GREATEST COMMON DIVISORS OF INTEGRAL POINTS OF NUMERICALLY EQUIVALENT DIVISORS

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Abstract. We generalize the G.C.D. results of Corvaja–Zannier and Levin on $\mathbb{G}^m_n$ to more general settings. More specifically, we analyze the height of a closed subscheme of codimension at least 2 inside an $n$-dimensional Cohen-Macaulay projective variety, and show that this height is small when evaluated at integral points with respect to a divisor $D$ when $D$ is a sum of $n+1$ effective divisors which are all numerically equivalent to some multiples of a fixed ample divisor. Our method is inspired by Silverman’s G.C.D. estimate as an application of Vojta’s conjecture, which is substituted by a more general version of Schmidt’s subspace theorem of Ru–Vojta in our proof.

1. Introduction and Statements

In a recent work [15], Levin obtained the following result which bounds the greatest common divisor of multivariable polynomials evaluated at $S$-unit arguments. This result is a generalization of results of Bugeaud-Corvaja-Zannier [1], Hernández-Luca [7] and Corvaja-Zannier [2], [3]. We refer to [15] for a survey of these related results.

**Theorem 1.1** ([15, Theorem 1.1]). Let $\Gamma \subseteq \mathbb{G}^r_m(\bar{\mathbb{Q}})$ be a finitely generated group and fix nonconstant coprime polynomials $f(x_1, \ldots, x_r), g(x_1, \ldots, x_r) \in \mathbb{Q}[x_1, \ldots, x_r]$ which do not both vanish at the origin $(0, \ldots, 0)$. Then, for each $\epsilon > 0$, there exists a finite union $Z$ of translates of proper algebraic subgroups of $\mathbb{G}^r_m$ so that

$$\log \gcd(f(u), g(u)) < \epsilon \max_i \{h(u_i)\}$$

for all $u = (u_1, \ldots, u_r) \in \Gamma \setminus Z$.

The greatest common divisor on the left-hand side of the above inequality is a generalized notion of the usual quantity for integers, adapted to algebraic numbers to also account for the archimedean contributions [15, Definition 1.4]. He then further elaborated Theorem 1.1 into the following geometric setting in projective spaces. Here, $h_Y$ is a height associated to the closed subscheme $Y$ (see Section 2).

**Theorem 1.2.** ([15, Theorem 1.16]) Let $Y$ be a closed subscheme of $\mathbb{P}^n$ of codimension at least 2, defined over a number field. Suppose that

$$P_0 := [1:0: \cdots :0], \ldots, P_n := [0:0: \cdots :1] \notin \text{Supp } Y.$$
Let $\Gamma \subseteq G_m^n(\mathbb{Q})$ be a finitely generated group. Then for all $\epsilon > 0$, there exists a finite union $Z$ of translates of proper algebraic subgroups of $G_m^n$ such that
\[
h_Y(P) \leq c\epsilon(P) + O(1),
\]
for all $P \in \Gamma \setminus Z \subset \mathbb{P}^n(\mathbb{Q})$.

The following is our main result that is a direct generalization of Theorem 1.2. The proof of Theorems 1.1 and 1.2 is a sophisticated, but a direct application of the classical Schmidt’s subspace theorem. Our proof of Theorem 1.3 below is based on an idea of Silverman in [24] and the recent work [22] of Ru and Vojta. Therefore, it provides a different proof for Theorem 1.2.

**Theorem 1.3.** Let $V$ be a Cohen–Macaulay projective variety of dimension $n$ defined over a number field $k$, and let $S$ be a finite set of places of $k$. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisors of $V$ defined over $k$ in general position. Suppose that there exist an ample Cartier divisor $A$ on $V$ and positive integers $d_i$ such that $D_i \equiv d_i A$ for all $1 \leq i \leq n+1$. Let $Y$ be a closed subscheme of $V$ of codimension at least 2 that does not contain any point of the set
\[
\bigcup_{i=1}^{n+1} \left( \bigcap_{1 \leq j \neq i \leq n+1} \text{Supp } D_j \right).
\]
Let $\epsilon > 0$. Then there exists a proper Zariski closed set $Z$ of $V(k)$ such that for any set $R$ of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $V$ we have
\[
h_Y(P) \leq c\epsilon A(P) + O(1)
\]
for all $P \in R \setminus Z$.

Here, $\equiv$ denotes numerical equivalence of divisors, and the notion of $(D, S)$-integral points will be discussed in more detail in Section 2.

Given a finitely generated subgroup $\Gamma \subseteq G_m^n(\mathbb{Q})$ and a closed subscheme $Y \subset \mathbb{P}^n$ defined over a number field, there exist a number field $k$ and a finite set of places $S$ of $k$ (containing the archimedean places) such that $\Gamma \subseteq G_m^n(\mathcal{O}_S)$ and $Y$ is defined over $k$. Therefore, we may replace $\Gamma$ in Theorem 1.2 by $G_m^n(\mathcal{O}_S)$ and assume that $Y$ is defined over $k$. Let $D_1 := \{x_0 = 0\}, \ldots, D_{n+1} := \{x_n = 0\}$ be the divisors given by the coordinate hyperplanes. Then the set (2) is exactly the set of points \( \left\{ P_0, \ldots, P_n \right\} \) in Theorem 1.2. Moreover, the set $G_m^n(\mathcal{O}_S)$ is a set of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $\mathbb{P}^n(k)$. Since the Zariski closure of a subset of $\Gamma$ is a finite union of translates of algebraic subgroups by Laurent [12], Theorem 1.2 is a direct consequence of Theorem 1.3. With a bit more work, the results of [21] can also be seen as special cases of Theorem 1.3.

We note that in [15], Levin divides his argument into the “counting function part” $N_{Y,S}$ and the “proximity function part” $m_{Y,S}$ (see Section 2 for the definitions of $N_{Y,S}$ and $m_{Y,S}$). The $N_{Y,S}$ part is the main work (see [15], Theorem 4.3), and for a closed subscheme $Y$ of $\mathbb{P}^n$ of codimension at least 2, he shows that
\[
N_{Y,S}(P) \leq c\epsilon(P) + O(1)
\]
for all $P \in G_m^n(\mathcal{O}_S)$ outside a Zariski-closed proper subset. Our main argument needs to consider $h_Y$, i.e. combining the $N_{Y,S}$ part and the $m_{Y,S}$ part together. Therefore, we recover (1) with the additional assumption that $Y$ does not intersect the set (2). However, when $Y$ contains only points, our method also works without
this assumption. In particular, it recovers a result of Corvaja-Zannier for $n = 2$ [3 Proposition 4], which is a consequence of the following.

**Theorem 1.4.** Let $V$ be a Cohen–Macaulay projective variety of dimension $n$ defined over a number field $k$, and let $S$ be a finite set of places of $k$. Let $D_1, \ldots, D_n$ be effective Cartier divisor of $V$ defined over $k$ and in general position. Suppose that there exists an ample Cartier divisor $A$ on $V$ and positive integer $d_i$ such that $D_i \equiv d_iA$ for all $1 \leq i \leq n$. Let $Y$ be a closed subscheme of $V$ of dimension 0. Let $\epsilon > 0$. Then there exists a proper Zariski closed set $Z$ of $V(k)$ such that for any set $R$ of $(\sum_{i=1}^{n} D_i, S)$-integral points on $V$ we have

$$N_{Y,S}(P) \leq ch_A(P) + O(1)$$

for all $P \in R \setminus Z$.

**Corollary 1.5** (Corvaja-Zannier [3]). Let $k$ be a number field and $S$ a finite set of places of $k$ containing the archimedean places. Let $f(X, Y), g(X, Y) \in k[X, Y]$ be nonconstant coprime polynomials. For all $\epsilon > 0$, there exists a finite union $Z$ of translates of proper algebraic subgroups of $\mathbb{G}_m^2$ such that

$$- \sum_{v \in M_k \setminus S} \log \max\{|f(x, y)|_v, |g(x, y)|_v\} < \epsilon \max\{h(x), h(y)\}$$

for all $(x, y) \in \mathbb{G}_m^2(O_S) \setminus Z$.

We note that Corvaja-Zannier also showed that the translate of one dimensional proper algebraic subgroups of $\mathbb{G}_m^2$ in $Z$ as stated in Corollary 1.3 can be determined effectively.

To further illustrate the ideas of our work, let us first recall the following conjecture of Vojta. (See [26 Chapter 15]).

**Conjecture 1.6** (Vojta’s Main Conjecture). Let $k$ be a number field, let $S$ be a finite set of places of $k$. Let $X$ be a smooth projective variety over $k$. Let $A$ be an ample divisor on $X$, let $D$ be a simple normal crossing divisor on $X$ and let $K_X$ be a canonical divisor on $X$. Then for every $\epsilon > 0$ there exists a proper Zariski closed subset $Z = Z(X, D, A, \epsilon)$ and a constant $C_\epsilon(X, D, A, \epsilon) \in \mathbb{R}$ the inequality

$$m_{D,S}(P) + h_{K_X}(P) \leq ch_A(P) + C_\epsilon$$

holds for all $P \in X(k) \setminus Z$.

In [24 Theorem 6], Silverman considered Vojta’s conjecture on the blowup $\pi : \tilde{X} \to X$ of $X$ along $Y$, where $X$ is a smooth variety over $k$, $Y \subset X$ is a smooth subvariety of codimension $r \geq 2$ under the assumption that $-K_X$ is a simple normal crossings divisor and $Y$ intersects the support of $K_X$ transversally (see also [17, 28, 30]). The key point of our method is to replace Vojta’s conjecture on $\tilde{X}$ by a more general version of Schmidt’s subspace theorem obtained recently by Ru and Vojta in [22].

To illustrate the idea, we consider the particular case when $X = \mathbb{P}^n$, $-K_X = \sum_{i=0}^{n} H_i$, where $H_i = \{X_i = 0\}$ is the divisor of the coordinate hyperplane. Applying Vojta’s conjecture with $D = -\pi^* K_X = \sum_{i=0}^{n} \pi^* H_i$ and ample divisor $\pi^* A - \frac{1}{\ell} E$ on $\tilde{X}$, where $A = H_0$ and $\ell$ is a sufficiently large integer, the equation (6) becomes

$$\sum_{i=0}^{n} m_{H_i, S}(Q) + (r - 1 + \frac{\epsilon}{\ell}) h_E(Q) \leq (n + 1 + \epsilon) h_{\pi^* A}(Q) + C_\epsilon.$$

(7)
In contrast, our computation in this situation (see equations (20) and (22)) shows that the result of [22] (see Theorem 4.4) implies that for a sufficiently large \( \ell \), there exists a proper closed subset \( \tilde{Z} \subset \tilde{X} \) such that

\[
\sum_{i=0}^{n} m_{i} \pi_{*} H_{i}, S(Q) + \frac{1}{\ell} h_{E}(Q) \leq (n + 1 + O(\frac{1}{\ell \sqrt{\ell}})) h_{\pi_{*} A}(Q) + C
\]

for all \( Q \in \tilde{X}(k) \setminus \tilde{Z} \) provided the supports of \( \pi_{*} H_{0}, \ldots, \pi_{*} H_{n} \) are in general position. We note that this assumption is weaker than what is required for [24] since for example \( Y \) does not have to be nonsingular at the intersection points with \( H_{i} \). While the inequality (8) is weaker than (7) deriving from Vojta’s conjecture (the coefficient for \( h_{E} \) is smaller), the estimate (8) is still sufficient for our goal of analyzing integral points. To see this, by functoriality and using the assumption that \( P = \pi(Q) \) is \( \left( \sum_{i=0}^{n} H_{i}, S \right) \)-integral,

\[
\sum_{i=0}^{n} m_{i} \pi_{*} H_{i}, S(Q) = \sum_{i=0}^{n} m_{i} H_{i}, S(P) + O(1) = (n + 1) h_{A}(P) + O(1).
\]

So it follows from (8) that

\[
h_{Y}(P) = h_{E}(Q) \leq O\left(\frac{1}{\sqrt{\ell}}\right) h_{A}(P) + O(1)
\]

for all \( P \) in a set of \( \left( \sum_{i=0}^{n} H_{i}, S \right) \)-integral points outside \( \pi(\tilde{Z}) \) of \( P \). We also note that the idea of replacing Vojta’s conjecture by the result of Ru and Vojta in [22] along the argument of Silverman [24] first appeared in [5], where they apply the complex part of Ru and Vojta in [22] for the blowup \( \tilde{X} \) of \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) along \( Y = (1, 1) \) and the ample divisor \( \tilde{A} := -\pi_{*} K_{X} - E = \pi_{*}((-0) \times \mathbb{P}^1) + \pi_{*}((\infty) \times \mathbb{P}^1) + \pi_{*}((\mathbb{P}^1 \times \{0\}) + \pi_{*}((\mathbb{P}^1 \times \{\infty\}) - E) \) (See [22, Corollary 1.11] for further discussion in the arithmetic side.)

Finally, we discuss the analogous topics and results in Nevanlinna theory. A starting point for such results comes from the study of holomorphic curves in semi-abelian varieties by Noguchi, Winkelmann, and Yamanoi, who proved the following:

**Theorem 1.7** (Noguchi, Winkelmann, Yamanoi [20, Th. 5.1] (see also [19, Section 6.5])). Let \( f: \mathbb{C} \rightarrow A \) be a holomorphic map to a semi-abelian variety \( A \) with Zariski-dense image. Let \( Y \) be a closed subscheme of \( A \) with \( \text{codim} Y \geq 2 \) and let \( \epsilon > 0 \).

(i) Then

\[
N_{f}(Y, r) \leq_{\text{exc}} \epsilon T_{f}(r).
\]

(ii) There exists a compactification \( A \) of \( A \), independent of \( \epsilon \), such that for the Zariski closure \( \overline{Y} \) of \( Y \) in \( A \),

\[
T_{\overline{Y}, f}(r) \leq_{\text{exc}} \epsilon T_{f}(r).
\]

Here \( N_{f}(Y, r) \) is a counting function associated to \( f \) and \( Y \), \( T_{\overline{Y}, f}(r) \) is a Nevanlinna characteristic (or height) function associated to \( f \) and \( \overline{Y} \), and \( T_{f}(r) \) is any characteristic function associated to an appropriate ample line bundle (see [16, Section 2] for the relevant definitions from Nevanlinna theory and [20] for more discussion). The notation \( \leq_{\text{exc}} \) means that the estimate holds for all \( r \) outside a set of finite Lebesgue measure, possibly depending on \( \epsilon \). More generally, Noguchi,
Winkelmann, and Yamanoi proved a result for $k$-jet lifts of holomorphic maps to semi-abelian varieties. The case when $A$ is an abelian variety was proved by Yamanoi [27].

In their recent work, Levin and the first named author obtain a refinement and a new proof of Theorem 1.7 when $A = (\mathbb{C}^*)^n$ is the complex algebraic torus [10, Theorem 1.2] by adapting the number-theoretical arguments of [12] to Nevanlinna theory. Their result in this case is a complex counter parts of Theorem 1.2. They also prove asymptotic gcd estimates for holomorphic maps in more general context (see [10, Theorem 1.8]). We refer to [10] and [21] for further discussion on the work of gcd problems. The proofs of our arithmetic results depend on the general version of Schmidt’s subspace theorem by Ru and Vojta in [22] and Levin’s Subspace theorem for algebraically equivalent divisors [14, Theorem 3.2]. The complex counter part of both results exist. (See [22, General theorem (analytic part)] and [14] for details.) The formal properties of Weil functions and height functions also carry over to the corresponding functions in Nevanlinna theory. Therefore, the following complex result can be derived in parallel to the arithmetic case. We refer to [16] for notation and basic results in Nevanlinna theory, and will omit details.

**Theorem 1.8.** Let $V$ be a Cohen–Macaulay complex projective variety of dimension $n$. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisor of $V$ in general position. Suppose that there exists an ample Cartier divisor $A$ on $V$ and positive integer $d_i$ such that $D_i \equiv d_i A$ for all $1 \leq i \leq n+1$. Let $Y$ be a closed subscheme of $V$ of codimension at least 2 that does not contain any point of the set $\bigcup_{i=1}^{n+1} (\cap_{1 \leq j \neq i \leq n+1} \text{Supp } D_j)$. Let $f : \mathbb{C} \rightarrow V \setminus (\bigcup_{i=1}^{n+1} \text{Supp } D_i)$ be a holomorphic mapping with Zariski-dense image. Let $\epsilon > 0$. Then for a given $\epsilon > 0$, we have

$$T_{Y,f}(r) \leq \epsilon T_{A,f}(r).$$

The relevant background material will be given in the next section. In Section 3 we formulate two main theorems: one (Theorem 3.1) for the part of the proximity function of the closed subscheme $Y$ in Theorem 1.3, and the other (Theorem 3.2) to estimate the entire height function of $Y$ with an extra assumption that $Y$ is not contained in any of the $D_i$. We will deduce our main results from these two theorems. The idea for the second theorem is what we have described above. The proof will be given in Section 4 after a quick introduction of the work of Ru–Vojta [22]. The first theorem dealing with the proximity function part relies on Levin’s generalization to a numerical equivalent class [14, Theorem 3.2] from a version of Schmidt’s subspace theorem of Evertse and Ferretti [4, Theorem 1.6] for a linear equivalence class of divisors. The proof will be given in Section 5.

2. Notation and Background Material

2.1. Heights and integral points. We refer to [11, Chapter 10], [8, B.8], [15, Section 2.3] or [23, Section 2] for more details about this section. Let $k$ be a number field and $M_k$ be the set of places, normalized so that it satisfies the product formula

$$\prod_{v \in M_k} |x|_v = 1, \quad \text{for } x \in k^\times.$$

For a point $[x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, the standard logarithmic height is defined by

$$h([x_0 : \cdots : x_n]) = \sum_{v \in M_k} \log \max\{|x_0|_v, \ldots, |x_n|_v\}.$$
This quantity is independent of the choice of the homogeneous coordinates \( x_0, \ldots, x_n \) by the product formula.

We also recall that an \( M_k \)-constant is a family \( \{ \gamma_v \}_{v \in M_k} \), where each \( \gamma_v \) is a real number with all but finitely many being zero. Given two families \( \{ \lambda_{1v} \} \) and \( \{ \lambda_{2v} \} \) of functions parametrized by \( M_k \), we say \( \lambda_{1v} \leq \lambda_{2v} \) holds up to an \( M_k \)-constant if there exists an \( M_k \)-constant \( \{ \gamma_v \} \) such that the function \( \lambda_{2v} - \lambda_{1v} \) has values at least \( \gamma_v \) everywhere. We say \( \lambda_{1v} = \lambda_{2v} \) up to an \( M_k \)-constant if \( \lambda_{1v} \leq \lambda_{2v} \) and \( \lambda_{2v} \leq \lambda_{1v} \) up to \( M_k \)-constants.

Let \( V \) be projective variety defined over a number field \( k \). The classical theory of heights associates to every Cartier divisor \( D \) on \( V \) a height function \( h_D : V(k) \to \mathbb{R} \) and a local Weil function (or local height function) \( \lambda_{D,v} : V(k) \setminus \text{Supp}(D) \to \mathbb{R} \) for each \( v \in M_k \), such that

\[
\sum_{v \in M_k} \lambda_{D,v}(P) = h_D(P) + O(1)
\]

for all \( P \in V(k) \setminus \text{Supp}(D) \).

We also recall some basic properties of local Weil functions associated to closed subschemes from [23, Section 2]. Given a closed subscheme \( Y \) on a projective variety \( V \) defined over \( k \), we can associate to each place \( v \in M_k \) a function \( \lambda_{Y,v} : V \setminus \text{Supp}(Y) \to \mathbb{R} \).

Intuitively, for each \( P \in V \) and \( v \in M_k \)

\[
\lambda_{Y,v}(P) = -\log(v\text{-adic distance from } P \text{ to } Y).
\]

To describe \( \lambda_{Y,v} \) more precisely, we need the following lemma:

**Lemma 2.1.** Let \( Y \) be a closed subscheme of \( V \). There exist effective divisors \( D_1, \ldots, D_r \) such that

\[
Y = \cap D_i.
\]

**Proof.** See Lemma 2.2 from [23]. \( \square \)

**Definition-Theorem 2.2** ([23, Lemma 2.5.2], [23, Theorem 2.1 (d)(h)]). Let \( k \) be a number field, and \( M_k \) be the set of places on \( k \). Let \( V \) be a projective variety over \( k \) and let \( Y \subset V \) be a closed subscheme of \( V \). We define the (local) Weil function for \( Y \) with respect to \( v \in M_k \) as

\[
\lambda_{Y,v} = \min_i \{ \lambda_{D_i,v} \},
\]

where \( D_i \)'s are as in the above lemma. This is independent of the choices of the \( D_i \)'s up to an \( M_k \)-constant, and satisfies

\[
\lambda_{Y_1,v}(P) \leq \lambda_{Y_2,v}(P)
\]

up to an \( M_k \)-constant whenever \( Y_1 \subseteq Y_2 \). Moreover, if \( \pi : \tilde{V} \to V \) is the blowup of \( V \) along \( Y \) with the exceptional divisor \( E \) (corresponding to the inverse image ideal sheaf of the ideal sheaf of \( Y \)), \( \lambda_{Y,v}(\pi(P)) = \lambda_{E,v}(P) \) up to an \( M_k \)-constant as functions on \( \tilde{V}(k) \setminus E \).

The height function for a closed subscheme \( Y \) of \( V \) is defined by

\[
h_Y(P) := \sum_{v \in M_k} \lambda_{Y,v}(P),
\]
for \( P \in V(k) \setminus Y \). We also define two functions related to the height function for a closed subscheme \( Y \) of \( V \), depending on a finite set of places \( S \) of \( k \): the proximity function

\[
m_{Y,S}(P) := \sum_{v \in S} \lambda_{Y,v}(P)
\]

and the counting function

\[
N_{Y,S}(P) := \sum_{v \in \mathcal{M}_k \setminus S} \lambda_{Y,v}(P) = h_Y(P) - m_{Y,S}(P)
\]

for \( P \in V(k) \setminus Y \).

**Definition 2.3.** Let \( k \) be a number field and \( \mathcal{M}_k \) be the set of places on \( k \). Let \( S \subset \mathcal{M}_k \) be a finite subset containing all archimedean places. Let \( X \) be a projective variety over \( k \), and let \( D \) be an effective divisor on \( X \). A set \( R \subseteq X(k) \setminus \text{Supp } D \) is a \((D,S)\)-integral set of points if there is a Weil function \( \{\lambda_{D,v}\} \) for \( D \) and an \( \mathcal{M}_k \)-constant \( \{\gamma_v\} \) such that for all \( v \notin S \), \( \lambda_{D,v}(P) \leq \gamma_v \) for all \( P \in R \).

It follows directly from the definition that if \( R \subseteq X(k) \setminus D \) is a \((D,S)\)-integral set of points, then \( N_{D,S}(P) \leq O(1) \) for \( P \in R \).

2.2. **Basic Propositions.** The following proposition is an immediate consequence of [8, Theorem B.3.2.(f)].

**Proposition 2.4.** Let \( X \) be a projective variety defined over a number field \( k \), and \( A \) be an ample Cartier divisor on \( X \) defined over \( k \). Let \( D \) be an effective divisor \( D \) defined over \( k \) with \( D \equiv A \). Let \( \epsilon > 0 \). Then there exists a constant \( c_\epsilon \) such that

\[
(1 - \epsilon)h_A(P) - c_\epsilon \leq h_D(P) \leq (1 + \epsilon)h_A(P) + c_\epsilon
\]

for \( P \in X(k) \).

**Proof.** Since \( A \) is ample and a nonzero multiple of \( A - D \) is algebraically equivalent to zero (see [13, Remark 1.1.20]), [8, Theorem B.3.2.(f)] shows that

\[
\lim_{h_A(P) \to \infty} \frac{h_{A-D}(P)}{h_A(P)} = 0
\]

In particular, for each \( \epsilon > 0 \), there exists \( C_\epsilon \) such that when \( h_A(P) \geq C_\epsilon \),

\[
-\epsilon h_A(P) < h_{A-D}(P) < \epsilon h_A(P),
\]

that is, \((1 - \epsilon)h_A(P) < h_D(P) < (1 + \epsilon)h_A(P)\). By Northcott, there are only finitely many points \( P \in X(k) \) with \( h_A(P) < C_\epsilon \), we can adjust by a suitable constant \( c_\epsilon \) to prove the assertion. \( \square \)

**Proposition 2.5.** Let \( V \) be a projective variety defined over a number field \( k \) and \( A \) be an ample divisor of \( V \). Let \( P_1, \ldots, P_\ell \) be distinct points of \( V(k) \). Let \( Y \) be a closed subscheme of \( V \) over \( k \) which does not contain any \( P_i \), \( 1 \leq i \leq \ell \). Then there exists an effective divisor \( D \) over \( k \) of \( V \) linearly equivalent to a multiple of \( A \) such that \( Y \) is contained in \( D \) (scheme-theoretically) and \( P_i \notin D \) for any \( 1 \leq i \leq \ell \).

**Proof.** Choose a positive integer \( N \) such that \( NA \) is a very ample divisor of \( V \). Let \( \phi : V \to \mathbb{P}^m \) be the closed embedding corresponding to \( NA \), i.e. \( NA = \phi^{-1}(H) \) for some hyperplane \( H \) in \( \mathbb{P}^m \). Let \( I_Y \) be the homogeneous ideals of \( k[x_0, \ldots, x_m] \) defining \( Y \) as closed subschemes of \( \mathbb{P}^m_k \). Let \( F_t \in I_Y \), \( 1 \leq t \leq r \), be a set of homogeneous polynomials of the same degree that generate the ideal \( I_Y \). Such generators
exist, since any set of generators can be made the same degree by multiplying each generator with suitable powers of \( x_0, \ldots, x_m \). Let
\[
F = a_1 F_1 + \cdots + a_r F_r \in I_Y,
\]
where \( a_1, \ldots, a_r \in k \) will be determined later. Since \( P_i \in V \setminus Y \), for \( 1 \leq i \leq \ell \), at least one of the \( F_i(P_i), \ldots, F_r(P_i) \) is not equal to 0. Therefore, each of the equation
\[
F_i(P_i)x_1 + \cdots + F_r(P_i)x_r = 0,
\]
for \( 1 \leq i \leq \ell \), determines a proper closed subvariety \( H_i \) in \( \mathbb{P}^{r-1} \). Choose \( (a_1, \ldots, a_r) \) from \( \mathbb{P}^{r-1}(k) \setminus \bigcup_{i=1}^{\ell} H_i \). Then the divisor \( D = [F = 0] \cap V \) contains \( Y \) as a subscheme and \( \text{Supp} \ D \) does not contain any \( P_i \), \( 1 \leq i \leq \ell \). Moreover, the divisor \( W := [F = 0] \) on \( \mathbb{P}^m \) is linearly equivalent to \( (\deg F) \cdot H \) and hence its pull-back \( \phi^{-1}(W) = D \) is linearly equivalent to \( (\deg F \cdot N) A \).

3. Proof of Theorem 1.3 and Theorem 1.4

Theorem 1.3 and Theorem 1.4 are consequences of the following two theorems.

**Theorem 3.1.** Let \( V \) be a projective variety of dimension \( n \) defined over a number field \( k \), and let \( S \) be a finite set of places of \( k \). Let \( D_1, \ldots, D_{n+1} \) be effective Cartier divisors of \( V \) defined over \( k \) in general position. Suppose that there exist an ample Cartier divisor \( A \) on \( V \) and positive integers \( d_i \) such that \( D_i \equiv d_i A \) for all \( 1 \leq i \leq n + 1 \). Let \( Y \) be a closed subscheme of \( V \) which does not contain any point of the set \( \bigcup_{i=1}^{n+1} (\bigcap_{1 \leq j \neq i \leq n+1} \text{Supp} \ D_j) \). Let \( \epsilon > 0 \). Then there exists a proper Zariski closed set \( Z \) such that for any set \( R \) of \( (\sum_{i=1}^{n+1} D_i, S) \)-integral points on \( V(k) \) we have
\[
\text{(12)} \quad m_{Y,S}(P) \leq \epsilon h_A(P) + O(1)
\]
for all \( P \in R \setminus Z \).

**Theorem 3.2.** Under the assumptions of Theorem 3.1, let us further assume that \( V \) is Cohen–Macaulay and that the support of \( Y \) is not contained in any support of \( D_i \), \( 1 \leq i \leq n+1 \). Then for any \( \epsilon > 0 \), there exists a proper Zariski closed set \( Z \) such that for any set \( R \) of \( (\sum_{i=1}^{n+1} D_i, S) \)-integral points on \( V(k) \) we have
\[
\text{(13)} \quad h_Y(P) \leq \epsilon h_A(P) + O(1)
\]
for all \( P \in R \setminus Z \).

**Proof of Theorem 3.1.** First, suppose that the support of \( Y \) is contained in \( \text{Supp} \ D_i \) for some \( 1 \leq i \leq n+1 \). Then \( Y \) with the reduced subscheme structure is contained in \( D_i \) scheme-theoretically. Moreover, since some suitable power of the ideal of the reduced structure is contained in the ideal of \( Y \), it follows from [23, Theorem 2.1(c)] that \( h_Y \) is less than or equal to a multiple of the height with respect to \( Y \) with the reduced structure. Therefore, we may assume that \( Y \) has the reduced structure and \( Y \subset D_i \) scheme-theoretically. Consequently, the definition of the local Weil function implies that
\[
N_{Y,S}(P) \leq N_{D_i,S}(P) + O(1) \leq O(1),
\]
for all \( P \in R \). On the other hand, we may apply Theorem 3.1 to find a proper Zariski closed set \( Z \) such that for any set \( R \) of \( (\sum_{i=1}^{n+1} D_i, S) \)-integral points on \( V(k) \) we have
\[
\text{(14)} \quad m_{Y,S}(P) \leq \epsilon h_A(P) + O(1)
\]
for all $P \in R \setminus Z$. Then the assertion is satisfied by combining the above two inequalities.

Suppose next that the support of $Y$ is not contained in any $\Supp D_i$ for $1 \leq i \leq n+1$. This case is covered by Theorem 3.2.

**Proof of Theorem 1.4.** Since $Y$ is a set of finitely many points, it suffices to show for $Y$ being a point $Q$ with a reduced subscheme structure. If $Q \in D_i$ for some $i$, then

$$N_{Q,S}(P) \leq N_{D_i,S}(P) + O(1) \leq O(1).$$

If $Q \notin D_i$ for each $1 \leq i \leq n+1$, then $Q$ is not a point in $\bigcup_{i=1}^{n+1}(\cap_{j\neq i}^{n+1} \Supp D_j)$. Then by Theorem 6.2 there exists a proper Zariski closed set $Z_Q$ such that for any set $R$ of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $V(k)$ we have

$$N_{Q,S}(P) \leq h_Q(P) \leq \epsilon h_A(P) + O(1)$$

for all $P \in R \setminus Z_Q$. □

**Proof of Corollary 1.5.** By replacing $f$ by $f^{\deg g}$ and $g$ by $g^{\deg f}$, we may assume that $\deg f = \deg g = d$. Let $F(X,Y,Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ and $G(X,Y,Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ be the homogenization of $f(X,Y)$. Let $\mathcal{F}$ and $\mathcal{G}$ be the projective curves defined by $F$ and $G$ respectively. Recall the following (standard) Weil functions for $v \in M_k$:

$$\lambda_{\mathcal{F},v}(x, y, z) = -\log \left\| F(x, y, z) \right\|_v \geq 0,$$

where $\left\| f \right\|_v$ is the maximum of the $v$-adic absolute value of the coefficients of $f$. In particular, for $v \in M_k \setminus S$ and $x, y \in \mathcal{O}_S^*$, we have

$$\lambda_{\mathcal{F},v}(x, y, 1) = -\log \left| F(x, y, 1) \right|_v + \log \left\| f \right\|_v = -\log \left| f(x, y) \right|_v + \log \left\| f \right\|_v,$$

and hence

$$-\log \left| f(x, y) \right|_v \leq \lambda_{\mathcal{F},v}(x, y, 1) - \log \left\| f \right\|_v$$

for $v \in M_k \setminus S$ and $x, y \in \mathcal{O}_S^*$. We note that the corresponding equations hold for $g$ respectively. Let $D_1 := \{X = 0\}, D_2 := \{Y = 0\}$, and $D_3 := \{Z = 0\}$ be the divisors given by the coordinate hyperplanes in $\mathbb{P}^2$. Then Theorem 1.4 for the set $W$ of points defined as the scheme-theoretic intersection of $\mathcal{F}$ and $\mathcal{G}$ implies that there exists a proper Zariski closed set $Z$ of $\mathbb{P}^2(k)$ such that

$$\sum_{v \in M_k \setminus S} -\log \max\{|f(x, y)|_v, |g(x, y)|_v\}$$

$$\leq \sum_{v \in M_k \setminus S} \min\{\lambda_{\mathcal{F},v}(x, y, 1), \lambda_{\mathcal{G},v}(x, y, 1)\} + O(1)$$

$$= \sum_{v \in M_k \setminus S} \lambda_{W}(x, y, 1) + O(1) \quad (\because \text{Definition-Theorem 2.2})$$

$$\leq \epsilon \max\{h(x), h(y)\} + O(1)$$

for all $(x, y) \in \mathbb{G}_m^2(\mathcal{O}_S) \setminus Z$. We note that the constant $O(1)$ can be eliminated by adding a finite number of points in $Z$. Finally, since $x$ and $y$ are $S$-units, we can conclude the proof by the standard reduction argument that the Zariski closure of a subset of $\mathbb{G}_m^2(\mathcal{O}_S)$ is a finite union of translates of algebraic subgroups (of dimension 1) by Laurent 12. □
4. Proof of Theorem 3.2

4.1. Theorem of Ru–Vojta and some basic propositions. We first recall the following definitions and geometric properties from [22].

**Definition 4.1.** Let $\mathcal{L}$ be a big line sheaf and let $D$ be a nonzero effective Cartier divisor on a projective variety $X$. We define

$$\gamma_{\mathcal{L}, D} := \limsup_{N \to \infty} \frac{N \cdot h^0(V, L^N)}{\sum_{m=1}^{\infty} h^0(V, L^m(-mD))}.$$ 

**Definition 4.2.** Let $D_1, \ldots, D_q$ be effective Cartier divisors on a variety $X$ of dimension $n$.

(i) We say that $D_1, \ldots, D_q$ are in general position if for any $I \subset \{1, \ldots, q\}$, we have $\dim(\cap_{i \in I} \text{Supp } D_i) \leq n - |I|$, with the convention $\dim \emptyset = -\infty$.

(ii) We say that $D_1, \ldots, D_q$ intersect properly if for any $I \subset \{1, \ldots, q\}$ and $x \in \cap_{i \in I} \text{Supp } D_i$, the sequence $(\phi_i)_{i \in I}$ is a regular sequence in the local ring $\mathcal{O}_{X, x}$, where $\phi_i$ are the local defining equations of $D_i$, $1 \leq i \leq q$.

**Remark 4.3.** If $D_1, \ldots, D_q$ intersect properly, then they are in general position. By [18, Theorem 17.4], the converse holds if $X$ is Cohen-Macaulay.

We will use the following result to prove Lemma 3.2.

**Theorem 4.4.** [22, General Theorem (Arithmetic Part)] Let $k$ be a number field and $M_k$ be the set of places on $k$. Let $S \subset M_k$ be a finite subset. Let $X$ be a projective variety defined over $k$. Let $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on $X$. Let $\mathcal{L}$ be a big line sheaf on $X$. Then for any $\epsilon > 0$

$$\sum_{i=1}^{q} m_{D_i} s(x) \leq \left( \max_{1 \leq i \leq q} \{ \gamma_{\mathcal{L}, D_i} \} + \epsilon \right) h_{\mathcal{L}}(x)$$

holds for all $x$ outside a proper Zariski-closed subset of $X(k)$.

The following proposition is well-known in the case when $X$ and $Y$ are nonsingular (and presumably even in this setting to the experts):

**Proposition 4.5.** Let $X$ be a Cohen-Macaulay scheme over $k$ and $Y \subset X$ be a locally complete intersection subscheme. Let $\pi : \tilde{X} \rightarrow X$ be the blowup of $X$ along $Y$. Then $\tilde{X}$ is a Cohen-Macaulay scheme. Moreover, if $Z$ is an irreducible subscheme of $Y$,

$$\dim \pi^{-1}(Z) = \dim Z + \text{codim } Y - 1.$$

**Proof.** Cohen-Macaulayness follows from [10, Proposition 5.5(i)]. For the second part, let $\mathcal{I}$ be the ideal sheaf corresponding to $Y$. Then [10, Proposition 5.5(ii)] shows that $\mathcal{I}/\mathcal{I}^2$ is locally free, and of rank $\text{codim } Y$. Since [10, Proposition 5.5(iii)] shows that over $Y$, $\pi$ is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{I}/\mathcal{I}^2) \rightarrow Y$, the conclusion follows.

We will use the following asymptotic Riemann-Roch theorem for nef divisors (see [13, Corollary 1.4.41]).

**Theorem 4.6.** Let $X$ be a projective variety of dimension $n$ and $D$ be a nef divisor on $X$. Then

$$h^0(X, O(mD)) = \frac{D^n}{n!} \cdot m^n + O(m^{n-1}).$$

For simplicity of notation, we let $h^0(X, D) := h^0(X, O(D))$. 

4.2. Proof of Theorem 3.2. We need a technical lemma.

Lemma 4.7. Let $V$ be a projective variety of dimension $n$. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisors of $V$ defined over $k$ in general position. Suppose that there exists an ample Cartier divisor $A$ on $V$ such that $D_i \equiv A$ for all $1 \leq i \leq n+1$. Let $Y$ be a closed subscheme of $V$ of codimension at least 2. Let $\pi : \tilde{V} \to V$ be the blowup along $Y$, and $E$ be the exceptional divisor. Let $D := D_1 + \cdots + D_{n+1}$. Then for all sufficiently large $\ell$, the divisor $\ell\pi^*D - E$ is ample and

$$
(18) \quad \gamma_{\mathcal{L}, \pi^*D_i} \leq \frac{1}{\ell} \left( 1 + O \left( \frac{1}{\ell^2} \right) \right) \leq \frac{1}{\ell} \left( 1 + \frac{1}{\ell \sqrt{\ell}} \right),
$$

where $\mathcal{L} = \mathcal{O}(\ell\pi^*D - E)$.

Proof of Theorem 3.2. Since the set of $D_i$ with $m_i \equiv 1$ for each $1 \leq i \leq n + 1$, we will assume that $D_i \equiv A$ for each $1 \leq i \leq n + 1$. By Lemma 4.5, we find an effective Cartier divisor $H_1$ which contains $Y$. Since the codimension of $Y$ is at least two, we can choose a point inside each irreducible component of $\text{Supp } H_1$ lying outside of Supp $Y$; let $T$ be the set of such points. Now, we apply Proposition 1.1 to obtain an effective Cartier divisor $H_2$ containing $Y$ while not containing any point of the set $T \cup \bigcup_{i=1}^{n+1} (\cap_{1 \leq j \neq i \leq n+1} \text{Supp } D_j)$. Since the $\text{Supp } (H_2)$ intersects with each irreducible component of $\text{Supp } (H_1)$, it follows that $Y' = H_1 \cap H_2$ has codimension 2 in $V$ and $H_1$ and $H_2$ are in general position. Consequently, $Y'$ is locally defined by the local equations of $H_1$ and $H_2$ which form a regular sequence, as $V$ is Cohen-Macaulay. Therefore, $Y'$ is of locally complete intersection. Moreover, $Y'$ is not contained in any of the $D_i$ since $Y' \cap Y$. As $h_Y \leq h_{Y'} + O(1)$, we may assume $Y$ is of locally complete intersection of codimension exactly 2 by replacing $Y$ with $Y'$.

Let $\pi : \tilde{V} \to V$ be the blowup along $Y$, and $E$ be the exceptional divisor. Since the support of $Y$ is not contained in any $\text{Supp } D_i$, $1 \leq i \leq n + 1$, then $\pi^*D_i$ is the strict transform $\tilde{D}_i$ of $D_i$, for each $1 \leq i \leq n + 1$. Moreover, $\pi|_{\tilde{D}_i} : \tilde{D}_i \to D_i$ is the blowup along $D_i \cap Y$ (see [13, Chapter II, Corollary 7.15]); in particular, $\pi(\tilde{D}_i) = D_i$.

Claim. $\tilde{D}_1, \ldots, \tilde{D}_{n+1}$ are in general position.

Proof. We first check their intersection is empty. Suppose that $q \in \tilde{D}_i$, for each $1 \leq i \leq n + 1$. Then $\pi(q) \in \cap_{i=1}^{n+1} \text{Supp } D_i$ which is not possible as $D_1, \ldots, D_{n+1}$ are in general position. Next, let $I \subset \{1, \ldots, n + 1\}$ with $\#I \leq n$. We show that $\dim(\cap_{i \in I} \text{Supp } \tilde{D}_i) \leq n - \#I$. By rearranging the index of the $D_i$, we let $I = \{1, \ldots, r\}$. By assumption, $\cap_{i=1}^{r} D_i$ has dimension at most $n - r$, but this dimension is also at least $n - r$ because each $D_i$ is ample (see [13, Corollary 1.2.24]). Now, if it has an irreducible component $W$ (of dimension $n - r$) lying inside $Y$, then $W.D_{r+1} \cdots .D_{n+1} = W.A^{n-r} > 0$, contradicting the assumption that $Y$ does not contain points in $\cap_{i=1}^{n} D_i$. Therefore, $\dim(\cap_{i=1}^{n} D_i) \setminus Y = \dim(\cap_{i=1}^{n} D_i) = n - r$. Since $(\cap_{i=1}^{r} \tilde{D}_i) \setminus \pi^{-1}(Y)$ and $(\cap_{i=1}^{r} D_i) \setminus Y$ are isomorphic, if we prove that any component $W$ of $(\cap_{i=1}^{r} \tilde{D}_i)$ which is contained in $(\cap_{i=1}^{r} D_i)$ satisfies $W \leq n - r$, then $\dim(\cap_{i=1}^{r} D_i) = \dim(\cap_{i=1}^{r} \tilde{D}_i) \setminus \pi^{-1}(Y)) = n - r$ and we are done. Now, let $Z = \pi(W) \subset (\cap_{i=1}^{r} D_i) \cap Y$. Since we have shown that no component of $(\cap_{i=1}^{r} D_i)$ is contained in $Y$, $\dim Z \leq n - r - 1$. By Proposition 1.5, $\dim \pi^{-1}(Z) = \dim Z + 1 \leq n - r - 1$. Therefore, $\dim \pi^{-1}(Z) \leq n - r - 1$, which implies that $\dim(\cap_{i=1}^{r} D_i) \setminus Y = n - r$.
Since $W \subset \pi^{-1}(Z)$, \( \dim W \leq n - r \). This completes the proof of the claim.

Since $Y$ is of locally complete intersection, $\tilde{V}$ is Cohen-Macaulay by Proposition 4.5 and hence $\tilde{D}_1, \ldots, \tilde{D}_{n+1}$ intersect properly. Let $D := D_1 + \cdots + D_{n+1}$. Let $\ell$ be a fixed positive integer satisfying $\sqrt{\frac{n+1}{\ell^2}} < \epsilon$ such that the line sheaf $\mathcal{L} = \mathcal{O}(\ell \pi^* D - E)$ is ample and Lemma 4.7 holds true, i.e.

\[
\gamma_{\mathcal{L}, \pi^* D_i} \leq \frac{1}{\ell} \left( 1 + \frac{1}{\ell \sqrt{\ell}} \right).
\]

Let $\epsilon' = \ell^{-5/2}$. Applying Theorem 4.4 to $\epsilon', \tilde{V}, \mathcal{L}$ and $\pi^* D_i$ (for $1 \leq i \leq n+1$), we have

\[
\sum_{i=1}^{n+1} m_{\pi^* D_i, S}(x) \leq \left( \frac{1}{\ell} \left( 1 + \frac{1}{\ell \sqrt{\ell}} \right) + \epsilon' \right) h_{\pi^* D - E}(x)
\]

\[
= \left( 1 + \frac{2}{\ell \sqrt{\ell}} \right) h_{\pi^* D}(x) - \left( \frac{1}{\ell} + \frac{2}{\ell^2 \sqrt{\ell}} \right) h_E(x)
\]

holds for all $x$ outside a proper Zariski-closed subset $Z$ of $\tilde{V}(k)$. By the functorial properties of the local height functions, $h_D = m_{D, S} + N_{D, S}$, and $h_E = h_Y \circ \pi$, we have

\[
\frac{1}{\ell} h_Y(\pi(x)) \leq \frac{2}{\ell \sqrt{\ell}} \cdot h_D(\pi(x)) + \sum_{i=1}^{n+1} N_{D_i, S}(\pi(x))
\]

holds for all $\pi(x)$ outside a proper Zariski-closed subset $Z'$ of $V(k)$. Since $D \equiv (n + 1)A$, by Proposition 2.4 we may replace

\[
h_D(\pi(x)) \leq (n + 1)(1 + \epsilon') h_A(\pi(x)) + c_1
\]

Let $R$ be a set of $(D, S)$-integral points on $V(k)$. Then there is a constant $c_2$ such that

\[
\sum_{i=1}^{n+1} N_{D_i, S}(P) \leq c_2
\]

for all $P \in R$. Consequently, (21) implies

\[
h_Y(P) \leq \frac{2(n + 1)(1 + \epsilon')}{\sqrt{\ell}} \cdot h_A(P) + \frac{2c_1}{\sqrt{\ell}} + \ell c_2
\]

\[
\leq \frac{4(n + 1)}{\sqrt{\ell}} h_A(P) + \frac{2c_1}{\sqrt{\ell}} + \ell c_2
\]

\[
< \varepsilon h_A(P) + O(1)
\]

holds for all all $P \in R \setminus Z'$.
Theorem 4.6 implies
\[ h^0(\mathcal{V}, \mathcal{L}^{\ell}) = h^0(\mathcal{V}, M\ell(\ell\pi*D - E)) = \frac{(M\ell)^n(\ell\pi*D - E)^n}{n!} + O((M\ell)^{n-1}) \]
\[ \geq \frac{((n + 1)\ell)^nA^n + C(M\ell)^n}{n!} + O((M\ell)^{n-1}), \]
where
\[ C = \sum_{j=1}^{n} \binom{n}{j} (\ell(n + 1))^{n-j}(\pi^*A)^{n-j}(-E)^j. \]

Let \( m \leq (n + 1)M\ell^2 - M \) be a positive integer. Similarly,
\[ h^0(\mathcal{V}, \mathcal{L}^{\ell}(-m\pi^*D_i)) = h^0(\mathcal{V}, M\ell(\ell\pi*D - E) - m\pi^*D_i) \]
\[ = h^0(\mathcal{V}, ((n + 1)M\ell^2 - m)\pi^*A - M\ell E) \]
\[ \geq \frac{((n + 1)\ell^2 - \lfloor \frac{m}{M} \rfloor - 1)^nA^n + C_{\lfloor \frac{m}{M} \rfloor, \ell}M^n + O(M^{n-1})}{n!} \]
\[ \geq \frac{((n + 1)\ell^2 - \lfloor \frac{m}{M} \rfloor - 1)^nA^n + C_{\lfloor \frac{m}{M} \rfloor, \ell}M^n + O(M^{n-1})}{n!} \]
\[ = \frac{((n + 1)M\ell^2 - M - m)^nA^n + C_{\lfloor \frac{m}{M} \rfloor, \ell}M^n + O(M^{n-1})}{n!} \]
where
\[ C_{\lfloor \frac{m}{M} \rfloor, \ell} = \sum_{j=1}^{n} \binom{n}{j} \ell^j((n + 1)\ell^2 - \lfloor \frac{m}{M} \rfloor - 1)^{n-j}(\pi^*A)^{n-j}(-E)^j. \]

Since \( 0 \leq \frac{m}{M} \leq (n + 1)\ell^2 - 1 \), we let
\[ C_\ell = \max_{0 \leq j \leq (n + 1)\ell^2 - 1} \{|C_{j, \ell}|\}. \]

Then we have
\[ n! \sum_{m \geq 1} h^0(\mathcal{V}, \mathcal{L}^{\ell}(-m\pi^*D_i)) \]
\[ \geq \sum_{m = 1}^{(n + 1)M\ell^2 - M} \left( \frac{((n + 1)M\ell^2 - M - m)^nA^n + C_{\lfloor \frac{m}{M} \rfloor, \ell}M^n + O(M^{n-1})}{n!} \right) \]
\[ \geq \frac{((n + 1)M\ell^2 - M - 1)^{n+1}A^n}{n + 1} - ((n + 1)M\ell^2 - M)C\ell M^n + O(M^n\ell^2) \]
\[ = ((n + 1)\ell - \frac{1}{\ell})^{n+1}A^n(M\ell)^{n+1} - ((n + 1)\ell - \frac{1}{\ell})(M\ell)^{n+1}\frac{C\ell}{\ell^n} + O(M^n\ell^2). \]
Consequently,
\[
\gamma_{\mathcal{L}, \pi^* D_i} = \limsup_{M \to \infty} \frac{M h^0(V, \mathcal{L}^M)}{\sum_{m \geq 1} h^0(V, \mathcal{L}^M(-m \pi^* D_i))} \leq \frac{((n + 1)\ell)^n A^n + C}{((n + 1)\ell - \frac{1}{\ell})^{n+1} A \frac{n\pi}{n+1} - ((n + 1)\ell - \frac{1}{\ell})^{n+1} A^n} = c_\ell \ell,
\]
where
\[
c_\ell = \frac{1 + \left(\frac{1}{n+1}\right)^n \frac{C}{A^n}}{\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} - \left(\frac{1}{(n+1)^{n+1}} - \frac{1}{(n+1)^{n+2}}\right) \frac{C}{A^n}}.\]

We note that \((\pi^* A)^{n-1} E = A^{n-1} Y = 0\) since the codimension of \(Y\) is at least 2. (See [9, Chapter VI. Proposition 2.11]). Therefore, (25) and (27) satisfy
\[
C = \left(\frac{n}{2}\right) (n+1)^{n-2} (\pi^* A)^{n-2} E^2 \cdot \ell^{n-2} + O(\ell^{n-3})
\]
\[
C_{(n^h), \ell} = \left(\frac{n}{2}\right) (n+1)^{n-2} (\pi^* A)^{n-2} E^2 \cdot \ell^{2n-2} + O(\ell^{2n-3}).
\]

Then for \(\ell\) sufficiently large, \(c_\ell = 1 + O(1/\ell^2)\), proving the claim. \(\square\)

5. Proof of Theorem 3.1

We recall the following theorem of Levin, which is a generalization of a result of Evertse and Ferretti [4, Theorem 1.6].

**Theorem 5.1.** [4] Theorem 3.2] Let \(X\) be a projective variety of dimension \(n\) defined over a number field \(k\). Let \(S\) be a finite set of places of \(k\). For each \(v \in S\), let \(D_{0,v}, \ldots, D_{n,v}\) be effective Cartier divisors on \(X\), defined over \(k\), in general position. Suppose that there exists an ample Cartier divisor \(A\) on \(X\) and positive integer \(d_{i,v}\) such that \(D_i,v = d_{i,v} A\) for all \(i\) and all \(v \in S\). Let \(\epsilon > 0\). Then there exists a proper Zariski-closed subset \(Z \subset X\) such that for all points \(P \in X(k) \setminus Z\),

\[
\sum_{v \in S} \sum_{i=0}^{n} \frac{\lambda_{D_{i,v}}(P)}{d_{i,v}} < (n + 1 + \epsilon) h_A(P).
\]

**Lemma 5.2.** Let \(D_1, \ldots, D_q\) be effective divisors on a projective variety \(V\) of dimension \(n\), defined over \(k\), in general position. Then

\[
\sum_{i=1}^{q} \lambda_{D_{i,v}} \leq \max_{I} \sum_{j \in I} \lambda_{D_{j,v}}
\]

up to an \(M_k\) constant, where \(I\) runs over all index subsets of \(\{1, \ldots, q\}\) with \(n\) elements.

**Proof.** Let \(i_1, \ldots, i_q\) be a rearrangement of \(1, \ldots, q\). Since the \(D_{i}, 1 \leq i \leq q\), are in general position, \(\cap_{i=1}^{n+1} \text{Supp } D_{i} = \emptyset\). Then

\[
\left\{ \min_{1 \leq i \leq n+1} \lambda_{D_{i,v}} \right\} = \left\{ \lambda_{\cap_{i=1}^{n+1} D_{i,v}} \right\}
\]

(30)
is bounded by an $M_k$-constant. Therefore, for points $P$ satisfying
\[
\lambda_{D_{i_1}, v}(P) \geq \lambda_{D_{i_2}, v}(P) \geq \cdots \geq \lambda_{D_{i_q}, v}(P),
\]
we have
\[
\sum_{i=1}^{q} \lambda_{D_{i}, v}(P) = \sum_{i=1}^{n} \lambda_{D_{i}, v}(P)
\]
up to an $M_k$-constant. Then the assertion (29) follows directly as the number of divisors under consideration is finite. \qed

**Corollary 5.3.** Let $V$ be a projective variety of dimension $n$ defined over a number field $k$, and let $S$ be a finite set of places of $k$. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisors of $V$ defined over $k$ in general position. Suppose that there exist an ample Cartier divisor $A$ on $V$ and positive integers $d_i$ such that $D_i \equiv d_i A$ for all $1 \leq i \leq n+1$. Let $D$ be an Cartier divisor of $V$ defined over $k$ so that $D \equiv d_0 A$ for some positive integer $d_0$ and its support does not contain any point of the set $\bigcup_{i=1}^{n+1} (\cap_{j \in I} \text{Supp } D_j)$. Then for a given $\epsilon > 0$, there exists a proper Zariski-closed set $Z$ such that for any set $R$ of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $V(k)$ we have
\[
(31) \quad m_{D,S}(P) \leq \epsilon h_A(P) + O(1)
\]
for all $P \in R \setminus Z$.

**Proof.** Let $D_0 := D$, and let $d$ be the least common multiple of $d_0, \ldots, d_{n+1}$. When we replace each $D_i$ by $d/d_i D$ and $A$ by $dA$, the set of integral points do not change and (31) only changes by a multiple. Therefore, we may assume that $d = d_0 = d_1 = \cdots = d_n = 1$. We observe that $D_0 := D, D_1, \ldots, D_{n+1}$ are in general position for the following reasons. Let $I \subset \{0, 1, \ldots, n + 1\}$. First of all, the assumption implies that $\cap_{j \in I} \text{Supp } D_j = \emptyset$ if $|I| > n$. If $i := |I| \leq n$, then $\dim(\cap_{j \in I} \text{Supp } D_j) \geq n - i$ because each $D_j$ is ample (see [13, Corollary 1.2.24]). Suppose that $\dim(\cap_{j \in I} \text{Supp } D_j) = n - r > n - i$. Let $j_1, j_2, \ldots, j_{n-r-i+2}$ be the distinct elements of $\{0, 1, \ldots, n+1\} \setminus I$. Since $D_0, \ldots, D_{n+1}$ are ample, $\dim(\cap_{j \in I} \text{Supp } D_j) \cap (\cap_{i=1}^{n-r-i+1} \text{Supp } D_{j_i}) \geq n - r - (n - i + 1) = i - r - 1 \geq 0$. This is impossible since $(\cap_{j \in I} \text{Supp } D_j) \cap (\cap_{i=1}^{n-r-i+1} \text{Supp } D_{j_i})$ is empty.

Combining Theorem 5.1 and Lemma 5.2 we find a proper Zariski-closed subset $Z \subset V$ such that for all points $P \in V(k) \setminus Z$
\[
(32) \quad \sum_{i=0}^{n+1} m_{D_i, S}(P) < (n + 1 + \epsilon) h_A(P).
\]

On the other hand, for a set $R$ of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $V(k)$, we have for $1 \leq i \leq n + 1$,
\[
N_{D_i, S}(P) = O(1)
\]
for $P \in R$. Therefore,
\[
(33) \quad \sum_{i=1}^{n+1} m_{D_i, S}(P) = \sum_{i=1}^{n+1} h_{D_i, S}(P) + O(1) \geq (n + 1) (1 - \epsilon) h_A(P) + O(1)
\]
for $P \in R$, by Proposition 2.4. Then by (32) and (33), we have
\[
(34) \quad m_{D, S}(P) = m_{D_0, S}(P) < (n + 2) \epsilon h_A(P) + O(1),
\]
for $P \in R \setminus Z$.

\[ \square \]

Proof of Theorem 3.1. By Proposition 2.5, there exists an effective divisor $D$ linearly equivalent to $dA$ for some positive integer $d$ with the property that $D \supset Y$ and its support does not contain any point of $\cup_{i=1}^{n+1} (\cap_{1 \leq j \neq i \leq n+1} \text{Supp } D_j)$. Then we arrive immediately that

\[ m_{Y,S}(P) \leq m_{D,S}(P) + O(1) \]

for any $P \notin \text{Supp } Y$, and by Corollary 5.3 there exists a proper Zariski closed set $Z$ such that for a set $R$ of $\sum_{i=1}^{n+1} D_i$, $S$)-integral points on $V(k)$

\[ m_{D,S}(P) \leq ch_A(P) + O(1) \]

for all $P \in R \setminus Z$. Hence, the assertion of the theorem is proved. \[ \square \]

Remark 5.4. As in the introduction, Corollary 5.3 implies \cite[Theorem 4.1]{15}, namely that

\[ m_{D,S}(P) \leq \epsilon h(P) + O(1) \]

for all $P \in \mathbb{G}^n_m(\mathcal{O}_S)$ outside a Zariski-closed proper set, as long as $D$ does not contain $P_0 := [1 : 0 : \ldots : 0], \ldots, P_n := [0 : 0 \cdot \cdot \cdot : 0 : 1]$. In fact, we can replace $D$ and extend to the case of a closed subscheme $Y$ on $\mathbb{P}^n$ not containing $P_0, \ldots, P_n$, by an argument similar to the beginning of the proof of Theorem 3.1 (see also \cite[Remark 4.2]{15}).

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