Newhouse Laminations

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Abstract

Any \((q+2)\)-dimensional unfolding of a non-degenerate homoclinic tangency contains a smooth codimension 2 lamination \(L_F\) of maps with infinitely many sinks. In particular, all sinks move smoothly along each \(q\)-dimensional leaf of the lamination.

The space \(\text{Poly}_d(\mathbb{C}^n)\) of degree at most \(d\) polynomial maps of \(\mathbb{C}^n\) contains a codimension 2 lamination, with transversal section homemorphic to \(\mathbb{R} \setminus \mathbb{Q}\), of maps with infinitely many sinks. The sinks move holomorphically along the leaves. The same holds also for the space \(\text{Poly}_d(\mathbb{R}^n)\) of degree at most \(d\) polynomial maps of \(\mathbb{R}^n\). In particular, the Hénon family \(F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2\),

\[
F_{a,b}(x,y) = \begin{pmatrix} a - x^2 - by \\ x \end{pmatrix}
\]

contains a set homeomorphic to \(\mathbb{R} \setminus \mathbb{Q}\) of maps with infinitely many sinks.

1 Introduction

One of the main question in dynamics is whether a system is structurally stable or not. Hyperbolic systems, which are known to be structurally stable, have been intensively studied and completely understood. The situation become much more complicated if a hyperbolic system is deformed until it ceases to be hyperbolic. In particular homoclinic tangencies can appear. Unfolding of homoclinic tangencies are very partially understood.

In the case of dissipative rank one systems, part of the dynamics of unfolding of homoclinic tangencies can be described by Hénon-like maps. Three phenomena have been detected in unfoldings of dissipative rank one homoclinic tangencies.

1. Newhouse phenomena (see [20]): there are maps near homoclinic tangencies which have infinitely many sinks.

2. Strange attractors (see [2] [19]): there are maps with an attractor having positive Lyapunov exponent. Moreover this attractor carries an SRB measure, see [3] [5].

3. Universality (see [18]): there are maps with a Cantor attractor having zero Lyapunov exponent. Moreover this Cantor attractor has universal geometrical properties.

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1 A rank one system is a system with only one unstable Lyapunov exponent.
We study here the stability of the Newhouse phenomenon. In particular we show that in an unfolding of a dissipative rank one homoclinic tangency there are codimension 2 laminations of maps with infinitely many sinks. The sinks move smoothly along the leaves of the lamination.

More specifically, we consider $C^\infty$ local diffeomorphisms on a $C^\infty$ manifold of any dimension with a strong homoclinic tangency, see Definition 2.6. One of the properties of a strong homoclinic tangency is that it is a tangency between the stable and unstable manifold of a rank one saddle point. Moreover if $\mu$ is the unstable eigenvalue and $\lambda_1$ is the spectral radius of the stable part, then a map with a strong homoclinic tangency satisfies also:

$$|\lambda_1||\mu|^3 < 1 \quad (1.1)$$

The collection of these maps is the complement of finitely many manifolds in the space of local $C^\infty$ maps satisfying also $|\lambda_1||\mu|^3 < 1$. Given a map with a strong homoclinic tangency $f$, we consider finite dimensional unfoldings of $f$, see Definition 2.14. The collection of these unfoldings is the complement of finitely many manifolds in the space of all families through $f$. We prove the following:

**Theorem A.** Let $F : \mathcal{P} \times M \rightarrow M$ be a two dimensional unfolding of a map $f$ with a strong homoclinic tangency, then there exists a set $NH \subset \mathcal{P}$ such that

- every map in $NH$ has infinitely many sinks,
- $NH$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$,
- the Minkowski dimension $MD(NH) \geq \frac{1}{2}$.

The conclusions of this theorem hold in particular for the following families.

- The two-dimensional real Hénon family $F_{a,b} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$F_{a,b}(x,y) = \left( a - x^2 - by, x \right).$$

It was already shown in [22, 13, 25] that there are real Hénon maps with infinitely many sinks.

- The real Hénon family of maps of $\mathbb{R}^n$. In [21] it is already indicated how to construct a map with infinitely many sinks in such families.

- The space $\text{Poly}_d(\mathbb{C}^n)$ of degree at most $d \geq 2$ polynomial maps of $\mathbb{C}^n$, $n \geq 1$. It was already shown that there are maps with infinitely many sinks in $\text{Poly}_d(\mathbb{C}^2)$ with $d$ large enough (see [11]) and in $\text{Poly}_d(\mathbb{C}^3)$ with $d \geq 2$ (see [8]).

Our main result describes the stability of the Newhouse phenomenon. There are maps with infinitely many sinks, which simultaneously move along codimension 2 families. This answers a question in [12].
Theorem B. Let $M$, $P$ and $T$ be $C^\infty$ manifolds and $F : (P \times T) \times M \to M$ a $C^\infty$ family with $\dim(P) = 2$. If $F_0 : (P \times \{\tau_0\}) \times M \to M$ is an unfolding of a map $f_{\tau_0}$ with a strong homoclinic tangency, then the set of maps with infinitely many sinks, $NH_F$, satisfies the following:

- $NH_F$ contains a codimension 2 lamination $L_F$,
- $L_F$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \times \mathbb{R}^{\dim(T)}$,
- the leaves of $L_F$ are $C^1$ codimension 2 manifolds.

An application of the main theorems to two and higher dimensional Hénon dynamics follows.

Theorem C. The real Hénon family contains a set $NH$, homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, of maps with infinitely many sinks. Moreover the space $\text{Poly}_d(\mathbb{R}^n)$ of real polynomials of $\mathbb{R}^n$ of degree at most $d$ contains a codimension 2 lamination of maps with infinitely many sinks. The lamination is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \times \mathbb{R}^m$ where $m = \dim \text{Poly}_d(\mathbb{R}^n) - 2$ and the sinks move analytically along the leaves.

Observe that the laminations mentioned in Theorem C are non trivial. Consider the two-dimensional Hénon family $F_{a,b}$. One can perturb this family by adding polynomial terms to obtain a new family $\tilde{F}_{a,b}$. According to [15], one can adjust the polynomial perturbation such that strongly dissipative Hénon maps in the boundary of chaos of $F$ are never topologically conjugate to strongly dissipative Hénon maps in the boundary of chaos of $\tilde{F}$. The two families are topologically different. Nevertheless, Theorem C says that the Newhouse points with their topological characteristics, persist.

Theorem D. Let $\text{Poly}_d(\mathbb{C}^n)$ be the space of polynomial maps of $\mathbb{C}^n$ of degree at most $d$. The space $\text{Poly}_d(\mathbb{C}^n)$ contains a (complex) codimension 2 lamination of maps with infinitely many sinks, whose transversal section is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. The sinks move holomorphically along the leaves.

Maps satisfying the Newhouse phenomenon have been constructed in different contexts. In particular, there are many examples of Baire sets of maps with infinitely many sinks in the space of systems, see [20, 9, 11, 17, 21, 6, 24]. In most studies the method behind these results relies on persistency of tangencies where the thickness of the stable and the unstable Cantor sets plays a crucial role.

In order to study the stability of maps with infinitely many sinks we needed to introduce a different method. This method does not rely on the persistency of tangency and the thickness condition is replaced by (1.1). Notice that in the Hénon family the set of maps satisfying the thickness condition and the one satisfying (1.1) intersect but they are not contained in each other.

The method is an inductive procedure and it is inspired by the construction of critical points and binding points in [2, 4]. From the given tangency we construct an almost periodic critical point which lies in the basin of a primary sink. Then we chose a secondary critical point, a binding point of the previous one. The location and the high speed of the return of the secondary critical point creates a secondary tangency. The method is very explicit and constructive. This allows to construct global curves of secondary tangencies and to have a precise control of the domains, where the primary sink occurs. This leads
to a transversality property which prevents the sink to disappear and allows to build the laminations.

As final remark we would like to stress that our method allows to have geometrical informations on the set of maps with infinitely many sinks. We can in particular estimate its Minkowski dimension. Other dimension estimates were obtained in [23, 26, 7].

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2 Preliminaries

The following well-known result is due to Steinberg.

**Theorem 2.1.** Given \((\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{R}^m\), there exists \(N (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{N}\) such that the following holds. Let \(M\) be a \(m\) dimensional \(C^\infty\) manifold and let \(f : M \to M\) be a diffeomorphism with a rank one saddle point \(p \in M\), with unstable eigenvalue \(|\lambda_0| > 1\) and stable eigenvalues \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})\). If for all \(j = 0, \ldots, m-1\),

\[
\lambda_j \neq \prod_{i \neq j} \lambda_i^{k_i}
\]

for \(k = (k_0, \ldots, k_{m-1}) \in \mathbb{N}^m\) with \(2 \leq |k| = k_0 + \cdots + k_{m-1} \leq N\) then \(f\) is \(C^4\) linearizable.

**Definition 2.3.** Let \(M\) be an \(m\)-dimensional \(C^\infty\) manifold and \(f : M \to M\) a diffeomorphism with a rank one saddle point \(p \in M\). We say that \(p\) satisfies the \(C^4\) non-resonance condition if (2.2) holds.

**Theorem 2.4.** Let \(M\) be a \(m\) dimensional \(C^\infty\) manifold and \(f : M \to M\) a diffeomorphism with a rank one saddle point \(p \in M\) which satisfies the \(C^4\) non-resonance condition. Let \(0 \in \mathcal{P} \subset \mathbb{R}^n\) and \(F : M \times \mathcal{P} \to M\) a \(C^\infty\) family with \(F_0 = f\). Then, there exists a neighborhood \(U\) of \(p\) and a neighborhood \(V\) of \(0\) such that, for every \(t \in V\), \(F_t\) has a saddle point \(p_t \in U\) satisfying the \(C^4\) non-resonance condition. Moreover \(p_t\) is \(C^4\) linearizable in the neighborhood \(U\) and the linearization depends \(C^4\) on the parameters.

The proofs of Theorem 2.1 and Theorem 2.4 can be found in [16, 10]. The following lemma is a direct consequence of Theorem 2.1.

**Lemma 2.5.** Let \(M\) be an \(m\)-dimensional \(C^\infty\) manifold and \(f : M \to M\) a diffeomorphism with a rank one saddle point \(p \in M\) satisfying the \(C^4\) non-resonance condition with \(|\lambda_1| = \max_{2 \leq i \leq m-1} |\lambda_i|\). If \(q \in W^u_p\), then

\[
E_q = \left\{ v \in T_q M \left| \lim_{n \to \infty} Df_q^{-n}(v)\lambda_1^n \text{ exists} \right\}
\]

is a two-dimensional vector space with \(T_q W^u_p \subset E_q\).

**Definition 2.6.** Let \(M\) be an \(m\)-dimensional \(C^\infty\) manifold and \(f : M \to M\) a local diffeomorphism satisfying the following conditions:
(f1) $f$ has a rank one saddle point $p \in M$, with unstable eigenvector $|\mu| > 1$ and stable eigenvectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, where $\lambda_1$ is the largest one, namely

$$|\lambda_1| > \max_{2 < i \leq m-1} |\lambda_i| = |\lambda_2|$$

(f2) $|\lambda_1||\mu|^3 < 1$,

(f3) $p$ satisfies the $C^4$ non-resonance condition,

(f4) $f$ has a non degenerate homoclinic tangency, $q_1 \in W^u(p) \cap W^s(p)$ in general position, namely

$$\lim_{n \to \infty} \frac{1}{n} \log d(f^n(q_1), p) = \log |\lambda_1|,$$

(f5) the direction $0 \neq B \in T_q W^u(p)$ is in general position, namely

$$\lim_{n \to \infty} \frac{1}{n} \log |Df^n_q(B)| = \log |\lambda_1|$$

and

$$E_{q_1} \cap W^s_{q_1}(p),$$

(f6) $f$ has a transversal homoclinic tangency, $q_2 \in W^u(p) \cap W^s(p)$ in general position, namely

$$\lim_{n \to \infty} \frac{1}{n} \log d(f^n(q_2), p) = \log |\lambda_1|,$$

(f7) the direction of $0 \neq v \in E_{q_2} \cap T_{q_2} W^s(q_2)$ is in general position, namely

$$\lim_{n \to \infty} \frac{1}{n} \log |Df^n_{q_2}(v)| = \log |\lambda_1|,$$

and

$$E_{q_2} \cap W^s_{q_2}(p),$$

(f8) let $[p, q_2]^u \subset W^u(p)$ be the arc connecting $p$ to $q_2$, then there exist arcs $W^u_{loc,n}(q_2) = [q_2, u_n]^u \subset W^u(q_2)$ such that $[p, q_2]^u \cap [q_2, u_n]^u = \{q_2\}$ and

$$\lim_{n \to \infty} f^n (W^u_{loc,n}(q_2)) = [p, q_2]^u,$$

(f9) there exist neighborhoods $W^u_{loc,n}(q_1) \subset W^u(q_1)$ such that

$$\lim_{n \to \infty} f^n (W^u_{loc,n}(q_1)) = [p, q_2]^u,$$

(f10) there exists $N \in \mathbb{N}$ such that

$$f^{-N}(q_1) \in [p, q_2]^u.$$

A map $f$ with these properties is called a map with a strong homoclinic tangency, see Figure 2.6.

\[^2\text{Observe that this is always verified because } q_2 \text{ is a transversal intersection.}\]
Remark 2.7. Observe that all conditions are open in the space of maps with an homoclinic and transversal tangency. Except for (f2), all others are also dense.

Remark 2.8. If the unstable eigenvalue is negative, \( \mu < -1 \), then (f8), (f9), and (f10) are redundant.

Remark 2.9. If \( \dim(M) = 2 \) and the unstable eigenvalue is negative, \( \mu < -1 \), then (f5), (f6), (f7), (f8), (f9), and (f10) are redundant. The condition (f4) reduces to have a non degenerate homoclinic tangency.

Let \( \mathcal{P} = [-r, r]^2 \) with \( r > 0 \). Given a map \( f \) with a strong homoclinic tangency, we consider a \( C^\infty \) family \( F : \mathcal{P} \times M \to M \) through \( f \) with the following properties:

(F1) \( F_{0,0} = f \),

(F2) \( F_{t,a} \) has a saddle point \( p(t,a) \) with unstable eighenvalue \( |\mu(t,a)| > 1 \), with largest stable eigenvalue \( \lambda_1(t,a) \), and

\[
\frac{\partial \mu}{\partial t} \neq 0 \text{ or } \frac{\partial \lambda_1}{\partial t} \neq 0.
\]

(F3) let \( \mu_{\text{max}} = \max_{(t,a)} |\mu(t,a)|, \lambda_{\text{max}} = \max_{(t,a)} |\lambda_1(t,a)| \) and assume

\[
\lambda_{\text{max}} \mu_{\text{max}}^3 < 1,
\]

(F4) there exists a \( C^2 \) function \( [-r, r] \ni t \mapsto q_1(t) \in W^u(p(t,0)) \cap W^s(p(t,0)) \) such that \( q_1(t) \) is a non degenerate homoclinic tangency and it is in general position, namely

\[
\lim_{n \to \infty} \frac{1}{n} \log d(F^n_{t,0}(q_1(t)), p(t,0)) = \log |\lambda_1(t,0)|.
\]
(F5) the direction $0 \neq B \in T_{q_1(t)} W^u(p(t,0))$ is in general position, namely

$$\lim_{n \to \infty} \frac{1}{n} \log |DF^n_{t,0}(B)| = \log |\lambda_1(t,0)|.$$ 

According to Theorem 2.4 we may assume without loss of generality that the family $F$ is $C^3$ and for all $(t, a) \in [-r_0, r_0]^2$ with $0 < r_0 < r$, $F_{t,a}$ is linear on the ball $[-2, 2]^m$, namely

$$F_{t,a} = \begin{pmatrix} 
\lambda_1(t, a) & 0 & \ldots & 0 & 0 \\
0 & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \lambda_{m-1}(t, a) & 0 \\
0 & 0 & \ldots & 0 & \mu(t, a) 
\end{pmatrix}.$$ 

Observe that, by (F3), for $t_0$ small enough,

$$0 < \frac{\log \mu_{\min}}{\log \lambda_{\max}} < \frac{3 \log \mu_{\min}}{2 \log \lambda_{\min}} < \frac{1}{2},$$

where $\mu_{\min} = \min_{(t,a)} |\mu(t,a)|$ and $\lambda_{\min} = \min_{(t,a)} |\lambda_1(t,a)|$. Moreover the saddle point $p(t,a) = (0,0)$ and the local stable and unstable manifolds satisfy:

- $W^s_{\text{loc}}(0) = [-2, 2]^{m-1} \times \{0\}$,
- $W^u_{\text{loc}}(0) = \{0\} \times [-2, 2]$,
- $q_1(t, a) \in [-1, 1]^{m-1} \times \{0\} \subset W^s_{\text{loc}}(0)$ and it has first coordinate equal to 1,
- $q_2(t, a) \in \{0\} \times \left(\frac{1}{\mu}, 1\right) \subset W^u_{\text{loc}}(0)$,
- there exists $N$ such that $f^N(q_3(t, a)) = q_1(t, a)$ where $q_3(t, a) = (0, 1)$,
- $Df^N_{q_3(e_1)} \notin T_{q_1} W^s(0)$ and points in the positive $y$ direction,
- the direction $B = T_{q_1} W^u(0)$ has a non zero first coordinate.

In the next lemma we prove that $q_3$ is contained in a curve of points whose vertical tangent vectors are mapped by $DF^N$ to horizontal ones. Let $(x, y)$ belong to a neighborhood of $q_3$ and consider the point

$$(X_{t,a}(x,y), Y_{t,a}(x,y)) = F^N_{t,a}(x,y).$$

Lemma 2.11. There exist $x_0, a_0 > 0$, a $C^2$ function $c : [-x_0, x_0]^{m-1} \times [-t_0, t_0] \times [-a_0, a_0] \to \mathbb{R}$ and a positive constant $Q$ such that

$$\frac{\partial Y_{t,a}}{\partial y}(x, c(x,t,a)) = 0$$

and

$$\frac{\partial^2 Y_{t,a}}{\partial y^2}(x, c(x,t,a)) \geq Q.$$ 

Moreover

$$|c(x,t,a) - c(0,t,a)| = O(|x|).$$

(2.12)
Proof. Let Φ : \([-\frac{1}{2}, \frac{3}{2}] \times [-1, 1]^{m-1} \times [-t_0, t_0] \times [-a_0, a_0] \) → \(\mathbb{R}\) defined by
\[
\Phi(y, x, t, a) = \frac{\partial Y_{t, a}}{\partial y}(x, y).
\]
Observe that \(\Phi\) is \(C^2\). Let \(q_3(t) = F_{t, 0}^{-N}(q_1(t))\). Because \(q_1(t)\) is an homoclinic tangency, see (\(F4\)), we have
\[
\Phi(q_3(t), 0, t, 0) = 0,
\]
and because \(q_1(t)\) is a non degenerate tangency, we get
\[
\frac{\partial \Phi}{\partial y}(q_3(t), 0, t, 0) > 0.
\]
For every \(t \in [-t_0, t_0]\) there exist, by the implicit function theorem, \(\epsilon > 0\) and a unique \(C^2\) function \(c : [-\epsilon, \epsilon]^{m-1} \times [-t_0 - \epsilon, t_0 + \epsilon] \times [-\epsilon, \epsilon] \to \mathbb{R}\) locally satisfying the requirements of the lemma. These local functions extend to a global one because of the compactness of the interval \([-t_0, t_0]\) and the local uniqueness.

Definition 2.13. Let \((t, a) \in [-t_0, t_0] \times [-a_0, a_0]\). We call the point
\[
c_{t, a} = (0, c(0, t, a))
\]
the primary critical point and
\[
z_{t, a} = F_{t, a}^N(c_{t, a}) = (z_x(t, a), z_y(t, a))
\]
the primary critical value of \(F_{t, a}\).

Observe that, near the saddle point, vertical vectors are expanding and horizontal ones are contracting. The critical point are defined to have the property that the expanding vertical vectors are sent to the contracting horizontal ones.

Definition 2.14. A family \(F_{t, a}\) is called an unfolding of \(f\) if it can be reparametrized such that

\((P1)\) \(z_y(t, 0) = 0\),

\((P2)\) \(\frac{\partial z_x(t, 0)}{\partial a} \neq 0\).

Remark 2.15. Without loss of generality we might assume that if \(F\) is an unfolding then \(z_y(t, a) = a\), the primary critical value is at height \(a\), see Figure 2.

Remark 2.16. A generic 2 dimensional family through \(f\) can locally be re-parametrized to become an unfolding.

Let \(\theta \in (0, \frac{1}{2})\) such that
\[
1 < \lambda_{\text{min}}^{2\theta} \mu_{\text{min}}^3 \quad \text{and} \quad \lambda_{\text{max}}^{\theta} \mu_{\text{max}} < 1.
\] (2.17)

Observe that
\[
0 < \theta_0 = \frac{\log \mu_{\text{max}}}{\log \lambda_{\text{max}}} < \theta < \frac{3 \log \mu_{\text{min}}}{2 \log \lambda_{\text{min}}} = \theta_1 < \frac{1}{2}
\] (2.18)

where we used (\(F3\)), the initial condition \(\lambda_{\text{max}}^{\theta_1} \mu_{\text{max}}^3 < 1\) and (2.10).
3 Cascades of sinks

In this section we are going to prove that an unfolding contains maps which have a sink of high period. Fix an unfolding $F$ and for each $(t, a) \in [-t_0, t_0] \times [-a_0, a_0]$ let

$$\Gamma_{t,a} = \{(x, c(x, t, a)) | x \in [-x_0, x_0]\}.$$  

**Lemma 3.1.** For $n$ large enough, there exists a $C^2$ function $a_n : [-t_0, t_0] \to (0, a_0]$ such that

$$F_{t,a_n(t)}^n \left(z_{(t,a_n(t))}\right) \in \Gamma_{(t,a_n(t))}.$$  

Moreover

$$\frac{da_n}{dt} = -n \frac{\partial \mu}{\partial t} \frac{1}{\mu^{n+1}} [1 + O(|\lambda_1|^n)].$$  

**Proof.** Let $\Gamma = \text{graph}(c)$, namely,

$$\Gamma = \{(c(x, t, a), x, t, a) | (x, t, a) \in [-x_0, x_0]^{m-1} \times [-t_0, t_0] \times [-a_0, a_0]\}.$$  

Then $\Gamma$ is a $C^2$ codimension 1 manifold transversal to $W^u(0)$. For $n \geq 0$ let

$$\Gamma_n = \{(y, x, t, a) | F_{t,a}^n (y, x) \in \Gamma\}$$  

and

$$\Gamma_\infty = \{(0, x, t, a) | (x, t, a) \in [-x_0, x_0]^{m-1} \times [-t_0, t_0] \times [-a_0, a_0]\}.$$ (3.2)
Because, for each \((t, a) \in [-t_0, t_0] \times [-a_0, a_0]\), \(F_{t,a}\) is linear, \(\Gamma_n\) converges to \(\Gamma_\infty\) in the \(C^2\) topology. Namely for large \(n\), \(\Gamma_n\) is a graph of a function also denoted by \(\Gamma_n\) and

\[
\|\Gamma_n - \Gamma_\infty\|_{C^2} \to 0
\]

Let \(z : [-t_0, t_0] \times [-a_0, a_0] \to \mathbb{R}^m \rightarrow [-t_0, t_0] \times [-a_0, a_0]\) be the \(C^2\) critical value function defined as

\[
z(t, a) = (z_y(t, a), z_x(t, a), t, a) = (a, z_x(t, a), t, a),
\]

where \(z_{l,a} = (z_x(t, a), z_y(t, a))\). Observe that

\[
\frac{\partial z_y}{\partial a} = 1 \text{ and } z_y(t, 0) = 0. \tag{3.3}
\]

Let \(Z = \text{Image}(z)\). Because of (3.3), \(Z\) is a manifold transversal to \(\Gamma_\infty\). Hence, there exists \(n_1 > 0\) such that, for all \(n \geq n_1\), \(Z\) is transversal to \(\Gamma_n\). As consequence, for all \(n \geq n_1\),

\[
A_n = z^{-1}(\Gamma_n)
\]

is a \(C^2\) codimension 1 manifold. We define \(a_n\) as a function whose graph is \(A_n\). Observe that, because \(c(0, t, a) = 1, c(x, t, a) = 1 + O(x(t + a))\) and that \(\mu^n a_n = c(\lambda^n z_x, t, a_n) = 1 + O(\lambda^n)\). Hence, by differentiating the previous formula, we have

\[
\mu^n \frac{da_n}{dt} + na_n \mu^{-1} \frac{d\mu}{dt} = O(\lambda^n).
\]

The lemma follows. \(\square\)

Choose \(\epsilon_0 > 0\) and define, for \(n\) large enough,

\[
A_n = \left\{(t, a) \in [-t_0, t_0] \times [-a_0, a_0] \mid |a - a_n(t)| \leq \frac{\epsilon_0}{|\mu(t, a_n(t))|^{2n}}\right\}.
\]

**Remark 3.4.** Observe that for all \((t, \tilde{a}) \in A_n\) we have

\[
\left(\frac{\mu(t, a_n(t))}{|\mu(t, \tilde{a})|}\right)^n = 1 + O\left(\frac{n}{|\mu(t, a_n(t))|^{2n}}\right), \quad \left(\frac{\lambda_1(t, a_n(t))}{|\lambda_1(t, \tilde{a})|}\right)^n = 1 + O\left(\frac{n}{|\lambda_1(t, a_n(t))|^{2n}}\right).
\]

In the following we prove, for a properly chosen \(\epsilon_0\), that for all parameters \((t, a) \in A_n\), \(F_{t,a}\) has a sink of period \(n + N\) which we call primary sink.

**Lemma 3.5.** There exist \(x' < x_0, a'_0 < a_0, b > 0\) and \(Q > 0\), such that, for all \((t, a) \in [-t_0, t_0] \times [-a_0, a'_0]\) and for every \((x, y) \in \Gamma_{t,a}\) with \(|x| < |x'|\) the following holds. There exist matrices \(A_{x,y}, B_{x,y} \neq 0\) and \(C_{x,y}\) such that \(F_{t,a}^N\) in coordinates centered in \((x, y)\) and \(F_{t,a}^N(x, y)\) has the form

\[
F_{t,a}^N \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} A_{x,y} & B_{x,y} \\ C_{x,y} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \begin{pmatrix} O(|\Delta x|^2 + |\Delta y|^2) \\ O(|\Delta x|^2 + |\Delta y|^3 + |\Delta x||\Delta y|) \end{pmatrix},
\]

where the matrices \(A_{x,y}, B_{x,y}, C_{x,y}\) are \(C^2\) dependent on \(x\) and \(y\), \(Q_{x,y} > Q\) and \(|B_{x,y}| > b > 0\).
Proof. The lemma gives the Taylor expansions of $F_{t,a}^N$ when $(x, y) \in \Gamma_{t,a}$ and it follows immediately from Lemma 2.11. In particular the horizontal tangency,

$$DF_{t,a}^N(x, y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} B_{x,y} \\ 0 \end{pmatrix}$$

is not degenerate for all $(x, y) \in \Gamma_{t,a}$. Let $t \in [-t_0, t_0]$. Because $F_{t,0}^N(q_3(t)) = q_1(t)$ is a non degenerate homoclinic tangency we know that the vector $B_{q_3(t)} \neq 0$ and $Q_{q_3(t)} > 0$. By taking $|a| < |a'_0|$, $|x| < |x'_0|$ small enough, the lower bounds on $Q_{x,y}$ and $B_{x,y}$ follow. 

For $n$ large enough, take $(t, a) \in A_n$ and denote by $c_n(t) = c_{t,a_n(t)}$ and by $z_n(t) = z_{t,a_n(t)} = (z_{n,x}(t), z_{n,y}(t))$. When the choice of $t$ is clear we just use the notation $c_n$ and $z_n$. Take $(t, a) \in A_n$. For $\delta > 0$ we define

$$B^n_\delta(t, a) = \left\{ (x, y) \mid \|x - z_{n,x}(t)\| \leq \frac{1}{3}, \|y - z_{n,y}(t)\| \leq \frac{\delta}{\mu(t, a_n(t))^{2n}} \right\}.$$ 

When the choice of $(t, a)$ is clear we just use the notation $B^n_\delta$. In the next lemma we prove that $B^n_\delta$ returns into itself, see Figure 3. Let $\tilde{Q} = \max Q_{x,y}$ where $Q_{x,y}$ as in Lemma 3.5.

**Lemma 3.6.** Choose $\delta = 1/2\tilde{Q}, \epsilon_0 = 1/6\tilde{Q}$. Then, for $n$ large enough and $(t, a) \in A_n,$

$$F^n_{t,a} (B^n_\delta) \subset B^n_\delta.$$
Proof. Fix \((t, a) \in A_n\). For \(n\) large enough, we write \(F_{t,a}^{n+N}\) in coordinates centred at \(z_n\), namely \(F_{t,a}^{n+N}(\Delta x, \Delta y) = (\Delta \tilde{x}, \Delta \tilde{y})\). Let \((\Delta x, \Delta y)\) such that \(z_n + (\Delta x, \Delta y) \in B_\delta^n\) and \(\Delta a = (a - a_n(t))\), then

\[
|\Delta x| \leq \frac{1}{3}, |\Delta y| \leq \frac{\delta}{||\mu(t, a_n(t))||^{2n}} \text{ and } |\Delta a| \leq \frac{\epsilon_0}{||\mu(t, a_n(t))||^{2n}}. \tag{3.7}
\]

Denote by \((\Delta x', \Delta y') = F_{t,a}^n(\Delta x, \Delta y) - c(t, a)\). Using that \(F_{t,a}^n\) is linear on \(B_\delta^n\) we get

\[
|\Delta x'| \leq 2|\lambda_1(t, a)|^n
\]

and

\[
|\Delta y'| = \left| 1 + O\left(\frac{n}{||\mu(t, a_n(t))||^{2n}}\right) \right| |\mu(t, a_n(t))|^{2n} z_{n,y} - c_{n,y} + \epsilon_n + \mu(t, a)^n| |\Delta y|
\leq \frac{\delta}{||\mu(t, a_n(t))||^{n}} \left[ 1 + O\left(\frac{n}{||\mu(t, a_n(t))||^{2n}}\right) \right] + O\left(\frac{n}{||\mu(t, a_n(t))||^{2n}}\right)
\]

where we used Remark 3.4 and Lemma 2.11. By Lemma 3.5 (center \(F_{t,a}^n\) in coordinates around \(c_{t,a}\)) extended to include also the Taylor expansion in \(\Delta a\) we get, for \(n\) large enough

\[
|\Delta \tilde{x}| = O(\Delta x') + O(\Delta y') + O(\Delta a) \leq \frac{1}{3}
\]

and

\[
|\Delta \tilde{y}| = |\Delta a| + Q_{x,y} |\Delta y'|^2 + O\left(\frac{n}{||\mu(t, a_n(t))||^{3n}}\right)
\leq \frac{\epsilon_0}{||\mu(t, a_n(t))||^{2n}} + \frac{\tilde{Q}_\delta}{||\mu(t, a_n(t))||^{3n}} + O\left(\frac{n}{||\mu(t, a_n(t))||^{3n}}\right)
\]

By (3.7) and our choice of \(\epsilon_0\) and \(\delta\) for \(n\) large enough, the lemma follows. \(\square\)

We fix \(\epsilon_0\), and \(\delta\) such that Lemm 3.6 holds. We are now ready to prove the existence of a sink.

**Proposition 3.8.** For \(n\) large enough and for all \((t, a) \in A_n\), \(B^n_\delta(t)\) has a unique periodic point which is a sink.

**Proof.** Because \(F_{t,a}^n\) is linear on \(B_\delta^n\), the image \(F_{t,a}^n(B_\delta^n)\) is contained in a neighborhood of \(c_{t,a}\) of diameter smaller than \(\frac{\delta}{||\mu||^n}\left(1 + O\left(\frac{n}{||\mu||^n}\right)\right)\). This implies that

\[
DF_{t,a}^{n+N} = \begin{pmatrix} O(1) & O(1) \\ O(1) & \frac{\Delta Q_{x,y}}{||\mu||^n} \left(1 + O\left(\frac{n}{||\mu||^n}\right)\right) \end{pmatrix}
\]

where we used Lemma 3.5. Using again that \(F_{t,a}^n\) is linear on \(B_\delta^n\) we obtain that

\[
DF_{t,a}^{n+N} = \begin{pmatrix} O(1) & O(1) \\ O(1) & \frac{\Delta Q_{x,y}}{||\mu||^n} \left(1 + O\left(\frac{n}{||\mu||^n}\right)\right) \end{pmatrix} \begin{pmatrix} O(\lambda_1^n) & 0 \\ 0 & \mu^n \end{pmatrix} \begin{pmatrix} O(1) & 0 \\ 0 & O(1) \end{pmatrix}
\]

where we used Lemma 3.5 again.
Let \( D = \begin{pmatrix} O(\vert \lambda_1 \vert^n) & O(\vert \mu \vert^n) \\ O(\vert \lambda_1 \vert^n) & 1 \end{pmatrix} \) and \( (DF_{t,a}^{N+1})^k (\Delta x, \Delta y) = (\Delta x_k, \Delta y_k) \), then

\[
\left( \frac{\vert \Delta x_{k+1} \vert}{\vert \Delta y_{k+1} \vert} \right) \leq D \left( \frac{\vert \Delta x_k \vert}{\vert \Delta y_k \vert} \right).
\]

Observe that \( \text{tr}(D) = \frac{1}{2} \) and \( \det(D) = O(\vert \lambda_1 \vert \vert \mu \vert^n) \). As consequence, for \( n \) large enough, \( \vert \Delta x_k \vert, \vert \Delta y_k \vert \to 0 \) exponentially fast. Hence, the periodic point in \( B_0^a \) is a sink whose basin of attraction contains \( B_0^a \).

\[\square\]

### 4 Curves of Secondary tangencies

In this section we are going to show that, for \( n \) large enough, there exists a curve \( B_n \) of parameters with a new homoclinic tangency which is again a strong homoclinic tangency. The curve \( B_n \) intersect \( A_n \). In particular, maps in this intersection have a sink and a strong homoclinic tangency. The following lemmas are a preparation for constructing these tangencies.

#### 4.1 Existence of the secondary tangencies

Choose \( \epsilon_1 > 0 \) and define, for large \( n \),

\[ B_n = \{ (t, a) \in [-t_0, t_0] \times [-a_0, a_0] \mid \vert a - a_n(t) \vert \leq \epsilon_1 \vert \lambda_1(t, a_n(t)) \vert^{\delta n} \}. \]

**Remark 4.1.** Observe that for \( n \) large enough, for all \( t \in [-t_0, t_0] \), \((\hat{t}, \hat{a}) \) with \( \vert \hat{t} - t \vert = O \left( \vert \lambda_1(t, a_n(t)) \vert^{\delta n} \right) \) and \( \vert \hat{a} - a \vert = O \left( \vert \lambda_1(t, a_n(t)) \vert^{\delta n} \right) \) we have

\[
\left( \frac{\vert \mu(t, a_n(t)) \vert}{\vert \mu(t, \hat{a}) \vert} \right)^n = 1 + O \left( n \vert \lambda_1(t, a_n(t)) \vert^{\delta n} \right), \quad \left( \frac{\vert \lambda_1(t, a_n(t)) \vert}{\vert \lambda_1(t, \hat{a}) \vert} \right)^n = 1 + O \left( n \vert \lambda_1(t, a_n(t)) \vert^{\delta n} \right).
\]

Moreover if \( (t, a) \in B_n \),

\[
\frac{\vert \mu(t, a_n(t)) \vert \vert a - a_n(t) \vert}{\left( \vert \lambda_1(t, a) \vert^{\delta n} \vert \mu(t, a) \vert \right)^n} \leq 2 \epsilon_1.
\]

Fix \( (t, a) \in B_n \) and in the sequel we will suppress this choice in the notation, for example, \( z = z(t, a) \), \( c = c(t, a) \), \( \lambda_1 = \lambda_1(t, a) \) and \( \mu = \mu(t, a) \). For \( n \) large enough and \( W^{u}_{\text{loc}}(z) \) small enough, \( F_{t,a}^{\delta n} (W^{u}_{\text{loc}}(z)) \) intersects \( \Gamma \) in exactly two points. Choose

\[ c' \in F_{t,a}^{\delta n} (W^{u}_{\text{loc}}(z)) \cap \Gamma \]

and a local unstable manifold \( W^{u}_{\text{loc}}(c') \) of diameter \( L \vert \lambda_1 \vert^{\delta n} \) with \( L \) a big constant. Let \( z^{(1)} = (z_x^{(1)}, z_y^{(1)}) \) be the lowest point of \( F^N (W^{u}_{\text{loc}}(c')) \), see Figure 4.1

**Lemma 4.2.** There exists a uniform constant \( K > 0 \) such that

\[
\frac{1}{K} \vert \lambda_1 \vert^{\delta n} \leq \vert z^{(1)}_y - z_y \vert \leq K \vert \lambda_1 \vert^{\delta n}
\]

and \( T_{z^{(1)}} W^{u}_{\text{loc}}(z^{(1)}) \subset \mathbb{R}^{m-1} \times \{0\} \) is horizontal. In particular \( \lim \frac{z^{(1)}_y - z_y}{\lambda_1^{\delta n}} \) exists.
Figure 4: $N + \theta n$ iterates

**Proof.** Use coordinates centred at the critical point $c$ of the parameter $(t, a)$ and let $(\Delta x, \Delta y) \in W^u_{\text{loc}}(c')$ with $(\Delta \tilde{x}, \Delta \tilde{y}) \in F^N_{t,a} (W^u_{\text{loc}}(c'))$ centred in $z$. By Lemma 3.5 we have

$$
\Delta \tilde{y} = C \Delta x + O((|\lambda_1|^2 + |\lambda_2|) \theta_n^2 + |\lambda_1|^2 |\lambda_2| |\theta_n|).
$$

Because the first coordinate of $q_1$ is equal to 1, see (F4), we have that $\Delta x = \lambda_1^\theta n e_1 + O(|\lambda_2|^{\theta n})$ and because $D_{f_{q_1}}(e_1) \notin T_{q_1} W^s(p)$ pointing in the positive $y$ direction we get that

$$
\frac{1}{K} |\lambda_1|^\theta n + O((|\lambda_1|^2 + |\lambda_2|) \theta_n^2 + |\lambda_1|^2 |\lambda_2| |\theta_n|) \leq \Delta \tilde{y} \leq K |\lambda_1|^\theta n + O((|\lambda_1|^2 + |\lambda_2|) \theta_n^2 + |\lambda_1|^2 |\lambda_2| |\theta_n|)
$$

with $K$ a positive uniform constant. The first claim of the lemma is then proved. For the second one, use the coordinates centered in $c'$. Let $(\Delta x, \Delta y) \in W^u_{\text{loc}}(c')$ with $(\Delta \tilde{x}, \Delta \tilde{y}) = F^N_{t,a} ((\Delta x, \Delta y))$ written in coordinates centered in $F^N(c')$. Consider $W^u_{\text{loc}}(c')$ as a graph over the $y$-axis and let $s$ be the slope at $c'$. Because $W^u_{\text{loc}}(c') \subset F^\theta n W^u_{\text{loc}}(z)$ which has a uniformly bounded derivatives, we get

$$
s = O\left(\frac{|\lambda_1|^2}{|\mu|} \theta_n \right)
$$

and

$$
\Delta x = s \Delta y + O\left(\frac{|\lambda_1|^2}{|\mu|} \theta_n |\Delta y|^2\right). \quad (4.3)
$$

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From Lemma 3.5
\[
\Delta \hat{x} = B \Delta y + O(\Delta y^2) \\
\Delta \hat{y} = O \left( \left( \frac{|\lambda_1|^2}{|\mu|} \right)^{\frac{\theta_n}{2}} |\Delta y| \right) + Q \Delta y^2
\]
where \( B, Q \neq 0 \). This implies that
\[
\frac{d \hat{y}}{d \hat{x}} = \frac{2Q}{B} \Delta y + O \left( \left( \frac{|\lambda_1|^2}{|\mu|} \right)^{\frac{\theta_n}{2}} \right).
\]
As consequence \( z^{(1)} \) corresponds to \( \Delta y = O \left( \left( \frac{|\lambda_1|^2}{|\mu|} \right)^{\frac{\theta_n}{2}} \right) \) in vertical size \( W \) with \( L \left( |\lambda_1|^\theta |\mu| \right)^n \). Indeed \( W_{\text{loc}}^u(z^{(1)}) \) has a horizontal tangency.

Let \( z^{(2)} = (z_x^{(2)}, z_y^{(2)}) = F_{t,a}^n(z^{(1)}) \) and take a local unstable manifold \( W_{\text{loc}}^u(z^{(2)}) \) of vertical size \( L \left( |\lambda_1|^\theta |\mu| \right)^n \) with \( L \) a large constant. The proof of the following lemma follows directly from Lemma 4.2.

**Lemma 4.4.** There exists a uniform constant \( K > 0 \) such that the following holds. For \( n \) large enough, if \( (x, y) \in W_{\text{loc}}^u(z^{(2)}), \mu_0 = \mu(t, a_n(t)) \) then
\[
\frac{1}{K} \left( |\lambda_1|^\theta |\mu| \right)^n \leq y - |\mu_0|^n z_{n,y} + O \left( |\mu_0|^n |a - a_n(t)| \right) \leq K \left( |\lambda_1|^\theta |\mu| \right)^n
\]
and
\[
x = O \left( |\lambda_1|^n \right)
\]
where \( z_n = z(t, a_n(t)) \).

**Lemma 4.5.** For \( n \geq 1 \) large enough, in coordinate centred in \( z^{(2)} \) the \( C^4 \) curve \( W_{\text{loc}}^u(z^{(2)}) \) is given by
\[
\Delta y = Q_2 \left( \frac{|\mu|}{|\lambda_1|^2} \right)^n |\Delta x|^2 + O \left( |\Delta x|^3 \left( \frac{|\mu|}{|\lambda_1|^2} \right)^{\theta_n} \right),
\]
where \( 0 \leq \Delta y < L \left( |\lambda_1|^\theta |\mu| \right)^n, |\Delta x| = O \left( |\lambda_1|^{2+\theta} \right)^{\frac{\theta_n}{2}} \) and \( Q_2 > 0 \) is a uniform constant.

**Proof.** Consider the \( W_{\text{loc}}^u(z^{(1)}) \) of vertical size \( L |\lambda_1|^\theta \) and let \( B \in T_{z^{(1)}} W_{\text{loc}}^u(z^{(1)}) \) be a unit vector. Then in coordinates centered in \( z^{(1)} \) the \( C^4 \) curve \( W_{\text{loc}}^u(z^{(1)}) \) is given by
\[
\Delta y = Q_1 |\Delta x|^2 + O(|\Delta x|^3),
\]
where \( Q_1 > 0 \) is a uniform constant, see (4.3). Given the fact that \( B \) has a non zero first coordinate, then the linear map \( F_{t,a}^n \) turns the curve \( W_{\text{loc}}^u(z^{(1)}) \) in a curve of the form as stated in the lemma. Observe also that for \( n \) large enough and for all \( (t, a) \in B_n \), the highest vertical point of \( W_{\text{loc}}^u(z^{(2)}) \) is still in the domain of linearization.

Let \( z^{(3)} = (z_x^{(3)}, z_y^{(3)}) \) be the lowest point of \( F_{t,a}^N \left( W_{\text{loc}}^u(z^{(2)}) \right) \).

**Lemma 4.6.** There exists a uniform constant \( H > 0 \) such that
\[
\frac{1}{H} \left( |\lambda_1|^\theta |\mu| \right)^{2n} \leq z_y^{(3)} - z_y \leq H \left( |\lambda_1|^\theta |\mu| \right)^{2n}.
\]
and \( T_{z^{(3)}} W_{\text{loc}}^u(z^{(3)}) \subset \mathbb{R}^{m-1} \times \{0\} \) is horizontal.
Proof. Use coordinates centered at \(c\) and let \((\Delta x, \Delta y) \in W^{u}_{\text{loc}}(z^{(2)}) = F_{t,a}^{n}(W^{u}_{\text{loc}}(z^{(1)}))\). From Lemma 4.4 we get

\[
K \left( |\lambda_1|^{\theta} |\mu| \right)^n \leq \Delta y \leq (K + L) \left( |\lambda_1|^{\theta} |\mu| \right)^n \tag{4.7}
\]

\[
|\Delta x| = O \left( |\lambda_1|^{n} \right). \tag{4.8}
\]

We apply Lemma 3.5 in image coordinates \((\Delta \tilde{x}, \Delta \tilde{y})\) centered in \(z\) and we find

\[
\Delta \tilde{y} = Q \Delta y^2 + O \left( \left( |\lambda_1|^{\theta} |\mu| \right)^{3n} \right) \tag{4.9}
\]

From (4.7) we get the stated bounds for \(z^{(3)}_y - z_y\). Moreover for \(L\) large enough the minimum is obtained in the interior of \(W^{u}_{\text{loc}}(z^{(2)})\) and \(T_{z^{(3)}}W^{u}_{\text{loc}}(z^{(3)}) \subset \mathbb{R}^{m-1} \times \{0\}\) is horizontal. \(\Box\)

**Lemma 4.9.** Let \((t, a) \in B_n\) then \(W^{u}_{\text{loc}}(z^{(3)}(t, a))\) is the graph of a function and its curvature satisfies

\[
\text{curv} \left( W^{u}_{\text{loc}}(z^{(3)}(t, a)) \right) \geq C \left( \frac{3}{|\lambda_1|^2 - 2\theta} \right)^n
\]

with \(C > 0\) a uniform constant.

**Proof.** Use coordinates centred at \(z^{(2)}\) and let \(m_5 = (\Delta x, \Delta y) \in W^{u}_{\text{loc}}(z^{(2)})\) such that \(F_{t,a}^{N}(\Delta x, \Delta y) = z^{(3)}\). Then \(0 \leq \Delta y \leq L \left( |\lambda_1|^{\theta} |\mu| \right)^n\). Let \(v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\) be the tangent vector at \(m_5\) to \(W^{u}_{\text{loc}}(z^{(2)})\). We use Lemma 4.5 and obtain that

\[
v_2 = \left[ 2Q_2 \left( \frac{|\mu|}{|\lambda_1|^2} \right)^n |\Delta x| + O \left( |\Delta x|^2 \left( \frac{|\mu|}{|\lambda_1|^3} \right)^n \right) \right] v_1. \tag{4.10}
\]

By Lemma 2.11 we have that \(\partial^2 Y / \partial Y^2\) is bounded away from zero in a neighbourhood of \(c\). As consequence, by Lemma 4.4 and Lemma 3.5 we have \(DF_{t,a}^{N}(m_5) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) where

\[
D \geq d \left( |\lambda_1|^{\theta} |\mu| \right)^n \quad \text{and} \quad d > 0 \quad \text{is a uniform constant.}
\]

Because \(F_{t,a}^{N}(m_5)\) is the minimum of the curve \(W^{u}_{\text{loc}}(z^{(3)})\), then

\[
Cv_1 + Dv_2 = 0. \tag{4.11}
\]

By (4.10), (4.11), Lemma 4.3 and the fact that \(D \geq d \left( |\lambda_1|^{\theta} |\mu| \right)^n\) we have

\[
|\Delta x| = O \left( \left( \frac{|\lambda_1|^2 - \theta}{|\mu|^2} \right)^n \right). \tag{4.12}
\]

We calculate now the second derivatives using again Lemma 4.5 and we get

\[
\frac{d^2 y}{dx^2} = 2Q_2 \left( \frac{|\mu|}{|\lambda_1|^2} \right)^n + O \left( |\Delta x| \left( \frac{|\mu|}{|\lambda_1|^3} \right)^n \right) \geq Q_2 \left( \frac{|\mu|}{|\lambda_1|^2} \right)^n \tag{4.13}
\]

where we used (4.12). Observe that \(W^{u}_{\text{loc}}(m_5)\) is a graph over an \(x\)-direction with second derivative at \(m_5\) given by (4.13). By (4.10) and (4.12), the slope satisfies

\[
\frac{dy}{dx} = O \left( \left( \frac{1}{|\lambda_1|^\theta |\mu|} \right)^n \right). \tag{4.14}
\]
Consider now the same curve as a graph over the \(y\)-axis. Then, by (4.13) and (4.14) we have
\[
\frac{d^2x}{dy^2} \geq C \left( \frac{|\mu|^3}{|\lambda_1|^{2-2\theta}} \right)^n.
\]
The map \(F_{t,a}^N\) will preserve this order of curvature.

**Proposition 4.15.** For \(n \geq 1\) large enough
\[
\frac{\partial z^{(3)}_y}{\partial t} = D_5 K (\lambda_1^n \mu)^n \frac{n \partial \mu}{\mu \partial t} + D_5 K n (\lambda_1^n \mu)^n \left[ C_3 X_2 \frac{\partial \lambda_1}{\partial t} + K \frac{\partial \mu}{\partial t} \right] + O \left( (|\lambda_1|^n |\mu|)^{2n} \right),
\]
\[
\frac{\partial z^{(3)}_x}{\partial a} = D_5 K (\lambda_1^n \mu)^n + 1 + D_5 K (\lambda_1^n \mu)^n \frac{n \partial \mu}{\mu \partial a} + D_5 K n (\lambda_1^n \mu)^n \left[ C_3 X_2 \frac{\partial \lambda_1}{\partial a} + K \frac{\partial \mu}{\partial a} \right] + O \left( (|\lambda_1|^n |\mu|)^{2n} \right),
\]
where \(|\lambda_1|^n |\mu|^2 > 1\), \(C_3 X_2, K, D_5 \neq 0\) and they converge to a non zero limit as \(n \to \infty\).

**Proof.** Choose \(t_1 \in [-t_0, t_0]\). Observe that, by Lemma 3.5, the definition of unfolding, by using coordinates \((x, y, t, a)\) centered around \((0, 1, t_1, 0)\) in the domain and coordinates centered around \(F_{t_1,0}^N(0, 1)\) we get the following expression for \(F_{t,a}^N(x, y)\),
\[
F_{t,a}^N(x, y) = \left( \frac{Ax + By + tE + aF}{Cx + Qy^2 + a [1 + x\phi_x + y^2\phi_y]} \right)
\]
where \(A, B, E, F, C, Q, \phi_x, \phi_y\) are \(C^4\) functions with \(B, Q \neq 0\). We fix the points \(m_6, m_5, m_4, m_3, m_2, m_1\) satisfying the following, see Figure 4.1.
- \(m_6 \in W^{nu}_{loc}(z^{(3)})\) sufficiently close to \(z^{(3)}\),
- \( m_3 = (m_{3,x}, m_{3,y}) \in W^u_{\text{loc}}(z^{(2)}), \) \( F_{t,a}^{-N}(m_6) = m_5. \) By Lemma 4.4 and Remark 4.1, there exists a uniform constant \( K > 0 \) such that, for \( \epsilon_1 \) small enough,

\[
\frac{1}{K} \left( \frac{|\lambda_1|^\theta |\mu|}{|\mu|^2} \right)^n \leq |m_{5,y} - c_{n,y}| \leq K \left( \frac{|\lambda_1|^\theta |\mu|}{|\mu|^2} \right)^n,
\]

and by (4.12)

\[
|m_{5,x}| = O \left( \frac{1}{|\mu|^n} \right) \tag{4.17}
\]

- \( m_4 = (m_{4,x}, m_{4,y}) \in W^u_{\text{loc}}(z^{(1)}), \) \( F_{t,a}^{-n}(m_5) = m_4. \) By Lemma 4.2 and the fact that \( z_n \) is proportional to \( \frac{1}{|\mu|^n} \), see Remark 4.1.

\[
|m_{4,y}| = O \left( \frac{1}{|\mu|^n} \right), \tag{4.20}
\]

and by (4.19)

\[
|m_{4,x} - z_n^{(2)}| = O \left( \left( \frac{|\lambda_1|^{1-\theta}}{|\mu|^2} \right)^n \right). \tag{4.19}
\]

- \( m_3 = (m_{3,x}, m_{3,y}) \in W^u_{\text{loc}}(c'), \) \( F_{t,a}^{-N}(m_4) = m_3. \) By (4.21),

\[
|m_3 - c| = O \left( \frac{|\lambda_1|^{\theta_n}}{|\mu|^n} \right) \tag{4.22}
\]

\[
|m_3 - c'| = O \left( \left( \frac{|\lambda_1|^{1-\theta}}{|\mu|^2} \right)^n \right). \tag{4.23}
\]

- \( m_2 = (m_{2,x}, m_{2,y}) \in W^u_{\text{loc}}(z), \) \( F_{t,a}^{-\theta_n}(m_3) = m_2. \) There exists a uniform constant \( K > 0 \) such that

\[
\frac{1}{K} \frac{1}{|\mu|^\theta_n} \leq |m_{2,y}| \leq K \frac{1}{|\mu|^\theta_n}. \tag{4.24}
\]

- \( m_1 = (m_{1,x}, m_{1,y}) \in W^u_{\text{loc}}(c), \) \( F_{t,a}^{-N}(m_2) = m_1. \) By (4.24), there exists a uniform constant \( K > 0 \) such that

\[
\frac{1}{K} \frac{1}{|\mu|^\theta_n} \leq |m_{1,y} - c_y| \leq K \frac{1}{|\mu|^\theta_n}. \tag{4.25}
\]

Take \((t_1 + t, a) \in B_n\) and consider the \(C^4\) map

\[
(\Delta t, \Delta a, \Delta y) \mapsto F_{t+\Delta t,a+\Delta a}^{3N+(\theta+1)n}(m_1 + (0, \Delta y)) = m_6 + (\Delta m_{6,x}, \Delta m_{6,y}).
\]

We will follow the perturbations of \(m_1 + (0, \Delta y) \in W^u_{\text{loc}}(m_1)\). Observe that, by (4.16), for \(i = 1, 5\), we have

\[
\Delta m_{i+1} = \begin{pmatrix} A_i \\ C_i \end{pmatrix} \begin{pmatrix} B_i \\ O(|m_{i,x}|) + D_i(m_{i,y} - 1) \end{pmatrix} \Delta m_i \tag{4.26}
\]

\[
+ \begin{pmatrix} O(\Delta t + \Delta a) \\ \frac{O(\Delta t + \Delta a)}{\Delta m} \end{pmatrix} \frac{O(\Delta t + \Delta a)}{\Delta m} \tag{4.27}
\]

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and for \( i = 3 \)

\[
\Delta m_4 = \left( \frac{A_3}{C_3} - O (|m_3 - c'|) \right) \Delta m_3
\]

\[
+ \left[ d_3 |m_{3,y} - 1|^2 + O (|m_{3,x}|) \right] \frac{O (\Delta t + \Delta a)}{\Delta t + [1 + O (|m_{3,y} - 1|^2 + |m_{3,x}|)] \Delta a}
\]  

(4.28)

where \( D_i, B_i, C_i \neq 0 \) (because \( q_1 \) is a non degenerate tangency in general direction and \( DF^N(0,1) \) is non singular). By using the fact that \( F_{t,a}^n \) is linear we get

\[
\Delta m_5 = \left( \begin{array}{cc} \lambda^n_1 & 0 \\ 0 & \mu^n \end{array} \right) \Delta m_4 + \left( X^n_4 \frac{\lambda^n_1}{\lambda_1} \frac{\partial \lambda_1}{\partial t} \Delta t + \frac{\partial \lambda_1}{\partial a} \Delta a \right)
\]

\[
\left( 1 + \frac{1}{|\lambda_1|^n} \right) \frac{\partial \lambda_1}{\partial t} + \frac{\partial \lambda_1}{\partial a} \Delta a
\]

(4.30)

with \( X_4 \neq 0 \). A similar formula holds going from \( \Delta m_2 \) to \( \Delta m_3 \). By (4.26) and (4.25),

\[
\Delta m_2, x = O (\Delta t + \Delta a + \Delta y)
\]

\[
\Delta m_2, y = \frac{d_1}{|\mu|^n} \Delta t + \left[ 1 + O \left( \frac{1}{|\mu|^n} \right) \right] \Delta a + D_1 \frac{|\mu|^n}{|\mu|^n} \Delta y.
\]

Since \( F_{t,a}^n \) is linear, see (4.30), and using (4.22) we get

\[
\Delta m_{3, x} = X_2 \frac{\theta n \lambda^n_1}{\lambda_1} \left[ \frac{\partial \lambda_1}{\partial t} \Delta t + \frac{\partial \lambda_1}{\partial a} \Delta a \right] + O \left( |\lambda_1|^n \Delta y \right)
\]

\[
\Delta m_{3, y} = \left[ \theta n \frac{\partial \mu}{\partial t} + d_1 + O \left( n |\lambda_1|^n \right) \right] \Delta t
\]

\[
+ \left[ |\mu|^n + \theta n \frac{\partial \mu}{\partial a} + O (1) \right] \Delta a + D_1 |\mu|^n \Delta y
\]

where \( X_2 \neq 0 \) pointing in the direction of \( e_1 \), see Figure 4.1. By (4.28) and (4.22), we get

\[
\Delta m_{4, x} = \left[ B_3 \frac{\theta n \partial \mu}{\partial t} + O (1) \right] \Delta t
\]

\[
+ \left[ B_3 |\mu|^n + B_3 \frac{\theta n \partial \mu}{\partial a} + O (1) \right] \Delta a
\]

\[
+ \left[ D_1 B_3 |\mu|^n + O \left( |\lambda_1|^n \right) \right] \Delta y
\]

\[
\Delta m_{4, y} = \left[ C_3 X_2 \frac{\theta n \lambda^n_1}{\lambda_1} \frac{\partial \lambda_1}{\partial t} + O \left( |\lambda_1|^n \right) \right] \Delta t
\]

\[
+ \left[ 1 + C_3 X_2 \frac{\theta n \lambda^n_1}{\lambda_1} \frac{\partial \lambda_1}{\partial a} + O \left( |\lambda_1|^n \right) \right] \Delta a
\]

\[
+ O \left( |\lambda_1|^n \Delta y \right).
\]
where $C_3X_2 \neq 0$, see (f5). By (4.30) we get

$$
\Delta m_{5,x} = \left[ B_3 \frac{\partial n \lambda_1^n \partial \mu}{\mu} + X_4 \frac{n \lambda_1^n \partial \lambda_1}{\lambda_1} + O \left( |\lambda_1|^n \right) \right] \Delta t \\
+ \left[ B_3 (\lambda_1 \mu^6)^n + B_3 \frac{\partial n \lambda_1^n \partial \mu}{\mu} + X_4 \frac{n \lambda_1^n \partial \lambda_1}{\lambda_1} + O \left( |\lambda_1|^n \right) \right] \Delta a \\
+ \left[ D_1 B_3 (\lambda_1 \mu^6)^n + O \left( (|\lambda_1|^\theta |\mu|)^n \right) \right] \Delta y
$$

$$
\Delta m_{5,y} = \left[ \frac{n \partial \mu}{\mu} + n (\lambda_1 \mu^6)^n \left[ C_3 X_2 \frac{\theta}{\lambda_1} \frac{\partial \lambda_1}{\partial t} + \frac{K \partial \mu}{\mu} \right] + O \left( (|\lambda_1|^\theta |\mu|)^n \right) \right] \Delta t \\
+ \left[ \mu^n + n \frac{\partial \mu}{\mu} + n (\lambda_1 \mu^6)^n \left[ C_3 X_2 \frac{\theta}{\lambda_1} \frac{\partial \lambda_1}{\partial a} + \frac{K \partial \mu}{\mu} \right] + O \left( (|\lambda_1|^\theta |\mu|)^n \right) \right] \Delta a \\
+ O \left( (|\lambda_1|^\theta |\mu|)^n \Delta y \right).
$$

By (4.26), (4.17) and (4.18) we get

$$
\Delta m_{6,x} = \left[ B_5 \frac{n \partial \mu}{\mu} + O \left( 1 \right) \right] \Delta t + \left[ B_5 \mu^n + B_5 \frac{n \partial \mu}{\mu} + O \left( 1 \right) \right] \Delta a \\
+ O \left( (|\lambda_1|^\theta |\mu|)^n \Delta y \right)
$$

$$
\Delta m_{6,y} = \left[ D_5 K (\lambda_1 \mu^6)^n + D_5 K n \left( \lambda_1 \mu^6 \right)^2 \left[ C_3 X_2 \frac{\theta}{\lambda_1} \frac{\partial \lambda_1}{\partial t} + \frac{K \partial \mu}{\mu} \right] + O \left( (|\lambda_1|^\theta |\mu|)^2n \right) \right] \Delta t \\
+ \left[ D_5 K (\lambda_1 \mu^6)^n + D_5 K \left( \lambda_1 \mu^6 \right)^n \frac{\partial \mu}{\mu} + D_5 K n \left( \lambda_1 \mu^6 \right)^2 \left[ C_3 X_2 \frac{\theta}{\lambda_1} \frac{\partial \lambda_1}{\partial a} + \frac{K \partial \mu}{\mu} \right] + O \left( (|\lambda_1|^\theta |\mu|)^n \right) \right] \Delta a \\
+ O \left( (|\lambda_1|^\theta |\mu|)^2n \right) \Delta y
$$

where $D_5, B_5, K \neq 0$ and $|\lambda_1|^\theta |\mu|^2 > 1$ (see (2.17)). Refer to Figure 4.1. The formula for $m_{6,y}$ holds generally for all $\Delta t$, $\Delta a$ and $\Delta y$. However at the point $(t_1, a)$ when $\Delta t = \Delta a = 0$ we have $\partial m_{6,y}/\partial y = 0$. Hence, the Taylor polynomial of second order for $\Delta m_{6,y}$ does not contain a linear term in $\Delta y$. As consequence $\frac{\partial^2 m_{6,y}}{\partial t^2} = \frac{\partial m_{6,y}}{\partial t}$ and $\frac{\partial^2 m_{6,y}}{\partial a^2} = \frac{\partial m_{6,y}}{\partial a}$. It is left to prove that $C_3 X_2, K, D_5$ converge. Observe that $D_5$ and $C_3$ converge because they are part of the derivative $DF^N$ which converges to $DF^N(0, 1)$. Moreover $\lim X_2 = \lim \frac{1}{\lambda_1^\nu} F^{\nu n}(q_1)$ which is not zero because $q_1$ is in general position. Finally $K$ converges because of Lemma 4.2. The proposition follows.

Let $W = W_{loc}^n(q_2)$ and for all $n \in \mathbb{N}$ let $W_n = F_{-a}^n(W)$. Because $W \cap W^n(p)$ we have that $W_n$ converges to $W_{loc}^n(0)$. In particular $W_n$ is the graph of a function which will also be denoted by $W_n$. Moreover

$$
\frac{1}{2|\mu|^n} |\mu|^n \leq |W_n| \leq 2|\mu|^n, \\
\frac{|\partial W_n|}{|x|} = O \left( \frac{|\lambda_1|^n}{|\mu|} \right) \\
\frac{|\partial^2 W_n|}{|x^2|} = O \left( \frac{|\lambda_1|^2 |\mu|^n}{|\mu|} \right)
$$

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Lemma 4.36. The function $W_n$ satisfies

$$
\frac{\partial W_n}{\partial t} = -\frac{n}{\mu} \frac{\partial \mu}{\partial t} W_n + \sum_i \left( \frac{\lambda_i}{\mu} \right)^n \frac{n}{\lambda_i} \frac{\partial \lambda_i}{\partial t} \frac{\partial W}{\partial x_i} x_i + \frac{1}{\mu^n} \frac{\partial W}{\partial t} = O \left( \frac{n}{\mu^n} \right),
$$

$$
\frac{\partial W_n}{\partial a} = -\frac{n}{\mu} \frac{\partial \mu}{\partial a} W_n + \sum_i \left( \frac{\lambda_i}{\mu} \right)^n \frac{n}{\lambda_i} \frac{\partial \lambda_i}{\partial a} \frac{\partial W}{\partial x_i} x_i + \frac{1}{\mu^n} \frac{\partial W}{\partial a} = O \left( \frac{n}{\mu^n} \right).
$$

Proof. Fix a point $x = (x_i) \in [-2, 2]^{m-1}$ and a parameter $(t, a) \in [-r_0, r_0]^2$. We denote by $W_n$ the manifold corresponding to $(t, a)$ and by $W_n + \Delta W_n$ the manifold to $(t, a + \Delta a)$. For all $i = 1, \ldots, m - 1$ let $\Delta \lambda_i = \frac{\partial \lambda_i}{\partial a} \Delta a$ and $\Delta \mu = \frac{\partial \mu}{\partial a} \Delta a$. Then, because the maps $F_{t,a}$ are linear, we obtain

$$(\mu + \Delta \mu)^n (W_n + \Delta W_n) (x) = \mu^n W_n (x) + \frac{\partial W}{\partial x} (\mu^n W_n (x) + \Delta \mu)^n x_i + \frac{\partial W}{\partial a} \Delta a.$$

Similarly, one gets the same bound for $\frac{\partial W_n}{\partial t}$. \hfill \Box

A similar proof as for the previous lemma gives the following.

Lemma 4.37. Let $\Gamma_n$ as in (3), then

$$
\frac{\partial \Gamma_n}{\partial t} = -\frac{n}{\mu} \frac{\partial \mu}{\partial t} \Gamma_n + \sum_i \left( \frac{\lambda_i}{\mu} \right)^n \frac{n}{\lambda_i} \frac{\partial \lambda_i}{\partial t} \frac{\partial \Gamma}{\partial x_i} x_i + \frac{1}{\mu^n} \frac{\partial \Gamma}{\partial t} = O \left( \frac{n}{\mu^n} \right),
$$

$$
\frac{\partial \Gamma_n}{\partial a} = -\frac{n}{\mu} \frac{\partial \mu}{\partial a} \Gamma_n + \sum_i \left( \frac{\lambda_i}{\mu} \right)^n \frac{n}{\lambda_i} \frac{\partial \lambda_i}{\partial a} \frac{\partial \Gamma}{\partial x_i} x_i + \frac{1}{\mu^n} \frac{\partial \Gamma}{\partial a} = O \left( \frac{n}{\mu^n} \right).
$$

Let

$$
S_n = \{(x, y) \in [-2, 2]^m | y \geq W_n (x) \}.
$$

Observe that $z \in S_n \setminus S_{n-1}$. Let $n_0$ be the maximal $n \in \mathbb{N}$ such that

$$
W_{n_0}^{\text{loc}} (z^{(3)}) \subset S_{n-n_0}. \tag{4.38}
$$

Let $\alpha = \frac{\log (|\lambda_1|^2 |\mu|^3)}{\log |\mu|}$. By (2.18), $\alpha \in (0, 1)$.

Lemma 4.39. The integer $n_0$ satisfies the following:

$$
n_0 = n \alpha + O(1).
$$

Proof. From Lemma 4.6 (4.33) and the fact that $z \in S_n \setminus S_{n-1}$, there exists a uniform constant $K > 0$ such that

$$
\frac{1}{K} \left( \frac{1}{|\mu|^n} + (|\lambda_1|^2 |\mu|^2)^n \right) \leq z_y^{(3)} \leq K \left( \frac{1}{|\mu|^n} + (|\lambda_1|^2 |\mu|^2)^n \right).
$$

Moreover the definition of $n_0$ implies that

$$
\frac{1}{K} \frac{1}{|\mu|^{n-n_0}} \leq z_y^{(3)} \leq K \frac{1}{|\mu|^{n-n_0}}.
$$

The two previous inequalities imply that

$$
\frac{1}{K^2} \leq \frac{1}{|\mu|^{n_0}} (1 + (|\lambda_1|^2 |\mu|^3)^n) \leq K^2
$$

The lemma follows from (2.17). \hfill \Box
Definition 4.40. A tangency between $W_{loc}^u(z^{(3)})$ and $W_{n-n_0}$ is called a secondary tangency of type $n_0$. We define

$$T_{n,n_0} = \{(t,a) \in B_n | F_{t,a} has a secondary tangency of type n_0\}.$$ 

In the next proposition we are going to prove that secondary tangencies exist for certain parameters in $A_n$, see Figure 4.1.

Proposition 4.41. Let $C$ be a positive constant. For $n$ large enough and for all $t \in (-t_0,t_0)$ there exists $t_n \in (-t_0,t_0)$ such that the following holds:

- $|t - t_n| = O \left(\frac{1}{n}\right)$,
- $F_{t_n,a_n(t_n)}$ has a secondary tangency $q_{1,n,n_0}$ and $(t_n,a_n(t_n)) \in A_n \cap T_{n,n_0}$,
- at the tangency point

$$\left| \frac{\partial W_{n-n_0}}{\partial t}(q_{1,n,n_0}) \right| \geq Cn \left(\lambda_1^2 \mu\right)^2n.$$ 

(4.42)

Proof. Fix $C > 0$ and choose $t^* \in (-t_0,t_0)$. Then, there exists $\theta \in (\theta_1,\theta_2)$ such that for all $(t,a)$ close enough to $(t^*,0)$,

$$\left| D_5K \left[ C_3x_2 \frac{\theta \partial \lambda_1}{\lambda_1} + \frac{K \partial \mu}{\mu} \right] \right| \geq v > 0.$$ 

Observe that $v$ can be chosen independently of $t^*$. Without loss of generality we may assume that $D_5K \left[ C_3x_2 \frac{\theta \partial \lambda_1}{\lambda_1} + \frac{K \partial \mu}{\mu} \right] > 0$. Take $n$ large enough and a point $(t,a_n(t)) \in$
$A_n$ near $(t^*, a_n(t^*)) \in A_n$. By Proposition 4.15 and Lemma 3.1

$$z_y^{(3)}(t, a_n(t)) = z_y^{(3)}(t^*, a_n(t^*)) + \int_0^{(t-t^*)} \left[ \frac{\partial z_y^{(3)}}{\partial t} + \frac{\partial z_y^{(3)}}{\partial a} \frac{da_n}{dt} \right] dt$$

$$\geq z_y^{(3)}(t^*, a_n(t^*)) + n \left( \lambda^3 |\mu| \right)^2 \frac{v}{2} (t-t^*),$$

where we used a cancellation of the dominant terms in the partial derivatives (see Proposition 4.15, Lemma 3.1). Let $n_0^*$ be such that $W_{ab}^u \left( z_y^{(3)}(t^*, a_n(t^*)) \right) \subset S_{n-n_0}$. Then, by (4.33),

$$z_y^{(3)}(t^*, a_n(t^*)) \geq \frac{2}{\mu} \frac{1}{\mu - n_0^*}.$$

Choose $\kappa \geq 2$. Then by (4.33), in a neighborhood of $(z_y^{(3)}(t^*, a_n(t^*))$, $t^*, a_n(t^*))$

$$\max W_{n-n_0^*} \leq \frac{2}{\mu^{n-n_0^*}}.$$

From the previous three inequalities, Lemma 4.39 and the definition of $\alpha$, we have that if

$$t - t^* \geq \frac{1}{n} \frac{4\mu^\kappa}{v}$$

then, $W_{ab}^u \left( z_y^{(3)}(t^*, a_n(t^*)) \right)$ is above $W_{n-n_0^*}$. Because $W_{ab}^u \left( z_y^{(3)}(t^*, a_n(t^*)) \right)$ contains a point below $W_{n-n_0^*-1}$, there exists a parameter between $(t, a_n(t))$ and $(t^*, a_n(t^*))$ for which a secondary homoclinic tangency of type $n_0 = n_0^* + \kappa$ occurs and $|t^* - t| = O \left( \frac{1}{n^\beta} \right)$. Moreover by Lemma 4.36 (4.33), Lemma 4.39 and the definition of $\alpha$ we get

$$\frac{\partial W_{n-n_0}^{ab} \left( q_{1,n,n_0} \right)}{\partial t} = -2n \left( \lambda^3 |\mu| \right)^2 \mu^{(3-n)\beta - 1} \left( 1 - \alpha - \frac{\kappa}{n} \right) \frac{\partial \mu}{\partial t} + O \left( \lambda^3 |\mu|^{2n} \right).$$

The last statement of the lemma follows by taking $\kappa$ large enough. □

### 4.2 Existence of tangency curves

**Lemma 4.43.** Let $(t, a) \in B_n$ such that $W_{ab}^u \left( z_y^{(3)}(t, a) \right)$ has a secondary tangency $q_{1,n,n_0}$ of type $n_0$, then $q_{1,n,n_0}$ is non degenerate. Namely, $W_{ab}^u \left( q_{1,n,n_0} \right)$ is the graph of a function and its curvature satisfies

$$\text{curv} \left( W_{ab}^u(q_{1,n,n_0}) \right) \geq C \left( \frac{|\mu|^3}{|\lambda|^2} \right)^n$$

with $C > 0$ a uniform constant.

**Proof.** Use coordinates centered at $z(2)$ and let $(\Delta x, \Delta y) \in W_{ab}^u \left( z_y^{(3)} \right)$ such that $F_{t,a}^N(\Delta x, \Delta y) = q_{1,n,n_0}$. Then $0 \leq \Delta y \leq L \left( |\lambda^3| |\mu| \right)^n$. Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be the tangent vector at $(\Delta x, \Delta y)$ to $W_{ab}^u \left( z_y^{(3)} \right)$. We use Lemma 4.5 and we obtain that

$$v_2 = \left[ 2Q_2 \left( \frac{|\mu|}{|\lambda|} \right)^n |\Delta x| + O \left( |\Delta x|^2 \left( \frac{|\mu|}{|\lambda|^3} \right)^n \right) \right] v_1. \quad \text{(4.44)}$$

By Lemma 2.11 we have that $\frac{\partial Y}{\partial x^2}$ is away from zero in a neighborhood of $c$. As consequence, by Lemma 4.4 and Lemma 3.5 we have $DF_{t,a}^N(\Delta x, \Delta y) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where
Figure 7: Curves of Secondary Tangencies

\[ D \geq d \left( |\lambda_1|^\beta |\mu| \right)^n \text{ and } d > 0 \text{ is a uniform constant. Because } F^N_{t,a}(\Delta x, \Delta y) \text{ is a tangency at } q_{1,n,0}, \text{ then} \]

\[ Cv_1 + Dv_2 = O \left( \left( \frac{|\lambda_1|}{|\mu|} \right)^{n-n_0} (|v_1| + |v_2|) \right) \]  

where we used (4.34). The proof of the lemma is completed by following exactly the argument in Lemma 4.9.

In the next proposition we prove the existence of curves of secondary tangencies, see Figure 4.2. Let \( C = 2 \max_{(t,a,\theta)} D_5 K \left[ C_3 \lambda_2 \frac{\partial \lambda_1}{\partial t} + \frac{K}{\mu} \frac{\partial \mu}{\partial t} \right] \) and

\[ T^*_{n,n_0} = \left\{ (t,a) \in T_{n,n_0} \mid \frac{\partial W_{n-n_0}}{\partial t}(q_{1,n,n_0}) \geq C_n (\lambda_1^\theta \mu)^{2n} \right\}. \]

Observe that the secondary tangencies in \( T^*_{n,n_0} \) are the ones for which the stable manifold moves faster than the local unstable manifold at the tangency when varying the \( t \) parameter. As consequence of Proposition 4.41, we get the following.

**Corollary 4.46.** For \( n \) large enough, the set of types of secondary tangencies,

\[ \mathcal{R}_n = \{ (n,n_0) \mid T^*_{n,n_0} \neq 0 \} \]

is non-empty.

**Proposition 4.47.** Each component of \( T^*_{n,n_0} \) is the graph of a \( C^2 \) function \( b_{n,n_0} : [t_{n,n_0}, t^*_{n,n_0}] \rightarrow \mathbb{R} \). Moreover

\[ \frac{db}{dt} = -\frac{n}{\mu^{n+1}} \left( Vn \lambda_1^{\theta_n} + O \left( |\lambda_1|^{\theta_n} \right) \right) \]

and, if \( (t,a) \in A_n \cap T^*_{n,n_0} \)

\[ \frac{db}{dt} = \frac{da}{dt} + Vn \lambda_1^{\theta_n} + O \left( |\lambda_1|^{\theta_n} \right) \]

where \( V \) is uniformly away from zero. In particular, the following holds:
- $\partial T_{n,n_0}^* \subset \partial B_n$,
- each component of $T_{n,n_0}^*$ has a unique transversal intersection with $A_n$.

**Proof.** Let $(t,a) \in T_{n,n_0}^*$. We start by constructing a local function whose graph is contained in $T_{n,n_0}^*$. Let $m_1 \in W_{\text{loc}}^u(c)$ such that $F_{t,a}^{3N+(1+\theta)n}(m_1) = q_{1,n,n_0}$. To describe the perturbation of $W_{\text{loc}}^u(q_{1,n,n_0})$ we define the following function by choosing coordinates centered in the image at $q_{1,n,n_0}$. Take some $\epsilon > 0$ and consider the $C^4$ function $(\tilde{x}, \tilde{y}) : (-\epsilon, \epsilon)^3 \rightarrow \mathbb{R}^{m-1} \times \mathbb{R}$ defined by

$$(\tilde{x}(\Delta y, \Delta t, \Delta a), \tilde{y}(\Delta y, \Delta t, \Delta a)) = F_{t+\Delta t, a+\Delta a}^{3N+(1+\theta)n}(m_1) + (\Delta y),$$

which describes $W_{\text{loc}}^u(F_{t+\Delta t, a+\Delta a}^{3N+(1+\theta)n}(m_1))$. Observe that the manifolds $W_{n-n_0}$ of $F_{t+\Delta t, a+\Delta a}$ are described locally, near $q_{1,n,n_0}$, as the graph of a $C^4$ function

$$w_{n-n_0} : (\Delta x, \Delta t, \Delta a) \mapsto w_{n-n_0}(\Delta x, \Delta t, \Delta a) \in \mathbb{R}. $$

Define now the $C^2$ function $\Psi : [-\epsilon, \epsilon]^3 \rightarrow \mathbb{R}^2$ as

$$\Psi(\Delta y, \Delta t, \Delta a) = \left( \tilde{y}(\Delta y, \Delta t, \Delta a) - w_{n-n_0}(\tilde{x}(\Delta y, \Delta t, \Delta a), \Delta t, \Delta a), \left( \frac{\partial \tilde{x}}{\partial \Delta y} (\Delta y, \Delta t, \Delta a) - \frac{\partial w_{n-n_0}}{\partial \Delta x}(\tilde{x}(\Delta y, \Delta t, \Delta a), \Delta t, \Delta a) \right) \right)$$

Observe that $\Psi^{-1}(0)$ describes locally the perturbation of the secondary tangency $q_{1,n,n_0}$ and that

$$D\Psi(0,0,0) = \begin{pmatrix}
0 & (\lambda_1^\theta \mu)^n \Psi_{1,2} & (\lambda_1^\theta \mu^2)^n \Psi_{1,3} \\
\frac{\mu_{|\mu|^3-2\theta}}{|\mu|^2} & 0 & 0 \\
\frac{\mu_{|\mu|^3-2\theta}}{|\mu|^2} & 0 & 0
\end{pmatrix}$$

(4.48)

with

$$\Psi_{1,2} = D_5 K \frac{n}{\mu} \frac{\partial \mu}{\partial t} + n (\lambda_1^\theta \mu)^n \left[D_5 K \left( C_3 X_{\lambda_1} + K \right) \frac{\partial \lambda_1}{\lambda_1} \frac{\partial \mu}{\partial t} + O \left((|\lambda_1|^\theta |\mu|)^n\right) \right]$$

and

$$\Psi_{1,3} = D_5 K \left[1 + O \left(\left(\frac{1}{|\lambda_1|^\theta |\mu|^2}\right)^\frac{1}{n}\right)\right],$$

where we used Lemma 4.36, 4.34, 4.31 and the definition of $T_{n,n_0}^*$. Observe that $\Psi_{1,2}, \Psi_{1,3} \neq 0$. Moreover $\Psi_{2,1} \neq 0$, see Lemma 4.43, 4.31 and (4.35). Because $D\Psi(0,0,0)$ is onto we get that the set of secondary tangencies $\Psi^{-1}(0)$ is locally the graph of a $C^2$ function $b$. Moreover the $T_{(0,0,0)}\Psi^{-1}(0) = \text{Ker} D\Psi$. Hence

$$(\lambda_1^\theta \mu)^n \Psi_{1,2} \Delta t + (|\lambda_1|^\theta |\mu|^2)^n \Psi_{1,3} \Delta b = 0$$

and

$$\frac{db}{dt} = - \frac{n}{\mu^\alpha+1} \frac{\partial \mu}{\partial t} + n \lambda_1^\theta V + O \left(|\lambda_1|^\theta_\nu\right).$$

Moreover, by the previous estimate on the slope of $b$ and by Lemma 3.1 we have

$$\frac{db}{dt} = \frac{da_n}{dt} + n \lambda_1^\theta V + O \left(|\lambda_1|^\theta_\nu\right),$$

(4.49)
for \((t, a) \in A_n \cap T^*_{n,n_0}\). By comparing \(\mu^n(t, a)\) and \(\frac{\partial \mu}{\partial t}(t, a)\) with \(\mu^n(t, a_n(t))\) and \(\frac{\partial \mu}{\partial t}(t, a_n(t))\) on \(B_n\), see Remark 4.1, one gets the same estimate as in (4.49) for any other point \((t, a) \in B_n\). This uniform bound on the difference of the slopes allows to extend \(b\) globally up to both boundaries of \(B_n\). In particular the length of a component of \(T^*_{n,n_0}\) is proportional to \(\frac{1}{n}\). Moreover each component of \(T^*_{n,n_0}\) intersects \(A_n\) transversally, see (4.49), in a unique point.

\[ \Box \]

5 Newhouse phenomenon

Let \(f_{n,n_0}\) be the map at the intersection point \(A_n \cap T^*_{n,n_0}\), see Proposition 4.47 and

\[ \mathcal{P}_{n,n_0} = \left\{ (t, a) \in b_{n,n_0}^{-1}(A_n) \times [-a_0, a_0] \middle| |a - a_n(t)| \leq \frac{\epsilon_0}{|\mu(t, a_n(t))|^{2\pi n}} \right\}, \]

**Lemma 5.1.** The domains \(\mathcal{P}_{n,n_0}\) are pairwise disjoint for \(n\) large enough and the diameter goes to zero.

**Proof.** By Proposition 4.47 there exists a uniform constant \(K > 0\) such that

\[ \frac{1}{K} \frac{1}{n (|\lambda_1|^2 |\mu|^2)^{\pi}} \leq |b_{n,n_0}^{-1}(A_n)| \leq K \frac{1}{n (|\lambda_1|^2 |\mu|^2)^{\pi}} \]

(5.2)

and the proof of Proposition 4.41 gives \(\text{dist}(f_{n,n_0}, f_{n,n_0+1})\) is proportional to \(\frac{1}{n}\). The disjointness follows from this estimates and the fact that \(A_n\) are pairwise disjoint. \( \Box \)

**Proposition 5.3.** For \(n\) large enough, the map \(f_{n,n_0}\) has a strong homoclinic tangency and the restriction \(F : \mathcal{P}_{n,n_0} \times M \to M\) can be reparametrized to become an unfolding. Moreover each map in \(\mathcal{P}_{n,n_0}\) has a sink of period \(n + N\).

**Proof.** Observe that conditions \((f1), (f2), (f3), (f6), (f7), (f8), (F1), (F2), (F3)\) are automatically satisfied. We need only to check the conditions involving the secondary tangency \(q_{1,n,n_0}\). Let \(T\) be the time such that \(f^T(q_2) \in [-2, 2]^{m-1} \times \{0\}\). From (6) we know that the first coordinate of \(f^T(q_2)\) is non zero. Recall now that \(q_{1,n,n_0} \in W_{n,n_0}\). As consequence, for \(n\) large enough, \(f_{n,n_0}^{-n}(q_{1,n,n_0})\) is close to \(q_2\). Hence, \(f_{n,n_0}^{-n-m+1}(q_{1,n,n_0}) \in [-2, 2]^{m-1} \times \{0\}\) has first coordinate non zero. This proves \((f4)\) for \(q_{1,n,n_0}\).

Use the notation \(m_1, m_2, m_3, m_4, m_5\) from Proposition 4.15 and observe that \(m_1 \in W^u_{\text{loc}}(q_2)\) close to \(q_3\). This proves \((f10)\) for \(q_{1,n,n_0}\).

Let \(B \in T_{q_2}W^u(p)\) the unit vector. From \((f5)\) we know that \(B\) has a non zero first coordinate. For \(n\) large, the direction of \(T_{m_2}W^u(p)\) is close to \(B\), hence it has a non zero first coordinate. This implies that the direction of \(T_{m_3}W^u(p)\) is close to \(E_{q_3}\) and again from \((f5)\), the direction of \(T_{m_2}W^u(p)\) is close to \(B\) and it has a non zero first coordinate. Hence the direction of \(T_{m_2}W^u(p)\) is close to \(E_{q_3}\) and from \((f5)\), the direction of \(T_{q_{1,n,n_0}}W^u(p)\) is close to \(B\) and it has a non zero first coordinate. The direction of \(T_{f_{n,n_0}^{-n-m}(q_{1,n,n_0})}W^u(p)\) is close to \(E_{q_2} \cap T_{q_2}W^s(p)\). Property \((f7)\) implies \((f5)\) for \(q_{1,n,n_0}\).

Observe that \(f_{n,n_0}^{-n}(q_{1,n,n_0})\) converges to \(q_2\). Property \((f8)\) implies \((f9)\) for \(q_{1,n,n_0}\).

For proving \((F4)\) use the \(C^2\) function \(b_{n,n_0}\) and observe that the maps on the graph of this curve, \(T^*_{n,n_0}\), have a non degenerate homoclinic tangency, see Lemma 4.43. The proof that these tangencies are in general position, it is the same as the one that we use to prove that \(q_{1,n,n_0}\) is in general position. Similarly one proves \((F5)\).

Observe that \((P1)\) follows by the fact that \(b_{n,n_0}\) is the curve of tangencies and \((P2)\) from the fact that \(\Psi_{1,3} \neq 0, 0\) see (4.48). \( \Box \)
Inductively we are going to construct parameters with multiple sinks of higher and higher periods and a strong homoclinic tangency. Let $\mathcal{N}^1 = \{(n, n_0) \mid n_0 \in \mathcal{N}_n\} \subset \mathbb{N}^2$. Suppose there exist sets $\mathcal{N}^1 \leftarrow \mathcal{N}^2 \leftarrow \cdots \leftarrow \mathcal{N}^g$ with $\mathcal{N}^k \in (\mathbb{N}^2)^k$, and countable to 1 natural projection having the following properties. For $n = \{(n^{(k)}, n_{0}^{(k)})\}_{k=1}^{g} \in \mathcal{N}^g$, there exists a set $\mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}} \subset \mathcal{P}$ with diameter $\text{diam}\left(\mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}}\right) \leq \frac{1}{k}$ such that

- $\mathcal{P}^{k+1}_{n^{(k+1)}, n_{0}^{(k+1)}} \subset \mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}}$ for $k = 1, \ldots, g - 1$,
- there exists a map $f^k_{n^{(k)}, n_{0}^{(k)}} \in \mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}}$ which has a strong homoclinic tangency,
- the restriction $F : \mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}} \times M \to M$ can be reparametrized to become and unfolding of $f^k_{n^{(k)}, n_{0}^{(k)}}$,
- every map in $\mathcal{P}^k_{n^{(k)}, n_{0}^{(k)}}$ has at least $k$ sinks of different periods.

By induction, using Proposition 5.3 we get an infinite sequence of sets $\mathcal{N}^k$.

**Definition 5.4.** The sets $\mathcal{P}^g_{n^{(g)}, n_{0}^{(g)}}$ are called Newhouse boxes of generation $g$, see Figure 8, and

$$NH = \bigcap_{g} \bigcup_{n \in \mathcal{N}^g} \mathcal{P}^g_{n^{(g)}, n_{0}^{(g)}}.$$  

**Lemma 5.5.**

$$NH \supset [-t_0, t_0] \times \{0\}.$$
In particular,
\[ \overline{NH} \cap P_{n,n_0} \supset \text{graph}(b_{n,n_0}). \]  

(5.6)

Proof. Given \((t, 0)\), by Proposition 4.41, for every \(n\) large enough, \(A_n\) has a secondary tangency at \((t_n, a_n(t_n))\) with \(|t - t_n| = O\left(\frac{1}{n}\right)\). Hence, there exists a sequence of Newhouse boxes in \(R_t\) accumulating at \((t, 0)\). By construction, each box in \(R_t\) contains points of \(NH\). The lemma follows.

Given any family \(F\) of diffeomorphisms, we define the Newhouse set \(NH_F\) as the set of parameters having infinitely many sinks. The Minkowski dimension is denoted by \(MD\).

**Theorem A.** Let \(F : \mathcal{P} \times M \to M\) be an unfolding of a map \(f\) with a strong homoclinic tangency, then

- \(NH \subset NH_F\), every map in \(NH\) has infinitely many sinks,
- \(NH\) is homeomorphic to \(\mathbb{R} \setminus \mathbb{Q}\),
- \(MD(NH) \geq \frac{1}{2}\).

**Proof.** The first two properties follow immediately from the definition of \(NH\), Proposition 5.3, Lemma 5.1 and Corollary 4.46. For the last property, let \((t^*, 0)\) such that

\[ \frac{\log \frac{1}{\lambda(t^*, 0)}}{\log \mu(t^*, 0)} \]

is the maximum of

\[ \frac{\log \frac{1}{\lambda(t, 0)}}{\log \mu(t, 0)} \]

Consider a sequence of first generation Newhouse boxes \(P_{n,n_0} \in R_t\) accumulating at \((t^*, 0)\). This is possible because of Lemma 5.5. Choose \(\epsilon > 0\) and let \(n\) be maximal such that \(\epsilon \leq \frac{\epsilon_0}{\mu(t^*, a^*)^{2n}}\). Because (5.2), (5.6) and the fact that the vertical size of \(P_{n,n_0}\) is \(\epsilon_0/\mu(t^*, a^*)^{2n}\), we need at least \(K\lambda(t^*, a^*)^{-\theta n}\) balls of radius \(\epsilon\) to cover \(\overline{NH} \cap P_{n,n_0}\). As a consequence

\[ MD(NH) \geq \frac{\theta}{2} \max_{NH} \left( \frac{\log \frac{1}{\lambda}}{\log \mu} \right) , \]

and \(MD(NH) \geq \frac{1}{2}\), where we used (2.18). \(\square\)

**Theorem B.** Let \(M, \mathcal{P}\) and \(\mathcal{T}\) be \(C^\infty\) manifolds and \(F : (\mathcal{P} \times \mathcal{T}) \times M \to M\) a \(C^\infty\) family with \(\dim(\mathcal{P}) = 2\). If \(F_0 : (\mathcal{P} \times \{\tau_0\}) \times M \to M\) is an unfolding of a map \(f_{\tau_0}\) with a strong homoclinic tangency, then

- \(NH_F\) contains a codimension 2 lamination \(L_F\),
- \(L_F\) is homeomorphic to \(\mathbb{R} \setminus \mathbb{Q} \times \mathbb{R}^{\dim(\mathcal{T})}\),
- the leaves of \(L_F\) are \(C^1\) codimension 2 manifolds.
Proof. Observe that there exists a small neighborhood $\tau_0 \in U \subset T$ and a $C^\infty$ function $U \ni \tau \to f_\tau$ such that, for all $\tau \in U$, $f_\tau$ has a strong homoclinic tangency and the family $F_\tau: (\mathcal{P} \times \{ \tau \}) \times M \to M$ is an unfolding of $f_\tau$.

Let $C = 2 \max_{(t,\tau,a,\theta)} D_\theta K \left[ C_3 X_2 \frac{\partial y}{\partial \tau} + \frac{K}{\mu} \frac{\partial y}{\partial \tau} \right]$ and reselect the secondary tangencies as follows:

$$T^*_n = \left\{ (t, a) \in T_{n, n_0} \left| \frac{\partial W_{n_0}}{\partial t}(q_1, n, n_0) \right| \geq C n \left( \lambda_1^a \mu \right)^{2n} \right\}.$$  

Fix $\tau \in U$ and denote the Newhouse boxes of the family $F_\tau$ by $\{ N^g_0(\tau) \}$. These boxes are defined in terms of the smooth functions $a$ and $b$. Because the angle between $a$ and $b$ is uniformly bounded by a constant independent of $\tau$, see Proposition 4.47, the boxes $P^k_{n, n_0}(\tau)$ move smoothly with $\tau$. Let

$$L_F = \bigcap_g \bigcup_{n \in \mathbb{N}} P^g_{n, n_0}(U)$$

and observe that $L_F$ is homeomorphic to $U \times NH(\tau_0)$.

It is left to prove that the leaves $L_F$ are $C^1$. Let $\text{Tang}^k_{n, n_0}(U)$ be the codimension 1 surface of tangencies contained in $P^k_{n, n_0}(U)$. Let $P^k_{n, n_0}(U) \subset P^{k-1}_{n, n_0}(U)$ and reparametrize $P^{k-1}_{n, n_0}(U)$ in coordinates, also denoted by $(t, a, \tau)$, such that the restriction of this reparametrization to a slice at $\tau$, using only the coordinates $(t, a)$, is an unfolding. In particular, $\text{Tang}^{k-1}_{n, n_0}(U) = \{ a = 0 \}$. Observe that there is no difference between parameters $t$ and parameters $\tau$ and Proposition 4.47 applies to both. According to Proposition 4.47 we have, in the coordinates of $P^{k-1}_{n, n_0}(U)$,

$$\frac{db}{dt} = -\frac{1}{\mu^{n(k)}} \left[ \frac{n(k)}{\mu} \frac{\partial \mu}{\partial t} + O \left( n(k) \left( |\lambda_1^a| \mu \right)^{n(k)} \right) \right],$$

$$\frac{db}{d\tau_i} = -\frac{1}{\mu^{n(k)}} \left[ \frac{n(k)}{\mu} \frac{\partial \mu}{\partial \tau_i} + O \left( n(k) \left( |\lambda_1^a| \mu \right)^{n(k)} \right) \right].$$

As consequence the tangent bundle of $\text{Tang}^k_{n, n_0}(U)$ is $O \left( n(k) / |\mu|^{n(k)} \right)$ close to the tangent bundle of $\text{Tang}^{k-1}_{n, n_0}(U)$. By Lemma 3.1 we have

$$\frac{da}{dt} = O \left( \frac{n(k)}{|\mu|^{n(k)}} \right),$$

$$\frac{da}{d\tau_i} = O \left( \frac{n(k)}{|\mu|^{n(k)}} \right).$$

By Proposition 4.47 the graphs of $a$ and $b$ intersect transversally in a manifold $\ell$. Notice that $\ell$ is the graph of a $C^2$ function $\ell: \tau \mapsto \ell(\tau) \in P^k_{n, n_0}(\tau)$. In particular $\ell$ is a
codimension 2 manifold. According to Proposition 4.15 we have
\[
\frac{db}{dt} - \frac{da}{dt} = Vn\lambda_1^\theta_1n + O\left(|\lambda_1|^{\theta_1n}\right),
\]
\[
\frac{db}{d\tau_i} - \frac{da}{d\tau_i} = V_i n\lambda_1^\theta_1n + O\left(|\lambda_1|^{\theta_1n}\right),
\]
where $V$ and $V_i$ are continuous and uniformly away from zero. By the previous formulas, the tangent space of $\ell$ is given by
\[
V\Delta t + \sum V_i \Delta \tau_i = 0
\]
\[
O \left( \frac{n^{(k)}}{|\mu|^{n^{(k)}}} \left( \Delta t + \sum \Delta \tau_i \right) \right) = \Delta a.
\]
By restricting $N^k$ to large values of $n^{(k)}$ the tangent spaces of $\ell$ converge to $\Delta a = 0$ and $V\Delta t + \sum V_i \Delta \tau_i = 0$. As consequence, the leaves of the lamination are smooth manifolds and the tangent spaces to the leaves of the lamination vary continuously.

6 The Hénon family

Consider the real Hénon family $F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$,
\[
F_{a,b}(x, y) = \left( a - x^2 - by, x \right).
\]

Theorem C. The real Hénon family contains a set $NH$, homemorphic to $\mathbb{R} \setminus \mathbb{Q}$, of maps with infinitely many sinks. Moreover the space $\text{Poly}_d(\mathbb{R}^n)$ of real polynomials of $\mathbb{R}^n$ of degree at most $d$ contains a codimension 2 lamination of maps with infinitely many sinks. The lamination is homemorphic to $\mathbb{R} \setminus \mathbb{Q} \times \mathbb{R}^m$ where $m = \dim \text{Poly}_d(\mathbb{R}^n) - 2$ and the sinks move analytically along the leaves.

Proof. Consider the map $f(x) = 2 - x^2$. Then $x = -2$ is an expanding fixed point and $f^2(0) = -2$ where 0 is the critical point. Moreover there exists an analytic curve $b \mapsto a(b)$ with $a(0) = 2$ such that the parameter $(a(b), b)$ corresponds to a Hénon map with an homoclinic tangency of the saddle point $p(b)$ which is a continuation of $p(0) = (-2, -2)$. For all $b$ positive and small enough $F_{a(b), b}$ has a strong homoclinic tangency and the Hénon family $F_{a,b}$ restricted to a small neighborhood of $(a(b), b)$ is an unfolding. The theorem follows for $\text{Poly}_d(\mathbb{R}^2)$ from Theorem A and Theorem B applied to the unfolding $F_{a,b}$.

Extend now the family $F_{a,b}$ to $F: \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ as
\[
F_{a,b}(x, y, y_3, \ldots, y_n) = \left( a - x^2 - by, x, b^3 y_3, \ldots, b^n y_n \right).
\]
Observe that this higher dimensional Hénon family is again an unfolding in the same neighborhood of $(a(b), b)$. As consequence Theorem A and Theorem B apply and the proof is complete. \qed
Observe that Theorem A and Theorem B have an holomorphic version. In particular, the main steps in the construction of the Newhouse laminations, these are Proposition 3.3, Proposition 4.41 and Proposition 4.15 can be obtained with exactly the same proof in the holomorphic setting. Then the holomorphic version of the proof of Theorem C gives the following.

**Theorem D.** Let $\text{Poly}_d(\mathbb{C}^n)$ be the space of polynomial maps of $\mathbb{C}^n$ of degree at most $d$. The space $\text{Poly}_d(\mathbb{C}^n)$ contains a (complex) codimension 2 lamination of maps with infinitely many sinks, whose transversal section is homemorphic to $\mathbb{R} \setminus \mathbb{Q}$. The sinks move holomorphically along the leaves.

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