IDENTITIES FOR THE Riemann Zeta Function

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1. Introduction

In this paper, we obtain several expansions for \( \zeta(s) \) involving a sequence of polynomials in \( s \), denoted by \( \alpha_k(s) \). These polynomials can be regarded as a generalization of Stirling numbers of the first kind and our identities extend some series expansions for the zeta function that are known for integer values of \( s \). The expansions also give a different approach to the analytic continuation of the Riemann zeta function.

The inspiration for our formulas comes from Kenter’s short note in the May 1999 Monthly \([K]\) where he derives a formula for Euler’s constant \( \gamma \) which can be regarded as the \( s \to 0 \) case of (1.8), after subtracting \( 1/s \) from both sides.

We start with Riemann’s formula

\[
\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx, \quad \Re s > 1,
\]

(1.1)

and substitute \( t = 1 - e^{-x} \) to get

\[
\int_0^1 (-\log(1 - t))^{s-1} \frac{dt}{t}, \quad \Re s > 1.
\]

(1.2)

Let

\[
g(t) = \frac{-\log(1 - t)}{t} = \sum_{k=0}^\infty \frac{t^k}{k+1}, \quad |t| < 1,
\]

(1.3)
and consider the Taylor expansion
\[ g(t)^{s-1} = \sum_{0}^{\infty} \alpha_k(s) t^k, \quad |t| < 1. \tag{1.4} \]

We will derive the following recursion:
\[
\begin{align*}
\alpha_0(s) &= 1, \\
\alpha_1(s) &= \frac{s - 1}{2}, \\
\alpha_{k+1}(s) &= \frac{1}{k(k+1)(k+2)} \sum_{j=1}^{k} \alpha_j(s) j(k + k^2 + s(2k + 2 - j)) \frac{1}{(k - j + 1)(k + j + 2)} , \quad k \geq 1, 
\end{align*}
\tag{1.5}
\]

and prove the following theorem concerning \( \alpha_k(s) \):

**Theorem 1.1.** For \( k \geq 1 \), \( \alpha_k(s)/(s - 1) \) is a polynomial in \( s \) with positive rational coefficients, and \( \alpha_k(s) \) satisfies:
\[
|\alpha_k(s)| \leq c_s \frac{(1 + \log(k + 1))|s|^{k+1}}{k+1}, \tag{1.6}
\]
where
\[
c_s = \frac{|s - 1|}{|s| + 1} (|s| + 2)^{|s|+1}. \tag{1.7}
\]

We then have the following identities:

**Theorem 1.2.**
\[
\begin{align*}
\Gamma(s) &= \sum_{0}^{\infty} \frac{\alpha_k(s)}{s + k}, \\
\Gamma(s)\zeta(s) &= \sum_{0}^{\infty} \frac{\alpha_k(s)}{s + k - 1}.
\end{align*}
\tag{1.8, 1.9}
\]

For positive integer \( \lambda \):
\[
\begin{align*}
\Gamma(s)\zeta(s - \lambda) &= \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=1}^{\lambda} (-1)^{\lambda + j} j! S(\lambda, j) \frac{1}{s + k - j - 1} \\
&= \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=0}^{\lambda-1} E(\lambda, j) \frac{(\lambda - j - 1)!}{(s + k - j - 2) \ldots (s + k - \lambda - 1)},
\end{align*}
\tag{1.10, 1.11}
\]
where \( S \) and \( E \) respectively denote the Stirling numbers of the second kind and the Eulerian numbers. We also have:

\[
\Gamma(s)\zeta(s + 1) = \sum_{k=0}^{\infty} \alpha_k(s) \Psi_1(s + k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha_k(s)}{(s + k + n)^2}, \tag{1.12}
\]

with \( \Psi_1(s + k) \) the trigamma function.

Equation (1.9) is known in the case that \( s \) is a positive integer where the coefficients \( \alpha_k(s) \) can be expressed in terms of the Stirling numbers of the first kind. Jordan \cite{Jordan} Sec. 68, (11), pg 194 credits it to Stirling. See also \cite{RS} and \cite{S}. This is described in Section 3.

2. Recursions and bound for \( \alpha_k(s) \)

To study the \( \alpha_k(s) \)'s in greater detail consider

\[
\frac{d}{dt} (g(t)^{s-1}) = (s - 1)g(t)^{s-2} \sum_{k=0}^{\infty} \frac{k + 1}{k + 2} t^k, \tag{2.1}
\]

so

\[
\frac{d}{dt} (g(t)^{s-1})g(t) = (s - 1)g(t)^{s-1} \sum_{k=0}^{\infty} \frac{k + 1}{k + 2} t^k. \tag{2.2}
\]

On the other hand, differentiating (1.4) term by term, the left hand side above equals

\[
\left( \sum_{k=0}^{\infty} (k + 1)\alpha_{k+1}(s)t^k \right) \left( \sum_{k=0}^{\infty} \frac{t^k}{k + 1} \right). \tag{2.3}
\]

Equating coefficients of \( t^k \) in (2.2) and (2.3), we get

\[
\sum_{j=0}^{k} \frac{(j + 1)\alpha_{j+1}(s)}{k - j + 1} = (s - 1) \sum_{j=0}^{k} \alpha_j(s) \frac{k - j + 1}{k - j + 2}. \tag{2.4}
\]

Using \( \alpha_0(s) = 1 \), we write this as

\[
\sum_{j=1}^{k+1} \frac{j - (s - 1)(k - j + 1)}{k - j + 2} \alpha_j(s) = (s - 1) \frac{k + 1}{k + 2}, \quad k \geq 0. \tag{2.5}
\]
The first few $\alpha_j(s)$’s are listed below

\begin{align*}
\alpha_0(s) &= 1 \\
\alpha_1(s) &= \frac{s - 1}{2} \\
\alpha_2(s) &= (s - 1) \left( \frac{1}{8}s + \frac{1}{12} \right) \\
\alpha_3(s) &= (s - 1) \left( \frac{1}{48}s^2 + \frac{1}{16}s + \frac{1}{24} \right) \\
\alpha_4(s) &= (s - 1) \left( \frac{1}{384}s^3 + \frac{7}{384}s^2 + \frac{23}{576}s + \frac{19}{720} \right)
\end{align*}

(2.6)

We also observe, from (2.5), that we can write

\[ \alpha_{k+1}(s) = \frac{s - 1}{k + 2} - \frac{1}{k + 1} \sum_{j=1}^{k} \frac{j - (s - 1)(k - j + 1)}{k - j + 2} \alpha_j(s) . \]  

(2.7)

Furthermore, substituting $k - 1$ for $k$ and moving the l.h.s. to the r.h.s., we have

\[ 0 = \frac{s - 1}{k + 2} - \frac{1}{k} \sum_{j=1}^{k} \frac{j - (s - 1)(k - j)}{k - j + 1} \alpha_j(s) . \]  

(2.8)

Subtracting $(k + 1)/(k + 2)$ times (2.8) from (2.7) we get

\[ \alpha_{k+1}(s) = \frac{1}{k(k + 1)(k + 2)} \sum_{j=1}^{k} \alpha_j(s) j(k + k^2 + s(2k + 2 - j)) \frac{(k + k^2 + s(2k + 2 - j))}{(k - j + 1)(k - j + 2)} , \quad k \geq 1. \]  

(2.9)

This last form for the recursion governing the $\alpha_k$’s is useful in that we easily obtain, inductively, the first part of Theorem 1.1, namely that $\alpha_k(s)/(s - 1)$, for $k \geq 1$, is a polynomial in $s$ with positive rational coefficients.

Next, we determine a bound for $|\alpha_k(s)|$. Let $M_s$ be the positive integer satisfying $|s| + 1 \leq M_s < |s| + 2$. From the positivity of the coefficients of $\alpha_k(s)/(s - 1)$ we have

\[ \left| \frac{\alpha_k(s)}{s - 1} \right| < \frac{\alpha_k(M_s + 1)}{M_s} , \quad k = 1, 2, \ldots . \]  

(2.10)

Here we are using the fact that $\alpha_k(s)/(s - 1)$ is a polynomial in $s$ with positive coefficients and also $M_s + 1 > |s|$. Rewriting (2.10), and using $|s| + 1 \leq M_s$, we find

\[ |\alpha_k(s)| \leq \frac{\alpha_k(M_s + 1)}{M_s} \leq \frac{|s - 1|}{|s| + 1} \alpha_k(M_s + 1) . \]  

(2.11)
Now,

**Lemma 2.1.**

\[ |\alpha_k(M + 1)| \leq \frac{M^{2M-1}(1 + \log(k + 1))^{M-1}}{k + 1}, \quad M = 1, 2, 3, \ldots \]

**Proof.**

\( \alpha_k(M + 1) \) is the coefficient of \( t^k \) in \( g(t)^M = \left(1 + \frac{t}{2} + \frac{t^2}{3} + \cdots\right)^M \).

When \( M = 1 \) the lemma is easily verified. Now assume that the lemma has been verified for \( M \) and consider the \( M+1 \) case. Writing \( g(t)^{M+1} = g(t)^M g(t) \) we get

\[ \alpha_k(M + 2) = \sum_{m=0}^{k} \frac{1}{m + 1} \alpha_{k-m}(M + 1). \quad (2.12) \]

By our inductive hypothesis,

\[ |\alpha_k(M + 2)| \leq \sum_{m=0}^{k} \frac{1}{m + 1} \frac{M^{2M-1}(1 + \log(k - m + 1))^{M-1}}{k - m + 1} \]

\[ = \frac{M^{2M-1}}{k + 2} \sum_{m=0}^{k} (1 + \log(k - m + 1))^{M-1} \left(\frac{1}{m + 1} + \frac{1}{k - m + 1}\right) \]

\[ \leq \frac{M}{k + 2} 2^{M-1}(1 + \log(k + 1))^{M-1} \sum_{m=0}^{k} \frac{2}{m + 1}. \quad (2.13) \]

But \( 2 \sum_{m=0}^{k} 1/(m + 1) \leq 2(1 + \log(k + 1)) \), and \( M/(k + 2) < (M + 1)/(k + 1) \), thus the lemma is proven. \( \square \)

Combining (2.11) with Lemma 2.1 and the assumption that \( M_s < |s| + 2 \) yields the bound in Theorem 1.1

\[ |\alpha_k(s)| \leq \frac{|s - 1|}{|s| + 1} \frac{(|s| + 2)|s|^{s+1}(1 + \log(k + 1))|s|^{s+1}}{k + 1}. \quad (2.14) \]

We conclude, by using the above bound to show that the sum below converges, that we may substitute (1.4) into (1.2) and integrate the power series term by term to obtain

\[ \Gamma(s)\zeta(s) = \sum_{0}^{\infty} \frac{\alpha_k(s)}{s + k - 1}. \quad (2.15) \]

Away from its poles, the above is, in bounded sets, a uniformly convergent sum of analytic functions. Thus, while we started in (1.1) with
\( \Re s > 1 \), formula (2.15) gives the meromorphic continuation of the left hand side.

Similarly, applying the same change of variable to the integral defining the Gamma function,

\[
\int_0^\infty x^{s-1} e^{-x} \, dx = \int_0^1 (- \log(1 - t))^{s-1} \, dt, \quad \Re s > 0 ,
\]

(2.16)
gives

\[
\Gamma(s) = \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s+k} .
\]

(2.17)

3. Connection to Stirling numbers of the first kind

The coefficients \( \alpha_k(s) \) defined by (1.4) are related to Stirling numbers of the first kind through the series expansion, for integer \( m \geq 0 \),

\[
\log(1 + t)^m = m! \sum_{n=m}^{\infty} \frac{s(n,m)}{n!} t^n .
\]

(3.1)

Thus, when \( s \) is a positive integer,

\[
\alpha_k(s) = (-1)^k \frac{(s-1)!}{(s+k-1)!} s(s+k-1, s-1) ,
\]

(3.2)

and equation (2.15) becomes

\[
\zeta(s) = \sum_{k=0}^{\infty} (-1)^k \frac{s(s+k-1, s-1)}{(s+k-1)(s+k-1)!} .
\]

(3.3)

As mentioned in the introduction, the latter formula, valid for positive integer \( s \), is known and (2.15) may be regarded as a generalization of (3.3) to \( s \in \mathbb{C} \). The expression (3.3) can be combined with identities that relate the Stirling numbers to the harmonic numbers, and from this various identities for zeta evaluated at positive integer values may be deduced. See for example Section 4 of [RS] or [S].

4. Connection to Stirling numbers of the second kind

Other related expansions of \( \zeta(s) \) are possible. For example, instead of (1.1), consider

\[
\Gamma(s)\zeta(s-\lambda) = \int_0^\infty x^{s-1} \sum_{n=1}^{\infty} n^\lambda e^{-nx} \, dx , \quad \Re s - \lambda > 1 .
\]

(4.1)
Substituting \( t = 1 - e^{-x} \), we can rewrite the above as
\[
\int_0^1 t^{s-1} \left( \frac{-\log(1-t)}{t} \right)^{s-1} \sum_{n=1}^{\infty} n^\lambda (1-t)^{n-1} dt . \tag{4.2}
\]

When \( \lambda \) is a positive integer, we can evaluate the sum over \( n \) by repeatedly multiplying by \( (1-t) \) and applying \(-d/dt\) to the geometric series \( \sum_{n=1}^{\infty} (1-t)^n = 1/t \).

Therefore, \[
\Gamma(s) \zeta(s-1) = \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s + k - 2} \tag{4.3}
\]
\[
\Gamma(s) \zeta(s-2) = \sum_{k=0}^{\infty} \alpha_k(s) \left( \frac{2}{s + k - 3} - \frac{1}{s + k - 2} \right)
\]
\[
\Gamma(s) \zeta(s-3) = \sum_{k=0}^{\infty} \alpha_k(s) \left( \frac{6}{s + k - 4} - \frac{6}{s + k - 3} + \frac{1}{s + k - 2} \right)
\]
\[
\Gamma(s) \zeta(s-4) = \sum_{k=0}^{\infty} \alpha_k(s) \left( \frac{24}{s + k - 5} - \frac{36}{s + k - 4} + \frac{14}{s + k - 3} - \frac{1}{s + k - 2} \right)
\]

Next, we derive the general form for the above expressions and give a connection to Stirling numbers of the second kind: For positive integer \( \lambda \),
\[
\sum_{n=1}^{\infty} n^\lambda (1-t)^{n-1} = \sum_{j=1}^{\lambda} (-1)^{\lambda+j} j! S(\lambda, j)/t^{j+1} . \tag{4.4}
\]
We can verify this formula inductively, multiplying by \((1-t)\), applying \(-d/dt\), and using the the recurrence relation for Stirling numbers of the second kind:
\[
S(\lambda + 1, j) = S(\lambda, j-1) + jS(\lambda, j) . \tag{4.5}
\]
Substituting into (4.2) and integrating gives the formula

$$\Gamma(s) \zeta(s - \lambda) = \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=1}^{\lambda} (-1)^{\lambda+j} \frac{j! S(\lambda, j)}{s + k - j - 1}$$

(4.6)

of Theorem 1.2.

We can also go in the opposite direction, taking, for example, \(\lambda = -1\) in (4.2). Now,

$$\sum_{1}^{\infty} n^{-1} (1 - t)^{n-1} = \frac{\log t}{t - 1}.$$  

(4.7)

Using

$$\int_{0}^{1} t^{s+k-1} \frac{\log t}{t - 1} dt = \Psi_1(s + k),$$

(4.8)

where \(\Psi_1\) is the trigamma function, we get

$$\Gamma(s) \zeta(s + 1) = \sum_{k=0}^{\infty} \alpha_k(s) \Psi_1(s + k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha_k(s)}{(s + k + n)^2}.$$  

(4.9)

5. Connection to Eulerian numbers

We can also expand the sum in (4.2) in terms of Eulerian numbers. For positive integer \(\lambda\),

$$\sum_{1}^{\infty} n^{\lambda} (1 - t)^{n-1} = t^{-\lambda} \sum_{j=0}^{\lambda-1} E(\lambda, j)(1 - t)^{\lambda-j-1},$$

(5.1)

where \(E(\lambda, j)\) are the Eulerian numbers, satisfying the recursion:

$$E(\lambda + 1, j) = (j + 1)E(\lambda, j) + (\lambda + 1 - j)E(\lambda, j - 1).$$  

(5.2)

Formula (5.1) can, again, be verified inductively, multiplying by \((1 - t)\), applying \(-d/dt\), and manipulating slightly before using the above recursion.

However,

$$\int_{0}^{1} t^{s+k-\lambda-2} (1 - t)^{\lambda-j-1} dt = \frac{\Gamma(s + k - \lambda - 1)\Gamma(\lambda - j)}{\Gamma(s + k - j - 1)},$$

(5.3)

i.e. the Beta function, gives, after applying \(\Gamma(z + 1) = z\Gamma(z)\),

$$\Gamma(s) \zeta(s - \lambda) = \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=0}^{\lambda-1} E(\lambda, j) \frac{(\lambda - j - 1)!}{(s + k - j - 2) \ldots (s + k - \lambda - 1)}. $$

(5.4)
6. Obtaining $\zeta(1 - \lambda)$, $\lambda = 1, 2, 3, \ldots$

We can apply formula (1.10) to obtain expressions for $\zeta(1 - \lambda)$, with $\lambda$ a positive integer. We first illustrate the technique for $\lambda = 1$.

By Theorem 1.1, $\alpha_k(s)$, for $k \geq 1$, is a polynomial in $s$ divisible by $s - 1$. Thus, $\alpha_k(1) = 0$, $k \geq 1$. We also have $\alpha_0(1) = 1$. Therefore, substituting $s = 1$ into the first equation in (1.3), all but the first two terms vanish. The $k = 0$ term gives $-1$, while the denominator of the $k = 1$ term cancels the $s - 1$ factor of $\alpha_1(s) = (s - 1)/2$, and the $k = 1$ term equals $1/2$. We thus get

$$\zeta(0) = -1 + 1/2 = -1/2 .$$  \hspace{1cm} (6.1)

The general situation is handled via equation (1.10). As in the $\lambda = 1$ case, we get two kinds of contributions- from the $k = 0$ term, and from the terms $k = j$, with $j = 1, \ldots, \lambda$. The latter terms are the ones for which the denominator $s + k - j - 1$ cancels the factor $s - 1$ of $\alpha_k(s)$.

The $k = 0$ term produces, on simplifying,

$$(-1)^{\lambda} \sum_{j=1}^{\lambda} (-1)^{j-1}(j - 1)!S(\lambda, j) .$$  \hspace{1cm} (6.2)

But, the Stirling numbers of the second kind are defined by the expansion

$$x^\lambda = \sum_{j=1}^{\lambda} S(\lambda, j)x(x - 1)\ldots(x - j + 1) , \hspace{1cm} \lambda \geq 1.$$  \hspace{1cm} (6.3)

Thus, dividing by $x$, and setting $x = 0$ shows that (6.2) equals $0$ if $\lambda > 1$, and $-1$ if $\lambda = 1$. Therefore, the $k = 0$ term only contributes when $\lambda = 1$, and we denote its contribution by $-\delta_\lambda$, equal to $0$ or $-1$ according to whether $\lambda > 1$ or equals $1$.

The terms $k = j$, with $j = 1, 2, \ldots, \lambda$, contribute to (1.10), on canceling the zero and pole at $s = 1$, the sum:

$$(-1)^{\lambda} \sum_{k=1}^{\lambda} \alpha_k(1)'(-1)^k k!S(\lambda, k) .$$  \hspace{1cm} (6.4)

Therefore, putting both contributions together:

$$\zeta(1 - \lambda) = -\delta_\lambda + (-1)^{\lambda} \sum_{k=1}^{\lambda} \alpha_k(1)'(-1)^k k!S(\lambda, k) .$$  \hspace{1cm} (6.5)

We thus need a formula for $\alpha_k(1)'$. We can differentiate (2.7), and use the fact from Theorem 1.1 that $(s - 1)\alpha_k(s)$ has a double order...
zero at $s = 1$ when $k \geq 1$, to get

$$\alpha_{k+1}(1)' = \frac{1}{k+2} - \frac{1}{k+1} \sum_{j=1}^{k} \frac{j}{k-j+2} \alpha_j(1)', \quad k \geq 0. \quad (6.6)$$

This recursion allows us, in conjunction with (6.5), to evaluate $\zeta(1 - \lambda)$ for any positive integer $\lambda$.

Equation (6.5) can also be used to derive Euler’s famous formula involving the Bernoulli numbers. To see the connection to Bernoulli numbers, we calculate the first few values of $\alpha_k(1)'$ from the above recursion, starting with $\alpha_1(1)' = 1/2$, and list them in Table 1.

| $k$ | $\alpha_k(1)'$ |
|-----|----------------|
| 0   | 0              |
| 1   | 1/2            |
| 2   | 5/24           |
| 3   | 1/8            |
| 4   | 251/2880       |
| 5   | 19/288         |
| 6   | 19087/362880   |
| 7   | 751/17280      |
| 8   | 1070017/29030400 |
| 9   | 2857/89600     |
| 10  | 26842253/958003200 |
| 11  | 434293/17418240 |
| 12  | 703604254357/31384184832000 |
| 13  | 8181904909/402361344000 |
| 14  | 1166309819657/62768369664000 |
| 15  | 5044289/295206912 |
| 16  | 8092989203533249/512189896458240000 |
| 17  | 5026792806787/342372925440000 |
| 18  | 12600467236042756559/91963695909769920000 |
| 19  | 69028763155644023/537799391281152000 |
| 20  | 8136836498467582599787/674400436666564608000000 |

Table 1. Values of $\alpha_k(1)'$

Googling some of the larger numerators in this table, for example 703604254357, immediately returns entries A002208 and A002657 from Sloane’s Online Encyclopedia of Integer Sequences [SI]. These entries deal with the numerators of the Norlund numbers, and the following formula is stated: ‘Numerator of integral of $x(x+1)...(x+n-1)$ from 0 to 1.’ Comparing a few of our numbers to the values of these integrals, we find that the denominators in our case are off by a factor of $k \; k!$
from the denominators of this formula. We are thus led to surmise that:

**Lemma 6.1.** The following formula holds for $k > 0$:

$$
\alpha_k(1)' = \frac{1}{k^k} \int_0^1 (x)_k dx ,
$$

where

$$(x)_k = x(x+1) \ldots (x+k-1) .
$$

**Proof.** We show that the r.h.s. in the lemma satisfies the same recursion as $\alpha_k(1)'$. Replacing, in (6.6), the $\alpha$’s by the r.h.s. of the lemma, rearranging and simplifying slightly, we wish to show that

$$
\sum_{j=1}^{k+1} \frac{1}{k-j+2} \frac{1}{j!} \int_0^1 (x)_j dx = \frac{k+1}{k+2} , \quad k \geq 0 .
$$

The l.h.s. above is the coefficient of $z^{k+2}$ in the product:

$$
\sum_{j=1}^{\infty} \frac{z^j}{j!} \times \sum_{j=1}^{\infty} \frac{z^j}{j!} \int_0^1 (x)_j dx .
$$

The first sum is the power series for $-\log(1-z)$, while the second sum equals the integral from $x = 0$ to 1 of

$$
\sum_{j=1}^{\infty} \frac{(x)_j}{j!} z^j ,
$$

which is the power series for $(1-z)^{-x} - 1$. Integrating from 0 to 1, we find that the product in (6.10) equals

$$
-\log(1-z) \left( \frac{z}{(z-1)\log(1-z)} - 1 \right) = \frac{z}{1-z} + \log(1-z) ,
$$

whose coefficient of $z^{k+2}$ is $1 - 1/(k+2) = (k+1)/(k+2)$.

Substituting Lemma 6.1 into (6.5) and rearranging summation and integration gives

$$
\zeta(1-\lambda) = -\delta_\lambda + (-1)^\lambda \int_0^1 \sum_{k=1}^{\lambda} (-1)^k \frac{S(\lambda,k)}{k} (x)_k dx .
$$

Moving the $(-1)^k$ into the $(x)_k$ and changing variables, $u = -x$, yields

$$
-\delta_\lambda + (-1)^\lambda \int_{-1}^{0} \sum_{k=1}^{\lambda} \frac{S(\lambda,k)}{k} u(u-1) \ldots (u-k+1) du .
$$
Next, apply the recursion (4.5) to split the above sum over $k$ into two sums. The second of these sums equals
\[
\sum_{k=1}^{\lambda-1} S(\lambda - 1, k)u(u - 1)\ldots(u - k + 1), \quad (6.15)
\]
which, by (6.3) is $u^{\lambda - 1}$, if $\lambda > 1$. If $\lambda = 1$ it equals 0. Integrating from $-1$ to 0, it contributes to (6.14):
\[
\begin{cases} 
-1/\lambda & \text{if } \lambda > 1, \\
0, & \text{if } \lambda = 1.
\end{cases}
\]
which we can write as
\[
-1/\lambda + \delta_\lambda. \quad (6.16)
\]
Therefore, (6.14) has been simplified to
\[
\zeta(1-\lambda) = -1/\lambda + (-1)\lambda \int_{-1}^{0} \sum_{k=1}^{\lambda} \frac{S(\lambda - 1, k - 1)}{k} u(u - 1)\ldots(u - k + 1) du. \quad (6.17)
\]
However, the sum above can be related to sums of powers, and hence to Bernoulli polynomials. Assume for now that $u$ is a positive integer. First, we use the operator $\Delta f(m) = f(m + 1) - f(m)$, applied to $m(m - 1)\ldots(m - k + 1)$ to get rid of the numerator above:
\[
(m+1)\ldots(m-k+2) - m(m-1)\ldots(m-k+1) = km(m-1)\ldots(m-k+2). \quad (6.18)
\]
Dividing by $k$, summing over $m = 0$ to $u - 1$, and telescoping yields the well known identity
\[
\frac{u(u - 1)\ldots(u - k + 1)}{k} = \sum_{m=0}^{u-1} m(m - 1)\ldots(m - k + 2), \quad k \geq 1. \quad (6.19)
\]
(The $m = 0$ term is needed if $k = 1$.) Therefore, the sum in (6.17) equals
\[
\sum_{k=1}^{\lambda} S(\lambda - 1, k - 1) \sum_{m=0}^{u-1} m(m - 1)\ldots(m - k + 2). \quad (6.20)
\]
Rearranging the two sums, and using
\[
\sum_{k=1}^{\lambda} S(\lambda - 1, k - 1)m(m - 1)\ldots(m - k + 2) = m^{\lambda - 1}, \quad (6.21)
\]
the integrand in (6.17) becomes, for positive integer \( u \),
\[
\sum_{m=0}^{u-1} m^{\lambda-1}.
\]  
(6.22)

But, sums of powers can be expressed in terms of the Bernoulli polynomials and numbers:
\[
\sum_{m=0}^{\lambda-1} m^{\lambda-1} = \frac{B_\lambda(u) - B_\lambda}{\lambda}, \quad \lambda \geq 1,
\]  
(6.23)
giving a formula for the integrand which is valid for all \( u \), and not just positive integer \( u \), since both the integrand in (6.17) and the above are polynomials in \( u \) agreeing on infinitely many values. We thus have
\[
\zeta(1-\lambda) = -1/\lambda + (-1)^\lambda \int_{-1}^{0} \frac{B_\lambda(u) - B_\lambda}{\lambda} du, \quad \lambda \geq 1.
\]  
(6.24)
Furthermore,
\[
\int_{-1}^{0} B_\lambda(u) du = (-1)^\lambda
\]  
(6.25)
which can be obtained by substituting \( t = u + 1 \), using the difference equation for the Bernoulli polynomials,
\[
\frac{B_\lambda(u+1) - B_\lambda(u)}{\lambda} = u^\lambda, \quad \lambda \geq 1,
\]  
(6.26)
and applying the third of the defining properties of the Bernoulli polynomials:
\[
B_0(t) = 1,
\]
\[
B_k^\prime(t) = kB_{k-1}(t), \quad k \geq 1
\]
\[
\int_{0}^{1} B_k(t) dt = 0, \quad k \geq 1.
\]  
(6.27)
Plugging (6.23) (6.25) into (6.24) therefore gives Euler’s formula:
\[
\zeta(1-\lambda) = (-1)^{\lambda+1} \frac{B_\lambda}{\lambda}, \quad \lambda \geq 1.
\]  
(6.28)

7. Further properties of \( \alpha_k \)

Because \( B_\lambda = 0 \) for odd \( \lambda > 1 \), Euler’s formula gives, as is well known, \( \zeta(-m) = 0 \) for positive even integer \( m \). Hence, the left hand side of (2.15) has poles at \( s = 1, 0, -1, -3, -5, -7, \ldots \). Therefore, to cancel the poles of the right hand side at \( s = -2, -4, -6, \ldots \), \( \alpha_k(s) \) must be divisible by \( s + k - 1 \) when \( k = 3, 5, 7, \ldots \).
Similarly, from the first equation in (4.3), \( \alpha_k(s) \) is divisible by \( s+k-2 \) when \( k = 1, 3, 5, 7, \ldots \), and from the second formula, \( 2\alpha_{k+1}(s) - \alpha_k(s) \) is divisible by \( s+k-2 \) when \( k = 0, 2, 4, 6, \ldots \).

By comparing the residue of (2.17) at \( s = -k, k \geq 0 \), we get
\[
\alpha_k(-k) = (-1)^k/k!.
\] (7.1)

Next, by considering the residue of (2.15) at the poles \( s = -1, -3, -5, \ldots \), we have
\[
\alpha_{2m+2}(-2m - 1) = \frac{B_{2m+2}}{(2m + 2)!}, \quad m \geq 0,
\] (7.2)

and from first formula in (4.3),
\[
\alpha_{2m+2}(-2m) = -(2m + 1)\frac{B_{2m+2}}{(2m + 2)!}, \quad m \geq 0.
\] (7.3)

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