Invariance Property of Orbifold Elliptic Genus for Multi-Fans

Akio Hattori

1. Introduction

The complex elliptic genus was first introduced by Witten [21] and then studied by several authors such as Hirzebruch [13], Bott and Taubes [5] mainly in connection with its rigidity property. It was further generalized in two ways; one way to complex orbifolds and the other to singular projective varieties.

Generalization to singular varieties was given by Borisov-Libgober [3, 4]. They called it singular elliptic genus; it is defined for Kawamata-log-terminal pairs \((X, D)\) of a variety \(X\) and a \(\mathbb{Q}\)-divisor \(D\). We shall denote it by \(\text{Ell}_{\text{sing}}(X, D)\). It has an invariant property with respect to blow-ups. Namely, if \(f : \tilde{X} \to X\) is a blow-up along a non-singular locus in \(X\) which is normal crossing to \(\text{Supp}(D)\) and \(\tilde{D}\) is a divisor on \(\tilde{X}\) such that

\[
K_{\tilde{X}} + \tilde{D} = f^*(K_X + D),
\]

then

\[
\text{Ell}_{\text{sing}}(\tilde{X}, \tilde{D}) = \text{Ell}_{\text{sing}}(X, D).
\]

The formula (1) is related to the work of Totaro [18]. He showed that the Chern numbers that can be extended to singular varieties, compatibly with \(\text{IH}\)-small resolutions, are at most linear combinations of the coefficients of the elliptic genus. The formula (1) implies that the elliptic genus can in fact be defined for projective varieties with Kawamata-log-terminal singularities, complementing the result of Totaro. In particular, if \(X\) is such a variety and \(f : \tilde{X} \to X\) is a crepant resolution, then

\[
\text{Ell}(\tilde{X}) = \text{Ell}_{\text{sing}}(X),
\]

where \(\text{Ell}_{\text{sing}}(X) = \text{Ell}_{\text{sing}}(X, 0)\) and \(\text{Ell}(\tilde{X})\) denotes the ordinary elliptic genus of \(\tilde{X}\).

Orbifold elliptic genus was introduced by Borisov-Libgober [2] for global quotient complex orbifolds, and then was generalized to general complex (or more generally to stably complex) orbifolds by Don-Liu-Ma [7]. Borisov and Libgober in [4] also defined orbifold elliptic genus \(\text{Ell}_{\text{orb}}(X, D, G)\) where \(G\) is a finite group.
and \((X, D)\) is a Kawamata-log-terminal \(G\)-normal pair with smooth \(X\) and showed a similar formula to (1). They also proved a formula

\[
\Ell_{\text{orb}}(X, D, G) = \Ell_{\text{sing}}(X/G, D_{X/G})
\]

for a suitably defined divisor \(D_{X/G}\). For example, when \(D = 0\), \(D_{X/G}\) is given by the following formula. Let \(\pi : X \to X/G\) be the quotient map and \(E = \sum (a_i - 1)E_i\) the ramification divisor of \(\pi\) where the sum runs over prime divisors \(E_i\). Then \(D_{X/G}\) is given by

\[
D_{X/G} = \sum \frac{a_i - 1}{a_i} \pi(E_i).
\]

A similar formula was already given by Batyrev [1] for \(E\)-function. The \(E\)-function is a generalization of Hirzebruch’s \(\chi_y\)-genus to singular varieties. The elliptic genus is also a generalization of \(\chi_y\)-genus. Suppose that the fixed point set \(X^g\) of the action of each element \(g \in G\) has codimension at least two. Then the ramification divisor is trivial, and the formula (3) for \(D = 0\) reduces to

\[
\Ell_{\text{orb}}(X, G) = \Ell_{\text{sing}}(X/G).
\]

If moreover \(X/G\) has a crepant resolution \(\tilde{X} \to X/G\), then we get

\[
\Ell_{\text{orb}}(X, G) = \Ell_{\text{orb}}(\tilde{X}),
\]

by (2). This sort of results goes back to [6] where stringy Euler number is considered instead of singular elliptic genus, and is related to an observation of Mckay concerning the relation between minimal resolutions of quotient singularities \(\mathbb{C}^2/G\) and the representations of \(G\).

Borizov and Libgober define in [4] not only the genus but a class \(\E_{\text{orb}}(X, D, G)\) for \(G\) normal pair \((X, D)\) in the Chow ring \(A_*(X)\) in such a way that the elliptic genus \(\Ell_{\text{orb}}(X, D, G)\) becomes the degree of the top component of \(\E_{\text{orb}}(X, D, G)\). They then prove the functorial property

\[
f_* \E_{\text{orb}}(\tilde{X}, \tilde{D}, G) = \E_{\text{orb}}(X, D, G),
\]

where \((\tilde{X}, \tilde{D})\) and \((X, D)\) are \(G\)-normal pairs related together as in (1). The formula (6) is sometimes called change of variables formula. The main result of [4] is the following formula

\[
\pi_* \E_{\text{orb}}(X, D, G) = \E_{\text{sing}}(X/G, D_{X/G}),
\]

where \(\E_{\text{sing}}(X/G, D_{X/G})\) is a class in \(A_*(X/G)\) defined in a similar way such that the degree of its top component coincides with \(\Ell_{\text{sing}}(X/G, D_{X/G})\). This immediately follows from (7). There is an equivariant version due to Waelder [19] which is a good reference for this subject. See also [20].

The (complex) orbifold elliptic genus is defined for compact, stably almost complex orbifolds in general. We shall write it \(\hat{\varphi}(X)\). It essentially depends on orbifold structures. There are many examples of orbifolds with the same underlying space but with different orbifold structures and different orbifold elliptic genus. This phenomenon is related to the above formulæ (3), (5) and (7). The singular elliptic genus is defined by using resolution of singularities. One might hope to get a direct, topological definition of the singular elliptic genus which can be extended to a larger class of singular spaces. In the case of orbifolds one already has orbifold elliptic genus. One wants to get a suitable notion of \(\mathbb{Q}\)-divisors and Chow ring which can be applied to formulate change of variables formula.
There is a class of orbifolds, called torus orbifolds, in which one can build a satisfactory theory. A torus orbifold $X$ of dimension $2n$ is, roughly speaking, a 2$n$-dimensional compact stably almost complex orbifold with an action of an $n$-dimensional torus $T$. Torus orbifolds can be considered as topological counterparts of $\mathbb{Q}$-factorial toric varieties. For a $\mathbb{Q}$-factorial toric variety $X$ of dimension $n$ there is associated a simplicial fan $\Delta$ in an $n$-dimensional lattice $L$. Algebro-geometric properties of a toric variety are translated to those of the fan associated to the variety. To each edge (one dimensional cone) of $\Delta$ there corresponds an irreducible $T$-divisor $D_i$ and there is an exact sequence

$$0 \rightarrow L^* \rightarrow \bigoplus_i \mathbb{Z} \cdot D_i \rightarrow A_{n-1}(X) \rightarrow 0,$$

where $A_{n-1}(X)$ is the $(n-1)$-th Chow group of $X$, see e.g. [8]. We note here that the dual lattice $L^*$ can be identified with the second cohomology $H^2(BT)$ of a classifying space of $T$, and the middle term of [8] is identified with the second equivariant cohomology $H^2_{\mathbb{Q}}(X; \mathbb{Q})$ after tensored by $\mathbb{Q}$. It is also identified with the degree two part of the Stanley–Reisner ring of $\Delta$ considered as a simplicial set.

To a torus orbifold there is associated a simplicial multi-fan, an analogue of fan, and an integral edge vector is assigned to each 1-dimensional cone of the multi-fan, see [16], [1]. These vectors are not primitive in general and they reflect the orbifold structure of the torus orbifold whereas one always takes primitive vectors when dealing with toric varieties. Moreover divisors over a torus orbifold can be defined as homogeneous elements of degree two in the Stanley-Reisner ring of the simplicial set associated to the multi-fan.

The (equivariant, stabilized) orbifold elliptic genus $\hat{\varphi}_{st}(\Delta, \mathcal{V}, \xi)$ is defined for triples of a simplicial multi-fan $\Delta$, a set of edge vectors $\mathcal{V}$ and a $\mathbb{Q}$-divisor $\xi$. We can go further to define orbifold elliptic class $\hat{\mathcal{E}}_{st}(\Delta, \mathcal{V}, \xi)$ of such triples in the Stanley-Reisner ring with $\mathbb{Q}$-coefficients. When $\Delta$ is the multi-fan associated to a torus orbifold $X$ the orbifold elliptic genus and orbifold elliptic class are the invariants of $X$ and the divisor $\xi$. The push-forward from the Stanley-Reisner ring to the complex numbers $\mathbb{C}$ sends the orbifold elliptic class to the orbifold elliptic genus.

Birational morphisms between multi-fans can be defined in such a way that they correspond to geometric birational morphisms between toric varieties. Moreover if $f : \Delta' \rightarrow \Delta$ is a birational morphism, and $\mathcal{V}$ and $\mathcal{V}'$ are sets of generating edge vectors for $\Delta$ and $\Delta'$ respectively, then the pull-back $f^*$ and the push-forward $f_*$ between the corresponding Stanley-Reisner rings are defined depending on not only $f$ but $\mathcal{V}$ and $\mathcal{V}'$.

The main theorem of the present paper can be stated in the following form.

**Theorem.** Let $(\Delta, \mathcal{V}, \xi)$ be a triple of simplicial multi-fan, a set of edge vectors and a $\mathbb{Q}$-divisor. Let $f : \Delta' \rightarrow \Delta$ be a birational morphism and $\mathcal{V}'$ a set of edge vectors for the multi-fan $\Delta'$. Then

$$f_* \hat{\mathcal{E}}_{st}(\Delta', \mathcal{V}', f^*(\xi)) = \hat{\mathcal{E}}_{st}(\Delta, \mathcal{V}, \xi).$$

**Corollary.** Under the above situation

$$\hat{\varphi}_{st}(\Delta', \mathcal{V}', f^*(\xi)) = \hat{\varphi}_{st}(\Delta, \mathcal{V}, \xi).$$

The canonical class $K_X$ of a $\mathbb{Q}$-factorial toric variety $X$ corresponds to $K_\Delta = -\sum_i x_i$ where $\Delta$ is the fan associated to $X$ and $x_i$ is the generator of the Stanley-Reisner ring corresponding to $D_i$. For a divisor $D = \sum_i a_i D_i$ we put $\xi = -K_\Delta - \sum_i a_i x_i.$
$\sum_i a_i x_i$. Then the singular elliptic genus $Ell_{\text{sing}}(X, D)$ is equal to $\hat{\phi}_{st}(\Delta, \mathcal{V}, \xi)$ up to a multiplicative constant depending only on the dimension of $X$ where all the vectors in $\mathcal{V}$ are taken primitive; cf. Remark 3.1.

As an example we take a complete non-singular toric variety (more generally a torus manifold) $X$. Let $G$ be a finite subgroup of the (compact) torus $T$ acting on $X$. Let $a_i$ be the order of the isotropy subgroup of $G$ at a generic point in $D_i$. Then the ramification divisor of the quotient map $X \to X/G$ is $\sum_i (a_i - 1) D_i$. Let $\Delta$ be the fan associated to the toric variety $X/G$, and set $\Delta' = \Delta$, $f = id : \Delta' \to \Delta$, $\mathcal{V} = \{v_i\}$ with all the $v_i$ primitive and $\mathcal{V}' = \{a_i v_i\}$. We then require $\xi$ to satisfy $f^* (\xi) = -K_{\Delta} = \sum_i x_i$. We have $\xi = \sum_i \frac{1}{a_i} x_i$; see Remark 6.1. In this case (9) is equivalent to (3) with $D = 0$ and $D_{X/G}$ given by (4). Note that $\mathcal{V}'$ corresponds to the orbifold structure of $X/G$ which has $X$ itself as an orbifold chart but $\mathcal{V}$ does not in general.

For the proof of Theorem we first prove Corollary. For that purpose we use an expression of equivariant orbifold elliptic genus as a character of the accompanying torus. That character formula was originally due to Borisov-Libgober in the case of Gorenstein toric varieties [2] and then was generalized to the case of multi-fans by [12]. The formula behaves well with respect to birational morphisms and reduces the invariance property of orbifold elliptic genus to its local version. The local invariance property is derived from the rigidity-vanishing property of orbifold elliptic genus which was exploited in [9] and [10]. Theorem itself follows from the functorial property of push-forward and the local version of the genus by using Mayer-Vietoris argument.

The paper is organized as follows. In Section 2 materials concerning multi-fans which are needed later are given. Orbifold elliptic class and orbifold elliptic genus of simplicial multi-fans are introduced in Section 3. The main theorem of this section is Theorem 3.3. It states that the (equivariant) orbifold elliptic genus is a character of the torus. In the case of multi-fans associated with torus orbifolds the orbifold elliptic genus is the index of a Dirac operator with bundle coefficients, from which Theorem follows. For general simplicial multi-fans a combinatorial proof is needed. In Section 4 vanishing theorems are given generalizing those of [10]. Section 5 is devoted to a character formula of the Borisov-Libgober type generalizing that of [12]. The invariance property of orbifold elliptic genus (Corollary above) is stated in Section 6 as Theorem 6.3 Its proof is given in Section 7 after the local version (Theorem 7.4) is established. The functorial property of orbifold elliptic class (Theorem above) is stated as Theorem 8.7 and is proved in Section 8. The final section is devoted to a generalization of orbifold elliptic genus to not necessarily simplicial multi-fans. It will be shown that the orbifold elliptic genus can be defined for triples $(\Delta, \mathcal{V}, \xi)$ with $\xi$ $\mathbb{Q}$-Cartier and, in particular, for $\mathbb{Q}$-Gorenstein pairs $(\Delta, \mathcal{V})$.

2. Preliminaries

We shall recall basic definitions and facts concerning simplicial multi-fans which will be used in the sequel. For details we refer to [11], [12].

Let $L$ be a lattice of rank $n$ (the notation $N$ is customary in literatures instead of $L$). A simplicial multi-fan in $L$ is a triple $\Delta = (\Sigma, C, w^\pm)$. Here $\Sigma$ is an augmented finite simplicial set, that is, $\Sigma$ is a finite simplicial set with empty set $*=\emptyset$ added as the unique $(-1)$-dimensional simplex. $\Sigma^{(k)}$ denotes the $k-1$ skeleton of $\Sigma$ so
that \( \ast \in \Sigma^{(0)} \). We assume that \( \Sigma = \bigcap_{k=0}^{n} \Sigma^{(k)} \), and \( \Sigma^{(n)} \neq \emptyset \). We further assume that any \( J \in \Sigma \) is contained in some \( I \in \Sigma^{(n)} \).

\( C \) is a map from \( \Sigma^{(k)} \) into the set of \( k \)-dimensional strongly convex rational polyhedral cones in the vector space \( L_{\mathbb{R}} = L \otimes \mathbb{R} \) for each \( k \) such that \( C(\ast) = \{0\} \), and if \( J \) is a face of \( I \), then \( C(J) \) is a face of \( C(I) \). We require the following condition to be satisfied. For any \( I \in \Sigma \) the map \( C \) restricted to \( \{J \in \Sigma \mid J \subset I\} \) is an isomorphism of ordered sets onto the set of faces of \( C(I) \).

\( w^\pm \) are maps \( \Sigma^{(n)} \to \mathbb{Z}_{\geq 0} \) which, when \( \Sigma \) is complete, satisfy certain compatibility conditions, as we shall explain below. We set \( w(I) = w^+ (I) - w^- (I) \).

For each \( K \in \Sigma \) we set

\[
\Sigma_K = \{ J \in \Sigma \mid K \subset J \}.
\]

It inherits the partial ordering from \( \Sigma \) and becomes an augmented simplicial set where \( K \) is the unique minimum element in \( \Sigma_K \). Let \( (L_K)_{\mathbb{R}} \) be the linear subspace of \( L_{\mathbb{R}} \) generated by \( C(K) \). Put \( L_K = L \cap (L_K)_{\mathbb{R}} \) and define \( L^K \) to be the quotient lattice of \( L \) by \( L_K \). For \( J \in \Sigma_K \) we define \( C_K(J) \) to be the cone \( C(J) \) projected on \( L^K \otimes \mathbb{R} \). We define two functions

\[
w^\pm_K : \Sigma^{(n-|K|)}_K \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}
\]

to be the restrictions of \( w^\pm \) to \( \Sigma^{(n-|K|)}_K \). The triple \( \Delta_K := (\Sigma_K, C_K, w^\pm_K) \) is a multi-fan in \( L^K \) and is called the projected multi-fan with respect to \( K \in \Sigma \). For \( K = \emptyset \), the projected multi-fan \( \Delta_K \) is nothing but \( \Delta \).

A vector \( v \in L_{\mathbb{R}} \) will be called generic if \( v \) does not lie on any linear subspace spanned by a cone in \( C(\Sigma) \) of dimension less than \( n \). For a generic vector \( v \) we set \( d_v = \sum_{I \in C(I)} w(I) \), where the sum is understood to be zero if there is no such \( I \).

**Definition.** A simplicial multi-fan \( \Delta = (\Sigma, C, w^\pm) \) is called pre-complete if the integer \( d_v \) is independent of generic vectors \( v \). In this case this integer will be called the degree of \( \Delta \) and will be denoted by \( d(\Delta) \). A pre-complete multi-fan \( \Delta \) is said to be complete if the projected multi-fan \( \Delta_K \) is pre-complete for any \( K \in \Sigma \).

A multi-fan is complete if and only if the projected multi-fan \( \Delta_J \) is pre-complete for any \( J \in \Sigma^{(n-1)} \). Let \( v \) be a vector such that its projection \( \bar{v} \) is generic for the multi-fan \( \Delta_K \). For a complete multi-fan we have

\[
deg(\Delta_K) = \sum_{I \in S_v(K)} w(I) \quad \text{where} \quad S_v(K) = \{ I \in \Sigma^{(n-k)}_K \mid \bar{v} \in C_K(I) \}.
\]

In the sequel we shall often consider a set \( \mathcal{V} \) consisting of non-zero edge vectors \( v_i \) for each \( i \in \Sigma^{(1)} \) such that \( v_i \in L \cap C(i) \). We do not require \( v_i \) to be primitive. For any \( J \in \Sigma \) let \( L_{J,\mathcal{V}} \) be the sublattice of \( L_J \) generated by \( \{v_i\}_{i \in J} \) and \( L^*_J,\mathcal{V} \) the dual lattice. Let \( \{u^I_i\}_{i \in J} \) be the basis of \( L^*_J,\mathcal{V} \) dual to \( \{v_i\}_{i \in J} \). For \( I \in \Sigma^{(n)} \) we put \( I(v) = \{ i \in I \mid \langle u^I_i, v \rangle < 0 \} \). Then it can be easily seen that \( S_v(K) \) is written as

\[
S_v(K) = \{ I \in \Sigma^{(n)} \mid I(v) \subset K \subset I \}.
\]

Let \( \Delta = (\Sigma, C, w^\pm) \) be a simplicial multi-fan in a lattice \( L \) and \( \mathcal{V} = \{v_i\}_{i \in \Sigma^{(1)}} \) a set of prescribed edge vectors as before. We denote the torus \( L_{\mathbb{R}}/L \) by \( T \). We define the equivariant cohomology \( H^*_T(\Delta) \) of a multi-fan \( \Delta \) as the Stanley-Reisner ring of the simplicial complex \( \Sigma \). Namely let \( \{x_i\} \) be indeterminates indexed by \( \Sigma^{(1)} \), and let \( R \) be the polynomial ring over the integers generated by \( \{x_i\} \). We
denote by $\mathcal{J}$ the ideal in $R$ generated by monomials $\prod_{i \in J} x_i$ such that $J \notin \Sigma$. $H^*_T(\Delta)$ is by definition the quotient $R/\mathcal{J}$.

The dual lattice $L^*$ is canonically identified with $H^2(BT)$ where $BT$ is a classifying space of $T$ so that $L^* \otimes \mathbb{Q}$ is identified with $H^2(BT) \otimes \mathbb{Q}$. We put $L_\mathcal{V} = \bigcap_{I \in \Sigma(n)} L_{I, \mathcal{V}}$. It is a sublattice of $L$ of the same rank $n$ and $L^*_\mathcal{V}$ contains $L^*$. $L^*_\mathcal{V} \otimes \mathbb{Q}$ is identified with $H^*(BT) \otimes \mathbb{Q}$. Let $S(L^*)$ be the symmetric algebra of $L^*$. It is identified with the polynomial algebra $S^*(L)$ over $L$. It is also identified with $H^*(BT)$. Similarly $S(L^*) \otimes \mathbb{Q}$ and $S(L^*_\mathcal{V}) \otimes \mathbb{Q}$ are identified with $H^*(BT) \otimes \mathbb{Q}$.

We regard $L^*_\mathcal{V}$ as a submodule of $H^*_T(\Delta)$ by the formula

$$u = \sum_{i \in \Sigma(\mathcal{V})} \langle u, v_i \rangle x_i.$$ 

This determines an $S^*(L_\mathcal{V})$-module structure of $H^*_T(\Delta)$ and an $H^*(BT) \otimes \mathbb{Q}$-module structure of $H^*_T(\Delta) \otimes \mathbb{Q}$. It should be noticed that these module structures depend on the choice of vectors $\mathcal{V}$ as above. In order to emphasize this dependence we shall write $H^*_T(\Delta, \mathcal{V})$ instead of $H^*_T(\Delta)$.

For $K \in \Sigma(k)$ let $\{u^K_i\}_{i \in K}$ be the basis of $L^*_K \otimes \mathbb{Q}$ dual to the basis $\{v_i\}_{i \in K}$ of $L_K$ as before. The restriction homomorphism $\iota^*_K : H^*_T(\Delta, \mathcal{V}) \to S^*(L_K, \mathcal{V})$ is determined by

$$\iota^*_K(x_i) = \begin{cases} u^K_i & \text{for } i \in K \\ 0 & \text{for } i \notin K. \end{cases}$$

For $J \subseteq K$ let $\iota^K_J : S^*(L_K, \mathcal{V}) \to S^*(L_J, \mathcal{V})$ be the induced homomorphism of the inclusion $\iota^K_J : J \to K$. If $J \subseteq K$ then $\iota^K_J \circ \iota^K_J = \iota^K_J$.

For $I \subseteq \Sigma(n)$ we have

$$\iota^*_I(u) = u \quad \text{for } u \in L^*_I.$$ 

In particular $\iota^*_I|L^*_\mathcal{V}$ is the identity map for any $I \subseteq \Sigma(n)$, and $\iota^*_I$ is an $S^*(L_\mathcal{V})$-module map. Note that $\bigoplus_{I \in \Sigma(n)} \iota^*_I$ embeds $H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q}$ into $(H^*(BT) \otimes \mathbb{Q})^{\Sigma(n)}$. Its image is given by the subset consisting of the elements $(u(I))_{I \in \Sigma(n)}$ satisfying

$$\iota^*_I(u(I_1)) = \iota^*_I(u(I_2)) \text{ for any } I_1, I_2 \subseteq \Sigma(n) \text{ with } I_1 \cap I_2 \neq \emptyset.$$ 

Let $S$ be the multiplicative subset of $S^*(L_\mathcal{V})$ consisting of non-zero elements in $S^*(L_\mathcal{V})$ and let $S^{-1}$ denote the localization by $S$. For $K \subseteq \Sigma(k)$ put

$$H_{K, \mathcal{V}} = L_K/L_{K, \mathcal{V}}.$$ 

$H_{K, \mathcal{V}}$ will be simply written $H_K$ when it is clear that $\mathcal{V}$ is understood in the context. We define the push-forward $\pi_* : H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q} \to S^{-1}H^*(BT) \otimes \mathbb{Q}$ by

$$\pi_*(x) = \sum_{I \in \Sigma(n)} \frac{w(I)x}{|H_{I, \mathcal{V}}| \prod_{i \in I} u_i^*}.$$ 

It is an $H^*(BT) \otimes \mathbb{Q}$-module homomorphism. When $\Delta$ is complete it is known that the image of $\pi_*$ is contained in $H^*(BT) \otimes \mathbb{Q}$, cf. [11].

Remark 2.1. For details concerning torus orbifolds and their associated multi-fans we refer to [9] and [10]. To a torus orbifold $X$ a complete simplicial multi-fan $\Delta$ and a set of edge vectors $\mathcal{V}$ are associated, and there is a canonical map

$$\kappa : H^*_T(X) \otimes \mathbb{Q} \to H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q}$$
which is an $H^*(BT) \otimes \mathbb{Q}$-homomorphism. It becomes an isomorphism in favorable cases. To each $i \in \Sigma^{(1)}$ there corresponds a $T$-invariant suborbifold of codimension two whose equivariant Poincaré dual is mapped to $x_i \in H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$ by $\kappa$. The first Chern class $c_1(X) \in H^2_T(X) \otimes \mathbb{Q}$ is mapped to $\sum_{i \in \Sigma^{(1)}} x_i$, cf. \cite{9}, Remark 3.2 and \cite{10}, Remark 2.5.

Remark 2.2. Suppose that $X$ is a $\mathbb{Q}$-factorial toric variety and $\Delta$ is its associated fan. In this case one usually takes all the vectors in $\mathcal{Y} = \{v_i\}$ primitive. Then $H^2_T(\Delta, \mathcal{Y})$ is identified with the module of all $T$-Weil divisors. Moreover $L^* = H^2(BT)$ is contained in the submodule $Div_T X$ of the $T$-Cartier divisors, and the quotients $Div_T X/L^*$ and $H^2_T(\Delta, \mathcal{Y})/L^*$ are identified with the Picard group $Pic(X)$ and the divisor class group $A_{n-1}(X)$ respectively. The element $-\sum_{i \in \Sigma^{(1)}} x_i$ corresponds to a canonical divisor $K_X$. See \cite{8} Sections 3.4, 4.3 and 4.4.

In this paper elements of $H^2_T(\Delta, \mathcal{Y})$ and $H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$ will be called divisors and $\mathbb{Q}$-divisors respectively. In the same spirit we adopt the following

Definition. A divisor $\xi$ is called $T$-Cartier if $\iota^*_T(\xi)$ is contained in $L^* = H^2(BT)$ for all $I \in \Sigma^{(n)}$.

We need another description of the group $H_K = H_{K, \mathcal{Y}}$. For simplicity identify the set $\Sigma^{(1)}$ with $\{1, 2, \ldots, m\}$ and consider a homomorphism $\eta : \mathbb{R}^m = \mathbb{R}^{\Sigma^{(1)}} \rightarrow L_\mathbb{R}$ sending $a = (a_1, a_2, \ldots, a_m)$ to $\sum_{i \in \Sigma^{(1)}} a_i v_i$. For $K \in \Sigma^{(k)}$ we define

$$\tilde{G}_{K, \mathcal{Y}} = \{a \mid \eta(a) \in L \text{ and } a_j = 0 \text{ for } j \notin K\}$$

and define $G_{K, \mathcal{Y}}$ to be the image of $\tilde{G}_{K, \mathcal{Y}}$ in $\tilde{T} = \mathbb{R}^m/\mathbb{Z}^m$. It will be written $G_K$ for simplicity. The homomorphism $\eta$ restricted on $\tilde{G}_{K, \mathcal{Y}}$ induces an isomorphism

$$\eta_K : G_K \cong H_K = H_{K, \mathcal{Y}} \subset T = L_\mathbb{R}/L.$$

Put

$$G_\Delta = \bigcup_{I \in \Sigma^{(n)}} G_I \subset \tilde{T} \quad \text{and} \quad DG_\Delta = \bigcup_{I \in \Sigma^{(n)}} G_I \times G_I \subset G_\Delta \times G_\Delta.$$

Let $v(g) = a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ be a representative of $g \in \tilde{T}$. The factor $a_i$ will be denoted by $v_i(g)$. It is determined modulo integers. If $g \in G_I$, then $v_i(g)$ is necessarily a rational number.

Let $g \in G_I$ and $h = \eta_I(g) \in H_I$. Then $\eta(v(g)) \in L_I$ is a representative of $h$ in $L_I$ which will be denoted by $v(h)$. Then

$$v_i(g) = \langle u_i^I, v(h) \rangle \quad \text{for } i \in I.$$

Define a homomorphism $\chi_I : \tilde{T} \rightarrow \mathbb{C}^*$ by

$$\chi_i(g) = e^{2\pi \sqrt{-1} v_i(g)} = e^{2\pi \sqrt{-1} \langle u_i^I, v(h) \rangle} \quad \text{for } g \in G_I \text{ and } i \in I.$$

This will also be written $\chi_I(u_i^I, h)$. It gives a character of $H_I$ for each $i \in I$.

3. Orbifold elliptic class and orbifold elliptic genus

Let $\Delta$ be a simplicial multi-fan in a lattice $L$ and $\mathcal{Y} = \{v_i\}_{i \in \Sigma^{(1)}}$ a set of prescribed vectors as in Section 2. Let also $\xi = \sum_{i \in \Sigma^{(1)}} d_i x_i \in H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor.

We shall define the orbifold elliptic class $\hat{\delta}_st(\Delta, \mathcal{Y}, \xi)$ in $(H^*_T(\Delta, \mathcal{Y}) \otimes \mathbb{C})[[q]]$ and the orbifold elliptic genus $\hat{\varphi}_st(\Delta, \mathcal{Y}, \xi)$ of the triple $(\Delta, \mathcal{Y}, \xi)$. The definition of
\( \hat{\varphi}_{st}(\Delta, \mathcal{Y}, \xi) \) is such that, if \( X \) is a complete \( \mathbb{Q} \)-factorial toric variety and \( \Delta(X) \) is its fan, then \( \hat{\varphi}_{st}(\Delta(X), \mathcal{Y}, \xi) \) coincides with the orbifold elliptic genus of \( X \) regarded as an orbifold, where all the vectors \( v_i \) in \( \mathcal{Y} \) are taken primitive and \( \xi = \sum_{i \in \Sigma^{(1)}} x_i \).

We first consider the function \( \Phi(z, \tau) \) of \( z \in \mathbb{C} \) and \( \tau \) in the upper half plane \( \mathcal{H} \) defined by the following formula.

\[
\Phi(z, \tau) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1 - tq^k)(1 - t^{-1}q^k)}{(1 - q^k)^2},
\]

where \( t = e^{2\pi \sqrt{-1} z} \) and \( q = e^{2\pi \sqrt{-1} \tau} \). It is meromorphic with respect to \((z, \tau)\). Note that \(|q| < 1\). Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), and put \( A(z, \tau) = \left( \frac{z}{\epsilon r + d}, \frac{\alpha r + b}{\epsilon r + d} \right) \). \( \Phi \) is a Jacobi form and satisfies the following transformation formulae, cf. \[14\].

\[
(12) \quad \Phi(A(z, \tau)) = (c\tau + d)^{-1} e^{\frac{czd}{\epsilon r + d}} \Phi(z, \tau),
\]

\[
(13) \quad \Phi(z + m\tau + n, \tau) = (-1)^{m+n} e^{-\pi \sqrt{-1} (2mz + m^2\tau)} \Phi(z, \tau)
\]

where \( m, n \in \mathbb{Z} \).

For \( \sigma \in \mathbb{C} \) we set

\[
\phi_{st}(z, \tau, \sigma) = \frac{\Phi(z + \sigma, \tau)}{\Phi(z, \tau)} \Phi(\sigma, \tau)
\]

\[
= \frac{1 - \zeta t}{(1 - \zeta)(1 - t)} \prod_{k=1}^{\infty} \frac{(1 - q^k)^2(1 - \zeta t q^k)(1 - \zeta^{-1} t^{-1} q^k)}{(1 - q^k)(1 - \zeta q^k)(1 - \zeta^{-1} q^{-k})(1 - t^{-1} q^k)},
\]

where \( \zeta = e^{2\pi \sqrt{-1} \tau} \). From \[12\] and \[13\] the following transformation formulae for \( \phi_{st} \) follow:

\[
(14) \quad \phi_{st}(A(z, \tau), \sigma) = e^{2\pi \sqrt{-1} cz \sigma} \phi_{st}(z, \tau, (c\tau + d)\sigma),
\]

\[
(15) \quad \phi_{st}(z + m\tau + n, \tau, \sigma) = e^{-2\pi \sqrt{-1} m\sigma} \phi_{st}(z, \tau, \sigma) = \zeta^{-m} \phi_{st}(z, \tau, \sigma).
\]

**Definition.** We define the (equivariant, stabilized) **orbifold elliptic class** \( \hat{\delta}_{st}(\Delta, \mathcal{Y}, \xi) \) of a triple \((\Delta, \mathcal{Y}, \xi)\) by \[16\]

\[
\hat{\delta}_{st}(\Delta, \mathcal{Y}, \xi) = \sum_{(g_1, g_2) \in DG_{\Delta} \in \Sigma^{(1)}} \prod_{x_i} x_i^{\xi_i} \phi_{st}(\xi_i \xi, \xi) + v_1(g_1) - v_1(g_2) - v_1(d) - v_1(\sigma).
\]

**Remark 3.1.** Let \( X \) be a complete \( \mathbb{Q} \)-factorial toric variety and \( \Delta \) the fan associated to \( X \). Put \( K_\Delta = -\sum_{i \in \Sigma^{(1)}} x_i \). For \( \xi = \sum_{i} x_i \) define \( D_\Delta = \sum_{i} a_i x_i \in H^2(\Delta, \mathcal{Y}) \otimes \mathbb{Q} \) by \( \xi = -K_\Delta - D_\Delta \), that is \( a_i = 1 - d_i \). For the divisor \( D = \sum_{i} a_i D_i \) in \( X \) the singular elliptic class \( \delta \ell_{\text{sing}}(X, D) \) defined by Borisov and Libgober is equal to \( (-\Psi(\sigma, \tau))^{\dim X} \delta_{st}(\Delta, \mathcal{Y}, \xi) \) where all vectors \( v_i \) in \( \mathcal{Y} \) are taken primitive.

The right hand side of \[16\] does not depend on the choice of representatives \( v(g_1), v(g_2) \) in view of \[13\] and \[15\]. It is sometimes useful to take a representative \( v(g) \) of \( g \in G_\Delta \) such that

\[
0 \leq v_i(g) < 1 \text{ for all } i \in I.
\]

Such a representative is unique. We denote the value \( v_i(g) \) by \( f_{g,i} \) for such a representative \( v(g) \). The sum \( \sum_{i \in \Sigma^{(1)}} d_i f_{g,i} \) will be denoted by \( f_g(\xi) \).
For $h = \eta_1(g) \in H_I$ we put $f_{h,i} = f_{g,s}$. It is equal to $(u^I_1, v(h))$ for a uniquely determined representative $v(h)$ of $h$. If $h$ lies in $H_K$ for $K \in \Sigma^{(k)}$ contained in $I$, then $f_{h,i} = 0$ for $i \notin K$, and $f_{h,i}$ depends only on $K$.

Note that

$$
\phi_{st}(-\frac{x_i}{2\pi\sqrt{-1}} + v_i(g_1)\tau - v_i(g_2), \tau, d, \sigma) = \frac{1}{1 - \xi_i} \prod_{k=1}^{\infty} (1 - \zeta^d, q^k(1 - \zeta^{-d}, q^k)) \frac{1 - \zeta^d, q^k(1 - \zeta^{-d}, q^k)}{1 - \xi_i} \prod_{k=1}^{\infty} (1 - \zeta^d, q^k(1 - \zeta^{-d}, q^k))
$$

where

$$
\xi_i = e^{-x_i} e^{-2\pi\sqrt{-1}v_i(g_2)} = e^{-x_i} e^{-2\pi\sqrt{-1}v_i(g_1)} \chi_i(g_2)^{-1}.
$$

e^{-x_i} is considered as an element of the completed ring $H^{**}_I(\Delta, \gamma) \otimes \mathbb{Q}$. Let $r$ be the least common multiple of $\{|H_I|\}_{I \in \Sigma^{(n)}}$. The right-hand side of \[19\] defines an element in $(H^{**}_I(\Delta, \gamma) \otimes \mathbb{C})[|q^k]|$. It will be shown in Theorem 3.3 that it in fact lies in $(H^{**}_I(\Delta, \gamma) \otimes \mathbb{C})[|q|]$.

**Definition.** The (equivariant, stabilized) orbifold elliptic genus $\hat{\phi}_{st}(\Delta, \gamma, \xi) \in (H^{**}(BT) \otimes \mathbb{C})[|q|]$ is defined as the image $\pi_*(\hat{\phi}_{st}(\Delta, \gamma, \xi))$.

Explicitly we have

**Proposition 3.2.**

\[18\]

$$
\hat{\phi}_{st}(\Delta, \gamma, \xi) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \prod_{h_1 \in H_I} \prod_{h_2 \in H_I} \phi_{st}\left(-\frac{x_i}{2\pi\sqrt{-1}} + v_i(h_1)\tau - v_i(h_2), \tau, d, \sigma\right).
$$

**Proof.** We take a representative $v(g)$ for each $g \in G_{\Delta}$ satisfying \[17\]. Let $(g_1, g_2) \in DG_\Delta$ and $I \in \Sigma^{(n)}$. Note that $x_j \phi_{st}(-\frac{x_j}{2\pi\sqrt{-1}} + v_j(g_1)\tau - v_j(g_2), \tau, d, \sigma)$ is of the form

$$
\frac{x_j}{1 - \xi_j} \hat{\phi}_j(x_j).
$$

If $g_1 \notin G_I$ or $g_2 \notin G_I$, then there is a $j \notin I$ such that $v_j(g_1) \neq 0$ or $v_j(g_2) \neq 0$. For such $j$ the Maclaurin expansion with respect to $x_j$ of the factor $\frac{x_j}{1 - \xi_j}$ has no constant term. Noting that $\hat{\phi}_j(x_j) = 0$ we have $\hat{\phi}_j(x_j) = 0$ and hence

$$
\hat{\phi}_j(x_j) \in \mathbb{C} \otimes \mathbb{C}.
$$

Therefore only elements $(g_1, g_2)$ in $G_I \times G_I$ contribute to $\hat{\phi}_{st}(\Delta, \gamma, \xi)$.

Next suppose $(g_1, g_2)$ lies in $G_I \times G_I$. If $j \notin I$, then $v_j(g_1) = v_j(g_2) = 0$ and $\hat{\phi}_j(x_j) = 0$. In particular $\hat{\phi}_j(x_j) = 1$ and $\hat{\phi}_j(x_j) = 1$, $\hat{\phi}_j(x_j) = 1$. Hence

$$
\hat{\phi}_j(x_j) \in \mathbb{C} \otimes \mathbb{C}.
$$

Therefore only elements $(g_1, g_2)$ in $G_I \times G_I$ contribute to $\hat{\phi}_{st}(\Delta, \gamma, \xi)$. 

Next suppose $(g_1, g_2)$ lies in $G_I \times G_I$. If $j \notin I$, then $v_j(g_1) = v_j(g_2) = 0$ and $\hat{\phi}_j(x_j) = 0$. In particular $\hat{\phi}_j(x_j) = 1$ and $\hat{\phi}_j(x_j) = 1$, $\hat{\phi}_j(x_j) = 1$. Hence

$$
\hat{\phi}_j(x_j) \in \mathbb{C} \otimes \mathbb{C}.
$$

Therefore only elements $(g_1, g_2)$ in $G_I \times G_I$ contribute to $\hat{\phi}_{st}(\Delta, \gamma, \xi)$.
It follows that
\[
\iota_!^* \left( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \right)
= \iota_!^* \left( \sum_{(g_1, g_2) \in G_1 \times G_2} \prod_{i \in I} x_i \zeta^{d_i v_i(g_1)} \phi_{st} \left( -\frac{x_i}{2\pi \sqrt{-1}} + v_i(g_1) \tau - v_i(g_2), \tau, d_i \sigma \right) \right)
= \sum_{(h_1, h_2) \in H_1 \times H_2} \prod_{i \in I} u_i^I \zeta^{d_i \langle u_i^I, v(h_1) \rangle} \phi_{st} \left( -\frac{u_i^I}{2\pi \sqrt{-1}} + \langle u_i^I, v(h_1) \rangle \tau - \langle u_i^I, v(h_2) \rangle, \tau, d_i \sigma \right).
\]

Here \( v(h) \) denotes a representative of \( h \in H_I \) such that \( 0 \leq \langle u_i^I, v(h) \rangle < 1 \). Then we have
\[
\pi_*(\hat{\phi}_{st}(\Delta, \mathcal{V}, \xi))
= \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_1 \times H_2} \prod_{i \in I} \zeta^{d_i \langle u_i^I, v(h_1) \rangle} \phi_{st} \left( -\frac{u_i^I}{2\pi \sqrt{-1}} + \langle u_i^I, v(h_1) \rangle \tau - \langle u_i^I, v(h_2) \rangle, \tau, d_i \sigma \right).
\]

\[\square\]

**Note.** The right hand side of (18) is independent of the choice of representatives \( v(h_1) \) and \( v(h_2) \) though we used representatives satisfying (17) in the proof above.

Put \( L_C = L \otimes \mathbb{C} \). We shall also consider a function \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \) from \( L_C \) into \( \mathbb{C}[[q]] \) defined by
\[
\hat{\phi}_{st}(\Delta, \mathcal{V}, \xi)(w)
= \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_1 \times H_2} \prod_{i \in I} \zeta^{d_i \langle u_i^I, v(h_1) \rangle} \phi_{st} \left( (u_i^I - w + \tau v(h_1) - v(h_2)), \tau, d_i \sigma \right).
\]

This function \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \) is also called (stabilized) orbifold elliptic genus. Later it will be shown that \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \) belongs to \( (R(T) \otimes \mathbb{C})[[q]] \) and \( \text{ch}(\hat{\phi}_{st}(\Delta, \mathcal{V}, \xi)) = \hat{\mathcal{E}}_{st}(\Delta, \mathcal{V}, \xi) \). Here \( R(T) \) is the character ring of the torus \( T \) and its elements are considered as functions on \( L_C \) via the projection \( L_C \rightarrow T_C \) where \( T_C \) is the complexification of \( T \). For \( \xi_0 = \sum_{i \in \Sigma(n)} x_i \) the genus \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi_0) \) was introduced in [12] in a non-stabilized form as the orbifold elliptic genus of the pair \( (\Delta, \mathcal{V}) \) and was denoted by \( \hat{\phi}(\Delta, \mathcal{V}) \), cf. also [10]. \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \) is sometimes written as \( \hat{\phi}(\Delta, \mathcal{V}, \xi; w, \tau, \sigma) \) to emphasize the variables. Also \( \tau \) and \( \sigma \) are often considered as parameters; in this case it is sometimes written as \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi; \tau, \sigma) \) to emphasize the parameters.

A vector \( v \in L \) can be considered as a homomorphism \( \mathbb{C} \ni z \mapsto zv \in L_C \). Let \( v^*(\hat{\phi}_{st}(\Delta, \mathcal{V}, \xi)) \) be the pull-back induced by \( v \). It will be denoted by \( \hat{\phi}_{st}^v(\Delta, \mathcal{V}, \xi) \), and will be called the (stabilized) orbifold elliptic genus along \( v \) of the triple \( (\Delta, \mathcal{V}, \xi) \). Explicitly it is given by
\[
\hat{\phi}_{st}^v(\Delta, \mathcal{V}, \xi)(z)
= \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_1 \times H_2} \prod_{i \in I} \zeta^{d_i \langle u_i^I, v(h_1) \rangle} \phi_{st} \left( (u_i^I - zv + \tau v(h_1) - v(h_2)), \tau, d_i \sigma \right).
\]
If $J \subset K$ then we have $L_J \cap L_K, \gamma = L_{J, \gamma}$, and hence $H_J$ is canonically embedded in $H_K$. We set

$$
\hat{H}_K = H_K \setminus \bigcup_{J \subset K} H_J.
$$

The subset $\hat{H}_K$ is characterized by

$$
\hat{H}_K = \{ h \in H_K \mid \langle u_i^K, v(h) \rangle \notin \mathbb{Z} \quad \text{for any} \quad i \in K \},
$$

where $v(h) \in L_K$ is a representative of $h \in H_K$. For the minimum element $\ast = \emptyset \in \Sigma^{(0)}$ we set $\hat{H}_\ast = H_\ast = 0$. Note that (22) can be rewritten as

$$
\hat{H}_K = \{ h \in H_K \mid f_{h,i} \neq 0 \quad \text{for any} \quad i \in K \}.
$$

If $K$ is contained in $I \in \Sigma^{(n)}$, then the canonical map $L_{I, \gamma}^* \rightarrow L_{K, \gamma}^*$ sends $u_i^I$ to $u_i^K$ for $i \in K$ and to 0 for $i \in I \setminus K$. Therefore, if $h$ is in $H_K$, then $\langle u_i^K, v(h) \rangle = 0$ for $i \in I \setminus K$, and $\langle u_i^I, v(h) \rangle = \langle u_i^K, v(h) \rangle$ for $i \in K$. Here $v(h) \in L_K$ is regarded as lying in $L$. This observation leads to the following expression of $\hat{\varphi}_{st}(\Delta, \gamma, \xi)$ which is sometimes useful.

$$
\hat{\varphi}_{st}(\Delta, \gamma, \xi)(w) = \sum_{k=0}^{n} \sum_{\substack{K \in \Sigma^{(k)} \setminus H_K}} \zeta^{(u_i^K, w, v(h_1))} \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{\lvert H_I \rvert} \prod_{h \in H_K} \phi_{st}(u_i^K, w + v(h_2)), \tau, d_i \sigma) \prod_{i \in K} \phi_{st}(-u_i^K, w - \tau v(h_1) + v(h_2)), \tau, d_i \sigma),
$$

where $u_i^K(\xi) = \iota_i^K(\xi) = \sum_{i \in K} d_i u_i^K$.

**Theorem 3.3.** $\hat{\varphi}_{st}(\Delta, \gamma, \xi)$ belongs to $(H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]]$. $\hat{\varphi}_{st}(\Delta, \gamma, \xi)$ belongs to $(H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]]$. Moreover, if $\Delta$ is complete, then $\hat{\varphi}_{st}(\Delta, \gamma, \xi)$ belongs to $(R(T) \otimes \mathbb{C})[[q]]$.

**Proof.** Since the map $\bigoplus_{I \in \Sigma^{(n)}} \ell_I^* : (H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]] \rightarrow ((H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]])^{\Sigma^{(n)}}$ is injective it is enough to prove that $\ell_I^*(\hat{\varphi}_{st}(\Delta, \gamma, \xi))$ belongs to $(H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]]$ for any $I \in \Sigma^{(n)}$.

Fix $I$ and $g_1 \in G_I$. In view of (19) it suffices to show that

$$
\ell_I^* \left( \sum_{g \in G_I} \prod_{i \in I} \phi_{st}(-u_i^I, w_1, \tau v_i(g_1) - v_i(g_2), \tau, d_i \sigma) \right)
$$

$$
= \sum_{g \in H_I} \prod_{i \in I} \phi_{st}(-u_i^I, w_1, \tau v_i(g_1) - v_i(g_2), \tau, d_i \sigma)
$$

belongs to $(H_{I, \gamma}^* (\Delta, \gamma) \otimes \mathbb{C})[[q]]$, where $v(g)$ is the unique representative of $g \in G_I$ satisfying (17). We introduce auxiliary variables $\tau_i$ with $\Im(\tau_i) > 0$ and put $q_1 = e^{2\pi v_1 \tau_1}$. We put

$$
\delta(q_1) = \sum_{g \in G_I} \prod_{i \in I} \phi_{st}(-u_i^I, w_1, \tau v_i(g_1) - v_i(g_2), \tau, d_i \sigma) = \sum_{s \in \mathbb{Z}_{\geq 0}} \delta_s(q_1)q^s.
$$

and expand it with respect to $q$:

$$
\delta(q_1) = \sum_{s \in \mathbb{Z}_{\geq 0}} \delta_s(q_1)q^s.
$$
Put $X_i(g_2) = \chi_i(g_2)^{-1} e^{-x_i q_1^i(g_1)}$. Since

$$
\phi_{st}(-\frac{x_i}{2\pi\sqrt{-1}} + \tau_1 v_1(g_1) - v_i(g_2), \tau, \sigma_1)
= \frac{1}{1 - \zeta^d} \prod_{k=1}^{\infty} \frac{(1 - q^k)^2}{(1 - \zeta^d q^k)(1 - \zeta^{-d} q^k)}.
$$

$$
\frac{1 - \zeta^d X_i(g_2)}{1 - X_i(g_2)} \prod_{k=1}^{\infty} \frac{(1 - \zeta^d X_i(g_2) q^k)(1 - \zeta^{-d} X_i(g_2)^{-1} q^k)}{(1 - X_i(g_2) q^k)(1 - X_i(g_2) q^{-1} q^k)},
$$

$\epsilon_s(q_1)$ is expanded in a Laurent series

$$
\epsilon_s(q_1) = \sum_{c=(c_1, \ldots, c_n)} a_c \sum_{g_2 \in G_I} X_1(g_2)^{c_1} \cdots X_n(g_2)^{c_n}
$$

where $I$ is indexed by $\{1, \ldots, n\}$. Moreover the sum of negative exponents $c_i$ is bounded below by $-s$, i.e., $-s \leq \sum_{c_i < 0} c_i$ for all $c$. If we put $u_c = \sum_{i \in I} c_i u_i$, then

$$
\iota_I^* (\sum_{g_2 \in G_I} X_1(g_2)^{c_1} \cdots X_n(g_2)^{c_n}) = \sum_{h_2} \chi_I(u_c, h_2) e^{-u_c q_1^i(u_{c,v(h_1)})}.
$$

Since $u_c$ lies in $L_I^*, \chi_I(u_c, \cdot)$ is a character of $H_I$, and we have

$$
\sum_{h \in H_I} \chi_I(u_c, h) = \begin{cases} |H_I|, & \text{if } u_c \in L_I^*, \\ 0, & \text{if } u_c \notin L_I^*. \end{cases}
$$

Hence $\iota_I^* (\epsilon_s(q_1)) = |H_I| \sum_{c \in \mathbb{Z}^n} a_c e^{-u_c q_1^i(u_{c,v(h_1)})}$. Since $(u_c, h_1)$ is an integer for $u_c \in L_I^*$, $\iota_I^* (\epsilon_s(q_1))$ is a Laurent series in $q_1$. Furthermore, since $0 < f_{h_1, i} < 1$ and $-s \leq \sum_{c_i < 0} c_i$, we have $-s \leq \sum_{c_i < 0} f_{h_1, i} = \langle u_c, v(h_1) \rangle$. Noting that $\epsilon(q) = \sum_{s \in \mathbb{Z}^n} \epsilon_s(q) q^s$, we see that $\iota_I^* (\epsilon(q))$ has no negative powers. Thus $\iota_I^* (\epsilon(q))$ is a power series in $q$. This proves that $\hat{\epsilon}_s(\Delta, \gamma, \xi)$ belongs to $(H_I^* \otimes \mathbb{C})[[q]]$. Since $\hat{\epsilon}_s(\Delta, \gamma, \xi) = \pi_s(\hat{\epsilon}_s(\Delta, \gamma, \xi)$ it belongs to $(H^{**} (BT) \otimes \mathbb{C})[[q]]$.

To prove the second part fix $K \in \Sigma(k)$ and $h_1 \in \hat{H}_K$. In view of $(20)$ it suffices to show that $\hat{\varphi}_h_1(\Delta, \gamma, \xi)$ defined by

$$
\hat{\varphi}_h_1(\Delta, \gamma, \xi)(w) = \sum_{I \in \Sigma^{(k-k)}_K} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi_{st}((u_i^I, -w + \tau v(h_1) - v(h_2)), \tau, \sigma_1)
$$

is expanded in a formal power series in $q$ with coefficients in $R(T) \otimes \mathbb{C}$.

Take $g_1 \in G_\Delta$ such that $\eta_K(g_1) = h_1$. Then $v_i(g_1) = 0$ for $i \notin K$ since $h_1 \in \hat{H}_K$.

Put $\Sigma^{(1)} = \bigcup_{l \geq K} I$ and consider the following two quantities

$$
\hat{\delta}(q_1) = \sum_{g_2 \in G_\Delta} \prod_{i \in \Sigma^{(1)}} \phi_{st}(-\frac{x_i}{2\pi\sqrt{-1}} + \tau_1 v_1(g_1) - v_i(g_2), \tau, \sigma_1),
$$

$$
\hat{\varphi}(q_1)(w) = \sum_{I \in \Sigma^{(k-k)}_K} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi_{st}((u_i^I, -w + \tau_1 v(h_1) - v(h_2)), \tau, \sigma_1),
$$

Consequently

$$
\hat{\varphi}_h_1(\Delta, \gamma, \xi)(w) = \sum_{I \in \Sigma^{(k-k)}_K} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi_{st}((u_i^I, -w + \tau_1 v(h_1) - v(h_2)), \tau, \sigma_1).
$$
and expand them with respect to \( q \):

\[
\tilde{\varphi}(q_1) = \sum_{s \in \mathbb{Z}_{\geq 0}} \tilde{\varphi}_s(q_1) q^s,
\]

\[
\varphi(q_1)(w) = \sum_{s \in \mathbb{Z}_{\geq 0}} \varphi_s(w, q_1) q^s.
\]

Note that

\[
\varphi(q)(w) = \varphi_{h_1}(\Delta, \mathcal{Y}, \xi)(w).
\]

As in the proof of the first part we see that \( \tilde{\varphi}_s(q_1) \) is expanded in the following form:

\[
\tilde{\varphi}_s(q_1) = \sum_{e=(c_1, \ldots, c_{m_{\Delta}})} b_e \sum_{g_{\Delta} \in \Gamma_{\Delta}} \frac{\prod_{i \in \Sigma_1} x_i^{c_i} (1 - x_i^{-1})}{\prod_{\mathcal{I} \in \Sigma_1 \setminus K} (1 - \chi_i(g_2)^{-1} e^{-x_i})}.
\]

If we put \( x_e = \sum_{i \in \Sigma_1} c_i x_i \), then, by a similar argument to the proof of Proposition 3.2, we see that \( \varphi_s(w, q_1) \) is expanded in the form

\[
\varphi_s(w, q_1) = \sum_e b_e \sum_{I \in \Sigma_1 - k} \frac{w(I)}{H_I} \sum_{h_2 \in H_I} \frac{\chi_I(\iota_e^*_w(x_e), h_2)^{-1} e^{-2\pi \sqrt{-1} \iota_e^*(x_e), w_1} q_1^\iota_e^*(x_e), v(h_1))}{\prod_{\mathcal{I} \in \Sigma_1 \setminus K} (1 - \chi_I(u_1^I, h_2)^{-1} e^{-x_1^I(w_1^I, w_1))}}.
\]

Corollary 2.4 of [12] says that

\[
\sum_{I \in \Sigma_1 - k} \frac{w(I)}{H_I} \sum_{h_2 \in H_I} \frac{\chi_I(\iota_e^*_w(x_e), h_2)^{-1} e^{-2\pi \sqrt{-1} \iota_e^*(x_e), w_1} q_1^\iota_e^*(x_e), v(h_1))}{\prod_{\mathcal{I} \in \Sigma_1 \setminus K} (1 - \chi_I(u_1^I, h_2)^{-1} e^{-x_1^I(w_1^I, w_1))}}
\]

belongs to \( R(T) \otimes \mathbb{C}[q_1, q_1^{-1}] \) as a function of \( w \). Hence \( \varphi_s(w, q_1) \) also does so.

Then, by a similar argument to the proof of the first part, \( \varphi_{h_1}(\Delta, \mathcal{Y}, \xi)(w) = \varphi(\tau)(w) = \sum_{s \in \mathbb{Z}_{\geq 0}} \varphi_s(w, q_1) q^s \) is expanded in a power series \( \sum_{s \in \mathbb{Z}_{\geq 0}} \varphi_s(w) q^s \) in \( q \) with \( \varphi_s \in R(T) \otimes \mathbb{C} \).

**Proposition 3.4.** Assume that \( \Delta \) is complete. Then the Chern character \( ch : (R(T) \otimes \mathbb{C})[[q]] \to H^{*}(BT) \otimes \mathbb{C}[[q]] \) sends \( \tilde{\varphi}_s(\Delta, \mathcal{Y}, \xi) \) to \( \tilde{\varphi}_s(\Delta, \mathcal{Y}, \xi) \).

**Proof.** The element \( t^u \) of the character ring \( R(T) \) corresponding to \( u \in L^* \) is considered as a function on \( L^*_\mathbb{C} \) defined by \( e^{2\pi \sqrt{-1} \iota(u, \cdot)} \). On the other hand \( ch(t^u) = e^u \) where \( L^* \) is identified with \( H^2(BT) \). Hence \( ch(e^{2\pi \sqrt{-1} \iota(u, \cdot)}) = e^u \) or \( ch(e^{i u}) = e^{\frac{2\pi}{i} u} \). Thus, comparing [15] with [20], we see that

\[
ch(\varphi_s(\Delta, \mathcal{Y}, \xi)) = \tilde{\varphi}_s(\Delta, \mathcal{Y}, \xi).
\]

**Remark 3.5.** For a not necessarily complete simplicial multi-fan \( \Delta \) we can consider the function \( \varphi_s(\Delta, \mathcal{Y}, \xi) \) as a formal power series in \( q \) whose coefficients are rational functions with denominators of the form \( \prod_i (1 - \alpha_i t^u_i) \), \( u_i \in L_{\mathbb{R}} \), \( \alpha_i \neq 0 \). The map \( ch(\cdot) \) is extended on such rational functions by \( ch(1 - \alpha t^u) = 1 - \alpha e^u \in H^{**}(BT) \otimes \mathbb{C} \). In this sense (24) holds for general simplicial multi-fans.
4. Rigidity and vanishing property

Let $N > 1$ be an integer. We shall consider the following condition for $\xi \in H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$:

\begin{equation}
\xi = N\eta + u, \text{ for some } \eta \in H^2_T(\Delta, \mathcal{Y}), \ u \in L^*_\mathcal{Y} \otimes \mathbb{Q}.
\end{equation}

If the condition (25) is satisfied, then the classes of $u^!(\xi) = \iota^*_T(\xi)$ and $u$ regarded as elements in the quotient $L^*_\mathcal{Y} \otimes \mathbb{Q}/NL^*_\mathcal{Y}$ are the same. Let $v$ be a vector in $L^*_\mathcal{Y}$. Since $\langle \iota^*_T(\eta), v \rangle$ is an integer for all $I \in \Sigma^{(n)}$, the values $\langle u^!(\xi), v \rangle$ and $\langle u, v \rangle$ regarded as elements in $\mathbb{Q}/\mathbb{Z}$ are equal. It will be denoted by $h(v) \in \mathbb{Q}/\mathbb{Z}$.

Note that if $\xi$ satisfies (25) with $\eta$ $T$-Cartier, then $\langle u^!(\xi), v \rangle \equiv \langle u, v \rangle \mod N\mathbb{Z}$ even for any $v \in L$, and hence $h(v) \in \mathbb{Q}/\mathbb{Z}$ is also defined for $v \in L$.

The following two theorems are the main results of this section. Prototypes of these theorems were already given in [7, 9] and [10]. The proofs given here are in the same line as those of the cited works.

**THEOREM 4.1.** Let $(\Delta, \mathcal{Y}, \xi)$ be a triple of complete simplicial multi-fan in a lattice $L$ of rank $n$, a set of edge vectors and an element of $H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$. Let $N > 1$ be an integer. Assume that $\xi$ satisfies (25) with $\eta$ $T$-Cartier. Then $\hat{\phi}_{st}(\Delta, \mathcal{Y}, \xi; \tau, \sigma)$ with $\sigma = \frac{k}{N}$, $1 < k < N$, is rigid, that is, it is a constant as a function of $w \in L\mathcal{Y}$. If, moreover, $\xi$ does not belong to $NK^2_L(\Delta, \mathcal{Y})$, then it constantly vanishes.

**THEOREM 4.2.** Let $(\Delta, \mathcal{Y}, \xi)$ be a triple of complete simplicial multi-fan in a lattice $L$ of rank $n$, a set of generating edge vectors and an element of $H^2_T(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$. If $\xi$ satisfies the equality $\xi = u$ with non-zero $u \in L^*_\mathcal{Y} \otimes \mathbb{Q}$, then $\hat{\phi}_{st}(\Delta, \mathcal{Y}, \xi; \tau, \sigma)$ constantly vanishes.

The rest of the section is devoted to the proofs of Theorem 4.1 and Theorem 4.2.

Let $v \in L^*_\mathcal{Y}$ be a generic vector. We put $H_I = L/L^*_\mathcal{Y}$ for $I \in \Sigma^{(n)}$ as before. For $A \in SL_2(\mathbb{Z})$ we set

$$
(\hat{\phi}_{st}^v)^A(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma) = \hat{\phi}_{st}(\Delta, \mathcal{Y}, \xi; A(z, \tau, \sigma)).
$$

**LEMMA 4.3.** Assume that $\xi$ satisfies (25). Then $(\hat{\phi}_{st}^v)^A(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma)$ with $\sigma = \frac{k}{N}$, $0 < k < N$, has the following expression.

\begin{equation}
(\hat{\phi}_{st}^v)^A(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma) = e^{-2\pi \varphi(\xi, \tau, \sigma)} \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_I \times H_I} e^{-2\pi \varphi(\xi, \tau, \sigma)} \phi_{st}\left(-\langle u'_1, zv - \tau v(h_1) + v(h_2), \tau, (\tau + d)\rangle \right).
\end{equation}

**PROOF.** By definition we have

\begin{equation}
(\hat{\phi}_{st}^v)^A(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{h_1 \in H_I} \prod_{i \in I} \phi_{st}\left(-\langle u'_i, zv - \tau v(h_1) + v(h_2), \tau, (\tau + d)\rangle \right).
\end{equation}
Using (13) we get

\[ \prod_{i \in I} \zeta^{\langle d, u^i \rangle, v(h_1)}(u^i, zv - (a\tau + b)v(h_1) + (c\tau + d)v(h_2)) \]

\[ = \zeta^{\langle u^i, v(h_1) \rangle} e^{2\pi \sqrt{-1}(c(u^i, v(h_1) - zv + (a\tau + b)v(h_1) - (c\tau + d)v(h_2)))} \]

\[ \prod_{i \in I} \phi_{st}(-u^i, zv - (a\tau + b)v(h_1) + (c\tau + d)v(h_2)), \tau, (c\tau + d)\sigma). \]

We have

\[ c((a\tau + b)v(h_1) - (c\tau + d)v(h_2)) = -v(h_1) + (c\tau + d)(av(h_1) - cv(h_2)). \]

Hence

\[ \zeta^{\langle u^i, v(h_1) \rangle} e^{2\pi \sqrt{-1}(c(u^i, v(h_1) - zv + (a\tau + b)v(h_1) - (c\tau + d)v(h_2)))} \]

\[ = e^{2\pi \sqrt{-1}(u^i, v(h_1))} e^{2\pi \sqrt{-1}(c(u^i, v(h_1) - zv))} e^{2\pi \sqrt{-1}(c(u^i, v(h_1) - cv(h_2)))}. \]

Since \( \xi \) satisfies (25), we get

\[ e^{2\pi \sqrt{-1}(u^i, v(h_1))} = e^{2\pi \sqrt{-1}(u, v)\sigma} e^{2\pi \sqrt{-1}(c(u, v)(h_1)v)}. \]

Let \( \rho : H_1 \times H_1 \to H_1 \times H_1 \) be the map defined by

\[ \rho(h_1, h_2) = (h_1, h_2) = (ah_1 - ch_2, -bh_1 + dh_2). \]

\( \rho \) is bijective and its inverse is given by

\[ \rho^{-1}(h_1, h_2) = (h_1, h_2). \]

Then \( av(h_1) - cv(h_2) \) and \( -bv(h_1) + dv(h_2) \) are representatives of \( h_1 \) and \( h_2 \) which we shall denote by \( v(h_1) \) and \( v(h_2) \) respectively.

In view of (28) and (29), the right hand side of (27) is equal to

\[ e^{-2\pi \sqrt{-1}(c(u, v)zv)\sigma} e^{-2\pi \sqrt{-1}(c(u, v)(h_1)v)} \]

\[ \prod_{i \in I} e^{2\pi \sqrt{-1}(d, u^i, (c\tau + d)\sigma)v(h_1))} \phi_{st}(-d, u^i, zv - (v(\tilde{h}_1))\tau + (v(\tilde{h}_2))\tau), (c\tau + d)\sigma). \]

Summing up over \( (h_1, h_2) \) is the same as summing up over \( (h_1, h_2) \). Hence from (30) we get (20) with \( h_1 \) replaced by \( h_i \) for \( i = 1, 2 \). This proves Lemma 4.3 \( \square \)

**Lemma 4.4.** Assume that \( \xi \) satisfies (25) with \( \eta \) \( T \)-Cartier. Then, for fixed \( \tau \), the meromorphic function \( (\hat{\phi}^\nu_{st})^4(\Delta, \mathcal{V}; z, \tau, \sigma) \) in \( z \) with \( \sigma = \frac{k}{N}, \) \( 0 < k < N, \) has no poles at \( z \in \mathbb{R}. \)

**Proof.** The expression (20) in Lemma 4.3 of the function \( (\hat{\phi}^\nu_{st})^4(\Delta, \mathcal{V}; z, \tau, \sigma) \) can be rewritten in the following form as in the case of (25).

\[ (\hat{\phi}^\nu_{st})^4(\Delta, \mathcal{V}; z, \tau, \sigma) \]

\[ = e^{-2\pi \sqrt{-1}(c(u, v)zv)\sigma} \sum_{k=0}^n \sum_{K \in \Sigma^k} e^{2\pi \sqrt{-1}(c(u, v)(h_1))}, \]

\[ \sum_{I \subseteq \Sigma_{N-k}} \frac{w(I)}{|H_1|} \sum_{k \in \Sigma_{N-k}} \prod_{i \in I} e^{2\pi \sqrt{-1}(c(u, v)(h_1))} \phi_{st}(-d, u^i, zv - (v(h_1))\tau + (v(h_2))\tau), (c\tau + d)\sigma). \]
Hence, in order to prove Lemma 4.4 it is sufficient to prove that
\[
\sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|H_I|} \prod_{h_2 \in H_I} e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),ckzv)} \prod_{i \in I} \phi_{st}(\langle -u_i^T, zv - \tau v(h_1) + v(h_2) \rangle, \tau, (ct + d)d\sigma),
\]
or, replacing \(ck\) by \(m\) and \((ct + d)\sigma\) by \(\sigma\),
\begin{equation}
\sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),mzv)} \prod_{i \in I} \phi_{st}(\langle -u_i^T, zv - \tau v(h_1) + v(h_2) \rangle, \tau, d_i\sigma),
\end{equation}
has no poles at \(z \in \mathbb{R}\) for any fixed \(K \in \Sigma^{(k)}\) and \(h_1 \in \tilde{H}_K\).

Note that \(\langle \epsilon_I^*(\eta), v(h_2) \rangle\) is an integer for all \(I \in \Sigma^{(n)}\) since \(\eta\) is \(T\)-Cartier. Therefore
\[
e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),mzv)} = e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),m\tau v(h_1))} e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),m(zv-\tau v(h_1)+v(h_2)))}.
\]
Note further that
\[
e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),m(zv-\tau v(h_1)+v(h_2)))} = \chi_I(\epsilon_I^*(\eta), h_2) e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta), m(zv-\tau v(h_1)+v(h_2)))}.
\]
By a similar argument to the proof of Theorem 3.3 we see that (31) can be expanded in the form
\[
e^{-2\pi \sqrt{-1}(\epsilon_I^*(\eta),m\tau v(h_1))} \sum_{s=0}^{\infty} (\hat{\phi}_{st})_K^{A}(z)^s,
\]
where \((\hat{\phi}_{st})_K^{A}(z)\) belongs to \(R(S^1) \otimes \mathbb{C}\). From this we can conclude that (31) has no poles at \(z \in \mathbb{R}\). We refer to Lemma in Section 7 of [13] for details. See also Lemma 4.3 of [7] whose argument can be applied to prove that the function \((\hat{\phi}_{st})_K^{A}(\Delta, \gamma, \xi, z, \tau, \sigma)\) with \(\sigma = \frac{\tau}{N}, 0 < k < N\), is holomorphic on \(\mathbb{R} \times \mathcal{H}\) as a function of \((z, \tau)\). This finishes the proof of Lemma 4.4

We now proceed to the proof of Theorem 4.1. We follow [15] for the idea of proof. We first show that \(\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma)\) is a constant function as a function of \(z\).

We regard \(\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma)\) as a meromorphic function of \(z\). By the transformation law (14) \(\phi_{st}(z, \tau, \sigma)\) is an elliptic function in \(z\) with respect to the lattice \(\mathbb{Z} \cdot N \tau \otimes \mathbb{Z}\) for \(\sigma = \frac{\tau}{N}\) with \(0 < k < N\). Hence \(\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma)\) with \(\sigma = \frac{\tau}{N}\), \(0 < k < N\), is also an elliptic function in \(z\). Thus, in order to show that \(\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma)\) is a constant function it suffices to show that it does not have a pole.

Assume that \(z\) is a pole. Then \(1 - t^mq^*\alpha = 0\) for some integer \(m \neq 0\), some rational number \(r\) and a root of unity \(\alpha\). Consequently there are integers \(m_1 \neq 0\) and \(k_1\) such that \(m_1z + k_1\tau \in \mathbb{R}\). Then there is an element \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\) such that
\[
\frac{z}{ct+d} \in \mathbb{R}.
\]
Since
\[
\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma) = \hat{\phi}_{st}^v(\Delta, \gamma, \xi; A^{-1}(z, \tau, \sigma), A\tau, A\sigma) = (\hat{\phi}_{st})^{A^{-1}}(\Delta, \gamma, \xi; \frac{z}{ct+d}, A\tau, A\sigma),
\]
the function \((\hat{\phi}_{st})^{A^{-1}}(\Delta, \gamma, \xi; w, A\tau, A\sigma)\) must have a pole \(w = \frac{z}{ct+d} \in \mathbb{R}\). But this contradicts Lemma 4.4. This contradiction proves that \(\hat{\phi}_{st}^v(\Delta, \gamma, \xi; z, \tau, \sigma)\) can not have a pole.
Since $\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma)$ is a constant function in $z$ for every generic vector $v \in L$, $\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma)$ is constant as a function on $T$. This proves the first part of Theorem 4.1.

To prove the second part note that

$$\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; \tau, \sigma) = \hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma)$$

for any $v$ as constants. On the other hand, using (13) and the fact that $\langle u^I(\xi), v \rangle \equiv h(v) \mod NZ$ for any $I \in \Sigma^{(n)}$, we have

$$\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; \tau, \sigma) = \zeta^{h(v)} \hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; z, \tau, \sigma)$$

(32)

We choose a generic vector $v$ such that $h(v) \neq 0$ in $\mathbb{Q}/NZ$, which is possible by Lemma 4.5 below because of the assumption $\xi \notin NH^2(\Delta, \mathcal{Y})$. Then $\zeta^{h(v)}$ is not equal to 1. Hence (32) implies that the constant $\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; \tau, \sigma)$ must vanish. This finishes the proof of Theorem 4.1.

**Lemma 4.5.** Assume that $d_i$ does not belong to $NZ$ for some $i$ and $\xi$ satisfies the condition (25). Then there exists a generic vector $v \in L$ such that $h(v)$ is non-zero in $\mathbb{Q}/NZ$.

**Proof.** Fix an element $I \in \Sigma^{(n)}$ containing $i_0$ with $d_{i_0} \notin NZ$. Since the $u^I_i$ form a basis of $L_{I, \mathcal{Y}}$ and $d_{i_0} \notin \mathbb{Q}/NZ$, there is a $v = \sum_{i \in I} a_i v_i \in L_{I, \mathcal{Y}} \subset L$ with $a_i \in \mathbb{Z}$, $a_i \neq 0$ for all $i \in I$ such that $\langle u^I(\xi), v \rangle = \sum_{i \in I} a_i d_i$ does not belong to $NZ$, i.e., $\langle u^I(\xi), v \rangle \neq 0$ in $\mathbb{Q}/NZ$. Since $\xi$ satisfies the condition (25), the value $\langle u^I, v \rangle \in \mathbb{Q}/NZ$ is independent of $I$ and equal to $h(v)$ as remarked above. 

For the proof of Theorem 4.2 we apply Theorem 4.1 with $\eta = 0$ and arbitrary $N > 1$ such that $\xi = u$ does not belong to $NL^2(\mathcal{Y})$. One sees that $\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; \tau, \sigma)$ vanishes for $\sigma = \frac{1}{k}$, $0 < k < N$. Since this is true for infinite many integers $N$ we see that $\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi; \tau, \sigma)$ must be equal to 0.

### 5. A character formula for orbifold elliptic genus

In this section we shall give a character formula for $\hat{\varphi}(\Delta, \mathcal{Y}, \xi)$. A similar formula was first given by Borisov and Libgober in [2]. The formula for the orbifold elliptic genus of a multi-fan was given in [12].

Recall that $f_{h,i}$ for $h \in H_J$ and $i \in J$ is given by $f_{h,i} = \langle u^I, v(h) \rangle$ where $v(h)$ satisfies

$$0 \leq \langle u^I, v(h) \rangle < 1 \text{ for all } i \in J.$$

**Theorem 5.1.** Let $(\Delta, \mathcal{Y}, \xi)$ be a triple of complete simplicial multi-fan in a lattice $L$ of rank $n$, a set of generating edge vectors and a $\mathbb{Q}$-divisor. Then

$$\hat{\varphi}_s(\Delta, \mathcal{Y}, \xi)$$

$$= \sum_{w \in L^*} q^{-w} \sum_{h \in H_J} \sum_{J \in \Sigma^{(n)}} (1 - \zeta^{\deg(\Delta_J)}) \zeta^{\xi, J(h)} q^{(u, v, J, h)} \prod_{i \in J} \frac{1}{1 - \zeta^{d_i} q^{(u, v, i)}},$$

where $f_{J,h}(\xi) = \sum_{i \in J} d_i f_{h,i}$ and $v_{J,h} = \sum_{i \in J} f_{h,i} v_i$. 

Note. \( v_{j,h} = \sum_{i \in J} \langle u_i^j, v(h) \rangle v_i = v(h) \in L_J \) for a particularly chosen representative \( v(h) \). Hence \( \langle u, v_{j,h} \rangle \in \mathbb{Z} \). This shows that \( \hat{\varphi}_{st}(\Delta, \mathcal{Y}, \xi) \) belongs to \((R(T) \otimes \mathbb{C})[[q]]\). This fact was already proved in Theorem 3.3. Note that
\[
\zeta_{l,j,h}(\xi) q^{(u,v_{j,h})} \prod_{i \in J} \frac{1}{1 - \zeta^{d_i} q^{(u,v_i)}} = \prod_{i \in J} \left( \zeta^{\cdot} q^{(u,v_i)} \right) f_{h,i}.
\]

We need the following three lemmas.

**Lemma 5.2.** Suppose that \(|q| < |t| < 1\). Then we have the equality
\[
\phi_{st}(z, \tau, \sigma) = \sum_{m \in \mathbb{Z}} \frac{t^m}{1 - \zeta q^m}.
\]

**Lemma 5.3.** Put \( \alpha = e^{2\pi \sqrt{-1} w} \). Suppose that \(|\alpha| = 1\) and \(|q| < |t| < 1\). If \( l \neq 0 \) is an integer, then we have the equality
\[
\phi_{st}(lz + w, \tau, \sigma) = \begin{cases} 
\sum_{m \in \mathbb{Z}} \alpha^m t^m \frac{1}{1 - \zeta q^m} & \text{if } l > 0, \\
\sum_{m \in \mathbb{Z}} \alpha^m t^m \left( \frac{1}{1 - \zeta q^m} - 1 \right) & \text{if } l < 0.
\end{cases}
\]

**Lemma 5.4.** Let \( f \) be a real number with \( 0 < f < 1 \) and \( l \neq 0 \) an integer. If
\[
|q^f|, |q^{1-f}| < |t|^{|l|}, |t| \leq 1
\]
then
\[
\phi_{st}(lz + f\tau + w, \tau, \sigma) = \sum_{m \in \mathbb{Z}} \alpha^m t^m \frac{q^m}{1 - \zeta q^m},
\]
where \( \alpha = e^{2\pi \sqrt{-1} w}, |\alpha| = 1 \) as before.

Lemma 5.2 is essentially the same as Lemma 3.5 of [12]. The equality (34) was first proved in [2]. Lemma 5.3 is essentially the same as Lemma 3.6 of [12]. Lemma 5.4 is essentially the same as Lemma 3.7 of [12].

We now proceed to the proof of Theorem 5.1. Take a generic vector \( v \in L_{\mathcal{Y}} \). Then \( \langle u_i^j, v \rangle \) is an integer for any \( i \in \Sigma^{(n)} \). For \( I \in \Sigma^{(n)} \) we put \( I(v) = \{ i \ | \ \langle u_i^j, v \rangle < 0 \} \). We take a representative \( v(h_1) \) of \( h_1 \in \tilde{H}_K \) such that
\[
\langle u_i^j, v(h_1) \rangle = \langle u_i^j, v(h_1) \rangle = f_{h_1,i}
\]
for each \( i \in K \). Then, by (23), \( \hat{\varphi}_{st}^{\mathcal{Y}}(\Delta, \mathcal{Y}, \xi) \) can be written in the form
\[
\hat{\varphi}_{st}^{\mathcal{Y}}(\Delta, \mathcal{Y}, \xi) = \sum_{k=0}^{n} \sum_{K \in \Sigma^{(k)}} \sum_{h_i \in \tilde{H}_K} \sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|H_I|} = \prod_{i \in I \setminus K} \phi_{st}(-\langle u_i^j, zv + v(h_2) \rangle, \tau, d_i \sigma) \prod_{i \in K} \zeta^{d_i} f_{h_1,i} \phi_{st}(-\langle u_i^j, zv - \tau v(h_1) + v(h_2) \rangle, \tau, d_i \sigma).
\]

If \( t \in \mathbb{C} \) satisfies
\[
|q^{f_{h_1,i}}|, |q^{1-f_{h_1,i}}| < |t|^{|\langle u_i^j, v \rangle|}, |t| < 1,
\]
then by Lemma 5.4, we have
\[ \phi_{st}(-\langle u'_1, zv - \tau v(h_1) + v(h_2) \rangle, \tau, d, \sigma) = \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, v(h_2))^{-m_i} q^{m_i f_{h_1} + \tau - m_i(u'_1, v)} \frac{1}{1 - \zeta d_i q^{m_i}} \]

Next fix \( i \in I \setminus K \). Then, by Lemma 5.3, we have
\[ \phi_{st}(-\langle u'_1, zv + v(h_2) \rangle, \tau, d, \sigma) = \begin{cases} \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, v) (-m_i, t^{-m_i(u'_1, v)})(1 - \frac{1}{1 - \zeta d_i q^{m_i}}) & \text{for } i \in I(v) \setminus K, \\ \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, v)(-m_i, t^{-m_i(u'_1, v)})(1 - \frac{1}{1 - \zeta d_i q^{m_i}}) & \text{for } i \in I \setminus (I(v) \cup K). \end{cases} \]

Now suppose that \(|t| < 1\) and \( q \) satisfy
\[ |q^{f_{h_1,i}}|, |q^{1-f_{h_1,i}}| < |t|^{(u'_1,v)}| \]
for all \( i \in K, h_1 \in \hat{H}_K, K \in \Sigma^{(k)}, \) and
\[ |q| < |t|^{(u'_1,v)}| \]
for all \( i \in I \setminus K, I \in \Sigma^{(n-k)}_K \) and \( K \in \Sigma^{(k)} \). Then we obtain
\[ \prod_{i \in I \setminus K} \phi_{st}(-\langle u'_1, zv + v(h_2) \rangle, \tau, d, \sigma) \prod_{i \in K} \chi^{d_i f_{h_1}, \phi_{st}(-\langle u'_1, zv - \tau v(h_1) + v(h_2) \rangle, \tau, d, \sigma)} = \zeta^{f_{K,i}}(\zeta) \prod_{i \in I(v) \setminus K} \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, h_2)^{-m_i} t^{-m_i(u'_1, v)} \frac{1}{1 - \zeta d_i q^{m_i}} \prod_{i \in I(v) \cup K} \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, h_2)^{-m_i} t^{-m_i(u'_1, v)} \frac{1}{1 - \zeta d_i q^{m_i}} \prod_{i \in K} \sum_{m_i \in \mathbb{Z}} \chi_I(u'_1, h_2)^{-m_i} q^{m_i f_{h_1} + t^{-m_i(u'_1, v)} \frac{1}{1 - \zeta d_i q^{m_i}}} \]
Furthermore
\[ \prod_{i \in I(v) \setminus K} \frac{1}{1 - \zeta d_i q^{m_i}} \prod_{i \in I(v) \cup K} \left( \frac{1}{1 - \zeta d_i q^{m_i}} - 1 \right) \prod_{i \in K} \frac{1}{1 - \zeta d_i q^{m_i}} \]
for each \( I \in \Sigma^{(n-i)}_K \).

If we put \( u = \sum_{i \in I} m_i u'_i \in L^*_I, \) then \( m_i = \langle u, v_i \rangle \). Hence \( \prod_{i \in I} t^{-m_i(u'_1, v)} = t^{-\langle u, v \rangle} \). Since \( \chi_I(u, v) = e^{2 \pi i \sqrt{-1} \langle u, v \rangle} \) we see that \( \prod_{i \in I} \chi_I(u'_1, h)^{-m_i} = \chi_I(u, h)^{-1} \).
The 1-dimensional representation \( \chi_I(u, h)^{-1} \) of \( H_I = L/L^*_I, \) \( y \) is trivial if and only if \( u \in L^* \). It follows that
\[ \sum_{h \in H_I} \chi_I(u, h)^{-1} = \begin{cases} |H_I| & \text{if } u \in L^*, \\ 0 & \text{if } u \notin L^*. \end{cases} \]
Combining these we have

\[ \sum_{h_j \in H_I, \nu_i \in I \wedge K} \prod_{i \in K} \phi_{st}(-\langle u_i^j, zv + v(h_2) \rangle), \tau, d, \sigma \prod_{i \in K} \zeta^{d, l_{h_1} i} \phi_{st}(-\langle u_i^j, zv - \tau v(h_1) + v(h_2) \rangle), \tau, d, \sigma) \]

\[ = |H_I| \sum_{u \in L^*} t^{-\langle u, v \rangle} \sum_{k=0}^{n} \sum_{J \in \Sigma(v), h \in H_J} \zeta^{f_{J, h}(\xi)} q^{\langle u, v, h \rangle} (-1)^{n-k} \sum_{I: I \cap J \subseteq J \subseteq I} w(I) \prod_{j \in J} \frac{1}{1 - \zeta q^{\langle u, v \rangle}}. \]

Fix $J \subseteq \Sigma$. It is easy to see that the union of $\{\hat{H}_K | K \in \Sigma, K \subseteq J\}$ is disjoint. Since any $h \in H_J$ is contained in $\hat{H}_{K_h}$ by (22) where $K_h = \{j \in J | f_{h,j} \neq 0\}$, we have $H_J = \sqcup \hat{H}_K$. Moreover we have

\[ f_{J,h}(\xi) = f_{K_h,h}(\xi) \quad \text{and} \quad v_{J,h} = v_{K_h,h}. \]

Taking these facts in account in (36) and using (35), we get

\[ \tilde{\phi}^v_{st}(\Delta, \nu', \xi) \]

\[ = \sum_{u \in L^*} t^{-\langle u, v \rangle} \left( \sum_{k=0}^{n} \sum_{J \in \Sigma(v), h \in H_J} \zeta^{f_{J, h}(\xi)} q^{\langle u, v, h \rangle} (-1)^{n-k} \sum_{I: I \cap J \subseteq J \subseteq I} w(I) \prod_{j \in J} \frac{1}{1 - \zeta q^{\langle u, v \rangle}} \right). \]

Since $\sum_{I \in \Sigma(v), J \subseteq I} w(I) = \deg(\Delta_J)$ by definition we have

\[ \tilde{\phi}^v_{st}(\Delta, \nu', \xi) \]

\[ = \sum_{u \in L^*} t^{-\langle u, v \rangle} \left( \sum_{k=0}^{n} \sum_{J \in \Sigma(v), h \in H_J} (-1)^{n-k} \deg(\Delta_J) \zeta^{f_{J, h}(\xi)} q^{\langle u, v, h \rangle} \prod_{i \in J} \frac{1}{1 - \zeta q^{\langle u, v \rangle}} \right). \]

Since $\tilde{\phi}^v_{st}(\Delta, \nu', \xi)$ belongs to $(R(T) \otimes \mathbb{C})[[q]]$ and (38) holds for any generic vector $v$, Theorem 5.1 follows.

REMARK 5.5. Stabilized orbifold elliptic genus along a vector $v$ is defined for a complete simplicial multi-fan by the formula (21). However the right-hand side of (21) still has meaning for general simplicial multi-fans. Thus we can define $\tilde{\phi}^v_{st}(\Delta, \nu', \xi)$ in general by (21). It can be written in the form (23) too. Note that Theorem 5.1 does not hold in general because it depends on an integrality theorem (Lemma 2.5 in [12]) that holds only for complete simplicial multi-fans. Neither holds Theorem 5.1 in general. But the following Proposition holds.

**Proposition 5.6.** Let $\Delta$ be a simplicial fan satisfying the condition that every $J \subseteq \Sigma$ is contained in some $I \in \Sigma^{(n)}$. Then (35) holds provided that the generic vector $v$ is contained in $\bigcup_{I \in \Sigma^{(n)}} C(I)$.

\[ \tilde{\phi}^v_{st}(\Delta, \nu', \xi) \]

\[ = \sum_{u \in L^*} t^{-\langle u, v \rangle} \left( \sum_{k=0}^{n} \sum_{J \in \Sigma(v), h \in H_J} (-1)^{n-k} \zeta^{f_{J, h}(\xi)} q^{\langle u, v, h \rangle} \prod_{i \in J} \frac{1}{1 - \zeta q^{\langle u, v \rangle}} \right). \]

**Proof.** Note first that $w(I) = 1$ for all $I \in \Sigma^{(n)}$ since $\Delta$ is a fan. It suffices to show that

\[ \#(S_v(J)) = \#\{I | I(v) \cap J \subseteq I\} = 1 \]

for all $J \subseteq \Sigma$ provided that $v$ is contained in $\bigcup_{I \in \Sigma^{(n)}} C(I)$ because the completeness of $\Delta$ was only used at the step from (37) to (38). Suppose that $J$ lies in $\Sigma^{(l)}$. Then
the set \( \{ I \mid I(v) \subset J \subset I \} \) is identified with the set \( S_v(J) = \{ I \in \Sigma_{j-1}^{(n-j)} \mid \bar{v} \in C^J(I) \} \) where \( \bar{v} \) is the image of \( v \) in \( L^J \). But \( \Delta J \) is also a fan and every simplex \( K \in \Sigma_{j} \) is contained in some \( I \in \Sigma_{j-1}^{(n-j)} \). Moreover \( \bar{v} \) is generic and contained in \( \bigcup_{J \in \Sigma_{j-1}^{(n-j)}} C^J(I) \). Hence \( \#(S_v(J)) = 1 \). □

Remark 5.7. If we consider \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \) as a function assigning to each generic vector \( v \in \bigcup_{I \in \Sigma^{(n)}} C(I) \) the value \( \hat{\phi}_{st}(\Delta, \mathcal{V}, \xi) \), then (33) holds in this case with the understanding that \( t^{-u} \) is a function assigning the value \( t^{(u,v)} \) to \( v \).

6. Invariance of orbifold elliptic genus under birational morphisms of multi-fans

Let \( \Delta \) be a simplicial multi-fan (not necessarily complete) in a lattice \( L \) and \( \mathcal{V} = \{ v_{I} \}_{I \in \Sigma^{(1)}} \) a set of prescribed edge vectors as before. We shall consider a multi-fan \( \Delta' = (\Sigma', C', w') \) in the same lattice \( L \) and a morphism \( \rho : \Delta' \to \Delta \) related to \( \Delta \) in the following way. \( \rho \) may be regarded as a generalization of birational morphism in toric theory. We require the following conditions for \( \Delta' \):

a) There is an injection \( \kappa : \Sigma^{(1)} \to \Sigma'^{(1)} \) satisfying \( C(i) = C'(\kappa(i)) \) for each \( i \in \Sigma^{(1)} \).

b) For each \( J' \in \Sigma' \) there is a simplex \( J \in \Sigma \) such that \( C'(J') \subset C(J) \). Moreover, for each \( J \in \Sigma \), the collection \( \{ C'(J') \mid J' \in \Sigma', C'(J') \subset C(J) \} \) gives a subdivision of the cone \( C(J) \). We shall denote by \( \rho(J') \) the minimal simplex \( J \in \Sigma \) such that \( C'(J') \subset C(J) \).

c) For \( I' \in \Sigma'^{(n)} \) \( w'^{\pm}(I') = w^{\pm}(\rho(I')) \).

In particular \( w'(I') = w(\rho(I')) \).

\( \rho(\{i'\}) \) is simply denoted by \( \rho(i') \). Note that \( \rho(\kappa(i)) = i \) for \( i \in \Sigma^{(1)} \). The map \( \rho : \Sigma' \to \Sigma \) is sometimes denoted by \( \rho : \Delta' \to \Delta \). Let \( \mathcal{V} = \{ v_{I} \}_{I \in \Sigma^{(1)}} \) and \( \mathcal{V}' = \{ v'_{I} \}_{I \in \Sigma'^{(1)}} \) be sets of edge vectors for \( \Delta \) and \( \Delta' \) respectively. Note that \( v_{\kappa(i)} \) and \( v_i \) lie on the same half line \( C(i) \) but they may be different.

The vector \( v_{i'} \) is written uniquely in the form
\[
v_{i'} = \sum_{i \in \rho(i')} a_{i'i} v_i
\]

with \( a_{i'i} \in \mathbb{Q}_{>0} \). We put \( a_{i'i} = 0 \) for \( i \not\in \rho(i') \). We then define a map \( \rho^* : H^2_{T}(\Delta, \mathcal{V}) \otimes \mathbb{Q} \to H^2_{T}(\Delta', \mathcal{V}') \otimes \mathbb{Q} \) by
\[
\rho^*(x_i) = \sum_{i' \in \Sigma'^{(1)}} a_{i'i} x_{i'}.
\]

Remark 6.1. As a special case related to (4) consider the following situation: \( \Delta' = \Delta \) and \( \mathcal{V}' = \{ v_{i} \}_{i \in \Sigma^{(1)}} \), \( \mathcal{V} = \{ v_i \}_{i \in \Sigma^{(1)}} \) with \( v_i' = a_i v_i \), \( a_i > 0 \). If \( \xi = \sum c_i x_i \), then \( \rho^*(\xi) = \sum a_i c_i x_i \). In particular \( \rho^*(\xi) = \sum x_i \) if and only if \( \xi = \sum a_i x_i \). In this case \( D_{\Delta} \) defined in Remark (4) is given by \( \sum a_i v_i \). As typical examples of such a situation one can quote weighted projective spaces \( \mathbb{P}(a_0, \ldots, a_n) \). We assume that the greatest common divisor of \( \{ a_0, \ldots, a_i, \ldots, a_n \} \) is equal to 1 for all \( 0 \leq i \leq n \). \( \mathbb{P}(a_0, \ldots, a_n) \) has two natural orbifold structures; one is given as the quotient space of \( \mathbb{C}^{n+1} \setminus \{ 0 \} \) by the action of \( \mathbb{C}^* \) given by
\[
z(z_0, \ldots, z_n) = (z_0^{a_0}, z_1^{a_1}, \ldots, z_n^{a_n}).
\]
and the other as a global quotient of \( \mathbb{P}^n \) by the standard action of the group \( \mathbb{Z}/a_0\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z} \). The former corresponds to \( \mathcal{V} \) and the latter to \( \mathcal{V}' \).

**Lemma 6.2.** \( \rho^* \) extends to a ring homomorphism \( \rho^* : H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q} \rightarrow H^*_T(\Delta', \mathcal{V}') \otimes \mathbb{Q} \). Moreover \( \rho^*(u) = u \) for \( u \in L^*_Q \) and \( \rho^* \) is an \( H^*_T(BT) \otimes \mathbb{Q} \)-module map. It satisfies

\[
\iota^*_{I'} \circ \rho^* = \iota^*_I
\]

for any \( I \in \Sigma^{(n)} \) and \( I' \in \Sigma'^{(n)} \) such that \( \rho(I') = I \).

**Proof.** Let \( I \) be a non-empty subset of \( \Sigma^{(1)} \). In order to show that \( \rho \) extends to a ring homomorphism it is enough to show that \( \prod_{i \in I} x_i = 0 \) implies

\[
\prod_{i \in I} \rho^*(x_i) = \prod_{i \in I} \sum_{i' \in \Sigma^{(1)}} a_{i'i} x_{i'} = 0.
\]

Assume that there is an element \( i'(i) \in \Sigma^{(1)} \) for each \( i \in I \) such that \( i \in \rho(i'(i)) \) and \( \prod_{i \in I} x_{i'(i)} \neq 0 \). Then \( I' = \{i'(i) \mid i \in I\} \) is a simplex and \( \rho(I') = I \). Hence \( I \) must be a simplex. This contradicts the fact that \( \prod_{i \in I} x_i = 0 \). Therefore, for all subsets \( \{i' \mid i \in \rho(i'), i \in I\} \), the product \( \prod x_{i'} \) must be equal to 0. Then \( \prod_{i \in I} \sum_{i' \in \Sigma^{(1)}} a_{i'i} x_{i'} \), as a linear combination of such elements, is equal to 0.

For \( u \in L^*_Q \) we have

\[
\rho^*(u) = \sum_i \langle u, v_i \rangle \rho^*(x_i)
\]

\[
= \sum_i \langle u, v_i \rangle \sum_{i'} a_{i'i} x_{i'}
\]

\[
= \sum_i \langle u, v_i \rangle x_{i'} = u.
\]

Then \( \rho^*(ux) = \rho^*(u)\rho^*(x) = up^*(x) \) for any \( x \in H^*_T(\Delta, \mathcal{V}) \). This shows that \( \rho^* \) is an \( H^*_T(BT) \otimes \mathbb{Q} \)-module homomorphism.

In order to prove (39) we may check it on the generators \( x_i \in H^*_T(\Delta) \) since \( \iota^*_I, \iota^*_{I'} \) and \( \rho^* \) are ring homomorphisms. If \( v_{i'} = \sum_{i \in I} a_{i'i} v_i \) for \( i' \in I' \), then \( u_{i'} = \sum_{i \in I} b_{i'i} u_i \) with \( \sum_{i' \in I'} a_{i'i} b_{i'i} = \delta_{ij} \). Hence

\[
\iota^*_{I'}(\rho^*(x_i)) = \iota^*_{I'}(\sum_{i'} a_{i'i} x_{i'})
\]

\[
= \sum_{i'} a_{i'i} u_{i'}
\]

\[
= \sum_{i'} \sum_{j} a_{i'i} b_{i'j} u_j = u_{i'} = \iota^*_I(x_i).
\]

This proves (39). \( \square \)

If \( \xi = \sum_{i \in \Sigma^{(1)}} d_i x_i \) and \( \rho^*(\xi) = \sum_{i' \in \Sigma'^{(1)}} d_{i'} x_{i'} \), then

\[
d_{i'} = \sum_{i \in \rho(i')} a_{i'i} d_i
\]

as is easily seen. If we put \( u^J(\xi) = \sum_{i \in J} d_i u_i^J \) with \( J = \rho(i') \), then \( d_{i'} \) is also written as

\[
(40) \quad d_{i'} = \langle u^J(\xi), v_{i'} \rangle.
\]
INVARIANCE PROPERTY OF ORBIFOLD ELLIPTIC GENUS FOR MULTI-FANS

Note also

\[ d_i = \langle u^i(\xi), v_i \rangle. \]

**Theorem 6.3.** Let \( \Delta \) be a complete simplicial multi-fan and let \( \rho : \Delta' \to \Delta \) be a map satisfying a), b) and c). We put \( \xi' = \rho^*(\xi) \). Then \( \Delta' \) is also complete and the following equality holds:

\[ \hat{\varphi}_{st}(\Delta', \mathcal{V}', \xi') = \hat{\varphi}_{st}(\Delta, \mathcal{V}, \xi). \]

The proof will be given in the next section by using a local version of the theorem.

**7. Local version of invariance**

For a triple \((\Delta, \mathcal{V}, \xi)\) with \( \xi = \sum_{i \in \Sigma(1)} d_i \xi_i \) we set

\[ b_J(\Delta, \mathcal{V}, \xi) = \sum_{h \in H_J} (-1)^{n-k} \zeta^{f_{J,h}(\xi)} q^{v_{J,h}} \prod_{i \in J} \frac{1}{1 - \zeta^{d_i} q^{v_i}}, \]

where \( q^v \) for \( v \in L \) is considered as a function assigning to each \(-u \in L^*\) the value \( q^{(u,v)} \). Thus \( q^{v_{J,h}} \prod_{i \in J} \frac{1}{1 - \zeta^{d_i} q^{(u,v)}} \) takes the value

\[ q^{v_{J,h}} \prod_{i \in J} \frac{1}{1 - \zeta^{d_i} q^{(u,v)}} \]

at \(-u \in L^*\). When \( \Delta \) is a complete simplicial multi-fan Theorem 5.1 says that

\[ \hat{\varphi}_{st}(\Delta, \mathcal{V}, \xi) = \sum_{u \in L^*} e^{-u} \left( \sum_{J \in \Sigma} \deg(\Delta_J) b_J(\Delta, \mathcal{V}, \xi) \right)(u). \]

In the sequel we shall write

\[ \hat{\varphi}_{st}(\Delta, \mathcal{V}, \xi) = \sum_{J \in \Sigma} \deg(\Delta_J) b_J(\Delta, \mathcal{V}, \xi) \]

to mean the equality (41).

Let \( I \in \Sigma^{(n)} \). For a while we shall concentrate on the part of \( \Sigma \) consisting of all faces of \( I \), which we shall denote by \( \Sigma(I) \). Similarly the part of \( \Delta \) restricted on \( \Sigma(I) \) will be denoted by \( \Delta(I) \). In this case we consider it as a fan forgetting the function \( w^\pm \). When there is no fear of confusion, we shall simply denote \( \Sigma(I) \) by \( \Sigma \) and \( (\Delta(I), \mathcal{V}\mid I, \xi\mid I) \) by \( (\Delta, \mathcal{V}, \xi) \).

If \( \Delta' \) is a fan and \( \rho : \Delta' \to \Delta(I) \) is a map satisfying a) and b), we put \( \xi' = \rho^*(\xi) \). If \( \xi' = \sum_{i' \in \Sigma^{(1)}} d_{i'} x_{i'} \), then

\[ d_{i'} = \sum_{i \in I} a_{i'i} d_i. \]

Define

\[ b_J(\Delta', \mathcal{V}', \xi'; \Delta, \mathcal{V}, \xi) := \sum_{J', \rho(J') = J} b_{J'}(\Delta', \mathcal{V}', \xi') - b_J(\Delta, \mathcal{V}, \xi). \]

for \( J \in \Sigma(I) \), i.e., for \( J \subset I \).

The following theorem can be considered as a local version of Theorem 6.3.

**Theorem 7.1.** The statement

\[ B_n : b_J(\Delta', \mathcal{V}', \xi'; \Delta, \mathcal{V}, \xi) = 0 \quad \text{for dim } L = n \text{ and for all } J \in \Sigma \]

holds for \( n \geq 1 \).
follows easily that $d_{J}$ in $C$ and the triple $(\tilde{\Sigma}, 1)$ is not contained in the boundary of $C(I)$. Moreover $H_{1} \cong \mathbb{Z}/a_{1}$ and $f_{h,1} = \tilde{h}/a_{1}$ for $h \in \mathbb{Z}/a_{1}$ where $\tilde{h}$ is the representative of $h$ with $0 \leq \tilde{h} < a_{1}$. Similarly $H_{1} \cong \mathbb{Z}/a_{1}$ and $f_{h,1} = \tilde{h}/a_{1}$ for $h' \in \mathbb{Z}/a_{1}$. Since $v_{1} = \frac{a_{1}'}{a_{1}}v_{1}$, we have $\frac{d_{1'}}{a_{1}} = \frac{d_{1}}{a_{1}}$ by definition. We denote this last number by $d$. Then

$$b_{(1)}(\Delta, \psi, \xi) = \frac{\sum_{k=0}^{d_{1}q_{1}} \xi_{1}^{-d_{1}q_{1}} q_{1}^{-d_{1}q_{1}}}{1 - \zeta_{1}^{-d_{1}q_{1}} q_{1}^{-d_{1}q_{1}}} = \frac{1}{1 - \zeta_{1}^{-d_{1}q_{1}} q_{1}^{-d_{1}q_{1}}} = \frac{1}{1 - \zeta_{1}^{-d_{1}q_{1}}}.$$  

Similarly we have

$$b_{(1')}(\Delta', \psi', \xi') = \frac{1}{1 - \zeta_{1}^{-d_{1}q_{1}}}.$$  

Hence

$$b_{(1)}(\Delta', \psi', \xi') : \Delta, \psi, \xi = b_{(1)}(\Delta', \psi', \xi') - b_{(1)}(\Delta, \psi, \xi) = 0.$$  

Clearly

$$b_{0}(\Delta', \psi', \xi') : \Delta, \psi, \xi = 1 - 1 = 0.$$ 

This proves $B_{1}$.

To illustrate the proof for general case we first give a proof of $B_{1}$ for the following special case, namely the case where $v_{\kappa(i)} = v_{i}$ for all $i \in \Sigma^{(1)}$ and $\rho(i') = I$ for all $i' \in \Sigma^{(1)} \setminus \kappa(\Sigma^{(1)})$. In this case we introduce a multi-fan $\tilde{\Delta}_{*}$ defined in the following way. Note that $\kappa$ induces an injective simplicial map $\kappa : \partial \Sigma = \Sigma \setminus \Sigma^{(n)} \rightarrow \Sigma' \cap \Sigma^{(n)}$ compatible with $C$ and $C'$. This simplicial map is realized as the identity map on the boundary of the cone $C(I)$. A simplicial set $\Sigma_{*}$ is defined as the sum of $\Sigma'$ and $\Sigma$ glued along $\partial \Sigma$ via $\kappa$. Note that $\Sigma_{*}^{(1)} = \Sigma^{(1)}$. $C$ and $C'$ determines a cone structure $C_{*} : \Sigma_{*} \rightarrow L_{R}$. We define the function $w^{\pm}$ for $\Sigma_{*}$ as follows:

$$w^{+}(I) = \begin{cases} 1, & I_{k} \in \Sigma^{(n)}, \\ 0, & I_{k} \in \Sigma(1), \text{i.e., } I_{k} = I. \end{cases}$$  

$$w^{-}(I) = \begin{cases} 0, & I_{k} \in \Sigma^{(n)}, \\ 1, & I_{k} = I. \end{cases}$$ 

Then

$$w(I) = \begin{cases} +1, & \text{for } I_{k} \in \Sigma^{(n)}, \\ -1, & \text{for } I_{k} = I. \end{cases}$$

The triple $(\Sigma_{*}, C_{*}, w^{\pm})$ defines a simplicial multi-fan $\tilde{\Delta}_{*}$.

We claim that $\tilde{\Delta}_{*}$ is complete and

$$\deg((\tilde{\Delta}_{*})_{J}) = \begin{cases} +1, & J \in \Sigma^{(n)} \setminus \partial \Sigma, \\ -1, & J \in \Sigma^{(n)}, \text{i.e., } J = I, \\ 0, & J \in \partial \Sigma. \end{cases}$$  

In fact take a generic vector $v$ in $L_{R}$ and consider $d_{v}(J) = \sum_{I_{k} \in S_{v}(J)} w(I_{k})$ for $J \in \tilde{\Sigma}_{*}^{n-1}$ where $S_{v}(J)$ is defined by (10). If $J$ is contained in $\Sigma^{(n)} \setminus \partial \Sigma$ then $C'(J)$ is not contained in the boundary of $C(I)$ and is two-sided in $L_{R}$. From this it follows easily that $d_{v}(J)$ is equal to 1 independently of generic $v$. This fact shows that $(\tilde{\Delta}_{*})_{J}$ is complete and $\deg((\tilde{\Delta}_{*})_{J}) = +1$. On the other hand, if $J$ is contained in $\partial \Sigma$, then $J$ is a face of $I$ and of exactly one simplex $I_{k}$ in $\Sigma^{(n)}$. Noting that $w(I) = -1$ and $w(I_{k}) = +1$ we see that $d_{v}(J) = 0$ independently of generic $v$.  


Hence \( (\tilde{\Delta}_*)_J \) is complete and \( \deg((\tilde{\Delta}_*)_J) = 0 \) in this case. Since \( (\tilde{\Delta}_*)_J \) is complete for all \( J \in \tilde{\Delta}_*^{(n-1)} \), the multi-fan \( \tilde{\Delta}_* \) is complete. Other statements concerning \( (\tilde{\Delta}_*)_J \) for \( J \in \tilde{\Delta}_* \) of arbitrary dimensions can be proved in a similar way.

We define \( \tilde{\mathcal{V}}_* \) and \( \tilde{\xi}_* \) to be equal to \( \mathcal{V}' \) and \( \xi' \) respectively. In view of the assumption \( v_{\kappa(i)} = v_i \) these definitions make sense. Then we have

\[
\tilde{\xi}_* = \sum_{i \in \Sigma_1^{(1)}} d_i(\tilde{x}_i) \tilde{x}_i = \sum_{i \in \Sigma_1^{(1)}} d_i' x_i' = \rho^* \left( \sum_{i \in \Sigma_1^{(1)}} d_i x_i \right) = \rho^*(u) = u \in H_T^2(\tilde{\Delta}_*, \tilde{\mathcal{V}}_*) \otimes \mathbb{Q}
\]

for \( u = u^I(\xi) = \sum_{i \in \Sigma_1^{(1)}} d_i u_i^I \in \mathbb{L}^*_\mathcal{V} \). We can apply Theorem 4.11 to obtain

\[
\tilde{\phi}_{st}(\tilde{\Delta}_*, \tilde{\mathcal{V}}_*, \tilde{\xi}_*) = 0.
\]

By 4.11 and 4.12 we have

\[
\tilde{\phi}_{st}(\tilde{\Delta}_*, \tilde{\mathcal{V}}_*, \tilde{\xi}_*) = \sum_{u \in \mathcal{L}} t^{-u} b_1(\Delta', \mathcal{V}', \xi'; \Delta, \mathcal{V}, \xi)(u)
\]

finally

\[
b_1(\Delta', \mathcal{V}', \xi'; \Delta, \mathcal{V}, \xi) = 0.
\]

In this case

\[
b_J(\Delta', \mathcal{V}', \xi'; \Delta, \mathcal{V}, \xi) = 0
\]

trivially holds for each \( J \in \Sigma^{(k)} \) with \( k < n \). Thus \( B_n \) is true in this case.

The proof of general case proceeds by induction starting from \( B_1 \):

\[
B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \cdots \rightarrow B_{n-1} \rightarrow B_n \rightarrow \cdots
\]

The statement \( C_n \) is described by using multi-fans \( \Delta' \) and \( \Delta_* \) which will be constructed below from \( \Delta' \) and \( \Delta \) respectively.

Let \( W \) be a rational affine hyperplane in \( L_R \) intersecting \( C(I) \) transversally and let \( S = W \cap C(I) \). We assume that there is an integral point \( v \in L \) in the interior of \( S \). Let \( S_m \) be the image of the homothety \( \psi \) of ratio \( m \) centered at \( v \) where \( m \) is a positive integer. \( S \) and \( S_m \) are geometric simplices. We give a triangulation of \( S_m \setminus S \) as follows. First the boundary \( \partial S \) is triangulated in the standard way, i.e., its simplices are of the form \( S_J = S \cap C(J) \) for \( J \in \Sigma(I)^{(k)} \), \( 0 < k < n \). The boundary \( \partial S_m \) is triangulated by the barycentric subdivision, i.e., its simplices are of the form \( (b_{J_1} \ldots b_{J_k}) \) with \( J_1 \subset \ldots \subset J_k \subset I \). Here \( b_J \) is the barycenter of the simplex \( \psi(S \cap C(J)) \). Simplices between \( \partial S \) and \( \partial S_m \) are of the form \( S_J \ast (b_{J_1} \ldots b_{J_k}) \) with \( J \subset J_1 \subset \ldots \subset J_k \subset I \) where \( \ast \) denotes the join. A triangulation of \( S_m \) is induced from this triangulation of \( S_m \setminus S \) together with the standard triangulation of \( S \). By projecting this into the cone over \( S_m \) centered at the origin we get a fan \( \Delta_* = (\Sigma_*, C_*) \). The simplicial set \( \Sigma_* \) can be described in the form

\[
\Sigma_* = \Sigma \cup \{ J \ast (J_1 \ldots J_k) \mid J \subset J_1 \subset \ldots \subset J_k \subset I \}.
\]

The case \( (J_1 \ldots J_k) = \emptyset \) is included. The simplex \( (J_1 \ldots J_k) \) will be denoted by \( K \) for simplicity. Then part of \( \Sigma_* \) with \( K = \emptyset \) is the subcomplex of \( \Sigma \) corresponding to the boundary of \( I \). Part of \( \Sigma_* \) with \( J = \emptyset \) will be denoted by \( \partial \Sigma_* \). It is the set of sequences \( (J_1 \ldots J_k) \) such that \( J \subset J_1 \subset \ldots \subset J_k \subset I \). The cone \( C_*(J \ast (J_1 \ldots J_k)) \) is the one generated by \( S_J \ast (b_{J_1} \ldots b_{J_k}) \).

The cone \( C(I) \) is triangulated by \( \{ C'(J') \} \) with \( \rho(J') \subset I \) and the simplex \( S \) is triangulated accordingly. If we replace the standard triangulation of \( S \) by this triangulation in the construction of \( \Delta_* \), above and \( S_J \ast (b_{J_1} \ldots b_{J_k}) \) by \( S_{J'} \ast (b_{J_1} \ldots b_{J_k}) \) with \( \rho(J') \subset J_1 \), then we obtain another fan \( \Delta'_* = (\Sigma'_*, C'_*) \) where

\[
\Sigma'_* = \Sigma' \cup \{ J' \ast (J_1 \ldots J_k) \mid \rho(J') \subset J_1 \subset \ldots \subset J_k \subset I \}.
\]
Part of $\Sigma^\prime$ with $J^\prime = \emptyset$ will be denoted by $\partial \Sigma^\prime$. It is identified with $\partial \Sigma_\ast$ in an obvious way.

We further assume that all the $b_J \in S_m$ are integral points. This is possible by choosing $m$ suitably. Moreover such $m$ can be taken arbitrarily large. We put $v_J = b_J$. Then we define $\mathcal{Y}_\ast$ to be the sum $\mathcal{Y} \sqcup \{v_J\}$ where $(J)$ ranges over $(\partial \Sigma_\ast)^{(1)} \subset \Sigma^{(1)}$. Putting $u^I(\xi) = \sum_{i \in I} d_i u_i^I$ define the number $d_J$ by

$$d_J = \langle u^I(\xi), v_J \rangle,$$

and the $\mathbb{Q}$-divisor $\xi_\ast$ by

$$\xi_\ast = \sum_{i \in \Sigma^{(1)}} d_i x_i + \sum_{(J) \in \partial \Sigma^{(1)}} d_J x_J,$$

where $x_J$ is the basis element in $H^2_F(\Delta_\ast, \mathcal{Y}_\ast)$ corresponding to $(J) \in (\partial \Sigma_\ast)^{(1)}$.

Similarly we define

$$\mathcal{Y}_\ast^\prime := \mathcal{Y} \sqcup \{v_J\}_{(J) \in \partial \Sigma^{(1)}}, \quad \xi_\ast^\prime := \sum_{i \in \Sigma^{(1)}} d_i x_i + \sum_{(J) \in \partial \Sigma^{(1)}} d_J x_J.$$

The map $\rho : \Delta' \to \Delta$ induces a map $\rho : \Delta_\ast^\prime \to \Delta_\ast$ by

$$\rho(J^\prime \ast (J_1 \cdots J_k)) = \rho(J^\prime) \ast (J_1 \cdots J_k)),$$

Then it is clear that

$$\rho^*(\xi_\ast) = \xi_\ast^\prime.$$

We put

$$b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime) = \sum_{J^\prime ; \rho(J') = J} b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime),$$

and

$$b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime, \Delta_\ast, \mathcal{Y}_\ast, \xi_\ast) = b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime) - b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast, \mathcal{Y}_\ast, \xi_\ast)$$

as before. We are ready now for the statement $C_n$:

$$C_n \quad b_{J^\prime \ast(J_1 \cdots J_k)}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime, \Delta_\ast, \mathcal{Y}_\ast, \xi_\ast) (u) \to 0 \text{ as } m \to +\infty$$

for dim $L = n$ and for all $J^\prime \ast (J_1 \cdots J_k)$ with $(J_1 \cdots J_k) \neq \emptyset$.

**Lemma 7.2.** The statement $C_n$ holds for $n \geq 2$.

**Proof.** We assume $B_k$ for $k < n$ and deduce $C_n$. The simplex $(J_1 \cdots J_k)$ will be denoted by $K$ as before. Its vertices are written as $(J_1, \ldots, J_k)$ to distinguish from the simplices $J_1, \ldots, J_k$ in $\Sigma$. In case dim $J_k < n - 2$ the dimension of $J \ast K$ is less than $n - 1$. Hence we can apply the inductive assumption for $\rho : \Sigma'((J \ast K) \to \Sigma(J \ast K)$ where $\Sigma'((J \ast K) = \{ J^\prime \in \Sigma' \mid \rho(J^\prime) \subset J \ast K \}$ and $\Sigma(J \ast K) = \{ J_\ast \mid J_\ast \subset J \ast K \}$. It follows that

$$b_{J^\prime \ast K}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime, \Delta_\ast, \mathcal{Y}_\ast, \xi_\ast) = 0.$$

Suppose next that dim $J_k = n - 2$, i.e. $J_k \in \Sigma^{n-1}$. Put

$$H^0_{J^\prime \ast K} = \{ h \in H_{J^\prime \ast K} \mid f_{h,J_k} = 0 \}.$$

It is a subgroup of $H_{J^\prime \ast K}$. Similarly we put

$$H^0_{J^\prime \ast K} = \{ h \in H_{J^\prime \ast K} \mid f_{h,J_k} = 0 \}.$$

Write

$$b_{J^\prime \ast K}(\Delta_\ast^\prime, \mathcal{Y}_\ast^\prime, \xi_\ast^\prime; \Delta_\ast, \mathcal{Y}_\ast, \xi_\ast)(u) = b_0 + b_1,$$
where

\[ b_0 = \left( \sum_{J' \in H_{J'K}} \sum_{h \in H_{J'K}} (-1)^{n-J} \prod_{j' \in J'} \frac{\zeta^{d_{j'}q(u,v_{j'})}f_{h,j'}}{1-\zeta^{d_{j'}q(u,v_{j'})}} \prod_{i<k} \frac{\zeta^{d_{j_i}q(u,v_{j_i})}f_{h,(j_i)}}{1-\zeta^{d_{j_i}q(u,v_{j_i})}} \right) - \sum_{h \in H_{J'K}} (-1)^{n-J} \prod_{j \in J} \frac{\zeta^{d_jq(u,v_j)}f_{h,j}}{1-\zeta^{d_jq(u,v_j)}} \prod_{i<k} \frac{\zeta^{d_{j_i}q(u,v_{j_i})}f_{h,(j_i)}}{1-\zeta^{d_{j_i}q(u,v_{j_i})}} \right), \]

and

\[ b_1 = \left( \sum_{J' \in H_{J'K}} \sum_{h \in H_{J'K}} (-1)^{n-J} \prod_{j' \in J'} \frac{\zeta^{d_{j'}q(u,v_{j'})}f_{h,j'}}{1-\zeta^{d_{j'}q(u,v_{j'})}} \prod_{i<k} \frac{\zeta^{d_{j_i}q(u,v_{j_i})}f_{h,(j_i)}}{1-\zeta^{d_{j_i}q(u,v_{j_i})}} \right) - \sum_{h \in H_{J'K}} (-1)^{n-J} \prod_{j \in J} \frac{\zeta^{d_jq(u,v_j)}f_{h,j}}{1-\zeta^{d_jq(u,v_j)}} \prod_{i<k} \frac{\zeta^{d_{j_i}q(u,v_{j_i})}f_{h,(j_i)}}{1-\zeta^{d_{j_i}q(u,v_{j_i})}} \right). \]

The subgroup \( H_{J'K} \) coincides with \( H_{J'K^0} \) where \( K^0 = (J_1 \cdots J_{k-1}) \). It follows that the term inside the parenthesis in the expression for \( b_0 \) is nothing but \( b_{J,K^0} = \left( \Delta, \gamma', \xi'; \Delta, \gamma, \xi \right) \) and is equal to zero by induction assumption. Thus \( b_0 = 0 \).

In order to estimate \( b_1 \) we first remark the following

**Lemma 7.3.** The orders of \( H_{J*K} = H_{J*K*\gamma} \) for \( J \in \Sigma \) and \( H_{J'*K} = H_{J'*K*\gamma} \) for \( J' \in \Sigma' \) are bounded by \( Cm^n \) for some constant \( C \) depending only on \( \Delta, W \) and \( v_0 \).

Proof of Lemma. If \( J \ast K \) is contained in \( J_1 \ast K_1 \) then \( |H_{J*K}| \leq |H_{J_1*K_1}| \). So we may assume \( J \ast K \in \Sigma_{n}^{(n)} \). Then \( H_{J*K} = L_{J,K} \). If we choose a basis of \( L \) we can express \( |H_{J*K}| \) as the determinant of a matrix \( A \) of degree \( n \). Note that \( v_i = v_0 + m(\bar{v}_i - v_0) \) where \( \bar{v}_i = C(i) \cap W \). It follows that components of \( A \) are linear functions of \( m \) with coefficients depending only on \( \Delta, W \) and \( v_0 \). Hence there exists a constant \( C \) such that \( |H_{J*K}| \leq Cm^n \). Similarly \( |H_{J'*K}| \leq Cm^n \) for some constant \( C \). This proves Lemma 7.3.

**Lemma 7.4.** Let \( g(x) \) be a polynomial in \( x \). The function \[ \frac{g(x)q^{fx}}{1-\zeta^2q^x} \] in real variable \( x \) tends to 0 as \( x \) tends to \( \pm \infty \) provided \( 0 < f < 1 \), where \( d \in \mathbb{Q} \). It is bounded for \( f = 0 \) and \( g(x) = constant \).

In fact, recall that \( \tau = a + b\sqrt{-1} \) with \( b > 0 \). Then

\[ \frac{g(x)q^{fx}}{1-\zeta^2q^x} = \frac{g(x)\alpha_1(x)e^{-2\pi bx}}{1-\zeta^2\alpha_2(x)e^{-2\pi bx}} \]

with \( |\alpha_i(x)| = 1, i = 1, 2 \). If \( 0 < f < 1 \) then the right-hand side tends to 0 as \( x \) tends to \( \pm \infty \). If \( f = 0 \) and \( g(x) = constant \) it is bounded. This finishes the proof of Lemma 7.4.

In view of Lemma 7.3 and Lemma 7.4 the absolute value of the term inside the parenthesis in the expression for \( b_1 \) is bounded by \( C'm^n \) with some constant \( C' \). Hence

\[ |b_1| \leq C'm^n \left| \frac{\zeta^{d_{j_k}q(u,v_{j_k})}f_{h,(j_k)}}{1-\zeta^{d_{j_k}q(u,v_{j_k})}} \right|. \]
\( \langle u, v, J \rangle \) is a linear function of \( m \) and its absolute value tends to \( \infty \) when \( m \) tends to \( \infty \). Then \( b_1 \) and hence \( b_{j*K}(\Delta', \varphi', \xi'; \Delta, \varphi, \xi) \) tend to 0 by Lemma 7.4.

This finishes the proof of Lemma 7.2. \( \square \)

We now prove that \( C_n \) implies \( B_n \) assuming \( B_k \) for \( k < n \). For that purpose we introduce the following multi-fan \( \tilde{\Delta}_n = (\tilde{\Sigma}_n, \tilde{C}_n, w^\pm) \). \( \tilde{\Sigma}_n \) is the union of \( \Sigma'_n \) and \( \Sigma_n \) glued along \( \partial \Sigma_n = \partial \Sigma'_n \), and \( \tilde{C}_n | \Sigma'_n = C'_n \) and \( \tilde{C}_n | \Sigma_n = C_n \).

We define the function \( w^\pm \) for \( \Sigma_n \) as follows:

\[
 w^+(J_n) = \begin{cases} 1, & J_n \in (\Sigma'_n)^{(n)}, \\ 0, & J_n \in (\Sigma_n)^{(n)}, \end{cases} \\
 w^-(J_n) = \begin{cases} 0, & J_n \in (\Sigma'_n)^{(n)}, \\ 1, & J_n \in (\Sigma_n)^{(n)}. \end{cases}
\]

Then

\[
 w(J_n) = \begin{cases} 1, & J_n \in (\Sigma'_n)^{(n)}, \\ -1, & J_n \in (\Sigma_n)^{(n)}. \end{cases}
\]

The triple \((\tilde{\Sigma}_n, \tilde{C}_n, w^\pm)\) defines a simplicial multi-fan \( \tilde{\Delta}_n \).

It can be proved in a similar way to (42) that \( \tilde{\Delta}_n \) is complete and the following equalities hold.

\[
 (44) \quad \text{deg}(\tilde{\Delta}_n, J_n) = \begin{cases} +1, & J_n \in \Sigma'_n \ \& \ \partial \Sigma_n, \\ -1, & J_n \in \Sigma_n \ \& \ \partial \Sigma_n, \\ 0, & J_n \in \partial \Sigma_n. \end{cases}
\]

\( \varphi' \) on \( \Sigma_n \) and \( \varphi'_n \) on \( \Sigma'_n \) define \( \tilde{\varphi}' \) on \( \tilde{\Sigma}_n \). Also \( \xi_n \) on \( \Sigma_n \) and \( \xi'_n \) on \( \Sigma'_n \) define \( \tilde{\xi}_n \) on \( \tilde{\Sigma}_n \). Put \( u = u^I(\xi) = \sum_{i \in I} t_i \psi_i \) as before. We claim that

\[
 \tilde{\xi}_n = u \in H^2(\tilde{\Delta}_n, \tilde{\varphi}_n) \otimes \mathbb{Q}.
\]

In fact \( \langle u^I(\xi), v_i \rangle = d_i \) by definition of \( u^I(\xi) \) and \( \langle u^I(\xi), v_i \rangle = d_i \) by (10). Similarly \( \langle u^I(\xi), v_J \rangle = d_J \) by (13). These equalities imply \( \tilde{\xi}_n = u^I(\xi) \).

We apply Theorem 4.2 and get

\[
 (45) \quad \tilde{\varphi}_{st}(\tilde{\Delta}_n, \tilde{C}_n, \tilde{\xi}_n) = 0.
\]

On the other hand by using (41) and (44) we have

\[
 \tilde{\varphi}_{st}(\tilde{\Delta}_n, \tilde{\varphi}'_n, \tilde{\xi}_n) \\
 = b_I(\Delta', \varphi', \xi' ; \Delta, \varphi, \xi) + \sum_{J \in \Sigma \setminus \Sigma'_n} b_{j*K}(\Delta'_n, \varphi'_n, \xi'_n ; \Delta, \varphi, \xi).
\]

For \( K = \emptyset \) the term \( b_{j*K}(\Delta'_n, \varphi'_n, \xi'_n ; \Delta, \varphi, \xi) = b_I(\Delta', \varphi', \xi' ; \Delta, \varphi, \xi) = 0 \) by induction assumption. In case \( K \neq \emptyset \), \( b_{j*K}(\Delta'_n, \varphi'_n, \xi'_n ; \Delta, \varphi, \xi) \) tends to 0 when \( m \) tends to \( \infty \) by Lemma 7.2. From this and (45) it follows that \( b_I(\Delta', \varphi', \xi' ; \Delta, \varphi, \xi) \) must be equal to zero. Together with inductive assumption this proves that \( B_n \) holds.

Thus Theorem 7.1 is proved.

The rest of this section is devoted to the proof of Theorem 6.3. We shall first show that \( \Delta' \) is complete and the following equality holds for every \( J \in \Sigma' \):

\[
 (46) \quad \text{deg}(\Delta'_J) = \text{deg}(\Delta_{p(J')}).
\]
We use the notation $V = L_\mathbb{R}$ and $V_K = (L_K)_\mathbb{R}$ where $(L_K)_\mathbb{R}$ is as in Section 2. Similar notations are used for $K' \in \Sigma'$. Let $v$ be a generic vector in $V/V_K$ and put
\[ S_v(K) = \{ I \in \Sigma^{(n-k)}_K \mid v \in C_K(I) \} \]
where $\Sigma^{(n-k)}_K = \{ I \in \Sigma^{(n)} \mid K \subset I \}$ and $C_K(I)$ is the image of $C(I)$ in $V/V_K$. Recall that $\deg(\Delta_K)$ is defined to be equal to
\[ \sum_{I \in S_v(K)} w(I). \]
Completeness of $\Delta$ implies that this is independent of generic vector $v$. We take a generic vector $v$ in $V$ and denote its image in $V/V_K$ also by $v$. In this sense $v$ may be considered as a vector in $V/V_J$ and in $V/V_{\rho(J')}$ at the same time. Suppose that $J'$ lies in $\Sigma(I')$ and $\rho(J')$ in $\Sigma(J)$.

We shall show
\[ \sum_{I' \in S_v(J')} w(I') = \sum_{I \in S_v(\rho(J'))} w(I). \]
Since the right-hand side is independent of $v$ (being equal to $\deg(\Delta_{\rho(J')})$) this would imply that $\Delta'$ is complete and the equality (46) holds. Since $w(I') = w(\rho(I'))$ it is enough to show that there is a bijection $S_v(J') \rightarrow S_v(\rho(J'))$.

The projection $p : V/V_{J'} \rightarrow V/V_{\rho(J')}$ maps every cone $C_{\rho(J')}(I')$ for $I' \in \Sigma(I')_{\rho(J')}$ to a cone contained in $C_{\rho(J')}(\rho(I'))$. In particular $p$ defines a map $S_v(J') \rightarrow S_v(\rho(J'))$. Take a simplex $I \in S_v(\rho(J'))$, and put $\Sigma(I) = \{ I' \in \Sigma(I_{\rho(J')}) \mid \rho(I') = I \}$. Then the cones $\{ C_{\rho(J')}(I') \}$ with $I' \in \Sigma(I)$ are contained in the image of $C(I)$ in $V/V_J$, and together with their faces, they form a fan in $V/V_{J'}$. Since $C(I)$ contains $v$ regarded as a point in $V/V_{\rho(J')}$, there is exactly one simplex $I' \in \Sigma(I)$ such that $C_{\rho(J')}(I')$ contains $v$ regarded as a point in $V/V_{J'}$, that is, there is one and only one simplex $I' \in S_v(J')$. This shows that the map $S_v(J') \rightarrow S_v(\rho(J'))$ is a bijection and proves (46).

Using (46) and Theorem 7.1 we have
\[ \hat{\varphi}^v_{st}(\Delta', \varphi', \xi') = \sum_{J' \in \Sigma'} \deg(\Delta_{J'} \rho(J))(\Delta', \varphi', \xi') \]
\[ = \sum_{J' \in \Sigma} \sum_{\rho(J') = J} \deg(\Delta_{J'} \rho(J))(\Delta', \varphi', \xi') \]
\[ = \sum_{J' \in \Sigma} \deg(\Delta_{J'} \rho(J))(\Delta', \varphi', \xi') \]
\[ = \hat{\varphi}^v_{st}(\Delta, \varphi, \xi). \]
This finishes the proof of Theorem 6.3

Remark 7.5. Let $\Delta$ be a fan such that every $J \in \Sigma$ is contained in some $I \in \Sigma^{(n)}$. As was pointed out in Proposition 5.6 the equality
\[ \hat{\varphi}^v_{st}(\Delta', \varphi', \xi') = \sum_{u \in L^*} t^{-u}(\Delta, \varphi') \sum_{J \in \Sigma} \deg(J \rho(J))(\Delta, \varphi', \xi')(u) \]
holds for $v \in \bigcup_{I \in \Sigma^{(n)}} C(I)$. If $\rho : \Delta' \rightarrow \Delta$ is a map satisfying a), b), then $\Delta'$ is a fan and $\hat{\varphi}^v_{st}(\Delta', \varphi', \xi')$ has a meaning for $v \in \bigcup_{I \in \Sigma^{(n)}} C(I)$. Theorem 7.1 implies
\[ \hat{\varphi}^v_{st}(\Delta', \varphi', \xi') = \hat{\varphi}^v_{st}(\Delta, \varphi, \xi). \]
and
\[ \hat{\varphi}_{st}(\Delta', \vartheta', \xi') = \hat{\varphi}_{st}(\Delta, \vartheta, \xi), \]
if \( \hat{\varphi}_{st}(\Delta', \vartheta', \xi') \) and \( \hat{\varphi}_{st}(\Delta, \vartheta, \xi) \) are defined as in Remark 5.7.

8. Invariance of orbifold elliptic class under push-forward \( \rho_* \)

Let \( \rho : \Delta' \to \Delta \) be a map satisfying a), b). We shall define a functorial map
\[ \rho_* : S^{-1}\mathbb{H}_*^\nu(\Delta', \vartheta') \otimes \mathbb{Q} \to S^{-1}\mathbb{H}_*^\nu(\Delta, \vartheta) \otimes \mathbb{Q}. \]
The following equality will hold if every simplex \( J \in \Sigma \) is contained in some \( J \in \Sigma^{(n)} \), cf. Theorem 8.7.

\[ \rho_*(\hat{\varphi}_{st}(\Delta', \vartheta', \xi')) = \hat{\varphi}_{st}(\Delta, \vartheta, \xi') \quad \text{for} \quad \xi' = \rho^*(\xi). \]

In order to define \( \rho_*(x) \) for \( x \in H_*^\nu(\Delta, \vartheta) \otimes \mathbb{Q} \) it is sufficient to determine \( \rho_*(x)_I = i_I^*(\rho_*(x)) \in S^{-1}S^*(L_{I,\vartheta'}) \otimes \mathbb{Q} = S^{-1}H^\nu(B\Sigma) \otimes \mathbb{Q} \) for each \( I \in \Sigma^{(n)} \) satisfying
\[ i_{I_1 \cap I_2}(\rho_*(x)_{I_1}) = i_{I_2 \cap I_2}(\rho_*(x)_{I_2}) \quad \text{for any} \quad I_1, I_2 \in \Sigma^{(n)} \text{ with } I_1 \cap I_2 \neq \emptyset \]
in view of (11). Note that the localized ring \( S^{-1}S^*(L) \otimes \mathbb{Q} \) is nothing but the algebra of rational functions on \( L_\mathbb{Q} \). Put \( u_I = \prod_{i \in I} u_i^! \). We then define
\[ (\rho_*(x)_I)_I = [H_I]\sum_{I' \in \Sigma^{(n)}, \rho(I') = I} \frac{i_{I'}^*(x)}{|H_{I'}|u_{I'}.} \]

In order to show that \( \rho_*(x)_I \) in fact satisfy (17), we introduce logarithmic forms \( \Omega^I \) and \( \omega^I \) on \( V = L_\mathbb{R} \) for each \( I \in \Sigma^{(n)} \). Give an orientation to \( I \) and let \( I = \{i_1, \ldots, i_n\} \) be the ordering of \( I \) concordant to the orientation. Put \( w_{i_v}^I = \frac{du_{i_v}^I}{u_{i_v}} \) and
\[ \Omega^I = w_{i_1}^I \wedge \cdots \wedge w_{i_n}^I, \quad \omega^I = \sum_{\nu=1}^n (-1)^{\nu-1} w_{i_1}^I \wedge \cdots \wedge \overset{\sim}{w_{i_\nu}} \wedge \cdots \wedge w_{i_n}^I \]
where \( \overset{\sim}{} \) means to delete the underlying symbol.

**Lemma 8.1.** Give the concordant orientation with \( I \) to \( I' \) such that \( \rho(I') = I \). Then
\[ \sum_{I' \in \Sigma^{(n)}, \rho(I') = I} i_{I'}^*(x)\Omega^{I'} = \rho_*(x)_I \Omega^I, \]
and
\[ \sum_{I' \in \Sigma^{(n)}, \rho(I') = I} i_{I'}^*(x)\omega^{I'} = \rho_*(x)_I \omega^I. \]

**Proof.** For simplicity we put \( I = \{1, 2, \ldots, n\} \) with orientation determined by this ordering. The ordering \( (v_1, v_2, \ldots, v_n) \) gives an orientation to the vector space \( V = L_\mathbb{R} \). Take an ordered integral basis of the lattice \( L \) which is concordant with the orientation of \( V = L_\mathbb{R} \) and let \( u_1, \ldots, u_n \) be the corresponding coordinates. Put \( \Theta = du_1 \wedge \cdots \wedge du_n \). Then
\[ du_1^I \wedge \cdots \wedge du_n^I = \frac{1}{|H_I|}du_1 \wedge \cdots \wedge du_n = \frac{1}{|H_I|}\Theta. \]
Hence
\[ \Omega^I = \frac{\Theta}{|H_I|u_I}. \]

Similarly
\[ \Omega^{I'} = \frac{\Theta}{|H_{I'}|u_{I'}.} \]

Then
\[ \sum_{\rho(J')=I} \iota_{\rho(J')}^*(x)\Omega^{I'} = \sum_{\rho(J')=I} \iota_{\rho(J')}^*(x) = \rho_*(x) \frac{\Theta}{|H_I|u_I} = \rho_*(x) \frac{\Theta}{|H_{I'}|u_{I'}}. \]

Put \( \theta = \sum_{i=1}^{n} (-1)^{i-1} u_i du_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge du_n \). Then we also have
\[ \omega^I = \frac{\Theta}{|H_I|u_I}. \]

From this (49) follows in an entirely similar way. \( \square \)

Fix \( I \in \Sigma^{(n)} \) and \( i \in I \) and put \( J = I \setminus \{i\} \in \Sigma^{(n-1)} \). Let \( \Sigma(J) \) be the simplicial set consisting of all faces of \( J \) and let \( \Sigma'(J) = \{J' \in \Sigma' \mid \rho(J') \subset J\} \). Fans \( \Delta(J) \) and \( \Delta'(J) \) are induced from \( \Delta \) and \( \Delta' \) by restriction on \( \Sigma(J) \) and \( \Sigma'(J) \) respectively. Put also
\[ \mathcal{V}_J = \{v_j \mid j \in \Sigma(J)^{(1)}\} \quad \text{and} \quad \mathcal{V}_J' = \{v_{j'} \mid j' \in \Sigma'(J)^{(1)}\}. \]

Then \( \rho \) induces \( \rho|J : (\Sigma(J), \mathcal{V}_J) \to (\Sigma(J), \mathcal{V}_J') \) satisfying a),b).

**Lemma 8.2.**

\[ ((\rho|J)_*(x|J))_J = (\rho_*(x)_I)|J. \]

In other words
\[ \sum_{J' \in \Sigma'(J)^{(n-1)}} \iota^{J'}_{\rho(J)}(x|J)\Omega^{J'} = (\rho_*(x)_I)|J \cdot \Omega^I. \]

as logarithmic forms in the hyperplane \((L_J)_\mathbb{R}\) containing \(C(J)\). Here \( x|J \) stands for the image of \( x \) by the map \( H^*_+(\Delta', \mathcal{V}') \to H^*_+(\Delta'(J), \mathcal{V}_J') \) sending \( x_{j'} \) to \( x_{j'} \) for \( j' \in \Sigma'(J)^{(1)} \) and to \( 0 \) for \( i' \not\in \Sigma'(J)^{(1)} \). Also \( u|J = \iota^{J'}(u) \) for \( u \in S^{-1}S^*(L_J, \mathcal{V}) \).

**Proof.** In general let \( V_1, \ldots, V_n \) be the hyperplanes spanned by \((n-1)\)-dimensional faces of a strongly convex \( n \)-dimensional simplicial cone \( C \) in an \( n \)-dimensional vector space \( V \). Let \( w_i = \frac{du_i}{u_i} \) be the logarithmic 1-form corresponding to \( V_i \). Here \( w_i \) is a linear form vanishing on \( V_i \). Note that \( w_i \) depends only on \( V_i \) but not on particular \( u_i \). Put
\[ \omega_i = (-1)^{i-1} w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_n \quad \text{and} \quad \omega = \sum_{i=1}^{n} \omega_i. \]

**Sublemma 8.3.** Let \( V_0 \) be a hyperplane defined by \( \sum_i a_i u_i = 0 \). If \( V_0 \) is different from \( V_i \), \( 1 \leq i \leq n \), then \( \omega|V_0 = 0 \).

Since none of the \( V_i \) coincides with \( V_0 \) there are at least two non-zero \( a_i \). We may suppose that \( a_n \neq 0 \). Then \( u_n = -\frac{\sum_{i=1}^{n-1} a_i u_i}{a_n} \) on \( V_0 \) and
\[ \omega_i|V_0 = (-1)^{i-1} w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_{n-1} \wedge \frac{-d(\sum_{i=1}^{n-1} a_i u_i)}{a_n u_n} \]
\[ = (-1)^{n+1} \frac{a_i u_1 \wedge \cdots \wedge du_{n-1} \wedge u_n}{a_n \prod_{j \neq i} u_j}. \]
for $i \neq n$. The last equality automatically holds for $i = n$. Therefore

$$\omega|V_0 = \sum_{i=1}^{n} \omega_i|V_0$$

$$= (-1)^{n+1} \frac{1}{a_n} \sum_{i=1}^{n} \prod_{j \neq i} a_j du_1 \wedge \cdots \wedge du_{n-1}$$

$$= (-1)^{n+1} \frac{1}{a_n} \prod_{j=1}^{n} a_j (\sum_{i=1}^{n} a_i u_i) du_1 \wedge \cdots \wedge du_{n-1} = 0$$

Thus Sublemma 8.3 holds.

We continue with the proof of Lemma 8.2. We may suppose without loss of generality that $I = \{1, \ldots, n\}$ and $i = n$ and $J = I \setminus \{n\}$. Put $V_0 = (L_{i})_R$ and $\omega''^I = (-1)^{i-1} u_i^I \wedge \cdots \wedge \omega^I_n \wedge \cdots \wedge \omega^I_n$. We also use similar notations for $I'$. Note that

$$(51) \quad \rho_*(x)\omega''^I|V_0 = (-1)^{n-1}(\rho_*(x)_I)|J : \Omega^I,$$

since $\omega''^I|V_0 = (-1)^{n-1}\Omega^I$. We next consider the contribution from the left hand side of (51) to the term $\rho_*(x)_I\omega''^I$ in $\rho_*(x)_I\omega^I = \sum_i \rho_*(x)_I\omega^I$. Sublemma 8.3 implies that it suffices to consider only such $I'$ that $\rho(I') = I$ and that some facet cone $C'(J')$ of $C'(I')$ is contained in $C(J) \subset V_0$. The set of such $I'$ will be denoted by $\Sigma'(I, J\setminus n)$. Let $J' = I' \setminus \{n'(I')\}$. In this case $\omega''_{n'(I')}$ for $i' \neq n'(I')$ contains $u_{n'(I')}$ when expressed in terms of $\omega^I_i$. Hence the contribution to $\rho_*(x)_I\omega''^I$ comes only from $\iota^*_I(x)\omega''^{n'(I')}_I$. It follows that

$$(52) \quad \rho_*(x)_I\omega''^I|V_0 = \sum_{I' \in \Sigma'(I, J\setminus n)} \iota^*_I(x)\omega''^{n'(I')}_I|V_0 = \sum_{I' \in \Sigma'(I, J\setminus n)} \iota^*_I(x_J)\omega''^{n'(I')}_I|V_0.$$  

Combining this with (51) and noting that $\omega''^{n'(I')}_I|V_0 = (-1)^{n-1}\Omega^{I'}$ we obtain (50). This finishes the proof of Lemma 8.2.

Lemma 8.2 shows that that $(\rho_*(x)_I)|J$ depends only on $J$. (17) follows from this when $I_1 \cap I_2$ has codimension 1. The general case is proved by induction on codimensions using Lemma 8.2. Thus $\rho_*(x)$ is well-defined. Once this is established one can rewrite Lemma 8.3 in the following

**Proposition 8.4.**

$$\sum_{I' \in \Sigma'(n), \rho(I') = I} \iota^*_I(x)\Omega''^{I'} = \iota^*_I(\rho_*(x))\Omega^I,$$

and

$$\sum_{I' \in \Sigma'(n), \rho(I') = I} \iota^*_I(x)\omega''^{I'} = \iota^*_I(\rho_*(x))\omega^I.$$  

**Proposition 8.5.** $\rho_*$ maps $H^*_\tau(\Delta', \mathcal{Y}') \otimes \mathbb{Q}$ into $H^*_\tau(\Delta, \mathcal{Y}) \otimes \mathbb{Q}$.

**Proof.** Take $x \in H^*_\tau(\Delta', \mathcal{Y}') \otimes \mathbb{Q}$. It is enough to show that $\iota^*_I(\rho_*(x))$ belongs to $S^*(L) \otimes \mathbb{R}$ for any $I \in \Sigma(n)$. Note that $\Omega^I$ has simple poles only along $V_j = (L_{j})_R$ for $J \in \Sigma(n-1)$ with $J \subset I$. In view of Proposition 8.4 it suffices to show that $\Omega'' = \sum_{I' \in \Sigma'(n), \rho(I') = I} \iota^*_I(x)\Omega''^{I'}$ has at most simple poles along the same loci $\{V_j\}$ as $\Omega^I$. 

Poles of $\Omega'$ appear along hyperplanes $(L_J')_\mathbb{R}$ where $J'$ runs over $(\Sigma')^{(n-1)}$ with $\rho(J') \subset I$. If $J'$ is such that $\rho(J') = I$, then there are exactly two $I_1', I_2' \in \Sigma^{(n)}$ that have $J'$ as common face. If $J' = (J_1', \ldots, J_{n-1}')$, $I_1' = J' \cup \{i_1\}$ and $I_2' = J' \cup \{i_2\}$, then $\omega_{i_1} = \frac{du_i}{u_0} = \omega_{i_2}$, where $u_0$ is a linear form vanishing on $(L_J')_\mathbb{R}$. Since $I_1'$ and $I_2'$ induce opposite orientations on $J'$,

$$\iota^*_I(x)(\Omega)_{I_1}' + \iota^*_I(x)(\Omega)_{I_2}' = \pm \frac{du_i}{u_0} \wedge \left( \iota^*_I(x)w_{i_1}' \wedge \cdots \wedge w_{i_{n-1}}' - \iota^*_I(x)w_{i_2}' \wedge \cdots \wedge w_{i_{n-1}}' \right).$$

The restriction of the form $\omega'$ in the parenthesis to the hyperplane $(L_J')_\mathbb{R}$ vanishes. Hence $\iota^*_I(x)(\Omega)_{I_1}' + \iota^*_I(x)(\Omega)_{I_2}' = \pm \frac{du_i}{u_0} \wedge \omega'$ has no pole along $(L_J')_\mathbb{R}$. This implies that $\Omega'$ has no pole along $(L_J')_\mathbb{R}$ for $J'$ such that $\rho(J') = I$.

For $J' \in (\Sigma')^{(n-1)}$ such that $\rho(J')$ is a facet $J$ of $I$, $\Omega'$ has at most simple pole along $(L_J')_\mathbb{R} = (L_J)_\mathbb{R} = V_J$. We have proved that $\Omega'$ has at most simple poles along the same loci as $\Omega$. Hence $\rho_*(x)_J$ has no pole and in fact it is a polynomial. □

Functorial properties of $\rho_*$ are expressed in the following

**Proposition 8.6.** $\rho_* : H^*_T(\Delta', \mathcal{V}') \otimes \mathbb{Q} \to H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q}$ satisfies the following properties:

(i) 

$$(\rho_2 \circ \rho_1)_* = (\rho_2)_* \circ (\rho_1)_* \text{ for } \rho_1 : (\Delta'', V'') \to (\Delta', \mathcal{V}') \text{ and } \rho_2 : (\Delta', \mathcal{V}') \to (\Delta, \mathcal{V}),$$

(id)$_* = \text{id}$ for the identity map $\text{id} : (\Delta, \mathcal{V}) \to (\Delta, \mathcal{V})$.

(ii) $\rho_*(1) = 1$ for $1 \in H^0_T(\Delta)$.

(iii) $\rho_* (x \rho^*(y)) = \rho_*(x)y$ for $y \in H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q}$.

(iv) $\rho_*$ is an $H^*(BT) \otimes \mathbb{Q}$-module map.

(v) Assume that $\rho$ satisfies the condition c) in addition to a) and b). Then $\rho_*$ is compatible with the push-forward $\pi_*$ to a point, i.e., the following diagram commutes:

$$\begin{array}{ccc}
H^*_T(\Delta', \mathcal{V}') \otimes \mathbb{Q} & \xrightarrow{\rho_*} & H^*_T(\Delta, \mathcal{V}) \otimes \mathbb{Q} \\
\pi_* & \downarrow \ & \uparrow \pi_* \\
S^{-1}H^*(BT) \otimes \mathbb{Q} & \xrightarrow{\rho_*} & S^{-1}H^*(BT) \otimes \mathbb{Q}.
\end{array}$$

**Proof.** By Proposition [8.3]

$$\iota^*_I(\rho_{2,*}(\rho_{1,*}(x)))\Omega^I = \sum_{\rho_2(L') = I} \iota^*_I(\rho_{1,*}(x))\Omega^{I'} = \sum_{\rho_2(L') = I} \sum_{\rho(L''') = I'} \iota^*_I(\rho_{1,*}(x))\Omega^I'' = \sum_{(\rho_2 \circ \rho_1)(L') = I} \iota^*_I((\rho_2 \circ \rho_1)_*(x))\Omega^I.$$

Thus $\rho_2 \circ \rho_1 = (\rho_2 \circ \rho_1)_*$, (id)$_* = \text{id}$ clearly holds.

Since $\iota^*_I(1) = 1$, $\iota^*_I(\rho_*(1))\omega^I = \sum_{\rho(L') = I} \omega^{I'} = \omega^I$. Thus $\rho_*(1) = 1$.

In order to prove the equality $\rho_*(x \rho^*(y)) = \rho_*(x)y$ we may assume $y = x_i \in H^2_T(\Delta, \mathcal{V})$ as is easily seen. Then

$$\iota^*_I(\rho_*(x \rho^*(x_i)))\Omega^I = \sum_{\rho(L') = I} \iota^*_I(\rho_{1,*}(x_i))\Omega^{I'} = \sum_{\rho(L') = I} \iota^*_I(x)\iota^*_I(\rho^*(x_i))\Omega^{I'}.$$
But \( \iota_T^*(\rho^*(x_v)) = \iota_T^*(x_v) \) by (59). Hence
\[
\iota_T^*(\rho_*(x\rho^*(x_v)))\Omega^I = \iota_T^*(x_v) \sum_{\rho'(I') = I} \iota_T^*(x_v)\Omega^I' = \iota_T^*(x_v)\iota_T^*(\rho_*(x_v))\Omega^I = \iota_T^*(\rho_*(x_v))\Omega^I.
\]

Thus \( \rho_* = \rho_* \) by Remark 7.5. Hence
\[
\rho_*(x\rho^*(x_v)) = \rho_*(x_v).
\]
Since \( \rho^*(u) = u \) for \( u \in L_\Omega \), we have
\[
\rho_*(u) = \rho_*(x\rho^*(u)) = \rho_*(x_u).
\]
This shows that \( \rho_* \) is an \( H^*_\Omega(BT) \otimes \mathbb{Q} \)-module map.

Finally, admitting the condition c),
\[
\pi_*(\rho_*(x_v)) = \sum_{I \in \Sigma(n)} \frac{w(I)\iota_T^*(\rho_*(x_v))}{H_I |u_I|} = \sum_{I \in \Sigma(n)} \frac{w'(I)\iota_T^*(\rho_*(x_v))}{H_{I'} |u_{I'}|} = \pi_*(x_v).
\]

The map \( \rho_* : H^*_\Omega(\Delta', \mathcal{V}') \otimes \mathbb{Q} \to H^*_\Omega(\Delta, \mathcal{V}) \otimes \mathbb{Q} \) extends in an obvious way to \( \rho_* : (H^*_\Omega(\Delta', \mathcal{V}') \otimes \mathbb{C})[[q]] \to (H^*_\Omega(\Delta, \mathcal{V}) \otimes \mathbb{C})[[q]] \).

**Theorem 8.7.** Let \( \Delta \) be a simplicial multi-fan in a lattice \( L \) of dimension \( n \). Assume that every simplex \( J \in \Delta \) is contained in some \( J \in \Sigma(n) \). Let \( \rho : \Delta' \to \Delta \) a map satisfying a) and b). Let \( \mathcal{V}' = \{ v_i \}_{i \in \Sigma(n)} \) and \( \mathcal{V}' = \{ v_i' \}_{i \in \Sigma(n)} \) be sets of edge vectors for \( \Delta \) and \( \Delta' \) respectively. Put \( \xi' = \rho^*(\xi) \). Then the following equality holds:
\[
\rho_*(\hat{\epsilon}_{st}(\Delta', \mathcal{V}', \xi')) = \hat{\epsilon}_{st}(\Delta, \mathcal{V}, \xi).
\]

**Proof.** It is enough to show that
\[
\iota_T^*(\rho_*(\hat{\epsilon}_{st}(\Delta', \mathcal{V}', \xi'))) = \iota_T^*(\hat{\epsilon}_{st}(\Delta, \mathcal{V}, \xi))
\]
for any \( I \in \Sigma(n) \). Let \( \Sigma(I) \) be the simplicial set consisting of faces of \( I \) and \( \Sigma'(I) = \{ J \in \Sigma \mid \rho(J') \subset I \} \). Cone structures of \( \Delta \) and \( \Delta' \) restricted on \( \Sigma(I) \) and \( \Sigma'(I) \) define fans \( \Delta(I) \) and \( \Delta'(I) \) respectively. We forget \( w \) and \( w' \) in \( \Delta \) and \( \Delta' \), and put \( w(I) = 1 \) and \( w'(I) = 1 \) for all \( I' \in \Sigma'(I) \). With this understanding, \( \rho : \Sigma'(I) \to \Sigma(I) \) satisfies the conditions a), b) and c). Then
\[
\iota_T^*(\hat{\epsilon}_{st}(\Delta, \mathcal{V}, \xi)) = \iota_T^*(\hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I))
\]
\[
\iota_T^*(\rho_*(\hat{\epsilon}_{st}(\Delta', \mathcal{V}', \xi'))) = \iota_T^*(\rho_*(\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I))).
\]
By definition
\[
\pi_*(\hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I)) = \hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I),
\]
and by Proposition 8.6
\[
\pi_*(\rho_*(\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I))) = \pi_*(\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I)) = \hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I).
\]
On the other hand, by Theorem 6.3 and Remark 7.5 one has
\[
\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I) = \hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I).
\]
From this equality and Remark 7.5 one gets
\[
\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I) = ch(\hat{\epsilon}_{st}(\Delta'(I), \mathcal{V}'|I, \xi'|I))
\]
\[
= ch(\hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I)) = \hat{\epsilon}_{st}(\Delta(I), \mathcal{V}'|I, \xi|I).
\]
It follows that
\[ \pi_*(\rho_*(\hat{\Delta}(I), \mathcal{V}|I, \xi|I))) = \pi_*(\hat{\Delta}(I), \mathcal{V}|I, \xi|I)). \]
Since \( \nu^*_I(x) = |H_I|u_I\pi_*(x) \) on \( \Delta(I) \) we get (52).

9. Generalization to \( \mathbb{Q} \)-Cartier triples

So far we dealt only with simplicial multi-fans. A (not necessarily simplicial) multi-fan in an \( n \)-dimensional lattice \( L \) is a triple \( \Delta = (\Sigma, C, w^\pm) \) where \( \Sigma \) is a partially ordered set with a unique minimum element \( * \). We denote the partial ordering by \( \leq \). \( C \) is a map from \( \Sigma \) to the set of strongly convex rational cones in \( L_\mathbb{R} \) satisfying the following three conditions:

(i) \( C(*) = \{0\} \);
(ii) If \( K \leq J \) for \( K, J \in \Sigma \), then \( C(K) \) is a face of \( C(J) \);
(iii) For any \( J \in \Sigma \) the map \( C \) restricted on \( \{ K \in \Sigma \mid K \leq J \} \) is an isomorphism of ordered sets onto the set of faces of \( C(J) \).

For an integer \( k \) with \( 0 \leq k \leq n \) we set
\[ \Sigma^{(k)} = \{ K \in \Sigma \mid \dim C(K) = k \}. \]

\( w^\pm \) are maps \( \Sigma^{(n)} \to \mathbb{Z}_{\geq 0} \).

A multi-fan is said complete, as in the case of simplicial multi-fans, if it satisfies the condition stated in Definition in Section 2. For \( K \in \Sigma^{(k)} \) the projected multi-fan \( \Delta_K \) and its degree are also defined in a similar way as in the case of simplicial multi-fans.

By a triangulation of \( \Delta \) we mean a simplicial multi-fan \( \Delta' = (\Sigma', C', w'^\pm) \) in the same lattice \( L \) related to \( \Delta \) in the following way:

a) There is a bijection \( \kappa : \Sigma^{(1)} \to \Sigma'^{(1)} \) satisfying \( C(\kappa(i)) = C'(\kappa(i)) \) for each \( j' \in \Sigma' \).

b) For each simplex \( J' \in \Sigma' \) there is an element \( J \in \Sigma \) such that \( C'(J') \subset C(J) \). Moreover, for each \( J \in \Sigma \), the collection \( \{ C'(J') \mid J' \in \Sigma', C'(J') \subset C(J) \} \) gives a subdivision of the cone \( C(J) \). We shall denote by \( \rho(J') \) the minimal element \( J \in \Sigma \) such that \( C'(J') \subset C(J) \).

c) For \( I' \in \Sigma'^{(n)} \)
\[ \rho^\pm(I') = w^\pm(\rho(I')). \]

In particular \( \rho'(I') = w(\rho(I')) \).

Returning to general multi-fans we shall assume that every \( J \in \Sigma \) is contained in some \( I \in \Sigma^{(n)} \) hereafter. Let \( \mathcal{V}' = \{ v_i \}_{i \in \Sigma^{(1)}} \) be a set of non-zero vectors \( v_i \in L \cap C(i) \). A set of rational numbers \( \xi = \{ d_i \}_{i \in \Sigma^{(1)}} \) is called \( \mathbb{Q} \)-Cartier if there is an element \( u(I) \in L^*_\mathbb{Q} \) for each \( I \in \Sigma^{(n)} \) such that
\[ \langle u(I), v_i \rangle = d_i \quad \text{for } i \in I. \]

The pair \( (\Delta, \mathcal{V}') \) is called \( \mathbb{Q} \)-Gorenstein if there is an element \( u(I) \in L^*_\mathbb{Q} \) for each \( I \in \Sigma^{(n)} \) such that
\[ \langle u(I), v_i \rangle = 1 \quad \text{for } i \in I. \]

When \( \Delta \) is simplicial every \( \xi \) is \( \mathbb{Q} \)-Cartier and every pair \( (\Delta, \mathcal{V}') \) is \( \mathbb{Q} \)-Gorenstein.

Take a triangulation \( \Delta' = (\Sigma', C', w'^\pm) \) of \( \Delta \). Then the collection \( \xi \) determines a \( \mathbb{Q} \)-divisor \( \xi' = \sum_i d_i x_i \in H^2_T(\Delta', \mathcal{V}') \otimes \mathbb{Q} \) on \( \Delta' \).
THEOREM 9.1. Let $(\Delta, \mathcal{V}, \xi)$ be as above. If $\xi$ is $\mathbb{Q}$-Cartier, then the orbifold elliptic genus $\varphi_{\mathbb{Q}}(\Delta, \mathcal{V}, \xi)$ does not depend on $\Sigma'$. It depends only on $(\Delta, \mathcal{V}, \xi)$.

COROLLARY 9.2. Let $(\Delta, \mathcal{V})$ be a $\mathbb{Q}$-Gorenstein pair. Then the orbifold elliptic genus $\varphi_{\mathbb{Q}}(\Delta, \mathcal{V})$ does not depend on $\Sigma'$ giving an invariant of $(\Delta, \mathcal{V})$.

PROOF. Take triangulations $\Delta'$ and $\Delta''$ of $\Delta$. Fix $I \in \Sigma^{(n)}$ and put $\Sigma(I) = \{J \in \Sigma \mid J \subset I\}$. Then $\Delta(I) = (\Sigma(I), C|\Sigma(I))$ determines a fan. Note that we are neglecting the functions $w^\pm$ defined on $\Delta(I)$ for the moment. We then define

$$b_J(\Delta', \mathcal{V}, \xi'; \Delta'', \mathcal{V}, \xi'') := \sum_{J', \rho(J') = J} b_{J'}(\Delta', \mathcal{V}, \xi') - \sum_{J'', \rho(J'') = J} b_{J''}(\Delta'', \mathcal{V}, \xi'').$$

for $J \in \Sigma(I)$. We shall prove the following fact by induction on $n$.

$$B_{n \text{top}} \quad b_J(\Delta', \mathcal{V}, \xi'; \Delta'', \mathcal{V}, \xi'') = 0 \text{ for } \dim L = n \text{ and for all } J \in \Sigma(I).$$

This will prove Theorem 9.1 in view of [38] as in the proof of Theorem [33].

The cases $n = 1$ and $n = 2$ is trivial since every cone is simplicial in these cases.

Suppose $n \geq 3$. Let $\Delta'(I)$ and $\Delta''(I)$ be the triangulations of $\Delta(I)$ induced by $\Delta'$ and $\Delta''$ respectively. We construct fans $\Delta'_* = (\Sigma'_*, C'_*)$ and $\Delta''_* = (\Sigma''_*, C''_*)$ from $\Delta'(I)$ and $\Delta''(I)$ respectively in a similar way to the proof of Theorem 7.1. Namely

$$\Sigma'_* = \Sigma' \cup \{J' + (1) \mid \rho(J') \subset J \subset \mathbb{R}^k \}.$$

and

$$\Sigma''_* = \Sigma'' \cup \{J'' + (1) \mid \rho(J'') \subset J \subset \mathbb{R}^k \}.$$

We set

$$\partial \Sigma'_* = \{(J_1 \cdots J_k) \in \Sigma'_* \}, \quad \partial \Sigma''_* = \{(J_1 \cdots J_k) \in \Sigma''_* \}.$$

They are isomorphic to the so-called order complex of the poset $\Sigma(I) \setminus \Sigma(I)^n$ as simplicial complexes. They will be identified with each other in the sequel. We have

$$\Sigma'_*(1) = \Sigma'(1) \cup (\partial \Sigma'_*)^{(1)} \text{ where } (\partial \Sigma'_*)^{(1)} = \{J \mid J \subset \mathbb{R}^k \},$$

and

$$\Sigma''_*^{(1)} = \Sigma''(1) \cup (\partial \Sigma''_*)^{(1)} \text{ where } (\partial \Sigma''_*)^{(1)} = \{J \mid J \subset \mathbb{R}^k \}.$$

Moreover $C'_*((J)) = C''_*((J))$ for $(J) \in (\partial \Sigma'_*)^{(1)} = (\partial \Sigma''_*)^{(1)}$. Define a vector $v_J \in C'_*((J)) = C''_*((J))$ as in the proof of Theorem 7.1 and set

$$\mathcal{V}_* = \mathcal{V} \cup \{v_J\}$$

where and $J$ ranges over $\Sigma^{(k)}$, $0 < k < n$.

The construction of $\Delta'_*$, $\Delta''_*$ and $v_J$ depends on an integer $m$. Similarly to the proof of Theorem 7.1 it can be shown that these vectors $v_J$ satisfy the property:

$$(53) \quad \text{The absolute value of } \langle u, v_J \rangle \text{ tends to } \infty \text{ as } m \text{ tends to } \infty \text{ for any } u \in L^*.$$ 

Then the number $d_J$ is defined by

$$d_J = \langle u^I(\xi), v_J \rangle,$$

and the $\mathbb{Q}$-divisor $\xi'_*$ by

$$\xi'_* = \sum_{v' \in \Sigma^{(1)}'} d_{v'} x_{v'} + \sum_{(J) \in (\partial \Sigma'_*)^{(1)}} d_J x_{J},$$

where $x_J$ is the basis element in $H^2_F(\Delta'_*, \mathcal{V}_*)$ corresponding to $(J) \in (\partial \Sigma'_*)^{(1)}$. The $\mathbb{Q}$-divisor $\xi''_*$ is similarly defined.
We glue $\Sigma'_s$ and $\Sigma''_s$ along the common boundary $\partial \Sigma'_s = \partial \Sigma''_s$ to obtain a multi-fan $\Delta_s$. The functions $w^\pm$ are defined in such a way that

$$w(J_s) = \begin{cases} 1, & \text{for } J_s \in (\Sigma'_s)^{(n)}; \\ -1, & \text{for } J_s \in \Sigma_s^{(n)}. \end{cases}$$

The multi-fan $\Delta_s$ is complete and the following equality holds.

$$\text{deg}((\Delta_s)_{\mathcal{J}}) = \begin{cases} +1, & J_s \in \Sigma'_s \setminus \partial \Sigma'_s; \\ -1, & J_s \in \Sigma''_s \setminus \partial \Sigma''_s; \\ 0, & J_s \in \partial \Sigma'_s = \partial \Sigma''_s. \end{cases}$$

The proof of these facts is similar to that of Theorem 7.1.

The $\mathbb{Q}$-divisors $\xi'_s$ on $\Sigma'_s$ and $\xi''_s$ on $\Sigma''_s$ define a $\mathbb{Q}$-divisor $\xi_s$ on $\Sigma_s$. Put $u = u'(\xi) = \sum_{i \in I} d_i u_i'$ as before. Then

$$\xi_s = u \in H^2(T(\Delta_s, \mathcal{V}_s) \otimes \mathbb{Q}).$$

We apply Theorem 4.2 and get

$$\hat{\varphi}_{st}(\Delta_s, \mathcal{V}_s, \xi_s) = 0.$$  

On the other hand by (54) we have

$$\hat{\varphi}_{st}(\Delta_s, \mathcal{V}_s, \xi_s) = b(\Delta', \mathcal{V}', \xi'; \Delta'', \mathcal{V}', \xi'') + b_2,$$

where

$$b_2 = \sum_{J \in \Sigma(I), J \not\in I} \left( \sum_{J' \in \mathcal{K}} b_{J' + K}(\Delta'_s, \mathcal{V}_s, \xi'_s) - \sum_{J'' \in \mathcal{K}} b_{J'' + K}(\Delta''_s, \mathcal{V}_s, \xi''_s) \right).$$

The term in the parenthesis for $K = \emptyset$ is equal to 0 by inductive assumption. For $K \neq \emptyset$ the term tends to 0 when $m$ tends to $\infty$ for any $u \in L^*$, as follows from a similar argument to the proof of Theorem 7.1 using (53). From this and (54) it follows that $b_J(\Delta', \mathcal{V}', \xi'; \Delta'', \mathcal{V}', \xi'')$ must be equal to zero. Together with inductive assumption this proves that

$$b_J(\Delta', \mathcal{V}', \xi'; \Delta'', \mathcal{V}', \xi'') = 0$$

for all $J \in \Sigma(I)$. Thus $B^f_{\text{top}}$ holds and Theorem 9.1 is proved. \qed

References

[1] V. V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc., 1 (1999), 5–33.

[2] L. A. Borisov and A. Libgober, *Elliptic genera of toric varieties and applications to mirror symmetry*, Invent. math., 140 (2000), 453–485.

[3] ________, *Elliptic genera of singular varieties*, Duke Math. J., 116 (2003), 319–351.

[4] ________, *McKay correspondence for elliptic genera*, Ann. of Math., 161 (2005), 1521–1569.

[5] R. Bott and C. Taubes, *On the rigidity theorem of Witten*, J. Amer. Math. Soc., 2 (1989), 137–186.

[6] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Strings on orbifolds I,II*, Nucl. Phys. 261 (1985), 678–686; 274 (1986), 285–314.

[7] C. Dong, K. Liu and X. Ma, *On orbifold elliptic genus*, Orbiﬁds in Mathematics and Physics, Contemp. Math., Amer. Math. Soc., 310 (2002), 87–105.

[8] W. Fulton, *Introduction to Toric Varieties*, Annals Math. Studies, No.131, Princeton UP, 1993.

[9] A. Hattori, *Orbifold elliptic genus and rigidity*, J. Math. Soc. Japan, 58 (2006), 419–425.

[10] ________, *Elliptic genera, torus orbifolds and multi-fans:II*, Intern. J. Math., 17 (2006), 957–998.
A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math., 40 (2003), 1–68.

F. Hirzebruch, *Elliptic genera of level N for complex manifolds*, Differential Geometrical Methods in Theoretical Physics, Kluwer, 1988, pp. 37–63; also reproduced with corrections and improvements in [14]

F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics, vol. E20, Vieweg, 1992.

K. Liu, *On elliptic genera and theta-functions*, Topology, 35 (1996), 617–640.

M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J., 51 (1999), 237–265.

B. Steer and A. Wren, *Grothendieck topology and the Picard group of a complex orbifold*, Contemp. Math., 239 (1999), 251–262.

B. Totaro, *Chern numbers for singular varieties and elliptic cohomology*, Ann. of Math., 151 (2000), 757–791.

R. Waelder, *Equivariant elliptic genera*, preprint, 2006, AG/0603521.

R. Waelder, *Equivariant elliptic genera and local McKay correspondence*, preprint, 2007, AG/0701336.

E. Witten, *The index of the Dirac operator in loop space*, Elliptic Curves and Modular Forms in Algebraic Geometry, Lecture Notes in Math., Springer, 1988, pp.161–181.

E-mail address: hattori@ms.u-tokyo.ac.jp