IMPROVED SUBCONVEXITY BOUNDS FOR $GL(2) \times GL(3)$ AND $GL(3)$ $L$-FUNCTIONS BY WEIGHTED STATIONARY PHASE

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Abstract. Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$, and $u$ an even Hecke-Maass form for $SL(2, \mathbb{Z})$ with Laplace eigenvalue $1/4 + k^2$, $k \geq 0$. A subconvexity bound $O((1 + k)^{4/3 + \varepsilon})$ in the eigenvalue aspect is proved for the central value at $s = 1/2$ of the Rankin-Selberg $L$-function $L(s, f \times u)$. Meanwhile, a subconvexity bound $O((1 + |t|)^{2/3 + \varepsilon})$ in the $t$ aspect is proved for $L(1/2 + it, f)$. These bounds improved corresponding subconvexity bounds proved by Xiaoqing Li (Annals of Mathematics, 2011). The main techniques in the proofs, other than those used by Li, are $n$th-order asymptotic expansions of exponential integrals in the cases of the explicit first derivative test, the weighted first derivative test, and the weighted stationary phase integral, for arbitrary $n \geq 1$. These asymptotic expansions sharpened the classical results for $n = 1$ by Huxley.

1. Introduction

Bounds for automorphic $L$-functions on the critical line $\text{Re}(s) = 1/2$ are central questions in number theory and have far-reaching applications (cf. Iwaniec and Sarnak [13] and Michel [26]). The ultimate conjectured bounds are predicted by the Lindelöf Hypothesis, while trivial bounds include the convexity bounds as a consequence of the Phragmén-Lindelöf principle. Any bound which have a power saving over the corresponding convexity bound is highly non-trivial and called a subconvexity bound.

The strength of a subconvexity bound is crucial. There are important applications which depend on the strength of the subconvexity bounds. A notable example is the number of real zeros of a holomorphic Hecke cusp form $f$ for $SL(2, \mathbb{Z})$ of weight $k$, i.e., zeros of $f$ on $\{iy | y \geq 1\}$. By Ghosh and Sarnak [7], the number of such zeros is $\gg \log k$. Their proof uses a Weyl-like, i.e., a $1/3$ power-saving, subconvexity bound for $L(s, f)$ proved by Peng [28] and Jutila and Motohashi [15]. Note that a subconvexity bound for $L(s, f)$ with a power saving less than $1/3$ does not suffice in [7].

In this paper, we will prove subconvexity bounds for certain Rankin-Selberg $L$-functions for $GL(3) \times GL(2)$ and automorphic $L$-functions for $GL(3)$ over $\mathbb{Q}$ which improve bounds established by Xiaoqing Li [20].

Theorem 1.1. Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$ normalized by $A(1, 1) = 1$, and $\{u_j\}$ an orthonormal basis of even Hecke-Maass forms for $SL(2, \mathbb{Z})$. Denote by $1/4 + t_j^2$, $t_j \geq 0$, the...
Laplace eigenvalue of $u_j$. Then for large $T$ and $T^{1/3+\varepsilon} \leq M \leq T^{1/2}$ we have
\begin{equation}
\sum_j e^{-(t_j - T)^2/M^2} L \left( \frac{1}{2}, f \times u_j \right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-(t-T)^2/M^2} \left| L \left( \frac{1}{2} - it, f \right) \right|^2 dt \ll_{\varepsilon,f} T^{1+\varepsilon} M
\end{equation}
for any $\varepsilon > 0$.

Note that in [20] the same (1.1) was proved for $T^{3/8+\varepsilon} \leq M \leq T^{1/2}$. As pointed out in [20], (1.2)
\begin{equation}
L \left( \frac{1}{2}, f \times u_j \right) \geq 0
\end{equation}
was proved by Lapid [17] because $f$ is orthogonal and $u_j$ is symplectic (Jacquet and Shalika [14]). The nonnegativity in (1.2) allows us to deduce a bound for individual terms from (1.1).

We remark that the normalization of $u_j$ is different from the normalization $\lambda_{u_j}(1) = 1$ as required in the definition of $L(s, f \times u_j)$, but the discrepancy is within $t_j^{\varepsilon}$ as proved in Hoffstein and Lockhart [10]. The smaller allowable power of $T$ for $M$ in Theorem 1.1 gives us a smaller subconvexity bound.

**Corollary 1.2.** Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$ normalized by $A(1,1) = 1$, and $u$ an even Hecke-Maass form for $SL(2, \mathbb{Z})$ normalized by $\lambda_u(1) = 1$. Denote by $1/4 + k^2$, $k > 0$, the Laplace eigenvalue of $u$.
\[
L \left( \frac{1}{2}, f \times u \right) \ll_{\varepsilon,f} k^{4/3+\varepsilon}.
\]

Note that Corollary 1.2 improved the bound $O(k^{11/8+\varepsilon})$ proved in [20]. The convexity bound is $O(k^{3/2+\varepsilon})$. Because of the nonnegativity (1.2), the bound in (1.1) implies a square moment bound for $L(s, f)$ over a short interval.

**Corollary 1.3.** Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$ normalized by $A(1,1) = 1$. Then for $T^{1/3+\varepsilon} \leq M \leq T^{1/2}$
\begin{equation}
\int_{-\infty}^{\infty} e^{-(t-T)^2/M^2} \left| L \left( \frac{1}{2} - it, f \right) \right|^2 dt \ll_{\varepsilon,f} T^{1+\varepsilon} M.
\end{equation}

Since $f$ is a $GL(3)$ form, the square moment in (1.3) is comparable to a sixth power moment of the Riemann zeta function. Similar arguments were carried out for a $GL(2)$ form in Ye [32] and Lau, Liu and Ye [18].

By a standard argument of analytic number theory (cf. Heath-Brown [21] or Ivić [12], p. 197), we derived a subconvexity bound for $L(s, f)$ in the $t$ aspect. Its improvement over [20]'s $O((1 + |t|)^{11/16+\varepsilon})$ is again based on the smaller allowable power of $T$ for $M$. The convexity bound is $O((1 + |t|)^{3/4+\varepsilon})$.

**Corollary 1.4.** Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$ normalized by $A(1,1) = 1$. Then
\[
L \left( \frac{1}{2} + it, f \right) \ll_{\varepsilon,f} (1 + |t|)^{2/3+\varepsilon}.
\]
Following Ye and Deyu Zhang [33], we can deduce the following result on zero density for $L(s, f)$ from (1.3). Let
$$N_f(\sigma, T, T+T^\delta) = \#\{\rho = \beta + i\gamma \mid L(\rho, f) = 0, \sigma < \beta < 1, T \leq \gamma \leq T + T^\delta\}$$
be the number of zeros of $L(s, f)$ in the box of $\sigma < \beta < 1$ and $T \leq \gamma \leq T + T^\delta$.

**Corollary 1.5.** Let $f$ be a fixed self-contragradient Hecke-Maass form for $SL(3, \mathbb{Z})$. Then for $1/3 < \delta \leq 1$, we have
$$N_f(\sigma, T, T+T^\delta) \ll_{\varepsilon,f, T} T^{\frac{12+4\delta}{3} - 2\sigma + \varepsilon} \text{ for } 1/2 \leq \sigma < \frac{2 + \delta}{2 + 2\delta};$$
$$\ll_{\varepsilon,f} T^{2(1+\delta)(1-\sigma)+\varepsilon} \text{ for } \frac{2 + \delta}{2 + 2\delta} \leq \sigma < 1.$$ (1.4)

We note that Corollary 1.5 shows that (1.4) is now valid on a shorter interval $[T, T+T^\delta]$ with $1/3 < \delta \leq 1$ than the interval with $3/8 < \delta \leq 1$ in [33] which uses Li [20].

As noted in [20], Theorem 1.1 can also be proved for $f$ being the minimal Eisenstein series on $GL(3)$. This has been carried out in Lu [23]. Our proof and improvement can also be applied to that case.

P. Sarnak pointed out to us that for a holomorphic cusp form $g$ for $SL(2, \mathbb{Z})$, the Dirichlet series for the $L$-functions $L(s, \text{Sym}^2 g)$ and $L(s, \text{Sym}^2 g \times u_j)$ have the same structure and properties as $L(s, f)$ and $L(s, f \times u_j)$, respectively, for $f$ being a self-dual Maass form for $SL(3, \mathbb{Z})$ (cf. Bump [4, 5] and Luo and Sarnak [24]). Consequently our theorem and corollaries are also valid for such $L(s, \text{Sym}^2 g)$ and $L(s, \text{Sym}^2 g \times u_j)$.

The main techniques of our proof, other than those used in [20], include an asymptotic expansion of exponential integrals
$$\int_{\alpha}^{\beta} g(x)e(f(x)) \, dx$$
when $f'(x)$ changes signs at a point $x = \gamma$ with $\alpha < \gamma < \beta$. Huxley [11] obtained the first-order asymptotic expansion of (1.5). His results [11] are used widely as standard techniques in analytic number theory and other branches of mathematics.

What we need in our proof, however, is an asymptotic expansion of (1.5) beyond the first order. Blomer, Khan and Young [3] proved such an asymptotic expansion for $f(x)$ being smooth and $g(x)$ being smooth of compact support. In [23] we proved a similar asymptotic expansion for $f(x)$ being continuously differentiable $2n + 3$ times and $g(x)$ being continuously differentiable $2n + 1$ times on a finite interval $[\alpha, \beta]$. Since the latter one is explicitly written, we will use it in the present paper:
$$\int_{\alpha}^{\beta} g(x)e(f(x)) \, dx = \frac{e(f(\gamma) \pm 1/8)}{\sqrt{|F''(\gamma)|}} \left( g(\gamma) + \sum_{j=1}^{n} \bar{\omega}_{2j} \frac{(-1)^j(2j-1)!!}{(2\pi i f''(\gamma))^j} \right) + \text{Boundary terms + Error terms.}$$

Here $\gamma$ is the only zero of $f'(x)$ in $(\alpha, \beta)$, and $\bar{\omega}_{2j}$ are given in (2.4). Note that the boundary terms do not appear in [3]. See Proposition 2.2 below for detail. We will apply Voronoi’s summation formula (Lemma 3.1) and its asymptotic expansion (Lemma 3.2) to the leading term of (2.4) for all $\bar{\omega}_{2j}$ the second time.
In the following sections, $\varepsilon$ is any arbitrarily small positive number. Its value may be different on each occurrence.

2. Oscillatory Integrals

The following proposition is the weighted first derivative test, which strengthens Lemma 5.5.5 of [11], p.113, with more boundary terms and smaller error terms. We can also use a similar formula proved in Jutila and Motohashi [15], Lemma 6.

**Proposition 2.1.** (McKee, Sun and Ye [25]) Let $f(x)$ be a real-valued function, $n + 2$ times continuously differentiable for $\alpha \leq x \leq \beta$, and $g(x)$ a real-valued function, $n + 1$ times continuously differentiable for $\alpha \leq x \leq \beta$. Suppose that there are positive parameters $M$, $N$, $T$, $U$, with $M \geq \beta - \alpha$, and positive constants $C_r$ such that for $\alpha \leq x \leq \beta$,

$$|f^{(r)}(x)| \leq C_r \frac{T}{M^r}, \quad |g^{(s)}(x)| \leq C_s \frac{U}{N^s},$$

for $r = 2, \ldots, n + 2$, and $s = 0, \ldots, n + 1$. If $f'(x)$ and $f''(x)$ do not change signs on the interval $[\alpha, \beta]$, then we have

$$\int_{\alpha}^{\beta} g(x)e(f(x))dx = \left[ e(f(x)) \sum_{i=1}^{n} H_i(x) \right]_{\alpha}^{\beta} + O\left( \frac{M}{N} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{UT^j}{\min |f^{[n+j+1]}(x)|} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t}M^t} \right)$$

$$+ O\left( \left( \frac{M}{N} + 1 \right) \frac{U}{N^n \min |f'|^{n+1}} \right) + O\left( \sum_{j=1}^{n} \frac{U}{\min |f'|^{n+j+1}M^j} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t}M^t} \right),$$

where

$$(2.1) \quad H_1(x) = \frac{g(x)}{2\pi i f'(x)}, \quad H_i(x) = -\frac{H_{i-1}'(x)}{2\pi i f'(x)}$$

for $i = 2, \ldots, n$.

The following proposition is for a weighted stationary phase integral and sharpens Lemma 5.5.6 of [11], p.114, with main terms up to the $n$th order, more boundary terms and smaller error terms. In [3], Proposition 8.2 Blomer, Khan and Young obtained the same main terms and the last big-$O$ term as in (2.4), under the assumptions that $f(x)$ and $g(x)$ are smooth and $g(x)$ is compactly supported on $\mathbb{R}$. We may use their version in the present paper.

**Proposition 2.2.** (McKee, Sun and Ye [25]) Let $f(x)$ be a real-valued function, $2n + 3$ times continuously differentiable for $\alpha \leq x \leq \beta$, and $g(x)$ a real-valued function, $2n + 1$ times continuously differentiable for $\alpha \leq x \leq \beta$. Let $H_k(x)$ be defined as in (2.1). Assume that there are positive parameters $M$, $N$, $T$, $U$ with

$$(2.2) \quad M \geq \beta - \alpha,$$
and positive constants \( C_r \) such that for \( \alpha \leq x \leq \beta \),
\[
|f^{(r)}(x)| \leq C_r \frac{T}{M^r}, \quad |f^{(2)}(x)| \geq \frac{T}{C_2 M^2}, \quad |g^{(s)}(x)| \leq C_s \frac{U}{N^s},
\]
for \( r = 2, \ldots, 2n + 3 \), and \( s = 0, \ldots, 2n + 1 \). Suppose that \( f'(x) \) changes signs only at \( x = \gamma \), from negative to positive, with \( \alpha < \gamma < \beta \). Let
\[
\Delta = \min \left\{ \log \frac{2}{C_2}, \max_{2 \leq k \leq 2n+3} \{ C_k \} \right\}.
\]
If \( T \) is sufficiently large such that \( T^{\frac{2n+3}{\Delta}} \Delta > 1 \), we have for \( n \geq 2 \) that
\[
\int_\alpha^\beta g(x) e(f(x)) \, dx = \frac{e(f(\gamma) + \frac{k}{\ell} t)}{\sqrt{f''(\gamma)}} (g(\gamma) + \sum_{j=1}^n \omega_j (-1)^j (2j - 1)!! (4\pi i \lambda_{2j})^j) + \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_\alpha^\beta + O \left( \frac{UM^{2n+4}}{T^{n+2} N^{n+2}} \left( \frac{1}{(\gamma - \alpha)^{n+2}} + \frac{1}{(\beta - \gamma)^{n+2}} \right) \right) + O \left( \frac{UM^{2n+4}}{T^{n+2} N^{n+2}} \left( \frac{1}{(\gamma - \alpha)^{2n+3}} + \frac{1}{(\beta - \gamma)^{2n+3}} \right) \right)
\]
where
\[
\lambda_j = \frac{f^{(j)}(\gamma)}{j!} \quad \text{for} \quad j = 2, \ldots, 2n + 2, \quad \eta_\ell = \frac{g^{(\ell)}(\gamma)}{\ell!} \quad \text{for} \quad \ell = 0, \ldots, 2n,
\]
and
\[
\omega_k = \eta_k + \sum_{\ell=0}^{k-1} \eta_\ell \sum_{j=1}^{k-\ell} C_{k\ell j} \lambda_j \sum_{3 \leq n_1, \ldots, n_j \leq 2n+3} \lambda_{n_1} \cdots \lambda_{n_j},
\]
with \( C_{k\ell j} \) being some constant coefficients.

3. Background on automorphic forms

We will follow the setting and notations in Li \cite{Li}. Recall for \( m, n \geq 1 \) the Kuznetsov trace formula (Kuznetsov \cite{Kuznetsov} and Conrey and Iwaniec \cite{ConreyIwaniec})
\[
\sum_{j \geq 1} \omega_j \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_{\mathbb{R}} h(t) \omega(t) \tilde{\eta} \left( m, \frac{1}{2} + it \right) \eta \left( n, \frac{1}{2} + it \right) \, dt = \delta(m, n) \frac{H}{2} + \sum_{c \geq 1} \frac{1}{2c} \left( S(m, n; c) H^+ \left( \frac{4\pi \sqrt{mn}}{c} \right) + S(-m, n; c) H^- \left( \frac{4\pi \sqrt{mn}}{c} \right) \right).
\]
Here \( \sum' \) in (3.1) means we are only summing over even Maass forms \( u_j \), \( \delta(m, n) \) is the Kronecker delta,
\[
\omega_j = \frac{4\pi \rho_j(1)^2}{\cosh \pi t_j}, \quad \omega(t) = \frac{|\phi(1/2 + it)|^2}{\cosh \pi t},
\]
\[
H = \frac{2}{\pi} \int_0^\infty h(t) \tanh(\pi t) t \, dt, \quad H^+(x) = 2i \int_{\mathbb{R}} J_{2it}(x) \frac{h(t)t}{\cosh \pi t} \, dt,
\]
we only deal with $\Psi_0$

Suppose Lemma 3.2.

\[ J_\nu(z) \]

is the standard Kloosterman sum. Above, $J_\nu$ is the $J$-Bessel function.

We let $f$ be a Maass form of type $\nu = (\nu_1, \nu_2)$ for $SL_3(\mathbb{Z})$ (cf. Goldfeld [8]). Then $f$ has a Whittaker function expansion

\[ f(z) = \sum_{\pm \Gamma \setminus SL_2(\mathbb{Z})} \sum_{m_1=0}^{\infty} \sum_{m_2 \neq 0} A(m_1, m_2) \frac{W_f(M \left( \begin{smallmatrix} 0 & 1 \\ \gamma & \nu \end{smallmatrix} \right) z, \nu, \psi_{1,1})}{m_1 m_2} \]

where $W_f$ is the Jacquet-Whittaker function, $M = \text{diag}(m_1 m_2, m_1, 1)$, and $\psi_{1,1}$ is a fixed specific generic character on the abelianization of the standard unipotent upper triangular subgroup of $SL_3(\mathbb{Z})$. Put $\alpha = -\nu_1 - 2\nu_2 + 1$, $\beta = -\nu_1 + \nu_2$, $\gamma = 2\nu_1 + \nu_2 - 1$. These are the Langlands parameters at $\infty$ of $f$. In the usual way, we put

\[ \tilde{\psi}(s) = \int_0^\infty \psi(x) x^{s-1} \, dx \]

to be the Mellin transform of $\psi$ which we assume is smooth and compactly supported on $(0, \infty)$.

For $k = 0, 1$ we define

\[ \Psi_k(x) = \int_{\text{Res} = \sigma} (\pi^3 x)^{-i} \frac{\Gamma(\frac{1+s+2k+a}{2}) \Gamma(\frac{1+s+2k+b}{2}) \Gamma(\frac{1+s+2k+c}{2})}{\Gamma(-\frac{s-\alpha}{2}) \Gamma(-\frac{s-\beta}{2}) \Gamma(-\frac{s-\gamma}{2})} \tilde{\psi}(-s - k) \, ds. \]

Here $\sigma$ is taken sufficiently large depending on $\alpha, \beta, \gamma$. We then define, for $k = 0, 1$,

\[ (3.3) \quad \Psi_{k,1}^1(x) = \Psi_{0,1}(x) + (-1)^k \frac{1}{x^{3/4}} \Psi_{1}(x). \]

Then the following is a crucial tool, the Voronoi formula for $GL(3)$.

**Lemma 3.1.** ([27]) Let $\psi \in C_c^\infty(0, \infty)$. Let $f$ be a $SL_3(\mathbb{Z})$ Maass form with corresponding Fourier coefficients $A(m, n)$ as in (3). Let $d, \bar{d}, c \in \mathbb{Z}$ with $c \neq 0$, $(d, c) = 1$, and $dd \equiv 1 (mod c)$. Then

\[ (3.4) \quad \sum_{n > 0} A(m, n) \left( \frac{cd}{c} \right) \psi(n) = \frac{1}{4\pi^{5/2}} \sum_{n_1 | cm, n_2 > 0} A(n_2, n_1) \frac{S(md, n_2; mc; n_1)}{n_1 n_2} \Psi_{0,1}^0 \left( \frac{n_1^2 n_2}{c^3 m} \right) \]

\[ \quad + \frac{1}{4\pi^{5/2}} \sum_{n_1 | cm, n_2 > 0} A(n_1, n_2) \frac{S(md, -n_2; mc; n_1)}{n_1 n_2} \Psi_{0,1}^1 \left( \frac{n_1^2 n_2}{c^3 m} \right). \]

To use this formula, asymptotics of $\Psi_0, \Psi_1$ are needed which were proved in Li [19] and Ren and Ye [29] for $GL(3)$. (For $GL(m)$ see Ren and Ye [30].) Since $x^{-1} \Psi_{1}(x)$ has similar asymptotics to $\Psi_0$, following [20], we only deal with $\Psi_0$. We will use the following Lemma ([19]):

**Lemma 3.2.** Suppose $\psi \in C_c^\infty([X, 2X])$. Then for any fixed integer $K \geq 1$ and $xX \gg 1$ we have

\[ \Psi_0(x) = 2\pi^3 x i \int_0^\infty \psi(y) \sum_{j=1}^K c_j \cos(6\pi(xy)^{1/3}) + d_j \sin(6\pi(xy)^{1/3}) \frac{dy}{(xy)^{1/3}} + O((xX)^{2-K}). \]

Here $c_j$ and $d_j$ are constants depending on the Langlands parameters with $c_1 = 0$ and $d_1 = -2/\sqrt{3\pi}$. 
We now assume \( f \) is a self-dual Hecke-Maass form for \( SL_3(\mathbb{Z}) \) of type \((\nu, \nu)\), normalized so that \( A(1, 1) = 1 \). The Rankin-Selberg \( L \)-function of \( f \) with itself is then defined by

\[
L(s, f \times f) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{|A(m, n)|^2}{(m^2 n)^s}
\]

for \( \text{Re} s \) large. \( L(s, f \times f) \) has meromorphic continuation to the complex plane, with a simple pole at \( s = 1 \).

By a standard analytic number theory argument using complex analysis, this gives

\[
\sum_{m^2 n \leq N} |A(m, n)|^2 \ll f N.
\]

Applying Cauchy-Schwartz, this gives

\[
(3.5) \sum_{n \leq N} |A(m, n)| \ll f |m| N.
\]

We will use (3.5) and summation by parts in the estimates below. Here \( f \) being self-dual also means \( A(m, n) = A(n, m) \) for all \( m, n \).

The Rankin-Selberg \( L \)-function of \( f \) with \( u_j \) is (for \( \text{Re} s \) sufficiently large)

\[
L(s, f \times u_j) = \sum_{m \geq 1} \sum_{n \geq 1} \lambda_j(n) A(m, n) \frac{m^2 n}{(m^2 n)^s}.
\]

\( L(s, f \times u_j) \) can be completed to \( \Lambda(s, f \times u_j) \) with six \( \Gamma \) factors at \( \infty \) (involving the Langlands parameters of \( f \), and \( t_j \)).

We now need to define the Rankin-Selberg \( L \)-function of \( f \) with the Eisenstein series. See Li [20] for the definition of \( E(z, s) \) and \( \eta(n, s) \).

\[
L(s, f \times E) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\bar{\eta}(n, 1/2 + it) A(m, n)}{(m^2 n)^s}.
\]

Following Goldfeld [8], comparing Euler products, we have

\[
L\left(\frac{1}{2}, f \times E\right) = \left| L\left(\frac{1}{2} - it, f\right) \right|^2.
\]

We need to set up the approximate functional equation. We define

\[
\gamma(s, t) = \pi^{-3s} \Gamma\left(\frac{s - it - \alpha}{2}\right) \Gamma\left(\frac{s - it - \beta}{2}\right) \Gamma\left(\frac{s - it - \gamma}{2}\right) \times \Gamma\left(\frac{s + it - \alpha}{2}\right) \Gamma\left(\frac{s + it - \beta}{2}\right) \Gamma\left(\frac{s + it - \gamma}{2}\right).
\]

Here \( \alpha = -3\nu + 1, \beta = 0, \) and \( \gamma = 3\nu - 1 \) are the Langlands parameters at \( \infty \) of \( f \). We define \( F(u) = (\cos(\pi u/A))^{-3A} \) for \( A \) a positive integer. For \( |\text{Im} t| \leq 1000 \) we now define

\[
(3.6) V(y, t) = \frac{1}{2\pi i} \int_{(1000)} y^{-u} F(u) \frac{\gamma(1/2 + u, t)}{\gamma(1/2, t)} \frac{du}{u}.
\]

By known bounds for the Langlands parameters, this integral converges. We have the important approximate functional equation (cf. [20]):
Lemma 3.3. For \( f \) a self-dual Maass form of type \((\nu, \nu)\) for \( SL_3(\mathbb{Z}) \) and \( u_j \) a Hecke-Maass form for \( SL_2(\mathbb{Z}) \) corresponding to the eigenvalue \( 1/4 + t_j^2 \) in an orthonormal basis, as above,

\[
L\left(\frac{1}{2}, f \times u_j\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n)A(m, n)}{\sqrt{m^2n}} V(m^2n, t_j).
\]

The point of using \( V \) in the expansion (3.7) is that \( V \) decays rapidly for \( m^2n \gg |t_j|^{3+\varepsilon} \), and so in an effective way, we can take both sums above to be finite. For the precise decay rate, see Lemma 2.3 of Li [20]. We also have the approximate functional equation for \( L(s, f \times E) \):

\[
L\left(\frac{1}{2}, f \times E\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, 1/2 + it)A(m, n)}{\sqrt{m^2n}} V(m^2n, t).
\]

Following Li [20] we now define

\[
W = \sum_j e^{-\frac{(t_j-t)^2}{m}} \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} e^{-\frac{(t-t')^2}{m}} |L\left(\frac{1}{2} - it, f\right)|^2 \, dt.
\]

Here \( \omega_j \) and \( \omega(t) \) are defined in (3.24). It is known that \( \omega_j \gg t_j^{-\varepsilon} \) and \( \omega(t) \gg t^{-\varepsilon} \). See the references in Li [20]. It follows that

\[
\sum_j e^{-\frac{(t_j-t)^2}{m}} L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} e^{-\frac{(t-t')^2}{m}} |L\left(\frac{1}{2} - it, f\right)|^2 \, dt \ll WT^{-\varepsilon}.
\]

Consequently, Theorem 1.1 will be proved if we show \( W \ll_{\varepsilon, f} T^{1+\varepsilon}M \). As Li [20] points out, the function \( e^{-\frac{(t-t)^2}{m}} \) cannot be used as a test function in the Kuznetsov trace formula simply because it is not even. Following Li [20] we will use the modified function

\[
k(t) = e^{-\frac{(t-t)^2}{m}} + e^{-\frac{(t+t)^2}{m}}
\]

which essentially captures the size of \( e^{-\frac{(t-t)^2}{m}} \) for \( t \) near \( T \). Thus, we define

\[
W = \sum_j k(t_j) \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} k(t) \omega(t) |L\left(\frac{1}{2} - it, f\right)|^2 \, dt.
\]

By plugging (3.7) and (3.8) into \( W \) in (3.9) we see that we need to analyze \( \mathcal{R} \) which we define by the equation

\[
\mathcal{R} = 2 \sum_j k(t_j) \omega_j \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n)A(m, n)}{\sqrt{m^2n}} V(m^2n, t_j) g\left(\frac{m^2n}{N}\right)
\]

\[
+ \frac{1}{2\pi} \int_{\mathbb{R}} k(t) \omega(t) \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, 1/2 + it)A(m, n)}{\sqrt{m^2n}} V(m^2n, t) g\left(\frac{m^2n}{N}\right) \, dt.
\]

Here, for the rest of this article we take \( N = T^{3+\varepsilon} \) and \( g \) is a fixed non-negative function with compact support in \([1, 2]\). This is the trick of using a dyadic partition of unity which is best outlined in Lau, Liu, and Ye [18].

Now, we apply the Kuznetsov trace formula (3.1) to \( \mathcal{R} \) (3.11). Consequently, we write

\[
\mathcal{R} = \mathcal{D} + \mathcal{R}^+ + \mathcal{R}^-;
\]
(3.13) \[ D = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \delta(n,1) H_{m,n}; \]

\[ H_{m,n} = \frac{2}{\pi} \int_{\mathbb{R}} k(t) V(m^2 n, t) \tanh(\pi t) t dt; \]

(3.14) \[ \mathcal{R}^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \sum_{c > 0} S(n,1; c) \frac{4\pi \sqrt{n}}{c} H_{m,n}^+ \left( \frac{4\pi \sqrt{n}}{c} \right); \]

(3.15) \[ H_{m,n}^+(x) = 2i \int_{\mathbb{R}} J_{2it}(x) \frac{k(t) V(m^2 n, t) t}{\cosh(\pi t)} dt; \]

(3.16) \[ \mathcal{R}^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \sum_{c > 0} S(n,-1; c) \frac{4\pi \sqrt{n}}{c} H_{m,n}^- \left( \frac{4\pi \sqrt{n}}{c} \right); \]

(3.17) \[ H_{m,n}^-(x) = \frac{4}{\pi} \int_{\mathbb{R}} K_{2it}(x) \sinh(\pi t) k(t) V(m^2 n, t) dt. \]

By the estimates in Section 3 of Li [20], we see easily that \( D \) in (3.13) is negligible for any \( M \) with \( T^c \leq M \leq T^{1-\varepsilon} \) and we leave the details for the reader. In the next two section we will estimate \( \mathcal{R}^+ \) in (3.14) and \( \mathcal{R}^- \) in (3.16).

4. Estimates for the J-Bessel function terms

In this section we provide estimates for \( \mathcal{R}^+ \) in (3.14). In this section and the next, we show estimates under the assumption \( T^{1/3+2\varepsilon} \leq M \leq T^{1/2} \). Following Li [20] we define the parameters

(4.1) \[ C_1 = T^{100}, \quad C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon} M}, \]

and we split \( \mathcal{R} = \mathcal{R}_1^+ + \mathcal{R}_2^+ + \mathcal{R}_3^+ \) with

(4.2) \[ \mathcal{R}_1^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \sum_{c \geq C_1/m} S(n,1; c) \frac{4\pi \sqrt{n}}{c} H_{m,n}^+ \left( \frac{4\pi \sqrt{n}}{c} \right), \]

(4.3) \[ \mathcal{R}_2^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \sum_{C_2/m \leq c \leq C_1/m} S(n,1; c) \frac{4\pi \sqrt{n}}{c} H_{m,n}^+ \left( \frac{4\pi \sqrt{n}}{c} \right), \]

(4.4) \[ \mathcal{R}_3^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g \left( \frac{m^2 n}{N} \right) \sum_{c \leq C_2/m} S(n,1; c) \frac{4\pi \sqrt{n}}{c} H_{m,n}^+ \left( \frac{4\pi \sqrt{n}}{c} \right). \]

For \( \mathcal{R}_1^+ \) in (4.2), Li [20] shifts the integral defining \( H_{m,n}^+ \) (see (3.13)), and uses an integral representation of the J-Bessel function and Stirling’s formula to conclude

(4.5) \[ H_{m,n}^+(x) \ll x^{\frac{1}{2}} T^\frac{1}{2} (m^2 n)^{-\frac{1}{2}} T^{1+\varepsilon} M. \]

Consequently (4.2) is bounded

(4.6) \[ \mathcal{R}_1^+ \ll T^{\frac{1}{2}+\varepsilon} M \sum_{m \leq \sqrt{2N}} \sum_{n \leq 2N/m^2} \left| A(m,n) \right| \sqrt{m} \sum_{c \geq C_1/m} \left| S(n,1;c) \right| \left( \frac{\sqrt{n}}{c} \right)^{\frac{1}{2}} (m^2 n)^{-\frac{3}{2}}. \]
Using Weil’s bound for $S(n, 1; c)$, we see

\[(4.7)\]

\[
\sum_{c \geq C_1/m} |S(n, 1; c)| e^{\frac{c \varepsilon}{m}} \ll \sum_{c \geq C_1/m} \frac{C_1}{m} \ll \left( \frac{C_1}{m} \right)^{-\frac{1}{2} + \varepsilon}.
\]

By (3.5) and summation by parts, we have

\[(4.8)\]

\[
\sum_{n \leq 2N/m^2} |A(m, n)| \ll m \left( \frac{N}{m^2} \right)^{\frac{1}{2}}.
\]

Inserting (4.7) and (4.8) into (4.6) we get

\[(4.9)\]

\[
R_i^+ \ll M N^+ C_1^{-\frac{1}{2}} \sum_{m \leq \sqrt{2N}} \frac{1}{m^{\frac{3}{2}}}. \]

Plugging in $C_1 = T^{100}$ from (4.1), $N = T^{3+\varepsilon}$ and noticing the sum on $m$ in (4.9) converges, we have $R_i^+ \ll 1$ for any $M$ with $T^\varepsilon \leq M \leq T^{1-\varepsilon}$. We now deal with $R_i^+$ in (4.3). We do not wish to reproduce all the estimates in Li [20] so we will summarize. As used in Liu and Ye [21] [22] and Li [20] we need an integral representation for

\[
J_{2it}(x) - J_{-2it}(x)
\]

from 1.13(69) of [2], vol.1, p.59. Using integration by parts, a change of variables, and the fact that $k(t)$ (recall (3.9)) is a Schwartz function, we define

\[
W_{m,n}(x) = T \int_\mathbb{R} \hat{k}(\zeta) \cos \left( x \cosh \left( \frac{\zeta \pi}{M} \right) \right) e \left( -\frac{T \zeta}{M} \right) d\zeta.
\]

Here

\[
k^*(t) = e^{-t^2} V(m^2 n, t M + T)
\]

is a Schwartz function, and $\hat{k}$ is its Fourier transform. We remark that derivatives of $k^*(t)$ are $\ll 1$. In fact, by (3.9) $\frac{\partial^t}{\partial t^t} V(y, t M + T)$ can be expressed in terms of derivatives of $\gamma(s, t M + T)$ and hence in terms of $\frac{d^s}{dz^s} \log \Gamma(z) =: \psi(z)$ and $\psi^{(t)}(z)$ (Bateman [1] p.15, 1.7(1), and p.45, 1.16(9)). By their asymptotic expansions in [1], p.47, 1.18(7), and p.48, 1.18(9), we can see

\[
\frac{\partial^t}{\partial t^t} V(y, t M + T) \ll \left( \frac{M}{T} \right)^t.
\]

We define

\[(4.10)\]

\[
W_{m,n}^*(x) = T \int_\mathbb{R} \hat{k}(\zeta) e \left( -\frac{T \zeta}{M} - \frac{x}{2\pi} \cosh \left( \frac{\zeta \pi}{M} \right) \right) d\zeta,
\]

so that

\[
W_{m,n}(x) = \frac{W_{m,n}^*(x) + W_{m,n}^*(-x)}{2}.
\]

The upshot here is that up to a lower order term (which can be handled in a similar way) and a negligible amount, we have $H^+_{m,n}(x) = 4W_{m,n}(x)$. 

The contribution to the integral in (4.10) from $|\zeta| \geq T^\varepsilon$ is a negligible amount, so in what follows we can assume $|\zeta| \leq T^\varepsilon$. The phase $\phi(\zeta)$ in the exponential (4.10) is

$$2\pi \phi(\zeta) = -\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh \left( \frac{\zeta x}{M} \right).$$

Looking at $\phi'(\zeta)$, we see $W_{m,n}^*(x)$ is negligible for $|x| \leq T^{1-\varepsilon} M$. So in what follows we assume $T^{1-\varepsilon} M \leq |x| \leq T^2$. Using a Taylor expansion in $\zeta$ (within the exponential) of

$$e\left( -\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh \left( \frac{\zeta x}{M} \right) \right)$$

in (4.10), using the Fourier transform of a Gaussian, using Parseval’s Theorem, completing the square, and the brackets in (4.11), as compared to [20], comes from a degree 2 Taylor expansion in $\zeta$, based on ideas in Sarnak [31]. For our purposes we can modify the proof of Proposition 4.1 of [20].

Lemma 4.1. 1) For $|x| \leq T^{1-\varepsilon} M$ we have $W_{m,n}^*(x) \ll_{\varepsilon,A} T^{-A}$.

2) For $T^{1-\varepsilon} M \leq |x| \leq T^2$, with $T^{1/3+2\varepsilon} M \leq T^{1/2}$ and $L_1, L_2 \geq 1$,

$$(4.11) \quad W_{m,n}^*(x) = \frac{TM}{\sqrt{|x|}} e\left( -\frac{x}{2\pi} + \frac{T^2}{\pi x} \right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{l_2 \leq L_2} c_{l_1,l_2} \frac{M^{2l_1} T^{4l_2-l_1}}{x^{l_1+3} l_2-l_1} \times \left[ \hat{k}^*(2l_1) - \frac{2MT}{\pi x} \right] - \frac{\pi^6 i x}{6! M^6} \left( y \hat{k}^*(y) \right)^{(2l_1-1)}

+ \frac{\pi^{12} i^2 x^2}{2! (6!)^2 M^2} \left( y \hat{k}^*(y) \right)^{(2l_1-1)} \left( -\frac{2MT}{\pi x} \right)

+ O\left( \frac{TM}{\sqrt{|x|}} \left( \frac{T^4}{|x|^3} \right)^{L_2+1} \right) + O\left( \frac{M}{\sqrt{|x|}} \left( \frac{T^4}{|x|^3} \right)^{2L_1+3} + \frac{T|x|}{M^{18}} \right),$$

where $c_{l_1,l_2}$ are constants depending only on the indices.

Note Part 1) is valid for $T^\varepsilon \leq M \leq T^{1-\varepsilon}$, and Part 2) is valid for $T^{1/3+\varepsilon} M \leq \sqrt{T}$ with the assumption of $T^{1-\varepsilon} M \leq |x| \leq T^2$. With our assumption $T^{1/3+2\varepsilon} M \leq \sqrt{T}$ on $M$, to acquire the desired decay rate of the

$$(4.12) \quad O\left( \frac{TM}{\sqrt{|x|}} \left( \frac{T^4}{|x|^3} \right)^{L_2+1} \right)$$

term, $L_2$ could depend on $\varepsilon$. From 1) of Lemma [11] and (4.3) we see $R^*_2$ is negligible. The extra term in the brackets in (4.11), as compared to [20], comes from a degree 2 Taylor expansion in $x$ (with remainder) of $e(-\pi^6 i x \zeta^6 / (2 \cdot 6! M^6))$.

In the rest of this section, we estimate $R^*_3$ as in (4.4). By choosing $L_1, L_2$ large enough (possibly depending on $\varepsilon$) in (4.11) the contribution to $R^*_3$ from the first two error terms in (4.11) can be made as small as desired. We need to estimate the contribution from the last error term in (4.11). By the support of $g$ we may assume $x^2 = 16\pi n/e^2 \ll N = T^{3+\varepsilon}$. By our assumptions on $M$ and $T$ we then have $T|x|^3/M^{18} \ll |x|/M^9$. Plugging in $x = 4\pi \sqrt{n}/e$ into $T|x|^3/M^9$, we estimate this error term contribution to $R^*_3$ in (4.11), using (4.5), Weil’s bound for the Kloosterman sum, and the compact support of $g$. This error can be seen to be bounded.
by \(O(TN/M^3)\) which is smaller than \(O(T^{1+\varepsilon}M)\) by a power of \(T\) with our assumption \(T^{1/3+2\varepsilon} \leq M \leq \sqrt{T}\).

In the finite series (4.11) with our assumptions we also have \(M^{2l-1}, T^{4l_2-1}, x^1, -l-3\eta, \ll 1\). All the terms in (4.11) are similar, and can be estimated in a similar way, so we will only work with the first term. Following Li [20] we define

\[
\tilde{R}_3^+ = \frac{i(i + 1)MT}{\sqrt{2\pi}} \sum_{m \geq 1} \sum_{n \geq 1} A(m, n) \frac{g(m^2n/N)}{mn^{3/4}} e\left(\frac{m^2n}{N}\right) \times \sum_{c \leq C_2/m} S(n, 1; c) \frac{2\sqrt{c}}{\sqrt{e}} \left(\frac{T^2c}{4\pi^2c^2}\right) \kappa^{*} \left(\frac{MTc}{2\pi^2c}\right).
\]

Li [20] points out here, that even with Weil’s bound for \(S(n, 1; c)\) simple estimates for \(\tilde{R}_3^+\) are too large. So we expand the Kloosterman sum \(S(n, 1; c)\) and use the Voronoi formula (Lemma 3.1) with

\[
\psi(y) = y^{-\frac{1}{2}} g\left(\frac{m^2y}{N}\right) e\left(\frac{2\sqrt{y}}{\sqrt{c}} \right) - \frac{T^2c}{4\pi^2\sqrt{y}} \kappa^{*} \left(\frac{MTc}{2\pi^2\sqrt{y}}\right).
\]

We get

\[
\tilde{R}_3^+ = \frac{(i - 1)MT}{\sqrt{2\pi}} \sum_{m \geq 1} \frac{1}{m} \sum_{c \leq C_2/m} \frac{1}{\sqrt{c}} \sum_{d \text{ (mod } c)} e\left(\frac{d}{c}\right) \sum_{n \geq 1} A(m, n) e\left(\frac{nd}{c}\right) c(e/n) \psi(n),
\]

where the innermost sum in (4.15) will be replaced by the right hand side of (3.4).

From the function \(g(m^2y/N)\) in (4.14) we can see that \(X = N/m^2\). Recall \(x = n_2n_1^2/(c^3m)\) from Lemma 3.1. Then by \(c \leq C_2/m\)

\[
x X = \frac{n_2n_1^2N}{c^3m^3} \geq \frac{n_2n_1^2N}{C_2^3} = \frac{n_2n_1^2T^{3-3\varepsilon}M^3}{\sqrt{N}} \geq n_2n_1^2T^{3/2-3\varepsilon}M^{3/2-1/2} \gg 1.
\]

Consequently we can apply Lemma 3.2 to (4.15) with (3.4) to get

\[
\Psi_0(x) = \pi^2d_1 x^{2/3} \int_0^\infty c(u_1(y))a(y) dy - \pi^2d_1 x^{2/3} \int_0^\infty c(u_2(y))a(y) dy
\]

with

\[
u_1(y) = \frac{2\sqrt{y}}{c} + 3(xy)^{1/3}, \quad u_2(y) = \frac{2\sqrt{y}}{c} - 3(xy)^{1/3}
\]

and

\[
a(y) = g\left(\frac{m^2y}{N}\right) \kappa^{*} \left(\frac{MTc}{2\pi^2\sqrt{y}}\right) e\left(\frac{-T^2c}{4\pi^2\sqrt{y}}\right) y^{-13/12}.
\]

Note that \(u_1\) has no stationary points; indeed simple calculus estimates give the first integral in (4.10) a negligible contribution to \(\tilde{R}_3^+\).

The second integral in (4.10) requires more analysis. As in [20], p.319, if \(x \geq 2\sqrt{N}/(c^3m)\) or \(x \leq 2\sqrt{N}/(3c^3m)\), then \(u_2'(y)\) will be effectively bounded away from zero, making the integral negligible by multiple integration by parts. Thus we assume the contrary in what follows, namely

\[
\frac{2\sqrt{N}}{3n_1^2} \leq n_2 \leq \frac{\sqrt{N}}{n_1^2}.
\]
We have

\[(4.20) \quad \int_0^\infty e(u_2(y))a(y) \, dy = \int_{\frac{2\pi^2 c^6}{x^2 c^6}}^{2\pi^2 c^6} e(u_2(y))a(y) \, dy. \]

We explain the limits of integration. The compact support of the integral on the right side of equation \((4.20)\) follows from the compact support of \(g\), and so that of \(a\). Further, recall \(x = n_2n_1^2/(c^3 m)\). As Li \([20]\) points out, the stationary phase point of the integral in \((4.20)\) is at \(y_0 = x^2 c^6\). The constants \(1/4\) and \(9/2\) in the limits of this integral give a segment that the support of \(a\) is contained in, since \(g \in C_c^\infty([1, 2])\). In \((4.21)\), from the support of \(g\), and since \(\hat{g}^*\) is a Schwartz function, we can assume

\[
\frac{N}{m^2} \leq y \leq \frac{2N}{m^2} \quad \text{and} \quad \frac{MTc}{2\pi^2 \sqrt{y}} \ll T^\varepsilon.
\]

Using this information, simple calculus estimates give us

\[(4.21) \quad u_2^{(r)}(y) \ll T_1 M_1^{-r} \quad \text{for} \quad r = 1, 2, \ldots, 2n_0 + 3\]

and

\[(4.22) \quad a^{(r)}(y) \ll U_1 N_1^{-r} \quad \text{for} \quad r = 0, 1, 2, \ldots, 2n_0 + 1\]

for \(y\) in the segment. Here \(n_0 \in \mathbb{N}\) will be chosen in terms of \(\varepsilon_0\) later, and

\[(4.23) \quad M_1 = \frac{N}{m^2}, \quad T_1 = \sqrt{\frac{N}{cm}}, \quad N_1 = \frac{N^{3/2}}{T^2 cm^3}, \quad U_1 = \left(\frac{N}{m^2}\right)^{-13/12}.\]

Further, \(u_2^{(2)}(y) \gg T_1 M_1^{-2} \quad \text{for} \quad y \in \left[\frac{1}{2}x^2c^6, \frac{9}{2}x^2c^6\right]\). The condition \(N_1 \geq M_1/\sqrt{T_1}\) is then consistent with our assumption \(c \leq C_2/m\) when \(M \geq T^{1/3+2\varepsilon}\).

Then, all assumptions \((2.2)\) and \((2.3)\) are satisfied for parameters in \((4.23)\), and we apply Proposition \((2.2)\) (where we take \(n = n_0\)). Or, one may use Blomer, Khan, and Young’s version in \([3]\). The main term of the integral in \((4.20)\) is

\[(4.24) \quad \frac{e(u_2(y_0) \pm 1/8)}{\sqrt{|u_2'(y_0)|}} (a(y_0) + \sum_{j=1}^{n_0} \varpi_{2j} \frac{(-1)^j (2j - 1)!!}{(4\pi^2 \lambda_2)^j}).\]

where \(\varpi_{2j}\) are defined above and \(\lambda_2 = u_2''(y_0)/2\). Notice we have used \(\gamma - \alpha = \beta - \gamma \simeq M_1\), with \(\alpha = \frac{1}{4}x^2c^6\), \(\beta = \frac{9}{2}x^2c^6\) and \(\gamma = y_0 = x^2c^6\). To save time in estimates, notice there are no boundary terms here. This is due to the compact support of \(a\), with itself and all of its derivatives zero at \(\frac{1}{4}x^2c^6\) and \(\frac{9}{2}x^2c^6\). The sum of the four error terms in Proposition \((2.2)\) can be simplified to

\[(4.25) \quad O\left(\frac{U_1 M_1^{2n_0+2}}{T_1^{n_0+1} N_1^{2n_0+1}}\right).\]

This estimate uses the current assumptions on \(c\) and \(m\), and the size of \(N\) compared to \(T\). Note that \(M_1 \gg N_1\).

We need to estimate this error term, as well as error terms coming from the \(\varpi_{2j}\) terms which will be very similar. First we need a nifty estimate from Li \([20]\). Using the basic definitions, as Li points out (equation...
Lemma 4.2. Assume \( \alpha \geq -1/2 \) and \( \delta - \alpha \geq 1/6 \). Suppose we have a term bounded by \( O(c^{\alpha T^\beta N^\gamma m^\delta}) \) with specific numbers \( \alpha, \beta, \delta, \) and \( \gamma \) for the integral in (4.30). Then the contribution of this term to \( \tilde{R}_3^+ \) is

\[
\ll M^{2/3-\delta-2\varepsilon}T^{13/6+\beta+3\gamma+\delta/2+\varepsilon_1}
\]

where \( \varepsilon \) is arbitrarily small from (4.28), and \( \varepsilon_1 = \varepsilon(11/6 + 3\delta/2 + \gamma) + 3\varepsilon^2 \).

Proof. By (4.30) the contribution of \( O(c^{\alpha T^\beta N^\gamma m^\delta}) \) to \( \tilde{R}_3^+ \) is

\[
\ll MT \sum_{m \leq C_2} m^{-2/3} \sum_{c \leq C_2/m} c^{-1/2+\varepsilon} \sum_{n_1 | cm} n_1^{-2/3} \sum_{n_2 \geq \sqrt{N/n_1^3}} \frac{|A(n_1, n_2)|}{n_2^{1/3}} e^{|A(n_1, n_2)| c^{\alpha T^\beta N^\gamma m^\delta}}.
\]

Note that the innermost sum in (4.31) is over (4.19). Also note Li [20] seems to have used the estimate \( (mc)^{1+\varepsilon} \) instead of the estimate \( mc^{1+\varepsilon}/n_1 \) from (4.28). Since the sum on \( n_1 \) is a divisor sum, this is not an
issue here. Using the estimates for $|A(n_1, n_2)|$ (see (4.35)), and partial summation one has

$$\sum_{n_2 \asymp \sqrt{N}/n_1^2} \frac{|A(n_1, n_2)|}{n_2^{1/3}} \ll n_1 \left( \frac{\sqrt{N}}{n_1^2} \right)^{2/3}.$$  

Since the number of divisors of $cm$ is $\ll (cm)^\epsilon$ this simplifies the contribution to (4.31) to

$$\ll MT^{1+\beta} N^{1/3+\gamma} \sum_{m \leq C_2} m^{-2/3+\epsilon+\delta} \sum_{c \leq C_2} c^{-1/2+2\epsilon+\alpha}.$$  

From a calculus estimate, we have

$$\sum_{c \leq C_2/m} c^{-1/2+2\epsilon+\alpha} \ll \left( \frac{C_2}{m} \right)^{1/2+2\epsilon+\alpha},$$

because $\alpha \geq -1/2$ and $m \leq C_2$. Plugging this into (4.32), and using $C_2 = \sqrt{N}/(T^{1-\epsilon} M)$ we have

$$\ll MT^{1+\beta} N^{1/3+\gamma} \left( \frac{\sqrt{N}}{T^{1-\epsilon} M} \right)^{1/2+2\epsilon+\alpha} \sum_{m \leq C_2} m^{-7/6+\delta-\alpha-\epsilon}.$$  

Now, since $\delta - \alpha \geq 1/6$, we have

$$\sum_{m \leq C_2} m^{-7/6+\delta-\alpha-\epsilon} \ll C_2^{-1/6+\delta-\alpha-\epsilon} + 1 \ll C_2^{-1/6+\delta-\alpha},$$

because $C_2 = \sqrt{N}/(T^{1-\epsilon} M) = T^{1/2+\epsilon}/M \geq T^{\epsilon}$. Inserting (4.34) into (4.33), we see (4.31) is bounded by

$$\ll M^{2/3-\delta-2\epsilon} T^{2/3+\beta-\delta+\epsilon(\delta-5/3+2\epsilon)} N^{1/2+\gamma+\delta/2+\epsilon}.$$  

Now plugging in $N = T^{3+\epsilon}$ gives our Lemma. \qed

Now let us turn back to the error term (4.25). By (4.28), (4.29) can be written as

$$O \left( e^{3n_0+2T^{4n_0+2} N^{1/2} m^{3n_0+13}} \right).$$

Since $(3n_0 + 13/6) - (3n_0 + 2) = 1/6$, we may apply Lemma 4.2 to (4.35) and get its contribution to $\tilde{R}_3^+$ as

$$O(M^{-3n_0-3/2-2\epsilon T^{n_0+2+\epsilon}}),$$  

where $\epsilon > 0$ is arbitrarily small as in (4.28) and $\epsilon_1 = \epsilon(3n_0 + 4) + 3\epsilon^2$. For any $\epsilon_0 > 0$ arbitrarily small, we want to make (4.36) $\ll T^{1+\epsilon_0} M$. This can be done if

$$M \geq T^{n_0+1+\epsilon_1 - \epsilon_0}. $$

We will choose $n_0$ later depending on $\epsilon_0$. Notice that if $n_0 = 1/2$, we pick up the 3/8 constant of Li [20] from (4.37). This concludes the estimation of contribution of error terms (4.29) in Proposition 2.2 to $\tilde{R}_3^+$. We now need to deal with the $\varpi_{2j}$ terms in (4.24) and their contribution to $\tilde{R}_3^+$. Recall the expression for $\varpi_{2j}$ in (2.1). Here we take $2 \leq 2j \leq 2n_0$. One can see from (2.1) that the main term from $\varpi_{2j}$ is $q^{(2j)}(y_0)$. (Here $a(y)$ given in (4.18) and $u_2(y)$ in (4.17) take the place of $g$ and $f$ in Proposition 2.2. Further $y_0$ takes
the place of $\gamma$.) Using the estimates in (4.21) and (4.22) along with $|u''_j(y_0)| \gg T_1/M_1^2$, and along with our current assumptions on $c$ and $m$ in (4.13) we have

$$\alpha_{2j} - a^{(2j)}(y_0) = O\left(\frac{U_1}{M_1^2 N_1^{-1}}\right)$$  

(4.38)

The constant ultimately depends on $n_0$ and we have used $M_1 \gg N_1$. To estimate the contribution of this error term to $\tilde{R}_3^+$, we must divide by $\lambda_j^{1+\frac{2}{c}}$ and sum over $j$. (See (4.24).) Since $y_0 \asymp N/m^2$, we have $\lambda_j \asymp m^3 N^{-3/2}/c$. We have then that this contribution is

$$\ll \left(\frac{N}{m^2}\right)^{-\frac{4}{3} \left(\frac{2}{c} T c m^3\right)^{2j-1} \left(\frac{N}{m^2}\right)^{j+\frac{1}{2}}} = O\left(\epsilon^{3j-1} T^{4j-2} N^{-\frac{3}{2} j + \frac{1}{3}} m^{3j - \frac{1}{3}}\right).$$

Since $(3j - 1/3) - (3j - 1/2) = 1/6$, by Lemma 4.2 the non-leading terms (4.38) of $\alpha_{2j}$ contribute the following to $\tilde{R}_3^+$:

$$O\left(M^{1 - 3j} - 2 \epsilon T^{j + 1/2 + \epsilon_1}\right) \text{ with } \epsilon_1 = \epsilon (3j + 3/2) + 3\epsilon^2,$$

which is

$$\ll T^{1 + \epsilon_0} M \text{ if } M \geq T^{\frac{j - 1/2 + \epsilon_1}{3j + 2\epsilon}}.$$

(4.40)

So we have

$$\frac{j - 1/2 + \epsilon_1 - \epsilon_0}{3j + 2\epsilon} \leq \frac{1}{3} \left(\frac{1}{6j} + 3\epsilon\right)$$

for $j \geq 1$. Thus the condition on $M$ in (4.40) is always true for $M \geq T^{1/3}$.

We must now estimate the $a^{(2j)}(y_0)$ term in $\alpha_{2j}$ in (4.24). Recall that $a(y)$ is given in (4.18). Then $a^{(2j)}(y)$ will consist of a sum of terms of the following form. Let $i_1$ be the number of times $g(m^2 y/N)$ is differentiated (with respect to $y$) plus the number of times a power of $y$ is differentiated. So at every differentiation either the factor $m^2/N$ comes out, or up to a constant, the factor $1/y$ comes out. Notice that $1/y \asymp m^2/N$. Let $i_2$ be the number of times $\hat{k}^+\left(\frac{M T c y}{\pi c m^2}\right)$ is differentiated, and put $i_3$ to be the number of times $e^{-\left(-\frac{2 T c y}{\pi T c m^2}\right)}$ is differentiated. (Note that we have no restriction on the order of differentiation, and that $a^{(2j)}(y)$ will be a sum of these terms over different possible orders of differentiation with various coefficients.) Then $i_1 + i_2 + i_3 = 2j$, and neglecting coefficients (which ultimately depend on $n_0$), $a^{(2j)}(y_0)$ is bounded by the sum over all combinatorial possibilities of

$$\left(\frac{N}{m^2}\right)^{-\frac{4}{3} \left(\frac{2}{c} T c m^3\right)^{i_1} \left(\frac{N}{m^2}\right)^{i_2} \left(\frac{2}{c} T c m^3\right)^{i_3}}.$$  

(4.41)

The main term is (4.41) when $i_3 = 2j$ and we will estimate this separately, below. So we can assume in all terms (4.41), now, that $i_1 + i_2 \geq 1$. To estimate this error term, which is all but one term in $a^{(2j)}(y_0)$, as before, in (4.24), we must divide by $\lambda_j^{1+\frac{2}{c}}$ where $\lambda_j \asymp m^3 N^{-3/2}/c$ with our assumption on $y_0$. We have then a sum of error terms which are all

$$O\left(M^{i_2 \epsilon^2 + i_1 + i_2 + \frac{1}{2} i_1 j - i_2 + \frac{1}{2} i_1} N^{2 \epsilon^2 + i_1 + i_2 + \frac{1}{2} i_1 j - i_2 + \frac{1}{2} i_1} m^{-3j - 2i_1 + 3i_2 + 3i_3 + \frac{1}{2}}\right).$$

(4.42)

Using $i_3 = 2j - i_1 - i_2$, by Lemma 4.2 this error term (4.42) can be seen to be

$$\ll M^{-3j + i_1 + i_2 - \epsilon} T^{j - i_1 - i_2 + \frac{1}{2} + \epsilon} \leq T^{1 + \epsilon_0} M \text{ if } M \geq T^{\frac{j - 1/2 + i_2 - \epsilon_0}{j - 1/2 + \epsilon_0}}.$$

(4.43)
Here $\varepsilon_1 = \varepsilon(3j - i_1 + 9/2) + 3\varepsilon^2$. Now
\[
\frac{j - i_1 - i_2 + \frac{1}{2} + \varepsilon_1 - \varepsilon_0}{3j - i_1 - i_2 + 1 + \varepsilon} \leq \frac{j - i_1 - i_2 + \frac{1}{2}}{3j - i_1 - i_2 + 1} + 10\varepsilon
\]
We are assuming $1 \leq i_1 + i_2 \leq 2j$ with $j \geq 1$, and so
\[
\frac{j - i_1 - i_2 + \frac{1}{2}}{3j - i_1 - i_2 + 1} + 10\varepsilon \leq \frac{1}{3} - \frac{1}{6j} + 10\varepsilon.
\]
Consequently, the latter condition on $M$ in (4.43) is always true for $M \geq T^{1/3}$.

This leaves the main term of $a^{(2j)}(y_0)$ (where $i_3 = 2j$ and $i_1 = i_2 = 0$) which is
\[
\alpha_j \left( \frac{T^2c}{y^2} \right)^{2j} g \left( \frac{MTe}{2\pi^2\sqrt{y}} \right) e \left( \frac{-T^2c\pi}{4\pi^2\sqrt{y}} \right) y^{-13/12} \mid_{y_0} =: a_{2j}(y_0).
\]
Here, the constant $\alpha_j$ depends on $j$ which ultimately can be bounded in terms of $n_0$. If we estimate this similarly, we will get an estimate similar to (4.37) with $2j$ replacing $n_0$. Instead, we will apply the Voronoi formula to (4.44). This is very similar to Li [20], in applying the Voronoi formula a second time, but only to the main term
\[
e \left( u_2(y_0) + 1/8 \right) \sqrt{n_2(y_0)} a(y_0)
\]
in (4.44). It appears that the term $(T^2c\pi y^{-1/2})^{2j}$ in (4.44) for $1 \leq j \leq n_0$ is on average $\approx 1$ in summing over $m$ and $c$, and so we do not improve upon the second application of Voronoi to the term for just $j = 0$.

Recall that in (4.16) we have
\[
x = \frac{n_2n_1^2}{c^3m}, \quad y_0 = x^2e^\delta = \frac{n_2n_1^2}{m^2}.
\]
Further, $\lambda_2 = \frac{1}{12} c^{-1} y_0^{-\frac{3}{2}}$. The contribution to $\tilde{R}_{3,j}^+$ of $a_{2j}(y_0)$ in (4.13) is then $\approx \tilde{R}_{3,j}^+$ where
\[
\tilde{R}_{3,j}^+ = \frac{MT}{m_2 \leq c_2} \sum_{m_2 \leq c_2/m_2} \frac{1}{c_2} \sum_{n_1 | cm_2 \geq 0} e(A(n_1, n_2) / n_1 n_2)
\]
\[
\times \sum_{u \mod{mc_1n_1^{-1}}}^* S(0, 1 + un_1; c)e \left( \frac{n_2u}{mc_1} \right) e \left( -xe^\delta x^2 \frac{n_2u}{mc} \frac{a_{2j}(y_0)}{\lambda_2^{j+1/2}} \right).
\]
Inserting what $x$, $y_0$, and $\lambda_2$ are in terms of $n_1$, $n_2$, $c$ and $m$ into (4.45) we have
\[
\tilde{R}_{3,j}^+ = \frac{MT^{i+1}}{m_2 \leq c_2} \sum_{m_2 \leq c_2/m_2} \sum_{n_1 | cm_2 \geq 0} \frac{1}{n_1^{i+1}} \sum_{n_2 > 0} A(n_2, n_1) / n_2^{i+1}
\]
\[
\times \sum_{u \mod{mc_1n_1^{-1}}}^* S(0, 1 + un_1; c)e \left( \frac{n_2u}{mc_1} \right) e \left( -n_2n_1^2 / cm \right)
\]
\[
\times g \left( \frac{n_2n_1^2}{N} \right) k^* \left( \frac{MTcm}{2\pi^2n_2n_1^2} \right) e \left( - \frac{T^2cm}{4\pi^2n_2n_1^2} \right).
\]
In (4.46) we can switch the sums over $n_2$ and $u$, pull out $S(0, 1 + un_1; c)$ which does not depend on $n_2$ and then the inner sum on $n_2$ is
\[
\sum_{n_2 > 0} A(n_2, n_1)e \left( \frac{n_2u}{c^2} \right) b_j(n_2)
\]
where
\begin{equation}
(4.48) \quad b_j(y) = \frac{1}{y^{3+j+T}} g \left( \frac{y^2 n_1^4}{N} \right) \tilde{k}^* \left( \frac{MTcm}{2\pi^2 y n_1^2} \right) e \left( - \frac{T^2 cm}{4\pi^2 y n_1^2} \right)
\end{equation}
and
\begin{equation}
(4.49) \quad \frac{u'}{c'} = \tilde{u} - \frac{n_1}{mcn_1}, \quad \text{with } (u'c') = 1 \text{ and } c'|mcn_1^{-1}.
\end{equation}

We now apply the Voronoi formula for GL(3) (Lemma 3.1) a second time to (4.47). (See (4.25) of Li [20].) We have
\begin{equation}
(4.50) \quad \sum_{n_2 \geq 1} A(n_1, n_2) c \left( \frac{n_2 u'}{c'} \right) b(n_2)
= \frac{c'}{4\pi^{5/2}i} \sum_{l_1 | c', n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \tilde{u}, l_2) B_{0,1}^0 \left( \frac{l_1^2 l_2}{c'^3 n_1} \right)
+ \frac{c'}{4\pi^{5/2}i} \sum_{l_1 | c', n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \tilde{u}, -l_2) B_{0,1}^1 \left( \frac{l_1^2 l_2}{c'^3 n_1} \right).
\end{equation}

(We followed Li [20] in using the notation \( B \) rather than \( \Psi \).) From (4.50) we have \( x = l_2 l_1^2 / (c'^3 n_1) \). From the function \( g(y^2 n_1^4/N) \) in (4.45) we have \( X = \sqrt{N}/n_1^2 \). Then
\begin{equation}
(4.51) \quad x X = \frac{l_2 l_1^2 \sqrt{N}}{c'^3 n_1} \geq \frac{l_2 l_1^2 \sqrt{N}}{c^3 m^3} \geq \frac{l_2 l_1^2 \sqrt{N}}{C_3} \geq \frac{l_2 l_1^2 \sqrt{N}}{c^3} \geq \frac{l_2 l_1^2 \sqrt{N}}{c^3} T^{3/2-3\epsilon} M^3 \gg 1
\end{equation}
by (4.49). Consequently we can apply Lemma 3.2 to \( B_0(x) \) in (4.50) which is, up to a negligible amount and lower order terms (up to a constant)
\begin{equation}
\begin{aligned}
(4.52) \quad v_2(y) &= -3(xy)^{1/3} - \frac{T^2 cm}{4\pi^2 y n_1^2} \\
(4.53) \quad q_j(y) &= y^{-3j-4} g \left( \frac{y^2 n_1^4}{N} \right) \tilde{k}^* \left( \frac{MTcm}{2\pi^2 y n_1^2} \right).
\end{aligned}
\end{equation}
See equation (4.26) of Li [20]. We need only consider the case
\begin{equation}
(4.54) \quad \frac{T^6 c^3 m^3 n_1^2}{10^3 \pi^6 N^2} \leq x \leq \frac{T^6 c^3 m^3 n_1^2}{10^6 \pi^6 N^2}.
\end{equation}
Thus
\begin{equation}
(4.55) \quad x = \frac{l_2 l_1^2}{c'^3 n_1} \geq \frac{T^6 c^3 m^3 n_1^2}{\pi^6 N^2}.
\end{equation}
By the compact support of \( g \), we may assume the integral (4.51) is taken over a compact segment in \( y \) so that \( 1 \leq y^2 n_1^4/N \leq 2 \). With these assumptions, differentiating (4.52) we have
\begin{equation}
|v_2'(y)| \gg \frac{T^2 cm n_1^4}{N^{3/2}}.
\end{equation}
By \([4.53]\) the variation of \(q_j\) over this interval can be seen to be \(\ll y^{-3j-\frac{3}{2}} T^\varepsilon\). This computation uses basic estimates with simple calculus. Also needed, is that

\[ y \approx \frac{\sqrt{N}}{n_1}, \quad n_1 \leq cm \leq C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}, \quad \text{and } M \geq T^{1/3+2\varepsilon}. \]

Then, by the second derivative test (see Huxley \([11]\)), we have by \((4.54)\) that

\[ (4.58) \]

\[ \tilde{R}_{3,j} \ll M^{4j+1} \sum_{m \leq C_2} m^{3j-1} \sum_{c \leq C_2/m} c^{3j-1} \sum_{n_1 | cm} \frac{1}{n_1^{6j+1}} \sum_{u \pmod{mcn_1^{-1}}} (1 + un_1, c)c' \]

\[ \times \sum_{l_1 | c' n_1 \geq L_2} \frac{|A(l_1, l_2)|}{l_1 l_2} \left( \frac{n_1 c'}{l_1} \right) \left( T^{3+\varepsilon} c^{\frac{3}{2}} N^{-\frac{3}{2}j-\frac{3}{2}n_1} m^{3/2} \right). \]

Here \(L_2 \approx L_2 \) means \(L_2/10^3 \leq L_2 \leq L_2/10\). Also, we have used the trivial bound for the Kloosterman sum:

\[ |S(n_1 u_1, l_2; n_1 c')| \leq n_1 c'. \]

Using the estimate \((4.28)\) and that \(c' \leq mc/n_1\), we deduce from \((4.60)\) that

\[ (4.57) \]

\[ \tilde{R}_{3,j} \ll N^{-\frac{3}{2}j-\frac{3}{2}} M^{4j+\varepsilon} \sum_{m \leq C_2} m^{3j+\frac{3}{2}} \sum_{c \leq C_2/m} c^{3j+\frac{3}{2}+\varepsilon} \sum_{n_1 | cm} \frac{1}{n_1} \sum_{l_1 | c' n_1 \geq L_2} \frac{|A(l_1, l_2)|}{l_2}. \]

Now

\[ (4.58) \]

\[ \sum_{l_2 \approx L_2} \frac{|A(l_1, l_2)|}{l_2} \ll l_1 L_2^{\varepsilon} \ll l_1^{1-2\varepsilon} T^{6\varepsilon} c^{6\varepsilon} m^{6\varepsilon} \frac{N^{2\varepsilon}}{N^{2\varepsilon}}. \]

\[ (4.59) \]

\[ \sum_{l_1 | c' n_1} \frac{1}{l_1} \leq O(\varepsilon^{-1}), \quad \sum_{n_1 | cm} \frac{1}{n_1} \leq \sum_{n_1 \leq cm} \frac{1}{n_1} \ll c^6 m^{6\varepsilon}. \]

Consequently by \((4.58)\) and \((4.59)\), \((4.57)\) is bounded by

\[ \tilde{R}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{3}{2}} M^{4j+4+7\varepsilon} \sum_{m \leq C_2} m^{3j+\frac{3}{2}+7\varepsilon} \sum_{c \leq C_2/m} c^{3j+\frac{3}{2}+8\varepsilon}. \]

Simple calculus and similar estimates then give us

\[ (4.60) \]

\[ \tilde{R}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{3}{2}} M^{4j+4+7\varepsilon} C_2^{3j+\frac{3}{2}+8\varepsilon}. \]

Plugging in \(N = T^{3+\varepsilon}\) and \(C_2 = \sqrt{N}/(T^{1-\varepsilon}M)\) into \((4.60)\), we see

\[ (4.61) \]

\[ \tilde{R}_{3,j}^+ \ll M^{-3j-\frac{3}{2}-8\varepsilon} T^j \frac{\varepsilon}{2}. \]
Here $\varepsilon = \varepsilon(3j + 33/2) + 12\varepsilon^2$. This final term (4.61) is $\leq MT^{1+\varepsilon_0}$ if

$$M \geq T^{\frac{j + \frac{3}{2} + \varepsilon_2 - \varepsilon_0}{3j + \frac{9}{2} + 8\varepsilon}}.$$  

Now $0 \leq j \leq n_0$, and (with $0 < \varepsilon \leq 1/2$)

$$\frac{j + \frac{3}{2} + \varepsilon_2 - \varepsilon_0}{3j + \frac{9}{2} + 8\varepsilon} \leq \frac{1}{3} + \frac{3j}{3j + \frac{9}{2}}\varepsilon + \frac{33/2}{3j + \frac{9}{2}}\varepsilon + \frac{12}{3j + \frac{9}{2}}\varepsilon^2 \leq \frac{1}{3} + 6\varepsilon.$$

Thus (4.62) is always true for $M \geq T^{1/3+6\varepsilon}$.

Now we have shown that $R_1^+ \ll 1$ after (4.9) and that $R_2^+$ is negligible after (4.12). For $R_3^+$, other than negligible terms, if we take arbitrarily small $\varepsilon_0 > 0$, we have proved the bound $O(T^{1+\varepsilon_0}M)$ for $M \geq T^{1/3}$ in (4.10) and (4.13), and for $M \geq T^{1/3+6\varepsilon}$ in (4.14) and (4.15), where $\varepsilon > 0$ is arbitrarily small independently. The only bound left is (4.36) which is $O(T^{1+\varepsilon_0}M)$ when (4.37) holds, where $\varepsilon > 0$ is arbitrarily small as in (4.28) and $\varepsilon_1 = \varepsilon(3n_0 + 4) + 3\varepsilon^2$. To have $O(T^{1+\varepsilon_0}M)$ for any $M \geq T^{1/3+\varepsilon_0}$ we require

$$n_0 + 1 + \varepsilon_1 - \varepsilon_0 \leq \frac{1}{3} + \varepsilon_0.$$

Solving (4.63) for $n_0$ we conclude that (4.36) is $\ll T^{1+\varepsilon_0}M$ for $M \geq T^{1/3+\varepsilon_0}$ provided we take $n_0$ sufficiently large, i.e., if we take sufficiently many main terms in (4.21) when we apply Proposition 2.2

$$n_0 \geq \frac{1}{\varepsilon_0 - \varepsilon} \left( \frac{1}{18} + \frac{11\varepsilon}{9} - \frac{7\varepsilon_0}{6} + \varepsilon^2 - \frac{\varepsilon_0}{3} \right).$$

Here we may simply take $\varepsilon = \varepsilon_0/6$.

Therefore, we have proved that $R^+$ in (3.14) is bounded by $T^{1+\varepsilon_0}M$ for $M \geq T^{1/3+\varepsilon_0}$ by choosing $n_0$ satisfying (4.64) and setting the $\varepsilon$ in (4.61) equal to $\varepsilon_0/6$.

5. K-Bessel function terms

Following Li [20] we split $R^-$ as in (3.10) into $R_1^- + R_2^-$ with

$$R_1^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{(m^2n)^{\frac{3}{4}}} \left( \frac{m^2n}{N} \right) \sum_{c \geq C/m} c^{-1} S(n, -1; c) H_m^{-}(\frac{4\pi\sqrt{n}}{c}),$$

$$R_2^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{(m^2n)^{\frac{3}{4}}} \left( \frac{m^2n}{N} \right) \sum_{c \leq C/m} c^{-1} S(n, -1; c) H_m^{-}(\frac{4\pi\sqrt{n}}{c}),$$

where $H_m^{-}$ is defined in (3.17) and $C = \sqrt{N + 1}$. In estimating $R_1^-$, one can express the $K$-Bessel function in terms of the $I$-Bessel function. Set $\sigma = 100$. Then the estimates for the $I$-Bessel function, along with Li’s previous estimates of $V$ (see (4.7) and (5.6) of Li [20]) give a bound for (5.1) (using the trivial bound for the Kloosterman sum)

$$R_1^- \ll MT^{\sigma+1+\varepsilon} \sum_{m \leq \sqrt{2N}} \frac{1}{m^{1+2\sigma}} \sum_{n \leq 2^m} \frac{A(m, n)}{n^{\frac{3}{2}}} \sum_{c \geq C/m} \frac{1}{c^{2\sigma}} e^{4\pi \frac{n}{c}}.$$
Using $n \leq 2N/m^2$ and $c \geq C/m$ we see that $e^{4\pi\sqrt{n}/c} \ll 1$. Further,

$$
\sum_{c \geq C/m} \frac{1}{c^{2\sigma}} \ll \left(\frac{C}{m}\right)^{1-2\sigma} \text{ and } \sum_{n \leq \frac{2N}{m^2}} \frac{A(m,n)}{n^{\sigma}} \ll m \left(\frac{2N}{m^2}\right)^{1/2}.
$$

Plugging this into (5.3), and noting the sum over $m$ converges, we have

(5.4) \[ R_{-1} \ll \sqrt{NMT^{\alpha+1+\varepsilon}C^{1-2\sigma}} \ll 1 \]

for $\varepsilon$ sufficiently small. Notice this bound holds for $T^\varepsilon \leq M \leq T^{1-\varepsilon}$.

Following the derivation in Li [20], up to a negligible term, we can write

(5.5) \[ H^{-j}_{m,n}(x) = H^{-j}_{m,n}(x) + H^{-2}_{m,n}(x) \]

where

$$
H^{-j}_{m,n}(x) = \frac{4M^jT^{2-j}}{\pi} \int_{R} \int_{|\zeta| \leq T^\varepsilon} t^{j-1}e^{-t^2V(m^2, tM + T)}
\times \cos(x \sinh \pi \zeta) e\left(-\frac{(tM + T)\zeta}{\pi}\right) dt d\zeta,
$$

for $j = 1, 2$. In (5.5) $H^{-2}_{m,n}(x)$ is a lower order term. We only work with $H^{-1}_{m,n}(x)$, since the analysis with $H^{-2}_{m,n}(x)$ is similar. Up to a negligible amount, we can write $H^{-1}_{m,n}(x) = 4Y_{m,n}(x)$, where

$$
Y_{m,n}(x) = \frac{Y^*_{m,n}(x) + Y^*_{m,n}(-x)}{2},
$$

with

(5.6) \[ Y^*_{m,n}(x) = T \int_{R} \hat{K}^*(\zeta) e\left(-\frac{T\zeta}{M} + \frac{\pi x}{2M} \frac{\zeta^2}{\pi M^2}\right) d\zeta. \]

The part of the integral over $|\zeta| \geq M^{\varepsilon/2}$ in (5.6) is negligible. Further, with this assumption, it can be shown by integration by parts, that $Y^*_{m,n}(x)$ is negligible unless

(5.7) \[ \frac{T}{100} \leq |x| \leq 100T \text{ and } \frac{x}{M^3} \ll T^{-\varepsilon}, \]

which we now assume. Recall $M \geq T^{1/2+\varepsilon}$. Thus, the sum over $c$ in (5.2) for which

$$
c \geq \frac{400\pi\sqrt{N}}{Tm} \text{ or } c \leq \frac{\sqrt{2\pi\sqrt{N}}}{25Tm}
$$

is negligible. We thus may assume

$$
\frac{\sqrt{2\pi\sqrt{N}}}{25Tm} \leq c \leq \frac{400\pi\sqrt{N}}{Tm}
$$

and we will denote this by $c \approx \sqrt{N}/(Tm)$.

Using one more nonzero term in the Taylor expansion than Li [20], estimating, we have

(5.8) \[ Y^*_{m,n}(x) = T \int_{R} \hat{K}^*(\zeta) e\left(-\frac{T\zeta}{M} + \frac{\pi x}{2M} \frac{\zeta^2}{12M^3} + \frac{\pi^4 x \zeta^5}{240M^5} + \frac{\pi^6 x \zeta^7}{2 \cdot 7!M^7}\right) d\zeta
+ O\left(T \int_{R} |\hat{K}^*(\zeta)| \frac{\zeta^9|x|}{M^9}\right). \]
Now, expanding
\[ e^{\left(\frac{\pi^2 x^5}{12 M^3} + \frac{\pi^4 x^5}{240 M^5} + \frac{\pi^6 x^7}{6 \cdot 7! M^7}\right)} \]
in (5.8) into a Taylor series of order \( L_2 \) (which could depend on \( \varepsilon \)) we have
\[
Y_{m,n}^*(x) = T \int_\mathbb{R} \tilde{K}^*(\zeta)e\left(-\frac{(2T-x)\zeta}{2M}\right) \times \sum_{j_1+j_2+j_3 \leq L_2} d_{j_1,j_2,j_3} \left(\frac{x\zeta}{M}\right)^{j_1} \left(\frac{x\zeta}{M}\right)^{j_2} \left(\frac{x\zeta}{M}\right)^{j_3} \text{d}\zeta + O\left(\frac{|x|^{L_2+1}}{M^{3L_2+3}} + \frac{|x|}{M^9}\right),
\]
where \( d_{j_1,j_2,j_3} \) are constants with \( d_{0,0,0} = 1 \) with the sum taken over \( j_1 \geq 0, j_2 \geq 0, \text{ and } j_3 \geq 0 \). It follows that
\[
Y_{m,n}^*(x) = T \sum_{j_1+j_2+j_3 \leq L_2} d_{j_1,j_2,j_3} \left(\frac{x\zeta}{M}\right)^{j_1} \left(\frac{x\zeta}{M}\right)^{j_2} \left(\frac{x\zeta}{M}\right)^{j_3} + O\left(\frac{|x|^{L_2+1}}{M^{3L_2+3}} + \frac{|x|}{M^9}\right).
\]
We take \( L_2 \) large enough (possibly depending on \( \varepsilon \)) so that the first error term in (5.9) is negligible, or rather has as fast inverse polynomial decay as desired. (Recall (5.7).) The contribution to \( R_2^* \) coming from the error term \( O(T|x|/M^9) \) can be seen to be bounded by
\[
\frac{T^2}{M^9} \sum_{m \leq \sqrt{2N}} \frac{1}{m} \sum_{n \leq 2N/m^2} \frac{|A(m,n)|}{n^2} \sum_{c \leq C/m} \frac{|S(n,-1;c)|}{c}.
\]
Using Weil’s bound for \( S(n,-1;c) \) we see
\[
\sum_{c \leq C/m} \frac{|S(n,-1;c)|}{c} \ll \left(\frac{C}{m}\right)^{\frac{1}{2}+\varepsilon}.
\]
Estimating similarly to the above, we see that (5.10) is bounded by
\[
\ll \frac{T^2}{M^9} C^{\frac{1}{2}+\varepsilon} \sqrt{N} = \frac{T^2 (\frac{1}{2}+\frac{1}{2}+\varepsilon)}{M^9}.
\]
The above is \( \ll T^{1+\varepsilon} M \) by a power of \( T \) for \( M \geq T^{\frac{1}{4}+2\varepsilon} \).

We take the leading term in the finite series for \( Y_{m,n}^*(x) \) in (5.9), as the terms with higher derivatives of \( k^* \) can be handled in the same way. It follows we need to bound
\[
\tilde{R}_2^* = T \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)}{(m^2 n)^{\frac{3}{2}}} g\left(\frac{m^2 n}{N}\right) \frac{S(n,-1;c)}{c} \frac{k^*}{\zeta} \frac{4\pi \sqrt{n}/c - 2T}{2M}.
\]
Denote
\[
r(y) = g\left(\frac{y}{\zeta}\right) \frac{k^*}{\zeta} \frac{4\pi \sqrt{n}/c - 2T}{2M} y^{-\frac{1}{2}},
\]
which is a smooth function of compact support. From \( x = n_2 n_1^2/(c^3 m) \) and \( X = N/m^2 \) we know
\[
x X = \frac{n_2 n_1^2 N}{c^3 m^3} \geq \frac{n_2 n_1^2 N}{C^3} \geq T^{\frac{1}{4}-\varepsilon} \gg 1.
\]
Consequently we may apply the Voronoi formula (Lemma 3.1) and its asymptotic expansion (Lemma 3.2) to the sum over $n$ in (3.11). As in Li [20] we only consider $R_0(x)$ (see (5.11) of [20]), which is (up to lower order terms)

$$R_0(x) = 2\pi^4 x^3 \int_0^\infty r(y) \frac{d_1 \sin(6\pi(xy)^{1/3})}{\pi(xy)^{2/3}} dy.$$  

Li [20] states that (in an equivalent form) if $n_2 \gg \frac{N^2 T^2}{M^2 n_1^2}$, then $r'(y) x^{-1/3} y^{\frac{1}{3}} \ll T^{-\varepsilon}$. For this assumption on $n_2$, the integral term in $R_0$ as well as the contribution to $\tilde{R}_2$ is found to be negligible.

Thus, we may assume $n_2 \ll \frac{N^2 T^2}{M^2 n_1^2}$. Now, $r(y)$ is negligible unless

$$\left| \frac{2\pi \sqrt{y/c-T}}{M} \right| \leq T^\varepsilon.$$  

This gives us an interval of width $\ll T^{1+\varepsilon}M c^2$ where $y \asymp N/m^2$, and so

$$R_0(x) \ll \left( \frac{n_2 n_1^2}{c^3 m} \right)^{1/3} \left( \frac{N}{m^2} \right)^{-\frac{4}{3}} T^{1+\varepsilon}M c^2.$$  

Using this estimate along with (4.28), it follows from (5.11) that

$$\tilde{R}_2 \ll T \sum_{m \leq \sqrt{2N}} \sum_{c \ll C/m} \sum_{n_1 | cm} \sum_{n_2 \ll \sqrt{N T^\varepsilon/(M^3 n_1^2)}} \frac{|A(n_1, n_2)|}{n_1 n_2} m c^{1+\varepsilon} n_1^{-\frac{1}{3}} n_2^{-\frac{4}{3}} T^{1+\varepsilon} M c^2.$$  

Estimating similarly to the last section, the inner sum in (5.12) is

$$\sum_{n_2 \ll \sqrt{N T^\varepsilon/(M^3 n_1^2)}} \frac{|A(n_1, n_2)|}{n_2^{1/3}} \ll n_1 \left( \frac{\sqrt{N T^\varepsilon}}{M^3 n_1^2} \right)^{2/3}.$$  

Plugging this and

$$\sum_{n_1 | cm} \frac{1}{n_1} \ll (cm)^{\varepsilon}.$$  

into (5.12) we see

$$\tilde{R}_2 \ll T^{2+5\varepsilon/3} M^{-1} N^{-1/2} \sum_{m \leq \sqrt{2N}} \sum_{c \ll \frac{C}{m}} m^{1+\varepsilon} c^{1+2\varepsilon}.$$  

Now

$$\sum_{c \ll \frac{C}{m}} c^{1+2\varepsilon} \ll \left( \frac{\sqrt{N}}{T^2} \right)^{2+2\varepsilon} \text{ and } \sum_{m \leq \sqrt{2N}} \frac{1}{m^{1+\varepsilon}} \ll \frac{1}{\varepsilon^2}.$$  

Consequently, $\tilde{R}_2 \ll T^{2+5\varepsilon/3} M^{-1}$. This is clearly smaller than $T^{1+\varepsilon} M$ if $M \geq T^{1/4+13\varepsilon/12-\varepsilon_0/2}$.

Together with (5.3) for $T^{\varepsilon} \leq M \leq T^{1-\varepsilon}$, we conclude that $\mathcal{R}^- \ll T^{1+\varepsilon_0} M$ if $M \geq T^{1/3}$. Recall that $D$ in (3.13) is negligible for $T^{\varepsilon} \leq M \leq T^{1-\varepsilon}$ as we pointed at the end of Section 3. Together with our conclusion at the end of Section 4 for $\mathcal{R}^+$, we have proved that $\mathcal{R}$ in (3.12) is bounded by $O(T^{1+\varepsilon_0} M)$ for $T^{1/3+\varepsilon_0} \leq M \leq T^{1/2}$. This implies Theorem 1.1. \qed
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