Families of stable and metastable solitons in coupled system of scalar fields

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In this paper, we obtain stable and metastable soliton solutions of a coupled system of two real scalar fields with five discrete points of vacua. These solutions have definite topological charges and rest energies and show classical dynamical stability. From a quantum point of view, however, the V-type solutions are expected to be unstable and decay to D-type solutions. The induced decay of a V-type soliton into two D-type ones is calculated numerically, and shown to be chiral, in the sense that the decay products do not respect left-right symmetry.

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I. INTRODUCTION

Relativistic solitons, including those of the well-known Sine-Gordon (SG) equation, exhibit remarkable similarities with classical particles. They exert short range forces on each other and collide, without losing their identities [1–5]. They are localized and do not disperse while propagating in the medium. Because of their field nature, they do tunnel a barrier in certain cases, although this tunnelling is different from the well-known quantum version [2, 6]. Topological solitons are stable, due to the boundary conditions at spatial infinity. Their existence, therefore, is essentially dependent on the presence of degenerate vacua [2, 7].

Topology provides an elegant way of classifying solitons in various sectors according to the mappings between the degenerate vacua of the field and the points at spatial infinity. For the Sine-Gordon system in 1 + 1 dimensions, these mappings are between \( \phi = 2n\pi, \ n \in \mathbb{Z} \) and \( x = \pm \infty \), which correspond to kinks and antikinks of the SG system. More complicated mappings occur
in solitons in higher dimensions \[3\]. Coupled systems of scalar fields with soliton solutions have found interesting applications in double-strand, long molecules like the DNA molecules \[8\]-\[11\] and bi-dimensional QCD \[12\].

Bazeia et al. \[13\] considered a system of two coupled real scalar fields with a particular self-interaction potential such that the static solutions are derivable from first order coupled differential equations. Riazi et al. \[14\] employed the same method to investigate the stability of the single-soliton solutions of a particular system of this type. Inspired by the coupled system introduced in \[2\], we propose a new coupled system of two real scalar fields which shows interesting types of solitons with well-defined topological charges and rest energies (masses).

In this paper, we focus on a system with the self-interaction potential

$$V(\phi, \psi) = \phi^2(\psi^2 - \psi_0^2)^2 + \psi^2(\phi^2 - \phi_0^2)^2,$$

in which \(\phi\) and \(\psi\) are real scalar fields, and \(\phi_0\) and \(\psi_0\) are constants. This potential is plotted in Fig. ?? for \(\phi_0 = 1\) and \(\psi_0 = 2\). Following the common terminology in field theory, this potential has absolute degenerate minima at \(\phi = \psi = 0\), \(\phi = \pm \phi_0\) and \(\psi = \pm \psi_0\) known as the true vacua.

In our proposed system, a spectrum of solitons with different rest energies exists which are stable or meta-stable, depending on their energies and boundary conditions.

The structure of this paper is as follows: in Section \[III\] we review some basic properties of the proposed system. In Section \[IV\] some exact solutions together with the corresponding charges and energies are derived. The necessary nomenclature and general behavior of the solutions due to the boundary conditions are also introduced in this section. Numerical solutions corresponding to different boundary conditions are presented, and properties of these solutions like their charges, masses, and stability status are addressed in this section. In order to investigate the stability of the numerical solutions, their evolution is worked out numerically. The classical dynamical stability of the solutions is investigated further in \[V\] Our conclusions and a summary of the results are given in the last section \[V\]

II. DYNAMICAL EQUATIONS, CONSERVED CURRENTS, AND TOPOLOGICAL CHARGES

Within a relativistic formulation, the Lagrangian density of the system is given by:

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \partial^\mu \psi \partial_\mu \psi - \left[ \phi^2(\psi^2 - \psi_0^2)^2 + \psi^2(\phi^2 - \phi_0^2)^2 \right].$$

(2)
From this Lagrangian density, we obtain the following equations for $\phi$ and $\psi$, respectively:

$$\Box \phi = -2\phi(\psi^2 - \psi_0^2)^2 - 4\phi^2(\phi^2 - \phi_0^2); \quad (3)$$

and

$$\Box \psi = -2\psi(\phi^2 - \phi_0^2)^2 - 4\psi^2(\psi^2 - \psi_0^2). \quad (4)$$

Since the lagrangian density Eq.(2) is Lorentz invariant, the corresponding energy-momentum tensor $T_{\mu\nu}$ is:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \partial_\mu \psi \partial_\nu \psi - g_{\mu\nu} \mathcal{L}; \quad (5)$$

which satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0. \quad (6)$$

In Equation (5), $g_{\mu\nu} = \text{diag}(1, -1)$ is the metric of the $(1 + 1)$ dimensional Minkowski spacetime.

The Hamiltonian (energy) density is obtained from Eq.(5) according to

$$\mathcal{H} = T^{00} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 + V(\phi, \psi). \quad (7)$$

It can be shown that the following topological currents can be defined, which are conserved independently, and lead to quantized charges:
\[ J_H^\mu = \frac{1}{2\phi_0} \varepsilon^{\mu\nu} \partial_\nu \phi, \]
\[ J_V^\mu = \frac{1}{2\psi_0} \varepsilon^{\mu\nu} \partial_\nu \psi. \]  
(8)

The subscripts H, V and D (to be used later), denote “horizontal”, “vertical” and “diagonal” which will be explained later. The currents $J_{H,V}^\mu$ are conserved, independent of each other:
\[ \partial_\mu J_H^\mu = 0, \]
\[ \partial_\mu J_V^\mu = 0. \]  
(9)

It is obvious that the total horizontal and vertical charges are conserved separately in any local dynamical evolution of the system. The corresponding topological charges are given by:
\[ Q_H = \int_{-\infty}^{+\infty} J_H^0 dx = \frac{1}{2}[\phi(+\infty) - \phi(-\infty)], \]
\[ Q_V = \int_{-\infty}^{+\infty} J_V^0 dx = \frac{1}{4}[\psi(+\infty) - \psi(-\infty)]. \]  
(10)

FIG. 2: Nomenclature of horizontal (H), vertical (V) and diagonal (D) solutions according to the boundary conditions on the $(\phi, \psi)$ plane.

III. SINGLE-SOLITON SOLUTIONS

Static solutions which correspond to transitions between neighboring vacua are symbolically shown in Fig. 2. Accordingly, we call the static solutions H (horizontal), V (vertical) and D
(diagonal) types, depending on the beginning and end points of the solution on the \((\phi, \psi)\) plane. We have called them “horizontal”, “vertical” and “Diagonal” simply because of their orientation in the \((\phi, \psi)\) plane. If we apply -in turn- the constraints \(\psi = \psi_0\), \(\phi = \phi_0\), and \(\psi = \frac{\phi}{\phi_0}\) in the Lagrangian density \([2]\), the resulting one degree of freedom equations will reduce to the well-known \(\phi^4\) and \(\phi^6\) systems which possess the following exact single-soliton solutions \([17, 18]\), summarized in Table 1. Of course, these are not the exact solutions of the coupled equations \([3, 4]\).

Table 1. Exact static solutions and the corresponding horizontal and vertical charges.

| Type | Solution | \(Q_H\) | \(Q_V\) |
|------|----------|--------|--------|
| H    | \(\phi = \pm \tanh(2\sqrt{2}x), \quad \psi_0 = \pm 2\) | \(\pm 1\) | 0 |
| V    | \(\phi_0 = \pm 1, \quad \psi = \pm 2 \tanh(2\sqrt{2}x)\) | 0 | \(\pm 1\) |
| D    | \(\phi = \pm (1/2)\psi, \quad \psi = \pm \frac{2}{(1+\exp(\pm 4\sqrt{2}x))^{1/2}}\) | \(\pm \frac{1}{2}\) | \(\pm \frac{1}{2}\) |

Since the functions given in Table 1 are not exact solutions of the coupled equations \([3, 4]\), we use them as initial guesses to find the minimum-energy solutions. Any static solution has an orbit \(f(\phi, \psi) = 0\) in the \((\phi, \psi)\) plane, which begins and ends at one of the true vacua. We therefore have

\[
\frac{df}{dx} = \frac{\partial f}{\partial \phi} \phi' + \frac{\partial f}{\partial \psi} \psi' = 0, \tag{11}
\]

where prime means differentiation with respect to \(x\). The field equations \([3, 4]\) can be integrated once to yield

\[
\frac{1}{2} (\phi')^2 = \int \frac{\partial V}{\partial \phi} d\phi + C_1, \tag{12}
\]

and

\[
\frac{1}{2} (\psi')^2 = \int \frac{\partial V}{\partial \psi} d\psi + C_2, \tag{13}
\]

where \(C_1\) and \(C_2\) are integration constants. Combining equations \([11]-[13]\), we obtain the following integro-differential equation for the orbit:

\[
\left( \frac{\partial f}{\partial \phi} \right)^2 \left( \int \frac{\partial V}{\partial \phi} d\phi + C_1 \right) = \left( \frac{\partial f}{\partial \psi} \right)^2 \left( \int \frac{\partial V}{\partial \psi} d\psi + C_2 \right). \tag{14}
\]

Bazeia et al. \([19]\) have shown that if the potential can be written in the form

\[
U = \frac{1}{2} \left( F + \phi \frac{\partial F}{\partial \phi} + \psi \frac{\partial G}{\partial \phi} \right)^2 + \frac{1}{2} \left( G + \phi \frac{\partial F}{\partial \psi} + \psi \frac{\partial G}{\partial \psi} \right)^2, \tag{15}
\]
then any solution to the following first order coupled equations

\[
\frac{d\phi}{dx} + F + \phi \frac{\partial F}{\partial \phi} + \psi \frac{\partial G}{\partial \phi} = 0,
\]

(16)

and

\[
\frac{d\psi}{dx} + G + \psi \frac{\partial G}{\partial \psi} + \phi \frac{\partial F}{\partial \psi} = 0,
\]

(17)

solves the corresponding second order equations. For the coupled system considered in this paper, we could use this method to obtain the diagonal (D-type) solutions, provided that \(\phi_0 = \psi_0\). In this case, the \(F\) and \(G\) functions will be

\[
F = -\frac{1}{2}\lambda a^2 \phi,
\]

(18)

and

\[
G = \frac{1}{2}\lambda (\phi^2 - a^2) \psi,
\]

(19)

where \(a = \phi_0\). For the symmetrical D-type solutions, we have \(\phi^2 = \psi^2\), and the exact solution will be

\[
\phi^2 = \psi^2 = \frac{1}{2}a^2 \left[ 1 + \tanh(\lambda a^2(x - x_0)) \right].
\]

(20)

The general D-type orbits are given by

\[
(\phi^2 - \phi_0^2) - \phi_0^2 \ln \frac{\phi^2}{\phi_0^2} = (\psi^2 - \psi_0^2) - \psi_0^2 \ln \frac{\psi^2}{\psi_0^2}.
\]

(21)

These orbits are plotted in Figure 3 for \(\phi_0 = 1\) and \(\psi_0 = 2\). It can be seen that the D-type numerical orbits of Figure 5 which minimize the energy functional closely resemble the analytical orbits of Figure 3.

In order to find other stable static solutions, we have used the expressions depicted in Table 1 as the starting "guesses" in a step-wise variational procedure. In this procedure, the bi-dimensional spacetime is represented by a grid of spatial and temporal step-sizes \(\epsilon\) and \(\delta\), respectively. All the derivatives in the energy density expression are then written in their discrete form (e.g. \(d\phi/dx \rightarrow (\phi_i - \phi_{i-1})/\epsilon\)). The initial guess is then repeatedly varied in steps, with the total energy checked at each step. Only variations which reduce the total energy are accepted and the rest are rejected. The fields are thus varied toward the minimum energy solution, where small variations no longer lead to a decrease in total energy. The minimum energy, static solutions obtained in this way are plotted in Figure 3. In Figure 4, we have mapped these solutions into the \((\phi, \psi)\) plane.
The conservation of $V$ and $H$ type charges (Equation 9) does not come from a continuous symmetry of the Lagrangian, but rather it stems from the boundary conditions and topological considerations [20]. The system under consideration respects various types of symmetries: 1) Lorentz symmetry: this symmetry comes from the invariance of the Lagrangian (2) under Lorentz transformations. Since $\phi$ and $\psi$ are scalar fields, we conclude that if $(\phi(x), \psi(x))$ is a static solution, then $(\phi(\gamma(x - vt), \psi(\gamma(x - vt)))$ where $v$ is a constant (soliton velocity) and $\gamma = (1 - v^2)^{-1/2}$, is also solution. 2) If $(\phi(x, t), \psi(x, t))$ is a solution, then $(\pm \phi(x, t), \pm \psi(x, t))$ are solutions, also. 3) For the special case $\phi_0 = \psi_0$, we have dual solutions $\phi \leftrightarrow \psi$. 4) Parity and time reversal are symmetries of the system. Therefore if $(\phi(x, t), \psi(x, t))$ is a solution, then $(\phi(\pm x, \pm t), \psi(\pm x, \pm t))$ are solutions, too.

IV. SOLITON STABILITY AND CHIRAL DECAY OF $V$-TYPE SOLITONS

Basic properties of the H, V, and D solitons are summarized in Table 1. It can be seen that all D solitons are degenerate (i.e. have the same rest energy). This degeneracy results from the $(\phi, \psi) \leftrightarrow (\pm \phi, \pm \psi)$ symmetry of the Lagrangian density (2).

Another interesting property of these solitons is that the mass (rest energy) of the $V_{EC}$ soliton is larger than the sum of the $D_{EA}$ and $D_{AC}$ solitons, while the topological charge of $V_{EC}$ is equal to the sum of the $D_{EA}$ and $D_{AC}$ charges. The same is for $V_{DB}$ which is more massive than the sum of $D_{DA}$ plus $D_{AB}$. The decay of $V_{EC}$ into $D_{EA} + D_{AC}$ (or $V_{DB}$ into $D_{DA} + D_{AB}$) is therefore possible with regard to energy and charge conservations. These decays were not observed to occur spontaneously in our dynamical simulations, implying that $V_{EC}$ and $V_{DB}$ are classically metastable configurations. Quantum tunnelling or external perturbations can -in principle- cause such a decay (see [2]). In order to show that stimulated decay is in fact possible in the system
under consideration, we have performed the following simulation: The static vertical soliton $V_{DB}$ is used as the initial condition in a program which calculates the dynamics of the system, by solving the coupled PDEs 3 and 4. This soliton is excited, by pumping some energy into the $\phi$ field, via an increase in its amplitude. The system is then observed to become unstable and decay into $D_{DA}$ and $D_{AB}$. This simulation is shown in Figure 6. Note that the excess energy is transferred into both the kinetic energy of the decay products and also emission of some low amplitude waves.

It is interesting to note that the decay of $V$-type solitons is chiral, in the sense that when $V_{EC}$ decays into $D_{EA}$ and $D_{AC}$, the former always moves to the left, while the latter always move to the right. The fact that the reverse never happens is due to the boundary conditions and topological reasons. Similarly, for the decay $V_{DB} \to D_{DA} + D_{AB}$, the decay product $D_{DA}$ always moves to the left and $D_{AB}$ always moves to the right. This property can be extended to an arbitrary array.
FIG. 5: The minimum energy, static solutions mapped on the \((\phi, \psi)\) plane: (a) \(H_{DE}, V_{DB}, H_{BC}\) and \(V_{EC}\) and (b) \(D_{DA}, D_{AC}, D_{EA}\) and \(D_{AB}\).

of solitons in a multi-soliton system. The consistency of boundary conditions between successive solitons permits certain sequences and forbids certain ones. Examples of allowed and forbidden arrays are the followings:

Allowed arrays: \(...D_{DA}D_{AB}H_{BC}V_{CE}H_{ED}\...\)

Forbidden arrays: \(...D_{DA}D_{BA}H_{BC}V_{EC}H_{EC}\...\)
FIG. 6: Stimulated decay of a $V_{DB}$ soliton into $D_{DA} + D_{AB}$. Note that the excess energy is transferred into the kinetic energy of daughter solitons and also emission of low amplitude waves. Also note that the decay is chiral, in the sense that $D_{DA}$ always moves to the left.

Table 2. General properties of different soliton families. Note that the soliton masses and stability status depend on the choice of $\phi_0$ and $\psi_0$. Here, we have assumed $\phi_0 = 1$ and $\psi_0 = 2$.

| Symbol | Mass | $Q_H$ | $Q_V$ | Stability | Decay Mode |
|--------|------|-------|-------|-----------|------------|
| $D_{AB}$ | 2.858 | -1/2  | +1/2  | stable    | –          |
| $D_{AC}$ | 2.858 | +1/2  | +1/2  | stable    | –          |
| $D_{DA}$ | 2.858 | +1/2  | +1/2  | stable    | –          |
| $D_{EA}$ | 2.858 | -1/2  | +1/2  | stable    | –          |
| $H_{BC}$ | 3.594 | +1    | 0     | stable    | –          |
| $V_{EC}$ | 6.383 | 0     | +1    | metastable | $V_{EC} \rightarrow D_{EA} + D_{AC}$ |
| $H_{DE}$ | 3.594 | +1    | 0     | stable    | –          |
| $V_{DB}$ | 6.383 | 0     | +1    | metastable | $V_{DB} \rightarrow D_{DA} + D_{AB}$ |
V. CONCLUDING REMARKS

We studied a nonlinear system of coupled scalar fields which had three (H, V, and D) types of stable and metastable soliton solutions. We started by deriving analytical, static solutions with definite topological charges. These static solutions, however, were shown to be unstable and the stable solutions were obtained numerically, by minimizing the total energy, using a step-wise variational method. The full, dynamical equations were used to ensure the stability of these minimum energy solutions. According to energy and topological charge conservations, the V-type solutions are expected to decay into D-type ones. This, however, does not happen spontaneously at the classical level. Either quantum tunnelling or external triggers are needed for decays like $V_{EC} \rightarrow D_{EA} + DAC$. Conversely, the formation of $V_{EC}$ from $D_{EA} + D_{AC}$ collision is endothermic, requiring -at least- the incoming kinetic energy $\Delta E = 6.38 - 2 \times 2.86 = 0.66$ in dimensionless units. The stimulated decay of a V-type soliton were calculated numerically, by pumping some energy into the $\phi$-field. This decay is exothermic and the excess energy was observed to be transferred into the kinetic energy of decay products and emission of low amplitude waves.

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