Abelian Group Clifford Algebras

Tim Neijens
University of Antwerp
tim.neijens@gmail.com
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Abstract
In [1] generalized Clifford algebras were introduced via Clifford representations; these correspond to projective representations of a finite group (Abelian), $G$ say, such that the corresponding twisted group ring has minimal center. The latter then translates to the fact that the corresponding 2-cocycle allows a minimal (none!) number of ray classes and this forces a decomposition of $G$ in cyclic components in a suitable way, cf. [10]. In this small paper, I will provide a way to represent an Abelian Group Clifford Algebra using a matrix, and then give a way to calculate whether or not the center is trivial.

1 Preliminaries and Notation

Definition 1.1 A projective representation of a finite group $G$ is a group morphism $T : G \rightarrow \text{PGL}_n(k)$ with $k$ some field or a commutative connected ring ($0$ and $1$ are the only idempotents).

Such projective representations define algebra homomorphisms (by $k$-linearly extending $T$) $kG^\alpha \rightarrow M_n(k)$ where $kG^\alpha$ is the twisted group ring over $k$ with respect to some 2-cocycle. It is defined as

$$kG^\alpha = \bigoplus_{\sigma \in G} ku_\sigma,$$

and multiplication is defined by $u_\sigma u_\tau = \alpha(\sigma, \tau)u_{\sigma\tau}$ for $\sigma, \tau \in G$.

Proposition 1.2 (see [1]) Let $R$ be a commutative connected (only idempotents are 0 and 1), $G$ a (finite) group and $\alpha$ a 2-cocycle, then the center
1 PRELIMINARIES AND NOTATION

$Z(RG^a)$ is freely generated over $R$ by the ray class sums. If $G$ is Abelian then $Z(RG^a) = R(G_{reg})^a$.

Let $G$ be a finite Abelian group. We can decompose this group as follows:

$$G \cong G_{p_1} \oplus \ldots \oplus G_{p_t},$$

where

$$G_{p_v} \cong \left( \frac{\mathbb{Z}}{p_v^{n_v}} \right)^{m_1} \oplus \ldots \oplus \left( \frac{\mathbb{Z}}{p_v^{n_v}} \right)^{m_{k_v}},$$

with $n_1 > n_2 > \ldots > n_{k_v}$.

Suppose for now that $t = 1$. (It will become clear that there is no loss of generality by doing this.) $G_p$ is generated by

$$e_1(n_1), \ldots, e_{m_1}(n_1)$$

$$\vdots$$

$$e_1(n_k), \ldots, e_{m_k}(n_k),$$

and denote $e_i(n_j)$ by $e_{ij}$, i.e. $e_{ij}$ has exact order $p^{n_j}$.

Now let $R$ be a commutative ring which contains a primitive $(p^{n_1})$-th root of unity and no idempotents other than 0 and 1.

Consider a 2-cocycle $\phi : G \times G \to U(R) \in Z^2(G, U(R))$. We can associate the map

$$f_\phi : G \times G \to U(R) : (A, B) \mapsto \frac{\phi(A, B)}{\phi(B, A)}.$$  

Since $G$ is Abelian, $f_\phi$ is a multiplicatively antisymmetric bipairing. This implies that $f_\phi$ is completely determined by the images of $e_{ij}$. Moreover, since the $e_{ij}$ have finite order, so do the images. In other words:

$$f_\phi(e_{ij}, e_{rs}) = \omega^{ex},$$

where $\omega$ is a $[p_{n_1}, p_{n_2}]$-th root of unity. ($[a, b] = \gcd(a, b)$). Remember that we assumed only one prime $p$. For a projective representation $T$, $T(e_{ij})T(e_{rs}) = f(e_{ij}, e_{rs})T(e_{rs})T(e_{ij})$. This means that generators corresponding to different primes commute and therefore we get a tensor of all those components, each component corresponding to a different prime. In other words, if $A_G$ is the Abelian group Clifford algebra corresponding to a group $G$, then we find for $G \cong G_{p_1} \oplus \ldots \oplus G_{p_t}$:

$$A_G \cong A_{G_{p_1}} \otimes \ldots \otimes A_{G_{p_t}}.$$  

This means that, like said in the beginning of the paragraph, we only need to study the case that $G \cong G_p$. 
2 The Matrix Method

Given a certain 2-cocycle $\phi \in Z^2(G, U(R))$ we construct a matrix $A_\phi$ with entries $x_{ijrs}$ determined by

$$f(e_{ij}, e_{rs}) = \omega_{[p^{n_j}, p^{n_s}]}^{x_{ijrs}}.$$

These entries are in the additively defined cyclic groups. So we find:

$$A_\phi = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix},$$

where

$$A_{ij} \in M_{m_i \times m_j} \left( \frac{Z}{p^{n_j}Z} \right).$$

By construction, $A_\phi$ is antisymmetric.

If we want to minimize the center, we need to minimize the number of elements that commute with the generators of our projective representation $A$. (We will from now on identify $t_{ij}$ with $T(e_{ij})$ for a projective representation $T$). We will denote $f_\phi$ as $f$ if we fix $\phi$. Let

$$b = t_{11}^{\alpha_{11}} \cdots t_{1m_1}^{\alpha_{1m_1}} \cdot t_{21}^{\alpha_{21}} \cdots t_{km_k}^{\alpha_{km_k}} \in Z(A).$$

Then we find $\forall i, j$

$$t_{ij} \cdot b \cdot t_{ij}^{-1} = b \quad (1)$$

$$\Rightarrow f^{\alpha_{11}}(e_{ij}, e_{11}) \cdots f^{\alpha_{km_k}}(e_{ij}, e_{km_k}) \cdot b = b \quad (2)$$

$$\Rightarrow \omega^{\alpha_{11}x_{ij11}}_{[p^{n_j}, p^{n_1}]} \cdots \omega^{\alpha_{km_k}x_{ijkkm_k}}_{[p^{n_j}, p^{n_1}]} \cdot b = b. \quad (3)$$

We want to minimize the number of solutions $(\alpha_{11}, \ldots, \alpha_{km_k})$ to this equation. We can express this minimality condition by using the matrix $A_\phi$ defined before. We use this matrix to calculate the exponents of the $\omega$ appearing in (3). If we want the left factor in (3) to be 1, we have to calculate the product of the $\omega$ appearing in (3). We do this by using a primitive $(p^{n_1})$-th root of unity, let us call it $\zeta$. We then get as the exponent of $\zeta$:

$$p^{n_1-n_j} \cdot \alpha_{11}x_{ij11} + \ldots + p^{n_1-\min(n_j, n_k)} \alpha_{km_k}x_{ijkkm_k}. \quad (4)$$
ζ to that power must be equal to 1 and that only happens if the exponent is 0 mod $p^{n_i}$. So we construct from $A_\phi$ a new matrix $\tilde{A}_\phi$ by multiplying each block $A_{ij}$ with $p^{n_i - \min(n, m_i)}$. The end result is that

$$\tilde{A}_\phi \in M_{m_1 + \ldots + m_k} \left( \frac{\mathbb{Z}}{p^{n_1} \mathbb{Z}} \right).$$

Since the condition is now focused on the exponents, we are working with the additive groups and the standard ring structure on $\mathbb{Z}/n\mathbb{Z}$. We will exploit this by using the matrix product with our newly formed matrix $\tilde{A}_\phi$ to find solutions or the lack thereof.

Let $g = (g_1, \ldots, g_k) \in G$ where $g_i$ is an $m_i$-dimensional vector with entries chosen in representatives in $\mathbb{Z}/p^{n_i}\mathbb{Z}$ of elements in $\mathbb{Z}/p^{n_i}\mathbb{Z}$. So each entry in $g$ modulo $p^{n_i}$ determines an element in the appropriate $\mathbb{Z}/p^{n_i}\mathbb{Z}$. Now we can calculate the matrix product $\tilde{A}_\phi \cdot g$. If this product is equal to the 0-vector, then we have found a solution $g$ to (4). In order to minimize the number of solutions of (4), we have to maximize the 'rank' of $\tilde{A}_\phi$.

### 2.0.1 Example

To make things clear, an actual example of how to solve the equations presented above. We first put the matrix in standard form, meaning that we construct $\tilde{A}_\phi$ from $A_\phi$. The solutions $\tilde{X}$ to $\tilde{A}_\phi \tilde{X} = 0$ are then reduced component per component with the corresponding prime power. We can find those solutions using simple invertible row operations. Let $G \cong (\mathbb{Z}_9)^2 \oplus (\mathbb{Z}_3)^2$:

$$A_\phi = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 8 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{pmatrix} \rightarrow \tilde{A}_\phi = \begin{pmatrix} 0 & 1 & 3 & 3 \\ 8 & 0 & 6 & 6 \\ 6 & 3 & 0 & 3 \\ 6 & 3 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and thus

$$\begin{cases} \tilde{x}_1 = 0 \\ \tilde{x}_2 = 0 \\ \tilde{x}_3 \in \{0, 3, 6\} \Rightarrow x_3 = 0 \mod 3 \\ \tilde{x}_4 \in \{0, 3, 6\} \Rightarrow x_4 = 0 \mod 3 \end{cases}$$

### 3 Diagonal Block Theorem

In this section we prove a theorem providing the necessary conditions on $A_\phi$ and so on the commutation relation of a projective representation $T$ to form an Abelian group Clifford algebra.
Theorem 3.1 \( R(G)^\phi \) has trivial center (i.e. \( Z(A) = R \)) if and only if each block \( A_{ii} \) in \( A_\phi \) is invertible.

Proof Since every matrix on the diagonal is invertible in its respective ring, we can use row operations to put those blocks into diagonal form. We can 'create zeroes' using the entries in the row diagonalized blocks. Fix such a diagonal block and as such a ring \( \mathbb{Z}/p^s\mathbb{Z} \). The entries in the blocks are invertible if we consider them in their 'original' ring (before we go from \( A_\phi \) to \( \tilde{A}_\phi \)) and the entries above or below these can be considered as in \( \mathbb{Z}/p^s\mathbb{Z} \) by construction of the matrix \( A_\phi \). We can use row operations to eliminate the corresponding variables. Repeating this we get a diagonal matrix. This matrix has only the 0-vector as solution, modulo the corresponding primes. In order to establish the other direction, suppose that one or more matrices are not invertible in their respective rings. Take the first such matrix, say \( A_{ii} \). Then consider \( \tilde{A}_\phi \) and do the row operations in \( A_{ii} \) that express the linear dependence of the entries. Let us say row \( s \) is the first row in \( \tilde{A}_\phi \) that is linearly dependent of the other rows in \( A_{ii} \). Up to row \( s \) we can put the matrix in diagonal form. This means that either \( x_s \) will be a fully random variable (the case when the rows below row \( s \) have zeroes on column \( s \)), or we get an entry from a row from an \( A_{jj} \) where \( j > i \). This entry is not invertible in the ring corresponding to \( A_{ii} \) and therefore will not yield one unique solution in the corresponding ring. So \( A_{ii} \) must be invertible. \( \square \)

References

[1] Caenenpeel, S.; Van Oystaeyen, F., A note on generalized Clifford algebras and representations, Comm. Algebra 17 (1989) no. 1, 93–102.

[2] Feit, W., The representation theory of finite groups, Dekker (1985), New York.

[3] Jagannathan, R., On projective representations of finite Abelian groups, Number theory (Ootacamund, 1984), Lecture Notes in Math. vol. 1122, Springer (1985), Berlin, 130–139.

[4] Morris, A.O., On a generalized Clifford algebra, Quart. J. Math. Oxford 18 (1967), 7–12.

[5] Morris, A.O., On a generalized Clifford algebra II, Quart. J. Math. Oxford 19 (1968), 289–299.

[6] Morris, A.O., Projective representations of finite groups, Proceedings of the Conference on Clifford Algebra, its Generalization and Application
[7] Morris, A.O., *Projective representations of Abelian groups*, J. London Math. Soc. (2) 7 (1973), 235–238.

[8] Neijens, T.; Van Oystaeyen, F., *Centers of certain crystalline graded rings*, Algebras and Representation Theory, to appear.

[9] Van Oystaeyen, F., *Azumaya strongly graded rings and ray classes*, J. of Algebra 103 (1986), no. 1, 228–240.

[10] Zmud, M., *Symplectic geometries and projective representations of finite Abelian groups*, (Russian) Mat. Sb. (N.S.) 87(129) (1972), 3–17.