} \frac{\partial v}{\partial \nu} g \, d\Gamma - \int_{\Gamma} \frac{\partial Dg}{\partial \nu} \, d\Gamma,
\]

(2.35)

where we have used \( Dg|_{\Gamma} = g \) by definition of \( D \), and \( v|_{\Gamma} = 0 \) as \( v \in W^{1,q'}_0(\Omega) \). Then (2.35) proves (2.31).

Step 2: Let \( v \in \mathcal{D}(A_{o,q}^*) = \mathcal{D}(A_q^{1/2}) = W^{1,q'}_0(\Omega) \cap L^{q'}(\Omega) \) by (2.15) and let \( g \in W^{1-1/q,q}(\Gamma) \) and \( g \cdot v = 0 \) on \( \Gamma \) so that \( Dg \in W^{1,q}(\Omega) \cap L^q(\Omega) \) by (2.8) with \( \theta = 0 \). Recall from Theorem 1.1 that \( y_e \in W^{2,q}(\Omega) \cap W^{1,q}(\Omega) \).

Our second step is to show that

\[
-D^* A_{o,q}^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \langle (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla)y_e, v \rangle_{L^q(\Omega), L^q'(\Omega)}
\]

(2.36)

Proof of (2.36).

Step (2a): Let initially \( h \in \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W^{1,q}_0(\Omega) \cap L^q(\Omega) \) by (2.15). Recalling (2.15) compute

\[
\langle A_{o,q} h, v \rangle_{L^q(\Omega), L^q'(\Omega)} = \langle P_q [(y_e \cdot \nabla)h], v \rangle_{L^q(\Omega), L^q'(\Omega)}
\]
\[ + \{ P_q[(h \cdot \nabla) y_e], v \} \big|_{L^q_\sigma(\Omega), L^{q'}_\sigma(\Omega)} \]  
\[ = \{ [(y_e \cdot \nabla) h], P_q^* v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ \quad + \{ [(h \cdot \nabla) y_e], P_q^* v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ = \{ [(y_e \cdot \nabla) h], P_q^* v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ \quad + \{ [(h \cdot \nabla) y_e], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ = \{ [(y_e \cdot \nabla) h], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ \quad + \{ [(h \cdot \nabla) y_e], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ = \{ [(y_e \cdot \nabla) h], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ \quad + \{ [(h \cdot \nabla) y_e], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
\[ = \{ [(y_e \cdot \nabla) h], v \big|_{L^q_\sigma, L^{q'}_\sigma} \} \]  
where we have recalled \( P_q^* = P_q^* \) from the statement above (1.7) and \( P_q^* v = v \), as \( v \in L^q_{\sigma'}(\Omega) \).

**Step (2b):** In the next lemma, we show that the terms in (2.40) are well-defined in an appropriate range of \( q \), at any rate for \( q > d \), which is our goal, \( d = 2, d = 3 \).

**Lemma 2.4.** With reference to (2.40) we have

(i)
\[ (y_e \cdot \nabla) h \in L^q(\Omega) = W^{0, q}(\Omega) \text{ for } \begin{cases} 
  d = 3, & q > \frac{3}{2} \\
  d = 2, & q > 1
\end{cases} \]  
(2.41)

(ii)
\[ (h \cdot \nabla) y_e \in W^{1, q}(\Omega) \text{ for } \begin{cases} 
  d = 3, & q > 3 \\
  d = 2, & q > 2
\end{cases} \]  
(2.42)

**Proof.** First way: We may use multiplier theory [59, Theorem 3, p 252]. We have by Theorem 1.1 on \( y_e \) and the assumption on \( h \in D(A_{a, q}) = W^{1, q}(\Omega) \cap L^q_\sigma(\Omega) \):

(i)
\[ y_e \in W^{2, q}(\Omega), \ |\nabla h| \in L^q(\Omega) = W^{0, q}(\Omega). \]  
(2.43)

Then [59, Theorem 3, p 252 with \( m = 2 > \ell = 0 \)] yields the multiplier space
\[ M(W^{2, q} \longrightarrow W^{0, q}) = W^{0, q}(\Omega). \]  
(2.44)

for \( mq = 2q > d \) or \( q > \frac{3}{2} \) for \( d = 3 \); \( q > 1 \) for \( d = 2 \); and part (i) of Lemma 2.4 is established.

(ii) We start with
\[ h \in W^{1, q}_0(\Omega), \ |\nabla y_e| \in W^{1, q}(\Omega). \]  
(2.45)

Then [59, Theorem 3, p 252; with \( m = \ell = 1 \)] yields the multiplier space
\[ M(W^{1, q} \longrightarrow W^{1, q}) = W^{1, q}(\Omega). \]  
(2.46)

for \( mq = 1.q > d \) or \( q > 3 \) for \( d = 3 \); \( q > 2 \) for \( d = 2 \) and part (ii) of Lemma 2.4 is established.
Second way: We use embedding theory [44, p 79]

\[ W^{m,q}(\Omega) \hookrightarrow C^k(\overline{\Omega}), \quad m > \frac{d}{q}, \quad k \text{ integer part of } \left[ m - \frac{d}{q} \right] \] (2.47)

Thus (i)

\[ \forall e \in W^{2,q}(\Omega) \hookrightarrow \forall e \in C^0(\overline{\Omega}) \quad \text{for} \quad \begin{cases} m = 2, d = 3, q > 3/2, k = 0 \\ m = 2, d = 2, q > 1, k = 0 \end{cases} \] (2.48)

and since \( |\nabla h| \in L^q(\Omega) \), then

\[ (\forall e \cdot \nabla)h \in L^q(\Omega), \quad d = 3, q > 3/2; \quad \text{or} \quad d = 2, q > 1 \] (2.49)

and (i) of Lemma 2.4 is reproved.

(ii) Similarly, (2.47) give for \( m = 1 \) that

\[ h \in W^{1,q}(\Omega) \hookrightarrow h \in C^0(\overline{\Omega}) \quad \text{for} \quad \begin{cases} d = 3, q > 3, k = 0 \\ d = 2, q > 2, k = 0 \end{cases} \] (2.50)

and since \( |\nabla ye| \in W^{1,q}(\Omega) \), then

\[ (h \cdot \nabla)ye \in W^{1,q}(\Omega), \quad d = 3, q > 3; \quad d = 2, q > 2, \] (2.51)

and (ii) of Lemma 2.4 is reproved.

Lemma 2.4 is proved. \( \square \)

Step (2c): Using Lemma 2.4 in (2.40) we see that the two terms are well-defined with \( v \in L^{q'}(\Omega) \). We rewrite (2.40) as

\[ \{ h, A_{o,q}^* v \}_{L^{q'}_D,L^q_D} = \{ (\forall e \cdot \nabla)h + (h \cdot \nabla)ye, v \}_{L^{q'}_D,L^q_D}, \] (2.52)

which shows that it can be extended to all \( h \in W^{1,q}(\Omega) \cap L^q_D(\Omega) \): the condition \( h|_\Gamma = 0 \) is not used. With \( g \in W^{1-1/q,q}(\Gamma), \quad g \cdot v = 0 \) on \( \Gamma \), so that \( Dg \in W^{1,q}(\Omega) \cap L^q_D(\Omega) \) by (2.9), we may apply such extended version (2.51) to \( Dg \) and obtain

\[ \{ Dg, A_{o,q}^* v \} = \{ g, D^* A_{o,q}^* v \}_{L^{q'}(\Gamma),L^q(\Gamma)} = \{ (\forall e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla)ye, v \}_{L^{q'}(\Omega),L^q(\Omega)} \] (2.53)

and (2.36) is established.

Step 3: In view of \( A_q = -(\nu_0 A_q + A_{o,q}) \) by (2.16), we now combine (2.31) of Step 1, with (2.36) of Step 2. Let again \( g \in W^{1-1/q,q}(\Gamma), \quad g \cdot v = 0 \) on \( \Gamma \), so that \( Dg \in W^{1,q}(\Gamma) \cap L^q_D(\Omega) \) by (2.8) with \( \theta = 0 \), and \( v \in D(A_q^*) = D(A_q^*) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^q_D(\Omega) \) via (2.25). We shall establish the following final relation

\[ \{ D^* A_q^* v, g \} = v_0 \int_\Gamma \frac{\partial v}{\partial \nu} g \, d\Gamma = v_0 \left[ \frac{\partial v}{\partial \nu}, g \right]_{L^q(\Gamma),L^{q'}(\Gamma)}. \] (2.54)
In fact, we start from

\[-A^* q = \nu_0 A^* q + A^* \partial_v q,\]

via part (iv) of Section 2.4 and next recall (2.31) and (2.36) to obtain

\[-\langle D^* A^* q v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \nu_0 \langle D^* A^* q v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} + \langle D G \cdot \nabla \rangle_{L^q'(\Omega), L^q(\Omega)} \]

\[-\langle \nabla \pi^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \nu_0 \langle \nabla \pi^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)},\]

recalling \( L_e(\psi = Dg) = (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla)y_e \) by (1.9). We next invoke the definition \( \psi = Dg \) in Equation (2.1a) of the Stationary Oseen Equation (2.1). This way we rewrite (2.57) as

\[-\langle D^* A^* q v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \nu_0 \langle D^* A^* q v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} + \langle D G \cdot \nabla \rangle_{L^q'(\Omega), L^q(\Omega)} \]

\[-\langle \nabla \pi^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)} = \nu_0 \langle \nabla \pi^* v, g \rangle_{L^q'(\Gamma), L^q(\Gamma)},\]

since

\[\int_\Omega v \cdot \nabla \pi^* = \int_\Gamma \pi^* v \cdot \nu d\Gamma - \int_\Omega \pi^* \text{div } v d\Omega \equiv 0,\]

where \( v|_\Gamma = 0 \) as \( v \in W^1,q_0(\Omega) \) and \( \text{div } v \equiv 0 \) since \( v \in L^q_0(\Omega) \), recall (1.4). Thus, (2.59) shows (2.30) so far for \( g \in W^{1-1/q,q}(\Gamma), \ g \cdot v = 0 \) on \( \Gamma \).

By density, we extend the validity of (2.30) to \( g \in L^q(\Gamma), \ g \cdot v = 0 \) on \( \Gamma \). Theorem 2.3 is proved.

**Proposition 2.5.** [12, Lemma 3.3.1 p35] Let \( \varphi \in C^1(\Omega) \) be a d-function satisfying the following properties:

(i) \( \varphi|_\Gamma = 0;\)

(ii) \( \text{div } \varphi = 0 \) in \( \Omega \) (actually only on an interior collar of \( \Gamma \))

Then we have that

\[
\begin{cases}
\text{the boundary vector } \nabla \varphi \cdot v = \frac{\partial \varphi}{\partial v} \text{ is tangential to } \Gamma \\
i.e. (\nabla \varphi \cdot v) \cdot v \equiv 0 \text{ on } \Gamma.
\end{cases}
\]
For $v \in \mathcal{D}(A^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0 \cap L^{q'}_\sigma(\Omega)$, we have $v|_\Gamma = 0$ and $\text{div} \, v = 0$ in $\Omega$. Thus extending Proposition 2.5 to $v \in W^{2,q'}(\Omega)$, we have $\frac{\partial v}{\partial v}|_\Gamma = \text{tangential}$ on $\Gamma$. Returning to Theorem 2.3, and recalling that $g$ is tangential, $g \cdot v = 0$ on $\Gamma$, we then obtain from (2.30) the following

**Corollary 2.6.** With reference to Theorem 2.3 we have

\[
\begin{align*}
\{ \text{tangential component of } D^* A^*_v \} \\
= (D^* A^*_q v) \tau = v_\sigma \frac{\partial v}{\partial \nu}, \ v \in \mathcal{D}(A^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0 \cap L^{q'}_\sigma(\Omega)
\end{align*}
\tag{2.62}
\]

$q > 3$ for $d = 3$; $q > 2$ for $d = 2$. 

We return to the Dirichlet map $D$ introduced in (2.9), and extract an important result [to be used example in (2.9) to claim that the feedback generator $A_F$ generates a s.c. analytic semigroup in $L^{q}_\sigma(\Omega)$]. All this is a perfect counterpart of results in Hilbert spaces ($q = 2$), which have been used in [53] etc. We first quote a known result.

**Proposition 2.7.** With reference to the Stokes operator $A_q$ in (2.14) on $L^{q}_\sigma(\Omega)$, we have, for $1 < q < \infty$, that

(i) 

\[
W^{s,q}(\Omega) = W^{s,q}_0(\Omega), \quad 0 \leq s \leq \frac{1}{q};
\tag{2.63a}
\]

(ii) 

\[
W^{2s,q}_0(\Omega) \cap L^{q}_\sigma(\Omega) \subset \mathcal{D}(A^*_q), \quad 0 \leq \gamma < s, \quad 0 \leq s \leq 1, \quad q \geq 2,
\]

\[
2s \neq \frac{1}{q}, \quad 2s \neq \frac{1}{q} + 1;
\tag{2.63b}
\]

(iii) In particular, for $\varepsilon > 0$ arbitrary, $q \geq 2$, via (i),

\[
W^{1/q,q}(\Omega) \cap L^{q}_\sigma(\Omega) = W^{1/q,q}_0(\Omega) \cap L^{q}_\sigma(\Omega) \subset \mathcal{D}(A^{1/2q-\varepsilon}_q).
\tag{2.63c}
\]

Indeed, for (i) we invoke [81, Thm 1, Eq (2a), p 318] or [82, (0.2.17) p XX1] which in turn quotes Triebel’s book. For $d = 2$, see [57, Thm 1.1, Eq (11.2), p 55]. For (ii), we quote [82, Theorem III.2.3 p 91] where, in this reference, the space $H^q(\Omega)$ is our space $L^q_\sigma(\Omega)$, and the space $\hat{H}^{2s,q}(\Omega)$ can be replaced (see proof) by the space $W^{2s,q}_0(\Omega)$ in our notation. For (iii), we apply (i) and (ii) with $2s = 1/q$, hence $\gamma < 1/2q$. 

\[\square\]
Corollary 2.8. For the Dirichlet map $D: g \rightarrow \psi$ defined in reference to problem (2.1) and the paragraph below it, we have complementing (2.9) = (2.22)

$$g \in U_q = \{ g \in L^q(\Gamma) ; \ g \cdot \nu = 0 \ on \ \Gamma \}$$

$$\rightarrow Dg \in W^{1/2,q}(\Omega) \cap L^q_\sigma(\Omega) \subset D\left( A^{1/2q-\varepsilon}_q \right)$$

or $A^{1/2q-\varepsilon}_q D \in \mathcal{L}(U_q, L^q_\sigma(\Omega)).$ (2.64a)

We shall invoke this property in Theorem 5.1, (5.7) and Proposition 6.2, (6.14b).

Remark 2.2. As noted in Remark 1.2, The literature reports physical situations where the volumetric force $f$ in (1.1a) or (1.2a), is actually replaced by $\nabla g(x)$; that is, $f$ is a conservative vector field. In this case, a solution to the stationary problem (1.2) is: $y_e \equiv 0, \pi_e = g$. Taking $y_e \equiv 0$ (hence $L_\sigma(\phi) = 0$) and returning to Eq (1.1a) with $f(x)$ replaced now by $\nabla g(x)$ and applying to the resulting equation the projection operator $P_q$, one obtains in this case the projected equation

$$y_t - \nu P_q \Delta y + P_q \left[ (y \cdot \nabla)y \right] = P_q(\mu u) \ in \ Q.$$ (2.65)

This, along with the solenoidal and boundary conditions (1.1b), (1.1c), yields the corresponding abstract form

$$y_t + \nu A_q (y - Dv) + N_q y = P_q(\mu u) \ in \ L^q_\sigma(\Omega).$$ (2.66)

Then $y$-problem (2.66) is the same as the $z$-problem (2.18) or (2.19), except without the Oseen term $A_{o,q}$ see (2.16). The linearized version of problem (2.66) is then

$$\eta_t + \nu A_q (\eta - Dv) = P_q(\mu u) \ in \ L^q_\sigma(\Omega),$$ (2.67)

which is the same as the $w$-problem (2.20), except without the Oseen term $A_{o,q}$ in (1.10). The s.c. analytic semigroup $e^{-\nu A_q t}$ driving the linear equation (2.67) is uniformly stable in $L^q_\sigma(\Omega)$, see (A.4) of Appendix A, as well as in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, (1.15b) with decay rate $-\delta$, (A.9) of Appendix A. Then, in the case of the present Remark and as anticipated in the Orientation, the present paper may be used to enhance at will the uniform stability of the corresponding problem by use only of the tangential boundary feedback finite dimensional control $v = F(y - y_e)$, as acting on the entire boundary $\Gamma$. The operator $F$ is of the form given by (5.6a), with boundary vectors $f_k$ now acting on the entire boundary $\Gamma$. Thus one can take the interior tangential-like control $u \equiv 0$, or vectors $q_k \equiv 0$ in (5.3). Given the original decay rate $-\delta$ of the Stokes semigroup in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ in (A.9) of Appendix A, and a preassigned desirable decay rate $-k^2$ (arbitrary), the procedure of the present paper can be adopted to construct such a tangential boundary finite dimensional feedback control $v$ on all of $\Gamma$ that yields the decay rate $-k^2$. Thus there is no need to perform the translation $y \rightarrow z$ of Section 1.5, when $f$ in (1.2a) is replaced by $\nabla g(x)$; that is $y_e \equiv 0$ in this case. The corresponding required “unique continuation property” holds true for the Stokes problem ($y_e \equiv 0$), see Problem #3 of Appendix C, [79,80].
3. Introducing the Problem of Feedback Stabilization of the Linearized $w$-Problem (2.20) on the Complexified $L^q_\sigma(\Omega)$-Space

**Preliminaries:** In this subsection we take $q$ fixed, $1 < q < \infty$ throughout. Accordingly, to streamline the notation in the preceding setting of Section 2, we shall drop the dependence on $q$ of all relevant quantities and thus write $P, A, A_\phi, A_\varepsilon$ instead of $P_q, A_q, A_{\phi,q}, A_{\varepsilon,q}$. We return to the linearized system (2.20). Moreover, as in [11,12], we shall henceforth let $L^q_\sigma(\Omega)$ denote the complexified space $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$, whereby then we consider the extension of the linearized problem (2.20) to such complexified space. Thus, henceforth, $w$ will mean $w + i\tilde{w}$, $u$ will mean $u + i\tilde{u}$, $v$ will mean $v + i\tilde{v}$, $w_0$ will mean $w_0 + i\tilde{w}_0$. Thus, henceforth, the abstract model (2.20) is rewritten with the same symbols as

$$\tag{3.1} w_t - A_\varepsilon w = -A_\varepsilon Dv + P((mu)\tau) \in \mathcal{D}(A^*)', \quad w(0) \in L^q_\sigma(\Omega), \quad v \cdot v = 0 \text{ on } \Sigma$$

to mean however the complexified version of (2.20). As noted in Theorem A.1(iii), the Oseen operator $A$ has compact resolvent on $L^q_\sigma(\Omega)$. It follows that $A$ has a discrete point spectrum $\sigma(A) = \sigma_p(A)$ consisting of isolated eigenvalues $\{\lambda_j\}_{j=1}^\infty$, which are repeated according to their (finite) geometric multiplicity $\ell_j$. However, since $A$ generates a $C_0$ analytic semigroup on $L^q_\sigma(\Omega)$, Theorem A.1(ii), its eigenvalues $\{\lambda_j\}_{j=1}^\infty$ lie in a triangular sector of a well-known type. We recall the underlying assumption (1.3) of instability of the equilibrium solution $y_\varepsilon$ under consideration, which is the prerequisite for investigating the present uniform stabilization problem. This means that the corresponding Oseen operator $A = A_q$ in (1.11) = (2.16) has a finite number, say $N$, of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ on the complex half plane $\{\lambda \in \mathbb{C} : Re \lambda \geq 0\}$ which we then order according to their real parts, so that (1.3) holds, repeated here for the convenience,

$$\tag{3.2} \ldots \leq Re \lambda_{N+1} < 0 \leq Re \lambda_N \leq \ldots \leq Re \lambda_1,$$

each $\lambda_i$, $i = 1, \ldots, N$ being an unstable eigenvalue repeated according to its geometric multiplicity $\ell_j$. Let $M$ denote the number of distinct unstable eigenvalues $\lambda_j$ of $A$. Denote by $P_N$ and $P^*_N$ the projections given explicitly by [11,42, p 178], [12]

$$\tag{3.3a} P_N = -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} d\lambda : L^q_\sigma(\Omega) \text{ onto } W^u_N \subset L^q_\sigma(\Omega)$$

$$\tag{3.3b} P^*_N = -\frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\lambda I - A^*)^{-1} d\lambda : (L^q_\sigma(\Omega))^* \text{ onto } (W^u_N)^* \subset L^q_\sigma(\Omega),$$

$$\frac{1}{q} + \frac{1}{q'} = 1,$$ recall (1.7), where $\Gamma$ (respectively, its conjugate counterpart $\tilde{\Gamma}$) is a smooth closed curve that separates the unstable spectrum from the stable spectrum of $A$ (respectively, $A^*$).

As in [12, Sect 3.4, p 37], following [76], we decompose the space $L^q_\sigma(\Omega)$ into the sum of two complementary subspaces (not necessarily orthogonal)

$$L^q_\sigma(\Omega) = W^u_N \oplus W^x_N; \quad W^u_N \equiv P_N L^q_\sigma(\Omega); \quad W^x_N \equiv (I - P_N)L^q_\sigma(\Omega);$$
dim $W^u_N = N$, \hspace{1cm} (3.4)

where each of the spaces $W^u_N$ and $W^s_N$ is invariant under $A$, and let

\[ A^u_N = P_NA = A|_{W^u_N}; \quad A^s_N = (I - P_N)A = A|_{W^s_N} \] \hspace{1cm} (3.5)

be the restrictions of $A$ to $W^u_N$ and $W^s_N$, respectively. The original point spectrum (eigenvalues) \( \{ \lambda_j \}_{j=1}^\infty \) of $A$ is then split into two sets

\[ \sigma(A^u_N) = \{ \lambda_j \}_{j=1}^N; \quad \sigma(A^s_N) = \{ \lambda_j \}_{j=N+1}^\infty, \] \hspace{1cm} (3.6)

and $W^u_N$ is the generalized eigenspace of $(A, \text{hence of }) A^u_N$, corresponding to its unstable eigenvalues. The system (3.1) on $L^q_0(\Omega)$ with $v \cdot v = 0$ on $\Sigma$ can accordingly be decomposed as

\[ w = w_N + \zeta_N, \quad w_N = P_N w, \quad \zeta_N = (I - P_N)w. \] \hspace{1cm} (3.7)

After applying $P_N$ and $(I - P_N)$ (which commute with $A$) to (3.1), we obtain via (3.5)

on $W^u_N$: \[ w'_N - A^u_N w_N = -P_N(ADv) + P_N (mu)\tau \]
\[ = -A^u_N P_N Dv + P_N ((mu)\tau); \quad w_N(0) = P_N w_0 \] \hspace{1cm} (3.8a)

on $W^s_N$: \[ \zeta'_N - A^s_N \zeta_N = -(I - P_N)(ADv) + (I - P_N)(mu)\tau \]
\[ = -A^s_N(I - P_N)Dv + (I - P_N)((mu)\tau); \quad \zeta_N(0) = (I - P_N)w_0 \] \hspace{1cm} (3.9a)

respectively. [In (3.8a), (3.8b), actually $P_N$ is the extension from original $L^q_0(\Omega)$ to $[D(A^*')]' [12, Appendix A.1]$. For each distinct $\lambda_i, i = 1, \ldots, M$, let $P_{N,i}, P_{N,i}$ be the projection corresponding to $\lambda_i$ and $\bar{\lambda}_i$, respectively, given by a similar integral of $(\lambda I - A)^{-1}$, or $(\lambda I - A^*)^{-1}$, respectively, as in (3.3a, 3.3b), this time over a curve that encircles only $\lambda_i, \text{ or } \bar{\lambda}_i$, respectively, and no other eigenvalue. Let $(W^u_N)_i = P_{N,i} L^q_0(\Omega)$, and $(A^u_N)_i = A^u|_{(W^u_N)_i}$.

We have that, for $1 < p, q < \infty$,

\[ W^u_N = \left\{ \text{space of generalized eigenfunctions of } A_q (= A^u_N) \right\} \right. \]
\[ \left. \text{corresponding to its distinct unstable eigenvalues} \right\} \subset \left\{ \left(L^q_0(\Omega), D(A_q) \right)_{1 - \frac{1}{p}, p} \right\} \subset \left\{ \left[D(A_q), L^q_0(\Omega) \right]_{1 - \alpha} = D(A^{\alpha}_q), \ 0 < \alpha < 1 \right\} \subset L^q_0(\Omega). \] \hspace{1cm} (3.10)
4. Uniform Stabilization with Arbitrary Decay Rate of the Finite Dimensional $w_N$-Dynamics (3.8) (= (1.28)) by Suitable Finite-Dimensional Tangent-Like Pair $\{v_N, u_N\}$ on $(\tilde{\Gamma}, \omega)$. Constructive Proof with $q \geq 2$

All the main results of this paper, Theorems 4.1 through 9.2, are stated (at first) in the complex state space setting $L^q(\Omega) + i L^q(\Omega)$. Thus, the finitely many stabilizing feedback vectors $p_k \in (W^u_N)^* \subset L^q(\Omega)$, $u_k \in W^u_N \subset L^q(\Omega)$ constructed in the subsequent proofs are related to the complex finite dimensional unstable subspace $W^u_N$. The question then arises as to transfer back these results into the original real setting. This issue was resolved in [11]. Here, the translation, taken from [11], from the results in the complex setting (Theorems 4.1 through 9.2) into corresponding results in the original real setting is given in Section 10. The following is the key desired control theoretic result of the dynamic $w_N$ in (3.8) over the finite dimensional space $W^u_N \subset L^q(\Omega)$. We shall henceforth impose the condition $q \geq 2$, due to requirement (B.5) in Appendix B.

Theorem 4.1. Let $\lambda_1, \ldots, \lambda_M$ be the unstable distinct eigenvalues of the Oseen operator $A (= A_q)$ as in (1.3) = (3.2), with geometric multiplicity $\ell_i$, $i = 1, \ldots, M$, and set $K = \sup \{ \ell_i; i = 1, \ldots, M \}$. Let $\tilde{\Gamma}$ be an open connected subset of the boundary $\Gamma$ of positive surface measure and $\omega$ be a localized collar supported by $\tilde{\Gamma}$ (Fig. 2). Let $q \geq 2$. Given $\gamma_1 > 0$ arbitrarily large, we can construct two $K$-dimensional controllers: a boundary tangential control $v = v_N$ acting with support on $\tilde{\Gamma}$, of the form given by

$$v = v_N = \sum_{k=1}^{K} \nu_k(t) f_k, \quad f_k \in F \subset W^{2-\frac{1}{q}, q}(\Gamma), \quad q \geq 2, \quad \text{so that} \quad f_k \cdot \nu = 0,$$

(4.1)

$F$ defined in (1.25), $q \geq 2$, $f_k$ supported on $\tilde{\Gamma}$, and an interior tangential-like control $u = u_N$ acting on $\omega$, of the form given by

$$u = u_N = \sum_{k=1}^{K} \mu_k(t) u_k, \quad u_k \in W^u_N \subset L^q(\Omega), \quad \mu_k(t) = \text{scalar},$$

(4.2)

thus with interior vectors $[u_1, \ldots, u_K]$ in the smooth subspace $W^u_N$ of $L^q(\Omega)$, $2 \leq q < \infty$, supported on $\omega$, such that, once inserted in the finite dimensional projected $w_N$-system in (3.8), yields the system

$$w'_N - A^u_N w_N = -A^u_N P_N D \left( \sum_{k=1}^{K} \nu_k(t) f_k \right) + P_N P \left( m \left( \sum_{k=1}^{K} \mu_k(t) u_k \right) \tau \right),$$

(4.3)

whose solution then satisfies the estimate

$$\|w_N(t)\|_{L^q(\Omega)} + \|v_N(t)\|_{L^q(\tilde{\Gamma})} + \|v'_N(t)\|_{L^q(\tilde{\Gamma})} + \|u_N(t)\|_{L^q(\omega)} + \|u'_N(t)\|_{L^q(\omega)} \leq C_{\gamma_1} e^{-\gamma_1 t} \|P_N w_0\|_{L^q(\Omega)}, \quad t \geq 0.$$
In (4.4) we may replace the $L^q_0(\Omega)$-norm, $2 \leq q < \infty$, alternatively either with the $(L^q_0(\Omega), \mathcal{D}(A_q))_{1-\frac{1}{p}, p}$ norm, $2 \leq q < \infty$, or else with the $[\mathcal{D}(A_q), L^q_0(\Omega)]_{1-\alpha} = \mathcal{D}(A^u_q)$-norm in (A.5), $0 \leq \alpha \leq 1$, $2 \leq q < \infty$. In particular, we also have

$$
\|w_N(t)\|_{\tilde B^{2-2/p}_{q,p}(\Omega)} + \|v_N(t)\|_{L^q(\tilde \Gamma)} + \|u'_N(t)\|_{L^q(\tilde \Gamma)}
$$

$$
+ \|u_N(t)\|_{\tilde B^{2-2/p}_{q,p}(\omega)} + \|u'_N(t)\|_{\tilde B^{2-2/p}_{q,p}(\omega)} \leq C \gamma_1 e^{-\gamma_1 t} \|P_N w_0\|_{\tilde B^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0,
$$

(4.5)

in the $\tilde B^{2-2/p}_{q,p}(\Omega)$-norm, $2 \leq q < \infty$, $p < 2q/2q-1$. [Estimate (4.4) (in the weaker form (5.20), that is without the derivative terms) will be invoked in the stabilization proof of Section 5.3.]

Moreover, such controllers $v = v_N$ and $u = u_N$ may be chosen in feedback form: that is, with references to the explicit expressions (4.1) for $v$ and (4.2) for $u$, of the form $v_k(t) = \{w_N(t), p_k\}_{W^u_N}$ and $u_k(t) = \{w_N(t), q_k\}_{W^u_N}$ for suitable vectors

$p_k \in (W^u_N)^* \subset L^q_0(\Omega), q_k \in (W^u_N)^* \subset L^{q'}_0(\Omega)$ depending on $\gamma_1$, where $\{ \, , \}$ denotes the duality pairing $W^*_N \times (W^u_N)^*$.

In conclusion, $w_N$ in (4.5) is the solution of the equation (4.3) on $W^u_N$ rewritten explicitly as

$$
w'_N - A^u_N w_N = -A^u_N P_N D \left( \sum_{k=1}^{K} \{w_N(t), p_k\}_{W^u_N} f_k \right)
$$

$$
+ P_N P \left( m \left( \sum_{i=1}^{K} \{w_N(t), q_k\}_{W^u_N} u_k \right) \right), \quad (4.6)
$$

$f_k$ supported on $\tilde \Gamma$, $u_k$ supported on $\omega$, rewritten in turn as

$$
w'_N = A^u_N w_N, \quad w_N(t) = e^{-A^u_N t} P_N w_0, \quad w_N(0) = P_N w_0 \quad \text{on } W^u_N. \quad (4.7)
$$

**Proof.** A (lengthy, technical) proof is given in [53] for $q = 2$. As the present Theorem 4.1 is the preliminary pillar upon which the present paper rests, we need to provide more insight. In the present general case $W^u_N$ is the space of generalized eigenfunctions of $A_q (= A^u_q)$ corresponding to its unstable eigenvalues, see (3.2). For $i = 1, \ldots, M$, we now denote by $\{\varphi_{ij}\}_{j=1}^{\ell_i}$, $\{\varphi_{ij}^*\}_{j=1}^{\ell_i}$ the (normalized) linearly independent (on $L^q_0(\Omega)$) eigenfunctions corresponding to the (assumed unstable) distinct eigenvalues $\lambda_1, \ldots, \lambda_M$ of $A_q (= A^u_q)$ and $\tilde \lambda_1, \ldots, \tilde \lambda_M$ of $A^* (= A^*_q)$, respectively:

$$
A_q \varphi_{ij} = \lambda_i \varphi_{ij} \in \mathcal{D}(A_q) = W^2,q(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_0(\Omega) \subset L^q_0(\Omega),
$$

$$
A^* \varphi_{ij}^* = \tilde \lambda_i \varphi_{ij}^* \in \mathcal{D}(A^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_0(\Omega) \subset L^{q'}_0(\Omega). \quad (4.8a)
$$

The eigenvectors $\varphi_{ij}$ and $\varphi_{ij}^*$ are in $\mathcal{D}(A^u_q)$ and $\mathcal{D}((A^*_q)^n)$, for any $n$, hence they are arbitrarily smooth in $L^q_0(\Omega)$ and $L^{q'}_0(\Omega)$, respectively. For our purposes, it
will suffice to take \( q \geq 2 \), hence \( q' = 2, \frac{1}{q} + \frac{1}{q'} = 1 \), and view eigenvectors henceforth as follows, see Appendix B, Eq (B.5)

\[
\varphi_{ij} \in W^{3,q}(\Omega) \cap L^q_\sigma(\Omega); \quad \varphi_{ij}^* \in W^{3,q}(\Omega) \cap L^q_\sigma(\Omega).
\] (4.8b)

Hence, for \( i = 1, \ldots, M \), we may view \( \varphi_{ij} \) and \( \varphi_{ij}^* \) as elements of the generalized eigenspace \( W^u_N \) in (3.10) and its dual \( (W^u_N)^* \), corresponding to the unstable eigenvalues as in (3.2). For \( h_1 \in W^u_N, h_2 \in (W^u_N)^* \), we set \( \langle h_1, h_2 \rangle_{W^u_N} = \int_\Omega h_1 h_2 \, d\Omega \), as a duality pairing.

Step 1: The challenging key step of the proof in [53] consists in showing that the \( \nu \)th matrix \( f \) will suffice to take \( 1610 \), Irena Lasiecka, Buddhika Priyasad & Roberto Triggiani

values as in (3.2). For \( W \), we introduce the following

\[
\begin{align*}
\ell &= (4.10) \\
\end{align*}
\]

Step 1: The challenging key step of the proof in [53] consists in showing that the \( N \)-dimensional \( w_N \)-problem (3.8) is controllable in \( W^u_N \) by using a finite dimensional pair \( \{ v, u \} \) of localized tangential controllers, in particular in feedback form. To this end, we introduce the following \( \ell_i \times K \) matrix \( W_i, i = 1, \ldots, M \):

\[
W_i = \begin{bmatrix}
(f_1, \partial_v \varphi_{i1}^* | r) \tilde{r}, & \cdots, (f_K, \partial_v \varphi_{i1}^* | r) \tilde{r} \\
(f_1, \partial_v \varphi_{i2}^* | r) \tilde{r}, & \cdots, (f_K, \partial_v \varphi_{i2}^* | r) \tilde{r} \\
\vdots & \ddots & \ddots & \ddots \\
(f_1, \partial_v \varphi_{i\ell_i}^* | r) \tilde{r}, & \cdots, (f_K, \partial_v \varphi_{i\ell_i}^* | r) \tilde{r}
\end{bmatrix}; \quad \ell_i \times K;
\]

as well as the \( \ell_i \times K \) matrix \( U_i, i = 1, \ldots, M; K \geq \ell_i, i = 1, \ldots, M \):

\[
U_i = \begin{bmatrix}
\{ u_1, \varphi_{i1}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i1}^* \cdot \tau \}_\omega \\
\{ u_1, \varphi_{i2}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i2}^* \cdot \tau \}_\omega \\
\vdots & \ddots & \ddots & \ddots \\
\{ u_1, \varphi_{i\ell_i}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i\ell_i}^* \cdot \tau \}_\omega \\
\{ \cdots \}_\omega = \{ \cdots \}_{L^q(\tilde{r}), L^q'(\tilde{r})}
\end{bmatrix}; \quad \ell_i \times K;
\]

\[
\begin{align*}
\partial_v &= \frac{\partial}{\partial v}, \quad (\cdots) \tilde{r} = (\cdots)_{L^q(\tilde{r}), L^q'(\tilde{r})}.
\end{align*}
\] (4.9)

The following is the main result of the present section—verification of the corresponding Kalman controllability criterion:

**Theorem 4.2.** With reference to (4.9), (4.10), it is possible to select boundary vectors \( f_1, \ldots, f_K \) in \( \mathcal{F} \subset W^{2-1/q,q}(\tilde{\Gamma}), \mathcal{F} \) defined in (1.25) with support on \( \tilde{\Gamma} \), and interior vectors \( u_1, \ldots, u_K \in L^q(\omega), K = \sup \{ \ell_i, i = 1 \ldots M \} \), such that for the matrix \( [-v_0 W_i U_i] \) of size \( \ell_i \times 2\ell_i, \) we have

\[
\text{rank } [-v_0 W_i U_i] = \text{full } = \ell_i, \quad i = 1, \ldots, M.
\] (4.11a)

In fact, explicitly and more precisely, for each \( i = 1, \ldots, M \), we have via (4.9), (4.10):

\[
\begin{align*}
\text{rank } &\begin{bmatrix}
(f_1, \partial_v \varphi_{i1}^* | r) \tilde{r}, & \cdots, (f_{\ell_i}, \partial_v \varphi_{i1}^* | r) \tilde{r} \\
(f_1, \partial_v \varphi_{i2}^* | r) \tilde{r}, & \cdots, (f_{\ell_i}, \partial_v \varphi_{i2}^* | r) \tilde{r} \\
\vdots & \ddots & \ddots & \ddots \\
(f_1, \partial_v \varphi_{i\ell_i}^* | r) \tilde{r}, & \cdots, (f_{\ell_i}, \partial_v \varphi_{i\ell_i}^* | r) \tilde{r}
\end{bmatrix}; \quad \ell_i, \\
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\{ u_1, \varphi_{i1}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i1}^* \cdot \tau \}_\omega \\
\{ u_1, \varphi_{i2}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i2}^* \cdot \tau \}_\omega \\
\vdots & \ddots & \ddots & \ddots \\
\{ u_1, \varphi_{i\ell_i}^* \cdot \tau \}_\omega, & \cdots, & \{ u_K, \varphi_{i\ell_i}^* \cdot \tau \}_\omega
\end{bmatrix}.
\end{align*}
\] (4.10)
where the matrix in (4.11b) is $\ell_i \times 2\ell_i$ and the boundary terms are only evaluated on $\tilde{\Gamma}$.

Step 2: Verification of the above algebraic rank conditions of Kalman and Hautus style rests critically on the following unique continuation property for the adjoint of the Oseen eigenvalue problem [53].

**Lemma 4.3.** Let $\tilde{\lambda}$ be an unstable eigenvalue of the adjoint Oseen operator, as in 4.8a. Let $(\varphi^*, p^*) \in W^{2,q}(\Omega) \cap W^{1,q}(\Omega)$ where

\[
\begin{align*}
-\nu_o \Delta \varphi^* - (L_e)^*(\varphi^*) + \nabla p^* &= \tilde{\lambda} \varphi^* \quad \text{in } \Omega; \quad (4.12a) \\
\text{div } \varphi^* &\equiv 0 \quad \text{in } \Omega; \quad (4.12b) \\
\varphi^*|_{\tilde{\Gamma}} &= 0; \quad \frac{\partial \varphi^*}{\partial \nu}|_{\tilde{\Gamma}} = 0; \quad \varphi^* \cdot \tau = 0 \text{ in } \omega; \quad (4.12c) \\
(L_e)^*(\varphi^*) &= (y_e \cdot \nabla)\varphi^* + (\varphi^* \cdot \nabla)^* y_e, \quad (4.13)
\end{align*}
\]

Then

$$\varphi^* \equiv 0, \text{ and } p^* \equiv \text{const} \quad \text{in } \Omega. \quad (4.14)$$

This result is equivalent to the UCP in the Problem #2, Appendix C: (C.6a, b, c) $\Rightarrow$ (C.7), of which it is the adjoint version. It is proved in [53, Lemma 6.2]. If one omits the over-determination $\varphi^* \cdot \tau \equiv 0$ in $\omega$ in (4.12c), then the corresponding UCP is false. This is documented in Appendix C, Problem #1, in view of the counterexample in [26]. Thus, it is Lemma 4.3 that justifies the necessity of using localized, tangential-like interior control $u$ on $\omega$, in addition to the localized boundary, even tangential, boundary control $v$ on $\tilde{\Gamma}$. Relying only on $v$ (that is $u \equiv 0$) would not establish the required controllability of the $N$-dimensional $w_N$-problem (3.8).

Step 2: Having established the controllability condition for the $N$-dimensional $w_N$-problem (3.8), then by the well-known Popov’s criterion in the finite-dimensional theory [[85, p44] under the name of complete stabilization] allows us to obtain the stabilizing controls in feedback form, thus completing the proof of Theorem 4.1. Details are in [53].

5. Global Well-Posedness and Uniform Exponential Stabilization of the Linearized $w$-Problem (3.8) ($= (1.28)$) in $L^q_\sigma(\Omega)$, $q \geq 2$, and $\tilde{B}_{q,p}^{2-2/p}(\Omega)$ by Means of the Same Feedback Controls $\{v, u\}$ Obtained for the $w_N$-Problem in Section 4

5.1. The Operator $A^{\text{f},q}$ Defining the Linearized $w$-Problem in Feedback Form

Let $q \geq 2$. Consider the same K-dimensional feedback controllers constructed in Theorem 4.1 and yielding estimate (4.4), (4.5) for the finite-dimensional pro-
jected \( w_N \)-system (3.8) in feedback form (4.6); that is, the tangential boundary controller \( v = v_N \) supported on \( \tilde{\Gamma} \), and the tangential-like interior controller \( u = u_N \) supported on \( \omega \)

\[
v(t) = \sum_{k=1}^{K} v_k(t) f_k, \quad f_k \in \mathcal{F} \subset W^{2-1/q,q}(\Gamma),
\]

\[p_k \in (W_N^w)^* \subset L_\sigma^q(\Omega), \quad q \geq 2\]

\[f_k \cdot v|_{\Gamma} = 0; \text{ hence } v \cdot v|_{\Gamma} = 0, \text{ } f_k \text{ supported on } \tilde{\Gamma} \] (5.1)

\[
u(t) = \sum_{k=1}^{K} u_k(t) q_k \in (W_N^w)^* \subset L_\sigma^q(\Omega), \text{ } u_k \text{ supported on } \omega. \] (5.2)

Once inserted, this time, in the full linear \( w \)-problem (1.28) or (2.20) = (3.1), such \( v \) and \( u \) in (5.1), (5.2) yield the linearized feedback dynamics \( w_N = P_N w \) driven by the dynamical feedback stabilizing operator \( A_{F,q} \) below

\[
\frac{dw}{dt} = A_{\epsilon,q}w - A_{\epsilon,q}D \left( \sum_{k=1}^{K} \langle w_N(t), p_k \rangle_{w_N^w} f_k \right)
\]

\[
+ P_q \left( m \left( \sum_{k=1}^{K} \langle w_N(t), q_k \rangle_{w_N^w} u_k \right) \tau \right) \equiv A_{F,q}w. \] (5.3)

More specifically, \( A_{F,q} \) is rewritten as in the subsequent Section 6, Equations (6.2), (6.9a) as

\[
A_{F,q} = A_{q}(I - DF) : L_\sigma^q(\Omega) \supset D(A_{F,q}) \to L_\sigma^q(\Omega), \quad q \geq 2 \] (5.4a)

\[
D(A_{F,q}) = \{ h \in L_\sigma^q(\Omega) : h - DFh \in D(A_{q}) \}
\]

\[
= W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L_\sigma^q(\Omega) = D(A_{F,q}) \] (5.4b)

\[
F(\cdot) = \sum_{k=1}^{K} \langle P_N \cdot, p_k \rangle_{w_N^w} f_k \in W^{2-1/q,q}(\tilde{\Gamma});
\]

\[
G(\cdot) = \langle P_q \left( m \left( \sum_{k=1}^{K} \langle P_N \cdot, q_k \rangle_{w_N^w} u_k \right) \tau \right) \in L_\sigma^q(\Omega) \] (5.6a)

\[
F \in \mathcal{L}(L_\sigma^q(\Omega), L_\sigma^q(\tilde{\Gamma})); \quad G \in \mathcal{L}(L_\sigma^q(\Omega)), \quad q \geq 2. \] (5.6b)
5.2. The Feedback Operator $A_{F,q}$ in (5.3) Generates a s.c Analytic Semigroup in $L^q_\sigma(\Omega)$, $2 \leq q < \infty$ or in $B^2_{q,p}(\Omega)$, $1 < p < \frac{2q}{2q-1}$, $q > d$, $d = 2, 3$

**Theorem 5.1.** Let $q \geq 2$. With reference to the feedback operator $A_{F,q}$ in (5.4) describing the feedback $w$-system in (5.3), with $D(\hat{A}_{F,q}) = D(A_{F,q}) = \{ \varphi \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega) : \varphi |_{\Gamma} = F \varphi \}$ we have: $\hat{A}_{F,q}$ generates a s.c. analytic semigroup $e^{\hat{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$, $t > 0$, $q \geq 2$; $A_{F,q}$ has a compact resolvent on $L^q_\sigma(\Omega)$, $q \geq 2$.

**Proof.** For $q \geq 2$ the finite dimensional feedback operators $F : L^q_\sigma(\Omega) \rightarrow L^q(\Gamma)$ and $G$ on $L^q_\sigma(\Omega)$ are bounded. This in turn, is due to Appendix B, in particular Eq (B.5): $\varphi_{ij} \in W^{3,q}(\Omega)$, $q \geq 2$ and

$$\frac{\partial \varphi_{ij}}{\partial \nu} |_{\Gamma} \in W^{2-1/q,q}(\Gamma) \subset L^q(\Gamma), \quad \mathcal{F} \subset W^{2-1/q,q}(\Gamma).$$

Thus, it suffices to consider the operator $A_{F,q}$, below in (5.5a), which differs from $\hat{A}_{F,q}$ by the bounded operator $G$. We give two proofs. First proof (after [B-L-T.1] $q = 2$): We shall critically use property (2.64) for the Dirichlet map $D$ in the $L^q$-setting, $1 < q < \infty$, just as it was done in these references in the Hilbert setting; namely that, with $\epsilon > 0$, recalling (2.64b) of Corollary 2.8, we have:

$$D : \text{continuous } U_q = \{ g \in L^q(\Omega) = W^{0,q}(\Omega), \ g \cdot v = 0 \text{ on } \Gamma \}$$

$$\longrightarrow W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset D\left(A^{1/2q-\epsilon}_q\right). \quad (5.7)$$

$1 < q < \infty$, where $A_q$ is the Stokes operator in (1.8). We next transfer relation (5.7) to the Oseen operator $A_q$ in (1.11). To do this, we just translate it. Let $k > 0$ be suitably large, then, via (5.7)

$$A^{1/2q-\epsilon}_q D = (kI - A_q)^{1/2q-\epsilon}$$

$$D : \text{continuous } \{ g \in L^q(\Gamma), \ g \cdot v = 0 \text{ on } \Gamma \} \longrightarrow L^q_\sigma(\Omega) \quad (5.8)$$

Both elements—that $A_{F,q}$ generates a s.c. analytic semigroup on $L^q_\sigma(\Omega)$ and has compact resolvent on it—rely on the perturbation formula [60] written for $A_{F,q}$ in (5.5a)

$$R(\lambda, A_{F,q}) = [I + R(\lambda, A_q)A_qDF]^{-1}R(\lambda, A_q) \quad (5.9)$$

where by property (5.8) $A^{1/2q-\epsilon}_q DF \in \mathcal{L}(L^q_\sigma(\Omega))$. Moreover, since $A_q$ generates a s.c. analytic semigroup in $L^q_\sigma(\Omega)$ (Theorem A.1.ii of Appendix A), a well-known formula [60] gives for $\epsilon > 0$, $\theta = 1 - \frac{1}{2q} - \epsilon$

$$\left\| R(\lambda, A_q)A^{\theta}_q \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq \frac{C}{|\lambda|^{1-\theta}} = \frac{C}{|\lambda|^{1/2q+\epsilon}} \rightarrow 0 \text{ as } |\lambda| \rightarrow \infty. \quad (5.10)$$
Then, (5.10) in (5.9) yields
\[ \| R(\lambda, A_{F,q}) \|_{L^q(\Omega)} \leq C_{\rho_0} \| R(\lambda, A_q) \|_{L^q(\Omega)}, \quad \forall \lambda, \ |\lambda| \geq \text{some } \rho_0 > 0 \]
(5.11)
and hence, via (5.11) the properties of \( R(\lambda, A_q) \) of Theorem A.1 [generation of s.c. analytic semigroup on \( L^q(\Omega) \) and, respectively, compact resolvent] transfer into corresponding properties for \( R(\lambda, A_{F,q}) \). Theorem 5.1 is proved.

**Analyticity: Second Proof:** One may provide a second proof that \( A_{F,q} \) generates a s.c. analytic semigroup on \( L^q(\Omega), q \geq 2 \). This is still a perturbation argument, however perturbation of an original analytic generator, not of the resolvent. In fact, the present perturbation argument applies to the adjoint operator \( A_{F,q}^* \) on \( L^{q'}(\Omega), \) not to \( A_{F,q} \) on \( L^q(\Omega), 1 < q' \leq 2, 2 \leq q. \) Equations (6.13), (6.14) in the argument below of Proposition 6.2 dealing with \( \tilde{A}_{F,q} \) show that \( \tilde{A}_{F,q}^* \) can be written as \( \tilde{A}_{F,q}^* = -A_q^* + \Pi, \) where the perturbation \( \Pi \) is \( (A_q^*)^\theta \)-bounded, with \( \theta = 1 - 1/2q + \varepsilon < 1, \) see (6.17). Thus, since \(-A_q^* \) generates s.c. analytic semigroup on \( L^q(\Omega) \) (by adjointness on Theorem A.1(i) on \(-A_q)), \) then a standard semigroup result [60] implies that the perturbed operator \( \tilde{A}_{F,q}^* \) is an analytic semigroup generator on \( L^q(\Omega), 1 < q' \leq 2. \) But this is equivalent (\( L^q(\Omega) \) being reflexive, \( 1 < q < \infty \)) to the original operator \( \tilde{A}_{F,q} \) being an analytic semigroup generator on \( L^q(\Omega), q \geq 2, \) as desired. The quoted Proposition 6.2 shows more. In fact: that is, that \( A_{F,q} \) (actually \( \tilde{A}_{F,q} = A_{F,q} + G \)) has \( L^p \)-maximal regularity on \( L^q(\Omega), \) in symbols, \( A_{F,q} \in M\text{Reg}(L^p(0, \infty; L^q(\Omega))), q \geq 2 (6.18). \) And maximal regularity implies analyticity [24]. The argument of Proposition 6.2 is of the perturbation type described above, however tuned to the notion of maximal regularity, which is stronger than analyticity.

We next extend Theorem 5.1 to the Besov space \( \tilde{B}^{2-2/q}_{q,p}(\Omega) \) in (1.15b) \( 1 < p < \frac{2q}{2q-1}, q \geq 2, \) of interest. To this end, we need the following result.

**Proposition 5.2.** Let \( 1 < p < \frac{2q}{2q-1}, q \geq 2. \) Then
\[ \left( L^q(\Omega), D(\tilde{A}_{F,q}) \right)_{1-1/p, p} = \tilde{B}^{2-2/q}_{q,p}(\Omega) \]
(5.12)
\[ = \left\{ g \in \tilde{B}^{2-2/q}_{q,p}(\Omega) : \text{div } g \equiv 0, \ g \cdot \nu|_{\Gamma} = 0 \right\}. \]
(5.13)

**Remark 5.1.** This formula should be compared with the original definition of \( \tilde{B}^{2-2/q}_{q,p}(\Omega) \) in (1.15b).

**Proof.** Step 1: From the characterization of \( D(\tilde{A}_{F,q}) = D(A_{F,q}) \) in (5.5b) we obtain for \( 0 < \theta < 1, p > 1, q \geq 2 \)
\[ \left( L^q(\Omega), D(\tilde{A}_{F,q}) \right)_{\theta, p} \subset \left( L^q(\Omega), W^{2,q}(\Omega) \cap L^q(\Omega) \right)_{\theta, p} = B^{2q}_{q,p}(\Omega) \cap L^q(\Omega) \]
recalling the definition/characterization (1.12) of $B^s_{q,p}(\Omega)$ with $m = 2, \frac{s}{2} = \theta$. Next we take $1 < p < \frac{2d}{2q - 1}, q \geq 2, \theta = 1 - \frac{1}{p}$, so that—for these parameters—(5.14) specializes to

$$\left( L^q_{\partial}(\Omega), \mathcal{D}(\tilde{A}_{F,q}) \right)_{1 - \frac{1}{p}, p} \subset \tilde{B}^{2 - \frac{\gamma}{p}}_{q,p}(\Omega) = \text{ defined in (1.15b)} \quad (5.15)$$

**Step 2:** But $\tilde{B}^{2 - \frac{\gamma}{p}}_{q,p}(\Omega)$ does not recognize boundary conditions [the conditions $\text{div } \frac{g}{g} = 0, g \cdot \nu |_{\partial} = 0$ are included in the definition of the underlying space $L^q_{\partial}(\Omega)$, see Remark 1.4]. Hence, the space in the LHS of (5.15) does not recognize boundary conditions. Thus (5.16) proves the desired conclusion (5.12).

**Theorem 5.3.** The operator $\tilde{A}_{F,q}$ in (5.4), where the bounded operators $F$ and $G$ are defined by (5.6), generates a s.c. analytic semigroup $e^{\tilde{A}_{F,q} t}$ on the Besov space $\tilde{B}^{2 - \frac{\gamma}{p}}_{q,p}(\Omega), 1 < p < \frac{2d}{2q - 1}, q \geq 2$ defined in (5.13).

**Proof.** The operator $\tilde{A}_{F,q}$ generates a s.c. analytic semigroup $e^{\tilde{A}_{F,q} t}$ on $L^q_{\partial}(\Omega)$ by Theorem 5.1 for $q \geq 2$. Then, it generates a s.c. analytic semigroup on $\mathcal{D}(\tilde{A}_{F,q})$. Hence the conclusion follows by (5.12).

5.3. The Analytic Semigroup $e^{\tilde{A}_{F,q} t}$ is Uniformly Stable on $L^q_{\partial}(\Omega)$ and $\tilde{B}^{2 - \frac{\gamma}{p}}_{q,p}(\Omega)$

**Theorem 5.4.** With vectors $f_k, p_k, q_k, u_k$ as selected in Theorem 4.1, the s.c. analytic semigroup $e^{\tilde{A}_{F,q} t}$ defining the feedback $w$-dynamics in (5.3) is uniformly stable with decay rate $\gamma_0 > 0$ in both the space $L^q_{\partial}(\Omega), 2 \leq q < \infty$, as well as in the space $\left( L^q_{\partial}(\Omega), \mathcal{D}(A_q) \right)_{1 - \frac{1}{p}, p}, 2 \leq q < \infty$; in particular, in the space $\tilde{B}^{2 - \frac{\gamma}{p}}_{q,p}(\Omega), 2 \leq q < \infty, 1 < p < \frac{2d}{2q - 1}$: there exists $C_{\gamma_0} > 0$ such that

$$\left\| e^{\tilde{A}_{F,q} t} w_0 \right\|_{L^q_{\partial}(\Omega)} = \left\| w(t; w_0) \right\|_{L^q_{\partial}(\Omega)} \leq C_{\gamma_0} e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_{\partial}(\Omega)} \, , \, t \geq 0, q \geq 2$$

or for $0 < \theta < 1, \delta > 0$ arbitrarily small, $q \geq 2$

$$\left\| A^\theta_q e^{\tilde{A}_{F,q} t} w_0 \right\|_{L^q_{\partial}(\Omega)} = \left\| A^\theta_q w(t; w_0) \right\|_{L^q_{\partial}(\Omega)} = \begin{cases} C_{\gamma_0, \theta} e^{-\gamma_0 t} \left\| A^\theta_q w_0 \right\|_{L^q_{\partial}(\Omega)} \, , \, t \geq 0, w_0 \in (5.18) \text{n} \, , \\ C_{\gamma_0, \theta, \delta} e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_{\partial}(\Omega)} \, , \, t \geq \delta > 0. \quad (5.18b) \end{cases}$$
Proof. On $L^q_\sigma(\Omega)$ A proof of (5.17) and (5.18) on the Hilbert space $H$ in (1.6b) (that is $q = 2$) is given in [53, Lemma 2.3]. Essentially, the same proof works on $L^q_\sigma(\Omega)$, using of course the estimate (4.4) now:

\[
\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|v_N(t)\|_{L^q(\tilde{\Gamma})} + \|u_N(t)\|_{L^q(\omega)} \leq C\gamma_0 e^{-\gamma_0 t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \ t \geq 0, \ q \geq 2 \tag{5.20}
\]

((4.4) also includes $v'_N$ and $u'_N$). We just sketch the strategy.

Next one examines the impact of such constructive feedback control pair $\{v_N, u_N \cdot \tau\}$ on the $\zeta_N$-dynamics (3.9), whose explicit solution is given by the variation of parameter formula

\[
\zeta_N(t) = e^{A_N t} \zeta_N(0) + (I_{\text{int}})(t) + (I_{\text{bry}})(t); \tag{5.21}
\]

\[
\left\|e^{A_N t}\right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C\gamma_0 e^{-\gamma_0 t}, \ 0 \leq t, \ 0 < \gamma_0 < |\text{Re} \lambda_{N+1}|; \tag{5.22}
\]

\[
(I_{\text{int}})(t) = - \int_0^t e^{A_N (t-r)} (I - P_N) P(m(u_N(r) \cdot \tau(r))) dr; \tag{5.23}
\]

\[
(I_{\text{bry}})(t) = - \int_0^t e^{A_N (t-r)} A_N^* (I - P_N) D v_N(r) dr; \tag{5.24}
\]

Here, $I_{\text{int}}$ is the integral term driven by the interior control $u_N$, while $I_{\text{bry}}$ is the integral term driven by the tangential boundary control $v_N$. We omit the details. See [53].

On $\tilde{B}^{2-2/p}_{q,p}(\Omega)$. The proof is similar using the fact that, by (A.3) of Appendix A, the s.c. analytic semigroup of the Oseen operator $e^{A_q t}$, once restricted on the stable subspace $W^s_N$, has the property

\[
e^{A_N q t}: \text{continuous} \ \tilde{B}^{2-2/p}_{q,p}(\Omega) \to X^\infty_{p,q,\sigma}, \tag{5.25}
\]

\[
\left\|e^{A_N q t}\right\|_{\mathcal{L}(\tilde{B}^{2-2/p}_{q,p}(\Omega))} \leq C e^{-\gamma_0 t}, \ t \geq 0 \]

counterpart of (5.21). We now repeat the above proof, except on the space $\tilde{B}^{2-2/p}_{q,p}(\Omega)$ rather than $L^q_\sigma(\Omega)$, by using (5.24) instead of (5.21). \hfill \Box
6. Maximal $L^p$-Regularity of $\hat{A}_{F,q}$ on $L^q_\sigma (\Omega)$. $q \geq 2$ up to $T = \infty$. Action of the s.c. Analytic Semigroup $e^{\hat{A}_{F,q} t}$ on $\tilde{B}^{2-2/p}_{q,p} (\Omega) \rightarrow X^\infty_{p,q,\sigma} (\hat{A}_{F,q})$

Preliminaries

1. We recall the tangential boundary feedback operator $F \in \mathcal{L} (L^q_\sigma (\Omega), L^q (\tilde{\Gamma}))$, for $q \geq 2$ and the interior tangential-like feedback operator $G \in \mathcal{L} (L^q_\sigma (\Omega))$ from (5.6), $q \geq 2$.

$$F(\cdot) = \sum_{k=1}^K \langle p_N \cdot, p_k \rangle_{W^q_N} f_k \in W^{2-1/q,q} (\tilde{\Gamma}); \quad G(\cdot) = P_q \left( m \left( \sum_{k=1}^K \langle p_N \cdot, q_k \rangle_{W^q_N} u_k \right) \tau \right) \in L^q_\sigma (\Omega),$$

with vectors $f_k, p_k, q_k, u_k$ chosen as in Theorem 4.1, so that we rewrite the feedback $w$-equation (5.3) in factor form as

$$\frac{dw}{dt} = A_q (I - DF) w + G w = \hat{A}_{F,q} w \quad (6.2a)$$

$$\begin{align*}
\hat{A}_{F,q} : L^q_\sigma (\Omega) &\supset \mathcal{D}(\hat{A}_{F,q}) \rightarrow L^q_\sigma (\Omega), \quad q \geq 2 \\
\mathcal{D}(\hat{A}_{F,q}) &\mathcal{D}(\hat{A}_{F,q}) = \{ h \in L^q_\sigma (\Omega) : (h - DFh) \in \mathcal{D}(A_q) = \mathcal{D}(A_q) \}, \quad (6.2b)
\end{align*}$$

since $G$ is a bounded operator $G \in \mathcal{L} (L^q_\sigma (\Omega))$. Recall from (1.25), (4.1), ultimately Appendix B, (B.7) that the boundary vectors $f_k$ (= linear combinations of normal traces of eigenfunctions of $A^* = A^*_q$, the adjoint of the Oseen operator (1.11), have the regularity $f_k \in W^{2-1/q,q} (\Gamma)$, so that $DFh \in W^{2,q} (\Omega) \cap L^q_\sigma (\Omega)$ for $h \in L^q_\sigma (\Omega), \quad q \geq 2$; in light of Corollary B.2(v) in Appendix B. Thus, we can more specifically describe $\mathcal{D}(\hat{A}_{F,q})$ as follows:

$$\begin{align*}
\hat{A}_{F,q} : L^q_\sigma (\Omega) &\supset \mathcal{D}(\hat{A}_{F,q}) \rightarrow L^q_\sigma (\Omega), \\
\mathcal{D}(\hat{A}_{F,q}) &\mathcal{D}(\hat{A}_{F,q}) = \{ \varphi \in W^{2,q} (\Omega) \cap L^q_\sigma (\Omega) : \varphi|_{\Gamma} = F \varphi \}, \quad q \geq 2; \quad (6.2c)
\end{align*}$$

see (5.4). Such characterization of $\mathcal{D}(\hat{A}_{F,q})$ will be critical in using maximal $L^p$ regularity of $\hat{A}_{F,q}$ in the analysis of the non-linear problem in Sections 7 and 8.

We also recall that $e^{\hat{A}_{F,q} t}$ is a s.c. analytic semigroup on $L^q_\sigma (\Omega)$ and on $\tilde{B}^{2-2/p}_{q,p} (\Omega)$ (Theorem 5.1), which, moreover, is uniformly stable here (Theorem 5.4, Equations (5.17), (5.19)):

$$\left\| e^{\hat{A}_{F,q} t} \right\|_{\mathcal{L}(L^q_\sigma (\Omega))} \leq C_{\gamma_0} e^{-\gamma_0 t}, \quad \left\| e^{\hat{A}_{F,q} t} \right\|_{\mathcal{L}(\tilde{B}^{2-2/p}_{q,p} (\Omega))} \leq C_{\gamma_0} e^{-\gamma_0 t}, \quad t \geq 0, \quad q \geq 2. \quad (6.3)$$
2. We consider the system

\[
\frac{d\eta}{dt} = \mathcal{A}_{F,q}\eta + f; \quad \eta(0) = \eta_0 \text{ in } L^q_\sigma(\Omega), \quad q \geq 2 \tag{6.4}
\]

\[
eta(t) = e^{\mathcal{A}_{F,q}t}\eta_0 + \int_0^t e^{\mathcal{A}_{F,q}(t-\tau)}f(\tau)\,d\tau. \tag{6.5}
\]

**Goal:** The goal of the present section is to establish maximal $L^p$ regularity on $L^q_\sigma(\Omega)$ and for $T = \infty$ of the feedback analytic generator $\mathcal{A}_{F,q}$ in (6.2), as described in the following result.

**Theorem 6.1.** Let $q \geq 2$. With reference to the dynamics (6.4), (6.5) with $\eta_0 = 0$, we have: the map

\[
f \mapsto \eta(t) = \int_0^t e^{\mathcal{A}_{F,q}(t-\tau)}f(\tau)\,d\tau \quad \text{continuous} \tag{6.6a}
\]

\[
L^p(0, \infty; L^q_\sigma(\Omega)) \hookrightarrow L^p(0, \infty; D(\mathcal{A}_{F,q})), \quad 1 < p < \infty, \tag{6.6b}
\]

\[
L^p(0, \infty; L^q_\sigma(\Omega)) \hookrightarrow X^\infty_{p,q,\sigma}(\mathcal{A}_{F,q}) \equiv L^p(0, \infty; D(\mathcal{A}_{F,q})) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega)), \quad q \geq 2 \tag{6.6c}
\]

by (6.2c), so that, there exists a constant $C = C_{p,q} > 0$ such that

\[
\|\eta_t\|_{L^p(0,\infty; L^q_\sigma(\Omega))} + \|\mathcal{A}_{F,q}\eta\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \leq C \|f\|_{L^p(0,\infty; L^q_\sigma(\Omega))}. \tag{6.7a}
\]

In short:

\[
\mathcal{A}_{F,q} \in M\text{Reg}(L^p(0, \infty; L^q_\sigma(\Omega))) \tag{6.7b}
\]

If we introduce the space of maximal regularity for \{\eta, \eta_t\}, with $\eta_0 = 0$, as

\[
X^\infty_{p,q,\sigma}(\mathcal{A}_{F,q}) \equiv L^p(0, \infty; D(\mathcal{A}_{F,q})) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega)) \tag{6.8a}
\]

\[
\subset X^\infty_{p,q} = L^p(0, \infty; W^{2,q}(\Omega)) \cap W^{1,p}(0, \infty; L^q(\Omega)), \tag{6.8b}
\]

we rewrite (6.7) as

\[
f \in L^p(0, \infty; L^q_\sigma(\Omega)) \quad \eta \in X^\infty_{p,q,\sigma}(\mathcal{A}_{F,q}) \quad C([0, \infty); B^{2-2/p}_{q,p}(\Omega)) \tag{6.8c}
\]

where to justify the continuous embedding in (6.8c), we recall [3, Theorem 4.10.2; p180] and the characterization (6.2c) for $D(\mathcal{A}_{F,q})$

**Proof.** Step 1: The operator $\mathcal{A}_{F,q}$ has a fractional power of the order of $\left(1 - \frac{1}{2q} + \epsilon\right)$ that is not absorbed by the Dirichlet map $D$ on the space $L^q_\sigma(\Omega)$, see (2.63c) of Proposition 2.7. In view of this, because of the intrinsic presence of the operator $DF$ (boundary feedback $F$ followed by the Dirichlet map $D$) as a right factor in

\[
\mathcal{A}_{F,q} = \mathcal{A}(I - DF) + G = (-\nu_o A_q - A_{\sigma,q})(I - DF) + G \tag{6.9a}
\]
By duality on Theorem 5.1 on a reflexive Banach space, the operator 
\[ q = (I - DF)^* A^*_q + G^* = -(I - DF)^* A^*_{q,0} - (I - DF)^* A^*_q + G^* \]  
(6.10a)

\[ \mathcal{D}(A^*_q) = \mathcal{D}(A^*_q) = \mathcal{D}(A^*_q) = \{ h \in W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^q(\Omega) \} \]  
(6.10b)

\[ A^*_q : L^q(\Omega) \supset \mathcal{D}(A^*_q) \rightarrow L^q(\Omega), \quad 1 < q' \leq 2 \]  
(6.10c)

Here \( \frac{1}{q} + \frac{1}{q'} = 1 \), where \( q \geq 2 \) for \( A^*_q \). Recall from (5.6b) that \( F \) is bounded \( L^q(\Omega) \rightarrow L^q(\Gamma) \) for \( q \geq 2 \). Thus we need to impose \( 1 < q' \leq 2 \), to have 
\[ (I - DF)^* \in \mathcal{L}(L^q_{\sigma}(\Omega)), \quad 1 < q' \leq 2 \], see Appendix B, Equation (B.11). We rewrite \( A^*_q \) in (6.10a) as 
\[ A^*_q = -A^*_q + \left[ F^* D^* A^*_{q/2q - \varepsilon} \right] A^*_{q^{1/2q} - \varepsilon} - \left[ (I - DF)^* \left( A^{-1/2}_{q/2} A_{q,0} \right)^* \right] A^*_{q^{1/2}} + G^* \]  
(6.11)

\[ : L^q(\Omega) \supset \mathcal{D}(A^*_q) \rightarrow L^q(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1, q \geq 2, 1 < q' \leq 2, \]  
(6.12)

whereby the adjoint of the right factor becomes now a left factor. In obtaining in (6.10a) the form of \( A^*_q \) from that of \( A^*_q \) in (6.9a), we have used that \( (I - DF) \in \mathcal{L}(L^q_{\sigma}(\Omega)) \) \( q \geq 2 \) [28, p 14]. Moreover, to go from (6.9) to (6.11), we use 
\[ A_{q,0} = A^{1/2}_{q/2} A^{-1/2}_{q,0}, \]  
(6.12)

Step 2: By duality on Theorem 5.1 on a reflexive Banach space, the operator \( A^*_q \) in (6.10) generates a s.c. analytic semigroup \( e^{A^*_q t} \) on \( L^q_{\sigma}(\Omega) \), which moreover is uniformly stable by Theorem 5.4 in \( \mathcal{L}(L^q_{\sigma}(\Omega)) \), \( 1 < q' \leq 2 \), with the same decay rate \( \gamma_0 > 0 \) in (6.3) = (5.17) as \( e^{A_{F,q} t} \) in \( \mathcal{L}(L^q_{\sigma}(\Omega)) \), \( q \geq 2 \) in Theorem 5.4.

Step 3:

**Proposition 6.2.** For the generator \( A^*_q \) in (6.10) of a s.c. analytic, uniformly bounded semigroup \( e^{A^*_q t} \) on \( L^q_{\sigma}(\Omega) \), we have: \( A^*_q \in M \text{Reg}(L^p(0, \infty; L^q_{\sigma}(\Omega))) \), \( 1 < q' \leq 2 \).

**Proof.** The proof is based on a perturbation argument. For \( q \geq 2 \), rewrite (6.11) as 
\[ A^*_q = -A^*_q + \Pi \]  
(6.13)

\[ \Pi = \left[ F^* D^* A^*_{q/2q - \varepsilon} \right] A^*_{q^{1/2q} - \varepsilon} \]
We now prove Theorem 6.1 that

\[-\left[(I - DF)^*(A_q^{-1/2} A_{\alpha,q})^*\right]A_q^{1/2} + G^*.\]  \hspace{1cm} (6.14a)

In (6.14), both terms in square brackets \([ \ )\] are bounded in \(L^q_q(\Omega)\), and so is \(G^*\), \(1 < q' \leq 2\). To this end we use critically and recall (2.64b):

\[A_q^{1/2 - \varepsilon} D \in \mathcal{L}(U_q, L^q_q(\Omega)), \text{ so } D^* A_q^{1/2 - \varepsilon} \in \mathcal{L}(L^q_q(\Omega), L^q_q(\Gamma)), 1 \leq q' \leq 2:\]  \hspace{1cm} (6.14b)

while \(A_q^{1/2} A_{\alpha,q} \in \mathcal{L}(L^q_q(\Omega))\) by (1.10) or (2.15).

The following estimates then hold, \(q \geq 2, 1 < q' \leq 2:\)

\[\begin{align*}
\text{i.} & \quad \left\| \left[ F^* D^* A_q^{1/2 - \varepsilon} \right] A_q^{1 - 1/2q + \varepsilon} x \right\|_{L^q_q(\Omega)} \leq C_q \left\| A_q^{1 - 1/2q + \varepsilon} x \right\|_{L^q_q(\Omega)}, \\
& \quad \forall x \in \mathcal{D}(A_q^{1 - 1/2q + \varepsilon}) \hspace{1cm} (6.15) \\
\text{ii.} & \quad \left\| \left[(I - DF)^* (A_q^{-1/2} A_{\alpha,q})^*\right] A_q^{1/2} x \right\|_{L^q_q(\Omega)} \leq C_q \left\| A_q^{1/2} x \right\|_{L^q_q(\Omega)}, \hspace{1cm} (6.16)
\end{align*}\]

Hence, the perturbation \(\Pi\) in (6.14) satisfies \(q \geq 2, 1 < q' \leq 2:\)

\[
\Pi x \bigg|_{L^q_q(\Omega)} \leq C_q \left\| A_q^{1 - 1/2q + \varepsilon} x \right\|_{L^q_q(\Omega)}, \quad x \in \mathcal{D}(A_q^{1 - 1/2q + \varepsilon}), \hspace{1cm} (6.17)
\]

\(1/q + 1/q' = 1, \varepsilon > 0\). We now draw the following consequences from (6.13), (6.17):

(a) The perturbation operator \(\Pi\) is \(A_q^{\ast \theta}\)-bounded on \(L^q_q(\Omega)\) with \(\theta = 1 - 1/2q + \varepsilon < 1, 1 < q' \leq 2 \leq q\).

(b) On the other hand \(A_q^{\ast} \in MReg(L^p(0, \infty; L^q_q(\Omega)))\), from Appendix A(d), in particular Theorem A.4. In fact, while \(A_q\) is the Stokes operator on \(L^q_q(\Omega)\), \(1 < q < \infty, A_q^{\ast}\) is the Stokes operator on \(L^q_q(\Omega)\) by (2.25), \(1/q + 1/q' = 1\). Then, properties (a), (b) imply -by the abstract perturbation theorem in the [50, Appendix B], see also [24,47, Theorem 6.2. p 311] and [48, SNP Remark 1i, p 426 for \(\beta = 1\)], for related results, that \(\mathring{A}_{F,q}^{\ast} \in MReg(L^p(0, \infty; L^q_q(\Omega)))\), \(1 < q' \leq 2\) and Proposition 6.2 is proved.

\(\Box\)

Step 4: We now prove Theorem 6.1 that \(\mathring{A}_{F,q}\) satisfies the maximal \(L^p\) regularity on \(L^q_q(\Omega)\):

\[
\mathring{A}_{F,q} \in MReg\left(L^p(0, \infty; L^q_q(\Omega))\right), \quad 2 \leq q < \infty. \hspace{1cm} (6.18)
\]

Step 4.i: We invoke the fundamental result of L. Weis [48, Theorem 1.11 p 76], [84, Theorem p 198]. Since \(\mathring{A}_{F,q}\) generates a bounded analytic semigroup \(e^{\mathring{A}_{F,q}}t\) on
Let $L^q_\sigma(\Omega)$, $2 \leq q < \infty$, on a UMD-space [48, p 75], then the sought-after property that $\mathbb{A}^*_{F,q} \in MReg\left(L^p(0, \infty; L^q_\sigma(\Omega))\right)$ is equivalent to the property that the family $\tau \in \mathcal{L}(L^q_\sigma(\Omega))$

$$\tau = \left\{ tR(it, \mathbb{A}^*_{F,q}), t \in \mathbb{R}\setminus\{0\} \right\} \text{ be } R\text{-bounded}$$  \hspace{1cm} (6.19)

Step 4.ii: By the complete duality for $R$-boundedness on $L^q(\Omega)$, $2 \leq q < \infty$, we have [48, Corollary 2.11 p90] that the family $\tau$ in (6.19) is $R$-bounded if and only if the corresponding dual family $\tau'$ in $\mathcal{L}(L^q_\sigma(\Omega))$, $(L^q_\sigma(\Omega))^* = L^q_\sigma(\Omega))$ by (1.7)

$$\tau' = \left\{ tR(it, \mathbb{A}^*_{F,q}), t \in \mathbb{R}\setminus\{0\} \right\} \text{ is } R\text{-bounded}$$  \hspace{1cm} (6.20)

(The equivalence between the $R$-boundedness condition (6.19) and its adjoint version (6.20) is true more generally in a UMD space [40, Proposition 8.4.1, p 211] for K-convex spaces, combined with [40, Ex 7.4.8, p 133] stating that a UMD space is K-convex.)

Step 4.ii: But the $R$-boundedness property in (6.20) is equivalent, by the same result [48] to the property that $\mathbb{A}^*_{F,q} \in MReg\left(L^p(0, \infty; L^q_\sigma(\Omega))\right)$, $1 < q' \leq 2$, $\frac{1}{q} + \frac{1}{q'} = 1$, and this is true by Proposition 6.2. In conclusion: $\mathbb{A}^*_{F,q} \in MReg\left(L^p(0, \infty; L^q_\sigma(\Omega))\right)$, and Theorem 6.1 is proved. \hfill $\Box$

We next examine the regularity of the term $e^{\mathbb{A}^*_{F,q}t}\eta_0$ due to the initial condition $\eta_0$ in (6.5) in $\tilde{B}^{2-2/p}_{q',p}(\Omega)$. For the same reasons noted in the Theorem 6.1, Equations (6.10) through (6.12), we shall equivalently examine the regularity of the adjoint semigroup $e^{\mathbb{A}^*_{F,q}t}$. To this end, we need the counterpart of Proposition 5.2 this time for the adjoint/dual operator $\mathbb{A}^*_{F,q}$ on $L^q_\sigma(\Omega)$, $1 < q' \leq 2$, $\frac{1}{q} + \frac{1}{q'} = 1$.

**Proposition 6.3.** Let $1 < p < \frac{2q'}{2q' - 1}$, $1 < q' \leq 2$, $q \geq 2$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then

$$\left(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\right)_{1/p, p} = \tilde{B}^{2-2/p}_{q', p}(\Omega)$$  \hspace{1cm} (6.21)

$$= \left\{ g \in \tilde{B}^{2-2/p}_{q', p}(\Omega) : \text{div } g \equiv 0, \text{ div } \mathbf{v} \mid_\Gamma = 0 \right\}$$  \hspace{1cm} (6.22)

**Proof.** From (6.10b) we have: $\mathcal{D}(\mathbb{A}^*_{F,q}) \subset \tilde{W}^{2,q'}(\Omega) \cap L^q_\sigma(\Omega)$, $1 < q' \leq 2$. Thus, as in (5.14)

$$\left(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\right)_{\theta, p} \subset \left(L^q_\sigma(\Omega), \tilde{W}^{2,q'}(\Omega) \cap L^q_\sigma(\Omega)\right)_{\theta, p}$$  \hspace{1cm} (6.23)

replacing the definition (1.14). Next we take $1 < p < \frac{2q'}{2q' - 1}$, $\theta = 1 - 1/p$, so that, for these parameters, (6.23) specializes to

$$\left(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\right)_{1/p, p} \subset \tilde{B}^{2-2/p}_{q', p}(\Omega) = \text{ defined in (1.15b)}.$$  \hspace{1cm} (6.24)
But $\tilde{B}_{q',p}^{2-2/p}(\Omega)$ does not recognize the boundary conditions so neither does the space on the LHS of (6.24). Thus for these parameters $\theta = 1 - \frac{1}{p}$, with $1 < p < 2q'/2q'-1$, we have

$$
(L_{q'}^{q}(\Omega), \mathcal{D}(A_{\sigma,F,q}^{*}))_{1-\frac{1}{p},p} = (L_{q'}^{q}(\Omega), \mathcal{D}(A_{\sigma,F,q}^{*}))_{1-\frac{1}{p},p} = \mathcal{B}_{q',p}^{2-2/p}(\Omega) \tag{6.25}
$$

recalling (2.25) ($A_{q}^{*}$ is the Stokes operator on $L_{q}^{q'}(\Omega)$) and (1.15b) as $\mathcal{D}(A_{\sigma,F,q}^{*})$ and $\mathcal{D}(A_{q}^{*})$ both consist of $W^{2,q'}(\Omega) \cap L_{q}^{q'}(\Omega)$ functions, subject only to possibly, different boundary conditions. Thus (6.26) proves the desired conclusion. \qed

We conclude this section with results for the semigroup $e^{A_{\sigma,F,q}^{*}t}$ that yield the solution in the space $X_{p,q,\sigma}(A_{\sigma,F,q})$ of maximal regularity for $\tilde{A}_{\sigma,F,q}$ up to $T = \infty$. This is the companion result of Theorem 6.1. It is done by duality on the adjoint semigroup $e^{A_{\sigma,F,q}^{*}t}$ as in the proof of Theorem 6.1.

**Theorem 6.4.** (i) Let $1 < p < 2q'/2q'-1$, $1 < q' \leq 2$, $q \geq 2$, $1/q + 1/q' = 1$. Consider the adjoint s.c. analytic semigroup $e^{A_{\sigma,F,q}^{*}t}$ on $\mathcal{B}_{q',p}^{2-2/p}(\Omega)$ (see (6.26)) which is uniformly stable here, by duality on (5.19) of Theorem 5.4. Then

$$
e^{A_{\sigma,F,q}^{*}t} : \text{continuous } \mathcal{B}_{q',p}^{2-2/p}(\Omega)

\rightarrow X_{p,q',\sigma}(A_{\sigma,F,q}^{*}) \equiv L^{p}(0, \infty; \mathcal{D}(A_{\sigma,F,q}^{*})) \cap W^{1,p}(0, \infty; L_{q}^{q'}(\Omega)) \tag{6.28a}

\leftarrow C \left[0, \infty; \tilde{B}_{q',p}^{2-2/p}(\Omega)\right] \tag{6.28b}

$$

recalling the embedding in (6.8c) [3, Theorem 4.10.2, p 180] in the range $1 < p < 2q'/2q'-1$, where moreover by (6.10b)

$$
\mathcal{D}(A_{\sigma,F,q}^{*}) = \mathcal{D}(A_{q}^{*}) = W^{2,q'}(\Omega) \cap W_{0}^{1,q'}(\Omega) \cap L_{\sigma}^{q'}(\Omega).

$$

(ii) Consider now the original s.c. analytic feedback semigroup $e^{A_{\sigma,F,q}t}$ on $\mathcal{B}_{q',p}^{2-2/p}(\Omega)$, which is uniformly stable here by (5.19). Let $1 < p < 2q'/2q'-1$, $q \geq 2$. Then, see (5.16)

$$
e^{A_{\sigma,F,q}t} : \text{continuous } \mathcal{B}_{q',p}^{2-2/p}(\Omega)

\rightarrow X_{p,q,\sigma}(A_{\sigma,F,q}) = L^{p}(0, \infty; \mathcal{D}(A_{\sigma,F,q})) \cap W^{1,p}(0, \infty; L_{\sigma}^{q}(\Omega)). \tag{6.30a}

$$
recalling the embedding in \((6.8c) [3, \text{Theorem 4.10.2, p 180}]\) in the range \(1 < p < \frac{2q}{2q-1}\), where \(\mathcal{D}(A_{F,q})\) is defined in \((6.2b)\) or \((6.2c)\).

**Proof.** We shall prove (i) and then (ii) will follow by duality.

**Step 1:** Thus, consider \(A_{F,q}^*\) in \(\widetilde{B}_{q',p}^{2-2/p}(\Omega)\), \(1 < q' \leq 2\). Write

\[
\chi(t) = e^{A_{F,q}^* t} \chi_o \in \mathcal{D}(A_{F,q}^*), \quad t > 0, \quad \chi_t = A_{F,q}^* \chi, \quad \chi(0) = \chi_o. \tag{6.31}
\]

Recalling \(A_{F,q}^* = -A_q^* + B_1 A_q^{1-1/2q+e} + B_2 A_q^{1/2} + G^*\) from \((6.11)\), where \(B_1, B_2\) are in \(\mathcal{L}(L_q^q(\Omega))\), we rewrite the equation in \((6.31)\) as

\[
\chi_t = -A_q^* \chi + B_1 A_q^{1-1/2q+e} \chi + B_2 A_q^{1/2} \chi + G^* \chi \tag{6.32}
\]

whose solution is

\[
\chi(t) = e^{-A_q^* t} \chi_o + \int_0^t e^{-A_q^* (t-\tau)} B_1 A_q^{1-1/2q+e} \chi(\tau) \, d\tau \\
+ \int_0^t e^{-A_q^* (t-\tau)} B_2 A_q^{1/2} \chi(\tau) \, d\tau \\
+ \int_0^t e^{-A_q^* (t-\tau)} G^* \chi(\tau) \, d\tau. \tag{6.33}
\]

Hence apply \(A_q^*\) throughout,

\[
A_q^* \chi(t) = A_q^* e^{-A_q^* t} \chi_o \\
+ A_q^* \int_0^t e^{-A_q^* (t-\tau)} B_1 A_q^{1-1/2q+e} \chi(\tau) \, d\tau \\
+ A_q^* \int_0^t e^{-A_q^* (t-\tau)} B_2 A_q^{1/2} \chi(\tau) \, d\tau \\
+ A_q^* \int_0^t e^{-A_q^* (t-\tau)} G^* \chi(\tau) \, d\tau. \tag{6.34}
\]

**Step 2:** We now recall from \((2.25)\) that \(A_q^*\) is nothing but the Stokes operator on the space \(L_q^q(\Omega)\). Thus, \(A_q^*\) enjoys the maximal regularity properties stated for \(A_q\) in Appendix A(d), in particular Theorem A.4 except on \((L_q^q(\Omega))^' = L_{q'}^q(\Omega)\), see \((1.7)\). We shall use these for each of the four terms of the RHS of \((6.34)\).

**First Term:** By use of estimate \((A.20b)\), or \((A.17)\) in Appendix A, we obtain changing \(q\) into \(q'\)

\[
\left\| A_q^* e^{-A_q^* t} \chi_o \right\|_{L_p^{q'}(0,\infty; L_{q'}^q(\Omega))} \leq C \left\| \chi_o \right\|_{\widetilde{B}_{q',p}^{2-2/p}(\Omega)}. \tag{6.35}
\]
Second Term: Again by the maximal regularity property of $A_q^*$ in (A.19), except in $L^{q'}_\sigma(\Omega)$, we estimate since $B_1 \in \mathcal{L}(L^{q'}_\sigma(\Omega))$

$$\left\|A_q^* \int_0^\tau e^{-A_q^*(\cdot-\tau)} B_1 A_q^{1-\frac{1}{2q}+\varepsilon} \chi(\tau) \, d\tau \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq C \left\|B_1 A_q^{1-\frac{1}{2q}+\varepsilon} \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq \tilde{C} \left\|A_q^{1-\frac{1}{2q}+\varepsilon} \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} + C\varepsilon_1 \left\|\chi\right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \quad (6.36)$$

after using an interpolation inequality [81, Thm, P59, Eq (3)], [1, p 74–75] as in [50, Appendix B], to go across (6.37).

Third Term: Similarly, since $B_2 \in \mathcal{L}(L^{q'}_\sigma(\Omega))$, via (A.17):

$$\left\|A_q^* \int_0^\tau e^{-A_q^*(\cdot-\tau)} B_2 A_q^{1/2} \chi(\tau) \, d\tau \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq C \left\|B_2 A_q^{1/2} \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq \tilde{C} \left\|A_q^{1/2} \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq \varepsilon_2 \left\|A_q^* \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} + C\varepsilon_2 \left\|\chi\right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \quad (6.38)$$

Fourth Term: Finally, since $G^* \in \mathcal{L}(L^{q'}_\sigma(\Omega))$, via (A.17)

$$\left\|A_q^* \int_0^\tau e^{-A_q^*(\cdot-\tau)} G^* \chi(\tau) \, d\tau \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq C \left\|G^* \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \quad (6.39)$$

Invoking (6.35), (6.37), (6.38), (6.39) in (6.34), we obtain

$$\left\|A_q^* \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq C \left\|\chi_0\right\|_{\tilde{B}^{2-2/p}_{q',p} (\Omega)} + (\varepsilon_1 + \varepsilon_2) \left\|A_q^* \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} + \tilde{C} \left\|\chi\right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \quad (6.40)$$

from which we obtain $1 < q' \leq 2$, $q \geq 2$, $1 < p < 2q'/2q'-1$, with $\varepsilon_1 + \varepsilon_2$ small,

$$\left\|A_q^* \chi \right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \leq C \left\|\chi_0\right\|_{\tilde{B}^{2-2/p}_{q',p} (\Omega)} + \tilde{C} \left\|\chi\right\|_{L^p(0,\infty;L^{q'}_\sigma(\Omega))} \quad (6.41)$$

Step 3: By returning to (6.31) with $\chi_0 \in \tilde{B}^{2-2/p}_{q',p} (\Omega)$, since $e^{Ah_q^*}$ is a.s. semigroup, uniformly stable in such space $\tilde{B}^{2-2/p}_{q',p} (\Omega) \subset L^{q'}_\sigma(\Omega)$, see (5.19) with $q$ replaced
by \( q' \), we obtain a fortiori
\[
\left\{ \begin{array}{l}
\chi_o \in \widetilde{B}_{q', p}^{2-2/p}(\Omega) \\
\| \chi \|_{L^p(0, \infty; L^q_{-1}(\Omega))} \leq C \| \chi_o \|_{\widetilde{B}_{q', p}^{2-2/p}(\Omega)}.
\end{array} \right.
\] (6.42a)

Substituting (6.42b) in (6.41) yields the desired estimate, see (6.9a), (6.10a).
\[
\| \tilde{A}_{F, q}^* \chi \|_{L^p(0, \infty; L^q_{-1}(\Omega))} = \| A_{q}^* \chi \|_{L^p(0, \infty; L^q_{-1}(\Omega))} \leq C \| \chi_o \|_{\widetilde{B}_{q', p}^{2-2/p}(\Omega)} (6.43)
\]

continuously see (6.10b). Consequently, (6.44a) gives
\[
e^{-\tilde{H}_{F, q} t} : \text{continuous } \widetilde{B}_{q', p}^{2-2/p}(\Omega) \rightarrow X_{q', p, o}^∞(\tilde{A}_{F, q}^*)
\]
\[
= L^p(0, \infty; D(\tilde{A}_{F, q}^*)) \cap W^{1,p}(0, \infty; L^q(\Omega))
\]
\[
= L^p(0, \infty; \tilde{W}^{2,q'}(\Omega)) \cap W^{1,q}(0, \infty; L^q(\Omega)) (6.45)
\]

Thus (6.45) shows part (i) for \( e^{-\tilde{H}_{F, q} t} \), based on \( \widetilde{B}_{q', p}^{2-2/p}(\Omega) \). As noted, part (ii) then follows by duality. Theorem 6.4 is proved. \( \square \)

7. Local Well-Posedness of the Translated nonlinear \( z \)-Problem (1.27) or (2.18) by Means of a Finite Dimensional, Tangential-like Feedback Control Pair \( \{v, u\} \) on \( \{\tilde{\Gamma}, \omega\} \). Case \( d = 3, q > 3 \)

Starting with the present section, the nonlinearity of problem (1.1) will impose for \( d = 3 \) the requirement \( q > 3 \), while \( q > 2 \) for \( d = 2 \), see (7.18) below. As our deliberate goal is to obtain the stabilization result in the space \( \widetilde{B}_{q, p}^{2-2/p}(\Omega) \) defined by (1.15b), which does not recognize boundary conditions (Remark 1.4), then the limitation \( p < 2q/2q-1 \), of this space applies. In conclusion, our well-posedness and stabilization results will hold under the restriction \( q > 3, 1 < p < 6/5 \) for \( d = 3 \); \( q > 2, 1 < p < 4/3 \) for \( d = 2 \). As throughout this paper, \( \tilde{\Gamma} \) is an open connected subset of the boundary \( \Gamma \) of positive surface measure and \( \omega \) in a localized collar, supported by \( \tilde{\Gamma} \) (Fig. 2).

Consider the nonlinear \( z \)-problem (1.27) or (2.18) in the following feedback form in the notation of (5.6):
\[
\frac{dz}{dr} - A_q(I - DF)z + N'_q z - Gz = 0; \quad z_0 = z(0) \] (7.1a)
in factor form, corresponding to (5.3) in additive form, explicitly
\[
\frac{dz}{dt} - A_q \left[ z - D \left( \sum_{k=1}^{K} \langle P_N z, p_k \rangle_{w_N^p} f_k \right) \right]
+ N_q z = P_q \left( m \left( \sum_{k=1}^{K} \langle P_N z, q_k \rangle_{w_N^p} u_k \right) \right); \quad z_0 = z(0) \tag{7.1b}
\]
that is subject to a feedback controls of the same structure as in the linear \(w\)-dynamics (5.3) of Theorem 5.1,
\[
v = F z = \sum_{k=1}^{K} \langle P_N z, p_k \rangle_{w_N^p} f_k, \quad u = G z = P_q \left( m \left( \sum_{k=1}^{K} \langle P_N z, q_k \rangle_{w_N^p} u_k \right) \right).
\tag{7.2}
\]
Here \(p_k, q_k, f_k, u_k\) are the same vectors as those constructed in Theorem 4.1, and appearing in (4.6), (5.1)–(5.3); \(f_k\) supported on \(\tilde{\Gamma}\), \(u_k\) supported on \(\omega\). Recalling from (5.4), (5.5) the feedback generator \(A_{F,q}\), we can rewrite (7.1a) as
\[
z_t = A_{F,q} z - N_q z; \quad z(0) = z_0 \tag{7.3}
\]
whose variation of parameters formula is
\[
z(t) = e^{A_{F,q} t} z_0 - \int_0^t e^{A_{F,q}(t-\tau)} N_q z(\tau) d\tau. \tag{7.4}
\]

**Theorem 7.1.** (Well-posedness) Let \(d = 3\), \(1 < p < \frac{6}{5}\) and \(q > 3\) (in order to satisfy the requirement \(p < \frac{2q}{2q-1}\)). There exists a positive constant \(r_1 > 0\) (identified in the proof below in (7.26)), such that if the initial condition \(z_0\) satisfies
\[
\|z_0\|_{B^{2,2}_{-\frac{2}{p}}(\Omega)} < r_1, \tag{7.5}
\]
then problem (7.3) defines a unique solution \(z\) in the space (see (6.8a)–(6.8c))
\[
X_{p,q,\sigma}^{\infty} (A_{F,q}) \equiv L^p(0, \infty; D(A_{F,q})) \cap W^{1,p}(0, \infty; L_q^q(\Omega)) \hookrightarrow C([0, \infty); B^{2,2}_{-\frac{2}{p}}(\Omega)) \tag{7.6}
\]
of maximal regularity of the operator \(A_{F,q}\). More precisely the operator \(F_q\) introduced below in (7.9) has a unique fixed point on \(X_{p,q,\sigma}^{\infty} (A_{F,q})\)
\[
F_q(z_0, z) = z, \text{ or } z(t) = e^{A_{F,q} t} z_0 - \int_0^t e^{A_{F,q}(t-\tau)} N_q z(\tau) d\tau \tag{7.7}
\]
which therefore is the unique solution of problem (7.4) = (7.1) in \(X_{p,q,\sigma}^{\infty} (A_{F,q})\), and defines a non-linear semigroup solution on \(C([0, \infty); B^{2,2}_{-\frac{2}{p}}(\Omega))\).
Proof. The proof will be critically based on the maximal regularity property of $A_{F,q}$ Section 6. We already know from (5.19) of Theorem 5.4. or (6.3) that for $z_0 \in \tilde{B}^{2-\gamma/p}_{q,p}(\Omega)$, $1 < q < \infty$, $1 < p < 2d/2q-1$ we have
\[
\left\| e^{\bar{A}_{F,q} t} z_0 \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)}, \quad t \geq 0,
\]
with $M_{\gamma_0}$ possibly depending on $p, q$. Maximal regularity properties corresponding to the solution operator formula in (7.4) were established in section 6. Accordingly, Proposition 7.2.

The above Claim will then follow from Proposition 7.2 after establishing Step 2.

Step 1:

Proposition 7.2. Let $d = 3$, $q > 3$ and $1 < p < 6/5$. There exists a positive constant $r_1 > 0$ (identified below in (7.26)) and a subsequent constant $r > 0$ (identified below in (7.24)) depending on $r_1 > 0$ and a constant $C > 0$ in (7.22), such that with $\left\| z_0 \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} < r_1$, the operator $F_q(z_0, f)$ maps the ball $B(0, r)$ in $X^{\infty}_{p,q,\sigma}$ into itself.

The above Claim will then follow from Proposition 7.2 after establishing Step 2:

Proposition 7.3. Let $d = 3$, $q > 3$ and $1 < p < 6/5$. There exists a positive constant $r_1 > 0$, such that if $\left\| z_0 \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} < r_1$ as in (7.5), then there exists a constant $0 < \rho_0 < 1$, depending on the constant $r$ of Proposition 7.2, such that the operator $F_q(z_0, f)$ in (7.9) defines a contraction in the ball $B(0, \rho_0)$ of $X^{\infty}_{p,q,\sigma}$ in (7.6).

The Banach contraction principle then establishes the Claim, once we prove Propositions 7.2 and 7.3.

Proof of Proposition 7.2. Step 1: We start from definition (7.9) of $F_q$ and invoke the maximal regularity properties (6.29), (??) for $e^{\bar{A}_{F,q} t}$ and (6.8c) for
\[
\int_0^t e^{\bar{A}_{F,q} (t-\tau)} \mathcal{N}_q f(\tau) d\tau.
\]
We obtain
\[
\left\| F_q(z_0, f)(t) \right\|_{X^{\infty}_{p,q,\sigma}} \leq \left\| e^{\bar{A}_{F,q} t} z_0 \right\|_{X^{\infty}_{p,q,\sigma}} + \left\| \int_0^t e^{\bar{A}_{F,q} (t-\tau)} \mathcal{N}_q f(\tau) d\tau \right\|_{X^{\infty}_{p,q,\sigma}}
\]
(7.10)
\[ \leq C \left[ \|z_0\|_{L^2_p(B_{\overline{\Omega}}^q)}^{\frac{1}{2-p}} + \|N_q f\|_{L^p(0,\infty;L^q_{\sigma}(\Omega))} \right]. \]  

(7.11)

**Step 2:** By the definition \( N_q f = P_q [(f, \nabla) f] \) in (2.17), we estimate ignoring \( \|P_q\| \) and using, \( \sup \| (g(\cdot)) \|^r = [\sup \| (g(\cdot)) \|^r]^r \)

\[ \|N_q f\|_{L^p(0,\infty;L^q_{\sigma}(\Omega))}^p \leq \int_0^\infty \|P_q [(f, \nabla) f]\|_{L^q_{\sigma}(\Omega)}^p \, dt \]

\[ \leq \int_0^\infty \left\{ \int_\Omega |f(t, x)|^q \|\nabla f(t, x)\|^q \, d\Omega \right\}^{\frac{p}{q}} \, dt \]  

(7.12)

\[ \leq \int_0^\infty \left\{ \left[ \sup_{\Omega} |\nabla f(t, \cdot)|^q \right] \left[ \int_\Omega |f(t, x)|^q \, d\Omega \right] \right\}^{\frac{1}{q}} \, dt \]  

(7.13)

\[ \leq \int_0^\infty \|\nabla f(t, \cdot)\|_{L^q(\Omega)} \|f(t, \cdot)\|_{L^q_{\sigma}(\Omega)}^p \, dt \]  

(7.14)

\[ \leq \sup_{0 \leq t \leq \infty} \|f(t, \cdot)\|_{L^q_{\sigma}(\Omega)}^p \int_0^\infty \|\nabla f(t, \cdot)\|_{L^q(\Omega)} \, dt \]  

(7.15)

\[ = \|f\|_{L^q(0,\infty;L^q_{\sigma}(\Omega))} \|\nabla f\|_{L^p(0,\infty;L^q_{\sigma}(\Omega))}^p \]  

(7.16)

**Step 3:** The following embeddings hold true:

(i) [36, Proposition 4.3, p 1406 with \( \mu = 0, s = \infty, r = q \)] so that the required formula reduces to \( 1 \geq 1/p \), as desired

\[ f \in X^\infty_{p,q,\sigma}(A_{F,q}) \hookrightarrow f \in L^\infty(0,\infty;L^q_{\sigma}(\Omega)) \]  

(7.17a)

so that, \( \|f\|_{L^\infty(0,\infty;L^q_{\sigma}(\Omega))} \leq C \|f\|_{X^\infty_{p,q,\sigma}} \)  

(7.17b)

(ii) [44, Theorem 2.4.4, p 74 requiring \( C^{1}\)-boundary]

\[ W^{1,q}(\Omega) \subset L^\infty(\Omega) \quad \text{for} \quad q > \dim \Omega = d, \quad d = 2, 3, \]  

(7.18)

so that, with \( p > 1, q > 3 \), in case \( d = 3 \):

\[ \|\nabla f\|_{L^p(0,\infty;L^q_{\sigma}(\Omega))}^p \leq C \|\nabla f\|_{L^p(0,\infty;W^{1,q}(\Omega))}^p \leq C \|f\|_{X^\infty_{p,q,\sigma}}^p \]  

(7.19)

\[ \leq C \|f\|_{X^\infty_{p,q,\sigma}}^p \]  

(7.20)

In going from (7.19) to (7.20) we have recalled the definition of \( f \in X^\infty_{p,q,\sigma}(A_{F,q}) \) in (6.8a). Then, the sought-after final estimate of the non-linear term \( N_q f, \ f \in X^\infty_{p,q,\sigma}, \) is obtained from substituting (7.17b) and (7.20) into the RHS of (7.16). We obtain

\[ \|N_q f\|_{L^p(0,\infty;L^q_{\sigma}(\Omega))} \leq C \|f\|_{X^\infty_{p,q,\sigma}}^2, \quad f \in X^\infty_{p,q,\sigma}. \]  

(7.21)

Returning to (7.11), we finally obtain, by (7.21), that
\[ \| \mathcal{F}_q(z_0, f) \|_{X_{p,q,\sigma}^\infty} \leq C \left\{ \| z_0 \|_{\tilde{B}^{2-2/p}_{q,p}(\Omega)} + \| f \|_{X_{p,q,\sigma}^\infty}^2 \right\}. \] (7.22)

**Step 4:** We now impose the restrictions on the data on the RHS of (7.22): \( z_0 \) is in a ball of radius \( r_1 > 0 \) in \( \tilde{B}^{2-2/p}_{q,p}(\Omega) \) and \( f \) is in a ball of radius \( r > 0 \) in \( X_{p,q,\sigma}^\infty \). We further demand that the final result \( \mathcal{F}_q(z_0, f) \) shall lie in a ball of radius \( r \) in \( X_{p,q,\sigma}^\infty \). Thus we obtain from (7.22) that

\[ \| \mathcal{F}_q(z_0, f) \|_{X_{p,q,\sigma}^\infty} \leq C \left\{ \| z_0 \|_{\tilde{B}^{2-2/p}_{q,p}(\Omega)} + \| f \|_{X_{p,q,\sigma}^\infty}^2 \right\} \leq C(r_1 + r^2) \leq r. \] (7.23)

This implies

\[ Cr^2 - r + Cr_1 \leq 0 \quad \text{or} \quad \frac{1 - \sqrt{1 - 4C^2r_1}}{2C} \leq r \leq \frac{1 + \sqrt{1 - 4C^2r_1}}{2C}, \] (7.24)

whereby

\[ \{ \text{range of values of } r \} \rightarrow \text{interval } \left[ 0, \frac{1}{C} \right], \text{ as } r_1 \searrow 0, \] (7.25)

a constraint which is guaranteed by taking

\[ r_1 \leq \frac{1}{4C^2}, \quad C \text{ being the constant in (7.22)}. \] (7.26)

We have thus established that by taking \( r_1 \) as in (7.26) and subsequently \( r \) as in (7.24), then the map \( \mathcal{F}_q(z_0, f) \) takes:

\[ \{ \text{ball in } \tilde{B}^{2-2/p}_{q,p}(\Omega) \} \times \{ \text{ball in } X_{p,q,\sigma}^\infty \} \rightarrow \{ \text{ball in } X_{p,q,\sigma}^\infty \}, \]

\[ 3 < q, \ 1 < p < \frac{2q}{2q - 1} \] (7.27)

This establishes Proposition 7.2.

**Proof of Proposition 7.3.** Step 1: For \( f_1, f_2 \) both in the ball of \( X_{p,q,\sigma}^\infty \) of radius \( r \) obtained in (7.24) of the proof of Proposition 7.2, subject to \( r_1 \) chosen as in (7.26), we estimate from (7.9):

\[ \| \mathcal{F}_q(z_0, f_1) - \mathcal{F}_q(z_0, f_2) \|_{X_{p,q,\sigma}^\infty} = \left\| \int_0^t e^{\lambda \mathcal{F}_q(t-\tau)} [\mathcal{N}_q f_1(\tau) - \mathcal{N}_q f_2(\tau)]d\tau \right\|_{X_{p,q,\sigma}^\infty} \]

\[ \leq \tilde{m} \left\| \mathcal{N}_q f_1 - \mathcal{N}_q f_2 \right\|_{L_p(0,\infty;L^q_3(\Omega))}. \] (7.28)

after invoking the maximal regularity property (6.6).

Step 2: Next recalling \( \mathcal{N}_q f = P_q[(f \cdot \nabla) f] \) from (2.17), we estimate the RHS of (7.29). In doing so, we add and subtract \( (f_2 \cdot \nabla) f_1 \), set \( A = (f_1 \cdot \nabla) f_1 - (f_2 \cdot \nabla) f_1 \),
\[\nabla f_1, B = (f_2 \cdot \nabla)f_1 - (f_2 \cdot \nabla)f_2,\] and use \(|A + B|^q \leq 2^q |A|^q + |B|^q\). [74, p 12] We obtain, again ignoring \(\|P_q\|\), that

\[
\|N_q f_1 - N_q f_2\|_{L^p(0,\infty; L^q_2(\Omega))} \leq \int_0^{\infty} \left\{ \left[ \int_\Omega |(f_1 \cdot \nabla)f_1 - (f_2 \cdot \nabla)f_2|^q \, d\Omega \right]^{1/q} \right\}^p \, dt
\]

(7.30)

\[
= \int_0^{\infty} \left[ \int_\Omega |A + B|^q \, d\Omega \right]^{p/q} \, dt
\]

(7.31)

\[
\leq 2^q \int_0^{\infty} \left\{ \int_\Omega \left[ |A|^q + |B|^q \right] \, d\Omega \right\}^{p/q} \, dt
\]

(7.32)

\[
= 2^q \int_0^{\infty} \left\{ \left[ \int_\Omega |A|^q \, d\Omega + \int_\Omega |B|^q \, d\Omega \right]^{1/q} \right\}^p \, dt
\]

(7.33)

Step 3: We now notice that regarding each of the integral term in the RHS of (7.37) we are structurally and topologically as in the RHS of (7.14), except that in (7.37) the gradient terms \(\nabla f_1, \nabla (f_1 - f_2)\) are penalized in the \(L^q_2(\Omega)\)-norm which is dominated by the \(L^\infty(\Omega)\)-norm, as it occurs for the gradient term \(\nabla f\) in (7.14).

Thus we can apply to each integral term on the RHS of (7.37) the same argument as in going from (7.14) to the estimates (7.17b) and (7.20) with \(q > \dim \Omega = 3\). We obtain

\[
\|N_q f_1 - N_q f_2\|_{L^p(0,\infty; L^q_2(\Omega))} \leq \text{RHS of (7.37)}
\]

(see (7.16)) \(\leq C_{p,q} \left\{ \|f_1 - f_2\|_{L^\infty(0,\infty; L^q(\Omega))}^p \|\nabla f_1\|_{L^p(0,\infty; L^\infty(\Omega))}^p + \text{norm } f_2^p_{L^\infty(0,\infty; L^q(\Omega))} \|\nabla (f_1 - f_2)\|_{L^p(0,\infty; L^\infty(\Omega))}^p \right\}
\]

(7.38)

(see (7.17b) and (7.20)) \(\leq C_{p,q} \left\{ \|f_1 - f_2\|_{X_{p,q,\sigma}}^p \|f_1\|_{X_{p,q,\sigma}}^p + \|f_2\|_{X_{p,q,\sigma}^\infty}^p \|f_1 - f_2\|_{X_{p,q,\sigma}^\infty}^p \right\}
\]

(7.39)
\[ C_{p,q} \left\{ \| f_1 - f_2 \|_{L^p(X^\infty_{p,q,q})} \left( \| f_1 \|_{X^\infty_{p,q,q}} + \| f_2 \|_{X^\infty_{p,q,q}} \right) \right\}, \tag{7.40} \]

with \( C_{p,q} = 2^{p+q+1/4} \), and finally (7.40) yields

\[
\| N_q f_1 - N_q f_2 \|_{L^p(0,\infty; L^2(\Omega))} \leq C_{p,q} \| f_1 - f_2 \|_{X^\infty_{p,q,q}} \left( \| f_1 \|_{X^\infty_{p,q,q}} + \| f_2 \|_{X^\infty_{p,q,q}} \right)^{1/p}, \tag{7.41} \]

\[
\leq 2^{1/p} C_{p,q} \| f_1 - f_2 \|_{X^\infty_{p,q,q}} \left( \| f_1 \|_{X^\infty_{p,q,q}} + \| f_2 \|_{X^\infty_{p,q,q}} \right). \tag{7.42} \]

Step 4: Using estimate (7.42) on the RHS of estimate (7.29) yields

\[
\| F_q(z_0, f_1) - F_q(z_0, f_2) \|_{X^\infty_{p,q,q}} \leq K_{p,q} \| f_1 - f_2 \|_{X^\infty_{p,q,q}} \left( \| f_1 \|_{X^\infty_{p,q,q}} + \| f_2 \|_{X^\infty_{p,q,q}} \right), \tag{7.43} \]

with \( K_{p,q} = \tilde{m} C_{p,q} = \tilde{m} 2^{p+1/p+q+1/4} \) (\( \tilde{m} \) as in (7.29)). Next, recall that \( f_1, f_2 \) are in the ball of \( X^\infty_{p,q,q} \) of radius \( r \) obtained in (7.24):

\[
\| f_1 \|_{X^\infty_{p,q,q}}, \| f_2 \|_{X^\infty_{p,q,q}} \leq r. \tag{7.44} \]

Then

\[
\| F_q(z_0, f_1) - F_q(z_0, f_2) \|_{X^\infty_{p,q,q}} \leq \rho_0 \| f_1 - f_2 \|_{X^\infty_{p,q,q}}, \tag{7.45} \]

and \( F_q(z_0, f) \) is a contraction on the space \( X^\infty_{p,q,q} \) as soon as

\[
\rho_0 \equiv 2 K_{p,q} r < 1 \text{ or } r < 1/2 K_{p,q}, \quad K_{p,q} = \tilde{m} 2^{p+1/p+q+1/4}, \tag{7.46} \]

where \( \rho_0 \) depends on \( r \), hence on \( r_1 \) in (7.26). In this case, the map \( F_q(z_0, f) \) defined in (7.9) has a fixed point \( z \) in \( X^\infty_{p,q,q} \)

\[
F_q(z_0, z) = z, \quad \text{or} \quad z = e^{\tilde{h}_{F,q}} z_0 - \int_0^t e^{\tilde{h}_{F,q}(t-\tau)} N_q z(\tau) d\tau, \tag{7.47} \]

and such fixed point \( z \in X^\infty_{p,q,q} = X^\infty_{p,q,q}(\tilde{h}_{F,q}) \) is the unique solution of the translated non-linear equation (7.1), or (7.6) with finite dimensional control \( u \) in feedback form, as described by the RHS of (7.1). The claim is proved. \( \square \)

8. Local Exponential Decay of the Non-linear Translated z-Dynamics (7.1) with Finite Dimensional, Localized, Tangential-Like, Feedback control \( \{v, u\} \) on \( (\Gamma, \omega) \). Case \( d = 3 \)

**Theorem 8.1.** (Uniform Stabilization) Let \( d = 3, 1 < p < 6/5, q > 3 \). Consider the setting of Theorem 7.1, which provides the solution of the z-problem (7.1) in the space \( X^\infty_{p,q,q}(\tilde{h}_{F,q}) \) in (7.6) provided the initial condition \( z_0 \) satisfies the smallness condition (7.5) with \( r_1 \) given by (7.26). If \( r_1 \) is, possibly, further smaller to satisfy
condition (8.18) below, then \( z(t; z_0) \) is uniformly stable on the space \( \widetilde{B}^{2-\gamma/p}_{q,p}(\Omega) \): there exist constants \( \gamma > 0, M_{\gamma} \geq 1 \), such that said solution satisfies

\[
\|z(t; z_0)\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq M_{\gamma} e^{-\gamma t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)}. \tag{8.1}
\]

Remark 8.1 will provide insight on the relationship between \( \gamma \) in the nonlinear case in (8.1) and \( \gamma_{0} \approx |Re\lambda_{N+1}| \) in the corresponding linear case in (5.13).

**Proof.** We return to the feedback problem (7.1) rewritten equivalently as in (7.4)

\[
z(t) = e^{h_{F,q}t}z_0 - \int_0^t e^{h_{F,q}(t-\tau)}N_qz(\tau)d\tau. \tag{8.2}
\]

For \( z_0 \) in a small ball of \( \tilde{B}^{2-\gamma/p}_{q,p}(\Omega) \), Theorem 7.1 provides a unique solution in a ball of \( X^{\infty}_{p,q,\sigma} \) in (7.6). We recall from (5.17) = (7.8)

\[
\left\| e^{h_{F,q}t}z_0 \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)}, \quad t \geq 0. \tag{8.3}
\]

Our goal now is to show that for \( z_0 \) in a small ball of \( \tilde{B}^{2-\gamma/p}_{q,p}(\Omega) \), problem (8.2) satisfies the exponential decay

\[
\left\| z(t) \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq C_\alpha e^{-\alpha t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)}, \quad t \geq 0,
\]

for some constants, \( \alpha > 0, C_\alpha \geq 1 \).

**Step 1:** Starting from (8.2) and using (8.3), we estimate

\[
\left\| z(t) \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq M e^{-\gamma_0 t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} + \sup_{0 \leq t \leq \infty} \left\| \int_0^t e^{h_{F,q}(t-\tau)}N_qz(\tau)d\tau \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \tag{8.4}
\]

\[
\leq M e^{-\gamma_0 t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} + C \left\| \int_0^t e^{h_{F,q}(t-\tau)}N_qz(\tau)d\tau \right\|_{X^{\infty}_{p,q,\sigma}} \tag{8.5}
\]

\[
\leq M e^{-\gamma_0 t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} + C \|N_qz\|_{L^p(0,\infty; L^q_{p,\sigma}(\Omega))} \tag{8.6}
\]

\[
\left\| z(t) \right\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} \leq M e^{-\gamma_0 t} \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} + C_1 \|z\|_{X^{\infty}_{p,q,\sigma}}^2 \tag{8.7}
\]

In going from (8.4) to (8.5) we have recalled the embedding \( X^{\infty}_{p,q,\sigma}(\tilde{h}_{F,q}) \hookrightarrow L^\infty(0,\infty; \tilde{B}^{2-\gamma/p}_{q,p}(\Omega)) \) from (6.8c) with \( T = \infty \). Next, in going from (8.5) to (8.6) we have used the maximal regularity property (6.6). Finally, to go from (8.6) to (8.7) we have invoked estimate (7.21).

**Step 2:** We shall next establish that

\[
\|z\|_{X^{\infty}_{p,q,\sigma}} \leq M \|z_0\|_{\tilde{B}^{2-\gamma/p}_{q,p}(\Omega)} + K \|z\|_{X^{\infty}_{p,q,\sigma}}^2, \quad \text{hence} \quad \|z\|_{X^{\infty}_{p,q,\sigma}} \leq (1 - K \|z\|_{X^{\infty}_{p,q,\sigma}}^2),
\]
In fact, to this end, we take the $X_{p,q,\sigma}^\infty$ estimate of equation (8.2). We obtain

$$\|z\|_{X_{p,q,\sigma}^\infty} \leq e^{h_F,q} \|z_0\|_{X_{p,q,\sigma}^\infty} + \left\| \int_0^t e^{h_F,q(t-\tau)} N_q z(\tau) d\tau \right\|_{X_{p,q,\sigma}^\infty}$$

(8.9)

from which then (8.8) follows by invoking the maximal regularity property (6.29), (6.30) on $e^{h_F,q}$ as well as the maximal regularity estimate (6.6) followed by use of (7.21), as in going from (8.5) to (8.7)

$$\left\| \int_0^t e^{h_F,q(t-\tau)} N_q z(\tau) d\tau \right\|_{X_{p,q,\sigma}^\infty} \leq \tilde{m} \|N_q z\|_{L_p(0,\infty; L_q^\sigma(\Omega))}$$

(8.10)

$$\leq \tilde{m} C \|z\|_{X_{p,q,\sigma}^\infty}^2.$$  

(8.11)

Thus (8.8) is proved with $K = \tilde{m} C$ where $C$ is the same constant occurring in (7.21) or (7.24), hence in (7.23), (7.24).

Step 3: The well-posedness Theorem 7.1 says that

$$\begin{align*}
\text{If } \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)} &\leq r_1 \\
\text{for } r_1 \text{ sufficiently small as in (7.26)}
\end{align*}$$

$$\implies \begin{cases} 
\|z\|_{X_{p,q,\sigma}^\infty} \leq r \\
\text{where } r \text{ satisfies the constraint (7.24) in terms of } r_1 \text{ satisfying (7.26) and some constant } C \text{ that occurs for } K = \tilde{m} C \text{ in (8.11). We seek to guarantee that we can obtain}
\end{cases}$$

$$\begin{align*}
\|z\|_{X_{p,q,\sigma}^\infty} &\leq r < \frac{1}{2K} = \frac{1}{2\tilde{m}C} \left( < \frac{1}{2C} \right) \\
n &\text{hence } \frac{1}{2} < 1 - K \|z\|_{X_{p,q,\sigma}^\infty},
\end{align*}$$

(8.13)

where w.l.o.g. we can take the maximal regularity constant $\tilde{m}$ in (7.29) to satisfy $\tilde{m} \geq 1$. Again, the constant $C$ arises from application of estimate (7.21). This is indeed possible by choosing $r_1 > 0$ sufficiently small. In fact, as $r_1 \searrow 0$, (7.25) shows that the interval $r_{\min} \leq r \leq r_{\max}$ of corresponding values of $r$ tends to the interval $\left[0, \frac{1}{C}\right]$. Thus (8.3) can be achieved as $r_{\min} \searrow 0$: $0 < r_{\min} < r < \frac{1}{2\tilde{m}C}$. Next, (8.3) implies that (8.8) holds true and yields then

$$\|z\|_{X_{p,q,\sigma}^\infty} \leq 2M \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)} \leq 2Mr_1.$$  

(8.14)

Substituting (8.14) in estimate (8.7) then yields

$$\|z(t)\|_{B_{q,p}^{2-2/p}((\Omega)} \leq Me^{-\gamma_0 t} \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)} + 4C_1 M^2 \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)}^2$$

(8.15)

$$= M \left[ e^{-\gamma_0 t} + 4MC_1 \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)} \right] \|z_0\|_{B_{q,p}^{2-2/p}((\Omega)}.$$  

(8.16)
\[ \|z(t)\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \leq M[e^{-\gamma_0 t} + 4MC_1r_1]\|z_0\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \]  

(8.17)

recalling the constant \( r_1 > 0 \) in (8.12).

Step 4: Now take \( T \) sufficiently large and \( r_1 > 0 \) sufficiently small such that

\[ \beta \equiv Me^{-\gamma_0 T} + 4M^2C_1r_1 < 1. \]  

(8.18)

Then (8.16) implies, by (8.18), that

\[ \|z(T)\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \leq \beta \|z_0\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \]  

(8.19a)

and hence

\[ \|z(nT)\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \leq \beta \|z((n-1)T)\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \leq \beta^n \|z_0\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)}. \]  

(8.19b)

Since \( \beta < 1 \), the semigroup property of the evolution implies that there are constants \( \tilde{M} \geq 1, \tilde{\gamma} > 0 \) such that

\[ \|z(t)\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)} \leq \tilde{M}e^{-\tilde{\gamma}t} \|z_0\|_{\tilde{B}_{q,p}^{2-2/p}(\Omega)}, \quad t \geq 0. \]  

(8.20)

This proves Theorem 8.1. \qed

Remark 8.1. The above computations—(8.18) through (8.20)—can be used to support qualitatively the intuitive expectation that “the larger the decay rate \( \gamma_0 \) in (5.13) of the linearized feedback \( w \)-dynamics (5.3), the larger the decay rate \( \tilde{\gamma} \) in (8.20) of the nonlinear feedback \( z \)-dynamics (1.29) = (7.1); hence the larger the rate \( \tilde{\gamma} \) in (1.22) of the original \( y \)-dynamics in feedback form as in (1.18)”.

The following considerations are somewhat qualitative. Let \( S(t) \) denote the nonlinear semigroup in the space \( \tilde{B}_{q,p}^{2-2/p}(\Omega) \), with infinitesimal generator \([\tilde{A}_F, q - N_q]\) describing the feedback \( z \)-dynamics (1.29) in the abstract form (7.1), as guaranteed by the well posedness Theorem B.(i) = Theorem 7.1. Thus, \( z(t; z_0) = S(t)z_0 \) on \( \tilde{B}_{q,p}^{2-2/p}(\Omega) \). By (8.18), we can rewrite (8.19a) as

\[ \|S(T)\|_{\mathcal{L}(\tilde{B}_{q,p}^{2-2/p}(\Omega))} \leq \beta < 1. \]  

(8.21)

It follows from [6, p 178] via the semigroup property that

\[ -\tilde{\gamma} \text{ is just below } \frac{\ln \beta}{T} < 0. \]  

(8.22)

Pick \( r_1 > 0 \) in (8.18) so small that \( 4M^2C_1r_1 \) is negligible, so that \( \beta \) is just above \( Me^{-\gamma_0 T} \), so \( \ln \beta \) is just above \( [\ln M - \gamma_0 T] \), hence

\[ \frac{\ln \beta}{T} \text{ is just above } (-\gamma_0) + \frac{\ln M}{T}. \]  

(8.23)

Hence, by (8.22), (8.23),

\[ \tilde{\gamma} \sim \gamma_0 - \frac{\ln M}{T}, \]  

(8.24)

and the larger \( \gamma_0 \), the larger is \( \tilde{\gamma} \), as desired.
The $z$-problem in feedback form: We return to the translated $z$ problem (1.29) = (7.1), with $L_c(z)$ given by (1.9)

\[ z_t - v \Delta z + L_c(z) + (z \cdot \nabla)z + \nabla \chi = m(\tilde{G}z)\tau \quad \text{in } Q \]  
\[ \text{div } z = 0 \quad \text{in } Q \]  
\[ z = Fz \quad \text{on } \Sigma \]  
\[ z(0, x) = y_0(x) - y_c(x) \quad \text{on } \Omega \]  

with $Fz$ and $m(\tilde{G}z)\tau$ given in the feedback form as in (5.6a, 5.6b)

\[ m(\tilde{G}z)\tau = m\left(\sum_{k=1}^{K}(P_N z, q_k)_{w_N} u_k\right)\tau, \quad Fz = \sum_{k=1}^{K}|P_N z, p_k|_{\Gamma} f_k, \]  

for which Theorem B(i) = Theorem 7.1 provides a local well-posedness result in (7.6), (7.7) for the $z$ variable. We now complement such well-posedness for $z$ with a corresponding local well-posedness result for the pressure $\chi$.

Here we recall maximal regularity result of the Stokes operator in Appendix (A.17) for problem (A.10a, b, c, d) which accounts for inhomogeneous no-slip Dirichlet boundary conditions [61]. We present it for convenience

\[
\begin{align*}
\psi_t - v_0 \Delta \psi + \nabla \pi &= F, & &\text{in } (0, T) \times \Omega = Q, \\
\text{div } \psi &= 0, & &\text{in } Q, \\
\psi|_{\Sigma} &= h_0, & &\text{in } (0, T) \times \Gamma = \Sigma, \\
\psi|_{t=0} &= \psi, & &\text{in } \Omega,
\end{align*}
\]

Then there exists a unique solution $\varphi \in X_{p,q,\sigma}^T, \pi \in Y_{p,q}^T$ to the dynamic Stokes problem (9.2) or Appendix (A.17), continuously on the data: there exist constants $C_0, C_1$ independent of $T, F_\sigma = F_q F, \varphi_0$ such that via Appendix (A.14), $0 < T \leq \infty$:

\[
\begin{align*}
C_0 \| \varphi \|_{C(0,T;B_{q,p}^{2-n/p}(\Omega))} &\leq \| \varphi \|_{X_{p,q,\sigma}^T} + \| \pi \|_{Y_{p,q}^T} \\
&= \| \varphi \|_{L^p(0,T;L^{\infty}_q(\Omega))} + \| A_q \varphi \|_{L^p(0,T;L^q_\sigma(\Omega))} + \| \pi \|_{Y_{p,q}^T} \\
&\leq C_1 \left\{ \| F_\sigma \|_{L^p(0,T;L^q_\sigma(\Omega))} + \| \varphi_0 \|_{L^q_\sigma(\Omega), D(A_q)} \right\}_{1-\frac{1}{p},p} \\
&\quad + \| h_0 \|_{L^p(0,\infty;W^{1-\frac{1}{q},q}(\Gamma))},
\end{align*}
\]  

\textbf{Theorem 9.1.} Consider the setting of Theorem A for problem (1.18). Then the following well-posedness result for the pressure $\chi$ holds true, where we recall the spaces $Y_{p,q}^\infty$ for $T = \infty$ and $W_{1,q}^q(\Omega)$ in Appendix (A.12), (A.13) as well as the steady state pressure $\pi_e$ from Theorem 1.1:

\[ \| \chi \|_{Y_{p,q}^\infty} \leq \tilde{C} \| \gamma_0 - \gamma_e \|_{B_{q,p}^{2-n/p}(\Omega)} \left\{ \| \gamma_0 - \gamma_e \|_{B_{q,p}^{2-n/p}(\Omega)} + 1 \right\}. \]
We first apply the full maximal-regularity up to $T = \infty$ (9.3) to the Stokes component of problem (9.1) with $F_q = P_q (mG(z) - L_e(z) - (z \cdot \nabla)z)$ and $h_0 = F z$ to obtain

\[
\|z\|_{X_{P,q,\sigma}} + \|X\|_{P,q} \leq C \left\{ \|P_q |m(Gz) - (z \cdot \nabla)z - L_e(z)\|_{L^p(0,\infty;L^q_\sigma(\Omega))} + \|z_0\|_{\widehat{B}^{2-2p}_{q,p}(\Omega)} + \|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \right\}
\]

\[
\leq C \left\{ \|P_q |m(Gz)\|_{L^p(0,\infty;L^q(\Omega))} + \|P_q (z \cdot \nabla)z\|_{L^p(0,\infty;L^q_\sigma(\Omega))} + \|P_q L_e(z)\|_{L^p(0,\infty;L^q_\sigma(\Omega))} + \|z_0\|_{\widehat{B}^{2-2p}_{q,p}(\Omega)} + \|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \right\}.
\]

(9.5)

But $P_q |m(Gz)\| = mG(z)$ as the vectors $u_k$ in the definition of $\tilde{G}$ in (9.1e) are $u_k \in W^\mu_N \subset L^q_\sigma(\Omega)$. Moreover $G \in L(L^q_\sigma(\Omega))$, we obtain

\[
\|P_q |m(Gz)\|_{L^p(0,\infty;L^q(\Omega))} \leq C_1 \|z\|_{X_{P,q,\sigma}},
\]

(9.6a)

\[
\|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \leq C_1' \|z\|_{X_{P,q,\sigma}}.
\]

(9.6b)

recalling the space $X_{P,q,\sigma}$ from Appendix (A.12). Next, recalling (7.21) for $\mathcal{N}_q \tilde{z} = P_q [(z \cdot \nabla)z]$, see (2.17), we obtain

\[
\|P_q (z \cdot \nabla)z\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq C_2 \|z\|_{X_{P,q,\sigma}}^2.
\]

(9.7)

The equilibrium solution $\{y_e, \pi_e\}$ is given by Theorem 1.1 as satisfying

\[
\|y_e\|_{W^2,q(\Omega)} + \|\pi_e\|_{\mathcal{H}^{q,1}} \leq c \|f\|_{L^q(\Omega)}, \quad 1 < q < \infty.
\]

(9.8)

We next estimate the term $P_q L_e(z) = P_q [(y_e \cdot \nabla)z + (z \cdot \nabla)y_e]$ in (9.5)

\[
\|P_q L_e(z)\|_{L^p(0,\infty;L^q_\sigma(\Omega))} = \|P_q (y_e \cdot \nabla)z + P_q (z \cdot \nabla)y_e\|_{L^p(0,\infty;L^q_\sigma(\Omega))}
\]

\[
\leq \|P_q (y_e \cdot \nabla)z\|_{L^p(0,\infty;L^q_\sigma(\Omega))}
\]

\[
+ \|P_q (z \cdot \nabla)y_e\|_{L^p(0,\infty;L^q_\sigma(\Omega))}
\]

\[
\leq \|y_e\|_{L^q(\Omega)} \|\nabla z\|_{L^p(0,\infty;L^q_\sigma(\Omega))}
\]

\[
+ \|z\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \|\nabla y_e\|_{L^q(\Omega)}
\]

\[
\leq 2C_2 \|f\|_{L^q(\Omega)} \|z\|_{L^p(0,\infty;L^q_\sigma(\Omega))}
\]

\[
\leq C_3 \|z\|_{X_{P,q,\sigma}}.
\]

(9.9-13)
with the constant $C_3$ depending on the $L^q(\Omega)$-norm of the datum $f$. Setting now $C_4 = C \cdot \{C_1, C_2, C_3\}$ and substituting (9.6a), (9.7), (9.13) in (9.5), we obtain
\[
\|z\|_{X_{p,q,\sigma}^\infty} + \|x\|_{Y_{p,q}^\infty} \leq C_4 \left\{ \|z\|_{X_{p,q,\sigma}^\infty}^2 + 2 \|z\|_{X_{p,q,\sigma}^\infty} + \|z_0\|_{B_q^{2-2/p}(\Omega)} \right\} \tag{9.14}
\]
Next we drop the term $\|z\|_{X_{p,q,\sigma}^\infty}$ on the left hand side of (9.14) and invoking Appendix (A.10) to estimate $\|z\|_{X_{p,q,\sigma}^\infty}$. Thus we obtain
\[
\|x\|_{Y_{p,q}^\infty} \leq C_5 \left\{ \|z_0\|_{B_q^{2-2/p}(\Omega)}^2 + 2 \|z_0\|_{B_q^{2-2/p}(\Omega)} + \|z_0\|_{B_q^{2-2/p}(\Omega)} + 1 \right\} \tag{9.15}
\]
\[
\leq \tilde{C} \|z_0\|_{B_q^{2-2/p}(\Omega)} \left\{ \|z_0\|_{B_q^{2-2/p}(\Omega)} + 1 \right\}, \quad \tilde{C} = 3C_5 \tag{9.16}
\]
and (9.16) proves (9.4), as desired, recalling (2.20).

The $y$-problem in feedback form We return to the original $y$-problem however in feedback form as in (1.18), (1.23), (1.24) for which Theorem A in Section 1.7 proves a local well-posedness result. We now complement such well-posedness feedback form as in (1.18), (1.23), (1.24) for which Theorem A in Section 1.7 proves a local well-posedness result. We now complement such well-posedness result for $y$ with the corresponding local well-posedness result for the pressure $\pi$.

**Theorem 9.2.** Consider the setting of Theorem A for the $y$-problem in (1.18), (1.23), (1.24). Then, the following well-posedness result for the pressure $\pi$ holds true:
\[
\|\pi - \pi_e\|_{Y_{p,q}^T} \leq \|\pi - \pi_e\|_{Y_{p,q}^\infty} \leq C \left\{ \|\pi_0 - \pi_e\|_{B_q^{2-2/p}(\Omega)}^2 + \|\pi_0 - \pi_e\|_{B_q^{2-2/p}(\Omega)} + 1 \right\} \tag{9.17}
\]
\[
\leq \tilde{C} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|\pi_e\|_{B_q^{2-2/p}(\Omega)} \right\} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|\pi_e\|_{B_q^{2-2/p}(\Omega)} + 1 \right\} \tag{9.18}
\]
\[
\leq \tilde{C} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} \right\} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} + 1 \right\} \tag{9.19}
\]
\[
\|\pi\|_{Y_{p,q}^T} \leq \tilde{C} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} \right\} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} + 1 \right\} + cT^{1/p} \|\pi_e\|_{H^{1,q}(\Omega)} \quad 0 < T < \infty \tag{9.20}
\]
\[
\leq \tilde{C} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} \right\} \left\{ \|\pi_0\|_{B_q^{2-2/p}(\Omega)} + \|f\|_{L_q^q(\Omega)} + 1 \right\} + cT^{1/p} \|f\|_{L_q^q(\Omega)} \quad 0 < T < \infty \tag{9.21}
\]

**Proof.** We return to the estimate (9.4) for $\chi$ and recall $\chi = \pi - \pi_e$ from (2.20) to obtain (9.17). We next estimate $y - y_e$ by
\[
\|y_0 - y_e\|_{B_q^{2-2/p}(\Omega)} \leq C \left\{ \|y_0\|_{B_q^{2-2/p}(\Omega)}^2 + \|y_e\|_{B_q^{2-2/p}(\Omega)} \right\} \tag{9.22}
\]
which, substituted in (9.17), yields (9.18). In turn, (9.18) leads to (9.19) by means of (9.8).
10. Results on the Real Space Setting

Here we shall complement the results of Theorems 5.1 through 9.2 by giving their version in the real space setting. We shall quote from [11]. In the complexified setting \( L^q \sigma (\Omega) + i L^q \sigma (\Omega) \) we have that the complex unstable subspace \( W_u^N \) is, recall (3.10):

\[
W_u^N = W_1^N + i W_2^N
\]

space of generalized eigenfunctions \( \{ \phi_j \}_{j=1}^N \) of the operator \( A_q (= A_q^u) \) corresponding to its \( N \) unstable eigenvalues in (1.12).

Set \( \phi_j = \phi_j^1 + i \phi_j^2 \) with \( \phi_j^1, \phi_j^2 \) real. Then:

\[
W_1^N = \text{Re} W_u^N = \text{span} \{ \phi_j^1 \}_{j=1}^N; \quad W_2^N = \text{Im} W_u^N = \text{span} \{ \phi_j^2 \}_{j=1}^N.
\]

The stabilizing vectors \( p_k, q_k, u_k, k = 1, \ldots, K \) are complex valued, with \( u_k \in W_u^N \subset L^q_\sigma (\Omega) \), and \( p_k, q_k \in (W_u^N)^* \subset L^{q'}_\sigma (\Omega) \) as in (5.1), (5.2) while \( f_k \in F \subset W^{1-1/q, q} (\Gamma) \).

The complex-valued uniformly stable linear \( w \)-system in (2.20) with \( K \) complex valued stabilizing vectors admits the following real-valued uniformly stable counterpart

\[
\frac{dw}{dt} = A_{e,q}w - A_{e,q}D \left( \sum_{k=1}^{K} \text{Re}(w_N(t), p_k)w_N^u \text{Re} f_k - \sum_{k=1}^{K} \text{Im}(w_N(t), p_k)w_N^u \text{Im} f_k \right) + P_q \left( \sum_{k=1}^{K} \text{Re}(w_N(t), q_k)w_N^u \text{Re} u_k - \sum_{k=1}^{K} \text{Im}(w_N(t), q_k)w_N^u \text{Im} u_k \right) \cdot \tau.
\]

with \( 2K \leq N \) real stabilizing vectors, see [11, Eq 3.52a, p 1472]. If \( K = \sup \{ \ell_i, i = 1, \ldots, M \} \) is achieved for a real eigenvalue \( \lambda_i \) (respectively, a complex eigenvalue \( \lambda_j \)), then the effective number of stabilizing controllers is \( K \leq N \), as the generalized functions are then real; respectively, \( 2K \leq N \), for, in this case, the complex conjugate eigenvalue \( \tilde{\lambda}_j \) contributes an equal number of components in terms of generalized eigenfunctions \( \tilde{\phi}_{\tilde{\lambda}_j} = \phi_{\lambda_j} \). In all cases, the actual (effective) upper bound \( 2K \) is \( 2K \leq N \). For instance, if all unstable eigenvalues were real and simple then \( K = 1 \), and only one stabilizing controller is actually needed.

Similarly, the complex-valued locally (near \( y_e \)) uniformly stable nonlinear \( y \)-system (1.18) with \( K \) complex-valued stabilizing vectors admits the following real-valued locally uniformly stable counterpart

\[
\frac{dy}{dt} - vA_{e,q}y + N_qy = -A_{e,q}D \left( \sum_{k=1}^{K} \text{Re}(w_N(t), p_k)w_N^u \text{Re} f_k - \sum_{k=1}^{K} \text{Im}(w_N(t), p_k)w_N^u \text{Im} f_k \right)
\]
with \( 2K \leq N \) real stabilizing vectors, see [12, p 43].

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Appendix A: Some Auxiliary Results for the Stokes and Oseen Operators: Analytic Semigroup Generation, Maximal Regularity, Domains of Fractional Powers

In this subsection we collect mostly known results to be used in the sequel.

(a) The Stokes and Oseen operators generate strongly continuous analytic semigroups on \( L^q_\sigma(\Omega) \), \( 1 < q < \infty \).

Theorem A.1. Let \( d \geq 2 \), \( 1 < q < \infty \) and let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) of class \( C^3 \). Then

(i) the Stokes operator \( -A_q = P_q \Delta \) in (2.14), repeated here as

\[
- A_q \psi = P_q \Delta \psi, \quad \psi \in \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \tag{A.1}
\]

generates a s.c analytic semigroup \( e^{-A_q t} \) on \( L^q_\sigma(\Omega) \). See [37] and the review paper [39, Theorem 2.8.5 p 17], and [68].

(ii) The Oseen operator \( A_q \) in (2.16)

\[
A_q = -(\nu_o A_q + A_{o,q}), \quad \mathcal{D}(A_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \tag{A.2}
\]

generates a s.c analytic semigroup \( e^{A_q t} \) on \( L^q_\sigma(\Omega) \). This follows as \( A_{o,q} \) is relatively bounded with respect to \( A_{q/2} \), to be formally defined in (A.6); thus a standard theorem on perturbation of an analytic semigroup generator applies [60, Corollary 2.4, p 81].

(iii)

\[
0 \in \rho(A_q) = \text{the resolvent set of the Stokes operator } A_q \tag{A.3a}
\]

\[
\left\{ A_q^{-1} : L^q_\sigma(\Omega) \rightarrow L^q_\sigma(\Omega) \text{ is compact} \right\} \tag{A.3b}
\]
(iv) The s.c. analytic Stokes semigroup $e^{-A_q t}$ is uniformly stable on $L^q_σ(Ω)$: there exist constants $M \geq 1, δ > 0$ (possibly depending on $q$) such that
\[
\|e^{-A_q t}\|_{L^q(L^q_σ(Ω))} \leq Me^{-δt}, \quad t > 0.
\] (A.4)

(b) Domains of fractional powers, $D(A^α_q), 0 < α < 1$ of the Stokes operator $A_q$ on $L^q_σ(Ω), 1 < q < ∞$.

Theorem A.2. For the domains of fractional powers $D(A^α_q), 0 < α < 1$, of the Stokes operator $A_q$ in (2.14) = (A.1), the following complex interpolation relation holds true [38, 39, Theorem 2.8.5, p 18]
\[
[D(A_q), L^q_σ(Ω)]_{1-α} = D(A^α_q), \quad 0 < α < 1, \quad 1 < q < ∞;
\] (A.5)
in particular, see (2.15)
\[
[D(A_q), L^q_σ(Ω)]_{1/2} = D(A^{1/2}_q) \equiv W_0^{1,q}(Ω) \cap L^q_σ(Ω).
\] (A.6)

Thus, on the space $D(A^{1/2}_q)$, the norms
\[
\|\nabla \cdot \|_{L^q(Ω)} \quad \text{and} \quad \|\cdot\|_{L^q(Ω)}
\] (A.7)
are equivalent via the Poincaré inequality.

c) The Stokes operator $-A_q$ and the Oseen operator $A_q, 1 < q < ∞$ generate s.c. analytic semigroups on the Besov space.

Recall (1.15a, 1.15b)
\[
\left( L^q_σ(Ω), D(A_q) \right)_{1-\frac{1}{p}, \frac{1}{p}} = \left\{ g \in B^2_{q,p}(Ω) : \text{div } g = 0, \ g|_Γ = 0 \right\}
\] if $\frac{1}{q} < 2 - \frac{2}{p} < 2;
\]
\[
\left( L^q_σ(Ω), D(A_q) \right)_{1-\frac{1}{p}, \frac{1}{p}} = \left\{ g \in B^2_{q,p}(Ω) : \text{div } g = 0, \ g \cdot ν|_Γ = 0 \right\}
\]
\[
\equiv \tilde{B}^2_{q,p}(Ω) \quad \text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}, \quad \text{or } 1 < p < \frac{2q}{2q - 1}.
\] (A.8a)

In fact, Theorem A.1(i) states that the Stokes operator $-A_q$ generates a s.c analytic semigroup on the space $L^q_σ(Ω), 1 < q < ∞$, hence on the space $D(A_q)$ in (A.1), with norm $\|\cdot\|_{D(A_q)} = \|A_q \cdot\|_{L^q_σ(Ω)}$ as $0 \in ρ(A_q)$. Then, one obtains that the Stokes operator $-A_q$ generates a s.c. analytic semigroup on the real interpolation spaces in (A.8). Next, the Oseen operator $A = -(ν_o A_q + A_o,q)$ likewise generates a s.c. analytic semigroup $e^{A q t}$ on $L^q_σ(Ω)$ by Theorem A.1(ii). Moreover $A_q$ generates a s.c. analytic semigroup on $D(A_q) = D(A_q)$ (equivalent norms). Hence $A_q$ generates a s.c. analytic semigroup on the real interpolation spaces (A.11). Here below, however, we shall formally state the result only in the case $2 - \frac{2}{p} < \frac{1}{q}$, that is $1 < p < \frac{2q}{2q - 1}$, in the space $\tilde{B}^2_{q,p}(Ω)$, as this does not contain B.C. The objective of the present paper is precisely to obtain stabilization results on spaces that do not recognize B.C.
Theorem A.3. Let \( 1 < q < \infty, 1 < p < \frac{2q}{2q-1} \)

(i) The Stokes operator \(-A_q\) in (A.1) generates a s.c analytic semigroup \(e^{-A_q t}\) on the space \(\tilde{B}^{2-2/q,p}_q(\Omega)\) defined in (1.15b) which moreover is uniformly stable, as in (A.4),

\[
\left\| e^{-A_q t} \right\|_{\mathcal{L}(\tilde{B}^{2-2/q,p}_q(\Omega))} \leq M e^{-\delta t}, \quad t > 0. \tag{A.9}
\]

(ii) The Oseen operator \(A_q\) in (A.2) generates a s.c analytic semigroup \(e^{A_q t}\) on the space \(\tilde{B}^{2-2/q,p}_q(\Omega)\) in (1.15b).

(d) Space of maximal \(L^p\) regularity on \(L^q(\sigma)(\Omega)\) of the Stokes operator \(-A_q\), \(1 < p < \infty\), \(1 < q < \infty\) up to \(T = \infty\). We shall use the notation of [24] and write

\(-A_q \in M Reg(L^p(0, \infty; L^q(\sigma)(\Omega)))\). We return to the dynamic Stokes problem in \(\{\phi(t,x), \pi(t,x)\}\)

\[
\begin{cases}
\phi_t - \Delta \phi + \nabla \pi = F & \text{in } (0, T] \times \Omega \equiv Q \tag{A.10a} \\
di \phi = 0 & \text{in } Q \tag{A.10b} \\
\phi|_{\Sigma} = h_0 & \text{in } (0, T] \times \Gamma \equiv \Sigma \tag{A.10c} \\
\phi|_{t=0} = \phi_0 & \text{in } \Omega, \tag{A.10d}
\end{cases}
\]

rewritten in abstract form, after applying the Helmholtz projection \(P_q\) to (A.10a) and recalling \(A_q\) in (A.1) as

\[
\phi' + A_q \phi = F_{\sigma} \equiv P_q F, \quad \phi_0 \in \left( L^q_{\sigma}(\Omega), \mathcal{D}(A_q) \right)_{1-\frac{1}{p},p}, \tag{A.11}
\]

recall (A.8). Next, we introduce the space of maximal regularity for \(\{\phi, \phi'\}\), that is for \(-A_q\), as [39, p 2; Theorem 2.8.5.iii, p 17], [36, p 1404-5], with \(T\) up to \(\infty\):

\[
X^{T}_{p,q,\sigma}(A_q) = L^p(0, T; \mathcal{D}(A_q)) \cap W^{1,p}(0, T; L^q_{\sigma}(\Omega)) \tag{A.12}
\]

(recall (2.14) for \(\mathcal{D}(A_q)\)) and the corresponding space for the pressure as

\[
Y^{T}_{p,q} = L^p(0, T; \tilde{W}^{1,q}(\Omega)), \quad \tilde{W}^{1,q}(\Omega) = W^{1,q}(\Omega)/\mathbb{R}. \tag{A.13}
\]

The following embedding, also called trace theorem, holds true [3, Theorem 4.10.2, p 180, BUC for \(T = \infty\)].

\[
X^{T}_{p,q,\sigma} \subset X^{T}_{p,q} \equiv L^p(0, T; W^{2,q}_{\sigma}(\Omega)) \cap W^{1,p}(0, T; L^q(\Omega))
\]

\[
\hookrightarrow C\left([0, T]; \tilde{B}^{2-2/q,p}_{q,p}(\Omega)\right). \tag{A.14}
\]
For a function \( g \) such that \( \text{div} \, g = 0, \, g|_\Gamma = 0 \) we have \( g \in X^T_{p,q,\sigma} \), by (1.4).

The solution of Equation (A.11) is

\[
\varphi(t) = e^{-A_q t} \varphi_0 + \int_0^t e^{-A_q (t-\tau)} F_\sigma(\tau) \, d\tau. \tag{A.15}
\]

The following is the celebrated result on maximal regularity on \( L^q_\sigma (\Omega) \) of the Stokes problem due originally to Solonnikov [70] reported in [39, 64, Theorem 2.8.5(iii) for \( \varphi_0 = 0 \) and Theorem 2.10.1 p 1405], [36, 61, Proposition 4.1, p 1405]. See also [12, Theorem 3.1 p 31 for \( p = q = 2 \) case]. See also [69, 71–73].

**Theorem A.4.** Let \( 1 < p, q < \infty, \, T \leq \infty. \) With reference to problem (A.10), (A.11), assume

\[
F_\sigma \in L^p(0, T; L^q_\sigma (\Omega)), \, \varphi_0 \in \left( L^q_\sigma (\Omega), D(A_q) \right)_{1-\frac{1}{p}, p}. \tag{A.16}
\]

Then there exists a unique solution \( \varphi \in X^T_{p,q,\sigma} \) continuously on the data: there exist constants \( C_0, C_1 \) independent of \( T, F_\sigma, \varphi_0 \) such that via (A.14)

\[
C_0 \left\| \varphi \right\|_{C([0,T]; \dot{H}^{2-2/p}_p(\Omega))} \leq \left\| \varphi \right\|_{X^T_{p,q,\sigma}} + \| \pi \|_{Y^T_{p,q}}
= \left\| \varphi' \right\|_{L^p(0,T; L^q_\sigma (\Omega))} + \left\| A_q \varphi \right\|_{L^p(0,T; L^q_\sigma (\Omega))} + \| \pi \|_{Y^T_{p,q}}
\leq C_1 \left\{ \left\| F_\sigma \right\|_{L^p(0,T; L^q_\sigma (\Omega))} + \left\| \varphi_0 \right\|_{\left( L^q_\sigma (\Omega), D(A_q) \right)_{1-\frac{1}{p}, p}} + \left\| \pi_0 \right\|_{L^p(0,\infty; W^{1,q}_\sigma(\Gamma))} \right\}. \tag{A.17}
\]

In particular,

(i) With reference to the variation of parameters formula (A.15) of problem (A.11) arising from the Stokes problem (A.10), we have recalling (A.12): the map

\[
F_\sigma \longrightarrow \int_0^t e^{-A_q (t-\tau)} F_\sigma(\tau) \, d\tau : : \text{continuous} \tag{A.18}
\]

\[
L^p(0, T; L^q_\sigma (\Omega)) \longrightarrow X^T_{p,q,\sigma}(A_q) \equiv L^p(0, T; D(A_q)) \cap W^{1,p}(0, T; L^q_\sigma (\Omega)). \tag{A.19}
\]

(ii) The s.c. analytic semigroup \( e^{-A_q t} \) generated by the Stokes operator \( -A_q \) (see (A.1)) on the space \( \left( L^q_\sigma (\Omega), D(A_q) \right)_{1-\frac{1}{p}, p} \) satisfies

\[
e^{-A_q t} : \text{continuous} \left( L^q_\sigma (\Omega), D(A_q) \right)_{1-\frac{1}{p}, p} \longrightarrow X^T_{p,q,\sigma}(A_q) \equiv L^p(0, T; D(A_q)) \cap W^{1,p}(0, T; L^q_\sigma (\Omega)). \tag{A.20a}
\]
In particular via (A.8b), for future use, for \(1 < q < \infty\), \(1 < p < \frac{2q}{2q-1}\), the s.c. analytic semigroup \(e^{-A_q t}\) on the space \(\tilde{B}^{2-2/p}_q(\Omega)\), satisfies
\[
e^{-A_q t} : \text{continuous} \quad \tilde{B}^{2-2/p}_q(\Omega) \rightarrow X^T_{p,q,\sigma}.
\] (A.20b)

(iii) Moreover, setting \(\nabla \pi = (I_d - P_q)(\Delta + F)\), it follows that \(\{\varphi, \pi\} \in X^T_{p,q,\sigma} \times Y^T_{p,q}\), see (A.13), solves problem (A.10) and there is a constant \(C > 0\) independent of \(T\), \(F_\sigma\), \(\phi_0\) s.t.
\[
\|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \leq C\left\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\tilde{B}^{2-2/p}_q(\Omega)} \right\}.
\] (A.21a)

while, for future use, for \(1 < q < \infty\), \(1 < p < \frac{2q}{2q-1}\), then (A.21a) specializes to
\[
\|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \leq C\left\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\tilde{B}^{2-2/p}_q(\Omega)} \right\}.
\] (A.21b)

(e) Maximal \(L^p\) regularity on \(L^q_\sigma(\Omega)\) of the Oseen operator \(A_q\), \(1 < p < \infty\), \(1 < q < \infty\): \(A_q \in M\text{Reg}(L^p(0,T;L^q_\sigma(\Omega)))\), \(T\) finite arbitrary. We next transfer the maximal regularity of the Stokes operator \((-A_q)\) on \(L^q_\sigma(\Omega)\)-asserted in Theorem A.4 into the maximal regularity of the Oseen operator \(A_q = -\nu_o A_q - A_o, q\) exactly on the same space \(X^T_{p,q,\sigma}\) defined in (A.12), except now only on \(T < \infty\).

Thus, consider the dynamic Oseen problem in \(\{\psi(t, x), \pi(t, x)\}\) with equilibrium solution \(y_e\), see Theorem 1.1 on (1.2):
\[
\begin{aligned}
\psi_t - \nu_o \Delta \psi + L_c(\psi) + \nabla \pi &= F & \text{in} & (0, T] \times \Omega \equiv Q \\
\text{div} \psi &= 0 & \text{in} & Q \\
\psi|_{\Gamma} &= 0 & \text{in} & (0, T] \times \Gamma \equiv \Sigma \\
\psi|_{t=0} &= \psi_0 & \text{in} & \Omega,
\end{aligned}
\] (A.22)

rewritten in abstract form, after applying the Helmholtz projector \(P_q\) to (A.22a) and recalling \(A_q\) in (A.2)
\[
\psi_t = A_q \psi + P_q F = -\nu_o A_q \psi - A_{o,q} \psi + F_\sigma, \quad \psi_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-p, p}^{-\frac{1}{p}, p}
\] (A.24)
whose solution is
\[\psi(t) = e^{A_q t} \psi_0 + \int_0^t e^{A_q (t-\tau)} F_s(\tau) d\tau.\] (A.25)
\[\psi(t) = e^{-\nu_o A_q t} \psi_0 + \int_0^t e^{-\nu_o A_q (t-\tau)} F_s(\tau) d\tau - \int_0^t e^{-\nu_o A_q (t-\tau)} A_{\rho, q} \psi(\tau) d\tau.\] (A.26)

**Theorem A.5.** Let \(1 < p, q < \infty, \) \(0 < T < \infty.\) Assume (as in (A.16))
\[F_\sigma \in L^p(0, T; L^q_\sigma(\Omega)), \quad \psi_0 \in \left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p}, p} \text{ (A.27)}\]
where
\[\mathcal{D}(A_q) = \mathcal{D}(A_q), \text{ see (A.2)}.\] Then there exists a unique solution \(\psi \in X^T_{p,q, \sigma}\) of the dynamic Oseen problem (A.22), continuously on the data: that is, there exist constants \(C_{0T}, C_{1T}\) independent of \(F_\sigma, \psi_0\) such that (recall (A.14)):
\[C_{0T} \parallel \psi \parallel_{C([0,T]; B_{2^{-2/p} p}(\Omega))} \leq \parallel \psi \parallel_{X^T_{p,q, \sigma}} + \parallel \pi \parallel_{Y^T_{p,q}} \]
\[\equiv \parallel \psi \parallel_{L^p(0,T; L^q(\Omega))} + \parallel A_q \psi \parallel_{L^p(0,T; L^q(\Omega))} + \parallel \pi \parallel_{Y^T_{p,q}} \] (A.28)
\[\leq C_{1T} \left\{ \parallel F_\sigma \parallel_{L^p(0,T; L^q(\Omega))} + \parallel \psi_0 \parallel_{\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p}, p}} \right\} \] (A.29)

Equivalently, for \(1 < p, q < \infty\)

i. The map
\[F_\sigma \longrightarrow \int_0^t e^{A_q (t-\tau)} F_s(\tau) d\tau : \text{ continuous} \]
\[L^p(0, T; L^q_\sigma(\Omega)) \longrightarrow L^p(0, T; \mathcal{D}(A_q) = \mathcal{D}(A_q)) \text{ (A.30)}\]
where then automatically, see (A.24)
\[L^p(0, T; L^q_\sigma(\Omega)) \longrightarrow W^{1,p}(0, T; L^q_\sigma(\Omega)) \text{ (A.31)}\]

and ultimately
\[L^p(0, T; L^q_\sigma(\Omega)) \longrightarrow X^T_{p,q, \sigma}(A_q) \equiv L^p(0, T; \mathcal{D}(A_q)) \cap W^{1,p}(0, T; L^q_\sigma(\Omega)). \] (A.32a)

Thus,
\[A_q \in MR_{e g}(L^p(0, T; L^q_\sigma(\Omega))), \quad 1 < T < \infty \quad \text{(A.32b)}\]
and the operator \(A_q\) has maximal \(L^p\) regularity on \(L^q_\sigma(\Omega)\).
ii. The s.c. analytic semigroup $e^{A_q t}$ generated by the Oseen operator $A_q$ (see (A.2)) on the space $(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$ satisfies for $1 < p, q < \infty$

$$e^{A_q t} : \text{continuous} \quad (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p} \longrightarrow L^p(0, T; D(A_q) = D(A_q))$$

(A.33)

and hence automatically

$$e^{A_q t} : \text{continuous} \quad (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p} \longrightarrow X_{p,q,\sigma}^T. \quad \text{(A.34)}$$

In particular, for future use, for $1 < q < \infty$, $1 < p < \frac{2q}{2q-1}$, we have that the s.c. analytic semigroup $e^{A_q t}$ on the space $\widetilde{B}_{q,p}^{2-2/p}(\Omega)$, satisfies

$$e^{A_q t} : \text{continuous} \quad \widetilde{B}_{q,p}^{2-2/p}(\Omega) \longrightarrow L^p(0, T; D(A_q) = D(A_q)), \quad T < \infty,$$

(A.35)

and hence automatically

$$e^{A_q t} : \text{continuous} \quad \widetilde{B}_{q,p}^{2-2/p}(\Omega) \longrightarrow X_{p,q,\sigma}^T(A_q), \quad T < \infty. \quad \text{(A.36)}$$

iii. An estimate such as the one in (A.21a) applies to the pressure $\pi$, where now $\nabla \pi(I - P_q)(\Delta - L_e + F)$.

A proof of Theorem A.5 is given in [50].

Appendix B: The Eigenvectors $\varphi_{ij}^* \in W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega) \cap L^{q'}(\Omega)$ of $A^*(= A_q^*)$ in $L^{q'}(\Omega)$ May Be Viewed Also as $\varphi_{ij}^* \in W^{3,q}(\Omega)$, so that

$$\left. \frac{\partial \varphi_{ij}^*}{\partial \nu} \right|_{\Gamma} \in W^{2-1/q-q}(\Gamma), \quad q \geq 2.$$  

The eigenvectors $\varphi_{ij}^*$ of $A^*(= A_q^*)$, defined in (4.8a), are in $D((A_q^*)^n)$ for any $n$, so the are arbitrarily smooth in $L^{q'}(\Omega)$, say $\varphi_{ij}^* \in W^{s,q'}(\Omega)$, with $s$ as large as we please. We seek to view $\varphi_{ij}^*$ in an $L^q(\Omega)$-based space. To this end, we recall a Sobolev embedding theorem.

Theorem B.1. [81, p328], For a more restricted version [1, p 97] Let $\Omega$ be an arbitrary bounded domain, $\dim \Omega = d$. Let $0 \leq t \leq s < \infty$ and $\infty > q \geq \tilde{q} > 1$. Then, the following embedding holds true:

$$W^{s,\tilde{q}}(\Omega) \subset W^{t,q}(\Omega), \quad s - \frac{d}{\tilde{q}} \geq t - \frac{d}{q}$$ \quad (B.1)

□

Corollary B.2. With $2 \leq q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, so that $1 < q' \leq 2 \leq q$, $0 \leq r$, we have
\[ \varphi_{ij}^* \in W^{r+m,q'}(\Omega) \subset W^{r,q}(\Omega), \quad m \geq d \left( \frac{1}{q'} + \frac{1}{q} \right) = \begin{cases} 0, & q' = q = 2 \\ d, & q' = 1, q = \infty \end{cases} \quad (B.2) \]

(ii)

\[ \frac{\partial \varphi_{ij}^*}{\partial \nu} \bigg|_\Gamma \in W^{r-1-1/q,q}(\Gamma), \quad r > 1 + \frac{1}{q} \quad (B.3) \]

(iii) With reference to the sub-space \( \mathcal{F} \) based on \( \Gamma \), as defined in (1.25), we have

\[ \mathcal{F} \equiv \text{span} \left\{ \frac{\partial}{\partial \nu} \varphi_{ij}^*, \ i = 1, \ldots, M; \ j = 1, \ldots, \ell_i \right\} \subset W^{r-1-1/q,q}(\Gamma), \quad r > 1 + \frac{1}{q} \quad (B.4) \]

In particular, for our purposes, if will suffice to take \( r = 3 \) in (B.2), so that (B.2)–(B.4) become

\[ \varphi_{ij}^* \in W^{3,q}(\Omega), \quad \frac{\partial \varphi_{ij}^*}{\partial \nu} \bigg|_\Gamma \in W^{2-1/q,q}(\Gamma), \quad \mathcal{F} \subset W^{2-1/q,q}(\Gamma). \quad (B.5) \]

(iv) Thus, with reference to the boundary vector \( v = v_N \) introduced in (5.1) = (6.10), we have

\[ v = \sum_{k=1}^{K} v_k(t) f_k \in W^{2-1/q,q}(\Gamma), \quad f_k \in \mathcal{F}, \quad f_k \cdot \nu|_\Gamma = 0, \quad v \cdot \nu|_\Gamma = 0 \quad (B.6) \]

(v) Recalling the Dirichlet map \( D \) introduced to describe the solution of problem (2.1), we have

\[ Dv \in W^{2,q}(\Omega), \quad 2 \leq q < \infty \quad (B.7) \]

**Proof.** (i) Apply Theorem B.1 with \( s = r + m \geq t = r, \quad \tilde{q} = q', \quad \frac{1}{q'} + \frac{1}{q} = 1, \quad q \geq 2, \) so that \( q' = \tilde{q} \leq q \), to verify that the required condition (B.1)

\[ s - \frac{d}{\tilde{q}} = r + m - \frac{d}{q'} \geq t - \frac{d}{q} = r - \frac{d}{q}, \quad \text{or} \quad m \geq d \left( \frac{1}{q'} - \frac{1}{q} \right) \geq 0 \quad (B.8) \]

can always be satisfied by taking \( m \geq 0 \) suitable as in (B.8). This is possible, since \( \varphi_{ij}^* \) is arbitrarily smooth.
(ii) Then (B.3) follows by the usual trace theory [1]. Then, (iii)–(v) readily follow, as $D$ improves regularity by $1/q$ from the boundary to the interior. □

Next, we return to the operator $F : L^q(\Omega) \subset L^q_\sigma(\Omega) \rightarrow L^q(\Gamma)$ in (5.6). Its adjoint $F^*$ is

$$F^* g = \sum_{k=1}^K (f_k, g)_\Gamma p_k \in (W^u_N)^* \subset L^q_\sigma(\Omega), \quad g \in L^q(\Gamma) \quad (B.9)$$

where we have seen in (2.64) that $D : L^q(\Gamma) \supset U_q \rightarrow W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset D\left(A^{1/2q-\epsilon}_q\right)$

$$F^* D^* h = \sum_{k=1}^K (f_k, D^* h)_\Gamma p_k = \sum_{k=1}^K [Df_k, h]_{W^u_N} p_k \in (W^u_N)^* \quad (B.10)$$

where we have conservatively: $f_k \in L^q(\Gamma)$, $f_k \cdot v = 0$ on $\Gamma$, thus by (2.9), $Df_k \in W^{1/q',q'}(\Omega) = W_0^{1/q',q'}(\Omega)$ by (2.63a) since $1/q' \leq q'$ for $1 < q' \leq 2$. Thus, in (B.10) we can take $h \in W^{-1/q,q}(\Omega)$. In particular

$$F^* D^* \in L\left(L^q(\Omega)\right), \quad 1 < q' \leq 2. \quad (B.11)$$

Appendix C: Relevant Unique Continuation Properties for Overdetermined Oseen Eigenvalue Problems

In this Appendix C, we assemble a comprehensive account of unique continuation problems for Oseen eigenproblems, as they pertain to the problem of controllability of finite dimensional projected system (4.8a, 4.8b) of the linearized $w$-problem (1.28) (with interior, tangential-like localized control $u \equiv 0$). Positive solution, or lack thereof, of this finite dimensional problem is a key step, or obstruction, for the uniform stabilization of the Navier Stokes equations. This issue has been known since the study of boundary feedback stabilization of a parabolic equation with Dirichlet boundary trace in the feedback loop, as acting on the Neumann boundary conditions [54]. We return to the bounded domain $\Omega$, $d = 2, 3$, with boundary $\Gamma = \partial\Omega$. As before, $\tilde{\Gamma}$ is a subportion of $\Gamma$.

Problem #1 (over-determination only on a portion $\tilde{\Gamma}$ of $\Gamma$) Let $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the over-determined problem

$$\begin{cases}
(-\nu_o \Delta)\varphi + L_e(\varphi) + \nabla \pi = \lambda \varphi & \text{in } \Omega \quad (C.1a) \\
\text{div } \varphi = 0 & \text{in } \Omega \quad (C.1b) \\
\varphi |_{\tilde{\Gamma}} = 0, \quad \frac{\partial \varphi}{\partial \nu} |_{\tilde{\Gamma}} = 0 & \text{on } \tilde{\Gamma} \quad (C.1c)
\end{cases}$$
with over-determination only on the portion $\Gamma$ of $\Gamma$. Does (C.1a, C.1b, C.1c) imply

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in} \quad \Omega ?$$  \hfill (C.2)

The answer is negative even in the Stokes case: $L_c(\varphi) \equiv 0$. This follows from [26], where the following counterexample is given in the 2-dimensional half-space $\Omega = \{(x, y) : x \in \mathbb{R}^+, y \in \mathbb{R}\}$ with boundary $\Gamma = \{x = 0\}$. On $\Omega$ take

$$u_1(x, y) \equiv 0, \quad u_2(x, y) = ax^2, \quad p = 2ay, \quad a \neq 0,$$  \hfill (C.3)

so that with $u = \{u_1, u_2\}$, it follows that

$$\Delta u = \nabla p \quad \text{in} \quad \Omega, \quad u|_{\Gamma} = \nabla u|_{\Gamma} = 0,$$  \hfill (C.4)

to obtain a nontrivial solution of the Stokes overdetermined eigenproblem with $\lambda = 0$. Such half-space example can then be transformed into a counterexample over the bounded domain $\Omega$ where the over-determination is active on any subset $\tilde{\Gamma}$ of the boundary $\Gamma = \partial \Omega$.

**Implications of failure of unique continuation under Problem #1:** A negative consequence of the lack of unique continuation (C.1) $\implies$ (C.2) with over-determination only in a portion $\tilde{\Gamma}$ of the boundary $\Gamma$ is as follows: that global uniform stabilization of the linearized $w$-problem (1.28) by means of a purely tangential (finite or infinite dimensional) feedback boundary control $v$ (as given by (5.1) in the finite dimensional case) acting only on a small subportion $\tilde{\Gamma}$ of the boundary $\Gamma$ (and thus with localized interior tangential-like control $u \equiv 0$) is not possible. This is so since the algebraic rank condition (4.11b) (with $u \equiv 0$) fails, as boundary traces

$$\left\{ \frac{\partial \varphi^*_{i1}}{\partial v} \bigg|_{\tilde{\Gamma}}, \frac{\partial \varphi^*_{i2}}{\partial v} \bigg|_{\tilde{\Gamma}}, \ldots, \frac{\partial \varphi^*_{\ell i}}{\partial v} \bigg|_{\tilde{\Gamma}} \right\}$$  \hfill (C.5)

fail to be linearly independent on $\tilde{\Gamma}$ since, equivalently, the implication (C.1) $\implies$ (C.2) fails. See Orientation.

**Problem #2** (dual of the statement of Lemma 4.3): necessity to complement the localized control $v$ on $\tilde{\Gamma}$ with a localized interior tangential-like control $u$ supported on $\omega$ in terms of $\tilde{\Gamma}$. Let now $(\varphi, p) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the problem

$$(-\nu_0 \Delta) \varphi \, + \, L_c(\varphi) \, + \, \nabla \pi \, = \, \lambda \varphi \quad \text{in} \quad \Omega$$  \hfill (C.6a)

$$\text{div} \, \varphi \, = \, 0 \quad \text{in} \quad \Omega$$  \hfill (C.6b)

$$\varphi|_{\tilde{\Gamma}} \equiv 0, \quad \frac{\partial \varphi}{\partial v} \bigg|_{\tilde{\Gamma}} \equiv 0, \quad \varphi \cdot \tau \equiv 0 \quad \text{in} \quad \omega$$  \hfill (C.6c)

Then, [53, Theorem 6.2],

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in} \quad \Omega.$$  \hfill (C.7)
It is as a consequence of such unique continuation property that the Kalman algebraic rank conditions (6.28b) are satisfied. This is the basic result upon which the uniform stabilization of the present paper relies. Thus we can conclude that the results of the present paper (as in [53]) are optimal in terms of the required extra condition of the localized interior, tangential-like control needed to supplement the insufficient role of the localized tangential boundary control $v$ on $\tilde{\Gamma}$. Optimality is in terms of the smallness of the required control action for $v$ and $u$.

**Problem #3** (over-determination on the entire boundary $\Gamma = \partial \Omega$). Let now $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the over-determined problem

\[
\begin{align*}
(-v_0 \Delta)\varphi + L_e(\varphi) + \nabla \pi &= \lambda \varphi & \text{in } \Omega \quad (\text{C.8a}) \\
\text{div } \varphi &= 0 & \text{in } \Omega \quad (\text{C.8b}) \\
\varphi|_{\Gamma} &\equiv 0, \quad \frac{\partial \varphi}{\partial \nu}|_{\Gamma} \equiv 0 & \text{on } \Gamma \quad (\text{C.8c})
\end{align*}
\]

with over-determination on all of $\Gamma$. Then, does (C.8a, C.8b, C.8c) imply

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega \quad (\text{C.9})$$

It seems that a general definitive answer is not known at present. Only partial results are known.

The desired unique continuation (C.8) \(\implies\) (C.9) holds true, if the equilibrium solution $y_e \equiv 0$ (Stokes eigenproblem) or, more generally, if $y_e$ is sufficiently small in the $W^{1,q}(\Omega)$-norm. Several different proofs are given in [79,80].

The case $y_e \equiv 0$ is actually physically quite important as it occurs for instance when the forcing function in (1.1a) or (1.2a) is a conservative vector field (say an electrostatic or gravitational field) $f = \nabla g$. In this case, a solution (1.2a, 1.2b, 1.2c) is: $y_e \equiv 0$, $\pi e = g$.

When $y_e \equiv 0$ (or $y_e$ small) the tangential boundary feedback control $v$ alone, in the form such as (5.1), as acting on the entire boundary $\Gamma$ produces enhancement of stability at will for the linearized $w$-problem.

Of course, with $y_e \equiv 0$, the corresponding Oseen problems reduces to the Stokes problem. The Stokes semigroup is already uniformly stable, see (3.7), with margin of stability $\delta > 0$. When $y_e \equiv 0$ a most valuable variation of the problem under investigation of the present paper is to enhance the original margin of stability $\delta > 0$ of the original linearized uncontrolled $w$-problem (1.11) (with $u \equiv 0$, $v \equiv 0$) to obtain an arbitrary decay rate, say $k^2$, by means of only a tangential boundary finite dimensional feedback control, of the same form as the operator $F$ in (5.6b) but applied to all of $\Gamma$. To this, it suffices to apply the procedure of the present paper to a finite dimensional projected space spanned by the eigenvectors of the Stokes operator corresponding to finitely many eigenvalues $\lambda_i$, $i = 1, \ldots, I$,

$$-k^2 \leq -Re \lambda_I \leq \cdots \leq Re \lambda_1 \leq -\delta \quad (\text{C.10})$$
Problem #4 over-determination on a portion of the boundary $\tilde{\Gamma}$ involving also the pressure $p$. Let $\{\varphi, p\} \in W^{2,q}(\Omega) \cap W^{1,q}(\Omega)$ solve the over-determined problem

$$
\begin{align}
\left\{ (-\nu_o \Delta) \varphi + L_e(\varphi) + \nabla \pi &= \lambda \varphi \quad \text{in } \Omega \\
\operatorname{div} \varphi &= 0 \quad \text{in } \Omega \\
\varphi|_{\Gamma} &\equiv 0, \quad \left[ \frac{\partial \varphi}{\partial \nu} - pv \right]|_{\Gamma} \equiv 0
\end{align}
$$

(C.11a) (C.11b) (C.11c)

Does this imply

$$
\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega?
$$

(C.12)

This answer is in the affirmative. The argument, given in the [79] is along more classical elliptic arguments [45]. Here however the new condition in (C.11c) contains the pressure, which must be viewed as unknown in general. Application of this result to the present paper will result in substituting $\partial_{\nu}^* \psi_i|_{\tilde{\Gamma}}$ with $[\partial_{\nu} \psi_i^* - p_i v]|_{\tilde{\Gamma}}$ in the matrix $W_i$ in (4.9), which then—with this modification—becomes full rank, as desired. Thus, the stabilizing control will be expressed in terms of the pressure on the boundary, which is typically unknown.

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IRENA LASIECKA & ROBERTO TRIGGIANI
Department of Mathematical Sciences,
The University of Memphis,
Memphis
TN
38152 USA.

and

IRENA LASIECKA
IBS,
Polish Academy of Sciences,
Warsaw,
Poland.
e-mail: lasiecka@memphis.edu

ROBERTO TRIGGIANI
e-mail: rtrggani@memphis.edu

and

BUDDHIKA PRIYASAD
621, Institute for Mathematics and Scientific Computing,
Heinrichstrasse 36,
8010 Graz
Austria.
e-mail: buddhikapriyasad@gmail.com

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