Free vibration and damping analysis of the cylindrical shell partially covered with equidistant multi-ring hard coating based on a unified Jacobi-Ritz method

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Abstract

In this study, the aim was to evaluate the vibration suppression performance of the partially covered equidistant multi-ring hard coating damping treatment for the cylindrical shell structure in aviation power equipment. A continuous rectangular pulse function was presented to describe the local thickness variation of arbitrary coating proportion and arbitrary number of coating rings. A semi-analytical unified solution procedure was established by combining the rectangular pulse function, the generalized Jacobi polynomials, and the Rayleigh-Ritz method. The stiffness coefficient $k = 10^{13}$ N/m² and the truncation number $N = 8$ were found to be large enough to achieve an accurate and efficient solution of the vibration analysis of the shell. The modal loss factor generally increased with the increase of the coating proportion ranging from 0.0 to 1.0 for all the circumferential wave numbers. The modal loss factor increased roughly linear with the coating proportion for all the circumferential wave numbers. And the modal loss factor was increased with the circumferential wave number, and the greater the number of circumferential waves, the greater the rate of change. The increase of the ring number was not always beneficial for vibration reduction

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of the shell, while the modal loss factor increased roughly linear with the coating proportion. The increased ring number and coating proportion tend more to exhibit an obvious incremental damping effect under larger circumferential wave number.

**Keywords**
Cylindrical shell, partial damping, free vibration, hard coating, Jacobi-Ritz method

**Introduction**

Being one kind of damping treatment with good performance in high temperature and strong corrosion environments, the hard coating has received extensive attention in vibration suppression of engine casing, blade, and blisk,\(^1\)–\(^3\) which can be used for vibration reduction of the thin cylindrical shell structures with promising application prospect.\(^4\)–\(^6\) However, due to the existence of appendages and the restrictions of structural weight,\(^7\)–\(^9\) it is impractical to fully cover the shell surface with a coating material. Hence the study on the vibration and damping characteristics of the cylindrical shell with partial hard-coating damping treatment is of important actual significance.

In recent years, many studies have been performed on the vibration analysis of partially coated composite cylindrical shells. In terms of vibro-acoustic problems, Zou et al.\(^10\) evaluated the effect of arbitrarily partially covered acoustic coatings on the underwater vibro-acoustic characteristics of the cylindrical shell structure, and the sound source levels of the cylindrical shell with acoustic coatings of different laying length were quantitatively discussed. Ferri et al.\(^11\) addressed the scattering of acoustic plane waves from infinite submerged cylindrical shells partially covered with two discontinuous axial surface coating, indicating that this partial treatment had little impact on the scattered far-field pressure, but would lead to large surface pressure and velocity gradient. Laulagnet and Guyader\(^12\) presented a vibro-acoustic model of a finite cylindrical shell partially covered with a compliant material layer to analyze the acoustic radiation in water and shell vibration, the results showed that the partial coating would exhibit considerable impact on the acoustic radiation of the shell. Xiang et al.\(^13\) proposed a novel matrix method for the liquid-filled circular cylindrical shell partially covered with a single-ring constrained damping layer. The forced vibration analysis of the shell with different damping thickness and coverage ratios under the harmonic liquid dynamic pressure was conducted.

Moreover, in terms of partially constrained viscoelastic damping treatment, Masti and Sainsbury\(^14\) explored the optimal damping treatment for the cylindrical shell partially covered with a double-ring standoff-layered viscoelastic coating to achieve a minimal distribution area and low added weight. It was indicated that the partial damping treatment can be more beneficial than using full coverage. Chen and Huang\(^15\) developed an analytical model for vibration analysis of the cylindrical shell partially covered with a single-ring constrained damping layer under the Donnell-Mushtari-Vlasov assumptions, and the effects of the axial coverage length and the thickness of the damping layer on the power dissipation coefficient of the
shell were discussed. The above researches lay a solid foundation for the vibration and damping analysis of the partially covered cylindrical shells. However, in the aspect of partial hard-coating damping treatment, to authors’ knowledge, only Sun et al.\textsuperscript{16} carried out research on the optimal configuration of vibration reduction characteristics of the cantilever plate partially covered with NiCrAlCoY + YSZ hard coating based on an analytical model, while the effects of the coverage ratio, optimum location, and modal loss factor of the hard coating were discussed.

As seen from the literature review, the vibration of a cylindrical shell partially covered with circumferential hard coating has not been studied, especially with the multi-ring hard coating. For the regular arrangement of the appendages (such as the rivets or pipelines) along the axial direction of the shell in aviation power equipment, the scheme of equidistant multi-ring hard coating damping treatment would be a targeted solution with high practical values. As the basic structures in practical engineering, the vibration characteristics of the cylindrical shells have been fully and systematically investigated in many studies,\textsuperscript{17–23} but there is still a lack of systematic and methodic research on the cylindrical shell partially coated with equidistant multi-ring hard coating for vibration reduction. Therefore, the study on vibration suppression performance of the partially covered equidistant multi-ring hard coating damping treatment for the cylindrical shell structure in aviation power equipment is carried out in this paper.

\section*{Methods}

\subsection*{Model description}

The cylindrical shell partially covered with an equidistant multi-ring hard coating with the clamped boundary condition is investigated in this work. The geometry of the shell in an orthogonal curvilinear coordinate system ($x$, $\theta$, $z$) is shown in Figure 1. $h$, $L$, and $R$ are the radial thickness, axial length, and nominal radius, respectively. The clamped boundary at the bottom of the shell is simulated by the artificial spring technology with an infinite stiffness $k$ for the translational and rotational springs.

\subsection*{Unified thickness formula for an arbitrary number of coating rings}

To achieve a unified semi-analytical solution for the free vibration problem of the cylindrical shell partially covered with equidistant multi-ring hard coating, a specially designed continuous rectangular pulse function is presented to describe the local thickness variation under arbitrary coating proportion and an arbitrary number of coating rings, defined by

$$T_c(x) = T_{c0} \left\{ 1 - \sqrt{[T_{c2}(x) - T_{c1}(x)]^2} \right\}$$  \hspace{1cm} (1)
where

\[
T_{c1}(x) = \frac{1}{1 + \exp\{a \cdot \sin[2\psi \pi (x + b)/(L + c)]\}}
\]

(2)

\[
T_{c2}(x) = \frac{1}{1 + \exp\{a \cdot \sin[2\psi \pi (x + (1 - \phi)L/\psi + b)/(L + c)]\}} , \quad 0 \leq \phi \leq 1
\]

(3)

where \( T_c \) is the general coating thickness. \( T_{c0} \) is the initial thickness. \( \psi \) is the ring number. \( \phi \) is the coating proportion. \( a = 1 \times 10^{-30}, b = 1 \times 10^{-6}, \) and \( c = 1 \times 10^{-5} \) are the introduced constants to meet various special situations. In fact, equation (1) can be regarded as a continuous rectangular pulse function with the axial coordinate, coating proportion, and ring number as its variables, which can precisely describe the thickness change of equidistant multi-ring hard coating with arbitrary coating proportion and ring number along the axial direction of the shell. Taking \( \psi = 4 \) and \( \phi = 0.5 \) as well as \( \psi = 12 \) and \( \phi = 0.2 \) for example, the coating thickness under different ring numbers and coating proportions is shown in Figure 2.

**Admissible displacements**

The axial, circumferential and radial displacements \( u, v, \) and \( w \) at the middle surface of the multi-ring hard-coating cylindrical shell can be expressed by\textsuperscript{24,25}

\[
\begin{align*}
    u &= U(x) \cos(n\theta) \sin(\omega t) \\
    v &= V(x) \sin(n\theta) \sin(\omega t) \\
    w &= W(x) \cos(n\theta) \sin(\omega t)
\end{align*}
\]

(4)
where \( n \) is the modal circumferential wave number, \( \omega \) is the angular frequency, and \( t \) is the time. \( U(x) \), \( V(x) \), and \( W(x) \) represent the axial modal functions of the shell, defined by

\[
U(x) = \sum_{m=1}^{N} \bar{U}_m J_m(x)
\]

\[
V(x) = \sum_{m=1}^{N} \bar{V}_m J_m(x)
\]

\[
W(x) = \sum_{m=1}^{N} \bar{W}_m J_m(x)
\]

where \( \bar{U}_m \), \( \bar{V}_m \), and \( \bar{W}_m \) refer to the undetermined weighting coefficients. \( J_m(x) \) is the admissible displacement function. Here \( m \) indicates the axial half-wave number. Here the Jacobi polynomials\(^{26,27} \) are introduced as the admissible displacement function of the partially covered shell, the recursion formula of the Jacobi polynomials is given by

\[
J_m^{(\alpha, \beta)}(\chi) = \frac{\gamma_{2m-1} \left( \gamma_{2m} \gamma_{2m-2} \chi + \alpha^2 - \beta^2 \right)}{2m\gamma_m \gamma_{2m-2}} J_{m-1}^{(\alpha, \beta)}(\chi)
\]

\[-\frac{2(\alpha + m - 1)(\beta + m - 1)\gamma_{2m} J_{m-2}^{(\alpha, \beta)}(\chi)}{2m\gamma_m \gamma_{2m-2}}\]

where

\[
J_0^{(\alpha, \beta)}(\chi) = 1
\]

\[
J_1^{(\alpha, \beta)}(\chi) = \frac{\gamma_2}{2} \chi + \frac{\alpha - \beta}{2}
\]

\[
\gamma_m = \alpha + \beta + m
\]
The Jacobi polynomials with \(\alpha > -1\) and \(\beta > -1\) are orthogonal on the interval \([-1, 1]\) with respect to the weight function \(w(\chi) = (1 - \chi)^{\alpha}(1 + \chi)^{\beta}\). As the axial coordinate of the shell \(x \in [0, L]\), the linear coordinate transformation from \(\chi\) to \(x\) should be made for the original Jacobi polynomial, that is, \(\chi = 2x/L - 1\). By setting \(\alpha\) and \(\beta\) with different values, the Jacobi polynomials can be directly changed into other types of orthogonal polynomials, such as the Gegenbauer, Legendre, and Chebyshev polynomials, which would make the admissible displacement function of the shell more generalized to achieve a unified solution.

**Energy expressions**

Based on Love’s first approximation theory,\(^{28}\) the strains at any point of the partially covered thin cylindrical shell with \(h/L \ll 1\) can be defined as:

\[
e_x = e_x^0 + z\kappa_x, \quad e_\theta = e_\theta^0 + z\kappa_\theta, \quad e_{x\theta} = e_{x\theta}^0 + z\kappa_{x\theta}
\]  

(8)

where \(e_x^0, e_\theta^0, e_{x\theta}^0\) and \(\kappa_x, \kappa_\theta, \kappa_{x\theta}\) are the strains and bending strains at the middle surface, respectively. The corresponding strain vector is given by

\[
\varphi = \begin{pmatrix}
e_x^0 \\
e_\theta^0 \\
e_{x\theta}^0 \\
\kappa_x \\
\kappa_\theta \\
\kappa_{x\theta}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial \theta} + w/R \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta}/R \\
-\partial^2 w/\partial x^2 \\
\frac{\partial v}{\partial \theta} - \partial^2 w/\partial \theta^2 / R^2 \\
\frac{\partial v}{\partial x} - 2\partial^2 w/\partial x \partial \theta / R
\end{pmatrix}
\]

(9)

Due to the unified treatment for the thickness of the multi-ring hard coating, the radius of the middle surface \(R\) of the shell is also transformed into the function of the axial coordinate, that is, varies with the thickness of the hard coating, given by\(^{16}\)

\[
R(x) = R_0 + \frac{E_s^x T_s^2 + 2E_s^x T_c(x) + E_c^x T_c^2(x)}{2\left[E_s^x T_s + E_c^x T_c(x)\right]}
\]

(10)

where \(R_0\) denotes the radius of the inner surface. For a cylindrical shell in the state of plane stress, the stress vector can be expressed as
\[ \sigma = \begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{bmatrix} = \begin{bmatrix} Q_{11}^x & Q_{12}^x & 0 \\ Q_{21}^x & Q_{22}^x & 0 \\ 0 & 0 & Q_{66}^x \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \end{bmatrix} \]

\[ = \begin{bmatrix} Q_{11}^x & Q_{12}^x & 0 & zQ_{11}^x & zQ_{12}^x & 0 \\ Q_{21}^x & Q_{22}^x & 0 & zQ_{21}^x & zQ_{22}^x & 0 \\ 0 & 0 & Q_{66}^x & 0 & 0 & zQ_{66}^x \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \end{bmatrix} \]

where \( Q_{ij}^x \) refers to the reduced stiffness of the substrate (\( x = s \)) and hard coating (\( x = c \)), which can be defined by

\[ Q_{11}^x = Q_{22}^x = \frac{E_x}{1 - \mu_x^2}, \quad Q_{12}^x = Q_{21}^x = \frac{E_x \mu_x}{1 - \mu_x^2}, \quad Q_{66}^x = \frac{E_x}{2(1 + \mu_x)} \]

where \( \mu_x \) is the Poisson’s ratio. \( \bar{E}_x \) is the complex-valued Young’s moduli, given by

\[ \bar{E}_x = E_x^\xi + iE_x^\eta \]

Here \( i = \sqrt{-1} \). \( E_x^\xi \) and \( E_x^\eta \) are the storage modulus and loss modulus, respectively. Then the internal force and moment vectors of the shell can be obtained as

\[ \{ N \ M \}^T = \int_{z_0}^{z_2} \{ \sigma \ z\sigma \}^T dz = \partial \varphi \]

\[ = \int_{z_0}^{z_2} \begin{bmatrix} Q_{11}^x & Q_{12}^x & 0 & zQ_{11}^x & zQ_{12}^x & 0 \\ Q_{21}^x & Q_{22}^x & 0 & zQ_{21}^x & zQ_{22}^x & 0 \\ 0 & 0 & Q_{66}^x & 0 & 0 & zQ_{66}^x \\ zQ_{11}^x & zQ_{12}^x & 0 & z^2Q_{11}^x & z^2Q_{12}^x & 0 \\ zQ_{21}^x & zQ_{22}^x & 0 & z^2Q_{21}^x & z^2Q_{22}^x & 0 \\ 0 & 0 & zQ_{66}^x & 0 & 0 & z^2Q_{66}^x \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \end{bmatrix} dz \]

To facilitate subsequent formula derivation, the so-called extensional, coupling and bending stiffness \( A_{ij}, B_{ij}, \) and \( D_{ij} \) are defined, respectively, by

\[ A_{ij} = \int_{z_0}^{z_2} Q_{ij}^x dz, \quad B_{ij} = \int_{z_0}^{z_2} zQ_{ij}^x dz, \quad D_{ij} = \int_{z_0}^{z_2} z^2Q_{ij}^x dz \]

where \( z_0 \) and \( z_2 \) represent the radial coordinate values of the inner surface and the outer surface of the shell, respectively. Then the strain energy \( U_e \) and the kinetic energy \( T \) of the covered and uncovered regions of the shell can be uniformly written as follows.
\[
U_e = \frac{1}{2} \int_{0}^{L} \int_{0}^{2\pi} \int_{z_0}^{z_2} (\sigma_x \varepsilon_x + \sigma_{\theta z} \varepsilon_{\theta z} + \sigma_{xz} \varepsilon_{xz}) R d\theta dz = \frac{1}{2} \int_{0}^{L} \int_{0}^{2\pi} \Phi^T \partial \Phi R d\theta dz \tag{16}
\]

\[
T = \frac{1}{2} \int_{0}^{L} \int_{0}^{2\pi} \int_{z_0}^{z_2} \rho_x \left[ (\partial u/\partial t)^2 + (\partial v/\partial t)^2 + (\partial w/\partial t)^2 \right] R d\theta dz \tag{17}
\]

Similarly, the potential energy introduced by the artificial springs applied to the bottom end of the shell can be expressed as the following form

\[
U_b = \frac{1}{2} k \left\{ \left[ u^2 + v^2 + w^2 + (\partial w/\partial x)^2 \right] R \right\}_{x = 0} \tag{18}
\]

**Unified solution via Rayleigh-Ritz procedure**

To determine the weighting coefficients and the displacement components in equations (4) and (5), the Rayleigh-Ritz method is employed to derive the motion equations of the shell. Substituting the detailed expressions of \(u, v, \) and \(w\) in equation (4) into equations (16) and (17), yields

\[
U = \frac{1}{2} \pi \sin (\omega t)^2 \int_{0}^{L} \frac{1}{R^3} \left[ R^4 U_x^2 A_{11} + 2R^2(V_n + W)U_x A_{12} 
+ R^2(V_n + W)^2 A_{22} + R^2(RV_x - U_n)^2 A_{66} - 2R^4 U_x W_{xx} B_{11} 
+ 2R^2(W_n^2 U_x - RVW_{xx} - RVnW_{xx} + VnU_x)B_{12} + 2Rn(W_n + V)(V_n + W)B_{22} 
+ 2R^2(2W_n + V)(RV_x - Un)B_{66} + R^4 W_{xx}^2 D_{11} - 2nR^2 W_{xx}(W_n + V)D_{12} 
+ n^2(W_n + V)^2 D_{22} + R^2(2W_n + V)^2 D_{66} \right] dx \tag{19}
\]

\[
T = \frac{1}{2} \pi \rho \omega^2 \cos (\omega t)^2 \int_{0}^{L} \left( U^2 + V^2 + W^2 \right) R dx \tag{20}
\]

where

\[
\bar{\rho} = \int_{z_0}^{z_2} \rho_x dz, \quad \alpha = c, s \tag{21}
\]

The potential energy introduced by the clamped boundary with artificial springs is given by

\[
U_b = \frac{1}{2} \pi k \sin (\omega t)^2 \left\{ R \left[ U^2 + V^2 + W^2 + W_x^2 \right] \right\}_{x = 0} \tag{22}
\]

Based on the above strain energy, kinetic energy, and potential energy, the Lagrangian energy function can be defined as the following
\[ L = U_i^{\max} + U_b^{\max} - T^{\max} \] (23)

where the superscript “max” refers to the maximum value. By differentiating the above Lagrangian energy function with respect to \( \bar{U}_s, \bar{V}_s, \) and \( \bar{W}_s, \) and set it to zero, respectively, then the equations of motion of the shell can be assembled into the following matrix form

\[ (\bar{K}_n + K_b - \omega_n^2 M) \chi_n = 0 \] (24)

where \( \bar{K} \) is the \( 3N \times 3N \) complex-valued stiffness matrix involving the real part \( K \) and the imaginary part \( D, \) defined by

\[
\bar{K} = \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\] (25)

Similarly, \( K_b \) is the \( 3N \times 3N \) stiffness matrix of the clamped boundary with artificial springs, \( M \) is the \( 3N \times 3N \) mass matrix of the shell, and \( \chi \) is the \( 3N \times 1 \) Ritz vector, given by

\[
K_b = \begin{bmatrix}
K_{b11} & & \\
& K_{b22} & \\
& & K_{b33}
\end{bmatrix}
\] (26)

\[
M = \begin{bmatrix}
M_{11} & & \\
& M_{22} & \\
& & M_{33}
\end{bmatrix}
\] (27)

\[ \chi = \{U_1, U_2, \cdots, U_N, \bar{V}_1, \bar{V}_2, \cdots, \bar{V}_N, \bar{W}_1, \bar{W}_2, \cdots, \bar{W}_N\}^T \] (28)

The detailed expressions of \( \bar{K}, M, \) and \( K_b \) are given in Appendices A–C, respectively. It is worth noting that due to the unified analytical rectangular pulse function is introduced to describe the local thickness variation of the partially covered shell, the values of more complex \( \bar{K}, M, \) and \( K_b \) are recommended to be evaluated by the numerical integral method instead of the analytical integral method.

By solving the eigenvalue problem in equation (24), the natural frequencies and the corresponding eigenvectors of the shell can be conveniently acquired. The natural frequency and the standardized eigenvector with respect to the circumferential wave number \( n \) are calculated, respectively, by

\[ f_n = \text{Re}(\omega_n)/2\pi \] (29)

\[
\hat{x}_n = \chi_n \sqrt{\chi_n^T M \chi_n}
\] (30)
Simultaneously, the modal loss factor of the shell for the certain circumferential wave number \( n \) can be obtained by using the real and imaginary part of the eigenvalues, expressed by

\[
\eta_n = \frac{\text{Im}(\omega_n)}{\text{Re}(\omega_n)}
\]  

(31)

**Results and discussion**

**Convergence analysis**

As the truncation terms of the Jacobi-type admissible function and the stiffness of the introduced artificial springs are directly related to the accuracy and efficiency of the semi-analytical model, the convergence of the natural frequency versus the truncation number \( N \) and the stiffness coefficient \( k \) should be carefully examined. Although the truncation terms of orthogonal polynomials for clamped-free shells have been investigated in many studies, the truncation number for achieving accurate results would be also slightly different due to the model differences including the geometric dimension, nonlinear factors, types of orthogonal polynomials, etc. Directly using the values provided in the above literature may cause certain accuracy problems. Moreover, since the general coating thickness \( T_c \) is described by a unified continuous rectangular pulse function with arbitrary coating proportion and ring number, the expressions of the strain energy, kinetic energy, and potential energy become extremely complicated. This makes the truncation number of the employed Jacobi orthogonal polynomials more important to the solution efficiency. Thus, based on optimal consideration of the computational accuracy and efficiency, the convergence analysis of truncation number on the natural frequency of the clamped-free cylindrical shell is still needed to achieve accurate results for the following analysis. Taking the natural frequencies under the circumferential wave number \( n = 1, 2, 3, 4, \) and 5 for example, the convergence analysis results for the truncation number \( N (\alpha = \beta = -1/2) \) and the stiffness coefficient \( k \) are shown in Figures 3 and 4, respectively.

From Figure 3 we can find that the natural frequencies increase in a nonlinear manner with the increasing stiffness coefficient ranging from \( 10^3 \) to \( 10^{14} \) N/m². Especially, in the range of \( 10^6 \)–\( 10^{11} \) N/m², the natural frequencies increase the fastest and then gradually converge to the stable values when the stiffness coefficient \( k \) is greater than or equal to \( 10^{13} \) N/m². The results indicate that the stiffness coefficient \( k = 10^{13} \) N/m² is large enough to simulate the clamped boundary condition of the shell. It is worth noting that the stiffness coefficient \( k = 10^{13} \) N/m² is only valid for the present model since the value of the stiffness coefficient for the clamped boundary condition also depends on the stiffness of the shell and other factors. Additionally, to evaluate the influence of the number of truncation terms of the admissible function on the natural frequency, the relative difference \( d_f \) is defined by taking the result with \( N = 9 \) as the benchmark. It can be seen from Figure 4 that the relative differences of the natural frequency decrease rapidly with the increase of the truncation number until \( N \) is up to 7. Based on synthesis consideration of
the computational accuracy and efficiency, the truncation number $N = 8$ for the Jacobi-type admissible function is finally employed in this study. The excellent convergence performance with a small truncation number is one of the advantages of this semi-analytical approach.

**Figure 3.** Convergence of the natural frequency versus the stiffness coefficient $k$.

**Figure 4.** Convergence of the natural frequency versus the truncation number $N$. 
Model validation

Since the research on free vibration of the thin cylindrical shell partially covered with equidistant multi-ring hard coating is seldom reported in existing literature, the natural frequencies and corresponding modal shapes of the shell calculated by the present semi-analytical method are compared with the numerical results from the finite element method (FEM) based on ABAQUS v6.13 for validation. The partially covered hard-coating cylindrical shell is modeled with a 4-node S4R shell element. The total number of nodes and elements are 86,524 and 85,632, respectively. In the FEM model, the cylindrical shell is split into eight equally-sized circular rings along the axial direction, while the composite and single-layer shell sections are respectively employed to simulate the coated and uncoated regions of the cylindrical shell partially covered by equidistant four-ring hard coating, as shown in Figure 5.

The adopted geometric parameters and material property of the shell are presented as below in detail: \( L = 0.095 \text{ m}, \ R_0 = 0.142 \text{ m}, \ T_s = 0.002 \text{ m}, \ T_{c0} = 0.00031 \text{ m}, \ \rho_s = 7850 \text{ kg/m}^3, \ \rho_c = 5560 \text{ kg/m}^3, \ \mu_s = \mu_c = 0.3, \ E_s = 2.1 \times 10^{11} \text{ Pa}, \ E_c = 5.45 \times 10^{10} \text{ Pa}, \ E_{sh} = 1.47 \times 10^9 \text{ Pa}, \ E_{ch} = 1.1554 \times 10^9 \text{ Pa}, \ \psi = 4, \) and \( \phi = 0.5. \) The natural frequencies predicted by the FEM and present semi-analytical method under the same conditions are listed in Table 1. From the table, we can see that the semi-analytical and FEM natural frequencies are in good agreement, and the relative differences are within 0.65%, which illustrates the validity of this semi-analytical model. Moreover, the comparison of the modal shapes predicted by the FEM and the present semi-analytical method in Figure 6 further confirms the rationality and validity of the model.

Effect analysis of the ring number

As a main coating parameter, the ring number would have a non-negligible impact on the vibration of the partially covered cylindrical shell. Therefore, further analysis should be carried out to better understand the effect of the ring number on the damping performance of this partial multi-ring hard coating treatment. For

![Figure 5. FEM modeling of the cylindrical shell partially covered by equidistant four-ring hard coating.](image)
different circumferential wave numbers, the effect of the ring number of hard coating with the modal loss factor as research object is investigated, as shown in Figure 7.

From Figure 7 we can see that the modal loss factor generally decreases with the increase of the ring number ranging from 1 to 20 when the circumferential wave number \( n \leq 4 \), while it increases with the increasing ring number when the

\[
\frac{f_S}{f_F} \leq 20
\]
circumferential wave number $n \geq 5$. Except the ring number $\psi = 1$, the modal loss factors with other ring numbers are all increased with the increase of the circumferential wave number. The results show that the increase of the ring number is not always beneficial for vibration reduction of the thin cylindrical shell, and the law of influence mainly depends on the circumferential wave number of the shell. The increased ring number tends more to exhibit an obvious incremental effect under larger circumferential wave numbers.

Furthermore, the effect of the ring number on the modal loss factor of each circumferential wave number is more clearly presented in Figure 8.

As can be seen from Figure 8, the variation of the ring number has little effect on the modal loss factor of the circumferential wave number $n = 1$. Meanwhile, the modal loss factor is decreased slightly by increasing the ring number when the circumferential wave number $n = 2, 3, \text{and} 4$. When the circumferential wave number $n > 5$, the modal loss factor is increased significantly at first and then gradually tends to stabilize with the increasing ring number. As a transition, the modal loss factor increases firstly and then gradually decreases to some degree when the circumferential wave number $n = 5$. The results again demonstrate the greater effect of the ring number on the larger circumferential wave number of the shell. Given this, the ring number $\psi = 20$ is good enough to achieve a reasonable and comprehensive partial multi-ring hard coating damping treatment.

**Effect analysis of the coating proportion**

Similar to the ring number, as an important parameter of the partial multi-ring hard coating damping treatment, the coating proportion also has a direct influence
on the modal loss factor of the shell. Hence the effect of the coating proportion on the modal loss factor under different circumferential wave numbers is investigated, as shown in Figure 9. From Figure 9, it can be found that the modal loss factor generally increases with the increase of the coating proportion ranging from 0.0 to 1.0 for all the circumferential wave numbers. When the coating proportion is
relatively small, the impacts of the coating proportion on the modal loss factors under different circumferential wave numbers are basically the same. With the increase of the coating proportion, a larger circumferential wave number tends to have a greater modal loss factor, and this phenomenon demonstrates that the increased coating proportion is more beneficial to suppress the vibration of the thin cylindrical shell with a large circumferential wave number.

Moreover, the effect of the coating proportion on the modal loss factor of each circumferential wave number is illustrated in Figure 10. It can be seen more directly and clearly that the modal loss factor increases roughly linear with the coating proportion for all the circumferential wave numbers. Meanwhile, the results also indicate that the modal loss factor is increased with the circumferential wave number, and the greater the number of circumferential waves, the greater the rate of change. Compared with the ring number, the effect of the coating proportion on the modal loss factor is relatively simple and the corresponding law of influence is only monotonic increasing. However, due to the existence of appendages (such as the rivets or pipelines) and the restrictions of structural weight, there often exists a limit for the coating proportion in practice. Hence the maximum coating proportion should be determined according to the actual working conditions of the shell.

**Conclusions**

In this paper, a continuous rectangular pulse function designed for arbitrary equidistant partial multi-ring hard coating treatment of the cylindrical shell is presented. By combining the rectangular pulse function, the generalized Jacobi polynomials and the Rayleigh-Ritz method, a unified Jacobi-Ritz semi-analytical solution
procedure is established for the free vibration and damping analysis of the shell. The convergence analysis and model validation demonstrate the feasibility, accuracy, and efficiency of the developed semi-analytical formula. The effects of the ring number and coating proportion on the vibration and damping characteristics are studied. The following conclusions are obtained:

(1) The stiffness coefficient \( k = 10^{13} \) N/m\(^2\) and the truncation number \( N = 8 \) are large enough to simulate the clamped boundary condition and to achieve the accurate and efficient solution of the vibration of the shell, respectively.

(2) The increase of the ring number is not always beneficial for vibration reduction of the shell, and the law of influence mainly depends on the circumferential wave number. The increased ring number tends more to exhibit an obvious incremental effect under a larger circumferential wave number. In particular, the ring number \( \psi = 20 \) is good enough to achieve a reasonable and comprehensive partial multi-ring hard coating damping treatment.

(3) The modal loss factor increases roughly linear with the coating proportion ranging from 0.0 to 1.0 for all the circumferential wave numbers, and the greater the circumferential wave number, the greater the modal loss factor and its rate of change. The increased coating proportion is more beneficial to suppress the vibration of the thin cylindrical shell with a large circumferential wave number, the results of which may be served as a theoretical basis for engineering application.

**Declaration of conflicting interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This project is supported by Youth Foundation of Educational Department of Liaoning Province, China (Grant No.2020LNQN16), the Youth Foundation of University of Science and Technology Liaoning, China (No.2019QN06).

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Appendix A: Detailed expressions of the complex-valued stiffness matrix $\tilde{K}$

The detailed expressions of the stiffness matrixes in equation (25) are given as follows

$$K_{11}(i,j) = \pi \int_0^L \left[ RA_{11}J_i'(x)J_j'(x) + n^2 A_{66} J_i(x) J_j(x) / R \right] dx$$ (A.1)

$$K_{12}(i,j) = n\pi \int_0^L \left[ (A_{12} + B_{12}/R) J_i'(x) J_j'(x) - (A_{66} + B_{66}/R) J_i(x) J_j'(x) \right] dx$$ (A.2)

$$K_{13}(i,j) = \pi \int_0^L \left[ (A_{12} + n^2 B_{12}/R) J_i'(x) J_j'(x) - RB_{11} J_i''(x) - 2n^2 B_{66} J_i(x) J_j'(x) / R \right] dx$$ (A.3)
\( \mathbf{K}_{21}(i,j) = n\pi \int_0^L \left[ (A_{12} + B_{12}/R)J_i(x)J'_j(x) - (A_{66} + B_{66}/R)J'_i(x)J_j(x) \right] dx \) \hspace{1cm} (A.4)

\( \mathbf{K}_{22}(i,j) = \pi \int_0^L \left[ (RA_{66} + 2B_{66} + D_{66}/R)J'_i(x)J'_j(x) + n^2(A_{22}/R + 2B_{22}/R^2 + D_{22}/R^3)J_i(x)J_j(x) \right] dx \) \hspace{1cm} (A.5)

\( \mathbf{K}_{23}(i,j) = n\pi \int_0^L \left\{ \left[ n^2D_{22}/R^3 + A_{22}/R + (n^2 + 1)B_{22}/R^2 \right]J_i(x)J_j(x) - (B_{12} + D_{12}/R)J''_i(x)J'_j(x) + 2(B_{66} + D_{66}/R)J'_i(x)J_j(x) \right\} dx \) \hspace{1cm} (A.6)

\( \mathbf{K}_{31}(i,j) = \pi \int_0^L \left[ (A_{12} + n^2B_{12}/R)J_i(x)J'_j(x) - RB_{11}J''_i(x)J'_j(x) - 2n^2B_{66}/RJ'_i(x)J_j(x) \right] dx \) \hspace{1cm} (A.7)

\( \mathbf{K}_{32}(i,j) = n\pi \int_0^L \left\{ \left[ n^2D_{22}/R^3 + A_{22}/R + (n^2 + 1)B_{22}/R^2 \right]J_i(x)J_j(x) - (B_{12} + D_{12}/R)J''_i(x)J_j(x) + 2(B_{66} + D_{66}/R)J'_i(x)J'_j(x) \right\} dx \) \hspace{1cm} (A.8)

\( \mathbf{K}_{33}(i,j) = \pi \int_0^L \left[ RD_{11}J''_i(x)J''_j(x) + (A_{22}/R + 2n^2B_{22}/R^2 + n^4D_{22}/R^3)J_i(x)J_j(x) - (B_{12} + n^2D_{12}/R)J''_i(x)J_j(x) - (B_{12} + n^2D_{12}/R)J'_i(x)J''_j(x) + 4n^2D_{66}/RJ'_i(x)J'_j(x) \right] dx \) \hspace{1cm} (A.9)

**Appendix B: Detailed expressions of the mass matrix \( M \)**

The detailed expressions of the mass matrix in equation (27) are given as follows

\[
\mathbf{M}_{11}(i,j) = \pi \rho \int_0^L R J_i(x) J_j(x) \, dx \tag{B.1}
\]

\[
\mathbf{M}_{22}(i,j) = \pi \rho \int_0^L R J_i(x) J_j(x) \, dx \tag{B.2}
\]

\[
\mathbf{M}_{33}(i,j) = \pi \rho \int_0^L R J_i(x) J_j(x) \, dx \tag{B.3}
\]

**Appendix C: Detailed expressions of the spring stiffness matrix \( K_b \)**

The detailed expressions of the stiffness matrix of the clamped boundary with artificial springs in equation (26) are given as follows
\[ K^{11}_{b}(i,j) = \pi kR(0)J_i(0)J_j(0) \]  \hspace{1cm} (C.1)

\[ K^{22}_{b}(i,j) = \pi kR(0)J_i(0)J_j(0) \]  \hspace{1cm} (C.2)

\[ K^{33}_{b}(i,j) = \pi kR(0) \left[ J_i(0)J_j(0) + J'_{i}(0)J'_{j}(0) \right] \]  \hspace{1cm} (C.3)

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