Holonomies of gauge fields in twistor space 2:  
Hecke algebra, diffeomorphism, and graviton amplitudes

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Abstract

We define a theory of gravity by constructing a gravitational holonomy operator in twistor space. The theory is a gauge theory whose Chan-Paton factor is given by a trace over elements of Poincaré algebra and Iwahori-Hecke algebra. This corresponds to a fact that, in a spinor-momenta formalism, gravitational theories are invariant under spacetime translations and diffeomorphism. The former symmetry is embedded in tangent spaces of frame fields while the latter is realized by a braid trace. We make a detailed analysis on the gravitational Chan-Paton factor and show that an S-matrix functional for graviton amplitudes can be expressed in terms of a supersymmetric version of the holonomy operator. This formulation will shed a new light on studies of quantum gravity and cosmology in four dimensions.
# Introduction

In 1989, Witten showed a remarkable relation between the Jones polynomial of knot theory and the topological field theory [1]. More concretely, it is shown that the knot invariants arise from partition functions of three-dimensional Chern-Simons theory, or the so-called Witten invariants. Among many important ideas in [1], there are two particular concepts to which we would like to give attention here. Firstly, the specific choice of Chern-Simons action has been made because it is a simple but yet nontrivial action that preserves general covariance. We can, in principle, use any generally covariant field theories in search of the knot invariants. Chern-Simons theory is convenient for this purpose since it does not contain a metric. If one is motivated to construct a theory of gravity and to compute physical quantities in terms of gravitons, it is, however, inevitable to define a metric or at least a frame field. In such a case, general covariance is achieved by an integration over all metrics along with a proper definition of metrics in some theory. This would be a key concept for the construction of gravitational theories.\(^1\)

The other concept of attention is that of braid trace, which arises from the understanding of knot polynomials. Braid trace is a trace over braid generators but its physical meaning in connection with a gravitational theory is yet to be clarified. In [1], it is argued that complete information about braid generators can be encoded by a choice of diffeomorphism on \(\mathbb{CP}^1 = S^2\). The \(S^2\) comes from a certain geometric surgery of a three-dimensional manifold (from \(S^2 \times S^1\) to \(S^3\)) where Chern-Simons theory is defined. This suggests that, in order to have diffeomorphism invariance in some theory, one has to sum over all possible braid structures. The summation is expected to be realized by a braid trace. Thus it is important to clarify the notion of braid trace and its relation to diffeomorphism in building a gravitational theory.

Bearing in mind these two concepts, in the present paper, we shall construct a theory of gravity in four dimensions. Owing to a dimensional discrepancy, these concepts may not be applicable at first glance. But use of twistor space can remedy the problem. For example, if we assume that a theory is given by a Chern-Simons action (or a variant of this action) which is defined in twistor space, we can obtain a four-dimensional theory as follows. We first note that partition functions of Chern-Simons theory on \(\mathcal{M}_3\) corresponds to current correlators of a Wess-Zumino-Witten (WZW) model on \(\Sigma = \partial \mathcal{M}_3\), where \(\mathcal{M}_3\) denotes a three-dimensional manifold and \(\Sigma = \partial \mathcal{M}_3\) denotes its boundary.\(^2\) Thus, the partition functions (or the generating functionals) of Chern-Simons theory on \(\mathcal{M}_3\) are encoded entirely by a two-dimensional WZW model on \(\Sigma\). Now twistor space \(\mathbb{CP}^3\) can be considered as a \(S^2\)-bundle over four-dimensional spacetime. Identification of \(\Sigma\) with the \(S^2\) fiber of twistor space then leads to a WZW model whose target space is the twistor space and, from this, one can extract four-dimensional physics á la Penrose.

The use of twistor space is also supported by recent developments in the so-called twistor space.

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\(^1\)Notice that this does not mean the exclusion of Chern-Simons theory from gravitational theories at all. In fact, it may be the case that Chern-Simons action emerges in some fashion after integration over all metrics. To answer this intriguing question is one of the objectives of the present paper.

\(^2\)As well-known, this is probably the most important mathematical concept developed in [1].
string theory [2, 3]. Twistor string theory, as part of string theory, contains supergravity theories but, due to the nature of twistor space, it turns out to be quite difficult to eliminate conformal invariance. This matter is first investigated in [4]. Some related work can also be found in [5, 6]. The elimination of conformal invariance from twistor string theory is proposed in the so-called new twistor string theories [7]. Even in this new approach, the extraction of Einstein supergravity (or general relativity) from twistor string theory is not yet satisfactory, as argued in [8, 9]. In the present paper, we shall not follow these lines of developments but take a more practical approach. Namely, we take advantage of the knowledge of graviton amplitudes and construct a gravitational theory such that it leads to correct amplitudes by a standard field theoretic technique.

There has been much progress in computations of both gluon amplitudes and graviton amplitudes, accompanied with the twistor-string developments. It is known that the graviton amplitudes can be obtained from the gluon counterparts. For four-dimensional theories, this was shown by Berends, Giele and Kuijf [10] who utilized the so-called Kawai-Lewellen-Tye (KLT) relation between tree amplitudes of closed and open string theories [11]. This relation means that, as in the gluon cases, the so-called maximally helicity violating (MHV) amplitudes for gravitons can also be described in a remarkably succinct form if we use a spinor-momenta formalism. Since the graviton amplitudes do, unlike the gluon ones, break conformal invariance, one of the peculiarities of gravity lies in the nonholomorphicity in terms of the spinor momenta. In this context, the MHV graviton amplitudes provide clues for the understanding of gravitational theories and particularly of $\mathcal{N} = 8$ supergravity. Analyses of the MHV graviton amplitudes along these lines can be found in [12]-[15]. Extensions of these ideas to loop calculations of graviton amplitudes have also been carried out (see, e.g., [16]-[21]). Remarkably, these calculational developments favorably support a long-pending question of the ultraviolet finiteness of $\mathcal{N} = 8$ supergravity [17, 22, 23]. This is a result of great significance for the study of $\mathcal{N} = 8$ supergravity. Recently, partly motivated by these results, there are new actions proposed for the $\mathcal{N} = 8$ theory [24, 25].

Relationships between $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 8$ supergravity in twistor space receive much attention recently. (For the very recent developments, see, e.g., [26]-[35] and for relatively earlier investigations, see also [36]-[39].) As mentioned above, the two theories have a structural similarity but they also have a mathematical difference in terms of a conformal property. Clarification of the similarity and difference will lead to a unified way of understanding the two theories in four dimensions. One way of having a unified picture is to regard a gravitational theory as a gauge theory and to introduce a notion of Chan-Paton factor in the former. This picture is in consistent with Berkovits’ open-string description of twistor string theory [3] and is first suggested by Nair in order to interpret a physical structure of the MHV graviton amplitudes [13]. Generalization of Nair’s interpretation to non-MHV amplitudes is carried out in [38]. One of the advantages of this approach is that we can encode the breaking of conformal invariance entirely in a Chan-Paton factor.

As we have postulated in an accompanying paper [40], any physical observables of gauge theories in twistor space can universally be generated by a “holonomy operator” in twistor space. In the present paper, we shall show that this idea also holds for a gravitational theory. If we make use of a spinor-momenta formalism in twistor space, Lorentz invariance
is manifest, and hence, in considering a certain representation of Poincaré algebra in this framework, the representation is essentially given by translational generators. This is in consistent with the fact that the Chan-Paton factor of graviton amplitudes is described by combinations of the translational generators. Diffeomorphism invariance suggests that these generators should be furnished with generators of braid groups or Hecke-algebra-valued quantities. (For mathematical backgrounds of Hecke algebra, or Iwahori-Hecke algebra, one may refer to [41]-[43].) A main objective of the present paper is to show that a Chan-Paton factor of a gravitational holonomy operator, which can be represented by a trace over Pincaré algebra and Iwahori-Hecke algebra, naturally leads to the Chan-Paton factor of graviton amplitudes. As in the case of gluon amplitudes, this allows us to express an S-matrix functional of graviton amplitudes in terms of the gravitational holonomy operator in twistor space.

The organization of this paper is as follows. In the next section, we recapitulate the results of the accompanying paper [40]. We review the definition of the above mentioned holonomy operator in twistor space. We see that Iwahori-Hecke algebra naturally arises from the construction of the holonomy operator. In section 3, we consider realization of diffeomorphism in the spinor-momenta formalism and discuss that diffeomorphism invariance can be represented by a braid trace. We also give an appropriate definition of metrics, following Nair’s interpretation of gravity as a gauge theory. In section 4, we construct and compute a gravitational holonomy operator in twistor space. A Chan-Paton factor of the holonomy operator is basically composed of two ingredients. One is a sum over all possible metrics and the other is a braid trace. We make a detailed analysis on this Chan-Paton factor and see that it has one-to-one correspondence with a certain combinatoric factor in graviton amplitudes. In section 5, utilizing the results of the previous sections, we give an explicit expression for an S-matrix functional of graviton amplitudes. As in the Yang-Mills case, the S-matrix functional is expressed in terms of a supersymmetric version of the gravitational holonomy operator. Lastly, we shall present some concluding remarks.

2 Review of holonomy formalism in twistor space

In this section, we review the construction of holonomy operators in twistor space, which has been developed in [40]. We shall also discuss the emergence of Iwahori-Hecke algebra in this formulation, following Kohno’s textbook [43].

Spinor momenta

A holonomy operator in twistor space is defined by use of a spinor-momenta formalism. Spinor momenta of massless particles, such as gluons and gravitons, are generally given by two-component complex spinors. In terms of four-momentum $p_\mu$ ($\mu = 0, 1, 2, 3$), which obey the on-shell condition $p^2 = \eta^{\mu\nu} p_\mu p_\nu = p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$ ($\eta^{\mu\nu}$ denoting the Minkowski metric), the spinor momenta can be expressed as

$$
u^A = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix}, \quad \bar{\nu}_A = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 + ip_2 \\ p_0 - p_3 \end{pmatrix}$$
where both $A$ and $\dot{A}$ take values of $(1, 2)$. With these, the four-momentum $p_\mu$ can be parametrized as a $(2 \times 2)$-matrix, $p_A^A = (\sigma^\mu)^A_\dot{A} p_\mu \equiv u^A \bar{u}_\dot{A}$ with $\sigma^i$ \textit{i.e.} $\bar{u}_\dot{A}$ a (2 \times 2) Pauli matrices and $1$ is the $2 \times 2$ identity matrix. Requiring that $p_\mu$ be real, we can take $\bar{u}_\dot{A}$ as a conjugate of $u^A$, \textit{i.e.}, $\bar{u}_\dot{A} = (u^A)^\ast$.

From the above parametrization of $p_\mu$, we see that $p_\mu$ is invariant under

$$u^A \rightarrow e^{i\phi} u^A, \quad \bar{u}_\dot{A} \rightarrow e^{-i\phi} \bar{u}_\dot{A}$$

(2)

where $\phi$ represents a $U(1)$ phase parameter. Thus there is a phase ambiguity in the definition of $u^A$ and $\bar{u}_\dot{A}$.

Lorentz transformations of $u^A$ are given by

$$u^A \rightarrow (gu)^A$$

(3)

where $g \in SL(2, \mathbb{C})$ is a $(2 \times 2)$-matrix representation of $SL(2, \mathbb{C})$; the complex conjugate of this relation leads to Lorentz transformations of $\bar{u}_\dot{A}$. Four-dimensional Lorentz transformations are realized by a combination of these, that is, the four-dimensional Lorentz symmetry is given by $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Scalar products of $u^A$'s or $\bar{u}_\dot{A}$'s, which are invariant under the corresponding $SL(2, \mathbb{C})$, are expressed as

$$u_i \cdot u_j \equiv (u_i u_j) = \epsilon_{AB} u^A_i u^B_j, \quad \bar{u}_i \cdot \bar{u}_j \equiv [\bar{u}_i \bar{u}_j] = \epsilon^{AB} \bar{u}^A_i \bar{u}^B_j$$

(4)

where $\epsilon_{AB}$ is the rank-2 Levi-Civita tensor. This can be used to raise or lower the indices, \textit{e.g.}, $u_B = \epsilon_{AB} u^A$. Notice that these products are zero when $i$ and $j$ are identical. In what follows, we can assume $1 \leq i < j \leq n$ without loss of generality.

For a theory with conformal invariance, such as a theory of electromagnetism or $N = 4$ super Yang-Mills theory, we can impose scale invariance on the spinor momentum, \textit{i.e.},

$$u^A \sim \lambda u^A, \quad \lambda \in \mathbb{C} - \{0\}$$

(5)

where $\lambda$ is non-zero complex number. With this identification, we can regard the spinor momentum $u^A$ as a homogeneous coordinate of the complex projective space $\mathbb{CP}^3$.

**Twistor space**

Twistor space is defined by a four-component spinor $Z_I = (\pi^A, v_{\dot{A}}) \ (I = 1, 2, 3, 4)$ where $\pi_A$ and $v_{\dot{A}}$ are two-component complex spinors. From this definition, it is easily understood that twistor space is represented by the complex homogeneous coordinates of $\mathbb{CP}^3$. Thus, $Z_I$ correspond to homogeneous coordinates of $\mathbb{CP}^3$ and satisfy the following relation.

$$Z_I \sim \lambda Z_I, \quad \lambda \in \mathbb{C} - \{0\}$$

(6)

In twistor space, the relation between $\pi^A$ and $v_{\dot{A}}$ is defined as $v_{\dot{A}} = x_{\dot{A}A} \pi^A$. With this relation, the condition (6) is realized by the scale invariance of $\pi^A$, as shown in (5) for $u^A$. $x_{\dot{A}A}$ are defined as the local coordinates on $S^4$. This can be understood from the fact that $\mathbb{CP}^3$ is a $\mathbb{CP}^1$-bundle over $S^4$. We consider that the $S^4$ describes a four-dimensional compact spacetime. A flat spacetime may be obtained by considering a neighborhood of this $S^4$. 

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Notice that in twistor space the spacetime coordinates $x_{\bar{A}A}$ are emergent quantities. Four-dimensional diffeomorphisms, \textit{i.e.}, general coordinate transformations, is therefore realized by
\begin{equation}
u^A \rightarrow u'^A \tag{7}\end{equation}
rather than $x_{\bar{A}A} \rightarrow x'_{\bar{A}A}$.

\textbf{What is essential in the spinor-momenta formalism in twistor space is to identify a CP$^1$ fiber of twistor space with a CP$^1$ on which the spinor momenta are defined. In other words, we identify $\pi_A$ with the spinor momenta $u_A$ so that we can essentially describe four-dimensional physics in terms of the coordinates of CP$^1$.}

In the spinor-momenta formalism, the twistor-space condition $v_{\bar{A}} = x_{\bar{A}A} \pi^A$ is then expressed as
\begin{equation}v_{\bar{A}} = x_{\bar{A}A} u^A \tag{8}\end{equation}

A helicity of a massless particle is generally determined by the so-called Pauli-Lubanski spin vector. In the spinor-momenta formalism, we can also define an analog of this spin vector, which can be used to define a helicity operator of massless particles as
\begin{equation}h = 1 - \frac{1}{2} u^A \frac{\partial}{\partial u^A} \tag{9}\end{equation}
This shows that the helicity of the particle is essentially given by the degree of homogeneity in $u$.

\textbf{Emergence of Iwahori-Hecke algebra}

We now consider the description of gluons in particular. The Hilbert space of the spinor-momenta formalism for the description of gluons is given by $V^\otimes n = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ where $V_i$ ($i = 1, 2, \ldots, n$) denotes a Fock space that creation operators of the $i$-th particle with helicity $\pm$ act on. Such operators can be expressed as $a_i^{(\pm)}$, with $(\pm)$ denoting helicities of the gluons. The index $i$ is called the numbering index in what follows. Notice that $a_i^{-}$ can be given by the conjugate of $a_i^{(+)}$, $a_i^{-} = (a_i^{(+)})^*$, and vice versa. These can be interpreted as ladder operators which form a part of the $SL(2, \mathbb{C})$ algebra. The algebra can be expressed as
\begin{equation}[a_i^{(+)}, a_j^{(-)}] = 2a_i^{(0)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(+)}] = a_i^{(+)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(-)}] = -a_i^{(-)} \delta_{ij} \tag{10}\end{equation}
where Kronecker’s deltas show that the non-zero commutators are obtained only when $i = j$. The remaining of commutators, those expressed otherwise, all vanish.

For a system of $n$ gluons or $n$ spinor-momenta, the physical configuration space is given by $\mathcal{C} = \mathbb{C}^n/\mathcal{S}_n$, where $\mathcal{S}_n$ is the rank-$n$ symmetric group. The $\mathcal{S}_n$ arises from the fact that gluons are bosons with invariance under permutations of the numbering indices. The complex number $\mathbb{C}$ corresponds to a local coordinate of each spinor-momenta defined on CP$^1$. It is well-known that the fundamental homotopy group of $\mathcal{C} = \mathbb{C}^n/\mathcal{S}_n$ is given by the braid group, $\Pi_1(\mathcal{C}) = \mathcal{B}_n$. The braid group $\mathcal{B}_n$ has generators, $b_1, b_2, \ldots, b_{n-1}$. Let $\rho(b_i)$ denote a linear representation of the braid generator $b_i$. An action of $\rho(b_i)$ on the Hilbert space $V^\otimes n$ can basically be carried out by transposition of the index $i$ with $i + 1$. 
Now, mathematically, a linear representation of a braid group is equivalent to a monodromy representation of the Knizhnik-Zamolodchikov (KZ) equation. The KZ equation is an equation that a function on \( \mathbb{C} \) satisfies in general. We can denote such a function as \( \Psi(z_1, z_2, \ldots, z_n) \), where \( z_i \) represents the local coordinate corresponding to the spinor momentum \( u_i \) (\( i = 1, 2, \ldots, n \)). The KZ equation is then expressed as

\[
\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j (j \neq i)} \frac{\Omega_{ij} \Psi}{z_i - z_j}
\]

(11)

where \( \kappa \) is a non-zero constant called the KZ parameter. We now introduce logarithmic differential one-forms

\[
\omega_{ij} = d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}.
\]

(12)

Notice that these satisfy the identity

\[
\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0
\]

(13)

where the indices are ordered as \( i < j < k \). \( \Omega_{ij} \) in the KZ equation is a bialgebraic operator. In terms of the operators of \( SL(2, \mathbb{C}) \) algebra in (10), this can be defined as

\[
\Omega_{ij} = a_i^{(+)} \otimes a_j^{(-)} + a_i^{(-)} \otimes a_j^{(+)} + 2a_i^{(0)} \otimes a_j^{(0)}
\]

(14)

Should we have \( i = j \), this would become the quadratic Casimir of \( SL(2, \mathbb{C}) \) algebra which acts on the \( i \)-th Fock space \( V_i \). Introducing the following one-form

\[
\Omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} \omega_{ij},
\]

(15)

we can rewrite the KZ equation (11) as a differential equation

\[
D \Psi = (d - \Omega) \Psi = 0
\]

(16)

where \( D = d - \Omega \) can be regarded as a covariant exterior derivative.

From an explicit form of (14), we can show the following relations.

\[
[\Omega_{ij}, \Omega_{kl}] = 0 \quad (i, j, k, l \text{ are distinct})
\]

(17)

\[
[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0 \quad (i, j, k \text{ are distinct})
\]

(18)

In mathematical literature, these relations are called infinitesimal braid relations. Remarkably, by use of these relations along with (13), the flatness of \( \Omega \), i.e., \( d\Omega - \Omega \wedge \Omega = 0 \), can be shown. (For a proof of this, see [40, 43].) Therefore, it is possible to define a holonomy of \( \Omega \), which gives a general linear representation of a braid group on the Hilbert space \( V^\otimes n \). This is the monodromy representation of the KZ equation. The Hilbert space \( V^\otimes n \) can then be identified as the space of conformal blocks for the KZ equation.

In physics, we need to use unitary irreducible representations (UIRs) of certain representations. In the case of the monodromy representation, this can be given by the so-called
Iwahori-Hecke algebra [41, 42]. In terms of elements \( \tilde{b}_i \) \((i = 1, 2, \cdots, n - 1)\), this algebra is defined by

\[
\begin{align*}
\tilde{b}_i \tilde{b}_{i+1} \tilde{b}_i &= \tilde{b}_{i+1} \tilde{b}_i \tilde{b}_{i+1}, \quad \text{if } |i - j| = 1 \\
\tilde{b}_i \tilde{b}_j &= \tilde{b}_j \tilde{b}_i, \quad \text{if } |i - j| > 1
\end{align*}
\]

(19)

where \( q = \exp(i2\pi/\kappa) \) and \( \tilde{b}_n \) is identified with \( \tilde{b}_1 \). The first two relations are equivalent to the relations satisfied by the generators \( b_i \) of a braid group. Remember that we denote a linear representation of \( b_i \) as \( \rho(b_i) \). We now introduce a scaled representation \( \tilde{\rho}(b_i) = \eta \rho(b_i) \), with \( \eta = q^{1/4} = \exp(i\pi/2\kappa) \). It is known that the elements of \( \tilde{\rho}(b_i) \) satisfy the last relation of (19). This can be shown by impositions of irreducibility on each of the Fock space \( V_i \). (For details of this fact, one may refer to [43].) Further, we may naturally impose a unitary condition \( \tilde{b}_i^{-1} = \tilde{b}_i^\dagger \). Thus, the Iwahori-Hecke algebra (19) forms a UIR of a linear representation of braid groups, and this algebra should be encoded in the definition of a holonomy operator.

### Comprehensive gauge fields and integrability

We now introduce a “comprehensive” gauge one-form for the description of \( n \) gluons in the spinor-momenta formalism. We define the comprehensive gauge field operator \( A \) as

\[
A = g \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij}
\]

(20)

\[
A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}
\]

(21)

\[
\omega_{ij} = d\log(u_i u_j) = \frac{d(u_i u_j)}{u_i u_j}
\]

(22)

where \( g \) is the coupling constant. Notice that, from the explicit form of \( A_{ij} \), we can also show that the bialgebraic quantity \( A_{ij} \) satisfy the relations (17) and (18). (For details of this fact, see [40].) These relations are the only conditions for the flatness or integrability of \( A \). Thus, as in the case of \( \Omega \), we can also obtain the expression

\[
DA = dA - A \wedge A = -A \wedge A = 0
\]

(23)

where \( D \) is now a covariant exterior derivative \( D = d - A \). This relation guarantees the existence of holonomies for the comprehensive gauge field \( A \).

Although the bialgebraic structures of \( \Omega \) and \( A \) are different, the constituents of these remain the same, i.e., they are given by \( a_i^{(0)} \) and \( a_i^{(\pm)} \). Thus, we can use the same Hilbert space \( V^\otimes n \) and physical configuration \( \mathcal{C} \) for both \( \Omega \) and \( A \). The KZ equation of \( A \) is then given by \( D\Psi = (d - A)\Psi = 0 \), where \( \Psi \) is a function of a set of spinor momenta \((u_1, u_2, \cdots u_n)\). This suggests that the coupling constant \( g \) is related to the KZ parameter \( \kappa \) by

\[
g = \frac{1}{\kappa}
\]

(24)

### Holonomy operators
A holonomy of \( A \) can be given by a general solution to the KZ equation \( D \Psi = (d - A) \Psi = 0 \). The construction is therefore similar to that of Wilson loop operators. In the present formalism, rank-\( n \) differential manifolds are physically relevant for the construction. Thus, we need differential \( n \)-forms in terms of \( A \) in order to define an appropriate holonomy operator. Further, an analog of Wilson loop should be defined on \( C \). These requirements lead to the following definition of the holonomy operator.

\[
\Theta_{R,\gamma}^{(A)}(u) = \text{Tr}_{R,\gamma} \exp \left[ \sum_{m \geq 2} \int_{\gamma} A \wedge A \wedge \cdots \wedge A \right] \tag{25}
\]

where \( \gamma \) represents a closed path on \( C \) along which the integral is evaluated and \( R \) denotes the representation of the gauge group. The color degree of freedom (or the Chan-Paton factor) can be attached to the physical operators \( a_i^{(\pm)} \) in (21) as

\[
a_i^{(\pm)} = t^{c_i} a_i^{(\pm)c_i} \tag{26}
\]

where \( t^{c_i} \)'s are the generators of the gauge group in the \( R \)-representation. Since here we are interested in the description of gluons, the relevant gauge groups are \( SU(N) \); we shall later consider gauge groups which are relevant to gravitons. The symbol \( P \) denotes an ordering of the numbering indices. The meaning of the action of \( P \) on the exponent of (25) can explicitly be written as

\[
P \sum_{m \geq 2} \int_{\gamma} A \wedge \cdots \wedge A = \sum_{m \geq 2} \int_{\gamma} A_{12} A_{23} \cdots A_{m1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m1}
\]

\[
= \sum_{m \geq 2} \frac{1}{2^{m+1}} \sum_{(h_1, h_2, \ldots, h_m)} (-1)^{h_1 + h_2 + \cdots + h_m}
\]

\[
\times a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_m^{(h_m)} \int_{\gamma} \omega_{12} \wedge \cdots \wedge \omega_{m1} \tag{27}
\]

where we use an ordinary definition of commutators for bialgebraic operators. In the above expression, \( h_i = \pm = \pm 1 \) \((i = 1, 2, \ldots, m)\) denotes the helicity of the \( i \)-th particle. From the above expression, we can easily find that the exponent of the holonomy operator in (25) vanishes if \( m \leq 1 \). This explains the condition \( m \geq 2 \) in (25).

The trace \( \text{Tr}_{R,\gamma} \) in the definition (25) means traces over the Lie-algebra-valued \( t^{c_i} \)'s and over the Hecke-algebra-valued \( \tilde{b}_i \)'s. Since the braid generators \( b_i \)'s essentially give the same algebra as \( \tilde{b}_i \)'s, except the last equation in (19), we can think of the trace over \( \tilde{b}_i \)'s as a braid trace. Information of braid generators along the loop \( \gamma \) can be characterized by orderings of the numbering indices. A braid trace is therefore realized by a sum over permutations of the indices. Thus the braid trace \( \text{Tr}_{\gamma} \) over the exponent of (25) can be expressed as

\[
\text{Tr}_{\gamma} P \sum_{m \geq 2} \int_{\gamma} A \wedge \cdots \wedge A = \sum_{m \geq 2} \sum_{\sigma \in S_{m-1}} \int_{\gamma} A_{1\sigma_2} A_{\sigma_2 \sigma_3} \cdots A_{\sigma_m1} \omega_{1\sigma_2} \wedge \omega_{\sigma_2 \sigma_3} \wedge \cdots \wedge \omega_{\sigma_m1} \tag{28}
\]

\(^{3}\)As discussed before, the difference between \( b_i \) and \( \tilde{b}_i \) is given by the numeric factor \( \eta = \exp(i\pi/2\kappa) \). This factor will be important for some particular solutions to a theory of gravity that we aim for, however, at the level of trace calculations, this difference seems irrelevant and we shall not discuss its effects in the present paper.
where the summation of $S_{m-1}$ is taken over the permutations of the elements $\{2, 3, \ldots, m\}$, with the permutations labeled by $\sigma = (\frac{2 3 \cdots m}{\sigma_2 \sigma_3 \cdots \sigma_m})$. Notice that the expression (28) is valid for a single distinct loop $\gamma$. For an $SU(N)$ gauge group, the color structure, i.e., the trace $Tr_R(t^{a_1} t^{a_2} t^{a_3} \cdots t^{a_n})$ with a sum over permutations, can be characterized by one index, say 1, due to cyclic invariance. Alternatively, this may be seen as a $U(1)$ invariance of the Chan-Paton factor for Yang-Mills theory. If we consider a gravitational theory in the same framework, a corresponding Chan-Paton factor is expected to have a larger symmetry, which will lead to two or more distinct loops associated with a braid trace. We shall clarify these points in section 4.

### 3 Diffeomorphism, braid trace and Chan-Paton factors

In this section, we start constructing a gravitational theory by use of the spinor-momenta formalism in twistor space. A main objective of the present paper is to express an $S$-matrix functional for graviton amplitudes in terms of a gravitational version of the holonomy operator (25), which we shall discuss in the following sections. In this section, we develop fundamental ingredients for the construction of a gravitational theory in the spinor-formalism. A basic idea we will follow is Nair’s interpretation that the so-called maximally helicity violating (MHV) graviton amplitudes can be understood as amplitudes of gauge theory (or open string theory) with an appropriate choice of a Chan-Paton factor. What to be clarified is then Chan-Paton factors of gravitons in the spinor-momenta formalism, which is a main theme of this section.

**Diffeomorphism**

We first notice that the holonomy operator (25) is described by differential forms. Generally, the antisymmetrization of covariant indices and the use of exterior derivatives are required for the invariance under general coordinate transformations (or diffeomorphism). So the use of holonomy operator seems to be more natural in gravitational theories than in Yang-Mills theories. In the present formalism, a situation is not so straightforward. Namely, the bases of the “covariant” differential forms are given by the Lorentz-invariant scalar products $(u_i u_j)$ as shown in (22). Thus an ordinary prescription for diffeomorphism by use of differential forms does not necessarily apply to the present case. Indeed, as discussed in (7), the four-dimensional diffeomorphism is given by changes of spinor momenta or permutations of the numbering indices for spinor momenta. Invariance under diffeomorphism is then realized by taking a sum over the all possible permutations. This is nothing but a braid trace we have discussed in the previous section. Therefore, in this sense, the use of holonomy operator (25) is appropriate for a gravitational theory as well. The difference from Yang-Mills theory is that gauge fields for gravitons are not given by one-forms but by two-forms so that they are to contain states of $\pm 2$ helicities. These fields may be constructed as a product of “comprehensive” frame fields which we consider are analogs of the comprehensive gauge fields $A$.

**Comprehensive frame fields in twistor space**
Gravitons are operators corresponding to metric tensors. The metric tensors are generally defined by products of frame fields or tetrad fields. In a conventional field theory, this can be expressed as

$$g_{\mu\nu} = e_a^\mu e_a^\nu$$  \hspace{1cm} (29)

where $g_{\mu\nu}$, $e_a^\mu$ are the metric tensors and the frame fields, respectively. The index $\mu$ (= 0, 1, 2, 3) denotes the Minkowski indices as before and $a$ (= 0, 1, 2, 3) denotes the coordinate indices for the tangent space. In view of gravity as a gauge theory, its gauge group is given by the Poincaré algebra. An ordinary covariant derivative is then expressed as

$$D_\mu = \partial_\mu + i e_\mu^a p^a + \Omega_{\mu}^{ab} J_{ab}$$  \hspace{1cm} (30)

where $\Omega_{\mu}^{ab}$ and $J_{ab}$ are the spin connection and the Lorentz generator, respectively. $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is a differential operator with respect to the spacetime coordinate $x^\mu$, while $i p^a = \frac{\partial}{\partial x^a}$ is a differential operator with respect to the tangent space coordinate. The latter can be interpreted as a Chan-Paton factor for the frame field.

In the spinor-momenta formalism, Lorentz invariance is manifest. Thus the last term in (30) is irrelevant in the calculations of physical quantities. Scalar fields are by definition Lorentz invariant. Thus another interpretation is that physical fields in the spinor-momenta formalism are described by scalar fields (or superfields) in twistor space. As discussed in [40], this is true in the Yang-Mills case and we shall use the same formalism for gravitational cases.

We now define the comprehensive frame field $E$ as an analog of the comprehensive gauge field $A$ in (20).

$$E = \sum_{1\leq i<j\leq n} E_{ij} \omega_{ij}$$  \hspace{1cm} (31)

$$E_{ij} = e_i^{(\pm)} \otimes e_j^{(0)} + e_i^{(-)} \otimes e_j^{(0)}$$  \hspace{1cm} (32)

$$\omega_{ij} = d \log(u_i u_j) = \frac{d(u_i u_j)}{(u_i u_j)}$$  \hspace{1cm} (33)

where $e_i^{(\pm)}$ and $e_i^{(0)}$ are operators which are algebraically the same as $a_i^{(\pm)}$ and $a_i^{(0)}$, obeying the SL(2, C) algebra in (10). $\omega_{ij}$ is a logarithmic one-form in terms of the Lorentz invariant product of spinor momenta $u_i$ and $u_j$. This one-form is the same as the Yang-Mills version in (22). We now consider a Chan-Paton factor of the frame field. Following the Yang-Mills case, we may impose this factor on the operators $e_i^{(\pm)}$ as

$$e_i^{(\pm)} = e_i^{(\pm)a} (\sqrt{2} p_i)^a = e_i^{(\pm)A\dot{A}} p_i^{A\dot{A}}$$  \hspace{1cm} (34)

where we split the tangent-space index $a$ (= 0, 1, 2, 3) into the two-component indices $A$ and $\dot{A}$. Generally, the tangent space is given by a copy of the coordinate space. So $p_i^{A\dot{A}}$ can be represented by the spinor momenta of interested particles. Explicitly, this can be written as

$$p_i^{A\dot{A}} = (\sigma^a)^{A\dot{A}} p_{ia} = u_i^A \bar{u}_i^{\dot{A}}$$  \hspace{1cm} (35)
where, as discussed before, $\sigma^a$ is given by $\sigma^a = (1, \bar{\sigma})$; $\bar{\sigma}$ and $1$ denote the ordinary $(2 \times 2)$ Pauli matrices and the $(2 \times 2)$ identity matrix, respectively. In order to define gravitons in analogy with (29), we need to consider products of the tangent-space translational operators. We denote these products as

$$p_i^{AA} p_j^{\dot{A}A} = (u_i u_j) [\bar{u}_i \bar{u}_j] = 2 p_i^a p_j^a \equiv \langle p_i \cdot p_j \rangle$$

(36)

where we use the expressions in (4).

So far, we have ignored an effect of a braid generator. Namely, we have not considered a Hecke-algebra-valued quantity in the expression (34). This is natural since a braid generator, by definition, emerges only in a multi-particle system. As long as we consider a single frame field, effects of braid generators are hidden. In the definition of the comprehensive frame field (31), however, we implicitly consider a multi-particle, if not multigraviton, system. Thus these effects are expected to be perceptible. This point may be obvious if we try to construct a graviton field in terms of the comprehensive frame field $E$. Following the relation (29), we can naively define a comprehensive graviton field as a product of $E$’s. This definition by itself however leads to a rather chaotic quantity because a particular frame field potentially couples to any other frame fields in tangent spaces. A Chan-Paton factor of a single graviton should therefore be determined by a certain rule for the couplings. Such a rule can and should be encoded by braid generators. In other words, an explicit form of a graviton Chan-Paton factor depends on a permutation of the numbering indices.

### Comprehensive fields for gravitons

From the above argument, we find that permutations of the numbering indices are involved in an explicit descriptions of comprehensive fields for gravitons. For the moment, we assume that such permutations are given by $\sigma = \left( \begin{array}{c} 2 3 \cdots r \\ \sigma_2 \sigma_3 \cdots \sigma_r \end{array} \right)$ like the Yang-Mills case. As we will see later, a way of taking a permutation in a gravitational theory is not as simple as in the Yang-Mills case. In the present section, for simplicity of discussion, we first assume the Yang-Mills type permutation. Notice that we are going to construct a theory of gravity in a holonomy formalism. This means that a theory, and hence a graviton field, is not well-defined until a gravitational holonomy operator is constructed. Thus a full form of a comprehensive graviton field becomes transparent, once we define a gravitational holonomy operator in the next section, where the full form is obtained in (56). To remind us of this fact, we shall use $r$ instead of $n$ as the number of gravitons for the rest of this section.

Using the notation of (36), we can define a comprehensive graviton field as

$$H = \sqrt{8\pi G_N} \langle E \cdot E \rangle = \sqrt{8\pi G_N} \sum_{1 \leq i < j \leq r} H_{ij} \omega_{ij}$$

(37)

$$H_{ij} = \sum_{\sigma \in S_{r-1}} \langle E_{ij} \cdot E_{\sigma(i)\sigma(j)} \rangle \omega_{\sigma(i+1)\sigma(j+1)}$$

4These effects are in fact buried in a multigluon system as well. In the Yang-Mills case, a multigluon Chan-Paton factor $T_R(t^c_1 t^c_{\sigma_2} t^c_{\sigma_3} \cdots t^c_{\sigma_n})$ is invariant in its form under permutations of $\sigma$’s. ($t^c$’s are generators of a gauge group in the $R$ representation.) Thus this factor is not affected by braid generators. This is a main reason why these effects are not transparent in Yang-Mills theory.
where $G_N$ is the Newton constant. In the natural unit ($c = \hbar = 1$), this is equivalent to an inverse square of the Planck mass $M_{Pl}$.

$$G_N = \frac{1}{M_{Pl}^2} = 6.7088 \times 10^{-39} \left( \frac{1}{\text{GeV}} \right)^2$$

In (38), $T^{\sigma_i}$ ($i = 2, 3, \ldots, r$) are defined as

$$T^{\sigma_i} = \left\langle \left( p_1 + p_{\sigma_{i+1} < i} + p_{\sigma_{i+2} < i} + \cdots + p_{\sigma_i < i} \right) \cdot p_{\sigma_i} \right\rangle = \left\langle \left( p_1 + \sum_{k=i+1}^r p_{\sigma_k < i} \right) \cdot p_{\sigma_i} \right\rangle$$

$$p_{\sigma_i < j} \equiv \begin{cases} p_{\sigma_i} & \text{for } \sigma_i < \sigma_j \\ 0 & \text{otherwise} \end{cases}$$

In the case of $i = 1$, we set $\sigma_1 = \sigma_{r+1} = 1$ and, for a reason we discuss soon, we can define $T^{\sigma_1}$ as

$$T^{\sigma_1} = T^1 = 1$$

What is important in the definition of $T^{\sigma_i}$’s is nothing but one-to-one correspondence between $T^{\sigma_i}$’s and the permutations $\sigma$, which define how a Chan-Paton factor of a particular frame field couples with Chan-Paton factors of the other frame fields. (As we discuss at the end of section 4, the definition of $T^{\sigma_i}$’s is inspired by an explicit form of graviton amplitudes.) A pattern of a particular permutation can diagramatically be shown as Figure 1.

![Figure 1: Braid diagram associated with a permutation of $\{\sigma_2, \sigma_3, \ldots, \sigma_r\}$](image)

The strands in Figure 1 connect the same numbering elements at the top and the bottom. There is correspondence between this two-dimensional diagram and the permutation $\sigma$. We call this diagram a braid diagram in what follows. Of course, there are many ways of drawing the strands with arbitrary twists and turns but it is possible to have an irreducible representation of the diagram for each permutation. In fact, as we shall see in the next section, the definition (40) corresponds to such an irreducible diagram under a condition...
Correspondingly, where a composite notation \(4\)-algebra, each copy acting on a distinct Hilbert space \(V\) can be represented by

\[
\omega_{\sigma_{i+1}\sigma_{j+1}} \cdot \text{identity matrix in the } \mathbb{R} \text{ resolved. The particular choice of } \omega_{\sigma_{i+1}\sigma_{j+1}} \text{ is constructed. Once a gravitational holonomy operator is defined, the arbitrariness in (38) is}
\]

Thus the Chan-Paton factor of this graviton can analogously be interpreted

As in the Yang-Mills case, a comprehensive frame field \(E\) is defined in the configuration space \(\mathcal{C} = \mathbb{C}^n / \mathbb{S}_n\). Thus the physical configuration space for \(H \sim \langle E \cdot E \rangle\) is given by \(\mathcal{C} \times \mathcal{C}\). Correspondingly, \(H_{ij}\) can be interpreted as two copies of bialgebraic operators, rather than \(4\)-algebra, each copy acting on a distinct Hilbert space \(V^\otimes n\). In the operator level, gravitons can be represented by

\[
g_{\sigma_i}^{(±±)} \equiv e_i^{(±a)} e_{σ_i}^{(±a)}
\]

where a composite notation \((±±)\) takes any pairs. Namely, we have \(g_{\sigma_i}^{(++)}\), \(g_{\sigma_i}^{(-+)}\), \(g_{\sigma_i}^{(-±)}\) and \(g_{\sigma_i}^{(−−)}\) among which the first and the last are relevant to gravitons with ±2 helicities. By use of these, we can express (38) as

\[
H_{ij} = E_{ij} \sum_{σ \in \mathbb{S}_{r-1}} T^σ E_{σ_iσ_j} \omega_{σ_iσ_j}
\]

where we should note that \(e_j^{(0)}\) and \(e_{σ_j}^{(0)}\) act on \(e_i^{(±)}\) and \(e_{σ_i}^{(±)}\) from the left, respectively.

We now consider the exceptional case in which \(i\) becomes \(i = 1\). In the Yang-Mills case, a multilgon Chan-Paton factor is given by \(\text{Tr}_R(t^{c_1} t^{c_2} t^{c_3} \cdots t^{c_n})\) where \(t^{c_i}\)'s are generators of a gauge group in the \(R\) representation. In this case, permutations are taken over \(t^{c_i}\)'s \((i = 2, 3, \cdots, n)\) and we can interpret \(t^{c_1}\) as a \(U(1)\) direction attached to the gauge group \(SU(N)\) or a \(U(1)\) generator of the gauge group \(U(N)\). Thus \(t^{c_1}\) can be expressed as the identity matrix in the \(R\) representation of the \(U(N)\) group. There are no braid generators associated with \(t^{c_1}\). This is consistent with the fact that the number of the elements of a braid group \(\mathbb{B}_n\) is given by \(n - 1\) rather than \(n\). We can make an analogous argument for a gravitational case. There are no braid generators associated with the graviton labeled by the index 1. Thus the Chan-Paton factor of this graviton can analogously be interpreted as an identity. This explains the definition in (42). In terms of \(H_{1j}\), this can explicitly be written as

\[
H_{1j} = \sum_{σ \in \mathbb{S}_{r-1}} \left[ g_{1σ_1}^{(++)} + g_{1σ_1}^{(-+)} + g_{1σ_1}^{(−±)} + g_{1σ_1}^{(−−)} \right] \otimes e_j^{(0)} e_{σ_j}^{(0)} \omega_{σ_2σ_{j+1}}
\]

where \(σ_1\) is fixed at \(σ_1 = 1\). As we shall discuss later, there are other exceptional indices that follow the expression (45); a symmetry analysis of a Chan-Paton factor in a gravitational holonomy operator would reveal that there are in fact two other such indices. (In terms of the numbering indices \(σ_i\) \((i = 1, 2, \cdots, n)\), the exceptional ones can be given by \(σ_1 = 1\), \(σ_{n-1} = n - 1\) and \(σ_n = n\).)
Chan-Paton factors of gravitons

From (38)-(45), we can rewrite the comprehensive field $H$ as

$$H = \sqrt{8\pi G N} \sum_{1 \leq i < j \leq r} \sum_{\sigma \in S_{r-1}} \langle E_{ij} \cdot E_{\sigma_i \sigma_j} \rangle \omega_{ij} \omega_{\sigma_{i+1} \sigma_{j+1}}$$

(46)

$$\langle E_{ij} \cdot E_{\sigma_i \sigma_j} \rangle = \sum_{h_i} T^{\sigma_i} g^{(h_i)}_{\sigma_i} \otimes e^{(0)}_j e^{(0)}_{\sigma_j} = \sum_{h_i} g^{(h_{\sigma_i})}_{\sigma_i} \otimes g^{(00)}_j$$

(47)

$$g^{(h_{\sigma_i})}_{\sigma_i} \equiv T^{\sigma_i} g^{(h_{\sigma_i})}_{\sigma_i} = T^{\sigma_i} (e^{(h_i)\alpha}_{\sigma_i}) (e^{(h_{\sigma_i})\alpha}_{\sigma_i})$$

(48)

$$g^{(00)}_j \equiv (1)^{\sigma_j} e^{(0)}_j e^{(0)}_{\sigma_j}$$

(49)

where the sum of $h_{\sigma_i}$ is taken over $h_{\sigma_i} \equiv h_i h_{\sigma_i} = (++,-+,+-,--)$. The expression (47) is analogous to that of (21) with (26). Thus we can naturally interpret $T^{\sigma_i}$ as a Chan-Paton factor of the graviton operators $g^{(++)}_i$ and $g^{(--)}_i$. Notice that the color indices are now given by the numbering indices to be permuted, which is natural since Chan-Paton factors of gravitons should be encoded by braid generators. We have not explicitly used the braid generators, however, as discussed earlier, information of braid generators is in one-to-one correspondence with permutations of the indices as long as we use an irreducible representation.

In the above expressions, the operators of $g^{(+-)}_{\sigma_i}$ and $g^{(-+)}_{\sigma_i}$ are naturally incorporated. These represent massless spin-zero particles with no charges. Life times of these should be the same as those of gravitons. So they are stable. We may therefore think of these spin-zero particles as a candidate for the origin of dark matter or something that couples to dark energy.

We find that $H$ in the form of (46) is basically given by a sum over non-vanishing products of the Chan-Paton factors of frame fields or the translational operators in tangent spaces. This is desirable in a view that we need to integrate over all metrics for the construction metric-free or topological theories. Notice that we also need to have another sum over the permutations so as to take care of diffeomorphism invariance. In the construction of a gravitational theory, this is realized by a braid trace in a holonomy operator. This provides another reason for considering a holonomy formalism in search of a theory of gravity. As mentioned elsewhere, a full definition of $H$ is then clarified after the construction and computation of a gravitational holonomy operator. (See the expression (56) for the full definition of $H$.)

Summary

In this section, we consider a comprehensive graviton field $H$ in a framework of the holonomy formalism which we have introduced in the previous section. Since we have not yet constructed a holonomy operator of $H$, a full-fledged definition of $H$ is to be obtained in the next section. The results of the present section are, however, quintessential for the construction of gravitational theories in twistor space. These results can be summarized as follows.

1. Diffeomorphism invariance in the spinor-momenta formalism is realized by a braid trace or a sum over permutations of the numbering indices.
2. A Chan-Paton factor of a frame field is given by a translational operator or a four-momentum in a tangent space.

3. A physical configuration space of gravitons is given by $C \times C$, where $C = C^n / S_n$. Here $C^n$ denotes complex number and $S_n$ denotes the rank-$n$ symmetric group.

4. Accordingly, a quantum Hilbert space of gravitons is given by $V \otimes V \otimes V \otimes V$, where $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ with $V_i$ ($i = 1, 2, \ldots, n$) representing a Fock space that one set of the frame field operators $e_i^{(\pm)}$ act on. The other half of the Hilbert space consists of Fock spaces for another set of frame fields $e_{\sigma_i}^{(\pm)}$.

5. Even though the Fock space of $e_i^{(\pm)}$ and that of $e_{\sigma_i}^{(\pm)}$ are different from each other, tangent spaces of the two operators are common since these are the constituents of a single graviton operator. A product of their Chan-Paton factors is defined on the common tangent space and is interpreted as a Chan-Paton factor of the graviton.

6. How to choose couplings of translational operators in tangent spaces is determined by braid generators. In other words, a Chan-Paton factor of a single graviton is encoded by a permutation of the numbering indices that label gravitons. An explicit form of this factor is essentially given by $T_{\sigma_i}$ in (40). For a full definition, we also need $T^n$ in (52) to be defined in the next section.

4 Gravitational holonomy operators

In analogy with the Yang-Mills case (25), we can construct a holonomy operator of the comprehensive graviton field $H$ as

$$\Theta^{(H)}_{R, \gamma}(u, \bar{u}) = \text{Tr}_{R, \gamma} P \exp \left[ \sum_{m \geq 5} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H \right]$$

(50)

where $\gamma$ represents a closed path on $C$ along which the integral is evaluated and $R$ denotes representations of Poincaré algebra and Iwahori-Hecke algebra. Since $H$ is defined on $C \times C$, the integral should be interpreted as a double integral. The loop $\gamma$ is commonly defined on each of $C$'s. This point will be clearer by the end of this section. The condition $m \geq 5$ will also be clarified later. As in the Yang-Mills case, the symbol $P$ denotes the ordering of the numbering indices. As discussed in the previous section, Chan-Paton factors of $H$ depend on a permutation of the numbering indices. In practical computations, we need to clarify this dependence. Thus, in the following, we consider an exact meaning of the Chan-Paton factor in (50).

Symmetries of Chan-Paton factors

We first consider the significance of the fact that the Chan-Paton factors of the comprehensive frame fields have vectorial properties. In the Yang-Mills case, the Chan-Paton factor has a cyclic property and we relate this to a $U(1)$ symmetry of the factor. In a gravitational case, the relevant symmetry can be given by a symmetry for a set of translational operators.
in tangent spaces. A tangent space is a copy of a coordinate space and the translational operators or the four-momenta can be encoded by the spinor momenta. Thus the relevant symmetry is given by a symmetry for a set of the spinor momenta. Since these spinor momenta are defined on \( \mathbb{CP}^1 \), we can map their local coordinates \( z_i \) \( (i = 1, 2, \cdots, m) \) on a complex \( z \)-plane by stereographic projections. It is well known that the conformal transformations of the complex \( z \)-plane (including the point at infinity) is given by an \( SL(2, \mathbb{C}) \) group. Thus, apart from gauge symmetries related to Pincaré algebra and Iwahori-Hecke algebra, we can identify the symmetry of the gravitational Chan-Paton factor as the \( SL(2, \mathbb{C}) \) symmetry. This symmetry is also related to the Lorentz invariance of \( u_i \)'s as shown in (3).

We now consider the effects of the \( SL(2, \mathbb{C}) \) symmetry on the braid trace. First of all, we note that the number of elements in the \( SL(2, \mathbb{C}) \) group is three. This suggests that there exist three indices which characterize the braid trace. Remember that in the Yang-Mills case the Chan-Paton factor has been characterized by one index due to the \( U(1) \) symmetry of the Chan-Paton factor. The index 1 has been chosen for a fixed numbering index for this reason. This also corresponds to the fact that we have a single closed loop along which a braid trace is defined. Notice that mathematically it is known that a loop (or a link) forms a braid group under *isotopy* of the loop. In this sense, the loop can be denoted as \( \gamma_1 \). If we have an \( SL(2, \mathbb{C}) \) symmetry for the Chan-Paton factor as in the present case, then the braid trace is characterized by three distinct loops or three disconnected links. Following a convention, we can choose the three numbering indices as \((1, m-1, m)\) so that corresponding loops are labeled by \( \gamma_1, \gamma_{m-1}, \gamma_m \). This means that we have permutations of the numbering elements \( \{2, 3, \cdots, m-2\} \) in the definition of the holonomy operator (50).

There must be correspondence between the loops \((\gamma_1, \gamma_{m-1}, \gamma_m)\) and the elements of \( SL(2, \mathbb{C}) \) algebra, say, a set of generators \((t^+, t^-, t^0)\). Since the \( SL(2, \mathbb{C}) \) symmetry is global or comprehensive in the present context, these generators are not labeled by a particular numbering index. Instead, a set of numbering indices can be used to define a “state” of a loop which is characterized by each of the generators. Such a characterization can be carried out as follows. We first regard the numbering index as something analogous to a quantum number of \( z \)-direction in the conventional angular momentum algebra. In terms of this number, the loop \( \gamma_m \) which corresponds to \( t^0 \) is trivial. The loop \( \gamma_m \) is then expected to have only one element for the numbering index, otherwise the \( SL(2, \mathbb{C}) \) symmetry would be enhanced to include more \( U(1) \) symmetries. Thus, basically, distinct loops of the Chan-Paton factor is characterized by the ladder generators \( t^{(\pm)} \) of \( SL(2, \mathbb{C}) \). One natural way of realizing this characterization is to make the assigned elements of numbering indices in a descending order for the loop \( \gamma_1 \) and in an ascending order for the loop \( \gamma_{m-1} \), along with certain orientations of the loops. Let us denote the elements of \( \gamma_1 \) by \( \{\sigma_2, \sigma_3, \cdots, \sigma_r\} \) and those of \( \gamma_{m-1} \) by \( \{\tau_{r+1}, \tau_{r+2}, \cdots, \tau_{m-2}\} \) \( (2 \leq r \leq m-3) \). Then the three disconnected loops can diagramatically be shown as Figure 2. In the figure, the numbering elements are ordered by \( \sigma_3 < \sigma_3 < \cdots < \sigma_r \) and \( \tau_{r+1} < \tau_{r+2} < \cdots < \tau_{m-2} \), with the union of these elements being \( \{2, 3, \cdots, m-2\} \).

Notice that an ordering of the numbering elements naturally arises from the characterization of loops by the \( SL(2, \mathbb{C}) \) algebra. This is a supportive fact for the appearance of the ordering symbol \( \mathrm{P} \) in the holonomy operator (50).
Figure 2: Braid diagrams for the calculation of Chan-Paton factors — to make a loop out of each diagram, we need to connect an index on the top with the one at the vertical bottom. The starting points can be chosen as the top indices $1, m-1, m$, with orientations of loops shown by arrows ($m \geq 5$). When two lines are crossing each other, we consider that a line with an arrow is closer to us, crossing over the other line without an arrow.

Explicit calculations

By use of the above analysis, we now calculate a Chan-Paton factor of the following quantity.

$$\text{Tr}_{R,\gamma} \mathcal{P} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H$$

This is essentially the exponent of the holonomy operator (50); to obtain a full form, we simply take a sum over $m \geq 5$. Probably, the simplest calculation is given by making an assignment of the numbering elements $\{2, 3, \cdots, r\}$ to $\sigma$’s and $\{r+1, r+2, \cdots, m-2\}$ to $\tau$’s, respectively. So the numbering elements are split into two parts. Under the ordering conditions, $\sigma_2 < \sigma_3 < \cdots < \sigma_r$ and $\tau_{r+1} < \tau_{r+2} < \cdots < \tau_{m-2}$, these elements are uniquely determined. There is another way of calculating the Chan-Paton factor in (51). This can be carried out by assigning $\sigma$’s and $\tau$’s to the overall elements $\{2, 3, \cdots, m-2\}$ homogeneously. Namely, the elements of both $\sigma$’s and $\tau$’s can take any values in the overall elements, given that they satisfy the ordering conditions. In the present paper, we shall leave this homogeneous case aside and consider that split case only.\(^5\)

There are essentially two important ingredients in an explicit calculation of the Chan-Paton factor, which can be stated as follows.

1. A sum over all possible metrics: This is necessary for the construction of a gravitational theory which preserves general covariance.

2. A braid trace or a sum over permutations of the numbering elements: This is necessary for diffeomorphism invariance.

\(^5\)Consideration of the homogeneous case will give an interpretation of $\Theta^{(H)}_{R,\gamma}$ as a square of $\Theta^{(E)}_{R,\gamma}$ in an intriguing way [38]. This point will be investigated in a separate paper.
In the present case, we have two independent permutations, \( i.e. \), \( \sigma = \left( \begin{array}{c} 2 \cdots r \\ \sigma_2 \cdots \sigma_r \end{array} \right) \) and \( \tau = \left( \begin{array}{c} r+1 \cdots m-2 \\ \tau_{r+1} \cdots \tau_{m-2} \end{array} \right) \). A sum over all possible metrics is then realized by a sum over these two permutations combined. On the other hand, as in the Yang-Mills case, a braid trace can be realized by a sum over a permutation of the overall elements \( \{2, 3, \ldots, m-2\} \). This sum (or trace) should be taken on top of the sum over the permutations of \( \sigma \)'s and \( \tau \)'s, which suggests that the Chan-Paton factor in (51) is independent of the choice of \( r \) (\( 2 \leq r \leq m-3 \)).

Figure 3: Braid diagrams that include permutations of \( \sigma \)'s and \( \tau \)'s — the symbol P denotes an ascending ordering of the arguments. A sum over all permutations corresponds to a sum over all possible metrics in a multigraviton system.

In Figure 3, we show braid diagrams that take account of the permutations of \( \sigma \)'s and \( \tau \)'s. In the figure, the elements of \( \sigma \)'s and \( \tau \)'s are in a random order, while the symbol P denotes an ascending ordering of the elements. A graviton labeled by a particular numbering index corresponds to a particular strand in the braid diagrams. A Chan-Paton factor of a graviton for a specific choice of the permutations can then be encoded by corresponding braid diagrams. Structures of the stands are schematically shown by thick down-arrows in Figure 3. As discussed in the previous section, explicit forms of Chan-Paton factors for gravitons are determined by these structures. A diagram on the left side in Figure 3 is the same as the one in Figure 1. Graviton Chan-Paton factors pertinent to this diagram are therefore given by (40). Similarly, we can define graviton Chan-Paton factors pertinent to a diagram in the center of Figure 3 as

\[
T^{\tau_i} = \left\langle p_{\tau_i} \cdot \left( p_{m-1} + p_{\tau_{i<r+1}} + p_{\tau_{i<r+2}} + \cdots + p_{\tau_{i<i-1}} \right) \right\rangle \\
= \left\langle p_{\tau_i} \cdot \left( p_{m-1} + \sum_{k=r+1}^{i-1} p_{\tau_{i<k}} \right) \right\rangle \quad \text{for } i = r + 1, r + 2, \ldots, m - 2 \\
T^{\tau_{m-1}} = T^{m-1} = 1
\]  

(52)  

(53)
where the definition of $p_{\tau_{i<j}}$ is the same as (41) except that we replace $\sigma_i$ ($i = 2, 3, \cdots, r$) by $\tau_i$ ($i = r + 1, r + 2, \cdots, m - 2$). A graviton Chan-Paton factor corresponding to a diagram on the right in Figure 3 is simple. Since there are no permutations involved, as in the cases of $i = 1$ and $i = m - 1$, the Chan-Paton factor of the $m$-th graviton is defined as

$$T^m = 1$$

Using these expressions, we now obtain a full form of the comprehensive graviton field $H$ as follows.

$$H = \sqrt{8\pi G_N} \sum_{1 \leq i < j \leq m} \sum_{\tau \in S_{m-r-2}} \left( \sum_{h_i} g_i^{(h_{i\mu_i})} \otimes g_j^{(00)} \right) \omega_{ij} \omega_{\lambda_i\lambda_j}$$

$$\mu_i = \begin{cases} \sigma_i & \text{for } i = 1, 2, \cdots, r \\ \tau_i & \text{for } i = r + 1, r + 2, \cdots, m - 1 \\ m & \text{for } i = m \end{cases}$$

$$\lambda_i = \begin{cases} \sigma_{i+1} & \text{for } i = 1, 2, \cdots, r \\ \tau_{i-1} & \text{for } i = r + 1, r + 2, \cdots, m - 1 \\ m & \text{for } i = m \end{cases}$$

$$g_i^{(h_{i\mu_i})} = T^{\mu_i} g_i^{(h_{i\mu_i})} = T^{\mu_i} e_i^{(h_{i\mu_i})} a$$

$$g_j^{(00)} = (1)^{\mu_j} e_j^{(0)} e_{\mu_j}^{(0)}$$

where the sum of $h_{i\sigma_i}$ is taken over $h_{i\mu_i} \equiv h_i h_{\mu_i} = (+++, --, --, --)$ as in (47). Notice that the index $\mu_i$ is a composite numbering index and that it should not be confused with a Minkowski index. From an SL$(2, \mathbb{C})$ symmetry of the comprehensive graviton field, we can fix the following indices.

$$\sigma_1 = 1, \tau_{m-1} = m - 1, \sigma_m = m, \lambda_r = 1, \lambda_{r+1} = m - 1, \lambda_{m+1} = \sigma_2$$

These choices are in accord with the braid diagrams in Figure 3. Information of $\lambda_{m+1}$ is necessary in defining the gravitational holonomy operator (50). Rigorously speaking, the number of gravitons should be represented by $n$ rather than $m$ in (56). If we substitute $H$ into the holonomy operator $\Theta^{(H)}(u, \bar{u})$, the number $n$ effectively becomes $m$ in the computations of the quantity (51). Thus, the expression (56), along with (40)-(42), (52)-(55) and (57)-(61), provides a full definition of the comprehensive fields for gravitons, complementing the arguments in the previous section.

Our particular choice of the graviton Chan-Paton factors, i.e., $T^{\mu_i}$'s in (59), are determined by the braid diagrams in Figure 3. These diagrams correspond an arbitrary permutation of the numbering indices which respects the SL$(2, \mathbb{C})$ symmetry of the Chan-Paton factor in the quantity (51). These interrelations arise from the fact that the Chan-Paton factors of gravitons are made of the Poincaré algebra and the Iwahori-Hecke algebra. Since
Lorentz invariance is manifest in the holonomy formalism, the Poincaré symmetry reduces to a symmetry of spacetime translations. An irreducible representation of this symmetry is given by translational operators or four-momenta, which we have identified with the Chan-Paton factors of frame fields. Needless to say, a quantum field theory is defined by a unitary irreducible representation (UIR) of physical observables. Irreducibility is crucial here to extract the pure four-momenta as basic ingredients of the Chan-Paton factors. The same argument applies to the Iwahori-Hecke algebra as well, that is, the Chan-Paton factors of gravitons should also be described by an irreducible representation of the braid generators. In terms of the braid diagrams, irreducibility means that the pattern of each diagram is uniquely determined up to isotopy or the so-called Reidemeister moves. There are in fact many irreducible representations in this regard. Our choice of $T^\mu_i$’s is one of them. In what follows, we shall see this point in a step-by-step manner, starting from the case of $m = 5$ to more general cases.

For $m = 5$

As discussed before, the graviton Chan-Paton factors are characterized by three distinct loops, due to an $SL(2, \mathbb{C})$ symmetry of the Chan-Paton factors. The condition $m \geq 5$ on the number of gravitons is imposed by the very distinctiveness of the three loops.

For $m = 5$, there is only one element for either $\sigma$ or $\tau$, i.e., $\sigma_2 = 2$ or $\tau_3 = 3$, so that there are no permutations involved in braid diagrams. This corresponds to the fact that there is only one choice of $r$ ($2 \leq r \leq m - 3$) for $m = 5$. A structure of the diagrams is therefore uniquely determined as in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{braid_diagram_m_5.png}
\caption{Braid diagrams for $m = 5$}
\end{figure}

In the case of $m = 5$, the three indices to be fixed are given by $\{1, 4, 5\}$. The graviton Chan-Paton factors labeled by these indices are trivial, i.e., $T^1 = T^4 = T^5 = 1$. Thus nontrivial factors arise from the gravitons labeled by $\sigma_2 = 2$ and $\tau_3 = 3$. From (40) and (52), we find that these are given by

$$f(\sigma_2) \equiv T^{\sigma_2} = \langle p_1 \cdot p_{\sigma_2} \rangle$$
$$f(\tau_3) \equiv T^{\tau_3} = \langle p_{\tau_3} \cdot p_4 \rangle$$

(62)

Notice that these can easily be read off from Figure 4. We first look at the strand of $\sigma_2$ and then interpret the crossing with the strand of 1 as a coupling between the Chan-Paton factors of the frame fields labeled by $\sigma_2$ and 1. A Chan-Paton factor relevant to the strand of $\tau_3$ can
similarly be determined by the middle diagram in Figure 4. Since there is no permutation, the diagrams are automatically irreducible at this level. It is, however, illustrative to express the quantity (51) for \( m = 5 \) in terms of (62). We can write down an explicit expansion as

\[
\text{Tr}_{R, \gamma} P \oint H \wedge H \wedge H \wedge H \wedge H = (8 \pi G_N)^{\frac{5}{2}} \text{Tr}_{R, \gamma} \oint H^{12} H^{23} \cdots H^{51} \omega^{12} \wedge \omega^{23} \wedge \cdots \wedge \omega^{51}
\]

\[
= (8 \pi G_N)^{\frac{5}{2}} \left( \frac{1}{26} \right)^2 f(\sigma_2) f(\tau_3) \sum_{(h_{11}, h_{2\sigma_2}, \cdots, h_{55})} g^{(h_{11})} \otimes g^{(h_{2\sigma_2})} \otimes g^{(h_{3\tau_3})} \otimes g^{(h_{44})} \otimes g^{(h_{55})}
\]

\[
\times \oint \omega^{12} \wedge \omega^{23} \wedge \omega^{34} \wedge \omega^{45} \wedge \omega^{51} \oint \omega^{\sigma_21} \wedge \omega^{14} \wedge \omega^{4\tau_3} \wedge \omega^{\tau_35} \wedge \omega^{5\sigma_2} \bigg|_{\sigma_2=2, \tau_3=3}
\]

\[
+ \mathcal{P}(23)
\]

where the sum of \( (h_{11}, h_{2\sigma_2}, \cdots, h_{55}) \) is taken over any combinations of \( h_{i\mu_i} \equiv h_i h_{\mu_i} = (++, +-, --, -+, -) \) for \( i = 1, 2, \cdots, 5 \), with \( \mu_i \) being defined as (57). \( \mathcal{P}(23) \) in the last line indicates terms obtained by the permutation of the numbering indices \( \{2, 3\} \) or \( \{\sigma_2, \tau_3\} \). This permutation arises from the braid trace \( \text{Tr}_{\gamma} \). As indicated in the second last line, \( \sigma_2 \) and \( \tau_3 \) are fixed. This can be interpreted as non-existence of a sum over possible metrics in the present case. Such a sum appears for \( m \geq 6 \) as we shall see in the following.

For \( m = 6 \)

In this case, the indices to be fixed are given by \( \{1, 5, 6\} \). Those that are relevant to a braid trace are given by \( \{\sigma_2, \sigma_3\} = \{2, 3\} \) and \( \tau_4 = 4 \). The braid diagrams for the calculation of the Chan-Paton factor in (51) are then shown as Figure 5.

![Braid diagrams for \( m = 6 \)](image)

In analogy with (61), we can define nontrivial factors as \( f(\sigma_2 \sigma_3) = T^{\sigma_2} T^{\sigma_3} \) and \( \tilde{f}(\tau_4) = T^{\tau_4} \). Explicit forms of these can be written as

\[
f(23) = \langle p_1 \cdot p_2 \rangle \langle p_1 \cdot p_3 \rangle
\]
\[
\begin{align*}
  f(32) &= \langle p_1 \cdot p_2 \rangle \langle (p_1 + p_2) \cdot p_3 \rangle \\
  \tilde{f}(4) &= \langle p_4 \cdot p_5 \rangle 
\end{align*}
\]

By use of the previous rules, it is obvious that we can reproduce these factors directly from Figure 5. It is also easy to see irreducibility of the diagrams, which can be understood as follows. We first notice that the characterization of braid diagrams in terms of an \( SL(2, \mathbb{C}) \) symmetry leads to preservation of a basic pattern shown on the left side in Figure 5. The right-hand side diagrams are build on the basic pattern with a subdiagram that represents a transposition of the indices 2 and 3. These diagrams, which is essentially given by the one labeled by \{1, 2, 3\}, are uniquely determined up to isotopy. In other words, the diagram of \{1, 2, 3\} in the right side of Figure 5 is irreducible up to the Reidemeister moves shown in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{reidemeister_moves.png}
\caption{Reidemeister moves corresponding to the raising operator \( t^{(+)} \) of the \( SL(2, \mathbb{C}) \) algebra}
\end{figure}

Notice that the diagrams labeled by the indices \{1, 2, 3\} correspond to the raising operator \( t^{(+)} \) of the \( SL(2, \mathbb{C}) \) algebra and that they have a crossing rule, i.e., if two strands are crossing each other, then a strand with the smaller index crosses over the other strand. We have actually drawn Figure 1, following this rule. On the other hand, the diagrams labeled by the indices \{4, 5\} correspond to the lowering operator \( t^{(-)} \) of the \( SL(2, \mathbb{C}) \) algebra and, in this case, a crossing rule can be stated as, “If two strands are crossing each other, then a strand with the larger index crosses over the other strand”. The Reidemeister moves for this type of braid diagrams can be given by Figure 7.

In both cases, the type-I moves are irrelevant in the present context. Only the type-II and type-III moves will be used for the irreducibility of the braid diagrams in general. In the case of \( m = 6 \), there are only two numbering elements to be permuted. Thus only the type-II move in Figure 6 is used to obtain irreducible diagrams. The type-III moves will be utilized for \( m \geq 7 \).

Without a notion of irreducibility, we can in principle put any subdiagrams that are made
of the Reidemeister moves on top of the basic diagrams indicated in the left side of Figure 5. Such procedures produce reducible diagrams. In reducible diagrams, there are crossings between the same strands more than once. Thus irreducibility in this context means a fact that a particular strand crosses with a specific strand once or none. This condition is in fact satisfied for any \( m \), and is reflected in the definitions of \( T^{\sigma_1} \) and \( T^{\sigma_2} \).

Using the expressions in (64), we can explicitly calculate the quantity (51) as

\[
\text{Tr}_{R,\gamma} P \oint_{\gamma} H \wedge H \wedge H \wedge H \wedge H
\]

\[
= (8\pi G_N)^3 \text{Tr}_{R,\gamma} \oint_{\gamma} H_{12} H_{23} \cdots H_{61} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{61}
\]

\[
= (8\pi G_N)^3 \left( \frac{1}{27} \right)^2 \sum_{\sigma \in S_2} f(\sigma_2 \sigma_3) \tilde{f}(\tau_4)
\]

\[
\times \sum_{(h_{11}, h_{2\sigma_2}, \ldots, h_{66})} g_{11}^{(h_{11})} \otimes g_{2\sigma_2}^{(h_{2\sigma_2})} \otimes g_{3\sigma_3}^{(h_{3\sigma_3})} \otimes g_{4\tau_4}^{(h_{4\tau_4})} \otimes g_{55}^{(h_{55})} \otimes g_{66}^{(h_{66})}
\]

\[
\times \oint_{\gamma} \omega_{12} \wedge \omega_{23} \wedge \omega_{34} \wedge \omega_{45} \wedge \omega_{56} \wedge \omega_{61} \oint_{\gamma} \omega_{\sigma_2 \sigma_3} \wedge \omega_{\sigma_3 \sigma_1} \wedge \omega_{15} \wedge \omega_{5\tau_4} \wedge \omega_{\tau_4 6} \wedge \omega_{6\sigma_2} \bigg|_{\tau_4 = 4}
\]

\[
+ P(234)
\]

Notice that we now have a sum over permutations of \( \sigma = \left( \begin{array}{cc} 2 & 3 \\ \sigma_2 & \sigma_3 \end{array} \right) \). This sum corresponds to a sum over possible metrics in a six-graviton system.

**For \( m = 7 \)**

At this stage, it is straightforward to extend our formalism to the case of \( m = 7 \). We choose \( r \) to be \( r = 4 \) so that the \( \tau \)-part of permutation is trivially fixed at \( \tau_5 = 5 \). The \( \sigma \)-part of the braid diagrams for \( m = 7 \) are then given by Figure 8.
In Figure 8, a link which is associated with each of the braid diagrams is also shown. Notice that the last diagrams with the factor of $f(432)$ (at the right-bottom corner) contains a subdiagram that is equivalent to the left-hand-side pattern of the type-III Reidemeister moves in Figure 6. Thus this subdiagram can be replaced by the other pattern of the same type-III moves. As is expected, such a replacement does not change the factor of $f(432)$ or $T^{\sigma_i}$'s, which shows another confirmation that $T^{\sigma_i}$'s correspond to an irreducible representation of the braid generators.

So far, we have not made direct use of the braid generators. This is because information of the braid generators is, at the level of trace calculations, encoded by a permutation of the numbering indices. An extraction of a specific braid generator labeled by a single numbering element therefore does not lead to physical quantities; we rather need information of full or comprehensive permutations of the indices in order to obtain physical quantities. In the present case, such information is given by $f(\sigma_2\sigma_3\sigma_4)$, which can be expressed as

$$f(\sigma_2\sigma_3\sigma_4) = T^{\sigma_2}T^{\sigma_3}T^{\sigma_4} \equiv \text{Tr}_R \oint_{\gamma_1} B_{\sigma_2\sigma_3\sigma_4}$$

where we introduce notation $B_{\sigma_2\sigma_3\sigma_4}$ to indicate dependence on braid generators. An irreducible representation of the braid generators are given by the elements of Iwahori-Hecke algebra $\hat{b}_i$ in (19). These elements depend on the numeric factor $\eta = \exp(i\pi/2\kappa)$. Thus we expect some contributions of this factor $\eta$ in $B_{\sigma_2\sigma_3\sigma_4}$; clarification of this point is currently under investigation.

In (66), $\text{Tr}_R$ denotes a trace over Poincaré algebra. This trace also implies a fact that
we can actually use any index among \(\{1, \sigma_2, \sigma_3, \sigma_4\}\) as a representing index of a loop corresponding to the raising operator of the \(SL(2, \mathbb{C})\) algebra. Of course, this trace does not mean cyclicity of the indices at all. In fact, as in the previous cases, explicit forms of (66) can easily be obtained from the definition of \(T^\sigma_\gamma\)'s as

\[
\begin{align*}
\mathcal{f}(234) &= \text{Tr}_R \int_{\gamma_1} B_{234} = \langle p_1 \cdot p_2 \rangle \langle p_1 \cdot p_3 \rangle \langle p_1 \cdot p_4 \rangle \\
\mathcal{f}(243) &= \text{Tr}_R \int_{\gamma_1} B_{243} = \langle p_1 \cdot p_2 \rangle \langle (p_1 + p_3) \cdot p_4 \rangle \langle p_1 \cdot \phi_3 \rangle \\
\mathcal{f}(324) &= \text{Tr}_R \int_{\gamma_1} B_{324} = \langle (p_1 + p_2) \cdot p_3 \rangle \langle p_1 \cdot p_2 \rangle \langle p_1 \cdot p_3 \rangle \\
\mathcal{f}(342) &= \text{Tr}_R \int_{\gamma_1} B_{342} = \langle (p_1 + p_2) \cdot p_3 \rangle \langle (p_1 + p_2) \cdot p_4 \rangle \langle p_1 \cdot p_2 \rangle \\
\mathcal{f}(432) &= \text{Tr}_R \int_{\gamma_1} B_{432} = \langle (p_1 + p_2 + p_3) \cdot p_4 \rangle \langle (p_1 + p_2) \cdot p_3 \rangle \langle p_1 \cdot p_2 \rangle \\
\mathcal{f}(423) &= \text{Tr}_R \int_{\gamma_1} B_{423} = \langle (p_1 + p_2 + p_3) \cdot p_4 \rangle \langle p_1 \cdot p_2 \rangle \langle p_1 \cdot p_3 \rangle \\
\end{align*}
\]

In terms of these factors, the quantity (51) can be calculated as

\[
\begin{align*}
\text{Tr}_{R, \gamma} P \int_{\gamma} H \wedge H \wedge H &\wedge H \wedge H \\
= (8\pi G_N)^\frac{m}{2} \text{Tr}_{R, \gamma} \int_{\gamma} H_{12} H_{23} \cdots H_{m1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m1} \\
= (8\pi G_N)^\frac{m}{2} \left( \frac{1}{2^m} \right)^2 \sum_{\sigma \in S_3} f(\sigma_2 \sigma_3 \sigma_4) \tilde{f}(\tau_4) \\
&\quad \times \sum_{(h_{11}, h_{2\sigma_2}, \ldots, h_{7\tau_7})} g_{h_{11}} \otimes g_{h_{2\sigma_2}} \otimes g_{h_{3\tau_3}} \otimes g_{h_{4\tau_4}} \otimes g_{h_{5\tau_5}} \otimes g_{h_{6\sigma_2}} \otimes g_{h_{7\tau_7}} \\
&\quad \times \int_{\gamma} \omega_{12} \wedge \omega_{23} \wedge \omega_{34} \wedge \omega_{45} \wedge \omega_{56} \wedge \omega_{67} \wedge \omega_{71} \\
&\quad \times \int_{\gamma} \omega_{\sigma_2 \sigma_3} \wedge \omega_{\sigma_3 \sigma_4} \wedge \omega_{\sigma_4 \tau_4} \wedge \omega_{\tau_4 \tau_5} \wedge \omega_{\tau_5 \tau_6} \wedge \omega_{\tau_6 \tau_7} \wedge \omega_{\tau_7 \sigma_2} \bigg|_{\tau_5 = 5} \\
&\quad + \mathcal{P}(2345) \\
\end{align*}
\]

**General cases**

For completion of the discussion, in the following we present an explicit calculation of the quantity (51) for arbitrary \(m\).

\[
\begin{align*}
\text{Tr}_{R, \gamma} P \int_{\gamma} H \wedge H \wedge \cdots \wedge H \\
= (8\pi G_N)^m \text{Tr}_{R, \gamma} \int_{\gamma} H_{12} H_{23} \cdots H_{m1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m1} \\
= (8\pi G_N)^m \left( \frac{1}{2^{m+1}} \right)^2 \sum_{\sigma \in S_{r-1}} \sum_{\tau \in S_{m-r-2}} f(\sigma) \tilde{f}(\tau)
\end{align*}
\]
\[
\sum_{(h_{11}, h_{22}, \ldots, h_{mm})} g_{h_{11}} \otimes g_{h_{22}} \otimes g_{h_{33}} \otimes \cdots \otimes g_{h_{rr}} \\
\otimes g_{h_{r+1}} \otimes g_{h_{r+2}} \otimes \cdots \otimes g_{(h_{m-2})} \otimes g_{(h_{m-1})} \otimes g_{(h_{mm})}
\times \oint \gamma \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m-1} \wedge \omega_{m1}
\times \oint \gamma \omega_{\sigma_2 \sigma_3} \wedge \omega_{\sigma_3 \sigma_4} \wedge \cdots \wedge \omega_{\sigma_{r-1} \sigma_r} \wedge \omega_{\sigma_1}
\wedge \omega_{1 \cdot \sigma_1} \wedge \omega_{m-1} \wedge \omega_{m-1 \cdot \tau_{r+1}} \wedge \omega_{r+1 \cdot \tau_{r+2}} \wedge \cdots \wedge \omega_{m-2} \wedge \omega_{m \sigma_2}
+ \mathcal{P}(23 \cdots m - 2) \tag{69}
\]

where \(f(\sigma)\) and \(\tilde{f}(\tau)\) are defined as

\[
f(\sigma) = \prod_{i=2}^{r} T^{\sigma_i}, \quad \tilde{f}(\tau) = \prod_{i=r+1}^{m-2} T^{\tau_i} \tag{70}
\]

Explicit forms of \(T^{\sigma_i}\)'s and \(T^{\tau_i}\)'s are defined in (40) and (52), respectively.

In (69), a sum over possible metrics is realized by the double sum over the permutations of \(\sigma = \left( \begin{array}{c} 2 \cdots r \\ \sigma_2 \cdots \sigma_r \end{array} \right)\) and \(\tau = \left( \begin{array}{c} r+1 \cdots m - 2 \\ \tau_{r+1} \cdots \tau_{m-2} \end{array} \right)\). On the other hand, a braid trace is realized by \(\mathcal{P}(23 \cdots m - 2)\), which indicates the terms obtained by permutations of the overall elements \(\{2, 3, \ldots, m - 2\}\).

An explicit description of the gravitational holonomy operator (50) in terms of the graviton operator \(g_{h_{\mu_i}}\), where \(\mu_i = (\sigma_i, \tau_i)\) denotes a composite index, can then be given by the expression (69). We consider that the gravitational holonomy operator defines a theory of gravity in twistor space and that any physical quantities, such as graviton amplitudes, are generated from this holonomy operator. Indeed, the structure of the Chan-Paton factor in (69) is the same as that of graviton amplitudes which has been obtained by Bern et al. in [36]. In fact, the definition of \(T^{\sigma_i}\) in (40) or \(T^{\tau_i}\) in (52) is inspired by the results of [10, 11] and [36]. In the next section, we shall use these relations to obtain an S-matrix functional of graviton amplitudes in terms of a supersymmetric version of the gravitational holonomy operator.

5 An S-matrix functional for graviton amplitudes

In this section, we obtain an S-matrix functional for graviton amplitudes, following a case of gluon amplitudes discussed in the accompanying paper [40]. For this purpose, we first review an \(\mathcal{N} = 4\) supersymmetric extension of the Yang-Mills holonomy operator (25) and how it can be used to describe an S-matrix functional for gluon amplitudes. (This review part can be omitted if the reader is already familiar with the material in [40].) We then apply these results to a gravitational theory which we have formulated in the previous two sections.

Supersymmetrization of \(\Theta_{R, \gamma}^{(A)}(u)\)
In the following, we simply present some key results in [40]. A supersymmetric extension of (25) can be expressed as

\[ \Theta^{(A)}_{R,\gamma}(u;x,\theta) = \text{Tr}_{R,\gamma} P \exp \left[ \sum_{m \geq 2} \oint_{\gamma} A \wedge A \wedge \cdots \wedge A \right] \]  

(71)

where, as in (20)-(21), \( A \) is defined by

\[ A = g \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij} \]  

(72)

\[ A_{ij} = \sum_{\hat{h}_i} a^{(\hat{h}_i)}_i(x,\theta) \otimes a^{(0)}_j \]  

(73)

\[ \omega_{ij} = d \log(u_i u_j) = \frac{d(u_i u_j)}{(u_i u_j)} \]  

(74)

These expressions are the same as the previous ones except that physical operators \( a^{(\hat{h}_i)}_i \) are now dependent on the four-dimensional chiral supercoordinate \( (x, \theta) \). Accordingly, the physical operators include the states of gluonic superpartners, so that the helicity index is extended from \( \hat{h}_i \) to \( \hat{h}_i \) which we shall specify in a moment. In the Yang-Mills case, we consider \( \mathcal{N} = 4 \) supersymmetry. So \( \theta \) is written as \( \theta^\alpha_A (A = 1, 2; \alpha = 1, 2, 3, 4) \). Projection of these Grassmann variables onto a \( \mathbb{CP}^1 \) fiber of supertwistor space is realized by

\[ \xi^\alpha = \theta^\alpha_A u^A \]  

(75)

In terms of these, an explicit form of \( a^{(\hat{h}_i)}_i(x,\theta) \) is given by

\[ a^{(\hat{h}_i)}_i(x,\theta) = \int d\mu(p_i) \left. a^{(\hat{h}_i)}_i(\xi_i) e^{i\xi_i \mu r_i} \right|_{\xi^\alpha_i = \theta^\alpha_A u^A} \]  

(76)

where \( a^{(\hat{h}_i)}_i(\xi_i) \)'s are defined as

\[ a^{(+)}_i(\xi_i) = a^{(+)}_i \]  

\[ a^{(+\frac{1}{2})}_i(\xi_i) = \xi^\alpha_i a^{(+\frac{1}{2})}_i \]  

\[ a^{(0)}_i(\xi_i) = \frac{1}{2} \xi^\alpha_i \xi^\beta_i a^{(0)}_i \]  

\[ a^{(-\frac{1}{2})}_i(\xi_i) = \frac{1}{3!} \xi^\alpha_i \xi^\beta_i \xi^\gamma_i \epsilon_{\alpha\beta\gamma} a^{(-\frac{1}{2})}_i \]  

\[ a^{(-)}_i(\xi_i) = \xi^1_i \xi^2_i \xi^3_i \xi^4_i a^{(-)}_i \]  

(77)

Notice that the helicity components are in accordance with the relation in (9). The measure \( d\mu(p) \) in (76) denotes the following Lorentz invariant measure.

\[ d\mu(p) \equiv \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} = \frac{1}{(2\pi)^3} \frac{(\bar{\alpha}\alpha) d(\bar{\alpha}\alpha) \ dz d\bar{z}}{2} \]  

\[ = \frac{1}{4} \left[ u \cdot du \frac{d^2 \bar{u}}{(2\pi)^2} - \bar{u} \cdot d\bar{u} \frac{d^2 u}{(2\pi)^2} \right] \]  

(78)
This is called the Nair measure. 

An S-matrix functional for gluon amplitudes

In the spinor-momenta formalism, the simplest way of describing gluon amplitudes is to factorize the amplitudes in terms of the maximally helicity violating (MHV) amplitudes. The MHV amplitudes are the scattering amplitudes of \((n - 2)\) positive-helicity gluons and 2 negative-helicity gluons. In a momentum-space representation, the MHV tree amplitudes are expressed as

\[
\mathcal{A}_{nMHV}^{(1, 2, \ldots, n)}(u, \bar{u}) \equiv \mathcal{A}_{nMHV}^{(r, s, \ldots)}(u, \bar{u})
\]

\[
= ig^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \tilde{A}_{nMHV}^{(r, s, \ldots)}(u) \tag{79}
\]

\[
\tilde{A}_{nMHV}^{(r, s, \ldots)}(u) = \sum_{\sigma \in S_{n-1}} \text{Tr}(t^{c_1} t^{c_2} t^{c_3} \cdots t^{c_n}) \frac{(u_r u_s)^4}{(u_1 u_{\sigma_2} (u_{\sigma_2} u_{\sigma_3}) \cdots (u_{\sigma_n} u_1)} \tag{80}
\]

where \(u_i\) denotes the spinor momentum of the \(i\)-th gluon \((i = 1, 2, \ldots, n)\). The elements \(r\) and \(s\) denote the numbering indices of the negative-helicity gluons. General amplitudes, the so-called non-MHV amplitudes, can be expressed in terms of the MHV amplitudes \(\tilde{A}_{nMHV}^{(r, s, \ldots)}(u)\). Prescription for these expressions is called the Cachazo-Svrcek-Witten (CSW) rules. For the next-to-MHV (NMHV) amplitudes, which contain three negative-helicity gluons, the CSW rules can be expressed as

\[
\tilde{A}_{NMHV}^{(r, s, t)}(u) = \sum_{(i, j)} \tilde{A}_{nMHV}^{(i_1 \ldots, r, \ldots, s_1 \ldots, j_1 \ldots, k_1 \ldots)}(u) \frac{\delta_{kl}}{q_{ij}} \tilde{A}_{nMHV}^{(l, s_2 \ldots, t \ldots, (i-1) \ldots)}(u) \tag{81}
\]

where the sum is taken over all possible choices for \((i, j)\) that satisfy the ordering \(i < r < s < j < t\). The momentum transfer \(q_{ij}\) between the two MHV vertices is given by

\[
q_{ij} = p_i + p_{i+1} + \cdots + p_r + \cdots + p_s + \cdots + p_j \tag{82}
\]

where \(p\)'s denote four-momenta of gluons as before. General non-MHV amplitudes are then obtained by an iterative use of the relation (81).

We now notice a structural similarity between (27) and (80). In terms of the supersymmetric holonomy operator in (71), an S-matrix functional \(\mathcal{F}\) of gluon amplitudes can then be expressed as follows.

\[
\mathcal{F}^{[a^{(h)c}]} = \hat{W}^{(A)} \mathcal{F}_{nMHV}^{[a^{(h)c}]} \tag{83}
\]

\[
\hat{W}^{(A)} = \exp \left[ \int d^4 x d^4 y \frac{\delta_{kl}}{q^2} \frac{\delta}{\delta a_k^{(+)}(x)} \otimes \frac{\delta}{\delta a_l^{(-)}(y)} \right] \tag{84}
\]

\[
\mathcal{F}_{nMHV}^{[a^{(h)c}]} = \exp \left[ \frac{i}{g^2} \int d^4 x d^8 \theta \theta_R^{(A)}(u; x, \theta) \right] \tag{85}
\]

where \(a^{(h)c}\) refers to a generic expression for \(a_i^{(h)c_i}(i = 1, 2, \ldots)\), with \(a_i^{(h)}\) being \(a_i^{(h)} = t^{c_i} a_i^{(h)c_i}\) as in (26). An \(x\)-space representation of the operator, \(a_i^{(h)}(x)\), is defined as

\[
a_i^{(h)}(x) = \int d\mu(p_i) a_i^{(h)}(x) e^{ix_{\mu} p_{i\mu}} \tag{86}
\]
where \( d\mu(p_i) \) denotes the Nair measure (78). An explicit expression for general gluon amplitudes \( \hat{A}^{(1, h_1, 2, h_2, \ldots, n_h)}(u) \) is written as

\[
\begin{align*}
\delta \frac{\delta}{\delta a^{(h_1)c_1}(x_1)} \otimes \frac{\delta}{\delta a^{(h_2)c_2}(x_2)} \otimes \cdots \otimes \frac{\delta}{\delta a^{(h_n)c_n}(x_n)} \mathcal{F}[a^{(h)c}] \bigg|_{a^{(h)c}(x) = 0} = 
ig^{n-2} \hat{A}^{(1, h_1, 2, h_2, \ldots, n_h)}(u)
\end{align*}
\]

where a set of \( h_i = \pm (i = 1, 2, \ldots, n) \) gives an arbitrary helicity configuration. The condition \( a^{(h)}(x) = 0 \) means that the remaining operators (or source functions) should be evaluated as zero in the end of the calculation.

There are few remarks in the above expressions. First of all, we choose the following normalization of the spinor momenta.

\[
\int d(u_1 u_2) \wedge d(u_2 u_3) \wedge \cdots \wedge d(u_m u_1) = 2^{m+1}
\]

Under a permutation of the numbering indices, a sign factor arises in the above expression. We omit this sign factor as well as the factor \( (-1)^{h_1 + h_2 + \cdots + h_n} \) in (27) since physical quantities are given by the squares of the amplitudes.

Secondly, we notice that the Grassmann integral over \( \theta \)'s picks up only the MHV amplitudes or vortices since the integral vanishes unless we have the following factor

\[
\int d^8 \theta \xi_1^1 \xi_2^2 \xi_3^3 \xi_4^4 \xi_1^1 \xi_2^2 \xi_3^3 \xi_4^4 \bigg|_{\xi^a_\alpha = \theta^{a}_\alpha u^A_\alpha} = (u_r u_s)^4
\]

We therefore find that the supersymmetric holonomy operator \( \Theta_{R, \gamma}(u; x, \theta) \) naturally describes an S-matrix functional \( \mathcal{F}_{MHV} \) for the MHV gluon amplitudes; an explicit form of \( \mathcal{F}_{MHV} \) is shown in (85). A Wick-like contraction operator \( \hat{W}^{(A)} \) in (84) is introduced so that we can obtain non-MHV amplitudes in terms of the MHV ones, following the CSW prescription, in a language of functional derivatives. This field theoretic description is convenient. For example, in the expression (87), the sum over \((i, j)\) in (81) is guaranteed by the functional derivatives acting on \( \mathcal{F} \) and the relation (89). This explains why the momentum transfer is denoted by \( q \) without the \((i, j)\) indices in (84). The relation (89) also suggests that gluon amplitudes vanish unless the helicity configuration can be factorized by the MHV helicity configurations. Thus the helicity index is given by \( h_i = (+, -) \), rather than the supersymmetric version \( \hat{h}_i = (0, \pm \frac{1}{2}, \pm) \).

Lastly, in obtaining the expression (87), we also use the following relations.

\[
a^{(\pm)}_1 \otimes a^{(h_2)}_2 \otimes \cdots \otimes a^{(h_m)}_m \otimes a^{(0)}_1 = \frac{1}{2} [a^{(0)}_1, a^{(\pm)}_1] \otimes a^{(h_2)}_2 \otimes \cdots \otimes a^{(h_m)}_m
\]

\[
= \pm \frac{1}{2} a^{(\pm)}_1 \otimes a^{(h_2)}_2 \otimes \cdots \otimes a^{(h_m)}_m
\]

This relation also holds under a permutation of the numbering indices. Notice that the operators \( a^{(\pm)}_i \) are, by construction, coupled with the logarithmic one form \( \omega_{ij} \). Thus the
indices \((1, 2, \cdots, m)\) have an antisymmetric property which we implicitly use in (90). This relation (90) has also been used in obtaining the expression (27).

**Supersymmetrization of \(\Theta^{(H)}_{R, \gamma}(u, \bar{u})\)**

In the following, we consider applications of the above expressions to a gravitational theory. In analogy with (71), supersymmetrization of \(\Theta^{(H)}_{R, \gamma}(u, \bar{u})\) in (50) can be expressed as

\[
\Theta^{(H)}_{R, \gamma}(u, \bar{u}; x, \theta) = \text{Tr}_{R, \gamma} \text{P exp} \left[ \sum_{m \geq 5} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H \right] \tag{91}
\]

where, as in (56)-(61), \(H\) is given by

\[
H = \sqrt{8 \pi G_N} \sum_{1 \leq i < j \leq n} H_{ij} \omega_{ij} \tag{92}
\]

\[
H_{ij} = \sum_{\sigma \in S_{r-1}} \sum_{\tau \in S_{n-r-2}} \left( \sum_{\hat{h}_{\mu i}} g_i^{(\hat{h}_{\mu i})}(x, \theta) \otimes g_j^{00} \right) \omega_{\lambda i, \lambda j} \tag{93}
\]

\[
\omega_{ij} = d \log(u_i u_j) = \frac{d(u_i u_j)}{(u_i u_j)} \tag{94}
\]

where \(\hat{h}_{\mu i}\) denotes supersymmetrization of \(h_{\mu i} \equiv h_i \hat{h}_{\mu i}, \ i.e., \ \hat{h}_{\mu i} \equiv \hat{h}_i \hat{h}_{\mu i}\). Either \(\hat{h}_i\) or \(\hat{h}_{\mu i}\) represents a helicity of a frame field with \(N = 4\) supersymmetry. Thus the operator \(g_i^{(\hat{h}_{\mu i})}(x, \theta)\) consists of \(N = 8\) supermultiplets. This corresponds to the fact that the gravitons are essentially given by two copies of a frame field which we regard as an analog of a gauge field in \(N = 4\) super Yang-Mills theory. Since gluons are expressed by the supertwistor variables, gravitons can be described by the variables on \(\mathbb{C}P^{3|4} \times \mathbb{C}P^{3|4}\). (This space is not super ambitwistor space, which is given by a product of \(\mathbb{C}P^{3|4}\) and its dual with a certain constraint, since here the two \(\mathbb{C}P^{3|4}\)'s are the same in nature, respectively corresponding to two frame fields by which a graviton is made of.) The Grassmann variables \(\theta\) are now expressed as \(\theta^A_\alpha\) with \(A = 1, 2\) and \(\alpha = 1, 2, \cdots, 8 = N\). As in (75), projection of these variables can be written as

\[
\xi^\alpha = \theta^A_\alpha u^A \quad (\alpha = 1, 2, \cdots, 8) \tag{95}
\]

Since the \(N = 8\) multiplets are obtained from those of \(N = 4\), we can split the index \(\alpha\) as

\[
\alpha = (\alpha_1, \alpha_2) \quad \alpha_1 = 1, 2, 3, 4, \quad \alpha_2 = 5, 6, 7, 8 \tag{96}
\]

The operator \(g_i^{(\hat{h}_{\mu i})}(x, \theta)\) in (93) is then defined as

\[
g_i^{(\hat{h}_{\mu i})}(x, \theta) = \int d\mu(p_i) g_i^{(\hat{h}_{\mu i})}(\xi_i) e^{ix \mu p_i^\mu} \bigg|_{\xi_i^\alpha = \theta^A_\alpha u^A} \tag{97}
\]

\[
g_i^{(\hat{h}_{\mu i})}(\xi_i) = T^{\mu i} g_{\mu i}^{(\hat{h}_{\mu i})}(\xi_i) = T^{\mu i} e^{(\hat{h}_{\mu i})\alpha}(\xi_i) e^{(\hat{h}_{\mu i})\alpha}(\xi_i) \tag{98}
\]
where \( e_i^{(\hat{h}) a}(\xi_i) \)'s are defined as

\[
e_i^{(+)} a(\xi_i) = e^{(+)} a_i
\]

\[
e_i^{(+\frac{1}{2})} a(\xi_i) = \xi^a_i e_i^{(+\frac{1}{2}) a_i}
\]

\[
e_i^{(0)} a(\xi_i) = \frac{1}{2} \xi^a_i \xi_i^{(0)} a_i
\]

\[
e_i^{(-\frac{1}{2})} a(\xi_i) = \frac{1}{3} \xi^a_i \xi_i^{(-\frac{1}{2}) a_i}
\]

\[
e_i^{(-)} a(\xi_i) = \xi^{(+)} e_i^{(+) a} \xi^{(+) a} e_i
\]

Similarly, \( \eta_i^{(\hat{h}) a}(\xi_i) \)'s are defined as

\[
\eta_i^{(+)} a(\xi_i) = \eta^{(+)} a_i
\]

\[
\eta_i^{(+\frac{1}{2})} a(\xi_i) = \xi^a_i \eta_i^{(+\frac{1}{2}) a_i}
\]

\[
\eta_i^{(0)} a(\xi_i) = \frac{1}{2} \xi^a_i \eta_i^{(0)} a_i
\]

\[
\eta_i^{(-\frac{1}{2})} a(\xi_i) = \frac{1}{3} \xi^a_i \eta_i^{(-\frac{1}{2}) a_i}
\]

\[
\eta_i^{(-)} a(\xi_i) = \xi^{(+) \eta_i^{(+) a} \eta^{(+) a} \eta_i}
\]

Notice that \( \xi_i \) is common in the above expressions. Namely, there appears no \( \xi_m \). This comes from the fact the graviton operator (97) is a point-like operator in \( \mathcal{N} = 8 \) chiral superspace. Alternatively, we can interpret \( \xi_m \) as chiral superpartners of the tangent-space coordinate \( x_a (a = 0, 1, 2, 3) \), with spacetime not being supersymmetrized. The latter interpretation can also be applied to Yang-Mills theories.

**An S-matrix functional for graviton amplitudes**

In the Yang-Mills case, general gluon amplitudes are expressed as (87), generated from the S-matrix functional in (83). In the following, we obtain an analogous expression for graviton amplitudes. Gluon amplitudes are represented by \( \hat{A}^{(1h_2 \cdots n_{h n})}(u) \) in (87). In a momentum-space representation, the amplitudes should be expressed as

\[
\mathcal{A}^{(1h_2 \cdots n_{h n})}(u, \tilde{u}) = i g^{n-2} (2\pi)^4 \delta(4) \sum_{i=1}^{n} p_i \hat{A}^{(1h_2 \cdots n_{h n})}(u)
\]

\[
\hat{A}^{(1h_2 \cdots n_{h n})}(u) = \sum_{\sigma \in S_{n-1}} \text{Tr}(t^{c_1} t^{c_{\sigma 2}} t^{c_{\sigma 3}} \cdots t^{c_{\sigma n}}) \mathcal{C}(1\sigma_2\sigma_3 \cdots \sigma_n)
\]

where \( \mathcal{C}(1\sigma_2\sigma_3 \cdots \sigma_n) \) are functions of the Lorentz-invariant scalar products \( (u_i u_j) \). For the MHV amplitudes, an explicit form of these can be written as (80). By use of the CSW rules, we can in principle obtain \( \mathcal{C}'s \) of any helicity configurations. In terms of such \( \mathcal{C}'s \), we can express tree-level graviton amplitudes as [36]

\[
\mathcal{M}^{(1h_1\mu_1 2h_2\mu_2 \cdots n_{h n\mu n})}(u, \tilde{u}) = i (8\pi G_N)^{\frac{n}{2}-1} (-1)^{n+1} (2\pi)^4 \delta(4) \sum_{i=1}^{n} p_i
\]
\[ \hat{M}^{(1h_{\mu_1} 2h_{\mu_2} \cdots n_{\mu_n})}(u, \bar{u}) = \sum_{\sigma \in \Sigma_{n-1}} \sum_{\tau \in \Sigma_{n-2}} f(\sigma) \tilde{f}(\tau) C(12 \cdots n) \times C(\sigma_2 \sigma_3 \cdots \sigma_r 1 n - 1 \tau_{r+1} \tau_{r+2} \cdots \tau_{n-2} n) + \mathcal{P}(23 \cdots n - 2) \] (104)

where \( f(\sigma) \) and \( \tilde{f}(\tau) \) are given by (70), with \( m \) replaced by \( n \).

In analogy with (85), an S-matrix functional for the MHV graviton amplitudes can be defined as
\[
\mathcal{F}_{\text{MHV}} \left[ g_{i_{\mu_1}}^{(h_{\mu_1})} \right] = \exp \left[ \frac{i}{8\pi G_N} \int d^4x \, d^4\theta \, \Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta) \right] (105)
\]
where \( g_{i_{\mu_1}}^{(h_{\mu_1})} \) (\( i = 1, 2, \cdots \)) denotes an operator or a source function associated with the expression \( g_{i_{\mu_1}}^{(h_{\mu_1})} = T^{\mu_1} g_{i_{\mu_1}}^{(h_{\mu_1})} \). As in (86), an \( x \)-space representation of the operator, \( g_{i_{\mu_1}}^{(h_{\mu_1})}(x) \), is defined as
\[
g_{i_{\mu_1}}^{(h_{\mu_1})}(x) = \int d\mu(p_i) \, g_{i_{\mu_1}}^{(h_{\mu_1})} \, e^{ix \cdot p_i} (106)
\]
where \( d\mu(p_i) \) is the Nair measure (78).

Labeling the two negative-helicity gravitons by \((s-t)\), the MHV graviton amplitudes \( \hat{M}^{(s-t--)}_{\text{MHV}}(u, \bar{u}) \) can be generated by (105) as follows.
\[
\frac{\delta}{\delta g_{i_{\mu_1}}^{(+)}(x)} \otimes \cdots \otimes \frac{\delta}{\delta g_{i_{\mu_1}}^{(-)}(x)} \otimes \cdots \frac{\delta}{\delta g_{i_{\mu_n}}^{(+)}(x)} \otimes \cdots \frac{\delta}{\delta g_{i_{\mu_n}}^{(-)}(x)} \mathcal{F}_{\text{MHV}} \left[ g_{i_{\mu_1}}^{(h_{\mu_1})} \right] \bigg|_{g_{i_{\mu_1}}^{(h_{\mu_1})}(x) = 0} = i(8\pi G_N) \hat{F}^{-1} \hat{M}^{(s-t--)}_{\text{MHV}}(u, \bar{u}) (107)
\]
where we use (69), (88) and the following Grassmann integral
\[
\int d^4\theta \, \prod_{\alpha=1}^{8} \xi_\alpha \prod_{\beta=1}^{8} \xi_\beta \, \prod_{a=1}^{4} \xi_a = (u_s u_t)^8 (108)
\]

By use of the CSW rules in \( \mathcal{C} \)'s, we can straightforwardly extend the above expressions to non-MHV cases. An S-matrix functional for general tree-level graviton amplitudes is then defined as
\[
\mathcal{F} \left[ g_{i_{\mu_1}}^{(h_{\mu_1})} \right] = \hat{W}^{(H)} \mathcal{F}_{\text{MHV}} \left[ g_{i_{\mu_1}}^{(h_{\mu_1})} \right] (109)
\]
\[
\hat{W}^{(H)} = \exp \left[ \int d^4x \, d^4y \, \frac{\delta_{kl}}{q^2} \frac{\delta}{\delta g_{k_{\mu_1}}^{(+)}(x)} \otimes \frac{\delta}{\delta g_{l_{\mu_1}}^{(-)}(y)} \right] (110)
\]

where the meaning of \( q \) in (110) is exactly the same as that in (84) except that gluon momenta are now generically replaced by graviton momenta. In terms of the S-matrix functional (109),
graviton amplitudes can generally be expressed as

$$\frac{\delta}{\delta g_{1\mu_1}(x_1)} \otimes \frac{\delta}{\delta g_{2\mu_2}(x_2)} \otimes \cdots \otimes \frac{\delta}{\delta g_{n\mu_n}(x_n)} \mathcal{F} \left[ g_{i\mu_i}^{(h_{i\mu_i})} \right] \bigg|_{g_{i\mu_i}^{(h_{i\mu_i})}(x_i) = 0}$$

where $h_{i\mu_i}$ ($i = 1, 2, \cdots, n$) can take any helicities in $(++, --)$. The rest of the helicity configurations are ruled out due to the Grassmann integral (108). Notice that the particular assignment for the index $\alpha$ in (96) is crucial to extract the helicities of $(++, --)$. Without such an assignment, the states with $(+, -)$ helicities would emerge. As mentioned before, the operators $g_{i\mu_i}^{(+)}$ and $g_{i\mu_i}^{(-)}$ represent stable and neutral particles without mass and spin, which can be regarded as a candidate for the origin of dark matter. Observational evidence of dark matter and dark energy strongly suggests that there should be operators like $g_{i\mu_i}^{(+)}$ and $g_{i\mu_i}^{(-)}$ to be incorporated in a full gravitational theory. In the present formalism, this can be carried out by relaxing the assignment (96) and construct a theory as a square of $N = 4$ theories to include terms of $g_{i\mu_i}^{(+)}$ and $g_{i\mu_i}^{(-)}$; this formulation is currently under study.

Few other remarks on the expression (111) is in order below. One may wonder why there is a tedious label $\mu_i$ for each of the helicity index $h_{i\mu_i}$. This label is nothing but a Chan-Paton index (57), playing the same role as $c_i$ in the Yang-Mills case. Thus it can actually be removed as in the expression (87). Lastly, the remaining $(-1)^{n+1}$ factor in (103) can easily be obtained by revising the definition of $e_i^{(\pm)}$ in (34), from $e_i^{(\pm)} = e_i^{(\pm)\alpha} (\sqrt{2}p_i)^\alpha$ to $e_i^{(\pm)} = e_i^{(\pm)\alpha} (\sqrt{-2}p_i)^\alpha$, which is the more consistent with the expression in (30). This revision is however immaterial since, as discussed elsewhere, physical quantities are obtained by the square of the amplitudes.

6 Concluding remarks

In the present paper, we construct a four-dimensional theory of gravity in terms of a holonomy operator in twistor space which has been introduced in the accompanying paper [40]. A gravitational holonomy operator $\Theta_{R,\gamma}^{(H)}$ is defined by (50) along with explicit expressions for $H$ in (56)-(60). We show that, as in the Yang-Mills case, an S-matrix functional for scattering amplitudes of gravitons is naturally described by a supersymmetric extension of $\Theta_{R,\gamma}^{(H)}$.

Use of a holonomy formalism means that we construct a gravitational theory as a gauge theory. A Chan-Paton factor of the holonomy operator is given by a trace over Poincaré algebra and Iwahori-Hecke algebra. Owing to manifest Lorentz invariance of the formalism, the Poincaré algebra is realized by translational operators in tangent spaces. A trace over Iwahori-Hecke algebra, or a braid trace, is realized by a sum over permutations of the numbering indices for gravitons. What is significant in this paper is that we clarify that the Chan-Paton factor of $\Theta_{R,\gamma}^{(H)}$ is in one-to-one correspondence with a certain combinatoric factor in graviton amplitudes.
In an analysis of the Chan-Paton factor, we find that it can be characterized by three distinct loops (up to isotopy), due to an $SL(2, \mathbb{C})$ symmetry which is relevant to the Lorentz invariance of the logarithmic one-form $\omega_{ij}$ in (94). The characterization can be made by a notion of ascending or descending order in the numbering indices assigned for each loop. Thus the very distinctiveness of the loops requires that the total number of gravitons should be more than or equal to five. This is reflected in the definition of $\Theta_{R,\gamma}^{(H)}$ as a condition of $m \geq 5$. Since graviton amplitudes exists for $m \geq 3$, we can in fact relax the above constraint to define the holonomy operator as

$$\Theta_{R,\gamma}^{(H)}(u, \bar{u}) = \text{Tr}_{R,\gamma} \exp \left[ \sum_{m \geq 3} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H \right]$$

with the same $H$ as defined in (56)-(60). One can easily check that this leads to correct graviton amplitudes for $n = 3$ and 4. It is then possible to factorize the holonomy operator by the following term.

$$\text{Tr}_{R,\gamma} \oint_{\gamma} H \wedge H \wedge H$$

As in the case of Yang-Mills theory (23), $H$ satisfies the integrability condition.

$$DH = dH - H \wedge H = 0$$

Thus the term in (113) can be interpreted as a gravitational Chern-Simons term, with (114) serving as an Einstein equation. This explains why Chern-Simons theory arises in so many areas of physics, including Yang-Mills theories, gravitational theories, and integrable models in general.\(^6\)

There are basically two ways of having a dual picture between Yang-Mills theory and a gravitational theory in the holonomy formalism. These can easily be seen by a simple dimensional analysis as follows. One way is to introduce a coupling constant $g_f$ for the frame fields $e_i^{(\pm)}$ and to express the Newton constant in terms of $g_f$, which leads to the relation $(8\pi G_N)^{\frac{1}{2}} = g_f$. Thus the mass dimension of $g_f$ is given by $-\frac{1}{2}$. This is not consistent with the fact that we have regarded $e_i^{(\pm)}$ as analogs of Yang-Mills fields which have dimensionless coupling constants. We need to interpret $e_i^{(\pm)}$ as five-dimensional Yang-Mills fields in order to overcome the discrepancy. This suggests that we need to have a notion which is similar to dimensional transmutation on $e_i^{(\pm)}$. Such a concept is necessary if we like to have an interpretation of gravity as a square of Yang-Mills theories. Taking account of Chan-Paton factors, an appropriate coupling constant can be written as $g_F = g_f M_{Pl}$ whose mass dimension is given by $\frac{1}{2}$. The square of this constant has mass dimension 1 and can be identified with a cosmological constant. Thus it may be more convenient to use $g_F$ than $g_f$ in applications to cosmology, however, there is still a dimensional discrepancy and we need to have a concept like dimensional transmutation as well.

The other way is a rather direct one. Namely, we follow our interpretation of gravity as a gauge theory and consider $\sqrt{8\pi G_N}$ multiplied by the Chan-Paton factor $T^{\mu_i} \sim M_{Pl}^2$ as a

\(^6\)This also gives an answer to a question we put in the first footnote of the present paper.
coupling constant of the gauge theory. Since the mass dimension of \( \sqrt{8\pi G_N M_P^2} \) is 1, we can not relate this quantity to the Yang-Mills coupling constant \( g \) unless we have a hidden Chen-Paton factor, with mass dimension 1, coupled to \( g \). Assuming such a factor, we can have the following correspondence

\[
\sqrt{8\pi G_N M_P^2} \Leftrightarrow g(\beta M_P) \quad (115)
\]

where \( \beta \) is some numerical constant. This relation shows an explicit weak-weak duality between Yang-Mills and gravitational theories. The right-hand side in (115) suggests a modification of Yang-Mills theory with a tangent-space contribution which is analogous to the gravitational theory constructed in this paper. Such a modification is natural since Yang-Mills theory is also invariant under spacetime translations and diffeomorphism in the holonomy formalism. The modification also implies breakings of conformal symmetry and holomorphicity or chiral symmetry. It is expected that this kind of symmetry breaking happens spontaneously, providing a potentially new mechanism to explain the origin of mass. We do not know anything about such an interesting direction of research yet but it is probably related to deep algebraic problems.

As discussed in (24), the Yang-Mills coupling constant \( g \) is related to the Knizhnik-Zamolodchikov parameter. For \( SU(N) \) gauge groups, \( g \) is given by \( g = \frac{1}{1+N} \). In such cases, an exact correspondence in (115) determines the value of \( \beta \) as

\[
\beta = \frac{\sqrt{8\pi \hbar c}}{g} = \sqrt{\frac{8\pi}{\alpha_{YM}}} = \sqrt{8\pi}(N + 1) \quad (116)
\]

where \( \alpha_{YM} = g^2/\hbar c \) is a fine structure constant for Yang-Mills theory.

As discussed in the Yang-Mills case, the S-matrix functional (109) may be utilized to generate loop amplitudes without further modifications. The ultraviolet finiteness of the theory, apart from Chan-Paton factors, is guaranteed since it is constructed as a product of \( N = 4 \) theories which are ultraviolet finite. The Chan-Paton factors of the theory are given by the left-hand side in (115) times the number of gravitons. Thus this part of the theory obviously diverges when the number of gravitons is infinite. In a practical calculation, however, we deal with a situation where the number of gravitons is finite so that this divergence is not physically relevant.

Lastly, we would like to remark that massless spin-zero particles, represented by \( g_i^{+-} \) and \( g_i^{--} \) in (59), are naturally incorporated in our construction of a gravitational theory. This shows that the holonomy formalism provides an interesting framework for a search of the origin of dark matter and dark energy.

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