OPTIMAL CONTROL OF UNDERACTUATED MECHANICAL SYSTEMS WITH SYMMETRIES

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ABSTRACT. The aim of this paper is to study optimal control problems for underactuated mechanical systems with symmetries using higher-order Lagrangian mechanics. We variationally derive the corresponding Lagrange-Poincaré equations for second-order Lagrangians with constraints defined on trivial principal bundles and apply them to study an optimal control problem for an underactuated vehicle.

1. Introduction. In the last years, the study of Lagrangian reduction and reconstruction of mechanical systems with Lie group symmetries has been of interest in different areas as engineering, economy, physics, etc. The goal of this paper is to study, from a variational point of view, optimal control problems of mechanical systems defined on a trivial principal bundles and their applications to underactuated vehicles. These types of mechanical systems exhibit more degrees of freedom than actuators.

Mechanical control systems are abundant in the real life for different reasons, for instance, as a result of design choices motivated by the search of less cost engineering devices or as a result of a failure regime in fully actuated mechanical systems. Underactuated systems include spacecrafts, underwater vehicles, helicopters, wheeled vehicles, mobile robots, underactuated manipulators, etc.

The structure of this paper is the following: first, we introduce the Euler-Poincaré equations and optimal control problems for underactuated mechanical system. In Section 3, we derive, from a variational point of view, second-order Lagrange-Poincaré equations on trivial principal bundles for second-order Lagrangians subject to second-order constraints. Finally, we apply these techniques to transform an optimal control problem for an underactuated mechanical system where the configuration space is the Lie group $SE(2)$, into an equivalent second-order variational problem with second-order (vakonomic) constraints. This technique may

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have several advantages, including the possibility of applying variational integrators to solve optimal control problems. The construction of variational integrators preserving geometric structures and the simulation of this kind of optimal control problems will be studied in a future work.

2. Preliminaries: Euler-Poincaré equations and optimal control problems. Let $G$ be a Lie group and consider the left-multiplication on itself

$$G \times G \to G, \quad (g, h) \mapsto L_g(h) = gh .$$

Obviously $L_g$ is a diffeomorphism (the same is valid for the right-translation, but in the sequel we only work with the left-translation, for sake of simplicity).

This left multiplication allows us to trivialize the tangent bundle $TG$

$$TG \to G \times \mathfrak{g}, \quad (g, \dot{g}) \mapsto (g, g^{-1}\dot{g}) = (g, \xi),$$

(1)

where $\mathfrak{g} = T_eG$ is the Lie algebra of $G$ and $e$ is the neutral element of $G$. In the same way, we have the identification $TTG \equiv T(G \times \mathfrak{g}) \equiv G \times 3\mathfrak{g}$, where $3\mathfrak{g}$ denotes 3 copies of the Lie algebra $\mathfrak{g}$, that is, $3\mathfrak{g} := \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$.

2.1. Euler-Poincaré equations. We consider a mechanical system determined by $L: TG \simeq G \times \mathfrak{g} \to \mathbb{R}$. The motion of the system is described by applying Lagrange-D’Alembert principle

$$\delta \int_0^1 L(g(t), \xi(t)) dt = 0,$$

for all variations $\delta \xi(t) = \dot{\eta}(t) + [\xi(t), \eta(t)]$ where $\eta$ is an arbitrary curve on $\mathfrak{g}$ with $\eta(0) = 0$, and $\eta(1) = 0$, and $\delta g = g \eta$.

The principle gives us the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} - ad^*_{\dot{\xi}} \frac{\partial L}{\partial \xi} - l_g^* \frac{\partial L}{\partial g} = u_a e^n$$

(3)

where $l_g^* = (T_eL_g)^*$ and $ad^*$ is the coadjoint representation of the Lie algebra $\mathfrak{g}$.

In the case when $L$ is $G$–invariant, that is, the Lagrangian does not depend on the first entry, we obtain the Euler-Poincaré equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} = ad^*_{\dot{\xi}} \frac{\partial L}{\partial \xi} .$$

(2)

2.2. Optimal control of underactuated mechanical systems. A control system is called underactuated if the number of control inputs is less than the dimension of the configuration space.

Let $L: G \times \mathfrak{g} \to \mathbb{R}$ be a Lagrangian function and consider the controlled Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}} \right) - ad^*_{\dot{\xi}} \left( \frac{\partial L}{\partial \xi} \right) - l_g^* \frac{\partial L}{\partial g} = u_a e^n$$

(3)

where $\{e^n\}$ are independent elements of $\mathfrak{g}^*$ and $\{u_a\}$ are admissible controls (see [2]). Complete $\{e^n\}$ to be a basis $\{e^a, e^A\}$ of $\mathfrak{g}^*$. Let $\{e_i\} = \{e_a, e_A\}$ be the dual basis in $\mathfrak{g}$ with the bracket relation

$$[e_i, e_j] = \delta^k_{ij} e_k .$$

The basis $\{e_i\} = \{e_a, e_A\}$ induces coordinates $(y^a, y^A) = (y^i)$ in $\mathfrak{g}$, that is, if $e \in \mathfrak{g}$ then $e = y^i e_i = y^a e_a + y^A e_A$. 
With this notation, the Euler-Lagrange equation with controls are re-written as
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^a} \right) - C^i_{ja} y^j \frac{\partial L}{\partial y^i} - \left\langle \frac{r}{g} \frac{\partial L}{\partial y^g}, e_a \right\rangle = u_a ,
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^A} \right) - C^j_{iA} y^j \frac{\partial L}{\partial y^i} - \left\langle \frac{r}{g} \frac{\partial L}{\partial y^g}, e_A \right\rangle = 0 .
\]

The optimal control problem consist on finding a trajectory \((g(t), y^i(t), u^a(t))\) of the state variables and control inputs satisfying Equations (3), given initial and final boundary conditions \((g(t_0), y^i(t_0)), (g(t_f), y^i(t_f))\), and minimizing the cost functional
\[
J = \int_{t_0}^{t_f} C(g(t), y^i(t), u^a(t)) \, dt .
\]

This optimal control problem is equivalent to a second-order variational problem with second-order (vakonomic) constraints (see [1] to see the proof of this equivalence) described by \(\bar{L} : G \times 2g \rightarrow \mathbb{R}\)
\[
\bar{L}(g, y^i, \dot{y}^i) = C \left( g, y^i, \frac{d}{dt} \left( \frac{\partial L}{\partial y^a} \right) - C^i_{ja} y^j \frac{\partial L}{\partial y^i} - \left\langle \frac{r}{g} \frac{\partial L}{\partial y^g}, e_a \right\rangle \right) ,
\]
subject to
\[
\Phi^A(g, y^i, \dot{y}^i) = \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - C^j_{iA} y^j \frac{\partial L}{\partial y^i} - \left\langle \frac{r}{g} \frac{\partial L}{\partial y^g}, e_A \right\rangle = 0 .
\]

3. Second-order Lagrange-Poincaré equations for systems with constraints on trivial principal bundles. In this section, we derive from a variational point of view, using Hamilton’s principle, second-order Lagrange-Poincaré equations on trivial principal bundles. It is well known that Lagrange-Poincaré equations are a generalization of Euler-Poincaré equations (see [3]). First, we derive Euler-Lagrange equations for Lagrangians defined on \(TM \times G \times g\), where \(G\) is a Lie group, \(g\) its associated Lie algebra and \(M\) an \(n\)-dimensional differentiable manifold. Secondly, using a left trivialization of the second-order tangent bundle \(T^{(2)}G\) we obtain the second-order Euler-Lagrange equations for Lagrangians defined on \(T^{(2)}M \times G \times g \times g\). Since the main application of this paper is optimal control of underactuated mechanical systems we obtain in (3.3) the second-order Lagrange-Poincaré equations for second-order Lagrangians subject to second-order (vakonomic) constraints.

3.1. Euler-Lagrange equations for trivial principal bundles. Now, we derive, from a variational point of view, the Euler-Lagrange equations on the trivial principal bundle \(Q = M \times G\) where \(M\) is an \(n\)-dimensional differentiable manifold with local coordinates \((q^i)\) \(1 \leq i \leq n\) and \(G\) is a Lie group.

Let \(L : TG \rightarrow \mathbb{R}\) be a Lagrangian function. Since \(TQ \simeq TM \times TG\) and \(TG \simeq G \times g\) from a left-trivialization, we consider our Lagrangian function as \(L : TM \times G \times g \rightarrow \mathbb{R}\).

The motion of the mechanical system is described by the variational principle
\[
\delta \int_{0}^{T} L(q(t), \dot{q}(t), g(t), \xi(t)) \, dt = 0 , \quad (4)
\]
for all variations \(\delta q\) and \(\delta \xi\) where \(\delta q(0) = \delta q(T) = 0\), \(q(t) \in M\) and \(\delta \xi\) verifies \(\delta \xi(t) = \dot{\eta}(t) + [\xi(t), \eta(t)]\), where \(\eta\) is an arbitrary curve on the Lie algebra with
\( \eta(0) = 0 = \eta(T) \) and \( \delta g = g_{\eta} \). This principle gives rise to the Euler-Lagrange equations on trivial principal bundles given by

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) &= \frac{\partial L}{\partial q^i}, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}} \right) &= \text{ad}_\xi^* \left( \frac{\partial L}{\partial \xi} \right) + l_g \frac{\partial L}{\partial g},
\end{align*}
\]

for \( i = 1, \ldots, n \), and \( \text{ad}^* \) is the coadjoint representation of the Lie algebra \( g \). If the Lagrangian \( L \) is \( G \)-invariant the above equations are the Lagrange-Poincaré equations:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial q^i} \right) &= \frac{\partial L}{\partial q^i}, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \xi} \right) &= \text{ad}_\xi^* \left( \frac{\partial L}{\partial \xi} \right).
\end{align*}
\]

**Remark 1.** Observe that if the Lagrangian does not depend on the variables in the manifold \( M \), the equations of motion are rewritten as the usual Euler-Poincaré equations \((2)\). Then, these last equations can be considered as a generalization of Euler-Poincaré equations \((2)\).

### 3.2. Second-order Lagrange-Poincaré equations for trivial principal bundles

In this subsection we deduce, from Hamilton’s principle, Euler-Lagrange equations for Lagrangians defined on \( T^{(2)}Q \cong T^{(2)}M \times G \times 2g \) from a left-trivialization; where \( 2g \) denotes two copies of the Lie algebra, that is \( 2g := g \times g \).

Let \( L : T^{(2)}Q \cong T^{(2)}M \times G \times 2g \to \mathbb{R} \) be a Lagrangian function, \( L(q, \dot{q}, \ddot{q}, g, \dot{g}, \ddot{g}) \equiv L(q, \dot{q}, \ddot{q}, g, \dot{\xi}, \ddot{\xi}) \) where \( \xi = g^{-1} \dot{g} \). The problem consists on finding the critical curves of the action defined by

\[
A(c(t)) = \int_0^T L(q(t), \dot{q}(t), \ddot{q}(t), g(t), \dot{\xi}(t), \ddot{\xi}(t)) dt,
\]

among all the \( C^2 \)-curves \( c(t) = (q(t), \dot{q}(t), \ddot{q}(t), g(t), \dot{\xi}(t), \ddot{\xi}(t)) \) satisfying the boundary conditions for arbitrary variations \( \delta c = (\delta q, \delta q^{(1)}, \delta q^{(2)}, \delta g, \delta \xi, \delta \dot{\xi}) \) where \( \delta q = \frac{d}{dt} |_{t=0} q_t, \delta q^{(l)} = \frac{d^l}{dt^l} \delta q, \) for \( l = 1, 2 \); and \( \delta g = \frac{d}{dt} |_{t=0} g_t \). Here, \( \epsilon \mapsto g_{\epsilon} \) and \( \epsilon \mapsto q_{\epsilon} \) are smooth curves in \( G \) and \( M \) respectively, for \( \epsilon \in (-a, a) \subset \mathbb{R} \) such that \( g_0 = g \) and \( q_0 = q \). We define, for any \( \epsilon, \xi := g_{-1} \dot{g}_{\epsilon} \). The corresponding variations \( \delta \xi \) induced by \( \delta g \) are given by \( \delta \xi = \eta + [\xi, \eta] \) where \( \eta := g^{-1} \delta g \in g \) (\( \delta g = g_{\eta} \)). Therefore

\[
\delta A(c(t)) = \delta \int_0^T L(q(t), \dot{q}(t), \ddot{q}(t), g(t), \dot{\xi}(t), \ddot{\xi}(t)) dt
\]

\[
= \frac{d}{dt} |_{t=0} \int_0^T L(q_t, \dot{q}_t, \ddot{q}_t, g_t, \dot{\xi}_t, \ddot{\xi}_t) dt
\]

\[
= \int_0^T \left( \left\langle \frac{\partial L}{\partial \dot{q}^i} \right| \delta \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial \dot{\xi}} \right| \delta \dot{\xi} \right\rangle + \left\langle \frac{\partial L}{\partial q^i} \right| \frac{d}{dt} \left\langle \delta q \right\rangle \right\rangle + \left\langle \frac{\partial L}{\partial g} \right| \delta g \right\rangle + \left\langle \frac{\partial L}{\partial \xi} \right| \delta \xi \right\rangle dt.
\]
Using twice integration by parts and the transversally condition \( q(0) = q(T) = \dot{q}(T) = \ddot{q}(T) = 0 \) and \( \eta(0) = \eta(T) = \dot{\eta}(0) = \dot{\eta}(T) = 0 \), the stationary condition \( \delta A = 0 \) implies

\[
\int_0^T \left< \left( -\frac{d}{dt} + ad\xi \right) \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right), \eta \right> dt + \int_0^T \left< \int_0^T \left( \frac{\partial L}{\partial g} \right) \eta, \eta \right> dt + \int_0^T \left< \frac{d}{dt} \left( \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}} \right), \eta \right> dt = 0.
\]

Therefore, \( \delta A(c(t)) = 0 \) if and only if \( c(t) \) is a solution of the Euler-Lagrange equations for \( L : T^{(2)}M \times G \times 2g \rightarrow \mathbb{R} \), given by

\[
\left( -\frac{d}{dt} + ad\xi \right) \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = -l_2^2 \frac{\partial L}{\partial q}, \tag{5}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}} \right) = -\frac{\partial L}{\partial q}. \tag{6}
\]

From the last computation we obtain the following theorem

**Theorem 3.1.** Let \( L : T^{(2)}Q \simeq T^{(2)}M \times G \times 2g \rightarrow \mathbb{R} \) be a left-trivialized second-order Lagrangian. The curve \( c(t) \in C^{(2)} \left( T^{(2)}M \times G \times 2g \right) \) satisfies \( \delta A(c(t)) = 0 \) for the action \( A : C^{(2)} \left( T^{(2)}M \times G \times 2g \right) \rightarrow \mathbb{R} \), given by

\[
A(c(t)) = \int_0^T L(q, \dot{q}, \ddot{q}, q, \dot{q}) dt,
\]

with respect to the variations \( \delta q, \delta \xi \), where \( \delta q^{(l)} = \frac{d}{dt} q(t) \) for \( l = 1, 2 \); \( \delta g \) and \( \delta \xi = \dot{\eta} + ad\xi \eta \), where \( \xi = g^{-1} \dot{q} \) and \( \eta \) a curve on \( g \) with fixed endpoints \( \eta(0) = \eta(T) = \dot{\eta}(0) = \dot{\eta}(T) = 0 \), if and only if \( c(t) \) is a solution of the trivialized Euler-Lagrange equations,

\[
l_2^2 \frac{\partial L}{\partial g} - \frac{d}{dt} \frac{\partial L}{\partial \xi} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \xi} + ad\xi \frac{\partial L}{\partial q} - ad\xi \left( \frac{d}{dt} \frac{\partial L}{\partial \xi} \right) = 0,
\]

\[
\frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}} = 0.
\]

**Remark 2.** If the Lagrangian is \( G \)-invariant, that is if \( L \) does not depend of \( g \in G \), the equations of motion are given by the second-order Lagrange-Poincaré equations:

\[
\frac{d^2}{dt^2} \frac{\partial L}{\partial \xi} - \frac{d}{dt} \frac{\partial L}{\partial \xi} + ad\xi \frac{\partial L}{\partial q} - ad\xi \left( \frac{d}{dt} \frac{\partial L}{\partial \xi} \right) = 0, \tag{7}
\]

\[
\frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}} = 0. \tag{8}
\]

**Remark 3.** As remark (1) if the Lagrangian does not depend on the variables on \( M \) these equations are rewritten as second-order Euler-Poincaré equations. In this sense, the last equations can be considered as a generalization of the equations derived in [4].

### 3.3. Mechanical systems defined on second-order trivial principal bundles subject to constraints

Now, we consider a second-order Lagrangian systems determined by \( L : T^{(2)}M \times G \times 2g \rightarrow \mathbb{R} \) subject to second-order (vakonomic) constraints \( \Phi^\alpha : T^{(2)}M \times G \times 2g \rightarrow \mathbb{R}, 1 \leq \alpha \leq m \). We denote by \( \hat{M} \) the constraint submanifold locally determined by the vanishing of these \( m \)-constraints.
The variational principle for this kind of second-order mechanical systems is given by

\[
\begin{aligned}
\min A(c(t)) \text{ with } c(t) = (q(t), \dot{q}(t), \ddot{q}(t), g(t), \xi(t), \dot{\xi}(t)), \\
\text{subject to } \Phi^\alpha(c(t)) = 0 \text{ with } 1 \leq \alpha \leq m
\end{aligned}
\]

and boundary conditions \( q(0) = q(T) = \dot{q}(0) = \dot{q}(T) = 0; \ g(0) = g(T) = \xi(0) = \xi(T) = 0 \) and \( \xi = g^{-1}\dot{q} \), where

\[
A(c(t)) = \int_0^T L(c(t))dt.
\]

**Definition 3.2.** A curve \( c(t) \in C^2(T^2M \times G \times 2\mathfrak{g}) \) will be called a solution of the second-order variational problem with second-order (vakonomic) constraints if \( c \) is a critical point of \( A|_{\mathcal{M}} \).

As in the case of systems with constraints on \( TM \), by using the Lagrange multipliers theorem, we can characterize the regular critical points of the second-order variational problem with second-order constraints as an unconstrained variational problem for an extended Lagrangian system (see [5] for a detailed proof).

**Proposition 1. Variational problem with second-order (vakonomic) constraints:** A curve \( c(t) \in C^2(T^2M \times G \times 2\mathfrak{g}) \) is a critical point of the variational problem with second-order (vakonomic) constraints if and only if \( c \) is a critical point of the functional

\[
\int_0^T \tilde{L}(c(t), \lambda(t))dt,
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_m) \) as regarded as generalized coordinates on \( \mathbb{R}^m \) and \( \tilde{L} : T^2M \times G \times 2\mathfrak{g} \times \mathbb{R}^m \to \mathbb{R} \) is defined by

\[
\tilde{L}(c(t), \lambda) = L(c(t)) - \lambda_\alpha \Phi^\alpha(c(t)).
\]

Therefore, the equations of motion for a Lagrangian system subject to second-order (vakonomic) constraints are:

\[
\begin{aligned}
0 &= \frac{\partial L}{\partial q^i} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \dot{q}^i} + \lambda_\alpha \frac{d}{dt} \left( \frac{\partial \Phi^\alpha}{\partial q^i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{q}^i} \right)
- \bar{\lambda}_\alpha \frac{\partial \Phi^\alpha}{\partial q^i} - 2\lambda_\alpha \frac{d}{dt} \left( \frac{\partial \Phi^\alpha}{\partial \dot{q}^i} \right) - \lambda_\alpha \frac{d^2}{dt^2} \left( \frac{\partial \Phi^\alpha}{\partial \dot{q}^i} \right)
\end{aligned}
\]

\[
\begin{aligned}
0 &= \left( \frac{d}{dt} - ad^* \right) \left( \frac{\partial L}{\partial \dot{\xi}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \dot{\xi}} - \frac{d}{dt} \frac{\partial L}{\partial \xi} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \xi} + \lambda_\alpha \frac{d}{dt} \left( \frac{\partial \Phi^\alpha}{\partial \xi} \right) \right)
- \bar{L}_g \left( \frac{\partial L}{\partial g} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial g} \right),
\end{aligned}
\]

\[
0 = \Phi^\alpha(c(t)), \quad 1 \leq \alpha \leq m, \quad i = 1, \ldots, m.
\]

\[
\dot{g} = g\xi,
\]

and are called trivialized second-order Euler-Lagrange equations with constraints defined on \( T^2M \times G \times \mathfrak{g} \times \mathbb{R}^m \).

If the Lagrangian is \( G \)-invariant (that is, \( L \) does not depend of the variables on \( G \)) these equations are called second-order Lagrange-Poincaré with (vakonomic)
4. Application to optimal control of underactuated mechanical systems. The aim of this section is to study optimal control problems for underactuated mechanical systems, that is, Lagrangian control systems such that the number of control inputs is less than the dimension of the configuration space. For this kind of mechanical control systems we consider as configuration space $Q = M \times G$, where $G$ is a Lie group and $M$ is an $n$–dimensional differentiable manifold. In what follows we assume that all the control systems in this work are controllable, that is, for any two points $q_0$ and $q_f$ in the configuration space, there exits an admissible control $u(t)$ defined on some interval $[0, T]$ such that the system with initial condition $q_0$ reaches the point $q_f$ in time $T$ (see for more details [1] and [2]).

4.1. Optimal control problem: Define the control manifold $U \subset \mathbb{R}^r$ where $u(t) \in U$ is the control parameter. Consider the left-trivialized Lagrangian $L: TQ \simeq TM \times \mathfrak{g} \to \mathbb{R}$, (where $\mathfrak{g}$ is the Lie algebra associated to the Lie group $G$). The equations of motion are the controlled Lagrange-Poincaré equations:

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial q^i} \right) - \frac{\partial L}{\partial q^i} = u_a \mu^a(q), 
$$

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial q^i} \right) - \text{ad}_\xi^* \left( \frac{\partial L}{\partial \xi^i} \right) = u_a \eta^a(q),
$$

(9a) (9b)

where we denote by $\mathcal{B}^a = \{ (\mu^a, \eta^a) \}$, $u^a(q) \in \mathcal{B}^a$ and $i = 1, \ldots, n$. Here, we are assuming that $\{ (\mu^a, \eta^a) \}$ are independent elements of $\Gamma(TM \times \mathfrak{g}^*)$ and $u_a$ are the admissible controls. Taking this into account, the optimal control problem consists on finding trajectories $g^a(q(t))$, $q(t)$, and final conditions $q(0)$, $q(0)$, $q(0)$) subject to initial conditions $(q(0), q(0), q(0))$ and extremizing the functional

$$
J(q, \dot{q}, \xi, u) = \int_0^T C(q(t), \dot{q}(t), q(t), u(t)) dt.
$$

(10)

We can reformulate this optimal control problem as a second-order order variational problem subject to second-order constraints in the following way: complete $\mathcal{B}^a$ to a basis $\mathcal{B}^a$, $\mathcal{B}^a$ of the vector space $T^*M \times \mathfrak{g}^*$. Take its dual basis $\{ B_\alpha, B_\alpha^* \}$ on $\Gamma(TM \times \mathfrak{g}) = \mathfrak{X}(M) \times C^\infty(M, \mathfrak{g})$. If we denote by $\mathcal{B}_\alpha = \{ (X_\alpha, \chi_\alpha) \} \in \Gamma(TM \times \mathfrak{g})$ (resp. $B_\alpha = \{ (X_\alpha, \chi_\alpha) \} \in \Gamma(TM \times \mathfrak{g})$, where $X_\alpha, X_\alpha \in \mathfrak{X}(M)$; $X_\alpha = X^a(q) \frac{\partial}{\partial q^a}$;
\[X_\alpha = X^i_\alpha(q) \frac{\partial}{\partial q^i}\] and \(\chi_\alpha(q); \chi_\alpha(q) \in \mathfrak{g}, q \in M\) then equations (9) are rewritten as

\[
\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) X^i_\alpha(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\xi}} \right) - \left(\text{ad}^*_\chi \frac{\partial L}{\partial \dot{\xi}} \right) \chi_\alpha(q) = 0,
\]

As mentioned before, the proposed optimal control problem is equivalent to a variational problem with second-order (vakonomic) constraints (see [1] and reference therein), where we define the Lagrangian \(\tilde{L} : T^{(2)}M \times 2\mathfrak{g} \to \mathbb{R}\) given, in the selected coordinates, by

\[
\tilde{L}(q^i, \dot{q}^i, \ddot{q}^i, \xi, \dot{\xi}) = C \left( q^i, \dot{q}^i, \xi, F_a(q^i, \dot{q}^i, \ddot{q}^i, \xi, \dot{\xi}) \right),
\]

where \(C\) is the cost function considered in (10) and

\[
F_a(q^i, \dot{q}^i, \ddot{q}^i, \xi, \dot{\xi}) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) X^i_\alpha(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\xi}} \right) - \left(\text{ad}^*_\chi \frac{\partial L}{\partial \dot{\xi}} \right) \chi_\alpha(q).
\]

Moreover, since the system is underactuated, the Lagrangian system determined by \(\tilde{L}\) is subjected to the second-order constraints:

\[
\Phi^\alpha(q^i, \dot{q}^i, \ddot{q}^i, \xi, \dot{\xi}) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) X^i_\alpha(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\xi}} \right) - \left(\text{ad}^*_\chi \frac{\partial L}{\partial \dot{\xi}} \right) \chi_\alpha(q).
\]

Thus, this kind of problems naturally fits in the setting introduced in §3.3 by considering \(k = 2\) and left invariance with respect to the Lie group \(G\) as it is illustrated by the following example.

### 4.2. Optimal control of an underactuated vehicle:
Consider a rigid body moving in \(SE(2)\) with a thruster to adjust its pose. The configuration of this system is determined by a tuple \((x, y, \theta, \gamma)\), where \((x, y)\) is the position of the center of mass, \(\theta\) is the orientation of the blimp with respect to a fixed basis, and \(\gamma\) the orientation of the thrust with respect to a body basis. Therefore, the configuration manifold is \(Q = SE(2) \times S^1\) (see [2]).

The Lagrangian of the system is given by its kinetic energy

\[
L(x, y, \theta, \gamma, \dot{x}, \dot{y}, \dot{\theta}, \dot{\gamma}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\theta} + \dot{\gamma})^2,
\]

and the input forces are

\[
F^1 = \cos(\theta + \gamma) dx + \sin(\theta + \gamma) dy - l \sin \gamma d\theta,
F^2 = d\gamma,
\]

where the control forces that we consider are applied to a point on the body with distance \(l > 0\) from the center of mass, along the body \(x\)-axis. Note that this system is an example of underactuated mechanical system where the configuration space is a trivial principal bundle.

The system is invariant under the left multiplication of the Lie group \(G = SE(2)\):

\[
\Phi : SE(2) \times SE(2) \times S^1 \to SE(2) \times S^1
((a, b, \alpha), (x, y, \theta, \gamma)) \mapsto (x \cos \alpha - y \sin \alpha + a, x \sin \alpha + y \cos \alpha + b, \theta + \alpha, \gamma).
\]

A basis of the Lie algebra \(\mathfrak{se}(2) \cong \mathbb{R}^3\) of \(SE(2)\) is given by

\[
e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]
with Lie algebra structure determined by the brackets
\[ [e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = 0. \]
Thus, we can write down the constant structure as
\[ C_{12}^1 = C_{31}^2 = 1, C_{21}^3 = C_{13}^2 = -1 \]
and all the others are equal to zero.
An element \( \xi \in \mathfrak{se}(2) \) is of the form \( \xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \); therefore the reduced Lagrangian \( \bar{L} : TS^1 \times \mathfrak{se}(2) \to \mathbb{R} \) is given by
\[
\bar{L}(\gamma, \dot{\gamma}, \dot{\xi_1}, \dot{\xi_2}, \dot{\xi_3}) = \frac{1}{2}m(\dot{\xi_1}^2 + \dot{\xi_2}^2) + \frac{J_1 + J_2}{2} \dot{\xi_3}^2 + J_2 \dot{\gamma}^2 + \frac{J_2}{2} \gamma^2. 
\]
Then the controlled Lagrange-Poincaré equations are given by
\[
\begin{align*}
\dot{m} \dot{\xi}_1 & = u_1 \cos \gamma, \\
\dot{m} \dot{\xi}_2 + (J_1 + J_2)\xi_1 \dot{\xi}_3 + J_2 \dot{\xi}_3 \dot{\gamma} - m \xi_1 \dot{\xi}_3 & = u_1 \sin \gamma, \\
(J_1 + J_2)\dot{\xi}_3 + J_2 \dot{\gamma} - m \xi_2 (\dot{\xi}_1 + \dot{\xi}_3) & = -u_1 l \sin \gamma, \\
J_2 (\dot{\xi}_3 + \dot{\gamma}) & = u_2. 
\end{align*}
\]
Using our techniques the equations of motion are now modified as follows
\[
\begin{align*}
\dot{m} (\cos \gamma \dot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3)) + (J_1 + J_2)\xi_1 \xi_3 \sin \gamma + J_2 \dot{\xi}_3 \dot{\gamma} \sin \gamma & = u_1, \\
\dot{m} (\cos \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3) - \sin \gamma \dot{\xi}_1) + \xi_1 \dot{\xi}_3 (J_1 + J_2) \cos \gamma + J_2 \dot{\xi}_3 \dot{\gamma} \cos \gamma & = 0, \\
\frac{J_1 + J_2}{l} (\dot{\xi}_3 + l \xi_1 \dot{\xi}_3) + \frac{J_2}{l} (\dot{\gamma} + l \dot{\xi}_3 \dot{\gamma}) + m (\dot{\xi}_2 - \xi_1 \dot{\xi}_3 - \frac{\xi_2 \dot{\xi}_1 + \xi_3 \dot{\xi}_2}{l}) & = 0, \\
J_2 (\dot{\xi}_3 + \dot{\gamma}) & = u_2. 
\end{align*}
\]
Now, we can study the optimal control problem that consists on finding trajectories of state variables and control inputs satisfying the previous equations from given initial and final conditions \((\gamma(0), \dot{\gamma}(0), \xi(0)), (\gamma(T), \dot{\gamma}(T), \xi(T))\) respectively and extremizing the cost functional
\[
\int_0^T (p_1 u_1^2 + p_2 u_2^2) \, dt, 
\]
where \(p_1\) and \(p_2\) are positive constants representing the weights of the cost functional.

The related optimal control problem is equivalent to the second-order Lagrangian problem with second-order constraints.

Extremize
\[
\bar{A} = \int_{t_0}^{t_f} \bar{L}(\gamma, \dot{\gamma}, \ddot{\gamma}, \dot{\xi}, \ddot{\xi}) \, dt, 
\]
subject to second-order constraints given by
\[
\begin{align*}
\Phi^1 & = m (\cos \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3) - \sin \gamma \dot{\xi}_1) + \xi_1 \dot{\xi}_3 (J_1 + J_2) \cos \gamma + J_2 \dot{\xi}_3 \dot{\gamma} \cos \gamma, \\
\Phi^2 & = \frac{J_1 + J_2}{l} (\dot{\xi}_3 + l \xi_1 \dot{\xi}_3) + \frac{J_2}{l} (\dot{\gamma} + l \dot{\xi}_3 \dot{\gamma}) + m (\dot{\xi}_2 - \xi_1 \dot{\xi}_3 - \frac{\xi_2 \dot{\xi}_1 + \xi_3 \dot{\xi}_2}{l}). 
\end{align*}
\]
Here, \( \bar{L} : T^{(2)} S^1 \times 2 \mathfrak{se}(2) \to \mathbb{R} \) is defined by
\[
\bar{L}(\gamma, \dot{\gamma}, \ddot{\gamma}, \dot{\xi}, \ddot{\xi}) = p_1 \left( \cos \gamma (\dot{\xi}_2 - \xi_2 \dot{\xi}_3) + \sin \gamma (\dot{\xi}_2 + \xi_1 \dot{\xi}_3) \right)^2 + p_2 J_2^2 (\dot{\xi}_3 + \dot{\gamma})^2. 
\]

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