A Remark on a Theorem by Kodama and Shimizu

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We prove a characterization theorem for the unit polydisc $\Delta^n \subset \mathbb{C}^n$ in the spirit of a recent result due to Kodama and Shimizu. We show that if $M$ is a connected $n$-dimensional complex manifold such that (i) the group $\text{Aut}(M)$ of holomorphic automorphisms of $M$ acts on $M$ with compact isotropy subgroups, and (ii) $\text{Aut}(M)$ and $\text{Aut}(\Delta^n)$ are isomorphic as topological groups equipped with the compact-open topology, then $M$ is holomorphically equivalent to $\Delta^n$.

1 Introduction

For a connected complex manifold $M$, let $\text{Aut}(M)$ denote the group of holomorphic automorphisms of $M$. Endowed with the compact-open topology, $\text{Aut}(M)$ is a topological group. We are interested in characterizing complex manifolds by their automorphism groups.

In general, two complex manifolds $M_1$ and $M_2$ need not be holomorphically equivalent if the topological groups $\text{Aut}(M_1)$ and $\text{Aut}(M_2)$ are isomorphic. A simple example of this kind with non-trivial automorphism groups is given by spherical shells

$$S_r := \{z \in \mathbb{C}^n : r < ||z|| < 1\}, \quad 0 \leq r < 1.$$  

It is straightforward to see that for $n \geq 2$ the group $\text{Aut}(S_r)$ coincides with the unitary group $U_n$ for all $r$. Next, every $S_r$ is a Kobayashi-hyperbolic Reinhardt domain. It is shown in [Kr], [S] that two such domains are holomorphically equivalent if and only if they are equivalent by means of an elementary algebraic map, i.e. a map of the form

$$z_j \mapsto \lambda_j z_1^{a_{j1}} \cdots z_n^{a_{jn}}, \quad j = 1, \ldots, n,$$

where $\lambda_j \in \mathbb{C}^*$ and $a_{jk}$ are integers satisfying $\det(a_{jk}) \neq 0$. An elementary algebraic map is holomorphic and one-to-one on $S_r$ only if it is linear (i.e.

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reduces to dilations and a permutation of coordinates). However, $S_{r_1}$ and $S_{r_2}$ are not equivalent by means of such a linear map for $r_1 \neq r_2$.

If the group $\text{Aut}(M)$ is sufficiently large, one can hope to obtain positive characterization results. For example, it was shown in [IK] that the space $\mathbb{C}^n$ is completely characterized by its holomorphic automorphism group as follows: if $M$ is a connected complex manifold of dimension $n$ and the groups $\text{Aut}(M)$ and $\text{Aut}(\mathbb{C}^n)$ are isomorphic as topological groups, then $M$ is holomorphically equivalent to $\mathbb{C}^n$. A similar characterization was obtained for the unit ball $B^n \subset \mathbb{C}^n$ in [I] (see also the erratum) and, under certain additional assumptions (that will be discussed below), for direct products $B^k \times \mathbb{C}^{n-k}$ in [BKS] as well as for the space $\mathbb{C}^n$ without some coordinate hyperplanes in [KS1, KS2].

Recently, in [KS3] Kodama and Shimizu obtained the following characterization of another classical domain, the unit polydisc $\Delta^n \subset \mathbb{C}^n$ (the direct product of $n$ copies of the unit disc $\Delta \subset \mathbb{C}$).

**THEOREM 1.1** [KS3] Let $M$ be a connected complex manifold of dimension $n$ that is holomorphically separable and admits a smooth envelope of holomorphy. If $\text{Aut}(M)$ and $\text{Aut}(\Delta^n)$ are isomorphic as topological groups, then $M$ is holomorphically equivalent to $\Delta^n$.

In particular, Theorem 1.1 holds for Stein manifolds and for all domains in $\mathbb{C}^n$.

The connected component of the identity $\text{Aut}(\Delta^n)^0$ of the group $\text{Aut}(\Delta^n)$ is isomorphic to the direct product of $n$ copies of the group $\text{Aut}(\Delta) \cong SU_{1,1}/\mathbb{Z}_2$, and therefore contains a subgroup (which is a maximal compact subgroup) isomorphic to the $n$-torus $\mathbb{T}^n$. A topological group isomorphism between $\text{Aut}(M)$ and $\text{Aut}(\Delta^n)$ yields a smooth action by holomorphic transformations of $\mathbb{T}^n$ on $M$. The assumptions of holomorphic separability and smoothness of the envelope of holomorphy in Theorem 1.1 are used by the authors to linearize this action thus representing the manifold $M$ as a Reinhardt domain in $\mathbb{C}^n$. This is possible due to a theorem by Barrett, Bedford and Dadok (see [BBD]). We note that similar assumptions were imposed on manifolds in [BKS], [KS1], [KS2] to guarantee the applicability of the result of [BBD].

It is anticipated that the assertion of Theorem 1.1 remains true if the assumptions of holomorphic separability and smoothness of the envelope of holomorphy are dropped. In this note we offer a version of Theorem 1.1 in
this direction. In particular, we do not refer to the linearization result of [BBD] in our proofs. Instead, we require that for every \( p \in M \) the isotropy subgroup
\[
\text{Aut}_p(M) := \{ g \in \text{Aut}(M) : g(p) = p \}
\]
is compact in \( \text{Aut}(M) \) and linearize the action of \( \text{Aut}_p(M) \) near \( p \), which is possible due to the results of Bochner in [B] (see also [Ka]). We note that the linearizability of actions of compact groups on complex manifolds with fixed points goes back to H. Cartan (see [M] for an account of Cartan’s results of this kind). In fact, we will only use the faithfulness of the isotropy representation (defined below); this statement is known as Cartan’s uniqueness theorem (see [C]). The local linearizability (as opposed to the global linearizability of the \( \mathbb{T}^n \)-action) is sufficient to characterize \( \Delta^n \). It is not clear at this time how one could avoid using linearization arguments altogether. One difficulty here is the low-dimensionality of the maximal compact subgroup of \( \text{Aut}(\Delta^n)_0 \). For comparison, the maximal compact subgroup of \( \text{Aut}(B^n) \) is isomorphic to \( U_n \) and thus has dimension \( n^2 \). This fact was of great help in [I] (see also [IK]).

Our result is the following theorem.

**THEOREM 1.2** Let \( M \) be a connected complex manifold of dimension \( n \) such that for every \( p \in M \) the isotropy subgroup \( \text{Aut}_p(M) \) is compact in \( \text{Aut}(M) \). If \( \text{Aut}(M) \) and \( \text{Aut}(\Delta^n) \) are isomorphic as topological groups, then \( M \) is holomorphically equivalent to \( \Delta^n \).

We remark that the assumption of compactness of the isotropy subgroups holds for large classes of manifolds a priori not covered by Theorem 1.1. For example, it holds whenever the action of the group \( \text{Aut}(M) \) on the manifold \( M \) is proper, i.e. the map
\[
\text{Aut}(M) \times M \to M \times M, \quad (g, p) \mapsto (g(p), p)
\]
is proper. It is shown in [Ka] that \( \text{Aut}(M) \) acts on \( M \) properly if and only if one can find a continuous \( \text{Aut}(M) \)-invariant distance on \( M \). In particular, the action of \( \text{Aut}(M) \) is proper for all Kobayashi-hyperbolic manifolds (see also [Ko]). Hence the following holds (cf. Remark 2.1).

**Corollary 1.3** Let \( M \) be a connected Kobayashi-hyperbolic manifold of dimension \( n \). If \( \text{Aut}(M) \) and \( \text{Aut}(\Delta^n) \) are isomorphic as topological groups, then \( M \) is holomorphically equivalent to \( \Delta^n \).
2 Proof of Theorem 1.2

Let $\text{Aut}(M)^0$ be the connected component of the identity of $\text{Aut}(M)$. Since $\text{Aut}(\Delta^n)^0$ is a Lie group of dimension $3n$ in the compact-open topology, so is $\text{Aut}(M)^0$. Furthermore, every maximal compact subgroup of $\text{Aut}(M)^0$ is $n$-dimensional and isomorphic to $\mathbb{T}^n$. For every $p \in M$ the subgroup $\text{Aut}_p(M)^c := \text{Aut}_p(M) \cap \text{Aut}(M)^0$ is compact and therefore is contained in some maximal compact subgroup of $\text{Aut}(M)^0$. Since the dimension of the $\text{Aut}(M)^0$-orbit of $p$ cannot exceed $2n$, it follows that $\dim \text{Aut}_p(M)^c = n$. Hence $\text{Aut}_p(M)^c$ is a maximal compact subgroup of $\text{Aut}(M)^0$ (thus $\text{Aut}_p(M)^c = \text{Aut}_p(M)^0$), and the action of $\text{Aut}(M)^0$ on $M$ is transitive.

Let $\alpha_p : \text{Aut}_p(M)^0 \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg(p)$

be the isotropy representation of $\text{Aut}_p(M)^0$, where $T_p(M)$ is the tangent space to $M$ at $p$ and $dg(p)$ is the differential of a map $g$ at $p$. Let further

$L_p := \alpha_p (\text{Aut}_p(M)^0)$

be the corresponding linear isotropy subgroup. By the results of [C], [B], [Ka] the isotropy representation is continuous and faithful. In particular, $L_p$ is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to $\text{Aut}_p(M)^0$. In some coordinates in $T_p(M)$ the group $L_p$ becomes a subgroup of the unitary group $U_n$. Since $L_p$ is isomorphic to $\mathbb{T}^n$, it is conjugate in $U_n$ to the subgroup of all diagonal unitary matrices. In particular, for every $p \in M$ the group $L_p$ contains the element $-\text{id}$.

Let $G$ be an $\text{Aut}(M)^0$-invariant Hermitian metric on $M$. Since $\text{Aut}(M)^0$ acts on $M$ transitively, such a metric can be constructed by choosing an $L_{p_0}$-invariant positive-definite Hermitian form on $T_{p_0}(M)$ for some $p_0 \in M$, and by extending it to a Hermitian metric on all of $M$ using the $\text{Aut}(M)^0$-action (see [P] for the existence of invariant metrics for not necessarily transitive proper actions). The manifold $M$ equipped with the metric $G$ is a Hermitian symmetric space.

The theorem now follows from the general theory of Hermitian symmetric spaces (see [H]). Indeed, since the group $\text{Aut}(M)^0$ acts on $M$ with compact isotropy subgroups, contains a symmetry at every point of $M$, is semi-simple and is isomorphic to the direct product of $n$ copies of the simple group $SU_{1,1}/\mathbb{Z}_2$, the manifold $M$ is holomorphically isometric to the product of $n$ one-dimensional irreducible Hermitian symmetric spaces (see Theorem 3.3 in
Chapter IV, Theorems 1.1 and 4.1 in Chapter V, Propositions 4.4, 5.5 and Theorem 6.1 in Chapter VIII of [H]). Clearly, each of the one-dimensional irreducible Hermitian symmetric spaces must be equivalent to the unit disc $\Delta$, and the proof is complete. □

**Remark 2.1** One can obtain Corollary 1.3 without referring to the theory of Hermitian symmetric spaces. Indeed, as in the proof of Theorem 1.2, we see that $M$ is homogeneous. Hence, by the (non-trivial) result of [N], the manifold $M$ is holomorphically equivalent to a bounded domain in $\mathbb{C}^n$. Corollary 1.3 now follows from Theorem 1.1.

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