Integrability of the Higher-Order Nonlinear Schrödinger Equation Revisited

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Abstract

Only the known integrable cases of the Kodama-Hasegawa higher-order nonlinear Schrödinger equation pass the Painlevé test. Recent results of Ghosh and Nandy add no new integrable cases of this equation.

Being a true high-technology application of a mathematical object, the optical soliton in fiber optical lines of communications is an ideal carrier of information because the integrability of its model equation, the nonlinear Schrödinger (NLS) equation, provides a guarantee on the input and output causal relation for light waves in fibers [1]. The integrable NLS equation, however, does not govern femtosecond light pulses which have much potential for the future technology. Kodama and Hasegawa [2] derived a model equation for ultra-short light pulses in optical fibers, the higher-order nonlinear Schrödinger (HNLS) equation:

\[ i w_z + \frac{1}{2} w_{yy} + |w|^2 w + i \alpha w_{yyy} + i \beta |w|^2 w_y + i \gamma w(|w|^2)_y = 0, \] (1)

where real parameters \( \alpha, \beta \) and \( \gamma \) are determined by spectral and geometric properties of a fiber. Eq. (1) admits one-soliton solutions of bright and dark
types in wide domains of its parameters [3], but this has no relation to its integrability. Only the following four integrable cases of the HNLS eq. (1) are known besides the NLS equation itself: the derivative NLS equation I [4] ($\alpha : \beta : \gamma = 0 : 1 : 1$), the derivative NLS equation II [5] (0 : 1 : 0), the Hirota equation [6] (1 : 6 : 0), and the Sasa-Satsuma equation [7] (1 : 6 : 3). According to the results of Nijhof and Roelofs [8], only these known integrable cases of the HNLS eq. (1) possess the infinite-dimensional prolongation structures.

The Painlevé analysis leads us to the same conclusion on the integrability of eq. (1). Let us remind in brief our results obtained in [9] in the framework of the Weiss-Kruskal algorithm (for details of the method see e.g. [10] and references therein).

The HNLS eq. (1) with $\alpha = 0$ lies in the class of derivative NLS equations which has been analyzed by Clarkson and Cosgrove [11]. We find from their results [11] that eq. (1) with $\alpha = 0$ has the Painlevé property if and only if $(\beta - \gamma)\gamma = 0$, i.e. exactly when eq. (1) is the derivative NLS equations I and II besides the NLS equation itself.

When $\alpha \neq 0$, we can transform eq. (1) by

$$w(y, z) = u(x, t) \exp\left(\frac{i}{6} \alpha^{-1} x - \frac{i}{216} \alpha^{-3} t\right),$$

$$y = x - \frac{1}{12} \alpha^{-2} t, \quad z = -\alpha^{-1} t$$

into the equivalent complex modified Korteweg-de Vries (CMKdV) equation

$$u_t = u_{xxx} + auu^* u_x + bu^2 u_x^* + icu^2 u^*,$$  

where * denotes the complex conjugation, $a = \alpha^{-1}(\beta + \gamma)$, $b = \alpha^{-1}\gamma$, and $c = \frac{1}{6} \alpha^{-2}(\beta - 6\alpha)$. (It is sometimes overlooked in the literature that the CMKdV eq. (3) with $c = 0$ is equivalent to the HNLS eq. (1) only if $\beta = 6\alpha$ in eq. (1).) Since eq. (3) is a complex equation, we supplement eq. (3) by its complex conjugation, introduce the new variable $v$, $v = u^*$, and then consider $u$ and $v$ as mutually independent. Thus, we have the following system of two nonlinear equations of total order six:

$$u_{xxx} + auu u_x + bu^2 u_x + icu^2 v - u_t = 0,$$

$$v_{xxx} + auv v_x + bu^2 u_x - icuv^2 - v_t = 0.$$  

A hypersurface $\varphi(x, t) = 0$ is non-characteristic for this system if $\varphi_x \neq 0$ (we take $\varphi_x = 1$), and the general solution of eq. (4) must contain six arbitrary functions of one variable. We substitute the expressions $u =$...
\(\varphi^\sigma[u_0(t) + \ldots + u_r(t) \varphi^r + \ldots]\) and \(v = \varphi^\tau[v_0(t) + \ldots + v_r(t) \varphi^r + \ldots]\) into eq. (4) for determining the exponents \(\sigma\) and \(\tau\) of the dominant behavior of solutions and the positions \(r\) of resonances, and find the following two branches if \(a^2 + b^2 \neq 0\) (we reject the case \(a = b = 0, c \neq 0\) because of inadmissible \(r = \frac{1}{2} (5 \pm i\sqrt{87})\)).

- **Branch (A):** \(\sigma = \tau = -1\), \(u_0v_0 = -6(a + b)^{-1}\), \(u_0/v_0\) is arbitrary, \(r = -1, 0, 3, 4, 3 - p, 3 + p\), where
  \[
p = 2 \left(\frac{a - 2b}{a + b}\right)^{1/2}.
  \tag{5}
\]

- **Branch (B):** \(\sigma = -1 \pm n, \tau = -1 \mp n\), \(u_0v_0 = -60(5a - 7b)^{-1}\), \(u_0/v_0\) is arbitrary, \(r = -1, 0, 4, 6, \frac{1}{2}(3 - q), \frac{1}{2}(3 + q)\), where
  \[
n = \left(\frac{5a + 17b}{5a - 7b}\right)^{1/2},
  \tag{6}
\]
  \[
q = \left(\frac{245a + 617b}{5a - 7b}\right)^{1/2}.
  \tag{7}
\]

We reject the following cases: \(a + b = 0\), when the branch (A) does not exist, because of inadmissible \(r = \frac{1}{2} (3 \pm i\sqrt{31})\) in the branch (B); \(5a - 7b = 0\), when the branch (B) does not exist, because of inadmissible \(r = 3 \pm i\) in the branch (A); and \(5a + 17b = 0\), when the branches (A) and (B) coincide, because the double resonance \(r = 0\) and the fact that \(u_0v_0\) is determined require logarithmic terms in the singular expansions. Thus, the two different branches, (A) and (B), exist for all the cases of eq. (3) we have to analyze farther. The existence of the branch (B) was ignored in several works, where some special cases of the CMKdV eq. (3) were tested. The branch (B) was lost as well in [12], where the Painlevé test of the HNLS eq. (1) missed all the integrable cases except the Sasa-Satsuma case.

Eqs. (5), (6) and (7) show that the dominant behavior of solutions in the branch (B) and the positions of two resonances in each of the branches (A) and (B) are determined only by the quotient \(a/b\). Elimination of \(a/b\) from eqs. (5), (6) and (7) leads to the following two equations:

\[
(1 + p^2) (1 + n^2) = 10,
\tag{8}
\]

\[
9 + 40n^2 = q^2.
\tag{9}
\]
The numbers \( p, n \) and \( q \) have to be integer for the CMKdV eq. (3) to possess the Painlevé property. Eqs. (8) and (9) admit three integer solutions: \((p, n, q) = (1, 2, 13), (2, 1, 7), (3, 0, 3)\), but the last one corresponds to the already rejected case \( a/b = -17/5 \). The solution \((2, 1, 7)\) leads to the Hirota case of the HNLS eq. (1): \( b = 0 \) in eq. (3) corresponds to \( \gamma = 0 \) in eq. (1), the usual way of constructing recursion relations and checking compatibility conditions at resonances gives us the condition \( c = 0 \) (i.e. \( \beta = 6\alpha \) in eq. (1)) at \( r = 1 \) in the branch (A), and then compatibility conditions become identities in both branches. The solution \((1, 2, 13)\) leads to the Sasa-Satsuma case of the HNLS eq. (1): \( \beta = 2\gamma \) since \( a = 3b \), the condition \( c = 0 \) (i.e. \( \beta = 6\alpha \)) arises at \( r = 3 \) in the branch (A), and other compatibility conditions are all satisfied.

Consequently, only the known integrable cases of the HNLS eq. (1) pass the Painlevé test for integrability [13]. This completely agrees with the results of Nijhof and Roelofs [8].

Recently, however, Ghosh and Nandy [13] reported that they found a parametric Lax representation for the CMKdV eq. (3) with \( c = 0 \) and any rational value of \( b/a \) from the interval \([0, 1]\). Let us consider in brief the intriguing results of [13].

The Lax pair, proposed in [13], is as follows:

\[
\Psi_x = U \Psi, \quad \Psi_t = V \Psi, \tag{10}
\]

\[
U = -i\lambda \Sigma + A, \tag{11}
\]

\[
V = A_{xx} + AA_x - A_xA - 2A^3 - 2i\lambda \Sigma (A_x - A^2) - 4\lambda^2 A + 4i\lambda^3 \Sigma, \tag{12}
\]

\[
\Sigma = \begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -1
\end{pmatrix}, \tag{13}
\]
where $\lambda$ is a parameter, $u = u(x,t)$, and the dimension of the block-form matrices $\Sigma$ and $A$ is $(l+m+1) \times (l+m+1)$ (note: the expression for $V$ contains some misprints in [13]). This form of $U$ and $V$ provides that the compatibility condition of the system (10), $U_t = V_x - UV + VU$, is as follows:

$$A_t = A_{xxx} - 3(A^2A_x + A_xA^2).$$

(15)

It is stated in [13] that the equation (15) with $A$ (14) is nothing but

$$u_t = u_{xxx} + (6l + 3m)uu^*u_x + 3mu^2u^*_x$$

(16)

and therefore the system (10) with (11), (12), (13) and (14) represents a Lax pair for eq. (3) with $c = 0$ and a rational value of $b/a$ determined by the dimension of the block matrices.

The real situation, however, is different. There are three distinct cases of what is in fact the equation (15) with $A$ (14), depending on values of $l$ and $m$.

- If $m = 0, l \neq 0$ or $l = 0, m \neq 0$, then eq. (15) is eq. (3) with $b = 0$. This is the Hirota case.
- If $m = l \neq 0$, then eq. (15) is eq. (3) with $a = 3b$. This is the Sasa-Satsuma case.
- If $m \neq 0, l \neq 0, m \neq l$, then eq. (15) is the following over-determined system of two complex evolution equations:

$$u_t = u_{xxx} + (6l + 3m)uu^*u_x + 3mu^2u^*_x,$$
$$u_t = u_{xxx} + (3l + 6m)uu^*u_x + 3lu^2u^*_x.$$  

(17)

Representing $u$ as $u = fe^{ig}$, where $f$ and $g$ are real functions of $x$ and $t$, we get the following equivalent form of the system (17):

$$f_t = f_{xxx} + 6(l + m)f^2f_x, \quad g_x = g_t = 0.$$  

(18)

Though integrable, the system (18) is only a reduction of the CMKdV eq. (3).
Consequently, no new integrable cases of the Kodama-Hasegawa HNLS eq. (1) were found in [13].

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