A GENERALIZED SEMI–INFINITE HECKE EQUIVALENCE AND THE LOCAL GEOMETRIC LANGLANDS CORRESPONDENCE

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Abstract. We introduce a class of equivalences, which we call generalized semi–infinite Hecke equivalences, between certain categories of representations of graded associative algebras which appear in the setting of semi–infinite cohomology for associative algebras and categories of representations of related algebras of Hecke type which we call semi–infinite Hecke algebras. As an application we obtain an equivalence between a category of representations of a non–twisted affine Lie algebra $\widehat{g}$ of level $-k - 2h^\vee$, where $h^\vee$ is the dual Coxeter number of the underlying semisimple Lie algebra $g$ and $k \in \mathbb{C}$, and the category of finitely generated representations of the W–algebra associated to $\widehat{g}$ of level $k$. When $k = -h^\vee$ this yields an equivalence between a category of representations of $\widehat{g}$ of central charge $-h^\vee$ and the category of coherent sheaves on the space $\text{Op}_{L^\mathbb{C}}(D \times \mathbb{C})$ of $L^\mathbb{C}$–opers on the punctured disc $D$, where $L^\mathbb{C}$ is the Langlands dual group to the algebraic group of adjoint type with Lie algebra $g$. This can be regarded as a version of the local geometric Langlands correspondence. The above mentioned equivalences generalize to the case of affine Lie algebras the Skryabin equivalence between the categories of generalized Gelfand-Graev representations of $g$ and the categories of representations of the corresponding finitely generated W–algebras, and Kostant’s results on the classification of Whittaker modules over $g$.

The purpose of this short note is to introduce a categorical equivalence, similar to the Skryabin equivalence between categories of generalized Gelfand-Graev representations of complex semisimple Lie algebras and categories of representations of finitely generated W–algebras (see the Appendix to [18]), in the setting of semi–infinite Hecke algebras defined in [21]. In particular, our result yields an equivalence between some categories of representations of affine Lie algebras and categories of representations of W–algebras associated to them.

Skryabin’s equivalence is in fact an example of a quite general and simple categorical equivalence which can be established in the following situation (see e.g. the introduction in [22] for a review of equivalences of this kind which occur in various settings). Let $A$ be an associative algebra over a field $k$, $A_0 \subset A$ a subalgebra with a character $\chi : A_0 \to k$. Denote by $k_\chi$ the corresponding rank one representation of $A_0$. Let $Q_\chi = A \otimes_{A_0} k_\chi$ be the induced representation of $A$.

Let $H_k(A,A_0,\chi) = \text{End}_A(Q_\chi)^{opp}$ be the algebra of $A$–endomorphisms of $Q_\chi$ with the opposite multiplication. The algebra $H_k(A,A_0,\chi)$ is called the Hecke algebra of the triple $(A,A_0,\chi)$. In [20] a homological generalization of Hecke algebras of this type was introduced. It is this homological generalization which is called in [20] the Hecke algebra of the triple $(A,A_0,\chi)$. In this paper we use a slightly different terminology.

The appearance of the term “Hecke” is justified by the fact that if $A$ is the group algebra of a Chevalley group over a finite field, $A_0$ is the group algebra of a Borel subgroup in it, and $\chi$ is the trivial complex representation of the Borel subgroup then $H_k(A,A_0,\chi)$ is the Iwahori–Hecke algebra (see [14]).

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The use of Hecke algebras is due to the observation that for any representation \( V \) of \( A \) the algebra \( H_k(A, A, \chi) \) naturally acts in the space of Whittaker vectors

\[
W^\chi(V) = \text{Hom}_A(Q, V) = \text{Hom}_{A_0}(k, V)
\]

by compositions of homomorphisms, and for any \( H_k(A, A, \chi) \)-module \( W \), \( Q \otimes_{H_k(A, A, \chi)} W \) is a left \( A \)-module. Let \( H_k(A, A, \chi) \) be the category of left \( H_k(A, A, \chi) \)-modules and \( - \mod \) the category of left \( A \)-modules. Then \( H_k(A, A, \chi) \)-modules and \( - \mod \) have dualities with morphisms.

**Theorem 1.** \( A - \mod \chi_{A_0} \) is a full subcategory in the category of left \( A \)-modules, and the functors \( \text{Hom}_A(Q, \cdot) \) and \( Q \otimes_{H_k(A, A, \chi)} \cdot \) yield mutually inverse equivalences of the categories,

\[
(1) \quad A - \mod \chi_{A_0} \simeq H_k(A, A, \chi) - \mod.
\]

**Proof.** Let \( W, W' \in H_k(A, A, \chi) - \mod \). Then by the Frobenius reciprocity and by the definition of the algebra \( H_k(A, A, \chi) \)

\[
\text{Hom}_A(Q, Q \otimes_{H_k(A, A, \chi)} W) = \text{Hom}_{A_0}(k, Q \otimes_{H_k(A, A, \chi)} W) = \text{Hom}_{A_0}(k, Q) \otimes_{H_k(A, A, \chi)} W = \text{Hom}_A(Q, Q \otimes_{H_k(A, A, \chi)} W) = H_k(A, A, \chi)^{\text{opp}} \otimes_{H_k(A, A, \chi)} W = W.
\]

This implies the second claim of this theorem.

By the formula above we also have

\[
\text{Hom}_A(Q \otimes_{H_k(A, A, \chi)} W', Q \otimes_{H_k(A, A, \chi)} W) = \text{Hom}_{H_k(A, A, \chi)}(W', \text{Hom}_A(Q, Q \otimes_{H_k(A, A, \chi)} W)) = \text{Hom}_{H_k(A, A, \chi)}(W', W),
\]

and hence \( A - \mod \chi_{A_0} \) is a full subcategory in the category of left \( A \)-modules.

\[\square\]

We call the equivalence established in Theorem 1 a generalized Hecke equivalence.

Skryabin considered a generalized Hecke equivalence in the following situation. Let \( g \) be a complex semisimple Lie algebra, \( e \in g \) a non–zero nilpotent element in \( g \). By the Jacobson–Morozov theorem there is an \( \mathfrak{sl}_2 \)-triple \((e, h, f)\) associated to \( e \), i.e. elements \( f, h \in g \) such that \([h, e] = 2e, [h, f] = -2f, [e, f] = h\). Fix such an \( \mathfrak{sl}_2 \)-triple.

Let \( \chi \) be the element of \( g^* \) which corresponds to \( e \) under the isomorphism \( g \simeq g^* \) induced by the Killing form. Under the action of \( \text{ad} \ h \) we have a decomposition

\[
(2) \quad g = \oplus_{i \in \mathbb{Z}} g(i), \quad \text{where } g(i) = \{ x \in g \mid [h, x] = ix \}.
\]

The skew–symmetric bilinear form \( \omega \) on \( g(-1) \) defined by \( \omega(x, y) = \chi([x, y]) \) is non–degenerate. Fix an isotropic Lagrangian subspace \( l \) of \( g(-1) \) with respect to \( \omega \).

Let

\[
(3) \quad m = l \oplus \bigoplus_{i \leq -2} g(i).
\]

Note that \( m \) is a nilpotent Lie subalgebra of \( g \) and \( \chi \in g^* \) restricts to a character \( \chi : m \to \mathbb{C} \). Denote by \( C_\chi \) the corresponding one–dimensional \( U(m) \)-module.

The associative algebra \( W^\chi(g) = \text{End}_{U(g)}(U(g) \otimes_{U(m)} C_\chi)^{\text{opp}} \) is called the \( W \)-algebra associated to the nilpotent element \( e \). The algebra \( W^\chi(g) \) was introduced in [15] in case when \( e \) is principal nilpotent and in [16] when the grading (2) is even. In paper [6] the algebras
$W^e(\mathfrak{sl}_n)$ are defined using cohomological BRST reduction, and the simple equivalent algebraic definition for arbitrary nilpotent element $e$ given above first appeared in [18]. The equivalence of these two definitions follows, for instance, from a general property of homological Hecke–type algebras (see [20, 21]). An explicit computation establishing this equivalence can also be found in the Appendix in [8].

Theorem 1 immediately yields a categorical equivalence

$$U(\mathfrak{g}) - \text{mod}^\chi_{U(\mathfrak{m})} \simeq \text{Hk}(U(\mathfrak{g}), U(\mathfrak{m}), \chi) - \text{mod} = W^\chi(\mathfrak{g}) - \text{mod}. $$

But in fact Skryabin proved a much stronger statement: the category $U(\mathfrak{g}) - \text{mod}^\chi_{U(\mathfrak{m})}$ can be described as the category of $\mathfrak{g}$–modules on which $x - \chi(x)$ acts locally nilpotently for any $x \in \mathfrak{m}$, and any module from this category is $\mathfrak{m}$–injective.

In the particular case when $\mathfrak{m} = \mathfrak{n} \subset \mathfrak{g}$ is a maximal nilpotent subalgebra, and $\chi : \mathfrak{n} \to \mathbb{C}$ is a non–singular character of $\mathfrak{n}$ which does not vanish on all simple root vectors in $\mathfrak{n}$, this result was already obtained by Kostant in [15] for the purpose of the study of Whittaker and principal series representations of $\mathfrak{g}$. This situation is particularly close to the definition of $W$–algebras associated to affine Lie algebras we are interested in.

Kostant showed that the algebra $\text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)^{\text{opp}}$ is canonically isomorphic to the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$,

$$\text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)^{\text{opp}} \simeq Z(U(\mathfrak{g})).$$

Using this result he proved that modules from the category $U(\mathfrak{g}) - \text{mod}^\chi_{U(\mathfrak{m})}$ are in one–to–one correspondence with representations of $Z(U(\mathfrak{g}))$ which is a polynomial algebra in rank $\mathfrak{g}$ generators. One of our objectives it to obtain a counterpart of this statement for affine Lie algebras. It will imply a version of the local geometric Langlands correspondence.

Firstly we are going to generalize the categorical equivalence established in Theorem 1 to the setting in which the semi–infinite cohomology of associative algebras and the corresponding versions of Hecke algebras are defined (see [1, 2, 3, 17, 21]). We start by recalling the initial setup.

Let $A$ be a $\mathbb{Z}$–graded associative algebra over a field $k$,

$$A = \bigoplus_{n \in \mathbb{Z}} A_n.$$ 

For $N \in \mathbb{N}$ let $I_N$ be the left ideal in $A$ generated by $A_n$, $n \geq N$. Define a topology on $A$ in which a basis of open neighborhoods of $0$ is formed by the left ideals $I_N$, $N \in \mathbb{N}$. The multiplication in $A$ is continuous in this topology, and hence one can define the restricted completion $\hat{A}$ of $A$ as the inverse limit

$$\hat{A} = \lim_{\leftarrow} A/I_N$$

with the multiplication induced from $A$.

The category of left (right) $A$–modules with morphisms being homomorphisms of $A$–modules is denoted by $A - \text{mod} \ (\text{mod} - A)$. For both of these categories the set of morphisms between two objects is denoted by $\text{Hom}_A(\cdot, \cdot)$. For $\mathbb{Z}$–graded $A$–modules $M, M' \in \text{Ob} A - \text{mod} \ (\text{Ob mod} - A)$, $M = \bigoplus_{n \in \mathbb{Z}} M_n, M' = \bigoplus_{n \in \mathbb{Z}} M'_n$ we shall also use the space of homomorphisms of all possible degrees with respect to the gradings on $M$ and $M'$ introduced by

$$\text{hom}_A(M, M') = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^0_A(M, M'(n)),$$

where the module $M'(n)$ is obtained from $M'$ by grading shift as follows:

$$M'(n)_{k} = M'_{k+n},$$

and $\text{Hom}^0_A(M, M')$ stands for set of $A$–homomorphisms from $M$ to $M'$ preserving the gradings.
In this paper we shall deal with the full subcategory of $A - \text{mod} (\text{mod} - A)$ whose objects are smooth modules $M \in \text{Ob} A - \text{mod} (\text{Ob} \text{mod} - A)$, i.e. for any $v \in M$ there exists $N \in \mathbb{N}$ such that $av = 0$ for any $n \geq N$ and any $a \in A_n$. This subcategory is denoted by $(A - \text{mod})_0 ((\text{mod} - A)_0)$.

The action of $A$ on any object of the category $(A - \text{mod})_0$ induces an action of the completion $\hat{A}$.

We also denote by $\text{Vect}_k$ the category of vector spaces over $k$.

Following [1] we shall impose additional restrictions on the algebra $A$. Namely, in the rest of this paper we suppose that $A$ satisfies the following conditions:

(i) $A$ contains two graded subalgebras $N$ and $B$.
(ii) $N$ is positively graded.
(iii) $N_0 = k$.
(iv) $\dim N_n < \infty$ for any $n \in \mathbb{N}$.

In particular, $N$ is naturally augmented.

(v) $B$ is negatively graded.

(vi) The multiplication in $A$ defines isomorphisms of graded vector spaces

\[ B \otimes N \rightarrow A \text{ and } N \otimes B \rightarrow A. \]

We call the decompositions (5) the triangular decompositions for the algebra $A$. Note that the compositions of the triangular decomposition maps and of their inverse maps yield linear mappings

\[ N \otimes B \rightarrow B \otimes N, \]

\[ B \otimes N \rightarrow N \otimes B. \]

(vii) Mappings (6) are continuous in the following sense: for every $m, n \in \mathbb{Z}$ there exist $k_+, k_- \in \mathbb{Z}$ such that

\[ N_m \otimes B_n \rightarrow \bigoplus_{k_- \leq k \leq k_+} B_{n-k} \otimes N_{m+k} \]

and

\[ B_n \otimes N_m \rightarrow \bigoplus_{k_- \leq k \leq k_+} N_{m-k} \otimes B_{n+k}. \]

Next, we recall the definition the semiregular bimodule for the algebra $A$. The notion of the semiregular bimodule was introduced by Voronov (see [23]) in the Lie algebra case and generalized in [1] to the case of graded associative algebras satisfying conditions (i)–(vii). In the semi–infinite version of the Hecke algebra theory this bimodule plays the role of the regular representation. In particular, the semiregular bimodule naturally appears in the definition of the semi–infinite modification of Hecke algebras.

First consider the right graded $N$-module $N^* = \text{hom}_k(N, k)$, where the action of $N$ on $N^*$ is defined by

\[ (n \cdot f)(n') = f(nn') \text{ for any } f \in N^*, \ n \in N. \]

The right $A$–module

\[ S_A = N^* \otimes_A A \]

is called the right semiregular representation of $A$ (see [23], Sect. 3.2; [1], Sect. 3.4).

Clearly, $S_A = N^* \otimes B$ as a right $B$-module. The space $S_A = N^* \otimes B$ is non–positively graded, and hence $S_A \in (\text{mod} - A)_0$. 

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\[ S_A = N^* \otimes_B A \]
Now we obtain another realization for the right semiregular representation. Consider another right $A$-module $S_A' = \text{hom}_B(A,\mathcal{B})$, where $\mathcal{B}$ acts on $A$ and $\mathcal{B}$ by right multiplication. The right action of $A$ on the space $S_A'$ is given by

$$(a \cdot f)(a') = f(aa'), \ f \in \text{hom}_B(A,\mathcal{B}), \ a \in A.$$ 

**Lemma 1.** ([1], Lemma 3.5.1) Fix a decomposition

$$(7) \quad A = N \otimes B$$ 

provided by the multiplication in $A$. Let $\phi : S_A \rightarrow S_A'$ be the map defined by

$$\phi(f \otimes a)(a') = f((aa')_N)(aa'_B),$$ 

where $f \otimes a \in S_A$, $a' \in A$ and $aa' = (aa')_N(aa')_B$ is decomposition (7) of the element $aa'$. Then $\phi$ is a morphism of right $A$–modules.

We shall suppose that the algebra $A$ satisfies the following additional condition:

**(viii)** The morphism $\phi : S_A \rightarrow S_A'$ constructed in the previous lemma is an isomorphism of right $A$–modules.

Finally we have two realizations of the right $A$–module $S_A$:

$$(8) \quad S_A = N^* \otimes_N A,$$ 

and

$$(9) \quad S_A = \text{hom}_B(A,\mathcal{B}).$$

Now we define the structure of a left module on $S_A$ commuting with the right semiregular action of $A$. First observe that using realizations (8) and (9) of the right semiregular representation one can define natural left actions of the algebras $N$ and $\mathcal{B}$ on the space $S_A$ induced by the natural left action of $N$ on $N^*$ and the left regular representation of $\mathcal{B}$, respectively. Clearly, these actions commute with the right action of the algebra $A$ on $S_A$. Therefore we have natural inclusions of algebras

$$N \hookrightarrow \text{hom}_A(S_A, S_A), \ \mathcal{B} \hookrightarrow \text{hom}_A(S_A, S_A).$$

Denote by $A^\sharp$ the subalgebra in $\text{hom}_A(S_A, S_A)$ generated by $N$ and $\mathcal{B}$.

**Proposition 1.** ([1], Corollary 3.3.3, Lemma 3.5.3 and Corollary 3.5.3) $A^\sharp$ is a $\mathbb{Z}$–graded associative algebra satisfying conditions (i)–(vii). Moreover, $S_A \in (A^\sharp - \text{mod})_0$ and

$$(10) \quad S_A = A^\sharp \otimes_N N^* = \text{hom}_B(A^\sharp, \mathcal{B})$$

as a left $A^\sharp$–module.

Using Proposition 1 the space $S_A$ is equipped with the structure of an $A^\sharp - A$ bimodule. This bimodule is called the semiregular bimodule associated to the algebra $A$. The left action of the algebra $A^\sharp$ on the space $S_A$ is called the left semiregular action.

Let $M \in \text{mod} - A$ be a right $A$–module and $M' \in A^\sharp - \text{mod}$ a left $A^\sharp$–module. Consider the subspace $M \otimes^N M'$ in the tensor product $M \otimes M'$ defined by

$$M \otimes^N M' = \{m \otimes m' \in M \otimes M' : \ mn \otimes m' = m \otimes nm' \text{ for every } n \in N\}.$$ 

Following [21], we define the semiproduct $M \otimes^N_M M'$ of modules $M \in \text{mod} - A$ and $M' \in A^\sharp - \text{mod}$ as the image of the subspace $M \otimes^N M' \subset M \otimes M'$ under the canonical projection $M \otimes M' \rightarrow M \otimes_B M'$,

$$(12) \quad M \otimes^N_B M' = \text{Im}(M \otimes^N M' \rightarrow M \otimes_B M').$$
The semiproduct \( \otimes^A_N \) is a mixture of the tensor product \( \otimes_B \) over \( B \) and of the functor \( \otimes^N \) of “\( N \)–invariants”. The semiproduct of modules naturally extends to a functor \( \otimes^N_B : (\text{mod} - A) \times (A^\sharp - \text{mod}) \to \text{Vect}_k \). The semiproduct functor is a generalization of the functor of semivariants (see [23], Sect. 3.8).

Let, as above, \( A \) be an associative \( \mathbb{Z} \)-graded algebra over a field \( k \). Suppose that the algebra \( A \) contains a graded subalgebra \( A_0 \), and both \( A \) and \( A_0 \) satisfy conditions (i)–(viii). We denote by \( N \), \( B \), \( B_0 \) the graded subalgebras in \( A \) and \( A_0 \), respectively, providing the triangular decompositions of these algebras (see condition (vi)).

Let \( S = \text{Ind}_{A_0}^A \) be the functor of semi-infinite induction

\[
S - \text{Ind}_{A_0}^A : A_0^\sharp - \text{mod} \to A^\sharp - \text{mod}
\]

defined on objects by

\[
S - \text{Ind}_{A_0}^A(V) = S_A \otimes_{B_0}^N V, \quad V \in A_0^\sharp - \text{mod},
\]

the structure of a left \( A^\sharp \)-module on \( S_A \otimes_{B_0}^N V \) being induced by the left semiregular action of \( A^\sharp \) on \( S_A \). In the Lie algebra case this functor was introduced in [24] and the definition given above first appeared in [21].

Note that the functor \( S - \text{Ind}_{A_0}^A \) sends objects of the category \( A_0^\sharp - \text{mod} \) to objects of the category \( (A^\sharp - \text{mod})_0 \) and \( \mathbb{Z} \)-graded \( A_0^\sharp \)-modules to \( \mathbb{Z} \)-graded \( A^\sharp \)-modules.

Now assume that the algebra \( A_0^\sharp \) is augmented, \( \varepsilon : A_0^\sharp \to k \). We denote this one–dimensional \( A_0^\sharp \)-module by \( k_e \).

**Definition 1.** The algebra

\[
Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) = \text{Hom}_{A_0^\sharp}(S - \text{Ind}_{A_0}^A(k_e), S - \text{Ind}_{A_0}^A(k_e))^{\text{opp}}
\]

is called the semi–infinite Hecke algebra of the triple \( (A, A_0, \varepsilon) \), and the \( \mathbb{Z} \)-graded algebra

\[
Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) = \text{hom}_{A_0^\sharp}(S - \text{Ind}_{A_0}^A(k_e), S - \text{Ind}_{A_0}^A(k_e))^{\text{opp}}
\]

is called the graded semi–infinite Hecke algebra of the triple \( (A, A_0, \varepsilon) \).

Obviously, \( Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) \subset Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) \), and \( Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) \) is a completion of \( Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) \).

In [21] a homological version of the algebra \( Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) \) is called a semi–infinite Hecke algebra. As shown in Proposition 3.1.1 in [21] in “good cases” the homological zero degree component of the homological version of the algebra \( Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) \) coincides with the algebra defined by formula (14).

Denote \( Q\tilde{\mathcal{F}}_e = S - \text{Ind}_{A_0}^A(k_e) \), so that

\[
Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) = \text{Hom}_{A_0^\sharp}(Q\tilde{\mathcal{F}}_e, Q\tilde{\mathcal{F}}_e)^{\text{opp}}.
\]

In complete analogy with the situation described in the beginning of the paper, for any representation \( V \in A^\sharp - \text{mod} \) the algebra \( Hk\tilde{\mathcal{F}}^\bullet(A, A_0, \varepsilon) \) naturally acts in the space

\[
\text{Wh}_{\tilde{\mathcal{F}}^\bullet}(V) = \text{Hom}_{A^\sharp}(Q\tilde{\mathcal{F}}_e, V)
\]

by compositions of homomorphisms. We call this space the space of semi–infinite Whittaker vectors. One can define an obvious graded version of this space for \( \mathbb{Z} \)-graded \( A \)-modules.

Let \( (Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) - \text{mod})^{fg} \) be the full subcategory in \( Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) - \text{mod} \) the objects of which are finitely generated modules from \( Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) - \text{mod} \).

For any \( Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) \)-module

\[
W \in Hk\tilde{\mathcal{F}}(A, A_0, \varepsilon) - \text{mod}
\]
$Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W$ is an object of the category $(A^I - \text{mod})_0$. Let $A^I - \text{mod}^{\infty}_{A^I_0}$ be the category of left $A^I$-modules of the form $Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W$, where $W \in (\text{Hk}^{\infty}(A, A_0, z) - \text{mod})^{fg}$, with morphisms induced by morphisms in the category $(\text{Hk}^{\infty}(A, A_0, z) - \text{mod})^{fg}$. Then we have the following analogue of Theorem [1]

**Theorem 2.** $A^I - \text{mod}^{\infty}_{A^I_0}$ is a full subcategory in the category $(A^I - \text{mod})_0$, and the functors $\text{Hom}_{A^I}(Q_z^{\infty}, \cdot)$ and $Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} \cdot$ yield mutually inverse equivalences of the categories,

$$A^I - \text{mod}^{\infty}_{A^I_0} \simeq (\text{Hk}^{\infty}(A, A_0, z) - \text{mod})^{fg}.$$  

**Proof.** Let $W, W' \in (\text{Hk}^{\infty}(A, A_0, z) - \text{mod})^{fg}$. First note that, since $Q_z^{\infty}$ is $\mathbb{Z}$-graded with zero positive degree components, we obviously have $Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W \in (A^I - \text{mod})_0$.

Next, since $W$ is finitely generated over $\text{Hk}^{\infty}(A, A_0, z)$ we have

$$\text{Hom}_{A^I}(Q_z^{\infty}, Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W) = \text{Hom}_{A^I}(Q_z^{\infty}, Q_z^{\infty}) \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W = \text{Hk}^{\infty}(A, A_0, z) \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W = W.$$

This implies the second claim of this theorem.

By the formula above we also have

$$\text{Hom}_{A^I}(Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W', Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W) = \text{Hom}_{\text{Hk}^{\infty}(A, A_0, z)}(W', \text{Hom}_{A^I}(Q_z^{\infty}, Q_z^{\infty} \otimes_{\text{Hk}^{\infty}(A, A_0, z)} W)) = \text{Hom}_{\text{Hk}^{\infty}(A, A_0, z)}(W', W),$$

and hence $A^I - \text{mod}^{\infty}_{A^I_0}$ is a full subcategory in the category $(A^I - \text{mod})_0$.

We call the categorical equivalence established in Theorem [2] a generalized semi–infinite Hecke equivalence.

Now we apply the above obtained results in the case of affine Lie algebras and their enveloping algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\hat{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}] + \mathbb{C}K$ the non–twisted affine Lie algebra corresponding to $\mathfrak{g}$. Recall that $\hat{\mathfrak{g}}$ is the central extension of the loop algebra $\mathfrak{g}[z, z^{-1}]$ with the help of the standard two–cocycle $\omega_{st}$,

$$\omega_{st}(x(z), y(z)) = \text{Res}(x(z), y(z)) \frac{dz}{z},$$

where $\langle \cdot, \cdot \rangle$ is the standard invariant normalized bilinear form of the Lie algebra $\mathfrak{g}$.

Let $\mathfrak{n} \subset \mathfrak{g}$ be a maximal nilpotent subalgebra in $\mathfrak{g}$ and $\hat{\mathfrak{n}} = \mathfrak{n}[z, z^{-1}]$ the loop algebra of the nilpotent Lie subalgebra $\mathfrak{n}$. Note that $\hat{\mathfrak{n}} \subset \hat{\mathfrak{g}}$ is a Lie subalgebra in $\hat{\mathfrak{g}}$ because the standard cocycle $\omega_{st}$ vanishes when restricted to the subalgebra $\hat{\mathfrak{n}} = \mathfrak{n}[z, z^{-1}] \subset \mathfrak{g}[z, z^{-1}]$. We denote by $U(\hat{\mathfrak{g}})$ and $U(\hat{\mathfrak{n}})$ the universal enveloping algebras of $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{n}}$, respectively.

Let $\chi$ be a character of $\mathfrak{n}$ which takes non–zero values on all simple root vectors of $\mathfrak{n}$. $\chi$ has a unique extension to a character $\hat{\chi}$ of $\hat{\mathfrak{n}} = \mathfrak{n}[z, z^{-1}]$, such that $\hat{\chi}$ vanishes on the complement $z^{-1}\mathfrak{n}[z^{-1}] + z\mathfrak{n}[z]$ of $\mathfrak{n}$ in $\mathfrak{n}[z, z^{-1}]$. We denote by $\mathbb{C}_{\chi}$ the left one–dimensional $U(\hat{\mathfrak{n}})$–module that corresponds to $\hat{\chi}$.

Let $U(\hat{\mathfrak{g}})_k$ be the quotient of the algebra $U(\hat{\mathfrak{g}})$ by the two–sided ideal generated by $K - k$, $k \in \mathbb{C}$. Note that for any $k \in \mathbb{C}$ $U(\hat{\mathfrak{n}})$ is a subalgebra in $U(\hat{\mathfrak{g}})_k$ because the standard cocycle $\omega_{st}$ vanishes when restricted to the subalgebra $\hat{\mathfrak{n}} \subset \mathfrak{g}[z, z^{-1}]$.
Next observe that the algebras $U(\hat{g})_k$ and $U(\hat{n})$ inherit $\mathbb{Z}$-gradings from the natural $\mathbb{Z}$-gradings of $\mathfrak{g}$ and $\mathfrak{n}$ by degrees of the parameter $z$, and satisfy conditions (i)–(viii), with the natural triangular decompositions $U(\hat{g})_k = U(\hat{g})_+ \otimes U(\hat{g})_-$ and $U(\hat{n}) = U(\hat{n})_+ \otimes U(\hat{n})_-$ provided by the decompositions $\hat{g} = \hat{g}_- + \hat{g}_+, \hat{n} = \hat{n}_- + \hat{n}_+$, where $\hat{g}_- = \mathfrak{g}[z^{-1}] + \mathcal{C}K$, $\hat{g}_+ = z\mathfrak{g}[z]$, $\hat{n}_\pm = \hat{n} \cap \hat{g}_\pm$. Hence one can define the algebras $U(\hat{g})_k^2$, $U(\hat{n})^2$.

The algebra $U(\hat{g})^2$ is explicitly described in the following proposition.

**Proposition 2.** (11, Proposition 4.6.7) The algebra $U(\hat{g})^2$ is isomorphic to $U(\hat{g})_{-2h^\vee -k}$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The algebra $U(\hat{n})^2$ is isomorphic to $U(\hat{n})$.

We shall identify the algebra $U(\hat{n})^2$ with $U(\hat{n})$ and $U(\hat{g})^2_k$ with $U(\hat{g})_{-2h^\vee -k}$.

Next observe that the algebra $U(\hat{g})^2_k$ and the graded subalgebra $U(\hat{n}) \subset U(\hat{g})^2_k$ satisfy the compatibility conditions (i)–(viii) under which the semi–infinite Hecke algebra of the triple ($U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi$) may be defined.

**Definition 2.** The algebra $W_k(\hat{g})$ defined by

\[
W_k(\hat{g}) = \text{Hom}_{U(\hat{g})_{-2h^\vee -k}}(S_{U(\hat{g})_k} \otimes_{U(\hat{n})^2} \mathcal{C}_\chi, S_{U(\hat{g})_k} \otimes_{U(\hat{n})^2} \mathcal{C}_\chi)^{opp} = \\
= \text{Hk}_k(U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi).
\]

is called the W–algebra associated to the affine Lie algebra $\hat{g}$ of level $k$.

As it is shown in [21], Proposition 3.2.2 the definition of the algebra $\text{Hk}_k(\hat{g})$ agrees with the definition of the W–algebra as the cohomology of the BRST complex which appears in [9]. This result implies that the completion $W_k(\hat{g})$ is $\text{Hk}_k(U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi)$ of $\text{Hk}_k(U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi)$ is the completed enveloping algebra, introduced in [5], Section 4.3.1, of the W–algebra defined in [9] as a vertex operator algebra.

From Theorem 2 we immediately obtain the following result which can be regarded as a counterpart of the Skryabin equivalence for affine Lie algebras.

**Theorem 3.** $U(\hat{g})_{-2h^\vee} - \text{mod}^{\hat{X}}_{U(\hat{n})}$ is a full subcategory in the category $U(\hat{g})_{-2h^\vee} - \text{mod}_0$, and the functors $\text{Hom}_{U(\hat{g})_{-2h^\vee}}(Q_{\chi}^{\hat{X}}, \cdot)$ and $Q_{\chi}^{\hat{X}} \otimes_{\text{Hk}_k(U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi)}^\cdot$ yield mutually inverse equivalences of the categories,

\[
U(\hat{g})_{-2h^\vee} - \text{mod}^{\hat{X}}_{U(\hat{n})} \simeq (\text{Hk}_k(U(\hat{g})_k, U(\hat{n}), \mathcal{C}_\chi) - \text{mod})^g. 
\]

Now recall that at the critical value of the parameter $k$, $k = -h^\vee$, the restricted completion $\hat{U}(\hat{g})_{-h^\vee}$ of the algebra $U(\hat{g})_{-h^\vee}$ has a large center. This center is canonically isomorphic to the W-algebra $W_{-h^\vee}(\hat{g})$ (see [9], Proposition 6, [10], Proposition 4.3.4 and [4], Theorem 3.7.7),

\[
Z(\hat{U}(\hat{g})_{-h^\vee}) \simeq W_{-h^\vee}(\hat{g}).
\]

Thus we obtain a canonical algebraic isomorphism

\[
Z(\hat{U}(\hat{g})_{-h^\vee}) \simeq \text{Hom}_{U(\hat{g})_{-h^\vee}}(S_{U(\hat{g})_{-h^\vee}} \otimes_{U(\hat{n})^2} \mathcal{C}_\chi, S_{U(\hat{g})_{-h^\vee}} \otimes_{U(\hat{n})^2} \mathcal{C}_\chi)^{opp} = \\
= \text{Hk}_k(U(\hat{g})_{-h^\vee}, U(\hat{n}), \mathcal{C}_\chi).
\]

Here using Proposition 2 we replaced the algebra $U(\hat{g})^2_{-h^\vee}$ with $U(\hat{g})_{-h^\vee}$ (We note that at the critical level of the parameter $k$, $k = -h^\vee$ the algebra $U(\hat{g})_{-h^\vee}$ is “self–dual” in the sense that the algebra $U(\hat{g})^2_{-h^\vee}$ is isomorphic to $U(\hat{g})_{-h^\vee}$).

Description (11) of the center $Z(\hat{U}(\hat{g})_{-h^\vee})$ is similar to realization (4) of the center $Z(U(\mathfrak{g}))$ of the algebra $U(\mathfrak{g})$ obtained by Kostant in [15].
As a corollary of Theorem 3 and of the last observation we have the following statement which can be regarded as an affine Lie algebra analogue of Kostant’s classification of Whittaker modules.

**Theorem 4.** $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ is a full subcategory in the category $(U(\hat{g})_{-h^\vee} - \text{mod})_0$, and the functors $\text{Hom}_{U(\hat{g})_{-h^\vee}}(Q_{-\infty}^+, \cdot)$ and $Q_{-\infty}^+ \otimes Z(U(\hat{g})_{-h^\vee})$ yield mutually inverse equivalences of the categories,

$$U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}} \simeq (\text{Hk}_{\hat{\mathfrak{h}}}(U(\hat{g})_{-h^\vee}, U(\hat{n}), \mathbb{C}_\chi) - \text{mod})^{fg} \simeq (Z(U(\hat{g})_{-h^\vee}) - \text{mod})^{fg}.$$

Note that the algebra $Z(\hat{U}(\hat{g})_{-h^\vee})$ is canonically isomorphic to the topological algebra of functions $\text{Fun}(\text{Op}_{L,G}(D^\times))$ on the space $\text{Op}_{L,G}(D^\times)$ of $L^G$–opers on the punctured disc $D^\times = \text{Spec} \mathbb{C}[[t]]$, where $L^G$ is the Langlands dual group to the semisimple algebraic group $G$ of adjoint type with Lie algebra $\mathfrak{g}$ (see e.g. [10], Theorem 4.3.6). Thus we obtain the following corollary of the previous theorem.

**Corollary 1.** The category $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ is equivalent to the category $\text{Coh}(\text{Op}_{L,G}(D^\times))$ of coherent sheaves on $\text{Op}_{L,G}(D^\times)$,

$$U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}} \simeq \text{Coh}(\text{Op}_{L,G}(D^\times)),$$

and irreducible objects in the category $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ are parametrized by $L^G$–opers on the punctured disc.

Equivalence (20) can be viewed as a version of the local geometric Langlands correspondence.

In conclusion we briefly compare this correspondence and correspondences of a similar kind established in [11][12]. Let $(U(\hat{g})_{-h^\vee} - \text{mod})_{\text{reg}}$ be the full subcategory of the category of $(U(\hat{g})_{-h^\vee} - \text{mod})_0$ with objects being modules on which the action of the center $Z(\hat{U}(\hat{g})_{-h^\vee})$ factors through the homomorphism

$$Z(\hat{U}(\hat{g})_{-h^\vee}) \simeq \text{Fun}(\text{Op}_{L,G}(D^\times)) \to \text{Fun}(\text{Op}_{L,G}(D)),$$

where $D$ is the disk $D = \text{Spec} \mathbb{C}[[t]]$. Let $(U(\hat{g})_{-h^\vee} - \text{mod})_{\text{reg}}^{G[[z]]}$ be the full subcategory in $(U(\hat{g})_{-h^\vee} - \text{mod})_0$ with objects being modules on which the action of the Lie subalgebra $\mathfrak{g}[[z]] \subset U(\hat{g})_{-h^\vee}$ is integrated to the action of the group $G[[z]]$. According to [11], Theorem 6.3 (see also [4], Section 8) there is an equivalence of categories,

$$U(\hat{g})_{-h^\vee} - \text{mod}_{\text{reg}}^{G[[z]]} \simeq \text{Qcoh}(\text{Op}_{L,G}(D)),$$

where $\text{Qcoh}(\text{Op}_{L,G}(D))$ is the category of quasicoherent sheaves on $\text{Op}_{L,G}(D)$.

In [12], Section 3, Main Theorem, a categorical equivalence of similar kind was established for the category $\text{Qcoh}(\text{Op}_{L,G}^{\text{unr}}(D^\times))$ of quasicoherent sheaves on the space $\text{Op}_{L,G}^{\text{unr}}(D^\times)$ of opers on $D^\times$ that are unramified as local systems.

Note that the objects of the category $\text{Coh}(\text{Op}_{L,G}(D^\times))$ associated to the punctured disk $D^\times$ exhaust all finitely generated representations of the center $Z(\hat{U}(\hat{g})_{-h^\vee}) \simeq \text{Fun}(\text{Op}_{L,G}(D^\times))$, while the objects of the category $\text{Qcoh}(\text{Op}_{L,G}(D))$ associated to the disk $D$ form a special class of representations of it.

Theorem 3 and geometric Langlands correspondence (20) look as very natural affine Lie algebra counterparts of Kostant’s result in [15] on the classification of Whittaker modules. Although, the category $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ which appears in (20) is quite different from the category $(U(\hat{g})_{-h^\vee} - \text{mod})_{\text{reg}}^{G[[z]]}$ in (21). Namely, the action of the positively graded Lie subalgebra $\mathfrak{g}[[z]] \subset U(\hat{g})_{-h^\vee}$ on objects of the category $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ is integrated to the action of the corresponding congruence subgroup $G(z \mathbb{C}[[z]]) \subset G[[z]]$ since $U(\hat{g})_{-h^\vee} - \text{mod}_{U(\hat{g})_{-h^\vee}}^{\hat{\mathfrak{h}}}$ is a subcategory in $(U(\hat{g})_{-h^\vee} - \text{mod})_0$. But this is not true for the Lie subalgebra $\mathfrak{g}[[z]] \subset U(\hat{g})_{-h^\vee}$. 
This agrees with the properties of the Whittaker modules which are objects of the category $U(\mathfrak{g}) - \text{mod}_{\Lambda}^\chi$. The action of the Lie subalgebra $\mathfrak{n}$ on them is not locally nilpotent. However, in the theory of Whittaker modules over $\mathfrak{g}$ developed in [15] integrable, i.e. finite–dimensional, representations of $\mathfrak{g}$ also appear. For instance, let $M'_\Lambda$ be the contragredient (full dual) module to a Verma module $M_\Lambda$. In the proof of Theorem 4.6 in [15] it is observed that the subspace $V \subset M'_\Lambda$ which consists of elements on which $x - \chi(x)$ acts locally nilpotently for any $x \in \mathfrak{n}$ is a submodule which is in fact an irreducible Whittaker module. Clearly, $M'_\Lambda$ contains an irreducible finite–dimensional submodule, and hence one can associate such modules to irreducible Whittaker modules.

More generally, Whittaker vectors and Whittaker representations appear in the study of Whittaker models for principal series representations of Lie groups (see [18, 19]). It would be natural to introduce and explore similar objects related to the modules from the category $U(\mathfrak{g}) - 2k^\chi - k - \text{mod}_{U(\mathfrak{g})}^{\chi - \frac{k}{2}}$.

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