Polynomial-valued constant hexagon cohomology

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Abstract

Hexagon relations are algebraic realizations of four-dimensional Pachner moves. ‘Constant’—not depending on a 4-simplex in a triangulation of a 4-manifold—hexagon relations are proposed, and their polynomial-valued cohomology is constructed. This cohomology yields polynomial mappings defined on the so called ‘coloring homology space’, and these mappings can, in their turn, yield piecewise linear manifold invariants. These mappings are calculated explicitly for some examples.

It is also shown that ‘constant’ hexagon relations can be obtained as a limit case of already known ‘nonconstant’ relations, and the way of taking the limit is not unique. This non-uniqueness suggests the existence of an additional structure on the ‘constant’ coloring homology space.

1 Introduction

A piecewise linear (PL) manifold may be specified combinatorially by its given triangulation. Most things we want to know about a PL manifold belong, however, to the manifold itself, and must be independent of a specific triangulation. This leads to the idea of, first, representing a transition from one triangulation to another as a combination of simple steps, and second, inventing an algebraic structure that corresponds to a triangulation but behaves under these steps in such a simple way that can produce quantities that do not change under these steps at all.

The mentioned simple steps are provided by the theorem of Pachner [6, 5]. In application to four-dimensional closed PL manifolds, it states that there are just three kinds of Pachner moves that can, together with their inverses, make up a chain connecting any two triangulations of a given manifold. Then, an algebraic structure is proposed based on hexagon relations and their cohomology.

Hexagon relations are ‘algebraic realizations’ of four-dimensional Pachner moves: this means, informally, that they imitate these moves algebraically. Somewhat similar things are known for three-dimensional manifolds and their quantum invariants [7]. It appears, however, that quandles and quandle cohomology—that
yield invariants of knots and their higher analogues [1]—are closer to our constructions.

Hexagon relations may be called analogues of quandles, and hexagon cohomology analogue of quandle cohomology, but applicable to 4-manifolds rather than knots. One essential difference is that, in the most general known case, hexagon relations (together with their cohomology) are not constant: they vary for different 4-simplices (pentachora) of the triangulation. In particular, a construction of invariants of a pair “manifold, middle cohomology class” has been proposed in [4], based on these nonconstant relations.

In the present paper, however, we work with constant relations, and just a little bit with their neighborhood in the ‘nonconstant’ space. The point is that constant relations are, first, already interesting in themselves, and second, calculations (see Subsection 8.3 below) suggest that there is a very nontrivial interplay between the ‘constant’ and ‘nonconstant’ cases.

The contents of the rest of this paper by sections is as follows:

• in Section 2, we introduce ‘permitted colorings’ of a simplicial complex—the basis for our hexagon relations. Interestingly, our permitted colorings immediately bring about some—their own—sort of homology [12];

• in Section 3, we describe four-dimensional Pachner moves in the form suitable for us, and introduce linear constant hexagon relations;

• in Section 4, we introduce our polynomial-valued hexagon cochain complex. This is a simple and general algebraic construction, dealing just with a sequence of ‘standard’ simplices, one for any dimension $n = 1, 2, \ldots$ (this may be contrasted with the ‘coloring homology’ mentioned above which depends on a chosen simplicial complex!);

• in Section 5, we discover the interplay between the coloring homology, hexagon cohomology, and usual simplicial cohomology: under some technical conditions, a hexagon cohomology class and a coloring homology class produce together a simplicial cohomology class [25]. The dependence of the latter on the coloring homology class is, however, nonlinear!

• In Section 6, we specialize these results for four-manifolds, introducing mappings (34) and (35) that are independent of a triangulation;

• in Section 7, we explain how our ‘constant’ hexagon can be obtained from the ‘nonconstant’ hexagon of [4]. This does not look completely trivial;

• in Section 8, we present some calculation results showing what the mentioned mappings (34) and (35)—from which PL manifold invariants can be extracted algebraically—can actually look like. Also, as we have already mentioned, we demonstrate a very nontrivial interplay between the ‘constant’ and ‘nonconstant’ cases.

• Finally, in Section 9, we briefly discuss our results and further work.
2 Permitted and edge-generated colorings of a simplicial complex

2.1 Definition of a coloring

A coloring of a simplicial complex $K$ means assigning a color to each of its simplices of a given dimension $n$. Color means here an element of a given color set $X$. In this paper, we take $n = 3$, and $X = F^2$ — a two-dimensional linear space over a fixed field $F$. Thus, our coloring is a map

$$\text{(set of all tetrahedra in } K) \rightarrow F^2.$$  \hspace{1cm} (1)

We write the color of an individual tetrahedron $t$—its image under the map (1)—as a two-column

$$x_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad x_t, y_t \in F.$$  \hspace{1cm} (2)

*Remark.* Of course, coloring simplices of other dimension(s) than three may be also of interest, as well as using a set of colors other than $F^2$. In this paper we, however, confine ourself to the case (1).

*Remark.* Also, using ring $\mathbb{Z}$ of integers instead of field $F$ might be of interest.

2.2 Vertex ordering

Typically, we will be considering finite simplicial complexes $K$ with vertices numbered from 1 through their total number $N_0(K)$. The subscript here stays for the fact that vertices are zero-dimensional; more generally, the number of $n$-simplices in a finite simplicial complex will be denoted $N_n$ or $N_n(K)$. Note that our “standard” simplex $\Delta^n$ has thus vertices $1, \ldots, n+1$, instead of probably more usual $0, \ldots, n$.

Consider, however, a situation where we deal with a complex $\mathcal{K}$ and its subcomplex $K \subset \mathcal{K}$ with vertices

$$i_1, \ldots, i_{N_0(K)}, \quad i_1 < \ldots < i_{N_0(K)},$$  \hspace{1cm} (3)

whose numbering is inherited from $\mathcal{K}$ (for instance, $\mathcal{K} = \Delta^n$ may be an $n$-simplex, and $K$ one of its $(n-1)$-faces). In a situation like this, $K$ can retain its vertex numbering (3), but the point is that if we have proved in this paper a theorem about a complex $K$ whose vertices have numbers $1, \ldots, N_0(K)$, then it can be always transferred to the same complex with vertices denoted as in (3), just by the obvious substitution

$$1 \rightarrow i_1, \quad \ldots, \quad N_0(K) \rightarrow i_{N_0(K)}.$$  \hspace{1cm} (4)
We usually denote triangles by the letter $s$, tetrahedra by the letter $t$, and pentachora by the letter $u$. When we write these in terms of their vertices, these latter go, by default, in the increasing order:

$$\text{if } t = ijkl, \text{ then } i < j < k < l.$$  

2.3 Permitted colorings of one pentachoron

Interesting structures appear when we declare some of the colorings permitted (which means of course that other colorings are prohibited). Our permitted colorings will form a linear subspace in the space of all colorings, described in terms of either edge functionals or edge vectors. Some motivation for introducing these functionals and vectors can be found in [4] (and see also Section 7 of the present work for explanation of how the structures introduced below can be obtained from those in [4]).

In this Subsection, we begin with introducing our permitted colorings for just one pentachoron.

Permitted colorings in terms of edge functionals

In this approach, permitted colorings of a pentachoron are singled out by linear relations. Namely, there is one linear relation associated with each pentachoron edge $ij$, formulated as the vanishing of a linear edge functional $\phi_{ij}$. This $\phi_{ij}$ can depend only on the colors of the three tetrahedra containing the edge: $t \supset ij$. Edge functionals are defined for unoriented edges: $\phi_{ij} = \phi_{ji}$.

The set (linear space) $V_u$ of permitted colorings for a pentachoron $u$ is, by definition, the intersection of kernels of all ten edge functionals:

$$V_u = \bigcap_{ij \subset u} \text{Ker } \phi_{ij}. \quad (5)$$

The colorings of a tetrahedron $t$ being written as two-columns [2], we can write the restriction of $\phi_{ij}$ onto $t$, or $t$-component of $\phi_{ij}$, as a two-row:

$$\phi_{ij}|_t = \begin{pmatrix} \phi^{(1)}_{t,ij} \\ \phi^{(2)}_{t,ij} \end{pmatrix}. \quad (6)$$

Consider pentachoron $u = 12345$ and its 3-face $t = 1 \ldots \hat{i} \ldots 5$—that is, tetrahedron $t$ lies opposite vertex $i$. The $t$-components of (nonvanishing on $t$) edge functionals are, by definition, as follows:

$$\begin{pmatrix} \phi_{k_1k_2} \\ \phi_{k_1k_3} \\ \phi_{k_1k_4} \\ \phi_{k_2k_3} \\ \phi_{k_2k_4} \\ \phi_{k_3k_4} \end{pmatrix}_t = (-1)^{i+1} \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad (7)$$
where $1 \leq k_1 < k_2 < k_3 < k_4 \leq 5$ are the four vertices of $t$, and we have written just one subscript $t$ meaning $t$-component for all $\phi_{ij}$. For other pentachora $i_1i_2i_3i_4i_5$, substitution (4) applies, that is, $1 \rightarrow i_1$, $\ldots$, $5 \rightarrow i_5$.

A direct calculation shows that

$$\dim V_u = 5. \quad (8)$$

Example. Relation $\phi_{12} = 0$ in pentachoron 12345 looks as follows:

$$y_{1234} - y_{1235} + y_{1245} = 0. \quad (9)$$

### Permitted colorings in terms of edge vectors

The same linear space $V_u$ of permitted colorings of one pentachoron can be described as the span of ten edge vectors. Given an edge $b$, edge vector $\psi_b$ is a permitted coloring of a simplicial complex—at this moment, one pentachoron—that has nonvanishing components only for tetrahedra $t \supset b$. Namely, by definition, for tetrahedron 1234 they are as follows:

$$\begin{pmatrix} \psi_{12} & \psi_{13} & \psi_{14} & \psi_{23} & \psi_{24} & \psi_{34} \end{pmatrix}_{1234} = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}, \quad (10)$$

while for other pentachora $i_1i_2i_3i_4$, substitution (4) again applies, that is, $1 \rightarrow i_1$, $\ldots$, $4 \rightarrow i_4$.

**Important Remark** 1. There are no additional signs in (10), in contrast with $(-1)^{i+1}$ in (7)!

The fact that edge vectors (10) generate the same five-dimensional space $V_u$ as in (5), is checked by a direct calculation.

### 2.4 Permitted and edge-generated colorings of a simplicial complex

By definition, a permitted coloring of a simplicial complex $K$ is such whose restriction onto any pentachoron $u \subset K$ is permitted. Permitted colorings of $K$ form a linear space denoted $V_K$.

And edge-generated colorings are, also by definition, linear combinations of edge vectors $\psi_b$ whose components are described by the same formula (10) as for one pentachoron (but there may be of course more than three tetrahedra containing a given edge in an arbitrary $K$). Edge-generated colorings form a linear subspace

$$V_K^{(0)} \subset V_K \quad (11)$$

in the space of permitted colorings, because edge vectors generate, according to the above, permitted colorings in every pentachoron.
2.5 Coloring homology

In this paper, we will understand *coloring homology* simply as the factor

\[ H_{\text{col}}(K, F) \overset{\text{def}}{=} V_K/V_K^{(0)}. \]  

(12)

This is the only homology group of the following very short chain complex:

\[ F^{N_1} \rightarrow F^{2N_3} \rightarrow F^{10N_4}. \]  

(13)

Recall (see the beginning of Subsection 2.2) that \( N_n = N_n^{(K)} \) is the number of \( n \)-simplices in complex \( K \). The first term in (13) consists of formal linear combinations of edges with coefficients in \( F \). The second term is the space of *all* (permitted or not) colorings of \( K \), and each edge \( b \) is sent by the first arrow to the edge vector \( \psi_b \). Finally, the second arrow is the direct sum of *all* edge functionals; these act in each pentachoron separately, and in each pentachoron there are ten of them. The rightmost term in (13) is thus the direct sum of copies of \( F \) for each pair \( u \supset b \), with \( u \) a pentachoron and \( b \) and edge.

*Important Remark 2.* Sequence (13), or a modification of it, can actually be extended both to the left and to the right, compare [4, Sections 5.2 and 5.3]. This means that more coloring, or ‘exotic’, homology groups can be defined. In the present paper we, however, work only with \( H_{\text{col}}(K, F) \) given by (12).

3 Four-dimensional Pachner moves and linear constant hexagon relations

3.1 Pachner moves

Consider a 5-simplex \( \Delta^5 \). Its boundary \( \partial \Delta^5 \) consists of six pentachora (= 4-simplices). Imagine that \( k \) of these pentachora, \( 1 \leq k \leq 5 \), enter in a triangulation of a four-dimensional piecewise linear (PL) manifold \( M \). Then we can replace them with the remaining \( 6 - k \) pentachora, without changing \( M \). This is called four-dimensional *Pachner move*, and there are five kinds of them: 1–5, 2–4, 3–3, 4–2, and 5–1; here the number before the dash is \( k \), while the number after the dash is, of course, \( 6 - k \).

We sometimes call the initial configuration—cluster of \( k \) pentachora—the *left-hand side* (l.h.s.) of the Pachner move, while its final configuration—cluster of \( 6 - k \) pentachora—its *right-hand side* (r.h.s.).

3.2 Linear constant hexagon relations

Consider one of the Pachner moves as described in Subsection 3.1 and denote its l.h.s. as \( C \), and its r.h.s. as \( \bar{C} \).
Important Remark 3. According to Subsection 2.2, there is a given order on the
vertices of our simplex $\Delta^5$, for instance, $\Delta^5 = 123456$. So, it must be noted
that $C$ is allowed to consist of any $k$ pentachora, whatever the numbers of their
vertices. Informally, we use the words full hexagon for combinatorial or algebraic
statements relating $C$ and $\bar{C}$ in such situation. This applies, in particular, to the
following Theorem 1.

Theorem 1.  (i) The restrictions of permitted colorings of the left-hand side $C$
of a Pachner move (described above) onto the common boundary $\partial C = \partial \bar{C}$
yield the same set of colorings of this common boundary as the restrictions
of permitted colorings of the right-hand side $\bar{C}$.

(ii) Moreover, all these permitted colorings of $\partial C = \partial \bar{C}$ can be generated by
edge vectors $\psi_b$ for edges $b \subset \partial C = \partial \bar{C}$ only.

(iii) There are fixed numbers $a_k$, $1 \leq k \leq 5$, such that, if we fix the zero coloring
on $\partial C = \partial \bar{C}$, the dimension of permitted coloring space of (inner tetrahedra
of) $C$ is $a_k$, and the same dimension for $C$ is $a_{6-k}$. Namely,

$$a_1 = a_2 = a_3 = 0, \quad a_4 = 1, \quad a_5 = 3.$$  

These permitted colorings of only inner tetrahedra are generated by vec-
tors $\psi_b$ for only inner edges $b$ in either $C$ or $\bar{C}$.

(iv) Consider a situation where $C$ was, and $\bar{C}$ has become, part of triangulation
of a PL 4-manifold $M$. Denote $K_{ini}$ and $K_{fin}$ the simplicial complexes
determined by the corresponding triangulations of $M$. Then, there is a
canonical isomorphism (see Subsection 2.4 for notations)

$$H_{col}(K_{ini}, F) \cong H_{col}(K_{fin}, F)$$  \hspace{1cm} (14)

that can be described as follows. We say that a permitted coloring of $K_{ini}$
corresponds to a permitted coloring of $K_{fin}$ if their restrictions onto the
closed complement of $C$ or $\bar{C}$ coincide. This correspondence is not gen-
erally one-to-one, but it becomes an isomorphism after factoring by edge-
generated colorings.

Proof. Items (i)–(iii) are proved by direct calculations. Item (iv) follows then
from the fact that the contribution of inner—with respect to $C$—edges is the
same in $V_{K_{ini}}$ and $V_{K_{fin}}$, and the same applies if we change $C$ to $\bar{C}$ and subscript
‘ini’ to ‘fin’. \hfill $\square$

Remark. It may make sense to remind once again that a motivation for the above
constructions can be found in [4], combined with Section 7 of the present paper.
4 Polynomial-valued hexagon cochain complex: Definition

By definition, a constant polynomial \( n \)-cochain \( c \), for \( n \geq 3 \), is an arbitrary polynomial defined on the linear space \( V_{\Delta^n} \) of all permitted colorings of the standard \( n \)-simplex \( \Delta^n = 1 \ldots (n+1) \).

The coboundary \( \delta c \) of \( c \) is the polynomial

\[
\delta c = \sum_{k=1}^{n+2} (-1)^{k+1} c_{1\ldots \hat{k}\ldots (n+2)}
\]

defined on the linear space \( V_{\Delta^{n+1}} \) of all permitted colorings of the standard \((n+1)\)-simplex \( \Delta^{n+1} = 1 \ldots (n+2) \). In (15), each \( n \)-face \( 1\ldots \hat{k}\ldots (n+2) \) of \( \Delta^{n+1} \) is identified with the standard \( \Delta^n \) in the natural way—that is, according to the general rule (4).

The complex can be written as follows:

\[
0 \to C^3_{\text{hex}} \xrightarrow{\delta} C^4_{\text{hex}} \xrightarrow{\delta} \ldots
\]

where \( C^n_{\text{hex}} \) means the linear space of \( n \)-cochains.

Bilinear cochains. One simple variation on the theme of polynomial-valued hexagon cochain complex may also be of interest. Namely, we define a constant bilinear \( n \)-cochain as a bilinear form

\[
V_{\Delta^n} \times V_{\Delta^n} \to F.
\]

This means that now a pair of permitted colorings comes into play. Definition (15) remains the same in this case, as well as the form (16) of the complex.

Our reasonings below in Sections 5 and 6 will apply to both the ‘polynomial’ and ‘bilinear’ cases. We will prefer, however, to formulate and prove our Theorems 2, 3 and 4 first for polynomial-valued hexagon cochain complexes as defined in the beginning of this Section, and then point out the changed necessary for the ‘bilinear’ case after their respective proofs. Hopefully, this will make our exposition less cumbersome.

5 Hexagon cohomology, coloring homology, and simplicial cohomology in a simplicial complex

In this Section, we work within a fixed finite simplicial complex \( K \), with the numbering of its vertices also fixed.
5.1 A permitted coloring produces a chain map from hexagon polynomial cochain complex to simplicial cochain complex

Standard \( n \)-simplex \( \Delta^n = 1 \ldots (n+1) \) is isomorphic to any \( n \)-simplex \( \sigma^n \subset K \). To be exact, we will be working with the isomorphism conserving the order of vertices, according to our general rule (1). This isomorphism yields, in particular, the (bijective) mapping

\[
\text{set of tetrahedra in } \Delta^n \rightarrow \text{set of tetrahedra in } \sigma^n.
\]  

A permitted coloring of \( \Delta^n \) or \( \sigma^n \) is a mapping from the l.h.s. or r.h.s. of (18), respectively, to \( F^2 \). Hence, (18) yields the mapping of these colorings, and in the opposite direction:

\[
\text{(permitted colorings of } \Delta^n) \leftarrow \text{(permitted colorings of } \sigma^n\).
\]  

Recalling now the polynomial cochain definition given in the beginning of Section 4, and taking into account that our polynomials can be understood as \( F \)-valued functions, we come to the following mapping (again from left to right):

\[
C^n_{\text{hex}} \rightarrow (F\text{-valued functions on permitted colorings of } \sigma^n),
\]  

where \( C^n_{\text{hex}} \) means the space of polynomial hexagon \( n \)-cochains, see (16). Taking mappings (20) for all \( \sigma^n \subset K \) at once, and assuming that a permitted coloring of \( K \) is given, we arrive finally at a linear mapping

\[
f_n: \quad C^n_{\text{hex}} \rightarrow C^n(K, F),
\]  

where \( C^n(K, F) \) is the linear space of usual simplicial \( F \)-valued \( n \)-cochains on \( K \).

To justify the header of this Subsection, it remains to prove the following theorem.

**Theorem 2.** Given a permitted coloring of \( K \), mappings \( f_n \) (21) form together a chain map \( f \). That is, the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C^3_{\text{hex}} & \rightarrow & C^4_{\text{hex}} & \rightarrow & \cdots \\
\downarrow & & \downarrow f_3 & & \downarrow f_4 & & \\
C^2(K, F) & \rightarrow & C^3(K, F) & \rightarrow & C^4(K, F) & \rightarrow & \cdots
\end{array}
\]  

is commutative. Here the first row is the complex (16), while the second row is a truncated simplicial cochain complex for \( K \), with 0- and 1-cochain spaces cut out.

**Proof.** It remains to note that the consistency of \( f \) with the codifferentials in the two complexes follows at once from the fact that these codifferentials are given by the alternated sums of the same kind (15) over \( (n-1) \)-faces of \( \Delta^n \).

**Important Remark 4.** For our polynomial cocycles, chain map \( f \) depends on a permitted coloring also polynomially.
Bilinear case. The constructions of this Subsection, including Theorem 2, are easily transferred to the bilinear case (see the paragraph containing (17)). In this case, a pair of permitted colorings must be given, and the first row in (22) will consist of bilinear hexagon cochains.

5.2 Induced cohomology map and coloring homology

Chain map $f$ (22) induces mappings of the cohomology spaces. Since $f$ depends on a chosen permitted coloring, so do the cohomology mappings. In some important cases a cohomology mapping depends actually only on the coloring homology class of the permitted coloring, and some—although not all!—of such cases are covered by the following theorem.

Recall (see, for instance, [5]) that the link $\text{lk}(A, K)$ of a simplex $A$ in a simplicial complex $K$ consists of all simplices $B$ such that the join $A \star B$ is also a simplex in $K$.

**Theorem 3.** Suppose that, for every edge $b$ of complex $K$, the link $\text{lk}(b, K)$ has trivial $(n-2)$-th simplicial cohomology group:

$$H^{n-2}(\text{lk}(b, K), F) = 0.$$  \hspace{1cm} (23)

Then, the cohomology map induced by chain map $f$ (22) in dimension $n \geq 3$,

$$f^{(n)}: \, H^n_{\text{hex}} \rightarrow H^n(K, F),$$  \hspace{1cm} (24)

depends only on the coloring homology class of the permitted coloring.

**Remark.** The situation where map (24) depends on a coloring homology element can of course be formulated as a map

$$H^n_{\text{hex}} \times H^\text{col}(K, F) \rightarrow H^n(K, F).$$  \hspace{1cm} (25)

**Proof.** It is enough to consider the change of the permitted coloring by one edge vector, with a coefficient from $F$, and corresponding to an edge $b$. We have to prove that the image $\phi = f(c)$ of a hexagon cocycle $c$ under $f$ can change only by a simplicial coboundary. We denote the ‘old’ and ‘new’ versions of $\phi$ as $\phi_{\text{old}}$ and $\phi_{\text{new}}$, and their difference as

$$\Delta = \phi_{\text{new}} - \phi_{\text{old}}.$$

As $c$ is a cocycle and $f$ a chain map, all of $\phi_{\text{old}}, \phi_{\text{new}}$ and $\Delta$ are simplicial cocycles, and we must show that $\Delta$ is, moreover, a coboundary.

In our situation, $\phi$ changes only locally, namely, only on simplices $\sigma^n$ containing $b$ and representable thus as

$$\sigma^n = b \star \tau^{n-2},$$  \hspace{1cm} (26)
where simplex $\tau^{n-2}$ belongs to $\text{lk}(b, K)$. Consider a simplicial $(n-2)$-cochain $\Xi$ on $\text{lk}(b, K)$ defined by
\[
\Xi(\tau^{n-2}) = \Delta(\sigma^n)
\]
for $\tau^{n-2}$ and $\sigma^n$ as in (26). As $\Delta$ is a cocycle, $\Xi$ is also a cocycle on the link $\text{lk}(b, K)$ and, due to (23), also a coboundary
\[
\Xi = \delta \hat{\Xi}
\]
of an $(n-3)$-cochain $\hat{\Xi}$ on $\text{lk}(b, K)$.

We now define $(n-1)$-cochain $\hat{\Delta}$ on $K$, vanishing outside $\text{lk}(b, K)$, as follows:
\[
\hat{\Delta}(\sigma^{n-1}) = \begin{cases} 
\hat{\Xi}(\tau^{n-3}) & \text{for } \sigma^{n-1} = b \star \tau^{n-3}, \\
0 & \text{for other } \sigma^{n-1}.
\end{cases}
\]

It follows from (27), (28) and (29) that $\Delta = \delta \hat{\Delta}$.

**Bilinear case.** The bilinear version of Theorem 3 says that the cohomology maps (24) depend on the pair of coloring homology classes of the permitted colorings taking part in the construction. As for the proof, it begins with the words ‘it is enough to consider the change of one permitted coloring by one edge vector’. Otherwise, everything goes the same way as in the ‘polynomial’ case.

### 6 Hexagon cohomology, coloring homology, and simplicial cohomology in a piecewise linear four-manifold

Theorem 3 can be reformulated (especially if we look at (25)) as follows: given a hexagon $n$-cocycle $c$ (up to a coboundary), and if the technical condition (23) is fulfilled, we obtain a polynomial mapping
\[
g_{\text{col}}^{(n)}: H_{\text{col}}(K, F) \to H^n(K, F).
\]

Recall that we worked in Section 5 within a fixed finite simplicial complex $K$, with the numbering of its vertices also fixed. Now we are going to consider a four-dimensional piecewise linear manifold $M$, and let $K$ represent a triangulation of $M$ (this is often written as $M = |K|$). As the link of an edge in a PL 4-manifold is a 2-sphere, (23) holds of course for $n = 3$, but not for $n = 4$. Nevertheless, mapping (30) can be defined quite naturally for $n = 4$ as well, using the following roundabout way.

Let $I = [0, 1]$, and consider the direct product $\mathcal{M} = M \times I$. In $\mathcal{M}$, the link of every edge has trivial second cohomology, hence, Theorem 3 does work for $\mathcal{M}$.
and \( n = 4 \). Also, \( M \times \{0\} \cong M \) is a deformation retract of \( M \), so there exists a canonical isomorphism \( H^4(M, F) \cong H^4(M, F) \).

The conclusion is that the cohomology map (24), for \( n = 4 \) and \( K \) a triangulation of \( M \), again depends only on the coloring homology class of the permitted coloring. Or, in other words, for a hexagon 4-cocycle given to within adding a coboundary, and \( n = 4 \), we have also defined the polynomial mapping (30).

Moreover, we are going to show that mapping (30), for \( n = \) either 3 or 4, does not actually depend on the specific triangulation of \( M \) or numbering of its vertices. As there is no problem with replacing \( H^n(K, F) \) with \( H^n(M, F) \), we will focus our attention on the term \( H_{\text{col}}(K, F) \) and on the mapping \( g_{\text{col}}^{(n)} \) itself.

**Theorem 4.** Let \( K_1 \) and \( K_2 \) be two simplicial complexes corresponding to two triangulations of a given PL 4-manifold \( M \), and let either \( n = 3 \) or \( n = 4 \). Then, there exists an isomorphism \( \iota: H_{\text{col}}(K_1, F) \to H_{\text{col}}(K_2, F) \) making the following diagram commutative:

\[
\begin{array}{ccc}
H_{\text{col}}(K_1, F) & \overset{\iota}{\longrightarrow} & H^n(M, F) \\
\downarrow & & \\
H_{\text{col}}(K_2, F) & \overset{\iota}{\longrightarrow} & 
\end{array}
\]  \tag{31}

Two non-vertical arrows in (31) are of course the versions of \( g_{\text{col}}^{(n)} \) for \( K_1 \) and \( K_2 \).

The proof of Theorem 4 is given in the next three subsections. Namely, in Subsections 6.1 and 6.2, we consider the case where the triangulation is changed by one Pachner move, for \( n = 3 \) and \( n = 4 \), respectively. We must pay attention to the vertex order, so we emphasize that our Pachner moves are as described in Subsection 3.1 (don’t forget also Important Remark 3), together with the possibility (4) of changing the vertex numbering without changing the order. As any two triangulations can be connected by a chain of Pachner moves, it will remain to show how the vertex order can be changed, and this is done in Subsection 6.3.

### 6.1 Dimension three

Let \( n = 3 \), and consider what happens with mapping (30) under a Pachner move. Returning to the notations of Theorem 1, we denote \( K_{\text{ini}} \) and \( K_{\text{fin}} \) the triangulations of \( M \) before and after this move, and use the isomorphism (14) between the coloring homology groups.

On the other hand, it is known that the \( \sim \)-product between \( n \)-th cohomology and \( n \)-th homology with coefficients in a field is nondegenerate. This implies that, in our situation, any element \( h \) of the third cohomology group is determined by the values that any 3-cocycle representing \( h \) takes on 3-cycles modulo 3-boundaries.
We can now take all simplicial 3-cycles intended for calculating the mentioned values such that they contain no inner tetrahedra of cluster $C$ or $\bar{C}$ (that is, with zero coefficients at such tetrahedra. For notations $C$ or $\bar{C}$ see Subsection 3.2).

We can thus describe mapping (30) avoiding the cluster changed by the Pachner move. This means, together with item (iv) of Theorem 1, that the commutative diagram (31), for $K_1 = K_{\text{ini}}$, $K_2 = K_{\text{fin}}$, and (14) as $\iota$, indeed takes place.

6.2 Dimension four

We now consider how a Pachner move affects the image of a given element $h_{\text{col}} \in H_{\text{col}}(K, F)$ under mapping (30), for $n = 4$. This Pachner move replaces (using the notations of Section 3) a cluster $C$ of $k$ pentachora, $1 \leq k \leq 5$, by $\bar{C}$—its closed complement in $\partial \Delta^5$.

Recall that there is a hexagon 4-cocycle $c$ behind our mapping (30). This $c$ is sent to a simplicial cocycle by $f_4$ in diagram (22), and $f_4(c)$ is nothing but a representative of the simplicial cohomology element $g_{\text{col}}^{(4)}(h_{\text{col}})$. In application to $\partial \Delta^5$, this means that

$$g_{\text{col}}^{(4)}(h_{\text{col}}) \sim \partial \Delta^5 = 0. \quad (32)$$

Now, if our PL manifold $M$ is orientable, we proceed as follows. Choose an orientation of $M$; it induces also an orientation of either $C$ or $\bar{C}$ as part of $M$. These orientations of $C$ and $\bar{C}$ are, however, not consistent if $C$ and $\bar{C}$ are regarded as part of $\partial \Delta^5$. Hence, if $C$ and $\bar{C}$ are oriented this way, (32) implies

$$g_{\text{col}}^{(4)}(h_{\text{col}}) \sim C - g_{\text{col}}^{(4)}(h_{\text{col}}) \sim \bar{C} = 0. \quad (33)$$

We see that replacing $C$ by $\bar{C}$ simply does not change the product $g_{\text{col}}^{(4)}(h_{\text{col}}) \sim c$ for any 4-cycle $c \in H_4(M, F)$, and hence the image of $h_{\text{col}}$ in $H^4(M, F)$ stays also the same.

In the most general case, $M$ may consist of several connected components. For the orientable components, we proceed as above. A non-orientable component yields a nontrivial 4-cycle only if our field $F$ is of characteristic 2, in which case (32) also surely implies (33).

6.3 Independence of vertex numbering

We have shown the invariance of mappings (30), for $n = 3$ and $n = 4$, under Pachner moves. One small problem that still remains is that we were always assuming that the vertices of any triangulation are ordered (see Subsection 2.2). We are now going to show that these mappings do not actually depend on the order of vertices.

Indeed, here is how we can change the position of any one vertex $v$ in this ordering. We do any chain of Pachner moves that removes $v$ from the triangulation; this removal is of course performed by a move 5–1. Then we do all this
chain backwards, but when doing the corresponding move 1–5, we change the position of \( v \) in the order of vertices into any other we like. Such possibility is of course ensured by the fact that we have a full hexagon, see Important Remark 3.

We have thus proved Theorem 4 and defined the following mappings for a PL 4-manifold \( M \):

\[
H_{\text{col}}(M, F) \rightarrow H^3(M, F), \quad \text{for a given hexagon 3-cocycle,} \quad (34)
\]

and

\[
H_{\text{col}}(M, F) \rightarrow H^4(M, F), \quad \text{for a given hexagon 4-cocycle.} \quad (35)
\]

**Bilinear case.** In the bilinear case, \( H_{\text{col}}(K, F) \) in (30) is replaced by \( H_{\text{col}}(K, F) \times H_{\text{col}}(K, F) \). Similarly, the spaces of color homologies are replaced with their Cartesian squares in the formulation of Theorem 4 and in its proof (namely, in Subsection 6.2). Otherwise, everything goes the same way, and we arrive at the bilinear versions of (34) and (35); these can be found below as (41) and (44).

## 7 Constant hexagon as a limiting case of non-constant hexagon

Our ‘constant’ edge functionals (7) and edge vectors (10) can be obtained as a limiting case (formal limit in the case of a finite characteristic) of ‘nonconstant’ edge functionals and edge vectors introduced in [4]. We will content ourself here with explaining how it works only for edge *functionals*; for edge *vectors*, the procedure is much the same and is left as an easy exercise for the reader. Also, we will do everything on the example of one tetrahedron \( t = 1234 \).

We first take edge functionals in the form [4, (31)]:

\[
\begin{pmatrix}
\phi_{12} \\
\phi_{13} \\
\phi_{14} \\
\phi_{23} \\
\phi_{34}
\end{pmatrix}_{1234} = \begin{pmatrix}
\omega_{234} - \omega_{134} & 0 \\
\omega_{124} & \omega_{234} \\
-\omega_{123} & -\omega_{234} \\
-\omega_{124} & -\omega_{134} \\
\omega_{123} & \omega_{134} \\
0 & \omega_{123} - \omega_{124}
\end{pmatrix}, \quad (36)
\]

where \( \omega \) is a \( F \)-valued simplicial 2-cocycle (which means, in application to the tetrahedron 1234, that its values entering (36) satisfy \( \omega_{123} - \omega_{124} + \omega_{134} - \omega_{234} = 0 \)).

We choose our \( \omega \) as follows:

\[
\omega_{ijk} = 1 + o \cdot \varrho_{ijk}, \quad i < j < k, \quad (37)
\]

where \( \varrho \) is a *given* simplicial 2-cocycle, and \( o \) is an infinitesimal parameter. To be exact, \( o \) is finite at this moment, but we are going to set \( o \to 0 \). Also, \( \varrho \) is
supposed to be generic enough so that we don’t encounter division by zero in our
expressions below (see expression for $A_2$ in (40)).

Now, we compose new edge functionals $\phi'_{ij}$ as follows: denote

$$A_o = \begin{pmatrix} 1 & o^{-1} \\ 0 & -o^{-1} \end{pmatrix}$$

(38)

—this matrix will be responsible for the invertible linear transformation in the
space $V_t = F^2$ of colorings of our tetrahedron $t$ for each finite $o$—and set

$$\begin{pmatrix} \phi'_{12} \\ \phi'_{13} \\ \phi'_{14} \\ \phi'_{23} \\ \phi'_{24} \\ \phi'_{34} \end{pmatrix}_{1234} = \lim_{o \to 0} \begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{23} \\ \phi_{24} \\ \phi_{34} \end{pmatrix}_{1234} A_o = \begin{pmatrix} 0 & \varrho_{234} - \varrho_{134} \\ 1 & \varrho_{124} - \varrho_{234} \\ -1 & -\varrho_{123} + \varrho_{234} \\ -1 & -\varrho_{124} + \varrho_{134} \\ 1 & \varrho_{123} - \varrho_{134} \\ 0 & -\varrho_{123} + \varrho_{124} \end{pmatrix}.$$  

(39)

Then we make the linear transform in the space of tetrahedron 1234 colorings,
corresponding to multiplying (39) from the right by the product $A_1 A_2$, where

$$A_1 = \begin{pmatrix} 1 & \varrho_{134} - \varrho_{124} \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & (\varrho_{123} - \varrho_{124})^{-1} \end{pmatrix},$$

(40)

that is, we set

$$\begin{pmatrix} \phi''_{12} \\ \phi''_{13} \\ \phi''_{14} \\ \phi''_{23} \\ \phi''_{24} \\ \phi''_{34} \end{pmatrix}_{1234} = \begin{pmatrix} \phi'_{12} \\ \phi'_{13} \\ \phi'_{14} \\ \phi'_{23} \\ \phi'_{24} \\ \phi'_{34} \end{pmatrix}_{1234} A_1 A_2.$$

Finally, we rename $\phi''_{ij} \mapsto \phi_{ij}$. We have arrived exactly at (7), for $k_1 = 1, \ldots, k_4 = 4$.

**Important Remark 5.** We could of course apply the linear transformation corre-
sponding to the whole product $A_o A_1 A_2$ before taking the limit $o \to 0$. We hope,
however, that our step-by-step approach makes things clearer.

**Important Remark 6.** As we see, the limiting process described above is far from
being unique, because it depends on a chosen cocycle $\varrho$.

## 8 Experimental results

We present here, just for illustration, some calculation results showing how map-
pings (34) and (35), or their bilinear analogues, can look explicitly. Then, in
Subsection 8.3 we briefly tell the reader what happens with the spaces $V_K$.
and $V_K^{(0)}$—recall formula (12)—at the point of passing from constant to non-constant hexagon. The conclusion is that, first, our ‘constant’ case is already very intriguing, and second, that the investigation of its neighborhood within the ‘nonconstant’ case—which will be a big and separate research—is extremely promising.

### 8.1 Some nontrivial hexagon cocycles

**Bilinear 3-cocycle.** Given a nontrivial hexagon bilinear 3-cocycle, we come to the bilinear analogue of (34), that is, a bilinear mapping

$$H_{\text{col}} \times H_{\text{col}} \rightarrow H^3(M, F). \quad (41)$$

Namely, we will use the following 3-cocycle:

$$c^{(3)} = -xy' - yx'. \quad (42)$$

Here $(x, y)$ and $(x', y')$ are a pair colorings of the tetrahedron 1234, as required in (17). Recall that all colorings of a separate tetrahedron are permitted.

**Bilinear 4-cocycle.** Similarly, there is also the following nontrivial hexagon bilinear 4-cocycle:

$$c^{(4)} = y_{2345} y'_{1234}. \quad (43)$$

which yields a mapping

$$H_{\text{col}} \times H_{\text{col}} \rightarrow H^4(M, F). \quad (44)$$

—the bilinear analogue of (35). Recall that cocycle (13)—as well as (15) and (16) below—belongs to the ‘standard’ pentachoron $\Delta^4 = 12345$, see Section 4.

**Two cubic 4-cocycles in characteristic 2.** There are two linearly independent modulo coboundaries cubic hexagon 4-cocycles in characteristic 2. The first of them is obtained from (13) by setting $y_i' = y_i^2$:

$$c_1^{(4)} = y_{2345} y_{1234}^2 \quad (45)$$

Recall that raising to the second power is a linear operation—*Frobenius endomorphism*—for a field of characteristic 2.

To make the structure of the second 4-cocycle more visible, we introduce, for the moment, the following notations:

$$a = x_{2345} + y_{2345}, \quad b = x_{1345} + y_{1345},$$

$$c = x_{1245} + y_{1245}, \quad d = x_{1235} + y_{1235}, \quad e = x_{1234} + y_{1234}.$$ 

The cocycle is then

$$c_2^{(4)} = bde + bce + ace + acd + abd. \quad (46)$$
Other nontrivial cocycles in finite characteristics. Many more nontrivial cocycles have been calculated in [3]. Note that a different basis in the two-dimensional space of colorings of one tetrahedron was used in [3]. Namely, if we denote, for a moment, the $x$ and $y$ of [3] as $\tilde{x}$ and $\tilde{y}$ (and our colors (2) as simply $x$ and $y$), then

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$  

8.2 Calculations for specific manifolds using constant cohomology

The first experimental result, and unexplained as yet, is that the dimension $d$ of coloring homology space $H_{\text{col}}(M, F)$ is the sum of the dimensions of two usual cohomology groups:

$$d \overset{\text{def}}{=} \dim H_{\text{col}}(M, F) = \dim H^2(M, F) + \dim H^3(M, F). \quad (47)$$

We introduce some basis in $H_{\text{col}}(M, F)$, and write polynomial functions on $H_{\text{col}}(M, F)$ in terms of coordinates $X_1, \ldots, X_d$ w.r.t. this basis. For the bilinear case, the coordinates of the second element in $H_{\text{col}}(M, F)$ are denoted by primed letters $X'_1, \ldots, X'_d$.

For mapping (34), we also introduce a basis in $H^3(M, F)$. Hence, mapping (34), corresponding to cocycle $c^{(3)}$ (12), is given by $d_3 = \dim H^3(M, F)$ polynomials; we denote them below as $p_1^{(3)}, \ldots, p_{d_3}^{(3)}$.

For mapping (35), we identify an element of $H^4(M, F)$ with its value on the fundamental class $[M]$ (because we are going to consider only connected manifolds). Hence, any of cocycles $c^{(4)}$, $c_1^{(4)}$ and $c_2^{(4)}$ of Subsection 8.1 gives just one polynomial, denoted respectively as $p^{(4)}$, $q^{(4)}$ and $r^{(4)}$.

Below our notations are as usual: $S^n$ is an $n$-dimensional sphere, $T^n$ is an $n$-dimensional torus, $\mathbb{R}P^n$ is an $n$-dimensional real projective space, $\mathbb{C}P^2$ is a complex two-dimensional projective space, and $S^2 \tilde{\times} S^2$ denotes the twisted product of two spheres $S^2$.

We present our results in the following form: the manifold $M$ and the field $F$ we are working with form a header highlighted by underlining, and then go the experimental results for them. We think it is enough to give here a few examples with just one field $F = \mathbb{F}_2$—the prime Galois field of two elements; the more so because $\mathbb{F}_2$ works well with both orientable and unorientable manifolds.

Important Remark 7. Experimental result (47) works of course in all characteristics, as far as we could check.
\[ M = \mathbb{C}P^2, \quad F = \mathbb{F}_2 \]

\[ d = 1, \]
\[ p^{(4)} = X_1X'_1, \]
\[ q^{(4)} = X_1^3, \]
\[ r^{(4)} = q^{(4)}. \]

\[ M = S^2 \times S^2, \quad F = \mathbb{F}_2 \]

\[ d = 2, \]
\[ p^{(4)} = X_1X'_2 + X_2X'_1, \]
\[ q^{(4)} = X_1^2X_2 + X_1X_2^2, \]
\[ r^{(4)} = q^{(4)}. \]

\[ M = S^2 \tilde{\times} S^2, \quad F = \mathbb{F}_2 \]

\[ d = 2, \]
\[ p^{(4)} = X_1X'_2 + X_2X'_1 + X_2X'_2, \]
\[ q^{(4)} = X_1^2X_2 + X_1X_2^2 + X_3^2, \]
\[ r^{(4)} = q^{(4)}. \]

\[ M = S^2 \times T^2, \quad F = \mathbb{F}_2 \]

\[ d = 4, \]
\[ p^{(3)}_1 = X_1X'_3 + X'_1X_3, \]
\[ p^{(3)}_2 = X_1X'_4 + X'_1X_4, \]
\[ p^{(4)} = X_1X'_2 + X_1X'_3 + X_2X'_1 + X'_1X_3, \]
\[ q^{(4)} = X_1^2X_2 + X_1^2X_3 + X_1X_2^2 + X_1X_3^2, \]
\[ r^{(4)} = q^{(4)}. \]
\[ M = \mathbb{R}P^2 \times S^2, \quad F = \mathbb{F}_2 \]

\[ d = 3, \]
\[ p_1^{(3)} = X_2 X'_3 + X'_2 X_3, \]
\[ p^{(4)} = X_1 X'_3 + X'_1 X_3, \]
\[ q^{(4)} = X'_1 X'_3 + X_1 X'_2, \]
\[ r^{(4)} = X'_1 X_3 + X_1 X'_2 + X'_2 X_3. \]

\[ M = \mathbb{R}P^2 \times T^2, \quad F = \mathbb{F}_2 \]

\[ d = 7, \]
\[ p_1^{(3)} = X_1 X'_6 + X'_1 X_6 + X_4 X'_5 + X_5 X'_4 + X_5 X'_6 + X_6 X'_5, \]
\[ p_2^{(3)} = X_1 X'_6 + X'_1 X_6 + X_2 X'_7 + X_2 X'_6 + X'_1 X_7 + X_3 X'_4 \]
\[ + X_3 X'_6 + X_4 X'_3 + X_4 X'_7 + X_5 X'_2 + X_6 X'_3 + X_7 X'_4 + X_7 X'_6, \]
\[ p_3^{(3)} = X_2 X'_6 + X'_2 X_5 + X_3 X'_6 + X_4 X'_7 + X_5 X'_6 + X_5 X'_7 + X_6 X'_4 \]
\[ + X_6 X'_5 + X_7 X'_4 + X_7 X'_5, \]
\[ p^{(4)} = X_1 X'_7 + X'_1 X_7 + X_3 X'_5 + X_5 X'_3 + X_5 X'_7 + X_7 X'_5, \]
\[ q^{(4)} = X'_7 X_7 + X_1 X'_7 + X_2 X'_5 + X_3 X'_5 + X'_2 X_7 + X_5 X'_7, \]
\[ r^{(4)} = X'_2 X_7 + X_1 X'_7 + X_2 X'_5 + X'_3 X_5 + X'_3 X_6 + X_3 X'_5 + X'_4 X_7 \]
\[ + X_5 X'_6 + X_5 X'_2 + X'_6 X_7 + X_6 X'_2. \]

\[ M = \mathbb{R}P^2 \times \mathbb{R}P^2, \quad F = \mathbb{F}_2 \]

\[ d = 5, \]
\[ p_1^{(3)} = X_1 X'_5 + X'_1 X_5 + X_1 X'_4 + X'_1 X_4 + X_2 X'_3 + X'_2 X_3, \]
\[ p_2^{(3)} = X_3 X'_5 + X_2 X'_5 + X'_3 X_5 + X'_2 X_5 + X_3 X'_4 + X'_3 X_4, \]
\[ p^{(4)} = X_1 X'_5 + X'_1 X_5 + X_3 X'_3, \]
\[ q^{(4)} = X_1 X'_5 + X'_1 X_5 + X'_3, \]
\[ r^{(4)} = X^2 X_5 + X^2 X_5 + X^2 X_5 + X'_3 X_4 + X^2 X_4 + X^2 X_3 + X_2 X^2. \]
\[ M = \mathbb{R}P^4, \quad F = \mathbb{F}_2 \]

\[ d = 2, \]
\[ p^{(3)} = X_{1}X'_{2} + X_{2}X'_{1}, \]
\[ p^{(4)} = X_{2}X', \]
\[ q^{(4)} = X^3_{2}, \]
\[ r^{(4)} = X^2_{1}X_{2}. \]

In the above examples, one can see that
\[ q^{(4)} = r^{(4)} \quad (48) \]
for all orientable manifolds. Interestingly, equality (48) may be violated for more complicated manifolds. Namely, define the simplest twisted tori as follows. First, we denote \( \tilde{T}_n^3 \) the fiber bundle with base \( S^1 \), fiber \( T^2 \), and monodromy matrix \( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \). Such fiber bundles are three-dimensional twisted tori. Then we consider four-dimensional twisted tori \( \tilde{T}_n^4 \) defined simply as direct products of \( \tilde{T}_n^3 \) with a circle:
\[ \tilde{T}_n^4 = \tilde{T}_n^3 \times S^1. \]

If \( n = 0 \), we get of course the usual torus \( \tilde{T}_0^4 = T^4 \).

The unexpected calculation result is: (48) holds for \( n = 0, 1, 3 \) and 4, but not for \( n = 2 \).

### 8.3 What happens at the point of passing to a nonconstant case

In the ‘nonconstant’ case of paper [4], there are also linear spaces \( W_K \) of permitted and \( W^{(0)}_K \) of edge-generated colorings—direct analogues of spaces \( V_K \) and \( V^{(0)}_K \) introduced in Subsection [2.4]. A very interesting question is what happens with them under the passage to the limit described in Section [7]. This is going to be the subject of a separate research; here we only explain some simple ideas and inform the reader of some experimental facts. Also, we restrict ourselves here to the field \( F = \mathbb{R} \) of real numbers, in order to be able to use simple analytic arguments.

Space \( W^{(0)}_K \) is the linear span of \( N_1 \) vectors in \( F^{2N_3} = \mathbb{R}^{2N_3} \) (see the first paragraph in Subsection [2.2] for notations). Suppose we pass to the limit in such way that \( W^{(0)}_K \) retains a constant dimension \( d^{(0)} \), then it has a limiting location (at least one, due to the compactness of the real Grassmannian \( \text{Gr}(d^{(0)}, \mathbb{R}^{2N_3}) \))—a subspace \( L^{(0)}_K \subset \mathbb{R}^{2N_3} \) of the same dimension \( d^{(0)} \).
Remark. The prelimit invertible linear transform—direct sum of transforms given by the product of matrices (38) and (40) (see also Important Remark 5), taken over all two-dimensional color spaces \( V_t \cong F^2 \) for all tetrahedra \( t \)—clearly does not affect the validity of this argument.

Every separate ‘nonconstant’ edge vector also has its ‘constant’ limit, given by (10) (recall the first paragraph of Section 7). The linear span of these limiting edge vectors is nothing but our ‘constant’ space \( V_K^{(0)} \), and of course \( V_K^{(0)} \subset L_K^{(0)} \), but it may—and does—happen that this inclusion is strict!

Space \( W_K \) is the opposite case: it is singled out by linear restrictions. So, its limit \( L_K \) may only be smaller than the ‘constant’ space \( V_K \). We come this way to the chain of inclusions:

\[
V_K^{(0)} \subset L_K^{(0)} \subset L_K \subset V_K.
\]  

(49)

Now the experimental facts:

(i) at least sometimes,

\[
\dim L_K^{(0)} = \dim V_K^{(0)} + 1,
\]

(50)

(ii) at least sometimes,

\[
\dim L_K = \dim V_K - 2k, \quad k = 1, 2, \ldots.
\]

(51)

9 Discussion

Polynomials calculated in Subsection 8.2, if taken as they are, are not PL manifold invariants, because their definition requires a basis in the coloring homology space \( H_{col}(M, F) \). Moreover, a basis in \( H^3(M, F) \) is also required for polynomials \( p_1^{(3)} , \ldots , p_{d_3}^{(3)} \). Invariant is, of course, the set of these polynomials taken up to linear transformations of the mentioned bases—but this is not the point where to stop!

Remark. In paper [3], some simple invariants were actually calculated. Namely, given one of our polynomials \( q^{(4)} , r^{(4)} \), or \( q^{(4)} + r^{(4)} \), we let its variables take values from a finite extension \( F_{2^k} \) of field \( F_2 \) and calculated, for each \( v \in F_{2^k} \), how many times the polynomial takes value \( v \). Polynomials \( p_1^{(3)} , \ldots , p_{d_3}^{(3)} \) were not considered in [3].

What looks much more interesting is the fact that the chain (49) of embeddings, taking into account experimental facts (50) and (51), brings about some additional structure on our ‘constant’ space \( V_K \) and hence—see (12)—on the coloring homology space, probably relating this latter to usual cohomologies. Given the existence of a great many nontrivial hexagon cocycles [3], this may lead to very interesting consequences.
Finally, it must be said that the constructions proposed in [3, 4] and the present paper, are not confined to just four-dimensional manifolds. Similar things surely can be done in three dimensions, based, for example, on the pentagon relations proposed in [2]. Moving in the opposite direction, there are indications of the existence of interesting heptagon relations for five dimensions.

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