Seiberg-Witten Curve for $E$-String Theory

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We construct the Seiberg-Witten curve for the $E$-string theory in six-dimensions. The curve is expressed in terms of affine $E_8$ characters up to level 6 and is determined by using the mirror-type transformation so that it reproduces the number of holomorphic curves in the Calabi-Yau manifold and the amplitudes of $\mathcal{N} = 4$ $U(n)$ Yang-Mills theory on $\frac{1}{2}K_3$. We also show that our curve flows to known five- and four-dimensional Seiberg-Witten curves in suitable limits. We further find new type of reduction to some particular four-dimensional theories such as the $SU(2)$ Seiberg-Witten theory with 4 flavors, without taking a degenerate limit of $T^2$ so that the $SL(2,\mathbb{Z})$ symmetry is left intact.

§1. Introduction

$E$-string was first discovered in the study of zero-size instantons in heterotic string theory.\(^1\),\(^2\) In the M-theory description\(^3\) $E$-string appears at the intersection of the (end-of-world) 9-brane and a membrane which connects the 9-brane to a 5-brane representing the small instanton. An $E$-string carries a level one $E_8$ current algebra and possesses one half of the degrees of freedom of the heterotic string. In fact an $E$-string may be obtained by wrapping an M5-brane around $\frac{1}{2}K_3$ surface\(^4\) while a heterotic string is obtained by wrapping an M5-brane around $K_3$. The second homology $b^+_2 = 1$, $b^-_2 = 9$ of $\frac{1}{2}K_3$ generates 1 (9) right-(left-)moving bosonic degrees of freedom in 2-dimensions. Together with the freedom representing the position of the M5-brane one obtains 4 (12) right (left) bosonic degrees of freedom of $E$-string which is one half of the heterotic string. $E$-string possesses one half of supersymmetry of the heterotic theory and the reduction of supersymmetry leads to threshold BPS bound states and non-trivial dynamics in low energy $E$-string theory.

It is known that the $\frac{1}{2}K_3$ can be obtained by blowing up 9 points of $\mathbb{P}^2$ and is also called as an almost del Pezzo surface $\mathcal{B}_9$. Elements in the second homology group of $\frac{1}{2}K_3$ span a unimodular lattice $\Gamma^{9,1}$ with signature (1, 9) which contains the lattice $I_8$ of $E_8$, $\Gamma^{9,1} = I_8 \oplus \Gamma^{1,1}$. The existence of the lattice $I_8$ is the origin of the $E_8$ symmetry in $E$-string theory.

It is known that the del Pezzo surface $\mathcal{B}_9$ possesses an elliptic fibration over a $\mathbb{P}^1$ with 12 singular fibers. The fibration is described by a family of elliptic curves written in a Weierstrass form

$$y^2 = 4x^3 - f(u, \tau; m_i)x - g(u, \tau; m_i),$$  \hspace{1cm} (1.1)

where the coefficient functions $f, g$ are degree 4 and 6 polynomials in $u$, respectively

$$f(u, \tau; m_i) = a_0(\tau)u^4 + a_1(\tau; m_i)u^3 + a_2(\tau; m_i)u^2 + a_3(\tau; m_i)u + a_4(\tau; m_i),$$  \hspace{1cm} (1.2)

$$g(u, \tau; m_i) = b_0(\tau)u^6 + b_1(\tau; m_i)u^5 + b_2(\tau; m_i)u^4 + b_3(\tau; m_i)u^3 + b_4(\tau; m_i)u^2 + b_5(\tau; m_i)u + b_6(\tau; m_i).$$  \hspace{1cm} (1.3)
u parametrizes the base $P^1$, $\tau$ gives the modulus of the elliptic fiber at $u = \infty$ and $m_i, i = 1, 2, \ldots, 8$ represent the coordinates of the points on this fiber where holomorphic sections intersect. Physically $\tau$ represents the modulus of the torus $T^2$ upon which $E$-string theory is compactified down to 4-dimensions while $\{m_i\}$ represent $E_8$ Wilson line parameters. In the picture of low-energy $\mathcal{N} = 2$ $SU(2)$ gauge theory $\tau$ is interpreted as the bare gauge coupling constant while $\{m_i\}$ are identified as the bare masses of 8 hypermultiplets.

With $\tau$ being the modulus at $u = \infty$, $a_0(\tau), b_0(\tau)$ are fixed to

$$a_0(\tau) = \frac{1}{12} E_4(\tau), \quad b_0(\tau) = \frac{1}{216} E_6(\tau),$$

where $E_{2n}$ is the Eisenstein series with weight $2n$,

$$E_{2n}(\tau) = 1 + \frac{(2\pi i)^{2n}}{(2n - 1)! \zeta(2n)} \sum_{m=1}^{\infty} \sigma_{2n-1}(m) q^m, \quad \sigma_k(m) = \sum_{d|m} d^k, \quad q = e^{2\pi i \tau}.$$  

By shifting the variable $u$ we can eliminate either one of the functions $a_1$ or $b_1$. In the following we choose a “gauge” where we put

$$a_1(\tau; m_i) = 0.$$  

We call Eq. (1.1) as the six-dimensional Seiberg-Witten (SW) curve: the six-dimensional curve determines the prepotential of $E$-string theory and gives a complete description of its low-energy dynamics. Previously there were attempts at deriving the six-dimensional curve, however, only partial results with a few non-vanishing Wilson line parameters $\{m_i\}$ were obtained. In this article we will determine all the coefficient functions $a_j, b_j$ and the curve (1.1) for arbitrary values of $\{m_i\}$. It turns out that the functions $a_j, b_j$ are expressed in terms of characters of affine $E_8$ at level $j$.

We shall show that the known five-dimensional $E_8$ SW curve is reproduced from the six-dimensional curve by taking the limit $\text{Im} \tau \rightarrow +\infty$. The four-dimensional $E_8$ curve is also obtained by further taking the remaining period of $T^2$ to 0. Seiberg-Witten curves with lower-rank symmetry groups can be easily obtained from these $E_8$ curves by taking some of the Wilson line parameters $\{m_i\}$ to $\infty$ or adjusting them to special values. Thus our six-dimensional curve serves as some kind of master theory encompassing all possible SW theories.

We also propose new type of reductions to four dimensions without taking the degenerate limit of $T^2$ so that the $SL(2, \mathbb{Z})$ symmetry is left intact. In fact, for instance, by setting four of the parameters $\{m_i\}$ to half-periods $(0, \pi, \pi + \pi \tau, \pi \tau)$ we obtain the curve for the $SU(2) N_f = 4$ gauge theory directly from our six-dimensional curve. As a remarkable by-product, the $SO(8)$ triality of the $N_f = 4$ theory is derived from the $SL(2, \mathbb{Z})$ symmetry of the six-dimensional curve in a natural manner.

Let us now recall the relation between BPS states of the $E$-string and holomorphic curves in the Calabi-Yau 3-fold and also the partition function of the $\mathcal{N} = 4$ $U(n)$ Yang-Mills theory on $\frac{1}{2}K3$. Consider an F-theory compactification down to 6-dimensions on a Calabi-Yau 3-fold $K$ which is elliptically fibered over a base $B$. 


We choose a curve $\Sigma$ inside the base $B$ so that the elliptic fibration restricted to $\Sigma$ gives the $\frac{1}{2}K_3$. Consider a D3-brane wrapped around $\Sigma$, which gives rise to a string in 6-dimensions. Low energy dynamics of such a string can be studied by looking at its BPS spectrum. We compactify the 5th dimension on a circle $R$ and count the BPS states of the string with winding number $n$ and momentum $k$. We denote the number of these states as $N_{n,k}^{\text{BPS}}$.

Next by using a duality between F- and M-theory we go to an M-theory description: a compactification of F-theory on $K \times S^1$, where the $S^1$ has radius $R$, is equivalent to a compactification of M-theory on $K$ with the Kähler parameter of the elliptic fiber being equal to $\frac{1}{R}$. In M-theory 5-dimensional BPS states are obtained by wrapping a membrane around holomorphic curves. Then the number of BPS states $N_{n,k}^{\text{BPS}}$ is given by the number $N_{n,k}^{\text{curve}}$ of holomorphic curves in the class $n[\Sigma] + k[E]$ where $[\Sigma]$ ($[E]$) denotes the class of the base (elliptic fiber) of $\frac{1}{2}K_3$. Thus the counting of BPS states of 6-dimensional $E$-string wrapped around a circle is related to the counting of holomorphic curves in $K$ which can be analyzed by using the technique of mirror symmetry.

If we introduce M5-branes, the number of BPS states becomes further related to the partition function of $U(n)$ Yang-Mills theory on $\frac{1}{2}K_3$. Consider an M5-brane wrapped around $\frac{1}{2}K_3$. One then obtains a string in 5-dimensions. In order to study its spectrum we may wrap the string around a circle and compute its toroidal partition function. Then the M5-brane becomes effectively wrapped around $\frac{1}{2}K_3 \times T^2$. If we consider the string $n$-times wound around a circle, M5-brane wraps around $T^2$ $n$-times and one obtains $\mathcal{N} = 4$ $U(n)$ Yang-Mills theory on $\frac{1}{2}K_3$. The gauge coupling constant is given by the modulus $\tau$ of the torus and the momentum $k$ is mapped to the instanton number. Thus $N_{n,k}^{\text{BPS}}$ also agrees with the $k$ instanton contribution $N_{n,k}^{\text{inst}}$ to the partition function of $U(n)$ gauge theory on $\frac{1}{2}K_3$, so we have

$$N_{n,k}^{\text{BPS}} = N_{n,k}^{\text{curve}} = N_{n,k}^{\text{inst}} = N_{n,k}.$$ (1.7)

We want to determine the prepotential which is defined as usual\textsuperscript{13)} by

$$\mathcal{F}(\phi, \tau) = \mathcal{F}_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} N_{n,k} \text{Li}_3(e^{2\pi i \phi + 2\pi i k \tau}),$$ (1.8)

where $\text{Li}_3(x)$ is the tri-logarithm function $\text{Li}_3(x) = \sum_{m=1}^{\infty} (x^m/m^3)$ and $\phi$ denotes the size of the base $\mathbb{P}^1$. Due to the global $E_8$ symmetry of the theory BPS states, etc., of $E$-string fall into $E_8$ Weyl orbits and thus $N_{n,k}$ may be expanded as

$$N_{n,k} = \sum_{\mathcal{O}} \text{dim}(\mathcal{O}) N_{n,k}^{\mathcal{O}},$$ (1.9)

where dim($\mathcal{O}$) denotes the dimension of the Weyl orbit $\mathcal{O}$. When we introduce the $E_8$ Wilson line parameters $m_i, i = 1, 2, \cdots, 8$, the prepotential is modified as

$$\mathcal{F}(\phi, \tau, \vec{m}) = \mathcal{F}_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\mathcal{O}} N_{n,k}^{\mathcal{O}} \sum_{\vec{\nu} \in \mathcal{O}} \text{Li}_3(e^{2\pi i \phi + 2\pi i k \tau + i\vec{\nu} \cdot \vec{m}}).$$ (1.10)
Here $\nu$ runs over the weights on the Weyl orbit $O$. Partition functions $Z_n$ of $U(n)$ gauge theories on $\frac{1}{2}K_3$ are then defined by

$$F(\phi, \tau, \vec{m}) = F_{\text{classical}} - \frac{1}{(2\pi i)^3} \sum_{n=1}^{\infty} q^{n/2} Z_n(\vec{m}; \tau) e^{2\pi i n \phi}, \quad q = e^{2\pi i \tau}. \quad (1.11)$$

(An extra factor of $q^{n/2}$ has been introduced so that $Z_n$ has a simpler modular property.) Prepotential (1.10), (1.11) is also interpreted as the generating function for the number of $E$-string BPS states or the holomorphic curves of $\frac{1}{2}K_3$. Variable $u$ and $\phi$ are related to each other by a mirror-type transformation of Seiberg-Witten theory.

Actually the functions $Z_n$ have already been computed recursively up to $Z_4$ by making use of the holomorphic anomaly and the gap condition ($N_{n,k} = 0$ for $0 < k < n$) in Ref. 4). One may continue this computation and obtain more data on $Z_n$. On the other hand, from the SW curve (1.1) one can compute the prepotential by making use of the mirror-type transformation and express $Z_n$ in terms of the coefficient functions $a_j, b_j$. It turns out that $Z_n$ is written as a polynomial in $a_j, b_j$, $0 \leq j \leq n$ and in particular linear in $a_n, b_n$. Hence given the data $\{Z_n\}$ it is easy to determine $\{a_j, b_j\}$. In our actual computation we have used the data up to $Z_8$ to determine the Seiberg-Witten curve.

This article is based on our earlier works Refs. 14), 15).

§2. Instanton expansion

Let us next describe the standard mirror-type transformation in Seiberg-Witten theory adopted to the present situation.\textsuperscript{16}) The coupling constant $\tilde{\tau}$ of $\mathcal{N} = 2$ gauge theory is given by the modulus of the elliptic curve

$$j(\tilde{\tau}) = \frac{1728 f(u, \tau; m_i)^3}{f(u, \tau; m_i)^3 - 27g(u, \tau; m_i)^2}. \quad (2.1)$$

Here $j$ is the modular $j$-function. We may expand the right-hand side of the above equation in $1/u$ and obtain

$$j(\tilde{\tau}) = \frac{1728 E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} + \frac{1}{4} \frac{E_4(\tau)^3 E_6(\tau) b_1(\tau; m_i) 1}{\Delta(\tau)^2} \frac{1}{u} + \cdots$$

$$= j(\tau) + \frac{1}{4} \frac{E_4(\tau)^3 E_6(\tau) b_1(\tau; m_i) 1}{\Delta(\tau)^2} \frac{1}{u} + \cdots, \quad (2.2)$$

where $\Delta = (E_4^3 - E_6^2)/1728 = \eta^{24}$. Thus $\tilde{\tau} = \tau$ at $u = \infty$. By inverting the $j$-function in (2.2) we can generate a Taylor series expansion of $\tilde{\tau}$ around $u = \infty$,

$$\tilde{\tau} = \tau + \frac{i}{8\pi} \frac{E_4 b_1 1}{\Delta} \frac{1}{u} + \frac{i}{384\pi} \frac{-4E_6 \Delta a_2 + 48E_4 \Delta b_2 + 5E_4 E_6 b_1^2 + E_2 E_4^2 b_1^2 1}{\Delta^2} \frac{1}{u^2} + \cdots \quad (2.3)$$

Periods of the torus $\omega_1, \omega_2$ are given by

$$\omega_1 = \frac{i}{2\pi} \left( \frac{E_4(\tilde{\tau})}{12 f(u, \tau; m_i)} \right)^{1/4}, \quad \omega_2 = \tilde{\tau} \omega_1. \quad (2.4)$$
Both of these $\omega_i$ are also expanded in Taylor series in $1/u$. We can then easily integrate them and obtain functions $\phi, \phi_D$

$$\phi = \int du \omega_1(u, \tau; m_i), \quad \phi_D = \int du \omega_2(u, \tau; m_i) = \int du \tilde{\tau}(u, \tau; m_i)\omega_1(u, \tau; m_i).$$  

(2.5)

Since $\partial\phi/\partial u = \omega_1, \partial\phi_D/\partial u = \omega_2$, $\phi, \phi_D$ correspond to the variables of the Coulomb branch $a, a_D$ in the 4-dimensional Seiberg-Witten theory. Prepotential is defined by

$$\frac{\partial F}{\partial \phi} = \phi_D$$

(2.6)

as usual.

Lower order computation goes as follows: $\omega_1$ has an expansion

$$\omega_1 = i \frac{1}{2\pi} \left( \frac{E_2 E_4 - E_6}{\Delta} \right) b_1 \frac{1}{u} + \cdots.$$  

(2.7)

$\phi$ is then given by

$$\phi = \phi_0 + i \frac{1}{2\pi} \ln u + i \frac{1}{96\pi} \left( \frac{E_2 E_4 - E_6}{\Delta} \right) b_1 \frac{1}{u} + \cdots,$$  

(2.8)

where $\phi_0$ is an integration constant. Thus

$$\frac{1}{u} = e^{2\pi i (\phi - \phi_0)} + \frac{1}{48} \left( \frac{E_2 E_4 - E_6}{\Delta} \right) e^{4\pi i (\phi - \phi_0)} + \cdots$$

(2.9)

and therefore

$$\frac{\partial^2 F}{\partial \phi^2} = \tilde{\tau} = \tau + \frac{i}{8\pi} \frac{E_4 b_1}{\Delta} e^{2\pi i (\phi - \phi_0)} + \frac{i}{192\pi} \frac{-2E_6 \Delta a_2 + 24E_4 \Delta b_2 + 2E_4 E_6 b_1^2 + E_2 E_4^2 b_1^2}{\Delta^2} e^{4\pi i (\phi - \phi_0)} + \cdots.$$  

(2.10)

By integrating twice in $\phi$ we obtain the prepotential. In the following we choose $\phi_0 = -(\ln \eta^{12})/2\pi i - 1/2$.

We have performed the above transformation up to higher orders and obtained the partition functions expressed in terms of the coefficient functions of the SW curve. We present lower order terms

$$Z_1 = -\frac{1}{4} \frac{E_4}{\eta^{12}} b_1,$$  

(2.11)

$$Z_2 = \frac{1}{8} \left( -\frac{1}{24} E_6 a_2 + \frac{1}{2} E_4 b_2 + \frac{1}{24} \frac{E_4 E_6}{\eta^{24}} b_1^2 + \frac{1}{48} \frac{E_2 E_4^2}{\eta^{24}} b_1^2 \right),$$  

(2.12)

$$Z_3 = \frac{1}{27} \left( \frac{1}{16} \eta^{12} E_6 a_3 - \frac{3}{4} \eta^{12} E_4 b_3 - \frac{9}{2} \eta^{12} a_2 b_1 + \frac{5}{512} \eta^{12} E_4^3 a_2 b_1 \right.$$  

$$+ \frac{3}{512} \frac{E_2 E_4 E_6}{\eta^{12}} a_2 b_1 - \frac{15}{128} \frac{E_4 E_6}{\eta^{12}} b_2 b_1 - \frac{9}{128} \frac{E_2 E_4^2}{\eta^{12}} b_2 b_1 \right.$$  

$$+ \frac{591}{64} \frac{E_4}{\eta^{12}} b_1^3 - \frac{17}{2048} \frac{E_4}{\eta^{36}} b_1^3 - \frac{3}{512} \frac{E_2 E_4^2 E_6}{\eta^{36}} b_1^3 - \frac{3}{2048} \frac{E_2 E_4^3}{\eta^{36}} b_1^3 \right).$$  

(2.13)
We have obtained similar formulas up to $Z_8$.

§3. Holomorphic anomaly and gap condition

Partition functions $Z_n$ of $U(n)$ gauge theory on $\frac{1}{2}K_3$ have been computed in Ref. 4) by imposing the relation of holomorphic anomaly and gap condition (see Ref. 17) for a different analysis). Holomorphic anomaly occurs in the gauge theory due to the appearance of reducible connections and corresponds to the existence of threshold bound states in the $E$-string spectrum. Holomorphic anomaly enters via the dependence on $E_2$ of $Z_n$ and is governed by the relation\(^{16}\)

\[
\frac{\partial Z_n}{\partial E_2} = \frac{1}{24} \sum_{m=1}^{n-1} m(n-m)Z_mZ_{n-m}.
\]  

(3.1)

The relation (3.1) corresponds to the reduction of the $U(n)$ connection down to that of $U(m) \times U(n-m)$. Holomorphic anomaly is tied with the fact that $\frac{1}{2}K_3$ does not possess a holomorphic 2-form and there exists no mass perturbation of $\mathcal{N} = 4$ theory which splits the location of $n$ $M5$-branes. Reduction of supersymmetry leads to a binding force among the branes.

Gap condition on the other hand requires that instantons with degree $k < n$ do not exist in $U(n)$ gauge theory. This comes from the positivity of intersection numbers among holomorphic curves in $\frac{1}{2}K_3$.

These two conditions and consideration of modular invariance uniquely determine the amplitudes. We list the first 3 instanton amplitudes rewritten slightly from Ref. 4).

\[
Z_1 = \frac{P(m_i; \tau)}{\eta^{12}},
\]

(3.2)

where

\[
P(m_i; \tau) = \frac{1}{2} \sum_{\ell=1}^{4} \prod_{j=1}^{8} \vartheta_{\ell}(m_j|\tau),
\]

(3.3)

\[
Z_2 = \frac{1}{\eta^{24}} \left[ f_{20}(\tau)P(2m_i; 2\tau) + f_{21}(\tau)P\left(m_i; \frac{\tau}{2}\right) + f_{21}(\tau + 1)P\left(m_i; \frac{\tau + 1}{2}\right) \right] + \frac{1}{24} E_2 Z_1^2,
\]

(3.4)

where

\[
f_{20}(\tau) = \frac{1}{24} \vartheta_3(\tau)^4 \vartheta_4(\tau)^4 \left( \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 \right),
\]

(3.5)

\[
f_{21}(\tau) = -\frac{1}{384} \vartheta_3(\tau)^4 \vartheta_2(\tau)^4 \left( \vartheta_3(\tau)^4 + \vartheta_2(\tau)^4 \right),
\]

(3.6)

\[
Z_3 = \frac{1}{\eta^{36}} \left[ f_{30}(\tau)P(3m_i; 3\tau) + f_{31}(\tau)P\left(m_i; \frac{\tau}{3}\right) + f_{31}(\tau + 1)P\left(m_i; \frac{\tau + 1}{3}\right) \right. \\
+ f_{31}(\tau + 2)P\left(m_i; \frac{\tau + 2}{3}\right) - \frac{1}{288} E_4(\tau)P(m_i; \tau) \bigg] \\
+ \frac{1}{6} E_2 Z_2 Z_1 - \frac{1}{288} E_2^2 Z_1^3,
\]

(3.7)
where \( f_{30}(\tau) = \frac{5}{216} \frac{\eta(\tau)^{36}}{\eta(3\tau)^{12}} + \frac{9}{8} \eta(\tau)^{24} \),
\( f_{31}(\tau) = \frac{5}{24} \frac{\eta(\tau)^{36}}{\eta(3\tau)^{12}} + \frac{1}{72} \eta(\tau)^{24} \). (3.8)

Note that the function \( P(m_i; \tau) \) is (up to \( \eta^8 \)) the character of the level-one representation of \( \hat{E}_8 \) and \( P(2m_i; 2\tau), P(m_i; \tau^2), P(m_i; \tau + 1/2) \) are linear combinations of characters of 3 level-two representations of \( \hat{E}_8 \). In general the amplitude \( Z_n \) is expressed in terms of level-\( n \) representations of \( \hat{E}_8 \). There exist 1, 3, 5, 10, 15, 27 distinct representations of \( \hat{E}_8 \) at the levels 1, 2, 3, 4, 5, 6, respectively. Coefficient functions \( f_{n0} \) are modular forms of \( \Gamma_1(n) \) which consists of matrices of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( ad - bc = 1 \) and \( a, d = 1, c = 0 \mod n \). Functions \( f_{n1} \) are the S-transform of \( f_{n0} \).

Using holomorphic anomaly and the gap condition we have obtained the partition functions up to \( Z_8 \).

§4. Six-dimensional curve

Now by using these data we can determine the coefficient functions of SW curve. We present the first few terms and relegate the rest to Refs. 14), 15).

\[
b_1 = -4 \frac{P(m_i; \tau)}{E_4},
\]

\[
a_2 = \frac{1}{E_4 \Delta} \left[ f_{a2,0}(\tau) P(2m_i; 2\tau) + f_{a2,1}(\tau) P\left( m_i; \frac{\tau}{2} \right) + f_{a2,1}(\tau + 1) P\left( m_i; \frac{\tau + 1}{2} \right) \right],
\]

where 
\[
f_{a2,0}(\tau) = \frac{2}{3} \left( E_4(\tau) - 9 \vartheta_2(\tau^8) \right),
\]

\[
f_{a2,1}(\tau) = \frac{1}{24} \left( E_4(\tau) - 9 \vartheta_4(\tau^8) \right),
\]

\[
b_2 = \frac{1}{E_4^2 \Delta} \left[ f_{b2,0}(\tau) P(2m_i; 2\tau) + f_{b2,1}(\tau) P\left( m_i; \frac{\tau}{2} \right) + f_{b2,1}(\tau + 1) P\left( m_i; \frac{\tau + 1}{2} \right) \right],
\]

where 
\[
f_{b2,0}(\tau) = \frac{1}{36} \left( \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 \right) \left( E_4(\tau)^2 + 60 E_4(\tau) \vartheta_2(\tau)^8 - 45 \vartheta_2(\tau)^{16} \right),
\]

\[
f_{b2,1}(\tau) = -\frac{1}{576} \left( \vartheta_3(\tau)^4 + \vartheta_2(\tau)^4 \right) \left( E_4(\tau)^2 + 60 E_4(\tau) \vartheta_4(\tau)^8 - 45 \vartheta_4(\tau)^{16} \right),
\]

\[
a_3 = \frac{1}{E_4^2 \Delta^2} \left[ f_{a3,0}(\tau) P(3m_i; 3\tau) + f_{a3,1}(\tau) P\left( m_i; \frac{\tau}{3} \right) + f_{a3,1}(\tau + 1) P\left( m_i; \frac{\tau + 1}{3} \right)
+ f_{a3,1}(\tau + 2) P\left( m_i; \frac{\tau + 2}{3} \right) + \frac{2}{3} E_6(\tau) P(m_i; \tau^3) \right],
\]
where \( f_{a3,0}(\tau) = \frac{1}{3} E_4(\tau) h_2(\tau)^2 \left( 7E_4(\tau) - 9h_0(\tau)^4 \right) \),
\[ f_{a3,1}(\tau) = -\frac{1}{3^3} E_4(\tau) h_3\left( \frac{\tau}{3} \right)^2 \left( 7E_4(\tau) - h_0\left( \frac{\tau}{3} \right)^4 \right), \]
\[ b_3 = \frac{1}{E_4^3 \Delta^2} \left[ f_{b3,0}(\tau) P(3m_i; 3\tau) + f_{b3,1}(\tau) P\left( m_i; \frac{\tau}{3} \right) + f_{b3,1}(\tau + 1) P\left( m_i; \frac{\tau + 1}{3} \right) + f_{b3,1}(\tau + 2) P\left( m_i; \frac{\tau + 2}{3} \right) \right] \]
\[ + \frac{1}{54} \left( 8E_4^3 - 5E_6^2 \right) P(m_i; \tau)^3, \]
where
\[ f_{b3,0}(\tau) = \frac{1}{18} E_4(\tau)^2 h_2(\tau)^2 \left( 32h_2(\tau)^2 + 48h_2(\tau)h_0(\tau)^3 - 81h_0(\tau)^6 \right), \]
\[ f_{b3,1}(\tau) = \frac{1}{2 \cdot 3^{12}} E_4(\tau)^2 h_3\left( \frac{\tau}{3} \right)^2 \left( 32h_3\left( \frac{\tau}{3} \right)^2 + 48h_3\left( \frac{\tau}{3} \right)h_0\left( \frac{\tau}{3} \right)^3 - 81h_0\left( \frac{\tau}{3} \right)^6 \right). \]

Here functions \( h_i \) are defined by
\[ h_0(\tau) = \sum_{n_1, n_2 = -\infty}^{\infty} q^{n_1^2 + n_2^2 - n_1 n_2} = \vartheta_3(2\tau) \vartheta_3(6\tau) + \vartheta_2(2\tau) \vartheta_2(6\tau), \]
\[ h_2(\tau) = \frac{\eta(3\tau)^9}{\eta(3\tau)^3}, \quad h_3(\tau) = 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3}. \]

It is fortunate that the six-dimensional curve can be written in a relatively compact expression. As we have mentioned, coefficient functions \( a_j, b_j \) are expressed in terms of affine \( E_8 \) characters at level-\( j \). Functions \( f_{\ast n,0} \) are modular forms of \( \Gamma_1(n) \) and \( f_{\ast n,1} \) are their S-transform. We have checked that our curve reduces to the results of Ref. 7) when six of the Wilson line parameters are set equal to zero.

In Ref. 15) we study the holomorphic sections of the elliptic fibration described by this six-dimensional curve. Their explicit form confirms the correctness of our curve and clarifies the geometric significance of the Wilson line parameters \( \{ m_i \} \).

The total six-dimensional curve exhibits several types of symmetries and interesting properties, which are extensively discussed in Ref. 15).

\[ \textbf{§5. Five-dimensional curve} \]

Let us next see how our result reproduces the five-dimensional curve in the limit \( q \to 0 \). Affine \( E_8 \) characters are reduced to those of finite-dimensional \( E_8 \) algebra in this limit. Let us first introduce some notations associated with the \( E_8 \) algebra. Let \( \bar{\Lambda} \) be some dominant weight and \( \bar{\mu}_1, \ldots, \bar{\mu}_8 \) be the fundamental weights of \( E_8 \). We then introduce a notation
\[ \bar{\Lambda} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 \end{bmatrix} = \sum_{i=1}^{8} n_i \bar{\mu}_i, \quad n_i \in \mathbb{Z}_{\geq 0}, \]
(5.1)
where \( \{n_i\} \) denote Dynkin indices of \( \tilde{A} \). We fix a labeling of the fundamental weights by placing eight indices at eight nodes of the Dynkin diagram (Fig. 1). Next we define a character for a Weyl orbit \( \mathcal{O} \) of weight \( \tilde{A} \) by

\[
w_{\mathcal{O}}(m_i) = w[n_1 n_3 n_4 n_5 n_6 n_7 n_8](m_i) \equiv \sum_{\tilde{\nu} \in \mathcal{O}} e^{i\tilde{m} \cdot \tilde{\nu}},
\]

where \( \tilde{\nu} \) runs over all weights on the Weyl orbit \( \mathcal{O} \). The variable \( \tilde{m} \) takes its values on the \( \mathbb{C} \)-extended root space.

With this preparation we can present the five-dimensional \( E_8 \) curve

\[
y^2 = 4x^3 - F(u; m_i)x - G(u; m_i),
\]

\[
F(u; m_i) = A_0 u^4 + A_2(m_i) u^2 + A_3(m_i) u + A_4(m_i),
\]

\[
G(u; m_i) = B_0 u^6 + B_1 u^5 + B_2(m_i) u^4 + B_3(m_i) u^3 + B_4(m_i) u^2 + B_5(m_i) u + B_6(m_i)
\]

with

\[
A_0 = \frac{1}{12}, \quad A_2 = -\frac{2}{3} w[1000000] + 12 w[0000001] - 1440,
\]

\[
A_3 = -2w[1000000] + 96 w[1000000] - 1152w[0000001] + 103680,
\]

\[
A_4 = \frac{4}{3} w[2000000] - \frac{4}{3} w[0100000] - \frac{8}{3} w[0001000] - \frac{16}{3} w[1000000] + 24 w[0000002] + \frac{328}{3} w[0000000] + 48 w[0000010] - \frac{896}{3} w[1000000] + 28128 w[0000001] - 2105280,
\]

\[
B_0 = \frac{1}{216}, \quad B_1 = -4, \quad B_2 = -\frac{1}{18} w[1000000] - 3w[0000001] + 840,
\]

\[
B_3 = -\frac{1}{8} w[0000000] - 4w[0000010] - 8w[1000000] + 528w[0000001] - 79680,
\]

\[
B_4 = -\frac{3}{2} w[2000000] + \frac{5}{3} w[0100000] - \frac{28}{9} w[0000100] - \frac{152}{9} w[1000000] - 92 w[0000002] - \frac{316}{9} w[0000000] + 416 w[0000010] + \frac{1308}{9} w[1000000] - 35536 w[0000001] + 3911520,
\]

\[
B_5 = \frac{2}{3} w[1000000] - \frac{2}{3} w[0001000] - \frac{16}{3} w[1000010] - \frac{74}{3} w[0000001] - 96 w[0000011]
\]
\[-\frac{160}{3} w^{0 \ldots 0}_{2000000} - \frac{280}{3} w^{0 \ldots 0}_{1010000} + 80 w^{0 \ldots 0}_{0000100} + 1088 w^{0 \ldots 0}_{0000010} \]
\[+ 7872 w^{0 \ldots 0}_{0000002} + \frac{9680}{3} w^{0 \ldots 0}_{0000000} - 15264 w^{0 \ldots 0}_{0000010} - \frac{196448}{3} w^{0 \ldots 0}_{1000000} \]
\[+ 1075776 w^{0 \ldots 0}_{0000001} - 97251840, \quad (5.12)\]

\[B_6 = -\frac{8}{27} w^{1 \ldots 0}_{3000000} + w^{2 \ldots 0}_{0000000} + \frac{4}{9} w^{0 \ldots 0}_{1100000} + \frac{2}{3} w^{0 \ldots 0}_{0010000} - \frac{16}{9} w^{0 \ldots 0}_{2000000} \]
\[+ \frac{8}{3} w^{0 \ldots 0}_{0000010} - 8 w^{0 \ldots 0}_{0000020} - \frac{88}{9} w^{0 \ldots 0}_{0100000} - \frac{112}{3} w^{0 \ldots 0}_{0000001} + \frac{1000}{9} w^{0 \ldots 0}_{1000000} \]
\[+ \frac{992}{3} w^{0 \ldots 0}_{0000003} - 70 w^{0 \ldots 0}_{1000000} - 64 w^{0 \ldots 0}_{0001000} + \frac{142}{9} w^{0 \ldots 0}_{1000010} + 636 w^{0 \ldots 0}_{0000010} \]
\[+ 3472 w^{0 \ldots 0}_{0000001} + 1584 w^{0 \ldots 0}_{2000000} + 3038 w^{0 \ldots 0}_{0100000} + \frac{7376}{9} w^{0 \ldots 0}_{0000010} \]
\[+ 18480 w^{0 \ldots 0}_{1000000} - 176608 w^{0 \ldots 0}_{0000002} - \frac{196384}{3} w^{0 \ldots 0}_{0000000} + 197968 w^{0 \ldots 0}_{0000010} \]
\[+ 936200 w^{0 \ldots 0}_{1000000} - 12291232 w^{0 \ldots 0}_{0000000} + 971250560. \quad (5.13)\]

(It is also possible to represent this curve using characters only for fundamental representations of \(E_8\).\(^{15}\) In this case products of characters appear in the formula.)

This curve has a manifest \(E_8\) symmetry and can be derived from the expression of Ref. 7 with \(SO(16)\) symmetry by a suitable shifting of variables and rearrangement of terms. Note that \(A_j, B_j\) contain representations with levels \(1 \leq l \leq j\) (constant term is interpreted as level-one representation). This is consistent with the fact that an affine representation of level \(j\) is reduced to finite-dimensional representations with levels \(l \leq j\) in the limit \(q \to 0\).

In fact we can explicitly show that

\[\lim_{q \to 0} a_j(\tau; m_i) = A_j(m_i), \quad j = 0, 1, 2, \ldots, 4, \quad (5.14)\]
\[\lim_{q \to 0} b_j(\tau; m_i) = B_j(m_i), \quad j = 0, 1, 2, \ldots, 6 \quad (5.15)\]

hold. Since

\[P(m_i; \tau), \quad E_4(\tau) \to 1 \quad \text{as} \quad q \to 0 \quad (5.16)\]

we can easily check the relation (5.15) for \(b_1\). In general it is convenient to use the basis of affine Weyl-orbit characters which are given by, for instance,

\[\hat{w}^{(2)}_{0 \ldots 0}(\tau; m_i) = P(2m_i; 2\tau), \quad (5.17)\]
\[\hat{w}^{(2)}_{1 \ldots 0}(\tau; m_i) = \frac{1}{2} \left( P\left(m_i; \frac{\tau}{2}\right) + P\left(m_i; \frac{\tau + 1}{2}\right) \right) - P(2m_i; 2\tau), \quad (5.18)\]
\[\hat{w}^{(2)}_{0 \ldots 0}(\tau; m_i) = \frac{1}{2} \left( P\left(m_i; \frac{\tau}{2}\right) - P\left(m_i; \frac{\tau + 1}{2}\right) \right) \quad (5.19)\]

at level-two. Then by taking the limit \(q \to 0\) we find

\[a_2(\tau; m_i) \]
\[= \frac{1}{24 E_4(\tau) \Delta(\tau)} \left( \left(-16 E_4(\tau) + 9 \vartheta_2(\tau)^8\right) \hat{w}^{(2)}_{1 \ldots 0}(\tau; m_i) \right. \]
\[+ 9 \left( \vartheta_3(\tau)^8 - \vartheta_4(\tau)^8 \right) \hat{w}^{(2)}_{0 \ldots 0}(\tau; m_i) \]
Seiberg-Witten Curve for E-String Theory

$$-135\vartheta_2(\tau)^8 \tilde{w}^{(2)}_{[0000000]}(\tau; m_i)$$

$$\rightarrow -\frac{2}{3} w^{(0)}_{[1000000]}(m_i) + 12 w^{(0)}_{[0000001]}(m_i) - 1440 = A_2(m_i),$$

$$b_2(\tau; m_i)$$

$$= \frac{1}{576 E_4(\tau)^2 \Delta(\tau)} \times \left( \left( \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 \right) \times \left( -16 E_4(\tau)^2 - 15 E_4(\tau) \vartheta_2(\tau)^8 + 45 \vartheta_2(\tau)^{16} \right) \tilde{w}^{(2)}_{[1000000]}(\tau; m_i) 
+ 9 \vartheta_2(\tau)^4 \left( -12 E_4(\tau)^2 + 25 E_4(\tau) \vartheta_2(\tau)^8 - 15 \vartheta_2(\tau)^{16} \right) \tilde{w}^{(2)}_{[0000001]}(\tau; m_i) 
+ 135 \vartheta_2(\tau)^8 \left( \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 \right) \left( 7 E_4(\tau) - 5 \vartheta_2(\tau)^8 \right) \tilde{w}^{(2)}_{[0000000]}(\tau; m_i) \right)$$

$$\rightarrow -\frac{1}{18} w^{(0)}_{[1000000]}(m_i) - 3 w^{(0)}_{[0000001]}(m_i) + 840 = B_2(m_i).$$

Flow of $E_8$ theory to theories with smaller symmetry groups has been extensively discussed in the literature 7),18)–20). In Refs. 14),15) we present five-dimensional curves with manifest $E_n$ symmetry for $n = 6, 7$.

§6. Four-dimensional curve

In order to derive four-dimensional curve we first have to reinstate the radius $R$ of the 5th dimension and redefine the mass parameter as

$$m_i \rightarrow R m_i, \quad i = 1, 2, \cdots, 8. \quad (6.1)$$

Now the new mass parameters carry the dimension of mass. Four-dimensional limit is obtained at $R \rightarrow 0$. Thus we expand the characters $w_{\Theta}$ in power series of $R$. Lower order terms of the series are canceled if we make a suitable shift of the parameter as

$$u \rightarrow u + 6c_2 R^2 - \frac{1}{4} c_2^2 R^4 + \frac{1}{240} c_2^3 R^6, \quad (6.2)$$

where $c_2$ is the second order Casimir invariant

$$c_2 = \sum_{i=1}^{8} m_i^2. \quad (6.3)$$

When we rescale variables as

$$x \rightarrow R^{10} x, \quad y \rightarrow R^{15} y, \quad u \rightarrow R^6 u, \quad (6.4)$$

all the terms in the curve cancel up to order $R^{29}$ and the $E_8$ curve of 4-dimensional theory\(^9),21\) appears as the coefficient of $R^{30}$. Flow of $E_8$ theory to theories with lower symmetry has been discussed in Refs. 9), 10).
§7. New reduction to four-dimensional curve

7.1. Extraction of the 4-dim $SU(2)$ $N_f=4$ curve and $SO(8)$ triality

Now let us discuss a new type of reduction of our six-dimensional curve to four-dimensional ones which maintains the manifest $SL(2,\mathbb{Z})$ symmetry. We claim that when we set four of the Wilson line parameters at half-periods

\[ m_5 = 0, \quad m_6 = \pi, \quad m_7 = \pi + \pi \tau, \quad m_8 = \pi \tau \quad (7.1) \]

we recover the four-dimensional $SU(2)$ Seiberg-Witten curve with $N_f = 4$ flavors.

Let us first recall the $N_f = 4$ curve, \(^{11}\) whose explicit form is given by

\[ \tilde{y}^2 = 4[W_1W_2W_3 + A(W_1T_1(e_2 - e_3) + W_2T_2(e_3 - e_1) + W_3T_3(e_1 - e_2)) - A^2N] \quad (7.2) \]

with

\[ W_i = \tilde{x} - e_i \tilde{u} - e_i^2 R, \quad (7.3) \]
\[ A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1), \quad (7.4) \]
\[ R = \frac{1}{2} \sum_i M_i^2, \quad (7.5) \]
\[ T_1 = \frac{1}{12} \sum_{i>j} M_i^2 M_j^2 - \frac{1}{24} \sum_i M_i^4, \quad (7.6) \]
\[ T_2 = -\frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2 M_j^2 + \frac{1}{48} \sum_i M_i^4, \quad (7.7) \]
\[ T_3 = \frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2 M_j^2 + \frac{1}{48} \sum_i M_i^4, \quad (7.8) \]
\[ N = \frac{3}{16} \sum_{i>j>k} M_i^2 M_j^2 M_k^2 - \frac{1}{96} \sum_{i \neq j} M_i^2 M_j^4 + \frac{1}{96} \sum_i M_i^6, \quad (7.9) \]

where $M_1, \cdots, M_4$ denote bare masses of matter hypermultiplets and

\[ e_1 = \frac{1}{12}(\vartheta_3^4 + \vartheta_4^4), \quad e_2 = \frac{1}{12}(\vartheta_2^4 - \vartheta_4^4), \quad e_3 = \frac{1}{12}(-\vartheta_2^4 - \vartheta_3^4). \quad (7.10) \]

We note that when four of the Wilson line parameters $\{m_i\}$ are set at zeros of the theta-function (7.1), all the odd coefficients $a_3, b_1, b_3, b_5$ vanish identically in our curve. This is due to the structure of Hecke-type transformation in the coefficient functions $\{a_i, b_i\}$.

Then if we redefine $u^2$ as $\tilde{u}$, the six-dimensional curve becomes

\[ y^2 = 4x^3 - (a_0u^2 + a_2u + a_4)x - (b_0u^3 + b_2u^2 + b_4u + b_6). \quad (7.11) \]

By comparing (7.2) with (7.11) one finds that these curves are in fact the same if one makes the following identifications of four-dimensional masses with Wilson line
parameters

\[ M_1 = \left( \prod_{j=1}^{4} \vartheta_1(m_j|\tau) - \prod_{j=1}^{4} \vartheta_2(m_j|\tau) \right) / \prod_{j=1}^{4} \vartheta_1(m_j|\tau), \quad (7.12) \]

\[ M_2 = \left( \prod_{j=1}^{4} \vartheta_1(m_j|\tau) + \prod_{j=1}^{4} \vartheta_2(m_j|\tau) \right) / \prod_{j=1}^{4} \vartheta_1(m_j|\tau), \quad (7.13) \]

\[ M_3 = \left( \prod_{j=1}^{4} \vartheta_3(m_j|\tau) - \prod_{j=1}^{4} \vartheta_4(m_j|\tau) \right) / \prod_{j=1}^{4} \vartheta_1(m_j|\tau), \quad (7.14) \]

\[ M_4 = \left( \prod_{j=1}^{4} \vartheta_3(m_j|\tau) + \prod_{j=1}^{4} \vartheta_4(m_j|\tau) \right) / \prod_{j=1}^{4} \vartheta_1(m_j|\tau). \quad (7.15) \]

(Common denominator of \( \{M_i\} \) can be chosen arbitrarily. We have fixed it to \( \prod_j \vartheta_1(m_j|\tau) \) so that transformation laws of \( \{M_i\} \) fit with the convention of Ref. 11.)

\( x, y, u \) of the curve (7.11) and \( \tilde{x}, \tilde{y}, \tilde{u} \) of the \( N_f = 4 \) curve (7.2) are related as

\[ u = L^2 \tilde{u} - \frac{1}{24\eta^{24}} \left( \frac{1}{12} E_4^2 a_2 - E_6 b_2 \right), \quad (7.16) \]

\[ x = L^2 \left( \tilde{x} - \frac{1}{144} E_4 \sum_{i=1}^{4} M_i^2 \right), \quad (7.17) \]

\[ y = L^3 \tilde{y} \quad (7.18) \]

with

\[ L = \frac{1}{q^{1/4} \eta^{18}} \prod_{j=1}^{4} \vartheta_1(m_j|\tau). \quad (7.19) \]

The parameter \( \tau \) was interpreted in the \( E \)-string theory as the modulus of the torus \( T^2 \) of fifth and sixth dimensions. Now we identify it with the bare coupling of four-dimensional gauge theory. We have derived the \( SL(2,\mathbb{Z}) \) symmetry of four-dimensional \( N_f = 4 \) theory from the geometry of six dimensions. This relationship has in fact been suggested before.\(^6\)

The structure of four-dimensional masses \( \{M_j\} \) (7.12)–(7.15) is quite interesting; they are invariant under the subgroup \( \Gamma(2) \) of the modular group which consists of matrices of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \) with \( ad - bc = 1 \). On the other hand they do transform under \( S \) and \( T \) transformations of the modular group. In fact the quotient \( SL(2,\mathbb{Z})/\Gamma(2) \) equals the symmetric group \( S_3 = \{I, S, T, ST, TST, T^{-1} S\} \). \( \{M_j\} \) transform under the action of \( S_3 \) as \( \tau \to \tau + 1 : \)

\[ M_1 \to M_1, \quad (7.20) \]

\[ M_2 \to M_2, \quad (7.21) \]
\[ M_3 \to M_3, \quad M_4 \to -M_4, \quad (7.22) \]

\[ \tau \to -\frac{1}{\tau}: \]

\[ M_1 \to \frac{1}{2}(M_1 + M_2 + M_3 - M_4), \quad (7.24) \]

\[ M_2 \to \frac{1}{2}(M_1 + M_2 - M_3 + M_4), \quad (7.25) \]

\[ M_3 \to \frac{1}{2}(M_1 - M_2 + M_3 + M_4), \quad (7.26) \]

\[ M_4 \to \frac{1}{2}(-M_1 + M_2 + M_3 + M_4). \quad (7.27) \]

This is exactly the action of \( SO(8) \) triality transformation of the \( N_f = 4 \) theory proposed in Ref. 11). Triality has been postulated for the consistency of physical interpretation of four-dimensional gauge theory. Here it has been derived naturally from a six-dimensional setting.

### 7.2. Donagi-Witten curve

Recall that if one sets the four masses \( M_i \) of the \( N_f = 4 \) curve at

\[ M_1 = M_2 = \frac{M}{2}, \quad M_3 = M_4 = 0, \quad (7.28) \]

one obtains the Seiberg-Witten curve for the \( SU(2) \) theory with an adjoint matter\(^{11} \)

\[ \tilde{y}^2 = 4 \prod_{k=1}^{3} \left( \tilde{x} - e_k \tilde{u} - \frac{1}{4} e_k^2 M^2 \right). \quad (7.29) \]

This is in fact the curve discussed by Donagi and Witten.\(^{22} \) To obtain this curve directly from the six-dimensional curve, one may set

\[ m_2 = 0, \quad m_3 = m_4 = \pi, \quad m_5 = m_6 = \pi + \pi \tau, \quad m_7 = m_8 = \pi \tau \quad (7.30) \]

and transform the variables as

\[ u^2 = L^2(\tilde{u} - \varphi(m_1)), \quad (7.31) \]

\[ x = L^2 \left( \tilde{x} - \frac{1}{72} E_4 \right), \quad (7.32) \]

\[ y = L^3 \tilde{y} \quad (7.33) \]

with

\[ L = \frac{2i}{q^{1/2} \eta^{15}} \vartheta_1(m_j | \tau). \quad (7.34) \]

Then one obtains a curve

\[ \tilde{y}^2 = 4 \prod_{k=1}^{3} \left( \tilde{x} - e_k \tilde{u} - e_k^2 \right), \quad (7.35) \]

which agrees with (7.29) after a rescaling.
§8. Discussion

In this article we have constructed the six-dimensional Seiberg-Witten curve which amounts to an equation describing the complex structure of $\frac{1}{2}K_3$: by applying the mirror symmetry technique one can generate the number of holomorphic curves and gauge theory partition functions on $\frac{1}{2}K_3$ at any higher order. It will be interesting to see if a similar construction of SW curves is possible for manifolds other than the $\frac{1}{2}K_3$ surface.

We have also presented a new way of reduction to four dimensions without taking the degenerate limit of $T^2$ so that the $SL(2,\mathbb{Z})$ symmetry is left intact. By setting some of the parameters $\{m_i\}$ to special values we have obtained the four-dimensional Seiberg-Witten theory with $N_f = 4$ flavors and also the curve by Donagi and Witten describing a perturbed $N=4$ theory. In this reduction four-dimensional masses are expressed in terms of theta functions and possess exactly the proposed triality properties. Thus our curve seems to serve successfully as a geometrical way of deriving $SL(2,\mathbb{Z})$ symmetry from higher dimensions.

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