DELAUNAY HYPERSURFACES WITH CONSTANT NONLOCAL MEAN CURVATURE

XAVIER CABRÉ, MOUHAMED MOUSTAPHA FALL, AND TOBIAS WETH

Abstract. We study hypersurfaces of $\mathbb{R}^N$ with constant nonlocal (or fractional) mean curvature. This is the equation associated to critical points of the fractional perimeter functional under a volume constraint. We establish the existence of a smooth branch of periodic cylinders in $\mathbb{R}^N$, $N \geq 2$, all of them with the same constant nonlocal mean curvature, and bifurcating from a straight cylinder. These are Delaunay type cylinders in the nonlocal setting. The proof uses the Crandall-Rabinowitz theorem applied to a quasilinear type fractional elliptic equation.

RÉSUMÉ. Nous étudions des hypersurfaces dans $\mathbb{R}^N$, $N \geq 2$, à courbure moyenne non-locale (ou fractionnaire) constante. Celà revient à étudier une équation associée aux points critiques du périmètre fractionnaire sous une contrainte de volume. Nous établissons l’existence d’une branche lisse d’hypersurfaces périodiques de type Delauney qui ont toutes la même courbure moyenne non-locale que celle d’un cylindre droit. La preuve utilise le théorème de bifurcation de Crandall-Rabinowitz appliqué à une équation elliptique fractionnaire de type quasilinéaire.

1. Introduction and main results

Let $\alpha \in (0, 1)$, $N \geq 2$, and let $E$ be an open set in $\mathbb{R}^N$ with $C^2$-boundary. For every $x \in \partial E$, the nonlocal or fractional mean curvature of $\partial E$ at $x$ (that we call NMC for short) is given by

$$H_{E}(x) = \text{PV} \int_{\mathbb{R}^N} \frac{1_{E^c}(y) - 1_{E}(y)}{|x - y|^{N+\alpha}} \, dy := \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{1_{E^c}(y) - 1_{E}(y)}{|x - y|^{N+\alpha}} \, dy$$ (1.1)

and is well defined. Here and in the following, $E^c$ denotes the complement of $E$ in $\mathbb{R}^N$ and $1_A$ denotes the characteristic function of a set $A \subset \mathbb{R}^N$. In the first integral PV denotes the principal value sense. For the asymptotics $\alpha$ tending to 0 or 1, $H_E$ should be renormalized with a positive constant factor $C_{N,\alpha}$. Since constant factors are not relevant for the results of this paper, we use the simpler expression in (1.1) without the constant $C_{N,\alpha}$.

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An alternative expression for the NMC is given by
\[ H_E(x) = -\frac{2}{\alpha} \int_{\partial E} |x - y|^{-(N+\alpha)} (x - y) \cdot \nu_E(y) \, dy, \] (1.2)

where \( \nu_E(y) \) denotes the outer unit normal to \( \partial E \) at \( y \). If \( \partial E \) is of class \( C^{1,\beta} \) for some \( \beta > \alpha \) and \( \int_{\partial E} (1 + |y|)^{1-N-\alpha} \, dy < \infty \), then the integral in (1.2) is absolutely convergent in the Lebesgue sense and the expression follows from (1.1) via the divergence theorem.

The notion of nonlocal mean curvature was introduced around 2008 by Caffarelli and Souganidis in [7] and by Caffarelli, Roquejoffre, and Savin in [6]. As first discovered in [6], the nonlocal mean curvature arises as the first variation of the fractional perimeter. For the notion of fractional perimeter and its convergence to classical perimeter as \( \alpha \to 1 \), see the papers [1,12,19]. The seminal paper [6] established the first existence and regularity theorems on nonlocal minimal surfaces, that is, (minimizing) hypersurfaces with zero NMC. Within these years, there have been important efforts and results concerning nonlocal minimal surfaces but still, apart from dimension \( N = 2 \), there is a lot to be understood, mainly for the classification of stable nonlocal minimal cones. See [3, Chapter 6] for a recent survey of known results.

The purpose of this paper is to establish a nonlocal analogue of the classical result of Delaunay [14] on periodic cylinders with constant mean curvature, the so called onduloids. In [4], a paper by the present authors and Solá-Morales, we accomplished this in the plane \( \mathbb{R}^2 \); that is, we proved the existence of a continuous branch of periodic bands, starting from a straight band, all of them with the same constant NMC. Here we establish the analogue result but in \( \mathbb{R}^N \) for \( N \geq 3 \). In addition, we show that the branch is not only continuous but smooth —and we prove this also in \( \mathbb{R}^2 \).

More precisely, we consider sets \( E \subset \mathbb{R}^N, N \geq 2 \), with constant nonlocal mean curvature which have the form
\[ E_u = \{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : |\zeta| < u(s)\}, \] (1.3)

where \( u : \mathbb{R} \to (0, \infty) \) is a positive function. We establish the existence of a smooth branch of sets as above (that we call cylinders) which are periodic in the variable \( s \) and have all the same constant nonlocal mean curvature; they bifurcate from a straight cylinder \( \{|\zeta| < R\} \). The radius \( R \) of the straight cylinder is chosen so that the periods of the new cylinders converge to \( 2\pi \) as they approach the straight cylinder. Our result is of perturbative nature and thus we find periodic cylinders which are all close to the straight one.

The following is the precise statement of our result. Throughout the paper, \( C^{k,\gamma}(\mathbb{R}) \) denotes the space of \( C^k(\mathbb{R}) \) bounded functions \( u \), with bounded derivatives up to order \( k \) and with \( u^{(k)} \) having finite Hölder seminorm of order \( \gamma \in (0, 1) \). The space is equipped with the standard norm (3.3).
**Theorem 1.1.** Let $N \geq 2$. For every $\alpha \in (0, 1)$ there exist $R > 0$, $a_0 > 0$, $\beta \in (\alpha, 1)$ and $C^\infty$-maps

$$(-a_0, a_0) \to C^{1,\beta}(\mathbb{R}), \quad a \mapsto u_a$$

$$(-a_0, a_0) \to (0, \infty), \quad a \mapsto \lambda(a)$$

with the following properties:

(i) $\lambda(0) = 1$ and $u_0 \equiv R$.

(ii) For every $a \in (-a_0, a_0) \setminus \{0\}$, the function $u_a : \mathbb{R} \to \mathbb{R}$ is even and periodic with minimal period $2\pi/\lambda(a)$.

(iii) For every $a \in (-a_0, a_0)$, the set

$$E_{u_a} = \{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : |\zeta| < u_a(s)\}$$

has positive constant nonlocal mean curvature equal to the nonlocal mean curvature of the straight cylinder

$$E_R := \{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : |\zeta| < R\}.$$

Moreover, $E_{u_a} \neq E_{u_{a'}}$ for $a, a' \in (-a_0, a_0)$, $a \neq a'$.

(iv) For every $a \in (-a_0, a_0)$, we have

$$u_a(s) = R + \frac{a}{\lambda(a)} \{\cos(\lambda(a)s) + v_a(\lambda(a)s)\}, \quad (1.4)$$

where $v_a \to 0$ in $C^{1,\beta}(\mathbb{R})$ as $a \to 0$ and $\int_{-\pi}^{\pi} v_a(t) \cos(t) \, dt = 0$ for every $a \in (-a_0, a_0)$. Moreover, we have

$$\lambda(-a) = \lambda(a) \quad \text{and} \quad u_{-a}(s) = u_a\left(s + \frac{\pi}{\lambda(a)}\right)$$

for every $a \in (-a_0, a_0)$, $s \in \mathbb{R}$, $\lambda'(0) = 0$ and $\partial_a u_a \big|_{a=0} = \cos(\cdot)$.

We prove that the branch is $C^\infty$ in the parameter $a$, extending our previous work [3] in $\mathbb{R}^2$ where we only proved continuous dependence.

The smoothness (i.e., the $C^\infty$-character) of our $C^{1,\beta}$ hypersurfaces $\partial E_{u_a}$, and in general of $C^{1,\beta}$ hypersurfaces in $\mathbb{R}^N$ with constant NMC which are, locally, Lipschitz graphs follows (since $\beta > \alpha$) from the methods and results of Barrios, Figalli, and Valdinoci [2] on nonlocal minimal graphs. This holds for all $N \geq 2$. More generally, to deduce the $C^\infty$ regularity, [2] needs to assume that the hypersurface is $C^{1,\beta}$ for some $\beta > \alpha/2$ and that it has constant nonlocal mean curvature in the viscosity sense; this fact can be found in Section 3.3 of [2]. Here, the notion of viscosity solution is needed since the expression (1.1) for the NMC is only well defined for $C^{1,\beta}$ sets when $\beta > \alpha$.

Regarding CNMC hypersurfaces, that is, hypersurfaces with constant nonlocal mean curvature, there have been three articles before this one (apart from the papers on zero NMC, that is, nonlocal minimal surfaces). In [4], besides finding the Delaunay bands in $\mathbb{R}^2$, the present authors and Solá-Morales also established the analogue of the Alexandrov rigidity theorem for bounded CNMC hypersurfaces in $\mathbb{R}^N$; these sets must be balls. At the same time and independently, Ciraolo, Figalli, Maggi, and Novaga [8]...
also proved the Alexandrov rigidity theorem for CNMC hypersurfaces in \( \mathbb{R}^N \) and, in addition, a strong quantitative version of this rigidity theorem.

A third paper, [13], by Dávila, del Pino, Dipierro, and Valdinoci, establishes variationally the existence of periodic and cylindrically symmetric hypersurfaces in \( \mathbb{R}^N \) which minimize a certain renormalized fractional perimeter under a volume constraint. More precisely, [13] establishes the existence of a 1-periodic minimizer for every given volume within the slab \( \{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : -1/2 < s < 1/2\} \). We have realized recently that these minimizers are in fact CNMC hypersurfaces in a certain weak sense. They would be CNMC hypersurfaces in the classical sense defined above if one could prove that they are of class \( C^{1,\beta} \) for some \( \beta > \alpha/2 \). However, [13] does not prove any regularity for the minimizers. The article also proves that for small volume constraints, the minimizers tend in measure (more precisely, in the so called Fraenkel asymmetry) to a periodic array of balls.

It is an open problem to establish the existence of global continuous branches of nonlocal Delaunay hypersurfaces as in Theorem 1.1 and to study their limiting configuration. In the case of classical mean curvature, embedded Delaunay hypersurfaces vary from a cylinder to an infinite compound of tangent spheres. However, it is easy to see that an infinite compound of aligned round spheres, tangent or disconnected, does not have constant NMC. In a forthcoming paper [5], we study nonlocal analogues of this periodic and disconnected CMC set.

Also related to our work, the papers [22] and [21] established the existence of periodic and cylindrical symmetric domains in \( \mathbb{R}^N \) whose first Dirichlet eigenfunction has constant Neumann data on the boundary. This is therefore a nonlinear and nonlocal elliptic operator of order 1 based on a certain Dirichlet to Neumann map.

The nonlocal mean curvature flow for the notion of NMC considered in this paper has been studied in strong sense in [20] and in viscosity sense in [9,10,16].

Let us describe the proof of Theorem 1.1 and its main difficulties. The first step is to write the NMC operator acting on graphs of functions —the functions \( u_a \) above. This leads to an integral operator of quasilinear type acting on functions \( u = u(s) \) and involving a double integral with respect to \( d\sigma ds \), where \( \sigma \in S^{N-2} \) takes into account the symmetry of revolution in the variable \( \zeta \in \mathbb{R}^{N-1} \). The presence of this new integral in \( d\sigma \) is the main difference and difficulty with respect to our previous paper where \( N = 2 \). In fact, changing the order of integration in \( d\sigma ds \), or making different changes of variables to simplify the integrands, will lead to quite different expressions for the nonlocal mean curvature of the set \( E_a \). We will present three of such expressions, namely (2.3), (2.5), and (2.7) below. Finding the second of these expressions was crucial to be able to prove the smoothness of the nonlocal mean curvature operator, which is stated in Proposition 2.3 and established in Proposition 4.4 below.

Another essential point in the proof is to have a simple expression for the linearized operator at the straight cylinder. This is given in Proposition 2.3 and we found it using our third expression (2.7) for \( H \). Even though we prove the formula for the linearized
operator by using our second expression for $H$ as given in [25], it would have been very difficult to guess it from this second expression.

The linearization gives rise to an integro-differential operator with a singular kernel close to (but different than) that of the fractional Laplacian. This is another difference with the previous 2D case. We use regularity theory both in Sobolev and Hölder spaces to analyze the linear operator and, thus, to be able to apply the Crandall-Rabinowitz theorem in [11], which will lead to our result.

The paper is organized as follows. Section 2 sets up the nonlinear nonlocal operator to be studied and states a simple expression for the linearized operator at a straight cylinder. It contains also some preliminary estimates concerning the linearized problem. These estimates are used in Section 3 to solve our nonlinear problem using the Crandall-Rabinowitz theorem. In Section 4 we establish the $C^\infty$ character of our nonlocal mean curvature operator and we prove the formula for the linearized operator at a straight cylinder. Since some expressions and estimates in the previous sections require $N \geq 3$, in Section 5 we treat the case $N = 2$.

2. The NMC operator acting on cylindrical graphs of $\mathbb{R}^N$

Let $\alpha \in (0,1)$ and $\beta \in (\alpha,1)$. In Section 3 we will also assume that $\beta < 2\alpha + 1/2$; see (3.2) below. This extra assumption will only be used at the end of the proof of Proposition 3.2. For a positive function $u \in C^{1,\beta}(\mathbb{R})$, we consider the set $E_u$ as defined in (1.3). We first recall the following expression for the NMC of $E_u$:

$$H_{E_u}(x) = -\frac{2}{\alpha} \int_{\partial E_u} |x - y|^{-(N+\alpha)}(x - y) \cdot \nu_{E_u}(y) \, dy;$$  \hfill (2.1)

see e.g. [4, Eqn. (1.2)]. Here $\nu_{E_u}(y)$ denotes the unit outer normal of $E_u$ and $dy$ is the volume element of $\partial E_u$. Next, we consider the open set

$$\mathcal{O} := \{ u \in C^{1,\beta}(\mathbb{R}) : \inf_{\mathbb{R}} u > 0 \}. \hfill (2.2)$$

For $u \in \mathcal{O}$, we consider the map $F_u : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}^N$ given by

$$F_u(s,\zeta) = (s,u(s)\zeta).$$

We have that the boundary of $E_u$,

$$\partial E_u = \{(s,u(s)\sigma) \in \mathbb{R} \times \mathbb{R}^{N-1} : \sigma \in S^{N-2}\},$$

is parameterized by the restriction of $F_u$ to $\mathbb{R} \times S^{N-2}$.

2.1. Two fundamental expressions for the NMC operator. The following results provide two expressions for the NMC of $E_u$ in terms of the above parametrization and the function $u$. Here, when $N = 2$, we have $S^{N-2} = S^0 = \{-1,1\} \subset \mathbb{R}$.

**Lemma 2.1.** Let $u \in \mathcal{O}$. Then the nonlocal mean curvature $H_{E_u}$ — that we will denote by $H(u)(s)$ — at a point $(s,u(s)\theta)$, with $\theta \in S^{N-2}$, does not depend on $\theta$ and is given...
by
\[
-\frac{\alpha}{2} H(u)(s) = \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\{u(s) - u(s - \tau) - \tau u'(s - \tau)\} u^{N-2}(s - \tau)}{\tau^2 + (u(s) - u(s - \tau))^2 + u(s)u(s - \tau)|\sigma - e_1|^2} d\tau d\sigma
\]
\[
- \frac{u(s)}{2} \int_{S^{N-2}} \int_{\mathbb{R}} \frac{|\sigma - e_1|^2 u^{N-2}(s - \tau)}{\tau^2 + (u(s) - u(s - \tau))^2 + u(s)u(s - \tau)|\sigma - e_1|^2} d\tau d\sigma,
\]
where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{N-1} \). Moreover, when \( N \geq 3 \) the two integrals above converge absolutely in the Lebesgue sense.

**Proof.** Starting from the absolutely converging integral representation \( \text{(2.1)} \), we deduce that
\[
-\frac{\alpha}{2} H(u)(s) = -\frac{\alpha}{2} H_{E_u}(F_u(s, \theta))
\]
\[
= \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\{F_u(s, \theta) - F_u(\bar{s}, \sigma)\} \cdot \nu_{E_u}(F_u(\bar{s}, \sigma))}{|F_u(s, \theta) - F_u(\bar{s}, \sigma)|^{N+\alpha}} J_{F_u}(\bar{s}, \sigma) d\bar{s} d\sigma,
\]
where the unit outer normal of \( \partial E_u \) at the point \( F_u(\bar{s}, \sigma) \) is given by
\[
\nu_{E_u}(F_u(\bar{s}, \sigma)) = \frac{1}{\sqrt{1 + (u')^2(\bar{s})}} (-u'(\bar{s}), \sigma) \quad \text{for } \bar{s} \in \mathbb{R}, \sigma \in S^{N-2},
\]
and
\[
J_{F_u}(\bar{s}, \sigma) = \sqrt{u^{2(N-2)}(\bar{s})(1 + (u')^2(\bar{s}))} = u^{N-2}(\bar{s}) \sqrt{1 + (u')^2(\bar{s})} \quad \text{for } \bar{s} \in \mathbb{R}, \sigma \in S^{N-2}.
\]
We also note that for \( s, \bar{s} \in \mathbb{R} \) and \( \theta, \sigma \in S^{N-2} \) we have
\[
|F_u(s, \theta) - F_u(\bar{s}, \sigma)|^2 = (s - \bar{s})^2 + |u(s)\theta - u(\bar{s})\sigma|^2
\]
\[
= (s - \bar{s})^2 + (u(s) - u(\bar{s}))^2 + 2u(s)u(\bar{s})(1 - \theta \cdot \sigma)
\]
and
\[
\{F_u(s, \theta) - F_u(\bar{s}, \sigma)\} \cdot \nu_{E_u}(F_u(\bar{s}, \sigma)) = \frac{-(s - \bar{s})u'(\bar{s}) + (u(s)\theta - u(\bar{s})\sigma) \cdot \sigma}{\sqrt{1 + (u')^2(\bar{s})}}
\]
\[
= \frac{u(s) - u(\bar{s}) - (s - \bar{s})u'(\bar{s}) - u(s)(1 - \theta \cdot \sigma)}{\sqrt{1 + (u')^2(\bar{s})}}.
\]
Inserting these identities in \( \text{(2.4)} \), we obtain
\[
-\frac{\alpha}{2} H(u)(s) = \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\{u(s) - u(\bar{s}) - (s - \bar{s})u'(\bar{s}) - u(s)(1 - \theta \cdot \sigma)\} u^{N-2}(\bar{s})}{(s - \bar{s})^2 + (u(s) - u(\bar{s}))^2 + 2u(s)u(\bar{s})(1 - \sigma_1)} d\bar{s} d\sigma
\]
\[
= \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\{u(s) - u(\bar{s}) - (s - \bar{s})u'(\bar{s}) - u(s)(1 - \sigma_1)\} u^{N-2}(\bar{s})}{(s - \bar{s})^{N+\alpha} + (u(s) - u(\bar{s}))^2 + 2u(s)u(\bar{s})(1 - \sigma_1)} d\bar{s} d\sigma
\]
\[
= \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\{u(s) - u(\bar{s}) - \tau u'(\bar{s}) - u(s)(1 - \sigma_1)\} u^{N-2}(\bar{s} - \tau)}{\tau^2 + (u(s) - u(\bar{s} - \tau))^2 + 2u(s)u(\bar{s} - \tau)(1 - \sigma_1)} d\tau d\sigma.
\]
Here, for the second equality, we note that the rotation invariance of the spherical integral allows to choose \( \theta = c_1 \in S^{N-2} \), whereas the third equality follows from the change of variable \( \tau = s - \tilde{s} \).

Since \( 1 - \sigma_1 = \frac{1 - \sigma}{2} \) for \( \sigma \in S^{N-2} \), the assertion of the lemma now follows once we have shown that both integrals in (2.3) converge absolutely in the Lebesgue sense. To prove this, we first note that

\[
|u(s) - u(s - \tau) - \tau u'(s - \tau)| \leq |\tau| \int_0^1 |u'(s - \rho \tau) - u'(s - \tau)| d\rho
\]

\[
\leq 2\|u\|_{C^{1, \beta}(\mathbb{R})} \min(\{\tau^{1+\beta}, |\tau|\})
\]

for \( s, \tau \in \mathbb{R} \). Using this, we get

\[
\int_{S^{N-2}} \int_{\mathbb{R}} \frac{|u(s) - u(s - \tau) - \tau u'(s - \tau)|u^{N-2}(s - \tau)}{\{\tau^2 + (u(s) - u(s - \tau))^2 + u(s)u(s - \tau)|\sigma - e_1|^2\}^{\frac{N+\alpha}{2}}} d\tau d\sigma
\]

\[
\leq 2\|u\|_{C^{1, \beta}(\mathbb{R})} \int_{S^{N-2}} \int_{\mathbb{R}} \frac{\min(\{\tau^{1+\beta}, |\tau|\})}{\{\tau^2 + (u(s) - u(s - \tau))^2 + u(s)u(s - \tau)|\sigma - e_1|^2\}^{\frac{N+\alpha}{2}}} d\tau d\sigma
\]

\[
\leq 2\|u\|_{C^{1, \beta}(\mathbb{R})} \int_{S^{N-2}} \int_{\mathbb{R}} \frac{d\sigma}{\{\tau^2 + \delta^2|\sigma - e_1|^2\}^{\frac{N+\alpha}{2}}} dt < \infty.
\]

Hence the first integral in (2.3) converges absolutely.

To see the absolute convergence of the second integral in (2.3), we again use the change of variable \( \tau = |\sigma - e_1|t \) to obtain the estimate

\[
\int_{S^{N-2}} \int_{\mathbb{R}} \frac{|\sigma - e_1|^2u^{N-2}(s - \tau)}{\{\tau^2 + (u(s) - u(s - \tau))^2 + u(s)u(s - \tau)|\sigma - e_1|^2\}^{\frac{N+\alpha}{2}}} d\tau d\sigma
\]

\[
\leq \|u\|_{C^{1, \beta}(\mathbb{R})} \int_{S^{N-2}} \int_{\mathbb{R}} \frac{|\sigma - e_1|^2}{\{\tau^2 + \delta^2|\sigma - e_1|^2\}^{\frac{N+\alpha}{2}}} d\tau d\sigma
\]

\[
\leq \|u\|_{C^{1, \beta}(\mathbb{R})} \int_{S^{N-2}} \int_{\mathbb{R}} \frac{d\sigma}{|\sigma - e_1|^{N+\alpha-2}} \int_{\mathbb{R}} \frac{dt}{(t^2 + \delta^2)^{\frac{N+\alpha}{2}}} < \infty.
\]

The proof is finished. \( \square \)

To prove the smoothness of the nonlocal mean curvature operator between appropriate Hölder spaces, it will be crucial to make a further transformation in the expression of \( H \) found in the previous Lemma 2.1. To describe this, we first introduce some notation. We denote

\[
p_\sigma := |\sigma - e_1|,
\]
and for $r \in \mathbb{R}$, we define

$$\mu_r(\sigma) = \frac{1}{|\sigma - e_1|^{N+r}} = p_\sigma^{-N-r}.$$

We define the maps $\Lambda_0, \Lambda : C^{1,\beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Lambda_0(\varphi, s, t, p) = \frac{\varphi(s) - \varphi(s - pt)}{pt} = \int_0^1 \varphi'(s - \rho pt) \, d\rho$$

and

$$\Lambda(\varphi, s, t, p) = \Lambda_0(\varphi, s, t, p) - \varphi'(s - pt) = \int_0^1 (\varphi'(s - \rho pt) - \varphi'(s - pt)) \, d\rho.$$

Recalling that $2(1 - \sigma \cdot e_1) = |\sigma - e_1|^2 = p_\sigma^2$, we make the change of variables

$$t = \frac{\tau}{|\sigma - e_1|} = \frac{\tau}{p_\sigma}$$

in the expression for $H(u)$ in Lemma 2.1. We immediately obtain

**Lemma 2.2.** With the notation above, for $u \in O$, we have

$$-\frac{\alpha}{2} H(u)(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} t \Lambda(u, s, t, p_\sigma) K_\alpha(u, s, t, p_\sigma) u^{N-2}(s - p_\sigma t) \, dt \, d\sigma \quad (2.5)$$

where the function $K_\alpha : C^{1,\beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$K_\alpha(u, s, t, p_\sigma) = \frac{1}{(t^2 + t^2 \Lambda_0(u, s, t, p_\sigma)^2 + u(s)u(s - pt))^{(N+\alpha)/2}}.$$

Moreover, when $N \geq 3$ the two integrals above converge absolutely in the Lebesgue sense.

We point out a very important and useful difference between this last expression for $H$ and that of Lemma 2.1. In (2.5) the dependence on the variable $|\sigma - e_1| = p_\sigma = p$ appears “inside the known variables for $u”$, that is, through $u(s - pt)$. This will allow us to establish in the following Proposition 2.3 a fundamental result on the smoothness of the nonlocal mean curvature map $H$. The result also states an expression for the differential of $H$ at a constant function (recall that a constant function corresponds to a straight cylinder in $\mathbb{R}^N$). The result will be proved further on in Section 4 (see Propositions 4.4 and 4.5) using the previous expression (2.5). Recall the definition of the set $O \subset C^{1,\beta}(\mathbb{R})$ defined in (2.2).

**Proposition 2.3.** For $N \geq 3$, the map $H : O \subset C^{1,\beta}(\mathbb{R}) \rightarrow C^{0,\beta-\alpha}(\mathbb{R})$ is of class $C^\infty$.

In addition, if $\kappa \in O$ is a constant function, then we have

$$DH(\kappa)v(s) = \kappa^{-1-\alpha} \left( PV \int_{\mathbb{R}} (v(s) - v(s - \kappa \tau)) G_\alpha(\tau) \, d\tau - b_\alpha v(s) \right) \quad (2.6)$$
for $v \in C^{1,\beta}(\mathbb{R})$, where
\[
G_\alpha : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad G_\alpha(\tau) = 2 \int_{S^{N-2}} \frac{1}{(\tau^2 + 2(1 - \sigma_1))^{\frac{N+\alpha}{2}}} d\sigma
\]
and
\[
b_\alpha = 2 \int_{\mathbb{R}} \int_{S^{N-2}} \frac{1 - \sigma_1}{(\tau^2 + 2(1 - \sigma_1))^{\frac{N+\alpha}{2}}} d\sigma d\tau.
\]

Expression (2.6) is rather simple and this will be crucial in order to analyze the associated linearized operator. We prove (2.6) in all detail in Proposition 4.5 using expression (2.5). The proof, however, involves collecting several groups of terms that could not have been guessed without knowing a priori the expression (2.6) that we want to establish. Indeed, we deduce the expression (2.6) for the linearized operator at a constant function from another, very different, expression for the NMC operator $H$ that we describe next.

2.2. A third expression for the NMC operator: finding the linearized operator. The following is another formula for $H$. We present it here only to show how we found expression (2.6) for the linearized operator. It will not be used in any proof of the paper.

The nonlocal mean curvature $H_{E_u}$ at a point $(s, u(s)\theta)$, with $\theta \in S^{N-2}$, does not depend on $\theta$ and is given by
\[
H(u)(s) = u(s)^{1-\alpha} \int_{\mathbb{R}} \left\{ 2I \left( \frac{u(\bar{s})}{u(s)}, \frac{s - \bar{s}}{u(s)} \right) - I \left( 0, \frac{s - \bar{s}}{u(s)} \right) \right\} d\bar{s}, \tag{2.7}
\]
where
\[
I(q, p) := \int_{q}^{+\infty} \int_{S^{N-2}} \frac{\tau^{N-2}}{(p^2 + 1 + \tau^2 - 2\sigma_1 \tau)^{(N+\alpha)/2}} d\sigma d\tau.
\]

Since we will not use (2.7) in any proof of the paper, we merely sketch a proof of this formula without looking in detail at the convergence of integrals. We define $\mu(s, \zeta) = |(s, \zeta)|^{-N-\alpha} = (s^2 + |\zeta|^2)^{-\frac{N+\alpha}{2}}$ for $(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Let $s \in \mathbb{R}$ and $\theta \in S^{N-2}$. At the point $(s, u(s)\theta) \in \partial E_u$, we then have
\[
H(u)(s) = \int \int_{\mathbb{R} \times \mathbb{R}^{N-1}} \left\{ 1_{E_{u}}(\bar{s}, \zeta) - 1_{E_u}(\bar{s}, \zeta) \right\} \mu(\bar{s}, \theta u(s)) d\zeta d\bar{s}.
\]

Using polar coordinates, we get
\[
H(u)(s) = \int \int_{S^{N-2}} \left( \int_{0}^{u(s)} - \int_{u(s)}^{+\infty} \right) r^{N-2} \mu\left( (s, \theta u(s)) - (\bar{s}, r\sigma) \right) dr d\sigma d\bar{s}.
\]

By direct computation, we have
\[
\mu\left( (s, \theta u(s)) - (\bar{s}, r\sigma) \right) = \left( (s - \bar{s})^2 + u^2(s) + r^2 - 2ru(s)\sigma \cdot \theta \right)^{-\frac{N+\alpha}{2}}.
\]
Using this, we can see that the integral \( \int_{S^{N-2}} \mu((s, \theta u(s)) - (\bar{s}, r \sigma)) \, d\sigma \) is independent of \( \theta \). Hence we may assume that \( \theta = e_1 \). We have

\[
H(u)(s) = \int_{\mathbb{R}} \int_{S^{N-2}} \int_{u(\bar{s})}^{+\infty} \frac{r^{N-2}}{(s - \bar{s})^2 + u^2(s) + r^2 - 2ru(s)\sigma_1 \frac{N+\alpha}{2}} \, dr \, d\sigma \, ds
\]

\[
- \int_{\mathbb{R}} \int_{S^{N-2}} \int_{0}^{u(\bar{s})} \frac{r^{N-2}}{(s - \bar{s})^2 + u^2(s) + r^2 - 2ru(s)\sigma_1 \frac{N+\alpha}{2}} \, dr \, d\sigma \, ds
\]

\[
= \int_{\mathbb{R}} \int_{S^{N-2}} 2 \int_{u(\bar{s})}^{+\infty} \frac{r^{N-2}}{(s - \bar{s})^2 + u^2(s) + r^2 - 2ru(s)\sigma_1 \frac{N+\alpha}{2}} \, dr \, d\sigma \, ds
\]

\[
- \int_{\mathbb{R}} \int_{S^{N-2}} \int_{0}^{+\infty} \frac{r^{N-2}}{(s - \bar{s})^2 + u^2(s) + r^2 - 2ru(s)\sigma_1 \frac{N+\alpha}{2}} \, dr \, d\sigma \, ds.
\]

By making the change of variable \( r = u(s)\tau \), we get (2.7).

We next find the simple expression for the linearized operator given in (2.6). Taking \( u \equiv \kappa \in \mathcal{O} \) a constant function, using (2.7), calling \( t = s - \bar{s} \), and denoting the partial derivatives of \( I \) by \( I_q \) and \( I_p \), we have

\[
\frac{d}{d\varepsilon} H(\kappa + \varepsilon v) \bigg|_{\varepsilon = 0} (s) = - (1 + \alpha)\kappa^{-2-\alpha} v(s) \int_{\mathbb{R}} \left\{ 2I \left( 1, \frac{t}{\kappa} \right) - I \left( 0, \frac{t}{\kappa} \right) \right\} dt
\]

\[
+ \kappa^{-2-\alpha} \int_{\mathbb{R}} 2I_q \left( 1, \frac{t}{\kappa} \right) (v(s - t) - v(s)) \, dt
\]

\[
+ \kappa^{-2-\alpha} v(s) \int_{\mathbb{R}} \left\{ 2I_p \left( 1, \frac{t}{\kappa} \right) - I_p \left( 0, \frac{t}{\kappa} \right) \right\} \frac{-t}{\kappa} \, dt.
\]

Using that \( I_p \left( \cdot, \frac{1}{\kappa} \right) \frac{1}{\kappa} = \partial_t I \left( \cdot, \frac{t}{\kappa} \right) \) and integrating by parts the third line in the previous expression, we find

\[
\frac{d}{d\varepsilon} H(\kappa + \varepsilon v) \bigg|_{\varepsilon = 0} (s) = - \alpha\kappa^{-2-\alpha} v(s) \int_{\mathbb{R}} \left\{ 2I \left( 1, \frac{t}{\kappa} \right) - I \left( 0, \frac{t}{\kappa} \right) \right\} dt
\]

\[
+ \kappa^{-2-\alpha} \int_{\mathbb{R}} 2I_q \left( 1, \frac{t}{\kappa} \right) (v(s - t) - v(s)) \, dt
\]

\[
- \kappa^{-2-\alpha} v(s) \left[ t2I \left( 1, \frac{t}{\kappa} \right) - tI \left( 0, \frac{t}{\kappa} \right) \right]_{t=-\infty}^{t=+\infty}
\]

\[
= - \alpha\kappa^{-2-\alpha} v(s) \int_{\mathbb{R}} \left\{ 2I \left( 1, \frac{t}{\kappa} \right) - I \left( 0, \frac{t}{\kappa} \right) \right\} dt
\]

\[
+ \kappa^{-2-\alpha} \int_{\mathbb{R}} 2I_q \left( 1, \frac{t}{\kappa} \right) (v(s - t) - v(s)) \, dt.
\]
Making the change of variable $\tau = \frac{t}{\kappa}$, we get
\[
\frac{d}{d\varepsilon} H(\kappa + \varepsilon v) \big|_{\varepsilon = 0} (s) = -\alpha \kappa^{-1-\alpha} v(s) \int_{\mathbb{R}} \{2I(1, \tau) - I(0, \tau)\} d\tau
+ \kappa^{-1-\alpha} \int_{\mathbb{R}} 2I_q(1, \tau) (v(s - \kappa \tau) - v(s)) d\tau
= \kappa^{-1-\alpha} \int_{\mathbb{R}} G_\alpha(\tau) (v(s) - v(s - \kappa \tau)) d\tau - \kappa^{-1-\alpha} b_\alpha v(s),
\]

since, by (2.7), $\int_{\mathbb{R}} \{2I(1, \tau) - I(0, \tau)\} d\tau = H(1)$ and on the other hand, by (2.3),
\[
\alpha H(1) = \int_{S^{N-2}} \int_{\mathbb{R}} \sigma - e_1|^2 \tau^2 + |\sigma - e_1|^2 (N+\alpha)/2 d\tau d\sigma = b_\alpha.
\]
We have also used that
\[
2I_q(1, \tau) = -2 \int_{S^{N-2}} \frac{1}{(\tau^2 + 2(1 - \sigma_1))^{(N+\alpha)/2}} d\sigma = -G_\alpha(\tau).
\]
Thus, we have obtained the expression (2.6) for the linearized operator.

### 2.3. Preliminary estimates on the linearized operator
The following lemma provides estimates for the function $G_\alpha$ appearing in Proposition 2.3.

**Lemma 2.4.** Let $N \geq 3$ and $\alpha > 0$. Then there exists a positive constant $C$ depending only on $N$ and $\alpha$ such that
\[
G_\alpha(\tau) \leq C \min\{|\tau|^{-2-\alpha}, |\tau|^{-N-\alpha}\} \quad \text{for } \tau \neq 0. \tag{2.8}
\]
Moreover, we have
\[
G_\alpha(\tau) = |\tau|^{-2-\alpha} g(\tau^2) \quad \text{for } \tau \neq 0, \tag{2.9}
\]
where $g : (0, +\infty) \to \mathbb{R}$ is a bounded function and it is given by
\[
g(\rho) = 2C_N \int_0^{\rho^2} \frac{t(2 - \rho t)^{N+1}}{(1 + 2t)^{N+2}} dt \quad \text{for } \rho > 0, \tag{2.10}
\]
with
\[
C_N = \frac{2\pi^{(N-2)/2}}{\Gamma((N-2)/2)}.
\]
Furthermore, we have
\[
g_0 := C'_N \int_0^{+\infty} \frac{t^{N+1}}{(1 + 2t)^{N+2}} dt = \lim_{\rho \to 0^+} g(\rho) \tag{2.11}
\]
with
\[
C'_N = 2^{N-4} C_N = 2^{N-4} \frac{2\pi^{(N-2)/2}}{\Gamma((N-2)/2)},
\]
and
\[
|g(\rho) - g_0| \leq C \rho \quad \text{for } \rho \in (0, 1) \tag{2.12}
\]
for some constant $C > 0$. 
Proof. In the following, the letter $C$ stands for different positive constants depending only on $N$ and $\alpha$. We have that

$$G_\alpha(\tau) = 2 \int_{S^{N-2}} \frac{1}{(\tau^2 + 2(1 - \sigma_1))^{N+\alpha}} d\sigma = 2C_N \int_{-1}^{1} \frac{(1 - \sigma_1^2)^{N-4}}{(\tau^2 + 2(1 - \sigma_1))^{N+\alpha}} d\sigma_1$$

$$= 2C_N \int_0^{2} \frac{r(2 - r) \alpha_{N-4}}{(\tau^2 + 2r)^{N+\alpha}} dr,$$

with $C_N = |S^{N-3}| = \frac{2\pi^{(N-2)/2}}{\Gamma((N-2)/2)}$. This leads to

$$G_\alpha(\tau) \leq 2C_N|\tau|^{-N-\alpha} \int_0^{2} (r(2 - r) \alpha_{N-4}) dr \leq C|\tau|^{-N-\alpha} \quad \text{for } \tau \neq 0. \quad (2.13)$$

Making the further change of variable $t = r/\tau^2$, we also find that

$$G_\alpha(\tau) = \frac{2C_N}{|\tau|^{N+\alpha}} \int_0^{2} \frac{(r(2 - r) \alpha_{N-4})}{(1 + 2r/\tau^2)^{N+\alpha}} dr = \frac{2C_N}{|\tau|^{2+\alpha}} \int_0^{2/\tau^2} \frac{(t(2 - t^2) \alpha_{N-4})}{(1 + 2t)^{N+\alpha}} dt = \frac{g(\tau^2)}{|\tau|^{2+\alpha}}$$

for $\tau \neq 0$, with $g$ defined in (2.10).

Next we prove (2.12). For this we write

$$g_0 - g(\rho) = 2C_N \int_0^{2/\rho} \frac{(2 \alpha_{N-4} - (2 - \rho t) \alpha_{N-4})}{(1 + 2t)^{N+\alpha}} dt + 2\alpha_{N-4} C_N \int_0^{+\infty} \frac{t^{N-2}}{(1 + 2t)^{N+\alpha}} dt$$

$$= 2C_N I_1(\rho) + 2\alpha_{N-4} C_N I_2(\rho). \quad (2.14)$$

We start with $I_2(\rho)$ and notice that

$$I_2(\rho) = \int_{2/\rho}^{+\infty} \frac{t^{N-2}}{(1 + 2t)^{N+\alpha}} dt \leq \frac{\rho}{2} \int_0^{+\infty} \frac{t^{N-2}}{(1 + 2t)^{N+\alpha}} dt,$$

which yields

$$I_2(\rho) \leq C\rho \quad \text{for } \rho > 0. \quad (2.15)$$

On the other hand, if $N = 4$, we have $I_1(\rho) = 0$ for $\rho \in (0,1)$, and thus follows. We now consider the case $N \geq 3$, $N \neq 4$, and we write $I_1(\rho)$ as follows:

$$I_1(\rho) = \int_0^{2/\rho} \frac{(2 \alpha_{N-4} - (2 - \rho t) \alpha_{N-4})}{(1 + 2t)^{N+\alpha}} dt$$

$$= \int_0^{2/\rho} \int_0^1 \frac{d}{d\varrho} \frac{(2 \alpha_{N-4} - (2 - \rho \varrho t) \alpha_{N-4})}{(1 + 2t)^{N+\alpha}} d\varrho dt$$

$$= \frac{N - 4}{2} \int_0^{2/\rho} \int_0^1 \frac{\rho t^{N-6} (2 - \rho \varrho t) \alpha_{N-4}}{(1 + 2t)^{N+\alpha}} d\varrho dt$$

$$= \frac{N - 4}{2} \rho \int_0^1 \left( I_{11}(\rho, \varrho) + I_{12}(\rho, \varrho) \right) d\varrho \quad (2.16)$$
with
\[ I_{11}(\rho, \varrho) := \int_0^{1/\rho} \frac{(2 - \varrho pt)^{\frac{N-6}{2}} t^{\frac{N-2}{2}}}{(1+2t)^{\frac{N+4}{2}}} dt \quad \text{and} \quad I_{12}(\rho, \varrho) := \int_{1/\rho}^{2/\rho} \frac{(2 - \varrho pt)^{\frac{N-6}{2}} t^{\frac{N-2}{2}}}{(1+2t)^{\frac{N+4}{2}}} dt. \]

To estimate \( I_{11} \), we observe that \( 2 - \varrho \leq 2 - \varrho pt \leq 2 \) if \( 0 \leq t \leq \frac{1}{\rho} \). Consequently,

\[ I_{11}(\rho, \varrho) \leq \max \left\{ (2 - \varrho)^{\frac{N-6}{2}}, 2^{\frac{N-6}{2}} \right\} \int_0^{+\infty} \frac{t^{\frac{N-2}{2}}}{(1+2t)^{\frac{N+4}{2}}} dt \leq C \quad (2.17) \]

for \( \rho, \varrho \in (0, 1) \). Moreover, for \( \rho, \varrho \in (0, 1) \), we have

\[ I_{12}(\rho, \varrho) \leq C \int_{1/\rho}^{2/\rho} (2 - \varrho pt)^{\frac{N-6}{2}} t^{-1-\alpha/2} dt = C(\varrho \rho)^{\alpha/2} \int_0^{2\varrho} (2-s)^{\frac{N-6}{2}} s^{-1-\alpha/2} ds \]

\[ \leq C \frac{\varrho}{\rho} \int_0^{2\varrho} (2-s)^{\frac{N-6}{2}} ds. \]

If \( \varrho \in \left(0, \frac{1}{2}\right] \), it thus follows that

\[ I_{12}(\rho, \varrho) \leq C \max_{0 \leq s \leq 1} (2-s)^{\frac{N-6}{2}} \leq C, \]

whereas in case \( \varrho \in \left[\frac{1}{2}, 1\right) \) we deduce, since \( N \geq 3, N \neq 4 \), that

\[ I_{12}(\rho, \varrho) \leq C |(2 - 2\varrho)^{\frac{N-6}{2}} - (2 - \varrho)^{\frac{N-6}{2}}| \leq C(1 - \varrho)^{-\frac{1}{2}}. \]

From the last two estimates and (2.17), we infer that

\[ \int_0^1 (I_{11}(\rho, \varrho) + I_{12}(\rho, \varrho)) d\varrho \leq C \quad \text{for} \ \rho \in (0, 1), \]

and together with (2.16) this yields

\[ I_1(\rho) \leq C \rho \quad \text{for} \ \rho \in (0, 1). \]

Combining this inequality with (2.14) and (2.15), we conclude that

\[ |g_0 - g(\rho)| \leq C \rho \quad \text{for} \ \rho \in (0, 1). \]

Therefore (2.12) follows, and (2.12) implies (2.11). Moreover, from (2.11) and (2.13) we deduce that (2.8) holds. Finally, using (2.11) and \( g(\tau^2) = |\tau|^{2+\alpha} G_\alpha(\tau) \) combined with (2.13), we see that \( g \) is bounded on \((0, \infty)\), as claimed. \( \square \)

Our next result will be important to derive estimates for the eigenvalues of the operator

\[ \varphi \mapsto \int_{\mathbb{R}} (\varphi(s) - \varphi(s-\tau)) G_\alpha(\tau) d\tau \]
acting on even $2\pi$-periodic functions, see Lemma 3.1 in Section 3. As we shall see in that lemma, the eigenvalues of this operator are expressed in terms of the function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(b) := \int_{\mathbb{R}} (1 - \cos(b\tau))G_\alpha(\tau) \, d\tau = 2 \int_{\mathbb{R}} \int_{S^{N-2}} \frac{1 - \cos(b\tau)}{(\tau^2 + 2(1 - \sigma_1))^{N+\alpha/2}} \, d\sigma d\tau. \quad (2.18)$$

We note that $h$ is well defined by (2.8), since $|1 - \cos(b\tau)| = 2\sin^2 \frac{b\tau}{2} \leq \frac{b^2\tau^2}{2}$ for $\tau \in \mathbb{R}$, $b \geq 0$.

**Lemma 2.5.** For $N \geq 3$, the function $h$ defined in (2.18) is differentiable. Moreover, it satisfies $h(0) = 0$,

$$h'(b) = \int_{\mathbb{R}} \tau \sin(b\tau)G_\alpha(\tau) \, d\tau > 0 \quad \text{for } b > 0 \quad (2.19)$$

and

$$0 < \lim_{b \to +\infty} \frac{h(b)}{b^{1+\alpha}} < +\infty. \quad (2.20)$$

**Proof.** We first note that $h(0) = 0$ holds trivially by definition. Next we prove that $h$ is differentiable. Indeed using Lemma 2.4, for $0 \leq b < b_0$ and $\tau \in \mathbb{R} \setminus \{0\}$ we have

$$|\tau \sin(b\tau)G_\alpha(\tau)| \leq C \min\{b_0, |\tau|\}^2 \frac{1}{|\tau|^{2+\alpha}} = e(\tau), \quad (2.21)$$

and the function $\tau \mapsto e(\tau)$ is integrable over $\mathbb{R}$. Hence a standard argument based on Lebesgue’s theorem and the mean value theorem shows that the limit

$$h'(b) = \lim_{\rho \to 0} \int_{\mathbb{R}} \frac{\cos(b\tau) - \cos((b + \rho)\tau)}{\rho}G_\alpha(\tau) \, d\tau = \int_{\mathbb{R}} \tau \sin(b\tau)G_\alpha(\tau) \, d\tau \quad (2.22)$$

exists. Hence $h$ is differentiable and satisfies the equality in (2.19).

To prove (2.20), using (2.9) we see that

$$h(b) = \frac{1}{b} \int_{\mathbb{R}} (1 - \cos(\tau))G_\alpha(\tau/b) \, d\tau = b^{1+\alpha} \int_{\mathbb{R}} \frac{1 - \cos(\tau)}{|\tau|^{2+\alpha}} g_0(\tau^2/b^2) \, d\tau \quad (2.23)$$

for $b > 0$, with $g_0$ given by (2.10). Since the function $g$ is bounded by Lemma 2.3, it follows from Lebesgue’s theorem that

$$\lim_{b \to +\infty} \frac{h(b)}{b^{1+\alpha}} = g_0 \int_{\mathbb{R}} \frac{1 - \cos(\tau)}{|\tau|^{2+\alpha}} \, d\tau \in (0, \infty)$$

with $g_0$ defined in (2.11).

To prove that $h'(b) > 0$ for $b > 0$, we note that from (2.10) we have, for $\rho > 0$,

$$g'(\rho) = -\frac{4C_N}{(1 + 4\rho^{N+\alpha}/\rho^2)} < 0 \quad \text{if } N = 4$$

and

$$g'(\rho) = -(N - 4)C_N \int_0^{2/\rho} \left(\frac{2 - \rho t}{\rho t} \right)^{N-6} \left(\frac{N-2}{t} \right)^{N+\alpha} \, dt < 0 \quad \text{if } N \geq 5.$$
Thus in case $N \geq 4$ the function $b \mapsto g(z^2)$ is increasing on $(0, \infty)$ for every $\tau \in \mathbb{R} \setminus \{0\}$. Consequently, by (2.23), the fact that $h$ is differentiable, and the strict positivity of the function $g$, we immediately deduce that $h'(b) > 0$ for $b > 0$. 

In case $N = 3$, we can use (2.22) and the estimate (2.21) to compute, by changing the order of integration, 

$$h'(b) = \int_\mathbb{R} \tau \sin(\tau b) G_\alpha(\tau) \, d\tau = 2 \int_\mathbb{R} \tau \sin(\tau b) \int_{S^1} \frac{1}{(\tau^2 + 2(1 - \sigma_1))^{\frac{3+\alpha}{2}}} \, d\sigma \, d\tau$$

$$= 2 \int_{S^1} \int_\mathbb{R} \frac{\tau \sin(\tau b)}{(\tau^2 + 2(1 - \sigma_1))^{\frac{3+\alpha}{2}}} \, d\tau \, d\sigma = \int_{S^1} V_\nu(\sqrt{2(1 - \sigma_1)}) \, d\sigma,$$

where

$$V_\nu(\xi) := 2 \int_\mathbb{R} \frac{\tau \sin(\tau b)}{(\tau^2 + \xi^2)^{\frac{3+\alpha}{2}}} \, d\tau = 4 \int_0^\infty \frac{\tau \sin(\tau b)}{(\tau^2 + \xi^2)^{\frac{3+\alpha}{2}}} \, d\tau = 4 \nu b^{1+\alpha/2} \xi^{-\alpha/2} K_{\alpha/2}(b \xi)$$

for $\xi, b > 0$. Here $K_\nu$ is the modified Bessel function of the second kind (also called Macdonald function), $\chi := 2^{-1-\alpha/2} \frac{\sqrt{\pi}}{\Gamma((3+\alpha)/2)} > 0$ and $\Gamma$ is the usual Gamma function. For the second equality in (2.24), we refer e.g. to [15, Page 442, 3.771, 5.] and note that $K_{\alpha/2} = K_{-\alpha/2}$ (see [15, Page 929, 8.486, 16.]). Since $K_{\alpha/2}(b \xi) > 0$ (by [15, Page 917, 8.432, 1.]) and therefore $V_\nu(\xi) > 0$ for $\xi, b > 0$, it thus follows that $h'(b) > 0$ for $b > 0$, completing the proof of the lemma. \hfill \square

3. Nonlinear problem to be solved and Proof of Theorem 1.1

To prove Theorem 1.1 we are looking for constants $R, a_0 > 0$ and functions $u_a$ of the form

$$u_a(s) = R + \frac{\varphi_a(\lambda(a)s)}{\lambda(a)}, \quad a \in (-a_0, a_0),$$

satisfying the equation

$$H(u_a)(s) = H(R) \quad \text{for all } s \in \mathbb{R}, \ a \in (-a_0, a_0).$$

(3.1)

Here we require that $\lambda : (-a_0, a_0) \to (0, \infty)$ is a smooth function such that $\lambda(0) = 1$. Moreover, we look for functions $\varphi_a \in C^{1,\beta}(\mathbb{R})$ with $a \in (-a_0, a_0)$ which are even, $2\pi$-periodic, and satisfy the expansion

$$\varphi_a = a(\cos(-\cdot) + v_a)$$

with $v_a \to 0$ in $C^{3,\beta}(\mathbb{R})$ as $a \to 0$ and $\int_{-\pi}^\pi v_a(t) \cos(t) \, dt = 0$ for $a \in (-a_0, a_0)$.

Note that we have rescaled the problem so that we can work with functions $\varphi_a$ with fixed period. For the rescaled function $\bar{u}_a(s) := \lambda(a) u_a \left( \frac{s}{\lambda(a)} \right)$, a change of variables gives

$$H(\bar{u}_a)(s) = \lambda(a)^{-\alpha} H(u_a) \left( \frac{s}{\lambda(a)} \right) \quad \text{for } s \in \mathbb{R}.$$
Therefore by (3.1) our problem becomes
\[ H(\lambda(a)R + \varphi_a)(s) = H(\tilde{u}_a)(s) = \lambda(a)^{-\alpha}H(R) = H(\lambda(a)R) \quad \text{for all } s \in \mathbb{R}. \]

For matters of convenience, we will use \( \mu(a) = \lambda(a)R \) as a new unknown. Our aim is to deduce Theorem 1.1 from the Crandall-Rabinowitz theorem \( [11] \) applied to the map
\[ (\mu, \varphi) \mapsto \Phi(\mu, \varphi) := \mu^{1+\alpha}\{H(\mu + \varphi) - H(\mu)\}, \]
since our equation has become \( \Phi(\mu, \varphi) = 0 \). The factor \( \mu^{1+\alpha} \) is introduced to simplify some expressions at a later stage.

We need to introduce the functional spaces in which we work. We fix \( \beta \) such that
\[ 0 < \alpha < \beta < \min\{1, 2\alpha + 1/2\}. \tag{3.2} \]
The condition \( \beta < 2\alpha + 1/2 \) is technical (to simplify a proof on regularity) and could be avoided. Consider the Banach spaces
\[ X := C^{1,\beta}_{p,e} = \{ \varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C^{1,\beta}(\mathbb{R}) \text{ is } 2\pi\text{-periodic and even} \} \]
and
\[ Y := C^{0,\beta-\alpha}_{p,e} = \{ \tilde{\varphi} : \mathbb{R} \to \mathbb{R} : \tilde{\varphi} \in C^{0,\beta-\alpha}(\mathbb{R}) \text{ is } 2\pi\text{-periodic and even} \}. \]
The norms of \( X \), respectively \( Y \), are the standard \( C^{1,\beta}(\mathbb{R}) \) and \( C^{0,\beta-\alpha}(\mathbb{R}) \)-norms, respectively, defined by
\[ \|u\|_{C^{k,\gamma}(\mathbb{R})} := \sum_{j=0}^{k} \|u^{(j)}\|_{L^\infty(\mathbb{R})} + \sup_{s,t \in \mathbb{R}} \frac{|u^{(k)}(s) - u^{(k)}(t)|}{|s - t|^\gamma}. \tag{3.3} \]

Since \( H : \mathcal{O} \subset C^{1,\beta}(\mathbb{R}) \to C^{0,\beta-\alpha}(\mathbb{R}) \) is smooth in \( \mathcal{O} \) by Proposition 2.3 (for \( N \geq 3 \)), and clearly \( H \) sends \( 2\pi \)-periodic and even functions to functions which are also \( 2\pi \)-periodic and even (for instance by expression (2.3)), we infer that
\[ \Phi : D_\Phi \to Y, \quad \Phi(\mu, \varphi) = \mu^{1+\alpha}\{H(\mu + \varphi) - H(\mu)\} \tag{3.4} \]
is a smooth map defined on the open set
\[ D_\Phi := \{(\mu, \varphi) : \mu > 0, \varphi \in X, \inf_{\mathbb{R}} \varphi > -\mu\} \subset \mathbb{R} \times X. \tag{3.5} \]
By definition, we have
\[ \Phi(\mu, 0) = 0 \quad \text{for every } \mu > 0. \]
Next we need to study the properties of the family of linearized operators
\[ L_\mu := D_\varphi \Phi(\mu, 0) = \mu^{1+\alpha}DH(\mu) \in \mathcal{L}(X,Y), \quad \mu > 0. \]
Here and in the following, \( \mathcal{L}(X,Y) \) denotes the space of bounded linear operators \( X \to Y \). By Proposition 2.3 \( L_\mu \) is given by
\[ L_\mu v(s) = PV \int_{\mathbb{R}} (v(s) - v(s - \mu \tau))G_\alpha(\tau) \, d\tau - b_\alpha v(s) \quad \text{for } v \in X. \tag{3.6} \]
Lemma 3.1. Let \( N \geq 3 \) and \( \mu > 0 \). The functions
\[
e_k \in X, \quad e_k(s) = \cos(ks), \quad k \in \mathbb{N} \cup \{0\}
\]
are eigenfunctions of \( L_\mu \) with corresponding eigenvalues
\[
\lambda_k(\mu) = h(k\mu) - b_\alpha, \quad k \in \mathbb{N} \cup \{0\},
\]
where the function \( h \) is defined in (2.18). Moreover,
\[
\lambda_0(\mu) < \lambda_1(\mu) < \lambda_2(\mu) < \ldots \quad \text{and}
\]
\[
0 < \lim_{k \to +\infty} \frac{\lambda_k(\mu)}{k^{1+\alpha}} < \infty. \tag{3.10}
\]
Proof. Let \( k \in \mathbb{N} \cup \{0\} \). We note that, by (3.6),
\[
L_\mu e_k(s) = PV \int \left\{ \cos(ks) - \cos(k(s - \mu \tau)) \right\} G_\alpha(\tau) \, d\tau - b_\alpha \cos(ks)
\]
\[
= PV \int \left\{ \cos(ks) - \cos(ks) \cos(k\mu \tau) - \sin(ks) \sin(k\mu \tau) \right\} G_\alpha(\tau) \, d\tau - b_\alpha \cos(ks)
\]
\[
= \left\{ PV \int \left\{ 1 - \cos(k\mu \tau) \right\} G_\alpha(\tau) \, d\tau - b_\alpha \right\} \cos(ks) = \left( h(k\mu) - b_\alpha \right) e_k(s),
\]
with \( h \) defined in (2.18). Here we used the oddness of \( \sin(\cdot) \) and the evenness of \( G_\alpha \). This shows that \( e_k \) is an eigenfunction of \( L_\mu \) with eigenvalue \( h(k\mu) - b_\alpha \). The properties (3.9) and (3.10) now follow readily from Lemma 2.5. \( \square \)

We are now in position to establish the following.

Proposition 3.2. Let \( N \geq 3 \). There exists a unique \( \mu^* > 0 \) such that the linear operator \( L := L_{\mu^*} : X \to Y \) has the following properties.

(i) The kernel of \( L \) is spanned by the function \( \cos(\cdot) \).

(ii) The range of \( L \) is given by
\[
R(L) = \left\{ v \in Y : \int_{-\pi}^{\pi} \cos(s)v(s) \, ds = 0 \right\}.
\]

Moreover, we have that
\[
\partial_\mu \bigg|_{\mu = \mu^*} L_{\mu^*} \cos(\cdot) \notin R(L). \tag{3.11}
\]

Proof. By Lemmas 2.5 and 3.1 there exists a unique \( \mu^* > 0 \) such that \( L_{\mu^*} \cos(\cdot) = 0 \). We put \( L := L_{\mu^*} \) in the following. Consider the spaces
\[
X_\perp := \left\{ v \in X : \int_{-\pi}^{\pi} \cos(s)v(s) \, ds = 0 \right\} \subset X,
\]
\[
Y_\perp := \left\{ v \in Y : \int_{-\pi}^{\pi} \cos(s)v(s) \, ds = 0 \right\} \subset Y.
\]
To show properties (i) and (ii), it clearly suffices to prove that
\( L \) defines an isomorphism between \( X_\perp \) and \( Y_\perp \). \( \square \)
To prove (3.13), we let
\[ H_\perp := \left\{ v \in H^{1+\alpha}_{\text{loc}}(\mathbb{R}) : v \text{ even, } 2\pi\text{-periodic with } \int_{-\pi}^{\pi} \cos(s)v(s) \, ds = 0 \right\}, \]  
\[ V_\perp := \left\{ v \in L^2_{\text{loc}}(\mathbb{R}) : v \text{ even, } 2\pi\text{-periodic with } \int_{-\pi}^{\pi} \cos(s)v(s) \, ds = 0 \right\}. \]

We note that the functions \( \cos(k\cdot), k \in \{0, 2, 3, 4, \ldots\} \) form an orthonormal basis of \( V_\perp \), and that \( H_\perp \) can be characterized in terms of Fourier coefficients as the subspace of all \( v \in V_\perp \) such that
\[ \sum_{k \in \mathbb{N}} \left( k^{1+\alpha} \int_{-\pi}^{\pi} v(s) \cos(ks) \, ds \right)^2 < \infty. \]

Since \( \cos(k\cdot) \) are eigenfunctions of \( L \) with eigenvalues \( \lambda_k(\mu_*) \), from (3.9), the fact that \( L \cos(\cdot) = 0 \), the asymptotics (3.10), and the characterization given above we deduce that
\[ L \text{ defines an isomorphism between } H_\perp \text{ and } V_\perp. \]  
(3.15)

Next, note that \( C^{1,\beta}(\mathbb{R}) \subset H^{1+\alpha}_{\text{loc}}(\mathbb{R}) \). This follows from the definition of \( H^{1+\alpha}_{\text{loc}}(\mathbb{R}) \) via the Gagliardo seminorm, see e.g. [18, Definition 1.3.2.1]. Indeed, let \( v \in C^{1,\beta}(\mathbb{R}) \). To see that \( v' \in H^{1+\alpha}_{\text{loc}}(\mathbb{R}) \) we need to ensure that
\[ \int_{\Omega} \int_{\Omega} |v'(s) - v'(\bar{s})|^2 |s - \bar{s}|^{-(1+2\alpha)} \, ds \, d\bar{s} < \infty \]
for any bounded interval \( \Omega \subset \mathbb{R} \). This is clearly true since \( v' \in C^{0,\beta}(\mathbb{R}) \) and \( \beta > \alpha \).

We deduce that \( X_\perp \subset H_\perp \). Since also \( Y_\perp = V_\perp \cap Y \), we see that \( L : X_\perp \to Y_\perp \) is well defined and one-to-one.

To establish surjectivity, let \( f \in Y_\perp \). Since \( Y_\perp \subset V_\perp \), by (3.15) there exists \( w \in H_\perp \) such that \( Lw = f \). Recall that, by (3.6) and a change of variable, \( L \) is given by
\[ Lw(s) = \int_{\mathbb{R}} \{w(s) - w(s-t)\} G_\alpha \left( \frac{dt}{\mu_*} \right) b_\alpha w(s). \]

Hence, \( Lw = f \) can be written as
\[ \int_{\mathbb{R}} \{w(s) - w(s-t)\} G_\alpha \left( \frac{dt}{\mu_*} \right) dt = \mu_0 b_\alpha w(s) + \mu_* f(s) \quad \text{for } s \in \mathbb{R}. \]

Moreover, \( w \in H_\perp \subset Y = C^{0,\beta-\alpha}_{\mu_0, \mu_*} \) by Morrey’s embedding, since \( 1 + \alpha - 1/2 = 1/2 + \alpha > \beta - \alpha \) as assumed in (3.2). Thus \( \mu_0 b_\alpha w + \mu_* f \in Y \), and Lemma 3.3 below yields \( w \in X \cap H_\perp = X_\perp \). The proof of (3.13) is complete.

It remains to prove (3.11), which is simply a consequence of the fact that
\[ \partial_{\mu} \bigg|_{\mu = \mu_*} L_{\mu} \cos(\cdot) = \partial_{\mu} \bigg|_{\mu = \mu_*} \lambda_1(\mu) \cos(\cdot) = h'(\mu_*) \cos(\cdot) \]
by (3.8) and that \( h'(\mu_*) > 0 \) by Lemma 2.5. \( \square \)
It remains to prove the regularity result that we have used at the end of the previous proof.

**Lemma 3.3.** Let $N \geq 3$, $\mu > 0$, $f \in Y$ and $v \in H_{\perp} \subset Y$ be such that
\[
\int_{\mathbb{R}} (v(s) - v(s - t))G_{\alpha}(t/\mu) \, dt = f(s) \quad \text{for all } s \in \mathbb{R},
\] (3.16)
where $H_{\perp}$ is defined in (3.14). Then $v \in X = C^{1,\beta}_{p,e}$.

**Proof.** Put $\Gamma_v(s, t) = v(s) - v(s - t)$ for $s, t \in \mathbb{R}$. Recalling Lemma 2.4 we write
\[
f(s) = \int_{\mathbb{R}} \Gamma_v(s, t)G_{\alpha}(t/\mu) \, dt = \mu^{2+\alpha}\int_{\mathbb{R}} \Gamma_v(s, t)|t|^{-2-\alpha}g(t^2/\mu^2) \, dt = \mu^{2+\alpha}g_0 \int_{\mathbb{R}} \Gamma_v(s, t)|t|^{-2-\alpha} \, dt + \tilde{f}(s),
\] (3.17)
where
\[\tilde{f} : \mathbb{R} \to \mathbb{R}, \quad \tilde{f}(s) := \mu^{2+\alpha}\int_{\mathbb{R}} \Gamma_v(s, t)|t|^{-2-\alpha}(g(t^2/\mu^2) - g_0) \, dt.\]

By Lemma 2.4, $g$ is a bounded function on $(0, \infty)$ which satisfies $|g(t^2/\mu^2) - g_0| \leq C t^2$ for $t \in (-\mu, \mu)$ by (2.12). Moreover, we have $|\Gamma_v(s, t) - \Gamma_v(s, \bar{s})| \leq 2\|v\|_Y|s - \bar{s}|^{\beta-\alpha}$ for $s, \bar{s}, t \in \mathbb{R}$. Consequently, we deduce that
\[
\|\tilde{f}\|_Y \leq C\|v\|_Y.
\]
with a constant $C > 0$ independent of $v$. Now thanks to (3.17), the equality (3.16) becomes
\[
\int_{\mathbb{R}} \frac{v(s) - v(s - t)}{|t|^{2+\alpha}} \, dt = \frac{1}{\mu^{2+\alpha}g_0}(f(s) - \tilde{f}(s)) \quad \text{for every } s \in \mathbb{R}.
\]
Since $f - \tilde{f} \in Y \subset C^{0,\beta-\alpha}(\mathbb{R})$ and $v \in Y \subset L^\infty(\mathbb{R})$, by [23, Proposition 2.8] we conclude that $v \in X$. \hfill \Box

We are now in a position to apply the Crandall-Rabinowitz theorem [11], which will give rise to the following bifurcation property.

**Proposition 3.4.** For $N \geq 3$, let $\mu_*$ be defined as in Proposition 3.2, let $X_{\perp} \subset X$ be the closed subspace given in (3.12), so that $X = X_{\perp} \oplus \langle \cos(\cdot) \rangle$. Moreover, let $D_\Phi \subset \mathbb{R} \times X$ be the open set defined in (3.12). Then there exists $a_0 > 0$ and a $C^\infty$ curve
\[
(-a_0, a_0) \to D_\Phi, \quad a \mapsto (\mu(a), \varphi_a)
\]
such that
(i) $\Phi(\mu(a), \varphi_a) = 0$ for $a \in (-a_0, a_0)$.
(ii) $\mu(0) = \mu_*$.
(iii) $\varphi_a = a(\cos(\cdot) + v_a)$ for $a \in (-a_0, a_0)$, and
\[
(-a_0, a_0) \to X_{\perp}, \quad a \mapsto v_a
\]
is a $C^\infty$ curve satisfying $v_0 = 0$. 
Proof. The claims (i)-(iii) follow by a direct application of the Crandall-Rabinowitz theorem as given in Theorems 1.7 and 1.18 of [11]. The assumptions of this theorem are satisfied by Proposition 3.2. To see (iv), we put

\[ \psi_a(s) := \varphi_a(s + \pi) = -a(\cos(s) + w_a) \quad \text{for } a \in (-a_0, a_0), \]

where \( w_a \in X \) is defined by \( w_a(s) := -v_a(s + \pi) \). We then have

\[ H(\mu(a) + \psi_a)(s) = H(\mu(a) + \varphi_a)(s + \pi) = H(\mu(a)) \quad \text{for } s \in \mathbb{R}, a \in (-a_0, a_0) \]

and thus \( \Phi(\mu(a), \psi_a) = 0 \) for \( a \in (-a_0, a_0) \). By the local uniqueness statement (1.8) in [11] Theorem 1.7], there exists \( \varepsilon \in (0, a_0) \) such that

\[ \{ \left( \mu(a), -a(\cos(\cdot) + w_a) \right) : |a| < \varepsilon \} \subset \{ \left( \mu(a), a(\cos(\cdot) + v_a) \right) : |a| < a_0 \} \quad (3.18) \]

By noting in addition that \( w_a \in X_\perp \) as a consequence of the fact that \( v_a \in X_\perp \), it follows that

\[ \mu(a) = \mu(-a) \quad \text{and} \quad w_a = v_{-a} \quad \text{for } a \in (-\varepsilon, \varepsilon), \]

hence also \( \varphi_{-a} = \psi_a \) for \( a \in (-\varepsilon, \varepsilon) \). Replacing \( a_0 \) by \( \varepsilon \), we thus conclude that properties (i)-(iv) hold. \( \square \)

Remark 3.5. As in our 2D paper [4], one could avoid using the Crandall-Rabinowitz theorem by considering the map \( (a, \varphi) \mapsto \left\{ \frac{1}{R}(H(\mu + a\varphi) - H(\mu)) \right\} \) instead of the map \( (3.2) \). In this way one uses the implicit function theorem at \( a = 0 \).

At the same time, we could have proved the 2D result in [4] using the Crandall-Rabinowitz theorem as in the present paper.

Proof of Theorem 1.1 (completed). Let \( \mu_* \) be given by Proposition 3.2 and consider \( a_0 > 0 \) and the smooth curve

\[ (-a_0, a_0) \to \mathcal{D}_\Phi, \quad a \mapsto (\mu(a), \varphi_a) \]

given by Proposition 3.4. We put \( R := \mu_* \) and consider the smooth maps

\[ (-a_0, a_0) \to (0, \infty), \quad a \mapsto \lambda(a) := \mu(a) \frac{R}{\mu_*} \]

\[ (-a_0, a_0) \to C^{1,\beta}(\mathbb{R}), \quad a \mapsto u_a = R + \frac{\varphi_a(\lambda(a) \cdot)}{\lambda(a)}. \]

With these definitions, all but two properties stated in Theorem 1.1 follow immediately from Proposition 3.4 and the remarks at the beginning of this section – note in particular that \( \partial_a u_a|_{a=0} = \cos(\cdot) \) follows from (1.4) and the fact that \( \lambda(0) = 1 \) and \( v_0 = 0 \). The following two statements still need to be justified:

Claim I. The minimal period of \( u_a \) is \( 2\pi/\lambda(a) \) if \( a \neq 0 \). Clearly this is equivalent, after the rescaling, to the statement that the function

\[ u(s) = \lambda R + a\{\cos(s) + v_a(s)\}. \]
with \( a \neq 0 \) and \( v_a \) orthogonal to \( \cos(\cdot) \) in \( L^2(-\pi, \pi) \). This is easily proved by expressing \( v_a(s) \) as a Fourier series \( a_0 + \sum_{k=2}^{\infty} a_k \cos(ks) \). If \( T \) is the minimal period of \( u \), we must have

\[
\cos(s) + v_a(s) = \cos(s + T) + v_a(s + T)
\]

\[
= \cos(s) \cos(T) - \sin(s) \sin(T) + a_0 + \sum_{k=2}^{\infty} a_k \{ \cos(ks) \cos(kT) - \sin(ks) \sin(kT) \}.
\]

Multiplying the first and last expressions in the above equalities by \( \cos(s) \) and integrating in \( (-\pi, \pi) \), we deduce that \( \cos(T) = 1 \). Hence the minimal period is \( T = 2\pi \).

Claim II. We have \( u_a \neq u_{a'} \) if \( a \neq a' \). Indeed, if \( u_a \equiv u_{a'} \), then the minimal periods of these functions coincide, and thus \( \lambda(a) = \lambda(a') \). By (1.4) we then have

\[
R + \frac{a}{\lambda(a)} \{ \cos(\cdot) + v_a(\cdot) \} = u_a(\frac{\cdot}{\lambda(a)}) = u_{a'}(\frac{\cdot}{\lambda(a)}) = R + \frac{a'}{\lambda(a)} \{ \cos(\cdot) + v_{a'}(\cdot) \},
\]

where the functions \( v_a \) and \( v_{a'} \) are orthogonal to \( \cos(\cdot) \) in \( L^2(-\pi, \pi) \). Multiplying (3.19) with \( \cos(\cdot) \) and integrating over \([-\pi, \pi]\), we obtain

\[
\frac{a}{\lambda(a)} \int_{-\pi}^{\pi} \cos^2(t) \, dt = \frac{a'}{\lambda(a)} \int_{-\pi}^{\pi} \cos^2(t) \, dt
\]

and therefore \( a = a' \). \( \square \)

4. Regularity of the NMC operator

The purpose of this section is to give the proof of Proposition 2.3. We first observe that obviously it suffices to consider \( \delta > 0 \) and to prove the regularity of the NMC operator as a map

\[
H : \mathcal{O}_\delta \rightarrow C^{0, \beta - \alpha}(\mathbb{R}), \quad \text{where } \mathcal{O}_\delta := \{ u \in C^{1, \beta}(\mathbb{R}) : \inf_{\mathbb{R}} u > \delta \}.
\]

To accomplish this, it will be crucial to use the expression of \( H \) given in Lemma 2.2.

For the readers convenience, let us first recall some notation introduced already in Subsection 2.1. We denote

\[
p_\sigma := |\sigma - e_1|
\]

and, for \( r \in \mathbb{R} \), we define

\[
\mu_r(\sigma) = \frac{1}{|\sigma - e_1|^{N+r}} = p_\sigma^{-N-r}.
\]

It is easy to see that

\[
\int_{S^{N-2}} \mu_r(\sigma) \, d\sigma < \infty \quad \text{for every } r < -2.
\]

We define the maps \( \Lambda_0, \Lambda : C^{1, \beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\Lambda_0(\varphi, s, t, p) = \frac{\varphi(s) - \varphi(s - pt)}{pt} = \int_0^1 \varphi'(s - ppt) \, dp
\]
and
\[ \Lambda(\varphi, s, t, p) = \Lambda_0(\varphi, s, t, p) - \varphi'(s - pt) = \int_0^1 (\varphi'(s - \rho pt) - \varphi'(s - pt))d\rho. \]

We observe that for every \( s, s_1, s_2, t, p \in \mathbb{R} \), we have
\[ |\Lambda(\varphi, s, t, p)| \leq 2\|\varphi\|_{C^1,\beta(\mathbb{R})} \min(|t|^\beta|p|^\beta, 1) \] (4.2)
and also
\[ |\Lambda(\varphi, s_1, t, p) - \Lambda(\varphi, s_2, t, p)| \leq 2\|\varphi\|_{C^1,\beta(\mathbb{R})} \min(|t|^\beta|p|^\beta, |s_1 - s_2|^\beta). \] (4.3)

Note also that for every \( s, s_1, s_2, t, p \in \mathbb{R} \), we have
\[ |\Lambda_0(u, s_1, t, p)^2 - \Lambda_0(u, s_2, t, p)^2| \leq 2\|u\|_{C^1,\beta(\mathbb{R})}^2 |s_1 - s_2|^\beta. \] (4.4)

In Lemma 2.2 we established that, for \( u \in O \), we have
\[ \mathcal{H}(u)(s) := -\frac{\alpha}{2} H(u)(s) \]
\[ = \int_{S_{N-2}} \mu\alpha_2(\sigma) \int_\mathbb{R} t\Lambda(u, s, t, p_\sigma)K_\alpha(u, s, t, p_\sigma)u^{N-2}(s - p_\sigma t)dtd\sigma \] (4.5)
\[ - \frac{u(s)}{2} \int_{S_{N-2}} \mu\alpha_3(\sigma) \int_\mathbb{R} K_\alpha(u, s, t, p_\sigma)u^{N-2}(s - p_\sigma t)dtd\sigma, \]
where the function \( K_\alpha : C^{1,\beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is defined by
\[ K_\alpha(u, s, t, p) = \frac{1}{(t^2 + t^2\Lambda_0(u, s, t, p)^2 + u(s)u(s - pt))^{(N+\alpha)/2}}. \] (4.6)

Using this expression (4.5) for the NMC, we shall show that \( H : O_\delta \to C^{0,\beta-\alpha}(\mathbb{R}) \) is of class \( C^\infty \) for every \( \delta > 0 \).

4.1. **Differential calculus toolbox.** For a finite set \( \mathcal{N} \), we let \( |\mathcal{N}| \) denote the length (cardinal) of \( \mathcal{N} \). It will be understood that \( |\emptyset| = 0 \). Let \( Z \) be a Banach space and \( U \) a nonempty open subset of \( Z \). If \( T \in C^k(U, \mathbb{R}) \) and \( u \in U \), then \( D^kT(u) \) is a continuous symmetric \( k \)-linear form on \( Z \) whose norm is given by
\[ \|D^kT(u)\| = \sup_{u_1, \ldots, u_k \in Z} \frac{|D^kT(u)[u_1, \ldots, u_k]|}{\prod_{j=1}^k \|u_j\|_Z}. \]

If \( T_1, T_2 \in C^k(U, \mathbb{R}) \), then also \( T_1T_2 \in C^k(U, \mathbb{R}) \), and the \( k \)-th derivative of \( T_1T_2 \) at \( u \) is given by
\[ D^k(T_1T_2)(u)[u_1, \ldots, u_k] = \sum_{\mathcal{N} \in \mathcal{J}_k} D^{\mathcal{N}}T_1(u)[u_n]_{n \in \mathcal{N}} D^{k-|\mathcal{N}|}T_2(u)[u_n]_{n \in \mathcal{N}^c}, \] (4.7)
where \( \mathcal{J}_k \) is the set of subsets of \( \{1, \ldots, k\} \) and \( \mathcal{N}^c = \{1, \ldots, k\} \setminus \mathcal{N} \) for \( \mathcal{N} \in \mathcal{J}_k \). If, in particular, \( L : Z \to \mathbb{R} \) is a linear map, we have
\[ D^{\mathcal{N}}(LT_2)(u)[u_i]_{i \in \mathcal{N}} = L(u)D^{\mathcal{N}}T_2(u)[u_i]_{i \in \mathcal{N}} + \sum_{j \in \mathcal{N}} L(u_j)D^{\mathcal{N} \setminus \{j\}}T_2(u)[u_i]_{i \neq j}. \] (4.8)
We also recall the Faá de Bruno formula. We let $T$ be as above, $V \subset \mathbb{R}$ open with $T(U) \subset V$ and $g : V \to \mathbb{R}$ be a $k$-times differentiable map. The Faá de Bruno formula states that

$$D^k(g \circ T)(u)[u_1, \ldots, u_k] = \sum_{\Pi \in \mathcal{P}_k} g^{(\#\Pi)}(T(u)) \prod_{P \in \Pi} D^{|P|}(T(u))[u_j]_{j \in P}, \quad (4.9)$$

for $u, u_1, \ldots, u_k \in U$, where $\mathcal{P}_k$ denotes the set of all partitions of $\{1, \ldots, k\}$, see e.g. [17].

### 4.2. Regularity of the nonlocal mean curvature operator.

For a function $u : \mathbb{R} \to \mathbb{R}$, we use the notation $[u; s_1, s_2] := u(s_1) - u(s_2)$ for $s_1, s_2 \in \mathbb{R}$, and we note the obvious equality

$$[uv; s_1, s_2] = [u; s_1, s_2]v(s_1) + u(s_2)[v; s_1, s_2] \quad \text{for } u, v : \mathbb{R} \to \mathbb{R}, s_1, s_2 \in \mathbb{R}. \quad (4.10)$$

We first give some estimates related to the kernel $K_\alpha$ as given in (4.6).

**Lemma 4.1.** Let $N \geq 3$ and $k \in \mathbb{N} \cup \{0\}$. Then, there exists a constant $c = c(N, \alpha, \beta, k, \delta) > 1$ such that for all $(s, s_1, s_2, t, p) \in \mathbb{R}^5$ and $u \in O_\delta$, we have

$$\|D^k_u K_\alpha(u, s, t, p)\| \leq \frac{c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c}{(1 + t^2)^{(N + \alpha)/2}}, \quad (4.11)$$

$$\|[D^k_u K_\alpha(u, \cdot, t, p); s_1, s_2]\| \leq \frac{c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c |s_1 - s_2|^{\beta}}{(1 + t^2)^{(N + \alpha)/2}}. \quad (4.12)$$

**Proof.** Throughout this proof, the letter $c$ stands for different constants greater than one and depending only on $N, \alpha, \beta, k$ and $\delta$. We define

$$Q : C^{1, \beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad Q(u, s, t, p) = t^2 + t^2 \Lambda_0(u, s, t, p)^2 + u(s)u(s - pt)$$

and

$$g_\alpha \in C^\infty(\mathbb{R}_+, \mathbb{R}), \quad g_\alpha(x) = x^{-(N + \alpha)/2},$$

so that

$$K_\alpha(u, s, t, p) = g_\alpha(Q(u, s, t, p)).$$

By (4.9) and recalling that $Q$ is quadratic in $u$, we have

$$D^k_u K_\alpha(u, s, t, p)[u_1, \ldots, u_k]$$

$$= \sum_{\Pi \in \mathcal{P}_k^2} g_\alpha^{(\#\Pi)}(Q(u, s, t, p)) \prod_{P \in \Pi} D^{|P|}_u Q(u, s, t, p)[u_j]_{j \in P}, \quad (4.13)$$
where $\mathcal{P}_k^2$ denotes the set of partitions $\Pi$ of $\{1, \ldots, k\}$ such that $|P| \leq 2$ for every $P \in \Pi$. Hence by (4.10), we have

$$
\begin{align*}
[D_u^k \mathcal{K}_\alpha(u, \cdot, t, p)[u_1, \ldots, u_k]; s_1, s_2] & = \sum_{\Pi \in \mathcal{P}_k^2} [g^{(1)}(Q(u, \cdot, t, p); s_1, s_2) \prod_{P \in \Pi} D_u^{|P|} Q(u, s_1, t, p)[u_j]_{j \in P} \\
& + \sum_{\Pi \in \mathcal{P}_k^2} g^{(2)}(Q(u, s_2, t, p)) \prod_{P \in \Pi} D_u^{|P|} Q(u, \cdot, t, p)[u_j]_{j \in P}; s_1, s_2].
\end{align*}
$$

For $P \in \Pi$ with $|P| \leq 2$, by using (4.14) and (4.10), we find that

$$
|[D_u^{|P|} Q(u, \cdot, t, p)[u_j]_{j \in P}; s_1, s_2]| \\
\leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})}^2) (1 + t^2) |s_1 - s_2|^\beta \prod_{j \in P} \|u_j\|_{C^{1, \beta}(\mathbb{R})}.
$$

For $\ell \in \mathbb{N}$ and $x > 0$, we have

$$
g^{(\ell)}(x) = (-1)^{\ell+1} \frac{\ell!}{2^{\ell}} \prod_{i=0}^{\ell-1} (N + \alpha + 2i)x^{-N+\alpha+2\ell}.
$$

Consequently, for every $u \in \mathcal{O}_\delta$, using (4.14) and (4.10), we have the estimates

$$
| [g^{(\ell)}(Q(u, \cdot, t, p)); s_1, s_2] | \\
= |Q(u, \cdot, t, p); s_1, s_2| \int_0^1 g^{(\ell+1)}(\tau Q(u, s_1, t, p) + (1 - \tau) Q(u, s_2, t, p)) d\tau \\
$$

for $\ell = 0, \ldots, k$. 

Therefore by (4.14), (4.15), (4.16), (4.17) and (4.18), we obtain
\[
| [D^k_u K_\alpha(u, \cdot, t, p)[u_1, \ldots, u_k]; s_1, s_2] | 
\leq c(1 + \|u\|_{C^1(\mathbb{R})}^c) |s_1 - s_2|^\beta \sum_{\Pi \in \mathbb{P}_k^2} \frac{1}{(1 + t^2)^{(N+2)} \prod_{\tau \in \Pi} (1 + t^2) \prod_{j \in P} \|u_j\|_{C^1(\mathbb{R})}} 
\]
\[
= c(1 + \|u\|_{C^1(\mathbb{R})}^c) |s_1 - s_2|^\beta \sum_{\Pi \in \mathbb{P}_k^2} \prod_{P \in \Pi} \prod_{j \in P} \|u_j\|_{C^1(\mathbb{R})}. 
\]
We then conclude that
\[
| [D^k_u K_\alpha(u, \cdot, t, p)[u_1, \ldots, u_k]; s_1, s_2] | \leq \frac{c(1 + \|u\|_{C^1(\mathbb{R})}^c) |s_1 - s_2|^\beta}{(1 + t^2)^{(N+\delta)/2}} \prod_{i=1}^k \|u_i\|_{C^1(\mathbb{R})}. 
\]
This yields (4.12). Furthermore we easily deduce from (4.13), (4.16) and (4.18) that
\[
|D^k_u K_\alpha(u, s, t, p)[u_1, \ldots, u_k]| \leq \frac{c(1 + \|u\|_{C^1(\mathbb{R})}^c) |s_1 - s_2|^\beta}{(1 + t^2)^{(N+\delta)/2}} \prod_{i=1}^k \|u_i\|_{C^1(\mathbb{R})},
\]
completing the proof. 

The following two lemmas provide the desired estimates for the formal candidates to be the derivatives of $H$.

**Lemma 4.2.** Let $N \geq 3$, $\delta > 0$, $u \in \mathcal{O}_\delta$ and $\varphi, u_1, \ldots, u_k \in C^{1,\beta}(\mathbb{R})$, $\psi \in C^{0,\beta}(\mathbb{R})$ and $k \in \mathbb{N}$. Define the functions $\mathcal{F}, \tilde{\mathcal{F}} : \mathbb{R} \to \mathbb{R}$ by
\[
\mathcal{F}(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} t \Lambda(\varphi, s, t, p_{\sigma}) D^k_u K_\alpha(u, s, t, p_{\sigma})[u_1, \ldots, u_k] \psi(s - p_{\sigma} t) dt d\sigma 
\]
and
\[
\tilde{\mathcal{F}}(s) = \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} D^k_u K_\alpha(u, s, t, p_{\sigma})[u_1, \ldots, u_k] \psi(s - p_{\sigma} t) dt d\sigma.
\]
Then $\mathcal{F} \in C^{0,\beta-\alpha}(\mathbb{R})$ and $\tilde{\mathcal{F}} \in C^{0,\beta}(\mathbb{R})$. Moreover, there exists a constant $c = c(N, \alpha, \beta, k, \delta) > 1$ such that
\[
\|\mathcal{F}\|_{C^{0,\beta-\alpha}(\mathbb{R})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})}^c) \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})} \quad (4.19)
\]
and
\[
\|\tilde{\mathcal{F}}\|_{C^{0,\beta}(\mathbb{R})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})}^c) \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}. \quad (4.20)
\]
**Proof.** Throughout this proof, the letter $c$ stands for different constants greater than one and depending only on $N, \alpha, \beta, k$ and $\delta$. We define
\[
F(s, t, p) := t \Lambda(\varphi, s, t, p) D^k_u K_\alpha(u, s, t, p)[u_1, \ldots, u_k] \psi(s - pt).
\]
We now use (4.10), the estimates (4.2), (4.3), (4.11), (4.12) and the fact \( \psi \in C^{0,\beta}(\mathbb{R}) \). We also assume that \(|s_1 - s_2| \leq 1 \leq 2\), which leads (since also \(|p| \leq 2\)) to \(|p|^\beta |s_1 - s_2|^{\beta} \leq 2^{\beta} \min\{|p|^\beta, |s_1 - s_2|^{\beta}\}\). We deduce that

\[
\left| [F(\cdot, t, p); s_1, s_2] \right| \leq C(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \times 
\left( \frac{\min\{|t|^\beta |p|^\beta, |s_1 - s_2|^{\beta}\}}{(1 + t^2)^{(N+\alpha)/2}} + \frac{\min\{|t|^\beta+1 |p|^\beta, |s_1 - s_2|^{\beta}\}}{(1 + t^2)^{(N+\alpha)/2}} \right) \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^{k} \|u_i\|_{C^{1,\beta}(\mathbb{R})}
\]

and consequently, since \(N - 1 + \alpha - \beta > 1\),

\[
\left| [F; s_1, s_2] \right| \leq \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} \left| [F(\cdot, t, p_\sigma); s_1, s_2] \right| dt d\sigma
\]

\[
\leq C(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^{k} \|u_i\|_{C^{1,\beta}(\mathbb{R})} \times 
\int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \min\{|p_\sigma|^\beta, |s_1 - s_2|^{\beta}\} d\sigma.
\]

We then have

\[
\int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \min\{|p_\sigma|^\beta, |s_1 - s_2|^{\beta}\} d\sigma
\]

\[
= |S^{N-3}| \int_{-1}^{1} \frac{(1 - \sigma^2)^{N-4}}{(2 - 2 \sigma)^{N+\alpha-2}} \min\{(2 - 2 \sigma_1)^{\beta/2}, |s_1 - s_2|^{\beta}\} d\sigma_1
\]

\[
\leq C \int_{-1}^{1} (2 - 2 \sigma_1)^{-\alpha/2-1} \min\{(2 - 2 \sigma_1)^{\beta/2}, |s_1 - s_2|^{\beta}\} d\sigma_1
\]

\[
\leq C \int_{0}^{4} \tau^{-\alpha/2-1} \min\{\tau^{\beta/2}, |s_1 - s_2|^{\beta}\} d\tau
\]

\[
\leq C \left( \int_{0}^{\sqrt{|s_1 - s_2|^2}} \tau^{\beta-\alpha} d\tau + \int_{\sqrt{|s_1 - s_2|^2}}^{4} \tau^{-\alpha/2-1} d\tau \right) \leq C |s_1 - s_2|^{-\beta-\alpha},
\]

so that

\[
\left| [F; s_1, s_2] \right| \leq C(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \|\psi\|_{C^{0,\beta}(\mathbb{R})} |s_1 - s_2|^{-\beta-\alpha} \prod_{i=1}^{k} \|u_i\|_{C^{1,\beta}(\mathbb{R})},
\]

Similarly but more easily, we also obtain the estimate

\[
\|F\|_{L^\infty(\mathbb{R})} \leq C(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^{k} \|u_i\|_{C^{1,\beta}(\mathbb{R})},
\]

and thus (4.19) follows.
To prove (4.20), we now set
\[
\tilde{F}(s, t, p) := D^k_u K_\alpha(u, s - pt, s, t)[u_1, \ldots, u_k] \psi(s - pt),
\]
and we get
\[
|[[\tilde{F}(\cdot, t, p); s_1, s_2]| \leq c (1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \frac{|s_1 - s_2|^\beta}{(1 + t^2)^{(N+\alpha)/2}} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}
\]
and
\[
\|[\tilde{F}(\cdot, t, p)]_{L^\infty(\mathbb{R})} \leq c \frac{(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c}{(1 + t^2)^{(N+\alpha)/2}} \|\psi\|_{C^{0,\beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}.
\]
By (4.1), we thus have (4.20).

With the aid of this lemma we can now prove the following result.

**Lemma 4.3.** Let \(N \geq 3\), \(k \in \mathbb{N} \cup \{0\}\), \(\delta > 0\), \(u \in \mathcal{O}_\delta\), and \(u_1, \ldots, u_k \in C^{1,\beta}(\mathbb{R})\).
Moreover, let \(\mathcal{M}, \tilde{\mathcal{M}} : \mathbb{R} \rightarrow \mathbb{R}\) be defined by
\[
\mathcal{M}(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^k_u M(u, s, t, p_\sigma)[u_1, \ldots, u_k] \ dt \ d\sigma,
\]
\[
\tilde{\mathcal{M}}(s) = \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} D^k_u \tilde{M}(u, s, t, p_\sigma)[u_1, \ldots, u_k] \ dt \ d\sigma,
\]
where \(M, \tilde{\mathcal{M}} : \mathcal{O}_\delta \times \mathbb{R}^3 \rightarrow \mathbb{R}\) are given by
\[
M(u, s, t, p) = t \Lambda(u, s, t, p) K_\alpha(u, s, t, p) u^{N-2}(s - pt) \quad \text{and} \quad \tilde{\mathcal{M}}(u, s, t, p) = u(s) K_\alpha(u, s, t, p) u^{N-2}(s - pt).
\]
Then \(\mathcal{M} \in C^{0,\beta-\alpha}(\mathbb{R})\) and \(\tilde{\mathcal{M}} \in C^{0,\beta}(\mathbb{R})\). Moreover, there exists a constant \(c = c(N, \alpha, \beta, k, \delta) > 1\) such that
\[
\|\mathcal{M}\|_{C^{0,\beta-\alpha}(\mathbb{R})} \leq c (1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}
\]
and
\[
\|\tilde{\mathcal{M}}\|_{C^{0,\beta}(\mathbb{R})} \leq c (1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}.
\]

**Proof.** We define \(T : \mathcal{O}_\delta \times \mathbb{R}^3 \rightarrow \mathbb{R}\) by
\[
T(u, s, t, p) = t \Lambda(u, s, t, p) K_\alpha(u, s, t, p),
\]
so that
\[
M(u, s, t, p) = T(u, s, t, p) u^{N-2}(s - pt).
\]
By (4.7), we thus have
\[ D^k u M(u, s, t, p)[u_1, \ldots, u_k] = \sum_{N \in S_k} \psi_N(s - pt) D_u^{[N]} T(u, s, t, p)[u_i]_{i \in N} \]
where \( \psi_N := u^{N-2} \) in case \( k = |N| \) and
\[ \Psi_N := \prod_{\ell=0}^{k-|N|-1} (N - 2 - \ell) u^{N-2-(k-|N|)} \prod_{i \in N^c} u_i \quad \text{in case } k > |N| \]
(noting that \( |N^c| = k - |N| \)). By (4.8) we have, if \( |N| \geq 1 \),
\[ D_u^{[N]} T(u, s, t, p)[u_i]_{i \in N} = t \Lambda(u, s, t, p) D_u^{[N]} K_\alpha(u, s, t, p)[u_i]_{i \in N} \]
\[ + \sum_{j \in N^c} t \Lambda(u_j, s, t, p) D_u^{[N]-1} K_\alpha(u, s, t, p)[u_i]_{i \in N, i \neq j} . \]
Consequently,
\[ D^k u M(u, s, t, p)[u_1, \ldots, u_k] = \sum_{N \in S_k} M_N(s, t, p) \]
with
\[ M_N(s, t, p) = t \Lambda(u, s, t, p) \left( \Lambda(u, s, t, p) D_u^{[N]} K_\alpha(u, s, t, p)[u_i]_{i \in N} \right. \]
\[ \left. + \sum_{j \in N^c} \Lambda(u_j, s, t, p) D_u^{[N]-1} K_\alpha(u, s, t, p)[u_i]_{i \in N, i \neq j} \right) . \]
Clearly we also have that
\[ \| \psi_N \|_{C^{0,\beta}(\mathbb{R})} \leq c(1 + \| u \|_{C^{1,\beta}(\mathbb{R})}) \prod_{i \in N^c} \| u_i \|_{C^{1,\beta}(\mathbb{R})} . \]
By Lemma 4.2 it thus follows that \( M \in C^{0,\beta-\alpha}(\mathbb{R}) \) and
\[ \| M \|_{C^{0,\beta-\alpha}(\mathbb{R})} \leq c(1 + \| u \|_{C^{1,\beta}(\mathbb{R})}) \prod_{N \in S_k} \| \psi_N \|_{C^{0,\beta}(\mathbb{R})} \prod_{i \in N} \| u_i \|_{C^{1,\beta}(\mathbb{R})} \]
\[ \leq c(1 + \| u \|_{C^{1,\beta}(\mathbb{R})}) k \prod_{i=1}^k \| u_i \|_{C^{1,\beta}(\mathbb{R})} , \]
as claimed. The proof of (4.21) is similar but easier. \( \square \)

We are now in position to prove that \( H : O_\delta \to C^{0,\beta-\alpha}(\mathbb{R}) \) given by (4.5) is smooth.

**Proposition 4.4.** For \( N \geq 3 \), the map \( H : O_\delta \subset C^{1,\beta}(\mathbb{R}) \to C^{0,\beta-\alpha}(\mathbb{R}) \) defined by (4.5) is of class \( C^\infty \), and for every \( k \in \mathbb{N} \) we have
\[ D^k H(u) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^k u M(u, \cdot, t, p_\sigma) dt d\sigma \]
\[ - \frac{1}{2} \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} \bar{D}^k u \bar{M}(u, \cdot, t, p_\sigma) dt d\sigma , \]
where $M$ and $\tilde{M}$ are defined in Lemma 4.3.

Proof. We can write $\mathcal{H} = \mathcal{H}_1 - \mathcal{H}_2$ with

$$\mathcal{H}_1(u)(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} M(u, s, t, p_{\sigma}) dtd\sigma$$

and

$$\mathcal{H}_2(u)(s) = \frac{1}{2} \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} \tilde{M}(u, s, t, p_{\sigma}) dtd\sigma.$$ 

We only prove that, for $k \in \mathbb{N} \cup \{0\}$,

$$D^k \mathcal{H}_1(u)(u_1, \ldots, u_k)(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^k_u M(u, \cdot, t, p_{\sigma})[u_1, \ldots, u_k] dtd\sigma.$$ 

The corresponding statement for $\mathcal{H}_2$ is similar but simpler to prove. Moreover, the continuity of $D^k \mathcal{H}$ is a well known consequence of the existence of $D^{k+1} \mathcal{H}$ in the Fréchet sense.

To prove (4.22), we proceed by induction. For $k = 0$, the statement is true by definition. Let us now assume that the statement holds true for some $k \geq 0$. Then $D^k \mathcal{H}_1(u)$ is given by

$$D^k \mathcal{H}_1(u)[u_1, \ldots, u_k](s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^k_u M(u, s, t, p_{\sigma})[u_1, \ldots, u_k] dtd\sigma.$$ 

We fix $u_1, \ldots, u_k \in C^{1,\beta}(\mathbb{R})$. For $u \in \mathcal{O}_\delta$ and $v \in C^{1,\beta}(\mathbb{R})$, we define

$$\Gamma(u, v, s) := \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^{k+1}_u M(u, s, t, p_{\sigma})[u_1, \ldots, u_k, v] dtd\sigma.$$ 

Let $u \in \mathcal{O}_\delta$ and $v \in C^{1,\beta}(\mathbb{R})$ with $\|v\|_{C^{1,\beta}(\mathbb{R})} < \delta/2$. We have

$$D^k \mathcal{H}_1(u + v)[u_1, \ldots, u_k](s) - D^k \mathcal{H}_1(u)[u_1, \ldots, u_k](s) - \Gamma(u, v, s)$$

$$= \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \times$$

$$\int_{\mathbb{R}} \int_0^1 \left\{ D^{k+1}_u M(u + \rho v, s, t, p_{\sigma}) - D^{k+1}_u M(u, s, t, p_{\sigma}) \right\}[u_1, \ldots, u_k, v] d\rho dtd\sigma$$

$$= \int_0^1 \rho \int_0^1 \mathcal{H}^{\rho, \tau}_1(s) d\tau d\rho,$$

with

$$\mathcal{H}^{\rho, \tau}_1(s) := \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D^{k+2}_u M(u + \tau \rho v, s, t, p_{\sigma})[u_1, \ldots, u_k, v, v] dtd\sigma.$$
Proposition 4.4 gives the formula

\[ \| H_1^{\rho,\tau} \|_{C^{0,\beta-\alpha}(\mathbb{R})} \leq c(1 + \| u + \tau \rho v \|_{C^{1,\beta}(\mathbb{R})}) \| v \|_{C^{1,\beta}(\mathbb{R})}^2 \prod_{i=1}^{k} \| u_i \|_{C^{1,\beta}(\mathbb{R})} \]

\[ \leq c(1 + \| u \|_{C^{1,\beta}(\mathbb{R})} + \| v \|_{C^{1,\beta}(\mathbb{R})}) \| v \|_{C^{1,\beta}(\mathbb{R})}^2 \prod_{i=1}^{k} \| u_i \|_{C^{1,\beta}(\mathbb{R})}, \]

with a constant \( c > 1 \) independent of \( \rho, \tau, u, u_1, \ldots, u_k \) and \( v \). Consequently,

\[ \| D^k H_1(u + v)[u_1, \ldots, u_k] - D^k H_1(u)[u_1, \ldots, u_k] - \Gamma(u, v, \cdot) \|_{C^{0,\beta-\alpha}(\mathbb{R})} \]

\[ \leq c(1 + \| u \|_{C^{1,\beta}(\mathbb{R})} + \| v \|_{C^{1,\beta}(\mathbb{R})}) \| v \|_{C^{1,\beta}(\mathbb{R})}^2 \prod_{i=1}^{k} \| u_i \|_{C^{1,\beta}(\mathbb{R})}. \]

This shows that \( D^{k+1} H_1(u) \) exists in the Frechét sense, and that

\[ D^{k+1} H_1(u)[u_1, \ldots, u_k, v] = \Gamma(u, v, \cdot) \in C^{0,\beta-\alpha}(\mathbb{R}). \]

We conclude that (4.22) holds for \( k + 1 \) in place of \( k \), and thus the proof is finished. \( \square \)

We finally establish the promised expression for the differential of \( H \) at constant functions. By this we complete the proof of Proposition 2.3.

**Proposition 4.5.** Let \( N \geq 3 \). If \( u \equiv \kappa \in \mathcal{O}_\delta \) is a constant function, we have

\[ -\frac{\alpha}{2} DH(\kappa)v(s) = D\mathcal{H}(\kappa)v(s) = -\frac{\alpha}{2} \kappa^{-\alpha} \left(PV \int_{\mathbb{R}} (v(s) - v(s - \kappa \tau)) G_\alpha(\tau) d\tau - b_\alpha v(s) \right) \]

with

\[ G_\alpha : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad G_\alpha(\tau) = \int_{S^{N-2}} \frac{2}{(\tau^2 + 2(1 - \sigma_1))^{\frac{N-1}{2}}} d\sigma \]

and

\[ b_\alpha = 2 \int_{\mathbb{R}} \int_{S^{N-2}} (1 - \sigma_1)^{\frac{N-1}{2}} d\sigma d\tau. \]

**Proof.** Proposition 4.4 gives the formula

\[ D\mathcal{H}(u)v(s) = \int_{S^{N-2}} \mu_{\alpha-2}(\sigma) \int_{\mathbb{R}} D_u M(u, s, t, p) v \, dt \, d\sigma \]

\[ -\frac{1}{2} \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} D_u \widetilde{M}(u, s, t, p) v \, dt \, d\sigma. \] (4.23)

In the case where \( u \equiv \kappa \in \mathcal{O}_\delta \) is a constant function, we have \( \Lambda(\kappa, s, t, p) = 0 \) and thus

\[ D_u M(\kappa, s, t, p)v = t \Lambda(v, s, t, p) \mathcal{K}_\alpha(\kappa, s, t, p) \kappa^{N-2} \]

\[ = t \left( \frac{v(s) - v(s - pt)}{pt} - v'(s - pt) \right) \mathcal{K}_\alpha(\kappa, s, t, p) \kappa^{N-2} \]

\[ = t \left( \frac{v(s) - v(s - pt)}{pt} - v'(s - pt) \right) \frac{\kappa^{N-2}}{(t^2 + \kappa^2)^{\frac{N-1}{2}}}. \]
Therefore, by substituting \( \tau = \frac{p_s}{\kappa} t \),

\[
\int_{S^{n-2}} \mu_{\alpha-2}(\sigma) \int_R D_u M(u, s, t, p_\sigma) v \, dt \, d\sigma = \\
\kappa^{-\alpha} \int_{S^{n-2}} \left( \frac{v(s) - v(s - \kappa \tau)}{\kappa \tau} - v'(s - \kappa \tau) \right) \frac{1}{(\tau^2 + \kappa^2)^{\frac{n+\alpha}{2}}} \, dt \, d\sigma,
\]

whereas, by integration by parts,

\[
\int_{|\tau| \geq \varepsilon} \kappa \tau v'(s - \kappa \tau) \, d\tau = - \int_{|\tau| \geq \varepsilon} \partial_\tau v(s - \kappa \tau) \frac{\tau}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha}{2}}} \, d\tau \tag{4.24}
\]

\[
= \varepsilon (v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon)) + \int_{|\tau| \geq \varepsilon} v(s - \kappa \tau) \partial_\tau \left( \frac{\tau}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha}{2}}} \right) \, d\tau 
\]

\[
= \varepsilon (v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon)) + \int_{|\tau| \geq \varepsilon} v(s - \kappa \tau) \left( \frac{1}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha}{2}}} - \frac{(N + \alpha) \tau^2}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha+2}{2}}} \right) \, d\tau 
\]

\[
= \varepsilon (v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon)) + \int_{|\tau| \geq \varepsilon} v(s - \kappa \tau) \left( \frac{(N + \alpha) p_\sigma^2}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha+2}{2}}} - \frac{N + \alpha - 1}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha}{2}}} \right) \, d\tau. 
\]

Hence, by Fubini's theorem,

\[
\int_{S^{n-2}} \mu_{\alpha-2}(\sigma) \int_R D_u M(\kappa, s, t, p_\sigma) v \, dt \, d\sigma = \\
\kappa^{-\alpha} \lim_{\varepsilon \to 0} \left( v(s) \int_{|\tau| \geq \varepsilon} \tilde{G}_\alpha(\tau) \, d\tau - \varepsilon \{ v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon) \} \tilde{G}_\alpha(\varepsilon) \right. \\
\left. + v(s - \kappa \tau) \left( (N + \alpha - 2) \int_{|\tau| \geq \varepsilon} \tilde{G}_\alpha(\tau) \, d\tau - (N + \alpha) \int_{|\tau| \geq \varepsilon} G_{1,\alpha}(\tau) \, d\tau \right) \right). 
\]

Here and in the following, we put

\[
\tilde{G}_\alpha(\tau) = \frac{G_\alpha(\tau)}{2} = \int_{S^{n-2}} \frac{1}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha}{2}}} \, d\sigma, \quad G_{\alpha,0}(\tau) := \int_{S^{n-2}} \frac{\tau^2}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha+2}{2}}} \, d\sigma \\
G_{1,\alpha}(\tau) = \int_{S^{n-2}} \frac{p_\sigma^2}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha+1}{2}}} \, d\sigma, \quad G_{\alpha,2}(\tau) := \int_{S^{n-2}} \frac{p_\sigma^4}{(\tau^2 + p_\sigma^2)^{\frac{n+\alpha+2}{2}}} \, d\sigma.
\]
We also have
\[ D_u\tilde{M}(\kappa, s, t, p)v = v(s)K_\alpha(\kappa, s, t, p)\kappa^{N-2} + \kappa K_\alpha(\kappa, s, t, p)(N-2)\kappa^{N-3}v(s - pt) + \kappa^{N-1}D_uK_\alpha(\kappa, s, t, p)v \]
\[ = \kappa^{N-2}((v(s) + (N-2)v(s - pt))K_\alpha(\kappa, s, t, p) + \kappa D_uK_\alpha(\kappa, s, t, p)v) \]
\[ = \kappa^{N-2}\left(\frac{v(s) + (N-2)v(s - pt)}{(t^2 + \kappa^2)^{\frac{N+\alpha}{2}}} - \frac{N + \alpha}{2} \frac{\kappa^2(v(s) + v(s - pt))}{(t^2 + \kappa^2)^{\frac{N+\alpha+2}{2}}}\right) \]
\[ = \kappa^{-\alpha}p^{N+\alpha}\left(\frac{v(s) + (N-2)v(s - pt)}{(\frac{2}{\kappa^2}t^2 + p^2)^{\frac{N+\alpha}{2}}} - \frac{N + \alpha}{2} \frac{p^2(v(s) + v(s - pt))}{(\frac{2}{\kappa^2}t^2 + p^2)^{\frac{N+\alpha+2}{2}}}\right), \]
so that we have, again by substituting \( \tau = \frac{\kappa}{\kappa}t, \)
\[ \int_{S^{N-2}} \mu_{\alpha-3}(\sigma) \int_{\mathbb{R}} D_u\tilde{M}(\kappa, s, t, p_a)v \, dt \, d\sigma = \kappa^{-1-\alpha} \times \]
\[ \int_{S^{N-2}} \int_{\mathbb{R}} \left(\frac{p_a^2(v(s) + (N-2)v(s - \kappa\tau))}{(\tau^2 + p_a^2)^{\frac{N+\alpha}{2}}} - \frac{N + \alpha}{2} \frac{p_a^2(v(s) + v(s - \kappa\tau))}{(\tau^2 + p_a^2)^{\frac{N+\alpha+2}{2}}}\right) \, d\tau \, d\sigma \]
\[ = \kappa^{-1-\alpha}\lim_{\varepsilon \to 0} \left(\int_{|\tau| \geq \varepsilon} \left\{v(s) + (N-2)v(s - \kappa\tau)\right\} G_{\alpha,0}(\tau) \, d\tau \right. \]
\[ \left. - \frac{N + \alpha}{2} \int_{|\tau| \geq \varepsilon} \left\{v(s) + v(s - \kappa\tau)\right\} G_{\alpha,2}(\tau) \, d\tau \right), \]
where in the last step we used Lebesgue's and Fubini's theorems. Collecting and reordering everything, and recalling (4.23), we thus get
\[ \kappa^{1+\alpha}D\mathcal{H}(\kappa)v(s) \]
\[ = \lim_{\varepsilon \to 0} \left[ \alpha \int_{|\tau| \geq \varepsilon} (v(s - \kappa\tau) - v(s))\tilde{G}_\alpha(\tau) \, d\tau + \frac{v(s)}{2} \int_{|\tau| \geq \varepsilon} G_{\alpha,0}(\tau) \, d\tau \right. \]
\[ + v(s) \int_{|\tau| \geq \varepsilon} \left(\frac{N + \alpha}{4} G_{\alpha,2}(\tau) - \frac{\alpha + 1}{2} G_{\alpha,0}(\tau) + (\alpha + 1)\tilde{G}_\alpha(\tau)\right) \, d\tau \]
\[ - \varepsilon \{v(s - \kappa\varepsilon) + v(s + \kappa\varepsilon)\} \tilde{G}_\alpha(\varepsilon) \]
\[ + \int_{|\tau| \geq \varepsilon} v(s - \kappa\tau)\left((N-2)\tilde{G}_\alpha(\tau) - \frac{N-2}{2} G_{\alpha,0}(\tau) \right. \]
\[ \left. - (N + \alpha)\{G_{1,\alpha}(\tau) - \frac{G_{\alpha,2}(\tau)}{4}\}\right) \, d\tau \right]. \]
We now claim that for every \( \tau \in \mathbb{R} \setminus \{0\} \) we have
\[ (N - 2)\tilde{G}_\alpha(\tau) - \frac{N-2}{2} G_{\alpha,0}(\tau) - (N + \alpha)\{G_{1,\alpha}(\tau) - \frac{G_{\alpha,2}(\tau)}{4}\} = 0. \quad (4.25) \]
Indeed,

\[
(N + \alpha) \{ G_{1,\alpha}(\tau) - G_{\alpha,2}(\tau) \} = (N + \alpha) \int_{S^{N-2}} \frac{p^2_{\sigma} - p^2_{\sigma}}{(\tau^2 + p^2_{\sigma})^{\frac{N+1}{2}}} d\sigma
\]

\[
= |S^{N-3}| (N + \alpha) \int_{-1}^{1} \frac{(1 - \sigma^2_{1})^{\frac{N-4}{2}} (2 - 2\sigma_1)(\frac{1}{2} + \frac{\sigma_1}{2})}{(\tau^2 + (2 - 2\sigma_1))^{\frac{N+1}{2}}} d\sigma_1
\]

\[
= |S^{N-3}| (N + \alpha) \int_{-1}^{1} \frac{(1 - \sigma^2_{1})^{\frac{N-4}{2}}}{(\tau^2 + (2 - 2\sigma_1))^{\frac{N+1}{2}}} d\sigma_1
\]

\[
= |S^{N-3}| \int_{-1}^{1} (1 - \sigma^2_{1})^{\frac{N-2}{2}} \partial_{\sigma_1} \frac{1}{(\tau^2 + (2 - 2\sigma_1))^{\frac{N+1}{2}}} d\sigma_1
\]

\[
= - |S^{N-3}| \int_{-1}^{1} (1 - \sigma^2_{1})^{\frac{N-2}{2}} \partial_{\sigma_1} d\sigma_1 = (N - 2) |S^{N-3}| \int_{-1}^{1} \frac{\sigma_1(1 - \sigma^2_{1})^{\frac{N-4}{2}}}{(\tau^2 + (2 - 2\sigma_1))^{\frac{N+1}{2}}} d\sigma_1
\]

\[
= - \frac{N - 2}{2} \int_{S^{N-2}} \frac{p^2_{\sigma} - 2}{(\tau^2 + p^2_{\sigma})^{\frac{N+1}{2}}} d\sigma = - \frac{N - 2}{2} G_{\alpha,0}(\tau) + (N - 2) \tilde{G}_{\alpha}(\tau).
\]

Hence we conclude that

\[
\kappa^{1+\alpha} D\mathcal{H}(\kappa) v(s) + \alpha \left( PV \int_{\mathbb{R}} (v(s) - v(s - \kappa \tau)) \tilde{G}_{\alpha}(\tau) d\tau - \frac{v(s)}{2} \int_{|\tau| \geq \varepsilon} G_{\alpha,0}(\tau) d\tau \right)
\]

\[
= \lim_{\varepsilon \to 0} R_{\varepsilon}(s),
\]

where

\[
R_{\varepsilon}(s) := v(s) \int_{|\tau| \geq \varepsilon} \Bigg( \frac{N + \alpha}{4} G_{\alpha,2}(\tau) - \frac{\alpha + 1}{2} G_{\alpha,0}(\tau) + (\alpha + 1) \tilde{G}_{\alpha}(\tau) \Bigg) d\tau
\]

\[
- \varepsilon \{ v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon) \} \tilde{G}_{\alpha}(\varepsilon).
\]

The proof of the proposition is finished once we have shown that \( \lim_{\varepsilon \to 0} R_{\varepsilon}(s) = 0. \) To see this, we note that by choosing \( v \equiv 1 \) in \( (4.24) \) we have the identity

\[
\int_{|\tau| \geq \varepsilon} \frac{N - 1}{2} \tau d\tau = \int_{|\tau| \geq \varepsilon} \frac{(N + \alpha)p^2_{\sigma}}{(\tau^2 + p^2_{\sigma})^{\frac{N+1}{2}}} d\tau + \frac{2\varepsilon}{(\varepsilon^2 + p^2_{\sigma})^{\frac{N+1}{2}}}
\]

for \( \sigma \in S^{N-2}. \) Integrating this identity over \( S^{N-2} \) yields

\[
(N + \alpha - 1) \int_{|\tau| \geq \varepsilon} \tilde{G}_{\alpha}(\tau) d\tau = (N + \alpha) \int_{|\tau| \geq \varepsilon} G_{1,\alpha}(\tau) d\tau + 2\varepsilon \tilde{G}_{\alpha}(\varepsilon).
\]

On the other hand, multiplying \( (4.26) \) with \( p^2_{\sigma} \) and integrating over \( S^{N-2} \) yields

\[
(N + \alpha - 1) \int_{|\tau| \geq \varepsilon} G_{\alpha,0}(\tau) d\tau = (N + \alpha) \int_{|\tau| \geq \varepsilon} G_{\alpha,2}(\tau) d\tau + 2\varepsilon G_{\alpha,0}(\varepsilon).
\]
Inserting the two previous identities successively gives

\[ R_\varepsilon(s) = v(s) \int_{|\tau| \geq \varepsilon} \left( \frac{N + \alpha}{4} G_{\alpha,2}(\tau) - \frac{\alpha + 1}{2} G_{\alpha,0}(\tau) + (\alpha + 1) \tilde{G}_\alpha(\tau) \right) d\tau \]

\[ - \varepsilon \{ v(s - \kappa \varepsilon) + v(s + \kappa \varepsilon) \} \tilde{G}_\alpha(\varepsilon) \]

\[ = v(s) \int_{|\tau| \geq \varepsilon} \left( (N + \alpha) \frac{G_{\alpha,2}(\tau)}{4} + (N + \alpha) G_{1,\alpha}(\tau) - \frac{\alpha + 1}{2} G_{\alpha,0}(\tau) - (N - 2) \tilde{G}_\alpha(\tau) \right) d\tau \]

\[ + \varepsilon \{ 2v(s) - v(s - \kappa \varepsilon) - v(s + \kappa \varepsilon) \} \tilde{G}_\alpha(\varepsilon) \]

\[ = v(s) \int_{|\tau| \geq \varepsilon} \left( (N + \alpha)(G_{1,\alpha}(\tau) - \frac{G_{\alpha,2}(\tau)}{4}) + \frac{N - 2}{2} G_{\alpha,0}(\tau) - (N - 2) \tilde{G}_\alpha(\tau) \right) d\tau \]

\[ + \varepsilon \{ 2v(s) - v(s - \kappa \varepsilon) - v(s + \kappa \varepsilon) \} \tilde{G}_\alpha(\varepsilon) - \varepsilon G_{\alpha,0}(\varepsilon) \]

\[ = \varepsilon \{ 2v(s) - v(s - \kappa \varepsilon) - v(s + \kappa \varepsilon) \} \tilde{G}_\alpha(\varepsilon) - \varepsilon G_{\alpha,0}(\varepsilon), \]

where we have used (4.25) again in the last step. Since

\[ \tilde{G}_\alpha(\varepsilon) = O(\varepsilon^{-2-\alpha}), \quad G_{\alpha,0}(\varepsilon) = O(\varepsilon^{-\alpha}) \]

and

\[ 2v(s) - v(s - \kappa \varepsilon) - v(s + \kappa \varepsilon) = O(\varepsilon^{1+\beta}) \]

as \( \varepsilon \to 0 \) since \( v \in C^{1,\beta}(\mathbb{R}) \), we conclude that

\[ \lim_{\varepsilon \to 0} R_\varepsilon(s) = 0, \]

as desired. \( \square \)

5. Smooth branch of periodic bands with constant nonlocal mean curvature

The aim of this section is to derive the regularity of the nonlocal mean curvature operator \( H \) when \( N = 2 \), thereby deducing the smoothness of the branch of CNMC bands bifurcating from the straight one. We proved that this branch is continuous in [4], but there we did not prove its smoothness. For this, we follow the approach of Section 4.
In case \( N = 2 \), from Lemma \( 5.1 \) (with \( S^{N-2} = S^0 = \{-1, 1\} \subset \mathbb{R} \)), we deduce that the nonlocal mean curvature \( H_{E_u} \) at the point \((s, u(s))\) is given by

\[
-\frac{\alpha}{2} H(u)(s) = \int_{\mathbb{R}} \frac{t \Lambda(u, s, t, 1)}{|t|^{2+\alpha} (1 + \Lambda_0(u, s, t, 1)^2)^{(2+\alpha)/2}} dt
+ \int_{\mathbb{R}} \frac{t \Lambda(u, s, t, 1)}{(t^2 + t^2 \Lambda_0(u, s, t, 1)^2 + 4u(s)u(s-t))^{(2+\alpha)/2}} dt
- 2u(s) \int_{\mathbb{R}} \frac{1}{|t|^{2+\alpha} (1 + \Lambda_0(u, s, t, 1)^2)^{(2+\alpha)/2}} dt.
\]

This is a quite different expression than the one we used in \( 4 \).

In Lemma \( 5.2 \) we will see that the integrals above converge absolutely in the Lebesgue sense. Changing \( \tau \) to \( t \) and using the notation from the beginning of Section \( 4 \) we have

\[
-\frac{\alpha}{2} H(u)(s) = \int_{\mathbb{R}} \frac{t \Lambda(u, s, t, 1)}{|t|^{2+\alpha} (1 + \Lambda_0(u, s, t, 1)^2)^{(2+\alpha)/2}} dt
+ \int_{\mathbb{R}} \frac{t \Lambda(u, s, t, 1)}{(t^2 + t^2 \Lambda_0(u, s, t, 1)^2 + 4u(s)u(s-t))^{(2+\alpha)/2}} dt
- 2u(s) \int_{\mathbb{R}} \frac{1}{|t|^{2+\alpha} (1 + \Lambda_0(u, s, t, 1)^2)^{(2+\alpha)/2}} dt.
\]

For \( \alpha > 0 \), we define the maps \( K_{\alpha,0} : C^{1,\beta}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( K_{\alpha,1} : \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
K_{\alpha,0}(u, s, t) := \frac{1}{(1 + \Lambda_0(u, s, t, 1)^2)^{(2+\alpha)/2}}
\]
and
\[
K_{\alpha,1}(u, s, t) := \frac{1}{(t^2 + t^2 \Lambda_0(u, s, t, 1)^2 + 4u(s)u(s-t))^{(2+\alpha)/2}}.
\]

Therefore, for every \( u \in \mathcal{O} \), we have

\[
-\frac{\alpha}{2} H(u)(s) = \int_{\mathbb{R}} \frac{t \Lambda(u, s, t, 1)}{|t|^{2+\alpha} K_{\alpha,0}(u, s, t) dt} + \int_{\mathbb{R}} t \Lambda(u, s, t, 1) K_{\alpha,1}(u, s, t) dt
- 2u(s) \int_{\mathbb{R}} K_{\alpha,1}(u, s, t) dt.
\]

As in Section \( 4 \) to prove the regularity of \( H \), it will be crucial to have estimates related to the maps \( K_{\alpha,0} \) and \( K_{\alpha,1} \).

**Lemma 5.1.** Let \( N = 2 \), \( k \in \mathbb{N} \cup \{0\} \), \( \delta, \alpha > 0 \) and \( \beta \in (0, 1) \).

(i) There exists a constant \( c = c(\alpha, \beta, k) > 1 \) such that for all \( (s, s_1, s_2, t) \in \mathbb{R}^4 \) and \( u \in C^{1,\beta}(\mathbb{R}) \), we have

\[
\|D_u^k K_{\alpha,0}(u, s, t)\| \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c,
\]
and

\[
\|D_u^k K_{\alpha,0}(u, s, t)\| \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c |s_1 - s_2|^\beta.
\]
(ii) There exists $c = c(\alpha, \beta, k, \delta) > 1$ such that for all $(s, s_1, s_2, t) \in \mathbb{R}^4$ and $u \in O_\delta$, we have

$$
\|D^k_u \mathcal{K}_{\alpha,1}(u, s, t)\| \leq \frac{c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c}{(1 + t^2)^{(2+\alpha)/2}},
$$

$$
\|[D^k_u \mathcal{K}_{\alpha,1}(u, \cdot, t); s_1, s_2]\| \leq \frac{c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c |s_1 - s_2|^\beta}{(1 + t^2)^{(2+\alpha)/2}}.
$$

Proof. The proof of (ii) is the same as the proof of Lemma 4.1.

The proof of (i) is very simple. Note simply that the function $y \mapsto (1 + y^2)^{-(2+\alpha)/2}$ is a bounded smooth function with bounded derivatives of all orders. At the same time, $\Lambda_0(w, s, t, 1)$ is a linear operator on $w$ satisfying $|\Lambda_0(w, s, t, 1)| \leq \|w\|_{C^{1,\beta}(\mathbb{R})}$ and $|\Lambda_0(w, s_1, t, 1) - \Lambda_0(w, s_2, t, 1)| \leq \|w\|_{C^{1,\beta}(\mathbb{R})}|s_1 - s_2|^\beta$. The claimed estimates follow easily from these two facts, applying the bounds for $\Lambda_0(w, \cdot, t, 1)$ at $w = u$ and/or $w = \varphi_i$ (when considering the $k$-th derivatives of $\mathcal{K}_{\alpha,0}$ at $u$ in directions $[\varphi_i]$).

The following two lemmas provide the desired estimates for the formal candidates to be the derivatives of $H$.

**Lemma 5.2.** Let $N = 2$, $\delta > 0$, $u \in O_\delta$ and $\varphi, u_1, \ldots, u_k \in C^{1,\beta}(\mathbb{R})$ and $k \in \mathbb{N} \cup \{0\}$. We define the functions $F_i : \mathbb{R} \to \mathbb{R}$ by

$$
F_0(s) = \int_\mathbb{R} \frac{t}{|t|^{2+\alpha}} \Lambda(\varphi, s, t, 1) D^k_u \mathcal{K}_{\alpha,0}(u, s, t)[u_1, \ldots, u_k]dt,
$$

$$
F_1(s) = \int_\mathbb{R} t \Lambda(\varphi, s, t, 1) D^k_u \mathcal{K}_{\alpha,1}(u, s, t)[u_1, \ldots, u_k]dt
$$

and

$$
F_2(s) = \int_\mathbb{R} D^k_u \mathcal{K}_{\alpha,1}(u, s, t)[u_1, \ldots, u_k]dt.
$$

Then $F_i \in C^{0,\beta-\alpha}(\mathbb{R})$, for $i = 0, 1$ and $F_2 \in C^{0,\beta}(\mathbb{R})$. Moreover, there exists a constant $c = c(\alpha, \beta, k, \delta) > 1$ such that

$$
\|F_i\|_{C^{0,\beta-\alpha}(\mathbb{R})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \|\varphi\|_{C^{1,\beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})},
$$

(5.3)

for $i = 0, 1$ and

$$
\|F_2\|_{C^{0,\beta}(\mathbb{R})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^c \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}.
$$

(5.4)

Proof. Throughout this proof, the letter $c$ stands for different constants greater than one and depending only on $\alpha, \beta, k$ and $\delta$. We define

$$
F_0(s, t) := \frac{t}{|t|^{2+\alpha}} \Lambda(\varphi, s, t, 1) D^k_u \mathcal{K}_{\alpha,0}(u, s, t)[u_1, \ldots, u_k]$$

and

$$
F_1(s, t) := \int_\mathbb{R} \frac{t}{|t|^{2+\alpha}} \Lambda(\varphi, s, t, 1) D^k_u \mathcal{K}_{\alpha,1}(u, s, t)[u_1, \ldots, u_k]dt
$$

respectively. The claimed estimates follow from the properties of $\mathcal{K}_{\alpha,0}$ and $\mathcal{K}_{\alpha,1}$, as well as the boundedness of $\Lambda$ and its derivatives.
and 
\[ F_i(s, t) := t \Lambda(\varphi, s, t, 1) D^k_u K_{\alpha, 1}(u, s, t)[u_1, \ldots, u_k]. \]

By (4.2) and Lemma 5.1 we have
\[ |F_i(s, t)| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c \frac{\min(|t|^{\beta}, 1)}{|t|^{1+\alpha}} \|\varphi\|_{C^{1, \beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R})}, \tag{5.5} \]
for \( i = 0, 1 \) and consequently,
\[ \|F_i\|_{L^\infty(\mathbb{R})} \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c \|\varphi\|_{C^{1, \beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R})}. \tag{5.6} \]

We now use (4.10), the estimates (4.2), (4.3) and Lemma 5.1 to get
\[ |[F_i(\cdot, t); s_1, s_2]| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c \times \]
\[ \left( \frac{\min(|t|^\beta, |s_1 - s_2|^\beta)}{|t|^{1+\alpha}} + \frac{\min(|t|^\beta, 1)|s_1 - s_2|^\beta}{|t|^{1+\alpha}} \right) \|\varphi\|_{C^{1, \beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R})}. \]

This leads to
\[ |[F_i; s_1, s_2]| \leq \int_{\mathbb{R}} |[F_i(\cdot, t); s_1, s_2]| dt \]
\[ \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c \|\varphi\|_{C^{1, \beta}(\mathbb{R})} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R})} \times \]
\[ \int_{\mathbb{R}} \left\{ \min(|t|^\beta, |s_1 - s_2|^\beta) + \min(|t|^\beta, 1)|s_1 - s_2|^\beta \right\} |t|^{-1-\alpha} dt. \tag{5.7} \]

Assuming \( |s_1 - s_2| \leq 1 \), we have
\[ \left\{ \int_{|t| \leq |s_1 - s_2|} + \int_{|t| \geq |s_1 - s_2|} \right\} \min(|t|^\beta, |s_1 - s_2|^\beta) + \min(|t|^\beta, 1)|s_1 - s_2|^\beta \right\} |t|^{-1-\alpha} dt \leq c \left( \int_{|t| \leq |s_1 - s_2|} |t|^{\beta-\alpha-1} dt + |s_1 - s_2|^\beta \int_{|t| \geq |s_1 - s_2|} |t|^{-1-\alpha} dt \right) \leq c |s_1 - s_2|^{\beta-\alpha}. \]

Using this in (5.7), we then conclude that, for \( i = 0, 1 \),
\[ |[F_i; s_1, s_2]| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R})})^c \|\varphi\|_{C^{1, \beta}(\mathbb{R})} |s_1 - s_2|^{\beta-\alpha} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R})}. \]

This together with (5.6) give (5.3).

To prove (5.4), we now set
\[ F_2(s, t) := D^k_u K_{\alpha, 1}(u, s, t)[u_1, \ldots, u_k], \]
and by Lemma 5.1 we have
\[ |F_2(\cdot, t; s_1, s_2)| \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^e \frac{|s_1 - s_2|^\beta}{(1 + t^2)(2+\alpha)/2} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})} \]
and
\[ \|F_2(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R})})^e \frac{1}{(1 + t^2)(2+\alpha)/2} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R})}. \]
We thus have (5.4), since \( F_2(\cdot) = \int_\mathbb{R} F_2(\cdot, t) dt \).

Next, we define
\[ M_0(u, s, t) = \frac{t}{|t|^{2+\alpha}} \Lambda(u, s, t, 1) \mathcal{K}_{\alpha,0}(u, s, t), \]
\[ M_1(u, s, t) = t\Lambda(u, s, t, 1) \mathcal{K}_{\alpha,1}(u, s, t) \]
and
\[ M_2(u, s, t) = -2u(s) \mathcal{K}_{\alpha,1}(u, s, t), \]
so that
\[ -\frac{\alpha}{2} H(u)(s) = \sum_{\ell=0}^2 \int_\mathbb{R} M_\ell(u, s, t) dt. \]
We also recall from (4.8) that, if \( k \geq 1 \),
\[ D_u^k M_0(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} = \frac{t}{|t|^{2+\alpha}} \Lambda(u, s, t, 1) D_u^k \mathcal{K}_{\alpha,0}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} \]
\[ + \sum_{j=1}^k \frac{t}{|t|^{2+\alpha}} \Lambda(u_j, s, t, 1) D_u^{k-1} \mathcal{K}_{\alpha,0}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}, i \neq j}, \]
\[ D_u^k M_1(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} = t\Lambda(u, s, t, 1) D_u^k \mathcal{K}_{\alpha,1}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} \]
\[ + \sum_{j=1}^k t\Lambda(u_j, s, t, 1) D_u^{k-1} \mathcal{K}_{\alpha,1}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}, i \neq j} \]
and
\[ D_u^k M_2(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} = -2u(s) D_u^k \mathcal{K}_{\alpha,1}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}} \]
\[ - 2 \sum_{j=1}^k u_j(s) D_u^{k-1} \mathcal{K}_{\alpha,1}(u, s, t)[u_i]_{i \in \{1,\ldots,k\}, i \neq j}. \]
With this and the estimates in Lemma 5.2 we can now follow step by step the arguments in Section 4 (noticing that the proof of Lemma 4.3 and Proposition 4.4 are
essentially algebraic) to deduce that $H : \mathcal{O} \subset C^{1,\beta}(\mathbb{R}) \to C^{\alpha,\beta}(\mathbb{R})$ is of class $C^\infty$. Moreover

$$-\frac{\alpha}{2} D^k H(u)[u_1, \ldots, u_k](s) = \sum_{\ell=0}^{2} \int_{\mathbb{R}} D^k M_{\ell}(u, s, t)[u_1, \ldots, u_k]dt.$$  

As remarked earlier, in our 2D paper [4], we applied the implicit function theorem to the $C^1$ map

$$\overline{\Phi} : \mathbb{R} \times \mathbb{R}_+ \times X \to Y, \quad \overline{\Phi}(a, \lambda, v) := \frac{1}{a} \left\{ H(\lambda R + a(\cos(\cdot) + v)) - H(\lambda R) \right\}$$

at the point $(0, 1, 0)$, where $R > 0$ was chosen in such anway that $\overline{\Phi}(0, 1, 0) = 0$ and that the linear maps $D_{\lambda} \overline{\Phi}(0, 1, 0) : \mathbb{R} \to \langle \cos(\cdot) \rangle$ and $D_{v} \overline{\Phi}(0, 1, 0) : X_\perp \to Y_\perp$ are invertible. Since, for every $s \in \mathbb{R},$

$$\overline{\Phi}(a, \lambda, v)(s) = \int_{0}^{1} DH(\lambda R + at(\cos(\cdot) + v))[\cos(\cdot) + v](s)dt,$$

it follows that $\overline{\Phi}$ is of class $C^\infty$ in a neighborhood of $(0, 1, 0)$, for every $R > 0$. Hence the curves $a \mapsto \lambda(a)$ and $a \mapsto v_a$ that we obtained in [4] Theorem 1.2 are smooth.

We recall that this branch in $\mathbb{R}^2$ could have been obtained also using the Crandall-Rabinowitz theorem as in the present paper.

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