Linear congruences and hyperbolic systems of conservation laws

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Abstract. S. I. Agafonov and E. V. Ferapontov have introduced a construction that allows naturally associating to a system of partial differential equations of conservation laws a congruence of lines in an appropriate projective space. In particular, hyperbolic systems of Temple class correspond to congruences of lines that place in planar pencils of lines. The language of Algebraic Geometry turns out to be very natural in the study of these systems. In this article, after recalling the definition and the basic facts on congruences of lines, Agafonov-Ferapontov’s construction is illustrated and some results of classification for Temple systems are presented. In particular, we obtain the classification of linear congruences in \( \mathbb{P}^5 \), which correspond to some classes of \( T \)-systems in 4 variables.

Introduction

Linear congruences of lines in \( \mathbb{P}^n \) are the (irreducible) subvarieties of dimension \( n-1 \) of the Grassmannian \( G(1,n) \), embedded in \( \mathbb{P}^{(\binom{n+1}{2})-1} \) by the Plücker embedding, obtained by the intersection with a linear space of dimension \( \binom{n+1}{2} \).

Such a congruence of lines \( \mathcal{B} \) has, in general, order one, i.e. through a general point in \( \mathbb{P}^n \) there passes only one line of \( \mathcal{B} \). Moreover, the family of lines parametrised by \( \mathcal{B} \) can be characterized as the set of the \( (n-1) \)-secant lines of the fundamental locus \( \Phi \subset \mathbb{P}^n \), where \( \Phi \) is defined by the property that through a point in it there pass infinitely many lines of \( \mathcal{B} \).

This article deals with a recent discovered application of congruences of lines to mathematical physics (precisely to hyperbolic systems of conservation laws) due to S. I. Agafonov and E. V. Ferapontov: see [AF96] and [AF99]. More precisely, to a system of conservation laws, which has the form
\[
\frac{\partial u_i}{\partial t} + \frac{\partial f_i(u)}{\partial x} = 0, \quad \text{with} \quad i = 1, \ldots, n-1,
\]
they associate an \( (n-1) \)-parameter family \( \mathcal{B} \) of lines in \( \mathbb{P}^n \), defined by the parametric equations
\[
y_i = u^i \lambda - f^i(u) \mu, \quad i = 1, \ldots, n-1 \quad \text{and} \quad y_0 = \lambda, \quad y_n = \mu,
\]
where \( (\lambda : \mu) \in \mathbb{P}^1 \) are the parameters of a line of \( \mathcal{B} \), \( u = (u^1, \ldots, u^{n-1}) \) are the (local) parameters of \( \mathcal{B} \) and \( (y_0 : \cdots : y_n) \) are the homogeneous coordinates on \( \mathbb{P}^n \). It turns out that with this correspondence the basic concepts of the theory of the systems of conservation laws acquire a clear and simple projective interpretation. For instance, for a particular class of systems of conservation laws, the so called \( T \)-systems (see [AF96] and [AF01]), their corresponding family of lines \( \mathcal{B} \) is characterized by the fact that the lines of \( \mathcal{B} \) passing through a point of its focal locus form a planar pencil of lines; moreover, a reciprocal transformation (see [AF01] for a definition)
of one of these systems of conservation laws corresponds to a projectivity in \( \mathbb{P}^n \), and vice versa. Therefore, the classification of the \( T \)-systems is equivalent to the study of these families of lines \( \mathcal{B} \).

Classically, the study of congruences of lines in \( \mathbb{P}^3 \) was started by E. Kummer in [Kum66], in which he gave a classification of those of order one. More recently Z. Ran in [Ran80] studied the surfaces of order one in a general Grassmannian \( \mathbb{G}(r, n) \) i.e. families of \( r \)-planes in \( \mathbb{P}^n \) for which the general \((n - r - 2)\)-plane meets only one element of the family. He gave a classification of such surfaces, obtaining in particular, in the case of \( \mathbb{P}^3 \), a modern and more correct proof of Kummer’s classification.

The congruences of lines in \( \mathbb{P}^4 \), with special regard to those of order one, have been considered by G. Marletta in [Mar09a] and [Mar09b]. The classification of the linear ones has been given by G. Castelnuovo in [C91], where a detailed description of these particular subvarieties of \( \mathbb{G}(1, 4) \) is obtained. These classical results in \( \mathbb{P}^4 \) have been analysed and extended by P. De Poi in [DP01], [DPi], [DPii], and [DPiii].

A general fact about linear congruences in \( \mathbb{P}^n \) is that the lines of the family passing through a general focus form a linear pencil and the plane of this pencil cuts the focal locus residually along a plane curve of degree \( n - 2 \). So linear congruences always define Temple systems. Conversely for \( n \leq 4 \) in [AF01] it has been proved that all families of \( T \)-systems are even algebraic, and more precisely linear congruences.

In higher dimensions nothing is known, in particular also a complete classification of the linear congruences of lines is still missing. So in this paper we have initiated a systematic study of the more simple unknown case, that of linear congruences in \( \mathbb{P}^5 \).

For general linear congruences in \( \mathbb{P}^5 \), the focal locus is a smooth Palatini threefold, which is a scroll over a cubic surface \( S \) in \( \mathbb{P}^3 \) (see [O92], [FM02]). This surface \( S \) can be realized as follows: let the congruence \( \mathcal{B} \) be defined as \( \mathbb{G}(1, 5) \cap \Delta \), for a 10-dimensional linear space \( \Delta \). The dual of the Grassmannian is a cubic hypersurface (the “Pfaffian”, see Section 4), then \( S \) is naturally identified with the intersection of \( \mathbb{G}(1, 5) \) with the dual of \( \Delta \). Classifying linear congruences in \( \mathbb{P}^5 \) amounts to describe all special positions of the 3-space \( \Delta \) with respect to \( \mathbb{G}(1, 5) \) and to its singular locus. For instance, when \( S \) meets \( \text{Sing} \mathbb{G}(1, 5) \), the focal locus acquires some linear irreducible components. Particularly interesting are the cases when \( S \) splits: the description of these congruences relies on a recent classification of the linear systems of \( 6 \times 6 \) skew-symmetric matrices of constant rank 4 up to the natural action of the projective linear group \( \mathbb{PGL}_6 \) ([MM04]).

This article is structured as follows: in Section 1 the basic definitions connected to congruences of lines in the algebraic setting are given. In Section 2 we recall some definitions and results about systems of conservation laws, reciprocal transformations and systems of Temple class. In Section 3 the correspondence between systems of conservation laws and families of lines is illustrated, with special regard to systems of conservation laws of Temple class. In Section 4 we collect some general facts about linear congruences. Finally, in Section 5 we study the linear congruences in \( \mathbb{P}^5 \). We first consider the case in which the cubic surface \( S \) has one or more singular points on the singular locus of \( \mathbb{G}(1, 5) \), these points correspond to some 3-dimensional linear spaces that enter in the focal locus. We then study the congruences such that the surface \( S \) is reducible, getting four types of congruences. In some cases the focal locus has a parasitic component, i.e. an irreducible, maybe embedded component, which is not met by a general line of \( \mathcal{B} \).
In a forthcoming paper we plan to apply these results to the classification of Temple systems in 4 variables. We have to point out that the classification considered here holds over an algebraically closed field, so it will be necessary to refine it over the real field.

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1. Notation, Definitions and Preliminary Results

In the realm of algebraic geometry, we will work with schemes and varieties over \( \mathbb{C} \), with standard notation and definitions as in [Har77]. A variety will always be projective. We refer to [DP01] and [DP03] for general results and references about families of lines, focal diagrams and congruences, and to [GH78] for notations about Schubert cycles. In particular we denote by \( \sigma_{a_0,a_1} \) the Schubert cycle of the lines in \( \mathbb{P}^n \) contained in a fixed \((n - a_1)\)-dimensional subspace \( H \subset \mathbb{P}^n \) and which meet a fixed \((n - 1 - a_0)\)-dimensional subspace \( \Pi \subset H \).

**Definitions 1.** A congruence of lines \( \mathcal{B} \) in \( \mathbb{P}^n \) is a flat family of lines of dimension \( n-1 \), and we can think of it as a \((n-1)\)-dimensional subvariety of the Grassmannian \( G(1,n) \). Its order \( a_0 \) is the number of lines passing through a general point in \( \mathbb{P}^n \).

Throughout this article, we will denote by \( \Lambda := \{(b, P) \mid P \in \Lambda(b)\} \) the incidence correspondence associated to the family \( \mathcal{B} \), with its natural projections \( f \) and \( p \) to \( \mathbb{P}^n \) and to \( \mathcal{B} \) respectively. If \( b \in \mathcal{B} \), then \( \Lambda(b) := f(\Lambda_b) \) denotes the line parametrised by \( b \) and \( \Lambda_b := p^{-1}(b) \) the corresponding subset of \( \Lambda \). We can summarize all in the diagram:

\[
\Lambda_b \subset \Lambda \subset \mathcal{B} \times \mathbb{P}^n \xrightarrow{f} \mathbb{P}^n \supset \Lambda(b)
\]

(1)

Note that \( f \) is surjective if and only if \( a_0 > 0 \). In this case \( f \) is a map of degree \( a_0 \).

The basic objects when one deals with congruences are the focal and fundamental loci.

**Definitions 2.** Let \( \mathcal{B} \) be a congruence of lines; then the focal divisor of the family \( \mathcal{B} \) is the ramification divisor \( R \subset \Lambda \) of

\[
f : \Lambda \to \mathbb{P}^n;
\]

the schematic image \( F \) of \( R \) under \( f \) is called the focal locus: \( F = f(R) \subset \mathbb{P}^n \). The foci of \( \mathcal{B} \) are the branch points of \( f \). The fundamental locus \( \Phi \) is the set of points \( y \) contained in more lines of the family than expected:

\[
\dim f^{-1}(y) > n - \dim f(\Lambda).
\]

**Example 1.** The family of the tangent lines to a curve \( C \) (which is not a line) in \( \mathbb{P}^2 \) is the most simple example of a congruence. Other examples are the family of the secant lines to a curve in \( \mathbb{P}^3 \) or the one of the tangent lines to a surface in \( \mathbb{P}^4 \).

**Remark.** The fundamental locus \( \Phi \) is in general properly contained in the focal locus \( F \), but if \( \mathcal{B} \) is a congruence of order one, then \( \Phi = F \) and the codimension of \( F \) in \( \mathbb{P}^n \) is \( \geq 2 \).

The following recent result gives to converse statement:
Theorem 1. (F. Catanese, P. De Poi, 2004, [DP04]) Let $\mathcal{B}$ be a congruence such that the fundamental locus $\Phi$ coincides (set-theoretically) with the focal locus $F$; then the order of $\mathcal{B}$ is zero or one.

Remark. It is important to note that the focal locus $F$ often has some unexpected components. For example, let $\mathcal{B}$ be the congruence of the secant lines to a curve $C$ in $\mathbb{P}^3$. If $L$ is a line meeting $C$ at two points $x, y$ such that the tangent lines to $C$ at $x$ and $y$ are incident, then the tangent plane to $\mathcal{B}$ at the point corresponding to $L$ is contained in the Grassmannian and $L \subset F$. Since a curve in $\mathbb{P}^3$ has in general a one-dimensional family of secant lines of this type (called stationary secants), the congruence $\mathcal{B}$ will have a focal surface. The only curve in $\mathbb{P}^3$ without stationary secant lines is the twisted cubic. For more details, see [ABT].

The focal locus of a congruence may also have a component which is not met by a general line of $\mathcal{B}$. Such a component is called a parasitic component. Some explicit examples will be described in Sections 4 and 5.

From now on, we assume that $f$ is surjective. In this case, the fundamental locus is the set of points $P \in \mathbb{P}^n$ for which the fibre of the map $f: \Lambda \to \mathbb{P}^n$ has positive dimension.

Theorem 2. (C. Segre, 1888, [Seg88], C. Ciliberto and E. Sernesi, 1992, [CS92]) With notations as above, given a congruence $\mathcal{B}$, for the general $b \in \mathcal{B}$, the corresponding line $\Lambda(b) \subset \mathbb{P}^n$ contains exactly $n-1$ foci (counting multiplicities) which are foci for the line $\Lambda(b)$. Otherwise $\Lambda(b) \subset F$.

Moreover, if $\dim(F) = n-1$, then $\Lambda(b)$ is tangent to $F$ at its (smooth) focal non-fundamental points.

2. Systems of Conservation Laws

In the realm of mathematical physics, we will work over $\mathbb{R}$. All the functions are—at least—$C^1$.

Definitions 3. A system of conservation laws is a quasi-linear system of first order partial differential equations of the form

$$\frac{\partial u_i}{\partial t} + \frac{\partial f^i(u)}{\partial x} = 0 \quad i = 1, \ldots, n-1$$

where $u(x, t) = (u^1(x, t), \ldots, u^{n-1}(x, t))$ are the unknown functions and the $f^i(u)$’s are functions defined over a domain $\Omega \subset \mathbb{R}^{n-1}$. 
The system can be written:

\[ u^i_t + \sum_i \frac{\partial f^i(u)}{\partial u} u^j_x u^j_x = u^i_t + J f(u) \cdot u_x = 0 \quad i = 1, \ldots, n - 1 \]  

where \( Jf \) denotes the Jacobian matrix of \( f \).

The system is called hyperbolic (resp. strictly hyperbolic) if all the eigenvalues of \( Jf \) are real (resp. real and distinct).

**Definitions 4.** If the system is strictly hyperbolic, the eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_{n-1}(u) \]

are called characteristic velocities.

The integral trajectories \( \gamma_i \) of the fields of eigenvectors \( \mathbf{v}_i \) are called the rarefaction curves:

\[ \dot{\gamma}_i(t) = \mathbf{v}_i(\gamma_i(t)). \]

**Remark.** Given a strictly hyperbolic system of conservation laws as above, through a point \( u \in \Omega \subset \mathbb{R}^{n-1} \) there pass \( n - 1 \) rarefaction curves. Indeed each eigenvector of the eigenvalue \( \lambda_i(u) \), for \( i = 1, \cdots, n - 1 \), has the direction of the tangent line to the curve \( \gamma_i \).

2.1. **Temple Systems.** These systems, which are strictly hyperbolic, were introduced by B. Temple in [Tem83]. They naturally arise in the theory of equations of associativity of 2D topological field theory (see [Dub96]).

**Definition 5.** A strictly hyperbolic system is said to be linearly degenerate if

\[ L_i(\lambda_i)(u) = 0 \quad \forall i \]

where \( L_i \) is the Lie derivative in the direction of \( \mathbf{v}_i \).

This means that the eigenvalue \( \lambda_i \) is constant along the rarefaction curve \( \gamma_i \).

**Definition 6.** A strictly hyperbolic system is said to be a Temple system or a T-system if it is linearly degenerate and the rarefaction curves are lines in the coordinates \( (u^1, \ldots, u^{n-1}) \) (see [Tem83]).

2.2. **Reciprocal Transformations.** Let us define two new independent variables \( (X, T) \) by

\[ dX := \sum_i \alpha_i(u^i dx + f^i(u) dt) + \mu dx + \nu dt \]

\[ dT := \sum_i \tilde{\alpha}_i(u^i dx + f^i(u) dt) + \tilde{\mu} dx + \tilde{\nu} dt. \]

By the two 1-forms on the right are closed, so takes the new form

\[ \frac{\partial U^i}{\partial T} + \frac{\partial F^i(U)}{\partial X} = 0 \quad i = 1, \ldots, n - 1 \]

**Definition 7.** Transformations of the form are called reciprocal.

Reciprocal transformations are known to preserve the class of T-systems (see [AF96]).
3. The Correspondence

Agafonov and Ferapontov ([AF96] and [AF99]) associate to a system of conservation laws \( \mathfrak{B} \) a \((n-1)\)-parameter family \( \mathcal{B} \) of lines in \( \mathbb{P}^n \), i.e. a congruence, defined by the parametric equations

\[
\begin{cases}
  y_0 = \lambda, \\
  y_i = u^i\lambda - f^i(u)\mu, \ i = 1, \ldots, n-1 \\
  y_n = \mu
\end{cases}
\]

where

- \( (\lambda : \mu) \in \mathbb{P}^1 \) are homogeneous coordinates on a line of \( \mathcal{B} \);
- \( u = (u^1, \ldots, u^{n-1}) \) are the (local) parameters of \( \mathcal{B} \);
- \( (y_0 : \cdots : y_n) \) are homogeneous coordinates on \( \mathbb{P}^n \).

So for every \( u \) in the domain \( \Omega \), the above equations define a line \( \Lambda(u) \).

A dictionary can be written translating properties of the system \( \mathfrak{B} \) to properties of the family of lines, and conversely. For example, reciprocal transformations of the system correspond to projectivities in \( \mathbb{P}^n \). The eigenvalues \( \lambda_1(u), \ldots, \lambda_{n-1}(u) \) are in natural bijection with the foci of \( \mathcal{B} \) on the line \( \Lambda(u) \). In particular, the system is strictly hyperbolic if and only if on a general line of \( \mathcal{B} \) there are \( n-1 \) distinct foci of the congruence. The rarefaction curves correspond to developable ruled surfaces in \( \mathbb{P}^n \).

3.1. Temple Systems again. In general for a hyperbolic system of conservation laws the corresponding focal locus of the congruence \( \mathcal{B} \) is a hypersurface \( F \) and the lines of the family are tangent to \( F \) at the \( n-1 \) foci. For a Temple system the situation is different.

**Theorem 3.** ([AF96]) The congruences of lines \( \mathcal{B} \) associated to Temple systems are characterized by the properties that every focus is a fundamental point and that the rarefaction curves correspond to planar pencils of lines of \( \mathcal{B} \).

By Theorem 1 it follows that \( \dim F = n-2 \) and the order of \( \mathcal{B} \) is one.

The classification of the \( T \)-systems up to reciprocal transformations is equivalent to the classification of non necessarily algebraic congruences of lines of order one with planar pencils.

**Example 2.** The wave equation. The equation \( f_{tt} = f_{xx} \) can be rewritten as a system of two conservation laws

\[
\begin{cases}
  u_t^1 = u_x^2 \\
  u_t^2 = u_x^1
\end{cases}
\]

The associated congruence \( \mathcal{B} \) in \( \mathbb{P}^3 \) is formed by the lines meeting two fixed skew lines \( L \) and \( L' \). On each line \( \Lambda \) of \( \mathcal{B} \) there are two distinct foci, its intersections with \( L \) and \( L' \).

This is also an example of a Temple system. The associated congruence of lines is linear. It is rather easy to prove (see Proposition 6) that all linear congruences, in any projective space, are associated to some Temple system. Conversely:

**Theorem 4.** ([AF96], [AF01]) All Temple systems in 2 and 3 variables give rise to linear congruences. In particular they are all algebraic.

Agafonov and Ferapontov have conjectured that congruences of lines whose developable surfaces are planar pencils of lines have always algebraic focal varieties (possibly reducible and singular).
4. Linear Congruences of lines

As we said in the introduction, a linear congruence of lines \( \mathcal{B} \) in \( \mathbb{P}^n \) has the form \( \mathcal{B} = \mathcal{G}(1, n) \cap \Delta \), where \( \Delta \) is a linear subspace of dimension \( \binom{n+1}{2} \) of \( \mathbb{P}(\wedge^2 V) \), with \( V := H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* \), the space of the Plücker embedding of the Grassmannian.

The \( \binom{n+1}{2} \) coordinates of a line \( \ell \in \mathcal{G}(1, n) \) can be interpreted as entries of a skew-symmetric \((n+1) \times (n+1)\) matrix \((p_{ij})_{i,j=0,\ldots,n}\) of rank 2. A hyperplane \( H \) in \( \mathbb{P}(\wedge^2 V) \) has an equation of the form \( \sum_{i,j=0}^n a_{ij} p_{ij} = 0 \), and the \( a_{ij} \)'s are coordinates of \( H \) as a point in the dual space \( \mathbb{P}(\wedge^2 V^*) \). Clearly, also the dual coordinates \( a_{ij} \) can be interpreted as entries of a skew-symmetric matrix \( A \). \( \Delta \) is the intersection of the \( n-1 \) hyperplanes \( H_1, \ldots, H_{n-1} \) with equations:

\[
\sum_{i,j=0}^n a_{ij}^1 p_{ij} = 0, \quad \ldots \quad \sum_{i,j=0}^n a_{ij}^{n-1} p_{ij} = 0
\]

associated to matrices \( A_1, \ldots, A_{n-1} \). In the dual space \( H_1, \ldots, H_{n-1} \) generate the dual \((n-2)\)-space \( \Delta \).

Some general results about fundamental varieties of linear congruences are given in [BM01]: in particular, it is proved that the focal locus of a linear congruence \( \mathcal{B} \) is the degeneracy locus \( F \) of a morphism of sheaves of the form

\[
\phi : \mathcal{O}_{\mathbb{P}^n}^{\otimes(n-1)} \to \Omega_{\mathbb{P}^n}(2).
\]

Explicitly, there is an isomorphism:

\[
H^0(\Omega_{\mathbb{P}^n}(2)) \cong (\wedge^2 V)^* ,
\]

and so a global section of \( \Omega_{\mathbb{P}^n}(2) \) is a skew-symmetric matrix of type \((n+1) \times (n+1)\) with entries in the base field. Then, the morphism \( \phi \) in (9) is defined by the \( n-1 \) skew-symmetric matrices \( A_1, \ldots, A_{n-1} \). The corresponding degeneracy locus \( F \) in \( \mathbb{P}^n \) is defined by the equations

\[
\sum_{i=1}^{n-1} \lambda_i A_i [X] = 0
\]

for some \( [\lambda] = (\lambda_1, \ldots, \lambda_{n-1}) \neq (0, \ldots, 0) \), where \([X]\) denotes the column matrix of the coordinates.

Since the matrices \( A_1, \ldots, A_{n-1} \) are skew-symmetric, the situation changes whether \( n \) is even or odd.

**Proposition 5.** ([DP03]) If \( F \) is the focal locus of a linear congruence \( \mathcal{B} \) in \( \mathbb{P}^n \), then

1. for \( \mathcal{B} \) general, \( F \) is smooth if \( \dim(F) \leq 3 \);
2. if \( n \) is even, for each \( [\lambda] \in \mathbb{P}^{n-2} \) equation (11) has at least one solution, and \( F \) is rational;
3. if \( n \) is odd, the vanishing of the Pfaffian of the matrix \( \sum_{i=1}^{n-1} \lambda_i A_i \) defines a hypersurface \( Z \) of degree \((n+1)/2\) in \( \mathbb{P}^{n-2} \) (in which \( \lambda_1, \ldots, \lambda_{n-1} \) are the coordinates). Furthermore, if \( \phi \) is general, for a fixed point \( [\lambda] \in Z \), equation (11) has a line contained in \( F \) as solution, and \( F \) results to be a scroll over (an open set of) \( Z \).

Besides, in both cases

\[
\deg(F) = \frac{n^2 - 3n + 4}{2}
\]

Another known result, which can be deduced by the above description of the focal locus of a linear congruence and which explains the fact that linear congruences give Temple systems is the following:
Proposition 6. Let $B$ be a linear congruence in $\mathbb{P}^n$, with focal locus $F \subset \mathbb{P}^n$; then $B$ has order one and is the closure of the family of the $(n-1)$-secant lines to $F$. Moreover, if $P$ is a general point in $F$, the family of the lines of $B$ through $P$ is a pencil, whose plane intersects $F$, out of $P$, in a curve of degree $n - 2$.

In particular, a linear congruence corresponds to a Temple system.

Proof. The first assertion follows from standard Schubert calculus and the second one can be deduced from Theorems [1] and [2]. We give here a different direct proof of both facts which relies on the previous observations. From Equation (11) (or Equations (8)) we infer that through the general point $P \in \mathbb{P}^n$ with coordinates $[X] = [x_0, \ldots, x_n]$ there passes only the line of $B$ whose coordinates are the maximal minors of

$$A := \begin{pmatrix} \sum_i a_{i0} x_i & \cdots & \sum_i a_{in} x_i \\ \vdots \\ \sum_i a_{ni} x_i & \cdots & \sum_i a_{ni} x_i \end{pmatrix}. \tag{12}$$

Moreover, if we think of $A$ as a matrix with linear entries in $\mathbb{P}^n$, the degeneracy locus of $A$ is $F$. Now, we can fix one line $\ell \in B$ and without loss of generality we can suppose that it does not intersect the $(n - 2)$-dimensional space defined by $x_0 = x_1 = 0$; then $F \cap \ell$ is defined by the determinant of the submatrix of $A$ formed by the last $(n - 1)$ columns, and therefore—if $\ell$ is general—$F \cap \ell$ is a zero dimensional scheme of length $(n - 1)$.

Now, if $P$ is a general focal point, by the linearity the lines of $B$ through it form a pencil $\ell_P$: in fact in Equation (11) we can suppose that $A_1[X] = 0$ and $A_2[X] \neq \ldots, A_{n-1}[X] \neq 0$, i.e. $P$ is the centre of the linear complex $A_1$. Then the lines of $\ell_P$ are contained in the plane whose Plücker coordinates are given by the $(n - 2) \times (n - 2)$-minors of

$$A' := \begin{pmatrix} \sum_i a_{i0}^2 x_i & \cdots & \sum_i a_{in}^2 x_i \\ \vdots \\ \sum_i a_{ni}^2 x_i & \cdots & \sum_i a_{ni}^2 x_i \end{pmatrix}.\tag{12}$$

The plane $\pi_P$ of the pencil intersects $F$ in $P$ and in a curve of degree $(n - 2)$: in fact $\pi_P$ intersects the hypersurface $V$ defined by a (fixed) minor of (12) in a curve of degree $(n - 1)$ which splits in the line of the congruence through $P$ and contained in $V$ and residually in a curve which must be contained in $F$. \hfill \Box

The dual variety of the Grassmannian $\mathbb{G}(1,n)$ parametrises the tangent hyperplanes to $\mathbb{G}(1,n)$. It is defined by the maximal Pfaffians of the matrix $A$, therefore if $n$ is odd, it is the hypersurface in $\mathbb{P}^n$ of degree $\frac{n+1}{2}$, defined by the Pfaffian, while if $n$ is even, it has codimension 3. In the case $n$ odd, using coordinates $(\lambda_1, \ldots, \lambda_{n-1})$ in $\Delta$, the intersection $S := \mathbb{G}(1,n) \cap \Delta$ is defined by Pfaff($\sum_i \lambda_i A_i$) = 0, hence it coincides with the hypersurface $Z$ of Proposition [5].

4.0.1. Linear Congruences in $\mathbb{P}^3$ and $\mathbb{P}^4$. We end this section by briefly recalling the classification of the linear congruences in low dimensional projective spaces.

In $\mathbb{P}^3$, the situation is very simple: $\mathbb{G}(1,3) \subset \mathbb{P}^5$ is the Klein quadric, its dual is again a quadric, and $\Delta$ is a line. Correspondently, we have the following cases: if $\Delta$ is general, it intersects $\mathbb{G}(1,3)$ at two distinct points and the congruence represents the join of the two corresponding lines. The line $\Delta$ can be tangent and therefore intersects $\mathbb{G}(1,3)$ in a double point. Correspondently, we have a congruence which has as focal locus a double line (and the congruence is a subset of the set of lines meeting its support). Finally, it can happen that $\Delta \subset \mathbb{G}(1,3)$: in this case, we do not have a congruence, since the corresponding family of lines in $\mathbb{P}^3$ has dimension greater than three.
In \( \mathbb{P}^4 \), the situation is more complicated: \( \mathbb{G}(1,4) \subset \mathbb{P}^9 \) has dimension six and degree five. \( \Delta \cong \mathbb{P}^2 \) and in the general case \( \Delta \cap \mathbb{G}(1,4) = \emptyset \): then the congruence is given by the trisecants to a projected Veronese surface. If \( \Delta \cap \mathbb{G}(1,4) \) is non empty and finite, then its length is at most 3. If it is a single point, this point gives us a focal plane: the congruence is given by the secant lines to a cubic scroll which meet this plane also. If \( \Delta \cap \mathbb{G}(1,4) \) is two points, then we have the lines meeting two fixed planes and which meet a quadric also. If \( \Delta \cap \mathbb{G}(1,4) \) is given by three points, we get the lines meeting three planes; the focal locus has also a fourth component, which is parasitic, the plane spanned by the 3 points of intersection of the three planes two by two. Of course also limit cases of these are possible, e.g. if \( \Delta \cap \mathbb{G}(1,4) \) is a double point, etc. Finally, \( \Delta \cap \mathbb{G}(1,4) \) can be a curve or \( \Delta \subset \mathbb{G}(1,4) \): several cases are possible but they don’t define a congruence, since the corresponding family of lines in \( \mathbb{P}^4 \) has dimension greater than four (see the original paper of Castelnuovo [C91]).

The article [AF01] contains the interpretation of this classification in terms of that of Temple systems in three variables up to reciprocal transformations.

5. Linear Congruences in \( \mathbb{P}^5 \)

A linear congruence of lines \( \mathcal{B} \) in \( \mathbb{P}^5 \) is of the form \( \mathcal{B} = \mathbb{G}(1,5) \cap \Delta \), where \( \Delta \) is a linear space of dimension 10, intersection of 4 hyperplanes with equations:

\[
\sum_{i,j=0}^{5} a_{ij}p_{ij} = 0, \quad \sum_{i,j=0}^{5} b_{ij}p_{ij} = 0, \quad \sum_{i,j=0}^{5} c_{ij}p_{ij} = 0, \quad \sum_{i,j=0}^{5} d_{ij}p_{ij} = 0
\]

associated to skew-symmetric \( 6 \times 6 \) matrices \( A, B, C, D \).

The dual variety of the Grassmannian is the cubic hypersurface \( \mathbb{G}(1,5) \) in \( \mathbb{P}^{14} \) defined by the Pfaffian. One can think of \( \mathbb{G}(1,5) \) as the locus of skew-symmetric matrices (in the \( (a_{ij}) \)'s coordinates) of rank at most four. Using coordinates \( (a, b, c, d) \) in \( \Delta \), the intersection \( S := \mathbb{G}(1,5) \cap \Delta \) is defined by \( \text{Pfaff}(aA + bB + cC + dD) = 0 \), hence, in general, it is a cubic surface.

The rational Gauss map \( \gamma : \mathbb{G}(1,5) \rightarrow \mathbb{G}(1,5) \) associates to a tangent hyperplane its unique tangency point. It is regular outside \( \text{Sing}(\mathbb{G}(1,5)) \), which is naturally isomorphic to \( \mathbb{G}(3,5) \) and corresponds to the matrices of rank two (which indeed is dually isomorphic to \( \mathbb{G}(1,5) \) also). If \( A \) is such a matrix, the corresponding 3-space \( \pi_A \) is the projectivised kernel of \( A \). As a hyperplane section of \( \mathbb{G}(1,5) \), \( A \) represents the lines in \( \mathbb{P}^5 \) meeting \( \pi_A \). In what follows we will always identify \( \text{Sing}(\mathbb{G}(1,5)) \) with \( \mathbb{G}(3,5) \).

For general \( \Delta \), the image \( \gamma(S) \) is a 2-dimensional family of lines, whose union is a smooth 3-fold \( F \) of degree 7 and sectional genus 4, called Palatini scroll. The lines of the congruence \( \mathcal{B} \) are the 4-secant lines of \( F \) and \( F \) is the focal locus of \( \mathcal{B} \) (see also [O92], [BM01], [FM02] and [DP03]). More in general:

**Proposition 7.** Let \( \Delta \) be a linear space of dimension 10 in \( \mathbb{P}^{14} \) such that \( S := \Delta \cap \mathbb{G}(1,5) \) is a reduced cubic surface and \( S \cap \mathbb{G}(3,5) \) is empty. The lines \( b \in \gamma(S) \) span (set-theoretically) the focal divisor \( F \) of the linear congruence \( \mathcal{B} = \mathbb{G}(1,5) \cap \Delta \), i.e. the map \( f \) of \( \mathcal{B} \) drops rank exactly at the pairs \( (b, P) \) such that \( b \in \gamma(S) \) and \( P \in \pi_A \). The lines of \( \mathcal{B} \) are the 4-secants of \( F \).

**Proof.** Since \( \gamma(S) \subset \mathcal{B} \) is the set of points whose tangent hyperplanes (to \( \mathbb{G}(1,5) \)) contain \( \Delta(= \langle \mathcal{B} \rangle) \), we have one inclusion recalling that for these points the global characteristic map—see for example Definition 1 of [DP01]—of the congruence drops rank. For the other inclusion, given a point \( P \in F \cap \pi_A \), by definition, through a smooth \( b \in \mathcal{B} \) there passes at least one tangent direction to \( \mathbb{G}(1,5) \setminus \mathcal{B} \).
contained in $\Delta$: we can find—by dimensional reasons—a tangent hyperplane of $P$ containing $\Delta$ also. The last assertion follows from Theorem 2.

Classifying linear congruences in $\mathbb{P}^5$ amounts to describing all special positions of the 3-space $\Delta$ with respect to $\mathcal{G}(1, 5)$ and to its singular locus. As we will see the situation is rather complicated and several different cases are possible.

**Remark.** Actually, we are interested mainly to the case of “true” linear congruences, i.e. if the intersection $\mathcal{B} = \Delta \cap \mathcal{G}(1, 5)$ is proper and the focal locus has dimension three. Therefore, in what follows we will not give many details on the cases such that $\mathcal{B}$ has dimension $> 4$ or $F$ has dimension $> 3$.

**Proposition 8.** With the above notation, if $\hat{\Delta}$ is contained in $\mathcal{G}(1, 5)$, then $\hat{\Delta}$ meets $\mathcal{G}(3, 5)$ and the focal locus $F$ of $\mathcal{B}$ has dimension $> 3$. If $\Delta \cap \mathcal{G}(1, 5)$ is a cubic surface $S$, then $\dim F > 3$ if and only if $S$ intersects $\mathcal{G}(3, 5)$ at least along a curve.

**Proof.** The first claim follows from Corollary 11 of [MM02]. The second and the third one are Theorem 4.3 and Theorem 4.9 of [FM02].

From now on, we will always assume that the congruence $\mathcal{B}$ is obtained from a 10-space $\Delta$ such that the intersection $\Delta \cap \mathcal{G}(1, 5)$ is proper i.e. a cubic surface $S$. From [FM02], Remark 4.4, it follows that every cubic surface in $\mathbb{P}^3$ can be realized in this way.

If $S$ is smooth, then $F$ is always a Palatini scroll. For any $S$, possibly singular and/or reducible, such that $\dim F = 3$, the Hilbert polynomial of $F$ is $P_F(t) = 7/6t^3 + 2t^2 + 11/6t + 1$. The equations of $F$ can be written explicitly as maximal minors of the following $4 \times 6$ matrix (see [FM02]):

\[
M = \begin{pmatrix}
\sum_i a_{0i} x_i & \cdots & \sum_i a_{5i} x_i \\
\sum_i b_{0i} x_i & \cdots & \sum_i b_{5i} x_i \\
\sum_i c_{0i} x_i & \cdots & \sum_i c_{5i} x_i \\
\sum_i d_{0i} x_i & \cdots & \sum_i d_{5i} x_i
\end{pmatrix}
\]

that we will also write in the form

\[
\begin{pmatrix}
L_{10} & L_{11} & \cdots & L_{15} \\
L_{20} & L_{21} & \cdots & L_{25} \\
L_{30} & L_{31} & \cdots & L_{35} \\
L_{40} & L_{41} & \cdots & L_{45}
\end{pmatrix}
\]

The lines of $\mathcal{B}$ through a general point $P$ in $F$ form a linear pencil contained in a plane cutting $F$, out of $P$, along a plane cubic curve. The coefficients of the equations of this plane are the lines of the matrix (14) computed at $P$.

It is well known that a cubic surface with isolated singularities can have at most 4 double points, or one triple point if it is a cone, and that an irreducible cubic with a singular curve is necessarily ruled with a double line. Observe also that if $A$ is a singular point of $S$, then either $A \in \mathcal{G}(3, 5)$ or $\hat{\Delta}$ is tangent to $\mathcal{G}(1, 5)$ at $A$. We will now consider the various possibilities for $S$, studying the linear congruences $\mathcal{B}$ and their focal loci. We will also give some explicit examples, constructed using CoCoA (see [CoCoA]).

We begin by studying two special situations for the singularities of $S$, i.e. the cases when one or more singular points belong to $\mathcal{G}(3, 5)$ and when $S$ is reducible not meeting $\mathcal{G}(3, 5)$.

5.1. $S$ with singular points on $\mathcal{G}(3, 5)$. 
5.1.1. Only one singular point. Let $A$ be a singular point of $S$ belonging to $G(3, 5)$. In this case $\Delta$ is contained in the linear complex of the lines meeting the $3$-space $\pi_A$, hence each line of the congruence $B$ intersects $\pi_A$. We get that $F$ splits as $F = \pi_A \cup Y$, where $Y$ is an arithmetically Cohen-Macaulay (aCM for short) threefold of degree 6 and sectional genus 3, defined by the maximal minors of a $3 \times 4$ matrix of linear forms. Indeed, if we choose a system of coordinates such that $\pi_A$ has equations $x_0 = x_1 = 0$, then the only non-zero coordinate of $A$ is $a_{01}$ and the matrix (16) becomes

\[
\begin{pmatrix}
x_1 & -x_0 & 0 & 0 & 0 \\
x_0 & 0 & 0 & 0 & 0 \\
L_{20} & L_{21} & \ldots & L_{25} \\
L_{30} & L_{31} & \ldots & L_{35} \\
L_{40} & L_{41} & \ldots & L_{45}
\end{pmatrix}
\]

One checks that $\pi_A$ is a component of $F$ and the residual is defined by the $3 \times 3$ minors of the matrix formed by the last 3 rows and 4 columns of (16).

If the other generators of $\Delta$ are general, then $S$ is smooth outside $A$. $Y$ results to be a singular Bordiga scroll, with 6 singular points all belonging to $Y \cap \pi_A$, which is a smooth quadric surface. The foci on a general line of the congruence are contained one in $\pi_A$ and the other three in $Y$, so the lines of $B$ are the trisecants of $Y$ meeting also $\pi_A$. The lines of $B$ through a point of $\pi_A$ cut $Y$ at the points of a plane cubic, whereas those through a point $P$ in $Y$ cut $Y$ residually along a conic $C$ and $\pi_A$ along a line $L$.

5.1.2. Two singular points. Assume now that $S$ has also a second singular point $B$ on $G(3, 5)$. Then $F$ is of the form $F = \pi_A \cup \pi_B \cup Z$, where $Z$ is an aCM threefold of degree 5 and sectional genus 2. To see this, let us note that the 3-spaces $\pi_A$ and $\pi_B$ are in general position (otherwise the line $\overline{AB}$ would be contained in $G(3, 5)$), so we can assume that $\pi_A$ is defined by $x_0 = x_1 = 0$ and $\pi_B$ by $x_2 = x_3 = 0$. So the matrix $M$ becomes

\[
\begin{pmatrix}
x_1 & -x_0 & 0 & 0 & 0 \\
0 & 0 & x_3 & -x_2 & 0 \\
L_{30} & L_{31} & \ldots & L_{35} \\
L_{40} & L_{41} & \ldots & L_{45}
\end{pmatrix}
\]

By developing its $4 \times 4$ minors, we get that the two 3-spaces $\pi_A$ and $\pi_B$ are irreducible components of $F$ and the residual is defined by the $2 \times 2$ minors of

\[
\begin{pmatrix}
x_0 L_{30} + x_1 L_{31} & L_{34} & L_{35} \\
x_0 L_{40} + x_1 L_{41} & L_{44} & L_{45}
\end{pmatrix},
\]

or, equivalently, of

\[
\begin{pmatrix}
x_2 L_{32} + x_3 L_{33} & L_{34} & L_{35} \\
x_2 L_{42} + x_3 L_{43} & L_{44} & L_{45}
\end{pmatrix}.
\]

$Z$ is therefore a Castelnuovo threefold. If the other generators of $\Delta$ are general, then $S$ is smooth outside $A$ and $B$. $Z$ results to be singular at 8 points, 4 of them belonging to $Q_A := Z \cap \pi_A$, and the other 4 to $Q_B := Z \cap \pi_B$. $Q_A$ and $Q_B$ are both quadric surfaces, linear sections of the unique quadric containing $Z$, of equation $L_{34} L_{45} - L_{35} L_{44} = 0$. They intersect along the line $\pi_A \cap \pi_B$. Clearly the lines of $B$ are the secants of $Z$ meeting also $\pi_A$ and $\pi_B$. The lines of $B$ through a point in $\pi_A$ or $\pi_B$ cut $Z$ along the points of a conic, whereas those through a point $P$ in $Z$ cut $Z$, $\pi_A$ and $\pi_B$ residually each along a line.
5.1.3. Three singular points. We assume now that $S$ has also a third singular point $C$ on $\mathbb{G}(3,5)$. Then $F$ is of the form $F = \pi_A \cup \pi_B \cup \pi_C \cup V$, where $V$ is a complete intersection of two quadrics. Indeed, as in the previous case the 3-spaces $\pi_A$, $\pi_B$ and $\pi_C$ are two by two in general position, so we can assume that $\pi_A$ is defined by $x_0 = x_1 = 0$, $\pi_B$ by $x_2 = x_3 = 0$ and $\pi_C$ by $x_4 = x_5 = 0$. The matrix $M$ takes the form

$$
\begin{pmatrix}
 x_1 & -x_0 & 0 & 0 & 0 & 0 \\
 0 & 0 & x_3 & -x_2 & 0 & 0 \\
 0 & 0 & 0 & 0 & x_5 & -x_4 \\
 L_0 & \ldots & \ldots & \ldots & L_5
\end{pmatrix}.
$$

It is easy to see that the residual of the 3 spaces $\pi_A$, $\pi_B$ and $\pi_C$ is defined by the equations $x_0L_0 + x_1L_1 = x_4L_4 + x_5L_5 = 0$, or equivalently by $x_2L_2 + x_3L_3 = x_5L_5 + x_4L_4 = 0$. $V$ is therefore a Del Pezzo threefold. Again, if the other generators of $\Delta$ are general, then $S$ is smooth outside $A$, $B$ and $C$. Also $V$ results to be smooth. The intersections of $V$ with the spaces $\pi_A$, $\pi_B$ and $\pi_C$ are quadric surfaces intersecting two by two along lines. The lines of $B$ are the lines meeting simultaneously $\pi_A$, $\pi_B$, $\pi_C$ and $V$.

5.1.4. Four singular points. We consider finally the case in which $S$ has also a fourth singular point $D$ on $\mathbb{G}(3,5)$. Then $F$ is of the form $F = \pi_A \cup \pi_B \cup \pi_C \cup \pi_D \cup W$, where $W$ is a rational normal cubic scroll.

It is clear that the four spaces $\pi_A$, $\pi_B$, $\pi_C$ and $\pi_D$ are all components of the focal locus $F$, and that the lines of $B$ are characterized by the property of meeting simultaneously all of them. The component $W$, that must exist by degree reasons, arises in the following way. Let $H$ be a hyperplane containing $\pi_A$, it intersects the other three spaces along planes, call them $\alpha_B$, $\alpha_C$ and $\alpha_D$, and $B_H$, the restriction of $B$ to $H$, is formed by the lines meeting them. There is a uniquely determined fourth plane meeting $\alpha_B$, $\alpha_C$ and $\alpha_D$ along lines, and it is necessarily contained in the focal locus of $B_H$ (it is a “parasitic plane”, see [DP], because its lines intersect all the three planes $\alpha_B$, $\alpha_C$ and $\alpha_D$ but a general line of $B_H$ does not meet it). If we let $H$ vary, we get a 1-dimensional family of planes, whose union is $W$.

5.1.5. Special degenerate cases. These 4 examples don’t exhaust the list of possibilities for the surfaces $S$ meeting $\mathbb{G}(3,5)$. For instance the space $\Delta$ could be contained in the tangent space to $\mathbb{G}(3,5)$ at one of the points of intersection, or intersect it along a plane or a line. In this situation four, three or two of the points of intersection of $\Delta$ with $\mathbb{G}(3,5)$ get identified. Hence some component of the focal locus appears with multiplicity greater than one and the 4 foci on a general line of $B$ are not distinct. A particular case is when $S$ is a cone of vertex a point $A$ of $\mathbb{G}(3,5)$, then the 3-space corresponding to the vertex counts twice as component of $F$.

5.2. $S$ with a double line not meeting $\mathbb{G}(3,5)$. Assume that $S$ is reducible and disjoint from $\mathbb{G}(3,5)$, therefore of the form $S = \pi \cup Q$ where $\pi$ is a plane and $Q$ a (possibly reducible) quadric.

In [MM04] the linear spaces contained in $\mathbb{G}(1,5)$ and not meeting $\mathbb{G}(3,5)$ are classified up to the natural action of $\text{PGL}_6$. They can be interpreted also as linear spaces of skew-symmetric matrices of constant rank 4.

It results that there are two orbits of lines, an open irreducible orbit of dimension 22, and a codimension one closed orbit. Note that the Gauss map $\gamma$, if restricted to such a line $\ell$, is regular and defined by the derivatives of the cubic Pfaffian polynomial (the equation of $\mathbb{G}(1,5)$), which have degree 2. Therefore $\gamma(\ell)$ is embedded in $\mathbb{P}^{14}$ as a conic. The lines parametrised by this conic represent a ruling of a smooth quadric or the the lines of a quadric cone, respectively for the two orbits.
The planes are distributed in 4 orbits, all of dimension 26. The Gauss image of such a plane $\pi$ is embedded in $\mathbb{P}^{14}$ as a Veronese surface $V$. The four orbits correspond precisely to the possible double Veronese embeddings of the plane in $\mathbb{G}(1,5)$. The lines represented by $V$ have the following geometrical interpretation, in the four cases (see [SU04]):

1. the secant lines of a skew cubic curve $C$ embedded in a 3-space $L$;
2. the lines contained in a quadric 3-fold $\Gamma$ and meeting a fixed line $r$ contained in $\Gamma$;
3. the lines joining the corresponding points in a fixed isomorphism between two disjoint planes;
4. the lines of a cone $C(V)$ over a projected Veronese surface.

If $\tilde{\Delta}$ is generated by one of these four types of planes plus one general point of $\tilde{\mathbb{P}}^{14}$, then the residual component of $\pi$ in $S$ is a quadric surface of maximal rank.

We describe now the corresponding congruences and their focal loci.

5.2.1. Case (1). $F = L' \cup X$, where $L'$ is a scheme whose support is $L$, having $C$ as embedded component, and $X$ is a singular threefold of degree 6 with Hilbert polynomial $P_X(t) = t^3 + 3t^2 + 2$. $X$ meets $L$ along a quartic surface with $C$ as singular locus. Through a general point of $L$ there passes a 1-dimensional family of lines, through a general point of $C$ a 2-dimensional family. Hence $r$ is a parasitic scheme. $X$ meets $\Gamma$ in a quartic surface which is singular along $r$. It follows that $X$ is an example of (singular) threefold containing a 3-dimensional family of plane cubic curves (see [MP96]).

An explicit example of this type of congruence is given by the following skew-symmetric matrix, which represents $\tilde{\Delta}$:

$$
\begin{pmatrix}
0 & -d & a & b & c & 0 \\
. & 0 & 0 & a & b & c \\
. & . & 0 & d & 0 & d \\
. & . & . & 0 & 0 & 0 \\
. & . & . & 0 & -d \\
. & . & . & . & . & 0
\end{pmatrix}
$$

5.2.2. Case (2). $F = \Gamma' \cup Y$, where $\Gamma'$ is a scheme whose support is $\Gamma$, having $r$ as embedded component, and $Y$ is a quintic threefold with Hilbert polynomial $P_Y(t) = 5/6t^3 + 5/2t^2 + 5/3t + 1$. Through a general point in $\Gamma$ there passes a 1-dimensional family of lines, through a general point in $r$ a 2-dimensional family. Hence $r$ is a parasitic scheme. $Y$ meets $\Gamma$ in a quartic surface which is singular along $r$.

An explicit example is provided by the matrix:

$$
\begin{pmatrix}
0 & a & b & c & d & 0 \\
. & 0 & 0 & d & c & b \\
. & . & 0 & d & 0 & 0 \\
. & . & 0 & 0 & d \\
. & . & . & 0 & a \\
. & . & . & . & . & 0
\end{pmatrix}
$$

5.2.3. Case (3). $F = Z_1 \cup Z_2$, where $Z_1$ is a rational cubic scroll, i.e. $\mathbb{P}^1 \times \mathbb{P}^2$, which is the union of the lines parametrised by $\Delta$, and $Z_2$ is a singular Del Pezzo threefold, complete intersection of two quadrics. $Z_1 \cap Z_2$ is a quartic surface, rational normal scroll of type $(2,2)$, which cuts a conic on each plane of $Z_1$. The singular locus of $Z_2$ is the union of two conics. The foci of a general line lie two on each component of $F$, so $B$ is formed by the lines which are secant both $Z_1$ and $Z_2$. 

An example of this congruence is the following:

\[
\begin{pmatrix}
0 & a & b & d & 0 & 0 \\
0 & c & d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

5.2.4. Case (4). \( F = C(V) \cup T \), where \( T \) is a cubic threefold contained in a hyperplane of \( \mathbb{P}^5 \) passing through the vertex of the cone. The lines of \( \mathcal{B} \) are the trisecants of \( F = C(V) \), meeting also \( T \).

An example of this congruence is the following:

\[
\begin{pmatrix}
0 & a & b & c & d \\
0 & c & d & a & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

5.2.5. Special degenerate cases. In all the four cases, for special choices of the 4th generator of \( \Delta \), the rank of the quadric \( Q \) can decrease, giving raise to various degenerations of the focal loci of the just considered congruences.

5.3. Classification of linear congruences in \( \mathbb{P}^5 \). Table II summarizes the sketch of classification of linear congruences of lines in \( \mathbb{P}^5 \) with 3-dimensional focal locus.

Note that the smooth Palatini threefolds, which are scroll over cubic surfaces, have a non-trivial moduli space. Indeed the moduli space of cubic surfaces is rational of dimension 4, and the moduli space of rank 2 vector bundles \( E \) on a cubic surface \( S \), such that \( \mathbb{P}(E) \) is a Palatini scroll, has dimension 5 (see [FM02]).

We finish this article with a few words about the singular Palatini scrolls and the corresponding congruences. Actually, the singular Palatini scrolls which come out from a surface \( S \) with (isolated) double points \( P_i \) (such that in fact \( \Delta \) is tangent to \((\mathbb{G}(1,5) \setminus \mathbb{G}(3,5)) at P_i\)) have only isolated singularities and can be described as the smooth ones. Therefore also the congruences associated to these can be described as in the case of the smooth Palatini scroll.

Let us pass now to the two more problematic Palatini scrolls, i.e. the ones which come from an \( S \) that is either a cubic cone or a ruled surface, singular along a line. In both cases, \( S \) has a 1-dimensional family of lines, and these lines correspond to quadric surfaces in \( \mathbb{P}^5 \).

A 3-space \( \tilde{\Delta} \) such that \( \tilde{\Delta} \cap \tilde{\mathbb{G}}(1,5) \) is a cubic cone of vertex \( A \) must be contained in the quadric \( \Pi_A \), the second fundamental form of \( \tilde{\mathbb{G}}(1,5) \) at \( A \), and intersect the vertex of \( \Pi_A \) only at \( A \). An example is the following:

\[
\begin{pmatrix}
0 & c & a & d - c & b & 0 \\
0 & d & a & 0 & b & 0 \\
0 & 0 & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The Palatini threefolds obtained in this way result to be singular along the line corresponding to \( A \). This Palatini scroll can indeed be constructed as the smooth one, and also the congruence does not give problems.
| Surface $S$ | Sing $S$ | Focal locus $F$ | Remarks |
|-------------|-----------|-----------------|---------|
| smooth      |           | smooth Palatini scroll | $F$ has moduli |
| irreducible with isolated singularities $P_1, \ldots, P_k$, $k \leq 4$ | $P_i \notin G(3,5)$ | singular Palatini scroll | $F$ can be singular at a point or along a line |
|             | $P_i \in G(3,5)$ | the 3-space $\pi_{P_i}$ is an irreducible component of $F$ | the residual intersects $\pi_{P_i}$ along a smooth quadric |
| irreducible ruled with a double line $r$ | $r \cap G(3,5) = \emptyset$ | singular Palatini scroll | $F$ is singular along a quadric cone $\hat{\Delta}$ is tangent to $\hat{G}(1,5)$ along $r$ |
| reducible $\pi \cup Q$, $\pi$ a conic in $\pi$, disjoint from $G(3,5)$ | $L' \cup X$, deg $X = 6$ | $L'$ is a parasitic component |
|             | $\Gamma' \cup Y$, deg $Y = 5$ | $\Gamma'$ has a parasitic line as embedded component |
|             | $Z_1 \cup Z_2$, deg $Z_2 = 4$, $Z_1 = \mathbb{P}^1 \times \mathbb{P}^2$ | the lines of $B$ are secant both $Z_1$ and $Z_2$ |
|             | $C(V) \cup T$, deg $T = 3$ | the lines of $B$ are the trisecants $C(V)$ meeting $T$ |

**Table 1.**

To obtain a ruled cubic $S$ with double line $\ell$, disjoint from $G(3,5)$, $\hat{\Delta}$ must be contained in the intersection of the tangent spaces to $\hat{G}(1,5)$ at $P$, for all $P \in r$. One checks that such a 3-space exists only if $\ell$ is a line of the closed orbit (see Subsection 5.2), whereas in the other case no 3-space is contained in the required intersection. The Palatini scroll constructed in such a way is singular along the quadric cone $Q$ which correspond to $\ell$. Now, if we consider the secant lines to $F$ which meet also $Q$ form—by dimensional reasons—a congruence. Therefore this is the congruence which we are looking for, since on a general line of it we have two distinct foci plus a double focus.

Note that this is only a sketch of classification. The task of writing a complete classification of linear congruences up to isomorphism seems to us out of reach, because of the multitude of special cases, due to the moduli of cubic surfaces (see for instance [BL98]) and to their several different embeddings in $\hat{G}(1,5)$.

In a forthcoming paper we plan to apply these results to the classification of Temple systems in 4 variables. We have to point out that this classification holds on an algebraically closed field, so it will be necessary to refine it over the real field.
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