Pointwise Convergence for sequences of Schrödinger means in $\mathbb{R}^2$

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Abstract

We consider pointwise convergence of Schrödinger means $e^{it_n \Delta} f(x)$ for $f \in H^s(\mathbb{R}^2)$ and decreasing sequences $\{t_n\}_{n=1}^\infty$ converging to zero. The main theorem improves the previous results of [Sjölin, JFAA, 2018] and [Sjölin-Strömberg, JMAA, 2020] in $\mathbb{R}^2$. This study is based on investigating properties of Schrödinger type maximal functions related to hypersurfaces with vanishing Gaussian curvature.

1 Introduction

The solution to the Schrödinger equation

\[ iu_t - \Delta u = 0, (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \quad (1.1) \]

with initial datum $u(x, 0) = f$, is formally written as

\[ e^{it \Delta} f(x) := \int_{\mathbb{R}^N} e^{i(x \cdot \xi + t|\xi|^2)} \hat{f}(\xi) \, d\xi. \]

The problem about finding optimal $s$ for which

\[ \lim_{t \to 0^+} e^{it \Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^N, \quad (1.2) \]

whenever $f \in H^s(\mathbb{R}^N)$, was first considered by Carleson [4], and extensively studied by Sjölin [20] and Vega [23], who proved independently the convergence for $s > 1/2$ in all dimensions. Dahlberg-Kenig [8] showed that the convergence does not hold for $s < 1/4$ in any dimension. In 2016, Bourgain [3] gave counterexample showing that convergence can fail if $s < \frac{N}{2(N+1)}$.

Very recently, Du-Guth-Li [9] and Du-Zhang [11] obtained the sharp results by the polynomial partitioning and decoupling method.

One of the natural generalizations of the pointwise convergence problem is to ask a.e. convergence of the Schrödinger means where the limit is taken over decreasing sequences $\{t_n\}_{n=1}^\infty$. This work is supported by the National Natural Science Foundation of China (No.11871452); Natural Natural Science Foundation of China (No.11701452); China Postdoctoral Science Foundation (No.2017M613193); Natural Science Basic Research Plan in Shaanxi Province of China (No.2017JQ1009).

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converging to zero. That is to investigate relationship between optimal $s$ and properties of \{\(t_n\)\}_{n=1}^\infty such that for each function \(f \in H^s(\mathbb{R}^N)\),
\[
\lim_{n \to \infty} e^{it_n \Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^N.
\] (1.3)

This problem was first considered by Sjölin [21] in general dimensions and later improved by Sjölin-Strömberg [22]. More recently, Dimou-Seeger [7] obtained a sharp characterization of this problem in the one-dimensional case. But in higher dimensional case, the sharp characterization of this problem still remains open.

In order to characterize the convergence of \(\{t_n\}^\infty_{n=1}\), the Lorentz space \(\ell^{r,\infty}(\mathbb{N})\), \(r > 0\) is involved. The sequence \(\{t_n\}^\infty_{n=1} \in \ell^{r,\infty}(\mathbb{N})\) if and only if
\[
\sup_{b > 0} \sup_{n \in \mathbb{N}} \left\{ n : t_n > b \right\} < \infty. \tag{1.4}
\]

Dimou-Seeger [7] proved that when \(N = 1\), \(s \geq \min\left\{ \frac{r}{1+r}, \frac{1}{3} \right\}\) is sufficient for \((1.3)\) to hold. For \(N > 1\), it follows from [21] that \(s > \min\{r, \frac{2N}{N+1}\}\) is sufficient. This was later improved by [22] where \(s > \min\left\{ \frac{r}{1+r}, \frac{N}{2(N+1)} \right\}\) is shown to be enough for pointwise convergence. The main theorem of this paper improves the previous results of [21] and [22] in dimension two.

By standard arguments, in order to obtain the convergence result, it is sufficient to show the maximal function estimate in \(\mathbb{R}^2\). Our main results are as follows.

**Theorem 1.1.** Given a decreasing sequence \(\{t_n\}^\infty_{n=1} \in \ell^{r,\infty}(\mathbb{N})\) converging to zero and \(\{t_n\}^\infty_{n=1} \subset (0,1)\), then for any \(s > s_0 = \min\{\frac{r}{1+r}, \frac{1}{3}\}\), we have
\[
\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f| \right\|_{L^2(B(0,1))} \leq C \|f\|_{H^s(\mathbb{R}^2)}, \tag{1.5}
\]
whenever \(f \in H^s(\mathbb{R}^2)\), where the constant \(C\) does not depend on \(f\).

By translating invariance in the \(x\)–variables, \(B(0,1)\) in Theorem 1.1 can be replaced by any ball of radius 1 in \(\mathbb{R}^2\). Then we obtain the following convergence result.

**Theorem 1.2.** Given a decreasing sequence \(\{t_n\}^\infty_{n=1} \in \ell^{r,\infty}(\mathbb{N})\) converging to zero and \(\{t_n\}^\infty_{n=1} \subset (0,1)\), then for any \(s > s_0 = \min\{\frac{r}{1+r}, \frac{1}{3}\}\), \((1.3)\) holds whenever \(f \in H^s(\mathbb{R}^2)\).

Theorem 1.2 improves the previous results of [21] and [22] when \(N = 2\) since \(\frac{r}{1+r} < \frac{r}{1+r} < \frac{r}{1+r}\) for \(r > 0\). We pose two examples for \(\{t_n\}^\infty_{n=1}\). It is not hard to check that \(\{t_n\}^\infty_{n=1} \in \ell^{r,\infty}(\mathbb{N})\) if we take (E1): \(t_n = \frac{1}{n^r}, n \geq 1\). It is obvious that when \(r < \frac{3}{5}\), there is a gain over the general pointwise convergence result for \(s > 1/3\) in \(\mathbb{R}^2\). Another example is the lacunary sequence (E2): \(t_n = 2^{-n}, n \geq 1\). For this example, it is worth to mention that \(\{t_n\}^\infty_{n=1} \in \ell^{r,\infty}(\mathbb{N})\) for each \(r > 0\). Therefore, inequality \((1.3)\) holds whenever \(f \in H^s(\mathbb{R}^2)\) for any \(s > 0\).

We briefly sketch the proof of Theorem 1.1, and leave the details to Section 2. Notice that when \(\frac{r}{1+r} \geq \frac{1}{3}\), Theorem 1.1 follows from Hölder’s inequality and the theorem below.

**Theorem 1.3.** ([9]) For any \(s > 1/3\), the following bound holds: for any function \(f \in H^s(\mathbb{R}^2)\),
\[
\left\| \sup_{0 < t < 1} |e^{it \Delta} f(x)| \right\|_{L^3(B(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^2)}.
\]
Therefore, we only need to consider the case when \( \frac{r}{4r+1} < \frac{1}{3} \).

By Littlewood-Paley decomposition, it is not hard to deal with the low frequency parts by standard argument. So next we just concentrate ourselves on the case when \( \text{supp} \hat{f} \subset \{ \xi : |\xi| \sim 2^k \} \), \( k \gg 1 \). We decompose \( \{t_n\}_{n=1}^{\infty} \) as

\[
A_k^1 := \left\{ t_n : t_n \geq 2^{-\frac{2k}{3r+1}} \right\}
\]

and

\[
A_k^2 := \left\{ t_n : t_n < 2^{-\frac{2k}{3r+1}} \right\}.
\]

Then we consider the maximal function

\[
\sup_{n \in \mathbb{N}, t_n \in A_k^1} |e^{it_n \Delta} f|
\]

and

\[
\sup_{n \in \mathbb{N}, t_n \in A_k^2} |e^{it_n \Delta} f|,
\]

respectively. We deal with the first term by the assumption that the decreasing sequence \( \{t_n\}_{n=1}^{\infty} \in \ell^r(\mathbb{N}) \) and Plancherel theorem. For the second term, since we only need to consider the case when \( \frac{r}{4r+1} < \frac{1}{3} \), we may assume that \( 0 < r < \frac{3}{5} \), which yields \( A_k^2 \subset (0, 2^{-\frac{2k}{3r+1}}) \) for \( k < 2k < 2k \). Hence we reduce our proof to the following theorem.

**Theorem 1.4.** If \( \text{supp} \hat{f} \subset \{ \xi : |\xi| \sim 2^k \} \), then for any small constant \( \epsilon > 0 \) and interval \( I = (0, 2^{-j}) \), where

\[ k < j < 2k, \quad j \in \mathbb{R}, \]

we have

\[
\left\| \sup_{t \in I} |e^{it \Delta} f(x)| \right\|_{L^2(B(0,1))} \leq C 2^{(k-\frac{4}{3})\frac{j}{2} + \epsilon k} \|f\|_{L^2(\mathbb{R}^2)},
\]

(1.6)

where the constant \( C \) does not depend on \( f \).

We will prove Theorem 1.4 in Section 3. In one-dimensional case, similar result was built in [7] by TT* and stationary phase method. But their method seems not to work well in the higher dimensional case. If \( j \geq 2k \) in Theorem 1.4, then the length of \( I \) is small enough so that we can apply Sobolev’s embedding to obtain

\[
\left\| \sup_{t \in I} |e^{it \Delta} f(x)| \right\|_{L^2(B(0,1))} \leq C \|f\|_{L^2(\mathbb{R}^2)}.
\]

(1.7)

When \( j = k \), Lee [13] applied the bilinear method to show that

\[
\left\| \sup_{t \in I} |e^{it \Delta} f(x)| \right\|_{L^2(B(0,1))} \leq C 2^{\frac{4k}{3} + \epsilon k} \|f\|_{L^2(\mathbb{R}^2)}.
\]

(1.8)
In our case, the exponent $2^{(k - \frac{j}{2}) + \epsilon}k$ is in the middle of 1 and $2^{\frac{j}{2}}k + \epsilon k$ when $k < j < 2k$. However, inequality (1.8) was improved by [9] and [11] using Broad-Narrow argument and polynomial partitioning. It was proved that if $j = k$, then

$$\left\| \sup_{t \in I} |e^{it\Delta} f(x)| \right\|_{L^2(B(0,1))} \leq C2^{2^{\frac{j}{2}}k + \epsilon k}\| f \|_{L^2(\mathbb{R}^2)}.$$ (1.9)

In view of technical difficulties, we could not improve our result to this exponent. Moreover, notice that (1.9) also holds for $j < k$ due to the localizing lemma in Lee-Rogers [14].

In our case $k < j < 2k$, by scaling, we actually have to treat the maximal function defined by

$$\sup_{t \in (0,1)} \left| \int_{\mathbb{R}^2} e^{i2^k(x \cdot \eta + 2^{k-j}t|\eta|^2)} \hat{g}(\eta) d\eta \right|,$$

where $\hat{g}$ is supported in the annular $\{ \eta : |\eta| \sim 1 \}$. It is obvious that the Gaussian curvature of the hypersurface $(\eta, 2^{k-j}|\eta|^2)$ vanishes as $k$ tends to infinity. Therefore, it seems difficult to apply the classical analysis of local smoothing estimate or Broad-Narrow argument directly. Inspired by [15], in which the authors established $L^p \rightarrow L^q$ estimates for Fourier integral operators related to hypersurfaces with vanishing Gaussian curvature, we try to use Whitney type decomposition and a bilinear result from [13] to prove Theorem 1.4. Theorem 1.5 below follows from Proposition 3.1 in [13] and rescaling.

**Theorem 1.5.** ([13]) Let $\lambda \gg 1, l : 2^l \ll \lambda^\frac{1}{4}$. If $\hat{g}_1, \hat{g}_2$ are supported in $B(\eta_0, 2^{-l}) \subset \{ \eta : |\eta| \sim 1 \}$ and $\text{dist}(\text{supp} \ \hat{g}_1, \ \text{supp} \ \hat{g}_2) \sim 2^{-l}$, then for any $\epsilon > 0$, we have

$$\left\| \sup_{t \in (0,1)} |T_\lambda g_1 T_\lambda g_2| \right\|_{L^2(B(0,1))} \leq C2^{-2\frac{l}{2}}\lambda^{4\frac{1}{4}} \lambda^{-2} \| g_1 \|_{L^2} \| g_2 \|_{L^2},$$ (1.10)

where for $i = 1, 2$,

$$T_\lambda g_i(x, t) = \int_{\mathbb{R}^2} e^{i\lambda(x \cdot \eta + t|\eta|^2)} \hat{g}_i(\eta) d\eta.$$ 

**Conventions:** Throughout this article, we shall use the well known notation $A \gg B$, which means if there is a sufficiently large constant $G$, which does not depend on the relevant parameters arising in the context in which the quantities $A$ and $B$ appear, such that $A \geq GB$. We write $A \sim B$, and mean that $A$ and $B$ are comparable. By $A \leq B$ we mean that $A \leq CB$ for some constant $C$ independent of the parameters related to $A$ and $B$.

# 2 Proof of Theorem 1.1

**Proof of Theorem 1.1.** Set

$$s_1 = \frac{r}{2^p + 1} + \epsilon$$

for some sufficiently small constant $\epsilon > 0$. We decompose $f$ as

$$f = \sum_{k=0}^{\infty} f_k,$$
where supp$f_0 \subset B(0, 1)$, supp$f_k \subset \{ \xi : |\xi| \sim 2^k \}, k \geq 1$. Then we have

\[
\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f| \right\|_{L^2(B(0,1))} \leq \sum_{k=0}^{\infty} \left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))}.
\] (2.1)

For $k \lesssim 1$, since for each $x \in B(0,1)$,

\[
|e^{it_n \Delta} f_k(x)| \lesssim \|f_k\|_{L^2(\mathbb{R}^2)},
\]

it is obvious that

\[
\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))} \lesssim \|f\|_{H^1(\mathbb{R}^2)}.
\] (2.2)

For each $k \gg 1$, we decompose $\{t_n\}_{n=1}^\infty$ as

\[
A_1^k := \left\{ t_n : t_n \geq 2^{-\frac{2k}{3r+1}} \right\}
\]

and

\[
A_2^k := \left\{ t_n : t_n < 2^{-\frac{2k}{3r+1}} \right\}.
\]

Then we have

\[
\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))} \leq \left\| \sup_{n \in \mathbb{N} : t_n \in A_1^k} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))} + \left\| \sup_{n \in \mathbb{N} : t_n \in A_2^k} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))} = I + II.
\] (2.3)

We firstly estimate $I$. Since $\{t_n\}_{n=1}^\infty \in \ell^{r,\infty}(\mathbb{N})$ and assumption $r \in (0,3/5)$, we have

\[
\# A_1^k \leq C 2^{\frac{2k}{3r+1}},
\] (2.4)

which implies that

\[
I \leq \left( \sum_{n \in \mathbb{N} : t_n \in A_1^k} \left\| e^{it_n \Delta} f_k \right\|_{L^2(B(0,1))}^2 \right)^{1/2} \leq 2^{\frac{2k}{3r+1}} \|f_k\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-rk} \|f\|_{H^1(\mathbb{R}^2)}.
\] (2.5)

For $II$, since

\[
A_2^k \subset \left( 0, 2^{-\frac{2k}{3r+1}} \right).
\]
By previous discussion, we have $k < \frac{2k}{2r+1} < 2k$. Then it follows from Theorem 1.4 that,

$$II \lesssim 2^{\frac{k}{2r+1} + \frac{j}{l} + 1} \|f_k\|_{L^2(\mathbb{R}^2)} \leq 2^{- \frac{j}{2k}} \|f\|_{H^s(\mathbb{R}^2)}. \quad (2.6)$$

Inequalities (2.3), (2.5) and (2.6) yield for $k \gg 1$,

$$\left\| \sup_{m \in \mathbb{N}} |e^{it_n \Delta} f_k| \right\|_{L^2(B(0,1))} \lesssim 2^{- \frac{j}{2k}} \|f\|_{H^s(\mathbb{R}^2)}. \quad (2.7)$$

Combining (2.1), (2.2) and (2.7), inequality (1.5) holds true for $s_1$. By the arbitrariness of $\epsilon$, in fact, we can get for any $s > s_0$, inequality (1.5) remains true.

## 3 Proof of Theorem 1.4

**Proof of Theorem 1.4.** If $2k - k \epsilon \leq j < 2k$, then $I \subset (0, 2^{-2k + k \epsilon})$, which can be covered by $2^{k \epsilon}$ intervals of length $2^{-2k}$, from inequality (1.7), we have that

$$\left\| \sup_{t \in I} |e^{it \Delta} f(x)| \right\|_{L^2(B(0,1))} \leq 2^{k \epsilon} \|f\|_{L^2}. \quad (3.1)$$

Next, we always assume that $k < j < 2k - k \epsilon$. By Whitney type decomposition, we have

$$(e^{it \Delta} f)^2 = \sum_{l: 1 \leq 2^l \leq 2^{(2k - j)/4}} \sum_{m, m': \text{dist}(\text{supp } \hat{f}_m, \text{supp } \hat{f}_{m'}) \sim 2^{-k - l}} e^{it \Delta} f_m \cdot e^{it \Delta} f_{m'},$$

where $l \in \mathbb{N}$, supp $\hat{f}_m$ and supp $\hat{f}_{m'}$ are contained in cubes with side length $2^{k-l}$. When $2^l \sim 2^{(2k - j)/4}$, we abuse the notation that by saying dist(supp $\hat{f}_m$, supp $\hat{f}_{m'}$) $\sim 2^{-k-l}$ we mean dist(supp $\hat{f}_m$, supp $\hat{f}_{m'}$) $\sim 2^{-k-l}$. Then we have

$$\left\| \sup_{t \in I} |e^{it \Delta} f(x)| \right\|_{L^2(B(0,1))} \leq \left\{ \sum_{l: 1 \leq 2^l \leq 2^{(2k-j)/4}} \sum_{m, m': \text{dist}(\text{supp } \hat{f}_m, \text{supp } \hat{f}_{m'}) \sim 2^{-k-l}} \left\| \sup_{t \in I} |e^{it \Delta} f_m| |e^{it \Delta} f_{m'}| \right\|_{L^1(B(0,1))} \right\}^{1/2}. \quad (3.2)$$

It is sufficient to prove the following two lemmas.

**Lemma 3.1.** For each $l$: $2^l \ll 2^{(2k-j)/4}$, we have

$$\left\| \sup_{t \in I} |e^{it \Delta} f_m| |e^{it \Delta} f_{m'}| \right\|_{L^1(B(0,1))} \leq C2^{- \frac{4}{7} 2^{(2k-j)/4} + 2k} \|f_m\|_{L^2} \|f_{m'}\|_{L^2}.$$

**Lemma 3.2.** For each $l$: $2^l \sim 2^{(2k-j)/4}$, we have

$$\left\| \sup_{t \in I} |e^{it \Delta} f_m| |e^{it \Delta} f_{m'}| \right\|_{L^1(B(0,1))} \leq C2^{(2k-j)/4} \|f_m\|_{L^2} \|f_{m'}\|_{L^2}.$$
Indeed, if Lemma 3.1 and Lemma 3.2 hold true, then by the Cauchy-Schwartz inequality, we obtain
\[
\left\| \sup_{t \in I} |e^{it\Delta} f(x)| \right\|_{L^2(B(0,1))} 
\leq \left\{ \sum_{l:1 \leq 2^l \leq 2^{2k-j}/4} C2^{-l/2}2^{(2k-j)/4 + 2k} \sum_{m,m' \text{dist}(\text{supp } \phi^m, \text{supp } \phi^{m'}) \sim 2^k} \|f^m_l\|_{L^2} \|f^{m'}_{l'}\|_{L^2} \right\}^{1/2} 
\leq \left\{ \sum_{l:2^l \sim 2^{2k-j}/4} C2^{3(2k-j)/4} \sum_{m,m' \text{dist}(\text{supp } \phi^m, \text{supp } \phi^{m'}) \sim 2^k-l} \|f^m_l\|_{L^2} \|f^{m'}_{l'}\|_{L^2} \right\}^{1/2} 
\lesssim 2^{(k-\frac{1}{4}) + 2k} \|f\|_{L^2},
\] (3.3)
which yields Theorem 1.4.

Let’s turn to prove Lemma 3.1 and Lemma 3.2. Lemma 3.2 follows from Hölder’s inequality and the following two estimates
\[
\left\| \sup_{t \in I} |e^{it\Delta} f^l_m| \right\|_{L^2(B(0,1))} \leq C2^{(k-\frac{1}{4})} \|f^l_m\|_{L^2},
\] (3.4)
\[
\left\| \sup_{t \in I} |e^{it\Delta} f^l_{m'}| \right\|_{L^2(B(0,1))} \leq C2^{(k-\frac{1}{4})} \|f^l_{m'}\|_{L^2}.
\] (3.5)
We only prove inequality (3.4) since the proof of inequality (3.5) is similar. Without loss of generality, we may assume that \( \text{supp } \phi^m \subset B(\xi_0, 2^{k/4}) \), where \(|\xi_0| \lesssim 2^k \). By changes of variables,
\[
\xi = \zeta + \xi_0, \quad |\zeta| \leq 2^{k/4},
\]
we have
\[
\sup_{t \in I} |e^{it\Delta} f^l_m| = \sup_{t \in I} \left| \int_{\mathbb{R}^2} e^{i[x \cdot \zeta + t(|\zeta|^2 + 2\zeta \cdot \xi_0)]} \phi^m_l(\zeta + \xi_0) \, d\zeta \right|.
\]
It follows from Sobolev’s embedding and Plancherel theorem that
\[
\left\| \sup_{t \in I} |e^{it\Delta} f^l| \right\|_{L^2(B(0,1))} 
\leq \|f^l_m\|_{L^2} + \left\| \int_{\mathbb{R}^2} e^{i[x \cdot \zeta + t(|\zeta|^2 + 2\zeta \cdot \xi_0)]} \phi^m_l(\zeta + \xi_0) \, d\zeta \right\|_{L^2(B(0,1) \times I)}^{1/2} 
\times \left\| \int_{\mathbb{R}^2} e^{i[x \cdot \zeta + t(|\zeta|^2 + 2\zeta \cdot \xi_0)]} \left(|\zeta|^2 + 2\zeta \cdot \xi_0\right) \phi^m_l(\zeta + \xi_0) \, d\zeta \right\|_{L^2(B(0,1) \times I)}^{1/2} 
\leq \|f^l_m\|_{L^2} + 2^{-\frac{k}{4}} \|f^m_l(\zeta + \xi_0)\|_{L^2}^{1/2} \left\| \left(|\zeta|^2 + 2\zeta \cdot \xi_0\right) \phi^m_l(\zeta + \xi_0) \right\|_{L^2}^{1/2} 
\leq \|f^l_m\|_{L^2} + 2^{-\frac{k}{4}} 2^{\frac{3k}{4} + \frac{1}{4}} \|f^m_l(\zeta + \xi_0)\|_{L^2}^{1/2} \left\| \left(|\zeta|^2 + 2\zeta \cdot \xi_0\right) \phi_l^m(\zeta + \xi_0) \right\|_{L^2}^{1/2}
\]
Then we arrive at inequality (3.4).

We will prove Lemma 3.1 in the rest of this section. By rescaling, we turn to estimate

$$\sup_{t \in I} |e^{it\Delta} f^l_m(x)| |e^{it\Delta} f^l_{m'}(x)|$$

$$= \sup_{t \in I} \left| \int_{\mathbb{R}^2} e^{i2^k(\eta x + 2^kJ(t)^2)} 2^{2^k} \hat{f}^l_m(2^k\eta) d\eta \right| \left| \int_{\mathbb{R}^2} e^{i2^k(\eta' x + 2^kJ(t)^2)} 2^{2^k} \hat{f}^l_{m'}(2^k\eta') d\eta' \right|$$

$$= \left| \int_{\mathbb{R}^2} e^{i2^k(\eta x + 2^kJ(t)^2)} \hat{F}^l_m(\eta) d\eta \right| \left| \int_{\mathbb{R}^2} e^{i2^k(\eta' x + 2^kJ(t)^2)} \hat{F}^l_{m'}(\eta') d\eta' \right|,$$

where

$$\hat{F}^l_m(\eta) = 2^{2^k} \hat{f}^l_m(2^k\eta),$$

$$\hat{F}^l_{m'}(\eta') = 2^{2^k} \hat{f}^l_{m'}(2^k\eta').$$

Here we notice that \( \text{supp} \, \hat{F}^l_m, \text{supp} \, \hat{F}^l_{m'} \) are contained in \( \{ \eta : |\eta| \sim 1 \} \). More concretely, \( \text{supp} \, \hat{F}^l_m, \text{supp} \, \hat{F}^l_{m'} \) are contained in cubes with side length \( 2^{-l} \) and \( \text{dist} (\text{supp} \, \hat{F}^l_m, \text{supp} \, \hat{F}^l_{m'}) \sim 2^{-l} \).

Next we will try to localize \( x \) into cubes with side length \( 2^{-j} \). We have

$$\int_{\mathbb{R}^2} e^{i2^k(\eta x + 2^{-J}(t)^2)} \hat{F}^l_m(\eta) d\eta = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i2^k(\eta x + 2^{-J}(t)^2) - iz\eta \phi(\eta)} d\eta F^l_m(z) dz,$$

where \( \phi \in C_c^\infty(\mathbb{R}^2) \) such that \( \phi(\eta) = 1 \) on \( \{ \eta : |\eta| \sim 1 \} \) and decays rapidly outside. Denote

$$K(x, z, t) = \int_{\mathbb{R}^2} e^{i2^k(\eta x + 2^{-J}(t)^2) - iz\eta \phi(\eta)} d\eta.$$

Integrating by parts shows that if \( |x - z|/2^k \gg 2^{-j} \), then for each \( t \in (0, 1) \) and any positive integer \( M \gg 1 \),

$$|K(x, z, t)| \leq \frac{C_M}{\left( 1 + 2^k \left| x - \frac{z}{2^k} \right| + O(2^{-j}) \right)^{M+2}}.$$

Notice that we may assume \( 2^j \leq 2^{2k-J} \) since (1.7) holds when \( j = 2k \), then there is a sufficiently large \( M < M \) such that

$$\int_{x : |x - z|/2^k \gg 2^{-j}} K(x, z, t) F^l_m(z) dz \leq C 2^{-kM} \| F^l_m \|_{L^2}.$$

By the same argument, if \( |x - z'|/2^k \gg 2^{-j} \), then

$$\int_{x : |x - z'|/2^k \gg 2^{-j}} K(x, z', t) F^l_{m'}(z') dz' \leq C 2^{-kM} \| F^l_{m'} \|_{L^2},$$

(3.8)
Pointwise Convergence of Schrödinger Means

where

\[ K(x, z', t) = \int_{\mathbb{R}^2} e^{i2^k(\eta' \cdot x + 2^k \cdot i(\eta')^2 - iz' \cdot \eta')} \phi(\eta') d\eta'. \]

Let \( \pi \) be a smooth function such that \( \sum_{h \in \mathbb{Z}^2} \pi(-h) = 1 \) and its Fourier transform is supported in \( B(0,1) \). Let \( \{Q\} \) be a collection of cubes of side length \( 2^{k-j} \) which partitions \( \mathbb{R}^2 \) and let \( a_Q \) be the affine map sending \( Q \) to the unit cube centered at the origin. Set \( \Pi_Q = \pi \circ a_Q \), here \( \Pi_Q \) is a smooth function essentially supported in \( Q \). Hence,

\[
\left| \int_{\mathbb{R}^2} e^{i2^k(\eta \cdot x + 2^k \cdot i(\eta)^2)} F_m(\eta') d\eta \right| \left| \int_{\mathbb{R}^2} e^{i2^k(\eta' \cdot x + 2^k \cdot i(\eta')^2)} F_{m'}(\eta') d\eta' \right|
\]

\[
= \left| \int_{\mathbb{R}^2} K(x, z, t) \left( \sum_{Q} \Pi_Q \left( \frac{z}{2^k} \right) F_m(z) dz \right) \left| \int_{\mathbb{R}^2} K(x, z', t) \left( \sum_{Q'} \Pi_Q' \left( \frac{z'}{2^k} \right) F_{m'}(z') dz' \right) \right|
\]

\[
\times \left| \int_{\mathbb{R}^2} K(x, z', t) \left( \sum_{Q: \text{dist}(x,Q) \leq 2^k-j} \Pi_Q(\frac{z'}{2^k}) F_{m'}(z') dz' \right) \right|
\]

\[
+ \left| \int_{\mathbb{R}^2} K(x, z, t) \left( \sum_{Q: \text{dist}(x,Q) \geq 2^k-j} \Pi_Q(\frac{z}{2^k}) F_m(z) dz \right) \right|
\]

\[
\times \left| \int_{\mathbb{R}^2} K(x, z', t) \left( \sum_{Q': \text{dist}(x,Q') \leq 2^k-j} \Pi_Q'(\frac{z'}{2^k}) F_{m'}(z') dz' \right) \right|
\]

\[
+ \left| \int_{\mathbb{R}^2} K(x, z, t) \left( \sum_{Q: \text{dist}(x,Q) \geq 2^k-j} \Pi_Q(\frac{z}{2^k}) F_m(z) dz \right) \right|
\]

\[
\times \left| \int_{\mathbb{R}^2} K(x, z', t) \left( \sum_{Q': \text{dist}(x,Q') \geq 2^k-j} \Pi_Q'(\frac{z'}{2^k}) F_{m'}(z') dz' \right) \right|
\]

\[
\leq \sum_{Q, Q': \text{dist}(cQ \cap cQ') \neq 0} \Pi_{cQ \cap cQ'}(x) \left| \int_{\mathbb{R}^2} K(x, z, t) \Pi_Q \left( \frac{z}{2^k} \right) F_m(z) dz \right| \left| \int_{\mathbb{R}^2} K(x, z', t) \Pi_Q' \left( \frac{z'}{2^k} \right) F_{m'}(z') dz' \right|
\]

\[
+ C2^{-kM} \left\| F_m \right\|_{L^2} \left\| F_{m'} \right\|_{L^2}. \tag{3.9}
\]

For fixed \( Q, Q': cQ \cap cQ' \neq \emptyset \), we may assume that \( cQ \cap cQ' \) is contained in a \( 2^{k-j} \times 2^{k-j} \) cube centered at the origin, then

\[
\left\| \sup_{t \in (0,1)} \left| \int_{\mathbb{R}^2} K(x, z, t) \Pi_Q \left( \frac{z}{2^k} \right) F_m(z) dz \right| \left| \int_{\mathbb{R}^2} K(x, z', t) \Pi_Q' \left( \frac{z'}{2^k} \right) F_{m'}(z') dz' \right| \right\|_{L^1(cQ \cap cQ')}
\]
Notice that by uncertainty principle and the assumption that $2^l \ll 2^{(2k-j)/4}$, the Fourier transform of $\Pi_Q(\frac{\cdot}{2^k})F_m^l$ and $\Pi_Q(\frac{\cdot}{2^k})F_m'^l$ are supported in $2^{-l}$ cubes with separation $\sim 2^{-l}$, and $2^l \ll 2^{\frac{2k-j}{2}}$. Therefore, we can apply Theorem 1.5 to get

\[
\left\| \sup_{t \in (0,1)} \left| T_{2^{2k-j}}(\Pi_Q(\frac{\cdot}{2^k})F_m^l)(x,t) \right| \right\|_{L^1(B(0,1))} \leq C 2^{-\frac{j}{2}2^{(2k-j)/4}(\frac{j}{2}+\epsilon)2^{-2(2k-j)}} \left\| \Pi_Q(\frac{\cdot}{2^k})F_m^l \right\|_{L^2} \left\| \Pi_Q(\frac{\cdot}{2^k})F_m'^l \right\|_{L^2}.
\]

Then inequalities (3.9) - (3.11) imply that

\[
\left\| \sup_{t \in I} |e^{it\Delta}f_m^l| \right\|_{L^1(B(0,1))} \leq C 2^{2k-2j}2^{-\frac{j}{2}2^{(2k-j)/4}2^{-2(2k-j)}} \sum_{Q, Q': Q \cap Q' \neq \emptyset} \left\| \Pi_Q(\frac{\cdot}{2^k})F_m^l \right\|_{L^2} \left\| \Pi_Q'(\frac{\cdot}{2^k})F_m'^l \right\|_{L^2}
\]

\[
+ C 2^{-kM} \left\| F_m^l \right\|_{L^2} \left\| F_m'^l \right\|_{L^2}
\]

\[
\leq C 2^{2k-2j}2^{-\frac{j}{2}2^{(2k-j)/4}2^{-2(2k-j)}} \left\| F_m^l \right\|_{L^2} \left\| F_m'^l \right\|_{L^2}
\]

\[
= C 2^{2k-2j}2^{-\frac{j}{2}2^{(2k-j)/4}2^{-2(2k-j)}}2^{2k} \left\| f_m^l \right\|_{L^2} \left\| f_m'^l \right\|_{L^2}
\]

\[
\leq C 2^{-\frac{j}{2}2^{(2k-j)/4}2^{2k}} \left\| f_m^l \right\|_{L^2} \left\| f_m'^l \right\|_{L^2}.
\]

This completes the proof of Lemma 3.1.

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