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A COMPACTNESS RESULT IN APPROACH THEORY
WITH AN APPLICATION TO THE CONTINUITY
APPROACH STRUCTURE

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Dedicated to Eva Colebunders on the occasion of her 65th birthday

Abstract. We establish a compactness result in approach theory which
we apply to obtain a generalization of Prokhorov’s Theorem for the
continuity approach structure.

1. Introduction

Measures of non-compactness ([BG80]) have been studied extensively in
the context of approach theory ([L15]), both on an abstract level ([BL94],
[BL95]) as in specific approach settings in e.g. hyperspace theory ([LS00*]),
functional analysis ([LS00]), function spaces ([L04]) and probability theory
([BLV11]). The presence of a vast literature on the interplay between com-
pactness and approach theory is explained by the fact that the latter is a
canonical setting which allows for a unified treatment of the classical concept
of measure of non-compactness ([L88]).

In this paper we contribute to the knowledge on the interplay between
compactness and approach theory. In Section 2 we provide a new compact-
ness result for a general approach space. In Section 3 we apply this result to
the specific setting of the so-called continuity approach structure ([BLV13],
[L15]) to obtain a quantitative generalization of Prokhorov’s Theorem.

2. A compactness result in approach theory

Let \( X \) be an approach space with approach system \( \mathcal{A} = (\mathcal{A}_x)_{x \in X} \). We
first recall some notions related to compactness in \( X \). For more details the
reader is referred to [L15].

We say that \( X \) is locally countably generated iff there exists a basis \( (\mathcal{B}_x)_{x \in X} \)
for \( \mathcal{A} \) such that each \( \mathcal{B}_x \) is countable.

For \( x \in X \), \( \phi \in \mathcal{A}_x \) and \( \epsilon > 0 \) we define the \( \phi \)-ball with center \( x \) and radius
\( \epsilon \) as the set \( B_\phi(x, \epsilon) = \{ y \in X \mid \phi(y) < \epsilon \} \). More loosely, we also refer to the
latter set as a ball with center \( x \) or a ball with radius \( \epsilon \).

Consider a point \( x \in X \), a sequence \( (x_n)_n \) in \( X \) and \( \epsilon > 0 \). We say that
\( (x_n)_n \) is \( \epsilon \)-convergent to \( x \) iff each ball \( B \) with center \( x \) and radius \( \epsilon \) contains

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(FWO).
Let $x_n$ for all $n$ larger than a certain $n_B$. We write $x_n \xrightarrow{\epsilon} x$ to indicate that $(x_n)_n$ is $\epsilon$-convergent to $x$. We define the limit operator of $(x_n)_n$ at $x$ as

$$\lambda (x_n \rightarrow x) = \inf \left\{ \alpha > 0 \mid x_n \xrightarrow{\alpha} x \right\}.$$ 

We call $X$ sequentially complete iff it holds for each sequence $(x_n)_n$ in $X$ that $\inf_{x \in X} \lambda_A (x_n \rightarrow x) = 0$ implies the existence of a point $x_0$ to which $(x_n)_n$ converges (in the topological coreflection).

Let $A \subseteq X$ be a set. We say that $A$ is $\epsilon$-relatively sequentially compact iff every sequence in $A$ contains a subsequence which is $\epsilon$-convergent and we define the relative sequential compactness index of $A$ as

$$\chi_{rsc}(A) = \inf \{ \alpha > 0 \mid A \text{ is } \alpha \text{-relatively sequentially compact} \}.$$ 

Notice that relatively sequentially compact sets (in the topological coreflection) have relative sequential compactness index zero, but that the converse does not necessarily hold.

If $(\Phi = (\phi_x)_x) \in \prod_{x \in X} A_x$, then a set $B \subseteq X$ is called a $\Phi$-ball iff there exist $x \in X$ and $\alpha > 0$ such that $B = B_{\phi_x}(x, \alpha)$. We call $A$ $\epsilon$-relatively compact iff it holds for each $\Phi \in \prod_{x \in X} A_x$ that $A$ can be covered with finitely many $\Phi$-balls with radius $\epsilon$ and we define the relative compactness index of $A$ as

$$\chi_{rc}(A) = \inf \{ \alpha > 0 \mid A \text{ is } \alpha \text{-relatively compact} \}.$$ 

We say that $X$ is $\epsilon$-Lindelöf iff it holds for each $\Phi \in \prod_{x \in X} A_x$ that $X$ can be covered with countably many $\Phi$-balls with radius $\epsilon$ and we define the Lindelöf index of $X$ as

$$\chi_L(X) = \inf \{ \alpha > 0 \mid X \text{ is } \alpha \text{-Lindelöf} \}.$$ 

Theorem 2.2, the main result of this section, interconnects the above notions. For its proof we use the following well-known lemma which belongs to the heart of approach theory ([L15]).

**Lemma 2.1** (Lowen). Let $\mathcal{D}_A$ be the set of quasi-metrics $d$ on $X$ with the property that $d(x, \cdot) \in A_x$ for each $x \in X$. Then the assignment of collections

$$\mathcal{B}_{\mathcal{D}_A} = \{d(x, \cdot) \mid d \in \mathcal{D}_A\}$$

is a basis for $A$.

**Theorem 2.2.** Let $X$ be locally countably generated. Then, for any set $A \subseteq X$,

$$\chi_{rsc}(A) \leq \chi_{rc}(A) \leq \chi_{rsc}(A) + \chi_L(X).$$

In particular, if $\chi_L(X) = 0$, then

$$\chi_{rsc}(A) = \chi_{rc}(A).$$

If, in addition, $X$ is sequentially complete, then

$$A \text{ is relatively sequentially compact } \iff \chi_{rsc}(A) = 0.$$

**Proof.** Suppose that $A$ is not $\epsilon$-relatively sequentially compact and fix $\epsilon_0 < \epsilon$. Then there exists a sequence $(a_n)_n$ in $A$ without $\epsilon$-convergent subsequence. But then, for each $x \in X$, there exists $\phi_x \in A_x$ such that the ball $B_{\phi_x}(x, \epsilon_0)$ contains at most finitely many terms of $(a_n)_n$. Indeed, if this was not the case, then the fact that $X$ is locally countably generated would allow us to
extract an $\epsilon$-convergent subsequence from $(a_n)_n$. Put $\Phi = (\phi_x)_x$. Now one easily sees that $A$ cannot be covered with finitely many $\Phi$-balls with radius $\epsilon_0$. We conclude that $A$ is not $\epsilon_0$-relatively compact. We have shown that

$$\chi_{rsc}(A) \leq \chi_{rc}(A).$$

Furthermore, let $X$ be $\delta$-Lindelöf and let $A$ fail to be $\epsilon$-relatively compact, with $\delta < \epsilon$, and fix $\delta < \epsilon_0 < \epsilon$. Then Lemma 2.1 enables us to choose $\Phi \in \Pi_{x \in X} A_x$ of the form

$$\Phi = (d_x(x, \cdot))_x,$$

where each $d_x$ is a quasi-metric in $\mathcal{D}_A$, such that $X$ cannot be covered with finitely many $\Phi$-balls with radius $\epsilon_0$. However, $X$ being $\delta$-Lindelöf, there is a countable cover $(B_n)_n$ of $X$ with $\Phi$-balls with radius $\delta$, say with centers $(x_n)_n$. Now construct a sequence $(a_n)_n$ in $A$ such that, for each $n$, the $\Phi$-ball with center $x_n$ and radius $\epsilon_0$ contains at most finitely many terms of $(a_n)_n$. But then $(a_n)_n$ has no $(\epsilon_0 - \delta)$-convergent subsequence. Indeed, if any subsequence $(a_{k_n})_n$ was $(\epsilon_0 - \delta)$-convergent to $x$, then we could choose $n_0$ such that $x \in B_{n_0}$, and then it is not hard to see that the $\Phi$-ball with center $x_{n_0}$ and radius $\epsilon_0$ would contain infinitely many terms of $(a_n)_n$. We conclude that $A$ is not $(\epsilon_0 - \delta)$-relatively sequentially compact. We have established that

$$\chi_{rc}(A) \leq \chi_{rsc}(A) + \chi_L(X).$$

Suppose, in addition, that $X$ is sequentially complete. Let $\chi_{rsc}(A) = 0$. Fix a sequence $(a_n)_n$ in $A$ and carry out the following construction:

Choose a subsequence $(a_{k_1(n)})_n$ and a point $x_1 \in X$ such that

$$\lambda (a_{k_1(n)} \to x_1) \leq 1.$$

Choose a further subsequence $(a_{k_1 \circ k_2(n)})_n$ and a point $x_2 \in X$ such that

$$\lambda (a_{k_1 \circ k_2(n)} \to x_2) \leq 1/2.$$

$$\vdots$$

Choose a further subsequence $(a_{k_1 \circ \cdots \circ k_m(n)})_n$ and a point $x_m \in X$ such that

$$\lambda (a_{k_1 \circ \cdots \circ k_m(n)} \to x_m) \leq 1/m.$$

$$\vdots$$

Then it holds for the diagonal subsequence

$$(a'_n = a_{k_1 \circ \cdots \circ k_n(n)})_n$$

that $\inf_{x \in X} \lambda (a'_n \to x) = 0$. Now the sequential completeness of $X$ allows us to conclude that $(a'_n)_n$ is convergent. We infer that $A$ is relatively sequentially compact. □
3. Compactness for the Continuity Approach Structure

We start by recalling some basic concepts. They can be found in any standard work on probability theory (e.g. [K02]).

A cumulative distribution function (cdf) is a non-decreasing and right-continuous map \( F : \mathbb{R} \to \mathbb{R} \) for which \( \lim_{x \to -\infty} F(x) = 0 \) and \( \lim_{x \to \infty} F(x) = 1 \). The collection of (continuous) cdf's is denoted as \( \mathcal{F}_c \).

The weak topology \( \mathcal{T}_w \) on \( \mathcal{F} \) is the initial topology for the source \( (\mathcal{F} \to \mathbb{R} : F \mapsto \int_{-\infty}^{\infty} h(x) dF(x))_{h \in C_b(\mathbb{R}, \mathbb{R})} \) with \( C_b(\mathbb{R}, \mathbb{R}) \) the set of bounded and continuous maps \( h : \mathbb{R} \to \mathbb{R} \).

The uniform distance between \( F \) and \( G \) in \( \mathcal{F} \) is \( D_u(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)| \).

The convolution product of \( F \) and \( G \) in \( \mathcal{F} \) is the cdf \( F \ast G = \int_{-\infty}^{\infty} F(\cdot - y) dG(y) \).

This product is commutative and \( F \ast G \in \mathcal{F}_c \) if \( F \in \mathcal{F}_c \).

The following classical result, which can be found in [B82], shows how the previous concepts are interconnected. We denote the underlying topology of \( D_u \) as \( \mathcal{T}_{D_u} \).

**Theorem 3.1** (Bergström). The source
\[
(\mathcal{T} \to \mathbb{R} : F \mapsto \int_{-\infty}^{\infty} h(x) dF(x))_{h \in C_b(\mathbb{R}, \mathbb{R})}
\]
with \( C_b(\mathbb{R}, \mathbb{R}) \) the set of bounded and continuous maps \( h : \mathbb{R} \to \mathbb{R} \).

The convolution product of \( F \) and \( G \) in \( \mathcal{T} \) is the cdf \( F \ast G = \int_{-\infty}^{\infty} F(\cdot - y) dG(y) \).

This product is commutative and \( F \ast G \in \mathcal{T}_c \) if \( F \in \mathcal{T}_c \).

The following classical result, which can be found in [B82], shows how the previous concepts are interconnected. We denote the underlying topology of \( D_u \) as \( \mathcal{T}_{D_u} \).

**Theorem 3.1** (Bergström). The source
\[
((\mathcal{T} \to \mathbb{R} : F \mapsto \int_{-\infty}^{\infty} h(x) dF(x))_{h \in C_b(\mathbb{R}, \mathbb{R})})_{G \in \mathcal{T}_c}
\]
is initial.

In order to lay the structural foundations of the quantitative central limit theory developed in [BLV13], approach theory was invoked. More precisely, the following definition was proposed in [BLV13] (appendix B). It is clearly inspired by Theorem 3.1. We denote the underlying approach structure of \( D_u \) as \( \mathcal{A}_{D_u} \).

**Definition 3.2.** The continuity approach structure \( \mathcal{A}_c \) on \( \mathcal{T} \) is the initial approach structure for the source
\[
(\mathcal{T} \to \mathcal{F}_c, \mathcal{T}_{D_u} : F \mapsto F \ast G)_{G \in \mathcal{T}_c}
\]
Define for each cdf \( F \) and each \( \alpha \in \mathbb{R}_0^+ \) the mapping
\[
\phi_{F,\alpha} : \mathcal{T} \to [0,1]
\]
by putting
\[
\phi_{F,\alpha}(G) = \sup_{x \in \mathbb{R}} \max \{F(x - \alpha) - G(x), G(x) - F(x + \alpha)\}.
\]
Furthermore, for \( F \in \mathcal{T} \), let \( \Phi(F) \) be the set of all maps \( \phi_{F,\alpha} \), where \( \alpha \) runs through \( \mathbb{R}_0^+ \).

The proofs of the following results can be found in [L15].

**Theorem 3.3.** The collection of sets \( (\Phi(F))_{F \in \mathcal{T}} \) is a basis for the approach system of \( \mathcal{A}_c \).
Theorem 3.4. The topological coreflection of \(A_c\) is \(T_{w}\). The metric coreflection of \(A_c\) is \(D_{w}\).

Theorem 3.5. The space \((\mathcal{F}, A_c)\) is locally countably generated and sequentially complete.

It is the aim of this section to express the relative sequential compactness index of a set \(D\) in the space \((\mathcal{F}, A_c)\) in terms of a canonical index measuring up to what extent \(D\) is tight ([K02]). We thus obtain a strong quantitative generalization of Prokhorov’s Theorem (Theorem 3.13). To this end, we make use of the compactness result obtained in the previous section. First some preparation is required.

Define, for \(\gamma \in \mathbb{R}^+_0\), the metric \(L_\gamma(F, G)\) between \(F\) and \(G\) in \(\mathcal{F}\) as the infimum of all \(\alpha \in \mathbb{R}^+_0\) for which the inequalities

\[
F(x - \gamma \alpha) - \alpha \leq G(x) \leq F(x + \gamma \alpha) + \alpha
\]

hold for all points \(x \in \mathbb{R}\). The metric \(L_\gamma\) is known as the Lévy metric with parameter \(\gamma\) ([K02]).

Theorem 3.6. The assignment of collections

\[
\left(\{L_\gamma(F, \cdot) \mid \gamma \in \mathbb{R}^+_0\}\right)_{F \in \mathcal{F}}
\]

is a basis for the approach system of \(A_c\).

Proof. It is easily seen that \(L_{\gamma_1} \leq L_{\gamma_2}\) whenever \(\gamma_2 \leq \gamma_1\), whence (1) is a basis for an approach structure which we denote \(A\). Now it is enough to prove that, for all \(F \in \mathcal{F}\) and \(D \subset \mathcal{F}\) nonempty, \(\delta_A(F, D) = \delta_{A_c}(F, D)\). We will do this in two steps, making use of Theorem 3.3.

1) \(\delta_A(F, D) \leq \delta_{A_c}(F, D)\): If \(\delta_{A_c}(F, D) < \theta\) with \(\theta > 0\), then for \(\gamma > 0\) there exists \(G \in D\) for which \(\phi_{F, \gamma}(G) < \theta\). But then we have for all real numbers \(x\) that \(F(x - \gamma \theta) - G(x) < \theta\) and \(G(x) - F(x + \gamma \theta) < \theta\), from which we deduce that \(L_\gamma(F, G) < \theta\) and hence \(\delta_A(F, D) \leq \theta\), which proves the desired inequality.

2) \(\delta_{A_c}(F, D) \leq \delta_A(F, D)\): If \(\delta_{A_c}(F, D) > \theta\) with \(\theta > 0\), then there exists \(\alpha > 0\) such that for all \(G \in D\) we have \(\phi_{F, \alpha}(G) > \theta\). If we put \(\gamma = \alpha \theta^{-1}\), then it follows that for every \(G \in D\) there exists \(x \in \mathbb{R}\) such that \(F(x - \gamma \theta) - G(x) > \theta\) or \(G(x) - F(x + \gamma \theta) > \theta\). We conclude that \(L_\gamma(F, G) \geq \theta\) and hence \(\delta_A(F, D) \geq \theta\), which proves the desired inequality.

We call a finite set of points at which \(F\) is continuous an \(F\)-net and we introduce for each \(F\)-net \(N\) the mapping \(\psi_{F,N}: \mathcal{F} \to [0, 1]\) by setting

\[
\psi_{F,N}(G) = \sup_{x \in N} |F(x) - G(x)|.
\]

Lemma 3.7. For every \(F \in \mathcal{F}\) the following hold.

1) For an \(F\)-net \(N\) and \(\epsilon > 0\) there exists \(\alpha \in \mathbb{R}^+_0\) so that

\[
\psi_{F,N}(G) \leq \phi_{F,\alpha}(G) + \epsilon
\]

for each \(G \in \mathcal{F}\).
2) For $\alpha \in \mathbb{R}_0^+$ and $\epsilon > 0$ there exists an $F$-net $N$ so that

$$\phi_{F,\alpha}(G) \leq \psi_{F,N}(G) + \epsilon$$

for each $G \in \mathcal{F}$.

**Proof.** Let $F \in \mathcal{F}$.

1) Fix an $F$-net $N$ and $\epsilon > 0$. Since all $x \in N$ are continuity points of $F$, we may choose $\alpha \in \mathbb{R}_0^+$ such that

$$\forall x \in N, \forall y \in X : |x - y| \leq \alpha \Rightarrow |F(x) - F(y)| \leq \epsilon.$$ 

Now, for $G \in \mathcal{F}$ and $x \in N$, we have on the one hand

$$F(x) - G(x) \leq F(x) - G(x) + \epsilon \leq \phi_{F,\alpha}(G) + \epsilon,$$

and on the other

$$G(x) - F(x) \leq G(x) - F(x + \alpha) + \epsilon \leq \phi_{F,\alpha}(G) + \epsilon,$$

from which it follows that

$$\psi_{F,N}(G) \leq \phi_{F,\alpha}(G) + \epsilon$$

and we are done.

2) Fix $\alpha \in \mathbb{R}_0^+$ and $\epsilon > 0$. The number of discontinuities of $F$ being at most countable, it is possible to construct an $F$-net $N$ consisting of points $x_0 < x_1 < \ldots < x_{n-1} < x_n$ such that $F(x_0) \leq \epsilon, x_{i+1} - x_i < \alpha$ for all $i \in \{0, \ldots, n-1\}$ and $F(x_n) \geq 1 - \epsilon$. Now fix $G \in \mathcal{F}$ and $x \in \mathbb{R}$. We distinguish between the following cases.

If there exists $i \in \{0, \ldots, n-1\}$ such that $x_i \leq x < x_{i+1}$, then

$$F(x - \alpha) - G(x) \leq F(x_i) - G(x_i) \leq \psi_{F,N}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq G(x_{i+1}) - F(x_{i+1}) \leq \psi_{F,N}(G) + \epsilon.$$

If $x < x_0$, then

$$F(x - \alpha) - G(x) \leq F(x_0) \leq \epsilon \leq \psi_{F,N}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq G(x) - F(x_0) \leq G(x) - (F(x_0) - \epsilon) \leq \psi_{F,N}(G) + \epsilon.$$

If $x \geq x_n$, then

$$F(x - \alpha) - G(x) \leq F(x) - G(x) \leq (F(x_n) + \epsilon) - G(x_n) \leq \psi_{F,N}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq \epsilon \leq \psi_{F,N}(G) + \epsilon.$$ 

Hence we conclude that

$$\phi_{F,\alpha}(G) \leq \psi_{F,N}(G) + \epsilon,$$

which finishes the proof. □
For \( F \in \mathcal{F} \), let \( \Psi(F) \) be the set of all maps \( \psi_{F,N} \), with \( N \) running through all \( F \)-nets.

**Theorem 3.8.** The collection of sets \( (\Psi(F))_{F \in \mathcal{F}} \) is a basis for the approach system of \( A_c \).

**Proof.** Combine Theorem 3.3 and Lemma 3.7. \( \square \)

The following result provides us with information about the Lindelöf index of the space \( (\mathcal{F}, A_c) \).

**Theorem 3.9.** We have \( \chi_\mathcal{L}(\mathcal{F}, A_c) = 0 \).

**Proof.** Fix a basis \( (\mathcal{B}_F)_{F \in \mathcal{F}} \) for \( A_c \), \( (\phi_F)_{F \in \mathcal{F}} \in \Pi_{F \in \mathcal{F}} \mathcal{B}_F \) and \( \epsilon > 0 \). The space \( (\mathcal{F}, \mathcal{T}_w) \) being separable ([P05]), we fix a countable set \( D \subset \mathcal{F} \) which is dense for the weak topology. Now, the assignment of collections

\[
\left( \{ L_{1/n}(F, \cdot) \mid n \in \mathbb{N}_0 \} \right)_{F \in \mathcal{F}}
\]

being a basis for \( A_c \) (Theorem 3.6), there exist for each \( F \in \mathcal{F} \):

1) a number \( n_F \in \mathbb{N}_0 \) such that

\[
\phi_F(G) < L_{1/n_F}(F, G) + \epsilon/3
\]

for each \( G \in \mathcal{F} \)

2) an element \( D_F \in D \) such that

\[
L_{1/n_F}(F, D_F) < \epsilon/3.
\]

Combining (2) and (3) we have, for \( F, G \in \mathcal{F} \),

\[
\phi_F(G) < L_{1/n_F}(F, G) + \epsilon/3
\]

\[
\leq L_{1/n_F}(F, D_F) + L_{1/n_F}(D_F, G) + \epsilon/3
\]

\[
< L_{1/n_F}(D_F, G) + 2\epsilon/3.
\]

Now consider the function

\[
\zeta : \mathcal{F} \to \mathbb{N}_0 \times \mathcal{D}
\]

defined by \( \zeta(F) = (n_F, D_F) \) and fix for each \( (n, D) \in \zeta(\mathcal{F}) \) a cdf \( H_{n,D} \in \mathcal{F} \) such that \( \zeta(H_{n,d}) = (n, D) \). Thus, by (4),

\[
\phi_{H_{n,d}}(G) < L_{1/n}(D, G) + 2\epsilon/3
\]

for each \( G \in \mathcal{F} \). Consider the countable set \( \mathcal{C} = \{ H_{n,D} \mid (n, D) \in \zeta(\mathcal{F}) \} \). We claim that

\[
\sup_{F \in \mathcal{F}} \inf_{C \in \mathcal{C}} \phi_C(H) \leq \epsilon.
\]

Indeed, for \( F \in \mathcal{F} \) it suffices to consider the point \( H_{n_F,D_F} \in \mathcal{C} \) since by (2) and (5)

\[
\phi_{H_{n_F,D_F}}(F) < L_{1/n_F}(D_F, F) + 2\epsilon/3 < \epsilon.
\]

This finishes the proof. \( \square \)

We call a set \( D \subset \mathcal{F} \) weakly relatively sequentially compact iff it is relatively sequentially compact under the weak topology on \( \mathcal{F} \), i.e. each sequence in \( D \) contains a weakly convergent subsequence.
Since the weak topology is the topological coreflection of $A_e$ (Theorem 3.4), the space $(\mathcal{F}, A_e)$ is locally countably generated and sequentially complete (Theorem 3.5) and $\chi_L(\mathcal{F}, A_e) = 0$ (Theorem 3.9), we may apply Theorem 2.2 to conclude that

**Theorem 3.10.** For a set $\mathcal{D} \subset \mathcal{F}$ we have

$$(\chi_{rsc})_{A_e}(\mathcal{D}) = (\chi_{rc})_{A_e}(\mathcal{D}).$$

Furthermore,

$$\mathcal{D} \text{ is weakly relatively sequentially compact } \iff (\chi_{rsc})_{A_e}(\mathcal{D}) = 0.$$ 

Recall that a collection $\mathcal{D} \subset \mathcal{F}$ is *tight* ([K02]) iff for each $\epsilon > 0$ there exists a constant $M \in \mathbb{R}_0^+$ such that $\max \{F(-M), 1 - F(M)\} \leq \epsilon$ for all $F \in \mathcal{D}$. We now define the number

$$\chi_e(\mathcal{D}) = \inf_{M > 0} \sup_{F \in \mathcal{D}} \max \{F(-M), 1 - F(M)\}.$$ 

We call $\chi_e(\mathcal{D})$ the *escape index* of $\mathcal{D}$ (not to be confused with the tightness indices discussed in [BLV11]). Notice that $\mathcal{D}$ is tight if and only if $\chi_e(\mathcal{D}) = 0$. The following simple example shows that the escape index produces meaningful non-zero values.

**Example 3.11.** Fix $0 < \alpha < 1$ and let $\mathcal{F}$ be the set of all probability distributions $F_n = (1 - \alpha)F_{\delta_0} + \alpha F_{\delta_n}, \ n \in \mathbb{N}_0$, $F_{\delta_n}$ standing for the Dirac probability distribution making a jump of height 1 at $x$. Then $\chi_e(\mathcal{D}) = \alpha$.

We finally come to a quantitative generalization of Prokhorov’s Theorem for the continuity approach structure. As in the classical case, the proof is based on Helly’s Selection Principle ([K02]).

**Theorem 3.12** (Helly’s Selection Principle). Fix a number $M \in \mathbb{R}_0^+$ and a sequence $(F_n : [-M, M] \rightarrow [0,1])_n$ of non-decreasing right-continuous functions. Then there exists a subsequence $(F_{k_n})_n$ and a non-decreasing right-continuous function $F : [-M, M] \rightarrow [0,1]$ such that $F_{k_n}(x) \rightarrow F(x)$ for each point $x$ at which $F$ is continuous.

**Theorem 3.13** (Quantitative Prokhorov’s Theorem). For $\mathcal{D} \subset \mathcal{F}$ we have

$$(\chi_{rsc})_{A_e}(\mathcal{D}) = \chi_e(\mathcal{D}).$$

**Proof.** Recall that, by Theorem 3.10,

$$(\chi_{rsc})_{A_e}(\mathcal{D}) = (\chi_{rc})_{A_e}(\mathcal{D}).$$

1) $(\chi_{rsc})_{A_e}(\mathcal{D}) \leq \chi_e(\mathcal{D})$: Fix $\epsilon > 0$ and a sequence $(F_n)_n$ in $\mathcal{D}$. Now we choose a constant $M \in \mathbb{R}_0^+$ in such a way that for each $F \in \mathcal{D}$ it holds that $\max \{F(-M), 1 - F(M)\} \leq \chi_e(\mathcal{D}) + \epsilon$. Then Helly’s Selection Principle furnishes a subsequence $(F_{k_n})_n$ and a non-decreasing right-continuous function $G : [-M, M] \rightarrow [0,1]$ such that $F_{k_n}(x) \rightarrow G(x)$ for all points $x$ at which $G$ is continuous. Finally, we define $\tilde{G} \in \mathcal{F}$ by

$$\tilde{G}(x) = \begin{cases} 
0 & \text{if } x < -M \\
n & 0 \leq x < M \\
1 & \text{if } x \geq M
\end{cases}.$$
But then, by Theorem 3.8, we clearly have \( \lambda_{\Delta_c} \left( F_n \to \tilde{G} \right) \leq \chi_e(\mathcal{D}) + \epsilon \) and hence \( (\chi_{rc})_{\Delta_c}(\mathcal{D}) \leq \chi_e(\mathcal{D}) + \epsilon \).

2) \( \chi_e(\mathcal{D}) \leq (\chi_{rc})_{\Delta_c}(\mathcal{D}) \): Let \( \epsilon > 0 \). Then for \( \alpha \in \mathbb{R}_0^+ \) there exists a finite collection \( \mathcal{E} \subset \mathcal{F} \) such that for all \( F \in \mathcal{D} \) we can find \( G \in \mathcal{E} \) for which \( \phi_{G,\alpha}(F) \leq (\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon/2 \). Since \( \mathcal{E} \) is finite we may choose a constant \( \tilde{M} \in \mathbb{R}_0^+ \) such that for each \( G \in \mathcal{E} \) we have \( G(-\tilde{M}) \leq \tilde{M} - \tilde{M} \leq 1/2 \) and \( G(\tilde{M}) \geq 1 - 1/2 \). Now put \( M = \tilde{M} + \alpha \), fix \( F \in \mathcal{D} \) and choose \( G \in \mathcal{E} \) in such a way that \( \phi_{G,\alpha}(F) \leq (\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon/2 \). Then we have on the one hand

\[
F(-M) = F(-\tilde{M} - \alpha) \\
\leq G(-\tilde{M}) + (\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon/2 \\
\leq (\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon,
\]

and on the other

\[
F(M) = F(\tilde{M} + \alpha) \\
\geq G(\tilde{M}) - (\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon/2 \\
\geq 1 - ((\chi_{rc})_{\Delta_c}(\mathcal{D}) + \epsilon),
\]

entailing that \( \chi_e(\mathcal{D}) \leq (\chi_{rc})_{\Delta_c}(\mathcal{D}) \).

\( \square \)

**Corollary 3.14** (Classical Prokhorov’s Theorem). For \( \mathcal{D} \subset \mathcal{F} \) the following are equivalent.

1. The collection \( \mathcal{D} \) is weakly relatively sequentially compact.
2. The collection \( \mathcal{D} \) is tight.

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