Lie point symmetries for biological magneto-Jeffrey fluid flow in expanding or contracting permeable walls: a blood vessel model

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ABSTRACT
This article addresses biological flow in expanding or contracting blood vessels. The alternate contraction and expansion of vessels is known to act as a physiological pump to generate flow. When muscles compress blood vessel walls, the valve at the end of the source is closed, and the downstream end opens; for this reason, blood is pumped in the downstream direction. To study this situation, a model has been developed here that consists of the unsteady two-dimensional flow of an incompressible magneto-Jeffrey fluid in a porous semi-infinite channel with expanding or contracting walls; the channel is closed from one end by a compliant membrane. The Lie group analysis method was used to transform a system of nonlinear partial differential equations into nonlinear ordinary differential equations that were solved using the perturbation method. The effects of Jeffrey parameters ($\lambda_1$, which is the ratio of relaxation time to retardation time, and Deborah parameter $D_\lambda$) and other physical parameters are plotted and discussed. We find that the axial velocity of a Newtonian fluid is greater than that of a non-Newtonian fluid if the walls are in contraction and vice versa if the walls are in expansion. If the walls are expanded, the blood is distributed in a large area, and thus, the normal pressure decreases, while if the walls are contracted, blood is distributed in less space, and thus, the pressure increases. Normal pressure $\Delta P_n$ in expanding blood vessels is less than that in contracting blood vessels.

1. Introduction

Biophysical flows through porous channels with expanding or contracting walls compose very important boundary value problems. There are many applications that can be considered as flow through porous channels with expanding or contracting walls in the petroleum industry, soil mechanics, air and blood circulation in the respiratory system, pulsating diaphragms, artificial diastasis, filtration, blood flow and binary gas diffusion. The earliest work on this topic was by Uchida and Aoki [1], who studied unsteady flows in a semi-infinite contracting or expanding pipe. In addition, Ohki [2] investigated unsteady flow in a porous, elastic, circular tube with contracting or expanding walls in an axial direction. Recently, many researchers have investigated expanding or contracting walls, Xinhu et al. [3] achieved multiple solutions for laminar flow in a porous pipe with suction at a slowly expanding or contracting wall. Jafaryar et al. [4] performed an analytical investigation of laminar flow through expanding or contracting gaps with porous walls. Hatami et al. [5] investigated a nanofluid flow conveying nanoparticles through expanding and contracting gaps between permeable walls via numerical analysis. Abo-elkhair [6] obtained the Lie point symmetries for a magneto couple stress fluid in a porous channel with expanding or contracting walls and a slip boundary condition. Additionally, Ojjela et al. [7] studied the chemically reacting micropolar fluid flow and heat transfer between expanding or contracting walls with ion slip and Soret and Dufour effects.

There are many applications for non-Newtonian fluids in various fields, such as the chemical and petroleum industries and geophysics and biological sciences; therefore, researchers are interested in studying the behaviour of non-Newtonian fluids for various applications. Non-Newtonian fluid flows have governing equations that are more complex than the Navier-Stokes equations. The governing equations for flows of non-Newtonian fluids are in fact the consequence of the constitutive relations that are used to predict the rheological behaviour of these fluids. There are various constitutive relations that have been considered in the modelling of non-Newtonian fluids. One of the various constitutive relations used for non-Newtonian fluids is the Jeffrey fluid model. The Jeffrey fluid model is a linear model that uses time derivatives instead of convected derivatives, which are used, for example, in the Oldroyd-B and Maxwell fluid models. Recently, many authors have studied the effect of Jeffrey parameters on various problems; see, for example, [8–15].

The Lie group analysis (Lie point symmetries) method is an important method for finding exact solutions of
ordinary and partial differential equations using transformation groups (similarity transformations), which were introduced first by Sophus Lie [16]. The groups of continuous transformations that yield a given family of invariant equations are defined as symmetries (isovector fields). Symmetry transformation reduces the number of independent variables from \( n \) to \( n-1 \) variables [17]. Many authors have obtained exact solutions for some problems in fluid mechanics using the Lie group analysis method. Mekheimer et al. obtained exact solutions for a couple stress fluid with heat transfer, a micropolar fluid through a porous medium and a hydro-magnetic Maxwell fluid through a porous medium [18–20]. Additionally, Shahzad et al. [21] used this method to find the analytical solution of a micropolar fluid.

The main goal of this paper is to find the analytical solution for magneto-Jeffrey fluid flow in a porous channel with expanding or contracting walls. Then, we study the effect of the Jeffrey parameters \( (\lambda_1 \text{ and } \lambda_2) \) in the presence of expanding or contracting walls on the axial velocity and the normal pressure. In Section 3, the basic roles of the Lie group analysis method are given and used to calculate the isovector field of our problem. The analytical solution (by the perturbation method) corresponding to the nonlinear ordinary differential equation is obtained in Section 4. Finally, the graphs for velocity components and the pressure distribution are presented, and for several values of the physical and geometric parameters, they are plotted and discussed.

2. Statement of the physical problem

Consider the unsteady two-dimensional flow of an incompressible Jeffrey fluid under the effect of a perpendicular magnetic field in a porous semi-infinite channel with expanding or contracting walls.

The distance \( 2\alpha(t) \) between the walls of the channel is very small with respect to the width and length of the channel. The channel is closed from one end by a compliant membrane. Walls have equal permeability \( V_w \) and expand or contract uniformly at a time-dependent rate \( \dot{\alpha}(t) \), as shown in Figure 1. We take \( \dot{x}_1 \) and \( \dot{x}_2 \) to be coordinate axes parallel and perpendicular to the channel walls and assume \( \dot{u}_1 \) and \( \dot{u}_2 \) to be the velocity components in the \( \dot{x}_1 \) and \( \dot{x}_2 \) directions, respectively. The governing equations are expressed as follows:

\[
\begin{align*}
\frac{\partial \dot{u}_1}{\partial \dot{x}_1} + \frac{\partial \dot{u}_2}{\partial \dot{x}_2} & = 0, \\
\rho \left( \frac{\partial \dot{u}_1}{\partial t} + \dot{u}_1 \frac{\partial \dot{u}_1}{\partial \dot{x}_1} + \dot{u}_2 \frac{\partial \dot{u}_1}{\partial \dot{x}_2} \right) & = -\frac{\partial \dot{p}}{\partial \dot{x}_1} + \lambda \frac{\partial \dot{s}_{11}}{\partial \dot{x}_1} + \lambda \frac{\partial \dot{s}_{12}}{\partial \dot{x}_2} - \sigma B_0^2 \dot{u}_1,
\end{align*}
\]

where \( \rho, t, \sigma, B_0 \text{ and } \mu \) are mass density, time, pressure, electrical conductivity of the fluid, magnetic field and coefficient of viscosity, respectively.

The boundary conditions of our problem will be

\[
\begin{align*}
& (i) \quad \dot{u}_1 = 0, \quad \dot{u}_2 = -V_w = -A\dot{\alpha}, \quad \text{at } \dot{x}_2 = \alpha(t), \\
& (ii) \quad \frac{\partial \dot{u}_1}{\partial \dot{x}_2} = 0, \quad \dot{u}_2 = 0, \quad \text{at } \dot{x}_2 = 0, \\
& (iii) \quad \dot{u}_1 = 0, \quad \text{at } \dot{x}_1 = 0.
\end{align*}
\]

Take the stream function \( \dot{\psi}(\dot{x}_1, \dot{x}_2, t) \) such that

\[
\dot{u}_1 = \frac{\partial \dot{\psi}}{\partial \dot{x}_2}, \quad \dot{u}_2 = -\frac{\partial \dot{\psi}}{\partial \dot{x}_1},
\]

which satisfies the continuity equation identically.

If we introduce the dimensionless perpendicular coordinate \( y = \dot{x}_2/\alpha(t) \), Equation (3) becomes

\[
\dot{u}_1 = \frac{1}{\alpha} \frac{\partial \dot{\psi}}{\partial y}, \quad \dot{u}_2 = -\frac{\partial \dot{\psi}}{\partial \dot{x}_1}.
\]

Substitute Equation (4) into (1); then, using the following dimensionless parameters,

\[
\begin{align*}
\frac{\dot{u}_1}{V_w}, \quad \frac{\dot{u}_2}{V_w}, \quad \frac{\dot{x}_1}{\alpha}, \quad \frac{\dot{t}}{tV_w}.
\end{align*}
\]
\[
\lambda_2 = \frac{\dot{z}_2 V_w}{\alpha} \quad \psi = \frac{\dot{y}}{\alpha V_w}, \quad p = \frac{\dot{P}}{\rho V_w^2}, \quad \beta = \frac{\alpha \dot{a}}{v}.
\]

System (1) becomes

\[
E_1 = \psi_y - \frac{B}{R} (\psi_y + \psi_{yy}) + \psi_y \psi_{xy} - \psi_x \psi_{yy} + p_x + \frac{M^2}{R} \psi_y - \frac{1}{(1 + \lambda_1)} \left[ \frac{1}{R} (\psi_{xxy} + \psi_{yy}) \right] \\
+ D_\varepsilon \left( \psi_{xyy} + \psi_{yy} + \frac{B}{R} (\psi_{xxy} + 3 \psi_y + y (\psi_{xxy} + \psi_{yy} + \psi_{yy} + \psi_{xxx}) + \psi_{xxy} + \psi_{yy}) \right) + y (\psi_{xxy} + \psi_{yy}) + \psi_y (\psi_{xxy} + \psi_{yy}) - 2 \psi_x \psi_{xyy} + 2 \psi_y \psi_{xyy} = 0,
\]

\[
E_2 = -\psi_x + \frac{B}{R} y \psi_{xy} - \psi_y \psi_{xx} + \psi_x \psi_{xy} + p_y - \frac{1}{(1 + \lambda_1)} \left[ \frac{1}{R} (-\psi_{xy} - \psi_{xxx}) \right] \\
+ D_\varepsilon \left( -\psi_{xyy} - \psi_{xx} + \frac{B}{R} (2 \psi_{xy} + y (\psi_{xxy} + \psi_{yy}) + \psi_y (3 \psi_{xxy} - \psi_{xxx} + \psi_{xx} + \psi_{xxy} - \psi_{xx} + \psi_{xxx}) \right) + \psi_y (\psi_{xxy} + \psi_{yy} + \psi_{xyy} + \psi_{xx}) - 2 \psi_x \psi_{xyy} + 2 \psi_y \psi_{xyy} = 0,
\]

where \( R = \alpha V_w / v \) is the permeation Reynolds number, \( D_\varepsilon = \lambda_2 / \sqrt{\alpha V_w} \) is the Deborah number, and \( M^2 = \sigma B_0^2 \alpha^2 / \mu \) is the Hartman number.

The wall permeance or injection coefficient \( \alpha \) is defined as \( \alpha = R / \alpha \), and it is a measure of wall permeability.

From Equations (4) and (5), we can write

\[
w_1 = \frac{\partial \psi}{\partial y}, \quad w_2 = -\frac{\partial \psi}{\partial x}.
\]

The boundary conditions (2) will be

(i) \( \psi_y = 0, \quad \psi_x = 1 \) at \( y = 1 \),

(ii) \( \psi_{yy} = 0, \quad \psi_x = 0 \) at \( y = 0 \),

(iii) \( \psi_y = 0, \quad \psi_x = 0 \) at \( x = 0 \).

From a physical standpoint, the idealization is based on a decelerating wall dilution rate that follows a plausible model according to which

\[
\alpha \dot{\psi} = \text{constant}.
\]

Therefore, the rate of dilution decreases as the channel height increases.

Since \( \beta = \alpha \dot{a} / v \), the integration of Equation (9) yields

\[
\frac{\alpha}{\alpha_0} = \sqrt{1 + \frac{2 \beta v t}{\alpha_0^2}},
\]

where \( \alpha_0 \) is the initial value of the channel height.

### 3. Lie group analysis and isovector fields

To obtain the analytical solution, we apply the Lie group analysis method on the equations of system (6) to transform our system of partial differential equations to ordinary differential equations. For this we write

\[
z^*_i = z_i + e_i \xi(z_j, w_m) + o(e^2), \quad i, j = 1, 2, 3, \quad l, m = 1, 2,
\]

\[
w^*_i = w_i + e_l \eta(z_j, w_m) + o(e^2),
\]

as the infinitesimal Lie point transformations. We have assumed that the system in Equation (6) is invariant under the transformations given in Equation (11). The corresponding infinitesimal generator of Lie groups is given by

\[
X = \xi_z \frac{\partial}{\partial z} + \eta_{w_l} \frac{\partial}{\partial w_l},
\]

with the summation convention over the repeated index and \( z_1 \equiv x, z_2 \equiv y, z_3 \equiv t, w_1 \equiv \psi, w_2 \equiv p \). The coefficients \( \xi_1, \xi_2, \xi_3, \eta_1 \) and \( \eta_2 \) are the functions of all independent and dependent variables and their coefficients are the components of the infinitesimals symmetries corresponding to \( x, y, t, \psi, \) and \( p \), respectively, to be determined from the invariance conditions

\[
Pr^{(4)}(X) |_{E_0=0} = 0, \quad a = 1, 2,
\]

where \( E_0 = 0, \) \( i = 1, 2 \) represent the system in Equations (6), and \( Pr^{(4)} \) is the fourth prolongation of the isovector field \( X \).

Since system (6) is of order four, our prolongation will be in the form

\[
Pr^{(1)} X = X + \eta_{w_{a_1}} \frac{\partial}{\partial w_{a_1}},
\]

\[
Pr^{(2)} X = Pr^{(1)} X + \eta_{w_{a_2}} \frac{\partial}{\partial w_{a_2}},
\]

\[
Pr^{(3)} X = Pr^{(2)} X + \eta_{w_{a_3}} \frac{\partial}{\partial w_{a_3}},
\]

\[
Pr^{(4)} X = Pr^{(3)} X + \eta_{w_{a_4}} \frac{\partial}{\partial w_{a_4}},
\]

where

\[
\eta_{a_1} = D_\varepsilon (\eta_{a_1} - \xi_j w_{a,j}) + \xi_j w_{a,j},
\]

\[
\eta_{a_2} = D_\varepsilon (\eta_{a_2} - \xi_k w_{a,k}) + \xi_k w_{a,k},
\]

\[
\eta_{a_3} = D_\varepsilon (\eta_{a_3} - \xi_l w_{a,l}) + \xi_l w_{a,l},
\]

\[
\eta_{a_4} = D_\varepsilon (\eta_{a_4} - \xi_m w_{a,m}) + \xi_m w_{a,m}.
\]

The operator \( D_{i_1 i_2 \cdots i_l} \) is called the total derivative (Hash operator) and has the following form:

\[
D_i = \frac{\partial}{\partial z_j} + w_{a,j} \frac{\partial}{\partial w_{a,j}} + w_{a,j} \frac{\partial}{\partial w_{a,j}} + w_{a,j} \frac{\partial}{\partial w_{a,j}}
\]

\[
+ w_{a,j} \frac{\partial}{\partial w_{a,j}} + w_{a,j} \frac{\partial}{\partial w_{a,j}},
\]

where \( D_i = D_i (D_i) = D_{ij} (D_{ij}) = D_{ijkl} \) and \( w_{a,j} = \partial w_a / \partial z_j \).
Expanding the system in Equations (13) (with the aid of Mathematica 7) and the original system in Equation (6) to eliminate $p_x, p_y$ and setting the coefficients involving $\psi_y, \psi_x, \psi_{x\dot{t}}, \psi_y, \psi_{x\dot{t}}, \psi_{yy}, \psi_{xy}, \psi_{xx}, \psi_{xy}, \psi_{x\dot{t}}, \psi_{yy}, \psi_{xy}, \psi_{xy}, \psi_{xx}, \psi_{xy}$ and various products to zero gives rise to the essential set of over-determined equations. Solving these sets of determining equations, we obtain the required components of the isovector field as follows:

$$
\begin{align*}
\xi_1 &= a_1, \quad \xi_2 = a_2(\dot{t}), \quad \xi_3 = a_3, \\
\eta_1 &= a_4(\dot{t}) - \frac{x}{R}(\beta a_2(\dot{t}) + Ra_2'\dot{t}), \\
\eta_2 &= a_5(\dot{t}) - \frac{y}{R}(\beta a_2'\dot{t} + Ra_2'\dot{t}).
\end{align*}
$$

If we take $a_2(\dot{t}) = a_2$, we get

$$
\begin{align*}
\xi_1 &= a_1, \quad \xi_2 = a_2, \quad \xi_3 = a_3, \\
\eta_1 &= a_4(\dot{t}) - \frac{1}{R}x\beta a_2, \quad \eta_2 = a_5(\dot{t}),
\end{align*}
$$

where $a_i, i = 1, 2, 3$ are arbitrary constants, and $a_4(\dot{t}), a_5(\dot{t})$ are arbitrary functions of variable $\dot{t}$ only. Therefore, the equations admit a five-parameters Lie group of transformations corresponding to the arbitrary constants $a_1, a_2, a_3$ and arbitrary functions $a_4, a_5$. The infinitesimal generator of Lie groups can be written in the form of Lie algebra as follows:

$$
X = \sum_{i=1}^{5} a_i X_i
$$

where

$$
\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} - \frac{x}{R} \frac{\partial}{\partial \psi}, \quad X_3 = \frac{\partial}{\partial \dot{t}}, \\
X_4 &= a_4(\dot{t}) \frac{\partial}{\partial \psi}, \quad X_5 = a_5(\dot{t}) \frac{\partial}{\partial p},
\end{align*}
$$

The one-parameter group generated by $X_1$ and $X_3$ consists of translations, whereas other symmetries are non-trivial. The commutator table of symmetries is given below, where the entry in the $ith$ row and $jth$ column is defined as $[X_i, X_j] = X_iX_j - X_jX_i$; see Table 1.

**Table 1.** Table of the commutators of the basis operators.

|   | $X_1$ | $X_2$ | $X_3$ | $X_4(a_4)$ | $X_5(a_5)$ |
|---|------|------|------|-----------|-----------|
| $X_1$ | 0 | -$\frac{\beta}{R}a_4$ | 0 | 0 | 0 |
| $X_2$ | -$\frac{\beta}{R}a_4$ | 0 | 0 | 0 | 0 |
| $X_3$ | 0 | 0 | 0 | $X_4(a_4')$ | $X_5(a_5')$ |
| $X_4(a_4)$ | 0 | 0 | -$X_4(a_4')$ | 0 | 0 |
| $X_5(a_5)$ | 0 | 0 | -$X_5(a_5')$ | 0 | 0 |

The solutions $\psi = \psi(x, y, \dot{t})$ and $p = p(x, y, \dot{t})$ are invariant under symmetry (12) if

$$
\begin{align*}
\phi_\psi &= X(\psi - \psi(x, y, \dot{t})) = 0 \quad \text{when } \psi = \psi(x, y, \dot{t}) \\
\phi_p &= X(p - p(x, y, \dot{t})) = 0 \quad \text{when } p = p(x, y, \dot{t})
\end{align*}
$$

For $X_2$, the characteristic has the components

$$
\phi_\psi = -\psi_\dot{t} = 0, \quad \phi_p = -p_\dot{t} = 0.
$$

Therefore, the general solutions of the invariant surface conditions (21) are

$$
\psi = \psi(x, y), \quad p = p(x, y).
$$

Suppose that the stream function in Equation (22) takes the following form:

$$
\psi = q(y)Q(x, y).
$$

Substituting Equation (23) into the first equation in (6) yields

$$
\begin{align*}
D_e(\beta y + Rq_{h1}) \frac{d^2 q}{dy^2} + (1 + D_e(3\beta) \quad &+ (4y\beta + 3Rq_{h1})h_2 + Rq_{h3})(\frac{d^3 q}{dy^3} - RD_eh_1(\frac{d^2 q}{dy^2})^2 \\
&+ \left((-3 + 9h_2)h_2 + RqD_e(h_1(5h_4 + 3h_5) \\
&- h_6 + h_7) + y\beta (-1 + D_e(6h_4 + h_5) - \lambda_1)
\end{align*}
$$

$$
\begin{align*}
&- Rq_{h1}(1 + \lambda_1) + RD_e(h_1h_2 - 5h_3) \frac{dq}{dy} \frac{d^2 q}{dy^2} \\
&+ R(h_1(1 + 3D_e(h_4 - h_5) + \lambda_1) - D_e(4h_2h_3 \\
&+ 3h_6 + h_7)) \left(\frac{dq}{dy}\right)^2 + (M^2 - \beta)(1 + \lambda_1)
\end{align*}
$$

$$
\begin{align*}
&- 3h_4 - h_5 + \beta D_e(9h_4 + h_5 + 2y(2h_8 + h_9)) \\
&+ Rq(D_e(h_1(5h_8 - h_9) - h_10 - h_11)) \\
&+ h_3(1 + D_e(h_4 + h_5) + \lambda_1) - h_2(Rq(D_e(5h_6 - h_7)) \\
&+ (2y\beta + Rq_{h1})(1 + \lambda_1)) \frac{dq}{dy} \\
&+ q^2R(D_e(2h_9h_6 - h_4(h_6 - h_7) \\
&+ h_3(3h_9) + h_2(h_9(1 + \lambda_1) - D_e(h_10 + h_11)) \\
&+ h_1(D_e(h_12 + h_13) - h_4(1 + \lambda_1))) \\
&+ q(M^2 - \beta)h_2(1 + \lambda_1) - h_8 - h_9 \\
&+ \beta(D_e(3h_8 + h_9 + y(h_12 + h_13)) - yh_4(1 + \lambda_1)) \\
&+ \frac{R(1 + \lambda_1)}{Q}p_x = 0
\end{align*}
$$

where

$$
\begin{align*}
h_1 &= Q_x, \quad h_2 = \frac{Q_y}{Q}, \quad h_3 = Q_{xy}, \quad h_4 = \frac{Q_{yy}}{Q}.
\end{align*}
$$
\begin{align}
    h_5 &= \frac{Q_{xx}}{Q}, \quad h_6 = Q_{xyy}, \quad h_7 = Q_{xxx} \\
    h_8 &= \frac{Q_{yy}}{Q}, \quad h_9 = \frac{Q_{xxy}}{Q}, \quad h_{10} = Q_{xyy} \\
    h_{11} &= Q_{xxy}, \quad h_{12} = \frac{Q_{yyy}}{Q}, \quad h_{13} = \frac{Q_{xyy}}{Q}.
\end{align}

(25)

Since \( q \) is a function of \( y \) only, whereas \( Q \) and \( p \) are functions of \( x \) and \( y \), from Equation (24), we conclude that each of \( h_i, \; i = 1, 2, \ldots, 13 \) must be a constant or a function of \( y \) only to obtain an expression in the single variable \( y \).

The solution of \( Q_x = h_1 \) in Equation (25) gives

\[ Q(x, y) = xh_1(y) + C_1(y). \]

(26)

Substituting Equation (26) into Equation (23) will give

\[ \psi = (xh_1(y) + C_1(y))q(y). \]

(27)

By using the boundary conditions Equation (8), we obtain

\[ C_1(y)q(y) = C_2, \]

(28)

where \( C_2 \) is a constant. Substituting Equation (28) into Equation (27) gives

\[ \psi = x\zeta(y) + C_2, \]

(29)

where \( \zeta(y) = h_1(y)q(y) \); substituting the second equation of (22) and Equation (29) into the first equation in (6) yields

\[ x((M^2 - \beta)(1 + \lambda_1)\zeta'(y) + \zeta''(y))^2 \\
+ -(1 + \lambda_1)(y\beta + R\zeta(y))\zeta''(y) - RDe\zeta''(y)^2 \\
+ (-1 + 3\beta D e)\zeta(3)(y) + De(y\beta + R\zeta(y))\zeta(4)(y)) \\
+ R(1 + \lambda_1)p_x = 0. \]

(30)

Substituting Equations (26) and (29) into the last term of Equation (24) yields \( C_1 = 0 \); then, \( Q(x,y) = xh_1(y) \), which satisfies the remaining \( h_i, \; i = 2, 3, \ldots, 13 \). Additionally, the stream function takes the form

\[ \psi = x\zeta(y). \]

(31)

Substituting Equation (31) into (7), we obtain

\[ u = x \frac{d\zeta}{dy}, \quad v = -\zeta. \]

(32)

By eliminating the pressure \( p \) from system (6) and then using Equation (31), we obtain

\[ De(y\beta + R\zeta) \frac{d^5\zeta}{dy^5} + \left(-1 + De \left( 4\beta + R \frac{d\zeta}{dy} \right) \right) \frac{d^4\zeta}{dy^4} \\
- (1 + \lambda_1)(y\beta + R\zeta) \frac{d^3\zeta}{dy^3} \\
+ \frac{d^2\zeta}{dy^2} \left( M^2 - 2\beta + (2 - R) \frac{d\zeta}{dy} \right) \\
+ \lambda_1(M^2 - 2\beta - R \frac{d\zeta}{dy} - 2RD e \frac{d^3\zeta}{dy^3}) = 0. \]

(33)

Boundary conditions (8) will be

\[ (i) \quad \frac{d\zeta}{dy} = 0, \quad \zeta = 1, \quad \text{at} \; y = 1 \]

(34)

\[ (ii) \quad \frac{d^2\zeta}{dy^2} = 0, \quad \zeta = 0 \quad \text{at} \; y = 0. \]

4. Analytical solution

Non-linear differential equation (33) with boundary conditions (34) will be solved analytically using the double perturbations method. For small \( R \) and \( \beta \), assume

\[ \zeta = \zeta_0 + R\zeta_1 + O(R^2), \]

\[ \zeta_0 = \zeta_{00} + \beta \zeta_{01} + O(\beta^2), \]

\[ \zeta_1 = \zeta_{10} + \beta \zeta_{11} + O(\beta^2). \]

The zero and first-order solutions with their boundary conditions are

\[ \zeta_{00}(y) = n(-y\gamma \cosh[\gamma] + \sinh[\gamma]), \]

\[ \zeta_{01}(y) = \frac{n^2}{4\gamma}(-y \cosh[\gamma]/[\gamma \cosh[\gamma] - \sinh[\gamma]]) \]

\[ \times (1 + 3D e \gamma^2 + \lambda_1) + \gamma(4D e \gamma^3 + \gamma \cosh[2\gamma]) \]

\[ \times (1 - D e \gamma^2 + \lambda_1) - \cosh[\gamma] \sinh[\gamma] \]

\[ \times (1 + 3D e \gamma^2 + \lambda_1) + \gamma \sinh[\gamma]/[(\gamma^2 - 1)\gamma \cosh[\gamma]] \]

\[ \times (1 - D e \gamma^2 + \lambda_1) - \sinh[\gamma] \]

\[ \times (\gamma^2 - D e (4 + \gamma^2) \gamma^2 + \gamma^2 \lambda_1)), \]

\[ \zeta_{10}(y) = \frac{n^3}{8} \cosh[\gamma]/[\gamma^2(-2y\gamma(2 + \cosh[2\gamma]))] \]

\[ + 6y \cosh[\gamma]/[\gamma \cosh[\gamma] - \sinh[\gamma]] + 3y \sinh[2\gamma] \]

\[ + 2\gamma(-(1 + y^2) \gamma \cosh[\gamma] - (-2 + y^2) \gamma \sinh[\gamma]/[\gamma \cosh[\gamma] + \sinh[\gamma]) - 7y \sinh[2\gamma] \]

\[ + 2\gamma((1 + y^2) \gamma \cosh[\gamma] - (-6 + y^2) \gamma \sinh[\gamma]/[\gamma \cosh[\gamma] + \sinh[\gamma]) + 1, \lambda_1)), \]

\[ \zeta_{11}(y) = \frac{n^4}{96y^3}(-4(-y \gamma \cosh[\gamma] + \sinh[\gamma]/[\gamma^2]) \]

\[ \times ((-1 + y^2) D e - \lambda_1)(-1 + 4y^2 D e - \lambda_1) \]

\[ - y\gamma \cosh[\gamma]/[\gamma \cosh[\gamma] - \sinh[\gamma)]/\gamma^2(\gamma^2 - 159) \]

\[ + 20y^2 \gamma^2 + (\gamma^2 - 69 + 20y^2 \gamma^2) \cosh[2\gamma] \]

\[ + 2(27 + 2y^2 \gamma^2) \sinh[2\gamma]/[\gamma \cosh[\gamma] + \sinh[\gamma]) \]

\[ + (\gamma + 4 \gamma^3) \cosh[2\gamma] \]

\[ + 2(9 + 11y^2) \gamma^2 \cosh[2\gamma] - (15y^2 + 4(-6 + 5y^2)) \gamma^3 \]

\[ + 2(9 + 11y^2) \cosh[2\gamma] + (15y^2 + 4(-6 + 5y^2)) \gamma^3 \]

\[ + (\gamma + 4 \gamma^3) \cosh[2\gamma] \]
\[ + y(-27 + 4(-6 + 5y^2)y^2 \cosh[2y] + 2(3 + (9 - 10y^2)y^2) \sinh[2y])(1 + \lambda_1)^2 \]
\[ + \frac{1}{2} \sinh[y\gamma](y^4((16 - 146y^2 + (116 - 9y^2 + 3y^2)y^4)\cosh[y] + (-16 + 82y^2 + (44 + 9y^2 - 3y^4)y^4 - 3(-1 + y^2)^2y^6)\cosh[3y] + y(-16 + (-244 + 45y^2)y^2 + 2(7 + 3y^4)y^4) \times \sinh[y] + y(48 - (124 + 15y^2)y^2 + 2(7 + 3y^4)y^4) \times \sinh[3y])D^2_x + 2y^2D_x(1 + \lambda_1)((-10 + 2(58 + 21y^2)y^2 + (212 - 3y^2(37 + y^2))y^4 + 9(-1 + y^2)^2\gamma \cosh[y] + (10 - 2(38 + 21y^2)y^2 + 3(-4 - 19y^2 + y^4)y^4 + 3(-1 + y^2)^2y^6)\cosh[3y] + y(10 + (166 + 57y^2)y^2 - 6(3 - 2y^2 + y^4)y^4) \times \sinh[y] + 3y(-10 + (2 + 31y^2)y^2 - 2(3 - 2y^2 + y^4)y^4) \times \sinh[3y]) + (4 + 2(5 + 6y^2)y^2 + 3(-52 - 19y^2 + y^4)y^4 - 9(-1 + y^2)^2y_6)\cosh[y] - (4 + 2(13 + 6y^2)y^2 + (20 - 9y^2 + 3y^4)y^4 + 3(-1 + y^2)^2\gamma \cosh[y] + y(12 + y^2(64 + 22y^2 + 3y^2(7 + 2(-4 + y^2)y^2))) \times \sinh[3y](1 + \lambda_1)^3 + \frac{1}{4}\gamma^2y(46 - 552y^2 + 80y^4 + 4(16 - 99y^2 + 20y^4)\cosh[2y] + 2(-65 + 18y^2)\cosh[4y] + 2y(371 - 16y^2 + (85 - 36y^2)\cosh[2y])\sinh[2y])D^2_x - 2y^2D_x(1 + \lambda_1)(24 - 8(45 + 2y^2)y^2 + 16(2 + y^2)y^4 + 4(10 + (-27 + 4y^2)y^2 + 4(2 + y^2)y^4)\cosh[2y] + 4(-16 + 9y^2)\cosh[4y] - 2y(-293 + 8(-19 + 2y^2)y^2 + (77 + 12y^2)\cosh[2y])\sinh[2y]) + (-18 + 8y^2)(15 - 2y^2 + 4y^2(-1 + y^2)^2 + 4(4 + y^2(-3 - 4y^2 + 8y^2(1 + y^2))\cosh[2y] + (2 - 60y^2)\cosh[4y] - 2y(25 + 32(-4 + y^2)y^2)\sinh[2y] + \gamma(1 + 12y^2)\sinh[4y])(1 + \lambda_1)^3) \right) \right) \right], \] \[ \text{where} \]
\[ n = \frac{1}{-y\cosh[y] + \sinh[y]}, \quad \gamma = \sqrt{M^2(1 + \lambda_1)}. \] \[ (37) \]

Then, the solution of (33) with boundary conditions (34) will be \[ \zeta(y) = \zeta_0 + \beta\zeta_0 + R(\zeta_0 + \beta\zeta_0). \] \[ (38) \]

The velocity components \( u \) and \( v \) can be obtained from Equation (32). To determine the normal pressure drop, substitute Equation (29) into the second equation in (6) to obtain
\[ P_y = \frac{1}{R(1 + \lambda_1)} \left( (1 + \lambda_1)(y\beta + R\zeta)\gamma + (1 - D_e(2\beta + 3R\zeta))\gamma \right) \left( \gamma - (y\beta + R\zeta)D\zeta_{yy} \right), \] \[ (39) \]

We can determine the normal pressure distribution as \[ \Delta p_n = \int_{\rho_c}^{p(y)} dp = p(y) - p_c \]
\[ = \frac{1}{R(1 + \lambda_1)} \int_{\rho_c}^{p(y)} \left( (1 + \lambda_1)(y\beta + R\zeta)\gamma + (1 - D_e(2\beta + 3R\zeta))\gamma \right) \left( \gamma - (y\beta + R\zeta)D\zeta_{yy} \right) dy, \] \[ (40) \]

if we set centreline pressure \( P_c \) equal to 0; then, using the boundary condition in Equation (34), we obtain
\[ \Delta p_n = -\frac{\mu}{R} \left( \gamma\zeta(y) - \int_{\rho_c}^{p(y)} \gamma \zeta(y) dy \right) - \frac{1}{2} \gamma^2(y) \left( \right) \] \[ - \frac{1}{R(1 + \lambda_1)} \left( (1 - 2D_e\beta)(\zeta(y) - \zeta(0)) \right) \] \[ + \frac{1}{4} \left( \right) \] \[ - \frac{3}{2} D_e R(\zeta^2(y) - \zeta^2(0)) - D_e (\beta(y\zeta_{yy}(y) - \zeta(y) + \zeta(0)) + R(\zeta(y)\zeta_{yy}(y) \right) - \frac{1}{2} \gamma^2(y) \left( \right) \] \[ \left( \right), \] \[ (41) \]

We can determine the axial pressure drop by substituting Equation (29) into the first equation in (6); we obtain the axial pressure as:
\[ \Delta \rho_a = \int_{\rho_c}^{\rho} \rho_x dx \]
\[ = \frac{\gamma^2}{2R(1 + \lambda_1)} \left( (M^2 - \beta)(1 + \lambda_1)\gamma + R(1 + \lambda_1)\gamma^2 \right) \] \[ - (1 + \lambda_1)(y\beta + R\zeta)\zeta_{yy} - R D_e \zeta_{yy}^2 \] \[ + (1 + 3D_e)\zeta_{yy} + D_e (y\beta + R\zeta)\zeta_{yyy} \]. \[ (42) \]

Note that the axial pressure drop takes a parabolic profile behaviour at any value of \( y \).

5. Special case

For Newtonian fluid flow without a magnetic field, i.e. \( M \to 0, \lambda_1 \to 0 \) and \( D_e \to 0 \), we obtain the same results as those obtained by Boutros et al. [22]:
\[ \zeta(y) = \frac{1}{2} y(-3 + y^2) + \frac{3}{40} y(-1 + y^2)^2 \beta \]
\[ - \frac{1}{100800} \left( R y(-1 + y^2)^2 \right) \left( -360(2 + y^2) \right) \] \[ + (681 + 454y^2 + 65y^4)\beta \), \[ (43) \]
\[ \Delta p_n = -\frac{\beta}{R} \left( \gamma \zeta(y) - \int_0^y \zeta(y') \, dy' \right) - \frac{1}{2} \zeta^2(y) - \frac{1}{R} \left( \zeta_y(y) - \zeta_y(0) \right), \]  

\( (44) \)

\[ \Delta p_a = \frac{\chi^2}{2R} \left( \beta \zeta_y - R \zeta_{yy} + (y \beta + R \zeta) \zeta_{yy} + \zeta_{yyy} \right). \]  

\( (45) \)

In addition, this result is shown in Figures 6(a) and 8(b).

6. Results and discussion

This section is divided into two subsections: the first subsection discusses the axial velocity and the second subsection discusses the normal pressure.

6.1. Velocity distributions

In this subsection, we discuss and interpret the effects of several physical parameters on velocity components \( u/x \) and \( v \).

Figure 2 shows the behaviour of the axial and normal velocities over a range of dimensionless wall dilation rate parameter \( \beta \) in an injection case \( (R = 0.2) \). This figure illustrates that, for the case of expanding walls \( (\beta > 0) \), the axial velocity increases at the middle region by increasing \( \beta \) while the inverse effect appears near the walls. This result is because the flow towards the centre becomes greater to make up for the space caused by the expansion of the wall, thus increasing the axial velocity near the centre. Additionally, this figure shows that, for the case of the contracting wall \( (\beta < 0) \), an increasing contraction ratio leads to lower axial velocity near the centre and higher axial velocity near the wall because the flow towards the wall becomes greater, and thus, the axial velocity near to the wall becomes greater. The second figure in Figure 2 shows that the normal velocity increases by increasing the expansion of the wall and decreases by increasing the contraction ratio. The same explanation applies for the suction case \( R = -0.2 \) in Figure 3. Figure 4 shows the variation of the axial and normal velocities for several values of permeation Reynolds number \( R \); this figure illustrates that increasing suction \( (R < 0) \) decreases velocity, while increasing injection \( (R > 0) \) increases velocity. Figure 4(b) has the same explanation as that in Figure 4(a), and these results agree physically. Figures 5 illustrates the behaviour of the axial velocity with non-Newtonian parameters \( \lambda_1 \) and \( D_e \). Figure 5(a) shows that in the expansion case, axial velocity increases as the ratio of relaxation time to retardation time \( (\lambda_1) \) increases, while it is a decreasing function of \( \lambda_1 \) if the walls are contracting. Physically, the Deborah number is dependent on retardation time \( \lambda_2 \). Retardation time increases with increasing Deborah numbers.

Figure 2. The variation of the axial and normal velocities over a range of dimensionless wall dilation rate parameter \( \beta \) for injection case \( R = 0.2 \) and fixed values of \( (M = 0.15, \lambda_1 = 5, D_e = 0.12) \).

Figure 3. The variation of the axial and normal velocities over a range of dimensionless wall dilation rate parameter \( \beta \) for suction case \( R = -0.2 \) at fixed values of \( (M = 0.2, \lambda_1 = 5, D_e = 0.12) \).
6.2. Normal Pressure distributions

This section addresses the effect of the various physical parameters on normal pressure distribution $\Delta p_n$.

Figure 7(a) shows normal pressure $\Delta p_n$ for expanding, fixed and contracting walls, and illustrates that the normal pressure is higher under contracting walls and lower under expanding walls. Physically, if the walls are expanded, the fluid is distributed in a larger

Figure 7. (a) Normal pressure $\Delta p_n$ for expanding, fixed and contracting walls and (b) the effect of M on the normal pressure at fixed value of $(M = 0.2, R = 0.2)$.
area, and thus, normal pressure decreases; additionally, under contracting walls, the fluid is distributed in less space, and thus, pressure increases. The effect of Hartman number $M$ on the normal pressure is shown in Figure 7(b). It is known that pressure is inversely proportional to velocity; thus, the normal pressure decreases at the walls when $M$ increases. Figure 8(a) shows the behaviour of normal pressure for several values of Deborah number $D_e$, and illustrates that the normal pressure is a decreasing function of $D_e$. Figure 8(b) shows that the normal pressure of a non-Newtonian fluid (Jeffrey model) is less than that of a Newtonian fluid; additionally, $\Delta \rho_n$ under expanding walls is less than that under contracting walls.

7. Conclusions

In this study, we design a mathematical model that describes the magnetohydrodynamic flow of a Jeffrey fluid in a porous channel with expanding or contracting walls. The channel is closed at one end by a complicated solid membrane. Graphs of the velocity components and the normal pressure are drawn for various values of the dimensionless wall dilation rate parameter $\beta$, the permeation Reynolds number $R$, the magnetic field $M$, the ratio of relaxation time to retardation time $\lambda_1$, and the Deborah number $D_e$. The main outcomes of the present study are concisely summarized as follows:

- The flow towards the center become greater to make up for the space caused by the expansion of the wall; as a result, the axial velocity also becomes greater near the centre.
- The velocity decreases as suction increases ($R < 0$), while increasing injection ($R > 0$) increases velocity.
- The axial velocity of Newtonian fluids is greater than that of non-Newtonian fluids if the walls are in contraction, while the axial velocity of a Jeffrey fluid is greater than that of a viscous fluid if the walls are in expansion.
- The magnetic field parameter $M$ reduces the fluid velocity and thus controls fluid movement.
- For expanding blood vessels, blood flow is distributed in a large area, and thus, normal pressure decreases; the opposite is true for contracting blood vessels.
- The normal pressure of a non-Newtonian fluid (Jeffrey model) is less than that of a Newtonian fluid.
- The normal pressure $\Delta \rho_n$ of expanding blood vessels is less than that of contracting blood vessels.

Figure 7. (a) The variation of the normal pressure with $\beta$ at fixed values of ($M = 0.5$, $R = 0.2$, $\lambda_1 = 0.2$, $D_e = 0.2$), and (b) the variation of the normal pressure with $M$ ($\beta = 0.1$, $R = 0.2$, $\lambda_1 = 2$, $D_e = 0.2$).

Figure 8. (a) The variation of the normal pressure with $D_e$ at fixed values of ($M = 2$, $R = 0.2$, $\lambda_1 = 1$, $\beta = 0.1$), and (b) the normal pressure of Newtonian and non-Newtonian fluids under expanding ($\beta = 0.12$) and contracting ($\beta = -0.12$) walls at fixed values of ($M = 0.2$, $R = 0.2$).
Disclosure statement

No potential conflict of interest was reported by the authors.

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