ASYMPTOTIC STABILITY
FOR A CLASS OF MARKOV SEMIGROUPS

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Abstract. Let $U \subset K$ be an open and dense subset of a compact metric space and let $\{\Phi_t\}_{t \geq 0}$ be a Markov semigroup on the space of bounded Borel measurable functions on $U$ with the strong Feller property. Suppose that for each $x \in \partial U$ there exists a barrier $h \in C(K)$ at $x$ such that $\Phi_t(h) \geq h$ for all $t \geq 0$. Suppose also that every real-valued $g \in C(K)$ with $\Phi_t(g) \geq g$ for all $t \geq 0$ and which attains its global maximum at a point inside $U$ is constant. Then for each $f \in C(K)$ there exists the uniform limit $F = \lim_{t \to \infty} \Phi_t(f)$. Moreover $F$ is continuous on $K$, agrees with $f$ on $\partial U$ and $\Phi_t(F) = F$ for all $t \geq 0$.

Let $E$ be a locally compact Hausdorff space and let $M_b(E)$ be the space of all complex-valued, bounded and Borel measurable functions on $E$. Let also $C_b(E)$ be the set of all continuous functions in $M_b(E)$.

A linear map $\Phi : M_b(E) \to M_b(E)$ is called a Markov operator if for each $x \in E$ there exists a probability Borel measure $\mu_x$ on $E$ such that

$$\Phi(f)(x) = \int f \, d\mu_x \quad \forall f \in M_b(E).$$

A Markov operator $\Phi$ is said to have the strong Feller property if $\Phi(f) \in C_b(E)$ for every $f \in M_b(E)$. The main result of this paper is the following:

**Theorem 1.** Let $K$ be a compact metric space and let $U \subset K$ be a dense open subset such that $\partial U$ contains at least two points. Let $C(K)$ be the space of all continuous complex-valued functions on $K$ and let $C(\partial U)$ be the corresponding space for $\partial U$.

Let $\Phi : M_b(U) \to M_b(U)$ be a Markov operator with the strong Feller property. Suppose that $\Phi$ satisfies the following conditions:

(A) For each point $x \in \partial U$ there exists $h \in C(K)$ such that $h(x) = 0$, $h(y) < 0$ for all $y \in K \setminus \{x\}$ and $\Phi(h_U) \geq h_U$ on $U$, where $h_U$ is the restriction of $h$ to $U$;

(B) If $g \in C(K)$ is a real valued function with $\Phi(g_U) \geq g_U$ and if there exists $z \in U$ such that $g(z) = \max\{g(x) : x \in K\}$ then $g$ is constant on $K$.

Then, for each $f \in C(\partial U)$ there exists a unique function $G \in C(K)$ such that $\Phi(G_U) = G_U$ and $G(x) = f(x)$ for all $x \in \partial U$. Moreover, if $F \in C(K)$ is an arbitrary continuous extension of $f$ to $K$ then the sequence $\{\Phi^n(F_U)\}_n$ converges uniformly on $U$ to $G_U$.

**Proof.** (1) First of all, it can be proved, exactly as in Proposition 1.3 from [1], that the boundary condition (A) implies the following. For each $f \in C(K)$ and for each
where \( i \) belongs to \( C \) generated by all the functions of the form \( \pi \) and let \( K \) be the norm closed subalgebra of \( C[0,1] \). By what we have already proved the first summand belongs to \( C \) and the second belongs, by the induction hypothesis, to \( C \). This shows that \( g \in C(K) \) and Dini's theorem shows that the convergence is uniform on \( K \).

(3) Let
\[
\mathcal{T}(\Psi) = \{ h \in C(K) : \Psi(h) = h \}
\]
and let \( A \) be the norm closed subalgebra of \( C(K) \) generated by \( \mathcal{T}(\Psi) \). Let \( C(\Psi) \) be the set of all \( f \in C(K) \) for which the sequence \( \{ \Psi^n(f) \} \) is uniformly convergent on \( K \) and denote \( \pi(f) \) its limit. Let also
\[
C(\Psi)_0 = \{ f \in C(\Psi) : \pi(f) = 0 \}.
\]
We will show that \( A \subset C(\Psi) \).

Let \( h \in \mathcal{T}(\Psi) \). Then \( |h|^2 \geq |h|^2 \) therefore (2) shows that \( |h|^2 \in C(\Psi) \) and also that \( \pi(|h|^2) - |h|^2 \geq 0 \). This also shows that the norm closed ideal of \( C(K) \) generated by all the functions of the form \( \pi(|h|^2) - |h|^2 \) with \( h \in \mathcal{T}(\Psi) \) is contained in \( C(\Psi)_0 \). Indeed it is easy to see that if \( f \in C(\Psi)_0 \) then \( f g \in C(\Psi)_0 \) for every \( g \in C(K) \).

We shall now prove that for any finite set of \( k \) functions from \( \mathcal{T}(\Psi) \) their product belongs to \( C(\Psi) \). Let \( k = 2 \). If \( h_1, h_2 \in \mathcal{T}(\Psi) \) then
\[
h_1h_2 = (1/4) \sum_{m=0}^{3} i^m |g_m|^2
\]
where \( i = \sqrt{-1} \) and \( g_m = (h_1 + i^m h_2) \). Since \( g_m \in \mathcal{T}(\Psi) \) we see that \( h_1h_2 \in C(\Psi) \).

Let \( k \geq 3 \) and assume that every product of at most \( k - 1 \) elements from \( \mathcal{T}(\Psi) \) belongs to \( C(\Psi) \). Let \( h_1, \ldots, h_k \) in \( \mathcal{T}(\Psi) \) and let \( g = h_1 \cdots h_k \). Then
\[
g = (h_1h_2 - \pi(h_1h_2)) \cdot h_3 \cdots h_k + \pi(h_1h_2) \cdot h_3 \cdots h_k.
\]
By what we have already proved the first summand belongs to \( C(\Psi)_0 \) and the second belongs, by the induction hypothesis, to \( C(\Psi) \). This shows that \( A \subset C(\Psi) \).

(4) Consider the map
\[
\rho : A \to C(\partial U)
\]
that takes any \( f \in A \) into its restriction to \( \partial U \). It turns out that \( \rho \) is onto. To see this, we first observe that the boundary condition (A) together with (2) implies that for each \( x \in \partial U \) there exists \( g \in C(K) \) such that \( g(x) = 0 \), \( g(y) < 0 \) for every \( y \in \partial U - \{x\} \) and \( \Phi(gu) = gu \). In particular \( g \in \mathcal{T}(\Psi) \). This shows that the range
of ρ separates the points of ∂U therefore ρ is onto. This proves the existence part of the theorem. Uniqueness follows easily from (B).

(5) We shall denote

\[ \theta : C(K) \to C(K) \]

the map which takes a function \( f \in C(K) \) into the uniquely determined function in \( T(\Psi) \) which agrees with \( f \) on \( \partial U \). It follows that for each \( f \in C(\Psi) \) we have \( \theta(f) = \pi(f) \). In particular, \( \theta(f) = f \) for every \( f \in T(\Psi) \).

Let

\[ L = \{ g \in C(K) : g = \pi(|h|^2) - |h|^2 \text{ for some } h \in T(\Psi) \}. \]

If \( g \in L \), then it follows from (3) that \( g \geq 0 \). Moreover \( \Psi(g) \leq g \) hence \( \Phi(gu) \leq gu \).

Suppose now that there exists a point \( z_0 \in U \) such that \( g(z_0) = 0 \) for every \( g \in L \). It then follows from (B) that \( g = 0 \) on \( U \) hence on \( K \) for every \( g \in L \). This easily implies that \( T(\Psi) \) is closed under multiplication hence it equals \( A \). Indeed if \( h_1 \) and \( h_2 \) are functions in \( T(\Psi) \) then \( \pi(h_1 h_2) - h_1 h_2 \) can be written as a linear combination of elements from \( L \) (see step 3). It then follows that \( \pi(h_1 h_2) = h_1 h_2 \) therefore \( T(\Psi) = A \).

This shows that the map \( \theta : C(K) \to C(K) \) defined above is multiplicative on \( C(K) \) and its range equals \( A \). Let \( M \) be the maximal ideal space of \( A \) and for each \( h \in A \) let \( \hat{h} \in C(M) \) be its Gelfand transform. Since \( A \) is self-adjoint, the Gelfand transform is an isometric isomorphism from \( A \) onto \( C(M) \) (the Banach algebra of all continuous, complex-valued functions on \( M \)). It then follows that there exists a continuous one-to-one map \( \gamma : M \to K \) such that \( \theta(f) = f \circ \gamma \) for every \( f \in C(K) \).

Moreover, since \( A \subset C(K) \) there exists a continuous surjective map \( \lambda : K \to M \) such that \( h = \hat{h} \circ \lambda \) for every \( h \in A \).

We now claim that \( \gamma(M) \subset \partial U \). Suppose there exists \( \alpha \in M \) such that \( z = \gamma(\alpha) \in U \). Then there exists a non-constant real valued function \( h \in A \) such that

\[ \hat{h}(\alpha) = \sup\{ h(x) : x \in K \}. \]

Every such \( h \) attains its maximum only on \( \partial U \). However

\[ \hat{h}(\alpha) = h(\gamma(\alpha)) = h(z) \]

This shows that \( \gamma(M) \subset \partial U \).

Let \( z \in U \) and let \( x = \gamma \circ \lambda(z) \). Then \( x \in \partial U \). Let \( h \in T(\Psi) \) which attains its maximum on \( K \) only at this point. Then

\[ h(z) = (h \circ \gamma \circ \lambda)(z) = (\hat{h} \circ \lambda)(z) = h(z). \]

We get a contradiction. This means that there is no \( z_0 \in U \) such that \( g(z_0) = 0 \) for all \( g \in L \).

(6) It follows from (5) that

\[ \partial U = \{ x \in K : g(x) = 0 \text{ for every } g \in L \}. \]

It then follows that the closed ideal of \( C(K) \) generated by \( L \) is precisely the ideal of all \( f \in C(K) \) which vanish identically on all points of \( \partial U \). Recall now that we already proved in part 3 that for any \( g \) in the closed ideal generated by \( L \) the sequence \( \{ \Psi^n(g) \} \) converges uniformly to 0. In particular, this holds true for all functions \( g \) of the form \( g = f - \theta(f) \) with \( f \in C(K) \). This completes the proof of this theorem.

□
This proof works as well for Markov semigroups with continuous parameter. Examples of Markov operators on complex domains which satisfy all the conditions in Theorem 1 are given in [1]. As a matter of fact, the results and the methods used in [1] strongly motivated and inspired our research. We have also used some ideas appearing in [3]. Theorem 1 can be used to give an alternate proof for the main result in [2].

REFERENCES

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