GENERALIZED DEFORMED su(2) ALGEBRAS,
DEFORMED PARAFERMIONIC OSCILLATORS AND FINITE W ALGEBRAS

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Abstract

Several physical systems (two identical particles in two dimensions, isotropic oscillator and Kepler system in a 2-dim curved space) and mathematical structures (quadratic algebra QH(3), finite W algebra W₀) are shown to posses the structure of a generalized deformed su(2) algebra, the representation theory of which is known. Furthermore, the generalized deformed parafermionic oscillator is identified with the algebra of several physical systems (isotropic oscillator and Kepler system in 2-dim curved space, Fokas–Lagerstrom, Smorodinsky–Winternitz and Holt potentials) and mathematical constructions (generalized deformed su(2) algebra, finite W algebras W₀ and W⁽²⁾₃). The fact that the Holt potential is characterized by the W⁽²⁾₃ symmetry is obtained as a by-product.

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1. Introduction

Quantum algebras (quantum groups)\(^1,2\), which are nonlinear generalizations of the usual Lie algebras to which they reduce for appropriate values of the deformation parameter(s), have been finding applications as the dynamical symmetry algebras of several physical systems. For the boson realization of these algebras, various kinds of deformed oscillators have been introduced\(^3-7\) and unification schemes for them have been suggested (see\(^8\) for a list of references).

Furthermore, generalized deformed su(2) algebras have been introduced\(^9,10\), in a way that their representation theory remains as close as possible to the usual su(2) one. It will be shown here that several physical systems (two identical particles in two dimensions\(^11\), isotropic oscillator and Kepler system in a 2-dim curved space\(^12\), as well as the quadratic algebra QH(3)\(^13\) and the finite W algebra \(\bar{W}_0\)\(^14\) can be accommodated within this scheme. The advantage this unification offers is that the representation theory of the generalized deformed su(2) algebras is known\(^9\).

In addition, generalized parafermionic oscillators have been introduced\(^15\), in analogy to generalized deformed oscillators\(^16\). It will be shown here that several physical systems (isotropic oscillator and Kepler system in a 2-dim curved space\(^12\), Fokas–Lagerstrom potential\(^17\), Smorodinsky–Winternitz potential\(^18\), Holt potential\(^19\)) and mathematical constructions (generalized deformed su(2) algebra\(^9,10\), finite W algebras \(\bar{W}_0\)\(^14\) and \(W_3^{(2)}\)\(^20-22\)) can be accommodated within this framework. As a by-product the fact that the Holt potential is characterized by the \(W_3^{(2)}\) symmetry occurs.

In section 2 the cases related to the generalized deformed su(2) algebra will be studied, while in section 3 the systems related to generalized deformed parafermionic oscillators will be considered. Section 4 will contain discussion of the present results and plans for further work.

2. Generalized deformed su(2) algebras

Generalized deformed su(2) algebras having representation theory similar to that of the usual su(2) have been constructed in\(^9\). It has been proved that it is possible to construct an algebra

\[
[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = \Phi(J_0(J_0 + 1)) - \Phi(J_0(J_0 - 1)),
\]  \(1\)
where $J_0$, $J_+$, $J_-$ are the generators of the algebra and $\Phi(x)$ is any increasing entire function defined for $x \geq -1/4$. Since this algebra is characterized by the function $\Phi$, we use for it the symbol $\text{su}_\Phi(2)$. The appropriate basis $|l, m\rangle$ has the properties

\begin{align}
J_0|l, m\rangle &= m|l, m\rangle, \\
J_+|l, m\rangle &= \sqrt{\Phi(l(l+1)) - \Phi(m(m+1))}|l, m+1\rangle, \\
J_-|l, m\rangle &= \sqrt{\Phi(l(l+1)) - \Phi(m(m-1))}|l, m-1\rangle,
\end{align}

where

\begin{align}
l &= 0, 1/2, 1, 3/2, 2, 5/2, 3, \ldots, \\
and \quad m &= -l, -l + 1, -l + 2, \ldots, l - 2, l - 1, l.
\end{align}

The Casimir operator is

\begin{align}
C = J_-J_+ + \Phi(J_0(J_0 + 1)) = J_+J_- + \Phi(J_0(J_0 - 1)),
\end{align}

its eigenvalues indicated by

\begin{align}
C|l, m\rangle &= \Phi(l(l+1))|l, m\rangle.
\end{align}

The usual $\text{su}(2)$ algebra is recovered for

\begin{align}
\Phi(x(x+1)) = x(x+1),
\end{align}

while the quantum algebra $\text{su}_q(2)$

\begin{align}
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q,
\end{align}

occurs for

\begin{align}
\Phi(x(x+1)) = [x]_q[x+1]_q,
\end{align}

with $q$-numbers defined as

\begin{align}
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.
\end{align}

The $\text{su}_\Phi(2)$ algebra occurs in several cases, in which the rhs of the last equation in (1) is an odd function of $J_0$. 

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2.1 Two identical particles in two dimensions

Let us consider the system of two identical particles in two dimensions. For identical particles observables of the system have to be invariant under exchange of particle indices. A set of appropriate observables in this case is

\begin{align}
    u &= (x_1)^2 + (x_2)^2, \quad v = (x_1)^2 - (x_2)^2, \quad w = 2x_1x_2, \\
    U &= (p_1)^2 + (p_2)^2, \quad V = (p_1)^2 - (p_2)^2, \quad W = 2p_1p_2, \\
    C_1 &= \frac{1}{4}(x_1p_1 + p_1x_1), \quad C_2 = \frac{1}{4}(x_2p_2 + p_2x_2), \quad M = x_1p_2 + x_2p_1,
\end{align}

where the indices 1 and 2 indicate the two particles. These observables are known to close an \(\text{sp}(4,\mathbb{R})\) algebra. A representation of this algebra can be constructed using one arbitrary constant \(\eta\) and three matrices \(Q, R,\) and \(S\) satisfying the commutation relations

\begin{align}
    [S, Q] &= -2iR, \quad [S, R] = 2iQ, \quad [Q, R] = -8iS(\eta - 2S^2). \tag{16}
\end{align}

The explicit expressions of the generators of \(\text{sp}(4,\mathbb{R})\) in terms of \(\eta, S, Q, R\) are given in \cite{11} and need not be repeated here. Defining the operators

\begin{align}
    X &= Q - iR, \quad Y = Q + iR, \quad S_0 = \frac{S}{2}, \tag{17}
\end{align}

one can see that the commutators of eq. (16) take the form

\begin{align}
    [S_0, X] &= X, \quad [S_0, Y] = -Y, \quad [X, Y] = 32S_0(\eta - 8(S_0)^2), \tag{18}
\end{align}

which is a deformed version of \(\text{su}(2)\). It is clear that the algebra of eq. (18) is a special case of an \(\text{su}_\Phi(2)\) algebra with structure function

\begin{align}
    \Phi(J_0(J_0 + 1)) = 16\eta J_0(J_0 + 1) - 64(J_0(J_0 + 1))^2. \tag{19}
\end{align}

The condition that \(\Phi(x)\) has to be an increasing function of \(x\) implies the restriction \(x < \eta/8\).

2.2 Kepler problem in 2-dim curved space

Studying the Kepler problem in a two-dimensional curved space with constant curvature \(\lambda\) one finds the algebra \cite{12}

\begin{align}
    [L, R_\pm] &= \pm R_\pm, \quad [R_-, R_+] = F \left( L + \frac{1}{2} \right) - F \left( L - \frac{1}{2} \right), \tag{20}
\end{align}
where
\[
F(L) = \mu^2 + 2HL^2 - \lambda L^2 \left(L^2 - \frac{1}{4}\right).
\] (21)

It is then easy to see that
\[
[R_+, R_-] = 2L \left(-2H + \frac{\lambda}{4}\right) + 4\lambda L^3,
\] (22)

which corresponds to an \(\text{su}_6(2)\) algebra with
\[
\Phi(J_0(J_0 + 1)) = \left(-2H + \frac{\lambda}{4}\right)J_0(J_0 + 1) + \lambda(J_0(J_0 + 1))^2.
\] (23)

For \(\Phi(x)\) to be an increasing function, the condition
\[
\lambda x > H - \frac{\lambda}{8}
\] (24)

has to be obeyed.

2.3 Isotropic oscillator in 2-dim curved space

In the case of the isotropic oscillator in a two-dimensional curved space with constant curvature \(\lambda\) one finds the algebra \(^{12}\)
\[
[L, S_\pm] = \pm 2S_\pm, \quad [S_-, S_+] = G(L + 1) - G(L - 1),
\] (25)

with
\[
G(L) = H^2 - \left(\omega^2 + \frac{\lambda^2}{4} + \lambda H\right)L^2 + \frac{1}{4}\lambda^2 L^4.
\] (26)

Using \(\tilde{L} = L/2\) one easily sees that
\[
[\tilde{L}, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 8\tilde{L} \left(\omega^2 - \frac{\lambda^2}{4} + \lambda H\right) - 16\lambda^2 \tilde{L}^3,
\] (27)

which corresponds to an \(\text{su}_6(2)\) algebra with
\[
\Phi(J_0(J_0 + 1)) = 4 \left(\omega^2 - \frac{\lambda^2}{4} + \lambda H\right)J_0(J_0 + 1) - 4\lambda^2(J_0(J_0 + 1))^2.
\] (28)

For \(\Phi(x)\) to be an increasing function, the condition
\[
x < \frac{1}{2\lambda^2} \left(\omega^2 - \frac{\lambda^2}{4} + \lambda H\right)
\] (29)

has to be satisfied.
2.4 The quadratic Hahn algebra QH(3)

The quadratic Hahn algebra QH(3)\(^\text{13}\)

\[
[K_1, K_2] = K_3, \tag{30}
\]

\[
[K_2, K_3] = A_2 K_2^2 + C_1 K_1 + D K_2 + G_1, \tag{31}
\]

\[
[K_3, K_1] = A_2 (K_1 K_2 + K_2 K_1) + C_2 K_2 + D K_1 + G_2, \tag{32}
\]

can be put in correspondence to an su\(\Phi(2)\) algebra in the special case in which \(C_1 = -1\) and \(D = G_2 = 0\). The equivalence can be seen \(^\text{24}\) by defining the operators

\[
J_1 = J_+ + J_- \quad \frac{1}{2}, \quad J_2 = J_+ - J_- \quad \frac{1}{2i}. \tag{33}
\]

Then the su\(\Phi(2)\) commutation relations can be written as

\[
[J_0, J_1] = i J_2, \quad [J_0, J_2] = -i J_1, \tag{34}
\]

\[
[J_1, J_2] = \frac{i}{2}(\Phi(J_0(J_0 + 1)) - \Phi(J_0(J_0 - 1))). \tag{35}
\]

Subsequently one can see that the two algebras are equivalent for

\[
K_1 = J_1 + A_2 J_0^2 + G_1, \quad K_2 = J_0, \quad K_3 = -i J_2, \tag{36}
\]

and

\[
\Phi(J_0(J_0 + 1)) = -(2A_2 G_1 + C_2)J_0(J_0 + 1) - A_2^2(J_0(J_0 + 1))^2. \tag{37}
\]

For \(\Phi(x)\) to be an increasing function, the condition

\[
x < -\frac{2A_2 G_1 + C_2}{2A_2^2} \tag{38}
\]

has to be obeyed.

2.5 The finite W algebra \(\bar{W}_0\)

It is worth remarking that the finite W algebra \(\bar{W}_0\)\(^\text{14}\)

\[
[U_0, L_0^\pm] = \pm L_0^\pm, \tag{39}
\]

\[
[L_0^+, L_0^-] = (-k(k-1) - 2(k+1)h)U_0 + 2(U_0)^3, \tag{40}
\]
is also an \( \text{su}_\Phi(2) \) algebra with
\[
\Phi(J_0(J_0 + 1)) = \left( -\frac{k(k - 1)}{2} - (k + 1)\hbar \right) J_0(J_0 + 1) + \frac{1}{2}(J_0(J_0 + 1))^2.
\] (41)

For \( \Phi(x) \) to be an increasing function, the condition
\[
x > \frac{k(k - 1)}{2} + (k + 1)h
\] (42)
has to be satisfied.

In all of the above cases the representation theory of the \( \text{su}_\Phi(2) \) algebra immediately follows from eqs. (2)–(4). In each case the range of values of the free parameters is limited by the condition that \( \Phi(x) \) has to be an increasing entire function defined for \( x \geq -1/4 \). The results of this section are summarized in Table 1.

### 3. Generalized deformed parafermionic oscillators

The relation of the above mentioned algebras, and of additional ones, to generalized deformed parafermions is also worth studying.

A deformed oscillator \(^8,^{16}\) can be defined by the algebra generated by the operators \( \{1, a, a^+, N\} \) and the structure function \( \Phi(x) \), satisfying the relations:
\[
[a, N] = a, \quad [a^+, N] = -a^+, \quad (43)
\]
and
\[
a^+a = \Phi(N) = [N], \quad aa^+ = \Phi(N + 1) = [N + 1], \quad (44)
\]
where \( \Phi(x) \) is a positive analytic function with \( \Phi(0) = 0 \) and \( N \) is the number operator. From eq. (44) we conclude that:
\[
N = \Phi^{-1}\left(a^+a\right), \quad (45)
\]
and that the following commutation and anticommutation relations are obviously satisfied:
\[
[a, a^+] = [N + 1] - [N], \quad \{a, a^+\} = [N + 1] + [N]. \quad (46)
\]

The structure function \( \Phi(x) \) is characteristic to the deformation scheme. In Table 2 (cases i–iv) the structure functions corresponding to various deformed oscillators are given.
The generalized deformed algebras possess a Fock space of eigenvectors $|0>, |1>, \ldots, |n>, \ldots$ of the number operator $N$

$$N|n> = n|n>, \ \ <n|m> = \delta_{nm},$$

if the vacuum state $|0>$ satisfies the following relation:

$$a|0> = 0.$$ (48)

These eigenvectors are generated by the formula:

$$|n> = \frac{1}{\sqrt{|n|!}} (a^+)^n |0>,$$ (49)

where

$$[n]! = \prod_{k=1}^{n} [k] = \prod_{k=1}^{n} \Phi(k).$$ (50)

The generators $a^+$ and $a$ are the creation and destruction operators of this deformed oscillator algebra:

$$a|n> = \sqrt{|n||n-1>}, \quad a^+|n> = \sqrt{|n+1||n+1>}.$$ (51)

It has been proved $^{15}$ that any generalized deformed parafermionic algebra of order $p$ can be written as a generalized oscillator with structure function

$$F(x) = x(p+1-x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots),$$ (52)

where $\lambda, \mu, \nu, \rho, \sigma, \ldots$ are real constants satisfying the conditions

$$\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \quad x \in \{1, 2, \ldots, p\}.$$ (53)

3.1 The $su_\Phi(2)$ algebra

Considering an $su_\Phi(2)$ algebra $^9$ with structure function

$$\Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + B(J_0(J_0 + 1))^2 + C(J_0(J_0 + 1))^3,$$ (54)

and making the correspondence

$$J_+ \rightarrow A^+, \quad J_- \rightarrow A, \quad J_0 \rightarrow N,$$ (55)
one finds by equating the rhs of the first of eq. (46) and the last of eq. (1) that the suΦ(2) algebra is equivalent to a generalized deformed parafermionic oscillator of the form

\[ F(N) = N(p + 1 - N) \]

\[ -(p^2(p + 1)C + pB) + (p^3C + (p - 1)B)N \]

\[ +((p^2 - p + 1)C + B)N^2 + (p - 2)CN^3 + CN^4], \]  

(56)

if the condition

\[ A + p(p + 1)B + p^2(p + 1)^2C = 0 \]  

(57)

holds. The condition of eq. (53) is always satisfied for \( B > 0 \) and \( C > 0 \).

In the special case of \( C = 0 \) one finds that the suΦ(2) algebra with structure function

\[ \Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + B(J_0(J_0 + 1))^2 \]  

(58)

is equivalent to a generalized deformed parafermionic oscillator characterized by

\[ F(N) = BN(p + 1 - N)(-p + (p - 1)N + N^2), \]  

(59)

if the condition

\[ A + p(p + 1)B = 0 \]  

(60)

is satisfied. The condition of eq. (53) is satisfied for \( B > 0 \).

Including higher powers of \( J_0(J_0 + 1) \) in eq. (54) results in higher powers of \( N \) in eq. (56) and higher powers of \( p(p + 1) \) in eq. (57). If, however, one sets \( B = 0 \) in eq. (58), then eq. (59) vanishes, indicating that no parafermionic oscillator equivalent to the usual su(2) rotator can be constructed.

3.2 The finite W algebra W0

The \( W_0 \) algebra \(^{14}\) of eqs (39)-(40) is equivalent to a generalized deformed parafermionic algebra with

\[ F(N) = N(p + 1 - N)\frac{1}{2}(-p + (p - 1)N + N^2), \]  

(61)

provided that the condition

\[ k(k - 1) + 2(k + 1)h = p(p + 1) \]  

(62)

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holds. One can easily check that the condition of eq. (53) is satisfied without any further restriction.

3.3 Isotropic harmonic oscillator in a 2-dim curved space

The algebra of the isotropic harmonic oscillator in a 2-dim curved space with constant curvature \( \lambda \) for finite representations can be put in the form \(^{25}\)

\[
F(N) = 4N(p + 1 - N) \left( \lambda(p + 1 - N) + \sqrt{\omega^2 + \lambda^2/4} \right) \left( \lambda N + \sqrt{\omega^2 + \lambda^2/4} \right),
\]

(63)

the relevant energy eigenvalues being

\[
E_p = \sqrt{\omega^2 + \frac{\lambda^2}{4}} (p + 1) + \frac{\lambda}{2}(p + 1)^2,
\]

(64)

where \( \omega \) is the angular frequency of the oscillator. It is clear that the condition of eq. (53) is satisfied without any further restrictions.

3.4 The Kepler problem in a 2-dim curved space

The algebra of the Kepler problem in a 2-dim curved space with constant curvature \( \lambda \) for finite representations can be put in the form \(^{25}\)

\[
F(N) = N(p + 1 - N) \left( \frac{4\mu^2}{(p + 1)^2} + \lambda \frac{(p + 1 - 2N)^2}{4} \right),
\]

(65)

the corresponding energy eigenvalues being

\[
E_p = -\frac{2\mu^2}{(p + 1)^2} + \lambda \frac{p(p + 2)}{8},
\]

(66)

where \( \mu \) is the coefficient of the \(-1/r\) term in the Hamiltonian. It is clear that the restrictions of eq. (53) are satisfied automatically.

3.5 The Fokas–Lagerstrom potential

The Fokas–Lagerstrom potential \(^{17}\) is described by the Hamiltonian

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{x^2}{2} + \frac{y^2}{18}.
\]

(67)

It is therefore an anisotropic oscillator with ratio of frequencies 3:1. For finite representations it can be seen \(^{25}\) that the relevant algebra can be put in the form

\[
F(N) = 16N(p + 1 - N) \left( p + \frac{2}{3} - N \right) \left( p + \frac{4}{3} - N \right)
\]

(68)
for energy eigenvalues \( E_p = p + 1 \), or in the form
\[
F(N) = 16N(p + 1 - N) \left( p + \frac{2}{3} - N \right) \left( p + \frac{1}{3} - N \right)
\] (69)
for eigenvalues \( E_p = p + 2/3 \), or in the form
\[
F(N) = 16N(p + 1 - N) \left( p + \frac{5}{3} - N \right) \left( p + \frac{4}{3} - N \right)
\] (70)
for energies \( E_p = p + 4/3 \). In all cases it is clear that the restrictions of eq. (53) are satisfied.

3.6 The Smorodinsky–Winternitz potential

The Smorodinsky–Winternitz potential \(^{18}\) is described by the Hamiltonian
\[
H = \frac{1}{2}(p_x^2 + p_y^2) + k(x^2 + y^2) + \frac{c}{x^2},
\] (71)
i.e. it is a generalization of the isotropic harmonic oscillator in two dimensions. For finite representations it can be seen \(^{25}\) that the relevant algebra takes the form
\[
F(N) = 1024k^2N(p + 1 - N) \left( N + \frac{1}{2} \right) \left( p + 1 + \frac{\sqrt{1 + 8c}}{2} - N \right)
\] (72)
for \( c \geq -1/8 \) and energy eigenvalues
\[
E_p = \sqrt{8k} \left( p + \frac{5}{4} + \frac{\sqrt{1 + 8c}}{4} \right), \quad p = 1, 2, \ldots
\] (73)
In the special case of \(-1/8 \leq c \leq 3/8 \) and energy eigenvalues
\[
E_p = \sqrt{8k} \left( p + \frac{5}{4} - \frac{\sqrt{1 + 8c}}{4} \right), \quad p = 1, 2, \ldots
\] (74)
the relevant algebra is
\[
F(N) = 1024k^2N(p + 1 - N) \left( N + \frac{1}{2} \right) \left( p + 1 - \frac{\sqrt{1 + 8c}}{2} - N \right).
\] (75)
In both cases the restrictions of eq. (53) are satisfied.

3.7 Two identical particles in two dimensions

Using the same procedure as above, the algebra of eq. (18) can be put in correspondence with a parafermionic oscillator characterized by
\[
F(N) = N(p + 1 - N)64(p + (1 - p)N - N^2),
\] (76)
if the condition
\[ \eta = 4p(p + 1) \]  
holds. However, the condition of eq. (53) is violated in this case.

3.8 The quadratic Hahn algebra QH(3)

For the quadratic Hahn algebra QH(3) of eqs (30)-(32) one obtains the parafermionic oscillator with
\[ F(N) = N(p + 1 - N)A_2\mathbf{2}(p + (1 - p)N - N^2), \]  
if the condition
\[ p(p + 1)A_2^2 + 2A_2G_1 + C_2 = 0 \]  
holds. Again, eq. (53) is violated in this case.

3.9 The finite W algebra $W^{(2)}_3$

The finite W algebra $W^{(2)}_3$\textsuperscript{20–22} is characterized by the commutation relations
\[ [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H^2 + C, \]  
\[ [C, E] = [C, F] = [C, H] = 0. \]
Defining $\tilde{H} = H/2$ these can be put in the form
\[ [\tilde{H}, E] = E, \quad [\tilde{H}, F] = -F, \quad [E, F] = 4\tilde{H}^2 + C, \]  
\[ [C, E] = [C, F] = [C, \tilde{H}] = 0. \]
This algebra is equivalent to a parafermionic oscillator with
\[ F(N) = \frac{2}{3}N(p + 1 - N)(2p - 1 + 2N), \]  
provided that the condition
\[ C = -\frac{2}{3}p(2p + 1) \]  
holds. One can easily see that the condition of eq. (53) is satisfied without any further restriction.
3.10 The Holt potential

The Holt potential
\[ H = \frac{1}{2}(p_x^2 + p_y^2) + (x^2 + 4y^2) + \frac{\delta}{x^2} \]  
(86)
is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. The relevant algebra can be put in the form of an oscillator with
\[ F(N) = 2^{3/2}N(p + 1 - N) \left(p + 1 + \frac{\sqrt{1 + 8\delta}}{2} - N\right), \]  
(87)
where \(1 + 8\delta \geq 0\), the relevant energies being given by
\[ E_p = \sqrt{8} \left(p + 1 + \frac{\sqrt{1 + 8\delta}}{4}\right). \]  
(88)
In this case it is clear that the condition of eq. (53) is always satisfied without any further restrictions.

In the special case \(-\frac{1}{8} \leq \delta \leq \frac{3}{8}\) one obtains
\[ F(N) = 2^{3/2}N(p + 1 - N) \left(p + 1 - \frac{\sqrt{1 + 8\delta}}{2} - N\right), \]  
(89)
the relevant energies being
\[ E_p = \sqrt{8} \left(p + 1 - \frac{\sqrt{1 + 8\delta}}{4}\right). \]  
(90)
The condition of eq. (53) is again satisfied without any further restrictions within the given range of \(\delta\) values.

The deformed oscillator commutation relations in these cases take the form
\[ [N, A^\dagger] = A^\dagger, \quad [N, A] = -A, \]  
(91)
\[ [A, A^\dagger] = 2^{3/2} \left(3N^2 - N \left(4p + 1 \pm \sqrt{1 + 8\delta}\right) + p^2 \pm \frac{1}{2}p\sqrt{1 + 8\delta}\right). \]  
(92)
It can easily be seen that they are the same as the \(W_3^{(2)}\) commutation relations with the identifications
\[ F = \sigma A^\dagger, \quad E = \rho A, \quad C = f(p), \quad H = -2N + k(p), \]  
(93)
where
\[ \rho \sigma = 2^{-19/2}/3, \quad k(p) = \frac{1}{3} \left( 4p + 1 \pm \sqrt{1 + 8\delta} \right), \]  
\[ f(p) = \frac{2}{9} \left( 14p^2 + 4p \pm (7p + 1)\sqrt{1 + 8\delta} + 1 + 4\delta \right). \]  
(94)  
(95)

It is thus shown that the Holt potential possesses the \( W_3^{(2)} \) symmetry.

The results of this section are summarized in Table 2.

4. Discussion

In conclusion, we have shown that several physical systems (two identical particles in two dimensions, isotropic oscillator and Kepler system in a 2-dim curved space) and mathematical structures (quadratic Hahn algebra \( \text{QH}(3) \), finite W algebra \( W_0 \)) are identified with a generalized deformed su(2) algebra, the representation theory of which is known. The results are summarized in Table 1. Furthermore, the generalized deformed parafermionic oscillator is found to describe several physical systems (isotropic oscillator and Kepler system in a curved space, Fokas–Lagerstrom, Smorodinsky–Winternitz and Holt potentials) and mathematical constructions (generalized deformed su(2) algebras, finite W algebras \( \bar{W}_0 \) and \( W_3^{(2)} \)). The results are summarized in Table 2. The framework of the generalized deformed parafermionic oscillator is more general than the generalized deformed su(2) algebra, since in the rhs of the relevant basic commutation relation in the former case (first equation in eq. 46) both odd and even powers are allowed, while in the latter case (eq. 1) only odd powers are allowed.

The relevance of deformed oscillator algebras, finite W algebras and quadratic algebras in the study of the symmetries of the anisotropic quantum harmonic oscillator in two and three dimensions is receiving attention.

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Table 1: Structure functions of generalized deformed su(2) algebras. For conditions of validity for each of them see the corresponding subsection of the text.

| \( \Phi(J_0(J_0 + 1)) \) | Reference |
|--------------------------|-----------|
| i: \( J_0(J_0 + 1) \) | usual su(2) |
| ii: \([J_0][J_0 + 1]_{q}\) | su\(_q\) (2) \(^{5-7}\) |
| iii: \( 16\eta J_0(J_0 + 1) - 64(J_0(J_0 + 1))^2 \) | 2 identical particles in 2-dim \(^{11}\) |
| iv: \( (-2H + \lambda/4) J_0(J_0 + 1) + \lambda(J_0(J_0 + 1))^2 \) | Kepler system in 2-dim curved space \(^{12}\) |
| v: \( 4(\omega^2 - \lambda^2 + \lambda H) J_0(J_0 + 1) - 4\lambda^2(J_0(J_0 + 1))^2 \) | isotropic oscillator in 2-dim curved space \(^{12}\) |
| vi: \(-2A_2G_1 + C_2)J_0(J_0 + 1) - A_2^2(J_0(J_0 + 1))^2 \) | quadratic Hahn algebra QH(3) \(^{13}\) |
| vii: \( \left(-\frac{k(k-1)}{2} - (k + 1)h\right) J_0(J_0 + 1) + \frac{1}{2}(J_0(J_0 + 1))^2 \) | finite W algebra \( \mathcal{W}_0 \) \(^{14}\) |
Table 2: Structure functions of deformed oscillators. For conditions of validity and further explanations in the case of the various generalized deformed parafermionic oscillators see the corresponding subsection in the text.

|   |   | Reference |
|---|---|-----------|
| i | $N$ | harmonic oscillator |
| ii | $q^N_q-q^{-N}_q = [N]_q$ | $q$-deformed harmonic oscillator $^5-^7$ |
| iii | $N(p + 1 - N)$ | parafermionic oscillator $^26$ |
| iv | $[N]_q[ p + 1 - N]_q$ | $q$-deformed parafermionic oscillator $^27$ |
| v | $N(p + 1 - N)(\lambda + \mu N + \nu N^2 + \rho N^3 + \sigma N^4 + \ldots)$ | generalized deformed parafermionic oscillator $^15$ |
| vi | $N(p + 1 - N)[(p^2(p + 1)C + pB) + (p^3C + (p - 1)B)N + (p^2 - p + 1)C + B]N^2 + (p - 2)C N^3 + C N^4]$ | 3-term su$_{\Phi}(2)$ algebra (eq. 54) |
| vii | $BN(p + 1 - N)(-p + (p - 1)N + N^2)$ | 2-term su$_{\Phi}(2)$ algebra (eq. 58) |
| viii | $N(p + 1 - N)\frac{1}{2}(-p + (p - 1)N + N^2)$ | finite W algebra $\tilde{W}_0$ $^{14}$ |
| ix | $4N(p + 1 - N)\left(\lambda(p + 1 - N) + \sqrt{\omega^2 + \lambda^2/4}\right)$ | isotropic oscillator in 2-dim curved space $^{12,25}$ |
| x | $N(p + 1 - N)\left(\frac{4\mu^2}{(p+1)^2} + \lambda(p+1-2N)^2\right)$ | Kepler system in 2-dim curved space $^{12,25}$ |
| xi | $16N(p + 1 - N)\left(p + \frac{3}{2} - N\right)\left(p + \frac{4}{3} - N\right)$ | Fokas–Lagerstrom potential $^{17,25}$ |
|   | or $16N(p + 1 - N)\left(p + \frac{2}{3} - N\right)\left(p + \frac{1}{3} - N\right)$ |   |
|   | or $16N(p + 1 - N)\left(p + \frac{5}{3} - N\right)\left(p + \frac{4}{3} - N\right)$ |   |
| xii | $1024k^2N(p + 1 - N)\left(N + \frac{1}{2}\right)\left(p + 1 \pm \frac{\sqrt{1+8k^2}}{2} - N\right)$ | Smorodinsky-Winternitz potential $^{18,25}$ |
| xiii | $\frac{2}{3}N(p + 1 - N)(2p - 1 + 2N)$ | finite W algebra $W_{3}^{(2)}$ $^{20-22}$ |
| xiv | $2^{3/2}N(p + 1 - N)\left(p + 1 \pm \frac{\sqrt{1+8k^2}}{2} - N\right)$ | Holt potential $^{19,25}$ |