APS INDEX THEOREM FOR EVEN-DIMENSIONAL MANIFOLDS WITH NON-COMPACT BOUNDARY

MAXIM BRAVERMAN† AND PENGSHUAI SHI

Abstract. We study the index of the APS boundary value problem for a strongly Callias-type operator \(D\) on a complete Riemannian manifold \(M\). We use this index to define the relative \(\eta\)-invariant \(\eta(A_1, A_0)\) of two strongly Callias-type operators, which are equal outside of a compact set. Even though in our situation the \(\eta\)-invariants of \(A_1\) and \(A_0\) are not defined, the relative \(\eta\)-invariant behaves as if it were the difference \(\eta(A_1) - \eta(A_0)\). We also define the spectral flow of a family of such operators and use it compute the variation of the relative \(\eta\)-invariant.

1. Introduction

In [12] we studied the index of the Atiyah–Patodi–Singer (APS) boundary value problem for a strongly Callias-type operator on a complete odd-dimensional manifold with non-compact boundary. We used this index to define the relative \(\eta\)-invariant \(\eta(A_1, A_0)\) of two strongly Callias-type operators on even-dimensional manifolds, assuming that \(A_0\) and \(A_1\) coincide outside of a compact set.

In this paper we discuss an even-dimensional analogue of [12]. Many parts of the paper are parallel to the discussion in [12]. However, there are two important differences. First, the Atiyah–Singer integrand was, of course, equal to 0 in [12], which simplified many formulas. In particular, the relative \(\eta\)-invariant was an integer. As opposed to it, in the current paper the Atiyah–Singer integrand plays an important role and the relative \(\eta\)-invariant is a real number. More significantly, the proof of the main result in [12] was based on the application of the Callias index theorem, [2][14]. This theorem is not available in our current setting. Consequently, a completely different proof is proposed in Section 3.

We now briefly describe our main results.

1.1. Strongly Callias-type operators. A Callias-type operator on a complete Riemannian manifold is an operator of the form \(D = D + \Psi\) where \(D\) is a Dirac operator and \(\Psi\) is a self-adjoint potential which anticommutes with the Clifford multiplication and satisfies certain growth conditions at infinity. In this paper we impose slightly stronger growth conditions on \(\Psi\) and refer to the obtained operator \(D\) as a strongly Callias-type operator. Our conditions on the growth of \(\Psi\) guarantee that the spectrum of \(D\) is discrete.

The Callias-type index theorem, proven in different forms in [2][7][13][15], computes the index of a Callias-type operator on a complete odd-dimensional manifold as the index of a certain operator induced by \(D\) on a compact hypersurface. Several generalizations and applications of the Callias-type index theorem were obtained recently in [8][11][16][21][22][27].

2010 Mathematics Subject Classification. 58J28, 58J30, 58J32, 19K56.

Key words and phrases. Callias, Atiyah–Patodi–Singer, index, eta, boundary value problem, relative eta.

†Partially supported by the Simons Foundation collaboration grant #G00005104.
P. Shi, proved a version of the Callias-type index theorem for the APS boundary value problem for Callias-type operators on a complete odd-dimensional manifold with compact boundary.

Fox and Haskell studied Callias-type operators on manifolds with non-compact boundary. Under rather strong conditions on the geometry of the manifold and the operator $D$ they proved a version of the Atiyah–Patodi–Singer index theorem.

In we studied the index of the APS boundary value problem on an arbitrary complete odd-dimensional manifold with non-compact boundary and defined a relative $\eta$-invariant for two strongly Callias-type operators which “coincide at infinity”. In the current paper, we obtain an even-dimensional analogue of . This version is closer in form to the original Atiyah–Patadi–Singer formula. While the results of can be considered as a generalization of the Callias-type index theorem to manifolds with boundary, the results of the current paper should be viewed as a generalization of the Atiyah–Patadi–Singer theorem to non-compact manifolds. The proof of our main result is quite different from the proof in since in even-dimensional case we can not use the Callias index theorem.

1.2. An almost compact essential support. A manifold $M$, whose boundary is a disjoint union of two complete manifolds $N_0$ and $N_1$, is called essentially cylindrical if outside of a compact set it is isometric to a cylinder $[0, \varepsilon] \times N'$, where $N'$ is a non-compact manifold. It follows that manifolds $N_0$ and $N_1$ are isometric outside of a compact set.

We say that an essentially cylindrical manifold $M_1$, which contains $\partial M$, is an almost compact essential support of $D$ if the restriction of $D^*D$ to $M \setminus M_1$ is strictly positive and the restriction of $D$ to the cylinder $[0, \varepsilon] \times N'$ is a product, cf. Definition . Every strongly Callias-type operator on $M$ which is a product near $\partial M$ has an almost compact essential support.

Theorem states that the index of the APS boundary value problem for a strongly Callias-type operator $D$ on a complete manifold $M$ is equal to the index of the APS boundary value problem of the restriction of $D$ to its almost compact essential support $M_1$.

1.3. Index on an essentially cylindrical manifold. Let $M$ be an essentially cylindrical manifold and let $D$ be a strongly Callias-type operator on $M$, whose restriction to the cylinder $[0, \varepsilon] \times N'$ is a product. Suppose $\partial M = N_0 \sqcup N_1$ and denote the restrictions of $D$ to $N_0$ and $N_1$ by $A_0$ and $-A_1$ respectively (the sign convention means that we think of $N_0$ as the “left boundary” and of $N_1$ as the “right boundary” of $M$). Let $D_B$ denote the operator $D$ with APS boundary conditions.

Let $\alpha_{AS}(D)$ denote the Atiyah–Singer integrand of $D$. This is a differential form on $M$ which depends on the geometry of the manifold and the bundle. Since all structures are product outside of the compact set $K$, this form vanishes outside of $K$. Hence, $\int_M \alpha_{AS}(D)$ is well-defined and finite. Our main result here is that

$$\text{ind } D_B = \int_M \alpha_{AS}(D)$$

depends only on the operators $A_0$ and $A_1$ and does not depend on the interior of the manifold $M$ and the restriction of $D$ to the interior of $M$, cf. Theorem.

1.4. The relative $\eta$-invariant. Suppose now that $A_0$ and $A_1$ are self-adjoint strongly Callias-type operators on complete manifolds $N_0$ and $N_1$. An almost compact cobordism between $A_0$ and $A_1$ is a pair $(M, D)$ where $M$ is an essentially cylindrical manifold with $\partial M = N_0 \sqcup N_1$ and
$D$ is a strongly Callias-type operator on $M$, whose restriction to the cylindrical part of $M$ is a product and such that the restrictions of $D$ to $N_0$ and $N_1$ are equal to $A_0$ and $-A_1$ respectively. We say that $A_0$ and $A_1$ are cobordant if there exists an almost compact cobordism between them. In particular, this implies that $A_0$ and $A_1$ are equal outside of a compact set.

Let $D$ be an almost compact cobordism between $A_0$ and $A_1$. Let $B_0$ and $B_1$ be the APS boundary conditions for $D$ at $N_0$ and $N_1$ respectively. Let $\text{ind } D_{B_0 \oplus B_1}$ denote the index of the APS boundary value problem for $D$. We define the relative $\eta$-invariant by the formula

$$\eta(A_1, A_0) = 2 \left( \text{ind } D_{B_0 \oplus B_1} - \int_M \alpha_{\text{AS}}(D) \right) + \dim \ker A_0 + \dim \ker A_1.$$ 

It follows from the result of the previous section, that $\eta(A_1, A_0)$ is independent of the choice of an almost compact cobordism.

If $M$ is a compact manifold, then the Atiyah-Patodi-Singer index theorem [3] implies that $\eta(A_1, A_0) = \eta(A_1) - \eta(A_0)$. In general, for non-compact manifolds, the individual $\eta$-invariants $\eta(A_1)$ and $\eta(A_0)$ might not be defined. However, $\eta(A_1, A_0)$ behaves like it were a difference of two individual $\eta$-invariants. In particular, cf. Propositions 4.9–4.10,

$$\eta(A_1, A_0) = -\eta(A_0, A_1), \quad \eta(A_2, A_0) = \eta(A_2, A_1) + \eta(A_1, A_0).$$

1.5. The spectral flow. Consider a family $\mathcal{A} = \{A^s\}_{0 \leq s \leq 1}$ of self-adjoint strongly Callias-type operators on a complete Riemannian manifold. We assume that there is a compact set $K \subset M$ such that the restriction of $A^s$ to $M \setminus K$ is independent of $s$. Since the spectrum of $A^s$ is discrete for all $s$, the spectral flow can be defined in a more or less usual way. By Theorem 5.10 if $A_0$ is a self-adjoint strongly Callias-type operator which is cobordant to $A^0$ (and hence, to all $A^s$), then

$$\eta(A^1, A_0) - \eta(A^0, A_0) - \int_0^1 \frac{d}{ds} \eta(A^s, A_0) \, ds = 2 \text{sf}(\mathcal{A}).$$

(1.2)

The derivative $\frac{d}{ds} \eta(A^s, A_0)$ can be computed as an integral of the transgression form — a differential form canonically constructed from the symbol of $A^s$ and its derivative with respect to $s$. Thus (1.2) expresses the change of the relative $\eta$-invariant as a sum of $2 \text{sf}(\mathcal{A})$ and a local differential geometric expression.

2. Boundary value problems for Callias-type operators

In this section we recall some results about boundary value problems for Callias-type operators on manifolds with non-compact boundary, [12], keeping in mind applications to even-dimensional case. The operators considered here are slightly more general than those discussed in [12], but all the definitions and most of the properties of the boundary value problems remain the same.

2.1. Self-adjoint strongly Callias-type operators. Let $M$ be a complete Riemannian manifold (possibly with boundary) and let $E \to M$ be a Dirac bundle over $M$, [23, Definition II.5.2]. In particular, $E$ is a Hermitian bundle endowed with a Clifford multiplication $c : T^* M \to \text{End}(E)$ and a compatible Hermitian connection $\nabla^E$. Let $D : C^\infty(M, E) \to C^\infty(M, E)$ be the Dirac operator defined by the connection $\nabla^E$. Let $\Psi \in \text{End}(E)$ be a self-adjoint bundle map (called a Callias potential). Then

$$D := D + \Psi$$

is a formally self-adjoint Dirac-type operator on $E$ and

$$D^2 = D^2 + \Psi^2 + [D, \Psi]_+, \quad (2.1)$$
where \([D, \Psi]_+ := D \circ \Psi + \Psi \circ D\) is the anticommutator of the operators \(D\) and \(\Psi\).

**Definition 2.2.** We call \(D\) a self-adjoint strongly Callias-type operator if

(i) \([D, \Psi]_+\) is a zeroth order differential operator, i.e. a bundle map;
(ii) for any \(R > 0\), there exists a compact subset \(K_R \subset M\) such that

\[
\Psi^2(x) - \|[D, \Psi]_+(x)\| \geq R
\]

for all \(x \in M \setminus K_R\). In this case, the compact set \(K_R\) is called an \(R\)-essential support of \(D\).

**Remark 2.3.** Condition (i) of Definition [2.2] is equivalent to the condition that \(\Psi\) anticommutes with the Clifford multiplication: \([c(\xi), \Psi]_+ = 0\), for all \(\xi \in T^*M\).

2.4. Graded self-adjoint strongly Callias-type operators. Suppose now that \(E = E^+ \oplus E^-\) is a \(\mathbb{Z}_2\)-graded Dirac bundle such that the Clifford multiplication \(c(\xi)\) is odd and the Clifford connection is even with respect to this grading. Then

\[
D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}
\]

is the \(\mathbb{Z}_2\)-graded Dirac operator, where \(D^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)\) are formally adjoint to each other. Assume that the Callias potential \(\Psi\) has odd grading degree, i.e.,

\[
\Psi = \begin{pmatrix} 0 & \Psi^- \\ \Psi^+ & 0 \end{pmatrix},
\]

where \(\Psi^\pm \in \text{Hom}(E^\pm, E^\mp)\) are adjoint to each other. Then we have

\[
\mathcal{D} = D + \Psi = \begin{pmatrix} 0 & D^- + \Psi^- \\ D^+ + \Psi^+ & 0 \end{pmatrix} =: \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.
\]

**Definition 2.5.** Under the same condition as in Definition [2.2] \(D\) is called a graded self-adjoint strongly Callias-type operator. In this case, we also call \(\mathcal{D}^+\) and \(\mathcal{D}^-\) strongly Callias-type operators. They are formally adjoint to each other. By an \(R\)-essential support of \(\mathcal{D}^\pm\) we understand as an \(R\)-essential support of \(D\).

**Remark 2.6.** When \(M\) is an oriented even-dimensional manifold there is a natural grading of \(E\) induced by the Hodge \(\ast\)-operator. We will consider this situation in the next section.

**Remark 2.7.** Suppose there is a skew-adjoint isomorphism \(\gamma : E^\pm \to E^\mp\), \(\gamma^* = -\gamma\), which anticommutes with multiplication \(c(\xi)\) for all \(\xi \in T^*M\), satisfies \(\gamma^2 = -1\), and is flat with respect to the connection \(\nabla^E\), i.e. \([\nabla^E, \gamma] = 0\). Then \(\xi \mapsto \gamma \circ c(\xi)\) defines a Clifford multiplication of \(T^*M\) on \(E^\pm\) and the corresponding Dirac operator is \(\tilde{D}^\pm = \gamma \circ D^\pm\). Suppose also that \(\gamma\) commutes with \(\Psi\). Then \(\Phi^+ = -i \gamma \circ \Psi^+\) is a self-adjoint endomorphism of \(E^+\). In this situation,

\[
\tilde{D}^+ + i \Phi^+ = \gamma \circ \mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^+)
\]

is a strongly Callias-type operator in the sense of [12, Definition 3.4].
2.8. **Restriction to the boundary.** Assume that the Riemannian metric $g^{M}$ is **product near the boundary**, that is, there exists a neighborhood $U \subset M$ of the boundary which is isometric to the cylinder

$$Z_r := [0, r) \times \partial M. \quad (2.4)$$

In the following we identify $U$ with $Z_r$ and denote by $t$ the coordinate along the axis of $Z_r$. Then the inward unit normal vector to the boundary is given by $\tau = dt$.

Furthermore, we assume that the Dirac bundle $E$ is **product near the boundary**. In other words we assume that the Clifford multiplication $c : T^* M \rightarrow \text{End}(E)$ and the connection $\nabla^E$ have product structure on $Z_r$, cf. [12, §3.7].

Let $D$ be a $\mathbb{Z}_2$-graded Dirac operator. In this situation the restriction of $D$ to $Z_r$ takes the form

$$D = c(\tau)(\partial_t + \hat{A}) = \begin{pmatrix} 0 & c(\tau) \\ c(\tau) & 0 \end{pmatrix} \begin{pmatrix} \partial_t + A & 0 \\ 0 & \partial_t + A^\sharp \end{pmatrix}, \quad (2.5)$$

where

$$A : C^\infty(\partial M, E^+_\partial M) \rightarrow C^\infty(\partial M, E^+_\partial M)$$

and

$$A^\sharp = c(\tau) \circ A \circ c(\tau) : C^\infty(\partial M, E^-_{\partial M}) \rightarrow C^\infty(\partial M, E^-_{\partial M}) \quad (2.6)$$

are formally self-adjoint operators.

**Remark 2.9.** It would be more natural to use the notation $A^+$ and $A^-$ instead of $A$ and $A^\sharp$. But since in the future we only deal with the operator $A : C^\infty(\partial M, E^+_\partial M) \rightarrow C^\infty(\partial M, E^+_\partial M)$ we remove the superscript “+” to simplify the notation.

Let $D = D + \Psi$ be a graded self-adjoint strongly Callias-type operator. Then the restriction of $D$ to $Z_r$ is given by

$$D = c(\tau)(\partial_t + \hat{A}) = \begin{pmatrix} 0 & c(\tau) \\ c(\tau) & 0 \end{pmatrix} \begin{pmatrix} \partial_t + A & 0 \\ 0 & \partial_t + A^\sharp \end{pmatrix}, \quad (2.7)$$

where

$$A := A - c(\tau)\Psi^+ : C^\infty(\partial M, E^+_\partial M) \rightarrow C^\infty(\partial M, E^+_\partial M). \quad (2.8)$$

and $A^\sharp = A^\sharp - c(\tau)\Psi^-$. By Remark [2.3] $c(\tau)\Psi^\pm \in \text{End}(E^\pm_{\partial M})$ are self-adjoint bundle maps. Therefore $A$ and $A^\sharp$ are formally self-adjoint operators. In fact, they are strongly Callias-type operators, cf. Lemma 3.12 of [12]. In particular, they have discrete spectrum. Also,

$$A^\sharp = c(\tau) \circ A \circ c(\tau).$$

**Definition 2.10.** We say that a graded self-adjoint strongly Callias-type operator $D$ is **product near the boundary** if the Dirac bundle $E$ is product near the boundary and the restriction of the Callias potential $\Psi$ to $Z_r$ does not depend on $t$. The operator $A$ (resp. $A^\sharp$) of (2.7) is called the **restriction of $D^+$ (resp. $D^-$) to the boundary**.

2.11. **Sobolev spaces on the boundary.** Consider a graded self-adjoint strongly Callias-type operator $D : C^\infty(M, E) \rightarrow C^\infty(M, E)$, cf. [2.3]. The restriction of $D^+$ to the boundary is a self-adjoint strongly Callias-type operator

$$A : C^\infty(\partial M, E^+_\partial M) \rightarrow C^\infty(\partial M, E^+_\partial M).$$

We recall the definition of Sobolev spaces $H^s_A(\partial M, E^+_\partial M)$ of sections over $\partial M$ which depend on the boundary operator $A$, cf. [12, §3.13].
Definition 2.12. Set
\[ C^\infty_A(\partial M, E^+_{\partial M}) := \left\{ u \in C^\infty(\partial M, E^+_{\partial M}) : \| (id + A^2)^{s/2} u \|_{L^2(\partial M, E_{\partial M})}^2 < +\infty \text{ for all } s \in \mathbb{R} \right\}. \]

For all \( s \in \mathbb{R} \) we define the Sobolev \( H^s_A \)-norm on \( C^\infty_A(\partial M, E^+_{\partial M}) \) by
\[ \| u \|_{H^s_A(\partial M, E_{\partial M})}^2 := \| (id + A^2)^{s/2} u \|_{L^2(\partial M, E_{\partial M})}^2 \] (2.9)
The Sobolev space \( H^s_A(\partial M, E^+_{\partial M}) \) is defined to be the completion of \( C^\infty_A(\partial M, E^+_{\partial M}) \) with respect to this norm.

2.13. Generalized APS boundary conditions. The eigensections of A belong to \( H^s_A(\partial M, E^+_{\partial M}) \) for all \( s \in \mathbb{R} \), cf. [12, §3.17]. For \( I \subset \mathbb{R} \) we denote by
\[ H^s_A(I) \subset H^s_A(\partial M, E^+_{\partial M}) \]
the span of the eigensections of A whose eigenvalues belong to \( I \).

Definition 2.14. For any \( a \in \mathbb{R} \), the subspace
\[ B = B(a) := H^{1/2}_{(-\infty, a]}(A). \] (2.10)
is called the the generalized Atiyah–Patodi–Singer boundary conditions for \( D^+ \). If \( a = 0 \), then the space \( B(0) = H^{1/2}_{(-\infty, 0]}(A) \) is called the Atiyah–Patodi–Singer (APS) boundary condition.

The spaces \( \bar{B}(a) := H^{1/2}_{(-\infty, a]}(A) \) and \( \bar{B}(0) := H^{1/2}_{(-\infty, 0]}(A) \) are called the dual generalized APS boundary conditions and the dual APS boundary conditions respectively.

The space \( B^{ad} = B^{ad}(a) := H^{1/2}_{(-\infty, -a]}(A^2) \) is called the adjoint of the generalized APS boundary condition for \( D^+ \). One can see that it is a dual generalized APS boundary condition for \( D^- \).

Definition 2.15. If \( B \) is a generalized APS boundary condition for \( D^+ \), we denote by \( D^+_B \) the operator \( D^+ \) with domain
\[ \text{dom } D^+_B := \{ u \in \text{dom } D^+_{\text{max}} : u|_{\partial M} \in B \}, \]
where \( \text{dom } D^+_{\text{max}} \) denotes the domain of the maximal extension of \( D^+ \). We refer to \( D^+_B \) as the generalized APS boundary value problem for \( D^+ \).

Recall that \( D^- \) is a formal adjoint of \( D^+ \). It is shown in Example 4.9 of [12] that the \( L^2 \)-adjoint of \( D^+_B(a) \) is given by \( D^- \) with the dual APS boundary condition \( B^{ad}(a) \):
\[ (D^+_B(a))^* = D^-_{B^{ad}(a)}. \] (2.11)

Theorem 2.16. Suppose that a graded strongly Callias-type operator \( [2.3] \) is product near \( \partial M \). Then the operator \( D^+_B : \text{dom } D^+_B \rightarrow L^2(M, E^-) \) is Fredholm. In particular, it has finite dimensional kernel and cokernel.

Proof. For the case discussed in Remark [2.7] this is proven in Theorem 5.4 of [12]. Exactly the same proof works in the general case. □
2.17. The index of generalized APS boundary value problems. By (2.4) the cokernel of $\mathcal{D}^+_{B(a)}$ is isomorphic to the kernel of $\mathcal{D}^{-}_{B(a)}$.

**Definition 2.18.** Let $\mathcal{D}^+$ be a strongly Callias-type operator on a complete Riemannian manifold $M$ which is product near the boundary. Let $B = H^{1/2}_{(-\infty,a)}(A)$ be a generalized APS boundary condition for $\mathcal{D}^+$ and let $B^\text{ad} = H^{1/2}_{(-\infty,-a)}(A^\ell)$ be the adjoint of the generalized APS boundary condition. The integer

$$\text{ind}\mathcal{D}^+_B := \dim\ker\mathcal{D}^+_B - \dim\ker(\mathcal{D}^-)_{B^\text{ad}} \in \mathbb{Z} \quad (2.12)$$

is called the index of the boundary value problem $\mathcal{D}^+_B$.

It follows directly from (2.11) that

$$\text{ind}(\mathcal{D}^-)_{B^\text{ad}} = -\text{ind}\mathcal{D}^+_B. \quad (2.13)$$

2.19. More general boundary value conditions. Generalized APS and dual generalized APS boundary conditions are examples of elliptic boundary conditions. [12, Definition 4.7]. In this paper we don’t work with general elliptic boundary conditions. However, in Section 5 we need a slight modification of APS boundary conditions, which we define now.

**Definition 2.20.** We say that two closed subspaces $X_1$, $X_2$ of a Hilbert space $H$ are finite rank perturbations of each other if there exists a finite dimensional subspace $Y \subset H$ such that $X_2 \subset X_1 \oplus Y$ and the quotient space $(X_1 \oplus Y)/X_2$ has finite dimension.

The relative index of $X_1$ and $X_2$ by

$$[X_1,X_2] := \dim(X_1 \oplus Y)/X_2 - \dim Y. \quad (2.14)$$

One easily sees that the relative index is independent of the choice of $Y$. We shall need the following analogue of Proposition 5.8 of [12]:

**Proposition 2.21.** Let $\mathcal{D}$ be a graded self-adjoint strongly Callias-type operator on $M$ and let $B$ be a generalized APS or dual generalized APS boundary condition for $\mathcal{D}^+$. If $B_1 \subset H^{1/2}_A(\partial M, E_{\partial M}^\ell)$ is a finite rank perturbation of $B$, then the operator $\mathcal{D}^+_{B_1}$ is Fredholm and

$$\text{ind}\mathcal{D}^+_B - \text{ind}\mathcal{D}^+_B = [B,B_1]. \quad (2.15)$$

The proof of the proposition is a verbatim repetition of the proof of Theorem 8.14 of [5]. As an immediate consequence of Proposition 2.21 we obtain the following

**Corollary 2.22.** Let $A$ be the restriction of $\mathcal{D}^+$ to $\partial M$ and let $B_0 = H^{1/2}_{(-\infty,0)}(A)$ and $\bar{B}_0 = H^{1/2}_{(-\infty,0)}(A)$ be the APS and the dual APS boundary conditions respectively. Then

$$\text{ind}\mathcal{D}^+_B = \text{ind}\mathcal{D}^+_B + \dim\ker A. \quad (2.16)$$

2.23. The splitting theorem. Let $M$ be a complete manifold. Let $N \subset M$ be a hypersurface disjoint from $\partial M$ such that cutting $M$ along $N$ we obtain a manifold $M'$ (connected or not) with $\partial M$ and two copies of $N$ as boundary. So we can write $M' = (M \setminus N) \sqcup N_1 \sqcup N_2$.

Let $E = E^+ \oplus E^- \to M$ be a $\mathbb{Z}_2$-graded Dirac bundle over $M$ and $\mathcal{D}^\pm : C^\infty(M,E^\pm) \to C^\infty(M,E^\pm)$ be strongly Callias-type operators as in Subsection 2.23. They induce $\mathbb{Z}_2$-graded Dirac bundle $E' = (E')^+ \oplus (E')^- \to M'$ and strongly Callias-type operators

$$(\mathcal{D}')^\pm : C^\infty(M',(E')^\pm) \to C^\infty(M',(E')^\pm).$$
on $M'$. We assume that all structures are product near $N_1$ and $N_2$. Let $A$ be the restriction of $(D')^+$ to $N_1$. Then $-A$ is the restriction of $(D')^+$ to $N_2$ and, thus, the restriction of $(D')^+$ to $N_1 \sqcup N_2$ is $A' = A \oplus (-A)$. The following Splitting Theorem is an analogue of Theorem 5.11 of [12] with the same proof.

**Theorem 2.24.** Suppose $M, D^+, M', (D')^+$ are as above. Let $B_0$ be a generalized APS boundary condition on $\partial M$. Let $B_1 = H_{(-\infty,0)}^{1/2}(A)$ and $B_2 = H_{(-\infty,0)}^{1/2}(-A) = H_{[0,\infty)}^{1/2}(A)$ be the APS and the dual APS boundary conditions for $(D')^+$ along $N_1$ and $N_2$, respectively. Then $(D')^+_{B_0 \oplus B_1 \oplus B_2}$ is a Fredholm operator and

$$\text{ind} D^+_{B_0} = \text{ind}(D')^+_{B_0 \oplus B_1 \oplus B_2}.$$

2.25. Reduction of the index to an essentially cylindrical manifold. Recall that in [12] Section 6, we introduce the notion of essentially cylindrical manifolds as following.

**Definition 2.26.** An essentially cylindrical manifold $M$ is a complete Riemannian manifold whose boundary is a disjoint union of two components, $\partial M = N_0 \sqcup N_1$, such that

(i) there exist a compact set $K \subset M$, an open manifold $N$, and an isometry $M \setminus K \simeq [0, \varepsilon] \times N$;

(ii) under the above isometry $N_0 \setminus K = \{0\} \times N$ and $N_1 \setminus K = \{\varepsilon\} \times N$.

Under the setting described earlier this section, one can construct (cf. [12] Lemma 6.5) a natural essentially cylindrical manifold called almost compact essential support.

**Definition 2.27.** An almost compact essential support of $D^+$ is a smooth submanifold $M_1 \subset M$ with smooth boundary, which contains $\partial M$ and such that

(i) $M_1$ contains an essential support for $D^+$, cf. Definition 2.5

(ii) there exist a compact set $K \subset M$ and $\varepsilon \in (0, r)$ such that

$$M_1 \setminus K = (\partial M \setminus K) \times [0, \varepsilon] \subset Z_r. \quad (2.17)$$

Let $M_1 \subset M$ be an almost compact essential support of $D^+$. Let $(D')^+$ be a compact perturbation of $D^+$ which is product near the boundary (cf. [12] §6.6). Let $A$ be the restriction of $D^+$ to $\partial M$. It is also the restriction of $(D')^+$. We denote by $-A_1$ the restriction of $(D')^+$ to $N_1$. Theorem 6.10 of [12] claims that the index of an elliptic boundary value problem on $M$ can be reduced to an index on an almost compact essential support:

**Theorem 2.28.** Suppose $M_1 \subset M$ is an almost compact essential support of $D^+$ and let $\partial M_1 = \partial M \sqcup N_1$. Let $(D')^+$ be a compact perturbation of $D^+$ which is product near $N_1$ and such that there is a compact essential support for $(D')^+$ which is contained in $M_1$. Let $B_0$ be a generalized APS boundary condition for $D^+$. View $(D')^+$ as an operator on $M_1$ and let

$$B_1 = H_{(-\infty,0)}^{1/2}(-A_1) = H_{[0,\infty)}^{1/2}(A_1)$$

be the APS boundary condition for $(D')^+$ at $N_1$. Then

$$\text{ind} D^+_{B_0} = \text{ind}(D')^+_{B_0 \oplus B_1}. \quad (2.18)$$
### 3. The Index of Operators on Essentially Cylindrical Manifolds

In this section we discuss the index of strongly Callias-type operators on even-dimensional essentially cylindrical manifolds. It is parallel to [12, Section 7], where the odd-dimensional case was considered.

From now on we assume that $M$ is an oriented even-dimensional essentially cylindrical manifold whose boundary $\partial M = N_0 \sqcup N_1$ is a disjoint union of two non-compact manifolds $N_0$ and $N_1$. Let $E$ be a Dirac bundle over $M$. As pointed out in Remark 2.3 there is a natural $\mathbb{Z}_2$-grading $E = E^+ \oplus E^-$ on $E$. Let $\mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$ be a strongly Callias-type operator as in Definition 2.5 (these data might or might not come as a restriction of another operator to its almost compact essential support. In particular, we don’t assume that the restriction of $\mathcal{D}^+$ to $N_1$ is invertible). Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to $N_0$ and $N_1$ respectively.

We first recall some definitions from [12, Section 7].

#### 3.1. Compatible data.

Let $M$ be an essentially cylindrical manifold and let $\partial M = N_0 \sqcup N_1$. As usual, we identify a tubular neighborhood of $\partial M$ with the product

$$ Z_r := (N_0 \times [0, r)) \cup (N_1 \times [0, r)) \subset M. $$

**Definition 3.2.** We say that another essentially cylindrical manifold $M'$ is compatible with $M$ if there is a fixed isometry between $Z_r$ and a neighborhood $Z'_r \subset M'$ of the boundary of $M'$.

Note that if $M$ and $M'$ are compatible then their boundaries are isometric.

Let $M$ and $M'$ be compatible essentially cylindrical manifolds and let $Z_r$ and $Z'_r$ be as above. Let $E \to M$ be a $\mathbb{Z}_2$-graded Dirac bundle over $M$ and let $\mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$ be a strongly Callias-type operator whose restriction to $Z_r$ is product and such that $M$ is an almost compact essential support of $\mathcal{D}^+$. This means that there is a compact set $K \subset M$ such that $M \setminus K = [0, \varepsilon] \times N$ and the restriction of $\mathcal{D}^+$ to $M \setminus K$ is product (i.e. is given by (2.7)). Let $E' \to M'$ be a $\mathbb{Z}_2$-graded Dirac bundle over $M'$ and let $(\mathcal{D}'^+ : C^\infty(M', (E')^+) \to C^\infty(M', (E')^-))$ be a strongly Callias-type operator, whose restriction to $Z'_r$ is product and such that $M'$ is an almost compact essential support of $(\mathcal{D}')^+$.

**Definition 3.3.** In the situation discussed above we say that $\mathcal{D}^+$ and $(\mathcal{D}')^+$ are compatible if there is an isomorphism $E|_{Z_r} \simeq E'|_{Z'_r}$ of graded Dirac bundles which identifies the restriction of $\mathcal{D}^+$ to $Z_r$ with the restriction of $(\mathcal{D}')^+$ to $Z'_r$.

Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to $N_0$ and $N_1$ respectively (the sign convention means that we think of $N_0$ as the “left boundary” and of $N_1$ as the “right boundary” of $M$). Let $B_0 = H^{1/2}_{(-\infty, 0)}(\mathcal{A}_0)$ and $B_1 = H^{1/2}_{(-\infty, 0)}(-\mathcal{A}_1) = H^{1/2}_{(0, \infty)}(\mathcal{A}_1)$ be the APS boundary conditions for $\mathcal{D}^+$ at $N_0$ and $N_1$ respectively. Since $\mathcal{D}^+$ and $(\mathcal{D}')^+$ are equal near the boundary, $B_0$ and $B_1$ are also APS boundary conditions for $(\mathcal{D}')^+$.

We denote by $\alpha_{\text{AS}}(\mathcal{D}^+)$ the Atiyah–Singer integrand of $\mathcal{D}^+$. It can be written as

$$ \alpha_{\text{AS}}(\mathcal{D}^+) := (2\pi i)^{-\dim M} \hat{A}(M) \text{ch}(E/S) $$

where $\hat{A}(M)$ and $\text{ch}(E/S)$ are the differential forms representing the $\hat{A}$-genus of $M$ and the relative Chern character of $E$, cf. [13, §4.1].

Since outside of a compact set $K$, $M$ and $E$ are product, the interior multiplication by $\partial/\partial t$ annihilates $\alpha_{\text{AS}}$. Hence, the top degree component of $\alpha_{\text{AS}}$ vanishes on $M \setminus K$. We conclude that the integral $\int_M \alpha_{\text{AS}}(\mathcal{D}^+)$ is well-defined and finite. Similarly, $\int_M \alpha_{\text{AS}}((\mathcal{D}')^+)$ is well-defined.
Theorem 3.4. Suppose $D^+$ is a strongly Callias-type operator on an oriented even-dimensional essentially cylindrical manifold $M$ such that $M$ is an almost compact essential support of $D^+$. Suppose that the operator $(D')^+$ is compatible with $D^+$. Let $\partial M = N_0 \sqcup N_1$ and let $B_0 = H^{1/2}_{(1,\infty)}(A_0)$ and $B_1 = H^{1/2}_{(-\infty,0)}(A_1)$ be the APS boundary conditions for $D^+$ (and, hence, for $(D')^+$) at $N_0$ and $N_1$ respectively. Then

$$\text{ind} D^+_{B_0 \sqcup B_1} - \int_M \alpha_{AS}(D^+) = \text{ind}(D')^+_{B_0 \sqcup B_1} - \int_M \alpha_{AS}((D')^+).$$

(3.1)

In particular, $\text{ind} D^+_{B_0 \sqcup B_1} - \int_M \alpha_{AS}(D^+)$ depends only on the restrictions $A_0$ and $-A_1$ of $D^+$ to the boundary.

The rest of the section is devoted to the proof of this theorem.

Remark 3.5. In [12] the odd dimensional version of Theorem 3.4 was considered. Of course, in this case $\alpha_{AS}$ vanishes identically and Theorem 7.5 of [12] states that the indexes of compatible operators are equal. The proof in [12] is based on application of a Callias-type index theorem and can not be adjusted to our current situation. Consequently, a completely different proof is proposed below.

3.6. Gluing the data together. We follow [12] §7.6, §7.7 to glue $M$ with $M'$ and $D^+$ with $(D')^+$.

Let $-M'$ denote the manifold $M'$ with the opposite orientation. We identify a neighborhood of the boundary of $-M'$ with the product

$$-Z'_r := (N_0 \times (-r,0]) \sqcup (N_1 \times (-r,0])$$

and consider the union

$$\tilde{M} := M \cup_{N_0 \sqcup N_1} (-M').$$

Then $Z_{(-r,r)} := Z_r \cup (-Z'_r)$ is a subset of $\tilde{M}$ identified with the product

$$\left( N_0 \times (-r,r) \right) \sqcup \left( N_1 \times (-r,r) \right).$$

We note that $\tilde{M}$ is a complete Riemannian manifold without boundary.

Let $E_{\partial M} = E_{\partial M}^+ \oplus E_{\partial M}^-$ denote the restriction of $E = E^+ \oplus E^-$ to $\partial M$. The product structure on $E|_{Z_r}$ gives a grading-respecting isomorphism $\psi : E|_{Z_r} \to [0,r) \times E_{\partial M}$. Recall that we identified $Z_r$ with $Z'_r$ and fixed an isomorphism between the restrictions of $E$ to $Z_r$ and $E'$ to $Z'_r$. By a slight abuse of notation we use this isomorphism to view $\psi$ also as an isomorphism $E'|_{Z'_r} \to [0,r) \times E_{\partial M}$.

Let $\tilde{E} \to \tilde{M}$ be the vector bundle over $\tilde{M}$ obtained by gluing $E$ and $E'$ using the isomorphism $c(\tau) : E|_{\partial M} \to E'|_{\partial M'}$. This means that we fix isomorphisms

$$\phi : \tilde{E}|_{\tilde{M}} \to E, \quad \phi' : \tilde{E}|_{\tilde{M}'} \to E',$$

(3.2)

so that

$$\psi \circ \phi \circ \psi^{-1} = \text{id} : [0,r) \times E_{\partial M} \to [0,r) \times E_{\partial M},$$

$$\psi \circ \phi' \circ \psi^{-1} = 1 \times c(\tau) : [0,r) \times E_{\partial M} \to [0,r) \times E_{\partial M}.$$

Note that the grading of $E$ is preserved while the grading of $E'$ is reversed in this gluing process. Therefore $\tilde{E} = \tilde{E}^+ \oplus \tilde{E}^-$ is a $\mathbb{Z}_2$-graded bundle.
We denote by \( c' : T^*M' \to \text{End}(E') \) the Clifford multiplication on \( E' \) and set \( c''(\xi) := -c'(\xi) \). Then \( \tilde{E} \) is a Dirac bundle over \( \tilde{M} \) with the Clifford multiplication
\[
\tilde{c}(\xi) := \begin{cases} 
  c(\xi), & \xi \in T^*M; \\
  c''(\xi) = -c'(\xi), & \xi \in T^*M'.
\end{cases}
\]

One readily checks that \( \tilde{c} \) defines a smooth odd-graded Clifford multiplication on \( \tilde{E} \). Let \( \hat{D} : C^\infty(\tilde{M}, \tilde{E}) \to C^\infty(\tilde{M}, \tilde{E}) \) be the \( \mathbb{Z}_2 \)-graded Dirac operator. Then the isomorphism \( \phi \) of \( 3.2 \) identifies the restriction of \( \hat{D}^\pm \) with \( D^\pm \), the isomorphism \( \phi' \) identifies the restriction of \( \hat{D}^\pm \) with \( -(D')^\mp \), and isomorphism \( \psi \circ \phi' \circ \psi^{-1} \) identifies the restriction of \( \hat{D}^\pm \) to \( -Z'_r \) with
\[
\hat{D}^\pm|_{Z'_r} = -c'(\tau) \circ (D')^\mp_{Z'_r} \circ c'(\tau)^{-1}.
\]

Let \( (\Psi')^\pm \) denote the Callias potentials of \( (D')^\pm \), so that \( (D')^\pm = (D')^\pm + (\Psi')^\pm \). Consider the bundle maps \( \tilde{\Psi}^\pm \in \text{Hom}(\tilde{E}^\pm, \tilde{E}^\mp) \) whose restrictions to \( \tilde{M} \) are equal to \( \tilde{\Psi}^\pm \) and whose restrictions to \( M' \) are equal to \( -(\Psi')^\mp \). The two pieces fit well on \( Z_{(-r,r)} \) by Remark 2.3. To sum up the constructions presented in this subsection, we have

**Lemma 3.7.** The operators \( \hat{D}^\pm := \hat{D}^\pm + \tilde{\Psi}^\pm \) are strongly Callias-type operators on \( \tilde{M} \), formally adjoint to each other, whose restrictions to \( M \) are equal to \( D^\pm \) and whose restrictions to \( M' \) are equal to \( -(\tilde{D}')^\mp - (\Psi')^\pm = -(\tilde{D}')^\mp \).

The operator \( \hat{D}^+ \) is a strongly Callias-type operator on a complete Riemannian manifold without boundary. Hence, \( \hat{D}^+ \) is Fredholm. We again denote by \( \alpha_{\text{AS}}(\hat{D}^+) \) the Atiyah–Singer integrand of \( \hat{D}^+ \). It is explained in the paragraph before Theorem 3.4 that the integral \( \int_M \alpha_{\text{AS}}(\hat{D}^+) \) is well defined.

**Lemma 3.8.** \( \text{ind } \hat{D}^+ = \int_M \alpha_{\text{AS}}(\hat{D}^+) \).

**Proof.** Since \( \tilde{M} \) is a union of two essentially cylindrical manifolds, there exists a compact essential support \( \tilde{K} \subset \tilde{M} \) of \( \hat{D} \) such that \( \tilde{M} \setminus \tilde{K} \) is of the form \( S^1 \times N \). We can choose \( \tilde{K} \) to be large enough so that the restriction of \( \hat{D} \) to a neighborhood \( W \) of \( \tilde{M} \setminus \tilde{K} \approx S^1 \times N \) is a product of an operator on \( N \) and an operator on \( S^1 \). Then the restriction of \( \alpha_{\text{AS}} \) to this neighborhood vanishes. We can also assume that \( \tilde{K} \) has a smooth boundary \( \Sigma = S^1 \times L \).

Let \( \hat{D}^+ \) be a compact perturbation of \( \hat{D}^+ \) (cf. Definition 6.7 of \[12\]) in \( W \) which is product both near \( \Sigma \) and on \( W \) and whose essential support is contained in \( \tilde{K} \). Then
\[
\text{ind } \hat{D}^+ = \text{ind } \hat{D}^+.
\]

We cut \( \tilde{M} \) along \( \Sigma \) and apply the Splitting Theorem 2.21 to get
\[
\text{ind } \hat{D}^+ = \text{ind } \hat{D}^+_K + \text{ind } \hat{D}^+_{\tilde{M} \setminus \tilde{K}},
\]
where \( \text{ind } \hat{D}^+_K \) stands for the index of the restriction of \( \hat{D}^+ \) to \( \tilde{K} \) with APS boundary condition, and \( \text{ind } \hat{D}^+_{\tilde{M} \setminus \tilde{K}} \) stands for the index of the restriction of \( \hat{D}^+ \) to \( \tilde{M} \setminus \tilde{K} \) with the dual APS boundary condition.

Since \( \hat{D}^+ \) has an empty essential support in \( \tilde{M} \setminus \tilde{K} \), by the vanishing theorem \[12\] Corollary 5.13, the second summand in the right hand side of \( 3.5 \) vanishes. The first summand in the right hand side of \( 3.5 \) is given by the Atiyah–Patodi–Singer index theorem \[3\] Theorem 3.10\footnote{Since \( \Sigma \) is compact we can also use the splitting theorem for compact hypersurfaces, \[5\] Theorem 8.17.}.
(Note that Σ is outside of an essential support of $\hat{D}^+$ and, hence, the restriction of $\hat{D}^+$ to Σ is invertible. Hence, its kernel of the restriction of $\hat{D}^+$ to Σ is trivial)

$$\text{ind } \hat{D}_K^+ = \int_K \alpha_{AS}(\hat{D}^+) - \frac{1}{2} \eta(0),$$

where $\eta(0)$ is the $\eta$-invariant of the restriction of $\hat{D}^+$ to Σ.

As $\alpha_{AS}(\hat{D}^+) \equiv 0$ on $W$ and $\hat{D}^+ \equiv \hat{D}^+$ elsewhere, we have

$$\int_K \alpha_{AS}(\hat{D}^+) = \int_K \alpha_{AS}(\hat{D}^+) = \int_M \alpha_{AS}(\hat{D}^+).$$

To finish the proof of the lemma it suffices now to show $\eta(0) = 0$.

Let $\omega$ be the inward (with respect to $\bar{K}$) unit normal vector field along Σ. Recall that $\Sigma = S^1 \times L$. We denote the coordinate along $S^1$ by $\theta$. Suppose that $\{\omega, d\theta, e_1, \cdots, e_m\}$ forms a local orthonormal frame of $T^*M$ on Σ. Then the restriction of $\hat{D}^+ = \hat{\mathcal{D}}^+ + \hat{\Psi}^+$ to Σ can be written as

$$\hat{\mathcal{A}}^+_\Sigma = -\sum_{i=1}^m \tilde{c}(\omega)\tilde{c}(e_i) \nabla_{e_i} - \tilde{c}(\omega)\tilde{c}(d\theta)\partial_\theta - \tilde{c}(\omega)\hat{\Psi}^+$$

which maps $C^\infty(\Sigma, \tilde{E}^+|_\Sigma)$ to itself. We define a unitary isomorphism $\Theta$ on the space $C^\infty(\Sigma, \tilde{E}|_\Sigma)$ given by

$$\Theta u(\theta, y) := -\tilde{c}(\omega)\tilde{c}(d\theta)u(-\theta, y).$$

One can check that $\Theta$ anticommutes with $\hat{\mathcal{A}}^+_\Sigma$. As a result, the spectrum of $\hat{\mathcal{A}}^+_\Sigma$ is symmetric about 0. Therefore $\eta(0) = 0$ and lemma is proved. □

3.9. **Proof of Theorem 3.4.** Recall that we denote by $B_0$ and $B_1$ the APS boundary conditions for $\mathcal{D}^+ = \tilde{D}^+|_M$ at $N_0$ and $N_1$ respectively. Let $(\mathcal{D}'')^+$ denote the restriction of $\bar{D}^+$ to $-M' = M \setminus M$. Let $B_0$ and $B_1$ be the dual APS boundary conditions for $(\mathcal{D}'')^+$ at $N_0$ and $N_1$ respectively.

By the Splitting Theorem 2.21

$$\text{ind } \bar{D}^+ = \text{ind } \mathcal{D}_B^+ \oplus B_1 + \text{ind } (\mathcal{D}'')_B^+ \oplus B_1.$$  

By Lemma 3.5 we obtain

$$\text{ind } \mathcal{D}_B^+ \oplus B_1 + \text{ind } (\mathcal{D}'')_B^+ \oplus B_1 = \int_M \alpha_{AS}(\mathcal{D}^+) + \int_{M'} \alpha_{AS}(\mathcal{D}'^+),$$

which means

$$\text{ind } \mathcal{D}_B^+ \oplus B_1 - \int_M \alpha_{AS}(\mathcal{D}^+) = -\text{ind } (\mathcal{D}'')_B^+ \oplus B_1 + \int_{M'} \alpha_{AS}(\mathcal{D}'^+). \quad (3.6)$$

By Lemma 3.7 $(\mathcal{D}'')^+ = -(\mathcal{D}')^−$. Thus $B_0 \oplus B_1$ is the adjoint of the APS boundary condition for $-\mathcal{D}'^+$ (cf. Definition 2.14). Therefore,

$$\text{ind } (\mathcal{D}'')_B^+ \oplus B_1 = \text{ind } (-\mathcal{D}')_B^+ \oplus B_1 = -\text{ind } (\mathcal{D}')_B^+ \oplus B_1 = -\text{ind } (\mathcal{D}')_B^+ \oplus B_1,$$

where we used (2.13) in the middle equality. Also by the construction of local index density,

$$\alpha_{AS}(\mathcal{D}'')^+ = \alpha_{AS}((-\mathcal{D}')^-) = \alpha_{AS}((\mathcal{D}')^-) = -\alpha_{AS}((\mathcal{D}')^+).$$

Combining these equalities with (3.6) we obtain (3.1). □
4. The relative $\eta$-invariant

In the previous section we proved that on an essentially cylindrical manifold $M$ the difference $\text{ind} \, D_{B_0 \cap \bar{B}_1} - \int_M \alpha_{\text{AS}}(\mathcal{D})$ depends only on the restriction of $\mathcal{D}$ to the boundary, i.e., on the operators $\mathcal{A}_0$ and $-\mathcal{A}_1$. In this section we use this fact to define the relative $\eta$-invariant $\eta(\mathcal{A}_1, \mathcal{A}_0)$ and show that it has properties similar to the difference of $\eta$-invariants $\eta(\mathcal{A}_1) - \eta(\mathcal{A}_0)$ of operators on compact manifolds. For special cases, [19], when the index can be computed using heat kernel asymptotics, we show that $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is indeed equal to the difference of the $\eta$-invariants of $\mathcal{A}_1$ and $\mathcal{A}_0$. In the next section we discuss the connection between the relative $\eta$-invariant and the spectral flow.

In the case when $\mathcal{A}_0, \mathcal{A}_1$ are operators on even-dimensional manifolds, an analogous construction was proposed in [12, §8]. Even though the definition of the relative $\eta$-invariant for operators on odd-dimensional manifolds proposed in this section is slightly more involved than the definition in [12], we show that most of the properties of $\eta(\mathcal{A}_1, \mathcal{A}_0)$ remain the same.

4.1. Almost compact cobordisms. Let $N_0$ and $N_1$ be two complete odd-dimensional Riemannian manifolds and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be self-adjoint strongly Callias-type operators on $N_0$ and $N_1$ respectively, cf. Definition 2.2.

**Definition 4.2.** An almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$ is a pair $(M, \mathcal{D})$, where $M$ is an essentially cylindrical manifold with $\partial M = N_0 \sqcup N_1$ and $\mathcal{D}$ is a graded self-adjoint strongly Callias-type operator on $M$ such that

(i) $M$ is an almost compact essential support of $\mathcal{D}$;

(ii) $\mathcal{D}$ is product near $\partial M$;

(iii) The restriction of $\mathcal{D}^+$ to $N_0$ is equal to $\mathcal{A}_0$ and the restriction of $\mathcal{D}^+$ to $N_1$ is equal to $-\mathcal{A}_1$.

If there exists an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$ we say that operator $\mathcal{A}_0$ is cobordant to operator $\mathcal{A}_1$.

**Lemma 4.3.** An almost compact cobordism is an equivalence relation on the set of self-adjoint strongly Callias-type operators, i.e.,

(i) If $\mathcal{A}_0$ is cobordant to $\mathcal{A}_1$ then $\mathcal{A}_1$ is cobordant to $\mathcal{A}_0$.

(ii) Let $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ be self-adjoint strongly Callias-type operators on odd-dimensional complete Riemannian manifolds $N_0, N_1$ and $N_2$ respectively. Suppose $\mathcal{A}_0$ is cobordant to $\mathcal{A}_1$ and $\mathcal{A}_1$ is cobordant to $\mathcal{A}_2$. Then $\mathcal{A}_0$ is cobordant to $\mathcal{A}_2$. 

**Proof.** The proof is a verbatim repetition of the proof of Lemmas 8.2 and 8.3 of [12]. \[\Box\]

**Definition 4.4.** Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are cobordant self-adjoint strongly Callias-type operators and let $(M, \mathcal{D})$ be an almost compact cobordism between them. Let $B_0 = H^{1/2}_{(-\infty, 0)}(\mathcal{A}_0)$ and $B_1 = H^{1/2}_{(-\infty, 0)}(-\mathcal{A}_1)$ be the APS boundary conditions for $\mathcal{D}^+$. The relative $\eta$-invariant is defined as

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = 2 \left( \text{ind} \, \mathcal{D}_{B_0 \cap \bar{B}_1}^+ - \int_M \alpha_{\text{AS}}(\mathcal{D}^+) \right) + \dim \ker \mathcal{A}_0 + \dim \ker \mathcal{A}_1. \quad (4.1)$$

Theorem 3.4 implies that $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is independent of the choice of the cobordism $(M, \mathcal{D})$. 

Remark 4.5. Sometimes it is convenient to use the dual APS boundary conditions $\bar{B}_0 = H^{1/2}_{(-\infty,0)}(\mathcal{A}_0)$ and $\bar{B}_1 = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_1)$ instead of $B_0$ and $B_1$. It follows from Corollary 2.22 that the relative $\eta$-invariant can be written as

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = 2 \left( \text{ind} \mathcal{D}^+_{\bar{B}_0 \oplus \bar{B}_1} - \int_M \alpha_{\text{AS}}(\mathcal{D}^+) \right) - \dim \ker \mathcal{A}_0 - \dim \ker \mathcal{A}_1. \quad (4.2)$$

4.6. The case when the heat kernel has an asymptotic expansion. In [19], Fox and Haskell studied the index of a boundary value problem on manifolds of bounded geometry. They showed that under certain conditions (satisfied for natural operators on manifolds with conical or cylindrical ends) on $M$ and $\mathcal{D}$, the heat kernel $e^{-t(D_B^+D_B)}$ is of trace class and its trace has an asymptotic expansion similar to the one on compact manifolds. In this case the $\eta$-function, defined by a usual formula

$$\eta(s; \mathcal{A}) := \sum_{\lambda \in \text{spec}(\mathcal{A})} \text{sign}(\lambda) |\lambda|^s, \quad \text{Re } s \ll 0,$$

is an analytic function of $s$, which has a meromorphic continuation to the whole complex plane and is regular at 0. So one can define the $\eta$-invariant of $\mathcal{A}$ by $\eta(\mathcal{A}) = \eta(0; \mathcal{A})$.

Proposition 4.7. Suppose now that $\mathcal{D}$ is an operator on an essentially cylindrical manifold $M$ which satisfies the conditions of [19]. We also assume that $\mathcal{D}$ is product near $\partial M = N_0 \sqcup N_1$ and that $M$ is an almost compact essential support for $\mathcal{D}$. Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to $N_0$ and $N_1$ respectively. Let $\eta(\mathcal{A}_j)$ ($j = 0, 1$) be the $\eta$-invariant of $\mathcal{A}_j$. Then

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = \eta(\mathcal{A}_1) - \eta(\mathcal{A}_0). \quad (4.3)$$

Proof. An analogue of this proposition for the case when $\dim M$ is odd is proven in [12, Proposition 8.8]. This proof extends to the case when $\dim M$ is even without any changes. \hfill $\square$

4.8. Basic properties of the relative $\eta$-invariant. Proposition 4.7 shows that under certain conditions the $\eta$-invariants of $\mathcal{A}_0$ and $\mathcal{A}_1$ are defined and $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is their difference. We now show that in general case, when $\eta(\mathcal{A}_0)$ and $\eta(\mathcal{A}_1)$ do not necessarily exist, $\eta(\mathcal{A}_1, \mathcal{A}_0)$ behaves like it were a difference of an invariant of $N_1$ and an invariant of $N_0$.

Proposition 4.9 (Antisymmetry). Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are cobordant self-adjoint strongly Callias-type operators. Then

$$\eta(\mathcal{A}_0, \mathcal{A}_1) = -\eta(\mathcal{A}_1, \mathcal{A}_0). \quad (4.4)$$

Proof. Let $-M$ denote the manifold $M$ with the opposite orientation and let $\tilde{M} := M \cup_{\partial M} (-M)$ denote the double of $M$. Let $\mathcal{D}$ be an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Using the construction of Section 3.6 (with $\mathcal{D}' = \mathcal{D}$) we obtain a graded self-adjoint strongly Callias-type operator $\tilde{\mathcal{D}}$ on $\tilde{M}$ whose restriction to $M$ is isometric to $\mathcal{D}$. Let $\mathcal{D}''$ denote the restriction of $\tilde{\mathcal{D}}$ to $-M = \tilde{M} \setminus M$. Then the restriction of $(\mathcal{D}'')^+$ to $N_1$ is equal to $\mathcal{A}_1$ and the restriction of $(\mathcal{D}'')^+$ to $N_0$ is equal to $-\mathcal{A}_0$.

Let

$$\tilde{B}_0 = H^{1/2}_{(0,\infty)}(\mathcal{A}_0) = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_0),$$

$$\tilde{B}_1 = H^{1/2}_{(0,\infty)}(-\mathcal{A}_1) = H^{1/2}_{(-\infty,0)}(\mathcal{A}_1)$$

be the dual APS boundary conditions for $(\mathcal{D}'')^+$. By (3.6),

$$\text{ind}(\mathcal{D}'')^+_{\tilde{B}_0 \oplus \tilde{B}_1} - \int_{\tilde{M}} \alpha_{\text{AS}}((\mathcal{D}'')^+) = -\text{ind}(\mathcal{D}_B^+_{\bar{B}_0 \oplus \bar{B}_1}) + \int_M \alpha_{\text{AS}}(\mathcal{D}^+). \quad (4.5)$$
Since $\mathcal{D}'$ is an almost compact cobordism between $\mathcal{A}_1$ and $\mathcal{A}_0$ we conclude from (1.2) that
\[
\eta(\mathcal{A}_0, \mathcal{A}_1) = 2 \left( \dim (\mathcal{D}''_{\partial \mathcal{B}_1}^+) - \int_{M'} \alpha_{AS}((\mathcal{D}'')^+) \right) - \dim \ker \mathcal{A}_0 - \dim \ker \mathcal{A}_1. \tag{4.6}
\]
Combining (4.6) and (4.5) we obtain (4.4).

Note that (4.4) implies that
\[
\eta(\mathcal{A}, \mathcal{A}) = 0 \tag{4.7}
\]
for every self-adjoint strongly Callias-type operator $\mathcal{A}$.

**Proposition 4.10 (The cocycle condition).** Let $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ be self-adjoint strongly Callias-type operators which are cobordant to each other. Then
\[
\eta(\mathcal{A}_2, \mathcal{A}_0) = \eta(\mathcal{A}_2, \mathcal{A}_1) + \eta(\mathcal{A}_1, \mathcal{A}_0). \tag{4.8}
\]

**Proof.** Let $M_1$ and $M_2$ be essentially cylindrical manifolds such that $\partial M_1 = N_0 \cup N_1$ and $\partial M_2 = N_1 \cup N_2$. Let $\mathcal{D}_1$ be an operator on $M_1$ which is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Let $\mathcal{D}_2$ be an operator on $M_2$ which is an almost compact cobordism between $\mathcal{A}_1$ and $\mathcal{A}_2$. Then the operator $\mathcal{D}_3$ on $M_1 \cup N_1, M_2$ whose restriction to $M_j$ ($j = 1, 2$) is equal to $\mathcal{D}_j$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_2$.

Let $B_0$ and $B_1$ be the APS boundary conditions for $\mathcal{D}_1^+$ at $N_0$ and $N_1$ respectively. Then $\bar{B}_1 = H^{1/2}_{0, \infty}(M_1)$ is equal to the dual APS boundary condition for $\mathcal{D}_2^+$. Let $B_2$ be the APS boundary condition for $\mathcal{D}_2^+$ at $N_2$. From Corollary 2.22 we obtain
\[
\eta(\mathcal{A}_2, \mathcal{A}_1) = 2 \left( \dim (\mathcal{D}_3^+)_{\bar{B}_0 \oplus B_1} - \int_M \alpha_{AS}(\mathcal{D}_3^+) \right) - \dim \ker \mathcal{A}_1 + \dim \ker \mathcal{A}_2. \tag{4.9}
\]
By the Splitting Theorem 2.24
\[
\dim (\mathcal{D}_3^+)_{\bar{B}_0 \oplus B_1} = \dim (\mathcal{D}_1^+)_{\bar{B}_0 \oplus B_1} + \dim (\mathcal{D}_2^+)_{\bar{B}_1 \oplus B_2}. \tag{4.10}
\]
Clearly,
\[
\int_{M_1 \cup M_2} \alpha_{AS}(\mathcal{D}_3^+) = \int_{M_1} \alpha_{AS}(\mathcal{D}_1^+) + \int_{M_2} \alpha_{AS}(\mathcal{D}_2^+). \tag{4.11}
\]
Combining (4.9), (4.10), and (4.11) we obtain (4.8).

5. THE SPECTRAL FLOW

Suppose $\mathcal{A} := \{\mathcal{A}^s\}_{0 \leq s \leq 1}$ is a smooth family of self-adjoint elliptic operators on a closed manifold $N$. Let $\bar{\eta}(\mathcal{A}^s) \in \mathbb{R}/\mathbb{Z}$ denote the mod Z reduction of the $\eta$-invariant $\eta(\mathcal{A}^s)$. Atiyah, Patodi, and Singer, [4], showed that $s \mapsto \bar{\eta}(\mathcal{A}^s)$ is a smooth function whose derivative $\frac{d}{ds} \bar{\eta}(\mathcal{A}^s)$ is given by an explicit local formula. Further, Atiyah, Patodi and Singer, [4], introduced a notion of spectral flow $\text{sf}(\mathcal{A})$ and showed that it computes the net number of integer jumps of $\eta(\mathcal{A}^s)$, i.e.,
\[
2 \text{sf}(\mathcal{A}) = \eta(\mathcal{A}^1) - \eta(\mathcal{A}^0) - \int_0^1 \left( \frac{d}{ds} \bar{\eta}(\mathcal{A}^s) \right) ds.
\]
In this section we consider a family of self-adjoint strongly Callias-type operators $\mathcal{A} := \{\mathcal{A}^s\}_{0 \leq s \leq 1}$ on a complete Riemannian manifold. Assuming that the restriction of $\mathcal{A}^s$ to a complement of a compact set $K \subset N$ is independent of $s$, we show that for any operator $\mathcal{A}_0$ cobordant to $\mathcal{A}^0$ the mod Z reduction $\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0)$ of the relative $\eta$-invariant depends smoothly on $s$ and
\[
2 \text{sf}(\mathcal{A}) = \eta(\mathcal{A}^1, \mathcal{A}_0) - \eta(\mathcal{A}^0, \mathcal{A}_0) - \int_0^1 \left( \frac{d}{ds} \bar{\eta}(\mathcal{A}^s, \mathcal{A}_0) \right) ds.
\]
5.1. A family of boundary operators. Let $E_N \to N$ be a Dirac bundle over a complete odd-dimensional Riemannian manifold $N$. We denote the Clifford multiplication of $T^*N$ on $E_N$ by $c_N : T^*N \to \text{End}(E_N)$. Let $A = \{A^s\}_{0 \leq s \leq 1}$ be a family of self-adjoint strongly Callias-type operators $A^s : C^\infty(N, E_N) \to C^\infty(N, E_N)$.

**Definition 5.2.** The family $A = \{A^s\}_{0 \leq s \leq 1}$ is called almost constant if there exists a compact set $K \subset N$ such that the restriction of $A^s$ to $N \setminus K$ is independent of $s$.

Consider the cylinder $M := [0, 1] \times N$ and denote by $t$ the coordinate along $[0, 1]$. Set

$$E^+ = E^- := [0, 1] \times E_N.$$

Then $E = E^+ \oplus E^- \to M$ is naturally a $\mathbb{Z}_2$-graded Dirac bundle over $M$ with

$$c(dt) := \begin{pmatrix} 0 & -\text{id}_{E_N} \\ \text{id}_{E_N} & 0 \end{pmatrix}$$

and

$$c(\xi) := \begin{pmatrix} 0 & c_N(\xi) \\ c_N(\xi) & 0 \end{pmatrix}, \quad \text{for} \quad \xi \in T^*N.$$

**Definition 5.3.** The family $A = \{A^s\}_{0 \leq s \leq 1}$ is called smooth if

$$D := c(dt) \left( \partial_t + \begin{pmatrix} A^t & 0 \\ 0 & -A^t \end{pmatrix} \right) : C^\infty(M, E) \to C^\infty(M, E)$$

is a smooth differential operator on $M$.

Fix a smooth non-decreasing function $\kappa : [0, 1] \to [0, 1]$ such that $\kappa(t) = 0$ for $t \leq 1/3$ and $\kappa(t) = 1$ for $t \geq 2/3$ and consider the operator

$$D := c(dt) \left( \partial_t + \begin{pmatrix} A^t(\kappa(t)) & 0 \\ 0 & -A^t(\kappa(t)) \end{pmatrix} \right) : C^\infty(M, E) \to C^\infty(M, E).$$

Then $D$ is product near $\partial M$. If $A$ is a smooth almost constant family of self-adjoint strongly Callias-type operators then (5.2) is a strongly Callias-type operator for which $M$ is an almost compact essential support. Hence it is a non-compact cobordism (cf. Definition 4.2) between $A^0$ and $A^1$.

5.4. The spectral section. If $A = \{A^s\}_{0 \leq s \leq 1}$ is a smooth almost constant family of self-adjoint strongly Callias-type operators then it satisfies the conditions of the Kato Selection Theorem [20, Theorems II.5.4 and II.6.8, [25, Theorem 3.2]. Thus there is a family of eigenvalues $\lambda_j(s)$ ($j \in \mathbb{Z}$) which depend continuously on $s$. We order the eigenvalues so that $\lambda_j(0) \leq \lambda_{j+1}(0)$ for all $j \in \mathbb{Z}$ and $\lambda_j(0) \leq 0$ for $j \leq 0$ while $\lambda_j(0) > 0$ for $j > 0$.

Atiyah, Patodi and Singer [4] defined the spectral flow $\text{sf}(A)$ for a family of operators satisfying the conditions of the Kato Selection Theorem ([20, Theorems II.5.4 and II.6.8, 25, Theorem 3.2]) as an integer that counts the net number of eigenvalues that change sign when $s$ changes from $0$ to $1$. Several other equivalent definitions of the spectral flow based on different assumptions on the family $A$ exist in the literature. For our purposes the most convenient is the Dai and Zhang’s definition [17] which is based on the notion of spectral section introduced by Melrose and Piazza [24].
**Definition 5.5.** A *spectral section* for $\mathcal{A}$ is a continuous family $\mathcal{P} = \{P^s\}_{0 \leq s \leq 1}$ of self-adjoint projections such that there exists a constant $R > 0$ such that for all $0 \leq s \leq 1$, if $A^s u = \lambda u$ then

$$P^s u = \begin{cases} 0, & \text{if } \lambda < -R; \\ u, & \text{if } \lambda > R. \end{cases}$$

If $\mathcal{A}$ satisfies the conditions of the Kato Selection Theorem, then the arguments of the proof of [24] Proposition 1 show that $\mathcal{A}$ admits a spectral section.

**5.6. The spectral flow.** Let $\mathcal{P} = \{P^s\}$ be a spectral section for $\mathcal{A}$. Set $B^s := \ker P^s$. Let $B_0^s := H_{(-\infty,0)}(A^s)$ denote the APS boundary condition defined by the boundary operator $A^s$. Since the spectrum of $A^s$ is discrete, it follows immediately from the definition of the spectral section that for every $s \in [0,1]$ the space $B^s$ is a finite rank perturbation of $B_0^s$, cf. Section 2.19.

Recall that the relative index $[B^s, B_0^s]$ was defined in Definition 2.20. Following Dai and Zhang [17] (see also [12, §9.8]), we give the following definition.

**Definition 5.7.** Let $\mathcal{A} = \{A^s\}_{0 \leq s \leq 1}$ be a smooth almost constant family of self-adjoint strongly Callias-type operators which admits a spectral section $\mathcal{P} = \{P^s\}_{0 \leq s \leq 1}$. Assume that the operators $A^0$ and $A^1$ are invertible. Let $B^s := \ker P^s$ and $B_0^s := H_{(-\infty,0)}(A^s)$. The *spectral flow* $\text{sf}(\mathcal{A})$ of the family $\mathcal{A}$ is defined by the formula

$$\text{sf}(\mathcal{A}) := [B^1, B_0^1] - [B^0, B_0^0]. \quad (5.3)$$

By Theorem 1.4 of [17] the spectral flow is independent of the choice of the spectral section $\mathcal{P}$ and computes the net number of eigenvalues that change sign when $s$ changes from 0 to 1.

**Lemma 5.8.** Let $-\mathcal{A}$ denote the family $\{-A^s\}_{0 \leq s \leq 1}$. Then

$$\text{sf}(-\mathcal{A}) = -\text{sf}(\mathcal{A}). \quad (5.4)$$

**Proof.** The lemma is an immediate consequence of Lemma 5.7 of [12].

**5.9. Deformation of the relative $\eta$-invariant.** Let $\mathcal{A} = \{A^s\}_{0 \leq s \leq 1}$ be a smooth almost constant family of self-adjoint strongly Callias-type operators on a complete odd-dimensional Riemannian manifold $N_1$. Let $A_0$ be another self-adjoint strongly Callias-type operator, which is cobordant to $A^0$. In Section 5.1 we showed that $A^0$ is cobordant to $A^s$ for all $s \in [0,1]$. Hence, by Lemma 4.3(ii), $A_0$ is cobordant to $A^1$. In this situation we say the $A_0$ is cobordant to the family $\mathcal{A}$.

The following theorem is the main result of this section.

**Theorem 5.10.** Suppose $\mathcal{A} = \{A^s : C^\infty(N_1, E_1) \to C^\infty(N_1, E_1)\}_{0 \leq s \leq 1}$ is a smooth almost constant family of self-adjoint strongly Callias-type operators on a complete odd-dimensional Riemannian manifold $N_1$. Assume that $A^0$ and $A^1$ are invertible. Let $A_0 : C^\infty(N_0, E_0) \to C^\infty(N_0, E_0)$ be an invertible self-adjoint strongly Callias-type operator on a complete Riemannian manifold $N_0$ which is cobordant to the family $\mathcal{A}$. Then the mod $\mathbb{Z}$ reduction $\bar{\eta}(A^s, A_0) \in \mathbb{R}/\mathbb{Z}$ of the relative $\eta$-invariant depends smoothly on $s \in [0,1]$ and

$$\eta(A^1, A_0) - \eta(A^0, A_0) - \int_0^1 \left( \frac{d}{ds} \bar{\eta}(A^s, A_0) \right) ds = 2 \text{sf}(\mathcal{A}). \quad (5.5)$$

The proof of this theorem occupies Sections 5.11–5.14.
5.11. **A family of almost compact cobordisms.** Let $M$ be an essentially cylindrical manifold whose boundary is the disjoint union of $N_0$ and $N_1$. First, we construct a smooth family $\mathcal{D}^r$ ($0 \leq r \leq 1$) of graded self-adjoint strongly Callias-type operators on the manifold

$$M' := M \cup_{N_1} ([0,1] \times N_1),$$

(5.6)
such that for each $r \in [0,1]$ the pair $(M', \mathcal{D}^r)$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^r$.

Let $\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$ be an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^0$. Let $E_0$ and $E_1$ denote the restrictions of $E$ to $N_0$ and $N_1$ respectively.

Let $M'$ be given by (5.6) and let $E' \to M'$ be the bundle over $M'$ whose restriction to $M$ is equal to $E$ and whose restriction to the cylinder $[0,1] \times N_1$ is equal to $[0,1] \times E_1$.

We fix a smooth function $\rho : [0,1] \times [0,1] \to [0,1]$ such that for each $r \in [0,1]$

- the function $s \mapsto \rho(r,s)$ is non-decreasing.

- $\rho(r,s) = 0$ for $s \leq 1/3$ and $\rho(r,s) = r$ for $s \geq 2/3$.

Consider the family of strongly Callias-type operators $\mathcal{D}^r : C^\infty(M', E') \to C^\infty(M', E')$ whose restriction to $M$ is equal to $\mathcal{D}$ and whose restriction to $[0,1] \times N_1$ is given by

$$\mathcal{D}^r := c(dt) \begin{pmatrix} \partial_t + (A^0(r,t) & 0 \\ 0 & -A^0(r,t) \end{pmatrix}.$$  

Then $\mathcal{D}^r$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^r$. In particular, the restriction of $\mathcal{D}^r$ to $N_1$ is equal to $-\mathcal{A}^r$.

Recall that we denote by $-\mathcal{A}$ the family $\{ -\mathcal{A}^s \}_{0 \leq s \leq 1}$. Let $\mathcal{P} = \{ P^s \}$ be a spectral section for $-\mathcal{A}$. Then for each $r \in [0,1]$ the space $B^r := \ker P^r$ is a finite rank perturbation of the APS boundary condition for $\mathcal{D}^r$ at $\{1\} \times N_1$. Let $B_0 := H^{1/2}_{(\infty,0)}(A_0)$ be the APS boundary condition for $\mathcal{D}^r$ at $N_0$. Then, by Proposition 2.21, the operator $\mathcal{D}^r_{B_0 \oplus B^r}$ is Fredholm. Recall that the domain $\text{dom} \mathcal{D}^r_{B_0 \oplus B^r}$ consists of sections $u$ whose restriction to $\partial M' = N_0 \sqcup N_1$ lies in $B_0 \oplus B^r$.

**Lemma 5.12.** $\text{ind} \mathcal{D}^r_{B_0 \oplus B^r} = \text{ind} \mathcal{D}^r_{B_0 \oplus B^r}$ for all $r \in [0,1]$.

**Proof.** For $r_0, r \in [0,1]$, let $\pi_{r_0 r} : B^{r_0} \to B^r$ denote the orthogonal projection. Then for every $r_0 \in [0,1]$ there exists $\varepsilon > 0$ such that if $|r - r_0| < \varepsilon$ then $\pi_{r_0 r}$ is an isomorphism. As in the proof of [12] Theorem 5.11, it induces an isomorphism

$$\Pi_{r_0 r} : \text{dom} \mathcal{D}^r_{B_0 \oplus B^{r_0}} \to \text{dom} \mathcal{D}^r_{B_0 \oplus B^r}.$$

Hence

$$\text{ind} \left( \mathcal{D}^r_{B_0 \oplus B^r} \circ \Pi_{r_0 r} \right) = \text{ind} \mathcal{D}^r_{B_0 \oplus B^r}. \quad (5.7)$$

Since for $|r - r_0| < \varepsilon$

$$\mathcal{D}^r_{B_0 \oplus B^r} \circ \Pi_{r_0 r} : \text{dom} \mathcal{D}^{r_0}_{B_0 \oplus B^{r_0}} \to L^2(M', E')$$

is a continuous family of bounded operators, $\text{ind} \mathcal{D}^r_{B_0 \oplus B^r} \circ \Pi_{r_0 r}$ is independent of $r$. The lemma follows now from (5.7). \qed
5.13. **Variation of the reduced relative $\eta$-invariant.** By Definition 4.3, the mod $\mathbb{Z}$ reduction of the relative $\eta$-invariant is given by

$$\bar{\eta}(A^r, A_0) := -2 \int_{M'} \alpha_{AS}(D^r).$$

(5.8)

It follows that $\bar{\eta}(A^r, A_0)$ depends smoothly on $r$ and

$$\frac{d}{dr} \bar{\eta}(A^r, A_0) = -2 \int_{M'} \frac{d}{dr} \alpha_{AS}(D^r).$$

(5.9)

A more explicit local expression for the right hand side of this equation is given in Section 5.15. For the moment we just note that (5.9) implies that

$$\int_0^1 \left( \frac{d}{ds} \bar{\eta}(A^s, A_0) \right) ds = -2 \int_{M'} \left( \alpha_{AS}(D^1) - \alpha_{AS}(D^0) \right).$$

(5.10)

5.14. **Proof of Theorem 5.10**

Since the operators $A_0, A^0,$ and $A^1$ are invertible, we have

$$\eta(A^j, A_0) = 2 \left( \text{ind} D_{B_0 \oplus B_0^j}^1 - \int_{M'} \alpha_{AS}(D^j) \right), \quad j = 0, 1.$$

Thus, using (5.10), we obtain

$$\eta(A^1, A_0) - \eta(A^0, A_0) - \int_0^1 \left( \frac{d}{ds} \bar{\eta}(A^s, A_0) \right) ds = 2 \left( \text{ind} D_{B_0 \oplus B_0^1}^1 - \text{ind} D_{B_0 \oplus B_0^0}^0 \right).$$

(5.11)

Recall that, by Proposition 2.21,

$$\text{ind} D_{B_0 \oplus B^r}^r = \text{ind} D_{B_0 \oplus B_0^r}^r + [B^r, B_0^r].$$

Hence, from (5.11) we obtain

$$\frac{1}{2} \left( \eta(A^1, A_0) - \eta(A^0, A_0) - \int_0^1 \left( \frac{d}{ds} \bar{\eta}(A^s, A_0) \right) ds \right) = \left( \text{ind} D_{B_0 \oplus B^1}^1 - [B^1, B_0^1] \right) - \left( \text{ind} D_{B_0 \oplus B^0}^0 - [B^0, B_0^0] \right) \text{Lemma 5.12} = -\text{sf}(-\mathfrak{A}) \text{ Lemma 5.3} \text{sf}(-\mathfrak{A}).$$

5.15. **A local formula for variation of the reduced relative $\eta$-invariant.** It is well known that there exists a family of differential forms $\beta_r$ ($0 \leq r \leq 1$), called the *transgression form* such that

$$d\beta_r = \frac{d}{dr} \alpha_{AS}(D^r).$$

(5.12)

The transgression form depends on the symbol of $D^r$ and its derivatives with respect to $r$. For geometric Dirac operators one can write very explicit formulas for $\beta_r$. For example, if $D^r$ is the signature operator (so that $A^r$ is the odd signature operator) corresponding to a family $\nabla^r$ of flat connections on $E$, then $\beta_r = L(M) \wedge \frac{d}{dr} \nabla^r$, where $L(M)$ is the $L$-genus of $M$, cf. for example, [9, Theorem 2.3]. For general Dirac-type operators, a formula for $\beta_r$ is more complicated, cf. [10, §6].

We note that since the family $A^r$ is constant outside of the compact set $K$, the form $\beta_r$ vanishes outside of $K$. Hence, $\int_{\partial M'} \beta_r$ is well defined and finite. Thus we obtain from (5.9) that

$$\frac{d}{dr} \bar{\eta}(A^r, A_0) = -2 \int_{M'} \beta_r = -2 \int_{\partial M'} \beta_r = 2 \left( \int_{\{1\} \times N_1} \beta_r - \int_{N_0} \beta_r \right).$$

(5.13)

Hence, (5.5) expresses $\eta(A^1, A_0) - \eta(A^0, A_0)$ as a sum of $2 \text{sf}(-\mathfrak{A})$ and a local differential geometric expression $2 \int_{\partial M'} (\beta_0^1 \beta_r) dr$. 

□
References

[1] N. Anghel, *$L^2$-index formulae for perturbed Dirac operators*, Comm. Math. Phys. **128** (1990), no. 1, 77–97.

[2] N. Anghel, *On the index of Callias-type operators*, Geom. Funct. Anal. **3** (1993), no. 5, 431–438.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), no. 1, 43–69.

[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99.

[5] C. Bär and W. Ballmann, *Boundary value problems for elliptic differential operators of first order*, Surveys in differential geometry. Vol. XVII, 2012, pp. 1–78.

[6] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, 1992.

[7] R. Bott and R. Seeley, *Some remarks on the paper of Callias: “Axial anomalies and index theorems on open spaces” [Comm. Math. Phys. 62 (1978), no. 3, 213–235; MR 80h:58045a]*, Comm. Math. Phys. **62** (1978), no. 3, 235–245.

[8] M. Braverman and S. Cecchini, *Callias-type operators in von Neumann algebras*, The Journal of Geometric Analysis **28** (2018), no. 1, 546–586.

[9] M. Braverman and T. Kappeler, *Refined Analytic Torsion*, J. Differential Geom. **78** (2008), no. 1, 193–267.

[10] M. Braverman and G. Mascher, *Equivariant APS index for dirac operators of non-product type near the boundary*, arXiv preprint [arXiv:1702.08105], to appear in Indiana University Mathematics Journal.

[11] M. Braverman and P. Shi, *Cobordism invariance of the index of Callias-type operators*, Comm. Partial Differential Equations **41** (2016), no. 8, 1183–1203. MR3532391

[12] M. Braverman and P. Shi, *The Atiyah-Patodi-Singer index on manifolds with non-compact boundary*, arXiv preprint [arXiv:1706.06737] (201706).

[13] J. Brüning and H. Moscovici, *$L^2$-index for certain Dirac-Schrödinger operators*, Duke Math. J. **66** (1992), no. 2, 311–336. MR1162192 (93g:58142)

[14] U. Bunke, *A $K$-theoretic relative index theorem and Callias-type Dirac operators*, Math. Ann. **303** (1995), no. 2, 241–279. MR1348799 (96e:58148)

[15] C. Callias, *Axial anomalies and index theorems on open spaces*, Comm. Math. Phys. **62** (1978), no. 3, 213–235.

[16] C. Carvalho and V. Nistor, *An index formula for perturbed Dirac operators on Lie manifolds*, The Journal of Geometric Analysis **24** (2014), no. 4, 1808–1843 (English).

[17] X. Dai and W. Zhang, *Higher spectral flow*, J. Funct. Anal. **157** (1998), no. 2, 432–469. MR1638328

[18] J. Fox and P. Haskell, *Heat kernels for perturbed Dirac operators on even-dimensional manifolds with bounded geometry*, Internat. J. Math. **14** (2003), no. 1, 69–104. MR1955511

[19] J. Fox and P. Haskell, *The Atiyah-Patodi-Singer theorem for perturbed Dirac operators on even-dimensional manifolds with bounded geometry*, New York J. Math. **11** (2005), 303–332. MR2154358

[20] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition. MR1335452

[21] C. Kottke, *An index theorem of Callias type for pseudodifferential operators*, J. K-Theory **8** (2011), no. 3, 387–417. MR2863418

[22] C. Kottke, *A Callias-type index theorem with degenerate potentials*, Comm. Partial Differential Equations **40** (2015), no. 2, 219–264. MR3277926

[23] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, New Jersey, 1989.

[24] R. B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the $b$-calculus*, J. Differential Geom. **46** (1997), no. 1, 99–180. MR1472895

[25] L. I. Nicolaescu, *The Maslov index, the spectral flow, and decompositions of manifolds*, Duke Math. J. **80** (1995), no. 2, 485–533. MR1369400

[26] P. Shi, *The index of Callias-type operators with Atiyah-Patodi-Singer boundary conditions*, Ann. Glob. Anal. Geom. **52** (2017), no. 4, 465–482. MR3735908

[27] R. Wimmer, *An index for confined monopoles*, Comm. Math. Phys. **327** (2014), no. 1, 117–149. MR3177934
