THE AUSLANDER-REITEN QUIVER OF A POINCARÉ DUALITY SPACE

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Abstract. In a previous paper, Auslander-Reiten triangles and quivers were introduced into algebraic topology. This paper shows that over a Poincaré duality space, each component of the Auslander-Reiten quiver is isomorphic to $\mathbb{Z}A_\infty$.

0. Introduction

In [5], the concepts of Auslander-Reiten triangles and Auslander-Reiten quivers from the representation theory of Artin algebras were introduced into algebraic topology.

The main theorem was that Auslander-Reiten triangles exist precisely over Poincaré duality spaces. On the other hand, the only concrete Auslander-Reiten quivers computed in [5] were those over spheres. An ad hoc computation showed that the Auslander-Reiten quiver over $S^d$, the $d$-dimensional sphere, consists of $d - 1$ components, each isomorphic to $\mathbb{Z}A_\infty$.

The purpose of this paper is to show that this result was no accident. Namely, let $k$ be a field of characteristic zero, and let $X$ be a simply connected topological space which has Poincaré duality of dimension $d \geq 2$ over $k$, that is, satisfies $\text{Hom}_k(H^*(X; k), k) \cong H^{d-*}(X; k)$, where $H^*(X; k)$ is singular cohomology of $X$ with coefficients in $k$. The main result here is the following.

Theorem 0.1. Let $C$ be a component of the Auslander-Reiten quiver of the category $D^c(C^*(X; k))$. Then $C$ is isomorphic to $\mathbb{Z}A_\infty$.

Here $C^*(X; k)$ is the singular cochain Differential Graded Algebra of $X$ with coefficients in $k$, and if $D(C^*(X; k))$ is the derived category of Differential Graded left-modules over $C^*(X; k)$, then $D^c(C^*(X; k))$ denotes the full subcategory of small objects, that is, objects $M$ for which the functor $\text{Hom}(M, -)$ preserves set indexed coproducts.

The ingredients in the proof are standard. First the Riedtmann structure theorem is invoked to show that the component $C$ has the

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form $\mathbb{Z}T/\Pi$ for a directed tree $T$ and an admissible group of automorphisms $\Pi \subseteq \text{Aut}(\mathbb{Z}T)$ (section 1). Then a certain function $\beta$ is constructed on $C$ (section 3). This induces an unbounded additive function $f$ on $T$ forcing its underlying graph to be $A_\infty$, and finally an elementary argument shows that $\Pi$ acts trivially (section 4). Hence $C$ is $\mathbb{Z}A_\infty$.

In fact, I will prove a more general result than theorem 0.1. Namely, I will not restrict myself to Differential Graded Algebras (DGAs) of the form $C^*(X; k)$, but will work over an abstract DGA denoted $R$.

**Setup 0.2.** Throughout, $R$ denotes a DGA over the field $k$ which satisfies the following.

(i) $R$ is a cochain DGA, that is, $R^i = 0$ for $i < 0$.
(ii) $R^0 = k$.
(iii) $R^1 = 0$.
(iv) $\dim_k R < \infty$. □

This setup is the same as in [5, sec. 3], so I can use the results about $R$ proved in [5].

For brief introductions to Auslander-Reiten triangles, Auslander-Reiten quivers, DGAs, Differential Graded modules (DG modules), and the way they interact, I refer the reader to [5]. My notation is mostly standard and identical to the notation of [5]; however, I do want to recapitulate a few ubiquitous items.

By $\mathcal{D}(R)$ is denoted the derived category of DG left-$R$-modules, and by $\mathcal{D}^c(R)$ the full subcategory of small objects.

By $R^{\text{op}}$ is denoted the opposite DGA of $R$ with multiplication

$$r \cdot_{\text{op}} s = (-1)^{|r||s|}sr.$$

DG left-$R^{\text{op}}$-modules are identified with DG right-$R$-modules, and so $\mathcal{D}(R^{\text{op}})$ is identified with the derived category of DG right-$R$-modules, and $\mathcal{D}^c(R^{\text{op}})$ with its full subcategory of small objects.

The DG left/right-$R$-modules

$$k = R/R^{\geq 1} \text{ and } DR = \text{Hom}_k(R, k)$$

are used frequently throughout.

When I wish to emphasize that I am viewing some $M$ as either a DG left-$R$-module or a DG right-$R$-module, I do so with subscripts, writing either $R M$ or $M_R$.

1. **Applying the Riedtmann structure theorem**

This section will apply the Riedtmann structure theorem to show under some conditions that a component $C$ of the Auslander-Reiten
quiver of $\mathcal{D}^c(R)$ has the form $\mathbb{Z}T/\Pi$ for a directed tree $T$ and an admissible group of automorphisms $\Pi \subseteq \text{Aut}(\mathbb{Z}T)$.

Let me start by recalling some facts from [5]. The triangulated category $\mathcal{D}^c(R)$ does not necessarily have Auslander-Reiten triangles, but if it does, then by [5, prop. 4.4(ii)] its Auslander-Reiten translation is

$$
\tau(-) = \Sigma^{-1}(DR \otimes_R -).
$$

Here $\Sigma$ denotes suspension of DG modules.

When $\mathcal{D}^c(R)$ has Auslander-Reiten triangles, $\tau$ induces a map from the Auslander-Reiten quiver of $\mathcal{D}^c(R)$ to itself which is also referred to as the Auslander-Reiten translation and denoted $\tau$.

The Auslander-Reiten quiver of $\mathcal{D}^c(R)$ is then a stable translation quiver with translation $\tau$; see [5, sec. 2]. This is even true in the strong sense that $\tau$ is an automorphism of the Auslander-Reiten quiver, under the extra assumption that $\mathcal{D}^c(R^{op})$ also has Auslander-Reiten triangles. This last fact was not mentioned in [5], but is proved in the following lemma.

**Lemma 1.1.** Suppose that $\mathcal{D}^c(R)$ and $\mathcal{D}^c(R^{op})$ have Auslander-Reiten triangles.

(i) The functor $\Sigma^{-1}(DR \otimes_R -)$ is an auto-equivalence of $\mathcal{D}^c(R)$, with quasi-inverse $\Sigma(P \otimes_R -)$ for a suitable DG left/right-$R$-module $P$.

(ii) The Auslander-Reiten translation $\tau$ is an automorphism of the Auslander-Reiten quiver of $\mathcal{D}^c(R)$.

(iii) For each integer $p \neq 0$, the map $\tau^p$ is without fixed points in the Auslander-Reiten quiver of $\mathcal{D}^c(R)$.

**Proof.** (i) Both $\mathcal{D}^c(R)$ and $\mathcal{D}^c(R^{op})$ have Auslander-Reiten triangles, so it follows from [5, cor. 5.2 and thm. 5.1 and its proof] that there are isomorphisms $R(DR) \cong R(\Sigma^d R)$ in $\mathcal{D}^c(R)$ and $(DR)_R \cong (\Sigma^d R)_R$ in $\mathcal{D}^c(R^{op})$. (Note that I do not know $R(DR)_R \cong R(\Sigma^d R)_R$!)

This makes it easy to check that with $P$ equal to $R\text{Hom}_R(DR, R)$, the functors $\Sigma^{-1}(DR \otimes_R -)$ and $\Sigma(P \otimes_R -)$ are quasi-inverse endofunctors on $\mathcal{D}^c(R)$.

(ii) This clearly follows from (i).

(iii) Let $M$ be an indecomposable object of $\mathcal{D}^c(R)$ with vertex $[M]$ in the Auslander-Reiten quiver of $\mathcal{D}^c(R)$. If $\tau^p$ had the fixed point $[M]$, then $p$ applications of equation (1) would give

$$
M \cong \Sigma^{-p}((DR) \otimes_R \cdots \otimes_R (DR) \otimes_R M).
$$
But this is impossible, as one proves for instance by checking
\[
\inf \{ i \mid H^i(M) \neq 0 \} \\
\neq \inf \{ i \mid H^i(\Sigma^{-p}((DR)_R \otimes_R \cdots \otimes_R (DR)_R M)) \neq 0 \},
\]
for which the general formula
\[
\inf \{ i \mid H^i(N \otimes_R M) \neq 0 \} \\
= \inf \{ i \mid H^i(N) \neq 0 \} + \inf \{ i \mid H^i(M) \neq 0 \}
\]
is handy. □

A loop in a quiver is an arrow that starts and ends at the same vertex.

**Lemma 1.2.** Suppose that \( \mathcal{D}^e(R) \) has Auslander-Reiten triangles. In this case, the Auslander-Reiten quiver of \( \mathcal{D}^e(R) \) has no loops.

**Proof.** Suppose that there were an indecomposable object \( M \) of \( \mathcal{D}^e(R) \) whose vertex \([M]\) in the Auslander-Reiten quiver of \( \mathcal{D}^e(R) \) had a loop \([M] \to [M]\). The arrow \([M] \to [M]\) would mean that there existed an irreducible morphism \( M \xrightarrow{\mu} M \) in \( \mathcal{D}^e(R) \). Such a morphism would be non-invertible, so it would be in the radical of the local artinian ring \( \text{End}_{\mathcal{D}^e(R)}(M) \) (cf. [5, lem. 3.6]). Hence some power \( \mu^n \) would be zero.

However, it is not hard to mimick the proof of [1, lem. VII.2.5] to show that if \( M \) is indecomposable and \( M \xrightarrow{\mu} M \) is irreducible, and the composition \( M \xrightarrow{\mu} \cdots \xrightarrow{\mu} M \) is zero, then \([M] = \tau[M]\) holds, and this is false by lemma 1.1(iii). □

In the following lemma and the rest of the paper, \( C \) is a component of the Auslander-Reiten quiver viewed as a stable translation quiver, so \( C \) is closed under the Auslander-Reiten translation \( \tau \) and its inverse. This implies that \( C \) is a connected stable translation quiver with translation the restriction of \( \tau \).

For the lemma’s terminology of directed trees, stable translation quivers of the form \( ZT \), and admissible groups of automorphisms, see [2, sec. 4.15].

**Lemma 1.3.** Suppose that \( \mathcal{D}^e(R) \) and \( \mathcal{D}^e(R^{\text{op}}) \) have Auslander-Reiten triangles. Let \( C \) be a component of the Auslander-Reiten quiver of \( \mathcal{D}^e(R) \).

Then there exists a directed tree \( T \) and an admissible group of automorphisms \( \Pi \subseteq \text{Aut}(ZT) \) so that \( C \) is isomorphic to \( ZT/\Pi \) as a stable translation quiver.
Proof. Lemma 1.1(ii) implies that the restriction of $\tau$ to $C$ is an automorphism of $C$, and lemma 1.2 implies that $C$ has no loops. Also, by construction, the Auslander-Reiten quiver has no multiple arrows, so neither has $C$.

Hence $C$ satisfies the conditions of the Riedtmann structure theorem, [2, thm. 4.15.6], and the conclusion of the lemma follows. □

2. Labelling the Auslander-Reiten Quiver

This section constructs a labelling of the Auslander-Reiten quiver of $D^c(R)$, that is, a pair of positive integers $(a_{m \to n}, b_{m \to n})$ for each arrow $m \to n$ in the quiver. The construction is classical, works in good cases such as when $D^c(R)$ and $D^c(R^{\text{op}})$ both have Auslander-Reiten triangles, and goes as follows.

Let $M$ and $N$ be indecomposable objects of $D^c(R)$ for which there is an arrow $[M] \to [N]$ in the Auslander-Reiten quiver, and let

$$M \to X \to \tau^{-1}M \to \tau N \to Y \to N \to$$

be Auslander-Reiten triangles in $D^c(R)$. Here $\tau^{-1}M$ makes sense because of lemma 1.1. Note that the triangles are determined up to isomorphism by [4, prop. 3.5(i)].

The arrow $[M] \to [N]$ means that there exists an irreducible morphism $M \to N$ in $D^c(R)$, and by [4, prop. 3.5] this means that $M$ is a direct summand of $Y$ and that $N$ is a direct summand of $X$. Let $a$ be the multiplicity of $M$ as a direct summand of $Y$ and let $b$ be the multiplicity of $N$ as a direct summand of $X$. These numbers are well-defined because $D^c(R)$ is a Krull-Schmidt category by [4, par. 3.1] and [5, lem. 3.6]. Now define the labelling by equipping $[M] \to [N]$ with the pair of positive integers $(a, b)$.

Lemma 2.1. Suppose that $D^c(R)$ and $D^c(R^{\text{op}})$ have Auslander-Reiten triangles. The above labelling of the Auslander-Reiten quiver of $D^c(R)$ satisfies the following.

(i) $a_{\tau[M] \to [N]} = b_{[N] \to [M]}$.

(ii) $b_{[M] \to [N]} = a_{[N] \to [M]}$.

(iii) $(a_{\tau p[M] \to \tau p[N]}, b_{\tau p[M] \to \tau p[N]}) = (a_{[M] \to [N]}, b_{[M] \to [N]})$ for each integer $p$.

Proof. (i) Let

$$\tau N \to Y \to N \to$$

be an Auslander-Reiten triangle. Then $a_{\tau[M] \to [N]}$ is defined as the multiplicity of $\tau M$ as a direct summand of $Y$. 

Proof. (ii) Let
Now, \( \tau \) is given by the formula \( \Sigma^{-1}(DR \otimes_R -) \) which also defines an auto-equivalence of \( D^c(R) \) by lemma 1.1(i). A quasi-inverse auto-equivalence is given by \( \Sigma(P \otimes_R -) \), and applying this to the Auslander-Reiten triangle (\ref{tr}) gives an Auslander-Reiten triangle

\[
N \longrightarrow \Sigma(P \otimes_R Y) \longrightarrow \tau^{-1}N \longrightarrow .
\]

Then \( b_{[N] \to [M]} \) is defined as the multiplicity of \( M \) as a direct summand of \( \Sigma(P \otimes_R Y) \).

Applying \( \Sigma^{-1}(DR \otimes_R -) \), this multiplicity equals the multiplicity of \( \Sigma^{-1}(DR \otimes_R M) \) as a direct summand of \( \Sigma^{-1}(DR \otimes_R \Sigma(P \otimes_R Y)) \), that is, the multiplicity of \( \tau M \) as a direct summand of \( Y \). And this again equals \( a_{\tau[M] \to [N]} \) by the above.

(ii) and (iii) are proved by similar means. \( \square \)

3. The function \( \beta \)

Let

\[
\beta(M) = \dim_k \text{Ext}_R(M, k) = \dim_k \text{Hom}_R(M, k)
\]

for \( M \) in \( D(R) \). Observe that \( \beta \) is constant on each isomorphism class in \( D^c(R) \), so induces a well-defined function on the vertices \([M]\) of the Auslander-Reiten quiver of \( D^c(R) \). I also denote the induced function by \( \beta \), so \( \beta([M]) = \beta(M) \).

The purpose of this section is to study \( \beta \).

**Lemma 3.1.** Suppose that \( D^c(R) \) and \( D^c(R^{\text{op}}) \) have Auslander-Reiten triangles and that the DG left-\( R \)-module \( nk \) is not in \( D^c(R) \).

Let \( N \) be an indecomposable object of \( D^c(R) \) with vertex \([N]\) in the Auslander-Reiten quiver of \( D^c(R) \).

(i) If \( \tau N \longrightarrow Y \longrightarrow N \longrightarrow \) is an Auslander-Reiten triangle in \( D^c(R) \), then \( \beta(\tau N) + \beta(N) - \beta(Y) = 0 \).

(ii) \( \beta([N]) + \beta([N]) - \sum_{[M] \to [N]} a_{[M] \to [N]} \beta([M]) = 0 \), where the sum is over all arrows in the Auslander-Reiten quiver of \( D^c(R) \) which end in \([N]\).

(iii) \( \beta([N]) = \beta([N]) \).

**Proof.** (i) The Auslander-Reiten triangle \( \tau N \longrightarrow Y \longrightarrow N \longrightarrow \) gives a long exact sequence consisting of pieces \( \text{Ext}_R^i(N, k) \longrightarrow \text{Ext}_R^i(Y, k) \longrightarrow \text{Ext}_R^i(\tau N, k) \overset{\delta}{\longrightarrow} \). Part (i) of the lemma will follow if the connecting maps \( \delta \) are zero.
And they are: By \([\text{3, lem. 4.2}]\), the Auslander-Reiten triangle is also an Auslander-Reiten triangle in \(\mathcal{D}^f(R)\), the full subcategory of \(\mathcal{D}(R)\) consisting of \(M\)'s with \(\dim_k \text{Ext}^1_M < \infty\). Moreover, no morphism \(\Sigma^{-i}(\tau N) \to R^k\) is a section, for any such section would clearly have to be an isomorphism, but \(\Sigma^{-i}(\tau N)\) is in \(\mathcal{D}^c(R)\) and \(R^k\) is not. The definition of Auslander-Reiten triangle now says that the composition \(\Sigma^{-i}(\tau N) \to \Sigma^{-i}(\tau N) \to R^k\) is zero, where the first arrow is a suspension of the connecting morphism from the Auslander-Reiten triangle.

In other words, the map

\[
\text{Hom}_{\mathcal{D}(R)}(\Sigma^{-i}(\tau N), k) \to \text{Hom}_{\mathcal{D}(R)}(\Sigma^{-i-1}N, k)
\]

is zero. But this map equals

\[
\text{Ext}^i_R(\tau N, k) \xrightarrow{\delta} \text{Ext}^{i+1}_R(N, k).
\]

(ii) Consider again \(\tau N \to Y \to N \to\), the Auslander-Reiten triangle from (i). From \([\text{4, prop. 3.5}]\) follows that \(Y\) is a coproduct of copies of those indecomposable objects \(M\) of \(\mathcal{D}^c(R)\) for which there are irreducible morphisms \(M \to N\). That is, \(Y\) is a coproduct of copies of those indecomposable objects \(M\) of \(\mathcal{D}^c(R)\) for which there are arrows \([M] \to [N]\) in the Auslander-Reiten quiver of \(\mathcal{D}^c(R)\). And by the definition at the beginning of section \([\text{2}]\) the multiplicity of \(M\) as a direct summand of \(Y\) is \(a_{[M] \to [N]}\). This clearly implies

\[
\beta(Y) = \sum_{[M] \to [N]} a_{[M] \to [N]} \beta([M]),
\]

and then (ii) follows from (i).

(iii) Since both \(\mathcal{D}^c(R)\) and \(\mathcal{D}^c(R^{op})\) have Auslander-Reiten triangles, it follows from \([\text{5, cor. 5.2 and thm. 5.1 and its proof}]\) that there is an isomorphism \(R(\mathcal{D}R) \cong_R \Sigma^d R\) in \(\mathcal{D}^c(R)\). So as \(k\)-vector spaces,

\[
H(R\text{Hom}_R(\mathcal{D}R, k)) \cong H(R\text{Hom}_R(\Sigma^d R, k)) \cong H(\Sigma^{-d} k),
\]

which easily implies \(R\text{Hom}_R(\mathcal{D}R, k) \cong \Sigma^{-d} k\) in \(\mathcal{D}(R)\). This again gives (a) in

\[
R\text{Hom}_R(\tau N, k) = R\text{Hom}_R(\Sigma^{-1}(\mathcal{D}R \otimes_R N), k)
\]

\[
\cong \Sigma R\text{Hom}_R(\mathcal{D}R \otimes_R N, k)
\]

\[
\cong \Sigma R\text{Hom}_R(N, R\text{Hom}_R(\mathcal{D}R, k))
\]

\[
\cong \Sigma R\text{Hom}_R(N, \Sigma^{-d} k)
\]

\[
\cong \Sigma^{1-d} R\text{Hom}_R(N, k),
\]

and taking cohomology and \(k\)-dimension shows \(\beta(\tau[N]) = \beta([N])\). \(\square\)
The following results leave $\beta$ for a while, but return to it in corollary 3.3.

There is an abelian category $\mathcal{C}(R)$ whose objects are DG left-$R$-modules and whose morphisms are homomorphisms of DG left-$R$-modules, that is, homomorphisms of graded left-modules which are compatible with the differentials. The following Harada-Sai lemma for $\mathcal{C}(R)$ is proved just like [2, lem. 4.14.1].

**Lemma 3.2.** Let $F_0, \ldots, F_{2p-1}$ be indecomposable objects of $\mathcal{C}(R)$ with $\dim_k F_i \leq p$ for each $i$, and let

$$F_{2p-1} \xrightarrow{\varphi_{2p-1}} F_{2p-2} \xrightarrow{\varphi_{2p-2}} \cdots \xrightarrow{\varphi_1} F_0$$

be non-isomorphisms in $\mathcal{C}(R)$.

Then $\varphi_1 \circ \cdots \circ \varphi_{2p-1} = 0$.

**Corollary 3.3.** Let $M_0, \ldots, M_{2p-1}$ be indecomposable objects of $\mathcal{D}^c(R)$ with $\beta(M_i) \leq \dim_k R$ for each $i$, and let

$$M_{2p-1} \xrightarrow{\mu_{2p-1}} M_{2p-2} \xrightarrow{\mu_{2p-2}} \cdots \xrightarrow{\mu_1} M_0$$

be non-isomorphisms in $\mathcal{D}^c(R)$.

Then $\mu_1 \circ \cdots \circ \mu_{2p-1} = 0$.

**Proof.** Pick minimal semi-free resolutions $F_i \xrightarrow{\cong} M_i$ (minimality means that the differential $\partial_{F_i}$ takes values inside $R^{\geq 1} \cdot F_i$; see [3, lem. 3.3]).

Then the morphisms $M_{2p-1} \xrightarrow{\mu_{2p-1}} M_{2p-2} \xrightarrow{\mu_{2p-2}} \cdots \xrightarrow{\mu_1} M_0$ in $\mathcal{D}^c(R)$ are represented by morphisms $F_{2p-1} \xrightarrow{\varphi_{2p-1}} F_{2p-2} \xrightarrow{\varphi_{2p-2}} \cdots \xrightarrow{\varphi_1} F_0$ in $\mathcal{C}(R)$.

If $\varphi_i$ were an isomorphism in $\mathcal{C}(R)$, then $\mu_i$ would be an isomorphism in $\mathcal{D}^c(R)$, so each $\varphi_i$ is a non-isomorphism in $\mathcal{C}(R)$. Also, since $F_i$ is minimal, it is easy to see that if $F_i$ decomposed non-trivially in $\mathcal{C}(R)$, then $M_i$ would decompose non-trivially in $\mathcal{D}^c(R)$, so each $F_i$ is indecomposable in $\mathcal{C}(R)$.

Now

$$\text{Ext}_R(M_i, k) = H(\text{RHom}_R(M_i, k)) \cong H(\text{Hom}_R(F_i, k))$$

$$(a) \cong \text{Hom}_R(F_i, k)^2 = \text{Hom}_{R^e}(F_i^\natural, k^\natural).$$

Here $\natural$ indicates the functor which forgets differentials, and therefore sends DGAs to graded algebras and DG modules to graded modules. The isomorphism (a) holds because $F_i$ is minimal, whence the differential of the complex $\text{Hom}_R(F_i, k)$ is zero, so taking cohomology amounts simply to forgetting the differential.
Taking \( k \)-dimensions in equation (3) shows that \( \beta(M_i) \) equals the \( k \)-dimension of \( \text{Hom}_{R^{\text{op}}}(F_i^z, k^z) \). However, since \( F_i \) is semi-free I have

\[
F_i^z \cong \bigoplus_j \Sigma^{\sigma_j}(R^z)
\]

for certain integers \( \sigma_j \), and the \( k \)-dimension of \( \text{Hom}_{R^{\text{op}}}(F_i^z, k^z) \) clearly equals the number of summands \( \Sigma^{\sigma_j}(R^z) \) in \( F_i^z \).

All in all, \( \beta(M_i) \) equals the number of summands \( \Sigma^{\sigma_j}(R^z) \) in \( F_i^z \) which gives \((b)\) in

\[
\dim_k F_i = \dim_k F_i^z = \beta(M_i) \dim_k R^z = \beta(M_i) \dim_k R \leq \frac{p}{\dim_k R} \dim_k R = p.
\]

But now lemma 3.2 gives \( \varphi_1 \circ \cdots \circ \varphi_{2p-1} = 0 \) which clearly implies \( \mu_1 \circ \cdots \circ \mu_{2p-1} = 0 \). \( \square \)

**Lemma 3.4.** Suppose that \( D^e(R) \) has Auslander-Reiten triangles and that \( rk \) is not in \( D^f(R) \).

Given an indecomposable object \( M_0 \) of \( D^e(R) \) and an integer \( q \geq 0 \), there exist indecomposable objects and irreducible morphisms

\[
M_q \xrightarrow{\mu_q} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0
\]

in \( D^e(R) \) with \( \mu_1 \circ \cdots \circ \mu_q \neq 0 \).

**Proof.** If \( M \) is a non-zero object in \( D^f(R) \), the full subcategory of \( D(R) \) consisting of \( M \)'s with \( \dim_k HM < \infty \), then the \( k \)-dual \( DM = \text{Hom}_k(M, k) \) is non-zero in \( D^f(R^{op}) \), so the minimal semi-free resolution \( F \) of \( DM \) is non-trivial. Hence \( \text{RHom}_{R^{op}}(DM, k) \cong \text{RHom}_{R^{op}}(F, k) \) has non-zero cohomology, so by dualization the same holds for

\[
\text{RHom}_R(Dk, DDM) \cong \text{RHom}_R(k, M).
\]

So there exists an \( i \) so that \( H^{-i}(\text{RHom}_R(k, M)) \cong \text{Hom}_{D(R)}(\Sigma^i k, M) \) is non-zero; that is,

For \( M \) non-zero in \( D^f(R) \), there exists a non-zero morphism \( R(\Sigma^i k) \rightarrow M \) in \( D^f(R) \).

\((4)\)

Moreover, since \( M \) is non-zero, each retraction \( R(\Sigma^i k) \rightarrow M \) in \( D^f(R) \) is clearly an isomorphism. If \( M \) is in \( D^e(R) \), then there is no such isomorphism because \( R(\Sigma^i k) \) is not in \( D^e(R) \), so

For \( M \) non-zero in \( D^e(R) \), no morphism \( R(\Sigma^i k) \rightarrow M \) is a retraction in \( D^f(R) \).

\((5)\)
Now for the proof proper. In fact, I shall prove slightly more than claimed, namely, there exists
\[ R(\Sigma^i k) \xrightarrow{\kappa_q} M_q \xrightarrow{\mu_q} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0 \]
with the \( M_i \) indecomposable and the \( \mu_i \) irreducible in \( \mathbb{D}^c(\mathbb{R}) \), with \( \kappa_q \) not a retraction in \( \mathbb{D}^f(\mathbb{R}) \), and with \( \mu_1 \circ \cdots \circ \mu_q \circ \kappa_q \neq 0 \). I do so by induction on \( q \).

For \( q = 0 \), existence of \( \kappa_0 \) holds by equations (4) and (5).

For \( q \geq 1 \), the induction gives data
\[ R(\Sigma^i k) \xrightarrow{\kappa_q} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0. \]

Let \( \tau M_{q-1} \xrightarrow{\kappa_q'} X_q \xrightarrow{\mu_q'} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0 \) be an Auslander-Reiten triangle in \( \mathbb{D}^c(\mathbb{R}) \). By [5, lem. 4.2] this is also an Auslander-Reiten triangle in \( \mathbb{D}^f(\mathbb{R}) \), so since \( R(\Sigma^i k) \xrightarrow{\kappa_q} M_{q-1} \) is not a retraction in \( \mathbb{D}^f(\mathbb{R}) \), it factors through \( X_q \) as \( R(\Sigma^i k) \xrightarrow{\kappa_q} X_q \xrightarrow{\mu_q} M_{q-1} \), and so \( \mu_1 \circ \cdots \circ \mu_{q-1} \circ \mu_q \circ \kappa_q = \mu_1 \circ \cdots \circ \mu_{q-1} \circ \kappa_{q-1} \) is non-zero.

Splitting \( X_q \) into indecomposable summands and \( \kappa_q \) and \( \mu_q \) into components gives that there exists \( R(\Sigma^i k) \xrightarrow{\kappa_q} M_q \xrightarrow{\mu_q} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0 \) in \( \mathbb{D}^c(\mathbb{R}) \) with \( \mu_1 \circ \cdots \circ \mu_q \circ \kappa_q \neq 0 \).

Here \( M_q \) is an indecomposable summand of \( X_q \) so is in \( \mathbb{D}^c(\mathbb{R}) \), so equation (5) gives that \( \kappa_q \) is not a retraction in \( \mathbb{D}^f(\mathbb{R}) \). And \( \mu_q \) is a component of \( \mu_q' \), so \( \mu_q \) is irreducible in \( \mathbb{D}^c(\mathbb{R}) \) by [4, prop. 3.5].

I now return to the function \( \beta \).

**Corollary 3.5.** Suppose that \( \mathbb{D}^c(\mathbb{R}) \) and \( \mathbb{D}^c(\mathbb{R}^{\text{op}}) \) have Auslander-Reiten triangles and that \( Rk \) is not in \( \mathbb{D}^c(\mathbb{R}) \). Let \( C \) be a component of the Auslander-Reiten quiver of \( \mathbb{D}^c(\mathbb{R}) \). Then \( \beta \) is unbounded on \( C \).

**Proof.** Let \( M_0 \) be an indecomposable object of \( \mathbb{D}^c(\mathbb{R}) \) for which \( [M_0] \) is a vertex of \( C \). Lemma 3.4 says that there exist arbitrarily long sequences of indecomposable objects and irreducible morphisms
\[ M_q \xrightarrow{\mu_q} M_{q-1} \xrightarrow{\mu_{q-1}} \cdots \xrightarrow{\mu_1} M_0 \]
in \( \mathbb{D}^f(\mathbb{R}) \) with \( \mu_1 \circ \cdots \circ \mu_q \neq 0 \). Each \([M_i]\) is clearly a vertex of \( C \). So corollary 3.3 implies that \( \beta \) cannot be bounded on \( C \). \( \square \)

4. **Structure of the Auslander-Reiten quiver**

This section combines the material of previous sections to prove the main result of this paper, theorem 0.1.

By lemma 1.3 if \( \mathbb{D}^c(\mathbb{R}) \) and \( \mathbb{D}^c(\mathbb{R}^{\text{op}}) \) have Auslander-Reiten triangles, then each component \( C \) of the Auslander-Reiten quiver of \( \mathbb{D}^c(\mathbb{R}) \)
is of the form $ZT/\Pi$, where $T$ is a directed tree and $\Pi \subseteq \text{Aut}(ZT)$ is an admissible group of automorphisms.

Recall from [2, sec. 4.15] that the vertices of $ZT$ are the pairs $(p, t)$ where $p$ is an integer and $t$ is a vertex of $T$, and that the vertices of $ZT/\Pi$ are the orbits $\Pi(p, t)$ under the group $\Pi$. By identifying $ZT/\Pi$ with $C$, I shall also view the $\Pi(p, t)$’s as vertices of $C$, so the following makes sense.

First, the start of section 2 constructs a labelling of the Auslander-Reiten quiver, and so in particular a labelling of $C$. Now, if there is an arrow $t \rightarrow u$ in $T$, then there is an arrow $\Pi(0, t) \rightarrow \Pi(0, u)$ in $ZT/\Pi$, that is, an arrow $\Pi(0, t) \rightarrow \Pi(0, u)$ in $C$. Hence the labelling of $C$ induces a labelling of $T$ by

\[(a_{t \rightarrow u}, b_{t \rightarrow u}) = (a_{\Pi(0, t) \rightarrow \Pi(0, u)}, b_{\Pi(0, t) \rightarrow \Pi(0, u)}).\] (6)

Secondly, the start of section 3 defines a function $\beta$ on the vertices of $C$. That is, $\beta$ is defined on the $\Pi(p, t)$’s, and so induces a function $f$ on $T$ by

\[f(t) = \beta(\Pi(0, t)).\] (7)

**Lemma 4.1.** Suppose that $\mathcal{D}^c(R)$ and $\mathcal{D}^c(R^{\text{op}})$ have Auslander-Reiten triangles.

(i) The function $f$ is additive with respect to the labelling of $T$ given by equation (6).

(ii) The function $f$ is unbounded on $T$.

To explain part (i), let me recall from [2, sec. 4.5] that when $T$ is a labelled directed tree, a function $f$ on the vertices of $T$ is called additive if it satisfies

\[2f(t) - \sum_{s \rightarrow t} a_{s \rightarrow t} f(s) - \sum_{t \rightarrow u} b_{t \rightarrow u} f(u) = 0\] (8)

for each vertex $t$.

**Proof of Lemma 4.1** (i) To show that the left hand side of equation (8) is zero, let me rewrite it as

\[2\beta(\Pi(0, t)) - \sum_{s \rightarrow t} a_{\Pi(0, s) \rightarrow \Pi(0, t)} \beta(\Pi(0, s))\]
\[- \sum_{t \rightarrow u} b_{\Pi(0, t) \rightarrow \Pi(0, u)} \beta(\Pi(0, u)),\] (9)

using equations (6) and (7).

Recall from [2, sec. 4.15] that the translation of $ZT/\Pi$ is given by $\tau(\Pi(p, t)) = \Pi(p + 1, t)$. Identifying $ZT/\Pi$ with $C$, lemma 2.1(i) gives
\[
b_{\Pi(0,t) \to \Pi(0,u)} = a_{\tau(\Pi(0,u)) \to \Pi(0,t)} = a_{\Pi(1,u) \to \Pi(0,t)}, \quad \text{and lemma 3.4(iii)}
\]
gives \(\beta(\Pi(0,v)) = \beta(\tau(\Pi(0,v))) = \beta(\Pi(1,v))\). Hence (9) is

\[
\beta(\tau\Pi(0,t)) + \beta(\Pi(0,t)) - \sum_{s \to t} a_{\Pi(0,s) \to \Pi(0,t)} \beta(\Pi(0,s))
- \sum_{t \to u} a_{\Pi(1,u) \to \Pi(0,t)} \beta(\Pi(1,u)).
\]

(10)

To rewrite further, recall also from [2, sec. 4.15] that the arrows in \(Z_T\) which end in \((0,t)\) are obtained as follows: There is an arrow \((0,s) \to (0,t)\) in \(Z_T\) for each arrow \(s \to t\) in \(T\), and there is an arrow \((1,u) \to (0,t)\) in \(Z_T\) for each arrow \(t \to u\) in \(T\).

Moreover, the canonical map \(Z_T \to Z_T/\Pi\) is a so-called covering, see [2, p. 156], so the map which sends the arrow \(m \to (0,t)\) to the arrow \(\Pi m \to \Pi(0,t)\) is a bijection between the arrows in \(Z_T\) which end in \((0,t)\) and the arrows in \(Z_T/\Pi\) which end in \(\Pi(0,t)\). All this implies that taken together, the two sums in (10) contain exactly one summand for each arrow in \(Z_T/\Pi\) which ends in \(\Pi(0,t)\), so (10) is

\[
\beta(\tau\Pi(0,t)) + \beta(\Pi(0,t)) - \sum_{m \to \Pi(0,t)} a_m \beta(m).
\]

Identifying \(Z_T/\Pi\) with \(C\), this is zero by lemma 3.4(ii).

(ii) I have

\[
\beta(\Pi(p,t)) = \beta(\tau^p(\Pi(0,t))) \overset{(a)}{=} \beta(\Pi(0,t)) = f(t),
\]

where (a) is by lemma 3.1(iii). So if \(f\) were bounded on \(T\), then \(\beta\) would be bounded on \(C\) which it is not by corollary 3.5.

The following is my abstract main result, which has theorem 0.1 as an easy consequence.

I emphasize that as above, \(C\) is a component of the Auslander-Reiten quiver viewed as a stable translation quiver, so \(C\) is closed under the Auslander-Reiten translation \(\tau\) and its inverse, and is a connected stable translation quiver with translation the restriction of \(\tau\).

**Theorem 4.2.** Suppose that \(D^c(R)\) and \(D^c(R^{\text{op}})\) have Auslander-Reiten triangles and that \(Rk\) is not in \(D^c(R)\). Let \(C\) be a component of the Auslander-Reiten quiver of \(D^c(R)\).

Then \(C\) is isomorphic to \(ZA_{\infty}\) as a stable translation quiver.

**Proof.** Lemma 1.3 says that \(C\) is of the form \(ZT/\Pi\).

Lemma 4.1 gives an additive function \(f\) on \(T\), and shows that \(f\) is unbounded. Consequently, the underlying graph of \(T\) must be \(A_{\infty}\) by [2, thm. 4.5.8(iv)]. So \(C\) is \(ZA_{\infty}/\Pi\). (Note that when the underlying graph of \(T\) is \(A_{\infty}\), the quiver \(ZT\) is determined up to isomorphism,
independently of the directions of the arrows in $T$. So it makes sense to denote $ZT$ by $ZA_\infty$.

Now, $\Pi$ must send vertices at the end of $ZA_\infty$ to other such vertices. If $\Pi$ acted non-trivially on the vertices at the end of $ZA_\infty$, then there would exist a $g$ in $\Pi$ and a vertex $m$ at the end of $ZA_\infty$ so that $gm$ was a different vertex at the end of $ZA_\infty$. As the other vertices at the end of $ZA_\infty$ have the form $\tau^p m$ for $p \neq 0$, this would mean $gm = \tau^p m$ for some $p \neq 0$. Then $m$ and $\tau^p m$ would get identified in $ZA_\infty/\Pi$, so $\Pi m$ would be a fixed point of $\tau^p$ in $ZA_\infty/\Pi$, that is, a fixed point of $\tau^p$ in $C$ contradicting lemma 1.1(iii).

So $\Pi$ acts trivially on the vertices at the end of $ZA_\infty$, and it is easy to see that this forces $\Pi$ to act trivially on all of $ZA_\infty$. So $ZA_\infty/\Pi$ is just $ZA_\infty$, and the theorem follows. □

**Proof of Theorem 0.1.** This proof uses a bit of rational homotopy theory for which my source is [3].

By [3, exam. 6, p. 146] I have that $C^*(X;k)$ is equivalent by a series of quasi-isomorphisms to a DGA, $R$, which satisfies setup 0.2. Hence the various derived categories of $C^*(X;k)$ are equivalent to those of $R$ by [6, thm. III.4.2], so theorem 4.2 can be applied to $C^*(X;k)$.

To get the desired conclusion, that each component $C$ of the Auslander-Reiten quiver of $D^c(C^*(X;k))$ is isomorphic to $ZA_\infty$, I must show that the premises of theorem 4.2 hold for $C^*(X;k)$. So I must show that $D^c(C^*(X;k))$ and $D^c(C^*(X;k)^{op})$ have Auslander-Reiten triangles, and that $C^*(X;k)$ is not in $D^c(C^*(X;k))$. The first two claims hold by the main result of [5], theorem 6.3, because $X$ has Poincaré duality over $k$. The last claim can be seen as follows.

By assumption, $X$ has Poincaré duality of dimension $d \geq 2$ over $k$, so I have $H^{\geq 2}(X;k) \neq 0$. Hence the minimal Sullivan model $\Lambda V$ of $C^*(X;k)$ has $V^{\geq 2} \neq 0$; see [31, chp. 12]. But $V$ is $\text{Hom}_{\mathbb{Q}}(\pi_*,X,k)$ by [31, chp. 15], where $\pi_*$ is the sequence of homotopy groups of $X$, so as $k$ has characteristic zero, $\pi_{\geq 2} X$ cannot be torsion, so $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\geq 2} X \neq 0$. Since [31, p. 434] gives $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\text{odd}} X) - \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\text{even}} X) \geq 0$, there follows $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\text{odd}} X \neq 0$. This implies that the denominator is non-trivial in the formula [3, (33.7)] from which follows $\dim_{\mathbb{Q}} H^*(\Omega X;\mathbb{Q}) = \infty$, where $\Omega X$ is the Moore loop space of $X$. So also
\[
\dim_k H^*(\Omega X;k) = \infty. \tag{11}
\]

Now consider the Moore path space fibration $\Omega X \rightarrow PX \rightarrow X$ from [3, exam. 1, p. 29]. The Moore path space $PX$ is contractible, and this implies that the DG left-$C^*(X;k)$-module $C^*(PX;k)$ is quasi-isomorphic to $C^*(X;k)$. Inserting the fibration into the Eilenberg-Moore theorem [3, thm. 7.5] therefore gives a quasi-isomorphism $C^*(\Omega X;k) \simeq
and taking cohomology shows
\[ H^*(\Omega X; k) \cong \text{Tor}_{-*}^{C^*(X; k)}(k, k). \]
Combining with equation (11) gives
\[ \dim_k \text{Tor}^{C^*(X; k)}(k, k) = \infty. \]
So \( C^*(X; k) \) cannot be finitely built from \( C^*(X; k) \) whence \( C^*(X; k) \) is not in \( \text{D}^c(C^*(X; k)) \) by [5, lem. 3.2].

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