STABILIZED FINITE ELEMENT METHODS BASED ON MULTISCALE ENRICHMENT FOR ALLEN-CAHN AND CAHN-HILLIARD EQUATIONS

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Abstract. In this paper, we investigate fully discrete schemes for the Allen-Cahn and Cahn-Hilliard equations respectively, which consist of the stabilized finite element method based on multiscale enrichment for the spatial discretization and the semi-implicit scheme for the temporal discretization. With reasonable stability conditions, it is shown that the proposed schemes are energy stable. Furthermore, by defining a new projection operator, we deduce the optimal $L^2$ error estimates. Some numerical experiments are presented to confirm the theoretical predictions and the efficiency of the proposed schemes.

1. Introduction. In this paper, we consider fully discretization schemes for the Allen-Cahn equation

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \Delta \varphi - \frac{1}{\epsilon^2} f(\varphi), \quad (x,t) \in \Omega \times (0,T], \\
\varphi \big|_{t=0} &= \varphi_0(x), \quad x \in \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0, \quad (x,t) \in \partial \Omega \times (0,T],
\end{aligned}
$$

(1.1)
and the Cahn-Hilliard equation

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= \Delta(-\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi)), \quad (x, t) \in \Omega \times (0, T), \\
\varphi \big|_{t=0} &= \varphi_0(x), \quad x \in \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0, \quad \frac{\partial (\Delta \varphi - \frac{1}{\epsilon^2} f(\varphi))}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times (0, T),
\end{align*}
\]

(1.2)

where \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) is a bounded domain with a Lipschitz continuous boundary \( \partial \Omega \), \( n \) is the outward normal on \( \Omega \), \( \varphi(x, t) \) is a scalar function represents the concentration of one of the two metallic components of the alloy, \( \varphi_0(x) \) is a given initial function, \( \epsilon \) is the interfacial width, \( T \) is a given time, the reaction term \( f(\varphi) = F'(\varphi) \) where \( F(\varphi) \) is a given energy potential. An important feature of the Allen-Cahn and Cahn-Hilliard equations is that they can be viewed as the gradient flow of the Liapunov energy function

\[
E(\varphi) = \int_{\Omega} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon^2} F(\varphi) \right) dx
\]

(1.3)

in \( L^2 \)-space and \( H^{-1} \)-space, respectively. By taking the inner product of (1.1) with \( -\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi) \), we can get the following energy law for (1.1):

\[
\frac{\partial E(\varphi)}{\partial t} = -\int_{\Omega} | -\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi) |^2 dx,
\]

(1.4)

and similarly, the energy law for (1.2) is

\[
\frac{\partial E(\varphi)}{\partial t} = -\int_{\Omega} | \nabla (-\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi)) |^2 dx.
\]

(1.5)

(1.4) and (1.5) imply that the free energy decreases monotonically with time. In 1979, the Allen-Cahn equation was originally proposed to describe the motion of antiphase boundaries in crystalline solids[2]. In 1958, Cahn and Hilliard introduced the Cahn-Hilliard equation to describe the complicated phase separation and coarsening phenomena in the solid[7]. The two boundary conditions imply that the mixture can not pass through the boundary walls.

The Allen-Cahn and Cahn-Hilliard equations have been used in many complicated moving interface problems in materials science and fluid dynamics extensively by the phase-field approach[8, 23, 9, 5, 6, 25, 10, 33, 19]. As we know that the Allen-Cahn and Cahn-Hilliard equations need to satisfy the energy law (1.4) and (1.5) respectively. To design accurate and efficient numerical schemes that satisfy the corresponding discrete energy law is very important for solving the Allen-Chan and Cahn-Hilliard equations.

Many numerical schemes had been proposed for solving the Allen-Cahn and Cahn-Hilliard equations recently. It is well-known that the explicit schemes lead to severe time step restrictions and also do not satisfy the discrete energy law. Hence, the semi-implicit (implicit/explicit) and fully implicit schemes attract much attention. The advantage of semi-implicit schemes is that it only involves to solve an elliptic equation with constant coefficients at each time step, making it easy to implement via fast elliptic solvers[4, 17]. However, the semi-implicit schemes usually have larger truncation errors and require smaller time steps to hold accuracy and the energy stability. On the other hand, the implicit schemes can be designed easily, which satisfy energy law and have smaller truncation errors. But the implicit schemes need to solve a nonlinear equation at each time step. Our aim is to design
a stabilized semi-implicit schemes based on multiscale enrichment method that can satisfy a discrete energy law, and partially alleviate the restriction on time steps.

It is remarkable that great developments about the stabilized semi-implicit numerical methods had been achieved for solving the Allen-Cahn and Cahn-Hilliard equations. In 2010, Shen and Yang proposed several stabilized semi-implicit numerical schemes for the Allen-Cahn and Cahn-Hilliard equations with the semi-implicit schemes in time and the spectral-Galerkin approximation in space, and gave the optimal error estimates and stability analysis[26]. In 2009, Yang designed a stabilized first-order semi-implicit scheme in time and the splitting scheme for the Allen-Cahn equation[31] and with the spectral-Galerkin approximation in space. In 2013, Feng et al. introduced the stabilized semi-implicit Crank-Nicolson/Adams-Bashforth schemes (in time) for the Allen-Cahn and Cahn-Hilliard equations with finite element method (FEM) approximation in space[13]. Liu and Shen in 2015 constructed a stabilized semi-implicit spectral deferred correction methods in time discretization for Allen-Cahn and Cahn-Hilliard equations with the Legendre-Galerkin approximation in space[20]. There are many other significant works that deal with the Allen-Cahn and Cahn-Hilliard equations[34, 15, 28, 16, 32, 24, 21, 12, 18, 14, 22, 27].

In this paper, we restrict our attention to the reaction term \( f(\varphi) \) satisfies the following condition: there exists a positive constant \( L \) such that

\[
\max_{\varphi \in \mathbb{R}} |f'(\varphi)| \leq L. \tag{1.6}
\]

We proposed a fully-discrete multiscale finite element method for the Allen-Cahn and Cahn-Hilliard equations. In detail, we use semi-implicit differential schemes in the time discretization. The nonlinear reaction term \( f(\varphi) \) is treated explicitly in order to improve the numerical computation and reduce iterations. But the nonlinear term \( f(\varphi) \) will get a severe stability limitation on the time step while adopting a usual explicit scheme. For overcoming the difficulty, we use a stabilized multiscale finite element method in space discretization. The multiscale finite element method is based on multiscale enrichment method[3]. Only a stabilized term needs to be introduced to enhance the stability while keeping the accuracy and simplicity. And the coefficient of the stabilized term is known exactly. The optimal error estimates for the fully-discrete multiscale finite element schemes are obtained.

The rest of the paper is organized as follows. In Section 2, we consider the Allen-Cahn equation with the multiscale finite element method in the space discretization and Euler semi-implicit scheme in the time discretization. We show the fully-discrete scheme is energy stable and also derive the optimal error estimates. In Section 3, we consider the Cahn-Hilliard equation and a similar error analysis and energy stability. Some numerical examples are presented in Section 4 and the concluding remarks are given in the final section.

2. Fully discrete multiscale finite element method for the Allen-Cahn equation. Firstly, we introduce some notations and the Sobolev spaces. Let \( L^2(\Omega) \) be endowed with \( L^2 \)-scalar product \( \langle \cdot , \cdot \rangle \) and \( L^2 \)-norm \( \| \cdot \|_0 \). Meanwhile, we use the standard Sobolev spaces \( W^{m,p}(\Omega) \) equipped with the norm \( \| \cdot \|_{m,p} \) where \( m \geq 0, p > 1 \). Especially, denote by \( \| \cdot \|_p = \| \cdot \|_{0,p} \) the norm on space \( L^p(\Omega) \). The space \( L^q(0, T; W) \) (where \( q \in [1, \infty), T > 0, W \) is a Hilbert space) is constructed by the set of functions defined on \( (0, T) \) into \( W \) that are \( q \)-integrable and \( L^q(0, T; W) \)
is equipped with the usual norm
\[ \| \cdot \|_{L^q(0,T;W)} = \left( \int_0^T \| \cdot \|_W^q \right)^{\frac{1}{q}} dt. \]

With above notations, the variational weak form for the Allen-Cahn equation (1.1) is:
find \( \varphi \in H^1(\Omega) \) for any \( t \in (0, T] \) such that
\[ (\frac{\partial \varphi}{\partial t}, v) + (\nabla \varphi, \nabla v) + \frac{1}{\varepsilon^2} (f(\varphi), v) = 0, \quad \forall v \in H^1(\Omega). \] (2.1)

2.1. Multiscale FEM for the Allen-Cahn equation. In this subsection, we extend the stabilized finite element method based on multiscale enrichment which is used in [3] to the Allen-Cahn equation. Consider the multiscale finite element method for the stationary Allen-Cahn equation of (1.1)
\[ \begin{cases} -\Delta \varphi + \frac{1}{\varepsilon^2} f(\varphi) = 0, & x \in \Omega, \\ \frac{\partial \varphi}{\partial n} = 0, & x \in \partial \Omega. \end{cases} \] (2.2)

Here, let \( \Gamma_h \) be a regular triangulation of \( \Omega \), formed by closed triangle elements \( K \) with boundary \( \partial K \). Denote by \( \{p_j, j = 1, 2, ..., N\} \) the set of vertices, denote by \( \varepsilon_h \) the set of interior edges of \( \Gamma_h \) that are no part of \( \partial \Omega \). Furthermore, \( h_K \) denotes the diameter of the element \( K \), and \( h = \max_{K \in \Gamma_h} h_K \) is the mesh parameter. We use the subspace \( V_h \) of \( H^1(\Omega) \) to denote the standard finite element space of continuous piecewise polynomials of degree \( l (1 \leq l \leq 2) \) for the scalar function \( \varphi \) on \( \Gamma_h \).

The standard variational formulation associated with the stationary Allen-Cahn equation (2.2) reads: find \( \varphi \in H^1(\Omega) \) such that
\[ (\nabla \varphi, \nabla v) + \frac{1}{\varepsilon^2} (f, v) = 0, \quad \forall v \in H^1(\Omega). \]

Let \( E_h \subset H^1_0(\Omega) \) be a finite-dimensional space (called multiscale space) satisfying \( V_h \cap E_h = 0 \). Then, we propose a Petrov-Galerkin approximation for the stationary Allen-Cahn equation: find \( \varphi + \varphi_e \in V_h \oplus E_h \) such that
\[ (\nabla (\varphi + \varphi_e), \nabla v_h) + \frac{1}{\varepsilon^2} (f, v_h) = 0, \quad \forall v_h \in V_h \oplus H^1_0(\Gamma_h), \] (2.3)

where \( H^1_0(K) \) is the space of functions whose restriction to \( K \in \Gamma_h \) belongs to \( H^1_0(K) \). And (2.3) is equivalent to the following system
\[ (\nabla (\varphi + \varphi_e), \nabla v) + \frac{1}{\varepsilon^2} (f, v) = 0, \quad \forall v \in V_h, \] (2.4)
\[ (\nabla (\varphi + \varphi_e), \nabla v_e)_K + \frac{1}{\varepsilon^2} (f, v_e)_K = 0, \quad \forall v_e \in H^1_0(K). \] (2.5)

(2.5) can be rewritten as
\[ (-\Delta \varphi_e, v_e)_K = (-\frac{1}{\varepsilon^2} f + \Delta \varphi, v_e)_K, \quad \forall v_e \in H^1_0(K), \] (2.6)

which, in strong form, follows
\[ -\Delta \varphi_e = -\frac{1}{\varepsilon^2} f + \Delta \varphi, \quad \text{in } K. \] (2.7)

To make the above differential problem well-posed, we will introduce the following boundary condition on \( \varphi_e \):
\[ \varphi_e = g_e, \quad \text{on each } Z \subset \partial K, \] (2.8)
where \( \frac{\partial g_e}{\partial n} = 0 \) if \( Z \subset \partial \Omega \) and \( g_e \) is the solution of the problem
\[
-\partial_{ss} g_e = \frac{1}{h_Z} [ [\partial_n \varphi] ] , \quad \text{in } Z, \\
\varphi_e = 0, \quad \text{at the nodes},
\]
on the internal edges, where \( h_Z = |Z|, n \) is the normal outward vector on \( \partial K \), \( \partial_s, \partial_n \) are the tangential and normal derivative operators, respectively, \( [ [\varphi] ] \) stands for the jump of \( \varphi \) across \( Z \).

Now, on each \( K \in \Gamma_h \), we can write \( \varphi_e|_K = \varphi^K_e + \varphi^{\partial K}_e \), where
\[
-\Delta \varphi^K_e = -\frac{1}{\epsilon^2} f + \Delta \varphi, \quad \text{in } K, \\
\varphi^K_e = 0, \quad \text{on } \partial K,
\]
and
\[
-\Delta \varphi^{\partial K}_e = 0, \quad \text{in } K, \\
\varphi^{\partial K}_e = g_e, \quad \text{on } \partial K.
\]
In this way, we can define two operators \( M_K : L^2(K) \rightarrow H^1_0(K) \) and \( N_K : L^2(\partial K) \rightarrow H^1(\Omega) \) such that
\[
\varphi^K_e = M_K(-\frac{1}{\epsilon^2} f + \Delta \varphi), \quad \forall K \in \Gamma_h,
\]
and
\[
\varphi^{\partial K}_e = N_K([ [\partial_n \varphi] ]), \quad \forall K \in \Gamma_h.
\]
Since the enriched part \( \varphi_e \) is well defined via (2.15) and (2.16), we can perform statical condensation to derive a stabilized finite element method for the Allen-Cahn equation (2.2). First, integrating by parts, we get
\[
(\nabla \varphi_e, \nabla v)_K = - (\varphi_e, \Delta v)_K + (\varphi_e, \partial_n v)_{\partial K}.
\]
Then, using the identity (2.17) we can rewrite (2.4) in the following way:
\[
(\nabla \varphi, \nabla v) + \sum_{K \in \Gamma_h} [ - (\varphi_e, \Delta v)_K + (\varphi_e, \partial_n v)_{\partial K} ] + \frac{1}{\epsilon^2} (f, v) = 0,
\]
which means by using (2.15) and (2.16)
\[
(\nabla \varphi, \nabla v) + \sum_{K \in \Gamma_h} [ - (M_K(-\frac{1}{\epsilon^2} f + \Delta \varphi) + N_K([ [\partial_n \varphi] ]), \Delta v)_K \\
+ (N_K([ [\partial_n \varphi] ]), \partial_n v)_{\partial K} ] + \frac{1}{\epsilon^2} (f, v) = 0.
\]
Selecting the finite element space
\[
V_h = \{ \varphi \in C^0(\Omega) : \varphi|_K \in P_1(K), \forall K \in \Gamma_h \} \cap H^1(\Omega),
\]
where \( P_1(K) \) denotes the space of linear functions in domain \( K \). We propose the following stabilized finite element method for stationary Allen-Cahn equation (2.2): find \( \varphi_h \in V_h \) such that
\[
B(\varphi, v_h) = 0, \quad \forall v_h \in V_h,
\]
where
\[
B(\varphi_h, v_h) = (\nabla \varphi_h, \nabla v_h) + \sum_{Z \in e_h} \tau_Z ([ [\partial_n \varphi_h] ], [ [\partial_n v_h] ])_Z + \frac{1}{\epsilon^2} (f(\varphi), v_h),
\]
here \( \tau_Z = \frac{\epsilon^2 h_Z}{12} \).
Remark 2. When deducing the multiscale finite element method for the Allen-Cahn /Cahn-Hilliard equations in this paper, the nonlinear reaction term $f(\varphi)$ is viewed as a whole term, which is a very simple and rough way. The better way to deal with the nonlinear reaction term $f(\varphi)$ will be studied in the further.

Remark 2. The stabilized coefficient $\tau_Z = \frac{c^2 h^2}{12}$ is selected according to [3] and our numerical experiments here, which can be adjusted for different cases.

2.2. Semi-implicit multiscale FEM for Allen-Cahn equation. Extending the above stabilized finite element method for stationary Allen-Cahn equation (2.2) to the transient Allen-Cahn equation (1.1), we define $\varphi_h^0 = \mathcal{P}_h \varphi_0$ and obtain the following semi-implicit fully multiscale finite element method for the transient case: find $\varphi_h^n \in V_h$ such that for $n \geq 1$

$$
\frac{1}{\tau} (\varphi_h^n - \varphi_h^{n-1}, v_h) + (\nabla \varphi_h^n, \nabla v_h) + \sum_{Z \in \mathcal{E}_h} \tau_Z ([[\partial_n \varphi_h^n]], [[\partial_n v_h]]) Z
$$

$$
+ \frac{1}{\epsilon^2} (f(\varphi_h^{n-1}), v_h) = 0,
$$

(2.23)

for any $v_h \in V_h$, $\tau \in (0, 1)$ is the time step, $\varphi_h^n$ denotes the fully discrete approximation of $\varphi(x,t,n)$ and $\mathcal{P}_h : L^2(\Omega) \rightarrow V_h$ stands for the usual $L^2$-projection such that

$$(\mathcal{P}_h v, v_h) = (v, v_h), \quad \forall v_h \in V_h.$$  

Furthermore, the following estimate holds for $\mathcal{P}_h$[29, 11]:

$$
\|v - \mathcal{P}_h v\|_0 \leq C h^m \|v\|_m, \quad \forall v \in H^m(\Omega), m \geq 0.
$$

(2.24)

Hereafter, $C$ denotes a positive constant, which may take different values at different places.

Lemma 2.1. If the condition (1.6) is satisfied and

$$
\tau \leq \frac{2\epsilon^2}{L},
$$

where $L$ is given by (1.6), then for the stabilized finite element method, the following energy stability property (2.23) holds

$$
\dot{E}(\varphi_h^n) \leq \dot{E}(\varphi_h^{n-1}), \quad \forall n \geq 1,
$$

(2.25)

where

$$
\dot{E}(\varphi_h^n) \triangleq E(\varphi_h^n) + \sum_{Z \in \mathcal{E}_h} \tau_Z \|[[\partial_n \varphi_h^n]]_{0,Z} \|_{0,Z}.
$$

(2.26)

Proof. Taking $v_h = \varphi_h^n - \varphi_h^{n-1}$ in (2.23) and using the formulation $(a - b, 2a) = a^2 - b^2 + (a - b)^2$, we get

$$
\frac{1}{\tau} \|\varphi_h^n - \varphi_h^{n-1}\|_0^2 + \frac{1}{2} \|\nabla \varphi_h^n\|_0^2 - \|\nabla \varphi_h^{n-1}\|_0^2 + \|\nabla (\varphi_h^n - \varphi_h^{n-1})\|_0^2
$$

$$
+ \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z ([[\partial_n \varphi_h^n]]_{0,Z} - [[\partial_n \varphi_h^{n-1}]]_{0,Z} + \|[[\partial_n (\varphi_h^n - \varphi_h^{n-1})]]_{0,Z} \|_{0,Z}^2
$$

$$
+ \frac{1}{\epsilon^2} (f(\varphi_h^{n-1}), \varphi_h^n - \varphi_h^{n-1}) = 0.
$$

(2.27)

By using the following Taylor’s expansion

$$
F(\varphi_h^n) - F(\varphi_h^{n-1}) = f(\varphi_h^{n-1})(\varphi_h^n - \varphi_h^{n-1}) + \frac{1}{2} f'(\xi^n)(\varphi_h^n - \varphi_h^{n-1})^2
$$

(2.28)
and the inequality (1.6) in the formulation (2.27), we get

\[
\frac{1}{dt}||\varphi^n_h - \varphi^{n-1}_h||_0^2 + \frac{1}{2}||\nabla\varphi^n_h||_0^2 - ||\nabla\varphi^{n-1}_h||_0^2 + ||\nabla(\varphi^n_h - \varphi^{n-1}_h)||_0^2 \\
+ \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z (||[\partial_n \varphi^n_h]||_{0,Z}^2 - ||[\partial_n \varphi^{n-1}_h]||_{0,Z}^2 + ||[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]||_{0,Z}^2) \\
+ \frac{1}{\epsilon^2} (F(\varphi^n_h) - F(\varphi^{n-1}_h), 1) \\
= \frac{1}{2\epsilon^2} f'(\xi) (\varphi^n_h - \varphi^{n-1}_h, \varphi^n_h - \varphi^{n-1}_h) \leq \frac{L}{2\epsilon^2} ||\varphi^n_h - \varphi^{n-1}_h||_0^2. \tag{2.29}
\]

(2.29) can be rewritten as

\[
\frac{1}{dt} - \frac{L}{2\epsilon^2} ||\varphi^n_h - \varphi^{n-1}_h||_0^2 + \frac{1}{2}||\nabla\varphi^n_h||_0^2 + \frac{1}{\epsilon^2} (F(\varphi^n_h), 1) \Omega + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z [||[\partial_n \varphi^n_h]||_{0,Z}^2 + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z [||[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]||_{0,Z}^2] \\
\leq \frac{1}{2}||\nabla\varphi^{n-1}_h||_0^2 + \frac{1}{\epsilon^2} (F(\varphi^{n-1}_h), 1) + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z [||[\partial_n \varphi^{n-1}_h]||_{0,Z}^2]. \tag{2.30}
\]

That is

\[
\frac{1}{dt} - \frac{L}{2\epsilon^2} ||\varphi^n_h - \varphi^{n-1}_h||_0^2 + \nabla \hat{E}(\varphi^n_h) + \frac{1}{2}||\nabla(\varphi^n_h - \varphi^{n-1}_h)||_0^2 \\
+ \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z [||[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]||_{0,Z}^2] \leq \hat{E}(\varphi^{n-1}_h). \tag{2.31}
\]

We have the desired result (2.25) under the condition \( dt \leq \frac{2\epsilon^2}{L} \).

### 2.3. Error analysis

In this subsection, we study the error estimate for the fully discrete multiscale stabilized finite element method. We firstly need to introduce the discrete Gronwall lemma and trace inequalities:

**Lemma 2.2.** [17] Let \( \tau, a_n, b_n \) and \( d_n \), for integers \( n \geq 0 \), be nonnegative numbers such that

\[
a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=1}^{m-1} a_n d_n + C, \quad m \geq 1, \tag{2.32}
\]

then it holds

\[
a_m + \tau \sum_{n=1}^{m} b_n \leq C \exp(\tau \sum_{n=1}^{m-1} d_n), \quad m \geq 1. \tag{2.33}
\]

**Lemma 2.3.** [1] For \( E \in \partial K \), the following trace inequalities hold

\[
||v||_{L^2(E)}^2 \leq C(h^{-1}_K ||v||_{L^2(K)}^2 + h_K ||v||_{H^1(K)}^2) \quad \forall v \in H^1(K),
\]

\[
||\partial_n v||_{L^2(E)}^2 \leq C(h_{-1}^{-1} ||v||_{H^1(K)}^2 + h_K ||v||_{H^1(K)}^2) \quad \forall v \in H^2(K).
\]

Furthermore, we define the following projection \( \mathcal{R}_h : H^1(\Omega) \to V_h \):

\[
(\nabla \mathcal{R}_h \varphi, v_h)_\Omega + \sum_{Z \in \mathcal{E}_h} \tau_Z ([\partial_n \mathcal{R}_h \varphi], [\partial_n v_h])_Z = (\nabla \varphi, \nabla v_h)_\Omega \quad \forall v_h \in V_h. \tag{2.34}
\]

**Lemma 2.4.** If \( \varphi \in H^2(\Omega) \), the projection \( \mathcal{R}_h \) satisfies

\[
||\varphi - \mathcal{R}_h \varphi||_0 + h ||\nabla(\varphi - \mathcal{R}_h \varphi)||_0 \leq C h^2 ||\varphi||_2. \tag{2.35}
\]
Proof. Let $I_h : H^2(\Omega) \to V_h$ be the usual Lagrange interpolation operator, which satisfies[11]

$$
\|v - I_h v\|_{L^2(\Omega)} + h \|v - I_h v\|_{H^1(\Omega)} \leq Ch^m \|v\|_{H^m(\Omega)} \quad m = 1,2.
$$

The formulation (2.34) can be rewritten as by using the definition of the operator $I_h$ and $[\partial_n \varphi] = 0$

$$(\nabla (R_h \varphi - I_h \varphi), \nabla v_h) + \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z, \[[\partial_n v_h]]_Z)$$

by using the definition of the operator $I_h$ and $[\partial_n \varphi] = 0$. Setting $v_h = R_h \varphi - I_h \varphi$ in (2.37), we get

$$
\|\nabla (R_h \varphi - I_h \varphi)\|_0^2 + \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 = (\nabla (\varphi - I_h \varphi), \nabla (R_h \varphi - I_h \varphi)) + \sum_{Z \in h} \tau_Z ([[\partial_n (\varphi - I_h \varphi)]]_Z, \[[\partial_n (R_h \varphi - I_h \varphi)]]_Z.
$$

Using the Hölder inequality, Lemma 2.3 and (2.36) in (2.38), it holds

$$
\|\nabla (R_h \varphi - I_h \varphi)\|_0^2 + \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 \leq Ch \|\varphi\|_2 \|\nabla (R_h \varphi - I_h \varphi)\|_0 + \left( \sum_{Z \in h} \tau_E ([[\partial_n (\varphi - I_h \varphi)]]_Z^2 \right)^{\frac{1}{2}}
$$

$$
\leq Ch \|\varphi\|_2 \|\nabla (R_h \varphi - I_h \varphi)\|_0 + \left( \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 \right)^{\frac{1}{2}} + h^2 \|\varphi - I_h \varphi\|_0^2, K \right)^{\frac{1}{2}} \left( \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 \right)^{\frac{1}{2}}
$$

$$
\leq Ch \|\varphi\|_2 \left( \|\nabla (R_h \varphi - I_h \varphi)\|_0 + \left( \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 \right)^{\frac{1}{2}} \right)
$$

$$
\leq Ch \|\varphi\|_2 \left( \|\nabla (R_h \varphi - I_h \varphi)\|_0 + \sum_{Z \in h} \tau_Z ([[\partial_n (R_h \varphi - I_h \varphi)]]_Z^2 \right)^{\frac{1}{2}},
$$

which follows

$$
\|\nabla (R_h \varphi - I_h \varphi)\|_0 \leq Ch \|\varphi\|_2.
$$

By using the triangular inequality, (2.36) and (2.40), we obtain

$$
\|\nabla (\varphi - R_h \varphi)\|_0 \leq \|\nabla (\varphi - I_h \varphi)\|_0 + \|\nabla (I_h \varphi - R_h \varphi)\|_0 \leq Ch \|\varphi\|_2.
$$

Next, we prove $\|\varphi - R_h \varphi\|_0$. Consider the following dual problem: find $\Phi \in H^1(\Omega)$ such that

$$(\nabla \Phi, \nabla v) = (\varphi - R_h \varphi, v), \quad \forall v \in H^1(\Omega).$$

Taking $v = \varphi - R_h \varphi$ in (2.42), we have

$$
(\nabla \Phi, \nabla (\varphi - R_h \varphi)) = \|\varphi - R_h \varphi\|_0^2, \quad \forall v \in H^1(\Omega).
$$
It is also easy to check that:
\[ \| \bar{\Phi} \|_2 \leq C\| \varphi - R_h \varphi \|_0, \]
(2.44)
which means that \[ (\partial_n \Phi) = 0. \]

The formulation (2.34) can be rewritten as
\[ (\nabla (\varphi - R_h \varphi), \nabla v_h) + \sum_{Z \in h} \tau_{Z} (\langle [\partial_n (\varphi - R_h \varphi)], [\partial_n v_h] \rangle)_{Z} = 0 \quad \forall \ v_h \in V_h. \]
(2.45)
Setting \( v_h = I_h \Phi \) in (2.45), we can get
\[ (\nabla (\varphi - R_h \varphi), \nabla I_h \Phi) + \sum_{Z \in h} \tau_{Z} (\langle [\partial_n (\varphi - R_h \varphi)], [\partial_n I_h \Phi] \rangle)_{Z} = 0. \]
(2.46)

Then subtracting (2.46) from (2.43), it holds
\[ \| \varphi - R_h \varphi \|_0^2 = (\nabla (\varphi - R_h \varphi), \nabla (\Phi - I_h \Phi)) \]
\[ + \sum_{Z \in h} \tau_{Z} (\langle [\partial_n (\varphi - R_h \varphi)], [\partial_n (\Phi - I_h \Phi)] \rangle)_{Z}. \]
(2.47)

By using the Hölder inequality, Lemma 2.3, (2.41) and (2.44) in the above formulation (2.47), we obtain
\[ \| \varphi - R_h \varphi \|_0^2 \leq \| \nabla (\varphi - R_h \varphi) \|_0 \| \nabla (\Phi - I_h \Phi) \|_0 + C \left( \sum_{K \in \Gamma_h} (\| \varphi - R_h \varphi \|_{1, K}^2 + h_K^2 \| \varphi - R_h \varphi \|_{0, K}^2) \right)^{\frac{1}{2}} \]
\[ \leq C h^2 \| \varphi \|_2 \| \Phi \|_2 \leq C h^2 \| \varphi \|_2, \]
(2.48)
which tells us that
\[ \| \varphi - R_h \varphi \|_0 \leq C h^2 \| \varphi \|_2. \]
(2.49)

Combining the formulation (2.41) and (2.49) leads to (2.35).

**Theorem 2.5.** Under the assumptions of Lemma 2.1, if the solution of (1.1) satisfying \( \varphi \in C(0, T; H^2(\Omega)), \varphi_t \in L^2(0, T; H^2(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) and \( \varphi_{tt} \in L^2(0, T; H^{-1}(\Omega)), \varphi_{tt}^n \) is the solution of the semi-implicit enrichment stabilized finite element scheme (2.23), then the following error estimate holds
\[ \| \varphi(t_n) - \varphi_{tt}^n \|_0 \leq C(dt + h^2), \]
(2.50)
where \( \varphi(t_n) \) denotes the exact solution \( \varphi(\cdot, t_n) \).

**Proof.** For convenience, we denote \( e^n = R_h \varphi(t_n) - \varphi_{tt}^n, \varphi(t_n) - R_h \varphi(t_n) \) and define the truncation error
\[ R^n = \varphi(t_n) - \varphi_{tt}^n. \]

By using the Taylor expansion with the integral residuals and the Cauchy-Schwarz inequality, we obtain
\[ \| R^n \|^2_i \leq \frac{1}{dt^2} \| \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \varphi_{tt}(t) dt \|^2_i \leq \frac{dt}{3} \int_{t_{n-1}}^{t_n} \| \varphi_{tt}(t) \|^2_i dt, \quad i = -1, 0. \]
(2.51)
Subtracting (2.23) from (2.1) at \( t_n \), it holds
\[
(\phi_t(t_n) - \frac{\varphi^n_h - \varphi^{n-1}_h}{dt}, v_h) + (\nabla(\phi(t_n) - \varphi^n_h), \nabla v_h)
- \sum_{Z \in \tau_h} \tau_Z([\partial_n \varphi^n_h], [\partial_n v_h])_Z + \frac{1}{\epsilon^2} (f(\phi(t_n)) - f(\varphi^{n-1}_h), v_h) = 0. \tag{2.52}
\]
By using the definition of the operator \( \mathcal{R}_h \), (2.52) is equivalent to
\[
(\phi_t(t_n) - \frac{\varphi^n_h - \varphi^{n-1}_h}{dt}, v_h) + (\nabla e^n, \nabla v_h) + \sum_{Z \in \tau_h} \tau_Z([\partial_n e^n], [\partial_n v_h])_Z
= - \frac{1}{\epsilon^2} (f(\phi(t_n)) - f(\varphi^{n-1}_h), v_h). \tag{2.53}
\]
The first term \( (\phi_t(t_n) - \frac{\varphi^n_h - \varphi^{n-1}_h}{dt}, v_h) \) on the left-hand side of (2.53) can be rewritten as
\[
(\phi_t(t_n) - \frac{\varphi^n_h - \varphi^{n-1}_h}{dt}, v_h)
= (\phi_t(t_n) - \frac{\varphi^n_h - \mathcal{R}_h \varphi(t_n) + \mathcal{R}_h \varphi(t_{n-1}) - \varphi^{n-1}_h}{dt}, v_h)
= (\phi_t(t_n), v_h) + (\frac{e^n - e^{n-1}}{dt}, v_h) - \frac{\mathcal{R}_h \varphi(t_n) - \mathcal{R}_h \varphi(t_{n-1})}{dt}, v_h)
= (\phi_t(t_n), v_h) + (\frac{e^n - e^{n-1}}{dt}, v_h)
- \frac{\mathcal{R}_h \varphi(t_n) - \varphi(t_n) + \varphi(t_{n-1}) - \mathcal{R}_h \varphi(t_{n-1}) + \varphi(t_{n-1}) - \varphi(t_{n-1})}{dt}, v_h)
= (\phi_t(t_n), v_h) + (\frac{e^n - e^{n-1}}{dt}, v_h) + \frac{1}{dt} (I - \mathcal{R}_h) (\varphi(t_n) - \varphi(t_{n-1}), v_h)
- (\frac{\varphi(t_n) - \varphi(t_{n-1})}{dt}, v_h)
= - (R^n, v_h) + (\frac{e^n - e^{n-1}}{dt}, v_h) + \frac{1}{dt} (I - \mathcal{R}_h) (\varphi(t_n) - \varphi(t_{n-1}), v_h). \tag{2.54}
\]
Combining (2.54) with (2.53) and using (2.34), we can yield
\[
(\frac{e^n - e^{n-1}}{dt}, v_h) + (\nabla e^n, \nabla v_h) + \sum_{Z \in \tau_h} \tau_Z([\partial_n e^n], [\partial_n v_h])_Z
= - \frac{1}{\epsilon^2}
(f(\phi(t_n)) - f(\varphi^{n-1}_h), v_h) + (R^n, v_h) - \frac{1}{dt} (I - \mathcal{R}_h) (\varphi(t_n) - \varphi(t_{n-1}), v_h). \tag{2.55}
\]
Setting \( v_h = 2dt e^n \) in (2.55), we can obtain
\[
\|e^n\|_0^2 - \|e^{n-1}\|_0^2 + \|e^n - e^{n-1}\|_0^2 + 2dt \|\nabla e^n\|_0^2 + \sum_{Z \in \tau_h} \tau_Z \|[\partial_n e^n]\|_0^2_Z
= - \frac{2dt}{\epsilon^2} (f(\phi(t_n)) - f(\varphi^{n-1}_h), e^n) + 2dt (R^n, e^n) - 2((I - \mathcal{R}_h) (\varphi(t_n) - \varphi(t_{n-1})), e^n)
\triangleq \sum_{i=1}^{\Delta} A_i. \tag{2.56}
\]
Now, we estimate the terms $A_i (i = 1, 2, 3)$ on the right-hand side of (2.56), respectively.

\[
A_1 = -\frac{2dt}{\epsilon^2}(f(\varphi(t_n)) - f(\varphi_h^{n-1}), e^n)
\]

\[
\leq \frac{dt}{\epsilon^2}\|e^n\|^2_0 + \frac{dt}{\epsilon^2}\|f(\varphi(t_n)) - f(\varphi_h^{n-1})\|^2_0
\]

\[
\leq \frac{dt}{\epsilon^2}\|e^n\|^2_0 + \frac{2dtL^2}{\epsilon^2}\|f(\varphi(t_n)) - f(\varphi(t_{n-1}))\|^2_0 + \|f(\varphi(t_{n-1})) - f(\varphi_h^{n-1})\|^2_0
\]

\[
\leq \frac{dt}{\epsilon^2}\|e^n\|^2_0 + \frac{2dtL^2}{\epsilon^2}\int_{t_{n-1}}^{t_n}\|\varphi_t(t)\|^2_0 dt + \frac{2dtL^2}{\epsilon^2}\|\varphi(t_{n-1}) - \varphi_h^{n-1}\|^2_0
\]

\[
\leq \frac{dt}{\epsilon^2}\|e^n\|^2_0 + \frac{2dtL^2}{\epsilon^2}\int_{t_{n-1}}^{t_n}\|\varphi(t)\|^2_0 dt + \frac{2dtL^2}{\epsilon^2}\|e^n - \tilde{e}^{n-1}\|^2_0,
\]

(2.57)

\[
A_2 \leq 2dt\|R^n\|_{-1}\|\nabla e^n\|_0 \leq \frac{dt}{2}\|\nabla e^n\|^2_0 + Cdt\|R^n\|^2_{-1}
\]

\[
\leq \frac{dt}{2}\|\nabla e^n\|^2_0 + Cdt\int_{t_{n-1}}^{t_n}\|\varphi_{tt}(t)\|^2_1 dt,
\]

(2.58)

\[
A_3 = -2((I - \mathcal{R}_h)(\varphi(t_n) - \varphi(t_{n-1})), e^n) \leq 2\|(I - \mathcal{R}_h)\varphi(t_n) - \varphi(t_{n-1})\|_0\|e^n\|_0
\]

\[
\leq \frac{dt}{\epsilon^2}\|e^n\|^2_0 + C\epsilon^2\int_{t_{n-1}}^{t_n}\|(I - \mathcal{R}_h)\varphi(t)\|^2_0 dt.
\]

(2.59)

Putting inequalities (2.57)-(2.59) into (2.56) yields

\[
\|e^n\|^2_0 - \|e^{n-1}\|^2_0 + \|e^n - e^{n-1}\|^2_0 + dt\|\nabla e^n\|^2_0 + \sum_{Z \in \mathcal{E}_h} \tau_Z \|([\partial_n e^n])\|^2_{0,Z}
\]

\[
\leq \frac{2dt}{\epsilon^2}\|e^n\|^2_0 + \frac{2dtL^2}{\epsilon^2}\int_{t_{n-1}}^{t_n}\|\varphi_t(t)\|^2_0 dt + \frac{2dtL^2}{\epsilon^2}\|e^n - \tilde{e}^{n-1}\|^2_0
\]

\[+ C\epsilon^2\int_{t_{n-1}}^{t_n}\|\varphi_{tt}(t)\|^2_1 dt + C\epsilon^2\int_{t_{n-1}}^{t_n}\|(I - \mathcal{R}_h)\varphi(t)\|^2_0 dt.
\]

(2.60)

Summing up the above inequality (2.60) for $n = 1, 2, \ldots, N$, we have

\[
\|e^N\|^2_0 - \|e^0\|^2_0 + dt\sum_{n=1}^{N}\|\nabla e^n\|^2_0
\]

\[
\leq C\frac{dtL^2}{\epsilon^2}\|\varphi_t(t)\|^2_{L^2(0,T;L^2(\Omega))} + \|\varphi_{tt}(t)\|^2_{L^2(0,T;H^{-1}(-1))} + C\epsilon^2h^4\|\varphi(t)\|^2_{L^2(0,T;H^2(\Omega))}
\]

\[+ \frac{Cdt}{\epsilon^2}\sum_{n=1}^{N}\|e^n\|^2_0 + \|e^{n-1}\|^2_0 + \|\tilde{e}^{n-1}\|^2_0.
\]

(2.61)

Applying the discrete Gronwall lemma (Lemma 2.2) and (2.24) to the above inequality (2.61), we can obtain the result (2.50) by utilizing the triangular inequality $\|\varphi(t_n) - \varphi_h^n\|_0 \leq \|e^n\|_0 + \|\tilde{e}^n\|_0$ and the inequality (2.35).

\[\square\]

3. Fully discrete multiscale finite element method for the Cahn-Hilliard equation. The mixed variational weak form for the Cahn-Hilliard equation (1.2)
is to find $\varphi \in H^1(\Omega)$, $w \in H^1(\Omega)$ for any $t \in (0,T]$ such that

$$
\begin{cases}
(\frac{\partial \varphi}{\partial t}, v) + (\nabla w, \nabla v) = 0, \ \forall v \in H^1(\Omega), \\
(\nabla \varphi, \nabla q) + \frac{1}{\epsilon^2} (f(\varphi), q) = (w, q), \ \forall q \in H^1(\Omega).
\end{cases}
$$

(3.1)

3.1. Multiscale FEM for the Cahn-Hilliard equation. Consider the stationary problem of (1.2)

$$
\begin{cases}
-\Delta(-\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi)) = 0, \ x \in \Omega, \\
\frac{\partial \varphi}{\partial n} = 0, \ -\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi) = 0, \ x \in \partial \Omega.
\end{cases}
$$

(3.2)

Setting $w = -\Delta \varphi + \frac{1}{\epsilon^2} f(\varphi)$, we obtain the mixed variational formulation as follows

$$
(\nabla w, \nabla v) = 0, \ \forall v \in H^1(\Omega),
$$

(3.3)

$$
(\nabla \varphi, \nabla q) + \frac{1}{\epsilon^2} (f(\varphi), q) = (w, q), \ \forall q \in H^1(\Omega).
$$

(3.4)

Now in order to get the multiscale finite element method for the stationary Cahn-Hilliard method, we apply the same multiscale Petrov-Galerkin aproximation for (3.3) and (3.4) as Section 2 respectively. The process for (3.3) is similar to Section 2, hence we omit it. Here, the detail for (3.4) is given as follows.

Find $\varphi + \varphi_e \in V_h \oplus E_h$ and $w \in V_h$ such that

$$
(\nabla (\varphi + \varphi_e), \nabla q_h) + \frac{1}{\epsilon^2} (f, q_h) = (w, q_h), \ \forall q_h \in V_h \oplus H^1_0(\Gamma_h).
$$

(3.5)

(3.5) is equivalent to the following system

$$
(\nabla (\varphi + \varphi_e), \nabla q) + \frac{1}{\epsilon^2} (f, q) = (w, q), \ \forall q \in V_h,
$$

(3.6)

$$
(\nabla (\varphi + \varphi_e), \nabla q_e)_K + \frac{1}{\epsilon^2} (f, q_e)_K = (w, q_e)_K, \ \forall q_e \in H^1_0(K).
$$

(3.7)

(3.7) can be rewritten as

$$
(-\Delta \varphi_e, q_e)_K = (-\frac{1}{\epsilon^2} f + \Delta \varphi + w, q_e)_K, \ \forall q_e \in H^1_0(K),
$$

(3.8)

which, in strong form, follows

$$
-\Delta \varphi_e = -\frac{1}{\epsilon^2} f + \Delta \varphi + w, \ \text{in } K.
$$

(3.9)

To make the above differential problem well-posed, we will introduce the following boundary condition on $\varphi_e$:

$$
\varphi_e = g_e, \ \text{on each } Z \subset \partial K, \quad \frac{\partial g_e}{\partial n} = 0 \text{ if } Z \subset \partial \Omega \text{ and } g_e \text{ is the solution of the problem}
$$

$$
-\partial_{ss} g_e = \frac{1}{h_Z} [\partial_n \varphi + w I \cdot n], \ \text{in } Z,
$$

$$
g_e = 0, \ \text{at the nodes}.
$$

(3.10)

Now, on each $K \in \Gamma_h$, we can write $\varphi_e|_K = \varphi^K_e + \varphi^{\partial K}_e$, where

$$
-\Delta \varphi^K_e = -\frac{1}{\epsilon^2} f + \Delta \varphi + w, \ \text{in } K,
$$

(3.11)
Then, using the identity (3.16) we can rewrite (3.6) in the following way:

\[
\frac{\partial \varphi_c^K}{\partial n} = 0, \text{ on } \partial K, \tag{3.12}
\]

and

\[
-\Delta \varphi_c^K = 0, \text{ in } K,
\]

\[
\varphi_c^K = g_c, \text{ on } \partial K. \tag{3.13}
\]

In this way, we can define two operators \( \mathcal{M}_K : L^2(K) \rightarrow H_0^1(K) \) and \( \mathcal{N}_K : L^2(\partial K) \rightarrow H^1(\partial K) \) such that

\[
\varphi_c^K = \mathcal{M}_K(-\frac{1}{\epsilon^2}f + \Delta \varphi + w), \ \forall K \in \Gamma_h, \tag{3.14}
\]

and

\[
\varphi_c^{\partial K} = \mathcal{N}_K([\partial_n \varphi + w], \forall K \in \Gamma_h. \tag{3.15}
\]

Since the enriched part \( \varphi_c \) is well defined via (3.14) and (3.15), we can perform statical condensation to derive a stabilized finite element method for the Cahn-Hilliard equation (3.2). First, integrating by parts, we get

\[
(\nabla \varphi_c, \nabla q)_K = -(\varphi_c, \Delta q)_K + (\varphi_c, \partial_n q)_{\partial K}. \tag{3.16}
\]

Then, using the identity (3.16) we can rewrite (3.6) in the following way:

\[
(\nabla \varphi, \nabla q) + \sum_{K \in \tau_h} \left[ -(\varphi_c, \Delta q)_K + (\varphi_c, \partial_n q)_{\partial K} \right] + \frac{1}{\epsilon^2}(f, q) = (w, q), \tag{3.17}
\]

which means by using (3.14) and (3.15)

\[
(\nabla \varphi, \nabla q) + \sum_{K \in \tau_h} \left[ -(\mathcal{M}_K(-\frac{1}{\epsilon^2}f + \Delta \varphi + w) + \mathcal{N}_K([\partial_n \varphi + w], \Delta q)_K 
\right.

+ \left. (\mathcal{N}_K([\partial_n \varphi + w]), \partial_n q)_{\partial K} \right] + \frac{1}{\epsilon^2}(f, q) = (w, q). \tag{3.18}
\]

Selecting the same finite element space \( V_h \) as (2.20) in Section 2, we propose the following stabilized finite element method for stationary Cahn-Hilliard equation (3.2): find \( \varphi_h \in V_h \) and \( w_h \in V_h \) such that

\[
\mathcal{C}(w_h, v_h) = 0, \quad \forall v_h \in V_h, \tag{3.19}
\]

\[
\mathcal{C}(\varphi_h, q_h) + \frac{1}{\epsilon^2}(f(\varphi), q_h) = (w_h, q_h), \quad \forall q_h \in V_h, \tag{3.20}
\]

where

\[
\mathcal{C}(\varphi_h, q_h) = (\nabla \varphi_h, \nabla q_h) + \sum_{\otimes \in \tau_h} \tau_Z([\partial_n \varphi_h], [\partial_n q_h])_Z. \tag{3.21}
\]

3.2. Semi-implicit multiscale FEM for Cahn-Hilliard equation. Extending the above stabilized finite element method for the stationary Cahn-Hilliard equation (3.2) to the transient Cahn-Hilliard equation (1.2), we obtain the following semi-implicit fully discrete multiscale mixed finite element method for the transient case: given \( \varphi_h^0 = \mathcal{P}_h \varphi_0 \), find \( \varphi_h^n \in V_h, w_h^n \in V_h \) such that for \( n \geq 1 \)

\[
\frac{1}{dt}(\varphi_h^n - \varphi_h^{n-1}, v_h) + (\nabla w_h^n, \nabla v_h) + \sum_{\otimes \in \tau_h} \tau_Z([\partial_n w_h^n], [\partial_n v_h])_Z = 0, \tag{3.22}
\]

\[
(\nabla \varphi_h^n, \nabla q_h) + \sum_{\otimes \in \tau_h} \tau_Z([\partial_n \varphi_h^n], [\partial_n q_h])_Z + \frac{1}{\epsilon^2}(f(\varphi_h^{n-1}), q_h) = (w_h^n, q_h), \tag{3.23}
\]

for any \( v_h \in V_h, q_h \in V_h \).
Lemma 3.1. If the condition (1.6) is satisfied and
\[ dt \leq \frac{4\epsilon^2}{L^2}, \]
the stabilized scheme (3.22) and (3.23) satisfies the following energy stability property:
\[ \hat{E}(\varphi^n_h) \leq \hat{E}(\varphi^{n-1}_h), \quad \forall n \geq 1. \]  
(3.24)

Proof. Taking \( v_h = dt w_h^n, q_h = \varphi^n_h - \varphi^{n-1}_h \) in (3.22) and (3.23) respectively, we obtain
\[ (\varphi^n_h - \varphi^{n-1}_h, w_h^n) + dt\|\nabla w_h^n\|^2_0 + dt \sum_{Z \in \mathcal{E}_h} \tau_Z \|[[\partial_n w_h^n]]\|^2_{0,Z} = 0, \]  
(3.25)
\[ \frac{1}{2}(\|\nabla \varphi^n_h\|^2_0 - \|\nabla \varphi^{n-1}_h\|^2_0 + \|\nabla(\varphi^n_h - \varphi^{n-1}_h)\|^2_0) + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z (\|[[\partial_n \varphi^n_h]]\|^2_{0,Z}
\] 
\[ \quad - \|[\partial_n \varphi^{n-1}_h]\|_{0,Z}^2 + \|[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]\|_{0,Z}^2 \frac{1}{dt\epsilon^2} (f(\varphi^{n-1}_h), \varphi^n_h - \varphi^{n-1}_h)
\] 
\[ \quad = (w_h^n, \varphi^n_h - \varphi^{n-1}_h). \]  
(3.26)
Furthermore, taking \( v_h = \sqrt{dt}(\varphi^n_h - \varphi^{n-1}_h) \) in (3.22), it holds
\[ \sqrt{dt}\|\varphi^n_h - \varphi^{n-1}_h\|_0^2 \leq \frac{dt}{2} \|\nabla w_h^n\|_0^2 + \frac{dt}{2} \|\nabla(\varphi^n_h - \varphi^{n-1}_h)\|_0^2 + \frac{dt}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z
\] 
\[ \quad \|[\partial_n w_h^n]\|_{0,Z}^2 + \frac{dt}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z \|[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]\|_{0,Z}^2 \]  
(3.27)
Adding the above formulations (3.25)–(3.27), and using (1.6) and (2.28), we get
\[ \sqrt{dt}\|\varphi^n_h - \varphi^{n-1}_h\|_0^2 + \frac{dt}{2} \|\nabla w_h^n\|_0^2 + \frac{dt}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z \|[\partial_n w_h^n]\|_{0,Z}^2 + \frac{dt}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z \|[\partial_n (\varphi^n_h - \varphi^{n-1}_h)]\|_{0,Z}^2 \]  
(3.28)
\[ \quad + \frac{1}{2} \|\nabla \varphi^n_h\|_0^2 + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z \|[\partial_n \varphi^n_h]\|_{0,Z}^2 + \frac{1}{dt\epsilon^2} (F(\varphi^n_h), 1)
\] 
\[ \leq \frac{1}{2} \|\nabla \varphi^n_h\|_0^2 + \frac{1}{2} \sum_{Z \in \mathcal{E}_h} \tau_Z \|[\partial_n \varphi^{n-1}_h]\|_{0,Z}^2 + \frac{1}{dt\epsilon^2} (F(\varphi^{n-1}_h), 1) + \frac{L}{2\epsilon^2} \|\varphi^n_h - \varphi^{n-1}_h\|_0^2. \]

The above formulation means the desired result (3.24) holds under the condition
\[ dt \leq \frac{4\epsilon^2}{L^2}. \]  
\[ \square \]

3.3. Error analysis.

Theorem 3.2. Under the assumptions of Lemma 3.1, if the solution of (1.2) satisfying \( \varphi, w \in C(0,T; H^2(\Omega)), \varphi_t \in L^2(0,T; H^2(\Omega)) \cap L(0,T; L^2(\Omega)) \) and \( \varphi_{tt} \in L^2(0,T; L^2(\Omega)) \), \( \varphi^n_h \) is the solution of the semi-implicit enrichment stabilized finite element scheme (3.22) and (3.23), then the following error estimate holds
\[ \|\varphi(t_n) - \varphi^n_h\|_0 \leq C(dt + h^2), \]  
(3.29)
where \( \varphi(t_n) \) is the same as Theorem 2.1 for the Cahn-Hilliard equation.

Proof. We denote \( e^n = \mathcal{R}_h \varphi(t_n) - \varphi^n_h, \tilde{e}^n = \varphi(t_n) - \mathcal{R}_h \varphi(t_n), \eta^n = \mathcal{R}_h w(t_n) - w^n_h, \tilde{\eta}^n = w(t_n) - \mathcal{R}_h w(t_n) \).
Subtracting (3.22) and (3.23) from (3.1) at $t_n$, we can obtain
\[
(\phi(t_n) - \frac{\phi^n_h - \phi^{n-1}_h}{dt}, v_h) + (\nabla(w(t_n) - w^n_h), \nabla v_h) - \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n w^n_h],[\partial_n v_h]) Z = 0, \quad (3.30)
\]
\[
(\nabla(\phi(t_n) - \phi^n_h), \nabla q_h) - \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n \phi^n_h],[\partial_n q_h]) Z
+ \frac{1}{c^2}(f(\phi(t_n)) - f(\phi^{n-1}_h), q_h) = (w(t_n) - w^n_h, q_h). \quad (3.31)
\]
By using (2.54) and (2.34) in (3.30) and (3.31), They are respectively equivalent to
\[
\left(\frac{e^n - e^{n-1}}{dt}, v_h\right) + (\nabla \eta^n, \nabla v_h) + \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n \eta^n],[\partial_n v_h]) Z
= \left(R^n, v_h - \frac{1}{dt}((I - I_h)(\phi(t_n) - \phi(t_{n-1})), v_h), \quad (3.32)
\right.
(\nabla e^n, \nabla q_h) + \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n e^n],[\partial_n q_h]) Z
= -\frac{1}{c^2}(f(\phi(t_n)) - f(\phi^{n-1}_h), q_h) + (\eta^n + \tilde{\eta}^n, q_h). \quad (3.33)
\]
Taking $v_h = 2dte^n$ in (3.32) and $q_h = -2dt\eta^n$ in (3.33), we can get
\[
\left(\frac{e^n - e^{n-1}}{dt}, 2te^n\right) + 2dt(\nabla \eta^n, \nabla e^n) + 2dt \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n \eta^n],[\partial_n e^n]) Z
= 2dt(R^n, e^n) - 2((I - I_h)(\phi(t_n) - \phi(t_{n-1})), e^n), \quad (3.34)
- 2dt(\nabla e^n, \nabla \eta^n) - 2dt \sum_{Z \in \mathcal{E}_h} \tau_Z([\partial_n e^n],[\partial_n \eta^n]) Z
= \frac{2dt}{c^2}(f(\phi(t_n)) - f(\phi^{n-1}_h), \eta^n) - 2dt(\eta^n + \tilde{\eta}^n, \eta^n). \quad (3.35)
\]
Adding (3.34) and (3.35), it holds
\[
\left\|e^n\right\|_0^2 - \left\|e^{n-1}\right\|_0^2 + \left\|e^n - e^{n-1}\right\|_0^2 + 2dt\|\eta^n\|_0^2 + 2dt\|\nabla \eta^n\|_0^2
= 2dt(R^n, e^n) - 2((I - I_h)(\phi(t_n) - \phi(t_{n-1})), e^n) + \frac{2dt}{c^2}(f(\phi(t_n))
- f(\phi^{n-1}_h), \eta^n) - 2dt(\hat{\eta}^n, \eta^n) \leq \frac{4}{\epsilon^2} \sum_{i=1}^{N} B_i. \quad (3.36)
\]
The terms $B_1$, $B_2$ had been estimated in (2.58) and (2.59) respectively. Here, the terms $B_3$ and $B_4$ can be estimated as follows
\[
B_3 = -\frac{2dt}{c^2} (f(\phi(t_n)) - f(\phi^{n-1}_h), \eta^n) \\
\leq \frac{dt}{8}\|\eta^n\|_0^2 + Cdt\epsilon^2 \left\|f(\phi(t_n)) - f(\phi^{n-1}_h)\right\|_0^2 \\
\leq \frac{dt}{8}\|\eta^n\|_0^2 + Cdt\epsilon^2 \left(\|f(\phi(t_n)) - f(\phi(t_{n-1}))\|_0^2 + \|f(\phi(t_{n-1})) - f(\phi^{n-1}_h)\|_0^2\right) \\
\leq \frac{dt}{8}\|\eta^n\|_0^2 + CdtL^2\epsilon^2 \left(\|\phi(t_n) - \phi(t_{n-1})\|_0^2 + \|\phi(t_{n-1}) - \phi^{n-1}_h\|_0^2\right)
\]
\[ \frac{d}{dt} \| \varphi(t) \|_0^2 + \frac{C dt^2 L^2}{\varepsilon^4} \int_{t_{n-1}}^{t_n} \| \varphi(t) \|_0^2 dt + \frac{C dt^2 L^2}{\varepsilon^4} \| \varphi(t_{n-1}) - \varphi_{n-1}^\varepsilon \|_0^2 \]

\[ \leq \frac{dt}{8} \| \varphi(t) \|_0^2 + \frac{C dt^2 L^2}{\varepsilon^4} \int_{t_{n-1}}^{t_n} \| \varphi(t) \|_0^2 dt + \frac{C dt^2 L^2}{\varepsilon^4} (\| e_{n-1}^\varepsilon \|_0^2 + \| \tilde{e}_{n-1}^\varepsilon \|_0^2). \quad (3.37) \]

Combining these inequalities (2.58), (2.59), (3.37), (3.38) into (3.36), it yields

\[ \| e_n^\varepsilon \|_0^2 - \| e_{n-1}^\varepsilon \|_0^2 + dt \| \tilde{e}_{n}^\varepsilon \|_0^2 \]

\[ \leq Cdt \| e_n^\varepsilon \|_0^2 + C \varepsilon dt^2 \int_{t_{n-1}}^{t_n} \| \varphi_{tt}(t) \|_0^2 dt + C \varepsilon \int_{t_{n-1}}^{t_n} \| (I - \mathcal{R}_h) \varphi(t) \|_0^2 dt \]

\[ + \frac{C dt^2 L^2}{\varepsilon^4} \int_{t_{n-1}}^{t_n} \| \varphi(t) \|_0^2 dt + \frac{C dt^2 L^2}{\varepsilon^4} (\| e_{n-1}^\varepsilon \|_0^2 + \| \tilde{e}_{n-1}^\varepsilon \|_0^2) + C \varepsilon dt \sum_{n=1}^{N} \| w(t_n) \|_2^2. \]

Summing up the above inequality for \( n = 1, 2, \ldots, N \), we have

\[ \| e_n^\varepsilon \|_0^2 - \| e_{n-1}^\varepsilon \|_0^2 + dt \sum_{n=1}^{N} \| \tilde{e}_{n}^\varepsilon \|_0^2 + \frac{dt}{2} \sum_{n=1}^{N} \| \nabla e_n^\varepsilon \|_0^2 \]

\[ \leq C \frac{dt^2 L^2}{\varepsilon^2} (\| \varphi(t) \|_0^2 + || \varphi_{tt}(t) \|_0^2 + || \varphi_{tt}(t) \|_0^2 + || \varphi_{tt}(t) \|_0^2) + C \varepsilon h^4 \| \varphi(t) \|_{L^2(0,T;H^1(\Omega))} \]

\[ + C \varepsilon h^4 \| w \|_{L^2(0,T;H^2(\Omega))} + \frac{C dt}{\varepsilon^2} \sum_{n=1}^{N} (\| e_n^\varepsilon \|_0^2 + || e_{n-1}^\varepsilon \|_0^2 + || \tilde{e}_{n-1}^\varepsilon \|_0^2). \quad (3.40) \]

Applying the discrete Gronwall lemma (Lemma 2.2) and (2.24) to the above inequality (3.40), we can obtain the result (3.29) by utilizing the triangular inequality \( \| \varphi(t_n) - \varphi_{n}^\varepsilon \|_0 \leq \| e_n^\varepsilon \|_0 + \| \tilde{e}_{n}^\varepsilon \|_0 \) and the inequality (2.35).

4. **Numerical examples.** In this section, we give some numerical experiments to confirm the theoretical results obtained in the previous sections and demonstrate the efficiency, accuracy and stability of our fully-discrete multiscale finite element methods for the Allen-Cahn and Cahn-Hilliard equations.

4.1. **The Allen-Cahn equation in 2D.** Example 1 (Accuracy test) We consider the Allen-Cahn equation with \( f(\varphi) = \varphi - \varphi^3 \), the periodic boundary and the following initial condition

\[ \varphi_0(x, y) = 0.05 \sin(x) \sin(y), (x, y) \in [0, 2\pi] \times [0, 2\pi]. \quad (4.1) \]

The parameter \( \varepsilon = 0.1 \) and \( T = 10^{-2} \). This example is used to confirm the efficiency and accuracy of the multiscale finite element method for the Allen-Cahn equation.

Since the exact solution is unknown, we take the numerical result obtained by using \( P_1 \)-conforming finite element \( (h = \frac{2\pi}{200}, dt = 10^{-4}) \) as the “exact” solution. Tables 1 shows the \( L^2 \)-error estimates and the optimal first-order convergence rate in \( L^2 \)-norm obtained by using different time steps for the Allen-Cahn equation. Then by fixing \( dt = 10^{-4} \), we confirm the convergence for the spatial discretization. Table 2 also shows the \( L^2 \)-error estimates and the expected optimal second-order convergence rates in \( L^2 \)-norm obtained by using different spatial steps. The numerical results listed in Tables 1 and 2 confirm the error estimates in Theorem 2.1.
Table 1. Convergence performance of the time discretization for the 2D Allen-Cahn equation.

| Time step | $dt = 10^{-2}$ | $dt/2$ | $dt/3$ | $dt/4$ | $dt/5$ | $dt/6$ | $dt/7$ | $dt/8$ | $dt/9$ |
|-----------|----------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\|u-uh\|_0$ | 0.257853 | 0.165833 | 0.121715 | 0.095751 | 0.078637 | 0.066504 | 0.057452 | 0.050440 | 0.044848 |
| $u_{L^2_{rate}}$ | \( \) | 0.636817 | 0.762828 | 0.834011 | 0.882443 | 0.919165 | 0.949093 | 0.974767 | 0.997637 |

Table 2. Convergence performance of the spatial discretization for the 2D Allen-Cahn equation.

| Mesh | $h = 2\pi/8$ | $h = 2\pi/16$ | $h = 2\pi/32$ | $h = 2\pi/64$ | $h = 2\pi/128$ |
|------|--------------|--------------|--------------|--------------|--------------|
| $\|u-uh\|_0$ | 0.122619 | 0.031633 | 0.007850 | 0.001841 | 0.000349 |
| $u_{L^2_{rate}}$ | \( \) | 1.954660 | 2.010630 | 2.092350 | 2.401100 |

Example 2 (Coarsening dynamics) In this example, we study phase separation behavior with the Allen-Cahn equation. The process of phase separation can be researched by considering an isotropic binary mixture quenched into the unstable part of its miscibility gap. Then, the spinodal decomposition takes place, which indicates in the spontaneous growth of the concentration fluctuations that leads the system from the mixing state to the two phase state. After the phase separation starts, the domains of the binary components are formed and the interface between different phases can be showed.

Here, we take the initial condition of this example corresponding to a random data with values between $-0.01$ and $0.01$. And we set $\epsilon^2 = 0.01$, the time step $dt = 10^{-4}$ and the domain $\Omega = [0, 2\pi] \times [0, 2\pi]$ with the periodic boundary. In Figs 1-4, we see that the final equilibrium solution is obtained after $t = 5.5$.

Fig. 1 The dynamic of the scheme for Allen-Cahn equation at the $t = 0.0001$(left), $t = 0.001$(middle)and $t = 0.01$(right).

Fig. 2 The dynamic of the scheme for Allen-Cahn equation at the $t = 0.1$ (left), $t = 1.0$(middle)and $t = 2.0$(right).
4.2. The Cahn-Hilliard equation in 2D. Example 3 (Accuracy test) We consider the Cahn-Hilliard equation with $f(\varphi) = \varphi - \varphi^3$, the periodic boundary and the following initial condition

$$\varphi_0(x, y) = 0.05 \sin(x) \sin(y), (x, y) \in [0, 2\pi] \times [0, 2\pi].$$

The parameter $\epsilon = 0.1$ and $T = 10^{-4}$. This example is used to confirm the efficiency and accuracy of the multiscale finite element method for the Cahn-Hilliard equation.

Since the exact solution is also unknown, we take the numerical result obtaining by using $P_1$–conforming finite element ($h = \frac{2\pi}{200}, dt = 10^{-6}$) as the “exact” solution. Tables 3 shows the $L^2$–error estimates and the optimal first-order convergence rate in $L^2$–norm obtained by using different time steps for the Cahn-Hilliard equation. Then by fixing $dt = 10^{-6}$, we confirm the convergence for the spatial discretization. Table 4 also shows the $L^2$–error estimates and the expected optimal second-order convergence rates in $L^2$–norm obtained by using different spatial steps. The numerical results listed in Tables 3 and 4 confirm the error estimates in Theorem 3.1.

Table 3. Convergence performance of the time discretization for the 2D Cahn-Hilliard equation.

| Time step | $dt = 10^{-4}$ | $dt/2$ | $dt/3$ | $dt/4$ | $dt/5$ | $dt/6$ | $dt/7$ | $dt/8$ | $dt/9$ |
|-----------|----------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\|u_{h,2,\text{rate}}\|$ | 0.019526 | 0.009760 | 0.006461 | 0.004804 | 0.003807 | 0.003141 | 0.002665 | 0.002308 | 0.002030 |
| $w_{h,2,\text{rate}}$ | 1.00045 | 1.01715 | 1.03078 | 1.04254 | 1.04493 | 1.04497 | 1.04261 |
| $\|u_{h,2,\text{rate}}\|$ | 1.940e-04 | 9.65e-05 | 6.38e-05 | 4.74e-05 | 3.76e-05 | 3.11e-05 | 2.65e-05 | 2.30e-05 | 2.04e-05 |
| $w_{h,2,\text{rate}}$ | 1.00706 | 1.02108 | 1.03078 | 1.04254 | 1.04493 | 1.04497 | 1.04261 |
Table 4. Convergence performance of the spatial discretization for the 2D Cahn-Hilliard equation.

| Mesh | $h = 2\pi/8$ | $h = 2\pi/16$ | $h = 2\pi/32$ | $h = 2\pi/64$ | $h = 2\pi/128$ |
|------|--------------|--------------|--------------|--------------|--------------|
| $\|u-u_h\|_0$ | 0.116925     | 0.030137     | 0.007479     | 0.001755     | 0.000337     |
| $\|u\|_0$     | 1.95296      | 2.01063      | 2.09162      | 2.38675      |              |
| $\|w-w_h\|_0$ | 0.120028     | 0.030952     | 0.007681     | 0.001802     | 0.000343     |
| $\|w\|_0$    | 1.95523      | 2.01063      | 2.09177      | 2.39460      |              |

Example 4 (Coarsening dynamics) Here we also give a example, which is similar to Example 2, to study phase separation behavior with the Cahn-Hilliard equation. We take the initial condition of this example corresponding to a random data with values between $-0.05$ and $0.05$. And we set $\epsilon^2 = 0.01$, the time step $dt = 10^{-4}$ and the domain $\Omega = [0, 2\pi] \times [0, 2\pi]$ with the periodic boundary. In Figs 5-8, we see that the final equilibrium solution is obtained after $t = 1.2$.

Fig. 5 The dynamic of the scheme for Cahn-Hilliard equation at the $t = 0.0001$ (left) and $t = 0.001$ (right).

Fig. 6 The dynamic of the scheme for Cahn-Hilliard equation at the $t = 0.01$ (left) and $t = 0.1$ (right).
5. Conclusion. In this paper, we developed the first-order fully-discrete multiscale finite element methods for the Allen-Cahn and Cahn-Hilliard equations, and proved that they are conditionally energy stable (under a reasonable stability condition) for Allen-Cahn equation and Cahn-Hilliard equation. By introducing a new projection operator, we also obtain the optimal first-order $L^2$–error estimates in time and second-order $L^2$–error estimates in space for the Allen-Cahn and Cahn-Hilliard equations respectively.

The fully-discrete multiscale finite element schemes, with the stabilized terms and an explicit treatment for the nonlinear reaction term $f(\varphi)$, are so efficient and stable as they lead to solve a sequence of linear equations at each time step. We provided several examples to demonstrate the efficiency and accuracy of the fully-discrete multiscale finite element schemes for the Allen-Cahn and Cahn-Hilliard equations.

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