Hom-center-symmetric algebras and bialgebras

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Abstract. In this work, the hom-center-symmetric algebras are constructed and discussed. Their bimodules, dual bimodules and matched pairs are defined. The relation between the dual bimodules of hom-center-symmetric algebras and the matched pairs of hom-Lie algebras is established. Furthermore, the Manin triple of hom-center-symmetric algebras is given. Finally, a theorem linking the matched pairs of hom-center-symmetric algebras, the hom-center-symmetric bialgebras and the matched pairs of sub-adjacent hom-Lie algebras is provided.

Keywords. hom-center-symmetric algebra; hom-Lie algebra; hom-Lie-admissible algebra; bialgebra; hom-center-symmetric bialgebra; Manin triple.

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1. Introduction

The Virasoro, Witt and Lie algebra deformations generate classes of nonassociative algebras [14] with some interesting algebraic identities (see [10] and references therein). The Lie admissible algebras [13] form a class of nonassociative algebras with a product commutator defining a Lie algebra. The hom-Lie algebras were first introduced in 2006 [6], using the $\sigma$-derivation of Virasoro and Witt algebras. Some $q$-deformations of Virasoro and Witt algebras also generate a hom-Lie algebra structure. The hom-Lie admissible algebras [9] constitute a class of nonassociative algebras with a product commutator giving a hom-Lie algebra.

The algebraic properties of hom-coalgebras, hom-coassociative coalgebras, and $G$ - hom - coalgebras, where $G$ is a subgroup of the permutation group $S_3$ [11] generalizing Lie-admissible coalgebras, were investigated in [5]. In these works, relevant definitions and properties of hom-Hopf algebras generalizing Hopf algebras, and giving the module and comodule structures over hom-associative algebras and hom-coassociative coalgebras, were given. Besides, the hom-Lie algebras were studied in terms of representation theory [16], hom-Lie bialgebras [15], hom-Lie 2-algebras [18], enveloping algebras [19], and Novikov algebras [21][24].

Nearly hom-Lie algebras, left-symmetric algebras (LSA), left-symmetric bialgebras (LSBA) were analysed by analogy to Lie bialgebras [4]. C. Bai [11] related left-symmetric algebras with symplectic Lie algebras, classical Yang-Baxter equations (CYBE), $\sigma$-operators, and para-Kähler Lie algebras. Similarly, hom-left-symmetric bialgebras and para-Kähler hom-Lie algebras were
equivalently introduced by Q. Sun and H. Li [17]. The center-symmetric algebras were investigated in [7], where their Lie-admissibility was also established, and their bimodules were constructed. Furthermore, their matched pairs were defined and linked to matched-pairs of Lie algebras and associated Manin triple.

The hom-classical Yang-Baxter equation (HCYBE) was established from a homeomorphism derivation of the CYBE. The Rota-Baxter hom-Lie admissible algebra identities were also derived and discussed in a series of works. For more details, see [12,20,22,23] and references therein.

The links between hom-associative algebras (HAA), hom-associative bialgebras (HABA), hom-Lie algebras (HLA), hom-Lie bialgebras (HLBA), hom-left-symmetric algebras (HLSA), and hom-left-symmetric bialgebras (HLSBA) can be represented by the following diagram:

\[ \text{HAA} \to \text{HLA} \to \text{HLA} \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ \text{HABA} \to \text{HLBA} \leftarrow \text{HLBA} \]

Our present work aims at investigating the relations existing between a hom-center-symmetric algebra (HCSA), a hom-center-symmetric bialgebra (HCSBA), a hom-Lie algebra (HLA) and a hom-Lie bialgebra (HLBA), as illustrated by the following diagram:

\[ \text{HCSA} \to \text{HLA} \]

\[ \downarrow \]

\[ \text{HCSBA} \to \text{HLBA} \]

We give basic definitions of hom-center-symmetric algebras and establish their properties. We show that the hom-center-symmetric algebras are deformations (homomorphism deformations) of center-symmetric algebras. In addition, the bimodules and matched pairs of hom-center-symmetric algebras are built and linked to the bimodules and matched pairs of hom-Lie algebras, respectively. Besides, the Manin triple of hom-center-symmetric algebras is defined, discussed, and analyzed with respect to the Manin triple of hom-Lie algebras. Finally, a theorem yielding the link between the hom-center-symmetric bialgebras, the matched-pairs of hom-center-symmetric algebras, and the matched-pairs of hom-Lie algebras is given and proved.

2. Preliminaries: main definitions and properties

We now develop the basic definitions and properties of hom-center-symmetric algebras.

**Definition 2.1.** A hom-center-symmetric algebra is a triple \((A, \mu, \alpha)\), where \(A\) is a vector space, \(\mu : A \otimes A \to A\) is a bilinear map, and \(\alpha \in \mathfrak{gl}(A)\) such that, for all \(x, y, z \in A\)

\[
\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)),
\]

\[
(x, y, z)_{\alpha, \mu} = (z, y, x)_{\alpha, \mu},
\]

where \((x, y, z)_{\alpha, \mu} := \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))\) is called \(\alpha\)-associator associated to the map \(\mu\).

**Remark 2.2.** The relation \((2.2)\) is equivalent to

\[
\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) = (\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \alpha - \alpha \otimes (\mu \otimes \tau)),
\]

which can be illustrated by the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\alpha \otimes \mu - \mu \otimes \alpha} & A \otimes A \\
(\mu \otimes \tau) \otimes \alpha - \alpha \otimes (\mu \otimes \tau) & \downarrow & \\
A \otimes A & \xrightarrow{\mu \otimes \tau} & A
\end{array}
\]

where \(\tau\) is the exchange map on \(A \otimes A\).
In the sequel, as matter of notation simplification, we denote \((A, \mu, \alpha)\) by \((A, \alpha)\) or by \(A\).

The left \(L\) and right \(R\) representations of the bilinear product on \(A\) are defined by

\[
L : A \rightarrow \text{gl}(A) \\
x \mapsto L_x : A \rightarrow A \quad \mu(x, y) := x \cdot y,
\]

\[
R : A \rightarrow \text{gl}(A) \\
x \mapsto R_x : A \rightarrow A \quad \mu(y, x) := y \cdot x.
\]

We infer the adjoint representation \(\text{ad} := L - R\) of the sub-adjacent Lie algebra \(G(A)\) of a center-

\(
\text{symmetric algebra} A \quad \forall\ \in A, \text{ad}(y) := (L_y - R_y)(y).\)

\[
\text{Definition 2.3. A hom-Lie algebra is a triple } (g, [\cdot, \cdot], \alpha_g) \text{ consisting of a vector space } g, \text{ an} \\
\text{algebra homomorphism } \alpha_g : g \rightarrow g, \text{ and a skew-symmetric bilinear map } [\cdot, \cdot] : g \otimes g \rightarrow g \text{ satisfying} \\
\text{the following relations, for all } x, y, z \in g:\
\]

\[
[a_g(x), [y, z]]_g + [a_g(y), [z, x]]_g + [a_g(z), [x, y]]_g = 0,
\]

called twisted Jacobi identity.

\[
\text{Definition 2.4. A representation of a hom-Lie algebra } (g, [\cdot, \cdot], \alpha_g) \text{ on a vector space } V \text{ with} \\
\text{respect to } \psi \in \text{gl}(V) \text{ is a linear map } \rho : g \rightarrow \text{gl}(V) \text{ such that for all } x, y \in g, \text{ the following relations:}
\]

\[
\rho(\alpha_g(x)) \circ \psi = \psi \circ \rho(x),
\]

\[
\rho([x, y]) \circ \psi = \rho(\alpha_g(x)) \circ \rho(y) - \rho(\alpha_g(y)) \circ \rho(x)
\]

are satisfied.

\[
\text{Proposition 2.5. Given a representation } (\rho, \psi, V) \text{ of a hom-Lie algebra } (g, [\cdot, \cdot], \alpha_g), \text{ there is} \\
a hom-Lie algebra on a semidirect sum vector space } g \oplus V \text{ given by the following identities, for all } x, y \in g \text{ and } u, v \in V:
\]

\[
(\alpha_g + \psi)(x + u) = \alpha_g(x) + \psi(u),
\]

\[
([x + u], (y + v))_{g \otimes \nu V} = [x, y]_g + \rho(x)v - \rho(y)u = -([y, x]_g - \rho(x)v + \rho(y)u) = -((y + v, x + u)_{g \otimes \nu V}).
\]

**Proof.**

According to the Definition 2.4, it is clear that the product \([\cdot, \cdot]_{g \otimes \nu V}\) is bilinear on \(g \oplus V\) and \(\forall x, y \in g \text{ and } u, v \in V,

\[
([x + u], (y + v))_{g \otimes \nu V} = [x, y]_g + \rho(x)v - \rho(y)u = -([y, x]_g - \rho(x)v + \rho(y)u) = -((y + v, x + u)_{g \otimes \nu V}).
\]

The bilinear product \([\cdot, \cdot]_{g \otimes \nu V}\) is skew symmetric. Besides, we have

\[
(\alpha_g + \psi)((x + u), (y + v))_{g \otimes \nu V} = (\alpha_g + \psi)([x, y]_g + \rho(x)v - \rho(y)u) = \alpha_g([x, y]_g) + \rho(\alpha_g(x)) \circ \psi(v) - \rho(\alpha_g(y)) \circ \psi(u) = \alpha_g(x) + \psi(u), \alpha_g(y) + \psi(v)_{g \otimes \nu V} = ([\alpha_g + \psi](x + u), (\alpha_g + \psi)(y + v))_{g \otimes \nu V}.
\]

In addition, we have for all \(x, y, z \in g \text{ and } u, v, w \in V:\

\[
([\alpha_g + \psi](x + u), [y + v, z + w]_{g \otimes \nu V})_{g \otimes \nu V} + ([\alpha_g + \psi](y + v), [z + w, x + u]_{g \otimes \nu V})_{g \otimes \nu V} + [([\alpha_g + \psi](z + w), [x + u, v]_{g \otimes \nu V})_{g \otimes \nu V} + ([\alpha_g(z + w)]_{g \otimes \nu V} + ([\alpha_g(z)]_{g \otimes \nu V}, [x + u, v]_{g \otimes \nu V})_{g \otimes \nu V} + ([\alpha_g(z)]_{g \otimes \nu V}, [x + u, v]_{g \otimes \nu V})_{g \otimes \nu V}.
\]
\[ [\alpha(x), [y, z]]_{\mu} + [\alpha(y), [z, x]]_{\mu} + [\alpha(z), [x, y]]_{\mu} = [\alpha(x), \mu(y, z) - \mu(z, y)]_{\mu} + [\alpha(y), \mu(z, x) + \alpha(z), \mu(x, y) - \mu(y, x)]_{\mu} + [\alpha(z), \mu(z, x) - \mu(x, z)]_{\mu} - [\alpha(x), \mu(\alpha, \alpha)]_{\mu} + [\alpha(y), \mu(\alpha, \alpha)]_{\mu} + [\alpha(z), \mu(\alpha, \alpha)]_{\mu} = \{\mu(\alpha, x), \mu(y, z) - \mu(z, y), \alpha\} + \{\mu(\alpha, y), \mu(z, x) - \mu(x, z), \alpha\} \] 

Therefore, for all \( x, y, z \in A \), the following equality holds

\[ [\alpha(x), [y, z]]_{\mu} + [\alpha(y), [z, x]]_{\mu} + [\alpha(z), [x, y]]_{\mu} = 0, \]

and shows that \( G(A) \) is a hom-Lie algebra. \( \square \)

**Remark 2.7.** The hom-Lie algebra \( G(A) := (A, [\cdot, \cdot], \alpha) \) given in the Proposition 2.6 is called the sub-adjacent hom-Lie algebra of \( (A, \alpha) \), and \( (A, \alpha) \) is said to be the compatible hom-center-symmetric algebra structure on the hom-Lie algebra \( G(A) \).

### 3. Bimodule and matched pair of hom-center-symmetric algebras

**Definition 3.1.** Consider a hom-center-symmetric algebra \( (A, \alpha) \), a vector space \( V \), two linear maps \( l, r : A \to gl(V) \), and \( \varphi \in gl(V) \). The quadruple \( (l, r, V, \varphi) \) is called bimodule of the hom-center-symmetric algebra \( (A, \alpha) \) if for all \( x, y \in A \) and \( v \in V \),

\[ \varphi \circ l_x = l_{\alpha(x)} \circ \varphi, \quad r_y \circ \varphi = r_{\alpha(y)} \circ \varphi, \tag{3.1} \]

\[ l_{\alpha(x)} \circ l_y - l_{\alpha(xy)} = r_{\alpha(y)} \circ \varphi - r_{\alpha(x)} \circ r_y, \tag{3.2} \]

\[ l_{\alpha(x)} \circ r_y - r_{\alpha(y)} \circ l_x = l_{\alpha(y)} \circ r_x - r_{\alpha(x)} \circ l_y. \tag{3.3} \]

In the particular case when \( \alpha = id \), the construction of the center-symmetric algebra structure on the semi-direct vector space \( A \oplus V \) is given in [7]. By analogy to this work, we have:
Proposition 3.2. Let \((A, \alpha)\) be a hom-center-symmetric algebra, \(V\) a vector space, \(l, r : A \to gl(V)\) two linear maps, and \(\varphi \in gl(V)\). Then the quadruple \((\varphi, l, r, V)\) is a bimodule of the hom-center-symmetric algebra \((A, \alpha)\) if and only if \((A \oplus V, *, \alpha \oplus \varphi)\) is a hom-center-symmetric algebra, where \(*\) and \(\alpha \oplus \varphi\) are defined as follows: for all \(x, y, z \in A, u, v, w \in V\),
\[
(x + u) * (y + v) = xy + l(x)v + r(y)u,
\]
\[
(\alpha \oplus \varphi)(x + u) = \alpha(x) + \varphi(u).
\]

Proof. For all \(x, y, z \in A\) and \(u, v, w \in V\), using the relations (3.1), we have:
\[
(\alpha \oplus \varphi)((x + u) * (y + v)) = (\alpha \oplus \varphi)(xy + l(x)v + r(y)u) = \alpha(xy) + \varphi(l(x)v) + \varphi(r(y)u) = \alpha(x)\alpha(y) + (\varphi \circ l(x))v + (\varphi \circ r(y))u,
\]
\[
= \alpha(x)\alpha(y) + (l \circ \alpha(x))\varphi(v) + (r \circ \varphi)(y)u = (\alpha(x) + \varphi(u)) * (\alpha(y) + \varphi(v)).
\]

The associator associated to the bilinear product \(*\) gives:
\[
(x + u, y + v, z + w) = ((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w) - (\alpha \oplus \varphi)((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)
\]
\[
= (\alpha \oplus \varphi)(x + u) * ((y + v) * (\alpha \oplus \varphi)(z + w)) - (\alpha \oplus \varphi)((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)
\]
\[
= (\alpha \oplus \varphi)((x + u) * ((y + v) * (\alpha \oplus \varphi)(z + w))) - (\alpha \oplus \varphi)((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)
\]
\[
= (\alpha \oplus \varphi)(((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)) - (\alpha \oplus \varphi)((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)
\]
\[
= (\alpha \oplus \varphi)(((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)) - (\alpha \oplus \varphi)((x + u) * (y + v)) * (\alpha \oplus \varphi)(z + w)
\]
\[
= 0.
\]

By using the relations (3.2) and (3.3), we have:
\[
(x + u, y + v, z + w)_{\alpha \oplus \varphi} = (x, y, z)_{\alpha} - (z, y, x)_{\alpha}
\]
\[
= (l \circ \alpha)(x) = (l \circ \alpha)(y) = (l \circ \alpha)(z)
\]
\[
= (l \circ \alpha)(x) = (l \circ \alpha)(y) = (l \circ \alpha)(z)
\]
\[
= 0.
\]

Therefore, \((\varphi, l, r, V)\) is a bimodule of the hom-center-symmetric algebra \((A, \alpha)\) if and only if the equations (3.1), (3.2), and (3.3) are satisfied.

We denote the hom-center-symmetric algebra \((A \oplus V, *, \alpha \oplus \varphi)\) by \(A \ltimes_{l,r} V\) or simply by \(A \ltimes V\).

Example 3.3. Let \((A, \alpha)\) be a hom-center-symmetric algebra. For all \(x, y, z \in A\), we have:
\[
(x, y, z)_{\alpha} = (z, y, x)_{\alpha} \iff (xy)\alpha(z) - \alpha(x)(yz) = (zy)\alpha(x) - \alpha(z)(yx)
\]
\[
= (L_{x} \circ \alpha \circ L_{y}(z)) = (R_{a} \circ \alpha \circ L_{y})(z)
\]
\[
= (L_{x} \circ \alpha \circ L_{y}) - (R_{a} \circ \alpha \circ L_{y})(z) = 0.
\]

On the other hand, we also have
\[
(x, y, z)_{\alpha} = (z, y, x)_{\alpha} \iff ((L_{x} \circ \alpha \circ L_{y}) - (R_{a} \circ \alpha \circ L_{y})(z) = 0.
\]

From the relations (3.2) and (3.3), where the right hand sides translate that \((\alpha, l, r, A)\) is a bimodule of \((A, \alpha)\).
Definition 3.4. A homomorphism of two hom-center-symmetric algebras \((\mathcal{A}_1, \mu_1, \alpha_1)\) and \((\mathcal{A}_2, \mu_2, \alpha_2)\) is a linear map \(f : \mathcal{A}_1 \to \mathcal{A}_2\) such that:
\[
\begin{align*}
  f \circ \mu_1 &= \mu_2 \circ (f \otimes f), \\
  f \circ \alpha_1 &= \alpha_2 \circ f,
\end{align*}
\] (3.6) (3.7)
as illustrated, respectively, by the following commutative diagrams:

\[
\begin{array}{ccc}
  \mathcal{A}_1 \otimes \mathcal{A}_1 & \xrightarrow{\mu_1} & \mathcal{A}_1 \\
  f \otimes f & \downarrow & \downarrow f \\
  \mathcal{A}_2 \otimes \mathcal{A}_2 & \xrightarrow{\mu_2} & \mathcal{A}_2 \\
\end{array}
\quad
\begin{array}{ccc}
  \mathcal{A}_1 & \xrightarrow{f} & \mathcal{A}_2 \\
  \downarrow \alpha_1 & & \downarrow \alpha_2 \\
  \mathcal{A}_1 & \xrightarrow{f} & \mathcal{A}_2 \\
\end{array}
\]

Lemma 3.5. Let \((\varphi, l, r, V)\) be a bimodule of a hom-center-symmetric algebra \((\mathcal{A}, \alpha)\). Then we have:

- The couple \((\rho = l - r, \varphi)\) is a representation of the sub-adjacent hom-Lie algebra \(\mathcal{G}(\mathcal{A})\) associated to \((\mathcal{A}, \alpha)\);
- For any representation \((\rho, \psi)\), with \(\rho : \mathcal{G}(\mathcal{A}) \to \mathfrak{gl}(\mathcal{A})\), of the underlying hom-Lie algebra \((\mathcal{G}(\mathcal{A}), \alpha)\) of the hom-center-symmetric algebra \((\mathcal{A}, \alpha)\), the triple \((\rho, 0, \psi)\) is a bimodule of \((\mathcal{A}, \alpha)\);
- The hom-center-symmetric algebras \(\mathcal{A} \ltimes^\rho V\) and \(\mathcal{A} \ltimes^0 V\) given by the bimodules \((l, r, \varphi)\) and \((l - r, 0, \varphi)\), respectively, have the same sub-adjacent hom-Lie algebra given by the semidirect sum \(\mathcal{G}(\mathcal{A})\ltimes^\rho V\) of the hom-Lie algebra \(\mathcal{G}(\mathcal{A})\), and its representation \((l - r, \alpha, V)\) as: \([x + u, y + v] = [x, y] + (l - r)(x)v - (l - r)(y)u, \quad \forall x, y \in \mathcal{A}, u, v \in V\).

Proof. Let \((\varphi, l, r, V)\) be a bimodule of a hom-center-symmetric algebra \((\mathcal{A}, \alpha)\). We have for all \(x, y, z, u, v, w \in V\),

\[
(\alpha(x)) \circ \varphi = \{(l - r)(x)\} \circ \varphi = \{l \circ l - \varphi \circ r = \varphi \circ (l - r)(x)\}
\]

Using the fact that \((\rho, \psi)\) is a representation of \(\mathcal{G}(\mathcal{A})\), and setting \(r = 0\), then the relations (3.3) and (3.1) are satisfied, and, in addition, we have, for all \(x, y \in \mathcal{A}\),

\[
\rho([x, y]) \circ \psi = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x)
\]

which implies
\[
\rho(xy) \circ \psi - \rho(\alpha(x)) \circ \rho(y) = \rho(xy) \circ \psi - \rho(\alpha(y)) \circ \rho(x).
\]

This is exactly the equation (3.2).

- The commutator of the bilinear product defined as \((x + u) \ast (y + v) := xy + lx + ry + uv\) is given by \((x + u) \ast (y + v) \ast (x + u) = [x, y] + (l - r)(x)v - (l - r)(y)u = [x + u, y + v]\).

Definition 3.6. Let \((\mathcal{A}, \alpha)\) be a hom-center-symmetric algebra. The dual linear maps \(l^*, r^*\) of the linear maps \(l, r : \mathcal{A} \to \mathfrak{gl}(V)\) are defined, respectively, as:

\[
\begin{align*}
  l^* : \mathcal{A} & \longrightarrow \mathfrak{gl}(V^*) \\
  x & \longmapsto l^*_x \quad \text{and} \quad u^* & \longmapsto l^*_x u^* \\
  V & \longrightarrow V^* \\
  V & \longrightarrow K \\
  (l^*_x u^*, v) & := (u^* (l^*_x v), v)
\end{align*}
\]
\[ r^* : \mathcal{A} \rightarrow \mathfrak{gl}(V^*) \]
\[ V^* \rightarrow V^* \]
\[ x \mapsto r_x^* : u^* \mapsto r_x^* u^* : V \mapsto \mathbb{K} \]
(3.9)

for all \( x \in \mathcal{A}, u^* \in V^*, v \in V \).

**Proposition 3.7.** Let \((\mathcal{A}, \alpha)\) be a hom-center-symmetric algebra, and let \(l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)\) and \(\varphi : V \rightarrow V\), where \(V\) is a finite dimensional vector space, be three linear maps. Then, the following conditions are equivalent:

- \((l, r, \varphi, V)\) is a bimodule of \(\mathcal{A}\).
- \((r^*, l^*, \varphi^*, V^*)\) is a bimodule of \(\mathcal{A}\) and \(\alpha^2 = \text{id}\).

**Proof.**

Consider a hom-center-symmetric algebra \((\mathcal{A}, \alpha)\) and two linear maps \(l^*, r^*\) satisfying the relations (3.8) and (3.9).

- \(\implies\) At first, let’s suppose that \((l, r, V)\) is a bimodule of the hom-center-symmetric algebra \((\mathcal{A}, \alpha)\). For all \(x, y \in \mathcal{A}, v \in V\), and \(u^* \in V^*\), we have:

\[
\langle (\varphi^* \circ r_x^*) u^*, v \rangle = \langle r_x^* u^*, \varphi(v) \rangle = \langle u^*, r_x \varphi(v) \rangle = \langle u^*, \varphi \circ r_{\alpha(x)} v \rangle
\]

and

\[
\langle (\varphi^* \circ l_x^*) u^*, v \rangle = \langle l_x^* u^*, \varphi(v) \rangle = \langle u^*, l_x \varphi(v) \rangle = \langle u^*, \varphi \circ l_{\alpha(x)} v \rangle
\]

Thus,

\[ \varphi^* \circ l_x^* = l_{\alpha(x)}^* \circ \varphi^*, \quad \varphi^* \circ r_x^* = r_{\alpha(x)}^* \circ \varphi^*. \]

Besides,

\[
\langle r_{\alpha(x)}^* r_y^* - r_{xy}^* \circ \varphi^*, u^*, v \rangle = \langle u^*, (r_y r_{\alpha(x)} - \varphi \circ r_{xy}) v \rangle = \langle u^*, (r_{\alpha^2(y)} r_{\alpha(x)} - r_{\alpha(xy)} \circ \varphi) v \rangle
\]

and

\[
\langle l_{\alpha(xy)}^* - l_{\alpha(x)}^* l_{\alpha(y)}^*, u^*, v \rangle = \langle u^*, (\varphi \circ l_{\alpha(xy)} - l_{\alpha(x)} l_{\alpha(y)}) v \rangle
\]

It follows that

\[ r_{\alpha(x)}^* r_y^* - r_{xy}^* \circ \varphi^* = l_y^* \circ \varphi^* - l_{\alpha(x)}^* l_y^*. \]

Setting \(\alpha^2 = \text{id}_\mathcal{A}\) leads to

\[
\langle r_{\alpha(x)}^* l_y^* - l_{\alpha(y)}^* r_x^*, u^*, v \rangle = \langle u^*, (l_y r_{\alpha(x)} - r_x l_{\alpha(y)}) v \rangle = \langle u^*, (l_{\alpha^2(y)} r_{\alpha(x)} - r_{\alpha(xy)} l_{\alpha(y)}) v \rangle
\]

and

\[
\langle l_{\alpha(y)}^* l_x^* - l_{\alpha(x)}^* l_y^*, u^*, v \rangle = \langle u^*, (l_x r_{\alpha(y)} - r_y l_{\alpha(y)}) v \rangle
\]

Then, using \(\alpha^2 = \text{id}_\mathcal{A}\), the relation

\[ r_{\alpha(x)}^* l_y^* - l_{\alpha(y)}^* r_x^* = r_{\alpha(y)}^* l_x^* - l_{\alpha(x)}^* r_y^* \]

is satisfied.

Finally, from the relations (3.10) and (3.11), we conclude that the triple \((r^*, l^*, V^*)\) is a bimodule of the hom-center-symmetric algebra \((\mathcal{A}, \alpha)\).

- \(\Longleftarrow\) Conversely, suppose that \(\alpha^2 = \text{id}_\mathcal{A}\) and \((r^*, l^*, V^*)\) is a bimodule of \(\mathcal{A}\). Then, it is straightforward to check that the triple \((l, r, V)\) is a bimodule of \(\mathcal{A}\).

Therefore, it is true that the triple \((l, r, V)\) is a bimodule of the hom-center-symmetric algebra \((\mathcal{A}, \alpha)\) if, and only if, the triple \((r^*, l^*, V^*)\), where \(r^*, l^*\) are given by the relations (3.8) and (3.9), respectively, is a bimodule of the hom-center-symmetric algebra \((\mathcal{A}, \alpha)\) with \(\alpha^2 = \text{id}_\mathcal{A}\). \(\square\)
Theorem 3.8. Let \((A, \cdot, \alpha_A)\) and \((B, \circ, \alpha_B)\) be two hom-center symmetric algebras. Suppose there are linear maps \(l_A, r_A : A \rightarrow \mathfrak{gl}(B)\) and \(l_B, r_B : B \rightarrow \mathfrak{gl}(A)\) such that \((l_A, r_A, \alpha_B)\) and \((l_B, r_B, \alpha_A)\) are bimodules of the hom-center-symmetric algebra \(A\) and \(B\), respectively, satisfying the following conditions for all \(x, y \in A\) and \(a, b \in B\):

\[
(l_B(a)x) \cdot (\alpha_A(y)) + l_B(r_A(x)a)(\alpha_A(y)) = -l_B(a)(\alpha_B(y))(x \cdot y) - r_B(\alpha_B(y))(y \cdot x)\]

\[
+r_B(y)(\alpha_B(x)) + r_B(l_A(x)a)(\alpha_A(y)) = 0,
\]

\[
(r_B(a)x) \cdot (\alpha_A(y)) + l_B(l_A(x)a)(\alpha_A(y)) = -\alpha_B(a)(x) \cdot (\alpha_A(y)) - (l_B(a)y)(\alpha_A(x)) - (l_B(a)y)(\alpha_A(x))\]

\[-(r_B(a)y)(\alpha_B(x)) - (l_B(a)y)(\alpha_A(x)) - (l_B(a)y)(\alpha_A(x)) = 0,
\]

\[
(l_B(a)x) \circ (\alpha_B(b)) + l_B(r_A(x)a)(\alpha_B(b)) = -l_B(a)(\alpha_B(b))(x \circ y) - r_B(\alpha_B(b))(y \circ x)\]

\[
+r_B(y)(\alpha_B(x)) + r_B(l_A(x)a)(\alpha_B(b)) = 0,
\]

Besides, there exists a hom-center-symmetric algebra structure on the vector space \(A \oplus B\) given by:

\[
(x + a) \circ (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a).
\]

Proof.

From its definition given in (3.16), the product \(*\) is bilinear on \(A \oplus B\), and we want to show that all \(x, y \in A\) and \(a, b \in B\):

\[
(x + a, y + b, z + c)_{\alpha_A \oplus \alpha_B} = (x, y, z)_{\alpha_A} + (x, y, c)_{\alpha_A \circ \alpha_B} + (x, b, z)_{\alpha_A \circ \alpha_B} + (x, b, c)_{\alpha_A \circ \alpha_B} + (a, y, z)_{\alpha_B} + (a, y, c)_{\alpha_B \circ \alpha_B} + (a, b, z)_{\alpha_B \circ \alpha_B} + (a, b, c)_{\alpha_B}.
\]

From the Definition 2.1, we can see that the hom-center-symmetric algebra structures, defined on \((A, \alpha_A)\) and \((B, \alpha_B)\), are \((x, y, z)_{\alpha_A} = (x, y, z)_{\alpha_A}\), and \((a, b, c)_{\alpha_B} = (a, b, c)_{\alpha_B}\), respectively.

Before showing that the remaining associators given in the equation (3.17) are center-symmetric, it is useful to remark that the trilinear identities \((x, y, z)_{\alpha_A} = (a, b, z)_{\alpha_B} = (a, b, z)_{\alpha_B}, \) \((x, y, c)_{\alpha_A \circ \alpha_B}\) play a symmetric role, which means that showing \((x, y, c)_{\alpha_A \circ \alpha_B}\) is center-symmetric is equivalent to showing that \((a, b, z)_{\alpha_B} \circ \alpha_B\) is center-symmetric too. Similarly, proving that \((a, y, z)_{\alpha_A \circ \alpha_B}\) is center-symmetric is equivalent to proving that \((x, y, c)_{\alpha_A \circ \alpha_B}\) is center-symmetric. Provided such an observation, it only remains to show that the associators \((x, y, c)_{\alpha_A \circ \alpha_B}, \) \((x, b, z)_{\alpha_A \circ \alpha_B}, \) \((x, b, c)_{\alpha_A \circ \alpha_B}, \) \((a, y, c)_{\alpha_A \circ \alpha_B}, \) \((a, b, z)_{\alpha_B} \circ \alpha_B, \) and \((a, b, c)_{\alpha_B} \circ \alpha_B\) are center-symmetric.

From the relation (3.10), we have:

\[
(x, y, c)_{\alpha_A \circ \alpha_B} = r_B(\alpha_B(c))(xy) - (\alpha_A(x)) \cdot (r_B(c)y) - r_B(l_A(y)c)(\alpha_A(x))
\]

\[
+ l_A(xy)(\alpha_B(c)) - l_A(\alpha_A(x)l_A(y)c),
\]

\[
(x, b, z)_{\alpha_A \circ \alpha_B} = (r_B(b)x) \cdot (\alpha_A(z)) + l_B(l_A(x)b)(\alpha_A(z)) - (\alpha_A(x)) \cdot (l_B(z))
\]

\[-r_B(r_A(z)b)(\alpha_A(x)) + r_A(\alpha_A(z))(l_A(x)b) - l_A(\alpha_A(x)r_A(z)b),
\]

\[
(x, b, c)_{\alpha_A \circ \alpha_B} = r_B(\alpha_B(c))(r_B(b)x) - r_B(b \circ c)(\alpha_A(x)) + l_A(x)b)(\alpha_B(c)
\]

\[+ l_A(r_B(b)(\alpha_B(c)) - l_A(\alpha_A(x)(bc)),
\]

\[
(a, y, c)_{\alpha_A \circ \alpha_B} = r_B(\alpha_B(c))(l_B(a)y) - l_B(\alpha_B(a))(r_B(c)y) + (r_A(y)a)(\alpha_B(c))
\]

\[+ l_A(l_B(a)y)(\alpha_B(c)) - l_A(\alpha_B(a)l_A(y)c) - r_A(r_B(c)y)(\alpha_B(a)).
\]
Besides,

\[(c, y, x)_{\alpha A \oplus \alpha B} = (l_B(c)y) \cdot (\alpha_A(x)) + l_B(r_A(y)c)(\alpha_A(x)) - l_B(\alpha_B(c))(xy) + r_A(\alpha(x))(l_B(y)c) - r_A(xy)(\alpha_B(c)),\]

\[(z, b, x)_{\alpha A \oplus \alpha B} = (r_B(b)z) \cdot (\alpha_A(x)) + l_B(l_A(z)b)(\alpha_A(x)) - (\alpha_A(z)) \cdot (l_B(x)) - r_B(r_A(x)b)(\alpha_A(z)) + l_B(l_A(z)b)(\alpha_A(x)) - l_B(\alpha_A(z))(r_A(x)b),\]

\[(c, b, x)_{\alpha A \oplus \alpha B} = l_B(c \circ b) \cdot (\alpha_A(x)) - l_B(\alpha_B(c))(l_B(b)x) + r_A(\alpha_A(x))(c \circ b) - \alpha_B(a) \circ (r_A(x)b) - r_A(l_B(b)x)(\alpha_B(c)),\]

\[(c, y, a)_{\alpha A \oplus \alpha B} = r_B(\alpha_B(a))(l_B(c)y) - l_B(\alpha_B(c))(r_B(b)y) + (r_A(y)c)(\alpha_B(a)) + l_A(l_B(c)y)(\alpha_B(a)) - \alpha_B(c)(l_A(y)a) - r_A(r_B(b)y)(\alpha_B(c)).\]

Hence, \((c, y, x)_{\alpha A \oplus \alpha B} = (c, y, x)_{\alpha A \oplus \alpha B}\) is equivalent to the relations \((3.12)\) and \((3.13)\), \((x, b, z)_{\alpha A \oplus \alpha B} = (z, b, x)_{\alpha A \oplus \alpha B}\) to the relations \((3.13)\) and \((3.3)\), \((c, b, c)_{\alpha A \oplus \alpha B} = (c, b, x)_{\alpha A \oplus \alpha B}\) to the relations \((3.14)\) and \((3.2)\), and finally, \((a, y, c)_{\alpha A \oplus \alpha B} = (c, y, a)_{\alpha A \oplus \alpha B}\) to the relations \((3.16)\) and \((3.3)\).

Conversely, if \((A, l_A, \rho_A)\) and \((B, \circ, \alpha_B)\) are two hom-center-symmetric subalgebras of the hom-center-symmetric algebra \((A \oplus B, *, \alpha_A + \alpha_B)\), then, by a direct computation, the linear maps \(l_A, r_A : A \to g(\bar{B})\) and \(l_B, r_B : B \to g(\bar{A})\), given by

\[x \ast B = l_B(xa) + r_B(xa), \quad a \ast x = l_B(a)x + r_A(xa), \quad \forall x \in A, a \in B,\]

satisfy the relations \((3.12)\), \((3.13)\), \((3.14)\) and \((3.15)\). In addition, \((l_A, r_A, B)\) and \((l_B, r_B, A)\) are bimodules of \((A, \alpha_A)\) and \((B, \alpha_B)\), respectively.

We denote the hom-center-symmetric algebra \((A \oplus B, *, \alpha_A \oplus \alpha_B)\) by \((A \bowtie_{l_A, r_A} B, \alpha_A \oplus \alpha_B)\). The octuple \((l_A, r_A, l_B, r_B, \alpha_A, \alpha_B, A, B)\), where \(l_A, r_A, l_B, r_B, \alpha_A\) and \(\alpha_B\) satisfy the conditions \((3.12)\), \((3.13)\), \((3.14)\), and \((3.15)\), is called a matched pair of the hom-center-symmetric algebras \(A\) and \(B\).

**Definition 3.9.** A matched pair of hom-Lie algebras \((G, H, \rho_G, \rho_H, \varphi_G, \varphi_H)\) consists of two hom-Lie algebras \((G, [\_, \_, \varphi_G])\) and \((H, [\_, \_, \varphi_H])\), together with the associated hom-Lie algebra representations \(\rho_G : G \to g(\bar{H})\) and \(\rho_H : H \to g(\bar{G})\) defined with respect to \(\varphi_H\) and \(\varphi_G\), respectively, satisfying the following relations

\[\rho_H(\varphi_H(a))[x, y]_G = [\rho_H(a)[x, \varphi_G(y)]_G + [\varphi_G(x), \rho_H(a)(y)]_G + \rho_H(\rho_G(\varphi_G(x)))(\varphi_G(y)) - \rho_H(\rho_G(\varphi_G(a)))(\varphi_G(y)), \]

\[\rho_G(\varphi_G(x))[a, b]_H = [\rho_G(x)[a, \varphi_H(b)]_H + [\varphi_H(a), \rho_G(x)(b)]_H + \rho_H(\rho_G(\varphi_G(b)))(\varphi_G(a)) - \rho_G(\rho_G(\varphi_G(a)))(\varphi_G(b)). \]

**Corollary 3.10.** Let \((A, B, l_A, r_A, l_B, r_B, \alpha_A, \alpha_B)\) be a matched pair of hom-center-symmetric algebras \((A, \cdot, \alpha_A)\) and \((B, \circ, \alpha_B)\). Then, \((G(A), G(B), l_A - r_A, l_B - r_B, \alpha_A, \alpha_B)\) is a matched pair of hom-Lie algebras \((G(A))_{\alpha_A} (G(B))_{\alpha_B}\).

**Proof.**

From Lemma 3.5, it comes that \((\rho_G(\_)) := l_A - r_A, \alpha_A(B)\) and \((\rho_G(\_)) := l_B - r_B, \alpha_B(A)\) are linear representations of the sub-adjoint Lie algebras \((G(A), \alpha_A)\) and \((G(B), \alpha_B)\), respectively. It only remains to show that the linear maps \((\rho_G = \rho_G(\_), \varphi_G = \alpha_A)\) and \((\rho_H = \rho_G(\_), \varphi_H = \alpha_B)\) satisfy the relations \((3.18)\) and \((3.19)\). In fact, for all \(x, y, z \in A\) and \(a, b, c \in B\), we have:

\[\alpha_A(x) + \alpha_B(a), [y + b, z + c] + [\alpha_A(y) + \alpha_B(b), [z + c, x + a]];\]

\[\alpha_A(z) + \alpha_B(c), [x + a, y + b] = [\alpha_A(x) + \alpha_B(a), [y, z] + (l_B - r_B)(b)(z) - (l_B - r_B)(c)(y)];\]

\[\beta_B(c) + (l_A - r_A)(y)c - (l_A - r_A)(z)b + \alpha_A(y) + \alpha_B(b), [z, x] + l_B - r_B)(c)(x);\]

\[- (l_B - r_B)(a)z + [c, a] + (l_A - r_A)(z)a - (l_A - r_A)(x)c + \alpha_A(z) + \alpha_B(c), [y, x];\]

\[- (l_B - r_B)(b)z + (l_B - r_B)(c)(y) + (l_B - r_B)(\alpha_B(a))(y, z) + (l_B - r_B)(b)(z) - (l_B - r_B)(c)(y);\]

\[- (l_B - r_B)(b)[c, l_A - r_A)(y)c - (l_A - r_A)(z)b(\alpha_A(x)) + \alpha_B(a), [b, c] + (l_A - r_A)(y)c.\]
Theorem 3.11. Let \((A, \cdot, \alpha)\) be a hom-center-symmetric algebra. Suppose there is a hom-center-symmetric algebra \(a \circ \alpha\) on its dual vector space \(A^*\). \((A, A^*, R^*, L^*, R^n, L^n, \alpha, \alpha^*)\) is a matched pair of hom-center-symmetric algebras \((A, \cdot, \alpha)\) and \((A^*, \circ, \alpha^*)\) if and only if the sixtuple \((G(A), G(A^*), -ad^{*} = R^{*} - L^{*}, -ad^{*}_B = R^{*}_B - L^{*}_B, \alpha, \alpha^*)\) is a matched pair of the hom-Lie algebras \(G(A)\) and \(G(A^*)\).

Proof.

By considering Theorem 3.8, setting \(B = A^*\), \(l_A = R^*\), \(r_A = L^*\), \(l_B = R^*_B\), \(r_B = L^*_B\), using Definition 3.9 by assuming that \(G(A) = G, G(A^*) = H, \rho_B = R^*_B - L^*_B, \rho_A = R^*_A - L^*_A, \varphi_B = \alpha, \varphi_A = \alpha^*\), and taking into account the relations (3.8) and (3.9), we get the following equivalences:

- The equation (3.18) is equivalent to both the equations (3.12) and (3.13), i.e. for all \(x, y \in A, a \in A^*\), we have:

\[
- (R^*_B - L^*_B)(a^*(a) \cdot y)_y + [\alpha, (R^*_B - L^*_B)(a)(y)]_y + [\alpha(x), (R^*_B - L^*_B)(a)(y)]_y = 0,
\]

- The equation (3.19) is equivalent to both the equations (3.14) and (3.15), i.e. for all \(a, b, c \in A^*\) and \(x, y \in A\), we have:

\[
- (R^*_B - L^*_B)(a \cdot b)_y + [\alpha, (R^*_B - L^*_B)(a)(b)]_y + [\alpha(a), (R^*_B - L^*_B)(a)(b)]_y = 0.
\]
\[-L^*((L^*_c(a)(a))(\alpha^*))(\alpha^* + b) + (L^*_c(x)(b)) \circ (\alpha^* + a) + (\alpha^* + a) \circ (R^*_c(x)(b))
+ R^*_c((R^*_c(b)(x))(\alpha^* + a)) + L^*_c((L^*_c(b)(x))(\alpha^* + a)) = 0\]

because the two first relations give zero from the equation (3.14), and the last brace yields zero from the equation (3.13).

Therefore, the equivalence is obtained. \[\square\]

4. Manin triple and hom-center-symmetric bialgebras

This section is devoted to the study of hom-center-symmetric bialgebras and to basic properties of the Manin triple of hom-center-symmetric algebras.

**Theorem 4.1.** Consider a hom-Lie algebra \((G,[\cdot],[\cdot],\alpha_\varphi)\), and its two representations \((\rho_u, U, \varphi_u)\) and \((\rho_v, V, \varphi_v)\) on the vector spaces \(U\) and \(V\), respectively. Let \(\rho_u : G \rightarrow \mathfrak{gl}(U)\) \(\rho_v : G \rightarrow \mathfrak{gl}(V)\), \(\varphi_u : U \rightarrow U\) and \(\varphi_v : V \rightarrow V\) be four linear maps such that \((\rho_u, \varphi)\) and \((\rho_v, \varphi_v)\) satisfy the relations (2.7) and (2.8). Then, the linear map \(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u : G \rightarrow \mathfrak{gl}(U \otimes V)\), defined by:

\[(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u)(x)u \otimes v = \rho_u(x)u \otimes \varphi_v + \rho_v \otimes \rho_u(x)v \quad \text{for all } x \in G, \ u \in U \text{ and } v \in V, \]

is a representation of the hom-Lie algebra \((G,[\cdot],[\cdot],\alpha_\varphi)\) on the vector space \(U \otimes V\).

**Proof.**

Let \((G,[\cdot],[\cdot],\alpha_\varphi)\) be a hom-Lie algebra, \((\rho_u, U, \varphi_u)\) and \((\rho_v, V, \varphi_v)\) be two representations of \((G,[\cdot],[\cdot],\alpha_\varphi)\) on \(U\) and \(V\), respectively. For all \(x \in G, \ u \in U\) and \(v \in V\), we have:

\[
(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u)(\alpha_\varphi(x)) = (\rho_u(\alpha_\varphi(x))) \circ (\varphi_v) + (\rho_v \otimes \rho_u)(\alpha_\varphi(x)) = (\rho_u(\alpha_\varphi(x))) \circ (\varphi_v) + (\rho_v \otimes \rho_u)(\alpha_\varphi(x)) \circ (\varphi_v)
\]

In addition, we have

\[
(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u)(\alpha_\varphi(x)) = (\rho_u(\alpha_\varphi(x))) \circ (\varphi_v) + (\rho_v \otimes \rho_u)(\alpha_\varphi(x)) \circ (\varphi_v)
\]

From this, we deduce that the linear map \(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u\) satisfies the relations (2.7) and (2.8), i.e. \(\rho_u \otimes \varphi_v + \rho_v \otimes \rho_u\) is a representation of hom-Lie algebra \((G,[\cdot],[\cdot],\alpha_\varphi)\).
For any linear map \( \phi : U \to V \), where \( U \) and \( V \) are finite dimensional vector spaces, we denote by \( \phi^* \) its dual map defined by:

\[
\phi^* : V^* \to U^* \\
\phi^*(v^*) : U \to \mathbb{K} \\
u \mapsto \langle \phi(u), v^* \rangle,
\]

where \( \langle, \rangle \) is a natural pairing between \( U \) and its dual space \( U^* \).

**Definition 4.2.** Consider a hom-Lie algebra \((G, [,], \alpha)\) and a representation \((\rho, V)\) of \( G \). A 1-hom-cocycle \( \delta : G \to G \) associated to the linear map \( \rho : G \to V \) satisfies the following relation:

\[
\delta(\alpha([x, y])) = \rho(x)\delta(y) - \rho(y)\delta(x), \forall x, y \in G; \quad \forall x, y \in G. \quad (4.1)
\]

**Theorem 4.3.** Let \((\mathcal{A}, \cdot, \alpha)\) be a hom-center-symmetric algebra given by the product \( \beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) such that \( \alpha^2 = \text{id} \). Suppose there is another hom-center-symmetric algebra structure on \( \mathcal{A} \) given by a linear map \( \gamma^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \). Then \( (\mathcal{G}(\mathcal{A}), \alpha^*), (\mathcal{G}(\mathcal{A}^*), \alpha^*) \) is a matched pair of hom-Lie algebras \((\mathcal{G}(\mathcal{A}), \alpha)\) and \((\mathcal{G}(\mathcal{A}^*), \alpha^*)\) if and only if \( \gamma : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) is a 1-hom-cocycle of the hom-Lie algebra \((\mathcal{G}(\mathcal{A}), \alpha)\) associated to \(-\text{ad}^\alpha \otimes \alpha + \alpha^* \otimes \text{ad}_\alpha\). 

**Proof.**

Let \( \{e_1, e_2, \cdots, e_n\} \) be a basis of \( \mathcal{A} \), and \( \{e_1^*, e_2^*, \cdots, e_n^*\} \) its dual basis.

For all \( i, j \in \{1, 2, \cdots, n\} \), consider \( e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k \) and \( e_i^* \circ e_j^* = \sum_{k=0}^n f_{ij}^k e_k^* \), where \( c_{ij}^k \) and \( f_{ij}^k \in \mathbb{K} \) are structure constants associated to the bilinear operations \( \cdot \) and \( \circ \), respectively. Then, for all \( i, j, k, l \in \{1, 2, \cdots, n\} \), we have:

\[
\langle \gamma(e_k), e_i^* \otimes e_j^* \rangle = \langle e_k, \gamma^*(e_i^* \otimes e_j^*) \rangle = \langle e_k, e_i^* \circ e_j^* \rangle = \left\langle e_k, \sum_{l=1}^n f_{ij}^l e_l \right\rangle = \sum_{l=1}^n f_{ij}^l \delta_l^k = f_{ij}^k.
\]

\[
\langle \beta(e_k^*), e_i \otimes e_j \rangle = \langle e_k^*, \beta^*(e_i \otimes e_j) \rangle = \langle e_k^*, e_i \circ e_j \rangle = \left\langle e_k^*, \sum_{l=1}^n c_{ij}^l e_l \right\rangle = \sum_{l=1}^n c_{ij}^l \delta_l^k = c_{ij}^k.
\]

It follows that \( \gamma(e_k) = \sum_{i,j=1}^n f_{ij}^k e_i \otimes e_j \) and \( \beta(e_k^*) = \sum_{i,j=1}^n c_{ij}^k e_i^* \otimes e_j^* \). In addition, let \( \alpha(e_i) = \sum_{k=1}^n d_i^k e_k \) and \( \alpha^*(e_i^*) = \sum_{k=1}^n d_i^k e_k^* \). From the identity \( \alpha^2 = \text{id} \), we get \( \sum_{k=1}^n d_i^k d_i^k e_l = \sum_{l=1}^n d_l e_i = e_i \), with \( d_l^k d_i^k = \delta_l^i \). Besides, we have: \( \langle \alpha^*(e_i^*), e_j \rangle = d_i^j \langle e_i^*, \alpha(e_j) \rangle = d_i^j \) which implies \( d_i^j = d_i^j \).

Furthermore, we also have

\[
\gamma(\alpha([e_i, e_j])) = \sum_{k,l,m,p=1}^n \{ f_{mp}^k (c_{ij}^k - c_{ij}^l) d_k^p \} e_m \otimes e_p. \quad (4.2)
\]

and

\[
\beta(\alpha^*([e_i^*, e_j^*])) = \sum_{k,l,m,p=1}^n \{ c_{mp}^k (f_{ij}^k - f_{ij}^l) d_k^p \} e_m \otimes e_p^*. \quad (4.3)
\]
Besides, we have for all \( i, j \in \{1, 2, \ldots, n\} \):

\[
\{(\text{ad} \ e_i) \otimes \alpha + \alpha \otimes (\text{ad} \ e_j)\} \gamma(e_j) - \{(\text{ad} \ e_j) \otimes \alpha + \alpha \otimes (\text{ad} \ e_i)\} \gamma(e_i)
\]

\[
= \{\text{ad} (e_j \otimes \alpha) \gamma(e_i) - (\text{ad} \ e_i \otimes \alpha) \gamma(e_j)\} + \{(\alpha \otimes \text{ad} \ e_j) \gamma(e_i) - (\alpha \otimes \text{ad} \ e_i) \gamma(e_j)\}
\]

\[
= \sum_{k,l=1}^{n} \{f_{kl}^i(\text{ad} \ e_j \otimes \alpha)e_k \otimes e_l - f_{kl}^j(\text{ad} \ e_i \otimes \alpha)e_k \otimes e_l\}
\]

\[
+ \sum_{l,k=1}^{n} \{f_{lk}^i(\alpha \otimes \text{ad} \ e_j)e_l \otimes e_k - f_{lk}^j(\alpha \otimes \text{ad} \ e_i)e_l \otimes e_k\}
\]

\[
= \sum_{k,l=1}^{n} \{f_{kl}^i[e_j, e_k] \otimes \alpha(e_i)] - f_{kl}^j[e_i, e_k] \otimes \alpha(e_l)]\}
\]

\[
= \sum_{l,k=1}^{n} \{f_{lk}^i(\alpha(e_l) \otimes [e_j, e_k] - f_{lk}^j(\alpha(e_l) \otimes [e_i, e_k])\}
\]

\[
= \sum_{k,l=1}^{n} \left\{ \sum_{m=1}^{n} \left\{ f_{kl}^i(e_{jk}^m - e_{kj}^m) e_m \otimes \alpha(e_i) - f_{kl}^j(e_{jk}^m - e_{kj}^m) e_m \otimes \alpha(e_l) \right\} \right\}
\]

\[
+ \sum_{l,k=1}^{n} \left\{ \sum_{p=1}^{n} \left\{ f_{lk}^i(e_{jk}^p - e_{kj}^p) \alpha(e_l) \otimes e_p - f_{lk}^j(e_{ik}^p - e_{kl}^p) \alpha(e_l) \otimes e_p \right\} \right\}
\]

\[
= \sum_{k,l=1}^{n} \left\{ \sum_{m=1}^{n} \left\{ f_{kl}^i(e_{jk}^m - e_{kj}^m) e_m \otimes \alpha(e_l) \right\} \right\}
\]

\[
+ \sum_{l,k=1}^{n} \left\{ \sum_{p=1}^{n} \left\{ f_{lk}^i(e_{jk}^p - e_{kj}^p) e_p \otimes \alpha(e_l) \right\} \right\}
\]

Therefore,

\[
\{(\text{ad} \ e_i) \otimes \alpha + \alpha \otimes (\text{ad} \ e_i)\} \gamma(e_j) - \{(\text{ad} \ e_j) \otimes \alpha + \alpha \otimes (\text{ad} \ e_i)\} \gamma(e_i)
\]

\[
= \sum_{k,l,m,p=1}^{n} \left\{ \sum_{m=1}^{n} \left\{ f_{kl}^i(e_{jk}^m - e_{kj}^m) e_m \otimes \alpha(e_l) \right\} \right\}
\]

By using the fact that \( \gamma \) is a 1-cocycle associated to the underlying hom-Lie algebra \((\mathcal{G}(\mathcal{A}), \alpha)\) with the representation \(-\text{ad} \ \otimes \alpha + \alpha \otimes \text{ad}\), and taking into account the relation (4.2), we get:

\[
\sum_{k,l,m,p=1}^{n} f_{mp}^i(c_{ij}^k - c_{ji}^k)d_k^p = \sum_{k,l,m,p=1}^{n} \left\{ \sum_{m=1}^{n} \left\{ f_{kl}^i(e_{jk}^m - e_{kj}^m) e_m \otimes \alpha(e_l) \right\} \right\}
\]

\[
+ \left\{ f_{lk}^i(e_{jk}^p - e_{kj}^p) e_p \otimes \alpha(e_l) \right\}
\]

Similarly, we obtain

\[
\{(\text{ad}^* \ e_i^*) \otimes \alpha^* + \alpha^* \otimes (\text{ad}^* \ e_i^*)\} \gamma(e_j^*) - \{(\text{ad}^* \ e_j^*) \otimes \alpha^* + \alpha^* \otimes (\text{ad}^* \ e_i^*)\} \gamma(e_i^*)
\]

\[
= \{(\text{ad}^* \ e_j^* \otimes \alpha^*) \gamma(e_i^*) - (\text{ad}^* \ e_i^* \otimes \alpha^*) \gamma(e_j^*)\} + \{(\alpha^* \otimes \text{ad}^* \ e_j^*) \gamma(e_i^*) - (\alpha^* \otimes \text{ad}^* \ e_i^*) \gamma(e_j^*)\}
\]

\[
= \sum_{k,l=1}^{n} \left\{ c_{kl}^i(\text{ad} \ e_j^* \otimes \alpha^*)e_k^* \otimes e_l^* - c_{kl}^j(\text{ad} \ e_i^* \otimes \alpha^*)e_k^* \otimes e_l^* \right\}
\]

\[
+ \sum_{l,k=1}^{n} \left\{ c_{lk}^i(\alpha^* \otimes \text{ad} \ e_j^*)e_l^* \otimes e_k^* - c_{lk}^j(\alpha^* \otimes \text{ad} \ e_i^*)e_l^* \otimes e_k^* \right\}
\]

\[
= \sum_{k,l=1}^{n} \left\{ c_{kl}^i(e_j^* e_k^* \otimes \alpha^*(e_i^*)) - c_{kl}^j(e_i^* e_k^* \otimes \alpha^*(e_j^*)) \right\}
\]
\[
+ \sum_{l,k=1}^{n} \left\{ f_{lk}^{i}(\alpha^{*}(e_l) \otimes [e_j, e_k]) - c_{lk}^{*}(\alpha^{*}(e_{l}^{*}) \otimes [e_{l}^{*}, e_{k}^{*}]) \right\}
= \sum_{k,l=1}^{n} \left\{ \sum_{m=1}^{n} \left( c_{kl}^{i}(f_{mk}^{m} - f_{lk}^{m})e_{m} \otimes \alpha(e_{l}^{*}) - c_{kl}^{i}(f_{mk}^{m} - f_{lk}^{m})e_{m} \otimes \alpha(e_{l}^{*}) \right) \right\}
+ \sum_{l,k=1}^{n} \left\{ \sum_{p=1}^{n} \left( c_{pk}^{i}(f_{pk}^{p} - f_{pk}^{p})e_{p}^{*} - c_{pk}^{i}(f_{pk}^{p} - f_{pk}^{p})e_{p}^{*} \right) \right\}
= \sum_{k,l=1}^{n} \sum_{m=1}^{n} \left\{ \sum_{p=1}^{n} \left( c_{kl}^{i}(f_{mk}^{m} - f_{lk}^{m}) - c_{kl}^{i}(f_{mk}^{m} - f_{lk}^{m}) \right) d_{l}^{p} e_{m}^{*} \right\}
+ \sum_{m=1}^{n} \left\{ \sum_{p=1}^{n} \left( c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) - c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) \right) d_{l}^{p} e_{p}^{*} \right\}
= \sum_{k,l,m,p=1}^{n} \left\{ \left( c_{lk}^{i}(f_{mk}^{m} - f_{lk}^{m}) - c_{lk}^{i}(f_{mk}^{m} - f_{lk}^{m}) \right) d_{l}^{p} + \left( c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) - c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) \right) d_{l}^{p} \right\} e_{m}^{*} \otimes e_{p}^{*}
\]

Therefore,
\[
\{(- \text{ad}_{e}^{*} e_{l}^{*}) \otimes \alpha^{*} + \alpha^{*} \otimes (- \text{ad}_{e}^{*} e_{l}^{*}) \} \gamma(e_{j}^{*}) = \{(- \text{ad}_{e}^{*} e_{l}^{*}) \otimes \alpha^{*} + \alpha^{*} \otimes (- \text{ad}_{e}^{*} e_{l}^{*}) \} \gamma(e_{j}^{*}) \sum_{k,l,m,p=1}^{n} \left\{ \left( c_{lk}^{i}(f_{mk}^{m} - f_{lk}^{m}) - c_{lk}^{i}(f_{mk}^{m} - f_{lk}^{m}) \right) d_{l}^{p} + \left( c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) - c_{lk}^{i}(f_{pk}^{p} - f_{lk}^{p}) \right) d_{l}^{p} \right\} e_{m}^{*} \otimes e_{p}^{*}
\]

By using the fact that \(\gamma\) is a 1-cocycle associated to the underlying hom-Lie algebra \((G(\mathcal{A}^{*}), \alpha^{*})\) with the representation \(- (\text{ad}_{e} \odot \alpha^{*} + \alpha^{*} \odot \text{ad}_{e})\), and using the relation \((13)\), we have:
\[
\sum_{k,l,m,p=1}^{n} \left\{ \sum_{l} d_{l}^{i}(c_{ik}^{m} - c_{ik}^{m})d_{l}^{p} \right\} = \sum_{k,l,m,p=1}^{n} \left\{ \sum_{l} d_{l}^{i}(c_{ik}^{m} - c_{ik}^{m})d_{l}^{p} \right\} + \left( c_{ik}^{i}(f_{pk}^{p} - f_{lk}^{p}) - c_{ik}^{i}(f_{pk}^{p} - f_{lk}^{p}) \right) d_{l}^{p} e_{m}^{*} \otimes e_{p}^{*}
\]

Besides, for all \(i, j, k \in \{1, 2, \ldots, n\}\), we obtain:
\[
\langle \text{ad}_{e}^{*}(\alpha(e_{i})) e_{j}^{*}, e_{k} \rangle = \langle e_{j}^{*}, [\alpha(e_{i}), e_{k}] \rangle = \sum_{l=1}^{n} d_{l}^{i} \langle e_{j}^{*}, [e_{l}, e_{k}] \rangle = \sum_{l,m,p=1}^{n} d_{l}^{i}(c_{ik}^{m} - c_{ik}^{m}) \langle e_{j}^{*}, e_{p} \rangle
\]
\[
= \sum_{l=1}^{n} d_{l}^{i}(c_{ik}^{m} - c_{ik}^{m}) = \sum_{l,m,p=1}^{n} d_{l}^{i}(c_{ip}^{p} - c_{ip}^{p})e_{p}^{*}, e_{k}
\]

and then we get
\[
\text{ad}_{e}^{*}(\alpha(e_{i})) e_{j}^{*} = \sum_{k,l=1}^{n} d_{l}^{i}(c_{ik}^{m} - c_{ik}^{m})e_{k}^{*}, \quad \text{ad}_{e}^{*}(\alpha^{*}(e_{i})) e_{j} = \sum_{k,l=1}^{n} d_{l}^{i}(f_{lk}^{k} - f_{lk}^{k})e_{l},
\]

\[
\text{ad}_{e}^{*}(e_{i}) e_{j}^{*} = \sum_{k=1}^{n} (c_{ik}^{m} - c_{ik}^{m})e_{k}^{*}, \quad (4.5)
\]

and
\[
\text{ad}_{e}^{*}(e_{i}) e_{j} = \sum_{k=1}^{n} (f_{lk}^{k} - f_{lk}^{k})e_{k}.
\]

(4.6)

In addition, we also have
\[
\text{ad}_{e}^{*}(\alpha^{*}(e_{i})) e_{k} e_{j} = \sum_{l,m,p=1}^{n} d_{l}^{i}(c_{jk}^{m} - c_{jk}^{m})(f_{pm}^{m} - f_{mp}^{m})e_{m}
\]
\[
\text{ad}_{e}^{*}(\alpha^{*}(e_{i})) e_{k} e_{j} = \sum_{l,m,p=1}^{n} d_{l}^{i}(c_{lk}^{m} - c_{lk}^{m})(f_{pm}^{m} - f_{mp}^{m})e_{m}
\]

(4.7)
\[
\text{ad}^*(\alpha(e_i))|e_k^*, e_j^*| = \sum_{l,m,p=1}^n d_p^l (f_{kj}^l - f_{jk}^l)(c_{pm}^l - c_{mp}^l)e_m^*
\]
\[
\text{ad}^*(\alpha(e_i))|e_k^*, e_j^*| = \sum_{l,m,p=1}^n d_p^l (f_{kj}^l - f_{jk}^l)(c_{pm}^l - c_{mp}^l)e_m^* \quad (4.8)
\]

Then, we find:
\[
\text{ad}^*_r(\text{ad}^*(e_k) e_i^*)\alpha(e_j) = \text{ad}^*_r(\text{ad}^*(e_k) e_i^*) - \text{ad}^*_r(\text{ad}^*_r(e_k) e_i^*)\alpha(e_k) - [\text{ad}^*_r(e_i^*) e_j, \alpha(e_k)] - [\alpha(e_j), \text{ad}^*_r(e_i^*) e_k]
\]
\[
+ \text{ad}^*_r(e_k) e_i^* \cdot \alpha(e_j) - \alpha(e_j) \cdot (\text{ad}^*_r(e_i^*) e_k)
\]
\[
+ \{\alpha(e_k) \cdot (\text{ad}^*_r(e_i^*) e_j) \text{ad}^*_r(e_i^*) \alpha(e_k) - (\text{ad}^*_r(e_i^*) e_j) \cdot \alpha(e_j)\}
\]
\[
= \sum_{i=1}^n \{d_j^l \{\text{ad}^*_r(e_k) e_i^*\} e_l - e_l \cdot (\text{ad}^*_r(e_i^*) e_k) + (\text{ad}^*_r(e_i^*) e_k) \cdot e_l\}
\]
\[
+ \sum_{i=1}^n \{d_j^l\{c^i_{kr} - c^i_{jk}\} \text{ad}^*_r(e_i^*) e_l - (f^l_{ir} - f^k_{ir})(e_l \cdot e_r) + (f^l_{ir} - f^k_{ir})(e_r \cdot e_l)\}
\]
\[
+ \sum_{i=1}^n \{d_j^l\{f^i_{ir} - f^i_{ir}\}(e_l \cdot e_r) - (c^i_{jr} - c^i_{ir}) \text{ad}^*_r(e_i^*) e_l - (f^i_{ir} - f^i_{ir})(e_r \cdot e_l)\}
\]
\[
= \sum_{l,r,m=1}^n d_k^l\{c^i_{kr} - c^i_{jr}\}(f^l_{ir} - f^l_{ir})(c^m_{mi} - c^m_{mi}) + d_j^l(f^i_{ir} - f^i_{ir})(c^m_{mi} - c^m_{mi}) + d_k^l(f^l_{ir} - f^l_{ir})(c^m_{mi} - c^m_{mi})
\]
\[
+ d_k^l(c^i_{ir} - c^i_{jr})(f^l_{ir} - f^l_{ir})\}
\]

and the following equalities hold:
\[
\sum_{l,m,p=1}^n d_p^l(c^i_{kj} - c^i_{jk})(f^l_{pm} - f^l_{mp}) = \sum_{l,r,m=1}^n \{d_j^l(c^i_{kr} - c^i_{jk})(f^l_{ir} - f^l_{ir})\}
\]
\[
+ d_j^l(f^l_{ir} - f^l_{ir})(c^m_{mi} - c^m_{mi}) + d_k^l(f^l_{ir} - f^l_{ir})(c^m_{mi} - c^m_{mi}) + d_k^l(c^i_{ir} - c^i_{jr})(f^l_{ir} - f^l_{ir})\}
\]
\[
= \sum_{l,r,m=1}^n \{d_j^l(c^i_{kr} - c^i_{jk})(f^l_{ir} + d_j^l f^k_{ir}(c^m_{mi} - c^m_{mi}) + f^k_{ir}(c^m_{mi} - c^m_{mi}) + c^i_{jr})(f^l_{ir} - f^l_{ir})\}
\]
\[
- \sum_{l,r,m=1}^n \{d_j^l(c^i_{kr} - c^i_{jr})(f^l_{ir} + d_j^l f^k_{ir}(c^m_{mi} - c^m_{mi}) + f^k_{ir}(c^m_{mi} - c^m_{mi}) + c^i_{jr})(f^l_{ir} - f^l_{ir})\}
\]
\[
= \sum_{l,m,p=1}^n d_p(c^i_{kj} - c^i_{jk})(f^l_{pm} - f^l_{mp}) - \sum_{l,r,m=1}^n \{d_j^l(c^i_{kr} - c^i_{jk})(f^l_{ir} + d_j^l f^k_{ir}(c^m_{mi} - c^m_{mi}) + f^k_{ir}(c^m_{mi} - c^m_{mi}) + c^i_{jr})(f^l_{ir} - f^l_{ir})\}
\]
\[
+ d_k^l(c^i_{ir} - c^i_{jr})(f^l_{ir})(f^l_{ir} - f^l_{ir})\}
\]

Therefore, we obtain
\[
\sum_{l,m,p=1}^n d_p^l(c^i_{kj} - c^i_{jk})(f^l_{pm}) = \sum_{l,r,m=1}^n d_j^l(c^i_{kr} - c^i_{jr})(f^l_{ir}) + d_j^l f^k_{ir}(c^m_{mi} - c^m_{mi})
\]
\[
+ d_k^l(c^i_{ir} - c^i_{jr})(f^l_{ir})\]
Similarly, we compute:
\[
\begin{align*}
\text{ad}^*(d^*_k) 
&= \sum_{l,m=1}^n d^l_k(f^l_{kr} - f^l_{rk})(c^l_{rm} - c^l_{mr}) + d^l_j(c^l_{ir} - c^l_{ri})(f^m_{rl} - f^m_{lr}) + d^l_k(c^l_{ir} - c^l_{ri})(f^m_{rl} - f^m_{lr}) \\
&\quad + d^l_k(f^l_{jr} - f^l_{jr})(c^l_{rm} - c^l_{mr}) \\
&= \sum_{l,m=1}^n d^l_j(f^l_{kr} - f^l_{rk})c^l_{rm} + d^l_j c^l_{ir}(f^m_{rl} - f^m_{lr}) + d^l_k(c^l_{ir} - c^l_{ri})(f^m_{rl} - f^m_{lr}) + d^l_k(f^l_{jr} - f^l_{jr})c^l_{rm} \\
&\quad - \sum_{l,m=1}^n \{d^l_j(f^l_{kr} - f^l_{rk})c^l_{mr} + d^l_j c^l_{ir}(f^m_{rl} - f^m_{lr}) + d^l_k(c^l_{ir} - c^l_{ri})(f^m_{rl} - f^m_{lr}) + d^l_k(f^l_{jr} - f^l_{jr})c^l_{mr}\} \\
&\quad + d^l_k(f^l_{jr} - f^l_{jr})c^l_{rm}.
\end{align*}
\]

Therefore, we get
\[
\sum_{l,m,p=1}^n d^l_j(f^l_{kr} - f^l_{rk})c^l_{pm} = \sum_{l,m=1}^n d^l_j(f^l_{kr} - f^l_{rk})c^l_{rm} + d^l_j c^l_{ir}(f^m_{rl} - f^m_{lr}).
\]

By adding the relation \(\alpha^2 = \text{id}\), we obtain that (4.9) is equivalent to (4.10), and using \(\alpha^* = \text{id}\), the equation (4.10) is equivalent to the equation (4.9). \(\square\)

**Definition 4.4.** Let \((A, \alpha)\) be a vector space. A hom-center-symmetric bialgebra structure on \(A\) is a pair of linear map \((\gamma, \beta)\) with \(\alpha \in \mathfrak{gl}(A)\), \(\gamma : A \to A \otimes A\), \(\beta : A^* \to A^* \otimes A^*\) and

- \(\gamma^* : A^* \otimes A^* \to A^*\) is a hom-center-symmetric algebra structure on \(A^*\),
- \(\beta^* : A \otimes A \to A\) is a hom-center symmetric algebra structure on \(A\),
- \(\gamma\) is a 1-hom-cocycle of \(G(A)\) associated to \(- (\text{ad} \otimes \alpha + \alpha \otimes \text{ad})\) with values in \(A \otimes A\),
- \(\beta\) is a 1-hom-cocycle of \(G(A^*)\) associated to \(- (\text{ad}_0 \otimes \alpha^* + \alpha^* \otimes \text{ad}_0)\) with values in \(A^* \otimes A^*\).

We denote this hom-center-symmetric bialgebra by \((A, A^*, \gamma, \beta, \alpha)\).

**Definition 4.5.** A Manin triple of hom-center-symmetric algebras \((A, \cdot, \alpha)\) and \((B, *, \alpha')\) is a triple \((A \oplus B, A, B)\), together with a nondegenerate symmetric bilinear form \(B(\cdot, \cdot)\) on the hom-center-symmetric algebra \((A \oplus B, *, \alpha \oplus \alpha')\) such that

- \(B\) is invariant, i.e., for all \(x, y, z \in A\) and \(a, b, c \in B\),
  \[
  B((x + a) \star (y + b), (z + c)) = B((x + a), (y + b) \star (z + c)),
  \]
  \[
  B((x + (\alpha \oplus \alpha'))(a + y), b) = B(x + (\alpha \oplus \alpha')(a + y), b).
  \]
- The hom-center-symmetric algebras \(A\) and \(B\) are isotropic hom-center-symmetric algebras of \(A \oplus B\).

In particular, if \((A, \cdot, \alpha)\) is a hom-center-symmetric algebra, and if there exists a hom-center-symmetric algebra structure on its dual space \(A^*\) denoted by \((A^*, \circ, \alpha^*)\), then there is a hom-center-symmetric algebra on the direct sum of the underlying vector space of \(A\) and its dual space \(A^*\) such that \((A \oplus A^*, \cdot, \circ, \alpha \oplus \alpha^*)\) is the associated Manin triple with the invariant bilinear symmetric form given by \(B_A(x + a, y + b) = \langle x, b \rangle + \langle y, a \rangle\) for all \(x, y \in A\) and \(a, b \in A^*\). It is called the standard Manin triple of the hom-center-symmetric algebra \(A\); \(\langle, \rangle\) is a natural paring between algebra and its dual space.

**Proposition 4.6.** Let \((A, \cdot, \alpha)\) and \((A^*, \circ, \alpha^*)\) be two hom-center-symmetric algebras. Then, \((A, A^*, R^*, L^2, R^2, L^2, \alpha, \alpha^*)\) is a matched pair of the hom-center-symmetric algebras \((A, \cdot, \alpha)\) and \((A^*, \circ, \alpha^*)\) if and only if \((A \oplus A^*, A, A^*)\) is a standard Manin triple.
Proof.
Let us compute and compare the following relations: \( \mathcal{B}_A((x + a) * (y + b), (z + c)) \) and \( \mathcal{B}_A((x + a), (y + b) * (z + c)) \) for all \( x, y, z \in A \) and \( \forall a, b, c \in A^* \).

\[
\begin{align*}
\mathcal{B}_A((x + a) * (y + b), (z + c)) &= \mathcal{B}_A(xy + R^*_c(a)y + L^*_c(b)x + a \circ b + R^*_x(y)b + L^*_x(y)a, z + c) \\
&= \langle xy + R^*_c(a)y + L^*_c(b)x, c \rangle + \langle z, a \circ b + R^*_x(y)b + L^*_x(y)a \rangle \\
&= \langle xy, c \rangle + \langle R^*_c(a)y, c \rangle + \langle L^*_c(b)x, c \rangle + \langle z, a \circ b \rangle + \langle z, R^*_x(y)b \rangle \\
&+ \langle z, L^*_x(y)a \rangle = \langle xy, c \rangle + \langle y, R_a(c) \rangle + \langle x, L_0(c) \rangle + \langle z, a \circ b \rangle \\
&+ \langle R_x(z), b \rangle + \langle L_y(z), a \rangle \\
&= \langle xy, c \rangle + \langle y, c \circ a \rangle + \langle x, b \circ c \rangle + \langle z, a \circ b \rangle + \langle zx, b \rangle + \langle yz, a \rangle.
\end{align*}
\]

It follows that

\[
\mathcal{B}_A((x + a) * (y + b), (z + c)) = \mathcal{B}_A((x + a), (y + b) * (z + c))
\]

which expresses the invariance of the standard bilinear form on \( A \oplus A^* \). Therefore, \( (A \oplus A^*, A, A^*) \) is the standard Manin triple of the center-symmetric algebras \( A \) and \( A^* \).

**Theorem 4.7.** Let \( (A, \cdot, \alpha) \) be a hom-center-symmetric algebra and let \( (A^*, \cdot, \alpha^*) \) be a hom-center-symmetric algebra structure on its dual space \( A^* \). Then the following conditions are equivalent:

(i) \( (A \oplus A^*, A, A^*) \) is the standard Manin triple of the hom-center-symmetric algebras \( (A, \cdot, \alpha) \) and \( (A^*, \cdot, \alpha^*) \);
(ii) \( (A, A^*, R^*, L^*, R^*_c, L^*_c, \alpha, \alpha^*) \) is a matched pair of the hom-center-symmetric algebras \( (A, \cdot, \alpha) \) and \( (A^*, \cdot, \alpha^*) \);
(iii) \( (\mathcal{G}(A), \mathcal{G}(A^*), -\text{ad}^*, -\text{ad}_c^*, \alpha, \alpha^*) \) is a matched pair of the sub-adjacent Lie algebras \( (\mathcal{G}(A), \cdot, \alpha) \) and \( (\mathcal{G}(A^*), \cdot, \alpha^*) \);
(iv) \( (A, A^*) \) is a center-symmetric bialgebra.

**Proof.**
By the Proposition 4.6 (i) \( \iff \) (ii). From Theorem 4.11 (ii) \( \iff \) (iii). According to Theorem 4.3 (iii) \( \iff \) (iv).

**Definition 4.8.** Let \( (A, A^*, \gamma_A, \beta_A, \alpha_1) \) and \( (B, B^*, \gamma_B, \beta_B, \alpha_2) \) be two hom-center-symmetric bialgebras. A homomorphism of a hom-center-symmetric bialgebra \( f : A \rightarrow B \) is a homomorphism of a hom-center-symmetric algebra such that \( f^* : B^* \rightarrow A^* \) is also a homomorphism of a hom-center-symmetric algebra, that is for all \( x \in A \), and \( a^* \in B^* \), \( f \) satisfies

\[
(f \otimes f)(\gamma_A(x)) = \gamma_B(f(x)), \quad f\alpha_1 = \alpha_2 f, \quad \alpha_2^* f = f\alpha_1^*, \quad (f^* \otimes f^*)(\beta_B(a^*)) = \beta_A(f^*(a^*)).
\]

**5. Concluding remarks**

In this work, we have constructed the hom-center-symmetric algebras, and discussed their main properties. We have defined their bimodules and matched pairs. Furthermore, we have established the dual bimodules of bimodules of hom-center-symmetric algebras, and linked them to the matched pairs of hom-Lie algebras. Besides, we have derived the Manin triple of hom-center-symmetric algebras. Finally, we have provided a theorem linking hom-center-symmetric algebras, associated matched pairs, hom-center-symmetric bialgebras, and matched pairs of sub-adjacent hom-Lie algebras.
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