Optimal Control of Hybrid Systems Using a Feedback Relaxed Control Formulation

Tyler Westenbroek
Humberto Gonzalez

Abstract—We present a numerically tractable formulation for computing the optimal control of the class of hybrid dynamical systems whose trajectories are continuous. Our formulation, an extension of existing relaxed-control techniques for switched dynamical systems, incorporates the domain information of each discrete mode as part of the constraints in the optimization problem. Moreover, our numerical results are consistent with phenomena that are particular to hybrid systems, such as the creation of sliding trajectories between discrete modes.

I. INTRODUCTION

Hybrid dynamical models are a powerful abstraction capable of describing the behavior of a wide range of systems such as legged locomotion robots [1], unmanned aerial vehicles [2], or industrial process control plants [3], [4], among many others. Yet, even though hybrid models allow us to mathematically describe complex systems, efficient and flexible numerical methods to analyze and control those systems are hard to develop due to inherent phenomena such as Zeno executions [5]. In this paper we present a formulation for the class of hybrid systems whose trajectories are continuous (i.e., hybrid systems without reset maps) based on the theory of relaxed controls [6], [7], [8], and we show that our formulation allows us to numerically solve hybrid optimal control problems.

Before we can describe our results in details we must lay out the context in which our research is found. Early results in optimality conditions for hybrid optimal control can be traced back to the work of Pontryagin et al. [9], who formulated a version of their Maximum Principle for systems whose vector field change discontinuously in predetermined stages. In his seminal work, Sussmann [10], [11] formulated an extension of the Maximum Principle for a general class of hybrid systems incorporating autonomous and controlled discrete variables. Yet, Sussmann’s work is based upon a collection of sufficient conditions limiting the possible trajectories of the hybrid system in questions. The Maximum Principle has been extended to other classes of hybrid systems, such as impulsive hybrid systems [12]. Shaikh and Caines [13], [14] also formulated an extension of the Maximum Principle for hybrid systems, and in their case they also found a numerical method capable of approximating the solution of those problems under some conditions. Shaikh and Caines’s optimality conditions impose restrictions to the optimal trajectories similar to Sussmann’s optimality conditions. In particular, the optimal trajectory cannot be Zeno (i.e., switching between discrete modes infinitely fast), nor non-orbitally stable (i.e., not continuous with respect to initial conditions), properties that cannot be easily checked in practice yet naturally occur in many applications [15]. Hence, these restrictions have limited the deployment of numerical algorithms based on these optimality conditions to a wide range of applications.

The case of optimal control algorithms for switched hybrid systems deserves particular attention, since the properties of these systems have enabled the development of a number of numerical algorithms. Xu and Antsaklis [16] formulated a general framework for the optimal control of switched systems dividing the problem in two parts, optimization of discrete sequences and optimization of discrete jump times. Boccadoro et al. [17] created a robust algorithm defining switching surfaces that are triggered by the continuous state variables, effectively creating a closed-loop solution. Axelson et al. [18] used needle variations to find an optimality condition that could be implemented using gradient descent algorithms. Their approach was later extended by Gonzalez et al. [19], [20] to systems with constraints and continuous inputs. Wardi and Egerstedt [21] used a measure theory approach to develop an switched optimal control algorithm.

Our work in this paper can be seen as an extension of the work by Morari et al. [22], who transformed a discrete-time hybrid optimal control problem into a mixed integer optimization problem, Heemels and Brogliato [23], who used linear complementary problems to formulate discrete hybrid modes, and Vasudevan et al. [24], [25], who used a relaxed control formulation to create an adaptive refinement optimal control algorithm for switched systems. In particular, we formulate the hybrid optimal control problem as a relaxed control switched optimal control problem (similar to [24]) with a set of extra constraints that use a complementary condition to disable certain discrete modes as a function of the continuous state values (similar to [22], [23]). In practice, our formulation behaves as a switched hybrid system with a set-valued feedback law that takes the continuous state and returns a set of allowed discrete modes, which can be numerically approximated using standard techniques as a nonlinear programming problem [26].

The paper is organized as follows. Section II presents the mathematical background necessary to describe our results. Section III presents our relaxed control formulation and the consequent hybrid optimal control problem. Section IV presents our numerical implementation of the hybrid optimal control. Section V presents three hybrid optimal control problems solved using our algorithm. Finally, Section VI...
presents our conclusions and future research directions.

II. MATHEMATICAL BACKGROUND

We begin by first formally defining the class of hybrid systems we will focus on.

Definition 1: A controlled hybrid system with continuous trajectories is a tuple $\mathcal{H} = (\mathcal{X}, \mathcal{I}, \mathcal{D}, \mathcal{U}, \mathcal{F})$, where:

- $\mathcal{X} = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$ is the continuous-state space, with $h: \mathbb{R}^n \to \mathbb{R}^p$;
- $\mathcal{I} = \{1, \ldots, M\}$ is the discrete-state space;
- $\mathcal{D} = \{\mathcal{D}_i\}_{i \in \mathcal{I}}$ is the set of domains, where for each $i \in \mathcal{I}$, $\mathcal{D}_i = \{x \in \mathcal{X} \mid d_i(x) \leq 0\}$ and $d_i: \mathcal{X} \to \mathbb{R}^p$;
- $\mathcal{U} \subseteq \mathbb{R}^m$ is the set control inputs; and,
- $\mathcal{F} = \{f_i\}_{i \in \mathcal{I}}$ is the set of vector fields, where $f_i: \mathcal{D}_i \times \mathcal{U} \to \mathbb{R}^n$ for each $i \in \mathcal{I}$.

Throughout this paper we use $\|\cdot\|_p$ to denote the finite-dimensional $p$-norm, i.e., given $x \in \mathbb{R}^n$, $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$. We will abuse notation and denote the 2-norm simply by $\|\cdot\|$. For each $i \in \mathcal{I}$.

Definition 2: Let $\mathcal{S}$ be a subset of a Banach space and $\gamma: [0, T] \to \mathcal{S}$. We say that $\gamma \in L^2([0, T], \mathcal{S})$ iff:

$$\|\gamma\|_{L^2} = \left( \int_0^T \|\gamma(t)\|^2 dt \right)^{\frac{1}{2}} < \infty. \tag{1}$$

Also, let $V(\gamma)$ be the total variation of $\gamma$, defined by:

$$V(\gamma) = \sup \left\{ \sum_{k=1}^N \|\gamma(t_{k+1}) - \gamma(t_k)\|_1 \mid \{t_k\}_{k=1}^N \in \mathcal{P}_{[0,T]}, N \in \mathbb{N} \right\}, \tag{2}$$

where $\mathcal{P}_{[0,T]}$ is the set of all finite partitions of $[0, T]$. We say that $\gamma$ is of bounded variation, denoted $\gamma \in BV([0, T], \mathcal{S})$, iff $V(\gamma) < \infty$.

We impose the following standard assumptions to ensure the existence and uniqueness of the continuous trajectories in each discrete mode, together with the well-posedness of the constraints in our optimization problem. Details about the practical consequences of these assumptions can be found in Chapter 5 of [26].

Assumption 3: The functions $\{f_i\}_{i \in \mathcal{I}}$, $h$, and $\{d_i\}_{i \in \mathcal{I}}$ are continuously differentiable. Moreover, all these functions are globally Lipschitz continuous, and so are all their partial derivatives.

Assumption 4: The set $\mathcal{U}$ is compact. Also, the sets $\mathcal{U}$, $\mathcal{X}$, and $\{\mathcal{D}_i\}_{i \in \mathcal{I}}$ are constraint qualified. More information regarding constraint qualified sets can be found in Chapter 12.2 of [27].

Now we state our definition of hybrid execution and hybrid trajectory.

Definition 5: Let $\mathcal{H}$ be a hybrid system as in Definition 1, $u \in L^2([0, T], \mathcal{U})$, and $x: [0, T] \to \mathcal{X}$. We say that the pair $(x, u)$ is a hybrid execution of $\mathcal{H}$ with length $T$ iff for almost every $t \in [0, T]$ there exists $i \in \mathcal{I}$ such that:

$$\dot{x}(t) = f_i(x(t), u(t)); \quad \text{and,} \quad d_i(x(t)) \leq 0. \tag{3}$$

Under these conditions we also say that $x$ is a trajectory of $\mathcal{H}$.

Note that Definition 5 implies that every trajectory of a hybrid system is absolutely continuous (as defined, for example, in [28]), which is consistent with the class of hybrid system in Definition 1. Also note that we make no assumptions regarding domains being disjoint, executions being unique, or Zeno phenomena not appearing. On the contrary, as we show in Section V, we can naturally formulate problems where domains overlap (therefore executions are not unique), as well as problems where the optimal solutions are either Zeno or sliding modes (as understood in the Filippov sense [29]).

As explained in Section II, our main result extends the relaxed control formulation for switched systems in [24] to hybrid systems with continuous trajectories. Hence, using the notation in [24], we introduce the $M$-simplex and the set of corners of the $M$-simplex.

Definition 6: Let $M \in \mathbb{N}$. We define the $M$-simplex, denoted $\Sigma^M$, and the set of corners of the $M$-simplex, denoted $\Sigma^M_p$, by:

$$\Sigma^M_r = \left\{ \omega \in \mathbb{R}^M \mid \sum_{i=1}^M \omega_i = 1, \omega_i \in [0, 1] \forall i \right\}; \quad \text{and,} \quad \Sigma^M_p = \left\{ \omega \in \mathbb{R}^M \mid \sum_{i=1}^M \omega_i = 1, \omega_i \in \{0, 1\} \forall i \right\}. \tag{5, 6}$$

Note that $\Sigma^M_p$ contains $M$ elements, thus if $\omega \in \Sigma^M_p$ then exactly one of its entries $\omega_i = 1$ while all the rest are zero. Thanks to this property we can use vectors in $\Sigma^M_p$ at each time to indicate the active discrete mode of a hybrid system. Also note that $\Sigma^M_p$ is in fact the space of measures (or, depending of the context, probability distributions) over the set $\Sigma^M$. Hence, if we choose a vector in $\Sigma^M_r$ to indicate the current discrete mode we are in fact choosing a relaxed control representation (as described in [6]) of a vector in $\Sigma^M$. This property inspires our notation, where the subscript $p$ stands for pure indicator vectors, while the subscript $r$ stands for relaxed indicator vectors.

We are now ready to present our main result involving the use of a relaxed control formulation to approximate hybrid optimal control problems.

III. RELAXED CONTROL FORMULATION

Our relaxed control formulation is based on two transformations, one for (3) and one for (4), aimed at enabling a simple numerical formulation to compute hybrid executions. First, note that if there exists $\omega \in L^2([0, T], \Sigma^M_p)$ such that:

$$\dot{x}(t) = \sum_{i=1}^M \omega_i(t) f_i(x(t), u(t)), \tag{7}$$

then, since exactly one entry $\omega_i(t) = 1$ for almost every $t \in [0, T]$, we recover the condition in (3). In other words, a function $\omega \in L^2([0, T], \Sigma^M_p)$ behaves as an indicator of the active discrete mode of the hybrid system at time $t$. As shown in Theorem 3.3 of [24], the differential equation in (7) is well defined with unique solutions in the interval $[0, T]$. 

Second, note that given \( \omega \in L^2([0, T], \Sigma^M) \), the condition in (3) implies that if \( d_i(x(t)) > 0 \) then \( \omega_i(t) = 0 \), i.e., if \( x(t) \) does not belong to \( D_i \), then indicator vector cannot set mode \( i \) as active. The contrapositive is therefore also true, i.e., if \( \omega_i(t) = 1 \) then \( d_i(x(t)) \leq 0 \). It is worth noting that the condition in (4) does not impose restrictions to \( \omega \) when \( d_i(x(t)) \leq 0 \), since in that case any other mode \( j \in \mathcal{I} \) might be active at time \( t \) provided \( d_j(x(t)) \leq 0 \). We summarize the relation between \( \omega_i(t) \) and \( d_i(x(t)) \) as follows:

\[
\omega_i(t) d_i(x(t)) \leq 0, \quad \forall i \in \mathcal{I}. \tag{8}
\]

In this way, when each of the inequalities in (8) is satisfied we get a complementarity condition where both \( \omega_i \) and \( d_i \) cannot occur at the same time \( t \).

The enforcement of the conditions in equations (7) and (8) can be theoretically interpreted as a switched hybrid system with a feedback rule disabling certain discrete modes as a function of \( x(t) \). Indeed, let:

\[
\mathcal{S}_p(x(t)) = \{ \omega \in \Sigma^M_p \mid \omega_i = 0 \text{ whenever } d_i(x(t)) > 0 \}, \tag{9}
\]

then the condition in (8) is equivalent to enforcing \( \omega \in \mathcal{S}_p(x(t)) \). This interpretation opens the doors to use algorithms computing the optimal control of switched systems in a bigger class of hybrid systems, as the ones in Definition 1.

In particular, we will use the approach defined in [24] to compute the optimal control of hybrid systems.

The proof of the following proposition follows directly from the argument described above.

**Proposition 7:** Let \( \mathcal{H} \) be a hybrid system as in Definition 1, \( u \in L^2([0, T], \mathcal{U}) \), and \( x: [0, T] \to \mathcal{X} \). Then \( (x, u) \) is a hybrid execution of \( \mathcal{H} \) with length \( T \) if and only if there exists \( \omega \in L^2([0, T], \Sigma^M) \) such that (7) and (8) are satisfied.

Using Proposition 7 we get the following optimal control problem.

**Definition 8:** Let \( \mathcal{H} \) be a hybrid system as in Definition 1.

A hybrid optimal control problem is defined by:

\[
\begin{align*}
\min_{x(t), u(t), \omega(t)} & \int_0^T \sum_{i=1}^M \omega_i(t) L_i(x(t), u(t)) \, dt + \varphi(x(T)), \\
\text{s.t.} & \quad x(0) = \xi, \\
& \quad \dot{x}(t) = \sum_{i=1}^M \omega_i(t) f_i(x(t), u(t)), \quad \forall t \in [0, T], \\
& \quad \omega_i(t) d_i(x(t)) \leq 0, \quad \forall i \in \mathcal{I}, \quad \forall t \in [0, T], \\
& \quad h(x(t)) \leq 0, \quad \forall i \in [0, T], \\
& \quad u(t) \in \mathcal{U}, \quad \forall t \in [0, T], \\
& \quad \omega(t) \in \Sigma^M, \quad \forall t \in [0, T].
\end{align*}
\]

(10)

where \( \{L_i\}_{i=1}^M \) and \( \varphi \) are continuously differentiable real-valued functions, and \( \xi \in \mathbb{R}^n \) is the initial condition of the continuous-state variables.

Note that since \( \Sigma^M \) is a discrete set with exactly \( M \) elements, the optimal control problem in Definition 8 is in practice a mixed-integer programming problem, which cannot be efficiently solved using gradient-based numerical algorithms.

### A. Optimal Control using Relaxed Controls

In view of the result in Proposition 7 and the relation between \( \Sigma^M_p \) and \( \Sigma^M_r \), we can extend Definition 1 to consider relaxed executions.

**Definition 9:** Let \( \mathcal{H} \) be a hybrid system as in Definition 1, \( u \in L^2([0, T], \mathcal{U}) \), and \( x: [0, T] \to \mathcal{X} \).

Then \((x, u)\) is a relaxed hybrid execution of \( \mathcal{H} \) with length \( T \) if and only if there exists \( \omega \in L^2([0, T], \Sigma^M) \) such that (7) and (8) are satisfied.

We can now avoid formulating the mixed-integer program in (10) by modifying its last constraint.

**Definition 10:** Let \( \mathcal{H} \) be a hybrid system as in Definition 1. A relaxed hybrid optimal control problem is defined as a relaxation of the hybrid optimal control problem in Definition 8 where we replace the last constraint \( \omega(t) \in \Sigma^M \) with \( \omega(t) \in \Sigma^M_r \) in (10).

Recall that \( \Sigma^M_p \subseteq \Sigma^M_r \), hence the feasible set of the relaxed problem is larger than that of the pure, and should therefore result in a lower optimal value in theory. Yet as shown in [24], and explained below, these two optimization problems produce exactly the same optimal value.

Consider the following operators:

**Definition 11:** Let \( b_{0,0}: [0, T] \to \mathbb{R} \) be the Haar wavelet, defined by:

\[
b_{0,0}(t) = \begin{cases} 1 & \text{if } t \leq \frac{T}{2}, \\
0 & \text{if } t > \frac{T}{2}. \tag{11}
\end{cases}
\]

Also, for each \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, 2^k - 1\} \) let \( b_{k,j}(t) = b_{0,0}(T (2^k t - j)) \).

We say that \( F_N: L^2([0, T], \Sigma^M) \to L^2([0, T], \Sigma^M) \) is the Haar wavelet operator, defined by:

\[
[F_N(\omega)]_i(t) = \frac{1}{T} \langle \omega_i, 1 \rangle + \sum_{k=0}^{N} \sum_{j=0}^{2^k - 1} \langle \omega_i, b_{k,j} \rangle \frac{b_{k,j}(t)}{\|b_{k,j}\|_{L^2}},
\]

(12)

for each \( i \in \mathcal{I} \).

In other words, given \( \omega \in L^2([0, T], \Sigma^M) \) the Haar wavelet operator returns a new function \( F_N(\omega) \in L^2([0, T], \Sigma^M) \) that is piecewise constant over a uniform partition of \( 2^N \) samples.

**Definition 12:** We define the pulse-width modulation operator, denoted \( P_N: L^2([0, T], \Sigma^M_r) \to L^2([0, T], \Sigma^M_r) \), by:

\[
[P_N(\omega)]_i(t) = \begin{cases} 1 & \text{if } t \in [S_{k,i-1}, S_{k,i}), \\
0 & \text{otherwise}, \tag{13}
\end{cases}
\]

where for each \( k \in \{0, \ldots, 2^N - 1\} \) and \( i \in \{1, \ldots, M\} \):

\[
S_{k,i} = \frac{T}{2^N} \left( k + \sum_{j=1}^{i} \omega_j \left( \frac{k}{2^N} \right) \right). \tag{14}
\]

Hence, given \( \omega \in L^2([0, T], \Sigma^M) \) the pulse-width modulation operator returns a new function \( P_N(\omega) \in L^2([0, T], \Sigma^M) \) where each coordinate switches between 0 and 1 with pulses whose width is proportional to the amplitude of \( \omega \) evaluated at the samples \( \frac{k}{2^N} \) for \( k \in \{0, \ldots, 2^N - 1\} \).
Using these two operators we get the following result. The proof is omitted since it is an extension of Theorem 5.10 in [24] for the case when \( \omega(t) \in \text{conv}(\mathcal{S}_p(x(t))) \), the convex hull of \( \mathcal{S}_p(x(t)) \), for each \( t \in [0, T] \).

**Theorem 13:** Let \( \mathcal{H} \) be a hybrid system as in Definition 1 and let \( \omega \in L^2([0, T], \Sigma^M_p) \).

Furthermore, let \( (x, u) \) be the hybrid execution associated to \( \omega \), and for each \( N \in \mathbb{N} \), let \( (x_N, u_N) \) be the hybrid execution associated to \( p_N(\mathcal{F}_N(\omega)) \in L^2([0, T], \Sigma^M_p) \).

If \( \omega \in BV([0, T], \Sigma^M_p) \), then:

\[
||x(t) - x_N(t)|| \leq C 2^{-\frac{N}{2}} (V(\omega) + 1),
\]

where \( C > 0 \) does not depend on \( \omega \).

**Theorem 13** implies that, provided the optimal indexing function \( \omega \) of the relaxed optimal control problem in Definition 10 is of bounded variation, then we can create a sequence of points in the feasible set of the optimization problem in Definition 8 that approximates the optimal relaxed execution. This fact implies that, in most situations, both optimization problems produce exactly the same value, and therefore by solving the relaxed hybrid optimal control problem we are in fact solving the original hybrid optimal control problem as well. It is worth noting that both projection operators, \( \mathcal{F}_N \) and \( \mathcal{P}_N \), can be numerically implemented using efficient algorithms (an implementation is provided in [30]).

A discussion regarding whether the optimal indexing function \( \omega \) of the relaxed hybrid optimal control problem is of bounded variation is beyond the scope of this paper. For the time being it is worth mentioning that all functions with weak derivatives (as defined in [31]) belong to the set of bounded variation, and that all functions belonging to a finite-dimensional subspace of \( L^2 \), such as the functions obtained using numerical methods, also belong to the set of bounded variation.

At this point we have shown that the optimal relaxed solutions resulting from the relaxed hybrid optimal control problem approximate solutions to the original hybrid optimal control problem with arbitrary accuracy. In the next section we detail the numerical implementation of the results above.

**IV. NUMERICAL IMPLEMENTATION**

Our numerical implementation of the optimization problem in Definition 10 uses the Forward Euler discretization to approximate the differential equation constraints and the integral term in the cost function. Even though other algorithms such as Runge-Kutta or Pseudospectral methods are more accurate than the Forward Euler discretization [32], we chose to use the latter due to its simple implementation and good convergence properties when used in optimal control problems (as shown in Chapter 4 of [26]).

Applying the Forward Euler discretization to our formulation, we get the following nonlinear programming optimization problem, which approximates the problem in

| Table I: Discrete modes of the hybrid system in Example V-A |
| --- | --- | --- |
| Mode | Domain | Vector Field |
| 1 | \( x_2 \geq \sin(x_1) - \frac{1}{2} \) | \( \hat{x} = [1, -1]^T \) |
| 2 | \( x_2 \leq \sin(x_1) + \frac{1}{2} \) | \( \hat{x} = [1, 1]^T \) |

**Definition 10**

\[
\min_{\{x_k\}, \{u_k\}, \{\omega_k\} \in \mathcal{X}_N \times \mathcal{U}_N \times \Sigma^M} \sum_{k=0}^{N-1} \sum_{i=1}^M \frac{T}{N} \omega_{k,i} L_i(x_k, u_k) + \varphi(x_N),
\]

\[\text{s.t. } \bar{x}_0 = \xi, \quad \bar{x}_{k+1} = \bar{x}_k + \frac{T}{N} \sum_{i=1}^M \omega_{k,i} f_i(\bar{x}_k, \bar{u}_k), \forall k \leq N-1, \]

\[\bar{\omega}_i d_i(\bar{x}_k) \leq 0, \forall i \in \mathcal{I}, \forall k \leq N, \]

\[h(\bar{x}_k) \leq 0, \forall k \leq N, \]

\[u_k \in \mathcal{U}, \forall k \leq N - 1, \]

\[\bar{\omega}_k \in \Sigma^M, \forall k \leq N. \]

The nonlinear programming problem in (16) results in a relaxed hybrid execution \((x, u)\) associated to the relaxed indexing mode function \( \omega \). Then, given \( N \in \mathbb{N} \) we can find a hybrid execution \((x_N, u_N)\) associated to the indexing function \( p_N(\mathcal{F}_N(\omega)) \), where \( \mathcal{F}_N \) and \( \mathcal{P}_N \) are as in Definitions 11 and 12 respectively.

In the next section we present two examples where we apply this numerical implementation to solve three hybrid optimal control problems.

**V. EXAMPLES**

**A. Sliding-Mode Trajectory**

In this example we show how our formulation produces optimal sliding-mode trajectories. Consider the hybrid system with continuous-state space \( \mathcal{X} = \mathbb{R}^2 \), no control inputs (i.e., \( \mathcal{U} = \emptyset \)), and with two overlapping domains as defined in Table I. Note that this hybrid system does not have unique trajectories, since any of the modes can be chosen in the intersection between their domains.

Using the notation in (16), the cost function is defined by \( L_1(x, u) = L_2(x, u) = \varphi(x) = ||x||^2 \), the initial condition is \( x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), the optimization horizon is \( T = 4.5 \), and the discretization has \( N = 100 \) samples. The results were obtained using the SNOPT solver [33], and the calculations took 6.28 seconds in an Intel Xeon E5-2680 processor running at 2.7 GHz.

As shown in Figure 1 the optimal solution of this problem is a sliding-mode trajectory, where modes 1 and 2 chatter in the intersection between their domains. Thus, as shown in Figure 16 the relaxed indexing function \( \omega(t) \) contains non-binary values, i.e., in the relaxed trajectory both discrete modes are applied simultaneously. Note that the execution generates two different types of sliding modes. In the first type, the trajectory follows the boundary between the two modes, and the modal weights converge to a sinusoidal pattern in order to keep the trajectory along this surface.
In the second type of sliding mode, \( \omega_1(t) \approx \omega_2(t) \approx \frac{1}{7} \) and the trajectory moves horizontally through the interior of the intersection of the two modes. This ability to transition between different sliding modes, under the influence of the objective function, allows our formulation to pick a relaxed optimal trajectory through regions of the state-space where Zeno phenomena readily occur.

Figures 1c and 1b show the effect of using the operators \( \mathcal{F}_N \) and \( \mathcal{P}_N \) to generate new hybrid indexing functions that, although sub-optimal, follow closely the behavior of the optimal relaxed solution, even for small values of \( N \).

### B. Partitioned Linear System

In this example we again consider \( \mathcal{X} = \mathbb{R}^2 \), but now we partition the plane along the lines \( x_1 = x_2 \) and \( x_1 = -x_2 \), creating four distinct modes with no overlap. The domains and vector fields are summarized in Table II where non-adjacent modes are endowed with the same vector field.

The vector field matrices are \( A_1 = \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & -0.1 \end{bmatrix} \), and \( A_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & -0.3 \end{bmatrix} \). The control input set is defined by \( \mathcal{U} = \{ u \in \mathbb{R} | u \in [-1,0.1] \} \), and the cost function is defined by \( L(x,u) = \|x\|^2 + \|u\|^2 \) for all modes, and \( \varphi(x) = \|x\|^2 \).

Figure 2 shows the optimal result with initial condition \( x_0 = [-1,0] \), horizon \( T = 40 \), and \( N = 100 \) samples. Note that hybrid systems without reset maps and non-overlapping domains are commonly used in control applications due to their easy design and intuitive formulation [34]. Our result shows that the problem in [16] can be used to efficiently solve this large class of hybrid systems.

### C. Quadrotor Helicopter Obstacle Avoidance

We now consider a simplified quadrotor helicopter model with six continuous states, similar to the one used in Section 5.3 of [25]. The first state is the horizontal displacement of the helicopter, the second state is its vertical displacement, and the third state is its roll angle. The remaining three states are the respective derivatives of the first three states. We model the helicopter as a hybrid system with three discrete modes: one whose task is to move the helicopter vertically (denoted \( U \)), the second to rotate to the left (denoted \( L \)), and the third to rotate to the right (denoted \( R \)). Each of these discrete modes has a single input.

Our control objective is to make the helicopter lift off the ground, fly over an obstacle, and land safely at the origin. Even though we could incorporate the obstacle as part of the domain constraints in [16] (i.e., using a nonlinear constraint function \( h \)), that approach typically results in longer computation times. Instead, we use a penalty function to increase the cost of flying through a region around the obstacle. We also add a fourth safety mode (denoted \( S \)), in which the helicopter is permitted to accelerate upwards very quickly, at the expense of higher battery usage, in order to pass over the obstacle. The input is not penalized in the mode, in order to incentivize the use of this extra mode when the helicopter is at risk of collision.

The discrete modes in this example are detailed in Table III with the following parameters: gravity constant \( g = 9.8 \), helicopter mass \( m = 1.3 \), horizontal span \( \ell = 0.305 \), moment of inertia \( I = 0.0605 \), and motor constant \( \lambda = 1530 \).

Our cost function is defined by \( L_U(x,u) = L_R(x,u) = 5u^2 + \rho(x) \), \( L_S(x,u) = \rho(x) \), and \( \varphi(x) = 10^3(x_1^2 + x_2^2 + x_3^2) + 10^2 x_3^2 \), where \( \rho \) is the bump function.
defined by \( \rho(x) = \exp\left(\frac{-1}{-2(x_1-4)^2-x_2^2}\right) \), if \( 2(x_1-4)^2 + x_2^2 \leq 1 \), and \( \rho(x) = 0 \) otherwise. Furthermore, we constraint the input to \( u(t) \in [0, 1] \). We also add the safety constraint \( x_2(t) \geq 0 \) so that the helicopter stays above ground. The time horizon is \( T = 6 \), with \( N = 100 \) samples, and the initial condition is \( x_1(0) = 6 \) with all other states set equal to zero. The simulation, also solved using SNOPT in the same computer as the previous examples, took 201 seconds to complete.

As shown in Figures 3a and 3c, the safety mode fires appropriately to ensure that the helicopter avoids the obstacle, successfully reaching the origin at the end of the simulation. This example shows that using a hybrid formulation to incorporate auxiliary modes of operation is a suitable strategy to help a control engineer getting the desired behavior in view of unsafe conditions. In Figure 3b we show the results of using the operators \( F_N \) and \( P_N \) to obtain hybrid indexing functions from the relaxed optimal solution, where \( N = 12 \) produces a very good approximation of the optimal result.

VI. CONCLUSION

We have presented a novel mathematical formulation that allows us to numerically solve hybrid optimal control problems for the class of systems whose trajectories are continuous. Our formulation uses the theory of relaxed controls together with a feedback rule which enables certain discrete modes as a function of the continuous state at any given time. We have shown that the formulation yields an accurate approximation to the optimal trajectories of a hybrid system in an efficient, and easy to compute, transformation that results in a nonlinear programming problem.

Our future efforts will be focused on extending this formulation to hybrid systems with discontinuous jumps, and to provide a simple computational interface to implement these optimal control problems built as a module of the OptWrapper library [30].

REFERENCES

[1] A. D. Ames, “Human-Inspired Control of Bipedal Walking Robots,” IEEE Transactions on Automatic Control, vol. 59, no. 5, pp. 1115–1130, 2014.

[2] J. H. Gillula, G. M. Hoffmann, H. Huang, M. P. Vitus, and C. J. Tomlin, “Applications of Hybrid Reachability Analysis to Robotic Aerial Vehicles,” The International Journal of Robotics Research, vol. 30, no. 3, pp. 335–359, 2011.

[3] K. Gokbayrak and C. G. Cassandras, “Hybrid Controllers for Hierarchically Decomposed Systems,” in Proceedings of the 3rd International Workshop on Hybrid Systems: Computation and Control, 2000, pp. 117–129.

[4] B. Li, L. Nie, C. Wu, H. Gonzalez, and C. Lu, “Incorporating Emergency Alarms in Wireless Process Control,” in Proceedings of the ACM/IEEE 6th International Conference on Cyber-Physical Systems, 2015.

[5] K. H. Johansson, M. B. Egerstedt, J. Lygeros, and S. S. Sastry, “On the Regularization of Zeno Hybrid Automata,” Systems & Control Letters, vol. 38, no. 3, pp. 141–150, 1999.

[6] J. Warga, Optimal Control of Differential and Functional Equations. Academic Press, 1972.

[7] L. J. Williamson and E. Polak, “Relaxed Controls and the Convergence of Optimal Control Algorithms,” SIAM Journal on Control and Optimization, vol. 14, no. 4, pp. 737–756, 1976.

[8] J. Warga, “Steepest Descent with Relaxed Controls,” SIAM Journal on Control and Optimization, vol. 15, no. 4, pp. 674–682, 1977.

[9] L. J. Williamson and E. Polak, “Relaxed Controls and the Convergence of Optimal Control Algorithms,” IEEE Transactions on Automatic Control, vol. 52, no. 9, pp. 1587–1603, 2007.

[10] A. Lamperski and A. D. Ames, “On the Existence of Zeno Behavior in Hybrid Systems with Non–Isolated Zeno Equilibria,” in Proceedings of the 47th IEEE Conference on Decision and Control, 2008, pp. 2776–2781.

[11] X. Xu and P. J. Antsaklis, “Optimal Control of Switched Autonomous Systems,” in Proceedings of the 41st IEEE Conference on Decision and Control, vol. 4, no. December. IEEE, 2002, pp. 4401–4406.

[12] M. Boccadoro, Y. Y. Wardi, M. B. Egerstedt, and E. I. Verriest, “Optimal Control of Switching Surfaces in Hybrid Dynamical Systems,” Discrete Event Dynamic Systems: Theory and Applications, vol. 15, no. 4, pp. 433–448, 2005.

[13] H. Axelsson, Y. Y. Wardi, M. B. Egerstedt, and E. I. Verriest, “Gradient Descent Approach to Optimal Mode Scheduling in Hybrid Dynamical Systems,” Journal of Optimization Theory and Applications, vol. 136, no. 2, pp. 167–186, 2008.

[14] H. Gonzalez, R. Vasudevan, M. Kamgarpour, S. S. Sastry, R. Bajcsy, and C. J. Tomlin, “A Descent Algorithm for the Optimal Control of Constrained Nonlinear Switched Dynamical Systems,” in Proceedings of the 13th International Conference on Hybrid Systems: Computation and Control, 2010, pp. 51–60.

[15] M. Morari, M. Bajt, and F. Borrelli, “Hybrid Systems Modeling and Control,” European Journal of Control, vol. 9, no. 2-3, pp. 177–189, 2003.

[16] W. P. M. H. Heemels and B. Brogliato, “The Complementarity Class of Hybrid Dynamical Systems,” European Journal of Control, vol. 9, no. 2-3, pp. 322–360, 2003.

[17] R. Vasudevan, P. Gonzalez, R. Bajcsy, and S. S. Sastry, “Consistent Approximations for the Optimal Control of Constrained Switched
(a) Optimal relaxed trajectory. The red ellipse shows the outline of the obstacle, the green ellipse shows the domain of Mode $S$. Red dots mark the use of Mode $S$. A sketch of the helicopter is shown in black for a handful of time samples.

(b) Hybrid trajectories using the projection operator $\mathcal{P}_N (F_N (\omega))$, with $N = 6$ (lighter), $N = 9$, and $N = 12$ (darker). Note how the approximation quickly converges to the optimal relaxed trajectory.

(c) Optimal relaxed indexing function $\omega$.

Fig. 3. Results of Example V-C.

---

[25] ——, “Consistent Approximations for the Optimal Control of Constrained Switched Systems—Part 2: An Implementable Algorithm,” SIAM Journal on Control and Optimization, vol. 51, no. 6, pp. 4484–4503, 2013.

[26] E. Polak, Optimization: Algorithms and Consistent Approximations, ser. Applied Mathematical Sciences. Springer, 1997.

[27] J. Nocedal and S. J. Wright, Numerical Optimization, 2nd ed. Springer, 2006.

[28] G. Folland, Real Analysis: Modern Techniques and Their Applications, 2nd ed., ser. Pure and Applied Mathematics. John Wiley & Sons, 1999.

[29] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides. Kluwer Academic Publishers, 1988.

[30] “OptWrapper Python Library,” March 2016. [Online]. Available: https://github.com/hgonzalez/optwrapper

[31] W. P. Ziemer, Weakly Differentiable Functions, ser. Graduate Texts in Mathematics. Springer, 1989.

[32] Q. Gong, W. Kang, and I. M. Ross, “A Pseudospectral Method for the Optimal Control of Constrained Feedback Linearizable Systems,” IEEE Transactions on Automatic Control, vol. 51, no. 7, pp. 1115–1129, 2006.

[33] P. E. Gill, W. Murray, and M. A. Saunders, “SNOPT: An SQP Algorithm for Large-Scale Constrained Optimization,” SIAM Journal on Optimization, vol. 12, no. 4, pp. 979–1006, 2002.

[34] M. S. Branicky, “Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems,” IEEE Transactions on Automatic Control, vol. 43, no. 4, pp. 475–482, 1998.