Polarization of the Cosmic Microwave Background
in a Reionized Universe

Marina Gibilisco
Universitá degli studi di Pavia,
Via Bassi 6, 20127 Pavia, Italy,
and Universitá degli studi di Milano,
Via Celoria 16, 20133, Milano, Italy.

Abstract:

In this work I study the polarization of the Cosmic Microwave Background (CMB) induced by a cosmological background of gravitational waves (here GWs), probably originated during the inflationary epoch. I discuss the influence of a possible reionization of the Universe, which should happen at late times ($z \sim 20 \div 30$): I show that, in the presence of such a reionization, we have a remarkable enhancement of the present polarization of the CMB. I also point out the role of a background of gravitational waves of large wavelength, re-entering the horizon very late, in producing the anisotropy and the polarization of the CMB through the Sachs-Wolf effect. Then, I use the standard formalism of the Stokes parameters to describe the properties of the CMB photons and I study their evolution with the time, the energy and the spatial coordinates through the radiative Boltzmann transfer equation. I solve such an equation in the form of a second-kind Volterra integral equation, by using the analytical method of the iterated kernels: this particular method can be successfully applied when we have an exponential kernel, as in the case when the Universe knew a fast and late reionization, due, for instance, to the Hawking evaporation of primordial black holes.

1. The Radiative Transfer Equation and the Sachs-Wolfe Effect.

The equation which expresses the evolution of the Stokes parameters with the time, the energy and the space coordinates is the Boltzmann radiative transfer equation:

$$\left( \frac{\partial \vec{n}}{\partial \eta} + \gamma^x \frac{\partial \vec{n}}{\partial x^x} \right) + \frac{\partial \nu}{\partial \eta} \frac{\partial \vec{n}}{\partial \nu} = \frac{\sigma T N_e R(\eta)}{4\pi} \times \left[ -4\pi\vec{n} + \int_{-1}^{1} \int_{0}^{2\pi} \vec{n} \cdot \mathbf{P}(\mu, \phi, \mu', \phi') \ d\mu' \ d\phi' \right]. \quad (1.1)$$

($\eta$ is the comoving time, defined as $\int [dt/R(t)]$, $R(t)$ being the scale factor of the Universe).

Here, the variables are the components of a 4-vector $n_\alpha = n_\alpha(\theta, \phi, \nu)$, which is a function of the polar angles ($\theta, \phi$) and of the photon frequency; these components are the four Stokes parameters which describe the polarization status of the radiation:

$$n_\alpha \equiv (I_I, I_r, U, V). \quad (1.2)$$

$I_I$, $I_r$ are the left and right intensities of the radiation, while $U$ and $V$ respectively represent the linear and the circular polarization of the CMB photons. In the Chandrasekhar formalism one has:

$$I = I_I + I_r \quad \quad Q = I_I - I_r. \quad (1.3)$$

In eq. (1.1) $\gamma^x$ are the components of an unit vector in the propagation direction of the photons (in particular, the z axis is taken parallel to the propagation direction of the gravitational waves we consider), $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$ is the Thomson scattering cross-section, $R(\eta)$ is the scale factor, $\mu = \cos \theta$, $P(\mu, \phi, \mu', \phi')$ is the Chandrasekhar scattering matrix and $N_e$ is the comoving number density of the free electrons: such a variable introduces in the Stokes parameters evolution an important dependence on the ionization history of the Universe. In fact, the Thomson scattering of an anisotropic radiation (as the CMB
is) gives it a linear polarization\(^2\): clearly, in a reionized Universe, the Thomson scattering processes are effective for a longer time, thus one obtains an enhancement of the resulting CMB polarization.

The relevant term in the left-hand side of eq. (1.1) is the third one; in fact, it expresses a very peculiar effect due to a cosmological background of gravitational waves, known as Sachs-Wolfe effect\(^3\). The presence of a ripple in the space-time metric produces a shift in the photons frequencies: in fact, a time-dependent perturbation in the metric, i.e. a gravitational wave, determines the non-conservation of the energy along the photons sight-line, thus causing a frequency shift:

\[
\frac{\partial \nu}{\partial \eta} = \frac{\nu}{2} \frac{\partial h_{\alpha\beta}}{\partial \eta} \epsilon^{\alpha\gamma\beta}.
\] (1.4)

If one writes the perturbation of the metric tensor as a plane wave

\[
h_{\alpha\beta} = h e^{-i(\vec{k} \cdot \vec{x} - k\eta)} \epsilon_{\alpha\beta},
\] (1.5)

with

\[
\epsilon_{\alpha\beta\gamma\delta} = (1 - \cos \theta^2) \cos 2\phi,
\] (1.6)

the frequency shift becomes

\[
\frac{\partial \nu}{\partial \eta} = \frac{\nu}{2} e^{-i\vec{k} \cdot \vec{x}} \left( \frac{\partial}{\partial \eta} h e^{i\eta} \right) (1 - \mu^2) \cos 2\phi.
\] (1.7)

Thus, the Sachs-Wolfe term obviously influences the evolution of the Stokes parameters and, as a consequence, there is a polarization induced in the CMB by cosmological gravitational waves.

2. THE SECOND-KIND VOLterra EQUATION

Here I will briefly recall the standard formalism of ref.\(^4\), useful to turn the transfer equation (1.1) into a second-kind Volterra equation.

By using two new variables, \(\alpha\) and \(\beta\), and two new vectors:

\[
\vec{a} = \frac{1}{2} (1 - \mu^2) \cos 2\phi \begin{pmatrix} 1, & 1, & 0 \end{pmatrix},
\] (2.1)

\[
\vec{b} = \frac{1}{2} \begin{pmatrix} (1 + \mu^2) \cos 2\phi, & -(1 + \mu^2) \cos 2\phi, & 4\mu \sin 2\phi \end{pmatrix},
\] (2.2)

the vector \(\vec{n}\) is written in the form

\[
\vec{n} = n_0 \begin{pmatrix} 1, & 1, & 0 \end{pmatrix} + e^{-i(\vec{k} \cdot \vec{x} - k\eta)} (\alpha\vec{a} + \beta\vec{b}).
\] (2.3)

In eq. (2.3) \(n_0 = (n_0, n_0, 0)\) is the unperturbed solution one has in absence of gravitational waves.

By expressing \(I, Q, U\) as functions of these variables, one obtains two coupled, first order differential equations\(^4\):

\[
\dot{\beta} + [q(\eta) - ik\mu] \beta = F(\eta, k),
\] (2.4)

\[
\dot{\xi} + [q(\eta) - ik\mu] \xi = H(\eta, k),
\] (2.5)

where \(q(\eta) = \sigma_T N_e R(\eta)\) depends, through the electron density \(N_e\), on the ionization history of the Universe.

The right-hand sides of eqs. (2.4) and (2.5) are

\[
F(\eta, k) = \frac{3q(\eta)}{16} \int_{-1}^{1} d\mu' \left[ (1 + \mu'^2) \beta(\mu', k) - \frac{1}{2} \xi(\mu', k) (1 - \mu'^2)^2 \right].
\] (2.6)
Eq. (2.6) comes from the calculation of the scattering integral in eq. (1.1), while \( H(\eta, k) \) is given by the time derivative of the metric tensor perturbation, calculated by taking into account the particular gravitational waves spectrum predicted by the theory. The form of their spectrum is expressed by the factor \( D(k) \), which is strongly model-dependent: the simplest choice, when quantum gravity effects are not involved, is

\[
D(k) = \sqrt{\frac{16 \pi G H^2(k)}{k^2}}.
\]

in eq. (2.8) \( H \) is the curvature of the de Sitter Universe at the epoch when the GWs enter the horizon.

More complex forms for \( D(k) \), referring to the various kinds of inflationary Universe, can also be considered: here, for simplicity, I will discuss the simplest case only.

By substituting the variables \( \beta \) and \( \xi \) in eq.(2.6), formally obtained by eqs. (2.4) and (2.5), the resulting equation is a 2-nd kind Volterra integral equation which reads:

\[
F(\eta, k) = 3q(\eta) \left\{ \int_0^\eta d\eta' \ F(\eta', k) \ K_+(\eta, \eta', k) - \frac{1}{2} \int_0^\eta d\eta'' \ H(\eta'', k) \ e^{-\tau(\eta'')} \ K_-(\eta', \eta'', k) \right\},
\]

where

\[
K_\pm(\eta, \eta', k) = \int_{-1}^1 d\mu \ (1 \pm \mu^2)^2 \ e^{ik\mu(\eta-\eta')}
\]

and

\[
\tau(\eta) = \int_\eta^1 q(\eta') \ d\eta'.
\]

is the optical depth.

3. THE SOLUTION WITH AN EXPONENTIAL KERNEL: THE METHOD OF THE ITERATED KERNELS

In Ref. I discussed a mixed analytical-numerical method to solve the radiative transfer equation, based on the resolvent method; such a method can be used when the kernel of the equation has a polynomial form. The interested people can find a detailed discussion of this technique in the quoted reference. Here I want rather to discuss a totally analytical way to solve the transfer equation, based on the properties of the Volterra integral equations. The method of the iterated kernel is a very simple and elegant way to solve a Volterra equation, but it can be used in few cases only: the fundamental task is to find a resolvent function by writing a convergent series built up with the kernel functions, after they have been convoluted and integrated. For a detailed discussion of this method see refs. 9,10; here, I will briefly recall the main results of my analysis, see ref. 10.

Look now at the general form of a Volterra equation:

\[
f(x) = G(x) + \int_0^x K(x, x') \ f(x') \ dx'.
\]

Here \( K(x, x') \) is the kernel and \( G(x) \) is the source function. Starting from eq. (3.1), the iterated kernel is given by

\[
K_m(x, x') = \int_x^{x'} K(x, t)K_{m-1}(t, x')dt.
\]

If the successive approximations of the kernel obtained in such a way really converge, the integral equation can be written in the form

\[
f(x) = G(x) + \sum_{m=1}^{\infty} \lambda^m \int K_m(x, x')G(x')dx'.
\]
By defining the resolvent as
\[
\Gamma(x, x'; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, x'),
\]
the final solution is
\[
f(x) = G(x) + \lambda \int G(x') \Gamma(x, x'; \lambda) \, dx'.
\] (3.5)

The applicability of this method clearly requires the convergence of the series in eqs. (3.3) and (3.4); moreover, the integral (3.2) should involve quite simple, analytical functions.

An exponential kernel satisfies these conditions very well: in our specific case, the kernel can be put in an exponential form if the function \(q(\eta)\) (see eq. (2.11)), specifying the ionization history of the Universe, has such a form. As I proved in \(^{11,12}\) a reionization of the Universe induced by the Hawking evaporation of Primordial Black Holes (PBHs) has just an exponential behaviour.

With this choice, the kernel of the Volterra equation can be approximated by
\[
S(k, \eta, \eta') = q(\eta)K_+(k, \eta, \eta') \sim \exp(c_1) \exp(c_2 k(\eta - \eta')).
\] (3.6)

The parameters \(c_1\) and \(c_2\) are obtained by a low \(\chi^2\) fit, whose results are shown in fig. 1 for the case \(z = 20, x = 1\), for which \(c_1 = 5.22, c_2 = -18.42\).

Finally, the original Volterra equation (2.9) can be turned into a very simple, linear, first order differential equation with a boundary condition \(F(k, 0) = G(k, 0)\); the final solution is
\[
\bar{F}(k, \eta) = G'(k, \eta) + \frac{3}{32} \frac{D(k)}{k} \lambda \frac{c^a}{b} \exp((\lambda + c_2 k)\eta)(\exp(b\eta) - 1),
\] (3.7)
where \(a\) and \(b\) are two constants coming from the calculation of the primitives.

\section{4. THE CALCULATION OF THE CMB POLARIZATION}

The exact solutions of eqs. (2.4) and (2.5) are obtained by substituting in eq. (2.9) the functions \(\bar{F}(\eta, k)\) and \(H(\eta, k)\) determined through the analytical solution of the Volterra equation previously found. Then, the present CMB polarization and anisotropy, induced by a superposition of \(k\)-fixed GWs, are given by \(^{10}\):

\[
P^2(k) = \int_{-1}^{1} |\beta(k)|^2 [(1 + \mu^2)^2 + 4\mu^2] \, d\mu,
\] (4.1)

\[
A^2(k) = \int_{-1}^{1} |\xi(k)|^2 (1 - \mu^2)^2 \, d\mu.
\] (4.2)

The normalization is carried out by defining the polarization degree as follows:
\[
p(k) = \frac{\sqrt{Q^2 + U^2}}{\sqrt{I^2}};
\] (4.3)

\[
\bar{T}^2(k) = |\alpha(k)|^2 (1 - \mu^2)^2 + 4I_0^2 + 4I_0 |\alpha(k)|(1 - \mu^2)^2;
\] (4.4)

here
\[
\bar{T}_0 = \frac{1}{\exp(h\nu/k_BT) - 1}
\] (4.5)
is the unperturbed intensity; I take also
\[
h\bar{\nu} \sim 10^{-13} \text{ GeV}, \quad k_BT \sim 2.35 \times 10^{-13} \text{ GeV}.
\] (4.6)
In the same way, the normalized anisotropy is defined as:

\[ a^2(k) = \frac{\int_{-1}^{1} |\xi(k)|^2 (1 - \mu^2)^2 d\mu}{T^2(k)} \]

where \( T^2(k) \) is still given by eq. (4.4).

In fig. 2 I showed the polarization peak obtained from the solution of the transfer equation in the case of a standard ionization history of the Universe \(^8\); in figs. 3 and 4 I showed my results respectively in the case of a reionization which happens at \( z = 10 \) and at \( z = 20 \). I plotted the natural logarithm of the polarization degree vs. the wave number of the considered gravitational waves. In fig.2 the maximum of the polarization corresponds to a wave-number \( k \sim 15 \), i.e. to gravitational waves re-entering the horizon at the end of the recombination epoch: their effect is dominant. The peak is the direct consequence of some mathematical suppressions: for \( \eta < 0.03 \), the source function \( G(\eta, k) \) is strongly suppressed by the exponential function; the physical reason for this suppression is that the polarizing effect cannot be relevant because the Thomson scattering has a low efficiency and thus the perturbations are overally damped. As a consequence, GWs re-entering the horizon too early are unimportant in order to produce a CMB polarization.

For \( 0.03 < \eta < 0.076 \), Thomson scattering just begins to have some importance, but the polarizing effect of GWs having a wavenumber \( k = 15 \) is dominant, thus masking that of GWs of different \( k \). Therefore, the polarization is relevant only near \( \eta = 0.066 \). Finally, if none reionization is considered, for \( \eta \geq 0.076 \) the ionization degree is very small: as a result, the polarizing effect is negligible because we have no more free electrons which scatter the CMB photons.

As one can see from figs. 3 and 4, a sharp secondary peak appears also at smaller \( k \) \( (k = 4) \), in the case of a total reionization coming at \( z = 20 \). That represents the enhancement of the CMB polarization in a reionized Universe: in this case, a polarization for the CMB photons should be observable at angular scales near 9°, slightly smaller than the ones tested by COBE. On the contrary, for a standard ionization history, one should expect to see the CMB polarization only at angular scales near 2°, thus requesting very high precision measurements.

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