Innocent Strategies are Sheaves over Plays

Deterministic, Non-deterministic and Probabilistic Innocence

Takeshi Tsukada
University of Oxford
JSPP Postdoctoral Fellow for Research Abroad
tskada@cs.ox.ac.uk

C.-H. Luke Ong
University of Oxford
Luke.Ong@cs.ox.ac.uk

Abstract
Although the HO/N games are fully abstract for PCF, the traditional notion of innocence (which underpins these games) is not satisfactory for such language features as non-determinism and probabilistic branching, in that there are stateless terms that are not innocent. Based on a category of P-visible plays with a notion of embedding as morphisms, we propose a natural generalisation by viewing innocent strategies as sheaves over (a site of) plays, echoing a slogan of Hirschowitz and Pous. Our approach gives rise to fully complete game models in each of the three cases of deterministic, non-deterministic and probabilistic branching. To our knowledge, in the second and third cases, ours are the first such factorisation-free constructions.

1. Introduction
Game semantics is a powerful paradigm for giving semantics to a variety of programming languages and logical systems. Both HO/N games [10, 14] (based on arenas and innocent strategies) and AJM games [2] (based on games equipped with a certain equivalence relation on plays, and history-free strategies) gave rise to fully complete game models in each of the three cases of deterministic, non-deterministic and probabilistic branching. Our approach gives rise to fully complete game models in each of the three cases of deterministic, non-deterministic and probabilistic branching. To our knowledge, in the second and third cases, ours are the first such factorisation-free constructions.

We are interested in the simply-typed lambda calculus because they have good algorithmic properties, notably, the decidability of compositional higher-order model checking [15, 18], which is proved using HO/N-style effect arenas and innocent strategies. Our study of the game semantics of non-deterministic lambda calculus was motivated, in particular, by a desire to introduce abstraction refinement to higher-order model checking based on the non-deterministic $\lambda Y$-calculus.

Let us begin with a quick overview of the HO/N-style games. Types are interpreted as arenas, and programs of a given type are interpreted as $P$-strategies for playing in the arena that denotes the type. Recall that an arena $A$ is a set of moves $M_A$ equipped with an enabling relation, $({\neg}A) \subseteq (M_A \cup \{\ast\} \times M_A)$, that gives $A$ the structure of a forest (whereby a move $m$ is a root, called initial, just if $* \vdash_A m$); furthermore, moves on levels 0, 2, 4, ... of the forest are O-moves, and those that are on levels 1, 3, 5, ... are P-moves. A justified sequence of $A$ is a finite sequence of OP-alternating moves, $m_1 m_2 m_3 \ldots m_n$, such that each non-initial move $m_i$ has a pointer to an earlier move $m_j$ (called the justifier of $m_i$) such that $m_i \vdash_A m_j$. A key notion of HO/N games is the view of a justified sequence: the $P$-view of a justified sequence $s$ is a certain justified subsequence, written $[s]$, consisting of moves-occurrences which $P$ considers relevant for determining his next move (similarly for the O-view $[s]$ of $s$). A play then is a justified sequence, $s_1 m_2 s_3 \ldots$, that satisfies Visibility: for every $i$, if $m_i$ is non-initial then its justifier appears in $[m_1 m_2 \ldots m_i]$ (respectively $[m_1 m_2 \ldots m_i]$) if $m_i$ is a P-move (respectively O-move). A strategy $\sigma$ over an arena $A$ is just a prefix-closed set of even-length plays $s, \sigma$ is said to be deterministic if whenever $s m_1^P, s m_2^P \in \sigma$, then $m_1^P = m_2^P$. We use superscript $P$ to indicate a P-move; similarly for O-move.) Recall that a strategy $\sigma$ is said to be innocent if it is view dependent i.e. for all $s \in \sigma$

$$s \in \sigma \land [s m_1^P m_2^P] \in \sigma \iff s m_1^P m_2^P \in \sigma \quad (1)$$

It is an important property of innocence that—in the sets-of-plays presentation of strategies—every deterministic innocent strategy can be generated by the set of P-views contained in it. The category of arenas and innocent strategies gives rise to a fully complete model of the simply-typed lambda calculus [10].

However, as Harmer observed in his thesis [7], the notion of innocence breaks down when one tries to use it to model (stateless) non-deterministic functional computation.

Example 1. Take simply-typed lambda terms $\tau := \lambda x y. x$ and $\mathbf{f} := \lambda x y z. x \mathbf{f} z$ of type $B = o \rightarrow o \rightarrow o$, and $M_1 := \lambda f f (\mathbf{f} + \mathbf{f})$ and $M_2 := (\lambda f f \mathbf{f} + \lambda f f \mathbf{f})$ of type $B = o \rightarrow (o \rightarrow o)$, where $+$ is the construct for non-deterministic branching. Assuming the call-by-name evaluation strategy, these terms can be separated by the term $N := \lambda g g (g \downarrow z) _\bot$, where $\downarrow$ is the divergence term, i.e. $M_1 N$ may converge but $M_2 N$ always diverges. In the HO/N game model (see, for example, [7]), $\sigma_1 := [M_1]$ are strategies over the arena ($$(d) \rightarrow (d') \rightarrow (c) \rightarrow (b) \rightarrow \{a\}$$, for $i = 1, 2$. Note that $\sigma_1$ and $\sigma_2$ are distinct as strategies: for example (we omit pointers from the plays as they can be uniquely reconstructible)
The preceding example shows the sets-of-plays approach works well for expressing, and even composing, non-deterministic strategies for stateless programs; the only problem is that, in general, the set of P-views cannot be a good generator for these strategies.

The problematic term is $M$. It applies the argument $f$ to $\pi$ or $\pi$, non-deterministically, but the branch has already been chosen when $M_2$ responds to the initial move. So $abc\, c\, cd'$ is not playable by $M_2$, although innocence requires it to.

Our approach is to admit that $M_2$ has two possible responses to the initial move; they give the same play $ab$ but have different internal states. Thus a strategy is formally a mapping from plays to sets that represent the internal states. For example, $[M_2](ab) = \{ \pi, \pi \}$, where $\pi$ means the left branch and $\pi$ the right branch. Now the P-views for $[M_2]$ are, say, $\{ a\implies b, a\implies b, a\implies b, a\implies b \}$. Notice that a $b\implies c\implies d\implies a$ and $a\implies b\implies c\implies d$ are no longer forced by innocence to be admissible plays. From this viewpoint, a deterministic strategy is a mapping from plays to empty or singleton sets.

In what follows, we discuss how to formalise this idea.

**Ideal-based innocence** Before we explain the main ideas behind our sheaf-theoretic approach to innocence, it is helpful to consider a category of plays $\mathcal{P}_A$, and an alternative view of deterministic innocence strategies as *ideals of a preorder presentation* [11]. The objects of the category $\mathcal{P}_A$ are (even-length) justified sequences of the arena $A$ satisfying O/P-alternation and P-visibility (but not necessarily O-visibility), which we shall henceforth call calls (by abuse of language). The morphisms $f: s \to s'$ are injective maps that preserve moves, justification pointers, and pairs of consecutive O/P moves. A morphism can permute such pairs, provided the pointers are respected. For example, for each play $s$, there are morphisms $[s] \to s$ and $s \to [s] \implies [s] \implies [s]$.

A preorder presentation is a triple $(\mathcal{P}, \leq, P)$ where $(\mathcal{P}, \leq)$ is a preorder and $P \subseteq \mathcal{P} \times \mathcal{P}$ is called a covering relation (we read $U \leq s$ as “$U$ covers $s$”). A subset $I \subseteq \mathcal{P}$ is called an ideal if (i) $I$ is lower-closed, i.e., if $I \in I$ and $s \leq I$ then $s \in I$, and (ii) for every covering $U \leq s$, if $\{ s \} \subseteq I$ then $s \in I$, and (ii) every covering $U \leq s$, if $\{ s \} \subseteq I$ then $s \in I$. A preorder presentation can be extracted from the category $\mathcal{P}_A$; namely, $(\text{Obj}(\mathcal{P}_A), \leq, P)$ whereby $s \leq s'$ just if there is a morphism $f: s \to s'$, and $U \leq s$ just if $U = \{ s \} \cup \{ \xi \} \subseteq I$. The objects of a preorder presentation are view-dependent, so there is a compelling sense in which sheaves on plays, $\mathcal{P}_A \to \text{Set}$, depend on (indeed, are determined by) sheaves on views, $\mathcal{P}_A \to \text{Set}$, whose action on objects is $u \mapsto \sigma_1(u|A,B) \times \sigma_2(u|A,B)$. Hence, the category whose objects are arenas and whose morphisms $\sigma: A \to B$ are (equivalence classes of isomorphic) sheaves $\sigma: \mathcal{P}_A \to \text{Set}$ is cartesian closed.

Just as innocent strategies are view dependent, so there is an obvious sense in which sheaves on plays, $\mathcal{P}_A \to \text{Set}$, depend on (indeed, are determined by) sheaves on views, $\mathcal{P}_A \to \text{Set}$, whose action on objects is $u \mapsto \sigma_1(u|A,B) \times \sigma_2(u|A,B)$. Hence, the category whose objects are arenas and whose morphisms $\sigma: A \to B$ are (equivalence classes of isomorphic) sheaves $\sigma: \mathcal{P}_A \to \text{Set}$ is cartesian closed.

Theorem 2.2. Let $\mathcal{P}_A \to \text{Set}$ be the category whose objects are arenas and whose morphisms $\sigma: A \to B$ are (equivalence classes of isomorphic) sheaves $\sigma: \mathcal{P}_A \to \text{Set}$.

Our contributions Our thesis is that sheaves $\mathcal{P}_A \to \text{Set}$ generalise innocent strategies of the arena $A$. (Indeed the sheaves approach seems more general than innocence, since it appears capable of capturing the computation of single-threaded (history-sensitive) strategies as well.)
notation of $M_t$. Then
\[
\tau_1(p_0) = \{x_1\} \quad \tau_2(p_0) = \{x_{21}, x_{22}\}
\]
\[
\tau_1(p_1) = \{y_1\} \quad \tau_2(p_1) = \{y_{21}\}
\]
\[
\tau_1(p_2) = \{z_1\} \quad \tau_2(p_2) = \{z_{22}\}
\]
Notice that in the set of plays $[M_2]$, there are two independent plays (which have the P-view) $p_0$.

Our approach gives rise to fully complete game models in each of the three cases of deterministic, non-deterministic and probabilistic branching. To our knowledge, in the second and third cases, ours are the first such factorisation-free constructions.

**Related work** The standard notion of innocence does not work well for certain language features, such as non-determinism. To address the deficiency, Levy [13] proposed a category of P-visible plays and viewing morphisms. This is essentially our category $F_A$ of plays. However in op. cit. an innocent strategy $\sigma$ is still defined to be a certain set of plays, namely, a lower-closed set of objects of the category: if $t \in \sigma$ and $s \to t$ is a morphism, then $s \in \sigma$. Because this definition captures only one of the two requirements of innocence (i.e. $\leq$ of (1)), Levy’s construction will likely not yield accurate (fully complete) models of the non-deterministic $\lambda$-calculus.

A related approach by Hirschowitz et al. [6, 9] does view strategies as presheaves (and sheaves) on a category of plays. However, in contrast to our focus on higher-type computation, they are concerned with CCS-style concurrent computation which they model as multi-player games. Strategies are presheaves on a category of plays $EX_X$ over a position $X$, and a strategy is deemed innocent if it is determined by its restriction to a subcategory of views $V_X \to EX_X$. A position is an undirected graph describing the channel-based communication topology connecting the players, and plays are certain “glueings” of moves over a position, with moves built-up using CCS constructs. Thus the connections with our work seem superficial.

Winiksel et al. [16, 17] have worked extensively on causal games as models of true concurrency, from the viewpoint of strategies as event structures with symmetries. Recently Clairambault et al. [4] built a conservative extension of HO/N games in a truly concurrent framework. An extensional quotient of their model yields a fully abstract model of PCF with parallel or.

Perhaps surprisingly, the question of what is the proper notion of innocence in the presence of non-determinism is still open. Harmer and McCusker [8] seem only concerned with stateful non-deterministic programs, namely non-deterministic Idealised Algol.

**Technical preliminaries** In the following we review the basic definitions of coverage, Grothendieck topology and sheaves, and refer the reader to the book [12] for an exposition.

A *coverage* on a category $C$ is a map $J$ assigning to each object $s$ of $C$ a collection $J(s)$ of families of maps $\{f_\xi : s_\xi \to s\}_{\xi \in \xi}$ of maps with codomain $s$, called *covering families*, such that the system of families is “stable under pullback”, meaning: if $\{f_\xi : s_\xi \to s\}_{\xi \in \xi}$ is a covering family and $g : t \to s$ is a map, then there is a covering family, $\{h_\xi : t_\xi \to f_\xi\}_{\xi \in \#}$, such that each $g \circ h_\xi$ factors through some $f_\xi$. A number of saturation conditions are often imposed on a coverage for convenience. A site is a category $C$ equipped with a coverage $J$, written $(C, J)$.

Given a family $S = \{f_\xi : s_\xi \to s\}_{\xi \in \xi}$ of maps with codomain $s$, and a presheaf $F : C^{op} \to Set$, a family of elements $x_{\xi} \in F(s_{\xi})_{\xi \in \xi}$ is said to be matching for $S$ if for all maps $g : t \to s_\xi$ and $h : t \to s_\zeta$ if $f_\xi \circ g = f_\zeta \circ h$ then $F(g)(x_\xi) = F(h)(x_\zeta)$. An amalgamation for the family $\{x_{\xi} \in F(s_{\xi})_{\xi \in \xi}\}$ is an $x \in F(s)$ such that $F(f_\xi)(x) = x_{\xi}$ for every $\xi \in \Xi$. A presheaf $F : C^{op} \to Set$ is a *sheaf* for a site $S = \{f_\xi : s_\xi \to s\}_{\xi \in \xi}$ of maps just if every matching family for $S$ has a unique amalgamation. A presheaf is a *sheaf* for a site if it is a sheaf for every covering family of the site.

A *sieve* on an object $s$ in a category $C$ is a family of maps with codomain $s$ that are closed under precomposition with maps in $C$. Given a family $\{f_\xi : s_\xi \to s\}_{\xi \in \xi}$, the sieve it generates is the family of all maps $g : t \to s$ with codomain $s$ that factor through some $f_\xi$. A presheaf is a sheaf for a family $\{f_\xi : s_\xi \to s\}_{\xi \in \xi}$ if, and only if, it is a sheaf for the sieve it generates. If $S$ is a sieve on $s$ and $g : t \to s$ is a map, we define $g^*\{S\}$ to be the sieve on $t$ consisting of all maps $h$ with codomain $t$ such that $g \circ h$ factors through some map in $S$.

A Grothendieck topology is a map $J$ that assigns to each object $s$ of $C$ a collection $J(s)$ of sieves on $s$, called *covering sieves*, that satisfies the following:

(i) The maximal sieve, $\{f \mid \text{cod}(f) = s\}$, is in $J(s)$.

(ii) (*Stability*) If $S \in J(s)$ then $h^*(S) \in J(t)$ for every map $h : t \to s$.

(iii) (*Transitivity*) If $S \in J(s)$ and $R$ is a sieve on $s$ such that $h^*(R) \in J(t)$ for every $h : t \to s$ in $S$, then $R \in J(s)$.

**Lemma 3.** For every coverage, there is a unique Grothendieck topology that has the same sieves.

**Notation** We write $N$ for the set of all positive integers. For an integer $n$, we define $[n] = \{k \mid 1 \leq k \leq n\}$ and $[0,n] = \{k \mid 0 \leq k \leq n\}$. For a category $C$, we write $x \in C$ to mean that $x$ is an object of $C$.

### 2. Sites of Plays

This section defines sites of plays over an arena. The innocent strategies are just sheaves over those sites. The category of plays has a subcategory of views. We prove that the sheaves over plays is equivalent to sheaves over views: this generalises view dependency to non-deterministic computation.

#### 2.1 Plays

The definition of arenas is standard (as in [10]) except that all moves are questions.

**Definition 4 (Arena).** An arena is a tuple $A = (M_A, \lambda_A, \vdash_A)$, where $M_A$ is a finite set of moves, $\lambda_A : M_A \to \{P, O\}$ is an ownership function and $\vdash_A \subseteq (\times) + M_A \times M_A$ is an enabling relation that satisfies the following conditions: (1) for every $m \in M_A$, there is a unique $x \in \{x\} + M_A$ such that $x \vdash_A m$, and (2) if $s \vdash_A m$ then $\lambda_A(m) = O$. If $m \vdash_A m'$, then $\lambda_A(m) \neq \lambda_A(m')$.

For an arena $A$, the set $M_A$ of $O$-moves is defined as $\{m \in M_A \mid \lambda_A(m) = O\}$. The set of $P$-moves is defined by $M_A^P := \{m \in M_A \mid \lambda(m) = P\}$. A move $m$ is initial if $s \vdash_A m$. An arena is prime if it has exactly one initial move.

We write $\{m\}$ for the arena that has one $O$-move $m$ and no $P$-moves. For a prime arena $A$ and an arena $B, B \to A$ is the arena whose moves are $M_A + M_B$ where the initial $B$-move is enabled by the unique initial $A$-move. For example, $\{m_1\} \to \{m_2\} \to \{m_3\}$ consists of an $O$-move $m_3$ and $P$-moves $m_1$ and $m_2$ with $s \vdash_A m_3, m_1 \vdash A, m_2 \vdash A$.

Unlike the standard formalisation, in which notions such as justified sequences and plays are parametrised by arenas, we parametrise them by a pair of arenas $(A, B)$, corresponding to the exponential arena $A \Rightarrow B$ in the standard formalisation. This change simplifies some definitions.

**Definition 5 (Arena pair).** Let $A = (M_A, \lambda_A, \vdash_A)$ and $B = (M_B, \lambda_B, \vdash_B)$ be arenas. The moves of $(A, B)$ is the disjoint union of moves, say $M_{A,B} := M_A + M_B$. We define $P$-moves
by $M_{A,B}^P := M_A^P + M_B^P$ and $O$-moves by $M_{A,B}^O := M_A^O + M_B^O$. For $m, m' \in M_{A,B}$, we write $m \vdash_{A,B} m'$ just if either (1) $m, m' \in M_A$ and $m \vdash_A m'$, or (2) $m, m' \in M_B$ and $m \vdash_B m'$, or (3) $\varnothing \vdash m \in M_B$ and $m \vdash_{A,B} m' \in M_A$. We write $\varnothing \vdash_{A,B} m$ just if $\varnothing \vdash m \in M_B$.

For a pair $(A, B)$, an initial $A$-move is a move $m \in M_A \subseteq M_{A,B}$ such that $\varnothing \vdash_{A,B} m$ do not confuse it with $\varnothing \vdash_{A,B} m$, which is impossible. An initial $B$-move is defined similarly.

**Definition 6 (Justified sequence).** Let $(A, B)$ be a pair of arenas. A justified sequence of $(A, B)$ is a finite sequence of moves equipped with justification pointers. Formally it is a pair of functions $s : [n] \rightarrow M_{A,B}$ and $\varphi : [n] \rightarrow [n]_0$ (for some $n$) such that

1. $\varphi(k) < k$ for every $k \in [n]$, and
2. $\varphi$ respects the enabling relation: $\varphi(k) \neq 0$ implies $s(\varphi(k)) \vdash_{A,B} s(k)$, and $\varphi(k) = 0$ implies $\varnothing \vdash_{A,B} s(k)$.

As usual, by abuse of notation, we often write $m_1 m_2 \ldots m_n$ for a justified sequence such that $s(i) = m_i$ for every $i$, leaving the justification pointers implicit. Further we use $m$ and $m_i$ as metavariables of occurrences of moves in justified sequences. We write $m_1 \cap m_2$ if $\varphi(j) = i > 0$ and $\varnothing \cap m_i$ if $\varphi(j) = 0$. We call $m_i$ the justifier of $m_1$ when $m_1 \cap m_i$.

We write $\cap^*$ for the transitive closure of $\cap$.

Write $[s]$ for the length of $s$.

It is convenient to relax the domain $[n]_0$ of justified sequences to arbitrary linearly-ordered finite sets such as a subset of $[n]$. For example, given a justified sequence $s : [n] \rightarrow M_{A,B}, \varphi : [n] \rightarrow [n]_0$, consider a subset $I \subseteq [n]$ that respects the justification pointers, i.e. $k \in I$ implies $\varphi(k) \in I \cup \{0\}$. Then the restriction $s\mid_I : I \rightarrow M_{A,B}, \varphi\mid_I : I \rightarrow \{0\} \cup I$ is a justified sequence in the relaxed sense. Through the unique monotone bijection $\alpha : I \rightarrow [n]$, we identify the restriction with the justified sequence in the narrow sense.

A justified sequence is alternating if $s(k) \in M_{A,B}^O$ iff $k$ is odd (so $s(k) \in M_{A,B}^P$ iff $k$ is even).

**Definition 7 (P-View/P-visibility).** Let $m_1 \ldots m_n$ be an alternating justified sequence over $(A, B)$. Its P-view $[m_1 \ldots m_n]$ (or simply view) is a subsequence defined inductively by:

**Lemma 14.** $\mathbb{P}_{A,B}$ has pullbacks.

Proof. Let $f : s_1 \rightarrow t$ and $g : s_2 \rightarrow t$. They are injective maps $f : [s_1] \rightarrow [t]$ and $g : [s_2] \rightarrow [t]$. Let $I = \text{img}(f) \cap \text{img}(g)$. The restriction of $I$ to $I$ is the pullback $s_1 \times_I s_2$.

We give another definition of morphisms via commutation.

**Definition 15 (Commutation of non-interfering blocks).** Let $s$ be an even-length alternating justified sequence over $(A, B)$. Let $m_1 m_2 m_3 m_4$ be an adjacent pair of $O$-blocks in $s$, i.e. $s = t m_1 m_2 m_3 m_4 t'$, where $m_1$ and $m_2$ are $O$-moves. We say that the pairs are non-interfering if the justifier of $m_2$ is not $m_1$. The commuted sequence $s'$ is defined by $s' := t m_2 m_3 m_1 m_4 t'$ (in which the justification pointers are modified accordingly).

A commuted sequence is not always a justified sequence: if $m_2'$ is justified by $m_1$, then $m_2'$ in the commuted sequence is not well-justified. If the justified sequence is P-visible, the commuted
sequence is a justified sequence. Furthermore the converse also holds.

**Lemma 16.** Let $P$ be a set of even-length alternating justified sequences over $(A, B)$. Suppose that $P$ is closed under commutations, i.e. for every sequence $s \in P$ and every non-interfering adjacent pairs of blocks in $s$, the commuted sequence is also in $P$. Then all justified sequences in $P$ are plays.

**Proof.** Let $s = m_1 \ldots m_n \in P$ and $m_k$ be a $P$-move occurrence in $s$. We prove that the justifier of $m_k$ is in the $P$-view $[m_1 \ldots m_k]$. By commuting pairs as much as required, we can reach a sequence, say $s' = m'_1 \ldots m'_k$, such that $m'_k$ is the move corresponding to $m_k$ in $s$ and $m'_{2i} \sqcap m'_{2i+1}$ for every $2i \leq l$. This means that $[m'_1 \ldots m'_l] = m'_1 \ldots m'_l$ and hence the justifier of $m'_k$ is in the view. Since P-visibility is preserved and reflected by the commutation of non-interfering blocks, $s$ is P-visible.

Every morphism can be expressed as the prefix embedding followed by commutations. This is insightful and technically useful.

**Lemma 17.** Every $f : s \to t$ in $P_{A,B}$ can be decomposed as $s \xrightarrow{1} t = s \xrightarrow{t_0} t_1 \xrightarrow{t_2} \ldots \xrightarrow{t_m} t_n$, where $n \geq 0$, $t_0 = t$ and $g_i$ is a commutation of adjacent $O$-blocks in $t_{i-1}$ for every $i \in [n]$. (This decomposition is not unique.)

**Proof.** Let $f : s \to t$ and $t = m_1 \ldots m_n$. If $f$ is induced by the prefix, then we complete the proof. Otherwise, there is an odd number $k \leq |s|$ such that either $f(k) = 2 \notin \text{img}(f)$ or $f(k) = f(k') - 2$ for some $k > k'$. Then we claim that $m_f(k) = m_f(k) - 2$ and $m_f(k)f(k) + 1$ in $t$ is a non-interfering pair. Suppose otherwise, i.e. the justifier of $m_f(k)$ is $m_f(k)$. Then $f(k) - 1 \notin \text{img}(f)$ since $f$ preserves the justification point. Let $l' \leq |s|$ be the index such that $f(l') = f(k') - 1$. Since $s(l') \sqcap s(k)$, we have $l' < k$. Because $f$ is even, we have $f(l') = f(k') - 1 = f(k') - 2$. In summary, we have $l < k$ such that $f(l) = f(k) - 2$, that contradicts the assumption. So the adjacent O-blocks $m_f(k) = m_f(k) - 2$ and $m_f(k)f(k) + 1$ in $t$ is non-interfering.

Consider the commutation $h : t \to t'$ and the inverse $h^{-1} : t' \to t$, which is also a commutation. By applying the same argument to $h \circ f : s \to t'$, $h \circ f$ can be decomposed as $g_0 \circ \ldots \circ g_i \circ g_0$, where $g_0$ is induced by the prefix and $g_i (i > 1)$ is a commutation. This inductive argument is justified by the same way as the termination of the bubble sort. Then $f = h^{-1} \circ g_0 \circ \ldots \circ g_i \circ g_0$.

**Remark 18.** Let $\sigma$ be an innocent strategy in the standard sense, i.e. an even-prefix closed subset of plays with a certain condition. Then $s \in \sigma$ and $f : s' \to s$ in $P_{A,B}$ implies $s' \in \sigma$. To see this, observe that a commutation of $s \in \sigma$ is in $\sigma$ and use Lemma 17.

2.3 **Topology of $P_{A,B}$**

As for the innocent strategies $\sigma$ for deterministic calculi, which is a set of plays, a play $s = m_1 \ldots m_k$ is in the strategy $\sigma$ iff $P$-views for (even-)prefixes are in $\sigma$, i.e. $\{m_1 \ldots m_k \ | \ k = 2, 4, \ldots \} \subseteq \sigma$. We use the Grothendieck topology to capture this condition.

**Definition 19** (Covering family / sieve). A family of morphisms $(f_x : s_x \to s)_{x \in E}$ is said to cover $s$ when they are jointly surjective, i.e. $\bigcup_{x \in E} \text{img}(f_x) = [n]$, where $n$ is the length of $s$. A covering sieve is a sieve that is a covering family. By abuse of notation, we write $P_{A,B}$ for the site associated with this topology.

**Example 20.** (i) For a play $s = m_1 \ldots m_n$, the family $(f : (m_1 \ldots m_{n-2}) \to s, g : [s] \to s)$ is a covering family. Here $f$ is induced by the prefix and $g$ by the P-view (see Example 11).

(ii) For a play $s = m_1 \ldots m_n$, the family $(f_k : [m_1 \ldots m_k] \to s)_{k \in \{2, \ldots , n\}}$ is a covering family. Here $f_k$ is the composite of the $P$-view embedding and the prefix embedding, i.e., $[m_1 \ldots m_k] \xrightarrow{f_k} s = [m_1 \ldots m_k] \longrightarrow (m_1 \ldots m_k) \longrightarrow s$.

The covering family generalises the set of $P$-views of the prefixes. (iii) The covering family is finer than the set of $P$-views. Let $s = m_1m_2m_1m_2$ (the repetition of $m_1m_2$ twice). Then $(f : m_1m_2 \to s)$, where $f(1) = 1$ and $f(2) = 2$, is not a covering family. However $(f : m_1m_2 \to s, g : m_1m_2 \to s)$, where $g(1) = 3$ and $g(2) = 4$, is a covering family. Notice that those two families have the same set of the domain, say $(m_1m_2)$, which is the set of $P$-views of $s$.

**Definition 21.** An innocent strategy is a sieve over $P_{A,B}$.

**Remark 22.** Let $\sigma$ be a functor $\mathbb{P}^p_{A,B} \to \mathbb{S}et$. It is pre-deterministic if $\sigma(s)$ is empty or singleton for every $s$. A pre-deterministic functor can be determined by the set $P_\sigma = \{s \in P_{A,B} \mid \sigma(s) \neq \emptyset \}$. Since $\sigma$ is a functor, the set $P_\sigma$ is lower closed, i.e. $s \in P_\sigma$ and $f : s' \to s$ in $P_{A,B}$ implies $s' \in \sigma$. A pre-deterministic functor $\sigma$ is a sieve if $s = m_1 \ldots m_n \in P_\sigma$ if $\{m_1 \ldots m_k \mid k = 2, 4, \ldots \} \subseteq P_\sigma$. To see this, observe that $(\{m_1 \ldots m_k \to s \mid k = 2, 4, \ldots \})$ is a covering family and the family of unique elements $\{x_k \in \sigma([m_1 \ldots m_k]) \mid k \}$ is a matching family and thus there is an amalgamation $x \in \sigma(s)$. In this sense, for pre-deterministic strategies, the innocence is equivalent to the sieve condition. However, if $\sigma(s)$ may have more than one element, innocence based on the set of views differs from the sieve condition.

2.4 **Sheaves over $P_{A,B}$ and its restriction to $P$-views**

In innocent game models for deterministic calculi (such as [10]), one often considers the restriction of strategies to $P$-views. A remarkable property is that an innocent strategy (qua set of plays) is completely determined by the subset of $P$-views it contains. After all, innocence means view dependence.

In this subsection, we shall see that a similar property holds for sheaves over plays $P_{A,B}$. This property comes from the topological structure of plays: every play is covered by $P$-views (see Example 20(ii)). This observation gives a justification of defining innocent strategies as sheaves.

**Definition 23** (Subcategory of $P$-views). A play $s \in P_{A,B}$ is a $P$-view if $|s| = n$ and $s$ is not empty. We use $p$ as a metariable ranging over $P$-views. The category of $P$-views $\mathbb{V}_{A,B}$ is the full subcategory of $P_{A,B}$ consisting of $P$-views. We write $\iota : \mathbb{V}_{A,B} \hookrightarrow P_{A,B}$ for the embedding. Henceforth we fix the topology for $\mathbb{V}_{A,B}$ to be that induced$^1$ from $P_{A,B}$: it is the trivial topology, i.e. every $P$-view has only one covering sieve, namely, the maximal sieve.

The category of $P$-views is a poset. We write $(p' \leq p)$ and $(p \geq p')$ for the unique morphism $f : p \to p$ (if it exists).

Because the topology is trivial, a sheaf over $\mathbb{V}_{A,B}$ is just a functor $\mathbb{V}_{A,B} \to \mathbb{S}et$. A sheaf $\sigma \in \mathbb{Sh}(\mathbb{V}_{A,B})$ induces a sheaf $\sigma \circ \iota$ over $P_{A,B}$. The strategy $\sigma$ can be reconstructed from the restriction to $P$-views $\sigma \circ \iota$ (up to natural isomorphism).

**Lemma 24** (Comparison). The functor $\iota' : \mathbb{Sh}(\mathbb{V}_{A,B}) \to \mathbb{Sh}(P_{A,B})$ induces an equivalence of categories.

Since every play has a covering by $P$-views, Lemma 24 follows from a standard result, known as the Comparison Lemma [19] (see, for example, [3, Prop. p. 721]) which generalises the classical

---

$^1$ Given a site $C$ and a full subcategory $\mathbb{D} \hookrightarrow C$, the induced topology on $\mathbb{D}$ is defined by: a sieve $S$ on $\mathbb{D}$ is covering iff the sieve $(S) := \{f \circ h \mid f \in S, \text{dom}(f) = \text{codom}(h)\}$ on $C$ generated from $S$ is covering.
result in SGA4). However an explicit description of the adjoint $i_* : \text{Sh}(\mathcal{V}_{\mathbb{A},B}) \to \text{Sh}(\mathcal{P}_{\mathbb{A},B})$ is insightful and worth clarifying.

Let $\tau \in \text{Sh}(\mathcal{V}_{\mathbb{A},B})$ be a sheaf over P-views. Let $s = m_1 \ldots m_n$ be a non-empty play and $p_k := [m_1 \ldots m_k]$ for every even number $k$. We define a set of $\tau$-annotations for a $\tau$-annotation is a sequence $e_2 e_4 \ldots e_n$, where $e_k \in \tau(p_k)$ for every even number $k$, such that the following condition: for every even number $k \leq n$, if $m_k \cap m_{k-1}$, then $e_k = \tau(p_k) \lor (p_k)$. For a non-empty play $s \in \mathcal{P}_{\mathbb{A},B}$, we write $(i_* \tau)(s)$ for the set of all $\tau$-annotations.

Given $s : s \to s'$, which is an injective map $f : \{s \} \to \{s'\}$, the morphism $(i_*(\tau))(f) : (i_* \tau)(s) \to (i_* \tau)(s')$ is defined by:

$$(i_*(\tau))(f) : e_2 e_4 \ldots e_n \mapsto e_f(e_2 e_4 \ldots e_n).$$

We define $(i_* \tau)(s) := \{ * \}$ for the empty play. Then $i_* \tau : \mathcal{P}_{\mathbb{A},B} \to \mathbb{A}$ is a functor.

**Example 25.** Consider an arena pair $\left(\{ d \} \to \{ d' \} \to \{ c \} \to \{ b \}, \{ a \} \right)$ and let $p_0 = a b, p_1 = a b c d$ and $p_2 = a b c d'$ (in which every move is justified by its predecessor). Define $\tau_1, \tau_2 : \text{Sh}(\mathcal{V}_{\mathbb{A},B}) \to \{0, 1\}$ as follows:

$$\begin{align*}
\tau_1(p_0) &= \{ x_1 \}, \\
\tau_1(p_0) &= \{ x_21, x_22 \}, \\
\tau_1(p_2) &= \{ z_1 \}, \\
\tau_1(f)(y_1) &= x_1, \\
\tau_1(g)(z_1) &= x_1, \\
\tau_2(p_0) &= \{ y_1 \}, \\
\tau_2(p_0) &= \{ y_21 \}, \\
\tau_2(p_2) &= \{ z_1 \}, \\
\tau_2(f)(y_1) &= x_21, \\
\tau_2(g)(z_1) &= x_22,
\end{align*}$$

Thus $f : (a b) \to (a b c d)$ and $g : (a b) \to (a b c d').$ Then

$$(i_* \tau_1)(a b c d c') = \{ x_1 y_1 z_1 \}, (i_* \tau_2)(a b c d c') = \{ \}.\$$

We write $P_\mathbb{A} := \{ s \in \mathcal{P}_{\mathbb{A},B} \mid \sigma(s) \neq 0 \}$ and $V_\mathbb{A} := \{ s \in \mathcal{V}_{\mathbb{A},B} \mid \sigma(s) \neq 0 \}$. Then $V_{\mathbb{A}} = V_\mathbb{A}$ but $P_\mathbb{A} \neq P_\mathbb{A}$. The set-of-views approach fails to distinguish $\tau_1$ from $\tau_2$.

**Proposition 26.** $i_* \tau$ is a sheaf for every $\tau \in \text{Sh}(\mathcal{V}_{\mathbb{A},B})$.

**Proof.** Let $S = \{ f_\mathbb{A} : s \to s \}$ be a covering sieve and $\{ x_1 \in (i_* \tau)(s) \}$ be a matching family. Each $x_1$ is a $\tau$-annotation for an annotation $e_2 e_4 \ldots e_n$. It suffices to give an annotation $e_2 e_4 \ldots e_n$ for $s$ (here $n = |s|$). Let $k \leq n$ be an even number. Since $S$ is a covering sieve, it must be jointly surjective, i.e., $k \in \text{img}(f_\mathbb{A})$ for some $\xi$. Then $f_\mathbb{A}(k) = k$, we define $e_k = e_\xi(k)$. This does not depend on the choice of $\xi$ since $x_1$ is a matching family. The resulting sequence $e_2 \ldots e_n$ satisfies the required conditions. The uniqueness is trivial.

**Proposition 27.** $i_*$ and $i^*$ form an adjoint equivalence.

**Proof.** Let $\tau \in \text{Sh}(\mathcal{V}_{\mathbb{A},B})$. For a P-view $p = m_1 \ldots m_n$, an annotation $a_2 a_4 \ldots a_n \in (i_* \tau)(p)$ is uniquely determined by $a_n$, since $a_n = \tau(f_k)(a_n)$ for the unique $f_k : \{ m_1 \ldots m_k \} \to \{ m_1 \ldots m_n \}$. This gives a bijection $\psi_p$ for each $p$ from $\tau(p)$ to $(i_* \tau)(p)$, and to $(i^* \tau)(p)$ through $(i^* \tau)(p) = (i_* \tau)(p)$.

For the other direction, let $s \in \text{Sh}(\mathcal{P}_{\mathbb{A},B})$. Let $s = m_1 \ldots m_n$ be a play. Then $x \in (i^* \tau)(s)$ is a sequence $e_2 e_4 \ldots e_n$ such that, for every even number $k \leq n$, $e_k \in \tau([m_1 \ldots m_k])$ and $e_1 = \tau(f_k)(e_k)$ if $m_k \cap m_{k-1}$. This means that $(\{ a \})_{k \in \{2, 4, \ldots, n\}}$ is a matching family of $\{ m_1 \ldots m_n \} \to s_{k \in \{2, 4, \ldots, n\}}$. Since $\tau$ is a sheaf, there exists a bijection $\varphi_s$ from $(i^* \tau)(s)$ to $\tau(s)$.

It is easy to see that $(i_*, i^*, \tau, \varphi)$ is an adjunction.

**3. Interaction and composition**

This section introduces the notion of interaction sequences and defines the composition $(\sigma_1, \sigma_2) \in \text{Sh}(\mathcal{P}_{\mathbb{A},C})$ of sheaves $\sigma_1 \in \text{Sh}(\mathcal{P}_{\mathbb{A},B})$ and $\sigma_2 \in \text{Sh}(\mathcal{P}_{\mathbb{B},C})$, generalising the composition of deterministic innocent strategies as in [10]. The composition is associative up to isomorphism, and the arenas and sheaves form a CCC (where isomorphic sheaves are identified).

**3.1 Interaction sequences**

**Definition 28** (Justified sequence). Let $(A, B, C)$ be a triple of arenas. The enabling relation $\vdash_{A, B, C}$ for the triple is defined by:

- For $X \in \{ A, B, C \}$, if $m \vdash_{X} m'$, then $m \vdash_{A, B, C} m'$.
- If $\vdash_{A} m \in M_{C}$, then $\vdash_{A, B, C} m$.
- If $\vdash_{B} m \in M_{C}$ and $\vdash_{B} m' \in M_{B}$, then $m \vdash_{A, B, C} m'$.
- If $\vdash_{B} m \in M_{B}$ and $\vdash_{A} m \in M_{A}$, then $m \vdash_{A, B, C} m'$.

A justified sequence of the triple is a sequence over $M_{A} + M_{B} + M_{C}$ equipped with justification pointers that respect the enabling relation $\vdash_{A, B, C}$.

A justified sequence $s$ of a triple $(A, B, C)$ induces justified sequences of $(A, B), (B, C)$ and $(A, C)$, basically by the restriction of moves. The projection to the component $(B, C)$, written $s |_{B, C}$, is just the restriction. The projection to the component $(C, B)$, written $s |_{C, B}$, is the restriction to moves in $M_{B}$, in which $\star \in M$ for an initial B-move $m$ (whereas $m' \not\in m$ for an initial C-move $m'$). The projection to the component $(A, C)$, written $s |_{A, C}$, is the restriction to moves in $M_{A}$, in which an initial A-move $m$ is justified by the move $m'$ such that $m' \not\in m$ and $m' \not\in m$ (so $m''$ is an initial B-move and $m'$ an initial C-move). A

**Definition 29** (Interaction sequence), Let $(A, B, C)$ be a triple of arenas. A justified sequence $s$ over $(A, B, C)$ is an interaction sequence if

- The last move is in $M_{A} + M_{B} + M_{C}$,
- $s |_{A, B}$ and $s |_{B, C}$ are plays of $(A, B)$ and $(B, C)$, respectively.

**Switching condition and basic blocks** Before defining the morphisms between interaction sequences, we introduce a useful tool to analyse the interaction sequences.

**Definition 30** (Switching condition), Let $(A, B, C)$ be a triple of arenas. A sequence over $M_{A} + M_{B} + M_{C}$ is said to satisfy the switching condition if it is accepted by the following automaton with the initial state $OOO$ of which all states are accepting.

A state express the owners of the next moves for components $(A, B)$, $(B, C)$ and $(A, C)$ in this order.

The switching condition generalises the O/P-alternation of justified sequences for a pair $(A, B)$.

**Lemma 31.** Interaction sequences satisfy the switching condition.

**Proof.** Observe that each state of the automaton is determined by the first two components. Thus the O/P-alternation for $(A, B)$ and $(B, C)$ components suffice for the switching condition.

Recall that basic constituents of plays are pairs of consecutive O/P-move occurrences, called O/P-blocks. Thanks to the switching condition (Lemma 31), we know that interaction sequences consist of what we shall call basic blocks: a basic block is a sequence of consecutive move occurrences in the interaction sequence, starting from a move in $M_{A}$ and ending with a move in $M_{A}$, possibly having moves in $M_{B}$ as intermediate moves.
The category of interaction sequences Given a triple (A, B, C), a generalised P-move is a move in $\mathcal{M}_A^A + \mathcal{M}_B + \mathcal{M}_C^C$. This can be written as $\mathcal{M}_A^{rA} + \mathcal{M}_B + \mathcal{M}_C^c$. An generalised O-move is a move in $\mathcal{M}_A^A + \mathcal{M}_B + \mathcal{M}_C^C$.

Definition 32. Let (A, B, C) be a triple of arenas and s, s′ be interaction sequences over (A, B, C). Suppose that $s = m_1 \ldots m_n$ and $s′ = m'_1 \ldots m'_{n'}$. A morphism between s and s′ is an injective map $f : [n] \to [n′]$ which satisfies:

- $m_i = m'_f(k)$ (as moves),
- $m_i \land m_k$ implies $m'_f(i) \land m'_f(k)$ (and similarly for $\land$), and
- if a generalised O-move $m_k$ is followed by $m_{k+1}$, $m'_f(k)$ is followed by $m'_f(k+1)$ (i.e. $f(k+1) = f(k) + 1$).

In other words, a morphism between interaction sequences is an injective map between the respective occurrence-sets that preserve moves, justification pointers and basic blocks.

Definition 33. Given arenas A, B and C, the category of interaction sequences, written as $\mathcal{I}_{A,B,C}$, has interaction sequences as objects and morphisms defined above.

Remark 34. One can introduce the topology to $\mathcal{I}_{A,B,C}$ as follows, though we shall not use them: A family of morphisms $\{f_k : s_k \to s\}_{k \in \mathbb{C}}$ in $\mathcal{I}_{A,B,C}$ is said to cover s if they are jointly surjective, i.e. $\bigcup_{k \in \mathbb{C}} \text{img}(f_k) = [n]$, where n is the length of s.

Projection to (A, C) component The projections of an interaction sequence onto (A, B) and (B, C) components are plays by definition. We show that the projection onto (A, C) component is also a play.

Definition 35 (Commuting an adjacent pair of non-interfering blocks). Let $u$ be an interaction sequence of (A, B, C). Let

$$m_1 v_1 m'_1 m_2 v_2 m'_2$$

be an adjacent pair of basic blocks in u, where $m_1$ and $m_2$ are moves in $\mathcal{M}_A^{rA}$, $m'_1$ and $m'_2$ are moves in $\mathcal{M}_C^c$, and $v_1$ and $v_2$ are sequences of moves in $\mathcal{M}_B$; i.e. $u = u_0 m_1 v_1 m'_1 m_2 v_2 m'_2 u_1$. We say that the pair of basic blocks are non-interfering if the justifier of $m_2$ is not $m_1$. The commuted sequence $u′$ is defined by $u′ := u_0 m_2 v_2 m'_2 m_1 v_1 m'_1 u_1$ (in which the justification pointers are modified accordingly).

Lemma 36. Let $u$ be an interaction sequence of (A, B, C) and let $v$ be obtained from $u$ by commuting an adjacent pair of non-interfering blocks. Then $v$ is an interaction sequence.

Proof. Let $u = s′ t_1 t_2 s′′$ and $v = s′ t_2 t_1 s′′$, where $t_1$ and $t_2$ are non-interfering basic blocks, i.e. the justifier of the first move in $t_2$ is not the last move in $t_1$. Let $t_2 = m_1 \ldots m_k$. We prove the following claim:

Let $m_i$ be a move in $t_2$. Then the justifier of $m_i$ is in $t_1$.

We prove this by induction on i.

We prove the base case $i = 1$. Since $m_1 \in \mathcal{M}_A^{rA}$, by the definition of the basic block, its justifier is in $\mathcal{M}_C^c$. Because $t_1$ is a basic block, the unique move in $\mathcal{M}_A^{rA}$ is the last move. By the assumption the justifier of $m_1$ differs from the last move of $t_1$, as desired.

We prove the induction step. Let $m_i$ be a move in $t_2$ ($i > 1$). Then $m_i$ is either in $\mathcal{M}_B^c$ or in $\mathcal{M}_A^{rA}$. Suppose that $m_i \in \mathcal{M}_B^c$. Since u is an interaction sequence, $u|_{[rA]}$ is a play. In particular the justifier of $m_i$ is in $[s′ t_1 t_1 \ldots m_i]_{[rA]}$. Let $m_1 \ldots m_i$ be the P-view. We show that no move in this sequence is in $t_1$. First $m_i = m_i$ and its immediate predecessor $m_{i-1}$ are in $t_2$. The preceding move $m_{i-2}$ is pointed by $m_{i-1}$, so by the induction hypothesis, $m_{i-2}$ is not in $t_1$. If $m_{i-2}$ is in $s′$, then all preceding moves are in $s′$. If $m_{i-2}$ is in $t_2$, by iterating the same argument, we conclude that $m_1 \ldots m_i$ does not contain moves in $t_1$. Since $u|_{[rA]}$ is a play, its justifier is in its P-view. Hence not a move in $t_2$.

We prove that $u|_{[rB]}$ is a play, using the above claim. Notice that $u|_{[rB]}$ is obtained by commuting adjacent O-P blocks in $u|_{[rB]}$ as much as required. The above claim implies that every O-P block in $t_1|_{[rB]}$ does not interfere to any O-P block in $t_2|_{[rB]}$. Since commutation of non-interfering O-P blocks preserves P-visibility, $v|_{[rB]}$ is a play. Similarly $v|_{[rA]}$ is a play.

\[\square\]

Lemma 37. For every interaction sequence $u$ of (A, B, C), the projection $u|_{[rA,C]}$ is a play.

Proof. Let $u$ be an interaction sequence of (A, B, C). We define the set $P$ of interaction sequences as the least set that satisfies (1) $u \in P$, and (2) if $v \in P$ and $v'$ is obtained from $v$ by commuting a non-interfering basic blocks, then $v' \in P$. In (2), $v'$ is an interaction sequence by Lemma 36. Consider $P\{A,C := \{v|_{[rA,C]} : v \in P\}$. This is a set of alternating justified sequences of (A, C) that is closed under the commutations. By Lemma 16, each element in $P\{A,C$ is a play. So $u|_{[rA,C]}$ is a play.

\[\square\]

Projection as functors Given an interaction sequence $u \in \mathcal{I}_{A,B,C}$, the projections $u|_{[rA,B]}$, $u|_{[rB,C]}$ and $u|_{[rA,C]}$ are plays of (A, B), (B, C) and (A, C), respectively. Those projections are naturally extended to functors: given interaction sequences $u, v \in \mathcal{I}_{A,B,C}$ and a morphism $f : u \to v$, the restriction $f|_{[rA,B}$ of $f$ is a morphism $f|_{[rA,B} : u|_{[rA,B} \to v|_{[rA,B}$.

Lemma 38. The projection $[rA,B : \mathcal{I}_{A,B,C} \to \mathcal{P}_{A,B} : [rB,C] \in \mathcal{I}_{A,B,C} \to \mathcal{P}_{B,C}$ and $[rA,C] : \mathcal{I}_{A,B,C} \to \mathcal{P}_{A,C}$ are functors.

Proof. Recall that $u|_{[rA,B}$ is the restriction of u to $I_{\lambda,\lambda}^A : \{i \in [u] | u(i) \in \mathcal{M}_A\}$, which is an injection $f : [u] \to [u′]$ on sets, is mapped to $f|_{[rA,B} : I_{\lambda,\lambda}^A \to I_{\lambda,\lambda}^A$. It is easy to see that this is functorial.

\[\square\]

Lemma 39. Let $f : s \to t$ in $\mathcal{P}_{A,C}$ and $v \in \mathcal{I}_{A,B,C}$ such that $v|_{[rA,C} = t$. Then there exists unique $f)v : u \to v$ in $\mathcal{I}_{A,B,C}$ such that $f|_{[rA,C} = f$.

Proof. Observe that the O-P blocks in t bijectively correspond to the basic blocks in v. Since a morphism $f : s \to t$ is an injective map between O-P blocks, the bijection between O-P blocks and basic blocks determines $f)v : u \to v$. So $f)v$ is unique if it exists. We prove the existence. If $f$ is a commutation, Lemma 36 suffices. If $f$ is an embedding induced by a prefix, existence of $f)v$ is trivial. Lemma 17 says that these cases are enough to prove the claim.

\[\square\]

In other words, $[rA,C] : \mathcal{I}_{A,B,C} \to \mathcal{P}_{A,C}$ is a fibration of which each fibre is a discrete category. We write $f^*(v)$ for the object $u$ in the lemma and $f)v$ for the morphism.

3.2 Composition

Let $\sigma_1 \in \text{Sh}(\mathcal{P}_{A,B})$ and $\sigma_2 \in \text{Sh}(\mathcal{P}_{B,C})$ be sheaves. We define the composite $(\sigma_1; \sigma_2) : \mathcal{P}_{A,C} \to \text{Sh}(\mathcal{P}_{A,B})$, which shall be proved to be a sheaf. For a play s $\in \mathcal{P}_{A,C}$, the set $(\sigma_1; \sigma_2)(s)$ is defined by

$$(\sigma_1; \sigma_2)(s) := \bigcup_{u \in \mathcal{I}_{A,B,C} : u|_{[rA,C} = s} \sigma_1(u|_{[rA,B}) \times \sigma_2(u|_{[rB,C})$.

So an element in $(\sigma_1; \sigma_2)(s)$ is represented by a triple $(u, e_1, e_2)$, where $u \in \mathcal{I}_{A,B,C}$ such that $u|_{[rA,C} = s$, $e_1 \in \sigma_1(u|_{[rA,B})$ and $e_2 \in \sigma_2(u|_{[rB,C})$. It follows that $(\sigma_1; \sigma_2)(s)$ is a play.
For a morphism \( f : s \to t \in \mathbb{P}_{A,C} \), \((\sigma_1; \sigma_2)(f)\) is a function given by

\[
(\sigma_1; \sigma_2)(f)(u) = (f^*(u), \sigma_1(f_*u_{A,B}), \sigma_2(f_*u_{B,C})).
\]

In the preceding, we use the common notation \( x \cdot f \) to mean \( F(f)(x) \) where \( F : \mathcal{C}^{op} \to \mathcal{Set} \), \( f : s \to t \) is a morphism of \( \mathcal{C} \), and \( x \in F(t) \). By this notation, the second component can be written as \( \epsilon_1 \cdot (f_*u_{A,B}) \) and the third component as \( \epsilon_2 \cdot (f_*u_{B,C}) \).

Categorically, the composite is the left Kan extension.

**Lemma 40.** Assume \( \sigma_1 \in \text{Sh}(\mathbb{P}_{A,B}) \) and \( \sigma_2 \in \text{Sh}(\mathbb{P}_{B,C}) \). Let \( F : \mathbb{P}_{A,B} \to \mathcal{Set} \) be a functor defined by \( F(u) := \sigma_1(u_{A,B}) \times \sigma_2(u_{B,C}) \). Then the composite \((\sigma_1; \sigma_2) \) is the left Kan extension of \( F \) along the projection \( \pi : \mathbb{P}_{A,B} \to \mathbb{P}_{A,C} \).

**Proof.** The universal natural transformation \( \alpha : F \to (\sigma_1; \sigma_2) \circ \pi \) is given by

\[
\alpha_u : F(u) \ni (e_1, e_2) \mapsto (u(e_1), e_2) \in (\sigma_1; \sigma_2)(\pi(u)).
\]

Assume a functor \( H : \mathbb{P}_{A,C} \to \mathcal{Set} \) and a natural transformation \( \beta : F \to H \circ \pi \). For each \( u \in \mathbb{P}_{A,B,C} \), define \( \beta_u : F(u) \to H(\pi(u)) \). Then \( \gamma_u := (\sigma_1; \sigma_2)(\beta_u) \) is defined by

\[
\gamma_u(u, e_1, e_2) := \beta_u(e_1, e_2)
\]

(recall that \( (\sigma_1; \sigma_2)(s) = [\pi(s) \mapsto \sigma_1(u_{A,B}) \times \sigma_2(u_{B,C})] \)). Then \( \gamma_u \) is natural and \( \alpha_u \circ \gamma_u = \beta_u \) for all \( u \). Uniqueness of \( \gamma_u \) comes from the universal property of coproducts.

**Remark 41.** In the traditional set-theoretic HO/N game semantics, the composite of strategies \( P_{A,B} \) and \( P_{B,C} \) (i.e., even-prefix closed subsets of plays over \( A,B \) and over \( B,C \), respectively) is defined by \((P_{A,B}; P_{B,C}) := \{ s \in \mathbb{P}_{A,C} \mid \exists u \in \mathbb{P}_{A,B,C} \text{ such that } u_{A,B} \in P_{A,B} \text{ and } u_{B,C} \in P_{B,C} \} \). Our composition satisfies \((P_{\sigma_1}); (P_{\sigma_2}) = P_{(\sigma_1; \sigma_2)} \), where \( P_\emptyset = \{ s \mid \sigma(s) \neq \emptyset \} \).

The composite of sheaves is again a sheaf.

**Theorem 42.** Let \( \sigma_1 \in \text{Sh}(\mathbb{P}_{A,B}) \) and \( \sigma_2 \in \text{Sh}(\mathbb{P}_{B,C}) \) be sheaves. Then \( (\sigma_1; \sigma_2) \) is a sheaf over \( \mathbb{P}_{A,C} \).

**Proof.** Let \( s = m_1 \ldots m_n \in \mathbb{P}_{A,C} \) be a play, \( \{ f : s_f \to s \}_{f \in S} \in J(s) \) be a covering sieve and \( \{ x_f \in (\sigma_1; \sigma_2)(s_f) \}_{f \in S} \) be a matching family. By the definition of \( (\sigma_1; \sigma_2) \), we have

\[
x_f = (u_f, y_f, z_f) \in \bigcup_u \sigma_1(u_{A,B}) \times \sigma_2(u_{B,C}).
\]

We claim that there exists \( u \) such that:

- \( u_{A,C} = s \), and
- \( u_f = f^*(u) \) for every \( f \in S \).

If such \( u \) exists, there is a bijective correspondence between basic blocks of \( u \) and O-P blocks of \( s \). This correspondence tells us the start and the last moves of each block. So it suffices to fill the intermediate B-moves for each basic block. Consider the \( k \)th basic block. Since \( S \) is a covering sieve, we have a morphism \( f : s_f \to s \in S \) such that \( 2k \in \text{img}(f) \) (recall that \( k \)th O-P block is \( m_{2k-1} \cdot m_{2k} \)). Let \( l \) be the index such that \( f(l) = 2k \). Recall that \( x_f = (u_f, y_f, z_f) \) with \( u_f(A,C) = s_f \). Then the basic block of \( u_f \) corresponding to the O-P block \( m_{2k-1} \cdot m_{2k} \) in \( s_f = m_1 \ldots m_{|s_f|} \) tells us the \( k \)th basic block of \( u \). This is independent of the choice of \( f \) since \( \{ f \}_{f \in S} \) is a matching family. Now by the construction, \( u_f = f^*(u) \).

Then we have a family \( T := \{ f_u : f^*(u) \to u \}_{f \in S} \). This family is jointly surjective, i.e., \( \bigcup_{f \in S} \text{img}(f_u) = [u] \), since \( S \) is jointly surjective on O-P blocks of \( s \), which bijectively correspond to basic blocks of \( u \). Hence \( T_{B,C} := \{ f_u \mid f \in S \} \) and \( T_{A,B} := \{ f_u \mid f \in S \} \) are covering families and \( \{ y \}_{f \in S} \) and \( \{ z_f \}_{f \in S} \) are matching families of them. Hence there exist amalgamations \( x \in \sigma_1(u_{A,B}) \) and \( y \in \sigma_2(u_{B,C}) \) such that \( (\sigma_1; \sigma_2)(s) \) is the amalgamation.

The uniqueness of \( u \) follows from the construction and the amalgamations \( x \) and \( y \) are unique since \( \sigma_1 \) and \( \sigma_2 \) are sheaves.

3.3 Associativity

The associativity of composition (up to natural isomorphism) is proved by studying “generalised” interaction sequences \( \|_{A,B,C,D} \) that have two internal components. This is a standard technique.

**Definition 43.** Given a quadruple \((A,B,C,D)\) of arenas, the enabling relation \( \|_{A,B,C,D} \) on \( \mathcal{M}_{A,B,C,D} := \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C + \mathcal{M}_D \) is defined by:

- If \( m \vdash X m' \) for some \( X \in \{A,B,C,D\} \), then \( m \vdash_{A,B,C,D} m' \).
- If \( \star \vdash m \) and \( \star \vdash m' \), then \( m \vdash_{A,B,C,D} m' \).
- If \( \star \vdash m \) and \( \star \vdash m' \), then \( m \vdash_{A,B,C,D} m' \).
- If \( m \vdash_{A,B,C,D} m' \).

A justified sequence over \((A,B,C,D)\) is a sequence of \( \mathcal{M}_{A,B,C,D} \) equipped with pointers that respect \( \|_{A,B,C,D} \). Given a justified sequence \( w \) over \((A,B,C,D)\), the projections \( w_{A,B} \to \text{interaction sequences and } w_{B,C} \) onto plays are defined in the obvious way. A justified sequence over \((A,B,C,D)\) is an interaction sequence if \( w_{A,B} \) and \( w_{B,C} \) are plays and its last move is in \( \mathcal{M}_D = \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_D \).

**Definition 44 (Switching condition).** Let \((A,B,C,D)\) be a quadruple of arenas and \( s \) be a sequence over \( \mathcal{M}_{A,B,C,D} \). It satisfies the switching condition if it is accepted by the following automaton from the initial state \( OOO \) (all states are accepting).

The three components of states correspond to \((A,B)\), \((B,C)\) and \((C,D)\) in this order.

**Lemma 45.** Every interaction sequence over \((A,B,C,D)\) satisfies the switching condition.

**Proof.** This is because the automaton checks if each component is O-P alternating.

A basic block consists of the start move in \( \mathcal{M}_A \), the last move in \( \mathcal{M}_B \) and intermediate moves in \( \mathcal{M}_B \). An morphism \( f : w \to w' \) between interaction sequences over \((A,B,C,D)\) is an injective map between move occurrences that preserve moves, the justification pointers and basic blocks. We write \( \|_{A,B,C,D} \) for the category of generalised interaction sequences.

**Lemma 46.**

- Projections from \( \|_{A,B,C,D} \) are functors.
- Composition of projections is a projection, e.g.,
  \[
  \|_{A,B,C,D} \to \|_{A,B,C} \to \|_{B,C} = \|_{A,B,C,D} \to \|_{B,C}.
  \]
- The projection \( \|_{A,D} : \|_{A,B,C,D} \to \|_{A,D} \) is a discrete fibration.
Proof. The first two claims are easy to see. The third claim can be proved by the same technique to Lemma 39.

Lemma 47. Let \( u \in I_{A,B,D} \) and \( v \in I_{B,C,D} \). If \( (w[A,B,D]) = (v[B,C,D]) \), there exists a unique \( w \in I_{A,B,C,D} \) such that \( u = w[A,B,D] \) and \( v = w[B,C,D] \). A similar statement holds for every \( u \in I_{A,C,D} \) and \( v \in I_{A,B,C} \).

Proof. Let \( u = m_1 \ldots m_M \in I_{A,B,D} \) and \( v = n_1 \ldots n_M \in I_{B,C,D} \), and suppose that \( u[B,D](w) = \pi_B[D](w) \). We construct \( w \in I_1 \ldots I_L \in I_{A,B,C,D} \). By the switching condition, \( u \) and \( v \) must be accepted by the left and right automata, respectively.

\[
\begin{array}{c}
| & M_A' \quad M_B' \quad M_C' \quad M_D' | \quad r_1 \quad r_2 \quad r_3 \quad r_4 |
\end{array}
\]

We construct a sequence of moves \( w \) such that \( w[A,B,D] = u \) and \( w[B,C,D] = v \). An intermediate state is a tuple \((i,j,k,l,p,q,r)\) such that \( i \leq M, j \leq N \) such that \( m_1 \ldots m_M | B,D = n_1 \ldots n_N | B,D \) is the current index of \( l \) and \( p,q \) are states of the above automata from which \( m_1 \ldots m_M, n_1 \ldots n_N \) and \( l_1 \ldots l_k \) are accepted, respectively.

- \((i,j,k,q_1,p_1,r_1)\): Then \( m_i \in M_{D}^0 + M_{A}^0 \). If \( m_i \in M_{D}^0 \), then let \( l_k = m_i = n_j \) and proceed to \((i+1,j+1,k+1,q_2,p_2,r_2)\). If \( m_i \in M_{A}^0 \), then let \( l_k = m_i \) and proceed to \((i+1,j,k+1,q_1,p_1,r_4)\).
- \((i,j,l,q_2,p_2,r_2)\): Then \( n_j \in M_{D}^0 + M_{B}^0 \). If \( n_j \in M_{D}^0 \), then let \( l_k = n_j \) and proceed to \((i,j+1,k+1,q_2,p_3,r_3)\). If \( n_j \in M_{B}^0 \), then let \( l_k = n_j = m_i \) and proceed to \((i+1,j+1,k+1,q_1,p_1,r_4)\).
- \((i,j,l,q_3,p_3,r_3)\): Then \( n_i \in M_{B}^0 + M_{C}^0 \). If \( n_i \in M_{B}^0 \), then let \( l_k = n_i \) and proceed to \((i+1,j+1,k+1,q_3,p_1,r_4)\). If \( n_i \in M_{C}^0 \), then let \( l_k = n_i = m_i \) and proceed to \((i+1,j,k+1,q_1,p_2,r_2)\).
- \((i,j,l,q_3,p_3,r_4)\): Then \( m_i \in M_{C}^0 + M_{A}^0 \). If \( m_i \in M_{C}^0 \), then let \( l_k = m_i \) and proceed to \((i+1,j,k+1,q_1,p_1,r_1)\). If \( m_i \in M_{A}^0 \), then let \( l_k = m_i = n_j \) and proceed to \((i+1,j+1,k+1,q_2,p_3,r_3)\).
- Other cases are never reached.

The justification pointer for \( A \)-moves are determined by \( u \) and others by \( v \).

Lemma 48. The simultaneous composition is naturally isomorphic to sequential compositions \( \sigma_1(\sigma_2; \sigma_3) \) and \( (\sigma_1; \sigma_2) \sigma_3 \).

Proof. Given \( s \in \mathcal{P}_{A,D} \), consider a function \( \psi_s \) that maps an element \((w,e_1,e_2,e_3)\) of

\[
\prod_{w,\pi_B[D](w) = s} \sigma_1(w[A,B]) \times \sigma_2(w[B,C]) \times \sigma_3(w[C,D])
\]

to \((w[A,B,D],e_1,((w[B,C,D],e_2,e_3))\)

\[
\prod_{w,\pi_B[D](w) = s} \sigma_1(w[A,B]) \times \sigma_2(v[B,C]) \times \sigma_3(v[C,D]).
\]

This is a bijection thanks to Lemma 47. It is easy to show the naturality of \( \psi \).

Let us write \( F \) for the simultaneous composition and \( G = (\sigma_1; \sigma_2; \sigma_3) \). Assume \( f : s \mapsto t \) in \( \mathcal{P}_{A,D} \). Then \( F(f) ; \psi_s \) maps \((w,e_1,e_2,e_3)\) to

\[
(w,e_1,e_2,e_3)
\]

\[
G(f) \psi_s \left( \left( w[A,B,D], e_1, (w[B,C,D], e_2), e_3 \right) \right)
\]

\[
= (f(w[A,B,D]), e_1, (f(w[B,C,D]), e_2), e_3)
\]

\[
= (f(w[A,B,D]), e_1, (f(w[B,C,D]), e_2), e_3)
\]

\[
= ((f(w[A,B,D]), b[D], \pi_B[D](w)), (f(w[B,C,D]), e_2), e_3).
\]

By Lemma 49, we have \( f^*(w)[A,B,D] = f^*(w)[A,B,D] \), so the first components coincide. As for the second components, again by Lemma 49, we have

\[
(f_w[A,B,D])_{x} = (f_w[A,B,D])_{x}.
\]

For the third components, recall that

\[
(f_w[A,B,D])[B,D] = (f_w[A,B,D])[B,D],
\]

and \( f_{w}[B,D] : (f^*(w)[B,D]) \mapsto (w[B,D]) \). Since \( f_{w}[B,C,D] = (f^*(w)[B,C,D]) \mapsto (w[B,C,D]) \) is projected onto \( (f_{w}[B,D]) \), we have

\[
((f_{w}[B,D])^*(w[B,C,D])) = f^*(w)[B,C,D].
\]

For the fourth components, by using Lemma 49, we have

\[
(f_{w}[A,B,D][B,D])_{w}[B,C,D] = (f_{w}[B,C,D][B,D])_{w}[B,C,D],
\]

\[
= (f_{w}[B,C,D][B,C])_{w}[B,C,D],
\]

\[
= (f_{w}[B,C,D][B,C] = f_{w}[B,C,D].
\]

(As desired. The fifth component is the same.)

Lemma 49. Let \( u \in I_{A,B,C,D} \) and \( f : s \mapsto (w[A,D]) \) in \( \mathcal{P}_{A,D} \). Then

\[
f^*(w)[A,B,D] = f^*(w)[A,B,D] = f_w[A,B,D].
\]
Proof. By definition, \( f_w : f^*(w) \mapsto w \) in \( \mathcal{I}_{A,B,D} \). Thus

\[
\tilde{f}_w|_{A,B,D} : (f^*(w)|_{A,B,D}) \mapsto (w|_{A,B,D}).
\]

Both claims follow from \( \tilde{f}_w|_{A,B,D}|_{A,D} = \tilde{f}_w|_{A,D} = f \). □

**Corollary 50.** Composition is associative up to isomorphism.

### 3.4 CCC of arenas and strategies

**Definition 51.** The category of arenas and strategies \( \mathcal{G} \) has arenas as objects and a sheaf \( \sigma \in \text{Sh}(\mathbb{P}_A,B) \) as a morphism from \( A \) to \( B \). We regard that isomorphic sheaves define the same morphism. The composition is defined in Section 3.2.

As usual, the identity morphisms are copycat strategies.

**Definition 52.** Let \( A \) be an arena. Let us write a move in \( \mathcal{M}_{A,A} = \mathcal{M}_A + \mathcal{M}_A \) as \( l(m) \) and \( r(m) \) for \( m \in \mathcal{M}_A \), in order to distinguish the component. The relation \( \sim \) is given by \( l(m) \sim l(m) \) and \( r(m) \sim r(m) \) (i.e., \( \sim \) relates the same move in the different component). A play \( s = m_1 \ldots m_n \in \mathcal{P}_A \) is copycat if, for every even number \( k < n \), \( m_k \sim m_{k-1} \) implies \( m_{k+1} \sim m_k \) and \( m_{k+2} \sim m_{k+1} \) implies \( m_{k+1} \sim m_k \). The copycat strategy \( \text{id}_A \in \text{Sh}(\mathbb{P}_A,B) \) is defined by: \( \text{id}_A(s) = \{ s \} \) if \( s \) is copycat and \( \text{id}_A(s) = \emptyset \) otherwise.

**Proposition 53.** \( (\text{id}_A; \sigma) \cong (\sigma; \text{id}_A) \forall \sigma \in \text{Sh}(\mathbb{P}_A,B) \).

In the rest of this subsection, we show that \( \mathcal{G} \) is a CCC. It is an adaptation of the standard arguments for HO/N game models.

**Products and terminal object**

Given arenas \( A \) and \( B \), the arena \( A \times B \) is defined by: \( \mathcal{M}_{A \times B} := \mathcal{M}_A + \mathcal{M}_B \), \( \mathcal{L}_{A \times B} := \{ \lambda_{A,B}, \lambda_{B,A} \} \) and \( \tau_{A \times B} := \tau_A \cup \tau_B \). We say a play \( s \in \mathcal{P}_{A \times B} \) is coproduct if \( s \) does not contain \( B \)-moves and it is copycat as a play of \( \mathcal{P}_A \). The projection \( \pi_1 \in \text{Sh}(\mathbb{P}_{A \times B}, B) \) is defined by: \( \pi_1(s) = \{ s \} \) if \( s \) is copycat and \( \pi_1(s) = \emptyset \) otherwise. The projection \( \pi_2 \in \text{Sh}(\mathbb{P}_{A \times B}, A) \) is defined similarly. For a play \( s \in \mathbb{P}_{A \times B} \), we write \( s\{A,B \} \) for the restriction of \( s \) to \( \{ m_1 \in m \} \) for some \( m \in \mathcal{M}_B \), where \( m \) is the reflexive and transitive closure of \( \tau \). The restriction is a functor.

**Terminal object**

The empty arena has no moves.

**Exponential laws**

Let \( A \) and \( B \) be arenas. The exponential arena \( A \Rightarrow B \) is defined by: \( 1 \Rightarrow A := \{ m \in \mathcal{M}_B \mid m \vdash B \} \times \mathcal{M}_A + \mathcal{M}_B \), \( 1 \Rightarrow A \Rightarrow B := \{ \lambda_{A,B}, \lambda_{B,A} \} \) and \( \tau_{A \Rightarrow B} := \tau_A \cup \tau_B \). We say a play \( s \in \mathcal{P}_{A \Rightarrow B} \) is coproduct if \( s \) contains only \( A \)-moves and it is copycat as a play of \( \mathcal{P}_A \). The operation \( \tau \) is the reflexive and transitive closure of \( \tau \). The restriction is a functor.

**Lemma 54.** \( \mathcal{G} \) is a cartesian closed category.

### 3.5 Key lemma for full completeness

 Basically the full completeness is achieved by establishing the correspondence between the paths of terms in normal form and \( P \)-views. This subsection describes the key lemma for full completeness, adapting the standard technique for HO/N game models.

An arena \( A \) is prime if it has a unique initial move. Then \( A = B \Rightarrow \{ m \} \) for some arena \( B \) and the initial \( A \)-move \( m \).

Let \( A = A_1 \times \ldots \times A_n \) be an arena, where \( A_i \) is prime for each \( i \), and \( i \in [n] \). Writing \( m_i \) for the unique initial \( A_i \)-move, \( (m_1,m_2) \in \mathcal{V}_{A_1}(m_1) \). We define \((m_1,m_2)\mathcal{V}_{A_1}(m_1)\) as the full subcategory of \( P \)-views \( p \mapsto (m_1,m_2) \). (Since \( \mathcal{V}_{A_1}(m_1) \) is a poset, this coincides with the standard definition of the under category.) Suppose \( A_i = B \Rightarrow \{ m_i \} \). There is an isomorphism \( \chi(m_1,m_2) : (m_1,m_2)\mathcal{V}_{A_1}(m_1) \cong \mathcal{V}_{A,B} \), given by \( m_1m_2m_3 \ldots m_i \Rightarrow m_i \Rightarrow \ldots m_1 \). Here we need to modify the justification pointer as follows:

- If \( m_2 \vdash m_k \) in LHS (then \( k = 3 \)), then \( m_2 \vdash m_k \) in RHS.
- If \( m_1 \vdash m_k \) in LHS, then \( m_1 \vdash m_k \) in RHS.
- If \( m_1 \vdash m_k \) in LHS (\( j \neq 1, 2 \)), then \( m_j \vdash m_k \) in RHS.

This isomorphism is the key to prove full completeness.

Let \( \tau \in \mathcal{V}_{A,B} \). Suppose that \( A = A_1 \times \ldots \times A_n \), where \( A_i \) is prime for each \( i \). Let \( i \in [n] \) and \( A_i = B \Rightarrow \{ m_i \} \). We define the operation \((m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau \) that “inserts” \( m_1m_2 \) before the \( P \)-views in \( \tau \), defined by:

\[
\begin{align*}
((m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau)(m_1,m_2) & := \{ s \} \\
((m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau)(m_1,m_2p) & := \tau(p) \\
((m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau)(p) & := \emptyset
\end{align*}
\]

To be precise, the second equation should be written as \((m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau)(m_1,m_2p) = \tau(\chi(m_1,m_2))(m_1,m_2p) \). Then \( (m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau \in \mathcal{V}_{A_1}(m_1) \).

**Lemma 55.** \( \tau \in \mathcal{V}_{A,B} \) and suppose that \( A = A_1 \times \ldots \times A_n \), \( A_i \) is prime for all \( i \), \( k \in [n] \), \( A_k = B \Rightarrow \{ m_k \} \). Then \( \tau \in (m_1,m_2)\mathcal{V}_{A,B} \Rightarrow \tau \cong \tau \).

where \( \pi_1 \in \mathcal{Sh}(\mathbb{P}_{A,A_1}) \) is the projection of the product and \( ev = \Lambda(\text{id}_A) \in \mathcal{Sh}(\pi_{B \Rightarrow \pi_{A \Rightarrow B}}(m_2), (m_2)) \) is the evaluation map.

### 4. Sheaves model for deterministic \( \lambda_\rightarrow \)

This section develops the sheaves model for simply-typed \( \lambda \)-calculus, the simplest functional programming language.

#### 4.1 The target language

The standard simply-typed call-by-name \( \lambda \)-calculus extended to have divergence \( \bot \). The syntax of terms is given by:

\[
M ::= x \mid \lambda x.M \mid M M \mid \bot.
\]

We consider simply-typed terms possibly having free variables. Types are type environments are given by the grammar:

\[
k ::= o \mid k \rightarrow k \quad \Gamma ::= \cdot \mid \Gamma, x : k.
\]

The typing rules are standard, expect that \( \bot \) is considered as a constant of the ground type \( o \).

We study the equational theory of terms, precisely \( \beta \eta \)-theory. The relation \( = \) is the least equivalence relation that satisfies

\[
(\lambda x.M) N = M[N/x] \\
\lambda x.M \equiv x = M \quad (x \text{ fresh})
\]

and the congruence rules: if \( M = M' \), then \( M N = M' N \) and \( N M = N M' \). The normal form is defined by:

\[
Q ::= \lambda x_1 \ldots x_k. y. Q_1 \ldots Q_n \mid \lambda x_1 \ldots x_k. \bot.
\]
where $y_1 Q_1 \ldots Q_n$ is fully applied, i.e. $y_1 Q_1 \ldots Q_n : o$. Every term has a unique normal form.

### 4.2 Deterministic strategies

**Definition 56.** An odd-length play is an odd-length alternating P-visible justified sequence. (It is not a play because a play is of even-length.) For an odd-length play $s$ over $(A, B)$, the immediate extension $i(e)(s)$ is a set of plays $\{ sm \mid sm \in P_A.C \}.

An odd-length play $s$ ends with an O-move and the immediate extension $i(e)(s)$ is the set of all possible Proponent’s responses.

**Definition 57.** An innocent strategy $\sigma \in Sh(P_{A,B})$ is deterministic if, for every odd-length play $s$, $\bigwedge_{t \in i(e)(s)} (t(s))$ is empty or singleton. It is finite if $\{ p \in V_{A,B} \mid \sigma(p) \neq \emptyset \}$ is a finite set.

**Remark 58.** If $\sigma$ is deterministic, then $\sigma(s)$ is empty or singleton for every $s \in P_{A,B}$. So it is completely determined by a set $\{ s \in P_{A,B} \mid \sigma(s) \neq \emptyset \}$. Through this translation, the sheaf-based definition of innocent strategies coincides with the standard one.

**Definition 59.** A category of deterministic strategies $G_{det}$ is a subcategory consisting of deterministic strategies.

$G_{det}$ is well-defined since the identity $id_A$ is deterministic and the composition preserves determinacy.

**Lemma 60.** Composition preserves determinacy.

**Proof.** Let $\sigma_1 \in Sh(P_{A,B})$ and $\sigma_2 \in Sh(P_{B,C})$ be deterministic strategies. Then for every odd-length play $s$ of $(A, C)$, there exists at most one $u$ such that $u[A, C] = sm, \sigma_1(u[A, B]) \neq \emptyset$ and $\sigma_2(u[B, C]) \neq \emptyset$ (see uncovering construction in [10]). Thus $\bigwedge_{u : u[A, C] \in i(e)(s)} \sigma_1(u[A, B]) \times \sigma_2(u[B, C])$ is empty or singleton.

Since projections $A \times B \rightarrow A$ and $A \times B \rightarrow B$ are deterministic and the isomorphism $Sh(P_{A,B,C}) \cong Sh(P_{A,B\rightarrow C})$ preserves determinacy, $G_{det}$ is a CCC.

### 4.3 Interpretation

Simple types are interpreted as objects by

$[\emptyset] := \{ m_k \}$

as well as type environments

$[x_1 : k_1, \ldots, x_n : k_n] := [k_1] \times \cdots \times [k_n]$. 

The interpretation of terms is fairly standard:

$[x_1 : k_1, \ldots, x_n : k_n \vdash t_1 : k_1] := \pi_t$

$[\Gamma \vdash \lambda x.M : k \rightarrow k'] := \lambda (\Gamma, x : k \vdash M : k')$

$[\Gamma \vdash M : k] := \langle [M], [N] \rangle$.

Proof. This is a special case of Theorem 66 below.

### 5. Sheaves model for nondeterministic $\lambda \rightarrow$

This section studies an extension of $\lambda \rightarrow$, having the non-deterministic branch and interprets the calculus using $G$. We shall prove the soundness of interpretation and the full completeness.

#### 5.1 The target language

Consider the simply-typed lambda calculus with $\bot$ extended to have the non-deterministic branch: $M_1 + M_2$. The additional axioms are:

$(M_1 + M_2) N = (M_1 N) + (M_2 N)$

$\lambda x.(M_1 + M_2) = (\lambda x.M_1) + (\lambda x.M_2)$

$M + (\lambda x_1 \ldots x_n .\bot) = M$ 

and the associativity and commutativity of $+$. These equations are sound with respect to the observational equivalence in the call-by-name evaluation strategy, where the observable may-convergence. (They are not sound for must-convergence because of the right equation.)

We define normal forms where $n, k \geq 0$:

$R := Q_1 + \cdots + Q_n 

Q := \lambda f_1 \ldots f_n \gamma R_1 \ldots R_k$

where $R_1 \ldots R_k$ is fully applied. Every term has a unique normal form (modulo the commutation of non-deterministic branches), or is equivalent to $\lambda x_1 \ldots x_n .\bot$. Note that $M + M \neq M$ in general.

#### 5.2 Interpretation and soundness

The term $\Gamma \vdash M + N : \kappa$ is interpreted as the coproduct $[M] + [N]$ in $Sh(P_{[1], [\kappa]})$. A simple way to describe the coproduct is to use sheaves over views: since the sheaves over views are just presheaves, the coproduct can be computed pointwise. So, given $\tau_1, \tau_2 \in Sh(V_{A,B})$, we have $\tau_1 + \tau_2 \in Sh(V_{A,B})$. Thus $\sigma_1 \sigma_2 \in Sh(P_{A,B\rightarrow C})$ over plays, we define $\sigma_1 \sigma_2 : = \tau_1(\tau_1 \sigma_1, \tau_2 \sigma_2) + (\tau_1 \sigma_2, \tau_2 \sigma_1))$ using the Comparison Lemma.
For a term in normal form, its view restriction can be computed by induction on the structure of \(s\). Let \(M\) be the unique initial move of \(\tau\). Then

\[
\Gamma \vdash x_1 R_1 \ldots R_n : \sigma
\]

for every \(i \in [n]\). Hence \(M\) is deterministic on initial response.

**Lemma 65.** \([Q]\) is deterministic on initial response.

**Proof.** By induction on the structure of \(Q\). If \(Q = x_1 R_1 \ldots R_n\), this follows from Lemma 64. If \(Q = \lambda x. Q'\), then \([Q]\) is deterministic on initial response and \(\lambda x. Q'\) is the restriction of \(\lambda x. Q\) on \(\{0\}\) and \(m_2\) be the unique initial move of \([\kappa_1]\). Then

\[
\Gamma \vdash x_1 R_1 \ldots R_n : \sigma
\]

for each \(i \in [n]\). Hence \(M\) is deterministic on initial response.

**Theorem 66 (Soundness).** \(M = N\) iff \([M] \equiv [N]\).

**Proof.** To prove the left-to-right direction, it suffices to show that all the equations are valid. The equation \([M_1 + M_2] N \equiv [M_1] N + [M_2] N\) follows from Lemma 63. Because \(\lambda x. Q'\) is the coproduct and commutative. Because \(\tau[\kappa_1]\) is the constant functor to \(\tau\), we have \(\sigma + [\kappa_1] \equiv \sigma + \tau[\kappa_1]\).

To prove the converse, assume that \([M] \equiv [N]\) for normal terms \(\Gamma \vdash M : \kappa\) and \(\Gamma \vdash N : \kappa\). Let \(m_1\) be the unique initial move of \(\Gamma\). Then, since \([M_1 + M_2] N \equiv [M_1] N + [M_2] N\), we have a bijection between \(\Pi_{i \in \{0\}} [M_i]_\gamma(s)\) and \(\Pi_{i \in \{0\}} [N_i]_\gamma(s)\). Let \(n\) be the number of elements of those sets. Then \(M \equiv Q_1 + \ldots + Q_n\), since \([Q]\) is deterministic on initial response for every \(i \in [n]\) by Lemma 65. Similarly \(N \equiv Q'_1 + \ldots + Q'_n\). Since \([M] \equiv [N]\), there is a bijection \(\varphi : [n] \rightarrow [n]\) such that \([Q_i] \equiv [Q'_{\varphi(i)}]\). By the induction hypothesis, \(Q_\varphi(\kappa_1)\). So \(M \equiv N\).

Suppose that

\[
M \equiv \lambda x_1 \ldots x_k. y R_1 \ldots R_n
\]

where \(\kappa\) is the initial move for \(y : \kappa' \in \Gamma\) and \(m_2\) is the initial move of \(y : \kappa' \in \Gamma\). Since \([M_1]_\gamma \equiv [N_1]_\gamma\), we have \(m_2 \equiv m_2\), which implies \(y = y\) and \(n = n'\). Furthermore \([M]_\gamma \equiv [N]_\gamma\) implies \([R_i]_\gamma \equiv [R'_i]_\gamma\) for all \(i \in [n]\) and thus \([R_i] \equiv [R'_i]\). By the induction hypothesis, \(R_i \equiv R'_i\) and hence \(M \equiv N\).

### 5.3 Full completeness

A sheaf \(\sigma \in \text{Sh}(\mathbb{P}_{A,B})\) is finite if \(\prod_{p \in \mathbb{P}_{A,B}} \sigma_\gamma(p)\) is finite.

**Lemma 67.** Every finite sheaf \(\sigma \in \text{Sh}(\mathbb{P}_{A,B})\) can be decomposed as \(\sigma \equiv \sigma_0 + \ldots + \sigma_n\), where \(\sigma_i\) is deterministic on initial response for all \(i\).

**Proof.** Let \(\tau = \iota^*\sigma\) be the restriction of \(\sigma\) to views. Consider the finite set \(\prod_{p \in \{0\}} \sigma(p)\), which we write as \(\{(p_1, a_1), \ldots, (p_n, a_n)\}\) (\(a_i \in \sigma(p_i)\) for each \(i \in [n]\)). We define \(\tau_i \in \text{Sh}(\mathbb{P}_{A,B})\). On objects,

\[
\tau_i(p) := \{a \in \sigma(p) \mid p \leq p_i \text{ and } a = a_i \cdot f \text{ where } f : p_i \rightarrow p\}
\]

Then \(\tau_i(p) \subseteq \sigma(p)\) for every \(i\) and \(p\). For \(f : p \rightarrow p'\), we define \(\tau_i(f)\) as the restriction of \(\tau(f) : \tau(t') \rightarrow \tau(t)\) to \(\tau_i(t') \subseteq \tau(t')\). It is easy to see that \(\tau_i\) is a functor. Then we have

\[
\tau \equiv \tau_1 + \ldots + \tau_n.
\]

To see this, consider \(\sigma \in \tau(p)\) for some \(p\). Let \(p'\) be the first two moves of \(p\) and let \(a' = a \cdot f\), where \(f : p_i \rightarrow p\) (unique). Then \((p', a')\) is \((p_i, a_i)\) for some \(i \leq n\). Hence \(a \in \tau_i(p)\). Furthermore such \(i\) is unique by the construction. So we have the claimed natural isomorphism. Letting \(\sigma_i := \iota^*\tau_i\), we obtain the statement.

**Theorem 68 (Full completeness).** Let \(\Gamma\) be a type environment, \(\kappa\) be a type and \(\sigma \in \text{Sh}(\mathbb{P}_{A,B}[\Gamma])\). If \(\sigma\) is finite, there exists a term \(\Gamma \vdash M : \kappa\) such that \(\sigma \equiv [M]\).

**Proof.** By induction on the number of elements in \(\prod_{p \in \mathbb{P}_{A,B}[\Gamma]} \sigma(p)\) and the structure of \(\kappa\). If \(\kappa = \kappa_1 \rightarrow \kappa_2\), consider \(\lambda x. \sigma\) \((\sigma \in \text{Sh}(\mathbb{P}_{A,B}[\Gamma][\xi]_{\kappa_1}[\xi]_{\kappa_2})\) and apply the induction hypothesis. Suppose that \(\kappa = [0]\). If \(\sigma\) has several initial responses, then by applying Lemma 67, we have \(\sigma = \sigma_1 + \ldots + \sigma_n\) (\(n \geq 2\)). By the induction hypothesis, we have \(\sigma_i \equiv [M_i]\) for every \(i\) and thus \(M_1 + \ldots + M_n\) is the required term. Suppose that \(\sigma\) is deterministic on initial response. Let \((m_1, m_2, a)\) be the unique response. Since \(\sigma \in \text{Sh}(\mathbb{P}_{A,B}[\Gamma]\{a\}), m_1\) is the unique initial move of \([\sigma]\) and \(m_2\) be the unique initial move of \([\sigma_\kappa]\), where \(x_k : \kappa_k \in \Gamma\) for some \(x_k\). Suppose that \(\kappa_\kappa = \kappa_1 \rightarrow \ldots \rightarrow \kappa_\kappa' \rightarrow \sigma\). We define the sheaf \(\tau' \equiv \text{Sh}(\Gamma, [\kappa_1] \times \ldots \times [\kappa_\kappa])\) by

\[
\tau'(p) := \sigma(\chi_{\kappa_1}^{-1}(p))\]

where \(\chi_{\kappa_1}^{-1} : V_{\kappa_1} \rightarrow \{0\} \times \{1\}\) is the projection. By the induction hypothesis, we have \(M_i\) for each \(i \in [n]\) such that \(\tau' \equiv [M_1, \ldots, M_n]\). Recall that \([x_k] \equiv \kappa_k\). Since \(\kappa\) is a CCC, the application of the product can be rewritten by the series of applications. Hence \(\sigma \equiv [x_k M_1, \ldots, M_n]\) as desired.

### Example 69.

Let \(\pi = \lambda x y . x \text{ and } \xi = \lambda x y . y\). Recall the example in Introduction, \(M_1 = \lambda f . (f \pi) + (f \xi)\) and \(M_2 = \lambda f . (f \pi) + (\lambda f . \xi)\). Then \(\iota^* [M_1] = \tau_1\) and \(\iota^* [M_2] = \tau_2\), where sheaves \(\tau_1\) and \(\tau_2\) over \(P\)-views can be found in Example 25.

### 6. Sheaves model for probabilistic \(\lambda\)-calculus

We have seen that a term of the non-deterministic \(\lambda\)-calculus is modelled by a sheaf \(\sigma\) which maps a play \(s\) to a (finite) set \(\sigma(s)\). An element of \(\sigma(s)\) represents a particular choice of branches by which the term behaves like \(s\).
In this section, we shall study a non-deterministic sheaf $\sigma$ equipped with a weight map $\mu$ which assigns each choice $(s,a)$ (where $s$ is a play and $a \in \sigma(s)$) with a positive real number $\mu(s,a)$.

### 6.1 The target calculus: weighted and probabilistic $\lambda \to$

The target language is an extension of the nondeterministic $\lambda \to$ studied in the previous section. The new feature is the term constructor $e \cdot M$, where $e$ is a positive real number. The additional equations are:

$$
\lambda x.(c \cdot M) = c \cdot (\lambda x.M) \quad c \cdot (M + N) = (c \cdot M) + (c \cdot N)
$$

$$
c_1 \cdot (c_2 \cdot M) = (c_1 c_2) \cdot M \quad (c \cdot M) N = c \cdot (MN)
$$

and $c \cdot \bot = \bot$. These equations are admissible in the sense that $M = N$ implies $M$ and $N$ are observably equivalent in the standard call-by-name operational semantics (where the observable is the probability of convergence). The probabilistic $\lambda \to$ is a fragment of this calculus in which nondeterministic branch and the weight construct are restricted to the form $(c_1 \cdot M_1) + \cdots + (c_n \cdot M_n)$, where $\sum_{i=1}^n c_i \leq 1$.

**Remark 70.** The rule $M \cdot N = c \cdot (MN)$ is unsound, because the application is not linear on the argument. For instance, if the argument is twice as in $(\lambda x.f(f(z))) (c \cdot \lambda x.x)$, the resulting coefficient is $c^2z$:

$$
(\lambda x.f(f(z)))(c \cdot \lambda x.x) = (c \cdot (\lambda x.x))(c \cdot (\lambda x.x))z
$$

$$
= c \cdot c \cdot ((\lambda x.x)(c \cdot ((\lambda x.x))))z
$$

Similarly, if the argument never be called as in $(\lambda x.x)(c \cdot N)$, the coefficient $c$ does not affect, e.g. $(\lambda x.z)(c \cdot N) = z = (\lambda x.z) N$.

### A normal form is defined by:

$$
R := c_1 \cdot Q_1 + \cdots + c_n \cdot Q_n \quad Q := \lambda x_1 \ldots x_k.g R_1 \ldots R_n,
$$

where $R_1 \ldots R_n$ is fully applied. Every term has a unique normal form (modulo commutation of the non-deterministic branches), or is equivalent to $\lambda x_1 \ldots x_k.\bot$. Note that $2 \cdot M + 2 \cdot M \neq 4 \cdot M$.

### 6.2 Sheaves with weight

**Definition 71 (Weight).** Let $F$ be a functor $\mathbb{D}^{op} \to \text{Set}$. A weight map $\mu$ assigns, for each $s \in \mathbb{D}$ and $a \in F(s)$, a positive real number $\mu(s,a) \in \mathbb{R}^+$. Let $\sigma \in \text{Sh}(PA,B)$ and $\mu$ be a weight map. Given a morphism $f : s \to t$ in $PA,B$ and an element $a \in \sigma(t)$, we define $\mu(f,a) := \mu(t,a)/\mu(s,a \cdot f)$. Notice that $\mu(g \cdot f,a) = \mu(g,a)\mu(f,a \cdot g)$.

**Definition 72 (Nondeterministic weight map).** Let $\sigma \in \text{Sh}(PA,B)$ be a sheaf and $\mu$ be a weight map. The weight map $\mu$ is **injective** if it satisfies the following conditions: (1) $\mu(\varepsilon,*) = 1$, and (2) given a covering family \{$(f : s \to u, g : t \to u)$ and $a \in \sigma(u)$\}, consider the pullback diagram

$$
\begin{array}{ccc}
s \times_u t & \xrightarrow{\mu(t,a)/\mu(s,a \cdot f)} & s \\
\downarrow{g^* f} & & \downarrow{f} \\
t & \xrightarrow{g} & u
\end{array}
$$

then $\mu(f,a) = \mu(g^* f, a \cdot g)$.

The typical case is that $u = v_0 v_1 v_2$, $s = v_0 v_1$, $t = v_0 v_3$ and $s \times_u t = v_0$. Intuitively $\mu(f,a)$ is the weight of playing $v_2$ from $s = v_0 v_1$ (that reaches to the state $a \in \sigma(s)$) and $\mu(g^* f, a \cdot g)$ is the weight of playing $v_2$ from $v_0$ (that reaches to the state $a \cdot g \in \sigma(t)$, the restriction of $a$ to $t$). The innocence of the weight map requires that the weight for playing $v_2$ is independent of the situation.

**Definition 73 (Weighted innocent strategy).** A weighted innocent strategy over pairs $(A,B)$ of arenas is a pair $(\sigma,\mu)$ of an innocent non-deterministic strategy $\sigma \in \text{Sh}(PA,B)$ and an innocent weight map $\mu$ for $\sigma$.

Similar to the deterministic/non-deterministic cases, a weighted innocent strategy is determined by its restriction on views.

**Lemma 74.** Assume $\sigma,\sigma' \in \text{Sh}(PA,B)$ and a natural isomorphism $\varphi : \sigma \xrightarrow{\sim} \sigma'$. Let $\mu$ and $\mu'$ are innocent weight maps for $\sigma$ and $\sigma'$, respectively. If $\mu(p,a) = \mu'(p,\varphi(a))$ for every $P$-view $p \in \mathbb{V}_{A,B}$, then $\mu(s,a) = \mu'(s,\varphi(a))$ for every play $s \in \mathbb{P}_{A,B}$.

**Proof.** By induction on the length of $s$. Let $s = s_0 m_1 m_2$ be a play and $e \in \sigma(s)$. If $s$ is a $P$-view, the claim is just assumed. Suppose that $s$ is not a $P$-view. We have a covering family $\{f : s_0 \to s, g : [s] \to s\}$. Since the pullback $g^* f : p_0 \to [s]$ is in $\mathbb{V}_{A,B}$,

$$
\mu(f,a) = \mu(g^* f, e \cdot g) = \mu'(g^* f, \varphi(e) \cdot g)
$$

$$
= \mu'(g^* f, \varphi(e) \cdot g) = \mu'(f, \varphi(e)).
$$

By the induction hypothesis, we have

$$
\mu(s_0, e \cdot f) = \mu'(s_0, \varphi(e) \cdot f).
$$

So we conclude

$$
\mu(s,e) = \mu(f,e)\mu(s_0, e \cdot f)
$$

$$
= \mu'(f, \varphi(e))\mu'(s_0, \varphi(e) \cdot f) = \mu'(s, \varphi(e))
$$

as desired.

**Lemma 75.** Let $\tau \in \text{Sh}(\mathbb{V}_{A,B})$. Every weight map $\mu_0$ for $\tau$ can be extended to an innocent weight map for $\tau \cup \tau$.

**Proof.** Given a non-empty $P$-view $p = p_0 m_1 m_2 \in \mathbb{V}_{A,B}$ and $e \in \tau(p)$, we define $\delta(e) := \mu_0(p_0 \to p, e)$ (if $p_0 \neq e$) and $\delta(e) := \mu_0(p, e)$ (if $p_0 = e$). We give a weight map $\mu$ for $\tau \cup \tau$. Let $s = m_1 m_2 \ldots m_n \in \mathbb{P}_{A,B}$ and $e \in \tau(s)$ where $e_k \in \tau([m_1 \ldots m_k])$ for every even number $k \leq n$. The weight for $x = e_2 e_4 \ldots e_n$ is defined by:

$$
\mu(s, e_2 e_4 \ldots e_n) := \delta(e_2)\delta(e_4)\ldots \delta(e_n).
$$

It is easy to see that $\mu$ is innocent.

So one can define a weighted innocent strategy as a pair of a sheaf over $P$-views and a weight function for it.

**Definition 76.** The category of weighted innocent strategies $\mathbb{G}_w$ has arenas as objects and weighted innocent strategies as morphisms. Here $(\sigma_1,\mu_1)$ and $(\sigma_2,\mu_2)$ are identified if there exists a natural isomorphism preserving weights. A composition of weighted innocent strategies $(\sigma,\mu)$ and $(\sigma',\mu')$ is $((\sigma;\sigma'),\mu'')$, where for each $s$ and $(u,e,e') \in (\sigma;\sigma')(s) = \{u_0 : \pi(u_0) = s \in \sigma(u[A,B] \times \sigma'(u[B,C])$, where $e \in \sigma(u[A,B])$ and $e' \notin \sigma'(u'[B,C])$, we define

$$
\mu''(s,(u,e,e')) = \mu(u[A,B],e)\mu'(u'[B,C],e').
$$

Associativity of the composition can be easily shown.

**Lemma 77.** $\mathbb{G}_w$ is a cartesian closed category.

**Proof.** Given a deterministic innocent strategy $\sigma \in \text{Sh}(PA,B)$, the trivial weight map $\mu$ is defined by $\mu(s,e) = 1$ for every $s$ and $e$. Then $\text{id}_A$ with the trivial weight map is the identity and $\pi_1 \in \text{Sh}(PA_{A,B},A)$ and $\pi_2 \in \text{Sh}(PA_{A,B},B)$ with the trivial weight maps are projections. The natural isomorphism $G(A \times B,C) = \text{Sh}(PA_{A,B,C}) \cong \text{Sh}(PA_{A,B} \to C) = G(A,B \to C)$ has obvious extension to weighted innocent strategies. Hence $\mathbb{G}_w$ is a CCC.
6.3 Semantics of weighted $\lambda \rightarrow$

Let $\tau$ and $\tau'$ be sheaves over $P$-views of $(A, B)$ and $\mu_0$ and $\mu_0'$ be weight maps for $\tau$ and $\tau'$, respectively. The weight map $[\mu_0, \mu_0']$ for $\tau + \tau'$ is defined by $[\mu_0, \mu_0'](p, e) := \mu_0(p, e)$ if $e \in \tau(p))$ and $[\mu_0, \mu_0'](p, e) := \mu_0'(p, e)$ if $e \in \tau'(p))$. We define $c \otimes \mu_0$ by $(c \otimes \mu_0)(p, e) := c \mu_0(p, e)$.

The same operations can be defined for weighted innocent strategies through Lemma 75. Given a weighted innocent strategy $(\sigma, \mu)$, we define $c \otimes \mu$ the unique extension of $c \otimes \mu$ to $\sigma$, where $\mu_0$ is the restriction of $\mu$ to $P$-views. Then $(c \otimes \mu)(s, e) = c \mu(s, e)$, where $k$ is the number of the moves in $s$ that point to $e$. It is easy to check that the equations about weights are sound for this interpretation, using the next lemma.

Lemma 78. Let $s$ be a well-opened play and $e \in \sigma(s)$. Then $(c \otimes \mu)(s, e) = c \mu(s, e)$.

Lemma 79. $M = N$ iff $[M] = [N]$.

Let $B$ be a prime arena. A weighted innocent strategy $(\sigma, \mu)$ of $(A, B)$ is deterministic on initial response if $\prod_{s \in \text{le}(s)} \sigma(s)$ is singleton and $\mu(s, e) = 1$ for its unique element $(s, e)$. The next lemma shows that it is determined on initial response.

Lemma 80. Every finite weighted innocent $(\sigma, \mu)$ strategy can be decomposed as $c_1 \otimes (\sigma_1, \mu_1) + \ldots + c_n \otimes (\sigma_n, \mu_n)$, where $(\sigma_i, \mu_i)$ is deterministic on initial response.

The full completeness for the weighted calculus is proved by the same technique as in the proof of Theorem 68, using Lemma 80.

Theorem 81 (Full completeness). Let $(\sigma, \mu)$ be a weighted innocent strategy for $[\Gamma]$, and suppose that $\sigma$ is finite. Then there exists a term $\Gamma \vdash M : \kappa$ such that $[\sigma, \mu] \equiv [M]$.

6.4 Semantics of probabilistic $\lambda \rightarrow$

A weighted innocent strategy $(\sigma, \mu)$ is probabilistic if, for every odd-length play $s = s_0m$ and $e_0 \in \sigma(s_0)$, the sum of weights of possible responses that extends $(s, e_0)$ is less than 1.

Definition 82. A weighted innocent strategy $(\sigma, \mu)$ over $(A, B)$ is probabilistic if, for every odd-length play $s = s_0m$ and $e_0 \in \sigma(s_0)$, we have

$$\sum_{t \in \text{le}(s)} \sum_{e : t = e_0} \mu(f, e) \leq 1$$

where $f : s_0 \rightarrow t$ is the prefix embedding. It can be strictly less than 1; the difference is the probability of divergence. A sheaf $\tau$ over views with a weight map $\mu_0$ is probabilistic when the same condition holds (but $s$ is restricted to $P$-views).

Lemma 83. $(\sigma, \mu)$ is probabilistic iff its restriction to views is.

Proof. Let $\sigma \in \text{Sh}(P_{A,B})$ and $\tau = e^* \sigma \in V_{A,B}$. Let $s = s_0m$ be an odd-length play and $e_0 \in \sigma(s_0)$. We prove

$$\sum_{t \in \text{le}(s)} \sum_{e : t = e_0} \mu(f, e) \leq 1$$

by induction on the length $s$, where $f : s_0 \rightarrow t$ is the prefix embedding. If $[s] = s$, then every $t \in \text{le}(s)$ is a P-view. Hence the claim follows from the assumption.

Assume that $[s] \neq s$. Let $s_0 = m_1 \ldots m_{2k}$ be the justifier of $m$ and $p_0 = [m]_0 \ldots [m]_k$. Consider the covering family $\{ f_t : s_0 \rightarrow t, g_t : [t] \rightarrow t \}$ for every $t$. Then we have $\mu(f, e) = \mu(g_t(f), e \cdot g_t)$ for every $t \in \text{le}(s_0)$. So it suffices to prove that

$$\sum_{t \in \text{le}(s)} \sum_{e : t = e_0} \mu(g_t(f), e \cdot g_t) \leq 1$$

Since the P-view of $t \in \text{le}(s)$ is given by $[t] = [s_0m^0m^1] = p_0m^0m^1$ (for some $m^0$), we have a bijection from $\text{le}(s)$ to $\text{le}(p_0m)$.

$$\sum_{t \in \text{le}(s)} \sum_{e : t = e_0} \mu(g_t(f), e \cdot g_t) = \sum_{p \in \text{le}(p_0m)} \sum_{e : (p \geq p_0, e) \leq 1}$$

where $h : s_0 \rightarrow s_0$ is the P-view embedding. □

Because the probabilistic $\lambda \rightarrow$ is a fragment of the weighted calculus, all the properties including soundness and adequacy are applicable from the probabilistic calculus. Full completeness can be proved by the same way as the weighted case.

Theorem 84 (Full completeness). Let $\kappa$ be a simple type, $\sigma \in \text{Sh}(P_{A,B})$ be finite and $\mu$ be a probabilistic weight. Then $[\sigma, \mu] = [M]$ for some probabilistic term $\Gamma \vdash M : \kappa$.

Concluding remarks

As presented, our model treats neither recursion nor primitive data types such as boolean. Further the target languages are restricted to simply-typed calculi. However we believe that these restrictions can be relaxed.

We will apply the sheaf-theoretic approach in the paper to study the model checking of non-deterministic calculi, such as non-deterministic PCF and its call-by-value version, and to develop a semantics of refinement dependent types.

References

[1] S. Abramsky and G. McCusker. Linearity, sharing and state: a fully abstract game semantics for Idealized Algol with active expressions. In Algol-like Languages, pages 297–329. Birkhauser, 1997.

[2] S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for pcf. Inf. Comput., 163(2):409–470, 2000.

[3] A. Bellinsson. P-adic periods and derived de Rham cohomology. J. AMS, 25(3):715–738, 2012.

[4] S. Castellan, P. Clairambault, and G. Winskel. Concurrent Hyland-Ong games. Lecture slides, IHP Workshop on Semantics of Proofs and Programs, 2014.

[5] V. Danos and R. Harmer. Probabilistic game semantics. ACM Trans. Comput. Log., 3(3):359–382, 2002.

[6] C. Eberhart, T. Hirschowitz, and T. Seiller. Fully abstract concurrent games for pi. CoRR, abs/1310.4306, 2013.

[7] R. Harmer. Games and Full Abstraction for Nondeterministic Languages. PhD thesis, Imperial College, 1999.

[8] R. Harmer and G. McCusker. A fully abstract game semantics for finite nondeterminism. In LICS, pages 422–430, 1999.

[9] T. Hirschowitz and D. Pouss. Innocent strategies as presheaves and interactive equivalences for ccs. Sci. Ann. Comput. Sci., 22(1):147–199, 2012.

[10] J. M. E. Hyland and C.-H. L. Ong. On full abstraction for PCF. I, II, and III. Inf. Comput., 163(2):285–408, 2000.

[11] A. Jung, M. A. Moshier, and S. J. Vickers. Presenting dcpos and dcpo Algol-like Languages. In In. Inf. Comput., pages 297–329. Birkhauser, 1997.

[12] S. M. Lane and I. Moerdijk. Sheaves in Geometry and Logic. Springer-Verlag, 1992.

[13] P. Levy. Morphisms between plays. Lecture Slides, GaLoP, 2013.
[14] H. Nickau. Hereditarily sequential functionals. In *LICS*, pages 253–264, 1994.

[15] C.-H. L. Ong. On model-checking trees generated by higher-order recursion schemes. In *LICS*, pages 81–90, 2006.

[16] S. Rideau and G. Winskel. Concurrent strategies. In *LICS*, pages 409–418, 2011.

[17] S. Staton and G. Winskel. On the expressivity of symmetry in event structures. In *LICS*, pages 392–401, 2010.

[18] T. Tsukada and C.-H. L. Ong. Compositional higher-order model checking via $\omega$-regular games over Bohm trees”. In *CSL/LICS*, 2014.

[19] J.-L. Verdier. Fonctorialité de catégories de faisceaux. In *Théorie des topos et cohomologie étale de schémas (SGA 4), Tome 1*, pages 265–298. Springer-Verlag, 1972. Lect. Notes in Math. 269.