Abstract: Exact solution of Einstein field equations (EFEs) is derived for \( \Lambda(t) \) cosmology in \( f(R, T) \) theory of gravity. The background metric considered here, is a homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time. A simple parametrization of the Hubble parameter \( H \) is considered for a deterministic solution of the field equations. The cosmological dynamics is consistent with the current observations, and are discussed here in some detail for our obtained model. We have analyzed the time evolution of the physical parameters e.g. energy density, pressure, equation of state parameter and cosmological constant and obtained their bounds analytically. Moreover, the behaviour of these parameters are shown graphically in terms of redshift \( z \) and their physical consequences are discussed. Also, the role of \( f(R, T) \) coupling constant \( \lambda \) is discussed in the evolution of equation of state parameter \( \omega \) separately. We have performed the statefiner and Om diagnostic analysis to distinguish our obtained model with other dark energy models and also discussed the energy conditions. The obtained model shows nice fit to the 28 points of Hubble dataset and 580 points of Supernovae Ia (SNIa) dataset. We have constrained the model parameters with theses datasets and found their values. Our obtained model is in good agreement with current observational results.

Keywords \( \Lambda(T) \) gravity, \( f(R, T) \) theory, Parametrization, Observational constraints
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1 Introduction

Globally, the most accepted theory of evolution of the Universe is undoubtedly the famous general theory of relativity which suggests that the Universe was condensed in a hotter and denser state with a very very small volume initially and expanded to everything, we can observe in the outer space. Moreover the expansion is a continuous process till today since the beginning of time. In recent times, the observations of type Ia supernovae pointed out that the current rate of expansion of the Universe is accelerating. Thereafter, so many observational
data supported the idea of the accelerating expansion \[12, 23, 41, 45\]. In order to explain the faster rate of expansion of the Universe, it is evident to consider a new form of energy in the Universe which has some anti-gravitational effect that drives the acceleration. This weird form of energy with repulsive force was termed as dark energy (DE) \[8, 9, 10\]. This form of energy having strange anti-gravitational effect is unlike anything we have encountered yet. According to the best estimate of planck mission team, it is estimated that the Universe is composed of three different substances, 4.9% ordinary matter, 26.8% dark matter (DM) and 68.3% dark energy. Even though DM and DE have similar names, but they are really very different in nature. DM is attractive and responsible for structure formation while DE seems to be some kind of energy intrinsic to the empty space and is keep getting stronger as the time passes by. There are multiple ideas of DE: one idea could be that DE is a property of space itself, or DE might be some kind of dynamic energy fluid which has some opposite effects on the Universe than ordinary energy and matter. Although DE is a popular explanation for the expansion mystery supported by many measurements, there remain many questions on its existence.

The two main models proposed in literature to explain the nature for DE are Cosmological constant $\Lambda$ (constant energy density filled in space homogeneously) and scalar fields (dynamic quantities with variable energy density in space-time). The simplest and favourable candidate of DE is the Einstein’s cosmological constant $\Lambda$ \[8, 9\] which act as a force that counteracted the force of gravity. Adding the cosmological constant $\Lambda$ to Einstein’s field equation of standard FLRW metric leads to $\Lambda$CDM model which cause the expansion of the Universe to accelerate. In spite of its theoretical and phenomenological problems \[10\], the $\Lambda$CDM model has been referred as the most efficient answer to the question of cosmic acceleration in many respects because of its precise agreement with observations. While DE is the constant energy density, another idea is quintessence scalar field which is also a contender of DE. The first scenario of quintessence model was proposed by Ratra and Peebles \[11\]. Quintessence model differ from $\Lambda$CDM in explanation of DE as quintessence model is dynamic that changes over time unlike $\Lambda$ which always stays constant \[12, 13\]. According to the theory of general relativity, the equation of state (EoS) in cosmology specifies the expansion rate in the Universe. Now a days the great attempt in observational cosmology is to analyse the EoS \(\omega = \frac{p}{\rho}\) of various DE models where $p$ and $\rho$ are the pressure and density of the fluid. Quintessence is a dynamic scalar field having EoS $\omega > -1$. The specific case of quintessence model is a phantom models of dark energy \[14, 15, 16\], whose EoS is $\omega < -1$ which could cause a big rip in the Universe due to the growing energy density of DE \[17, 18, 19\]. Also a number of other scalar field DE models have been proposed such as spintessence \[20\], k-essence \[21, 22\], quintom \[23\], tachyon \[24, 25\], chameleon \[26\] having EoS parameter $\omega \in (-1, 0)$. Another class of alternative idea is to come up with the theory of dark fluid that unifies both DM and DE as a single phenomenon \[27\]. In addition to such alternatives, Holographic DE is one of the contender which has been suggested that DE might originated from quantum fluctuation of space-time. For a detailed review on DE and it’s alternative, see \[28, 29, 30, 31\].

In the other direction, the cause of accelerating expansion of the Universe can be explained by modifying the Einstein-Hilbert action. The standard Einstein Lagrangian can be modified by replacing Ricci scalar $R$ with some arbitrary function of $R$ known as $f(R)$ gravity. Moreover, the replacement of Ricci scalar $R$ with scalar torsion $T$ is known as $f(T)$ gravity, and with Gravitational constant $G$ is known as $f(G)$ gravity. Many other modifications of underlying geometry can cause a different modified theory to GR. Among the wide range of alternative ideas of modified gravity, $f(R)$ theory of modified gravity is served as the most viable alternative \[32\]. $f(R)$ gravity is considered good on large scales, but fails to hold good on some of the observational tests, e.g. on rotation of the curved spiral galaxies \[33, 34\], on solar system regime \[35, 36\]. A more generic extension of $f(R)$ gravity can be considered as $f(R, Sm)$ where the gravitational Lagrangian $S_m$ is a function of trace $T$ of energy momentum tensor and is named as $f(R, T)$ gravity \[37\]. The main reason to introduce the term $T$ is to take the quantum effects and exotic imperfect fluids into account, and also $f(R, T)$ gravity can describe the late time cosmic speed up. Some observational tests \[38, 39\] have been applied to $f(R, T)$ gravity in order to resolve the issues mentioned in $f(R)$ gravity. For a detailed work on $f(R, T)$ theory in the area of cosmology and astrophysics see \[40, 41, 42, 43\].

In palatini formalism of $f(R)$ theory of gravity, called $\Lambda(T)$ gravity which was first proposed by Nikodem J. Pop lawski \[44\] is considered as the most general case where $\Lambda$-term present in the general gravitational Lagrangian, is taken as a function of $T$ ($T$ being the trace of stress energy momentum tensor). Moreover, the palatini $f(R)$ gravity can be brought back if we ignore the pressure dependent term from $\Lambda(T)$ gravity. Also, the dynamical cosmological constant $\Lambda$ is supported by theory to solve the cosmological constant problem \[45\] and it is in good agreement with $\Lambda(T)$ gravity. For a review on $\Lambda(T)$ cosmology in $f(R, T)$ modified gravity see
The paper is organized as follows: section 1 provides a brief introduction on dark energy and alternative ideas to the fact of cosmic acceleration. In section 2, we have reviewed the derivation of the field equations in \( f(R, T) \) formalism with variable cosmological constant and obtained the exact solutions to EFEs by considering a parametrization of Hubble parameter. In section 3, we have discussed the dynamics of the obtained model and briefly analysed the behavior of geometrical and physical parameters with the help of some graphical representation. In section 4, we have analysed the energy conditions and performed the diagnostic analysis for our model. and in section 5, we analyse our model with some cosmological observations. The final conclusion is included in section 6.

2 Basic equations and Solution

2.1 Field equations in \( f(R, T) \) gravity

The \( f(R, T) \) gravity \([37]\) is a more generic extended theory of \( f(R) \) gravity or more precisely general relativity which explains the coupling between matter and geometry in the Universe. The formalism of \( f(R, T) \) model depends on a source term which is a function of Lagrangian matter density \( S_m \). The action of \( f(R, T) \) gravity is given by

\[
S = \int \left( \frac{1}{16\pi G} f(R, T) + S_m \right) \sqrt{-g} dx^4. \tag{1}
\]

In the above action, we consider the functional form of \( f(R, T) = f_1(R) + f_2(T) \), sum of two independent functions of Ricci scalar and trace of energy momentum tensor respectively. We assume the forms of \( f_1(R) = \lambda R \) and \( f_2(T) = \lambda T \), where \( \lambda \) is any arbitrary coupling constant of \( f(R, T) \) gravity.

Taking a variation of action (1) w.r.t. \( g_{ij} \) and neglecting the boundary terms, we have

\[
f'_1(R) R_{ij} - \frac{1}{2} f'_1(R) g_{ij} + (g_{ij} \Box - \nabla_i \nabla_j) f'_1(R) = 8\pi T_{ij} - f'_2(T) T_{ij} - f'_2(T) \theta_{ij}, \tag{2}
\]

where prime indicates the derivative w.r.t. argument, operator \( \Box \) defined above is De Alembert’s operator (\( \Box \equiv \nabla^i \nabla_i \)). Also if the matter content filled in the Universe follows the perfect fluid behavior then in that case \( \theta_{ij} \) becomes \( \theta_{ij} = -2T_{ij} - pg_{ij} \), matter Lagrangian density \( S_m \) can be considered as \( S_m = -p \) and energy momentum tensor takes the form \( T_{ij} = (\rho + p) u_i u_j - p g_{ij} \). Here, \( u^i = (0, 0, 0, 1) \) is the 4-velocity vector which satisfies the condition \( u^i u_i = 1 \) and \( u^i \nabla_j u_i = 0 \) in the co-moving coordinate system. Also \( \rho \) and \( p \) indicated in the above definition of \( T_{ij} \) are fluid energy density and pressure respectively.

Using the above considered form of \( f(R) \) and \( f(T) \) functions in equation (2), for which \((g_{ij} \Box - \nabla_i \nabla_j) \lambda = 0\), the field equation (2) after rearranging the terms takes the form

\[
G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = \left( \frac{8\pi + \lambda}{\lambda} \right) T_{ij} + (p + \frac{1}{2} T) g_{ij}. \tag{3}
\]

If we recall the Einstein field equations with cosmological constant of general theory of relativity

\[
G_{ij} + \Lambda g_{ij} = 8\pi T_{ij}, \tag{4}
\]

and comparing equations (3) and (4) by considering a non-negative small value of the arbitrary coupling constant \( \lambda \) so that the sign of RHS of equations (3) and (4) remain same then we have

\[
\Lambda \equiv \Lambda(T) = -(p + \frac{1}{2} T), \tag{5}
\]

which regards as the effective cosmological constant \( \Lambda \) as a function of the trace \( T \) of energy momentum tensor \([44]\). Thus, for the above mentioned energy momentum tensor, we have

\[
\Lambda = \Lambda(T) = \frac{1}{2} (\rho - p). \tag{6}
\]
We consider the background metric which expresses a curvature-less homogeneous and isotropic Universe, the famous FLRW metric characterized by
\[ ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \]

where \( a(t) \) is the expansion scale factor.

In the background of the above metric (7) in the \( f(R, T) \) gravity for \( \Lambda(T) \) cosmology, Einstein’s field equations (3) yield the following two independent equations:
\[ 3H^2 = \left( A - \frac{1}{2} \right) \rho + \frac{1}{2} p, \]  
\[ -2\frac{\ddot{a}}{a} = \left( A + \frac{1}{3} \right) \rho + \left( A - \frac{1}{3} \right) p, \]

where \( A = \frac{8\pi G\rho}{3}, \) \( H = \frac{\dot{a}}{a} \) is the Hubble parameter which measure the fractional rate of change of scale factor \( a(t) \) and an overhead dot indicates the time derivative. In the next section, we solve the above cosmological equations with a simple parametrization of Hubble parameter.

2.2 Parametrization of \( H \) and exact solution

The composition of the above two evolution equations (8) and (9) involve three unknowns \( a, \rho \) and \( p \). In order to accomplish a unique and consistent solution of the field equations, an additional constrain equation is needed to close the system completely. In general, the EoS for the matter content in the Universe is considered as the supplementary condition. But, there are other approaches too, which have been discussed by many authors e.g. the parametrization of cosmological parameter involved in the field equations (in particular one can parametrize, Hubble parameter, deceleration parameter, EoS parameter, energy density, pressure, cosmological constant, for a detailed summary on parametrization, see [50]). From equations (8) and (9), \( \rho \) and \( p \) can also be represented in terms of \( H \) and \( q \) as,
\[ \rho = \frac{3}{A(3A - 4)} [(3A - 1) - q] H^2; \]  
\[ p = -\frac{3}{A(3A - 4)} [(3A + 1) - (2A - 1)q] H^2. \]

From above equations, we can see that for a known \( q \) or \( H \), we obtain the solution explicitly. As the recent astronomical observations acknowledge the accelerating expanding phase of the Universe with prior period of decelerating phase, we will take care of this scenario in our present study and adopt an appropriate parametrization of Hubble parameter used by J. P. Singh \( H \) [51] and Banerjee et al. [52], which describe both the scenario of the early deceleration and present accelerations in the form
\[ H(a) = \alpha (1 + a^{-n}), \]

where \( \alpha > 0 \) and \( n > 1 \) are constants (better call them model parameters to be constrained through observations). Integrating equation (12), we obtain the explicit form of scale factor as,
\[ a(t) = (e^{\text{nat}} - 1)^{\frac{1}{n}} + c, \]

where, we have used the initial big bang condition (at \( t = 0, a = 0 \)) which make the constant of integration \( c \) to be zero. Further the deceleration parameter \( q \) in terms of ‘\( t \)’ turn out to be
\[ q(t) = \frac{n}{e^{\text{nat}}} - 1. \]

Using equations (12) and (13) in equations (10) and (11), we obtain the explicit forms of the physical parameters in terms of \( t \) as
\[ \rho(t) = \frac{3\alpha^2}{A(3A - 4)} \left[ 3A - \frac{n}{e^{\text{nat}}} \right] \frac{e^{2\text{nat}}}{(e^{\text{nat}} - 1)^2}. \]
\[ p(t) = -\frac{3\alpha^2}{A(3A - 4)} \left[ 5A - (2A - 1)n \right] \frac{e^{2n\alpha t}}{(e^{n\alpha t} - 1)^2}, \]  
\[ (16) \]

\[ \omega(t) = -\frac{(5Ae^{n\alpha t} - (2A - 1)n)}{3Ae^{n\alpha t} - n}, \]
\[ (17) \]

and
\[ \Lambda(t) = \frac{3\alpha^2}{A(3A - 4)} \left[ 8A - 2A \right] \frac{e^{2n\alpha t}}{(e^{n\alpha t} - 1)^2}. \]
\[ (18) \]

2.3 Bounds on the parameters

We can evaluate these parameters at two extremes, \( t \rightarrow 0 \) and \( t \rightarrow \infty \) to see their behavior near singularity and in far future. We can discuss these in tabular form as follows:

| Time (t) | a | H | q | \( \rho \) | \( \omega \) | \( \Lambda \) |
|----------|---|---|---|--------|---------|--------|
| \( t \rightarrow 0 \) | 0 | \( \infty \) | \( n - 1 \) | \( \infty \) | \( -\infty \) | \( -\frac{5A + n(2A - 1)}{4A - n} \) |
| \( t \rightarrow \infty \) | \( \infty \) | \( \alpha \) | \( -1 \) | \( \frac{9A\alpha^2}{A(3A - 4)} \) | \( \frac{-15A\alpha^2}{A(3A - 4)} \) | \( -\frac{5}{3} \) | \( \frac{12A\alpha^2}{A(3A - 4)} \) |

From the above Table 1, we can have a rough estimate of the range of these cosmological parameters which depends on the parameter \( n \) and the \( f(R, T) \) coupling constant \( \lambda \) (we have \( A = \frac{8\pi G\Lambda}{3} \)). By choosing suitable values of \( n \) and \( \lambda \), we can regulate the expansion history of the various cosmological parameter. The role of \( f(R, T) \) coupling constant \( \lambda \) can be seen clearly from the above Table 1. The Universe starts with infinite density and pressure reduces from their dense state to constant values in the infinite future. The EoS parameter \( \omega \) vary in the range \( \left[ -\frac{5A + n(2A - 1)}{4A - n}, -\frac{5}{3} \right] \). The role of the \( f(R, T) \) coupling constant bounds the limits for EoS parameter that will be discussed in a forthcoming subsection.

We shall examine the behavior of physical and geometrical parameters in the following section more explicitly with the help of graphical representation by expressing the cosmological parameters in terms of redshift \( z \).

3 Dynamics of the model

In this study, we are trying to serve a cosmological model mathematically which can determine the dynamics of the Universe by explaining the behavior of its geometrical parameters and physical parameters on large scale. There are around four to twenty cosmological parameters through which dynamical behaviour of the Universe can be quantified. Among these the most fundamental cosmological parameters are Hubble parameter \( H(t) \) and deceleration parameter \( q(t) \). The other geometrical parameters can be determined by expanding the scale factor \( a(t) \) in the neighbourhood of \( t_0 \) by Taylor series theorem, where the subscript ‘0’ indicates the quantity at present time.

\[ a(t) = a(t_0 + t - t_0) = a_0 + \frac{(t - t_0)}{1!} a_0 + \frac{(t - t_0)^2}{2!} a_0 + \frac{(t - t_0)^3}{3!} \ddot{a}_0 + \cdots, \]  
\[ (19) \]

where \( a_0 \) represents the value of \( a(t) \) at present time \( t_0 \). The parameters \( H \) and \( q \) which specify the significance of EFEs and explain the recent astronomical observations can be accomplished by equation \( (19) \). Also the involvement of higher derivative terms of scale factor \( a(t) \) in equation \( (19) \) extend the cosmographic analysis of the geometrical parameters \[ 53, 54 \]. From equation \( (19) \), one can define some geometrical parameters such as jerk, snap, lerk parameters including Hubble and deceleration parameters through the higher derivatives of the scale factor as

\[ H = \frac{\dot{a}}{a}, \quad q = -\frac{\ddot{a}}{aH^2}, \quad j = \frac{\dddot{a}}{aH^3}, \quad s = \frac{\ddddot{a}}{aH^4}, \quad l = \frac{\ddddddot{a}}{aH^5}. \]  
\[ (20) \]
In the following subsections, we discuss the behavior of all these geometrical parameter for our model in
details. Moreover, we express the cosmological parameters in terms of redshift \((1 + z = \frac{a}{a_0})\) with normalized
scale factor \((a_0 = 1)\). To do that, we establish the \(t - z\) relationship here which comes out to be
\[
  t(z) = \frac{1}{H_0} \log \left( 1 + (1 + z)^{-n} \right).
\]
So, the most important geometrical parameter, the Hubble parameter \(H\) that explains the cosmological dynamics
can be written in terms of redshift as
\[
  H(z) = \alpha (1 + (1 + z)^n), \quad (21)
\]
or,
\[
  H(z) = \frac{H_0}{2} (1 + (1 + z)^n), \quad (22)
\]

3.1 Phase transition from deceleration to acceleration

Among the various cosmological parameters that describe the dynamics of the Universe, deceleration pa-
rameter is examined as one of the influential geometrical parameter. In this section, we examine the different
phases of evolution of deceleration parameter. Cosmological observations indicate that the Universe experiences
a cosmic speed up at late time imlying that the Universe must have passes through a slower expansion phase in the past [1, 2]. Moreover, a decelerating phase is also necessary for the structure formation. The cosmic
transit from deceleration to acceleration or the ‘Phase transition’ may be treated as a necessary phenomena
while describing the dynamics of the Universe. The above considered parametrization of the Hubble parameter
in equation (12) which yield a time dependent expression of the deceleration parameter in equation (14) is
rational with a phase transition. Present cosmic accelerating behaviour can be estimated through the values
of the deceleration parameter \(q\) that belong to negative domain. Keeping all these things in mind, here, we
plot \(q\) w.r.t redshift \(z\) and choose the model parameter \(n\) carefully so that we have a phase transition redshift \((z_{tr})\) exhibiting early deceleration to late acceleration. The deceleration parameter in terms of redshift \(z\) can be
written as
\[
  q(z) = \frac{(n - 1)(1 + z)^n - 1}{1 + (1 + z)^n}. \quad (23)
\]
From this expression, we can find the range of \(q\) to be \(q \in [(n - 1), -1]\). As \(n > 1\), we have, the lower
limit is positive and the upper limit is negative, showing a signature flip. At \(z = 0\), we have \(q_0 = \frac{n}{2} - 1\). The
model parameter \(n\) can be suitably choosen and \(q(z)\) can be plotted for a close view to discuss the behavior of
deceleration parameter and is shown in Fig. 1.

![Figure 1: The plots of deceleration parameter Vs. redshift z for different n.](image-url)
In the above figure 1, it can be clearly observed that for high redshift $z$, deceleration parameter $q$ is positive and for low redshift $z$, deceleration parameter $q$ is negative. One can explicitly observe the decelerating to accelerating regimes of the Universe depend on the variation of model parameter $n$. The plot shows phase transition redshift ($z_{tr}$) for various values of $n$ in the feasible range, $n \in (1, 2)$. For $n = 1.25$, $q = 0$ at $z_{tr} = 1.988$, for $n = 1.45$, $q = 0$ at $z_{tr} = 0.73$, for $n = 1.65$, $q = 0$ at $z_{tr} = 0.29$ and for $n = 1.85$, $q = 0$ at $z_{tr} = 0.091$. The present value of deceleration parameter $q_0$ corresponding to $n = 1.25$, $n = 1.45$, $n = 1.65$ and $n = 1.85$ are $-0.371$, $-0.275$, $-0.179$ and $-0.077$ respectively. We will see in one of our subsequent section the best fit value of the model parameter $n$ lies in the neighbourhood of 1.45 for which $z_{tr} = 0.73$.

### 3.2 Physical parameters and their evolution

In this section, we will examine the evolution of energy density $\rho$, pressure $p$, EoS parameter $w$ and the cosmological constant $\Lambda$ for our obtained model resulting from the parametrization \cite{12}. Using the $t - z$ relationship, we obtain the expressions for $\rho$, $p$, $\omega$ and $\Lambda$ in terms of redshift as follows

$$\frac{\rho(z)}{H_0^2} = \frac{3}{4A(3A - 4)} [1 + (1 + z)^n] [3A + A(n - A)(1 + z)^n],$$  \hspace{1cm} (24)

$$\frac{p(z)}{H_0^2} = -\frac{3}{4A(3A - 4)} [1 + (1 + z)^n] [10A - (2A - 1)n(1 + z)^n],$$  \hspace{1cm} (25)

$$\omega(z) = -\frac{5A + (5A - n(2A - 1))(1 + z)^n}{3A + (3A - n)(1 + z)^n},$$  \hspace{1cm} (26)

$$\frac{\Lambda(z)}{H_0^2} = \frac{3}{8A(3A - 4)} [1 + (1 + z)^n] [16A - 2A(n + 1 + z)^n].$$  \hspace{1cm} (27)

The evolution of the physical parameters in equation (24)-(27) are shown in the following figures.

![Figure 2: The plots of energy density $\frac{\rho}{H_0^2}$ and pressure $\frac{p}{H_0^2}$ Vs. redshift $z$ for different $n$ with $\lambda = 0.1$.](image)

The left panel in figure 2 depicts the evolution of energy density $\frac{\rho}{H_0^2}$ with redshift $z$ for different values of model parameter $n$ mentioned in plot. For a high redshift, energy density is also very high as expected then energy density falls down as time unfolds and later on it approaches to $\frac{9A\omega_0^2}{A(3A - 4)}$ as $z \to -1$. The right panel of the figure 2 highlights the picture of the cosmic pressure for the said values of $n$. In the initial phases of the early Universe for a very high redshift, pressure $\frac{p}{H_0^2}$ attains a very large negative value, then $\frac{p}{H_0^2}$ varies as time goes by and in the near future ($z \to -1$), pressure $p$ approaches to $\frac{-15A\omega_0^2}{A(3A - 4)}$ and remains negative throughout.
the cosmic evolution. The negative value of cosmic pressure, according to the standard cosmology is subjected to the cosmic acceleration. Hence our obtained model shows eternal acceleration.

\[ \omega_n = 1.85, \quad n = 1.65, \quad n = 1.45, \quad n = 1.25 \]

Figure 3: The plots of EoS parameter \( \omega \) and cosmological constant \( \frac{\Lambda}{H_0^2} \) Vs. redshift \( z \) for different \( n \) with \( \lambda = 0.1 \).

The profile of EoS parameter \( \omega \) and cosmological constant \( \lambda \) has been investigated in figure 3. The left panel of the above figure shows the evolution of \( \omega \) w.r.t redshift \( z \). For all the four values of model parameter \( n \) and a fixed value of \( f(R, T) \) coupling constant \( \lambda \), \( \omega \in \) quintessence region, and \( (\omega \to -\frac{5}{3}) \) in late time \( (z \to -1) \) which suggest that our dark energy model is similar to the quintessence model initially and later on it behaves like phantom model. The right panel of the above figure depict the variation of cosmological constant \( (\frac{\Lambda}{H_0^2}) \) w.r.t redshift \( z \). It has been observed that cosmological constant remains positive throughout the cosmic evolution, decreasing in nature and reaches a small positive value at present epoch \( (z \to 0) \) which is in favored with the observations \[1, 2, 58, 59\] and \( (\frac{\Lambda}{H_0^2}) \to \frac{12Aa^2}{A(3A-4)} \) as \( z \to -1 \). The outcome from these observations advises a very minute positive value having magnitude \( \sim 10^{-123} \).

### 3.3 Role of \( \lambda \) in the evolution of \( \omega \)

Here, in this subsection, we fix the value of the model parameter \( n \) to 1.45 (for which the phase transition redshift will be \( z_{tr} \approx 0.7 \)) and observe the role of \( f(R, T) \) coupling constant \( \lambda \) on the evolution of EoS parameter \( \omega \) by providing different values to \( \lambda \). The behavior of \( \omega \) is represented in the graph as follows.

As we have shown earlier that the non negativity condition of energy density \( \rho \) holds good whenever \( f(R, T) \) constant \( \lambda \) takes value in the range \((0, 2\pi)\). In this way, we restrict our domain of \( \lambda \) to be same for the plot of EoS parameter \( \omega \). From the above figure, we investigate that for small value of \( \lambda \) (e.g. \( \lambda = 0.1 \)), EoS \( \omega \) lies in the quintessence region initially and crosses the phantom divide line, and therefore enters into the phantom regime. As we start providing large values of \( \lambda \) (within the said range), EoS parameter \( \omega \) belong to the phantom regime only and it has nothing to do with quintessence region. Eventually \( \omega \to -\frac{5}{3} \) as \( \lambda \) increases which may lead to cause big rip singularity. This is how \( f(R, T) \) gravity contributes in this model (on considering the wide acceptable range of \( \lambda \)) by analysing the contrast nature of EoS parameter.

### 3.4 jerk, snap, lerk parameters

For our model, the expressions for jerk parameter \( (j) \), snap parameter \( (s) \) and lerk parameter \( (l) \) are obtained in terms of redshift \( z \) and are given by:

\[ j(z) = (1 + n(n - 3))[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}], \]  

\[ s(z) = (1 + n(n - 3))[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}] + \frac{n^2(n - 3)^2}{3}[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}], \]

\[ l(z) = (1 + n(n - 3))[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}] + \frac{n^2(n - 3)^2}{3}[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}] + \frac{n^3(n - 3)^3}{45}[(1 + (1 + z)^{-n})^{-1} + n^2(1 + (1 + z)^{-n})^{-2}]. \]
Figure 4: The plot of $\omega$ Vs. redshift $z$ for different $\lambda$ with $n = 1.45$.

\[
\begin{align*}
    s(z) &= 1 + n[-n^2(1 + (1 + z)^{-n})^{-3} - n(4n - 7)(1 + (1 + z)^{-n})^{-2} \nonumber \\
    & \quad \quad \quad - (6 + n(n - 4))(1 + (1 + z)^{-n})^{-1}], \\
    l(z) &= [1 + n^4(1 + (1 + z)^{-n})^{-4} + n^3(11n - 15)(1 + (1 + z)^{-n})^{-3} \nonumber \\
    & \quad \quad \quad + n^2(25 + n(11n - 30))(1 + (1 + z)^{-n})^{-2}n(-10 + n(10 + n(n - 5)))](1 + (1 + z)^{-n})^{-1}. \quad (29)
\end{align*}
\]

Figure 5: The plots of jerk $j$, snap $s$ and lerk $l$ parameters Vs. redshift $z$ for different $n$.

Figure 4 depict the evolution of jerk $j$, snap $s$ and lerk $l$ parameters w.r.t redshift $z$. In Fig. 4(a), the evolution of jerk parameter is represented for all the four values of $n$, and it can be observed that $j$ parameter lies in the positive range throughout its course. Also $j \to 1$ as $z \to -1$, $\forall n$ which matches with observations of standard ΛCDM but at present $z = 0$, $j$ is different from 1 $\forall n$, and that means, at present time our model is similar to another dark energy model other than ΛCDM $\forall n$. Fig. 4(b) enact the profile of snap $s$ parameter during its evolution. In the early Universe $s$ assume value in the negative range $\forall n$ then as Universe evolves, $s$ take values in the positive range, i.e. in the entire evolution of $s$, there is one transition from negative to positive range. Also it can be directly seen from the figure that transition of $s$ depends on the model parameter $n$ i.e. transition redshift of $s$ is delayed as $n$ takes value from 1.25 to 1.85. Figure 4(c) shows
the detailed variation of lerk $l$ parameter over the whole redshift. $l$ assumes only positive values without any redshift transition. In addition to $j$, both $s$ and $l$ also approaches to 1 as $z \rightarrow -1$ which is in good agreement with $\Lambda$CDM.

4 Physical analysis and geometrical diagnostic

4.1 Energy conditions

In general theory of relativity, energy conditions (EC) have great advantage for the broad understanding of singularity theorem of spacetime. EC are considered as the basic ingredient to describe the role of different geodesics e.g. null geodesics, spacelike, timelike, or lightlike geodesics. The additional privilege of EC is to provide the elementary tool for study certain ideas about Black holes and Wormholes. There are several different ways in which EC can be formulated, e.g. geometric way, physical way, or in effective way. The viability of various types of pointwise EC could be discussed by widely known Raychaudhuri’s equation [60]. The situation of exploring EC in GR is to relate cosmological geometry with general energy momentum tensor in a way so that energy remains positive [61]. But generally this is not the case in modified gravity theories. Therefore, in modified gravity one has to extra concerned while expressing such relation. For the literature review of EC that have been already examined in general theory of relativity see [62, 63, 64]. Several issues to explore the ideas of EC have been proposed in modified gravity also. For a brief and recent reviews see ([65, 66] for $f(R)$ gravity, [67, 68, 69] for $f(G)$ gravity). The expression of four types of EC in $f(R,T)$ gravity (with effective energy density $\rho$ and pressure $p$) can be represented as follows.

- **NEC**\(\Leftrightarrow\) $\rho + p_i \geq 0$, $\forall i = 1, 2, 3$.
- **WEC**\(\Leftrightarrow\) $\rho \geq 0$ and $\forall i = 1, 2, 3$, $\rho + p_i \geq 0$.
- **SEC**\(\Leftrightarrow\) $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho + p_i \geq 0$.
- **DEC**\(\Leftrightarrow\) $\rho \geq 0 \forall i$, $|p_i| \leq \rho$.

Also, if the energy density ($\rho$) and pressure ($p$) are described in terms of scalar field $\phi$ (real), then energy conditions in terms of scalar field $\phi$ satisfies:

- **NEC**: $\forall V(\phi)$.
- **WEC**: $V(\phi) \geq \frac{\dot{\phi}^2}{2}$.
- **SEC**: $V(\phi) \leq \frac{\dot{\phi}^2}{2}$.
- **DEC**: $V(\phi) \geq 0$.

Here, we have presented the graphs of NEC, SEC and DEC by fixing the value of the model parameter $n$ to 1.45 for which the phase transition redshift is $z_{tr} \approx 0.7$. Also we will observe in the subsequent section, the best fit value of $n$ according to the Hubble dataset lies in the neighbourhood of $n = 1.45$.

From the figure 5, we can observe that NEC and SEC violate within the acceptable non-negative range of $\lambda$ (Fig. 5a, Fig. 5b) and DEC holds good in the given domain of $\lambda$ (Fig. 5c).

4.2 Statefinder diagnostic

As we know the role of geometric parameters in order to study the dynamics of a cosmological model has of great importance. In what follows, in subsection 3.1, we have discussed the different phases of evolution of deceleration parameter and concluded that, deceleration parameter turn its sign from positive to negative for corresponding high redshift to low redshift respectively. The phase transition of deceleration parameter provides a hope to discover the source of recent acceleration. The requirement of a more general dark energy model other than $\Lambda$CDM and the development in the accuracy of current cosmological observational data, demands a question to look into the quantities involving higher derivatives of scale factor $a$.

In order to have a general study of different dark energy models, a geometrical parameter pair technique, known
Figure 6: The plots of NEC, WEC and DEC for the model with \( n = 1.45 \).

as statefinder diagnostic (SFD), have been proposed \[70, 71\] and is denoted by \( \{r, s\} \), where \( r \) and \( s \) are defined as

\[
 r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r - 1}{3(q - \frac{1}{2})},
\]  
(31)

where \( q \neq \frac{1}{2} \). This \( s \) is different from the snap parameter we have discussed in subsection 3.4 and this \( r \) is same as the jerk parameter \( j \) we have discussed in section 3.

Various dark energy scenarios can be examined by the distinct evolutionary trajectories of the geometric pair \( \{r, s\} \) emerging in the \( r - s \) plane diagram. A symbolic feature of the SFD is that, the standard ΛCDM model of the cosmology is represented by the pegged point \( \{r, s\} = \{1, 0\} \), whereas the standard matter dominated Universe SCDM corresponds to a fixed point \( \{r, s\} = \{1, 1\} \). Other than the ΛCDM and SCDM model, the SFD analysis can successfully discriminate among the several dark energy candidates such as quintessence, braneworld dark energy models, Chaplygin gas and some other interacting dark energy models by locating some particular region in the said diagram in the distinctive trajectories. For a brief review on SFD, see \[72, 73, 74, 75\].

Now we implement the SFD approach in our dark energy model to discuss the behavior of our obtained model and study its converging and diverging nature with respect to the SCDM or ΛCDM model. The expression for \( r, s \) parameters for our model are as follows

\[
 r = 1 + n[e^{-2n\alpha t}(n - 3) + n],
\]  
(32)

and

\[
 s = \frac{2n[3 - n(1 + e^{-n\alpha t})]}{9e^{n\alpha t} - 6n}.
\]  
(33)

The left panel in figure 4 represents the time evolution of four trajectories for different values of \( n \) in \( \{r, s\} \) plane diagram. All the trajectories corresponding to different \( n \) evolving with time but deviate from the point SCDM i.e. \( \{r, s\} = \{1, 1\} \) which correspond to matter dominated Universe. The directions of \( r - s \) trajectories in the plane diagram are represented by the arrows. Initially, we have examined that corresponding to \( n = 1.25 \) and \( n = 1.45 \), trajectories remain in domain \( \{r < 1, s > 0\} \) which relate our dark energy model to quintessence model. Also trajectories corresponding to \( n = 1.65 \) and \( n = 1.85 \) start evolving from the region \( r > 1, s < 0 \) which resemble the behavior of dark energy with Chaplygin gas and this region is highlighted by CG in the top leftmost part of the plot. The downward pattern of trajectories (those representing CG behavior) and upward trend of the trajectories (those representing quintessence behavior) are eventually met at the point \( \{r, s\} = \{1, 0\} \) in the top most part of the plot. This suggest us that our model behaves like ΛCDM in the late time of cosmic evolution. In addition we have presented one more horizontal line in the above diagram that shows the transformation of trajectories from matter dominated Universe SCDM to ΛCDM as time unfolds. The point having coordinates \( \{r, s\} = \{1, \frac{3}{2}\} \) on the horizontal line labelled HDE represents holographic dark energy model with future event horizon as IR cutoff , begins the evolution from the point \( \{r, s\} = \{1, \frac{3}{2}\} \) and ultimately ends its evolution at ΛCDM \[76, 77, 78\]. So the plot of \( \{r, s\} \) for our model effectively discriminant among other dark energy model for differnt \( n \).
Figure 7: The $s - r$ and $q - r$ diagrams for the model for different $n$.

The right panel in figure 4 represents the time evolution of four trajectories for different values of $n$ in $\{r, q\}$ plane diagram. Since we have seen the complete description of the phase transition of deceleration parameter in subsection 3.2, we can again observe the phase transition of our model by looking into the trajectories of $r - q$ diagram (as $q$ changes its sign from positive to negative). The evolution of the trajectories, for different values of $n$, commence in the vicinity of matter dominated Universe $SCDM$ but never converges to $SCDM$. As time evolves, the value of $r$ and $q$ start decline and attain their minima, after that both $r$ and $q$ start increases towards the steady state model which is located in the diagram at $(1, -1)$ and donated by $SS$. The progression of the trajectories to $SS$ suggest us that our dark energy model may behave like $SS$ in late-time.

4.3 Om diagnostic

In this section, we have bring out one more technique to differentiate standard $\Lambda$CDM model from other dark energy models. This approach have been developed to examine the dynamics of the dark energy models, by connecting the geometric parameter $H$ with redshift $z$, and is known as Om diagnostic [79, 80, 81]. It is worth mentioning that Om diagnostic can make distinction among various dark energy models without actually referring the exact present value of density parameter of matter and without comprising EoS parameter. Also Om diagnostic yields a null test for cosmological constant $\Lambda$ as Om takes same constant value irrespective of the redshift $z$ for $\Lambda$CDM, which exhibits the non evolving behavior of Om, if dark energy is cosmological constant. Also Om diagnostic is a single parameter evaluation technique, therefore it is quite simple to formulate, as compare to SFD. Om diagnostic is defined as

$$Om(z) = \left(\frac{H(z)}{H_0}\right)^2 - 1 \frac{z^2 + 3}{z^2 + 3z + 3}.$$  \hspace{1cm} (34)

The contrasting behavior of dark energy models from $\Lambda$CDM depend on the slope of Om(z) diagnostic. A quintessence ($\omega > -1$) type behavior of dark energy can be identified by its negative curvature with respect to $z$, and a phantom type behavior ($\omega < -1$) can be diagnosed by its positive curvature with respect to $z$ and zero curvature of $Om(z)$ represents the standard $\Lambda$CDM.

Fig. 7 exhibits the evolution of different trajectories of function $Om(z)$ w.r.t redshift $z$, corresponding to different values of model parameter $n$. From the graph we can observe that all the trajectories showing negative slope, means all the trajectories moves in an upward direction as time increases or redshift decrease. This negative curvature pattern suggest us that our model is behaving similar to quintessence for all $n$. 

12
Observational constraint

In this section, we fit our model with the 28 points of Hubble datasets [82] in the redshift range $0.1 \leq z \leq 2.3$ and 580 points of Union 2.1 compilation supernovae datasets [83] and compare with the ΛCDM model. We choose the value of the current Hubble constant from Plank 2015 result [84] as $H_0 = 67.8$ $Km/s/Mpc$ to complete the datasets.

The left panel of the following figure shows the best fitting curve of our model compared with the ΛCDM model for $H(z)$ datasets and the right panel shows best fitting curve of our model compared with the ΛCDM model for $SN$ datasets.

We constrain the model parameter $n$ with the observational Hubble data and Union 2.1 compilation data respectively. The mean value of model parameter $n$ is determined by minimizing the corresponding Chi-square value as follows:

$$\chi^2_{CHD}(p_s) = \sum_{i=1}^{28} \frac{[H_{th}(p_s, z_i) - H_{obs}(z_i)]^2}{\sigma^2_{H(z_i)}}.$$  

Here, $H_{th}$ refers to the model based theoretical value for the $H$ and $H_{obs}$ refers to the observed value while $p_s$ refers to the parameter of the model. The standard error in the observed value is denoted by $\sigma_{H(z_i)}$.

$$\chi^2_{OSN}(\mu_0, p_s) = \sum_{i=1}^{580} \frac{[\mu_{th}(\mu_0, p_s, z_i) - \mu_{obs}(z_i)]^2}{\sigma^2_{\mu(z_i)}},$$

where $\mu_{th}$ refers to the model based theoretical distance modulus and $\mu_{obs}$ refers to the observed distance modulus. The standard error in the observed value is denoted by $\sigma_{\mu(z_i)}$. $\mu(z)$ refers to the distance modulus and is defined by $\mu(z) = m - M = 5\log D_l(z) + \mu_0$ where both $m$ and $M$ indicate the apparent magnitude and absolute magnitude of a standard candle respectively. $D_l(z)$ and $\mu_0$ are the luminosity distance and the nuisance parameter which are defined as $D_l(z) = (1 + z)H_0 \int_0^z \frac{1}{H(z^*)} dz^*$ and $\mu_0 = 5\log \left( \frac{H_0^{-1}}{Mpc} \right) + 25$ respectively.

The likelihood contours for the parameter $n$ with $1\sigma$ and $2\sigma$ error in the $n - H_0$ plane are shown in Fig 10. The best fit values of $n$ are found to be $1.427$ and $1.3$ according to the Hubble dataset and Union 2.1 compilation data for which the corresponding values of $H_0$ are $67.2224$ and $68.554$ respectively.
Figure 9: Comparision of our model and ΛCDM model with error bar plots of Hubble and SN datasets. Red lines indicate our model and dashed black lines indicate ΛCDM model in both the plots.

Figure 10: Figures show the contour plot in the $n$-$H_0$ for $H(z)$ and Union 2.1 SNIa datasets. The dark shaded regions shows the 1-$σ$ error and light shaded region shows 2-$σ$ error. Black dot represent the best fit values of model parameter $n$ in both the plots and the values of $H_0$. 
6 Conclusion

In this article, we have studied a $\Lambda(t)$ cosmology obtained by a simple parametrization of Hubble parameter in a flat FLRW space-time in $f(R,T)$ theory of gravity. We have considered the most simplest form of $f(R,T)$ function that can explain the non-minimal coupling between geometry and matter present in the Universe. The field equations have been derived by taking the functional form of $f(R,T) = f(R) + f(T)$ in to consideration which lead to general relativistic field equations with a trace $T$ dependent term, which we termed as cosmological constant $\Lambda(T)$ in this study. To obtain the exact solution of the cosmological field equations, we have endorsed a parametrization of Hubble parameter $H$ that yields a time dependent deceleration parameter $q(t)$. A comprehensive observations have been recorded for our obtained model based on the above mentioned information.

- In order to study a cosmological model capable of explaining the recent observations of accelerating expansion of the Universe with a decelerating phase of evolution in the past, we have considered a geometrical parametrization of the Hubble parameter $H$ used by J. P. Singh [51] and Banerjee et al. [52], that lead to a variable deceleration parameter $q$. The obtained form of $q$ describes both the scenario of early deceleration and present acceleration. The behavior of geometrical parameters $a$, $H$ and $q$ at two extremities $(t \to 0, t \to \infty)$ have been analysed in Table 1.

- For the considered parametrization of $H$, the different phases of evolution of the deceleration parameter has been examined. From the expression of $q(z)$, we have found the range of deceleration parameter, i.e. $q \in [n - 1, -1]$ which clearly shows the signature flipping behavior because the model parameter $n > 1$. For a close view on $q$, we can observe the decelerating to accelerating regimes of the Universe depending on the variation of the model parameter $n$ in Fig. 1. As the values of $n$ increases from 1.25 to 1.85, phase transition redshift $z_{tr}$ come closer to the present time ($z = 0$).

- The behavior of physical parameters $\rho$, $p$, $\omega$, and $\Lambda$ have been investigated at $t \to 0$ and $t \to \infty$ and shown in Table 1. From Table 1, we can have a rough estimate on the behavior of these cosmological parameters and also, we can see the role of $f(R,T)$ coupling constant on these physical parameters. Also in subsection 3.2, we have discussed the evolution of physical parameters in terms of redshift $z$. Energy density and pressure reduces from their dense state to a constant value, depends on $f(R,T)$ coupling constant $\lambda$ in near future (see Fig. 2). $\rho$ remains positive throughout the cosmic evolution whereas $p$ assumes only negative values which favor the existence of new form of energy with anti-gravitational effects that drives acceleration in the Universe. By taking $\lambda = 0.1$, we have presented the plot of EoS in Fig. 3(a). For a fixed $\lambda = 0.1$, $\omega \in (-1, -0.5)$ initially which relate our dark energy model to quintessence region, thereafter $\omega$ enters in to the phantom region for all model parameters $n$ and lastly $\omega \to -\frac{3}{2}$. The role of $\Lambda$ bounds the limits for EoS parameter which we have observed in subsection 3.3. The variation of cosmological constant $\Lambda$ has been observed throughout the cosmic evolution. $\Lambda$ remains positive, decaying in nature and reaches a small positive value at present epoch ($z \to 0$) which is in favored with the observations [1] [2] [58] [59].

- To discuss the role of $f(R,T)$ coupling constant $\lambda$ on the evolution of EoS parameter $\omega$, we fix the value of $n$ and vary $\lambda$. From figure 4, we have examined the special character of $f(R,T)$ coupling constant $\lambda$, and observed that as $\lambda$ takes high values from the said range $(0, 24\pi)$, EoS parameter moves from quintessence region to phantom region and ultimately approaches to $-\frac{5}{3}$ (see Fig. 4). This is the contribution of $f(R,T)$ gravity in this model on considering the complete acceptable range of $\lambda$.

- Next, we have compared our dark energy model with standard $\Lambda$CDM model by examine the behavior of other geometrical parameters i.e. jerk $j$, snap $s$ and lerk $l$ parameters. It has been recorded that $\forall$ values of $n$, our model behaves different from $\Lambda$CDM model at present time $z = 0$, but in the late future $j \to 1$ as accordance with $\Lambda$CDM. In addition to $j$, the behavior of snap $s$ and lerk $l$ parameters have been graphically demonstrated in Fig. 5. The snap parameter $s$ shows one transition from negative to positive throughout its evolution w.r.t redshift $z$ while $l$ is decaying in nature with no transition.

- In section 4, some physical analysis and geometrical diagnostics of the model have been studied. Physical viability of the model have been analysed by verifying the energy conditions of the obtained model. From Fig. 6, the infringement of NEC and SEC can be seen easily for the acceptable wide non negative range of $\lambda$ and Fig. 6c represented that DEC holds good in the given domain of $\lambda$. In subsection 4.2, Fig. 7 represents the time evolution of four trajectories for different values of $n$ in $\{r, s\}$ and $\{r, q\}$ plane diagram. The directions of $r - s$ trajectories in the plane diagram are represented by the arrows showing different
dark energy models and ultimately approaches to ΛCDM (see Fig. 7a). In the $r - q$ plane diagram, the evolution of the trajectories, for different values of $n$, commence in the vicinity of SCDM and as time evolves, the trajectories of $r$ and $q$ move towards steady state model SS (see Fig. 7b).

- Also, one more geometrical diagnostic have been interpreted to understand different dark energy models ∀ values of $n$. Plot of $Ω_m(z)$ against redshift $z$ has been displayed in Fig. 8. All the trajectories of $Ω_m(z)$ exhibits negative slope, which suggest us that our model is behaving similar to quintessence model ∀ $n$ and in the late future, i.e. $z \to -1$, $Ω_m(z) \to$ finite positive quantity, that means in late future our model may correspond to ΛCDM.

- The model parameter $n$ is constrained using the 28 point of $H(z)$ dataset and Union 2.1 compilation dataset. The obtained model fits well with the $H(z)$ and SN datasets and nearly follow the ΛCDM behaviour. The constrained values of the model parameter $n$ comes out to be 1.427 with $H(z)$ data and 1.30 with SNIa data.

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