Spatio-temporal Chaos and Vacuum Fluctuations of Quantized Fields

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Abstract

We consider deterministic chaotic models of vacuum fluctuations on a small (quantum gravity) scale. As a suitable small-scale dynamics, nonlinear versions of strings, so-called ‘chaotic strings’ are introduced. These can be used to provide the ‘noise’ for second quantization of ordinary strings via the Parisi-Wu approach of stochastic quantization. Extensive numerical evidence is presented that the vacuum energy of chaotic strings is minimized for the numerical values of the observed standard model parameters, i.e. in this extended approach to second quantization concrete predictions for vacuum expectations of dilaton-like fields and hence on masses and coupling constants can be given. Low-energy fermion and boson masses are correctly obtained with a precision of 3-4 digits, the electroweak and strong coupling strengths with a precision of 4-5 digits. In particular, the minima of the vacuum energy yield high-precision predictions of the Higgs mass (154 GeV), of the neutrino masses ($1.45 \cdot 10^{-5}$ eV, $2.57 \cdot 10^{-3}$ eV, $4.92 \cdot 10^{-2}$ eV) and of the GUT scale ($1.73 \cdot 10^{16}$ GeV).
Preface

This book is written for an interdisciplinary readership of graduate students and researchers interested in nonlinear dynamics, stochastic processes, statistical mechanics on the one hand and high energy physics, quantum field theory, string theory on the other. In fact, one of the goals that I had in mind when writing this book was to make particle physicists become interested in nonlinear dynamics, and nonlinear physicists become interested in particle physics. Why that? Didn’t so far these two subjects evolve quite independently from each other? So what is this book about?

Mathematically, the subject of the book are coupled map lattices exhibiting spatio-temporal chaotic behaviour. Physically, the subject is a topic that lies at the heart of elementary particle physics: There are about 25 free parameters in the standard model of electroweak and strong interactions, namely the coupling strengths of the three interactions, the fermion and boson masses, and various mass mixing angles. These parameters are not fixed at all by the standard model itself, they are just measured in experiments, and a natural question is why these free parameters take on the numerical values that we observe in nature and not some other values. It will turn out that the answer is closely related to certain distinguished types of coupled map lattices that we will consider in this book as suitable models of vacuum fluctuations. These dynamical systems, called ‘chaotic strings’ in the following, are observed to have minimum vacuum energy for the observed standard model parameters. They yield an extension of ordinary quantization schemes which can account for the free parameters.

In this sense this book deals with both, nonlinear dynamics and high energy physics. So far only very few original papers have been published on this very new subject. With the current book I hope to make these important new applications for coupled chaotic dynamical systems accessible to a broad readership.

The book consists of 12 chapters. The first few chapters will mainly concentrate onto the theory of the relevant class of coupled map lattices, their
use for second quantization purposes, and their physical interpretation in terms of vacuum fluctuations. In the later chapters concrete numerical results are presented and these are then related to standard model phenomenology. Sections marked with an asterisk can be omitted at a first reading, these sections deal with interesting side issues which, however, are not necessary for the logical development of the following chapters. In view of the fact that (unfortunately!) many readers may not have the time to read this book from the beginning to the end, I included a very detailed summary as a self-contained chapter 12. This summary contains the most important concepts and results of this book and is written in a self-consistent way, i.e. no knowledge of previous chapters is required.

The research described in this book developed over a longer period of time at various places. I started to work on the relevant types of coupled map lattices during my stay at the Niels Bohr Institute, Copenhagen, in 1992 and continued during a stay at the University of Maryland in 1993. Some important numerical results, now described in section 7.2 and 8.5, were obtained at the RWTH Aachen in 1994 as well as during a visit to the Max Planck Institute for Physics of Complex Systems, Dresden, in 1996. The main part of the work was done at my home institute, the School of Mathematical Sciences at Queen Mary, University of London, as well as during long-term research visits to the Institute for Theoretical Physics at the University of California at Santa Barbara in 2000 and to the Newton Institute for Mathematical Sciences at Cambridge in 2001. The hospitality that I enjoyed during these visits was very pleasant, and the nice research atmosphere was really inspiring.

The number of people from which I learned during the past years and who thus indirectly contributed to this book is extremely large—too large to list all these individuals separately here! So at this point let me just thank all of them in one go.

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Christian Beck
Introduction

This book deals with new applications for coupled map lattices in quantum field theories and elementary particle physics. We will introduce appropriate classes of coupled map lattices (so-called ‘chaotic strings’) as suitable spatio-temporal chaotic models of vacuum fluctuations.

From a mathematical point of view, coupled map lattices are high-dimensional nonlinear dynamical systems with discrete space, discrete time and continuous state variables. They were for the first time introduced by Kaneko in 1984 [Kaneko (1984)]. The dynamics is generated by local maps that are situated at the sites of a lattice. There can be various types of couplings between the maps at the lattice sites, for example global coupling, exponentially decreasing coupling or diffusive coupling. For globally coupled systems, typically each lattice site is connected to all others with the same coupling strength. In the exponentially decreasing case the coupling strength decays exponentially with distance. For diffusively coupled map lattices there is just nearest-neighbor coupling, corresponding to a discrete version of the Laplacian. The latter one is the most relevant coupling form for applications in quantum field theories. Very complicated periodic, quasi-periodic or spatio-temporal chaotic behaviour is possible in all these cases (see the color plates in chapter 2 and 4 for some illustrations).

Generally, the spectrum of possibilities of spatio-temporal structures that can be generated by coupled map lattices is extremely rich and has been extensively studied in the literature, the emphasis being on the bifurcation structure [Bunimovich et al. (1996), Just (1995), Amritkar et al. (1993), Gade et al. (1993), Amritkar et al. (1991), Pikovsky et al. (1991)], Liapunov exponents [Yang et al. (1996), Torcini et al. (1997a), Kaneko (1986b)], Isola et al. (1990), traveling waves [Carretero-González (1997)], He et al. (1997), phase transition-like phenomena [Grassberger et al. (1991), Cuche et al. (1997), Blank (1997), Marcq et al. (1996), Boldrighini et al. (1995), Keller et al. (1992a), Houlrik et al. (1992), Miller et al. (1993), Gielis et al. (2000)], the existence of smooth invariant measures [Baladi et al. (1998), Jiang et al. (1998a), Chaté et al. (1997), Mackey et al. (1995)], synchronization [Lemaitre et al. (1999), Bagnoli et al. (1999), de San Roman et al. (1998), Jiang et al. (1998b)], Wang et al. (1998), Ding et al. (1997), control [Gade (1998), Egolf et al. (1998), Parekh et al. (1998), Mondragon et al. (1997), Ohishi et al. (1995)] and many other properties. Applications for coupled
map systems have been pointed out for various subjects, among them hydro-
dynamic turbulence [Beck (1994), Hilgers et al. (1997b), Hilgers et al. (1999a),
Bottin et al. (1998)], chemical waves [Kapral (1993)], financial markets
[Hilgers et al. (1997a)], biological systems [Bevers et al. (1999),
Losson et al. (1995), Martinezmekler et al. (1992), Dens et al. (2000)] and,
at a much more fundamental level, for quantum field theories [Beck (1998),
Beck (1995c)]. In this book we will concentrate on the quantum field theo-
retical applications.

A possible way of embedding coupled map lattices into a general quantum
field theoretical context is via the Parisi-Wu approach of stochastic quantiza-
tion [Parisi et al. (1981), Damgaard et al. (1988), Damgaard et al. (1984),
Gozzi (1983), Namiki et al. (1983), Batrouni et al. (1985), Rumpf (1986),
Ryang et al. (1985), Breit et al. (1984), Beveria (1997)]. In this approach
quantized field is described by a stochastic differential equation evolv-
ing in a fictitious time coordinate. Essentially, spatio-temporal Gaussian
white noise is added to the classical field equation in order to second quan-
tize it. The fictitious time is different from the physical time; it is an
additional parameter that is a useful tool for the quantization of classical
fields. Quantum mechanical expectations can be calculated as expectations
with respect to the realizations of the stochastic process. It is now pos-
sible to generate the spatio-temporal Gaussian white noise of the Parisi-
Wu approach by a weakly coupled chaotic dynamics on a very small scale.
In particular, if we choose e.g. Tchebyscheff maps to locally generate the
‘chaotic noise’, the convergence to Gaussian white noise under rescaling can
be proved rigorously [Beck et al. (1987), Billingsley (1968), Chernov (1995),
Beck (1990b), Beck (1995a), Chew et al. (2002), Zygmund (1959)]. If we
quantize by means of such a chaotic dynamics, no difference occurs on large
(standard-model) scales, since on large scales the chaotic behavior of the
maps is very well approximated by Gaussian white noise, leading to or-
dinary quantum field theoretical behavior. However, on very small scales
(e.g. the Planck scale or below) there are interesting differences and new
remarkable features. The view that the ultimate theory underlying quantum
mechanical behaviour on a small scale is a deterministic one exhibiting com-
plex behaviour has also been advocated by ‘t Hooft [‘t Hooft et al. (1992),
‘t Hooft (1997a), ‘t Hoot (1997b)].

How can a discrete chaotic noise dynamics arise from an ordinary field
theory? How can there be a dynamical origin of the noise? We will show
that ordinary continuum field theories with formally infinitely large self in-
teraction directly and intrinsically lead to diffusively coupled map lattices exhibiting spatio-temporal chaos. This limit of large couplings stands in certain analogy to the anti-integrable limit of Frenkel-Kontorova-like models [Aubry et al. (1990), Baesens et al. (1993)]. One of our main examples is a self-interacting scalar field of \( \phi^4 \)-type, which leads to diffusively coupled cubic maps in the anti-integrable limit. A discrete dynamics with strongest possible chaotic properties can then be obtained, which can be used for stochastic quantization. One can then consider coupled string-like objects in the noise space, which, to have a name in the following, will be called ‘chaotic strings’. We will use this model and some related ones as dynamical models of vacuum fluctuations. The chaotic dynamics will be scale invariant, similar as fully developed turbulent states in hydrodynamics exhibit a selfsimilar dynamics on a large range of scales [Bohr et al. (1998), Frisch (1995), Arad et al. (2001), Pope (2000), Ruelle (1982)]. In fact, chaotic strings behave very much like a turbulent quantum state. The probabilistic aspects of our model can be related to a generalized version of statistical mechanics, the formalism of nonextensive statistical mechanics [Tsallis (1988), Tsallis et al. (1998), Abe (2000), Abe et al. (2001), Beck (2001b), Beck et al. (2001), Plastino et al. (1995), Wilk et al. (2000), Pennini et al. (1995), Johal (1999), Cohen (2002)].

What can we learn from these types of statistical models? We will show that the assumption of a dynamical origin of vacuum fluctuations, due to chaotic strings on a small scale, can help to explain and reduce the large number of free parameters of the standard model. The guiding principle for this is the minimization of vacuum energy of the chaotic string. We will provide numerical evidence that the vacuum energy is minimized for certain distinguished string coupling constants. These couplings are numerically observed to coincide with running standard model couplings as well as with gravitational couplings, taking for the energy scales the masses of the known quarks, leptons, and gauge bosons. In this way our approach can help to understand many of the free parameters of the standard model, using concepts from generalized statistical mechanics.

The approach described in this book is new and different from previous attempts to calculate, e.g., the fine structure constant [Eddington (1948), Gilson (1996)]. It is much more in line with a suggestion made by R.S. MacKay in his book [MacKay (1993)] (p. 291), namely that the fine structure constant might be derived as a property of a fixed point of an appropriate renormalization operator. As we shall see in chapter 7, the relevant dynamical systems are indeed the chaotic strings, the renormalization operator is a
scale transformation, and the renormalization flow corresponds to an evolution equation for possible standard model couplings in the fictitious time of the Parisi-Wu approach. This renormalization flow is not only relevant for the fine structure constant but provides information on all the other standard model parameters as well.

The minima of the vacuum energy of chaotic strings can be determined quite precisely and allow for high-precision predictions of various running electroweak, strong, Yukawa and gravitational coupling constants. These can then be translated into high-precision estimates of the masses of the particles involved. Moreover, evolving the couplings to higher energies grand unification scenarios can be constructed. In this sense the approach described in this book yields an interesting amendment of the usual formulation of the standard model. Based on the assumption that chaotic noise strings exist in addition to the continuous standard model fields, we obtain high-precision predictions of the free parameters of the standard model (see Tab. 4 in chapter 12), which can be checked by experiments. Our chaotic models yield rapidly evolving dynamical models of vacuum fluctuations which, as we will show in detail in the following chapters, have minimum vacuum energy for the observed standard model parameters.

Can we further embed the chaotic strings into other theories, for example superstring and M-theory [Green et al. (1987), Kaku (1988), Polchinski (1998), Polchinski (1999), Witten (1997), Banks et al. (1997), Gauntlett (1998), Susskind (1996), Antoniadis et al. (1999a), Gubser et al. (2001)], or relate them to models of 2-dimensional quantum gravity [Gross et al. (1990)] or string cosmology [Ghosh et al. (2000), Melchiorri et al. (1999), Veneziano (1997), Lidsey (1998)]? Could the very recently established contact between string field theory and stochastic quantization yield a suitable embedding [Polyakov (2001), Baulieu et al. (2001), Periwal (2000), Ennyu et al. (1999)]? All this is possible but open at the moment. Generally it should be clear that chaotic strings are very different from superstrings. The latter ones evolve in a regular way, the former ones in a chaotic way. Still it is reasonable to look for possible connections with candidate theories of quantum gravity, such as superstring theory or M-theory. These theories require an extension of ordinary 4-dimensional space-time to 10 (or 11) space-time dimensions. The 6 extra dimensions are thought to be ‘compactified’, i.e. they are curled up on small circles with periodic boundary conditions. One possible way to embed chaotic strings is to assume that they live in the compactified space of superstring theory. The couplings of the chaotic
strings can then be regarded as a kind of inverse metric in the compactified space, determining the strength of the Laplacian coupling. The analogue of the Einstein equations as well as suitable scalar field equations then lead to the observed standard model coupling constants, fixed and stabilized as equilibrium metrics in the compactified space.

Let us give an overview over the following chapters. In chapter 1 we will generalize the stochastic quantization method to a chaotic quantization method, where the noise is generated by a discrete chaotic dynamics on a small time scale. In chapter 2 we will introduce chaotic strings and discuss some of their symmetry properties. Two types of vacuum energies associated with chaotic strings are discussed in chapter 3, namely the self energy and the interaction energy of chaotic strings. Spontaneous symmetry breaking phenomena for chaotic strings and their higher-dimensional extensions will be investigated in chapter 4. In chapter 5 we will show why chaotic strings can be regarded as simple selfsimilar dynamical models of vacuum fluctuations, and introduce webs of Feynman graphs that describe this physical interpretation. In chapter 6 we will relate the chaotic string dynamics to a thermodynamic description of the vacuum, using concepts from generalized statistical mechanics and information theory. In chapter 7 we will consider analogues of Einstein field equations that make a priori arbitrary standard model couplings evolve to the stable zeros of the interaction energy of chaotic strings. We will provide extensive numerical evidence that the smallest stable zeros of the interaction energy numerically coincide with running electroweak and strong coupling strengths, evaluated at the smallest fermionic and bosonic mass scales. In chapter 8 we will consider suitable self-interacting scalar field equations for possible standard model couplings, which make a priori arbitrary couplings evolve to the local minima of the self energy of the chaotic strings. We will present extensive numerical evidence that the self energy has local minima that numerically coincide with various Yukawa, gravitational, electroweak and strong couplings at energy scales given by masses of the three families of quarks and leptons. In chapter 9 we extend the analysis to bounded quark states, and provide numerical evidence that the total vacuum energy has minima for running strong coupling constants that correspond to the mass spectrum of light mesons and baryons. The precision results of chapter 7 and 8 will be used in chapter 10 to evolve the standard model couplings to much higher energies and to construct grand unification scenarios. In chapter 11 we will discuss the connection with extra dimensions and describe possible scenarios at the Planck
scale and beyond. Finally, chapter 12 is a detailed, self-contained summary of the most important concepts and results described in chapter 1-11.

CHAPTER 12 SUMMARY

12.1 Motivation and main results

A fundamental problem of elementary particle physics is the fact that there are about 25 free fundamental constants which are not understood on a theoretical basis. These constants are essentially the values of the three coupling constants, the quark and lepton masses, the $W$ and Higgs boson mass, and various mass mixing angles. An explanation of the observed numerical values is ultimately expected to come from a larger theory that embeds the standard model. Prime candidates for this are superstring and M-theory \cite{Green:1987, Kaku:1988, Polchinski:1998, Witten:1997, Banks:1997}. However, so far the predictive power of these and other theories is not large enough to allow for precise numerical predictions.

In this book we have developed a dynamical theory of vacuum fluctuations that may provide possible answers to this problem. We have found that there is a simple class of 1+1-dimensional strongly self-interacting discrete field theories (which, in order to have a name, we have called ‘chaotic strings’) that have a remarkable property. The expectation of the vacuum energy of these strings is minimized for string couplings that numerically coincide with running standard model or gravitational couplings $\alpha(E)$, the energy $E$ being given by the masses of the known quarks, leptons, and gauge bosons. Chaotic strings can thus be used to provide theoretical arguments why certain standard model parameters are realized in nature, others are not. It is natural to assume that the \textit{a priori} free parameters evolve to the local minima of the effective potentials generated by the chaotic strings. Out of the many possible vacua, chaotic strings seem to select the physically relevant states.

The dynamics of the chaotic strings is discrete in both space and time and exhibits strongest possible chaotic behaviour \cite{Beck:1995c}. It can be regarded as a dynamics of vacuum fluctuations that can be used to 2nd-quantize other fields, for example ordinary standard model fields, or ordinary strings, by dynamically generating the noise of the Parisi-Wu approach of stochastic quantization \cite{Parisi:1981, Damgaard:1987} on a very small scale. Mathematically, chaotic strings are 1-dimensional cou-
pled map lattices [Kaneko (1984)] of diffusively coupled Tchebyscheff maps $T_N$ of order $N$. The dynamics describes a kind of ‘turbulent quantum state’. It turns out that there are six different relevant chaotic string theories — similar to the six components that make up M-theory in the moduli space of superstring theory [Gauntlett (1998)]. We have labeled these six chaotic string theories as $2A, 2B, 2A^-, 2B^-, 3A, 3B$. Here the first number denotes the index $N$ of the Tchebyscheff polynomial and the letter A, B distinguishes between a forward and backward coupling form. The index $-$ denotes anti-diffusive coupling (alternating signs of Tchebyscheff polynomials in spatial direction). In principle one can study these string theories for arbitrary $N$, but for stochastic quantization only the cases $N = 2$ and $N = 3$ yield non-trivial behaviour in a first and second order perturbative approach [Hilgers et al. (1999b), Hilgers et al. (2001)].

Chaotic strings can be used to generate effective potentials for possible standard model couplings, regarding the $a$ priori free couplings as suitable scalar fields. The chaotic dynamics can be embedded into ordinary physics in various ways, ranging from a generalization of stochastic quantization (chapter 1) to an extension of statistical mechanics (chapter 6) and to a quantum gravity setting (chapter 11). Assuming that the $a$ priori free standard model couplings evolve to the minima of the effective potentials generated by the chaotic strings, one can obtain a large number of very precise predictions. The smallest stable zeros of the expectation of the interaction energy of the chaotic $3A$ and $3B$ strings are numerically observed to coincide with the running electroweak couplings at the smallest fermionic mass scales. Inverting the argument, the chaotic $3A$ string can be used to theoretically predict that the low-energy limit of the fine structure constant has the numerical value $\alpha_{el}(0) = 0.0072979(17) = 1/137.03(3)$, to be compared with the experimental value $1/137.036$. The $3B$ string predicts that the effective electroweak mixing angle is numerically given by $s_{12}^2 = \sin^2 \theta_{eff}^{\text{eff}} = 0.23177(7)$, in perfect agreement with the experimental measurements, which yield the value $s_{12}^2 = 0.23185(23)$ [Groom et al. (2000)]. The smallest stable zeros of the interaction energy of the $N = 2$ strings are observed to coincide with strong couplings at the smallest bosonic mass scales. In particular, the smallest stable zero of the interaction energy of the $2A$ string yields a very precise prediction of the strong coupling at the $W$ mass scale, which, if evolved to the $Z^0$ scale, corresponds to the prediction $\alpha_s(m_{Z^0}) = 0.117804(12)$. The current experimentally measured value is $\alpha_s(m_{Z^0}) = 0.1185(20)$ [Groom et al. (2000)].

Besides the coupling strengths of the three interactions, also the fermion
mass spectrum can be obtained with high precision from chaotic strings. Here the expectation of the self energy of the chaotic strings is the relevant observable. One observes a large number of string couplings that locally minimize the self energy and at the same time numerically coincide with various running electroweak, strong, Yukawa and gravitational couplings, evaluated at the mass scales of the higher fermion families. The highest precision predictions for fermion masses comes from the self energy of the $2A$ and $2B$ strings, which is observed to exhibit minima for string couplings that coincide with gravitational and Yukawa couplings of all known fermions. These minima of the vacuum energy yield the free masses of the six quarks as $m_u = 5.07(1) \text{ MeV}$, $m_d = 9.35(1) \text{ MeV}$, $m_s = 164.4(2) \text{ MeV}$, $m_c = 1.259(4) \text{ GeV}$, $m_b = 4.22(2) \text{ GeV}$ and $m_t = 164.5(2) \text{ GeV}$. Note that a free top mass prediction of 164.5(2) GeV corresponds to a top pole mass prediction of 174.4(3) GeV, in very good agreement with the experimentally measured value $M_t = 174.3 \pm 5.1 \text{ GeV}$. The masses of the charged leptons come out as $m_e = 0.5117(8) \text{ MeV}$, $m_\mu = 105.6(3) \text{ MeV}$ and $m_\tau = 1.782(7) \text{ GeV}$. All these theoretically obtained values of fermion masses are in perfect agreement with experimental measurements. To the best of our knowledge, there is no other theoretical model that has achieved theoretical predictions of similar precision. Chaotic strings also provide evidence for massive neutrinos, and yield concrete predictions for the masses of the neutrino mass eigenstates $\nu_1, \nu_2, \nu_3$. These are $m_{\nu_1} = 1.452(3) \cdot 10^{-5} \text{ eV}$, $m_{\nu_2} = 2.574(3) \cdot 10^{-3} \text{ eV}$, $m_{\nu_3} = 4.92(1) \cdot 10^{-2} \text{ eV}$ (our symmetry considerations would also allow the mass $m_{\nu_2}$ to be smaller by a factor $1/8$).

Not only fermion masses, but also boson masses can be obtained from chaotic strings. The $2A$ string correctly reproduces the masses of the $W$ and $Z$ boson, and a suitable interpretation of the $2B^-$ string dynamics provides evidence for the existence of a scalar particle of mass $m_H = 154.4(5) \text{ GeV}$, which could be identified with the Higgs particle. The latter mass prediction is slightly larger than supersymmetric expectations but well within the experimental bounds based on the ordinary standard model. We also obtain estimates of the lightest glueball masses, which are consistent with estimates from lattice QCD.
12.2 The chaotic string dynamics

From a nonlinear dynamics point of view, chaotic strings are easily introduced. They are 1-dimensional coupled map lattices of diffusively coupled Tchebyscheff maps. In fact, for somebody with a background in dynamical systems the dynamics is a straightforward standard example of a spatially extended dynamical system exhibiting chaotic behaviour. On the other hand, for somebody working in high energy physics the equation may first look somewhat unfamiliar, but the results summarized in the previous section certainly indicate that it is worth learning new things.

Consider a 1-dimensional lattice with lattice sites labeled by an integer \( i \). At each lattice site \( i \) there is a variable \( \Phi_i \) that takes on values in the interval \([-1, 1] \). \( n \) is a discrete time variable. Given some initial value \( \Phi_i^0 \) the time evolution is given by deterministic recurrence relation

\[
\Phi_{i+1}^n = T_N(\Phi_i^n).
\]  

(1)

Here \( T_N(\Phi) \) is the \( N \)-th order Tchebyscheff polynomial. One has \( T_2(\Phi) = 2\Phi^2 - 1 \) and \( T_3(\Phi) = 4\Phi^3 - 3\Phi \), generally \( T_N(\Phi) = \cos(N \arccos \Phi) \). The Tchebyscheff maps \( T_N \) with \( N \geq 2 \) are known to exhibit strongly chaotic behaviour. There is sensitive dependence on initial conditions: Small perturbations in the initial values will lead to completely different trajectories in the long-term run. The maps are conjugated to a Bernoulli shift with an alphabet of \( N \) symbols. This means, in suitable coordinates the iteration process is just like shifting symbols in a symbol sequence (see section 1.6 or any textbook on dynamical systems for more details). As shown in [Beck (1991a)] Tchebyscheff maps have least higher-order correlations among all systems conjugated to a Bernoulli shift, and are in that sense closest to Gaussian white noise, though being completely deterministic. A graph theoretical method for this type of ‘deterministic noise’ has been developed in [Beck (1991a), Hilgers et al. (2001)].

We now couple the Tchebyscheff dynamics with a small coupling \( a \) in the spatial direction labeled by \( i \), obtaining the chaotic string dynamics:

\[
\Phi_{i+1}^n = (1 - a)T_N(\Phi_i^n) + s \frac{a}{2}(T_N^b(\Phi_{i-1}^n) + T_N^b(\Phi_{i+1}^n))
\]  

(2)

We consider both the positive and negative Tchebyscheff polynomial \( T_{b,N}(\Phi) = \pm T_N(\Phi) \), but have suppressed the index \( \pm \) in the above equation. The variable \( a \) is a coupling constant taking values in the interval \([0, 1]\). \( s \) is a sign
variable taking on the values ±1. The choice \( s = +1 \) is called ‘diffusive coupling’, but for symmetry reasons it also makes sense to study the choice \( s = -1 \), which we call ‘anti-diffusive coupling’. The integer \( b \) distinguishes between the forward and backward coupling form, \( b = 1 \) corresponds to forward coupling \((T_N^1(\Phi) := T_N(\Phi))\), \( b = 0 \) to backward coupling \((T_N^0(\Phi) := \Phi)\).

We consider periodic boundary conditions and large lattices of size \( i_{\text{max}} \).

The dynamics (2) is deterministic chaotic, spatially extended, and strongly nonlinear. The field variable \( \Phi^i_n \) is physically interpreted in terms of rapidly fluctuating virtual momenta in units of some arbitrary maximum momentum scale. There are some analogies with velocity fluctuations in fully developed turbulent flows, which are also deterministic chaotic, spatially extended, and induced by strong nonlinearities. For this reason it makes sense to think of eq. (2) as describing a turbulent state of vacuum fluctuations, or in short a ‘turbulent quantum state’. These states appear to have physical relevance, since they reproduce the observed values of the standard model parameters.

We may also write the coupled dynamics as

\[
\Phi^i_{n+1} = T_N(\Phi^i_n) + \frac{a}{2} \left( sT_N^b(\Phi^i_{n-1}) - 2T_N(\Phi^i_n) + sT_N^b(\Phi^{i+1}_n) \right). 
\] (3)

This way of writing illustrates that the effect of the coupling is similar to the action a Laplacian operator. Since \( a \) determines the strength of the Laplacian coupling, and since in quantum field theories this role is usually attributed to a metric, \( a^{-1} \) can be regarded as a kind of metric in the 1-dimensional string space indexed by \( i \).

It is easy to see that for odd \( N \) the statistical properties of the coupled map lattice are independent of the choice of \( s \) (since odd Tchebysheff maps satisfy \( T_N(-\Phi) = -T_N(\Phi) \)), whereas for even \( N \) the sign of \( s \) is relevant and a different dynamics arises if \( s \) is replaced by \( -s \). Hence, restricting ourselves to \( N = 2 \) and \( N = 3 \), in total 6 different chaotic string theories arise, characterized by \((N, b, s) = (2, 1, +1), (2, 0, +1), (2, 1, -1), (2, 0, -1)\) and \((N, b) = (3, 1), (3, 0)\). For easier notation, we have labeled these string theories as \( 2A, 2B, 2A^-, 2B^-, 3A, 3B \), respectively.

If the coupling \( a \) is sufficiently small, the chaotic variables \( \Phi^i_n \) can be used to generate the noise of the Parisi-Wu approach of stochastic quantization [Parisi et al. (1981), Damgaard et al. (1988)] on a very small scale. That is to say, we assume that on a very small (quantum gravity) scale the noise used for quantization purposes is not structureless but evolves in a deterministic chaotic way. Just on a large scale it looks like Gaussian white noise, and there
is convergence to ordinary path integrals using this ‘deterministic noise’, as can be rigorously proved for $a = 0$ [Beck et al. (1987), Beck (1995a), Beck (1995b)]. In this interpretation the discrete time variable $n$ corresponds to the fictitious time of the Parisi-Wu approach, an artificial time coordinate that is just introduced for quantization purposes (see chapter 1 for details).

The chaotic string dynamics (3) formally originates from a 1-dimensional continuum $\phi^{N+1}$-theory in the limit of infinite self-interaction strength (see section 2.2 for details). In this sense, chaotic strings can also be regarded as degenerated Higgs-like fields with infinite self-interaction parameters, which are constraint to a 1-dimensional space. Another way to physically motivate chaotic strings is to emphasize certain analogies with ordinary strings (section 2.4), to connect them with fluctuating momenta that are allowed due to the uncertainty relation (sections 5.1-5.3), and to relate them to a 1+1 dimensional model of quantum gravity (section 5.7). One can also embed them into the compactified space of string theories (section 11.1).

### 12.3 Vacuum energy of chaotic strings

Though the chaotic string dynamics is dissipative, one can formally introduce potentials that generate the discrete time evolution. For $a = 0$ we may write

$$\Phi_{n+1} - \Phi_n = \pm T_N(\Phi_n) = -\frac{\partial}{\partial \Phi} V_\pm(\Phi_n).$$  

(4)

For $N = 2$ the (formal) potential is given by

$$V_\pm^{(2)}(\Phi) = \pm \left(-\frac{2}{3} \Phi^3 + \Phi\right) + \frac{1}{2} \Phi^2 + C,$$

(5)

for $N = 3$ by

$$V_\pm^{(3)}(\Phi) = \pm \left(-\Phi^4 + \frac{3}{2} \Phi^2\right) + \frac{1}{2} \Phi^2 + C.$$

(6)

Here $C$ is an arbitrary constant. The uncoupled case $a = 0$ is completely understood. The dynamics is ergodic and mixing. Any expectation of an observable $A(\Phi)$ can be calculated as

$$\langle A \rangle = \int_{-1}^{1} \rho(\phi) A(\phi).$$

(7)
where
\[ \rho(\phi) = \frac{1}{\pi \sqrt{1 - \phi^2}} \] (8)
is the natural invariant density describing the probability distribution of iterates of Tchebyscheff maps (see, e.g., Beck et al. (1993) for an introduction). In more general versions of statistical mechanics [Tsallis (1988)], this probability density can be regarded as a generalized canonical distribution (see section 6.4 for more details).

If a spatial coupling $a$ is introduced, things become much more complicated, and the invariant 1-point density deviates from the simple form (8). A spatial coupling is formally generated by the interaction potential $aW_{\pm}(\Phi, \Psi)$, with
\[ W_{\pm}(\Phi, \Psi) = \frac{1}{4}(\Phi \pm \Psi)^2 + C. \] (9)
Here $\Phi$ and $\Psi$ are neighbored noise field variables on the lattice. One has
\[ -\frac{\partial}{\partial \Phi} W_{\pm}(\Phi^i, \Phi^{i+1}) - \frac{\partial}{\partial \Phi} W_{\pm}(\Phi^i, \Phi^{i-1}) = \pm \frac{1}{2}\Phi^{i+1} - \Phi^i \pm \frac{1}{2}\Phi^{i-1}. \] (10)
This generates diffusive (+), respectively anti-diffusive (−) coupling. Anti-diffusive coupling can equivalently be obtained by keeping $W_-$ but replacing $T_N \rightarrow -T_N$ at odd lattice sites. The coupled map dynamics (2) is obtained by letting the action of $V$ and $W$ alternate in discrete time $n$, then regarding the two time steps as one.

The expectations of the potentials $V$ and $W$ yield two types of vacuum energies $V_{\pm}(a) := \langle V_{\pm}^{(N)}(\Phi^i) \rangle$ (the self energy) and $W_{\pm}(a) := \langle W_{\pm}(\Phi^i, \Phi^{i+1}) \rangle$ (the interaction energy). Here $\langle \cdots \rangle$ denotes the expectation with respect to the coupled chaotic dynamics. Numerically, any such expectation can be determined by averaging over all $i$ and $n$ for random initial conditions $\Phi^i_0 \in [-1, 1]$, omitting the first few transients. Note that in the stochastic quantization approach the chaotic noise is used for 2nd quantization of standard model fields (or ordinary strings) via the Parisi-Wu approach [Beck (1995c)]. Hence generally expectations with respect to the chaotic dynamics correspond to expectations with respect to 2nd quantization. The expectation of the vacuum energy of the string, given by the above functions $W_{\pm}(a)$ and $V_{\pm}(a)$, depends on the coupling $a$ in a non-trivial way. Moreover, it also depends on the integers $N, b, s$ that define the chaotic string theory.

Since negative and positive Tchebyscheff maps essentially generate the same dynamics, up to a sign, any physically relevant observable should be
invariant under the transformation $T_N \to -T_N$. The vacuum energies $V_\pm(a)$ and $W_\pm(a)$ of the various strings exhibit full symmetry under the transformation $T_N \to -T_N$ (respectively $s \to -s$) if the additive constant $C$ is chosen to be

$$C = -\frac{1}{2} \langle \Phi^2 \rangle. \quad (11)$$

For that choice of $C$, the expectations of $V_+$ and $V_-$ as well as those of $W_-$ and $W_+$ are the same, up to a sign.

Choosing (by convention) the $+$ sign one obtains from eq. (5), (6) and (9) for the expectations of the potentials

$$V^{(2)}(a) = -\frac{2}{3} \langle \Phi^3 \rangle + \langle \Phi \rangle \quad (12)$$

$$V^{(3)}(a) = -\langle \Phi^4 \rangle + \frac{3}{2} \langle \Phi^2 \rangle, \quad (13)$$

and

$$W(a) = \frac{1}{2} \langle \Phi^i \Phi^{i+1} \rangle. \quad (14)$$

The above ± symmetry can actually be used to cancel unwanted vacuum energy and to avoid problems with the cosmological constant. If one assumes that strings with both $T_N$ and $-T_N$ are physically relevant, the two contributions $V(a)$ and $-V(a)$ (respectively $W(a)$ and $-W(a)$) may simply add up to zero. This reminds us of similarly good effects that supersymmetric partners have in ordinary quantum field and string theories. In fact, as a working hypothesis we may regard the above $Z_2$ symmetry as representing a kind of supersymmetry in the chaotic noise space.

Similarly as for ordinary strings it also makes sense to consider certain conditions of constraints for the chaotic string. For ordinary strings (for example bosonic strings in covariant gauge [Green et al. (1987)]) one has the condition of constraint that the energy momentum tensor should vanish. The first diagonal component of the energy momentum tensor is an energy density. For chaotic strings, the evolution in space $i$ is governed by the potential $W_\pm(\Phi, \Psi)$ and the corresponding expectation of the energy density is $\pm W(a)$. We should thus impose the condition of constraint that $W(a)$ should vanish for physically observable states. Moreover, the evolution in fictitious time $n$ is governed by the self-interacting potential $V^{(N)}(\Phi)$. This potential generates a shift of information, since the Tchebyscheff maps $T_N$ are conjugated to a Bernoulli shift of $N$ symbols. Hence $V(a)$ can be regarded as the expectation
of a kind of information potential or entropy function, which, motivated by thermodynamics, should be extremized for physically observable states. Note that the action of $V$ and $W$ alternates in $n$ and $i$ direction. Both types of vacuum energies describe different relevant observables of the chaotic string and are of equal importance.

12.4 Fixing standard model parameters

In order to construct a link to standard model phenomenology it is useful to introduce a simple physical interpretation of the chaotic string dynamics in terms of fluctuating virtual momenta. Suppose we regard $\Phi^i_n$ to be a fluctuating virtual momentum component that can be associated with a hypothetical particle $i$ at time $n$ that lives in the constraint 1-dimensional string space. $n$ can be either interpreted as fictitious time or as physical time, it doesn’t really matter for our purposes. Neighbored particles $i$ and $i-1$ can exchange momenta due to the Laplacian coupling of the coupled map lattice. To make this model more concrete we may assume that at each time step $n$ a fermion-antifermion pair $f_1, \bar{f}_2$ is spontaneously created in cell $i$. In units of some arbitrary energy scale $p_{\text{max}}$, the particle has momentum $\Phi^i_n$, the antiparticle momentum $-\Phi^i_n$. They interact with particles in neighbored cells by exchange of a (hypothetical) gauge boson $B_2$, then they annihilate into another boson $B_1$ until the next vacuum fluctuation takes place. This can be symbolically described by the Feynman graph in Fig. 1. We called this graph a ‘Feynman web’, since it describes an extended spatio-temporal interaction state of the string, to which we have given a standard model-like interpretation. Note that in this interpretation $a$ is a (hypothetical) standard model coupling constant, since it describes the strength of momentum exchange of neighbored particles. At the same time, $a$ can also be regarded as an inverse metric in the 1-dimensional string space, since it determines the strength of the Laplacian coupling.

What is now observed numerically for the various chaotic strings is that the interaction energy $W(a)$ has zeros and the self energy $V(a)$ has local minima for string couplings $a$ that numerically coincide with running standard model couplings $\alpha(E)$, the energy being given by

$$ E = \frac{1}{2} N \cdot (m_{B_1} + m_{f_1} + m_{f_2}). $$

Here $N$ is the index of the chaotic string theory considered, and $m_{B_1}, m_{f_1}, m_{f_2}$.
Figure 1: Interpretation of the coupled map dynamics in terms of fluctuating momenta exchanged by fermions $f_1, \bar{f}_2$ and bosons $B_1, B_2$. 
Figure 2: Alternative interpretation of the coupled map dynamics. Black holes (denoted by $B$) emit low-energy particles $f_1, \bar{f}_2$ that interact with gauge bosons $B_2$ at low temperatures.

denote the masses of the standard model particles involved in the Feynman web interpretation. The surprising observation is that rather than yielding just some unknown exotic physics, the chaotic string spectrum appears to reproduce the masses and coupling constants of the known quarks, leptons and gauge bosons of the standard model with very high precision. Gravitational and Yukawa couplings are observed as well.

We thus have the possibility to fix and predict the free standard model parameters by simply assuming that the entire set of parameters is chosen in such a way that it minimizes the vacuum energy of the chaotic strings.

Although eq. (15) looks like a low-energy formula, the chaotic strings may well describe a scenario that takes place at very large energies well above the Planck scale. The reason is that alternatively the Feynman web dynamics can also be interpreted in terms of black holes that emit low-energy particles by Hawking radiation, similar to the process sketched in Fig. 2. In this case a very large energy is given by something of the order of the mass $M$ of
the black holes, but what is physically relevant is the Hawking temperature 
\[ kT_H \sim \frac{1}{GM} \]
by which the black holes radiate. Here \( G \) is the gravitational constant. Thus very large energies \( M \) can be associated with very small Hawking temperatures \( kT_H \) and allow for fixing of low-energy parameters in a pre-Planck epoch (see section 11.5 for more details).

At a very early stage of the universe, where standard model parameters are not yet fixed and ordinary space-time may not yet exist as well, pre-standard model couplings may be realized as coupling constants \( a \) in the chaotic string space. The parameters are then fixed by an evolution equation (a renormalization flow) of the form

\[ \dot{a} = \text{const} \cdot W(a) + \text{noise}, \tag{16} \]

respectively

\[ \dot{a} = -\text{const} \cdot \frac{\partial V}{\partial a} + \text{noise}, \tag{17} \]

where we assume that the constant \( \text{const} \) is positive. It can also depend on \( a \), this doesn’t matter as long as it does not switch sign. The equations make the expectations of \( a \) evolve to the stable zeros of \( W(a) \), respectively to the local minima of \( V(a) \). Eq. (16) is a kind of Einstein equation and eq. (17) a kind of scalar self-interacting field equation for \( a \) (see sections 7.1, 8.1, 11.2 for more details).

### 12.5 Numerical findings

Our numerical results for zeros of \( W(a) \) and local minima of \( V(a) \) and the corresponding physical interpretations are described in detail in chapter 7-10. Let us here just summarize the most important points.

The smallest non-trivial zeros of the interaction energy \( W(a) \) with negative slope at the zero, describing stable stationary states of couplings under the evolution equation (16), are listed in Tab. 1. The \( N = 3 \) zeros are observed to numerically coincide with the coupling constants of electroweak interactions, evaluated at four different energy scales. The four relevant energy scales are given by the masses of the lightest fermions \( d, e, u, \nu_e \). To calculate the concrete value of the energy scale, one uses eq. (15). For example, the zero \( a_2^{(3A)} \) is associated with a Feynman web of the form \( f_1 = e^-, \bar{f}_2 = e^+, B_1 \) massless, \( B_2 = \gamma \). This yields \( E = \frac{3}{2}(2m_e + m_{B_1}) = 3m_e \). Tab. 1 shows that there is coincidence of all four observed stable zeros with the corresponding
Table 1: The smallest stable zeros of the interaction energies of the $3A, 3B, 2A, 2B, 2A^-, 2B^-$ strings and comparison with experimentally measured standard model couplings (where known).

| string | stable zero | running SM coupling |
|--------|-------------|---------------------|
| $3A$   | $a_1^{(3A)} = 0.0008164(8)$ | $\alpha_{el}^d(3m_d) = 0.0008166$ |
| $3A$   | $a_2^{(3A)} = 0.0073038(17)$ | $\alpha_{el}^e(3m_e) = 0.007303$ |
| $3B$   | $a_1^{(3B)} = 0.0018012(4)$ | $\alpha_{weak}^u(3m_u) + \alpha_{el}^d(3m_d) = 0.001800$ |
| $3B$   | $a_2^{(3B)} = 0.017550(1)$ | $\alpha_{weak}^\nu(3m_\nu) + \alpha_{el}^e(3m_e) = 0.01755$ |
| $2A$   | $a_1^{(2A)} = 0.120093(3)$ | $\alpha_s(m_W + 2m_d) = 0.1208(20)$ |
| $2B$   | $a_1^{(2B)} = 0.3145(1)$ | $\alpha_s(m_{gg}^{+++} + 2m_u) = ?$ |
| $2A^-$ | $a_1^{(2A^-)} = 0.1758(1)$ | $\alpha_s(m_{gg}^{+++} + 2m_b) = ?$ |
| $2B^-$ | $a_1^{(2B^-)} = 0.095370(1)$ | $\alpha_s(m_H + 2m_t) = ?$ |

Symmetry considerations with the $N = 3$ strings suggest that the smallest stable zeros of the $N = 2$ strings fix strong couplings at the smallest bosonic mass scales. These are given by the $W$ boson, the Higgs boson, and the lightest glueballs of spin 0 and 2. Since only the $W$ boson mass is precisely known, we can only compare the zero $a_1^{(2A)}$ with experimental data. It indeed coincides with $\alpha_s(m_W)$. The other zeros yield predictions for the Higgs and glueball masses (details in section 7.6 and 7.7).

The observed zeros of the $N = 2$ and $N = 3$ strings allow for high-precision predictions of the fine structure constant, the Weinberg angle and the strong coupling constant. The fine structure constant and the effective Weinberg angle for $Z^0$-lepton coupling are correctly obtained with a precision of 4-5 digits. The strong coupling constant at the $W$ mass scale is predicted with about 5 valid digits, a much higher precision than can be confirmed by experiments at the present stage. $\alpha_s(E)$ can then be easily evaluated at other energy scales as well, using the well-known QCD formulas.

Generally, we observe interesting symmetries between the various zeros and the corresponding standard model interactions. These kind of symmetries often help to fill ‘gaps’ in the table, i.e. to find the right Feynman web interpretation for a given zero. For the $N = 3$ strings, proceeding from the
smallest stable zeros \(a_1^{(3A/3B)}\) to the next larger stable zeros \(a_2^{(3A/3B)}\) means replacing quarks by leptons, whereas for the \(N = 2\) string it means replacing ordinary gauge and Higgs bosons \((W^\pm\) and \(H\)) by their Planck-scale analogues \((\tilde{W}^\pm\) and \(\tilde{H}\), see section 10.6). For the \(N = 3\) strings, the forward coupling form \((3A)\) describes just one species of fermions (either \(d\) or \(e\)). These fermions certainly have spin. On the other hand, the backward coupling form \((3B)\) describes simultaneous states of two fermions \(((u_R, d_L)\) and \((\nu_L, e_R))\) of opposite handedness, which coexist independently. Hence proceeding from forward to backward coupling means proceeding from a state with spin to a state with total spin 0, since \(R + L = 0\). For the \(N = 2\) strings proceeding from forward to backward coupling also means going from a state with spin (the \(W\)-boson, the glueball with spin 2) to a state without spin (the Higgs boson, the glueball with spin 0).

Now let us look at the other relevant type of vacuum energy, the self energy \(V(a)\). The functions \(V(a)\) have plenty of local minima for the various strings. For small \(a\), oscillating scaling behaviour sets in and all local minima of \(V(a)\) are only fixed modulo \(N^2\) (details in section 8.6). Also, in this limit there is no visible difference between the self energies of the forward and backward coupling form. The self energy of the \(2A/B\) string turns out to have local minima for Yukawa couplings of heavy fermions \((t, b, \tau, c)\) and gravitational couplings of light fermions \((\mu, s, d, u, e)\). This is shown in Tab. 2. The Yukawa coupling of the \(t\) quark falls out of the scaling region and is separately listed in Tab. 3.

Clearly, since the minima of the vacuum energy can be determined with high precision, and since gravitational and Yukawa couplings depend quadratically on the mass, the minima allow for high-precision predictions of masses modulo 2, in particular of quark masses. Note that most of the values listed in column 1 of Tab. 2 have a much higher precision than the experimental values in column 3. There are further minima (not listed in Tab. 2) that can be used to make neutrino mass predictions (see section 8.7). For the attribution of the various minima to the various particles one uses simple discrete symmetry considerations.

For all states in the scaling region, the relevant power of 4 of the coupling (equivalent to a power of 2 for the mass) is \textit{a priori} undetermined. It can be theoretically related to a folding number of the string in a quantum gravity epoch (section 11.6). In this epoch all masses are only fixed modulo 2. For standard model states, however, the relevant power of 2 can be deduced by
Table 2: Local minima of the self energy of the 2A/B string in the scaling region and comparison with gravitational and Yukawa couplings of standard model particles. For the experimental quark mass values that lead to the numbers and error bars in column 3 we have chosen $m_t = 164(5)$ GeV, $m_b = 4.22(4)$ GeV, $m_c = 1.26(3)$ GeV, $m_s = 167(7)$ MeV, $m_d = 9.1(6)$ MeV, $m_u = 4.7(9)$ MeV.

Outside the scaling region, lots of other minima are observed that can directly be identified with standard model interaction strengths (Tab. 3). For the $N = 3$ strings we observe minima describing strongly interacting heavy quarks $q$, i.e. $f_1 = q$, $f_2 = q$, $B_1$ massless, $B_2 = g$ (gluon), where $q = t, b, c$, respectively. Pure flavor states are described by forward coupling and mixed flavor states ($t, b$) and ($c, s$) by backward coupling. Further minima can be identified with weak interaction states of right-handed fermions, fixing the three different charges of charged quarks and leptons. Small differences between the forward and backward coupling form may possibly provide information on mass mixing angles, i.e. the entries of the Kobayashi-Maskawa matrix. Between small (weak) and large (strong) couplings we have identified the remaining two minima as describing unified couplings at the GUT and Planck scale (see chapter 10).

Summarizing, the two types of vacuum energy and the six types of chaotic strings considered seem to be sufficient to fix the most relevant parameters of the standard model such as masses, charges and coupling constants. The standard model appears to have evolved to a state of minimum vacuum...
Table 3: Various observed local minima of the self energy of the $3A, 3B, 2A, 2B$ strings outside the scaling region and comparison with running standard model couplings. The running strong coupling $\alpha_s(E)$ can be very precisely evaluated using the zero $a_1^{(2A)}$ listed in Tab. 1.

One can also study the total vacuum energy of the strings as given by $H_+(a) = V(a) + aW(a)$ and $H_-(a) = V(a) - aW(a)$. It turns out that for the $N = 3$ strings local minima of $H_-(a)$ essentially reproduce the spectrum of light mesons (Tab. 5) and local minima of $H_+(a)$ essentially that of light baryons (Tab. 6).

As explained in section 9.1, the relevant energy scales for mesons $M$ are
Table 4: Most important standard model parameters as obtained from the chaotic string spectrum and as measured in experiments. All quark masses denote free quark masses. $m_t = 164.5(2)$ GeV corresponds to a top pole mass of $M_t = 174.4(3)$ GeV. Symmetry considerations also allow $m_{\nu_2}$ to be smaller by a factor $\frac{1}{8}$ (see section 8.7).
Table 5: Local minima of the total vacuum energy $H_-(a)$ of the $N = 3$ strings and comparison with strong couplings at energy levels given by mesonic states.

| string | local minimum of $H_-(a)$ | strong coupling |
|--------|---------------------------|-----------------|
| 3A     | 0.163                     | $\alpha_s\left(\frac{3}{2}m_{\Upsilon(1s)}\right) = 0.164$ |
| 3A     | 0.186                     | $\alpha_s\left(\frac{3}{2}m_B\right) = 0.188$ |
| 3A     | 0.282                     | $\alpha_s\left(\frac{3}{2}m_\Phi\right) = 0.282$ |
| 3A     | 0.375                     | $\alpha_s\left(\frac{3}{2}m_{K^+}\right) = 0.374$ |
| 3A     | 0.416                     | $\alpha_s\left(\frac{3}{2}m_\rho\right) = 0.418$ |
| 3A     | 0.615                     | $\alpha_s\left(\frac{3}{2}m_\eta\right) = 0.60$ |
| 3A     | 0.676                     | $\alpha_s\left(\frac{3}{2}m_K\right) = 0.70$ |
| 3B     | 0.161                     | $\alpha_s\left(\frac{3}{2}m_{\Upsilon(2s)}\right) = 0.161$ |
| 3B     | 0.182                     | $\alpha_s\left(\frac{3}{2}m_{B_c}\right) = 0.179$ |
| 3B     | 0.224                     | $\alpha_s\left(\frac{3}{2}m_D\right) = 0.224$ |
| 3B     | 0.290                     | $\alpha_s\left(\frac{3}{2}m_{\eta'}\right) = 0.291$ |
| 3B     | 0.341                     | $\alpha_s\left(\frac{3}{2}m_\Phi\right) = 0.344$ |
| 3B     | 0.356                     | $\alpha_s\left(\frac{3}{2}m_{\eta'}\right) = 0.356$ |
| 3B     | 0.412                     | $\alpha_s\left(\frac{3}{2}m_\omega\right) = 0.413$ |

Table 6: Local minima of the total vacuum energy $H_+(a)$ of the $N = 3$ strings and comparison with strong couplings at energy levels given by baryonic states.

| string | local minimum of $H_+(a)$ | strong coupling |
|--------|---------------------------|-----------------|
| 3A     | 0.233                     | $\alpha_s\left(\frac{3}{2}m_{\Delta_b}\right) = 0.233$ |
| 3A     | 0.282                     | $\alpha_s\left(\frac{3}{2}m_{\Lambda_b}\right) = 0.282$ |
| 3A     | 0.508                     | $\alpha_s\left(\frac{3}{2}m_{p,n}\right) = 0.508$ |
| 3A     | 0.609                     | $\alpha_s\left(\frac{3}{2}m_{\Delta}\right) = 0.60$ |
| 3A     | 0.628                     | $\alpha_s\left(\frac{3}{2}m_{\Sigma}\right) = 0.63$ |
| 3A     | 0.70                      | $\alpha_s\left(\frac{3}{2}m_{\Lambda}\right) = 0.68$ |
| 3B     | 0.356                     | $\alpha_s\left(\frac{3}{2}m_{N(1440)}\right) = 0.356$ |
| 3B     | 0.367                     | $\alpha_s\left(\frac{3}{2}m_{\Sigma^*}\right) = 0.366$ |
| 3B     | 0.378                     | $\alpha_s\left(\frac{3}{2}m_{\Xi}\right) = 0.379$ |

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given by $\frac{3}{2}m_M$ and $\frac{9}{4}m_M$; those for baryons $B$ by $\frac{2}{3}m_B$ and $m_B$. Apparently, for confined states the total vacuum energy $H_\pm(a)$ is the relevant quantity to look at, rather than the single vacuum energies $V(a)$ and $W(a)$.

Another interesting observation is that from the chaotic string spectrum there is no straightforward evidence for supersymmetric particles with masses in the region 100-1000 GeV. There are simply no minima observed in the relevant energy region! Also, the observed minima of the vacuum energy of the strings seem to support grand unification scenarios based on the non-supersymmetric beta functions, rather than the supersymmetric ones. More on supersymmetry and possible grand unification scenarios can be found in chapter 10.

In total, we found more than 30 zeros and minima that could be identified with known standard model and gravitational interaction strengths in a straightforward way. Moreover, more than 20 minima could be identified with hadronic states. Could all this be a random coincidence? It can’t. Let us estimate the probability to obtain all this by a pure random coincidence. In Tab. 1–3 we observe some 20 string couplings that coincide with experimentally measured standard model coupling constants with a precision of 3-4 digits. The joint probability for this being the result of a pure random coincidence is of the order $(10^{-3})^{20} = 10^{-60}$. Here we still haven’t taken into account the hadronic minima of Tab. 5 and 6. Even if we allow for different attributions of the minima to various possible standard model interaction states, the joint probability of obtaining randomly a joint coincidence of this order of magnitude is still extremely small. In other words, a random coincidence can be excluded. For this reason the chaotic string dynamics needs to be integrated into future theories of particle physics in one way or another, there is no way around it. The question of real interest is whether chaotic strings are already the exact and complete theory, or whether they are just the beginning (and perhaps the perturbative approximation) of a more advanced theory.

12.6 Physical embedding

We saw that the chaotic strings resemble the mass spectrum of elementary particles in a correct way. That means, although a complete picture certainly requires the field equations of QED, QCD, weak interactions, Einstein’s gravity, and possibly superstring theory, the information on masses and coupling
strengths at a certain energy scale is correctly encoded by the strings. The chaotic strings seem to contain the most important information on what is going on at a certain temperature. In fact they yield information on parameters that is not directly given by the other theories, thus providing a nice and necessary amendment.

How can we understand the general role of these strings? Let us become a bit philosophical and make a simple comparison. Suppose you want to build a house. Before you can start, an architect has to draw a plan of the house. The plan contains the most important information about the house, but is certainly not the house itself. Moreover, the plan is 2-dimensional, whereas the house is 3-dimensional. Also, the plan is drawn before the house is being built. In our case, the plan are the chaotic strings and the house is the universe. The strings are 1+1 dimensional, the universe is 4 or 11-dimensional. The chaotic strings presumably fix low-energy standard model parameters of the universe already at an extremely early stage, long before standard model field equations and a 4-dimensional space-time become relevant. The different dimensionalities somewhat remind us of the holographic principle [Susskind (1995), Maldacena (1998), Polyakov (2001)], which relates degrees of freedom of quantum field theories in different dimensions.

Keeping on being philosophical, we may even see some analogies with biological systems. The DNA string encodes the most important information about a living being, in fact already long before this living being is being born. This information is encoded using symbol sequences made up of 4 different symbols, which are the chemical substances called Adenine, Cytosine, Guanine, and Thymine. So for biological systems a kind of \( N = 4 \) string is realized. The DNA string is certainly not the living being itself, but it contains the most important information about it. In that sense, we may regard the chaotic string as a kind of ‘DNA’ string of the universe. It encodes the most important information on the universe, and this may be relevant already long before ordinary space-time is being created. What chemistry is for biology, is information theory for physics.

Still one may ask how to concretely embed the chaotic string dynamics into ordinary physics and possible theories of quantum gravity. Several approaches are possible, all of which are somewhat related.

The first and simplest approach is to regard the chaotic dynamics as a new and \textit{a priori} standard model-independent dynamics of virtual momenta. We have shown in chapter 1 that it effectively reduces to spatio-temporal Gaussian white noise if seen on a large scale. This noise can then be coupled
to the classical standard model field equations and, in fact, can be used to generate the noise fields of the stochastic quantization method. So the strings are embedded as a tool for quantization. The new thing is that the dynamics used for quantization is a deterministic chaotic one, rather than a purely random one.

The second approach is to relate the chaotic dynamics to an effective thermodynamic description of vacuum fluctuations allowed by the uncertainty relation. Clearly ordinary statistical mechanics is not valid for a description of vacuum fluctuations, but the chaotic strings and their expectation values with respect to the invariant measures of the coupled dynamics may yield the correct tools for a statistical mechanics of vacuum fluctuations. In fact, free \((a = 0)\) chaotic strings generate invariant densities that can be regarded as generalized canonical distributions in the formalism of non-extensive statistical mechanics, with an entropic index \(q\) given by either \(q = -1\) or \(q = 3\). So the strings are embedded as a generalization of statistical mechanics (more details in chapter 6).

The third approach is to relate the chaotic strings to the structure of space-time itself. There are 6 relevant chaotic strings, which we labeled as \(2A, 2B, 2A^-, 2B^-, 3A, 3B\). Also, there are 6 compactified dimensions necessary for the formulation of superstring theory or M-theory. If we let the 6 chaotic strings wind around (or even span up) the 6 compactified dimensions, then they do not ‘disturb’ our usual understanding of 4-dimensional space-time physics. Rather, they yield a very relevant amendment. Each coupling constant \(a\) can then be regarded as a kind of metric in the compactified space, and the analogue of the Einstein field equations makes the observed standard model parameters evolve to the minima of the effective potentials of the chaotic strings. In this picture the chaotic strings are related to a higher-dimensional extension of ordinary 4-dimensional space-time, in fact to kind of ‘excited states’ of ordinary 4-dimensional space-time. The ‘ground state’ (4-dimensional Minkowski space) is represented by the (non-chaotic) \(N = 1\) strings, whereas the chaotic strings with \(N \geq 2\) span up higher states (higher dimensions), similar to the energy levels of a quantum mechanical harmonic oscillator with quantum number \(N\) (more details on this kind of approach in chapter 11).

Which of these different physical embeddings is the most relevant one is not clear at the moment. Perhaps there is some truth in a combination of all three of them.
Figure 3: The way in which the interaction energies $W(a)$ of the six chaotic string theories fix the coupling strengths of the four interactions.

### 12.7 Conclusion

Instead of considering the standard model alone and putting in about 25 free parameters by hand, in this book we have postulated the existence of chaotic strings. The chaotic string dynamics can be physically interpreted as a 1-dimensional strongly fluctuating dynamics of vacuum fluctuations. It generates effective potentials which distinguish the observed standard model couplings from arbitrary ones. The dynamics may have already determined the standard model parameters at a very early stage of the universe (in a pre-Planck scenario) and it may still evolve today and stabilize the observed values of the parameters.

There are 6 relevant chaotic string theories (Figs. 3, 4). Whereas for standard model fields, as well as for superstrings after compactification, continuous gauge symmetries such as $U(1)$, $SU(2)$ or $SU(3)$ are relevant, for the chaotic strings a discrete $Z_2$ symmetry is relevant. In fact, the total theory may be regarded as a $SU(3) \times SU(2) \times U(1) \times Z_2$ theory. Whereas standard
Figure 4: The way in which the self energies $V(a)$ of the six chaotic string theories fix the charges, mass mixing angles and masses of the standard model particles.
model fields or ordinary strings usually evolve in a regular way, the chaotic strings obtained for $N > 1$ evolve in a deterministic chaotic (turbulent) way. They arise out of strongly self-interacting 1-dimensional field theories and correspond to a Bernoulli shift of information for vanishing spatial coupling $a$. The constraint conditions on the vacuum energy (or the analogues of the Einstein and scalar field equations) fix certain equilibrium metrics in string space, which determine the strength of the Laplacian coupling. We have provided extensive numerical evidence that these equilibrium metrics reproduce the free standard model parameters with very high precision (see Tab. 4). Essentially coupling constants are fixed by the interaction energy $W(a)$, and masses, mass mixing angles and charges by the self energy $V(a)$. This is summarized in Fig. 3 and 4.

The simplest physical interpretation is to regard the chaotic string dynamics as a dynamics of vacuum fluctuations, which is present everywhere but which is unobservable due to the uncertainty relation. Only expectations of the dynamics can be measured, in terms of the fundamental constants of nature. The strings can be related to a generalized statistical mechanics description of vacuum fluctuations and may possibly wind around the compactified space of superstring theory.

Generically, chaotic strings exhibit symmetry under the replacement $V \rightarrow -V$ (or $T_N \rightarrow -T_N$), which can be formally associated with a kind of supersymmetry transformation. However, when introducing the evolution equations (16) and (17) of the couplings one has to decide on the sign of the constant $\text{const}$. This choice effectively breaks the symmetry. Generally, with such a choice of sign the expectation of the vacuum energy of the chaotic strings singles out the physically relevant vacua, in the sense of stability. Supersymmetric partners of ordinary particles, if they exist at all, can be formally described by maxima rather than minima of the effective potentials. But they are unstable with respect to the fictitious time evolution, at least in our world. The instability might indicate that supersymmetric partners, though formally there to cancel divergences in the Feynman diagrams as well as unwanted vacuum energy, may turn out to be unobservable in our world.

In any case, chaotic dynamics of the type studied in this book seems to significantly enrich our understanding of standard model parameters and of quantum fluctuations in general. Assuming that on a very small scale quantum fluctuations are a deterministic chaotic process rather than a pure random process the most important free parameters of the standard model
can be understood with high precision. Embedding the chaotic strings into the compactified space of a 10- or 11-dimensional theory we seem to be looking at an extremely early stage of the universe where neither matter nor radiation but information is the relevant concept.

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