An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups.

by

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Introduction

We consider here the moduli space $\mathcal{M}$ of stable principal $G$-bundles over a compact Riemann surface $C$, with $G$ an algebraic complex group. We denote by $K$ the canonical bundle over $C$. In [Hi] N.Hitchin defined an analytic map $\mathcal{H}$ from the cotangent bundle $T^*\mathcal{M}$ to the "characteristic space" $\mathcal{K}$ by associating to each $G$-bundle $P$ and section $s \in H^0(C, \text{ad}P \otimes K)$ the spectral invariants of $s$. Hitchin showed for $G = \text{Gl}(n), \text{SO}(n), \text{Sp}(n)$ that the generic fibre of $\mathcal{H}$ is an open set in an abelian variety $\mathcal{A}$. In fact, he considers in each case a non-singular spectral curve $S$ covering $C$: when $G = \text{Gl}(n)$, $\mathcal{A}$ is identified with the Jacobian $J(S)$; in the other cases, there is a naturally defined involution on $S$ and $\mathcal{A}$ is the associated Prym variety. More recently, Faltings extended these results and described an abelianization procedure for the moduli space of Higgs $G$-bundles, with $G$ any reductive group (see [F]).

If $T \subset G$ is a fixed maximal torus with Weyl group $W$, one may construct for each given generic element $\phi \in \mathcal{K}$ a ramified covering $\hat{C}$ of $C$ having $|W|$ sheets. The combined action of $W$ on $\hat{C}$ and on the group of one parameter subgroups of $T$ induces an action on the space of all principal $T$-bundles $\tau$ over $\hat{C}$ and we may consider the subvariety $\hat{\mathcal{P}}$ of those $\tau$ which are $W$-invariant in this sense. The connected component $\mathcal{P}_0$ of $\hat{\mathcal{P}}$ which contains...
the trivial $T$-bundle is an abelian variety. In [F] it is shown that the generic fibre of the Hitchin map is a principal homogeneous space with respect to a group (namely the first étale cohomology group of $C$ with coefficients in a suitably defined group scheme) which is isogenous to $\hat{P}$. In the present paper, by means of mostly elementary techniques, we explicitly construct a map $\mathcal{F}$ from each connected component $\mathcal{H}^{-1}(\phi)_c$ of $\mathcal{H}^{-1}(\phi)$ to $P_0$ and show that $\mathcal{F}$ has finite fibres. We use the classical theory of representations of finite groups to compute $\dim P_0 = \dim \mathcal{M}$ and conclude that the image under $\mathcal{F}$ of $\mathcal{H}^{-1}(\phi)$ contains a Zariski open set in $P_0$.

In case $G = PGL(2)$ one can check directly that the generic fibre of $\mathcal{F}_c : \mathcal{H}^{-1}(\phi)_c \to P_0$ is a principal homogeneous space with respect to a product of $(2 \cdot \deg K - 2)$ copies of $\mathbb{Z}/2\mathbb{Z}$. However in case the Dynkin diagram of $G$ does not contain components of type $B_l$, $l \geq 1$ or when the commutator subgroup $(G, G)$ is simply connected the map $\mathcal{F}_c$ is injective.

Such results were announced in our previous paper [S], in which we showed that $P_0$ is isogenous to a "spectral" Prym-Tjurin variety $P_\lambda$ for each given dominant weight $\lambda$. Results concerning the description of the Hitchin fibre in terms of generalized Prym varieties were also announced in R. Donagi, *Spectral covers*, preprint, alg-geom/9505009 (1995).

1 The Hitchin map for any reductive group

We denote by $C$ a compact Riemann surface of genus $g \geq 2$ and by $G$ a reductive algebraic group over the field of complex numbers. We also write $\mathfrak{g}$ as the Lie algebra of $G$. The moduli space of stable principal $G$-bundles over $C$ is a quasi-projective complex variety $\mathcal{M}$ with $\dim \mathcal{M} = (g-1)\dim G + \dim Z(G)$, $Z(G)$ being the center of $G$. Note here that semistability for a principal $G$-bundle $P$ corresponds to semistability for the holomorphic vector bundle $adP$ associated to the adjoint representation $Ad : G \to \mathfrak{gl}(\mathfrak{g})$ ([A-B], [F]).

We denote by $K$ the canonical line bundle over $C$. By deformation theory and Serre duality, a point in the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$ is a pair $(P, s)$ with $P$ a stable principal $G$-bundle over $C$ and $s$ a section of the vector bundle $adP \otimes K$. The ring of polynomials on $\mathfrak{g}$ which are invariant with respect to the adjoint action is freely generated by homogeneous polynomials $h_1, \ldots, h_k$. Each $h_i$ induces a map $\mathcal{H}_i : adP \otimes K \to K^{d_i}$ where $d_i = \deg h_i$, and the
Hitchin map
\[ \mathcal{H} : T^*\mathcal{M} \longrightarrow \mathcal{K} = \bigoplus_{i=1}^{k} H^0(C, K^{d_i}) \]
takes \((P, s)\) to the element in \(\mathcal{K}\) whose \(i\)-th component is the composition of \(\mathcal{H}_i\) with \(s\). It is a remarkable fact that the dimension of \(\mathcal{K}\) is equal to the dimension of \(\mathcal{M}\). Moreover the map \(\mathcal{H}\) is surjective. This fact can be deduced from the existence of very stable \(G\)-bundles (see \([L]\), \([BR]\), \([KP]\) Lemma 1.4).

We fix once and for all a maximal torus \(T \subset G\) with associated root system \(R = R(G, T)\) and Weyl group \(W = N_G(T)/T\). We also fix a subset \(R^+ \subset R\) (or equivalently a Borel subgroup \(B \supset T\)). If \(t\) denotes the Lie algebra of \(T\), the differential of each root \(\alpha \in R\) induces a map \(d\alpha : t \otimes K \rightarrow K\) and the homogeneous \(W\)-invariant polynomials \(\sigma_1, \ldots, \sigma_k\) on \(t\) obtained by restriction of \(h_1, \ldots, h_k\) define a Galois covering
\[ \sigma = (\sigma_1, \ldots, \sigma_k) : t \otimes K \longrightarrow \bigoplus_{i=1}^{k} K^{d_i} \]
whose discriminant \(\Xi\) is given by the zeroes of the \(W\)-invariant function \(\prod_{\alpha \in R} d\alpha\). For generic \(\phi \in \mathcal{K} = H^0(C, \bigoplus_i K^{d_i})\), we consider the curve \(\tilde{C} := \phi^*(t \otimes K)\). This is a ramified covering of \(C\) having \(m = |W|\) sheets, whose branch locus \(Ram\) satisfies by construction
\[ \mathcal{O}(Ram) \cong K^{[R]} \cong K^{(\dim G - \text{rank} G)}. \] (1)

If we indicate by \(\iota : \tilde{C} \rightarrow t \otimes K\) the natural inclusion map, we have by definition, for each \(w \in W\),
\[ \iota(w \eta) = \text{Ad}(n_w) \iota(\eta) \] (2)
where \(n_w \in N_G(T)\) is any representative of \(w\). Note also that, if \(\pi : \tilde{C} \rightarrow C\) denotes the projection map, \(da \circ \iota\) is a holomorphic section of \(\pi^*K\).

\[ \begin{array}{ccc}
\tilde{C} & \xrightarrow{\iota} & t \otimes K \\
\downarrow \pi & & \downarrow \\
C & \xrightarrow{\phi} & \bigoplus_i K^{d_i}
\end{array} \]

As a consequence of our genericity hypothesis, \(\tilde{C}\) has the following properties:
\(a\) it is smooth and irreducible.
\(b\) each ramification point \(p \in \pi^{-1}(Ram)\) has index 1; i.e. is a simple zero
for the section $\prod_{a \in R^+} (d\alpha \circ t) : \check{C} \to \pi^* K^{[R]}/2$.

This may be checked as follows. Let us denote by $\pi_i : K^{d_i} \to C$, $i = 1, \ldots, k$ and $q : t \otimes K \to C$ the projections. Moreover for every $i = 1, \ldots, k$ let us denote by $\gamma_i : K^{d_i} \to \pi_i^* K^{d_i}$ the tautological section. For each $i$ we consider those sections of $q^* K^{d_i}$ that have the form $s = c \cdot \sigma_i^* \gamma_i + q^* a_i$ for some $c \in \mathbb{C}$ and $a_i \in H^0(C, K^{d_i})$. As $c$ varies in $\mathbb{C}$ and $a_i$ in $H^0(C, K^{d_i})$ the zero divisor of $s$ forms a linear system $\delta_i$ of divisors in $t \otimes K$ that has no base points since the linear system $| K^{d_i} |_C$ on $C$ has no base points. For $\phi = (a_1, \ldots, a_k) \in K$, the curve $\check{C}$ is defined by the equations $\sigma_i^* \gamma_i = q^* a_i$, $i = 1, \ldots, k$. One immediately checks that the map

$$K^{d_i} \longrightarrow \mathbb{P}^{ \dim H^0(C, K^{d_i}) }$$

$$x \longmapsto [\gamma_i(x), \pi_i^* a_{i,1}(x), \ldots, \pi_i^* a_{i,m_i}(x)]$$

where the $a_{i,j}$'s form a basis of $H^0(C, K^{d_i})$ has image of dimension 2 and that $\sigma_1 : t \otimes K \to K^{d_i}$ is dominant. By Bertini’s theorem (see [1], theorem 6.3) the divisor $X_1 \in \delta_1$ of the section $\sigma_1^*(\gamma_1 - \pi_1^* a_1) = \sigma_1^* \gamma_1 - q^* a_1$ with $a_1$ generic in $H^0(C, K^{d_i})$ is smooth and irreducible. If $k \geq 2$, we next consider the linear system on $X_1$ given by the restriction of $\delta_2$. Since the polynomial $\sigma_2$ is algebraically independent from $\sigma_1$ the map $\sigma_2 |_{X_1} : X_1 \to K^{d_2}$ is dominant. We use the same argument as above and from Bertini’s theorem we obtain that the divisor $X_2 \subset X_1$ of the section $\sigma_2^* \gamma_2 - q^* a_2 |_{X_1}$ with generic $a_2$ is smooth and irreducible. We can repeat the same argument for the linear system $\delta_i |_{X_{i-1}}$ for every $i \leq k$ (since the map $\sigma_i |_{X_{i-1}} : X_{i-1} \to K^{d_i}$ is dominant) and thus prove a) . As for the statement b) one may consider the restriction of the linear systems above both to the discriminant locus $\Xi$ and to the locus $\mathcal{Z} \subset \Xi$ where $\prod_{a \in R^+} d\alpha$ vanishes with multiplicity $\geq 2$ ($\mathcal{Z} = \text{Sing} \ \Xi$). Again from Bertini’s theorem one obtains that $\check{C}$ does not intersect $\mathcal{Z}$ and intersects $\Xi \setminus \mathcal{Z}$ transversely.

Remark 1.1. For each $\alpha \in R^+$, let $s_\alpha \in W$ denote the corresponding reflection. As a consequence of condition b) above we may consider the ramification locus in $\check{C}$ as a disjoint union: $\mathcal{D} = \prod_{a \in R^+} \mathcal{D}_\alpha$, with $\mathcal{D}_\alpha = \{ \text{zeroes of } d\alpha \circ t \} = \{ \eta \in \check{C} \mid s_\alpha \eta = \eta \}$. By our previous considerations $\mathcal{D}_\alpha$ belongs to the linear system $| \pi^* K |$. In case $G$ is simple and simply laced, i.e. $W$ acts transitively on the set of roots $R$, we may write for each $y \in \text{Ram}$

$$\pi^{-1}(y) = \prod_{a \in R^+} \mathcal{D}_\alpha^y.$$
where \( \mathcal{D}_\alpha^y := \mathcal{D}_\alpha \cap \pi^{-1}(y) \) is nonempty for every \( \alpha \in R^+ \).

If \( G \) is not simply laced and has connected Dynkin diagram, \( R \) is the union of two \( W \)-orbits \( R_1, R_2 \), each one consisting of roots having the same length. Then we have

\[
\pi^{-1}(y) = \prod_{\alpha \in R_1 \cap R^+} \mathcal{D}_\alpha^y \quad \text{or} \quad \pi^{-1}(y) = \prod_{\alpha \in R_2 \cap R^+} \mathcal{D}_\alpha^y
\]

(3)

depending on whether \( y \) corresponds to a short or a long root.

More generally, if the Dynkin diagram of \( G \) has more than one connected component, we have as many different "kinds" of fibers

\[
\pi^{-1}(y) = \prod_{\alpha \in R_j \cap R^+} \mathcal{D}_\alpha^y
\]

as are the \( W \)-orbits \( R_j \subset R \). Since for each \( \alpha \in R^+ \) we have \( |\mathcal{D}_\alpha| = |W| \cdot \deg K \) and each fibre over a branch point consists of \( |W|/2 \) points, the number of fibres which correspond to the same orbit \( R_j \) is equal to

\[
n_j = |R_j^+| \cdot |W| \cdot \deg K \cdot \frac{1}{2} |W| = |R_j| \cdot \deg K.
\]

(4)

Let now \( X(T) \) be the group of characters on \( T \) and consider the group \( H^1(\tilde{C}, T) \) of isomorphism classes of holomorphic principal \( T \)-bundles over \( \tilde{C} \). Each pair \((\tau, \mu)\) with \( \tau \) a principal \( T \)-bundle, \( \mu \in X(T) \) defines a line bundle \( \tau_\mu \equiv \tau \times_\mu \mathcal{C} \) and this way \( H^1(\tilde{C}, T) \) is identified with

\[
Pic(\tilde{C}) \otimes X(T)^*,
\]

\( X(T)^* \equiv Hom(X(T), \mathbb{Z}) \) being the dual group. For the same reason, the group of isomorphism classes of topologically trivial principal \( T \)-bundles is a tensor product

\[
J(\tilde{C}) \otimes X(T)^*
\]

(here, as usual, \( J(\tilde{C}) \) denotes the group of divisors with zero degree modulo linear equivalence ). Now, the action of \( W \) on the sheets of \( \tilde{C} \) induces an
action on $J(\tilde{C})$. On the other hand, $W$ acts by conjugation on $T$, hence on $X(T)^*$. If $\tau = D_1 \otimes \chi_1 + \cdots + D_l \otimes \chi_l$ is a principal $T$-bundle over $\tilde{C}$ and $w \in W$ an element of the Weyl group, we set
\[ w\tau = wD_1 \otimes w\chi_1 + \cdots + wD_l \otimes w\chi_l. \]

**Definition 1.1** The generalized Prym variety $P = [J(\tilde{C}) \otimes X(T)^*]^W$ consists of those isomorphism classes of topologically trivial $T$-bundles $\tau$ which satisfy $w\tau \equiv \tau$ for each $w \in W$.

Note that $P$ is an algebraic group whose connected component of the identity $P_0$ is an abelian variety.

## 2 Computing the dimension of $P$

The following can be deduced from the above mentioned Faltings’ result describing the generic Hitchin fibre as isogenous to $\hat{P} = [\text{Pic}(\tilde{C}) \otimes X(T)^*]^W$ ([F], theorem III.2) and the fact (due to G. Laumon and proved in [F], theorem II.5) that all Hitchin fibers have the same dimension:

**Proposition 2.1** The dimension of $P$ is equal to the dimension of $M$.

In this section we want to give a direct proof of such statement. If we set $S = X(T) \otimes \mathbb{Z} \mathbb{C}$ and denote by $H^1$ the first cohomology $W$-representation $H^1(\tilde{C}, \mathbb{C})$, by Doulbault theorem we have
\[ \dim P = \frac{1}{2} \dim [H^1 \otimes S^*]^W = \frac{1}{2} \dim \text{Hom}_W(S, H^1). \]

We will compute $M = \dim \text{Hom}_W(S, H^1)$ by use of the classical theory of representations of finite groups and associated characters (for more details about this subject, see for example [Se]).

For any $W$-representation $V$ considered here, we denote by $\chi_V : W \to \mathbb{C}$ its character (for $\rho : W \to Gl(V)$ the homomorphism defining the representation, we have by definition $\chi_V(w) = \text{trace}(\rho(w)) \forall w \in W$). By the theory of characters of finite groups we have
\[ M = \langle \chi_S, \chi_{H^1} \rangle \]

(5)
where $< , >$ is the usual scalar product between characters. If $N$ is the number of connected components of the Dynkin diagram $\Pi$ of $G$ and $h = dimZ(G)$ we have a decomposition

$$S = B \oplus \cdots \oplus B \oplus S_1 \oplus \cdots \oplus S_N$$

where $B$ is the 1-dimensional trivial representation and $S_i$ the irreducible reflection representation corresponding to the $i$-th component of $\Pi$, $i = 1, ..., N$.

Then we may rewrite (5) as

$$M = h < \chi_B, \chi_{H^1} > + \sum_{i=1}^{N} < \chi_{S_i}, \chi_{H^1} > .$$

(6)

We observe that $W$ acts trivially on the cohomology groups $H^0(\tilde{C}, C) \cong H^2(\tilde{C}, C) \cong C$. Hence the Lefschetz character $\chi_L = \chi_{H^0} - \chi_{H^1} + \chi_{H^2}$ satisfies $\chi_L = 2\chi_B - \chi_{H^1}$ and we have

$$< \chi_B, \chi_{H^1} > = 2 - < \chi_B, \chi_L >$$

(7)

$$< \chi_{S_i}, \chi_{H^1} > = - < \chi_{S_i}, \chi_L > .$$

(8)

On the other hand, it is well known (Hopf trace formula, see e.g. [CR]) that the Lefschetz character satisfies

$$\chi_L = \chi_{\tilde{C}^0} - \chi_{\tilde{C}^1} + \chi_{\tilde{C}^2}$$

$\tilde{C}^n$ being the free $C$-module generated by the $n$-cells of some cellular decomposition of $\tilde{C}$ ($\tilde{C}^n \cong H_n(K^n, K^{n-1}; C)$, with $K^j$ the $j$-th skeleton of $\tilde{C}$, $j = n, n-1$).

We choose one finite triangulation $\Delta$ of $C$ whose set of vertices contains all branch points. We denote by $C^m$ the free module generated by the $n$-cells of $\Delta$ for $n = 1, 2$, and by $C^0_0$ and $D_j$ the free modules whose generators are respectively all vertices not lying in the branch locus $Ram$ and all branch points corresponding to the same $W$-orbit $R_j \subset R$ (see Remark 1.1.). Let $N'$ be the number of $W$-orbits in $R$, and for each $j = 1, \ldots, N'$ let us fix one positive root $\alpha_j \in R^+_j$ and set $H_j = \{1, s_{\alpha_j}\} \subset W$. We denote by $Ind_W^{H_j}(B_j)$ the $W$-representation induced by the 1-dimensional trivial representation $B_j$.
of $H_j$ (by definition, $\text{Ind}^{W}_{H_j}(B_j) = \bigoplus_{[u] \in W/H_j} C[v_{[u]}]$ with $W$ acting by $u \circ v_{[u]} = v_{[uw]}$). We have the following isomorphisms of $W$-modules:

\[
\begin{align*}
\tilde{C}^2 & \cong C[W] \otimes C^2 \\
\tilde{C}^1 & \cong C[W] \otimes C^1 \\
\tilde{C}^0 & \cong C[W] \otimes C^0 \oplus \bigoplus_{j=1}^{N'} \text{Ind}^{W}_{H_j}(B_j) \otimes D_j \\
& \equiv C[W] \otimes C^0 \oplus \bigoplus_{j=1}^{N'} (\text{Ind}^{W}_{H_j}(B_j))^{n_j}
\end{align*}
\]

where $C[W]$ denotes as usual the regular representation and the $n_j$'s are defined as in (4).

By Frobenius reciprocity formula we have

\[< \chi_B, \chi_{\text{Ind}^{W}_{H_j}(B_j)} > = < \chi_{B_j}, \chi_{B_j} > = 1 ;\]

and since from the general theory each irreducible $W$-representation occurs as a subrepresentation of $C[W]$ as many times as is its dimension, we obtain

\[< \chi_B, \chi_L > = rk C^2 - rk C^1 + rk C^0 + |\text{Ram}| = (2 - 2g). \quad (9)\]

Analogously, we have

\[< \chi_{S_i}, \chi_L > = (rk C^2 - rk C^1 + rk C^0) \dim S_i + \sum_{j=1}^{N'} n_j < \chi_{B_j}, \chi_{\text{res}_j S_i} > \]

where $\text{res}_j S_i$ denotes the representation obtained by restriction to $H_j$.

Now, given some positive root $\alpha \in R^+$, the corresponding reflection $s_\alpha \in W$ acts trivially on $S_i$ whenever $\alpha \notin S_i$, otherwise it acts trivially on one subspace of codimension 1 in $S_i$. Thus we get

\[< \chi_{S_i}, \chi_L > = (rk C^2 - rk C^1 + rk C^0) \dim S_i + \sum_{R_j \subseteq S_i} n_j (\dim S_i - 1) + \]

\[+ \sum_{R_j \not\subseteq S_i} n_j \cdot \dim S_i \]

\[= (2 - 2g) \dim S_i - \sum_{R_j \subseteq S_i} n_j. \quad (10)\]
By substituting (9) and (10) respectively in (7) and (8) and then (7) and (8) in (6), we finally obtain

\[ M = 2h + (2g - 2)\left( h + \sum_{i=1}^{N} \dim S_i \right) + \sum_{j=1}^{N'} n_j \]

Since \( \dim T + |R| = \dim G \), by (11) we get

\[ \dim \mathcal{P} \equiv \frac{1}{2} M = (g - 1)\dim G + h. \]

## 3 The main results

In this section we will define a map \( \mathcal{F} \) from each component of the generic Hitchin fibre to the abelian variety \( \mathcal{P}_0 \) and study its properties. We first show how one can associate to each given pair \( (P, s) \in \mathcal{H}^{-1}(\phi) \) a \( T \)-bundle \( \mathcal{T} = \mathcal{T}(P, s) \) which satisfies \( w\mathcal{T} \cong \mathcal{T} \forall w \in W \).

For \( \phi \in \mathcal{K} \) generic, let then \( P \) be a principal \( G \)-bundle and \( s \in H^0(C, adP \otimes K) \) such that \( (P, s) \in \mathcal{H}^{-1}(\phi) \). We first consider the restriction \( P_0 \) of \( P \) to the open set \( C_0 \). Since for every \( \xi \in C_0 \), \( s(\xi) \in \mathfrak{g} \) is regular semisimple (for an analysis of the regular elements in \( \mathfrak{g} \), see for example [K]), we have a morphism of vector bundles

\[ [s, ] : adP_0 \longrightarrow adP_0 \otimes K \]

whose kernel \( \mathcal{N} \) is a bundle of Cartan subalgebras in \( \mathfrak{g} \). We thus have a section

\[ \gamma : C_0 \rightarrow P/N_G(T) \equiv P \times_G G/N_G(T) \]

locally defined by \( \gamma(\xi) = \nu(\xi)N_G(T) \) where \( \nu(\xi) \in G \) satisfies \( Ad \nu(\xi)t = N_\xi \equiv c\mathfrak{g}(s(\xi)) \). If we pull back \( P_0 \) over \( \tilde{C}_0 \) we actually have a section

\[ \varphi : \tilde{C}_0 \rightarrow \pi^*P_0/T \]

locally defined by \( \varphi(\eta) = \mu(\eta)T \) where \( \mu(\eta) \in G \) satisfies

\[ Ad \mu(\eta)(\nu(\eta)) = s(\pi(\eta)). \]
Thus over $\tilde{C}_0$ the bundle $\pi^*P$ has a reduction of its structure group to $T$. Moreover, from (2) we have for each $w \in W$

$$\varphi(w\eta) = \mu(\eta)n_w^{-1}T$$

which implies that such $T$-reduction $\tau_0 = \varphi^*(\pi^*P_0)$ is $W$-invariant with respect to the action previously defined. Now if we consider a Borel subgroup $B \subset G$ containing $T$, the inclusion map $T \hookrightarrow B$ and $\varphi$ define a section $: \tilde{C}_0 \to \pi^*P \times_G G/B$. Since $G/B$ is a complete variety, by the valuative criterion of properness this section can be extended to the whole curve $\tilde{C}$ and we thus obtain (uniquely up to isomorphisms) a $B$-reduction $P_B$ of the $G$-bundle $\pi^*P$ such that $P_B|_{\tilde{C}_0}$ is the $B$-extension of $\tau_0$.

If $(\ , \ )$ denotes a $W$-invariant scalar product on $X(T)\otimes \mathbb{R}$ and $\beta \in R$, we define as usual the one parameter subgroup $\beta' \in \text{Hom}(X(T), \mathbb{Z})$ by

$$\beta'(\lambda) = \langle \lambda, \beta \rangle \equiv \frac{2(\lambda, \beta)}{(\beta, \beta)} \forall \lambda \in X(T).$$

We want to prove the following:

**Theorem 3.1** Let $\tau_B = \tau(P, s)$ be the $T$-bundle associated to $P_B$ via the natural projection $B \to T$. Let us fix one theta characteristic $\frac{1}{2}K$ and consider the $T$-bundle $K_\rho = \frac{1}{2}\pi^*K \otimes \sum_{\beta \in R^+} \beta'$, where $R^+ \subset R$ is the subset of positive roots that corresponds to $B$. Then $T(P, s) := \tau_B + K_\rho$ is $W$-invariant.

The proof will be organized in a few lemmas. We first observe that since $W$ is generated by the simple reflections it suffices to show

$$s_\alpha \tau_B \cong \tau_B + \pi^*K \otimes \alpha'$$

for every simple root $\alpha$. In fact we have

$$\sum_{\beta \in R^+} s_\alpha(\beta') = \sum_{\beta \in R^+} \beta' - \alpha',$$

so, if relation (13) holds, one has $s_\alpha(\tau_B + K_\rho) \cong \tau_B + K_\rho$. In terms of line bundles associated to characters on $T$, relation (13) can be rewritten as

$$\left(s_\alpha \tau_B - \tau_B\right) \times X \cong < \lambda, \alpha > \pi^*K \forall \lambda \in X(T).$$

(16)

Given a simple root $\alpha$, let us denote by $s_\alpha(B)$ the Borel subgroup $n_\alpha B n_\alpha^{-1}$, where $n_\alpha \in N_G(T)$ represents $s_\alpha$. One analogously obtains another $T$-bundle $\tau_{s_\alpha(B)}$ such that $\tau_{s_\alpha(B)}|_{\tilde{C}_0} \cong \tau_0$ from the completion of $\tau_0$ to an $s_\alpha(B)$-reduction $P_{s_\alpha(B)}$. The first lemma treats the relationship between $\tau_B$ and $\tau_{s_\alpha(B)}$. 10
Lemma 3.2 We have $\tau_{s_\alpha(B)} \cong s_\alpha \tau_B$.

Proof. We consider an open covering $\{V_h\}_{h \in H}$ of $C$ over which $P$ and the canonical bundle $K$ can be trivialized and with the property that each $V_h$ contains at most one branch point. We choose a Čech covering $\mathcal{U} = \{U_h\}_{h \in H}$ of $\tilde{C}$ to be given by all open sets $U_h = \pi^{-1}(V_h)$ (by definition each $U_h$ is stable with respect to the action of $W$). For $h \in H$ we choose frames $e_1^h, \ldots, e_q^h$ for the vector bundle $adP \otimes K$ over $V_h \subset C$, $q$ being equal to the dimension of $g$. With respect to this choice the section $s : C \to adP \otimes K$ is locally given by ”coordinates” $s_h : V_h \to g$ satisfying

$$s_h = Ad g_{hl} \cdot k_{hl}s_l \text{ for } V_h \cap V_l \neq \emptyset,$$  \hspace{1cm} (17)

$g_{hl}$ and $k_{hl}$ being transition functions for $P$, $K$ respectively. Let $\iota_h : U_h \to t$ be coordinates for $i : \tilde{C} \to t \otimes K$. We define $J \subset H$ to be the subset of those indices $j$ such that $V_j$ contains a branch point and set $I = H \setminus J$. For each $h \in H$ we fix maps $\mu_h : U_h \to G$ such that for each $i \in I$ $\mu_i$ satisfies

$$Ad \mu_i(\eta)(\iota_i(\eta)) = s_i(\pi(\eta))$$ \hspace{1cm} (18)

(compare with (12) ) and the 0-chain $\{\mu_h(\eta)_B\}_{h \in H}$ defines the section $\tilde{\varphi}_B : \tilde{C} \to \pi^*P/B$ completing $\varphi$ in (11). By definition, the $B$-bundle $P_B$ is represented by the cocycle $\{b_{hl}\} \in Z^1(\mathcal{U}, B)$ where $b_{hl}(\eta) \equiv \mu_h(\eta)^{-1}g_{hl}(\pi(\eta))\mu_l(\eta)$. Define $\{b_{hl}'\} \in Z^1(\mathcal{U}, s_\alpha(B))$ by $b_{hl}'(\eta) = n_\alpha b_{hl}(s_\alpha\eta)n^{-1}_\alpha \forall \eta \in U_h \cap U_l$. We have $b_{hl}'(\eta) \equiv n_\alpha \mu_h(s_\alpha\eta)^{-1}g_{hl}(\pi(\eta))\mu_l(s_\alpha\eta)n^{-1}_\alpha$, hence $\{b_{hl}'\}$ represents an $s_\alpha(B)$-reduction of $\pi^*P$. On the other hand, from (13) we have $\{\mu_i(s_\alpha\eta)n^{-1}_\alpha T\}_{i \in I} = \{\mu_i(\eta)T\}_{i \in I}$ hence $\{b_{hl}'\}$ represents $P_{s_\alpha(B)}$. Now, if we denote by $p : B \to T$, $p' : s_\alpha(B) \to T$ the natural projections we have $p' \circ b_{hl}'(\eta) = n_\alpha(p \circ b_{hl}(s_\alpha\eta))n^{-1}_\alpha$ (since every Borel subgroup is a semidirect product of its maximal torus and its maximal unipotent subgroup). Since $\{n_\alpha(p \circ b_{hl}(s_\alpha\eta))n^{-1}_\alpha\}$ are by definition transition functions for $s_\alpha \tau_B$, we thus have an isomorphism $	au_{s_\alpha(B)} \cong s_\alpha \tau_B$. \hfill $\Box$

We keep the notations of the proof of lemma 3.2. For each positive root $\beta \in R^+$, we shall denote by $\beta_h : U_h \to C$ the coordinates of the section of $\pi^*K$ over $\tilde{C}$ given by the composition $d\beta \circ \iota$ (see (11)). Our next step consists in finding suitable transition functions $b_{ji}$ for $P_B$ on intersections $U_i \cap U_j$ with $j \in J$. Indeed, we will find suitable maps $\mu_j : U_j \to G$ with $j \in J$ defining the completed section $\tilde{\varphi}_B$. We fix nilpotent generators $\{X_\gamma\}_{\gamma \in R^+}$ in the Lie
algebra \( b \) of \( B \) with \( \text{ad} \ t(X_\gamma) = \gamma(t)X_\gamma, \forall t \in \mathfrak{t}, \forall \gamma \in R^+ \). In general, the completion \( \hat{\varphi}_B : \hat{\mathcal{C}} \to \pi^*P/B \) of our \( \varphi \) above is locally given by holomorphic maps \( f_j : U_j \to G \) with \( j \in J \) such that

\[
\text{Ad} \ f_j^{-1}(t)(\pi) = \iota_j(t) + \sum_{\gamma \in R^+} a_\gamma(t)X_\gamma.
\]

By Remark 1.1, for \( j \in J \) the set \( U_j \) is a union of open sets \( \bigcup_{\beta \in R(j) \cap R^+} U_{j,\beta} \) where \( R(j) \) is some \( W \)-orbit of roots depending on \( j \) and each \( U_{j,\beta} \) contains only those ramification points that are zeroes for \( \beta \).

**Lemma 3.3** There exists a holomorphic map \( \mu_j : U_j \to G \) satisfying for each \( \beta \in R(j) \cap R^+ \) and \( \eta \in U_{j,\beta} \)

\[
\text{Ad} \ \mu_j(\eta)^{-1}s_j(\pi(\eta)) = \iota_j(\eta) + X_\beta.
\]

**Proof.** We construct \( \mu_j \) separately on each open set \( U_{j,\beta} \). By our genericity hypothesis we may assume for a ramification point \( p \in U_{j,\beta} \)

\[
\text{Ad} \ f_j(p)^{-1}s_j(\pi(p)) = \iota_j(p) + X_\beta
\]

with \( \beta_j(p) \equiv d\beta(t_j(p)) = 0 \).

Let \( \alpha \) be the root with minimal height in \( R^+ \setminus \{\beta\} \) such that \( a_\alpha(\eta) \) in (19) is not identically zero. The map \( c_j(\eta) = \exp^{a_\alpha(\eta)}X_\alpha : U_{j,\beta} \to G \) is holomorphic on each fixed connected component of \( U_{j,\beta} \) and by evaluating \( \text{Ad} \ c_j(\eta) \) on the right-hand side of (19) we get

\[
\text{Ad} \ c_j(\eta)(\iota_j(\eta) + \sum_{\gamma \in R^+} a_\gamma(\eta)X_\gamma) = \iota_j(\eta) + a_\beta(\eta)X_\beta + \sum_{\gamma \in R^+ \setminus \{\beta\}} a_\gamma(\eta)X_\gamma.
\]

By an induction argument we can then assume

\[
\text{Ad} \ f_j(\eta)^{-1}s_j(\pi(\eta)) = \iota_j(\eta) + a_\beta(\eta)X_\beta
\]

where \( a_\beta(p) = 1 \) (since we may multiply \( f_j \) by a suitable constant in \( T \)). Consider now the map \( d_j(\eta) = \exp^{a_\beta(\eta)^{-1}}X_\beta : U_j \to G \). Since \( p \) is a simple zero for \( \beta_j(\eta) \), \( d_j(\eta) \) is holomorphic on each chosen connected component of \( U_{j,\beta} \). We have

\[
\text{Ad} \ d_j(\eta)(\iota_j(\eta) + a_\beta(\eta)X_\beta) = \iota_j(\eta) + X_\beta
\]
and the claim of our lemma is proved. □

For each $j \in J$, define $u_j : U_j \to B$ by $u_j(\eta) = \exp \frac{X_\beta}{\beta(\eta)}$ whenever $\eta \in U_{j,\beta}$. We have

$$Ad u_j(\eta)^{-1}t_j(\eta) = t_j(\eta) + X_\beta. \quad (23)$$

We may represent the completed section $\hat{\varphi}_B$ by $\{\mu_h(\eta)B\}$ where the $\mu_i$’s are as in (18) for every $i \in I$ and the $\mu_i$’s satisfy (24) for every $j \in J$. By substituting (18) and (20) in (17) and replacing $t_j(\eta) + X_\beta$ with $Ad u_j(\eta)^{-1}t_j(\eta)$ we obtain transition functions on each nonempty intersection $U_j \cap U_i$

$$b_{ji}(\eta) \equiv \mu_j(\eta)^{-1}g_{ji}(\pi(\eta))\mu_i(\eta) = u_j^{-1}(\eta)t_{ji}(\eta) \quad (24)$$

where $t_{ji}(\eta) : U_i \cap U_j \to T$ is holomorphic (as $u_j$ is holomorphic on $U_i \cap U_j$).

Since each element in $B$ can be written uniquely as a product of a unipotent element by an element in $T$ we have $t_{ji} = p \circ b_{ji}$.

We now compare $P_B$ with $P_{s_\alpha(B)}$. By definition we only need to compare them around the ramification points. As set of nilpotent generators in the Lie algebra of $s_\alpha(B)$ we may choose $\{X_\beta\}_{\beta \in R^\alpha} \cup \{Ad n_\alpha(X_\beta)\}$. Thus from lemma 3.3 we may define a section $\hat{\varphi}_{s_\alpha(B)} : \hat{\mathcal{C}} \to \pi^*P/s_\alpha(B)$ completing $\varphi$ by

$$\hat{\varphi}_{s_\alpha(B)}(\eta) = \mu_j(\eta)s_\alpha(B) \text{ for } \eta \in U_j \setminus U_{j,\alpha}$$

$$\hat{\varphi}_{s_\alpha(B)}(\eta) = \mu_j(s_\alpha(\eta)n_\alpha^{-1}s_\alpha(B)) \text{ for } \eta \in U_{j,\alpha}$$

where the $G$-valued maps $\mu_j$ satisfy (24). From this we see that $P_{s_\alpha(B)}$ and $P_B$ are isomorphic on $\mathcal{C} \setminus D_\alpha$ and that on all intersection sets $U_{j,\alpha} \cap U_i$ with $j \in J$ we have transition functions for $P_{s_\alpha(B)}$ of the form

$$b_{ji}'(\eta) = n_\alpha \mu_j(s_\alpha(\eta))^{-1}\mu_j(\eta)b_{ji}(\eta). \quad (25)$$

If we apply lemma 3.3 to the set $s_\alpha(R^+)$ of positive roots corresponding to $s_\alpha(B)$ we obtain on $U_{j,\alpha} \cap U_i$ a factorization $b_{ji}'(\eta) = u_j^{-1}(\eta)t_{ji}(\eta)$ with $u_j(\eta) = \exp \frac{Ad n_\alpha(X_\beta)}{-\alpha_j(\eta)} = n_\alpha u_j^{-1}(\eta)n_\alpha^{-1}$ and $t_{ji}(\eta) = p' \circ b_{ji}(\eta)$ (compare with (24)). Let us denote by $I$ the identity element in $G$. From (25) and lemma 3.2 a meromorphic section of $s_\alpha\tau_B - \tau_B$ is given by a 0-cochain $\{t_h\}_{h \in H} \in C^0(\mathcal{U}, T)$ where

$$t_h(\eta) = I \text{ whenever } h \in I \text{ or } h \in J \text{ and } \eta \notin U_{j,\alpha} \quad (26)$$

$$t_j(\eta) = n_\alpha u_j(\eta)^{-1}\mu_j(s_\alpha(\eta))^{-1}\mu_j(\eta)u_j(\eta)^{-1} \forall \eta \in U_{j,\alpha}, \ j \in J. \quad (27)$$
By (20) on each $U_{j,\alpha}$ the map $h_j(\eta) = \mu_j(s_\alpha \eta)^{-1}\mu_j(\eta)$ satisfies

$$Ad h_j(\eta)(\iota_j(\eta) + X_\alpha) = \iota_j(s_\alpha \eta) + X_\alpha = Ad n_\alpha(\iota_j(\eta)) + X_\alpha.$$  \hspace{1cm} (28)

Choose $X_{-\alpha} \in g$ so that $X_\alpha, X_{-\alpha}, h_\alpha := [X_\alpha, X_{-\alpha}] \in t$ generate a Lie subalgebra $h_\alpha \subset g$ with $h_\alpha \cong sl(2)$ and $d\alpha(h_\alpha) = 2$. Define

$$F_j(\eta) = exp(\alpha_j(\eta)X_{-\alpha}) \ \forall \eta \in U_{j,\alpha}.$$ 

Since $F_j(\eta)$ satisfies $Ad F_j(\eta)(\iota_j(\eta) + X_\alpha) = Ad n_\alpha(\iota_j(\eta)) + X_\alpha$, by (28) we have on $U_{j,\alpha}$

$$\mu_j(s_\alpha \eta)^{-1}\mu_j(\eta) = F_j(\eta) \cdot L_j(\eta)$$ \hspace{1cm} (29)

where for each $\eta \in U_{j,\alpha}$, $L_j(\eta) \in B$ lies in the centralizer of $\iota_j(\eta) + X_\alpha \in b$.

Note that for $q$ any ramification point in $U_{j,\alpha}$ we have by definition

$$L_j(q) = I.$$ \hspace{1cm} (30)

In particular the map $L_j$ is holomorphic. Since when $\eta \in U_{j,\alpha}$ is not a ramification point $\iota_j(\eta) + X_\alpha$ is regular semisimple and by (23) one has $c_g(\iota_j(\eta) + X_\alpha) = Ad u_j(\eta)^{-1}t$, the holomorphic $T$-valued map $l_j(\eta) = p \circ L_j(\eta)$ has the form

$$l_j(\eta) = u_j(\eta)L_j(\eta)u_j(\eta)^{-1}.$$ \hspace{1cm} (31)

Relation (27) becomes

$$t_j(\eta) = z_j(\eta) \cdot l_j(\eta)$$ \hspace{1cm} (32)

where the map $z_j(\eta) \equiv n_\alpha u_j(\eta)^{-1}F_j(\eta)u_j(\eta)^{-1}$ has values in $T$ and is holomorphic everywhere in $U_{j,\alpha}$ but on the ramification points. The connected subgroup $H_\alpha \subset G$ generated by $exp(X_\alpha), exp(X_{-\alpha}), exp(h_\alpha)$ is isomorphic to a copy of $Sl(2)$ or $PGl(2)$ in $G$ and one can compute $z_j(\eta)$ directly in terms of two by two matrices. In the $Sl(2)$ case, denoting by $\varrho$ the isomorphism $H_\alpha \to Sl(2)$, one has for some $c \in \mathbb{C}^*$

$$\varrho(z_j(\eta)) = \varpsilon \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & -c/\alpha_j(\eta) \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \alpha_j(\eta)/c & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -c/\alpha_j(\eta) \\ 0 & 1 \end{array} \right)$$

$$= k \cdot diag(\alpha_j(\eta), \alpha_j(\eta)^{-1})$$ \hspace{1cm} (33)
where \( k \in T \) is a constant and \( \alpha_j(\eta) \) are the coordinates of the section \( da \equiv \iota \), according to our previous notations. As for \( H_\alpha \cong PGL(2) \) one gets
\[
\phi(z_j(\eta)) = \frac{k \cdot \text{diag}(\alpha_j(\eta), \alpha_j(\eta)^{-1})}{(34)}
\]
where the bar indicates the image under the factor map : \( GL(2) \to PGL(2) \).

Let now \( T_\alpha \subset T \) be the identity component of the subgroup \( \text{Ker}(\alpha) = \{ t \in T \mid \alpha(t) = 1 \} \). The centralizer \( Z_\alpha \) in \( G \) of \( T_\alpha \) is a reductive group of semisimple rank 1 having Lie algebra \( z = t \oplus CX_\alpha \oplus CX_{-\alpha} \), and it is known that such a group is a product \( T \times H \), \( T \) being a torus and \( H \) being a copy of \( S\ell(2) \), \( PGL(2) \) or \( GL(2) \). The case \( H = S\ell(2) \) is characterized by the group of characters \( X(T) \) being an orthogonal direct sum \( Z\chi_1 \oplus X' \), with \( \chi_1 = \sqrt{\alpha} \). If we compose any \( \lambda \in X' \) with the 0-chain \( \{ t_h \}_{h \in H} \) defined by (20) and (P7) we obtain a nowhere vanishing holomorphic section of the line bundle \( (s_\alpha \tau_B - \tau_B) \times \lambda C \). If instead we compose \( \chi_1 \) to \( \{ t_h \}_{h \in H} \), by (22) and (33) we get an holomorphic section for \( (s_\alpha \tau_B - \tau_B) \times \chi_1 C \) having simple zeroes exactly on the locus \( D_\alpha \). Thus relation (14) is satisfied (see Remark 1.1).

The case \( H = PGL(2) \) is characterized by \( X(T) \) being an orthogonal direct sum \( Z\alpha \oplus X' \). For \( \lambda \in X' \), we get the same result as for the \( S\ell(2) \) case. For \( \lambda = \alpha \) we find instead an holomorphic section for \( (s_\alpha \tau_B - \tau_B) \times \chi_1 C \) having zeroes of multiplicity two on \( D_\alpha \). This proves (16).

In case \( H = GL(2) \), we have an orthogonal direct sum \( X(T) = X' \oplus Z\chi_1 \oplus Z\chi_2 \) with \( \alpha = \chi_1 \cdot \chi_2^{-1} \). Composing \( \lambda \in X' \) gives us again \( s_\alpha \tau_B \times \chi_1 C \cong \tau_B \times \chi_1 C \) as in the previous cases. If we compose \( \chi_1 \) we obtain an holomorphic section of \( (s_\alpha \tau_B - \tau_B) \times \chi_1 C \) having simple zeroes exactly on \( D_\alpha \). If we compose \( \chi_2 \) we obtain a meromorphic section of \( (s_\alpha \tau_B - \tau_B) \times \chi_2 C \) having simple poles exactly on \( D_\alpha \). Thus relation (16) holds also in this case and theorem 3.1 is proved.

We thus have a map
\[
\mathcal{T} : \mathcal{H}^{-1}(\phi) \to \hat{\mathcal{P}} \equiv [\text{Pic}(\hat{C}) \otimes X(T)^*]^W
\]
\[
(P, s) \mapsto (\tau(P, s) + K_P, \mathcal{T}(P, s)).
\]

Note that from (13) and lemma 3.2 \( \mathcal{T} \) does not depend on the choice of the Borel subgroup \( B \supset T \) (or of the subset of positive roots in \( R \)).

**Definition 3.4** Let \( \mathcal{H}^{-1}(\phi)_c \) be some connected component of \( \mathcal{H}^{-1}(\phi) \). For a fixed point \( (P', s') \in \mathcal{H}^{-1}(\phi)_c \) we define \( \mathcal{F}_c : \mathcal{H}^{-1}(\phi)_c \to \mathcal{P}_0 \) by
\[
\mathcal{F}_c(P, s) = \mathcal{T}(P, s) - \mathcal{T}(P', s') \equiv \tau(P, s) - \tau(P', s').
\]
Such definition does not depend on our previous choice of the theta characteristic $\frac{1}{2}K$. We now want to study the fibers of $\mathcal{F}_c$. First we make the following

**Remark 3.1** For $i \in I$, the maps $\mu_i(\eta)$ in (12) are defined up to multiplication to the right by some holomorphic map $m_i : U_i \to T$. As for $j \in J$, any other holomorphic map $\mu'_j(\eta)$ satisfying (20) has the form $\mu'_j(\eta) = \mu_j(\eta)M_j(\eta)$ where for every $\alpha \in R(j) \cap R^+$, $M_j(\eta) : U_{j, \alpha} \to B$ is holomorphic and such that $M_j(\eta) \in c_G(t_j(\eta) + X_\alpha)$. If we replace $\mu_j$ and $\mu_i$ with the new maps $\mu'_j(\eta)$ and $\mu'_i(\eta) = \mu_i(\eta)m_i(\eta)$, we obtain from $(P, s)$ and $B$ an equivalent cocycle $\{m_h^{-1}t_hm_i\}$ representing $\tau_B$. Since for every $j \in J$ and $q \in U_j \cap \mathcal{D}_\alpha$, $t_j(q) + X_\alpha \in b$ is regular, we have $c_G(t_j(q) + X_\alpha) = T_\alpha U_\alpha$, where $T_\alpha$ is the identity component of $\text{Ker}(\alpha : T \to \mathbb{C}^*)$ and $U_\alpha$ is the unipotent 1-dimensional subgroup corresponding to the root $\alpha$. Hence the $T$-valued map $m_j(\eta) := p \circ M_j(\eta) \equiv u_j(\eta)M_j(\eta)u_j(\eta)^{-1}$ satisfies for every $\alpha \in R(j) \cap R^+$

$$\alpha(m_j(q)) = 1 \ \forall q \in U_j \cap \mathcal{D}_\alpha \ .$$

**Lemma 3.5** Let $(P, s), (Q, v)$ be pairs in $\mathcal{H}^{-1}(\phi)$ such that $\tau(P, s)$ and $\tau(Q, v)$ are isomorphic. Let $\{t_{hl}\}$ and $\{\tilde{t}_{hl}\}$ with $h, l \in H$ be cocycles representing $\tau(P, s)$ and $\tau(Q, v)$ respectively and suppose

$$\tilde{t}_{hl} = m_h^{-1}t_hm_l \tag{36}$$

where the maps $m_h : U_h \to T$ are holomorphic and satisfy condition (13) for every $j \in J$ and $\alpha \in R(j) \cap R^+$. Then $Q$ is isomorphic to $P$ and $v = s$.

**Proof.** For what concerns $P$ and the construction of $\tau(P, s)$ we keep the notations used in the proof of theorem 3.1. In particular we still consider a Čech covering $\mathcal{U} = \{U_h\}_{h \in H}$ of $\tilde{C}$ consisting of $W$-invariant open sets as it was first defined in the proof of lemma 3.2. For each nonempty intersection $U_h \cap U_l$ we have transition functions for the $B$-reduction $Q_B$ of $\pi^*Q$ having the form:

$$\tilde{b}_{ji}(\eta) = \tilde{\mu}_j(\eta)^{-1}\tilde{g}_{ji}(\pi(\eta))\tilde{\mu}_i(\eta) = u_j(\eta)^{-1}\tilde{t}_{ji}(\eta) \ \forall j \in J, i \in I \tag{37}$$

$$\tilde{b}_{hi}(\eta) = \tilde{\mu}_h(\eta)^{-1}\tilde{g}_{hi}(\pi(\eta))\tilde{\mu}_i(\eta) = \tilde{t}_{hi}(\eta) \ \forall i, h \in I \tag{38}$$
where \( \{ g_{hl} \}_{h,l \in H} \) are transition functions for the \( G \)-bundle \( Q \) and \( \tilde{\mu}_i, \tilde{\mu}_j \) are defined analogously as \( \mu_i \) and \( \mu_j \) in (24). For \( j \in J \), define \( M_j : U_j \rightarrow B \) by

\[
M_j := u_{j}^{-1}m_{j}u_{j} \quad \text{(see Remark 3.1).} \tag{39}
\]

The hypothesis of the lemma provide that \( M_j \) is holomorphic on \( U_{j,\alpha} \) for each \( \alpha \in R(j) \cap R^+ \) and we have \( M_j(\eta) \in c_G(\gamma_j(\eta) + X_\alpha) \) for \( \eta \in U_{j,\alpha} \) by definition of \( u_j \). Define the holomorphic maps

\[
\Gamma_i = \mu_i m_i \tilde{\mu}_i^{-1} \quad \forall i \in I \quad \text{and} \quad \Gamma_j = \mu_j M_j \tilde{\mu}_j^{-1} \quad \forall j \in J.
\]

From (37), (24) and (36) we obtain the equivalence condition between cocycles on \( \tilde{C} \):

\[
\tilde{g}_{hl}(\pi(\eta)) = \Gamma_h(\eta)^{-1}g_{hl}(\pi(\eta)) \Gamma_l(\eta) \quad \forall \eta \in U_h \cap U_l \forall h, l \in H.
\]

The claim of the lemma is then proved provided we show that the maps \( \Gamma_l \) are invariant with respect to the action of \( W \) on the sheets of \( \tilde{C} \). In fact if we indicate by \( \{ v_h \}_{h \in H} \) the coordinates of \( v \) so that \( v_h = Ad \tilde{g}_{hl} \cdot k_{hl}v_l \), by our definition of the maps \( \tilde{\mu}_i, \tilde{\mu}_h \) we have:

\[
Ad \Gamma_l v_l = s_l \quad \forall l \in H.
\]

Since \( W \) is generated by the simple reflections, it suffices to show \( \Gamma_i(s_\alpha \eta) = \Gamma_i(\eta) \) for every simple reflection \( s_\alpha \). From (13) we have for each \( i \in I \)

\[
\mu_i(s_\alpha \eta)^{-1}n_\alpha l_i(\eta) \quad \text{for suitable holomorphic maps} \ l_i : U_i \rightarrow T.
\]

By evaluating the transition functions \( t_{hi} = \mu_i^{-1} g_{hl} \mu_i \) with \( h, i \in I \) on \( s_\alpha \eta \) and replacing \( \mu_i(s_\alpha \eta) \) with \( \mu_i(\eta)l_i(\eta)^{-1}n_\alpha^{-1} \) and \( \mu_h(s_\alpha \eta) \) with \( \mu_h(\eta)l_h(\eta)^{-1}n_\alpha^{-1} \) we obtain

\[
t_{hi}(s_\alpha \eta) = n_\alpha l_h(\eta) t_{hi}(\eta) l_i(\eta)^{-1}n_\alpha^{-1}. \tag{41}
\]

Analogously, if we define \( \tilde{l}_i : U_i \rightarrow T \) by

\[
\tilde{\mu}_i(s_\alpha \eta)^{-1}n_\alpha l_i(\eta) \quad \text{for suitable holomorphic maps} \ l_i : U_i \rightarrow T
\]

we have

\[
\tilde{t}_{hi}(s_\alpha \eta) = n_\alpha \tilde{l}_h(\eta) \tilde{t}_{hi}(\eta) \tilde{l}_i(\eta)^{-1}n_\alpha^{-1}. \tag{43}
\]
By replacing $\tilde{t}_{hi}$ with $m_h^{-1}t_{hi}m_i$ in both sides of (18) and substituting (14) in the left-hand side, we obtain an equality both sides of which contain only factors with values in $T$. We cancel $t_{hi}(\eta)$ and obtain

$$m_h(\eta) \cdot n_\alpha^{-1} m_h(s_\alpha \eta)^{-1} n_\alpha \cdot \tilde{l}_h(\eta)^{-1} \cdot l_h(\eta) = m_i(\eta) \cdot n_\alpha^{-1} m_i(s_\alpha \eta)^{-1} n_\alpha \cdot \tilde{l}_i(\eta)^{-1} \cdot l_i(\eta)$$

for every $\eta \in U_h \cap U_i$, $i, h \in I$. We can repeat the same calculation on intersection sets $U_i \cap U_j$ with $j \in J$ and $i \in I$. What we need is the analog for $j \in J$ of the relations (10) and (12). On each open set $U_{j,\alpha}$ the map $\mu_j(\eta)$ is related with $\mu_j(s_\alpha \eta)$ via the identity (29). If for each $\beta \in R^+ \setminus \{\alpha\}$ we define $n_{\alpha,\beta} \in N(R)$ to be the representative of $s_\alpha$ satisfying $Ad n_{\alpha,\beta}(X_\beta) = X_{s_\alpha(\beta)}$, by construction of the maps $\mu_j$ in lemma (3,3) we have for $\eta \in U_{j,\beta}$

$$\mu_j(s_\alpha \eta)^{-1} \mu_j(\eta) = n_{\alpha,\beta} L_j(\eta)$$

(44)

where $L_j(\eta)$ is a suitable element in the centralizer of $\iota_j(\eta) + X_\beta$. We analogously define $\tilde{L}_j : U_j \to B \forall j \in J$ by

$$\tilde{\mu}_j(s_\alpha \eta)^{-1} \tilde{\mu}_j(\eta) = F_j(\eta) \tilde{L}_j(\eta) \quad \text{for} \quad \eta \in U_{j,\alpha}$$

(45)

$$\tilde{\mu}_j(s_\alpha \eta)^{-1} \tilde{\mu}_j(\eta) = n_{\alpha,\beta} \tilde{L}_j(\eta) \quad \text{for} \quad \eta \in U_{j,\beta} \text{ with } \beta \neq \alpha$$

(46)

and set for each $\eta \in U_j$

$$l_j(\eta) := p \circ L_j(\eta) = u_j(\eta)L_j(\eta)u_j(\eta)^{-1}$$

(47)

$$\tilde{l}_j(\eta) := p \circ \tilde{L}_j(\eta) = u_j(\eta)\tilde{L}_j(\eta)u_j(\eta)^{-1}.$$ 

(48)

One uses (24), (13) and the fact that the map $z_j(\eta) = n_\alpha u_j^{-1}(\eta)F_j(\eta)u_j^{-1}(\eta)$ (see (32) ) is holomorphic $T$-valued outside the ramification points (hence it commutes with any other map with values in $T$), to obtain by the same procedure described above for all pairs of indices $h, i \in I$

$$m_j(\eta) \cdot n_\alpha^{-1} m_j(s_\alpha \eta)^{-1} n_\alpha \cdot \tilde{l}_j(\eta)^{-1} \cdot l_j(\eta) = m_i(\eta) \cdot n_\alpha^{-1} m_i(s_\alpha \eta)^{-1} n_\alpha \cdot \tilde{l}_i(\eta)^{-1} \cdot l_i(\eta)$$

for each $\eta \in U_{j,\alpha} \cap U_i$. One uses (14) and (10) to prove the same identity for all $\eta \in U_{j,\beta} \cap U_i$ with $\beta \neq \alpha$. In conclusion, the maps

$$m_h(\eta) \cdot n_\alpha^{-1} m_h(s_\alpha \eta)^{-1} n_\alpha \cdot \tilde{l}_h(\eta)^{-1} \cdot l_h(\eta) : U_h \to T$$

with $h \in H$ are the restriction to $U_h$ of a global holomorphic map on $\tilde{C}$, hence are equal to some constant $c$. We compute such map on one ramification point $q \in U_{j,\alpha}$. Since we have
\( l_j(q) = \tilde{l}_j(q) = 1 \) (compare with (30)) and \( \alpha(m_j(q)) = 1 \) by hypothesis, we obtain \( c = 1 \), i.e.

\[
\begin{align*}
m_h(s_\alpha \eta) &= n_\alpha m_h(\eta) \cdot l_h(\eta) \cdot \tilde{l}_h(\eta)^{-1} n_{\alpha}^{-1} & \forall h \in H.
\end{align*}
\]

(49)

By use of (40), (42) and this last identity we find \( \Gamma_i(s_\alpha \eta) = \Gamma_i(\eta) \) for each \( \eta \in U_i, i \in I \). As for \( j \in J \), if \( \eta \) is in \( U_j, \alpha \) we have by (29) and (45), by the definition of \( M_j, l_j \) and \( \tilde{l}_j \) and by (49)

\[
\begin{align*}
\Gamma_j(s_\alpha \eta) &= \mu_j(\eta) u_j(\eta)^{-1} l_j(\eta)^{-1} z_j(\eta)^{-1} m_j(\eta) l_j(\eta) \tilde{l}_j(\eta)^{-1} z_j(\eta) \tilde{l}_j(\eta) u_j(\eta) \tilde{\mu}_j(\eta)^{-1} = \\
&= \Gamma_j(\eta).
\end{align*}
\]

If \( \eta \) is in \( U_j, \beta \), one proves \( \Gamma_j(s_\alpha \eta) = \Gamma_j(\eta) \) by using (44), (46), (49) and the identity (following from the above definition of \( n_{\alpha, \beta} \))

\[
\begin{align*}
\alpha(m_j(q)) = 1 \forall q \in U_j \cap D_\alpha.
\end{align*}
\]

(51)

Proof. Choose one ramification point \( q_\alpha \in D_\alpha \) for each \( \alpha \in \Delta, q_\alpha \in U_j(\alpha) \) for suitable \( j(\alpha) \in J \). Up to multiplying the maps \( \{ m_h \}_{h \in H} \) by a suitable element in \( T \) we may assume

\[
\alpha(m_j(\alpha)(q_\alpha)) = 1 \forall \alpha \in \Delta.
\]

(52)

We keep the same notation as before. We consider the maps \( \{ l_h \} \) and \( \{ \tilde{l}_h \}, h \in H \) as in (40), (42), (47) and (48) and let \( \alpha \) be some simple root. From
the proof of lemma (3.5) one has that the maps $m_h(\eta) \cdot n_\alpha^{-1}m_h(s_\alpha \eta)^{-1}n_\alpha \cdot \tilde{l}_h(\eta)^{-1} \cdot l_h(\eta) : U_h \to T$ are the restriction of a global holomorphic map on $\tilde{C}$. Computing such map on $q_\alpha$ gives us by (52) and the fact that we have $l_j(q) = \tilde{l}_j(q) = I \ \forall q \in D_\alpha \cap U_j$

$$m_j(q) \cdot n_\alpha^{-1}m_j(s_\alpha q)^{-1}n_\alpha \cdot \tilde{l}_j(q)^{-1} \cdot l_j(q) = I \ \forall q \in D \cap U_j, j \in J \quad (53)$$

and

$$m_j(q) = n_\alpha^{-1}m_j(s_\alpha q)n_\alpha \ \forall q \in D_\alpha \cap U_j, j \in J.$$ 

By evaluating $\alpha : T \to \mathbb{C}^*$ on both sides of this last identity we obtain

$$\alpha^2(m_j(q)) = 1.$$ 

If moreover $\alpha$ satisfies condition (51), evaluating $\lambda$ on both sides of the same identity gives $\lambda(m_j(q)) = \lambda(m_j(q)) \cdot \alpha^{-1}(m_j(q))$, or

$$\alpha(m_j(q)) = 1.$$ 

The claim of the theorem is thus proved for every simple root. Consider now $q \in D_\beta$ with $\beta \in R^+ \setminus \Delta$. Note that for $q \in U_j$, from the definition of $l_j$ and $\tilde{l}_j$ and the fact that $L_j(q)$ and $\tilde{L}_j(q)$ belong to the centralizer in $G$ of $t_j(q) + X_\beta$ we have

$$\beta(l_j(q)) = \beta(\tilde{l}_j(q)) = 1 \quad (54)$$

(compare with (55) in Remark 3.1). By evaluating $\beta : T \to \mathbb{C}^*$ on both sides of (53) as $\alpha$ runs over all simple roots we obtain $\beta(m_j(q)) = \beta(n_\alpha^{-1}m_j(s_\alpha q)n_\alpha)$ $\forall \alpha \in \Delta$, hence

$$\beta(m_j(q)) = \beta(n_w^{-1}m_j(wq)n_w) \ \forall w \in W.$$ 

On the other hand, we know that there exist $\alpha \in \Delta$ and $u \in W$ with $u(\alpha) = \beta$. We thus have

$$\beta(m_j(q)) = \beta(n_w m_j(u^{-1}q)n_w^{-1}) = \alpha(m_j(u^{-1}q)) = \mp 1.$$

\[\square\]

**Theorem 3.7** Suppose $G$ has one of the following properties:

a) the commutator group $(G, G)$ is simply connected;

b) the Dynkin diagram of $G$ has no component of type $B_l$, $l \geq 1$.

Then the map $T : \mathcal{H}^{-1}(\phi) \to \hat{P}$ is injective.
Proof. In case $(G, G)$ is simply connected the fundamental weights are elements in $X(T)$; in particular condition $(\ref{eq:51})$ in lemma \ref{lemma:3.6} is satisfied for every root $\alpha \in R^+$ and our claim follows from lemma \ref{lemma:3.5}. As for the case $G$ satisfies condition $b_0$), we see from the Dynkin diagram of all simple groups of type different from $B_l$, $l \geq 1$ and $G_2$ that for every $\alpha \in R^+$ there exists another root $\beta$ with $<\beta, \alpha> = 1$. On the other hand the type $G_2$ is simply connected. \qed

**Theorem 3.8** Let $a \geq 1$ be the cardinality of the subset $A \subset R^+$ of those roots which do not satisfy condition $(\ref{eq:51})$ in lemma \ref{lemma:3.6}. If $d$ denotes the degree of $\pi^*K$, the fibre of $T$ consists of at most $2a(d-1)$ points.

Proof. Let $(P, s) \in H^{-1}(\phi)$, $\tau(P, s)$ be as in theorem \ref{theorem:3.7} and suppose there exists a pair $(Q, v) \in H^{-1}(\phi)$ such that $\tau(Q, v) \equiv \tau(P, s)$. Let $\{t_{hl}\}_{h, l \in H}$ and $\{\tilde{t}_{hl}\}_{h, l \in H}$ be cocycles representing $\tau(P, s)$ and $\tau(Q, v)$ respectively and write $t_{hl} = m_h^{-1}t_{hl}m_l$ for suitable holomorphic maps $m_h : U_h \to T$ with $h \in H$. From the proof of lemma \ref{lemma:3.6} we can assume that for a chosen ramification points $q \in D_\beta$, one for each $\beta \in A$, and every other ramification point $q \not\in D_\beta$ with $\beta \not\in A$, condition $\beta(m_j(q)) = 1$ (for suitable $j \in J$) holds. If $(Q, v)$ is distinct from $(P, s)$, by lemmas \ref{lemma:3.5} and \ref{lemma:3.6} there exists some $\alpha \in A$ and some $p_\alpha \in U_j \cap D_\alpha$ (with suitable $j \in J$) such that condition

$$\alpha(m_j(p_\alpha)) = -1 \quad (\ref{eq:55})$$

is satisfied. Moreover, two pairs for which relation $(\ref{eq:55})$ holds for exactly the same set of ramification points coincide by Remark \ref{remark:3.1}. \qed

From theorems \ref{theorem:3.7} and \ref{theorem:3.8} and from proposition \ref{proposition:2.1} we obtain the following

**Corollary 3.9** The image under $\mathcal{F}$ of the generic Hitchin fibre $H^{-1}(\phi)$ contains a Zariski open set in $\mathcal{P}_0$.

**3.1 The $PGL(2)$ case.**

Let $\phi \in H^0(C, K^2)$ be generic. Let $P$ be a $PGL(2)$-bundle over $C$ and $s \in H^0(C, adP \otimes K)$ such that $\mathcal{H}(P, s) = \phi$. We indicate by $pr : GL(2) \to PGL(2) = GL(2)/C^*$ the factor map and as maximal torus $T \subset PGL(2)$ we choose the one obtained by restricting $pr$ to the maximal torus $\tilde{T} \subset GL(2)$.
given by all diagonal matrices. We also set $t = \text{Lie } T$, $\mathfrak{i} = \text{Lie } \tilde{T}$. In this setting, $\tilde{C} = \phi^*(t \otimes K)$ is a ramified double covering of $C$ whose ramification divisor $\mathcal{D}$ satisfies by definition $O(\mathcal{D}) \cong \pi^*K$.

Let $\{V_h\}_{h \in H}$ and $\{U_h\}_{h \in H}$ be open coverings of $C$ and $\tilde{C}$ defined as before. If $\{g_{hl} : V_h \cap V_l \to PGL(2)\}_{h,l \in H}$ are transition functions for $P$, it is known that there exists some rank 2 vector bundle $F$, hence some principal $GL(2)$-bundle $\tilde{P}$, with transition functions $\tilde{g}_{hl}$ satisfying

$$pr \circ \tilde{g}_{hl} = g_{hl} \quad \forall h, l \in H. \quad (56)$$

Moreover, any other rank 2 vector bundle $F'$ has the same property if and only if $F' \cong F \otimes L$ for some line bundle $L \in Pic(C)$. Note also that this implies $\deg F \equiv \deg F' \mod{2}$ (since $\deg(F \otimes L) = \deg F \cdot \deg L^2$). For the sake of simplicity for any $F$ satisfying relation (56) we write $P = pr(F)$. For $\tilde{P}$ as above, we clearly have an isomorphism $ad \tilde{P} \otimes K \cong (ad P \otimes K) \oplus K$ and given some fixed generic section $x : C \to \tilde{K}$ we may define $\tilde{s} \in H^0(ad \tilde{P} \otimes K)$ by $\tilde{s} = s \oplus x$. We set $\phi = \mathcal{H}_{GL(2)}(\tilde{P}, \tilde{s}) \in H^0(C, K \oplus K^2)$ (the subscript indicating that we are in the $GL(2)$ setting) and observe that the covering $\phi^*(\mathfrak{i} \otimes K)$ of $C$ coincides with $\tilde{C}$. Then it is clear from the argument above that we have a surjective map

$$"pr" : \mathcal{H}_{GL(2)}^{-1}(\phi) \to \mathcal{H}_{PGL(2)}^{-1}(\phi).$$

This also shows that $\mathcal{H}_{PGL(2)}^{-1}(\phi)$ has two components $\mathcal{H}_{PGL(2)}^{-1}(\phi)_0$, $\mathcal{H}_{PGL(2)}^{-1}(\phi)_1$: namely $(Q, v) \in \mathcal{H}_{PGL(2)}^{-1}(\phi)$ is contained in $\mathcal{H}_{PGL(2)}^{-1}(\phi)_0$ or $\mathcal{H}_{PGL(2)}^{-1}(\phi)_1$ depending on the parity of the degree of those $F$ which satisfy $pr(F) = Q$.

We now look at our construction in the $GL(2)$ case. If we indicate by $\chi_1$ and $\chi_2$ the coordinate functions on $\tilde{T}$ and set $\tilde{\alpha} = \chi_1 \cdot \chi_2^{-1}$, $\sigma = s_{\tilde{\alpha}}$, we have by definition

$$\mathcal{P}_{GL(2)} = \{Q \otimes \chi'_1 \oplus \sigma^*Q \otimes \chi'_2 \mid Q \in J(\tilde{C})\} \equiv J(\tilde{C})$$

(the one parameter subgroups $\chi'_i$ being defined by $\chi'_i(\chi'_j) = (\chi_i, \chi_j)$, $j = 1, 2$) and

$$\hat{\mathcal{P}}_{GL(2)} = Pic(\tilde{C}).$$

The map $\mathcal{T} : \mathcal{H}_{GL(2)}^{-1}(\phi) \to Pic(\tilde{C})$ is injective (see theorem [3.7]), dominant and by Hitchin’s theory (see [3]) it preserves the parity of the degrees. By
the argument above the generic fibre of the map ”pr” is a principal homogeneous space with respect to \( \Lambda = \{ M \in Pic(C) \mid M = \pi^*L, \ L \in Pic(C) \} \). In this setting the map \( \pi^* : Pic(C) \to Pic(\tilde{C}) \) is injective (since \( \tilde{C} \to C \) is a ramified covering; see e.g [M]), hence \( \Lambda \) coincides with \( Pic(C) \). Since \( Pic(\tilde{C})^{\text{even}}/Pic(C) \) and \( Pic(\tilde{C})^{\text{odd}}/Pic(C) \) are both principal homogeneous spaces with respect to the connected group \( J(\tilde{C})/J(C) \), it follows that the components \( \mathcal{H}_{PGL(2)}^{-1}(\phi)_0, \mathcal{H}_{PGL(2)}^{-1}(\phi)_1 \) are connected. Now, let \( \chi' \) be the one parameter subgroup in \( T \subset PGL(2) \) given by composing \( pr \) with \( \chi'_1 \) (we have \( X(T)^* = \mathbb{Z}\chi' \)). By definition, we have \( \tilde{P}_{PGL(2)} = P_{PGL(2)} = \{ Q \otimes \chi' \mid Q \in J(\tilde{C}), \sigma^*Q \cong Q^{-1} \} \) and, since \( \pi^* : J(C) \to J(\tilde{C}) \) is injective, this is just the Prym variety \( P(C, \sigma) \subset J(\tilde{C}) \). From theorem 3.1 the \( \tilde{T} \)-bundle \( \tilde{\tau} = \tau(\tilde{P}, \tilde{s}) \) has transition functions \( t_{hl} : U_h \cap U_l \to \mathbb{C}^* \)

\[
t_{hl}(\eta) = \text{diag}(q_{hl}(\eta), \sigma^*q_{hl}(\eta) \cdot k_{hl}(\pi(\eta)))
\]

One can easily check that the maps

\[
pr \circ t_{hl}(\eta) = q_{hl}(\eta) \cdot \sigma^*q_{hl}(\eta)^{-1} \cdot k_{hl}(\pi(\eta))^{-1} : U_h \cap U_l \to \mathbb{C}^*
\]

are transition functions for \( \tau = \tau(P, s) \). In other words, if we use the additive notation, we have \( T_{PGL(2)}(P, s) = (1 - \sigma^*) \circ T_{GL(2)}(\tilde{P}, \tilde{s}) \). Moreover, if \( \tilde{P}' \) is another \( GL(2) \)-bundle inducing via the factor map \( pr \) the same \( PGL(2) \)-bundle \( P \), we have that \( \tau(\tilde{P}', \tilde{s}) \) has transition functions \( t_{hr}(\eta) \cdot l_{hr}(\pi(\eta)) \), where \( \{ l_{hr} : V_h \cap V_r \to \mathbb{C}^* \} \) define some line bundle \( L \) over \( C \). We thus have the following commutative diagram:

\[
\begin{array}{ccc}
Pic(\tilde{C}) & \xrightarrow{(1-\sigma^*)} & P(\tilde{C}, \sigma) \\
\tau_{GL(2)} & \uparrow & \uparrow \\
\mathcal{H}_{GL(2)}^{-1}(\tilde{\phi}) & \xrightarrow{pr} & \mathcal{H}_{PGL(2)}^{-1}(\phi)_0 \boxplus \mathcal{H}_{PGL(2)}^{-1}(\phi)_1 \\
\mathcal{T}_{PGL(2)} & & \\
\end{array}
\]

If we set \( \Lambda' = \{ N \in Pic(\tilde{C}) \mid N = \sigma^*N \} \), we see that all sufficiently general fibres of the dominant map \( T_{PGL(2)} \) are principal homogeneous spaces with respect to \( \Lambda'/\Lambda \). It is known (see [M]) that \( \Lambda'/\Lambda \) is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^{(d-1)} \), \( d \) being the number of ramification points of \( C \) or, in this setting, the degree of \( \pi^*K \). Note here that the number of \( \mathbb{Z}/2\mathbb{Z} \) factors reaches its maximum with respect to the estimate given in theorem 3.8. Since each component \( \mathcal{H}_{PGL(2)}^{-1}(\phi)_c \), \( c = 0,1 \), is connected, we have that the generic fibre
of $\mathcal{F}_c : \mathcal{H}^{-1}_{\text{PGl}(2)}(\phi)_c \rightarrow P(\tilde{C}, \sigma)$ consists of $2^{(d-2)}$ points.

Acknowledgments.
I wish to express my big debt to my advisor Corrado De Concini for sharing his ideas on the subject and my gratefulness to Vassil Kanev for discussions of crucial importance concerning the algebro-geometric aspects of the problem. Also, it is a pleasure for me to thank one of the referees for his interesting remarks and his contribution in improving the paper.

References

[A-B] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A 308 (1982), 523-615.

[BR] I. Biswas and S. Ramanan, *An infinitesimal study of the moduli of Hitchin pairs*, preprint.

[CR] C.W. Curtis and I. Reiner, *Methods of Representation Theory* vol. II, New-York, John Wiley & Sons, 1987.

[F] G. Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. 2 (1993), 507-568.

[Hi] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. 54 (1987), no. 1, 91-114.

[Hu] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag (1972).

[J] J.P. Jouanolou, *Théorèmes de Bertini et Applications*, Boston, Basel, Stuttgart: Birkhäuser (1983).

[K] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. 85 (1963), 327-404.

[KP] A. Kouvidakis and T. Pantev, *The automorphism group of the moduli space of semi stable vector bundles*, Math. Ann. 302 (1995), 225-268.
[L] G. Laumon, *Un analogue global du cône nilpotent*, Duke Math. J. **57** (1988), 647-671.

[M] D. Mumford, *Prym Varieties I*. In Ahlfors, L.V. et al. (eds.) Contributions to analysis, pp.325-350. New York London: Academic Press (1974).

[R] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129-152.

[Sc] R. Scognamillo, *Prym-Tjurin varieties and the Hitchin map*, Math. Ann. **303** (1995), 47-62.

[Se] J.P. Serre, *Linear Representations of Finite Groups*, New-York, Springer-Verlag, 1977.