Logarithm-free $A$-hypergeometric series

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Abstract

We give a dimension formula for the space of logarithm-free series solutions to an $A$-hypergeometric (or a GKZ hypergeometric) system. In the case where the convex hull spanned by $A$ is a simplex, we give a rank formula for the system, characterize the exceptional set, and prove the equivalence of the Cohen-Macaulayness of the toric variety defined by $A$ with the emptiness of the exceptional set. Furthermore we classify $A$-hypergeometric systems as analytic $\mathcal{D}$-modules.

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1 Introduction

Given a finite set $A$ of integral vectors on a hyperplane off the origin and given a parameter vector, Gel’fand, Kapranov and Zelevinskii defined a system of differential equations, called an $A$-hypergeometric (or a GKZ hypergeometric) system. The rank of an $A$-hypergeometric system is greater than or equal to the volume of the convex hull $\text{conv}(A)$ spanned by $A$ (Theorem 3.5.1 in [4]). The set of parameters where the rank is strictly greater than the volume is called the exceptional set. In this paper, we give a dimension formula for the space of logarithm-free series solutions to an $A$-hypergeometric system. Furthermore, in the case where $\text{conv}(A)$ is a simplex, we give a rank formula, characterize the exceptional set, and prove the equivalence of the
Cohen-Macaulayness of the toric variety defined by $A$ with the emptiness of the exceptional set.

In the paper [6], we defined a finite set $E_{\tau}(\beta)$ associated to a parameter $\beta$ and a face $\tau$ of the cone generated by $A$, to classify parameters according to isomorphism classes of their corresponding algebraic $A$-hypergeometric systems. In this paper, we define a finite set $E_{\tau}(\beta)$ in the same way as in [6], associated to a parameter $\beta$ and a face $\tau$ of the regular triangulation $\Delta_w$ of $\text{conv}(A)$ determined by a generic weight vector $w$, to investigate logarithm-free $A$-hypergeometric series.

Roughly speaking, each element of the set $E_{\tau}(\beta)$ gives volume-of-$\tau$-many linearly independent logarithm-free series solutions converging in the direction of $w$ (Theorems 3.1 and 7.2). Using this, we prove that if $E_{\tau}(\beta) \neq E_{\tau}(\beta')$ for some face $\tau$, then the analytic $A$-hypergeometric systems with parameter $\beta$ and with parameter $\beta'$ are not isomorphic (Theorem 7.3). By counting canonical series solutions associated with $\tau$ but not with smaller faces, we obtain a dimension formula for the space of logarithm-free series solutions converging in the direction of $w$ (Theorem 4.6).

When the convex hull $\text{conv}(A)$ is a simplex, we take a generic weight vector $w$ such that $\Delta_w$ coincides with $\text{conv}(A)$ itself. Then the faces of the cone generated by $A$ and those of $\Delta_w$ have a natural one-to-one correspondence. In this situation, we prove that all canonical series solutions with respect to $w$ are logarithm-free (Propositions 6.1 and 6.2). Hence the formula in Theorem 6.3 is also a rank formula for $A$-hypergeometric systems (Theorem 6.3). From the rank formula we characterize the exceptional set (Theorem 6.7), and then we prove the equivalence of the Cohen-Macaulayness of the affine toric variety defined by $A$ with the emptiness of the exceptional set (Theorem 6.7).

Matushevich [5] has recently proved the equivalence in the codimension 2 case, and has given examples in which the exceptional sets are infinite.

2 Canonical $A$-hypergeometric series

In this section, we recall logarithm-free canonical $A$-hypergeometric series. For details, see [7].

Let $A = (a_1, \ldots, a_n) = (a_{ij})$ be a $d \times n$-matrix of rank $d$ with coefficients in $\mathbb{Z}$. Throughout this paper, we assume that all $a_j$ belong to one hyperplane off the origin in $Q^d$. Let $N$ be the set of nonnegative integers, and $k$ a field.
of characteristic zero. Let $I_A$ denote the toric ideal in the polynomial ring $k[\partial] = k[\partial_1, \ldots, \partial_n]$, i.e.,

$$I_A = \langle \partial^u - \partial^v : Au = Av, u, v \in \mathbb{N}^n \rangle \subset k[\partial].$$

Here and hereafter we use the multi-index notation; for example, $\partial^u$ means $\partial_1^{u_1} \cdots \partial_n^{u_n}$ for $u = (u_1, \ldots, u_n)$. Given a column vector $\beta = (\beta_1, \ldots, \beta_d) \in k^d$, let $H_A(\beta)$ denote the left ideal of the Weyl algebra $D = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ generated by $I_A$ and $\sum_{j=1}^n a_{ij} \theta_j - \beta_i$ ($i = 1, \ldots, d$), where $\theta_j = x_j \partial_j$. The quotient $M_A(\beta) = D/H_A(\beta)$ is called the $A$-hypergeometric system with parameter $\beta$, and a formal series annihilated by $H_A(\beta)$ an $A$-hypergeometric series with parameter $\beta$.

Fix a generic weight vector $w \in \mathbb{R}^n$. The ideal of the polynomial ring $k[\theta] = k[\theta_1, \ldots, \theta_n]$ defined by

$$\widetilde{\text{fin}}_w(H_A(\beta)) := D \cdot \text{fin}_w(I_A) \cap k[\theta] + \langle A\theta - \beta \rangle$$

is called the fake indicial ideal, where $\text{fin}_w(I_A)$ denotes the initial ideal of $I_A$ with respect to $w$, and $\langle A\theta - \beta \rangle$ denotes the ideal generated by $\sum_{j=1}^n a_{ij} \theta_j - \beta_i$ ($i = 1, \ldots, d$). Each zero of $\widetilde{\text{fin}}_w(H_A(\beta))$ is called a fake exponent.

Let $w \cdot u$ denote $w_1 u_1 + \cdots + w_n u_n$ for $u \in \mathbb{Q}^n$. Let

$$L = \{ u \in \mathbb{Z}^n : Au = 0 \}.$$  (2)

An $A$-hypergeometric series

$$x^v \cdot \sum_{u \in L} g_u (\log x) x^u \quad (g_u \in k[x])$$

is said to be in the direction of $w$ if there exists a basis $u^{(1)}, \ldots, u^{(n)}$ of $\mathbb{Q}^n$ with $w \cdot u^{(j)} > 0$ ($j = 1, \ldots, n$) such that $g_u = 0$ whenever $u \notin \sum_{j=1}^n \mathbb{Q}_{\geq 0} u^{(j)}$. A fake exponent $v$ is called an exponent if there exists an $A$-hypergeometric series $\mathbb{3}$ in the direction of $w$ with nonzero $g_0$. Let $\prec$ be the lexicographic order on $\mathbb{N}^n$. Suppose that $u^{(1)}, \ldots, u^{(n)}$ is a basis as above. Then a monomial like $x^v \cdot \text{fin}_<(g_0)(\log x)$ in the $A$-hypergeometric series

$$x^v \cdot \sum_{u \in L \cap \sum_{j=1}^n \mathbb{Q}_{\geq 0} u^{(j)}} g_u (\log x) x^u \quad (g_u \in k[x])$$

(4)
with nonzero \( g_0 \) is called a starting monomial. The \( A \)-hypergeometric series (4) is said to be canonical with respect to \( w \) if no starting monomials other than \( x^v \cdot \ln_{<}(g_0)(\log x) \) appear in the series.

Next we recall logarithm-free \( A \)-hypergeometric series \( \phi_v \). For \( v \in k^n \), its negative support \( n\mathrm{supp}(v) \) is the set of indices \( i \) with \( v_i \in \mathbb{Z}_{<0} \). When \( n\mathrm{supp}(v) \) is minimal with respect to inclusions among \( n\mathrm{supp}(v + u) \) with \( u \in L \), \( v \) is said to have minimal negative support. For \( v \) satisfying \( Av = \beta \) with minimal negative support, we define a formal series

\[
\phi_v = x^v \psi_v = x^v \cdot \sum_{u \in N_v} \frac{[v]_{u-}}{[v + u]_{u+}} x^u. \tag{5}
\]

Here \( N_v = \{ u \in L : n\mathrm{supp}(v) = n\mathrm{supp}(v + u) \} \), \( u_+, u_- \in N^n \) satisfy \( u = u_+ - u_- \) with disjoint supports, and \( [v]_t = \prod_{j=1}^n v_j (v_j - 1) \cdots (v_j - t_j + 1) \) for \( t \in N^n \). Proposition 3.4.13 and Theorem 3.4.14 in [4] respectively state that the series \( \phi_v \) is \( A \)-hypergeometric, and that if \( v \) is a fake exponent of \( M_A(\beta) \), then \( \phi_v \) is canonical, and \( v \) is an exponent. Let \( \mathrm{Minex}_{\beta,w} \) denote the set of fake exponents with minimal negative support, of \( M_A(\beta) \) with respect to \( w \). Then \( \mathrm{Minex}_{\beta,w} \) is a subset of the set of exponents; moreover by Corollary 3.4.15 in [4], under the correspondence of \( v \) with \( \phi_v \), the set \( \mathrm{Minex}_{\beta,w} \) corresponds to the set of logarithm-free canonical series solutions to \( M_A(\beta) \) with respect to \( w \). Let \( S_{\beta,w} \) denote the space spanned by logarithm-free \( A \)-hypergeometric series in the direction of \( w \). Then the set of logarithm-free canonical \( A \)-hypergeometric series with respect to \( w \) is a basis of \( S_{\beta,w} \). Hence we have

\[
\dim S_{\beta,w} = |\mathrm{Minex}_{\beta,w}|. \tag{7}
\]

We shall compute \( |\mathrm{Minex}_{\beta,w}| \) in the following sections.

### 3 Finite sets \( E_\tau(\beta) \) and logarithm-free canonical \( A \)-hypergeometric series

We denote the set \( \{ a_1, \ldots, a_n \} \) of the column vectors of the matrix \( A \) by \( A \) as well. Let \( \mathrm{conv}(A) \) denote the convex hull of the set \( A \), and \( \Delta_w \) the regular triangulation of \( \mathrm{conv}(A) \) determined by \( w \). In this section, we define
a finite set $E_r(\beta)$ associated to a parameter $\beta$ and a face $\tau$ of $\Delta_w$, and prove that each element of $E_r(\beta)$ gives volume-of-$\tau$-many logarithm-free canonical series solutions with respect to $w$.

Let $NA$ denote the monoid generated by $A$. Given a face $\tau$ of $\Delta_w$, $\mathbb{Z}(A \cap \tau)$ and $\mathbb{k}(A \cap \tau)$ respectively denote the additive group and the vector space generated by $A \cap \tau$. Here we agree that $\mathbb{k}(A \cap \tau) = \mathbb{Z}(A \cap \tau) = \{ 0 \}$ when $\tau = \emptyset$. Associated to a parameter $\beta \in \mathbb{k}^d$ and a face $\tau \in \Delta_w$, we define a finite set $E_r(\beta)$ in the same way as in [3]:

$$E_r(\beta) := \{ \lambda \in \mathbb{k}(A \cap \tau)/\mathbb{Z}(A \cap \tau) : \beta - \lambda \in NA + \mathbb{Z}(A \cap \tau) \}. \quad (8)$$

An algorithm for $E_r(\beta)$ is given in [3]. For fundamental properties of $E_r(\beta)$ such as the finiteness, see [8].

Let vert$(\tau)$ denote the set of vertices of a face $\tau$. For a face $\tau$ of $\Delta_w$ or for a subset $\tau$ of $A$, we abuse the notation to denote the set $\{ j : a_j \in \tau \}$ by $\tau$ again; for example, the symbol vert$(\tau)$ could mean the set $\{ j : a_j \text{ is a vertex of } \tau \}$. For a face $\tau$ of $\Delta_w$, let $\text{vol}(\tau)$ denote the index $[\mathbb{Z}(A \cap \tau) : \sum_{i \in \text{vert}(\tau)} \mathbb{Z}a_i]$.

**Theorem 3.1** Let $w$ be a generic weight vector. Given a face $\tau \in \Delta_w$, there exist at least $|E_r(\beta)|\text{vol}(\tau)$-many logarithm-free canonical series solutions $\phi_v$ to $M_A(\beta)$ with respect to $w$, satisfying $\sum_{j \in \tau} v_j a_j \in E_r(\beta)$, $\text{nsupp}(v) \subset \text{vert}(\tau)$, and $\{ j : v_j \notin \mathbb{Z} \} \subset \text{vert}(\tau)$.

**Proof.** Put $I := \text{vert}(\tau)$. Given $\lambda \in E_r(\beta)$, there exists $v \in \mathbb{k}^n$ such that $\beta = Av$, $\lambda = \sum_{j \in \tau} v_j a_j$, $\text{nsupp}(v) \subset \tau$, and $\{ j : v_j \notin \mathbb{Z} \} \subset \tau$. Since $I$ is a base of $\tau$, we may further assume $\text{nsupp}(v) \subset I$ and $\{ j : v_j \notin \mathbb{Z} \} \subset I$. There are exactly $\text{vol}(\tau)$-many such vectors $v$ modulo $L$ (see (2) for the definition of $L$), which we prove below as Lemma 3.2. We can take the representatives $v$ such that they have minimal negative supports. Then the series $\phi_v$ (see (3) for the definition of $\phi_v$) are formal solutions, and

$$N_v = \{ u \in L : u_j < -v_j \ (j \in \text{nsupp}(v)), \ u_j \geq -v_j \ (j \in \text{psupp}(v)) \}.$$ 

Here, similarly to the negative support, $\text{psupp}(v)$ denotes the set of indices $i$ with $v_i \in \mathbb{N}$.

Let $L_\mathbb{Q}$ be the $\mathbb{Q}$-vector space generated by $L$, and consider a polyhedron

$$N_{v, \mathbb{Q}} := \{ u \in L_\mathbb{Q} : u_j \leq -v_j \ (j \in \text{nsupp}(v)), \ u_j \geq -v_j \ (j \in \text{psupp}(v)) \}.$$
Since the vectors \( a_i \ (i \in I) \) are linearly independent, the characteristic cone of \( N_{v,Q} \)

\[
K_{v,I} = \{ u \in L_Q : u_j \leq 0 \ (j \in \text{nsupp}(v)), \ u_j \geq 0 \ (j \in \text{psupp}(v)) \}
\]

is pointed, or strongly convex. Hence the polyhedron \( N_{v,Q} \) has vertices and it is decomposed as \( N_{v,Q} = P + K_{v,I} \), where \( P \) is the convex hull of the set of vertices of \( N_{v,Q} \) (for example, see [9]). Note that the characteristic cone \( K_{v,I} \) of \( N_{v,Q} \) for any such \( v \) is contained in the cone \( K_I := \{ u \in L_Q : u_j \geq 0 \ (j \notin I) \} \).

\[
K_I \subset K_{\tilde{I}} = \{ u \in L_Q : u_j \geq 0 \ (j \notin \tilde{I}) \} \quad \text{where} \quad c_{ji} \text{ are given by} \quad a_j = \sum_{i \in \tilde{I}} c_{ji} a_i, \quad \text{and} \quad e_1, \ldots, e_n \text{ is the standard basis for} \ Z^n.
\]

Since \( \tilde{\tau} \in \Delta_w \) means that \( w \cdot u > 0 \) for all nonzero \( u \in K_{\tilde{I}} \), and since \( w \) is generic, there exists a unique optimal solution \( u \) for the integer programming problem:

**Minimize** \( w \cdot u \) subject to \( u \in N_v \).

Replacing \( v \) by \( v + u \), we may assume 0 is the unique optimal solution. Then we claim that \( v \) is a fake exponent, i.e., \( x^v \) is annihilated by \( \sum_{j=1}^n a_j \theta_j - \beta_i \ (i = 1, \ldots, d) \) and \( \text{in}_w(I_A) \). Since \( Av = \beta \), we only need to verify that \( x^v \) is annihilated by \( \text{in}_w(I_A) \). Let \( Au_+ = Au_- \) and \( w \cdot u_+ > w \cdot u_- \). Among all the terms appearing in \( \partial^{u^+}(\phi_v) - \partial^{u^-}(\phi_v) \), \( \partial^{u^+}(x^v) = [v]u_+ x^{v-u_+} \) is the unique term with cost \( w \cdot (v - u_+) \). Since \( \phi_v \) is a formal solution, it implies \( \partial^{u^+}(x^v) = 0 \). We have thus proved that \( v \) is a fake exponent. It is furthermore an exponent, and \( \phi_v \) is canonical by Theorem 3.4.14 in [7].

For a subset \( I \) of \( \{1, \ldots, n\} \), let \( I^c \) denote the complement of \( I \), i.e., \( I^c = \{1, \ldots, n\} \setminus I \). Given \( \lambda \in E_{v}(\beta) \), put

\[
\mathcal{E}xp_w(\tau, \lambda) := \{ v \in k^n : \beta = Av, \ \lambda = \sum_{j \in \tau} v_j a_j, \ v_j \in N \text{ for all } j \in \text{vert}(\tau)^c \}.
\]

We denote by \( \mathcal{E}xp_w(\tau, \lambda)/L \) the image of \( \mathcal{E}xp_w(\tau, \lambda) \) in \( \{ v \in k^n : \beta = Av \}/L \).
Lemma 3.2 We keep the notation in the proof of Theorem 3.1. Let $\lambda \in E_{\tau}(\beta)$. Then

$$Z(A \cap \tau)/\sum_{i \in I} Z a_i \simeq \{ u \in L_\tau : u_i \in Z \text{ for all } i \in \tau \setminus I \}/L_\tau$$

(11)

$$\simeq \{ u \in L_{\tau,Q} : u_i \in Z \text{ for all } i \in \tau \setminus I \}/L_\tau$$

(12)

$$\simeq Exp_w(\tau, \lambda)/L,$$

(13)

where $L_\tau := \{ u \in L : u_j = 0 \text{ for all } j \in \tau^c \}$, and $L_{\tau,Q}$ is the $Q$-vector space generated by $L_\tau$.

Proof. As we saw in the proof of Theorem 3.1, the set (13) is not empty.

First we prove the bijectivity between the sets (11) and (12). For $u \in L_{\tau,Q}$ with $u_i \in Z$ ($i \in \tau \setminus I$), we define $g(u) := \sum_{i \in \tau \setminus I} u_i a_i \in Z(A \cap \tau)$. For the surjectivity, suppose $u \in Z$ for all $i \in \tau \setminus I$. Since $I$ is a base of $\tau$, we can find $b_i \in Q$ ($i \in I$) such that $\sum_{i \in \tau \setminus I} u_i a_i = \sum_{i \in I} b_i a_i$, where $\bar{u}_i = u_i$ for $i \in \tau \setminus I$ and $\bar{u}_i = -b_i$ for $i \in I$. For the injectivity of $g$, suppose $g(u) = g(v)$. Then $\sum_{i \in I} u_i a_i - \sum_{i \in I} v_i a_i = \sum_{i \in \tau \setminus I} v_i a_i - \sum_{i \in \tau \setminus I} u_i a_i \in \sum_{i \in I} Z a_i$. Since $a_i$ ($i \in I$) are linearly independent, $u_i - v_i$ is an integer for all $i \in I$. Hence $u - v \in L$.

Next let $u$ belong to the set (12). Clearly we may assume that $u_i \in N$ for all $i \in \tau \setminus I$. Suppose that $v$ is an element of the set (13). Then $v + u$ is again an element of the set (12). Thus the set (12) can be embedded into the set (13).

Next we show that the set (13) can be embedded into the set (11). Suppose that $v$ and $v'$ belong to the set (13). Then $\sum_{i \in I} (v_i - v'_i) a_i \in Z(A \cap \tau)$. If it belongs to $\sum_{i \in I} Z a_i$, then $v_i - v'_i \in Z$ for all $i \in I$. Hence $v - v' \in L$.

Finally we remark that the set (11) is finite, to complete the proof. [ ]

4 Dimension formula for the vector space of logarithm-free $A$-hypergeometric series

In this section, we give a dimension formula for the vector space $S_{\beta,w}$. Recall that the set $\text{Minex}_{\beta,w}$ corresponds to the set of logarithm-free canonical series solutions to $M_A(\beta)$ with respect to $w$, and hence $\dim S_{\beta,w} = |\text{Minex}_{\beta,w}|$.

We denote by $\text{vert}(\Delta_w)$ the set of vertices of $\Delta_w$.

Lemma 4.1 If $v$ is a fake exponent, then $v_j \in N$ for $j \notin \text{vert}(\Delta_w)$. 


Proof. If $a_j \notin \text{vert}(\Delta_w)$, then there exist nontrivial $m_j, m_{ji} \in \mathbb{N}$ such that $m_j a_j = \sum_{i \in \text{vert}(\Delta_w)} m_{ji} a_i$. Hence we have $\partial^m_j \in \text{in}_w(I_A)$ for $j \notin \text{vert}(\Delta_w)$. Since $v$ is a fake exponent, we see $v_j \in \mathbb{N}$ for $j \notin \text{vert}(\Delta_w)$. 

Suppose $v \in \text{Minex}_{\beta,w}$. Define a set $I_v$ by

$$I_v := \{ j : v_j \notin \mathbb{N} \}. \tag{14}$$

By Lemma 4.1, the set $I_v$ is a subset of $\text{vert}(\Delta_w)$. Since $v$ has minimal negative support, $I_v$ is linearly independent, i.e., $a_j (j \in I_v)$ are linearly independent.

Lemma 4.2 Let $v \in \text{Minex}_{\beta,w}$, $I_v := \{ j : v_j \notin \mathbb{N} \}$, and $\tau_v := \text{conv}\{a_j : j \in I_v\}$. Then $\tau_v$ is a face of $\Delta_w$, $I_v$ is the vertex set of $\tau_v$, and $\sum_{a_i \in \tau_v} v_i a_i$ represents an element of $E_{\tau_v}(\beta)$.

Proof. We prove only the statement $\tau_v \in \Delta_w$. The others follow easily. Suppose $\tau_v \notin \Delta_w$. Then there exist a subset $I \subset I_v$, integers $m_i \in \mathbb{N}$ ($i \in I$) and $n_j \in \mathbb{N}$ ($j \notin I$) with the properties $\sum_{i \in I} m_i a_i = \sum_{j \notin I} n_j a_j$ and $\sum_{i \in I} m_i w_i > \sum_{j \notin I} n_j w_j$. Since $v$ is a fake exponent, $\prod_{i \in I} \partial^m_i \in \text{in}_w(I_A)$ implies $v_i \in \mathbb{N}$ for some $i \in I$. This contradicts the definition of $I_v$. 

Lemma 4.2 shows that $v \in \text{Minex}_{\beta,w}$ implies $v \in E_{x_p}(\tau_v, \lambda_v)$, where

$$\tau_v := \text{conv}\{a_j : j \in I_v\} \quad \text{and} \quad \lambda_v := \sum_{a_i \in \tau_v} v_i a_i \in E_{\tau_v}(\beta). \tag{15}$$

Define a map $\Lambda$ from $\text{Minex}_{\beta,w}$ to the set $\prod_{\tau \in \Delta_w, \lambda \in E_{\tau}(\beta)} E_{x_p}(\tau, \lambda)/L$ by

$$\Lambda(v) = [v] \in E_{x_p}(\tau_v, \lambda_v)/L, \tag{16}$$

where $[v]$ denotes the image of $v$ in $\{ v \in k^n : \beta = Av \}/L$.

Lemma 4.3 Let $v, v' \in \text{Minex}_{\beta,w}$. Suppose $\tau_v = \tau_{v'}$ and $v + L = v' + L$. Then $v = v'$.

Proof. Since $I_v$ (resp. $I_{v'}$) is the vertex set of $\tau_v$ (resp. $\tau_{v'}$), we have $I_v = I_{v'}$, which implies $\text{psupp}(v) = \text{psupp}(v')$. Then by the equality $v + L = v' + L$, we have $\text{nsupp}(v) = \text{nsupp}(v')$. If $v \neq v'$, then without loss of generality, we may assume $w \cdot (v - v') < 0$. Since $\phi_v$ is canonical by Theorem 3.4.14 in [], the term $x^{v'}$ does not appear in $\phi_v$, which contradicts the definition of $\phi_v$. 

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The following is immediate from Lemma 4.3.

**Corollary 4.4** The map $\Lambda$ is injective, and

$$|\text{Minex}_{\beta,w}| = \sum_{\tau \in \Delta_w, \lambda \in E_\tau(\beta)} |\Lambda(\Lambda^{-1}(\mathcal{E}_{\tau,\lambda}\setminus L))|.$$ (17)

We introduce a partial order $\preceq$ in the set $\{ (\tau, \lambda) : \tau \in \Delta_w, \lambda \in E_\tau(\beta) \}$. Note that there exists a natural map from $E_{\tau'}(\beta)$ to $E_\tau(\beta)$ if $\tau' \preceq \tau$, i.e., if $\tau'$ is a face of $\tau$. We define a partial order $\preceq$ by $(\tau', \lambda') \preceq (\tau, \lambda)$ if $\tau' \preceq \tau$, and the image of $\lambda'$ under the natural map from $E_{\tau'}(\beta)$ to $E_\tau(\beta)$ coincides with $\lambda$. Note that $\mathcal{E}_{\tau,\lambda}/\lambda$ is a subset of $\mathcal{E}_{\tau,\lambda}$ when $(\tau', \lambda') \preceq (\tau, \lambda)$.

**Lemma 4.5**

$$\Lambda(\Lambda^{-1}(\mathcal{E}_{\tau,\lambda}\setminus L)) = \mathcal{E}_{\tau,\lambda}\setminus \bigcup_{(\tau', \lambda') \preceq (\tau, \lambda)} \mathcal{E}_{\tau,\lambda}/\lambda.$$ (18)

**Proof.** The definition of $\Lambda$ implies that the LHS is contained in the RHS. By the proof of Theorem 4.4, there exists $v \in \text{Minex}_{\beta,w} \cap \mathcal{E}_{\tau,\lambda}$ such that $c = [v]$ in $\{ v : \beta = Av \}/L$. Since $c$ does not belong to $\bigcup_{(\tau', \lambda') \preceq (\tau, \lambda)} \mathcal{E}_{\tau,\lambda}/\lambda$, neither does $[v]$, which means $\Lambda(v) \in \mathcal{E}_{\tau,\lambda}$, and thus $\Lambda(v) = c$. 

For a face $\tau$ of $\Delta_w$, let $\text{facet}(\tau)$ denote the set of one-codimensional faces of $\tau$. The following is the main theorem in this section.

**Theorem 4.6**

1. Let $\tau \in \Delta_w$ and $\lambda \in E_\tau(\beta)$. Then

$$|\Lambda(\Lambda^{-1}(\mathcal{E}_{\tau,\lambda}\setminus L))| = \text{vol}(\tau) - \sum_{i} \sum_{\tau_1, \ldots, \tau_i \in \text{facet}(\tau) \atop E_{\tau_i}(\beta) \ni \lambda} (-1)^{i-1} \text{vol}(\tau_1 \cap \cdots \cap \tau_i).$$ (19)

2. \[
\dim S_{\beta,w} = \sum_{\tau \in \Delta_w} \left( \sum_{\lambda \in E_\tau(\beta)} \text{vol}(\tau) \right. \\
- \left. \sum_{i} \sum_{\tau_1, \ldots, \tau_i \in \text{facet}(\tau) \atop E_{\tau_i}(\beta) \ni \lambda} (-1)^{i-1} \text{vol}(\tau_1 \cap \cdots \cap \tau_i) \right). \] (20)
Proof. It is enough to prove the first statement by Corollary 4.4. The combination of Lemma 3.2, Corollary 4.4, and Lemma 4.5 implies that the number \(|\Lambda^{-1}(Ex_pw(\tau, \lambda)/L)|\) equals the number of the equivalence classes of \(\mathbb{Z}(A \cap \tau)/\sum_{i \in \text{vert}(\tau)} \mathbb{Z}a_i\) which cannot be represented by an element of those for \((\tau', \lambda') \prec (\tau, \lambda)\). An equivalence class of \(\mathbb{Z}(A \cap \tau)/\sum_{i \in \text{vert}(\tau)} \mathbb{Z}a_i\) which can be represented by an element of \(\mathbb{Z}(A \cap \tau')\) for \((\tau', \lambda') \prec (\tau, \lambda)\), is actually represented by one for \(\tau'\) of codimension one to \(\tau\). The number of such equivalence classes for \(\tau'\) is \(\text{vol}(\tau')\). Here we remark that \(\mathbb{Z}(A \cap \tau') \cap \mathbb{Z}(\text{vert}(\tau)) = \mathbb{Z}(\text{vert}(\tau'))\) since \(\text{vert}(\tau)\) is linearly independent. An equivalence class which can be represented by equivalence classes for \(\tau_1, \ldots, \tau_i\) comes from an equivalence class for \(\tau_1 \cap \cdots \cap \tau_i\). Hence the first statement holds. \(\square\)

Example 4.7 Let

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix},
\]

and \(\beta = a_1 = ^t(1, 1)\). Then there are two regular triangulations \(\Delta_w\) and \(\Delta_w'\) of \(\text{conv}(A)\) (see Figure 1), and for all \(\tau\) of either triangulation, \(E_{\tau}(\beta) = \{0\}\).

![Figure 1: \(\Delta_w\) and \(\Delta_w'\)](image)

By Theorem 4.4, we obtain

\[
\dim S_{\beta,w} = (2 - 2 \times 1 + 1) + 2(1 - 1) + 1 = 1 + 0 + 0 + 1 = 2,
\]

(21)

and

\[
\dim S_{\beta,w'} = 2(1 - 2 \times 1 + 1) + 3(1 - 1) + 1 = 0 + 0 + 0 + 1 = 1.
\]

(22)

In both cases, the polynomial solution \(\phi_{(0,1,0)} = x_2\) is associated with face \(\emptyset\). In the case \(\Delta_w\), there exists a logarithm-free \(A\)-hypergeometric series \(\phi_{(1/2,0,1/2)}\) associated with face \(\text{conv}(A)\).
A generalization of the case $\Delta_w$ in Example 4.7 will be considered in Section 6. The case $\Delta^\prime_w$ can be generalized as the following corollary. A triangulation $\Delta_w$ is said to be unimodular if $a_j (j \in \text{vert}(\tau))$ is a basis of $ZA$ for all $d$-dimensional faces $\tau \in \Delta_w$.

**Corollary 4.8** Suppose that $\beta \in NA$, and $\Delta_w$ is unimodular. Then $S_{\beta,w}$ is one-dimensional, and coincides with the space of polynomial solutions.

**Proof.** In this case, $\text{vol}(\tau) = 1$ and $E_\tau(\beta) = \{0\}$ for all faces $\tau \in \Delta_w$. For a face $\tau \neq \emptyset$, we have

$$\text{vol}(\tau) - \sum_i \sum_{\tau_1, \ldots, \tau_i \in \text{facet}(\tau)} (-1)^{i-1}\text{vol}(\tau_1 \cap \cdots \cap \tau_i) = 0.$$ 

The face $\emptyset$ gives one canonical series. It is a polynomial. 

## 5 Logarithm coefficients of $A$-hypergeometric series

In the next section, we shall prove that there exists a fundamental system of solutions to $M_A(\beta)$ consisting of logarithm-free canonical series with respect to a suitable weight $w$ in the case where $\text{conv}(A)$ is a simplex. In order to prove this, we consider logarithm coefficients of $A$-hypergeometric series in this section.

We denote by $k[L]$ the symmetric algebra of the vector space spanned by $\sum_{j=1}^n c_j x_j$ ($c \in L$), which is a subring of the polynomial ring $k[x] = k[x_1, \ldots, x_n]$.

**Lemma 5.1** Let $f \in k[x]$. Then $\sum_{j=1}^n a_{ij} \partial_j(f) = 0$ for all $i = 1, \ldots, d$ if and only if $f \in k[L]$.

**Proof.** We prove the only-if-direction. The if-direction is straightforward. Without loss of generality, we may assume that the vectors $a_1, \ldots, a_d$ are linearly independent. Then $k[x] = k[L][x_1, \ldots, x_d]$. Let $f = \sum_v g_v x^v$ with $g_v \in k[L]$ and $v \in \mathbb{N}^d$. Suppose that there exists $u \neq 0$ with $g_u \neq 0$. Without loss of generality we may assume that $u_1 \neq 0$. Then by the equality $\sum_{j=1}^n a_{ij} \partial_j(f) = 0$ we have $\sum_{j=1}^d a_{ij}(u_j + 1 - \delta_{ij})g_{u-e_i+e_j} = 0$ for all $i$, where $e_1, \ldots, e_d$ is the standard basis for $Z^d$. This contradicts the linear independence of the vectors $a_1, \ldots, a_d$. 

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Proposition 5.2 Suppose that

\[ \phi = \sum_u g_u (\log(x))x^u \]  \hspace{1cm} (23)

is an $A$-hypergeometric series. Then $g_u \in k[L]$ for all $u$.

**Proof.** Suppose that $\phi$ is a solution to $M_A(\beta)$. Then $(\sum_{j=1}^n a_{ij}\theta_j - \beta_i)(g_u(\log(x))x^u) = 0$, which leads to $Au = \beta$ and $\sum_{j=1}^n a_{ij}\partial_j(g_u) = 0$. Lemma 5.1 finishes the proof. \[ \]

Let $v \in k$ and $n \in \mathbb{N}$. For $j \in \mathbb{N}$ with $j \leq n$, put

\[ c_j(v, n) := \sum_{1 \leq i_1 < \cdots < i_j \leq n} \prod_{k=1}^j (v - i_k + 1). \]

For example, $c_n(v, n) = v(v-1) \cdots (v-n+1)$, $c_{n-1}(v, n) = \sum_{i=1}^n \prod_{k \neq i} (v-k+1)$, $c_1(v, n) = \sum_{i=1}^n (v-i+1) = nv - n(n-1)/2$. We agree that $c_0(v, n) = 1$. Then $c_{n-j}(v, n) = (1/j!)(d/dv)^j(c_n(v, n))$. Note that for $n > 0$

\[ c_{n-1}(v, n) \neq 0 \quad \text{if} \quad v \in \mathbb{N}. \]  \hspace{1cm} (24)

Let $v \in k^n$ and $u \in \mathbb{N}^n$. For $u' \in \mathbb{N}^n$ with $u' \leq u$, i.e., with $u'_j \leq u_j$ for all $j$, put

\[ c_{u'}(v, u) := \prod_{j=1}^n c_{u'_j}(v_j, u_j). \]

The proof of the following lemma is straightforward.

**Lemma 5.3** Let $p \in k[x]$. Then

\[ \partial^n(x^v p(\log x)) = \sum_{0 \leq u' \leq u} c_{u'}(v, u)x^{v-u}(\partial^{u-u'}p)(\log x). \]

**Proposition 5.4** Let $w$ be a generic weight vector. Let $v$ be an exponent of $M_A(\beta)$ with respect to $w$. Assume that an $A$-hypergeometric series

\[ \phi = \sum_{u \in L} g_u (\log(x))x^{v+u} \]

satisfies $g_u = 0$ for $w \cdot u < 0$. Then $g_u(x) \in k[x_j : j \in \text{vert}(\Delta_w)]$ for all $u \in L$. 

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Proof. Put $V := \text{vert}(\Delta_w)$, and let $j \notin V$. As in the proof of Lemma 4.1, there exist $m_j, m_{ji} \in \mathbb{N}$ with the properties $m_j a_j = \sum_{i \in V} m_{ji} a_i$ and $w_j m_j > \sum_{i \in V} w_i m_{ji}$. Suppose that there exists $u \in I$ such that the degree $d_j$ of $g_u$ in $x_j$ is positive. Let $N$ be a positive integer, and $h$ a polynomial satisfying

$$\partial_j^{Nm_j}(g_u(\log x)x^{v+u}) = h(\log x)x^{v+u-Nm_je_j}.$$  

By (24) and Lemma 5.3, we see that the degree of $h$ in $x_j$ is at least $d_j - 1$. Hence the coefficient of $x^{v+u-Nm_je_j+\sum_{i \in V} Nm_{ji}e_i}$ in $\phi$ is not zero. For large $N$, $w \cdot (u - Nm_je_j + \sum_{i \in V} Nm_{ji}e_i) < 0$. This contradicts our assumption. 

6 Simplicial case

Throughout this section, we assume that the convex hull $\text{conv}(A)$ is a simplex, and that $w$ is a generic weight vector such that the corresponding regular triangulation $\Delta_w$ is $\text{conv}(A)$ itself. Then $\tau \mapsto Q_{\geq 0}^\tau$ gives a one-to-one correspondence between the faces of $\Delta_w$ and those of the cone

$$Q_{\geq 0}A = \{ \sum_{j=1}^n c_ja_j : c_j \in Q_{\geq 0} \}.$$  

Hence we consider the same sets $E_\tau(\beta)$ as the ones in [3]. In this section, first we prove that all canonical series solutions with respect to $w$ are logarithm-free. Second we give a rank formula for $M_A(\beta)$ in terms of the finite sets $E_\tau(\beta)$. Third we characterize the exceptional set in terms of $E_\tau(\beta)$ again. Finally we prove the equivalence of the Cohen-Macaulayness of the affine toric variety defined by $A$ with the emptiness of the exceptional set.

Let $V$ denote the set of vertices of $\text{conv}(A)$, i.e., $V = \text{vert}(\text{conv}(A))$. Recall that a fake exponent is a zero of the fake indicial ideal (see (1) for the definition).

Proposition 6.1 The fake indicial ideal $\tilde{\text{fin}}_w(H_A(\beta))$ is radical.

Proof. Let $\theta$ be a fake exponent. It is enough to prove that the localization $\tilde{\text{fin}}_w(H_A(\beta))_\theta$ at $\theta$ is radical. As seen in the proof of Lemma 4.1, if $j \notin V$, then there exists $m_j \in \mathbb{N}$ such that $\partial_j^{m_j} \in \text{fin}_w(I_A)$. Hence $\theta_j - v_j \in \tilde{\text{fin}}_w(H_A(\beta))_\theta$ for $j \notin V$. Since each component of $A\theta - \beta$ belongs to $\text{fin}_w(H_A(\beta))$, we see that each component of $\sum_{j \in V} (\theta_j - v_j)a_j$ belongs to
Then we conclude \( \theta_j - v_j \in \widetilde{\fin}_w(H_A(\beta))_{v_j} \) for \( j \in V \) as well because the square matrix \( (a_j : j \in V) \) is nonsingular. []

**Proposition 6.2** A fake exponent which does not have minimal negative support is never an exponent.

**Proof.** Let \( v \) be an exponent and \( v' \) a fake exponent such that \( v - v' \in L \), and that \( \text{nsupp}(v) \) is a proper subset of \( \text{nsupp}(v') \). Put \( u := v - v' \). We first prove that \( w \cdot u_+ > w \cdot u_- \). Assume the contrary. Since \( v' \) is a fake exponent, \( \partial^{u-}(x^{v'}) = 0 \). Hence there exists \( i \) such that \( v'_i \in N \) and \( v'_i < -u_i = v'_i - v_i \). This implies \( i \in \text{nsupp}(v) \), and thus \( i \in \text{nsupp}(v') \). This contradicts the fact that \( v'_i \in N \).

Suppose that \( \phi \) is a canonical solution with starting monomial \( x^{v'} \). Since \( V = \text{vert(\text{conv}(A))} \) is linearly independent, \( \phi \) is logarithm-free by Propositions 5.2 and 5.4. As seen in the first paragraph, \( \partial^{u-}(x^{v'}) \neq 0 \). Hence the coefficient of \( x^{v'} \) in \( \phi \) is not zero. This contradicts the fact that \( \phi \) is canonical. []

By Propositions 6.1 and 6.2, Theorem 4.6 gives a rank formula in this case.

**Theorem 6.3** Assume that \( \text{conv}(A) \) is a simplex. Let \( w \) be a generic weight vector such that the corresponding regular triangulation \( \Delta_w \) is \( \text{conv}(A) \) itself. Then there exists a fundamental system of solutions to \( M_A(\beta) \) consisting of logarithm-free canonical series with respect to \( w \), and the rank of \( M_A(\beta) \), denoted by \( \text{rank}(M_A(\beta)) \), is given by the following formula:

\[
\text{rank}(M_A(\beta)) = \sum_{\tau} \left( \sum_{\lambda \in E_{r}(\beta)} \text{vol}(\tau) \right) - \sum_{i} \left( \sum_{E_{r}(\beta) \ni \lambda \atop \tau_1, \ldots, \tau_i \in \text{facet}(\tau)} (-1)^{i-1} \text{vol}(\tau_1 \cap \cdots \cap \tau_i) \right). \tag{26}
\]

**Remark 6.4** Theorem 6.3 generalizes Theorem 4.2.4 and Corollary 4.2.5 in [7], which are concerned with the case \( d = 2 \).
For a generic parameter \( \beta \), the rank of the \( A \)-hypergeometric system \( M_A(\beta) \) is known to equal the volume \( \text{vol}(A) \) (see \([1]\) and \([2]\)). We characterize such a generic parameter by considering the following equivalent conditions on \( \beta \):

**Condition 6.5**

1. If \( \lambda \in k(A \cap \tau_1 \cap \cdots \cap \tau_l) \) belongs to \( E_{\tau_i}(\beta) \) for all \( i = 1, \ldots, l \), then \( \lambda \in E_{\tau_1 \cap \cdots \cap \tau_l}(\beta) \).

2. If \( \lambda \in k(A \cap \tau_1 \cap \tau_2) \) belongs to \( E_{\tau_i}(\beta) \) for \( i = 1, 2 \), then \( \lambda \in E_{\tau_1 \cap \tau_2}(\beta) \).

The set

\[
E(A) := \{ \beta : \text{rank}(M_A(\beta)) > \text{vol}(A) \}
\]

is called the \textit{exceptional set}.

**Theorem 6.6** The exceptional set \( E(A) \) is the set of parameters which does not satisfy Condition 6.5.

**Proof.** Note that \( \text{rank}(M_A(\beta)) = |\text{Minex}_{\beta,w}| \) by Propositions 6.1 and 6.2, and \( \text{vol}(A) = |Z A / \sum_{j \in V} Z a_j| \). Recall the proof of Theorem 4.6. Given an exponent \( v \in \text{Minex}_{\beta,w} \), we put \( I_v := \{ j : v_j \notin N \} \subset V \), \( \tau_v := \sum_{j \in I_v} Q_{2^a} a_j \), and \( \lambda_v := \sum_{j \in I_v} v_j a_j \in E_{\tau_v}(\beta) \). Fix an exponent \( v(0) \in \text{Minex}_{\beta,w} \), and define a map \( C : \text{Minex}_{\beta,w} \to Z A / \sum_{j \in V} Z a_j \) by

\[
C(v) = \sum_{j \in V} (v_j - (0)_j a_j) = \sum_{j \notin V} (v_j - (0)_j) a_j.
\]

Then \( C \) is surjective by Theorem 3.1.

We show that Condition 6.5 is equivalent to the injectivity of \( C \). Suppose that \( \beta \) satisfies Condition 6.5, and let \( v(1), v(2) \in \text{Minex}_{\beta,w} \) satisfy \( C(v(1)) = C(v(2)) \). Then \( \sum_{j \in V} (v(1)_j - v(2)_j) a_j \in \sum_{j \in V} Z a_j \). The linear independence of \( V \) implies that \( v(1) - v(2) \in L \). Put \( \lambda := \sum_{j \in I_v(1) \cap I_v(2)} (v(1)_j a_j) \in k(A \cap \tau_v(1) \cap \tau_v(2)) \). Then \( \lambda = \lambda_{v(i)} \) in \( E_{\tau_v(i)}(\beta) \) \((i = 1, 2)\), and thus by Condition 6.5, \( \lambda \in E_{\tau_v(1) \cap \tau_v(2)}(\beta) \). By Theorem 3.1, there exists \( v \in \text{Minex}_{\beta,w} \) such that \( I_v \subset I_v(1) \cap I_v(2) \), and \( \sum_{j \in I_v(1) \cap I_v(2)} (v_j - (1)_j a_j) \in \sum_{j \in I_v(1) \cap I_v(2)} Z a_j \). Then \( C(v) = C(v(1)) = C(v(2)) \). This leads to the fact that \( v - v(1), v - v(2) \in L \) as above, and thus nsupp(\( v \)) \( \subset \) nsupp(\( v(i) \)) \((i = 1, 2)\). Since \( v(1), v(2), \) and \( v \) have minimal negative support, we obtain \( v(1) = v = v(2) \).

Next we assume the injectivity of \( C \), and suppose that \( \lambda \in k(A \cap \tau_1 \cap \tau_2) \) belongs to \( E_{\tau_1}(\beta) \) and \( E_{\tau_2}(\beta) \). By Theorem 3.1, there exists \( v(i) \in \text{Minex}_{\beta,w} \)
such that \( I_v(i) \subseteq \text{vert}(\tau_i) \), and \( \lambda = \sum_{j \in I_v(i)} v(i)_j a_j \subseteq \sum_{j \in \text{vert}(\tau_i)} Z a_j \) (\( i = 1, 2 \)). Hence \( C(v(1)) = C((v(2))) \), and thus \( v(1) = v(2) \) by the injectivity. Put \( v := v(1) = v(2) \). Then we have \( I_v \subseteq \text{vert}(\tau_1) \cap \text{vert}(\tau_2) \), and \( \beta - \lambda = \sum_{j \notin I_v} v_j a_j + (\sum_{j \in I_v} v_j a_j - \lambda) \in N A + \sum_{j \in \text{vert}(\tau_1) \cap \text{vert}(\tau_2)} Z a_j \). Hence we obtain \( \lambda \in E_{\tau_1 \cap \tau_2}(\beta) \).

**Theorem 6.7** The ring \( k[N A] = k[\partial]/I_A \) is Cohen-Macaulay if and only if \( \mathcal{E}(A) = \emptyset \).

**Proof.** Since \( \{ \partial_j \}_{j \in V} \) is a linear system of parameters, the Cohen-Macaulayness means that \( k[N A] \) is a free \( k[\partial_j : j \in V] \)-module. Hence \( k[N A] \) is not Cohen-Macaulay if and only if there exist minimal generators \( \partial^{u(1)} \) and \( \partial^{u(2)} \), and \( \partial^{m(1)}, \partial^{m(2)} \in k[\partial_j : j \in V] \) with \( \text{supp}(m(1)) \cap \text{supp}(m(2)) = \emptyset \) such that \( \partial^{m(1)} \partial^{u(1)} - \partial^{m(2)} \partial^{u(2)} \in I_A \), where \( \text{supp}(m(i)) \) denotes the support of \( m(i) \). Suppose that \( k[N A] = k[\partial]/I_A \) is not Cohen-Macaulay. Let \( \tau_i := \sum_{j \in \text{supp}(m(i))} Q_{\geq 0} a_j \) for \( i = 1, 2 \), and \( \beta := Au(1) - Am(2) = Au(2) - Am(1) \). Then \( 0 \in E_{\tau_i}(\beta) \) (\( i = 1, 2 \)) and \( 0 \notin E_{\{0\}}(\beta) \), and hence \( \beta \in \mathcal{E}(A) \).

Replacing \( \beta \) by \( \beta - \lambda \) in Condition 3.5 if necessary, we see that the condition \( \mathcal{E}(A) \neq \emptyset \) is equivalent to the condition that there exist \( \beta \in Z A \), and faces \( \tau_1, \tau_2 \) such that \( 0 \in E_{\tau_i}(\beta) \) (\( i = 1, 2 \)) and \( 0 \notin E_{\tau_1 \cap \tau_2}(\beta) \). The latter condition is equivalent to the condition that there exist \( \beta \in Z A \), and \( m(1), m(2) \in N^V \) such that \( \beta + Am(i) \in N A \) (\( i = 1, 2 \)), and no \( m(3) \in N^V \) with \( \text{supp}(m(3)) \subseteq \text{supp}(m(1)) \cap \text{supp}(m(2)) \) such that \( \beta + Am(3) \in N A \). Let \( m' \) be the greatest common divisor of \( m(1) \) and \( m(2) \). By considering \( \beta + Am' \) instead of \( \beta \), we see that the above condition is equivalent to the condition that there exist \( \beta \in Z A \setminus N A \), and \( m(1), m(2) \in N^V \) with \( \text{supp}(m(1)) \cap \text{supp}(m(2)) = \emptyset \) such that \( \beta + Am(i) \in N A \) (\( i = 1, 2 \)).

Suppose \( \mathcal{E}(A) \neq \emptyset \). Let \( \beta \in Z A \setminus N A \), and \( m(1), m(2) \in N^V \) be the ones in the last condition. Then for \( i = 1, 2 \), there exist \( u(i) \in N^n \) and \( v(i) \in N^V \) such that \( \partial^{u(i)} \) is a minimal generator, and \( \beta + Am(i) = Au(i) + Au(i) \). We may assume that \( \text{supp}(m(i)) \cap \text{supp}(v(i)) = \emptyset \) (\( i = 1, 2 \)). Then \( \partial^{m(2)+\nu(1)} \partial^{u(1)} - \partial^{m(1)+\nu(2)} \partial^{u(2)} \in I_A \). Since the supports of \( m(2) + v(1) \) and \( m(1) + v(2) \) are different, \( \partial^{u(1)} \neq \partial^{u(2)} \). We conclude that \( k[N A] = k[\partial]/I_A \) is not Cohen-Macaulay. \( \square \)

**Remark 6.8** In the general situation, the only-if-direction of Theorem 6.7 was proved by Gel’fand, Zelevinskii, and Kapranov (\cite{6}, see also \cite{2}).
if-direction was conjectured by Sturmfels, and proved when $d = 2$ (see [2]). Recently Matushevich [5] proved it when $n - d = 2$.

Example 6.9 Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \end{pmatrix},$$

and $\beta_0 = t(1, 1, 1)$. We prove that $E(A) = \{ \beta_0 \}$ and $\text{rank}(M_A(\beta_0)) = 11$. Note that $N_A$ has clear $\mathbb{Z}/3\mathbb{Z}$-symmetry (see Figure 2).

Figure 2: The set $A$

Given a subset $\{i_1, \ldots, i_r\}$ of $\{1, 4, 9\}$, let $\tau_{i_1, \ldots, i_r}$ denote the face $Q_{\geq 0}a_{i_1} + \cdots + Q_{\geq 0}a_{i_r}$ of $Q_{\geq 0}A$. Then there are one 3-dimensional face $\tau_{149} = Q_{\geq 0}A$, three 2-dimensional faces $\tau_{14}, \tau_{19}, \tau_{49}$, three 1-dimensional faces $\tau_1, \tau_4, \tau_9$, and one 0-dimensional face $\tau_{\emptyset} = \{ 0 \}$. The volumes corresponding to the faces are

$$\text{vol}(\tau_{149}) = \text{vol}(A) = 9,$$
$$\text{vol}(\tau_{14}) = \text{vol}(\tau_{19}) = \text{vol}(\tau_{49}) = 3,$$
$$\text{vol}(\tau_1) = \text{vol}(\tau_4) = \text{vol}(\tau_9) = 1,$$
$$\text{vol}(\{ 0 \}) = 1.$$ \hfill (27)

For each face $\tau$, we have $\langle Q(A \cap \tau) \rangle \cap \mathbb{Z}A = \mathbb{Z}(A \cap \tau)$, and thus $|E_\tau(\beta)| = 0$ or 1. Given $\beta$, let $\text{minface}(\beta)$ denote the set of minimal faces among faces $\tau$ with nonempty $E_\tau(\beta)$. Then the equivalence class of $M_A(\beta)$ is determined by $\text{minface}(\beta)$ (see Theorem 2.1 in [6]).

Suppose $\beta \in E(A)$. Theorem 6.6 implies that the set $\text{minface}(\beta)$ contains at least two elements. If $\text{minface}(\beta)$ contains $\tau_1, \tau_4$, then it has to contain $\tau_9$.
as well. Hence minface(\(\lambda\)) = \{\(\tau_1, \tau_4, \tau_9\)\}, \(\beta = \beta_0\), and
\[
\text{rank}(M_A(\beta_0)) = (9 - 3 \times 3 + 3 \times 1 - 1) \\
+ 3(3 - 2 \times 1 + 1) \\
+ 3 \times 1 = 2 + 6 + 3 = 11 > 9 = \text{vol}(A).
\]
If \(E_{\tau_4}(\beta) \neq \emptyset\) and \(E_{\tau_9}(\beta) \neq \emptyset\), then we see \(E_{\tau_1}(\beta) \neq \emptyset\). Hence the cases minface(\(\beta\)) = \{\(\tau_4, \tau_9\)\}, \{\(\tau_4, \tau_9, \tau_4\)\}, and \{\(\tau_1, \tau_4\)\} do not occur. This finishes the proof of the equality \(\mathcal{E}(A) = \{\beta_0\}\).

7 Analytic Classification

We gave a criterion for two \(A\)-hypergeometric systems to be isomorphic as algebraic \(D\)-modules in [3]. In this section, by using the argument in the proof of Theorem 3.1, we show that the same criterion remains valid in the analytic category.

Lemma 7.1 Let \(w\) be a generic weight vector. Let \(\tau\) be a face of \(Q_{\geq 0} A\), and \(I\) the vertex set \(\text{vert}(\tau')\) of a face \(\tau' \in \Delta_w\) with \(Q\tau = \sum_{i \in I} Qa_i\). Then there exists a natural map
\[
\nu : E_\tau(\beta) \longrightarrow E_\tau(\beta).
\]
The map \(\nu\) is surjective and a \([Z(A \cap \tau) : Z(A \cap \tau')] : 1\)-map, unless both \(E_\tau(\beta)\) and \(E_\tau(\beta')\) are empty. In particular,
\[
|E_\tau(\beta)| = |E_\tau(\beta)||Z(A \cap \tau) : Z(A \cap \tau')|.
\]

Proof. First we show the existence of \(\nu\). Note that \(k(A \cap \tau) = k(A \cap \tau')\), and let \(\lambda\) belong to \(k(A \cap \tau) = k(A \cap \tau')\). Then \(\lambda \in E_\tau(\beta)\) means \(\beta - \lambda \in NA + Z(A \cap \tau')\), which implies \(\beta - \lambda \in NA + Z(A \cap \tau)\), which in turn means \(\lambda \in E_\tau(\beta)\). This gives the natural map \(\nu\).

Second we prove the surjectivity of \(\nu\). Suppose \(\lambda \in E_\tau(\beta)\). Then as in the first two sentences of the proof of Theorem 3.1, there exists \(v \in k^n\) such that \(\beta = Av\), \(\lambda = \sum_{a_j \in \tau} v_j a_j\), \(\text{nsupp}(v) \subset \tau'\), and \(\{ j : v_j \notin Z \} \subset \tau'\). Then \(\sum_{a_j \in \tau'} v_j a_j \in E_\tau(\beta)\) and \(\nu(\sum_{a_j \in \tau'} v_j a_j) = \lambda\).

Finally we prove that \(\nu\) is a \([Z(A \cap \tau) : Z(A \cap \tau')] : 1\)-map. Suppose that \(\lambda, \lambda' \in E_\tau(\beta)\) and \(\nu(\lambda) = \nu(\lambda')\). Then \(\lambda - \lambda' \in Z(A \cap \tau)\). Conversely suppose \(\lambda \in E_\tau(\beta)\), and \(v \in Z^n\) with \(v_j = 0\) for \(j \notin \tau\). We prove that \(\lambda - Av \in E_\tau(\beta)\). Since \(I\) is a base of \(\tau\), there exists \(l \in L_\tau\) such that \((v + l)_j \in N\) for all \(j \notin I\). Then \(\beta - (\lambda - A(v + l)) \in NA + Z(A \cap \tau')\). Hence \(\lambda - Av = \lambda - A(v + l) \in E_\tau(\beta)\).
Theorem 7.2 Let \( w \) be a generic weight vector. Let \( \tau \) be a face of \( \mathbb{Q}_{\geq 0}A \), and \( I \) the vertex set \( \operatorname{vert}(\tau') \) of a face \( \tau' \in \Delta_w \) with \( Q^\tau = \sum_{i \in I} Q a_i \). Then there exist at least \( |E_\tau(\beta)|\) \( \frac{1}{\nu_r(\tau')}-\)many linearly independent logarithm-free canonical series solutions \( \phi_v \) to \( M_A(\beta) \) with respect to \( w \), satisfying \( \sum_{a_j \in \tau} v_j a_j \in E_\tau(\beta) \), \( \nu_{\tau}(v) \subset I \), and \( \{ j : v_j \notin \mathbb{Z} \} \subset I \). Here \( \nu_{\tau}(\tau') \) denotes the index \( \nu(\mathbb{Z}(A \cap \tau) : \sum_{i \in I} \mathbb{Z} a_i) \).

Proof. This is immediate from Theorem 3.1 and Lemma 7.1. [6]

The algebraic version of the following is the main theorem in the paper [6].

Theorem 7.3 Let \( k \) be the field of complex numbers \( \mathbb{C} \), and \( D \) the sheaf of rings of analytic linear differential operators on \( \mathbb{C}^n \). Put \( M_A(\beta) = D \otimes \mathbb{C} M_A(\beta) \), and \( M_A(\beta') \) likewise. The \( A \)-hypergeometric systems \( M_A(\beta) \) and \( M_A(\beta') \) are isomorphic as \( D \)-modules if and only if \( E_\tau(\beta) = E_\tau(\beta') \) for all faces \( \tau \) of the cone \( \mathbb{Q}_{\geq 0}A \).

Proof. Here we prove the only-if-part of the theorem. The if-part is immediate from Theorem 2.1 in [8].

We suppose that \( \lambda \in E_\tau(\beta) \setminus E_\tau(\beta') \) for some face \( \tau \), and then we prove that \( M_A(\beta) \) and \( M_A(\beta') \) are not isomorphic.

Take \( w, v \) with \( \lambda = \sum_{j \in \tau} v_j a_j \), and \( I \) as in Theorem 7.2. For \( j \in \tau \setminus I \), let \( u(j) = e_j - \sum_{i \in I} c_i e_i \) in the notation of [8]. Since \( \tau \) is a face of \( \mathbb{Q}_{\geq 0}A \), and since \( I \) is a base of \( \tau \), we have \( K_I = \sum_{j \in \tau \setminus I} \mathbb{Q}_{\geq 0}u(j) \), and \( u(j) (j \in \tau \setminus I) \) is a basis of \( L_{\tau, \mathbb{Q}} \) satisfying \( L \cap K_I \subset \sum_{j \in \tau \setminus I} \mathbb{N} u(j) \). The series \( \psi_v \) (see (3)) converges in \( U_r \) for small enough, where

\[
U_r := \left\{ x : |x^u(j)| < r (j \in \tau \setminus I), \quad x_i \neq 0 \ (i \in I) \right\},
\]

since \( M_A(\beta) \) is regular holonomic (for example, see Theorem 2.4.9 in [8]). Hence \( \phi_v \) is a nonzero element of \( \operatorname{Hom}_{D(U_r)}(M_A(\beta)(U_r), x^v \mathcal{O}(U_r)) \). Here note that \( x^v \mathcal{O}_{U_r} \) is a \( D_{U_r} \)-module.

Next we prove that \( \operatorname{Hom}_{D(U_r)}(M_A(\beta')(U_r), x^v \mathcal{O}(U_r)) = 0 \), which completes the proof of the theorem. Suppose that \( \phi = x^v \psi \) belongs to the solution space \( \operatorname{Hom}_{D(U_r)}(M_A(\beta')(U_r), x^v \mathcal{O}(U_r)) \). Since \( \psi \) is holomorphic on \( U_r \), it has the Laurent expansion

\[
\psi(x) = \sum_{l \in \mathbb{N}^{\nu_v} \times \mathbb{Z}^r} c_l y^l,
\]
where \( y_j = x^{u(j)} \) if \( j \in \tau \setminus I \), and \( y_j = x_j \) otherwise. By the equality
\[
(A\theta - \beta')x^v\psi = 0,
\]
each \( l \in \mathbb{N}^\tau \times \mathbb{Z}^\tau \) with \( c_l \neq 0 \) must satisfy \( \beta' = \sum_{i \in I} (v_i + l_i)a_i + \sum_{j \in \tau \setminus I} v_ja_j + \sum_{j \notin \tau} (v_j + l_j)a_j \). The condition that \( \lambda \notin E_{\tau}(\beta') \), implies that all \( c_l \) are zero. Therefore \( \mathcal{M}_A(\beta) \) and \( \mathcal{M}_A(\beta') \) are not isomorphic. 

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