Non-equatorial circular geodesics for the Painlevé–Gullstrand form of Lense–Thirring spacetime

Joshua Baines\textsuperscript{1}, Thomas Berry\textsuperscript{2}, Alex Simpson\textsuperscript{1}, and Matt Visser\textsuperscript{1}

1 School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand.
2 Robinson Institute, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand.
E-mail: joshua.baines@sms.vuw.ac.nz, thomas.berry@vuw.ac.nz, alex.simpson@sms.vuw.ac.nz, matt.visser@sms.vuw.ac.nz

Abstract:

Herein we explore the non-equatorial circular geodesics (both timelike and null) in the Painlevé–Gullstrand variant of the Lense–Thirring spacetime recently introduced by the current authors. Even though the spacetime is not spherically symmetric, shells of circular geodesics still exist. While the radial motion is (by construction) utterly trivial, determining the allowed locations of these circular geodesics is decidedly non-trivial, and the stability analysis is equally tricky. Regarding the angular motion, these circular orbits will be seen to exhibit both precession and nutation — typically with incommensurate frequencies. Thus circular geodesic motion, though integrable in the technical sense, is generically surface-filling, with the orbits completely covering an equatorial band which is a segment of a sphere, and whose extent is governed by an interplay between the orbital angular momentum and the Carter constant. The situation is qualitatively similar to that for the (exact) Kerr spacetime — but we now see that any physical model having the same slow-rotation weak-field limit as general relativity will still possess non-equatorial circular geodesics.

DATE: Friday 18 February 2022; Thursday 3 March 2022; \LaTeX-ed March 4, 2022

KEYWORDS: Painlevé–Gullstrand metrics; Lense–Thirring metric; Killing tensor; Carter constant; integrability; geodesics; circular orbits.

PHYSH: Gravitation
1 Introduction

The Kerr spacetime [1–6], perhaps the pre-eminent exact solution of the Einstein equations of vacuum general relativity, is both a standard textbook exemplar [7–14], and is increasingly of central importance to both observational and theoretical astrophysics [15–21]. One key issue of particular importance is a full understanding of the geodesics — and the fact that despite the lack of spherical symmetry (the Kerr spacetime is merely stationary and axisymmetric, so that the Birkhoff theorem does not apply [22–26]), there are still a multitude of circular geodesics which are not confined to the equatorial plane. (Contrast, for example, [27] with [28–32].)
It should be noted that the non-equatorial circular null geodesics are particularly important tools for studying photon rings and black hole silhouettes [33–36]. Similarly non-equatorial circular timelike geodesics are particularly important tools for studying off-axis accretion disks and their related ISCOs and OSCOs [37–41].

In the current article we shall be interested is seeing how much of this qualitative structure survives once one moves away from the exact Kerr spacetime, specifically once one considers the Painlevé–Gullstrand version of the weak-field slow-rotation Lense–Thirring spacetime. The weak-field slow-rotation Lense–Thirring spacetime was originally introduced in 1918 [42, 43], while the current authors have recently introduced, and extensively explored, a novel Painlevé–Gullstrand variant [44–49] of the Lense–Thirring spacetime [50–52].

2 Basic framework

The line-element of interest is [50–52]:

$$\text{d}s^2 = -\text{d}t^2 + \left\{ \text{d}r + \sqrt{\frac{2m}{r}} \text{d}t \right\}^2 + r^2 \left\{ \text{d}\theta^2 + \sin^2 \theta \left( \text{d}\phi - \frac{2J}{r^3} \text{d}t \right) \right\}^2. \tag{2.1}$$

This line-element is somewhat related to the “river” model for black holes [53] — it exhibits both unit lapse [54], and flat spatial 3-slices [50–52] — the presence of flat spatial 3-slices being incompatible with the exact Kerr spacetime [55–58]. Furthermore this line element possesses a non-trivial Killing tensor [51, 52]:

$$K_{ab} \text{d}x^a \text{d}x^b = r^4 \left\{ \text{d}\theta^2 + \sin^2 \theta \left( \text{d}\phi - \frac{2J}{r^3} \text{d}t \right) \right\}^2. \tag{2.2}$$

This Killing tensor was found by applying the algorithm presented in [59–61]. Once found, one can easily verify that $\nabla (cK_{ab}) = K_{(abcc)} = 0$. For any affine parameter $\lambda$, the (generalized) Carter constant is then [51, 52]:

$$C = K_{ab} \frac{\text{d}x^a}{\text{d}\lambda} \frac{\text{d}x^b}{\text{d}\lambda} = r^4 \left[ \left( \frac{\text{d}\theta}{\text{d}\lambda} \right)^2 + \sin^2 \theta \left( \frac{\text{d}\phi}{\text{d}\lambda} - \frac{2J}{r^3} \frac{\text{d}t}{\text{d}\lambda} \right) \right]^2. \tag{2.3}$$

By construction $C \geq 0$. Without loss of generality we choose $\lambda$ to be future-directed, $\text{d}t/\text{d}\lambda > 0$. 
In addition to the Carter constant, we have three other conserved quantities. Two (the energy and azimuthal component of angular momentum) come from the time-translation and axial Killing vectors \[51, 52\]:

\[
E = \left(1 - \frac{2m}{r} - \frac{4J^2 \sin^2 \theta}{r^4}\right) \frac{dt}{d\lambda} - \sqrt{\frac{2m}{r^3}} \frac{d\theta}{d\lambda} + \frac{2J \sin^2 \theta}{r} \frac{d\phi}{d\lambda};
\]

\[
L = r^2 \sin^2 \theta \frac{d\phi}{d\lambda} - \frac{2J \sin^2 \theta}{r} \frac{dt}{d\lambda}.
\]

The final conserved quantity, the “mass-shell constraint”, \(\epsilon \in \{0, -1\}\) for null and timelike geodesics respectively, comes from the trivial Killing tensor (the metric):

\[
\epsilon = g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = - \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{dr}{d\lambda} + \sqrt{\frac{2m}{r}} \frac{dt}{d\lambda} \right)^2 + \frac{r^2}{\sin^2 \theta} \left[ \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\lambda} - \frac{2J}{r^3} \frac{dt}{d\lambda} \right)^2 \right].
\]

Simplify these four conserved quantities by re-writing them as follows \[51, 52\]:

\[
L = r^2 \sin^2 \theta \left( \frac{d\phi}{d\lambda} - \frac{2J}{r^3} \frac{dt}{d\lambda} \right);
\]

\[
C = r^4 \left( \frac{d\theta}{d\lambda} \right)^2 + \frac{L^2}{\sin^2 \theta};
\]

\[
\epsilon = - \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{dr}{d\lambda} + \sqrt{\frac{2m}{r}} \frac{dt}{d\lambda} \right)^2 + \frac{C}{r^2};
\]

\[
E = \left(1 - \frac{2m}{r}\right) \frac{dt}{d\lambda} - \sqrt{\frac{2m}{r}} \frac{d\theta}{d\lambda} + \frac{2J}{r^3} \frac{L}{L^2}.
\]

In particular \(L^2 \leq C\). For (generic) geodesic trajectories we have \[51, 52\]:

\[
\frac{dr}{d\lambda} = S_r \sqrt{X(r)};
\]

\[
\frac{dt}{d\lambda} = \frac{E - 2JL/r^3 + S_r \sqrt{(2m/r)X(r)}}{(1 - 2m/r)};
\]

\[
\frac{d\theta}{d\lambda} = S_\theta \sqrt{C - L^2/\sin^2 \theta/r^2};
\]

\[
\frac{d\phi}{d\lambda} = \frac{L}{r^2 \sin^2 \theta} + 2J \frac{E - 2JL/r^3 + S_\phi \sqrt{(2m/r)X(r)}}{r^3(1 - 2m/r)}.
\]
Here

\[
S_r = \begin{cases} 
+1 & \text{outgoing geodesic} \\
-1 & \text{ingoing geodesic} 
\end{cases} ; 
\]  

(2.15)

\[
S_\theta = \begin{cases} 
+1 & \text{increasing declination geodesic} \\
-1 & \text{decreasing declination geodesic} 
\end{cases} ; 
\]  

(2.16)

\[
S_\phi = \begin{cases} 
+1 & \text{prograde geodesic} \\
-1 & \text{retrograde geodesic} 
\end{cases} . 
\]  

(2.17)

Furthermore, \(X(r)\) is explicitly given by the sextic Laurent polynomial:

\[
X(r) = \left( E - \frac{2JL}{r^3} \right)^2 - \left( 1 - \frac{2m}{r} \right) \left( -\epsilon + \frac{C}{r^2} \right) ; \quad \lim_{r \to \infty} X(r) = E^2 + \epsilon . 
\]  

(2.18)

In terms of the roots of this polynomial we can in the generic case write

\[
X(r) = \frac{E^2 + \epsilon}{r^6} \prod_{i=1}^{6} (r - r_i) . 
\]  

(2.19)

We shall now restrict attention to the circular orbits, \(r \to r_0\).

3 Radial location of possible circular geodesics

First let us analyze the lack of radial motion; this is not entirely trivial.

3.1 Generalities

Fix our \(r\) coordinate to take some fixed value \(r = r_0\). Hence, since \(dr/d\lambda = S_r \sqrt{X(r)}\), we must have \(X(r_0) = 0\). Furthermore, using the chain rule and the fact that \(S_r^2 = +1\), we have

\[
\frac{d^2r}{d\lambda^2} = S_r \frac{d\sqrt{X(r)}}{d\lambda} = \frac{1}{2} X'(r) . 
\]  

(3.1)

So to remain at \(r_0\) we must also have \(X'(r_0) = 0\). The two conditions

\[
X(r_0) = 0 \quad \text{and} \quad X'(r_0) = 0 
\]  

(3.2)

imply that \(r_0\) is a repeated root of \(X(r)\). The existence of a repeated root will put some constraint on the four geodesic constants \(E, L, \epsilon, \text{ and } C\), (and the spacetime parameters \(m\) and \(J\)); they cannot all be functionally independent.
Higher derivatives do not lead to extra constraints, since

\[ \frac{d^3r}{d\lambda^3} = S \frac{1}{2} X''(r) \sqrt{X(r)} ; \quad (3.3) \]

\[ \frac{d^4r}{d\lambda^4} = \left\{ \frac{1}{2} X'''(r) X(r) + \frac{1}{4} X''(r) X'(r) \right\} , \quad (3.4) \]

and one sees inductively that all terms in all higher-order derivatives contain either \( \sqrt{X(r)} \) or \( X'(r) \) as a factor; quantities which we have already seen vanish at \( r \to r_0 \).

Finally we note that stability of the circular orbit is determined by considering

\[ \frac{d}{dr} \left( \frac{d^2r}{d\lambda^2} \right) = \frac{1}{2} X''(r) . \quad (3.5) \]

Thence if \( X''(r_0) > 0 \) the circular orbit is unstable, if \( X''(r_0) = 0 \) the circular orbit is marginal, and if \( X''(r_0) < 0 \) the circular orbit is stable. So we are interested in evaluating sign \( (X''(r_0)) \). Let us now see what more we can say about the radial location of possible circular orbits.

### 3.2 Circular null geodesics

For massless particles following null geodesics we have \( \epsilon \to 0 \), and without any loss of generality we can set \( E \to 1 \). That implies that we can write

\[ X(r) = \left( 1 - \frac{2JL}{r^3} \right)^2 - \left( 1 - \frac{2m}{r} \right) \frac{C}{r^2} \]

\[ = \frac{r^6 - Cr^4 + 2(Cm - 2JL)r^3 + 4J^2 L^2}{r^6} , \quad (3.6) \]

while

\[ X'(r) = 2 \frac{Cr^4 - 3(Cm - 2JL)r^3 - 12J^2 L^2}{r^7} , \quad (3.8) \]

and

\[ X''(r) = 6 \frac{-Cr^4 + 4(Cm - 2JL)r^3 + 28J^2 L^2}{r^8} . \quad (3.9) \]

Thence we are interested in solving

\[ r_0^6 - Cr_0^4 + 2(Cm - 2JL)r_0^3 + 4J^2 L^2 = 0 , \quad (3.10) \]

\[ Cr_0^4 - 3(Cm - 2JL)r_0^3 - 12J^2 L^2 = 0 , \quad (3.11) \]

and evaluating the sign of \( X''(r_0) \):

\[ \text{sign} \left( X''(r_0) \right) = \text{sign} \left( -Cr_0^4 + 4(Cm - 2JL)r_0^3 + 28J^2 L^2 \right) . \quad (3.12) \]
The non-negativity of \( C \geq 0 \), applied to \( X(r_0) = 0 \), from equation (3.6) immediately implies that \( r_0 \geq 2m \). We also recall that \( L^2 \leq C \). The four quantities \( C, m, JL, \) and \( r_0 \), are subject to two constraints, so only two of these four quantities are functionally independent. More on this point below.

Since we are interested in the sign of \( X''(r_0) \) at a location where \( X'(r_0) = 0 \), we can use that extra information to deduce

\[
\text{sign} \left( X''(r_0) \right) = \text{sign} \left( Cr_0^4 + 36J^2L^2 \right) = +1. \tag{3.13}
\]

Thus there are no stable circular null geodesics. (And, as we shall soon see, there are no marginal circular null geodesics either, all of the circular null geodesics are unstable.)

Let us now consider several special case solutions to the radial part of the circular null geodesic conditions, \( X(r_0) = 0 = X'(r_0) \):

(i) If \( JL = 0 \), corresponding either to a non-rotating source, or to a zero angular momentum geodesic (ZAMO), then one has the unique unstable circular null geodesic:

\[
r_0 = 3m; \quad C = 27m^2; \quad X''(r_0) = \frac{2}{3m^2} = \frac{6}{r_0^2} > 0. \tag{3.14}
\]

This is the situation familiar from Schwarzschild spacetime; an unstable photon orbit at \( r = 3m \).

(ii) If \( C = 0 \), then \( L = 0 \), and there are no circular null orbits.

(iii) If \( r_0 = 2m \) then this implies \( C = 0 \). This is a sub-case of (ii) above.

(iv) If \( r_0 = 3m \) then: either \( JL = 0 \) which is a sub-case of (i) above, or \( C = 0 \) which is a sub-case of (ii) above.

(v) If \( r_0 = \sqrt[3]{2}JL \neq 0 \) then \( C < 0 \), which is non-viable, and there are no circular null orbits.

(vi) The generic case is \( JL \neq 0, C > 0 \) and \( 2m < r_0 \notin \{3m, \sqrt[3]{2}JL\} \).

Now let us consider the generic case:

Treat \( m \) and \( r_0 \) as the two independent variables; then we can explicitly solve for \( C(m, r_0) \) and \( 2JL(m, r_0) \). Let us proceed as follows: If \( JL \neq 0 \) then first solve \( X'(r_0) = 0 \) to find \( C(JL, m, r_0) \). We find

\[
C(JL, m, r_0) = \frac{6(r_0^3 - 2JL)JL}{r_0^3(3m - r_0)} \neq 0. \tag{3.15}
\]
Using this value of $C(JL, m, r_0)$, solve $X(r_0) = 0$ for $2JL(m, r_0)$:

$$2JL(m, r_0) = -\frac{r_0^3}{2} \frac{(r_0 - 3m)}{(2r_0 - 3m)}. \quad (3.16)$$

Third, substitute these values of $JL(m, r_0)$ back into $C(JL, m, r_0)$ to yield $C(m, r_0)$:

$$C(m, r_0) = 9r_0^3 \frac{(r_0 - 2m)}{(2r_0 - 3m)^2}. \quad (3.17)$$

Since we must always have $C > 0$ this limits the generic circular photon orbits to the range $r_0 \in (2m, \infty)$. Finally, inserting this back into $X''(r_0)$ we see:

$$X''(r_0) = \frac{18}{r_0^2} \frac{2(r_0 - 2m)^2 + m^2}{(2r_0 - 3m)^2} > 0. \quad (3.18)$$

Since $X''(r_0) > 0$, we again see that all of these circular photon orbits are unstable. That is, instead of just having one unstable photon orbit at $r = 3m$, once we allow $JL \neq 0$ we can arrange unstable photon orbits at arbitrary $r_0 \in (2m, \infty)$.

### 3.3 Circular timelike geodesics

For massive particles following timelike geodesics $\epsilon \to -1$, and $E$ is unconstrained. That implies that we can write

$$X(r) = \left( E - \frac{2JL}{r^3} \right)^2 - \left( 1 - \frac{2m}{r} \right) \left( 1 + \frac{C}{r^2} \right); \quad (3.19)$$

$$X'(r) = \frac{12JL}{r^4} \left( E - \frac{2JL}{r^3} \right) + \frac{2C(r - 3m)}{r^4} - \frac{2m}{r^2}; \quad (3.20)$$

$$X''(r) = -\frac{24JL}{r^5} \left( 2E - \frac{7JL}{r^3} \right) - \frac{6C(r - 4m)}{r^5} + \frac{4m}{r^3}. \quad (3.21)$$

Rewrite this as

$$X(r) = \frac{(E^2 - 1)r^6 + 2mr^5 - Cr^4 + 2(Cm - 2EJL)r^3 + 4J^2L^2}{r^6}; \quad (3.22)$$

$$X'(r) = -2 \frac{mr^5 - Cr^4 + 3(Cm - 2EJL)r^3 + 12J^2L^2}{r^7}; \quad (3.23)$$

$$X''(r) = 2 \frac{2mr^5 - 3Cr^4 + 12(Cm - 2EJL)r^3 + 84J^2L^2}{r^8}. \quad (3.24)$$
As before we are interested in solving \( X(r_0) = 0 = X'(r_0) \), and determining the sign of \( X''(r_0) \). So we are interested in studying

\[
(E^2 - 1)r_0^6 + 2mr^5 - Cr^4 + 2(Cm - 2EJL)r^3 + 4J^2L^2 = 0; \tag{3.25}
\]

\[
3mr^5 - Cr^4 + 3(Cm - 2EJL)r^3 + 12J^2L^2 = 0; \tag{3.26}
\]

\[
\text{sign}\{2mr^5 - 3Cr^4 + 12(Cm - 2EJL)r^3 + 84J^2L^2\}. \tag{3.27}
\]

The five quantities \( E, C, m, JL, \) and \( r_0 \) are subject to two constraints, so only three of these quantities can be functionally independent. The positivity of \((1 + C/r^2) > 0\), applied to \( X(r_0) = 0 \), immediately implies \( r_0 \geq 2m \). There are several ways of proceeding.

Let us now consider several special case solutions to \( X(r_0) = 0 = X'(r_0) \):

(i) If \( JL = 0 \), corresponding either to a non-rotating source, or to a zero angular momentum geodesic (ZAMO), then for circular orbits one has:

\[
C = \frac{mr_0^2}{r_0 - 3m}; \quad E^2 = \frac{(r_0 - 2m)^2}{r_0(r_0 - 3m)}; \tag{3.28}
\]

and

\[
X''(r_0) = -\frac{2m}{r_0^3} \frac{r_0 - 6m}{r_0 - 3m}. \tag{3.29}
\]

Positivity of \( C \) and/or \( E^2 \) implies \( r_0 \geq 3m \), and \( X''(r_0) \) changes sign at \( r_0 = 6m \). This is the situation familiar from Schwarzschild spacetime; an ISCO at \( r = 6m \), stable orbits for \( r_0 \in (6m, \infty) \), and unstable orbits for \( r_0 \in (3m, 6m) \).

Note that for these circular orbits \( E < 1 \) for \( r_0 > 4m \), \( E = 1 \) for \( r_0 = 4m \), and \( E > 1 \) for \( r_0 \in (3m, 4m) \). Indeed \( E \to \infty \) as \( r_0 \to (3m)^+ \).

(ii) If \( C = 0 \), then automatically \( L = 0 \), and there is no consistent solution.

(iii) If \( r_0 < 2m \) then \( X(r_0) \) is a sum of positive and non-negative terms, so there is no consistent solution.

(iv) If \( r_0 = 2m \), then from \( X(r_0) = 0 \) we have \( E = JL/(4m^2) \), but then from \( X'(r_0) = 0 \) we have \( C = -4m^2 < 0 \), and so there is no consistent solution.

(v) If \( r_0 = \sqrt[3]{2JL/E} \neq 0 \) and \( r_0 > 2m \) then \( X(r_0) = 0 \) implies \( 1 + C/r_0^2 = 0 \), so that \( C = -r_0^2 < 0 \) and there is no consistent solution.

(vi) The generic case is \( JL \neq 0, C > 0 \) and \( 2m < r_0 \neq \sqrt[3]{2JL/E} \).
Now consider the general case:
Choose the three independent variables to be \( m \), \( E \), and \( r_0 \). Let us solve for \( C(m, E, r_0) \) and \( JL(m, E, r_0) \). First take linear combinations of (3.25) and (3.26) to obtain:

\[
3(E^2 - 1)r_0^3 + 5mr_0^2 - C(2r_0 - 3m) - 6EJL = 0; \tag{3.30}
\]
\[
3(E^2 - 1)r_0^6 + 4mr_0^5 - Cr_0^4 - 12J^2L^2 = 0. \tag{3.31}
\]

Solve the first of these equations for \( C \) to find

\[
C(JL, m, E, r_0) = \frac{3(E^2 - 1)r_0^3 + 5mr_0^2 - 6EJL}{2r_0 - 3m}. \tag{3.32}
\]

Inserting this back into the second equation and solving for \( JL \) one finds

\[
JL(m, E, r_0) = \left( \frac{Er_0^2 \pm (r_0 - 2m)\sqrt{r_0\{[9E^2 - 8]r_0 + 12m\}}}{2r_0 - 3m} \right)^2. \tag{3.33}
\]

Since \( JL \) must be real one in turn deduces \( E^2 > \frac{8}{9} - 4m^2 > \frac{4}{9} \). However, the sign of \( JL \) is not constrained; so both roots (\( \pm \)) are valid.

Inserting this back into \( C(JL, m, E, r_0) \) we see

\[
C(m; E, r_0) = \frac{(9E^2 - 12)r_0^4 - 2(9E^2 - 19)mr_0^3 - 30m^2r_0^2}{2(2r_0 - 3m)^2} \pm \frac{3Er_0^2(r_0 - 2m)\sqrt{r_0\{[9E^2 - 8]r_0 + 12m\}}}{2(2r_0 - 3m)^2}. \tag{3.34}
\]

The reality conditions for \( C(m; E, r_0) \) are the same as they were for \( JL(m; E, r_0) \), that is, \( E^2 > \frac{8}{9} - \frac{4m}{3r_0} > \frac{2}{9} \). To enforce positivity of \( C(m; E, r_0) \), both roots (\( \pm \)) are acceptable when \( E^2 \leq \frac{(3r_0-5m)^2}{9r_0(r_0-2m)} \), but only the + root is acceptable outside this range.

Then to determine stability one must determine the sign of:

\[
X''(m; E, r_0) = \frac{18E^2(3r_0^2 - 10mr_0 + 9m^2)}{r_0^6(2r_0 - 3m)^2} - \frac{2(12r_0^2 - 37mr_0 + 30m^2)}{r_0^6(2r_0 - 3m)} \pm \frac{6E(r_0 - 2m)\sqrt{r_0\{[9E^2 - 8]r_0 + 12m\}}}{r_0^6(2r_0 - 3m)^2}. \tag{3.35}
\]
That is:

\[
\text{sign}(X''(m; E, r_0)) = \text{sign}\left\{18E^2(3r_0^2 - 10mr_0 + 9m^2)r_0 - 2(12r_0^2 - 37mr_0 + 30m^2)(2r_0 - 3m) \right. \\
\left. \mp 6Er_0(r_0 - 2m)\sqrt{r_0\{[9E^2 - 8]r_0 + 12m\}}\right\}. \quad (3.36)
\]

In short, there will be many circular orbits, but determining the stability of these circular orbits as functions of the independent parameters \((m, E, r_0)\) will be extremely tedious.

4 General angular motion for circular orbits

Now that we have investigated acceptable values of the parameters \(\{C, JL, E, m\}\) and the radius \(r_0\) for circular orbits, we note that two of the the four constants of the motion reduce to

\[
\epsilon = -\left(1 - \frac{2m}{r_0}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{C}{r_0^2}; \quad (4.1)
\]

\[
E = \left(1 - \frac{2m}{r_0}\right)\frac{dt}{d\lambda} + \frac{2JL}{r_0^3}. \quad (4.2)
\]

Thence for the circular geodesic trajectories

\[
\frac{dr}{d\lambda} = 0 = \frac{d^2r}{d\lambda^2}; \quad (4.3)
\]

\[
\frac{dt}{d\lambda} = \frac{E - 2JL/r_0^3}{(1 - 2m/r_0)}; \quad (4.4)
\]

\[
\frac{d\theta}{d\lambda} = S_\theta \sqrt{C - \frac{L^2}{r_0^2}} \frac{1}{\sin^2 \theta}; \quad (4.5)
\]

\[
\frac{d\phi}{d\lambda} = \frac{L}{r_0^2 \sin^2 \theta} + \frac{2J}{r_0^3} \frac{E - 2JL/r_0^3}{(1 - 2m/r_0)}. \quad (4.6)
\]

We immediately see that \(t\) is an affine parameter, that the declination \(\theta(\lambda)\) evolves independently of the azimuth \(\phi(\lambda)\), and that the azimuthal motion depends on a constant drift and a fluctuating term driven by the declination. Note that the angular motion is qualitatively unaffected by the difference between timelike and null.

5 Declination for circular orbits \((L \neq 0)\)

Consider the ODE controlling the evolution of the declination \(\theta(\lambda)\).
5.1 Forbidden declination range

The form of the Carter constant, equation (2.8), gives a range of forbidden declination angles for any given, non-zero values of $C$ and $L$. We require that $d\theta/d\lambda$ be real, and from equation (2.8) this implies the following requirement:

$$\left( r^2 \frac{d\theta}{d\lambda} \right)^2 = C - \frac{L^2}{\sin^2 \theta} \geq 0 \implies \sin^2 \theta \geq \frac{L^2}{C} .$$

Then provided $C \geq L^2$, which is automatic in view of (2.8), we can define a critical angle $\theta_\ast \in [0, \pi/2]$ by setting

$$\theta_\ast = \sin^{-1} \left( \frac{|L|}{\sqrt{C}} \right) .$$

Then the allowed range for $\theta$ is the equatorial band:

$$\theta \in \left[ \theta_\ast, \pi - \theta_\ast \right] .$$

- For $L^2 = C$ we have $\theta = \pi/2$; the motion is restricted to the equatorial plane.
- For $L = 0$ with $C > 0$ the range of $\theta$ is a priori unconstrained; $\theta \in [0, \pi]$.
- For $L = 0$ with $C = 0$ the declination is fixed $\theta(\lambda) = \theta_0$, and the motion is restricted to a constant declination conical surface.

5.2 Evolution of the declination

As regards the declination angle $\theta$, from equation (4.5), we find

$$\frac{d \cos \theta}{d\lambda} = -S_0 \frac{\sqrt{C} \sin^2 \theta - L^2}{r_0^2}$$

$$= -S_0 \frac{\sqrt{C}}{r_0^2} \sqrt{\sin^2 \theta - \sin^2 \theta_\ast}$$

$$= -S_0 \frac{\sqrt{C}}{r_0^2} \sqrt{\cos^2 \theta_\ast - \cos^2 \theta} ,$$

implying

$$\frac{d \cos \theta}{\sqrt{\cos^2 \theta_\ast - \cos^2 \theta}} = -S_0 \frac{\sqrt{C}}{r_0^2} d\lambda .$$
From this we see

\[ d \cos^{-1} \left( \frac{\cos \theta}{\cos \theta_*} \right) = S_\theta \frac{\sqrt{C}}{r_0^2} d\lambda , \]  

(5.6)

that is

\[ \cos^{-1} \left( \frac{\cos \theta}{\cos \theta_*} \right) = \cos^{-1} \left( \frac{\cos \theta_0}{\cos \theta_*} \right) + S_\theta \frac{\sqrt{C}}{r_0^2} (\lambda - \lambda_0) . \]  

(5.7)

Without loss of generality we may allow the geodesic to reach the critical angle \( \theta_* \) at some affine parameter \( \lambda_* \), and then use that as our new initial data. This effectively sets \( \theta_0 = \theta_* \), and gives us the following simple result:

\[ \cos^{-1} \left( \frac{\cos \theta}{\cos \theta_*} \right) = S_\theta \frac{\sqrt{C}}{r_0^2} (\lambda - \lambda_*) . \]  

(5.8)

Thence, using the fact that cosine is an even function of its argument:

\[ \cos \theta = \cos \theta_* \cos \left( S_\theta \frac{\sqrt{C}}{r_0^2} (\lambda - \lambda_*) \right) = \cos \theta_* \cos \left( \frac{\sqrt{C}}{r_0^2} (\lambda - \lambda_*) \right) . \]

Note the motion is periodic, with period

\[ \Delta \lambda = \frac{2\pi r_0^2}{\sqrt{C}} . \]  

(5.9)

In terms of the Killing time coordinate the period is

\[ T_\theta = \frac{2\pi (E - 2JL/r_0^3)}{\sqrt{C}(1 - 2m/r_0)} . \]  

(5.10)

6 Azimuth for circular orbits \( (L \neq 0) \)

Now consider the ODE for the evolution of the azimuthal angle \( \phi(\lambda) \). We have

\[ \frac{d\phi}{d\lambda} = \frac{2J}{r_0^3} \left( \frac{E - 2JL/r_0^3}{1 - 2m/r_0} \right) + \frac{L}{r_0^2 \sin^2 \theta} , \]  

(6.1)

and

\[ \cos \theta = \cos \theta_* \cos \left( \frac{\sqrt{C}}{r_0^2} (\lambda - \lambda_*) \right) . \]

Thence

\[ \phi(\lambda) = \phi_* + \frac{2J}{r_0^3} \left( \frac{E - 2JL/r_0^3}{1 - 2m/r_0} \right) (\lambda - \lambda_*) + \frac{L}{r_0^2} \int_{\lambda_*}^{\lambda} \frac{d\lambda}{1 - \cos^2(\lambda)} . \]  

(6.2)
The only tricky item here is evaluation of the integral
\[
\int_{\lambda_*}^{\lambda} \frac{d\tilde{\lambda}}{1 - \cos^2(\lambda)} = \int_{\lambda_*}^{\lambda} \frac{d\tilde{\lambda}}{1 - \cos^2(\theta_*) \cos^2 \left( (\sqrt{C}/r_0^2) (\tilde{\lambda} - \lambda_*) \right)}.
\]  
(6.3)

But it is easy to check that
\[
\int \frac{d\lambda}{1 - (A \cos(B + F \lambda))^2} = \frac{1}{F \sqrt{1 - A^2}} \arctan \left( \frac{\tan(B + F \lambda)}{\sqrt{1 - A^2}} \right).
\]  
(6.4)

Thence
\[
\int_{\lambda_*}^{\lambda} \frac{d\tilde{\lambda}}{\sin^2(\theta(\lambda))} = \frac{r_0^2}{\sqrt{C} \sin \theta_*} \arctan \left( \frac{\tan((C/r_0^2) (\lambda - \lambda_*))}{\sin \theta_*} \right).
\]  
(6.5)

Finally, using \( C = L^2 / \sin^2 \theta \), we have
\[
\phi = \phi_* + \frac{2J}{r_0^3} \frac{E - 2JL/r_0^3}{1 - 2m/r_0} (\lambda - \lambda_*) + \arctan \left( \frac{1}{\sin \theta_*} \tan \left( \frac{\sqrt{C} (\lambda - \lambda_*)}{r_0^3} \right) \right).
\]  
(6.6)

So the azimuthal motion is a constant drift (growing linearly in the affine parameter) with a superimposed oscillation. Note the sensible limit for equatorial motion as \( \sin \theta_* \to 1 \). The oscillatory contribution to the azimuthal evolution has the same period as the evolution in declination
\[
\Delta \lambda_{\text{oscillation}} = \frac{2\pi r_0^2}{\sqrt{C}},
\]  
(6.7)

but the drift component has periodicity
\[
\Delta \lambda_{\text{drift}} = \frac{2\pi r_0^3}{2J} \frac{1 - 2m/r_0}{E - 2JL/r_0^3},
\]  
(6.8)

so that
\[
(T_\phi)_{\text{drift}} = \frac{2\pi r_0^3}{2J}.
\]  
(6.9)

This drift in azimuth periodicity is typically incommensurate with the periodicity in declination, so the geodesics are surface filling and will eventually cover the entire equatorial band \( \theta \in [\theta_*, \pi - \theta_*] \).
7 Angular motion for $L = 0$ (Circular ZAMOS)

If we now consider the special case of circular orbits where $L = 0$, then $\sin \theta \rightarrow 0$, so we need to be careful. The equations of motion reduce even further to:

\[
\left( \frac{d\phi}{d\lambda} \right) = \frac{2J}{r_0^3} \frac{E}{1 - 2m/r_0}, \quad \left( \frac{d\theta}{d\lambda} \right) = \pm \sqrt{C} \frac{r_0}{r_0^3} .
\] (7.1)

So in this special case we find

\[
\phi = \phi_0 + \frac{2J}{r_0^3} E (\lambda - \lambda_0) ; \quad \theta = \theta_0 \pm \sqrt{C} \frac{r_0}{r_0^3} (\lambda - \lambda_0) .
\] (7.2)

Now $\phi$ is defined only modulo $2\pi$, but $\theta$ is naively in $[0, \pi]$. However if we formally drive it outside this range we just need to reset $\phi$ by $\pi$. That is, we can identify the points $(\theta + \pi, \phi) \equiv (\pi - \theta, \phi + \pi)$.

In view of the fact that for circular orbits with $L = 0$ the quantity

\[
\frac{dt}{d\lambda} = \frac{E}{1 - 2m/r_0}
\] (7.3)

is a constant, we can also rewrite angular dependence as

\[
\phi = \phi_0 + \frac{2J}{r_0^3} (t - t_0) ; \quad \theta = \theta_0 \pm \frac{\sqrt{C}(1 - 2m/r_0)}{E r_0^2} (t - t_0) .
\] (7.4)

Note the periodicities in azimuth and declination are

\[
T_\phi = \frac{\pi r_0^3}{J}, \quad \text{and} \quad T_\theta = \frac{\pi E r_0^2}{\sqrt{C}(1 - 2m/r_0)} .
\] (7.5)

These are typically incommensurate, so these ZAMO curves are surface filling and will eventually cover the entire angular 2-sphere.

8 Limit as $J \to 0$

Physically the limit $J \to 0$ corresponds to switching off the angular momentum of the central object generating the gravitational field, so that the spacetime becomes Schwarzschild in Painlevé–Gullstrand coordinates; so for circular orbits we must recover the unstable photon sphere at $r = 3m$ and the ISCO at $r = 6m$. If not, something is very wrong.

For $J \to 0$ the quantity $X(r)$ simplifies to

\[
X(r) \to E^2 - \left( 1 - \frac{2m}{r} \right) \left( -\epsilon + \frac{C}{r^2} \right) .
\] (8.1)
8.1 Photon spheres

For massless particles $\epsilon \to 0$, and without loss of generality we can set $E \to 1$. This implies

$$X(r) \to 1 - \left(1 - \frac{2m}{r}\right) \frac{C}{r^2};$$  \hfill (8.2)

$$X'(r) \to \frac{2C(r - 3m)}{r^4};$$  \hfill (8.3)

and

$$X''(r) \to -\frac{6C(r - 4m)}{r^5}. \quad (8.4)$$

There is a unique photon sphere at $r_0 = 3m$. Then $X(r_0) = 0 = 1 - C/(3r_0^2)$, that is $C = 3r_0^2$. We then see that $X''(3m) = 2C/(243m^4) = 2/(81m^2) > 0$, these photon orbits are unstable. This is exactly as it should be.

8.2 Massive particle spheres

For massive particles $\epsilon \to -1$, that implies

$$X(r) \to E^2 - \left(1 - \frac{2m}{r}\right) \left(1 + \frac{C}{r^2}\right);$$  \hfill (8.5)

$$X'(r) \to \frac{2C(r - 3m) - 2mr^2}{r^4}. \quad (8.6)$$

and

$$X''(r) \to \frac{-6C(r - 4m) + 4mr^2}{r^5}. \quad (8.7)$$

Solve $X'(r_0) = 0$ to find $C(m, r_0)$:

$$C(m, r_0) = \frac{mr_0^2}{r_0 - 3m}. \quad (8.8)$$

Since $C \geq 0$, there will now be many circular orbits, all the way from $r_0 = \infty$ down to $r_0 = 3m$. Use this to evaluate $X''(m, r_0)$:

$$X''(r_0, m) \to -\frac{2m(r_0 - 6m)}{r_0^3(r_0 - 3m)}. \quad (8.9)$$

Inspecting the sign of $X''(m, r_0)$, the circular orbits are stable for $r_0 > 6m$, marginal for $r_0 = 6m$, and unstable for $r_0 < 6m$. This is exactly as it should be.
9 Conclusions

We have explored the existence and properties of circular geodesics in the recently introduced Painlevé–Gullstand variant of the Lense–Thirring spacetime [50–52]. We emphasize that although the underlying spacetime is not spherically symmetric, (only stationary and axisymmetric), so that the Birkhoff theorem does not apply [22–26], one nevertheless encounters (partial) spherical shells of circular geodesics; notably this behaviour is not limited to the (exact) Kerr spacetime, but also persists in the Painlevé–Gullstand variant of the Lense–Thirring spacetime. Overall, we see that the Painlevé–Gullstand variant of the Lense–Thirring spacetime [50–52] exhibits many useful and interesting properties, and is well-adapted to direct confrontation with observational astrophysics.

From a wider perspective, these considerations can be viewed as an element of the study of modified black holes — alternative black holes to the standard Schwarzschild–Kerr family that are nevertheless carefully formulated so as to pass the most obvious observational tests, and so provide useful templates for driving observational astrophysics [62–77].

Acknowledgements

JB was supported by a MSc scholarship funded by the Marsden Fund, via a grant administered by the Royal Society of New Zealand.

TB was supported by a Victoria University of Wellington MSc scholarship, and was also indirectly supported by the Marsden Fund, via a grant administered by the Royal Society of New Zealand.

AS was supported by a Victoria University of Wellington PhD Doctoral Scholarship, and was also indirectly supported by the Marsden fund, via a grant administered by the Royal Society of New Zealand.

MV was directly supported by the Marsden Fund, via a grant administered by the Royal Society of New Zealand.
References

[1] Roy Kerr, “Gravitational field of a spinning mass as an example of algebraically special metrics”, Physical Review Letters 11 237-238 (1963).

[2] Roy Kerr, “Gravitational collapse and rotation”, published in: Quasi-stellar sources and gravitational collapse: Including the proceedings of the First Texas Symposium on Relativistic Astrophysics, edited by Ivor Robinson, Alfred Schild, and E.L. Schücking (University of Chicago Press, Chicago, 1965), pages 99–102. The conference was held in Austin, Texas, on 16–18 December 1963.

[3] E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, “Metric of a Rotating, Charged Mass”, J. Math. Phys. 6 (1965) 918.

[4] M. Visser, “The Kerr spacetime: A brief introduction”, [arXiv:0706.0622 [gr-qc]]. Published in [5].

[5] D. L. Wiltshire, M. Visser and S. M. Scott (editors), The Kerr spacetime: Rotating black holes in general relativity, (Cambridge University Press, Cambridge, 2009).

[6] Barrett O'Neill, The geometry of Kerr black holes, (Peters, Wellesley, 1995). Reprinted (Dover, Mineloa, 2014).

[7] Charles Misner, Kip Thorne, and John Archibald Wheeler, Gravitation, (Freeman, San Francisco, 1973).

[8] Robert Wald, General relativity, (University of Chicago Press, Chicago, 1984).

[9] Steven Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, (Wiley, Hoboken, 1972).

[10] Ronald J. Adler, Maurice Bazin, and Menahem Schiffer, Introduction to General Relativity, Second edition, (McGraw–Hill, New York, 1975). [It is important to acquire the 1975 second edition, the 1965 first edition does not contain any discussion of the Kerr spacetime.]

[11] M. P. Hobson, G. P. Estathiou, and A N. Lasenby, General relativity: An introduction for physicists, (Cambridge University Press, Cambridge, 2006).

[12] Ray D’Inverno, Introducing Einstein’s Relativity, (Oxford University Press, 1992).

[13] James Hartle, Gravity: An introduction to Einstein’s general relativity, (Addison Wesley, San Francisco, 2003).
[14] Sean Carroll, An introduction to general relativity: Spacetime and Geometry, (Addison Wesley, San Francisco, 2004).

[15] E. Berti, E. Barausse, V. Cardoso, L. Gualtieri, P. Pani, U. Sperhake, L. C. Stein, N. Wex, K. Yagi and T. Baker, et al. “Testing General Relativity with Present and Future Astrophysical Observations”, Class. Quant. Grav. 32 (2015), 243001 doi:10.1088/0264-9381/32/24/243001 [arXiv:1501.07274 [gr-qc]].

[16] N. Yunes, K. Yagi and F. Pretorius, “Theoretical Physics Implications of the Binary Black-Hole Mergers GW150914 and GW151226”, Phys. Rev. D 94 (2016) no.8, 084002 doi:10.1103/PhysRevD.94.084002 [arXiv:1603.08955 [gr-qc]].

[17] V. Cardoso and P. Pani, “Testing the nature of dark compact objects: a status report”, Living Rev. Rel. 22 (2019) no.1, 4 doi:10.1007/s41114-019-0020-4 [arXiv:1904.05363 [gr-qc]].

[18] L. Barack and C. Cutler, “Using LISA EMRI sources to test off-Kerr deviations in the geometry of massive black holes”, Phys. Rev. D 75 (2007), 042003 doi:10.1103/PhysRevD.75.042003 [arXiv:gr-qc/0612029 [gr-qc]].

[19] C. Bambi, K. Freese, S. Vagnozzi and L. Visinelli, “Testing the rotational nature of the supermassive object M87* from the circularity and size of its first image”, Phys. Rev. D 100 (2019) no.4, 044057 doi:10.1103/PhysRevD.100.044057 [arXiv:1904.12983 [gr-qc]].

[20] L. Barack and A. Pound, “Self-force and radiation reaction in general relativity”, Rept. Prog. Phys. 82 (2019) no.1, 016904 doi:10.1088/1361-6633/aae552 [arXiv:1805.10385 [gr-qc]].

[21] E. Barausse, E. Berti, T. Hertog, S. A. Hughes, P. Jetzer, P. Pani, T. P. Sotiriou, N. Tamanini, H. Witek and K. Yagi, et al. “Prospects for Fundamental Physics with LISA”, Gen. Rel. Grav. 52 (2020) no.8, 81 doi:10.1007/s10714-020-02691-1 [arXiv:2001.09793 [gr-qc]].

[22] Garret Birkhoff, Relativity and Modern Physics, (Harvard University Press, Cambridge, 1923).

[23] Jørg Tofte Jebsen, “Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum”, Ark. Mat. Astr. Fys. (Stockholm) 15 (1921) nr.18.

[24] Stanley Deser and Joel Franklin, “Schwarzschild and Birkhoff a la Weyl”, Am. J. Phys. 73 (2005) 261 [arXiv:gr-qc/0408067 [gr-qc]].

[25] Nils Voje Johansen, Finn Ravndal, “On the discovery of Birkhoff’s theorem”, Gen.Rel.Grav. 38 (2006) 537-540 [arXiv:physics/0508163 [physics.hist-ph]].

[26] J. Skakala and M. Visser, “Birkhoff-like theorem for rotating stars in (2+1) dimensions”, [arXiv:0903.2128 [gr-qc]].
[27] A. Edery and J. Godin, “Second order Kerr deflection”, Gen. Rel. Grav. 38 (2006), 1715-1722 doi:10.1007/s10714-006-0347-5

[28] S. Hod, “The fastest way to circle a black hole”, Phys. Rev. D 84 (2011), 104024 doi:10.1103/PhysRevD.84.104024 [arXiv:1201.0068 [gr-qc]].

[29] N. Warburton, L. Barack and N. Sago, “Isofrequency pairing of geodesic orbits in Kerr geometry”, Phys. Rev. D 87 (2013) no.8, 084012 doi:10.1103/PhysRevD.87.084012 [arXiv:1301.3918 [gr-qc]].

[30] S. Hod, “Spherical null geodesics of rotating Kerr black holes”, Phys. Lett. B 718 (2013), 1552-1556 doi:10.1016/j.physletb.2012.12.047 [arXiv:1210.2486 [gr-qc]].

[31] E. Teo, “Spherical geodesics around a Kerr black hole”, Gen. Rel. Grav. 53 (2021) no.1, 10 doi:10.1007/s10714-020-02782-z [arXiv:2007.04022 [gr-qc]].

[32] A. Tavlayan and B. Tekin, “Exact Formulas for Spherical Photon Orbits Around Kerr Black Holes”, Phys. Rev. D 102 (2020) no.10, 104036 doi:10.1103/PhysRevD.102.104036 [arXiv:2009.07012 [gr-qc]].

[33] A. E. Broderick, T. Johannsen, A. Loeb and D. Psaltis, “Testing the No-Hair Theorem with Event Horizon Telescope Observations of Sagittarius A*”, Astrophys. J. 784 (2014), 7 doi:10.1088/0004-637X/784/1/7 [arXiv:1311.5564 [astro-ph.HE]].

[34] T. Johannsen, “Sgr A* and General Relativity”, Class. Quant. Grav. 33 (2016) no.11, 113001 doi:10.1088/0264-9381/33/11/113001 [arXiv:1512.03818 [astro-ph.GA]].

[35] A. Broderick and A. Loeb, “Imaging the Black Hole Silhouette of M87: Implications for Jet Formation and Black Hole Spin”, Astrophys. J. 697 (2009), 1164-1179 doi:10.1088/0004-637X/697/2/1164 [arXiv:0812.0366 [astro-ph]].

[36] S. E. Gralla, D. E. Holz and R. M. Wald, “Black Hole Shadows, Photon Rings, and Lensing Rings”, Phys. Rev. D 100 (2019) no.2, 024018 doi:10.1103/PhysRevD.100.024018 [arXiv:1906.00873 [astro-ph.HE]].

[37] C. Bambi, “Astrophysical Black Holes: A Compact Pedagogical Review”, Annalen Phys. 530 (2018), 1700430 doi:10.1002/andp.201700430 [arXiv:1711.10256 [gr-qc]].

[38] F. H. Vincent, M. Wielgus, M. A. Abramowicz, E. Gourgoulhon, J. P. Lasota, T. Paumard and G. Perrin, “Geometric modeling of M87* as a Kerr black hole or a non-Kerr compact object”, Astron. Astrophys. 646 (2021), A37 doi:10.1051/0004-6361/202037787 [arXiv:2002.09226 [gr-qc]].

[39] A. Chael, M. D. Johnson and A. Lupcasca, “Observing the Inner Shadow of a Black Hole: A Direct View of the Event Horizon”, Astrophys. J. 918 (2021) no.1, 6 doi:10.3847/1538-4357/ac09ee [arXiv:2106.00683 [astro-ph.HE]].

[40] T. Berry, A. Simpson and M. Visser, “Photon spheres, ISCOs, and OSCOs: Astrophysical observables for regular black holes with asymptotically Minkowski cores”, Universe 7 (2020) no.1, 2 doi:10.3390/universe7010002 [arXiv:2008.13308 [gr-qc]].
[41] P. Boonserm, T. Ngampitipan, A. Simpson and M. Visser, “Innermost and outermost stable circular orbits in the presence of a positive cosmological constant”, Phys. Rev. D 101 (2020) no.2, 024050 doi:10.1103/PhysRevD.101.024050 [arXiv:1909.06755 [gr-qc]].

[42] Hans Thirring and Josef Lense, “Über den Einfluss der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie”, Physikalische Zeitschrift, Leipzig Jg. 19 (1918), No. 8, p. 156–163.

English translation by Bahram Mashoon, Friedrich W. Hehl, and Dietmar S. Theiss: “On the influence of the proper rotations of central bodies on the motions of planets and moons in Einstein’s theory of gravity”, General Relativity and Gravitation 16 (1984) 727–741.

[43] Herbert Pfister, “On the history of the so-called Lense–Thirring effect”, http://philsci-archive.pitt.edu/archive/00002681/01/lense.pdf

[44] Paul Painlevé, “La mécanique classique et la théorie de la relativité”, C. R. Acad. Sci. (Paris) 173, 677–680(1921).

[45] Paul Painlevé, “La gravitation dans la mécanique de Newton et dans la mécanique d’Einstein”, C. R. Acad. Sci. (Paris) 173, 873–886(1921).

[46] Allvar Gullstrand, “Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie”, Arkiv för Matematik, Astronomi och Fysik. 16 (8): 1–15 (1922).

[47] K. Martel and E. Poisson, “Regular coordinate systems for Schwarzschild and other spherical space-times”, Am. J. Phys. 69 (2001), 476-480 doi:10.1119/1.1336836 [arXiv:gr-qc/0001069 [gr-qc]].

[48] V. Faraoni and G. Vachon, “When Painlevé–Gullstrand coordinates fail”, Eur. Phys. J. C 80 (2020) no.8, 771 doi:10.1140/epjc/s10052-020-8345-4 [arXiv:2006.10827 [gr-qc]].

[49] P. Boonserm, T. Ngampitipan and M. Visser, “Near-horizon geodesics for astrophysical and idealised black holes: Coordinate velocity and coordinate acceleration”, Universe 4 (2018) no.6, 68 doi:10.3390/universe4060068 [arXiv:1710.06139 [gr-qc]].
[50] Joshua Baines, Thomas Berry, Alex Simpson, and Matt Visser, “Painlevé-Gullstrand form of the Lense-Thirring spacetime”, Universe 7 # 4 (2021) 105, doi:10.3390/universe704010 [arXiv:2006.14258 [gr-qc]].

[51] Joshua Baines, Thomas Berry, Alex Simpson, and Matt Visser, “Killing tensor and Carter constant for Painlevé–Gullstrand form of Lense–Thirring spacetime”, Universe 7 # 12 (2021) 473, doi:10.3390/universe7120473 [arXiv:2111.01814 [gr-qc]].

[52] Joshua Baines, Thomas Berry, Alex Simpson, and Matt Visser, “Geodesics for Painlevé–Gullstrand form of Lense–Thirring spacetime”, Universe 8 #2 (2022) 115, doi:10.3390/universe8020115 [arXiv:2112.05228 [gr-qc]].

[53] A. J. Hamilton and J. P. Lisle, “The river model of black holes”, Am. J. Phys. 76 (2008), 519-532 doi:10.1119/1.2830526 [arXiv:gr-qc/0410060 [gr-qc]].

[54] J. Baines, T. Berry, A. Simpson and M. Visser, “Unit-lapse versions of the Kerr spacetime”, Class. Quant. Grav. 38 (2021) no.5, 055001 doi:10.1088/1361-6382/abd071 [arXiv:2008.03817 [gr-qc]].

[55] J. A. Valiente Kroon, “On the nonexistence of conformally flat slices in the Kerr and other stationary space-times”, Phys. Rev. Lett. 92 (2004), 041101 doi:10.1103/PhysRevLett.92.041101 [arXiv:gr-qc/0310048 [gr-qc]].

[56] J. A. Valiente Kroon, “Asymptotic expansions of the Cotton-York tensor on slices of stationary space-times”, Class. Quant. Grav. 21 (2004), 3237-3250 doi:10.1088/0264-9381/21/13/009 [arXiv:gr-qc/0402033 [gr-qc]].

[57] J. L. Jaramillo, J. A. Valiente Kroon and E. Gourgoulhon, “From geometry to numerics: Interdisciplinary aspects in mathematical and numerical relativity”, Class. Quant. Grav. 25 (2008), 093001 doi:10.1088/0264-9381/25/9/093001 [arXiv:0712.2332 [gr-qc]].

[58] Joshua Baines, Thomas Berry, Alex Simpson, and Matt Visser, “Darboux diagonalization of the spatial 3-metric in Kerr spacetime”, Gen.Rel.Grav. 53 (2021) 1, 3 doi:10.1007/s10714-020-02765-0 [arXiv:2009.01397 [gr-qc]].

[59] G. O. Papadopoulos and K. D. Kokkotas, “On Kerr black hole deformations admitting a Carter constant and an invariant criterion for the separability of the wave equation”, Gen. Rel. Grav. 53 (2021) no.2, 21 doi:10.1007/s10714-021-02795-2 [arXiv:2007.12125 [gr-qc]].

[60] G. O. Papadopoulos and K. D. Kokkotas, “Preserving Kerr symmetries in deformed spacetimes”, Class. Quant. Grav. 35 (2018) no.18, 185014 doi:10.1088/1361-6382/aad7f4 [arXiv:1807.08594 [gr-qc]].
[61] S. Benenti and M. Francaviglia, “Remarks on Certain Separability Structures and Their Applications to General Relativity”, General Relativity and Gravitation 10 (1979) 79–92.

[62] R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, “Phenomenological aspects of black holes beyond general relativity”, Phys. Rev. D 98 (2018) no.12, 124009 doi:10.1103/PhysRevD.98.124009 [arXiv:1809.08238 [gr-qc]].

[63] R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, “Geodesically complete black holes”, Phys. Rev. D 101 (2020), 084047 doi:10.1103/PhysRevD.101.084047 [arXiv:1911.11200 [gr-qc]].

[64] R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, “Geodesically complete black holes in Lorentz-violating gravity”, JHEP 2022 (in press). [arXiv:2111.03113 [gr-qc]].

[65] R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, “Opening the Pandora’s box at the core of black holes”, Class. Quant. Grav. 37 (2020) no.14, 14 doi:10.1088/1361-6382/ab8141 [arXiv:1908.03261 [gr-qc]].

[66] A. Simpson and M. Visser, “The eye of the storm: A regular Kerr black hole”, JCAP (in press), [arXiv:2111.12329 [gr-qc]].

[67] A. Simpson and M. Visser, “Astrophysically viable Kerr-like spacetime – into the eye of the storm”, [arXiv:2112.04647 [gr-qc]].

[68] M. Visser, C. Barceló, S. Liberati and S. Sonego, “Small, dark, and heavy: But is it a black hole?”, PoS BHGRS (2008), 010 doi:10.22323/1.075.0010 [arXiv:0902.0346 [gr-qc]].

[69] M. Visser, “Black holes in general relativity”, PoS BHGRS (2008), 001 doi:10.22323/1.075.0001 [arXiv:0901.4365 [gr-qc]].

[70] J. Mazza, E. Franzin and S. Liberati, “A novel family of rotating black hole mimickers”, JCAP 04 (2021), 082 doi:10.1088/1475-7516/2021/04/082 [arXiv:2102.01105 [gr-qc]].

[71] E. Franzin, S. Liberati, J. Mazza, A. Simpson and M. Visser, “Charged black-bounce spacetimes”, JCAP 07 (2021), 036 doi:10.1088/1475-7516/2021/07/036 [arXiv:2104.11376 [gr-qc]].

[72] T. De Lorenzo, C. Pacilio, C. Rovelli and S. Speziale, “On the Effective Metric of a Planck Star”, Gen. Rel. Grav. 47 (2015) no.4, 41 doi:10.1007/s10714-015-1882-8 [arXiv:1412.6015 [gr-qc]].

[73] S. A. Hayward, “Formation and evaporation of regular black holes”, Phys. Rev. Lett. 96 (2006), 031103 doi:10.1103/PhysRevLett.96.031103 [arXiv:gr-qc/0506126 [gr-qc]].

[74] K. A. Bronnikov, “Regular magnetic black holes and monopoles from nonlinear electrodynamics”, Phys. Rev. D 63 (2001), 044005 doi:10.1103/PhysRevD.63.044005 [arXiv:gr-qc/0006014 [gr-qc]].
[75] K. A. Bronnikov and J. C. Fabris, “Regular phantom black holes”, Phys. Rev. Lett. 96 (2006), 251101 doi:10.1103/PhysRevLett.96.251101 [arXiv:gr-qc/0511109 [gr-qc]].

[76] T. Johannsen and D. Psaltis, “A Metric for Rapidly Spinning Black Holes Suitable for Strong-Field Tests of the No-Hair Theorem”, Phys. Rev. D 83 (2011), 124015 doi:10.1103/PhysRevD.83.124015 [arXiv:1105.3191 [gr-qc]].

[77] J. M. Bardeen, “Non-singular general relativistic gravitational collapse”, Abstracts of the 5th international conference on gravitation and the theory of relativity (GR5), eds. V. A. Fock et al. (Tbilisi University Press, Tblisi, Georgia, former USSR, 1968), pages 174–175.