The global attractors and their Hausdorff and fractal dimensions

estimation for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms*

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Abstract

In this paper, we study the long time behavior of solution to the initial boundary value problems for higher -order Kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_t + (-\Delta)^m u_t + \left( \int_{\Omega} |\nabla u|^p \right) \nabla (-\Delta)^m u + h(u) = f(x). \]

At first, we prove the existence and uniqueness of the solution by priori estimate and Galerkin method then we establish the existence of global attractors, at last, we consider that estimation of upper bounds of Hausdorff and fractal dimensions for the global attractors are obtain.

Keywords: Higher-order nonlinear Kirchhoff wave equation; The existence and uniqueness; The Global attractors; Hausdorff dimensions; Fractal dimensions

1 Introduction

In this paper we concerned with the long time behavior of solution to the initial boundary value problems for Higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_t + (-\Delta)^m u_t + \left( \int_{\Omega} |\nabla u|^p \right) \nabla (-\Delta)^m u + h(u) = f(x) \quad (1.1) \]

\[ u(x,t) = 0, \quad \frac{\partial^i u}{\partial v^i} = 0, \quad i = 1, 2, \ldots, m - 1, x \in \partial \Omega, t > 0; \quad (1.2) \]

\[ u(x,0) = u_0, \quad u_1(x) = 0, \quad x \in \partial \Omega \quad (1.3) \]

Where \( \Omega \subset \mathbb{R}^2 \) is bounded open domain with smooth boundary; \( v \) is the outer norm vector; \( m \geq 1 \) is a positive integer, and \( q > 0 \) is a positive constants, \( h(u) \) is a nonlinear forcing, \( (-\Delta)^m u \) is a strongly dissipation.

There have been many researches on the well-positive and the longtime dynamics for Kirchhoff equation. We can see [1-6], FUCAI Li [5] deals with the higher-order kirchhoff-type equation with nonlinear dissipation:

\[ u_t + \left( \int_{\Omega} |\nabla u|^p \right) \nabla (-\Delta)^m u + h(u) = f(x) \quad (1.4) \]
In a bounded domain, where \( m > 1 \) is a positive integer, \( p, q, r > 0 \) are positive constants and obtain that the solution exists globally if \( p \leq r \), while if \( p > \max\{r, 2q\} \), then for any initial data with negative initial energy, the solution blows up at finite time in \( L^{p+2} \) norm.

Yang Zhijian, Wang Yunqing [6] also studied the global attractor for the Kirchhoff type equation with a strong dissipation:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - M(\|\Delta u\|_{L^2})\Delta u - \Delta u_i + h(u_i) + g(u) &= f(x) \\
\text{in } \Omega \times \mathbb{R}^+, (1.7) \\
\left. u(x,t) \right|_{\partial \Omega} &= 0, \quad t > 0, \\
\left. u(x,0) \right|_{\partial \Omega} &= u_0(x), \quad u_i(x,0) = u_i(x), \quad x \in \Omega.
\end{align*}
\]

Where \( M(s) = 1 + s^2 \), \( 1 \leq m \leq \frac{4}{N-2} \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), with smooth boundary \( \partial \Omega \), \( h(x) \) and \( g(s) \) are nonlinear functions, and \( f(x) \) is an external force term. It proves that the relative continuous semigroup \( S(t) \) possesses in the phase space with low regularity a global attractor which is connected.

Yang Zhijian, Cheng Jianling [7] studies the asymptotic behavior of solutions to the Kirchhoff-type equation:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - M(\|\Delta u\|_{L^2})\Delta u - \Delta u_i + h(u_i) + g(u) &= f(x) \\
\text{in } \Omega \times \mathbb{R}^+, (1.10) \\
\left. u(x,t) \right|_{\partial \Omega} &= 0, \quad t > 0, \\
\left. u(x,0) \right|_{\partial \Omega} &= u_0(x), \quad u_i(x,0) = u_i(x), \quad x \in \Omega.
\end{align*}
\]

They prove that the related continuous semigroup \( S(t) \) possesses in phase \( X = (H^2(\Omega) \cap H^1_0) \times H^1_0(\Omega) \) a global attractor. At the end of the paper, an example is shown, which indicates the existence of nonlinear functions \( g(x,u) \) and \( h(u_i) \).

Recently, Meixia Wang, Cuicui Tian, Guoguang Lin [8] studied the global attractor and dimension for a 2D generalized Anisotropy Kuramoto-Sivashinsky equation:
where \( \Omega \subset \mathbb{R}^2 \) is bounded set; \( \partial \Omega \) is the bound of \( \Omega \); \( \varphi(u) \) and \( g(u) \) are considered as smooth function of \( u(x, y, t) \). Under the existence of the global solution, it proves that the global attractor and Hausdorff dimensions and fractal dimension.

The paper is arranged as follows. In section 2, we state some preliminaries under the assume of Lemma 1 and Lemma 2, we get the existence and uniqueness of solution; in section 3, we obtain the global attractors for the problems (1.1)-(1.3); in section 4, we consider that the global attractor of the the above mentioned problems (1.1)-(1.3) has finite Hausdorff dimensions and fractal dimensions.

### 2 Preliminaries

For convenience, we denote the norm and scalar product in \( L^2(\Omega) \) by \( \| \| \) and \( (,..) \); \( f = f(x) \),

\[
H^k = H^k(\Omega), \quad H^k_0 = H^k_0(\Omega), \quad H^{-k} = H^{-k}(\Omega), \quad \| \| = \| \|_{L^2}, \quad C_i(i = 0 \ldots 8), \quad \kappa \quad \text{are constants},
\]

\[
K_o \geq \max\{ q e - q, \frac{2 a e^2}{q + 1}, \frac{4 C_s e (2q - 2p + 2)}{2p + 2} \}.
\]

In this section, we present some materials needed in the proof of our results, state a global existence result, and prove our main result. For this reason, we assume that and notations needed in the proof of our results. For this reason, we assume that

\[
(G_i) \quad \text{there exist}
\]

\[
0 < \delta < \frac{1}{2}, \quad \| h(u) \|_{L^\infty} \leq C_0 (h(u_i, u)^{1-\delta}, \quad \forall u \in H^\infty_0,
\]

\[
(G_2) \quad \text{there exist constant}
\]

\[
0 < \sigma < 1, \quad \| h(v) \| \leq C_1 (R)(1 + \| \Delta v \|^{1-\sigma}, \quad \forall v \in H^{1 \infty} \cap H^\infty_0, \quad \| v \| \leq R,
\]

\[
(G_3) \quad \| h'(s) \| \leq C_2
\]

**Lemma 1.** Assume \((G_1)\) hold, and \((u_0, u_i) \in H^\infty(\Omega) \times L^\infty(\Omega), \quad f \in L^\infty(\Omega), \quad v = u_i + \varepsilon u_0, \) then the solution
\[(u, v) \in H^m(\Omega) \times L^2(\Omega), \text{ and}\]
\[
\|u, v\|_{H^m + L^2} = \|v^m u\| + \|v\| \leq \frac{W(0)e^{-\epsilon\delta}}{1 - \epsilon} + \frac{C_1(1 - e^{-\epsilon\delta}) - K_0}{1 - \epsilon} + \frac{q}{q + 1}(2.1)\]
Where \(v = u + \varepsilon u,\)
\[
0 < \epsilon < \min\{1, \sqrt[2]{1 + 4{\lambda_i}^{2} - 1}, \frac{q + 1}{2}, \frac{q + 1}{2\alpha}\} \quad \text{W(0) = \|v\|} + 
\]
\[
\frac{q}{q + 1}\|v^m u\| - \epsilon\|v\| + K_0, \quad v_0 = u + \varepsilon u_0, \text{ thus there exist } R_0 \text{ and } t_i \equiv t_i(\Omega) > 0.
\]

Such that
\[
\|u, v\|_{H^m + L^2} = \|v^m u\| + \|v\| \leq R_0(t > t_i)
\]

**Proof.** Let \(v = u + \varepsilon u\) we multiply \(v\) with both sides of equation (1.1) and obtain
\[
(u + (-\Delta)^m u, (\int_\Omega v^m u) (-\Delta)^m v + h(u), v) = (f(x), v), (2.2)
\]
\[
(u, v) = (v, v - \varepsilon u, v)
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_\Omega v^m u - \varepsilon(v - \varepsilon u, v) = \frac{1}{2} \frac{d}{dt} \|v^m u\|^2 - \varepsilon(v - \varepsilon u, v) (2.4)
\]
\[
\geq \frac{1}{2} \frac{d}{dt} \|v^m u\|^2 - \varepsilon(v - \varepsilon u, v) - \varepsilon^2 \|v^m u\|^2 - \varepsilon^2 \|u\|^2
\]
\[
((-\Delta)^m u, v) = ((-\Delta)^m (v - \varepsilon u), v) (2.5)
\]
\[
\geq \|v^m u\|^2 - \frac{1}{2} \frac{d}{dt} \|v^m u\|^2 - \varepsilon^2 \|v^m u\|^2
\]
\[
\left(\int_\Omega v^m u\right)^2 (-\Delta)^m u, v) = (\int_\Omega v^m u\right)^2 \left((-\Delta)^m u, v + \varepsilon u\right) (2.6)
\]
\[
= \frac{1}{2} \frac{d}{dt} \|v^m u\|^2 + \varepsilon \|v^m u\|^{q+2}
\]
\[
(h(u), v) = (h(u), u + \varepsilon u) (2.7)
\]
\( (f(x), v) \leq \frac{\varepsilon}{2} \|f\|^2 + \frac{1}{2\varepsilon^2} \|f\|^2 \) (2.9)

For above, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|^2 + \frac{1}{q+1} \|\nabla u\|^{q+2} - \varepsilon \|\nabla u\|^2 \right) + \frac{1}{q+1} \|\nabla u\|^{q+2} + \varepsilon \|\nabla u\|^2 - \varepsilon^2 \|u\|^2 \leq 0.
\]

(2.10)

By using Poincare inequality, we obtain:

\( \|\nabla u\| \geq \lambda_1 \|u\| \), then we have

\[
\frac{d}{dt} \left( \|u\|^2 + \frac{1}{q+1} \|\nabla u\|^{q+2} - \varepsilon \|\nabla u\|^2 + K_0 + (2\lambda_1 + 2\varepsilon - 2\varepsilon^2) \right) \leq 0.
\]

(2.11)

By using Young’s inequality, we obtain

\[
\|\nabla u\| \leq \frac{1}{q+1} \|\nabla u\|^{q+2} + \frac{q}{q+1} \varepsilon^2.
\]

(2.12)

\[
-\varepsilon \|\nabla u\| \geq -\frac{\varepsilon}{q+1} \|\nabla u\|^{q+2} - \frac{q}{q+1} \varepsilon^2.
\]

(2.13)

Then we have
\[ \| \| + \frac{1}{q+1} \| \nabla^m u \|^2_{q+2} - \frac{\varepsilon}{q+1} \| \nabla^m u \|^2_{q+2} + K_0 \geq 0. \] (2.14)

Taking \( \alpha = 2 + \frac{1}{\lambda^*} \) and \( p \leq q + 1 \) using Young's inequality, we obtain

\[ -\alpha \varepsilon \| \nabla^m u \|^2_{q+2} \geq -\frac{\alpha \varepsilon^2}{q+1} \| \nabla^m u \|^2_{q+2} - \frac{\alpha \varepsilon^2}{q+1}, \] (2.15)

\[ \| \nabla^m u \|^p \leq \frac{2p}{2q+2} \| \nabla^m u \|^2_{q+2} + \frac{2q - 2p + 2}{2p + 2}, \] (2.16)

\[ -2C_0 \varepsilon^2 \| \nabla^m u \|^p \leq \frac{4\varepsilon^2}{2q+2} \| \nabla^m u \|^2_{q+2} + \frac{2C_0 \varepsilon^2 (2q - 2p + 2)}{2p + 2}. \] (2.17)

So, we obtain

\[ \frac{\varepsilon}{2} \| \nabla^m u \|^2_{q+2} - \alpha \varepsilon \| \nabla^m u \|^2_{q+2} + K_0 \geq \frac{\varepsilon}{2} - \frac{\alpha \varepsilon^2}{q+1} \| \nabla^m u \|^2_{q+2} + \frac{K_0}{2} - \frac{\alpha \varepsilon^2}{q+1} \geq 0. \] (2.18)

\[ \frac{\varepsilon}{2} \| \nabla^m u \|^2_{q+2} - 2C_0 \varepsilon^2 \| \nabla^m u \|^2_{q+2} + K_0 \geq 0. \] (2.19)

So, we have

\[ \frac{d}{dt} \left( \| \nabla^m u \|^2_{q+2} - \varepsilon \| \nabla^m u \|^2_{q+2} + K_0 \right) + (2\lambda_1 + 2\varepsilon - 2\varepsilon^2) \| \nabla^m u \|^2_{q+2} \leq \frac{1}{\varepsilon^2} \| f \|^2 + 2K_0. \] (2.20)

Assume Next we take \( \alpha_0 = \{ 2\lambda^*, -2\varepsilon - 2\varepsilon^2, (q+1)\varepsilon, 1 \} \), so we can obtain

\[ \frac{d}{dt} \left( \| \nabla^m u \|^2_{q+2} - \varepsilon \| \nabla^m u \|^2_{q+2} + K_0 \right) + \alpha_0 \left( \| \nabla^m u \|^2_{q+2} - \varepsilon \| \nabla^m u \|^2_{q+2} + K_0 \right) \leq \frac{1}{\varepsilon^2} \| f \|^2 + 2K_0. \] (2.21)
Then we have
\[
\frac{d}{dt} W(t) + \alpha \gamma W(t) \leq C_3 ,
\] (2.22)
where \( W(t) = \left\| v \right\| + \frac{1}{q+1} \left\| v^{\alpha} u \right\|^{\alpha+2} - \varepsilon \left\| v^{\alpha} u \right\|^2 + K_0 \), \( C_3 = \frac{1}{\varepsilon^2} \left\| f \right\| + 2 K_0 \), by using Gronwall inequality, we obtain
\[
W(t) \leq W(0)e^{-\varepsilon t} + \frac{C_3(1-e^{-\alpha t})}{\alpha_0},
\] (2.23)
From (2.12), we know
\[
\left\| v^{\alpha} u \right\|^2 - \frac{q}{q+1} \leq \frac{1}{q+1} \left\| v^{\alpha} u \right\|^{\alpha+2}.
\] (2.24)
So
\[
\left\| v \right\|^2 + \frac{1}{q+1} \left\| v^{\alpha} u \right\|^{\alpha+2} - \varepsilon \left\| v^{\alpha} u \right\|^2 + K_0 \geq \left\| v \right\|^2 + (1-\varepsilon) \left\| v^{\alpha} u \right\|^2 + K_0 - \frac{q}{q+1} \geq 0 ,
\] (2.25)
From (2.23), we obtain
\[
\left\| v \right\|^2 + (1-\varepsilon) \left\| v^{\alpha} u \right\|^2 + K_0 - \frac{q}{q+1} \leq W(0)e^{-\varepsilon t} + \frac{C_3(1-e^{-\alpha t})}{\alpha_0},
\] (2.26)
Then we have
\[
(1-\varepsilon)(\left\| v \right\|^2 + \left\| v^{\alpha} u \right\|^2) \leq W(0)e^{-\varepsilon t} + \frac{C_3(1-e^{-\alpha t})}{\alpha_0} - K_0 + \frac{q}{q+1},
\] (2.27)
so, we obtain
\[
\left\| (u,v) \right\|^2_{L^\infty L^1} = \left\| v^{\alpha} u \right\|^2 + \left\| v \right\|^2 \leq \frac{W(0)e^{-\varepsilon t} - K_0 + \frac{q}{q+1}}{1-\varepsilon} \frac{C_3(1-e^{-\alpha t})}{\alpha_0} + \frac{q}{q+1} + \frac{C_3(1-e^{-\alpha t})}{\alpha_0} - K_0 + \frac{q}{q+1},
\] (2.28)
Then
\[
\lim_{t \to \infty} \left\| (u,v) \right\|^2_{L^\infty L^1} \leq \frac{C_3 - K_0 + \frac{q}{q+1}}{1-\varepsilon}.
\] (2.29)
So, there exist \( \alpha_0 \) and \( t_0 = t_0(\Omega) > 0 \), such that
Lemma 2. In addition to the assumptions \((G_2)\) hold, \(f \in H^m(\Omega)\), then the solution \((u,v)\) of the problems (1.1)-(1.3) satisfies \((u,v) \in H^m(\Omega) \times H^m(\Omega)\), and

\[
\|u,v\|_{\mathcal{C}([t,t_2], H^m(\Omega) \times H^m(\Omega))} = \|\nabla^m u\| + \|\nabla^m v\| \leq R_\delta (t > t_1)
\]

(2.30)

Proof. Let \((-\Delta)^m v = (-\Delta)^m u_i + \varepsilon (-\Delta)^m u\), we multiply \((-\Delta)^m v\) with both sides of equation (1.1), and obtain

\[
\begin{align*}
(u_i + (-\Delta)^m u_i, (-\Delta)^m v) + \int_0^t \nabla^m u \nabla^m (-\Delta)^m u + h(u_i, (-\Delta)^m v) &= (f(x), (-\Delta)^m v), \\
(u_i, (-\Delta)^m v) &\geq \frac{1}{2} \frac{d}{dt} \|\nabla^m v\|^2 - \varepsilon \|\nabla^m v\|^2 - \frac{\varepsilon^2}{2\lambda_i} \|(-\Delta)^m u\|^2 - \frac{\varepsilon^2}{2} \|\nabla^m v\|^2
\end{align*}
\]

(2.33)

(2.34)

Form \((G_2)\), we have

\[
\|h(u_i)\| \leq C_1 \|R(1 + \|\nabla^m u_i\|^2)^{1-\sigma}\),
\]

(2.38)

By using Young's inequality
\[
\| h(u, \cdot) \| \leq \frac{\sigma}{\mu^s} (C_1^s(R))^\frac{1}{s} + (1 - \sigma) \mu^{s - 1} (1 + \| (-\Delta)^n u \|) \left(1 + \| (-\Delta)^n u \|\right)^{\frac{1}{1 - \sigma}}, \tag{2.39}
\]
and
\[
\| h(u, \cdot) \| \leq C_s + \frac{1}{4} \| (-\Delta)^n v \| + \varepsilon^2 \left(\frac{1}{4} \| (-\Delta)^n u \|\right)^{2}, \tag{2.40}
\]

Where \( C_s := \frac{\sigma}{\mu^s} (C_1^s(R))^\frac{1}{s} + 2(1 - \sigma) \mu^{s - 1} \), we take proper \( \mu \), such that: \( 4(1 - \sigma) \mu^{s - 1} = \frac{1}{4} \),

\[ K_1 = C_s. \]

\[
\left\| f(x), (-\Delta)^n v \right\| \leq \left\| \nabla^n f \right\|^2 + \frac{\varepsilon^2 \left\| \nabla^n v \right\|^2}{2} \tag{2.41}
\]

Form above, we have
\[
\frac{d}{dt} (\| \nabla^n v \|^2 + \| \nabla^n u \|^2 - \varepsilon \| (-\Delta)^n u \|) + \left(\frac{3\lambda_1^n}{4} - 2\varepsilon - 2\varepsilon^2\right) \| \nabla^n v \|^2
\]
\[+ (\frac{\varepsilon^2}{4} - 9\varepsilon^2 - \frac{\varepsilon^2}{\lambda_1^n}) \| (-\Delta)^n u \|^2 \leq \frac{1}{\varepsilon^2} \| \nabla^n f \|^2 + 2K_1. \tag{2.42}
\]

Then we take proper \( \varepsilon \), let \( \frac{3\lambda_1^n}{4} - 2\varepsilon - 2\varepsilon^2 \geq 0 \) and \( \| \nabla^n u \|^2 - \varepsilon > 0 \), next we assume exist a positive constant \( K > 0 \), let \( K - 2\varepsilon \geq 0 \), satisfies
\[
0 \leq K (\| \nabla^n u \|^2 - \varepsilon) \leq \frac{d}{dt} (\| \nabla^n u \|^2 + 2\varepsilon \| \nabla^n u \|^2) - \frac{9\varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_1^n} \tag{2.43}
\]

where \( C_s := -\frac{9\varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_1^n} + K\varepsilon \) such that

\[
(K - 2\varepsilon) \| \nabla^n u \|^2 + \frac{d}{dt} \| \nabla^n u \|^2 \leq C_s, \tag{2.44}
\]

Multiplying (2.44) by \( e^{(K - 2\varepsilon)t} \) then
\[
\| \nabla^n u \|^2 + \frac{d}{dt} (e^{(K - 2\varepsilon)t} \| \nabla^n u \|^2) + e^{(K - 2\varepsilon)t} \frac{d}{dt} \| \nabla^n u \|^2 \leq C_s e^{(K - 2\varepsilon)t}, \tag{2.45}
\]

we integrate (2.45) with respect to time \( t \) and get that
\[ \varepsilon < \| \nabla^m u \|^{2\varepsilon} \leq \frac{C_{\varepsilon}}{K - 2\varepsilon} (1 + ke^{-K-2\varepsilon^2}), \quad (2.46) \]

So, (2.43) exists a positive constant \( K \).

From above, we have

\[ \frac{d}{dt} \left( \| \nabla^m v \|^{2\varepsilon} + \| \nabla^m u \|^{2\varepsilon} - \varepsilon \| (-\Delta)^m u \|^{2\varepsilon} \right) + \left( \frac{3\lambda_1^m}{4} - 2\varepsilon - 2\varepsilon^2 \right) \| \nabla^m v \|^{2\varepsilon} + K \left( \| \nabla^m u \|^{2\varepsilon} - \varepsilon \| (-\Delta)^m u \|^{2\varepsilon} \right) \leq \frac{1}{\varepsilon^2} \| \nabla^m f \|^{2\varepsilon} + 2K_1, \quad (2.47) \]

he Taking \( \beta_u = \min \left\{ \frac{3\lambda_1^m}{4} - 2\varepsilon - 2\varepsilon^2, K \right\}, \quad C_1 = \frac{1}{\varepsilon^2} \| \nabla^m f \|^{2\varepsilon} + 2K_1 \), then

\[ \frac{d}{dt} Y(t) + \beta_u Y(t) \leq C_1, \quad (2.48) \]

assumptions where \( Y(t) = \| \nabla^m v \|^{2\varepsilon} + \| \nabla^m u \|^{2\varepsilon} - \varepsilon \| (-\Delta)^m u \|^{2\varepsilon} \) by using Gronwall inequality, then

\[ Y(t) \leq Y(0)e^{-\beta_u t} + \frac{C_1}{\beta_u} (1 - e^{-\beta_u t}). \quad (2.49) \]

Let \( \beta_1 = \min \left\{ 1, \inf_{t>0} \| \nabla^m u \|^{2\varepsilon} - \varepsilon \right\} \), we get \( \beta_1 (\| \nabla^m v \|^{2\varepsilon} + \| (-\Delta)^m u \|^{2\varepsilon}) \leq Y(t) \).

so we get

\[ \| (u, v) \|^2_{H^{\varepsilon, H^m}} = \| (-\Delta)^m u \|^2 + \| \nabla^m v \|^2 \leq \frac{Y(0)e^{-\beta_1 t}}{\beta_1} + \frac{C_1 (1 - e^{-\beta_1 t})}{\beta_u \beta_1}. \quad (2.50) \]

where \( Y(0) = \left( \| (-\nabla)^m u_0 \|^{2\varepsilon} - \varepsilon \right) \| (-\Delta)^m u_0 \|^{2\varepsilon} + \| \nabla^m v_0 \|^{2\varepsilon}, \quad u_0 = u_1 + \varepsilon u_0 \), then

\[ \lim_{t \to \infty} \| (u, v) \|^2_{H^{\varepsilon, H^m}} \leq \frac{C_1}{\beta_u \beta_1}. \quad (2.51) \]

So, there exist \( R_1 \) and \( t_1 = t_1(\Omega) > 0 \), such that

\[ \| (u, v) \|^2_{H^{\varepsilon, H^m}} = \| (-\Delta)^m u \|^2 + \| \nabla^m v \|^2 \leq R_1(t > t_1). \quad (2.52) \]

3. Global attractor

3.1 the existence and uniqueness of solution

**Theorem 3.1** Assume \((G_1), (G_2), (G_3)\), holds and Lemma1 Lemma2 holds; the problem (1.1)-(1.3) exists a unique
smooth solution

\[(u, v) \in L^2(0, +\infty); H^{2m} \times H^{2m}\]  
(3.1)

Proof. By the Galerkin method, Lemma 1 and Lemma 2, we can easily obtain the existence of solution. Next, we prove the uniqueness of Solutions in detail

Assume \(u, v\) are two solutions of the problems (1.1)-(1.3). Let \(w = u - v\), then

\[w(x, 0) = w_0(x) = 0, \quad w_t(x, 0) = w_t(x) = 0\]

Then two equations subtract and obtain

\[w_n + (-\Delta)^m w_t + \|\nabla^n u\|^{2\theta} (-\Delta)^m u - \|\nabla^n v\|^{2\theta} (-\Delta)^m v + h(u) - h(v) = 0 \]  
(3.2)

By multiplying \(3.2\) by \(w_t\), we get

\[(w_n + (-\Delta)^m w_t + \|\nabla^n u\|^{2\theta} (-\Delta)^m u - \|\nabla^n v\|^{2\theta} (-\Delta)^m v + h(u) - h(v), w_t) = 0\]  
(3.3)

\[\langle \|\nabla^n u\|^{2\theta} (-\Delta)^m u - \|\nabla^n v\|^{2\theta} (-\Delta)^m v, w_t \rangle\]

\[= \langle \|\nabla^n u\|^{2\theta} (-\Delta)^m u - \|\nabla^n v\|^{2\theta} (-\Delta)^m v + \|\nabla^n v\|^{2\theta} (-\Delta)^m v - \|\nabla^n v\|^{2\theta} (-\Delta)^m v, w_t \rangle\]

\[= \langle \|\nabla^n u\|^{2\theta} (-\Delta)^m w, w_t \rangle + \langle \|\nabla^n u\|^{2\theta} (-\Delta)^m v - \|\nabla^n v\|^{2\theta} (-\Delta)^m v, w_t \rangle\]  
(3.4)

\[\frac{1}{2} \frac{d}{dt} \langle \|\nabla^n u\|^{2\theta} - \|\nabla^n v\|^{2\theta} \rangle - \theta \langle \|\nabla^n u\|^{2\theta} - \|\nabla^n v\|^{2\theta} \rangle \langle \nabla^n u, \nabla^n w \rangle\]

\[+ \langle \|\nabla^n u\|^{2\theta} (-\Delta)^m v - \|\nabla^n v\|^{2\theta} (-\Delta)^m v, w_t \rangle\]

\[\leq 2\theta \langle \|\nabla^n u\|^{2\theta} - \|\nabla^n v\|^{2\theta} \rangle \langle \nabla^n w \rangle \langle (-\Delta)^m v, w_t \rangle\]  
(3.5)

According to Lemma 1, Lemma 2, we have

\[2\theta \langle \|\nabla^n u\|^{2\theta} - \|\nabla^n v\|^{2\theta} \rangle \langle \nabla^n w \rangle \langle (-\Delta)^m v, w_t \rangle \leq C_{\gamma}\]  
(3.6)
\[ q \| \nabla^n u \|^{q-1} \leq C_q. \]

Then
\[ \left( \| \nabla^n u \|^{q} - \| \nabla^n v \|^{q} \right) (-\Delta)^{n} v, w \rangle \leq C_j \| \nabla^n w \| \| w \|. \quad (3.9) \]

According to Young’s inequality, we get
\[ \left| \left( \| \nabla^n u \|^{q} - \| \nabla^n v \|^{q} \right) (-\Delta)^{n} v, w \rangle \right| \leq C_j \| \nabla^n w \|^{q} + C_j \| w \|. \quad (3.10) \]

Form \( G_j \), we have
\[ \| h(u_j) - h(v_j), w \| \rangle \| h'(s) w, w \| \rangle \leq C_j \| w \|. \quad (3.11) \]

Form above, we have
\[ \frac{d}{dt} (\| u \|^2 + \| \nabla^n u \|^q \| \nabla^n w \|^q) + 2 \| \nabla^n w \| \leq 0. \quad (3.12) \]

Then
\[ \frac{d}{dt} (\| u \|^2 + \| \nabla^n u \|^q \| \nabla^n w \|^q) \leq (C_j + 2C\| \nabla^n u \| \| \nabla^n w \| \leq 0. \quad (3.13) \]

According to \( \| \nabla^n u \|^{q} \| \nabla^n w \| \geq \varepsilon \| \nabla^n w \| \) then
\[ \frac{d}{dt} (\| u \|^2 + \| \nabla^n u \|^q \| \nabla^n w \|^q) \leq \left( \frac{C_j + 2C\| \nabla^n u \| \| \nabla^n w \| \leq \gamma (\| \nabla^n u \|^{q} \| \nabla^n w \| + \| w, \|) \right) \quad (3.14) \]

Taking \( \gamma = \max\{ \frac{C_j + 2C\| \nabla^n u \| \| \nabla^n w \| \leq \varepsilon \} \), we have
\[ \frac{d}{dt} (\| w \|^2 + \| \nabla^n u \|^q \| \nabla^n w \|^q) \leq \gamma (\| \nabla^n u \|^q \| \nabla^n w \| + \| w, \|) \quad (3.15) \]

By using Gronwall inequality, we obtain
\[ \| u \|^2 + \| \nabla^n u \|^q \| \nabla^n w \|^q \leq \gamma (\| \nabla^n u \|^q \| \nabla^n w(0) \|^q + \| w(0) \|) e^\gamma. \quad (3.16) \]

Therefore
\[ u = v \]

So we get the uniqueness of the solution.

### 3.2 Global attractor
Theorem 3.1. [11] Let $E_i$ be a Banach space, and $(S(t)) (t \geq 0)$ are the semigroup operator on $E_i$. 

$S(t) : E_i \rightarrow E_i$, $S(t+s) = S(t)S(s)$ $(\forall t, s \geq 0)$, $S(0) = I$, where $I$ is a unit operator. set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is uniformly bounded, namely $\forall R > 0, \Box u \sqsubseteq R$, it exists a constant $C(R)$, so that $\Box S(t)u \sqsubseteq C(R) (t \in [0, +\infty))$.

2) It exists a bounded absorbing set $B_0 \subset E_i$, namely, $\forall B \subset E_i$, it exists a constant $t_0$, so that $S(t)B \subset B_0 (t \geq t_0)$.

Where $B_0$ and $B$ are bounded sets.

3) When $t > 0$, $S(t)$ is a completely continuous operator $A$.

Therefore, the semigroup operator $S(t)$ exists a compact global attractor.

Theorem 3.2. [12] Under the assume of Lemma 1, Lemma 2, Theorem 3.1, equations have global attractor $\omega(B_0) = \bigcap_{t \geq 0} \bigcup_{s \geq 0} S(t)B_0$.

Where $B_0 = \{(u, v) \in H^{2m}_0(\Omega) \times H^m_0(\Omega) | \Box u \Box^2 \mu^{2m} + \Box v \Box^2 \mu^m \leq R_0 + R_1\}$, $B_0$ is the bounded absorbing set of $H^{2m}_0(\Omega) \times H^m_0(\Omega)$ and satisfies

1) $S(t)A = A$, $t > 0$;

2) $\lim_{t \to \infty} \text{dist}(S(t)B, A) = 0$ , here $B \subset H^{2m}_0(\Omega) \times H^m_0(\Omega)$ and it is a bounded set,

$\lim_{t \to \infty} \text{dist}(S(t)B, A) = \sup \left( \inf_{x \in B} \sup_{y \in A} \Box (S(t)x - y) \Box^2 \mu^{2m} \right) \to 0, t \to \infty$.

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup $S(t)$, $S(t) : H^{2m}_0(\Omega) \times H^m_0(\Omega) \rightarrow H^{2m}_0(\Omega) \times H^m_0(\Omega)$, here $E_i = H^{2m}_0(\Omega) \times H^m_0(\Omega)$.

(1) from Lemma 2.1 to Lemma 2.2, we can get that $\forall B \subset H^{2m}_0(\Omega) \times H^m_0(\Omega)$ is a bounded set that includes in the ball $\Box (u, v) \Box^2 \mu^{2m} \leq R$.

$\Box S(t)(u_0, v_0) \Box^2 \mu^{2m} \leq u_0 \Box^2 \mu^{2m} + v \Box^2 \mu^m \leq u_0 \Box^2 \mu^{2m} + v \Box^2 \mu^m + C \leq R^2 + C$ ($t \geq 0, (u_0, v_0) \in B$)

This shows that $S(t) (t \geq 0)$ is uniformly bounded $H^{2m}_0(\Omega) \times H^m_0(\Omega)$.

(2) Furthermore, for any $(u_0, v_0) \in H^{2m}_0(\Omega) \times H^m_0(\Omega)$, when $t \geq \max \{t_1, t_2\}$, we have,
\[ \Box S(t)(\nu, t) \leq \frac{2}{\mu} \mathcal{M}_M^m \nu + \Box \nu \leq R_n + R \]

So we get \( B_n \) is the bounded absorbing set.

(3) Since \( E_1 = H^m(\Omega) \times H^m(\Omega) \rightarrow E_0 = H^m(\Omega) \times L^2(\Omega) \) is compact embedded, which means that the bounded set in \( E_1 \) is the compact set in \( E_0 \), so the semigroup operator \( S(t) \) exists a compact global attractor \( \mathcal{A} \).

4 The estimates of the upper bounds of Hausdorff and fractal dimensions for the global attractor

We rewrite the problems (1.1)-(1.3):

\[
\begin{align*}
\dot{u} + A^m u + \left| A^{-1} u \right|^2 A^m u + h(u) &= f(x), \\
U_0 &= \xi, U_0(0) = \zeta.
\end{align*}
\]

Let \( A u = -\Delta u \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). The linearized equations of the above equation as follows

\[
U_t + AU = FU, \quad (4.2)
\]

Let \( U_0 \in H^m_0(\Omega), U(t) \) is the solution of problems (4.20)-(4.21). We can prove that the problems (4.20)-(4.21) have a unique solution \( U \in L^\infty(0, T, H^m_0(\Omega)) \), \( U_t \in L^\infty(0, T, L^2(\Omega)) \). The equation (4.20) is the linearized equation by the equation (4.17). Define the mapping \( Ls(t) = Ls(t) \xi = U(t) \), here \( u(t) = s(t)u_0 \), let \( \varphi = (u_0, u_1) \), \( \varphi_0 = \varphi_0 + [\xi, \zeta] = (u_0 + \xi, u_1 + \zeta) \) let :

\[
\|\varphi\|_{C^1} \leq R_1, \quad \|\varphi\|_0 \leq R_2, \quad E_0 = \nu \times \mathbb{R}_+, \quad E_0 = \nu \times \mathbb{R}_+ \mathcal{N} := H^m_0(\Omega) \mathcal{N} := L^2(\Omega), \quad S(t)\varphi = \varphi(t) = \{u(t), u_1(t)\},
\]

Lemma 4.1. Assume \( H \) is a Hilbert space, \( E_0 \) is a compact set of \( H \), \( S(t) : E_0 \rightarrow H \) is a continuous mapping, satisfy the following conditions.

1) \( S(t)E_0 = E_0, t > 0 \);

2) if \( S(t) \) is Frechet differentiable, it exists a bounded linear differential operator \( L(t, \varphi) \in C^\infty(\mathbb{R}^+; L(E_0, E_0)) \), \( \forall t > 0 \). that is
The proof of lemma 4.1 see ref.[9]. is omitted here. According to Lemma 4.1, we can get the following theorem:

**Theorem 4.1** Let $A$ is the global attractor that we obtain in section 3. In that case, $A$ has finite Hausdorff dimensions and fractal dimensions in $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$, that is $d_H(A) \leq n$, $d_{\phi}(A) \leq \frac{6n}{5}$.

**Proof.** Firstly, we rewrite the equations (4.17), (4.18) into the first order abstract evolution equations in $E_o$.

Let $\Psi = R_{\tau} \varphi = \{u, u_i + \varepsilon u\}$, let $R_{\tau} : \{u, u_i\} \rightarrow \{u, u_i + \varepsilon u\}$, is an isomorphic mapping. So let is the global attractor of \{S(t)\}, then $R_{\tau} A$ is also the global attractor of \{S_{\tau}(t)\}, then $\Psi$ satisfies as follows:

$$\Psi_t + A_{\tau} \Psi + \tilde{h}(\Psi) = f, \quad \Psi(0) = \{u_o, u_i + \varepsilon u_o\}^T \quad (4.4)$$

$$\Psi(0) = \{u_o, u_i + \varepsilon u_o\}^T \quad (4.5)$$

Where $\Psi = \{u, u_i + \varepsilon u\}^T$, $\tilde{h}(\Psi) = \{0, h(u_i)\}^T$, $f = \{0, f(x)\}^T$.

$$A_{\tau} = \begin{pmatrix} 2^n & -I \\ A^\tau & \varepsilon I \\ \varepsilon I & -I \end{pmatrix} \right) \quad (4.6)$$

$$\Psi_t := F(\Psi) = f - A_{\tau} \Psi - \tilde{h}(\Psi) \quad (4.7)$$

$$P_t = F_t(\Psi) \quad (4.8)$$

$$P_t + A_{\tau} P + \tilde{h}(\Psi) = 0 \quad (4.9)$$

Where $P = \{U, U_i + \varepsilon U\}^T$, $\tilde{h}(\Psi) P = \{0, h_i(U)\}^T$. The initial condition (4.3) can be written in the following form:

$$P(0) = w, w = \{\xi, \zeta\} \in E_o \quad (4.10)$$

We take $n \in N$, then consider the corresponding $n$ solution: $(P = P_1, P_2 \ldots P_n, P_i \in E_o)$ of the initial values:

$$(w = w_1, w_2 \ldots w_n, w_i \in E_o) \quad \text{in the equations (4.8)-(4.10)}$$

So there is

$$\begin{align*}
P_1(t) A P_2(t) A \cdots A P_n(t) & = \left| \begin{array}{c} w_1 \varepsilon w_2 \cdots \varepsilon w_n \end{array} \right|_{k_{x_0}^n} \varepsilon i^{r} \int_{t_0}^{t} TrF_i(S_{\tau}(\tau) \Psi_o)Q_i(\tau) d \tau 
\end{align*}$$
Form $\psi(\tau) = S_x(\tau)\Psi_0$, we get $S_x(\tau) : \{u_0, v_i = u_i + \varepsilon u_0\} \rightarrow \{u(\tau), v(\tau) = u_i(\tau) + \varepsilon u(\tau)\}$,

$\psi(\tau) = \{u(\tau), v_i(\tau) = u_i(\tau) + \varepsilon u(\tau)\}$, here $u$ is the solution of problems (4.1); $\Lambda$ represents the outer product,

$Tr$ represents the trace, $Q_x(\tau) = Q_x(\tau, \Psi_0; w_1, w_2 \cdots w_n)$ is an orthogonal projection from the space $E_u = \nu \times H$ to the subspace spanned by $\{P_j(\tau), P_2(\tau), \cdots P_n(\tau)\}$. For a given time $\tau$, let $\phi_j(\tau) = \{\xi_j(\tau), \zeta_j(\tau)\}$, $j = 1, 2, \cdots n$.

$\{\phi_j(\tau)\}_{j=1,2,\cdots n}$ is the standard orthogonal basis of the space $Q_x(\tau)_{E_u} = span\{P_1(\tau), P_2(\tau), \cdots P_2(\tau)\}$.

From the above, we have

$$Tr F(\psi(\tau)) \cdot Q_x(\tau) = \sum_{j=1}^{n} F_i(\psi(\tau)) \cdot Q_x(\tau) \phi_j(\tau) \phi_j(\tau)_{E_u}$$

$$= \sum_{j=1}^{n} F_i(\psi(\tau)) \phi_j(\tau) \phi_j(\tau)_{E_u}$$

(4.11)

Where $(\cdot, \cdot)_{E_u}$ is the inner product in $E_u$. Then

$$((\xi, \xi) \phi_j(\tau), \phi_j(\tau))_{E_u} = (\xi, \xi) + (\zeta, \zeta);$$

$$(F_i(\psi) \phi_j, \phi_j)_{E_u} = -(\Lambda \phi_j, \phi_j)_{E_u} - (h_i(u) \xi_j, \xi_j);$$

$$(\Lambda \phi_j, \phi_j)_{E_u} = \varepsilon \|\xi_j\|^2 + \left(\frac{n}{A^2 - \varepsilon I} - \frac{n}{A^2} - \varepsilon A - \varepsilon I\right)\zeta_j, \zeta_j)$$

$$- (\zeta_j, \zeta_j) + (A^2 - \varepsilon I)\zeta_j, \zeta_j$$

(4.12)

$$\geq a \left(\xi_j, \xi_j\right) + \left(\zeta_j, \zeta_j\right).$$

Where $a := \min\left\{\frac{2 \varepsilon + \varepsilon^2 - \left(\frac{n}{A^2 - \varepsilon I}\right)^2}{2}, \frac{2 \varepsilon + \varepsilon^2 - \left(\frac{n}{A^2 - \varepsilon I}\right)^2}{2}, \frac{2 \varepsilon + \varepsilon^2 - \left(\frac{n}{A^2} - \varepsilon I\right)^2}{2}\right\}.$

Now, suppose that $\{u_0, u_j\} \in A$, according to theorem 3.3, $A$ is a bounded absorbing set in $E$.

$\Psi(\tau) = \{u(t), u_j(t) + \varepsilon u(t)\} \in D(A)$; $D(A) = \{u \in \nu, Au \in H\}$.

Then there is a $s \in [0, 1]$ to make the mapping $h_i : D(A) \rightarrow \rho(v, H)$ At the same time, there are the following results:
\[
R_A = \sup_{\xi, \zeta \in A} |A\xi| < \infty
\]
\[
\sup_{a \in D(A)} \|h_a(u_j)\|_{\mu(\tau, \mathbb{N})} \leq r < \infty \quad (4.13)
\]

Where \( \|h_a(u_j)\xi, \zeta\| \) meets: \( \|h_a(u_j)\xi, \zeta\| \leq r\|\xi\|\|\zeta\| \). Comprehensive above can be obtained

\[
(F_i'\Psi)\phi_j(\tau) \leq -a\left(\|\xi\|^2 + \|\zeta\|^2\right) + r\|\zeta\|^2 \quad (4.14)
\]

Where \( \|\xi\|^2 + \|\zeta\|^2 = \|\xi\|_{E_n} = 1 \), due to \( \{\phi_j(\tau)\}_{j=1,2,...,n} \) is a standard orthogonal basis in \( Q_\tau(\tau)_{E_n} \)

So

\[
\sum_{j=1}^n (F_i'\Psi)\phi_j(\tau)\phi_j(\tau)_{E_n} \leq -\frac{na}{2} + \frac{r^2}{2a}\|\zeta\|_{E_n} \quad (4.15)
\]

Almost to all \( \tau \), making

\[
\sum_{j=1}^n \|\zeta\|^2 \leq \sum_{j=1}^{n-1} \lambda_j^{-1} \quad (4.16)
\]

So

\[
TrF_i'\Psi(\tau) \cdot Q_\tau(\tau) \leq -\frac{na}{2} + \frac{r^2}{2a}\sum_{j=1}^{n-1} \lambda_j^{-1} \quad (4.17)
\]

Let us assume that \( \{u_0, u_j\} \in A \) is equivalent to \( \Psi_0 = \{u_0, u_1 + \varepsilon u_0\} \in R_A \)

Then

\[
q_n(t) = \sup_{\Psi_0 \in R_A, u \in E_n} \sup_{I} \int_0^t TrF_i'\Psi_0(\tau) \cdot Q_\tau(\tau) d \tau, j = 1,2,...,n.
\]

\[
q_n = \lim_{n \to \infty} \sup_{I} q_n(t) \quad (4.18)
\]

According to (4.17), (4.18), so

\[
q_n(t) \leq -\frac{na}{2} + \frac{r}{2a}\sum_{j=1}^{n-1} \lambda_j^{-1}
\]
Therefore, the Lyapunov exponent of \( A \) (or \( R_z A \)) is uniformly bounded

\[
\mu_1 + \mu_2 + \cdots + \mu_n \leq -\frac{na}{2} + \frac{r^2}{2a} \sum_{j=1}^{n} \lambda_j^{-1}
\]

\[(4.20)\]

From what has been discussed above, it exists \( n > 1 \) and \( s \in [0,1] \), \( a, r \) are constants, then

\[
\frac{1}{n} \sum_{j=1}^{n} \lambda_j^{-1} \leq \frac{a^2}{6r^2}
\]

\[(4.21)\]

\[
q_n \leq -\frac{na}{2} (1 - \frac{r^2}{a} \sum_{j=1}^{n} \lambda_j^{-1}) \leq -\frac{5na}{12}
\]

\[(4.22)\]

\[
(\frac{q}{j})_{+} \leq -\frac{r^2}{2a} \sum_{j=1}^{n} \lambda_j^{-1} \leq \frac{na}{12}, j = 1, 2 \ldots n.
\]

\[(4.23)\]

So, we immediately to the Hausdorff dimension and fractal dimension are respectively

\[d_n(A) < n, \quad d_f(A) < \frac{5}{3} n.\]

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