On the $O(1/N^2)$ $\beta$-function of the
Nambu–Jona-Lasinio model with non-abelian chiral
symmetry.

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Abstract. We present the formalism for computing the critical exponent

\[ \text{corresponding to the } \beta\text{-function of the Nambu-Jona-Lasinio model with } SU(M) \times SU(M) \text{ continuous chiral symmetry at } O(1/N^2) \text{ in a large } N \text{ expansion, where } N \text{ is the number of fermions. We find that the equations can only be solved for the case } M = 2 \text{ and subsequently an analytic expression is then derived. This contrasting behaviour between the } M = 2 \text{ and } M > 2 \text{ cases, which appears first at } O(1/N^2), \text{ is related to the fact that the anomalous dimensions of the bosonic fields are only equivalent for } M = 2.\]
1 Introduction.

Quantum field theories with a four fermi interaction are known to possess an interesting range of properties. For example, the two dimensional model, introduced by Gross and Neveu in [1], is an asymptotically free theory where the fundamental particles of the massless classical theory acquire a mass dynamically in the quantum theory. Therefore, it mimics several of the features of more involved four dimensional field theories such as quantum chromodynamics and so has been used as a simple laboratory for testing out ideas which are harder to examine there. More recently, four dimensional models possessing a four fermi interaction have received intense study following the observation of Nambu, [2], that such models could provide a realistic alternative to the Higgs boson for generating a mass for the particles seen in nature, [3, 4].

One technique which can be used to analyse models with a four fermi interaction is the large $N$ expansion, [1], where $N$ is the number of fermions. For instance, computing the effective potential of the two dimensional model in the saddle point approximation when $N$ is large one observes that the perturbative vacuum of the theory is not the correct one but that where the vacuum expectation value of a bosonic auxiliary field is non-zero which in turn generates the fermion mass. This behaviour is also preserved in the three dimensional model and it is important to note this property is non-perturbative and therefore would never be accessed in a perturbative approach. However, whilst the large $N$ expansion has been powerful in revealing such structure the conventional method possesses one particular drawback in that it is virtually impossible to calculate beyond the leading order. This is primarily due to the nature of the (dynamically generated) propagators of the bosonic $\bar{\psi}\psi$ bound states which exist in the model and are non-fundamental in structure, [1]. Thus computing Feynman integrals with such massive propagators becomes impossible beyond leading order.

Clearly there is a need to probe such models to higher orders in $1/N$ and to have a technique which interpolates between the spacetime dimensions already discussed. One such method currently does exist which achieves these aims is the critical point self consistency method of Vasil’ev et al which was developed for the $O(N)$ bosonic $\sigma$ model in [5, 6]. The beauty of that approach is that one calculates within the theory at the fixed point which is defined as the non-trivial zero of the $d$-dimensional $\beta$-function. At this point the theory is conformal and therefore massless which allows one to overcome the difficulty of computing previously intractable integrals.
beyond the leading order. Moreover, one solves the model by determining the critical exponents of the fields and Green’s functions \[5\]. These can then be related to the perturbative renormalization group functions through an examination of the renormalization group equation in the critical region. Examples of this approach can be found in \[7\]. Further, the \(\beta\)-function of any theory carries all the important information on the evolution of the coupling constant with the renormalization scale, and it too can be computed order by order in \(1/N\) by considering the appropriate critical exponent, \[9\]. As these are determined as functions of the spacetime dimension one can deduce information on models simultaneously in several different dimensions.

In this paper, we complete the examination of the four fermi models at \(O(1/N^2)\) by computing the \(\beta\)-function exponent of the Nambu–Jona-Lasinio model with a non-abelian continuous global chiral symmetry. Previously various exponents had been calculated in the \(O(N)\) Gross Neveu model in \[8, 9, 10, 11\] using the original large \(N\) approach. The breakthrough to \(O(1/N^2)\) was achieved in \[12\] with the computation of the fermion anomalous dimension \(\eta\) and latterly the mass, \[13, 14\], vertex, \[13, 14\], and \(\beta\)-function exponents, \[14, 15\] have been determined as well as \(\eta\) at \(O(1/N^3)\), \[14, 16\]. Several of these exponents were subsequently determined to \(O(1/N^2)\), \[17\], for the generalization of the \(O(N)\) model, which possesses a discrete chiral symmetry, to the case where it has a \(U(1) \times U(1)\) or \(SU(M) \times SU(M)\) continuous global chiral symmetry, \[1, 18\]. Several leading order results had earlier been presented in \[11, 19\] using the saddle point approach. By computing the exponent \(\lambda\), where \(2\lambda = - \beta'(g_c)\), at \(O(1/N^2)\) in the \(SU(M) \times SU(M)\) case here the model can then be said to have been solved thermodynamically since knowledge of two independent exponents means that the remaining ones can be deduced through hyperscaling laws which have been checked at leading order in \[11\] and their consistency merely reflects the renormalizability of the model. Several additional motivations for considering the \(SU(M) \times SU(M)\) model in particular include the provision of analytic results for the three dimensional model which will provide key estimates for numerical simulations of the model on the lattice. Recently, estimates for several exponents have been determined using Monte Carlo methods in the \(O(N)\) Gross Neveu model for relatively small values of \(N\) in \[20\] and they are in agreement with \(O(1/N^2)\) results, \[13\]. Further the provision of expressions valid in \(d\)-dimensions as a function of \(N\) is important for demonstrating the equivalence of various models. For example, it is known that the \((4 - \epsilon)\)-dimensional Yukawa model and the \((2 + \epsilon)\)-dimensional Gross Neveu model lie in the same universality classes.
which has been established by examining the $\epsilon$-expansions of the perturbative renormalization group functions to as many orders as they are presently known. In our case the equivalence of the $(2 + \epsilon)$-dimensional non-abelian Nambu–Jona-Lasinio or chiral Gross Neveu model will probably be with the generalized $(4 - \epsilon)$-dimensional Gell–Mann-Lévy $\sigma$ model introduced in [21] and currently of interest due to its relation to hadronic physics. (The original model possessed an $SU(2) \times SU(2)$ chiral symmetry and by generalized we mean its natural extension to the case of $SU(M) \times SU(M)$.) The provision of critical exponents to $O(1/N^2)$ which we carry out here will be the first step in such a proof.

Finally, we mention that there is some uncertainty in the literature concerning the behaviour of the $SU(M) \times SU(M)$ model for $M > 2$ and for $M = 2$, [11]. It has been suggested in [11] that the chiral symmetry is only realised in the case $M = 2$ and not for higher $M$, [1]. So far in the large $N$ self consistency approach there has been no indication of distinct behaviour between either case. This is normally observed in the breakdown of the computation of the $\beta$-function exponent at some order in large $N$ which has been discussed in the context of other models in [22]. At leading order we were able in [17] to compute $\eta$, $\lambda$ and the vertex anomalous dimensions, $\chi$, at $O(1/N)$ and $\eta$ and $\chi$ at $O(1/N^2)$ without complications for all $M$. So it will be interesting to see if it is possible to deduce $\lambda$ at $O(1/N^2)$ for all $M$.

The paper is organised as follows. In section 2 we introduce the details of the model we will examine at criticality and illustrate the method by computing $\eta$ at $O(1/N)$. The leading order analysis is continued in section 3 where the calculation of $\lambda$ at $O(1/N)$ is detailed in preparation for the $O(1/N^2)$ analysis which is presented in section 4. We conclude the paper by solving the master equation in section 5 where we discuss the results for $M = 2$ and $M > 2$ separately.

## 2 Preliminaries.

To begin with we describe the model we are interested and introduce the notation and formalism we will use in solving it. Its lagrangian can be written in several ways but we choose to use the auxiliary field version since it involves three point vertices which are essential to the technique of uniqueness used to compute the Feynman graphs, [23]. Thus we take [1, 18]

$$L = i\bar{\psi}^i I \psi^i I + \sigma \bar{\psi}^i I \gamma^I \psi^i I + i\pi^a \bar{\psi}^i I \lambda^a_{IJ} \gamma^I \psi^j I - \frac{1}{2g^2} (\sigma^2 + \pi^a)^2 \quad (2.1)$$
where $\psi^{iI}$ is the fermion field with $1 \leq i \leq N$, $1 \leq I \leq M$, $1 \leq a \leq (M^2 - 1)$ and $g$ is the coupling constant. Eliminating the auxiliary bosonic fields $\sigma$ and $\pi^a$ from (2.1) would yield the four fermion interaction explicitly and according to [1] (2.1) possesses an $SU(M) \times SU(M)$ continuous global chiral symmetry. The matrices $\lambda^a_{IJ}$ are the generalized Pauli matrices of the group $SU(M)$, where $[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c$ and $f^{abc}$ are its structure constants, and we normalize them in as general a way as possible by taking

$$\text{Tr}(\lambda^a \lambda^b) = 4T(R) \delta^{ab}$$

(2.2)

The Casimirs which will appear throughout the paper are

$$\lambda^a \lambda^a = 4C_2(R) I, \quad f^{acd} f^{bcd} = C_2(G) \delta^{ab}$$

(2.3)

where for $SU(M)$, $T(R) = \frac{1}{2}$, $C_2(R) = (M^2 - 1)/2M$ and $C_2(G) = M$. Also, in order to carry out intermediate checks on our calculation we will retain $C_2(R)$ and $C_2(G)$ in their general form to allow us to take the limits to either the case with $U(1) \times U(1)$ chiral symmetry, $(C_2(R) = T(R) = \frac{1}{2}, C_2(G) = 0, M = 1)$ or the $O(N)$ model with discrete chiral symmetry, $(C_2(R) = C_2(G) = 0)$ and compare with results we already know.

We now introduce the basics to enable us to solve (2.1) to $O(1/N^2)$ in the critical point self consistency approach. First, the fundamental idea is to examine the scaling behaviour of the fields of (2.1) in the critical region defined as the non-trivial zero of the $\beta$-function in $d$-dimensions, [5, 6, 12]. Since the fields are massless at this fixed point where there is a conformal symmetry the structure of the Green’s functions take a particularly simple form. For instance, in coordinate space they are

$$\psi(x) \sim \frac{A^d}{(x^2)^\alpha}, \quad \sigma(x) \sim \frac{B}{(x^2)^\beta}, \quad \pi(x) \sim \frac{C}{(x^2)^\gamma}$$

(2.4)

as $x^2 \to 0$ where we use the same letter to denote the propagator and $A$, $B$ and $C$ are $x$-independent amplitudes. By analogy with ideas in statistical mechanics the properties of the model in the critical neighbourhood are described totally by the critical exponents $\alpha$, $\beta$ and $\gamma$ of each of the individual fields, [7]. By the universality principle they are functions only of the space-time dimension, $d = 2\mu$, and any internal parameters of the theory. For our case, these will be $N$ and $M$ through the appearance of the Casimirs of (2.3). Moreover, the exponents can be related to the appropriate renormalization group functions which are ordinarily calculated perturbatively as a
series in $g$, by examining the renormalization group equation at criticality, \[7\]. For instance, if we examine the canonical dimension of each field in the action which is a dimensionless quantity then we can define their anomalous dimension as follows, \[17\].

\[
\alpha = \mu + \frac{1}{2} \eta, \quad \beta = 1 - \eta - \chi \sigma, \quad \gamma = 1 - \eta - \chi \pi \tag{2.5}
\]

where $\eta$ is the fermion anomalous dimension which is related through the critical renormalization group equation to the fermion wave function renormalization constant. The other exponents, $\chi \sigma$ and $\chi \pi$, are the anomalous dimensions of the respective 3-vertices in (2.1) and these can also be related to the analogous renormalization constants.

In the critical point approach we discuss throughout this paper, the exponents are expanded order by order in $1/N$, where $N$ is large, and the exponent $\eta_i$, say, is calculated at the $i$th order where $\eta = \sum_{i=1}^{\infty} \eta_i/N^i$ and $\eta_i = \eta_i(\mu, M)$. The method to do this has been discussed extensively before, \[12\], but for completeness sake and for preparation to calculating the $\beta$-function exponent $\lambda$ we review the calculation of $\eta_1$, \[17\]. This is achieved by examining the skeleton Dyson equations with dressed propagators in the critical region, truncated to the appropriate order in $1/N$, which for the moment will be leading order. The relevant Feynman graphs for this are illustrated in fig. 1, where the quantities $\psi^{-1}$, $\sigma^{-1}$ and $\pi^{-1}$ are the 2-point functions of the fields. Their asymptotic scaling forms in the critical region in coordinate space can be easily deduced from the forms of the propagators (2.4) by mapping (2.4) first to momentum space performing the inversion there before inverting the map to coordinate space, \[5, 12\]. This is facilitated by use of the Fourier transform

\[
\frac{1}{(x^2)^{a}} = \frac{a(\alpha)}{2^{2\alpha-\alpha}} \int_k e^{ikx} \tag{2.6}
\]

where, for simplicity, $a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha)$. The functions are then

\[
\psi^{-1}(x) \sim \frac{r(\alpha - 1)^{\frac{1}{\mu - \alpha + 1}}}{A(x^2)^{2\mu - \alpha + 1}}, \quad \sigma^{-1}(x) \sim \frac{p(\beta)}{B(x^2)^{2\mu - \beta}}
\]

\[
\pi^{-1}(x) \sim \frac{p(\gamma)}{C(x^2)^{2\mu - \gamma}} \tag{2.7}
\]

as $x^2 \to 0$ where

\[
p(\beta) = \frac{a(\beta - \mu)}{\pi^{2\mu} a(\beta)} , \quad r(\alpha - 1) = \frac{\alpha a(\alpha - \mu)}{\pi^{2\mu} (\mu - \alpha) a(\alpha)} \tag{2.8}
\]
To represent the Dyson equations of fig. 1 in the critical region, one merely substitutes (2.4) in the lines of the one loop graphs to obtain

\[
0 = r(\alpha - 1) + z + 4C_2(R)y \quad (2.9)
\]

\[
0 = p(\beta) + 2zNM \quad (2.10)
\]

\[
0 = p(\gamma) + 8T(R)Ny \quad (2.11)
\]

where \(z = A^2B\) and \(y = A^2C\) and we have used the relations (2.2) and (2.3).

In (2.9)-(2.11) one has three unknowns, \(z\), \(y\) and \(\eta_1\) through the appearance of \(\Gamma(\mu - \alpha)\) in (2.9). Thus eliminating \(z_1\) and \(y_1\) after expanding to leading order in \(1/N\) one obtains

\[
\eta_1 = \frac{\tilde{\eta}_1}{2} \left[ \frac{1}{M} + \frac{C_2(R)}{T(R)} \right] \quad (2.12)
\]

where \(\tilde{\eta}_1 = -2\Gamma(2\mu - 1)/[\Gamma(\mu + 1)\Gamma(1 - \mu)\Gamma(\mu - 1)]\) and consequently

\[
z_1 = \frac{\mu \Gamma^2(\mu) \tilde{\eta}_1}{4\pi^2\mu M}, \quad y_1 = \frac{Mz_1}{4T(R)} \quad (2.13)
\]

which will be required later. Whilst (2.12) is a result for all \(M\) and agrees in the limits mentioned earlier with [8, 9, 11] and [12], it also checks with a leading order calculation in [11] for \(M = 2\), using canonical techniques. However, as has been mentioned this latter method is not powerful enough to probe to \(O(1/N^2)\).

The vertex anomalous dimensions of (2.5) have also been computed to the same order in \(1/N\) by examining the scaling behaviour of the 3-point functions in an analogous way at criticality by extending the earlier work of [24, 25] and we record the results in preparation for computing \(\lambda\), as

\[
\chi_{\sigma 1} = \frac{\mu \tilde{\eta}_1}{2(\mu - 1)} \left[ \frac{1}{M} - \frac{C_2(R)}{T(R)} \right] \quad (2.14)
\]

\[
\chi_{\pi 1} = \frac{\mu \tilde{\eta}_1}{2(\mu - 1)} \left[ \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right] \quad (2.15)
\]

Again each agrees in the appropriate limit with earlier results and for \(M = 2\), \(\chi_{\sigma 1}\) agrees with [11].

3 \(\beta\)-function exponent.

Having indicated the simplicity of the method to deduce the exponents of all the fields at leading order, we now present the extensions to the formalism
to deduce $\lambda_1$ which will serve as the foundation for calculating $\lambda_2$ where $\lambda = \mu - 1 + \sum_{i=1}^{\infty} \lambda_i/N^i$. To achieve this one considers the corrections to the asymptotic scaling forms (2.4) by including $O(x^2)$ terms, \[3, 5, 6, 12\].

\[
\psi(x) \sim \frac{A' f}{(x^2)^{\alpha}} [1 + A'(x^2)^{\lambda}] \quad (3.1)
\]

\[
\sigma(x) \sim \frac{B}{(x^2)^{\beta}} [1 + B'(x^2)^{\lambda}] \quad (3.2)
\]

\[
\pi(x) \sim \frac{C}{(x^2)^{\gamma}} [1 + C'(x^2)^{\lambda}] \quad (3.3)
\]

Additional higher order corrections, which will involve other exponents such as the specific heat, could also be computed by including the appropriate forms. However, as there exist hyperscaling laws relating such exponents, by doing this one would merely succeed in demonstrating their consistency which is not in doubt, \[5, 6\]. The quantities $A'$, $B'$ and $C'$ are the respective amplitudes associated with each correction and the analogous 2-point functions can be deduced in a similar way to (2.7) to obtain, \[12\],

\[
\psi^{-1}(x) \sim \frac{r(\alpha - 1) f}{A(x^2)^{2\mu - \alpha + 1}} [1 - A's(\alpha - 1)(x^2)^{\lambda}] \quad (3.4)
\]

\[
\sigma^{-1}(x) \sim \frac{p(\beta)}{B(x^2)^{2\mu - \beta}} [1 - B'q(\beta)(x^2)^{\lambda}] \quad (3.5)
\]

\[
\pi^{-1}(x) \sim \frac{p(\gamma)}{C(x^2)^{2\mu - \gamma}} [1 - C'q(\gamma)(x^2)^{\lambda}] \quad (3.6)
\]

as $x^2 \rightarrow 0$ where

\[
q(\alpha) = \frac{a(\alpha - \mu + \lambda)a(\alpha - \lambda)}{a(\alpha - \mu)a(\alpha)} , \quad s(\alpha) = \frac{q(\alpha)a(\alpha - \mu)}{(\alpha - \mu + \lambda)(\alpha - \lambda)} \quad (3.7)
\]

With these corrections one can now reconsider the Dyson equations in the critical region. However, when one is dealing with a theory where the fundamental field is fermionic in this large $N$ approach, as we have here, several higher order graphs have to be included in the $\sigma$ and $\pi$ Dyson equations, \[22\], which are illustrated in fig. 2. The reason for this is quite simple and can be seen, for instance, by examining the $N$-dependence of each term in the $\sigma$-consistency equation. First, we have from figs 1 and 2, in the critical region

\[
0 = \frac{p(\beta)}{N} [1 - B'q(\beta)(x^2)^{\lambda}] + 2zM[1 + 2A'(x^2)^{\lambda}]
\]
\[
- z^2 M [\Pi + (x^2)\lambda (A'\Pi_{1A} + B'\Pi_{1B})] \\
+ 4yz MC_2(R) [\Pi + (x^2)\lambda (A'\Pi_{1A} + C'\Pi_{1C})]
\]

(3.8)

where the subscripts \(A, B\) and \(C\) denote the respective insertion of \((x^2)\lambda\) on a \(\psi, \sigma\) or \(\pi\) line. As in \[6, 12\] the equation decouples into two pieces one of which is relevant for the computation of \(\eta_2\), which we ignore here, and the other which involves the \((x^2)\lambda\) terms, since each is of differing dimension in \(x^2\), ie

\[
0 = 4zA' - B' \left[ p(\beta)q(\beta) \frac{NM}{N} + z^2 \Pi_{1B} \right] + 4C_2(R) yz \Pi_{1C}
\]

(3.9)

In (3.9), we have neglected the graphs where there is an insertion on the fermion line of the 2-loop graphs of fig. 2 as these do not contribute at \(O(1/N)\). Further, the expressions \(\Pi_{1A}, \Pi_{1B}\) and \(\Pi_{1C}\) denote the value of the integral which is \(O(1)\), without symmetry factors. Now if one substitutes the leading order values for \(\alpha, \beta\) and \(\gamma\) into (3.9) to examine the location of \(N\) in each term, it is easy to see that both terms of the coefficient of \(B'\) are of the same order. Therefore, neglecting the graph \(\Pi_{1B}\) would omit a contribution and lead to an erroneous result and so it must be included. Similarly, the \(\pi\) consistency equation gives

\[
0 = 16T(R)yA' + 4T(R)y zA_{1B} B' - C' \left[ p(\gamma)q(\gamma) N \Pi^{'} + 8T(R)y^2 (2C_2(R) - C_2(G)) \Pi_{1C} \right]
\]

(3.10)

Finally, in the \(\psi\) equation one needs only to consider the one loop graphs of fig. 1 since there is no reordering as there is in (3.9) and (3.10) which can be seen by inspection. Thus

\[
0 = A' [ -r(\alpha - 1) s(\alpha - 1) + z + 4yC_2(R)] + zB' + 4yC_2(R) C'
\]

(3.11)

To proceed one forms a \(3 \times 3\) matrix with \(A', B'\) and \(C'\) as the basis vectors and sets its determinant to zero for consistency whence an expression for \(\lambda_1\) will emerge since it is the only unknown. For the moment the matrix is

\[
\begin{pmatrix}
  r(\alpha - 1) s(\alpha - 1) & z & \frac{C_2(R)y}{16T(R)N} \\
  4z & \frac{p(\beta)q(\beta) NM}{N} + z^2 \Pi' & \frac{p(\gamma)q(\gamma) NM}{16T(R)N} \\
  4y & -yz \Pi' & \frac{y^2 (2C_2(R) - C_2(G))}{2}
\end{pmatrix}
\]

(3.12)

where the explicit calculation of the 2-loop graphs gives \(\Pi_{1B} = \Pi_{1C} = \Lambda_{1B} = \Lambda_{1C} \equiv \Pi' = 2\pi^2 \mu / [\Gamma^2(\mu)(\mu - 1)^2], [12, 22]\). Again it is easy to see by examining the \(N\)-dependence in each element of (3.12) the necessity of including
the graphs of fig. 2. To solve for $\lambda_1$ by setting the determinant of (3.12) to zero one makes several row and column transformations. Subtracting row two from row three and then column three from column two one obtains the following matrix whose determinant is set to zero

$$
\begin{pmatrix}
\alpha - 1 & s(\alpha - 1) & t(\alpha - 1) \\
-4 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

where we have used (2.13). Thus setting (3.12) to zero yields $\lambda_1$ as

$$
\lambda_1 = -\frac{(2\mu - 1)\tilde{\eta}^1}{2} \left[ \frac{1}{M} + \frac{C_2(R)}{T(R)} \right]
$$

This agrees with previous results of the $O(N)$ model [12] and the $U(1) \times U(1)$ case [11, 26] in the appropriate limits as well as the $SU(2) \times SU(2)$ calculation of [11]. However, it is worth noting that an alternative second solution might appear to emerge from the lower right element of (3.13). We discard this as it is not consistent with previous results and, moreover, the row and column transformations we have made do not induce any contribution from the $\psi$ equations which appear as the upper row in (3.12). Indeed this type of factorization of the final determinant has been observed in other contexts, [22], where one had also to ignore a spurious solution. Also, we remark that together with (2.12), (2.14) and (2.15) we now have a complete set of exponents for the $SU(M) \times SU(M)$ model at leading order.

We conclude this section by stating that it is now possible to proceed beyond (3.14) and attempt to calculate the $O(1/N^2)$ corrections, which has been achieved recently for the $O(N)$ model [14, 15] and $U(1) \times U(1)$ case [26] and use can be made of results calculated there. Something which is necessary for this is the $O(1/N^2)$ corrections to $\eta$, $\chi_\sigma$ and $\chi_\pi$ since these will appear in the $1/N$ expansion of the functions of (3.4)-(3.6). They have been deduced in [17] by considering the scaling behaviour of the higher order graphs in the Dyson equations of the 2 and 3 point functions with dressed propagators and we record the results here. We have, [17],

$$
\eta_2 = \frac{\tilde{\eta}^2}{4} \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right)^2 \left( \Psi(\mu) + \frac{2}{\mu - 1} + \frac{1}{2\mu} \right) \right]
$$

$$
+ \frac{\mu}{(\mu - 1)} \left( \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right)^2 - \frac{C_2(G)C_2(R)}{2T^2(R)} \right) \left( \Psi(\mu) + \frac{3}{2(\mu - 1)} \right)
$$

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(the sign of the coefficient of the $C_2(G)$ term was incorrectly given in [17])

$$\chi_{\pi 2} = \frac{\mu \eta_r^2}{4(\mu - 1)^2} \left[ (2\mu - 1) \left( \frac{1}{M^2} - \frac{C_2^2(R)}{T^2(R)} \right) \left( \Psi(\mu) + \frac{1}{(\mu - 1)} \right) \right]
+ \frac{\mu C_2(R)C_2(G)}{2T^2(R)} \left( \Psi(\mu) + \frac{1}{(\mu - 1)} \right) + \frac{3\mu}{2(\mu - 1)} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right)^2
+ \frac{5\mu C_2(R)}{(\mu - 1)T(R)} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) - \frac{2\mu}{(\mu - 1)} \left( \frac{1}{M} - \frac{C_2^2(R)}{T^2(R)} \right)
+ \frac{\mu}{M} \left( 3(\mu - 1)\Theta(\mu) - \frac{2\mu - 3}{(\mu - 1)} \right) \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) \right] \ (3.16)$$
where $\Psi(\mu) = \psi(2\mu - 1) - \psi(1) + \psi(2 - \mu) - \psi(\mu)$ and $\Theta(\mu) = \psi'(\mu) - \psi'(1)$ where $\psi(x)$ is the logarithmic derivative of the $\Gamma$-function. Further, from the $\eta_2$ consistency equations

\begin{align}
\eta_2 &= \frac{\mu\Gamma^2(\mu)\tilde{\eta}_1^2}{8\pi^2\mu M} \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} + \frac{\mu}{(\mu - 1)} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) \right) \times \left( \Psi + \frac{2}{(\mu - 1)} \right) - \frac{\mu}{(\mu - 1)^2} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) \right] \quad (3.17)
\end{align}

\begin{align}
\eta_2 &= \frac{\mu\Gamma^2(\mu)\tilde{\eta}_1^2}{32\pi^2\mu T(R)} \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} + \frac{\mu}{(\mu - 1)} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right) \times \left( \Psi + \frac{2}{(\mu - 1)} \right) - \frac{\mu}{(\mu - 1)^2} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right] \quad (3.18)
\end{align}

\begin{align}
\eta_2 &= \frac{\mu\Gamma^2(\mu)\tilde{\eta}_1^2}{32\pi^2\mu T(R)} \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} + \frac{\mu}{(\mu - 1)} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right) \times \left( \Psi + \frac{2}{(\mu - 1)} \right) - \frac{\mu}{(\mu - 1)^2} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right] \quad (3.19)
\end{align}

\section{Beyond $O(1/N)$.}

The main effort required to go beyond (3.14) and determine $\lambda_2$ involves computing the higher order Feynman graphs which appear in the Dyson equations of each field. First, though we need to derive the formal corrections to each critical representation of the Dyson equations which is a non-trivial exercise due to the appearance of divergent graphs which have a vertex subgraph and finite three and four loop graphs. The renormalization techniques required to handle the infinities have been discussed in other places, [12], but for completeness we detail the procedure here and concentrate on the $\psi$ equation since there are fewer corrections to be considered compared to the $\sigma$ and $\pi$ fields because of the extra graphs which arise. The higher order graphs with dressed propagators for the $\psi$ Dyson equations are illustrated in fig. 3. Including them in (2.9) and (3.11) one obtains

\begin{align}
0 &= r(\alpha - 1)[1 - A'(\alpha - 1)(x^2)\chi]\partial_t\psi + z[1 + (A' + B')(x^2)\chi]\partial_t^2\psi + \frac{4C_2(R)[1 + (A' + C')](x^2)\chi}{\partial_t^2\psi} + \frac{z^2}{\partial_t^2\psi} + \frac{8yR_2(R)[1 + (A' + C')(x^2)\chi]}{\partial_t^2\psi} + \frac{16y^2C_2(R)[C_2(R) - \frac{1}{2}C_2(G)]}{\partial_t^2\psi} \quad (4.1)
\end{align}

Unlike at leading order the powers of $x^2$ do not cancel here and, moreover, the graphs $\Sigma, \Sigma_A, \Sigma_B$ and $\Sigma_C$ are divergent with each having simple poles.
This infinitesimal quantity is a regularization introduced to control such infinities by shifting the exponents of $\sigma$ and $\pi$ by $\beta \rightarrow \beta - \Delta, \gamma \rightarrow \gamma - \Delta, [8, 12]$. If we formally define the $\Delta$-finite part of the two loop corrections to be $\Sigma'_A$ etc the simple poles in the $(x^2)^\lambda$ sector of (4.1) are absorbed by choosing the respective vertex counterterms $u$ and $v$ in such a way that (4.1) is $\Delta$-finite at $O(1/N^2)$. After this the $\ln x^2$ terms which remain and which prevent one approaching the $x^2 \rightarrow 0$ region are removed by choosing $\chi_{\sigma 1}$ and $\chi_{\pi 1}$ appropriately. It turns out that the choices already found in the analysis of the 3-point functions, (2.14) and (2.15), achieve this which is a consistency check on our renormalization. Decoupling the equation as before yields

$$0 = A'[r(\alpha - 1)s(\alpha - 1) + z + 4yC_2(R)] + \Sigma'_A(z^2 - 8yzC_2(R) + 16y^2(C_2(R) - \frac{1}{2}C_2(G))] + B'[z + z\Sigma'_B(z - 8yC_2(R))] + C'[4C_2(R)y + 8yC_2(R)\Sigma'_C(y(2C_2(R) - C_2(G)) - z)]$$

which is the $O(1/N^2)$ correction to (3.11). However, examining the $N$-dependence of the terms in the coefficient of $A'$, $\Sigma'_A$ is $O(1/N^2)$ relative to $r(\alpha - 1)s(\alpha - 1)$ and can therefore be neglected in the overall $3 \times 3$ determinant. Moreover, the explicit calculation of $\Sigma'_B$ and $\Sigma'_C$ reveals that both are zero [12, 15] and therefore (4.2) simplifies to

$$0 = A'[r(\alpha - 1)s(\alpha - 1) + z + 4yC_2(R)] + zB' + 4C_2(R)yC'$$

where of course now the $O(1/N^2)$ terms of $z$ and $y$ are needed. We have detailed the renormalization of the $\psi$ equation since it has a simpler structure than that of the $\sigma$ and $\pi$ equations which are complicated by the higher order graphs which have to be included. The relevant graphs, with dressed propagators, for the $\sigma$ equation are illustrated in fig. 4 where the label beside each graph will be used in the following. We have only displayed the distinct topologies in fig. 4 and the solid internal lines, without dots, denote either a $\sigma$ or $\pi$ field. As at leading order we need only consider $(x^2)^\lambda$ insertions on the bosonic lines, since within the final $3 \times 3$ matrix graphs with insertions on the fermionic lines will contribute to $\lambda_3$ only. The dot is intended to indicate the location of the $(x^2)^\lambda$ insertion when one substitutes the asymptotic scaling forms of the propagators (3.1)-(3.3) in the graphs. The correction graphs for the $\pi$ equations are formally equivalent to those of fig. 4. It is clear from the structure of $\Pi_{2B1}$ and $\Pi_{3B}$ that they are $\Delta$-divergent due to the presence of a vertex subgraph. Within the consistency
equation the same renormalization procedure and vertex counterterms we discussed for the \( \psi \) equation removes the simple poles in \( \Delta \) and it is therefore a straightforward matter to write down the finite \( O(1/N^2) \) correction to (3.9) in the critical region as

\[
0 = zM A'[4 - z\Pi_{1A}] - B' \left[ \frac{p(\beta)q(\beta)}{N} + z^2 M \Pi_{1B} + \Pi_{B2} \right] \\
+ C'[4C_2(R)Myz\Pi_{1C} - \Pi_{C2}] \quad (4.4)
\]

where \( \Pi_{B2} \) and \( \Pi_{C2} \) are formally defined to be

\[
\Pi_{B2} = 2(\Pi_{2B1} + \Pi_{2B2}) + \Pi_{3B} + \Pi_{4B} \\
- 2(2\Pi_{5B1} + \Pi_{5B2}) - 4(\Pi_{6B1} + 2\Pi_{6B2}) \quad (4.5)
\]

\[
\Pi_{C2} = 2(\Pi_{2C1} + \Pi_{2C2}) + \Pi_{3C} + \Pi_{4C} \\
- 2(2\Pi_{5C1} + \Pi_{5C2}) - 4(\Pi_{6C1} + 2\Pi_{6C2}) \quad (4.6)
\]

with the \( \Delta \)-finite part only understood as contributing. The factors of \( z \) and \( y \) as well as group factors are contained within each formal term of (4.5) and (4.6) and will be computed explicitly later. Whilst \( \Pi_{1B} \) and \( \Pi_{1C} \) contributed at leading order it is important to remember that each has an \( O(1/N) \) correction which will appear in \( \lambda_2 \). Further, explicit calculation has shown that \( \Pi_{1A}' = 0 \), [12]. The formal corrections to the \( \pi \) equation are also now straightforward to write down and we find

\[
0 = 16T(R)yA' + B'[4T(R)y\Lambda_{1B} - \Lambda_{B2}] \\
- C' \left[ \frac{p(\gamma)q(\gamma)}{N} + 8y^2 T(R)(2C_2(R) - C_2(G))\Lambda_{1C} + \Lambda_{C2} \right] \quad (4.7)
\]

where the higher order corrections, \( \Lambda_{B2} \) and \( \Lambda_{C2} \), have the same formal definitions as (4.5) and (4.6). Thus, with (4.2), (4.4) and (4.7) one can now form the corrected version of the \( 3 \times 3 \) matrix (3.12) and deduce the master equation which yields \( \lambda_2 \). All that remains is the explicit evaluation of the higher order graphs.

These fall into two classes. First, we need to consider the \( O(1/N) \) corrections to \( \Pi_{1B}, \Pi_{1C}, \Lambda_{1B} \) and \( \Lambda_{1C} \) since, for example, the fermionic lines of the 2-loop graphs each have exponents \( \mu + \frac{1}{2}\eta \) which therefore means they have to be expanded in powers of \( 1/N \). An algorithm to do this was presented in [27] and the analogous graph has also been computed in the \( O(N) \) model, [12, 22]. Whilst the graphs are of a similar structure the exponents of the internal bosonic lines differ in each case and we therefore had
to recompute $\Pi_{1B} = \Pi_{1C}$ and $\Lambda_{1B} = \Lambda_{1C}$ explicitly. Using the techniques of [27] we obtained

$$
\Pi_{1B} = \frac{2\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)} \left[ 1 - \frac{\bar{\eta}_1}{(\mu - 1)N} \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right) \right]
+ \frac{3\mu(\mu - 1)\bar{\eta}_1}{2N} \left( \Theta(\mu) + \frac{1}{(\mu - 1)^2} \right)
\times \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{1}{2(\mu - 1)} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right) \right] \quad (4.8)
$$

$$
\Lambda_{1B} = \frac{2\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)} \left[ 1 - \frac{\bar{\eta}_1}{(\mu - 1)N} \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right) \right]
+ \frac{3\mu(\mu - 1)\bar{\eta}_1}{2N} \left( \Theta(\mu) + \frac{1}{(\mu - 1)^2} \right)
\times \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{1}{2(\mu - 1)} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right) \right] \quad (4.9)
$$

The second class of graphs are those of fig. 4. Since the integral structure of each is completely the same as the graphs of the $O(N)$ model we needed only to compute the $SU(M)$ factors multiplying each which arise from the appearance of $\lambda^a\gamma^5$ at various vertices of the graphs. Making use of the relations, which have been discussed in [28],

$$
\lambda^a\lambda^b = \frac{4T(R)}{M} \delta^{ab} I + d^{abc}\lambda^c + if^{abc}\lambda^c \quad (4.10)
$$

which defines the totally symmetric tensor $d^{abc}$ in $SU(M)$ and

$$
d^{apq}d^{bpq} = \left[ 4C_2(R) - \frac{4T(R)}{M} - C_2(G) \right] \delta^{ab} \quad (4.11)
$$

$$
d^{apq}f^{brp}f^{crq} = \frac{C_2(G)}{2} d^{abc} \quad (4.12)
$$

$$
d^{apq}d^{brp}d^{crq} = \left[ 4C_2(R) - \frac{8T(R)}{M} - \frac{3C_2(G)}{2} \right] d^{abc} \quad (4.13)
$$

we managed to compute $\Pi_{B2}, \Pi_{C2}, \Lambda_{B2}$ and $\Lambda_{C2}$ relative to the $O(N)$ model graphs, which are denoted by a superscript $^o$ in the following. Making use of the relation between $y_1$ and $z_1$ we have

$$
\Pi_{B2} = 2z^3M^2 \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right) [\Pi_{2B2} + \Pi_{4B}^o]
$$
\[- z_1 M (2 \Pi_{5B1}^0 + \Pi_{5B2}^0 + 2 \Pi_{6B1}^0) \]
\[
+ 2 z^3 M^2 \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) \left[ \Pi_{2B1}^0 + \Pi_{3B}^0 - 4 z_1 M \Pi_{6B2}^0 \right] \quad (4.14)
\]

\[
\Pi_{C2} = \frac{2 z^3 M^3 C_2(R)}{T(R)} \left[ \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) + \frac{C_2(G)}{2T(R)} \right] \Pi_{2B1}^0
\]
\[- \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \Pi_{2B2}^0 \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) \Pi_{3B}^0 \right]
\[
+ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \Pi_{4B}^0 - 2 z_1 (2 \Pi_{5B1}^0 + \Pi_{5B2}^0 - 2 \Pi_{6B1}^0) \quad (4.15)
\]

\[
\Lambda_{B2} = -2 M^2 z_1 \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \Pi_{2B2}^0 \right.
\[- \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \Pi_{4B}^0 + 2 z_1 \left( \Pi_{5B1}^0 + \Pi_{5B2}^0 - 2 \Pi_{6B1}^0 \right) \]
\quad \left. \right] \quad (4.16)
\]

\[
\Lambda_{C2} = 2 z^3 M^3 \left[ \left( \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \Pi_{2B1}^0 \right.
\[
+ \left( \frac{C_2(R)}{MT(R)} + \left( \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right)^2 \right) \Pi_{2B2}^0 \right.
\[
+ \left( \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \Pi_{3B}^0 \right]
\[
+ \left( \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \Pi_{4B}^0 \right.
\[
- z_1 M \left( \frac{1}{M^2} + \frac{C_2(R)}{MT(R)} + \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} \right) \right. \left. \right] \left( \frac{C_2(G)}{4T(R)} \right)^2 + \frac{C_3^2(G)}{16T^2(R)} \right)
\[
\times (2 \Pi_{5B1}^0 + \Pi_{5B2}^0) \right.
\[
+ \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{2T(R)} \right) \right.
\[
+ \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{4T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \Pi_{6B1}^0 \right.
\[
+ 4 \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{C_2(G)}{4T(R)} \right) \Pi_{6B2}^0 \right]\] \quad (4.17)
The explicit values of the basic graphs of the $O(N)$ model are, \[13\],

\[
\Pi_{o2B1} = - \frac{2\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu) \Delta} \left[ 1 - \Delta(\mu - 1) \left(3\Theta + \frac{2}{(\mu - 1)^2}\right) \right] \quad (4.18)
\]

\[
\Pi_{o2B2} = - \frac{2\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu)} \quad (4.19)
\]

\[
\Pi_{o3B} = - \frac{2\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu) \Delta} \left[ 1 - \Delta(\mu - 1) \left(\frac{1}{2} \left(3\Theta + \frac{1}{(\mu - 1)^2}\right) \right) \right] \quad (4.20)
\]

\[
\Pi_{o4B} = \frac{\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu)} \left[ 3\Theta + \frac{1}{(\mu - 1)^2}\right] \quad (4.21)
\]

\[
\Pi_{o5B1} = \frac{(2\mu - 3)\pi^{6\mu}a(2\mu - 2)}{2(\mu - 1)^5(\mu - 2)^3 \Gamma^3(\mu)} \left[ 6\Theta - \Phi^2 + \frac{5}{2(\mu - 1)^2} - \frac{8}{(2\mu - 3)} \right.
\]

\[+ \left. \frac{1}{(2\mu - 3)(\mu - 1)} + \frac{1}{(2\mu - 3)(\mu - 1)} + \frac{2(\mu - 2)\Psi}{(\mu - 1)} + \frac{(\mu - 2)}{(\mu - 1)^2}\right] \quad (4.22)
\]

\[
\Pi_{o5B2} = - \frac{\pi^{6\mu}a(2\mu - 2)}{(\mu - 1)^6 \Gamma^3(\mu)} \left[ \frac{2(\mu - 3)}{(\mu - 2)} \left(\Phi + \Psi^2 - \frac{1}{2(\mu - 1)^2}\right) \right.
\]

\[+ \left. \frac{1}{(\mu - 1)^2} + \frac{1}{(\mu - 1)^2} + \frac{2\pi^{6\mu}a^2(2\mu - 2)}{(\mu - 1)^6(\mu - 2)^2}\right] \quad (4.23)
\]

\[
\Pi_{o6B1} = \frac{\pi^{6\mu}a(2\mu - 2)}{(\mu - 1)^8 \Gamma^3(\mu)} \left[ \frac{1}{(\mu - 1)} - \frac{5}{2(\mu - 1)^2} - \frac{2(\mu - 1)\Psi}{(\mu - 1)} \right.
\]

\[+ \left. \frac{\pi^{6\mu}a^2(2\mu - 2)}{(\mu - 1)^8}\right] \quad (4.24)
\]

where $\Phi(\mu) = \psi'(2\mu - 1) - \psi'(2 - \mu) - \psi'(\mu) + \psi'(1)$. It is worth recalling that in the $U(1) \times U(1)$ case, \[26\], only three of the above were relevant for $\lambda_2$ due to the high degree of symmetry present in that model.

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5 Discussion.

We are now in a position to substitute for all the quantities in the overall $3 \times 3$ matrix and set its determinant to zero. If one follows the same formal transformations which were made at leading order to simplify the evaluation we notice that the factorization of the matrix into two solutions, one of which is irrelevant, does not occur at $O(1/N^2)$. For instance, the (13) element is, up to factors,

$$4T(R)y - Mz = \frac{\mu^2\Gamma^2(\mu)\tilde{\eta}_1}{8(\mu - 1)N^2} \left[ \frac{2C_2(R)}{T(R)} - \frac{2}{M} - \frac{C_2(G)}{2T(R)} \right] \times \left[ \Psi(\mu) + \frac{1}{(\mu - 1)} \right]$$

(5.1)

which is zero at leading order but not at $O(1/N^2)$. Similarly the (23) element also vanishes at $O(1/N)$ but not at the subsequent order for all $M$. This is a rather unfortunate situation especially given the fact that it was possible to write down $O(1/N^2)$ expressions for $\eta$, $\chi_\sigma$ and $\chi_\pi$ and $\lambda$ at $O(1/N)$ for all $M$ for this model. The absence of a solution can, however, be related to what we will term as the non-realisation of the chiral symmetry of the model. For instance, it is easy to see that when $M = 2$ in the complicated expressions (3.16) and (3.17), $\chi_\sigma = \chi_\pi$ to $O(1/N^2)$, but this simple relation does not hold for $M > 2$, [23]. In the $U(1) \times U(1)$ model, [17], $\chi_\sigma$ is also equivalent to $\chi_\pi$ at $O(1/N^2)$ which indicated that the chiral symmetry is preserved in that instance. Now if one examines (5.1) it is zero at $M = 2$ and no other (positive) value of $M$ for all $\mu$ since the group factor is proportional to $(M^2 - 4)$. Whilst the different behaviour of the $M = 2$ and $M > 2$ models has been suggested in [11] where the non-realisation of the chiral symmetry was postulated this is as far as we are aware the first observation of it in an explicit exponent calculation.

We now restrict attention to $M = 2$ as this is the only case we can obtain $\lambda_2$ in all dimensions. (Of course, it would be possible to derive an expression for $\lambda_2$ for $M > 2$ as the solution of a quadratic equation, but the complicated and non-linear nature of such an exponent would not be in keeping with the structure of exponents found in other models. Indeed a similar situation has also arisen before in [24] when the non-factorization of the consistency determinant indicated that that exponent was not a sensible quantity to compute.) It is worth recording the explicit values of the various exponents when $M = 2$ in order to solve (4.2), (4.4) and (4.7) as, [17],

$$\eta_1 = \tilde{\eta}_1$$

(5.2)
\[
\eta_2 = \eta_1^2 \left[ \frac{(\mu - 2)\Psi}{2(\mu - 1)} + \frac{1}{2\mu} + \frac{2}{(\mu - 1)} - \frac{3\mu}{4(\mu - 1)^2} \right] \tag{5.3}
\]
\[
\chi_{\sigma 1} = \chi_{\pi 1} = -\frac{\mu \eta_1}{2(\mu - 1)} \tag{5.4}
\]
\[
\chi_{\sigma 2} = \chi_{\pi 2} = -\frac{\mu \eta_1^2}{8(\mu - 1)^2} \left[ 3\mu(\mu - 1)\Theta + 2(\mu - 2)\Psi 
+ \frac{(5\mu - 1)(2\mu^2 - 5\mu + 4)}{\mu - 1} \right] \tag{5.5}
\]
\[
\lambda_1 = -(2\mu - 1)\eta_1 \tag{5.6}
\]

Moreover,
\[
\Pi_{B2} = 8z^3[2(\Pi^0_{B2} + \Pi^0_{4B} - 2z_1(2\Pi^0_{5B} + \Pi^0_{6B} + 2\Pi^0_{B1})
- \Pi^0_{5B} + \Pi^0_{3B} - 8z_1\Pi^0_{B2}])] \tag{5.7}
\]
\[
\Pi_{C2} = 24z^3[\Pi^0_{2B1} + \Pi^0_{5B} - 2z_1(2\Pi^0_{5B} + \Pi^0_{B2} - 2\Pi^0_{B1})] \tag{5.8}
\]
\[
\Lambda_{B2} = 8z^3[\Pi^0_{2B1} + \Pi^0_{3B} - 2z_1(2\Pi^0_{5B} + \Pi^0_{B2} - 2\Pi^0_{B1})] \tag{5.9}
\]
\[
\Lambda_{C2} = 8z^3[\Pi^0_{2B1} + 2\Pi^0_{B2} + \Pi^0_{3B} + 2\Pi^0_{B} 
- 8z_1(2\Pi^0_{5B} + \Pi^0_{B2} - \Pi^0_{B2})] \tag{5.10}
\]

from which it is easy to verify that
\[
\Pi_{B2} + \Pi_{C2} - \Lambda_{B2} - \Lambda_{C2} = 0 \tag{5.11}
\]

which implies that the (23) element of the transformed determinant is zero at \(O(1/N^2)\) as at leading order and as in the \(U(1) \times U(1)\) case, \([26]\), which again reinforces our point of view on the importance of the chiral symmetry at \(O(1/N^2)\). Hence the formal correction to (3.13) is
\[
[r(\alpha - 1)s(\alpha - 1) - 4z] \left[ \frac{p(\beta)q(\beta)}{N} - 4z^2\Pi_{1B} + \Pi_{B2} + \Pi_{C2} \right] = 32z^2 \tag{5.12}
\]

With the values given earlier, some tedious algebra leads to
\[
\lambda_2 = \frac{\mu \eta_1^2}{(\mu - 1)} \left[ \frac{(6\mu^2 - 12\mu + 5)}{2(\mu - 2)^2\eta_1} \frac{(6\mu^2 - 12\mu + 5)}{2(\mu - 2)} - \frac{\mu(2\mu - 3)(\Phi + \Psi^2)}{2(\mu - 2)} \right] + \frac{3\mu(2\mu - 3)(2\mu - 1)\Theta(\mu)}{8(\mu - 2)} + \frac{5}{2\mu} - \frac{1}{2\mu^2} - \frac{5}{8(\mu - 2)^2} + \frac{1}{4(\mu - 1)^2} \frac{37\mu}{4} + \frac{23}{16(\mu - 1)} + 4\mu^2 + \frac{3}{8} - \frac{55}{16(\mu - 2)} \tag{5.13}
\]
\[
+ \Psi(\mu) \left( 3 - \frac{1}{\mu} + \frac{3}{8(\mu - 1)} + \frac{25\mu}{4} - 4\mu^2 \\
+ \frac{\mu(2\mu - 3)}{8} \left( \frac{5}{(\mu - 2)^2} - \frac{4}{(\mu - 2)} + \frac{9}{(\mu - 1)} \right) \right) \tag{5.13}
\]

Also, from (5.13) one can restrict to three dimensions to find

\[
\lambda = \frac{1}{2} - \frac{16}{3\pi^2 N} + \frac{8(27\pi^2 + 125)}{27\pi^4 N^2} \tag{5.14}
\]

Indeed, it is worth comparing (5.14) to the \(O(1/N^2)\) versions of \(\lambda\) in other four fermi models to observe a reassuring similarity in their structure. For instance, in the \(O(N)\) model, \[14, 15\],

\[
\lambda = \frac{1}{2} - \frac{16}{3\pi^2 N} + \frac{32(27\pi^2 + 632)}{27\pi^4 N^2} \tag{5.15}
\]

and in the \(U(1) \times U(1)\) case, \[26\],

\[
\lambda = \frac{1}{2} - \frac{16}{3\pi^2 N} + \frac{16(27\pi^2 + 472)}{27\pi^4 N^2} \tag{5.16}
\]

Thus, numerically the exponents differ first at \(O(1/N^2)\) in each model.

To conclude with we remark that we have discovered that the first occurrence of distinct behaviour between the \(M = 2\) and \(M > 2\) models arises at \(O(1/N^2)\) which could therefore not be noticed in a canonical large \(N\) approach. This feature is intimately related to the equivalence or otherwise of the anomalous dimensions of the \(\sigma\) and \(\pi\) fields, and therefore the realisation of a chiral symmetry. However, the deeper implications that this important observation has in relation to four dimensional field theories is beyond the scope of the present paper but does deserve consideration.

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Figure Captions.

**Fig. 1.** Leading order skeleton Dyson equations with dressed propagators.

**Fig. 2.** Additional graphs for $\lambda_1$.

**Fig. 3.** Higher order corrections for $\psi$ equation.

**Fig. 4.** $O(1/N^2)$ contributions to $\sigma$ equation.