CONSTRUCTING TREE-DECOMPOSITIONS THAT DISPLAY ALL TOPOLOGICAL ENDS

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Abstract. We give a short, topological proof that all graphs admit tree-decompositions displaying their topological ends.

1. Introduction

Historically, one of the strongest driving force behind investigating tree structure of infinite graphs has been Halin’s end-faithful spanning tree conjecture from the 1960’s [6], until it was refuted in the early 1990’s independently by Seymour & Thomas [8] and by Thomassen [9].

However, in a 50-page breakthrough from 2014 (published in 2019), Carmesin [2] proved a general theorem how to separate ends in graphs that generalised corresponding results from Cayley graphs of finitely generated groups, and applied it to establish that all graphs admit tree-decompositions displaying their topological ends. The latter result also implies that every connected graph has a spanning tree that is end-faithful for all topological ends. Together, these results settled a problem of Diestel from 1992 [3] and gave the first satisfying answer to the above conjecture of Halin’s in amended form.

The purpose of this note is to offer a short proof for both of Carmesin’s applications that is motivated by topological instead of algebraic considerations. It is based on a technique that we call enveloping a given set of vertices, which enables one to expand any set of vertices without changing the number of ends in its closure, and which we expect to have further applications in the study of tree-decompositions and end structure of infinite graphs.

2. Background on ends and topological ends

2.1. Ends. For graph theoretic terms we follow [4], and in particular [4, Chapter 8] for ends of graphs. A 1-way infinite path is called a ray and the subrays of a ray are its tails. Two rays in a graph $G = (V, E)$ are equivalent if no finite set of vertices separates them; the corresponding equivalence classes of rays are the ends of $G$. If $X \subseteq V$ is finite and $\varepsilon$ is an end, there is a unique component of $G - X$ that contains a tail of every ray in $\varepsilon$, which we denote by $C(X, \varepsilon)$. Then $\varepsilon$ lives in the component $C(X, \varepsilon)$.

An end $\varepsilon$ of $G$ is contained in the closure of $M$, where $M$ is either a subgraph of $G$ or a set of vertices of $G$, if for every finite vertex set $X \subseteq V$ the component $C(X, \varepsilon)$ meets $M$. We write $\partial M$ for the set of ends of $G$ lying in the closure of $M$.

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2.2. **Star-comb-lemma.** A *comb* is the union of a ray $R$ (the comb’s spine) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on $R$. The last vertices of those paths are the *teeth* of this comb. Given a vertex set $U$, a *comb attached to $U$* is a comb with all its teeth in $U$, and a *star attached to $U$* is a subdivided infinite star with all its leaves in $U$.

**Lemma 2.1** (Star-comb lemma). Let $U$ be an infinite set of vertices in a connected graph $G$. Then $G$ contains either a comb attached to $U$ or a star attached to $U$.

2.3. **Dominated ends.** A vertex $v$ *dominates* an end $\varepsilon$ if there is a star with center $v$ attached to some (equivalently: any) ray of $\varepsilon$. In this case, we say that $\varepsilon$ is *dominated*. Diestel and Kühn proved that the undominated ends of a graph $G$ correspond precisely to the topological ends in the sense of Freudenthal [5], and hence the undominated ends are also called *topological* ends of a graph.

We will need the following standard result isolated from the proof of [5, Theorem 2.2]:

**Lemma 2.2.** Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of disjoint finite sets of vertices in a graph $G$ and suppose $C_n$ are connected components of $G - X_n$ such that $C_n \supseteq C_{n+1} \cup X_{n+1}$ for all $n \in \mathbb{N}$. Then there is a unique end $\varepsilon$ of $G$ that lives in all $C_n$, and this end is undominated.

We will also need the following routine result relating the closure operator to dominated ends.

**Lemma 2.3.** Let $W$ be a finite set of vertices in a graph $G$, and $U = N(W)$ its neighbourhood. Then $\partial U$ consists of precisely those ends of $G$ that are dominated by a vertex in $W$.

**Proof.** This follows from the well-known fact that $\varepsilon \in \partial U$ if and only if there is a comb attached to $U$ with spine in $\varepsilon$. \qed

2.4. **End-faithful spanning trees and tree-decompositions.** A rooted spanning tree $T$ of a graph $G$ is *end-faithful* for a set $\Psi$ of ends of $G$ if for each end $\varepsilon \in \Psi$ there is a unique rooted ray $R$ in $T$ with $R \in \varepsilon$.

A *tree-decomposition* of a graph $G$ is a pair $T = (T, \mathcal{V})$ where $T$ is a tree and $\mathcal{V} = (V_t : t \in T)$ is a family of vertex sets of $G$ called *parts* such that the following hold (see also [4, §12.3]):

- (T1) for every vertex $v$ of $G$ there exists $t \in T$ such that $v \in V_t$;
- (T2) for every edge $e$ of $G$ there exists $t \in T$ such that $e \in G[V_t]$; and
- (T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_2$ lies on the $t_1 - t_3$ path in $T$.

Let $e = xy$ be any edge of $T$ and let $T_x$ and $T_y$ be the two components of $T - e$ with $x \in T_x$ and $y \in T_y$. Each edge $e = xy$ of $T$ in a tree-decomposition gives rise to a separator $X_e := V_x \cap V_y$ called the separator *induced by the edge $e$*, which separates $A_x = \bigcup_{t \in T_x} V_t$ from $A_y = \bigcup_{t \in T_y} V_t$. The tree-decomposition has *finite adhesion* if all separators of $G$ induced by the edges of $T$ are finite.

Given a tree-decomposition $T = (T, \mathcal{V})$ of finite adhesion of $G$, any end $\varepsilon$ of $G$ orients each edge $e = xy$ of $T$ according to whether $\varepsilon$ lives in a component of $G[A_x] - X_e$ or $G[A_y] - X_e$. This orientation of $T$ points towards a node of $T$ or to an end of $T$, and $\varepsilon$ *lives* in that part for that node or that end, respectively.
Then $T$ displays a set $Ψ$ of ends of $G$ if in every end of $T$ there lives a unique end and it is in $Ψ$, and conversely every end of $Ψ$ lives in some end of $T$.

Finally, a rooted tree-decomposition is $T = (T, V)$ where the decomposition tree $T$ is rooted. A rooted tree-decomposition is said to have upwards disjoint separators if the induced separators $X_e$ and $X_{e'}$ for any two distinct edges $e < e'$ comparable in the tree order of $T$ are disjoint.

2.5. Rooted trees containing a set of vertices cofinally. Recall that a subset $X$ of a poset $P = (P, \leq)$ is cofinal in $P$, and $\leq$, if for every $p \in P$ there is an $x \in X$ with $x \geq p$. We say that a rooted tree $T \subseteq G$ contains a set $U$ cofinally if $U \subseteq V(T)$ and $U$ is cofinal in the tree order of $T$. The main assertion of the following is an immediate corollary of [1, Lemma 2.13].

Lemma 2.4. Let $G$ be any graph, and let $U \subseteq V(G)$ be a set of vertices. If $T \subseteq G$ is a rooted tree that contains $U$ cofinally, then $\partial T = \partial U$.

3. Envelopes for sets of vertices

Let $G$ be a connected graph. Given a subgraph $C \subseteq G$, write $N(C)$ for the set of vertices in $G - C$ with a neighbour in $C$. An adhesion set of a set of vertices or a subgraph $U \subseteq G$ is any subset of the form $N(C)$ for a component $C$ of $G - U$. The set or subgraph $U$ is said to have finite adhesion in $G$ if all its adhesion sets are finite.

An envelope for a set of vertices $U \subseteq V(G)$ is a set of vertices $U^* \supseteq U$ of finite adhesion such that $\partial U^* = \partial U$. The following theorem has been developed by Kurkofka and the author in [7, Theorem 3.1]. We take the opportunity here to present a somewhat different proof.

Theorem 3.1. Any set of vertices in a connected graph admits a connected envelope.

Proof. We use the following concepts. Let $W$ be any set of vertices. An external comb attached to $W$ is the union of a ray $R$ that avoids $W$ together with infinitely many disjoint $R$-$W$ paths. The last vertices of those paths in $W$ form the attachment set of this external comb. An external star attached to $W$ is a subdivided infinite star with precisely its leaves in $W$. Its set of leaves is its attachment set. The interior of an external star or comb attached to $W$ is obtained from it by deleting $W$. We call a collection of external stars and combs attached to $W$ internally disjoint if all its elements have pairwise disjoint interior.

We recursively construct a sequence $(U_i; i < \omega_1)$ of sets of vertices in $G$ as follows. Let $T \subseteq G$ be a rooted tree that includes $U$ cofinally, and put $U_0 := V(T)$. If $U_i$ is already defined, we use Zorn’s lemma to choose a maximal collection $\mathcal{C}_i$ consisting of internally disjoint external stars and combs in $G$ attached to $U_i$, and let $U_{i+1} := U_i \cup V[\bigcup \mathcal{C}_i]$. For limits $\ell < \omega_1$ we define $U_\ell := \bigcup_{i < \ell} U_i$. We claim that $U^* := \bigcup_{i < \omega_1} U_i$ is a connected envelope for $U$.

First, $U_0$ is connected, and it follows by induction on $i$ that every $U_i$ is connected, too. Hence, so is $U^*$. Similarly, $\partial U_0 = \partial U$ by choice of $T$ and Lemma 2.4, and it follows once again by induction on $i$ that $\partial U_i = \partial U$ for every $i < \omega_1$. Indeed, consider an end $\varepsilon \notin \partial U_0$. Then there is a finite set of vertices $X$
such that $C(X, \varepsilon)$ avoids $U_0$. But then throughout the whole process, we will attach at most $|X|$ external
combs or stars that intersect $C(X, \varepsilon)$, as every one also has to intersect $X$ internally. Since every such star
or comb will intersect $C(X, \varepsilon)$ finitely, also $U^*$ intersects $C(X, \varepsilon)$ finitely, witnessing $\varepsilon \notin \partial U^*$. This gives
$\partial U^* = \partial U$ as desired.

To see that $U^*$ has finite adhesion, suppose for a contradiction that there is a component $C$ of $G - U^*$
with infinite neighbourhood. Then by a routine application of the star-comb lemma (Lemma 2.1), we find
either an external star or comb attached to $U^*$ whose interior is completely contained in $C$. Its countable
attachment set, however, already belongs to some $U_i$ with $i < \omega_1$ (for $\omega_1$ has uncountable cofinality). But
then the existence of this external star or comb contradicts the maximality of $\mathcal{C}_i$. \hfill \Box

4. TREE-DECOMPOSITIONS THAT DISPLAYS ALL TOPOLOGICAL ENDS

**Lemma 4.1.** Every connected graph $G$ admits a sequence of induced connected subgraphs $H_0 \subseteq H_1 \subseteq \cdots$
in $G$ all of finite adhesion such that

(i) $N(H_n) \subseteq H_{n+1}$ for all $n \in \mathbb{N}$ (implying $G = \bigcup_{n \in \mathbb{N}} H_n$),

(ii) for all $n \in \mathbb{N}$, every topological end lives in a unique component of $G - H_n$, and

(iii) for all $n \in \mathbb{N}$ and every component $C$ of $G - H_n$, the set $C \cap H_{n+1}$ is connected.

**Proof.** Let $H_0$ consist of some arbitrarily chosen singleton. If $H_n$ is already defined, consider some com-
ponent $C$ of $G - H_n$. Let $W = N(C)$, and let $U_C$ be the neighbourhood of $W$ in $C$. Use Theorem 3.1
inside $C$ to find a connected envelope $U_C^*$ of $U_C$. Define $H_{n+1}$ to consist of $H_n$ together with all connected
envelopes $U_C^*$ for all components $C$ of $G - H_n$. Then $H_{n+1}$ has finite adhesion.

Now properties (i) and (iii) hold by construction. For (ii), consider a topological end $\varepsilon$ of $G$. By induction
assumption, $\varepsilon$ lives in a unique component $C$ of $G - H_n$. By Lemma 2.3, $\varepsilon$ does not belong to $\partial U_C$, and
hence also not to $\partial U_C^*$. By finite adhesion of $U_C^*$, there is a unique component $C'$ of $C - U_C^*$ in which $\varepsilon$
lives. By definition of $H_{n+1}$, this component $C'$ is also a component of $G - H_{n+1}$ as desired. \hfill \Box

**Theorem 4.2.** Every connected graph has a rooted tree-decomposition, of finite adhesion and into connected
parts with upwards disjoint separators, that displays all topological ends.

**Proof.** The sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ from Lemma 4.1 gives rise to a tree decomposition $(T, \mathcal{V})$ of
finite adhesion and into connected parts as follows: Write $\mathcal{C}_n$ for the collection of components of $G - H_n$.
The reverse inclusion relation ‘⊇’ defines a tree order on the set $T := \{G\} \bigcup \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ with root $G$; this will
be our decomposition tree. The part corresponding to the root of $T$ will be $H_0$. The part corresponding
to a node $C \in \mathcal{C}_n$ of $T$ will be $N(C) \bigcup (C \cap H_{n+1})$, which is connected by (i) and (iii). Then it is readily
checked that all properties (T1) – (T3) of a tree-decomposition are implied by (i), as is the property that
this tree-decomposition has upwards disjoint separators.

To see that $(T, \mathcal{V})$ displays all topological ends, observe first that every rooted ray $t_0 t_1 t_2 t_3, \ldots$ in $T$ gives
rise to a nested sequence of non-empty components $C_{t_1} \supseteq C_{t_2} \supseteq \cdots$ with $C_{t_n} \in \mathcal{C}_n$ for $n \in \mathbb{N}$, such that
$C_n \supseteq C_{n+1} \cup N(C_{n+1})$ for all $n \in \mathbb{N}$ by property (i). Hence by Lemma 2.2 there is a unique end that lives in all $C_n$, which is undominated. Conversely, every topological end of $G$ lives in a unique connected component $C_n$ of $G - H_n$ by (ii), and so $t_1t_2t_3\ldots$ is a ray in $T$ corresponding to this end. □

**Theorem 4.3.** Every connected graph admits a spanning tree that is end-faithful for the topological ends.

**Proof.** Given the sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ from Lemma 4.1 we construct rooted trees $T_0 \subseteq T_1 \subseteq \cdots$ such that $T_n$ is a spanning tree of $H_n$ as follows: Let $T_0$ be a rooted spanning tree of $H_0$. If $T_n$ is already defined, extend $T_n$ to a spanning tree of $H_{n+1}$ by choosing, for every component $C$ of $G - H_n$ a spanning tree of $C \cap H_{n+1}$ by (iii), and attaching it to $T_n$ via a single edge $e_C$.

We show that $T = \bigcup_{n \in \mathbb{N}} T_n$ is a spanning tree of $G$ that is end-faithful for the topological ends of $G$. First, every topological end $\varepsilon$ of $G$ lives, for all $n$, in a unique connected component $C_n$ of $G - H_n$ by (ii). Since $C_{n+1} \cup N(C_{n+1}) \subseteq C_n$ by (i), the edges $e_{C_n}$ lie on a rooted ray $R$ of $T$, which satisfies $R \in \varepsilon$. To see that $R$ is the unique ray of $T$ which belongs to $\varepsilon$, suppose for a contradiction there was another rooted ray $R' \subseteq T$ belonging to $\varepsilon$. Then $R$ and $R'$ are eventually disjoint, so choose $n \in \mathbb{N}$ such that $R \cap R' \subseteq H_n$. Since $e_{C_n} \in E(R) \setminus E(R')$, it follows that $R$ eventually belongs to $C_n$ while $R' \cap C_n = \emptyset$, contradicting that $R$ and $R'$ are equivalent. □

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