ZERO-POLE INTERPOLATION FOR MATRIX MEROMORPHIC
FUNCTIONS ON A COMPACT RIEMANN SURFACE AND A
MATRIX FAY TRISECANT IDENTITY

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ABSTRACT: This paper presents a new approach to constructing a meromorphic bundle map between flat vector bundles over a compact Riemann surface having a prescribed Weil divisor (i.e. having prescribed zeros and poles with directional as well as multiplicity information included in the vector case). This new formalism unifies the earlier approach of Ball-Clancey (in the setting of trivial bundles over an abstract Riemann surface) with an earlier approach of the authors (where the Riemann surface was assumed to be the normalizing Riemann surface for an algebraic curve embedded in $\mathbb{C}^2$ with determinantal representation, and the vector bundles were assumed to be presented as the kernels of linear matrix pencils). The main tool is a version of the Cauchy kernel appropriate for flat vector bundles over the Riemann surface. Our formula for the interpolating bundle map (in the special case of a single zero and a single pole) can be viewed as a generalization of the Fay trisecant identity from the usual line bundle case to the vector bundle case in terms of Cauchy kernels. In particular we obtain a new proof of the Fay trisecant identity.

1. Introduction

The following zero-pole interpolation problem is one of the main objects of study in the recent monograph [6]. We state here the simplest case where all zeros and poles are assumed to be simple and disjoint. Given a finite collection $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n_\infty}$ of distinct points in the complex plane $\mathbb{C}$, nonzero column vectors $u_1, \ldots, u_{n_\infty} \in \mathbb{C}^{r \times 1}$ and nonzero row vectors $x_1, \ldots, x_{n_0} \in \mathbb{C}^{1 \times r}$, find (if possible) an $r \times r$ matrix function $T(z)$ having value equal to the identity matrix $I$ at infinity such that (1) $T(z)$ is analytic on $(\mathbb{C} \cup \{\infty\}) \setminus \{\mu_1, \ldots, \mu_{n_\infty}\}$ and $T(z)^{-1}$ has analytic continuation to $(\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_1, \ldots, \lambda_{n_0}\}$, (2) for $i = 1, \ldots, n_0$, $T(\lambda_i)$ has rank $r - 1$ and $x_i T(\lambda_i) = 0$, and (3) for $j = 1, \ldots, n_{n_\infty}$, $T(\mu_j)^{-1}$ (i.e., the analytic continuation of $T(z)^{-1}$ to $z = \mu_j$) has rank $r - 1$ and $T(\mu_j)^{-1} u_j = 0$. A complete solution, along with numerous applications to problems in factorization, matrix interpolation and $H_\infty$-control, is given in [6]. The solution (for this simple
A solution exists if and only if the \( n_0 \times n_{\infty} \) matrix
\[
\Gamma = [\Gamma_{ij}] \text{ with } \Gamma_{ij} = \frac{x_i u_j}{\mu^j - \lambda^i}
\] is square and invertible. In this case the unique solution is given by
\[
T(z) = I + \sum_{j=1}^{n_{\infty}} u_j (z - \mu^j)^{-1} c_j
\] (1.2)
where \( c = [c_1 \ldots c_{n_{\infty}}]^T \) is given by
\[
c = \Gamma^{-1} \begin{bmatrix} x_1^T \ldots x_{n_0}^T \end{bmatrix}^T.
\] (1.3)

The solution in [6] uses system theory ideas, especially the state space similarity theorem specifying the level of uniqueness for two realizations of the same rational matrix function as the transfer function of a linear system. Later work (see [5]) handles the nonregular case (where \( \det T(z) \) vanishes identically and the nature of the zero structure must be enlarged) by elementary linear algebra, without recourse to the state space similarity theorem.

There have now appeared two seemingly distinct generalizations of this result to higher genus. In [3], the problem is posed to construct a (global, single-valued) meromorphic matrix function on the compact Riemann surface \( X \) satisfying conditions as in (1), (2) and (3) above. A matrix analogous to \( \Gamma \) appears, but the solution criterion is not as simple; nevertheless, an explicit formula analogous to (1.2) and (1.3) was found for the solution when it exists. The approach in [3] can be seen as an analogue of that in [6] (i.e., system theory ideas are avoided and a simple ansatz is used to reduce the problem to an analysis of a linear system of equations). The paper [7], on the other hand, while formulating a more general problem (involving bundle maps between certain types of flat vector bundles rather than global meromorphic matrix functions) in an abstract setting, works primarily in a more concrete setting, where the Riemann surface \( X \) is taken to be the normalizing Riemann surface for an algebraic curve \( C \) having a determinantal representation
\[
C = \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \det(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma) = 0 \}
\]
where \( \sigma_1, \sigma_2, \gamma \) are \( M \times M \) matrices and \( \lambda = (\lambda_1, \lambda_2) \) are affine coordinates), and the input and output bundles \( E \) and \( \tilde{E} \) are assumed to have kernel representations, e.g.,
\[
E(\lambda) = \{ v \in \mathbb{C}^M : (\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma)v = 0 \}.
\]
In this setting, a non-metric version of the several-variable system theory connected with the model theory for commuting operators due to Livsic (i.e., a version with all Hilbert space inner products dropped) applies, any meromorphic bundle map satisfying appropriate conditions at the points at infinity can be realized as the transfer function, or the joint characteristic function, of a Livsic-Kravitsky 2D system, and the zero-pole bundle-map interpolation problem can be solved using the state-space similarity theorem for this setting in a manner completely parallel to that of [6] for the genus 0 case. (For a recent systematic treatment of the Livsic theory, we refer to [17] and [28]). In this solution there is a matrix \( \Gamma \) analogous to the \( \Gamma \) in (1.1) along with an explicit formula for the solution (when it exists) as in (1.2). In this setting, one specifies an output bundle \( \tilde{E} \) having a kernel bundle representation as well as the zero-pole interpolation data. The invertibility of \( \Gamma \) is then equivalent
to the existence of an input bundle $E$ also having a kernel representation together with a bundle map $T: E \to \tilde{E}$ meeting the zero-pole interpolation conditions.

The purpose of this paper is to synthesize these two approaches. We obtain a generalization of the approach of [3] which handles the vector bundle problem, and clarify the solution criterion as well as the role of the invertibility of $\Gamma$ in this abstract setting. To obtain an analogue of the basic ansatz in [3] used for the form of the solution for the general bundle case, we need a version of the Cauchy kernel $(z, w) \mapsto \frac{1}{z-w}$ for sections of a flat vector bundle $\chi$ satisfying $h^0(\chi \otimes \Delta) = 0$ where $\Delta$ is a line bundle of differentials of order 1/2 (a theta characteristic or a spin structure). This object was introduced in [7] (see also [2], [27] for the line bundle case) but a proof of its existence for the vector bundle case relied on the theory of determinantal representations of algebraic curves and of kernel representations of bundles over such curves. Here we give a simple, direct existence proof using only some cohomology theory of vector bundles and the Riemann-Roch theorem. A similar proof of the same result for the line bundle case is given in [20] and [21]; the general case is also handled in [11] but by using completely different techniques (involving the theory of the Green’s function for the heat equation over $X$). Various other forms of the Cauchy kernel for a Riemann surface have appeared earlier in the literature, in particular in connection with the Riemann-Hilbert problem (see [22], [29]). However, these are developed within the framework of meromorphic differentials whereas our Cauchy kernel is defined as a multiplicative meromorphic differential of order 1/2. The use of half-order differentials has the advantage that no extraneous poles are introduced in the Cauchy kernel. We mention that the paper [1] applies our Cauchy kernel to the study of indefinite Hardy spaces on a finite bordered Riemann surface.

Taking the constant term in the Laurent expansion of the Cauchy kernel around the diagonal allows us to define a certain flat connection on the flat bundle $\chi$. In the concrete setting of the determinantal representations, this connection was already introduced in [3]. This flat connection is determined canonically up to a choice of a bundle $\Delta$ of half-order differentials.

We also make explicit the mappings between the concrete and abstract settings; in this way we are able to see explicitly the equivalence between the solution in [3] and the solution in [3]. The main ingredient is the explicit formula for the determinantal representation of an algebraic curve with a given kernel bundle in terms of the Cauchy kernel of the bundle.

We also specialize the results to the line bundle case. For this case, both the solution to the zero-pole interpolation problem and the Cauchy kernel can be expressed explicitly in terms of theta functions (see [4], [8] and [3] for background material on theta functions). When this is done, the equality between these two forms of the solution of the interpolation problem leads to a new proof of the trisecant identity due to Fay (see [10] Corollary 2.19 or [18] Volume II page 3.214). In the general vector bundle case, the formula for the solution of the interpolation problem in terms of the Cauchy kernels (in the case of a single zero and a single pole—see (3.13)) can be viewed as a matrix version of the Fay trisecant identity. We mention that in the genus 1 case one can obtain an explicit formula for the Cauchy kernel for the general case of flat vector bundles (see [3]).

We close the introduction by mentioning three other possible further applications of our Cauchy kernel.
First of all, as will be shown in Section 5, the vector bundle \( \chi \) is completely determined by the values \( K(\chi; x^i, x^j) \) of the Cauchy kernel at a certain finite collection of points \( x^1, \ldots, x^m \) (forming a line section in a birational planar embedding of \( X \)). This suggests that these values can be used as affine coordinates for the bundle \( \chi \) in the corresponding moduli space of semistable bundles on the complement of the generalized theta divisor. A very similar construction for line bundles on hyperelliptic curves is due to Jacobi (see [15]) and has been given a modern treatment by Mumford (see Volume II of [18]).

Secondly, in the line bundle case consideration of the Fay trisecant identity when some of the points come together leads to very interesting identities, showing in particular that the theta function satisfies the KP equations (see [10], Volume II of [18] and [24]). An interesting line of research is to consider similar limiting versions of our matrix Fay trisecant identity (3.13). A related problem is to find the relation between the Cauchy kernel and the matrix Baker-Akhiezer function of Krichever and Novikov [16] whose definition involves the so called Tjurin parameters [25, 26] of the vector bundle.

Thirdly, the absence of explicit formulas for the Cauchy kernel makes it interesting to try to find formulas for the Cauchy kernel of one vector bundle in terms of the Cauchy kernel for another. In particular it would be interesting to find how the Cauchy kernel behaves under pullback and direct image.

The paper is organized as follows. Section 1 is this introduction. Section 2 develops the Cauchy kernel for a flat line bundle. Section 3 then formulates and solves the zero-pole interpolation problem in the abstract setting. Section 4 obtains explicit formulas for all the results in the line bundle case and obtains the new proof of the Fay trisecant identity. Section 5 explains how to use Cauchy kernels to obtain a canonical map from the abstract to the concrete setting. Finally, Section 6 explains the connections with the concrete interpolation problem solved in [7].

2. The Cauchy kernel for a flat vector bundle

We assume that we are given a compact Riemann surface \( X \). Let \( \Delta \) be a line bundle of differentials of order \( \frac{1}{2} \) on \( X \), i.e., a line bundle satisfying \( \Delta \otimes \Delta \cong K \), where \( K \) is the canonical line bundle (i.e., the line bundle with local holomorphic sections equal to local holomorphic differentials on \( X \)). Note that since \( \deg(K) = 2g - 2 \), \( \deg(\Delta) = g - 1 \) where \( g \) is the genus of \( X \). In addition we assume that we are given a holomorphic complex vector bundle \( \chi \) of degree 0 (and of rank \( r \), say) over \( X \) such that

\[
h^0(\chi \otimes \Delta) = 0,
\]

i.e., \( \chi \otimes \Delta \) has no nonzero global holomorphic sections. The condition implies that \( \chi \) is necessarily semistable, and means that the (equivalence class of) \( \chi \) lies on the complement of the generalized theta divisor in the moduli space of semistable vector bundles of rank \( r \) and degree 0 on \( X \) (see [11], [23] and [8]). It also follows immediately from Weil’s criterion for flatness [13] that \( \chi \) is actually a flat vector bundle (see [9] page 275 for details). By definition, since \( \chi \) is flat, sections \( h \) of \( \chi \) have the property that they lift to \( \mathbb{C}^r \)-vector functions \( \tilde{h} \) defined on the universal cover \( \tilde{X} \) of \( X \) such that

\[
h(R\tilde{p}) = \chi(R)h(\tilde{p})
\]
for all \( \tilde{p} \in \tilde{X} \) where \( R \) is any element of the group of deck transformations \( \text{Deck}(\tilde{X}/X) \cong \pi_1(X) \) and where \( R \to \chi(R) \in GL(r) \) is a (constant) factor of automorphy associated with the bundle \( \chi \). (We somewhat abuse the notation denoting a constant factor of automorphy and the corresponding flat vector bundle by the same letter.)

The main object of this section is to define an object \( K(\chi; \cdot, \cdot) \) associated with any such bundle \( \chi \) which we shall call the Cauchy kernel for the bundle \( \chi \). Let \( M \) denote the Cartesian product \( M = X \times X \) and let \( \pi_1 \colon M \to X \) be the projection map onto the first coordinate and \( \pi_2 \colon M \to X \) the projection onto the second coordinate. The defining property of \( K(\chi; \cdot, \cdot) \) is that \( K(\chi; \cdot, \cdot) \) be a meromorphic mapping of the vector bundles \( \pi_2^*\chi \) and \( \pi_1^*\chi \otimes \pi_1^*\Delta \otimes \pi_2^*\Delta \) on \( M \) which is holomorphic outside of the diagonal \( D = \{(p, p) \in M : p \in X \} \), where it has a simple pole with residue \( I_r \). More precisely, the latter condition means the following: for any \( \tilde{p}_0 \in \tilde{X} \) and any local parameter \( t \) on \( X \), if we let \( \sqrt{dt} \) be the corresponding local holomorphic frame for \( \Delta \) lifted to a neighborhood of \( \tilde{p}_0 \) on \( \tilde{X} \), then near \( (\tilde{p}_0, \tilde{p}_0) \in \tilde{X} \times \tilde{X} \) the lift of \( K(\chi; \cdot, \cdot) \) to \( \tilde{X} \times \tilde{X} \) has the form

\[
\frac{K(\chi; \tilde{p}, \tilde{q})}{\sqrt{dt(\tilde{p})}\sqrt{dt(\tilde{q})}} = \frac{1}{t(\tilde{p}) - t(\tilde{q})} \left[ I_r + O \left( \sqrt{|t(\tilde{p})|^2 + |t(\tilde{q})|^2} \right) \right].
\]

Thus, if \( e \) is in the fiber of \( \chi \) at a point \( q \) on \( X \), and \( t \) is a local parameter of \( X \) centered at \( q \), then \( K(\chi; \cdot, q) \frac{t}{\sqrt{dt(q)}} \) is a meromorphic section of \( \chi \otimes \Delta \) that has a single simple pole at \( q \), with a residue (in terms of the local parameter \( t \)) equal to \( e\sqrt{dt(q)} \). Note that since \( \chi \otimes \Delta \) has no nontrivial global holomorphic sections, such a meromorphic section is unique whenever it exists. Note that when \( X \) is the Riemann sphere and \( \chi \) is (necessarily) trivial, then

\[
K(\chi; p, q) = \frac{I_r}{t(p) - t(q)} \sqrt{dt(p)\sqrt{dt(q)}},
\]

where \( t \) is the standard coordinate on the complex plane, i.e., we get the usual Cauchy kernel. Existence was shown in \( [3] \) by exhibiting an explicit formula for \( K(\chi; \cdot, \cdot) \); the construction involved using a representation of \( X \) as the normalized Riemann surface for an algebraic curve \( C \) embedded in \( \mathbb{P}^2 \) and representing the bundle \( E = \chi \otimes \Delta \otimes \mathcal{O}(1) \) as the kernel bundle associated with a determinantal representation of the curve \( C \)

\[
C = \{[\mu_0, \mu_1, \mu_2] \in \mathbb{P}^2 : \det(\mu_1\sigma_2 - \mu_2\sigma_1 + \mu_0\gamma) = 0 \}
\]

\[
E(\mu) = \ker(\mu_1\sigma_2 - \mu_2\sigma_1 + \mu_0\gamma).
\]

When the rank \( r \) of the vector bundle \( \chi \) is 1, one can get an explicit formula (in terms of the Abel-Jacobi map and classical theta functions on the Jacobian variety of \( X \)) for the Cauchy kernel (see \( [3] \)). Details of this formula will be reviewed in Section \( 3 \) of this paper, where other special aspects of the line bundle case will also be discussed.

Our purpose in this section is to give an alternative proof of the existence of such a Cauchy kernel \( K(\chi; \cdot, \cdot) \) for a flat vector bundle \( \chi \) by a direct, simple, more abstract argument (without relying on representing \( X \) as the normalizing Riemann surface for a curve \( C \) having a determinantal representation as in \( [3] \)). This is the content of the following theorem.

**Theorem 2.1.** Let \( \chi \) be a flat vector bundle over the Riemann surface \( X \) with \( h^0(\chi \otimes \Delta) = 0 \) as above. Then the Cauchy kernel \( K(\chi; \cdot, \cdot) \) exists, i.e., there is a
unique meromorphic mapping of the vector bundles \( \pi_2^*\chi \) and \( \pi_1^*\chi \otimes \pi_1^*\Delta \otimes \pi_2^*\Delta \) on \( M = X \times X \) which is holomorphic outside of the diagonal \( D = \{(p, p) \in M: p \in X\} \), where it has a simple pole with residue \( I_r \).

Proof. Note that we have the following exact sequence of vector bundles over \( M = X \times X \):

\[
0 \to O(-D) \to O \to O|_D \to 0.
\]

(2.1)

For ease of notation, define vector bundles \( F \) and \( W \) over \( M \) and \( V \) and \( K \) over \( X \) by

\[
F = \pi_1^*\chi \otimes \pi_1^*\Delta \otimes \pi_2^*\Delta
\]

\[
W = F \otimes O(D)
\]

\[
V = \chi \otimes \Delta
\]

\[
K = \text{the canonical line bundle on } X
\]

where \( \chi^\vee \) is the dual bundle of \( \chi \). Tensoring the exact sequence (2.1) with \( W \) gives us

\[
0 \to F \to W \to W \otimes O|_D \to 0.
\]

(2.2)

The map of taking the residue along the diagonal defines a linear mapping

\[
\mathcal{R}: H^0(M, W) \to H^0(X, End V).
\]

Our goal is to show that there exists a unique element \( K(\chi; \cdot, \cdot) \) of \( H^0(M, W) \) so that \( \mathcal{R}(K(\chi; \cdot, \cdot)) = I_V \).

Note that the bundle \( W \otimes O|_D \) can be identified with the bundle \( End V \) of endomorphisms of \( V \). Moreover the residue mapping \( \mathcal{R} \) is exactly the mapping from \( H^0(M, W) \) into \( H^0(X, End V) \) induced by the mapping \( W \to W \otimes O|_D \) in (2.2).

The vector bundle exact sequence (2.2) induces (see the Basic Fact on page 40 of [12]) the exact cohomology sequence

\[
0 \to H^0(M, F) \to H^0(M, W) \to H^0(X, End V) \to
\]

\[
H^1(M, F) \to \cdots.
\]

(2.3)

Next we argue that (i) \( h^0(F) = 0 \) and (ii) \( h^1(F) = 0 \). The statement (i) follows easily from our assumption that \( h^0(V) = 0 \). As for statement (ii) it follows from the Kunneth formulas (see page 58 of [12]) that

\[
H^1(M, F) = H^1(M, \pi_1^*(V) \otimes \pi_2^*(V^\vee \otimes K))
\]

\[
\cong (H^0(M, \pi_1^*(V)) \otimes H^1(M, \pi_2^*(V^\vee \otimes K)) \otimes (H^1(M, \pi_1^*(V)) \otimes H^0(M, \pi_2^*(V^\vee \otimes K))).
\]

(2.4)

(2.5)

By our assumption that \( h^0(V) = 0 \) it follows that the first term on the right hand side of (2.5) is 0. Since \( h^0(V) = 0 \) we also have \( h^0(V^\vee \otimes K) = 0 \) as well, by the Riemann-Roch Theorem for vector bundles on an algebraic curve (see [13]) and the assumption that \( \deg V = r(g - 1) \). Hence the second term in (2.5) is zero as well. This verifies the desired fact (ii).

Hence the exact sequence (2.3) collapses to

\[
0 \to H^0(M, W) \xrightarrow{\mathcal{R}} H^0(X, End V) \to 0.
\]

(2.6)
It follows that $\mathcal{R} : H^0(M, W) \to H^0(X, \text{End} V)$ is an isomorphism and the Theorem follows.

Let $\bar{p}_0$ be an arbitrary point of $\bar{X}$, and let $t$ be a local coordinate for $\bar{X}$ near $\bar{p}_0$. Then by definition the Cauchy kernel $K(\chi; \cdot, \cdot)$ is such that

$$(t(\bar{p}) - t(\bar{q})) \frac{K(\chi; \bar{p}, \bar{q})}{\sqrt{dt(\bar{p})} \sqrt{dt(\bar{q})}}$$

is analytic in $(\bar{p}, \bar{q})$ near $(\bar{p}_0, \bar{q}_0)$ with value at $(\bar{p}_0, \bar{q}_0)$ equal to $I_r$; hence, for $(\bar{p}, \bar{q})$ close to $(\bar{p}_0, \bar{q}_0)$, $K(\chi; \cdot, \cdot)$ has a representation of the form

$$K(\chi; \bar{p}, \bar{q})$$

for appropriate holomorphic matrix $K$-valued coefficients $A(\bar{p}_0)$ and $A_\ell(\bar{p}_0)$. An important point for us is the following result.

**Lemma 2.2.** Let $A_t$ and $A$ be the linear coefficients appearing in the Laurent expansion of the Cauchy kernel $K(\chi; \cdot, \cdot)$ as in (2.7). Then

$$A(\bar{p}_0) + A_\ell(\bar{p}_0) = 0$$

for all $\bar{p}_0 \in \bar{X}$.

**Proof.** Let $\bar{p}_0$ be an arbitrary point of $\bar{X}$ and let $t$ be a local coordinate for $\bar{X}$ near $\bar{p}_0$. For $\bar{p} \in X$ near $\bar{p}_0$, define

$$f(\bar{p}) = (t(\bar{p}) - t(\bar{q})) \cdot \frac{K(\chi; \bar{p}, \bar{q})}{\sqrt{dt(\bar{p})} \sqrt{dt(\bar{q})}},$$

From (2.7) we see that

$$f(\bar{p}) = I_r + \frac{A_t}{dt}(\bar{p})t(\bar{p}) + \frac{A}{dt}(\bar{p}_0)t(\bar{q}) + O((t(\bar{p}))^2 + (t(\bar{q}))^2).$$

In particular,

$$\frac{df}{dt}(\bar{p}) \bigg|_{\bar{p} = \bar{p}_0} = \frac{A_t}{dt}(\bar{p}_0) + \frac{A}{dt}(\bar{p}_0). \tag{2.8}$$

On the other hand, we can use $t'(\bar{p}') = t'(\bar{p}) - t(\bar{p})$ as a local coordinate for the variable $\bar{p}'$ near the point $\bar{p} \in X$. From (2.7) again we have

$$[t(\bar{p}') - t(\bar{p}) - t(\bar{q}') - t(\bar{p})] \frac{K(\chi; \bar{p}', \bar{q}')}{\sqrt{dt(\bar{p}') \sqrt{dt(\bar{q}')}}},$$

$$= I_r + \frac{A_t}{dt}(\bar{p})(t(\bar{p}) - t(\bar{p})) + \frac{A}{dt}(\bar{p})(t(\bar{q}') - t(\bar{p})) + O((t(\bar{p}') - t(\bar{p}))^2 + (t(\bar{q}') - t(\bar{p}))^2).$$

Evaluation of both sides of this equation at $\bar{p}' = \bar{q}' = \bar{p}$ yields $f(\bar{p}) = I_r$ from which we get

$$\frac{df}{dt}(\bar{p}) = 0 \tag{2.10}$$

for all $\bar{p} \in X$. Comparison of (2.8) and (2.10) now gives $A(\bar{p}_0) + A_\ell(\bar{p}_0) = 0$ as asserted. □
We can use these coefficients \( A(\tilde{p}) \) and \( A(\tilde{q}) \) defined by (2.7) to define connections \( \nabla_\chi \) on \( \chi \) and \( \nabla_\chi^* \) on \( \chi^\vee \) according to the formulas

\[
\nabla_\chi y = Ay + dy \\
\nabla_\chi^* x = A^T x + dx.
\]

(2.11)

for local holomorphic sections \( y \) of \( \chi \) and \( x \) of \( \chi^\vee \). The result \( A_\chi + A_\chi^* = 0 \) from Lemma 2.2 is equivalent to the fact that \( \nabla_\chi \) and \( \nabla_\chi^* \) are dual to each other, i.e.

\[
d(x^T y) = x^T(\nabla_\chi y) + (\nabla_\chi^* x)^T y
\]

for local holomorphic sections \( y \) of \( \chi \) and \( x \) of \( \chi^\vee \), where \((y, x) \to x^T y\) is the pairing between \( \chi \) and \( \chi^\vee \). Moreover, from the formula for \( \nabla_\chi \) we see that the connection matrix associated with \( \nabla_\chi \) is a matrix of holomorphic \((1, 0)\)-forms. Hence, \( \nabla_\chi \) is compatible with the complex structure of \( X \) and moreover, since we are in complex dimension 1, the connection \( \nabla_\chi \) is flat, i.e., \( \nabla_\chi \) has zero curvature (see Section 5 of Chapter 0 of [12] for all relevant definitions). The existence of such a flat connection on \( \chi \) in turn implies that \( \chi \) itself is a flat vector bundle (see [13] pages 294–295). In general there are many choices of distinct flat connections on a flat vector bundle; our construction via the Cauchy kernel provides a canonical choice of such a flat connection (up to a choice of a bundle \( \Delta \) of half-order differentials). An explicit formula for \( \nabla \) in the line bundle case is given in Section 4.

Remark: The proof of Theorem 2.1 used only the fact that \( \deg \chi = 0 \) and did not use the flatness of \( \chi \). Since the existence of a flat connection (compatible with the complex structure) implies that the bundle is flat, our construction gives a direct proof of the flatness of \( \chi \) independent of Weil’s theorem.

3. The abstract interpolation problem

In this section we consider as given two flat vector bundles \( \chi \) and \( \tilde{\chi} \) over the Riemann surface \( X \) for which both \( h^0(\chi \otimes \Delta) = 0 \) and \( h^0(\tilde{\chi} \otimes \Delta) = 0 \), where again, \( \Delta \) is a line bundle of half-order differentials over \( X \). We are interested in studying pole-zero interpolation conditions imposed on a bundle map of \( \chi \) to \( \tilde{\chi} \). The data for the interpolation problem is as follows. We assume that we are given \( n_\infty \) distinct points \( \mu^1, \ldots, \mu^{n_\infty} \) (the prescribed poles) together with \( n_0 \) distinct points \( \lambda^1, \ldots, \lambda^{n_0} \) (the prescribed zeros). For each fixed index \( j \) \((j = 1, \ldots, n_\infty)\) we specify a linearly independent set \( \{u_{j1}, \ldots, u_{js_j}\} \) of \( s_j \) vectors in the fiber \( \tilde{\chi}(\mu^j) \) of \( \tilde{\chi} \) over \( \mu^j \) (the prescribed pole vectors) and for each fixed index \( i \) \((i = 1, \ldots, n_0)\) we specify a linearly independent set \( \{x_{i1}, \ldots, x_{it_i}\} \) of \( t_i \) vectors in the fiber \( \chi^\vee(\lambda^i) \) of the dual bundle \( \chi^\vee \) of \( \chi \) (the prescribed null vectors). Also, for each pair of indices \( (i, j) \) for which \( \lambda^i = \mu^j \equiv \xi^0 \), we specify a collection \( \{\rho_{ij, \alpha, \beta} : 1 \leq \alpha \leq t_i, 1 \leq \beta \leq s_j\} \) of numbers that depend on the choice of the local parameter at the point \( \xi^0 \).

The Abstract Interpolation Problem (ABSINT) which we study in this section is the following: determine if there exists a bundle map \( T: \chi \to \tilde{\chi} \) with transpose \( T^\vee: \tilde{\chi} \to \chi^\vee \) such that:

(i) \( T \) has poles only at the points \( \{\mu^1, \ldots, \mu^{n_\infty}\} \); for each \( j = 1, \ldots, n_\infty \), the pole of \( T \) at \( \mu^j \) is simple, and the residue \( R_j = \text{Res}_{p=\mu^j} T: \chi(\mu^j) \to \tilde{\chi}(\mu^j) \) of \( T \) at \( \mu^j \) is such that \( \{u_{j1}, \ldots, u_{j{s_j}}\} \) spans the image space \( \text{im} R_j \) of \( R_j \).

(ii) The bundle map \( (T^\vee)^{-1}: \chi^\vee \to \tilde{\chi}^\vee \) has poles only at \( \{\lambda^1, \ldots, \lambda^{n_0}\} \); for each \( i = 1, \ldots, n_0 \), the pole of \( (T^\vee)^{-1} \) at \( \lambda^i \) is simple and the residue \( \tilde{R}_i = \)}
where \( u \) and \( r \chi \) and natively as

\[ T'(p)x_{\alpha}(p) \text{ has analytic continuation to } p = \xi^{ij} \text{ with value at } p = \xi^{ij} \text{ equal to } 0. \]

Thus the interpolation condition in part (iii) of (ABSINT) can be expressed alternatively as

\[ \text{Res}_{p=\lambda^i}(T'(p)^{-1}) : \chi'(\lambda^i) \to \chi'(\lambda^i) \text{ of } (T'(p)^{-1} \text{ at } \lambda^i \text{ is such that } \{x_{i1}, \ldots, x_{i,t_i}\} \text{ spans the image space } \im \hat{R}_i \text{ of } \hat{R}_i. \]

(iii) For each pair of indices \((i, j)\) for which \( \lambda^i = \mu^j =: \xi^{ij} \) and for \( \alpha = 1, \ldots, t_i \), let \( x_{\alpha}(p) \) be a local holomorphic section of \( \chi' \) with \( x_{\alpha}(\xi^{ij}) = x_{\alpha} \) such that

\[ (t^{ij}(p)x_{\alpha}(p)) \text{ has analytic continuation to } p = \xi^{ij} \text{ with value at } p = \xi^{ij} \text{ equal to } 0. \]

Then

\[ (\nabla^\omega x_{\alpha}(\xi^{ij}))^T u_{ij} = \rho_{ij,\alpha} \]

for \( \beta = 1, \ldots, s_j \).

When such a bundle map \( T \) exists, give an explicit formula for the construction of \( T \).

In order for solutions to exist, the compatibility condition

\[ x_{\alpha} u_{ij} = 0 \text{ whenever } \lambda^i = \mu^j. \quad (3.1) \]

must hold. This follows from the requirement that the meromorphic section

\[ x_{\alpha}(p)T(p) \]

be analytic at the point \( p = \xi^{ij} := \lambda^i = \mu^j \). Hence we shall always assume that our data collection

\[ \omega = \{ (x_{\alpha}, \lambda^i), (u_{ij}, \mu^j), \rho_{ij,\alpha} \} \quad (3.2) \]

also satisfies this compatibility condition.

It will be convenient to work with an alternate form of the interpolation condition (iii) in (ABSINT). Suppose that \( u(p) = T(p)\phi(p) \), where \( \phi \) is a local holomorphic section of \( \chi \) near \( \xi^{ij} \) chosen so that \( \text{Res}_{p=\xi^{ij}} u(p) = u_{ij} \) with respect to the local coordinate \( t^{ij} \) centered at \( \xi^{ij} \). Then

\[ (x_{\alpha}(p))^T (t^{ij}(p)u(p)) = t^{ij}(p) \cdot (x_{\alpha}(p))^T T(p)\phi(p) \]

\[ = (t^{ij}(p)T'(p)x_{\alpha}(p))^T \phi(p) \]

has a double order zero at \( p = \xi^{ij} \), and hence

\[ d \left( x_{\alpha}(p)^T (t^{ij}(p)u(p)) \right)_{p=\xi^{ij}} = 0. \]

Since \( \nabla^\omega x_{\alpha} \) and \( \nabla^\omega x_{\alpha} \) are dual connections as a consequence of Lemma 2.2, we therefore have

\[ \left( \nabla^\omega x_{\alpha}(\xi^{ij}) \right)^T u_{ij} + x_{\alpha} T^{ij}(p)u(p) \big|_{p=\xi^{ij}} = 0. \]

Thus the interpolation condition in part (iii) of (ABSINT) can be expressed alternatively as

\[ x_{\alpha} T^{ij}(p)u(p) \big|_{p=\xi^{ij}} = -\rho_{ij,\alpha} \]

where \( u(p) = T(p)\phi(p) \) for a local holomorphic section \( \phi \) of \( \chi \) near \( \xi^{ij} \) such that \( \text{Res}_{p=\xi^{ij}} u(p) = u_{ij} \).

In the scalar case \( (r = 1) \), the compatibility condition (3.1) can never be satisfied in a nontrivial way and hence the third set of interpolation conditions is absent under our assumptions; this corresponds to the fact that a scalar meromorphic function cannot have a zero and a pole at the same point \( \xi^{ij} \). Moreover, for the case \( r = 1, \) necessarily \( t_i = 1 \) for all \( i \) and \( s_j = 1 \) for all \( j \). In the case where both \( \chi \) and \( \chi' \) are trivial (or more generally if we use coordinates with respect to a local holomorphic frame for \( \chi' \), near \( \mu^j \) or \( \chi' \) near \( \lambda^i \)), there is no loss of generality in taking \( x_i := x_{i1} = 1 \) and \( u_j := u_{j1} = 1 \) for all \( i \) and \( j \). Thus the only remaining
relevant data are the zeros $\lambda^1, \ldots, \lambda^m$ and the poles $\mu^1, \ldots, \mu^n$ (all assumed here to be distinct). As is standard in algebraic geometry, the formal sum

$$\lambda - \mu := \lambda^1 + \cdots + \lambda^m - \mu^1 - \cdots - \mu^n$$

is said to be a divisor on $X$. If $f$ is a meromorphic function, the associated principal divisor $(f)$ is defined to be the formal sum $p^1 + \cdots + p^m - q^1 - \cdots - q^n$ where the $p^i$’s are the zeros of $f$ and the $q^j$’s are the poles of $f$ (with repetitions according to respective multiplicities). Associated with any divisor $\lambda - \mu$ as above is the vector bundle $\mathcal{O}(\mu - \lambda)$ whose holomorphic sections can be identified with global meromorphic functions $h$ such that

$$(h) \geq \lambda - \mu,$$

i.e., such that the zeros of $h$ include the points $\lambda^1, \ldots, \lambda^m$ (all with multiplicity at least 1) and the poles of $h$ are a subset of $\mu^1, \ldots, \mu^n$ (all with multiplicity at most 1).

It is convenient for us to introduce matrix analogues of these ideas. Let $\omega$ be an interpolation data set as in (3.2). Let us introduce the notation

$$(\mu, u) = \{(\mu^j, u_{ij}) : 1 \leq j \leq n, 1 \leq \beta \leq s_j\} \quad (3.6)$$

for the pole part of $\omega$. We let $\mathcal{M}(\chi \otimes \Delta)$ be the sheaf of meromorphic sections of $\chi \otimes \Delta$. For $U$ an open subset of $X$ we define

$$\mathcal{O}(\chi \otimes \Delta)(\mu, u)(U) = \{u \in \mathcal{M}(\chi \otimes \Delta)(U) : u \text{ has poles only at } \mu^j, u_{-1} := \text{Res}_{p=\mu^j} \in \text{span} \{u_{ij} : 1 \leq \beta \leq s_j\}\}$$

and

$$\mathcal{O}(\chi \otimes \Delta)(\omega)(U) = \{u \in \mathcal{O}(\chi \otimes \Delta)(\mu, u)(U) : \text{if } \lambda^i \neq \mu^j, \text{ then } x_{i\alpha}u(\lambda^i) = 0; \text{ if } \lambda^i = \mu^j =: \xi^j, \text{ there is a local holomorphic section } x_{i\alpha}(p) \text{ of } \bar{\chi}^\vee \}$$

such that (i) $x_{i\alpha}(\xi^j) = x_{i\alpha}$, (ii) $x_{i\alpha}(p) \frac{u}{\sqrt{dt^j}}(p)$ has analytic continuation to $p = \xi^j$ with value 0 there, and

$$\text{if } \text{Res}_{p=\mu^j} u = \sum_{\beta=1}^{s_j} u_{j\beta} \xi^j.$$

It is obvious that $\mathcal{O}(\chi \otimes \Delta)(\mu, u)$ and $\mathcal{O}(\chi \otimes \Delta)(\omega)$ are locally free sheaves of rank $r$, and we denote the corresponding rank $r$ vector bundles by $(\chi \otimes \Delta)(\mu, u)$ and $(\chi \otimes \Delta)(\omega)$. It is also obvious that $T$ is a solution of (ABSINT) if and only if $T$ is an isomorphism from $\chi \otimes \Delta$ to $(\chi \otimes \Delta)(\omega)$.

The solution of the zero-pole interpolation problem introduced at the beginning of this section is as follows.

**Theorem 3.1.** Define an $n_0 \times n_\infty$ block matrix $\Gamma = [\Gamma_{ij}]$ $(1 \leq i \leq n_0$, $1 \leq j \leq n_\infty)$ where the block entry $\Gamma_{ij}$ in turn is a $t_i \times s_j$ matrix $\Gamma_{ij} = [\Gamma_{ij,\alpha\beta}]$ $(1 \leq \alpha \leq t_i$, $1 \leq \beta \leq s_j)$ with matrix entries $\Gamma_{ij,\alpha\beta}$ given by

$$\Gamma_{ij,\alpha\beta} = \begin{cases} -x_{i\alpha}^T K(\chi; \lambda^i, \mu^j)u_{j\beta}, & \text{if } \lambda^i \neq \mu^j; \\ -\rho_{ij,\alpha\beta} & \text{if } \lambda^i = \mu^j. \end{cases} \quad (3.7)$$
In addition we introduce the block matrices
\[ u_i = [u_{i1} \ldots u_{in}], \quad x_j = [x_{j1} \ldots x_{j1}], \]
\[ K_{\mu, u}(p) = [K(\tilde{x}; p, \mu)] u_1 \ldots K(\tilde{x}; p, \mu)u_n, \]
(3.8)
\[ K_{\mu, \lambda}(q) = \begin{bmatrix} x^T K(\tilde{x}; \lambda, q) \\ \vdots \\ x^T K(\tilde{x}; \lambda, q) \end{bmatrix}. \]
(3.9)

Let \( q \) be a point of \( X \) disjoint from all the interpolation nodes \( \lambda^1, \ldots, \lambda^{n_0}, \mu^1, \ldots, \mu^{n_\infty} \) and let \( Q \) be an invertible linear map of the fiber space \( \chi(q) \) to the fiber space \( \tilde{\chi}(Q) \). Then the abstract interpolation problem (ABSINT) has a solution \( T \) with value \( Q \) at the point \( q \) if and only if the matrix \( \Gamma \) is square and invertible and
\[ [K(\tilde{x}; p^i, q) + K_{\mu, u}(p^i)\Gamma^{-1}K_{\mu, \lambda}(q)]Q(\text{Res}_{q^i}, K(q; q)^{-1}) = 0 \]
(3.10)
at each pole \( p^i \) of \( K(q; q)^{-1} \). In this case the unique solution \( T \) of the interpolation problem (ABSINT) with value \( Q \) at \( q \) is given by
\[ T(p) = [K(\tilde{x}; p, q) + K_{\mu, u}(p)\Gamma^{-1}K_{\mu, \lambda}(q)]QK(p; q)^{-1} \]
(3.11)
with inverse given by
\[ T^{-1}(p) = K(p; q)^{-1}T^{-1}(q)[K(\tilde{x}; q, p) + K_{\mu, u}(q)\Gamma^{-1}K_{\mu, \lambda}(p)]. \]
(3.12)

Two special cases of formula (3.11) deserve to be mentioned. The first is the case where \( n_0 = n_\infty = 1 \), \( t_1 = s_1 = 1 \) and \( \lambda^1 \neq \mu^1 \). If we set \( x = x_1 \), \( \lambda = \lambda^1 \), \( \mu = \mu^1 \) and \( u = u_1 \), then \( \Gamma = -xK(\tilde{x}; \lambda, \mu)u \) is just a number and the formula (3.11) becomes
\[ T(p)K(\tilde{x}; p, q)T(q)^{-1} = \frac{K(\tilde{x}; p, q) - K(\tilde{x}; p, \mu)ux^TK(\tilde{x}; \lambda, q)}{x^TK(\tilde{x}; \lambda, \mu)u}. \]
(3.13)
In the line bundle case, the identity (3.13) reduces to the Fay trisecant identity and will be discussed in Section 4. The second special case of interest is the case where the given zero and pole vectors at each interpolation node span the whole fiber space. In this case there is an explicit multiplicative formula for the interpolant \( T \) in terms of the prime form \( E(p, q) \); this will be discussed in detail at the end of Section 4.

A second version of the abstract interpolation problem (ABSINT) has the same form (i), (ii) and (ii) as (ABSINT), but with the input bundle \( \chi \) left also as an unknown to be found, subject to the proviso that it also be flat and have \( h^0(\chi \otimes \Delta) = 0 \). This version of the problem was studied in in a more concrete setting where \( X \) is the normalizing Riemann surface for an algebraic curve \( C \) embedded in \( \mathbb{P}^2 \) having a maximal rank \( r \) determinantal representation; we will discuss the connections of this setup with ours in Sections 5 and 6. At this time we also state the solution to the modified (ABSINT).

**Theorem 3.2.** Let \( \omega \) be a data set for (ABSINT) as above and form the matrix \( \Gamma \) as in (3.3). Then there exists a flat bundle \( \chi \) with \( h^0(\chi \otimes \Delta) = 0 \) and a meromorphic bundle map \( T : \chi \rightarrow \tilde{\chi} \) satisfying the interpolation conditions (i), (ii) and (iii) of (ABSINT) if and only if \( \Gamma \) is square and invertible.
To prove Theorem 3.1 we need some preliminary lemmas.

**Lemma 3.3.** A global meromorphic section of \( \overline{X} \otimes \Delta \) is in \( \mathcal{O}(\overline{X} \otimes \Delta)(\mu, u)(X) \) if and only if \( h \) has the form
\[
h = \sum_{j=1}^{n} \sum_{\beta=1}^{s_j} K(\overline{X}; p, \mu^j) u_{j\beta} c_{j\beta}
\]
for some scalars \( c_{j\beta} \).

**Proof.** Suppose \( h \in \mathcal{O}(\overline{X} \otimes \Delta)(\mu, u)(X) \). Choose scalars \( c_{j\beta} \) so that
\[
\text{Res}_{p=\mu} h(p) = \sum_{\beta=1}^{s_j} u_{j\beta} c_{j\beta}
\]
and set
\[
\hat{h}(p) = \sum_{j=1}^{n} \sum_{\beta=1}^{s_j} K(\overline{X}; p, \mu^j) u_{j\beta} c_{j\beta}.
\]
Then \( \hat{h} \in \mathcal{M}(\overline{X} \otimes \Delta)(X) \) and \( h - \hat{h} \in \mathcal{O}(\overline{X} \otimes \Delta)(X) \). Thus \( h = \hat{h} \) since \( \mathcal{O}(\overline{X} \otimes \Delta)(X) = H^0(X, \overline{X} \otimes \Delta) = 0 \) by our standing assumptions on \( \overline{X} \). The converse direction follows easily from the defining properties of the Cauchy kernel \( K(\overline{X}; \cdot, \cdot) \).

**Lemma 3.4.** The map
\[
[[c_{j\beta}|1 \leq \beta \leq s_j]|_{1 \leq j \leq n_\infty} \to h(p) = \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{t_j} K(\overline{X}; p, \mu^j) u_{j\beta} c_{j\beta}
\]
establishes a one-to-one correspondence between \( \ker \Gamma \) and \( \mathcal{O}(\overline{X} \otimes \Delta)(\omega)(X) \). In particular
\[
\dim \ker \Gamma = h^0(\overline{X} \otimes \Delta)(\omega)).
\]

**Proof.** Suppose first that \( h \in \mathcal{O}(\overline{X} \otimes \Delta)(\omega)(X) \). In particular \( h \in \mathcal{O}(\overline{X} \otimes \Delta)(\mu, u)(X) \), so by Lemma 3.3 there exists a collection of complex numbers \( \{c_{j\beta}\}_{1 \leq j \leq n_\infty, 1 \leq \beta \leq s_j} \) so that
\[
h(p) = \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{t_j} K(\overline{X}; p, \mu^j) u_{j\beta} c_{j\beta}.
\]
We next see what conditions the other requirements on \( h \) for admission to the class \( \mathcal{O}(\overline{X} \otimes \Delta)(\omega)(X) \) impose on the scalars \( c_{j\beta} \).

If \( i \) is any index for which \( \lambda^i \neq \mu^j \) for all \( j \), we must have
\[
0 = x_{i\alpha} h(\lambda^i) = \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{t_j} x_{i\alpha} K(\overline{X}; \lambda^i, \mu^j) u_{j\beta} c_{j\beta} = -\sum_{j=1}^{n_\infty} \sum_{\beta=1}^{t_j} \Gamma_{ij, \alpha, \beta} c_{j\beta}. \tag{3.14}
\]
If, on the other hand, \( i \) is an index such that \( \lambda^i = \mu^j =: \xi^j \) for some index \( j \), then we write the Laurent series expansion for \( \frac{h}{\sqrt{|dt^j|}} \) near \( \xi^j \) (where \( t^{ij}(p) \) is a local coordinate for \( X \) centered at \( \xi^j \))
\[
\frac{h}{\sqrt{|dt^j|}} = \left[ \frac{h}{\sqrt{|dt^j|}} \right]_{-1} \frac{1}{t^j} + \left[ \frac{h}{\sqrt{|dt^j|}} \right]_0 + O(|t^j(p)|).
\]
We compute
\[
\left[ \frac{h}{\sqrt{dt^{ij}}} \right]_{-1} = \text{Res}_{p=\xi^{ij}} \frac{h}{\sqrt{dt^{ij}}}(p) = \sum_{\beta=1}^{s_j} u_{j\beta} \sqrt{dt^{ij}}(\xi^{ij}) c_{j\beta}
\]
and hence
\[
\text{Res}_{p=\xi^{ij}} \frac{h}{\sqrt{dt^{ij}}}(p) = \sum_{\beta=1}^{s_j} u_{j\beta} c_{j\beta}
\]
where \( c_{j\beta} = \sqrt{dt^{ij}}(\xi^{ij}) c_{j\beta} \). Moreover, we have
\[
\left[ \frac{h}{\sqrt{dt^{ij}}} \right]_0 = \sum \sum_{k \neq j}^{s_k} K(\chi; \xi^{ij}, \mu^k) u_{k\beta} c_{k\beta}
\]
\[+ \sum_{\beta=1}^{s_j} A_t(\xi^{ij}) \sqrt{dt^{ij}}(\xi^{ij}) u_{j\beta} c_{j\beta}.
\]

By Lemma 2.2, the first term in the braces vanishes. Combining this fact with the formula (3.16) for the coefficients \( c_{j\beta} \), we see that the interpolation condition
\[
x_{\alpha}^T(A(\xi^{ij})) \left[ \frac{h}{\sqrt{dt^{ij}}} \right]_{-1} + \left[ \frac{h}{\sqrt{dt^{ij}}} \right]_0 dt^{ij}(\xi^{ij}) = \sum_{\beta=1}^{s_j} x_{\alpha}^T A(\xi^{ij}) u_{j\beta} \sqrt{dt^{ij}}(\xi^{ij}) c_{j\beta}
\]
\[+ \sum_{k \neq j}^{s_k} x_{\alpha}^T K(\chi; \xi^{ij}, \mu^k) u_{k\beta} c_{k\beta} dt^{ij}(\xi^{ij}) + \sum_{\beta=1}^{s_j} x_{\alpha}^T A_t(\xi^{ij}) \sqrt{dt^{ij}}(\xi^{ij}) u_{j\beta} c_{j\beta} dt^{ij}(\xi^{ij})
\]
\[= \left\{ \sum_{\beta=1}^{s_j} x_{\alpha}^T(A(\xi^{ij}) + A_t(\xi^{ij})) u_{j\beta} c_{j\beta} + \sum_{k \neq j}^{s_k} \sum_{\beta=1}^{s_j} x_{\alpha}^T K(\chi; \xi^{ij}, \mu^k) u_{k\beta} c_{k\beta} \right\} \sqrt{dt^{ij}}(\xi^{ij}).
\]

By Lemma 2.2, the first term in the braces vanishes. Combining this fact with the formula (3.16) for the coefficients \( c_{j\beta} \), we see that the interpolation condition
\[
x_{\alpha}^T(A(\xi^{ij})) \left[ \frac{h}{\sqrt{dt^{ij}}} \right]_{-1} + \left[ \frac{h}{\sqrt{dt^{ij}}} \right]_0 dt^{ij}(\xi^{ij}) = \sum_{\beta=1}^{s_j} x_{\alpha}^T A(\xi^{ij}) u_{j\beta} \sqrt{dt^{ij}}(\xi^{ij}) c_{j\beta}
\]
becomes
\[
\left\{ \sum_{k \neq j}^{s_k} x_{\alpha}^T K(\chi; \xi^{ij}, \mu^k) u_{k\beta} c_{k\beta} \right\} \sqrt{dt^{ij}}(\xi^{ij}) = \sum_{\beta=1}^{s_j} \rho_{ij, \alpha \beta} c_{j\beta} \sqrt{dt^{ij}}(\xi^{ij}).
\]
After canceling off \( \sqrt{dt^{ij}}(\xi^{ij}) \) and recalling that \( \xi^{ij} = \lambda^j \) we see that this can be rewritten as
\[
\sum_{j=1}^{s_j} \sum_{\beta=1}^{s_j} \rho_{ij, \alpha \beta} c_{j\beta} = 0
\]
and this equation holds for all pairs of indices \((i, \alpha)\) such that \( \lambda^i = \mu^j \) for some \( j \).

Combining (3.18) and (3.14) we see that the column vector \([c_{j\beta}]_{1 \leq \beta \leq s_j} | 1 \leq j \leq n_{\infty}\) is in \( \ker \Gamma \) as claimed.
Conversely, if \([c_{j\beta}]_{1 \leq \beta \leq s_j} | 1 \leq j \leq n_{\infty}\) is in \( \ker \Gamma \) and we set
\[
h(p) = \sum_{j=1}^{n_{\infty}} \sum_{\beta=1}^{s_j} K(\chi; p, \mu^j) c_{j\beta},
\]
then one can verify that $h \in \mathcal{O}(\tilde{\chi} \otimes \Delta)(\omega)(X)$ by reversing the steps of the above argument.

The analogue of Lemma 3.3 at the level of bundle endomorphisms is the following.

Lemma 3.5. Suppose that $T$ is a holomorphic bundle map from $\chi \otimes \Delta$ to $(\tilde{\chi} \otimes \Delta)(\mu, u)$ such that $T(q) = Q: \chi(q) \to \tilde{\chi}(q)$. Then there exists a unique choice of operators $\tilde{x}_{j\beta} : \tilde{\chi}(q) \to \mathbb{C}$ such that

$$T(p) = \left[ K(\tilde{\chi}; p, q) + \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{s_j} K(\tilde{\chi}; p, \mu^j) u_{j\beta} \tilde{x}_{j\beta} \right] QK(\chi; p, q)^{-1}.$$  

Proof. Choose operators $\tilde{x}_{j\beta} : \tilde{\chi}(q) \to \mathbb{C}$ so that

$$\text{Res}_{\mu^j} T(\cdot) K(\chi; \mu^j, q) Q^{-1} = \sum_{\beta} u_{j\beta} \tilde{x}_{j\beta}.$$  

Set

$$\tilde{T}(p) = \left[ K(\tilde{\chi}; p, q) + \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{s_j} K(\tilde{\chi}; p, \mu^j) u_{j\beta} \tilde{x}_{j\beta} \right] Q K(\chi; p, q)^{-1}.$$  

Then, for any vector $v \in \chi(q)$ we have

$$T(\cdot)K(\chi; \cdot, q)v - \tilde{T}(\cdot)K(\chi; \cdot, q)v = T(\cdot)K(\chi; \cdot, q)v - \left[ K(\tilde{\chi}; \cdot, q)Q + \sum_{j=1}^{n_\infty} \sum_{\beta=1}^{s_j} K(\tilde{\chi}; \cdot, \mu^j) u_{j\beta} \tilde{x}_{j\beta} Q \right] v$$

(3.19)

is an element of $\mathcal{O}(\tilde{\chi} \otimes \Delta)(X)$. By our standing assumption that $h^0(\tilde{\chi} \otimes \Delta) = 0$, we conclude that $(T - \tilde{T})(\cdot)K(\chi; \cdot, q)v = 0$ for all $v \in \chi(q)$. This is enough to force $T = \tilde{T}$, and the lemma follows.

Proof of Theorem 3.4. We first argue that necessarily $\Gamma$ is square and invertible if a solution $T$ to the interpolation problem (ABSINT) exists. To do this, we show first that $\ker \Gamma = \{0\}$ and secondly, that $\Gamma$ is square.

To see that $\ker \Gamma = \{0\}$, we proceed as follows. If $T$ is a solution of (ABSINT), then multiplication by $T$ induces a biholomorphic bundle map between $\chi \otimes \Delta$ and $(\tilde{\chi} \otimes \Delta)(\omega)$. By assumption, $h^0(\chi \otimes \Delta) = 0$. Hence we also have $h^0((\tilde{\chi} \otimes \Delta)(\omega)) = 0$. Now it follows from Lemma 3.4 that $\ker \Gamma = \{0\}$.

Next we argue that $\Gamma$ is square. Since $\chi$ and $\tilde{\chi}$ by assumption are both flat, both $\chi$ and $\tilde{\chi}$ have degree 0, as do $\det \chi$ and $\det \tilde{\chi}$. Then

$$\deg(\det T) = \deg(\det \tilde{\chi}) - \deg(\det \chi) = 0.$$  

If $T$ is a solution of (ABSINT), then the total number of zeros $n_0(T)$ of $T$ (counted with multiplicities as appropriate for meromorphic matrix functions—see Chapter 3 of [3]) is equal to $\sum_{i=1}^{n_0} t_i =: N_0$ which is the number of rows of $\Gamma$, while the total number of poles $n_\infty(T)$ (again counted with multiplicities) is equal to $\sum_{j=1}^{n_\infty} s_j =: N_\infty$ which is equal to the number of columns of $\Gamma$. In general we have $\deg(\det T) = n_0(T) - n_\infty(T)$. Hence the equality $\deg(\det T) = 0$ for $T$ a solution of (ABSINT) implies that $\Gamma$ is invertible as well. Combining this with the result of the previous paragraph, we see that $\Gamma$ is invertible as well.
If \( T \) is a solution of \( \text{ABSINT} \), then in particular \( T \) satisfies the hypotheses of Lemma 3.5 and hence \( T \) has the form

\[
T(p) = K(\chi; p, q) + \sum_{j=1}^{n_{\chi}} \sum_{\beta=1}^{s_j} K(\chi; p, \mu_j) u_{j\beta} \tilde{x}_{j\beta}
\]

for appropriate operators \( \tilde{x}_{j\beta} : \tilde{X}(q) \to \mathbb{C} \). We now find what additional restrictions on \( \tilde{x}_{j\beta} \) are forced by the zero and coupled zero-pole interpolation conditions (ii) and (iii) in \( \text{ABSINT} \).

Suppose that \( i \) is an index for which \( \lambda^i \neq \mu^j \) for any \( j \). Then the zero interpolation condition \( x_{\alpha \beta}^T T(\lambda^i) = 0 \) for all \( \alpha \) between 1 and \( t_i \),

\[
x_{\alpha \beta}^T K(\chi; \lambda^i, q) Q K(\chi; \lambda^i, q)^{-1} + \sum_{j=1}^{n_{\chi}} \sum_{\beta=1}^{s_j} x_{\alpha \beta}^T K(\chi; \lambda^i, \mu_j) u_{j\beta} \tilde{x}_{j\beta} Q K(\chi; \lambda^i, q)^{-1} = 0.
\]

Recalling the definition of \( \Gamma_{ij,\alpha\beta} \), we can rewrite this as

\[
\sum_{j=1}^{n_{\chi}} \sum_{\beta=1}^{s_j} \Gamma_{ij,\alpha\beta} \tilde{x}_{j\beta} = x_{\alpha \beta} K(\chi, \lambda^i, q)
\]

for all index pairs \((i, \alpha)\) such that \( \lambda^i \neq \mu^j \) for any \( j \).

We next consider an index \( i \) for which \( \lambda^i = \mu^j \) for some \( j \). Let \( x_{\alpha \beta}(p) \) be a local holomorphic section of \( \chi \) as in the third set of interpolation conditions. Let \( \varphi(p) \) be the meromorphic local section of \( \chi \) given by

\[
\varphi(p) = \frac{K(\chi; p, q)}{\sqrt{dt^{ij}(p)}} e
\]

for a vector \( e \in \chi(q) \) where \( t^{ij}(p) \) is a local coordinate on \( X \) centered at \( \xi^{ij} \), and let \( u(p) \) be the local meromorphic section of \( \tilde{\chi} \) given by \( u(p) = T(p) \varphi(p) \). From (3.20) we have then

\[
u(p) = \frac{K(\chi; p, q)}{\sqrt{dt^{ij}(p)}} Q e + \sum_{j=1}^{n_{\chi}} \sum_{\beta=1}^{s_j} \frac{K(\chi; p, \mu_j)}{\sqrt{dt^{ij}(p)}} u_{j\beta} \tilde{x}_{j\beta} Q e.
\]

Then the coefficients \([u]_{-1}\) and \([u]_{0}\) in the Laurent expansion of \( u(p) \) centered at \( \xi^{ij} \) with respect to local coordinate \( t^{ij} \) are given by

\[
[u]_{-1} = \sum_{\beta=1}^{s_j} u_{j\beta} \sqrt{dt^{ij}(\xi^{ij})} \tilde{x}_{j\beta} Q e,
\]

\[
[u]_{0} = \frac{K(\chi; \xi^{ij}, q)}{\sqrt{dt^{ij}(\xi^{ij})}} Q e + \sum_{k \neq j} \sum_{\beta=1}^{s_k} \frac{K(\chi; \xi^{ij}, \mu_k)}{\sqrt{dt^{ij}(\xi^{ij})}} u_{k\beta} \tilde{x}_{k\beta} Q e + \sum_{\beta=1}^{t_{ij}} A_\beta(\xi^{ij}) \sqrt{dt(\xi^{ij})} u_{j\beta} \tilde{x}_{j\beta} Q e
\]

so the alternate coupled interpolation condition (iii) given by (3.5) implies that

\[
x_{\alpha \beta}^T (A(\xi^{ij}) [u]_{-1} + [u]_{0} dt^{ij}(\xi^{ij})) = \sum_{\beta=1}^{s_j} \rho_{ij,\alpha\beta} \tilde{x}_{j\beta} Q e \cdot \sqrt{dt^{ij}(\xi^{ij})}.
\]
Substitution of the expressions (3.22) and (3.23) for \([u]_{-1}\) and \([u]_0\) gives
\[
\sum_{\beta=1}^{s_j} x_{ia}^T A(\xi_{ij}) u_{j\beta} \hat{x}_{j\beta} Q e \sqrt{dt_{ij}(\xi_{ij})} + x_{ia}^T K(\hat{x}; \xi_{ij}, q) Q e dt_{ij}(\xi_{ij})
\]
\[+ \sum_{k \neq j, \beta=1}^{s_k} x_{ia}^T K(\hat{x}; \xi_{ij}, \mu_{k}) u_{k\beta} \hat{x}_{k\beta} Q e \cdot dt_{ij}(\xi_{ij}) + \sum_{\beta=1}^{s_j} x_{ia}^T A(\xi_{ij}) u_{j\beta} \hat{x}_{j\beta} Q e \cdot dt_{ij}(\xi_{ij})
\]
\[= - \sum_{\beta=1}^{s_j} \rho_{ij,\alpha\beta} \hat{x}_{j\beta} Q e \cdot \sqrt{dt_{ij}(\xi_{ij})}.
\]
By using the result of Lemma 2.2 and recalling the definition \([3.7]\) of \(\Gamma_{ij,\alpha\beta}\) we see that this expression collapses to
\[
\sum_{k=1}^{n_{\infty}} \sum_{\beta=1}^{s_k} \Gamma_{ik,\alpha\beta} \hat{x}_{k\beta} Q e = x_{ia}^T K(\hat{x}; \xi_{ij}, q) Q e.
\]
Since this must hold for all \(e \in \chi(q)\), we arrive at the operator equation
\[
\sum_{k=1}^{n_{\infty}} \sum_{\beta=1}^{s_k} \Gamma_{ik,\alpha\beta} \hat{x}_{k\beta} = x_{ia}^T K(\hat{x}; \xi_{ij}, q)
\]  \hspace{1cm} (3.24)
which must hold for all index pairs \((i, \alpha)\) for which \(\lambda^i = \mu^j\) for some \(j\). Combining \((3.21)\) and \((3.24)\) gives us

\[\Gamma \hat{x} = K^{\lambda x}(q)\]
where we have set \(\hat{x}\) equal to the column vector \([\hat{x}_{j\beta}]_{1 \leq \beta \leq s_j, 1 \leq j \leq n_{\infty}\}\). Plugging this value into \((3.20)\) leaves us with the formula \((3.1)\) for the solution \(T\). This also establishes the uniqueness of the solution of (ABSINT) whenever it exists.

Since \(T\) is analytic at the points \(p^1, \ldots, p^n\) where \(K(\chi; \cdot, q)^{-1}\) has poles, necessarily the residue conditions \([3.10]\) must hold as well. The necessity and uniqueness parts of the theorem are now established.

Conversely, assume that \(\Gamma\) is invertible and that the residue conditions \([3.10]\) hold. We define \(T(p)\) by the formula \((3.11)\). Then \(T\) is a meromorphic bundle map of \(\chi\) and \(\hat{\chi}\) with only simple poles which occur at most at the points \(\mu^1, \ldots, \mu^{n_{\infty}}\) with

\[\text{im } \text{Res}_{\mu^j} T(p) \subseteq \text{span } \{u_{j\alpha}; 1 \leq \alpha \leq s_j\}\]
for \(j = 1, \ldots, n_{\infty}\). Since \(T\) has the form \((3.20)\) with operators \(\hat{x}_{j\beta}\) \((1 \leq j \leq n_{\infty}, 1 \leq \beta \leq s_j)\) satisfying \((3.21)\) and \((3.24)\), we see that also \(T\) satisfies the interpolation conditions (ii) and (iii) in (ABSINT) as well. Hence, the number of poles \(n_{\infty}(T)\) of \(T\) (counting multiplicities for a meromorphic matrix function as in Chapter 3 of \((6)\)) is at most \(\sum_{j=1}^{s_j} s_j =: N_{\infty}\) and the number of zeros \(n_{0}(T)\) of \(T\) (counting multiplicities) is at least \(\sum_{i=1}^{n_{\infty}} t_i =: N_0\). As \(T\) is a bundle map of the flat bundles \(\chi\) and \(\hat{\chi}\), we know that \(n_{0}(T) = n_{\infty}(T)\). On the other hand, since \(\Gamma\) is square we have \(N_0 = N_{\infty}\). From the chain of inequalities

\[N_0 \leq n_{0}(T) = n_{\infty}(T) \leq N_{\infty}\]
combined with the equality \(N_0 = N_{\infty}\), we get that \(n_{0}(T) = N_0\) and \(n_{\infty}(T) = N_{\infty}\). This implies that necessarily

\[\text{im } \text{Res}_{\mu^j} T(p) = \text{span } \{u_{j1}, \ldots, u_{js_j}\}\]
and
\[ \text{im Res}_{p=\lambda} (T^\vee)^{-1}(p) = \text{span}\{x_{i1}, \ldots, x_{it}, \} \]
and that \( T(p) \) is analytic and invertible at every point \( p \in X \) outside of \( \mu^1, \ldots, \mu^{n_0}, \lambda^1, \ldots, \lambda^{n_0} \). This verifies that \( T \) is a bona fide solution of the interpolation problem (ABSINT).

It remains only to verify the formula (3.12) for the inverse of \( T \). To see this, we note that \( (T^{-1})^T \) is also the solution of an interpolation problem of the type (ABSINT), namely the one with data set \( [\omega] \) given by
1. \( (\lambda, x) \) in place of \( (\mu, u) \),
2. \( (\mu, u) \) in place of \( (\lambda, x) \), and
3. \( -\rho_{j_1, \alpha} \) in place of \( \rho_{j_1, \beta} \).

The matrix \( \Gamma \) associated with this interpolation problem turns out to be exactly \( -\Gamma^T \) where \( \Gamma \) is as in (3.7). Hence by Theorem 3.1 the bundle map \( (T^{-1})^T \) must be given by
\[
(T^{-1})^T = [K(\tilde{\chi}^\vee; p, q) - K_{\lambda, x}(p)(\Gamma^{-1})^T K^{u, \mu}(q)](T(q)^{-1})^T K(\chi^\vee; p, q)^{-1}.
\]

By the uniqueness property of Cauchy kernels, it is easy to see that
\[
K(\chi^\vee; p, q)^T = -K(\chi; q, p).
\]

Hence, taking transpose on both sides of (3.25) gives
\[
T^{-1}(p) = - K(\chi; q, p)^{-1} T(q)^{-1}[\hat{K}(\chi; q, p) - K_{\mu, u}(q) \Gamma^{-1} K^{x, \lambda}(p)]
\]
and the formula (3.12) follows.

\[ \Box \]

Remark: In case \( \tilde{\chi} = \chi = \chi_0 \) are both taken to be the trivial bundle of rank \( r \), the Cauchy kernel \( K(\chi_0; \cdot, \cdot) \) has the scalar form \( k_0(\cdot, \cdot)I_r \) where \( k_0(\cdot, \cdot) \) is the Cauchy kernel for the trivial line bundle over \( X \). In this case the ansatz (3.20) simplifies to
\[
T(p) = Q + \sum_{j=1}^{r_0} \sum_{\beta=1}^{n} f_{\mu}(p) u_{j\beta} \tilde{x}_{j\beta}
\]
where we have set
\[
f_{\mu}(p) = \frac{k_0(\mu, p)}{k_0(p, q)}
\]
and the row vectors \( \tilde{x}_{j\beta} = \tilde{x}_{j\beta} Q \) are now taken to be the unknowns. Note that \( k_0(\cdot, q) \) is a half-order differential with divisor of degree \( q - 1 \) and a pole at \( q \); if we assume that the zeros are distinct, this divisor has the form \( p^1 + \cdots + p^g - q \) for distinct points \( p^1, \ldots, p^g, q \in X \). If the image of the divisor \( p^1 + \cdots + p^g \) under the Abel-Jacobi map is not on the classical theta divisor in the Jacobian (i.e. if \( p^1 + \cdots + p^g \) is a non-special divisor), then there are no nonzero constant meromorphic functions with only poles equal to at most simple poles at the points \( p^1, \ldots, p^g \); this corresponds to our assumption that \( h^0(\chi_0 \otimes \Delta) = h^0(\Delta) = 0 \). Furthermore, in this case, the global scalar meromorphic function \( f_{\mu}(\cdot) \) on \( X \) (for \( \mu \) a point of \( X \) disjoint from \( p^1, \ldots, p^g, q \)) is uniquely determined (up to a nonzero scalar multiple) by the condition that it have a pole at \( \mu \) and that its divisor \( (f_{\mu}) \) satisfy
\[
(f_{\mu}) \geq q - \mu - p^1 - \cdots - p^g.
\]
In this way our results and analysis on the (ABSINT) problem reduce to the work in \[3\] for the trivial bundle case. Notice that \(\chi_0\) can be replaced here by \(\xi \otimes \chi_0\) for any line bundle \(\xi\) of degree 0 satisfying \(h^0(\xi \otimes \Delta) = 0\), replacing the Cauchy kernel \(k_0(\cdot, \cdot)\) for the trivial line bundle by the Cauchy kernel for \(\xi\); this corresponds to letting \(p^1 + \cdots + p^g\) be any non-special effective divisor of degree \(g\).

One remaining piece of business in this section is the proof of Theorem 3.2. The problem of identifying the unknown input bundle in a more explicit fashion will be addressed in Section 6.

**Proof of Theorem 3.2.** If there exists such an input bundle \(\chi\) and meromorphic bundle map \(T\), then \(T\) implements a biholomorphic bundle map between \(\chi \otimes \Delta\) and \((\overline{\chi} \otimes \Delta)(\omega)\). Since \(h^0(\chi \otimes \Delta) = 0\), it then follows from Lemma 3.4 that \(\Gamma\) is injective. Since \(\deg(\chi \otimes \Delta) = r(g - 1)\), it must be the case that \(\deg(\overline{\chi} \otimes \Delta)(\omega) = r(g - 1)\) as well. This means that \(\Gamma\) is square, and hence invertible.

Conversely, suppose that \(\Gamma\) is square and invertible. Define a bundle \(\chi\) so that \(\chi \otimes \Delta \cong (\overline{\chi} \otimes \Delta)(\omega)\). Since \(\Gamma\) is square, it follows that

\[
\deg((\overline{\chi} \otimes \Delta)(\omega)) = \deg(\chi \otimes \Delta) = r(g - 1),
\]

and hence \(\deg(\chi \otimes \Delta) = r(g - 1)\). Since \(\ker \Gamma = \{0\}\), we know by Lemma 3.4 that \(h^0((\overline{\chi} \otimes \Delta)(\omega)) = 0\); thus \(h^0(\chi \otimes \Delta) = 0\). It follows from these two facts as in the proof of Theorem 3.1 in \[3\] that \(\chi\) is flat.

Let now \(S: \chi \otimes \Delta \to (\overline{\chi} \otimes \Delta)(\omega)\) be an implementation of the holomorphic bundle isomorphism between \(\chi \otimes \Delta\) and \((\overline{\chi} \otimes \Delta)(\omega)\). Define \(T: \chi \to \overline{\chi}\) so that \(S = T \otimes I_{O(\Delta)}\). Then \(T\) is a meromorphic bundle map from \(\chi\) to \(\overline{\chi}\) which solves the interpolation problem (ABSINT).

\[\square\]

4. **The line bundle case and Fay’s identity**

In this section we specialize the work of the preceding sections to the line bundle case.

We shall need here some basic facts concerning the Jacobian variety, the Abel-Jacobi map and associated theta functions (theta function, theta functions with characteristics and prime form) for the Riemann surface \(X\). The review here is quite sketchy; for complete details the reader should consult \[11\], \[3\] or \[18\].

When the rank \(r\) of the vector bundle \(\chi\) is 1, one can get an explicit formula for \(K(\chi; \cdot, \cdot)\) in terms of the Abel-Jacobi map for the surface \(X\) and various variants of the classical theta function associated with the Jacobian variety of \(X\) (see \[3\]). Specifically, in this case we may assume that \(\chi\) is a flat unitary line bundle with factor of automorphy (also called \(\chi\)) given by \(\chi(A_j) = \exp(-2\pi i a_j), \chi(B_j) = \exp(2\pi i b_j)\), \(j = 1, \ldots, g\), where \(A_1, \ldots, A_g, B_1, \ldots, B_g\) form a canonical integral homology basis on \(X\). Let \(\Omega\) be the corresponding period matrix, let \(J(X) = C^g/Z^g + \Omega Z^g\) be the Jacobian variety of \(X\) and let \(\phi: X \to J(X)\) be the Abel-Jacobi map. As is standard, we extend \(\phi\) by linearity to any divisor on \(X\), and, using the correspondence between linear equivalence classes of divisors and isomorphism classes of line bundles, we consider \(\phi\) to be defined on any line bundle on \(X\) as well. One can verify that then \(\phi(z) = z\) where \(z = \Omega a + b\) and \(a, b \in R^g\) have respective coordinates \(a_j, b_j\). Then the explicit formula for the Cauchy kernel (as given in \[3\]) is the following. The verification is straightforward, once one has in hand the properties and factors of automorphy for the various objects involved.
Theorem 4.1. For the case where \( \chi \) is a flat unitary line bundle as above, the Cauchy kernel as defined in Section \ref{sec:cauchy} is given explicitly by

\[
K(\chi; p, q) = \frac{\theta\left[ \begin{array}{c} a \\ b \end{array} \right] (\phi(q) - \phi(p))}{\theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) E(q, p)}.
\] (4.1)

In the statement of Theorem 4.1 \( \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (\cdot) \) is the associated theta function with characteristics \( \left[ \begin{array}{c} a \\ b \end{array} \right] \), \( E(\cdot, \cdot) \) is the prime form on \( X \times X \), and we assume the line bundle \( \Delta \) of differentials of order \( \frac{1}{2} \) has been chosen so that \( \phi(\Delta) = -\kappa \), where \( \kappa \in J(X) \) is Riemann’s constant (see \( [10] \) and \( [18] \)). Note that a consequence of Riemann’s theorem is that \( \theta(z) \neq 0 \) if and only if \( h^0(\chi \otimes \Delta) = 0 \), and hence \( \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) \neq 0 \) in (4.1) and the formula makes sense. No such explicit formula is known at present for the higher rank case except in genus 1 (see \( [4] \)).

In the line bundle case one can also give an explicit formula for the canonical connections \( \nabla_\chi, \nabla_\chi^* \) associated with the flat unitary line bundle \( \chi \). This is the content of the following Proposition.

Proposition 4.2. For the case where \( \chi \) is a flat unitary line bundle with normalizations as above, then the canonical connections \( \nabla_\chi \) and \( \nabla_\chi^* \) are given by

\[
\nabla_\chi y = \left[ \sum_{j=1}^{g} \frac{\partial}{\partial z_j} \log \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) \omega_j(p) \right] y + dy
\] (4.2)

\[
= \sum_{j=1}^{g} [2\pi i a_j + \frac{\partial}{\partial z_j} \log \theta(z) \omega_j(p)] y + dy,
\] (4.3)

\[
\nabla_\chi x = - \left[ \sum_{j=1}^{g} \frac{\partial}{\partial z_j} \log \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) \omega_j(p) \right] x + dx
\] (4.4)

\[
= - \sum_{j=1}^{g} [2\pi i a_j + \frac{\partial}{\partial z_j} \log \theta(z) \omega_j(p)] x + dx.
\] (4.5)

Proof. This follows directly by comparing the general expansion for the Cauchy kernel

\[
\frac{K(\chi; p, p_0)}{\sqrt{dt(p) dt(p_0)}} = \frac{1}{t(p) - t(p_0)} \left[ I_e + \frac{A(t)}{dt(p_0)} t(p) + O(|t(p)|^2) \right]
\]
on the one hand and substituting the expansion of the theta function

\[
\theta\left[ \begin{array}{c} a \\ b \end{array} \right] (\phi(p_0) - \phi(p)) = \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) - \sum_{j=1}^{g} \frac{\partial}{\partial z_j} \theta\left[ \begin{array}{c} a \\ b \end{array} \right] (0) \omega_j(p_0) t(p) + O(|t(p)|^2)
\]
and the expansion of the prime form (see Corollary 2.5 in \( [10] \))

\[
E(p_0, p) = t(p) + O(|t(p)|^3)
\]
into (4.1).

Remark. From the formula for $\nabla \chi$ and $\nabla^* \chi$ it follows that the coefficients $A(p)$ and $A_\ell(p)$ are independent of the choice of homology bases (i.e., marking) on the Riemann surface $X$ as long as the bundle $\Delta$ of half-order differentials defined by $\phi(\Delta) = \kappa$ remains the same, since the unitary flat representative for a flat line bundle is unique. It is an amusing exercise to verify this independence directly by using the transformation law for theta functions (see [18] and [14]).

We next specialize the work of Section 3 to the scalar (or line bundle) case, where $\chi$ and $\tilde{\chi}$ are flat unitary line bundles. As explained in Section 3, necessarily the multiplicities $s_j$ and $t_i$ are all 1 and without loss of generality we may take $u_1 = 1$, $x_1 = 1$ for all $i$ and $j$. Then the compatibility condition (3.1) forces the third interpolation condition to be absent. The data of the problem consists simply of the set of $n_\infty + n_0$ distinct points $\mu_1, \ldots, \mu_{n_\infty}, \lambda_1, \ldots, \lambda_{n_0}$ together with the flat unitary line bundles $\chi$ and $\tilde{\chi}$. The problem then is to produce a bundle map $T: \chi \rightarrow \tilde{\chi}$ with divisor equal to $\lambda - \mu$ (where we have set $\lambda = \lambda_1 + \cdots + \lambda_{n_0}$ and $\mu = \mu_1 + \cdots + \mu_{n_\infty}$). If we view the bundles in terms of factors of automorphy, we can view $T$ simply as a multivalued function on $X$ having divisor equal to $\lambda - \mu$ and factor of automorphy $\chi_T$ given by

$$\chi_T(A_j) = e^{-2\pi i a_j}, \quad \chi_T(B_j) = e^{2\pi i b_j} \text{ for } j = 1, \ldots, g$$

where $\phi(\tilde{\chi}) - \phi(\chi) = \Omega a + b$ (where we have set $a = [a_1 \ldots a_g]^T$ and $b = [b_1 \ldots b_g]^T$).

In the genus zero case where $X = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, any flat unitary line bundle is trivial and the problem is to produce a global meromorphic function with divisor equal to $\lambda - \mu$. Trivially a solution exists if and only if $n_0 = n_\infty$ and then the unique solution with value 1 at infinity is given in the multiplicative form

$$T(z) = \prod_{i=1}^{n_\infty} (z - \lambda_i) \prod_{j=1}^{n_0} (z - \mu_j). \quad (4.6)$$

or in the partial fraction form

$$T(z) = 1 + \sum_{j=1}^{n_\infty} c_j (z - \mu_j)^{-1} \quad (4.7)$$

where $c^T = [c_1 \ldots c_{n_\infty}]$ is the unique solution of the linear system of equations $Sc = [1 \ldots 1]^T$ with $S$ equal to the Sylvester matrix

$$S = [S_{ij}] \text{ with } S_{ij} = \frac{1}{\mu^j - \lambda^i},$$

or, in other words,

$$c = S^{-1} [1 \ldots 1]^T. \quad (4.8)$$

It is possible to evaluate the vector $c$ explicitly from (4.8) once one knows the entries of $S^{-1}$ explicitly. This in turn can be done once one knows an explicit formula for the determinant of a Sylvester matrix $S$, since the cofactor matrices are again of the same form. In this way one can verify directly the equivalence of the two formulas (4.6) and (4.7). For details on the algebra of this computation, we refer to Theorem 4.3.2 of [1].
We shall see that an analogous pair of formulas holds for the solution of the abstract interpolation problem (ABSINT) for the higher genus case (for the line bundle setting). This is the content of the next Theorem.

**Theorem 4.3.** Consider the problem (ABSINT) for the case where $\chi$ and $\bar{\chi}$ are flat unitary line bundles and given data set equal to $\lambda - \mu = \lambda_1 + \ldots + \lambda^{n_0} - \mu_1 - \ldots - \mu^{n_\infty}$ as above. Then a solution exists if and only if

$$n_0 = n_\infty \text{ and } \phi(\bar{\chi}) - \phi(\chi) = \phi(\lambda) - \phi(\mu).$$

(4.9)

In this case, if $q$ is a point of $X$ disjoint from the set $\{\lambda^1, \ldots, \lambda^{n_0}, \mu^1, \ldots, \mu^{n_\infty}\}$ of prescribed zeros and poles and $Q$ is any invertible fiber map from $\chi(q)$ onto $\bar{\chi}(q)$, then a solution of (ABSINT) having value $Q$ at $q$ is given in multiplicative form as

$$T(p) = \frac{\prod_{j=1}^{n_0} E(p, \lambda^j)/E(q, \lambda^j)}{\prod_{j=1}^{n_\infty} E(p, \mu^j)/E(q, \mu^j)} \exp(-2\pi ia^T(\phi(p) - \phi(q))Q$$

(4.10)

where $a^T = [a_1 \ldots a_g]$ and $\phi(\lambda) - \phi(\mu) = \Omega a + b$ with $a, b \in \mathbb{R}^g$.

If $\chi \otimes \Delta$ and $\bar{\chi} \otimes \Delta$ have no nontrivial holomorphic sections, then the solution $T$ with $T(q) = Q$ is unique and alternatively is given by the partial fraction formula

$$T(p) = \left\{ \frac{\theta [\bar{\chi}](\phi(q) - \phi(p))}{\theta [\bar{\chi}](0)E(q, p)} + \sum_{j=1}^{n_0} \sum_{k=1}^{n_\infty} \frac{\theta [z](\phi(\mu^j) - \phi(p))}{\theta [z](0)E(\mu^j, p)} \cdot [\Gamma^{-1}]_{jk} \cdot \frac{\theta [\bar{\chi}](\phi(q) - \phi(\lambda^j))}{\theta [\bar{\chi}](0)E(\lambda^j, \lambda^j)} \right\}$$

$$\times Q \cdot \frac{\theta [z](0)E(q, p)}{\theta [z](\phi(q) - \phi(p))}$$

(4.11)

where the $n_0 \times n_\infty$ matrix $\Gamma$ is given by

$$\Gamma = [\Gamma_{ij}] \text{ with } \Gamma_{ij} = -\frac{\theta [z](\phi(\mu^j) - \phi(\lambda^j))}{\theta [z](0)E(\mu^j, \lambda^j)}.$$

and where we have set

$$z = \phi(\chi), \quad \bar{z} = \phi(\bar{\chi}).$$

Here we write $\theta [\lambda](\lambda)$ rather than $\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (\lambda)$ if $z = \Omega a + b \in \mathbb{C}^g$ with $a, b \in \mathbb{R}^g$.

**Proof.** If such a bundle map exists, then the bundle $\chi \otimes \mathcal{O}(\lambda - \mu)$ and $\bar{\chi}$ are holomorphically equivalent. In particular, the divisor $\lambda - \mu$ must have degree 0 since both $\chi$ and $\bar{\chi}$ are flat bundles. The equality $\phi(\bar{\chi}) = \phi(\chi) + \phi(\lambda) - \phi(\mu)$ then follows from the correspondence between flat bundles and linear equivalence classes of divisors mentioned above. This verifies the necessity condition (4.9).

Conversely, assume that (4.9) holds and define $T$ by the right-hand side of (4.10). That the zero-pole divisor of $T$ is $\lambda - \mu$ follows directly from the fact that the divisor of the prime form $(p, q) \to E(p, q)$ is the diagonal $\{(p, p) : p \in X\} \subset X \times X$. One can next check from the known period relations of $E$ that the right-hand side of (4.10) has the factor of automorphy $\chi_T$

$$\chi_T(A_j) = \exp(-2\pi ia_j), \quad \chi_T(B_j) = \exp(2\pi ib_j) \text{ for } j = 1, \ldots, g$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ are respective components of $a, b \in \mathbb{R}^g$ chosen so that $\Omega a + b = \phi(\lambda) - \phi(\mu)$. The second condition in (4.9) now guarantees that $T(\cdot)$ so defined is a bundle map from $\chi$ into $\bar{\chi}$. The uniqueness assertion is a consequence of Lemma 3.3.
The alternative formula (4.11) is simply a rewriting of the formula (4.11) from Theorem 4.1, specialized to the line bundle case, where we have substituted the explicit formula (4.1) for the Cauchy kernel from Theorem 4.1.

Note that part of the content of Theorem 4.3 is that the matrix $\Gamma$ is invertible whenever $\chi$ and $\bar{\chi}$ have no nontrivial holomorphic sections and (ABSINT) has a solution.

It is of interest to specialize Theorem 4.3 to the case of one prescribed zero and pole $\lambda - \mu = \lambda^1 - \mu^1$. When this is done we obtain the following result.

**Theorem 4.4.** If $\chi$ and $\bar{\chi}$ are two flat unitary line bundles such that neither $\chi \otimes \Delta$ nor $\bar{\chi} \otimes \Delta$ have nontrivial holomorphic sections and $\lambda$ and $\mu$ are two distinct points of $X$, then the unique meromorphic bundle map from $\chi$ to $\bar{\chi}$ with zero-pole divisor equal to $\lambda - \mu$ and value $Q \neq 0$ at the point $q \in X$ is given by either

$$T(p) = \frac{E(p, \lambda)}{E(p, \mu)} \exp(-2\pi ia^T(\phi(p) - \phi(q))) Q$$

or

$$T(p) = \exp(-2\pi ia^T(\phi(p) - \phi(q))) \left\{ \frac{\theta(z + \phi(\lambda) - \phi(\mu) + \phi(q) - \phi(p))\theta(z)}{\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))} \right. - \left. \frac{\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))E(\mu, \lambda)}{\theta(z + \phi(\mu)\theta(z + \phi(q) - \phi(p))E(\mu, \lambda)}\right\} Q.$$  

(4.13)

**Proof.** The starting point of course is formula (4.10) and (4.11) specialized to the case $\lambda - \mu = \lambda - \mu$. The formula (4.12) is an immediate consequence of (4.10). Derivation of (4.13) requires a little bit of algebra. We use the definition

$$\theta[z](\lambda) = \exp(\pi ia^T\Omega a + 2\pi ia(\lambda + b))\theta(\lambda + z)$$

(where $z = \Omega a + b$ with $a, b \in \mathbb{R}^g$) to express all theta functions with characteristic $\theta[z](\cdot)$ in terms of the theta function itself $\theta(\cdot)$. When this is plugged into (4.11) and little bit of algebra is used to collect the exponential factor (noting that $a = a_z - a_z$ if $z - z(= \phi(\lambda) - \phi(\mu)) = \Omega a + b$ and $z = \Omega a_z + b_z$, $z = \Omega a_z + b_z$ with $a, b, a_z, b_z, a_z, b_z$ in $\mathbb{R}^g$), we get

$$T(p) = \exp(-2\pi ia^T(\phi(p) - \phi(q))) \left\{ \frac{\theta(z + \phi(\lambda) - \phi(\mu) + \phi(q) - \phi(p))}{\theta(z + \phi(\lambda) - \phi(\mu))E(\mu, \lambda)} \right. - \left. \frac{\theta(z + \phi(\lambda) - \phi(\mu))}{\theta(z + \phi(q) - \phi(p))}\right\} Q.$$

The formula (4.13) now follows by simple algebraic manipulation.

As a Corollary we obtain a version of Fay’s Trisecant Identity (see [10] formula (45) page 34 or [18] Volume II page 3.214).

**Corollary 4.5.** For $X$ a compact Riemann surface, $\phi$ its Abel-Jacobi map, $\theta(\lambda)$ and $E(p, q)$ its associated respective theta function and prime form, $p, q, \lambda, \mu$ points...
of $X$ and $z \in \mathbb{C}^g$, the following identity holds:

$$
\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))E(p, \lambda)E(q, \mu)
+ \theta(z + \phi(\lambda) - \phi(p))\theta(z + \phi(q) - \phi(\mu))E(\lambda, \mu)E(q, p)
= \theta(z + \phi(\lambda) - \phi(\mu) + \phi(q) - \phi(p))\theta(z)E(p, \mu)E(q, \lambda).
$$

(4.14)

Proof. From the identity of the two expressions (4.12) and (4.13) for $T(p)$ in Theorem 4.4, we have equality of the following two expressions for $\exp(2\pi ia^T(\phi(p) - \phi(q)))T(p)Q^{-1} E$:

$$
\frac{E(p, \lambda)E(q, \mu)}{E(p, \mu) E(q, \lambda)} \frac{\theta(z + \phi(\lambda) - \phi(\mu) + \phi(q) - \phi(p))\theta(z)}{\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))}
- \frac{\theta(z + \phi(\lambda) - \phi(p))\theta(z + \phi(q) - \phi(\mu))E(\mu, \lambda)E(q, p)}{\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))E(\mu, \mu)E(q, \lambda)}
$$

Multiplication of both sides by $\theta(z + \phi(\lambda) - \phi(\mu))\theta(z + \phi(q) - \phi(p))E(\mu, \mu)E(q, \lambda)$ along with a liberal use of the general identity $E(p, q) = -E(q, p)$ along with some algebra now leads to Fay’s identity (4.14) as desired.

The identity (4.14) is actually a special case of a more general identity (see Corollary 2.19 in [10]) which gives an explicit expression for the determinant of a matrix $M$ of the form

$$
M = [M_{ij}] \text{ where } M_{ij} = \frac{\theta(z + \phi(\mu^j) - \phi(\lambda^i))}{E(\mu^j, \lambda^i)}.
$$

Since the cofactor matrices of such a matrix are of the same form, one can then compute explicitly (in terms of theta functions and prime forms) the entries of the inverse of the matrix $\Gamma$ appearing in Theorem 4.3. In this way one can verify by direct computation the identity of the two expressions (4.10) and (4.11) for $T(p)$ in Theorem 4.3. This then is a canonical higher genus generalization of Theorem 4.3.2 in [3].

Note that this proof of Fay’s identity arises from equating a multiplicative formula for the solution of a zero-pole interpolation problem to a partial-fraction expression for the same solution. Formula (3.11) in Theorem 3.1 gives an analogue of the partial fraction expression for the solution of a zero-pole interpolation problem for a vector bundle endomorphism. Formula (3.11), giving a connection between the Cauchy kernel $K(\chi; p, q)$ and $K(\tilde{\chi}; p, q)$, can be viewed as a matrix-valued version of the Fay trisecant identity.

There is one case in higher rank when a multiplicative representation does exist, namely the case of full rank zero-pole interpolation (where the given pole vectors $\{u_{j\beta} : 1 \leq \beta \leq s_j = r\}$ span the fiber space $\tilde{\chi}(\mu^j)$ and the given null vectors $\{x_{i\alpha} : 1 \leq \alpha \leq t_i = r\}$ span the fiber space $\tilde{\chi}^i(\lambda^i)$ for each $i$ and $j$. Then $T(p)$ is again given by (4.10) where $Q$ now is a product of a scalar from the fiber of $O(\lambda - \mu)(q)$ and a value at $q$ of an automorphism of $\tilde{\chi}$. Without loss of generality we may assume that $\{x_{i\alpha} : 1 \leq \alpha \leq r\}$ and $\{u_{j\beta} : 1 \leq \beta \leq r\}$ consist of the standard basis vectors for each $i$ and $j$. Then in the partial fraction expansion (3.11) of $T(p)$ we have that $\Gamma$ has the block matrix form

$$
\Gamma = -[K(\tilde{\chi}; \lambda^i, \mu^j)]_{i=1,\ldots,n_0; j=1,\ldots,n_0},
$$

(4.15)
and that $K_{\mu,u}(p)$ and $K^{x,\lambda}(q)$ are block row and column matrices respectively
\[
K_{\mu,u}(p) = \begin{bmatrix} K(\bar{x};p,\mu^1) & \cdots & K(\bar{x};p,\mu^{\infty}) \end{bmatrix},
\]
\[
K^{x,\lambda}(q) = \begin{bmatrix} K(\bar{x};\lambda^1, q) \\ \vdots \\ K(\bar{x};\lambda^{\infty}, q) \end{bmatrix}.
\]
Equating this multiplicative formula to the partial fraction expansion leads to the same result as in formula (2.16) in [13].

5. Determinantal representations of algebraic curves and kernel bundles via Cauchy kernels

In [8] zero-pole interpolation problems of the sort discussed here were studied in a more concrete setting of vector bundles over an algebraic curve embedded in projective space with fiber space given as the kernel of a two-variable matrix pencil. In this section we make the connections between that setting and the abstract compact Riemann surface setting of Section 3 of this paper explicit. As we shall see, the link between the two settings is provided by the Cauchy kernels introduced in Section 3.

We first review the setting from [8]. Suppose that we are given three $M \times M$ matrices $\sigma_1, \sigma_2, \gamma$ and let $U_0(z) = U_0(z_1, z_2)$ be the two-variable linear matrix pencil
\[
U_0(z) = z_1 \sigma_2 - z_2 \sigma_1 + \gamma, \quad z = (z_1, z_2).
\]
We will also often consider the homogenization $U(\mu)$ (where $\mu = [\mu_0, \mu_1, \mu_2]$ are projective coordinates in $\mathbb{P}^2$) given by
\[
U(\mu) = \mu_0 U_0(\mu_1, \mu_2) = \mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_0 \gamma.
\]
Although $\det U(\mu)$ is not well-defined as a function of the projective variable $\mu = [\mu_0, \mu_1, \mu_2]$, nevertheless its zero set is well-defined and defines a curve $C \subset \mathbb{P}^2$ by
\[
C = \{ \mu = [\mu_0, \mu_1, \mu_2] \in \mathbb{P}^2 : \det U(\mu) = 0 \}.
\]
We shall assume that $U(\mu)$ defines a maximal irreducible determinantal representation of rank $r$; this means that $\det U(\mu) = F(\mu)^r$ where $F$ is an irreducible homogeneous polynomial of degree $m$ (so $M = rm$), and that $\ker U(\mu) = r$ for all smooth points $\mu$ of $C$, i.e., points $\mu^0$ where at least one of $\frac{\partial F}{\partial \mu_0}(\mu^0)$, $\frac{\partial F}{\partial \mu_1}(\mu^0)$, and $\frac{\partial F}{\partial \mu_2}(\mu^0)$ is not zero. In case of a singular point $\mu^0$, we assume that $\dim \ker U(\mu^0)$ is as large as possible, namely $sr$ where $s$ is the multiplicity of $\mu^0$. Under these conditions $E(\mu) = \ker U(\mu)$ lifts to a vector bundle $E$ of rank $r$ over the normalizing Riemann surface $X$ of $C$; note that the bundle $E$ is realized concretely as a rank $r$ subbundle of the trivial bundle of rank $M$ over $X$. The normalizing Riemann surface $X$ is a Riemann surface such that there is a holomorphic mapping $\pi : X \to \mathbb{P}^2$ whose image equals $C$ such that $\pi$ is a one-to-one immersion on the inverse image of smooth points of $C$; we call $\pi : X \to \mathbb{P}^2$ a birational embedding of $X$ in $\mathbb{P}^2$. For more details, see [8]. As in [8], we shall assume for simplicity that all the singular points of $C$ are nodes (i.e., $\pi^{-1}(q) = \{p^1, p^2\}$ where $p^1$ and $p^2$ are distinct points on $X$ with neighborhoods $U_1$ and $U_2$ such that $\pi$ is an immersion at both $p^1$ and $p^2$ and the analytic arcs $\pi(U_1)$ and $\pi(U_2)$ intersect transversally at $q$). We also assume that the line at infinity $\{\mu_0 = 0\}$ is nowhere tangent to $C$. 
The holomorphic vector bundle $E_\ell$ which is dual to $E$ can be realized concretely as a subbundle of the trivial rank $M$ bundle over $X$ (with fibers now written as row vectors) via

$$E_\ell(\mu) = \ker_\mu U(\mu) = \{ x \in \mathbb{C}^{1 \times M} : xU(\mu) = 0 \}.$$ 

A concrete pairing between $E_\ell \otimes \mathcal{O}(1) \otimes \Delta$ and $E \otimes \mathcal{O}(1) \otimes \Delta$ is given by

$$\{ u_\ell, u \} = \frac{u_\ell}{\mu_0} \frac{\xi_1 \sigma_1 + \xi_2 \sigma_2}{d \lambda_1 + d \lambda_2} \frac{u}{\mu_0}.$$

(5.1)

Here $u_\ell$ and $u$ are local holomorphic sections of $E_\ell \otimes \mathcal{O}(1) \otimes \Delta$ and $E \otimes \mathcal{O}(1) \otimes \Delta$ respectively, $\lambda_1$ and $\lambda_2$ are meromorphic functions on $X$ given by $\lambda_1 = z_1 \circ \pi$ and $\lambda_2 = z_2 \circ \pi$, and $\xi_1, \xi_2$ are arbitrary (not both zero) complex parameters.

If $E$ and $E_\ell$ are right and left kernel bundles determined by a rank $r$ maximal determinantal representation $U(\mu)$ of a curve $C$ as above, then it can be shown that necessarily $E \otimes \mathcal{O}(1) \otimes \Delta$ is isomorphic to a flat bundle $\chi$ over $X$ with the property that $h^0(\chi \otimes \Delta) = 0$, and that $E_\ell \otimes \mathcal{O}(1) \otimes \Delta$ is isomorphic to the dual $\chi^\vee$ of $\chi$. This isomorphism of $E \otimes \mathcal{O}(1) \otimes \Delta$ with $\chi$ is implemented explicitly by a matrix of normalized sections $u^\chi(\xi)$. Explicitly, $u^\chi$ is an $M \times r$ matrix whose columns are meromorphic sections of the pullback of $E \otimes \Delta$ to the universal cover $\tilde{X}$ of $X$ such that:

1. $\frac{1}{\sqrt{dt(\tilde{\ell})}} u^\chi(\tilde{\xi}) = \frac{1}{\sqrt{dt(\tilde{\ell})}} u^\chi(\tilde{\xi}) \chi^{-1}(R)$ for all $\tilde{\xi} \in \tilde{X}$ and all $R \in \text{Deck}(\tilde{X}/X)$
   $\cong \pi_1(X)$, where $t$ is a local parameter on $X$ and $\sqrt{dt}$ is the corresponding local holomorphic frame for $\Delta$ lifted to the neighborhoods of $\tilde{\xi}$ and $\tilde{\ell}$ on $\tilde{X}$.
2. Each column of $u^\chi$ has first order poles at (the points of $\tilde{X}$ over) the points of $C$ at infinity, and is holomorphic everywhere else.
3. For each $\xi \in X$, the columns of $u^\chi(\tilde{\xi})$ form a basis for the fiber $(E \otimes \Delta)(\xi)$, where $\tilde{\xi} \in \tilde{X}$ is over $\xi$ (if $\xi$ is a point of $C$ at infinity we have first to multiply $u^\chi$ by a local parameter centered at $\xi$).

Simply speaking, $u^\chi$ consists of a multiplicative $\Delta$-valued meromorphic frame for $E$, normalized to have poles exactly at the points of $C$ at infinity. An isomorphism $\chi : E \otimes \mathcal{O}(1) \otimes \Delta$ is now given explicitly by $y \to \mu_0 u^\chi y$ where $y$ is a local holomorphic section of $\chi$. An $r \times M$ matrix of normalized sections $u^\chi_\ell$ of $E_\ell$, whose rows are meromorphic sections of the pullback of $E_\ell \otimes \Delta$ to the universal covering $\tilde{X}$ of $X$, is defined similarly, with item (1) replaced by

$$1. \frac{1}{\sqrt{dt(\tilde{\ell})}} u^\chi_\ell(\tilde{\xi}) = \frac{1}{\sqrt{dt(\tilde{\ell})}} \chi(\tilde{\xi}) u^\chi_\ell(\tilde{\xi})$$

for all $\tilde{\xi} \in \tilde{X}$ and all $R \in \text{Deck}(\tilde{X}/X) \cong \pi_1(X)$, where $t$ and $\sqrt{dt}$ are as before.

An isomorphism $\chi^\vee : E_\ell \otimes \mathcal{O}(1) \otimes \Delta$ is given explicitly by $x \to \mu_0 x^T u^\chi_\ell$, where $x$ is a local holomorphic section of $\chi^\vee$. Given $u^\chi$, the dual matrix of normalized section $u^\chi_\ell$ is determined uniquely by

$$u^\chi_\ell = \frac{\xi_1 \sigma_1 + \xi_2 \sigma_2}{\xi_1 d \lambda_1 + \xi_2 d \lambda_2} u^\chi = I_r$$

(where $I_r$ is the $r \times r$ identity matrix), so that under the isomorphisms $\chi \cong E \otimes \mathcal{O}(1) \otimes \Delta$ and $\chi^\vee \cong E_\ell \otimes \mathcal{O}(1) \otimes \Delta$ the natural duality pairing between $\chi^\vee$ and $\chi$ equals the pairing (5.1).
If we now define $K(\chi; \tilde{p}, \tilde{q})$ by

$$K(\chi; \tilde{p}, \tilde{q}) = u_\ell^\chi(\tilde{p}) \frac{\xi_1 \sigma_1 + \xi_2 \sigma_2}{\xi_1 (\lambda_1(\tilde{p}) - \lambda_1(\tilde{q})) + \xi_2 (\lambda_2(\tilde{p}) - \lambda_2(\tilde{q}))} u^\chi(\tilde{q}), \quad (5.2)$$

then $K$ has all the properties of the Cauchy kernel as defined in Section 2. This method of constructing the Cauchy kernel, via a dual pair of normalized sections of the kernel bundles associated with a maximal determinantal representation of an algebraic curve $C$ embedded in $\mathbb{P}^2$ which has the Riemann surface $X$ as its normalizing surface, was presented in [7].

Here we wish to make explicit the reverse path. We start with a compact Riemann surface $X$ and a flat holomorphic vector bundle $\chi$ over $X$ for which $h^0(\chi \otimes \Delta) = 0$. We assume as given the associated Cauchy kernel as developed in Section 2. We then construct a birational embedding of $X$ into $\mathbb{P}^2$ with image equal to the curve $C$ together with a rank $r$ maximal determinantal representation of $C$ in such a way that we recover the Cauchy kernel $K(\chi; \cdot, \cdot)$ from a dual pair of normalized sections for the associated left and right kernel bundles associated with this determinantal representation of $C$, as in (5.2).

We first need some preparations. Let $\chi$ be a flat vector bundle over the Riemann surface $X$ such that $h^0(\chi \otimes \Delta) = 0$. In addition choose two scalar meromorphic functions $\lambda_1, \lambda_2$ on $X$ such that $\mathcal{M}(X) = \mathbb{C}(\lambda_1, \lambda_2)$, i.e., rational functions in $\lambda_1, \lambda_2$ generate the whole field of (scalar) meromorphic functions on $X$. Assume that all poles of $\lambda_1$ and $\lambda_2$ are simple, and denote the set of poles by $x^1, \ldots, x^m \in X$. Define complex numbers $c_{ik}$ ($1 \leq i \leq m$, $k = 1, 2$) by

$$c_{ik} = -\text{Res}_{p=x^i} \lambda_k(p)$$

where the residue is with respect to some fixed local coordinate $t^i = t^i(p)$ centered at $p = x^i$. On occasion we shall also need the next coefficient $-d_{ik}$ in the Laurent expansion of $\lambda_k$ at $x^i$:

$$\lambda_k(p) = -\frac{c_{ik}}{t^i} - d_{ik} + O(|t^i|).$$

Define $M \times M$ matrices (where $M = mr$) $\sigma_1, \sigma_2, \gamma$ by

$$\sigma_1 = \text{diag.} (c_{i1} I_r), \quad \sigma_2 = \text{diag.} (c_{i2} I_r) \quad \gamma = [\gamma_{ij}]_{i,j=1,\ldots,m} \quad (5.3)$$

where

$$\gamma_{ij} = \begin{cases} d_{i1} c_{j2} - d_{i2} c_{j1}, & \text{if } i = j \\ (c_{i1} c_{j2} - c_{i2} c_{j1}) K(\chi; x^i, x^j) dt^j(x^j), & \text{if } i \neq j. \end{cases}$$

Also define

$$u_\ell^\chi(p) = \begin{bmatrix} K(\chi; x^1, p) \\ \vdots \\ K(\chi; x^m, p) \end{bmatrix}, \quad u_\ell^\chi(p) = -\begin{bmatrix} K(\chi; p, x^1) & \ldots & K(\chi; p, x^m) \end{bmatrix} \quad (5.4)$$

Then we have the following result.

**Theorem 5.1.** Let $\chi$ be a flat vector bundle over the Riemann surface $X$ such that $h^0(\chi \otimes \Delta) = 0$ with associated Cauchy kernel $K(\chi; \cdot, \cdot)$ and use a pair of meromorphic functions $\lambda_1(p), \lambda_2(p)$ on $X$ which generate the field $\mathcal{M}(X)$ of meromorphic functions on $X$ to define matrices $\sigma_1, \sigma_2$ and $\gamma$ as in (5.3). Then:
(i) The map \( \pi_0 : X \to \mathbb{C}^2 \) given by
\[
\pi_0(p) = (\lambda_1(p), \lambda_2(p))
\]
maps \( X \setminus \{x^1, \ldots, x^m\} \) onto the affine part \( C_0 \) of an algebraic curve \( C \subseteq \mathbb{P}^2 \) and extends to a birational embedding \( \pi : X \to C \) of \( X \) in \( \mathbb{P}^2 \). The defining irreducible homogeneous polynomial \( F(\mu_0, \mu_1, \mu_2) \) of \( C \) is such that \( \det(\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_0 \gamma) = F(\mu_0, \mu_1, \mu_2) \).

(ii) Denote by \( E \) the kernel bundle over \( C \) given in affine coordinates by
\[
E(\lambda) = \ker(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma), \quad \lambda = (\lambda_1, \lambda_2).
\]
Then \( \chi \cong E \otimes \mathcal{O}(1) \otimes \Delta \) with \( u^\times \) and \( u^\times_\ell \) given by \([5.4]\) equal to the dual matrices of normalized sections of \( E \) and \( E_\ell \).

We shall prove Theorem [5.1] under the assumption that all the singular points of \( C \) are nodes.

**Proof.** We define the curve \( C \) as the compactification in projective space of the image of the map \( \pi \) in the statement of the theorem
\[
C_0 = \{ (\lambda_1(p), \lambda_2(p)) : p \in X \}.
\]
Then \( X \) is the normalizing Riemann surface for the curve \( C \) and the degree of \( C \) is equal to the number of intersections with the line at infinity, namely, \( \deg C = m \).

Let \( f(z_1, z_2) = 0 \) be an irreducible polynomial of degree \( m \) such that \( f(z_1, z_2) = 0 \) is the defining equation for \( C \) (in affine coordinates). Thus \( f(\lambda_1(p), \lambda_2(p)) = 0 \) for all \( p \in X \). We must show that \( z_1 \sigma_2 - z_2 \sigma_1 + \gamma \) is a maximal determinantal representation of \( f(z_1, z_2)^r = 0 \), that \( E \otimes \mathcal{O}(1) \otimes \Delta \cong \chi \) and that \( u^\times \) and \( u^\times_\ell \) are dual matrices of normalized sections of \( E \) and \( E_\ell \).

The first step is to prove the identities
\[
(\lambda_1(p) \sigma_2 - \lambda_2(p) \sigma_1 + \gamma) u^\times(p) = 0
\]
\[
u^\times_\ell(p) (\lambda_1(p) \sigma_2 - \lambda_2(p) \sigma_1 + \gamma) = 0
\]
\[
\frac{u^\times_\ell(p)(\xi_1 \sigma_1 + \xi_2 \sigma_2) u^\times(p)}{\xi_1 d \lambda_1(p) + \xi_2 d \lambda_2(p)} = 1.
\]

To prove (5.5), set
\[
h(p) = (\lambda_1(p) \sigma_2 - \lambda_2(p) \sigma_1 + \gamma) u^\times(p).
\]
Note that \( h(p) \) is a meromorphic section of \( \chi \otimes \Delta \). To check that \( h = 0 \) it suffices to check that \( h \) has no poles (since \( h^0(\chi \otimes \Delta) = 0 \)). The only possible poles in the formula for \( h \) occur at the points \( x^1, \ldots, x^m \) and these are at most double poles. For \( \alpha = 1, \ldots, m \), let us write down the Laurent expansion for \( h \) near \( x^\alpha \) as
\[
h(p) = [h]^\alpha, -2 (t^\alpha)^{-2} + [h]^\alpha, -1 (t^\alpha)^{-1} + [\text{analytic at } x^\alpha].
\]

We must show that \( [h]^\alpha, -2 = 0 \) and \( [h]^\alpha, -1 = 0 \) for \( 1 \leq \alpha \leq m \).

Each of \([h]^\alpha, -2 \) and \([h]^\alpha, -1 \) in turn is a block \( m \times 1 \) column matrix: \([h]^\alpha, -2 = [h]^\alpha, -2_i \) and \([h]^\alpha, -1 = [h]^\alpha, -1_i \) with \( i = 1, \ldots, m \). We compute
\[
[h]^\alpha, -2_i = \sum_{j=1}^{m} (-c_{\alpha 1} c_{\alpha 2} \delta_{ij} + c_{\alpha 2 c_{\alpha 1}} \delta_{ij}) \delta_{ij} \cdot (-dt^\alpha(x^\alpha))
\]
\[
= (c_{\alpha 1} c_{\alpha 2} - c_{\alpha 2 c_{\alpha 1}} dt^\alpha(x^\alpha)) = 0
\]
where $\delta_{ij}$ is the Kronecker delta. Similarly,
\[
[h]_{i}^{\alpha,-1} = \sum_{j, j \neq \alpha} (-c_{\alpha 1} c_{\alpha 2} \delta_{ij} + c_{\alpha 2} c_{1 \alpha} \delta_{ij}) K(\chi, x^j, x^\alpha) \\
+ \sum_{j} \{ -d_{\alpha 1} c_{\alpha 2} \delta_{ij} + d_{\alpha 2} c_{1 \alpha} \delta_{ij} + \gamma_{ij} \} \delta_{j\alpha} (-dt^\alpha(x^\alpha)).
\] (5.8)

For $i = \alpha$ (5.8) becomes
\[
(-d_{\alpha 1} c_{\alpha 2} + d_{\alpha 2} c_{1 \alpha} + d_{\alpha 1} d_{\alpha 2} (-dt^\alpha(x^\alpha)) = 0
\]
while, for $i \neq \alpha$, (5.8) becomes
\[
(-c_{\alpha 1} c_{\alpha 2} + c_{\alpha 2} c_{1 \alpha}) K(\chi; x^i, x^\alpha) + (c_{\alpha 1} c_{\alpha 2} - c_{\alpha 1} c_{\alpha 2}) \frac{K(\chi; x^i, x^\alpha)}{dt^\alpha(x^\alpha)} (-dt^\alpha(x^\alpha)) = 0.
\]

Thus $h = 0$ and (5.5) follows.

By a similar calculation of Laurent series coefficients one can verify (5.6).

To verify (5.7), by a standard lemma (see Proposition 2.3 in [7]), it suffices to show that
\[
\frac{u^x_i(p) \sigma_k u^x(p)}{d\lambda_k(p)} = 1 \text{ for } k = 1, 2.
\]
The numerator of the expression on the left is given by
\[
[K(\chi; p, x^1) \ldots K(\chi; p, x^m)] \begin{bmatrix}
-c_{ik} \\
\ddots \\
-c_{mk}
\end{bmatrix} K(\chi; x^i, p) = -\sum_{i=1}^{m} K(\chi; p, x^i) [\text{Res}_{p=x^i} \lambda_k(p)] K(\chi; x^i, p)
\]
and hence
\[
\frac{u^x_i(p) \sigma_k u^x(p)}{d\lambda_k(p)} = -\frac{\sum_{i=1}^{m} K(\chi; p, x^i) [\text{Res}_{p=x^i} \lambda_k(p)] K(\chi; x^i, p)}{d\lambda_k(p)}.
\]
The double pole at each $x^i$ in the numerator is cancelled by a double pole at each $x^i$ in the denominator. Note also that the product of two half-order differentials in the numerator is cancelled by the differential in the denominator. The resulting quotient is a well-defined holomorphic section of $Hom(x, \chi)$ which has the value $I$ at $x^1, \ldots, x^m$, and hence must equal $I$ at all $p \in X$. Equation (5.7) now follows.

Denote by $d(z_1, z_2)$ the polynomial $d(z_1, z_2) = \det(z_1 \sigma_2 - z_2 \sigma_1 + \gamma)$. By (5.5) or (5.6) we know that $d(\lambda_1(p), \lambda_2(p)) = 0$ identically in $p \in X$. Since $f$ is by assumption the irreducible defining polynomial for $X$, it follows that $f(z_1, z_2)|d(z_1, z_2)$, and hence,
\[
d(z_1, z_2) = f(z_1, z_2)^s g(z_1, z_2)
\] (5.9)
for some positive integer $s$ and some polynomial $g$ relatively prime with respect to $f$. By inspection we see that $d$ has degree equal to $M = nr$, while, as already mentioned, $f$ has degree equal to $m$. From the factorization we see that $\deg d \geq s(\deg f)$, or $r \geq s$. On the other hand, at a smooth point $(z_1, z_2) \in C$, we know that $\dim E(z) \geq r$ since the $r$ linearly independent columns of $u^x(p)$ are in $\ker(\lambda_1(p) \sigma_2 - \lambda_2(p) \sigma_1 + \gamma)$. But also, from the factorization (5.9) and an inductive argument working with the minors the matrix pencil $z_1 \sigma_2 - z_2 \sigma_1 + \gamma$ (see the proof of Theorem
3.2 in [7] one can show that \( \dim E(z) \leq s \). From \( r \geq s \) and \( r \leq \dim E(z) \leq s \) we conclude that \( r = s \). From (5.3) and degree counting we conclude that the polynomial \( g \) is a constant, and without loss of generality, \( d = f^r \). Thus \( z_1 \sigma_2 - z_2 \sigma_1 + \gamma \) is a maximal determinantal representation of \( f(z_1, z_2)^r = 0 \) as asserted, except that we still must check that for any node \( q \) on \( C \) the columns of \( u^x(p^1) \) and \( u^x(p^2) \) are linearly independent, where \( \pi^{-1}(q) = \{p^1, p^2\} \). It then follows from (5.5), (5.6) and (5.7) that \( u^x(p) \) and \( u^x_\gamma(p) \) form the associated dual pair of normalized cross-sections for \( E \) and \( E_\ell \) respectively.

We first claim if \( L \) is a straight line nowhere tangent to \( C \) and \( y^1, \ldots, y^m \) are the preimages on \( X \) of the points of intersection of \( C \) with \( L \), then the block matrix \( [K(\chi; x^i, y^j)]_{i,j=1,\ldots,m} \) is invertible. This follows immediately from the discussion of the full rank zero-pole interpolation problem at the end of Section 4, as we now show. Since the divisor \( x^1 + \cdots + x^m - y^1 - \cdots - y^m \) is equivalent to \( 0 \), the input bundle in the full rank zero-pole interpolation problem with zeros \( x^1, \ldots, x^m \) and poles \( y^1, \ldots, y^m \) and output bundle \( \chi \) is again (isomorphic to) \( \chi \). Since \( h^0(\chi \otimes \Delta) = 0 \), the matrix \( \Gamma \) \( [4.13] \) is invertible. Next, by taking any line \( L \) through the node \( q \) which is nowhere tangent to \( C \), we see that the columns of \( u^x(p^1) \) and \( u^x(p^2) \) are columns in a \( M \times M \) invertible matrix (namely, the associated matrix \( \Gamma \)), and hence are linearly independent.

We remark that the same proof works when the singularities of \( C \) are any ordinary singular points, or more generally, are such that a singular point of multiplicity \( s \) has \( s \) distinct preimages on \( X \).

6. The concrete interpolation problem for meromorphic bundle maps between kernel bundles of determinantal representations of an algebraic curve

In the paper [7] the following problem was considered. We are given an irreducible algebraic curve \( C \) in \( \mathbb{P}^2 \) together with its normalizing compact Riemann surface \( X \) and the normalization map \( \pi: X \to C \). We assume that the defining polynomial for \( C \) is an irreducible polynomial \( f \) of degree \( m \) (in affine coordinates). For simplicity we assume again that the only singularities of \( C \) are nodes and that \( C \) intersects the line at infinity in \( m \) distinct smooth points. We suppose in addition that \( f^r \) has a maximal determinantal representation

\[
f^r(z_1, z_2) = \det(z_1 \sigma_2 - z_2 \sigma_1 + \gamma)
\]

where \( \sigma_1, \sigma_2 \) and \( \gamma \) are \( M \times M \) matrices \( (M = mr) \), with which is associated the kernel bundle \( \tilde{E} \) of rank \( r \) over \( C \setminus C_{\text{sing}} \) \( (C_{\text{sing}} \) is the set of the singular points of \( C \)) with fibers (over affine points) given by

\[
\tilde{E}(z) = \ker(z_1 \sigma_2 - z_2 \sigma_1 + \gamma). \tag{6.1}
\]

As explained in Section 3, we may consider the pullback of \( \tilde{E} \) to \( X \setminus \pi^{-1}(C_{\text{sing}}) \) as extended to a rank \( r \) vector bundle over all of \( X \). We also have the left kernel bundle \( \tilde{E}_\ell \) where

\[
\tilde{E}_\ell(z) = \ker_{\ell}(z_1 \sigma_2 - z_2 \sigma_1 + \gamma) \tag{6.2}
\]

with pullback under \( \pi \) also extendable to a rank \( r \) vector bundle defined over all of \( X \). These bundles, or more precisely their twists \( \tilde{E} \otimes O(1) \otimes \Delta \) and \( \tilde{E}_\ell \otimes O(1) \otimes \Delta \), have the canonical pairing (5.2) with each other, as explained in Section 3.
The data for the concrete interpolation (CONINT) problem to be considered in this section consists of:

(D1) $n_\infty$ distinct smooth, finite points $\mu^1 = (\mu^1_1, \mu^1_2), \ldots, \mu^{n_\infty} = (\mu^{n_\infty}_1, \mu^{n_\infty}_2)$ of $C$ (the preassigned poles),

(D2) for each $j = 1, \ldots, n_\infty$, a linearly independent set $\{\varphi_{j1}, \ldots, \varphi_{js_j}\}$ of $s_j$ vectors in the fiber $\tilde{E}(\mu^j)$ (the preassigned pole vectors),

(D3) $n_0$ distinct smooth, finite points $\lambda^1 = (\lambda^1_1, \lambda^1_2), \ldots, \lambda^{n_0} = (\lambda^{n_0}_1, \lambda^{n_0}_2)$ of $C$ (the preassigned zeros),

(D4) for each $i = 1, \ldots, n_0$, a linearly independent set $\{\psi_{i1}, \ldots, \psi_{it_i}\}$ of $t_i$ vectors in the fiber $\tilde{E}_i(\lambda^i)$ (the preassigned null vectors), and

(D5) for each pair of indices $(i, j)$ for which $\lambda^i = \mu^j =: \xi^{ij}$, a choice of a local coordinate $t^{ij}$ on $X$ centered at $\xi^{ij}$ and a collection of numbers $\{\rho_{ij, \alpha\beta} : 1 \leq \alpha \leq t_i, 1 \leq \beta \leq s_j\}$ (the preassigned coupling numbers with respect to the chosen local coordinate).

The interpolation problem then is to find an $M \times M$ matrix $\gamma$ defining a maximal determinantal representation of $f^r$

$$f^r(z_1, z_2) = \det(z_1\sigma_2 - z_2\sigma_1 + \gamma)$$

(6.3)

giving the kernel bundle $E$ over $X$ with fiber over a smooth finite point $z \in C$ given by

$$E(z) = \ker(z_1\sigma_2 - z_2\sigma_1 + \gamma)$$

(6.4)

and the left kernel bundle $E_\ell$ given by

$$E_\ell(z) = \ker(\ell_1\sigma_2 - \ell_2\sigma_1 + \gamma)$$

(6.5)

together with meromorphic bundle maps

$$S : E \to \tilde{E}, \quad S_\ell : \tilde{E}_\ell \to E_\ell$$

(where we write bundle maps on left kernel bundles as acting from the right), where $S \otimes I_{O(1) \otimes \Delta}$ and $S_\ell \otimes I_{O(1) \otimes \Delta}$ are transposes of each other with respect to the pairing (6.1), so that $S$ (and $S_\ell$) act as the identity operator $I$ on the corresponding fibers at the points at infinity, and the following set of interpolation conditions is satisfied:

(I1) $S$ has poles only at $\mu^1, \ldots, \mu^{n_\infty}$; for each $j = 1, \ldots, n_\infty$, the pole of $S$ at $\mu^j$ is simple, and the vectors $\{\varphi_{j1}, \ldots, \varphi_{j, n_\infty}\}$ span the image space of the residue $R_j : E(\mu^j) \to \tilde{E}(\mu^j)$ of $S$ at $\mu^j$.

(I2) The bundle map $S^{-1}_\ell : E_\ell \to \tilde{E}_\ell$ has poles only at the points $\{\lambda^1, \ldots, \lambda^{n_0}\}$; for each $i = 1, \ldots, n_0$, the pole of $S^{-1}_\ell$ at $\lambda^i$ is simple and the vectors $\{\psi_{i1}, \ldots, \psi_{it_i}\}$ span the image space of the residue $R_i : E_\ell(\lambda^i) \to \tilde{E}_\ell(\lambda^i)$ of $S^{-1}_\ell$ at $\lambda^i$.

(I3) For each pair of indices $(i, j)$ where $\lambda^i = \mu^j =: \xi^{ij}$, and for $\alpha = 1, \ldots, t_i$, let $\psi_{i\alpha}(p)$ be a local holomorphic section of $\tilde{E}_\ell$ near $\xi^{ij}$ with

$$\psi_{i\alpha}(t^{ij}) = \psi_{i\alpha} + \psi_{i\alpha t^{ij}} + o(t^{ij})$$

such that $\psi_{i\alpha}S_\ell(p)$ has analytic continuation to $p = \xi^{ij}$ with value there equal to 0. Then, for any choice of complex parameters $\xi_1$ and $\xi_2$

$$\psi_{i\alpha} = \frac{\xi_1\sigma_1 + \xi_2\sigma_2}{\xi_1\lambda^1_1(\xi^{ij}) + \xi_2\lambda^2_1(\xi^{ij})}\varphi_{j\beta} - \psi_{i\alpha}(\xi_1\sigma_1 + \xi_2\sigma_2)\varphi_{j\beta} + \frac{\xi_1\lambda^1_1(\xi^{ij}) + \xi_2\lambda^2_1(\xi^{ij})}{2(\xi_1\lambda^1_1(\xi^{ij}) + \xi_2\lambda^2_1(\xi^{ij}))^2} \rho_{ij, \alpha\beta}.$$
Here \( t' = \frac{d}{dt}\). It can be shown that a necessary consistency condition on the data set for the problem to have a solution is that

\[
\psi_{\alpha}(\xi_1 \sigma_1 + \xi_2 \sigma_2) \varphi_{\beta} = 0.
\]  

(6.6)

Given the data set (D1)–(D5) we form the \( n_0 \times n_\infty \) block matrix \( \Gamma^0 = [\Gamma^0_{ij}] \) (1 \( \leq i \leq n_0, 1 \leq j \leq n_\infty \)) where \( \Gamma^0_{ij} = [\Gamma^0_{ij,\alpha\beta}] \) (1 \( \leq \alpha \leq t_i, 1 \leq \beta \leq s_j \)) in turn is the \( t_i \times s_j \) matrix with entries given by

\[
\Gamma^0_{ij,\alpha\beta} = \begin{cases} 
\psi_{\alpha} \frac{\xi_1 \sigma_1 + \xi_2 \sigma_2}{\xi_1 (\mu_1 - \lambda_1) + \xi_2 (\mu_2 - \lambda_2)} \varphi_{\beta} & \text{if } \lambda^i \neq \mu^j \\
-\rho_{ij,\alpha\beta} & \text{if } \lambda^i = \mu^j.
\end{cases}
\]  

(6.7)

Additional matrices which we shall need are

\[
A_1 = \begin{bmatrix} 
\mu_1 I_{s_1} & \cdots & \mu_1 \infty I_{s_\infty} \\
\mu_2 I_{s_1} & \cdots & \mu_2 \infty I_{s_\infty} \\
\lambda_1 I_{t_0} & \cdots & \lambda_1 \infty I_{t_\infty} \\
\lambda_2 I_{t_0} & \cdots & \lambda_2 \infty I_{t_\infty}
\end{bmatrix}, \\
A_2 = \begin{bmatrix} 
\mu_1 I_{s_1} \\
\mu_2 I_{s_1} \\
\lambda_1 I_{t_0} \\
\lambda_2 I_{t_0}
\end{bmatrix},
\]

\[
Z_1 = \begin{bmatrix} 
1 & \cdots & 1 \\
\varphi_j & \cdots & \varphi_j \\
\varphi_{j+1} & \cdots & \varphi_{j+1} \\
\varphi_{j+n} & \cdots & \varphi_{j+n}
\end{bmatrix}, \\
Z_2 = \begin{bmatrix} 
1 & \cdots & 1 \\
\varphi & \cdots & \varphi \\
\varphi & \cdots & \varphi
\end{bmatrix},
\]

\[
\varphi = \begin{bmatrix} 
\varphi_1 \\
\varphi_2 \\
\vdots \\
\varphi_n
\end{bmatrix}, \\
\psi = \begin{bmatrix} 
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_n
\end{bmatrix}.
\]  

(6.8)

The solution of the concrete interpolation problem (CONINT) obtained in [7] is as follows.

**Theorem 6.1.** (See Theorem 4.1 of [7].) Assume that we are given a curve \( C \) with defining irreducible polynomial \( f \), a maximal determinantal representation for \( f^r \) as in (6.3) together with associated kernel bundle \( \tilde{E} \) (6.1) and left kernel bundle \( \tilde{E}_t \) (6.2) and a data set (D1)–(D5) for the interpolation problem (II)–(I3). Then the interpolation problem has a solution if and only if the interpolation data satisfy the compatibility conditions (6.6) at the overlapping zeros and poles, and the matrix \( \Gamma^0 \) given by (6.7) is square and invertible. In this case the unique solution of the interpolation problem (II)–(I3) is given by

\[
\gamma = \tilde{\gamma} - \sigma_1 \varphi(\Gamma^0)^{-1} \psi \sigma_2 + \sigma_2 \varphi (\Gamma^0)^{-1} \psi \sigma_1
\]  

(6.9)

with associated kernel bundle \( E \) (6.4) and left kernel bundle \( E_t \) (6.5), with \( S(z) \) given by

\[
S(z) = [I + \varphi(\xi_1 (z_1 I - A_1) + \xi_2 (z_2 I - A_2))^{-1} (\Gamma^0)^{-1} \psi (\xi_1 \sigma_1 + \xi_2 \sigma_2)]\big|_{E(z)}
\]  

(6.10)

and with \( S_t^{-1}(z) \) given (as a right multiplication operator) by

\[
S_t^{-1}(z) = [I - (\xi_1 \sigma_1 + \xi_2 \sigma_2) \varphi (\Gamma^0)^{-1} (\xi_1 (z_1 I - Z_1) + \xi_2 (z_2 I - Z_2))^{-1} \psi]\big|_{E_t(z)}.
\]  

(6.11)

Here the matrices \( A_1, A_2, Z_1, Z_2, \psi, \varphi \) are as in (6.8).
The main goal of this section is to use the machinery developed in Section 3 to make explicit the connections between Theorem 5.1 and Theorem 6.1 of Section 6.

Suppose therefore that $\chi$ and $\tilde{\chi}$ are two flat bundles over the Riemann surface $X$ with $h^0(\chi \otimes \Delta) = 0 = h^0(\tilde{\chi} \otimes \Delta)$. We use a fixed pair $(\lambda_1(p), \lambda_2(p))$ of meromorphic functions on $X$ generating $\mathcal{M}(X)$ to produce a map $\pi: X \to C$. The respective Cauchy kernels $K(\chi; \cdot, \cdot)$ and $K(\tilde{\chi}; \cdot, \cdot)$ generate corresponding maximal determinantal representations $z_1\tilde{\sigma}_2 - z_2\sigma_1 + \tilde{\gamma}$ and $z_1\sigma_2' - z_2\sigma_1' + \gamma'$ for $f(z_1, z_2)^r = 0$ as in Theorem 5.1. Note however that the formulas for $\sigma_1$ and $\sigma_2$ in (5.3) depend only on the choice of embedding functions $\lambda_1(p)$ and $\lambda_2(p)$, and not on the particular flat bundle $\chi$; hence we may and shall write simply $\sigma_i$ in place of $\tilde{\sigma}_i$ and $\sigma'_i$ for $i = 1, 2$.

We also have associated dual pairs of matrices of normalized section $s$: $\tilde{\chi}$ and $\chi$ for $E = \ker(z_1\sigma_2 - z_2\sigma_1 + \gamma')$ and $\tilde{E} = \ker(z_1\sigma_2 - z_2\sigma_1 + \tilde{\gamma})$ respectively, and $u^\times, u^\times_\ell$ for $E = \ker(z_1\sigma_2 - z_2\sigma_1 + $ $\gamma')$ and $E_\ell = \ker(z_1\sigma_2 - z_2\sigma_1 + \gamma')$. Since $\tilde{u}^\times$ implements an isomorphism between $\tilde{\chi}$ and $\tilde{E} \otimes \mathcal{O}(1) \otimes \Delta$ and $u^\times_\ell$ implements an isomorphism between $\chi$ and $E' \otimes \mathcal{O}(1) \otimes \Delta$, any meromorphic bundle map $T: \chi \to \tilde{\chi}$ induces a meromorphic bundle map $S: E' \to \tilde{E}$ determined by

$$S(p)u^\times(p) = \tilde{u}^\times(p)T(p).$$

However, in the solution of (CONINT) from 6 stated in Theorem 6.1 the solution $S$ is normalized to act as the identity operator over the points of $C$ at infinity. In order for the map $S$ constructed as above from the abstract bundle map $T: \chi \to \tilde{\chi}$ to achieve this normalization, we must make an adjustment

$$\alpha(z_1\sigma_2 - z_2\sigma_1 + \gamma')\beta = z_1\sigma_2 - z_2\sigma_1 + \gamma \quad (6.12)$$

on the input determinantal representation, where $\alpha, \beta \in GL(M^{mr}, \mathbb{C})$. If $u^\times, u^\times_\ell$ is the dual pair of normalized sections for $E = \ker(z_1\sigma_2 - z_2\sigma_1 + \gamma)$ and $E_\ell = \ker(z_1\sigma_2 - z_2\sigma_1 + \gamma)$, then

$$u^\times(p) = \beta^{-1}u^\times_\ell(p), \quad u^\times_\ell(p) = u^\times_\ell(p)\alpha^{-1}$$

and we seek to solve instead the equation

$$S(p)u^\times(p) = \tilde{u}^\times(p)T(p),$$

or equivalently

$$S(p)\beta^{-1}u^\times_\ell(p) = \tilde{u}^\times(p)T(p) \quad (6.13)$$

for $S, \alpha, \beta$ subject to the proviso that $S(x^i) = I_{E(x^i) = \tilde{E}(x^i)}$ for $i = 1, \ldots, m$. Since the columns of $u^\times_\ell(p)$ and $\tilde{u}^\times(p)$ (after multiplication by a local parameter on $X$ at $p$) simply form a standard basis in $\mathbb{C}^M$ when evaluated at $x^1, \ldots, x^m$, we see that we should take

$$\beta = \begin{bmatrix} T(x^1) \\ \vdots \\ T(x^m) \end{bmatrix}^{-1}.$$

In order to guarantee $\alpha\sigma_k\beta = \sigma_k$ for $k = 1, 2$ as required in (6.12) we then take

$$\alpha = \begin{bmatrix} T(x^1) \\ \vdots \\ T(x^m) \end{bmatrix}.$$
Thus \( \gamma = \alpha \gamma' \beta \) is given by \( \gamma = [\gamma_{ij}]_{i,j = 1, \ldots, m} \) with

\[
\gamma_{ij} = \begin{cases} 
  d_{1i}c_{1j} - c_{1i}d_{1j} & \text{if } i = j \\
  (c_{1i}c_{1j} + d_{1i}d_{1j})T(x^i)K(x^i, x^j)T(x^j)^{-1} & \text{if } i \neq j.
\end{cases}
\]

We remark that the "adjustment" of \( \gamma \) by the values of a bundle map at the points of \( X \) over the points of \( C \) at infinity plays a central role in the construction of triangular models for commuting nonselfadjoint operators; see [27] and [17], Chapter 12.

We now suppose that we are given an abstract interpolation data set \( \omega \) as in (3.2) for an Abstract Interpolation Problem (ABSINT) as in Section 3 with output \( \gamma \). We consider that the "adjustment" of \( \gamma \) from \( X \) to \( \tilde{X} \) is given by (5.4), all with \( \lambda \) functions \( \tilde{\chi} \) given by (5.3) and \( \tilde{\chi}' \) given by (3.2), all with \( \tilde{\chi} \) in place of \( \chi \). We assume that the pair of meromorphic functions \( \lambda_1(p) \) and \( \lambda_2(p) \) is chosen in such a way that the set of poles \( x^1, \ldots, x^m \) is disjoint from the preassigned poles \( \mu^1, \ldots, \mu^{n_\infty} \) and the set of preassigned zeros \( \lambda^1, \ldots, \lambda^{n_0} \). Define a data set \( \omega_0 \) for a (CONINT) problem as follows:

1. The preassigned poles consist of the points \( \pi(\mu^1) = (\mu^1_1, \mu^1_2, \ldots, \pi(\mu^{n_\infty}) = (\mu^{n_\infty}_1, \mu^{n_\infty}_2) \) with associated pole vectors \( \varphi_{j \beta} \in \tilde{E}(\pi(\mu^j)) \) given by \( \varphi_{j \beta} = \tilde{\omega}^\mu(x^j)u_{j \beta} \) for \( j = 1, \ldots, n_\infty \) and \( \beta = 1, \ldots, s_j \).
2. The preassigned zeros consist of the points \( \pi(\lambda^1) = (\lambda^1_1, \lambda^1_2, \ldots, \pi(\lambda^{n_0}) = (\lambda^{n_0}_1, \lambda^{n_0}_2) \) with associated sets of null vectors \( \psi_{i \alpha} \in \tilde{E}(\pi(\lambda)) \) given by \( \psi_{i \alpha} = x^i_\alpha \tilde{\omega}^\lambda(x^i) \) for \( i = 1, \ldots, n_0 \) and \( \alpha = 1, \ldots, t_i \).
3. For those pairs of indices \((i, j)\) for which \( z^i = w^j \) \( := \xi^{ij} \) we take the associated coupling numbers \( \rho_{ij, \alpha \beta} \) to be the same as those specified for the (ABSINT) problem.

With this choice of data set, the reader can check that the matrices \( \varphi \) and \( \psi \) as defined in (6.8) reduce to

\[
\varphi = \begin{bmatrix} 
K_{\mu, u}(x^1) \\
\vdots \\
K_{\mu, u}(x^m)
\end{bmatrix}, \quad \psi = -\begin{bmatrix} 
K^{x^\cdot \lambda}(x^1) & \ldots & K^{x^\cdot \lambda}(x^m)
\end{bmatrix}
\]

where the notation \( K_{\mu, u}(p) \) and \( K^{x^\cdot \lambda}(p) \) is as in the statement of Theorem 3.1.

It turns out that the (ABSINT) problem with data set \( \omega \) is equivalent to the (CONINT) problem with data set \( \omega_0 \) under the identifications \( \pi: X \to C \) and \( \tilde{\omega}^\mu: \tilde{X} \to \tilde{E} \otimes O(1) \otimes \Delta \) and \( \tilde{\omega}^\lambda: \tilde{X}' \to \tilde{E} \otimes O(1) \otimes \Delta \) sketched above, in that a solution \( T \) of (ABSINT) corresponds to a solution \( S \) of (CONINT) under the correspondence (including the normalization at the points over infinity) between abstract bundle maps \( T \) and concrete bundle maps \( S \) discussed above. (For the first two interpolation conditions, this observation is rather transparent. For the third interpolation condition (I3), this requires the relation between the interpolation condition (I3) for the (CONINT) problem with a flat connection on the bundle \( \tilde{E} \otimes O(1) \otimes \Delta \) and the correspondence of this connection with the coefficient \( A_t(p) \) in the Laurent expansion of the Cauchy kernel \( K(\tilde{\chi}^\cdot, \cdot) \) along the diagonal; this is explained in Section 3.2 of [17].) Hence the formula for the solution \( T \) of (ABSINT) in (3.11) must correspond to the formula for the solution \( S \) of (CONINT) in (5.10).
under the correspondence \( (6.13) \). The point of the next result is to verify this directly; in addition we see that the matrix \( \Gamma \) appearing in Theorem \( 1.1 \) is identical to the matrix \( \Gamma^0 \) appearing in Theorem \( 6.1 \).

**Theorem 6.2.** Let \( \omega \) be the data set for an (ABSINT) problem with \( \omega_0 \) the corresponding data set for a (CONINT) problem. Then \( \Gamma = \Gamma_0 \). Furthermore, if \( \Gamma \) is invertible and \( \chi \) is the input bundle for which (ABSINT) has a solution, then a solution \( T \) of (ABSINT) is related to the unique solution \( S \) of (CONINT) having value identity on the fibers over the points at infinity according to the intertwining condition \( (6.13) \).

**Proof.** The fact that \( \Gamma = \Gamma_0 \) is a simple consequence of the definitions and of the formula \( (6.12) \) expressing the Cauchy kernel \( K(\chi; \cdot, \cdot) \) in terms of a dual pair \( \tilde{u}^\times, \tilde{u}^{\times'} \) of normalized sections of \( E \) and \( \tilde{E}_c \).

It remains to verify the intertwining relation \( (6.13) \)

\[
S(p)\beta^{-1}u^{\times'}(p) = \tilde{u}^\times(p)T(p)
\]

where \( S \) is given by \( (6.10) \), \( T \) by \( (3.11) \) and \( \beta^{-1} = \text{diag.} \{T(x^i)\} \). We compute

\[
[S(p)\beta^{-1}u^{\times'}(p)]_i = \sum_{j=1}^m S_{ij}(p)T(x^j)K(\chi; x^j, p)
\]

\[
= T(x^i)K(\chi; x^i, p) - K_{\mu, u}(x^i) \cdot \text{diag.} \left\{ \left( \xi_1\lambda_1(p) + \xi_2\lambda_2(p) - \xi_1\mu_{1^i}^1 - \mu_{2^i}^2 \right)^{-1}I_{s^i} \right\} \cdot \Gamma^{-1} \cdot \sum_{j=1}^m \left( \xi_1c_{j1} + \xi_2c_{j2} \right) K^{\times, \lambda}(\lambda^j)T(x^j)K(\chi; x^j, p).
\]

From \( (6.11) \) with \( x^i \) in place of \( p \) and \( p \) in place of \( q \) (and hence \( T(p) \) in place of \( Q \) we see that

\[
T(x^i)K(\chi; x^i, p) = [K(\tilde{\chi}; x^i, p) + K_{\mu, u}(x^i)\Gamma^{-1}K^{\times, \lambda}(p)]T(p).
\]

We use this identity both in the form indicated and with \( x^j \) in place of \( x^i \) to convert \( (6.15) \) to

\[
[S(p)\beta^{-1}u^{\times'}(p)]_i = [K(\tilde{\chi}; x^i, p) + K_{\mu, u}(x^i)\Gamma^{-1}K^{\times, \lambda}(p)]T(p) - K_{\mu, u}(x^i) \cdot \text{diag.} \left\{ \left( \xi_1\lambda_1(p) + \xi_2\lambda_2(p) - \xi_1\mu_{1^i}^1 - \mu_{2^i}^2 \right)^{-1}I_{s^i} \right\} \cdot \Gamma^{-1} \cdot \sum_{j=1}^m \left( \xi_1c_{j1} + \xi_2c_{j2} \right) K^{\times, \lambda}(\lambda^j)[K(\tilde{\chi}; x^i, p) + K_{\mu, u}(x^j)\Gamma^{-1}K^{\times, \lambda}(p)]T(p).
\]

Next we use the general identity (see also \( \| \))

\[
\sum_{j=1}^m \left( \xi_1c_{j1} + \xi_2c_{j2} \right) K(\tilde{\chi}; p, x^j)K(\tilde{\chi}; x^j, q) =
\]

\[
= \left( \xi_1\lambda_1(q) + \xi_2\lambda_2(q) - \xi_1\lambda_1(p) - \xi_2\lambda_2(p) \right) K(\tilde{\chi}; p, q)
\]

which is valid for all distinct points \( p, q \) in \( X \) which are disjoint from \( x^1, \ldots, x^m \). To prove this “collection formula” \( (6.17) \), consider each side as a function of \( p \) with \( q \) fixed. Since \( h^0(\tilde{\chi} \otimes \Delta) = 0 \), it suffices to show that the local principal part in the Laurent series expansion at each pole of each side matches with the local principal part of the other side. One can check that the only possible poles are all simple
and occur at \( x^i, \ldots, x^m \) with residue of each side at \( x^i \) equal to the common value 
\[-(\xi_1 c_{1i} + \xi_2 c_{2j}) dt(x^i) K(\tilde{\chi}; x^i, q) \]. Immediate consequences of the identity (6.17)
which are important for our context here are:

\[
\sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K^{x, \lambda}(\lambda^i) K(\tilde{\chi}; x^i, p) = \\
= \text{diag}.\{ (\xi_1 \lambda_1(p) + \xi_2 \lambda_2(p) - \xi_1 \lambda^j_1 - \xi_2 \lambda^j_2) I_{i,j} \} \cdot K^{x, \lambda}(p),
\]

(6.18)

and, if \((i', j')\) is a pair of indices for which \( \lambda^i \neq \mu^j \) then the \((i', j')\)-matrix entry
of \( \sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K^{x, \lambda}(x^j) K_{\mu, u}(x^j) \) is given by

\[
\left[ \sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K^{x, \lambda}(x^j) K_{\mu, u}(x^j) \right]_{i', j'} = \\
= (\xi_1 \mu^i_1 + \xi_2 \mu^i_2 - \xi_1 \lambda^j_1 - \xi_2 \lambda^j_2) x^i_{i', j'} K(\tilde{\chi}; \lambda^i, \mu^j) u_{j'} \\
= -(\xi_1 \mu^i_1 + \xi_2 \mu^i_2 - \xi_1 \lambda^j_1 - \xi_2 \lambda^j_2) \Gamma_{i', j'}.
\]

Hence in matrix form we have

\[
\sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K^{x, \lambda}(x^j) K_{\mu, u}(x^j) = \\
= \text{diag}.\{ (\xi_1 \lambda^i_1 + \xi_2 \lambda^i_2) I_{i', j'} \} \cdot \Gamma - \Gamma \cdot \text{diag}.\{ (\xi_1 \mu^i_1 + \xi_2 \mu^i_2) I_{i', j'} \}.
\]

(6.19)

In the case where \( p = q \), the collection formula (6.17) takes the limiting form

\[
\sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K(\tilde{\chi}; p, x^j) K(\tilde{\chi}; x^j, p) = \\
= -(\xi_1 \lambda^i_1(p) + \xi_2 \lambda^i_2(p)) \left. dt(p) \right|_{i', j'} 
\]

(6.20)

where \( \cdot = \frac{d}{dt} \) where \( t \) is a local coordinate centered at \( p \). An application of this
degenerate collection formula (6.20) gives, for \( \lambda^i = \mu^j \),

\[
\left[ \sum_{j=1}^{m} (\xi_1 c_{j1} + \xi_2 c_{j2}) K^{x, \lambda}(x^j) K_{\mu, u}(x^j) \right]_{i', j'} = \\
= -(\xi_1 \lambda^i_1(\xi^i_1) + \xi_2 \lambda^i_2(\xi^i_2)) \left. dt(\xi^i_1) x^i_{i', j'} u_{j'} \right|_{i', j'} = 0
\]

where we used the compatibility condition (6.1) for the last step. We conclude that
(6.19) continues to be valid even in the case where \( \lambda^i = \mu^j \).
Moreover, as explained in [7], it is possible to construct a matrix of normalized biholomorphic equivalence between $\chi$ and the intertwining relation (6.13) follows. In the context of the (CONINT) problem, Theorem 6.1 solves the problem of identifying the input bundle. Namely, the input bundle $\omega$ computing explicitly the unknown input bundle $\omega^\gamma$ such that

$$
\sigma_1 \omega^\gamma = \ker(z_1 \sigma_2 - z_2 \sigma_1 + \gamma)
$$

where $\sigma_1, \sigma_2$ and $\gamma$ are given by (5.3) and $\gamma$ is given by (13.3). The vector bundle $\chi$ is then determined up to biholomorphic equivalence by the condition

$$
\chi \cong E \otimes \mathcal{O}(1) \otimes \Delta.
$$

Moreover, as explained in [4], it is possible to construct a matrix of normalized sections $u^\gamma$ for $E$ from working with the minors of the matrix pencil $z_1 \sigma_2 - z_2 \sigma_1 + \gamma$. Such a matrix of normalized sections $u^\gamma$ in turn implements concretely the biholomorphic equivalence between $\chi$ and $E \otimes \mathcal{O}(1) \otimes \Delta$.

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