Towards a unified theory of the fundamental physical interactions based on the underlying geometric structure of the tangent bundle

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This paper pursues the hypothesis that the tangent bundle (TB) with the central extended little groups of the SO(3,1) group as gauge group is the underlying geometric structure for a unified theory of the fundamental physical interactions. Based on this hypothesis as a first step recently I presented a generalized theory of electroweak interaction which includes hypothetical dark matter particles (Eur. Phys. J C 79, 779 (2019). The vertical Laplacian of the tangent bundle possesses the same form as the Hamiltonian of a 2D semiconductor quantum Hall system. This explains fractional charge quantization of quarks and the existence of lepton and quark families. As will be shown the SU(3) colour symmetry for strong interaction arises in the TB as an emergent symmetry similar as Chern-Simon gauge symmetries in quantum Hall systems. This predicts a signature of quark confinement as an universal large-scale property of the Chern-Simon fields and induces a new understanding of the vacuum as the ground state occupied with a condensate of quark-antiquark pairs. The gap for quark-antiquark pairing is calculated in the mean-field approximation which allows a numerical estimation of the characteristic parameters of the vacuum such as its chemical potential, the quark condensation parameter and the vacuum energy. Note that previously a gauge theoretical understanding of gravity has been achieved by considering the translation group T(3,1) in the TB as gauge group. Therefore the theory presented here can be considered as a new type of unified theory for all known fundamental interactions linked with the geometricization program of physics.

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1. Introduction

The development of the Standard Model (SM) of elementary particles was a great triumph of modern physics. However despite its enormous success in explaining a large number of experiments, it cannot be considered as the ultimate theory of particle physics. The SM is incomplete and contains several open problems that cannot be solved on the basis of this model, as well as phenomenological elements that have not yet been explained on a microscopic basis. A fundamental problem in the SM is the question of the physical origin of the internal symmetry groups assumed a priori for phenomenological reasons. Several other unanswered questions are the mysterious existence of three families of leptons and quarks (also called generations) that differ only by their masses and the origin of the hierarchy of fermion masses, the microscopic understanding of the spontaneous symmetry breaking by the Higgs mechanism and the hierarchy problem, the lack of understanding of fractional quark charges, the missing explanation of dark matter and dark energy, and others.

There have been many attempts to formulate a unified theory beyond the SM that could solve the above puzzles. In Grand Unified theories, the electroweak and strong interactions are embedded in a larger gauge group, such as the SU(5) group first proposed by Georgi and Glashow or in the SO(10) group. There have also been some attempts to understand the origin of families in the SM by using a so-called family symmetry. Despite many subsequent attempts, no unified model exists that is able to solve all these problems on a unified basis and none of them are considered to be universally accepted.

The SM of particle physics is based on the gauge principle, with local symmetry transformations of the variables of the phenomenologically determined continuous gauge groups $SU(2) \otimes U(1)$ in electroweak interaction and $SU(3)$ in strong interaction (quantum chromodynamics) acting in an "internal space" related to the symmetry group. The geometric or physical origin of the internal symmetries given by the gauge transformations remains unknown. Here we follow the hypothesis that the gauge principle is not a complete independent postulate but is connected with a deeper principle which allows one to determine the internal symmetries without phenomenological assumptions. A possible way to realize this aim is by revisiting the geometrization program of unified field theory in the 1920s to find a more general geometric structure than the Riemann geometry of the world space-time manifold for the unification of gravity with electromagnetism and its combination with the knowledge in modern particle physics. This requires a synthesis of the principle of general relativity and gauge transformations in the frame of an unified geometric structure. Note that the general type of such geometric structure has been recognized since the 1960s by the discovery of a formal equivalence of gauge theories with the mathematical formalism of fiber bundles. In the fiber bundle interpretation of gauge theories, the gauge potentials are understood as a geometric entity, the connection on the principal bundle; matter fields are described by the associated fiber bundles and gauge transformations are identified as transformations of the fiber variables along the
fiber axis at fixed space points $x$. But up to now one key question remained unanswered: which specific fibre manifold could be the basis for a unified theory of the fundamental forces? In previous studies the transformation groups of the fibers were taken from the phenomenologically determined gauge groups of the SM. Therefore the fiber bundle approach mainly delivered a geometrization and reinterpretation of the gauge potentials but could not be used as a bridge to a theory beyond the SM.

The present paper pursues the hypothesis [1] that the fundamental interactions are linked with geometric symmetries in the underlying extended geometric structure of the most fundamental fiber bundle—the tangent bundle. A tangent bundle is a more general manifold than the pseudo-Riemann manifold $M$ in the GTR. It associates to every point $x$ of the space-time manifold $M$ a 4-dimensional tangent space $T_x(M)$ which is the set of all tangent vectors at point $x$. In $T_x(M)$ one can define the addition of tangent vectors and multiplication between a tangent vector and a real number. The union of all tangent spaces at all points $x$ of the space-time manifold $M$ is called the tangent bundle $TM$. Tangent vectors in $T_x(M)$ can be transformed by the special affine group $G = SO(3, 1) \rtimes T(3, 1)$, but there is generally no natural choice of isomorphism between $T_x(M)$ and $T_y(M)$ for $x \neq y$. Any point can be mapped to the base manifold by a projection map $\pi$. In a Riemann geometry the differential calculus requires the existence of a connection between the tangent spaces of neighboring points. Such isomorphism can be constructed by the parallel transport using the Levi-Civita connection which depends on the metric of the space-time manifold. In the more general geometric structure of fiber bundles the connection is independent on the metric and can be axiomatically defined as a matrix-valued 1-form [27, 28].

The geometry of the TB is closely linked to the conceptual basis of gravity theories and its extensions to gravity gauge theories. Considerable efforts have been made in the construction of a gauge theory for gravity by analogy with gauge theories in the SM of particle physics. The theory proposed by Kibble (19) and Sciama (20) marks the earliest attempts to formulate a gauge theory of gravity based on the localization of the Poincare group as gauge group (for reviews see [21, 22]). The Kibble-Sciama theory incorporates besides curvature, Cartan's torsion as a dynamical variable and predicts that the torsion tensor is related to the spin density through a linear, algebraic equation, so that torsion does not propagate. This theory leads to a hypothetical generalized gravity theory, the Einstein-Cartan gravity theory. Although the present paper consider the same mathematical group for a gauge-theoretical construction, the theory presented differs significantly from the Kibble-Sciama theory in the main technical details, in the aim and in the described physical object. The Kibble-Sciama theory uses a heuristic scheme within the Minkowski space of special relativity with the Poincare group as a global symmetry for coordinate transformations in the flat Minkowsky spacetime manifold. Applying the gauge principle to this symmetry leads to Riemann-Cartan geometry with curvature and torsion due to the intrinsic spin of matter. It is not apparent from this procedure what is actually transformed by the localized Poincare transformation. Particularly problematic is the interpretation of local transformations which act not only on fields, but also on spacetime points. A precise analysis of the gauge content reveals certain structural differences with respect to the other SM gauge transformations. As many authors have remarked (see e.g. [21–23]) more advantageous is to work in the framework of fiber bundles, which avoids various pitfalls and confusions arising from mixing up of basic transformations in the tangent space and that of general coordinate transformations in the spacetime manifold. So far the theory of Kibble and Sciama has not found general acceptance as a viable theory of gravity because there has been no experimental evidence supporting the existence of gravitational torsion.

There exists a second class of a gauge theory of gravity which has found large attention by considering instead the full Poincare group the translation sub-group $T(3, 1)$ in the tangent bundle as gauge group [13–15] leading to the teleparallel gravity theory. In this theory the connection takes the form of the Weizenböck connection, the curvature is identical zero and gravity is described in terms of torsion instead of the curvature. The action only differ by a boundary term from the Einstein-Hilbert action and thus teleparallel gravity and Einsteins theory of gravity are classical equivalently. Consequently this theory is just as experimentally viable as Einsteins gravity theory.

There is a further essential point in this context in that the Poincare group is not semi-simple. The action of this group on a vector space is not transitive, but the vector space decomposes into different orbits in which the group acts transitively. Wigner developed the method of induced representations to find the unitary representation of the Poincare group by using projective representations. For the case of simply connected groups like the rotation group $SO(3)$ projective representations are obtained by replacing the group $SO(3)$ by its universal cover $SU(2)$. However the Euclidean group $E(2)$ is not semi-simple and the covering group $\tilde{E}(2)$ is not sufficient. One has to use a larger group: the universal central extension $E^c(2)$.

In the present paper it is assumed that gravity can be described by the translational gauge theory with the structure group $T(3, 1)$ based on the translational connection as gauge field. However, the restriction of the symmetry group $G = SO(3, 1) \rtimes T(3, 1)$ in the TB to the subgroup $T(3, 1)$ seems to be somewhat dissatisfying and introduced several controversies if spinor fields are present (see e.g. [24, 25]). One can avoid these prob-
lems if the spin-connection is not only related with inertia. This raises the question of the physical meaning of the other subgroup $SO(3,1)$ which is not connected with gravity. The basic hypothesis in the present manuscript is that sub-group $SO(3,1)$ is related with the other fundamental interactions. As a first step based on this hypothesis in $[1]$ a generalized theory of electroweak interaction without a priory assumption of a phenomenological determined gauge group but with internal (gauge) symmetries arising from geometric transformations of tangent vectors along the tangent vector axis is presented. The little groups $SU(2) \otimes E^c(2)$ of the non-transitive group $SO(1,3)$ were defined as structure groups (gauge groups) of the fiber bundle. The eigenfunctions of the Laplacian on this product group yields the known internal quantum numbers of iso-spin and hypercharge, but in addition the $E^c$-charge $x$ and the family quantum number $n$. The connection coefficients (gauge potentials) describe the known SM gauge bosons but additional extra gauge bosons are predicted (which can be interpreted as dark vector bosons). Besides the SM particles candidate stable and instable dark matter fermions and dark matter scalars without additional a priory phenomenological model assumptions arise from a common origin with SM particles. On the other hand the color group $SU(3)$ of QCD cannot be described as a geometric symmetry in an analog way as the $SU(2) \otimes E^c(2)$ group. However the TB delivers an interesting link which could pave the path for the inclusion of the color symmetry $SU(3)$ into a unified theory based on the TB geometry. A surprising feature in strong interaction is the fact that quarks carry fractional electric charges similar to electrons in a quantum Hall system. Up to now this analogy in the property of quarks and electrons in a quantum Hall system has not found a convincing explanation. On the other hand, the eigenfunctions of the Laplacian of the $E^c(2)$ group have the same form as the solution of the 2D Schrödinger equation for electrons in a perpendicular external magnetic field $[32, 33, 34]$. Combining all tangent fibers at all spacetime points, the vertical (internal) bundle Laplacian of the TB obtains a form analogous as the multi-particle Hamiltonian of a 2D quantum Hall system. The eigenstates in a quantum Hall system are called Landau levels, they show the same form as the solutions with internal quantum numbers (IQN) $n$ of the Laplacian of the $E^c(2)$ group (here denoted with TB Landau levels, TB-LL). In the SM members of different families have identical IQNs and properties except of its masses. In contrast in the TB the existence of three families of leptons and quarks can be distinguished by the different family numbers $n = 1, 2, 3$ while the vacuum state (ground state) carry the TB family quantum number $n = 0$. The analogy of fractional quark charges with the quantum Hall Effect (QHE) requires the additional hypothesis that the vacuum with the IQN $n = 0$ is filled with seeleptons and composite seequarks and all higher levels $n = 1, 2, 3$ are empty. Further an energy gap exists between the lowest TB-LL and a higher TB-LL. If a seequark is excited into a higher TB-LL it creates an excited state which will be denoted as valence quark and leaves a hole in the old state. A lepton or quark hole carry the opposite hypercharge and opposite isospin IQN, but a positive energy and can be interpreted as anti-particle. Taking into account the three isospin components of quarks with $I_3 = 1/2, 0, -1/2$ the vertical Laplacian obtains the analog form as the Hamiltonian of a three-layer quantum Hall system. In this approach, effective gauge fields (denoted as Chern-Simon fields) with a local $SU(3)$ symmetry arise in the vertical TB Laplacian in an internal way. An emergent phenomenon is a collective effect of a large number of particles that cannot be deduced from the microscopic theory in a rigorous way $[35]$. The fractional QHE is a prototype of such phenomenon.

In the model of a completely filled vacuum state due to the presence of attractive interaction by gluon exchange the vacuum is instable with respect to the formation of a quark condensate caused by the pairing of quarks with anti-quarks. This phenomenon is analogous to exciton condensation in solid states $[36–39]$ where pairs of electrons and holes form a condensate due to the weak attractive force. The condensed TB vacuum is characterized as a completely filled state with a finite particle density, therefore we have to introduce the Fermi-energy of the vacuum $E_{\text{Fermi}}$ or a vacuum chemical potential $\mu_{\text{vac}}$ with $\mu_{\text{vac}} \simeq E_{\text{Fermi}}$. To explore the dynamics of quark condensation by quark-antiquark pairing we will calculate the energy gap in the mean-field approximation of the relativistic Hamiltonian formalism taking into account the vacuum expectation for quark-antiquark pairing. The numerical parameter of the energy gap $\Delta_{\text{gap}}$ is determined by a relation of the quark condensation parameter with the gap parameter and the chemical potential $\mu_{\text{vac}}$.

The paper is organized as follows. In chapter 2 fundamentals of differential geometry on the TB are briefly described. Chapter 3 describes how the color $SU(3)$ symmetry for strong interaction arises as an emergent symmetry similar as Chern-Simon gauge fields in quantum Hall systems. In chapter 4 the condensed vacuum structure in the TB with a quark condensate is described and the gap equation for quark-antiquark pairing is derived.

2. Basics of the tangent bundle - geometry and central extensions of the little groups

The tangent bundle is one of the most important concept in differential geometry on curved manifolds. A tangent vector can be defined as linear differential operators $v_P f(x(t)) = \frac{df}{dt}(x(t)) P$ which acts on functions on a spacetime manifold $M$ tangent to the curve $x(t)$ at the space point $P$. The set of all tangent vectors $v_P$ spanned by frame vectors in a given basis is the tangent space $T_P(M)$. This means tangent vectors at the point $P$ are
placed in their own space $T_P(M)$. The tangent bundle $TM$ is the disjoint union of all tangent spaces at all points $P$:

$$TM := \bigcup_{P \in M} T_P(M) = \{(x, v) : x \in M, v \in T_x(M)\},$$  \hspace{1cm} (2.1)

In coordinate description a point in the tangent bundle is given by the pairs $X = (x, v)$ with $x = (x_0, x_1, x_2, x_3)$ as the coordinate of the spacetime manifold and $v = (v_0, v_1, v_2, v_3)$ the coordinate of tangent vectors, i.e. the TB is a 8-dimensional manifold. We introduce tetrads $e^\mu_a(x)$ which form an orthonormal basis $g_{\mu\nu}(x)e^\mu_a(x)e^\nu_b(x) = \eta_{ab}$ where $g_{\mu\nu}(x)$ is the metric of the spacetime manifold and $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ the metric of the Minkowski space. The four tetrads $e_a = e^\mu_a(x)\partial_\mu$ form the basis for the tangent space at each spacetime point $x$. The subscript $a,b$ numbers the vectors $(a,b = 0, 1, 2, 3)$ and $\mu = 0, 1, 2, 3$ their components in the coordinate basis $e_\mu = \partial_\mu (\mu = 0, 1, 2, 3)$. Each vector $v$ described in the coordinate basis $v^\mu$ can be expressed by a vector with respect to the frame basis $e_a$ according to the rule $v^a = e_a^\mu(x)v^\mu$. Cotangent 1-forms $\theta^a = \theta^a_\mu dx^\mu$ can be defined satisfying the orthogonality relation $e^\mu_a(x)\theta^a_\nu(x) = \delta^\mu_\nu$. The tetrads induce a spacetime pseudo-Rieman metric $g_{\mu\nu}(x) = \theta^a_\mu \theta^b_\nu \eta_{ab}$. The scalar product of a vector and a co-vector is defined as

$$(v, u) = g_{\mu\nu}(x)v^\mu u^\nu = \eta_{ab}v^a u^b, \hspace{1cm} (2.2)$$

The scalar product (2.2) is the governing structure relation of the tangent bundle, its invariance with respect to certain transformations determines the geometry of the TB. There exist two types of transformations which leaves (2.2) invariant. First, general spacetime coordinate transformations $x^\mu \rightarrow y^\mu = y^\mu(x)$ with vectors which are transformed as $v^\mu = (\partial y^\mu/\partial x^\nu)v^\nu(x)$ do not change the scalar product (2.2). Besides at a fixed spacetime point $x$ the tangent vectors can be transformed by a second type of transformation along the tangent vector axis which also preserve the scalar product (2.2):

$$v'^a = T^a_b(x)v^b, \hspace{1cm} (2.3)$$

where $T^a_b(x)$ are matrices satisfying the condition $\eta_{ab}T^a_cT^c_b = \eta_{cd}$. This means that the matrix elements $T^a_b(x)$ are elements of the group $SO(3, 1)$ of special linear local transformations with positive determinant depending on the spacetime point $x$ as a parameter.

Note that the scalar product (2.2) has also to be invariant with respect to infinitesimal tangent vector line elements. This allows us to add translations to the transformations in (2.3):

$$v'^a = T^a_b(x)v^b + a^a(x), \hspace{1cm} (2.4)$$

and leads to the more general transformation group of a semi-direct product $SO(3, 1) \rtimes T(3, 1)$.

The Poincare group and the transformation group (2.4) of tangent vectors in the TB are described by the same group $SO(3, 1) \rtimes T(3, 1)$. However both have a principal different geometrical and physical meaning: the first transforms the coordinates of a flat spacetime manifold while the second describes transformations within the tangent fiber $F = T_x(M)$ leaving the spacetime point $x$ unchanged. In order to avoid confusions we denote exclusively Poincare or Lorentz transformations as coordinate transformations of the flat spacetime manifold. In mathematics the transformation group (2.4) is denoted as a special affine group (SAG).

The TB is a special fiber bundle with the structure group $G = SO(3, 1) \rtimes T(3, 1)$. In general, gauge theories with a gauge group $G$ can be described using the mathematical formalism of fiber bundles with the group $G$ as structure group. In this formalism gauge fields correspond to the connection on the principle bundle while matter fields are described by representations of the group $G$ on the associated vector bundle. A principle bundle $P(M) = (P, M, \pi, G)$ is a geometric structure over the base manifold $M$ (here the spacetime manifold) with $G$ as the typical fiber, but $G$ act also as structure (gauge) group on the fiber, $\pi$ is the projection from the bundle $P(M)$ to the base manifold $M$. In quantum theory there is a fundamental relationship between the irreducible representations of a symmetry group $G$ of a system and the space of quantum states of the system. Therefore group representations must be included into the fiber bundle formalism. This is done through the concept of vector bundles associated to the principle fiber bundles. An associated fiber bundle $E(M, V, \pi, G, P)$ includes the vector space $V$ into the bundle structure which is the vector representation of $G$ and has the same structure group $G$ as the principle bundle $P(M)$.

Differential geometry on a fiber bundle can be executed by using the definition of connections and covariant derivatives on the bundle. The definition of a covariant derivative demands to consider vectors which point from one fiber to the other at neighboring points $x$ and $x'$ of the spacetime manifold. The generators $L_a$ of the group $G$ are vertical vectors pointing along the fibers and therefore belong to the vertical subspace $V_u(P)$. Horizontal vectors in the subspace $H_u(P)$ point away from the fibers and are elements of the tangent space of the fiber bundle $T_u(P)$ that complement the vertical vectors in $V_u(P)$. They can be constructed by [27, 28]

$$T_u(P) = H_u(P) \oplus V_u(P). \hspace{1cm} (2.5)$$

The complementary splitting of the tangent bundle into the vertical and horizontal sub-bundle is an important concept in the theory of fiber bundles. In the SM the vertical bundle describes the internal degree of freedom arising by the gauge group. On the other hand the horizontal sub-bundle is a concept to formulate the notion of a connection on the fiber bundle.
Let us now return to the specifics of the TB. The affine group \( G = SO(3, 1) \times T(3, 1) \) is a semi-direct product and not semi-simple. As explained in the introduction here it is assumed that gravity can be described by the translational gauge theory with the structure group \( T(3, 1) \) based on the translational connection as gauge field leading to the teleparallel gravity theory \[\textbf{13} \textbf{18}]. In contrast to this theory we assume that the sub-group \( SO(3, 1) \) is related with the other fundamental interactions of particle physics and the spin-connection do not vanish but is related with the generalized electroweak interaction \[\textbf{1}\]. The generators of the group \( SO(3, 1) \) are given by the 6 generators \( M_{ab} \) of the group \( SO(3, 1) \) with

\[ J_a = \epsilon_{abc} M_{bc}, K_a = -M_{da} \]

and the generators \( P_a \) of the translational group \( T(3, 1) \). This groups has two Casimir operators \( P^2 = P^a P_a \) and \( w^2 = w^a w_a \) where the Pauli-Lubanski vector \( w_a \) is defined by

\[
 w_a = \frac{1}{2} \epsilon_{abcd} M^{bc} P^d \tag{2.6}
\]

The action of the affine group \( G \) on a vector space \( X \) is not transitive, but the vector space decomposes into different orbits in which \( G \) acts transitively. A group action of \( G \) on \( X \) is transitive if for every pair of elements \( x \) and \( y, x, y \in X \) there is a group element \( g \in G \) such that \( gx = y \). The orbit of \( X \) is the sub-manifold of \( X \) consisting of all points that can be reached by acting on \( X \) with \( G \). The space \( X \), which has a transitive group action, is called a homogeneous space. Wigner developed the method of induced representations to find the unitary representation of the Poincare group. The Poincare group for spacetime transformations and the SAG for tangent vector transformations are mathematical identical, therefore we can use this method here. Since the translational generator \( P^a \) commute with each other the physical states can be expressed in term of the eigenvalues of the action of \( P^a \Psi = k^a \Psi \). The subgroup of \( SO(3, 1) \) with group elements \( T^a_b(x) \) which leaves \( k^a \) invariant is determined by

\[
 T^a_b(x) k^b = k^a \tag{2.7}
\]

and is called Wigner’s little group, \( k^a \) is denoted as the standard vector. There exist 6 orbits according to the eigenvalues of \( P^2 \) and \( P^0 \). For \( k^2 \prec 0 \) with \( k^0 \succ 0 \) and \( k^0 \prec 0 \) the little group is \( SO(3) \) with the generators \( J_1, J_2, J_3 \). For \( k^2 = 0 \) with \( k^0 \succ 0 \) and \( k^0 \prec 0 \) the little group is the Euclidian group \( E(2) \) with the generators \( T_1 = K_1 + J_2, T_2 = K_2 - J_1 \) and \( T_3 = J_3 \). For \( k^2 \succ 0 \) the little group is \( SO(2, 1) \).

The description of matter fields in the quantum field theory (QFT) requires the knowledge of the unitary representations \( T_L(\psi) \) of these groups. The composition law of the so-called vector representation \( T_L(\psi) \) satisfy the functional equation \( T_L(g_1) T_L(g_2) = T_L(g_1g_2) \) and encodes the law of group transformations on the set of vector states. This leads to certain pathologies, as for example the Dirac equation for massive particles is not invariant under the Poincare group, but under its universal covering group. Wigner solved this problem by using projective representations of the Poincare group. In quantum theory the physical symmetry of a group of transformations on a set of vector states has to preserve the transition probability between two states \( | \Phi, T_L(\psi) \rangle |^2 = | \langle \Phi, \Psi \rangle |^2 \). Therefore, generalized representations (denoted as projective representations) are allowed which satisfy the more general composite law:

\[
 T_L(g_1) T_L(g_2) = \varepsilon(g_1, g_2) T_L(g_1g_2) \tag{2.8}
\]

where \( \varepsilon(g_1, g_2) \) is a complex-valued antisymmetric function of the group elements with \( |\varepsilon(g_1, g_2)| = 1 \). Any projective representation of a Lie group \( G \) is equivalent to the unitary representation of the central extension of the group \( G^c \). In general a central extension \( G^c \) of a group \( G \) with elements \( (g, \xi) \in G^c \) and \( g \in G, \xi \in U(1) \) satisfy the group law

\[
 (g, \xi) = (g_1, \xi_1) * (g_2, \xi_2) = ((g_1 * g_2, \xi_1 \xi_2 \exp[i\xi(g_1, g_2)]) \tag{2.9}
\]

where \( \xi(g_1, g_2) \) is the 2-cocyle satisfying the relation

\[
 \xi(g_1, g_2) + \xi(g_1 * g_2, g_3) = \xi(g_1, g_2 * g_3) + \xi(g_2, g_3), \xi(e, g) = \xi(g, e) \tag{2.10}
\]

For the case of simply connected groups like the rotation group \( SO(3) \) projective representations are obtained by replacing the group \( SO(3) \) by its universal cover \( SU(2) \). However the Euclidean group \( E(2) \) is not semi-simple and the covering group \( E(2) \) is not sufficient.

One has to use a larger group: the universal central extension \( E^c(2) \). This group includes in addition to the group elements of \( E(2) \) the group \( U(1) \) of phase factors \( \varepsilon(g_1, g_2) \) with \( |\varepsilon(g_1, g_2)| = 1 \). The group \( E^c(2) \) has been studied previously as e.g. in \[\textbf{10} \textbf{12} \] and consists of elements \( (\alpha, a, \omega) \) with \( (\alpha, a) \in E(2), \omega \in R \).

The action of a group on a vector space determines the representation of the group. The action of the group \( E^c(2) \) on a vector \( Z = (\xi_1, \xi_2, \beta) \) is described by \[\textbf{10} \textbf{12} \].

\[
 (\alpha, a, \omega)(\xi_1, \xi_2, \beta) = (\xi_1 \cos \alpha + \xi_2 \sin \alpha + a^1, \xi_1 \sin \alpha + \xi_2 \cos \alpha + a^2, \beta + \omega + \frac{1}{2} m(\alpha, a, \xi_1, \xi_2)), \tag{2.11}
\]

where \( m(\alpha, a, \xi_1, \xi_2) \) is the 2-cocyle which gives the desired central extension parametrized as

\[
 m(\alpha, a, \xi_1, \xi_2) = (a_1 \xi_1 + a_2 \xi_2) \sin \alpha - (a_1 \xi_2 - a_2 \xi_1) \cos \alpha. \tag{2.11}\]

From (2.10) and (2.11) the generators of the central extended \( E^c(2) \) group can be calculated which are given by:
\[
T^1 = -i(\frac{\partial}{\partial \xi_1} + \frac{1}{2}\xi_2 \frac{\partial}{\partial \beta}), \quad (2.12)
\]
\[
T^2 = -i(\frac{\partial}{\partial \xi_2} - \xi_1 \frac{\partial}{\partial \beta}),
\]
\[
T^3 = -i(\xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1}), \quad E = -i \frac{\partial}{\partial \beta},
\]

which satisfy the following commutation rules \([T^1, T^2] = i\xi_E, [T^1, T^3] = -i T^2, [T^1, T^3] = -i T^2, [T^3, E] = 0\).

In general the (vertical) Laplacian (Casimir operator) of a gauge group plays an important role in gauge theories, it delivers the description of the "internal" space of elementary particles. Its eigenfunctions provide a complete basis for the description of the basic states \(\Phi_{\text{vert}}(u)\) of the vertical subspace which are a part of the total wavefunction \(\Phi(x, u) = \Phi_{\text{vert}}(u) \Phi_{\text{hor}}(x)\). The horizontal part \(\Phi_{\text{hor}}\) depend on the spacetime variable and is determined by the covariant derivative with the connection on the associated bundle. The eigenvalue of the Laplacian depend on the internal quantum numbers (IQN) such as hypercharge and iso-spin in the SM. The vertical Laplacian of the group \(E^c(2)\) is determined by

\[
\Delta_{E^c} = (T^1)^2 + (T^2)^2 + 2T^3E. \quad (2.13)
\]

Using polar coordinates \(\xi_1 = \xi \cos \phi, \xi_2 = \xi \sin \phi\) and \(h_{nm\kappa}(\xi, \beta, \phi) = \exp(i\phi\beta)\exp(im\phi)g_{nm\kappa}(\xi)\) for the eigenfunctions of the Laplacian we find with (2.12),(2.13) the following equation for the functions \(g_{nm\kappa}(\xi)\):

\[
[(\frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{1}{2}m^2 + x^2\xi^2 - 2\kappa m)g_{nm\kappa}(\xi) = \epsilon_{nm\kappa}g_{nm\kappa}(\xi). \quad (2.14)
\]

The solutions of (2.14) for \(h_{nm\kappa} = h_{nm\kappa}(\xi, \beta, \phi)\) are given by

\[
h_{nm\kappa} = N_{nm\kappa} \exp(i\kappa\beta) \exp(im\phi)g_{nm\kappa} \quad (2.15)
\]
\[
g_{nm\kappa} = (\exp(-\frac{|\kappa|}{2})(\xi^2)^{m}|L^m_n|(|\kappa| \xi^2)\]

with \(N_{nm\kappa} = \sqrt{\frac{1}{2^n(2m+1)!}}\), \(\epsilon_{nm\kappa} = 4\kappa(n + \frac{1}{2} + \frac{1}{2}(m + |m|)), m = 0, \pm1, \pm2, \kappa = \pm1, \pm2, \cdots, L^m_n(x)\) are the associated Legendre polynomials and the IQN \(m\) can be interpreted as hypercharge known from the SM, but here two additional IQNs arise: the \(E^c\)-charge \(\kappa\) and the family quantum number \(n\) which could elucidate the existence of families in the SM.

Note that the here considered problem of the central extension of the Euclidian group \(E(2)\) for symmetry transformations on the tangent bundle differ from the analog problem for massless representations of the Poincare group for spacetime transformations. Since massless particles are not observed to have continuous degree of freedom, one put the requirement for the physical states \(\Psi_{\kappa\sigma}\)

\[
T^1\Psi_{\kappa\sigma} = T^2\Psi_{\kappa\sigma} = 0 \quad (2.16)
\]

and physical states are only characterized by the operator \(T^3\Psi_{\kappa\sigma} = \sigma\Psi_{\kappa\sigma}\), where the operators \(T^i\) are the corresponding operators for massless representations of the Poincare group and \(\sigma\) is the helicity. Even though the Poincare group and the affine group (2.4) are mathematical identical, there exist a significant difference in physical applications.

The solutions \(h_{nm\kappa}\) in (2.15) form an orthonormal set and have the analogus form like the solutions of the Schrödinger equation in two space dimensions for electrons in a perpendicular external magnetic field \(B\) where the entity \(\kappa\) is substituted by \(\kappa \rightarrow eB/\pi\). In the Schrödinger equation, the different levels with quantum number \(n = 0, 1, 2, \cdots\) are denoted as Landau levels. Each Landau level is highly degenerate, the degeneracy is \(BA/\pi\) where \(A\) is the area of the system. An important entity in the quantum Hall effect is the filling factor \(\nu\) of a Landau level defined as the ratio of the density of electrons \(n_{el}\) to the density of states in a Landau level \(n_{d} = eB/2\pi\) (in the unit system \(c = 1, h = 2\pi\), with \(h\) as the Planck constant) given by \(\nu = 2\pi n_{el}/eB\).

The analogy of the solutions (2.15) with the solutions of the 2D Schrödinger equation in an external magnetic field suggests to define a TB filling factor \(\nu_{TB}\) as a characteristic entity. The variables \(\xi_1, \xi_2\) are dimensionless variables of the representation of the group \(E^c(2)\). It is thus natural to define a dimensionless "density" \(\nu_{TB}\) of quarks in the vertical subspace. In analogy with the QHE the filling factor of a TB-LL is given by

\[
\nu_{TB} = \frac{\pi n_{TB}}{|\kappa|}. \quad (2.17)
\]

The generators \(J_1, J_2, J_3\) of the semi-simple group \(SU(2)\) are well known (see e.g. [44]) and, by using the Laplacian \(\Delta = (J_1^2 + J_2^2 + J_3^2)\) of this group all finite-dimensional eigenfunctions of the Laplacian depending on the iso-spin IQN \(I\) and \(\beta\) can be found.

In the traditional quantum field theory, the internal degrees of freedom (such as isospin or color) are described by spacetime- depend multi-component fields taking into account the vertical structure given by the gauge groups by the corresponding Lie-algebra representation. In the TB description, in contrast to the multi-component formalism, the basic objects are functions \(\Phi(x, u)\) depending both on the coordinate \(x\) of the spacetime manifold and the variables \(u\) of the gauge group. For final dimensional representations of the group \(SU(2)\), complex coordinates \(z^1\) and \(z^2\) can be used for its description. This is
equivalent to the usual two-component description by isospinors \((\phi_1(x), \phi_2(x))^T\). However, for a non-semi-simple group like the group \(E^*(2)\), a multi-component representation is not favorable, and the tangent bundle approach using coordinates of the tangent space is more convenient. Therefore we use here in (2.12) and (2.13) the generators of the group and not those of the Lie algebra as commonly done in the SM.

The remaining little group \(SU(1, 1)\) is non-compact, and we do not consider here whether it has a physical meaning as symmetry group for tangent fiber transformations.

3. Emergent SU(3) symmetry and strong interaction in the tangent bundle geometry

3.1 Analogy of the vertical TB Laplacian with the Hamiltonian of a fractional quantum Hall system

The Laplacian \(\Delta_E\) on the group \(E^*(2)\) given by (2.13) refer to a single tangent fiber at a fixed space point \(x\). Now we consider the whole space with a finite volume \(V = L^3\) and present the fields arranges in regular cubic lattice of unit cells which are defined by a set of position vectors \(R = n_1a_1 + n_2a_2 + n_3a_3\) in which \(n_i = 0, \pm 1, \pm 2\) run over all integers and \(a_i\) are linear independent basis vectors. The reciprocal lattice is defined by \(G = l_1b_1 + l_2b_2 + l_3b_3\). Using periodic boundary conditions, the Fourier coefficients take discrete values \(\mathbf{k} \rightarrow \mathbf{k}_i = \frac{2\pi}{L}(l_1, l_2, l_3)\), \(l_i = 0, \pm 1, \pm 2\)...

In the limit \(L \rightarrow \infty\) we associate a tangent fiber with an elementary cell with volume \(\Delta V_i = (\frac{2\pi}{L})^3\). Due to the fermion character of quarks and leptons each of the state in a lattice cell can be occupied with one quark and one lepton of every possible internal quantum number and helicity. The tangent bundle \(TM\) associates to every point \(x \in M\) a vector space \(T_xM\) which is described by the variables of the representation of the groups \(E^*(2)\) and \(SU(2)\). To obtain the vertical Laplacian of the TB we combine the fibres attached at all space points \(x_i\). For the \(E^*(2)\) part of the vertical Laplacian of the bundle, we obtain from (2.13)

\[
\Lambda_{E^*(2)} = \sum_i \left[ -i \frac{\partial}{\partial \xi^{(i)}_1} + \frac{\kappa}{2} \xi^{(i)}_1 \right]^2 + \left[ -i \frac{\partial}{\partial \xi^{(i)}_2} + \frac{\kappa}{2} \xi^{(i)}_2 \right]^2
\]

(3.1)

where \(\xi^{(i)}_1, \xi^{(i)}_2\) are the corresponding variables of the representation of the \(E^*(2)\) group attached to the space cell \(i\). The Laplacian (3.1) has an analog form as the multi-particle Hamiltonian of a 2D quantum mechanical many electron system in an external uniform magnetic field. This Laplacian contains a rich internal structure, and describes a highly correlated system characterized as a topological quantum liquid with closely analogous to the integer and fractional quantum Hall effect.

The integer QHE has been discovered in 1980 by Klitzing et al. [43] in a 2D layer of a semiconductor at low temperature and strong magnetic fields. In this experiment it was found that the Hall conductance takes quantized values of \(\sigma_{xy} = e^2/2\pi h\), where \(\nu\) is precisely an integer number, \(\nu = 1, 2, ...\). The fractional QHE was discovered by Tsui, Stormer and Gossard in 1982 [46], who reported that \(\nu\) is not only restricted to take integer values, but can take values at \(\nu = 1/3\) and \(\nu = 2/3\).

The integer QHE can be understood because the 2D electron gas forms an incompressible liquid at the filling factors \(\nu = n = 1, 2, 3\) due to the Landau level structure with a finite energy gap for all charged excitations. The fractional QHE was first explained by a theory of Laughlin [47]. He proposed a trial ground-state many-body wavefunction in a partially filled Landau level with filling fraction \(\nu = 1/(2p + 1), p = 1, 2, 3\)... which include strong Coulomb interaction and correlations among the electrons.

Even though the issue of the QHE is now well understood and described in excellent reviews and books (see e.g. [30, 33, 34]) we choose a self-contained presentation accessible to readers who are not specialized in solid state theory.

In the TB-QFT the ground state (vacuum state) with the lowest possible energy is described by the family IQN \(n = 0\). In traditional QFT the vacuum state is a Fock state which contains no physical particles but quantum fluctuations of virtual particles related with the non-commutation of the quantized fields. In this case the vacuum expectation value of any field operator vanishes; in particular, the particle density of the vacuum is zero.

The close analogy to a QH system requires a redefinition of the vacuum in the TB-QFT. As explained in more detail in chapter 4 the analog form of the Laplacian with that of a quantum Hall system can be interpreted by the hypothesis that the vacuum with the IQN \(n = 0\) is filled with see leptons and composite see quarks and all higher levels \(n = 1, 2, 3\) are empty. Furthermore, there is an energy gap between the lowest TB-LL \(n = 0\) and a higher TB-LL. If a seecquark is excited into a higher TB-LL, it leaves a hole in the old state. A quark hole carry the opposite hypercharge and opposite isospin IQN, but a positive energy, and can be interpreted as an anti-quark. There exists also several other physical well established facts in QCD as the existence of quark condensation, Higgs condensation and others which hint on a non trivial vacuum structure similar, as a condensate in solid state theory.

Let us first study the eigenvalue equation of the vertical Laplacian for the \(E^*\) group of the TB

\[
\Lambda_{E^*} \Phi_{vert} = \epsilon \Phi_{vert}
\]

(3.2)

where \(\Phi_{vert}\) is the vertical part of the total wavefunction \(\Phi = \Phi(x)\Phi_{vert}\) describing the internal degree of freedom of the group \(SU(2) \otimes E^*(2)\). To simplify the analysis we first ignore the iso-spin degree of freedom
and assume that all iso-spins are polarized and frozen into one direction. For the completely filled ground state \( n = 0 \) the wave function of \( N \) fermions can be described by a Slater determinant. In every cell \( i \) particles with different hypercharge numbers \( m \) are placed which run from \( 0 \) to \( N - 1 \). Using the single-particle wave function (apart from a normalization constant) and variables \( \eta_i = \xi_1^{(i)} + i \xi_2^{(i)} = \xi_i \exp(i \varphi_i) \) the many-particle solution can be determined from (3.2) by

\[
\Phi^0_{\text{vert}}(\eta) = S \exp(i \xi \beta) \exp\left[-\frac{1}{4 l_\xi^2} \sum_{r=1}^{N} | \eta_r |^2 \right] \quad (3.3)
\]

with the Slater determinant \( S \) as

\[
S = \det \begin{bmatrix}
1 & 1 & 1 & \ldots \\
\eta_1 & \eta_2 & \eta_3 & \ldots \\
\eta_1^2 & \eta_2^2 & \eta_3^2 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\eta_1^{N-1} & \eta_2^{N-1} & \eta_3^{N-1} & \ldots
\end{bmatrix}
\]

and \( l_\xi^2 = | \varphi | / 2 \). Since the highest power of any particle coordinate \( \eta_i \) is \( N - 1 \) the maximum eigenvalue \( m = m_{\text{max}} = N - 1 \) and the solution describes a fully occupied lowest TB-LL with a filling fraction \( \nu_{TB} = 1 \). The wavefunction vanishes whenever two fermions come together and the exponential factor decreases quickly with increasing \( | \eta_r |^2 \). When we add one fermion state to the completely filled system it is placed into a higher TB-LL level \( n = 1, 2, \ldots \) because of the Pauli exclusion principle.

In a 2D semiconductor in an external uniform magnetic field the analogous solution (3.3) describes a remarkable macroscopic quantum phenomenon: the integer quantum Hall effect which is explained by a fixed and well-defined density of the system with a filling fraction \( \nu \) given by \( \nu = n / (2p + 1) \). One can use (3.3) as the solution for the ground state \( n = 0 \) of the vertical part of the wavefunction of leptons.

Laughlin made a brilliant ansatz for a trial fermion wave function (known as the Laughlin wave function) that describes the fractional quantum Hall effect. Since the vertical Laplacian (3.1) has the same form as the 2D multi-particle Hamiltonian in a quantum Hall system we can use the analog Laughlin wave function \( \Phi^L_{\text{vert}}(\eta) \) for the description of the vertical part of the quark wavefunction in the vacuum state \( n = 0 \):

\[
\Phi^L_{\text{vert}}(\eta) = \prod_{i<j} (\eta_i - \eta_j)^k \exp\left( -\frac{1}{4 l_\xi^2} \sum_i | \eta_i |^2 \right) \quad (3.4)
\]

where \( k = 2p + 1 \) must be an odd integer for \( \Phi^L_{\text{vert}}(\eta) \) to be totally anti-symmetric. This wavefunction describes a uniform "density" state with a partially filled lowest TB-LL with filling factor \( \nu_{TB} = 1/(2p + 1) \). Due to a particle-hole symmetry there exist also states at the filling factor \( \nu_{TB} = 1 - 1/(2p + 1) \). In particular for \( p = 1 \) we find states with the filling factor \( \nu_{TB} = 1/3 \) and \( \nu_{TB} = 2/3 \).

Laughlin achieved a physical understanding of the wavefunction (3.4) by an analogy with a 2D plasma. The joint probability distribution \( P = | \Phi^L_{\text{vert}} |^2 = \exp(-\beta U(\eta_1 \ldots \eta_N)) \) can be described by a potential \( U \). By using (3.4) and choosing \( \beta = 2/k \) the potential \( U \) becomes

\[
U = -k^2 \sum_{i<j} \ln \left( \frac{1}{\xi} | \eta_i - \eta_j | \right) + \frac{k}{4 l_\xi^2} \sum_i | \eta_i |^2 \quad (3.5)
\]

Eq.(3.5) has the form of a Coulomb potential of electrons in a 2D one-component plasma. The first term has the analog form as the Coulomb interaction term between charged particles in 2D characterized by a logarithmic potential with a charge \( q = -k \). The second term describes their interaction with a neutralizing background of positive charges similar as in the yellowium model. The analog for the "density" of the background charge is \( \rho_B = 1/2 \pi l_\xi^2 = | \varphi | / \pi \) (note that \( \rho_B \) describes a dimensionless entity in the \( E \) group manifold). The complete screening property of the hypercharge in the ground state now requires, similar as in a plasma, that the "charge" density of the plasma particles of see quarks in the ground state is equal to the background charge density \( \rho_B = | \varphi | / \pi \). Each particle carry the charge \( q = -k \), therefore the compensating "density" of particles in the ground state \( n = 0 \) is \( k \theta_{\text{vac}} = \rho_B \), or

\[
\theta_{\text{vac}} = \frac{| \varphi |}{\pi (2p + 1)} \quad (3.6)
\]

This is the "density" of a state at filling fraction

\[
\nu = \frac{1}{2p + 1} \quad (3.7)
\]

The effect of adding or removing one particle from the ground state corresponds to the creation of a quasi-particle or a quasi-hole. A quasi-hole located at \( \eta_0 \) can be described by the wavefunction generated by acting on the vacuum state as follows:

\[
\Phi^h_{\text{vert}}(\eta) = \prod_{i=1}^{N} (\eta_i - \eta_0) \Phi^L_{\text{vert}}(\eta) \quad (3.8)
\]

obtained from Laughlin's wave function \( \Phi^L_{\text{vert}}(\eta) \) by introducing the first factor \( (\eta_i - \eta_0) \). It vanishes at the position corresponding to the complex parameter \( \eta_0 \). This means it describes a state where the quark "density" is zero at \( \eta_0 \) and therefore characterizes a hole. It is an excited state because it has an additional vortices and the hypercharge is boosted by one unit. To calculate the quasi-hole charge
one can use the plasma analogy with the wave function (3.8). The joint probability distribution of the quasihole is given by

$$P = |\Phi_{\text{vert}}(\eta)|^2 = \exp\left\{-\frac{1}{2k}(U - k \ln(1 / \phi_{\text{vert}} | \eta_i \neq \eta_0))\right\}$$

(3.9)

where $U = U(\eta_1, \eta_N)$ is given in (3.5). Since a system like a plasma attempts to maintain charge neutrality according to (3.9) it will screen the hole by the charge of $k$ plasma particles. The resulting screening cloud has a net deficit of $1/k$ vacuum particles. This means that the quasihole represents an excitation with a fractional charge $e_{\text{h}} = e/k = e/(2p + 1)$. The above described analogy with the fractional QHE here arise in a natural way from the bundle Laplacian (3.1) of the group $E^c(2)$ and can explain charge quantization of leptons with $m_{\text{lep}} = 1$ and fractional charges of quarks with a quasi-hole hypercharge given by $m_q = 1/(2p + 1)$. The fractional charges of quasi-holes and quasi-particles can also be proven by other methods (see e.g. [33]).

The hypercharge of an excited valence quark should carry the opposite sign as in (3.8). The corresponding wavefunction should contain an antivortex which involves $\eta^*_i$, but then the wavefunction do not sit in the lowest TB-LL, and rather one has to project it into it. According this procedure the wavefunction proposed by Laughlin for the quasiparticle state can be used for excited valence quarks as [47]

$$\Phi_{\text{vert}}^q = \prod_{i=1}^N \left(2i_\varepsilon^2 \frac{\partial}{\partial \eta_i} - \eta_i^* \right) \Phi_{\text{vert}}^L(\eta)$$

(3.10)

where the derivatives act only on the polynomial part of $\Phi_{\text{vert}}^L(\eta)$.

Another important feature of the wavefunction is the existence of vortices—here we denote it as TB vortices. A TB vortex is a winding in the phase of the variable $\eta_r = \xi^r_1 + i \xi^r_2 = \xi \exp(i \varphi_r)$ which means the phase $\varphi_r$ changes by $2\pi$ if the particle moves around $\eta_r$.

In the discussion above it has been assumed that in the vacuum, all states carry the same isospin number. Generalization of this situation is possible to systems where the quarks are not isospin polarized but the lowest TB-LL is occupied by both isospin states, forming an isospin singlet. The simplest two-component wavefunction describing a topological state filled with particles with spin in both directions was given by Halperin [48]. Here we use it for the description of the vertical part of wavefunctions for isospin singlet states of quarks with two iso-spin components. According [48] this solution has the form

$$\Phi_{\text{vert}}^H = \prod_{i<j} \exp\left[-\frac{1}{4\xi} \left(\sum_i (|\eta^+_i|^2 + |\eta^-_i|^2)\right)^p\right] \times (\eta_i^+ - \eta_j^+)^{p+1} (\eta_i^- - \eta_j^-)^{p+1} (\eta_i^+ - \eta_j^-)^p$$

(3.11)

where the set of coordinates $\eta^+_i, i = 1...N$ corresponds to isospin up and $\eta^-_i, i = 1...N$ to isospin down quarks, $p$ is an integer number. The filling factors are given by

$$\nu_+ = \nu_\downarrow = \frac{1}{1 + 2p}.$$  

(3.12)

Note that in experimental studies of the anomalous QHE also filling factors with other rational numbers than (3.7) or (3.12) has been observed and several theoretical explanations has been developed.

### 3.2 Composite quarks with fractional hypercharges

The composite-fermion picture [34, 49] is a convenient way to provide an intuitive idea for the fractionally charged quasi-particles and various other aspects of the quantum Hall effect. This concept can be transferred to the understanding of fractional charged quarks in the TB geometry. When all particles are confined in the lowest TB-LL the wave function is a polynomial of the complex variable $\eta = \xi_1 + i \xi_2 = \xi \exp(i \phi)$ which has a vortex at the origin because a complete loop around the origin changes $\phi$ by $2\pi$. A composite fermion is simply envisioned as a bound state of a fermion carrying an even number of quantized vortices of the many particle wave function. This can be understood as a screening of the fermions, since vortices create holes around each fermion. Composite quarks capture two TB vortices and as a consequence each fermion can be described qualitatively by a simple reformulation of the Laughlin function (3.4) by [49]

$$\Phi_{\text{vert}}^J = \prod_{i<j} (\eta_i - \eta_j)^2 \Phi_{\text{vert}}^L(\eta)$$

(3.13)

Here $\Phi_{\text{vert}}^L(\eta)$ is the Laughlin state given by (3.4) and the first factor represents two TB vortices that are attached to each coordinate $\eta_i$ where a quark is present. Thus the Laughlin ground state $\Phi_{\text{vert}}^J(\eta)$ for partially filled quarks of the state $n = 0$ is now understood as a Jain ground state $\Phi_{\text{vert}}^J$ completely filled with composite quarks.

A composite quark experience a reduced IQN $\kappa_{eff}$. This can be calculated in an analogous way as in the case of the anomalous QHE in semiconductors taking into account the magnetic field of the $2p$ attached vortices $B_{\text{vert}} = -2p \Phi_0 \Phi_0$ pointing antiparallel to the external magnetic field $B$, where $\Phi_0 = h/e$ is the elementary quantum of magnetic flux. Thus with the effective magnetic field $B_{\text{eff}} = B - 2p \Phi_0 \Phi_0$ the composite fermions experience a reduced effective magnetic field. In analogy, the fractional filling factor of bare quarks $\nu_{\text{eff}} = 1/(2p + 1)$ gives for the composite quarks a completely filled ground state $\nu_{eff} = 1$. Using the analogy of $\kappa$ with the magnetic field $B$, we can substitute $eB_{\text{eff}} \rightarrow 2\kappa_{eff}$ and
with \( \rho_0 = \nu/\pi, \Phi_0 = 2\pi \) (for \( c = 1, h = 2\pi \)) and
\( \nu_{TB} = 1/(2p + 1) \) we obtain
\[
x_{eff} = x - \frac{2p}{2p + 1} = x - \frac{1}{2p + 1}. \quad (3.14)
\]

This means that in TB-QFT not only the effective hypercharge \( m_{eff} = (2p + 1)^{-1} \) receive a fractional number, but also the \( E_c \) charge \( x_{eff} \).

### 3.3 Quark and lepton families, fractional hypercharges and fractional \( E_c \) charges of quarks

In the SM the members arranged in different families of leptons and quarks have identical IQNs and propercharges except for their masses. In contrast in the TB-QFT of leptons and quarks have identical IQNs and propercharges and fractional \( E_c \) charges of quarks. The Yukawa interaction Lagrangian can be written as
\[
\mathcal{L}_{Yuk} = \sum_{i,j} [c^{D}_{ij} \Phi_H(D_j)R + c^{U}_{ij} \Phi_H^*(U_j)R + h.c.]
\]
with \( c^{D}_{ij} \) and \( c^{U}_{ij} \) as constant coupling coefficients and \( \Phi_H = i\sigma_2 \Phi_H^* \). Inserting the expansion (2.15) into \( \mathcal{L}_{Yuk} \) the Yukawa interaction Hamiltonian is easy to build with analog expressions as in the SM but including additional matrix elements
\[
\mathcal{I}^{Y}_{QHU_R} = \int d\mu \chi_H(u)\chi_Q^* (u)\chi_{U_R}(u), \quad (3.16a)
\]
and
\[
\mathcal{I}^{Y}_{QH+D_R} = \int d\mu \chi_H(u)\chi_Q^* (u)\chi_{D_R}(u), \quad (3.16b)
\]
with the integration measure \( d\mu(u) = dp_{SU(2)} dp_{E^c} \).

From (3.16) we see that the Yukawa interaction is nonzero only if in addition to leptons and quarks the Higgs particle also carry a nonzero \( E_c \) charge \( x_{H} \). The matrix elements are non-zero if the following selection rules for the hypercharges \( m_{eff} \) of quarks are fulfilled
\[
-m_Q + m_H + m_{D_R} = 0 \quad (3.17)
\]
\[
-m_Q + m_H + m_{U_R} = 0 \quad (3.18)
\]
where \( m_H = m_{H^c} = 1 \). Besides we have also the analog relations for the \( E_c \) charge \( x \):
\[
-xq + xH + xD_R = 0 \quad (3.18a)
\]
\[
-xq + xH^c + xU_R = 0 \quad (3.18b)
\]
The solutions of the equations for the unknown \( E_c \) charges \( x \) are not unique. Since interaction processes favor fractions with small denominators, a possible special solution of the above equation system is given by
\[
m_{eff} = x_{eff}. \quad (3.19)
\]
This means fractional \( E_c \) charges \( x \) of quarks are required by the Yukawa interaction. Note that this results agrees with the composite fermion picture discussed in caption 3.2.

Let us summarize the extension of the IQNs in the TB-QFT and its interpretation by the composite quark picture. For the left-handed quarks \( (D_L) \) we have the IQNs \( M_{D_L} = \{m_{eff} = \frac{1}{3}, x_{eff} = \frac{1}{3}, I_3 = -\frac{1}{2} \} \) and \( M_{U_L} = \{m_{eff} = \frac{1}{2}, x_{eff} = \frac{1}{2}, I_3 = \frac{1}{2} \} \). For the right-handed down-quarks \( (D_R) \) we have \( M_{D_R} = \{m_{eff} = -\frac{2}{3}, x_{eff} = -\frac{2}{3}, I_3 = 0 \} \) and for the right-handed up-quark states \( M_{U_R} = \{m_{eff} = \frac{2}{3}, x_{eff} = \frac{2}{3}, I_3 = 0 \} \). On the other hand the IQN of the Higgs are \( M_H = \{m_H, m = 1, x = 1, I_3 = -\frac{1}{2} \} \).

Now one can explain the IQNs of different quarks according its composition by bare quarks and attached TB vortices. As noted above, left-handed quarks \( (U_L, D_L) \) have the IQN \( m_{eff} = 1/3 \) and \( x_{eff} = 1/3 \) which can be interpreted as excitations of composite holes with two attached TB vortices. Right-handed \( D_R \) quarks are isospin singles with \( I_3 = 0 \) and can be interpreted by the assumption that the lowest TB-LL is occupied with isospin up and isospin down quarks with total filling fraction \( \nu_{eff} = 2 \). Since every isospin component carries the hypercharge \( -1/3 \) the hypercharge of the \( D_R \) quark is \( m_{eff} = -2/3 \). The right-handed \( U_R \) quarks with \( I_3 = 0 \) can be identified as excited hole states with two isospin components, each of which carries the hypercharge \( m_{eff} = 2/3 \). This means in agreement with the above remark the total hypercharge of
the $U_R$ quarks is $m_{eff} = 4/3$ and of the $E^c$ charge is $\kappa_{eff} = 4/3$.

The left-handed leptons $(N,E)_L$ are interpreted as fermions without attached TB vortices with $m = -1$ and $\kappa = -1$ and a completely filled lowest TB-LL $\nu = 1$. Right-handed leptons $E_R$ are isospin singlets with $m = -2$ and $\kappa = -2$ and filling factor $\nu = 2$.

### 3.4. Emergent Chern-Simon U(1) gauge fields on the tangent bundle

In the composite fermion picture of the QHE, the Laughlin wavefunction of the TB determined by the bundle Laplacian as given in (3.1). In the Chern-Simon approach a singular gauge transformation of the Laughlin (3.1) is considered given by (see e.g. [49])

$$\Psi(\xi_1, ..., \xi_N) = U \Psi(\xi_1, ..., \xi_N)$$

with

$$U = \prod_{j > k} \exp(-ik\theta(\xi_j - \xi_k))$$

with $\xi_i = (\xi^1_i, \xi^2_i)$ and $\theta(\xi_j - \xi_k) = \text{Im} \ln(\xi_j - \xi_k)$. An even number $k = 2p$ is required for a transformation from the bare fermions to composite fermions. Performing the unitary transformation $U^{-1} \Delta_{E^c} U$ the transformed Laplacian of (3.1) can be written as

$$\tilde{\Delta}_{E^c} = \sum_i \left[ -i \frac{\partial}{\partial \xi^1_i} + \frac{\kappa}{2} \frac{\partial^2}{\partial \xi^2_i} - A_1(\xi^1_i) \right]^2 + \left[ -i \frac{\partial}{\partial \xi^2_i} - \frac{\kappa}{2} \frac{\partial^2}{\partial \xi^1_i} - A_2(\xi^2_i) \right]^2$$

where $A_a(\xi^a)$ (with $a = 1, 2$) is an auxiliary (emergent) field arising by the unitary transformation (3.20) and defined by

$$A_a(\xi^a) = k \sum_{j \neq i} \frac{\partial}{\partial \xi^a_i} \theta(\xi_j - \xi_i).$$

Therefore we find

$$\oint_i A_a(\xi^a_i) d\xi^a_i = 2\pi k$$

where the integral is around any closed contour surrounding only the particle $i$. The two-component vector potential $A_\nu(\xi)$ is related with an emergent field $B(\xi)$ (which is the analog of the "magnetic field strength" in the 2-dimensional $E^c$ manifold)

$$B(\xi) = \varepsilon^{ab} \partial_a A_b(\xi) = 2\pi k \rho(\xi).$$

where $\rho(\xi) = \sum_k \delta^2(\xi - \xi_k)$ is the local "particle density" in the $E^c(2)$ manifold and $\varepsilon^{ab}$ is the antisymmetric 2D Levi-Civita symbol.

The Laplacian (3.21) can be written in field-theoretical formalism

$$\tilde{\Delta}_{E^c} = \Phi_{vert}^1(\xi) \left[ -i \frac{\partial}{\partial \xi^1} - \frac{\kappa}{2} \frac{\partial^2}{\partial \xi^2} - A_1(\xi) \right]^2 + \left[ -i \frac{\partial}{\partial \xi^2} - \frac{\kappa}{2} \frac{\partial^2}{\partial \xi^1} - A_2(\xi) \right]^2 \Phi_{vert}(\xi)$$

where $\Phi_{vert}(\xi)$ is the $E^c(2)$ part of the vertical quark wavefunction. The "density" $\rho_{vert}(\xi)$ in the field-theoretical formalism is given by

$$\rho_{vert}(\xi) = \Phi_{vert}^1(\xi) \Phi_{vert}(\xi).$$

We obtain a dynamic description if we introduce the time variable $\xi_0 = t$ together with $\xi_1$ and $\xi_2$ and a new time component $A_0$ of the emergent potential $A_a$. One can use the gauge condition $\partial^a A_a = 0 \,(a = 0, 1, 2)$. If we take the time derivative of (3.24) and use the equation of conserved current $\partial_1 \rho + \partial_2 \rho = 0$ one obtain

$$\varepsilon^{ab} \partial_1 \rho_a A_b(\xi) = 2\pi k \partial_1 \rho(\xi)$$

and consequently

$$\varepsilon^{ab} \partial_1 A_b = -2\pi k j^a.$$
evolution, because the time derivation of (3.24) implies (3.28).

Alternatively the constrains (3.24) and (3.28) can also be taken into account if we add an additional term to the Laplacian (3.25) (denoted as the Chern-Simon action)

\[
L_{CS} = \Phi_{\text{vert}}^\dagger(\xi)|i\frac{\partial}{\partial \xi^0} - A_0(\xi)|\Phi_{\text{vert}}(\xi) + \Lambda_{E^c} + \frac{1}{4\pi k}\varepsilon^{abc} A_a(\xi)\partial_b A_c(\xi) \tag{3.29}
\]

(with \(\Lambda_{E^c}\) given in (3.25), \(a, b = 0, 1, 2\)). The Chern-Simon action is described by the last term in (3.29). In \((2 + 1)D\) dimensions a particle current coupled to a Chern-Simon field produces states with fractional charges and the binding of particles to fluxes. This can be seen by the variation of (3.29) with respect of \(A_0\) yielding (3.28) and with respect to \(A_a\) \((a = 1, 2)\) which delivers

\[
\varepsilon^{ab}(\partial_b A_0 - \partial_0 A_b) = 2\pi k j^b(\xi). \tag{3.30}
\]

(3.30) agrees with (3.28) using the special gauge \(A_0 = 0\). The coupling constant \(k\) has to be an even number: \(k = 2n\).

The Chern-Simons theory is in \((2 + 1)D\) a new type of gauge theory, completely different from Maxwell theory in \((2 + 1)D\). This theory is particular interesting for its practical application in the QHE and describes a deep analogy with quantum mechanical Landau lev- 

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The Chern-Simons theory is in \((2 + 1)D\) a new type of gauge theory, completely different from Maxwell theory in \((2 + 1)D\). This theory is particular interesting for its practical application in the QHE and describes a deep analogy with quantum mechanical Landau levels and fractional charge quantization. The effect of the Chern-Simons term in (3.29) is to tie the magnetic field \(B(\xi)\) to the number density as described in (3.24).

A simple way to analyze the Chern-Simons approach is to make the mean field approximation in the gauge \(A_0 = 0\). Using an average over the variables \(\xi\) of the \(E^c\) manifold \(\langle \rho(\xi) \rangle = \rho_0\) according to (3.24) the TB Chern-Simons field is smeared out to \(\langle B(\xi) \rangle = 2\pi k \rho_0\). The effective field \(B_{\text{eff}}\) is reduced to

\[
B_{\text{eff}} = B - \langle B \rangle = B - 2\pi k \rho_0 \tag{3.31}
\]

Using the analogy of \(\pi\) with the magnetic field \(B\) \((eB \to 2\pi)\) we can substitute \(eB_{\text{eff}} \to 2\pi e_{\text{eff}}\) and find with \(\rho_0 = \frac{2}{\pi} \nu\) and \(\nu = \frac{1}{(2p + 1)}\) a fractional \(E^c\) charge \(e_{\text{eff}} = \pi/(2p + 1)\). This relation for \(e_{\text{eff}}\) agrees with the composite quark interpretation explained in chapter 3.2 and the rules derived from the Yukawa interaction Lagrangian in chapter 3.3.

3.5 Emergent SU(3) symmetry in the tangent bundle

The Chern-Simon formalism for the fractional QHE with an emergent \(U(1)\) gauge group has been generalized for the inclusion of spin or for a bilayer quantum Hall system \([51, 54]\). In \([51, 52]\) the fractional QHE was described by two-component Abelian Chern-Simon fields, one for each spin or each layer, and magnetic fluxes are attached to electrons in their own layer and to the opposite layer. A detailed description of this approach for the bilayer QHE is given e.g. in \([30]\). Note that this abelian two-component Chern-Simon approach has a \(U(1) \otimes U(1)\) gauge symmetry, and the \(SU(2)\) spin symmetry is broken from the external construction. In \([53]\) a non-Abelian Chern-Simon approach for \(SU(2)\) spin singlets and in \([54]\) for the \(U(1) \otimes SU(2)\) symmetry was constructed in which a non-Abelian Chern-Simon field \([53]\) is used as the source of an emergent additional gauge field which is invariant with respect to the \(SU(2)\) transformation of the wavefunction. We apply some of the results of these papers for a treatment of the vertical TB Laplacian and the description of the way, in which emergent color \(SU_c(3)\) gauge fields can arise together with fractional hypercharges by an \(U_{\text{em}}(1)\) field.

Quarks can occupy three different isospin states: two states are arranged as an isospin dublet with \(I_3 = 1/2\) and \(I_3 = -1/2\) and the third is the isospin singlet \(I_3 = 0\). In the Laplacian of the product group \(E^c(2) \otimes SU(2)\) one can assume that in the ground state the three isospin states are degenerate and equally active. Therefore the system possesses an underlying \(SU(3)\) symmetry and the vertical part of the TB Laplacian shows certain analog features of a quantum Hall three-layer system.

In the case of a three layer system the eigenfunctions of the Laplacian are described by three separate eigenfunctions \(g_I(u)\) \((I = 1, 2, 3)\) with \(u\) as the variables of the group. The eigenfunctions of the vertical Laplacian at a fixed spacepoint \(x\) are represented as a \(SU(3)\) spinor which will be later identified as the quark spinor in the internal color space of QCD. We denote the three components by red (r), green (g) and blue (b)

\[
g(u) = \begin{bmatrix} g_r(u) \\ g_g(u) \\ g_b(u) \end{bmatrix} \tag{3.32}
\]

which possesses the underlying symmetry of the color \(SU(3)\) group

\[
g_I(u) = \exp(i\omega_A \lambda_A) g_I(u) \tag{3.33}
\]

with \(\omega_A = \omega_A(u)\). The \(SU(3)\) group has 8 generators \(\lambda_A\) presented as \(3 \otimes 3\) Lie algebra matrixes with the commutation rule

\[
[\lambda_A, \lambda_B] = f^C_{AB} \lambda_C \tag{3.34}
\]

where \(f^C_{AB}\) are the structure constants of the \(SU(3)\) group. The normalization is

\[
Tr(\lambda_A \lambda_B) = 2\delta_{AB} \tag{3.35}
\]

In the last section we described in the Chern-Simon approach for the Laplacian of the \(E^c(2)\) group the existence of an emergent \(U(1)\) gauge field connected with fractional hypercharges. As explained above due to the three isospin components of quark states an additional
SU_c(3) symmetry arises which leads to emergent color SU(3) gauge fields.

For simplification we consider the case of a vacuum state with aligned spin and isospin states. Similar to that in the Abelian Chern-Simon approach in section 3.4, an Abelian and a non-Abelian Chern-Simon term is used to describe the attachment of charge density $\varrho(\xi)$ and color SU_c(3) spin density to the quarks. The variables of number density $\varrho(\xi)$ and color SU_c(3) spin density in the ground state $n = 0$ are defined as:

$$\varrho(\xi) = \Phi_{\text{vert}}^\dagger(\xi)\Phi_{\text{vert}}(\xi),$$

$$S_A(\xi) = \Phi_{\text{vert}}^\dagger(\xi)\lambda_A\Phi_{\text{vert}}(\xi)$$

We denote the emergent SU_c(3) Chern-Simon gauge vector potential as $G^A_a(\xi)$, $(a = 0, 1, 2, A = 1 - 8)$ and the emergent $U_{em}(1)$ gauge potential as $A_0(\xi)$. Using the analogous approach as in chapter 3.4 the coupling of SU(3) spinor quark fields to the emergent gauge fields $A_0(\xi)$ and $G^A_a(\xi)$ is described by the generalized bundle Laplacian

$$\mathcal{L} = \Phi_{\text{vert}}^\dagger[i\partial_0\Phi - (G^A_0(\xi) + A_0(\xi))\Phi_{\text{vert}}] - \mathcal{L}_E + \mathcal{L}_{CS}$$

$$\mathcal{L}_{CS} = \frac{1}{4\pi k_2} \varepsilon^{abc} [G^A_a\partial_bG^A_c +$$

$$+ \frac{1}{3} f_{ABC}G^A_aG^B_cG^C_e]$$

where

$$\mathcal{L}_E = \sum_{A=1}^8 \Phi_{\text{vert}}^\dagger(\xi)[(-i\frac{\partial}{\partial \xi^1} + \alpha\xi^2 - \lambda_A G^A_1(\xi) - A_1(\xi))^2$$

$$+ (-i\frac{\partial}{\partial \xi^2} - \alpha\xi^1 - \lambda_A G^A_2(\xi) - A_2(\xi))^2]$$

(3.38)

is the transformed effective bundle Laplacian of the $E^c(2)$ group, $\mathcal{L}_{CS}$ is the non-Abelian Chern-Simon action of the group $U_{em}(1)\otimes SU_c(3)$ (compare [53, 55])

$$B(\xi) = \varepsilon^{ab}\partial_aA_b(\xi)$$

and for the SU(3) group

$$G^A(\xi) = \varepsilon^{ab}(\partial_aG^A_b(\xi) - f_{ABC}G^B_cG^C_e)$$

(3.41)

The equation of motion can be obtained by the variation of the Lagrangian $L$ over $A_0(\xi)$ given by

$$B(\xi) = \varepsilon^{ab}\partial_aA_b = 2\pi k_1\varrho(\xi)$$

(3.42)

Variation over $G^A_0(\xi)$ yields the constraint

$$G^A(\xi) = \varepsilon^{ab}(\partial_aG^A_b - f_{ABC}G^B_cG^C_e)$$

$$= 2\pi k_2 S^A(\xi)$$

(3.43)

The invariant Chern-Simon SU(3) color-magnetic field $G^A(\xi)$ is according to (3.43) directly proportional to the color SU(3) spin density $S^A(\xi)$. An important property of the (2+1)D Chern-Simon approach is that the large-scale physics of an incompressible 2D system (this means that there is an energy gap above the ground state) is determined purely by the Chern-Simon action $L_{SC}$ in (3.39) [50]. Other interaction terms in (3.37) are short range and invisible in the large-distance scale. This leads to an interesting conclusion. The field equation derived from the pure Chern-Simon action $L_{SC}$ in (3.39) is given by

$$G^A(\xi) = 0$$

(3.44)

Since the Chern-Simon color-magnetic field $G^A(\xi)$ in the large-scale limit of the variables of the $E^c$ group vanishes we find from (3.43) for the averaged color SU(3) spin density

$$\langle S^A(\xi) \rangle = 0$$

(3.45)

The vanishing of the average $\langle S^A(\xi) \rangle$ over the variables $\xi$ implies that the ground state is a color singlet. This can be interpreted as a signature of quark confinement in the TB-QFT and explains why only colorless quark-antiquark pairs and colorless bound three-quark systems forming mesons and baryons exist in nature. This surprising and unexpected result in this paper follows from general universal physical principles in the vertical TB Laplacian independent on the microscopic dynamics of quarks in QCD. Note that similar universal properties in a quantum Hall system, such as the Hall conductivity, are known where the theoretical understanding of properties are encoded into the large-scale Chern-Simon Lagrangian which do not involve a detailed understanding of the microscopic quantum mechanics of such systems [50]. However the prediction of quark confinement must be confirmed by a complete microscopic analysis of QCD in the TB which is beyond the present paper but will be studied later.

In such a way the SU(3) color symmetry is hidden in the vertical TB Laplacian of the group $E^c(2)$ arising as an emergent symmetry. To describe the full dynamics in the TB we have to include the spacetime-depending horizontal part $\Phi_{hor}$ of the wavefunctions; therefore, the multiplets of quark fields in the TB depend both on the
The dependence of the gluon fields on the spacetime variables and quark fields in dependence on the spacetime variables requires us to include the horizontal part of the wavefunctions and the treatment of gauge potentials and quark fields in dependence on the spacetime variables. The dependence of the field strength tensor of the gluon fields on the spacetime variables \( x \) is defined as

\[
G_{\mu
u}^a = \frac{\partial}{\partial x^\mu} G_a^{\nu} - \frac{\partial}{\partial x^\nu} G_a^\mu + g_s f_{abc} G_b^\mu G_c^\nu
\]

with \( g_s \) as the coupling coefficient for strong interaction.

We can combine the left-handed (L) and right-handed (R) quark fields into Dirac spinors

\[
q_f = \begin{pmatrix} q_{\text{L}f} \\ q_{\text{R}f} \end{pmatrix}
\]

where \( f = (u, d, c, s, t, b) \) is the flavour index \( q_{\text{R}f}, q_{\text{L}f} \) refer to the right or left handed particles. The Lagrangian density of the system is \( \mathcal{L} = \mathcal{L}_q + \mathcal{L}_g \) where \( \mathcal{L}_q \) is the Lagrangian density for the quark fields

\[
\mathcal{L}_q = \sum_f \overline{q}_f(x, u) \begin{pmatrix} i \gamma^\mu \frac{\partial}{\partial x^\mu} - m_f - \gamma^0 \mu_0 \\
-g_s G_a^\mu(x) \lambda^A q_f(x, u) \end{pmatrix} \]

The sum is over all flavours of quarks, \( m_f \) is the quark mass and \( q_f(x, u) \) is a color triplet. The gluon Lagrangian density is given by

\[
\mathcal{L}_g = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}
\]

In the one-loop approximation the interaction Lagrangian density is determined by

\[
\mathcal{L}_{\text{int}} = \frac{g_s^2}{2} \int d^4y \, d\mu(u) J^A_{\mu}(x, u) D_{\mu\nu}^{AB}(x-y) J^{B\nu}(y, u)
\]

where \( D_{\mu\nu}^{AB}(x-y) \) is the gluon propagator. The current is given by

\[
J^A = \sum_f \frac{1}{2} g_s \gamma^\mu \lambda^A q_f
\]

From the Lagrange density one can obtain the Hamiltonian density via the Legendre transformation. The Hamiltonian density is determined by

\[
H(t) = \int d^3x \, d\mu(u) \left[ \frac{\partial \overline{q}_f(x, u)}{\partial t} - \mathcal{L}(x, u) \right]
\]

where \( d\mu(u) \) is the invariant measure of the group and

\[
\pi(x, u) = \frac{\partial \mathcal{L}(x, u)}{\partial \frac{\partial q_f(x, u)}{dt}}
\]

is the conjugate momentum of the field \( q_f(x, u) \). The above given equations constitute the starting point for the construction of a generalized theory for strong interaction based on the underlying geometry of the TB.

4. The condensed vacuum structure in the TB, quark condensation and the gap equation for quark-antiquark pairing

4.1 The completely filled vacuum state with family IQN \( n = 0 \)

The understanding of the physical nature of the vacuum is a central problem in modern quantum field theory rising some most profoundly mysterious unanswered questions. Several well established factors hint that the vacuum -i.e. the empty space- is in reality a complex structured medium. According the rules of QED, the vacuum is not empty but actively populated by virtual particle-antiparticle pairs that appear, annihilate and disappear during a very short time. But even though in QED the vacuum is defined as the zero-particle state in the Fock space, and the expectation values of quantized matter fields and gauge fields are zero, due to the quantization these virtual pairs have measurable effects in a shift of the spectrum of atomic systems and the mass of elementary particles. In the SM of particle physics additional phenomena arise which demonstrate that the vacuum may not be empty. The classical Lagrangian of the Higgs field has a nonzero, constant value in its lowest energy state, which means that the vacuum do not have zero field. This resembles the Ginsburg-Landau model for superconductivity but up to now a microscopic derivation of the Higgs mechanism has not been found which could explain this phenomenon. The vacuum state of the Higgs field is a central entity in the SM responsible for the spontaneous symmetry breaking of the gauge symmetry and for the mass generation of gauge particles. In addition, in QCD exist well established facts of the existence of a quark condensate described by a...
non-vanishing vacuum expectation value of the composite quark fields $\langle \bar{q} q \rangle$. This means that the vacuum is populated by quark-antiquark pairs leading to measurable effects in QCD and connected with a finite density of vacuum quark-antiquark pairs.

A further open question is whether the vacuum contains any energy. The zero-point energy of all normal modes of quantized scalar or fermion fields deliver a finite vacuum energy density. This has a profound effect for the evolution of the universe, as the energy of the vacuum acts like a cosmological constant in Einstein’s equation of GTR and therefore it is a central entity for the evolution of the universe. However, there is a huge discrepancy of 120 orders of magnitudes between the theoretical prediction and the observed value obtained from cosmological measurements. Moreover, the finite vacuum energy in QFT could be connected with the dark energy problem in the expanding universe.

For the understanding of these problems significant changes in the understanding of the vacuum in QFT seems to be required. Some outstanding problems in elementary particle physics such as the Higgs condensate, chiral quark condensation, fractional charges of quarks and the analogy of quark confinement with the behavior of vortices in superconducting solid states seem to find a counterpart in well understood phenomena in condensed matter physics. However, these analogies arise in a heuristically phenomenological way and there do not exists a consistent derivation of such relationships from the fundamentals of the theory. Moreover significant differences exist between particle physics and condensed matter physics, which seems to be an insuperable obstacle to finding a basis for a consistent foundation of these relationships. A central element in solid state theory is the existence of well defined quantum energy bands, which differentiate between the ground state and the excited states and arise due to the potential of periodic arrangement of atoms or molecules in the solid state. The ground state in semiconductors and isolators is a completely filled energy band denoted as the Fermi sphere and is separated from the excited states by an energy gap. In the SM of particle physics such difference between the ground state (vacuum) and excited states do not exist; virtual particles in the vacuum are differentiated from real particles only by the zero-particle number in Fock space and can take energies continuously distributed up to a cut-off, typically assumed to be the Plank energy.

In TB-QFT with the generalized gauge group $G = SU(2) \otimes E_c(2)$ significant conceptional changes arise in comparison with standard QFT. As shown in chapter 3.1 the Laplacian on the TB takes the form of the multi-particle Hamiltonian of 2D electrons in an external magnetic field, predicting two additional IQNs denoted here as the $E^\prime$-charge $\kappa$ and the family quantum number $n$, which characterize different states analogous to Landau levels for electrons in an external magnetic field. The lowest number $n = 0$ describes the vacuum, which means that the vacuum differs from the excited states with different IQNs $n = 1, 2, 3$. The multi-electron solid-state quantum Hall system shows a remarkable macroscopic quantum phenomenon: the integer and the fractional quantum Hall effect (QHE). Experimental observations indicate that such system undergoes a phase transition into a peculiar ordered quantum state in which the Hall conductivity is accurately quantized. In the integer QHE the ground state is a completely filled level with a fixed and well-defined density, while in the anomalous QHE the ground state is completely filled with composite electrons which are bound objects of electrons paired with an even number of vortices.

The analog form of the vertical Laplacian with the Hamiltonian of a quantum Hall system gives rise to the hypothesis that the vacuum with the IQN $n = 0$ is completely filled with seeleptons and composite seequarks, and all higher levels $n = 1, 2, 3$ are empty. In this picture, the background charge of vortices cancel the charge with opposite sign of the composite quarks. When we add one quark state to the system of the completely occupied ground state it is placed into a higher energy level $n = 1, 2, 3$ because of the Pauli exclusion principle and leave an unoccupied state (hole) in the lowest energy state. This resembles the understanding of an electron hole in a semiconductor crystal lattice. In solid-state physics, an electron hole is simply the absence of an electron from a full valence band. Similarly, the quark-hole introduced here is a way to conceptualize the interaction of composite quarks (with $n = 1, 2, 3$) with the full vacuum state $n = 0$, which leads to a redefinition of antiparticles as holes in the completely filled vacuum state.

The understanding of the vacuum as a completely filled band is a hypothesis introduced in the present paper, but there exist similar concepts in physics. Inspired by the exclusion principle Dirac proposed that the negative energy states in his relativistic equation were in fact occupied states, and excitations in this state can make holes interpreted as positrons. Even though the Dirac interpretation was later revised, the status of a completely filled ground state found in solid-state physics a deep foundation. A completely filled solid-state band is inert, it remains a filled band at all times even in the presence of space and time-dependend external fields. It cannot contribute to electric and thermal currents, and only partially filled bands contribute to conduction. Since a completely filled band carries no current, an unoccupied state in a band evolve in time under the influence of external fields precisely as it were occupied by real particles with opposite charge. These fictitious particles are called holes in solid state theory, but in the present meaning of the ground state as the vacuum, they have precisely the same physical meaning as antiparticles in the standard QFT.

The new understanding of the vacuum as a completely
filled band with the family IQN $n = 0$ differs significantly from the interpretation of the vacuum in QFT as a Fock state with zero-particle number. Rather, it associates a finite particle number density or a chemical potential with the vacuum state.

A very simple model of a completely filled vacuum state could be that of an ideal Fermi gas characterized by its density $\rho = \frac{N}{V}$ where $N$ is the total number of particles that occupy the lowest band and $V$ the volume. The ideal Fermi gas is characterized by the Fermi energy $E_F$ or by the Fermi momentum $k_F$. The particle number density of the vacuum is related with $k_F$ to

$$\rho_{\text{vac}} = g \frac{k_F^3}{3\pi^2},$$

(4.1)

where $g$ is the degeneracy factor given by the spin and internal quantum number degree of freedom in the vacuum. Neglecting interactions, the Fermi energy $E_F$ is approximately equal the chemical potential of the vacuum $E_F \approx \mu_{\text{vac}}$.

### 4.2 Vacuum quark-antiquark condensation in QCD

In the model of an ideal Fermi gas the interaction of particles is neglected. In the presence of attractive interaction by gluon exchange, the vacuum is unstable with respect to the formation of a quark condensate due to the pairing of quarks with antiquarks. This phenomenon shows some analogies with superconductivity in solid states where pairs of electrons form a condensate due to the weak attractive force mediated by phonons. Many phenomenological observations leave no doubt that in QCD a vacuum condensate of quark-antiquark pairs is formed dynamically, leading to spontaneous breakdown of the chiral symmetry. The quark condensate preserves Lorentz invariance of the vacuum, but it does not preserve the global chiral symmetry.

An early understanding of the formation of a quark condensate and chiral symmetry breaking was achieved by the Nambu-Jona-Lasinio model [64] based on an interaction potential with delta-function behavior. According to this model the formation of a quark condensate occurs similar as in the microscopic theory of superconductivity in solid states developed by Bardeen, Cooper and Schrieffer [65], where superconductivity is explained by the formation of a condensate of electron-electron pairs (Cooper pairs). In contrast the chiral condensate in QCD is formed by a condensate of quark-antiquark pairs.

The quark condensate is responsible for the spontaneous breaking of the chiral symmetry in QCD. The order parameter for chiral symmetry breaking is given by the vacuum expectation value $\langle 0 | \mathbf{q} \mathbf{q} | 0 \rangle$. The value of this parameter can be estimated by the Gell-Mann-Oakes-Renner relation [66]

$$\langle 0 | \mathbf{q} \mathbf{q} | 0 \rangle = -\frac{f^2 m^2}{2(m_u + m_d)}$$

(4.2)

Estimation of the quark condensate parameter can be directly done through analysis of correlation functions in QCD using an operator product expansion where the value of the condensate is dictated by the value of the correlation function [67]. In this approach QCD sum rules use the analytical properties of current correlation functions to relate low-energy (non-perturbative) hadronic quantities to calculable perturbative QCD contributions at high energies.

The formation of a quark condensate by quark-antiquark pairing shows analogies with a superconductor in solid state theory, but differ in one important aspect. Electron-electron Cooper pairs responsible for superconductivity are bosons and carry a charge $2e$. In contrast quark-antiquark pairs are colorless and are electrically neutral. Therefore they more resemble excitons in a semiconductor which consists of electrons in the conduction band bound to holes (realized as an unfilled electronic state in the valence band). The possible condensation of excitons has been studied theoretically beginning in the 1960th [36–39]. Despite the formal similarity of an exciton condensate and a superconductor, the physical properties of these two states are very different. In an exciton condensate, the conductivity vanishes and there is no Meisner effect and no long-range order like in superconductors. Studies in semiconductor bilayer systems has provided experimental evidence for the existence of exciton-condensation [60].

Note that there exist an analog to solid-state superconductivity in QCD characterized by the formation of bi-quark condensate in dense quark matter [for reviews see e.g. 57, 59]. At very high densities quark matter could behave as a color superconductor, with possible phenomenological applications for the description of neutron star interiors, neutron star collisions or the physics of collapsing stars. While in these studies of color superconductivity in QCD, extreme conditions of high-density quark matter are assumed as e.g. in the interior of neutron stars in the context of the present paper, dense quark matter is assumed to form the completely filled vacuum state $n = 0$ in the TB-QFT.

### 4.3 The gap equation for quark-antiquark pairing

Relativistic QFT is commonly formulated on the basis of Green functions using the Lagrange density, which depends on the four spacetime coordinates $x^\mu$ and includes retardation effects. To explore the dynamics of quark condensation by quark-antiquark pairing we will apply in the mean-field approximation a relativistic Hamiltonian formalism describing the two-body interaction of quarks
in the TB by gluon exchange. The TB-vacuum is characterized as a completely filled state with a finite particle density; therefore we have to consider a contribution to the Hamiltonian arising from the finite particle number term \( \mu N \) where \( \mu \approx E_{\text{Fermi}} \) is the chemical potential and \( E_{\text{Fermi}} \) the Fermi-energy of the vacuum which has to be determined later. The Hamiltonian density (3.54) can be obtained from the Lagrangian density (3.50) via the Legendre transformation. The Hamiltonian in the single-gluon exchange approximation takes the form

\[
H = H_0 + H_I \tag{4.3}
\]

with

\[
H_0 = \int d^4x \rho \delta(\mu - \gamma^0 \rho_0) q_f^\dagger(x, u) \gamma^\mu q_f(x, u) \tag{4.4}
\]

and

\[
H_I = \frac{g^2}{2} \int d^4x d^4y \delta(\mu - \gamma^0 \rho_0) q_f^\dagger(x, u) q_f^\dagger(y, u)
\times J^{A\mu}(x, y) \tilde{J}^{A\nu}(y, u). \tag{4.5}
\]

Hereafter, the summation convention concerning repeated color and flavor indices is used. \( A, B \) are the indices of the Gell-Mann matrices. The single-gluon exchange is attractive in the color-singlet channel for quark-antiquark interaction. In relativistic treatment physical entities depend on the four-momentum \( k^\mu = (k_0, \mathbf{k}) \). The interaction Hamiltonian (4.5) depending on the gluon Green function \( D_{\mu\nu}(x - y) \) is not instantaneous. In dense quark matter as discussed later in detail, the screening of gluons plays an important role and dominant contributions to the interaction are at the edge of the Fermi-sphere. This allows to neglect retardation effects.

In order to avoid problems, arising from the mixing of color, flavor and Dirac indices in the interaction Hamiltonian, it is more convenient to introduce Fiertz transformations for the sum of the Gell-Mann matrices:

\[
\sum_{i=1}^8 (\lambda^A)_{ab} (\lambda^A)_{cd} = \frac{16}{9} (\delta)_{ad} (\delta)_{bc} \tag{4.6}
\]

where the first term describes color singlets and the second color octet terms. Since the vacuum contains only color singlets, we consider only the first term in (4.6). Therefore we can substitute in (4.5)

\[
J^{A\mu}(x, u) \tilde{J}^{A\nu}(y, u) = \frac{g^2}{9} q_f^\dagger(x, u) \gamma^\mu q_f(x, u)
\times q_f^\dagger(y, u) \gamma^\nu q_f(y, u) \tag{4.7}
\]

The Fourier expansion of the quark field operator is determined by

\[
q_f^\dagger(x, u) = \sum_{p, s} \frac{1}{2\sqrt{p}} [e^{ipx} a_f^0(p) u_s(p) h_{f,s}(z) + e^{-ipx} b_f^a(p) v_s(p) h_{f,s}(z)] \tag{4.8}
\]

where \( f \) denotes the flavor degree of freedom with the IQN \( M_f = \{n, m, j, s, z\} \). The hole (anti-quark) is an un-filled state in the lowest TB-LL \( n = 0 \) and the quark state is in an excited state with the IQN \( n = 1, 2, 3 \). \( s = R, L \) is the helicity, \( u_s(p) \) with \( p = (e(p), \mathbf{p}) \) is the plane-wave solution of the Dirac equation for a particle and \( v_s(p) \) that of an antiparticle, and \( h_{f,s}(z) \) are the eigenfunctions of the Laplacian of the SU(2) \( \otimes \) \( E \) (2) group. \( V \) is the volume of the system. The annihilation operators \( a_f^a(p) \) and \( b_f^a(p) \) for particles and antiparticles (holes) satisfying the anti-commutation relations destroy the unpaired vacuum \( |0\rangle \), i.e. \( a_f^a(p) |0\rangle \) \( \neq |0\rangle \) and \( b_f^a(p) |0\rangle \) \( \neq |0\rangle \). In the absence of spin-orbit coupling we can assume that only spin-singlet condensation takes place and spin-triplet pairing can be neglected. The subsystem of particles of left-handed quarks (and right-handed anti-quarks) can be treated independently from the subsystem of right-handed quarks (and left-handed anti-quarks). Both are dynamically decuplet and may be considered as a non interacting mixture. This allow to simplify the notations and to drop the spin-index.

Substituting (4.8) into (4.4) we obtain

\[
H_0 = \sum_k (\epsilon_q(p) - \mu_q)(a_f^q(p) a_f^q(p) + \epsilon_h(p) - \mu_h)(b_f^q(p) b_f^q(p)) \tag{4.9}
\]

where \( \epsilon_q(p) \) is the kinetic energy of a quark and \( \epsilon_h(p) \) that of the holes, \( \mu_q \) and \( \mu_h \) are the chemical potentials of quarks and holes, respectively. For the interaction Hamiltonian (4.5) we obtain

\[
H_I = \frac{1}{2V} \sum_{p, p_1, p_2, p_3, P_4} [V_{qq}^{p_1p_2p_3p_4} a_f^q(p_1) \gamma^\mu a_f^q(p_2)
+ V_{hh}^{p_1p_2p_3p_4} b_f^q(p_1) \gamma^\mu b_f^q(p_2)
+ 2V_{qh}^{p_1p_2p_3p_4} \gamma^\mu a_f^q(p_1) b_f^q(p_2)] \tag{4.10}
\]

where \( p_2 - p_3 = p_4 - p_1 = q \). The interaction matrix elements are given by

\[
V_{qq}^{p_1p_2p_3p_4} = \frac{4}{9} g^2 D_{\mu\nu}^{p_4}(q) K^{\mu}(p_1, p_4) K^{\nu}(p_2, p_3) \tag{4.11}
\]

\[
V_{hh}^{p_1p_2p_3p_4} = \frac{4}{9} g^2 D_{\mu\nu}^{p_4}(q) Q^{\mu}(p_1, p_4) Q^{\nu}(p_2, p_3) \tag{4.10}
\]

\[
V_{qh}^{p_1p_2p_3p_4} = \frac{4}{9} g^2 D_{\mu\nu}^{p_4}(q) J_2^{\mu}(p_1, p_3) J_2^{\nu}(p_2, p_4) \tag{4.11}
\]
In such approximation we use

\[ K^\mu(p_1, p_4) = \frac{1}{2\sqrt{p_1 \cdot p_4}} \eta^\mu u(p_4), \]

\[ Q^\mu(p_1, p_4) = \frac{1}{2\sqrt{p_1 \cdot p_4}} \eta^\mu v(p_4), \]

\[ L^\mu(p_1, p_3) = \frac{1}{2\sqrt{p_1 \cdot p_3}} \eta^\mu v(p_3). \] (4.12)

The initial 4-momenta are taken on the mass-shell. We also introduced the integrals over the variables of the gauge group \( G = SU(2) \otimes \mathcal{E}_c(2) \):

\[ J^{\rho \sigma}_F = \int d\mu(z) d\mu(\bar{z}) | h^{\rho}_f(z) |^2 | h^{\sigma}_f(\bar{z}) |^2, \] (4.13)

\[ J^{qh}_2 = \int d\mu(z) d\mu(\bar{z}) | h^{q}_f(z) |^2 | h^{h}_g(\bar{z}) |^2, \] (4.14)

with \( q, \sigma = q, h \). If we excite a certain number of fermionic particles from the fully occupied ground state across the Fermi surface, an equal number of holes below the Fermi surface is obtained. Since the eigenfunctions \( h^q_f(z) \) for a hole with \( n = 0 \) and \( h^q_f(z) \) for an excited valence quark with \( n = 1, 2, 3 \) are orthogonal and normalized, we obtain for the overvol integral in (4.13),(4.14) \( J_{\rho \sigma}^F = 1, J_{qh}^q = 0 \). In the Feynman gauge the gluon propagator is given by

\[ D^{\mu \nu} = \frac{P^{\mu \nu}_F}{q^2 - G_T(q)} + \frac{P^{\mu \nu}_L}{q^2 - G_L(q)}, \] (4.15)

where \( P^{\mu \nu}_F \) and \( P^{\mu \nu}_L \) are the transverse and the longitudinal projectors

\[ P^{T}_{i j} = \delta_{i j} - \frac{q_i q_j}{| q |^2}, P^{T}_{0 0} = 0, \]

\[ P^{L}_{\mu \nu} = -g_{\mu \nu} + \frac{q_\mu q_\nu}{q^2} - P^{T}_{\mu \nu}, \]

\[ \cong \delta_{\mu \nu} \delta_{\nu 0}. \] (4.16)

The functions \( G_T(q) \) and \( G_L(q) \) describe the screening of gluons in the dense quark matter. In the dense hard loop approximation one can approximate \( G_T(p - q) = \frac{|e_{qa}^-| N_f g^2 \mu_a^2}{8 \pi} \) and \( G_L(p - q) = N_f g^2 \mu_a^2 \). [60, 61]

Similar as in the BCS theory or in the theory of exciton condensation, we apply the mean-field approach for particle-particle interaction and particle-antiparticle pairing. In this approximation mean-field decoupling can be used to generate terms containing the VEV of two of the four operators in the interaction Hamiltonian (4.10). In such approximation we use

\[ \langle a_f^{q \dagger}(k) a_f^q(p) \rangle = n_f^q(k) \delta_{k p} \delta_{f g}, \] (4.17)

\[ \langle b_f^{q \dagger}(k) b_f^q(p) \rangle = n_f^h(k) \delta_{k p} \delta_{f g}. \]

Taking into account only the pair attractive interaction with opposite momenta \( k \), the vacuum expectation for quark-hole pairing is

\[ \langle b_f^{q \dagger}(k) a_f^q(k) \rangle = F_g^{q q}(k) \] (4.18)

\[ \langle a_f^{q \dagger}(k) b_f^q(k) \rangle = F_g^{q h}(k) \]

With this approximation the quark-quark interaction term in (4.10) is given by

\[ H^{qq} = \frac{1}{2V} \sum_{p k q q} V_{p k q q}^{qq} [b_f^{q \dagger}(k) b_f^{q \dagger}(p) a_f^q(p) a_f^q(p)] \] (4.19)

A corresponding expression can be derived for the hole Hamiltonian \( H^{hh} \). The first term in (4.19) describes direct (Hartree) interaction and the second the exchange interaction. Note that the Hartree terms of quarks and holes are cancelled by the corresponding terms in the quark-hole interaction Hamiltonian \( H^{qh} \) which guarantees the charge neutrality. Therefore in \( H^{qh} \) we can include only the quark-hole pairing term

\[ H^{qh} = - \frac{1}{V} \sum_{p k q q} V_{p k q q}^{qh} [b_f^{q \dagger}(k) a_f^{h \dagger}(p) b_f^{h}(p)] \]

\[ + F_g^{h q}(k) a_f^q(p) \] (4.20)

\[ - F_g^{h q}(k) b_f^q(p) \]

With local charge neutrality in the vacuum we have

\[ n_f^q(k) = \langle a_f^{q \dagger}(k) a_f^q(k) \rangle = n_f^q(k) = \langle b_f^{q \dagger}(k) b_f^q(k) \rangle \] (4.21)

Including the exchange energy into renormalized chemical potentials \( \mu_{ren}^q = \mu^q + \frac{1}{2} \sum_k (V_{p k q q}^{qh} n_f^q(k), \mu_{ren}^h = \mu^h + \frac{1}{2} \sum_k (V_{p k q q}^{hh} n_f^h(k) \) and neglecting terms quadratic in fluctuations, the Hamiltonian (4.10) takes the quadratic form

\[ H = \sum_k \left[ (\epsilon_f - \mu_{ren}^q) a_f^{q \dagger}(k) a_f^q(k) \right] \] (4.22)

\[ + (\epsilon_h - \mu_{ren}^h) b_f^{q \dagger}(k) b_f^q(k) \]

\[ - \sum_k V_{p k q q}^{qh} [b_f^{q \dagger}(k) a_f^q(p) a_f^q(p)] \]

\[ + F_g^{h q}(k) b_f^q(p) a_f^{h \dagger}(p) + E_{vac} \]

with the vacuum energy

\[ E_{vac} = \frac{1}{V} \sum_{k. p} V_{p k q q}^{qh} n_f^q(k) n_f^q(p) \]

\[ + V_{p k q q}^{hh} n_f^h(k) n_f^h(p) \] (4.23)

\[ + F_g^{h q}(k) F_g^{h q}(p) \]
In order to diagonalize the Hamiltonian, we perform a Bogoljubov-Valentin transformation and introduce new fermi-operators $A_{ij}^{\alpha}(k)$ and $B_{ij}^{\alpha}(k)$ by the following linear combination for the operators $a_{ij}^{\alpha}(k)$ and $b_{ij}^{\alpha}(k)$

\[
a_{ij}^{\alpha}(k) = u(k)A_{ij}^{\alpha}(k) + v(k)B_{ij}^{\alpha}(k)
\]

\[
b_{ij}^{\alpha}(k) = -v(k)A_{ij}^{\alpha}(k) + u(k)B_{ij}^{\alpha}(k)
\]

(4.24)

The Bogoljubov-Valentin transformation offers a simple intuitive physical insight for the physics of quark condensation and explicitly displays the quark-antiquark pairing mechanism that is responsible for chiral symmetry breaking.

The Bogoljubov-Valentin transformation (4.24) preserves the fermionic anti-commutation rule if the coefficients $u(k)$ and $v(k)$ satisfy the condition

\[
u(k)^2 + v(k)^2 = 1
\]

(4.25)

Taking into account the pairing effect the ground state is now defined as $A_{ij}^{\alpha}(k) \mid \Phi_0 \rangle = A_{ij}^{\alpha}(k) \mid \phi_0 \rangle = 0$. The mean quark and hole occupation numbers are then given by

\[
n^{\alpha}(p) = n^{\beta}(p) = v^2(p)
\]

(4.26)

satisfying the charge neutrality condition. By the request that terms proportional to $A_{ij}^{\alpha}(k)B_{ij}^{\alpha}(k)$ and $B_{ij}^{\alpha}(k)A_{ij}^{\alpha}(k)$ vanish, we obtain the condition

\[
2\xi(k)v(k)u(k) + \Delta(k)(v(k)^2 - u(k)^2) = 0
\]

(4.27)

where the entity

\[
\xi(k) = [(\epsilon(k) - \mu) - \frac{1}{\Omega} \sum_{k}(V^{gg}_{kkpp} + V^{hh}_{kkpp})v(k)^2]
\]

(4.28)

is introduced with $\mu^{ren} = \mu^{ren}_q + \mu^{ren}_h$, $\epsilon(p) = \epsilon_q(p) + \epsilon_h(p)$. The relation (4.25) together with (4.27) are both satisfied by the solution

\[
u(k)^2 = \frac{1}{2}(1 + \frac{\xi(k)}{E(k)})
\]

(4.29)

\[
u(k)^2 = \frac{1}{2}(1 - \frac{\xi(k)}{E(k)})
\]

(4.29)

where $E(k)$ is the quasi-particle energy $E^2(k) = \xi^2(k) + |\Delta(k)|^2$.

Now we define the gap parameters $\Delta_{gf}^{ba}(p)$ as follows

\[
\Delta_{gf}^{ba}(p) = \frac{2}{V} \sum_{k,p} V_{kkpp}^{eh} F_{gf}^{ba}(k)
\]

(4.30)

We consider the chiral limit with vanishing quark masses $m_u = m_d = m_s = 0$ (and all heavy quarks neglected). In QCD, the chiral symmetry is spontaneously broken down to the vectorial flavor subgroup $H = SO(3)_{L+R}U(1)_{L+R}$ of isospin and hypercharge generated by the vector currents. This means that in the limit of massless quarks the different flavours obtain the same vacuum expectation value $\langle 0 \mid \bar{u}u \mid 0 \rangle = \langle 0 \mid \bar{d}d \mid 0 \rangle = \langle 0 \mid \bar{s}s \mid 0 \rangle$ and pairing of quarks and anti-quarks with different flavors can be neglected. In agreement with this consideration the gap $\Delta_{gf}^{ba}(p)$ in Equ. (4.30) is independent on the flavor and color index. Then equation (4.30) takes the form

\[
\Delta(p) = \frac{-N_g}{V} \sum_{k} V_{kkpp}^{eh} \frac{1}{E(k)} \Delta(k)
\]

(4.31)

with $N_g = N_c N_f$, where $N_c$ is the number of colors and $N_f$ the number of flavors.

After substitution of (4.24) with (4.25) and (4.27) into (4.22) the Hamiltonian is

\[
H = H_{vac} + \sum_{k} \langle E(k)[A_{ij}^{\dagger}(k)A_{ij}^{\alpha}(k)
\]

\[
+ B_{ij}^{\dagger}(k)B_{ij}^{\alpha}(k)]
\]

(4.32)

The vacuum energy (4.23) is given by

\[
E_{vac} = \frac{N_g}{V} \sum_{k,p} (\epsilon_k - \mu) \mid v_k \mid^2
\]

\[
- V_{kkpp}(v_k u_k v_p u_p + v_p^2 v_p)
\]

(4.33)

\[
= \frac{N_g}{V} \sum_k \left[ (\epsilon_k - \mu^{ren}) \frac{1}{2} (1 - \frac{\xi_k}{E_k}) - \frac{1}{2} \frac{\Delta_k^2}{E_k} \right]
\]

The gap equation (4.31) takes the form

\[
\Delta(k) = \frac{-4}{9 g^2 N_g} \int \frac{1}{(2\pi)^d} d^d p W^{\mu\nu}(p,k,k,p)
\]

\[
D_{\mu\nu}(p-q) \frac{1}{E(q)} \Delta(q)
\]

(4.34)

with $W^{\mu\nu}(p,k,k,p) = K^{\mu}(p,k)Q^\nu(k,p)$.

The integral in (4.34) can be solved using analog approximations as in [61, 62] in the study of color superconductivity by bi-quark condensation. The gap function does not depend on the orientation of $k$. Due to the factor $E^{-1}(q)$ in (4.34) for large chemical potential $\mu$ the main contribution in the integral arise at the renormalized Fermi surface $|k| \approx \mu^{ren}$. The integral in (4.34) can be solved using analog approximations as in [21, 22] in the study of color superconductivity by bi-quark condensation. The gap function does not depend on the orientation of $k$. Due to the factor $E^{-1}(q)$ in (4.34) for large chemical potential $\mu$ the main contribution in the integral arise at the renormalized Fermi surface $|k| \approx \mu^{ren}$. We find

\[
\Delta(p) = \frac{-b_{gh}^2}{V} \int \frac{\pi}{d} d\cos\theta \frac{1}{\mu^{ren}\theta + \mu^{ren}\Delta(k)}
\]

(4.35)

\[
\left[ \frac{\mu^{ren}(1 - \cos \theta) - (\epsilon_p - \epsilon_q)^2 + G(k)}{(\mu^{ren}(1 - \cos \theta) - (\epsilon_p - \epsilon_q)^2 + F^2 E(k)} \right] \Delta(k)
\]

(4.35)
with \( \nu_{p\bar{p}} = \frac{1}{4} g^2 N_f N_c \frac{1}{\pi^2} \) and \( k_\perp = |\mathbf{k}| \). Here we restricted the integration over \( \mathbf{k} \) to a region near the Fermi surface with a cut-off factor \( \delta \). The matrix elements \( Q_{T,L} \) are defined as
\[
Q_{T,L} = \langle \mathbf{v}(\mathbf{k}) \gamma^\mu (u(p)) (\mathbf{v}(\mathbf{p})) \gamma^\nu v(\mathbf{k}) \rangle_{\mu \nu}^{P_{T,L}} \frac{1}{4 \epsilon_p \epsilon_{\mathbf{k}}} \tag{4.36}
\]

Let us choose \( p \) into the z-direction with \( p = \epsilon_p (1,0,0,1) \) and \( k = \epsilon_k (1,\sin \theta, 0, \cos \theta) \). For the matrix elements (4.12) one gets
\[
K_{RR}^\mu(p,k) = K_{LL}^\mu(p,k) = Q_{LL}^\mu(p,k) = 2 \sqrt{\epsilon_p \epsilon_k} (\cos \theta/2, \sin \theta/2, i \sin \theta/2, \cos \theta/2)
\]
\[
K_{RL}^\mu(p,k) = Q_{RL}^\mu(p,k) = 0 \tag{4.37}
\]

The matrix elements \( Q_{T,L} \) in (4.36) take the form
\[
Q_T = \frac{1}{2}(3 - \cos \theta), \quad Q_L = \frac{1}{2}(1 + \cos \theta) \tag{4.38}
\]

In (4.35) one can use the approximations \(|\epsilon_p - \epsilon_k| \ll \mu_{ren} \) and \(|p-k| \gg \sqrt{2} \mu_{ren} (1 - \cos \theta)^{1/2} \). In leading order we can solve the integral over \( \theta \) by setting in the numerator \( \cos \theta \approx 1 \). For the gap equation (4.35) we find
\[
\Delta(p) = \frac{2}{3} b_{gh} \int_0^\delta \frac{\mu^2 dk}{\Delta^2(k) + \xi(k)} \left( 2 \ln(1 + \frac{8 \pi^2}{N_f g^2}) + \ln(1 + \frac{64 \pi \mu}{N_f g^2} |\epsilon_p - \epsilon_k|) \right) \Delta(k) \tag{4.39}
\]

with \( p = |\mathbf{p}|, k = |\mathbf{k}| \). Analogous as in the studies of color superconductivity \cite{61,63}, the integral equation (4.39) can be rewritten as a differential equation. Using the same approach, an approximate solution of (4.39) is given by
\[
\Delta(k_0) = \Delta_0 \sin \frac{2}{3} b_{gh} \ln \left( \frac{\epsilon_p}{\epsilon_k} \right) \tag{4.40}
\]
\[
\Delta_0 = 2c \mu \exp \left( - \frac{3 \pi^2}{\sqrt{2g}} \right) \tag{4.41}
\]

with
\[
2c = \frac{512 \pi^4}{g^2} \left( \frac{2}{N_f} \right)^{5/2}.
\]

Using these solutions one can approximately calculate the vacuum energy (4.33) as
\[
E = \frac{N_g \mu^2}{2 \pi^2} \int_0^\delta d\xi \left[ \frac{1}{2} \xi (1 - \frac{\xi}{E_k}) - \frac{1}{2} \frac{\Delta_0^2}{\sqrt{\xi^2 + \Delta_0^4}} \right] = - \frac{N_g}{16(\pi)^2} \mu^2 \Delta_0^2 \tag{4.42}
\]

In the standard BCS theory of superconductivity there exist an internal ultraviolet cut-off related with the Debye frequency \( \omega_D \) because the interaction potential of electrons due to phonon exchange is only attractive in the interval \(|\epsilon(p) - \mu| \leq \omega_D \). In QCD, such cut-off frequency is absent. However, in the weak-coupling approximation the results given in (4.40),(4.41) do not depend on the momentum cut-off \( \delta \).

For a check of the results we compare the gap equation for quark-anti-quark condensation with the gap in bi-quark condensation in color superconductivity in dense quark matter. Several fully microscopic calculations of the gap for color superconductivity have been reported in the literature (see e.g. \cite{61,62}) by using Green functions in the Nambu-Gorkov formalism. Here we applied the different method as described above: the mean-field approach using the Bogolyubov transformation in the Hamiltonian formalism. In the mean-field approximation of the Hamiltonian formalism for bi-quark condensation we find the same expression (4.40) and (4.41) for the gap as for quark-anti-quark pairing but with the substitution of the pre-factor \( b_{gh}^2 \) in (4.39) by \( b_{gh}^2 = g^2/18 \pi^2 \). This agrees with the results in \cite{61,63}.

Finally let us compare the gap equation for the quark condensate with the results of the simplified Nambu-Jona-Lasino model with a point-like interaction potential \cite{64}. In this approach the gluon Green function in (4.34) has to be substituted by \( D(q) = g^2/M_g^2 \) (where \( M_g \) is an effective gluon mass). The gap equation (4.34) with this point-like Green function is given by
\[
\Delta(p) = \frac{G_s}{(2\pi)^3} \int d^3 k \frac{\Delta(k)}{\sqrt{\xi(k)^2 + \Delta^2(k)}} \tag{4.43}
\]

with \( G_s = \frac{8}{9 \pi^2} g^2 N_c N_f \). The integration in (4.43) can be performed if we regulate the ultraviolet divergence by a cutoff \( \bar{\mu} \) and substitute \( \Delta(k) = \Delta_0 \) by a constant. This solution agrees with the Nambu-Jona-Lasino model. For \( \Delta_0/\mu_{ren} \ll 1 \) one gets from (4.43)
\[
\Delta_0 = 2 \bar{\mu} \exp \left( - \frac{8 \pi^2}{G_s} \right) \tag{4.44}
\]

Note that the gap has an exponential proportional to \( 1/g^2 \), while the exponential behavior in (4.41) is like 1/g. Similar to the theory of color superconductivity in dense quark matter, the dominant contribution to the formation of quark-antiquark pairs comes from the collinear scattering through the long-range magnetic gluon exchange \cite{61}.

### 4.4 Numerical estimations

Finally, it is interesting to obtain a few numerical estimates. Due to the analogy with the fractional QHE, we introduced in the TB geometry a chemical potential for the vacuum state as a completely filled fermion state.
We understand \( \mu_{\text{vac}} \) as a new constant of nature. For the determination of the parameter \( \mu_{\text{vac}} \), we can use the relations of the gap parameter and the known quark condensation parameter in dependence on the gap.

The quark field operator \( \Psi_{f,s}^a(x,u) \) in (4.8) can be expressed by the new operators \( A_{f,s}^a(p) \) and \( B_{f,s}^a(p) \) defined in (4.24):

\[
\Psi_{f,s}^a(x,u) = \sum_p \frac{1}{\sqrt{2\pi p}} \left[ e^{i px} A_{f,s}^a(p) U_s(p) h_{f,s}(z) + e^{-i px} B_{f,s}^a(p) V_s(p) h_{f,s}(z) \right] \tag{4.45}
\]

where the new spinor functions \( U_s(p) \) and \( V_s(p) \) are given by

\[
U_s(p) = u(p) u_s(p) - v(p) v_s(p) \tag{4.46}
\]

\[
V_s(p) = u(p) v_s(p) + v(p) u_s(p) \tag{4.47}
\]

which retain the orthogonality and normalization conditions. The new paired ground state \( | \Phi > \) is annihilated by \( A_{f,s}^a(p) \) and \( B_{f,s}^a(p) \) and \( | \Phi > \) can be constructed from the unpaired vacuum \( | 0 > \) as

\[
| \Phi > = \prod_{s,f,a} (| (u(p) - v(p)a_{f,s}^a(p))b_{f,s}^a(p) > | 0 >) \tag{4.47}
\]

Using (4.45) with (4.46) and (4.47), one can determine the quark condensate parameter \( C_q \) as

\[
C_q = \langle \Phi | \bar{\Psi}_f(x,u) \Psi_f(x,u) | \Phi > \tag{4.48}
\]

\[
= -\frac{N_g}{V} \sum_p u(p) v(p) = -N_g \frac{1}{V} \sum_p \frac{1}{2E_p} \Delta(p) \tag{4.49}
\]

\[
= -\frac{N_g^2}{2(2\pi)^2} \int d^3p \frac{1}{\sqrt{p^2 + \Delta_0}} \simeq -N_g^2 \Delta_0 \frac{\mu_{\text{ren}}^3}{8\pi^2} \tag{4.49}
\]

where the normalization of the factor \( h_{f,s}(z) \) is used. Using the gap \( \Delta_0 \) from (4.41), \( C_q \) from (4.48) and (4.4) we find for the renormalized chemical potential \( \mu_{\text{ren}} \) the approximate relation

\[
\mu_{\text{ren}}^3 = -C_q \frac{4\pi^2}{N_g} \exp\left(\frac{3\pi^2}{\sqrt{2}g}\right) \tag{4.49}
\]

We need the behavior of the fundamental coupling constant \( g(Q^2) \) in the low momentum transfer domain \( Q \ll 1 \text{GeV} \). Various theoretical models in the non-perturbative strongly coupled regime has been developed. Here we use the effective coupling from light-front holography \cite{64}, leading to a non-perturbative coupling \( \alpha_s(Q^2) = \pi \exp(-Q^2/4\alpha_s) \) with \( \alpha_s = 0.54\text{GeV}^2 \). For \( Q = 0.3\text{GeV} \) we find \( g = \sqrt{4\pi\alpha_s} \simeq 6.0 \). With this value we find a vacuum chemical potential \( \mu \simeq 0.8\text{GeV} \) and a gap \( \Delta_0 = 2\mu_{\text{ren}} \exp(-\frac{3\pi^2}{\sqrt{2}g}) \simeq 0.2\text{GeV} \). For the vacuum energy density we obtain \( E_{\text{vac}} \simeq -10^{-21}\text{GeV}^4 \). Galactic observations deliver an upper bound on the cosmological constant which is usually interpreted as a bound on the vacuum energy in the order of \( 10^{-47}\text{GeV}^4 \). In contrast the zero-point energy in QFT is roughly \( 10^{76}\text{GeV}^4 \) if we insert the Plank energy as cut-off. This huge difference of more than 120 orders constitutes the so-called cosmological constant problem.

With these rough estimations we want to emphasis the underlying physics, but a more exact numerical will be undertaken in future.

5. Summary

This paper is a follow-up of \cite{1} with a study of the hypothesis that the tangent bundle (TB) with the structure group \( SO(3,1) \times T(3,1) \) is the underlying geometric structure for a unified theory of the fundamental interactions, explaining their common origin and enabling a deeper understanding of the relationship between them. Based on this assumption in \cite{1} a generalized theory of electroweak interaction (including hypothetical dark matter particles) with the little groups \( G = SU(2) \otimes E^7(2) \) of the \( SO(3,1) \) group as gauge group was presented. The present paper describes a possible way that strong interaction can emerge in the tangent bundle geometry. The group \( SU(3) \) cannot be described as a geometrical symmetry in the TB but this symmetry is hidden in the fundamentals of the tangent bundle geometry arising as an emergent internal symmetry similar as Chern-Simon gauge symmetries in quantum Hall systems. This assumption is based on the fact that the vertical Laplacian of the TB has the same form as the multi-particle Hamiltonian of a quantum Hall system. The eigensolutions of the vertical Laplacian exhibit two additional internal quantum numbers (IQN) which explain the existence of lepton and quark families: the \( E^f \)-charge \( x \) and the family quantum number \( n \). The family quantum number \( n \) characterizes different states analogous to Landau levels of electrons in an external magnetic field. The lowest quantum number describes a completely occupied vacuum state (filled with seeleptons and seequarks). This means the vacuum state (with \( n=0 \)) differs from excited states (with \( n = 1, 2, 3 \)) describing valence quarks by a different IQN. The analogy with a quantum Hall system allows us to use the Laughlin wave function for the description of quarks with fractional hypercharges which can be interpreted as composite quarks formed from bare quarks and two attached hypercharge vortices. Taking into account the three iso-spin components \( I_3 = -1/2, 0, 1/2 \) the color \( SU(3) \) symmetry arises as an emergent gauge symmetry described by \( (2+1) \)-D Chern-Simon gauge fields. The field equation of the vertical Laplacian including the emergent Chern-Simon fields implies that in the large-scale limit of the variables of the TB Laplacian the ground state is a color singlet demonstrating a signature of quark confinement. This result follows from general universal principle in the TB verti-
cal Laplacian independent on the microscopic dynamics of quarks in QCD. In addition in the TB geometry, a new understanding for the vacuum as the ground state is introduced that is occupied by a condensate of quark-antiquark pairs with finite density (or a chemical potential). The gap for quark-antiquark pairing is calculated in the mean-field approximation, which allows a numerical calculation of the characteristic parameters of the vacuum such as its chemical potential, the chiral condensation parameter and the vacuum energy.

Recently at CERN new exotic particles were observed formed as tetraquarks containing two quarks and two antiquarks and pentaquark containing four quarks and one antiquark (for a review see e.g. [69]). The analogy with the anomalous QHE could hint to the possible existence of other types of exotic particles formed from exotic quark states with hypercharges of $e/5$ for up and down quarks and the exotic up quark had an electric charge of $(7/10)e$. The exotic down quark had a charge of $(-3/10)e$. Pairs of exotic quark and exotic antiquarks with the same flavour can form neutral flavorless exotic mesons. The real existence of $e/5$ charged quasiparticles has been proven by shot-noise measurements in a quantum Hall system [70].

Since the tangent bundle is also the geometric fundament for a gauge theory of gravity based on translational transformations $T(3, 1)$ of tangent fibers [13, 14, 16–18] one can identify the TB as the underlying geometric structure for a new type of unified geometrized field theory. However the hardest unsolved problem in the unification of fundamental interactions is the quantization of gravity. While there has exciting developments in this field, to date the ongoing intense research efforts did not resulted in a general accepted consistent theory. In the best case, the underlying unified geometric structure of the TB with conceptual changes in comparison with GTR, as described in this paper, could provide a new perspective to this problem to allow a deeper understanding of the nature of quantum gravity.

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