Synchrotron radiation inside a dielectric waveguide

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Abstract. We investigate the electromagnetic fields generated by a charged particle rotating inside a dielectric cylinder immersed into a homogeneous medium. The radiation intensity for the modes propagating inside the cylinder is evaluated by using two different ways: by evaluating the work done by the radiation field on the charge and by evaluating the energy flux through the cross-section of the cylinder. The relation between these two quantities is discussed. We investigate the relative contributions of the bound modes and the modes propagating at large distances from the cylinder to the total radiation intensity. The presence of the cylinder can lead to the considerable increase of the synchrotron radiation intensity.

1. Introduction
Unique characteristics of the synchrotron radiation have resulted in its extensive applications. Related to this, the investigation of mechanisms for the control of the radiation parameters is of great interest. Particularly important is the study of the influence of a medium on the spectral and angular characteristics. It is well-known that the medium can essentially change the characteristics of the high-energy electromagnetic processes. Moreover, new types of phenomena arise, such as the Cherenkov and transition radiations.

In [1]-[3] we have investigated the synchrotron radiation from a charge rotating around/inside a dielectric cylinder enclosed by a homogeneous medium. It has been shown that under the Cherenkov condition for the material of the cylinder and the particle velocity, strong narrow peaks appear in the angular distribution of the radiation intensity in the exterior medium. At these peaks the radiated energy exceeds the corresponding quantity in the case of a homogeneous medium by several orders of magnitude. Similar features for the radiation generated by a charge moving along a helical orbit have been discussed in [4, 5].

In the previous investigations the radiation intensity is considered in the exterior region at large distances from the cylinder. In addition to this part there is radiation which propagates inside the cylinder and it is of interest to investigate the relative contribution of these modes to the total intensity. Here we discuss the radiation intensity inside a dielectric cylinder emitted by a charge rotating inside the cylinder.

2. Radiation intensity inside a dielectric cylinder
Consider a dielectric cylinder of radius $\rho_1$ and with dielectric permittivity $\varepsilon_0$ and a point charge $q$ rotating inside a cylinder. The radius of the rotation orbit and the velocity of the charge will be denoted by $\rho_0$, $\rho_0 < \rho_1$, and $v$ respectively. We assume that the system is immersed in a
homogeneous medium with permittivity \( \varepsilon_1 \). In the cylindrical coordinate system \((\rho, \phi, z)\) with the \( z \)-axis directed along the cylinder axis, the only nonzero component of the current density is given by the expression

\[
j_\phi = \frac{q}{\rho} v \delta(\rho - \rho_0) \delta(\phi - \omega_0 t) \delta(z),
\]

where \( \omega_0 = v/\rho_0 \) is the angular velocity of the charge.

Inside the cylinder the radiation fields can be expanded over the eigenmodes of the dielectric cylinder. The latter are the solutions of the equation

\[
U_m \equiv V_m \left( \varepsilon_0 |\lambda_1| \rho_1 J'_m(J_m) + \varepsilon_1 \lambda_0 \rho_1 K'_m(K_m) \right) - m^2 \lambda_0^2 + \frac{|\lambda_1|^2}{\lambda_0^2 |\lambda_1|^2} \left( \varepsilon_1 \lambda_0^2 + \varepsilon_0 |\lambda_1|^2 \right) = 0,
\]

where \( J_m = J_m(\lambda_0 \rho_1) \) and \( K_m = K_m(|\lambda_1| \rho_1) \) are the Bessel and MacDonald functions, the prime means the derivative with respect to the argument of the function, and

\[
V_m = |\lambda_1| \rho_1 J'_m(J_m) + \lambda_0 \rho_1 K'_m(K_m).
\]

In (2), we use the notations

\[
\lambda_0^2 = m^2 \omega_0^2 \varepsilon_0 / c^2 - k_z^2, \quad j = 0, 1,
\]

with \( k_z \) being the projection of the wave vector on the \( z \)-axis. For the modes propagating inside the cylinder one has \( \lambda_0^2 < 0 \). We denote by \( k_z = \pm k_{m,s}, k_{m,s} > 0, s = 1, 2, \ldots, s_m, \) the solutions to the equation (2) and define

\[
\lambda_{m,s} = \rho_1 \sqrt{m^2 \omega_0^2 \varepsilon_0 / c^2 - k_{m,s}^2}, \quad \lambda_{m,s}^{(1)} = \rho_1 \sqrt{k_{m,s}^2 - \varepsilon_1 m^2 \omega_0^2 / c^2}.
\]

The radiation intensity propagating inside the dielectric cylinder can be computed in two different ways. In the first one we evaluate the work done by the radiation field on the charged particle:

\[
I = - \int d\rho d\phi dz j_\phi E_\phi,
\]

with \( E_\phi \) being the azimuthal component of electric field. This leads to the result (details can be found in [6])

\[
I = \sum_{m=1}^{\infty} I_m = \frac{q^2 v_q^2}{2 \varepsilon_0 \omega_0} \sum_{m=1}^{\infty} \sum_{s=1}^{s_m} \frac{\lambda_{m,s}^{(1)} J'_{m-2}(\lambda_{m,s} \rho_0 / \rho_1)}{\alpha_{m,s}(k_{m,s}) K_m} \sum_{l=\pm 1} J_{m+l}(\lambda_{m,s} \rho_0 / \rho_1) \frac{J_{m+p}(\lambda_{m,s} \rho_0 / \rho_1)}{V_{m-l} - l \mu u} \frac{V_{m+p} - l \mu u}{\rho_1^2 V_{m+p} + l \mu u},
\]

where

\[
\alpha_m = \frac{U_m}{(\varepsilon_1 - \varepsilon_0) (V_m^2 - m^2 \mu^2)}, \quad u = \frac{\lambda_0}{|\lambda_1|} + \frac{|\lambda_1|}{\lambda_0},
\]

with \( \lambda_0 = \lambda_{m,s} \) and \( |\lambda_1| = \lambda_{m,s}^{(1)} \). In the limit \( \rho_1 \rightarrow \infty \), from (7) we find the intensity for the synchrotron radiation in a homogeneous medium with dielectric permittivity \( \varepsilon_0 \).

Alternatively, the radiation intensity inside the dielectric cylinder can be obtained by evaluating the energy flux through the cross-section of the dielectric cylinder:

\[
I_{f}^{(in)} = \int_{0}^{\rho_1} dp \int_{0}^{2\pi} d\phi \rho n_z : \mathbf{S}
\]
where $\mathbf{n}_z$ is the unit vector along the $z$-axis and $\mathbf{S}$ is the Poynting vector. This leads to the formula:

$$I^{(in)}_f = \sum_{m=1}^{\infty} I^{(in)}_{f,m} = \frac{q^2 v^2 \rho_1^2}{8 \varepsilon_0 \omega_0} \sum_{m=1}^{\infty} k_{m,s} \lambda_{m,s}^{(1)} J^{-1}_{m} \left[ \sum_{l=\pm 1} J_{m+l}(\lambda_{m,s} \rho_0/\rho_1) \right]^2 \times \sum_{p=\pm 1} \frac{K_{m+p}}{V_m - p \mu} \left( \frac{m^2 \omega_0^2}{c^2} \varepsilon_0 + k_{m,s}^2 \right) \frac{K_{m+p}}{V_m - p \mu} - \frac{\lambda_{m,s}^2}{\rho_1^2} \frac{K_{m-p}}{V_m + p \mu} \times \left[ J_{m+p}^2 + \left( 1 - \frac{(m+p)^2}{\lambda_{m,s}^2} \right) J_{m+p}^2 \right].$$

(10)

This formula gives the flux in the region $z > 0$. By the symmetry of the problem we have the same flux in the region $z < 0$ and the total flux will be $2I_f^{(in)}$. Again, it can be seen that in the limit $\rho_1 \to \infty$, from (10) the expression for the radiation intensity in a homogeneous medium is obtained.

In figure 1, by the black points we present the number of the radiated quanta at a given harmonic $m$ per period of the charge rotation,

$$N_m = \sum_s N_{m,s} = \frac{T I_m}{\hbar \omega_0}, \quad T = \frac{2\pi}{\omega_0},$$

(11)

for the electron energy $E_e = 2$ MeV and for the values of the parameters $\varepsilon_1 = 1$, $\varepsilon_0 = 3$, $\rho_1/\rho_0 = 1.2$. The red points correspond to the number of the radiated quanta evaluated by the flux through the cylinder cross-section:

$$N^{(in)}_{f,m} = \frac{2TI^{(in)}_{f,m}}{h \omega_0}.$$  

(12)

The factor 2 in the last expression corresponds to that we have the same amount of the energy flux in the region $z < 0$. The number of the radiated quanta increases with the appearance of new modes. The numerical data in figure 1 show that $N_m \neq N^{(in)}_{f,m}$, i.e., the number of the radiated quanta evaluated by the energy flux through the cross-section of the cylinder and by the work done by the radiation field differ. As it will be shown in the next section, the reason for this difference is that for the eigenmodes of the dielectric cylinder there is also energy flux in the exterior region located near the surface of the cylinder.

3. Radiation intensity outside a cylinder

For a charge rotating inside a dielectric cylinder, in Ref. [2] we have evaluated the average energy flux per unit time through the cylindrical surface of large radius $\rho (\rho \gg \rho_1$) coaxial with the cylinder. The expression for the angular density of the radiation intensity on a given harmonic $m$ with the frequency $m \omega_0$ is given by the expression

$$\frac{d I^{(ex)}_m}{d\theta} = \frac{q^2 v^2 m^2 \omega_0^2 \sqrt{\varepsilon_1}}{\pi^2 c^3 \rho_1^2} \sin \theta \left[ |C_m^{(1)} - C_m^{(1)}|^2 + |C_m^{(1)} + C_m^{(1)}|^2 \cos^2 \theta \right],$$

(13)

where we have defined

$$C_m^{(p)} = \frac{J_{m+p}(\lambda_0^{(ex)} \rho_0)}{W_{m+p}} + p\lambda_0^{(ex)} H_m(\lambda_1^{(ex)} \rho_1) \frac{J_{m+p}(\lambda_0^{(ex)} \rho_1)}{2\alpha_m W_{m+p}} \sum_{l=\pm 1} J_{m+l}(\lambda_0^{(ex)} \rho_0) W_{m+l}^{(ex)},$$

(14)
The number of the radiated quanta, per period of the rotation, on a given mode \( m \) for the electron energy \( E_e = 2 \) MeV and for the values of the parameters \( \varepsilon_1 = 1, \varepsilon_0 = 3, \rho_1/\rho_0 = 1.2 \). The black/red points correspond to the number of the quanta evaluated by formula (11)/(12).

and

\[
\lambda_0^{(\text{ex})} = \frac{m\omega_0}{c} \sqrt{\varepsilon_0 - \varepsilon_1 \cos^2 \theta}, \quad \lambda_1^{(\text{ex})} = \frac{m\omega_0}{c} \sqrt{\varepsilon_1 \sin \theta},
\]

with \( \theta \) being the angle between the radiation direction and the axis of the cylinder. In (14),

\[
W^{J_m}_m = \lambda_1^{(\text{ex})} J_m(\lambda_0^{(\text{ex})} \rho_1) H_m'(\lambda_1^{(\text{ex})} \rho_1) - \lambda_0^{(\text{ex})} H_m(\lambda_1^{(\text{ex})} \rho_1) J_m'(\lambda_0^{(\text{ex})} \rho_1),
\]

and \( H_m(x) = H_m^{(1)}(x) \) is the Hankel function of the first kind.

The properties of the radiation corresponding to (13) are described in detail in [2] (see also [4] for a more general case of helical motion inside a dielectric cylinder). In particular, it was shown that under the Cherenkov condition for dielectric permittivity of the cylinder and the velocity of the particle image on the cylinder surface, strong narrow peaks are present in the angular distribution for the number of radiated quanta. At these peaks the radiated energy exceeds the corresponding quantity for a homogeneous medium by several orders of magnitude. For the illustration, in figure 2 we plot the angular density of the number of the quanta radiated in the exterior region per period of the charge rotation,

\[
\frac{dN^{(\text{ex})}_m}{d\theta} = \frac{T}{h\omega_0} \frac{dI^{(\text{ex})}_m}{d\theta},
\]

with the radiation intensity given by (13). The corresponding parameters are as follows: \( E_e = 2 \) MeV, \( \varepsilon_1 = 1, \varepsilon_0 = 3, m = 16, \rho_1/\rho_0 = 1.2 \). The dashed curve corresponds to the synchrotron radiation in the vacuum in the absence of the dielectric cylinder (\( \varepsilon_0 = 1 \)). We see the presence of the strong narrow peak at \( \theta \approx 42.63 \) (in degrees). The height of this peak is \( \approx 463.4 \) and the width \( \approx 0.05 \). At large distances from the cylinder for the total number of the radiated quanta one has \( N^{(\text{ex})}_m = 1.748 \). Note that for the number of quanta radiated on the eigenmodes of the cylinder we have \( N_m = 1.897 \) (see figure 1). For the synchrotron radiation from an electron in the vacuum (\( \varepsilon_0 = 1 \)) one has \( N^{(0)}_m = 0.352 \). As it is seen, the presence of the cylinder considerably increases the radiation intensity.
Figure 2. The angular density of the number of the quanta radiated on the harmonic $m = 16$ in the exterior region per period of the charge rotation as a function of $\theta$ (in degrees). The full curve is plotted for the values of parameters $E_e = 2$ MeV, $\varepsilon_1 = 1$, $\varepsilon_0 = 3$, $m = 16$, $\rho_1/\rho_0 = 1.2$. The dashed curve is for the radiation in the vacuum ($\varepsilon_0 = 1$).

In addition to the radiation corresponding to (13), in the exterior region there are also radiation fields localized near the surface of the dielectric cylinder. These fields are the tails of the eigenmodes for the dielectric cylinder and exponentially decay in the exterior region for $\rho/\rho_1 \gg 1/\lambda_{m,s}^{(1)}$. For a fixed value of $\rho > \rho_1$ and in the limit $z \to \infty$ the corresponding energy flux through a plane perpendicular to the axis of the cylinder is given by the expression

$$I_f^{(\text{ex})} = \sum_{m=1}^{\infty} I_f^{(\text{ex})}_m = \frac{q^2 c^2 \rho_1^2}{8 \varepsilon \omega} \sum_{m=1}^{\infty} \sum_{s} k_{m,s} \lambda_{m,s}^{(1)2} J_m^{-4} \left[ \sum_{j=\pm 1} J_{m+(\lambda_{m,s}\rho_0/\rho_1)} \frac{1}{V_m - j \mu} \right]^2 \times \sum_{p=\pm 1} \frac{J_{m+p}}{V_m - p \mu} \left[ \frac{m^2 \omega^2}{c^2} \varepsilon_1 + k_{m,s}^2 \right] \frac{J_{m+p}}{V_m - p \mu} - \frac{\lambda_{m,s}^{(1)2}}{\rho_1^2} \frac{J_{m-p}}{V_m + p \mu} \right] \times \left[ K_{m+p}^2 - \left( 1 + \frac{(m+p)^2}{\lambda_{m,s}^{(1)2}} \right) K_{m+p}^2 \right]. \quad (18)$$

Now it can be explicitly checked that

$$I_m = 2 \left( I_f^{(\text{in})}_m + I_f^{(\text{ex})}_m \right).$$

In figure 3 we have plotted the number of the radiated quanta per period of the rotation on a given mode $m$ for different values of $m$. The values of the parameters are as follows: $E_e = 2$ MeV, $\varepsilon_1 = 1$, $\varepsilon_0 = 3$, $\rho_1/\rho_0 = 1.2$. The black points correspond to the radiation in the exterior region at large distances from the cylinder, $N_m^{(\text{ex})} = \int_0^{\pi/2} d\theta (\theta N_m^{(\text{ex})}/d\theta)$ (see (17)), and the red points are for the radiation on the eigenmodes of the dielectric cylinder, $N_m^{(\text{in})} = \sum_s N_{m,s}$ (see (11)). For the comparison with the synchrotron radiation in the vacuum, in figure 4, for the same values of the parameters, we plot the total number of the radiated quanta, $N_m = N_m^{(\text{in})} + N_m^{(\text{ex})}$, in the presence (black points) and in the absence (red points) of the dielectric cylinder. As it is seen from these graphs, the presence of the cylinder essentially increases the radiation intensity.
**Figure 3.** The number of the radiated quanta per period of the rotation on a given mode $m$ for different values of $m$ and for the values of the parameters $E_e = 2$ MeV, $\varepsilon_1 = 1$, $\varepsilon_0 = 3$, $\rho_1/\rho_0 = 1.2$. The black points correspond to the radiation in the exterior region and the red points are for the radiation emitted on the eigenmodes of the dielectric cylinder.

**Figure 4.** The total number of the radiated quanta, $N_m = N_m^{\text{in}} + N_m^{\text{ex}}$, in the presence (black points) and in the absence (red points) of the dielectric cylinder. The values of the parameters are the same as those for figure 3.

**4. Conclusion**

We have investigated the synchrotron radiation from a charged particle rotating inside a dielectric cylinder surrounded by a homogeneous medium. Firstly we have evaluated the work done by the radiation field on the charged particle. Further, we have computed the energy flux through the cross-section of the cylinder and it is shown that these quantities differ.

The radiation field in the exterior region consists of two parts. The first one corresponds to the radiation propagating at large distances from the cylinder. The angular density of the corresponding radiation intensity on a given harmonic $m$ is given by formula (13). For large values of $m$ and under the Cherenkov condition for dielectric permittivity of the cylinder and the velocity of the particle image on the cylinder surface, strong narrow peaks appear in the angular distribution for the radiation intensity. The second part of the radiation field in the
exterior region corresponds to the modes of the dielectric cylinder. The corresponding fields are located near the surface of the cylinder. These fields exponentially decay at distances from the cylinder surface larger than the radiation wavelength. The energy flux in the exterior region corresponding to the eigenmodes of the cylinder is given by the expression (18). We have explicitly checked that the radiation intensity evaluated by the work done by the radiation field on the charged particle is equal to the radiation intensity evaluated by the energy flux through the plane perpendicular to the cylinder axis if we take into account the contribution of the radiation field in the exterior region.

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