We provide high probability sample complexity guarantees for non-parametric structure learning of tree-shaped graphical models whose nodes are discrete random variables with a finite or countable alphabet, both in the noiseless and noisy regimes. First, we introduce a new, fundamental quantity called the (noisy) information threshold, which arises naturally from the error analysis of the Chow-Liu algorithm and characterizes not only the sample complexity, but also the inherent impact of the noise on the structure learning task, without explicit assumptions on the distribution of the model. This allows us to present the first non-parametric, high-probability finite sample complexity bounds on tree-structure learning from potentially noise-corrupted data. In particular, for number of nodes $p$, success rate $1 - \delta$, and a fixed value of the information threshold, our sample complexity bounds for exact structure recovery are of the order of $O\left(\log^{1+\zeta}(p/\delta)\right)$, for all $\zeta > 0$, for both noiseless and noisy settings. Subsequently, we apply our results on two classes of hidden models, namely, the $M$-ary erasure channel and the generalized symmetric channel, illustrating the usefulness and importance of our framework. As a byproduct of our analysis, this paper resolves the open problem of tree structure learning in the presence of non-identically distributed observation noise, providing explicit conditions on the convergence of the Chow-Liu algorithm under this setting as well.

**Keywords** Structure learning · Hidden Markov trees · Information Threshold · Chow-Liu Algorithm

## 1 Introduction

Graphical models are a widely-used and powerful tool for analyzing high-dimensional structured data. Variables are represented as nodes of a graph whose edges indicate conditional dependencies between the node variables. In this paper, we study the problem of learning acyclic undirected graphs, also known as tree structured Markov Random Fields (MRFs). In particular, we are interested in the **Chow-Liu Algorithm** [1], which, given a dataset of samples from the distribution, returns an estimate of the original tree structure. The Chow-Liu algorithm is efficient and has low computational complexity. Furthermore, it has been shown to be optimal in terms of sample complexity (matching information-theoretic limits) for Gaussian data [2], binary data [3], as well as on non-parametric models with discrete alphabets [4, 5].

In many applications, the underlying phenomenon may be well-modeled by an MRF but the data acquisition device or sensor may itself introduce noise. A good estimator should be robust against noise: it is important to understand how sensitive the performance of an algorithm is to noisy inputs. Two recent works study the impact of corrupted
observations on binary and Gaussian models [6,7]. Goel et al. [6] extend the Interaction Screening Objective of Vuffray et al. [8] for Ising models while [7] analyzes the performance of the Chow-Liu algorithm for both noisy Ising and Gaussian models.

Noise-corrupted structured data appear in various applications for major branches of science as physics, computer science, biology and finance. The assumption of discrete and (arbitrarily) large alphabets is suitable for many of these scenarios. One recent application domain is the analysis of data protected by local differential privacy [9,10,11,12,13,14]. One approach to local differential privacy perturbs each sample at the time it is collected: our results can characterize the increase in sample complexity for a given privacy guarantee. Finally, modern distributed machine learning application often entail communication constraints. In these settings the data are often quantized, resulting in quantization noise on the samples (c.f. Tavassolipour et al. [15]).

In this paper, we study the problem of learning non-parametric tree-shaped distributions from original tree structured $X^{1:n}$ and noise-corrupted data $Y^{1:n}$. In particular, the underlining distributions extend the discrete models which were considered in prior; they do not belong to any parametric family and they support finite and countably infinite alphabet size. For both noiseless and noisy data cases, we provide finite sample complexity bounds for exact recovery. Our results characterize the sample complexity of the Chow-Liu algorithm, show its consistency for general classes of channels and quantify how measurement noise affects its performance.

**Notation.** We denote vectors or tuples by using boldface and we reserve calligraphic face for sets. For an integer $M$, let $[M] \triangleq \{1, 2, \ldots, M\}$ and $[M]^2 \equiv [M] \times [M]$ denote the Cartesian product. The indicator function of a set $A$ is denoted as $\mathbf{1}_A$. The cardinality of the set of nodes $\mathcal{V}$ is assumed to be equal to $p$, $|\mathcal{V}| = p$. The node variables of the tree are denoted by $X = (X_1, X_2, \ldots, X_p)$. If $X$ has a finite alphabet we write $X \in [M]^p$, and $X \in X^p$ for countable size alphabets. The probability mass function of $X$ is denoted as $p(\cdot)$ and the probability mass function of the pair $X_i, X_j$ is written as $p(x_i, x_j)$ or $p_{ij}(\cdot, \cdot)$, for any $i, j \in \mathcal{V}$. $T = (\mathcal{V}, \mathcal{E})$ is a tree (acyclic graph) with set of nodes and edges $\mathcal{V}$ and $\mathcal{E}$ respectively. $\hat{p}(\cdot)$ is the plug-in estimate of a distribution $p(\cdot)$. The symbol $X^{1:n}$ denotes the dataset of $n$ i.i.d samples of $X$. The estimated mutual information of a pair $X_1, X_2$ is $\hat{I}(X_1; X_2)$ and the resulting tree structure of the Chow-Liu algorithm is denoted by $T^{\text{CL}}$.

**1.1 Contribution**

Our work considers general tree-structured graphical models with variables taking values in finite or countable sets. We provide the first results on non-parametric hidden tree-structure learning, where the distributions of the hidden and observable layers are unknown and the mappings between the hidden and observed variables are general. Specifically, the main goal of the paper is to quantify how the noise affects the sample complexity of the Chow-Liu algorithm under minimal regularity assumptions on the graphical model at hand. In fact, we provide computable bounds on the sample size needed for exact structure recovery for both noiseless and noisy settings. We introduce convergence conditions of the Chow-Liu algorithm for generalized classes of noisy models, as the case of non-identically distributed noise, which is currently an open problem. Our contributions are as follows:

- We define a new quantity called the information threshold $\Gamma^p$ (or $\Gamma^o_1$ for the noisy case), which fundamentally characterizes the complexity of tree structure learning in non-parametric models. In the noisy case, structure recovery may not always be possible. However, we show that the positivity of $\Gamma^o_1$ implies the hidden structure can be recovered from the observable data.

- We provide sample complexity bounds for exact structure recovery of non-parametric noiseless (Theorem 1) and noisy models (Theorem 2). In particular, the sample complexity scales almost as $1/(\Gamma^p)^2$ in the noiseless case and $1/(\Gamma^o_1)^2$ in the noisy case. These bounds generalize the prior work of [4,5] and [3]. Both results hold for finite and countable alphabets, with the sample complexity depending almost logarithmically in the number of variables. Essentially, we show that the Chow-Liu algorithm is an effective universal estimator for structure learning from noisy data. If $\Gamma^o_1 > 0$, then by applying Chow-Liu algorithm directly on the noisy observations $Y^{1:n}$, exact recovery is guaranteed for sufficiently large $n$; the dependence of the noise is captured in the noisy information threshold $\Gamma^o_1$. However, in case $\Gamma^o_1 < 0$, post-processing is required on the noisy observations, before applying the Chow-Liu algorithm.

- We illustrate how our results capture certain scenarios of interest involving generalized $M$-ary erasure and symmetric channels. Our framework extends and unifies recent results for the binary case [6,7]. We further study the open problem of learning tree structures from data which are corrupted by non-identically distributed noise, and we provide conditions for the convergence of the Chow-Liu algorithm.
While this paper focuses on the analysis of Chow-Liu algorithm, our approach is more generally applicable to the analysis of δ-PAC Maximum Spanning Tree (MST) algorithms. At its root, our work shows how the error probability of MST algorithms (for example, Kruskal’s algorithm or Prim’s algorithm) behaves when edge weights are uncertain, i.e., when only some (random) estimates of the true edge weights are known. For example, in network planning we may wish to find a sub-network with maximum capacity, but the edge weights must be guessed, because they vary over time, or because some noise affects their value. Identically to the information threshold, we can define an equivalent capacity threshold for this type of problems. The latter will quantify the condition for accurate convergence and the statistical performance of greedy MST algorithms on graphs with random (variables as) edge weights.

1.2 Related Work

Structure learning from node observations is a fundamental and well-studied problem in the context of graphical models. For general graph structures, the complexity of the problem has been studied by Karger and Srebro [16]. Under the assumption of bounded degree the problem becomes tractable, leading to a large body of work in the last decade [8, 17, 18, 19, 20, 21, 22]. These approaches employ greedy algorithms, l_1 regularization methods, or other efficient optimization techniques. The sample complexity of each approach is evaluated based on information theoretic bounds [23, 24, 25]. For acyclic graphs, the Chow-Liu algorithm has been shown to be optimal in terms of sample complexity with respect to the number of variables. The error analysis and convergence rates for trees and forests were studied by Tan et al. [2, 4, 5] and Bresler and Karzand [3]. In particular, for finite alphabets the number of samples needed by the Chow-Liu algorithm is logarithmic with respect to the ratio p/δ for tree structures and polylogarithmic in the case of forests: O(log^{1+ζ}(p/δ)), for all ζ > 0, (see Theorem 5 of Tan et al. [4]). Comparatively, our results extend to countable alphabets and noisy observations (hidden models). Our sample complexity bounds are consistent with prior work: the order is poly-logarithmic but remains arbitrarily close to logarithmic (O(log^{1+ζ}(p/δ)), for all ζ > 0). In the special case of Ising models, hidden structures were recently considered by Goel et al. [6] and by Nikolakakis et al. [7, 26]. Goel et al. [6] consider bounded degree graphs and require the noise model (or an approximation of it) to be known with the noise i.i.d. on each variable (see their Section 6). Our results are more general in the sense that we apply to settings with non-identically distributed noise (Sections 4, 4), but under the tree structure model assumption. Nikolakakis et al. [7] solve the problem of hidden tree structure learning for (binary) Ising models with (modulo-2) additive noise. Our results generalize those in [2] to finite and countable alphabets, encompassing non-parametric tree structures and general noise models.

2 Model and Problem Statement

First, we provide a complete description of the model which includes definitions, properties and assumptions of the underlying distributions.

2.1 Non-Parametric Tree-Structured Models

We consider graphical models over p nodes with variables \{X_1, X_2, \ldots, X_p\} = X and finite or countable alphabet \mathcal{X}^p. We assume that the distribution \( p(\cdot) \) of X is given by a tree-structured Markov Random Field, (MRF). According to prior work by Lauritzen [27], we know that any distribution \( p(\cdot) \) which is Markov with respect to a tree \( T = (V, E) \) factorizes as

\[
p(x) = \prod_{i \in V} p(x_i) \prod_{(i,j) \in E} \frac{p(x_i, x_j)}{p(x_i)p(x_j)}, \quad x \in \mathcal{X};
\]

we call such distributions \( p(\cdot) \) tree-structured.

The noisy node variables Y = \{Y_1, Y_2, \ldots, Y_p\} are generated by a randomized mapping (noisy channel) \( \mathcal{F} : \mathcal{X}^p \rightarrow \mathcal{Y}^p \). We restrict attention here to mappings that act independently on each component: \( \mathcal{F}(\cdot) = \{F_1(\cdot), F_2(\cdot), \ldots, F_p(\cdot)\} \) and each \( F_i(\cdot) : \mathcal{X} \rightarrow \mathcal{Y} \) so \( Y_i = F_i(X_i) \) for all \( i \in V \). Let \( P(Y_i = y_i | X_i = x_i) \) be the transition kernel associated with the randomized mapping \( F_i(\cdot) \), then the distribution of the output is given by

\[
p_f(y_i) = \sum_{x_i \in \mathcal{X}} P(Y_i = y_i | X_i = x_i)p(x_i)
\]

for \( y_i \in \mathcal{Y} \). Note that while the distribution of X is tree-structured, the distribution of Y does not factorize according to any tree. In general the Markov random field of Y is a complete graph, which makes the hidden structure learning non-trivial (see [27, 28]).

Given n i.i.d observations \( X_1:n \sim p(\cdot) \), our goal is to learn the underlying structure \( T \). To do this, we use a plug-in estimate of the mutual information \( I(X_i; X_j) \) between pairs of variables. In similar fashion, when noise-corrupted
observations $Y^{1:n}$ are available, we aim to learn the hidden tree structure $T$ of $X$, by estimating the mutual information $I(Y_i; Y_j)$ between pairs of observable variables. Unfortunately, the plug-in mutual information estimate $\hat{I}(X_i; X_j)$ can converge slowly to $I(X_i; X_j)$ in certain cases of countable alphabets (see Corollary 5 of \cite{29}). To avoid such ill-conditioned cases, we make the following assumption.

**Assumption 1.** Assume that for some $c > 1$ there exist $c_1, c_2 > 0$ such that $c_1/k^c \leq p_i(k) \leq c_2/k^c$, for $k \in \mathcal{X}$, and $c_1/(k\ell)^c \leq p_{i,j}(k, \ell) \leq c_2/(k\ell)^c$, for $k, \ell \in \mathcal{X}^2$ and for all $i, j \in \mathcal{V}$. That is, the tuple $\{c, c_1, c_2\}$ satisfies the assumption for all marginal and pairwise joint distributions.

The next assumption guarantees there is a unique tree structure $T$ with exactly $p$ nodes.

**Assumption 2.** The tree $T$ is connected \(I(X_i; X_j) > 0\) for all $i, j \in \mathcal{V}$ and the distribution $p(\cdot)$ of $X$ is not degenerate.

Assumption 2 guarantees convergence for the Chow-Liu algorithm; $T^{\text{CL}} \rightarrow T$. We use the notation $\mathcal{P}_T(c_1, c_2)$ to denote the set of tree-structured distributions which satisfy Assumption 1 and Assumption 2. In particular, we assume that $X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2)$.

### 2.2 Chow-Liu Algorithm

We consider the classical version of the algorithm proposed by Chow and Liu \cite{1}, due to the non-parametric nature of our model. Given $n$ i.i.d samples of the node variables, we first find the plug-in estimates of the pairwise joint distributions and then we evaluate the plug-in mutual information estimates. Finally, the maximum spanning tree algorithm returns the estimated tree. For the rest of the paper, we refer to Algorithm 1 by explicitly mentioning the input data set; if $D = X^{1:n}$, then the input consist of noiseless data and we consider $Z \equiv X$ (see Algorithm 1), furthermore the estimated structure $T^{\text{estimate}}$ is denoted by $T^{\text{CL}}$. Equivalently, if $D = Y^{1:n}$, then the input consists of noisy data, $Z \equiv Y$, and the estimated structure $T^{\text{estimate}}$ is named as $T^{\text{CL}}$. Note that the estimates $T^{\text{CL}}$ and $T^{\text{CL}}$ depend on the value $n$, but for brevity, we write $T^{\text{CL}} \rightarrow T$ and $T^{\text{CL}} \rightarrow T$ instead of $\lim_{n \rightarrow \infty} T^{\text{CL}} \rightarrow T$ and $\lim_{n \rightarrow \infty} T^{\text{CL}} \rightarrow T$ respectively. We continue by analyzing the event of incorrect reconstruction $T^{\text{CL}} \neq T$ (or $T^{\text{CL}} \neq T$), which yields a sufficient condition for exact structure recovery.

**Proposition 1.** The estimated tree $T^{\text{CL}} \neq T$ if and only if there exist two edges $e \equiv (u, \bar{u}) \in T$ and $g \equiv (u, \bar{u}) \in T^{\text{CL}}$ such that $e \notin T^{\text{CL}}, g \notin T$ and $e \in \text{path}_T(u, \bar{u})$.

Intuitively, exact recovery fails when there is at least one edge in the original tree $T$ which does not appears in the estimated tree $T^{\text{CL}}$. We refer the reader to \cite{8} for a proof of Proposition 1. For sake of space, we define the set $E^2$.

**Definition 1.** (The feasibility set $E^2$) Let $e \equiv (u, \bar{u}) \in \mathcal{E}_T$ be an edge and $u, \bar{u} \in \mathcal{V}_T$ be a pair of nodes such that $e \in \text{path}_T(u, \bar{u})$. The set of all such tuples $(e, u, \bar{u})$, is denoted as $E^2$, that is,

$$E^2 \triangleq \{(u, \bar{u}), u, \bar{u} \in \mathcal{E}_T \times \mathcal{V}_T \times \mathcal{V}_T : (u, \bar{u}) \in \text{path}_T(u, \bar{u})\}.$$  \hspace{1cm} (2)

Notice that $T \neq T^{\text{CL}}$ if and only if a tuple $(e, u, \bar{u}) \in E^2$ exists such that $\hat{I}(X_u; X_{\bar{u}}) < I(X_u; X_{\bar{u}})$. That is, if the inequality $\hat{I}(X_u; X_{\bar{u}}) < I(X_u; X_{\bar{u}})$ is satisfied by at least one tuple $(e, u, \bar{u}) \in E^2$, then an error occurs. Under the error event the following holds:

$$\hat{I}(X_u; X_{\bar{u}}) \leq I(X_u; X_{\bar{u}}) \iff \hat{I}(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}}) \leq [\hat{I}(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}})] - [\hat{I}(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}})].$$  \hspace{1cm} (3)

Equivalently, $T = T^{\text{CL}}$ if and only if for all tuples $(e, u, \bar{u}) \in E^2$,

$$I(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}}) > [\hat{I}(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}})] - [\hat{I}(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}})].$$  \hspace{1cm} (4)

The latter yields the following sufficient condition for exact structure recovery: if for all $\ell, \ell' \in \mathcal{V}$

$$\left|\hat{I}(X_{\ell}; X_{\ell'}) - I(X_{\ell}; X_{\ell'})\right| < \frac{1}{2} \min_{(e, u, \bar{u}) \in E^2} \{I(X_u; X_{\bar{u}}) - I(X_u; X_{\bar{u}})\},$$  \hspace{1cm} (5)

then $T^{\text{CL}} = T$. The condition in \cite{8} shows that if the error of mutual information estimates is less than a threshold statistic then exact structured recovery is guaranteed.

\hspace{1cm} \cite{4} Tan et al. have studied the Chow-Liu algorithm for learning forests. Learning hidden forests is an interesting problem but we defer it for future work.
2.3 Information Threshold and Information Order Preservation

We now define a new quantity for tree structured distributions, which we call the information threshold. The sample complexity bounds on exact structure recovery only depend on the distribution through the information threshold, \( I^o \). We first define \( I^o \) and then show how it affects the performance of the structure estimation.

**Definition 2. (Information Threshold)** Let \( e \equiv (w, \tilde{w}) \in \mathcal{E}^2 \) be an edge and \( u, \tilde{u} \in \mathcal{V}_T \) be a pair of nodes such that \( e \in \text{path}_T(u, \tilde{u}) \). The information threshold associated with the model \( p(\cdot) \in \mathcal{P}_T(c_1, c_2) \) is defined as

\[
I^o \triangleq \frac{1}{2} \min_{(e,u,\tilde{u}) \in \mathcal{E} \mathcal{V}^2} \left( I(X_w; X_{\tilde{w}}) - I(X_u; X_{\tilde{u}}) \right).
\]

The minimization in (6) is with respect to all tuples \( (e,u,\tilde{u}) \in \mathcal{E} \mathcal{V}^2 \), while the values \( I(X_i; X_j) > 0 \) are considered fixed for all \( i, j \in \mathcal{V} \). Note that the definition in (6) comes form the sufficient condition in (5), which implies that

\[
|\bar{I}(X_i; X_{\ell}) - I(X_i; X_{\ell'})| < I^o \quad \text{for all } \ell, \ell' \in \mathcal{V} \text{ then } T = T_{\mathcal{CL}}.
\]

**Proposition 2.** If Assumption 2 holds then \( I^o > 0 \).

We prove Proposition 2 in Section A.1, Appendix. Note that under the reasonable assumption that the values \( \min_{(i,j) \in \mathcal{E}} I(X_i; X_j) \) and \( \max_{(i,j) \in \mathcal{E}} I(X_i; X_j) \) are constant relative to \( p \) (see [4]), \( I^o \) is also constant relative to \( p \). In fact, the minimum value in (6) is always attained by an element \( \{(w, \tilde{w}), u, \tilde{u}\} \) of the set \( \mathcal{E} \mathcal{V}^2 \) such that \( w = u \) and \( \tilde{w}, \tilde{u} \) are neighbors (see Appendix, A.1). Thus \( I^o \) characterizes a local property of the graph and it does not depend on \( p \). When the data are noisy, the gap between mutual informations that defines the information threshold will change. If the errors of the mutual information estimates of the noisy variables satisfy the condition

\[
|\bar{I}(Y_i; Y_{\ell}) - I(Y_i; Y_{\ell'})| < \frac{1}{2} \min_{(e,u,\tilde{u}) \in \mathcal{E} \mathcal{V}^2} \left( I(Y_w; Y_{\tilde{w}}) - I(Y_u; Y_{\tilde{u}}) \right),
\]

for all \( \ell, \ell' \in \mathcal{V} \) then \( T_{\mathcal{CL}} = T \), and (8) is derived similarly to (5). The definition of the noisy information threshold naturally results from the previous condition.

**Definition 3. (Noisy Information Threshold)** The noisy information threshold is defined as

\[
I^o \triangleq \frac{1}{2} \min_{(e,u,\tilde{u}) \in \mathcal{E} \mathcal{V}^2} \left( I(Y_w; Y_{\tilde{w}}) - I(Y_u; Y_{\tilde{u}}) \right)
\]

The minimization in (9) is with respect to the feasible set \( \mathcal{E} \mathcal{V}^2 \) of the hidden tree structure \( T \) of \( X \). Note that the distribution of \( Y \) does not factorize according to any tree [28]. In general, learning structures for the case of non-identically distributed noise or for unknown values of failure probability are open problems [6]. Here, for tree structured models, we show that both tasks are feasible. We proceed by introducing a sufficient condition for structure recovery from a dataset \( D = Y_{1:n} \) that can be used as an alternative to (5).

**Definition 4. (Information Order Preservation)** Consider random vectors \( X \in \mathcal{X}^p \) and \( Y \in \mathcal{Y}^p \), such that \( Y = F(X) \), with \( F \) defined as in Section 2.1. We say that the randomized mapping \( F \) is information order-preserving (IOP) relative to \( X \) if and only if, for every tuple \( \{k, l, m, r\} \subset \mathcal{V} \),

\[
I(X_k; X_l) < I(X_m; X_r) \iff I(F_k(X_k); F_l(X_l)) < I(F_m(X_m); F_r(X_r)).
\]

**Example:** If the set \( F \) models an \( M \)-ary erasure channel with the same probability of erasure for all node variables then (10) holds. However, if the probability of erasures are not equal (noise is not identically distributed) then (10) does not hold in general, because

\[
I(Y_i; Y_j) = (1 - q_i)(1 - q_j)I(X_i; X_j) \quad \text{for all } i, j \in \mathcal{V}, \text{ and } q_i, q_j \in [0, 1].
\]

The supporting calculations are given in the Appendix, Section B.2.
3 Recovering the Structure from Noisy Data

First, we introduce a computable bound on the sample complexity for exact noiseless structure learning with high probability. The structure learning condition in (7), combined with results of mutual information estimates by Antos and Kontoyiannis [29], yields the sample complexity bound.

**Theorem 1.** Assume that \( X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2) \) for some \( c \in (1, 2) \). Fix a number \( \delta \in (0, 1) \). There exists a constant \( C > 0 \), independent of \( \delta \) such that, if the number of samples of \( X \) satisfies the inequalities

\[
\frac{n}{\log^2 n} \geq \frac{72 \log \left( \frac{p}{\delta} \right)}{\left( I^o - Cn^{1-c} \right)^2} \text{ and } I^o > Cn^{1-c}, \tag{12}
\]

then Algorithm \( \hat{\alpha} \) with input \( D = X^{1:n} \) returns \( T_{CL} = T \) with probability at least \( 1 - \delta \).

The constant \( C \) depends on the values of constants \( c, c_1, c_2 \) which are defined in Assumption [1]. Specifically, \( C = 3c_2 \left[ c_2^{-1} - 1 \right]/c_1 + c_2^{-1} \int_1^{c_1} u^1/c_2 \log (eu/c_1) + 1/c_1 \right] \). The derivation of \( C \) has been given by Antos and Kontoyiannis [29, Theorem 7]. Note that \( n/\log^2 n = \Omega(\log n^\delta) \) for all \( \epsilon \in (0, 1) \), thus the order of sufficient number samples \( n \) with respect to \( p \) and \( \delta \) for fixed \( I^o \) can be expressed as \( O(\log n^{1+c}(p/\delta)) \), for all \( \zeta > 0 \). In contrast, for fixed \( p \) and \( \delta \) the order relative to \( \zeta \) is \( O(I^o^{-2(1+c)}) \), for all \( \zeta > 0 \). The proof of Theorem [1] is given in Appendix A.2. Theorem 5 (Appendix A.2) provides the corresponding bound of Theorem 1 when Assumption [1] holds for \( c \geq 2 \).

4 Recovering the Hidden Structure from Noisy Data

We continue by providing conditions which guarantee that structure learning is feasible by applying Algorithm \( \hat{\alpha} \) with input \( D = Y^{1:n} \), that is, \( T_{CL}^{Y} \rightarrow T \). The importance of the results lies on the fact that we can simply run the Chow-Liu algorithm on the noisy observations in the same way as we would do if noiseless observations \( X^{1:n} \) were given. In particular, the model of noise might be unknown, or we might be unaware of the existence of noise; however, we can still learn the hidden structure efficiently!

**Convergence of Algorithm \( \hat{\alpha} \)** If \( \mathcal{F} \) is information order-preserving (IOP), then Algorithm \( \hat{\alpha} \) with input \( D = Y^{1:n} \) converges to the true tree \( (T^{1:n} \rightarrow T) \) (see Theorem 2). For IOP randomized mappings, the following ordering with respect to all pairs of nodes

\[
I(X_{i_1}; X_{j_1}) < I(X_{i_2}; X_{j_2}) < \ldots < I(X_{i_r}; X_{j_r}), \text{ and } \rho = \left( \frac{p}{2} \right), i_s, j_s \in \mathcal{V}, s \in [r], \tag{13}
\]

remains unchanged for the observable node variables \( Y \), that is,

\[
(13) \implies I(Y_{i_1}; Y_{j_1}) < I(Y_{i_2}; Y_{j_2}) < \ldots < I(Y_{i_r}; Y_{j_r}), \text{ and } \rho = \left( \frac{p}{2} \right), i_s, j_s \in \mathcal{V}, s \in [r]. \tag{14}
\]

Therefore, the maximum spanning tree algorithm with input weights in (13) returns the same tree structure \( T \) even if the input weights are changed to the corresponding mutual information values of \( Y \)’s in (14). Therefore, for sufficient large number of samples the order is also preserved for the estimates \( \hat{I}(Y_i; Y_j) \), which ensures that Algorithm \( \hat{\alpha} \) is consistent for \( D = Y^{1:n} \), that is, \( T = \lim_{n \to \infty} T_{CL}^{Y} = \lim_{n \to \infty} T_{CL}^{X} \). Thus, the information order preservation property is sufficient, and constitutes an alternative to the condition in (8). Recall that the condition in (8) also shows consistency of Algorithm \( \hat{\alpha} \) because if \( I^o > 0 \), Algorithm \( \hat{\alpha} \) returns \( T_{CL}^{Y} = T \) for sufficiently large \( n \), with high probability. Given that the algorithm converges, we are interested in finding an explicit (computable) bound on the sample complexity. Our main result provides a bound on \( n \) which guarantees exact structure structure recovery \( T_{CL}^{Y} = T \) with probability at least \( 1 - \delta \in (0, 1) \).

**Theorem 2.** Assume that \( X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2) \). Assume that noisy data \( Y \sim p_Y(\cdot) \) are generated by a randomized set of mappings \( \mathcal{F} = \{ F_i : i \in [p] \} \), and \( p_Y(\cdot) \) satisfies the Assumption [1] for some \( c' \in (1, 2), c_1', c_2' > 0 \). Fix \( \delta \in (0, 1) \). There exists a constant \( C' > 0 \) independent of \( \delta \) such that, if \( 1/4 > 0 \) and the number of samples \( n \) of \( Y \) satisfies the inequalities

\[
\frac{n}{\log^2 n} \geq \frac{72 \log \left( \frac{p}{\delta} \right)}{\left( I^o - C'n^{1-c'} \right)^2} \text{ and } I^o > C'n^{1-c'}, \tag{15}
\]

then Algorithm \( \hat{\alpha} \) with input the noisy data \( D = Y^{1:n} \) returns \( T_{CL}^{Y} = T \) with probability at least \( 1 - \delta \). The relationship between \( C' \) and \( c', c_1', c_2' \) is identical to that between \( C \) and \( c, c_1, c_2 \) in Theorem 1.
Notice that the distribution $p_t(\cdot)$ of $Y$ does not factorize according to any tree structure (see [25] [29]). Nevertheless, if $I^o_i > 0$, then we are still able to learn the tree $T$ of the hidden variables $X$. Let $n$ and $n_t$ denote the sufficient number of samples of $X$ and $Y$ respectively and consider $p, \delta, \Gamma, I^o_i$ fixed, then the ratio $n/n_t$ converges to $(I^o/o^o)^{-2(1+\zeta)}$ for all $\zeta > 0$. The latter shows how $n_t$ changes relative to $n$ under the same probability of success for both settings (noiseless and noisy). Although we are primarily interested in conditions ensuring a positive $I^o_i$, a negative $I^o_i$ is also informative. As we will show later, if we can find an appropriate post-processing $Y \rightarrow Z$ such that $I^o_{i,z} > 0$, Theorem 2 applies with $I^o_{i,z} > 0$ as the new threshold, and $I^o_{i,z}$ is defined by replacing $Y$ variables with $Z$ in [26].

Theorem 2 can be applied on a wide class models which satisfies the general Assumptions 1 and 2. To illustrate the effect of noise on the structure learning complexity, we consider two classical noisy channels in the hidden model: the $M$-ary erasure and the (generalized) symmetric channel. We show that a simple comparison of $I^o_i$ and $I^o_i$ determines the impact of noise on the sample complexity, and we present the relationship between the two. Additionally, we provide sufficient conditions for structure recovery in the presence of non-identically distributed noise.

First, we consider the erasure channel and then an extension of the binary symmetric channel to alphabets of size $M$. For these examples, we assume uniform marginal distributions for the hidden node variables, that is, $X_i \sim p_i(k) = 1/M$, for $k \in [M]$ and $M \in \mathbb{N}$, while the assumption $X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2)$ still holds. In the next example, we define the $M$-ary erasure channel and we show that if the randomized mappings $F_1(\cdot), F_2(\cdot), \ldots, F_n(\cdot)$ satisfy appropriate conditions that we introduced earlier, then the structure learning is always feasible from noisy data.

Example 1: $M$-ary Erasure Channel. Let $X \sim p(\cdot) \in \mathcal{P}(c_1, c_2)$. Assume that the randomized mappings $F_i(\cdot)$ “erase” each variable independently with probability $q$, so for all $i \in [p]$, we have $Y_i = X_i$ with probability $1 - q$ and $Y_i = M + 1$ (an erasure) with probability $q$. Then

$$I(Y_i; Y_j) = (1 - q)^2 I(X_i; X_j)$$

(see Appendix B.2). Therefore, $I^o_i = (1 - q)^2 \Gamma^o \leq I^o$ and the information order preservation holds for a fixed value of $q \in [0, 1]$ and $\Gamma > 0$. Given the values of $p, \delta, q$ and $I^o$, Theorem 2 provides the sample complexity for exact structure recovery from noisy observations. For fixed values of $p, \delta$ and $q$, the ratio $n/n_t$ converges to $(1 - q)^{2(1+\zeta)}$ for all $\zeta > 0$. In contrast, if the parameter $q$ is not the same for every node; each $F_i$ erases the $i^{th}$ node value with probability $q_i \in [0, 1)$ then

$$I(Y_i; Y_j) = (1 - q_i)(1 - q_j) I(X_i; X_j)$$

and the condition [3] shows that Algorithm 1 with input $D = Y^{1:n}$ converges; $T^{\text{CL}}_1 \rightarrow T$ if for all tuples $(w, w, u, \bar{u}) \in \mathcal{E}_{1}^{2}$

$$(1 - q_w)(1 - q_u) \geq \frac{I(X_u; X_{\bar{u}})}{I(X_w; X_{\bar{u}})}.$$  

This inequality provides the following sufficient condition; $T^{\text{CL}}_1 \rightarrow T$ if for all $i \in \mathcal{V}$

$$\frac{1 - q_i}{1 - q_j} \in \left(\max_{(w, w, u, \bar{u}) \in \mathcal{E}_{1}^{2}} \frac{I(X_u; X_{\bar{u}})}{I(X_w; X_{\bar{u}})}\right)^{1/2} \cdot \left(\max_{(w, w, u, \bar{u}) \in \mathcal{E}_{1}^{2}} \frac{I(X_u; X_{\bar{u}})}{I(X_w; X_{\bar{u}})}\right)^{-1/2}.$$  

(19)

We continue by showing a feasibility condition for the binary symmetric channel with non-identically distributed noise. The identically distributed noise case was studied earlier by [27].

Example 2: Binary Symmetric Channel. Assume that the hidden variables $X \in \{-1, +1\}$ follow a tree-structured Ising model, the noisy observable variables $Y \in \{-1, +1\}$ are generated by setting $Y_i = X_i$ with probability $1 - q_i$ and $Y_i = -X_i$ with probability $q_i \in [0, 1/2]$, $i \in [p]$, and $q_i$ is the probability of a value to change sign. Structure recovery directly from noisy observations $Y^{1:n}$ is feasible, because the information order preservation property holds when the crossover probability values $q_i$ are equal for all $i \in \mathcal{V}$. The sample complexity of the Chow-Liu algorithm for the i.i.d noise setting has been studied by [27]. To guarantee convergence of Algorithm 1 ($D = Y^{1:n}$) for more general cases, the condition $I^o_i > 0$ should hold. As an example, consider the case of non identically distributed noise, that is, the probability $q_i$ of a flip may differ for each node. In this case, the information order preservation property does not hold for all the possible sequences $(q_1, q_2, \ldots, q_p) \in [0, 1/2]^p$ and we would like to know for which values of $q_i$, $i \in [p]$, Algorithm 1 with $D = Y^{1:n}$ learns the hidden structure. The condition [3] implies that if

$$\frac{1 - 2q_i}{1 - 2q_j} \in \left(\max_{(i,j) \in \mathcal{E}_{T}} |\mathbb{E}[X_iX_j]|\right)^{1/2} \cdot \left(\max_{(i,j) \in \mathcal{E}_{T}} |\mathbb{E}[X_iX_j]|\right)^{-1/2}$$

(20)

Examples of models for which the distribution $p_t(\cdot)$ factorizes according to the hidden structure are trivial models, for instance single edge forests (see [29], [7]).
Figure 1: Estimating the probability $\mathbb{P}(T^{\text{CL}} \neq T)$ for different values of $I^o$ and $n$ through $10^3$ independent runs.

Figure 2: Estimating the probability $\mathbb{P}(T^{\text{CL}} \neq T)$ through $10^4$ samples and $10^3$ independent runs.

and the proof of (20) is given in Appendix B.3. Note that for identical noise $q_i = q_j = q$ it is true that $(1 - 2q_i)/(1 - 2q_j) = 1$ for all $i, j \in \mathcal{Y}$, thus (20) is always satisfied because $|\mathbb{E}[X_i X_j]| \in (0, 1)$, and structure learning is always feasible for this regime (see [1]). If the condition (20) is not satisfied then structure recovery is still feasible by applying an appropriate post-processing on the data $Y^{1:n}$. The post processing procedure requires the values $q_i$ (or estimates of them) to be known. In contrast, Algorithm 1 does not require any information related to the values $q_i$, and its convergence is guaranteed under the condition (20). In the next example we study an extension of the binary symmetric channel to alphabets of size $M$.

**Example 3: Generalized Symmetric Channel.** We define the generalized symmetric channel as follows, assume $X \sim \mathcal{P}_T(c_1, c_2)$, $X \in [M]^p$ and let $Z_i$ for $i \in [p]$ be i.i.d uniform random variables, such that $\mathbb{P}(Z_i = k) = 1/M$, for all $k \in [M]$. Also assume that $Z$ and $X$ are independent, then the $i$th variable of the channel output $Y \in [M]^p$ is defined as

$$ Y_i = F_i(X_i) = \begin{cases} X_i, & \text{with probability } 1 - q \\ Z_i, & \text{with probability } q, \text{ and } q \in [0, 1). \end{cases} $$

Note that the probability of a symbol to remain unchanged is $\mathbb{P}(Y_i = X_i) = (1 - q) + q/M$, and for $M = 2$ it reduces to the binary symmetric channel. Theorem 2 can be directly applied for given values of $p$, $\delta$ and $I^o_i$. Nevertheless, closed form expressions of the quantity $I^o_i$ relative to $I^o$ are unknown. Thus, we consider the regime of sufficiently small $q$ while the hidden tree-shaped distribution is considered fixed. Remark 1 (B.4, Appendix) shows that under
We analyzed the problem of recovering the structure of a tree-shaped graphical model from samples in both the noiseless and noisy settings. To quantify the difference between the necessary number of samples for the noisy and noiseless case, we find the relationship between the information order preservation property holds, thus Algorithm 1 with $D = Y^{1:n}$ returns $T_{CL} \rightarrow T$. To quantify the difference between the necessary number of samples for the noisy and noiseless case, we find the relationship between each $\delta$ and number of samples $n$.

### Proposition 2

**Proof of Proposition 2.** Note that the tuple $(e^*, u^*, \bar{u}^*) \in \mathcal{E}^2$ which minimizes $I^o$ (non-trivially) has the following properties $e^* = (w^*, \bar{w}^*) \in \text{path}_T(u^*, \bar{u}^*)$, also $u^* \equiv w^*$ and $(\bar{w}^*, \bar{u}^*) \in \mathcal{E}_T$ or $(u^*, w^*) \in \mathcal{E}_T$ and $\bar{u}^* \equiv \bar{w}^*$. The latter comes from the data processing inequality [30]. Without loss of generality, assume that the first holds, $u^* \equiv w^*$ and $(\bar{w}^*, \bar{u}^*) \in \mathcal{E}_T$, which results to the three node sub-tree $w^* - \bar{w}^* - \bar{u}^*$. Then $I(X_{w^*}; X_{\bar{w}^*}) = I(X_{w^*}; X_{\bar{u}^*})$ if and only if $I(X_{w^*}; X_{\bar{w}^*} | X_{\bar{u}^*}) = 0$, which implies that $w^* \equiv \bar{w}^* \equiv \bar{u}^*$. The latter is impossible because of the uniqueness of the structure (Assumption 2). 

---

3We consider as trivial the case $\text{path}_T(u^*, \bar{u}^*) = (w^*, \bar{w}^*) \in \mathcal{E}_T$, which does not affect Algorithm 1 because an edge can not be mismatched with itself.

### 5 Simulations

To demonstrate the relationships between $\delta$, $n$ and $I^o$, $I^o$, we estimate $T_{CL}$, $T_{CL}^\dagger$ and $\delta$ through $10^3$ independent runs on tree-structured synthetic data, for different values of $n$ in the interval $[10^3, 10^4]$ and $p = 10$. The variable nodes are binary. Figure 1 (Left) illustrates the exponential decay of probability of failure while $I^o$ increases and (Right) the relationship between the log-probability of error as a function of the squared information threshold (see Theorem 1 for a comparison with the theoretical results). Finally, Figure 2 presents the effect of noise of a BSC for four different values of the crossover $q$ and number of samples $n = 10^4$. Our experimental results support and confirm the theoretical bounds (scaling) in Theorem 1 and Theorem 2.

### 6 Conclusion & Future Directions

We analyzed the problem of recovering the structure of a tree-shaped graphical model from samples in both the noiseless and noisy settings. We defined a new quantity, the information threshold $I^o$, which characterizes when structure recovery is possible. This quantity arises naturally in the error analysis for the Chow-Liu algorithm. We prove novel bounds on the sample complexity of structure recovery that depend on the number of nodes $p$, the probability of failure $\delta$ and the fundamental statistic $I^o$, which captures the complexity of the learning task. As a main result, we introduced the first sample complexity bound for learning hidden tree structured non-parametric models. Our results can be used to show how noise affects the sample complexity of learning for a variety of standard models, including models for which the noise is not identically distributed.

The non-parametric nature of the model raises further theoretical questions. The relationship of $I^o$ and $I^o$ is related with open problems in information theory related to Strong Data Processing Inequalities (SDPIs), for which tight characterizations are only known for a few channels. In our situation a general analytical relationship may be similarly challenging. Additionally, from a practical standpoint, we may wish to estimate the sample size needed to guarantee recovery with a pre-specified error probability. To do so would require knowing $I^o$ before collecting the full data; since $I^o$ depends on the noise model, we could find such a bound by considering a reasonable class of underlying models and taking the worst case. An interesting open question for future work is how to effectively estimate $I^o$ from (auxiliary) training data rather than relying on such a priori modeling assumptions.

### A Proofs for the noiseless case

#### A.1 Proposition 2: Positivity of the information threshold

Proposition 2 states that if Assumption 2 holds then $I^o > 0$.

**Proof of Proposition 2.** We consider as trivial the case $\text{path}_T(u^*, \bar{u}^*) = (w^*, \bar{w}^*) \in \mathcal{E}_T$, which does not affect Algorithm 1 because an edge can not be mismatched with itself.
A.2 Sample complexity for the noiseless setting: Theorem 1 and Theorem 3

The corresponding result of Theorem 1 for the case of $c \geq 2$ (Assumption 1) follows.

**Theorem 3.** Assume that $X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2)$ for some $c \geq 2$. Fix a number $\delta \in (0, 1)$. If the number of samples of $X$ satisfies the inequalities

$$\frac{n}{\log^2 n} \geq \frac{72 \log \left( \frac{p}{\delta} \right)}{\left( I^0 - C \log n / \sqrt{n} \right)^2} \quad \text{and} \quad I^0 > C \log n / \sqrt{n}, \quad (22)$$

for a constant $C > 0$, then Algorithm 1 with input $D = X^{1:n}$ returns $T^{\text{CL}} = T$ with probability at least $1 - \delta$.

We prove Theorem 1 and Theorem 3 in parallel.

**Proof.** To calculate the probability of the exact structure recovery we use a concentration inequality and rate of convergence for entropy estimates by Antos and Kontoyiannis [29]. Their Corollary 1 shows how the estimate $\hat{H}_n$ is distributed around its mean $E[\hat{H}_n]$. For every $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\mathbb{P} \left[ \left| \hat{H}_n - E[\hat{H}_n] \right| > \epsilon \right] \leq 2e^{-n \epsilon^2 / 2 \log^2 n}. \quad (23)$$

The plug-in entropy estimate is a biased estimator and $H \geq E[\hat{H}_n]$. Under their Assumption 1 in Theorem 7 [29] they characterize the bias as follows:

$$H - E[\hat{H}_n] = \Theta \left( n^{1-\epsilon} \right) \quad \text{for } c \in (1, 2), \quad (24)$$

and

$$\Omega \left( n^{1-\epsilon} \right) = H - E[\hat{H}_n] = \mathcal{O} \left( n^{-1/2} \log n \right) \quad \text{for } c \in [2, \infty). \quad (25)$$

We continue by considering the case of $c \in (1, 2)$. The case of $c \geq 2$ can be similarly derived.

$$\mathbb{P} \left[ \left| \hat{H}_n - H \right| > \epsilon \right] = \mathbb{P} \left[ \left| \hat{H}_n - E[\hat{H}_n] + E[\hat{H}_n] - H \right| > \epsilon \right]$$

$$\leq \mathbb{P} \left[ \left| \hat{H}_n - E[\hat{H}_n] \right| + \left| E[\hat{H}_n] - H \right| > \epsilon \right]$$

$$= \mathbb{P} \left[ \left| \hat{H}_n - E[\hat{H}_n] \right| > \epsilon - \left| E[\hat{H}_n] - H \right| \right]$$

$$\leq \mathbb{P} \left[ \left| \hat{H}_n - E[\hat{H}_n] \right| > \epsilon - Cn^{1-\epsilon} \right]$$

$$\leq 2e^{-n \left( \epsilon - Cn^{1-\epsilon} \right)^2 / 2 \log^2 n}, \quad (26)$$

the last inequality comes from (23) and (24).

Notice that for non-trivial bounds we have to assume that $\epsilon > Cn^{1-\epsilon}$. In addition, $\epsilon$ is error of estimation and we choose $\epsilon = I^0 / 3$, to satisfy property (7). Thus the following assumption is necessary

$$I^0 > 3Cn^{1-\epsilon}. \quad (27)$$

The last property can be always satisfied for a sufficiently large value of $n$.

By combining the above and applying the property $I(X; Y) = H(X) + H(Y) - H(X, Y)$ we have

$$\mathbb{P} \left[ \left| I(X; X^t) - I(X; X^t) \right| > I^0 \right]$$

$$= \mathbb{P} \left[ \left| \hat{H}(X^t) + \hat{H}(X^t) - \hat{H}(X^t, X^t) - H(X^t, X^t) + H(X^t, X^t) \right| > I^0 \right]$$

$$\leq \mathbb{P} \left[ \left| \hat{H}(X^t) - H(X^t) \right| + \left| \hat{H}(X^t) - H(X^t) \right| + \left| H(X^t, X^t) - \hat{H}(X^t, X^t) \right| > I^0 \right]$$

$$\leq \mathbb{P} \left\{ \left| \hat{H}(X^t) - H(X^t) \right| > \frac{I^0}{3} \right\} \bigcup \left\{ \left| \hat{H}(X^t) - H(X^t) \right| > \frac{I^0}{3} \right\}$$

$$\bigcup \left\{ \left| H(X^t, X^t) - \hat{H}(X^t, X^t) \right| > \frac{I^0}{3} \right\}$$

$$\leq 6e^{-n \left( \frac{I^0}{3} - Cn^{1-\epsilon} \right)^2 / 2 \log^2 n}, \quad (28)$$
and the last inequality is a consequence of (26).

To guarantee (7) we apply union bound on the events \(\{\left|\hat{I}(X_\ell; X_{\bar{\ell}}) - I(X_\ell; X_{\bar{\ell}})\right| > I^o\}\) for all \(\ell, \bar{\ell} \in \mathcal{V}\), since there are \(\binom{p}{2}\) pairs we choose

\[
\delta = \left(\frac{p}{2}\right)6e^{-n\left(\frac{4n}{3} - Cn^{1-\frac{1}{\omega}}\right)/2\log^2 n}.
\] (29)

To conclude, if

\[
\frac{n}{\log^2 n} \geq \frac{2\log \left(\frac{6(\delta)}{\delta}\right)}{\left(\frac{1}{3} - Cn^{1-\frac{1}{\omega}}\right)^2} \quad \text{and} \quad I^o > 3Cn^{1-\frac{1}{\omega}},
\] (30)

then the probability of exact recovery is at least \(1 - \delta\).

Note that the inequality \(8\log (p/\delta) > 2\log (6(p)/\delta)\) holds for \(p \geq 3\). We use this to simplify the statement of our bounds. The statement of Theorem 3 is similarly derived.

\[\square\]

\section{B Proofs for the noisy case}

\subsection{B.1 Proof of Theorem 2}

The proof of Theorem 2 is similar to the proof of Theorem 1 in Section A.2, Appendix. The difference is introduced by the convergence condition in (8), that is the error on the mutual information estimates should be less than the noisy information threshold \(I^o\).

\textbf{Proof.} Note that for the entropy estimates of \(Y\), equations (23) up to (26) hold, with possibly different constant \(c', c_1', c_2', C'\) (see Assumption 1), and consider \(c' \in (1, 2)\) (the case \(c' \geq 2\) is similar, see also the proof of Theorem 3). Nevertheless, corresponding error of estimation \(\epsilon\) has to be \(\epsilon = I^o/3\) (see also the condition (8)). Thus, (27) becomes

\[
I^o > 3C'n^{1-\frac{1}{\omega}}.
\] (31)

and (28) now is written as

\[
P\left[\left|\hat{I}(Y_i; Y_{\bar{i}}) - I(Y_i; Y_{\bar{i}})\right| > I^o\right] \leq 6e^{-n\left(\frac{4n}{3} - C'\left(n^{1-\frac{1}{\omega'}}\right)\right)/2\log^2 n}.
\] (32)

Finally, by applying union bound over the pairs \(\ell, \bar{\ell} \in \mathcal{V}\) we derive the statement of Theorem 2 similarly to the proof of Theorem 1 in (29) and (30) (see Appendix A.2).

\(\square\)

\subsection{B.2 M-ary erasure channel}

For the \(M\)-ary erasure channel, we have the following relation for all \(i, j \in \mathcal{V}\) and \(q_i, q_j \in [0, 1)\):

\[
I(Y_i; Y_j) = (1 - q_i)(1 - q_j)I(X_i; X_j).
\] (33)
Proof. We start by expanding the mutual information from the definition and pulling out the erasure event:

\[
I(Y_i; Y_j) = \sum_{y_i, y_j \in [M+1]^2} p_t(y_i, y_j) \log \frac{p_t(y_i, y_j)}{p_t(y_i)p_t(y_j)}
\]

\[
= \sum_{y_i, y_j \in [M]^2} p_t(y_i, y_j) \log \frac{p_t(y_i, y_j)}{p_t(y_i)p_t(y_j)} + \sum_{y_j \in [M]} p_t(y_j, M+1) \log \frac{p_t(y_j, M+1)}{p_t(y_j)p_t(M+1)}
\]

\[
+ \sum_{y_j \in [M]} p_t(M+1, y_j) \log \frac{p_t(M+1, y_j)}{p_t(M+1)p_t(y_j)} + p_t(M+1, M+1) \log \frac{p_t(M+1, M+1)}{p_t(M+1)p_t(M+1)}
\]

\[
= \sum_{y_i, y_j \in [M]^2} p_t(y_i, y_j) \log \frac{p_t(y_i, y_j)}{p_t(y_i)p_t(y_j)}
\]

\[
= \sum_{x_i, x_j \in [M]^2} (1-q_i)(1-q_j)p(x_i, x_j) \log \frac{(1-q_i)(1-q_j)p(x_i, x_j)}{(1-q_i)(1-q_j)p(x_i)p(x_j)}
\]

\[
= (1-q_i)(1-q_j)I(X_i; X_j).
\]

An erasure occurs independently on each node variable observable and independently with respect to the \( X \), thus \( p_t(y_i, M+1) = p_t(y_i)p_t(M+1) \), for any \( y_i \in [M+1] \) and \( p_t(M+1, y_j) = p_t(M+1)p_t(y_j) \) for any \( y_j \in [M+1] \). The last gives \( (34) \).

\[\square\]

### B.3 Binary Symmetric Channel with non-identically distributed noise

We wish to show if the following holds then \( I \) which guarantee that \( I \) is increasing for \( x > 0 \) to \( T = T^{\text{CL}} \):

\[
I(Y_i, Y_j) = \left(1 - \frac{2q_i}{1 - 2q_i}\right) \left(1 - \frac{1}{\tanh(\beta)}\right)
\]

for all \( i, j \in \mathcal{V} \).

Proof. To prove the condition \( \text{(20)} \) we find the values of the sequence of crossover probabilities \( q_1, q_2, \ldots, q_k \in [0,1/2) \) which guarantee that \( I_t > 0 \). The mutual information of two binary random variables \( Y_i, Y_j \in \{-1, +1\} \) is

\[
I(Y_i, Y_j) = \frac{1}{2} \log_2 \left( (1 - \mathbb{E}[Y_iY_j])^{1-\mathbb{E}[Y_i][Y_j]} (1 + \mathbb{E}[Y_iY_j])^{1+\mathbb{E}[Y_i][Y_j]} \right).
\]

(36)

The definition of \( I_t \) (Definition \( \text{(3)} \) and \( \text{(36)} \) give

\[
I_t = \frac{1}{2} I(Y_w; Y_{\bar{w}}) - I(Y_u; Y_{\bar{u}})
\]

(37)

\[
= \frac{1}{2} \log_2 \frac{(1 - \mathbb{E}[Y_wY_{\bar{w}}])^{1-\mathbb{E}[Y_w][Y_{\bar{w}}]} (1 + \mathbb{E}[Y_wY_{\bar{w}}])^{1+\mathbb{E}[Y_w][Y_{\bar{w}}]}}{(1 - \mathbb{E}[Y_uY_{\bar{u}}])^{1-\mathbb{E}[Y_u][Y_{\bar{u}}]} (1 + \mathbb{E}[Y_uY_{\bar{u}}])^{1+\mathbb{E}[Y_u][Y_{\bar{u}}]}}.
\]

(38)

Define the symmetric function \( f(\cdot) \) by

\[
f(x) \triangleq (1 - x)^{1-x} (1 + x)^{1+x} \equiv f(|x|),
\]

(39)

then

\[
I_t = \frac{1}{2} \log_2 \frac{f(\mathbb{E}[Y_wY_{\bar{w}}])}{f(\mathbb{E}[Y_uY_{\bar{u}}])}
\]

(40)

\[
= \frac{1}{2} \log_2 \frac{f((1 - 2q_w)(1 - 2q_{\bar{w}}))\mathbb{E}[X_wX_{\bar{w}}]}{f((1 - 2q_u)(1 - 2q_{\bar{u}}))\mathbb{E}[X_uX_{\bar{u}}] \prod_{(i,j) \in \text{path}_T(u, \bar{u}) \setminus \{w, \bar{w}\}} \mathbb{E}[X_iX_j]}
\]

(41)

for the last equality we used the correlation decay property (see \( \text{(3), (7)} \) and the fact that for \( \pm 1 \)-valued variables the binary symmetric channel can be consider as multiplicative binary noise \( \text{(7)} \). Note that \( f(x) \) is increasing for \( x > 0 \). To guarantee that \( I_t > 0 \) we need

\[
(1 - 2q_w)(1 - 2q_{\bar{w}})|\mathbb{E}[X_wX_{\bar{w}}]| > (1 - 2q_u)(1 - 2q_{\bar{u}})|\mathbb{E}[X_uX_{\bar{u}}]|
\]

(42)

\[
\prod_{(i,j) \in \text{path}_T(u, \bar{u}) \setminus \{w, \bar{w}\}} |\mathbb{E}[X_iX_j]|<
\]

(43)
This holds if and only if
\[
\frac{(1 - 2q_w)(1 - 2\bar{q})}{(1 - 2q_u)(1 - 2\bar{q}_u)} > \prod_{(i,j) \in \text{path}_T(u, \bar{u}) \setminus (w, \bar{w})} |\mathbb{E}[X_iX_j]|. \tag{44}
\]
Recall that (44) should hold for all \((w, \bar{w}) \in \mathcal{E}\) and for all \(u, \bar{u} \in \mathcal{V}\) such that \((w, \bar{w}) \in \text{path}_T(u, \bar{u})\). In addition,
\[
\prod_{(i,j) \in \text{path}_T(u, \bar{u}) \setminus (w, \bar{w})} |\mathbb{E}[X_iX_j]| \leq \left( \max_{(i,j) \in \mathcal{E}_T} |\mathbb{E}[X_iX_j]| \right)^{|\text{path}_T(u, \bar{u})| - 1}. \tag{45}
\]
The last two inequalities give the sufficient condition
\[
\frac{(1 - 2q_w)(1 - 2\bar{q})}{(1 - 2q_u)(1 - 2\bar{q}_u)} > \left( \max_{(i,j) \in \mathcal{E}_T} |\mathbb{E}[X_iX_j]| \right)^{|\text{path}_T(u, \bar{u})| - 1}. \tag{46}
\]
As a consequence, if
\[
\frac{(1 - 2q_i)}{(1 - 2q_j)} < \left( \frac{\max_{(i,j) \in \mathcal{E}_T} |\mathbb{E}[X_iX_j]|}{\max_{(i,j) \in \mathcal{E}_T} |\mathbb{E}[X_iX_j]|} \right) \quad \forall i, j \in \mathcal{V}, \tag{47}
\]
then none of the edges will be missed, \(T = T^{CL}\). Note that for the case of i.i.d. noise \((q_i = q_j\) for all \(i, j \in \mathcal{V}\)) the inequality always holds because \(\max_{(i,j) \in \mathcal{E}_T} |\mathbb{E}[X_iX_j]| \in (0, 1)\). \(\square\)

B.4 \(M\)-ary Symmetric Channel

**Lemma 1.** Let \(A, B \in [L]\) be two discrete random variables, such that \(A \sim p_A(\cdot)\), \(B \sim p_B(\cdot)\) and \(H(A) < H(B)\). Assume \(A', B' \in [L]\) and \(A' \sim p_{A'}(\cdot)\) and \(B' \sim p_{B'}(\cdot)\) and \(q \in [0, 1/2)\) such that
\[
p_{A'}(\ell) = (1 - q)^2 p_A(\ell) + \frac{1 - (1 - q)^2}{L}, \quad \text{for all } \ell \in [L], \tag{48}
\]
\[
p_{B'}(\ell) = (1 - q)^2 p_B(\ell) + \frac{1 - (1 - q)^2}{L}, \quad \text{for all } \ell \in [L]. \tag{49}
\]
Then \(H(A') < H(B')\) for sufficiently small values of \(q > 0\).

**Proof.** Define \(\epsilon \triangleq (1 - (1 - q)^2) / L\), so \(0 < \epsilon < q\) for any \(L \geq 2\).
\[
p_{A'}(\ell) \log_2 p_{A'}(\ell) = (1 - q)^2 p_A(\ell) \log_2 (1 - q)^2 p_A(\ell) + \epsilon (1 - q)^2 p_A(\ell) + \epsilon \log_2 (1 - q)^2 p_A(\ell) + \epsilon \log_2 ((1 - q)^2 p_A(\ell)) + O(\epsilon^2) \tag{50}
\]
\[
= (1 - q)^2 p_A(\ell) \log_2 ((1 - q)^2 p_A(\ell)) + \epsilon \log_2 ((1 - q)^2 p_A(\ell)) + O(\epsilon^2) \tag{51}
\]
\[
= (1 - q)^2 p_A(\ell) \log_2 ((1 - q)^2 p_A(\ell)) + \epsilon + O(\epsilon^2) \tag{52}
\]
to derive (52) recall that \(\log(1 + x) = x + O(x^2)\) for \(x \to 0^+\). \(x = \epsilon / (1 - q)^2 p_A(\ell)\) and the last gives
\[
p_{A'}(\ell) \log_2 p_{A'}(\ell) = (1 - q)^2 p_A(\ell) \log_2 ((1 - q)^2 p_A(\ell)) + \epsilon (1 + \log_2 ((1 - q)^2 p_A(\ell))) + O(\epsilon^2). \tag{53}
\]
Also,
\[
p_{B'}(\ell) \log_2 p_{B'}(\ell) = (1 - q)^2 p_B(\ell) \log_2 ((1 - q)^2 p_B(\ell)) + \epsilon (1 + \log_2 ((1 - q)^2 p_B(\ell))) + O(\epsilon^2). \tag{54}
\]
Expanding both sides of the inequality \(H(A) < H(B)\) we obtain the following:
\[
- \sum_{\ell=1}^{L} p_A(\ell) \log_2 p_A(\ell) < - \sum_{\ell} p_B(\ell) \log_2 p_B(\ell) \tag{55}
\]
\[
- \sum_{\ell=1}^{L} (1 - q)^2 p_A(\ell) \log_2 ((1 - q)^2 p_A(\ell)) < - \sum_{\ell} (1 - q)^2 p_B(\ell) \log_2 ((1 - q)^2 p_B(\ell)) \tag{56}
\]
We consider the extension an extension of the binary symmetric channel to alphabets of size \( M \). Also assume that

\[ I (X; Y) \]

The last inequality together with (54) and (55) give \( H (A') < H (B') \).

We consider the extension an extension of the binary symmetric channel to alphabets of size \( M \) as follows. Assume \( X \sim \mathcal{P} (c_1, c_2) \), \( X \in [M] \) and let \( Z_i \) for \( i \in [p] \) be i.i.d uniform random variables, \( \mathbb{P} (Z_i = k) = 1/M \), for all \( k \in [M] \). Also assume that \( Z \) and \( X \) are independent, then the noisy output variable \( Y \in [M]^p \) of the channel is defined for \( q \in [0, 1] \) as

\[
Y_i = F_i (X_i) = \begin{cases} X_i, & \text{with probability } 1 - q \\ Z_i, & \text{with probability } q \end{cases}
\]

**Lemma 2.** The distribution of the two output variables \( Y_i, Y_j \) of the M-ary symmetric channel can be expressed as

\[
\mathbb{P} (Y_i = y_i, Y_j = y_j) = (1 - q)^2 \mathbb{P} (X_i = y_i, X_j = y_j) + \frac{1 - (1 - q)^2}{M^2}.
\]

**Proof.** This is a straightforward calculation:

\[
p_{ij} (y_i, y_j) = \mathbb{P} (Y_i = y_i, Y_j = y_j) = (1 - q)^2 \mathbb{P} (X_i = y_i, X_j = y_j) + q (1 - q) \mathbb{P} (Z_i = y_i, X_j = y_j)
\]

\[
= (1 - q)^2 \mathbb{P} (X_i = y_i, X_j = y_j) + q^2 \mathbb{P} (Z_i = y_i, Z_j = y_j) + 2 q (1 - q) \mathbb{P} (Z_i = y_i, Z_j = y_j)
\]

\[
= (1 - q)^2 \mathbb{P} (X_i = y_i, X_j = y_j) + 2 q^2 \mathbb{P} (Z_i = y_i, Z_j = y_j) + \frac{1 - (1 - q)^2}{M^2}
\]

**Remark 1.** Consider \( X_k, X_\ell, X_m, X_r \) as four distinct inputs variables of the M-ary symmetric channel (defined in Section 2) with corresponding outputs \( Y_k, Y_\ell, Y_m, Y_r \). If the crossover probability \( q \) is sufficiently small and \( I (X_k; X_\ell) < I (X_m; X_r) \) then \( I (Y_k; Y_\ell) < I (Y_m; Y_r) \).

**Proof.** Note that the assumption of uniform marginal distributions for all four \( X_k, X_\ell, X_m, X_r \), implies that \( Y_k, Y_\ell, Y_m, Y_r \) also have uniform marginal distributions. Thus, it is sufficient to show that

\[
H (X_k, X_\ell) > H (X_m, X_r) \Rightarrow H (Y_k, Y_\ell) > H (Y_m, Y_r).
\]

**Lemma 2** shows that

\[
p_{ij} (y_k, y_\ell) = (1 - q)^2 p (x_k, x_\ell) + \frac{1 - (1 - q)^2}{M^2},
\]

\[
p_{ij} (y_m, y_r) = (1 - q)^2 p (x_m, x_r) + \frac{1 - (1 - q)^2}{M^2}.
\]

Then we consider \( M^2 = L \) and Lemma 1 gives (67).

**Lemma 3.** Consider \( X_w, X_\bar{w}, X_u, X_\bar{u} \) as inputs variables of the M-ary symmetric channel (defined in Section 2) with corresponding outputs \( Y_w, Y_\bar{w}, Y_u, Y_\bar{u} \). \( X \sim \mathcal{P} (c_1, c_2) \) and \( ((w, \bar{w}), (u, \bar{u})) \in \mathcal{E} V^2 \). If the crossover probability \( q \) is sufficiently small \( q \to 0 \) then

\[
I (Y_w; Y_\bar{w}) - I (Y_u; Y_\bar{u}) = (1 - q)^2 \left[ I (X_w; X_\bar{w}) - I (X_u; X_\bar{u}) \right] - (1 - q)^2 \left[ KL (U || p (x_w, x_\bar{w})) - KL (U || p (x_u, x_\bar{u})) \right] + \mathcal{O} (\epsilon^2),
\]

where \( \epsilon = (1 - q)^2 / M^2 \) and \( U \) is the uniform distribution on the alphabet \([M]^2\).
Proof. Recall that the marginal distributions of each node variable $X_w$, $X_{\bar{w}}$, $X_u$, $X_{\bar{u}}$ is uniform and this implies that the marginal distributions of the corresponding outputs $Y_w$, $Y_{\bar{w}}$, $Y_u$, $Y_{\bar{u}}$ are uniform as well. Note that $\mathbf{Y} \in [M]^p$ and $\mathbf{X} \in [M]^p$. In addition, the pairwise joint distributions of $\mathbf{Y}$ in terms of the corresponding joint pairwise distributions of $\mathbf{X}$

$$\mathbb{P}
(Y_w = x_w, Y_{\bar{w}} = x_{\bar{w}}) = (1 - q)^2 p(x_w, x_{\bar{w}}) + \frac{1 - (1 - q)^2}{M^2}, \quad x_w, x_{\bar{w}} \in [M]^2, \quad (70)$$

$$\mathbb{P}
(Y_u = x_u, Y_{\bar{u}} = x_{\bar{u}}) = (1 - q)^2 p(x_u, x_{\bar{u}}) + \frac{1 - (1 - q)^2}{M^2}, \quad x_u, x_{\bar{u}} \in [M]^2. \quad (71)$$

and we denote the probability mass function of the pairs $X_w, X_{\bar{w}}$ and $X_u, X_{\bar{u}}$ by $p(x_w, x_{\bar{w}}) \triangleq \mathbb{P}(X_w = x_w, X_{\bar{w}} = x_{\bar{w}})$ and $p(x_u, x_{\bar{u}}) \triangleq \mathbb{P}(X_u = x_u, X_{\bar{u}} = x_{\bar{u}})$ for $(x_w, x_{\bar{w}}), (x_u, x_{\bar{u}}) \in [M]^2$ and similarly, for the noisy versions $Y_w, Y_{\bar{w}}$ and $Y_u, Y_{\bar{u}}$, we use $p_\epsilon(x_w, x_{\bar{w}}) \triangleq \mathbb{P}(Y_w = x_w, Y_{\bar{w}} = x_{\bar{w}})$ and $p_\epsilon(x_u, x_{\bar{u}}) \triangleq \mathbb{P}(Y_u = x_u, Y_{\bar{u}} = x_{\bar{u}})$ for $(x_w, x_{\bar{w}}), (x_u, x_{\bar{u}}) \in [M]^2$. Thus,

$$I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}}) = -H(X_w, X_{\bar{w}}) + H(X_u, X_{\bar{u}}) \quad (72)$$

$$= \sum_{x_w, x_{\bar{w}} \in [M]^2} p(x_w, x_{\bar{w}}) \log p(x_w, x_{\bar{w}}) - \sum_{x_u, x_{\bar{u}} \in [M]^2} p(x_u, x_{\bar{u}}) \log p(x_u, x_{\bar{u}}) \quad (73)$$

$$= \sum_{x_w, x_{\bar{w}} \in [M]^2} p(x_w, x_{\bar{w}}) \log (1 - q)^2 p(x_w, x_{\bar{w}}) - \sum_{x_u, x_{\bar{u}} \in [M]^2} p(x_u, x_{\bar{u}}) \log (1 - q)^2 p(x_u, x_{\bar{u}}) \quad (74)$$

$$= \frac{1}{(1 - q)^2} \sum_{x_w, x_{\bar{w}} \in [M]^2} (1 - q)^2 p(x_w, x_{\bar{w}}) \log (1 - q)^2 p(x_w, x_{\bar{w}}) \quad (75)$$

$$- \frac{1}{(1 - q)^2} \sum_{x_u, x_{\bar{u}} \in [M]^2} (1 - q)^2 p(x_u, x_{\bar{u}}) \log (1 - q)^2 p(x_u, x_{\bar{u}}), \quad (76)$$

where (73) holds because the marginal distributions are uniform. Define $\epsilon \triangleq (1 - (1 - q)^2) / M^2$, so $0 < \epsilon < q$ for any $M^2 \geq 2$. Similarly to Lemma[1] for any $x_w, x_{\bar{w}} \in [M]^2$

$$p_\epsilon(x_w, x_{\bar{w}}) \log_2 p_\epsilon(x_w, x_{\bar{w}}) = ((1 - q)^2 p(x_w, x_{\bar{w}}) + \epsilon) \log_2 ((1 - q)^2 p(x_w, x_{\bar{w}}) + \epsilon) \quad (77)$$

$$= (1 - q)^2 p(x_w, x_{\bar{w}}) \log_2 ((1 - q)^2 p(x_w, x_{\bar{w}})) + \epsilon \log_2 (1 - q)^2 p(x_w, x_{\bar{w}}) + \epsilon \log_2 ((1 - q)^2 p(x_w, x_{\bar{w}})) + \mathcal{O}(\epsilon^2). \quad (78)$$

$$= (1 - q)^2 p(x_w, x_{\bar{w}}) \log_2 ((1 - q)^2 p(x_w, x_{\bar{w}})) + \epsilon \log_2 ((1 - q)^2 p(x_w, x_{\bar{w}})) + \epsilon \log_2 (1 - q)^2 p(x_w, x_{\bar{w}}) + \mathcal{O}(\epsilon^2). \quad (79)$$

Here the last equality holds because $\log (1 + \epsilon / (1 - q)^2 p(x_w, x_{\bar{w}})) = \epsilon / (1 - q)^2 p(x_w, x_{\bar{w}}) + \mathcal{O}(\epsilon^2)$ for $\epsilon \to 0$, while $p(x_w, x_{\bar{w}})$ and $M$ are considered fixed. Also,

$$p_\epsilon(x_u, x_{\bar{u}}) \log_2 p_\epsilon(x_u, x_{\bar{u}}) = (1 - q)^2 p(x_u, x_{\bar{u}}) \log_2 ((1 - q)^2 p(x_u, x_{\bar{u}})) + \epsilon \log_2 ((1 - q)^2 p(x_u, x_{\bar{u}})) + \mathcal{O}(\epsilon^2). \quad (81)$$

Now, we add and subtract the terms $(1 - q)^2 \sum x_w, x_{\bar{w}} \epsilon \log_2 (2(1 - q)^2 p(x_w, x_{\bar{w}})) + \mathcal{O}(\epsilon^2)$ and $(1 - q^2) \sum x_u, x_{\bar{u}} \epsilon \log_2 (2(1 - q)^2 p(x_u, x_{\bar{u}})) + \mathcal{O}(\epsilon^2)$ in (76), which gives

$$I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}}) \quad (82)$$

$$= \frac{1}{(1 - q)^2} \sum_{x_w, x_{\bar{w}} \in [M]^2} [(1 - q)^2 p(x_w, x_{\bar{w}}) \log (1 - q)^2 p(x_w, x_{\bar{w}}) + \epsilon \log_2 (2(1 - q)^2 p(x_w, x_{\bar{w}})) + \mathcal{O}(\epsilon^2)]$$

$$- \frac{1}{(1 - q)^2} \sum_{x_u, x_{\bar{u}} \in [M]^2} [(1 - q)^2 p(x_u, x_{\bar{u}}) \log (1 - q)^2 p(x_u, x_{\bar{u}}) + \epsilon \log_2 (2(1 - q)^2 p(x_u, x_{\bar{u}})) + \mathcal{O}(\epsilon^2)]$$

$$+ \frac{1}{(1 - q)^2} \sum_{x_u, x_{\bar{u}} \in [M]^2} \epsilon \log_2 (2(1 - q)^2 p(x_u, x_{\bar{u}})) + \mathcal{O}(\epsilon^2). \quad (83)$$

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The last combined with (80) and (81) gives
\[
I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}})
= \frac{1}{1 - q^2} [H(Y_w, Y_{\bar{w}}) - H(Y_u, Y_{\bar{u}})]
+ \frac{\epsilon}{1 - q^2} \left[ \sum_{x_u, x_{\bar{u}} \in [M]^2} \log_2 \left( \frac{p(x_u, x_{\bar{u}})}{q(x_u)} \right) \right] + O(\epsilon^2)
= \frac{1}{1 - q^2} \left[ I(Y_w; Y_{\bar{w}}) - I(Y_u; Y_{\bar{u}}) \right]
+ \frac{\epsilon}{1 - q^2} \left[ \sum_{x_u, x_{\bar{u}} \in [M]^2} \log_2 \left( \frac{p(x_u, x_{\bar{u}})}{q(x_u)} \right) \right] + O(\epsilon^2),
\]
where the last line holds because the marginal distribution of each $Y$ is uniform. The definition of $\epsilon$, $\epsilon \doteq (1 - (1 - q)^2)^{-1}$, together with (85) give
\[
I(Y_w; Y_{\bar{w}}) - I(Y_u; Y_{\bar{u}})
= (1 - q)^2 [I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}})]
- \frac{1 - (1 - q)^2}{M^2} \left[ \sum_{x_u, x_{\bar{u}} \in [M]^2} \log_2 \left( \frac{p(x_u, x_{\bar{u}})}{1/M^2} \right) - \sum_{x_u, x_{\bar{u}} \in [M]^2} \log_2 \left( \frac{p(x_u, x_{\bar{u}})}{1/M^2} \right) \right] + O(\epsilon^2)
= (1 - q)^2 [I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}})]
- \frac{1 - (1 - q)^2}{M^2} \left[ \sum_{x_u, x_{\bar{u}} \in [M]^2} \frac{1}{M^2} \log_2 \left( \frac{1/M^2}{p(x_u, x_{\bar{u}})} \right) + \sum_{x_u, x_{\bar{u}} \in [M]^2} \frac{1}{M^2} \log_2 \left( \frac{1/M^2}{p(x_u, x_{\bar{u}})} \right) \right] + O(\epsilon^2)
= (1 - q)^2 [I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}})] - (1 - (1 - q)^2) [\text{KL}(U || p(x_w, x_{\bar{w}})) - \text{KL}(U || p(x_u, x_{\bar{u}}))] + O(\epsilon^2).\]
This completes the proof.

\[\square\]

C Generalized Definitions

We extend the definitions of information threshold and information order preservation relative to a set of distributions as follows,

**Definition 5. (Information Threshold)** Let $e = (w, \bar{w}) \in E$ be an edge and $u, \bar{u} \in V$ a pair of nodes such that $e \in \text{path}_T(u, \bar{u})$. Fix $q \in (1, 2)$, $0 < c_1 < c_2 < \infty$. The information threshold relative to a set of distribution $\mathcal{P}_T(c_1, c_2)$ is defined as follows:

\[
\Gamma \doteq \frac{1}{2} \min_{p(\cdot) \in \mathcal{P}_T(c_1, c_2)} \min_{(e, u, \bar{u}) \in EV^2} \left( I(X_w; X_{\bar{w}}) - I(X_u; X_{\bar{u}}) \right).
\]

Correspondingly, we extend the definition of the (IOP) property.

**Definition 6. (Information Order Preservation)** Consider random vectors $X \in \mathcal{X}^v$ and $Y \in \mathcal{Y}^v$, such that $Y = \mathcal{F}(X)$, with $\mathcal{F}$ defined as in Section 2.7. We say that the randomized mapping $\mathcal{F}$ is information order-preserving relative to set of tree-shaped distributions $\mathcal{P}_T(c_1, c_2)$ if and only if, for every tuple $\{k, l, m, r\} \subset V$, and for all $X \sim p(\cdot) \in \mathcal{P}_T(c_1, c_2)$,

\[
I(X_k; X_l) < I(X_m; X_r) \iff I(F_k(X_k); F_l(X_l)) < I(F_m(X_m); F_r(X_r)).
\]

The above definitions are closely related to the analysis of parametric structure learning problems, which were recently studied by Bresler and Karzand [3] and by Nikolakakis et al. [7]. In these papers, a set of Ising and Gaussian models is considered.
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