The relationship between the matrices of the two analyses (SVD) and (GSVD)

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Abstract. In this paper we will search in analysis the matrices (SVD) and (GSVD) in terms of the relationship between their matrices after touching and briefly about the idea of each. Where we will two analyze matrices \( A(\alpha,\beta) \) and \( B(\gamma,\delta) \) using the techniques above and we compare between each of the two analyzes for the purpose of reaching a final relationship connecting between the matrices of the two analyze. This relationship can be utilized it in image processors, encryption and image enhancement as well as text and video. Perhaps in a later research we will focus on determining the relationships between the matrices of other analyzes such as (QR) and (LU) analysis.

1. Introduction
The matrix analyzes is as an inexhaustible rhombus in terms of scientific value, research space and practical applications. These analyzes and this particular science entered into all modern sciences in the field of mathematics, physics, chemistry and medicine, especially astronomy, images and their branches. May serious on us mathematicians throughout history in discovering this complex analyzes. Certainly, they made extraordinary efforts in the time of the discovery of these analyzes because of the weakness of scientific tools such as computers and the internet. These efforts will remain immortal throughout history and embodied in the service they provide throughout history for science, scientists, researchers and students.

From these matrixes analyzes (SVD), (GSVD), (LU), (QR)... and others. Note that for each analysis, multiple types and special cases can be go back for sources to see it.

2. Research Idea
The idea in our research is to find the relationship between the matrices of each of the two analyzes above. That is, between \((SVD) \& (GSVD)\), \((SVD) \& (LU)\), \((SVD) \& (QR)\), \((GSVD) \& (LU)\), \((GSVD) \& (QR)\), and \((LU) \& (QR)\) and we will allocate this research on the first relationship only \((SVD)\) and \((GSVD)\), and we hope that life will remain to write later research about other relationships.

3. Brief overview of the two analyzes (SVD) & (GSVD)
3.1. Singular Values Decomposition (SVD)[1][2][3][4][5][6]:-
Definition:-
For all \( B(\ell,p) \) is real matrix
There exist two matrices with specifications below:

\[ M_{(r,r)} \text{ and } N_{(p,p)} \] are orthogonally such that:

\[ [M, C, N] = \text{svd}(B) \]

\[ B_{(r,p)} = M_{(r,r)} * C_{(r,p)} * N_{(p,p)}^T \]

Where \( C_{(r,p)} \) is an \( r \times p \) matrix with nondiagonal entries all zero and:

\[ c_{11} \geq c_{22} \geq \cdots \geq c_{tt} \geq 0 \text{ where } t = \min\{r, p\} \]

The matrices above are called as follows:

i) The diagonal entries of \( C \) are called the singular values of \( B \).

ii) The columns of \( M \) are called the left singular vectors of \( B \).

iii) The columns of \( N \) are called the right singular vectors of \( B \).

iv) The components of the diameter of the matrix \( C_{(r,p)} \) are the square roots of the eigenvalues of the symmetrical square matrix \( (B_{(r,p)})^T * B_{(r,p)} \).

v) The columns of the matrix \( N_{(p,p)} \) are the eigenvectors of the symmetrical square matrix \( (B_{(r,p)})^T * B_{(r,p)} \) corresponding to the eigenvalues for same matrix referred to in (iv) above arranged in form consensual with the components \( C_{(r,p)} \). In means, the first column of the matrix \( N_{(p,p)} \) corresponds to the first eigenvalue of the symmetrical square matrix \( (B_{(r,p)})^T * B_{(r,p)} \) that it square root is \( c_{11} \). The second column of the matrix \( N_{(p,p)} \) corresponds to the second eigenvalue of the symmetrical square matrix \( (B_{(r,p)})^T * B_{(r,p)} \) that it square root is \( c_{22} \)… thus the rest of the columns. After processed by the Gram-Schmidt method to convert it into an orthogonal matrix and then divided each column on its length to be the matrix at the end orthonormal.

vi) The columns of the matrix \( M_{(r,r)} \) are calculated by the following equation:

\[ m_j = \frac{1}{c_{jj}} B_{nj} \]

Where \( m_j \) is column \( j \) from matrix \( M_{(r,r)} \),

\( n_j \) is column \( j \) from matrix \( N_{(p,p)} \)

Also after processed by the Gram-Schmidt method to convert it into an orthogonal matrix and then divided each column on its length to be the matrix at the end orthonormal.

Possible go back to the sources below for increase identify on details, and know method Gram-Schmidt and how to apply them through illustrative examples.

Analysis \( B_{(r,p)} = M_{(r,r)} * C_{(r,p)} * N_{(p,p)}^T \) is call (Singular Values Decomposition SVD). Some text.

### 3.1.1. Numerical Example of (svd)

Example for (SVD):

Let \( B = \begin{bmatrix} 25 & 47 & 32 & 41 \\ 87 & 75 & 96 & 78 \\ 20 & 10 & 90 & 80 \end{bmatrix} \)

Then \([M, C, N] = \text{svd}(B)\):

\[ M = \begin{bmatrix} -0.3306 & 0.2748 & 0.9029 \\ -0.7827 & 0.4547 & -0.4250 \\ -0.5273 & -0.8472 & 0.0647 \end{bmatrix} \]

\[ C = \begin{bmatrix} -0.3738 & 0.6645 & 0.5495 & -0.3417 \\ -0.6261 & -0.4103 & -0.2979 & -0.5924 \\ -0.5490 & -0.3628 & 0.4436 & 0.6085 \end{bmatrix} \]

\[ N = \begin{bmatrix} -0.4085 & 0.5083 & -0.6423 & 0.4027 \\ -0.5083 & 0.5083 & 0.6423 & -0.4027 \\ 0.4027 & -0.6423 & -0.5083 & -0.5083 \end{bmatrix} \]
3.2. General Singular Values Decomposition (GSVD)[7][8][9][10][11][12][13]:

For all two matrices \(M_{(r,t)} \) and \(N_{(q,t)} \) have the same number of columns and the number of rows in them may not be equal.

Therefore, it the matrix analysis below is called (General Singular Values Decomposition (GSVD)).

\[
\begin{bmatrix}
H_{(r,t)}, K_{(q,t)}, Z_{(t_{\min}(r+q,t)), E_{(r,t)}}, F_{(q,t)}
\end{bmatrix} = \text{gsvd}(M_{(r,t)}, N_{(q,t)})
\]

Such that:

\[
(\begin{bmatrix}
H_{(r,t)} \times E_{(r,t)} \times (Z_{(t_{\min}(r+q,t))})^T
\end{bmatrix})^T = M_{(r,t)}
\]

\[
(\begin{bmatrix}
K_{(q,t)} \times F_{(q,t)} \times (Z_{(t_{\min}(r+q,t))})^T
\end{bmatrix})^T = N_{(q,t)}
\]

And \( (E_{(r,t)})^T \times (E_{(r,t)}) + (F_{(q,t)})^T \times (F_{(q,t)}) = I_{(tt)} \)

The matrix \(Z_{(t_{\min}(r+q,t))}\) is the matrix of its columns representing the eigenvectors of the symmetrical square matrix \(\text{COV}(M, N)\) after converted to orthonormal.

The matrix \(E_{(r,t)}\) is a diagonal matrix be the values of the main diameter in it the square roots of the eigenvalues of the symmetrical square matrix \(\begin{bmatrix}(M_{(r,t)})^T + (M_{(r,t)})\end{bmatrix}\) corresponding to the eigenvectors mentioned in the matrix \(Z_{(t_{\min}(r+q,t))}\) ordered descending, which: -

\[e_{(1,1)} \leq e_{(2,1)} \leq \cdots \leq e_{(n,1)} \text{ where } y_1 = \min(r,t)\]

The matrix \(F_{(q,t)}\) is a diagonal matrix be the values of the main diameter in it the square roots of the eigenvalues of the symmetrical square matrix \(\begin{bmatrix}(N_{(q,t)})^T + (N_{(q,t)})\end{bmatrix}\) corresponding to the eigenvectors mentioned in the matrix \(Z_{(t_{\min}(r+q,t))}\) ordered descending, which:

\[f_{(1,1)} \leq f_{(2,1)} \leq \cdots \leq f_{(n,1)} \text{ where } y_2 = \min(q,t)\]

The matrix \(H_{(r,t)}\) is a square matrix whose columns are calculated as follows:

\[h_j = \frac{1}{e_{jj}} M_{ji}\]

It is converted to orthonormal.

In addition, the matrix \(K_{(q,t)}\) is a square matrix whose columns are calculated as follows:

\[k_j = \frac{1}{f_{jj}} N_{ji}\]

And it is converted it to orthonormal.

3.2.1. Numerical example of (gsvd):-

Example for (GSVD):

\[
M = \begin{bmatrix}
25 & 47 & 32 & 41 \\
87 & 75 & 96 & 78 \\
20 & 10 & 90 & 80 \\
12 & 32 & 45 & 65
\end{bmatrix}
\]

\[
N = \begin{bmatrix}
78 & 98 & 14 & 41 \\
80 & 89 & 94 & 56
\end{bmatrix}
\]

\[
[H, K, Z, E, F] = \text{gsvd}(M, N)
\]

\[
H = \begin{bmatrix}
0.4379 & -0.8990 & 0.0000 \\
-0.5171 & -0.2519 & 0.8180 \\
0.7354 & 0.3582 & 0.5752 \\
-0.4223 & 0.9065 & -0.4223
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
-0.9065 & -0.4223 \\
-0.4223 & 0.9065
\end{bmatrix}
\]
4. Extract the relationship between the matrices of (SVD) and matrices of (GSVD)

Let's have the two matrices \((A_{m \times p})\) and \((B_{n \times p})\),

\[
\begin{bmatrix}
U_{(m \times m)} & V_{(n \times n)} & X_{(p \times \min(m+n,p))} & C_{(m \times p)} & S_{(n \times p)}
\end{bmatrix} = \text{gsvd}(A_{m \times p},B_{n \times p})
\]

From analysis GSVD:

\[
U_{(m \times m)} \ast C_{(m \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T = A_{(m \times p)} \text{ ........... (1)}
\]

\[
V_{(n \times n)} \ast S_{(n \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T = B_{(n \times p)} \text{ ........... (2)}
\]

From analysis SVD:

\[
U_{(m \times m)} \ast S_{(m \times m)} \ast \left(V_{(1p)}\right)^T = A_{(m \times p)} \text{ ........... (3)}
\]

\[
U_{(n \times n)} \ast S_{(n \times p)} \ast \left(V_{(2p)}\right)^T = B_{(n \times p)} \text{ ........... (4)}
\]

Multiply both sides of the equation (5) from the left hand side by the inverse of the matrix \((U_{(m \times m)})\) and from the right hand side with the inverse of the matrix \((V_{(1p)}\)). And because these two matrices \((U_{(m \times m)})\) & \((V_{(1p)}\)) Therefore, the inverse of each of them is his transpose.

\[
(U_{(m \times m)})^T \ast U_{(m \times m)} \ast C_{(m \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T \ast \left((V_{(1p)})^T\right) = \left((V_{(1p)}\right)^T \text{ ........... (7)}
\]

\[
(U_{(n \times n)})^T \ast V_{(n \times n)} \ast S_{(n \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T \ast \left((V_{(2p)}\right)^T) = \left((V_{(2p)}\right)^T \text{ ........... (8)}
\]

From the functions (7)

\[
(U_{(m \times m)})^T \ast U_{(m \times m)} \ast C_{(m \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T \ast \left((V_{(1p)}\right)^T) = I \ast S_{(m \times p)} \ast I \text{ ........... (9)}
\]

From the functions (8)

\[
(U_{(n \times n)})^T \ast V_{(n \times n)} \ast S_{(n \times p)} \ast \left(X_{(p \times \min(m+n,p))}\right)^T \ast \left((V_{(2p)}\right)^T) = I \ast S_{(n \times p)} \ast I \text{ ........... (10)}
\]

Therefore, from the last two functions (9) and (10) we conclude the relationship between the matrices of the two analyzes (SVD & GSVD) as follows:-
With regard to the matrix $A$:

$$
S_1(m \ p) = (U_{1(m \ m)}^T \ C_{(m \ p)} * (X_{(p \ \min(m+n,p))})^T * (V_{1(p \ p)})^T
\frac{1}{2} - \frac{1}{2}
$$

With regard to the matrix $B$:

$$
S_2(n \ p) = (U_{2(n \ n)}^T \ V_{(n \ n)} \ S_{(n \ p)} \ C_{(n \ p)} \ X_{(p \ \min(m+n,p))})^T * (V_{2(p \ p)})^T
\frac{1}{2} - \frac{1}{2}
$$

### 4.1. Experimental result

#### Illustrative example

Let $\begin{bmatrix} a & b \end{bmatrix} = \text{gsvd}(a,b)$

Let $[u,v,x,c,s]=\text{svd}(a)$

Let $[u1,s1,v1]=\text{svd}(a)$

Let $[u2,s2,v2]=\text{svd}(b)$

Let $u = \begin{bmatrix} 0.9359 & 0.3522 & -0.0000 \\ 0.0772 & -0.2052 & -0.9757 \\ 0.3436 & -0.9132 & 0.2193 \end{bmatrix}$

Let $v = \begin{bmatrix} -0.8158 & 0.8158 \\ -21.3870 & 4.4818 & -8.8922 & -2.9050 \\ -22.7811 & 5.8591 & -9.6584 & -3.6614 \end{bmatrix}$

Let $x = \begin{bmatrix} -24.1753 & 7.2365 & -10.4245 & -5.1742 \end{bmatrix}

Let $\begin{bmatrix} 0.9850 & 0 \end{bmatrix}$

Let $c = \begin{bmatrix} 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$

Let $s = \begin{bmatrix} 0.1723 & 0 \\ 0 & -0.2067 & 0.8892 & 0.4082 \end{bmatrix}$

Let $u1 = \begin{bmatrix} -0.5183 & 0.2544 & -0.8165 \\ -0.8298 & -0.3804 & 0.4082 \end{bmatrix}$

Let $s1 = \begin{bmatrix} 25.4368 & 0.0000 & 0.0000 & 0 \\ 0.0000 & 1.7226 & 0.0000 & 0.0000 \end{bmatrix}$

Let $v1 = \begin{bmatrix} -0.4036 & -0.7329 & 0.5135 & 0.1906 \\ -0.4647 & -0.2989 & -0.8267 & 0.1287 \\ -0.5259 & 0.1532 & 0.1129 & -0.8290 \end{bmatrix}$

Let $u2 = \begin{bmatrix} -0.6173 & -0.7867 \\ 0.1726 & 0.0000 \end{bmatrix}$

Let $s2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.3797 \end{bmatrix}$

Let $v2 = \begin{bmatrix} -0.4542 & 0.7026 & -0.4196 & 0.3521 \\ -0.4840 & 0.2564 & 0.2970 & 0.7822 \\ -0.5138 & -0.1898 & 0.6647 & 0.5081 \end{bmatrix}$

Let $u1^T * u * c * x^T * (v1)^T = \begin{bmatrix} 25.4368 & 0.0000 & 0.0000 & 0 \\ 0.0000 & 1.7226 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}$

Let $u2^T * v * s * x^T * (v2)^T = \begin{bmatrix} 47.1153 & 0 & 0 & 0 \\ 0 & 0.3797 & 0 & 0 \end{bmatrix}$
5. Conclusions
We conclude from the relationship (11) above between the matrices of the analyzes of matrix in general and between the analysis of (SVD) and (GSVD) in particular attached to relationships mathematical Systemic. Where we can produce researches other looking at the rest of the relationships between the matrices of the rest of the analyzes as described in Future Research below.

6. Discussion
- After the conclusion of the two relationships (11) above can be manipulated the two ends of relationships and extract the value of any matrix in terms of the rest of the matrices other where we will have six relationships for each matrix, which with total 12 relationships.
- Can be used these relationships in algorithms image processing such as optimization, coloring, encryption, etc. Because these relationships have sufficient flexibility allow to be used in this field.
- The relationships above represent the possibility of conducting a compound analysis of the matrices and regrouping parts and retrieve the original matrix and this allows the use of these techniques in text encryption, where it the encryption of texts does not bear any error or loss of information in the step of retrieving the original data.
- Because we are very confident in the results of program MATLAB[14], we have used this program in the application of the examples received in the above.
- We can use the same analytical method in deduce the relationship between any two matrices analyzes such as:
  - The relationship between the matrices of the two analyzes ((SVD) and (LU)).
  - The relationship between the matrices of the two analyzes ((SVD) and (QR)).
  - The relationship between the matrices of the two analyzes ((GSVD) and (LU)).
  - The relationship between the matrices of the two analyzes ((GSVD) and (QR)).
  - The relationship between the matrices of the two analyzes ((LU) and (QR)).
And other from matrix analyzes.

7. Future Research:-
There are many research ideas that been generated in our mind mention some of them in below, can be accomplished this ideas in the case of the availability of material and logistical support for us. Which has a great role in facilitating the tasks on the scientific researcher and we hope we can accomplish them with the success from God.
- The relationship between the matrices of the two analyzes (SVD) and (LU).
- The relationship between the matrices of the two analyzes (SVD) and (QR).
- The relationship between the matrices of the two analyzes (GSVD) and (LU).
- The relationship between the matrices of the two analyzes (GSVD) and (QR).
- The relationship between the matrices of the two analyzes (LU) and (QR).
- Image Encryption with relationship between the matrices of the two analyzes (SVD) and (GSVD).
- Image Encryption with relationship between the matrices of the two analyzes (SVD) and (LU).
- Image Encryption with relationship between the matrices of the two analyzes (SVD) and (QR).
- Image Encryption with relationship between the matrices of the two analyzes (GSVD) and (LU).
- Image Encryption with relationship between the matrices of the two analyzes (GSVD) and (QR).
- Image Encryption with relationship between the matrices of the two analyzes (LU) and (QR).
• Text Encryption with relationship between the matrices of the two analyzes (SVD) and (GSVD).
• Text Encryption with relationship between the matrices of the two analyzes (SVD) and (LU).
• Text Encryption with relationship between the matrices of the two analyzes (SVD) and (QR).
• Text Encryption with relationship between the matrices of the two analyzes (GSVD) and (LU).
• Text Encryption with relationship between the matrices of the two analyzes (GSVD) and (QR).
• Text Encryption with relationship between the matrices of the two analyzes (LU) and (QR).
In addition to other many research ideas.

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