The Capacity Region of the MIMO Interference Channel and its Reciprocity to Within a Constant Gap

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Abstract

The capacity region of the 2-user multi-input multi-output (MIMO) Gaussian interference channel (IC) is characterized to within a constant gap that is independent of the signal-to-noise ratio (SNR) and all channel parameters for the general case of the MIMO IC with an arbitrary number of antennas at each node. For a class of MIMO ICs characterized by a certain relationship between the numbers of antennas at the different nodes this gap is strictly smaller than the gap in a previous result obtained by Telatar and Tse. For instance, the gap for the SIMO IC with single antenna transmitters and \(N\)-antenna receivers obtained here is 1 bit, instead of \(N\) bits. Moreover, in contrast to that previous work, a simple and an explicit achievable coding scheme are given here that have the constant-gap-to-capacity property and in which the sub-rates of the common and private messages of each user are explicitly specified for each achievable rate pair. The constant-gap-to-capacity results are thus proved in this work by first establishing explicit upper and lower bounds to the capacity region. A reciprocity result is also proved which is that the capacity of the reciprocal MIMO IC is within a constant gap of the capacity region of the forward MIMO IC.

Index Terms

Capacity region, Interference channel, MIMO.

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I. INTRODUCTION

The 2-user interference channel is a model for a single-hop, multi-flow wireless network in which multiple transmit-receive pairs communicate over a common noisy channel in that it captures the fundamental interactions between the multiple transmitted signals in such networks, namely, broadcast, superposition and interference. This model was first mentioned in [1], and was studied in a series of works in [2], [3], [4], [5], [6], [7], [8] that considered certain special classes of the IC where the capacity regions of the so-called very strong IC, the strong IC and certain classes of degraded and deterministic ICs, respectively, were established. Different sets of inner and outer bounds considering the embedded multiple-access and broadcast and Z channels were derived in [9], [6], [10], [11], [12], [13]. However, the Han-Kobayashi (HK) [4] coding scheme that improves that of the rate-splitting strategy of [11] remains the best known achievable scheme for this channel. In spite of over 3 decades of research, the capacity region in the general case remained unsolved.

Recent results include the simplified description of the HK rate region due to Chong et al. in [14] (see also [15], [16]) and the capacity regions of new and/or more general classes of channels than for which capacity was previously known, e.g., the sum capacity of the so-called noisy interference channels was found in [17], [18], [19] and the capacity region of the very strong and aligned strong MIMO IC were found in [20]. The common feature of this line of work is that it focuses on a small subset of channel parameters but seeks to solve the challenging problem of obtaining the exact capacity of the channel.

A different line of research was initiated by Etkin et al. [21], where the authors find an approximation of the capacity region of the two-user scalar Gaussian IC where the criterion of approximation is to specify the capacity region to within a constant gap independently of SNR and the direct and cross channel coefficients. Moreover, they obtain that result through a simple HK scheme, i.e., by identifying a single, channel parameter dependent, joint distribution of input and auxiliary random variables among the infinitely many possible specifications including time-sharing that together contribute to the general HK rate region. The key feature of this simple HK scheme is that each user employs independent Gaussian superposition coding of private and public messages with the private message power set so that it reaches the unintended receiver at the noise level. A 1 bit gap to capacity was proved in [21] using the simplified description of the HK rate region of [14]. Thus, the result of [21] characterizes the capacity region to within a constant gap that is independent of the SNR and all channel coefficients. Moreover, it identifies a simple HK scheme that has this property thereby also providing an explicit expression for the achievable rate region in terms of channel parameters.
Since most modern wireless communication systems feature multiple antennas at some or all terminals it is of interest to study the 2-user Gaussian MIMO IC. However, multiple antennas at different nodes make it harder to obtain results similar to those available for the SISO IC. For instance, the deterministic model developed in [22] for the 2-user SISO IC which was shown to reproduce the constant gap to capacity approximation result of [21] doesn’t extend to MIMO channels. Moreover, as compared to the result on the capacity of the strong SISO IC, the capacity of the MIMO IC is known [20] only for the so-called aligned strong interference regime, where the direct and cross link channel matrices satisfy a matrix equation. Two important constraints involved in this result are (a) a covariance constraint on the inputs and (b) the direct link’s channel matrix is a matrix multiple of the cross link’s channel, where the multiple satisfies some particular constraint. The covariance constraint can be relaxed at the cost of making the expression for the capacity region much more complicated, but the matrix relation between the two channel matrices may seldom, if ever, hold. In general, the problem of characterizing the exact capacity of a MIMO IC even for small and special classes can be challenging; this point is also illustrated by [23] where the capacity region of a class of very strong MISO ICs was characterized.

In [24], Telatar and Tse consider an interesting class of two-user semi-deterministic discrete memoryless ICs which generalizes the class of deterministic ICs of [8] and is also applicable to the Gaussian MIMO IC. They obtain outer bounds to the capacity region that are within a gap specified in terms of certain conditional mutual informations to the general HK achievable region [14]. The implication of this work to the 2-user MIMO IC is that the union of all the achievable atomic rate sub-regions of the general HK scheme (one sub-region for each input distribution), is within a constant gap (of $N_i$ bits, where $N_i$ is the number of antennas at receiver $i$) to the outer bound developed therein (which in turn is given as a union over all input and “time-sharing” distributions), and hence, to the capacity region. However, no specific achievable scheme is identified with the constant-gap-to-capacity property among the infinitely many possibilities that make up the the general HK scheme. In fact, it is unclear from that work if there exists a simple HK scheme in general (corresponding to a single input distribution, as it does for the SISO case [21]) or even an explicit HK scheme (whose rate region is the union of rate regions achievable by a finite number of input distributions) with the constant-gap-to-capacity property. Moreover, since the upper and lower bounds are not given explicitly as functions of the channel matrices in [24] they cannot be used for further analysis such as for example, for finding the generalized degrees of freedom (GDoF) analysis, as mentioned in [25].
In this paper, we consider the 2-user Gaussian MIMO IC with an arbitrary number of antennas at each node. Without restricting the channel matrices in any way, we obtain constant-gap-to-capacity characterizations through a simple and an explicit HK scheme, neither of which involves time-sharing. The approach we adopt is as follows: starting from a set of genie-aided strategies that are similar to, but are not the same as, those of [24] (see Remark 1), we establish a set of explicit channel-matrices-dependent upper bounds to the capacity region of the 2-user MIMO IC under input power constraints, i.e., the resulting explicit outer bound on the capacity region does not involve a union over input distributions as does the outer bound in [24]. Consequently, inspired by a novel interpretation of this outer bound, we propose a simple HK coding scheme which involves independent Gaussian linear superposition coding with certain explicit channel dependent covariance matrix assignments for the private and public messages of each user and show that this input distribution produces a rate region that is within a constant gap to the capacity region. Moreover, the explicit bounds obtained here were used by the authors to obtain the GDoF region of the MIMO IC in the companion paper [26].

The above specification of coding scheme does not conform to the specification of [24] on the choice of the conditional distribution of the auxiliary random variables given the inputs. It is thus distinct from any achievable scheme that might result as a consequence of the prescription of [24] (see Remark 8). Moreover, since in the HK coding scheme the public message of a user gets decoded at the receiver of the other user, it is important to choose the sub-rates of the private and public messages of each user carefully because an arbitrary rate for the public message might not be supported if the corresponding cross-link is weak. We thus also specify explicitly the set of these sub-rates for the private and public messages for which such a scenario never arises. In fact, a two-dimensional projection of this latter set actually yields the achievable rate region of the simple HK coding scheme.

The gap to capacity of the aforementioned simple HK coding scheme is then improved by proposing an explicit HK scheme where the transmitters are allowed to use one of three simple superposition coding schemes depending on the operating rate pair. Interestingly, for a large class of MIMO ICs, this latter gap is smaller than the gap of \( N_i \) bits of [24]. This class includes, for example, SIMO ICs (with single-antenna transmitters and multiple antenna receivers) for which the gap is 1 bit, instead of \( N_i \) bits.

Using the explicit expressions for both the achievable rate region and the set of upper bounds to the capacity region of the MIMO IC, we then derive an interesting reciprocity result which is that the capacity of a 2-user MIMO IC is within a constant gap to that of the channel obtained by interchanging the roles of the transmitters and the receivers.

The rest of the paper is organized as follows. Following a description of the notations used in this
paper in Section III we specify the system model. In Section III we derive a set of upper bounds to the capacity region and two different rate regions achievable by one simple and one explicit HK coding scheme. Comparing the set of upper and lower bounds, the capacity region of the MIMO IC is characterized within a constant number of bits. As a byproduct of this analysis, we also prove the reciprocity of the capacity region of the MIMO IC in the approximate capacity sense in Section III-D. Finally, Section IV concludes the paper. In order that the paper is easy to read, many of the proofs are given in the Appendices.

**Notations:** Let $\mathbb{C}$ and $\mathbb{R}^+$ represent the field of complex numbers and the set of non-negative real numbers, respectively. An $n \times m$ matrix with entries in $\mathbb{C}$ will be denoted as $A \in \mathbb{C}^{n \times m}$. The conjugate transpose of the matrix $A$ is denoted as $A^\dagger$ and its determinant as $|A|$. Let $||z||^2$ represents the square of the absolute value of the complex number, i.e., if $z = (x + iy)$ then $||z||^2 = x^2 + y^2$. The trace of the matrix $A \in \mathbb{C}^{n \times n}$ is denoted as $\text{Tr}(A)$, i.e., $\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. $I_n$ represents the $n \times n$ identity matrix, $0_{m \times n}$ represents an all zero $m \times n$ matrix and $\mathbb{U}^{n \times n}$ represents the set of $n \times n$ unitary matrices. The $k^{th}$ column of the matrix $A$ will be denoted by $A[k]$ whereas $A[k_1:k_2]$ represents a matrix whose columns are same as the $k_1^{th}$ to $k_2^{th}$ columns of matrix $A$. $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. The fact that $(A - B)$ is a positive semi-definite (p.s.d.) (or positive definite (p.d.)) matrix is denoted by $A \succeq B$ (or $A \succ B$). $A \otimes B$ denotes the tensor or Kronecker product of the two matrices. If $x_t \in \mathbb{C}^{m \times 1}, \forall$ $1 \leq t \leq n$, then $x^n \triangleq [x_1^\dagger, \ldots, x_n^\dagger]^\dagger$. $\{A, B, C, D\}$ will represent an ordered set of matrices. Moreover, $I(X; Y)$, $I(X; Y|Z)$, $h(X)$ and $h(X|Y)$ represents the mutual information, conditional mutual information, differential entropy and conditional differential entropy of the random variable arguments, respectively. The quantities $x \wedge y$, $x \vee y$ and $x^+$ denote the minimum and maximum between $x$ and $y$ and the $\max\{x, 0\}$, respectively. All the logarithms in this paper are with base 2. The distribution of a complex circularly symmetric Gaussian random vector with zero mean and covariance matrix $Q$ is denoted as $\mathcal{CN}(0, Q)$.

II. CHANNEL MODEL AND MATHEMATICAL PRELIMINARIES

The 2-user MIMO IC is considered where transmitter $i$ ($T x_i$) has $M_i$ antennas and receiver $i$ ($R x_i$) has $N_i$ antennas, respectively, for $i = 1, 2$. Such a MIMO IC will be referred to henceforth as the $(M_1, N_1, M_2, N_2)$ MIMO IC. Let the matrix $H_{ij} \in \mathbb{C}^{N_j \times M_i}$ denote the channel between $T x_i$ and $R x_j$. We shall consider a time-invariant or fixed channel where the channel matrices remain fixed for the entire duration of communication. The $(M_1, N_1, M_2, N_2)$ MIMO IC is depicted in Fig. II. We also incorporate a real-valued attenuation factor, denoted as $\eta_{ij}$, for the signal transmitted from $T x_i$ to receiver $R x_j$. At
time $t$, $Tx_i$ chooses a vector $X_{it} \in \mathbb{C}^{M_i \times 1}$ and sends $\sqrt{P_i}X_{it}$ over the channel, where we assume the following average input power constraint at $Tx_i$,

$$\frac{1}{n} \sum_{t=1}^{n} \text{Tr}(Q_{it}) \leq 1,$$

for $i \in \{1, 2\}$, where $Q_{it} = \mathbb{E}(X_{it}X_{it}^\dagger)$. Note that in the above power constraint $Q_{it}$’s can depend on the channel matrices.

![Fig. 1: The $(M_1, N_1, M_2, N_2)$ MIMO IC.](image)

The received signals at time $t$ can be written as

$$Y_{1t} = \sqrt{\rho_{11}} H_{11} X_{1t} + \sqrt{\rho_{21}} H_{21} X_{2t} + Z_{1t},$$
$$Y_{2t} = \sqrt{\rho_{22}} H_{22} X_{2t} + \sqrt{\rho_{12}} H_{12} X_{1t} + Z_{2t},$$

where $Z_{it} \in \mathbb{C}^{N_i \times 1}$ are i.i.d $\mathcal{CN}(0, I_{N_i})$ across $i$ and $t$, $\rho_{ii} = \eta_{ii}\sqrt{P_i}$ represents the signal-to-noise ratio (SNR) at receiver $i$ and $\rho_{ij} = \eta_{ij}\sqrt{P_i}$ represents the interference-to-noise ratio (INR) at receiver $j$ for $i \neq j \in \{1, 2\}$. In what follows, the MIMO IC with channel matrices, SNRs and INRs as described above will be denoted by $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, where $\mathcal{H} = \{H_{11}, H_{12}, H_{21}, H_{22}\}$ and $\bar{\rho} = [\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}]$. The capacity region of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ will be denoted by $\mathcal{C}(\mathcal{H}, \bar{\rho})$ and is defined as follows.

Let us assume that user $i$ transmits information at a rate of $R_i$ to $Rx_i$ using the codebook $C_{i,n}$ of $n$-length codewords with $|C_{i,n}| = 2^{nR_i}$. Given a message $m_i \in \{1, \ldots, 2^{nR_i}\}$, the corresponding codeword $X_{i}^{n}(m_i) \in C_{i,n}$ must satisfy the power constraint given in equation (1). From the received signal $Y_i^n$, the receiver obtains an estimate $\hat{m}_i$ of the transmitted message $m_i$ using a decoding function $f_{i,n}$, i.e., $f_{i,n}(Y_i^n) = \hat{m}_i$. Let the average probability of error be denoted by $e_{i,n} = \mathbb{E} (\text{Pr}(\hat{m}_i \neq m_i))$.

A rate pair $(R_1, R_2)$ is achievable if there exists a family of codebooks $\{C_{i,n}, 1 \leq i \leq 2\}_n$ and decoding functions $\{f_{i,n}(\cdot), 1 \leq i \leq 2\}_n$ such that $\max_i(e_{i,n})$ goes to zero as the block length $n$ goes to infinity. The capacity region $\mathcal{C}(\mathcal{H}, \bar{\rho})$ of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ is defined as the closure of the set of achievable rate pairs.
**Definition 1:** An achievable rate region is said to be within \(n_i\) bits of the capacity region if for any given rate pair \((R_1, R_2) \in C(H, \bar{\rho})\) the rate pair \(((R_1 - n_1)^+, (R_2 - n_2)^+)\) lies in the achievable region.

### III. Capacity to within a Constant Gap

In this section, we shall characterize the capacity region of the 2-user MIMO IC to within a constant number of bits where the constant is independent of SNRs, INRs and the channel matrices. Such a characterization involves establishing a rate region and showing that no rate pair in the capacity region can be further from all the points in the achievable region by more than this constant. Such a characterization of the capacity region will sometimes be referred as the approximate capacity of the channel and the constant as the gap of approximation. A coding scheme which can achieve a rate region that is within a constant number of bits will be called an approximate capacity optimal (or constant-gap-to-capacity optimal) coding scheme.

In what follows, we shall first obtain a set of explicit upper bounds to the capacity region in terms of the channel matrices. We then give an operational interpretation of these bounds which in turn helps us identify a particular input distribution and linear superposition scheme (by specifying the covariance matrices for the private and public message of each user) leading to a simple HK coding scheme. The achievable rate region of this coding scheme and the corresponding gap to approximate capacity is computed in Section III-B. Comparing these set of upper and lower bounds we prove that the two bounds are within \(n_i\) bits of each other – thus proving the constant gap capacity result – where

\[
n_i = \max \left\{ (m_{ii} \log(M_i) + m_{ij} \log(M_j + 1)), \min\{N_i, M_s\} \log(M_s) \right\} + \hat{m}_{ji}, \quad \text{for } 1 \leq i \neq j \leq 2 \quad (4)
\]

with \(M_s = \max\{M_1, M_2\}, M = (M_1 + M_2), m_{ij} = \min\{M_i, N_j\}\), and \(\hat{m}_{ij} = m_{ij} \log \left( \frac{(M_i + 1)}{M_i} \right) \).

In Section III-C an improvement is proposed by allowing the transmitters to select one of three carefully chosen superposition strategies depending on the rate pair to be achieved. It will be shown that the achievable region of this explicit HK coding scheme is within \(n_i^*\) bits to the capacity region, where

\[
n_i^* = \min\{N_i, M_s\} \log(M_s) + \hat{m}_{ji}, \quad \text{for } 1 \leq i \neq j \leq 2 \quad (5)
\]

Note that on a SIMO IC, \(n_i^* = 1\). Finally, in Section III-D we prove the constant gap reciprocity of the MIMO IC, i.e., the capacity of the 2-user MIMO IC does not change by more than a constant number of bits if the roles of the transmitters and receivers are interchanged.
A. A new outer bound to the capacity region

The set of outer bounds to the capacity region for $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, derived in this paper, will be denoted by $\mathcal{R}^w(\mathcal{H}, \bar{\rho})$. For economy of notation, we define the matrices

$$K_i \triangleq \left( I_{M_i} + \rho_{ij} H_{ij}^H H_{ij} \right)^{-1} \quad 1 \leq i \neq j \leq 2. \quad (6)$$

**Lemma 1 (The Outer Bound):** For a given $\mathcal{H}$ and $\bar{\rho}$ the capacity region, $\mathcal{C}(\mathcal{H}, \bar{\rho})$ of a $2$-user MIMO Gaussian IC, with input power constraint (1), is contained within the set of rate pairs $\mathcal{R}^w(\mathcal{H}, \bar{\rho})$, i.e.,

$$\mathcal{C}(\mathcal{H}, \bar{\rho}) \subseteq \mathcal{R}^w(\mathcal{H}, \bar{\rho}),$$

where $\mathcal{R}^w(\mathcal{H}, \bar{\rho})$ represents the set of rate pairs $(R_1, R_2)$, satisfying the following constraints:

$$R_1 \leq \log \det \left( I_{N_i} + \rho_{11} H_{11}^H H_{11} \right); \quad (7)$$

$$R_2 \leq \log \det \left( I_{N_2} + \rho_{22} H_{22}^H H_{22} \right); \quad (8)$$

$$R_1 + R_2 \leq \log \det \left( I_{N_2} + \rho_{12} H_{12}^H H_{12} + \rho_{22} H_{22}^H H_{22} \right) + \log \det \left( I_{N_i} + \rho_{11} H_{11}^H H_{11} \right); \quad (9)$$

$$R_1 + R_2 \leq \log \det \left( I_{N_i} + \rho_{21} H_{21}^H H_{21} + \rho_{11} H_{11}^H H_{11} \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22}^H H_{22} \right); \quad (10)$$

$$R_1 + R_2 \leq \log \det \left( I_{N_i} + \rho_{21} H_{21}^H H_{21} + \rho_{11} H_{11}^H H_{11} \right) + \log \det \left( I_{N_2} + \rho_{12} H_{12}^H H_{12} + \rho_{22} H_{22}^H H_{22} \right); \quad (11)$$

$$2R_1 + R_2 \leq \log \det \left( I_{N_i} + \rho_{21} H_{21}^H H_{21} + \rho_{11} H_{11}^H H_{11} \right) + \log \det \left( I_{N_i} + \rho_{11} H_{11}^H H_{11} \right) + \log \det \left( I_{N_2} + \rho_{12} H_{12}^H H_{12} + \rho_{22} H_{22}^H H_{22} \right); \quad (12)$$

$$R_1 + 2R_2 \leq \log \det \left( I_{N_i} + \rho_{12} H_{12}^H H_{12} + \rho_{22} H_{22}^H H_{22} \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22}^H H_{22} \right) + \log \det \left( I_{N_i} + \rho_{21} H_{21}^H H_{21} + \rho_{11} H_{11}^H H_{11} \right). \quad (13)$$

**Proof of Lemma 1** The proof is given in Appendix A.

In the following three remarks we describe how the above lemma is distinct from related results in the literature in [21], [24], [27].

**Remark 1:** There are two major differences between the set of upper bounds given in [24] and those provided in Lemma 1. 1) The genie aided signal or side information provided at the different output nodes are different from those of [24]. For instance, to derive the first three sum-rate upper bounds we provide $Rx_i$ with the interference plus noise received at $Rx_j$, i.e., $S_i^n = (I_n \otimes H_{ij}) X_i^n + Z_j^n$ as the genie signal whereas the corresponding genie signal of [24] is some signal which has the same distribution as $S_i^n$ but is conditionally independent of $S_j^n$ given $X_i^n$; and 2) the set of upper bounds of [24] are specified...
as a function of input (and time-sharing) distributions whereas the bounds of Lemma 1 are explicit and valid for any 2-user MIMO IC with the input power constraints in (1).

Remark 2: It was mentioned in [24] that obtaining the approximate capacity results for the MIMO case starting from the result of [21] appears difficult. It seems that the difficulty lies in deriving a tight upper bound for the capacity region in the case when either \( M_i > N_i \) or \( M_i > N_j \neq i \). For example, consider the first sum rate upper bound in [21], where the second user’s codeword is given to the receiver of the first user. Using Fano’s inequality for the MIMO channel we have

\[
n(R_1 + R_2) \leq I(X^n_1; Y^n_1) + n\epsilon_n, \quad \text{[here, } \epsilon_n \to 0 \text{ as } n \to \infty]\]

\[
\leq I(X^n_1; Y^n_2) + I(X^n_2; Y^n_2) + n\epsilon_n,\]

\[
= h(Y^n_1 | X^n_2) - h(Z^n_1) + h(Y^n_2) - h(Y^n_2 | X^n_2) + n\epsilon_n,
\]

which simplifies to

\[
n(R_1 + R_2) \leq h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X^n_1 + Z^n_1) - h(\sqrt{\rho_{12}}(I_n \otimes H_{12})X^n_1 + Z^n_2) + h(Y^n_2) - h(Z^n_1) + n\epsilon_n.
\]

Now \( h(Y^n_2) \) in the above equation can be easily upper bounded by Gaussian inputs and upper bounding the difference between the first two differential entropies is relatively easy when both \( H_{11} \) and \( H_{12} \) are square. A similar concept can be extended to the case when both of these matrices have a larger row dimension than the column dimension, using the singular value decomposition (SVD) of these matrices. In fact, this approach was used in [28] to extend just the generalized degrees of freedom result for SISO Gaussian IC of [21] to the class of symmetric \((M, N, M, N)\) MIMO ICs with \( N \geq M \). However, it can not be applied when \( M_1 > N_1 \) or \( M_1 > N_2 \) because it is not clear how to upper bound the above difference in such a way that it is tight enough to yield a constant gap to capacity or even tight in the weaker sense of generalized degrees of freedom. In the genie-based model, used in the proof in Appendix A, this problem does not arise.

Remark 3: Two sets of explicit upper bounds to the capacity region of the MIMO IC, denoted therein respectively as \( R_0(H, G) \) and \( R_{00}(H, G) \), were derived in [27] from the result in [24]. It can be easily verified that the first four bounds in Lemma 1 are identical to those in [27] and the fifth bound (on \( R_1 + R_2 \)) can be shown to be equivalent to the 7th bound of \( R_0(H, G) \) and \( R_{00}(H, G) \). However, the bounds on \( 2R_1 + R_2 \) and \( R_1 + 2R_2 \) in [27] are incorrect. Figure 4 illustrates this fact by showing that on the SISO IC specified in Example 1 the bound on \( 2R_1 + R_2 \) in \( R_{00}(H, G) \) contradicts the achievability of some of the rate pairs. We elaborate this point further.
Consider the bound on \((2R_1 + R_2)\) in \(R_{00}(H, G)\), which for the SISO channel, can be written as
\[
(2R_1 + R_2) \leq \log(1 + P \|H_{11}\|^2 + P \|H_{21}\|^2) + \log \left( 1 + \frac{P \|H_{11}\|^2}{1 + P \|H_{21}\|^2} \right) \\
+ \log \left( 1 + P \|H_{21}\|^2 + \frac{P \|H_{22}\|^2}{1 + P \|H_{12}\|^2} \right). \tag{14}
\]
By the notation of [21], \(\text{SNR}_i = P_i \|H_{ii}\|^2\) and \(\text{INR}_j = P_i \|H_{ij}\|^2\) for \(i \neq j \in \{1, 2\}\). Using this notation in the above equation we get
\[
(2R_1 + R_2) \leq \log(1 + \text{SNR}_1 + \text{INR}_1) + \log \left( 1 + \frac{\text{SNR}_1}{1 + \text{INR}_2} \right) \\
+ \log \left( 1 + \text{INR}_1 + \frac{\text{SNR}_2}{1 + \text{INR}_2} \right) \triangleq B_{UB}^{AOC}. \tag{15}
\]
The corresponding bound in the achievable region for the weak SISO IC (\(\text{INR}_1 < \text{SNR}_2\) and \(\text{INR}_2 < \text{SNR}_1\)) derived in [21] is given as (e.g., see Theorem 5, equation (61) in [21])
\[
(2R_1 + R_2) \leq \log(1 + \text{SNR}_1 + \text{INR}_1) + \log \left( 2 + \frac{\text{SNR}_1}{\text{INR}_2} \right) \\
+ \log \left( 1 + \text{INR}_1 + \frac{\text{SNR}_2}{\text{INR}_1} \right) - 3 \triangleq B_{LB}^{ETW}. \tag{16}
\]
Comparing the two bounds in equation (15) and (16) we see
\[
B_{UB}^{AOC} < B_{LB}^{ETW} + \log(1 + \text{INR}_1) - \log(1 + \text{INR}_2) + 3,
\]
which implies that if \(\text{INR}_1\) is sufficiently smaller than \(\text{INR}_2\), i.e., \(\log(1 + \text{INR}_1) \leq \log(1 + \text{INR}_2)\) then the upper bound in (15) can be strictly smaller than the lower bound (16). Suppose there exists an IC on which, in addition to the fact that \(B_{UB}^{AOC} < B_{LB}^{ETW}\), the bound (16) is active. This would imply that there exists achievable rate pairs \((R_1, R_2)\) which satisfy equation (16) but violate (15) since \(B_{UB}^{AOC} < B_{LB}^{ETW}\).

**Example 1:** Consider a real SISO IC with \(H_{11} = 45, H_{12} = 25, H_{21} = 3, H_{22} = 30\) and \(P_1 = P_2 = P = 1\). For this channel we have \(\text{SNR}_1 = 2025 > \text{INR}_2 = 625\) and \(\text{SNR}_2 = 900 > \text{INR}_2 = 9\). Clearly, this is a weak interference channel (by the definition of [21]), for which an achievable rate region can be easily computed using equation (61) in Theorem 5 of [21] and is depicted in Fig. 2 as the polygon ABCDEF. On the other hand, putting the values of \(\text{SNR}_i\)'s and \(\text{INR}_j\)'s in equation (15) we get the upper bound on \((2R_1 + R_2)\) from [27], which is also depicted in the figure. According to this bound any point (e.g., point S in the figure), above the line segment PQ is not achievable, which evidently is not true.
B. A new inner bound on the capacity region via a simple achievable scheme

The achievable region of [24], which is specified as a union of sub-regions over all possible input distributions and time-sharing schemes, was shown to be within a constant gap (of $N_i$ bits) of the capacity region. It is not clear from that result as to whether a coding scheme corresponding to a single input distribution, or one that corresponds to time sharing between a few carefully chosen input distributions, can achieve the capacity of the Gaussian MIMO IC within a constant number of bits. In this section, we develop a simple HK coding scheme corresponding to a single joint distribution of input and auxiliary random variables – that does not belong to the class of distributions that would be consistent with the prescription of [24] – but that nevertheless has the desirable property of having a rate region that is within a constant gap to the outer bound of Lemma 1 and hence to the capacity region. First, in Section III-B1 we briefly review the original HK coding scheme [4] and some recent developments in [15], [14] for the discrete memoryless interference channel (DM-IC) and then apply those results to the Gaussian MIMO IC. In Section III-B2, we give a novel interpretation for the outer bound of Lemma 1 which is used as the basis for the specification of a simple achievable scheme in Section III-B3. The description of the simple achievable scheme includes not only the specification of the joint distribution of the input and auxiliary random variables but also the rate split between private and common sub-messages and this is detailed in Section III-B4. An inner bound on the achievable rate region of the simple HK scheme of this section is then obtained in Section III-B5. This inner bound is seen to resemble the outer bound of Lemma 1 from which the constant gap to capacity result is easily deduced.
1) A review of HK achievable region and related work: A concise description of known results in the literature on the achievable regions of the IC is given here which will later be used to connect and contrast them with the results of this work.

On a DM-IC with transition probability \( P(Y_1, Y_2|X_1, X_2) \), for any set \( \mathcal{P}^* \) of probability distributions \( P^* \) which factors as

\[
P^*(Q, U_1, U_2, W_1, W_2, X_1, X_2) = P(Q) P(U_1|Q) P(U_2|Q) P(W_1|Q) P(W_2|Q) P(X_1|U_1, W_1, Q) P(X_2|U_2, W_2, Q),
\]

where \( P(X_1|U_1, W_1, Q) \) and \( P(X_2|U_2, W_2, Q) \) equals either 0 or 1, let

\[
\mathcal{R}_{\text{HK}}^{\alpha}(P^*) \triangleq \mathcal{R}_{\text{HK}}^{(\alpha,1)}(P^*) \cap \mathcal{R}_{\text{HK}}^{(\alpha,2)}(P^*)
\]

represents a set of sub-rate 4-tuples, where

\[
\mathcal{R}_{\text{HK}}^{(\alpha,ij)}(P^*) = \left\{ (r_{1u}, r_{1w}, r_{2u}, r_{2w}) : r_{iu} \leq I(U_i; Y_i|W_i, W_j, Q) \triangleq I_{a_i}; \\
\quad r_{iw} \leq I(W_i; Y_i|U_i, W_j, Q) \triangleq I_{b_i}; \\
\quad r_{jw} \leq I(W_j; Y_i|U_i, W_i, Q) \triangleq I_{c_i}; \\
\quad (r_{iu} + r_{iw}) \leq I(U_i, W_i; Y_i|W_j, Q) \triangleq I_{d_i}; \\
\quad (r_{iu} + r_{jw}) \leq I(U_i, W_j; Y_i|U_i, Q) \triangleq I_{e_i}; \\
\quad (r_{iw} + r_{jw}) \leq I(W_i, W_j; Y_i|U_i, Q) \triangleq I_{f_i}; \\
\quad (r_{iu} + r_{iw} + r_{jw}) \leq I(U_i, W_i, W_j; Y_i|Q) \triangleq I_{g_i} \right\}
\]

for \( i \neq j \in \{1, 2\} \). Further, for a set \( \mathcal{S} \) of 4-tuples \( (r_{1u}, r_{1w}, r_{2u}, r_{2w}) \), let \( \Pi(\mathcal{S}) \triangleq \{(R_1, R_2) : 0 \leq R_i \leq (r_{iu} + r_{iw}), 1 \leq i \leq 2, \text{ for some } (r_{1u}, r_{1w}, r_{2u}, r_{2w}) \in \mathcal{S} \} \). \( \Pi(\mathcal{S}) \) is hence a particular 2-dimensional projection of \( \mathcal{S} \). Then from [4] we have the following theorem.

**Theorem 1 ([4])**: The set

\[
\mathcal{R}_{\text{HK}}^\alpha = \Pi \left( \bigcup_{P^* \in \mathcal{P}^*} \mathcal{R}_{\text{HK}}^{\alpha}(P^*) \right)
\]

is an achievable region for the DM-IC.

**Remark 4**: Let \( U_1 \) \((U_2)\) and \( W_1 \) \((W_2)\) represent the private and common parts of the message to be transmitted by \( T x_1 \) \((T x_2)\), which hereafter will be referred to as the private and common message of \( T x_1 \) \((T x_2)\), respectively. Also, let \( r_{iu} \) and \( r_{iw} \) represent the rates of information carried by \( U_i \) and \( W_i \), respectively, for \( i \in \{1, 2\} \) and let \( X_i \) be constructed from \( U_i \) and \( W_i \) in such a manner that the
The Han-Kobayashi achievable region is given by pairs of the form of \( (r_{1u}, r_{1w}, r_{2u}, r_{2w}) \) \( \in \mathcal{R}^o_{\text{HK}}(P^*) \). Then, Theorem 1 essentially states that, for any \((r_{1u}, r_{1w}, r_{2u}, r_{2w}) \in \mathcal{R}^o_{\text{HK}}(P^*)\) the rate pair \((r_{1u} + r_{1w}, r_{2u} + r_{2w})\) is achievable on the DM-IC.

Thus, for any given \( P^* \), Theorem 1 not only provides a set of achievable rate pairs of the channel in the form of \( \Pi(\mathcal{R}^o_{\text{HK}}(P^*)) \), but also provides the set of 4-tuples from which the rates of their private and public messages can be determined. However, to determine the achievable rate region of the channel, it is necessary to obtain the auxiliary sets \( \mathcal{R}^{(o,1)}_{\text{HK}}(P^*) \) and \( \mathcal{R}^{(o,2)}_{\text{HK}}(P^*) \) first. This indirect method can be avoided by using the equivalent description, denoted as \( \mathcal{R}^e_{\text{HK}}(P^*) \), of \( \Pi(\mathcal{R}^o_{\text{HK}}(P^*)) \) \(^1\) that was obtained by Chong et al. in Lemma 1 of [14], stated below for easy reference.

**Lemma 2 (Lemma 1 in [14]):** For a fixed \( P^* \in \mathcal{P}^* \), let \( \mathcal{R}^e_{\text{HK}}(P^*) \) be the set of rate pairs \((R_1, R_2)\) satisfying:

\[
\begin{align*}
R_1 &\leq I(X_1; Y_1|W_2, Q); \\
R_1 &\leq I(X_1, Y_1|W_1, W_2, Q) + I(W_1; Y_2|X_2, Q); \\
R_2 &\leq I(X_2; Y_2|W_1, Q); \\
R_2 &\leq I(X_2, Y_2|W_1, W_2, Q) + I(W_2; Y_1|X_1, Q); \\
R_1 + R_2 &\leq I(X_2, W_1; Y_2|Q) + I(X_1; Y_1|W_1, W_2, Q); \\
R_1 + R_2 &\leq I(X_1, W_2; Y_1|Q) + I(X_2; Y_2|W_1, W_2, Q); \\
R_1 + R_2 &\leq I(X_1, W_2; Y_1|W_1, Q) + I(X_2, W_1; Y_2|W_2, Q); \\
2R_1 + R_2 &\leq I(X_1, W_2; Y_1|Q) + I(X_1; Y_1|W_1, W_2, Q) + I(X_2, W_1; Y_2|W_2, Q); \\
R_1 + 2R_2 &\leq I(X_2, W_1; Y_2|Q) + I(X_2; Y_2|W_1, W_2, Q) + I(X_1, W_2; Y_1|W_1, Q).
\end{align*}
\]

The Han-Kobayashi achievable region is given by \( \mathcal{R}^e_{\text{HK}} = \bigcup_{P^* \in \mathcal{P}^*} \mathcal{R}^e_{\text{HK}}(P^*) \).

Note that the achievable rate region \( \mathcal{R}^e_{\text{HK}}(P^*) \) in Lemma 2 is now specified directly as a set of rate pairs \((R_1, R_2)\) defined through constraints (20a)-(20i).

The set of rate pairs \((R_1, R_2)\) constrained by all the bounds of equation (20) except (20b) and (20d) was defined and called the “compact rate region” in [14]. In the rest of this paper, we shall denote this set with input distribution \( P^* \) by \( \mathcal{R}^c_{\text{HK}}(P^*) \). Clearly, \( \mathcal{R}^c_{\text{HK}}(P^*) \subseteq \mathcal{R}^e_{\text{HK}}(P^*) \) and it can be shown that there exists channels \( P(Y_1, Y_2|X_1, X_2) \) such that \( \mathcal{R}^c_{\text{HK}}(P^*) \neq \mathcal{R}^e_{\text{HK}}(P^*) \) (e.g., see Fig. 6(b)).

\(^1\)We use the superscript “o” to refer to the original HK coding scheme [4] and “e” to emphasize that \( \mathcal{R}^e_{\text{HK}}(P^*) \) is an equivalent description of \( \Pi(\mathcal{R}^o_{\text{HK}}(P^*)) \).
**Remark 5:** Lemma was proved by showing that for any given $P^* \in \mathcal{P}^*$,

$$
\mathcal{R}_{\text{HK}}^c(P^*) = \Pi \left( (\mathcal{R}_{\text{HK}}^o(P^*)) \right) = \Pi \left( (\mathcal{R}_{\text{HK}}^{(o,1)}(P^*) \cap \mathcal{R}_{\text{HK}}^{(o,2)}(P^*)) \right). 
$$

(21)

An equivalent description for $\Pi(\mathcal{R}_{\text{HK}}^o(P^*))$ was derived earlier in [15] using the Fourier-Motzkin elimination method on the set of constraints given in equation (18) which have two additional constraints on $(2R_1 + R_2)$ and $(R_1 + 2R_2)$ besides those in Lemma 2. Later, in [14] these bounds were shown to be redundant resulting in Lemma 2.

The above discussion is summarized in the following schematic diagram.

$$
\mathcal{R}_{\text{HK}}^{(o,1)}(P^*) \cap \mathcal{R}_{\text{HK}}^{(o,2)}(P^*) \xrightarrow{\Pi()} \Pi(\mathcal{R}_{\text{HK}}^o(P^*)) = \mathcal{R}_{\text{HK}}^c(P^*) \cup_{P^* \in \mathcal{P}^*} \mathcal{R}_{\text{HK}}^c(P^*)
$$

**Remark 6:** Recently, an alternative proof of Theorem 2 of [14] was given in [29]. That proof is based on the fact that the rate region $\mathcal{R}_{\text{in}}(P_1^*)$ is achievable by a single input distribution of the form $P_2^*(W_1, X_1, W_2, X_2, Q) = P(Q)P(X_1|Q)P(X_2|Q)P(W_1|X_1, Q)P(W_2|X_2, Q)$, where $\mathcal{R}_{\text{in}}(P_1^*)$ is the two-dimensional projection, i.e.,

$$
\mathcal{R}_{\text{in}}(P_1^*) = \left\{ (R_1, R_2) : R_i = (r_{iu} + r_{iw}), \text{ and } (r_{1u}, r_{1w}, r_{2u}, r_{2w}) \in \mathcal{R}_{\text{in}}^{(4)}(P_1^*) \right\},
$$

of the set of 4-tuples, denoted as $\mathcal{R}_{\text{in}}^{(4)}(P_1^*)$, defined as (see equations (224)-(233) of [29])

$$
\mathcal{R}_{\text{in}}^{(4)}(P_1^*) = \left\{ (r_{1u}, r_{1w}, r_{2u}, r_{2w}) : r_{iu} \leq I(X_i; Y_i | W_i, W_j, Q); (r_{iu} + r_{iw}) \leq I(X_i; Y_i | W_j, Q); (r_{iu} + r_{jw}) \leq I(U_i, W_j; Y_i | W_i, Q); (r_{iu} + r_{iw} + r_{jw}) \leq I(X_i, W_j; Y_i | Q) \text{ for } i = 1, 2 \right\}.
$$

The expression for $\mathcal{R}_{\text{in}}(P_1^*)$ is denoted as $\mathcal{R}_{\text{CMG}}(P_1^*)$ in Lemma 4 of [14], and was computed in Theorem D of [15]. The expression for $\mathcal{R}_{\text{CMG}}(P_1^*)$ has two additional constraints than those that define $\mathcal{R}_{\text{HK}}^c(P_1^*)$ (e.g., see Lemma 4 of [14] or Theorem D in [15]). In other words, $\mathcal{R}_{\text{in}}(P_1^*) \neq \mathcal{R}_{\text{HK}}^c(P_1^*)$ for all inputs of the form $P_1^*$. However, it was proved in [14] that $\cup_{P_1^* \in \mathcal{P}^*_1} \mathcal{R}_{\text{in}}(P_1^*) = \cup_{P_1^* \in \mathcal{P}^*_1} \mathcal{R}_{\text{HK}}^c(P_1^*)$. This result can be pictorially represented as shown in the following diagram.
Using standard techniques (cf. Chapter 7 of [30]) these discrete-alphabet results can be applied to the Gaussian IC with continuous alphabets. To distinguish them from each other, the rate regions corresponding to \( R_{o}^{c} \text{HK}(P) \), \( R_{o}^{G} \text{HK}(P) \), \( R_{e}^{c} \text{HK}(P) \) and \( R_{e}^{G} \text{HK}(P) \) in the Gaussian IC will be denoted as \( R_{c}^{e} \text{HK}(P) \), \( R_{c}^{G} \text{HK}(P) \), \( R_{c}^{e} \text{MG}(P) \) and \( R_{c}^{G} \text{MG}(P) \), respectively.

Evidently, both the original description of Theorem 1 and the alternative description of Lemma 2 of the HK coding scheme are given as a union of an infinite number of atomic sub-regions, each corresponding to a particular input distribution and time sharing strategy. Since a complete characterization of this region is prohibitively complicated, we seek in some sense a single good input distribution and time sharing strategy. Indeed, we provide a novel and important operational interpretation of the bounds of Lemma 1 through which such a good choice of input distribution becomes apparent, leading to a simple HK coding scheme. Moreover, this simple HK coding scheme has a property of being universally good in that it achieves a rate region that is within a constant number of bits to the set of upper bounds of Lemma 1 independently of SNR and the channel parameters.

2) An interpretation of the outer bound of Lemma 1: The first two bounds in \( \mathcal{R}^u(\mathcal{H}, \rho) \) come from the rate bound on a point-to-point channel. The first term of the third bound given in (9) represents the sum rate upper bound of a 2-user multiple-access channel (MAC) having channel matrices \( H_{i2} \), for \( i = 1, 2 \) and Gaussian input with zero mean and scaled identity matrix as covariance. The second term represents the mutual information on a point-to-point channel whose input covariance matrix is \( K_1 \) (see (6) for the definition of \( K_1 \)). These terms can be given the following operational interpretation. The entire message of \( T_{x2} \) has to be decoded at \( R_{x2} \) and some part of \( T_{x1} \) might be decoded at \( R_{x2} \). Let us call this the public message of the first user, denoted as \( W_1 \) having rate \( R_{1u} \). Subsequently, let us denote the remaining part of the first user’s message by \( U_1 \) having rate \( R_{1u} \) which will be referred to as the private message of the first user. Thus we have \( R_1 = R_{1u} + R_{1u} \). Now, with respect to \( W_1 \) and \( X_2 \), \( R_{x2} \) acts as a MAC and thus has the following upper bound

\[
R_{1w} + R_2 \leq \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right).
\]
On the other hand, since \( U_1 \) has to be decoded at \( R_{x1} \), it has the following point-to-point channel upper bound

\[
R_{1u} \leq \log \det \left( I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger \right),
\]

where \( K_1 \) is the covariance matrix of \( U_1 \). These two bounds together imply the third bound in Lemma 1.

The 4th bound can also be interpreted similarly just by interchanging the role of transmitters. The first term of the fifth bound can be thought as a bound on the private message of \( Tx_1 \) and the public message of \( Tx_2 \) which are to be decoded at \( R_{x1} \), i.e.,

\[
R_{1u} + R_{2w} \leq \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger \right),
\]

where the private message has the same covariance matrix as before. Similarly, the second term in the 5th bound can be interpreted as an upper bound on \( (R_{1u} + R_{2u}) \), and together, they imply the fifth bound. The other terms of the remaining bounds can be similarly interpreted. This interpretation motivates a simple HK scheme, where \( Q \) is a deterministic number (no time-sharing), the \( i \)th user’s message is divided into a private and a public message and the private message has an input covariance matrix proportional to \( K_i \).

3) The simple HK coding scheme: The interpretation of the outer bound in the previous section inspires the following simple coding scheme.

**Definition 2 (Input distribution for the simple coding scheme):** Let the private and public messages of the users be encoded using mutually independent random Gaussian codewords and the overall codeword is a linear or additive superposition of the two, i.e., the transmit signals for any particular channel use can be written as

\[
\begin{align*}
X_1^g &= U_1^g + W_1^g; \\
X_2^g &= U_2^g + W_2^g,
\end{align*}
\]

(22)

where \( U_i^g \sim \mathcal{CN}(0, K_{iu}) \) and \( W_i^g \sim \mathcal{CN}(0, K_{iw}) \), represent symbols of the codewords of the private and public messages of user \( i \), respectively and

\[
\begin{align*}
K_{iu}(\mathcal{H}) &\triangleq \mathbb{E}(U_i^g U_i^g) = \frac{K_i}{M_i} \left( I_{M_i} + \rho_{ij} H_{ij} H_{ij}^\dagger \right)^{-1} \\
K_{iw}(\mathcal{H}) &\triangleq \mathbb{E}(W_i^g W_i^g) = \frac{1}{M_i} (I_{M_i} - K_i).
\end{align*}
\]

(23) (24)

The scaling by \( \frac{1}{M_i} \) is required to satisfy the power constraint \( \mathbb{I} \). In the sequel, we shall refer to such a superposition coding scheme where the covariance matrices of the private and public messages of user \( i \) are given by \( K_{iu} \) and \( K_{iw} \) will be referred to as the \( \mathcal{HK}(\{K_{iu}, K_{iw}, K_{iu}, K_{iw}\}) \) scheme. In particular,
when $K_{iu}$ and $K_{iw}$ are as in equation (23) and (24), respectively the coding scheme will be denoted as $\mathcal{HK}^s$ (the superscript $s$ stands for simple). Let us denote the distribution of the random variables as defined above by $P_s(U^g_1, W^g_1, X_1, U^g_2, W^g_2, X_2)$. Clearly, $P_s(U^g_1, W^g_1, X_1, U^g_2, W^g_2, X_2) \in \mathcal{P}^*$. 

Fig. 3: The equivalent virtual channel for the simple HK coding scheme.

Remark 7: The above choice ensures that the private message of user $i$, the covariance of the contribution of which at $Rx_j$ (namely, $\sqrt{\rho_{ij}^H H_{ij} U^g_i}$), is given by

$$\rho_{ij}^H H_{ij} K_{iu} H_{ij}^\dagger = \frac{\rho_{ij}}{M_i} H_{ij} \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} H_{ij}^\dagger \leq I_{M_i},$$

and thus reaches the unintended receiver below the noise floor. Thus the simple achievable scheme here when specialized to the SISO IC embodies the key principle in the achievable scheme of [21]. It is of course applicable much more generally to MIMO ICs and cannot be as such inferred in its general form from just that principle alone. Moreover, we will also soon see that even when specialized to the SISO IC the details of the power split between the public and private messages resulting from this work are different from that in [21].

The $\mathcal{HK} (\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme thus effectively divides each user into two virtual users as shown in Fig. 3. Note that the interference links from the first virtual user to $Rx_2$ and the fourth virtual user to $Rx_1$ are made very weak so that any signal along those links always reaches the receivers below noise floor. As shown in the figure, the channel can be thought as two interfering MACs where $Rx_i$ jointly decodes $U^g_i$, $W^g_i$ and $W^g_{j \neq i}$, treating $U^g_j$ as noise for $1 \leq i \neq j \leq 2$. 

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Applying Theorem 1 and Lemma 2 for the Gaussian IC and evaluating it for the distribution, $P_s(.)$ of Definition 2 we get the following achievable region for the 2-user MIMO Gaussian IC.

**Lemma 3:** On a 2-user Gaussian MIMO IC, the simple $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme can achieve the rate region, $\mathcal{R}_\mathcal{HK}^G(P_s)$, which is a set of rate pairs ($R_1, R_2$) where $R_i$'s satisfy the following constraints

\[
\begin{align*}
R_1 &\leq I(X_1^g; W_1^g | Y_1^g) ; \\
R_1 &\leq I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) ; \\
R_2 &\leq I(X_2^g; Y_2^g | W_2^g) ; \\
R_2 &\leq I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g) ; \\
R_1 + R_2 &\leq I(X_1^g, W_1^g; Y_1^g) + I(X_2^g, Y_2^g | W_1^g, W_2^g) ; \\
R_1 + R_2 &\leq I(X_1^g, W_2^g; Y_1^g) + I(X_2^g, Y_2^g | W_1^g, W_2^g) ; \\
R_1 + R_2 &\leq I(X_1^g, W_1^g; Y_2^g) + I(X_2^g, W_1^g, W_2^g) + I(Y_1^g | X_1^g) ; \\
2R_1 + R_2 &\leq I(X_1^g, W_2^g; Y_1^g) + I(X_1^g, Y_1^g | W_1^g, W_2^g) + I(X_2^g, W_1^g, W_2^g) ; \\
R_1 + 2R_2 &\leq I(X_1^g, W_2^g; Y_2^g) + I(X_2^g, Y_2^g | W_1^g, W_2^g) + I(Y_1^g | X_1^g) ,
\end{align*}
\]

where $Y_i^g$'s are the outputs of the 2-user MIMO IC when its inputs are Gaussian as stated in Definition 2.

Further,

\[
\mathcal{R}_\mathcal{HK}^G(P_s) = \Pi \left( \mathcal{R}_\mathcal{HK}^{G_{\alpha,1}}(P_s) \cap \mathcal{R}_\mathcal{HK}^{G_{\alpha,2}}(P_s) \right) ,
\]

where

\[
\begin{align*}
\mathcal{R}_\mathcal{HK}^{G_{\alpha,i}}(P_s) &= \{ r_{1u}, r_{1w}, r_{2u}, r_{2w} : r_{i} &\leq I(X_i^g; Y_i^g | W_j^g, W_j^g) ; \\
r_{i} &\leq I(W_i^g; Y_i^g | U_j^g, W_j^g) ; \\
r_{j} &\leq I(W_j^g; Y_i^g | W_j^g, U_i^g) ; \\
(r_{i} + r_{i}) &\leq I(X_i^g; Y_i^g | W_j^g) ; \\
(r_{i} + r_{j}) &\leq I(U_i^g, W_j^g; Y_i^g | W_j^g) ; \\
(r_{i} + r_{j}) &\leq I(W_i^g, W_j^g, Y_i^g | U_i^g) ; \\
(r_{i} + r_{i} + r_{j}) &\leq I(U_i^g, W_i^g, W_j^g, Y_i^g) \}
\end{align*}
\]
for $i \neq j \in \{1, 2\}$ and

$$I(X_i^g; Y_i^g | W_{ij}^g, W_{ij}^g) = \log \det \left( I_{N_i} + \rho_{11} H_{11} K_{1u} H_{11}^\dagger + \rho_{21} H_{21} K_{2u} H_{21}^\dagger \right) - \tau_{21}; \quad (27)$$

$$I(W_1^g; Y_1^g | W_2^g, U_i^g) = \log \det \left( I_{N_i} + \rho_{11} H_{11} K_{1u} H_{11}^\dagger + \rho_{21} H_{21} K_{2u} H_{21}^\dagger \right) - \tau_{21}; \quad (28)$$

$$I(W_2^g; Y_1^g | X_i^g) = \log \det \left( I_{N_i} + \frac{\rho_{21}}{M_2} H_{21} H_{21}^\dagger \right) - \tau_{21}; \quad (29)$$

$$I(X_1^g; Y_1^g | W_2^g) = \log \det \left( I_{N_i} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} K_{2u} H_{21}^\dagger \right) - \tau_{21}; \quad (30)$$

$$I(X_2^g, W_2^g; Y_1^g | W_1^g) = \log \det \left( I_{N_i} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_{1u} H_{11}^\dagger \right) - \tau_{21}; \quad (31)$$

$$I(W_1^g, W_2^g; Y_1^g | U_i^g) = \log \det \left( I_{N_i} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_{1u} H_{11}^\dagger \right) - \tau_{21}; \quad (32)$$

$$I(X_1^g, W_2^g; Y_1^g) = \log \det \left( I_{N_i} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger \right) - \tau_{21}, \quad (33)$$

$\tau_{ij} = \log \det (I_{N_i} + \rho_{ij} H_{ij} K_{iu} H_{ij}^\dagger)$ for $i \neq j \in \{1, 2\}$ and $I(X_i^g, Y_i^g | W_i^g, W_i^g)$ through $I(X_2^g, W_1^g; Y_1^g)$ are obtained by swapping the indexes 1 and 2 in the above set of equations, where $K_{iu}$ and $K_{iw}$ are given by equation (23) and (24), respectively for $1 \leq i \leq 2$.

Proof: Equations (25) and (26) result from a simple application of the DM-IC result of Lemma 2 and Theorem 1 to the Gaussian channel. Equations (27)–(33) are obtained by evaluating the different mutual information terms in equations (18) for the given distribution of $U_1^g, U_2^g, W_1^g, W_2^g, X_1^g$ and $X_2^g$ in Definition 2.

Remark 8: In this remark we substantiate the claim that the simple coding scheme of Definition 2 does not fall into the set of admissible coding schemes of 24 the union of whose rate regions constitutes the achievable rate region therein. In particular, suppose we consider the input to be distributed as Gaussian, i.e., $X_i \sim CN(0, \Sigma_i)$ where $\Sigma_i$ satisfies the power constraint (1) for both $i = 1, 2$. Then according to the coding scheme of 24 the random variable $W_i$ (note this would be denoted as $U_i$ in 24) which corresponds to the common message of the $i$-th user, is conditionally independent of, but identically distributed as, $S_i$ given $X_i$. This in turn implies that the conditional distribution of $W_i$ is $CN(\sqrt{\rho_{ij}} H_{ij} X_i, I_{N_i})$ and its marginal distribution is hence $CN \left( 0, (I + \rho_{ij} H_{ij} \Sigma_i H_{ij}^\dagger) \right)$. Note that, depending on the channel matrices, the covariance matrix of $W_i$ can itself be larger (in partial order) than the covariance of the input $X_i$ (e.g., when $H_{ij} = I$). Therefore, $W_i$ cannot be a signal that can propagate through the channel. In the admissible coding schemes of 24 therefore, unlike in the coding scheme of Definition 2 $W_i$ serves as an auxiliary random variable which is used to generate the transmitted codeword for the user. In other words, the prescription of 24 can not lead to any explicit additive
superposition coding scheme specified by Definition 2.

4) The rate splitting strategy for the simple HK coding scheme: Note that in the simple HK coding scheme, each user has 2 messages: a private message $U_i^g$, which is to be decoded at its own receiver, and a common message $W_i^g$, which is to be decoded at both the receivers. Therefore, to achieve a rate pair $(R_1, R_2) \in \mathcal{R}_{HK}^G(P_s)$ the rates of these messages have to be chosen in such a way that each can be decoded at their respective receivers with arbitrarily small probability of error and

$$(r_{iu} + r_{iw}) = R_i, \forall i \in \{1, 2\},$$

(34)

where $r_{iu}$ and $r_{iw}$ are the rates of the private and public messages of user $i$, respectively. The second part of Lemma 3 provides such a set (namely, $\mathcal{R}_{HK}^G(P_s)$) from which these sub-rates can be chosen. Since, $\mathcal{R}_{HK}^G(P_s) = \Pi \left( \mathcal{R}_{HK}^G(P_s) \right)$, for every $(R_1, R_2) \in \mathcal{R}_{HK}^G(P_s)$, by the definition of $\Pi(\cdot)$, there exists at least one 4-tuple $(r_{iu}, r_{iw}, r_{2u}, r_{2w}) \in \mathcal{R}_{HK}^G(P_s)$ such that $(r_{iu} + r_{iw}) = R_i$ for both $i = 1, 2$. On the other hand, by Theorem 1, for any $(r_{iu}, r_{iw}, r_{2u}, r_{2w}) \in \mathcal{R}_{HK}^G(P_s)$, if $r_{iu}$ and $r_{iw}$ represent the rates of information carried by $U_i^g$ and $W_i^g$, respectively, then the simple HK scheme can achieve the rate pair $(r_{iu} + r_{iw}, r_{2u} + r_{2w})$ (recall Remark 3), i.e., $U_i^g$, $W_i^g$ and $W_j^g$ can be decoded at $Rx_i$ with arbitrarily small probability of error, for $i \neq j \in \{1, 2\}$. So, the rate splitting strategy of the HK($\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\}$) scheme can be summarized as follows.

For any $(R_1, R_2) \in \mathcal{R}_{HK}^G(P_s)$ and the simple HK coding scheme of Definition 2 choose $(r_{iu}, r_{iw}, r_{2u}, r_{2w})$ from the atomic sub-region $\mathcal{R}_{HK}^G(P_s)$ in such a way that $(r_{iu} + r_{iw}) = R_i$ (the existence of which is now guaranteed by Lemma 3), assign rate $r_{iu}$ to the private message $U_i^g$ and $r_{iw}$ to the public message $W_i^g$ of user $i$ and then transmit the signals using the additive superposition coding scheme specified by Definition 2. On the decoding side, $Rx_i$ can jointly decode $U_i^g$, $W_i^g$ and $W_j^g$ treating $U_j^g$ as noise for $i \neq j \in \{1, 2\}$, with vanishing probability of error (e.g., see Theorem 1).

Example 2: Consider a 2-user Gaussian $(2, 3, 2, 2)$ IC with $\rho = [20, 8, 12, 20]$ dB, where the channel matrices are given as follows

$$H_{11} = \begin{bmatrix} 1.1975 - 0.4385i & -0.0902 + 0.1895i \\ 0.3234 - 1.3614i & 0.1330 - 0.2564i \\ 0.7546 - 1.0080i & -0.3205 - 0.6958i \end{bmatrix} \quad H_{21} = \begin{bmatrix} 0.3816 - 0.8508i & 0.4450 - 0.4386i \\ -0.4892 - 0.2179i & -0.5346 - 0.1519i \\ 0.7665 - 1.0875i & 0.1689 + 0.7651i \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} 0.9652 - 0.8085i & -0.3033 + 0.0055i \\ 0.6130 + 1.4479i & 0.6872 + 0.5280i \end{bmatrix} \quad H_{22} = \begin{bmatrix} -0.1209 - 0.4575i & -0.0040 + 0.0921i \\ -0.5730 + 1.1118i & -0.8223 - 0.5687i \end{bmatrix}$$

In Fig. 4, the dotted line represents the rate region achievable by the simple HK scheme and the solid line represents the superset $\mathcal{R}^u(\mathcal{H}, \rho)$ which contains the capacity region of the channel.
5) The constant gap result for the simple HK scheme: It is not unreasonable to imagine that the gap between the boundaries of the achievable rate region and the set $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ can behave arbitrarily, including becoming unbounded sometimes, as a function of the channel matrices. However, in what follows we shall show that this gap actually remains bounded and can not be larger than a constant which is independent of the SNR, INR or the channel coefficients. This fact will be proved by showing that $\mathcal{R}_{HK}^{G_e}(P_s)$ contains a subset which is within a constant number of bits to the set of upper bounds. The following lemma specifies this subset.

Lemma 4: The achievable rate region of the simple $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme employed on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, contains the region $\mathcal{R}_a(\mathcal{H}, \bar{\rho})$, which is a set of non-negative rate pairs satisfying

![Fig. 4: An achievable rate region of the simple HK scheme.](image-url)
the following constraints:

\[
R_1 \leq \left( \log \det \left( I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger \right) - n_1 \right)^+; \\
R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger \right) - n_2 \right)^+; \\
R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) - (n_1 + n_2) \right)^+; \\
R_1 + R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (n_1 + n_2) \right)^+; \\
R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (n_1 + n_2) \right)^+; \\
2R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (2n_1 + n_2) \right)^+; \\
R_1 + 2R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right) + \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) - (n_1 + 2n_2) \right)^+, \\
\]

where \( K_i \)'s are as specified before (see equation (6)) and \( n_i \)'s are given by equation (4) for \( 1 \leq i \leq 2 \).

**Proof:** The proof is given in Appendix C. It is shown there that \( \mathcal{R}_a(\mathcal{H}, \bar{\rho}) \) describes a polygon which is completely inside \( \mathcal{R}_{HK}^G(P_a) \), and is hence achievable by \( \mathcal{HK}(\mathcal{s}) \). The idea is to show that the bounds in (35)-(46) are obtained by replacing the right hand sides of (25a)-(25i) by their respective lower bounds (which in turn are the right hand sides of the bounds describing \( \mathcal{R}_a(\mathcal{H}, \bar{\rho}) \) in this lemma). Appendix C establishes these lower bounds.

Note that each bound of Lemma 6 differs from the corresponding bound in Lemma 1 only by a constant, from which we get the following constant gap to capacity result.

**Theorem 2:** The rate region \( \mathcal{R}_a(\mathcal{H}, \bar{\rho}) \) which is achievable by the simple HK scheme \( \mathcal{HK}(\{K_{1w}, K_{1w}, K_{2u}, K_{2w}\}) \), is within \( n_i \) bits to the capacity region of the Gaussian MIMO IC, where \( n_i \) is given by equation (4).

**Proof:** We need to prove that for any given \((R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\rho})\), there exists a rate pair \((\hat{R}_1, \hat{R}_2) \in \mathcal{R}_a(\mathcal{H}, \bar{\rho})\) such that \( \hat{R}_i \geq R_i - n_i \) for \( 1 \leq i \leq 2 \), or equivalently, \( ((R_1 - n_1)^+, (R_2 - n_2)^+) \in \mathcal{R}_a(\mathcal{H}, \bar{\rho}) \).

This can be proved using Lemma 1 and 4 as follows. The proof is by contradiction. Using Lemma 1
we have

\[(R_1, R_2) \in C(\mathcal{H}, \rho) \implies (R_1, R_2) \in \mathcal{R}^u(\mathcal{H}, \rho)\].

Now, denoting \(\hat{R}_i = (R_i - n_i)^+\) for \(i = 1, 2\), let us assume that \((\hat{R}_1, \hat{R}_2) \notin \mathcal{R}^u(\mathcal{H}, \rho)\). This implies that one or more of the bounds of Lemma 4 are not satisfied by the rate pair \((\hat{R}_1, \hat{R}_2)\).

Without loss of generality, we assume that \(R_i \geq n_i, \forall i\) because the other case follows trivially and the 3\(^{rd}\) bound is not satisfied, i.e.,

\[
(\hat{R}_1 + \hat{R}_2) = (R_1 + R_2 - (n_1 + n_2)) ,
\]

\[
> \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{11} H_{11} K H_{11}^\dagger \right) - (n_1 + n_2); \]

\[
\implies (R_1 + R_2) > \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{11} H_{11} K H_{11}^\dagger \right). \]

However, this implies that \((R_1, R_2) \notin \mathcal{R}^u(\mathcal{H}, \rho)\), which clearly is a contradiction. \(\blacksquare\)

C. An explicit coding scheme to achieve a smaller gap

In this section, we propose what we call an explicit HK coding scheme that can be guaranteed to have a smaller gap to the capacity region than is possible with the simple HK coding scheme of the previous section.

We begin with a heuristic discussion of what may be limiting the performance of the simple HK scheme. In particular, we argue that it is possible to choose the input covariance matrices for the private and public messages of each user in a rate-dependent manner so as to achieve a rate region that is larger than that of the simple HK scheme which is given as in Lemma 3 but without the 2\(^{nd}\) and 4\(^{th}\) constraints (i.e., the bounds of equation (25b) or (25d)). To this end, we shall first identify the scenarios in which these bounds can be tighter than the corresponding bounds of equations (25a), (25c), (25e)-(25i). It will be shown that by suitably choosing the covariance matrices for the private and public messages of each user, it is possible to ensure that such scenarios never arise. Thus we get a new rate region that is achievable by an explicit coding scheme and in which the rate pairs are constrained only by (25a), (25c) and (25e)-(25i). It will also be shown that this new bigger rate region contains a subset which is within just \(n_i^*\) bits (recall (5)) to the capacity region of the channel, thereby improving the constant gap result of the previous section.
Suppose there exists a rate pair \((R'_1, R'_2)\), such that \(R'_i\)'s satisfy all the constraints of equation (25) but (25b), i.e.,

\[
I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) < R'_1 \leq I(X_1^g; Y_1^g | W_2^g). \tag{47}
\]

The maximum value of \(R_1 \in \mathcal{R}_H^G(P_s)\) in such a scenario is restricted only by the bound in equation (25b). However, comparing the two sides of equation (47) we see that, the first term on the left hand side of equation (47) differs from that on the right hand side only due to the extra \(W_1^g\) in the conditioning. If all of the power is allocated to the private message only (i.e., \(X_1^g = U_1^g\) and \(W_1^g = \phi\)) then the first term on the left hand side alone is equal to the right hand side and equation (47) can not be true. So, some fraction of the total power available at \(T x_1\) is being used to send \(W_1^g\) which decreases the term \(I(X_1^g; Y_1^g | W_1^g, W_2^g)\). However, this decrease is more than the corresponding increase in the second term on the left hand side which in turn also suggests that the cross link from \(T x_1\) to \(R x_2\) is weaker in some sense than the direct link.

Clearly, when equation (47) is true the rate pair \((R'_1, R'_2)\) is not achievable by the simple HK scheme. The main flaw of the encoding technique in the above scenario is therefore that a significant fraction of the power is spent to send some common information \((W_1^g)\) through a weak channel to a receiver \((R x_2)\) where the message is not even desirable. Intuitively it seems that, instead of wasting power on a weak channel, it is better if \(T x_1\) chooses not to send any public information at all, i.e., set \(K_{1w} = 0\) and assign all of its power to the private message. As mentioned earlier, if we put \(X_1^g = U_1^g\) and \(W_1^g = 0\) in equation (47), the strict inequality becomes an equality, i.e.,

\[
I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) = I(U_1^g; Y_1^g | W_2^g) = I(X_1^g; Y_1^g | W_2^g), \tag{48}
\]

and the two bounds in equation (25a) and (25b) become identical. With such a power split it might turn out that the rate pair \((R'_1, R'_2)\) is actually achievable. That this is indeed the case is proved in Lemma 5.

**Example 3 (A case with no common message):** Consider the 2-user Gaussian \((2, 3, 2, 2)\) IC of Example 2. Computing the right hand sides of the bounds in equations (25a) and (25b) for this channel we get

\[
I(X_1^g, Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) = 9.6572 < I(X_1^g, Y_1^g | W_2^g) = 11.8524.
\]

\(^2\)Because, \(I(X_1^g; Y_1^g | W_1^g, W_2^g)\) represents the information carried by the private message of the first user, on the direct link.

\(^3\)\(I(W_1^g; Y_2^g | X_2^g)\), which represents the information carried by only the public message of user 1 on the cross link in the absence of \(X_2\).
In Fig. 5, the dotted line represents the rate region achievable by the simple HK scheme of the previous section and the solid line represents the rate region achievable by the simple HK coding scheme when $T x_1$ uses all its power to send the private message only, i.e., $K_{1u} = 0$. This figure illustrates that it is indeed possible to achieve a rate pair outside the rate region $R_{\text{HK}}^G(P_s)$. For example, on the particular channel of Fig. 5, point A is achievable by the coding scheme $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\})$ but not by the simple $\mathcal{HK}(s)$ scheme.

Fig. 5: Comparison of the achievable rate regions of the simple HK scheme and the HK scheme with no public message for the first user.

**Remark 9:** Fig. 5 points out another salient but important point regarding the usage of full power for the private message only. Note that the achievable rate region of the $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\})$ (the region marked by the solid line in the figure) is not strictly larger than $R_{\text{HK}}^G(P_s)$. For instance, point B in Fig. 5 can not be achieved by the $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\})$ scheme but it can be achieved by the simple HK scheme of the previous section. Thus, it is not helpful to set $W_1 = 0$ whenever (25b) is tighter than (25a).

The above discussion motivates a rate (and channel) dependent covariance splitting strategy for the private and public messages of each user. Before describing it, let us define a rate region $\mathcal{R}_2$ to be a set of rate pairs $(R_1, R_2)$ which satisfy all but the bounds (25b) and (25d) in the achievable rate region of Lemma 3.

**Definition 3 (The rate region $\mathcal{R}_2$):** We call the set of rate pairs $(R_1, R_2)$ that satisfy the following
set of equations, $R_2$:

\begin{align}
R_1 & \leq I(X_1^g; Y_1^g | W_2^g) ; \\
R_2 & \leq I(X_2^g; Y_2^g | W_2^g) ; \\
R_1 + R_2 & \leq I(X_1^g, W_2^g; Y_2^g) + I(X_1^g; Y_1^g | W_1^g, W_2^g) ; \\
R_1 + R_2 & \leq I(X_2^g, W_2^g; Y_2^g) + I(X_2^g; Y_1^g | W_1^g, W_2^g) ; \\
R_1 + 2R_2 & \leq I(X_1^g, W_2^g; Y_2^g) + I(X_2^g; Y_1^g | W_1^g, W_2^g) + I(X_2^g, W_1^g; Y_2^g | W_1^g) ;
\end{align}

and $I(\cdot;\cdot)$’s are given by equation (27)-(33).

Evidently, $R_2 = R_{IIK}^G(P_s)$ (e.g., see Subsection III-B1).

**Definition 4 (The new explicit coding scheme (with rate dependent covariance split))**: Consider a modification of the simple HK coding scheme of Definition 2 where each of the transmitters chooses its covariance split between its private and public messages depending on the rate pair to be achieved in the following manner:

1) $(R_1, R_2) \in R_2$ but $R_i$’s violate constraint (25b): $T_{x_1}$ assigns all its available energy to its private message only, i.e., the coding scheme $\mathcal{HK}((\frac{1}{M_1} I_{M_1}, 0, K_{2u}, K_{2w}) \oplus \mathcal{HK}(s_1)$ is used.
2) $(R_1, R_2) \in R_2$ but $R_i$’s violate constraint (25d): $T_{x_2}$ assigns all its available energy to its private message only, i.e., the coding scheme $\mathcal{HK}((K_{1u}, K_{1w}, \frac{1}{M_2} I_{M_2}, 0) \oplus \mathcal{HK}(s_2)$ is used.
3) $(R_1, R_2) \in R_{IIK}^G(P_s)$, i.e., $R_i$’s violate neither (25b) nor (25d) of the simple HK coding scheme $\mathcal{HK}((K_{1u}, K_{1w}, K_{2u}, K_{2w})$ is used, where $K_{iu}$ and $K_{iw}$ are chosen according equation (23) and (24) for both $i = 1, 2$.

An HK coding scheme which uses mutually independent Gaussian codewords to encode the private $(U_i)$ and public $(W_i)$ messages of each user, where the covariance matrices for the different messages are chosen as described above, will be referred to as the explicit HK coding scheme and will be denoted by $\mathcal{HK}$.

**Lemma 5**: The explicit HK coding scheme $\mathcal{HK}$ employed on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ has an achievable rate region that contains the rate region $R_2$.

\(^4\)It will be shown in the proof of Lemma 5 that there does not exist any rate pair $(R_1, R_2) \in R_2$ which violates both (25b) and (25d) simultaneously.
**Proof outline:** The detailed proof is given in Appendix D which makes precise the heuristic argument in the beginning of this section that lead to the definition of the explicit HK scheme of Definition 4. We provide here an outline.

We know that every point in $\mathcal{R}_{HK}^G(P_s)$ is achievable by the $\mathcal{HK}^{(s)}$ scheme. Using this result we prove the lemma by showing that every rate pair that lies in $\mathcal{R}_2$ but not in $\mathcal{R}_{HK}^G(P_s)$ can also be achieved by the explicit coding scheme $\hat{\mathcal{HK}}$ of Definition 4.

It can be easily seen that (see Appendix D), when $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}_{HK}^G(P_s)$ both of (25a) and (25d) can not be violated simultaneously because this would imply that $(R_1 + R_2)$ violates equation (29e), thereby contradicting the assumption that $(R_1, R_2) \in \mathcal{R}_2$. On the other hand it can be shown that when (25b) is violated for a given $(R_1, R_2) \in \mathcal{R}_2$, the $\mathcal{HK}^{(s)}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\})$ scheme – in which user one assigns all its power to the private message – can achieve the rate pair. When (25d) is violated, the $\mathcal{HK}^{(s)}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, 0\})$ scheme can achieve the corresponding rate pair. This however is exactly the coding scheme in Definition 4 Therefore, the $\hat{\mathcal{HK}}$ scheme can achieve any rate point in $\mathcal{R}_2$.

It is clear that, depending on the rate pair to be achieved, the explicit HK scheme uses one of the three simple HK coding schemes specified in Definition 4. The corresponding input distribution when $T x_i$ spends all its power to send the private message only, was denoted by $P_{s_i}$ (e.g., see Appendix D) in the proof of Lemma 5 where $P_{s_i}(.) \in \mathcal{P}^s$, for $i = 1, 2$. The achievable rate region of the simple HK scheme with input distribution $P_{s_i}$ is given by $\mathcal{R}_{HK}^G(P_{s_i})$, where $\mathcal{R}_{HK}^G(P_{s_i})$ can be computed as in Lemma 5. As was argued earlier, to achieve a point in this rate region, it is important to choose the sub-rates carefully. In particular, to achieve any rate pair $(R_1, R_2) \in \mathcal{R}_{HK}^G(P_{s_i})$, the corresponding sub-rates for the private and public messages can be chosen from $\mathcal{R}_{HK}^G(P_{s_i})$ since $\mathcal{R}_{HK}^G(P_{s_i}) = \Pi\left(\mathcal{R}_{HK}^G(P_{s_i})\right)$ by Lemma 5. This suggest the following rate splitting strategy for the explicit HK scheme, $\hat{\mathcal{HK}}$.

**Rate splitting strategy for the explicit HK coding scheme:** Depending on the rate pair to be achieved when the input distribution of the coding is $P$, where $P \in \{P_s, P_{s_1}, P_{s_2}\}$, the sub-rates for the different private and public messages are chosen from $\mathcal{R}_{HK}^G(P)$, where $\mathcal{R}_{HK}^G(P)$ can be computed from equation (26) by using distribution $P$ in place of $P_s$.

**Remark 10:** Since the sub-rates are chosen from $\mathcal{R}_{HK}^G(P)$, when $T x_i$ spends all its power to send the private message only, i.e., $W_i^\phi = \phi$, it is expected that $\mathcal{R}_{HK}^G(P)$ should not allow any positive rate for the common message. Putting $W_i^\phi = \phi$ in equation (26b) it can be easily seen that it is indeed the case, i.e., $T_i \leq 0$.

**Remark 11:** It is worth pointing out the differences between the explicit coding scheme of this
paper specialized to the SISO IC and that in [21], where a simple coding scheme was also suggested to characterize the capacity region of the SISO IC within one bit. The authors in [21] use a linear superposition coding scheme where each users private and public messages are encoded using independent Gaussian random codewords with powers \( P_{iu} \) and \( (P_i - P_{iu}) \), respectively for \( i = 1, 2 \). Here, \( P_i \) is the total average power of \( T x_i \) and \( P_{iu} \) depends on the cross channel coefficients as follows (see equation (57) and (58) of [21])

\[
P_{iu} = \min\{P_i, \frac{1}{\|H_{ij}\|^2}\}, \quad i \neq j \in \{1, 2\}.
\] (50)

In the notation of the present paper this coding scheme is identical to \( \mathcal{H}\mathcal{K}(\{\frac{P_{11}}{P_1}, (1 - \frac{P_{11}}{P_1}) \frac{P_{22}}{P_2}, \left(1 - \frac{P_{22}}{P_2}\right)\}) \), when \( P_i \geq \frac{1}{\|H_{ij}\|^2} \). On the other hand, it is identical to \( \mathcal{H}\mathcal{K}(\{1, 0, \frac{P_{11}}{P_1}, (1 - \frac{P_{11}}{P_1})\}) \), when only \( P_1 < \frac{1}{\|H_{12}\|^2} \), \( \mathcal{H}\mathcal{K}(\{\frac{P_{22}}{P_2}, (1 - \frac{P_{22}}{P_2}), 1, 0\}) \), when only \( P_2 < \frac{1}{\|H_{21}\|^2} \) and it is identical to \( \mathcal{H}\mathcal{K}(\{1, 0, 1, 0\}) \), when \( P_i \leq \frac{1}{\|H_{ij}\|^2} \) for both \( i = 1, 2 \). So, depending on the channel coefficients the coding scheme is equivalent to one of the four schemes just described. However, for a given channel the coding scheme and power allocation of [21] is fixed and does not change with the rate pair to be achieved. However, the explicit coding scheme of this paper utilizes one of the three different power splitting schemes depending on the rate pair to be achieved.

Moreover, in contrast to [21], the description of the \( \tilde{\mathcal{H}\mathcal{K}} \) scheme here explicitly specifies the sub-rates of the different messages for each rate pair to be achieved in \( \mathcal{R}_2 \).

**Remark 12:** There is also a subtle difference between the \( \tilde{\mathcal{H}\mathcal{K}} \) scheme and a coding scheme which time shares between the three simple HK schemes of Definition 4. In general, the latter can achieve a larger rate region. This is the case because the \( \tilde{\mathcal{H}\mathcal{K}} \) scheme does not use \( \mathcal{H}\mathcal{K}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\}) \) or \( \mathcal{H}\mathcal{K}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, 0\}) \) to achieve any point that is not inside \( \mathcal{R}_2 \). Whereas, both of \( \mathcal{H}\mathcal{K}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\}) \) and \( \mathcal{H}\mathcal{K}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, 0\}) \) may achieve points which do not lie in \( \mathcal{R}_2 \) but are achievable by a time sharing scheme. Fig. 6 illustrates this point through an example. In particular, Fig. 6(a) depicts the achievable rate regions of the three simple HK schemes for the channel of Example 2. In Fig. 6(b) the rate region bounded by the dotted-dashed line represents the achievable region \( \mathcal{R}_2 \) of the explicit HK scheme and the dashed line represents the rate region, \( \mathcal{R}_{TS} \), achievable by time sharing. Point A in the latter represents a rate pair which lies in the achievable region of \( \mathcal{H}\mathcal{K}(\{\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w}\}) \) and hence also lies in \( \mathcal{R}_{TS} \) but it is clearly outside \( \mathcal{R}_2 \) and hence not achievable by \( \tilde{\mathcal{H}\mathcal{K}} \). There are of course rate pairs in \( \mathcal{R}_{TS} \) that don’t even lie in the union of the three regions of Fig. 6(a) as seen in Fig. 6(b). The advantage of \( \tilde{\mathcal{H}\mathcal{K}} \) however is of course that it does not use time-sharing and moreover, it is not clear that the time-sharing scheme here necessarily leads to improved performance in terms of guaranteeing a
smaller gap to the capacity region.

Fig. 6: Comparison of the achievable rate regions of the explicit scheme and the region achievable by
time sharing among the component schemes, on the channel of Ex. 2

**Remark 13:** Recently in [29], an alternative proof of Theorem 2 of [14] was given (e.g., see Remark 6). The distinguishing aspect of this result from that in [14] is that, the rate region $R_{in}(P^*_1)$ for each $P^*_1$ is achievable by a single input distribution. It might therefore might appear that $R_{HK}^{c}(P^*_1)$ is achievable by a single input distribution and therefore $R_2$ is also achievable by a single input distribution. However, as explained in Remark 6, the expression for $R_{in}(P^*_1)$ has two extra constraints than those that define $R_{HK}^{c}(P^*_1)$. In other words, $R_{in}(P^*_1)$ may not be equal to $R_{HK}^{c}(P^*_1)$ for all inputs of the form $P^*_1$. Therefore, the alternative proof of [29] does not suggest that $R_2$ is achievable by a single input distribution.

From the fact that the rate region $R_2$ is larger than $R_{HK}^{G_e}(P_3)$, it might be possible to identify a subset of $R_2$ whose boundary is also at most a constant number of bits from the set of upper bounds in Lemma 1. The interesting point here is that this constant can now be smaller than $n_i$. Indeed, the following lemma provides such a subset of rate pairs.

**Lemma 6:** Let $R^*_a(H, \bar{\rho})$ be a set of non-negative rate pairs $(R_1, R_2)$ which satisfy the following
constraints:

\[ R_1 \leq \left( \log \det \left( I_{N_i} + \rho_{11} H_{11}^1 H_{11}^\dagger \right) - n_i^* \right)^+; \]
\[ R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{22} H_{22}^1 H_{22}^\dagger \right) - n_2^* \right)^+; \]
\[ R_1 + R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{12} H_{12}^1 H_{12}^\dagger + \rho_{22} H_{22}^1 H_{22}^\dagger \right) \right. \]
\[ + \log \det \left( I_{N_1} + \rho_{11} H_{11}^1 K_1 H_{11}^\dagger \right) - (n_1^* + n_2^*) \left. \right)^+; \]
\[ R_1 + R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{21} H_{21}^1 H_{21}^\dagger + \rho_{11} H_{11}^1 H_{11}^\dagger \right) \right. \]
\[ + \log \det \left( I_{N_1} + \rho_{12} H_{12}^1 H_{12}^\dagger + \rho_{22} H_{22}^1 H_{22}^\dagger \right) - (n_1^* + n_2^*) \left. \right)^+; \]
\[ 2R_1 + R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{21} H_{21}^1 H_{21}^\dagger + \rho_{11} H_{11}^1 H_{11}^\dagger \right) \right. \]
\[ + \log \det \left( I_{N_1} + \rho_{12} H_{12}^1 H_{12}^\dagger + \rho_{22} H_{22}^1 H_{22}^\dagger \right) - (2n_1^* + n_2^*) \left. \right)^+; \]
\[ R_1 + 2R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{12} H_{12}^1 H_{12}^\dagger + \rho_{22} H_{22}^1 H_{22}^\dagger \right) \right. \]
\[ + \log \det \left( I_{N_1} + \rho_{21} H_{21}^1 H_{21}^\dagger + \rho_{11} H_{11}^1 K_1 H_{11}^\dagger \right) - (n_1^* + 2n_2^*) \left. \right)^+; \]

where \( n_i^* \) as defined in (5). Then, \( \mathcal{R}_n^\ast (\mathcal{H}, \bar{\rho}) \) is an achievable rate region on \( \mathcal{IC} (\mathcal{H}, \bar{\rho}) \) and is achievable by the explicit HK coding scheme \( \bar{\mathcal{H}} \mathcal{K} \), i.e.,

\[ \mathcal{R}_n^\ast (\mathcal{H}, \bar{\rho}) \subseteq \mathcal{R}_2. \]

**Proof:** It is sufficient to prove that the set of bounds on the different linear combinations of the rate tuples \( (R_1, R_2) \) in the above lemma are stricter than those given in the expression for \( \mathcal{R}_2 \), i.e., equations (49). Now, on one hand, the bounds of equation (49) are same as those in (25a), (25c) and (25e) - (25i). On the other hand, in the proof of Lemma 4 given in Appendix C, a stricter set of bounds than those in equation (25a) - (25i) was computed (see equations (68) in Appendix C). Clearly, equations in (68a), (68c) and (68e) - (68i) represent a set of stricter bounds than those in (49). The set of rate tuples defined by (68a), (68c) and (68e) - (68i) is a subset of \( \mathcal{R}_2 \). The Lemma then follows from the fact that these bounds are exactly same as those in the lemma.

**Theorem 3:** The achievable rate region \( \mathcal{R}_n^\ast (\mathcal{H}, \bar{\rho}) \), given by Lemma 6, is within \( n_i^* \) bits to the capacity region of the Gaussian MIMO IC, where \( n_i^* \) is given by equation (5).

**Proof:** The proof is identical to that of Theorem 2.
Corollary 1: The achievable rate region $R^*_a(H, \hat{\rho})$ is within $n^*_i$ bits to $R^u(H, \hat{\rho})$.

Proof: Follows from the proof of Theorem 2 with $R_a(H, \hat{\rho})$ replace by $R^*_a(H, \hat{\rho})$ and $n_i$ by $n^*_i$.

Example 4: Capacity of the SIMO IC within 1 bit: On a $(1, N_1, 1, N_2)$ IC, $n^*_1 = n^*_2 = 1$, thus the explicit $\hat{HK}$ scheme can achieve a rate region which is within 1 bit of the capacity region for any SNRs, INRs and the channel vectors. This result is different from that reported in [23] where the exact sum capacity of the strong SIMO IC with $\|H_{ii}\| \leq \|H_{ij}\|$ for $1 \leq i \neq j \leq 2$, was characterized. While [23] provides the exact sum capacity for the strong SIMO IC, our 1 bit approximation is valid for all channel coefficients. Further, this approximation is tighter than that reported in [24] and [25], where the capacity approximation within $N_i$ bits was proved.

Remark 14: Although the above approximate characterization is not always better than that reported in [24], it provides a better approximation for a large class of interference channels. In particular, for all the interference channels on which $n^*_i < N_i$ the explicit $\hat{HK}$ scheme provides a tighter approximation.

Among the other interesting aspects of the approximate characterization of this section are (a) we have a set of explicit expressions for the achievable region and upper bounds to the capacity region, which for instance, can be used for a further analysis such as the evaluation of the generalized degrees of freedom region (which is reported in the companion paper [26]) and the diversity-multiplexing tradeoff (DMT) analysis obtained by the authors in [31] and [32]; and (b) an explicit coding scheme, involving just three linear superposition strategies was shown to be approximate capacity optimal in contrast to the result of [24] where no light is shed on what simple or explicit scheme, if any, out of all possible input distributions and all possible time sharing schemes, would be approximate capacity optimal.

D. Reciprocity of the approximate capacity region

For a communication channel with an unequal number of antennas at the source and destination nodes, how does the capacity (or any other performance metric) change if the information flows in the opposite direction (i.e., the roles of the transmitters and the receivers are interchanged)? The property of maintaining the same performance even if the direction of flow of information is reversed is widely known as the reciprocity of the channel. For instance, the following reciprocity of the point-to-point MIMO channel was proved in [33]: the capacity of a MIMO point-to-point channel is unchanged when the roles of the transmitters and receivers are interchanged provided the power constraint is appropriately scaled. In [34] the degrees of freedom (DoF) region of a $(M_1, N_1, M_2, N_2)$ MIMO IC was shown to be the same as that of a $(N_1, M_1, N_2, M_2)$ IC. In this section, we prove a reciprocity result for the $(M_1, N_1, M_2, N_2)$ MIMO IC by showing that reciprocity actually holds in the much stronger constant-gap-to-capacity sense.
Fig. 7: Information flowing in the reverse direction on an 2-user MIMO IC and its corresponding forward information flow model.

Fig. 7(a) illustrates an \((M_1, N_1, M_2, N_2)\) MIMO IC with channel parameters \(H\) and \(\tilde{\rho}\) with roles of the transmitters and receivers interchanged so that information flows in the reverse direction. Fig. 7(b) shows its equivalent model where the information flows in the forward direction. Clearly, the capacity of the reverse channel is the same as that of \(\mathcal{IC}(H^r, \tilde{\rho}^r)\) where \(H^r = \{H_{11}^T, H_{21}^T, H_{12}^T, H_{22}^T\}\) and \(\tilde{\rho}^r = [\rho, \rho_{21}, \rho_{12}, \rho_{22}]\). The capacity region of the reverse channel is denoted as \(C(H^r, \tilde{\rho}^r)\).

Let us define the counterparts in the reverse channel of the capacity gap parameters of the forward channel in (5) as

\[
m^*_i \triangleq \min \{M_i, N_s\} \log(N_x) + \tilde{m}_{ij}, \quad 1 \leq i \neq j \leq 2,
\]

where \(N_s = (N_1 + N_2)\), \(N_x = \max\{N_1, N_2\}\) and \(\tilde{m}_{ij} = m_{ij} \log \left( \frac{(N_i+1)}{N_j} \right) \) for \(1 \leq i \neq j \leq 2\).

To prove the reciprocity in the constant gap to capacity sense, the capacity regions of \(\mathcal{IC}(H, \tilde{\rho})\) and \(\mathcal{IC}(H^r, \tilde{\rho}^r)\) must be shown to be within a constant number of bits to each other. We start with a result on the outer bounds.

**Lemma 7:** The outer bound \(\mathcal{R}^u(H, \tilde{\rho})\) from Lemma I of the forward channel \(\mathcal{IC}(H, \tilde{\rho})\) and the outer bound \(\mathcal{R}^u(H^r, \tilde{\rho}^r)\) (obtained in the same way as in Lemma I but for the reverse channel \(\mathcal{IC}(H^r, \tilde{\rho}^r)\)) define the same set of rate pairs, i.e.,

\[
\mathcal{R}^u(H, \tilde{\rho}) = \mathcal{R}^u(H^r, \tilde{\rho}^r).
\]
Proof: The proof is given in Appendix E.

Corollary 1 proves that the explicit HK scheme, \( \widehat{HK} \), achieves a rate region on \( IC(\mathcal{H}, \bar{\rho}) \) which is within \( n_i^* \) bits to a set of rate pair \( \mathcal{R}_u^{u}(\mathcal{H}, \bar{\rho}) \) which contains its capacity region. Clearly, the counterpart of this explicit HK coding scheme for the reverse channel (with suitable changes in the channel matrices, INRs and the number of antennas) can achieve a rate region on \( IC(\mathcal{H}^r, \bar{\rho}^r) \) which is within \( m_i^* \) bits to \( \mathcal{R}_u^{u}(\mathcal{H}^r, \bar{\rho}^r) \), where \( m_i^* \) is given by equation (51). However, from Lemma 7 we know that \( R_u^u(\mathcal{H}, \bar{\rho}) = R_u^u(\mathcal{H}^r, \bar{\rho}^r) \).

Thus the capacity regions of the two interference channels can not differ by more than \( \max\{m_i^*, n_i^*\} \) bits proving the following theorem.

**Theorem 4:** The capacity regions of \( IC(\mathcal{H}, \bar{\rho}) \) and \( IC(\mathcal{H}^r, \bar{\rho}^r) \) are within \( \max\{m_i^*, n_i^*\} \) bits to each other, i.e., if \( (R_1, R_2) \in C(\mathcal{H}, \bar{\rho}) \), then there exists a rate pair \( (R_1^r, R_2^r) \in C(\mathcal{H}^r, \bar{\rho}^r) \), the capacity region of the reverse channel, such that

\[
|\left(R_i - R_i^r\right)| \leq \max\{m_i^*, n_i^*\}, \ \forall \ 1 \leq i \leq 2.
\]

Proof: Let \( (R_1, R_2) \in C(\mathcal{H}, \bar{\rho}) \). From Corollary 1 there exist a rate pair \( (\hat{R}_1, \hat{R}_2) \in \mathcal{R}_u^u(\mathcal{H}, \bar{\rho}) \) such that

\[
0 \leq (\hat{R}_i - R_i) \leq n_i^*, \ \forall \ 1 \leq i \leq 2.
\] (53)

Further, from Lemma 7 we have \( (\hat{R}_1, \hat{R}_2) \in \mathcal{R}_u^u(\mathcal{H}^r, \bar{\rho}^r) \). Next, applying Corollary 1 for \( IC(\mathcal{H}^r, \bar{\rho}^r) \), we have that there exists a rate pair \( (R_1^r, R_2^r) \in C(\mathcal{H}^r, \bar{\rho}^r) \) such that

\[
0 \leq (\hat{R}_i - R_i^r) \leq m_i^*, \ \forall \ 1 \leq i \leq 2.
\] (54)

Note that equations (53) and (54) provide ranges of \( R_i \) and \( R_i^r \) and the magnitude of the difference between them is maximum when one takes its largest value and the other its smallest, i.e.,

\[
|\left(R_i - R_i^r\right)|_{\max} = m_i^* \text{ or } n_i^*,
\]

which proves the theorem.

Remark 15: Note that reciprocity holds for the 2-user MIMO IC without power scaling and this may seem counter-intuitive given the point-to-point MIMO channel result of [33]. The reason is that reciprocity was shown here in the approximate capacity sense. The difference due to not scaling power gets absorbed in the gap that already exists between the exact capacity and the achievable region.
IV. Conclusion

An approximate capacity region of the 2-user MIMO IC with an arbitrary number of antennas at each node is characterized. It is shown that a simple and an explicit HK coding schemes which can be seen to inherently perform a form of joint interference alignment in the signal space and in the signal level (see Section II.C of the companion paper [26] for this interpretation) can achieve the capacity region within a constant gap. For a class of ICs, this gap is the tightest approximation to the capacity region of the MIMO IC found to date and this includes the SIMO ICs for which the gap is 1 bit independently of the number of antennas at the receivers. The explicit upper and lower bounds to the capacity region are used to prove the reciprocity of the MIMO IC in the constant-gap-to-capacity sense.

Appendix A

Proof of Lemma [1]

Proof: The set of upper bounds to the achievable rate region will be derived in two steps. In the first step, the different mutual information terms in Fano’s inequality are expanded in terms of the corresponding differential entropies. The genie-aided signaling strategies of [24] are employed albeit with different side information (see Remark [1] to the receivers so that no negative entropy term involving inputs appear in the upper bound, thereby allowing for a single-letterization of the resulting bounds. The positive differential entropies are then upper bounded using Lemma 8 and its corollaries proved in Appendix B.

1) From Fano’s lemma we have

\[ nR_1 \leq I(X^n_1; Y^n_1) + n\epsilon_n \]
\[ \leq I(X^n_1; Y^n_1, X^n_2) + n\epsilon_n, \]  \hspace{1cm}(a)
\[ = I(X^n_1; X^n_2) + I(X^n_1; Y^n_1 | X^n_2) + n\epsilon_n, \] \hspace{1cm}[I(X^n_1; X^n_2) = 0, since \( X_{1t} \) and \( X_{2t} \) are independent]
\[ = h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X^n_1 + Z^n_1) - h(Z^n_1) + n\epsilon_n \] \hspace{1cm}(b)
\[ \leq n \log \det \left( I_{N_1} + \rho_{11} H_{11} H_{11}^H \right) - \log \det(I_{N_1}) + n\epsilon_n, \]
\[ = n \log \det \left( I_{N_1} + \rho_{11} H_{11} H_{11}^H \right) + n\epsilon_n, \]

where step (a) follows from the fact that the extra information \( X^n_2 \) at the receiver does not reduce mutual information and step (b) follows from Corollary [4] which in turn is proved in Appendix B.

Now dividing both sides by \( n \) and taking the limit as \( n \to \infty \), we have

\[ R_1 \leq \log \det \left( I_{N_1} + \rho_{11} H_{11} H_{11}^H \right), \]
2) The second bound can be obtained similarly.

3) The third upper bound is derived with the genie providing the side information \((S_1^n, X_2^n)\) to \(R_x_1\) where \(S_{it}\) is the interference plus noise at \(R_x_j\) \(j \neq i\) (as shown in Fig. 8). Since additional information does not reduce mutual information we have from Fano’s lemma

\[
n(R_1 + R_2) \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n \epsilon_n
\]

\[
\leq I(X_1^n; Y_1^n, S_1^n, X_2^n) + I(X_2^n; Y_2^n) + n \epsilon_n,
\]

\[
= I(X_1^n; X_2^n) + I(X_1^n; S_1^n|X_2^n) + I(X_1^n; Y_1^n|S_1^n, X_2^n) + I(X_2^n; Y_2^n) + n \epsilon_n.
\]

Again \(I(X_1^n; X_2^n) = 0\), so that

\[
n(R_1 + R_2) \leq h(S_1^n|X_2^n) - h(Z_2^n) + h(Y_1^n|S_1^n, X_2^n) - h(Z_1^n) + h(Y_2^n) - h(Y_2^n|X_2^n) + n \epsilon_n,
\]

\[
= h(S_1^n) - h(Z_2^n) + h(Y_1^n|S_1^n, X_2^n) - h(Z_1^n) + h(Y_2^n) - h(S_1^n) + n \epsilon_n,
\]

\[
= h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X_1^n + Z_1^n|S_1^n) + h(Y_2^n) - n(N_1 + N_2) \log(2\pi e) + n \epsilon_n,
\]

\[
\leq n \log \det \left( I_{N_1} + \rho_{11}H_{11} \left( I_{M_1} + \rho_{12}H_{12}^H H_{12} \right)^{-1} H_{11}^H \right) + n \epsilon_n + 
\]

\[
n \log \det \left( I_{N_2} + \rho_{12}H_{12}H_{12}^H + \rho_{22}H_{22}H_{22}^H \right),
\]

where \((b)\) follows from Corollaries 2 and 3 in Appendix B. Now, dividing both sides by \(n\) and taking the limit as \(n \to \infty\), we get the third bound since \(\epsilon_n \to 0\) as \(n \to \infty\).
4) The fourth bound can be obtained similarly assuming the genie provides the side information $(S_2^n, X_1^n)$ to $R_2$.

5) To derive the fifth upper bound we assume that the genie gives side information $S_1^n$ to $R_1$. Using Fano’s lemma we have

$$n(R_1 + R_2) \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon_n,$$

$$\leq I(X_1^n; Y_1^n, S_1^n) + I(X_2^n; Y_2^n, S_2^n) + n\epsilon_n,$$

$$= I(X_1^n; S_1^n) + I(X_2^n; S_2^n) + I(X_2^n; Y_2^n | S_2^n) + n\epsilon_n,$$

$$= h(S_1^n) + h(Y_1^n | S_1^n) - h(Y_1^n | S_1^n, X_1^n) + h(S_2^n) + h(Y_2^n | S_2^n)$$

$$- h(Z_2^n) - h(Y_2^n | S_2^n, X_2^n) + n\epsilon_n,$$

$$= h(S_1^n) + h(Y_1^n | S_1^n) - h(S_1^n | S_1^n) + h(S_2^n) + h(Y_2^n | S_2^n) - h(S_2^n | S_2^n)$$

$$- h(Z_2^n) - h(Z_1^n) + n\epsilon_n,$$

$$= h(Y_1^n | S_1^n) + h(Y_2^n | S_2^n) - n(N_1 + N_2) \log(2\pi e)$$

$$+ n\epsilon_n, \quad [... S_i^n \text{ is independent of } S_j^n, \text{ for } i \neq j]$$

$$(d) \leq n \log \text{det} \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} \left( I_{M_1} + \rho_{12} H_{112} H_{12} \right)^{-1} H_{11}^\dagger \right) +$$

$$n \log \text{det} \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} \left( I_{M_2} + \rho_{21} H_{212} H_{21} \right)^{-1} H_{22}^\dagger \right) + n\epsilon_n$$

where in step $(d)$ we used Lemma 8 of Appendix B twice.

6) Again from Fano’s lemma we have

$$n(2R_1 + R_2) \leq I(X_1^n; Y_1^n) + I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon_n,$$

$$\leq I(X_1^n; Y_1^n) + I(X_1^n; Y_1^n, S_1^n, X_2^n) + I(X_2^n; Y_2^n, S_2^n) + n\epsilon_n,$$

where the last step again follows from the fact that extra information does not reduce mutual
Similarly, the output at receiver \( X \) is
\[
Y \triangleq X + \sqrt{\rho_{ij}} H_{ij} X + Z,
\]
where the additive noise at receiver \( j \) at time \( t \) and \( S_{ij}^t = [ S_{i1}^t, S_{i2}^t, \ldots, S_{in}^t ]^\top \) can be written as
\[
S_{ij}^t = \sqrt{\rho_{ij}} \begin{bmatrix}
    H_{ij} & 0 & \cdots & 0 \\
    0 & H_{ij} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & H_{ij}
\end{bmatrix} \begin{bmatrix}
    X_{i1} \\
    X_{i2} \\
    \vdots \\
    X_{in}
\end{bmatrix} = \sqrt{\rho_{ij}} (I_n \otimes H_{ij}) X_i^t + Z_{ij}^t.
\]
Similarly, the output at \( Rx_i \) over \( n \) channel uses can be written as
\[
Y_i^t = \sqrt{\rho_{ii}} (I_n \otimes H_{ii}) X_i^t + \sqrt{\rho_{jj}} (I_n \otimes H_{jj}) X_j^t + Z_{ii}^t,
\]
where \( X_{it} \in \mathbb{C}^{M \times 1}, \forall t \leq n \) satisfies the power constraints of equation (1).

**Lemma 8:** For the \( n \)-length vector sequences \( Y_i^n \) and \( S_i^n \) as described above,
\[
\begin{align*}
    h(Y_i^n | S_i^n) & \leq n \log \det \left( I_{N_i} + \rho_{ji} H_{jj} H_{ji}^\top + \rho_{ii} H_{ii} \left( I_{M_i} + \rho_{ij} H_{ij} H_{ij}^\top \right)^{-1} H_{ii}^\top \right) + nN_i \log(2\pi e),
\end{align*}
\]
where \( h(X | Y) \) is the mutual information between \( X \) and \( Y \), \( \rho_{ii} \) is the channel gain, and \( N_i \) is the noise power at receiver \( i \).
for $i \neq j \in \{1, 2\}$.

Proof: We shall prove the Lemma for $i = 1$ with $i = 2$ case being identical. Denoting the covariance matrix of a zero-mean random vector $V$ by $\text{Cov}(V)$, i.e., $\text{Cov}(V) = \mathbb{E}(VV^\dagger)$ and the composite vector $V_t$ by

$$V_t = \begin{bmatrix} S_{1t} \\ Y_{1t} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_{12}} H_{12} X_{1t} + Z_{2t} \\ \sqrt{\rho_{11}} H_{11} X_{1t} + \sqrt{\rho_{21}} H_{21} X_{2t} + Z_{1t} \end{bmatrix}, \quad \forall \ t \leq n, \quad (55)$$

it can be easily verified that

$$K_{J_t} \triangleq \mathbb{E} \left( V_t V_t^\dagger \right) = \begin{bmatrix} \rho_{12} H_{12} Q_{1t} H_{12}^\dagger + I_{N_2} & \sqrt{\rho_{12}} \sqrt{\rho_{11}} H_{12} Q_{1t} H_{11}^\dagger \\ \sqrt{\rho_{11}} \sqrt{\rho_{12}} H_{11} Q_{1t} H_{12}^\dagger & \rho_{11} H_{11} Q_{1t} H_{11}^\dagger + \rho_{21} H_{21} Q_{2t} H_{21}^\dagger + I_{N_1} \end{bmatrix}, \quad \forall \ t \leq n. \quad (56)$$

Note that in the above computation we assumed that input distribution has zero mean, which is standard since a non-zero mean only contributes to power inefficiency. Let us define

$$\hat{S}^*_1 = \sqrt{\rho_{12}} H_{12} X_1^G + Z_2,$$
$$\hat{Y}^*_1 = \sqrt{\rho_{11}} H_{11} X_1^G + \sqrt{\rho_{21}} H_{21} X_2^G + Z_1,$$

where $X_i^G \sim \mathcal{CN}(0, \frac{1}{n} \sum_{i=1}^n Q_{it})$ for $1 \leq i \neq j \leq 2$ and $X_1^G$ and $X_2^G$ are mutually independent. It can be easily verified that $\hat{S}^*_1$ and $\hat{Y}^*_1$ are Gaussian vectors with covariance matrices $\bar{K}$ and $\bar{K}_J$, respectively, where $\bar{K} = \rho_{12} H_{12} \bar{Q}_1 H_{12}^\dagger + I_{N_2}$, $\bar{Q}_1 = \frac{1}{n} \sum_{t=1}^n Q_{1t}$ and

$$\bar{K}_J = \begin{bmatrix} \rho_{12} H_{12} \bar{Q}_1 H_{12}^\dagger + I_{N_2} & \sqrt{\rho_{12}} \sqrt{\rho_{11}} H_{12} \bar{Q}_1 H_{11}^\dagger \\ \sqrt{\rho_{11}} \sqrt{\rho_{12}} H_{11} \bar{Q}_1 H_{12}^\dagger & \rho_{11} H_{11} \bar{Q}_1 H_{11}^\dagger + \rho_{21} H_{21} \bar{Q}_2 H_{21}^\dagger + I_{N_1} \end{bmatrix} \quad (57)$$

which follows from the fact that any linear transformation of a Gaussian vector is also Gaussian (Proposition 5.2, [35]) and the sum of several mutually independent Gaussian vectors is also Gaussian. In other words,

$$\text{Cov} \left[ \begin{bmatrix} \hat{S}^*_1 \\ \hat{Y}^*_1 \end{bmatrix} \right] = \frac{1}{n} \sum_{t=1}^n \text{Cov} \left[ \begin{bmatrix} S_{1t} \\ Y_{1t} \end{bmatrix} \right].$$

Under the constraints satisfied on $\hat{S}^*_1$, $\hat{Y}^*_1$ and $V_t$, $1 \leq t \leq n$, it was proved in Lemma 2 of [20] that

$$h(Y_1^n|S_1^n) \leq n h(\hat{Y}^*_1|\hat{S}^*_1)$$
$$= n \left( h \left( \begin{bmatrix} \hat{S}^*_1 \\ \hat{Y}^*_1 \end{bmatrix} \right) - h(\hat{S}^*_1) \right),$$
$$= n \left( \log \det(\bar{K}_J) - \log \det(\bar{K}) \right).$$
Putting the values of $\bar{K}_J$ and $\bar{K}$ in the above equation and after some simplification, we get

$$
\begin{align*}
&h(Y_1^n|S^n_1) \leq n \log \det \left( I_{N_i} + \rho_{11} H_{11} \bar{Q}_1 H_{11}^\dagger + \rho_{21} H_{21} \bar{Q}_2 H_{21}^\dagger - nN_1 \log(2\pi e) \right) ,
\end{align*}
$$

where step (c) follows from the Woodbury identity and the last step follows from the fact that $\log \det(.)$ is a monotonically increasing function on the cone of p.d. matrices and Lemma 9 (below) with $G_1 = Q_1^\dagger$, $A = H_{12}^\dagger H_{12}$ and $G_2 = I_{M_i}$.

**Lemma 9:** Let $0 \preceq G_1 \preceq G_2$ and $0 \preceq A$ are p.s.d. matrices of size $n$, then for any given $\pi \in \mathbb{R}^+$

$$
G_1 (I + \pi G_1 AG_1)^{-1} G_1 \preceq G_2 (I + \pi G_2 AG_2)^{-1} G_2.
$$

**Proof:** Let $\epsilon \in \mathbb{R}^+$, $G_{1\epsilon} = (G_1 + \epsilon I)$ and $G_{2\epsilon} = (G_2 + \epsilon I)$. For any such $\epsilon$, we have

$$
\begin{align*}
G_{2\epsilon} &\succeq G_{1\epsilon} \succ 0, & \text{or} \\
G_{1\epsilon}^{-2} &\succeq G_{2\epsilon}^{-2} \succ 0, & \text{or} \\
(G_{1\epsilon}^{-2} + \pi A) &\succeq (G_{2\epsilon}^{-2} + \pi A) \succ 0, & \text{or} \\
(G_{1\epsilon}^{-2} + \pi A)^{-1} &\preceq (G_{2\epsilon}^{-2} + \pi A)^{-1}, & \text{or} \\
G_{1\epsilon} (I + \pi G_{1\epsilon} AG_{1\epsilon})^{-1} G_{1\epsilon} &\preceq G_{2\epsilon} (I + \pi G_{2\epsilon} AG_{2\epsilon})^{-1} G_{2\epsilon}.
\end{align*}
$$

From the definition of partial order between p.s.d. matrices we get

$$
\begin{align*}
x \left( G_{1\epsilon} (I + \pi G_{1\epsilon} AG_{1\epsilon})^{-1} G_{1\epsilon} \right) x^\dagger &\preceq x \left( G_{2\epsilon} (I + \pi G_{2\epsilon} AG_{2\epsilon})^{-1} G_{2\epsilon} \right) x^\dagger, \quad \forall \ x \in \mathbb{C}^{1 \times n}; \\
\Rightarrow \lim_{\epsilon \to 0} x \left( G_{1\epsilon} (I + \pi G_{1\epsilon} AG_{1\epsilon})^{-1} G_{1\epsilon} \right) x^\dagger &\preceq \lim_{\epsilon \to 0} x \left( G_{2\epsilon} (I + \pi G_{2\epsilon} AG_{2\epsilon})^{-1} G_{2\epsilon} \right) x^\dagger, \quad \forall \ x \in \mathbb{C}^{1 \times n},
\end{align*}
$$

where the last step follows from the fact that for any $G_i$, $A$ and $\pi$ as defined above and for any sequence of positive real numbers $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon \to 0$ as $n \to \infty$ we have

$$
\lim_{n \to \infty} \left( G_{i\epsilon_n} (I + \pi G_{i\epsilon_n} AG_{i\epsilon_n})^{-1} G_{i\epsilon_n} \right) = \left( G_i (I + \pi G_i AG_i)^{-1} G_i \right).
$$
Substituting their limits in the last equation we get

\[
x \left( G_1 (I + \pi G_1 A G_1)^{-1} G_1 \right) x^\dagger \leq x \left( G_2 (I + \pi G_2 A G_2)^{-1} G_2 \right) x^\dagger, \quad \forall \ x \in \mathbb{C}^{1 \times n}.
\]

Invoking the definition of partial ordering once again, the lemma is proved.

The following corollaries can be proved from Lemma 8. In particular, Corollary 2 by setting \( H_{ji} = 0 \), Corollary 3 by setting \( H_{ij} = 0 \) and Corollary 4 by putting \( H_{ji} = 0 \) and \( H_{ij} = 0 \).

**Corollary 2:**

\[
h(\sqrt{\rho_{ii}} (I_n \otimes H_{ii}) X_i^n + Z_i^n | S_i^n) \leq n \log \det \left( I_{N_i} + \rho_{ii} H_{ii} \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} H_{ii}^\dagger \right),
\]

\[
+ n N_i \log(2\pi e), \quad i \neq j \in \{ 1, 2 \}.
\]

**Corollary 3:**

\[
h(Y_i^n) \leq n \log \det \left( I_{N_i} + \rho_{ij} H_{ji} H_{ji}^\dagger + \rho_{ii} H_{ii} H_{ii}^\dagger \right) + n N_i \log(2\pi e), \quad i \neq j \in \{ 1, 2 \}.
\]

**Corollary 4:**

\[
h(\sqrt{\rho_{ii}} (I_n \otimes H_{ii}) X_i^n + Z_i^n) \leq n \log \det \left( I_{N_i} + \rho_{ij} H_{ii} H_{ii}^\dagger \right) + n N_i \log(2\pi e), \quad i \neq j \in \{ 1, 2 \}.
\]

**APPENDIX C**

**PROOF OF LEMMA 8**

As stated in the outline, it is only required to show that the right hand sides of the different bounds in the lemma are actually lower bounds to the corresponding terms in (58)–(59). However, first we derive some common inequalities which will be used throughout the proof. From the definition of p.s.d. matrices \([36]\) we get

\[
\left( \rho_{ij} H_{ij} K_{iu} H_{ij}^\dagger \right) = \left( \frac{\rho_{ij}}{M_i} H_{ij} \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} H_{ij}^\dagger \right) \geq 0, \quad \left[ \vdots \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} \right] \geq 0 \quad (58)
\]

On the other hand, for any given \( 0 \neq x \in \mathbb{C}^{1 \times N_i} \), we have

\[
x \left( \frac{\rho_{ij}}{M_i} H_{ij} \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} H_{ij}^\dagger \right) x^\dagger = \frac{\rho_{ij}}{M_i} (x U_{ij}) \Sigma_{ij} \left( I_{M_i} + \rho_{ij} \Sigma_{ij} \Sigma_{ij} \right)^{-1} \Sigma_{ij}^\dagger (x U_{ij})^\dagger \leq \frac{1}{M_i} (x U_{ij})^\dagger (x U_{ij}) \leq \frac{x x^\dagger}{M_i},
\]

where \( \Sigma_{ij} \in \mathbb{C}^{N_i \times M_i} \) is the singular value matrix of \( H_{ij} \), i.e., \( H_{ij} = U_{ij} \Sigma_{ij} V_{ij}^\dagger \) and step (a) follows from the fact that \( \rho_{ij} \Sigma_{ij} \left( I_{M_i} + \rho_{ij} \Sigma_{ij} \Sigma_{ij} \right)^{-1} \Sigma_{ij}^\dagger \Sigma_{ij} \leq I_{N_j} \). However, the last inequality along with the definition of p.s.d. matrices imply

\[
\rho_{ij} H_{ij} K_{ij} H_{ij}^\dagger = \left( \frac{\rho_{ij}}{M_i} H_{ij} \left( I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij} \right)^{-1} H_{ij}^\dagger \right) \leq \frac{1}{M_i} I_{N_j}.
\]
Each of the eigenvalues of the matrix on the left hand side of equation (59) is smaller than or equal to 1. However, that matrix has only \( \min\{m_i, n_j\} = m_{ij} \) non-zero eigenvalues since the rank of \( H_{ij} \) is \( m_{ij} \), which in turn implies that

\[
\log \det \left( I_{N_j} + \rho_{ij} H_{ij} K_{iu} H_{ij}^\dagger \right) \leq m_{ij} \log \left( \frac{1 + M_i}{M_i} \right) = \hat{m}_{ij},
\]

for all \( 1 \leq i \neq j \leq 2 \).

As a first step towards deriving the lower bounds, in what follows, we shall first derive lower bounds for the different mutual information terms of equation (27)-(33). From equation (27) we obtain

\[
I(X_1^g; Y_1^g | W_1^g, W_2^g) = \log \det \left( \frac{\rho_{11} H_{11} K_{11} H_{11}^\dagger}{M_1} + \frac{\rho_{21} H_{21} K_{21} H_{21}^\dagger}{M_2} + I_{N_1} \right) \\
- \log \det \left( \rho_{21} H_{21} K_{2u} H_{21}^\dagger + I_{N_1} \right),
\]

\[
\geq \log \det \left( \frac{\rho_{11} H_{11} K_{11} H_{11}^\dagger}{M_1} + \frac{1}{M_1} I_{N_1} \right) - \hat{m}_{21},
\]

\[
= \log \det \left( \rho_{11} H_{11} K_{11} H_{11}^\dagger + I_{N_1} \right) - (m_{11} \log(M_1) + \hat{m}_{21}),
\]

where step (a) follows from the fact that \( \log \det(.) \) is a monotonically increasing function over the cone of positive-definite matrices with respect to the partial ordering and equations (58) and (60). The last equality follows from the fact that \( H_{11} \) is a rank \( m_{11} \) matrix. Similarly,

\[
I(W_2^g; Y_2^g | W_2^g, W_1^g) \geq \log \det \left( \rho_{22} H_{22} K_{22} H_{22}^\dagger + I_{N_2} \right) - (m_{22} \log(M_2) + \hat{m}_{12}).
\]

From equation (29) we obtain

\[
I(W_2^g; Y_1^g | X_1^g) = \log \det \left( \frac{\rho_{21} H_{21} H_{21}^\dagger}{M_2} + I_{N_1} \right) - \log \det \left( \rho_{21} H_{21} K_{2u} H_{21}^\dagger + I_{N_1} \right),
\]

\[
\geq \log \det \left( \frac{\rho_{21} H_{21} H_{21}^\dagger}{M_2} + I_{N_1} \right) - \hat{m}_{21}, \quad [\because (60)]
\]

\[
\geq \log \det \left( \rho_{21} H_{21} H_{21}^\dagger + \frac{1}{M_2} I_{N_1} \right) - \hat{m}_{21},
\]

\[
= \log \det \left( \rho_{21} H_{21} H_{21}^\dagger + I_{N_1} \right) - (m_{21} \log(M_2) + \hat{m}_{21}),
\]

where the last step follows from the fact that the rank of \( H_{21} \) is \( m_{21} \). Similarly, we have

\[
I(W_1^g; Y_2^g | X_2^g) \geq \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger \right) - (m_{12} \log(M_1) + \hat{m}_{12}).
\]
From equation (50)
\[
I(X^g_2; Y^g_2 | W^g_2) = \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + \rho_{21} H_{21} K_{2u} H_{21}^\dagger + I_{N_1} \right) - \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + \rho_{21} H_{21} K_{2u} H_{21}^\dagger + I_{N_1} \right) ,
\]
\[
\geq \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + I_{N_1} \right) - \hat{m}_{21}, \quad \geq \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + \frac{1}{M_1} I_{N_1} \right) - \hat{m}_{21}, \quad \geq \log \det \left( \rho_{11} H_{11} H_{11}^\dagger + I_{N_1} \right) - (m_{11} \log(M_1) + \hat{m}_{21}),
\]
where steps (b) follows from equations (58) and (60) and the fact that \( \log \det(.) \) is a monotonically increasing function over the cone of positive-definite matrices. The last equality follows from the fact that \( H_{11} \) is a rank \( m_{11} \) matrix. Similarly, we have
\[
I(X^g_2; Y^g_2 | W^g_2) \geq \log \det \left( \rho_{22} H_{22} H_{22}^\dagger + I_{N_2} \right) - (m_{22} \log(M_2) + \hat{m}_{12}). \quad (66)
\]

Next, we focus on the term \( I(X^g_2, W^g_1; Y^g_2) \).
\[
I(X^g_2, W^g_1; Y^g_2) = \log \det \left( \frac{\rho_{12}}{M_1} H_{12} H_{12}^\dagger + \frac{\rho_{22}}{M_2} H_{22} H_{22}^\dagger + I_{N_2} \right) - \log \det \left( \rho_{12} H_{12} K_{1u} H_{12}^\dagger + I_{N_2} \right) ,
\]
\[
\geq \log \det \left( \frac{\rho_{12}}{M_1} H_{12} H_{12}^\dagger + \frac{\rho_{22}}{M_2} H_{22} H_{22}^\dagger + \frac{1}{M_x} I_{N_2} \right) - \hat{m}_{12}, \quad \geq \log \det \left( \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger + I_{N_2} \right) - \min\{N_2, M_s\} \log(M_x) + \hat{m}_{12} ,
\]
where the last step follows from the fact that the matrix \( \left( \frac{\rho_{12}}{M_1} H_{12} H_{12}^\dagger + \frac{\rho_{22}}{M_2} H_{22} H_{22}^\dagger \right) \) has rank \( \min\{N_2, M_s\} \), \( M_s \) and \( M_x \) are as defined in Section III. Similarly, we have
\[
I(X^g_1, W^g_2; Y^g_1) \geq \log \det \left( \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger + I_{N_1} \right) - (\min\{N_1, M_s\} \log(M_x) + \hat{m}_{21}).
\]

From equation (31)
\[
I(X^g_1, W^g_2; Y^g_1 | W^g_1) = \log \det \left( \rho_{11} H_{11} K_{1u} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger + I_{N_1} \right) - \log \det \left( \rho_{21} H_{21} K_{2u} H_{21}^\dagger + I_{N_1} \right) ,
\]
\[
\geq \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + \frac{\rho_{21}}{M_2} H_{21} H_{21}^\dagger + I_{N_1} \right) - \hat{m}_{21}, \quad \geq \log \det \left( \frac{\rho_{11}}{M_1} H_{11} H_{11}^\dagger + \frac{\rho_{21}}{M_2} H_{21} H_{21}^\dagger + \frac{1}{M_x} I_{N_1} \right) - \hat{m}_{21}, \quad \geq \log \det \left( \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger + I_{N_1} \right) - (\min\{N_1, M_s\} \log(M_x) + \hat{m}_{21}) \leq \frac{I^L}{e},
\]
where step (a) follows from the fact that the sum of the first two matrices inside \( \log \det(\cdot) \) has a rank of \( \min\{N_1, M_s\} \), and similarly,

\[
I(X^g_2, W^g_1; Y^g_2 | W^g_2) \geq \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (\min\{N_2, M_s\} \log(M_x) + \hat{m}_{12}).
\]  

(67)

Note that each of the lower bounds are difference of a channel dependent term and a constant. To denote the constants by a common notation we define \( n^*_i = \min\{N_i, M_s\} \log(M_x) + \hat{m}_{ji} \), for \( i \neq j \in \{1, 2\} \).

Using this notations and equations (61)-(67) we get the following set of stricter bounds.

\[
R_1 \leq \left( \log \det \left( I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger \right) - n^*_1 \right)^+; 
\]  

(68a)

\[
R_1 \leq \left( \log \det \left( I_{M_1} + \rho_{11} H_{11} H_{11} + K_1^{-1} \right) - (m_{11} \log(M_1) + m_{12} \log(M_1 + 1) - \hat{m}_{21}) \right)^+; 
\]  

(68b)

\[
R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger \right) - n^*_2 \right)^+; 
\]  

(68c)

\[
R_2 \leq \left( \log \det \left( I_{M_2} + \rho_{22} H_{22} H_{22} + K_2^{-1} \right) - (m_{22} \log(M_2) + m_{21} \log(M_2 + 1) - \hat{m}_{12}) \right)^+; 
\]  

(68d)

\[
R_1 + R_2 \leq \left( \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger \right) 
+ \log \det \left( I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) - (n^*_1 + n^*_2) \right)^+; 
\]  

(68e)

\[
R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger \right) 
+ \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (n^*_1 + n^*_2) \right)^+; 
\]  

(68f)

\[
R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) 
+ \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (n^*_1 + n^*_2) \right)^+; 
\]  

(68g)

\[
2R_1 + R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger \right) 
+ \log \det \left( I_{N_2} + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) 
+ \log \det \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) - (2n^*_1 + n^*_2) \right)^+; 
\]  

(68h)

\[
R_1 + 2R_2 \leq \left( \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger \right) 
+ \log \det \left( I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger \right) 
+ \log \det \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger \right) - (n^*_1 + 2n^*_2) \right)^+, 
\]  

(68i)

\[\text{\textsuperscript{5}}\text{It is both necessary and sufficient to keep the non-negative parts of the lower bounds because } R_i's \text{ can not be negative and the rate pair } (0, 0) \text{ is trivially achievable, respectively.}\]
Since the right hand sides of each of the bounds in equation (68) is smaller than those in the corresponding bound in equation (25), the set of rate tuples defined by the above set of bounds is a subset of $\mathcal{R}^{G \rightarrow}_{\text{HK}}(P_s)$. However, in equation (68) there are two bounds on each $R_i$. To combine them into one we define

$$n_i = \max\{(m_{ii} \log(M_i) + m_{ij} \log(M_i + 1)) + \hat{m}_{ji}, n_i^*\}, \forall i \neq j \in \{1, 2\}.$$ 

Using this notation the first four bounds on $R_1$ and $R_2$ in equation (68) can be replaced by following two further stricter bounds.

$$R_1 \leq \left(\log \det \left(I_{N_1} + \rho_{11} H_{11} \right)^\dagger - n_1\right)^+;$$
$$R_2 \leq \left(\log \det \left(I_{N_2} + \rho_{22} H_{22} \right)^\dagger - n_2\right)^+,$$

where we have used the fact that $K_i^{-1}$ is a p.d. matrix and $\log \det(\cdot)$ is a monotonically increasing function in the cone of p.s.d. matrices. Substituting the above two equations in place of the first four in (68) we get the bounds of the Lemma since $n_i \geq n_i^*$, for all $i \in \{1, 2\}$.

**APPENDIX D**

**PROOF OF LEMMA 5**

As stated in the outline, first we show that when $(R_1, R_2) \in \mathcal{R}_2$, equations (25b) and (25d) both can not be violated simultaneously. Suppose, it is not true, i.e., $(R_1, R_2) \in \mathcal{R}_2$ but

$$R_1 > I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g);$$
$$R_2 > I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g).$$

Adding the above two equations we get

$$R_1 + R_2 > I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g),$$
$$\geq I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | W_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | W_1^g),$$
$$= I(X_1^g, W_2^g; Y_1^g | W_2^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g),$$

which violates equation (49c) and contradicts the assumption that $(R_1, R_2) \in \mathcal{R}_2$. Therefore, whenever $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}^{G \rightarrow}_{\text{HK}}(P_s)$ only one among (25b) and (25d) is violated and not both.

Next we shall show that such a rate pair is always achievable by the explicit HK coding scheme. Suppose $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}^{G \rightarrow}_{\text{HK}}(P_s)$ because it violates equation (25b), i.e.,

$$R_1 > I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_2^g; Y_2^g | X_2^g). \quad (69)$$
From Definition 4 we know that in this case, $T_{x_1}$ transmits only its private message, i.e., the coding scheme $\mathcal{HK}\{(\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w})\}$ is used. To distinguish the codewords of the first user in this case to those used in the simple HK scheme we use the following notations:

$$\tilde{X}_1^g = \tilde{U}_1^g \sim \mathcal{CN}(0, \frac{1}{M_1}I_{M_1}), \tilde{W}_1^g = \phi, \tilde{W}_2^g = W_2^g$$ and $\tilde{U}_2 = U_2^g$, \hspace{1cm} (70)

are mutually independent and $\tilde{X}_2^g = \tilde{W}_2^g + \tilde{U}_2^g$, where $\tilde{Y}_i^g$'s are the outputs of the channel (e.g., see equations (2)) when $\tilde{X}_i^g$'s are the inputs. Evidently, if the joint distribution of these random vectors be denoted by $P_s(\tilde{X}_1^g, \tilde{U}_1^g, \tilde{W}_1^g, \tilde{X}_2^g, \tilde{U}_2^g, \tilde{W}_2^g)$, then $P_s(.) \in \mathcal{P}^*$. Putting $P^* = P_{s_1}$ in the expression for $\mathcal{R}_G^c(P^*)$ we get the achievable region of the coding scheme $\mathcal{HK}\{(\frac{1}{M_1}I_{M_1}, 0, K_{2u}, K_{2w})\}$ as

$$\mathcal{R}_G^c(P_{s_1}) = \{(R_1, R_2) : R_1 \leq I(\tilde{X}_1^g; \tilde{Y}_1^g|\tilde{W}_2^g);$$

$$R_2 \leq I(\tilde{X}_2^g; \tilde{Y}_2^g);$$

$$R_2 \leq I(\tilde{W}_2^g; \tilde{Y}_1^g|\tilde{X}_1^g) + I(\tilde{X}_2^g; \tilde{Y}_2^g|\tilde{W}_2^g);$$

$$R_1 + R_2 \leq I(\tilde{X}_2^g; \tilde{W}_2^g; \tilde{Y}_1^g) + I(\tilde{X}_2^g; \tilde{Y}_2^g|\tilde{W}_2^g)\}$$

which can easily be shown to be equivalent to

$$\mathcal{R}_G^c(P_{s_1}) = \{(R_1, R_2) : R_1 \leq I(X_1^g; Y_1^g|W_2^g);$$

$$R_2 \leq I(X_2^g; Y_2^g);$$

$$R_2 \leq I(W_2^g; Y_1^g|X_1^g) + I(X_2^g; Y_2^g|W_2^g);$$

$$R_1 + R_2 \leq I(X_2^g; W_2^g; Y_1^g) + I(X_2^g; Y_2^g|W_2^g)\}$$

using the relations between $\tilde{Y}_i^g$'s and $V_i^g$'s defined earlier, where $V \in \{U, W, X, Y\}$. In what follows, we shall shown that $(R_1, R_2) \in \mathcal{R}_G^c(P_{s_1})$ and hence achievable by $\overline{\mathcal{HK}}$.

From (49a) we know that

$$R_1 \leq I(X_1^g; Y_1^g|W_2^g).$$ \hspace{1cm} (75)

From (69) and (49c) we obtain

$$R_2 \leq I(X_2^g; Y_2^g).$$ \hspace{1cm} (76)

From (69) and (49e) we obtain

$$R_2 < I(W_2^g; Y_1^g|W_1^g) + I(X_2^g; Y_2^g|W_2^g);$$

$$\leq I(W_2^g; Y_1^g|X_1^g) + I(X_2^g; Y_2^g|W_2^g);$$ \hspace{1cm} (78)
and from (69) and (49)
\[ R_1 + R_2 \leq I(X_1^g, W_1^g; Y_1^g) + I(X_2^g; Y_1^g | W_2^g). \]  
(79)

This proves that \((R_1, R_2) \in \mathcal{R}_{HK}^{G}(P_{s_1})\) and hence achievable by \(\mathcal{H}\mathcal{K}\).

The other case when \((R_1, R_2) \in \mathcal{R}_2\) but equation (25c) is violated, i.e.,
\[ R_2 > I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g), \]
(80)
can be similarly proved. Again from Definition 4 we know that in this case the coding scheme \(\mathcal{H}\mathcal{K}\) \((\{K_{1u}, K_{1w}, \frac{1}{M_2} I_{M_2}, 0\})\) is used. Defining
\[
\hat{X}_2^g = \hat{U}_2^g \sim \mathcal{CN}(0, \frac{1}{M_2} I_{M_2}), \hat{W}_2^g = \phi, \hat{W}_1^g = W_1^g, \hat{U}_1^g = U_1^g,
\]
(81)
and \(\hat{X}_1^g = \hat{W}_1^g + \hat{U}_1^g\), where \(\hat{Y}_i^g\)'s are the outputs of the channel (e.g., see equations (2)) when \(\hat{X}_i^g\)'s are the inputs, it is clear that the joint distributions of these variables, \(P_{x_2}(\hat{X}_1^g, \hat{U}_1^g, \hat{W}_1^g, \hat{X}_2^g, \hat{U}_2^g, \hat{W}_2^g) \in \mathcal{P}^*\). Finally, putting \(P^* = P_{s_2}\) in the expression for \(\mathcal{R}_{HK}^{G}(P^*)\) the achievable region of the coding scheme \(\mathcal{H}\mathcal{K}\) \((\{K_{1u}, K_{1w}, \frac{1}{M_2} I_{M_2}, 0\})\) can be computed. Now, combining equation (80) and (49) in the same way as the previous case, it can be shown that \((R_1, R_2) \in \mathcal{R}_{HK}^{G}(P_{s_2})\) and is hence achievable by \(\mathcal{H}\mathcal{K}\).

Finally, if \((R_1, R_2) \in \mathcal{R}_{HK}^{G}(P_s)\), then by Lemma 5 we know that \(\mathcal{H}\mathcal{K}\) \((\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})\) or the simple HK scheme can achieve this rate pair.

**APPENDIX E**

**PROOF OF LEMMA 7**

We shall prove this lemma in two steps. In step one, we shall prove
\[
\mathcal{R}^u(\mathcal{H}, \tilde{\rho}) = \mathcal{R}^u\left(\mathcal{H}, \tilde{\rho}^r\right),
\]
where \(\mathcal{H} = \{H_{11}^t, H_{21}^t, H_{12}^t, H_{22}^t\}\) and in the second step we shall prove that
\[
\mathcal{R}^u\left(\mathcal{H}, \tilde{\rho}^r\right) = \mathcal{R}^u\left(\mathcal{H}^r, \tilde{\rho}^r\right).
\]

Clearly, the above two equalities prove the lemma.

**Step 1:** Let us consider the interference channel \(\mathcal{IC}(\mathcal{H}, \tilde{\rho}^r)\). Following a similar method as in Lemma 1 we can derive an upper bound to the capacity region of this IC. Let the corresponding bounds of \(\mathcal{R}^u\left(\mathcal{H}, \tilde{\rho}^r\right)\) be denoted by \(I_k^r\), \(1 \leq k \leq 7\). In what follows, we shall first prove that \(I_{k3} = I_4^r\), \(I_{b4} = I_3^r\) and \(I_{b6} = I_k^r\) for \(k \in \{1, 2, 5, 6, 7\}\).
Towards proving the first equality, from equation (9) we get
\[ I_{b_3} = \log \det \left( I_{N_z} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger + \rho_{22} \mathbf{H}_{22} \mathbf{H}_{22}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{11} \mathbf{H}_{11} \left( \mathbf{I}_{M_1} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger \right)^{-1} \mathbf{H}_{11}^\dagger \right), \]
\[ = \log \det \left( I_{N_z} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger + \rho_{22} \mathbf{H}_{22} \mathbf{H}_{22}^\dagger \right) + \log \det \left( I_{M_1} + \rho_{12} \mathbf{H}_{12}^\dagger \mathbf{H}_{12} + \rho_{11} \mathbf{H}_{11}^\dagger \mathbf{H}_{11} \right) \]
\[ - \log \det \left( I_{M_1} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger \right), \]
\[ = \log \det \left( I_{N_z} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger + \rho_{22} \mathbf{H}_{22} \mathbf{H}_{22}^\dagger \right) + \log \det \left( I_{M_1} + \rho_{12} \mathbf{H}_{12}^\dagger \mathbf{H}_{12} + \rho_{11} \mathbf{H}_{11}^\dagger \mathbf{H}_{11} \right), \]
\[ = I_{b_5}. \]

Similarly, it can be proved that \( I_{b_4} = I_{b_5}. \) The equality of the first two bounds follow trivially from the identity \( \log \det (I + AB) = \log \det (I + BA). \) Now, towards proving the fifth bound we see
\[ I_{b_5}(1) = \log \det \left( I_{N_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger + \rho_{11} \mathbf{H}_{11} \mathbf{K}_{11} \mathbf{K}_{11}^\dagger \right) \]
\[ = \log \det \left( I_{N_1} + \rho_{11} \left( I_{N_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right)^{-1} \mathbf{H}_{11} \mathbf{K}_{11} \mathbf{K}_{11}^\dagger \right) + \log \det \left( I_{N_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right) \]
\[ = \log \det \left( I_{M_1} + \rho_{11} \mathbf{H}_{11} \left( I_{M_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right)^{-1} \mathbf{H}_{11} \mathbf{K}_{11} \right) + \log \det \left( I_{M_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right) \]
\[ = \log \det \left( \mathbf{K}_{11}^{-1} + \rho_{11} \mathbf{H}_{11} \left( I_{N_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right)^{-1} \mathbf{H}_{11} \right) + \log \det (\mathbf{K}_{11}) + \log \det (\mathbf{K}_{22}^{-1}) \]
\[ = \log \det \left( I_{M_1} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12} + \rho_{11} \mathbf{H}_{11} \left( I_{N_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21}^\dagger \right)^{-1} \mathbf{H}_{11} \right) \]
\[ + \log \det (\mathbf{K}_{11}) - \log \det (\mathbf{K}_{22}). \]

Similarly, it can be easily proved that
\[ I_{b_5}(2) = \log \det \left( I_{N_z} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger + \rho_{22} \mathbf{H}_{22} \mathbf{H}_{22}^\dagger \right) \]
\[ = \log \det \left( I_{M_1} + \rho_{21} \mathbf{H}_{21} \mathbf{H}_{21} + \rho_{22} \mathbf{H}_{22} \left( I_{N_z} + \rho_{12} \mathbf{H}_{12} \mathbf{H}_{12}^\dagger \right)^{-1} \mathbf{H}_{22} \right) \]
\[ + \log \det (\mathbf{K}_{22}) - \log \det (\mathbf{K}_{11}). \]
Combining the last two equations we get

\[ I_{b5} = I_{b5}(1) + I_{b5}(2), \]

\[ = \log \det \left( I_{M_1} + \rho_{12} H_{12}^\dagger H_{12} + \rho_{11} H_{11}^\dagger \left( I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger \right)^{-1} H_{11} \right) \]

\[ + \log \det \left( I_{M_2} + \rho_{21} H_{21}^\dagger H_{21} + \rho_{22} H_{22}^\dagger \left( I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger \right)^{-1} H_{22} \right), \]

\[ = I_{r5}. \]

Proving the equality of the other two bounds is similar. Hence, the set upper bounds for the capacity region of \( IC \left( \tilde{H}, \tilde{\rho} \right) \) defines the same set of rate pairs as \( R^u (H, \bar{\rho}) \).

Step 2: Suppose \( S \) is a p.s.d. matrix and \( S^* \) represents its complex conjugate, i.e., the matrix obtained by replacing all its entries by the corresponding complex conjugates. Then, using the fact that its eigen-values are real, it can be easily be proved that

\[ \log \det (I + S) = \log \det (I + S^*). \]

However, note that all the terms in the different bounds of Lemma 1 are of the form just described. This in turn proves that if we replace all the channel matrices of a 2-user MIMO IC by their complex conjugates the set of upper bounds remain the same. From this fact, it easily follows that

\[ R^u \left( \tilde{H}, \tilde{\rho} \right) = R^u \left( H^r, \bar{\rho}^r \right). \]

REFERENCES

[1] C. E. Shannon, “Two-way communication channel,” in Proc. of 4th Berkeley symp. Mathematical statistics and probability, Berkeley, CA, vol. 1, Mar, 1961, pp. 611–644.
[2] A. B. Carleial, “A case where interference does not reduce the capacity,” IEEE Trans. on Inform. Th., vol. 21, pp. 569–570, Sep, 1975.
[3] H. Sato, “The capacity of Gaussian interference channel under strong interference,” IEEE Trans. on Inform. Th., vol. 27, pp. 786–788, Nov, 1981.
[4] T. S. Han and K. Kobayashi, “A new achievable region for the interference channel,” IEEE Trans. on Inform. Th., vol. 27, pp. 49–60, Jan, 1981.
[5] M. H. M. Costa and A. A. E. Gamal, “The capacity region of the discrete memoryless interference channel with strong interference,” IEEE Trans. on Inform. Th., vol. 33, pp. 710–711, Sep, 1987.
[6] H. Sato, “On degraded Gaussian two-user channels,” IEEE Trans. on Inform. Th., vol. 24, pp. 637–640, Nov, 1978.
[7] R. Benzel, “The capacity region of a class of discrete additive degraded interference channels,” IEEE Trans. on Inform. Th., vol. 25, pp. 228–231, Mar, 1979.
[8] A. A. E. Gamal and M. H. M. Costa, “The capacity region of a class of deterministic interference channels,” IEEE Trans. on Inform. Th., vol. 28, pp. 343–346, Mar, 1982.
[9] H. Sato, “Two-user communication channels,” IEEE Trans. on Inform. Th., vol. 23, pp. 295–304, May, 1977.
[10] A. B. Carleial, “Interference channels,” IEEE Trans. on Inform. Th., vol. 24, pp. 60–70, Jan, 1978.
[11] ——, “Outer bounds on the capacity of the interference channels,” IEEE Trans. on Inform. Th., vol. 29, pp. 602–606, Jul, 1983.
[12] M. H. M. Costa, “On the gaussian interference channel,” IEEE Trans. on Inform. Th., vol. 31, pp. 607–615, Sep, 1985.
[13] G. Kramer, “Outer bounds on the capacity of Gaussian interference channels,” IEEE Trans. on Inform. Th., vol. 50, pp. 581–586, Feb, 2004.
[14] H. Chong, M. Motani, H. Garg, and H. E. Gamal, “On the Han-Kobayashi region for the interference channel,” IEEE Trans. on Inform. Th., vol. 54, pp. 3188–3195, Jul, 2008.
[15] M. Kobayashi and T. S. Han, “A further consideration of the HK and CMG regions for the interference channel,” Jan, 2007, preprint, available at http://ita.ucsd.edu/workshop/07/files/paper/paper5f133.pdf.
[16] G. A. Hodtani, “Improvement of the Han-Kobayashi rate region for the general interference channel,” Aug. 2010, available online: http://arxiv.org/abs/1008.4153.
[17] V. S. Annapureddy and V. V. Veeravalli, “Sum capacity of the Gaussian interference channels in the low interference regime,” IEEE Trans. on Inform. Th., vol. 55, pp. 3032 – 3050, Jul. 2009.
[18] A. S. Motahari and A. K. Khandani, “Capacity bounds for the Gaussian interference channel,” IEEE Trans. on Inform. Th., vol. 55, pp. 620–643, Feb, 2009.
[19] X. Shang, G. Kramer, and B. Chen, “A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels,” IEEE Trans. on Inform. Th., vol. 55, pp. 689–699, Feb, 2009.
[20] X. Shang, B. Chen, G. Kramer, and H. V. Poor, “Capacity regions and sum-rate capacities of vector Gaussian interference channels,” IEEE Trans. on Inform. Th., vol. 56, no. 10, pp. 5030 – 5044, Oct, 2010.
[21] R. Etkin, D. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” IEEE Trans. on Inform. Th., vol. 54, pp. 5534–5562, Dec, 2008.
[22] G. Bresler and D. Tse, “The two-user Gaussian interference channel: A deterministic view,” European Trans. Telecommun., vol. 19, no. 4, pp. 333–354, Jun, 2008.
[23] S. Vishwanath and S. A. Jafar, “On the capacity of vector Gaussian interference channels,” in Proc. of Inform. Th. Workshop, Oct, 2004, pp. 365 – 369, San Antonio, TX.
[24] E. Telatar and D. N. C. Tse, “Bounds on the capacity region of a class of interference channels,” in Proc. IEEE Int. Symp. on Inform. Th., Jun, 2007, pp. 2871 – 2874.
[25] I.-H. Wang and D. N. C. Tse, “Gaussian interference channels with multiple receive antennas: Capacity and generalized degrees of freedom,” in Proc. of Annu. Allerton Conference on Communication, Control and Computing, Monticello, IL, 2004.
[26] S. Karmakar and M. K. Varanasi, “The generalized degrees of freedom region of the MIMO interference channel,” submitted, IEEE Transactions on Information Theory, Mar, 2011, available online at http://arxiv.org/abs/1103.2560.
[27] E. Akuiyibo, O. Leveque, and C. Vignat, “High SNR analysis of the MIMO interference channel,” in Proc. IEEE Int. Symp. on Inform. Th., July, 2008, pp. 905–909.
[28] P. A. Parker, D. W. Bliss, and V. Tarokh, “On the degrees-of-freedom of the MIMO interference channel,” in Proc. of Inform. Sciences and Systems, Mar, 2008, pp. 62–67.
[29] A. Raja, V. M. Prabhakaran, and P. Viswanath, “The two-user compound interference channel,” IEEE Trans. on Inform. Th., vol. 55, pp. 5100–5120, Nov, 2009.
[30] R. G. Gallager, *Information Theory and Reliable Communications*. New York: Wiley, 1968.

[31] S. Karmakar and M. K. Varanasi, “The diversity-multiplexing tradeoff of the symmetric MIMO 2-user interference channel,” in *Proc. IEEE Int. Symp. on Inform. Th., Austin, Texas*, Jun, 2010, pp. 2213 – 2217.

[32] ———, “The diversity-multiplexing tradeoff of the MIMO Z interference channel,” in *Proc. IEEE Int. Symp. on Inform. Th., Austin, Texas*, Jun, 2010.

[33] E. Telatar, “Capacity of multi-antenna Gaussian channels,” *European Trans. on Telecomm. ETT*, vol. 10(6), pp. 585–596, Nov, 1999.

[34] S. A. Jafar and M. J. Fakhereddin, “Degrees of freedom for the MIMO interference channel,” *IEEE Trans. on Inform. Th.*, vol. 53, pp. 2637–2642, Jul, 2007.

[35] M. Bilodeau and D. Brenner, *Theory of Multivariate statistics*. Springer-Verlag New York, Inc, 1999.

[36] R. A. Horn and C. R. Jhonson, *Matrix analysis*. Cambridge Univ. Press, 1990, vol. 1st.