Degeneracy of energy cross sections in scattering of light

Jeng Yi Lee

1 Department of Opto-Electronic Engineering, National Dong Hwa University, Hualien 974301, Taiwan

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We rigorously prove that under an electromagnetic plane wave with linear polarization incident normally to cylindrical passive scatterers, a single energy diagram can integrate absorption, scattering, and extinction cross sections for general scattering states. At the boundary of the energy diagram, not only the magnitudes of scattering coefficients, but also its phases, are required the same, corresponding to superabsorption or superscattering. When systems consist of a larger order of scattering resonances, its energy diagram can completely cover systems with lower one. This result displays the degenerate property of energy distribution in scattering of light. Hence, systems with different resonant orders may have the same energy performances, with potential applications on offering more degrees of freedom in designs of devices associated with energy issues. We demonstrate various systems based on real materials to support this finding.

Scattering and absorption of light by a subwavelength isotropic and homogeneous scatterer can not go beyond a single resonant order \[1,3\]. To overcome this limit, a composite system with proper multi-layered coatings can induce the overlapping of multiple partial waves. It has been shown that excitation of a confined polariton can lead to a system with a degenerate superscattering in the framework of whispering gallery condition \[4,6\]. On another side example, superabsorber \([7]\), the underlying mechanism is also by a means of multiple partial waves excited, formed by balanced partial wave resonances, but without need of degenerate resonances.

We note that for these functional devices \([8–11]\), due to passive materials embedded, the energy conservation law must be satisfied \([12]\). With this law, the corresponding partial absorption and scattering cross sections are bounded, irrespective of inherent system configurations and material parameters \([12,13]\). However, the answer to whether there can exist a single integrated energy map to accommodate all overlapping resonances is far from obvious and remains open.

In this work, following the approach of differential calculus as well as the employed of Lagrange multiplier, we rigorously prove that under an electromagnetic plane wave with linear polarization excitation, for a cylindrical system, a energy diagram can integrate all possible scattering states. The energy diagram can indicate all cross sections in absorption, scattering, and extinction, but, on the contrary, it can correspond to a variety of constitutions of scattering coefficient states, reflecting the degenerate property in light scattering. Quite interestingly, only at the boundary of energy diagram, the magnitudes of scattering coefficients and its phase are required, which corresponds to superabstraction or superscattering situations. Moreover, the system domain with lower scattering orders is just the subspace of higher ones. Therefore, one can design a variety of systems with specific energy performances, but the constitution of scattering states can be different. We demonstrate some examples to support this finding. We believe our results can provide more degrees of freedom for designs of functional devices, especially related to energy harvesting issues \([13,16]\).

Consider a cylindrically symmetric scatterer is normally impinged by a plane wave with electric field oscillated along the z-axis direction, i.e., z mode. The scattering, extinction, and absorption powers can be expressed in the following \([1,4,17]\):

\[
P_{\text{ext}} = P_{\text{scat}} + P_{\text{abs}} = -\frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{n=-\infty}^{\infty} \text{Re}(a_n^s)\]

\[
P_{\text{abs}} = -\frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{n=-\infty}^{\infty} [\text{Re}(a_n^s) + |a_n^s|^2]
\]

where \(a_n^s\) is complex scattering coefficient, \(k_0\) is environmental wave number, and \(\epsilon_0\) and \(\mu_0\) are free space permittivity and permeability. We note that due to cylindrical symmetry, the scattering coefficients have a symmetry for \(a_n^s = a_{-n}^a\). As \(n\) = \([0,1,2]\), they correspond to electric dipole, magnetic dipole, and magnetic quadrupole, respectively. The extinction power is also related to the optical theorem, which links the forward scattering amplitude \([18,21]\). Thus, systems with non-zeros of energy dissipation and radiation, they must possess a scattering component along the forward direction. Now, due to passive material embedded, the partial absorption cross section at each orders would be restricted \(\text{Re}(a_n^s) + |a_n^s|^2 \geq 0\) by energy conservation, here \(\text{Re}(a_n^s) + |a_n^s|^2\) is defined as normalized absorption power. Following this inequality, we obtain a clear physical bound on the scattering coefficient for orders, i.e., \(|a_n^s| \leq 1\) and \(\text{Arg}[a_n^s] \in [\frac{\pi}{2}, \frac{3\pi}{2}]\) \([8,12]\).

For each order, the normalized absorption power can not go beyond 0.25, while the upper limit for normalized scattering power, defined as \(|a_n^s|^2\), is 1. Therefore, if a system has a N orders excited, the maximum absorption power can not be larger than \((2N + 1) \times \frac{1}{2k_0} \sqrt{\frac{\mu_0}{\epsilon_0}}\) while the corresponding scattering power would be \((2N + 1) \times \frac{1}{2k_0} \sqrt{\frac{\mu_0}{\epsilon_0}}\). In an extreme situation, the maximum scattering power in N orders can reach \((2N + 1) \times \frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}}\), while its corresponding absorption is zero. Therefore, it is naturally extended to seek for the existence of the en-
ergy diagram when more complicated resonance orders interfere.

To address such a fundamental problem, we use the differential calculus as well as the Lagrange multiplier to figure out the energy cross section diagram under given dissipated power \[\text{[22 23]}\]. Suppose our system has a fixed dissipation power \(P_{\text{abs}} = \text{Const.}\), we expect to evaluate the extreme scattering power \(P_{\text{scat}}\). Now, \(P_{\text{abs}} = \text{Const.}\) is a constraint with primary \(a_{-N}^s\) to \(a_N^s\) involved, i.e.,

\[
P_{\text{abs}}(a_{-N}^s, \ldots, 0, a_{-N}^s) = -\frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{n=-N}^N |a_n^s|^2 = \text{const.} \tag{2}
\]

We have no assumption for given specific scattering coefficients.

Then, we define a energy function \(L\) related to scattering and absorption powers as follows

\[
L(a_{-N}^s, \ldots, 0, a_{-N}^s) = \frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{n=-N}^N |a_n^s|^2 - \lambda (\text{Re}(a_n^s) + |a_n^s|^2) = \frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{n=-N}^N |a_n^s|^2 - \lambda |a_n^s|^2 \tag{3}
\]

where \(\lambda\) is a Lagrange multiplier, and we have used the phasor representation for complex scattering coefficients, \(a_n^s = |a_n^s e^{i\theta_n}|\) here \(\theta_n\) is the argument of \(a_n^s\).

To have the extreme minimum or maximum of the scattering powers, the energy function is required to meet,

\[
\frac{\partial L}{\partial |a_n^s|^2} = \frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} [2|a_n^s| - \lambda \cos \theta_n - 2\lambda |a_n^s]| = 0, \tag{4}
\]

\[
\frac{\partial L}{\partial \theta_n} = \frac{2}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} [\lambda |a_n^s| \sin \theta_n] = 0,
\]

which are valid for \(n = -N\) to \(N\). In the latter expression, to obtain the non-trivial solution, we have \(\theta_n = 0\) or \(\theta = \pi\). However, by energy conservation, the only applicable solution would be \(\theta_n = \pi\). By this outcome, the first expression in Eq.(4) would be

\[
2|a_n^s| + \lambda - 2\lambda |a_n^s| = 0. \tag{5}
\]

Thus, we have \(|a_n^s| = \frac{\lambda}{2\lambda - 2} = s\). Further, \(a_n^s = -s\), that the magnitudes of scattering coefficients are identical and the corresponding phases are \(\pi\) at each orders.

Consequently, we can express the scattering coefficients in terms of absorption power,

\[
a_n^s = -1 \pm \sqrt{1 - \frac{2P_{\text{abs}}k_0}{(2N+1)\sqrt{\frac{\mu_0}{\epsilon_0}}}},
\]

that there are two different scattering performances under given absorption.

In the square root, it can guarantee the real value, because \(P_{\text{abs}} \leq \frac{2N+1}{2k_0} \sqrt{\frac{\mu_0}{\epsilon_0}}\) for \(N\) orders dominant. The ultimate limit would be maxima absorption, i.e., \(\frac{2N+1}{2k_0} \sqrt{\frac{\mu_0}{\epsilon_0}}\), corresponding to coherent perfect absorption \[\text{[24]}\]. More appealingly, our results imply that under constant absorption power, there can support two solutions for scattering powers:

\[
P_{\text{scat}}^{\text{Max}} = \frac{1}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} (2N+1)[1 + \sqrt{1 - \frac{2P_{\text{abs}}k_0}{(2N+1)\sqrt{\frac{\mu_0}{\epsilon_0}}}}] - \frac{P_{\text{abs}}k_0}{(2N+1)\sqrt{\frac{\mu_0}{\epsilon_0}}},
\]

\[
P_{\text{scat}}^{\text{Min}} = \frac{1}{k_0} \sqrt{\frac{\mu_0}{\epsilon_0}} (2N+1)[1 - \sqrt{1 - \frac{2P_{\text{abs}}k_0}{(2N+1)\sqrt{\frac{\mu_0}{\epsilon_0}}}}] - \frac{P_{\text{abs}}k_0}{(2N+1)\sqrt{\frac{\mu_0}{\epsilon_0}}}. \tag{7}
\]

Now, we depict an energy diagram involving normalized scattering, absorption, and extinction cross sections.
FIG. 2: (Color online)(a) Energy distribution evolutions of two different configuration systems by tuning the geometry sizes. Red line is from a core-shell system made by gold in shell and silicon in core. Here the radius ratio by this core-shell (inner to outer) is fixed constant, 0.2. The blue is from a homogenous silicon system. We can see two systems have the same energy distribution in absorption, scattering, and extinction, highlighted by a black star. The inset figure enlarges its energy distribution details. The magnitudes and phases of scattering coefficients for each order for the gold-shell and silicon-core system by tuning the outer radius from 50nm to 70nm are shown in Left and Right sides of (b). The relative permittivity parameter of gold is $\varepsilon = -2.81 + 3.19i$ and that of silicon is $18.5 + 0.63i$. In (c), we analyze the magnitudes and phases from a silicon-based system with the $n = 0, 1, 2, 3$ as shown in Fig. 1. The boundary of the energy diagram is followed by $[(\frac{2}{k_0} \sqrt{\varepsilon_0})^{-1} P_{\text{abs}}; (\frac{2}{k_0} \sqrt{\mu_0})^{-1} P_{\text{scat}}] = [(s - s^2)(2N + 1), s^2(2N + 1)]$ where $s \in [0, 1]$ in each $N$ resonance orders. We construct the density plot for normalized extinction power, defined as $(\frac{2}{k_0} \sqrt{\varepsilon_0})^{-1} P_{\text{ext}}$, in Fig. 1. It clearly reveals the limit of normalized absorption, scattering, and extinction powers. Here $N = 0$ represents a system with only electric dipole resonance supported, $N = 1$ has electric and magnetic dipoles resonances, while a system with $N = 2$ consists of electric, magnetic dipoles, and magnetic quadrupole resonances. The maximum normalized extinction power is at the maxima of normalized scattering power, while the corresponding normalized absorption is zero. Additionally, with the constant normalized extinction power, the normalized absorption (dissipation) can reduce the output scattering performance. Interestingly, within the regime of lower scattering, as indicated by red dashed line box in Fig. 1, the energy diagram reflects the existence of a higher absorption power as well as a lower scattering and extinction, corresponding to a superabsorption [7]. Through it has a definite domain boundary for $N$ resonance orders, it is just a subspace of a higher orders system, which implies that the scattering states are degenerate in energy cross sections.

As a result, the scattering system can have the interference of different resonant orders with the same energy performance. Moreover, the existence of a boundary for the corresponding resonant system requires a definite constraint in scattering coefficients, but there have no such correspondences for that inside the domain.

To verify our finding, in Fig. 2, we first discuss the energy degenerate property of systems based on experimental material dispersions [25]. We choose two systems with different geometry sizes, materials, and inherent configurations, but at the same operating wavelength 500nm. In Fig. 2 (a), the result by a red line represents that a core-shell nanowire system is constituted by golden in shell and silicon in core, while the blue line is from a homogenous silicon nanowire. We tune the outer radius a from 50nm to 70nm for the core-shell system, while tune the radius from 69nm to 80nm for the homogenous system. Initially, the rad line residues in $N = 0$ region when the outer radius is 50nm, its energy distribution then evolves through $N = 1$ region when the outer radius is larger than 55nm marked by a yellow point in the below part of Fig. 2 (a). In order to understand its behind mechanism, we investigate the magnitudes of scattering coefficients at each orders in Fig. 2 (b). We can observe that the
FIG. 3: (Color online) (a) Evolutions of energy distribution of two different configuration systems by tuning the wavelength. The cyan color line represents the silicon-shell and gold-core system with an outer radius of 132.98 nm and a ratio of the inner to outer radius of 0.42. The magnitudes and phases of scattering components are shown in (b). There is a single intersection at $N = 1$ boundary at wavelength 530 nm, marked by a black star. In (c), we consider another core-shell system with silicon in shell and silver in core with outer radius being 51.6 nm and ratio of outer-inner radius being 0.89. The evolution of energy distribution is from wavelength from 430 nm to 450 nm shown in a brown color line. There is a intersection point in $N = 0$ boundary, marked by a black cross. The magnitudes and phases of scattering components by this wavelength range are shown in (d). A marked cross denotes the intersection point is at 440 nm.

**dominant order is by $n = 0$ at $a = 50$ nm, however, with the increasing the geometrical size, the $n = 1$ order would gradually become another primary contribution. We also find that the phase from the primarily contributed order is nearly $\pi$ when close to the boundary with $N = 0$.**

For the system made by a homogeneous silicon, we depict a blue line for a relation between the normalized absorption and scattering powers by geometry size from 69 nm to 80 nm in Fig. 2 (a). We can see, in Fig. 2 (c), that the contributed orders are $n = [-2, -1, 0, 1, 2]$, which obviously belong to scattering space $N = 2$. This outcome implies that the domain should be $N = 2$, but its location is at $N = 1$, which reflects the subspace of high orders. Moreover, in the case of radius being 74 nm of the homogeneous silicon nano-wire, we find the whole powers including absorption, scattering, and extinction, are identical to that of the core-shell system at outer radius being 66 nm, which marked by a black star symbol. Although their constitution of scattering components are totally different, they can provide the same energy distributions, reflecting the degeneracy in scattering of light.

Now, we turn to discuss the existence of the superab- sorption, in which the system can absorb more energy while maintaining lower scattering power, as highlights in red dashing box of Fig. 1. We consider a core-shell nanowire system made by silicon in shell and gold in core, where the ratio of inner to outer radius is fixed to 0.42 and the outer radius is chosen as 132.98 nm. In Fig. 3 (a), we study energy distribution of the core-shell system with respect to varying wavelengths from 520 nm to 540 nm, as denoted by cyan color line. To understand its inherent scattering components, we also plot the scattering coefficients for each orders from $n = [-2, -1, 0, 1, 2]$. In this wavelength range, the dominant orders are $n = [-1, 0, 1]$. We note that only at 530 nm, the cyan line would intersect with the boundary $N = 1$ marked by a black star, revealing a system with lower scattering but with large absorption. From the right side of Fig. 3 (b), it is interesting to see that the phases from dominant orders at 530 nm would be $\pi$, satisfying the superabsorption requirement.

Furthermore, we consider another system configuration by silicon in shell and silver in core with an outer radius of 51.6 nm and the ratio of inner to outer radius being 0.89. Fig. 3 (c) shows a brown color trajectory within the wavelength of 430 nm to 450 nm. Here the sys-
tem obviously resides in $N = 0$ domain. We mark an operating 440nm by a black cross, corresponding to an intersection at $N = 0$ boundary, also meeting a system with lower scattering as well as larger absorption. However, when analyzing the components of the scattering coefficients in Fig. 3 (d), we find the primary orders to be $n = 0$ and $n = 1$. As for its phase analysis in Fig. 3 (d), the phases of primary orders are not needed to be $\pi$. We stress that this system certainly possesses the desirable energy distribution to superabsorption, but its components of dominant orders has no the required of work in [7], because this system belongs to the high scattering space domain. This outcome reveals that when designing a subwavelengthly system with functional energy property, it can have an opportunity to relax the constraints on inherent scattering states when considering a system with high scattering space.

In the superscattering case, systems can support multiple partial wave resonances at same wavelength, beyond single order limit. In appendix A, we consider an alternative system to support the similar result, but without the need of phase matching on scattering. Before conclusion, we want to remark that although our discussion is limited to cylindrical systems, but there has a similar result to a spherical system which supports a degenerate energy characteristic. The finding of the general energy diagram, irrespective of inherent system configurations, material choices and operating wavelength, would provide complete information in designing functional energy systems.

In conclusion, with the approach of differential calculus as well as the use of Lagrange multiplier, we can develop the general energy diagram involving scattering, absorption, and extinction, for any passive systems. We observe that at the boundary of energy diagram, it requires scattering coefficients the same for each orders. However, inside the diagram, there has no such correspondence to scattering coefficients. When a system consists of higher resonant orders, the corresponding diagram can completely cover system with lower ones. This result reflects a degenerate property of energy distribution in light scattering. As a result, system could have the same energy responses, but could possess different scattering states. We discuss systems with quasi superabsorption and super-scattering, but without the phase-matching constraints. This work not only provides the complete information for energy distribution in the process of light scattering, but also highlights degrees of freedom in practical energy designs.

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Appendix A

By exciting degenerate resonances at same operating wavelength, the overall system can have a strong scattering power, overcoming the single order limit. This mechanism can result from inducing confined surface waves in multi-layered system [4-6]. Here we release this constraint by employing a higher resonance order. In Fig. 4, we choose a core-shell system with gold embedded in shell and silicon in core by tuning the ratio of inner to outer radius, $\gamma$, under the constant operating wavelength 500 nm. We limit the materials in this core-shell system lossless, in order to clearly observe its overlapping resonances phenomenon. The blue line denotes this system under the radius ratio from from 0.9 to 1. We observe that when $\gamma = 0.912$, there is an intersection with $N = 1$ boundary, but its dominant orders are by $n = [-2, -1, 0, 1, 2]$ as shown in Fig. 4 (b). Moreover, its phases are not required to be $\pi$ as shown in Fig. 4 (b). This case displays a quasi-super scattering result, but is achieved from high scattering space.
FIG. 4: (Color online) (a) Energy distribution evolutions for a system with gold in shell and silicon in core. Here we ignore the materials loss, although the material dispersions are based on data \[25\]. With outer radius being 140.4nm and operating wavelength being 500nm, by tuning ratio of inner to outer radius $\gamma$ from 0.9 to 1, the blue line represents this energy distribution evolution. The blue star denotes the satisfaction of superscattering for $N = 1$ space. The magnitudes of phases of scattering coefficients for this system are shown in (b). The dominant orders in the blue star are from $n = 0, 1, 2$, while its phases are not required to be resonances. Here the relative permittivities for lossless gold being $-2.8$ and lossless silicon being 18.5.