ON SINGULAR CUBIC SURFACES

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Abstract. We study global log canonical thresholds of cubic surfaces with canonical singularities, and we prove the existence of a Kähler–Einstein metric on two singular cubic surfaces.

1. Introduction.

Let $X$ be a Fano variety with log terminal singularities, and $G$ be a finite subgroup in $\text{Aut}(X)$.

Definition 1.1. Global $G$-invariant log canonical threshold of the variety $X$ is the number

$$\text{lct}(X,G) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the log pair } (X,\lambda D) \text{ has log canonical singularities} \right. $$

$$\left. \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \equiv -K_X \right\}.$$ 

We put $\text{lct}(X) = \text{lct}(X,G)$ in the case when $G$ is a trivial group.

Example 1.2. Let $X$ be a smooth hypersurface in $\mathbb{P}^n$ of degree $n$. Then $\text{lct}(X) \geq 1 - 1/n$ by [2].

Example 1.3. The simple group $\text{PGL}(2, \mathbb{F}_7)$ is a group of automorphisms of the quartic curve

$$x^3y + y^3z + z^3x = 0 \subseteq \mathbb{P}^2 \cong \text{Proj}(\mathbb{C}[x,y,z]),$$

which induces an embedding $\text{PGL}(2, \mathbb{F}_7) \subseteq \text{Aut}(\mathbb{P}^2)$. Then $\text{lct}(\mathbb{P}^2, \text{PGL}(2, \mathbb{F}_7)) = 4/3$ by [5].

The number $\text{lct}(X,G)$ plays an important role in birational geometry (see Section 5).

Example 1.4. Let $X$ be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, \ldots, a_4)$ of degree $\sum_{i=1}^4 a_i$ with terminal singularities such that $-K_X^2 \leq 1$. Then $\text{lct}(X) = 1$ by [4], which implies that

$$\text{Bir} \left( \underbrace{X \times \cdots \times X}_{m \text{ times}} \right) = \left\langle \prod_{i=1}^m \text{Bir}(X), \text{ Aut} \left( \underbrace{X \times \cdots \times X}_{m \text{ times}} \right) \right\rangle,$$

and the variety $X \times \cdots \times X$ is non-rational (see [7], [15], [4]).

The number $\text{lct}(X,G)$ plays an important role in Kähler geometry.

Example 1.5. Suppose that $X$ has at most quotient singularities, and the inequality

$$\text{lct}(X,G) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then $X$ has a Kähler–Einstein metric (see [8]).

Let $S$ be a del Pezzo surface with canonical singularities. Put $\Sigma = \text{Sing}(S)$.

Remark 1.6. It follows from [16], [9], [13], [11], [10], [5] that

- the surface $S$ has a Kähler–Einstein metric in the following cases:
  - when $\Sigma = \emptyset$, $S \not\cong \mathbb{F}_1$ and $K_S^2 \neq 7$;
  - when $S$ is a complete intersection
    $$\sum_{i=0}^4 x_i^2 = \sum_{i=0}^4 \lambda_i x_i^2 = 0 \subseteq \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_4]),$$
    and $\Sigma$ consists of points of type $\mathbb{A}_1$, where $\lambda_i \in \mathbb{C}$;
  - when $K_S^2 = 2$, and $\Sigma$ consists of points of points of types $\mathbb{A}_1$ and $\mathbb{A}_2$;

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1 We assume that all varieties are projective, normal, and defined over $\mathbb{C}$. 

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is not log canonical if (S, D) is not log canonical at the point α.

**Example 1.7.** Suppose that S is a cubic surface in \( \mathbb{P}^3 \) and Σ = ∅. Then

\[
\text{lct}(S) = \begin{cases} 
2/3 \text{ when } S \text{ has an Eckardt point}, \\
3/4 \text{ when } S \text{ does not have Eckardt points}.
\end{cases}
\]

We prove the following result in Sections 3.

**Theorem 1.8.** Suppose that S is a cubic surface in \( \mathbb{P}^3 \) and Σ ≠ ∅. Then

\[
\text{lct}(S) = \begin{cases} 
2/3 \text{ when } Σ = \{A_1\}, \\
1/3 \text{ when } Σ \supseteq \{A_4\}, \\
1/3 \text{ when } Σ = \{D_4\}, \\
1/3 \text{ when } Σ \supseteq \{A_2, A_2\}, \\
1/4 \text{ when } Σ \supseteq \{A_5\}, \\
1/4 \text{ when } Σ = \{D_5\}, \\
1/6 \text{ when } Σ = \{E_6\}, \\
1/2 \text{ in other cases}.
\end{cases}
\]

The group \( S_4 \) naturally acts on the cubic surface \( \hat{S} \subset \mathbb{P}^3 \) that is given by the equation

\[
xyz + xyt + xzt + yzt = 0 \subseteq \mathbb{P}^3 \cong \text{Proj}\left(\mathbb{C}[x, y, z, t]\right),
\]

the group \( S_3 \times \mathbb{Z}_3 \) naturally acts on the cubic surface \( \hat{S} \subset \mathbb{P}^3 \) that is given by the equation

\[
xyz = t^3 \subseteq \mathbb{P}^3 \cong \text{Proj}\left(\mathbb{C}[x, y, z, t]\right),
\]

and \( \text{lct}(\hat{S}, S_4) = \text{lct}(\hat{S}, S_3 \times \mathbb{Z}_3) = 1 \) (see Section 3). But both surfaces \( \hat{S} \) and \( \hat{S} \) are singular.

**Corollary 1.11.** The surfaces \( \hat{S} \) and \( \hat{S} \) have Kähler–Einstein metrics.

It is very likely that the method in [16] can be applied to prove the existence of a Kähler–Einstein metric on every singular cubic surface having only singular points of type \( A_1 \) and \( A_2 \).

2. **Basic Tools.**

Let S be a surface with canonical singularities, and D be an effective \( \mathbb{Q} \)-divisor on it.

**Remark 2.1.** Let B be an effective \( \mathbb{Q} \)-divisor on S such that (S, B) is log canonical. Then

\[
\left( S, \frac{1}{1-\alpha}(D - \alpha B) \right)
\]

is not log canonical if (S, D) is not log canonical, where \( \alpha \in \mathbb{Q} \) such that \( 0 \leq \alpha < 1 \).

Let LCS(S, D) ⊂ S be a subset such that P ∈ LCS(S, D) if and only if (S, D) is not log terminal at the point P. The set LCS(S, D) is called the locus of log canonical singularities.

**Remark 2.2.** The set LCS(S, D) is connected if \( -(K_S + D) \) is ample (see Theorem 17.4 in [12]).

Let P be a point of the surface S such that (S, D) is not log canonical at the point P.

**Remark 2.3.** Suppose that S is smooth at P. Then \( \text{mult}_P(D) > 1 \).
Let $C$ be an irreducible curve on the surface $S$. Put
\[D = mC + \Omega,\]
where $m \in \mathbb{Q}$ such that $m \geq 0$, and $\Omega$ is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \text{Supp}(\Omega)$.

**Remark 2.4.** Suppose that $C \subseteq \text{LCS}(S, D)$. Then $m \geq 1$.

Suppose that $C$ is smooth at $P$, the inequality $m \leq 1$ holds and $P \in C$.

**Remark 2.5.** Suppose that $S$ is smooth at $P$. Then it follows from Theorem 17.6 in [12] that
\[C \cdot \Omega \geq \text{mult}_P \left( \Omega \mid_C \right) > 1.\]

**Remark 2.6.** The log pair $(S, D)$ is log canonical if and only if $(\bar{S}, \bar{D} + \sum_{i=1}^r a_i E_i)$ is log canonical.

Suppose that $r = 1$, $\pi(E_1) = P$, and $P$ is a singular point of the surface $S$ of type $\mathbb{A}_1$.

**Remark 2.7.** Suppose that $n = 1$, and $\bar{S}$ is smooth along $E_1$. Then $a_1 > 1/2$.

Suppose that $n > 1$, and $E_1 \cap \text{Sing}(S)$ consists of one singular point of type $\mathbb{A}_{n-1}$.

**Remark 2.8.** It follows from Theorem 17.6 in [12] that $a_1 > 1/(n + 1)$.

Most of the described results are valid in much more general settings (see [12]).

### 3. Main result.

Let us use the assumptions and notations of Theorem 1.8. Put
\[
\omega = \begin{cases} 
2/3 \text{ when } \Sigma = \{\mathbb{A}_1\}, \\
1/3 \text{ when } \Sigma \supseteq \{\mathbb{A}_4\}, \\
1/3 \text{ when } \Sigma = \{\mathbb{D}_4\}, \\
1/3 \text{ when } \Sigma \supseteq \{\mathbb{A}_2, \mathbb{A}_2\}, \\
1/4 \text{ when } \Sigma \supseteq \{\mathbb{A}_5\}, \\
1/4 \text{ when } \Sigma = \{\mathbb{D}_5\}, \\
1/6 \text{ when } \Sigma = \{\mathbb{E}_6\}, \\
1/2 \text{ in other cases.}
\end{cases}
\]

**Remark 3.1.** It follows from [17] that
\[w = \sup \left\{ \mu \in \mathbb{Q} \mid \text{the log pair } (S, \mu D) \text{ is log canonical for every } D \in \left\lfloor -K_S \right\rfloor \right\}.\]

Take $\lambda \in \mathbb{Q}$ such that $\lambda < \omega$. Let $D$ be any effective $\mathbb{Q}$-divisor on $S$ such that $D \equiv -K_S$.

**Lemma 3.2.** Suppose that $\lambda < 1/3$. Then $\text{LCS}(S, \lambda D) \subseteq \Sigma$.

**Proof.** Suppose that $(S, \lambda D)$ is not log terminal at a smooth point $P \in S$. Then
\[3 = -K_S \cdot D \geq \text{mult}_P (D) > 1/\lambda > 3,
\]
which is a contradiction. \hfill \qed

**Lemma 3.3.** Suppose that $|\text{LCS}(S, \lambda D)| < +\infty$. Then $\text{LCS}(S, \lambda D) \subseteq \Sigma$.

**Proof.** The necessary assertion follows from [2] or [5]. \hfill \qed
Let $O$ be the worst singular point of the surface $S$, and $\alpha: \tilde{S} \to S$ be a partial resolution of singularities that contracts smooth rational curves $E_1, \ldots, E_k$ to the point $O$ such that 

$$\tilde{S} \setminus \left( \cup_{i=1}^k E_i \right) \cong S \setminus O,$$

the surface $\tilde{S}$ is smooth along $\cup_{i=1}^k E_i$, and $E_i^2 = -2$ for every $i = 1, \ldots, k$. Then 

$$\tilde{D} \equiv \alpha^* (D) - \sum_{i=1}^k a_i E_i,$$

where $\tilde{D}$ is the proper transform of $D$ on the surface $\tilde{S}$, and $a_i \in \mathbb{Q}$. Let $L_1, \ldots, L_r$ be lines on the surface $S$ such that $O \in L_i$, and $\tilde{L}_i$ be the proper transform of $L_i$ on the surface $\tilde{S}$.

**Lemma 3.4.** Suppose that $\Sigma = \{ \mathbb{A}_1 \}$. Then $\text{lct}(S) = 2/3$.

**Proof.** There is a conic $C_i \subset S$ such that the singularities of the log pair $(S, \lambda^i (L_i + C_i))$ are log canonical and not log terminal. So, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there is an irreducible curve $Z \subset S$ such that $D = \mu Z + \Omega$, where $\mu$ is a rational number such that $\mu \geq 1/\lambda$, and $\Omega$ is an effective $\mathbb{Q}$-divisor such that $Z \not\subseteq \text{Supp}(S)$. Then 

$$3 = -K_S \cdot D = \mu \text{deg}(Z) - K_S \cdot \Omega \geq \mu \text{deg}(Z) > 3 \text{deg}(Z)/2,$$

which implies that $Z$ is a line. Let $C$ be a general conic on $S$ such that $-K_S \sim Z + C$. Then 

$$2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geq \mu C \cdot Z \geq \frac{3}{2} \mu,$$

which is a contradiction. Then $	ext{LCS}(S, \lambda D) = O$ by Lemma 3.3. We have $3 - 2a_1 = \tilde{H} \cdot \tilde{D} \geq 0$, where $\tilde{H}$ is a general curve in $| -K_S - E_1|$. Thus, it follows from the equivalence 

$$K_S + \lambda \tilde{D} + \lambda a_1 E_1 \equiv \alpha^* (K_S + \lambda D)$$

that there is a point $Q \in E_1$ such that $(\tilde{S}, \lambda \tilde{D} + \lambda a_1 E_1)$ is not log canonical at the point $Q$.

Suppose that $Q \not\in \cup_{i=1}^6 \tilde{L}_i$. Let $\pi: \tilde{S} \to \mathbb{P}^2$ be a contraction of the curves $\tilde{L}_1, \ldots, \tilde{L}_6$. Then 

$$\pi (\tilde{D} + a_1 E_1) \equiv \pi (-K_S) \equiv -K_{\mathbb{P}^2},$$

and $\pi$ is an isomorphism in a neighborhood of $Q$. Let $L$ be a general line on $\mathbb{P}^2$. Then the locus 

$$\text{LCS} \left( \mathbb{P}^2, L + \pi (\lambda \tilde{D} + \lambda a_1 E_1) \right)$$

is not connected, which is impossible by Remark 2.2.

Therefore, we may assume that $Q \in \tilde{L}_1$. Put $D = a L_1 + \Upsilon$, where $a$ is a non-negative rational number, and $\Upsilon$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line $L_1$. Then 

$$\Upsilon \equiv \alpha^* (\Upsilon) - \epsilon E_1,$$

where $\epsilon = a_1 - a/2$, and $\Upsilon$ is the proper transforms of the divisor $\Upsilon$ on the surface $\tilde{S}$.

The log pair $(\tilde{S}, \lambda a \tilde{L}_1 + \lambda \Upsilon + \lambda (a/2 + \epsilon) E_1)$ is not log canonical at $Q$. Then 

$$1 + a/2 - \epsilon = \tilde{L}_1 \cdot \Upsilon > 1/\lambda - a/2 - \epsilon$$

by Remark 2.5 because $\lambda a \ll 1$. Hence, we have $a > 1/2$.

We may assume that $\text{Supp}(D)$ does not contain the conic $C_1$ due to Remark 2.1. Then 

$$2 - 3a/2 - \epsilon = \tilde{C}_1 \cdot \Upsilon \geq \text{mult}_Q (\Upsilon) > 1/\lambda - a/2 - \epsilon,$$

where $\tilde{C}_1$ be the proper transforms of $C_1$ on the surface $\tilde{S}$. Hence, we see that $a < 1/2$. \qed

**Lemma 3.5.** Suppose that $\Sigma = \{ \mathbb{A}_1, \ldots, \mathbb{A}_k \}$ and $|\Sigma| \geq 2$. Then $\text{lct}(S) = 1/2$.

**Proof.** Let $P$ be a point in $\Sigma$ such that $P \neq O$. We may assume that $P \in L_1$. Then 

$$2L_1 + L' \sim -K_S$$

for some line $L' \subset S$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there is an irreducible curve $Z$ on the surface $S$ such that 

$$D = \mu Z + \Omega,$$
where $\mu$ is a rational number such that $\mu \geq 1/\lambda$, and $\Omega$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the curve $Z$. Then $Z$ is a line (see the proof of Lemma 3,4). We have
\[
2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geq \mu C \cdot Z \geq \mu \geq 1/\lambda > 2,
\]

where $C$ is a general conic on $S$ that intersects $Z$ in two points.

We see that $\text{LCS}(S, \lambda D) = O$ and $a_1 > 1$ (see Lemma 3,3 and Remarks 2,2 and 2,7).

Arguing as in the proof of Lemma 3,4, we see that there is a point $Q \in E$ such that the singularities of the log pair $(\bar{S}, \lambda D + \lambda a_1 E_1)$ are not log canonical at the point $Q$.

Suppose that $Q \in \bar{L}_1$. Let $a$ be a non-negative rational number such that
\[
D = aL_1 + \Upsilon,
\]

where $\Upsilon$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line $L_1$. Then
\[
\bar{\Upsilon} \equiv a^*(\Upsilon) - \epsilon E_1,
\]

where $\bar{\Upsilon}$ is the proper transforms of $\Upsilon$ on the surface $\bar{S}$, and $\epsilon = a_1 - a/2$. The log pair
\[
\left(\bar{S}, \lambda a L_1 + \lambda \bar{\Upsilon} + \lambda (a/2 + \epsilon) E_1\right)
\]

is not log canonical at the point $Q$. We have $\bar{L}_1^2 = -1/2$. Then
\[
1 - \epsilon = \bar{L}_1 \cdot \bar{\Upsilon} > 1/\lambda - a/2 - \epsilon
\]

by Remark 2,5. We have $a > 1/\lambda$, which is impossible. Hence, we see that $Q \notin \bar{L}_1$.

There is a unique reduced conic $Z \subset S$ such that $O \in Z \ni P$ and $Q \in Z$, where $\bar{Z}$ is the proper transform of the conic $Z$ on the surface $\bar{S}$. Then $L_1 \notin \text{Supp}(Z)$, because $Q \notin \bar{L}_1$.

Suppose that $Z$ is irreducible. Put $D = eZ + \Delta$, where $e$ is a non-negative rational number, and $\Delta$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the conic $C$. Then
\[
\bar{\Delta} \equiv a^*(\Delta) - \delta E_1,
\]

where $\bar{\Delta}$ is the proper transforms of $\Delta$ on the surface $\bar{S}$, and $\delta = a_1 - e/2$. Then
\[
2 - e - \delta = \bar{Z} \cdot \bar{\Delta} > 1/\lambda - e/2 - \delta > 2 - e/2 - \delta
\]

by Remark 2,5, because $\bar{C}^2 = 1/2$. We have $e < 0$, which is impossible.

We see that the conic $Z$ is reducible. Then
\[
Z = L_2 + L'_2,
\]

where $L'_2$ is a line on $S$ such that $P \in L'_2$ and $L_2 \cap L'_2 \neq \emptyset$.

The intersection $L_2 \cap L'_2$ consists of a single point. The impossibility of the case $Q \in \bar{L}_1$ implies that the surface $S$ is smooth at the point $L_2 \cap L'_2$. There is a rational number $c \geq 0$ such that
\[
D = cL_2 + \Xi,
\]

where $\Xi$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line $L_2$. Then
\[
\bar{\Xi} \equiv a^*(\Xi) - \nu E_1,
\]

where $\bar{\Xi}$ is the proper transforms of $\Xi$ on the surface $\bar{S}$, and $\nu = a_1 - c/2$. The log pair
\[
\left(\bar{S}, \lambda c L_2 + \lambda \bar{\Xi} + \lambda (c/2 + \nu) E_1\right)
\]

is not log canonical at $Q$. We have $Q \in \bar{L}_2$ and $\bar{L}_2^2 = -1$. Then
\[
1 + c/2 - \nu = \bar{L}_2 \cdot \bar{\Xi} > 1/\lambda - c/2 - \nu > 2 - c/2 - \nu
\]

by Remark 2,5. Therefore, the inequality $c > 1$ holds.

There is a unique hyperplane section $T$ of the surface $S$ such that $T = C_2 + L_2$ and
\[
Q = \bar{C}_2 \cap \bar{L}_2 = O,
\]

where $C_2$ is a conic, and $\bar{C}_2$ is the proper transforms of $C_2$ on the surface $\bar{S}$.

The conic $C_2$ is irreducible. We may assume that $C_2 \not\subseteq \text{Supp}(D)$ (see Remark 2,1). Then
\[
2 - 3c/2 - \nu = C_2 \cdot \Xi \geq \text{mult}_Q(\Xi) > 1/\lambda - c/2 - \nu,
\]

which implies that $c < 0$. The obtained contradiction completes the proof. \(\Box\)
Lemma 3.6. Suppose that $\Sigma = \{ \mathbb{D}_4 \}$. Then $\text{lct}(S) = 1/3$.

Proof. We have $r = 3$, and $L_1, L_2, L_3$ lie in a single plane. Then

$$\left( S, \frac{1}{3}(L_1 + L_2 + L_3) \right)$$

is log canonical and not log terminal. We may assume that $L_3 \not\subset \text{Supp}(D)$ due to Remark 2.1.

Let $\beta: \tilde{S} \to S$ be a birational morphism such that the morphism $\alpha$ contracts one irreducible rational curve $E$ that contains three singular points $O_1, O_2, O_3$ of type $\mathbb{A}_1$.

Let $\tilde{D}$ and $\tilde{L}_i$ be proper transforms of $D$ and $L_i$ on the surface $\tilde{S}$, respectively. Then

$$\tilde{L}_i \equiv \beta^*(L_i) - E, \quad \tilde{D} \equiv \beta^*(D) - \mu E,$$

where $\mu$ is a rational number. We have

$$0 \leq \tilde{D} \cdot \tilde{L}_3 = \left( \beta^*(D) - \mu E \right) \cdot \tilde{L}_3 = 1 - \mu E \cdot \tilde{L}_3 = 1 - \mu/2,$$

which implies that $\mu \leq 2$. Therefore, we may assume there is a point $Q \in E$ such that the singularities of the log pair $(\tilde{S}, \lambda \tilde{D} + \mu E)$ are not log canonical at the point $Q$ (see Lemma 3.2).

Suppose that $\tilde{S}$ is smooth at $Q$. The log pair $(\tilde{S}, \lambda \tilde{D} + E)$ is not log canonical at $Q$. Then

$$1 \geq \mu/2 = -\mu E^2 = E \cdot \tilde{D} > 1/\lambda > 3$$

by Remark 2.5. We see that $Q = O_j$ for some $j$.

The curves $L_1, \tilde{L}_2, \tilde{L}_3$ are disjoined, and each of them passes through a singular point of the surface $\tilde{S}$. Therefore, we may assume that $O_i \in \tilde{L}_i$ for every $i$.

Let $\gamma: \tilde{S} \to \tilde{S}$ be a blow up of the point $O_j$, and $G$ be the exceptional curve of $\gamma$. Then

$$\tilde{L}_j = \gamma^*(\tilde{L}_j) - \frac{1}{2}G \equiv (\beta \circ \gamma)^*(L_1) - \hat{E} - G,$$

where $\hat{L}_j$ and $\hat{E}$ are proper transforms of the curves $L_j$ and $E$ on the surface $\tilde{S}$, respectively.

Let $\hat{D}$ be the proper transform of the divisor $D$ on the surface $\tilde{S}$. Then

$$\hat{E} \equiv \gamma^*(E) - \frac{1}{2}G, \quad \hat{D} \equiv \gamma^*(\tilde{D}) - \epsilon G \equiv (\beta \circ \gamma)^*(D) - \mu \hat{E} - (\mu/2 + \epsilon)G,$$

where $\epsilon$ is a rational number. Then $\lambda \epsilon + \lambda \mu/2 > 1/2$ (see Remark 2.7).

Suppose that $j = 3$. Then

$$0 \leq \tilde{D} \cdot \tilde{L}_3 = \left( (\beta \circ \gamma)^*(D) - \mu \hat{E} - (\mu/2 + \epsilon)G \right) \cdot \tilde{L}_3 = 1 - \mu/2 - \epsilon$$

which implies that $\epsilon + \mu/2 < 1$. But we know that $\epsilon + \mu/2 > 3/2$.

We may assume that $Q = O_1$, and the support of the divisor $D$ contains the line $L_1$. Put

$$D = aL_1 + \Omega \equiv -K_S,$$

where $a \in \mathbb{Q}$ and $a \geq 0$, and $\Omega$ is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \text{Supp}(\Omega)$. Then

$$\hat{\Omega} \equiv (\beta \circ \gamma)^*(\Omega) - m \hat{E} - (m/2 + b)G,$$

where $\hat{\Omega}$ is the proper transform of $\Omega$, and $m$ and $b$ are non-negative rational numbers. Then

$$(\beta \circ \gamma)^*(D) - \mu \hat{E} - (\mu/2 + \epsilon)G \equiv \hat{D} = aL_1 + \hat{\Omega} \equiv (\beta \circ \gamma)^*(aL_1 + \Omega) - (a + m)\hat{E} - (a + m/2 + b)G,$$

which implies that $\mu = a + m \leq 2$ and $\epsilon = a/2 + b$. We have

$$L_i^2 = -1, \quad \hat{E}^2 = -1, \quad G^2 = -2, \quad \hat{L} \cdot \hat{E} = 0, \quad \hat{L} \cdot G = \hat{E} \cdot G = 1$$

on the surface $\tilde{S}$. The surface $\tilde{S}$ is smooth along the curve $G$. Then

$$-a \leq -a + \hat{\Omega} \hat{L}_1 = \left( aL_1 + \hat{\Omega} \right) \cdot \hat{L}_1 = \left( (\beta \circ \gamma)^*(-K_S) - (a + m)\hat{E} - (a + m/2 + b)G \right) \cdot \hat{L}_1 = 1 - a - m/2 - b,$$

which implies that $m/2 + b \leq 1$ and $a + m/2 + b \leq 1 + a \leq 3$. Thus, the equivalence

$$K_S + \lambda aL_1 + \lambda \hat{\Omega} + \lambda(a + m)\hat{E} + \lambda(a + m/2 + b)G \equiv (\beta \circ \gamma)^*(K_S + \lambda aL_1 + \lambda \Omega)$$
implies the existence of a point $A \in G$ such that the log pair
\[
(\hat{S}, \lambda a\hat{L}_1 + \lambda\hat{\Omega} + \lambda(a + m)\hat{E} + \lambda(a + m/2 + b)G)
\]
is not log canonical at the point $A$.

Suppose that $A \notin \hat{L}_1 \cup \hat{E}$. Then $(\hat{S}, \lambda\hat{\Omega} + \lambda(a + m/2 + b)G)$ is not log canonical at $A$, and
\[
2b + a = \left(a\hat{L}_1 + \hat{\Omega}\right) \cdot G = a + \hat{\Omega} \cdot G > a + 3,
\]
by Remark 2.5. We see that $b > 3/2$. But $m/2 + b \leq 1$. We see that $A \notin \hat{L}_1 \cup \hat{E}$.

Suppose that $A \notin \hat{L}_1$. The log pair $(\hat{S}, \lambda\hat{\Omega} + \lambda(a + m)\hat{E} + \lambda(a + m/2 + b)G)$ is not log canonical at the point $A$. Arguing as in the previous case, we see that
\[
m/2 - b = \left(a\hat{L}_1 + \hat{\Omega}\right) \cdot \hat{E} = \hat{\Omega} \cdot \hat{E} \geq \text{mult}_A\left(\hat{\Omega}\big|_{\hat{E}}\right) > 3 - (a + m/2 + b),
\]
which implies that $a + m > 3$. But $a + m \leq 2$. We see that $A \in \hat{L}_1$.

The log pair $(\hat{S}, \lambda a\hat{L}_1 + \lambda\hat{\Omega} + \lambda(a + m/2 + b)G)$ is not log canonical at the point $A$. Then
\[
1 - a - m/2 - b = \left(a\hat{L}_1 + \hat{\Omega}\right) \cdot \hat{L}_1 = -a + \hat{\Omega} \cdot \hat{L}_1 > -a + 3 - (a + m/2 + b)
\]
by Remark 2.5. We have $a > 2$. But $a + m \leq 2$. \hfill \square

**Lemma 3.7.** Suppose that $\Sigma = \{D_5\}$. Then $\text{lct}(S) = 1/4$.

*Proof.* We have $r = 2$. We may assume that $-K_S \sim 2L_1 + L_2$. Then the log pair
\[
\left(S, \frac{1}{4}(2L_1 + L_2)\right),
\]
is not log terminal. We may assume $(S, \lambda D)$ is not log canonical at $O$ (see Lemma 3.2).

It follows from [11] that $S$ contains a line $L$ such that $O \notin L$. Projecting from $L$, we see that there is a conic $C \subset S$ such that $O \notin C$, $-K_S \sim C + L$, and $C \cdot L = 2$. Put $P = C \cap L$. Then
\[
P \cup O \subseteq \text{LCS}\left(S, \frac{3}{4}(C + L) + \lambda D\right) \subseteq P \cup O \cup C \cup L,
\]
which is impossible by Remark 2.5. \hfill \square

**Lemma 3.8.** Suppose that $\Sigma = \{E_6\}$. Then $\text{lct}(S) = 1/6$.

*Proof.* We have $r = 1$. The log pair $(S, \frac{1}{2}L_1)$ is not log terminal. The surface $S$ contains a plane cuspidal curve $C$ such that $O \notin C$. The proof of Lemma 3.6 implies that $\text{lct}(S) = 1/6$. \hfill \square

**Lemma 3.9.** Suppose that $\Sigma = \{A_2\}$. Then $\text{lct}(S) = 1/2$.

*Proof.* We may assume that $-K_S \sim L_1 + L_2 + L_3 \sim L_4 + L_5 + L_6$. The log pair
\[
\left(S, \frac{1}{2}(L_1 + L_2 + L_3)\right)
\]
is log canonical and not log terminal. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

The proof of Lemma 3.4 implies that LCS$(S, \lambda D) = O$.

Let $\bar{H}$ be a proper transform on $S$ of a general hyperplane section that contains $O$. Then
\[
0 \leq \bar{H} \cdot \bar{D} = 3 - a_1 - a_2, \quad 2a_1 - a_2 = E_1 \cdot \bar{D} \geq 0, \quad 2a_2 - a_1 = E_2 \cdot \bar{D} \geq 0,
\]
which implies that $a_1 \leq 2$ and $a_2 \leq 2$. There is a point $Q \in E_1 \cup E_2$ such that the singularities of the log pair $(\bar{S}, \lambda(\bar{D} + a_1 E_1 + a_2 E_2))$ are not log canonical at $Q$. We may assume that $Q \in E_1$, and
\[
\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_6 \cdot E_2 = 1,
\]
which implies that $\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = \bar{L}_6 \cdot E_1 = 0$.

It follows from Remark 2.1 that we may assume that $\bar{L}_1 \not\subseteq \text{Supp}(D) \not\supseteq \bar{L}_4$. Then
\[
1 - a_1 = D \cdot \bar{L}_1 \geq 0, \quad 1 - a_2 = D \cdot \bar{L}_4 \geq 0,
\]
which implies that $a_1 \leq 1$ and $a_2 \leq 1$. \hfill \square
Suppose that $Q \not\in E_2$. Then $(S, \lambda D + E_1)$ is not log canonical at $Q$. We have
\[ 2a_1 - a_2 = D \cdot E_1 > 1/\lambda > 2, \]
by Remark 2.5. Then $a_1 \geq 4/3$, which is impossible. Hence, we see that $Q \in E_2$.

The log pairs $(S, \lambda D + E_1 + E_2)$ and $(S, \lambda D + a_3 E_1 + E_2)$ are not log canonical at $Q$. Then
\[ 2a_1 - a_2 = D \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \]
by Remark 2.5. Then $a_1 > 1$ and $a_2 > 1$, which is impossible.

\begin{lemma}
Suppose that $\Sigma = \{\lambda_5\}$. Then $\text{lct}(S) = 1/2$.
\end{lemma}

\begin{proof}
We have $r = 5$. Then $\bar{L}_i^2 = -1$ and $\bar{L}_i \cdot L_j = 0$ for $i \neq j$. We may assume that
\[ \bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_3 = \bar{L}_5 \cdot E_3 = 1, \]
which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = 0$ and
\[ \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = 0, \]
\[ \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0. \]
We have $-K_S \sim L_1 + L_2 + L_3$. But it follows from elementary calculations that
\[ \bar{L}_1 \equiv \alpha^*(L_1) - \frac{3}{4} E_1 - \frac{1}{2} E_2 - \frac{1}{4} E_3, \]
\[ \bar{L}_2 \equiv \alpha^*(L_2) - \frac{3}{4} E_1 - \frac{1}{2} E_2 - \frac{1}{4} E_3, \]
\[ \bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{2} E_1 - E_2 - \frac{1}{2} E_3, \]
which implies that $\text{lct}(S) \leq 1/2$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there are a line $L \subset S$ and a rational number $\mu \geq 1/\lambda$ such that $D = \mu L + \Omega$, where $\Omega$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line $L$. Then
\[ 2 = C \cdot D = \mu C \cdot L + C \cdot \Omega \geq \mu C \cdot L > 2C \cdot L, \]
where $C$ is a general conic on the surface $S$ such that the divisor $C + L$ is a hyperplane section of the surface $S$. Then $|L \cap C| = 1$, which implies that $L = L_3$. But $L_3 \cdot C = 1$.

It follows from Remark 2.2 and Lemma 3.3 that $\text{LCS}(S, \lambda D) = O$.

Let $\bar{H}$ be a general curve in $| - K_S - \sum_{i=1}^3 E_i |$. Then
\[ a_1 + a_3 \leq 3, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2, \]
because $\bar{H} \cdot \bar{D} \geq 0, E_1 \cdot \bar{D} \geq 0, E_2 \cdot \bar{D} \geq 0, E_3 \cdot \bar{D} \geq 0$, respectively.

We may assume that either $L_1 \not\subseteq \text{Supp}(D)$ or $L_3 \not\subseteq \text{Supp}(D)$. But
\[ \bar{L}_1 \cdot \bar{D} = 1 - a_1, \quad \bar{L}_3 \cdot \bar{D} = 1 - a_2, \]
which implies that either $a_1 \leq 1$ or $a_2 \leq 1$. Similarly, we assume that either $a_3 \leq 1$ or $a_2 \leq 1$.

We have $a_1 \leq 2, a_2 \leq 3$. Then there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the log pair $(S, \lambda(D + a_1 E_1 + a_2 E_2 + a_3 E_3))$ is not log canonical at $Q$. We may assume that $Q \not\in E_3$.

Suppose that $Q \not\in E_2$. Then $(S, \lambda D + E_1)$ is not log canonical at the point $Q$. We have
\[ 2a_1 - a_2 = D \cdot E_1 > 1/\lambda > 2, \]
by Remark 2.5. Then $a_1 > 3/2$ and $a_2 > 1$. But either $a_1 \leq 1$ or $a_2 \leq 1$. Contradiction.

Suppose that $Q \in E_2 \cap E_1$. Arguing as in the proof of of Lemma 3.3, we see that
\[ 2a_1 - a_2 = D \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \]
by Remark 2.5. Then $a_1 > 1$ and $2a_2 > 2 + a_3$, which is impossible.

We see that $Q \in E_2$ and $Q \not\in E_1$. Then $(S, \lambda D + E_2)$ is not log canonical at $Q$. We have
\[ 2a_2 - a_1 - a_3 = D \cdot E_2 > 1/\lambda > 2, \]
which implies that $a_1 > 3/2$ and $a_2 > 2$. The latter is impossible.

\begin{lemma}
Suppose that $\Sigma = \{\lambda_4\}$. Then $\text{lct}(S) = 1/3$.
\end{lemma}

\begin{proof}
We have $r = 4$. Then $\bar{L}_i^2 = -1$ and $\bar{L}_i \cdot L_j = 0$ for $i \neq j$. We may assume that
\[ \bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_3 = \bar{L}_5 \cdot E_4 = 1, \]
which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = 0$ and
\[ \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_4 \cdot E_2 = \bar{L}_4 \cdot E_3 = 0. \]
The equivalence $-K_S \sim 2L_3 + L_4$ holds. Similarly, we have
\[ \bar{L}_3 = \alpha^*(L_3) - \frac{2}{5}E_1 - \frac{4}{5}E_2 - \frac{6}{5}E_3 - \frac{3}{5}E_4, \quad \bar{L}_4 = \alpha^*(L_4) - \frac{1}{5}E_1 - \frac{2}{5}E_2 - \frac{3}{5}E_3 - \frac{4}{5}E_4, \]
which implies that $\text{ltc}(S) \leq 1/3$. Thus, we may assume that $(S, \lambda D)$ is not log canonical, which implies that $\text{LCS}(S, \lambda D) = O$ by Lemma 3.11. Let $\bar{H}$ be a general curve in $|\bar{K}_S - \sum_{i=1}^4 E_i|$. Then
\[ 3 \geq a_1 + a_4, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3, \]
because $\bar{H} \cdot \bar{D} \geq 0$, $E_1 \cdot \bar{D} \geq 0$, $E_2 \cdot \bar{D} \geq 0$, $E_3 \cdot \bar{D} \geq 0$, $E_4 \cdot \bar{D} \geq 0$, respectively.

We have $-K_S \sim L_1 + L_2 + L_3$ and $-K_S \sim 2L_3 + L_4$. But the log pairs
\[ \left( S, \frac{1}{2}(L_1 + L_2 + L_3) \right) \quad \text{and} \quad \left( S, \frac{1}{3}(L_4 + 2L_3) \right) \]
are log canonical. So, we may assume that either $L_3 \not\subseteq \text{Supp}(D)$ or $L_1 \not\subseteq \text{Supp}(D) \not\supseteq L_4$. But
\[ L_3 \cdot \bar{D} = 1 - a_3, \quad L_1 \cdot \bar{D} = 1 - a_1, \quad L_4 \cdot \bar{D} = 1 - a_4, \]
which implies that there is a point $Q \in \bigcup_{i=1}^4 E_i$ such that $(\bar{S}, \lambda(\bar{D} + \sum_{i=1}^4 a_i E_i))$ is not log canonical at the point $Q$. Arguing as in the proof of Lemma 3.10, we see that
\[
\begin{aligned}
Q \in E_1 \setminus (E_1 \cap E_2) &\Rightarrow 2a_1 > a_2 + 3, \\
Q \in E_1 \cap E_2 &\Rightarrow 2a_1 > 3 \quad \text{and} \quad 2a_2 > 3 + a_3, \\
Q \in E_2 \setminus ((E_1 \cap E_2) \cup (E_2 \cap E_3)) &\Rightarrow 2a_2 > a_1 + a_3 + 3, \\
Q \in E_2 \cap E_3 &\Rightarrow 2a_2 > 3 + a_1 \quad \text{and} \quad 2a_3 > 3 + a_4, \\
Q \in E_3 \setminus ((E_2 \cap E_3) \cup (E_3 \cap E_4)) &\Rightarrow 2a_3 > 3 + a_2 + a_4, \\
Q \in E_3 \cap E_4 &\Rightarrow 2a_3 > 3 + a_2 \quad \text{and} \quad 2a_4 > 3, \\
Q \in E_4 \setminus (E_4 \cap E_3) &\Rightarrow 2a_4 > 3,
\end{aligned}
\]
which leads to a contradiction, because either $a_3 \leq 1$ or $a_1 \leq 1$ and $a_4 \leq 1$. \hfill \Box

**Lemma 3.12.** Suppose that $\Sigma = \alpha_5$. Then $\text{ltc}(S) = 1/4$.

**Proof.** We have $r = 3$. We may assume that $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_4 = 1$. Then
\[ \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_1 \cdot E_5 = \bar{L}_2 \cdot E_5 = 0 \]
and $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_3 \cdot E_5 = 0$. But $-K_S \sim 3L_3$. Then $\text{ltc}(S) \leq 1/4$, because
\[ \bar{L}_3 = \alpha^*(L_3) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{2}{3}E_5. \]

We may assume that $(S, \lambda D)$ is not log canonical. Then $\text{LCS}(S, \lambda D) = O$. By Lemma 3.11, let $\bar{H}$ be a proper transform on $\bar{S}$ of a general hyperplane section that contains $O$. Then
\[ 3 \geq a_1 + a_5, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3 + a_5, \quad 2a_5 \geq a_4, \]
because $\bar{H} \cdot \bar{D} \geq 0$, $E_1 \cdot \bar{D} \geq 0$, $E_2 \cdot \bar{D} \geq 0$, $E_3 \cdot \bar{D} \geq 0$, $E_4 \cdot \bar{D} \geq 0$, $E_5 \cdot \bar{D} \geq 0$, respectively.

We may assume that $L_3 \not\subseteq \text{Supp}(D)$ due to Remark 2.7. Then $1 - a_4 = L_3 \cdot \bar{D} \geq 0$, which easily implies that $a_1 \leq 5/2$, $a_2 \leq 2$, $a_3 \leq 3/2$, $a_4 \leq 1$, $a_5 \leq 5/4$.

There is a point $Q \in \bigcup_{i=1}^5 E_i$ such that the log pair $(\bar{S}, \lambda(\bar{D} + \sum_{i=1}^5 a_i E_i))$ is not log canonical at the point $Q$. Arguing as in the proof of Lemma 3.10, we see that
\[
\begin{aligned}
Q \in E_1 \setminus (E_1 \cap E_2) &\Rightarrow 2a_1 > a_2 + 4, \\
Q \in E_1 \cap E_2 &\Rightarrow 2a_1 > 4 \quad \text{and} \quad 2a_2 > 4 + a_3, \\
Q \in E_2 \setminus ((E_1 \cap E_2) \cup (E_2 \cap E_3)) &\Rightarrow 2a_2 > a_1 + a_3 + 4, \\
Q \in E_2 \cap E_3 &\Rightarrow 2a_2 > 4 + a_1 \quad \text{and} \quad 2a_3 > 4 + a_4, \\
Q \in E_3 \setminus ((E_2 \cap E_3) \cup (E_3 \cap E_4)) &\Rightarrow 2a_3 > 4 + a_2 + a_4, \\
Q \in E_3 \cap E_4 &\Rightarrow 2a_3 > 4 + a_2 \quad \text{and} \quad 2a_4 > 4 + a_5, \\
Q \in E_4 \setminus ((E_3 \cap E_4) \cup (E_4 \cap E_5)) &\Rightarrow 2a_4 > 4 + a_3 + a_5, \\
Q \in E_4 \cap E_5 &\Rightarrow 2a_4 > 4 + a_3 \quad \text{and} \quad 2a_5 > 4, \\
Q \in E_5 \setminus (E_4 \cap E_5) &\Rightarrow 2a_5 > a_4 + 4.
\end{aligned}
\]
Now taking into account the inequalities \(3.13\) the inequalities \(3.14\), the inequality \(a_4 \leq 4\), and the inequality \(a_1 + a_5 \leq 3\), we see that either \(Q = E_3 \cap E_4\) or \(Q = E_4 \cap E_5\).

Let \(H_1\) and \(H_3\) be general curves in \(|-K_S|\) that contain \(L_1\) and \(L_3\), respectively. Then

\[
H_1 = L_1 + C_1, \quad H_3 = L_3 + C_3,
\]

where \(C_1\) and \(C_3\) are irreducible conics such that \(C_1 \not\subset \text{Supp}(D) \supset C_3\).

Let \(\bar{C}_1\) and \(\bar{C}_3\) be the proper transforms of \(C_1\) and \(C_3\) on the surface \(\bar{S}\), respectively. Then

\[
\bar{C}_1 \cdot E_1 = \bar{C}_1 \cdot E_2 = \bar{C}_1 \cdot E_3 = \bar{C}_1 \cdot E_4 = \bar{C}_3 \cdot E_1 = \bar{C}_3 \cdot E_2 = \bar{C}_3 \cdot E_3 = \bar{C}_3 \cdot E_4 = \bar{C}_3 \cdot E_5 = 0
\]

and \(\bar{C}_1 \cdot E_5 = \bar{C}_3 \cdot E_2 = 1\). Therefore, we see that

\[
0 \leq \bar{C}_1 \cdot \bar{D} = 2 - a_5, \quad 2 - a_2 = \bar{C}_3 \cdot \bar{D} \geq 0,
\]

which implies that \(a_2 \leq 2\) and \(a_5 \leq 2\). Now we can easily obtain a contradiction.

Lemma 3.15. Suppose that \(\Sigma = \{A_1, A_3\}\). Then \(\text{lct}(S) = 1/4\).

Proof. Let \(P\) be a point in \(\Sigma\) of type \(A_1\). Then \(r = 2\). We may assume that \(P \in L_1\). Then

\[
\bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = \bar{L}_2 \cdot E_5 = \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_1 \cdot E_5 = 0,
\]

and \(\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_4 = 1\). The equivalence \(-K_S \sim 3L_2\) holds. Then \(\text{lct}(S) \leq 1/4\), because

\[
\bar{L}_2 \equiv \alpha^*(L_2) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{2}{3}E_5.
\]

We may assume that \((S, \lambda D)\) is not log canonical. Then \(\text{LCS}(S, \lambda D) \subseteq \{O, P\}\) by Lemma 3.12. Suppose that \((S, \lambda D)\) is not log terminal at \(P\). Let \(\beta : \tilde{S} \to S\) be a blow up of \(P\). Then

\[
\tilde{D} \equiv \beta^*(-K_S) - mF,
\]

where \(F\) is the \(\beta\)-exceptional curve, \(\tilde{D}\) is the proper transform of the divisor \(D\), and \(m \in \mathbb{Q}\). Then

\[
0 \leq \tilde{H} \cdot \tilde{D} = \left(\beta^*(-K_S) - mF\right) \cdot \left(\beta^*(-K_S) - F\right) = 3 - 2m,
\]

where \(\tilde{H}\) is general curve in \(|-K_{\tilde{S}} - F|\). Thus, we have \(m \leq 3/2\). But \(m > 2\) by Remark 2.7. We see that \(\text{LCS}(S, \lambda D) = O\). Let \(C_1\) and \(C_2\) be general conics on the surface \(S\) such that

\[
L_1 + C_1 \sim L_2 + C_2 \sim -K_S,
\]

and let \(\bar{C}_1\) and \(\bar{C}_2\) be the proper transforms of \(C_1\) and \(C_2\) on the surface \(\bar{S}\), respectively. Then

\[
2 - a_1 = \bar{C}_1 \cdot \bar{D} \geq 0, \quad 2 - a_5 = \bar{C}_2 \cdot \bar{D} \geq 0,
\]

because \(C_1 \not\subset \text{Supp}(D) \supset C_2\). We may assume that \(L_2 \not\subset \text{Supp}(D)\) due to Remark 2.11.

Arguing as in the proof of Lemma 3.12, we obtain the inequalities

\[
3 \geq a_1 + a_5, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3 + a_5, \quad 2a_5 \geq a_4, \quad 2 \geq a_2, \quad 2 \geq a_5, \quad 1 \geq a_4,
\]

which imply that there is a point \(Q \in \bigcup_{i=1}^{5} E_i\) such that \((\bar{S}, \lambda(\bar{D} + \sum_{i=1}^{5} a_iE_i))\) is not log canonical at the point \(Q\). Arguing as in the proof of Lemma 3.10, we obtain a contradiction.

Lemma 3.16. Suppose that \(\Sigma = \{A_1, A_4\}\). Then \(\text{lct}(S) = 1/3\).

Proof. We have \(r = 3\). Let \(P\) be a point in \(\Sigma\) of type \(A_1\). We may assume that \(P \in L_1\). Then

\[
\bar{L}_1 \cdot E_1 = 1, \quad \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = 0,
\]

and we may assume that \(L_3 \cdot E_3 = \bar{L}_2 \cdot E_4 = 1\). Then

\[
\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = 0.
\]

The equivalence \(-K_S \sim L_2 + 2L_3\) holds. But

\[
\bar{L}_2 \equiv \alpha^*(L_2) - \frac{1}{5}E_1 - \frac{2}{5}E_2 - \frac{3}{5}E_3 - \frac{4}{5}E_4, \quad \bar{L}_3 \equiv \alpha^*(L_3) - \frac{2}{5}E_1 - \frac{4}{5}E_2 - \frac{6}{5}E_3 - \frac{3}{5}E_4,
\]

which implies that \(\text{lct}(S) \leq 1/3\). Thus, we may assume that \((S, \lambda D)\) is not log canonical.

We may assume that either \(L_3 \not\subset \text{Supp}(D)\) or \(L_1 \not\subset \text{Supp}(D) \not\supset L_2\) (see Remark 2.11). Arguing as in the proof of Lemma 3.15, we see that the log pair \((S, \lambda D)\) is log canonical outside of the point \(O\). Now arguing as in the proof of Lemma 3.11, we obtain a contradiction. □
Lemma 3.17. Suppose that $\Sigma = \{A_1, A_3\}$. Then $\text{lc}(S) = 1/2$.

Proof. Let $P$ be a point in $\Sigma$ of type $A_1$. We may assume that $P \in L_1$. Then $r = 4$, and it easily follows from (1) that the surface $S$ contains lines $L_5, L_6, L_7$ such that

$L_5 \ni P \in L_6$, $O \notin L_7 \ni P$, $L_3 \cap L_5 \neq \emptyset$, $L_4 \cap L_6 \neq \emptyset$, $L_7 \cap L_2 \neq \emptyset$, $L_7 \cap L_5 \neq \emptyset$, $L_7 \cap L_6 \neq \emptyset$,

which implies that $L_7 \cap L_1 = L_7 \cap L_3 = L_7 \cap L_4 = \emptyset$. Then $-K_S \sim L_1 + L_3 + L_5$

$L_1 + L_3 + L_5 \sim L_1 + L_4 + L_6 \sim L_5 + L_6 + L_7 \sim L_2 + 2L_1 \sim L_2 + L_3 + L_4 \sim 2L_2 + L_7$,

which implies that $\text{lc}(S) \leq 1/2$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Put $D = \mu_1 L_1 + \Omega_1$, where $\mu_1$ is a non-negative rational number, and $\Omega_1$ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line $L_i$. Let us show that that $\mu_i < 1/\lambda$ for $i = 1, \ldots, 7$.

Suppose that $\mu_1 \geq 1/\lambda$. We may assume that $L_1 \nsubseteq \text{Supp}(D)$ by Remark 2.1. Then

$1 = L_1 \cdot D = L_1 \cdot (\mu_2 L_2 + \Omega_2) \geq \mu_2 L_1 \cdot L_2 = \mu_2/2 > 1$,

which is a contradiction. Similarly, we see that $\mu_i < 1/\lambda$ for $i = 1, \ldots, 7$.

Arguing as in the proof of Lemma 3.14, we see that $\text{LCS}(S, \lambda D)$ does not contain curves and smooth points of the surface $S$. Then either $\text{LCS}(S, \lambda D) = O$ or $\text{LCS}(S, \lambda D) = P$ by Remark 2.2.

Suppose that $\text{LCS}(S, \lambda D) = P$. Put

$D = \mu_5 L_5 + \mu_6 L_6 + \Upsilon$,

where $\Upsilon$ is an effective $\mathbb{Q}$-divisor such that $L_5 \nsubseteq \text{Supp}(\Upsilon) \nsubseteq L_6$. Then $\mu_5 > 0$ and $\mu_6 > 0$. But

$1 = L_7 \cdot D = L_7 \cdot (\mu_5 L_5 + \mu_6 L_6 + \Upsilon) \geq L_7 \cdot (\mu_5 L_5 + \mu_6 L_6) = \mu_5 + \mu_6$,

because we may assume that $L_7 \nsubseteq \text{Supp}(\Upsilon)$. Let $\beta : \tilde{S} \rightarrow S$ be a blow up of the point $P$. Then

$\mu_5 \tilde{L}_5 + \mu_6 \tilde{L}_6 + \tilde{\Upsilon} \equiv \beta^*(\mu_5 L_5 + \mu_6 L_6 + \Upsilon) - (\mu_5/2 + \mu_6/2 + \epsilon) G$,

where $\epsilon$ is a rational number, $G$ is the exceptional curve of $\beta$, and $\tilde{L}_5, \tilde{L}_6, \tilde{\Upsilon}$ are proper transforms of the divisors $L_5, L_6, \Upsilon$ on the surface $\tilde{S}$, respectively. Then

$0 \leq (\mu_5 \tilde{L}_5 + \mu_6 \tilde{L}_6 + \tilde{\Upsilon})\tilde{H} = 3 - \mu_5 - \mu_6 - 2\epsilon$,

where $\tilde{H}$ is a general curve in $|-K_{\tilde{S}} - G|$. There is a point $Q \in G$ such that the singularities of the log pair $(\tilde{S}, \mu_5 \tilde{L}_5 + \mu_6 \tilde{L}_6 + \tilde{\Upsilon}) + (\lambda(\mu_5/2 + \mu_6/2 + \epsilon) G)$ are not log canonical at $Q$. We have

$2 - 2\epsilon = \tilde{\Upsilon} \cdot (\tilde{L}_5 + \tilde{L}_6) \geq 0$,

which implies that $\epsilon \leq 1$. Then $2\epsilon = \tilde{\Upsilon} \cdot G > 2$ in the case when $Q \nsubseteq \tilde{L}_5 \cup \tilde{L}_6$ by Remark 2.5, which implies that we may assume that $Q \subseteq \tilde{L}_5$. Then

$1 + \mu_5/2 - \mu_6 - \epsilon = \tilde{\Upsilon} \cdot \tilde{L}_5 > 2 - \mu_5/2 - \mu_6/2 - \epsilon$,

due to Remark 2.5. Thus, we see that $\mu_5 > 1$. But $\mu_5 \leq \mu_5 + \mu_6 \leq 1$.

We see that $\text{LCS}(S, \lambda D) = O$. We may assume that

$\tilde{L}_1 \cdot E_1 = \tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_3 = \tilde{L}_4 \cdot E_4 = 1$,

and $\tilde{L}_1 \cdot E_2 = \tilde{L}_1 \cdot E_3 = \tilde{L}_2 \cdot E_1 = \tilde{L}_2 \cdot E_3 = \tilde{L}_3 \cdot E_1 = \tilde{L}_3 \cdot E_2 = \tilde{L}_4 \cdot E_1 = \tilde{L}_4 \cdot E_2 = 0$. The log pairs

$\left( S, \frac{1}{2} \left( 2L_1 + L_2 \right) \right)$ and $\left( S, \frac{1}{2} \left( L_2 + L_3 + L_4 \right) \right)$

are log canonical. So, we may assume that either $L_2 \nsubseteq \text{Supp}(D)$ or $L_1 \nsubseteq \text{Supp}(D) \nsubseteq L_3$, which easily leads to a contradiction (see the proof of Lemma 3.10).

□

Lemma 3.18. Suppose that $\Sigma = \{A_1, A_2\}$. Then $\text{lc}(S) = 1/2$.

Proof. Let $P$ be a point in $\Sigma$ of type $A_1$. We may assume that $P \in L_1$. Then $r = 5$, and it easily follows from (1) that the surface $S$ contains lines $L_6, L_7, L_8, L_9, L_{10}, L_{11}$ such that

$L_5 \ni P \in L_6$, $O \notin L_7 \ni P$, $L_3 \cap L_5 \neq \emptyset$, $L_4 \cap L_6 \neq \emptyset$, $L_7 \cap L_2 \neq \emptyset$, $L_7 \cap L_5 \neq \emptyset$, $L_7 \cap L_6 \neq \emptyset$,

and $L_9 \cap L_7 \neq \emptyset$, $L_{10} \cap L_7 \neq \emptyset$, $L_{10} \cap L_8 \neq \emptyset$, $L_{11} \cap L_6 \neq \emptyset$, $L_{11} \cap L_8 \neq \emptyset$. Then

$L_2 \nsubseteq P \notin L_3$, $L_4 \nsubseteq P \notin L_5$, $L_6 \nsubseteq O \notin L_7$, $L_8 \nsubseteq O \notin L_9$, $L_{10} \nsubseteq O \notin L_{11}$,
which implies that \( -K_S \sim L_3 + L_4 + L_5 \sim 2L_1 + L_2 \sim L_3 + L_4 + L_5 \) and
\[ 2L_1 + L_2 \sim L_1 + L_3 + L_6 \sim L_1 + L_4 + L_7 \sim L_1 + L_5 + L_8 \sim L_6 + L_7 + L_9 \sim L_7 + L_8 + L_{10} \sim L_6 + L_8 + L_{11}. \]

We see that \( \text{lct}(S) \leq 1/2 \). Therefore, we may assume that \((S, \lambda D)\) is not log canonical.

Arguing as in the proof of Lemma 3.17, we see that \( \text{LCS}(S, \lambda D) = O \). We may assume that \( \bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_5 \cdot E_2 = 1, \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0. \)

It follows from elementary calculations that
\[
\bar{L}_1 \equiv \alpha^*(L_1) - \frac{2}{3}E_1 - \frac{1}{3}E_2, \quad \bar{L}_2 \equiv \alpha^*(L_2) - \frac{2}{3}E_1 - \frac{1}{3}E_2,
\]
which implies that we may assume that either \( L_1 \not\subseteq \text{Supp}(D) \) or \( L_2 \not\subseteq \text{Supp}(D) \). But
\[
\bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{3}E_1 - \frac{2}{3}E_2, \quad \bar{L}_4 \equiv \alpha^*(L_4) - \frac{1}{3}E_1 - \frac{2}{3}E_2, \quad \bar{L}_5 \equiv \alpha^*(L_5) - \frac{1}{3}E_1 - \frac{2}{3}E_2,
\]
which easily implies that we may assume that the support of the divisor \( D \) does not contain one of the lines \( L_3, L_4, L_5 \). Arguing as in the proof of Lemma 3.9, we obtain a contradiction. \( \square \)

**Lemma 3.19.** Suppose that \( \Sigma = \{\mathbb{A}_2, \ldots, \mathbb{A}_2\} \) and \(|\Sigma| \geq 2\). Then \( \text{lct}(S) = 1/3 \).

**Proof.** Let \( P \) be a point in \( \Sigma \) such that \( P \neq O \). We may assume that \( P \in L_1 \). Then \( -K_S \sim 3L_1 \), which implies that \( \text{lct}(S) \leq 1/3 \). Thus, we may assume that \((S, \lambda D)\) is not log canonical.

We may assume that \((S, \lambda D)\) is not log canonical at the point \( O \) by Lemma 3.2. Then
\[
\bar{L}_1 \equiv \alpha^*(L_1) - \frac{1}{3}E_1 - \frac{2}{3}E_2,
\]
where we assume that \( \bar{L}_1 \cap E_2 \neq \emptyset \). Thus, we may assume that \( L_1 \not\subseteq \text{Supp}(D) \) due to Remark 2.1, which implies that \( a_2 \leq 1 \), because \( \bar{D} \cdot L_1 \geq 0 \). Arguing as in the proof of Lemma 3.9, we see that
\[
3 \geq a_1 + a_2 \leq 3, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1, \quad 1 \geq a_2,
\]
which implies that there is a point \( Q \in E_1 \cup E_2 \) such that the log pair \((\bar{S}, \lambda(D + a_1E_1 + a_2E_2))\) is not log canonical at the point \( Q \). Arguing as in the proof of Lemma 3.9, we see that
\[
\begin{cases}
Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 3, \\
Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 3 \text{ and } 2a_2 > 3, \\
Q \in E_2 \setminus (E_2 \cap E_1) \Rightarrow 2a_2 > a_1 + 3,
\end{cases}
\]
which easily leads to a contradiction. \( \square \)

**Lemma 3.20.** Suppose that \( \Sigma = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_2\} \). Then \( \text{lct}(S) = 1/3 \).

**Proof.** Let \( P \neq O \) be a point in \( \Sigma \) of type \( \mathbb{A}_2 \). We may assume that \( P \in L_1 \). Then \( -K_S \sim 3L_1 \), which implies that \( \text{lct}(S) \leq 1/3 \). Thus, we may assume that \((S, \lambda D)\) is not log canonical.

We may assume that \( L_1 \not\subseteq \text{Supp}(D) \) due to Remark 2.1. But \( \text{LCS}(S, \lambda D) \subseteq \Sigma \) by Lemma 3.2. Arguing as in the proof of Lemma 3.15, we see that \( \text{LCS}(S, \lambda D) \subseteq O \cup P \), which easily leads to a contradiction (see the proof of Lemma 3.19). \( \square \)

**Lemma 3.21.** Suppose that \( \Sigma = \{\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_3\} \). Then \( \text{lct}(S) = 1/2 \).

**Proof.** Let \( P_1 \) and \( P_2 \) be points in \( \Sigma \) of type \( \mathbb{A}_1 \). Then we may assume that \( P_1 \in L_1 \) and \( P_2 \in L_2 \), while we have \( r = 3 \). The surface \( S \) contains lines \( L_4 \) and \( L_5 \) such that
\[
P_1 \in L_4 \ni P_2, \quad O \not\subseteq L_4, \quad P_1 \not\subseteq L_3 \ni P_2, \quad L_5 \cap \Sigma = \emptyset,
\]
which implies that \( L_5 \cap L_3 \neq \emptyset, \) \( L_5 \cap L_4 \neq \emptyset, \) \( L_5 \cap L_1 = \emptyset, \) \( L_5 \cap L_2 = \emptyset. \) Then
\[
-K_S \sim L_1 + L_2 + L_4 \sim L_3 + 2L_1 \sim L_3 + 2L_2 \sim 2L_3 + L_5 \sim 2L_4 + L_5,
\]
which implies that \( \text{lct}(S) \leq 1/2 \). We may assume that \((S, \lambda D)\) is not log canonical.

Put \( D = \mu_1 L_1 + \Omega_1 \), where \( \mu_1 \) is a non-negative number, and \( \Omega_1 \) is an effective \( \mathbb{Q} \)-divisor, whose support does not contain the line \( L_i \). Let us show that \( \mu_i < 1/\lambda \) for every \( i = 1, \ldots, 5 \).

Suppose that \( \mu_1 \geq 1/\lambda > 2 \). It follows from equivalences 3.22 and Remark 2.1 that we may assume that \( L_3 \not\subseteq \text{Supp}(D) \). Therefore, we have
\[
1 = L_3 \cdot D = L_3 \cdot (\mu_1 L_1 + \Omega_1) \geq \mu_1 L_3 \cdot L_1 = \mu_1/2 > 1,
\]
which is a contradiction. Similarly, we see that $\mu_2 < 1/\lambda$, $\mu_3 < 1/\lambda$, $\mu_4 < 1/\lambda$, $\mu_5 < 1/\lambda$.

Arguing as in the proof of Lemma 3.21, we see that $\text{LCS}(S, \lambda D)$ does not contain curves and smooth points of $S$. It follows from Remark 2.2 that $\text{LCS}(S, \lambda D)$ consist of one point in $\Sigma$.

Suppose that $\text{LCS}(S, \lambda D) = P_1$. Let $\beta: \tilde{S} \to S$ be a blow up of the point $P_1$. Then

$$\mu_4 \tilde{L}_4 + \tilde{\Omega} \equiv \beta^* (\mu_4 L_4 + \Omega) - (\mu_4/2 + \epsilon) G,$$

where $G$ is the exceptional curve of the birational morphism $\beta$, $\tilde{L}_4$ and $\tilde{\Omega}$ are proper transforms of the divisors $L_4$ and $\Omega$ on the surface $\tilde{S}$, respectively, and $\epsilon$ is a positive rational number. Then

$$0 \leq \left( \mu_4 \tilde{L}_4 + \tilde{\Omega} \right) \tilde{H} = \left( \beta^* (\mu_4 L_4 + \Omega) - (\mu_4/2 + \epsilon) G \right) \cdot \left( \beta^* (\lambda S) - G \right) = 3 - \mu_4 - 2\epsilon,$$

where $\tilde{H}$ is a general curve in $|-K_{\tilde{S}} - G|$. Thus, there is a point $P \in G$ such that the log pair

$$\left( \tilde{S}, \mu_4 \tilde{L}_4 + \tilde{\Omega} + (\mu_4/2 + \epsilon) G \right)$$

is not log canonical at $P$. We have $1 - \epsilon = \tilde{\Omega} \cdot \tilde{L}_4 > 0$, which implies that $\epsilon \leq 1$. Then

$$2\epsilon = \tilde{\Omega} \cdot G > 2$$

in the case when $P \not\in \tilde{L}_4$ (see Remark 2.5). Thus, we see that $P \in \tilde{L}_4$. Then

$$1 - \epsilon = \tilde{\Omega} \cdot \tilde{L}_4 > 2 - \mu_4/2 - \epsilon,$$

due to Remark 2.5. Thus, we see that $\mu_4 > 2$, which is a contradiction.

Similarly, we see that $P_2 \not\in \text{LCS}(S, \lambda D)$. Then $\text{LCS}(S, \lambda D) = O$. We may assume that

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_3 = 1, \quad \tilde{L}_1 \cdot E_2 = \tilde{L}_1 \cdot E_3 = \tilde{L}_2 \cdot E_1 = \tilde{L}_2 \cdot E_3 = \tilde{L}_3 \cdot E_1 = \tilde{L}_3 \cdot E_3 = 0.$$

It follows from elementary calculations that

$$\tilde{L}_1 \equiv \alpha^*(L_1) - \frac{3}{4} E_1 - \frac{1}{2} E_2 - \frac{1}{4} E_3, \quad \tilde{L}_2 \equiv \alpha^*(L_2) - \frac{1}{4} E_1 - \frac{1}{2} E_2 - \frac{3}{4} E_3, \quad \tilde{L}_3 \equiv \alpha^*(L_3) - \frac{1}{2} E_1 - E_2 - \frac{1}{2} E_3,$$

which implies that the singularities of the log pairs

$$\left( S, \frac{1}{2} (2L_1 + L_3) \right) \quad \text{and} \quad \left( S, \frac{1}{2} (2L_2 + L_3) \right)$$

are log canonical. But we may assume that either $L_1 \not\subset \text{Supp}(D) \not\subset L_2$ or $L_3 \not\subset \text{Supp}(D)$, because the equivalences 3.22 hold. Now the proof of Lemma 3.10 leads to a contradiction. \hfill \Box

**Lemma 3.23.** Suppose that $\Sigma = \{A_1, A_1, A_2\}$. Then $\text{lct}(S) = 1/2$.

**Proof.** Let $P_1 \neq P_2$ be points in $\Sigma$ of type $A_1$. Then we may assume that $P_1 \in L_1$ and $P_2 \in L_4$, while we have $r = 4$. The surface $S$ contains lines $L_5, L_6, L_7, L_8$ such that

$$P_1 \in L_5, \quad P_2 \in L_6, \quad P_1 \in L_7 \ni P_2, \quad O \not\in L_8, \quad P_1 \not\in L_8 \ni P_2,$$

which implies that $L_8 \cap L_7 \neq \emptyset$, $L_8 \cap L_2 \neq \emptyset$, $L_8 \cap L_3 \neq \emptyset$, $L_2 \cap L_7 = \emptyset$, $L_3 \cap L_7 = \emptyset$. Then

$$L_1 + L_4 + L_7 \sim L_2 + 2L_1 \sim L_3 + 2L_4 \sim 2L_7 + L_8 \sim L_2 + L_3 + L_8 \sim L_1 + L_3 + L_5 \sim L_4 + L_2 + L_6,$$

and $-K_S \sim L_1 + L_4 + L_7$. Then $\text{lct}(S) \leq 1/2$. We may assume that $(S, \lambda D)$ is not log canonical. Arguing as in the proof of Lemma 3.21, we see that $\text{LCS}(S, \lambda D) = O$. We may assume that

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_2 \cdot E_1 = \tilde{L}_3 \cdot E_2 = \tilde{L}_4 \cdot E_2 = 1, \quad \tilde{L}_1 \cdot E_2 = \tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_1 = \tilde{L}_4 \cdot E_1 = 0.$$

The log pair $(S, L_1 + \frac{1}{2} L_2)$ is log canonical, because the equivalences

$$\tilde{L}_1 \equiv \alpha^*(L_1) - \frac{2}{3} E_1 - \frac{1}{3} E_2, \quad \tilde{L}_2 \equiv \alpha^*(L_2) - \frac{2}{3} E_1 - \frac{1}{3} E_2$$

hold. So, we may assume that either $L_1 \not\subset \text{Supp}(D)$ or $L_2 \not\subset \text{Supp}(D)$, because $-K_S \sim 2L_1 + L_2$.

Similarly, we may assume that either $L_3 \not\subset \text{Supp}(D)$ or $L_4 \not\subset \text{Supp}(D)$, which very easily leads to a contradiction (see the proof of Lemma 3.9). \hfill \Box

Therefore, it follows from [1] that the assertion of Theorem 1.8 is proved.
4. Invariant thresholds.

In this section we prove the following two lemmas.

Lemma 4.1. Let $S$ be a cubic surface in $\mathbb{P}^3$ given by the equation $3x + y + z + t = 0$. Then $\text{lct}(S, S_4) = 1$.

Proof. Let $O_1, \ldots, O_4$ be singular points of the surface $S_1$, and let $L_{ij}$ be a line in $S$ that contains the points $O_i$ and $O_j$, where $i \neq j$. Then $S_4$ acts transitively on $\{O_1, \ldots, O_4\}$ and $\{L_{12}, \ldots, L_{34}\}$.

Let $T$ be a curve that is cut out on $S$ by the equation $x + y + z + t = 0$. Then $T$ is $S_4$-invariant, which implies that $\text{lct}(S, S_4) \leq 1$. Suppose that $\text{lct}(S, S_4) < 1$. Then there is an effective $S_4$-invariant $\mathbb{Q}$-divisor $D$ such that $D \equiv -K_S$, and $(S, \lambda D)$ is not log canonical, where $\lambda \in \mathbb{Q}$ and $\lambda < 1$.

The surface $S$ does not contain $S_4$-invariant points, because the group $S_4$ does not have faithful two-dimensional linear representations. Then $\text{LCS}(S, \lambda D)$ contains a curve by Remark 2.2.

There are a reduced $S_4$-invariant curve $C \subset S$ and a rational number $m \geq 1/\lambda$ such that

$$D = mC + \Omega,$$

where $\Omega$ is an effective divisor, whose support does not contain components of $C$. Then

$$3 = -K_S \cdot D = m\deg(C) - K_S \cdot \Omega \geq m\deg(C) > \deg(C),$$

which implies that either $C$ is a line, or $C$ is a conic.

Suppose that the curve $C$ is not an irreducible conic. Let $L$ be any irreducible component of the curve $C$. Then $L$ is a line. Let $M$ be a general hyperplane section of $S$ that contains $L$. Then

$$M = L + \bar{L} \sim -K_S,$$

where $\bar{L}$ is an irreducible conic. We have

$$2 = \bar{L} \cdot D = m\bar{L} \cdot L + \bar{L} \cdot \Omega \geq m\bar{L} \cdot C > \bar{L} \cdot L,$$

which implies that $L \cap \{O_1, \ldots, O_4\} \neq \emptyset$. Then $L \in \{L_{12}, \ldots, L_{34}\}$, which is impossible, because the curve $C$ contains at most two components.

We see that $\text{LCS}(S, \lambda D)$ does not contain lines, and $C$ is an irreducible conic.

Let $R$ be a hyperplane section of the surface $S$ that contains the conic $C$. Then

$$R = C + \bar{C} \sim -K_S,$$

where $\bar{C}$ is a $S_4$-invariant line. The intersection $\bar{C} \cap C$ consists of two points.

The log pair $(S, C + C)$ is log canonical. We may assume $C \not\subseteq \text{Supp}(\Omega)$ by Remark 2.1. Then

$$1 = \tilde{S} \cdot D = m\tilde{C} \cdot C + \tilde{C} \cdot \Omega \geq m\tilde{C} \cdot C > \tilde{C} \cdot C,$$

which implies that $\tilde{C} \cap C \subset \{O_1, \ldots, O_4\}$. Then $\tilde{C} \in \{L_{12}, \ldots, L_{34}\}$, which is impossible. \hfill $\Box$

Lemma 4.2. Let $S$ be a cubic surface in $\mathbb{P}^3$ given by the equation $3x + y + z + t = 0$. Then $\text{lct}(S, S_3 \times \mathbb{Z}_3) = 1$.

Proof. Put $G = S_3 \times \mathbb{Z}_3$. Let $O_1, O_2, O_3$ be singular points of the surface $S$, and $L_i \subset S$ be a line such that $O_i \not\in L_i$. Then $\text{lct}(S, G) \leq 1$, because the curve $L_1 + L_2 + L_3$ is $G$-invariant.

We suppose that $\text{lct}(S, G) < 1$. Then there is an effective $G$-invariant $\mathbb{Q}$-divisor $D$ such that the equivalence $D \equiv -K_S$ holds, and $(S, \lambda D)$ is not log canonical, where $\lambda \in \mathbb{Q}$ and $\lambda < 1$.

The surface $S$ does not contain $G$-invariant points. Then $|\text{LCS}(S, \lambda D)| = +\infty$ by Remark 2.2, which implies that there is a $G$-invariant curve $C \subset S$ and a rational number $m \geq 1/\lambda$ such that

$$D = mC + \Omega,$$

where $\Omega$ is an effective $\mathbb{Q}$-divisor, whose support does not contain components of $C$. Then

$$3 = -K_S \cdot D = m\deg(C) - K_S \cdot \Omega \geq m\deg(C) > \deg(C),$$

which implies that either $C$ is a line, or $C$ is a conic.

The only lines contained in $S$ are the lines $L_1, L_2, L_3$. The group $G$ acts on the set

$$\{L_1, L_2, L_3\}$$

transitively. Hence, the curve $C$ is neither a line, nor conic. \hfill $\Box$
5. Fiberwise maps.

Let $Z$ be a smooth curve. Suppose that there is a commutative diagram

\[(5.1) \quad \begin{array}{ccc}
V & \xrightarrow{\rho} & \tilde{V} \\
\pi \downarrow & & \tilde{\pi} \\
Z & \xrightarrow{\pi} & \tilde{Z}
\end{array} \]

such that $\pi$ and $\tilde{\pi}$ are flat morphisms, and $\rho$ is a birational map that induces an isomorphism

\[(5.2) \quad \rho|_{V \setminus X} : V \setminus X \longrightarrow \tilde{V} \setminus \tilde{X}, \]

where $X$ and $\tilde{X}$ are scheme fibers of $\pi$ and $\tilde{\pi}$ over a point $O \in Z$, respectively. Suppose that

- the varieties $V$ and $\tilde{V}$ have terminal $\mathbb{Q}$-factorial singularities,
- the divisors $-K_V$ and $-K_{\tilde{V}}$ are $\pi$-ample and $\tilde{\pi}$-ample, respectively,
- the fibers $X$ and $\tilde{X}$ are irreducible.

The following example is due to [6].

**Example 5.3.** Suppose that $X$ is a smooth cubic surface that contains lines $L_1, L_2, L_3$ such that the intersection $L_1 \cap L_2 \cap L_3$ consists of a point $P \in X$. There is commutative diagram

\[U \xrightarrow{\psi} V \xrightarrow{\rho} Z \]

where $\alpha$ is a blow up of $P$, $\psi$ is an antiflip in the proper transforms of $L_1, L_2, L_3$, and $\beta$ is a contraction of the proper transform of the fiber $X$. Then $\tilde{X}$ is a cubic surface that has one singular point of type $\mathbb{D}_4$. We have \(\text{lct}(X) = 2/3\) and \(\text{lct}(\tilde{X}) = 1/3\) (see Example 1.7 and Lemma 3.6).

Which kind of conditions on the fibers $X$ and $\tilde{X}$ imply that $\rho$ is biregular?

**Example 5.4.** Suppose that both fibers $X$ and $\tilde{X}$ are nonsingular del Pezzo surfaces such that the inequality $K_X^2 = K_{\tilde{X}}^2 \leq 4$ holds. Then $\rho$ is an isomorphism (see [14]).

The question we asked is local by the curve $Z$. Thus, in the following, we will not assume that the curve $Z$ is projective. Let us consider two examples with $Z = \mathbb{C}^1$ (see [14]).

**Example 5.5.** Let $V$ be $\tilde{V}$ subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

\[x^3 + y^2z + z^2w + t^{12}w^3 = 0\]

respectively, where $t$ is a coordinate on $\mathbb{C}^1$, and $(x, y, z, w)$ are coordinates on $\mathbb{P}^3$. The projections

\[\pi : V \longrightarrow \mathbb{C}^1 \quad \text{and} \quad \tilde{\pi} : \tilde{V} \longrightarrow \mathbb{C}^1\]

are fibrations into cubic surfaces. Let $O$ be the point on $\mathbb{C}^1$ given by $t = 0$. Then $\tilde{X}$ is smooth, the surface $X$ has one singular point of type $\mathbb{D}_6$. Put $Z = \mathbb{C}^1$. Then the map

\[(x, y, z, w) \longrightarrow (t^2x, t^3y, z, t^6w)\]

induces a birational map $\rho : V \longrightarrow \tilde{V}$ such that the diagrams (5.1) and isomorphism (5.2) exist, and $\rho$ is not biregular. But $\text{lct}(X) = 1/6$ and $\text{lct}(\tilde{X}) = 2/3$ (see Example 1.7 and Lemma 3.8).

**Example 5.6.** Let $V$ be $\tilde{V}$ subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

\[wz^2 + xz^2 + y^2x + t^6w^3 = 0\]

respectively, where $t$ is a coordinate on $\mathbb{C}^1$, and $(x, y, z, w)$ are coordinates on $\mathbb{P}^3$. The projections

\[\pi : V \longrightarrow \mathbb{C}^1 \quad \text{and} \quad \tilde{\pi} : \tilde{V} \longrightarrow \mathbb{C}^1\]

are fibrations into cubic surfaces. Let $O$ be the point on $\mathbb{C}^1$ given by $t = 0$. Then $\tilde{X}$ is smooth, the surface $X$ has one singular point of type $\mathbb{D}_5$. Put $Z = \mathbb{C}^1$. Then the map

\[(x, y, z, w) \longrightarrow (t^2x, ty, z, t^4w)\]
induces a birational map \( \rho: V \rightarrow \bar{V} \) such that the diagrams \([5.1]\) and isomorphism \([5.2]\) exist, and \( \rho \) is not biregular. But \( \text{lct}(X) = 1/4 \) and \( \text{lct}(\bar{X}) = 2/3 \) (see Example \([1.7]\) and Lemma \([3.7]\)).

The following result holds (see Examples \([1.7]\) and \([5.4]\)).

**Theorem 5.7.** The map \( \rho \) is an isomorphism if one of the following conditions hold:

- the varieties \( X \) and \( \bar{X} \) have log terminal singularities, and \( \text{lct}(X) + \text{lct}(\bar{X}) > 1 \);
- the variety \( X \) has log terminal singularities, and \( \text{lct}(X) \geq 1 \).

**Proof.** Suppose that the variety \( X \) has log terminal singularities, the inequality \( \text{lct}(X) \geq 1 \) holds, and \( \rho \) is not an isomorphism. Let \( D \) be a general very ample divisor on \( Z \). Put

\[
\Lambda = \lfloor -nK_V + \pi^*(nD) \rfloor, \quad \Gamma = \lfloor -nK_{\bar{V}} + \bar{\pi}^*(nD) \rfloor, \quad \bar{\Lambda} = \rho(\Lambda), \quad \bar{\Gamma} = \rho^{-1}(\Gamma),
\]

where \( n \) is a natural number such that \( \Lambda \) and \( \Gamma \) have no base points. Put

\[
M_V = \frac{2\varepsilon}{n} \Lambda + \frac{1-\varepsilon}{n} \Gamma, \quad M_{\bar{V}} = \frac{2\varepsilon}{n} \bar{\Lambda} + \frac{1-\varepsilon}{n} \bar{\Gamma},
\]

where \( \varepsilon \) is a positive rational number.

The log pairs \((V, M_V)\) and \((\bar{V}, M_{\bar{V}})\) are birationally equivalent, and \( K_V + M_V \) and \( K_{\bar{V}} + M_{\bar{V}} \) are ample. The uniqueness of canonical model (see Theorem 1.3.20 in \([3]\)) implies that \( \rho \) is biregular if the singularities of both log pairs \((V, M_V)\) and \((V, M_{\bar{V}})\) are canonical.

The linear system \( \Gamma \) does not have base points. Thus, there is a rational number \( \varepsilon \) such that the log pair \((\bar{V}, M_{\bar{V}})\) is canonical. So, the log pair \((V, M_V)\) is not canonical. Then the log pair

\[
\left( V, X + \frac{1-\varepsilon}{n} \bar{\Gamma} \right)
\]

is not log canonical, because \( \Lambda \) does not have not base points, and \( \bar{\Gamma} \) does not have base points outside of the fiber \( X \), which is a Cartier divisor on the variety \( V \). The log pair

\[
\left( X, \frac{1-\varepsilon}{n} \Gamma|_X \right)
\]

is not log canonical by Theorem 17.6 in \([12]\), which is impossible, because \( \text{lct}(X) \geq 1 \).

To conclude the proof we may assume that the varieties \( X \) and \( \bar{X} \) have log terminal singularities, the inequality \( \text{lct}(X) + \text{lct}(\bar{X}) > 1 \) holds, and \( \rho \) is not an isomorphism.

Let \( \Lambda, \Gamma, \bar{\Lambda}, \bar{\Gamma} \) and \( n \) be the same as in the previous case. Put

\[
M_V = \frac{\text{lct}(X) - \varepsilon}{n} \Lambda + \frac{\text{lct}(X) - \varepsilon}{n} \Gamma, \quad M_{\bar{V}} = \frac{\text{lct}(\bar{X}) - \varepsilon}{n} \bar{\Lambda} + \frac{\text{lct}(\bar{X}) - \varepsilon}{n} \bar{\Gamma},
\]

where \( \varepsilon \) is a sufficiently small positive rational number. Then it follows from the uniqueness of canonical model that \( \rho \) is biregular if both log pair \((V, M_V)\) and \((V, M_{\bar{V}})\) are canonical.

Without loss of generality, we may assume that the singularities of the log pair \((V, M_V)\) are not canonical. Arguing as in the previous case, we see that the log pair

\[
\left( X, \frac{\text{lct}(X) - \varepsilon}{n} \Gamma|_X \right)
\]

is not log canonical, which is impossible, because \( \bar{\Gamma}|_X \equiv -nK_X \).

The assertion of Theorem \([5.7]\) can not be improved (see Examples \([5.3]\) \([5.5]\) \([5.6]\)).

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