When Janson meets McDiarmid: Bounded difference inequalities under graph-dependence

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Abstract

We establish concentration inequalities for Lipschitz functions of dependent random variables, whose dependencies are specified by forests. We also give concentration results for decomposable functions, improving Janson’s Hoeffding-type inequality for the summation of graph-dependent bounded variables. These results extend McDiarmid’s bounded difference inequality to the dependent cases.

1. Introduction

Concentration inequalities bound the deviation of a function of random variables from some value that is usually the expectation, see [2] for a good reference. One of the well-known ones, bounded difference inequality (also called McDiarmid’s inequality or Azuma-Hoeffding inequality) gives exponential concentration bound for Lipschitz functions of independent random variables.

McDiarmid’s inequality requires independence, thus is restrictive in certain applications. We extend it to the dependent cases via dependency graph, which is a common combinatorial tool for modelling the dependencies among random variables. The dependency graph has been widely used in the probability and statistics to establish normal approximation or Poisson approximation via the Stein’s method, cumulants, etc. (see, for example, [1, 12, 11, 10]). It is also heavily used in probabilistic combinatorics, such as Lovász local lemma [8], Janson’s inequality [14], etc.

In this note, we use the standard graph-theoretic notations. All graphs considered are finite, undirected and simple. Given a graph $G = (V, E)$, let $V(G)$ be the vertex set and $E(G)$ be the edge set. The edge connecting a pair of vertices $u, v$ is denoted as $\{u, v\}$, which is assumed to be unordered. For every $S \subseteq V(G)$, the induced subgraph of $G$ by $S$ is denoted as $G[S]$, that is, for any two vertices $u, v \in S$, $u, v$ are adjacent in $G[S]$ if and only if they are adjacent in $G$. A tree is a connected, acyclic graph, and a forest is a disjoint union of trees.

Throughout this note, let $n$ be a positive integer and $[n]$ be the set $\{1, 2, \ldots, n\}$. Let $\Omega_i$ be a Polish space for all $i \in [n]$, $\Omega = \prod_{i \in [n]} \Omega_i = \Omega_1 \times \ldots \times \Omega_n$ be the product space, $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}_+$ be the set of non-negative real numbers. Let $\| \cdot \|_p$ denote the standard $\ell_p$-norm of a vector. We use uppercase letters for random variables, lowercase letters for their realizations, and bold letters for vectors. For every set $V \subseteq [n]$, let $\Omega_V = \prod_{i \in V} \Omega_i$, $X_V = (X_i)_{i \in V}$, and $x_V = (x_i)_{i \in V}$.

We first introduce the definition of a Lipschitz function.

Definition 1.1 ($c$-Lipschitz). Given a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}_+^n$, a function $f : \Omega \to \mathbb{R}$ is $c$-Lipschitz if for all $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n) \in \Omega$,

$$\|f(x) - f(x')\| \leq \sum_{i=1}^n c_i 1_{\{x_i \neq x'_i\}},$$

where $c_i$ is the $i$-th Lipschitz coefficient of $f$ (with respect to the Hamming metric).

McDiarmid’s inequality states that a Lipschitz function of independent random variables concentrates around its expectation.
Theorem 1.2 (McDiarmid’s inequality [18]). Let \( f : \Omega \to \mathbb{R} \) be \( c \)-Lipschitz and \( X = (X_1, \ldots, X_n) \) be a vector of independent random variables that takes values in \( \Omega \). Then for every \( t > 0 \),

\[
P( f(X) - \mathbb{E}[f(X)] \geq t) \leq \exp\left(-\frac{2t^2}{\|c\|^2_2}\right),
\]

We extend McDiarmid’s inequality to the graph-dependent case, where the dependencies among random variables are characterized by a dependency graph.

Definition 1.3 (Dependency graph). Given a graph \( G = (V, E) \), we say that a random vector \( X = (X_i)_{i \in V} \) is \( G \)-dependent if for any disjoint \( S, T \subset V \) such that \( S \) and \( T \) are non-adjacent in \( G \) (that is, no edge in \( E \) has one endpoint in \( S \) and the other in \( T \)), random variables \( \{X_i\}_{i \in S} \) and \( \{X_j\}_{j \in T} \) are independent.

The above dependency graph is a strong version; there are ones with weaker assumptions, such as the one used in Lovász local lemma. Let \( K_n \) denote the complete graph on \([n]\), that is, every two vertices are adjacent. Then \( K_n \) is a dependency graph for any set of variables \( \{X_i\}_{i \in [n]} \). Note that the dependency graph for a set of random variables may not be necessarily unique and the sparser ones are the more interesting ones. The term ‘\( G \)-dependent’ (graph-dependent) first appeared in [20], and various other notions such as ‘locally dependent’ [6], ‘partly dependent’ [13], etc. are essentially referring to the graph-dependence.

Janson obtained a Hoeffding-type inequality for graph-dependent random variables by breaking up the sum into sums of independent variables.

Theorem 1.4 (Janson’s concentration inequality [13]). Let random vector \( X \) be \( G \)-dependent such that for every \( i \in V(G) \), random variable \( X_i \) takes values in a real interval of length \( c_i \geq 0 \). Then, for every \( t > 0 \),

\[
P\left( \sum_{i \in V(G)} X_i - \mathbb{E}\left[ \sum_{i \in V(G)} X_i \right] \geq t \right) \leq \exp\left(-\frac{2t^2}{\chi_f(G)\|c\|^2_2}\right),
\]

where \( c = (c_i)_{i \in V(G)} \) and \( \chi_f(G) \) is the fractional chromatic number of \( G \).

A fractional coloring of a graph \( G \) is a mapping \( g \) from \( \mathcal{I}(G) \), the set of all independent sets of \( G \), to \([0, 1]\) such that \( \sum_{I \in \mathcal{I}(G) : v \in I} g(I) \geq 1 \) for every vertex \( v \in V(G) \). The fractional chromatic number \( \chi_f(G) \) of \( G \) is the minimum of the value \( \sum_{I \in \mathcal{I}(G)} g(I) \) over fractional colorings of \( G \).

2. Results

Here we introduce our concentration results for Lipschitz functions of dependent random variables, whose dependencies are specified by forests and for decomposable Lipschitz functions of general graph-dependent variables.

2.1. Concentration under forest-dependence

Our first result is for the case where the dependency graph is a tree.

Theorem 2.1. Let function \( f : \Omega \to \mathbb{R} \) be \( c \)-Lipschitz and \( \Omega \)-valued random vector \( X \) be \( G \)-dependent. If \( G \) is a tree, then for every \( t > 0 \),

\[
P\left( f(X) - \mathbb{E}[f(X)] \geq t \right) \leq \exp\left(-\frac{2t^2}{c_{\min}^2 + \sum_{(i,j) \in E(G)} (c_i + c_j)^2}\right),
\]

where \( c_{\min} \) is the minimum entry of \( c \).

A simple extension of the proof leads to our second result, in which the dependency graph is a forest.

Theorem 2.2. Let function \( f : \Omega \to \mathbb{R} \) be \( c \)-Lipschitz and \( \Omega \)-valued random vector \( X \) be \( G \)-dependent. If \( G \) is a disjoint union of trees \( \{T_i\}_{i \in [k]} \), then for \( t > 0 \),

\[
P\left( f(X) - \mathbb{E}[f(X)] \geq t \right) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^k c_{\min,i}^2 + \sum_{\{i,j\} \in E(G)} (c_i + c_j)^2}\right),
\]

where \( c_{\min,i} := \min\{c_j : j \in V(T_i)\} \) for all \( i \in [k] \).
Remark 2.3. If random variables \((X_1, \ldots, X_n)\) are independent, then the empty graph \(\overline{K}_n = ([n], \emptyset)\) is a valid dependency graphs for \((X_i)_{i \in [n]}\). In this case, inequality (4) becomes the McDiarmid’s inequality (Theorem 1.2), since each vertex is treated as a tree.

If all Lipschitz coefficients are of the same value \(c\), then the denominator of the exponent in (4) becomes \(k c^2 + 4(n - k) c^2 = (4n - 3k) c^2\), since the number of edges in the forest is \(n - k\). The denominator in Janson’s bound (3) is \(2n c^2\), since the fractional chromatic number of any tree is 2. Thus if \(k \geq 2n/3\), then (4) is better than Janson’s bound in this case.

2.2. Concentration of decomposable functions via fractional vertex coverings

For Lipschitz functions of general graph-dependent random variables, we give concentration results under certain decomposability constraints.

First we introduce the forest vertex covering and independent vertex covering of a graph. Formally, given a graph \(G\), we introduce the following.

(a) A family \(\{S_k\}_k\) of subsets of \(V(G)\) is a vertex cover of \(G\) if \(\bigcup S_k = V(G)\).

(b) A family \(\{(S_k, w_k)\}_k\) of pairs \((S_k, w_k)\), where \(S_k \subseteq V(G)\) and \(w_k \in [0, 1]\) is a fractional vertex cover of \(G\) if \(\sum_{k: v \in F_k} w_k = 1\) for every \(v \in V(G)\).

(c) A fractional forest vertex cover \(\{(F_k, w_k)\}_k\) of \(G\) is a fractional vertex cover such that each set \(F_k\) in it induces a forest of \(G\). We denote the set of (vertex sets of) disjoint trees in forest \(F_k\) as \(T(F_k)\). The set of all fractional forest vertex cover of graph \(G\) is denoted as \(FFC(G)\).

(d) A fractional independent vertex cover \(\{(I_k, w_k)\}_k\) of \(G\) is a fractional vertex cover such that \(I_k \in I(G)\) for every \(k\). The set of all fractional independent vertex cover of graph \(G\) is denoted as \(FIC(G)\).

Note that the fractional chromatic number \(\chi_f(G)\) of graph \(G\) is the minimum of \(\sum_k w_k\) over \(FIC(G)\) (see, for example, [13]).

Next, we introduce the decomposable Lipschitz functions. Given a graph \(G\) on \(n\) vertices and a vector \(c = (c_i)_{i \in [n]} \in \mathbb{R}^n_+\), a function \(f : \Omega \to \mathbb{R}\) is forest-decomposable \(c\)-Lipschitz with respect to graph \(G\) if for all \(x = (x_1, \ldots, x_n) \in \Omega\) and for all \(\{(F_k, w_k)\}_k \in FFC(G)\), there exist \((c_i)_{i \in F_k}\)-Lipschitz functions \(\{f_k : \Omega F_k \to \mathbb{R}\}_k\) such that \(f(x) = \sum_k w_k f_k(x_{F_k})\).

Theorem 2.4. Let \(\Omega\)-valued random vector \(X\) be \(G\)-dependent and function \(f : \Omega \to \mathbb{R}\) be forest-decomposable \(c\)-Lipschitz with respect to \(G\). Then for every \(t > 0\),

\[
\mathbb{P}(f(X) - \mathbb{E}[f(X)] \geq t) \leq \exp \left( -\frac{2t^2}{D(G, c)} \right),
\]

where

\[
D(G, c) := \min_{\{(F_k, w_k)\}_k \in FFC(G)} \left( \sum_k w_k \sqrt{\sum_{(i, j) \in E(G[F_k])} (c_i + c_j)^2 + \sum_{T \in T(F_k)} c_{\text{min}, k, T}^2} \right)^2,
\]

and \(c_{\text{min}, k, T} := \min\{c_i : i \in T\}\) for all \(T \in T(F_k)\).

Remark 2.5. An upper bound for \(D(G, c)\) via fractional chromatic number follows an approach by Janson [13]. Let \(\{(I_k, w_k)\}_k \in FIC(G)\) be a fractional independent vertex cover of \(G\). Since \(FIC(G) \subseteq FFC(G)\) for all graph \(G\), then

\[
D(G, c) \leq \left( \sum_k w_k \sqrt{\sum_{i \in I_k} c_i^2} \right)^2 = \left( \sum_k w_k \sqrt{\sum_{i \in I_k} c_i^2} \right)^2 \leq \left( \sum_k w_k \right) \left( \sum_k w_k \sum_{i \in I_k} c_i^2 \right) = \left( \sum_k w_k \right) \left( \sum_{i \in V(G)} \sum_{k: i \in F_k} c_i^2 \right) = \left( \sum_k w_k \right) \sum_{i \in V(G)} c_i^2.
\]

Then by choosing \(\{(F_k, w_k)\}_k \in FFC(G)\) with \(\sum_k w_k = \chi_f(G)\), we have \(D(G, c) \leq \chi_f(G)\|c\|_2^2\), which is exactly the denominator in Janson’s bound in (3).
2.2.1. Concentration of the sum of graph-dependent random variables

Here we give an application. This improves Janson’s Hoeffding-type inequality for graph-dependent random variables.

**Corollary 2.6.** Let random vector $\mathbf{X}$ be $G$-dependent. If for every $i \in V(G)$, random variable $X_i$ takes values in a real interval of length $c_i \geq 0$, then for every $t > 0$,

$$
\mathbb{P} \left( \sum_{i \in V(G)} X_i - \mathbb{E} \left[ \sum_{i \in V(G)} X_i \right] \geq t \right) \leq \exp \left( - \frac{2t^2}{D(G, c)} \right),
$$

where $c = (c_v)_{v \in V(G)}$ and $D(G, c)$ is defined by (6).

**Proof.** It suffices to show that the summation is forest-decomposable $c$-Lipschitz. Since for every $\{ (F_k, w_k) \} \in \text{FFC}(G)$, we have $\sum_{i \in V(G)} X_i = \sum_{i \in V(G)} \sum_{k : i \in F_k} w_i X_i = \sum_k w_i \sum_{i \in F_k} X_i$, then Corollary 2.6 follows from Theorem 2.4.

Next we give an example in which our bound is better than Janson’s.

**Example 2.7.** Let $(X_1, X_2, X_3)$ be dependent random variables with $K_3$ as their dependency graph, and $(X_i)_{i \in [9]}$ be independent variables that are also independent from $(X_i)_{i \in [9]}$. Then the vertex-disjoint union of $K_3$ and 6 copies of $K_1$ is a dependency graph for $(X_i)_{i \in [9]}$. If for every $i \in [9]$, random variable $X_i$ takes values in a real interval of length $c > 0$, then for every $t > 0$, Janson’s bound gives

$$
\mathbb{P} \left( \sum_{i \in [9]} X_i - \mathbb{E} \left[ \sum_{i \in [9]} X_i \right] \geq t \right) \leq \exp \left( - \frac{2t^2}{27c^2} \right),
$$

since $\chi_f = 3$. We give a slightly better bound

$$
\mathbb{P} \left( \sum_{i \in [9]} X_i - \mathbb{E} \left[ \sum_{i \in [9]} X_i \right] \geq t \right) \leq \exp \left( - \frac{8t^2}{81c^2} \right).
$$

This is by giving equal weight $1/2$ to vertex covers $F_1 = \{ 1, 2, 4, 5, 6, 7 \}$, $F_2 = \{ 1, 3, 4, 5, 8, 9 \}$ and $F_3 = \{ 2, 3, 6, 7, 8, 9 \}$. Then the subgraph induced by $F_k$ in $G$ is a vertex-disjoint union of $K_2$ and 4 copies of $K_1$ for all $k \in [3]$, thus we have

$$
\left( \sum_k w_k \sqrt{\sum_{\{i, j\} \in E(G[F_k])}(c_i + c_j)^2 + \sum_{T \in T(F_k)} c_{\text{min}, k, T}^2} \right)^2 = \left( \frac{3}{2} \sqrt{2^2 + 1 + 4} \right)^2 = 81c^2/4.
$$

2.3. Concentration under local dependence

A sequence of random variables $(X_i)_{i=1}^n$ is said to be $f(n)$-dependent if subsets of variables separated by some distance $f(n)$ are independent. This was introduced by Hoeffding and Robbins [9] and has been studied extensively (see, for example, [22, 5]). This is usually the canonical application for the results based on the dependency graph model. A special case of $f(n)$-dependence when $f(n) = m$ is the following $m$-dependent model.

**Definition 2.8** $(m$-dependence [9]). A sequence of random variables $(X_i)_{i=1}^n$ is $m$-dependent for some $m \geq 1$ if $(X_j)_{j=1}^m$ and $(X_j)_{j=m+i+1}^n$ are independent for all $i > 0$.

The $m$-dependent sequences usually appear as block factors. Let $k \in \mathbb{N}$, the sequence $(X_i)$ is an $k$-block factor if there are an independent and identically distributed sequence $(Y_i)_{i=1}^\infty$ and a function $g : \mathbb{R}^k \to \mathbb{R}$ such that $X_i = g(Y_i, \ldots, Y_{i+k-1})$. Note that every such sequence $(X_i)$ is $(k-1)$-dependent, and there are $m$-dependent sequences that are not block factors, see [3].

**Corollary 2.9.** Let $f : \Omega \to \mathbb{R}$ be $c$-Lipschitz and $\Omega$-valued random vector $\mathbf{X}$ be $m$-dependent. Then for every $t > 0$,

$$
\mathbb{P} \left( f(\mathbf{X}) - \mathbb{E} [f(\mathbf{X})] \geq t \right) \leq \exp \left( - \frac{2t^2}{\sum_{i \in [\frac{n}{m}]} \left( \sum_{j \in B_i \cup B_{i+1}} c_j \right)^2 \min_{i \in [\frac{n}{m}]} \left( \sum_{j \in B_i} c_j \right)^2} \right),
$$

where $c = (c_v)_{v \in V(G)}$ and $D(G, c)$ is defined by (6).
where for every $j \in \left[\frac{n}{m}\right]$, 
\begin{equation}
B_j := \{k : (j - 1)m + 1 \leq k \leq jm\}, \quad \text{and} \quad B_{\left\lceil \frac{n}{m} \right\rceil} := [n] \setminus \bigcup_{j \in \left[\frac{n}{m}\right]} B_j.
\end{equation}

**Remark 2.10.** Corollary 2.9 improves the following bound obtained by Paulin in [19, Example 2.14]:

\[
\mathbb{P} \left( f(X) - \mathbb{E} [f(X)] \geq t \right) \leq \exp \left( -\frac{2\ell^2}{\sum_{i \in \left[\frac{n}{m}\right]} \left( \sum_{j \in B_i \cup B_{i+1}} c_j \right)^2 + \left( \sum_{j \in B_{\left\lceil \frac{n}{m} \right\rceil}} c_j \right)^2} \right),
\]

where the second summand in the denominator of the exponent is without taking minimum over blocks. Note that Corollary 2.9 does not assume stationarity, and it may be slightly improved by choosing better grouping schemes.

If all Lipschitz coefficients are of the same value $c$ and w.l.o.g. assume that $n$ is divisible by $m$, then the denominator of the exponent in (8) becomes $(2mc)^2 (n/m - 1) + m^2 c^2 \leq 4mnc^2$, thus we have
\begin{equation}
\mathbb{P} \left( f(X) - \mathbb{E} [f(X)] \geq t \right) \leq \exp \left( -\frac{\ell^2}{2mnc^2} \right),
\end{equation}
which is $4m$ times worse than the independent case, see (2).

**Proof of Corollary 2.9.** A dependency graph for $m$-dependent random variables $(X_i)_{i=1}^n$ is $D_{n,m} = ([n], \{\{i,j\} : i, j \in [n], |i - j| \in [m]\})$. Note that $D_{n,m}$ is not a forest, nevertheless, via the following transformation, we can apply our results. We group the $m$-dependent random variables $(X_i)_{i=1}^n$ into $\left[\frac{n}{m}\right]$ blocks such that each block $(X_i)_{i \in B_j}$ contains $m$ consecutive random variables except for the last one, which might contain less than $m$ ones, where $(B_j)$ are defined in (9). The resulting dependency graph for the blocks $((X_i)_{i \in B_j} : j \in \left[\frac{n}{m}\right])$ is a path $P$ on $\left[\frac{n}{m}\right]$ vertices. Since the Lipschitz coefficient $\tilde{c}_j$ of each block $(X_i)_{i \in B_j}$ is at most $\sum_{i \in B_j} c_i$ due to the triangle inequality, then we have $\tilde{c}_{\min}^2 + \sum_{(i,j) \in E(P)} (\tilde{c}_i + \tilde{c}_j)^2 = \sum_{i \in \left[\frac{n}{m}\right]} \left( \sum_{j \in B_i \cup B_{i+1}} c_j \right)^2 + \min_{i \in \left[\frac{n}{m}\right]} \left( \sum_{j \in B_i} c_j \right)^2$. The result then follows from the Theorem 2.1. \qed

3. Proofs

We first introduce some additional notations. Given a graph $G$, for every vertex $v \in V(G)$, let $N_G(v) := \{v \in V(G) : \{u,v\} \in E(G)\}$ denote the neighbours of $v$, and $N_G^+(v) := N_G(v) \cup \{v\}$ denote the inclusive neighborhood. The neighborhood of a set of vertices $V$ is $N_G^+(V) := \cup_{v \in V} N_G^+(v)$, and the neighbours of $V$ is $N_G(V) := N_G^+(V) \setminus V$. The subscript $G$ might be omitted if it is clear from context. Given $\Omega = \prod_{i \in [n]} \Omega_i$, let $x = (x_1, \ldots, x_n)$ be an arbitrary vector in $\Omega$ and $x^{(i)} := (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)$, where $x'_i \in \Omega_i$.

For tree-dependent random variables $X$, without loss of generality, we assume that the dependency tree $G$ satisfies the following assumptions:

**Rooted:** $G$ is rooted at the vertex $n$ and $c_n = c_{\min}$.

**Ordered:** for every pair of vertices $i, j \in V(G)$, $j$ is a descendant of $i$ only if $j < i$.

Notice the above assumptions are just for the simplicity of the statement of the vertex exposure martingale in the proofs, and there is no such requirement for the ordering of the vertices. Such ordering of tree vertices exists and can be obtained via a topological sort.

We briefly explain the idea before the formal proof. The proof of Theorem 2.1 relies on Lemma 3.1, which states that the small deviation of $\mathbb{E} [f(X) \mid X_{[i-1]} = x_{[i-1]}, X_i = x_i]$ with respect to $x_i \in \Omega_i$ for all $i \in [n]$ leads to the concentration of $f(X)$ around its expectation. Our task is thus to bound the difference of the conditional expectations $\mathbb{E} [f(X) \mid X_{[i-1]} = x_{[i-1]}, X_i = \alpha] - \mathbb{E} [f(X) \mid X_{[i-1]} = x_{[i-1]}, X_i = \beta]$ for any $\alpha, \beta \in \Omega_i$ and given $X_{[i-1]}$ (Lemma 3.4). This is by the coupling constructions, namely, jointly distributed variables $(Y^{(i)}, Z^{(i)})$ whose marginal distributions are distributions of $X$ conditioned on $\{X_{[i-1]} = x_{[i-1]}, X_i = x_i\}$ and on $\{X_{[i-1]} = x_{[i-1]}, X_i = x_i^{(i)}\}$, respectively. Hence, the main part of the proof is to construct such
recursive couplings of the conditional probability distribution (Lemma 3.3) whose feasibility relies on the independence among \( X \) (Lemma 3.2).

First of all, recall a lemma by McDiarmid. By this lemma, it suffices to bound the deviation of \( \mathbb{E} [f(X) \mid X_{[i-1]} = x_{[i-1]}, X_i = x_i] \) with respect to \( x_i \in \Omega_i \) for all \( i \in [n] \).

**Lemma 3.1** ([18]). If for every \( i \in [n], \omega_{[i-1]} \in \Omega_{[i-1]} \), there is a constant \( c_i \geq 0 \) such that

\[
\sup_{\alpha \in \Omega_i} \mathbb{E} [f(X) \mid X_{[i-1]} = \omega_{[i-1]}, X_i = \alpha] - \inf_{\beta \in \Omega_i} \mathbb{E} [f(X) \mid X_{[i-1]} = \omega_{[i-1]}, X_i = \beta] \leq c_i,
\]

then for \( s > 0 \),

\[
\mathbb{E} [\exp (s f(X) - \mathbb{E} [f(X)])] \leq \exp \left( \frac{s^2}{8} \sum c_i^2 \right).
\]

Moreover, the bound on moment-generating function \( (12) \) implies that for \( t > 0 \),

\[
\mathbb{P} (f(X) - \mathbb{E} [f(X)] \geq t) \leq \exp \left( -\frac{2t^2}{\sum n_i c_i^2} \right).
\]

For every non-root vertex \( i \in V(G) \), let \( p_i \) be the parent vertex of \( i \). For the rest of the section, define \( S_i := [i+1,n] \setminus \{p_i\} \), where \( [j,k] \) stands for the integer set \( \{j, \ldots, k\} \) for all \( j < k \). The following lemma indicates that the distribution of \( X_{S_i} \) is independent of the realization of \( X_i \) when \( X_{[i-1]} \) is given.

**Lemma 3.2.** For every \( i \in [n-1], \omega_{S_i} \in \Omega_{S_i} \), we have

\[
\mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{[i]}) = \mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{(i)[i]})
\]

**Proof.** Let \( T_i \) be the subtree rooted at vertex \( i \) of the dependency tree \( G \) (such \( T_i \) is also called fringe subtree in the literature). Since \( G \) is assumed to be Ordered, we have \( V(T_i) \subseteq [i] \), and \( [i] = V(T_i) \cup ([i-1] \setminus V(T_i)) \).

We will actually show stronger results: \( X_{V(T_i)} \) is independent of \( \{X_{S_i}, X_{[i-1]} \setminus V(T_i)\} \), which follows from the following two observations:

**Observation 1:** \( N^-_G(T_i) \cap ([i-1] \setminus V(T_i)) = \emptyset \). Since \( G \) is a dependency tree, then \( N^-_G(T_i) = V(T_i) \cup \{p_i\} \), and \( p_i \in [i+1,n] \) because \( G \) is Ordered, then \( p_i \not\in [i-1] \setminus V(T_i) \). Thus \( N^-_G(T_i) \cap ([i-1] \setminus V(T_i)) = \emptyset \).

**Observation 2:** \( N^+_G(T_i) \cap S_i = \emptyset \). From observation 1, \( N^+_G(T_i) = V(T_i) \cup \{p_i\} \). Then observation 2 follows from the definition \( S_i = [i+1,n] \setminus \{p_i\} \).

Observations 1 and 2 indicate that \( X_{V(T_i)} \) is independent of \( \{X_{S_i}, X_{[i-1]} \setminus V(T_i)\} \), due to the definition of the dependency graphs in Definition 1.3. Then

\[
\mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i-1]} \setminus V(T_i) = x_{[i-1]} \setminus V(T_i)) = \mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i-1]} \setminus V(T_i) = x_{[i-1]} \setminus V(T_i), X_{V(T_i)} = x_{V(T_i)})
\]

\[
= \mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{[i]})
\]

Similarly, we also have

\[
\mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i-1]} \setminus V(T_i) = x_{[i-1]} \setminus V(T_i))
\]

\[
= \mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i-1]} \setminus V(T_i) = x_{[i-1]} \setminus V(T_i), X_{V(T_i)} \setminus \{i\} = x_{V(T_i) \setminus \{i\}}, X_i = x'_i) = \mathbb{P} (X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{(i)[i]})
\]

The lemma follows from the combinations of the above. \( \square \)

Then we introduce a Marton-type coupling [17, 19], more precisely, we construct the joint distribution of random vector \( (Y^{(i)}, Z^{(i)}) \) taking values in \( \Omega \times \Omega \), with respect to all \( i \in [n-1], x = (x_1, \ldots, x_n) \in \Omega \), and \( x^{(i)} = (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n) \), where \( x'_i \in \Omega_i \). Specifically, the distributions of \( Y^{(i)} := (Y^{(i)}_1, \ldots, Y^{(i)}_n) \) and \( Z^{(i)} := (Z^{(i)}_1, \ldots, Z^{(i)}_n) \) are set as follows.

(C1) \( Y^{(i)}_{[i]} = x_{[i]} \).

(C2) For every \( \omega_{[i+1,n]} \in \Omega_{[i+1,n]} \),

\[
\mathbb{P} (Y^{(i)}_{[i+1,n]} = \omega_{[i+1,n]} \mid X_{[i]} = x_{[i]}) = \mathbb{P} (X_{[i+1,n]} = \omega_{[i+1,n]} \mid X_{[i]} = x_{[i]})
\]
(C3) \( Z_{[i]}^{(i)} = x_{[i]}^{(i)} \), \( Z_{S_i}^{(i)} = Y_{S_i}^{(i)} \).

(C4) For every \( \omega_{S_i} \in \Omega_{S_i} \) and \( \omega_{p_i} \in \Omega_{p_i} \),

\[
\mathbb{P}\left( Z_{p_i}^{(i)} = \omega_{p_i} \mid Z_{S_i}^{(i)} = \omega_{S_i} \right) = \mathbb{P}\left( X_{p_i} = \omega_{p_i} \mid X_{[i]} = x_{[i]}^{(i)}, X_{S_i} = \omega_{S_i} \right).
\]

The next lemma states that \((Y^{(i)}, Z^{(i)})\) has the desired marginal distributions.

**Lemma 3.3.** For every \( i \in [n-1] \), \( \omega_{[i+1,n]} \in \Omega_{[i+1,n]} \), we have

(A1) \( \mathbb{P}\left( Y_{[i+1,n]}^{(i)} = \omega_{[i+1,n]} \right) = \mathbb{P}\left( X_{[i+1,n]} = \omega_{[i+1,n]} \mid X_{[i]} = x_{[i]}^{(i)} \right) \),

(A2) \( \mathbb{P}\left( Z_{[i+1,n]}^{(i)} = \omega_{[i+1,n]} \right) = \mathbb{P}\left( X_{[i+1,n]} = \omega_{[i+1,n]} \mid X_{[i]} = x_{[i]}^{(i)} \right). \)

**Proof.** (A1) is by the constructions of \( Y^{(i)} \). For (A2), we arbitrarily choose \( \omega_{[i+1,n]} = (\omega_{i+1}, \ldots, \omega_n) \in \Omega_{[i+1,n]} \), then

\[
\mathbb{P}\left( Z_{[i+1,n]}^{(i)} = \omega_{[i+1,n]} \right) = \mathbb{P}\left( Z_{S_i}^{(i)} = \omega_{S_i} \right) \mathbb{P}\left( Z_{p_i}^{(i)} = \omega_{p_i} \mid Z_{S_i}^{(i)} = \omega_{S_i} \right) = \mathbb{P}\left( Y_{S_i}^{(i)} = \omega_{S_i} \right) \mathbb{P}\left( Z_{p_i}^{(i)} = \omega_{p_i} \mid Z_{S_i}^{(i)} = \omega_{S_i} \right).
\]

Combining with (14) and (15) gives

\[
\mathbb{P}\left( Z_{[i+1,n]}^{(i)} = \omega_{[i+1,n]} \right) = \mathbb{P}\left( X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{[i]}^{(i)} \right) \mathbb{P}\left( X_{p_i} = \omega_{p_i} \mid X_{[i]} = x_{[i]}^{(i)}, X_{S_i} = \omega_{S_i} \right).
\]

By Lemma 3.2, we have

\[
\mathbb{P}\left( Z_{[i+1,n]}^{(i)} = \omega_{[i+1,n]} \right) = \mathbb{P}\left( X_{S_i} = \omega_{S_i} \mid X_{[i]} = x_{[i]}^{(i)} \right) \mathbb{P}\left( X_{p_i} = \omega_{p_i} \mid X_{[i]} = x_{[i]}^{(i)}, X_{S_i} = \omega_{S_i} \right) = \mathbb{P}\left( X_{[i+1,n]} = \omega_{[i+1,n]} \mid X_{[i]} = x_{[i]}^{(i)} \right).
\]

This completes the proof. \( \square \)

**Lemma 3.4.** For every \( i \in [n-1] \), we have

\[
\mathbb{E}\left[ f(X) \mid X_{[i]} = x_{[i]}^{(i)} \right] - \mathbb{E}\left[ f(X) \mid X_{[i]} = x_{[i]}^{(i)} \right] \leq c_i + c_{p_i}.
\]

**Proof.** By the construction of random vectors \( Y^{(i)}, Z^{(i)} \) and Lemma 3.3, we have

\[
\mathbb{E}\left[ f(X) \mid X_{[i]} = x_{[i]}^{(i)} \right] - \mathbb{E}\left[ f(X) \mid X_{[i]} = x_{[i]}^{(i)} \right] = \mathbb{E}\left[ f(Y^{(i)}) \right] - \mathbb{E}\left[ f(Z^{(i)}) \right].
\]

By the linearity of expectation and the Lipschitz assumption (1), we get

\[
\mathbb{E}\left[ f(Y^{(i)}) \right] - \mathbb{E}\left[ f(Z^{(i)}) \right] = \mathbb{E}\left[ f(Y^{(i)}) - f(Z^{(i)}) \right] \leq \mathbb{E}\left[ \sum_{j=1}^{n} c_j \mathbb{1}_{\{Y_j \neq Z_j\}} \right] \leq c_i + c_{p_i},
\]

where the last inequality is because the only different variables of \( Y^{(i)}, Z^{(i)} \) are the \( i \)-th and \( p_i \)-th ones due to the coupling construction. \( \square \)

We are now ready to prove Theorem 2.1 and Theorem 2.2.

**Proof of Theorem 2.1.** Combining Lemma 3.1 and Lemma 3.4, we have

\[
\mathbb{P}\left( f(X) - \mathbb{E}\left[ f(X) \right] \geq t \right) \leq \exp\left( -\frac{2t^2}{c_n^2 + \sum_{i \in V(G) \setminus \{i\}} (c_i + c_{p_i})^2} \right) = \exp\left( -\frac{2t^2}{c_{\min} + \sum_{\{i,j\} \in E(G)} (c_i + c_j)^2} \right),
\]

where the equality is due to the Rooted and Ordered assumptions. \( \square \)

**Proof of Theorem 2.2.** The proof is similar to that of Theorem 2.1. Without loss of generality, we assume that each tree \( T_i \) of the forest \( G \) is Rooted and Ordered. Then the proofs of Lemmas 3.2 - 3.4 remain valid, since variables in different connected components are independent. Then, the theorem follows from Lemma 3.1. \( \square \)
Proof of Theorem 2.4. Let \( \{(F_k, w_k)\}_{k \in [K]} \) be a fractional forest cover of \( G \). Since function \( f \) is forest-decomposable \( c \)-Lipschitz with respect to \( G \), then for \( s > 0 \), we have \( \mathbb{E} \left[ \exp \left( s (f(X) - \mathbb{E}[f(X)]) \right) \right] = \mathbb{E} \left[ \exp \left( \sum_{k \in [K]} sw_k f(X_{F_k}) \right) \right] \). Let \( z_1, \ldots, z_K \) be any set of \( K \) positive reals that sum to 1. By the Jensen’s inequality,

\[
\mathbb{E} \left[ \exp \left( s (f(X) - \mathbb{E}[f(X)]) \right) \right] \leq \mathbb{E} \left[ \sum_{k \in [K]} z_k \exp \left( \frac{sw_k}{z_k} (f(X_{F_k}) - \mathbb{E}[f(X_{F_k})]) \right) \right] = \sum_{k \in [K]} z_k \mathbb{E} \left[ \exp \left( \frac{sw_k}{z_k} (f(X_{F_k}) - \mathbb{E}[f(X_{F_k})]) \right) \right],
\]

where the equality is due to the linearity of expectation. Then Lemma 3.4 and Lemma 3.1 give

\[
(16) \quad \mathbb{E} \left[ \exp \left( s (f(X) - \mathbb{E}[f(X)]) \right) \right] \leq \sum_{k \in [K]} z_k \exp \left( \frac{s w_k^2}{8 z_k^2} \left( \sum_{\{i,j\} \in E(G[F_k])} (c_i + c_j)^2 + \sum_{T \in \mathcal{T}(F_k)} c_{\min,k,T}^2 \right) \right).
\]

Next, for all \( k \in [K] \), we choose

\[
z_k = \frac{w_k}{Z} \sqrt{\sum_{\{i,j\} \in E(G[F_k])} (c_i + c_j)^2 + \sum_{T \in \mathcal{T}(F_k)} c_{\min,k,T}^2},
\]

with

\[
Z = \sum_{k \in [K]} w_k \sqrt{\sum_{\{i,j\} \in E(G[F_k])} (c_i + c_j)^2 + \sum_{T \in \mathcal{T}(F_k)} c_{\min,k,T}^2}.
\]

Hence we have \( \mathbb{E} \left[ \exp \left( s (f(X) - \mathbb{E}[f(X)]) \right) \right] \leq \exp \left( s^2 Z^2 / 8 \right) \). Combining with Lemma 3.1 completes the proof. \( \square \)

4. Discussions

We establish bounded difference inequalities for forest-dependent random variables; it is unclear whether the proof based on coupling technique can be adapted to the case of general graph-dependent case without imposing decomposability constraints to the functions. Some heuristic ideas of transforming the general graph to a forest are given in [23], however, various ad hoc constructions are needed for different graphs. For the summation, which satisfies the decomposability constraint, we obtain a better bound under graph-dependence than Janson’s bound. The direct application of Theorem 2.2 under forest-dependence for summation may also give better results than Janson’s, see Remark 2.3.

The (fractional) forest vertex covering used in Subsection 2.2 closely relates to (fractional) vertex arboricity. Given a graph \( G \), the vertex arboricity \( a(G) \) is the minimum number of subsets into which the vertex set \( V(G) \) can be partitioned so that each subset induces an acyclic subgraph, and the fractional vertex arboricity \( a_f(G) \) is the minimum of \( \sum_k w_k \) over \( FFC(G) \). Other upper bounds on \( D(G, c) \) in (6) can also be obtained via fractional vertex arboricity following Janson’s approach using the Cauchy-Schwarz inequality, see Remark 2.5.

Other dependence characterizations for concentration widely used in random fields and statistical physics are various dependency matrices, which quantify the strength of dependence among variables. These matrices include Dobrushin interdependence matrix [7, 4] and other mixing-based dependency matrices [21, 16, 19]. To employ their results, suitable estimates of the mixing coefficients are required for specific applications, which might not be handy in the combinatorial applications. On the other hand, the coupling construction in this note can be used for the estimation of mixing coefficients, see [15].
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