The canonical superenergy tensors and stability of the solutions to the Einstein equations

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Here we present a new method to study stability of the solutions to the Einstein equations. This method uses the canonical superenergy tensors which have been introduced in our papers.

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I. INTRODUCTION

In the paper we propose a new approach to study stability of a solution to the Einstein equations. This approach uses the canonical superenergy tensors which were introduced to general relativity in our papers [3]. Namely, we assert that when the total superenergy density, matter and gravitation,

$$\epsilon_s$$

is non-negative, i.e., when

$$\epsilon_s \geq 0$$,

then the solution is stable. Contrary, when

$$\epsilon_s$$

is negative-definite, i.e., when

$$\epsilon_s < 0$$,

then the solution is unstable.

The paper is organized as follows. In Section II we remind problems with local energy-momentum in general relativity and our proposition to avoid them – the canonical superenergy tensors.

In Section III we give examples of an intriguing correlation between stability of the very known solutions to the Einstein equations and sign of the total canonical superenergy density, $$\epsilon_s$$, for them. We claim there that these exciting correlations are consequences of the Proposition, which we have formulated and proved in this Section.

Finally, the short Section IV, contains our conclusion.

In Appendix we present some results of the last our calculations.

In the paper we use the same signature and notation as used in the last editions of the famous book by Landau and Lifshitz.

II. THE CANONICAL SUPERENERGY TENSORS

In the framework of general relativity (GR), as a consequence of the Einstein Equivalence Principle (EEP), the gravitational field has non-tensorial strengths $$\Gamma^i_{kl} = \{_{kl}^i\}$$ and admits no energy-momentum tensor. One can only attribute to this field gravitational energy-momentum pseudotensors. The leading object of such a kind is the canonical gravitational energy-momentum pseudotensor $$E_t^k$$ proposed already in past by Einstein. This pseudotensor is a part of the canonical energy-momentum complex $$EK^k$$ in GR.

The canonical complex $$EK^k$$ can be easily obtained by rewriting Einstein equations to the superpotential form

$$EK^k := \sqrt{|g|}(T^{ik} + E_t^k) = F U_{[kl]}^i$$

where $$T^{ik} = T^{ki}$$ is the symmetric energy-momentum tensor for matter, $$g = det[g_{ik}]$$, and

$$E_t^k = \frac{c^4}{16\pi G} \{ \delta^k_s g^{ms} (\Gamma^l_{mr} \Gamma^r_{sl} - \Gamma^r_{ms} \Gamma^l_{rl}) + g^{ms}_{,r} (\Gamma^l_{ms} - \frac{1}{2}(\Gamma^k_{tp} g^{tp} - \Gamma^l_{tl} g^{kl}) g_{ms} - \frac{1}{2} (\delta^k_s \Gamma^l_{m} + \delta^l_s \Gamma^l_{m}) \}$$

and

$$F U_{[kl]}^i = \frac{c^4}{16\pi G} g_{ia} (\sqrt{|g|})^{-1} \left[ (-g) (g^{ka} g^{lb} - g^{la} g^{kb}) \right]_{,b}.$$
\( E t^k_i \) are components of the canonical energy-momentum pseudotensor for gravitational field \( \Gamma^i_{kl} = \{^i_{kl}\} \), and \( \mu U^{[kl]}_i \) are von Freud superpotentials.

\[
E K^k_i = \sqrt{|g|} (T^k_i + E t^k_i)
\]

are components of the Einstein canonical energy-momentum complex, for matter and gravity, in GR.

In consequence of (1) the complex \( E K^k_i \) satisfies local conservation laws

\[
E K^k_i, k = 0.
\]

In very special cases one can obtain from these local conservation laws the reasonable integral conservation laws.

Despite that one can easily introduce in GR the canonical (and others) superenergy tensor for gravitational field. This was done in past in a series of our articles (See, e.g., [3] and references therein). It appeared that the idea of the superenergy tensors is universal: to any physical field having an energy-momentum tensor or pseudotensor one can attribute the corresponding superenergy tensor.

So, let us give a short reminder of the general, constructive definition of the superenergy tensor \( S^b_a \) applicable to gravitational field and to any matter field. The definition uses locally Minkowskian structure of the spacetime in GR and, therefore, it fails in a spacetime with torsion, e.g., in Riemann-Cartan spacetime. In the normal Riemann coordinates \( \text{NRC}(P) \) we define (pointwise)

\[
S^{(b)}_a(P) = S^b_a := (-) \lim_{\Omega \to P} \frac{\int_{\Omega} \left[ T^{(b)}_a(y) - T^{(b)}_a(P) \right] d\Omega}{1/2 \int_{\Omega} \sigma(P; y) d\Omega},
\]

where

\[
T^{(b)}_a(y) := T^k_i(y) e^i_a(y) e_k^{(b)}(y),
\]

\[
T^{(b)}_a(P) := T^k_i(P) e^i_a(P) e_k^{(b)}(P) = T^k_i(P) = T_a^b(P)
\]

are physical or tetrad components of the pseudotensor or tensor field which describes an energy-momentum distribution, and \( \{y^i\} \) are normal coordinates. \( e^i_a(y) \), \( e_k^{(b)}(y) \) mean an orthonormal tetrad \( e^i_a(P) = \delta^i_a \) and its dual \( e_k^{(a)}(P) = \delta_k^a \) parallelly propagated along geodesics through \( P \) (\( P \) is the origin of the \( \text{NRC}(P) \)). We have

\[
e^i_a(y) e_k^{(b)}(y) = \delta^b_a.
\]

For a sufficiently small 4-dimensional domain \( \Omega \) which surrounds \( P \) we require

\[
\int_{\Omega} y^i d\Omega = 0, \quad \int_{\Omega} y^i y^k d\Omega = \delta^{ik} M,
\]

where

\[
M = \int_{\Omega} (y^0)^2 d\Omega = \int_{\Omega} (y^1)^2 d\Omega = \int_{\Omega} (y^2)^2 d\Omega = \int_{\Omega} (y^3)^2 d\Omega,
\]

is a common value of the moments of inertia of the domain \( \Omega \) with respect to the subspaces \( y^i = 0, \quad (i = 0, 1, 2, 3) \). We can take as \( \Omega \), e.g., a sufficiently small analytic ball centered at \( P \):

\[
(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2,
\]

which for an auxiliary positive-definite metric

\[
h^{ik} := 2v^i v^k - g^{ik},
\]

can be written in the form

\[
h_{ik} y^i y^k \leq R^2.
\]
A fiducial observer $O$ is at rest at the beginning $P$ of the used Riemann normal coordinates $\mathbf{NRC}(P)$ and its four-velocity is $v^i = \ast \delta^i_0$. $\ast$ means that an equations is valid only in special coordinates. $\sigma(P; y)$ denotes the two-point world function introduced in past by J.L. Synge [4]

$$\sigma(P; y) = \frac{1}{2} \left( y^0 v^2 - y^1^2 - y^2^2 - y^3^2 \right).$$

(12)

The world function $\sigma(P; y)$ can be defined covariantly by the eikonal-like equation [4]

$$g^{ij} \sigma_i \sigma_j = 2\sigma, \quad \sigma_i := \partial_i \sigma,$$

(13)

together with

$$\sigma(P; P) = 0, \quad \partial_i \sigma(P; P) = 0.$$

(14)

The ball $\Omega$ can also be given by the inequality

$$h^{ik} \sigma_i \sigma_k \leq R^2.$$  

(15)

Tetrad components and normal components are equal at $P$, so, we will write the components of any quantity attached to $P$ without tetrad brackets, e.g., we will write $S_{(a)}^b(P)$ instead of $S_{(a)}(b)(P)$ and so on.

If $T_{ik}^l(y)$ are the components of an energy-momentum tensor of matter, then we get from (5)

$$m S_{a}^b(P; v^l) = (2g^{lm} - g^{lm}) \nabla_l \nabla_m \hat{T}_{a}^b = \hat{h}^{lm} \nabla_l \nabla_m \hat{T}_{a}^b.$$  

(16)

Hat over a quantity denotes its value at $P$, and $\nabla$ means covariant derivative. Tensor $m S_{a}^b(P; v^l)$ is the canonical superenergy tensor for matter.

For the gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as $T_{ik}^l$ in (5) gives

$$g S_{a}^b(P; v^l) = \hat{h}^{lm} W_{a}^b \hat{m},$$

(17)

where

$$W_{a}^b \hat{m} = \frac{2\alpha}{9} \left[ B_{alm}^b P_{alm}^b \right] - \frac{1}{2} \delta^b_a R_{ijkl} (R_{ijkl} + R_{ikjl}) + 2 \delta_a^b \beta^2 E_{(i|g|l} E_{j|m)} - 3 \beta^2 E_{a(i|g|l} E_{j|m)} + 2 \beta R_{a|g|l} E_{j|m}].$$

Here $\alpha = \frac{\epsilon^2}{16 \pi G} = \frac{1}{2\beta}$, and

$$E_{ik} := T_{ik} - \frac{1}{2} \delta_{ik} T$$

(18)

is the modified energy-momentum tensor of matter [7]. On the other hand

$$B_{alm}^b := 2R_{ijkl} R_{aklm} - \frac{1}{2} \delta_a^b R_{ijkl} R_{ijkl}$$

(19)

are the components of the Bel-Robinson tensor (BRT), while

$$P_{alm}^b := 2R_{ijkl} R_{aklm} - \frac{1}{2} \delta_a^b R_{ijkl} R_{ijkl}$$

(20)

is the Bel-Robinson tensor with “transposed” indices $(ik)$. Tensor $g S_{a}^b(P; v^l)$ is the canonical superenergy tensor for gravitational field $\left\{ \hat{l} \right\}$. In vacuum $g S_{a}^b(P; v^l)$ takes the simpler form

$$g S_{a}^b(P; v^l) = \frac{8\alpha}{9} \hat{h}^{lm} (C_{a}^{bik}(l) \hat{C}_{aik|m} - \frac{1}{2} \delta_a^b \hat{C}^{ik}(l) \hat{C}_{ikp|m}).$$

(21)

Here $C_{b}^{a} \hat{blm}$ denote components of the Weyl tensor.

Some remarks are in order:
1. In vacuum the quadratic form $g^{ab}v^av_b$, where $v^av_a = 1$, is positive-definite giving the gravitational superenergy density $\epsilon_g$ for a fiducial observer $O$.

2. In general, the canonical superenergy tensors are uniquely determined only along the world line of the observer $O$. But in special cases, e.g., in Schwarzschild spacetime or in Friedman universes, when there exists a physically and geometrically distinguished four-velocity $v^i(x)$, one can introduce in an unique way the unambiguous fields $g^S_i^k(x; v^i)$ and $m_i^S_k(x; v^i)$.

3. We have proposed in our previous papers to use the tensor $g^{S_i^k}(P; v^i)$ as a substitute of the non-existing gravitational energy-momentum tensor.

4. It can easily seen that the superenegy densities $\epsilon_g := g^{S_i^k}v^i v_k$, $\epsilon_m := m^{S_i^k}v^i v_k$ for an observer $O$ who has the four-velocity $v^i$ correspond exactly to the energy of acceleration $\frac{1}{2}m\ddot{a}$ which is fundamental in Appel’s approach to classical mechanics.

In past we have used the canonical superenergy tensors $g^{S_i^k}$ and $m^{S_i^k}$ to local (and also, in some cases, to global) analysis of well-known solutions to the Einstein equations like Schwarzschild and Kerr solutions; Friedman and Goedel universes, and Kasner and Bianchi I, II universes. The obtained results were interesting (See [3]).

We have also studied the transformational rules for the canonical superenergy tensors under conformal rescalling of the metric $g_{ik}(x)$ [3, 6].

The idea of the superenergy tensors can be extended on angular momentum also [3]. The obtained angular superenergy tensors do not depend on a radius vector and they depend only on spinorial part of the suitable gravitational angular momentum pseudotensor [8].

**III. STABILITY OF THE SOLUTIONS TO THE EINSTEIN EQUATIONS AND CANONICAL SUPERENERGY TENSORS**

Recently we have observed an exciting correlation between the total superenergy density, $\epsilon_s := \epsilon_m + \epsilon_g$, and stability of solutions to the Einstein equations. Namely, we have noticed that when a solution is stable, then $\epsilon_s \geq 0$, and when the solution is unstable, then $\epsilon_s < 0$.

The examples

1. Exterior Schwarzschild —— stable ——- $\epsilon_s > 0$;
2. Einstein static universe —- unstable —– $\epsilon_s < 0$;
3. Kerr solution ————- stable —— $\epsilon_s > 0$;
4. de Sitter universe ——— unstable — $\epsilon_s < 0$;
5. Anti-de Sitter universe —— unstable —- $\epsilon_s < 0$;
6. Friedman universes ———- stable — $\epsilon_s > 0$;
7. Bianchi I universe ———- stable — $\epsilon > 0$;
8. Kasner universe ——— stable — $\epsilon_s > 0$;
9. Exterior Reissner-Nordstroem - stable — $\epsilon_s > 0$;
10. Minkowski spacetime ——— stable — $\epsilon_s = 0$.

Instability of the de Sitter and anti-de Sitter universes was proved recently [1, 2].

One can easily see that the above mentioned correlation follows from the Proposition.

**Proposition** If a solution to the Einstein equations is stable, then $\epsilon_s > 0$, and if the solution is unstable, then $\epsilon_s < 0$.

**Proof.** Our proof lies on the constructive definition (5). It is easily seen from it that the sign of the superenergy density $S_a^b(P)v^av_b = \epsilon_s$ is determined by the sign of the integral in nominator because (-) denominator is always positive. Let us apply the definition (5) to the component $K_0^0(y)$ of the canonical energy-momentum complex, matter and gravitation. This component gives us the total energy density, gravitation and matter, for analyzed solution to the Einstein equations.
Let the analyzed solution will stable at point \( P \). Then, \( K_0^0(y) \) has minimum at this point.

One can see from (5) and (12) that \( S_0^0(P) = g S_0^0(P) + m S_0^0(P) = \epsilon_s(P) > 0 \) in the case because \( K_0^0(y) - K_0^0 > 0 \). Contrary, if \( P \) is an instability point of the analyzed solution, then \( K_0^0(y) \) has maximum at this point, and it follows that \( S_0^0(P) = g S_0^0(P) + m S_0^0(P) = \epsilon_s(P) < 0 \) because \( K_0^0(y) - K_0^0(P) < 0 \) in this case.

If the analyzed solution is stable (or unstable) in a 4-dimensional domain \( \Omega \), then one has in this domain \( S_0^0(y) = \epsilon_s(y) > 0 \) (or, respectively, \( S_0^0(y) = \epsilon_s(y) < 0 \)).

The direct calculation shows that for the stable Minkowskian spacetime one has limiting case with value \( \epsilon_s(y) = 0 \).

We also conjecture that the inverse is correct: if for a solution to the Einstein equations \( \epsilon_s > 0 \), then the same should be correct in all the cases in which \( \epsilon_s > 0 \). (In this case the stabilizing positive differences \( K_0^0(y) - K_0^0(P) \) in the formula (5) overbalance destabilizing negative differences \( K_0^0(y) - K_0^0(P) \)). On the other hand, if \( \epsilon_s < 0 \), then the destabilizing negative differences \( K_0^0(y) - K_0^0(P) \) overbalance stabilizing positive differences \( K_0^0(y) - K_0^0(P) \) and, in the case, the analyzed solution should be unstable.

IV. CONCLUSION

On the superenergy level we have no problem with suitable tensor for gravity, e.g., one can introduce gravitational canonical superenergy tensor. The canonical superenergy tensors, gravitation and matter, are useful to local analysis of the solutions to the Einstein equations, especially to analyze of their singularities\(^\text{[3]}\).

In this paper we have proposed an application of these tensors to study stability of the solution to the Einstein equations.

We think that this new application of the superenergy tensors can be very useful.

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V. APPENDIX

We give here the canonical superenergy densities \( \epsilon_s \) for de Sitter, anti-de Sitter, static Einstein and Reissner-Nordström spacetimes. For simplicity we will use in here the geometrized units in which \( G = c = 1 \).

1. De Sitter spacetime —– \( \epsilon_s = (-) \frac{3}{2} \alpha \Lambda^2 < 0 \);
2. Anti-de Sitter spacetime —– \( \epsilon_s = (-) \frac{3}{2} \alpha \Lambda^2 < 0 \);
3. Einstein static universe —– \( \epsilon_s = (-) \frac{4 \pi}{9 \alpha} < 0 \), where \( \frac{1}{9 \alpha} = 4 \pi (\rho + p) = \Lambda - 8 \pi p > 0 \);  
4. Exterior Reissner-Nordström spacetime —–

\[
\epsilon_s = \frac{2 \alpha}{9 r^8} \left[ 3 (2 Q^2 - r_s r) + 5 (Q^2 - r_s r)^2 + 2 (3 Q^2 - r_s r)^2 \\
+ 2 (3 Q^2 - r_s r) (2 Q^2 - r_s r) \right] \\
+ \frac{2 Q^2}{r^8} \left( r_s r - 2 Q^2 \right) + \frac{12 Q^2 \Lambda_{RN}}{r^6}.
\]

(22)

The last expression is positive for \( r \geq r_H = m + \sqrt{m^2 - Q^2} \), i.e., outside and on horizon \( H \) of the Reissner-Nordström black hole.

Here \( r_s := 2m, \quad \Lambda_{RN} := 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \), and \( m^2 > Q^2 \).
The total superenegy densities for the other solutions to the Einstein equations mentioned in this paper have been already given in past \[3\].

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[7] In terms of $E_i^k$ Einstein equations read $R_{i}^{\ k} = \beta E_i^k$.
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