A small cosmological constant due to non-perturbative quantum effects

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Abstract
We propose an explanation for the ‘unnatural smallness’ of the cosmological constant, arguing that the stress–energy tensor of the Standard Model should be given by \( \langle T_{\mu\nu} \rangle = \rho_{\text{vac}} \eta_{\mu\nu} \), with a vacuum energy \( \rho_{\text{vac}} \) that differs from the usual ‘dimensional analysis’ result by an exponentially small factor associated with non-perturbative effects. We substantiate our proposal by a rigorous analysis of a toy model, namely the two-dimensional Gross–Neveu model. The stress energy operator is constructed concretely via the operator-product-expansion, and the inherent ambiguities in its construction are carefully examined. Our result for the vacuum energy is then obtained from the assumptions that (a) the OPE-coefficients have an analytic dependence on \( g \), which we propose to be a generic feature of QFT, and that (b) the vacuum energy vanishes to all orders in perturbation theory. Our result can also be interpreted as saying that, while the semi-classical Einstein’s equation can be fulfilled in Minkowski space at the perturbative level, it cannot at the non-perturbative level. Extrapolating our result from the Gross–Neveu model to the Standard Model, one would expect to find \( \rho_{\text{vac}} \sim \Lambda^4 e^{-\Omega(1)/g^2} \), where \( \Lambda \) is an energy scale such as \( \Lambda = M_{\text{Pl}} \), and \( g \) is a gauge coupling such as \( g^2/4\pi = \alpha_{\text{EW}} \). Assuming this extrapolation is justified, the exponentially small factor due to non-perturbative effects would explain why this quantity is tiny, instead of strictly zero.

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1. Introduction

One of the major puzzles in modern cosmology is the origin of Dark Energy, and its apparently ‘unnatural’ magnitude. Many, and very diverse, explanations have been proposed in this direction, see e.g. [1] for a review. Many of these proposals involve highly speculative features such as hypothetical new fields or dynamical mechanisms that have neither been observed, nor have been explored thoroughly from the theoretical viewpoint.

A very economical, and perhaps the most natural, hypothesis is that Dark Energy is simply quantum field theoretic vacuum energy. In other words, it is simply the expectation value of the quantum field theoretic stress energy operator, \( T_{\mu\nu} \), of the Standard Model of particle physics. The quantum state should in principle contain the approximately \( 10^{80} \) hadronic particles in the universe distributed onto stars, galaxies, dust clouds, etc. But for the problem at hand, we are not really interested in the detailed functional form of \( T_{\mu\nu} \) on smaller scales arising from these features, but rather in the contribution from the vacuum itself, in particular since the universe is mostly empty. Hence, one may take the state to be the vacuum state. Also, although our universe is expanding, its expansion rate is so small compared to the scales occurring in particle physics that we may safely do our analysis in Minkowski spacetime. Since the Minkowski vacuum state is Poincaré invariant, the vacuum expectation value (VEV) must automatically have the form

\[
\langle T_{\mu\nu} \rangle = \rho_{\text{vac}} \eta_{\mu\nu}
\]

of a cosmological constant.

A natural guess for \( \rho_{\text{vac}} \) in the Standard Model, based essentially on dimensional analysis, is \( \rho_{\text{vac}} \sim \Lambda^4 \), where \( \Lambda \) is a characteristic energy scale of the Standard Model, such as, perhaps, \( \Lambda = M_{\text{H}} \sim 125 \text{ GeV} \). This is well-known to be in striking conflict\(^3\) with the observed value \( \rho_{\text{vac}} \sim (10^{-12} \text{ GeV})^4 \). In this paper, we propose that a proper QFT-calculation of \( \rho_{\text{vac}} \) may rather result in a value of the type \( \rho_{\text{vac}} \sim \Lambda^4 e^{-O(1)/g^2} \), with \( \Lambda \) a typical energy scale of the Standard Model, such as perhaps \( \Lambda = M_{\text{H}} \), and with \( g \) a gauge coupling such as perhaps \( g^2/4\pi = \alpha_{\text{EW}} \sim \frac{1}{137} \). This can give the right order of magnitude for \( \rho_{\text{vac}} \) for a suitable constant \( O(1) \) of order unity, to be calculated in principle from the Standard Model. The essential point is that our proposal differs from dimensional analysis by an exponentially small, dimensionless factor, which we attribute to non-perturbative effects.

To justify our proposal rigorously, one would have to overcome the following two fundamental problems:

1. The huge complexity of the Standard Model, and in particular, the difficulty of making non-perturbative calculations.
2. The fact that, as is well-known, ‘the’ stress energy operator, like any other ‘composite operator’ in QFT, i.e. polynomial in the ‘basic fields’, is an intrinsically ambiguous object.

Point (1) requires no further comment, except maybe that one cannot expect to be able to calculate a ‘form factor’ like \( \langle T_{\mu\nu} \rangle \) by perturbative methods. To substantiate our proposal in a clean setup, we therefore consider a toy model which is tractable, and at the same time displays some of the non-perturbative effects characteristic for the Standard Model. This model is the well-known Gross–Neveu model in two dimensions [5, 6] (we expect very similar results to hold also for the two dimensional \( O(N) \) sigma model, treated along the lines of [7]). However, before we describe the Gross–Neveu model, let us say more clearly what we mean by (2).

Given some quantum field operator \( A \), we are free in general to make a field redefinition

\[
A(x) \to \sum_B Z_B^A \cdot B(x)
\]

3 The same conclusion is drawn if \( M_{\text{H}} \) is replaced by other natural scales associated with the Standard Model such as the scale of electro-weak symmetry breaking.
and to consider the right side as our new, equally legitimate (1), definition of that operator. In the context of standard renormalized perturbation theory around a Gaussian fixed point, the ambiguity can be attributed to the necessity of imposing ‘renormalization conditions’, a change of which can be seen to correspond to field redefinitions (1). The ‘mixing matrix’ of complex numbers, $Z_B^A$, is somewhat restricted by various obvious requirements. For example, we want $\sum Z_B^A \cdot B$ to have the same tensor/spinor character as $A$. Also the field redefinition should not be in conflict with Poincaré invariance, and it should respect the quantum numbers of fields associated with any other symmetry of the theory. If we are near a Gaussian fixed point (e.g. in perturbation theory), we can naturally assign a dimension $\Delta_A$ to each composite operator, and the field redefinition should not increase the dimension, so $\Delta_B \leq \Delta_A$ in the sum (1). If the theory depends on a coupling constant $g$ (so that we not only have one QFT, but a 1-parameter family), then $Z_B^A (g)$ can be a function of $g$, but it is reasonable to require it should have a smooth dependence on $g$. Also, if $A(x)$ satisfies a differential relation such as a conservation law, then so should the right side of (1). If $A = A^\dagger$ then $Z_B^A = (Z_B^A)^\star$, etc.

To illustrate these restrictions, suppose $A$ is a conserved current $J^\mu$ associated with a symmetry of the theory. If there is no other conserved current in the theory, then the only possible field redefinition is $J^\mu \rightarrow Z J^\mu$ for $Z$ real. The corresponding conserved charge $Q = \int J^\mu d^3 x$ should furthermore generate the symmetry, $[Q, B(x)] = ig q_B B(x)$, where $q_B$ is the charge quantum number of the operator $B$. Since $q_B$ is fixed, we must have $Z = 1$ in this example. Thus, the current $J^\mu$ is uniquely defined as an operator. Consider next the case when $A$ is the stress energy operator $T_{\mu\nu}$ of the theory. This operator should satisfy $\partial_\mu T^{\mu\nu} = 0$, so the stress tensor can only mix with other conserved operators that are symmetric tensors. A possible field redefinition is now

$$T_{\mu\nu} \rightarrow Z T_{\mu\nu} + c \eta_{\mu\nu} \mathbf{1},$$

where $\mathbf{1}$ is the identity operator, and $c$ a dimensionful constant. For example, if the microscopic Lagrangian of the theory contains a single mass parameter, $M$, then $c \propto M^2$. Similarly to the previous example, we want $P_\mu = \int T^\mu d^3 x$ to generate translations, $[P_\mu, B(x)] = ig q_B B(x)$, so we must have $Z = 1$. But, unfortunately, no restriction is obtained on the real constant $c$ that way. Since (1) $= 1$, our field redefinition changes $\langle T_{\mu\nu} \rangle \rightarrow \langle T_{\mu\nu} \rangle + c \eta_{\mu\nu}$, so we can set $\rho_{\text{vac}}$ to any value we want. Furthermore, in a theory depending on a coupling constant, $g$, we may let $c (g)$ be any (smooth) function of $g$ that we want, so we can even give the expected stress tensor an essentially arbitrary dependence on the coupling constant. Therefore, unless we impose other reasonable conditions to cut down the ambiguity, we simply cannot predict what $\langle T_{\mu\nu} \rangle$ is within the framework of quantum field theory.

In order to motivate such a condition, we must better understand the true nature of ‘products’ of operators in quantum field theory. The only natural definition of product is in fact provided by the operator product expansion (OPE), which states

$$A(x) B(0) \prod_i \phi(z_i) \sim \sum_C C_{AB}^C (x) \prod_i \phi(z_i).$$

The $\phi(z_i)$ are ‘spectator fields’, and the sum over the composite fields $C$ is organized by their dimension, in the sense that the numerical coefficients $C_{AB}^C (x)$ are most singular in $x$ for the operator $C$ with the smallest dimension, and become more and more regular as the dimension of $C$ increases. The $\sim$ sign means that if we subtract the partial sum up to a large dimension of $C$ from the right side, then we get a quantity that goes to 0 fast as $x \rightarrow 0$. In this sense, the OPE is a short distance expansion. The OPE coefficients encode the dynamics of the theory, and depend in particular on the coupling constants in the Lagrangian. We may indicate this by writing $C_{AB}^C (x; g)$, where $g$ is the (or possibly several) coupling constant. Clearly, if we make a $g$-dependent field redefinition (1) with mixing matrix $Z_B^A (g)$, then the OPE coefficients will
change accordingly. Suppose, now, that there exists some definition of the composite fields such that \( C^{CA}(x, g) \) is an analytic function of \( g \), i.e. has a convergent Taylor expansion in \( g \) for small, but finite, \( g \). Then it is natural to allow only field redefinitions \( Z^B_A(g) \) preserving this property, i.e. ones which are likewise analytic in \( g \). Therefore, for example, we would only be allowed to make a redefinition (2) for an analytic function \( c(g) \). Such analytic field redefinitions could therefore not cancel out any non-analytic (= non-perturbative) dependence on \( g \) of the VEV \( \langle T_{\mu\nu} \rangle \). Thus, if the theory has non-perturbative effects showing up in the VEV \( \langle T_{\mu\nu} \rangle \), these cannot be removed by a, necessarily analytic, field redefinition. In other words, the non-analytic contribution to this VEV is thereby uniquely specified [8].

Our analyticity requirement does not restrict the analytic contributions to VEV’s of composite operators, which still prevents us from making concrete quantitative predictions for the vacuum energy density. We can eliminate this remaining ambiguity by making the additional, reasonable sounding, assumption that \( \rho_{\text{vac}} \) should vanish to all orders in perturbation theory\(^4\). This is the same as demanding that, at the perturbative level, Minkowski space is a solution to the semi-classical Einstein equations.

We shall analyze the vacuum energy under these two assumptions (analyticity of the OPE, and vanishing of VEV to all perturbative orders) in the context of the Gross–Neveu model in two spacetime dimensions. As we shall see, in that model, our assumptions necessarily imply that the VEV must be given by a non-perturbatively small quantity, which is exponentially small in the coupling constant of the model. In this sense, our proposal may naturally explain why the value of the cosmological constant is tiny, instead of strictly zero. The existence of a non-perturbatively small, non-zero, VEV of the stress energy operator which cannot be removed by field redefinitions under our assumptions can be viewed as saying that, while Minkowski spacetime is a solution to the semi-classical Einstein equations to all perturbative orders, it is not a solution at the non-perturbative level.

Before we present our considerations in some detail, let us make some final remarks about the analyticity assumptions of the OPE and field redefinitions. The meaning/status of this assumption can maybe be explained by thinking of field redefinitions as being similar to coordinate transformations in geometry. Just as there is in general no natural choice of coordinates, there are many, equally reasonable, choices of field variables to ‘parameterize’ the given QFT. Having said that, there are some conditions which naturally restrict the possible field redefinitions, just as there are basic restrictions on coordinate transformations (e.g. invertible, smooth). Such conditions in QFT (and also in geometry!) often have to do with symmetries, e.g. if you have an action of the Poincare group in your QFT, then the definition of fields should be compatible with that, etc. The restriction to analytic field redefinitions is maybe best thought of as being analogous to complex manifolds. On such manifolds, one can locally choose holomorphic coordinates, in which the natural geometric objects become analytic functions. Holomorphic coordinates should be viewed as analogous to a choice of field coordinates in which the OPE is analytic in the coupling constant(s). Of course, in geometry, one can make a coordinate transformation that is not analytic, but then working with such non-analytic coordinates obscures the features of the geometry having to do with holomorphicity. Similarly, in QFT, if there is a definition of fields such that the OPE coefficients are analytic in the coupling, then in our view such a definition is preferred and should be used to describe the observables of the theory.

\(^4\) We would like to stress that making this assumption is not the same as fine-tuning the cosmological constant. Only the perturbative contributions are set to zero by hand, while the non-perturbative terms naturally yield a small value for the vacuum energy. As remarked in [1] (see pages 2, 26 and 33), it seems much easier to imagine a mechanism that makes the perturbative contributions to the vacuum energy vanish altogether than one which would suppress \( \rho_{\text{vac}} \) by just the right amount to agree with observations.
2. The model

The massless (classically) Gross–Neveu model in \( d = 2 \) dimensions is described by the Lagrangian

\[
\mathcal{L} = N \left[ i \tilde{\psi} \partial_\mu \psi + \frac{g^2}{2} (\tilde{\psi} \psi)^2 \right],
\]

where \( \tilde{\psi} \) and \( \psi \) are row/column vectors of \( N \) flavors of a 2-component spinor field. Relative to the usual presentation of the Lagrangian [5, 6], the fields have been rescaled by \( 1/\sqrt{N} \), which is convenient in view of the large \( N \) limit taken later. In the Lagrangian, and in similar expressions below, the flavor index is summed over in the obvious way. The expression for the classical stress–energy tensor per flavor, i.e. divided by \( N \), is:

\[
T_{\mu\nu} = \frac{1}{2} \tilde{\psi} \gamma_\mu \partial_\nu \psi + \tilde{\psi} \gamma_\nu \partial_\mu \psi - \eta_{\mu\nu} \left[ i \tilde{\psi} \partial_\mu \psi + \frac{g^2}{2} (\tilde{\psi} \psi)^2 \right].
\]

Here we use the standard notation \( A \overset{\rightarrow}{\partial} B := (1/2) [\partial DB - (\partial A)B] \). The \('t\) Hooft coupling constant \( g \) is dimensionless, and the model is conformally invariant at the classical level. By contrast, the corresponding quantum field theory is not conformally invariant, but exhibits the phenomenon of ‘dynamical mass generation’. This means concretely that, at large space-like separation \( x \), the 2-point correlation function has an exponential fall-off, \( \langle \tilde{\psi}(x) \psi(0) \rangle \sim \exp(-K(g)\sqrt{-x^2/\ell^2}) \), where \( K(g) > 0 \) is a numerical constant, and where \( \ell \) is a constant of dimension (length). The dynamically generated mass is accordingly given by \( m(g) := K(g)/\ell \). The constant \( \ell \) corresponds to a choice of ‘units’ (fm, mm, km, etc), which are obviously not provided by the classical, scale invariant, Lagrangian (4). Its value is therefore best viewed as part of the definition of the quantum field theory.

The exponential fall-off was rigorously proven, for sufficiently large but finite \( N \), by [9]. The effect of mass generation is non-perturbative, in the sense that all \( g \)-derivatives of \( K(g) \), corresponding to the various perturbation orders, vanish. It was first discovered in a large \( N \) analysis of the model by Gross and Neveu [5]. The model becomes essentially solvable in this limit, and we have, in fact (compare equation (6)) \( K(g) = e^{-\pi/\ell^2} + O(\frac{1}{\sqrt{N}}) \). In spite of this characteristic non-perturbative dependence on \( g \), we will see that the OPE coefficients have a perfectly analytic dependence on \( g \), see equations (9)–(11), as proposed in the previous section. The full quantum field theory is defined by the collection of all \( n \)-point correlation functions of the basic fields \( \psi, \tilde{\psi} \), but we will only need the 2- and 4-point functions. The 2-point function is (here and below we assume \( x \) to be space-like):

\[
\langle \tilde{\psi}_\alpha(x) \psi_\beta(0) \rangle = -\frac{i(\slashed{D} + m)_{\beta\alpha}}{2\pi} K_0(\sqrt{-x^2m^2}) + O \left( \frac{1}{\sqrt{N}} \right),
\]

where \( \alpha, \beta \) are spinor indices. The 4-point function is written most conveniently as

\[
\langle \tilde{\psi}_\alpha(x_1) \psi_\beta(x_2) \tilde{\psi}_\gamma(z_1) \psi_\delta(z_2) \rangle = -\frac{1}{2N} \int d^2q d^2p \frac{1}{(2\pi)^4} \frac{1}{(p^2 - m^2)(q^2 - m^2)|B(q - p)|} \times \left[ (\slashed{q} - \slashed{p} + m - i\slashed{\gamma}_5)(m - i\slashed{\gamma}_5)_{\alpha\beta} \right] \times \left[ (\slashed{q} - \slashed{p} + m - i\slashed{\gamma}_5)(m - i\slashed{\gamma}_5)_{\gamma\delta} \right] \times \int_0^1 \frac{d\alpha}{\sqrt{-\alpha(1 - \alpha)(q - p)^2}} \int \frac{d\alpha}{\sqrt{\alpha(1 - \alpha)(q - p)^2}} \times \left[ \frac{1}{N} \langle \tilde{\psi}_\alpha(x) \psi_\beta(0) \rangle \langle \tilde{\psi}_\gamma(z_1) \psi_\delta(z_2) \rangle \right] - \frac{1}{N} \langle \tilde{\psi}_\alpha(x_1) \psi_\beta(x_2) \rangle \langle \tilde{\psi}_\gamma(z_1) \psi_\delta(z_2) \rangle + O \left( \frac{1}{N^2} \right),
\]

\(5\) Our conventions are (cf [6]): signature \((+, -)\), Dirac matrices \(\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), \(\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\). Dirac conjugate \(\bar{\psi} = \psi^\dagger \gamma^0\).
for our purposes. $K_\alpha$ are modified Bessel functions, and we use the short-hand
\[
B(k) := \sqrt{\frac{4m^2 - k^2}{k^2}} \log \frac{\sqrt{4m^2 - k^2} + \sqrt{-k^2}}{\sqrt{4m^2 - k^2} - \sqrt{-k^2}}.
\] (8)

Note that the correlation functions have a non-analytic dependence on $g$ through $m = e^{-\pi/8}/\ell$.

Correlation functions of composite operators can be obtained from the correlation functions of the basic fields by means of the OPE. We will need the following OPE’s in this paper:
\[
\bar{\psi}(x)\psi(0) = O\left(\frac{1}{N}\right) 1 + \left[1 - \frac{g^2}{2\pi} \log \frac{-x^2e^{2}\gamma}{4\ell^2} + O\left(\frac{1}{N}\right)\right] \bar{\psi}\psi(0) + \cdots
\] (9)
\[
\bar{\psi}(x)\bar{\psi}(0) = O\left(\frac{1}{N}\right) 1 + O\left(\frac{1}{N}\right) \bar{\psi}\psi(0)
+ \frac{g^2}{i} \left[1 - \frac{g^2}{2\pi} \log \frac{-x^2e^{2}\gamma}{4\ell^2} + O\left(\frac{1}{N}\right)\right] \bar{\psi}\psi(0) + \cdots
\] (10)
\[
\bar{\psi}(x)\gamma_{\mu} \rightarrow_{\nu} \bar{v}\psi(0) = \left[-\frac{2\epsilon_{\mu\nu} x_{\gamma}}{i2\pi x^2} + O\left(\frac{1}{N}\right)\right] 1 + O\left(\frac{1}{N}\right) \bar{\psi}\psi(0)
+ \left[1 + 0\left(\frac{1}{N}\right)\right] \bar{\psi} \gamma_{\mu}\bar{\nabla}_{\nu} \psi(0) - \left[\frac{2g^2}{2\pi i x^2} + O\left(\frac{1}{N}\right)\right] (\bar{\psi}\psi)^2(0) + \cdots.
\] (11)

Here $\Gamma_E$ is the Euler–Mascheroni constant, $t_{\mu
u} = \frac{1}{2}(\bar{t}_{\mu\nu} + t_{\nu\mu}) - \frac{1}{2}\eta_{\mu\nu}\eta^\sigma$ is the symmetric traceless part and dots are terms of order $O(x)$. The OPE coefficients in these expressions were calculated using standard $\frac{1}{N}$-expansion techniques and e.g. the methods of [10]. Terms not written explicitly are not needed later. We also have $g^2(\bar{\psi}\psi)^2 = i\bar{\psi}\bar{\psi}$ as an operator equation—in fact, one may consistently view this as the definition of the operator $(\bar{\psi}\psi)^2$—which is (formally) a consequence of the equation of motion. We see explicitly that the OPE coefficients are analytic in $g$, in contrast to the correlation functions.

3. VEV of $T_{\mu\nu}$

We would like to calculate the VEV of $T_{\mu\nu}$ (cf (5)), which is evidently a composite operator. VEV’s of renormalized composite operators are calculated from the correlation functions of the basic field $\bar{\psi}\psi$ by means of the OPE, and are subject to the intrinsic renormalization ambiguities mentioned above. As a warm-up, let us illustrate the procedure for the VEV $\langle \bar{\psi}\psi(0) \rangle$. First, we take an expectation value of equation (9), solve for $\langle \bar{\psi}\psi(0) \rangle$, and take $x \to 0$:6
\[
\langle \bar{\psi}\psi(0) \rangle = \lim_{x \to 0} \frac{\langle \bar{\psi}(x)\psi(0) \rangle - O\left(\frac{1}{x}\right) (1)}{1 - \frac{x^2}{2\pi} \log \frac{-x^2e^{2}\gamma}{4\ell^2} + O\left(\frac{1}{N}\right)}.
\] (12)

6 Note that this construction automatically yields finite expectation values for the composite operators. In the limit $g \to 0$ (free fermions) this procedure corresponds to ‘normal ordering’, i.e. $\bar{\psi}\psi$ as defined via the OPE as in (12) is equal to : $\bar{\psi}\psi$ : for a free field. Normal ordered products in free field theory automatically have a vanishing VEV. The same applies also to our definition of the stress tensor via the OPE, which at the level of free fields ($g = 0$) corresponds to the normal ordered stress tensor having vanishing VEV, $\langle T_{\mu\nu} \rangle = 0$. Thus, for a free field, our prescription yields an expected stress tensor which obeys the semi-classical Einstein-equations for Minkowski space. But below we argue that it is consistent, in this model, to have Minkowski spacetime as a solution to the semi-classical Einstein equation if non-perturbative effects (in particular effects of interaction) are taken into account.
We now substitute equation (6) for the 2-point function, and ignore terms of $O(1/N)$. Making use of the standard expansion of the Bessel-function $K_0$ for small argument, we find
\begin{equation}
\langle \bar{\psi} \psi(0) \rangle = -\frac{1}{g^2 \ell} e^{-\pi/\ell^2} + O \left( \frac{1}{N} \right) .
\end{equation}
Of course, the VEV is the same at any other spacetime point $x$ by translation invariance. Thus, we see that the VEV has a non-analytic dependence on $g$, and we cannot make the VEV zero for all $g$ by any, necessarily analytic, field redefinition (1) of $\bar{\psi} \psi$. In fact, the VEV shows that not only conformal-, but also (discrete) chiral symmetry is broken in the quantum theory.

Let us now determine the VEV of the stress tensor (5) by this method. We have to be more careful here, because we need to make sure our definition of this composite operator obeys $\partial^\mu T_{\mu\nu} = 0$ as an operator equation. Our strategy is to define separately the composite operators appearing in formula (5) by the same method as just described. Their sum defines a composite operator, which actually turns out not to be conserved. But fortunately we can add another operator of the same dimension (field redefinition) to it such that it now is conserved (up to order $O(1/N)$). The resulting conserved operator is then the physical stress energy operator, which is seen to have a non-zero VEV. Let us now describe this in some more detail.

Since we can consistently assume that $i \bar{\psi} \gamma_\mu \partial \nu \psi$ is defined using the OPE (11).

Since we would like to check whether it is conserved as an operator, we need to calculate the divergence $\langle \partial^\mu T_{\mu\nu}(0) \bar{\psi}(y_1) \psi(y_2) \rangle$ in terms of the 4-point function (7). We also need $(\bar{\psi} \psi)^2(0) \bar{\psi}(z_1) \psi(z_2)$ which is obtained in terms of the 4-point function (7) in a similar way, using (10) this time. Then using the concrete form of (7), one derives, after a somewhat lengthy calculation, the relationship
\begin{equation}
\langle \bar{\psi} \psi(z_1) \psi(z_2) \rangle = -\frac{g^2}{4\pi} \langle \partial_\nu (\bar{\psi} \psi)^2(0) \bar{\psi}(z_1) \psi(z_2) \rangle + O \left( \frac{1}{N^2} \right).
\end{equation}
Since the rhs is not zero, it follows that the composite operator $T_{\mu\nu}$, as defined, is not conserved. However, it follows that the operator $\theta_{\mu\nu} := T_{\mu\nu} - (g^2/4\pi) \eta_{\mu\nu}(\bar{\psi} \psi)^2$ is conserved (up to order $O(1/N)$). We consequently define $\theta_{\mu\nu}$ to be the physical stress–energy tensor up to that order. Its VEV is found by using the now familiar OPE method, as
\begin{equation}
\langle \theta_{\mu\nu} \rangle = -\frac{1}{4\pi \ell^2} e^{-2\pi/\ell^2} \eta_{\mu\nu} + O \left( \frac{1}{N} \right).
\end{equation}
This corresponds to a negative vacuum energy of $\rho_{\text{vac}} = -1/(4\pi \ell^2) e^{-2\pi/\ell^2}$ to leading order in $1/N$. The negative sign is related to the negative sign of the $\beta$-function in the Gross–Neveu model.

We must finally discuss the ambiguity of our result. According to the general discussion above, equation (1), we are still free to change $\theta_{\mu\nu} \rightarrow \theta_{\mu\nu} + \ell^2 c(g) \eta_{\mu\nu}$, where $c(g) = c_0 + c_1 g + c_2 g^2 + \cdots$ is analytic. This will result in a corresponding change $\rho_{\text{vac}} \rightarrow \rho_{\text{vac}} + \ell^2 c(g)$. As we have already explained in the introduction, we eliminate this remaining ambiguity by making the additional assumption that $\rho_{\text{vac}}$ should vanish to all orders in perturbation theory. Indeed, since all derivatives (i.e. perturbative contributions) of $-1/(4\pi \ell^2) e^{-2\pi/\ell^2}$ at $g = 0$ vanish, this requirement means that all derivatives of $c(g)$ at $g = 0$ must vanish, too, which, since $c(g)$ must be analytic, means that $c(g) = 0$ for all $g$ derivatives. Thus, under the assumptions (a) of admitting only analytic field redefinitions, and
found to be equal to (15), corresponding to the vacuum energy

\[ \rho_{\text{vac}} = -1/(4\pi \ell^2) \ e^{-2\pi/\ell^2} \]

is unique in the Gross–Neveu model. This is the main result of this section.

4. Conclusions

We have defined the stress tensor of the Gross–Neveu model as a renormalized composite operator which obeys the conservation law. Its expectation value in the vacuum state was found to be equal to (15), corresponding to the vacuum energy

\[ \rho_{\text{vac}} = -1/(4\pi \ell^2) \ e^{-2\pi/\ell^2}. \]

We have argued that this expectation value cannot be removed by (necessarily analytic) field redefinitions, although it can be removed to any finite order in perturbation theory. Whence, our result can be interpreted as saying that it is inconsistent to have a vanishing VEV of the stress tensor in the Gross–Neveu model, or alternatively, that it is inconsistent, at the non-perturbative level, to assume that Minkowski spacetime is a solution to the semi-classical Einstein equations

\[ G_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu} \rangle \]

in this model.

The present model contains, as part of its definition at the quantum level, the dimensionful constant \( \ell \) which corresponds to the units of length, and which are not provided by the classical Lagrangian (4). It would be more satisfactory to have a model wherein all dimensionful parameters are already part of the fundamental Lagrangian defining the theory in the ultraviolet. This can be achieved, in principle, by coupling our model to other fields with dimensionful couplings. For example, we could add to the Lagrangian (4) an interaction with some massive (by hand) scalar field \( \phi \) such as in

\[ \mathcal{L} \to \mathcal{L} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{4} M^2 \phi^2 + \gamma M \phi \overline{\psi} \psi, \]

where \( M, \gamma \) are new coupling parameters. The constant \( \ell \) can then be related to the dimensionful parameter \( M \) by a renormalization condition, e.g. by demanding that the physical (renormalized) mass of \( \phi \) (as determined by the exponential decay of the \( \phi \) 2-point function) at some value \( g = O(1) = \gamma \) equals \( M \), where \( M \) is the physical (renormalized) mass of \( \phi \) (as determined by the exponential decay of the \( \phi \) 2-point function). In a large \( N \) analysis, one would also take \( M = O(N) \). Although we will not carry out such an analysis here, one would expect that the result for the vacuum energy is now modified to

\[ \rho_{\text{vac}} \sim M^2 e^{-O(1)/\ell^2}, \]

i.e. \( \ell \) is simply set by \( M \), which is now a parameter appearing explicitly in the Lagrangian. The renormalization ambiguity of \( \theta_{\mu\nu} \) now consists in adding \( c(g, \gamma) M^2 \eta_{\mu\nu} \), where \( c(g, \gamma) \) is analytic. Again, we can eliminate this ambiguity by demanding that Minkowski space is a solution to the semi-classical Einstein equations to all perturbation orders in \( \gamma, g \), i.e. that \( \theta_{\mu\nu} \) vanishes to all orders in perturbation theory in \( \gamma, g \).

The real world, of course, is not described by the Gross–Neveu model, but by the Standard Model of elementary particle physics. It should be noted, however, that the Gross–Neveu model, along with the closely related nonlinear sigma models in two dimensions, is commonly used as a testing ground for non-perturbative effects in the Standard Model [7], due to the fact that it shares some crucial features, such as asymptotic freedom, spontaneous breaking of chiral symmetry and dimensional transmutation, with quantum chromodynamics. It therefore seems plausible that a similar effect to the one described in this paper also exists in the Standard Model. If we pursue this analogy, \( M \) would perhaps be replaced by a mass scale associated with the Standard Model Lagrangian, such as the Higgs mass \( M \to M_H \). Furthermore, the coupling would perhaps be replaced by a gauge coupling such as \( g^2/4\pi \to g_{\text{EW}} \sim \frac{1}{137} \). Assuming that these speculations are correct, we obtain an analogue formula

\[ \rho_{\text{vac}} \sim M^2_H e^{-O(1)/g_{\text{EW}}} \]

for some constant of order unity. The smallness of \( \rho_{\text{vac}} \) is then achieved by the characteristic non-perturbative dependence on the dimensionless coupling constant. In our model example,
the sign of \( \rho_{\text{vac}} \) is negative, whereas vacuum energy in our universe is positive. The sign in our model can be traced back to the negative sign of the corresponding \( \beta \)-function. We do not know what the sign of \( \rho_{\text{vac}} \) may be in the Standard Model, but we note that there are gauge couplings with either sign of the \( \beta \)-function. Future research should aim at applying the strategy described in this paper to more realistic models.

To summarize, we believe that non-perturbative effects are a potential explanation for the order of magnitude of Dark Energy.

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