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Multiscale heat transport with inertia and thermal vortices

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Abstract

In this paper, we present a Hamiltonian and thermodynamic theory of heat transport on various levels of description. Transport of heat is formulated within kinetic theory of polarized phonons, kinetic theory of unpolarized phonons, hydrodynamics of polarized phonons, and hydrodynamics of unpolarized phonons. These various levels of description are linked by Poisson reductions, where no linearizations are made. Consequently, we obtain a new phonon hydrodynamics that contains convective terms dependent on vorticity of the heat flux, which are missing in the standard theories of phonon hydrodynamics. Within the zero-order Chapman-Enskog reduction, the resulting hydrodynamic equations are hyperbolic and Galilean invariant, while the first Chapman-Enskog expansion gives additional viscous-like terms. The vorticity-dependent terms violate the alignment of the heat flux with the temperature gradient even in the stationary state, which is expressed by a Fourier-Crocco equation. Those terms also cause that temperature plays in heat transport a similar role as pressure in aerodynamics, which is illustrated on numerical simulations of flow past a cylinder. In particular, we show that the vorticity-dependent terms lead to a colder spot just behind the cylinder, and for high-enough Reynolds numbers they lead to the von Kármán vortex street.

1. Introduction

This manuscript contains a new theory of phonon hydrodynamics with corrected behavior of thermal vortices. Phonon hydrodynamics is of particular importance for instance in nanoscale heat transport in semiconductors, thermal diodes, or two-dimensional materials [1–3].

Heat transport in solids is represented by transport of energy by lattice vibrations [4, 5]. Since the lattice vibrations can be seen as quasiparticles obeying Hamilton canonical equations (phonons), evolution equation of their distribution function obeys kinetic theory [6, 7], which is also a Hamiltonian system [8, 9]. Here, we start with kinetic theory of phonons with three polarizations (two transverse and one longitudinal). This description is then reduced to hydrodynamics with momenta and entropies of the particular polarizations, and to the overall kinetic theory of phonons (disregarding polarizations). Both these levels of description are then further reduced to phonon hydrodynamics, where state variables are the overall momentum density and entropy density. Diagram 1 summarizes the various levels of description and connections between them.

It is advantageous to further transform phonon momentum density to the conjugate entropy flux \( \mathbf{w} = \mathbf{m}/s \) (the ratio of phonon momentum and entropy, see equation (27)). Thermal vorticity, \( \omega = \nabla \times \mathbf{w} \), then satisfies an analogue of the Crocco equation known from aerodynamics [12], relating vorticity with temperature gradient and field \( \mathbf{w} \) itself. In contrast with other approaches, our approach consistently describes the motion of thermal vorticity because we neither linearise the reversible part of the equations nor are we restricted to one-dimensional systems [1, 13–16]. The terms that depend on \( \omega \) make the equations Galilean invariant, unlike the usual Maxwell-Cattaneo-Vernotte or Guyer-Krumhansl equations [13, 17]. Although evidence for the presence of thermal vorticity in heat transport is provided by kinetic theory [18–20], standard phonon hydrodynamics can not correctly propagate such vortices because it lacks convective terms [13, 16]. Such terms are lost in the usual linearized Chapman-Enskog reduction, while our approach (based on reduction of the underlying Poisson
brackets) does not linearise the reversible evolution and thus recovers the convective terms. In terms of field \( \mathbf{w} \), convective terms translate to vorticity-dependent terms.

The vorticity-dependent terms cause that the heat flux no longer needs to be aligned with the gradient of temperature even in the stationary state. Instead, in the low-dissipation case, the stationary state is described by a Fourier-Crocco equation \((56)\), where the heat flux, thermal vorticity, and temperature gradient can form a triplet of mutually perpendicular vectors, see figure 2. In case of high dissipation, on the other hand, temperature gradient becomes proportional to the Laplacian of phonon momentum density, see figures 3(a) and 4.

In \([21]\), it is shown based on kinetic theory of phonons that heat conductivity (in the normally parabolic heat equation) may depend on the gradient of temperature, causing non-local behavior, similar as \([3]\). In phonon hydrodynamics, similar behavior is achieved by traveling waves of phonon momentum, but the underlying dynamical equations are different (hyperbolic in the first approximation, not parabolic). In \([22]\), the stationary Guyer-Krumhansl equation is verified experimentally, showing indeed that heat flux does not need to be aligned with the temperature gradient. However, here we discuss a more delicate feature of heat transfer, propagation of phonon vortices (not contained in the Guyer-Krumhansl equation).

In \([23,24]\), a linearized Boltzmann equation for phonons is given a hydrodynamic approximation that leads to the drifting and drift-less second-sound velocities. The drifting velocity corresponds to the distribution function when phonons are drifting with a collective velocity, which is the MaxEnt estimate of the hydrodynamic distribution used also in this manuscript. The drifting and drift-less second sound velocities are discussed in \([25]\) in the context of two-dimensional materials, resulting in disappearance of the drifting velocity in the thermodynamic limit. On the other hand, second sound was not observed in Germanium at temperatures from 50K to 300K \([26]\).

When carrying out on the zero-th order Chapman-Enskog reduction, phonon hydrodynamics in this manuscript represents a system of quasilinear first-order hyperbolic equations, so it describes hyperbolic heat conduction \([27–30]\). In the one-dimensional approximation and after linearization, the equations simplify to the standard Maxwell-Cattaneo-Vernotte equation \([17]\). Finally, when we go beyond the zero-th Chapman-Enskog approximation in the irreversible evolution of phonon momentum density, we obtain higher-order (Laplacian) terms corresponding to the dissipation in the Guyer-Krumhansl equation. Therefore, our phonon hydrodynamics can be seen as a non-linear generalization of the (linear) Guyer-Krumhansl equation \([13]\). Dynamics of thermal vorticity based on solutions of the phonon kinetic theory was already observed in \([19,20]\), but here we also formulate the corresponding phonon hydrodynamics.

Convective terms in the equation for the drift velocity of phonons have already been found in \([31]\) by means of Chapman-Enskog expansion of the Callaway model \([32]\) (kinetic theory for phonons with two relaxation times). The same authors have later generalized the theory by the Maximum Entropy (MaxEnt) principle \([33]\), obtaining a formulation equivalent with equations obtained by projecting the Hamiltonian structure of kinetic theory to phonon hydrodynamics \([11]\). In \([34]\), effects of a nonlinear dispersion relation (namely the Dubey relation \([k] = \omega e^{-\mu (1 + b \omega^2)}\) are discussed (while replacing the whole space by the first Brillouin zone). In particular, MaxEnt (expanded to the first order in the phonon momenta square) gives an approximate dependence of the free energy on phonon momentum, which affects the speed of second sound propagation, albeit not affecting the overall structure of the equations (for instance being first-order and hyperbolic). In the present manuscript, we use the same Hamiltonian projection as in \([11]\), but we go beyond by taking a concrete formula for the dispersion relation (though the simplest possible one—linear), by carrying out the first Chapman-Enskog expansion to obtain also the viscous-like terms, and by showing how phonon vorticity is propagated. Another work where a Navier–Stokes-like equation for phonon hydrodynamics was derived is \([35]\), and here we enrich the discussion by showing the effects of phonon vorticity on the hydrodynamic level.

A theory of hyperbolic heat transport was developed in the context of Extended Rational Thermodynamics (RET) \([7,16,36]\), including comparison with the heat-pulse experiment \([37]\). In the present manuscript, we show

\[
\begin{align*}
\mathbf{f}_T^+, \mathbf{f}_T^-, \mathbf{f}_L & \rightarrow \mathbf{m}_T^+, \mathbf{m}_T^-, \mathbf{m}_T^L; \mathbf{s}_T^+, \mathbf{s}_T^-, \mathbf{s}_T^L \\
\mathbf{f} & \rightarrow \mathbf{s}, \mathbf{m}
\end{align*}
\]

**Figure 1.** Distribution functions of positively and negatively transversely and longitudinally polarized phonons, \(f_+, f_-, f_L\), can be summed to the overall distribution function \(f\). They can be also projected to the momentum and entropy densities of the particular polarizations, \(m_T^+, m_T^-, m_T^L\) and \(s_T^+, s_T^-, s_T^L\). These densities can be then summed to the overall momentum density and entropy density of phonons, \(m\) and \(s\). Finally, the overall distribution function \(f\) can be projected to \(m\) and \(s\). Arrows indicate Poisson bracket projections \([10,11]\).
the Hamiltonian structure of the theory without linearization, which keeps also the non-linear vorticity-dependent terms, that are important for inertial effects, for effects of thermal vorticity, and for Galilean invariance.

Another description of hyperbolic heat transport was developed within the Internal Variables Theory (IVT) [15, 38]. This theory is not constructed by reduction from a more microscopic description (in contrast with kinetic theory), but by extending the Fourier heat equation by including additional internal variables and parameters, which remain unspecified. IVT shows a better agreement with the experimental data than the kinetic theory for the same number of state variables, but it also includes more fitting parameters. Comparison of IVT with the GENERIC framework (used in this paper) can be found in [39].

Apart from RET as IVT, there are other theories of hyperbolic heat transport coming from non-equilibrium thermodynamics (NET). NET was actually founded by Boltzmann [40], Onsager [41, 42], and others [43–45]. Subsequent theories of NET, as Rational (Extended) Thermodynamics [7, 46], or Extended Irreversible Thermodynamics [47, 48], bring the possibility to derive evolution equations for additional state variables.

There is plenty of successful applications of these theories, ranging from non-Newtonian fluids and solids to theories of charged mixtures [49]. However, it is often difficult to fully determine the reversible terms that are not visible in the formula for entropy production, which is why we use a different framework, that builds upon Hamiltonian mechanics and has already been applied to radiation hydrodynamics [50]. Here, we rely on the General equation for Non-Equilibrium Reversible-Irreversible Coupling (GENERIC) [10–51, 53].

The plan of this paper follows. Section 2 recalls kinetic theory of polarized phonons, in particular the evolution of the distribution function of the phonon quasiparticles. Section 2.2 then simplifies the description to the overall distribution of phonons, disregarding the polarizations, and it adds irreversible evolution in the form of the Callaway model (collisions of phonons between themselves and with impurities of the lattice). Section 3 represents the main part of this paper as it contains the phonon hydrodynamics that emerges by reduction of the kinetic theory. The reversible part of the hydrodynamic theory is expressed by equations (26b) or (30a) while their full form (in the zero-th Chapman-Enskog approximation) is in equation (37). Equation (45) then show phonon hydrodynamics within the first Chapman-Enskog approximation, where higher-order dissipative terms (Laplacian-like) appear. The Fourier-Crocco equation (50) relates thermal vorticity, temperature gradient, and heat flux, which leads to a similarity between temperature in heat transfer and pressure in aerodynamics (figure 5). In order to obtain the enhanced precision, we rely on geometric mechanics and Poisson reductions, which represent a cornerstone of the GENERIC framework, summarized in the following section.

1.1. Introduction to GENERIC

The General equation for Non-Equilibrium Reversible-Irreversible Coupling (GENERIC) framework combines (reversible) Hamiltonian evolution with (irreversible) gradient dynamics. The framework was established in [51, 52] and further developments can be found in monographs [10, 53]. An advantage of GENERIC is that it determines the reversible part of the evolution of given state variables because it better exploits the results of geometrical mechanics. Evolution of that functional is then given within GENERIC in two equivalent forms,

\[ \dot{x}^i = L^i + \frac{\delta E}{\delta x^i} + \frac{\delta \Xi}{\delta x^i} \]  \

\[ \dot{A} = \{A, E\} + \left( \frac{\delta A}{\delta x^i} \frac{\delta \Xi}{\delta x^i} \right) \bigg|_{x^* = x^*}, \tag{1} \]

where the latter is a sort of weak formulation. The evolution consists of two parts, a reversible one and an irreversible one. Note that \( \delta^*/\delta^* \) is the functional derivative, see [10, 53, 57]. \( \langle \cdot, \cdot \rangle \) is a duality (also represented by contraction over repeated indexes), \( E \) is the total energy of the isolated system under consideration, \( \Xi(x, x^*) \) is a dissipation potential, and \( S \) is the total entropy. We denote by \( x^* \) the dual variables to the state variables \( x \), which are also called conjugate entropic variables, eventually identified with derivatives of the total entropy \( S \).

If the dissipation potential is quadratic,

\[ \Xi = \frac{1}{2} \langle x^* \rangle M^{ij}(x, x^*) \langle x_j^* \rangle, \tag{2} \]

the GENERIC evolution (1) can be rewritten as

\[ \partial_t x^i = L^i + M^{ij} S_{x^j}, \tag{3} \]

where the symmetric operator \( M^{ij} \) (called a dissipative matrix) works as in the Ginzburg-Landau theory [59, 60].

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4 It is equivalent with the metriplectic systems if the dissipation potential is considered quadratic [54, 55]. However, in cases with non-quadratic dissipation potentials, both frameworks are different [56]. The GENERIC framework has also a precursor in the so called one-generator formalism [57] and in the nonlinear Onsager-Casimir (NOC) equations [38].
The reversible part is generated by a Poisson bivector $L$ or a Poisson bracket $\{\cdot, \cdot\}$, which are related through
\[
\{A, B\} = \langle dA, L^i, dB \rangle = \frac{\delta A}{\delta x^i} L^j \frac{\delta B}{\delta x^j} \quad \text{and} \quad L^j = \{x^i, x^j\}.
\]  
(4)

The bivector is a twice contravariant skew-symmetric tensor field on the space of the state variables, which is Lie-dragged by the Hamiltonian evolution \[61\]. The latter property then ensures validity of the Jacobi identity,
\[
\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \quad \forall A, B, C
\]  
(5)

while the skew-symmetry of $L$ gives skew-symmetry of the bracket,
\[
\{A, B\} = -\{B, A\} \quad \forall A, B.
\]  
(6)

A particular implication of the skew-symmetry is conservation of energy, $\dot{E} = \{E, E\} = 0$, by the Hamiltonian mechanics. Finally, an equivalent way to express the reversible part of the evolution is
\[
(\partial_t \mathbf{x})_{\text{rev}} = \{\mathbf{x}, E\},
\]  
(7)

as can be verified by the first relation of (4). Once we have recalled the GENERIC framework, we are now in position to formulate kinetic theory of phonons within that framework, which is the subject of the following section.

2. Kinetic theory of phonons

Phonons are quasiparticles obtained by diagonalization of the Hamiltonian of a crystal \[62\], and their motion is described by Hamilton canonical equations \[4\], constructed from the canonical Poisson bracket and from the energy of phonons. The purpose of this section is to formulate Hamiltonian kinetic theory that governs the distribution functions of the phonons. Actually, phonons have three possible polarizations, two transverse ($T+\,$ and $T-\,$) and one longitudinal. Therefore, there are three distribution functions $f_{T+}(t, \mathbf{r}, \mathbf{p}), f_{T-}(t, \mathbf{r}, \mathbf{p})$, and $f_L(t, \mathbf{r}, \mathbf{p})$.

2.1. Kinetic theory of phonons with polarization

Similarly as the evolution of the distribution function of classical particles is Hamiltonian, and thus generated by a Poisson bracket and energy, evolution of the phononic distribution functions is also Hamiltonian, and generated by the sum of the Boltzmann Poisson brackets
\[
\{A, B\}^{f_{\rho+}, f_{\rho-}, f_L} = \sum_{\alpha \in \{T+, T-, L\}} \int d\mathbf{r} \int d\mathbf{p} f_\alpha \cdot \left( \frac{\partial A_{f_\alpha}}{\partial \mathbf{r}} \cdot \frac{\partial B_{f_\alpha}}{\partial \mathbf{p}} - \frac{\partial B_{f_\alpha}}{\partial \mathbf{p}} \cdot \frac{\partial A_{f_\alpha}}{\partial \mathbf{r}} \right),
\]  
(8)

where $A$ and $B$ are two arbitrary functionals of the distribution functions \[10, 63\]. Symbols $A_{f_\alpha}$ stands for the functional derivative of $A$ with respect to $f_\alpha$. The reversible evolution equations implied by this Poisson bracket are (using equation (7))
\[
(\partial_t f_\alpha)_{\text{rev}} = - \frac{\partial f_\alpha}{\partial \mathbf{r}} \cdot \frac{\partial E_{f_\alpha}}{\partial \mathbf{p}} + \frac{\partial f_\alpha}{\partial \mathbf{p}} \cdot \frac{\partial E_{f_\alpha}}{\partial \mathbf{r}} \quad \forall \alpha \in \{T+, T-, L\}.
\]  
(9)

What is the energy $E(f_{T+}, f_{T-}, f_L)$? In general, we need the whole band structure of the crystal to find the precise formula for energy \[4\]. However, if we only restrict to low temperatures, to the first Brillouin zone, and acoustic phonons, we can approximate the energy as
\[
E = \int d\mathbf{r} \int d\mathbf{p} |\mathbf{p}| (f_{T+} + f_{T-} + f_L),
\]  
(10)

where $c$ is the Debye velocity and $|\mathbf{p}|$ is the Eulerian norm of the momentum \[64\].

The evolution equations obtained from a Poisson bracket are reversible both in the sense of the time-reversal transformation \[65\] and in the sense that they preserve the bosonic entropy \[66\],
\[
S^{f_{\rho+}, f_{\rho-}, f_L} = \sum_{\alpha} S^{\text{kinetic}, \alpha}(f_\alpha)
\]
\[
\text{with} \quad S^{\text{kinetic}, \alpha}(f_\alpha) = -\frac{k_B}{h^3} \int d\mathbf{r} \int d\mathbf{p} (h^3 f_\alpha \ln(h^3 f_\alpha) - (1 + h^3 f_\alpha) \ln(1 + h^3 f_\alpha)).
\]  
(11)

Indeed, any integral of a function of the distribution functions is a Casimir of the Poisson bracket, that is $\{S, B\}^{f_{\rho+}, f_{\rho-}, f_L} = 0 \forall B$, which can be used also to stability analysis of the Hamiltonian equations \[67\].

Instead of the distribution functions, which contain a large amount of information, we can reduce the description to the hydrodynamic fields of the distribution functions, namely the momentum densities and entropy densities.
where $\theta_0$ stands for a real-valued function, for instance the integrands in entropy (11).

The Poisson bracket for the hydrodynamic fields is obtained by plugging functionals that depend only on the hydrodynamic fields, $\mathbb{A}(m_\alpha(f_{\alpha}), s(f_\alpha))$, to the Poisson bracket (8). The resulting Poisson bracket reads

$$\{A, B\} = \sum_\alpha \int ds_\alpha \left( \partial_t A_{\alpha \alpha} B_{\alpha \alpha} - \partial_t B_{\alpha \alpha} A_{\alpha \alpha} \right) + \sum_\alpha \int dm_{\alpha \alpha} \left( \partial_t A_{\alpha \alpha} B_{\alpha \alpha} - \partial_t B_{\alpha \alpha} A_{\alpha \alpha} \right),$$

and the implied reversible evolution equations are

$$\begin{align*}
\partial_t s_\alpha &= - \partial_j (s_\alpha E_{\alpha \alpha}) \\
\partial_t m_{\alpha \alpha} &= - \partial_j (m_{\alpha \alpha} E_{\alpha \alpha}) - s_\alpha \partial_i E_{\alpha \alpha} - m_{\alpha \alpha} \partial_i E_{\alpha \alpha}
\end{align*}$$

2.2. Overall kinetic theory of phonons

When the various polarizations of phonons are not important, we may reduce the kinetic theory of polarized phonons to the overall kinetic theory that describes evolution of the overall distribution function

$$f = f_{T+} + f_{T-} + f_L.$$  

The Poisson bracket for the overall distribution function $f$ follows from the Poisson bracket for state variables $f_{T+}, f_{T-},$ and $f_L$ (equation (8)). When we plug functionals that depend only on $f(f_{T+}, f_{T-}, f_L)$ to the Poisson bracket, we obtain bracket

$$\{A, B\} = \int dr \int df \cdot \left( \frac{\partial A_f}{\partial r} \frac{\partial B_f}{\partial p} - \frac{\partial B_f}{\partial r} \frac{\partial A_f}{\partial p} \right),$$

which is the standard Poisson bracket for the one-particle distribution functions (also called Boltzmann bracket), see for instance [10, 68]. The reversible evolution equation implied by this Poisson bracket is

$$\begin{align*}
\partial_t f &= \mathcal{E}_f + \frac{\partial E_f}{\partial r} + \frac{\partial f}{\partial p} \frac{\partial E_f}{\partial r}
\end{align*}$$

In order to write the evolution equation in a closed form, we need the energy $E(f)$. In our simplified situations (restriction to the first Brillouin zone, acoustic phonons with small $k$ vectors), the energy is

$$E = \int dr \int dp |p| f,$$

where $c$ is the Debye speed [4]. Once the Debye velocity $c$ is determined, the reversible part of the evolution equation for $f$ is fully specified. In particular, the constant $c$ can be obtained from measurement of the second sound speed. For instance, for a NaF crystal, equation (49) relates the Debye velocity with the speed of second sound, theoretical limit of which at $T = 0$ is $c = 2020.9$ m/s [69]. The heat capacity of low-temperature NaF is $c_v = 234N_A \theta_0(T/\theta_D)^3$ for temperature between 0.6K and 15K; the Debye temperature is found to be $\theta_D = (466 \pm 5)$K [70]. The density is approximated by $\rho \approx 2.851 g/cm^3$ [71], and the molar mass is $M \approx 42g/mol$.

However, the evolution of $f$ actually contains two parts, a reversible (Hamiltonian) part, which is given by the Poisson bracket and energy, and an irreversible part, which is given by a dissipation potential and entropy. The entropy can be derived from the entropy of phonons with polarization, equation (11). By maximization of the entropy subject to the constraint (15), we obtain the entropy

$$S(f) = \frac{3k_B}{h^2} \int dr \int dp \left( \frac{h^2 f}{3} \ln \left( \frac{h^2 f}{3} \right) - \left( 1 + \frac{h^2 f}{3} \right) \ln \left( 1 + \frac{h^2 f}{3} \right) \right).$$

The irreversible evolution of the distribution function consists of collisions between phonons (normal process, conserving both energy and momentum) and collisions with impurities of the crystal (resistive process, conserves energy, but not momentum) [7]. These two processes are described by the so called...
subject to the constraints given by the overall hydrodynamic momentum and energy that only the hydrodynamic energy is known, respectively, models of hyperbolic heat conduction as the linear Guyer-Krumhansl convective term, which expresses inertial effects of the motion of phonons and which is missing in the usual conjugate entropy Similarly as in mass density Here, Poisson bracket expressing kinematics of the hydrodynamic Let us now further reduce the Boltzmann Poisson bracket, that expresses kinematics of kinetic theory, to the 3. Overall hydrodynamics of phonons The next section contains a further reduction of this kinetic theory to phonon hydrodynamics. Here, \( m^* \) and \( e^* \) are the derivatives of the hydrodynamic entropy (34) with respect to the momentum density and energy density, respectively, \( m^* = \frac{\partial E}{\partial m} \) and \( e^* = \frac{\partial E}{\partial e} \). A feature of distribution functions \( f_R \) and \( f_N \) is that the Callaway irreversible evolution conserves momentum and energy (in the normal processes) or only the energy (in the resistive processes), \[ \int d\mathbf{p}f(f - f_N) = 0, \quad \int d\mathbf{p}|f - f_N| = 0, \quad \int d\mathbf{p}|f - f_R| = 0. \] The normal and resistive relaxation times, \( \tau_N \) and \( \tau_R \), have typically values around \( \tau_R \approx 3 \times 10^{-6} \) s and \( \tau_N \approx 2 \times 10^{-7} \) s [16]. The dependence of relaxation times on temperature has been measured in solid helium and LiF crystals [72], in NaF [73], or in Bi [74]. The final evolution equation is the sum of the reversible part (17) and irreversible part (20), \[ \partial_t f = \left( \frac{\partial f}{\partial r} \right) \cdot \nabla E_f + \left( \frac{\partial f}{\partial p} \right) \cdot \frac{\partial E_f}{\partial r} + \frac{1}{\tau_R}(f - f_R) - \frac{1}{\tau_N}(f - f_N). \] The next section contains a further reduction of this kinetic theory to phonon hydrodynamics. 3. Overall hydrodynamics of phonons Let us now further reduce the Boltzmann Poisson bracket, that expresses kinematics of kinetic theory, to the Poisson bracket expressing kinematics of the hydrodynamic fields of momentum density and entropy density, \[ m_i(t, \mathbf{r}) = \int d\mathbf{p}f(t, \mathbf{r}, \mathbf{p}) \quad (24a) \]
\[ s(t, \mathbf{r}) = \int d\mathbf{p}|f(t, \mathbf{r}, \mathbf{p})|. \quad (24b) \] Here, \( \eta(f) \) is the entropy density of phonons \( s^{HF} \) (equation (34)), but in general it can be an arbitrary function of \( f \). 3.1. Reversible evolution Plugging arbitrary functionals \( A \) and \( B \) dependent only on momentum and entropy densities, \( A(m, s) \) and \( B(m, s) \), into bracket (16), we obtain a part of the Poisson brackets of fluid mechanics, \[ \{A, B\}^{(m, s)} = \int d\mathbf{r}(\partial_i A m_j B_m - \partial_j B_i A_m) + \int d\mathbf{r}m_i(\partial_j A m_j B_m - \partial_j B m_j A_m), \quad (25) \] which is actually the same as the part of the Poisson bracket for fluid mechanics of normal particles (without mass density), see [10, 11] for details of the calculation. The reversible evolution implied by this bracket reads \[ \partial_t s = -\partial_j(s E_m) \quad (26a) \]
\[ \partial_t m_i = -\partial_j(m_i E_m) - s \partial_j E_m - m_j \partial_j E_m. \quad (26b) \] Similarly as in fluid mechanics, the momentum equation implied by this Poisson bracket contains a non-linear convective term, which expresses inertial effects of the motion of phonons and which is missing in the usual models of hyperbolic heat conduction as the linear Guyer-Krumhansl model [1, 13]. These equations, however, are equivalent with those found in [11, 34]. Note that although the heat flux is proportional to the momentum density (equation (46)), the convective terms can not be written as \( \mathbf{q} \cdot \nabla \mathbf{q} \), since the derivative \( E_m \) also depends on the field of energy density (see equation (47)). It is advantageous to transform the momentum density to the field \[ \mathbf{w} = \mathbf{m}/s, \quad (27) \] called conjugate entropy flux, because then the subsequent reduction to the Fourier heat conduction becomes simpler, see section 3.6. Derivatives of functionals of \( \mathbf{m} \) and \( s \) transform as

\[ \frac{\partial f}{\partial m_j} = \frac{\partial f}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial m_j}, \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial s}, \]
\[ F_m \rightarrow \frac{1}{s} F_w \quad \text{and} \quad F_i \rightarrow F_i - \frac{1}{s} F_w \cdot \omega, \]  
\( \text{(28)} \)

and when we plug functionals dependent only on \((s, w)\) into the Poisson bracket for \((m, s)\), we obtain a Poisson bracket for \(s\) and \(w\),

\[ \{ A, B \}^{\text{(Cattaneo)}} = \int d\tau (\partial_i A_i B_w - \partial_i B_i A_w) + \int d\tau \frac{1}{s} (\partial_i w_j - \partial_j w_i) A_i B_w. \]  
\( \text{(29)} \)

We refer to this Poisson bracket as the Cattaneo Poisson bracket \([10]\) because the resulting model simplifies to the usual Cattaneo equations when we consider only one-dimensional processes in the linear approximation, see appendix \(B\). The resulting equations are also close (except to the vorticity-dependent terms) to the description of heat conduction in the SHTC (or Godunov-Peshkov-Romenski) framework \([75]\). The reversible part of the evolution equations, stemming from this bracket, is obtained by formula \((7)\),

\[ (\partial_t s)_{\text{rev}} = \{ s, E \}^{\text{(Cattaneo)}} = \bar{\partial}_i E_m, \]  
\( \text{(30a)} \)

\[ (\partial_t w_i)_{\text{rev}} = \{ w_i, E \}^{\text{(Cattaneo)}} = -\bar{\partial}_i E_s + \frac{1}{s} (\partial_i w_j - \partial_j w_i) E_w. \]  
\( \text{(30b)} \)

The latter equation can be also rewritten as

\[ (\partial_t w)_{\text{rev}} = -\nabla E_s + \frac{1}{s} E_w \times \omega, \]  
\( \text{(30c)} \)

where \(\omega = \nabla \times w\) is the vorticity of the field \(w\). From the evolution equation for entropy, we can see that the entropy flux is given by \(E_w\), and the field \(w\) can be thus called the conjugate entropy flux.

What is the heat flux implied by equation \((30c)\)? If we define the heat flux as the flux of energy, we have to first find the evolution equation for the total energy density \(E(s, w)\). And since we know evolution equations for both \(s\) and \(w\), we obtain the balance of total energy by chain rule,

\[ \partial_t E = E_i \partial_i s + E_w \partial_i w_i = -\partial_i (E_i E_w), \]  
\( \text{(31)} \)

which means that the heat flux (or flux of energy) is

\[ \mathbf{q} = E_i E_w = -(T^{(w)})^2 S_w, \]  
\( \text{(32)} \)

where \(T^{(w)} = \left(\frac{E}{s}ight)_w\). In other words, the heat flux is equal to the product of temperature \(T^{(w)}\) and entropy flux \((E_w)\), which is a consequence of the model, not an assumption. Another (equivalent) relation between the heat flux and momentum density is derived later (equation \((46)\)). Anyway, field \(w\) is referred to as the conjugate entropy flux because the derivative of energy with respect to that field is the reversible part of the entropy flux.

Alternatively, temperature can be defined as \(T = \left(\frac{E}{s}ight)_m\). Note, however, that in general \(T^{(w)} \neq T\) because \(\left(\frac{E}{s}\right)_w \neq \left(\frac{E}{s}\right)_m\). The relation between the two temperatures reads

\[ T^{(w)} = T + \frac{\sqrt{T^2 + 4c^2w^2}}{2}. \]  
\( \text{(33)} \)

This is connected with the various notions of temperature in non-equilibrium thermodynamics \([76]\). From the theoretical point of view, both temperatures are equivalent, but in practical measurements, one of them might be advantageous, depending on whether the thermometer interacts with the \(w\) field or the \(m\) field. Assuming that the thermometer measures the energy flux, which is closely related with the \(m\) field, temperature \(T\) should be more pertinent.

What is the energy \(E(s, w)\), that is necessary to write the evolution equations in a closed form? It can again be derived by maximization of the entropy for the distribution function \((19)\) subject to the constraints given by the knowledge of the hydrodynamic fields (equation \((24)\)). Appendix \(A\) contains details of the calculation, resulting in entropy density

\[ s^H = \frac{4}{3} \left(\frac{6\sigma}{c^2 c} \right)^{1/4} e^{3/4} \frac{3 \chi^{1/4}}{2^{7/4}} (3 - \chi)^{3/4} (1 - \chi)^{1/2}, \]  
\( \text{(34)} \)

where \(\sigma = \frac{2\pi^2 k_b^2}{16\sqrt{\pi}}\) is a Stefan-Boltzmann constant (dependent on the particular \(c\) for a given material) and \(\chi = \frac{s}{T} = \frac{\sigma c^2 w^2}{\pi} - \frac{1}{7} \left(1 - \frac{1}{2} c^2 w^2\right)^{7/4}\) is the Eddington factor \([77]\). This entropy is the same as in \([7, 78]\). In the following section, the reversible evolution is equipped with an irreversible counterpart.

### 3.2. Zero-th order irreversible evolution

By integration of the Callaway model (equation \((20)\)) with respect to momentum, we obtain the following zero-th approximation of the irreversible evolution of the \(s\) and \(m\) fields,
the evolution equation for the distribution function \( f \) in rarefied gases leads to the Navier–Stokes equations, which contain the viscous terms, on top of the hyperbolic Euler equations [35, 44]. Similarly, the BGK equation, which is actually a form of the here used Callaway model, gives the Navier–Stokes equations, similarly as in [83, 84]. Our goal is now to obtain the viscous-like terms that appear in the momentum equation in the first Chapman–Enskog approximation of the Callaway model also in the phonon hydrodynamics, similarly as in [85] but in our state variables.

The Chapman–Enskog expansion starts with the projection of the equation for \( f \) (equation (23)) to the momentum density \( m \),

\[
(\partial_t m_{ij})_{irr} = -\frac{1}{\tau_R} m_{ij},
\]

\[
(\partial_t s)_{irr} = \frac{1}{\tau_R E_i} m \cdot E_w,
\]

where the latter equation is obtained from the former by the requirement of energy conservation, \((\partial_t e)_{irr} = 0\).

These equations can be transformed into the \( s, w \) variables as follows,

\[
(\partial_t s)_{irr} = \frac{1}{\tau_R E_i} w \cdot E_w,
\]

\[
(\partial_t w)_{irr} = -\frac{1}{\tau_R} w - \frac{w^2}{\tau_R E_i} - \frac{E_w}{\tau_E E_i} \cdot w.
\]

The final evolution equations for the hydrodynamic fields \( s \) and \( w \) are the sum of the reversible (Hamiltonian) evolution and irreversible evolution (here, \( E_i = T(w) \)),

\[
\partial_t s = -\partial_t E_w + \frac{1}{\tau_R E_i} w \cdot E_w,
\]

\[
\partial_t w = -\partial_t E_i + \frac{1}{s} (\partial_t w_j - \partial_j w_i) E_{w_i} - \frac{1}{\tau_R} w - \frac{w^2}{\tau_R E_i} - \frac{E_w}{\tau_E E_i} \cdot w,
\]

which is a nonlinear generalization of the Maxwell–Cattaneo–Vernotte (MCV) equation, see appendix B. Equation (37) are a system of quasilinear first-order partial differential equations, and they are also symmetric hyperbolic because they can be rewritten as a system of conservation laws in the \( s, m \) variables, which is suitable for the Godunov–Boillat theorem [36, 79–81]. The dissipative part (terms that contain \( \tau_k \)) can be seen as the zero-th order approximation in the Chapman–Enskog expansion of equation (23). In the following section, we go for the first Chapman–Enskog approximation, which brings viscous-like terms to the evolution equations.

### 3.3. First-order irreversible evolution

The first-order Chapman–Enskog approximation of the Boltzmann equation for rarefied gases leads to the Navier–Stokes equations, which contain the viscous terms, on top of the hyperbolic Euler equations [44, 82]. Similarly, the BGK equation, which is actually a form of the here used Callaway model, gives the Navier–Stokes equation as the first Chapman–Enskog approximation [83, 84]. Our goal is now to obtain the viscous-like terms that appear in the momentum equation in the first Chapman–Enskog approximation of the Callaway model also in the phonon hydrodynamics, similarly as in [85] but in our state variables.

The Chapman–Enskog expansion starts with the projection of the equation for \( f \) (equation (23)) to the momentum density \( m \),

\[
\partial_t m_{ij} = -\frac{\partial}{\partial t^i} \left( \int d\mathbf{p} \frac{C_{ij}}{|\mathbf{p}|} f \right) - \frac{1}{\tau_R} m_{ij}.
\]

For simplicity, we now disregard the upper/lower positions of the indexes, which means that the metric tensor is implicitly present in the calculations. In order to provide an approximation for the unknown integral on the right hand side, \( Q_{ij} = \int d\mathbf{p} p_i p_j/|\mathbf{p}| f \), the evolution of that integral is obtained again from the evolution equation for \( f \),

\[
\partial_t Q_{ij} = -\frac{\partial}{\partial t^k} \left( \int d\mathbf{p} \frac{P_i P_j P_k}{|\mathbf{p}|} f \right) + \frac{1}{\tau_N} \int d\mathbf{p} \frac{P_i P_j}{|\mathbf{p}|} (f_N - f) + \frac{1}{\tau_R} \int d\mathbf{p} \frac{P_i P_j}{|\mathbf{p}|} (f_R - f).
\]

Appendix E contains approximate values of the integrals on the right hand side of the latter equation, with which the evolution equation for \( Q_{ij} \) becomes

\[
\partial_t Q_{ij} \approx \frac{\partial}{\partial t^k} \left( \frac{16 \pi^3 k_B^4}{75 h^4 (c_s)^3 c_i} (m^+ \delta_i^j + (m^+)^2 \delta_i^j + m^+ \delta_j^i) \right)
\]

\[
- \left[ \frac{1}{\tau_R} + \frac{1}{\tau_N} \right] \left( Q_{ij} - \frac{4 \pi^3 k_B^4}{15 h^4 (c_s)^3 c_i} \delta_i^j \right).
\]

In order to solve approximately (39), we first cast the equation into a dimensionless form. The underlying evolution equation for the distribution function (23) has the dimensionless form
\[
\frac{\partial \tilde{f}}{\partial t} = \frac{-St}{|p|} \cdot \tilde{\nabla} \tilde{f} + \frac{1}{De_N} (\tilde{f}_N - \tilde{f}) + \frac{1}{De_R} (\tilde{f}_R - \tilde{f}),
\]  

(40)

where the quantities with hats are dimensionless, \(St = \frac{d}{\tau}\) is the Strouhal number, \(\bar{t}\) is a typical time value, \(\tau\) is a typical spatial dimension, and \(De_N = \tau_N / \bar{t}\) and \(De_R = \tau_R / \bar{t}\) are the Deborah numbers corresponding to the two relaxation times. Note that the Deborah numbers are close to the Knudsen numbers used for instance in [85, 86]. Because the phonon gas first reaches the quasi-equilibrium manifold (defined by the momentum density and energy density) before it reaches the equilibrium (defined only by the energy), the normal relaxation time is assumed to be much smaller than the resistive one, \(\tau_N \ll \tau_R\), which means that \(De_N \ll De_R\). Therefore, the second moment \(Q_4\) can be expanded in powers of the normal Deborah number,

\[
Q_4 = Q_4^{(0)} + De_N Q_4^{(1)} + O(De_N^2).
\]

(41)

Plugging this expansion into the equation for \(Q_4\) (equation (39)), the \(De_N^2\) order gives

\[
Q_4^{(0)} = \frac{4 \pi^4 k_B^4}{15 \hbar^2 (e^*)^4 e^4}
\]

(42a)

and the order \(De_N^0\) gives

\[
\partial_t Q_4^{(0)} = \partial_t \left( \frac{16 \pi^4 k_B^4}{75 \hbar^3 (e^*)^3 e^4} (m_i^* \delta_{ij} + (m^*)^j \delta_{ij} + m_i^* \delta_{ij}) \right) - \bar{t}^{-1} \left( 1 + \frac{De_N}{De_R} \right) Q_4^{(1)}.
\]

(42b)

These two orders then combine to

\[
Q_4 = \frac{4 \pi^4 k_B^4}{15 \hbar^2 (e^*)^4 e^4} + \tau \partial_t \left( \frac{16 \pi^4 k_B^4}{75 \hbar^3 (e^*)^3 e^4} (m_i^* \delta_{ij} + (m^*)^j \delta_{ij} + m_i^* \delta_{ij}) \right) - \tau^4 \frac{4 \pi^4 k_B^4}{15 \hbar^2 e^4} \partial_t (e^*)^{-4} \delta_{ij}
\]

(43)

where \(\tau^{-1} = \tau_R^{-1} + \tau_N^{-1}\) is the effective relaxation time.

The approximate \(Q_4\) from equation (43) can be now plugged into the momentum equation (38a). However, since we have already obtained the reversible part of the latter equation precisely, using the Poisson reduction, we use the approximate \(Q_4\) only in the irreversible part of the equation,

\[
(\partial_t m_i)_{irr} = -\frac{1}{\tau_R} m_i - \partial_t \left( \frac{16 \pi^4 k_B^4}{75 \hbar^3 (e^*)^3 e^4} (m_i^* \delta_{ij} + (m^*)^j \delta_{ij} + m_i^* \delta_{ij}) \right) - \tau \partial_t (e^*)^{-4} \delta_{ij}.
\]

(44)

The reversible part of the momentum equation is contained in equation (26b), and the overall evolution of the momentum density is the sum of the reversible and irreversible parts,

\[
\partial_t m_i = \partial_t \left( m_i \frac{m_i^*}{e^*} \right) - s(m, e) \partial_t \left( \frac{1}{e^*} \right) + m_j \partial_j m_i^* \frac{m_i^*}{e^*} - \frac{1}{\tau_R} m_i - \partial_t \left( \frac{16 \pi^4 k_B^4}{75 \hbar^3 (e^*)^3 e^4} (m_i^* \delta_{ij} + (m^*)^j \delta_{ij} + m_i^* \delta_{ij}) \right) - \tau \partial_t (e^*)^{-4} \delta_{ij},
\]

(45a)

where derivatives of energy were replaced with the corresponding derivatives of entropy, \(E_m = -S_m / S_p\) \(E_e = (S_e)^{-1}\). The right hand side of the momentum equation contains the conjugate variables \(m^* = \frac{\partial S}{\partial m}\) and \(e^* = \frac{\partial S}{\partial e}\), but we know the entropy (equation (34)), which makes the right hand side explicit in \(e\) and \(m\).

Finally, the momentum equation needs to be supplemented with the energy equation,

\[
\partial_t e = \int d\textbf{p} \text{p} \cdot \partial_t \textbf{f} = -\partial_t (e^2 m_i),
\]

(45b)

which also shows that the heat flux is proportional to the momentum density,

\[
\textbf{q} = e^2 \textbf{m}.
\]

(46)

Equation (45) governs evolution of momentum density and energy density of phonons in the first Chapman-Enskog approximation.

In order to better see the viscous-like character of the irreversible terms in the momentum equation, let us use the approximate values of the Lagrange conjugate variables (A9), which give that \(m^* / e^* \approx -\frac{3 \sigma}{4} \text{m} \) and \(\frac{16 \pi^4 k_B^4}{75 \hbar^3 (e^*)^3 e^4} = \frac{8 \sigma}{3 \xi (e^* e^*)} \approx \frac{4}{15} e^*.\) Neglecting higher-order terms in \(\text{m} / e\), we get that \(2 \sigma e^{-1} (e^*)^{-4} \approx e / 3\), and the momentum equation then simplifies to
Here, we neglect the second irreversible term in the equation for $m$ and $\epsilon$. However, they still represent a more precise version of the Guyer-Krumhansl equation, since they still contain the convective terms in the phonon momentum density. The precision is enhanced due to the Poisson reduction, which makes it possible to obtain the reversible part of the equation without linearizations. The viscous terms in the momentum equation are similar to those found in [85].

\[
\partial_t m_i = -\partial_j p - \partial_j \left( \frac{3c^2}{4e} m_i m_j \right) - \frac{\tau_c}{5} m_i + \frac{\tau_c^2}{5} \left( \partial_j m_i + \partial_i m_j - \frac{2}{3} \nabla \cdot \mathbf{m} \partial_j \right)
\]

with pressure

\[
p = -e + \frac{4}{3} e (2\sqrt{2} - 1) + \frac{3c^2}{2e} \frac{m^2}{1 + \sqrt{2}} = \frac{e}{3} - \frac{c^2 m^2}{4e} + \mathcal{O}(m^4) \approx \frac{e}{3} - \frac{c^2 m^2}{4e}.
\]

Equations (2) and (47) are a partial linearization of the full equations for $m$ and $\epsilon$. However, they still represent a more precise version of the Guyer-Krumhansl equation, since they still contain the convective terms in the phonon momentum density. The precision is enhanced due to the Poisson reduction, which makes it possible to obtain the reversible part of the equation without linearizations. The viscous terms in the momentum equation are similar to those found in [85], except for the terms with $\nabla \cdot \mathbf{m}$ (here, the viscous part of the Cauchy stress tensor is traceless). Equation (47) is similar to the evolution equation for heat flux in [87], but it still differs by the presence of the convective term. Numerical solutions to equations (2) and (47) can be seen in figures 3(a), (b), 4 while an analytical Poiseuille-like solution can be found in appendix F. Comments on when the convective terms are non-negligible through analysis of the non-dimensional form of equation (47) are added in 3.5.

Equations (2) and (47) in one dimension can be linearized around a stationary homogeneous state, $e \approx e_0 + \epsilon e$ and $m = \delta m$, which leads to a wave equation (neglecting the dissipative terms),

\[
\frac{\partial^2 e}{\partial x^2} = \frac{c^2}{3} \frac{\partial^2 e}{\partial t^2}.
\]

Therefore, the speed of the second sound can be related to the Debye velocity through $c_{(2nd)} = c/\sqrt{3}$, as reported in [11].

Although boundary conditions are out of scope of the present paper, let us briefly mention at least some results from the literature. Apart from no heat flux across an insulated wall $q \cdot n = 0$, one could consider zero heat flux also along the insulating wall (non-slip condition, $q = 0$), or a non-zero value (slip condition) analogous to the no-slip condition of velocity for rarefied gases [88, 89]. These conditions are especially relevant in narrow channels [90, 91]. Boundary conditions for the heat flux have also been explored within IVT [92], and a derivation of boundary conditions from the BGK equation for phonons can be found in [85].

In the following section, we discuss some features of equation (37) (obtained by the Poisson reduction and the zero-th Chapman-Enskog approximation).

3.4. Galilean invariance, inertia, and thermal vortices

Equation (37) have several features that are not present in the usual models for hyperbolic heat conduction, Galilean invariance, inertial effects, and effects of thermal vorticity. First, their reversible part (the Hamiltonian part) is Galilean invariant, as shown in appendix C. Galilean invariance of the irreversible part of the entropy equation is achieved if we count with the possible motion of the crystal (with velocity $\mathbf{V}$) by taking energy $e = e(s, w) + s \mathbf{V} \cdot \mathbf{w}$.

Second, the evolution equation for $w$ contains inertial terms, which are proportional to $E_w$. In other words, if we abruptly change the temperature gradient, it takes some time to change the direction of $w$.

The zero-th Chapman-Enskog approximate evolution of $w$ (37) also expresses how the thermal vorticity $\omega = \nabla \times w$ affects the evolution of $w$ itself. In a stationary state with negligible effects of the viscosity, the evolution equation for $w$ yields an analogy of the Crocco theorem [12]

\[
\nabla T^{(w)} = \frac{1}{s} E_w \times \omega - \frac{1}{\tau_c} w,
\]

here called the Fourier-Crocco equation. Figure 2 shows this equation in the case of two dimensions. In particular, for the limit of weak dissipation, $\tau_c \rightarrow \infty$, the heat flux is perpendicular to the temperature gradient. On the other hand, in the limit of strong dissipation, $\tau_c \rightarrow 0$, the heat flux aligns with the temperature gradient, as expected from the Fourier theory of heat conduction.

Note, however, that this description neglects the first-order Chapman-Enskog dissipation (viscosity). If $\tau$ is high, viscosity terms are non-negligible and break the Fourier-like behavior, as seen in figure 4.

How can be thermal vorticity $\omega$ created? Consider for instance a body with a gradient of thermal conductivity in the $x$-direction. Thermal conductivity is proportional to $\tau^{-1}$ (see the following section). When we cast the curl operator on the evolution equation for $w$ (37b), we obtain that

\footnote{Here, we neglect the second irreversible term in the equation for $w$ in (37) because it is of higher order in $w$.}
\[ \partial_t \omega \big|_{\omega=0} = - \nabla \left( \frac{1}{\tau_R} \right) \times \mathbf{w}, \]  

(51)
even in the case where \( \omega = 0 \). Note that now we have to allow for \( \tau_R \) not to be a constant. This means that thermal vorticity \( \omega \) is created when the gradient of \( \tau_R \) is not aligned with \( \mathbf{w} \). For instance, \( \mathbf{w} \) is created in the \( y \)-direction (by applying a temperature gradient to the body), some amount of thermal vorticity is created. We expect this behavior in thermal diodes \[93, 94\]. The thermal vorticity can be also created at the walls (by boundary conditions) and at sharp corners as in fluid mechanics \[12\].

What effects can be expected in the presence of thermal vorticity? Since the Fourier-Crocco equation \(50\) is similar to the classical Crocco equation from fluid mechanics, we can find inspiration in aerodynamics. Figure 3(a) shows the flow of phonons around an insulated cylinder. Since the cylinder is insulated, there is no heat flux across its boundary, so that \( \mathbf{w} \cdot \mathbf{n} = 0 \) at the boundary. In the stationary state, when the Fourier-Crocco equation applies, we have, moreover, that \( \nabla \cdot \mathbf{m} = 0 \), which brings us close to dynamics of incompressible fluids. From the Fourier-Crocco equation, we can expect that the temperature behind the cylinder is lower than the temperature in front of the cylinder. The analogy in classical fluid mechanics would be that the hydrodynamic drag force (or lower pressure behind the cylinder). In other words, an insulated obstacle in a flow of phonons should create a region of lower temperature behind the obstacle. Figures 3 and 4 show finite-element numerical simulations of two flows past a cylinder, using the FEniCS framework \[95\], Taylor–Hood elements in time, backward Euler time-stepping and enforcing slip boundary condition on the cylinder weakly with Nitsche’s method \[96\].

Similarly, figure 5 shows that flow of phonons along an insulated airfoil should create a region or lower temperature above the airfoil and a region of higher temperature below the airfoil. No such effects could be obtained without the non-linear vorticity-dependent terms, that are usually neglected in standard theories of hyperbolic heat conduction \[13\].

### 3.5. Reynolds number and Von Kármán vortex street

In this section, we rewrite equation \(47\) into a non-dimensional form in order to classify when the heat transport is dominated by convection and when the dissipation terms dominate. Moreover, we show that based on an analogue of Reynolds number, phonon hydrodynamics exhibits laminar flows or von Kármán vortex street.

We denote the characteristic length by \( L \), characteristic momentum by \( M \), characteristic energy by \( E \), and non-dimensional quantities are decorated by tilde. Finally, the characteristic time is computed as \( T = LE/M \). This leads to

\[
\frac{M}{T} \partial_t \tilde{m}_i = -E \frac{L}{3} \partial_t \tilde{m}_i + \frac{c^2 M^2}{4LE} \partial_t \tilde{m}_i - \frac{3c^2 M^2}{4LE} \partial_t \left( \frac{1}{\tau_R} \tilde{m}_i \tilde{m}_j \right) \\
- \frac{M}{\tau_R} \tilde{m}_i + \frac{\tau M c^2}{5L^2} \partial_t \left( \partial_j \tilde{m}_i + \partial_i \tilde{m}_j - \frac{2}{3} \nabla \cdot \tilde{m} \tilde{n}_j \right). 
\]

(52)

The ratio of the last three coefficients (namely \( \frac{3c^2 M^2}{4LE} \), \( \frac{M}{\tau_R} \) and \( \frac{\tau M c^2}{5L^2} \)) shows whether the convection or dissipation give more effect. Dividing the whole equation by \( \frac{3c^2 M^2}{4LE} \) gives
Figure 3. Flow past a cylinder with non-dominant dissipation (low $\tau$ and high $\tau_R$). Figures (a) and (b) show phonon vorticity $\omega$, temperature gradient $\nabla T^{(w)}$, conjugate entropy flux $w$, and temperature $T^{(w)}$ itself. (a) Phonon vorticity $\omega$ is depicted in color, black arrows show $\nabla T^{(w)}$ (out of scale), and white arrows indicate $w$ (out of scale). We can see that vorticity is negative above the cylinder and negative below. Moreover, $\nabla T^{(w)}$ is almost perpendicular to $w$, which coincides with the Fourier-Crocco equation (50). In regions where phonon vorticity is almost zero (green color), the orthogonal term is lost and Fourier-law emerges. (b) Temperature field for the fields in figure 3(a). As expected from the Fourier-Crocco equation, the temperature behind the cylinder is lower than in front of it.

Figure 4. Flow past a cylinder high dissipation (high $\tau$ and low $\tau_R$). Color and arrows depict the same fields as in figure 3(a), that is color indicates $\omega$, black arrows $\nabla T^{(w)}$, and white arrows $w$. In this high-dissipation regime, gradient of temperature $T^{(w)}$ is aligned with contours of $\omega$, which are nearly with $\nabla \times \omega = -\Delta \omega$ (in the stationary case, $\nabla \cdot w \approx 0$) due to the viscosity terms however, in regions where phonon vorticity and its derivatives are almost zero (green color), the viscosity-terms become negligible and Fourier-like behavior dominates.
\[
\frac{4LE}{3c^2M} \partial_t \delta_i \delta_i = - \frac{4E^2}{3c^2M^2} \partial_t \frac{\nabla T}{\nabla T} + \frac{1}{3} \partial_i \delta_i + \partial_j \left( \frac{1}{2} \delta_i \delta_i \right) - \partial_j \left( \frac{1}{2} \frac{\nabla T}{\nabla T} \nabla \delta_i \right).
\]

(53)

Inversion of the last two coefficients then leads to an analogue of Reynolds number

\[ Re_{\text{heat}} = \frac{15LM}{4\tau E} \]

(54)

while the ratio of the viscous and algebraic coefficients gives an analogue of Deborah number

\[ De_{\text{heat}} = \frac{3c^2\tau_R}{4T} = \frac{3c^2\tau_R M}{4LE}. \]

(55)

If the two numbers are large \((Re_{\text{heat}}, De_{\text{heat}} > 1)\), the convective term is dominant. If, however, one of them is small, \((Re_{\text{heat}} < 1 \text{ or } De_{\text{heat}} < 1)\), the respective dissipation (viscous-like or algebraic) dominate and convective terms can be neglected.

Figure 6 shows phonon flow past a cylinder with the no-slip boundary conditions on the cylinder and full slip on walls. Two different flows are presented with \(De_{\text{heat}} \approx 56000\) in both cases and \(Re_{\text{heat}} \approx 15\) and \(Re_{\text{heat}} \approx 150\), respectively. High Deborah number means that the algebraic dissipation is negligible. The higher Reynolds number, the weaker effect of diffusion, which causes a qualitative difference in the numerical solution—laminar flow for the lower Reynolds number and von Kármán vortex street for the higher one. Note that numerical edge stabilization has been used to enforce convergence of the solver.
3.6. Reduction to Fourier heat transport
When we asymptotically expand the conjugate entropy flux \( \mathbf{w} = w^{(0)} + \mathcal{O}(\epsilon^2) \), then from the evolution equation for \( \mathbf{w} \) we obtain that \( w^{(0)} = 0 \), \( w^{(1)} = -\nabla T^{(w)} \), and \( \mathbf{w} \approx -\tau_T \nabla T^{(w)} \). The evolution equation for entropy then becomes

\[
\partial_t s = \nabla \cdot \left( \tau_T \kappa \nabla T^{(w)} \right) + \frac{\tau_T}{T^{(w)}} (\nabla T^{(w)})^2,
\]

where \( \kappa = \frac{3\epsilon^2 \kappa_0}{\epsilon} \) is the coefficient in approximated energy \( e \approx e^{(0)} + \frac{1}{2} \kappa \mathbf{w}^2 \). In terms of energy density, this equation reads

\[
\partial_t e = -\nabla \cdot \left( -\tau_T \kappa T^{(w)} \nabla T^{(w)} \right),
\]

which means that the heat flux, or flux of energy, is \( \mathbf{q} = -\tau_T \kappa T^{(w)} \nabla T^{(w)} \), that represents Fourier law with thermal conductivity \( \lambda = \tau_T T^{(w)} \kappa \). Fourier heat conduction can be thus seen as the first-order approximation of the asymptotic expansion of equation (37) in the limit of strong dissipation.

3.7. A hypothetical dependence of energy on vorticity
In this section, we add to the energy a hypothetical dependence on the thermal vorticity \( \omega \), and replace the energy density derived by maximization of the entropy of bosons by

\[
\hat{E} = \int d\mathbf{r} \hat{\varepsilon}(s, \mathbf{w}, \omega),
\]

where \( \hat{\varepsilon}(s, \mathbf{w}, \omega) \) is a function of entropy, the conjugate entropy flux, and thermal vorticity. In case of \( \omega = 0 \), \( \hat{\varepsilon} \) should turn to the standard formula (34), but we leave it undetermined because it is not clear whether such (hypothetical) dependence is possible from the physical point of view. Phonons express collective vibrations of atoms of a lattice, which means that \( \omega \) encodes a sort of angular momentum of the vibrations. A similar energy density can be seen for instance in superfluids, where dependence of energy on vorticity of the superfluid velocity is due to the presence of topological defects (quantum vortices) [97–100]. We are not aware, however, of any experimental observation of similar vortices in the case of phonons.

Anyway, let us close the paper with exposition of the consequences of the hypothetical dependence of energy density on thermal vorticity. The functional derivative of energy with respect to the \( \mathbf{w} \) field turns to

\[
\frac{\delta \hat{E}}{\delta \mathbf{w}} = \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} + \nabla \times \frac{\partial \hat{\varepsilon}}{\partial \omega}.
\]

The evolution equation for \( \mathbf{w} \) (equation (37b)) and the equation for curl of \( \mathbf{w} \) become

\[
\partial_t \mathbf{w} = -\nabla T^{(w)} + \frac{1}{s} \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \times \omega + \frac{1}{s} \left( \nabla \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \cdot \omega - \omega \cdot \nabla \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \right) - \frac{1}{\tau_T} \mathbf{w}
\]

\[
\partial_t \omega = \nabla \cdot \left( \frac{1}{s} \left( \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \times \omega - \omega \times \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \right) + \nabla \times \left( \frac{1}{s} \left( \nabla \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \cdot \omega - \omega \times \nabla \frac{\partial \hat{\varepsilon}}{\partial \mathbf{w}} \right) \right) \right) + \mathbf{w} \times \nabla \left( \frac{1}{\tau_T} \right) - \frac{1}{\tau_T} \omega.
\]

The equation for thermal vorticity has thus two fluxes (terms under the \( \nabla \) operator), a source term (with gradient of \( \tau_T \)), and a sink term (proportional to \( \omega \)). In particular, if \( \omega \) is zero initially and if there is no gradient in \( \tau_T \), then \( \omega \) remains zero. On the other hand, if there is a gradient in \( \tau_T \) as in case of a thermal diode [93, 94], then some amount of \( \omega \) is being created until a balance with the sink term is reached.

Heat flow also exhibits hydrodynamics behavior in superfluid helium-4; for instance, in the laminar regime in counterflow along a cylindrical tube or along tubes or rectangular or square cross section, the behavior of the heat flow is analogous to that of viscous fluids [98, 101, 102]. In the two-fluid model [97, 100, 103, 104] this may be easily interpreted, as the heat flow is transported by the viscous component of the fluid [105]. Exploration of a possible von Kármán vortex street when an insulating cylinder perpendicular to the heat flux is introduced in a channel could be of interest. However, in superfluid helium the circulation of the vortices is quantized (due to the requirement that the \( \nabla \times \mathbf{w} = 0 \) almost everywhere), which makes it different from the phonon hydrodynamics explored in the present paper.

4. Conclusion

In this paper, we begin with the evolution equation of distribution functions of phonons with two transversal and one longitudinal polarizations (actually kinetic theory of mixtures for \( f_{_{T^+}}, f_{_{T^-}}, \) and \( f_{_0} \)). The energy is
simplified to the energy of acoustic phonons in the first Brillouin zone, and the entropy is obtained as the entropy of an ideal bosonic gas in kinetic theory.

This description is then reduced to hydrodynamics with momentum densities and entropy densities of the particular polarizations \((m_\alpha, s_\alpha)\), for \(\alpha \in \{T^+, T^-, L\}\). The underlying Poisson bracket is obtained by reduction from the kinetic Poisson bracket and the entropy is derived from the entropy in kinetic theory by means of MaxEnt. The kinetic theory of polarized phonons is also reduced to the overall kinetic theory of phonons with distribution function \(f = \sum \alpha f_\alpha\) (disregarding polarizations), where the Poisson bracket is again obtained by reduction from the kinetic Poisson bracket, and entropy is obtained again by MaxEnt.

Finally, these levels of description are further reduced to phonon hydrodynamics described by state variables \(m = \sum \alpha m_\alpha, s = \sum \alpha s_\alpha\) (momentum density and entropy density). The Poisson bracket and entropy are again derived by reduction from their more detailed counterparts. Apart from the zero-th Chapman-Enskog approximation, which leads to hyperbolic equations, we also carry out the first Chapman-Enskog approximation, which adds higher-order (Laplacian) terms to the evolution equation for phonon momentum density. It turns out more practical to reformulate the hydrodynamic equations on this level of description to variables \(w = m/s\) and \(s\), since then we can better see the relation with standard theories of heat transport. The obtained evolution equations are Galilean invariant, unlike usual theories of heat transport, and they depend on the thermal vorticity, \(\omega = \nabla \times w\).

In particular, even in the stationary case, the heat flux turns out to be not aligned with the gradient of temperature, becoming proportional to it only in the limit of strong dissipation. In the one-dimensional case and after linearization, the theory simplifies to the Maxwell-Cattaneo-Vernotte equation. The thermal vorticity is created if gradients in thermal conductivity are present that are not parallel with the heat flux. In the presence of thermal vorticity, we can expect various phenomena similar to aerodynamics, like a Fourier-Crocco equation (50), drop of temperature (instead of pressure) behind and obstacle, or temperature gradient perpendicular to the flow around an airfoil (instead of aerodynamic lift). When Reynolds number is high enough, the vorticity-dependent terms lead to the von Kármán vortex street.

The reduction of phonon kinetic theory to phonon hydrodynamics is built upon the Hamiltonian character of both theories, where the reduction of the underlying Poisson bracket depends neither on the dispersion relation nor on the relaxation times. However, in order to obtain the energy on the hydrodynamic level of description by analytical means, we are constrained to the linear phonon dispersion relation. Moreover, constantness of the relaxation times is also useful when deriving the dissipation on the hydrodynamic level, but is not essential. It is possible to go beyond the linear relation in two ways: (i) by applying the MaxEnt principle numerically (the most precise way), or (ii) by slightly perturbing the linear dispersion relations [34]. The character of the resulting equation and propagation of phonon vorticity would not be affected by such generalization, so we postpone it to future. Another possibility is to use a non-linear dispersion relation in phonon kinetic theory [106].

In future, we would also like to extend the set of numerical examples and to compare the results with experimental results [19]. This entails, however, an implementation of pertinent boundary conditions [85]. In dimension one, behavior of phonon propagation seems to be different from higher dimensions. Two- and three-dimensional hydrodynamic simulations that at some time and place, the temperature will be lower than the lowest initial temperature [86]. Perhaps, this phenomenon could be connected with phonon vorticity, which disappears in dimension one.

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**Data availability statement**

No new data were created or analysed in this study.
Appendix A. Maximum entropy principle for energy and entropy

Entropy of phonons, which are bosons, is expressed in terms of their distribution function [66]

\[
S_{\text{kinetic}}(f) = -\frac{k_B}{\hbar^3} \int d\mathbf{r} \int d\mathbf{p} (h^3 f) \ln(h^3 f) - (1 + h^3 f) \ln(1 + h^3 f)),
\]

where \(h\) is the Planck constant. The goal is now to maximize the entropy with the constraints that we know the energy density and momentum density fields. This procedure is equivalent to that in [7], with the difference that here we obtain also the entropy for an intermediate level of description where the various polarizations of phonons have their own hydrodynamic entropies.

Indeed, because there are two transverse and one longitudinal polarizations of phonons, we actually have three distribution functions \(f_{T+}, f_{T-},\) and \(f_L\), to which correspond three entropies, given by formula (A1) for each of the distribution functions. The total entropy is then the sum of the entropies for each polarization, equation (11). In order to obtain analytical results, we neglect phonon-phonon interactions, as well as phonon-crystal interactions. We shall be constrained to the first Brillouin zone, where all the polarizations have the same formula for energy. Moreover, we stay restricted only to the case of acoustic phonons. This will give us the hydrodynamic entropy of the mixture of the three types of acoustic phonons. Eventually, we shall apply MaxEnt once more to obtain the overall hydrodynamic entropy of all acoustic phonons, which turns out to be the same as in [7].

The projection to the fields of momentum density and energy density is

\[
m(r) = \int d\mathbf{p} f,
\]

\[
e(r) = \int d\mathbf{p} \mathbf{p} |f|,
\]

where \(c\) is the speed of sound and \(|\mathbf{p}|\) is the Euclidean norm of the momentum, \(\mathbf{p} = \hbar \mathbf{k}\). By maximization of entropy (11) subject to the constraints given by fields \(e\) and \(m\),

\[
\frac{\delta}{\delta f} \left( -S_{\text{kinetic}} + \int d\mathbf{r} m^*(\mathbf{r}) \cdot \int d\mathbf{p} p^* + \int d\mathbf{r} e^*(\mathbf{r}) \int d\mathbf{p} |\mathbf{p}| |f| \right) = 0,
\]

we obtain the MaxEnt estimate of the distribution function

\[
f_{\text{MaxEnt}}(m, e) = \frac{1}{\hbar^3} e^{m / k_B} e^{c e / k_B - 1},
\]

where \(m^*\) and \(e^*\) are the corresponding Lagrange multipliers. The hydrodynamic formula for entropy of phonons is then given by \(S_{\text{MaxEnt}}(f_{\text{MaxEnt}})\), but in order to express it in terms of state variables \(m\) and \(e\), we have to first express the Lagrange multipliers in terms of the state variables.

By integration of equation (A2) with the MaxEnt estimate of the distribution function (A4), we obtain that

\[
m_i = -\frac{16}{45} \frac{k_B^4}{\hbar^3 (c e)^2} \frac{m_i^*}{1 - \gamma^2},
\]

\[
e = \frac{4}{45} \frac{\pi k_B^2}{(c e)^2 \hbar^3 (1 - \gamma^2)^3},
\]

where \(\gamma = \sqrt{\frac{m}{c e}}\). Note that is not trivial to carry out the integration and several tricks are necessary. Apart from standard techniques like swapping integrals and series or transformation to the spherical coordinates, it is useful to rotate the coordinates so that \(m^*\) becomes \((0, 0, |m^*|)\). Equation (A5) have to be solved so that the multipliers, \(m^*\) and \(e^*\), are expressed in terms of the state variables, \(m\) and \(e\). In particular, the equation imply that

\[
\beta^2 = \frac{16}{16} (\gamma^4 + 6 \gamma^2 + 9) = \gamma^2,
\]

where \(\beta = \frac{m}{e}\), which gives that

\[
\gamma^4 = \frac{1 - \sqrt{\gamma}}{1 + \sqrt{\gamma}}
\]

with \(\gamma = 1 - \frac{3}{4} \beta^2\). Note that this quadratic equation has two roots, one of which is unphysical because it would lead to a singularity at \(\beta = 0\) (equilibrium). Since \(\beta\) is a function of the state variables, we the Lagrange multipliers are now expressed in terms of the state variables, \(m\) and \(e\), as

\[
(e^*)^4 = \frac{3 \sigma (1 + \sqrt{\gamma})^2}{2e (2\sqrt{\gamma} - 1)}
\]

(A8a)
\[ \mathbf{m}^* = -\frac{3}{2}\left(\frac{3}{2}\right)^{1/4} \frac{1}{(2\sqrt{2})^{3/4}} \frac{1}{1 + \sqrt{2}} \mathbf{m}. \] (A8b)

Near the equilibrium, where \( y = 1 - \frac{3}{2}\frac{m_e^2}{c^2} \approx 1 \), the multipliers are approximated by
\[ e^* \approx \left(\frac{6\alpha}{c^2}\right)^{1/4} \quad \text{and} \quad \mathbf{m}^* \approx -\frac{3}{2}\left(\frac{3}{2}\right)^{1/4} \frac{1}{(2\sqrt{2})^{3/4}} \mathbf{m}. \] (A9)

The hydrodynamic entropy is then obtained by plugging \( f_{\text{MaxEnt}}(\mathbf{m}, e) \) to the kinetic formula for entropy \( S_{\text{kinetic}} \), which after some calculations results in
\[ S_{\text{kinetic},\alpha}(f_{\text{MaxEnt}}(\mathbf{m}_\alpha, e_\alpha)) = \int d\mathbf{r} \left( \frac{2\alpha}{c^2} e_\alpha \right)^{3/4} \left( 3 - \chi_\alpha \right)^{1/2} (1 - \chi_\alpha)^{1/4}, \] (A10)
\[ \alpha \in \{T^+, T^-, T, L\}, \sigma = \frac{2\pi c^2}{15h^2} \text{ is the Stefan-Boltzmann constant, and} \quad \chi_\alpha = \frac{5}{3} - \frac{4}{3} \sqrt{1 - \frac{3c^2m_e^2}{4}} \text{ is the Eddington factor.} \]

Because we actually have three polarizations, \( T^+, T^-, \) and \( L \), we also have three entropies \( S_{\text{kinetic},\alpha} \), while the total entropy is their sum,
\[ S^H = \sum_\alpha S_{\text{kinetic},\alpha}(f_{\text{MaxEnt}}(\mathbf{m}_\alpha, e_\alpha)), \] (A11)
referred to as the hydrodynamic (H) entropy of polarized (P) phonons. In other words, we have obtained the entropy for the level of description with state variables \( \mathbf{m}_{T^+}, \mathbf{m}_{T^-}, \mathbf{m}_L, e_{T^+}, e_{T^-}, \) and \( e_L \).

The level of description where the various polarizations of acoustic phonons have their own momentum and energy allows for description of processes like change of polarization of the phonons. However, when such processes are not important, one may opt out for a simpler level of description, where only the overall momentum and energy,
\[ \mathbf{m} = \mathbf{m}_{T^+} + \mathbf{m}_{T^-} + \mathbf{m}_L \]
\[ e = e_{T^+} + e_{T^-} + e_L, \]
are known. What is the entropy on such reduced level of description? The answer is given again by maximization of the total entropy (A11) subject to the constraints given by projection (A12), that is by
\[ \frac{\delta S^H_{\text{kinetic},\alpha}}{\delta \mathbf{m}_\alpha} \bigg|_{\text{MaxEnt}} = \mathbf{m}^* \quad \text{and} \quad \frac{\delta S^H_{\text{kinetic},\alpha}}{\delta e_\alpha} \bigg|_{\text{MaxEnt}} = e^* \quad \forall \alpha. \] (A13)

Since all the formulas for \( S_{\text{kinetic},\alpha} \) share the same form, their solutions are the same and the MaxEnt estimates of \( \mathbf{m}_\alpha \) and \( e_\alpha \) are the same for all polarizations \( \alpha \). Finally, the overall hydrodynamic entropy is obtained by plugging the MaxEnt estimates to the formula for the total entropy (A11),
\[ S^H(\mathbf{m}, e) = S^H(\mathbf{m}_{\text{MaxEnt}}, e_{\text{MaxEnt}}) = \int d\mathbf{r} \left( \frac{2\alpha}{c^2} e_{\text{MaxEnt}} \right)^{3/4} \left( 3 - \chi \right)^{1/2} (1 - \chi)^{1/4}, \] (A14)
where \( \chi = \frac{5}{3} - \frac{4}{3} \sqrt{1 - \frac{3c^2m_e^2}{4}} \) is the overall Eddington factor.

In equilibrium, where \( \mathbf{m} = 0 \), we obtain the equilibrium volumetric entropy density
\[ s^E = \left( \frac{6\alpha}{c^2} \right)^{1/4} e^{3/4}, \] (A15)
which means that \( e = \frac{6\alpha}{c^2} T^4 \) with temperature \( T = \left( \frac{\partial s^E}{\partial e} \right)^{-1} \).

**Appendix B. Simplification to the Cattaneo equation**

How does the dynamics for \( s \) and \( w \) (37) correspond to the classical Cattaneo equation [17]? Because Cattaneo equation is a linear equation for temperature in one dimension. Therefore, we shall be restricted to the one-dimensional case, \( w = (w, 0, 0) \), we shall transform the \( (s, w) \) variables to \( (T, w) \), and finally we will neglect all nonlinear terms.

The derivative of energy \( e \) transform to the derivatives of the free energy \( f = -Ts \) as
\[ \frac{\partial e}{\partial s} = T, \quad \frac{\partial e}{\partial w} = \frac{\partial f}{\partial w} T. \] (B1)

Derivatives of the free energy (in the \( (T, w) \) variables) will be denoted by subscripts. Using the chain rule, equation (37) become
\[
\left( \frac{\partial s}{\partial T} \right)_w \frac{\partial^2 s}{\partial T^2} \cdot \partial_t T + \left( \frac{\partial s}{\partial w} \right)_T \cdot \partial_t w = - \partial_t f_w + \frac{1}{\tau_R} f_w^2
\]  
\[
\partial_t w = - \partial_t T - \frac{1}{\tau_R} f_w.
\]  

(B2a)

Let us assume, for simplicity, that we are restricted to a narrow temperature range around a temperature \(T_0\) and low values of \(w\), so that the free energy is quadratic as well

\[
f(T, w) \approx f_0 - s_0(T - T_0) - \frac{c_V}{2T_0} (T - T_0)^2 + \frac{1}{2}\kappa w^2,
\]  

(B3)

where \(f_0 = f(T_0, 0), s_0 = -f/T\) at \(T_0\) and \(w = 0\), and \(c_V = T \partial s/\partial T\) is the heat capacity. Then we have, in particular, that \(f_{TT} = -c_V/T_0 = -\beta, \beta\) being a positive constant, \(f_{ww} = 0\), and \(f_{ww} = 0\). Note that the non-negative coefficient \(\kappa\) is a measure of the non-Fourier behavior. Equation (B2) can be then rewritten as

\[
\beta \partial_t T = - \partial_t f_w + \frac{1}{\tau_R} f_w^2 = -\kappa \partial_t w + \frac{\kappa^2}{\tau_R} w^2
\]  

(B4a)

\[
\partial_t w = - \partial_t T - \frac{1}{\tau_R} f_w = - \partial_t T - \frac{\kappa}{\tau_R} w.
\]  

(B4b)

Partial-time-derivative of equation (B4a) then yields, assuming that \(\tau_R = \text{const}\),

\[
\beta \partial_t \partial_t T = -\kappa \partial_t w - \frac{\kappa}{\tau_R} w + \frac{\kappa^2}{\tau_R} \left( \frac{\partial_t T}{T} w^2 + 2 w \partial_t w \right)
\]  

\[
= \kappa \partial_t \partial_t T - \frac{\kappa \beta}{\tau_R} \partial_t T + \mathcal{O}(\kappa^2 \cdot w).
\]  

(B5)

Assume that the terms of order \(\kappa^2 w\) (not necessarily the terms or order \(\kappa \partial_t w\)) are smaller than the remaining terms, we obtain the classical Maxwell-Cattaneo-Vernotte telegrapher’s equation,

\[
\beta \partial_t \partial_t T + \frac{\kappa \beta}{\tau_R} \partial_t T = \kappa \partial_t \partial_t T.
\]  

(B6)

In summary, after the transformation to variables \(T\) and \(w\), we obtain in the one-dimensional case and after linearization the classical Maxwell-Cattaneo-Vernotte equation. However, equation (37) are more general.

### Appendix C. Galilean invariance of phonon hydrodynamics

Consider an observer \(O\) in an inertial laboratory frame, who measures time and space positions in \(t\) and \(r\). Consider, moreover, another observer \(O’\) passing by the laboratory with a constant velocity \(V\). Observer \(O’\) measures time and space positions \(t’\) and \(r’ = r - Vt\).

An object with mass \(M\) at position \(r’\) is at position \(r = r’ + Vt\) in the laboratory frame. By differentiation with respect to time, the velocities become related by \(v = v’ + V\). Momentum of the object in the laboratory frame, \(p = Mv\), relates to the momentum in the \(O’\) frame as \(p’ = p’ + MV\). In particular, if the object has no mass, as a phonon, then its momentum does not change, that is \(m = m’\). The \(w\) field is equal to \(m/s\), and entropy itself is Galilean invariant (\(s = s’\)), which means that the \(w\) field does not change under Galilean transformations, \(w = w’\).

Energy of the object transforms as \(e = \frac{1}{2} M v^2 = e’ + p’ \cdot V + \frac{1}{2} M V^2\), which is a general formula for Galilean transformations of energy [107]. In particular, if the object has no mass (as a phonon), then the formula simplifies to \(e’ = e - m \cdot V\), where \(e\) stands for the volumetric energy density and \(m\) for the volumetric momentum density of phonons. In terms of the \(w\) field, the energy density transforms as \(e’ = e - sw \cdot V\), which means that derivatives of energy transform as

\[
\frac{\partial e}{\partial s} = \frac{\partial e’}{\partial s’} + w’ \cdot V \quad \text{and} \quad \frac{\partial e}{\partial w} = \frac{\partial e’}{\partial w’} + sV.
\]  

(C1)

How do the time-and-space derivatives transform? For any function \(s(t, r)\) it holds that

\[
\left( \frac{\partial s}{\partial t} \right)_t = \left( \frac{\partial s}{\partial t} \right)_{t’} + \left( \frac{\partial s}{\partial r} \right)_{r’} \left( \frac{\partial r’}{\partial t} \right)_{t’}, \quad \left( \frac{\partial s}{\partial r} \right)_r = \left( \frac{\partial s}{\partial r} \right)_{r’} \left( \frac{\partial r’}{\partial r} \right)_{r’}.
\]  

(C2a)

\[
\left( \frac{\partial s}{\partial t} \right)_{t’} = \left( \frac{\partial s}{\partial t} \right)_{t’} - \left( \frac{\partial s}{\partial r} \right)_{r’} \left( \frac{\partial r’}{\partial t} \right)_{t’}, \quad \left( \frac{\partial s}{\partial r} \right)_{r’} = \left( \frac{\partial s}{\partial r} \right)_{r’} \left( \frac{\partial r’}{\partial r} \right)_{r’}.
\]  

(C2b)
The reversible part of the evolution equations for the fields $s$ and $w$ then transform to

$$
\left( \frac{\partial s}{\partial t} \right)_r = - \frac{\partial}{\partial \tau^i} \left( \frac{\partial e^l}{\partial w^i} + V^l s \right) = - \frac{\partial}{\partial \tau^i} \left( \frac{\partial e^l}{\partial w^i} \right) - \frac{V^l}{\partial \tau^i} \left( \frac{\partial e^l}{\partial w^i} + V^l s \right),
$$

(C3)

and

$$
\left( \frac{\partial w^l}{\partial t} \right)_r = - \frac{\partial}{\partial \tau^i} \left( \frac{\partial e^l}{\partial s} + V^l w^i \right) + \frac{1}{s^i} \left( \frac{\partial w^l}{\partial \tau^i} - \frac{\partial w^l}{\partial \tau^i} \right) \left( \frac{\partial e^l}{\partial w^i} + V^l s \right),
$$

(C4)

which means that the reversible part of the equations is Galilean invariant. Note, however, that without the last term on the right hand side, the evolution equation for $w$ would not be Galilean invariant. The convective terms that we get by the Poisson reduction thus make the equations Galilean invariant.

Appendix D. Reversibility, irreversibility, and Onsager-Casimir reciprocal relations

When talking about reversible and irreversible evolution, we shall first define what we mean by (ir)reversibility. We use the definition based on the time-reversal transformation (TRT) [43, 108]. TRT inverts velocities of all particles and the magnetic field, and a quantity is called even with respect to TRT (parity $\mathcal{P} = 1$) if it is not affected by that inversion. On the other hand, a quantity is odd (parity $\mathcal{P} = -1$) if its sign flips under TRT. Note that there are also quantities that are neither even nor odd, as for instance distribution functions, but the concept of parity still works in a geometrical sense [65]. Most state variables, however, have a definite parity, for instance energy, entropy, and density are even, whereas momentum and velocity are odd.

Also, parity of a quantity can change. For instance, parity of the heat flux, or the conjugate entropy flux $w$, is odd if $w$ is among the state variables. On the other hand, after the reduction to the Fourier heat conduction (section 3.6), where only the energy density is the state variable and $w$ is proportional to the temperature gradient, parity of $w$ becomes even (the same as parity of the temperature gradient).

Consider now an evolution equation written in such a form that the time-derivative of a state variable is on the left hand side. The right hand side of the evolution equation is then interpreted as a component of a vector field $\mathbf{X}$. The part of the right hand side that transforms under TRT in the same way as the left hand side is called reversible, while the part that flips the sign is irreversible. In order to generate reversible dynamics, the Poisson bivector must fulfill that

$$
\text{TRT}(L^{ij}) = -\mathcal{P}(x^i)\mathcal{P}(x^j)L^{ij},
$$

(D1)

Since energy is even, this property indeed ensures that the Hamiltonian part of the GENERIC evolution transforms as the left hand side of the evolution equations and is thus reversible in the above sense. Property (D1) is fulfilled by the canonical Poisson bracket, from which the reversibility is inherited by the Boltzmann Poisson bracket and all brackets obtained by its projection (brackets within this manuscript) [65].

Entropy is required to be a Casimir of the Poisson bracket, that is $\{S, A\} = 0$ for all $A$, and the Hamiltonian evolution then does not produce entropy (it can only redistribute it). The dissipative part is usually required not to produce or destroy energy, so that $\langle E_\alpha, \Xi_{v^i} \rangle_{v^i = S_\alpha} = 0$. These degeneracy requirements ensure that energy is conserved by the irreversible part while entropy being conserved by the reversible part.

The dissipation potential is then typically required to be convex in order to produce entropy,

$$
\tilde{S} = \langle S_\alpha, \Xi_{v^i} \rangle_{v^i = S_\alpha} \geq 0, \text{ see e.g. [68]}.
$$

Note, however, that convexity is just a sufficient condition of the positivity of entropy production, but not necessary [109].

The dissipation potential is supposed to be even with respect to TRT, which means that it generates irreversible evolution. A part of the nonlinear Onsager-Casimir reciprocal relations (OCRR) can be seen as the symmetry of the Hessian of the dissipation potential [110]. However, to be closer to the historical formulation of OCRR, let us assume, for a moment, that the dissipation potential is quadratic as in equation (2).

The requirement that $\Xi$ be even with respect to TRT means that

$$
\text{TRT}(M^{ij}) = \mathcal{P}(x^i)\mathcal{P}(x^j)M^{ij}
$$

(D2)

and the gradient part of the GENERIC evolution then indeed generates irreversible evolution in the above sense. Operator $M$ is often referred to as the dissipative matrix, it is symmetric, and it can be obtained as the second functional derivative of the dissipation potential,

$$
M^{ij} = \frac{\delta^2 \Xi}{\delta x^i \delta x^j}.
$$

We are now in position to recall a generalized version of OCRR [10, 53]. GENERIC evolution can be rewritten in the form
\[ \partial_t x^i = \left( T_0 U^j - M^{0i} \right) \frac{\partial \Phi}{\partial x^j} \]  \hspace{1cm} (D4)

with thermodynamic potential \( \Phi = -S + \frac{1}{T} E \left( T_0 \right) \) (a constant). The operator \( K \) in front of the derivatives of the potential \( \Phi \) provides coupling between the state variables and it plays the role of the Onsager-Casimir matrix of phenomenological coefficients. Let us now explicitly discuss particular cases of the possible couplings. Due to properties (D1) and (D2), we obtain the OCRR [53]:

- Variables with the same parities, \( \mathcal{P}(x^i) = \mathcal{P}(x^j) \), are coupled through an operator symmetric with respect to the simultaneous TRT and transposition.
- Variables with opposite parities, \( \mathcal{P}(x^i) = -\mathcal{P}(x^j) \), are coupled through an operator skew-symmetric with respect to the simultaneous TRT and transposition.

These properties are satisfied by both the reversible and irreversible parts of GENERIC, in particular by the Hamiltonian mechanics, as can be checked directly. Since GENERIC can be seen as a consequence of Hamiltonian mechanics [111], OCRR can be seen as a consequence of Hamiltonian mechanics as well.

Let us now illustrate TRT on phonon hydrodynamics. The Poisson bivector for the \( (s, w) \) variables is

\[ L^s = \{ s(r'), s(r'') \} = 0 \]  \hspace{1cm} (D5a)

\[ L^{ws} = \{ w_k(r'), w_l(r'') \} = \int d\tau \partial_t \delta(r' - r) \delta_{kl} \delta(r'' - r) = \partial_{k} \delta(r' - r) \]  \hspace{1cm} (D5b)

\[ L^w = \{ w_k(r'), s(r'') \} = -\int d\tau \partial_t \delta(r'' - r) \delta_{kl} \delta(r' - r) = -\partial_{k} \delta(r'' - r) \]  \hspace{1cm} (D5c)

\[ L^{w^2} = \{ w_k(r'), w_l(r'') \} = \int d\tau \frac{1}{s(r)} (\partial_t w_j - \partial_j w_k) \delta_{kl} \delta(r' - r) \delta(r'' - r) = \frac{1}{s(r)} (\partial_{k} w_l - \partial_{l} w_k) \delta(r'' - r). \]  \hspace{1cm} (D5d)

We can check that the evolution generated by the bivector is reversible, since it satisfies condition (D1). Coupling between variables of the same parity, for instance \( L^{w^2} \), is symmetric with respect to the simultaneous transposition and time-reversal. Coupling between variables with opposite parities, here \( L^w \) and \( L^{ws} \), is skew-symmetric in the same sense. Hamiltonian mechanics thus provides both the symmetric and skew-symmetric coupling.

The irreversible evolution in equation (37) can be considered as gradient dynamics with dissipation potential

\[ \Xi = \frac{1}{2} \int d\tau \xi (w^w)^2, \]  \hspace{1cm} (D6)

\( (xi \) being a constant) and the dissipative matrix can be obtained as the second derivative of the dissipation potential,

\[ M = \xi \delta(r' - r) \begin{pmatrix} \left( \frac{\partial s}{\partial w} \right)^2 & \frac{\partial s}{\partial w} \\ \frac{\partial s}{\partial s} & 1 \end{pmatrix}. \]  \hspace{1cm} (D7)

Both \( L \) and \( M \) operators form the Onsager-Casimir operator (D4) and satisfy OCRR in the above sense. That is, the variables with the same parity \( (s, s, w, w) \) are coupled through operators that are symmetric with respect to the simultaneous transposition (adjoint operator) and time reversal. Variables with opposite parities \( (s, w) \) are coupled through operators that are skew-symmetric in the same sense. Note that the Hamiltonian coupling contributes also to the symmetric coupling \( (w, w) \) while the dissipative matrix also contributes to the skew-symmetric coupling \( (s, w) \).

Appendix E. Details of the Chapman-Enskog approximation

This section contains details of the calculations needed in the first Chapman-Enskog approximation from section 3.3. In particular, it contains calculations of the following integrals. The equilibrium value of the \( Q_{ij} \) moment is obtained by
\[ \int \frac{dp_i p_j}{|p|} \frac{1}{e^{\varepsilon (p)/k_B} - 1} = \frac{4\pi^3 k_B^4}{45 (e^\varepsilon)^4 c^4} \delta_{ij}, \]  

(E1)

where which is calculated by observing that the integral vanishes if \( i \neq j \), by transformation to the spherical coordinates, and by expansion of the fraction into a geometric series.

Integral

\[ \int \frac{dp_i p_j}{|p|} \frac{1}{e^{\varepsilon (p)/k_B e^{m^2/p/k_B}} - 1} \approx \frac{4\pi^3 k_B^4}{45 (e^\varepsilon)^4 c^4} \delta_{ij} \]  

(E2)

is approximated by the Taylor expansion \( \frac{1}{e^{\varepsilon (p)/k_B} - 1} \approx \frac{1}{\varepsilon (p)/k_B} + \frac{\varepsilon (p)/k_B^2}{2} \), where the zero-th order term represents the equilibrium distribution function, \( f_R \), and the first-order term disappears after the integration.

Integral

\[ \int \frac{dp_i p_j p_k}{|p|} \frac{1}{e^{\varepsilon (p)/k_B e^{m^2/p/k_B}} - 1} \approx -\frac{16\pi^3 k_B^4}{225 (e^\varepsilon)^8} (m^2_{x} \delta_{ij} + m^2_{y} \delta_{ji} + m^2_{z} \delta_{ik}) \]  

(E3)

also starts with the above Taylor expansion, where the zero-th order term does not contribute to the overall integral. The first-order term vanishes if an odd power of \( p_x, p_y, p_z \), or \( p_i \) is present, which means that the momenta must come either in two distinct pairs or as the fourth power of a particular component of the momentum. Calculation in the spherical coordinates then leads to the above result.

**Appendix F. Poiseuille-like flow in phonon hydrodynamics**

In the stationary case, equations (2) and (47) for the energy density and momentum density admit an approximate solution that resembles the Poiseuille flow in hydrodynamics. Let us assume a two-dimensional channel \((x \in (0, L), y \in (-h, h))\) with zero Dirichlet boundary conditions on the top and bottom boundaries and with a non-zero Dirichlet condition at the left boundary (the inlet).

We assume, moreover, that \( m_y(y) \) is just a function of the \( y \) coordinate (transverse to the channel), that \( e(x) \) depends only on the \( x \) coordinate (along the channel) and that \( m_x = 0 \). The stationary energy equation is then automatically satisfied while the momentum equation simplifies to

\[ 0 = -\frac{1}{3} \partial_y e - \frac{c^2 m_x^2}{4 e^2} \partial_x e + \frac{3 c^2}{4 e^2} m_x \partial_y e - \frac{1}{\tau_R} m_x - \frac{\tau c^2}{5} \partial_y \partial_y m_x, \]  

(F1)

which can be simplified (upon neglecting terms of order \( m_x^2 c^2/e^2 \)) to

\[ \frac{1}{3} \partial_y e = -\frac{1}{\tau_R} m_x - \frac{\tau c^2}{5} \partial_y \partial_y m_x. \]  

(F2)

Since the left hand side does not depend on \( y \) and the right hand side does not depend on \( x \), both sides are equal to a constant \( \lambda < 0 \) (assuming negative energy gradient along the channel). Hence, we obtain that

\[ e = e_{|x=0} + 3 \lambda x \]  

(F3a)

\[ m_x = \frac{\lambda \tau_R}{\cosh \left( \frac{5}{\sqrt{\tau_R \tau c^2}} y \right)} \]  

(F3b)

where \( \lambda \) can be determined for instance from the energy drop along the channel. The \( m_x \) solution is approximately quadratic in \( y \), thus resembling the Poiseuille profile.

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