A new sequence of topological terms at any spacetime dimensions

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Abstract

We investigate a sequence of quadratic topological terms of the Chern-Simons type in different spacetime dimensions, related by dimensional compactification and sharing the properties of topological mass generation and statistical transmutation. The implications for bosonization in several dimensions are also analyzed.

1. The Chern-Simons term is usually considered at spacetime dimension $D = 2 + 1$ and is given by (up to a mass dimension constant) \[ S_{CS} = \int d^3x \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho \] (1)

We are going to refer to the Abelian case only. Considering that the 0+1 dimensional case is just given by $\int dt A$, we may generalize the form of Chern-Simons term for any dimensions as

\[ S_{CS1} = \int dx A \]

\[ S_{CS2} = \int d^2x \epsilon^{\mu\nu} \partial_\mu A_\nu \]

\[ S_{CS3} = \int d^3x \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho \]

\[ S_{CS4} = \int d^4x \epsilon^{\mu\nu\rho\lambda} \partial_\mu A_\nu \partial_\rho A_\lambda \]

\[ S_{CS5} = \int d^5x \epsilon^{\mu\nu\rho\lambda\eta} \partial_\mu A_\nu \partial_\rho A_\lambda A_\eta \]

\[ \vdots \] (2)

The case of D=3, given by (1), however, is special for many reasons. Firstly, it is the only one that is quadratic in $A_\mu$ (note that in D=4 and actually in any even dimension it is a total derivative and, therefore, classically trivial). In this sense, D=3 is the only case where it can be used alone as a “kinetic” term for $A_\mu$. Secondly, it is well-known that in this case it produces a change in the statistics of the particles that couple to $A_\mu$. Another feature of $S_{CS3}$ is that, if we consider it together with a Maxwell term, it generates a mass for $A_\mu$, the so-called topological mass generation.

In view of the special characteristics of $S_{CS3}$ and the fact that the number of gauge fields is not the same for all $D$ we also conclude that these terms cannot be linked each other by any kind of dimensional compactification. It would be very interesting, on the other hand, to be able to generate a whole sequence of topological terms by dimensional reduction. For this purpose, we consider the possibility of constructing topological terms involving gauge fields of different ranks. These terms are not necessarily trivial for even spacetime dimensions. In fact, for $D = 4$, Cremer and J. Scherk \[ S_4 = \int d^4x \epsilon^{\mu\nu\rho\lambda} \partial_\mu A_\nu B_{\rho\lambda} \] (3)

where $B_{\mu\nu}$ is the antisymmetric Kalb-Ramon field \[ B_{\mu\nu} \] whose gauge transformation is expressed in terms of a vector parameter $\xi_\mu$ as

\[ \delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \] (4)

This term can be associated to a mechanism of mass generation for the gauge fields $A_\mu$ or $B_{\mu\nu}$. Indeed, considering (3) together with Maxwell terms for $A_\mu$ and $B_{\mu\nu}$, namely

\[ S = \int d^4x \left( \frac{1}{12} H_{\mu\nu\rho}^2 - \frac{M^2}{2} \epsilon^{\mu\nu\rho\lambda} A_\mu \partial_\nu B_{\rho\lambda} - \frac{1}{4} F_{\mu\nu}^2 \right) \] (5)

we get

\[ S_{eff}[A_\mu] = -\frac{1}{4} \int d^4x F_{\mu\nu} \left( 1 + \frac{M^2}{\Box} \right) F^{\mu\nu} \] (6)

or

\[ S_{eff}[B_{\mu\nu}] = \frac{1}{12} \int d^4x H_{\mu\nu\rho} \left( 1 + \frac{M^2}{\Box} \right) H^{\mu\nu\rho} \] (7)

respectively, upon integration over $B_{\mu\nu}$ or $A_\mu$. It is opportune to mention that its non-Abelian version \[ F_{\mu\nu} \] can
also be used as an alternative mechanism of mass generation for the gauge fields in the electroweak theory without Higgs bosons.\footnote{\textsuperscript{3}}

The topological term given by (\textsuperscript{3}) presents the three features closely related to the ones of (\textsuperscript{1}). It is quadratic in the fields, it generates a mass for these fields and it is also related to the change of statistics of extended objects, namely, strings, in $D = 3 + 1$. It is, therefore, natural, attempting to find another sequence of terms containing (\textsuperscript{3}) and (\textsuperscript{1}), and presenting the common feature of being quadratic. We are going to see that some other features are also shared.

2. The motivation of the present paper is to make a general study of a new sequence for topological terms, by considering that these terms always have two fields (with the same rank or not). Since even dimensional terms are not necessarily zero, our guide will be that a term from a dimension $D$ can be obtained from compactification of a term higher than $D$. The only natural constraint we impose is that the usual Chern-Simons term in $D = 2 + 1$ takes part in the sequence.

In order to have a better comprehension of the problem, let us start from a specific spacetime dimension, say $D = 5$. At first sight, the corresponding Chern-Simons term could be written by means of two gauge fields of rank two, i.e. $\int d^5 x \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \partial_\mu A_\nu C_{\mu_\nu \lambda}$. However, this term corresponds to a total derivative and consequently is not a good candidate for a Chern-Simons at spacetime dimension $D = 5$. The next one to play this role is formed by a totally antisymmetric gauge field of rank three and a vector field, namely

$$S_5 = \int d^5 x \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \partial_\mu A_\nu C_{\mu_\nu \lambda}(\mathbf{x})$$

(8)

The gauge transformation for the rank three gauge field $C_{\mu_\nu \rho}$ should be

$$\delta C_{\mu_\nu \rho} = \partial_\mu \zeta_{\nu \rho} + \partial_\nu \zeta_{\mu \rho} + \partial_\rho \zeta_{\mu \nu}$$

(9)

where $\zeta_{\mu \nu}$ is an antisymmetric gauge parameter.

The consistency of the term given by (\textsuperscript{1}) can be verified by performing the compactification of one space dimension and see if the Chern-Simons term at $D = 4$, given by (\textsuperscript{3}), is obtained. We use the spontaneous compactification procedure (\textsuperscript{4}). Let us compactify the coordinate $x^4$. Using a more convenient notation, where capital Roman indices correspond to the spacetime dimension $D = 5$, we have

$$\int d^5 x \epsilon^{MNPQR} \partial_M A_N C_{PQR} = \epsilon^{\mu_\nu \rho \lambda} \int d^4 x \int_0^R d^4 x \left( \partial_\lambda A_\mu C_{\nu \mu \rho} \right) + \partial_\lambda A_\mu C_{\nu \mu \rho} + 3 \partial_\mu A_\nu C_{\rho \lambda} \right)$$

(10)

where $\epsilon^{\mu_\nu \rho \lambda} = \epsilon^{A_4 \mu_\nu \rho \lambda}$. We next take the (Fourier) expansions:

$$A_\mu(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} A_{(n)\mu}(x) \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

$$A_4(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} \phi_{(n)}(x) \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

$$C_{\mu \nu \rho}(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} C_{(n)\mu \nu \rho}(x) \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

$$C_{\mu \nu 4}(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} B_{(n)\mu \nu}(x) \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

(11)

Since all fields in (\textsuperscript{11}) are assumed to be real, we must have $A_{(n)\mu} = A_{(-n)\mu}$, $\phi_{(n)} = \phi_{(-n)}$, and so on.

Considering similar expansions for the parameters related to the gauge transformations of $A_M$ and $C_{MN}$

$$\alpha(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} \alpha_{(n)} \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

$$\zeta_{MN}(x, x^4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{\infty} \zeta_{(n)MN} \exp \left( 2i n \pi x^4 \frac{R}{R} \right)$$

(12)

we obtain the following gauge transformations for the mode expansions that appear in (\textsuperscript{11})

$$\delta A_{(n)\mu}(x) = \partial_\mu \alpha_{(n)}(x)$$

$$\delta \phi_{(n)}(x) = \frac{2i n \pi}{R} \alpha_{(n)}(x)$$

$$\delta C_{(n)\mu \nu \rho}(x) = \partial_\mu \zeta_{(n)\nu \rho}(x) + \partial_\nu \zeta_{(n)\mu \rho}(x) + \partial_\rho \zeta_{(n)\mu \nu}(x)$$

$$\delta B_{(n)\mu \nu}(x) = \partial_\nu \tilde{\zeta}_{(n)\mu}(x) - \partial_\mu \tilde{\zeta}_{(n)\nu}(x) + 2i n \pi \zeta_{(n)\mu \nu}(x)$$

(15)

(16)

where

$$\tilde{\zeta}_{(n)\mu}(x) = \zeta_{(n)\mu 4}(x)$$

(17)

Comparing (\textsuperscript{11}) and (\textsuperscript{12}), we observe that just $\tilde{B}_{(0)\mu \nu}$ can be identified with $B_{\mu \nu}$ where one takes $\tilde{\zeta}_{(0)\mu \nu} = \zeta_{\mu \nu}$.

Let us now insert the expansions given by (\textsuperscript{11}) into (\textsuperscript{12}). The final result is

$$\int d^5 x \epsilon^{MNPQR} \partial_M A_N C_{PQR}$$

$$= \epsilon^{\mu_\nu \rho \lambda} \sum_{n=-\infty}^{\infty} \int d^4 x \left( \frac{2i n \pi}{R} A_{(n)\mu} C_{(n)\nu \rho \lambda} \right)$$

$$\partial_\lambda \phi_{(n)} C_{(n)\mu \nu \rho} + 3 \partial_\mu A_{(n)\nu} B_{(n)\rho \lambda}$$

(18)
Since $\tilde{B}_{(0)\mu\nu}$ can be identified with the tensor gauge field $B_{\mu\nu}$ and there is no problem in identifying $A_{(0)\mu}$ with $A_{\mu}$, the Cremer and Scherk topological term given by (13) is actually present in the compactified expression (18) for $n = 0$. Further, there is another kind of topological term that can also be identified in (13) for $n = 0$, involving a real scalar field and a three-form gauge field, namely

$$S'_{4} = \int d^{4}x \epsilon^{\mu\nu\rho\lambda} \partial_{\mu} \phi C_{\nu\rho\lambda}$$  \hspace{1cm} (19)$$

where $\phi = \phi_{(0)}$. In fact, we could also have written a term like this in $D = 5$, involving a scalar and a four-form gauge field,

$$S'_{5} = \int d^{5}x \epsilon^{MNPQR} \partial_{M} \phi D_{NPQR}$$  \hspace{1cm} (20)$$

It is not difficult to see that the spontaneous compactification of this term also leads to $S'_{4}$ given by (19), but not to (3).

3. We now proceed in a similar way and go to lower dimensions. Having the same care in identifying vector, tensor and scalar fields for $n = 0$, we get from $D = 4$ to $D = 3$ the usual Chern-Simons terms given by (1) and also another one involving a scalar field,

$$S'_{3} = \int d^{3}x \epsilon^{\mu\nu\rho} \partial_{\mu} \phi B_{\nu\rho}$$  \hspace{1cm} (21)$$

This term can be reached by compactification of both $S_{4}$ and $S'_{4}$, and has also been considered in a recent literature [13]. It is easily seen that topological terms involving scalar field is a kind of residual Chern-Simons term that appears at any spacetime dimensions. However, in $D = 2$ and $D = 1$ these terms are the only ones that remain. Indeed,

$$S_{2} = \int d^{2}x \epsilon^{\mu\nu} \partial_{\mu} A_{\nu} \phi$$ \hspace{1cm} (22)$$

$$S_{1} = \int dx \phi \partial_{\mu} \phi$$  \hspace{1cm} (23)$$

In the obtainment of (23), the two different scalar fields come from $A_{(0)1}$ and $\phi_{(0)}$.

The term (21) also has been shown to produce a mass generation for the corresponding fields [11]. For the one given by (22) and considering the kinetic terms for $\phi$ and $A_{\nu}$, we have

$$S = \int d^{2}x \left[ \frac{1}{2} (\partial_{\mu} \phi)^{2} - M \epsilon^{\mu\nu} A_{\mu} \partial_{\nu} \phi - \frac{1}{4} F_{\mu\nu}^{2} \right]$$  \hspace{1cm} (24)$$

After integrating over $\phi$ and $A_{\nu}$, we respectively obtain

$$S_{\text{eff}}[A_{\mu}] = -\frac{1}{4} \int d^{2}x F_{\mu\nu} \left( 1 + \frac{M^{2}}{\Box} \right) F^{\mu\nu}$$  \hspace{1cm} (25)$$

and

$$S_{\text{eff}}[\phi] = \int d^{2}x \left[ \frac{1}{2} (\partial_{\mu} \phi)^{2} - \frac{M^{2}}{2} \phi^{2} \right]$$  \hspace{1cm} (26)$$

and we, once again, identify the mass generation mechanism. A similar procedure also happens to (23).

We observe that the topological term in $D = 1$ does not match the corresponding one of the sequence given by (3). Even though that term appears naturally in the framework of quantum field theory, in the sense that it can be generated by quantum corrections of fermionic loops [2], the $S_{CS3}$ term is not related to mass generation. This property on the other hand is fulfilled by the term given by (23). Concerning the quantum generation for the terms of the sequence we are studying, we remark that they are also consistent from the quantum point of view. What happens is that they emerge in an almost trivial way. For example, for the term given by (3), the $\epsilon^{\mu\nu\rho\lambda}$ tensor appears directly from the interaction vertices involving fermion and tensor fields, while the usual Chern-Simons term in $D = 3$ comes from the trace of gamma matrices in a one-loop calculation.

4. In the previous section we have considered topological terms by making spontaneous compactification starting from $D = 5$. From the results we have by now, it is feasible to infer those terms for spacetime dimensions higher than five. For example, at $D = 6$ these terms are

$$S_{5} = \int d^{6}x \epsilon^{\mu\nu\rho\lambda\xi} \partial_{\mu} \phi E_{\nu\rho\lambda\xi}$$

$$S'_{5} = \int d^{6}x \epsilon^{\mu\nu\rho\lambda\xi} \partial_{\mu} A_{\nu} D_{\rho\lambda\xi}$$

$$S''_{5} = \int d^{6}x \epsilon^{\mu\nu\rho\lambda\xi} \partial_{\mu} B_{\nu\rho} C_{\lambda\xi}$$ \hspace{1cm} (27)$$

Let us also write down the terms for $D = 7$

$$S_{7} = \int d^{7}x \epsilon^{\mu\nu\rho\lambda\xi\zeta} \partial_{\mu} \phi F_{\nu\rho\lambda\xi\zeta}$$

$$S'_{7} = \int d^{7}x \epsilon^{\mu\nu\rho\lambda\xi\zeta} \partial_{\mu} A_{\nu} E_{\rho\lambda\xi\zeta}$$

$$S''_{7} = \int d^{7}x \epsilon^{\mu\nu\rho\lambda\xi\zeta} \partial_{\mu} B_{\nu\rho} D_{\lambda\xi\zeta}$$

$$S'''_{7} = \int d^{7}x \epsilon^{\mu\nu\rho\lambda\xi\zeta} \partial_{\mu} C_{\nu\rho\lambda} C_{\xi\zeta}$$ \hspace{1cm} (28)$$

and it is not difficult to infer the general case. We observe that the number of terms increases with the spacetime dimension. There is just one term for $D = 1$ and $D = 2$. For $D = 3, 4, 5$ there are two terms. In fact, there would be three terms for $D = 5$, but one of them, involving two fields of rank two, is a total derivative. The same does
not occur for example for \( D = 7 \), where the corresponding term involving two gauge fields of rank three is not a total derivative. This just occurs when the two equal gauge fields have even rank.

The number of topological terms for a specific dimension \( D \) is \( D/2 \) for \( D \) even. In the case of odd \( D \) we have that the number is \( D/2 - 1/2 \) if the term with two equal gauge fields is a total derivative and \( D/2 + 1/2 \) when it is not.

An important point to be emphasized is that even though the number of topological terms increases with \( D \), there is one term at each specific spacetime dimension that is more important than the others in the sense that it generates all the other terms for lower dimensions. It is not difficult to identity these terms. For example, at \( D = 5 \) this term is \( S_5 \), in \( D = 6 \), \( S''_6 \), and in \( D = 7 \), \( S''_7 \) (notice that it is not \( S''_6 \) because it would not generate \( S''_7 \)).

In what follows, we analyze physical properties of some of the topological terms in the sequence, as far as statistical transmutation and bosonization are concerned.

5. Some of the topological terms generated in the sequence studied in this paper play an important role in connection to the statistical transmutation of different objects. In the case of \( D = 3 \), let us consider the coupling of a point-particle, associated to a current density \( j^\mu \), to the topological field:

\[
\mathcal{L}_3 = \frac{\theta}{2} \epsilon^{\mu\nu\alpha} A_\alpha \partial_\mu A_\nu - j^\mu A_\mu
\]  

(29)

In this equation,

\[
j^\mu = \int_L d\xi^\mu \delta^3(x - \xi)
\]  

(30)

where \( L \) is the universe-line of the particle. The 0-component of the field equation associated to expression (29), for a static point-particle is

\[
j^0 = \delta^2(\vec{x} - \vec{x}_0) = \theta \epsilon^{ij} \partial_i A_j = \theta B
\]  

(31)

where \( B = \epsilon^{ij} \partial_i A_j \) is the magnetic field, which is a scalar in \( D = 3 \). We see that the topological term imparts a point magnetic flux to any point particle that couples to the \( A_\mu \) field. This fact is responsible for the change in the statistics of the particle since, through an Aharonov-Bohm type effect, its wave function will acquire a phase when interchanged with another particle [12].

Let us consider now the case of a string in \( D = 4 \). This is associated to the 2-tensor current density given by

\[
j^{\mu\nu} = \int_S d^2 \sigma^{\mu\nu} \delta^4(x - \xi)
\]  

(32)

where \( S \) is the universe-sheet of the string. The coupling of the string with the topological term \( S_4 \) is given by

\[
\mathcal{L}_4 = \theta \epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\nu B_{\alpha\beta} - j^{\mu\nu} B_{\mu\nu}
\]  

(33)

The \((0i)\)-component of the field equation associated with the Lagrangian density (33), corresponding to a static string along the spatial curve \( \Gamma \), is

\[
j^{0i} = \int_\Gamma d\xi^i \delta^3(x - \xi) = \theta B^i
\]  

(34)

where \( B^i = \epsilon^{ijk} \partial_j A_k \) is the magnetic field. For a straight string along the \( x^3 \)-direction, piercing the \((12)\)-plane, for instance, we have \( \theta B^3 = \delta^2(\vec{x} - \vec{x}_0) \).

We see that the topological term produces a constant magnetic field along the string. It is not difficult to infer that charged strings in the presence of the topological term \( S_4 \) will suffer a statistical transmutation determined by \( \theta \), again as a consequence of an Aharonov-Bohm like effect. This fact has been already identified before [9] in a different framework. Nevertheless, it becomes especially transparent here and can be unified with what happens in other dimensions as well.

Consider now the case of membranes in \( D = 5 \). The current density associated to a membrane is the 3-tensor

\[
j^{\mu\nu\alpha} = \int_V d^3 \sigma^{\mu\nu\alpha} \delta^3(x - \xi)
\]  

(35)

where \( V \) is the universe-volume of the membrane. The membrane couples to the topological term \( S_5 \), Eq. (31), in the following way:

\[
\mathcal{L}_5 = \theta \epsilon^{\mu\nu\alpha\beta\gamma} A_\mu \partial_\nu C_{\alpha\beta\gamma} - j^{\mu\nu\alpha\beta\gamma} C_{\mu\nu\alpha\beta\gamma}
\]  

(36)

The \((0ij)\)-component of the field equation associated to (34), and corresponding to a static membrane along the spatial surface \( \Sigma \) is

\[
j^{0ij} = \int_\Sigma d^2 \xi^{ij} \delta^3(x - \xi) = \theta B^{ij}
\]  

(37)

where \( B^{ij} = \epsilon^{ijkl} \partial_k A_l \) is the magnetic field, which in \( D = 5 \) is a 2-tensor. We see that the topological term \( S_5 \) attaches a constant magnetic field along a membrane that couples to it through the vector field \( A_\mu \) in \( D = 5 \). We may immediately conclude that, in analogy to what happens in \( D = 3 \) and \( D = 4 \), a charged membrane in the presence of the topological term \( S_5 \) will undergo statistical transmutation determined by the parameter \( \theta \). This fact can certainly be also inferred through a study of the membrane propagator or by the membrane creation operator formalism, as has been done for the string in \( D = 4 \) [9]. We are presently investigating this point.

6. Another important feature associated with the sequence of topological terms in an arbitrary dimension
is bosonization. In this case the basic starting point is $D = 2$. Consider a point particle associated to a current density $j^\mu$, which couples to the topological term $S_2$, given by (32), as follows

$$\mathcal{L}_2 = \theta \epsilon^{\mu\nu} A_\nu \partial_\mu \phi - j^\mu A_\mu$$  \hfill (38)

The 0-component of the field equation obtained by varying (38) with respect to $A_\mu$, for a static point particle in position $x_0$ is

$$j^0 = \delta(x - x_0) = \theta \partial_\nu \phi$$  \hfill (39)

This implies that corresponding to the point particle described by the current $j^\mu$, we have a solution for the field $\phi$ given by

$$\phi_0(x) = \theta \tilde{\theta}(x - x_0)$$  \hfill (40)

where $\tilde{\theta}(x - x_0)$ is the step function. This is a soliton profile and we are led to infer that corresponding to the point particles associated with the current $j^\mu$, we will have soliton excitations of the $\phi$-field, associated to the identically conserved current $J^\mu = \epsilon^{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta}$. This fact is at the very basis of the process of bosonization, whenever $j^\mu$ is a fermionic current. The fermionic particles are identified with the soliton excitations of the associated bosonic theory [13].

The previous observation allows us to look at the possibility of bosonization in dimensions higher than $D = 2$ and its possible connection with the topological terms studied here. Indeed, from (41) we see that corresponding to a point particle associated to a current $j^\mu$, minimally coupled to a Chern-Simons field in $D = 3$, we will have a magnetic vortex solution for the $A_\mu$-field at the same point. These vortex configurations, however, are topological solitons corresponding to the identically conserved current $J^\mu = \epsilon^{\mu\nu\alpha\beta} \partial_\nu A_\alpha$. Once more we see that bosonization may be achieved by identifying the fermions associated to a current $j^\mu$ with magnetic vortices in the corresponding bosonic vector gauge field theory. This has actually been pursued in the case of a free massless Dirac fermion [14] but still there is a vast area to explore in bosonization in $D = 3$.

Some conclusions can also be drawn by extending the present analysis for $D = 4$. Coupling a fermionic vector current to the vector field of the topological term $S_4$, namely

$$\mathcal{L}'_4 = \theta \epsilon^\mu \epsilon^\nu A_\mu \partial_\nu B_{\alpha\beta} - j^\mu A_\mu$$  \hfill (41)

We see that the field equation obtained by varying with respect to $A_\mu$ is

$$j^\mu = \theta \epsilon^{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta}$$  \hfill (42)

For a static point particle at $\tilde{x}_0$, we have

$$j^0 = \delta^3(\tilde{x} - \tilde{x}_0) = \theta \epsilon^{ijk} \partial_i B_{jk}$$  \hfill (43)

We see that the point fermionic particle may be identified with a configuration of the bosonic Kalb-Ramond field $B_{\alpha\beta}$ with a nonzero topological charge corresponding to the identically conserved current $J^\mu = \epsilon^{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta}$. This observation should be at the basis of any attempt to extend bosonization to $D = 4$.

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