The extended vs standard Holstein model; results in two and three dimensions

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We present numerically exact solutions to the problem of a single electron interacting through a long range interaction with optical phonons in two and three dimensions. Comparisons are made with results for the standard Holstein model, and with perturbative approaches from both the weak coupling and strong coupling sides. We find, in agreement with earlier work, that the polaron effective mass increases (decreases) in the weak (strong) coupling regime, respectively. However, in two dimensions, the decrease in effective mass still results in too large an effective mass to be relevant in realistic models of normal metals. In three dimensions the decrease can be more relevant, but exists only over a very limited range of coupling strengths.

I. INTRODUCTION

The standard theoretical framework for superconductivity is known as BCS-Eliashberg theory, and, as catalogued in the cited reviews, accurately describes many experimentally known properties of the so-called conventional superconductors, like Pb. BCS theory itself is almost universal, and confirmation of this theory, achieved with unprecedented accuracy for weak coupling superconductors like Aluminium, serves to vindicate the “pairing formalism”, utilized to construct BCS theory, but does little to confirm the mechanism.

The mechanism for pairing in conventional superconductors is believed to be virtual phonon exchange, in complete analogy to the virtual photon exchange which is responsible for the direct Coulomb interaction between two charged particles. The primary evidence for this belief comes from a comparison of tunnelling data with the deviations from BCS theory captured in Eliashberg theory, and again, a considerable body of evidence that confirms the virtual phonon exchange mechanism for pairing is described in Refs. [1–6].

At a more microscopic level, for the past several decades the Holstein model has served as the chief paradigm to describe electron-phonon interactions in solids. In part this paradigm choice has been driven by the physics, and the realization that in the single polaron problem the interaction can be very local and the (optical) phonons are well-described by Einstein oscillators. In addition, however, computational techniques for understanding the properties of a polaron have evolved in a manner conducive to lattice models. They concluded that with extended range interactions the effective mass can be much smaller than for the Holstein polaron. Thus, a microscopic model with long-range electron-phonon interactions is a possible means of reconciling exact single electron “building-block” calculations with the Migdal approximation that underlies the Eliashberg theory of electron-phonon-mediated superconductivity.

In an effort to draw comparisons between the short-range Holstein model and the longer-range Fröhlich model, Alexandrov and Kornilovitch defined a Fröhlich polaron problem on a discrete lattice. They examined the behaviour of the effective mass as a function of coupling strength, primarily on one and two dimensional lattices. They concluded that with extended range interactions the effective mass can be much smaller than for the Holstein polaron. Thus, a microscopic model with long-range electron-phonon interactions is a possible means of reconciling exact single electron “building-block” calculations with the Migdal approximation that underlies the Eliashberg theory of electron-phonon-mediated superconductivity.

However, as mentioned earlier, the single electron longer-range interaction studies were carried out with an adiabatic ratio of order unity. Here we wish to re-examine this problem with more physical values of the
adiabatic ratio, and extend their calculations\textsuperscript{26} to three dimensions. We find that while their conclusion that the effective mass can be much smaller than for the Holstein model is correct, this statement applies for a very restrictive range of the coupling strength. We also note that the behaviour in two dimensions is not representative of what occurs in either one or three dimensions. In fact, even in the perturbative regime, low order perturbation theory is not very accurate. Their initial conclusions are actually more representative for the three dimensional case.

The rest of the paper is as follows. First, following Ref. \textsuperscript{20}, we define the model, and we outline the method of solution. We have exact results for all our calculations, based on refinements of the method introduced by Bonča et al\textsuperscript{23} This controlled method of solution becomes somewhat more difficult for three dimensions, but we present converged results for phonon frequencies as low as $\omega_E/t = 0.3$. Considering that this is achieved in three dimensions, where the electronic bandwidth is $W \equiv 12t$, this phonon energy scale represents 2.5% of the electronic bandwidth.

Results are first presented in two dimensions. We first re-assess some older results\textsuperscript{25} and note that, even for the standard Holstein model, perturbative calculations, either in weak or strong coupling, are actually not very accurate. In weak coupling for example, multi-phonon excitations lead to a significantly enhanced effective mass. This phenomenon is amplified when longer range interactions are included, so in fact we find the conclusions of Ref. \textsuperscript{25} somewhat misleading. The effective mass does decrease due to longer range interactions in the strong coupling regime, but in two dimensions, the resulting effective mass is still much too high to be relevant for normal (i.e. non-polaronic) metals.

The following section treats the three dimensional case, where we find in fact that a lower effective mass, to realistic values, is indeed achieved by including longer range interactions. However, even here the range of coupling strengths over which this is achieved is very narrow; in terms of the dimensionless coupling constant $\lambda$ (to be defined below), this range is very close to unity, and not in the range associated with so-called Eliashberg strong coupling superconductors. We conclude with a Summary.

\section{Model and Method}

The lattice Fröhlich ("extended Holstein") model is defined as \textsuperscript{20}

\begin{equation}
H = -t \sum_{i,j} \left[ c_i^\dagger c_j^\dagger + c_i^\dagger c_j \right] - g \omega_E \sum_{(i,j)} f(j)n_i a_i^\dagger a_j^\dagger + \omega_E \sum_i a_i^\dagger a_i,
\end{equation}

where the range of the interaction is given by

\begin{equation}
 f(j) = \frac{1}{(|j|^2 + 1)^{3/2}}.
\end{equation}

In Eq. (1) $t$ is the electron hopping parameter, $\omega_E$ is the characteristic phonon frequency, taken to be a constant here, and $g \omega_E f(j)$ is the coupling strength between an electron at a particular site and a phonon at a site a distance $a_0|j|$ away, where $a_0$ is the lattice spacing (taking to be unity hereafter) and $j$ is the vector connecting the electron and phonon. The case $f(j) = \delta_{j,0}$ reduces to the usual Holstein model. The other symbols are defined as follows: $c_i^\dagger$ (c$_i$) creates (annihilates) an electron at site i, and $a_i^\dagger$ ($a_i$) creates (annihilates) a phonon at site i. The electron number operator is given by $n_i = c_i^\dagger c_i$, and the spin index has been suppressed since we are dealing with only one electron. Note that the sum over $\delta$ in the electron hopping part of the Hamiltonian is over nearest-neighbour sites on the positive side only, to avoid double counting, whereas the sum over $j$ in the interaction term in principle includes the on-site term ($j = 0$ along with neighbouring sites in all directions. In practice, following Ref. \textsuperscript{24} we will terminate the sum at nearest neighbour interactions.

We will use Eq. (2) for all dimensions. In reality this form is motivated by the three dimensional case, where the long range interaction follows a $1/r^3$ decay. At short distances a potential divergence is cutoff by the constant ‘1’ in the denominator of Eq. (2); this corresponds to a characteristic decay length of the lattice spacing, and in principle this can be varied as well. Here, for simplicity, we keep the constant fixed at unity. This Hamiltonian contains a dimensionless coupling constant, namely $g$, which becomes most important in the strong coupling regime. In practice we also define another dimensionless coupling constant, $\lambda$, which becomes important in the weak coupling regime.

Specifically for the coupling of the electron to the phonons at the same site we use, following Li et al\textsuperscript{10}, $\lambda_c \equiv \omega_E g^2/(W/2)$ in one and three dimensions, where $W$ is the bare electron bandwidth, $W = 4t$ in 1D and $W = 12t$ in 3D for the tight binding model on a linear or cubic lattice with nearest neighbour hopping, respectively. In two dimensions we use a definition where the electron density of states at the bottom of the bare band is used [$\equiv 1/(4\pi t)$] instead of the average value of the density of states across the entire band [$\equiv 1/(8t)$], so $\lambda_{c} \equiv \omega_E g^2/(2\pi t)$. For the Holstein model the entire coupling would be that of the on-site coupling.

To define the coupling strength for the extended model, $\lambda_{\text{tot}}$, we follow the definitions of Alexandrov and Kornilovitch\textsuperscript{20} with the total coupling being the sum of the couplings to the different sites. For example, in two dimensions we obtain the on-site contribution along with four equally weighted nearest neighbour contributions,
reduced by $[1/2^{3/2}]^2 = 1/8$ compared to the on-site value:

$$\lambda_{tot} = \sum_j \lambda_c f^2(j) = \lambda_c \left(1 + 4 \frac{1}{8}\right). \quad (3)$$

The single polaron problem is solved here with the variational exact diagonalization method described in Bonča et al.\cite{21} with the same refinements for low frequency calculations as developed by Li et al.\cite{10} The modifications required for the extended model fit nicely into this computational framework, with the cost of a denser Hamiltonian matrix, and a Hilbert space that grows faster with each application of Hamiltonian, compared to similar calculations for the standard (i.e. on-site) Holstein model. This rapid growth makes it difficult to converge results using an extended version of the adaptive method of Li et al.

We therefore further refined the method by producing a list of the most important basis states for each point in parameter space. Starting with a list of basis states from a nearby parameter point previously diagonalized, or a truncated coherent state from the strong coupling Lang-Firsov solution\cite{10,21} we diagonalized the Hamiltonian in this basis. These basis states in turn were ranked according to the magnitude of their contribution to the ground state, and the top $N_1$ contributions were kept. We then acted on these $N_1$ states with the Hamiltonian to produce more basis states and diagonalized the Hamiltonian in this new space. The resulting eigenvector was then sorted, the top $N_1$ contributions were kept, and the process repeated. Once this procedure converged, we sorted one last time, and kept the top $N_2$ states ($N_2 > N_1$) and did the final diagonalization. While this was a time-consuming calculation, it allowed for much better results with a finite amount of computer memory since it selected out the basis states that were the most important for describing the ground state.

One limit of this method should reduce to weak coupling 2nd order perturbation theory. Using straightforward Rayleigh-Schrödinger perturbation theory in 2D with on-site and nearest-neighbour interactions only results in a second-order correction to the ground state energy,

$$E^{(2)}(k_x, k_y) = -\frac{2\pi \lambda_{tot} \hbar \omega_E}{1 + 4 f^2(1)} \sum_{k'} \left( \frac{1 - f(1)}{4} \epsilon_{k'} - \epsilon_k \right)^2,$$

where $\epsilon_k \equiv -2t[\cos(k_x a_0) + \cos(k_y a_0)]$ is the bare energy for the nearest-neighbour tight-binding model. This expression can be evaluated in terms of complete elliptic integrals — for example $[f_1 \equiv f(1)]$,

$$E^{(2)}(0, 0) = -\lambda_{tot} \hbar \omega_E \left\{ \frac{(1 + 4z f_1)^2}{1 + 4 f_1^2} \frac{1}{z} \left[ 1 - K \left( \frac{1}{z^2} \right) \right] - 4\pi f_1 \frac{1 + 2zf_1}{1 + 4f_1^2} \right\} \quad (5)$$

where $z \equiv 1 + \omega_E/(4t)$, but for most of our perturbation results we have simply evaluated the effective mass numerically. Results are shown for the effective mass in the $k_x$ direction although of course there is a $k_x - k_y$ symmetry.

**III. RESULTS IN 2D**

While others have studied the standard Holstein model in detail\cite{10,21,24} there are a few results here that should be emphasized, and which are important for a more complete understanding of the extended Holstein model. The perturbative regime for the 2D Holstein model is actually very small. While the ground state energy from perturbation theory matches the exact ground state energy well, the wavefunction and effective mass do not, as seen in Fig. 1. Perturbation theory does not work very well because the wavefunction, even at weak coupling, needs to include states with multiple phonons, especially when the phonon frequency is small compared to the electron hopping parameter, $t$. Even for the standard Holstein model, restricting our exact diagonalizations to a subspace with a limited number of phonons gives quantitatively accurate results for the effective mass, even if the energy was not strongly affected, as illustrated in Fig. 2A. For
FIG. 2a. \( m^*/m \) decreases as the accuracy of the calculation decreases. The curves with 5 and 3 phonons include states with phonons far from the electron, but never more than 5 or 3 phonons total respectively. 2nd order perturbation theory includes excited states with at most a single phonon. \( \omega_E = 0.2t \).

FIG. 2b. For the 2D extended Holstein model, \( m^*/m \) again decreases as the accuracy of the calculation decreases similar to the standard model. Here the effect is even more pronounced: many phonons are required even for moderate coupling strengths. Again, 2nd order perturbation theory includes at most a single phonon. \( \omega_E = 0.2t \).

The extended model, the discrepancies are even more pronounced, as shown in Fig. 2b.

A second point we wish to make concerning the standard model is that, contrary to what the (fairly accurate) results for the ground state energy might imply, the strong coupling solution is not simply the single Lang-Firsov coherent state:

\[
|\psi\rangle = e^{-g^2/2} \frac{1}{\sqrt{N}} \sum_{\ell} e^{ikR_{\ell}g\hat{a}_\ell^\dagger\hat{c}_\ell^\dagger}|0\rangle.
\]  

We have done an exact calculation and compare the resulting exact wavefunction to the Lang-Firsov coherent state in Fig. 3. The exact solution has a large coherent state component which has a peak somewhat shifted from the Lang-Firsov coherent state. The really important difference is the contribution of states with phonons that are not on the same site as the electron. While these states do not have a large probability in the overall wavefunction, they are important for calculating an accurate effective mass. These states increase the size of the phonon cloud so that neighboring sites are “more prepared” to receive the electron and thus lower the overall effective mass. Using only the Lang-Firsov coherent state gives an effective mass that is too high by orders of magnitude. The same principle reduces the effective mass in the extended model; a nearest neighbour coupling produces phonons on the neighbouring sites, preparing them to receive the electron and lowering the effective mass.

As illustrated in Fig. 2, the crucial addition of extended phonons can be grouped with great precision into multiple coherent states, and coherent states modified with a few other phonons. So, in general, the wave function could be expanded into multiple coherent components:

\[
|\psi\rangle \approx b_0 e^{-g^2/2} \frac{1}{\sqrt{N}} \sum_{\ell} e^{ikR_{\ell}g\hat{a}_\ell^\dagger\hat{c}_\ell^\dagger}|0\rangle
+ b_1 e^{-g^2/2} \frac{1}{\sqrt{4N}} \sum_{\ell,\delta} e^{ikR_{\ell}g\hat{a}_{\ell+\delta}^\dagger\hat{c}_{\ell}^\dagger}|0\rangle
+ b_2 e^{-g^2/2} \frac{1}{\sqrt{4N}} \sum_{\ell,\delta} e^{ikR_{\ell}g\hat{a}_{\ell+\delta}^\dagger\hat{a}_{\ell+\delta}^\dagger\hat{c}_{\ell}^\dagger}|0\rangle + \ldots,
\]

where \( \delta \) designates neighbour sites in all directions. Based on how orderly Fig. 3 looks one could imagine using a variational approach with these coherent states as well. For this paper, however, we kept with the simple Bloch states which though far more numerous are easier to handle as they are guaranteed to be orthogonal:

\[
|\psi\rangle = \sum_n d_n \left( \frac{1}{\sqrt{N}} \sum_{\ell} e^{ikR_{\ell}|\phi_{n\ell}\rangle} \right),
\]

where the \( |\phi_{n\ell}\rangle \) are orthonormal product states consisting of an electron at site \( \ell \) and phonons at sites \( \ell + \delta \).
FIG. 3. Probability \((|d_n|^2)\) of various basis states from the Lang-Firsov approximation and the standard model in 2D, with \(\lambda = 0.62\) and \(\omega_E = 0.2t\). The legend shows the distribution of phonons around the electrons for each ‘coherent state’, while the Lang-Firsov wavefunction only has phonons on the same site as the electron. Lines are a guide to the eye only.

For example, for the Lang-Firsov state given by Eq. (6),
\[ d_n = e^{-g^2/2} g^n / \sqrt{n!}. \]

The main message of this is that while the standard Holstein model may be the simplest polaron model, it’s solution is still a fairly complicated many body wavefunction, even in the weak and strong coupling limits. Without accurate characterization of this wavefunction it may still be possible to calculate some expectation values, like the ground state energy, accurately, but others, such as the effective mass, need a more precise wavefunction. When measuring expectation values with any numerical method, it is important to converge these values on their own as they may need a much larger basis space than, for example, the ground state energy, to be accurately described.

The first qualitative difference found between the extended and on-site Holstein models is at weak coupling. In the standard model, we find a very slight increase of the effective mass with increasing phonon frequency (see Fig. 4), while in the extended model, we find the opposite: the effective mass decreases as a function of increasing frequency. Note that the analytical result for the effective mass in the adiabatic limit is obtained from perturbation theory, and not from the semi-classical adiabatic calculations. The adiabatic calculation reveals that for definiteness, the value of \(\lambda_{\text{tot}}\) is 0.1. In the large \(\omega_E\) limit all the curves approach 1.0 but the Extended model always has a larger \(m^*/m\) than the Standard model in both 2D and 3D.

Quantitatively, the extended Holstein model has a larger effective mass at weak coupling, and a smaller effective mass at strong coupling compared to the standard

\[ m^*/m = 1 + \frac{\lambda_{\text{tot}}}{2} \left[ 1 + 4f(1)^2 \right] / \left[ 1 + 4f^2(1) \right]. \] (9)

This agrees with the limiting value as \(\omega_E \to 0\), obtained through numerical integration in Fig. 4.

Since the exact results and perturbation theory agree for the effective mass in very weak coupling the latter calculations can be trusted in this regime. Therefore we plot in Fig. 4 only the perturbation theory results for the effective mass vs. \(\omega_E/t\) to highlight the differences between the extended and standard models (for definiteness, we use \(\lambda_{\text{tot}} = 0.1\)). At low values of \(\omega_E/t\) the extended model’s effective mass decreases monotonically with increasing \(\omega_E/t\) while the on-site model has a peak near \(\omega_E/t = 1\). Both models have an effective mass ratio of unity (\(m^*/m = 1\)) in the anti-adiabatic limit, \(\omega_E \to \infty\).

FIG. 4. \(m^*/m\) from 2nd order perturbation theory vs \(\omega_E\). For definiteness, the value of \(\lambda_{\text{tot}}\) is 0.1. In the large \(\omega_E\) limit all the curves approach 1.0 but the Extended model always has a larger \(m^*/m\) than the Standard model in both 2D and 3D.
Holstein model (for the same phonon frequency). Previous workers have examined both the ordinary and/or extended models in the past, but due to computational considerations did not examine the model in the physically most important parameter regime. In many real materials the electronic bandwidth is in the eV range while the phonon modes tend to be in the meV range, so the physical adiabatic ratio regime is $\omega_E/t \approx 0.2$.

For reasons stated earlier, calculations with small phonon frequencies are difficult to converge, so we have utilized a number of small frequencies. As a compromise, we will present most results for $\omega_E/t = 0.2$, as these are well converged, and also representative of the low frequency regime. In Figs. 5a and 5b we show the effective mass ratio vs coupling strength for $\omega_E/t = 0.2$ and $\omega_E/t = 1.0$, respectively, for the standard vs extended Holstein models. Several points should be made. First, the crossover from the so-called weak to the so-called strong coupling regime for both standard and extended models is not so clear when $\omega_E/t \approx 1$, especially for the extended model. In contrast, the crossover is much better delineated for low phonon frequencies, as is clear in Fig. 5a. The point made in Ref. 23, that the extended model results in a lower effective mass is also clear, for $\lambda$ values beyond some intermediate coupling strength. At high phonon frequency (Fig. 5a) this means a reduction from $m^*/m \approx 30$ to $m^*/m \approx 7$, for example. By any measure this reduction is significant, but somewhat irrelevant, since the effective masses involved, even after reduction, are too high to describe normal state properties. However, for more realistic (lower) phonon frequencies (Fig. 5a), both crossovers are sharpened as a function of coupling strength, although less so for the extended model, with the net result that a regime of effective mass reduction remains, and the reduction is enormous, but now the mass is ‘lowered’ to values of 40 or higher. In fact, the clear effect of extended interactions is to raise the effective mass for most of the parameter regime that is physically relevant.

To summarize this subsection, the extended Holstein model is more realistic than the standard model insofar as it includes longer range interactions. This does give rise to a coupling regime where the effective mass is lowered, compared to the standard model, but we argue that lowering the effective mass ratio from 100 to 40 is not so relevant. Instead, the clear result of increasing the range of interaction is to enhance the effective mass so that in the so-called weak coupling regime the effective mass is increased due to the longer range interactions. Extended range interactions is therefore not seen as a means to lower the electron effective mass to reasonable levels in the 2D polaron problem. However, the effect of
FIG. 6. For the three dimensional model, $m^*/m$ as a function of $\lambda_{\text{tot}}$, with $\omega_E = 0.3t$, to compare exact and perturbative results. Similar to the 2D results the region of validity for perturbation theory is much smaller in the extended model though it is not particularly large for the standard model either. Inset: Ground state energy vs. $\lambda_{\text{tot}}$. Again, perturbation theory works better for the ground state energy than for the effective mass.

FIG. 7. For the three dimensional case, $m^*/m$ vs. $\lambda_{\text{tot}}$, with $\omega_E = 0.3t$. The effective mass increases as more and more states with more than one phonon excitation are included. The curves with 4 and 5 phonons include states with phonons far from the electron, but there are never more than a total of 4 or 5 phonons, respectively. The calculation with 2nd order perturbation theory includes basis states with at most a single phonon.

long range interactions can be different in three dimensions, and we turn to that question next.

IV. RESULTS IN 3D

We applied the same technique to an extended Holstein model in three dimensions. We limited ourselves again to nearest neighbour and on site electron phonon coupling, simply extending the two-dimensional Hamiltonian to three dimensions. We then then computed the ground state energy and effective mass with perturbation theory and with the numerical method outlined above.

The perturbation theory was done with straightforward Rayleigh-Schrödinger perturbation theory with the integrals performed numerically as in the 2D case. We used $\omega_E/t = 0.3$, since the electronic bandwidth with nearest neighbour hopping only is $12t$ in 3D (as opposed to $8t$ in 2D). As in the 2D case there is good agreement between the exact results and perturbation theory only for a small region of weak coupling, $\lambda_{\text{tot}} \lesssim 0.4$, for the standard model, as seen in Fig. 6. In the extended model, the good agreement is only achieved for an even more restricted range of $\lambda_{\text{tot}} \lesssim 0.2$, as seen in the same figure.

These results are plotted for a ‘low’ phonon frequency, $\omega_E/t \approx 0.3$; however, in 3D there is no ambiguity about the adiabatic limit. Perturbation theory, semi-classical adiabatic calculations, and limiting trends from exact diagonalization all agree that for small $\lambda$, the effective mass ratio approaches unity, i.e. $m^*/m \to 1$ as $\omega_E \to 0$, and there is no polaron formation in this limit. In Fig. 4 it is clear from the 3D results that the behaviour of the effective mass ratio for the extended model is quite similar to that for the standard Holstein model. Quantitatively, the effective mass is somewhat larger for the extended model, but not enormously so. The results in Fig. 6 are both representative of the adiabatic limit, which has no polaron formation.

In Fig. 7 we show the so-called perturbative regime to illustrate that also in three dimensions this regime is confined to very small values of $\lambda$ only. The crossover region, shown for the effective mass in Fig. 8, becomes very sharp as the phonon frequency is decreased to coincide with the point in semiclassical calculations where the ground state abruptly becomes polaronic, after being free-electron-like up to that point. The impact of longer
FIG. 8. For the three dimensional case, $m^*/m$ vs. $\lambda_{\text{tot}}$ with various values of $f(1)$. These results are from numerically exact calculations. The extended model can shift the onset of unphysically high effective masses to stronger coupling, though the extra range of tenable coupling strengths is rather small.

range interactions is clear from the figure; the crossover region is definitely moved to higher coupling strengths as the range and strength of the nearest neighbour interaction increases. An obvious limiting case is where the interaction becomes infinitely long ranged, with the same strength independent of distance from the electron. In this case the electron will remain free-electron-like for all coupling strengths. Note that for realistic nearest neighbour interactions there is now a small range of coupling strengths where the effective mass is reduced to realistic values through longer-range interactions, consistent with the original conclusions of Ref. [26]. That is, in contrast to the two-dimensional case, extended range interactions seem to shift the regime of coupling strengths wherein the effective mass is low, without at the same time increasing significantly the effective mass in this regime.

V. SUMMARY

We have presented exact and perturbative results for the extended Holstein model. This model was conceived in an effort to realize the Fröhlich model on a lattice. Our primary purpose was to re-assess the conclusions of Ref. [26] when a smaller and more realistic adiabatic ratio, $\omega_E/t$, is used. We found that, in two dimensions, the effective mass in the so-called weak coupling regime is enhanced by longer range interactions, while the effective mass in strong coupling is suppressed. This is in agreement with the results of Ref. [26], but we nonetheless find this assessment misleading. In particular, in strong coupling, being able to achieve an effective mass reduction from 100 to 40 is wonderful, but does not serve to reconcile the qualitative results of the Migdal description with the single polaron results. It still remains that, over most of the range of coupling strengths, the effective mass is increased by longer range interactions. Moreover, it is clear that a perturbative description, which, for a single electron problem actually coincides (in a technical sense) with the Migdal approximation, is woefully inaccurate when it comes to describing the details (wave function, electron effective mass, etc.) of the solution, even for much weaker coupling strengths. In three dimensions this problem is slightly ameliorated, in that, at least for a very limited range of coupling strengths, the effective mass can be vastly reduced by many orders of magnitude by longer range interactions (see Fig. 8, just to the right of the standard Holstein results, i.e. the left-most almost vertical line). Even in three dimensions, however, the perturbative weak coupling regime seems to require considerably more phonon excitations then second order perturbation theory would suggest.

Further attempts to reconcile Migdal-based approximations versus exact single electron calculations can proceed along a number of paths, several of which are currently under investigation. For example, one can attempt to develop controlled approximations for more than one electron (the bipolaron, for example, has been already investigated) [33,34]. One would expect phonon and coupling strength renormalization to occur as the increasing number of electrons will have a more significant impact on the phonons. Another direction involves more sophisticated electron-phonon couplings, and the possible importance of acoustic modes.

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