ON SURFACES WITH ZERO VANISHING CYCLES

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Abstract. We show that using an idea from a paper by Van de Ven one can obtain a simple proof of Zak’s classification of smooth projective surfaces with zero vanishing cycles. This method of proof allows one to extend Zak’s theorem to the finite characteristic case.

Introduction

In his paper [Zak73], Fyodor Zak obtained a complete classification of smooth projective surfaces over \( \mathbb{C} \) for which “Condition (A)” from Exposé XVIII of SGA7 fails to be satisfied (see Definition 1.1 below). It is well known (see, for example, [Lan84, Section 1] or [Lv94, Proposition 6.1]) that for surfaces the violation of Condition (A) is equivalent to triviality of vanishing cycles or (in characteristic zero) to the emptiness of the adjoint linear system.

Zak’s elegant proof, being based ultimately on the theory of degeneration of isolated singularities, does not appear to be directly applicable to the case of finite characteristic.

The aim of this paper is to show that, using an idea from Van de Ven’s article [VdV79], one can produce a simple proof of Zak’s result that is valid over an arbitrary base field.

It would be interesting to learn something about higher-dimensional varieties not satisfying Condition (A), at least in characteristic zero to begin with.

Zak’s result was reproved later (and independently: Antonio Lanteri communicated to me in a letter that he and Palleschi were unaware of Zak’s paper while preparing their articles) by Lanteri and Palleschi [LP81, Proposition 3.1]. Their proof also depends on the char = 0 assumption. In the paper [Lan80] Lanteri uses a construction resembling our proof of Theorem 3.1 to obtain a characterisation of projectively ruled surfaces, but the condition the author imposes on the vanishing cycles (see [Lan80, Section 3, Condition (T)]) is much harder to check than just vanishing.

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has carefully read the first version of this text and made me aware of the related works of Lanteri and Palleschi, as well as of the paper [Fär57]. I am grateful to Antonio Lanteri for sending me scans of his papers and for useful email discussions. Last but not least, I would like to thank Sergei Tabachnikov for providing me with a copy of the paper [Ram74].

1. The condition (A)

Suppose that $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension $n$ over an algebraically closed field and $Y \subset X$ is its smooth hyperplane section. For the case in which the embedding $X \hookrightarrow \mathbb{P}^r$ is Lefschetz (in characteristic zero every embedding is Lefschetz, in characteristic $p$ it means that $X$ is reflexive or the dual variety $X^* \subset (\mathbb{P}^r)^*$ has codimension greater than 1; see Remark 1.2 below), N. Katz introduced in SGA7 the following “Condition (A)”.

**Definition 1.1** (N. Katz). If $X \subset \mathbb{P}^r$ is a Lefschetz embedding and $Y \subset X$ is a smooth hyperplane section, then one says that condition (A) is satisfied for this embedding if either $\dim X^* < r - 1$, where $X^* \subset (\mathbb{P}^r)^*$ is the dual variety, or $\dim X^* = r - 1$ and the homomorphism $i^*: H^{n-1}(X) \to H^{n-1}(Y)$ induced by the embedding $i: Y \hookrightarrow X$ is not an isomorphism.

Here, $H^k(Z)$ means $H^k(Z, \mathbb{Q})$ (singular cohomology) if the base field is $\mathbb{C}$, and $H^k(Z, \mathbb{Q}_\ell)$ ($\ell$-adic cohomology, where $\ell$ is different from characteristic) in general. Lefschetz hyperplane theorem asserts that $i^*$ is always injective; if Condition (A) is satisfied for at least one smooth hyperplane section $Y \subset X$, it is satisfied for any smooth hyperplane section. Finally, Condition (A) is always satisfied if $\dim X$ is odd and $X$ is not a linear subspace. If $\text{codim} X^* = 1$, Condition (A) is equivalent to the assertion that vanishing cycles with respect to the generic (equivalently: at least one) Lefschetz pencil corresponding to the embedding $X \subset \mathbb{P}^r$ are not zero. See [DK73, Exposé XVIII, §5.3 and Theorem 6.3]. Katz and Deligne write that Condition (A) has a strong tendency to hold (“cette condition (A) a une nette tendance à être vérifiée”), so it is interesting to describe the exceptional cases where it is not satisfied.

Denote by $C(X) = \{(x, t) \in X \times (\mathbb{P}^r)^* : H_t \text{ is tangent to } X \text{ at } x\}$ the conormal variety of $X$ ($H_t \subset \mathbb{P}^r$ stands for the hyperplane in $\mathbb{P}^r$ corresponding to the point $t \in (\mathbb{P}^r)^*$ in the dual projective space). Recall (see, for example, [Kaj09, Theorem 1.1]) that a projective variety is called reflexive if the projection $C(X) \to X^* \subset (\mathbb{P}^r)^*$ is separable (so, in characteristic zero everything is reflexive).

**Remark 1.2.** The definition of Lefschetz embeddings in SGA7 (see [DK73, Exposé XVII, Definitions 2.2 and 2.3]) is equivalent to the following. An embedding $X \subset \mathbb{P}^r$, where $X$ is smooth, is Lefschetz if for a general line $L \subset (\mathbb{P}^r)^*$ the following conditions are satisfied.
The $(r - 2)$-dimensional linear subspace $\perp L \subset \mathbb{P}^r$ corresponding to the line $L \subset (\mathbb{P}^r)^*$, is transversal to $X$.

(b) There exists a non-empty Zariski open subset $U \subset L$ such that for each $t \in U$ the hyperplane $H_t$ is transversal to $X$.

(c) If $t_0 \in L$ is such that $H_{t_0}$ is not transversal to $X$, then the scheme $H_{t_0} \cap X$ is a reduced variety with only one singular point, and this point is the ordinary quadratic singularity.

Here, condition (a) follows from a simple version of Bertini theorem, condition (b) means that $L$ is not contained in $X^*$, and condition (c) is satisfied if either $\dim X^* \leq r - 2$ or $\dim X^* = r - 1$ and the set of points $t \in X^*$ over which derivative of the projection morphism $q: C(X) \to X$ has maximal rank everywhere is non-empty. The latter condition is equivalent to the separability of the morphism $q$.

It is well known that if $X$ is a smooth reflexive surface and not a linear subspace, then $X^*$ is a hypersurface. See, for example, [Zak73, Proposition 1], where the proof is valid in arbitrary characteristic provided that $X$ is reflexive, or Landman’s “parity theorem” [Kle86, Theorem II(21)]. Thus, in the two-dimensional case the condition “$X \hookrightarrow \mathbb{P}^r$ is a Lefschetz embedding” is equivalent to the reflexivity of $X$. Note also that in the definition of Condition (A) the ambient projective space can be safely replaced by the linear span of $X$, so in the sequel we may and will assume that $X$ is not contained in a hyperplane.

It is worth mentioning that the non-triviality of Condition (A) had already been observed in the pre-Grothendieck epoch. At least, this “exceptional case” is mentioned explicitly in the paper [Far57] (see the note at the end of p. 37), where the author indicates that the existence of embedded varieties for which Condition (A) does not hold had been known to J. Leray.

2. Two auxiliary results

In this section we state two well-known folklore results about projective surfaces. For the sake of completeness, we sketch the proofs.

**Definition 2.1.** Let us say that a smooth projective surface $X \subset \mathbb{P}^r$ is **projectively ruled** if $X$ is swept by a 1-dimensional family of disjoint lines.

**Proposition 2.2.** If a smooth projective surface $X \subset \mathbb{P}^r$ contains a line $L$ with self-intersection index $(L, L) = 0$, then $X$ is projectively ruled. If $X$ is projectively ruled, then there exists a smooth projective curve $C$ and a locally free sheaf $E$ on $C$ of rank 2 such that $X \cong \mathbb{P}(E)$ and $\mathcal{O}_X(1) \cong \mathcal{O}_{X|C}(1)$.

**Sketch of proof.** It is easy to see that the result of any flat deformation of a line $L \subset X$ is again a line. If $C$ is the connected component of the Hilbert scheme of lines on $X$ which (the component) contains the point corresponding to $L$, then for any deformation $L'$ of the line $L \subset X$ one has $(L', L') = 0$, whence $h^0(N_X|L') = 1$, $h^i(N_X|L') = 0$ for $i > 0$, so $C$ is a smooth projective curve. If $\pi: T \to C$ is the family of lines on $X$ corresponding to
$C$ and $p: T \to X$ is the canonical projection, then it is easy to see that $p$ is separable. Indeed, if $y \in T$ is a closed point and $L = p(\pi^{-1}(y)) \subset X$, then the restriction of derivative of $p$ to the tangent space $T_y T$ is isomorphic onto its image. If $z = \pi(y) \in C$, then in the commutative diagram

$$
\begin{array}{ccc}
T_y T/\mathcal{T}_y\pi^{-1}(z) & \longrightarrow & H^0(L, N_{X|L}) \\
\downarrow & & \downarrow \\
\mathcal{T}_z C & \longrightarrow & 
\end{array}
$$

both the vertical and the diagonal arrow are isomorphisms, whence the horizontal arrow is also an isomorphism, so the mapping $T_y T \to \mathcal{T}_p(y) X$ is non-degenerate. Moreover, $p$ is generically one to one since self-intersection index of each line in the family is 0. Since $X$ is smooth, it follows that $p: T \to X$ is an isomorphism. Identifying $X$ with $T$, it suffices to put $E = \pi_* \mathcal{O}_X(1)$. □

**Proposition 2.3.** Suppose that $X$ is a smooth projective surface and $C \subset X$ is a curve such that $C \cong \mathbb{P}^1$, the self-intersection index $(C, C)$ equals 1, and $C \subset X$ is an ample divisor. Then $X \cong \mathbb{P}^2$.

*Sketch of proof.* Any flat deformation $C' \subset X$ of the curve $C \subset X$ is isomorphic to $\mathbb{P}^1$. Indeed, $\chi(\mathcal{O}_{C'}) = 1$ and $C'$ is irreducible since $(C', C) = 1$ and $C$ is ample. Hence, $h^0(N_{X|C'}) = 2$, $h^i(N_{X|C'}) = 0$ for $i > 0$, so if $B$ is the connected component of the Hilbert scheme of curves on $X$ which (the component) contains the point corresponding to $C$, then $B$ is a (smooth) projective surface. If

$$
\begin{array}{ccc}
T & \overset{q}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
B & & \end{array}
$$

is the standard diagram representing the family of curves on $X$ parametrized by $B$, then, for a general (closed) point $x \in X$, one has $\dim q^{-1}(x) = 1$.

Let $\sigma: \tilde{X} \to X$ be the blowup of $X$ at $x$. Proper transforms (with respect to $\sigma$) of the curves from the family $B$ passing through $x$, are isomorphic to $\mathbb{P}^1$ and have zero self-intersection. Arguing as in the proof of Proposition 2.2 we conclude that $\tilde{X}$ admits a morphism $\pi: \tilde{X} \to C$ onto a smooth curve $C$ such that the fibers of $\pi$ are the above-mentioned proper transforms, all isomorphic to $\mathbb{P}^1$. Restricting $\pi$ to the exceptional curve $E = \sigma^{-1}(x) \subset \tilde{X}$, one concludes that there exists a surjective morphism $E \to C$, whence $C \cong \mathbb{P}^1$ by Lüroth’s theorem. Besides, $E$ is a section of the morphism $\pi$, so $\tilde{X}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \cong C$, so $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ for some $d \geq 0$. Since this $\mathbb{P}^1$-bundle has a section $E$ with self-intersection equal to $-1$, one concludes that $d = 1$; the blowdown of such a section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to $\mathbb{P}^2$, and we are done. □
3. Statement and proof

The following theorem was first proved by Zak [Zak73] over \( \mathbb{C} \).

**Theorem 3.1.** Suppose that \( X \subset \mathbb{P}^r \) is a smooth reflexive surface not lying in a hyperplane. Then \( X \) fails to satisfy Condition (A) if and only if one of the following conditions holds.

(i) \( X = \mathbb{P}^2 \).

(ii) \( X \) is projectively ruled.

(iii) \( X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5 \) (the second Veronese image of \( \mathbb{P}^2 \)).

(iv) \( X \subset \mathbb{P}^4 \) is an isomorphic projection of the surface \( v_2(\mathbb{P}^2) \subset \mathbb{P}^5 \).

Since it is clear that Condition (A) is not satisfied for the plane \( \mathbb{P}^2 \subset \mathbb{P}^r \), from now on we assume that \( X \) is not a linear subspace of \( \mathbb{P}^r \).

Proof of the theorem is based on the following observation.

**Proposition 3.2.** Suppose that \( X \subset \mathbb{P}^r \) is a smooth projective reflexive surface and not a linear subspace of \( \mathbb{P}^r \). Then the following two conditions are equivalent.

(i) Condition (A) fails for the embedding \( X \hookrightarrow \mathbb{P}^r \).

(ii) There exists a hyperplane \( H \subset \mathbb{P}^r \) for which \( H \cap X \) is reduced, reducible, and smooth except for one ordinary quadratic singularity.

Moreover, if (ii) holds, then \( H \cap X = Y_1 \cup Y_2 \), where \( Y_1 \) and \( Y_2 \) are smooth irreducible curves intersecting transversally at one point, and this is also the case for any hyperplane \( H' \subset \mathbb{P}^r \) for which \( H' \cap X \) is smooth except for one ordinary quadratic singularity.

Proof of the proposition. To begin with, recall that any hyperplane section \( Y \subset X \) is connected (see, for example, [Har77, Corollary III.7.9]). Choose a hyperplane \( H \subset \mathbb{P}^r \) for which \( Y = H \cap X \) is smooth except for one ordinary quadratic singularity. The hyperplane \( H \) can be included in a Lefschetz pencil \( L \subset (\mathbb{P}^r)^* \) (see [DK73, Exposé XVII, Definition 2.2]; see also [Lam81], §1.6) for the case of varieties over \( \mathbb{C} \). If \( \tilde{X} \) is the blow-up of \( X \) at the finite set \( L \cap X \) (where \( L \subset \mathbb{P}^r \) is the linear space of codimension 2 corresponding to \( L \subset (\mathbb{P}^r)^* \)), then this Lefschetz pencil is a morphism \( \pi: \tilde{X} \to L \) such that for each (closed) point \( t \in L \), its fiber over \( t \) is isomorphic to \( X \cap H_t \). Recall some basic facts from Picard–Lefschetz theory.

If the base field is \( \mathbb{C} \) and \( Y_0 = X \cap H_t \) has an ordinary quadratic singularity, then for all \( t' \) close enough to \( t \) the intersection \( Y' = X \cap H_{t'} \) is smooth and contains an embedded circle \( c \subset Y' \) (“vanishing cycle”) such that \( Y_0 \) is homeomorphic to \( Y'/c \) and the class \( \delta \) of \( c \) in \( H^1(Y', \mathbb{Q}) \) equals zero if and only if \( b_1(X) = b_1(Y) \). Thus, Condition (A) fails if and only if \( c = 0 \); now it follows from the cohomology exact sequence

\[
H^1(Y', \mathbb{Z}) \to H^1(c, \mathbb{Z}) \to H^2(Y_0, \mathbb{Z}) \to H^2(Y', \mathbb{Z}) \to H^2(c, \mathbb{Z}),
\]

in which the leftmost arrow is zero since the class of \( c \) is zero, that \( b_2(Y_0) = 2 \), whence \( Y_0 = Y_1 \cup Y_2 \) is union of two smooth components intersecting transversally at one point. If, on the other hand, Condition (A) is satisfied,
then the leftmost arrow in (1) is injective, whence \( b_2(Y_0) = b_2(Y) = 1 \) and \( Y_0 \) is irreducible.

In arbitrary characteristic the same argument requires a slightly different wording. As usual, \( \ell \) will denote a prime different from the characteristic; since the base field is algebraically closed, we may and will identify \( H^*(\cdot, \mathbb{Q}_\ell(j)) \) with \( H^*(\cdot, \mathbb{Q}_\ell) \). If a point \( t \in L \) is such that \( X \cap H_t \) has an ordinary quadratic singularity, put, according to SGA7, \( \hat{\mathcal{O}}_{L,t} \) (completion of the local ring) and \( S = \text{Spec} A; \) by \( \bar{\eta} \) denote Spec of the algebraic closure of the field of fractions of \( A \). If \( Y_0 = H_t \cap X, \) \( \pi: \hat{X} \to S \) is the pullback of the morphism \( \pi: \tilde{X} \to S \) with respect to the morphism \( S \to L, \) and \( Y_{\bar{\eta}} \) is the general geometric fiber of \( \hat{\pi} \), then there exists a class \( \delta \in H^1(Y_{\bar{\eta}}, \mathbb{Q}_\ell) \) (the vanishing cycle) and an exact sequence
\[
0 \to H^1(Y_0, \mathbb{Q}_\ell) \to H^1(Y_{\bar{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\cdot, \delta} \mathbb{Q}_\ell \to H^2(Y_0, \mathbb{Q}_\ell) \to H^2(Y_{\bar{\eta}}, \mathbb{Q}_\ell) \to 0
\]
(see [DK73] Exposé XV, Theorem 3.4] or [Del74, 4.3.3]). Now the following conditions are equivalent.

(a) \( \delta = 0. \)
(b) The arrow \( H^1(Y_{\bar{\eta}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell \) in (2) is zero.
(c) \( b_2(Y_s) = b_2(Y_{\bar{\eta}}) + 1. \)
(d) Condition (A) fails for the embedding \( X \subset \mathbb{P}^r. \)

Indeed, since \( Y_{\bar{\eta}} \) is a smooth and connected projective curve, Poincaré duality shows that (a) \( \Leftrightarrow \) (b), the equivalence (a) \( \Leftrightarrow \) (c) follows from (2), and the equivalence (a) \( \Leftrightarrow \) (d) follows from main results of Picard–Lefschetz theory (cokernel of the injection \( H^1(X, \mathbb{Q}_\ell) \to H_1(X \cap H, \mathbb{Q}_\ell) \) is generated by “the” vanishing cycles and all the vanishing cycles are conjugate). Thus, Condition (A) fails if and only if \( b_2(Y_0) = 2, \) so the curve \( Y_0 \) has two irreducible components; since the only singular point of this curve is ordinary quadratic, these components intersect transversally at one point.

If, on the other hand, \( \delta \neq 0, \) then the exact sequence (2) shows that \( b_2(Y_s) = b_2(Y_{\bar{\eta}}) = 1, \) so \( Y_s \) is irreducible.

Now we pass to the proof of Theorem 3.1.

Proof of the “if” part of Theorem 3.1. We are to check that Condition (A) fails for projectively ruled surfaces, the Veronese surface, and its projection. If \( X \subset \mathbb{P}^r \) is projectively ruled, \( p \in X, \) and \( L \) is the line of the ruling passing through \( p, \) then \( L \) is contained in the embedded tangent space \( T_pX \subset \mathbb{P}^r, \) so each hyperplane \( H \) that is tangent to \( X \) at \( p \) must contain \( L. \) Thus, if \( H \cap L \) has only an ordinary quadratic singularity at \( p, \) then the curve \( H \cap X \) contains \( L, \) and since \( L \) has zero self-intersection, \( H \cap X \) must have other components, and Proposition 3.2 shows that Condition (A) fails.

If \( X \) is the Veronese surface or its projection, observe that if \( D \in |O_{\mathbb{P}^2}(2)| \) is a curve with one singularity, then \( D \) is union of two different lines, so \( D \) is reducible and Proposition 3.2 completes the proof again.
Proof of the “only if” part of Theorem 3.1. The main idea of this proof is borrowed from Van de Ven’s paper [VdV79] (see proof of Theorem I therein).

Suppose that Condition (A) fails for a smooth surface $X \subset \mathbb{P}^r$, where $X$ is not a linear space. Proposition 3.2 implies that $X$ has a hyperplane section of the form $Y_1 + Y_2$, where $Y_1$ and $Y_2$ are smooth, irreducible, and intersect transversally at one point. Since $Y_1 + Y_2$ is a hyperplane section of $X$, one has, for $j = 1$ or 2,

$$(Y_j, Y_1 + Y_2) = \deg Y_j > 0;$$

observing that $(Y_1, Y_2) = 1$ (since $Y_1$ and $Y_2$ intersect transversally at one point), one concludes from (3) that $(Y_j, Y_j) > -1$, so

$$(Y_1, Y_1) \geq 0, \quad (Y_2, Y_2) \geq 0.$$

Denote by $V \subset \text{Num}(X) \otimes \mathbb{Q}$, where $\text{Num}(X)$ is the group of divisors on $X$ modulo numeric equivalence, the subspace generated by the classes of $Y_1$ and $Y_2$. Since the curve $Y_1 + Y_2$ has positive self-intersection, it follows from Hodge index theorem that only the following two cases are possible:

(a) $\dim V = 2$ and

$$(Y_1, Y_1)(Y_2, Y_2) < 0;$$

(b) $\dim V = 1$ and classes of $Y_1$ and $Y_2$ are proportional. In case (a), inequality (5) implies that

$$(Y_1, Y_1)(Y_2, Y_2) < 1,$$

so it follows from (4) that at least one of the self-intersection indices $(Y_1, Y_1)$ or $(Y_2, Y_2)$ must be zero. If one of them (say, $(Y_1, Y_1)$) equals zero and the other equals 1, then $\deg Y_1 = (Y_1, Y_1 + Y_2) = 1$, so $Y_1$ is a line with self-intersection index 0, and Proposition 2.2 shows that $X$ is projectively ruled. If they both are zero, then $\deg X = (Y_1 + Y_2, Y_1 + Y_2) = 2$, so $X$ is a quadric, and the smooth two-dimensional quadric is projectively ruled.

In case (b), $Y_2$ is numerically equivalent to $rY_1$, where $r$ must be positive since both $Y_1$ and $Y_2$ are effective divisors. Since

$$(Y_1, Y_2) = r(Y_1, Y_1) = r^{-1}(Y_2, Y_2)$$

and both $(Y_1, Y_1)$ and $(Y_2, Y_2)$ are integers, it follows that $r = (Y_1, Y_1) = (Y_2, Y_2) = 1$ and $Y_1 \sim Y_2$, where $\sim$ means numeric equivalence. Since $Y_1 + Y_2$ is an ample divisor, it follows that $Y_1$ is ample. Now (6) implies that $\deg Y_1 = (Y_1, Y_1) + (Y_1, Y_2) = 2$, so $Y_1$ is a conic, whence $Y \cong \mathbb{P}^1$. Proposition 2.2 implies that $X$ is isomorphic to $\mathbb{P}^2$, and this isomorphism takes $Y_1$ and $Y_2$ to lines since self-intersection indices of these curves equal 1. Thus, $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, whence $X$ is projectively isomorphic to $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or to an isomorphic projection of this surface. It is well known that secant variety of $v_2(\mathbb{P}^2)$ has dimension 4, so an isomorphic projection of $v_2(\mathbb{P}^2)$ must lie in $\mathbb{P}^4$. This completes the proof. \hfill \square
Observe finally that it follows from Theorem 3.1 that if \( \dim X = 2 \) and \( X \) is not a linear subspace, one can put \( d_0 = 2 \) in Corollary 6.4 from [DK73, Exposé XVIII]. Indeed, if \( d \geq 2 \) and \( X \) is a surface satisfying one of the conditions (ii)–(iv) of the theorem, then \( v_d(X) \) does not satisfy any of them.

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