SHIFTED POLYHARMONIC MAASS FORMS FOR $\text{PSL}(2, \mathbb{Z})$

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ABSTRACT. We study the vector space $V^m_k(\lambda)$ of shifted polyharmonic Maass forms of weight $k \in 2\mathbb{Z}$, depth $m \geq 0$, and shift $\lambda \in \mathbb{C}$. This space is composed of real-analytic modular forms of weight $k$ for $\text{PSL}(2, \mathbb{Z})$ with moderate growth at the cusp which are annihilated by $(\Delta_k - \lambda)^m$, where $\Delta_k$ is the weight $k$ hyperbolic Laplacian. We treat the case $\lambda \neq 0$, complementing work of the second and third authors on polyharmonic Maass forms (with no shift). We show that $V^m_k(\lambda)$ is finite-dimensional and bound its dimension. We explain the role of the real-analytic Eisenstein series $E_k(z, s)$ with $\lambda = s(s + k - 1)$ and of the differential operator $\frac{d}{ds}$ in this theory.

1. INTRODUCTION

The second and third authors initiated a study of polyharmonic Maass forms for $\text{PSL}(2, \mathbb{Z})$ in [16]. The present paper extends this study to a more general class of such forms. Fix $k \in 2\mathbb{Z}$ and let

$$\Delta_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy \in \mathbb{C},$$

denote the weight $k$ hyperbolic Laplacian. A classical Maass form of weight $k$ and eigenvalue $\lambda$ for $\text{PSL}(2, \mathbb{Z})$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ with moderate (at worst polynomial) growth at $i\infty$ which satisfies

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \quad \text{for all} \quad (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{PSL}(2, \mathbb{Z})$$

and for which

$$(\Delta_k - \lambda)f = 0. \quad (1.1)$$

A theory for more general Fuchsian groups $\Gamma \subseteq \text{SL}(2, \mathbb{Z})$ which do not include $-I$ would allow odd integer weights as well, but no new forms are gained here since all odd integer weight modular forms on $\text{PSL}(2, \mathbb{Z})$ must vanish identically.

In this work we study the situation when the Laplacian eigenfunction condition (1.1) is relaxed to require only that

$$(\Delta_k - \lambda)^m f(z) = 0 \quad (1.2)$$
for some non-negative integer \( m \). We denote the vector space of such functions by \( V_k^m(\lambda) \), and we call such \( f \) **shifted polyharmonic Maass forms of depth \( m \)** with eigenvalue \( \lambda \) (or **shifted \( m \)-harmonic Maass forms with eigenvalue \( \lambda \)**). The integer parameter \( m \) in (1.2) is termed the (shifted) harmonic depth rather than order (as in PDEs) because the term order is used in conflicting ways in the literature (see [16]).

Our object in this paper is to establish properties of the vector spaces \( V_k^m(\lambda) \) for eigenvalue shifts \( \lambda \neq 0 \). The case \( \lambda = 0 \) was previously treated in [16]. We show that \( V_k^m(\lambda) \) is finite-dimensional, determine an upper bound for the dimension, and exhibit linearly independent forms in these spaces. The finite dimensionality of the spaces \( V_k^m(\lambda) \) has entirely to do with the moderate growth condition; if this is relaxed, then the resulting space of solutions can be infinite dimensional. As in [16] new members of the vector spaces over the harmonic depth 1 case involve derivatives in the \( s \)-variable of (properly scaled) non-holomorphic Eisenstein series \( E_k(z, s) \).

This case \( \lambda = 0 \) has exceptional properties which justify its separate treatment in [16]. It includes all the holomorphic modular forms; the paper [16] explains that holomorphic forms should be assigned harmonic depth a half-integer. In that paper, functions in \( V_k^m(0) \) are named **polyharmonic Maass forms**, in parallel with the literature on polyharmonic functions, which are functions annihilated by a power \( \Delta^m \) of the Euclidean Laplacian. (For work on Euclidean polyharmonic functions see Almansi [1], Aronszajn, Crease and Lipkin [2], Render [24], Mitrea [21, Chap. 7].)

Our results involve the non-holomorphic Eisenstein series \( E_k(z, s) \) of weight \( k \) for \( \text{PSL}(2, \mathbb{Z}) \) given by the series

\[
E_k(z, s) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^s}{|n z + m|^2 (m z + n)^k},
\]

which converges absolutely for \( \text{Re} \, (s) > 1 - \frac{k}{2} \), and has a meromorphic continuation in the \( s \)-variable (see Section 2.3). The series is well-defined for all \( k \in \mathbb{Z} \), but vanishes identically for odd \( k \). It is shifted 1-harmonic with eigenvalue \( \lambda = s(k+1) \). We will consider the **doubly-completed Eisenstein series**

\[
\mathring{E}_k(s) := (s + \frac{k}{2}) (s + k \sqrt{-1}) \sum_{n \in \mathbb{Z}} \frac{\sqrt{-1} \pi s - \frac{k}{2}}{(s + \frac{k}{2} + \sqrt{-1} n) E_k(z, s)},
\]

which one can show is an entire function of \( s \) for each \( z \in \mathbb{H} \) (see Section 2.3). We write the Taylor series expansions of the doubly-completed Eisenstein series at \( s = s_0 \in \mathbb{C} \) as

\[
\mathring{E}_k(z, s) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathring{E}_k^{[j]}(z; s_0) (s - s_0)^j.
\]

**Theorem 1.1.** Fix \( k \in 2\mathbb{Z} \). For \( \lambda \in \mathbb{C} \) fix \( s_0 \in \mathbb{C} \) such that \( \lambda = s_0(s_0 + k - 1) \).

1. The complex vector space \( V_k^m(\lambda) \) is finite dimensional, with

\[
\dim V_k^m(\lambda) \leq m + m \dim S_k^m(\lambda),
\]

where \( S_k^m(\lambda) \) is the space of Maass cusp forms of weight \( k \) and eigenvalue \( \lambda \).

2. This space decomposes as

\[
V_k^m(\lambda) = E_k^m(\lambda) \oplus S_k^m(\lambda),
\]

in which the Eisenstein series space \( E_k^m(\lambda) \) is spanned by certain Taylor coefficients of shifted Eisenstein series and \( S_k^m(\lambda) \) is a recursively defined space of “generalized \( m \)-harmonic Maass cusp forms.” Both vector spaces \( E_k^m(\lambda) \) and \( S_k^m(\lambda) \) are closed under the action of \( \Delta_k \).

3. For all \( \lambda \in \mathbb{C} \) the space \( E_k^m(\lambda) \) has dimension \( m \).
For $\lambda \neq -(1 - k)^2$ it has a basis consisting of the Taylor coefficient functions
\[
\hat{E}^{[j+r]}(z; s_0) := \frac{\partial^j \partial^r}{\partial s^j \partial s^r} \hat{E}_k(z; s)|_{s=s_0} \quad \text{for} \quad 0 \leq j \leq m - 1,
\]
where $r$ is minimal such that $\hat{E}_k^{[r]}(z; s_0) \neq 0$. Here $r = 0$ unless $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ and $k \neq 0$, in which case $r = 1$.

For $\lambda = -(1 - k)^2$ and $s_0 = \frac{1-k}{2}$, a basis is given by the even-indexed Taylor coefficient functions $\hat{E}^{[2j]}(z; s_0)$ for $0 \leq j \leq m - 1$. All odd-indexed functions $\hat{E}^{[2j+1]}(z; s_0) \equiv 0$.

(4) For $m \geq 1$ one has
\[
\dim \left( S^m_k(\lambda) \right) \leq m \dim \left( S^1_k(\lambda) \right).
\]

While the space $S^1_k(\lambda)$ comprises the usual Maass cusp forms, Theorem 1.1 leaves open the question of determining the dimension of the space of “generalized $m$-harmonic Maass cusp forms” $S^m_k(\lambda)$ for $m \geq 2$. In [16, Sect. 6.3] the second and third authors showed for $\lambda = 0$ (the holomorphic cusp form case) that $S^1_k(0) = S^1_k(0)$, which in turn forces $S^m_k(0) = S^1_k(0)$ for all $m \geq 1$. The proof given there was specific to the assumption $\lambda = 0$. One may ask:

**Question.** Is it true for general $\lambda \in \mathbb{C}$ that $S^m_k(\lambda) = S^1_k(\lambda)$ for all $m \geq 1$?

This question asks whether there is a nonzero element of $S^1_k$ that has a “liftability” property, i.e. is the image of some element in $V^2_k(\lambda)$. If some cusp forms can be “lifted”, then we warn the reader that the term “cusp form” could be a misnomer in our recursive definition of $S^m_k(\lambda)$; that is, we do not know whether all the forms in $S^m_k(\lambda)$ will have identically zero constant term in their Fourier expansion.

The next two results concern the preservation of the spaces $V^m_k(\lambda)$ under various differential operators. The first of these concerns the actions of the weight $k$ Maass raising operator
\[
R_k := 2i \frac{\partial}{\partial z} + \frac{k}{y} = i \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{k}{y},
\]
and weight $k$ Maass lowering operator
\[
L_k := 2iy^2 \frac{\partial}{\partial z} = iy^2 \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
on these vector spaces. Note that the operator $L_k$ does not depend on $k$.

**Theorem 1.2.**

(1) There holds
\[
R_k \left( V^m_k(\lambda) \right) \subset V^m_{k+2}(\lambda + k)
\]
and
\[
L_k \left( V^m_k(\lambda) \right) \subset V^m_{k-2}(\lambda + 2 - k).
\]

(2) The map \([1.4]\) is an isomorphism when $\lambda + k \neq 0$. The map \([1.5]\) is an isomorphism when $\lambda \neq 0$.

This result is proved as Propositions [4.5] and [4.7]. The key property established is that these operators acting on individual Fourier coefficients preserve the moderate growth property.

Our second result for differential operators determines the action of the Bruinier-Funke differential operator $\xi_k := 2iy^k \frac{\partial}{\partial z}$ on shifted polyharmonic forms of weight $k$.

**Theorem 1.3.** For $k \in 2\mathbb{Z}$ and all $m \geq 1$, the antiholomorphic differential operator $\xi_k$ maps $V^m_k(\lambda)$ to $V^m_{2-k}(\lambda)$. For $\lambda \neq 0$ this map is a vector space isomorphism.
This result is proved as Proposition 5.5. Again a key property established is that the operator $\xi_k$ preserves the moderate growth property of these functions at the cusp.

Figure 1 pictures the action of these maps, along with $\Delta_k - \lambda$, acting on these spaces. It has a “tower” and “ladder” structure for weights $k \geq 2$, paired with the dual weight $2 - k \leq 0$, at the level of vector spaces.

| shifted harmonic depth | weight $2 - k \leq 0$ | weight $k \geq 2$ | shifted harmonic depth |
|------------------------|-------------------------|-------------------|-----------------------|
| $\geq 3$               | $\vdots$                | $\vdots$          | $\geq 3$              |
| $\Delta_{2-k} - \lambda$ | $\xi_k$                 | $\Delta_k - \lambda$ |
| 2                      | $V^2_{2-k}(\lambda)$    | $V^2_k(\lambda)$  | 2                     |
| $\Delta_{2-k} - \lambda$ | $\xi_{2-k}$             | $\Delta_k - \lambda$ |
| 1                      | $V^1_{2-k}(\lambda)$    | $V^1_k(\lambda)$  | 1                     |
| $\Delta_{2-k} - \lambda$ | $\xi_{2-k}$             | $\Delta_k - \lambda$ |
| 0                      | $V^0_{2-k}(\lambda) \equiv \{0\}$ | $V^0_k(\lambda) \equiv \{0\}$ | 0                     |

Figure 1. Tower and ladder structure for shifted polyharmonic vector spaces (eigenvalue $\lambda \neq 0$).

When $\lambda = 0$ the action of $\xi_k : V^m_k(0) \to V^m_{2-k}(0)$ is never an isomorphism. In that case there is, in contrast, a “tower” and “ramp” structure between weights $k$ and $2 - k$ for the action of these maps, cf. [16, Figures 1 and 2].

The finite-dimensionality results in Theorem 1.1 are proved using general results on the form of Fourier expansions of polyharmonic functions $f$ having eigenvalue shift $\lambda$ that are invariant under $z \mapsto z + 1$. The relevant property of Fourier coefficients of harmonic depth $m$ is that the $2m$-dimensional space of harmonic depth $m$ eigenfunctions for an individual (non-constant term) Fourier coefficient contains an $m$-dimensional subspace comprising the full set of eigenfunctions having moderate growth (in fact, fast decay) at the cusp. Our proof of this fact, given in Appendix A, rests on explicit calculation of asymptotics of the eigenfunction families of the associated ordinary differential operators. The proof of Theorem 1.1 does not make use of the Bruinier-Funke anti-holomorphic differential operator $\xi_k$, in contrast to [16].

These results on the allowed form of polyharmonic Fourier coefficients apply more generally to such expansions for subgroups of the modular group at cusps of any width, with rescaling. They also will apply to finite covering groups of $\text{SL}(2, \mathbb{R})$, such as the metaplectic group $\text{Mp}(2, \mathbb{R})$, thus extending to the case of half-integer weight modular forms on suitable discrete subgroups. That is, one may expect derivatives in the $s$-variable of (non-holomorphic) Eisenstein series to give polyharmonic modular forms more generally.

Section 2 contains known results on Maass cusp forms (Section 2.1), results concerned with Maass’s calculations related to non-holomorphic Eisenstein series (Section 2.2), and basic facts on non-holomorphic Eisenstein series, including their Fourier expansions and functional equations (Section 2.3). Section 3 discusses the Fourier expansions of polyharmonic Maass forms. Section 4 studies the action of Maass raising and lowering operators on the vector spaces $V^m_k(\lambda)$. Section 5
presents results about the Bruinier-Funke non-holomorphic differential operator $\xi_k$, showing that it preserves the property of moderate growth for shifted polyharmonic Maass forms for any eigenvalue $\lambda$. In Section 5.2 we compute the action of $\xi_k$ on the Eisenstein series. In Section 6 we give recursions for the Taylor series coefficients of $\hat{E}_k(z; s_0)$ in the $s$-variable, which are functions of $z$, and determine recursion relations for the action of $\Delta_k$ on the Taylor series coefficients. In Section 7 we give the proof of Theorem 1.1. There are two appendices treating subsidiary topics. Appendix A gives asymptotic expansions of derivatives of the Whittaker $W$-functions $W_{\kappa,\mu}(y)$ in the second index $\mu$. Appendix B sketches an alternate proof of Proposition 5.4 giving the action of the $\xi_k$-operator on the non-holomorphic Eisenstein series.

2. Background Results

The definition of Maass forms given in Section 1 includes classical holomorphic modular forms of weight $k \in 2\mathbb{Z}$ when $\lambda = 0$; this definition was adopted for much recent work treating mock theta functions and other mock modular forms. We note however that the original treatment of Maass [19], [20, Chap. IV] and other later treatments (e.g. Bump [5, Sect. 2.1], Duke, Friedlander, Iwaniec [7, Sect. 4]) use a different definition of Maass form. They define a Maass form of weight $k$ as a function $f^{[M]}$ with at worst polynomial growth at $i\infty$ which satisfies the modified modular invariance condition

$$f^{[M]} \left( \frac{cz + d}{cz + d} \right) = \left( \frac{cz + d}{cz + d} \right)^k f^{[M]}(z) \quad \text{for all } (a \ b \ c \ d) \in \text{PSL}(2, \mathbb{Z})$$

and is annihilated by $\Delta^{[M]}_k - \lambda$, where

$$\Delta^{[M]}_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \frac{\partial}{\partial x}.$$  

The two definitions coincide for weight $k = 0$ but differ otherwise. For $k \neq 0$ Maass forms in this sense are never holomorphic functions of $z$. Given a Maass form $f$ of weight $k$ in our sense, one can transfer results to the definition above by setting $f^{[M]}(z) := y^k f(z)$.

We begin by discussing Maass cusp forms of weight 0 in Section 2.1. We then recall properties of Maass’s Eisenstein series as discussed in [20] in Section 2.2, and in Section 2.3 we derive parallel properties for the non-holomorphic Eisenstein series $E_k(z, s)$.

2.1. Maass cusp forms. A Maass cusp form is a Maass form with rapid decay at the cusp. In 1949 Maass [18] introduced the weight 0 case for certain congruence subgroups of SL(2, $\mathbb{Z}$), so we begin there.

Suppose that $F$ is a Maass cusp form of weight 0 with eigenvalue $\lambda$. Writing $\lambda = s_0(s_0 - 1)$, such an $F$ has a Fourier series expansion of the form

$$F(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n \sqrt{y} K_{s_0 - \frac{1}{2}}(2\pi |n| y) e^{2\pi inx},$$

in which $K_{\nu}(z)$ is a $K$-Bessel function. This expansion can also be given in terms of Whittaker functions $W_{\kappa,\mu}(y)$ via the relation

$$2\sqrt{|n| y} K_{s_0 - \frac{1}{2}}(2\pi |n| y) = W_{0,s_0 - \frac{1}{2}}(4\pi |n| y)$$

(see Section 3). We let $S^1_0(\lambda)$ denote the space of Maass cusp forms of eigenvalue $\lambda$, consistent with the notation of Theorem 1.1.
Theorem 2.1. The space $S^1_0(\lambda)$ is finite-dimensional for every $\lambda$. The values of $\lambda$ for which $S^1_0(\lambda)$ is nontrivial satisfy $\lambda = -(1/4 + r^2)$ for some $r > 0$, corresponding to $s_0 = 1/2 \pm ir$ with $\lambda = s_0(s_0 - 1)$.

Proof. This result is a basic consequence of Selberg’s theory. The condition that $\lambda$ is real follows from self-adjointness of the Maass Laplacian $\Delta_0$ on the modular surface. The fact that $\lambda \leq 0$ follows from a positivity property for $-\Delta_0$. The fact that there are no eigenvalues $-1/4 < \lambda \leq 0$ follows from a computation of Maass [20].

The finite multiplicity follows from the a version the Weyl law for eigenvalues for $X(1) = \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}$. Let $N(T)$ count the number of eigenvalues $s_0 = 1/2 + ir_j$ having $|r_j| \leq T$ (with multiplicity). Then we have

$$N_{X(1)}(T) = \frac{1}{12} T^2 + O(T \log T),$$

see [14, Sec. 15.5].

The space $S^1_0(\lambda)$ may be divided into subspaces of even and odd Maass forms: even forms have Fourier coefficients $a_{-n} = a_n$ and odd forms have $a_{-n} = -a_n$. The coefficients $a_n$ are in general complex-valued. Maass shows that a theory of Hecke operators is applicable to the coefficients $a_n$ and that $S^1_0(\lambda)$ has a basis consisting of Hecke eigenforms.

It is conjectured that the spectrum of cusp forms for $X(1)$ is simple [26, Conjecture 3], when given in terms of the value of $s_0 = 1/2 + ir$ (the eigenfunctions occur in complex conjugate pairs, with $s_0 = 1/2 - ir$, so that $\dim S^1_0(\lambda)$ is always even). Sarnak attributes this conjecture to Cartier [6] (who raised it as Problem (B) in [6, p. 39]). It is supported by the existing computational evidence. The following is the best upper bound on the multiplicity of eigenvalues currently known. It is stated in Sarnak [26, Sect. 4, (36)], with a proof being outlined in [25].

Theorem 2.2. Let $m_{X(1)}(\lambda) = \dim S^1_0(\lambda)$ denote the multiplicity of Maass cusp forms for $\text{PSL}(2, \mathbb{Z})$ with eigenvalue $\lambda$. Then

$$\limsup_{|\lambda| \to \infty} m_{X(1)}(\lambda) \frac{\log |\lambda|}{\sqrt{|\lambda|}} \leq \frac{\text{Vol}(X(1))}{4} = \frac{\pi}{12}.$$
A method of computing Fourier coefficients of Hecke-equivariant Maass cusp forms was given by Stark [23].

2.2. Maass’s Eisenstein series. For later reference we record results and calculations of Maass [20] Chapter IV concerning the Eisenstein series

\[ G(z, \bar{z}; \alpha, \beta) := \sum_{(m,n) \neq (0,0)} (mz+n)^{-\alpha}(\bar{m} \bar{z}+n)^{-\beta} = \sum_{(m,n) \neq (0,0)} (mz+n)^{-(\alpha-\beta)}|mz+n|^{-2\beta}, \] (2.1)

in which \( \text{Im}(z) > 0, \alpha - \beta \in 2\mathbb{Z}, \text{Re}(\alpha + \beta) > 2, \) and the sum runs over all pairs of integers not equal to \((0,0)\). We define the function by the second summation, taking the principal branch of the logarithm, noting that \( \text{im}(z) = \text{imag}(\text{arg}(z)) \).[2]

Maass introduces the notation \( q := \alpha + \beta, r := \alpha - \beta \in 2\mathbb{Z}; \) in terms of our variables we have \( q = 2s + k, r = k \).[3]

**Theorem 2.3** (Maass [20] page 210). (Fourier expansion for \( G(z, \bar{z}; \alpha, \beta) \)) For fixed \( \alpha - \beta \in 2\mathbb{Z}, \) the Fourier expansion of the function \( G(z, \bar{z}; \alpha, \beta) \) with respect to periodicity under \( x \mapsto x + 1 \) is:

\[ G(z, \bar{z}; \alpha, \beta) = \varphi_{\frac{2}{r}}(y, q) + 2(-1)^{\frac{r}{2}}(\sqrt{2\pi})^q \sum_{n \in \mathbb{Z}} \frac{\sigma_{q-1}(n)}{(y, q)^{1/2} + \text{sgn}(n)q^{1/2}} W(2\pi ny; \alpha, \beta)e^{2\pi i nx}. \]

In this formula the constant term is

\[ \varphi_{\frac{2}{r}}(y, q) := 2\zeta(q) + (-1)^{\frac{r}{2}}(\sqrt{2\pi})^{2-1-q} \frac{\Gamma(q - 1)}{\Gamma(\frac{r+1}{2}) \Gamma(\frac{q-r}{2})} \zeta(q - 1)y^{1-q}. \]

This formula also contains the \( s \)-th power divisor function \( \sigma_s(n) := \sum_{d|n, d > 0} d^s \), for \( s \in \mathbb{C} \), and the modified Whittaker function, defined for \( y > 0 \) by

\[ W(\pm y; \alpha, \beta) := y^{-\frac{r}{2}}W_{\pm \frac{r}{2}(q-1)}(2y), \]

where \( W_{\kappa, \mu}(y) \) is the \( \mathcal{W} \)-Whittaker function (see Section 3).

**Proof.** The main formula appears in [20] p.210, with constant term written

\[ \varphi_{\frac{2}{r}}(y, q) := 2\zeta(q) + (-1)^{\frac{r}{2}}(\sqrt{2\pi})^{2-1-q} \frac{\Gamma(q - 1)}{\Gamma(\frac{r+1}{2}) \Gamma(\frac{q-r}{2})} ((1 - q)u(y, q) + 1). \]

We use \( u(y, q) = \frac{y^{1-q} - 1}{1-q} \) if \( q \neq 1 \) (defined on [20] p. 181) whence \((1 - q)u(y, q) + 1 = y^{1-q}\). \( \square \)

This Eisenstein series was initially defined for \( \text{Re}(\alpha + \beta) > 2 \) but the Fourier expansion yields an analytic continuation of \( G(z, \bar{z}; \alpha, \beta) \) in \( q = \alpha + \beta \) to all \( q \in \mathbb{C} \), using the fact that \( W_{\kappa, \mu}(y) \) is an entire function of \( \mu \) for fixed \( \kappa, y \), and has rapid decay as \( y \to \infty \).

Maass [20] p. 213] introduced completed versions of these Eisenstein series, setting

\[ G^*(z, \bar{z}; \alpha, \beta) := \frac{q}{2} \left( 1 - \frac{q}{2} \right) \pi^{-\frac{2}{q}} \Gamma \left( \frac{q}{2} + \frac{|r|}{2} \right) G(z, \bar{z}; \alpha, \beta) \] (2.2)

and defining its completed Fourier series constant term by

\[ \varphi_{\frac{2}{r}}^*(y, q) := \frac{q}{2} \left( 1 - \frac{q}{2} \right) \pi^{-\frac{2}{q}} \Gamma \left( \frac{q}{2} + \frac{|r|}{2} \right) \varphi_{\frac{2}{r}}(y, q). \] (2.3)

Maass wishes to view the variables \( z \) and \( \bar{z} \) as independent, and requires a convention on the branch of the logarithm defining \( (mz+n)^{-\alpha} = e^{-\alpha \log(mz+n)} \), requiring a compatibility condition to hold between \( mz+n \) and \( m\bar{z}+n \).

On page 209 Maass defines a parameter \( k \) by \( r = 2k \in 2\mathbb{Z} \). This notation conflicts with our notation; his \( k \) equals our \( k/2 \). In the statement of Theorem [23], we replace the Maass parameter \( k \) by \( \frac{k}{2} \) wherever it occurs.
These functions have the following properties.

**Theorem 2.4. (Properties of \( G(z, \overline{z}; \alpha, \beta) \))** For fixed \( r = \alpha - \beta \in 2\mathbb{Z} \), the functions \( G(z, \overline{z}; \alpha, \beta) \) and their completions have the following properties.

(1) (Analyticity in \( q \)-variable) The completed function \( G^*(z, \overline{z}; \alpha, \beta) \) and its completed constant term \( \varphi_{r/2}^*(y, q) \) both analytically continue to entire functions of the variable \( q = \alpha + \beta \).

(2) (Laplacian Eigenfunction) In the variables \( z = x + iy \), \( \overline{z} = x - iy \) it is a solution of the (elliptic) partial differential equation

\[
\Omega_{\alpha,\beta}G(z, \overline{z}; \alpha, \beta) = 0
\]

where

\[
\Omega_{\alpha,\beta} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i(\alpha - \beta)y \frac{\partial}{\partial x} - (\alpha + \beta)y \frac{\partial}{\partial y}. \tag{2.4}
\]

The function \( G(z, \overline{z}; \alpha, \beta) \) is real-analytic in the variables \( x, y \) on \( \mathbb{H} \). These properties also hold for \( G^*(\overline{z}, \alpha, \beta) \).

(3) (\( \{\alpha, \beta\} \)-Modular Invariance) For each \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \) we have

\[
G \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) = (cz + d)^{\alpha-\beta}|cz + d|^{2\beta}G(z, \overline{z}; \alpha, \beta). \tag{2.5}
\]

(4) (Moderate Growth) For some constant \( K > 0 \), \( G(z, \overline{z}; \alpha, \beta) = o(y^K) \) for \( y \to \infty \), uniformly in \( x \).

(5) (Functional Equations) \( G^*(z, \overline{z}; \alpha, \beta) \) satisfies the following:

\[
G^*(z, \overline{z}; \alpha, \beta) = G^*(-\overline{z}, -z; \alpha, \beta)
\]

and

\[
G^*(z, \overline{z}; 1 - \alpha, 1 - \beta) = y^{q-1}G^*(z, \overline{z}; \beta, \alpha). \tag{2.6}
\]

**Proof.** Results (1), (3), (4), and (5) are given on [20, pp. 213–214] (in (5) we have corrected a typo in the first equation). In particular (3) and (4) are obtained by unwinding the definition of \( [\text{SL}(2, \mathbb{Z}), \alpha, \beta, 1] \), the space of automorphic functions of type \( \{\text{SL}(2, \mathbb{Z}), \alpha, \beta, 1\} \) on [20, p. 185]. The differential operator in (2) is defined on [20, pp. 175–176], and that (2) holds for \( G(z, \overline{z}; \alpha, \beta) \) appears on [20, pp. 177, equation (14) and preceding equation]. The result (2) then holds for \( G^*(z, \overline{z}; \alpha, \beta) \) because the prefactor is a constant with respect to the differential equation and for real-analyticity. \( \square \)

### 2.3. Properties of non-holomorphic Eisenstein series.

Recall that the weight \( k \) non-holomorphic Eisenstein series \( E_k(z, s) \) is defined as

\[
E_k(z, s) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{y^s}{|mz + n|^{2s}(mz + n)^k}. \tag{2.6}
\]

This series converges absolutely for \( \text{Re} (s) > 1 - \frac{k}{2} \). The resulting function transforms under elements of \( \text{SL}(2, \mathbb{Z}) \) as

\[
E_k \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z, s \right) = (cz + d)^k E_k(z, s).
\]
At $s = 0$ for even weights $k \geq 4$ this function specializes to an unnormalized version of the holomorphic Eisenstein series $E_k(z)$, with

$$E_k(z, 0) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(mz + n)^k} = \frac{1}{2} \zeta(k) E_k(z).$$

With our scaling $E_k(z, 0) = \frac{1}{2} G_k(z)$, where $G_k(z)$ is the usual unnormalized holomorphic Eisenstein series in Serre [27].

The non-holomorphic Eisenstein series $E_k(z, s)$ has a simple relation to Maass’s Eisenstein series $G(z, \bar{z}; \alpha, \beta)$. Take $\alpha = s + k$ and $\beta = s$ with $k \in 2\mathbb{Z}$. Then in the absolute convergence region $\text{Re}(s) > 1 - \frac{1}{2}k$ we have the identity

$$E_k(z, s) = \frac{1}{2} y^s G(z, \bar{z}; s + k, s),$$

(2.7) which then holds for all $s \in \mathbb{C}$ under analytic continuation.

One can use Maass’s result for $G(z, \bar{z}; \alpha, \beta)$ to obtain Fourier expansions for the completed non-holomorphic Eisenstein series $\hat{E}_k(z, s)$ defined by

$$\hat{E}_k(z, s) := \pi^{-s-k} \Gamma \left( s + k \frac{k}{2} \right) y^{-\frac{k}{2}} E_k(z, s),$$

(2.8) as follows. We denote by $\hat{\zeta}(s)$ the completed Riemann zeta function

$$\hat{\zeta}(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

**Proposition 2.5.** (Fourier expansion of $\hat{E}(z, s)$) For $k \in 2\mathbb{Z}$, the completed non-holomorphic Eisenstein series $\hat{E}(z, s)$ has the Fourier expansion

\begin{align*}
\hat{E}_k(z, s) &= C_0(y, s) + (-1)^{\frac{s}{2}} \Gamma \left( s + k \frac{k}{2} \right) y^{-\frac{k}{2}} \\
&\quad \times \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sigma_{2s+k-1}(n)}{\Gamma \left( s + k \frac{k}{2} (1 + \sgn(n)) \right)} W_{\sgn(n) \frac{k}{2}, s + k \frac{k}{2} \frac{1}{2}} (4\pi |n| y) e^{2\pi i n x},
\end{align*}

in which the Fourier constant term is

$$C_0(y, s) = \frac{\Gamma \left( s + k \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma \left( s + k \frac{k}{2} \right)} \hat{\zeta}(2s + k) y^s + (-1)^{\frac{s}{2}} \frac{\Gamma \left( s + k \frac{k}{2} \right) \Gamma \left( s + k \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma(s+k)\Gamma(s)} \hat{\zeta}(2 - 2s - k) y^{1-s-k}$$

and $\sigma_s(n) = \sum_{d|n, d>0} d^s$.

**Proof.** The Fourier expansion follows from that of $G(z, \bar{z}; \alpha, \beta)$ in Theorem 2.3 via the relation

$$\hat{E}_k(z, s) = \pi^{-s-k} \Gamma \left( s + k \frac{k}{2} + \frac{|k|}{2} \right) \frac{1}{2} y^s G(z, \bar{z}; s + k, s).$$

The duplication formula for the gamma function is used, as well as the functional equation of the Riemann zeta function $\hat{\zeta}(2s + k - 1) = \hat{\zeta}(2 - 2s - k)$.

For later use we give a more detailed formula for the constant term in the Fourier series.

---

5The corresponding Proposition 3.5 of [16 p. 304] has misprints. In the nonconstant terms with $e^{2\pi i n x}$ the multiplicative factor $(\sqrt{2\pi})^{2s+k} \pi^{-s-k} \frac{1}{2}$ should read $(2\pi)^{\frac{k}{2}} |n|^{-s}$ as in Proposition 2.5 above.
Proposition 2.6. For \( k \in 2\mathbb{Z} \) and \( s \in \mathbb{C} \) the Fourier constant term of the completed Eisenstein series \( \tilde{E}_k(z, s) \) is as follows.

1) Suppose \( s \neq \frac{1+k}{2} \). Then for weights \( k \geq 2 \) we have

\[
C_0(y, s) = \left( s + \frac{k}{2} \right) \left( s + \frac{k+2}{2} \right) \cdots \left( s + \frac{2k-2}{2} \right) \tilde{\zeta}(2s + k) y^s + (-1)^{\frac{k}{2}} s(s+1) \cdots \left( s + \frac{k-2}{2} \right) \tilde{\zeta}(2 - 2s - k) y^{1-s-k}.
\]

For weights \( k \leq -2 \), we have

\[
C_0(y, s) = (s-1)(s-2) \cdots \left( s - \frac{|k|}{2} \right) \tilde{\zeta}(2s + k) y^s + (-1)^{\frac{k}{2}} (s-1)(s-|k|) \cdots \left( s - \frac{|k|}{2} - 1 \right) \tilde{\zeta}(2 - 2s - k) y^{1-s-k}.
\]

For weight \( k = 0 \) and \( s \neq 0 \) or 1, \( C_0(y, s) = \tilde{\zeta}(2s) y^s + \tilde{\zeta}(2 - 2s) y^{1-s} \).

2) Suppose \( s = \frac{1+k}{2} \). Then for weights \( k \neq 0 \),

\[
C_0 \left( y, \frac{1-k}{2} + \frac{\epsilon}{2} \right) = \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{|k|-1}{2} \left( \gamma - \log 4\pi + \log y + 2 \left( 1 + \frac{1}{3} + \cdots + \frac{1}{|k|-1} \right) \right) y^{\frac{1+k}{2}},
\]

where \( \gamma \) is Euler’s constant. For weight \( k = 0 \), \( C_0 \left( y, \frac{1}{2} \right) = \frac{1}{2} \left( \gamma - \log 4\pi + \log y \right) \).

Note that Proposition 2.6 immediately gives the relation \( C_0(y, s) = C_0(y, 1 - k - s) \).

Proof of Proposition 2.6

1) The products of Gamma factors in the constant term formula in (2.5) simplify to polynomials using the identity \( s\Gamma(s) = \Gamma(s+1) \).

2) We write \( s = \frac{1-k}{2} + \frac{\epsilon}{2} \) and find that

\[
C_0 \left( y, \frac{1-k}{2} + \frac{\epsilon}{2} \right) = \frac{1}{\sqrt{\pi}} \tilde{\zeta}(1 + \epsilon) \left( 1 + \psi(a) \epsilon + O(\epsilon^2) \right)
\]

\[
\left( \frac{1 + \frac{|k|}{2} + \frac{\epsilon}{2}}{2} \right) y^{\frac{1+k}{2}} + (-1)^{\frac{k}{2}} \frac{\Gamma(\frac{1+k}{2} + \frac{\epsilon}{2}) \tilde{\zeta}(1 - \epsilon)}{\Gamma(\frac{1+k}{2} + \frac{\epsilon}{2} + \frac{1}{2})} y^{-\frac{\epsilon}{2}}.
\]

Let \( \psi(s) = \frac{d}{ds} \Gamma(s) \Gamma(1-s) \). Using the series expansions

\[
\Gamma(a + \epsilon) = \Gamma(a) \left( 1 + \psi(a) \epsilon + O(\epsilon^2) \right),
\]

\[
\tilde{\zeta}(1 + \epsilon) = \frac{1}{\epsilon} + \frac{1}{2} \left( \gamma - \log 4\pi \right) + O(\epsilon),
\]

\[
y^\epsilon = 1 + (\log y)\epsilon + O(\epsilon^2),
\]

and the reflection formulas \( \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \) and \( \psi(s) - \psi(1 - s) = \frac{\pi}{\tan \pi s} \), we find that

\[
C_0 \left( y, \frac{1-k}{2} + \frac{\epsilon}{2} \right) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1 + \frac{|k|}{2} + \frac{\epsilon}{2}}{2} \right) \frac{2\gamma - \log \pi + \log \psi(1 + \frac{k}{2}) + O(\epsilon)}{\Gamma(\frac{1+k}{2} + \frac{\epsilon}{2} + \frac{1}{2})} y^{-\frac{\epsilon}{2}}.
\]

When \( k = 0 \) we have \( \Gamma(\frac{1+|k|}{2}) = \sqrt{\pi} \) and \( \psi(\frac{1+k}{2}) = -\gamma - \log 4 \). For \( k \neq 0 \) we use [22] (5.4.2) and (5.5.4) to obtain

\[
\frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1 + \frac{|k|}{2}}{2} \right) = \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{|k|-1}{2},
\]

\[
\psi \left( \frac{1+k}{2} \right) = -\gamma - \log 4 + 2 \left( 1 + \frac{1}{3} + \cdots + \frac{1}{|k|-1} \right).
\]

The proposition follows. \( \square \)

We now collect various analytic properties of the Eisenstein series.
**Theorem 2.7.** (Properties of $E_k(z, s)$) Let $k \in 2\mathbb{Z}$.

1. (Analytic Continuation) For fixed $z \in \mathbb{H}$, the completed weight $k$ Eisenstein series $\widehat{E}_k(z, s)$ analytically continues to the $s$-plane as a meromorphic function. For $k = 0$ its has two singularities, which are simple poles at $s = 0$ and $s = 1$ with residues $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. For $k \neq 0$ it is an entire function.

2. (Functional Equation) For fixed $z \in \mathbb{H}$, the completed weight $k$ Eisenstein series satisfies the functional equation

$$\widehat{E}_k(z, s) = \widehat{E}_k(z, 1 - k - s).$$

The doubly-completed series

$$\widehat{E}_k(z, s) := \left(s + \frac{k}{2}\right) \left(s + \frac{k}{2} - 1\right) \widehat{E}_k(z, s)$$

is an entire function of $s$ for all $k \in 2\mathbb{Z}$ and satisfies the same functional equation

$$\widehat{E}_k(z, s) = \widehat{E}_k(z, 1 - k - s).$$

The center line of these functional equations is $\text{Re}(s) = \frac{1-k}{2}$.

3. ($\Delta_k$-Eigenfunction) $E_k(z, s)$ is a (generalized) eigenfunction of the hyperbolic Laplacian operator $\Delta_k$ with eigenvalue $\lambda = s(s + k - 1)$. That is, for all $s \in \mathbb{C}$,

$$\Delta_k E_k(z, s) = s(s + k - 1) E_k(z, s).$$

This eigenfunction property holds for the completed functions $\widehat{E}_k(z, s)$ and $\widehat{E}'_k(z, s)$.

**Proof.** (1) and (2). We have $E_k(z, s) = y^q G(z, \overline{z}; s + k, s)$. For the doubly-completed Eisenstein series, a calculation from the definition \eqref{eq:converg} verifies the identity

$$\widehat{E}_k(z, s) = -\frac{1}{2} y^q G^*(z, \overline{z}; s + k, s).$$

The latter is an entire function by Theorem 2.4(1). The functional equation \eqref{eq:fe2} for $\widehat{E}_k(z, s)$ follows from unwinding the second functional equation

$$G^*(z, \overline{z}; 1 - \alpha, 1 - \beta) = y^{q-1} G^*(z, \overline{z}; \beta, \alpha)$$

in Theorem 2.4(4), now making the assignment $\alpha = s, \beta = s + k$, so $q = 2s + k$, and multiplying both sides by $y^{1-k-s}$. Now the relation \eqref{eq:doubly} implies that $\widehat{E}_k(z, s)$ is meromorphic in $s$ and that its only possible singularities are at most simple poles at $s = -\frac{k}{2}$ and at $s = 1 - \frac{k}{2}$. It inherits the functional equation $\widehat{E}_k(z, s) = \widehat{E}_k(z, 1 - k - s)$, which interchanges the two possible polar points and implies the sum of the residues of the (possible) poles at these two points is 0.

It remains to determine when poles are present for general $k \in 2\mathbb{Z}$ which can be done using its Fourier series expansion given in Proposition 2.5. The only singularities in $s$ can be contributed by the constant term of the Fourier expansion. The case $k = 0$ is well treated in the literature; where it has poles at $s = 0$ and $s = 1$ with residues as specified, e.g. \cite{15} pp. 45–46, \cite{17} pp. 98–100. For all $k \neq 0$ the gamma factors in the formula for the constant term $C_0(y, s)$ contribute a canceling zero at all the possible pole location points $s = \frac{2 - k}{2}, s = \frac{k}{2}$ and $s = \frac{1-k}{2}$ associated to $\zeta(2s + k)$ and $\zeta(2 - 2s - k)$, and the functions $\widehat{E}_k(z, s)$ are entire functions.

3. From Theorem 2.4(2) we have

$$\Omega_{s+k,s} G(z, \overline{z}; s + k, s) = 0,$$
where
\[
\Omega_{s+k,s} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x} - (k + 2s)y \frac{\partial}{\partial y} = -\Delta_k - 2sy \frac{\partial}{\partial y}. \tag{2.14}
\]
By (2.7) it follows that
\[
(\Delta_k - s(s + k - 1)) E_k(z, s) = 0.
\]

3. Polyharmonic Fourier Series Expansions

In this section we give the general form of the Fourier expansions of shifted polyharmonic Maass forms, with and without growth restrictions. These expansions involve the Whittaker functions and their derivatives in the auxiliary parameter \(s\). We begin by discussing a certain family of linearly independent solutions to Whittaker’s differential equation.

3.1. Linearly independent Whittaker function solutions. Given parameters \(\kappa, \mu \in \mathbb{C}\), the Whittaker differential equation \([22, (13.14.1)]\) is
\[
\frac{d^2 F}{dz^2} + \left( -\frac{1}{4} + \frac{\kappa}{z} + \frac{1}{z^2} - \frac{\mu^2}{z^2} \right) F = 0. \tag{3.1}
\]
The standard solutions to (3.1) are the \(M\)-Whittaker function \(M_{\kappa,\mu}(z)\) and the \(W\)-Whittaker function \(W_{\kappa,\mu}(z)\). For all parameters \((\kappa, \mu) \in \mathbb{C}^2\) the Whittaker \(W\)-function \(W_{\kappa,\mu}(z)\) is uniquely determined up to a multiplicative constant by its property of having rapid decay along the positive real axis. The function \(M_{\kappa,\mu}(z)\) does not exist for \(2\mu + 1 \in \mathbb{Z} \leq 0\); furthermore, its Wronskian with \(W_{\kappa,\mu}(z)\) is given by \([22, (13.14.26)]\)
\[
\mathcal{W}\{M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)\} = -\frac{\Gamma(1 + 2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)},
\]
so it is linearly dependent on \(W_{\kappa,\mu}\) whenever \(\frac{1}{2} + \mu - \kappa \in \mathbb{Z} < 0\).

A second independent solution of the Whittaker differential equation is obtained by analytic continuation of the function \(W_{\kappa,\mu}(z)\) in the \(z\)-variable to the negative real axis. The analytic continuation of the function is multi-valued with a branch point at \(z = 0\), so we get two different functions, according as we continue along a path in the upper half plane,
\[
\mathcal{M}^+_\kappa(\mu)(z) := W_{-\kappa,\mu}(ze^{\pi i}), \tag{3.2}
\]
or along a path in the lower half plane,
\[
\mathcal{M}^-\kappa(\mu)(z) := W_{-\kappa,\mu}(ze^{-\pi i}).
\]
The notation \(\mathcal{M}^\pm_{\kappa,\mu}(z)\) is introduced here and is not standard. The functions \(\mathcal{M}^\pm_{\kappa,\mu}(z)\) are linearly independent with \(W_{\kappa,\mu}(z)\) since the Wronskian \([22, (13.14.30)]\)
\[
\mathcal{W}\{W_{\kappa,\mu}(z), W_{-\kappa,\mu}(ze^{\pm\pi i})\} = e^{\mp \pi i \kappa}
\]
is everywhere nonzero. For the sake of concreteness, in this paper we always choose \(\mathcal{M}^+_\kappa(\mu)(z)\) as the second linearly independent solution to (3.1).
3.2. **Shifted harmonic Fourier coefficients.** The following well-known result gives allowable functional forms of Fourier coefficients for periodic functions that satisfy \((\Delta_k - \lambda)f_n(z) = 0\), with no restriction on the growth of \(f_n\) at the cusp.

**Theorem 3.1.** (Shifted harmonic Fourier coefficients of unrestricted growth) Let \(k \in 2\mathbb{Z}\) and suppose that \(f_n(z) = h_n(y)e^{2\pi \imath nx}\) satisfies

\[
(\Delta_k - \lambda)f_n(z) = 0 \quad \text{for all} \quad z = x + iy \in \mathbb{H}.
\]

Write \(\lambda = s_0(s_0 + k - 1)\) for some \(s_0 \in \mathbb{C}\) (there are generally two choices for \(s_0\)). Then the complete set of such functions \(h_n(y)\) are given in the following cases.

1. If \(n \neq 0\) then

\[
h_n(y) = y^{-\frac{k}{2}} \left( a_n^- W_{\text{sgn}(n)\frac{k}{2},s_0 + \frac{k-1}{2}}(4\pi |n|y) + a_n^+ M_{\text{sgn}(n)\frac{k}{2},s_0 + \frac{k-1}{2}}^+(4\pi |n|y) \right)
\]

for some constants \(a_n^-, a_n^+ \in \mathbb{C}\), where \(W_{\kappa,\mu}\) and \(M_{\kappa,\mu}^+\) are the Whittaker functions discussed in Section 3.1.

2a) Suppose \(n = 0\) with \(s_0 \neq \frac{-1-k}{2}\) (equivalently, \(\lambda \neq -(\frac{1-k}{2})^2\)). Then

\[
h_n(y) = a_0^- y^{1-k-s_0} + a_0^+ y^{s_0},
\]

for some constants \(a_0^-, a_0^+ \in \mathbb{C}\).

2b) Suppose \(n = 0\) with \(s_0 = \frac{-1-k}{2}\) (equivalently, \(\lambda = -(\frac{1-k}{2})^2\)). Then

\[
h_n(y) = a_0^- y^{\frac{1+k}{2}} + a_0^+ \frac{\partial}{\partial s} y^s|_{s=\frac{-1-k}{2}} = a_0^- y^{\frac{1+k}{2}} + a_0^+ y^{\frac{1+k}{2}} \log y,
\]

for some constants \(a_0^-, a_0^+ \in \mathbb{C}\).

**Proof.** (1). Let \(\epsilon \in \{-1, 1\}\). The requirement that \((\Delta_k - \lambda)h_\epsilon(y)e^{\pm \epsilon \pi x} = 0\) leads to a second-order linear ordinary differential equation that \(h_\epsilon(y)\) must satisfy which simplifies if we set \(h_\epsilon(y) = g_\epsilon(y)y^{-\frac{k}{2}}\). We obtain

\[
\Delta_k \left( g_\epsilon(y)y^{-\frac{k}{2}} e^{\pm \epsilon \pi x} \right) = y^{-\frac{k}{2}} e^{\pm \epsilon \pi x} \left( -\frac{1}{4} \left( e^2 y^2 - 2\epsilon ky + k^2 - 2k \right) g_\epsilon(y) + y^2 g_\epsilon''(y) \right),
\]

which leads to the differential equation

\[
g_\epsilon''(y) + \left( -\frac{1}{4} + \frac{\epsilon k}{y} + \frac{\frac{k^2}{4} + \frac{k}{2} - \lambda}{y^2} \right) g_\epsilon(y) = 0.
\]

We now set \(\lambda = s_0(s_0 + k - 1)\), which yields \(-\frac{k^2}{4} + \frac{k}{2} - \lambda = \frac{1}{4} - (s_0 + \frac{k-1}{2})^2\), from which we conclude that \(g_\epsilon(y)\) satisfies Whittaker’s differential equation

\[
F''(w) + \left( -\frac{1}{4} + \frac{\kappa}{w} + \frac{\frac{1}{4} - \mu^2}{w^2} \right) F(w) = 0
\]

with parameters \((\kappa, \mu) = \left( \frac{\epsilon k}{2}, s_0 + \frac{k-1}{2} \right)\). By the discussion in Section 3.1 for a fixed \(s_0\) we obtain two linearly independent solutions

\[
f_\epsilon^+(z) = y^{-\frac{k}{2}} W_{\frac{k}{2},s_0 + \frac{k-1}{2}}(y) e^{\pm \epsilon \pi x},
\]

and

\[
f_\epsilon^-(z) = y^{-\frac{k}{2}} M_{\frac{k}{2},s_0 + \frac{k-1}{2}}^+(y) e^{\pm \epsilon \pi x}.
\]

We obtain the given solutions by replacing \(z\) with \(4\pi |n| z\), noting that the operator \(\Delta_k\) is invariant under rescaling by a multiplicative constant.
(2a) and (2b). For \( n = 0 \), again writing \( h_0(y) = g_0(y)y^{-\frac{k}{2}} \), we obtain instead the differential equation

\[
g_0''(y) + \frac{1}{y^2} - \left(s_0 + \frac{k-1}{2}\right)g_0(y) = 0,
\]

which has two linearly independent solutions \( y^{s_0 + \frac{k}{2}} \) and \( y^{1-s_0 - \frac{k}{2}} \) provided \( s \neq \frac{1-k}{2} \); this gives (2a).

In the exceptional case \( s = \frac{1-k}{2} \) the differential equation has two linearly independent solutions

given by \( y^{\frac{k}{2}} \) and \( y^{\frac{k}{2}} \log y \), as given in (2b).

We next treat the special case of Fourier coefficients for shifted harmonic functions having moderate growth at the cusp. In Appendix A (see (A.1)) we show that \( W_{\kappa,\mu}(y) \) decays exponentially as \( y \to \infty \), while \( M_{\kappa,\mu}^+(y) \) grows exponentially as \( y \to \infty \). This immediately gives the following theorem.

**Theorem 3.2.** (Shifted harmonic Fourier coefficients of moderate growth) Let \( k \in 2\mathbb{Z} \) and suppose that \( f_n(z) = h_n(y)e^{2\pi i ny} \) satisfies

\[
(\Delta_k - \lambda)f_n(z) = 0 \quad \text{for all} \quad z = x + iy \in \mathbb{H},
\]

and has at most polynomial growth in \( y \) approaching the cusp \( i\infty \). Then the complete set of such \( h_n(y) \) are those functions in Theorem 3.1 which omit \( M_{\kappa,\mu}^+(z) \), i.e. those for which \( a_n^+ = 0 \) for all \( n \neq 0 \).

### 3.3. Shifted polyharmonic Fourier coefficients of unrestricted growth

We now characterize for general \( m \geq 2 \) the individual Fourier coefficients \( f_n(z) \) which satisfy \( (\Delta_k - \lambda)^mf_n(z) = 0 \), with no growth restriction at the cusp. The space of allowable Fourier coefficient functions always has dimension \( 2m \). The basic mechanism leading to new functions is that the operator \( \frac{\partial}{\partial s} \) commutes with \( \Delta_k \), but does not commute with the multiplication operator \( \lambda I = s(s + k - 1)I \) when \( s \) is regarded as variable. Letting \([A, B] = AB - BA\), the commutation relation \([sI, \frac{\partial}{\partial s}] = -I\) yields

\[
\left[\lambda I, \frac{\partial}{\partial s}\right] = \left[s(s + k - 1)I, \frac{\partial}{\partial s}\right] = (1 - k - 2s)I. \tag{3.3}
\]

We first treat the case of the non-constant Fourier coefficients, which involve derivatives in the second parameter of the Whittaker functions discussed in Section 3.1. We introduce a new notation for these functions. For \( n \neq 0 \) and for each \( m \geq 0 \) define

\[
u^{[m]}_{k,n}(-y; s_0) := \begin{cases} 
\left.y^{-\frac{k}{2}} \frac{\partial^m}{\partial s^m} W_{\text{sgn}(n)^{\frac{k}{2}}, s + \frac{k-1}{2}}(4\pi |n| y)\right|_{s = s_0} & \text{if } s_0 \neq \frac{1-k}{2}, \\
\left.y^{-\frac{k}{2}} \frac{\partial^m}{\partial s^m} W_{\text{sgn}(n)^{\frac{k}{2}}, s + \frac{k-1}{2}}(4\pi |n| y)\right|_{s = \frac{1-k}{2}} & \text{if } s_0 = \frac{1-k}{2},
\end{cases}
\]

\[
u^{[m]}_{k,n}(y; s_0) := \begin{cases} 
\left.y^{-\frac{k}{2}} \frac{\partial^m}{\partial s^m} M^+_{\text{sgn}(n)^{\frac{k}{2}}, s + \frac{k-1}{2}}(4\pi |n| y)\right|_{s = s_0} & \text{if } s_0 \neq \frac{1-k}{2}, \\
\left.y^{-\frac{k}{2}} \frac{\partial^m}{\partial s^m} M^+_{\text{sgn}(n)^{\frac{k}{2}}, s + \frac{k-1}{2}}(4\pi |n| y)\right|_{s = \frac{1-k}{2}} & \text{if } s_0 = \frac{1-k}{2}.
\end{cases}
\]

**Theorem 3.3.** (Shifted polyharmonic Fourier coefficients of unrestricted growth, \( n \neq 0 \)) Let \( k \in 2\mathbb{Z} \). Suppose that \( n \neq 0 \) and that \( f_n(z) = h_n(y)e^{2\pi i ny} \) is a shifted polyharmonic function for \( \Delta_k \) on \( \mathbb{H} \) with eigenvalue \( \lambda \in \mathbb{C} \), i.e. it satisfies

\[
(\Delta_k - \lambda)^mf_n(z) = 0 \quad \text{for all} \quad z = x + iy \in \mathbb{H}. \tag{3.6}
\]
Write $\lambda = s_0(s_0 + k - 1)$ for some $s_0 \in \mathbb{C}$ (there are generally two choices for $s_0$). Then

$$h_n(y) = \sum_{j=0}^{m-1} \left( a_{n,j}^- u_{k,n}^{[j],-}(y; s_0) + a_{n,j}^+ u_{k,n}^{[j],+}(y; s_0) \right)$$

for some constants $a_{n,j}^-, a_{n,j}^+ \in \mathbb{C}$.

**Proof.** We know a priori from (3.6) that $h_n(y)$ must satisfy a linear differential equation of order $2m$, whose solutions will form a a vector space of $2m$ linearly independent functional solutions. By (3.4), (3.5), (3.2), and Corollary A.2 from Appendix A the set

$$\left\{ u_{k,n}^{[j],-}(y; s_0) : 0 \leq j \leq m - 1 \right\} \cup \left\{ u_{k,n}^{[j],+}(y; s_0) : 0 \leq j \leq m - 1 \right\}$$

is linearly independent. It remains to show that

$$(\Delta_k - s_0(s_0 - k - 1))^m u_{k,n}^{[j],\pm}(y; s_0)e^{2\pi inx} = 0$$

for all $m \geq 1$ and all $j \leq m - 1$.

We proceed by induction on $m$. Theorem 3.1 shows that (3.7) is true for the base case $j = 0$, $m = 1$. Suppose that (3.7) is true for all $m \leq r$ and all $j \leq m - 1$ for some $r \geq 1$. Then clearly (3.7) holds for $m = r + 1$ and $j \leq r - 1$, so it remains to show that it holds for $m = r + 1$ and $j = r$. For brevity, we write $U^{[j]} := u_{k,n}^{[j],\pm}(y; s)e^{2\pi inx}$ (thinking of $s$ as a variable) and write $\lambda = s(s + k - 1)$. Then by (3.3) we have

$$(\Delta_k - s)(\Delta_k - \lambda) U^{[r]} = (\Delta_k - \lambda) \frac{\partial}{\partial s} U^{[r-1]}$$

$$= \frac{\partial}{\partial s} (\Delta_k - \lambda) U^{[r-1]} + (1 - k - 2s) U^{[r-1]} = \ldots$$

$$= \frac{\partial^r}{\partial s^r} (\Delta_k - \lambda) U^{[0]} + (1 - k - 2s) \sum_{i=1}^{r} U^{[r-i]}$$

$$= (1 - k - 2s) \sum_{i=1}^{r} U^{[r-i]}$$

by Theorem 3.1. Applying the operator $(\Delta_k - \lambda)^r$ to both sides of (3.8) we find that

$$(\Delta_k - \lambda)^{r+1} U^{[r]} = (1 - k - 2s) \sum_{i=1}^{r} (\Delta_k - \lambda)^r U^{[r-i]} = 0$$

by the induction hypothesis. The theorem follows. \hfill $\square$

It remains to treat the constant term case, whose solutions involve power functions and logarithms. This is straightforward, but note that the value $s = \frac{k-1}{2}$ is exceptional.

**Theorem 3.4.** (Shifted polyharmonic Fourier coefficients of unrestricted growth, $n = 0$) Let $k \in 2\mathbb{Z}$. Suppose that $f_0(z) = h_0(y)$ satisfies

$$(\Delta_k - \lambda)^m f_0(z) = 0 \quad \text{for all } z = x + iy \in \mathbb{H}.$$ 

Write $\lambda = s_0(s_0 + k - 1)$ for some $s_0 \in \mathbb{C}$ (there are generally two choices for $s_0$).
(1) Suppose that $s_0 \neq \frac{1-k}{2}$ (equivalently, $\lambda \neq -(\frac{1-k}{2})^2$). Then

\[
h_n(y) = \sum_{j=0}^{m-1} a_{0,j}^+ \frac{\partial^j}{\partial s^j} y^s \bigg|_{s=s_0} + \sum_{j=0}^{m-1} a_{0,j}^- \frac{\partial^j}{\partial s^j} y^{1-k-s} \bigg|_{s=s_0}
= \sum_{j=0}^{m-1} a_{0,j}^+ (\log y)^j y^{s_0} + \sum_{j=0}^{m-1} a_{0,j}^- (\log y)^j y^{1-k-s_0}
\]

for some constants $a_{0,j}^+, a_{0,j}^- \in \mathbb{C}$.

(2) Suppose that $s_0 = \frac{1-k}{2}$ (equivalently, $\lambda = -(\frac{1-k}{2})^2$). Then

\[
h_n(y) = \sum_{j=0}^{m-1} \frac{\partial^j}{\partial s^j} \left( a_{0,j}^- y^s + a_{0,j}^+ y^s \log y \right) \bigg|_{s=\frac{1-k}{2}}
= \sum_{j=0}^{m-1} a_{0,j}^- (\log y)^{2j} y^{\frac{1-k}{2}} + \sum_{j=0}^{m-1} a_{0,2j+1}^- (\log y)^{2j+1} y^{\frac{1-k}{2}}
\]

for some constants $a_{0,j}^+, a_{0,j}^- \in \mathbb{C}$.

**Proof.** The proof mirrors that of Theorem 3.3 using the commutation relation (3.3). \qed

### 3.4. Shifted polyharmonic Fourier coefficients of moderate growth

We now characterize the vector spaces of shifted polyharmonic Fourier coefficients of depth $m$ of moderate growth. These vector spaces have dimension $m$ for all Fourier coefficients with index $n \neq 0$ but have dimension $2m$ for the constant term coefficient $n = 0$. We immediately obtain the following theorem.

**Theorem 3.5.** (Shifted polyharmonic Fourier coefficients of moderate growth) Let $k \in 2\mathbb{Z}$. Suppose that $f_n(z) = h_n(y)e^{\imath \pi n x}$ is a shifted-polyharmonic function for $\Delta_k$ on $\mathbb{H}$ with eigenvalue $\lambda \in \mathbb{C}$, i.e. it satisfies

\[(\Delta_k - \lambda)^m f_n(z) = 0 \quad \text{for all} \quad z = x + iy \in \mathbb{H}.
\]

Suppose also that $f_n(z)$ has at most polynomial growth in $y$ at the cusp. Then $h_n(y)$ is of the form given in Theorem 3.3 with the extra requirement that for $n \neq 0$ all coefficients $a_{n,j}^+ = 0$, i.e. no $M_{k,\mu}^+(z)$-functions appear in the expansion.

**Proof.** This result for $n \neq 0$ is an immediate consequence of the asymptotics given in Corollary A.5 from Appendix A. The result for $n = 0$ follows from Theorem 3.4. \qed

### 3.5. Shifted polyharmonic Fourier expansions

We show that Theorem 3.5 implies a Fourier expansion formula valid for all $m$-harmonic Maass forms with shifted eigenvalue $\lambda$. For $n \neq 0$ define $u_{k,n}^{[m],\pm}(y; s_0)$ by (3.4) and (3.5). For $n = 0$ with $s_0 \neq \frac{1-k}{2}$, for each $m \geq 0$ set

\[
u_{k,0}^{[m],-}(y; s_0) = \begin{cases} (\log y)^m y^{1-k-s_0} & \text{if } s_0 \neq \frac{1-k}{2}, \\ (\log y)^2 m y^{1-k} & \text{if } s_0 = \frac{1-k}{2}, \end{cases}
\]

\[
u_{k,0}^{[m],+}(y; s_0) = \begin{cases} (\log y)^m y^{s_0} & \text{if } s_0 \neq \frac{1-k}{2}, \\ (\log y)^{2m+1} y^{1-k} & \text{if } s_0 = \frac{1-k}{2}. \end{cases}
\]

Then we have the following result, which concerns Fourier expansions for functions of moderate growth.
**Theorem 3.6.** (Fourier expansion in $V^m_k(\lambda)$) Let $f(z) \in V^m_k(\lambda)$ for some $k \in 2\mathbb{Z}$. Let $m \geq 1$, and fix an $s_0 \in \mathbb{C}$ with $\lambda = s_0(k + 1 - k)$. Then the Fourier expansion of $f(z)$ exists and has the form

$$f(z) = \sum_{j=0}^{m-1} \left( c_{0,j}^+ u_{k,0}^+[y; s_0] + c_{0,j}^- u_{k,0}^-[y; s_0] \right) + \sum_{n=-\infty}^{\infty} \sum_{j=0}^{m-1} c_{n,j}^- u_{k,n}^-[y; s_0] e^{2\pi inz},$$

in which $c_{n,j}^\pm$ are constants. This Fourier expansion converges absolutely and uniformly to $f(z)$ on compact subsets of $\mathbb{H}$.

**Proof.** See the proof of Theorem 4.3 of [16].

**Remark.** There is a more explicit version of the Fourier expansion for the case $m = 1$ and eigenvalue $\lambda = 0$, which is a special case of a Fourier expansion for 1-harmonic Maass forms that appears in the literature. It contains incomplete Gamma functions instead of Whittaker functions, see [16] Lemma 4.4.

### 4. MAASS RAISING AND LOWERING OPERATORS

**4.1. Properties of the Maass operators.** Recall that the the weight $k$ hyperbolic Laplacian is defined as

$$\Delta_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

Maass [20, Chap. 4.1] introduced raising and lowering operators$^6$ which relate eigenfunctions of $\Delta_k$ with eigenfunctions of $\Delta_{k+2}$ and $\Delta_{k-2}$, respectively. We follow the convention of [4, Sec. 2] and define the weight $k$ Maass raising operator by

$$R_k := 2i \frac{\partial}{\partial z} + \frac{k}{y} = i \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{k}{y},$$

and the weight $k$ Maass lowering operator by

$$L_k := 2iy^2 \frac{\partial}{\partial z} = iy^2 \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Note that $L_k$ is independent of the weight $k$. As the following well-known lemma shows, $R_k$ raises the weight by 2 and $L_k$ lowers the weight by 2.

**Lemma 4.1.** For any $\gamma \in \text{SL}(2, \mathbb{R})$ we have

$$R_k \left( f \big|_{k\gamma} \right) = (R_k f) \big|_{k+2\gamma} \quad (4.1)$$

and

$$L_k \left( f \big|_{k\gamma} \right) = (L_k f) \big|_{k-2\gamma}. \quad (4.2)$$

**Proof.** For $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ we write $w = \gamma z = \frac{az + b}{cz + d}$, so that

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z} \frac{\partial}{\partial w} + \frac{\partial \bar{w}}{\partial z} \frac{\partial}{\partial \bar{w}} = (cz + d)^{-2} \frac{\partial}{\partial w}.$$
We compute

\[ R_k(f | \gamma) = -2ik(cz + d)^{-k-1} f(w) + 2i(cz + d)^{-k-2} \frac{\partial f}{\partial w} + \frac{k}{y}(cz + d)^{-k} f(w) \]

\[ = (cz + d)^{-k-2} \left[ 2i \frac{\partial f}{\partial w} + \frac{k(cz + d)}{y} (-2icy + cz + d) f(w) \right]. \]

Since \(-2icy + cz + d = c\bar{z} + d\) this becomes

\[ R_k(f | \gamma) = (cz + d)^{-k-2} \left[ 2i \frac{\partial f}{\partial w} + \frac{k|cz + d|^2}{y} f(w) \right] = (R_k f | \gamma). \]

For \(L_k\) the proof is similar, and uses the relation

\[ \frac{\partial}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} \frac{\partial}{\partial w} + \frac{\partial \bar{w}}{\partial \bar{z}} \frac{\partial}{\partial \bar{w}} = (c\bar{z} + d)^{-2} \frac{\partial}{\partial \bar{w}}. \]

We recall the following relations ([4, Sec.2]).

**Lemma 4.2.** (1) The Laplacian \(\Delta_k\) can be expressed in terms of \(R_k\) and \(L_k\) in two ways:

\[-\Delta_k = L_{k+2} R_k + k, \quad -\Delta_k = R_{k-2} L_k.\]

(2) The operators \(R_k\) and \(L_k\) satisfy the commutation relations

\[ R_{k-2} L_k - L_{k+2} R_k = k. \]

(3) If \(f\) is an eigenfunction of \(\Delta_k\) satisfying \((\Delta_k - \lambda)f = 0\), then \(R_k f\) and \(L_k f\) are also eigenfunctions with shifted eigenvalues:

\[ (\Delta_{k+2} - (\lambda + k)) R_k f = 0, \]

\[ (\Delta_{k-2} - (\lambda + 2 - k)) L_k f = 0. \]

**Proof.** Relation (1) is a straightforward calculation, and relation (2) follows immediately from (1). For (3) we obtain using (1) and (2) that

\[ \Delta_{k+2} R_k = (-R_k L_{k+2}) R_k = R_k (\Delta_k + k). \]

Thus we have the operator identity

\[ (\Delta_{k+2} - (\lambda + k)) R_k = R_k (\Delta_k - \lambda). \quad (4.3) \]

If \((\Delta_k - \lambda)f = 0\) then (4.3) gives

\[ (\Delta_{k+2} - (\lambda + k)) R_k f = R_k (\Delta_k - \lambda) f = 0. \]

A similar calculation gives the operator identity

\[ (\Delta_{k-2} - (\lambda + 2 - k)) L_k = L_k (\Delta_k - \lambda), \quad (4.4) \]

from which the second part of (3) follows. \(\square\)

The following lemma generalizes part (3) of Lemma 4.2 to shifted polyharmonic functions.

**Lemma 4.3.** If \((\Delta_k - \lambda)^m f = 0\) then

\[ (\Delta_{k+2} - (\lambda + k))^m R_k f = 0, \]

\[ (\Delta_{k-2} - (\lambda + 2 - k))^m L_k f = 0. \]
Proof. For the first equation it suffices to iterate the operator identity (4.3) to obtain
\[(\Delta_{k+2} - (\lambda + k))^m R_k = (\Delta_{k+2} - (\lambda + k))^{m-1} R_k (\Delta_k - \lambda) = \cdots = R_k (\Delta_k - \lambda)^m.\]
The second equation follows by iterating (4.4) similarly. \qed

4.2. Maass operator action on shifted polyharmonic vector spaces.

Lemma 4.4. For \( n \neq 0 \), let \( u_{k,n}^{[m]}(y; s_0) \) be as in (3.4). Then we have
\[ R_k \left( u_{k,n}^{[m]}(y; s_0) e^{2\pi i n x} \right) = u_{k+1,n}^{[m]}(y; s_0 - 1) e^{2\pi i n x} \times \begin{cases} -1 & \text{if } n > 0, \\ (s_0 + k)(1 - s_0) & \text{if } n < 0, \end{cases} \]
\[ L_k \left( u_{k,n}^{[m]}(y; s_0) e^{2\pi i n x} \right) = u_{k-1,n}^{[m]}(y; s_0 + 1) e^{2\pi i n x} \times \begin{cases} s_0(s_0 + k - 1) & \text{if } n > 0, \\ 1 & \text{if } n < 0. \end{cases} \]

Proof. Since \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial s} \) commute with \( \frac{\partial}{\partial y} \), it suffices to prove the lemma for the case \( m = 0 \). To simplify notation, we let \( \epsilon = \text{sgn}(n) \) and we write \( \mu = s_0 + \frac{k-1}{2} \) and \( w = 4\pi |n| y \).

We begin with the \( R_k \) formula. We have
\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) = -\frac{i}{2} \frac{\partial w}{\partial y} \frac{\partial}{\partial w} = -2\pi i |n| \frac{\partial}{\partial w}. \]
Hence
\[ R_k \left( u_{k,n}^{[m]}(y; s_0) e^{2\pi i n x} \right) = (4\pi |n|)^{\frac{k}{2}} \left( \frac{2i}{y} \frac{\partial}{\partial z} + \frac{k}{y} \right) \left( w^{-\frac{k}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k}{2},\mu}(w) e^{2\pi i n x} \right) \]
\[ = (4\pi |n|)^{\frac{k}{2}+1} e^{2\pi i n x} \left[ \left( \frac{k}{w} - \epsilon + \frac{\partial}{\partial w} \right) w^{-\frac{k+1}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k+1}{2},\mu}(w) \right]. \] (4.7)

If \( \epsilon = +1 \) then \([22] (13.15.23) \) and \((13.15.11)\) give
\[ \left( \frac{k}{w} - 1 + \frac{\partial}{\partial w} \right) w^{-\frac{k}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k}{2},\mu}(w) = -w^{-\frac{k+1}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k+1}{2},\mu}(w), \]
while if \( \epsilon = -1 \) then \([22] (13.15.26) \) and \((13.15.11)\) give
\[ \left( \frac{k}{w} + 1 + \frac{\partial}{\partial w} \right) w^{-\frac{k}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k}{2},\mu}(w) = \left( \frac{1}{2} + \mu + \frac{k}{2} \right) \left( \frac{1}{2} - \mu + \frac{k}{2} \right) w^{-\frac{k+1}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k+1}{2},\mu}(w). \] (4.8)

Equation (4.5) follows from (4.7), (4.2), and (4.8) after replacing \( w \) by \( 4\pi |n| y \) and writing \( \mu = s_0 - 1 + \frac{k+2-1}{2} \).

The \( L_k \) formula is similar. Using the fact that \( \frac{\partial}{\partial z} = 2\pi i |n| \frac{\partial}{\partial w} \) we obtain
\[ L_k \left( u_{k,n}^{[m]}(y; s_0) e^{2\pi i n x} \right) = - (4\pi |n|)^{\frac{k}{2}+1} e^{2\pi i n x} y^2 \frac{\partial}{\partial w} \left( w^{-\frac{k}{2}} e^{\frac{\pi i n}{2}} W_{\frac{k}{2},\mu}(w) \right) \]
\[ = (4\pi |n|)^{\frac{k}{2}+1} e^{2\pi i n x} y^2 e^{\frac{\pi i n}{2}} w^{-\frac{k}{2}} - \frac{k}{2} - 1 W_{\frac{k+1}{2},\mu}(w) \]
\[ \times \begin{cases} - \left( \frac{1}{2} + \mu - \frac{k}{2} \right) \left( \frac{1}{2} - \mu - \frac{k}{2} \right) & \text{if } \epsilon = 1, \\ 1 & \text{if } \epsilon = -1, \end{cases} \]
where we used \([22] (13.15.23) \) and \((13.15.26)\) in the last line. Equation (4.5) follows after replacing \( w \) by \( 4\pi |n| y \) and writing \( \mu = s_0 + 1 + \frac{k+2-1}{2} \). \qed

Lemmas [4.1] and [4.3] show that the Maass operators \( R_k \) and \( L_k \) preserve modularity and shift eigenvalues, and Lemma [4.4] implies that they preserve moderate growth of the individual Fourier coefficients. These facts lead to the following proposition (illustrated schematically in Figure 2).
Proposition 4.5. There holds

\[ R_k \left( V^m_k(\lambda) \right) \subset V^m_{k+2}(\lambda + k) \]

and

\[ L_k \left( V^m_k(\lambda) \right) \subset V^m_{k-2}(\lambda + 2 - k). \]

Proof. By the comments above, it suffices to show that if \( f \in V^m_k(\lambda) \) then \( R_k f \) and \( L_k f \) have moderate growth as \( y \to \infty \). We first obtain an estimate for the size of the Fourier coefficients \( c_{n,j}^- \) of \( f \) for \( n \neq 0 \) (we can safely ignore the index \( n = 0 \) terms since they clearly contribute at most polynomial growth).

Fix \( y_0 \geq 1 \). By assumption \( |f(z)| \ll y^\alpha \) for some \( \alpha \) uniformly for \( x \in [0,1] \) and \( y \geq y_0 \). Thus for each \( n \neq 0 \) and every \( j \leq m \) we have

\[ c_{n,j}^- u_{k,n}^{-}[j]; \sigma_0, y \) → \( f(x) e^{2\pi i n x} dx \ll y^\alpha. \]

Using the asymptotic formula from Proposition A.1 we obtain

\[ c_{n,j}^- \ll y^\beta e^{2\pi |n| y} \]

as \( |n| y \to \infty \), where the exponent \( \beta \) and the implied constant are allowed to depend on \( m \) (but not on \( n \) or \( j \)). Setting \( y = \frac{\log |n|}{|n|} \) we find, for some \( A \in \mathbb{R} \), that

\[ c_{n,j}^- \ll |n|^A. \]  \hspace{1cm} (4.9)

Hence, for any fixed \( k' \in 2\mathbb{Z} \), \( s'_0 \in \mathbb{C} \), any \( j \leq m \), and any constants \( a, b \in \mathbb{C} \), we have the estimate

\[ a \sum_{n > 0} c_{n,j}^- u_{k',n}^{-}[j]; \sigma_0, y \) + \( b \sum_{n < 0} c_{n,j}^- u_{k,n}^{-}[j]; \sigma_0, y \) \ll \( y^B \sum_{n \neq 0} |n|^A e^{-2\pi |n| y_0} \ll y^B \]

for some \( B \in \mathbb{R} \), as \( y \to \infty \). It follows that both \( R_k f \) and \( L_k f \) satisfy the moderate growth condition, and this completes the proof. \( \square \)

We conclude this section by establishing in Proposition 4.7 below that, under certain conditions, the maps \( R_k \) and \( L_k \) are isomorphisms. First, we prove the following lemma.

Lemma 4.6. (1) If \( \lambda + k \neq 0 \) then the map

\[ L_k R_k : V^m_k(\lambda) \to V^m_k(\lambda) \]

is an isomorphism.

(2) If \( \lambda \neq 0 \) then the map

\[ R_k L_k : V^m_k(\lambda) \to V^m_k(\lambda) \]

is an isomorphism.
Proof. We prove statement (1); the proof of (2) is analogous.

Suppose that \( \lambda + k \neq 0 \). By Lemma 4.1(1) we have the relation

\[
L_{k+2} R_k = -\Delta_k - k.
\]

It follows that, for \( f \in V_k^1(\lambda) \), we have

\[
L_{k+2} R_k f = -(\lambda + k) f.
\]

Thus \( L_{k+2} R_k : V_k^1(\lambda) \to V_k^1(\lambda) \) is surjective. We proceed by induction. Suppose that \( m \geq 1 \) and that \( L_{k+2} R_k : V_k^{m-1}(\lambda) \to V_k^{m-1}(\lambda) \) is surjective. If \( f \in V_k^m(\lambda) \), then \((\Delta_k - \lambda) f \in V_k^{m-1}(\lambda)\). So by the induction hypothesis, \((\Delta_k - \lambda) f = L_{k+2} R_k g \) for some \( g \in V_k^{m-1}(\lambda) \). We compute

\[
L_{k+2} R_k (f + g) = -(\Delta_k + k) f + (\Delta_k - \lambda) f = -(\lambda + k) f,
\]

hence \( L_{k+2} R_k : V_k^m(\lambda) \to V_k^m(\lambda) \) is surjective. It follows that \( L_{k+2} R_k \) is an isomorphism. \(\Box\)

**Proposition 4.7.**

1. The map \( R_k : V_k^m(\lambda) \to V_{k+2}^m(\lambda + k) \) is an isomorphism when \( \lambda + k \neq 0 \).
2. The map \( L_k : V_k^m(\lambda) \to V_{k-2}^m(\lambda + 2 - k) \) is an isomorphism when \( \lambda \neq 0 \).

Proof. (1) Suppose that \( \lambda + k \neq 0 \). Applying Lemma 4.6(1) to \( V_k^m(\lambda) \), we find that

\[
\dim V_{k+2}^m(\lambda + k) \geq \dim V_k^m(\lambda).
\]

Similarly, Lemma 4.6(2) applied to \( V_{k+2}^m(\lambda + k) \) gives

\[
\dim V_k^m(\lambda) \geq \dim V_{k+2}^m(\lambda + k).
\]

It follows that \( R_k \) is an isomorphism.

(2) Suppose that \( \lambda \neq 0 \). Arguing as in (1), we apply Lemma 4.6(2) to \( V_k^m(\lambda) \) and Lemma 4.6(1) to \( V_{k-2}^m(\lambda + 2 - k) \) to conclude that \( L_k \) is an isomorphism. \(\Box\)

5. The \( \xi \)-Operator and its Properties

5.1. **Properties of the differential operators** \( \xi_k \). Bruinier and Funke [3, Proposition 3.2] introduce the operator

\[
\xi_k := 2iy^k \overline{\partial}_{\overline{z}}.
\]

This operator is essentially “half” of a Laplacian; that is, we have (by a straightforward computation) the relation

\[
\Delta_k = \xi_{2-k} \xi_k.
\]

(5.1)

The operator \( \xi_k \) is related to the Maass operator \( L_k \) of Section 4 by

\[
\xi_k = -y^{k-2} L_k,
\]

(5.2)

and from this it inherits many of its important properties, as the following lemmas show.

**Lemma 5.1.** Let \( f : \mathbb{H} \to \mathbb{C} \) be any \( C^1 \)-function. Then for \( k \in \mathbb{Z} \) we have

\[
\xi_k (f|_k \gamma) = (\xi_k f)|_{2-k} \gamma \quad \text{for all} \ \gamma \in \text{SL}(2, \mathbb{R}).
\]

Proof. From Lemma 4.1 and (5.2) we have

\[
\xi_k (f|_k \gamma) = -y^{k-2} L_k (f|_k \gamma) = -y^{k-2} (L_k f)|_{k-2} \gamma = y^{k-2} \Im (\gamma z)^{2-k} (cz+d)^{2-k} (\xi_k f)(\gamma z) = (\xi_k f)|_k \gamma.
\]
Lemma 5.2 implies that if \( f \) is a weight \( k \) (holomorphic or non-holomorphic) modular form for a discrete subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \), with no growth conditions imposed on any cusp, then \( \xi_k f \) is a weight \( 2 - k \) modular form for \( \Gamma \), again imposing no growth condition at any cusp.

**Lemma 5.2.** Suppose that \( f : \mathbb{H} \to \mathbb{C} \) satisfies \((\Delta_k - \lambda)^m f(z) = 0\). Then we have
\[
(\Delta_{2-k} - \bar{\lambda})^m (\xi_k f) = 0.
\]

**Proof.** By (5.1) and the fact that \( \xi_k \lambda = \bar{\lambda} \xi_k \) we have
\[
(\Delta_{2-k} - \bar{\lambda}) \xi_k = (\xi_k \xi_{2-k}) \xi_k - \bar{\lambda} \xi_k = \xi_k (\xi_{2-k} \xi_k) - \xi_k \lambda = \xi_k (\Delta_k - \lambda).
\]
Iterating this, we find that
\[
(\Delta_{2-k} - \bar{\lambda})^m (\xi_k f) = \xi_k (\Delta_k - \lambda)^m f = 0. \quad \Box
\]

**Lemma 5.3.** Let \( u_{k,n}^{[m],-}(y; s_0) \) be as in (3.4). Then
\[
\xi_k \left( u_{k,n}^{[m],-}(y; s_0) e^{2\pi i n x} \right) = u_{2-k,-n}^{[m],-}(y; s_0) e^{-2\pi i n x} \times \begin{cases} \bar{s}_0 (1 - k + s_0) & \text{if } n > 0, \\ -1 & \text{if } n < 0. \end{cases}
\]

**Proof.** This follows immediately from Lemma 4.4 and (5.2), together with the relations
\[
W_{\kappa,\mu}(y) = W_{\kappa,-\mu}(y)
\]
and (for \( \kappa, y \in \mathbb{R} \))
\[
W_{\kappa,\mu}(y) = W_{\kappa,\mu}(y).
\]
The latter relation follows from the integral representation [22 (13.16.5)] when \( \text{Re } (\mu) + \frac{1}{2} > \text{Re } (\kappa) \) and by analytic continuation otherwise.

### 5.2. Action of \( \xi_k \) on non-holomorphic Eisenstein series

We compute the action of the \( \xi_k \)-operator on the completed and doubly-completed non-holomorphic Eisenstein series.

**Proposition 5.4.** Let \( k \in 2\mathbb{Z} \). Then
\[
\xi_k \hat{E}_k(z, s) = \begin{cases} \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \leq 0, \\ \bar{s}(\bar{s} + k - 1) \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \geq 2. \end{cases}
\]

In addition
\[
\xi_k \hat{E}_k(z, s) = \begin{cases} \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \leq 0, \\ \bar{s}(\bar{s} + k - 1) \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \geq 2. \end{cases}
\]

Proposition 5.4 is proved in [16 Sect. 7], which applies \( \xi_k \) directly to the series expansion (1.3). This proof is initially justified in the half-plane \( \text{Re } (s) > 1 - \frac{k}{2} \), then requires analytic continuation in the \( s \)-variable to hold in general. We outline in Appendix B a second proof of Proposition 5.4 which is based on term-by-term calculation of the Fourier series, using Lemma 5.3. This alternate proof uses Whittaker function identities and is longer, but has the merit of working directly for all \( s \in \mathbb{C} \) since the Fourier series converge absolutely and uniformly on compact subsets (avoiding the poles for the case \( k = 0 \)). Appendix B gives details of the calculation only for the constant term of the Fourier series.

The following proposition shows that the action of the \( \xi_k \) operator on shifted polyharmonic vector spaces preserves the moderate growth condition.
Proposition 5.5. For every $m \geq 1$ we have

$$\xi_k(V^m_k(\lambda)) \subseteq V^m_{2-k}(\lambda).$$

If $\lambda \neq 0$, this map is an isomorphism.

Proof. This result follows from Lemmas 5.1 and 5.2 and using the relation $\xi_k = y - y^{k-2}L_k$.

Remark. On the space $V^m_k(\lambda)$ the operator $\Delta_k = \xi_{2-k}\xi_k : V^m_k(\lambda) \to V^m_k(\lambda)$ has minimal polynomial dividing $(T - \lambda)^m$ and is invertible when $\lambda \neq 0$. For such $\lambda$ we obtain a structure of towers for the $\Delta_k$-action, and a ladder in which the maps $\xi_k$ and $\xi_{2-k}$ together comprise the rungs. Figure II depicts the tower and ladder structure of these maps.

For $\lambda = 0$ the action of $\Delta_k$ on $V^m_k(0)$ is nilpotent, yielding the tower and ramp structure exhibited in [16 Tables 1 and 2]. In the $\lambda = 0$ case there always exist weight $k$ real-analytic modular forms $f$ that do not have moderate growth at the cusp which nevertheless have the property that $\xi_k f$ has moderate growth at the cusp. The simplest examples are the weakly holomorphic modular forms in $M^!_k \setminus M_k$ which have linear exponential growth at the cusp but are annihilated by $\xi_k$ since they are holomorphic functions. See [16 Sect. 6].

6. Shifted Polyharmonic Maass Forms from Eisenstein Taylor Coefficients

We show that the Taylor series coefficients of the doubly completed Eisenstein series $\hat{E}_k(z, s)$ in the $s$-variable at a point $s = s_0 \in \mathbb{C}$ define shifted polyharmonic Maass forms of eigenvalue $\lambda = s_0 + k - 1$. We consider doubly-completed Eisenstein series rather than the singly completed series $\hat{E}_k(z, s)$ because they are entire functions of $s$ for all $k \in 2\mathbb{Z}$; the series $\hat{E}_0(z, s)$ has simple poles at $s = 0, 1$.

6.1. Taylor series expansions for weight $k$ non-holomorphic Eisenstein series. The doubly-completed Eisenstein series has a Taylor series expansion in the $s$-variable around any point $s_0 \in \mathbb{C}$ given by

$$\hat{E}_k(z, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{E}^{[n]}_k(z; s_0)(s - s_0)^n,$$

in which the Taylor coefficients

$$\hat{E}^{[n]}_k(z; s_0) := \frac{\partial^n}{\partial s^n} \hat{E}_k(z, s) \big|_{s=s_0}$$

are viewed as functions of $z \in \mathbb{H}$ to $\mathbb{C}$.

Theorem 6.1. (Taylor Series Recursion) Fix $s_0 \in \mathbb{C}$ and set $\lambda = s_0 + k - 1$.

1. The functions $\hat{E}^{[n]}_k(z; s_0)$ given by (6.1) obey a recursion

$$(\Delta_k - \lambda)\hat{E}^{[n]}_k(z; s_0) = n(2s_0 + k - 1)\hat{E}^{[n-1]}_k(z; s_0) + n(n-1)\hat{E}^{[n-2]}_k(z; s_0),$$

in which any terms on the right are omitted whenever their superscript $[m]$ has $m < 0$.

2. One has $\hat{E}^{[n]}_k(z; s_0) \in V^{n+1}_k(\lambda)$ for each $n \geq 0$. That is, $\hat{E}^{[n]}_k(z; s_0)$ is a shifted polyharmonic function for $\Delta_k$ with (shifted) harmonic depth at most $n + 1$, with eigenvalue $\lambda$ and moderate growth at the cusp.
Proof. (1) The \( n = 0 \) case of (6.2) asserts that

\[
\Delta_k \hat{E}_k^{[0]}(z; s_0) = s_0(s_0 + k - 1) \hat{E}_k^{[0]}(z; s_0),
\]

which follows from Theorem 2.7 (3). For \( n = 1 \) we have

\[
\Delta_k \hat{E}_k^{[1]}(z; s_0) = \Delta_k \left( \frac{\partial}{\partial s} \hat{E}_k(z; s) \big|_{s=s_0} \right) = \frac{\partial}{\partial s} \left[ \Delta_k \hat{E}_k(z, s) \right]_{s=s_0} = s(s + k - 1) \hat{E}_k(z, s) + (n - 1)(2s + k - 1) \hat{E}_k^{[n-2]}(z; s) + (n - 1)(n - 2) \hat{E}_k^{[n-3]}(z; s),
\]

as required. The cases \( n \geq 2 \) are proved by induction on \( n \). For the induction step, assuming (6.2) holds for \( n - 1 \) and \( n - 2 \), we observe that

\[
\Delta_k \hat{E}_k^{[n]}(z; s_0) = \frac{\partial}{\partial s} \left[ \Delta_k \hat{E}_k^{[n-1]}(z; s) \right]_{s=s_0} = s(s + k - 1) \hat{E}_k^{[n-1]}(z; s) + (n - 1)(2s + k - 1) \hat{E}_k^{[n-2]}(z; s) + (n - 1)(n - 2) \hat{E}_k^{[n-3]}(z; s),
\]

which verifies (6.2) for \( n \).

(2) The Taylor coefficients \( \hat{E}_k^{[n]}(z; s_0) \) inherit the property of transforming as weight \( k \) Maass forms from \( \hat{E}_k(z, s) \). The recursion (6.2) for \( n = 0 \) states that \( \hat{E}_k^{[0]}(z; s_0) \) is shifted polyharmonic with eigenvalue \( \lambda \) of shifted harmonic depth at most 1. By induction on \( n \) this recursion establishes that \( \hat{E}_k^{[n]}(z; s_0) \) is shifted polyharmonic of depth at most \( n + 1 \). Finally, by Proposition 2.5 and (6.1), the functions \( \hat{E}_k^{[n]}(z; s_0) \) are of moderate growth at the cusp since, aside from the constant term, their Fourier expansions only involve the functions \( u_{k,n}^{-}(y; s_0) \), which decay exponentially as \( y \to \infty \). We conclude that \( \hat{E}_k^{[n]}(z; s_0) \) belongs to \( V_k^{n+1} \).

\( \square \)

Remark 6.2. In the special case of the central point \( s_0 = \frac{1-k}{2} \) of the functional equation, where \( \lambda = -(\frac{1-k}{2})^2 \), the recursion (6.2) degenerates to

\[
\left( \Delta_k + \left( \frac{1-k}{2} \right)^2 \right) \hat{E}_k^{[n]}(z; \frac{1-k}{2}) = n(n - 1) \hat{E}_k^{[n-2]}(z; \frac{1-k}{2}).
\]

The functional equation \( \hat{E}_k(z; s) = \hat{E}_k(z; 1 - k - s) \) implies that \( \hat{E}_k(z; \frac{1-k}{2} + s) \) is an even function of \( s_1 \), so its odd-indexed Taylor coefficients at \( s_1 = 0 \) vanish identically.

6.2. Shifted polyharmonic Eisenstein series vector spaces. We define vector spaces of shifted polyharmonic Maass forms spanned by Taylor coefficients of non-holomorphic Eisenstein series.

Definition 6.3. The shifted \( m \)-harmonic Eisenstein space \( E_k^{[m]}(\lambda) \) is the vector space generated by all the Taylor coefficient functions \( \hat{E}_k^{[n]}(z; s_0) \) for those \( n \geq 0 \) such that \( \hat{E}_k^{[n]}(z; s_0) \in V_k^{n} \). As the following theorem shows, these spaces are finite-dimensional and are generated by all the Taylor coefficient functions up to some depth \( n_0 \) depending on \( m \) and \( \lambda \).
Theorem 6.4. Let $k \in 2\mathbb{Z}$. Then for all $\lambda \in \mathbb{C}$ the shifted $m$-harmonic Eisenstein space $E^m_k(\lambda)$ has dimension $m$. In addition, letting $\lambda = s_0(s_0 + k - 1)$, we have

1. If $\lambda \notin \{(\frac{k}{2} - \frac{1}{2}), -((\frac{k}{2})^2\}}$ then there are two choices for $s_0$. For each choice we have $\widehat{E}^{[0]}_k(z; s_0) \neq 0$, and a basis of $E^m_k(\lambda)$ is given by

$$\left\{ \widehat{E}^{[n]}_k(z; s_0) : 0 \leq n \leq m - 1 \right\}.$$

2. If $\lambda = \frac{k}{2}(1 - \frac{k}{2})$, so that $s_0 = -\frac{k}{2}$ or $s_0 = 1 - \frac{k}{2}$, then for all $k \neq 0$, we have $\widehat{E}^{[0]}_k(z; s_0) \equiv 0$, and a basis of $E^m_k(\lambda)$ is given by

$$\left\{ \widehat{E}^{[n]}_k(z; s_0) : 1 \leq n \leq m \right\}.$$

If $\lambda = k = 0$, we have $\widehat{E}^{[0]}_k(z; s_0) \neq 0$, and a basis of $E^m_k(\lambda)$ is given by

$$\left\{ \widehat{E}^{[n]}_k(z; s_0) : 0 \leq n \leq m - 1 \right\}.$$

3. If $\lambda = -(\frac{k-1}{2})^2$ so that $s_0 = \frac{1-k}{2}$ is unique, a basis of $E^m_k(\lambda)$ is given by

$$\left\{ \widehat{E}^{[2n]}_k(z; s_0) : 0 \leq n \leq m - 1 \right\}.$$

Proof. It is easy to check via the functional equation relating $s$ to $1 - k - s$ that the two values of $s_0$ with $\lambda = s_0(s_0 + k - 1)$ correspond to the same spaces $E^m_k(\lambda)$ for all $m$, so we may fix one such value.

1. Suppose that $\lambda \notin \{-((\frac{k}{2})^2), \frac{k}{2}(1 - \frac{k}{2})\}$. It suffices to show that

$$\widehat{E}^{[n]}_k(z; s_0) \in V^{n+1}_k(\lambda) \setminus V^n_k(\lambda) \tag{6.4}$$

holds for all $n \geq 0$, taking $V^0_k(\lambda) = \{0\}$. We proceed by induction on $n \geq 0$. By Proposition 2.6 the Fourier constant term of $\widehat{E}^{[0]}_k(z; s_0)$ is nonzero since the $s$-polynomial factors in front of the terms $y^{s_0}$ and $y^{1-k-s_0}$ have no common roots. Hence $\widehat{E}^{[0]}_k(z; s_0) \neq 0$, which verifies the $n = 0$ case of (6.4). By hypothesis $2s_0 + k - 1 \neq 0$, so the recursion (6.2) certifies the induction step.

2. Suppose $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ and that $k \neq 0$, so $s_0 = -\frac{k}{2}$ or $1 - \frac{k}{2}$ are nonzero. We can see directly from Theorem 2.7 that the completed Eisenstein series $\widehat{E}_k(z, s)$ is an entire function, hence

$$\widehat{E}_k(z, s) = (s + \frac{k}{2})(s + \frac{k}{2} - 1)\widehat{E}_k(z, s) = 0,$$

so $\widehat{E}_k^{[0]}(z; s_0) \equiv 0$. Now

$$\widehat{E}_k^{[1]}(z; s_0) = \frac{\partial}{\partial s} \left[ (s + \frac{k}{2})(s + \frac{k}{2} - 1)\widehat{E}_k(z, s) \right]_{s=s_0} = \pm \widehat{E}_k(z, s_0),$$

and by Proposition 2.6 we see that the Fourier constant term of $\widehat{E}_k^{[1]}(z; s_0) \neq 0$. Again $2s_0 + k - 1 \neq 0$, so it follows by induction on $m \geq 1$ using the recursion (6.2) that $\left\{ \widehat{E}_k^{[n]}(z; s_0) : 1 \leq n \leq m \right\}$ is a basis of $E^m_k(\lambda)$.

For $\lambda = k = 0$ and $s_0 \in \{0, 1\}$, Theorem 2.7 shows $\widehat{E}_k(z; s)$ has simple poles at $s_0$, coming from the Fourier constant term, so $\widehat{E}_k^{[0]}(z; s_0) \neq 0$. The basis result is again proved by induction on $m \geq 1$. (The case $\lambda = 0$ is treated in detail in Theorems 1.1 and 2.2 of [16].)
induction on $m$, having Fourier constant term equal to $\frac{1-k}{2}$. By Proposition 2.6 we have the constant term of $\hat{E}_k^0(z; s_0) \neq 0$, and by Theorem 6.4 we have $\hat{E}_k^0(z; s_0) \in V_k^1(\lambda)$. By Remark 6.2 all Taylor coefficients $\hat{E}_k^{2n+1}(z; s_0)$ vanish identically, and we have the recursion
\[
(\Delta_k + \left(\frac{1-k}{2}\right)^2) \hat{E}_k^{2n}(z; s_0) = 2n(2n-1) \hat{E}_k^{2n-2}(z; s_0).
\]
By induction on $n \geq 0$ we conclude that $\hat{E}_k^{2n}(z; s_0) \in V_k^{n+1}(\lambda) \setminus V_k^n(\lambda)$, the base case $n = 0$ having been established.

Remark 6.5. The vector space $E_k^m(\lambda)$ varies continuously as a function of $\lambda$ for all $\lambda \notin \{-\frac{1-k}{2}, \frac{k}{2}(1-\frac{k}{2})\}$, in the sense that the functions $\hat{E}_k^{|s|}(z; s_0)$ vary continuously in the parameter $s_0$ (restricting $z$ to any compact subset of $\mathbb{H}$). It has discontinuous “jumps” at the point $\lambda = -\frac{1-k}{2}$ for all weights and at the point $\lambda = \frac{k}{2}(1-\frac{k}{2})$ for all nonzero weights.

7. Proof of Theorem 1.1

For the reader’s convenience, we restate Theorem 1.1 here.

**Theorem 1.1** Fix $k \in 2\mathbb{Z}$. For $\lambda \in \mathbb{C}$ fix $s_0 \in \mathbb{C}$ such that $\lambda = s_0(s_0 + k - 1)$.

1. The complex vector space $V_k^m(\lambda)$ is finite dimensional, with
\[
\dim V_k^m(\lambda) \leq m + m \dim S_k^1(\lambda),
\]
where $S_k^1(\lambda)$ is the space of Maass cusp forms of weight $k$ and eigenvalue $\lambda$.

2. This space decomposes as
\[
V_k^m(\lambda) = E_k^m(\lambda) \oplus S_k^m(\lambda),
\]
in which the Eisenstein series space $E_k^m(\lambda)$ is spanned by certain Taylor coefficients of shifted Eisenstein series and $S_k^m(\lambda)$ is a recursively defined space of “generalized $m$-harmonic Maass cusp forms.” Both vector spaces $E_k^m(\lambda)$ and $S_k^m(\lambda)$ are closed under the action of $\Delta_k - \lambda$.

3. For all $\lambda \in \mathbb{C}$ the space $E_k^m(\lambda)$ has dimension $m$.
   (i) For $\lambda \neq -\frac{1-k}{2}$ it has a basis consisting of the Taylor coefficient functions
   \[
   \hat{E}_k^{(j+r)}(z; s_0) := \left. \frac{\partial^{j+r}}{\partial s_j^{r+r}} \hat{E}_k(z; s) \right|_{s=s_0} \quad \text{for} \quad 0 \leq j \leq m-1,
   \]
   where $r$ is minimal such that $\hat{E}_k^r(z; s_0) \neq 0$. Here $r = 0$ unless $\lambda = \frac{k}{2}(1-\frac{k}{2})$ and $k \neq 0$, in which case $r = 1$.
   (ii) For $\lambda = -\frac{1-k}{2}$ and $s_0 = \frac{1-k}{2}$, a basis is given by the even-indexed Taylor coefficient functions $\hat{E}_k^{(2j)}(z; s_0)$ for $0 \leq j \leq m-1$. All odd-indexed functions $\hat{E}_k^{(2j+1)}(z; s_0)$ $\equiv 0$.

4. For $m \geq 1$ one has
\[
\dim \left( S_k^m(\lambda) \right) \leq m \dim \left( S_k^1(\lambda) \right).
\]

**Proof.** (1) We prove the upper bound (7.1) by induction on $m \geq 1$. The base case $m = 1$ holds since $V_k^1(\lambda) = E_k^1(\lambda) \oplus S_k^1(\lambda)$, where $S_k^1(\lambda)$ is the space of Maass cusp forms (those forms having Fourier constant term equal to 0), and since $\dim E_k^1(\lambda) = 1$ by Theorem 6.4. We verify by induction on $m \geq 1$ the hypothesis
\[
\dim V_k^m(\lambda) - \dim V_k^{m-1}(\lambda) \leq \dim E_k^1(\lambda) + \dim S_k^1(\lambda),
\]
(7.2)
where we set \( V_k^0(\lambda) = \{0\} \). Since \( V_k^{m-1}(\lambda) \subset V_k^m(\lambda) \) we may define the quotient space \( W_k(\lambda) := V_k^m(\lambda)/V_k^{m-1}(\lambda) \). Since \((\Delta_k - \lambda) : V_k^m(\lambda) \to V_k^{m-1}(\lambda)\) for all \( m \geq 1 \), the map
\[
(\Delta_k - \lambda) : W_k^m(\lambda) \to W_k^{m-1}(\lambda)
\] (7.3)
is well-defined. We claim this map is injective. Given \( f \in V_k^m(\lambda) \) let \([f]\) denote the coset in \( W_k^m(\lambda) \) containing \( f \). Let \( f, g \in V_k^m(\lambda) \) and suppose that
\[
(\Delta_k - \lambda)[f] = (\Delta_k - \lambda)[g] \quad \text{in} \quad W_k^{m-1}(\lambda).
\]
Then \((\Delta_k - \lambda)(f - g) \in V_k^{m-2}(\lambda)\). It follows that
\[
(\Delta_k - \lambda)^{m-1}(f - g) = (\Delta_k - \lambda)^{m-2}((\Delta_k - \lambda)(f - g)) = 0,
\]
whence \( f - g \in V_k^{m-1} \), i.e. \([f] = [g]\). This shows that the map (7.3) is injective, from which it follows that
\[
\dim W_k^m(\lambda) \leq \dim W_k^{m-1}(\lambda).
\]
The latter inequality verifies the induction hypothesis (7.2). We conclude that
\[
\dim V_k^m(\lambda) \leq m + m \dim S_k^1(\lambda).
\]

(2) Given \( f \in V_k^m(\lambda) \) let
\[
CT(f) = \sum_{j=0}^{m-1} \left( c_{0,j}^+(f) u_{k,0}^+(y, s_0) + c_{0,j}^-(f) u_{k,0}^-(y, s_0) \right)
\]
denote the Fourier constant term of \( f \) (see Theorem 3.6). We define a Hermitian scalar product on the vector space \( V_k^m(\lambda) \) by
\[
\langle F, G \rangle_{s_0} := \sum_{j=0}^{\infty} \overline{c_{0,j}^+(F)} c_{0,j}^+(G) + \sum_{j=0}^{\infty} \overline{c_{0,j}^-(F)} c_{0,j}^-(G).
\] (7.4)
This sum is finite because \( c_{0,j}^+(f) = c_{0,j}^-(f) = 0 \) for \( j \) sufficiently large.

The spaces \( E_k^m(\lambda) \) were defined in Section 6.2. We start for \( m = 1 \) with the usual decomposition
\[
V_k^1(\lambda) = E_k^1(\lambda) + S_k^1(\lambda)
\]
in which \( E_k^1(\lambda) \) is spanned by a single Eisenstein series, \( \tilde{E}_k(z) := \tilde{E}_k^r(z; s_0) \), where \( r \in \{0, 1\} \) (see Theorem 6.4). The proof of Theorem 6.4 showed that \( \tilde{E}_k(z) \) has a non-vanishing Fourier constant term. The space \( S_k^1(\lambda) \) consists of the cusp forms, which have identically zero Fourier constant term.

For the Eisenstein series \( \tilde{E}_k(z) \), we have all \( c_{0,j}^+(\tilde{E}_k) = 0 \) if \( j \geq 1 \) (since it is annihilated by \( \Delta_k - \lambda \) hence for all \( G \in V_k^m(\lambda) \),
\[
\langle \tilde{E}_k, G \rangle_{s_0} = c_{0,0}^+(\tilde{E}_k) c_{0,0}^+(G) + c_{0,0}^-(\tilde{E}_k) c_{0,0}^-(G).
\] (7.5)
Here at least one of \( c_{0,0}^+(\tilde{E}_k), c_{0,0}^-(\tilde{E}_k) \) is nonzero.

We now recursively construct for \( m \geq 2 \) a decomposition
\[
V_k^m(\lambda) = E_k^m(\lambda) + S_k^m(\lambda),
\]
using the Hermitian scalar product as follows. Let \( S_k^m(\lambda) \) denote the space
\[
S_k^m(\lambda) := \{ G \in V_k^m(\lambda) : (\Delta_k - \lambda) G \in S_k^{m-1}(\lambda) \},
\]
and define, for $m \geq 2$, the subspace
\[
S_k^m(\lambda) := \left\{ G \in S_k^m(\lambda) : \langle \tilde{E}_k, G \rangle_{s_0} = 0 \right\}.
\]
Note that $S_k^1(\lambda)$ satisfies the property above as well. This definition ensures that $\tilde{E}_k(z) \notin S_k^m(\lambda)$, and that
\[
(\Delta_k - \lambda)S_k^m(\lambda) \subseteq S_k^{m-1}(\lambda).
\] (7.6)
In the following claims, we prove that the “generalized cusp form” space $S_k^m(\lambda)$ has the required properties.

Claim 1: $S_k^{m-1}(\lambda) \subseteq S_k^m(\lambda)$.

The definition of $S_k^m(\lambda)$ includes all elements of $S_k^{m-1}(\lambda)$ because the orthogonality condition for $\tilde{E}_k(z)$ in the product uses only the first two coefficients in the constant term (see (7.5)). This proves Claim 1.

Claim 2: $E_k^m(\lambda) \cap S_k^m(\lambda) = \{0\}$.

Suppose, by way of contradiction, that $0 \neq G(z) \in E_k^m(\lambda) \cap S_k^m(\lambda)$. By (7.6) and Claim 1, $S_k^m(\lambda)$ is closed under the action of $\Delta_k - \lambda$. We first treat the case $\lambda \notin \{-(1-k)^2, \frac{k}{2}(1 - \frac{k}{2})\}$. By Theorem 6.4 (1) we have
\[
G(z) = \sum_{j=0}^{m-1} \alpha_j \tilde{E}_k^{[j]}(z, s_0).
\]
Let $\ell$ denote the largest integer such that $\alpha_\ell \neq 0$. Now $G \in S_k^m(\lambda)$ so $(\Delta_k - \lambda)^\ell G \in S_k^m(\lambda)$. However, using the recursion of Theorem 6.1 we have
\[
(\Delta_k - \lambda)^\ell (G(z)) = \alpha_\ell (\ell)! (2s_0 + k - 1)^\ell \tilde{E}_k^{[\ell]}(z, s_0).
\]
Since $2s_0 + k - 1 \neq 0$ this is a nonzero scalar multiple of $\tilde{E}_k(z)$, contradicting $\tilde{E}_k(z) \notin S_k^m(\lambda)$.

The cases $\lambda \in \{-(1-k)^2, \frac{k}{2}(1 - \frac{k}{2})\}$ are handled by similar arguments using Theorem 6.4 (2) and (3). This proves Claim 2.

Claim 3: $V_k^m(\lambda) = E_k^m(\lambda) + S_k^m(\lambda)$.

By Claim 2 it suffices to show that
\[
\dim (E_k^m(\lambda)) + \dim (S_k^m(\lambda)) \geq \dim (V_k^m(\lambda)).
\] (7.7)
We first show via an inductive argument on $m \geq 1$ that
\[
\dim \left( S_k^m(\lambda) \right) \geq \dim (V_k^m(\lambda)) - \dim (E_k^m(\lambda)) + 1.
\] (7.8)
This bound holds for the base case $m = 1$ since $S_k^1(\lambda) = V_k^1(\lambda)$ and $\dim (E_k^1(\lambda)) = 1$. Now let $F \in V_k^m(\lambda)$ so $(\Delta_k - \lambda)F \in V_k^{m-1}(\lambda)$, where $V_k^{m-1}(\lambda) = E_k^{m-1}(\lambda) + S_k^{m-1}(\lambda)$ by the inductive hypothesis. Now $(\Delta_k - \lambda)E_k^m(\lambda)$ has image $E_k^{m-1}(\lambda)$ and $(\Delta_k - \lambda)\tilde{E}_k(z) = 0$. Therefore there is a subspace $\tilde{E}_k^m(\lambda) \subseteq E_k^m(\lambda)$ of codimension at least 1 whose image under $(\Delta_k - \lambda)$ is the full space $E_k^{m-1}(\lambda)$. Thus we can find an element $E \in \tilde{E}_k^m(\lambda)$ such that $(\Delta_k - \lambda)(F - E) \in S_k^{m-1}(\lambda)$. Therefore
\[
\dim \left( S_k^m(\lambda) \right) \geq \dim (V_k^m(\lambda)) - \dim (\tilde{E}_k^m(\lambda)) \geq \dim (V_k^m(\lambda)) - \dim (E_k^m(\lambda)) + 1,
\]
which proves (7.8). To prove (7.7) it suffices to show that
\[
\dim \left( S_k^m(\lambda) \right) \leq \dim (S_k^m(\lambda)) + 1.
\] (7.9)
This bound follows from the definition of $S^m_k(\lambda)$, which shows that every element of $\tilde{S}^m_k(\lambda)$ differs from an element of $S^m_k(\lambda)$ by a multiple of $E_k(z)$. The inequalities (7.8) and (7.9) together prove (7.7). This proves Claim 3.

(3) This result summarizes the content of Theorem 6.4.

(4) This follows immediately from statements (1) and (3). □

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Appendix A. Derivatives of Whittaker Functions in the Second Index

We give asymptotic expansions for Whittaker functions and for the derivatives with respect to $\mu$ of parametrized families of Whittaker functions, used in Section 3.4. Fix $\delta > 0$. For $z \in \mathbb{C}$ with $|z| \to \infty$ in $|\arg z| \leq \frac{3\pi}{2} - \delta$ we have the asymptotic expansion [22] (13.19.3))

$$W_{\kappa,\mu}(z) \sim e^{-\frac{i}{4}z^2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} + \mu - \kappa\right)_n \left(\frac{1}{2} - \mu - \kappa\right)_n}{n!} z^{-n},$$

(A.1)

where $(a)_n$ is the Pochhammer symbol

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1) \cdots (a + n - 1).$$

When one of $\frac{1}{2} \pm \mu - \kappa$ is a negative integer or zero, the series (A.1) terminates; in that case $W_{\kappa,\mu}(z)$ is equal to its asymptotic expansion.

Differentiating asymptotic expansions (especially with respect to parameters) requires special care. In the following proposition we show that the asymptotic expansions of the $\mu$-derivatives of $W_{\kappa,\mu}(z)$ are obtained by differentiating (A.1) term-by-term.

**Proposition A.1.** Fix $\delta > 0$, and for $\mu_0 \in \mathbb{C}$ fix a small open neighborhood $U \subset \mathbb{C}$ of $\mu_0$. For $\mu \in U$ and for $z \in \mathbb{C}$ with $|z| \to \infty$ in $|\arg z| \leq \frac{3\pi}{2} - \delta$ we have the asymptotic expansion

$$\frac{\partial^m}{\partial \mu^m} W_{\kappa,\mu}(z) \sim e^{-\frac{i}{4}z^2}\sum_{n=0}^{\infty} \frac{\partial^m}{\partial \mu^m} \frac{\left(\frac{1}{2} + \mu - \kappa\right)_n \left(\frac{1}{2} - \mu - \kappa\right)_n}{n!} z^{-n}. $$

(A.2)

**Proof.** We begin with the Mellin-Barnes integral representation [32] Section 16.4 for $W_{\kappa,\mu}(z)$, valid for $|\arg z| \leq \frac{3\pi}{2} - \delta$ and all $\kappa, \mu$ such that neither of $\frac{1}{2} \pm \mu + \kappa$ is a positive integer:

$$M_{\kappa,\mu}(z) = \frac{e^{-\frac{i}{4}z^2}}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2} - \mu - \kappa - s\right) \Gamma\left(\frac{1}{2} + \mu - \kappa - s\right) \frac{z^s}{s} ds. $$

(A.3)

The contour in (A.3) loops if necessary so that it separates the poles of $\Gamma(s)$ and

$$G_{\kappa,\mu}(s) := \frac{\Gamma\left(\frac{1}{2} - \mu - \kappa - s\right) \Gamma\left(\frac{1}{2} + \mu - \kappa - s\right)}{\Gamma\left(\frac{1}{2} - \mu - \kappa\right) \Gamma\left(\frac{1}{2} + \mu - \kappa\right)}.$$  

Fix a positive integer $N$ such that for each $\mu \in U$ (with neither of $\frac{1}{2} \pm \mu + \kappa$ a positive integer), the poles of $G_{\kappa,\mu}(s)$ are to the right of the line $\text{Re}(s) = -N - \frac{1}{2}$. Following Section 16.4 of [32] we find that

$$W_{\kappa,\mu}(z) = e^{-\frac{i}{4}z^2} \sum_{n=0}^{N} \frac{G_{\kappa,\mu}(-n)}{n!} (-z)^{-n} + \frac{1}{2\pi i} \int_{-N - \frac{1}{2} - i\infty}^{-N - \frac{1}{2} + i\infty} \Gamma(s) G_{\kappa,\mu}(s) z^s ds.$$  

(A.4)
As above, the contour loops if necessary to avoid poles of the integrand. Note that for \( \kappa, \mu \) with one of \( \frac{1}{2} \pm \mu + \kappa \) a positive integer, we have \( G_{\kappa, \mu}(s) \equiv 0 \), so \((A.4)\) holds for all \( \mu \in U \).

We can differentiate with respect to \( \mu \) under the integral sign as long as

\[
\int_{-N - \frac{1}{2} + i \infty}^{N - \frac{1}{2} + i \infty} \frac{\partial^m}{\partial \mu^m} \Gamma(s)G_{\kappa, \mu}(s)z^s \, ds \tag{A.5}
\]

is absolutely convergent. Let \( \psi(s) \) denote the digamma function \( \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} \), and define

\[
H_{\kappa, \mu}(s) := -\psi \left( \frac{1}{2} - \mu - \kappa - s \right) + \psi \left( \frac{1}{2} + \mu - \kappa - s \right) - \psi \left( \frac{1}{2} - \mu - \kappa \right) + \psi \left( \frac{1}{2} + \mu - \kappa \right),
\]

so that

\[
\frac{\partial}{\partial \mu} G_{\kappa, \mu}(s) = G_{\kappa, \mu}(s)H_{\kappa, \mu}(s).
\]

Iterating the latter equation, we find that

\[
\frac{\partial^m}{\partial \mu^m} G_{\kappa, \mu}(s) = G_{\kappa, \mu}(s) (H_{\kappa, \mu}(s)^m + R), \tag{A.6}
\]

where \( R \) is a polynomial of degree \( m - 1 \) in \( H_{\kappa, \mu}(s) \) and its \( \mu \)-derivatives. For large \( |s| \) with \( \arg s \leq \pi - \delta \) we have (by differentiating the asymptotic expansion \([22] \, (5.11.2)\), see \([22] \, Section \, 2.1(ii)\)) the estimates

\[
\psi(s) \sim \log |s| \quad \text{and} \quad \psi^{(j)}(s) \asymp \frac{1}{|s|^j}, \quad j \geq 1. \tag{A.7}
\]

Setting \( s = -N - \frac{1}{2} + it \), it follows from \((A.6), (A.7), \text{and} \, [22] \, (5.11.9)\) that

\[
\frac{\partial^m}{\partial \mu^m} \Gamma(s)G_{\kappa, \mu}(s)z^s \ll_{m, \kappa, \mu} |z|^{-N + \frac{1}{2}} e^{-\frac{3}{2}|s|t} |t|^{N - 2\Re(\kappa)} (\log |t|)^m \quad \text{as} \, t \to \pm \infty. \tag{A.8}
\]

Thus the integral \((A.5)\) is absolutely convergent and is \( O(|z|^{-N - \frac{1}{2}}) \). It follows that for \( \mu \in U \), the function \( \frac{\partial^m}{\partial \mu^m} W_{\kappa, \mu}(z) \) has asymptotic expansion

\[
\frac{\partial^m}{\partial \mu^m} W_{\kappa, \mu}(z) \sim e^{-\frac{1}{2} z^\kappa} \sum_{n=0}^{\infty} \frac{\partial^m}{\partial \mu^m} G_{\mu, \kappa}(-n) \frac{1}{n!} (-z)^{-n}.
\]

This completes the proof since

\[
G_{\mu, \kappa}(-n) = \left( \frac{1}{2} + \mu - \kappa \right) \left( \frac{1}{2} - \mu - \kappa \right)^n. \quad \square
\]

The following corollary of Proposition \([A.1]\) is vital for the results of Section \([3]\)

**Corollary A.2.** Fix \( \mu_0 \in \mathbb{C} \). If \( \mu_0 \neq 0 \) then for each \( m \geq 0 \) the set

\[
\left\{ \frac{\partial^j}{\partial \mu^j} W_{\kappa, \mu}(z) \bigg|_{\mu = \mu_0} : 0 \leq j \leq m \right\} \cup \left\{ \frac{\partial^j}{\partial \mu^j} W_{-\kappa, \mu}(z) \bigg|_{\mu = \mu_0} : 0 \leq j \leq m \right\} \tag{A.9}
\]

is linearly independent. If \( \mu_0 = 0 \) then for each \( m \geq 0 \) the set

\[
\left\{ \frac{\partial^j}{\partial \mu^j} W_{\kappa, \mu}(z) \bigg|_{\mu = 0} : 0 \leq j \leq m \right\} \cup \left\{ \frac{\partial^j}{\partial \mu^j} W_{-\kappa, \mu}(z) \bigg|_{\mu = 0} : 0 \leq j \leq m \right\} \tag{A.10}
\]

is linearly independent.
Proof. Suppose first that $\mu_0 \neq 0$. For each $j \geq 0$ and $\ell = 1, 2$ let
\[ F_{2j+\ell}(z) := \frac{\partial^{2j+\ell}}{\partial \mu^{2j+\ell}} \left( \frac{1}{2} + \mu - \kappa \right)_{j+1} (\frac{1}{2} - \mu - \kappa)_{j+1} \right)_{\mu=\mu_0}, \]

Since $\left( \frac{1}{2} + \mu - \kappa \right)_n (\frac{1}{2} - \mu - \kappa)_n$ is a polynomial in $\mu^2$ of degree $n$, we have
\[ F_{2j+1}(z) = \frac{(2j+2)!}{(j+1)!} \frac{\mu_0}{z^{j+1}} + \frac{(2j+4)!}{6(j+2)!} \frac{\mu_0^3}{z^{j+2}} + \frac{\alpha_j \mu_0}{z^{j+2}} \]
for some $\alpha_j \in \mathbb{C}[\kappa]$. Hence
\[ \mu_0 F_{2j+2}(z) - F_{2j+1}(z) = \frac{(2j+4)!}{3(j+2)!} \frac{\mu_0^3}{z^{j+2}}, \]
from which it follows that $F_{2j+1}(z)$ and $F_{2j+2}(z)$ are linearly independent (since $\mu_0 \neq 0$).

By Proposition [A.4] and the fact that $\frac{\partial^{2j+1}}{\partial \mu^{2j+1}} \left( \frac{1}{2} + \mu - \kappa \right)_n (\frac{1}{2} - \mu - \kappa)_n = 0$ for $n \leq j$ we have the asymptotic formula
\[ \frac{\partial^{2j+\ell}}{\partial \mu^{2j+\ell}} W_{\kappa,\mu}(z) \sim e^{-\frac{s}{2}} z^s F_{2j+\ell}(z). \]
It follows that the set (A.9) is linearly independent.

If $\mu_0 = 0$ then since the right-hand side of (A.2) is an even function of $\mu$, all of the odd-order derivatives $\frac{\partial^{2j+1}}{\partial \mu^{2j+1}} W_{\kappa,\mu}(z) |_{\mu=0}$ are identically zero. But by Proposition [A.4] we have the asymptotic formula
\[ \frac{\partial^{2j}}{\partial \mu^{2j}} W_{\kappa,\mu}(z) \sim \frac{(2j)!}{j!} e^{-\frac{s}{2}} z^{\kappa-j}, \]
from which it follows that the set (A.10) is linearly independent. \(\square\)

The proof of Corollary [A.2] has the following immediate consequence.

Corollary A.3. Suppose that $s > 0$. All $\mathbb{C}$-linear combinations of the functions $\frac{\partial^{j}}{\partial \mu^{j}} W_{\kappa,\mu}(y)$ decay exponentially as $y \to \infty$, while all nonzero $\mathbb{C}$-linear combinations of the functions $\frac{\partial^{j}}{\partial \mu^{j}} W_{-\kappa,\mu}(-y)$ grow exponentially as $y \to \infty$.

APPENDIX B. ACTION OF $\xi_k$-OPERATOR ON NON-HOLOMORPHIC EISENSTEIN SERIES

This appendix sketches an alternate proof of Proposition [5.4] which works directly for all $s \in \mathbb{C}$.

Proposition B.1. Let $k \in 2\mathbb{Z}$. Then
\[ \xi_k \mathcal{E}_k(z, s) = \left\{ \begin{array}{ll} \mathcal{E}_{2-k}(z, -\overline{s}) & \text{if } k \leq 0, \\ \overline{s}(\overline{s} + k - 1) \mathcal{E}_{2-k}(z, -\overline{s}) & \text{if } k \geq 2. \end{array} \right. \]

Proof. We compute the action of $\xi_k$ on the Fourier series of $E_k(z, s)$ term by term, using the formulas in Lemma [5.3]. We assert that under the action of $\xi_k$ the $n$-th the Fourier term maps to the $-n$-th term of $\mathcal{E}_{2-k}(z, -\overline{s})$, multiplied by the appropriate constant (1 or $\overline{s}(\overline{s} + k - 1)$). Futhermore, for the constant term the coefficients of $y^s$ and $y^{1-s-k}$ are interchanged, again multiplied by the appropriate constant. Here we supply details proving the assertion for one specific case, sufficient to uniquely determine the multiplying constants. We write
\[ C_0(y, s) = C T_k^+(s) y^s + C T_k^-(s) y^{1-s-k} \]
for the constant term of $\hat{E}_k(z, s)$. By Proposition 2.5 we have

$$CT_k^+(s) = \frac{\Gamma\left(s + \frac{k}{2} + \frac{|k|}{2}\right)}{\Gamma\left(s + \frac{k}{2}\right)} \zeta(2s + k)$$

and

$$CT_k^-(s) = (-1)^{\frac{k}{2}} \frac{\Gamma\left(s + \frac{k}{2}\right) \Gamma\left(s + \frac{k}{2} + \frac{|k|}{2}\right)}{\Gamma(s + k)\Gamma(s)} \zeta(2 - 2s - k).$$

We will show that

$$\xi_k\left(CT_k^+(s)y^s\right) = \begin{cases} CT_{2-k}^-(\bar{s})y^{\bar{s}+k-1} & \text{if } k \leq 0, \\ \bar{s}(\bar{s} + k - 1)CT_{2-k}^-(\bar{s})y^{\bar{s}+k-1} & \text{if } k \geq 2. \end{cases}$$

The proof depends on the value of $k$ and uses identities for the Gamma function. First, suppose that $k \leq 0$. We compute that

$$\xi_k\left(CT_k^+(s)y^s\right) = y^k CT_k^+(s) \frac{\partial}{\partial y} y^s = \frac{s\Gamma(s)}{\Gamma\left(s + \frac{k}{2}\right)} \bar{s}(\bar{s} + k) y^{\bar{s}+k-1}.$$ 

On the other hand,

$$CT_{2-k}^-(\bar{s})y^{\bar{s}+k-1} = (-1)^{1-\frac{k}{2}} \frac{\Gamma(1 - \bar{s} - \frac{k}{2})}{\Gamma(-\bar{s})} \zeta(2\bar{s} + k)y^{\bar{s}+k-1}.$$ 

It remains to show (replacing $\bar{s}$ by $s$) that

$$\frac{s\Gamma(s)}{\Gamma(s + \frac{k}{2})} = (-1)^{1-\frac{k}{2}} \frac{\Gamma(1 - s - \frac{k}{2})}{\Gamma(-s)}.$$ 

Indeed, using $(-z)\Gamma(-z) = \Gamma(1 - z)$ and $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$ we find that

$$\frac{s\Gamma(-s)}{\Gamma\left(s + \frac{k}{2}\right)\Gamma(1 - s - \frac{k}{2})} = -\frac{\Gamma(s)\Gamma(1 - s)}{\Gamma\left(s + \frac{k}{2}\right)\Gamma(1 - s - \frac{k}{2})} = -\frac{\sin \pi(s + \frac{k}{2})}{\sin \pi s} = (-1)^{1-\frac{k}{2}},$$

as desired.

Now suppose that $k \geq 2$. We compute

$$\xi_k\left(CT_k^+(s)y^s\right) = \frac{s\Gamma(\bar{s} + k)}{\Gamma(\bar{s} + \frac{k}{2})} \zeta(2\bar{s} + k)y^{\bar{s}+k-1}.$$ 

On the other hand,

$$CT_{2-k}^-(\bar{s})y^{\bar{s}+k-1} = (-1)^{1-\frac{k}{2}} \frac{\Gamma(1 - \bar{s} - \frac{k}{2})}{\Gamma(2 - \bar{s} - k)} \zeta(2\bar{s} + k)y^{\bar{s}+k-1}.$$ 

It remains to show that

$$\frac{\Gamma(s + k)}{\Gamma(s + \frac{k}{2})} = (-1)^{1-\frac{k}{2}} (s + k - 1) \frac{\Gamma(1 - s - \frac{k}{2})}{\Gamma(2 - s - k)}.$$ 

Indeed,

$$\frac{\Gamma(s + k)\Gamma(2 - s - k)}{(s + k - 1)\Gamma(s + \frac{k}{2})\Gamma(1 - s - \frac{k}{2})} = -\frac{\Gamma(s + k)\Gamma(1 - s - k)}{\Gamma(s + \frac{k}{2})\Gamma(1 - s - \frac{k}{2})} = -\frac{\sin \pi(s + \frac{k}{2})}{\sin \pi(s + k)} = (-1)^{1-\frac{k}{2}},$$

as desired. This completes the proof of (B.1).
We omit the details of similar assertions for $\xi_k(CT^-(s)y^{1-k-s})$ and for all other Fourier terms of index $n \neq 0$. For the cases $n \neq 0$ various Whittaker function identities are required.

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