Cubic Vertices for Symmetric Higher-Spin Gauge Fields in \((A)dS_d\)

M.A. Vasiliev

I.E. Tamm Department of Theoretical Physics, Lebedev Physical Institute of RAS, Leninsky prospect 53, 119991, Moscow, Russia

Abstract

Cubic vertices for symmetric higher-spin gauge fields of integer spins in \((A)dS_d\) are analyzed. \((A)dS_d\) generalization of the previously known action in AdS4, that describes cubic interactions of symmetric massless fields of all integer spins \(s \geq 2\), is found. A new cohomological formalism for the analysis of vertices of higher-spin fields of any symmetry and/or order of nonlinearity is proposed within the frame-like approach. Using examples of spins two and three it is demonstrated how nontrivial vertices in \((A)dS_d\), including Einstein cubic vertex, can result from the AdS deformation of trivial Minkowski vertices. A set of higher-derivative cubic vertices for any three bosonic fields of spins \(s \geq 2\) is proposed, which is conjectured to describe all vertices in \(AdS_d\) that can be constructed in terms of connection one-forms and curvature two-forms of symmetric higher-spin fields. A problem of reconstruction of a full nonlinear action starting from known unfolded equations is discussed. It is shown that the normalization of free higher-spin gauge fields compatible with the flat limit relates the noncommutativity parameter \(\hbar\) of the higher-spin algebra to the \((A)dS\) radius.
# Contents

1 Introduction 4

2 $(A)dS_d$ covariant formalism 6
   2.1 $(A)dS_d$ gravity with compensator 6
   2.2 Free symmetric higher-spin gauge fields 9
      2.2.1 Fields and action 9
      2.2.2 First On-Shell Theorem 10
      2.2.3 Central On-Shell Theorem 12

3 Cubic interactions 13
   3.1 General setup 13
   3.2 Abelian vertices 16
   3.3 Current vertices 16
   3.4 Non-Abelian vertices 17
   3.5 Chern-Simons vertices 18
   3.6 Minkowski versus $AdS$ 19

4 Vertex tri-complex 20
   4.1 Vertex differentials 21
   4.2 Vertex cohomology 23
   4.3 $AdS$ deformation and vertex sets 26
      4.3.1 Off-shell case 28
      4.3.2 On-shell case 29

5 Examples 30
   5.1 Quadratic action 31
   5.2 Spin two cubic vertices 31
   5.3 Spin three cubic vertices 33
      5.3.1 Vertex with three derivatives 33
      5.3.2 Vertex with five derivatives 35

6 Non-Abelian vertices in $AdS_d$ from HS algebra 36
   6.1 Higher-spin algebra in $AdS_d$ 36
   6.2 The case of $AdS_4$ 40
   6.3 Cubic action in $AdS_d$ 44
   6.4 Properties of the cubic action 45

7 Vertex generating functions 47
   7.1 Primitive vertices 47
      7.1.1 $sp(2)$ invariance 48
      7.1.2 Graph interpretation 49
      7.1.3 Cubic vertices 51
1 Introduction

During several decades, significant progress in understanding the structure of higher-spin (HS) gauge theories has been achieved. In the papers [1, 2, 3, 4, 5] it was shown that some consistent cubic vertices, that involve HS gauge fields, do exist in Minkowski background. Consistent interactions of massless fields of all spins $s > 1$ in $AdS_4$ were constructed in [6], where it was shown in particular how the Aragone-Deser argument [7] against compatibility of diffeomorphisms with HS gauge symmetries is avoided in presence of nonzero cosmological constant.

Peculiarities of HS interactions originate from the fact that they contain higher derivatives of orders increasing with spin [1, 2, 6]. Hence, the respective coupling constant should be dimensionful. For massless fields with no mass scale parameter, higher derivatives can only appear in the dimensionless combination $\rho \partial$, where $\rho = \lambda^{-1}$ is the radius of background space-time. Clearly, the higher derivative terms contain negative powers of $\lambda$ and diverge in the flat limit $\lambda \to 0$ [6].

There are three main approaches to the study of HS vertices.

Historically first was the light-cone approach of [1] further developed in application to HS fields in [9, 10, 8]. In particular, an important restriction on the number of derivatives $N$ in consistent cubic HS interaction vertices for symmetric fields of any three spins $s_1, s_2, s_3$ in Minkowski space of dimension $d > 4$ was obtained by Metsaev in [8]:

$$N_{\text{min}} \leq N \leq N_{\text{max}}, \quad N_{\text{min}} = s_1 + s_2 + s_3 - 2s_{\text{min}}, \quad N_{\text{max}} = s_1 + s_2 + s_3,$$

where $s_{\text{min}} = \min(s_1, s_2, s_3)$ and $N_{\text{max}} - N = 2k$, $k \in \mathbb{N}$.

The covariant metric-like approach to the study of HS interactions used in [2] operates with the set of fields introduced by Fronsdal [11] and was further developed in a number of papers [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In particular, in the recent papers [23, 24, 25] the covariant HS vertices, associated with the list of Metsaev, were found in Minkowski space of any dimension. (Let us also mention an interesting construction of massless HS amplitudes from higher-picture sectors of string theory proposed in [26].)

However, using the minimal set of covariant fields and hence missing important geometric ingredients of the theory, the covariant metric-like approach quickly gets involved in the $(A)dS$ background. (See, however, [27].)

The covariant frame-like approach, which at the free field level was originally developed for the case of symmetric fields in [28, 29, 30, 31], reproduces the metric-like approach as a particular gauge. Its application to the study of HS interactions generalizes that of [6] and was further developed for the case of symmetric fields in [31, 32, 33] as well as in this paper where, in particular, we extend the results of [6] to any $d$. Moreover, very recently, it was also successfully applied to certain examples of mixed-symmetry fields in [34, 35, 36, 37, 38].

The frame-like approach is geometric, operating with HS gauge connections associated with HS symmetry which plays as fundamental role in the HS theory as SUSY in supergravity. Since the frame-like formalism uncovers the structure of HS symmetry it has great potential for the construction of HS interactions. Its natural generalization via unfolded dynamics
led to the full nonlinear field equations for HS fields in $AdS_4$ \cite{10} and $AdS_d$ \cite{11}. As argued in \cite{12} and in this paper, the unfolded dynamics approach is appropriate both for the construction of field equations and actions. Nevertheless, the complete nonlinear action for massless fields remains unknown. (Interesting recent proposals of \cite{43, 44, 45} generalizing an old comment of \cite{39} unlikely fully resolve this problem as long as it is not clear how they reproduce standard actions for fields of lower and higher spins at the free field level.)

In this paper we study cubic interactions of symmetric massless HS fields using the $(A)dS_d$ covariant frame-like approach developed in \cite{31}. Its extension to generic gauge fields in $(A)dS_d$ was worked out in \cite{46, 47, 48, 49, 50}. Essentials of this formalism are recalled in Section 2.

Different types of gauge invariant HS interactions are discussed in Section 3. These include Abelian interactions that are formulated in terms of gauge invariant field strengths (Section 3.2), current interactions that have the form of Noether interactions of HS gauge connections with gauge invariant conserved HS currents (Section 3.3), non-Abelian interactions with at least two HS connections entering directly rather than through the gauge invariant HS field strengths (Section 3.4) and Chern-Simons interactions that contain three connections (Section 3.5). Peculiarities of the deformation of Minkowski vertices to $AdS_d$ are discussed in Section 3.6 where it is shown in particular that although nontrivial Chern-Simons vertices can exist in Minkowski space, in all cases, except for genuine Chern-Simons vertices, their $(A)dS_d$ deformations are equivalent to some non-Abelian vertices.

In Section 4, we develop a vertex tri-complex cohomological formalism which controls nontrivial vertices (i.e., those that are not total derivatives and cannot be removed by a local field redefinition) both in Minkowski and $AdS_d$ backgrounds in the frame-like approach and is applicable to HS gauge fields of any symmetry type as well as to higher-order interactions. In particular, the appropriate cohomology controls which Minkowski vertices admit a deformation to $AdS_d$. An interesting option widely used throughout this paper is that a trivial Minkowski vertex may deform to a nontrivial lower derivative vertex in $AdS_d$.

As an illustration, the vertex complex formalism is applied in Section 5.1 to the derivation of the free HS action and, respectively, in Sections 5.2 and 5.3 to the analysis of cubic spin two and spin three vertices in $AdS_d$. From these examples we learn important peculiarities of general HS interactions studied in the subsequent sections. In particular, we show how the cubic Einstein vertex can be interpreted as $AdS$ deformation of a trivial Minkowski vertex. Also, it is shown that both the Berends, Burgers, van Dam vertex with three derivatives \cite{3} and the Bekaert, Boulaoner, Cnockaert vertex with five derivatives \cite{14} admit deformations to $AdS_d$ which can also be interpreted as $AdS$ deformations of trivial Minkowski vertices.

In Section 6 the action, which extends the $AdS_4$ results of \cite{6} to symmetric HS gauge fields in $AdS_d$ with any $d \geq 4$, is presented. The construction is based on the properties of the HS symmetry algebra recalled in Section 6.1. In Section 6.2, we recall the construction of \cite{6} for the HS vertices in $AdS_4$. The cubic action in $AdS_d$ is presented in Section 6.3 where we also derive the relation between the noncommutativity parameter of the HS algebra and the cosmological constant. Properties of the constructed action are discussed in Section 6.4.

In Section 7 we develop the formalism of generating functions that greatly simplifies
analysis of cubic interactions in Section 8, where we consider both non-Abelian vertices associated with constituents of the cubic action of Section 6 and Abelian vertices. In particular, we show in this section that non-Abelian vertices differ from certain Abelian vertices by lower-derivative vertices. It is also shown here that, up to lower-derivative vertices, a vertex, that contains $s_1 + s_2 + s_3 - 2$ derivatives, is associated with certain quotient of the space of vertices in the formalism of Section 7.

Results of Section 8 provide systematics of a class of cubic vertices in $AdS_d$, called strictly positive. In particular, vertices of this class can only involve spins $s_1, s_2, s_3$ that obey the triangle inequality

$$s_1 + s_2 + s_3 - 2s_{\text{max}} > 0.$$ (1.2)

These form a subclass of independent vertices in Minkowski space (1.1). As argued in Section 9, those vertices of the list (1.1), that are not in the strictly positive class, should involve explicitly the generalized Weyl zero-form and its derivatives. Extension of the formalism to general vertices which can depend directly on Weyl zero-forms is briefly discussed.

In Section 10, the problem of reconstruction of a full nonlinear action starting from known unfolded equations is briefly discussed. In particular, a bi-complex is introduced, that generalizes the vertex tri-complex designed for the analysis of cubic vertices to abstract dynamical systems with known nonlinear unfolded equations.

Section 11 contains conclusions and discussion.

In Appendices A and B we collect, respectively, some technicalities of the analysis of spin-three vertices with three derivatives and on-shell relations relevant to the analysis of spin three vertices with five derivatives.

Since the paper happened to be fairly long, let us mention that Section 6 is relatively independent and can be skipped by a reader not interested in the respective issues. On the other hand, one can read Section 6 just after Section 3.1.

2 $(A)dS_d$ covariant formalism

2.1 $(A)dS_d$ gravity with compensator

It is well known that gravity can be formulated in terms of the gauge fields associated with one or another space-time symmetry algebra [51, 52, 53, 54, 55, 56]. Gravity with nonzero cosmological term in any space-time dimension can be described in terms of the one-form gauge fields $w^{AB} = -w^{BA} = dx^a w^{AB}_a$ associated either with the $AdS_d$ algebra $h = O(d-1, 2)$ or $dS_d$ algebra $h = O(d, 1)$ ($\mu, \nu = 0, \ldots, d - 1$ are indices of differential forms on the $d$-dimensional base manifold; $A, B, \ldots = 0, \ldots, d$ are tangent vector indices of $O(d-1, 2)$ or $O(d, 1)$). Both of these algebras have basis elements $t_{AB} = -t_{BA}$. Let $r^{AB}$ be the field strength of $O(d-1, 2)$ or $O(d, 1)$

$$r^{AB} = dw^{AB} + w^{AC} \wedge w_C^B,$$ (2.1)

where indices are contracted by the invariant metric $\eta_{AB}$ of either $O(d-1, 2)$ or $O(d, 1)$. For definiteness, in the rest of this paper we focus on the $AdS_d$ case of $O(d-1, 2)$. The $dS_d$ case
of $o(d,1)$ is considered analogously.

One can use the decomposition
\[ w = w^{AB} t_{AB} = \omega^{L}{}_{ab} L_{ab} + \lambda e^{a} P_{a} \]  
(2.2)
\((a,b = 0, \ldots, d-1)\). Here $\omega^{L}{}_{ab}$ is the Lorentz connection associated with a Lorentz subalgebra $o(d-1,1) \subset o(d-1,2)$. The frame one-form $e^{a}$ is associated with the $AdS_{d}$ translations $P_{a}$ parametrizing $o(d-1,2)/o(d-1,1)$. Provided that $e^{a}$ is nondegenerate, the zero-curvature condition
\[ r^{AB}(w) = 0 \]  
(2.3)
implies that $\omega^{L}{}_{ab}$ and $e^{a}$ identify with the gravitational fields of $AdS_{d}$. $\lambda^{-1}$ is the $AdS_{d}$ radius. (Note that the factor of $\lambda$ in (2.2) is introduced to make $e^{a}$ dimensionless.)

It is useful to covariantize these definitions with the help of the compensator field \[ V^{A}(x) \] being a time-like $o(d-1,2)$ vector $V^{A}$ normalized to
\[ V^{A}V_{A} = \lambda^{-2}. \]  
(2.4)
Identification of the length of $V^{A}$ with the $AdS_{d}$ radius is convenient for the analysis of the $\lambda$-dependence of HS interactions and differs from that of [31], where $V^{A}$ had unit length. In the de Sitter case, $\lambda^{-2}$ in (2.4) has to be replaced by $-\lambda^{-2}$.

Sometimes it may be useful to replace the compensator restricted by the condition (2.4) by a field $W^{A}(x)$
\[ V^{A}(x) = \frac{W^{A}(x)}{\lambda \sqrt{W^{B}(x)W_{B}(x)}} \]  
(2.5)
with $W^{A}(x)W_{A}(x) > 0$ but otherwise arbitrary. With this identification, the space of functions of $V^{A}$ turns out to be equivalent to the space of homogeneous functions $F(W)$ that satisfy
\[ W^{A} \frac{\partial}{\partial W^{A}} F(W) = 0 \]  
(2.6)
and are allowed to be nonpolynomial only with respect to $W^{A}(x)W_{A}(x)$.

Lorentz algebra is the stability subalgebra of $V^{A}$. This allows for the covariant definition of the frame field and Lorentz connection \[ E^{A} \] and \[ \omega^{L}{}_{AB} \] :
\[ E^{A} := DV^{A} \equiv dV^{A} + w^{AB} V_{B}, \]  
(2.7)
\[ \omega^{L}{}_{AB} := w^{AB} - \lambda^{2}(E^{A}V^{B} - E^{B}V^{A}). \]  
(2.8)
According to these definitions
\[ E^{A}V_{A} = 0, \quad D^{L}V^{A} := dV^{A} + \omega^{L}{}_{AB} V_{B} \equiv 0. \]  
(2.9)
When the matrix $E_{A}^{B}$, that is orthogonal to $V_{A}$, has the maximal rank $d$, it can be identified with the frame field giving rise to the nondegenerate space-time metric tensor
\[ g_{\mu\nu} = E_{A}^{B} E_{\mu}^{B} \eta_{AB} \] in $d$ dimensions. The torsion two-form is
\[ t^{A} := DE^{A} \equiv r^{AB} V_{B}. \]  
(2.10)
The zero-torsion condition

\[ t^A = 0 \]  \hspace{1cm} (2.11)

expresses the Lorentz connection via (derivatives of) the frame field in a usual manner.

With the help of \( V_A \) it is straightforward to build a \( d \)-dimensional generalization \[31\] of the 4d MacDowell-Mansouri-Stelle-West action for gravity \[54, 55, 56\]

\[
S = \left( -1 \right)^{d+1} \frac{4}{k^2 \lambda^2} \int_{M^d} G_{A_1 A_2 A_3 A_4} \wedge r^{A_1 A_2} \wedge r^{A_3 A_4} ,
\]  \hspace{1cm} (2.12)

where we use notation

\[
G^{A_1 \ldots A_q} = \epsilon^{A_0 \ldots A_d} V_{A_0} E_{A_{q+1}} \wedge \ldots \wedge E_{A_d} .
\]  \hspace{1cm} (2.13)

The \((d - q)\)-form \( G^{A_1 \ldots A_q} \) has a number of useful properties. From the identity

\[
\epsilon^{A_1 \ldots A_{d+1}} = \lambda^2 (V^{A_1} V_B \epsilon^{B A_2 \ldots A_{d+1}} + \ldots + V^{A_{d+1}} V_B \epsilon^{A_1 \ldots A_d B})
\]  \hspace{1cm} (2.14)

along with the definitions of \( E^A \) \[2.7\] and torsion \[2.11\] it follows that

\[
D(G^{A_1 \ldots A_q}) \simeq (-1)^q q \lambda^2 V^{[A_q} G^{A_2 \ldots A_{q-1}]} ,
\]  \hspace{1cm} (2.15)

where \( [\ ] \) denotes antisymmetrization with the projector normalization \( [[\ ]] = [\ ] \) and \( \simeq \) implies equality up to terms that are zero by virtue of the zero-torsion condition \[2.11\]. Another relevant formula is

\[
G^{A_1 \ldots A_q} \wedge E^C = \frac{q}{d+1 - q} G^{[A_1 \ldots A_{q-1}] (\eta^{A_q]C} - \lambda^2 V^{A_q]} V^C)} .
\]  \hspace{1cm} (2.16)

Taking into account that

\[
\delta r^{AB} = D \delta w^{AB} , \quad \delta E^A = \delta w^{AB} V_B + D \delta V^A , \quad V_A \delta V^A = 0 ,
\]  \hspace{1cm} (2.17)

with the help of these relations we obtain

\[
\delta S = \frac{1}{k^2 \lambda^2} \int_{M^d} \left( G_{A_1 A_2 A_3} - \frac{(d - 4)}{4 \lambda^2} G_{A_1 A_2 A_3 A_4 A_5} \wedge r^{A_4 A_5} \right) \wedge r^{A_1 A_2} \wedge \delta w^{B A_3} V_B + \delta_1 S ,
\]  \hspace{1cm} (2.18)

where \( \delta_1 S \) is the part of the variation proportional to torsion. Using so-called 1.5 order formalism, we will assume that the zero-torsion constraint is imposed to express the Lorentz connection via derivatives of the frame field, hence neglecting \( \delta_1 S \).

The second term in \[2.18\], which is nonzero for \( d > 4 \), results from the variation of the factors of \( E^A \) in \( G_{A_1 A_2 A_3 A_4} \) and hence contributes to the nonlinear corrections of the field equations. The generalized Einstein equations resulting from \[2.18\] are \[31\]

\[
\left( G_{A_1 A_2 A_3} - \frac{(d - 4)}{4 \lambda^2} G_{A_1 A_2 A_3 A_4 A_5} \wedge r^{A_4 A_5} \right) \wedge r^{A_1 A_2} = 0 .
\]  \hspace{1cm} (2.19)

The first term is the left-hand-side of the Einstein equations with the cosmological term. The second term describes interaction terms bilinear in the curvature \( r^{AB} \), that do not contribute to the linearized equations. In the 4d case they are absent because the corresponding part of the action is topological having the Gauss-Bonnet form. Note that the additional interaction terms contain higher derivatives together with the factor of \( \lambda^{-2} \) that diverges in the flat limit \( \lambda \to 0 \). In HS theory, such terms play important role to preserve HS gauge symmetries.
2.2 Free symmetric higher-spin gauge fields

2.2.1 Fields and action

Analogously to the case of gravity with the connection $w^{AB}$ as dynamical field, a spin $s \geq 2$ massless field can be described [31] by a one-form $dx \omega^{a_1...a_{s-1},b_1...b_t}$ carrying the irreducible representation of the $AdS_d$ algebra $o(d-1,2)$ described by the traceless two-row rectangular Young diagram of length $s-1$, i.e.,

$$\omega^{(A_1...A_{s-1},A_s)B_2...B_{s-1}} = 0, \quad \omega^{A_1...A_{s-3}C,B_1...B_{s-1}} = 0. \quad (2.20)$$

That $\omega^{A_1...A_{s-1},B_1...B_{s-1}}$ is described by a Young diagram of length $s-1$ rather than $s$ is because it carries a one-form index, cf. the case of spin one.

The linearized HS curvature $R_1$ has simple form

$$R_1^{A_1...A_{s-1},B_1...B_{s-1}} := D_0(\omega^{A_1...A_{s-1},B_1...B_{s-1}})$

$$+(s-1)\left(\omega^{A_1,C} \omega_{1C}^{A_2...A_{s-1},B_1...B_{s-1}} + \omega_{1C}^{B_1} \omega^{A_1...A_{s-1},C} \omega^{B_2...B_{s-1}}\right), \quad (2.21)$$

where $w_0$ is the $AdS_d$ background gauge field satisfying the zero curvature condition [28] equivalent to

$$D_0^2 = 0. \quad (2.22)$$

By its definition, $R_1$ satisfies the Bianchi identities

$$D_0 R_1 = 0. \quad (2.23)$$

In the sequel we will skip the label 0 of the background field $w_0^{AB}$ and vielbein $E^a_b$.

In these terms, Lorentz covariant irreducible fields $dx \omega^{a_1...a_{s-1},b_1...b_t}$ used in [30] identify with those components of $dx \omega^{A_1...A_{s-1},B_1...B_{s-1}}$ that are parallel to $V^A$ in some $s-t-1$ indices and transversal in the remaining $t$ indices. The dynamical frame-like and auxiliary Lorentz-like fields are, respectively, those with $t = 0$ and $t = 1$

$$e^{A_1...A_{s-1}} = \omega^{A_1...A_{s-1},B_1...B_{s-1}} V_{B_1} \cdots V_{B_{s-1}}, \quad \omega^{A_1...A_{s-1},C} = \Pi_V \omega^{A_1...A_{s-1},C} V_{B_1} \cdots V_{B_{s-1}}, \quad (2.24)$$

where $\Pi_V$ is the projector to the $V^A$-transversal part of a tensor (note that $e^{A_1...A_{s-1}}$ is automatically $V^A$-transversal because of the Young properties (2.20)). More generally,

$$\omega_{A_1...A_{s-1},C_1...C_t} = \Pi_V \omega^{A_1...A_{s-1},C_1...C_t} V_{B_{t+1}} \cdots V_{B_{s-1}}. \quad (2.25)$$

The HS gauge fields $\omega^{A_1...A_{s-1},C_1...C_t}$, which for $t > 1$ are called extra fields, are expressed via up to order $t$ derivatives of the frame-like field $e^{A_1...A_{s-1}}$ by generalized zero-torsion constraints [30,31] that constitute a part of the First On-Shell Theorem discussed below. Equivalently, we can say that a HS connection $\omega^{A_1...A_{s-1},B_1...B_{s-1}}$ contains up to $s-1$ derivatives of $e^{A_1...A_{s-1}}$, while contraction of any its index with the compensator removes one derivative.

The normalizations of the vielbein in (2.2) and of the compensator $V^A$ in (2.4) are adjusted in such a way that, upon resolving the generalized zero-torsion constraints, the
expressions of extra fields in terms of derivatives of the frame-like field contain only non-negative powers of \( \lambda \), having a \( \lambda \)-independent coefficient in front of the leading derivative term. Indeed, contracting the expression for the curvature (2.21) with \( s - p - 1 \) compensators and using (2.8) we obtain

\[
\Pi_y V_{B_{p+1}} \cdots V_{B_{s-1}} R_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{s-1}} = \\
D^L \omega_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{p}} + (p + 1) E_D \omega_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{p} D} + \lambda^2 P(E_{B_{p} \omega_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{p-1}})} (2.26)
\]

where \( P \) projects the third term to the Lorentz representation carried by the l.h.s. The second and third terms on the r.h.s. of this formula contain extra fields that carry up to \( p + 1 \) and \( p - 1 \) derivatives of the dynamical field, respectively. It is important that the coefficient in the second term on the r.h.s. of (2.26) is \( \lambda \)-independent because the factor of \( \lambda^2 \) in (2.8) is compensated by the factor of \( \lambda^{-2} \) in (2.4) resulting from a contraction of two compensators. Setting to zero a set of independent components of the HS curvature, that contain the extra fields \( \omega_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{q+1}} \) (see also Section 11) for explicit form of the constraints we refer the reader to [30], we express the latter via derivatives of the fields \( \omega_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{q}} \) with \( q \leq p \) in such a way that the resulting expressions only contain non-negative powers of \( \lambda \) and the leading derivative term is \( \lambda \)-independent. Thus, for the chosen normalization of \( V^A \), resolution of the constraints is regular in \( \lambda \).

In the frame-like formalism, the free MacDowell-Mansouri–like action, that describes spin-s HS gauge fields in AdS\(_d\), is [31]

\[
S^2_s = \frac{1}{2} \int_{M^d} \sum_{p=0}^{s-2} a(s,p) V_{C_{1}} \cdots V_{C_{2(s-2-p)}} G_{A_{1} A_{2} A_{3} A_{4}} \wedge R_{A_{1} B_{1} \cdots B_{s-2}} \wedge \wedge \wedge \wedge R_{A_{1}} B_{1} \cdots B_{s-2}, A_{4} C_{s-1-p} \cdots C_{2(s-2-p)} D_{1} \cdots D_{p},
\]

(2.27)

where

\[
a(s,p) = b(s) \lambda^{-2p} \frac{(d - 5 + 2(s - p - 2))!! (s - p - 1)}{(d - 5)!! (s - p - 2)!!}
\]

(2.28)

and \( b(s) \) is an arbitrary spin-dependent normalization coefficient. The coefficients (2.28) are adjusted so that the variation of the action (2.27) over all extra fields be identically zero (see also Section 5.1). This implies that, at the linearized level, only the frame-like and Lorentz-like fields (2.24) do contribute to the action, i.e., all potentially dangerous higher-derivative terms in the action (2.27), (2.28) combine to a total derivative.

### 2.2.2 First On-Shell Theorem

The key fact of the theory of free massless fields is the First On-Shell Theorem (FOST) which states that constraints on the auxiliary and extra fields can be chosen so that [30] [31]

\[
R_{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{s-1}} = E_{A_{s} \wedge E_{B_{s}} C_{A_{1} \cdots A_{s}, B_{1} \cdots B_{s}} + X^{A_{1} \cdots A_{s-1}, B_{1} \cdots B_{s-1}} \frac{\delta S^2_s}{\delta \omega_{dyn}},}
\]

(2.29)
where the second term vanishes on the mass shell $\frac{\delta S^2}{\delta \omega_{\text{dyn}}}$ = 0 ($\omega_{\text{dyn}}$ denotes the frame-like and Lorentz-like HS fields) while the generalized Weyl tensor $C^{A_1...A_s,B_1...B_s}$ parametrizes those components of the curvatures that may remain nonzero when the field equations and constraints on extra fields are imposed. $C^{A_1...A_s,B_1...B_s}$ generalizes the Weyl tensor in gravity to any spin. (For HS fields in four-dimensional Minkowski space, generalized Weyl tensor was originally introduced by Weinberg [58] in the two-component spinor formalism discussed in Section 6.2 where Eq. (6.45) provides the 4d spinor version of Eq. (2.29)). It is described by a traceless $V^A$–transversal two-row rectangular Young diagram of length $s$, i.e.,

$$C^{(A_1...A_s,A_{s+1})B_2...B_s} = 0, \quad C^{A_1...A_{s-2}CD,B_1...B_s} \eta_{CD} = 0, \quad C^{A_1...A_{s-1}C,B_1...B_s} V_C = 0. \quad (2.30)$$

Equivalently, FOST can be written in the form

$$R_1^{A_1...A_{s-1},B_1...B_{s-1}} \sim E_{A_s} \wedge E_{B_s} C^{A_1...A_s,B_1...B_s}, \quad (2.31)$$

where $\sim$ implies equivalence up to terms that are zero by virtue of constraints and/or field equations, i.e., on-shell. The equations (2.31) can be interpreted as constraints that express all auxiliary and extra fields, as well as the Weyl tensors, via derivatives of the dynamical frame-like fields modulo terms that vanish on-shell.

As a consequence of FOST the linearized HS curvature is on-shell $V^A$-transversal with respect to any its fiber index

$$R_1^{A_1...A_{s-1},B_1...B_{s-1}} V^A \sim 0. \quad (2.32)$$

Using the definition of the frame one-form (2.7) and Bianchi identities (2.23), Eq. (2.32) implies

$$R_1^{A_1...A_{s-1}} \wedge E_A \sim 0. \quad (2.33)$$

By virtue of (2.16), from (2.33) it follows that

$$G^{[A_1...A_q} \wedge R_1^{A_{q+1}]} \sim 0, \quad (2.34)$$

where $A_{q+1}$ is any fiber index of the curvature. This relation means that, on shell, the $(d - q + 2)$–form

$$G^{A_1...A_q} \wedge R_1^{B_1...B_{s-1},C_1...C_{s-1}} \quad (2.35)$$

has the properties of the Lorentz tensor described by the Young diagram

$$\begin{array}{|c|c|c|c|}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
q & s & & \\
\hline
\end{array} \quad (2.36)$$

(Recall that on-shell it is $V$-transversal with respect to all indices $A$, $B$ and $C$.) However this tensor is not irreducible due to possible contractions between indices of $G^{A_1...A_q}$ and $R^{B_1...B_{s-1},C_1...C_{s-1}}$. 

11
Contracting two pairs of fiber indices, we introduce a dual curvature \((d - r)\)-form
\[
\tilde{R}_{[D_{1}...D_{r}],A_{1}...A_{s-2},B_{1}...B_{s-2}} = G_{D_{1}...D_{r}}^{FG} R_{F,A_{1}...A_{s-2},G,B_{1}...B_{s-2}}.
\] (2.37)

From tracelessness of the curvature with respect to fiber indices together with [2.34] it follows that \(\tilde{R}_{[D_{1}...D_{r}],A_{1}...A_{s-2},B_{1}...B_{s-2}}\) is Lorentz irreducible, being traceless, \(V^{A}\)-transversal and having the Young symmetry
\[
\begin{array}{|c|c|c|}
\hline
& & \scriptstyle{s-1} \\
\hline
& & \\
\hline
\end{array}
\quad .
\] (2.38)

This has an important consequences that \(\tilde{R} \sim 0\) for \(r < 2\), \(i.e.,\)
\[
G^{A,B,C} \wedge \tilde{R}_{A,...B,...} \sim 0.
\] (2.39)

In the case of \(r = 2\), the \((d - 2)\)-form \(\tilde{R}\) is valued in the same two-row Young diagram as \(R\), \(i.e.,\)
\[
\tilde{R}_{A_{1}...A_{s-1},B_{1}...B_{s-1}} = \tilde{R}_{[A_{1},B_{1}],A_{2}...A_{s-2},B_{2}...B_{s-2}},
\] (2.40)
where \(R_{A_{1}...A_{s-1},B_{1}...B_{s-1}}\) represents the dual curvature in the symmetric basis, having the symmetry properties \((2.20)\).

Consider a \(d\)-form \(F(R', R)\) bilinear in the curvature \(R\) and its dual \((d - 2)\)-form \(R'\)
\[
F(R', R) = R_{A_{1}...A_{s-1},B_{1}...B_{s-1}} \wedge R_{C_{1}...C_{s-1},D_{1}...D_{s-1}} A^{A_{1}...A_{s-1},B_{1}...B_{s-1};C_{1}...C_{s-1},D_{1}...D_{s-1}}
\] (2.41)
with some coefficients \(A^{\cdot,\cdot}\). An important consequence of FOST is the \textit{on-shell} symmetry relation
\[
F(R'_{1}, R_{1}) \sim F(R_{1}, R'_{1})
\] (2.42)
for any \(F(R', R)\). Indeed, substitution of \((2.29)\) into \(F(R', R)\) gives
\[
F(R'_{1}, R_{1}) \sim G C_{F,A_{1}...A_{s-1},G,B_{1}...B_{s-1}} C^{F}_{C_{1}...C_{s-1},G,D_{1}...D_{s-1}} A^{A_{1}...A_{s-1},B_{1}...B_{s-1};C_{1}...C_{s-1},D_{1}...D_{s-1}}
\] (2.43)
Obviously, the antisymmetric part of the coefficients \(A^{A_{1}...A_{s-1},B_{1}...B_{s-1};C_{1}...C_{s-1},D_{1}...D_{s-1}}\) does not contribute to \(F(R'_{1}, R_{1})\). This is equivalent to the property \((2.42)\) which underlies the proof of gauge invariance of the cubic HS action in Section \(6.3\).

### 2.2.3 Central On-Shell Theorem

Extension of FOST \((2.31)\) to the full free unfolded system for massless fields of all spins \([31]\) is provided by the Central On-Shell Theorem (COST) which supplements \((2.31)\) with the field equations on the generalized Weyl tensors and all their derivatives described by a set of zero-forms \(C^{A_{1}...A_{s},B_{1}...B_{s}}\) that consists of all two-row traceless \(V^{A}\)-transversal Young diagrams with the second row of length \(s\), \(i.e.,\)
\[
C^{(A_{1}...A_{s},A_{s+1})B_{2}...B_{s}} = 0,
\] (2.44)
\[ C^{A_1\ldots A_{u-2}C,B_{1}\ldots B_{s}} \eta_{CD} = 0, \quad C^{A_1\ldots A_{u-1}C,B_{1}\ldots B_{s}} V_C = 0. \]  

(2.45)

Here the bottom member of the set \( C^{A_1\ldots A_{u-2}C,B_{1}\ldots B_{s}} \) is the generalized Weyl tensor while \( C^{A_1\ldots A_{u},B_{1}\ldots B_{s}} \) identify with on-shell nontrivial derivatives of the latter by virtue of the equations

\[ \tilde{D} C^{A_1\ldots A_{u},B_{1}\ldots B_{s}} \sim 0, \quad u \geq s, \]  

(2.46)

where

\[ \tilde{D}_0 = D_0^L + \sigma_- + \sigma_+, \]  

(2.47)

\( D_0^L \) is the vacuum Lorentz covariant derivative and the operators \( \sigma_{\pm} \) have the form

\[ \sigma_-(C)^{A_1\ldots A_{u},B_{1}\ldots B_{s}} = (u - s + 2)E_C C^{A_1\ldots A_{u}C,B_{1}\ldots B_{s}} + sE_C C^{A_1\ldots A_{u}B_{1},B_{1}\ldots B_{s-1}C} \]  

(2.48)

\[ \sigma_+(C)^{A_1\ldots A_{u},B_{1}\ldots B_{s}} = u \lambda^2 \left( \frac{d + u + s - 4}{d + 2u - 2} E^{(d)} C^{A_2\ldots A_{u}B_{1}\ldots B_{s}} \right. \]  

\[ - \frac{s}{d + 2u - 2} \eta^{A_1B_1} E_C C^{A_2\ldots A_{u}B_{2}\ldots B_{s}} \]  

\[ - \frac{(u - 1)(d + u + s - 4)}{(d + 2u - 2)(d + 2u - 4)} \eta^{A_1A_2} E_C C^{A_3\ldots A_{u}C,B_{1}\ldots B_{s}} \]  

\[ + \frac{s(u - 1)}{(d + 2u - 2)(d + 2u - 4)} \eta^{A_1A_2} E_C C^{A_3\ldots A_{u}B_{1},CB_{2}\ldots B_{s}} \right) \]  

(2.49)

(total symmetrization within the groups of indices \( A_i \) and \( B_j \) is assumed).

Equations (2.46) on the fields \( C^{A_1\ldots A_{u},B_{1}\ldots B_{s}} \) can be interpreted as the covariant constancy condition in an appropriate infinite-dimensional \( o(d-1,2) \)–module called Weyl module. The fields \( C^{A_1\ldots A_{u},B_{1}\ldots B_{s}} \) are often referred to as zero-forms in the Weyl module or Weyl zero-forms.

COST is the system of equations (2.31) and (2.46). For spins \( s \geq 2 \), equations (2.46) are consequence of FOST (2.31) along with constraints that express an infinite set of higher tensors \( C^{A_1\ldots A_{u},B_{1}\ldots B_{s}} \) via derivatives of the generalized Weyl tensor. The equation of motion of a massless scalar is described by Eq. (2.46) with \( s = 0 \) \[59\]. Analogously, equations (2.46), (2.31) with \( s = 1 \) impose Maxwell equations on the spin one one-form \( \omega \).

COST plays the key role in many respects and, in particular, for the analysis of interactions as was originally demonstrated in \[39\] where it was proved for the 4d case. As discussed in Sections 9 and 10, it is also useful for the analysis of Lagrangian interactions.

### 3 Cubic interactions

#### 3.1 General setup

Looking for an action and gauge transformations in the form

\[ S = S^2 + S^3 + \ldots, \quad \delta \omega_\alpha = \delta_0 \omega_\alpha + \delta_1 \omega_\alpha + \ldots, \]  

(3.1)
where \( \alpha \) labels all gauge fields \( \omega_{\alpha} \) under consideration and, schematically,

\[
\delta_0 \omega_{\alpha} = D_0 \xi_{\alpha},
\]

\[
\delta_1 \omega_{\alpha} = \Delta_\alpha^\beta \omega_{\beta} \xi_{\beta},
\]

the standard observation (see, e.g., [2]) is that the gauge variation, that leaves invariant the action (3.1) up to terms cubic in fields, exists provided that \( \delta_0 S^3 \) is proportional to \( \frac{\delta S^2}{\delta \omega_{\alpha}} \), i.e., vanishes on the free equations

\[
\frac{\delta S^2}{\delta \omega_{\alpha}} = 0.
\]

Indeed, all such terms can be compensated by an appropriate deformation of the gauge transformation law \( \delta_1 \omega_{\alpha} \). Below we will not be interested in the explicit form of \( \delta_1 \omega_{\alpha} \) because we believe that it should be determined by other, more systematic and geometric means like, e.g., HS algebra and unfolded equations. To prove that such a deformation exists, it is enough to check that \( \delta_0 S^3 \sim 0 \).

A specific property of the analysis of gauge invariance at the cubic level (in fact, at the lowest level with respect to any independent coupling constant) is that it uses free field equations (3.4). Any system of free field equations decomposes into independent subsystems for irreducible fields (e.g., massless fields of different spins). Because setting to zero any set of irreducible fields is consistent with the free field equations, the cubic level analysis can be done independently for any subset of irreducible fields in the system. Hence, at the cubic level, vertices associated with different subsets of elementary fields are separately consistent. For this reason, the cubic level analysis neither determines a full set of fields necessary to introduce consistent higher-order interactions nor relates coupling constants of different cubic couplings. Both the full spectrum and truly independent coupling constants can only be determined by the analysis of higher-order interactions. We will come back to this issue in Section 3.4.

Noether current interactions are most conveniently analyzed within the frame-like formalism where all fields \( \omega_{\alpha} \) are realized as differential forms valued in some module \( V \) of the space-time symmetry algebra. For example, in the \( AdS_d \) case, the index \( \alpha \) refers to some \( o(d-1,2) \)-module.

Cubic interactions of gauge fields are usually associated with current interactions of the form

\[
S^3 = \int_{M^d} \omega_{\alpha} \wedge \Omega^{\alpha}.
\]

For a \( p \)-form gauge field \( \omega_{\alpha} \), \( \Omega^{\alpha} \) is a \( (d-p) \)-form dual to the usual conserved current. It is called conserved if it is covariantly closed on shell

\[
D_0 \Omega^{\alpha} \sim 0,
\]

where \( D_0 \) is the background covariant differential that obeys (2.22).

Assuming that the contraction of indices \( \alpha \) in Eq. (3.5) is \( o(d-1,2) \) invariant, i.e.,

\[
d(\xi_{\alpha} \wedge \Omega^{\alpha}) = D_0(\xi_{\alpha})\Omega^{\alpha} + (-1)^p \xi_{\alpha} D_0 \Omega^{\alpha},
\]

where \( \xi_{\alpha} \) is a \( p_{\xi} \)-form, (3.6) implies that the
gauge variation of the action with respect to the field $\omega_\alpha$ is weakly zero and hence can be compensated by a field-dependent deformation of the free gauge transformation law.

This consideration is not fully satisfactory, however, if $\Omega^\alpha$ itself depends on the gauge fields $\omega_\alpha$. In this case, the proper condition is that current $\Omega^\alpha$ defined via

$$\delta S^3 = \int_{M^d} \delta \omega_\alpha \wedge \Omega^\alpha(\omega) , \tag{3.7}$$

where $\delta \omega_\alpha$ is the generic variation, i.e.,

$$\Omega^\alpha(\omega) = \left. \frac{\delta S^3}{\delta \omega_\alpha} \right|_{\delta \omega_\alpha = 0} , \tag{3.8}$$

should obey the conservation condition (3.6). Clearly, to introduce gauge invariant cubic interactions in the case where the current itself depends on the gauge fields, it is not sufficient to construct a conserved current $\Omega^\alpha(\omega)$ but is also necessary to fulfill the integrability condition

$$\frac{\delta \Omega^\beta}{\delta \omega_\alpha} = (-1)^{p_\alpha p_\beta} \frac{\delta \Omega^\alpha}{\delta \omega_\beta} . \tag{3.9}$$

It is this condition which makes it difficult to introduce Noether current interactions for a system of gauge fields. This difficulty is most relevant to the situation where a current $\Omega_1(\omega_2)$, that is conserved with respect to the gauge transformation of $\omega_1$, is not gauge invariant under the gauge transformations of $\omega_2$. Indeed, in this case the naive action (3.5) may not be invariant under the gauge transformation for $\omega_2$. This is equivalent to the fact that the form $\Omega_2$ associated with $\omega_2$ via (3.8) is not conserved. On the other hand, if a conserved current $\Omega_1(\omega_2)$ is built from the gauge invariant curvatures, i.e., $\Omega_1 = \Omega_1(D_0(\omega_2))$, which is the case of Section 3.3, there is no problem with the compatibility condition because $\Omega_2$ defined via (3.8) and hence satisfying (3.9) is $D_0$ exact, thus being automatically conserved.

For example, as discussed in [60], the stress tensor of HS gauge fields is not invariant under the HS gauge transformations. Naively, this can be interpreted as a no-go result against consistent interactions of HS gauge fields with gravity. However, as was originally demonstrated in [6], as well as in [13, 19] and in this paper (see Section 6.4), the way out is to consider interactions that contain higher derivatives rather than just two derivatives normally expected from the stress tensor. In other words, the current, that satisfies the compatibility condition (3.9), contains higher derivatives. In the case of flat space it indeed has no relation to the stress tensor in accordance with the no-go statement that HS fields do not admit consistent interactions in Minkowski background [7]. In the AdS case, or in the massive case in the Stueckelberg formalism [19, 33, 61], the conventional two-derivative stress tensor gets supplemented with the higher-derivative terms, that contain negative powers of the mass scale and/or cosmological constant, to form a conserved current obeying the condition (3.9).

Let us discuss possible types of cubic vertices.
3.2 Abelian vertices

The simplest option is to consider cubic interactions of the form

\[ S^3 = \int_{M^d} V^{\alpha_1 \alpha_2 \alpha_3} \wedge R_{1\alpha_1} \wedge R_{1\alpha_2} \wedge R_{1\alpha_3}, \quad R_{1\alpha} = D_0 \omega_\alpha, \tag{3.10} \]

where the differential form \( V^{\alpha_1 \alpha_2 \alpha_3} \), which contracts indices between the three gauge fields, is built from the background frame one-form and compensator. Such interactions are manifestly gauge invariant under (3.2) as a consequence of (2.22). The respective currents \( \Omega^{\alpha_1} \) (3.7) are \( D_0 \) exact and hence are trivially conserved. Correspondingly, Abelian interactions of this type do not give rise to nontrivial conserved charges.

More generally, one can consider Abelian actions of the form

\[ S^3 = \int_{M^d} UC_{s_1} C_{s_2} C_{s_3}, \tag{3.11} \]

where \( C_{s_i} \) are Weyl zero-forms, that parametrize on-shell nontrivial components of the linearized curvatures \( R_{1\alpha} \) and their derivatives. The operator \( U \), which contracts indices between Weyl zero-forms, is composed from the background fields.

Interactions of this type do not require a deformation of the gauge transformation and hence are called Abelian. Such interactions may appear in the full nonlinear HS action. In the case of symmetric HS fields, which is of most interest in this paper, the cubic actions constructed from three Weyl tensors contain just the maximal number of derivatives \( N_{\text{max}} \) (1.1). Note that from the results of Metsaev [8] confirmed in [23, 24, 25] it follows that possible actions (3.11) of symmetric HS gauge fields, that contain derivatives of the Weyl tensors, are trivial on the mass shell.

We distinguish between two types of Abelian vertices. Vertices (3.10) will be called quasi exact because, as will be shown in Section 8.2, they describe total derivatives in the flat limit. Abelian vertices of the form (3.11), that remain nontrivial in the flat limit, cannot be represented in the form (3.10). Such nontrivial Abelian vertices in Minkowski space were considered, e.g., in [12]. In this paper we mostly focus on quasi exact vertices (3.10) to argue in Section 8.2 that they turn out to be related to the variety of lower-derivative HS vertices both in \( AdS_d \) and in Minkowski space.

3.3 Current vertices

Another class of cubic actions describes interactions between HS gauge fields and currents built from gauge invariant Weyl zero-forms

\[ S^3 = \int_{M^d} \omega_\alpha \wedge \tilde{\Omega}^\alpha(C), \tag{3.12} \]

where, for a \( p \)-form \( \omega_\alpha \), \( \tilde{\Omega}^\alpha(C) \) is a \( (d - p) \)-form bilinear in the Weyl zero-forms, that is \( D_0 \)-closed on shell

\[ D_0 \tilde{\Omega}^\alpha(C) \sim 0. \tag{3.13} \]
Note that $\tilde{\Omega}^\alpha(C)$ may differ from the current $\Omega^\alpha$ resulting from the action (3.12) by virtue of (3.7) by $D_0$–exact terms resulting from the variation of $\tilde{\Omega}^\alpha(C)$.

A subclass of current vertices equivalent to Abelian vertices is associated with $D_0$–exact forms $\Omega^\alpha(C)$

$$\Omega^\alpha(C) = D_0\tilde{\beta}^\alpha(C).$$

(3.14)

Exact gauge invariant currents, often called improvements, do not give rise to nontrivial charges. Nontrivial (i.e., different from improvements) current interactions usually require a modification of the linearized Abelian gauge transformation by $C$–dependent terms.

Manifestly gauge invariant conserved currents $\tilde{\Omega}^\alpha(C)$ were originally constructed in [2] for the case of $d = 4$, using the formalism of two-component spinors. In [2] they were called generalized Bell-Robinson tensors. Such interactions can contribute to the full HS action. To the best of our knowledge, except for the HS conserved currents built from a scalar field [4, 62, 63, 64, 65, 67], generalized Bell-Robinson currents were not constructed for $d > 4$.

### 3.4 Non-Abelian vertices

Vertices of the form $\omega^2 \times C$ will be called non-Abelian. They are typical for the actions constructed from bilinears of some non-Abelian curvatures $R = R_1 + \omega^2$ since the cubic part of the Lagrangian

$$L = \frac{1}{2}RR$$

(3.15)

has the structure $R_1\omega^2$. In particular, such cubic vertices result from the gravity action in the MacDowell-Mansouri-Stelle-West form considered in Section 2.1. The HS actions constructed in Section 6 are of this class.

Writing schematically

$$R_\alpha = d\omega_\alpha + f^\beta_\alpha \omega_\beta \wedge \omega_\gamma,$$

(3.16)

where $f^\beta_\alpha$ are some structure coefficients, we observe that $f^\beta_\alpha$ contribute linearly to the action (3.15) while the on-shell analysis involves only Abelian (i.e., free) gauge transformation law. This means that for the cubic order analysis it does not matter whether or not $f^\beta_\alpha$ satisfy Jacobi identities. As will be shown in Section 5.1, what does matter is the symmetry properties of the coefficients, i.e., the existence of such a metric $g^{\alpha\beta}$ that the structure coefficients

$$f^{\alpha\beta\gamma} = g^{\alpha\rho} f^\beta_\rho$$

(3.17)

are totally antisymmetric

$$f^{\alpha\beta\gamma} = -f^{\alpha\gamma\beta} = -f^{\beta\alpha\gamma}.$$ 

(3.18)

In the analysis of cubic HS actions in Section 6, this property is a consequence of the existence of supertrace which induces the Killing metric $g^{\alpha\beta}$ on the HS algebra.

That the detailed structure of HS algebra is not needed at the cubic level simplifies the analysis because at this level any (graded) antisymmetric coefficients $f^{\alpha\beta\gamma}$ can be used to construct a consistent cubic vertex. In particular, one can consider such $f^{\alpha\beta\gamma}$ which are nonzero only for given three spins. Moreover, even for a given set of spins there may exist a
number of independent consistent cubic vertices associated with independent antisymmetric tensors $\tilde{f}^{\alpha \beta \gamma}$. To classify non-Abelian vertices one has to classify totally antisymmetric rank-three tensors on the space of $o(d - 1, 2)$ tensors valued in various two-row Young diagrams as discussed in Section 8.1. The fact of existence of different vertices for given spins is anticipated to match the list (1.1) of [8]. (See also [22] where different types of HS vertices in Minkowski space were analyzed in terms of invariant contractions of two-row Lorentz Young diagrams within the Batalin-Vilkovisky formalism.)

Let us, however, stress that, beyond the cubic level, the Jacobi identities for $f_{\alpha \beta \gamma}$ play crucial role, relating coupling constants of different cubic vertices.

### 3.5 Chern-Simons vertices

$\omega^3$-type vertices we call Chern-Simons. Chern-Simons vertices where $\omega^3$ is a $d$-form will be called genuiner. (In other words, genuine Chern-Simons vertices do not contain the background vielbein one-form $E^A$.) Using the compensator formalism explained in Section 2 it is easy to see that all non-genuine gauge invariant Chern-Simons vertices in $AdS_d$ are equivalent up to total derivatives to some non-Abelian vertices.

Indeed, consider a cubic action

$$S = \frac{1}{3} \int_{M^d} U^{\alpha \beta \gamma}(V, E) \wedge \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma,$$

where $\omega_\alpha$ are some $p_\alpha$-forms and a $(d - p)$-form $U^{\alpha \beta \gamma}(V, E)$ with $p = p_\alpha + p_\beta + p_\gamma$ is built from $V^A$ and $E^A$. All contractions in (3.19) are demanded to be $o(d - 1, 2)$ covariant.

The gauge variation of the action (3.19) is

$$\delta S = \int_{M^d} \left( (-1)^{d+p+1} D_0(U^{\alpha \beta \gamma}(V, E)) \wedge \varepsilon_\alpha \wedge \omega_\beta \wedge \omega_\gamma + 2(-1)^{pa} U^{\alpha \beta \gamma}(V, E) \wedge \varepsilon_\alpha \wedge D_0(\omega_\beta) \wedge \omega_\gamma \right).$$

(3.20)

The condition that it is zero requires each of the two terms vanish on-shell. The first term should vanish as it is because it does not contain HS curvatures while in the analysis of the second term one can use FOST.

Thus, the necessary condition for gauge invariance of $S$ is

$$D_0(U^{\alpha \beta \gamma}(V, E)) = 0.$$  

(3.21)

To solve this condition we observe that the equations

$$E^A = D_0 V^A, \quad D_0 E^A = 0$$

(3.22)

imply that $U^{\alpha \beta \gamma}(V, E)$ is closed with respect to de Rham operator

$$\tilde{d} = E^A \frac{\partial}{\partial V^A},$$

(3.23)

---

1I am grateful to E.Skvortsov for drawing my attention to this aspect of [22] and related discussion.
where $V^A$ and $E^A$ are interpreted as coordinates and differentials, respectively. Since $V^A$ satisfies (2.4), $\tilde{d}$ can be identified with the exterior differential on the $AdS_d$ hyperboloid. The analysis of equation (3.21) is insensitive to the signature of the metric $\eta^{AB}$ and hence is equivalent to that on the sphere $S^d$. As a result, the general solution of (3.21) is

$$U^{\alpha\beta\gamma}(V, E) = D_0(\tilde{U}^{\alpha\beta\gamma}(V, E)) + H^{\alpha\beta\gamma}(V, E), \quad H^{\alpha\beta\gamma}(V, E) \in H^{d-\rho}(S^d).$$  

(3.24)

Since the sphere cohomology $H^{d-\rho}(S^d)$ is nonzero only at $d = p$ or $p = 0$, a nontrivial solution of (3.21) at $p > 0$ only exists at $d = p$ with $U^{\alpha\beta\gamma}(V, E) = \text{const}$, which is the case of genuine Chern-Simons vertices. At $d = 3$ they correspond to usual Chern-Simons vertices. In higher dimensions, genuine Chern-Simons vertices can be constructed from mixed-symmetry fields described by higher differential forms [46, 49, 50] of total degree $d$ with all their $o(d-1, 2)$ indices contracted with the $o(d-1, 2)$ invariant metric and epsilon symbol. One can consider $p$-odd or even vertices that are free or contain the $o(d-1, 2)$ epsilon symbol, respectively (recall that another epsilon symbol is hidden in the wedge product).

Thus, unless $p = d$, which for the case of symmetric fields described by one-forms implies $d = 3$, gauge invariance of a Chern-Simons vertex leads to

$$U^{\alpha\beta\gamma}(V, E) = D_0(\tilde{U}^{\alpha\beta\gamma}(V, E)).$$  

(3.25)

Substitution of this expression into (3.19) and integration by parts implies that the action (3.19) is equivalent to some $C \times \omega^2$-type action.

It should be stressed that the property that non-genuine gauge invariant Chern-Simons vertices in $AdS_d$ are equivalent to non-Abelian vertices is not true in Minkowski geometry where nontrivial Chern-Simons vertices can exist beyond the genuine class as illustrated by the spin three example of Section 5.3.1.

### 3.6 Minkowski versus $AdS$

The issue of deformation of a Minkowski vertex to $AdS_d$ is quite interesting. One option is that some of the Minkowski vertices may admit no gauge invariant $AdS_d$ deformations. In particular, it may happen that the integrability conditions (3.9) are not respected at $\lambda \neq 0$. Somewhat analogous phenomenon occurs for free mixed-symmetry fields [68, 69], in which case different types of gauge symmetries require different $\lambda$-dependent deformations incompatible to each other. However, for the case of symmetric fields considered in this paper, we found no obstructions for deformations of Minkowski vertices to $AdS_d$.

A kind of opposite option, which plays a role in the sequel, is that $AdS$ deformation of a nontrivial Minkowski vertex may be equivalent to the $AdS$ deformation of a trivial Minkowski vertex with higher derivatives.

Indeed, a trivial $d$–form vertex $V^{\text{triv}}$ can be represented in the form

$$V^{\text{triv}} \sim dV'.$$  

(3.26)

In the case of massless fields in Minkowski space, the field equations contain no dimensionful parameters and hence have definite dimension. So, in Minkowski case it is possible to consider...
basis vertices that have definite scaling, carrying a definite number $N$ of derivatives of the dynamical fields from which they are constructed. In other words, a number of derivatives forms a grading in the massless Minkowski case. Hence, for a trivial massless Minkowski vertex with $N$ derivatives, the “potential” $V’$ necessarily carries $N - 1$ derivatives.

The situation in $AdS_d$ is different because of presence of the dimensionful $AdS_d$ radius $\rho = \lambda^{-1}$. Indeed, consider a deformation of a nontrivial Minkowski vertex $V_M$ to a vertex $V_{AdS}$ in $AdS_d$. The point is that it may admit a representation

$$V_{AdS} \sim \lambda^{-2}(dU_{AdS} + \tilde{V}_{AdS})$$  \hspace{1cm} (3.27)

with $U_{AdS}$ and $\tilde{V}_{AdS}$ that contain, respectively, $N + 1$ and $N + 2$ derivatives of the dynamical fields. Once the representation (3.27) takes place, the vertex $V_{AdS}$ is equivalent to $\tilde{V}_{AdS}$ for all $\lambda$ except for $\lambda = 0$. Hence, $AdS_d$ vertices that contain different numbers of derivatives can belong to the same equivalence class. (Note that, generally, the field equations in $AdS_d$ are inhomogeneous in derivatives, containing factors of $\lambda$ in lower-derivative terms.) Similar phenomenon can take place in massive theories.

On the other hand, taking the flat limit of $V_{AdS}$ and $\tilde{V}_{AdS}$ gives, respectively, two consistent vertices $V_M$ and $\tilde{V}_M$ in Minkowski space. Since $V_M$ and $\tilde{V}_M$ carry different numbers of derivatives they cannot be equivalent. Moreover, multiplying the both sides of Eq. (3.27) by $\lambda^2$ we see that the vertex $\tilde{V}_M$ should be trivial in Minkowski space. Vertices $V_{AdS}$ that are not exact in $AdS_d$ but lead to trivial vertices $\tilde{V}_M$ in the flat limit we call quasi exact.

Note that, as discussed in more detail in Section 4.3 in principle it may happen that $\tilde{V}_{AdS} = 0$, i.e., a vertex that is nontrivial in Minkowski geometry becomes trivial in $AdS_d$.

That vertices with different numbers of derivatives can be equivalent in $AdS$ will be used in Section 8 where we will argue that a large class of HS vertices in $AdS_d$, that respect the condition (1.2), can be represented as combinations of a set of non-Abelian and Abelian vertices, that all contain $s_1 + s_2 + s_3 - 2$ derivatives in $AdS_d$. This analysis is based on the cohomological techniques which controls both $AdS$ and Minkowski vertices.

4 Vertex tri-complex

Interactions of massless gauge fields in the frame-like formalism can be analyzed in terms of the forms $F(\omega, C|V, E)$ that depend on the gauge $p$-forms $\omega$ and Weyl zero-forms $C$ as well as on the compensator $V^A$ and background frame one-form $E^A$. Important subclass of interactions is described by vertices $F(\omega, R_1|V, E)$, where the Weyl zero-forms $C$ enter through the curvatures $R_1 = D_0\omega$ in accordance with FOST (2.31). Being constructed in terms of differential forms of positive degrees $\omega$ and $R_1 = D_0\omega$, such vertices will be called strictly positive. An important feature simplifying analysis of this subclass is that both $\omega$ and $R_1$ are valued in finite-dimensional tensor modules of $o(d-1,2)$, while Weyl zero-forms $C$ are valued in the infinite-dimensional Weyl module as explained in Section 2.2.3. In this paper we mostly consider strictly positive vertices, deriving in this section a vertex tri-complex that controls their structure in $AdS_d$ background and contains a sub
bi-complex that controls Minkowski vertices. That AdS and Minkowski vertex complexes have different structures manifests itself in the peculiarities of the correspondence between AdS and Minkowski vertices.

### 4.1 Vertex differentials

Consider a strictly positive differential form

\[ F(\omega, R_1|V, E) = V^{C_1} \cdots V^{C_p} G^{A_1 \cdots A_q} \wedge F_{[A_1 \cdots A_q],C_1\cdots C_p}(\omega, R_1) = G^{A_1 \cdots A_q} \wedge F_{[A_1 \cdots A_q]}(\omega, R_1|V) , \]

where \( F_{[A_1 \cdots A_q],C_1\cdots C_p}(\omega, R) \) is built in a \( o(d-1,2) \) covariant way from the wedge products of the \( p \)-form connections \( \omega \) and \( (p+1) \)-form (linearized) curvatures \( R_1 \) valued in arbitrary tensor \( o(d-1,2) \)-modules. The method presented in this section applies to HS gauge fields of most general mixed symmetry type. In the case of symmetric HS fields, which is of most interest in the rest of this paper, HS connections are one–forms and curvatures are two-forms, both valued in the irreducible \( o(d-1,2) \)-modules described by two-row rectangular Young diagrams.

We require all \( V^C \) in (4.1) be contracted directly to \( \omega \) or \( R_1 \), i.e., the \( o(d-1,2) \) tensor \( F_{[A_1 \cdots A_q],C_1\cdots C_p}(\omega, R_1) \) is not allowed to contain explicitly the \( o(d-1,2) \) metric tensor that carries indices \( A \) and/or \( C \). Let \( F \), that have this property, be called basic and \( \mathcal{F} \) be the space of strictly positive basic forms. Any \( F \) can be represented as a sum of basic forms with the help of (2.4) and (2.9). In the sequel we only consider basic forms \( F \).

Using Bianchi identities (2.23) and definition of the frame one-form (2.7), we obtain

\[ dF = \left( D_0(G^{A_1 \cdots A_q}) + (-1)^{d-1} G^{A_1 \cdots A_q} \wedge E^C \frac{\partial}{\partial \omega^C} + R_{1a} \wedge \frac{\partial}{\partial \omega_a} \right) \wedge F_{A_1 \cdots A_q}(\omega, R_1|V) , \]

where the derivative \( \frac{\partial}{\partial \omega_a} \) is left with respect to the wedge algebra and \( \alpha \) accumulates all indices carried by the gauge forms. With the help of (2.15) and (2.16) this gives

\[ dF = QF , \]

where

\[ Q = Q^{\text{top}} + \lambda^2 Q^{\text{sub}} + Q^{\text{cur}} , \]

\[ Q^{\text{top}} F = (-1)^{d-q} \frac{q}{d+1-q} G^{A_1 \cdots A_{q-1}} \wedge \frac{\partial}{\partial V_{A_q}} F_{A_1 \cdots A_q}(\omega, R_1|V) , \]

\[ Q^{\text{sub}} F = (-1)^d \frac{q}{d+1-q} G^{A_2 \cdots A_q} \wedge V^A (d+1-q + V^E \frac{\partial}{\partial V^E}) F_{A_1 \cdots A_q}(\omega, R_1|V) , \]

\[ Q^{\text{cur}} F = (-1)^{d-q} R_{1a} \frac{\partial}{\partial \omega_a} F(\omega, R_1|V) . \]

Here \( \frac{\partial}{\partial V^C} \) is the usual derivative for unrestricted \( V^A \), i.e.,

\[ \frac{\partial}{\partial V^E} \left( V^{C_1} \cdots V^{C_p} F_{[A_1 \cdots A_q],C_1\cdots C_p}(\omega, R_1) \right) = p V^{C_2} \cdots V^{C_p} F_{[A_1 \cdots A_q],EC_2\cdots C_p}(\omega, R_1) . \]
In the case of interest with the compensator field restricted by the normalization condition (2.4), this definition is still well defined for the class of basic forms simply because they never contain $V^A V_A$ explicitly. If desired, the restriction to basic forms can be relaxed via replacement of the compensator $V^A$ by the unrestricted field $W^A$ according to (2.5). In practice, it is however simpler to work with basic forms. It should also be noted that $Q_{sub}$ is well defined for $q \leq d$ and cannot be used for $q > d$.

From Eqs. (4.5) and (4.6) it is easy to see that, in agreement with $d^2 = 0$, 

\[
(Q_{top})^2 = 0, \quad (Q_{sub})^2 = 0, \quad (Q_{cur})^2 = 0, 
\]

\[
\{Q_{top}, Q_{sub}\} = 0, \quad \{Q_{top}, Q_{cur}\} = 0, \quad \{Q_{cur}, Q_{sub}\} = 0. 
\]

Thus, $Q_{top}$, $Q_{sub}$ and $Q_{cur}$ form a tri-complex called vertex tri-complex.

In terms of the generating function

\[
F(\omega, R_1|V, \psi) = \psi^{Aq} \ldots \psi^{A1} F_{A1 \ldots Aq}(\omega, R_1|V), \tag{4.11}
\]

where $\psi^A$ are auxiliary anticommuting variables \{

$\psi^A, \psi^B\} = 0$, $Q_{top}$, $Q_{sub}$ and $Q_{cur}$ acquire the form

\[
Q_{top} = -(-1)^{d+N_\psi} \frac{1}{d - N_\psi} \frac{\partial}{\partial V_A} \frac{\partial}{\partial \psi^A}, \tag{4.12}
\]

\[
Q_{sub} = (-1)^{d+N_\psi} \frac{d - 1 - N_\psi + N_V}{d - N_\psi} V^A \frac{\partial}{\partial \psi^A}, \tag{4.13}
\]

\[
Q_{cur} = (-1)^{d-N_\psi} R_1 \alpha \frac{\partial}{\partial \omega^\alpha}, \tag{4.14}
\]

where

\[
N_\psi = \psi^A \frac{\partial}{\partial \psi^A}, \quad N_V = V^A \frac{\partial}{\partial V_A}. \tag{4.15}
\]

Up to the $N_\psi$ and $N_V$–dependent factors in Eqs. (4.12), (4.13) and (4.14), resulting from the definition (4.3), the operators $Q_{top}$, $Q_{sub}$ and $Q_{cur}$ admit the following interpretation. $Q_{top}$ is the Batalin-Vilkovisky odd differential [70, 71]. $Q_{sub}$ is the Koszul differential. $Q_{cur}$ is the de Rham differential on the space of connections $\omega$, where the linearized curvature $R_1$ is interpreted as the differential for $\omega$.

It is also useful to introduce the de Rham–like nilpotent operator

\[
P_{sub} = -(-1)^{d+N_\psi} \frac{d - N_\psi + 1}{(d - N_\psi + N_V + 1)(N_\psi + N_V)} \psi^A \frac{\partial}{\partial V^A}, \tag{4.16}
\]

\[
(P_{sub})^2 = 0. \tag{4.17}
\]

The coefficients in (4.16) are adjusted in such a way that $P_{sub}$ forms a homotopy for the operator $Q_{sub}$, i.e.,

\[
\{Q_{sub}, P_{sub}\} = Id. \tag{4.18}
\]

22
More precisely, this is true provided that the denominators in (4.16) are non-zero. In agreement with the discussion of Section 3.5, singularities of \( P_{\text{sub}} \) are associated with cohomology of sphere. In this paper, they are irrelevant because we consider gauge \( p \)-forms with \( p > 0 \) and do not consider genuine Chern-Simons vertices that, beyond \( d = 3 \), require higher differential forms associated with mixed symmetry fields. Hence in the sequel we assume that \( Q_{\text{sub}} \) has trivial cohomology:

\[
\forall \quad Q_{\text{sub}} X = 0 \quad \Rightarrow \quad X = Q_{\text{sub}} P_{\text{sub}} X. \tag{4.19}
\]

Note that \( \{Q_{\text{cur}}, P_{\text{sub}}\} = 0 \) but \( \{Q_{\text{top}}, P_{\text{sub}}\} \) is nontrivial.

### 4.2 Vertex cohomology

Let \( F(\omega, R_1) \) be a \( d \)-form. Consider an action

\[
S = \int_{M^d} F(\omega, R_1). \tag{4.20}
\]

Using the convention that \( \frac{\partial}{\partial \omega_\alpha}(R_1) = 0 \), which expresses the fact that \( R_1 \) is gauge invariant and agrees with the interpretation of \( R_1 \alpha \) as differentials in the space of connections \( \omega_\alpha \), it is easy to see that the gauge variation of \( S \) with gauge parameters \( \varepsilon_\alpha \) is

\[
\delta \int_{M^d} F(\omega, R_1) = \int_{M^d} \varepsilon_\alpha \frac{\partial}{\partial \omega_\alpha}(QF(\omega, R_1)), \tag{4.21}
\]

where it is used that \( \varepsilon_\alpha \frac{\partial}{\partial \omega_\alpha} \) anticommutates to \( Q_{\text{cur}} \) and hence to \( Q \).

Thus, the necessary condition for the action (4.20) to be gauge invariant is

\[
\frac{\partial}{\partial \omega_\alpha}(QF(\omega, R_1)) \sim 0 \tag{4.22}
\]

which implies that all \( \omega \)-dependent terms in \( QF \) should vanish. This requires

\[
QF(\omega, R_1) \sim G(R_1), \tag{4.23}
\]

where \( G(R_1) \) is \( \omega \)-independent. For \( F(\omega, R_1) \), that is at least bilinear in \( \omega \), this implies

\[
QF(\omega, R_1) \sim 0. \tag{4.24}
\]

Since all \( Q \)-exact \( F \) are \( d \)-exact by virtue of (4.13), \( Q \)-exact \( F \) give rise to trivial actions (4.20). As a result, the space of nontrivial strictly positive gauge invariant vertices \( F(\omega, R_1) \) identifies with \( Q \)-cohomology (with the convention that \( \omega \)-independent functionals have to be factored out). An interesting feature of this formalism is that \( Q \)-cohomology controls simultaneously the issues of gauge invariance and space-time exactness of the vertices. In this respect it differs from the Batalin-Vilkovisky formalism [70, 71] where these two aspects
are independent (see also [72] and references therein). On the other hand, the vertex complex formalism does not control the issue of on-shell equivalence.

Indeed, one has to distinguish between the off-shell and on-shell cases. In the off-shell case, dynamical field equations are not used. In accordance with the general discussion of Section 3.1 this means that there is no room for the deformation of the transformation law (3.3). Hence, the off-shell $Q$-cohomology describes those vertices that are gauge invariant under undeformed Abelian gauge transformations. On the other hand, if the gauge invariance is achieved on-shell, i.e., up to terms that vanish on the free field equations, this implies that noninvariant terms can be compensated by a field-dependent deformation (3.3) of the Abelian gauge transformation. Clearly, being related to a non-Abelian deformation of the HS symmetry, the on-shell case is most interesting.

In the on-shell analysis, all forms $F(\omega, R_1)$, that are zero by virtue of the free field equations, have to be ignored, i.e., we can use FOST (2.31) or its mixed symmetry analogues [49, 50]. For symmetric HS fields we will use it in the form of relations (2.32), (2.34), (2.39) and (2.42). Let $\mathcal{O}$ be the space of forms $F(\omega, R_1)$ that are zero by virtue of FOST. The space $\mathcal{V}$ of on-shell nontrivial forms $F$ is

$$
\mathcal{V} = \mathcal{F}/\mathcal{O}.
$$

(4.25)

It is not hard to check that $\mathcal{O}$ is invariant under the action of any of the operators $Q^{\text{top}}, Q^{\text{sub}}$ and $Q^{\text{cur}}$ (i.e., application of any of these operators to the left-hand-sides of the on-shell conditions gives zero by virtue of COST). Hence their action is well defined on $\mathcal{V}$.

Thus, the $Q$-cohomologies relevant to the off-shell and on-shell analysis are, respectively, $H(Q, \mathcal{F})$ and $H(Q, \mathcal{V})$. In practice, the analysis of the on-shell cohomology is more subtle than that of the off-shell one. In particular, having a simple form of exterior differential on the space of connections in the off-shell case, $Q^{\text{cur}}$ loses this property in the on-shell case because most of the curvature components are zero on shell by virtue of (2.31). Also the on-shell analysis of $H(Q^{\text{sub}}, \mathcal{V})$ becomes nontrivial because the homotopy operator $P^{\text{sub}}$ (4.16) does not leave invariant $\mathcal{O}$ and hence does not act on $\mathcal{V}$.

The factor of $\lambda^2$ in front of $Q^{\text{sub}}$ in (4.4) signals that $Q^{\text{sub}}F$ contains at least two less space-time derivatives of the HS fields than $Q^{\text{top}}F$ and $Q^{\text{cur}}F$. Since the term with $Q^{\text{sub}}$ disappears in the flat limit, the structure of vertices in Minkowski space is controlled by the operator

$$
Q^{\text{fl}} = Q^{\text{top}} + Q^{\text{cur}}.
$$

(4.26)

The space of strictly positive gauge invariant vertices $F^{\text{fl}}$ in Minkowski space is represented by $H^{\text{fl}}(Q^{\text{fl}}, \mathcal{V})$, i.e.,

$$
Q^{\text{fl}} F^{\text{fl}} \sim 0, \quad F^{\text{fl}} \not\sim Q^{\text{fl}} G^{\text{fl}}.
$$

(4.27)

A remarkable property of the vertex cohomology formalism is that neither the form of the vertex $F$ in terms of connections and curvatures, nor the form of $Q^{\text{top}}$ and $Q^{\text{cur}}$ depend on whether the problem is considered in Minkowski or $AdS$ space. The difference is only that the term with $\lambda^2 Q^{\text{sub}}$ contributes to the full differential $Q$ and also that explicit expressions for connections in terms of dynamical (i.e., Fronsdal-like) fields, resulting from the solution
of constraints contained in equations (2.29), are $\lambda$-dependent because the linearized curvatures depend on $\lambda$ via background connections (2.2) that satisfy (2.3). However, because of the additional term $\lambda^2Q_{\text{sub}}$ in the $AdS$ case, the $Q^{fl}$-cohomology may differ from the $Q$-cohomology at $\lambda \neq 0$. Hence classification of nontrivial Minkowski and $AdS$ vertices may be essentially different.

Mixture of terms with different numbers of derivatives in the $AdS$ case makes it impossible to say that a particular vertex contains a definite number of derivatives. Naively, the best one can do is to replace the grading associated with a number of derivatives in the massless Minkowski case by the filtration that characterizes the maximal number of derivatives in a vertex. Nevertheless the proposed formalism suggests a natural $AdS$ substitute for the notion of order of derivatives in Minkowski space provided by the grading $G$

$$G = N_R - N_V,$$  \hspace{1cm} (4.28)

that counts the difference between the number of curvatures and the number of compensators $V^A$ in a basic form $F(\omega, R_1|V, E)$. To explain the meaning of $G$ recall that replacement of a connection by its curvature adds one derivative while addition of the compensator $V^A$ removes one derivative. More precisely, as explained in Section 2.2.1 for a spin $s$ HS gauge form, the curvature $R_1$ (2.21) contains up to $s$ derivatives of the spin $s$ frame-like field (2.24) while a connection (2.25), that contains $k$ contractions with the compensator $V^A$, contains up to $s - 1 - k$ derivatives. $G$ effectively counts a number of derivatives in $F$ shifted by a constant that depends on the spins of fields under consideration.

The space of strictly positive basic forms $\mathcal{F}$ can be represented as a direct sum of homogeneous subspaces $\mathcal{F}_g$ of definite $G$-grades $g$,

$$\mathcal{F} = \oplus \mathcal{F}_g, \quad F_g \in \mathcal{F}_g : \quad G F_g = g F_g.$$  \hspace{1cm} (4.29)

$\mathcal{F}_g$ can be interpreted as the $(A)dS$ generalization of the space of vertices with definite number of derivatives. Important properties of this definition are that the on-shell conditions (2.32) and (2.34) have definite grades and that $Q^{\text{top}}, Q^{\text{cur}}$ and, hence, $Q^{fl}$ have $G$-grade $+1$ while $Q_{\text{sub}}$ has $G$-grade $-1$. In the case $\lambda = 0$, $G$ is equivalent to the Minkowski derivative grading. Note that

$$G_\lambda = G + \lambda \frac{\partial}{\partial \lambda}$$  \hspace{1cm} (4.30)

forms a true grading of the system with the convention that $\frac{\partial}{\partial \lambda}$ acts on the explicit dependence on $\lambda$ (i.e., not on the $\lambda$-dependence of connections $\omega$ expressed in terms of the frame-like or Fronsdal fields, that is $\frac{\partial}{\partial \lambda} \omega = 0$).

For any consistent $AdS_d$ cubic vertex $F^{AdS}$, that contains a finite number of derivatives, the leading derivative term is associated with some Minkowski vertex $F^{fl}$. Indeed, the highest derivative part of $F^{AdS}$ (i.e., the part with the highest value of $G$) is $Q^{fl}$ closed and hence is related to some Minkowski vertex. If $F^{fl}$ is $Q^{fl}$ exact, i.e., $F^{fl} = Q^{fl}W$, the redefinition

$$F^{AdS} \rightarrow F^{flAdS} = F^{AdS} - QW$$  \hspace{1cm} (4.31)
shows that $F^{AdS}$ is equivalent to another lower-derivative $AdS$ vertex $F^{lAdS}$.

Note that the analysis of $Q$–cohomology makes sense for various $p$ including $p > d$. This is because $Q$ is defined on the space of basic forms $F$ that depend both on the frame one–forms $E^A$ and on the gauge connections and curvatures. Although, a product of $d + 1$ frame one-forms $E^A$ is zero (recall that due to (2.9) $E^A$ has $d$ independent components), one can consider their product with an arbitrary number of gauge forms $\omega$ and curvatures $R$ which can be higher differential forms for mixed symmetry fields.

4.3 AdS deformation and vertex sets

Let $F^{fl} = F^0$ be a nontrivial Minkowski vertex that satisfies (4.27). The process of AdS deformation of $F^0$ goes as follows. From (4.27) and (4.4) we find that

$$Q F^0 \sim \lambda^2 F^1, \quad F^1 = Q^{sub} F^{fl}.$$ (4.32)

From (4.10) it follows that $F^1$ is $Q^{fl}$–closed, $Q^{fl} F^1 \sim 0$. Let $F^1$ be $Q^{fl}$–exact

$$F^1 \sim Q^{fl} F^2.$$ (4.33)

In this case it is possible to deform $F^{fl}$

$$F^{fl} \to F^{2,AdS} = F^{fl}(\omega, R_1) - \lambda^2 F^2(\omega, R_1)$$ (4.34)

to achieve $Q$–closure of $F^{2,AdS}$ up to $\lambda^4$ terms

$$Q F^{2,AdS} \sim -\lambda^4 F^3,$$ (4.35)

where

$$F^3 = Q^{sub} F^2.$$ (4.36)

Using (4.36), (4.33) and (4.32), we obtain that $Q^{fl} F^3 \sim 0$. Again, if $F^3$ is $Q^{fl}$–exact,

$$F^3 \sim Q^{fl} F^4,$$ (4.37)

$F^{fl}$ can be further deformed to

$$F^{fl} \to F^{4,AdS} = F^{fl}(\omega, R_1) - \lambda^2 F^2(\omega, R_1) + \lambda^4 F^4(\omega, R_1)$$ (4.38)

to achieve $Q$–closure up to terms of order $\lambda^6$.

The process continues with the end result that the AdS deformation of the flat vertex

$$F^{fl} \to F^{AdS} = \sum_{n=0}^\infty (-\lambda^2)^n F^{2n}$$ (4.39)

exists provided that there exists a set of $F^{2n}$, $0 \leq m \leq n_{max}$ with $F^0 = F^{fl}$, that satisfy

$$Q^{sub} F^{2n} \sim Q^{fl} F^{2n+2}, \quad Q^{sub} F^{2n_{max}} \sim 0, \quad Q^{fl} F^0 \sim 0.$$ (4.40)
These relations can be equivalently written in the form

\[ F^{2n+1} = Q^{\text{sub}} F^{2n}, \quad F^{2n+1} \sim Q^{\text{fl}} F^{2n+2} \] (4.41)

with new forms \( F^{2n+1} \). A set of forms (4.41) where \( F_{2n} \) are \( p \)-forms and \( F_{2n+1} \) are \( (p + 1) \)-forms will be referred to as even \( p \)-set since the number of the last form \( F_{2n_{\text{max}}} \) is even.

Alternatively, if the process leads to some \( F^{2n+1} \in H(Q^{\text{fl}}) \), the relations (4.41) are replaced by

\[ F^{2n+1} = Q^{\text{sub}} F^{2n}, \quad F^{2n+1} \sim Q^{\text{fl}} F^{2n+2}, \quad 0 \leq n < n_{\text{max}} \]

\[ F^{2n_{\text{max}}+1} = Q^{\text{sub}} F^{2n_{\text{max}}}, \quad Q^{\text{fl}} F^{2n_{\text{max}}+1} \sim 0, \quad F^{2n_{\text{max}}+1} \in H(Q^{\text{fl}}). \] (4.42)

Such set of forms will be referred to as odd \( p \)-set.

The above analysis implies that if a Minkowski Lagrangian \( F^0 \) is the lowest element of an odd \( d \)-set, no its \( \text{AdS} \) deformation regular at \( \lambda = 0 \) exists.

Let \( G^{2n+1} \) be arbitrary forms. An even set of \( F^m \) of the form

\[ \tilde{F}^{2n} = Q^{\text{fl}} G^{2n+1} - Q^{\text{sub}} G^{2n-1}, \quad \tilde{F}^{2n+1} = Q^{\text{sub}} Q^{\text{fl}} G^{2n+1} \] (4.43)

we call exact. The ambiguity in exact sets reflects the ambiguity in \( Q \)-exact forms

\[ \delta F^{\text{AdS}} = Q \sum_{n=0}^{2n} \lambda^{2n} G^{2n+1}. \] (4.44)

An even set is called nontrivial if it is not exact.

Let \( S^e \), \( S^o \) and \( S^{\text{ex}} \) be, respectively, the spaces of even, odd and exact sets. We say that two sets are equivalent if they differ by an exact set. Note that a number of nonzero terms in any even or odd set is always finite since a number of derivatives in \( F^{2n} \) decreases with \( n \) (equivalently, a number of compensators increases with \( n \)).

By adding an exact set it is always possible to achieve that the lowest element of a non-exact set, \( i.e., F^0 \), belongs to \( H(Q^{\text{fl}}) \), hence representing a nontrivial Minkowski vertex. Thus, every non-exact \( Q \)-closed set of forms is associated with some Minkowski vertex form \( F^0 \). The space \( S^{\text{fl}} = S^e / S^{\text{ex}} \) represents those flat vertices that allow an \( \text{AdS} \) deformation while \( S^o / S^e \) (note that odd sets are defined up to even sets) represents those flat vertices that allow no \( \text{AdS} \) deformation.

Let a form \( F \) be both \( Q^{\text{sub}} \) and \( Q^{\text{fl}} \) closed

\[ Q^{\text{fl}} F = 0, \quad Q^{\text{sub}} F = 0. \] (4.45)

Then it can be interpreted as an even set that consists of a single element. Such a form (vertex) will be called pure. If a Minkowski vertex \( F^{\text{fl}}(\omega, R_1) \) is pure, it also provides a consistent \( \text{AdS} \) vertex, \( i.e., \) its form in terms of HS connections and curvatures remains unchanged upon deformation to \( \text{AdS} \). This does not mean of course that its expression in terms of dynamical or frame-like fields remains unchanged because the expressions for the higher connections in terms of the frame-like fields are sensitive to the \( \text{AdS} \) deformation.
Note that, *a priori*, it is not guaranteed that, being nontrivial in Minkowski space, a pure form $F$ remains non-trivial in $AdS$. In principle, it may happen that $F = Q^{fl} \tilde{F}$, where $\tilde{F}$ satisfies $Q^{fl} \tilde{F} = 0$. Then $F$ is $Q^{fl}$ closed and can be represented in the form $F = \lambda^{-2} Q^{fl} \tilde{F}$. In fact, according to Theorem 4.1 below, this is what happens to any strictly positive off-shell Minkowski vertex that admits an $AdS$ deformation. We are not aware, however, of a physically interesting realization of such a mechanism in the on-shell case. In all examples available to us a nontrivial vertex in Minkowski space remains nontrivial in $AdS_d$.

Another interesting property of pure forms is that they decompose into a sum of $G$-homogeneous pure forms. Indeed, let a vertex $F$ be pure. Then any its homogeneous component $F_p$ is also pure, i.e., satisfies (4.45). This is because $Q^{sub}$ and $Q^{fl}$ have definite grades so that $Q^{sub} F = 0$ and $Q^{fl} F = 0$ imply $Q^{sub} F_n = 0$ and $Q^{fl} F_n = 0 \quad \forall n$.

The decomposition of a pure vertex $F$ into a sum of homogeneous pure vertices $F_p$ provides an $(A)dS$ generalization of the simple fact that a massless vertex in Minkowski space, that contains terms with different numbers of derivatives, is a sum of consistent vertices, each carrying a definite number of derivatives.

Now we are in a position to analyze in some more detail properties of vertex complex cohomology in the off-shell and on-shell cases.

### 4.3.1 Off-shell case

In the off-shell case, where the field equations are not used, $Q^{sub}$ has trivial cohomology, except for the case of genuine Chern-Simons vertices not considered in this paper. This considerably simplifies the analysis since the homotopy operator $P^{sub} (4.16)$ makes it possible to reconstruct an odd set up to a trivial set starting from a $Q^{sub}$ closed $F^{2n_{max}+1}$

$$F^{2m+1} = (Q^{fl} P^{sub})^{n_{max} - m} F^{2n_{max}+1},$$

$$F^{2m} = P^{sub} (Q^{fl} P^{sub})^{n_{max} - m} F^{2n_{max}+1}. \quad (4.46)$$

Similarly, an even set is reconstructed from the last $Q^{sub}$ closed term $F^{2n_{max}}$ as follows

$$F^{2m+1} = (Q^{fl} P^{sub})^{n_{max} - m - 1} Q^{fl} F^{2n_{max}},$$

$$F^{2m} = (P^{sub} Q^{fl})^{n_{max} - m} F^{2n_{max}}. \quad (4.47)$$

A useful criterium for existence of a $\lambda$-regular off-shell $AdS$ deformation of an off-shell Minkowski vertex $F^{fl}$ is provided by

**Lemma 4.1**

Any even off-shell set $F^n$ is $\lambda$-regularly equivalent to some pure off-shell vertex $F^0_{rep}$, where $\lambda$ regularity implies that the $Q$-exact difference between the sets $F^n$ and $F^0_{rep}$ is a sum of vertices with coefficients carrying strictly non-negative powers of $\lambda$.

Indeed, if $F^0$ is $Q^{fl}$ exact, i.e., $F^0 = Q^{fl} G$, it can be removed by a trivial shift by $QG$. Assume that an even set starts with a $Q^{fl}$ nontrivial $F^0$ and ends up with a $Q^{sub}$ closed element $\lambda^{2n_{max}} F^{2n_{max}}$. Since $H(Q^{sub}) = 0$, it is $Q^{sub}$ exact

$$F^{2n_{max}} = Q^{sub} F^{2n_{max} - 1}$$

(4.50)
and hence can be removed by a trivial shift by $Q\lambda^{2n_{\text{max}}-2}F^{2n_{\text{max}}-1}$. The $\lambda$-regularity condition stops the process when there remains a single element $F^0_{\text{rep}}$ that may differ from $F^0$ by some $Q^{fl}$ exact form and satisfies

$$Q^{fl}F^0_{\text{rep}} = 0, \quad Q^{sub}F^0_{\text{rep}} = 0, \quad F^0_{\text{rep}} \in H(Q^{fl}).$$

(4.51)

By construction, $F^n$ and $F^0_{\text{rep}}$ are $\lambda$-regularly equivalent. □

**Corollary 4.1**

A nontrivial off-shell Minkowski vertex $F^{fl} \in H(Q^{fl})$ admits a $\lambda$–regular deformation to $AdS_d$ if and only if its $H(Q^{fl})$ cohomology class contains a $Q^{sub}$–closed element. In other words, by adding total derivatives in Minkowski space, represented by $Q^{fl}$ exact forms, it should be possible to achieve that $Q^{sub}F^{fl} = 0$. In this case, the $AdS_d$ deformation is given by $F^{fl}$ itself. If a Minkowski vertex $F^{fl}$ is not equivalent to some pure vertex, no its $\lambda$–regular off-shell deformation to $AdS_d$ exists, i.e., $F^{fl}$ belongs to an off-shell odd set.

The following useful Lemmas hold

**Lemma 4.2**

Any off-shell pure $F^{fl}$ can be interpreted as a top element of an off-shell odd set. Indeed, the conditions that $F^{fl}$ is $Q^{sub}$ closed and belongs to $H(Q^{fl})$ implies that it can be identified with some $F^{2n_{\text{max}}+1}$. The existence of $F^{2n_{\text{max}}}$ follows from the triviality of $H(Q^{sub})$.

The rest relations in (4.42) is easy to obtain by induction. □

**Lemma 4.3**

Let $F^n$ form an even or odd off-shell set. Then $F^{2m+1}$ is $Q$-exact for any $m$.

This follows from the relations

$$F^{2n+1} = Q^{sub}F^{2n} = \lambda^{-2}(QF^{2n} - Q^{fl}F^{2n}) = \lambda^{-2}(QF^{2n} - Q^{sub}F^{2n-2})$$

$$= Q(\lambda^{-2}F^{2n} - \lambda^{-4}F^{2n-2}) + \lambda^{-2}F^{2n-3} = \ldots$$

(4.52)

along with the fact that for the case of forms valued in finite-dimensional $o(d-1,2)$–modules, all odd and even sets can contain at most a finite number of non-zero terms. □

It should be stressed that **Lemma 4.3** relaxes the $\lambda$-regularity condition.

From **Corollary 4.1** along with **Lemmas 4.2, 4.3** we obtain

**Theorem 4.1**

Except for vertices, that belong to $H(Q^{sub})$, if an off-shell $AdS_d$ deformation of a strictly positive off-shell Minkowski vertex exists, it is trivial in $AdS_d$.

This conclusion agrees with the related discussion of the structure of cohomology of hyperboloid in Section 3.5, extending it to any off-shell vertices including the curvature-dependent ones. Note that the $Q$–exact representation of the $AdS$ deformation of an off-shell Minkowski vertex is not $\lambda$–regular and hence does not apply to Minkowski space.

Although the off-shell analysis may seem to have little relation to interesting physical problems, in practice it does since it controls the curvature–independent part of any vertex.

### 4.3.2 On-shell case

On-shell analysis should be performed in the space of on-shell vertices $V$ where all terms, that are zero by virtue of FOST, are ignored. Although the operators $Q^{sub}$ and $Q^{fl}$ properly
act in $\mathcal{V}$, the cohomology of $H(Q_{\text{sub}}, \mathcal{V})$ becomes nontrivial.

Let us first consider the case of curvature-independent vertices $F(\omega)$. By (4.26), $Q^{fl} = Q^{\text{top}} + Q^{\text{cur}}$. $Q^{\text{top}}$ does not change the number of connections and curvatures, while $Q^{\text{cur}}$ decreases the number of connections and increases the number of curvatures by one. Writing

$$Q = Q' + Q^{\text{cur}}, \quad Q' = Q^{\text{top}} + \lambda^2 Q_{\text{sub}}$$

we see that $Q'$ acts on the space of curvature-independent forms, while $Q^{\text{cur}}$ maps the curvature-independent forms to curvature-dependent ones. On the other hand, on-shell conditions may only affect the analysis of the curvature-dependent terms. Hence, the analysis of the curvature-independent part is fully off-shell with the substitution of $Q^{\text{top}}$ in place of $Q^{fl}$. Applying the results of the previous section to curvature-independent vertices, we conclude that any nontrivial curvature-independent vertex, that admits an AdS deformation, should be represented by a fundamental curvature-independent form $F(\omega)$ that satisfies

$$F(\omega) \in H(Q^{\text{top}}), \quad Q_{\text{sub}} F(\omega) = 0.$$  \hspace{1cm} (4.54)

From Theorem 4.1 it follows that

$$F(\omega) = Q' \left( \sum \lambda^{-2n} F_{2n} \right)$$

for some set of forms $F_{2n}$. Hence

$$F(\omega) = (Q - Q^{\text{cur}}) \left( \sum \lambda^{-2n} F_{2n} \right) = \left( d - Q^{\text{cur}} \right) \left( \sum \lambda^{-2n} F_{2n} \right).$$

Since $Q^{\text{cur}}$ brings in a power of curvature, this implies that any gauge invariant non-genuine curvature-independent vertex, that admits a deformation to $AdS_d$, is equivalent to the curvature-dependent vertex

$$- \sum \lambda^{-2n} Q^{\text{cur}} F_{2n}.$$ \hspace{1cm} (4.57)

This provides an alternative proof of the statement of Section 3.5.

Another interesting consequence of Theorem 4.1 is that any nontrivial vertex $F(\omega)$ must be $Q^{\text{cur}}$-closed on-shell but not off-shell,

$$Q^{\text{cur}} F(\omega) \neq 0, \quad Q^{\text{cur}} F(\omega) \sim 0.$$ \hspace{1cm} (4.58)

The first condition avoids that $F(\omega)$ is trivial in AdS$_d$ by Theorem 4.1. The second one implies that $F(\omega)$ is on-shell $Q$-closed.

Gauge invariant vertices of all orders in AdS$_d$ and Minkowski space are classified by $H(Q|\mathcal{V})$ and $H(Q^{fl}|\mathcal{V})$ which identify vertices of any order of nonlinearity associated with lower powers of independent coupling constants which can be introduced in the theory. Comprehensive analysis of this problem is, however, beyond the scope of this paper.

5 Examples

Vertex complex is useful for various problems, including analysis of the free HS action. In the rest of this paper wedge products are implicit.
5.1 Quadratic action

General variation of the quadratic action (2.27) is
\[ \delta S^2_s = (-1)^{d+1} \int_{M^d} \sum_{p} a(s, p) \left( Q^{top} + \lambda^2 Q^{sub} \right) V_{C_1} \cdots V_{C_{2(s-2-p)}} \]
\[ G^{A_1A_2A_3A_4} \delta \omega_{A_1B(s-2)}, \ A_2 \ C^{(s-2-p)D(p)} R_{A_3} B(s-2), \ A_4 \ C^{(s-2-p)D(p)} . \]  
\[ (5.1) \]

With the help of Eqs. (4.5) and (4.6) and using the Young and tracelessness properties of the HS connections this gives
\[ \delta S^2_s = -\frac{1}{d-3} \int_{M^d} \left( \sum_{p=1}^{s-2} a(s, p-1)(s-1-p) - \lambda^2 \sum_{p=0}^{s-2} a(s, p)(d-7+2(s-p)) \frac{s-p}{s-p-1} \right) \]
\[ V_{C_1} \cdots V_{C_{2(s-p)-3}} G^{A_1A_2A_3} \left( \delta \omega_{A_1B(s-2)}, \ A_2 \ C^{(s-2-p)D(p)} R_{A_3} B(s-2), \ A_4 \ C^{(s-1-p)D(p)} + \delta \omega \leftrightarrow R \right) . \]
\[ (5.2) \]

The extra field decoupling condition \[ [29, 31] \] requires the variation over all extra fields (those that are contracted with less than \( s-2 \) compensator fields) be identically zero. This is the case provided that all terms with \( p > 0 \) in the variation (5.2) vanish, which is achieved for the coefficients (2.28). In this case the nonzero part of the variation is
\[ \delta S^2_s = \lambda^2 a(s, 0) \frac{s(d-7+2s)}{(s-1)(d-3)} \int_{M^d} V_{C_1} \cdots V_{C_{(2s-3)}} G^{A_1A_2A_3} \left( \delta \omega_{A_1B(s-2)}, \ A_2 \ C^{(s-2)} R_{A_3} B(s-2), \ A_4 \ C^{(s-1)} + \delta \omega \leftrightarrow R \right) . \]
\[ (5.3) \]

Nontrivial field equations associated with the variations over the Lorentz-like and frame-like fields give, respectively, the torsion-like constraint, which expresses the Lorentz-like field via derivatives of the frame-like field, and dynamical equations equivalent to those of the Fronsdal model \[ [11] \] (for more detail see \[ [31] \]).

Note that the action \( S^2_s \) with the coefficients (2.28) is “almost topological” in the sense that its variation over almost all connections, namely extra field connections, is identically zero. This property, known since \[ [29] \], suggests the idea that, in a more general setup, the HS action is a kind of anomalous topological action.

5.2 Spin two cubic vertices

Let us apply the vertex complex techniques to the analysis of spin two cubic vertices. The \( d \)-form vertices
\[ B_1 = G^{A_1\ldots A_5} V^C R_{A_1,A_2} R_{A_3,A_4} \omega_{A_5,C} \]
\[ (5.4) \]
and
\[ B_2 = G^{A_1\ldots A_4} R_{A_1,A_2} \omega_{A_3,A_4} \omega_{A_5,B} \]
\[ (5.5) \]
are on-shell pure hence being gauge invariant both in Minkowski space and in \( AdS_d \).
Indeed, using the on-shell conditions (2.32), $B_1$ can be represented in the form

$$B_1 = Q^{\text{sub}} U,$$

$$U = -\frac{1}{2}(-1)^d G^{A_1\ldots A_6} R_{A_1,A_2} R_{A_3,A_4} \omega_{A_5,A_6}. \quad (5.6)$$

Obviously,

$$Q^{\text{top}} U = 0, \quad Q^{\text{cur}} U = -\frac{1}{2} G^{A_1\ldots A_6} R_{A_1,A_2} R_{A_3,A_4} R_{A_5,A_6}. \quad (5.7)$$

From anticommutativity of the differentials (4.10) and on-shell conditions (2.32) it follows that $B_1$ is $Q^{\text{fl}}$–closed. Since $B_1$ is $Q^{\text{sub}}$–exact, it is pure. $B_2$ is also $Q^{\text{fl}}$–closed and $Q$–closed. Indeed, $B_2$ is $Q^{\text{top}}$–closed because it does not contain the compensator field. It is on-shell $Q^{\text{sub}}$–closed by virtue of the on-shell conditions (2.39) and (2.32). On the other hand,

$$Q^{\text{cur}} B_2 = 2(-1)^d G^{A_1\ldots A_4} R_{A_1,A_2} R_{A_3,B} \omega_{A_4,B}. \quad (5.8)$$

This is zero by virtue of the on-shell conditions (2.34). Indeed, the identity

$$G[^{[A_1\ldots A_4} R_{A_1,A_2} R_{A_3,A_5],\omega_{A_4,A_5} \sim 0 \quad (5.9)$$

just gives the right-hand side of Eq. (5.8).

A particular combination of $B_1$ and $B_2$ results from the cubic part of the action (2.12). Namely, $B_1$ comes from the part of the action (2.12) that contains the linearized part of the vielbein in $G^{A_1\ldots A_4}$ while $B_2$ originates from the nonlinear part of the curvature. Each of the vertices $B_1$ and $B_2$ contains up to four space-time derivatives of the vielbein. $B_1$, each curvature (which is the linearized Riemann tensor) contains two derivatives while the vielbein $V^{C\omega_{A_5,C}}$ contains no derivatives. In $B_2$, the curvature contains two derivatives while each of the connections contains one derivative.

In Minkowski space we expect to have a vertex with four derivatives, resulting from the highest derivative term of the action (2.12) but, according to general results of Metsaev [8], there is no room for two independent vertices with four derivatives. This implies that some combination of $B_1$ and $B_2$ should be $Q^{\text{fl}}$ exact. Indeed, it is not hard to see that

$$(d - 4)B_1 - 3B_2 \sim (-1)^d Q^{\text{fl}}(E_2 - (d - 4)E_1), \quad (5.10)$$

where

$$E_1 = G^{A_1\ldots A_5} V^C R_{A_1,A_2} \omega_{A_3,A_4} \omega_{A_5,C}, \quad E_2 = G^{A_1\ldots A_4} \omega_{A_1,B} R_{A_2,A_3} \omega_{A_4,B}. \quad (5.11)$$

However, being equivalent in Minkowski space, the vertices $B_1$ and $B_2$ are not equivalent in $AdS_d$. Indeed, rewriting (5.10) in the form

$$(d - 4)B_1 - 3B_2 \sim (-1)^d (Q - \lambda^2 Q^{\text{sub}})(E_2 - (d - 4)E_1), \quad (5.12)$$

we see that $AdS$ deformation of the vertex $(d - 4)B_1 - 3B_2$, that was trivial in Minkowski case, gives rise to the $Q$-closed vertex

$$V_3 = \frac{1}{2}(-1)^d Q^{\text{sub}}(E_2 - (d - 4)E_1). \quad (5.13)$$
Being $Q_{\text{sub}}$ exact, $V_3$ is pure. Its explicit form is

$$V_3 = (d-3)G^{A_1...A_4}V^C R_{A_1,A_2,A_3,A_4} + G^{A_1A_2A_3}V^C (\omega_C, D\omega_{A_1}, A_2\omega_{A_3}, D + \omega_{A_1}, D\omega_{A_2}, \omega_{A_3}).$$

V$_3$ contains up to two derivatives, representing the cubic vertex of the Einstein action with the cosmological term.

Thus, in accordance with (1.1), we have one spin two cubic vertex with two derivatives and another one with four derivatives. (The vertex with six derivatives is not on the list because it is not strictly positive being built from three Weyl tensors.)

The spin two example illustrates the general phenomenon that AdS deformation of a higher-derivative vertex that was trivial in Minkowski space may give rise to a nontrivial lower-derivative vertex in AdS$_d$.

### 5.3 Spin three cubic vertices

In this section, we present results of the vertex complex analysis of spin three cubic vertices. We let the spin three gauge connections $\omega _{AA,BB}$ carry matrix indices, looking for vertices of the form

$$tr(\omega ^{-} \{\omega ^{-}, \omega ^{-}\}), \quad tr(R^{-}_{1} \{\omega ^{-}, \omega ^{-}\}), \quad tr([R^{-}_{1}, R^{-}_{1}] \omega ^{-}),$$

where trace is over matrix indices. In this section all objects are multiplied as matrices. Note that the exterior algebra anticommutativity of one-forms transforms the anticommutators in $\omega$ into matrix commutators of components of the spin three gauge fields.

#### 5.3.1 Vertex with three derivatives

Consider the following $d$-form vertex built from three spin three fields

$$F_3 = G^{A_1A_2A_3}V^C (3)tr\left(\omega_{A_1B,A_2E}(-2(\omega_{A_3B}, C^D, \omega^E_D, C^{(2)}) - (\omega_{A_3D, C_D^B, \omega^E_D, C^{(2)})}ight) + \omega_{A_1B, FC}(2(\omega_{A_2B}, C_G, \omega_{A_3F}, G_C) + \frac{4}{3}(\omega_{A_2G}, B_C, \omega_{A_3F}, G_C)).$$

As explained in Appendix A, $F_3$ belongs to $Q^{\top}$ cohomology. It is obviously $Q^\text{cur}$ closed on shell by virtue of (2.32) and (2.39). Thus, $F_3$ (5.16) represents on-shell $Q^{\text{fl}}$ cohomology,

$$Q^{\text{fl}} F_3 \sim 0.$$

As explained in Appendix A, $F_3$ belongs to $Q^{\top}$ cohomology. It is obviously $Q^\text{cur}$ closed on shell by virtue of (2.32) and (2.39). Thus, $F_3$ (5.16) represents on-shell $Q^{\text{fl}}$ cohomology,

$$Q^{\text{fl}} F_3 \sim 0.$$
precisely what is expected from the Berends, Burgers, van Dam (BBD) vertex [3], which we therefore identify with $F_3$. Note that, being of Chern-Simons type, the vertex $F_3$ is nontrivial in Minkowski space.

A simple computation, that only uses the symmetry properties of connections, gives

$$Q^{\text{sub}} F_3 = 0.$$  \hspace{1cm} (5.18)

Hence, the vertex $F_3$ (5.19) is pure, remaining gauge invariant in $AdS_d$. Although, the $AdS_d$ deformation of the BBD vertex keeps the same form in the frame-like formalism, this does not imply that this is true in the metric-like formalism. In terms of Fronsdal fields, the vertex should necessarily be corrected by the $\lambda$-dependent lower-derivative terms resulting from (2.29) due to dependence of the linearized curvatures on $\lambda$.

The question whether or not $F_3$ remains nontrivial in $AdS_d$ has to be reconsidered, however. The $Q^{\text{sub}}$ closure condition (5.18) implies that it is $Q^{\text{sub}}$ exact. Indeed, it is easy to check that

$$F_3 = Q^{\text{sub}} F_4,$$  \hspace{1cm} (5.19)

where

$$F_4 = -(-1)^d \frac{d-3}{6(d-1)} G^{A_1 A_2 A_3 A_4} V^{C(2)} \tr \left( \omega_{A_1 B, A_2 F} \left( 4 \{ \omega_{A_3 C, B G}, \omega_{A_4 C, F G} \} + 2 \{ \omega_{A_3 B, C G}, \omega_{A_4 F, C G} \} - \{ \omega_{A_3 B, A_4 D}, \omega_{F D, C(2)} \} \right) \right).$$  \hspace{1cm} (5.20)

The label 4 refers to the maximal number of derivatives of frame-like fields in $F_4$.

From Eq. (4.3) we have

$$F_3 = \lambda^{-2} d F_4 - \lambda^{-2} Q^{\text{fl}} F_4.$$  \hspace{1cm} (5.21)

This implies that the vertex $F_3$ with three derivatives is equivalent to the $AdS$ deformation of a trivial ($Q^{\text{fl}}$-exact) Minkowski vertex $Q^{\text{fl}} F_4$ with five derivatives. That $Q^{\text{fl}} F_4$ indeed describes an $AdS$ deformation of a trivial flat vertex follows from the fact that it is pure because

$$Q^{\text{sub}} Q^{\text{fl}} F_4 = -Q^{\text{fl}} F_3 \sim 0.$$  \hspace{1cm} (5.22)

We see that choosing different representatives in the space of Minkowski vertices and deforming them to (A)dS may give rise to essentially different vertices that differ by lower-derivative terms. Another important interpretation of the relation (5.21) is that a combination of two pure vertices with different $G$-grades can constitute an exact form.

Reconstruction of the odd vertex set starting from $F_3$ continues as follows. One obtains

$$F_5 = Q^{\text{top}} F_4 = -\frac{1}{3(d-1)} G^{A_1 A_2 A_3} V^{C} \tr \left( \omega^{B(2)}_{C, G} \left( 2 \{ \omega_{A_1 D, A_2 B}, \omega_{A_3 B, G D} \} + 4 \{ \omega_{A_1 D, A_2 B}, \omega_{A_3 G, B D} \} \right) \right)$$

$$+ \omega_{A_1 B, C G} \left( \{ \omega_{A_3 B, F D}, \omega_{A_4 G, F D} \} + 2 \{ \omega_{A_2 F, B D}, \omega_{A_4 F, G D} \} \right),$$  \hspace{1cm} (5.23)

$$Q^{\text{cur}} F_4 = -\frac{(d-3)}{3(d-1)} G^{A_1 A_2 A_3 A_4} V^{C(2)} \tr \left( R_{A_1 B, A_2 F} \left( 2 \{ \omega_{A_3 C, B G}, \omega_{A_4 C, F G} \} + \{ \omega_{A_3 B, C G}, \omega_{A_4 F, C G} \} - \{ \omega_{A_3 B, A_4 D}, \omega_{F D, C(2)} \} \right) \right).$$  \hspace{1cm} (5.24)
From (5.18) and (5.19) it follows that $Q_{\text{sub}} F_5 = 0$. Hence, $F_5 = Q_{\text{sub}} F_6$. One can check that $F_6$ can be chosen in the form

$$F_6 = (-1)^d \frac{1}{6(d-1)} G^{A_1 A_2 A_3 A_4} tr \left( \omega_{A_1}^D A_2 B \left( 2 \{ \omega_{A_3G,F,D,B}^G , \omega_{A_4}^G , B \} + \{ \omega_{A_3D,F,G} , \omega_{A_4B} , F \} \right) \right).$$

(5.25)

At this stage the process stops since $Q_{\text{top}} F_6 = 0$.

As a result, we obtain a particular realization of the formula (4.57)

$$F_3 = d(\lambda^{-2} F_4 - \lambda^{-4} F_6) - \lambda^{-2} Q_{\text{cur}} F_4 + \lambda^{-4} Q_{\text{cur}} F_6. \quad (5.26)$$

Manifest form of $Q_{\text{cur}} F_6$ results from direct differentiation of (5.25) according to (1.7).

Thus, the BBD vertex in $AdS_d$ is equivalent to the higher-derivative curvature-dependent vertex $Q_{\text{cur}} \left( -\lambda^{-2} F_4 + \lambda^{-4} F_6 \right)$. This is a particular realization of the general phenomenon explained in Sections 3.3 and 4.3.2 that all non-genuine Chern-Simons vertices, that are $Q$-closed in $AdS_d$, are equivalent to some curvature-dependent vertices.

### 5.3.2 Vertex with five derivatives

The nontrivial spin three vertex with five derivatives in Minkowski space was analyzed by Bekaert, Boulanger and Cnockaert in [14]. We argue that its $AdS_d$ deformation has the following $\lambda$-independent form in the frame-like formalism

$$H_5 = G^{A_1 A_2 A_3 A_4} V^C V^C tr \left( R_{A_1}^G A_2^D \left( 2 \{ \omega_{A_3G,C,D}^C , \omega_{A_4}^C \} - 3 \{ \omega_{A_3C,D}^C , \omega_{A_4C} \} \right) + 6 \{ \omega_{A_3}^C , \omega_{A_4}^C \} + 8 \{ \omega_{A_3}^C , \omega_{A_4}^C \} + \{ \omega_{A_3}^C , \omega_{A_4}^C \} \right). \quad (5.27)$$

Clearly $H_5$ contains up to $3 \times 2 + 1 - 2 = 5$ derivatives (+1 for the curvature and -2 for two compensators). Using the on–shell conditions of the type (2.32) and (2.34) listed in Appendix B, it is straightforward, although somewhat lengthy, to check that

$$Q_{\text{top}} H_5 \sim 0, \quad Q_{\text{sub}} H_5 \sim 0, \quad Q_{\text{cur}} H_5 \sim 0. \quad (5.28)$$

That $H_5$ is not $Q^R$ exact one can see as follows. Suppose that $H_5$ can be represented in the form $Q^R(U_{\text{CS}} + U_{\text{cur}})$ where $U_{\text{CS}}$ is curvature independent while $U_{\text{cur}}$ is linear in the curvature. Since $H_5$ is linear in curvatures, it is necessary that $Q_{\text{top}} U_{\text{CS}} = 0$. If $U_{\text{CS}} = Q_{\text{top}} V_{\text{CS}}$, it can be represented in the form $U_{\text{CS}} = (Q - Q_{\text{cur}}) V_{\text{CS}}$ and hence its contribution to $Q^R U_{\text{CS}}$ is equivalent to the contribution of $U_{\text{cur}} = -Q_{\text{cur}} V_{\text{CS}}$. As a result, a nontrivial contribution can only result from those $U_{\text{CS}}$, that belong to the cohomology of $Q_{\text{top}}$. An elementary few page computation shows that the respective cohomology is zero. Hence it is enough to consider curvature-dependent $U = U_{\text{cur}}$. There is a unique option

$$U_{\text{cur}} = G^{A_1 \ldots A_5} V^C V^C tr \left( R_{A_1}^F A_2^G \{ \omega_{A_3F,A_4} , \omega_{A_5G,C} \} \right). \quad (5.29)$$
It is important that $Q^{\text{top}}U^\text{cur}$ contains the term

$$G^{A_1A_2A_3A_4}V^C V^C \text{tr}\left(R_{A_1} G^{DF} \omega_{A_2A_3}^{GA_4} \omega_{A_4FCC}\right)$$

(5.30)

with some nonzero coefficient. Such a term can never appear in the consequences of on-shell conditions (2.34) which all have the structure

$$G[A_1...A_4V^C V^C \text{tr}\left(R_{A_1...A_5} \omega_{A_2...A_3} \omega_{A_4...A_5}\right)$$

(5.31)

not allowing any of the connections $\omega$ to carry two indices $C$ because of the antisymmetrization over $A_2, A_3$ or $A_4, A_5$. This proves that the vertex $H_5$, that contains no terms with the curvature carrying a single index $A_i$, belongs to $H^\text{fl}(Q_{\text{top}}|V)$ and, since $U^\text{CS} = 0$, to $H^\text{fl}(Q^\text{fl}|V)$.

Thus, $H_5$ is a nontrivial pure vertex, providing the $AdS$ deformation of the flat BBC vertex [14]. Again, one can check that $H_5$ is equivalent to the $AdS$ deformation of a trivial Minkowski vertex with seven derivatives.

6 Non-Abelian vertices in $AdS_d$ from HS algebra

In this section we generalize the construction of non-Abelian vertices proposed originally for the case of $AdS_4$ in [6] to $AdS_d$ with any $d$. This construction deforms Abelian HS gauge transformations of free HS theory to the non-Abelian HS symmetry.

6.1 Higher-spin algebra in $AdS_d$

The simplest HS algebra in any dimension was originally found by Eastwood in [73] where it was realized as conformal HS algebra in $d-1$ dimensions. Here we recall an equivalent alternative definition of the HS algebra along with its further generalizations that include inner symmetries [41, 74], which is most relevant to the bulk HS theory in $AdS_d$.

Consider the associative Weyl algebra $A_{d+1}$ generated by the oscillators $\hat{Y}_i^A$, where $i = 1, 2$ and $A = 0, 1, \ldots d$, that satisfy the commutation relations

$$[\hat{Y}_i^A, \hat{Y}_j^B] = \hbar \epsilon_{ij} \eta^{AB},$$

(6.1)

where $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{12} = \epsilon^{12} = 1$. The invariant metrics $\eta_{AB} = \eta_{BA}$ and symplectic form $\epsilon^{ij}$ of $o(d-1, 2)$ and $sp(2)$, respectively, are used to raise and lower indices in the usual manner

$$A^A = \eta^{AB} A_B, \quad a^i = \epsilon^{ij} a_j, \quad a_i = a^j \epsilon_{ij}.$$  

(6.2)

We use mostly minus signature of the $o(d-1, 2)$ invariant metric $\eta_{AB}$. The “Planck constant” $\hbar$ is an arbitrary parameter introduced for the future convenience. We will work with Weyl-ordered basis of $A_{d+1}$ represented by the operators totally symmetric under the exchange of $\hat{Y}_i^A$'s. A generic element of $A_{d+1}$ then has the form

$$f(\hat{Y}) = \sum_{p=0}^{\infty} \phi_{A_1...A_p}^{i_1...i_p} \hat{Y}_{i_1}^{A_1} ... \hat{Y}_{i_p}^{A_p}.$$ 

(6.3)
where $\phi_{i_1...i_p}$ is symmetric under the exchange $(i_k, A_k) \leftrightarrow (i_l, A_l)$ for any $k$ and $l$.

Equivalently, one can define basis elements $S^{A_1...A_m,B_1...B_n}$ that are completely symmetrized products of $m$ $\hat{Y}^A_1$'s and $n$ $\hat{Y}^B_2$'s (e.g. $S^{A,B} = \frac{1}{2} \{ \hat{Y}^A_1 , \hat{Y}^B_2 \}$), writing the generic element as

$$f(\hat{Y}) = \sum_{m,n} f_{A_1...A_m,B_1...B_n} S^{A_1...A_m,B_1...B_n},$$

(6.4)

where the coefficients $f_{A_1...A_m,B_1...B_n}$ are symmetric in the indices $A_i$ and (separately) $B_j$.

As a consequence of (6.1),

$$T^{AB} = -T^{BA} = \frac{1}{4} \{ \hat{Y}^A_1 , \hat{Y}^B_1 \}$$

(6.5)

form the $o(d-1,2)$ algebra

$$[T^{AB}, T^{CD}] = \frac{\hbar}{2} \left( \eta^{BC} T^{AD} - \eta^{AC} T^{BD} - \eta^{BD} T^{AC} + \eta^{AD} T^{BC} \right).$$

The operators

$$t_{ij} = t_{ji} = \frac{1}{2} \{ \hat{Y}^A_i , \hat{Y}^B_j \} \eta_{AB}$$

(6.6)

generate $sp(2)$. $T^{AB}$ and $t_{ij}$ commute

$$[T^{AB}, t_{ij}] = 0,$$

(6.7)

forming a Howe dual pair $o(d-1,2) \oplus sp(2)$.

Consider the subalgebra $\mathcal{S}$ of those elements $f(\hat{Y})$ of the complex Weyl algebra $A_{d+1}(\mathbb{C})$, that are invariant under $sp(2)$, i.e. $f(\hat{Y}) \in \mathcal{S}$: $[f(\hat{Y}), t_{ij}] = 0$. Replacing $f(\hat{Y})$ by its Weyl symbol $f(Y)$, which is the ordinary function of commuting variables $Y$ that has the same power series expansion as $f(\hat{Y})$ in the Weyl ordering, the $sp(2)$ invariance condition takes the form

$$\left( \epsilon_{kj} Y^A_i \frac{\partial}{\partial Y^A_k} + \epsilon_{ki} Y^A_j \frac{\partial}{\partial Y^A_i} \right) f(Y^A) = 0.$$  

(6.8)

This condition implies that the coefficients $f_{A_1...A_n,B_1...B_m}$ vanish unless $n = m$, and the nonvanishing coefficients carry irreducible representations of $gl(d+1)$ corresponding to two-row rectangular Young diagrams. The $sp(2)$ invariance condition means in particular that (the symbol of) any element of $\mathcal{S}$ is an even function of $Y^A_i$.

The associative algebra $\mathcal{S}$ is not simple, containing the two-sided ideal $\mathcal{I}$ spanned by elements of $\mathcal{S}$ that can be represented in the form $g = t_{ij} g^{ij} = g^{ij} t_{ij}$. Due to the definition of $t_{ij}$ (6.6), all traces of two-row Young tableaux are contained in $\mathcal{I}$. As a result, the associative algebra $\mathcal{A} = \mathcal{S}/\mathcal{I}$ contains only traceless two-row rectangular tableaux. Its general element is

$$f(\hat{Y}) = \sum_n f_{A_1...A_n,B_1...B_n} S^{A_1...A_n,B_1...B_n},$$

(6.9)

where

$$f(A_1...A_n,A_{n+1})B_{2...B_n} = 0, \quad f^{A_1...A_{n-2}C}_{B_1...B_n} = 0.$$  

(6.10)
Let $\circ$ be the product law in $\mathcal{A}$. Consider the complex Lie algebra $h_C$ resulting from the associative algebra $\mathcal{A}$ with the commutator
\[
[f, g] = f \circ g - g \circ f
\] (6.11)
as a Lie bracket. It admits several inequivalent real forms $h_R$. The particular real form, that corresponds to a unitary HS theory, is denoted $hu(1|2;[d - 1, 2])$. This notation refers to the Howe dual pair $sp(2) \oplus o(d - 1, 2)$ and to the fact that the related spin one Yang-Mills subalgebra is $u(1)$. The algebra $hu(1|2;[d - 1, 2])$ consists of elements that satisfy the reality condition
\[
(f(\hat{Y}))^\dagger = -f(\hat{Y})
\] (6.12)
where $\dagger$ is the involution of the complex Weyl algebra defined by the relation
\[
(\hat{Y}^A_i)^\dagger = i\hat{Y}^A_i.
\] (6.13)

For the Weyl ordering prescription, reversing the order of the oscillators has no effect so that $(\bar{f}(\hat{Y}))^\dagger = \bar{f}(i\hat{Y})$ where the bar means complex conjugation of the coefficients in the expansion (6.3)
\[
\bar{f}(\hat{Y}) = \sum_{p=0}^{\infty} \sum_{i_1,...,i_p} \theta_{A_1...A_p}^{i_1...i_p} \hat{Y}^A_{i_1} ... \hat{Y}^A_{i_p}.
\] (6.14)

As a result, the reality condition (6.12) implies that the coefficients in front of the spin-$s$ generators $S_{A_1...A_{s-1},B_{1}...B_{s-1}}$ with even and odd $s$ are, respectively, real and pure imaginary. In particular, the spin two generator enters with real coefficient.

The structure coefficients of the product law
\[
(A \circ B)^{A(n),B(n)} = \sum_{kl} f_{C(k),D(k);F(l),G(l)}^{A(n),B(n)}(\hbar) A^{C(k),D(k)} B^{F(l),G(l)}
\] (6.15)
can be systematically derived from the definition of $\mathcal{A}$ (we use notations where the numbers of different groups of symmetrized indices are indicated in brackets). We will not focus on their precise form because, as explained in Section 3.3, it does not matter much at the cubic level. The important property is that the structure coefficients have the following scaling with respect to $\hbar$
\[
f_{C(k),D(k);F(l),G(l)}^{A(n),B(n)}(\hbar) = \hbar^{k+n-l} f_{C(k),D(k);F(l),G(l)}^{A(n),B(n)}
\] (6.16)
as follows from the rescaling $Y^A_i \to \hbar^{\frac{1}{2}} Y^A_i$ of the commutation relations (6.1) at $\hbar = 1$.

The construction of the action presented in this paper is based on general properties of the HS algebra $hu(1|2;[d - 1, 2])$, discussed in detail in [74]. The most important one is that $hu(1|2;[d - 1, 2])$ possesses the trace operation
\[
tr(A) = A_0 , \quad A_0 = A_{A(0),B(0)}.
\] (6.17)
where \(A_0\) is the \(o(d-1,2)\) singlet component of \(A\). This follows from the simple fact that 
\(hu(1|2;d-1,2)\) possesses the involutive antiautomorphism \(\rho\) that changes a sign of elements 
\(A_{A(n),B(n)}\) with odd \(n\), i.e., 
\[
\rho(A_{A(n),B(n)}) = (-1)^n A_{A(n),B(n)}.
\] (6.18)

From here it follows that the commutator \(A \circ B - B \circ A\) never contains a constant element and hence 
\[
\text{tr}(A \circ B) = \text{tr}(B \circ A),
\] (6.19)

which is the defining property of the trace operation. Indeed, a singlet representation of 
\(o(d-1,2)\) can only appear in the tensor product of equivalent irreducible 
\(o(d-1,2)\)–tensor modules. However, the commutator of two elements valued in equivalent modules is \(\rho\)–odd and hence does not contain a constant. (For example, that 
\(o(d-1,2)\) is a subalgebra of 
\(hu(1|2;d-1,2)\) implies that the commutator of two elements with 
\(n = 1\) gives another element of the same class.)

This gives 
\[
\text{tr}(A \circ B) = \sum_n f_n(d, \hbar) A_{C(n),D(n)} B_{C(n),D(n)},
\] (6.20)

where 
\[
f_{C(n),D(n);F(n),G(n)}^{A(0),B(0)}(\hbar) = f_n(d, \hbar) \Pi \underbrace{\eta_{CF} \eta_{DG} \cdots \eta_{CF} \eta_{DG}}_{n}
\] (6.21)

and \(\Pi\) is the projector to the two-row irreducible (i.e., traceless) Young diagrams. (Here we used that trace of the product of any two elements, that belong to inequivalent irreducible 
\(o(d-1,2)\)–tensor modules, is zero.) From (6.16) it follows that 
\[
f_n(d, \hbar) = \hbar^{2n} f_n(d, 1).
\] (6.22)

Explicit form of the coefficients \(f_n(d, 1)\) is not difficult to obtain from the identity 
\[
\text{tr}\{T_{CD}, A\}_o \circ B = \text{tr}(A \circ \{T_{CD}, B\}_o), \quad \{A, B\}_o = A \circ B + B \circ A.
\] (6.23)

Indeed, using the Weyl star-product and factoring out terms containing \(t_{ij}\), it is not hard to check that 
\[
\{g_{CD} S_{A_{1} \ldots A_{n}, B_{1} \ldots B_{n}}^{A_{1} \ldots A_{n}, B_{1} \ldots B_{n}}\}_o = 2 g_{A_{0} B_{0}} f_{A_{1} \ldots A_{n}, B_{1} \ldots B_{n}}^{A_{0} \ldots A_{n}, B_{0} \ldots B_{n}} + \frac{1}{2} n^2 \frac{d + 2n - 6}{d + 2n - 3} g_{CD} f_{CA_{2} \ldots A_{n}, DB_{2} \ldots B_{n}}^{A_{2} \ldots A_{n}, B_{2} \ldots B_{n}}.
\] (6.24)

Then, from (6.23) it follows that 
\[
f_n(d, 1) = \frac{(n!)^2 (\frac{d}{2} + n - 3)!(\frac{d}{2} - \frac{3}{2})!}{2^{2n} (\frac{d}{2} + n - \frac{3}{2})!(\frac{d}{2} - 3)!},
\] (6.25)

where the normalization is chosen so that \(f_0(d, 1) = 1.\)
The algebra $hu(1|2;[d-1,2])$ can be generalized to three series of HS algebras $hu(n|2;[d-1,2])$, $husp(2n|2;[d-1,2])$ and $ho(n|2;[d-1,2])$ with $n \geq 1$ that contain Yang-Mills algebras $u(n)$, $usp(2n)$ and $o(n)$, respectively, in the spin one sector $[11]$. Gauge fields of $hu(n|2;[d-1,2])$, $husp(2n|2;[d-1,2])$ and $ho(n|2;[d-1,2])$ are one-forms $\omega_{A(n),B(n)}(x)$ that are valued in the traceless two-row Young diagrams of $o(d-1,2)$ and carry additional matrix indices so that all fields of odd spins are in the adjoint representation of the Yang-Mills algebra while all fields of even spins are in the second rank tensor representation of the opposite symmetry, i.e., symmetric and antisymmetric for the orthogonal and symplectic algebras, respectively, and adjoint for the unitary algebra. In particular, $ho(1|2;[d-1,2]) \subset hu(1|2;[d-1,2])$ contains only generators of even spins, each in one copy. As such, $ho(1|2;[d-1,2])$ is the minimal HS algebra. It is a subalgebra of any other HS algebra.

The HS curvatures and gauge field transformations have the standard form

$$R_{A(n),B(n)} = d\omega_{A(n),B(n)}(x) + (\omega(x) \circ \omega(x))_{A(n),B(n)}, \quad (6.26)$$

$$\delta\omega_{A(n),B(n)} = d\varepsilon_{A(n),B(n)}(x) + ([\omega(x),\varepsilon(x)]_\circ)_{A(n),B(n)}, \quad (6.27)$$

where the product law $\circ$ combines the product in $Mat_n(\mathbb{C})$ with that in $Mat_n(\mathbb{C})$.

As explained in Section 2.2, a spin $s$ massless field is described by the gauge connections $\omega_{A(s-1),B(s-1)}$. Let $f_1$ and $f_2$ carry spins $s_1$ and $s_2$. The structure constants of $hu(1|2;[d-1,2])$ are such that the element $f_1 \circ f_2 - f_2 \circ f_1$ contains spins in the range

$$s_{\text{min}} \leq s \leq s_{\text{max}} \quad (6.28)$$

with

$$s_{\text{min}} = |s_1 - s_2| + \sigma, \quad s_{\text{max}} = s_1 + s_2 - \sigma \quad (6.29)$$

where $\sigma = 2$ and $\sigma = 1$ for HS algebras with Abelian and non-Abelian Yang-Mills subalgebras, respectively. This means that the bilinear terms in the spin $s$ curvature (6.26) contain the fields of spins $s_1$ and $s_2$ that respect (6.28), (6.29).

### 6.2 The case of AdS$_4$

The AdS$_4$ analysis of [6] used spinorial realization of the HS algebra [75] with a spin $s$ field described by a one-form $\omega_{\Phi_1,..\Phi_{2(s-1)}}(x)$ carrying $4d$ Majorana spinor indices $\Phi, \Lambda, \ldots = 1, 2, 3, 4$. (Equivalence of the spinorial and tensorial realizations of $4d$ HS algebras was shown in [74].) These are raised and lowered by the symplectic form $C_{\Phi\Lambda} = -C_{\Lambda\Phi}$

$$A^\Phi = C^{\Phi\Lambda} A_\Lambda, \quad A_\Phi = A^\Lambda C_{\Lambda\Phi}, \quad C^{\Phi\Phi} C_{\Pi\Lambda} = \delta^\Phi_\Lambda, \quad C_{\Phi\Lambda} = C_{\Phi,\Lambda}, \quad (6.30)$$

which is the charge conjugation matrix in four dimension. The AdS$_4$ symmetry algebra $sp(4|\mathbb{R}) \sim o(3,2)$ leaves $C_{\Phi\Lambda}$ invariant.

The form of the curvatures and gauge transformations is analogous to (6.26) and (6.27)

$$R_{\Phi(n)} = d\omega_{\Phi(n)}(x) + (\omega(x) \circ \omega(x))_{\Phi(n)}, \quad (6.31)$$

40
\[ \delta \omega \Phi(n) = d\epsilon \Phi(n)(x) + ([\omega(x), \epsilon(x)]_\Phi)\Phi(n) . \] (6.32)

However, because of absence of the factorization procedure in the spinorial realization of the AdS\(_4\) HS algebra, the structure coefficients in the product law have simple form, being expressed in terms of factorials so that

\[ R \Phi(n) = d\omega \Phi(n) + \sum_{p+q=n} \frac{1}{p!q!s!} \omega \Phi(p)\Lambda(s)\omega \Phi(q)^\Lambda(s) . \] (6.33)

(For simplicity we skip factors of \(i\) usually introduced to simplify the reality conditions.)

In the spinorial language, the compensator field can be realized as a non-degenerate imaginary symplectic form \(V\Phi\Lambda\) obeying the conditions

\[ V\Phi\Lambda = -V\Lambda\Phi, \quad V\Phi = 0, \quad \Omega V = \delta^\Lambda, \quad \sigma(V^\Phi\Lambda) = -V^\Phi\Lambda, \] (6.34)

where \(\sigma\) is an involutive antilinear map (i.e., \(\sigma\) conjugates complex numbers and \(\sigma^2 = I_d\)). This makes it possible to introduce two mutually conjugated projectors

\[ \Pi^\pm\Phi = \frac{1}{2}(\delta^\Lambda \pm V\Phi^\Lambda), \quad \Pi^\pm\Phi = \sigma(\Pi^\mp\Phi) . \] (6.35)

Conventional two-component spinor formalism results from \(C\Phi\Lambda\) and \(V\Phi\Lambda\) with the nonzero components

\[ C_{\alpha\beta} = V_{\alpha\beta} = \epsilon_{\alpha\beta}, \quad C_{\dot{\alpha}\dot{\beta}} = -V_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} \] (6.36)

(in this section lower case Greek indices are used for two-component spinors, \(\alpha, \beta = 1, 2\); \(\dot{\alpha}, \dot{\beta} = 1, 2\)). In these term, two-component spinors are projected out by \(\Pi_+\) and \(\Pi_-\)

\[ A_\alpha \sim \Pi_+ A_\Phi, \quad A_\dot{\alpha} \sim \Pi_- A_\Phi . \] (6.37)

As in the tensorial case, the Lorentz subalgebra \(sl_2(\mathbb{C}) \subset sp(4|\mathbb{R})\) leaves invariant \(V\Phi\Lambda\).

In two-component spinor notation, HS gauge connections are one-forms \(\omega_{\alpha_1...\alpha_n\beta_1...\beta_m}\) with \(n + m = 2(s - 1)\). The HS curvatures are

\[ R_{\alpha(n),\beta(m)} = d\omega_{\alpha(n),\beta(m)} + \sum_{p+q=n, u+v=m} \frac{1}{p!q!s!u!v!} \omega_{\alpha(p)\gamma(s),\beta(u)\dot{\gamma}(t)}\omega_{\alpha(q)\gamma(s),\beta(v)\dot{\gamma}(t)} . \] (6.38)

The AdS\(_4\) HS algebras with non-Abelian Yang-Mills symmetries were introduced in [76].

In [8] it was shown that cubic interactions for massless fields of all spins \(s > 1\) in AdS\(_4\) can be concisely described in terms of HS curvatures (6.38) by the action of the form

\[ S = \frac{1}{2} \sum_{n,m=0}^\infty \frac{\alpha(n, m)}{n! m!} \int_{M^4} R_{\alpha(n),\beta(m)} R^\alpha_{\alpha_1...\alpha_n,\beta_1...\beta_m} R^{\alpha_1...\alpha_n,\beta_1...\beta_m} . \] (6.39)

The idea was to obey two conditions by adjusting the coefficients \(\alpha(n, m)\).
First, at the linearized level, the action should amount to a sum of free frame-like HS actions of [29] equivalent to the Fronsdal actions [11] in AdS4. As shown in [29], this condition determines the coefficients $\alpha(n, m)$ up to an arbitrary $n + m$-dependent factor reflecting the ambiguity in spin-dependent overall coefficients in front of the free spin $s$ actions.

The second condition is that the part of gauge variation of the action under (6.27) bilinear in the linearized curvatures

$$R_{1,\alpha(n),\dot{\alpha}(m)} = d\omega_{\alpha(n),\dot{\alpha}(m)}(x) + [\omega_0(x), \omega(x)]_{\alpha(n),\dot{\alpha}(m)}$$

(6.40)

($\omega_0$ is the background connection of AdS4 which, skipping the label 0, consists of the background vierbein $e^{\alpha\dot{\alpha}}$ and Lorentz connection $\omega^{\alpha\beta}$, $\bar{\omega}^{\dot{\alpha}\dot{\beta}}$) should vanish on solutions of the free HS field equations. This condition implies that the gauge transformation (6.27) can be deformed by some curvature-dependent terms

$$\delta \omega \rightarrow \delta^{cor} \omega = \delta \omega + \Delta(R_1, \varepsilon)$$

(6.41)

in such a way that

$$\delta^{cor} S = O(\omega_1)^3.$$ (6.42)

In this case the action $S$ is said to be consistent up to the cubic order since, to cancel the remaining terms in the variation, one has to account quartic corrections to the action.\footnote{Note that the curvature-dependent deformation of the gauge transformation law resulting from gauging of the global symmetry algebra is typical for models that contain gravity (see, e.g., [77]). In particular, in pure gravity, diffeomorphisms can be represented as a combination of the gauged Poincare’ or AdS transformations with some curvature-dependent terms. This phenomenon is closely related to the form of gauge transformation law in the unfolded dynamics approach as discussed in Section [10]. (For a more detailed discussion see, e.g., [78] and references therein.)}

The second condition fixes the leftover ambiguity in the coefficients $\alpha(n, m)$ up to an overall constant

$$\alpha(n, m) = \alpha \varepsilon(n - m),$$

(6.43)

where

$$\varepsilon(n) = -\varepsilon(-n), \quad \varepsilon(n) = 1 \quad n > 0.$$ (6.44)

Its analysis in [6] was based on the 4d FOST of [29] which states that it is possible to impose such constraints on the higher connections $\omega_{\alpha_1...\alpha_n,\dot{\alpha}_1...\dot{\alpha}_m}$ with $|n - m| \geq 2$ which express them via derivatives of the dynamical fields and, together with the free field equations, imply

$$R_{1,\alpha_1...\alpha_n,\dot{\alpha}_1...\dot{\alpha}_m} \sim \delta^0_{\alpha} e^{\gamma_1}_{\beta} e^{\gamma_2}_{\dot{\beta}} C_{\alpha_1...\alpha_2s-2\gamma_1\gamma_2} + \delta^0_{n} e^{\gamma_1}_{\beta} e^{\gamma_2}_{\dot{\beta}} \bar{C}_{\dot{\alpha}_1...\dot{\alpha}_2s-2\gamma_1\gamma_2}, \quad n + m = 2(s - 1),$$

(6.45)

where the generalized Weyl tensors $C_{\alpha_1...\alpha_2s}$ and $\bar{C}_{\dot{\alpha}_1...\dot{\alpha}_2s}$ denote those components of the linearized curvatures $R_1$ that may remain non-zero when the field equations and constraints hold. (For example, in the case of spin two, $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ and $\bar{C}_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}$ are, respectively, the selfdual and antiselfdual parts of the 4d Weyl tensor.)
FOST (6.45) along with the fact that the HS algebra possesses a supertrace make the analysis of gauge invariance of the cubic HS action really simple. Indeed, the part of the gauge variation of the action (6.39) bilinear in fields is

$$\delta S = \sum_{n,m=0}^{\infty} \frac{\alpha(n,m)}{n! m!} \int_{M^4} [R_1, \varepsilon]_{\alpha_1...\alpha_n, \dot{\beta}_1...\dot{\beta}_m} R_1^{\alpha_1...\alpha_n, \dot{\beta}_1...\dot{\beta}_m}. \tag{6.46}$$

Using FOST (6.45) we see that there are three types of terms:

- holomorphic terms where linearized curvatures and gauge parameters carry only undotted indices
- antiholomorphic terms where linearized curvatures and gauge parameters carry only dotted indices
- mixed terms where linearized curvatures carry different types of indices.

The mixed terms vanish on-shell for any coefficients $\alpha(n,m)$ because the substitution of FOST into the variation contains the four-form $e^{\alpha_i} e^{\beta_j} e^q e^\gamma$ that is zero because, on the one hand, being built from the vierbein it is Lorentz invariant, while on the other hand it carries a nontrivial representation of the Lorentz algebra.

The holomorphic and antiholomorphic terms vanish because in these cases $\epsilon(n-m)$ can be replaced by a constant. The key fact is that the substitution of a constant in place of $\epsilon(n-m)$ brings the bilinear functional (6.39), (6.43) to the form $\text{str}(R \circ R)$. Indeed, as shown in [79], the star-product algebra admits a unique supertrace operation

$$\text{str} f = f_0, \tag{6.47}$$

where $f_0$ is the singlet component in the set $f = \{ f = f_{\alpha(n)\dot{\beta}(m)} \}$, such that

$$\text{str}(a \circ b) = \text{str}(b \circ a) \tag{6.48}$$

for any $a = \{ a^{\alpha(n)\dot{\beta}(m)} \}$, $b = \{ b^{\alpha(n)\dot{\beta}(m)} \}$ that are (anti)commuting for $n + m$ (odd)even. Explicitly,

$$\text{str}(a \circ b) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} a_{\alpha_1...\alpha_n, \dot{\beta}_1...\dot{\beta}_m} b^{\alpha_1...\alpha_n, \dot{\beta}_1...\dot{\beta}_m}, \tag{6.49}$$

which is just the expression (6.39) at $\alpha(n,m) = 1$.

As a result, in the holomorphic and antiholomorphic cases, the variation takes the form

$$\delta S \sim \text{str}([R_1, \varepsilon]\circ R_1), \tag{6.50}$$

which is zero by virtue of (6.48). Note that the action (6.39) describes interactions of bosons and fermions of all spins $s > 1$. The supertrace structure is important in the fermion sector.

Now we are in a position to show that the cubic HS action in any dimension can be analyzed analogously by virtue of FOST (2.29).

\footnote{In fact, the relation of this part of the proof to the existence of supertrace of the HS algebra was not explicitly mentioned in [6].}
6.3 Cubic action in $AdS_d$

Proceeding along the lines of the $AdS_4$ case, consider the action of the form (2.27) with the non–Abelian HS curvatures instead of the linearized ones

\[
S = \frac{1}{2} \int_{M^d} \sum_s \sum_{p=0}^{s-2} a(s, p) V_{C_1} \cdots V_{C_{2(s-2-p)}} G_{A_1 A_2 A_3 A_4} \nonumber \\
tr \left( R_{A_1 B_1 \cdots B_{s-2}, A_2 C_1 \cdots C_{s-2-p} D_1 \cdots D_p} A_4 C_{s-1-p} \cdots C_{2(s-2-p)} D_1 \cdots D_p \right). \quad (6.51)
\]

(Here $tr$ is the trace over matrix indices in the case of HS algebras with non-Abelian Yang-Mills symmetries.) The gauge variation of $S$ is

\[
\delta S = \int_{M^d} \sum_s \sum_{p=0}^{s-2} a(s, p) V_{C_1} \cdots V_{C_{2(s-2-p)}} V_{C_1} \cdots V_{C_{2(s-2-p)}} G_{A_1 A_2 A_3 A_4} \nonumber \\
tr \left( \left( [R, \varepsilon] \circ A_1 B_1 \cdots B_{s-2}, A_2 C_1 \cdots C_{s-2-p} D_1 \cdots D_p \right) R_{A_3 B_1 \cdots B_{s-2}, A_4 C_{s-1-p} \cdots C_{2(s-2-p)} D_1 \cdots D_p} \right). \quad (6.52)
\]

The part of the gauge variation bilinear in HS fields results from the variation (6.52) via replacement of the full HS curvatures $R$ by the linearized ones $R_1$. Since all terms proportional to the free massless field equations can be compensated by a deformation of the transformation law (6.41), to see whether or not some deformation of this type leaves the action invariant we can use FOST (2.29) which implies in particular that all components of the linearized curvatures, that are not orthogonal to the compensator $V_A$, can be neglected. This means that we should only consider the term with $p = s - 2$, i.e.,

\[
\delta S \sim \int_{M^d} \sum_s a(s, s-2) tr \left( \left( [R_1, \varepsilon] \circ b_1 \cdots b_{s-1}, c_1 \cdots c_{s-1} \right) R'_1 b_1 \cdots b_{s-1}, c_1 \cdots c_{s-1} \right), \quad (6.53)
\]

where $R'_1$ is the dual curvature (2.40). From here and the on-shell symmetry relation (2.42) it is obvious that (6.53) is zero on shell provided that it can be represented in the form

\[
\delta S \sim \int_{M^d} Tr \left( [R_1, \varepsilon] \circ R'_1 \right), \quad (6.54)
\]

where $Tr$ is the trace operation of the corresponding HS algebra, that includes usual trace over the matrix indices. This is the case provided that

\[
a(s, s-2) = \tilde{a} f_{s-1}(d, \hbar), \quad (6.55)
\]

where $\tilde{a}$ is some $s$–independent constant. This condition implies that we should set in (2.28)

\[
b(s) = \tilde{a} \lambda^{2(s-2)} f_{s-1}(d, \hbar). \quad (6.56)
\]

The scaling property (6.22) implies that setting $\tilde{a} = 1$ and

\[
\hbar = \lambda^{-1} = \rho \quad (6.57)
\]
we achieve that
\[ b(s) = f_{s-1}(d, 1)\lambda^{-2}. \]

With this convention we finally obtain
\[ a(s, p) = f_{s-1}(d, 1)\lambda^{-2(p+1)} \frac{(d - 5 + 2(s - p - 2))!! \left( s - p - 1 \right) (d - 5)!!(s - p - 2)!}{(d - 5)!!(s - p - 2)!}, \]

with the coefficients \( f_{n}(d, 1) \) (6.25).

As follows from the variation (5.3), convention (6.57), (6.58) leads to the \( \lambda \)-independent normalization of the kinetic terms of the free HS actions (2.27) resulting from (6.51). As discussed in Section 2.2, the normalization (2.4) of the compensator is adjusted so that the expressions for auxiliary and extra field connections via derivatives of the frame-like HS field, that result from the constraints contained in (2.29), involve only non-negative powers of \( \lambda \) with \( \lambda \)-independent leading derivative terms. As a result, with the normalizations (2.4) and (6.57), singular in \( \lambda \) terms in the nonlinear action originate entirely from the coefficients \( a(s, p) \) (6.59) and the dependence of structure coefficients of the HS algebra on \( \hbar \).

The relation (6.57) of the scale of noncommutativity of the star-product HS algebra to the radius of the most symmetric vacuum space is very intriguing.

### 6.4 Properties of the cubic action

The action (6.51), (6.59) is gauge invariant under the appropriately deformed HS gauge transformations (6.41) at the cubic order. Hence it describes consistent cubic interactions of triplets of massless fields of integer spins \( s_1, s_2, s_3 > 1 \). Since our consideration includes HS algebras with non-Abelian Yang-Mills symmetries, the constructed action contains cubic vertices antisymmetric in some of the fields involved. Note that the action (6.51) contains traces over matrix indices of HS fields just as spin three vertices of Section 5.3.

According to the general counting, given triplet of spins \( s_1, s_2, s_3 \), the action (6.51) contains at most \( (s_1 - 1) + (s_2 - 1) + (s_3 - 1) + 1 \) derivatives (one derivative is due to exterior differential in the curvature tensor). In the case of \( AdS_4 \), the part of the action, that only depends on the top HS connections, is topological being the HS generalization of the Euler characteristics. This is because, as discussed in Section 6.2, the terms with highest derivatives correspond to purely holomorphic and antiholomorphic connections, in which case the action amounts to \( str(R \circ R) \) which is obviously topological. Hence, the highest derivative terms in the action (6.51), (6.59) are of orders \( s_1 + s_2 + s_3 - 4 \) and \( s_1 + s_2 + s_3 - 2 \) in the cases of \( d = 4 \) and \( d > 4 \), respectively. This conclusion agrees with the fact that, beyond four dimensions, the variation of the Gauss-Bonnet term is non-zero.

Eq. (6.28) implies the following triangle-like restrictions on spins in cubic vertices of the action (6.51)
\[ |s_{i_1} - s_{i_2}| + \sigma \leq s_{i_3} \leq s_{i_1} + s_{i_2} - \sigma, \]

where \( i_1, i_2 \) and \( i_3 \) are pairwise different and \( \sigma = 2 \) or \( 1 \) for HS algebras with Abelian or non-Abelian (i.e., with the HS fields carrying inner indices) Yang-Mills symmetries, respectively.
This is equivalent to
\[ s_{\min} \geq s_{\max} - s_{\mathrm{mid}} + \sigma, \quad (6.61) \]
where spins are arranged so that \( s_{\max} \geq s_{\mathrm{mid}} \geq s_{\min} \). An interesting consequence of this restriction is that a colorless spin two field (which implies \( \sigma = 2 \)) only interacts with fields of equal spins as anticipated for the graviton. On the other hand, colorful massless spin two fields, which can appear in non-Abelian HS gauge theories with \( \sigma = 1 \), can interact with the fields whose spins differ by one.

Consider general variation of the action (6.51). Using
\[ \delta R = D_0 \delta \omega + [\omega, \delta \omega], \quad (6.62) \]
we obtain
\[
\delta S = \int_{M^d} \sum_{s} \sum_{p=0}^{s-2} a(s, p) G_{A_1 \ldots A_4} V_{C_1} \cdots V_{C_{2(s-2-p)}} \\
\times \text{tr} \left( [\omega \circ \omega] A_1 B_1 \ldots B_{s-2}, A_2 C_1 \ldots C_{s-2-p} D_1 \ldots D_p \delta \omega A_3 B_1 \ldots B_{s-2}, A_4 C_{s-1-p} \ldots C_{2(s-2-p)} D_1 \ldots D_p \right) \\
+ (D_0 \omega)^{A_1 B_1 \ldots B_{s-2}, A_2 C_1 \ldots C_{s-2-p} D_1 \ldots D_p} \left( [\delta \omega, \omega] A_3 B_1 \ldots B_{s-2}, A_4 C_{s-1-p} \ldots C_{2(s-2-p)} D_1 \ldots D_p \right) \quad (6.63)
\]
Integration by parts gives
\[
\delta S = \int_{M^d} \sum_{s} \sum_{p=0}^{s-2} a(s, p) \left( G_{A_1 \ldots A_4} V_{C_1} \cdots V_{C_{2(s-2-p)}} \\
\times \text{tr} \left( [\omega, R_1] A_1 B_1 \ldots B_{s-2}, A_2 C_1 \ldots C_{s-2-p} D_1 \ldots D_p \delta \omega A_3 B_1 \ldots B_{s-2}, A_4 C_{s-1-p} \ldots C_{2(s-2-p)} D_1 \ldots D_p \right) \\
+ R_1^{A_1 B_1 \ldots B_{s-2}, A_2 C_1 \ldots C_{s-2-p} D_1 \ldots D_p} \left( [\delta \omega, \omega] A_3 B_1 \ldots B_{s-2}, A_4 C_{s-1-p} \ldots C_{2(s-2-p)} D_1 \ldots D_p \right) \right) \\
- D_0 \left( G_{A_1 \ldots A_4} V_{C_1} \cdots V_{C_{2(s-2-p)}} \right) \\
\times \left( \text{tr} \left( [\omega \circ \omega] A_1 B_1 \ldots B_{s-2}, A_2 C_1 \ldots C_{s-2-p} D_1 \ldots D_p \delta \omega A_3 B_1 \ldots B_{s-2}, A_4 C_{s-1-p} \ldots C_{2(s-2-p)} D_1 \ldots D_p \right) \right) \quad (6.64)
\]
By virtue of (6.5) this gives the conserved current associated with the action (6.51), (6.56). Apart from the \( R_1 \)-dependent terms, that contain higher derivatives, it contains subleading \( \omega^2 \)-type terms, that carry lower derivatives. The structure of the current (6.64) manifests the effect of nontrivial mixture between lower and higher derivatives in presence of non-zero cosmological constant. In particular, the \( \omega^2 \) terms contain usual minimal gravitational interactions in the spin two sector. However, HS gauge invariance requires additional higher-derivative terms with negative powers of the cosmological constant.

The proposed action describes cubic vertices for symmetric bosonic massless fields of all spins \( s \geq 2 \) in \( AdS_d \). The normalization of the coefficients (6.59) is such that kinetic terms of the quadratic part of the action are \( \lambda \)-independent while higher-derivative interaction terms contain negative powers of \( \lambda \), becoming divergent in the flat limit. As discussed in Section 3.1, at the cubic level, an overall normalization of a vertex can be changed arbitrarily.
without breaking its consistency. (This property is specific for the lowest-order analysis.)
By multiplication with an appropriate $\lambda$-dependent factor it is possible to achieve that the
highest derivative part of the cubic action will remain finite in the limit $\lambda \to 0$ while all
subleading terms disappear at $\lambda \to 0$. The resulting vertex in Minkowski space contains
higher derivatives and is expected to reproduce the flat space vertices with the same number
of derivatives found by different methods in [9, 10, 11, 8, 16, 20, 21, 23, 24, 25].
Naively, it looks problematic to establish relation of the zoo of Minkowski vertices,
that contain different numbers of derivatives in accordance with (1.1), with the constructed
higher-derivative cubic vertices in $AdS_d$. A peculiar feature of this correspondence is that, as
we have seen already from particular examples, vertices with lower derivatives in $AdS_d$ can
be equivalent to those with higher derivatives. In Section 8, we argue that lower derivative
vertices in $AdS_d$ result from combinations of higher-derivative non-Abelian vertices, con-
tained in the action (6.51), with certain Abelian vertices. The lower-derivative vertices in
Minkowski space then result from their flat limit.

7 Vertex generating functions

To work with general vertices it is most convenient to use the language of generating functions

\[ A(Y) = \sum_n A_{A_1...A_n,B_1...B_n} Y_1^{A_1} \ldots Y_n^{A_n} Y_2^{B_1} \ldots Y_n^{B_n}. \] (7.1)

As discussed in Section 6.1 (for more detail see e.g. [82]), that $A_{A_1...A_n,B_1...B_n}$ has properties of
a traceless rectangular two-row Young diagram (2.20) is concisely encoded by the constraints

\[ \tau_{ij} A(Y) = 0, \quad \Delta^{ij} A(Y) = 0, \] (7.2)

where

\[ \tau^i_j = Y_i^A \frac{\partial}{\partial Y_j^A} - \frac{1}{2} \delta^i_j Y_k^A \frac{\partial}{\partial Y_k^A}, \quad \Delta^{ij} = \frac{\partial^2}{\partial Y_i^A \partial Y_j^A}. \] (7.3)

$\tau_{ij} = \tau_{ji}$ generate the Lie algebra $sp(2)$ with the invariant symplectic form $\epsilon_{ij}$ which raises
and lowers indices $i, j, \ldots = 1, 2$ according to [6,2]. Hence, the first of the conditions (7.2)
implies that $A(Y)$ associated with a two-row Young diagram is $sp(2)$ invariant. A tensor,
described by a two-row rectangular diagram of length $l$, obeys

\[ Y_i^A \frac{\partial}{\partial Y_i^A} A(Y) = 2l A(Y). \] (7.4)

7.1 Primitive vertices

Consider a vertex $V(A)$ built from a set of differential forms $A_\mu(Y)$ enumerated by $\mu =
1, 2, \ldots N_e$ that are all valued in $o(d-1,2)$ tensors described by traceless two-row rectangular
Young diagrams. We use convention that any index, that carries an upper label $\mu$, say $i^{\mu}$, is
associated with the variable $Y_\mu$ of $A_\mu(Y)$, often writing $Y_i^{\mu}$ instead of $Y^A_\mu$. 

47
For simplicity, in this section we consider *primitive* vertices free of the \((d-q)\)-form \(G^{A_1\ldots A_q}\) and compensator \(V^A\). Requiring a primitive vertex \(V(A)\) to be \(\sigma(d-1,2)\) invariant, we can represent it in the form:

\[
V(A) = V(\Delta) \prod_{\rho=1}^{N} A_{\rho}(Y_{\rho}) \big|_{Y_{\rho}=0},
\]

(7.5)

where

\[
\Delta^{i^\mu j^\mu} = \Delta^{j^\mu i^\mu} = \frac{\partial^2}{\partial Y_{i^\mu}^A \partial Y_{j^\mu}^A}.
\]

(7.6)

The condition (7.2) that \(A_{\nu}(Y_{\nu})\) describes traceless tensors implies

\[
\Delta^{i^\mu j^\mu} A_{\mu}(Y_{\mu}) = 0.
\]

(7.7)

### 7.1.1 \(sp(2)\) invariance

That \(A_{\nu}(Y_{\nu})\) for all \(\nu\) belong to the space of two-row rectangular Young diagrams implies that nontrivial vertices \(V\) are associated with \(\oplus_{\mu=1}^{N} sp_{\mu}(2)\) invariant coefficients \(V(\Delta)\). Indeed, the vertex \(V(A)\) is invariant under the algebra \(gl(2N)\) that acts on indices \(i^\mu\) carried by both \(Y^A_{i^\mu}\) and \(\frac{\partial}{\partial Y_{i^\mu}^A}\) (the condition \(Y_{i^\mu}=0\) in (7.5) is \(gl(2N)\) invariant). This means that the result of the action of \(gl(2N)\) on \(V(\Delta)\) differs by a sign from the result of the action of \(gl(2N)\) on \(\prod_{\rho=1}^{N} A_{\rho}(Y_{\rho})\). Since \(\prod_{\rho=1}^{N} A_{\rho}(Y_{\rho})\) is invariant under \(\oplus_{\rho=1}^{N} sp_{\rho}(2)\), vertices associated with those \(V(\Delta)\), that can be represented in the form

\[
V(\Delta) = \tau^{i^\mu j^\mu} V^{i^\mu j^\mu}(\Delta)
\]

(7.8)

with some \(V^{i^\mu j^\mu}(\Delta)\), vanish. On the other hand, any \(V(\Delta)\) decomposes into a sum of finite-dimensional \(gl(2N)\)-modules formed by various homogeneous differential operators over \(Y\). Hence, it decomposes into a sum of finite-dimensional \(sp_{\mu}(2)\)-modules. Since any element of a nontrivial finite-dimensional \(sp_{\mu}(2)\)-module can be represented in the form (7.8), all \(V(\Delta)\), that belong to nontrivial \(sp_{\mu}(2)\)-modules, represent zero vertices. This gives

**Lemma 7.1**

*Non-zero vertices are represented by those \(V(\Delta)\) that are \(sp_{\mu}(2)\) singlets for all \(\mu\).*

and

**Corollary 7.1**

*Nontrivial vertices \(V(\Delta)\) are generated by such polynomials of \(\Delta^{i^\mu j^\mu}\) where all indices \(i^\mu\) are contracted by the symplectic form \(\epsilon^{i^\mu j^\mu}\) with indices \(i^\mu\), \(j^\mu\) carrying the same label \(\mu\).*

For example, in the case of \(N=2\), a most general primitive \(sp(2) \oplus sp(2)\) singlet vertex is given by a function of one variable

\[
V(\Delta) = W(\Phi_{12}), \quad \Phi_{12} = \Delta_{i^1 j^2} \Delta_{j^1 i^1}.
\]

(7.9)

\^6Equivalently one can use the Fock space language where \(Y\)-derivatives are realized as annihilation operators while the Fock vacuum sets \(Y = 0\).
Expansion of $W(\Phi_{12})$ in powers of $\Phi_{12}$ reproduces contractions of indices between pairs of equivalent two-row rectangular Young diagrams of various lengths.

In the cubic case $N = 3$, a most general $sp(2) \oplus sp(2) \oplus sp(2)$ singlet vertex is given by an arbitrary function

$$V(\Delta) = W(\Phi_{\mu\nu}, \Phi) \tag{7.10}$$

of four variables

$$\Phi_{\mu\nu} = \Delta_{i\mu}^\nu \Delta_{j\nu}^\mu, \quad \mu \neq \nu, \quad \Phi \equiv \Phi_{123} = \Delta_{i1}^2 \Delta_{j2}^3 \Delta_{k3}^i. \tag{7.11}$$

From these examples we observe that general primitive vertex for any $N$ is a function of combinations of $\Delta_{i\mu}^\nu$ of the form

$$\Phi_{\mu_1\mu_2...\mu_p} = \Delta_{i_1}^{\mu_2} \Delta_{i_2}^{\mu_3} \ldots \Delta_{i_p}^{\mu_1}. \tag{7.12}$$

Note that for $N = 3$

$$\Phi_{\mu_1\mu_2\mu_3} = \epsilon_{\mu_1\mu_2\mu_3} \Phi, \tag{7.13}$$

where $\epsilon_{\mu_1\mu_2\mu_3}$ is totally antisymmetric and $\epsilon_{123} = 1$.

### 7.1.2 Graph interpretation

Invariant vertices can be depicted by graphs where nodes and edges are associated, respectively, with the fields $A_\mu(Y_\mu)$

$$\bullet \mu \tag{7.14}$$

and operators $\Delta^{\mu\nu}$

$$\Delta_{\mu\nu} \tag{7.15}$$

(recall that $\Delta_{\mu\mu} = 0$ for traceless tensors). Since $\Delta^{\mu\nu} = \Delta^{\nu\mu}$, primitive graphs associated with primitive vertices are undirected ($i.e.$, edges are undirected) and such that every node belongs to an even number of edges because indices of each $sp_\mu(2)$ should be contracted in pairs. Moreover, edges related to a given node decompose into pairs of linked edges associated with the contraction of two $sp(2)$ indices. This is visualized by associating every pair of linked edges with two sides of a line passing through a node.

$$\mu \tag{7.16}$$

Non-primitive vertices considered in the next section can be associated with non-primitive graphs containing additional types of nodes and edges.

Let a primitive graph, that contains no primitive subgraphs, be called path. Clearly, every path is closed. Length $p$ of a path $P$ equals to the number of its edges. Various paths encode
operators $\Phi_{\mu_1\mu_2...\mu_p}$ (7.12). It is convenient to endow every closed path with orientation so that every outcoming (incoming) line at node $\mu$ corresponds to an upper (lower) index $i^\mu$ in (7.12).

In agreement with the relation

$$\Phi_{\mu_1\mu_2...\mu_p} = (-1)^p \Phi_{\mu_p\mu_{p-1}...\mu_1},$$

changing the orientation of a path $P$ gives a sign factor $(-1)^p$.

Since every graph is a function of the $\oplus_{\mu=1}^N sp_\mu(2)$ invariant operators $\Phi_{\mu_1\mu_2...\mu_p}$ (7.12), it is convenient to endow the space of graphs with the commutative product law $\bullet$

$$\Gamma_1 \bullet \Gamma_2 = \Gamma_2 \bullet \Gamma_1$$

which, along with the linear structure on the space of graphs, endows the latter with the structure of commutative algebra $\mathcal{G}$ of closed paths $\Phi_{\mu_1\mu_2...\mu_p}$.

Although the algebra of primitive graphs is generated by closed paths, not all different graphs and paths give rise to independent vertices because of further relations between the variables $\Phi_{\mu_1\mu_2...\mu_p}$. The true algebra $\mathcal{P}$ of $sp_\mu(2)$ invariant primitive vertices is $\mathcal{Y} = \mathcal{G}/\mathcal{I}$ where $\mathcal{I}$ is the ideal of $\mathcal{G}$ generated by these relations. In particular, the following fact holds Lemma 7.2

A path of length $p > 2$, containing two identical edges, is a function of shorter paths. Indeed, a path with two identical segments corresponds to

$$tr(ABAC) = A_{i^\nu_1}^{i^\nu}B_{j^\nu_1}^{j^\nu}A_{j^\nu_1}^{j^\nu}C_{j^\nu_1}^{j^\nu_1}.$$ (7.20)

Using that

$$A_{i^\nu_1}^{i^\nu}A_{j^\nu_1}^{j^\nu} = A_{j^\nu_1}^{j^\nu}A_{i^\nu_1}^{i^\nu} - \frac{1}{2} \epsilon_{i^\nu_1 j^\nu_1} \epsilon_{i^\nu_1 j^\nu_1} A_{k^\nu_1}^{k^\nu}A_{k^\nu_1}^{k^\nu}$$ (7.21)

we obtain that

$$tr(ABAC) = tr(AB)tr(AC) - \frac{1}{2} tr(AA^t)tr(BC^t),$$ (7.22)

where $t$ denotes the transposition $\square$. 

50
For example,

\[
\begin{align*}
\mu_1 & \quad \mu_2 \\
\mu_3 & \quad \mu_4 \\
B & \quad C
\end{align*}
\]

\[
\begin{align*}
A & = 1/2(-1)^{p(A)+p(C)+1} \\
B & \quad C
\end{align*}
\]

where \(p(X)\) is the length of a segment \(X\).

Let a closed path be called elementary if it has length two or contains no pair of edges connecting the same pair of nodes. Lemma 7.2 implies that the algebra of graphs is generated by elementary paths. Note that if the length of \(A\) in (7.22) is larger than one, there exist several options for the decomposition of the same expression in the form \(tr(\tilde{A}\tilde{B}\tilde{A}\tilde{C})\) which lead to different relations (7.22) and, eventually, to relations on functions of elementary paths. Generally, the ideal \(I\) is generated by the relations

\[
\Delta_{p_1}^{i_1} \Delta_{j_1}^{j_1} = \Delta_{j_1}^{j_1} \Delta_{p_1}^{i_1} + \epsilon_{i_1,j_1} \Delta_{k_1}^{k_1} \Delta_{p_1}^{i_1} \Delta_{j_1}^{j_1}.
\]

(7.23)

The analysis of the algebra \(\mathcal{Y} = \mathcal{G}/I\) for general \(N\) is rather nontrivial being beyond the scope of this paper. The \(N = 3\) case of cubic vertices is, however, elementary.

### 7.1.3 Cubic vertices

In the \(N = 3\) case of cubic vertices the set of variables (7.11) is complete. This follows from Lemma 7.3

At \(N = 3\), any path \(\Phi_{\mu_1\mu_2...\mu_p}\) of length \(p > 3\) is a function of shorter paths.

This follows from Lemma 7.2. Indeed, in the case of three nodes, the list of elementary paths contains one path of length three and three of length two

\[
\begin{align*}
\mu_1 & \quad \mu_2 \\
\mu_3 & \quad \mu_4
\end{align*}
\]

Thus, a general primitive cubic vertex has the form

\[
V(A) = W(\Phi_{\mu\nu}, \Phi)A_1(Y_1)A_2(Y_2)A_3(Y_3)\left|_{Y_\sigma=0}\right.
\]

(7.25)
Strictly speaking, one has to prove that $\Phi_{\mu\nu}$ and $\Phi$ are algebraically independent. We were not able to find any relations between $\Phi_{\mu\nu}$ and $\Phi$ and believe that the algebra $\mathcal{Y}$ is freely generated by $\Phi_{\mu\nu}$ and $\Phi$, i.e., different $W(\Phi_{\mu\nu}, \Phi)$ give rise to different cubic vertices.

The vertex generating formalism greatly simplifies analysis of cubic vertices.

For instance, as a function of four variables, a general primitive cubic vertex depends on four parameters. Fixing spins of $A_{\mu}$ by the condition (7.4) imposes three conditions, i.e., the remaining ambiguity is one-parametric. Let $A_{\mu}(\psi_{i\mu})$ carry two-row Young diagrams of lengths $l_1$, $l_2$ and $l_3$, respectively. Expanding

$$W(\Phi_{\mu\nu}, \Phi) = \sum_{p,q,r,v \geq 0} w(p, q, r, v)(\Phi_{12})^p(\Phi_{13})^q(\Phi_{23})^r\Phi^v$$

(7.26)

we see that the nonzero contribution is due to the terms that satisfy

$$p + q + v = l_1, \quad p + r + v = l_2, \quad q + r + v = l_3.$$  

(7.27)

That $p, q, r, v \geq 0$ requires $l_1, l_2, l_3$ to obey the triangle conditions

$$l_1 + l_2 \geq l_3, \quad l_1 + l_3 \geq l_2, \quad l_2 + l_3 \geq l_1.$$  

(7.28)

From (7.27) it is easy to obtain that

$$2p = l_1 + l_2 - l_3 - v, \quad 2q = l_1 + l_3 - l_2 - v, \quad 2r = l_2 + l_3 - l_1 - v,$$

(7.29)

which requires

$$0 \leq v \leq N, \quad v + l_1 + l_2 + l_3 = 2k, \quad k \in \mathbb{N}, \quad N = \min(l_1 + l_2 - l_3, l_1 + l_3 - l_2, l_2 + l_3 - l_1).$$  

(7.30)

(7.31)

(In particular, it follows that $v \leq l_i$.) These formulas imply that the number of independent vertices $N(l_1, l_2, l_3)$ contained in the vertex $V$ (7.5) is

$$N(l_1, l_2, l_3) = 1 + \left[\frac{1}{2} \min(l_1 + l_2 - l_3, l_1 + l_3 - l_2, l_2 + l_3 - l_1)\right], \quad [k] = [k + \frac{1}{2}] = k.$$  

(7.32)

From Eq.(7.30) it follows that, for a set of diagrams of definite lengths $l_1, l_2, l_3$,

$$W(\Phi_{\mu\nu}, -\Phi) = (-1)^{l_1+l_2+l_3}W(\Phi_{\mu\nu}, \Phi).$$  

(7.33)

To classify totally (anti)symmetric vertices $V$ it is enough to observe that it is impossible to construct a totally antisymmetric polynomial from $\Phi_{\mu\nu}$ while symmetric polynomials are functions of

$$\Psi_1 = \Phi_{12} + \Phi_{23} + \Phi_{13}, \quad \Psi_2 = \Phi_{12}\Phi_{23} + \Phi_{31}\Phi_{12} + \Phi_{13}\Phi_{32}, \quad \Psi_3 = \Phi_{12}\Phi_{23}\Phi_{13}.$$  

(7.34)

Hence, the vertices (7.5) with

$$V(\Delta) = W(\Psi_1, \Phi)$$

(7.35)
are totally (anti)symmetric if they are (odd)even in $\Phi$. In particular, this gives the realization of totally antisymmetric structure coefficients (3.18) with no reference to the HS algebra.

Suppose now that $l_2 = l_3$. Eq. (7.27) gives $p = q$. From (7.33) it follows that the elementary monomials

\[(\Phi_{12}\Phi_{13})^p\Phi_{23}^{r}\Phi^v\]  

are (anti)symmetric under the permutations $2 \leftrightarrow 3$ for (odd)even $l_1$.

Consider a non-Abelian vertex written as a primitive vertex for the wedge product of the dual $(d-2)$–form curvature $R'_{A_1...A_{s-1}B_1...B_{s-1}}$ (2.40) as $A_1$ and the HS connection one-forms as $A_2$ and $A_3$. As such it should be antisymmetric under the exchange of $A_2$ and $A_3$. Hence, the elementary structure coefficients should be (anti)symmetric with respect to the inner indices associated with $A_2$ and $A_3$ for (odd)even $s_1$ and $s_2 = s_3$. (Recall that $s = l + 1$ for the fields $A(Y)$ identified with HS connections or curvatures.) Analogously one can see that the Abelian vertices of Section 8.2 have similar symmetry properties.

### 7.2 Non-primitive vertices

Now consider a vertex $F$ (4.1) that contains $G^{A_1...A_q}$ and $V^A$. It is convenient to represent $F$ in the form (4.11), replacing $G^{A_1...A_q}$ by a product of anticommuting variables $\psi^{A_1}...\psi^{A_q}$. To contract indices with $\psi^A$ (equivalently, $G^{A_1...A_q}$) and $V^A$, we introduce operators

\[p_{i\mu} = V^A \frac{\partial}{\partial Y_{A_i\mu}},\]  
\[\sigma_{j\mu} = \psi^A \frac{\partial}{\partial Y_{A_j\mu}}.\]  

A non-primitive vertex has the form

\[F(A) = F(\Delta, p, \sigma) \prod_{\rho=1}^{N} A_{\rho}(Y_{\rho})\bigg|_{Y_{\sigma}=0},\]  

where, again, the function $F(\Delta, p, \sigma)$ should be $sp_{\mu}(2)$ invariant for any $\mu$.

Non-primitive vertices is also convenient to describe in terms of graphs. In this case elementary graphs are not necessarily closed since they can contain two additional nodes $\blacksquare$ and $\blacklozenge$ associated, respectively, with $V^A$ and $\psi^A$ and two new types of edges associated with $p_{i\mu}$ or $\sigma_{j\mu}$, that connect new nodes to the old ones. Note that, because of anticommutativity of $\psi^A$, a $\psi$-node can be connected to another node by at most two edges $\sigma_{j\mu}$.
In these terms, vertex complex acquires the following simple form

\[ Q^{\text{top}} = -(-1)^{d+N_{\sigma}} \frac{1}{d-N_{\sigma}} \sum_{\mu\nu} \frac{\partial}{\partial p_{\mu\nu}} \frac{\partial}{\partial \sigma_{j\nu}} \Delta_{\mu\nu}, \]  

(7.41)

\[ Q^{\text{sub}} = (-1)^{d+N_{\sigma}} \frac{d-1-N_{\sigma}+N_{p}}{d-N_{\sigma}} \sum_{\nu} p_{\nu} \frac{\partial}{\partial \sigma_{j\nu}}, \]  

(7.42)

\[ Q^{\text{cur}} = (-1)^{d-N_{\sigma}} R_{1\alpha} \frac{\partial}{\partial \omega_{\alpha}}, \]  

(7.43)

where

\[ N_{\sigma} = \sum_{\nu} \sigma_{\nu} \frac{\partial}{\partial \sigma_{j\nu}}, \quad N_{p} = \sum_{\nu} p_{\nu} \frac{\partial}{\partial \sigma_{j\nu}}. \]  

(7.44)

These operators have simple interpretation in terms of graphs. \( Q^{\text{top}} \) and \( Q^{\text{sub}} \) change the graphs, while the operator \( Q^{\text{cur}} \) replaces a connection by the curvature. This is most conveniently realized by introducing connection nodes and curvature nodes so that \( Q^{\text{cur}} \) replaces a connection node by the curvature node, giving zero when acting on the curvature node.

We will use marked labels \( \dot{\nu} \) for the curvature nodes, i.e., if \( A(Y_{\nu}) = R_{1}(Y_{\nu}) \). The on-shell conditions (2.32) and (2.34) then take the form

\[ p_{\nu\dot{\nu}} F \sim 0, \]  

(7.45)

\[ \rho_{\nu\dot{\nu}} F \sim 0, \]  

(7.46)

where

\[ \rho_{\nu\dot{\nu}} = \frac{\partial}{\partial Y^{A\mu}} \frac{\partial}{\partial \psi^{A}}. \]  

(7.47)

A useful property is

\[ \{\rho_{\nu\dot{\nu}}, \sigma_{j\nu}\} = \Delta_{\nu\nu}. \]  

(7.48)

### 7.3 Relation with tensor notation

Being very efficient, the language of generating functions differs from the \( o(d-1,2) \) tensor language of previous sections because contraction of indices with \( \epsilon_{ij} \) combines contractions of \( o(d-1,2) \) indices from different rows of Young diagrams. Consider a cubic vertex

\[ U(a|A_{\mu}) = \sum_{n,m,k,l,p,q=0}^{\infty} a(n,m,k,l,p,q) \text{tr} \left( A_{1}^{B(n)D(m)}, E(k)F(l) A_{2E(k)}^{G(p)}, B(n)H(q), A_{3F(l)H(q),D(m)G(p)} \right), \]  

(7.49)

where \( a(n,m,k,l,p,q) \) are some coefficients that can be nonzero provided that

\[ n+m = k+l, \quad k+p = n+q, \quad l+q = m+p. \]  

(7.50)
Since only two of these three conditions are independent, $a(n, m, k, l, p, q)$ depends on four free parameters. In the case where $A_i$ describe rectangular two-row tensors of definite lengths $l_1, l_2, l_3$, the remaining ambiguity is one-parametric in accordance with the relations

$$n + m = k + l = l_1, \quad k + p = n + q = l_2, \quad l + q = m + p = l_3.$$  \hspace{1cm} (7.51)

These equations admit as many solutions as the conditions (7.27) in terms of generating functions. Indeed, let $l_3 = \max(l_1, l_2, l_3)$. Then general solution of (7.51) is

$$n = i, \quad k = N - i, \quad q = l_2 - i, \quad p = l_3 - l_1 + i, \quad l = l_3 - l_2 + i, \quad m = l_3 - l_2 + N - i,$$  \hspace{1cm} (7.52)

where

$$0 \leq i \leq N, \quad N = l_1 + l_2 - l_3.$$  \hspace{1cm} (7.53)

Using that exchange of the first and second rows of all diagrams produces a sign factor $(-1)^{l_1 + l_2 + l_3}$, we observe that vertices resulting from the replacement $i \to N - i$ are equivalent up to a sign. As a result, the number of vertices described by (7.49) is $1 + \left[\frac{1}{2}(l_1 + l_2 - l_3)\right]$ which just coincides with $N(l_1, l_2, l_3)$ (7.31).

In terms of generating functions, the vertex (7.49) can be represented as follows. Introduce two auxiliary $sp(2)$ vectors $S_i$ and $T_j$ such that

$$S^1 = 1, \quad S^2 = 0, \quad T^1 = 0, \quad T^2 = -1,$$  \hspace{1cm} (7.54)

$$T_i S^i = 1.$$  \hspace{1cm} (7.55)

Introduce operators

$$\Theta_{\mu\nu} = \Delta_{\mu;\nu} S^{\mu; T^{\nu}},$$  \hspace{1cm} (7.56)

which describe contraction of an index of the first row of the $\mu^{th}$ diagram with an index of the second row of the $\nu^{th}$ diagram. The operator form of the vertex (7.49) is

$$U = U(\Theta_{\mu\nu}) \prod_{\rho=1}^{3} A_\rho(Y_\rho) \bigg|_{Y_\sigma=0}.$$  \hspace{1cm} (7.57)

To reduce this vertex to the form (7.5) one has to single out the singlet part of $U$ with respect to $sp_1(2) \oplus sp_2(2) \oplus sp_3(2)$. To this end one should expand $U$ in powers of $S^{\mu;\nu}$ and $T^{\mu;\nu}$ with various $\mu$ and $\nu$ dropping all nonsinglet components in the products of $S^{\mu;\nu}$ and $T^{\mu;\nu}$ with $\mu = 1, 2, 3$. This is equivalent to (appropriately weighted) contraction of indices between pairs of $S^{\mu;\nu}$ and $T^{\mu;\nu}$ for $\mu = 1, 2, 3$ using (7.55). In particular, this implies that nonzero vertices can result only from those functions $U(\Theta_{\mu\nu})$ that contain equal numbers of $S^{\mu;\nu}$ and $T^{\mu;\nu}$ with the same $\mu$. For a homogeneous vertex

$$U(\Theta_{\mu\nu}) = \prod_{\mu \neq \nu, \mu, \nu=1}^{3} (\Theta_{\mu\nu})^{n_{\mu\nu}}.$$  \hspace{1cm} (7.58)
this imposes the conditions
\[ n_{12} + n_{13} = n_{21} + n_{31}, \quad n_{21} + n_{23} = n_{12} + n_{32}, \quad n_{31} + n_{32} = n_{13} + n_{23}, \] (7.59)
which just encode the relations (7.50).

Explicit expression for the vertex generating function \( F(\Delta) \) equivalent to \( U(\Theta_{\mu\nu}) \) is
\[
F(\Delta) = \prod_{\mu=1}^{3} G \left( \frac{\partial^2}{\partial S_{\mu} \partial T_{\mu}} U(\Theta) \right) \bigg|_{S=T=0}, \quad G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}. \quad (7.60)
\]

Indeed, the only \( sl_2 \) singlet component in \( S_i \ldots S_i T_j \ldots T_j \) is represented by its complete trace, i.e., taking into account (7.55) along with \( \delta_i^i = 2 \), the \( sp(2) \) invariant vertex results from the substitution
\[
\underbrace{S_i \ldots S_i}_{n} T_j \ldots T_j \rightarrow \frac{1}{n+1} \underbrace{\delta_i^i \ldots \delta_j^j}_{n} \quad (7.61)
\]
for any \( \mu \). This is just what Eq. (7.60) implements.

8 Abelian and non-Abelian vertices in \( AdS_d \)

From examples of Section 5 we learned that \( AdS \) deformation of a vertex in Minkowski space may be equivalent to a higher-derivative vertex in \( AdS_d \). This observation suggests that various HS vertices may admit a uniform higher-derivative realization in \( AdS_d \). In this section we argue that this is indeed the case. Namely, for \( d \geq 6 \) there are two classes of cubic vertices with \( s_1 + s_2 + s_3 - 2 \) derivatives, which we call non-Abelian and Abelian. Non-Abelian vertices, which are contained in the action (6.51), are considered in Section 8.1. Abelian vertices are considered in Section 8.2. Details of reduction of higher-derivative vertices to lower-derivative ones are discussed in Section 8.3. Specificities of \( d < 6 \) models are considered in Section 8.4.

8.1 Non-Abelian vertices for symmetric HS fields

The cubic part of the action (6.51) has the following form on-shell
\[
S^3 \sim \int_{M^d} a(s, s - 2) G_{A_1 A_2 A_3 A_4} tr \left( R^{A_1 B_1 \ldots B_{s-2}, A_2 D_1 \ldots D_{s-2}} (\omega \circ \omega)^{A^i_{B_1 \ldots B_{s-2}, A_2 D_1 \ldots D_{s-2}}} \right), \quad (8.1)
\]
where all terms, that contain contractions with the compensator \( V^A \), are omitted as they are on-shell trivial by virtue of (2.32). Let us reconsider actions of this type in terms of generating functions. To this end we introduce a set of one-form connections \( \omega^\alpha(Y) \) where \( \alpha \) is an inner index. (If \( \omega(Y) \) is a matrix, then \( \alpha \) encodes a pair of matrix indices.)

Consider the following primitive \( d \)-form vertex (7.5)
\[
L(V) = V^{\alpha_1 \alpha_2 \alpha_3}_{123} \left( \Delta \right) R^{\alpha_1}_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y_i=0}, \quad (8.2)
\]
Since the HS connection $\omega_\alpha$ is a one-form, $V_{123}^{\alpha_1^2\alpha_2^3}(\Delta)$ has the symmetry property

$$V_{123}^{\alpha_1^2\alpha_2^3}(\Delta) = -V_{132}^{\alpha_3^1\alpha_2^3}(\Delta).$$

(8.3)

Recall that $R'(Y)$ is the dual curvature $(d-2)$-form $(2.40)$ that itself carries a two-row Young diagram traceless representation of $o(d-1,2)$. Vertices $(8.2)$ make sense for $d \geq 4$.

$L(V)$ is $Q^{top}$ closed because it is free of the compensator $V^A$. Also, it is on-shell $Q^{sub}$-closed as a consequence of $(2.32)$ and $(2.39)$. The on-shell symmetry relation $(2.42)$ gives

$$Q^{cur} L \sim (-1)^d \left( V_{123}^{\alpha_1^2\alpha_2^3}(\Delta) - V_{231}^{\alpha_2^3\alpha_1^1}(\Delta) \right) R'_{\alpha_1}(Y_1) R_{\alpha_2}(Y_2) \omega_{\alpha_3}(Y_3) \bigg|_{Y_3=0}.\quad (8.4)$$

Thus, $L(V)$ is on-shell $Q^{cur}$ closed provided that $V_{123}^{\alpha_1^2\alpha_2^3}(\Delta)$ is cyclically symmetric

$$V_{123}^{\alpha_1^2\alpha_2^3}(\Delta) = V_{312}^{\alpha_3^1\alpha_2^2}(\Delta) = V_{231}^{\alpha_2^3\alpha_1^1}(\Delta).\quad (8.5)$$

Taking into account $(8.3)$, this means that, as discussed in Section $3.4$, to describe a consistent vertex, $V_{123}^{\alpha_1^2\alpha_2^3}(\Delta)$ should be totally antisymmetric.

Thus, the vertex $L(V)$ $(8.2)$ is pure

$$Q^{fl} L(V) \sim 0, \quad Q^{sub} L(V) \sim 0\quad (8.6)$$

for any totally antisymmetric $V_{123}^{\alpha_1^2\alpha_2^3}(\Delta)$. The number $N_{nab}$ of different non-Abelian cubic vertices equals to the number of independent $sp(2) \oplus sp(2) \oplus sp(2)$ invariant totally antisymmetric coefficients $V_{123}^{\alpha_1^2\alpha_2^3}(\Delta)$. In turn this means that, for any singlet in the tensor product of three two-row rectangular traceless Young diagrams of lengths $s_1-1$, $s_2-1$ and $s_3-1$ with $s_i \geq 2$ (the latter condition has to be imposed to respect the definition of the dual curvature $(2.40)$), there exists a non-Abelian vertex with appropriately adjusted symmetry properties with respect to the inner indices (In this counting, vertices, that form an irreducible inner tensor of the symmetric group $S_3$, are regarded as a single vertex.)

Since $(8.2)$ is a primitive vertex, by $(7.32)$ a number of independent vertices $L(V)$ is

$$N_{nab} = 1 + \left[ \frac{1}{2} (s_{min} + s_{mid} - s_{max} - 1) \right], \quad [k] = \left[ k + \frac{1}{2} \right] = k, \quad k \in \mathbb{Z}.\quad (8.7)$$

Alternatively, non-Abelian vertices can be written in the form

$$L'(V) = \sigma_i \sigma_j \sigma_k \sigma_\mu \sigma_\nu \sigma_\kappa \lambda^{(1^2\alpha_2^3)} \Delta R_{\alpha_1}(Y_1) \omega_{\alpha_2}(Y_2) \omega_{\alpha_3}(Y_3) \bigg|_{Y_3=0},\quad (8.8)$$

where $\mu, \nu = 2, 3$ (if $\mu = 1$ or $\nu = 1$, the vertex is zero as containing a product of three anticommutative operators $\sigma_i \sigma_i$). One can see that, up to an overall numerical factor, $L'(V)$ is equivalent to $(8.2)$ with

$$V_{123}^{\alpha_1^2\alpha_2^3}(\Delta) = \Delta n_1^1 n_2^2 \Delta n_3^1 \lambda^{(1^2\alpha_2^3)} \Delta.\quad (8.9)$$

---

7 This result fits an independent observation of Boulanger and Skvortsov (I am grateful to Evgeny Skvortsov for communication of this unpublished result) that a number of vertices, that admit non-Abelian deformation of HS symmetry transformations, coincides with the number of singlets in the triple tensor product of two-row Young diagrams of lengths $s_1 - 1$, $s_2 - 1$ and $s_3 - 1$. 57
Clearly, the action $S^3$ is a linear combination of the vertices $L(V)$. The specific form of the operator $V(\Delta)$ in (8.1), that fixes relative coefficients between different non-Abelian vertices, is determined by the structure coefficients of the HS algebra. It is this choice of $V(\Delta)$ that should allow a higher-order extension.

8.2 Abelian vertices for symmetric HS fields

8.2.1 Symmetric Abelian vertices

Consider the following $d$–form vertex

$$\mathcal{L}(a|R) = \sum_{n,m,k,l,p,q=0}^{\infty} G^{A_1\ldots A_6} a(n,m,k,l,p,q)$$

$$tr\left( R_{A_1} B(n) D(m) , A_2 E(k) F(l) , A_3 E(q) H(p) , A_4 B(n) H(q) , A_5 F(l) H(q) , A_6 D(m) R\right) \quad (8.10)$$

where $a(n,m,k,l,p,q) \in \mathbb{C}$ are some coefficients. Being constructed in terms of curvature two-forms, $\mathcal{L}(a|R)$ makes sense for $d \geq 6$. It is manifestly invariant under Abelian HS gauge transformations.

In terms of generating functions it can be written in the form

$$\mathcal{L}(U) = \sigma_{i_1}^{i_2} \sigma_{j_1}^{j_2} \sigma_{k_1}^{k_2} U^{a_1 a_2 a_3}_{123} (\Delta) R_{a_1}(Y_1) R_{a_2}(Y_2) R_{a_3}(Y_3) \bigg|_{Y_i=0} ,$$

(8.11)

where $U^{a_1 a_2 a_3}_{123}(\Delta)$ is totally symmetric

$$U^{a_1 a_2 a_3}_{123}(\Delta) = U^{a_1 a_2 a_3}_{132}(\Delta) , \quad U^{a_1 a_2 a_3}_{123}(\Delta) = U^{a_2 a_3 a_1}_{312}(\Delta) = U^{a_2 a_3 a_1}_{231}(\Delta) . \quad (8.12)$$

The form $\mathcal{L}(U)$ satisfies

$$Q^{top} \mathcal{L}(U) = 0 , \quad Q^{sub} \mathcal{L}(U) \sim 0 , \quad Q^{cur} \mathcal{L}(U) = 0 . \quad (8.13)$$

The first and the third properties hold because $\mathcal{L}(U)$ is independent of both the compensator $V^A$ and the connection $\omega$. The $Q^{sub}$–closure follows from the on-shell conditions (2.32).

Naively, $\mathcal{L}(U)$ contains $s_1 + s_2 + s_3$ derivatives. However it is quasi exact since

$$\mathcal{L}(U) = Q^{fl} T(U) , \quad (8.14)$$

where

$$T(U) = (-1)^d \sigma_{i_1}^{i_2} \sigma_{j_1}^{j_2} \sigma_{k_1}^{k_2} U^{a_1 a_2 a_3}_{123} (\Delta) R_{a_1}(Y_1) R_{a_2}(Y_2) \omega_{a_3}(Y_3) \bigg|_{Y_i=0} . \quad (8.15)$$

Hence,

$$\lambda^{-2} \mathcal{L}(U) = \lambda^{-2} Q T(U) + F(U) , \quad F(U) = -Q^{sub} T(U) . \quad (8.16)$$

58
Using (7.42) along with the on-shell conditions (7.45) and (7.46) we find that \( \lambda^{-2} \mathcal{L}(U) \) is equivalent to

\[
F(U) \sim 2\sigma_1 \sigma_2 \sigma_{k3} p^k \mathcal{U}_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y_i = 0}.
\] (8.17)

\( F(U) \) is a particular case of a broader class of vertices

\[
F(\tilde{U}) = -Q^{\text{sub}} T(\tilde{U}) = 2\sigma_1 \sigma_2 \sigma_{k3} p^k \tilde{\mathcal{U}}_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y_i = 0},
\] (8.18)

where \( \tilde{\mathcal{U}}_{123}^{\alpha_1 \alpha_2 \alpha_3} \) is not necessarily totally symmetric in its indices, satisfying a weaker condition

\[
\tilde{\mathcal{U}}_{123}^{\alpha_1 \alpha_2 \alpha_3} = \tilde{\mathcal{U}}_{213}^{\alpha_2 \alpha_1 \alpha_3}.
\] (8.19)

Coefficient functions \( \tilde{\mathcal{U}}_{123}^{\alpha_1 \alpha_2 \alpha_3} \) satisfying (8.19) belong either to the totally symmetric or to hook-type representations of the group \( S_3 \) which permutes indices 1, 2, 3 of different fields in the vertex. Since \( Q^{\text{top}} T(\tilde{U}) = 0 \) and \( Q^{\text{cur}} T(\tilde{U}) \) is trilinear in curvatures (or zero), it follows that \( F(\tilde{U}) \) is pure, i.e.,

\[
Q^{\mu\nu} F(\tilde{U}) \sim 0, \quad Q^{\text{sub}} F(\tilde{U}) \sim 0.
\] (8.20)

As shown below, the vertices with the hook-type \( \tilde{\mathcal{U}}_{123}^{\alpha_1 \alpha_2 \alpha_3} \) are \( Q \)-exact, hence describing total derivatives. The remaining non-trivial vertices (8.17) with totally symmetric vertex functions \( U_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \) we call Abelian since they are equivalent to the vertices (8.11) which are off-shell invariant under the Abelian HS gauge transformations. A number \( N_{ab} \) of Abelian vertices is determined by the formula analogous to (8.7) up to the shift \( s_i \rightarrow s_i - 1 \)

\[
N_{ab} = \left[ \frac{1}{2} (s_{\text{min}} + s_{\text{mid}} - s_{\text{max}}) \right].
\] (8.21)

Note that the vertices (8.17) have a form of current vertices discussed in Section 3.3. It would be interesting to see how the \( \text{AdS}_d \) deformation of Abelian vertices (8.11) generates (strictly positive) current vertices. As demonstrated in Section 8.3.2, the process of derivative reduction of the Abelian vertices indeed gives rise to lower-derivative current vertices.

It is easy to construct Abelian vertices of any order \( k \) in the form

\[
\mathcal{L}_k (U) = U_{1...k}^{\alpha_1...\alpha_k} (\Delta) \sigma_{i1} \sigma_{i2} \cdots \sigma_{ik} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \cdots R_{\alpha_k} (Y_k) \bigg|_{Y_i = 0}.
\] (8.22)

Again, these vertices are quasi exact, hence giving rise to lower-derivative vertices

\[
F(U) = -Q^{\text{sub}} T(U),
\] (8.23)

\[
T(U) = (-1)^d \sigma_{i1} \sigma_{i2} \cdots \sigma_{ik} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \cdots R_{\alpha_k} (Y_k) \bigg|_{Y_i = 0}.
\] (8.24)

The number of different order-\( k \) Abelian vertices of this type equals to the number of singlets in the tensor product of \( k \) two-row rectangular Young diagrams.
8.2.2 Triviality of hook-type vertices

Consider a $Q$-exact vertex $QP$ with

$$P = \sigma_{i1}\sigma^{i2}\sigma^{j2}\sigma^{k3}p^iP_{123}^{a_1a_2a_3}(\Delta)R_{a_1}(Y_1)\omega_{a_2}(Y_2)\omega_{a_3}(Y_3)\bigg|_{Y_i=0},$$

where

$$P_{123}^{a_1a_2a_3}(\Delta) = -P_{132}^{a_1a_3a_2}(\Delta).$$

The latter condition along with (7.42) and the on-shell condition (7.43) imply that

$$Q_{\text{sub}}P \sim 0.$$ (8.27)

On the other hand, using again (8.26), we obtain

$$Q_{\text{top}}P = (-1)^d\frac{2}{d-4}\sigma^{i2}\sigma^{j2}\sigma^{k2}\Delta_{ij3}p^iP_{123}^{a_1a_2a_3}(\Delta)R_{a_1}(Y_1)\omega_{a_2}(Y_2)\omega_{a_3}(Y_3)\bigg|_{Y_i=0}. \quad (8.28)$$

Now consider the on-shell relations resulting from (7.46)

$$0 \sim \rho_{ij}\gamma_{123}^{a_1a_2a_3}(\Delta)\sigma^{i1}\sigma^{j2}\sigma^{k3}R_{a_1}(Y_1)\omega_{a_2}(Y_2)\omega_{a_3}(Y_3)\bigg|_{Y_i=0}, \quad (8.29)$$

with

$$\gamma_{123}^{a_1a_2a_3} = -\gamma_{132}^{a_1a_2a_3}. \quad (8.30)$$

With the help of (7.38) this gives

$$\gamma_{123}^{a_1a_2a_3}(\Delta)(\sigma^{i1}\Delta_{ij3}\sigma^{j2}\sigma^{k3} + \sigma^{i1}\Delta_{ij3}\sigma^{j2}\sigma^{k3})R_{a_1}(Y_1)\omega_{a_2}(Y_2)\omega_{a_3}(Y_3)\bigg|_{Y_i=0} \sim 0. \quad (8.31)$$

This implies that $Q_{\text{top}}P \sim 0$.

Finally, $Q_{\text{cur}}P$ gives a vertex (8.18) with general hook-type coefficients with respect to $S_3$ because of superposition of the antisymmetrization condition (8.26) with the symmetrization due to presence of two curvatures. Hence, the hook-type vertices (8.18) are on-shell $Q$-exact.

8.3 Derivative reduction

All non-Abelian vertices $L(V)$ (8.2) and Abelian vertices $F(U)$ (8.17) contain up to $s_1 + s_2 + s_3 - 2$ derivatives. Being pure, they are gauge invariant both in Minkowski space and in $AdS_d$. On the other hand, as shown by Metsaev [8], there exists a single nontrivial Minkowski vertex of this order in derivatives. Hence, most of vertices $L(V)$ and $F(U)$ should be quasi exact. Let us explain how this can be seen using the formalism of generating functions. We start with quasi exact relation between Abelian and non-Abelian vertices which generalizes the relation (5.10) for spin two.
8.3.1 Quasi exact equivalence of Abelian and non-Abelian vertices

Consider \(Q^f\)–exact vertex \(Q^f(H + W)\) with

\[
H = h_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \omega_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0},
\]

\[
W = w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \omega_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0},
\]

where \(w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta)\) and \(h_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta)\) are totally symmetric

\[
h_{123}^{\alpha_1 \alpha_2 \alpha_3} = h_{213}^{\alpha_2 \alpha_1 \alpha_3} = h_{132}^{\alpha_1 \alpha_3 \alpha_2},
\]

\([8.32]\)

\(\Phi w_{123}^{\alpha_1 \alpha_2 \alpha_3} = w_{213}^{\alpha_2 \alpha_1 \alpha_3} = w_{132}^{\alpha_1 \alpha_3 \alpha_2}.\)

(Recall that the operators \(\sigma^i \) \([7.38]\) anticommute as well as the one-forms \(\omega_\alpha (Y)\), while the operators \(\Delta^{i \mu} \) \([7.6]\) are symmetric.)

Being independent of the compensator, \(H\) is \(Q^{top}\)-closed. As a result,

\[
Q^f H = Q^{cur} H = (-1)^d (h_{123}^{\alpha_1 \alpha_2 \alpha_3} \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \Delta^{j k l} + (1 \to 3 \rightarrow 2 \to 1) + (1 \to 2 \to 3 \rightarrow 1)) \quad R_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0}.
\]

With the help of the on-shell conditions \([7.45]\) we obtain

\[
Q^{cur} W = -(-1)^d w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \omega_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0},
\]

while the computation of \(Q^{top} W\) gives

\[
Q^{top} W = \frac{2 (-1)^d}{d-4} w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) (\sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \Delta^{j k l} + \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \Delta^{j k l}) R_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0}.
\]

Require that the terms in \(Q^{cur} H + Q^{top} W\), that contain either \(\sigma_{k_2} \sigma_{k_3}\) or \(\sigma_{k_1} \sigma_{k_3}\), cancel out. Using \([8.31]\) we find that this happens provided that

\[
w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) = (d-4) h_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta).
\]

As a result

\[
(-1)^d Q^f (H + W) \sim I + J,
\]

where

\[
I = -w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \omega_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y = 0},
\]

\[
J = \frac{3}{d-4} w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_i \sigma_j \sigma_k \sigma_l \sigma_m \omega_{\alpha_1} (Y_1) \omega_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3).
\]

Note that in the form \([8.23]\), the vertex \(J\) is represented by the vertex coefficient \(U_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sim \Phi w_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta)\).

Thus it is shown that the sum of the Abelian vertex \(I\) and non-Abelian vertex \(J\) is quasi exact, hence reducing to a lower-derivative vertex. This phenomenon we have seen in the
example of spin two in Section 5.2 where the relation (5.10) is a particular case of (8.39). Since the vertex $I$ (8.40) has most general structure, we conclude that all Abelian vertices (8.18) differ from non-Abelian ones by lower-derivative vertices. However, generally, the relation (8.39) is not sufficient to reduce all but one vertex to lower derivatives because, relating Abelian and non-Abelian vertices, it does not help to decrease derivatives within each of these classes if there are several non-Abelian and/or Abelian vertices. Now we are in a position to consider the derivative reduction within the class of Abelian vertices.

8.3.2 Derivative reduction of Abelian vertices

Let

$$N = N_{123}^{\alpha_1 \alpha_2 \alpha_3} (\Delta) \sigma_{i_1} \sigma^{i_1} \sigma_{j_2} \sigma^{j_2} \sigma_{k_3} \sigma^{k_3} p_{n_1} p_{m_2} \Delta^{n_1 u_1} \Delta^{u_2 m_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0}. \tag{8.42}$$

Since $Q^{\text{cur}} N \sim 0$ by virtue of the on-shell condition (7.45), $Q^{\text{top}} N$ is $Q^{\text{fl}}$-exact. An elementary computation gives

$$Q^{\text{top}} N = -2 \frac{(-1)^d}{d-5} N_{123}^{\alpha_1 \alpha_2 \alpha_3} \left( \sigma_{j_2} \sigma^{j_2} \sigma_{k_3} \sigma^{k_3} (\Phi_{13} \sigma_{i_1} \Delta^{i_1 n_2} \Delta^{u_2 m_3} p^{i_3} + \Phi_{i_1} \Delta^{i_1 n_3} p^{n_3}) 
+ \sigma^{j_2} \sigma_{j_2} \sigma_{k_3} \sigma^{k_3} (\Phi_{23} \sigma_{i_1} \Delta^{i_1 n_2} \Delta^{u_2 m_3} p^{k_3} - \Phi_{i_1} \Delta^{i_1 n_3} p^{n_3}) \right) R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0}. \tag{8.43}$$

From the on-shell conditions (7.46) of the form

$$\rho_{\mu_1} \sigma_{i_1} \sigma^{i_1} \sigma_{j_2} \sigma^{j_2} \sigma_{k_3} \sigma^{k_3} D^{\mu_1} p^{j_2} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0, \tag{8.44}$$

where $\mu = 1$ or 2 and $D^{\mu_1}$ is some $sp(2)^3$ covariant operator, by virtue of (7.48) we have

$$\left( \sigma^{j_1} \sigma_{j_2} \sigma^{j_2} \sigma_{k_3} \sigma^{k_3} \Delta_{i_2 j_1} + \sigma^{j_1} \sigma_{j_1} \sigma^{j_1} \sigma_{k_2} \sigma^{k_2} \Delta_{j_2 j_3} \right) D^{i_1} p^{j_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0 \tag{8.45}$$

and

$$\left( \sigma^{j_1} \sigma_{j_1} \sigma^{j_1} \sigma_{k_3} \sigma^{k_3} \Delta_{i_2 j_2} + \sigma^{j_2} \sigma_{j_2} \sigma^{j_2} \sigma_{k_1} \sigma^{k_1} \Delta_{i_2 j_3} \right) D^{i_1} p^{j_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0. \tag{8.46}$$

For $D^{i_1} = \Delta^{i_1 j_3}$, this gives two relations

$$\left( \sigma^{j_1} \sigma_{j_1} \sigma^{j_1} \sigma_{k_3} \sigma^{k_3} \Delta_{i_2 j_2} \Delta^{i_1 j_3} - \frac{1}{2} \Phi_{13} \sigma_{j_2} \sigma^{j_2} \sigma_{k_3} \sigma^{k_3} \right) p^{j_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0, \tag{8.47}$$

$$\left( \sigma^{j_2} \sigma_{j_2} \sigma^{j_2} \sigma_{k_1} \sigma^{k_1} \Delta_{i_1 j_1} \Delta^{i_2 j_3} - \frac{1}{2} \Phi_{23} \sigma_{j_1} \sigma^{j_1} \sigma_{k_3} \sigma^{k_3} \right) p^{j_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0. \tag{8.48}$$

Setting $D^{i_1} = \Delta^{i_1 j_3} F_{j_3 j_2} \Phi_{i_1} \sigma_{j_2} \sigma^{j_2} \sigma_{k_1} \sigma^{k_1} \sigma_{k_2} \sigma^{k_2} \sigma^{n_2}$ and using that antisymmetrization over any two indices $i_1$ and $j_2$ is equivalent to their contraction, we obtain

$$\left( \sigma^{j_3} \sigma_{k_3} \sigma^{k_3} \sigma_{j_2} \sigma^{j_2} \sigma_{k_2} \sigma^{k_2} \Delta_{i_1 j_3} \Delta^{i_1 n_2} - \frac{1}{2} \Phi_{12} \sigma_{k_1} \sigma^{k_1} \sigma_{j_3} \sigma^{j_3} \sigma^{n_2} \right) F_{n_2 i_3} p^{j_3} R_{\alpha_1} (Y_1) R_{\alpha_2} (Y_2) \omega_{\alpha_3} (Y_3) \bigg|_{Y=0} \sim 0, \tag{8.49}$$
\[
\left( \sigma^j \sigma^k \sigma^j \sigma^j \Delta_{ij} \Delta_{ij} \Delta_{ij} \right) \left. F_{i,j} \Phi^j \right|_{Y=0} \sim 0.
\]

Plugging these relations with appropriate \( F_{i,j} \) into (8.43) makes it possible to reduce \( Q^{\text{top}} N \) to the form where all terms contain a factor of \( \sigma_k \sigma^k \sigma_j \sigma^j \). Using again that antisymmetrization over any two indices \( i^\mu \) and \( j^\mu \) is equivalent to their contraction we obtain

\[
Q^{\text{top}} N = \frac{(-1)^d}{d-5} \sigma_k \sigma^k \sigma_j \sigma^j \sigma_j \sigma^j \Delta_{ij} \Delta_{ij} \Delta_{ij} \left( \frac{2 \Phi_{12} \Phi_{13} \Phi_{23} + \Phi^2}{\Phi_{12}} N^{a_1 a_2 a_3} (\Delta) R_{a_1} (Y_1) R_{a_2} (Y_2) \omega_{a_3} (Y_3) \right) \bigg|_{Y=0}.
\]

Setting

\[
N^{a_1 a_2 a_3} (\Delta) = \Phi_{12} U^{a_1 a_2 a_3} (\Delta)
\]

we finally obtain that every vertex \( F(U) \) (8.17) with

\[
U^{a_1 a_2 a_3} (\Delta) = \left( 2 \Phi_{12} \Phi_{13} \Phi_{23} + \Phi^2 \right) U^{a_1 a_2 a_3} (\Delta)
\]

is quasi exact, hence being equivalent the lower derivative vertex \(-\lambda^2 Q^{\text{sub}} N\).

Thus, Abelian vertices, that contain \( s_1 + s_2 + s_3 - 2 \) derivatives, are represented by the vertex functions \( U^{a_1 a_2 a_3} (\Delta) \) in (8.17) that belong to the quotient space over the ideal of vertices of the form (8.53).

The vertex \( \lambda^2 Q^{\text{sub}} N \), shown to be equivalent to the vertex \( F(U) \) (8.17), (8.53), again has the form of a current vertex, containing two curvatures and one connection. Continuation of the process of derivative reduction is expected to give rise to a list of lower-derivative current vertices equivalent to combinations of Abelian vertices (8.11).

### 8.4 Lower dimensions

Analysis of this section was so far applicable to sufficiently large space-time dimension \( d \). Not all constructed vertices are independent in lower dimensions where one has to take into account that \((d+1)\)-forms vanish. Similarly, the antisymmetrization over any \( d+1 \) Lorentz indices gives zero. The Metsaev’s classification (1.1) applies to \( d > 4 \), while for \( d = 4 \) it is shown that there are just two vertices, one of which is that with \( s_1 + s_2 + s_3 \) derivatives [8].

Although, containing explicitly a six-form, the Abelian vertex (8.11) makes sense starting from \( d = 6 \), the vertex (8.17) equivalent to (8.11) at \( d \geq 6 \) is well defined for \( d \geq 5 \). There is, however, a subtlety that, in accordance with the comment below Eq. (4.8), the \( Q^{\text{sub}} \) exact representation (8.16) does not apply at \( d = 5 \) because the denominator in (4.13) vanishes at \( d = 5 \) and \( q = 4 \). Note that this prevents the vertex (8.17) from being trivial because, otherwise, it would admit trivial representation \(-\lambda^{-2} QT\) given that \( Q^{\text{top}} T = 0 \), and \( Q^{\text{cur}} T = 0 \) because exterior product of three curvatures gives zero at \( d = 5 \). Thus, in accordance with [8], the list of nontrivial vertices at \( d = 5 \) is the same as at \( d \geq 6 \).

At \( d = 4 \), Abelian vertices (8.11) and (8.17) trivialize. The non-Abelian vertices also greatly simplify. Namely, there exists just one nontrivial non-Abelian vertex for any triple of spins that satisfy the triangle inequality. This is most easily seen in the spinorial language.
where a two-row irreducible $o(d - 1, 2)$ Young diagram of length $l A_{(l)}, B_{(l)}$ is represented by a symmetric multispinor $A_{\Lambda(2)}$ of $sp(4)$. Indeed, there exists a single possible $sp(4)$ invariant contraction of indices between three symmetric multispinors of given lengths

$$A_{1\Omega(n)\Lambda(m)\phi(k)} A_{2\phi(k)}^{\Omega(n)} A_{3\Lambda(m)}.$$

These non-Abelian vertices are reproduced by the action (6.39), (6.43). We believe that this action properly describes all strictly positive $4d$ vertices, fixing relative couplings for different vertices in a unique way consistent with higher-order corrections. To complete analysis of $4d$ cubic action it remains to analyze semipositive vertices along the lines of the next section.

9 Missed interactions and Weyl module

Although being quite rich, the list of $AdS_d$ vertices of Section 8 is smaller than the Metsaev’s list (1.1) which contains

$$N_{\text{Min}} = s_{\text{min}} + 1$$

vertices. An important restriction on the vertices considered in Section 8 is that they exist at the condition that $s_i - 1$ satisfy the triangle inequality (1.2) which is not necessarily respected by a general Minkowski vertex. Using the formulae (8.7) and (8.21), we obtain that

$$N_{\text{nab}} + N_{\text{ab}} = s_{\text{min}} + s_{\text{mid}} - s_{\text{max}}.$$

As a result, the number of vertices not reproduced by the scheme of this paper is

$$N_{\text{missed}} = N_{\text{Min}} - N_{\text{nab}} - N_{\text{ab}} = 1 + s_{\text{max}} - s_{\text{mid}}.$$

One missed vertex is the top Abelian vertex with $s_1 + s_2 + s_3$ derivatives, built from three Weyl tensors. In the case of $s_{\text{max}} = s_{\text{mid}}$, this is the only vertex not represented in our list.

The reason why some of vertices turned out to be missed in this paper is the restriction to strictly positive vertices that do not contain Weyl zero-forms directly. The missed vertices should explicitly contain (derivatives of) Weyl tensors. We call such vertices *semipositive*.

In particular, the vertex with the maximal number of derivatives $s_1 + s_2 + s_3$ is semipositive since all $R^3_1$ vertices are total derivatives in flat space, hence being subleading in $AdS_d$. Indeed, substituting (2.31) into (8.11) and using (2.16) we obtain that

$$\mathcal{L}(U) \sim a G \Phi U_{123}^{\alpha^1 \alpha^2 \alpha^3} (\Delta) C_{\alpha^1} (Y_1) C_{\alpha^2} (Y_2) C_{\alpha^3} (Y_3) \bigg|_{Y_i = 0} \hspace{1cm} (9.4)$$

where $\Phi$ is the operator (7.11) and $a$ is some nonzero constant. Hence those Abelian vertices

$$\mathcal{L}(U) \sim a G R_{123}^{\alpha^1 \alpha^2 \alpha^3} (\Delta) C_{\alpha^1} (Y_1) C_{\alpha^2} (Y_2) C_{\alpha^3} (Y_3) \bigg|_{Y_i = 0},$$

where $R_{123}^{\alpha^1 \alpha^2 \alpha^3} (\Delta)$ contains a factor of $\Phi$ are strictly positive while those from the quotient over the ideal generated by $\Phi$ are semipositive. From the analysis of Section 7.1.3 it follows that, in agreement with (1.1), there is just one vertex with $s_1 + s_2 + s_3$ derivatives.
In terms of generating functions this analysis straightforwardly extends to the higher-order vertices (8.22). Indeed, substituting again (2.31) into (8.22) and using (2.16) we obtain that
\[ \mathcal{L}_k(U) = G \Phi_k U_{1...k}^{\alpha_1...\alpha_k} (\Delta) C_{\alpha_1}(Y_1) \ldots C_{\alpha_k}(Y_k) \bigg|_{Y_i=0}, \tag{9.6} \]
where
\[ \Phi_k = \Delta_{2k}^{[A_1^i B_1^i ... A_k^i B_k^i], [C_1^i D_1^i ... C_k^i D_k^i]} \frac{\partial^{4k}}{\partial Y_{i_1}^{A_1^i} \ldots \partial Y_{i_k}^{A_k^i} \partial Y_{j_1}^{B_1^i} \ldots \partial Y_{j_k}^{B_k^i} \partial Y_{j_1}^{D_1^i} \ldots \partial Y_{j_k}^{D_k^i}}, \tag{9.7} \]
and
\[ \Delta^{[A_1 ... A_l], [B_1 ... B_l]} = \eta^{[A_1 B_1] ... \eta^{A_l B_l]}, \tag{9.8} \]
where brackets imply total antisymmetrization. Hence, a set of semipositive Abelian vertices of order \( k \) is described by the quotient space of all coefficients \( U_{1...k}^{\alpha_1...\alpha_k} (\Delta) \) over the subspace of the coefficients proportional to \( \Phi_k \). Note that higher-order Abelian vertices in flat space were considered recently in [80] using the conformal anomaly techniques. Presumably, upon appropriate reduction of the number of derivatives, the flat limit of Abelian vertices (8.22) and (9.6) should reproduce the vertices of [80].

To see why the triangle inequality has to be respected by strictly positive vertices it suffices to analyze symmetry properties of rectangular two-row Young diagrams carried by connections and curvatures. Indeed, although indices of one of the rows of a two-row diagram associated with one of the fields in the vertex can be contracted with the compensator, all indices of the other row should be contracted with those of the other two fields in the vertex. If the length of the diagram associated, say, with the first field is larger than the total length of the other two diagrams, \( i.e., s_1 > s_2 + s_3 - 1 \), this will imply symmetrization over either more than \( s_2 - 1 \) indices of the second field or/and more than \( s_3 - 1 \) indices of the third, thus giving zero. For semipositive vertices this restriction is circumvented because, as explained in Section 2.2.3 elements of the spin \( s \) Weyl module are described by (Lorentz) Young diagrams with the first row of any length \( l \geq s \). Hence, vertices violating the triangle inequality are necessarily semipositive.

For instance, consider a vertex \( V_{s00} \) for a spin-\( s \) gauge field and two scalars. The latter are described by the zero-forms \( C(x) \) and their derivatives which are elements of the spin zero Weyl module. Hence, \( V_{s00} \) is semipositive. Clearly, having two spin zero fields in a vertex does not respect the triangle inequality. On the other hand, \( V_{s00} \) just describes standard current interactions between a spin \( s \) gauge field and HS currents built from (derivatives) of a scalar field [41, 62, 63, 64, 65, 66, 67]. In agreement with (1.1), \( V_{s00} \) contains \( s \) derivatives.

Thus, to incorporate general vertices, semipositive vertices have to be included into the scheme. This is achieved via extension of the First On-Shell Theorem to the Central On-Shell Theorem which supplements Eq. (2.29) by equation (2.46) on the Weyl zero-forms. This generalization requires a modification of the vertex complex in the Weyl zero-form sector. The most important modification is that the Weyl module is infinite dimensional.

\[ \text{The presence of } G^{A_1 A_2 A_3 A_4} \text{ does not affect this conclusion as is most obvious from the primitive form of the vertex (8.2) where } G^{A_1 A_2 A_3 A_4} \text{ is hidden in the dual curvature } (d-2)-\text{form } R'. \]
10 Towards full nonlinear action

Extension of the analysis of cubic interactions of this paper to semipositive vertices requires consideration of the full set of fields of the free unfolded formulation of massless HS fields, i.e., connection one-forms and Weyl zero-forms. In these terms, analysis of cubic HS interactions reduces to the analysis of $d$-form vertices that are closed on-shell by virtue of free unfolded field equations. This suggests the idea that one can analogously search HS interactions in all orders. Namely, having the full nonlinear unfolded equations in various dimensions [40, 41] (see also [81] and references therein for $2d$ and $3d$ systems), we can look for a gauge invariant nonlinear action that gives rise to these equations, reproducing in the lowest-order standard free massless HS actions \[2.7\] and their lower-spin analogues [59].

Let $W^\alpha(x)$ be a set of differential forms serving as dynamical variables. General unfolded equations have the form [39]

$$\mathcal{R}^\alpha = 0,$$

where

$$\mathcal{R}^\alpha(x) = dW^\alpha(x) - G^\alpha(W(x))$$

with some exterior algebra functions of differential forms $G^\alpha(W)$ required to satisfy the conditions

$$G^\beta(W) \wedge \frac{\partial G^\alpha(W)}{\partial W^\beta} \equiv 0,$$

which guarantee compatibility of equations (10.1) with $d^2 = 0$ for unrestricted differential forms $W^\alpha$. We consider unfolded systems that belong to the universal class [82], which means that the condition (10.3) holds independently of space-time dimension $d$, i.e., without using that any $(d + 1)$-form is zero. (All known HS unfolded systems are universal.)

By virtue of (10.3), the generalized curvatures $\mathcal{R}^\alpha$ satisfy Bianchi identities

$$d\mathcal{R}^\alpha = -\mathcal{R}^\beta \frac{\partial G^\alpha(W)}{\partial W^\beta}.$$

In the universal class, equation (10.1) is invariant under the gauge transformation

$$\delta W^\alpha = d\varepsilon^\alpha + \varepsilon^\beta \frac{\partial G^\alpha(W)}{\partial W^\beta},$$

where the derivative $\frac{\partial}{\partial W^\beta}$ is left and the gauge parameter $\varepsilon^\alpha(x)$ is a $(p_\alpha - 1)$-form (0-forms have no associated gauge parameters). Indeed, using (10.3), one can see that

$$\delta \mathcal{R}^\alpha = \mathcal{R}^\gamma \frac{\partial}{\partial W^\gamma} \varepsilon^\beta \frac{\partial}{\partial W^\beta} G^\alpha(W).$$

In the case of symmetric HS fields, $W^\alpha$ encodes HS connection one-forms $\omega$ and Weyl zero-forms $C$. The curvatures $\mathcal{R}^\alpha$ provide a nonlinear generalization of

$$R_{A_1 \ldots A_{s-1},B_1 \ldots B_{s-1}}^{A_s} = E_{A_s} \wedge E_{B_s} C^{A_1 \ldots A_{s},B_1 \ldots B_s}.$$
and \( \tilde{D}_0 C \), while equation (10.1) describes a nonlinear deformation of equations (2.31) and (2.46). As such, the curvatures \( \mathcal{R}^\alpha \) should not be confused with the HS curvatures \( R \) used in the rest of this paper.

Unfolded formalism provides a far going generalization of the vertex complex formalism of Section 4. Indeed, from the definition (10.2) it follows that

\[
dF(\mathcal{R}(x), W(x)) = Q(F(\mathcal{R}(x), W(x)),
\]

where

\[
Q = Q^\text{cur} + Q', \quad Q^\text{cur} = \mathcal{R}^\alpha \frac{\partial}{\partial W^\alpha}, \quad Q' = -\mathcal{R}^\beta \frac{\partial G^\alpha(W)}{\partial W^\beta} \frac{\partial}{\partial \mathcal{R}^\alpha} + G^\alpha(W) \frac{\partial}{\partial W^\alpha},
\]

where \( Q^\text{cur} \) and \( Q' \), respectively, increases and leaves intact a number of curvatures. We again use the convention

\[
\frac{\partial \mathcal{R}^\gamma}{\partial W^\beta} = 0.
\]

By construction, \( Q \) is a degree one nilpotent differential on the space of fields \( W^\alpha \) and curvatures \( \mathcal{R}^\alpha \),

\[
Q^2 = 0.
\]

It is easy to check that \( Q^\text{cur} \) and \( Q' \) form a bi-complex

\[
(Q^\text{cur})^2 = 0, \quad (Q')^2 = 0, \quad \{Q^\text{cur}, Q'\} = 0.
\]

The operators \( Q^\text{cur} \) and \( Q' \) are analogues of \( Q^\text{cur} \) (4.7) and \( Q' = Q^\text{top} + \lambda^2 Q^\text{sub} \) (4.53) of the vertex complex of Section 4.1.

On the unfolded equations (10.1), \( Q \) reproduces the differential \( Q \) of [42]

\[
Q \sim Q, \quad Q = G^\alpha(W) \frac{\partial}{\partial W^\alpha},
\]

which should not be confused with that of Section 4.

Let a differential form \( L(\mathcal{R}(x), W(x)) \) be some function of \( \mathcal{R} \) and \( W \). Its gauge variation under the gauge transformation (10.5) is

\[
\delta L(\mathcal{R}, W) = d(\varepsilon^\alpha \frac{\partial L(\mathcal{R}, W)}{\partial W^\alpha}) + \varepsilon^\alpha \frac{\partial}{\partial W^\alpha} (QL(\mathcal{R}, W)).
\]

Analogously to the analysis of Section 4 it follows that gauge invariance of the nonlinear action

\[
S = \int L(\mathcal{R}, W)
\]

is equivalent to the condition that \( L(\mathcal{R}, W) \) is \( Q \)-closed up to \( W \)-independent terms

\[
QL(\mathcal{R}, W) = V(\mathcal{R}).
\]
By virtue of (10.8), $Q$-exact Lagrangians are total derivatives. This means that, in all orders in interactions, on-shell nontrivial functionals invariant under the on-shell gauge transformation are classified by $Q$-cohomology on the space of $W$-dependent functionals.

It should be stressed that the action $S$ (10.15) can be nontrivial only if $V(R) \neq 0$. Indeed, if $L(R, W)$ is $Q$-closed, by virtue of (10.8) it is $d$-closed. But since so far no field equations were imposed, this is only possible if $L(R, W)$ is $d$-exact, hence being a total derivative. Thus a nontrivial Lagrangian $L(R, W)$ cannot be $Q$-closed.

General variation of the action $S$ (10.15)

$$\delta S = \int \delta W^\alpha \left( \frac{\partial L(R, W)}{\partial W^\alpha} - (-1)^\alpha d \frac{\partial L(R, W)}{\partial R^\alpha} - \frac{\partial G^\beta}{\partial W^\alpha} \frac{\partial L(R, W)}{\partial R^\beta} \right)$$  \hspace{1cm} (10.17)\

can be cast into the following remarkable form

$$\delta S = \int \delta W^\alpha \frac{\partial}{\partial R^\alpha} (QL(R, W)).$$  \hspace{1cm} (10.18)\

Hence,

$$\delta S = \int \delta W^\alpha \frac{\partial V(R)}{\partial R^\alpha}.$$  \hspace{1cm} (10.19)\

As a result, provided that $V(R)$ does not contain a linear term in $R$, the variation of the gauge invariant action $S$ is zero on the unfolded field equations (10.8). For a particular dynamical system, the action $S$ has to reproduce appropriate free field equations. In the most of physically interesting models this cannot be true however, because, being a $d$-form dependent only on curvatures, $V(R)$ turns out to be too nonlinear for sufficiently large $d$ to reproduce properly the field equations that start with terms linear in $R$. (Recall that the HS curvatures for symmetric fields are $p$-forms with $p \leq 2$.) One option for circumventing this obstacle may consist of introducing auxiliary fields associated with higher differential forms similarly to the approach of [43]. Alternatively, this problem can be circumvented via deformation of the gauge transformations by on-shell trivial curvature-dependent terms.

Indeed, analysis of a most general unfolded formulation of a given dynamical system reproduces all possible nonlinear deformations of dynamical equations (i.e., coupling constants) and, simultaneously, determines the on-shell form of the nonlinear gauge transformations (10.5). However, in the off-shell case, the transformation law can differ from (10.5) by terms that are themselves proportional to $R^\alpha$

$$\delta^{off} W^\alpha = \delta W^\alpha + \varepsilon^\beta \Delta^\alpha_\beta(W, R), \quad \Delta^\alpha_\beta(W, 0) = 0.$$  \hspace{1cm} (10.20)\

In fact, such a deformation seems to be necessary. Indeed, a proper definition for the extra fields, that contribute to the full nonlinear action $S$ but not to its free part, requires appropriate constraints that express the extra and auxiliary fields in terms of derivatives of the dynamical fields without using the differential field equations. Such constraints can be put into the form

$$\Phi^i := V^i_\alpha(W, R) R^\alpha = 0,$$  \hspace{1cm} (10.21)
where $V^i_\alpha$ maps the space of all curvatures $\mathcal{R}$ to the space of extra and auxiliary fields. However, constraints of this form can hardly be invariant under the gauge transformation (10.5). Indeed, the variation $\delta(V^i_\alpha(W, \mathcal{R})\mathcal{R}^\alpha)$ is proportional to $\mathcal{R}$. By construction, the equations $\mathcal{R}^\alpha = 0$ are equivalent to constraints on auxiliary and extra fields along with dynamical field equations. Hence, the variation of the constraints (10.21) with respect to the gauge transformation (10.5) is proportional not only to constraints but also to field equations. This implies that constraints (10.21) require some deformation (10.20) of the transformation law.

Expanding $L(\mathcal{R}, W)$ in powers of $\mathcal{R}$

$$L(\mathcal{R}, W) = \sum_{n \geq 0} L_n(W, \mathcal{R}), \quad L_n(W, \mu \mathcal{R}) = \mu^n L_n(W, \mathcal{R}), \quad (10.22)$$

we obtain that the necessary condition for $L(\mathcal{R}, W)$ to be gauge invariant under the deformed transformation (10.20) is

$$QL_0(W) = 0, \quad (10.23)$$

where $Q$ is the differential (10.13). $L_0(W)$ is the on-shell part of the Lagrangian while $Q$ is the on-shell part of the differential $Q$.

Although $L_0(W)$ may look analogous to Chern-Simons vertices of Section 3, this is not quite the case for two related reasons. First, the curvatures $\mathcal{R}$ differ from the curvatures $R$ of Section 3 according to (10.7). Second, $L_0(W)$ depends not only on the one-forms $\omega$ but also on the Weyl zero-forms $C$. $L_0(W)$ provides a starting point for the construction of the off-shell action. The form of the condition (10.23) on $L_0(W)$ is analogous to the conditions studied in [42] where it was proposed to look for an action in terms of off-shell unfolded systems. We hope to come back to these issues elsewhere.

11 Conclusion

In this paper, parity even cubic actions for symmetric HS fields in $AdS_d$, which generalize the previously known actions in $4d$ [6], $5d$ [31] [32] [33] and any $d$ for particular symmetric and mixed symmetry fields [33] [34] [35] [36] [37] [38], are found. It is argued that the variety of HS vertices in flat space [8] [21] [23] [24] [25] contains two parts. Strictly positive vertices, that can be formulated in terms of gauge connection one-forms and curvature two-forms, constitute a subclass of the full set of vertices in Minkowski space [8]. In particular, spins of three fields in a strictly positive cubic vertex have to satisfy the triangle inequality (1.2). Those vertices from the list (1.1), that do not belong to the strictly positive class, are called semipositive and contain explicitly HS Weyl zero-forms. Simplest examples of semipositive vertices are provided by the Abelian vertices with the maximal number of derivatives $s_1 + s_2 + s_3$, built from three HS Weyl tensors, and vertices that describe HS current interactions with scalars. Semipositive vertices not considered in this paper can also be analyzed within the frame-like approach by an appropriate extension of the formalism outlined in Section 9 as, in fact,
was demonstrated long ago in [85] where semipositive cubic HS interactions with spin one massless field in $AdS_4$ were found.

Systematics of strictly positive HS vertices in $AdS_d$ is quite uniform. They all carry $s_1 + s_2 + s_3 - 2$ derivatives and form two classes of Abelian and non-Abelian vertices. There are as many independent Abelian and non-Abelian vertices as independent singlets in the tensor products of three $o(d-1,2)$–modules depicted by two-row rectangular Young diagrams of lengths $s_i - 2$ and $s_i - 1$, respectively, $(i = 1, 2, 3)$. We believe that uniformity of HS vertices in $AdS_d$ indicates that, beyond the cubic order, HS interactions should be largely fixed by the gauge symmetry principle. In particular, the relative coefficients of different non-Abelian vertices are determined by structure coefficients of the HS algebra within the construction of Section 6.3. Being independent at the cubic level, Abelian vertices of Section 8 are expected to be related to the non-Abelian vertices by the higher-order analysis.

We hope that the results of this paper provide a good piece of illustration of the efficiency of the frame-like and unfolded formalisms that operate with well-organized sets of fields associated with the HS algebra. Of course, the metric-like formulation should also be well organized in terms of the corresponding geometric structures found for free fields in [86] (see also [87], [88], [89]) provided that their non-Abelian extension explored recently in [90] is available. However, as Cartan geometry contains the Riemann one, the non-linear HS curvatures in the frame-like formalism will reproduce those of the metric-like.

The main tool of this paper is the vertex tri-complex techniques resulting from the frame-like formalism in $AdS_d$. In the form presented in Section 4.1 it applies to the variety of problems including higher-order vertices (Abelian vertices for any number of HS fields are presented in Sections 8.2 and 9) and general mixed symmetry HS fields. It is tempting to see whether the vertex complex analysis of non-Abelian vertices of mixed symmetry HS fields may help to find still mysterious HS symmetries underlying a nonlinear theory of mixed symmetry HS gauge fields. Vertices in Minkowski space are characterized by the cohomology of the Minkowski subcomplex of the $AdS_d$ tri-complex. As shown in Section 10, the vertex complex admits a natural generalization to arbitrary nonlinear unfolded systems.

As is by now well known [19], the role of cosmological constant in a massless HS theory is analogous to that of the parameter of mass for massive fields both in Minkowski space and $AdS_d$. Hence, the developed approach may also be useful for the analysis of nontrivial vertices of massive fields of various kinds. Note that the manifestly gauge invariant frame-like formalism for symmetric massive fields has been worked out in [61].

Although $AdS_d$ vertices constructed in this paper admit consistent flat limit, due to peculiarities of the relation between interactions in Minkowski and $AdS$ spaces, explicit correspondence between the known lists of Minkowski and $AdS_d$ vertices is not at all obvious since, as shown in Section 8.3, the naive flat limit of most of $AdS$ vertices of this paper gives total derivatives. Correspondingly, the systematics of general HS Minkowski vertices [8, 23, 24, 25] is very different from that in $AdS_d$. Hopefully, elaboration of the precise correspondence between the Minkowski and $AdS$ lists of vertices may help to understand better the correspondence between HS gauge theory, which is most naturally formulated in the frame-like formalism, and string theory which naturally leads to the list of HS vertices
in Minkowski background [23, 24]. (See, however, an interesting recent paper on the reformulation of string theory in the frame-like formalism [91] .) This program requires however a generalization to massive fields since the possibility of taking a flat limit in the interacting theory of massless HS fields is an artifact of the lowest-order analysis, i.e., the full consistent action for massless HS fields can only be constructed in the AdS background.

The important remaining question is, of course, how to find the complete nonlinear action. Hopefully, this problem can be analyzed similarly to the analysis of cubic vertices in this paper with the free unfolded equations replaced by the full nonlinear unfolded equations known in various dimensions (see, e.g., [64, 82] and references therein). The idea expressed in Section 10 is to reverse the problem by looking for an action whose structure is dictated by known unfolded equations instead of deriving an action, that leads to one or another unfolded system, as proposed e.g. in [42, 83, 84, 43, 44]. We believe that in this setup the action problem fits nicely the analysis of HS theory performed in [93, 94, 95] in the context of AdS/CFT interpretation of HS theory as suggested in [96] (see also [97]), which is based on the unfolded dynamics approach. (For related work see also [98, 99, 100, 101, 102, 103, 104, 105, 106] in the context of $AdS_3/CFT_2$ and [107, 108, 109] in the context of $AdS_{d+1}/CFT_d$.)

Finally, we would like to mention an intriguing fact obtained in Section 6.3 that the noncommutativity parameter $\hbar$ of the vector realization of the HS symmetry algebra turns out to be related to the AdS radius. It is tempting to speculate that this may indicate a new kind of interplay between geometry and quantization.

Acknowledgments

I would like to thank Konstantin Alkalaev and Nicolas Boulanger for stimulating discussions in the beginning of 2008 and Evgeny Skvortsov for reminding me about the problem in November of 2010 and numerous communications. I am grateful to Nicolas Boulanger, Olga Gelfond, Ruslan Metsaev, Evgeny Skvortsov and Per Sundell for useful discussions and wish to thank for hospitality Theory Group of Physics Department of CERN, where this work was initiated, and Quantum Gravity Group of AEI Max-Planck-Institute, where it was finished. Also I am grateful to Vyatcheslav Didenko, Evgeny Skvortsov and Dimitry Sorokin for numerous important comments on the original version of this paper. This research was supported in part by RFBR Grant No 11-02-00814-a and Alexander von Humboldt Foundation Grant No PHYS0167.

9Note that an attempt to construct nonlocal HS interactions in flat space was undertaken recently in [92]. However, being applicable to any set of HS fields, i.e., not necessarily associated with a HS symmetry multiplet, the approach of [92] does not automatically respect global HS symmetries. (Technically, the reason seems to be that the resulting nonlocal global transformation law blows up on-shell.) In this respect, the models of [92] are quite different from HS theories in $AdS_d$, which respect global HS symmetries.
Appendix A. Frame-like derivation of the BBD vertex

The most general $d$-form $F$, that contains at most three derivatives and has the form (5.15), is

$$F_3 = F_{31} + F_{32} + F_{33} + F_{34},$$  \hfill (A.1)

where

$$F_{31} = G^{A_1 A_2 A_3 V C(3)} \text{tr} \left( \omega_{A_1 B, A_2 E} \left( a \{ \omega_{A_3 B, CD, \omega^{ED}, C(2)} \} + b \{ \omega_{A_3 D, C B, \omega^{ED}, C(2)} \} \right) \right),$$ \hfill (A.2)

$$F_{32} = G^{A_1 A_2 A_3 V C(3)} \text{tr} \left( \omega_{A_1 B, FC} \left( \frac{1}{2} \{ \omega_{A_2 B, CG, \omega^{F G}, C} \} + \sigma \{ \omega_{A_2 G, B C, \omega^{F G}, C} \} \right) \right),$$ \hfill (A.3)

$$F_{33} = G^{A_1 A_2 A_3 V C(3)} \text{tr} \left( \omega_{A_1 B, A_2 C} \left( \alpha \{ \omega_{A_3 F, CG, \omega^{F G}, C} \} + \beta \{ \omega_{A_3 F, CG, \omega^{F B, G C}} \} \right) \right.$$ \hfill (A.4)

$$+ \omega_{A_1 B, C(2)} \left( f \{ \omega_{A_3 C, EE, \omega_{A_3 B, EE}} \} + g \{ \omega_{A_2 D, E C, \omega_{A_3 D, BB}} \} \right)$$

$$+ \kappa \omega_{A_1 B, C(2)} \left( \omega_{A_1 D, A_2 C, \omega_{A_3 D, BB}} \right) \right) \right),$$

$$F_{34} = \rho G^{A_1 A_2 A_3 A_4 V C(4)} \text{tr} \left( R_{A_1 B, A_2 D} \{ \omega_{A_3 B, C(2)} \} \omega_{A_4 D, C(2)} \right).$$  \hfill (A.5)

Here $a$, $b$, $\gamma$, $\sigma$, $\alpha$, $\beta$, $f$, $g$ and $\kappa$ are some coefficients to be determined.

Our goal is to find $F_3 \in H(Q^{top})$, i.e., such solutions of the equation $Q^{top} F_3 = 0$ that $F_3 \neq Q^{top} G$. Taking $G$ of the form

$$G = G^{A_1 A_2 A_3 A_4 V C(4)} \text{tr} \left( u \omega^{BG, C(2)} \{ \omega_{A_1 B, A_2 C, \omega_{A_3 G, A_4 C}} \} + l \omega_{A_1 B, A_2 D} \{ \omega_{A_3 B, C(2), \omega_{A_4 B, C(2)} \} \right)$$

$$+ \omega_{A_1 C, A_2 C} \left( v \{ \omega_{A_3 B, DC, \omega_{A_4 B, C(2)} \} + k \{ \omega_{A_3 D, BC, \omega_{A_4 B, C(2)} \} \} \right),$$  \hfill (A.6)

it is not hard to see that, using the ambiguity in exact $F_3$, it is possible to achieve $f = g = \beta = \rho = 0$. It is also elementary to see that the condition $Q^{top} F_3 = 0$ requires $\alpha = \kappa = 0$. Namely, $Q^{top}$ of the $\kappa$ and $\alpha$ terms produces, respectively, the curvature-dependent term and the term

$$G^{A_1 A_2 V C(2)} \text{tr} \left( \omega^{BD, FF} \{ \omega_{A_1 B, \omega_{A_2 C}, \omega_{C D, FF} \} \right),$$ \hfill (A.7)

that cannot be canceled against any other terms. (Note that the on-shell relation (B.13) does not affect the analysis of the curvature-dependent term.) As a result, it remains to check which $F_3 = F_{31} + F_{32}$ is $Q^{top}$ closed. An elementary straightforward analysis shows that $F_3$ is $Q^{top}$ closed provided that

$$F_3 = F_{31} + F_{32}, \quad a = -2, \quad b = -1, \quad \gamma = 4, \quad \sigma = \frac{4}{3}. \hfill (A.8)$$

This gives the BBD vertex $F_3$ (5.16).
Appendix B. On-shell identities

In this appendix we collect consequences of the on-shell identities \([2.32]\) and \([2.34]\) relevant to the analysis of the spin three vertex with five derivatives.

\[
G^{A_1 A_2 A_3 A_4} V^C V^C tr \left( R_{A_1} F^{,GG} \left( 2 \{ \omega_{A_2 G, A_3 C}, \omega_{A_4 G, FC} \} + \{ \omega_{A_2 F, A_3 C}, \omega_{A_4 C, GG} \} \right) \right) = 0, \quad (B.1)
\]

\[
G^{A_1 A_2 A_3 A_4} V^C V^C tr \left( R_{A_1} B^{,A_2} G \left( \{ \omega_{A_3 B, A_4 D}, \omega_{A_4 D, CC} \} - \{ \omega_{A_3 B, GD}, \omega_{A_4 D, CC} \} \right) \right) = 0, \quad (B.2)
\]

\[
G^{A_1 A_2 A_3 A_4} V^C V^C tr \left( R_{A_1} B^{,A_2} G \left( \{ \omega_{A_3 D, GC}, \omega_{A_4 B, C^D} \} - \{ \omega_{A_3 C, GD}, \omega_{A_4 B, C^D} \} \right) \right) = 0, \quad (B.3)
\]

\[
G^{A_1 A_2 A_3 A_4} V^C V^C tr \left( R_{A_1} B^{,A_2} G \left( \{ \omega_{A_3 C, A_4 D}, \omega_{A_4 D, CB} \} + 2 \{ \omega_{A_3 D, GC}, \omega_{A_4 D, CB} \} \right) + \{ \omega_{A_3 G, C^D}, \omega_{A_4 D, CB} \} \right) = 0, \quad (B.4)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} F^{,GG} \left( 2 \{ \omega_{A_2 F, GD}, \omega_{A_3 D, CG} \} + \{ \omega_{A_2 D, GG}, \omega_{A_3 D, CF} \} \right) \right) = 0, \quad (B.5)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} F^{,GG} \left( 3 \{ \omega_{A_2 D, GG}, \omega_{A_3 F, C^D} \} + \{ \omega_{A_2 D, A_3 F}, \omega_{A_3 G, C^D} \} \right) \right) = 0, \quad (B.6)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} F^{,GG} \left( \{ \omega_{A_2 D, GG}, \omega_{A_3 F, C^D} \} + 2 \{ \omega_{A_2 D, A_3 F}, \omega_{A_3 G, C^D} \} \right) + 2 \{ \omega_{A_2 F, GD}, \omega_{A_3 G, C^D} \} \right) = 0, \quad (B.7)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} D^{,FF} \left( 4 \{ \omega_{A_2 G, FC}, \omega_{A_3 D, GF} \} + 2 \{ \omega_{A_2 F, C^G}, \omega_{A_3 D, GF} \} \right) \right) = 0, \quad (B.8)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} D^{,FF} \left( \{ \omega_{A_2 G, A_3 C}, \omega_{A_3 G, FF} \} + 2 \{ \omega_{A_2 G, DC}, \omega_{A_3 G, FF} \} \right) \right) = 0, \quad (B.9)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{A_1} D^{,FF} \left( \{ \omega_{A_2 G, A_3 F}, \omega_{A_3 G, FF} \} + \frac{3}{2} \{ \omega_{A_2 G, DC}, \omega_{A_3 G, FF} \} \right) \right) = 0, \quad (B.10)
\]

\[
G^{A_1 A_2 A_3} V^C tr \left( R_{BB}^{,GG} \left( \{ \omega_{A_1 B, A_2 G}, \omega_{A_3 G, CB} \} - \frac{3}{2} \{ \omega_{A_1 G, A_2 C}, \omega_{A_3 G, BB} \} \right) \right) = 0, \quad (B.11)
\]

\[
G^{A_1 A_2 A_3} V^C V^C tr \left( [ R_{A_1} B^{,GG} , R_{A_2 B, A_3 G} ] \omega_{A_3 G, CC} \right) = 0, \quad (B.12)
\]

\[
G^{A_1 A_2 A_3} V^C V^C tr \left( R_{A_1} D^{,FF} \left( 3 \{ \omega_{A_2 D, CC}, \omega_{A_3 C, FF} \} - \{ \omega_{A_2 F, CC}, \omega_{A_3 C, DD} \} \right) \right) = 0. \quad (B.13)
\]
References

[1] A. K. H. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B 227 (1983) 31, 41.
[2] F. A. Berends, G. J. H. Burgers and H. Van Dam, Z. Phys. C 24 (1984) 247.
[3] F. A. Berends, G. J. H. Burgers and H. Van Dam, Nucl. Phys. B 260 (1985) 295.
[4] F. A. Berends, G. J. H. Burgers and H. Van Dam, Nucl. Phys. B 271 (1986) 429.
[5] A. K. H. Bengtsson and I. Bengtsson, Class. Quant. Grav. 3 (1986) 927.
[6] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B 189 (1987) 89.
[7] C. Aragone and S. Deser, Phys. Lett. B 86 (1979) 161.
[8] R. R. Metsaev, Nucl. Phys. B 759 (2006) 147 [arXiv:hep-th/0512342].
[9] E. S. Fradkin and R. R. Metsaev, Class. Quant. Grav. 8 (1991) L89.
[10] R. R. Metsaev, Mod. Phys. Lett. A 6 (1991) 359; Mod. Phys. Lett. A 8 (1993) 2413; Phys. Lett. B 309 (1993) 39.
[11] C. Fronsdal, Phys. Rev. D 18 (1978) 3624; Phys. Rev. D 20 (1979) 848.
[12] L. Cappiello, M. Knecht, S. Ouvry and J. Stern, Annals Phys. 193 (1989) 10.
[13] D. Sorokin, AIP Conf. Proc. 767 (2005) 172 [arXiv:hep-th/0405069].
[14] X. Bekaert, N. Boulanger and S. Cnockaert, JHEP 0601 (2006) 052 [arXiv:hep-th/0508048].
[15] N. Boulanger, S. Leclercq and S. Cnockaert, Phys. Rev. D 73 (2006) 065019 [arXiv:hep-th/0509118].
[16] N. Boulanger and S. Leclercq, JHEP 0611 (2006) 034 [arXiv:hep-th/0609221].
[17] R. R. Metsaev, Phys. Rev. D 77 (2008) 025032 [arXiv:hep-th/0612279].
[18] A. Fotopoulos and M. Tsulaia, Int. J. Mod. Phys. A 24 (2009) 1 [arXiv:0805.1346 [hep-th]].
[19] Yu. M. Zinoviev, Class. Quant. Grav. 26 (2009) 035022 [arXiv:0805.2226 [hep-th]].
[20] N. Boulanger, S. Leclercq and P. Sundell, JHEP 0808 (2008) 056 [arXiv:0805.2764 [hep-th]].
[21] R. Manvelyan, K. Mkrtchyan and W. Ruhl, Nucl. Phys. B 826 (2010) 1 [arXiv:0903.0243 [hep-th]]; [arXiv:1002.1358 [hep-th]]; Nucl. Phys. B 836 (2010) 204 [arXiv:1003.2877 [hep-th]].
[22] X. Bekaert, N. Boulanger and S. Leclercq, J. Phys. A 43 (2010) 185401 [arXiv:1002.0289 [hep-th]].
[23] A. Sagnotti and M. Taronna, Nucl. Phys. B 842 (2011) 299 [arXiv:1006.5242 [hep-th]].
[24] A. Fotopoulos and M. Tsulaia, JHEP 1011 (2010) 086 [arXiv:1009.0727 [hep-th]].
[25] R. Manvelyan, K. Mkrtchyan and W. Ruehl, Phys. Lett. B 696 (2011) 410 [arXiv:1009.1054 [hep-th]].
[26] D. Polyakov, Phys. Rev. D 82 (2010) 066005 [arXiv:0910.5338 [hep-th]].
[27] A. Sagnotti and M. Tsulaia, Nucl. Phys. B 682 (2004) 83 [arXiv:hep-th/0311257].
[28] M. A. Vasiliev, Yad. Fiz. 32 (1980) 855.
[29] M. A. Vasiliev, Fortsch. Phys. 35 (1987) 741.
[30] V. E. Lopatin and M. A. Vasiliev, Mod. Phys. Lett. A3 (1988) 257.
[31] M. A. Vasiliev, Nucl. Phys. B 616 (2001) 106 [arXiv: hep-th/0106200].
[32] K. B. Alkalaev and M. A. Vasiliev, Nucl. Phys. B 655 (2003) 57 [arXiv:hep-th/0206068].
[33] Yu. M. Zinoviev, JHEP 1008 (2010) 084 [arXiv:1007.0158 [hep-th]].
[34] K. Alkalaev, JHEP 1103 (2011) 031 [arXiv:1011.6109 [hep-th]].
[35] Yu. M. Zinoviev, JHEP 1103 (2011) 082 [arXiv:1012.2706 [hep-th]].
[36] N. Boulanger, E. D. Skvortsov and Yu. M. Zinoviev, J. Phys. A 44 (2011) 415403 [arXiv:1107.1872 [hep-th]].
[37] Yu. M. Zinoviev, Class. Quant. Grav. 29 (2012) 015013 [arXiv:1107.3222 [hep-th]].
[38] N. Boulanger and E. D. Skvortsov, JHEP 1109 (2011) 063 [arXiv:1107.5028 [hep-th]].
[39] M. A. Vasiliev, Annals Phys. 190 (1989) 59.
[40] M.A. Vasiliev, Phys. Lett. B243 (1990) 378; Phys. Lett. B285 (1992) 225.
[41] M. A. Vasiliev, Phys. Lett. B 567 (2003) 139 [arXiv:hep-th/0304049].
[42] M. A. Vasiliev, Int. J. Geom. Meth. Mod. Phys. 3 (2006) 37 [arXiv:hep-th/0504090].
[43] N. Boulanger and P. Sundell, J. Phys. A 44 (2011) 495402 [arXiv:1102.2219 [hep-th]].
[44] E. Sezgin and P. Sundell, arXiv:1103.2360 [hep-th].
[45] N. Doroud and L. Smolin, arXiv:1102.3297 [hep-th].
[46] K. B. Alkalaev, O. V. Shaynkman and M. A. Vasiliev, Nucl. Phys. B 692 (2004) 363 [arXiv:hep-th/0311164].
[47] N. Boulanger, C. Iazeolla and P. Sundell, JHEP 0907 (2009) 013 [arXiv:0812.3615 [hep-th]].
[48] N. Boulanger, C. Iazeolla and P. Sundell, JHEP 0907 (2009) 014 [arXiv:0812.4438 [hep-th]].
[49] E. D. Skvortsov, J. Phys. A 42 (2009) 385401 [arXiv:0904.2919 [hep-th]].
[50] E. D. Skvortsov, JHEP 1001 (2010) 106 [arXiv:0910.3334 [hep-th]].
[51] R. Utiyama, Phys. Rev. D101 (1956) 1597.
[52] T. W. B. Kibble, J. Math. Phys. 2 (1961) 212.
[53] A. Chamseddine and P. West, Nucl. Phys. B129 (1977) 39.
[54] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. 38 (1977) 739.
[55] F. Mansouri, Phys. Rev. D16 (1977) 2456.
[56] K. Stelle and P. West, Phys. Rev. D21 (1980) 1466.
[57] C. Preitschopf and M. A. Vasiliev, hep-th/9805127.
[58] S. Weinberg, Phys. Rev. D16 (1977) 2456.
[59] O. V. Shaynkman and M. A. Vasiliev, Theor. Math. Phys. 123 (2000) 683 [Teor. Mat. Fiz. 123 (2000) 323] [arXiv:hep-th/0003123].
[60] S. Deser and A. Waldron, arXiv:hep-th/0403059.
[61] D. S. Ponomarev and M. A. Vasiliev, Nucl. Phys. B 839 (2010) 466 [arXiv:1001.0062 [hep-th]].
[62] D. Anselmi, Nucl. Phys. B 541 (1999) 323 [arXiv:hep-th/9808004].
[63] D. Anselmi, Class. Quant. Grav. 17 (2000) 1383 [arXiv:hep-th/9906167].
[64] M. A. Vasiliev, arXiv:hep-th/9910096.
[65] S. E. Konstein, M. A. Vasiliev and V. N. Zaikin, JHEP 0012 (2000) 018 [arXiv:hep-th/0010239].
[66] F. Kristiansson and P. Rajan, JHEP 0304 (2003) 009 [arXiv:hep-th/0303202].
[67] X. Bekaert and E. Meunier, JHEP 1011 (2010) 116 [arXiv:1007.4384 [hep-th]].
[68] R. R. Metsaev, Phys. Lett. B354 (1995) 78.
[69] L. Brink, R. R. Metsaev and M. A. Vasiliev, Nucl. Phys. B 586 (2000) 183 [arXiv:hep-th/0005136].
[70] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B 102 (1981) 27.
[71] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D 28 (1983) 2567 [Erratum-ibid. D 30 (1984) 508].
[72] G. Barnich and M. Henneaux, Phys. Lett. B 311 (1993) 123 [arXiv:hep-th/9304057].
[73] M. G. Eastwood, Annals Math. 161 (2005) 1645 [arXiv:hep-th/0206233].
[74] M. A. Vasiliev, JHEP 0412 (2004) 046 [arXiv:hep-th/0404124].
[75] E. S. Fradkin and M. A. Vasiliev, Annals Phys. 177 (1987) 63.
[76] S. E. Konstein and M. A. Vasiliev, Nucl. Phys. B 331 (1990) 475.
[77] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189; “Supergravity as a Yang-Mills theory,” in G. ’t Hooft ed., 50 Years of Yang-Mills Theory (World Scientific), hep-th/0408137.
[78] M. A. Vasiliev, Nucl. Phys. B793 (2008) 469 [arXiv:0707.1085 [hep-th]].
[79] M. A. Vasiliev, Fortsch. Phys. 36 (1988) 33.
[80] W. Ruehl, arXiv:1108.0225 [hep-th].
[81] M. A. Vasiliev, hep-th/9910096.
[82] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, hep-th/0503128.
[83] M. A. Vasiliev, Nucl. Phys. B 829 (2010) 176 [arXiv:0909.5226 [hep-th]].
[84] M. Grigoriev, JHEP 1107 (2011) 061 [arXiv:1012.1903 [hep-th]].
[85] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. B291 (1987) 141.
[86] B. de Wit and D. Z. Freedman, Phys. Rev. D21 (1980) 358.
[87] T. Damour and S. Deser, Annales Poincare Phys. Theor. 47 (1987) 277.
[88] D. Francia and A. Sagnotti, Phys. Lett. B 543 (2002) 303 [arXiv:hep-th/0207002].
[89] D. Francia and A. Sagnotti, Class. Quant. Grav. 20 (2003) S473 [arXiv:hep-th/0212185].
[90] R. Manvelyan, K. Mkrtchyan, W. Ruhl and M. Tovmasyan, Phys. Lett. B 699 (2011) 187 [arXiv:1102.0306 [hep-th]].
[91] D. Polyakov, Phys. Rev. D 84 (2011) 126004 [arXiv:1106.1558 [hep-th]].
[92] M. Taronna, arXiv:1107.5843 [hep-th].
[93] S. Giombi and X. Yin, JHEP 1009 (2010) 115 [arXiv:0912.3462 [hep-th]].
[94] S. Giombi and X. Yin, JHEP 1104 (2011) 086 [arXiv:1004.3736 [hep-th]].
[95] S. Giombi and X. Yin, [arXiv:1105.4011 [hep-th]].
[96] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 550 (2002) 213 [arXiv:hep-th/0210114].
[97] E. Sezgin and P. Sundell, Nucl. Phys. B 644 (2002) 303 [Erratum-ibid. B 660 (2003) 403] [arXiv:hep-th/0205131].
[98] M. Henneaux and S. J. Rey, JHEP 1012 (2010) 007 [arXiv:1008.4579 [hep-th]].
[99] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, JHEP 1011 (2010) 007 [arXiv:1008.4744 [hep-th]].
[100] M. R. Gaberdiel and R. Gopakumar, Phys. Rev. D 83 (2011) 066007 [arXiv:1011.2986 [hep-th]].
[101] M. R. Gaberdiel and T. Hartman, JHEP 1105 (2011) 031 [arXiv:1101.2910 [hep-th]].
[102] C. Ahn, JHEP 1110 (2011) 125 [arXiv:1106.0351 [hep-th]].
[103] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, JHEP 1108 (2011) 077 [arXiv:1106.1897 [hep-th]].
[104] C. M. Chang and X. Yin, arXiv:1106.2580 [hep-th].
[105] M. R. Gaberdiel and C. Vollenweider, JHEP 1108 (2011) 104 [arXiv:1106.2634 [hep-th]].
[106] A. Campoleoni, S. Fredenhagen and S. Pfenninger, JHEP 1109 (2011) 113 [arXiv:1107.0290 [hep-th]].

[107] R. d. M. Koch, A. Jevicki, K. Jin and J. P. Rodrigues, Phys. Rev. D 83 (2011) 025006 [arXiv:1008.0633 [hep-th]].

[108] M. R. Douglas, L. Mazzucato and S. S. Razamat, Phys. Rev. D 83 (2011) 071701 [arXiv:1011.4926 [hep-th]].

[109] A. Jevicki, K. Jin and Q. Ye, arXiv:1106.3983 [hep-th].