Moser-Tardos resampling algorithm, entropy compression method and the subset gas

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Abstract

We establish a connection between the entropy compression method and the Moser-Tardos algorithmic version of the Lovász local lemma through the cluster expansion of the subset gas. We also show that the Moser-Tardos resampling algorithm and the entropy compression backtracking algorithm produce identical bounds.

Keywords: Probabilistic Method in combinatorics; Lovász Local Lemma; Randomized algorithms.

MSC numbers: 05D40, 68W20.

1 Introduction

1.1 The Lovász Local Lemma

The Lovász Local Lemma (LLL), originally formulated by Erdős and Lovász in [19], is a powerful tool in the framework of the probabilistic method used in an impressive quantity of applications in combinatorics such as graph coloring problems, K-sat, latin transversal, etc., (see [5] and references therein for a review). Its basic idea is to prove the existence of some combinatorial object with certain desired property (e.g. such a proper coloring of the vertices of a graph) by identifying a family \( \mathcal{F} \) of (bad) events in some probability space \( \Omega \) whose presence, even of only one of them, spoils the object under analysis and whose simultaneous non-occurrence guarantees that the object under analysis is actually present. Denoting by \( \overline{\mathcal{F}} \) the complement event of \( \varepsilon \in \mathcal{F} \), the Lovász local lemma provides a condition on the probabilities \( \text{Prob}(\varepsilon) \) in order to ensure that \( \text{Prob}(\bigcap_{\varepsilon \in \mathcal{F}} \overline{\varepsilon}) > 0 \). To formulate explicitly this condition one need to identify a so-called dependency graph for the family \( \mathcal{F} \). That is to say, a graph \( \mathcal{G} \) with vertex set \( \mathcal{F} \) and edge set such that each event \( \varepsilon \in \mathcal{F} \) is independent of the \( \sigma \)-algebra generated by the collection of events \( \mathcal{F} \setminus \Gamma^\varepsilon_\mathcal{G}(\varepsilon) \), where \( \Gamma^\varepsilon_\mathcal{G}(\varepsilon) = \Gamma_\varepsilon(\varepsilon) \cup \{\varepsilon\} \) and \( \Gamma_\varepsilon(\varepsilon) \) is the set of all the events of \( \mathcal{F} \) adjacent to \( \varepsilon \) in \( \mathcal{G} \). According to the usual terminology, \( \Gamma^\varepsilon_\mathcal{G}(\varepsilon) \) is called the neighborhood of \( \varepsilon \) in \( \mathcal{G} \) while \( \Gamma_\varepsilon(\varepsilon) \) is called the punctured neighborhood of \( \varepsilon \) in \( \mathcal{G} \).

Once the dependency graph of the family \( \mathcal{F} \) has been determined, the LLL can be stated as follows.
Theorem 1.1 (Lovász local Lemma) Let $\mathcal{F}$ be a finite family of events in a probability space $\Omega$ and let $\mathcal{G}$ be a dependency graph for $\mathcal{F}$. Let $\mathbf{\mu} = \{\mu_e\}_{e \in \mathcal{F}}$ be a collection of non-negative numbers. If, for each $e \in \mathcal{F},$

$$\text{Prob}(e) \leq \frac{\mu_e}{\prod_{e' \in \Gamma^*_G(e)} (1 + \mu_e')},$$

(1.1)

then

$$\text{Prob}\left(\bigcap_{e \in \mathcal{F}} \neg e\right) > 0.$$  

(1.2)

1.2 The abstract polymer gas

The abstract polymer system (APS) is a discrete model originally proposed by Kotecky and Preiss [36] as a generalization of a lattice polymer model introduced by Gruber and Kunz [27] in 1968. Its relevance in statistical mechanics is very important since it is a widely used tool to analyse a large number of systems in physics, such as discrete spin systems, continuous and discrete particle systems, percolative models and even quantum field theories.

The APS is defined by a triple $(\mathcal{P}, w, \mathcal{W})$ where $\mathcal{P}$ is a countable (possibly infinite) set whose elements are called polymers, $w : \mathcal{P} \to \mathbb{C}$ is a function which associates to each polymer $\gamma \in \mathcal{P}$ a complex number $w_\gamma$, called the activity of the polymer $\gamma$, and $\mathcal{W} : \mathcal{P} \times \mathcal{P} \to \{0, 1\}$ is a function called the Boltzmann factor, such that $\mathcal{W}(\gamma, \gamma') = 0$ and $\mathcal{W}(\gamma', \gamma) = 1$ for all $\{\gamma, \gamma'\} \subset \mathcal{P}$. Usually the pair $\{\gamma, \gamma'\}$ is called incompatible when $\mathcal{W}(\gamma, \gamma') = 0$ and compatible when $\mathcal{W}(\gamma, \gamma') = 1$.

Let $\mathcal{G}$ be the simple graph with vertex set $\mathcal{P}$ and edge set formed by the pairs $\{\gamma, \gamma'\} \subset \mathcal{P}$ such that $\mathcal{W}(\gamma, \gamma') = 0$. The graph $\mathcal{G}$, which is uniquely determined by the Boltzmann factor $\mathcal{W}$, is sometimes called the support graph of $\mathcal{W}$. The neighborhood of the vertex $\gamma$ in the graph $\mathcal{G}$ is the set $\Gamma^*_G(\gamma) = \{\gamma' \in \mathcal{P} : \mathcal{W}(\gamma, \gamma') = 0\}$ formed by all polymers incompatible with $\gamma$. An independent set of the support graph $\mathcal{G}$ is a set $Y$ of polymers such that each pair $\{\gamma, \gamma'\} \subset Y$ is compatible. We denote by $I(\mathcal{G})$ the set formed by all finite independent sets of $\mathcal{G}$.

Given a finite collection of polymers $\Lambda \subset \mathcal{P}$, the grand canonical partition function of the APS at “finite volume” $\Lambda$ is given by

$$Z_\Lambda(w) = \sum_{\substack{S \subset \Lambda \mid \gamma \in S}} \prod_{\gamma \in S} w_\gamma.$$  

This is a key quantity since the thermodynamic properties of the system can be derived from it. In particular, a fundamental question physicists are interested in, is to find radii $\mathbf{R} = \{R_\gamma\}_{\gamma \in \mathcal{P}}$ (with $R_\gamma \geq 0$ for all $\gamma \in \mathcal{P}$) such that the partition function $Z_\Lambda(w)$, for any $\Lambda$ finite, is free of zeros for all complex activities $w$ within the polydisk $\{|w_\gamma| < R_\gamma\}_{\gamma \in \mathcal{P}}$ (shortly $w \leq \mathbf{R}$). This would guarantee that the logarithm of the partition function, which is related to the pressure of the system, is analytic in such regions, so that no phase transitions occur. The best current lower bound for such radii $\mathbf{R}$ is due to Fernández and Procacci [21] who improved the older bounds due to Kotecky and Preiss [36] and Dobrushin [16] proving the following theorem.

Theorem 1.2 (Fernández-Procacci criterion) Let $\mathbf{\mu} = \{\mu_\gamma\}_{\gamma \in \mathcal{P}}$ be a collection of non-negative numbers such that

$$|w_\gamma| \leq R^{FP}_\gamma \equiv \frac{\mu_\gamma}{\Xi_\gamma(\mathbf{\mu}, \mathcal{G})}, \quad \forall \gamma \in \mathcal{P}$$  

(1.3)
with
\[ \Xi_\gamma(\mu, G) = \sum_{S \subseteq \Gamma^*_G(\gamma)} \prod_{\gamma' \in S} \mu_{\gamma'}. \] (1.4)

Then, for all finite \( \Lambda \subset \mathcal{P}, Z_\Lambda(w) \neq 0 \).

The Kotecky-Preiss and the Dobrushin criteria can be formulated analogously with the only difference that function \( \Xi_\gamma(\mu, G) \) appearing in the r.h.s. of (1.3) is replaced respectively by
\[ \varphi^{KP}_\gamma(\mu) = e^{\sum_{\gamma' \in \Gamma^*_G(\gamma)} \mu_{\gamma'}}, \] (1.5)
and
\[ \varphi^{D}_\gamma(\mu) = \sum_{S \subseteq \Gamma^*_G(\gamma)} \prod_{\gamma' \in S} \mu_{\gamma'} = \prod_{\gamma' \in \Gamma^*_G(\gamma)} (1 + \mu_{\gamma'}). \] (1.6)

The bound on radii \( R \) given by the Fernández-Procacci criterion (1.3) is always greater than the bounds on the same radii given by the Kotecky-Preiss and the Dobrushin criteria since
\[ \exp \left\{ \sum_{\gamma' \in \Gamma^*_G(\gamma)} \mu_{\gamma'} \right\} \geq \prod_{\gamma' \in \Gamma^*_G(\gamma)} (1 + \mu_{\gamma'}) = \sum_{S \subseteq \Gamma^*_G(\gamma)} \prod_{\gamma' \in S} \mu_{\gamma'} \geq \sum_{S \subseteq \Gamma^*_G(\gamma)} \prod_{\gamma' \in S} \mu_{\gamma'}. \]

1.3 The connection between the LLL and the APS

It is important to remark that the LLL criterion (1.1) is a sufficient condition in order to the thesis (1.2) to hold. In 1985, Shearer [47] presented a necessary and sufficient condition for (1.2) to hold. The Shearer condition was actually constituted by a set of several conditions which were very difficult (if not impossible) to be checked in practical applications. Probably for this reason Shearer’s result went somehow overseen until 2005. In this year Scott and Sokal [44], inspired by Shearer’s work, showed that there was a quite surprising connection between the Lovász Local Lemma and the abstract polymer gas. Scott and Sokal showed that, given the family of events \( \mathcal{F} \) and their dependency graph \( G \), the Shearer criterion is equivalent to require that the probabilities of the bad events, \( \text{Prob}(\mathcal{e}) \), fall in the zero-free region of the partition function of the APS whose support graph coincides with the dependency graph \( G \) of the family \( \mathcal{F} \). So, once rephrased in the statistical mechanics lingo, it is no surprise that the Shearer criterion was unusable in practice.

On the other hand Scott and Sokal observed that this equivalence implies that the LLL criterion (1.1) coincides with the aforementioned Dobrushin criterion. Later, Bissacot et al. [12], via the connection disclosed in [44] and Theorem 1.2, improved the LLL criterion (1.1) as follows.

**Theorem 1.3 (Cluster expansion local lemma (CELL))** With the same hypothesis of the Theorem 1.1 if, for each event \( \mathcal{e} \in \mathcal{F} \)
\[ \text{Prob}(\mathcal{e}) \leq \frac{\mu_\mathcal{e}}{\Xi_\mathcal{e}(\mu, G)}, \] (1.7)

with
\[ \Xi_\mathcal{e}(\mu, G) = \sum_{S \subseteq \Gamma^*_G(\mathcal{e})} \prod_{\mathcal{e}' \in S} \mu_{\mathcal{e}'}, \] (1.8)

then
\[ \text{Prob}\left( \bigcap_{\mathcal{e} \in \mathcal{F}} \overline{\mathcal{e}} \right) > 0. \]
As observed above this is clearly an improvement w.r.t. Theorem 1.1 since
\[
\sum_{S \subseteq \Gamma^*_G(e)} \prod_{e' \in S} \mu_{e'} \leq \sum_{S' \subseteq \Gamma^*_G(e)} \prod_{e' \in S'} \mu_{e'} = \prod_{e' \in \Gamma^*_G(e)} (1 + \mu_{e'}).
\]
Condition (1.7) has been shown to be effective in several applications of the LLL (see e.g. [38] and [13]). This new criterion is nowadays known as “Cluster Expansion (CE) criterion”.

Formulas (1.1) and (1.7) on one hand and formulas (1.3) and (1.6) on the other hand show in a crystal way the evident connection between the LLL and the APS:

- events \( e \in \mathcal{E} \) in LLL correspond to polymers \( \gamma \in \mathcal{P} \) in APS;
- dependents events correspond to incompatible polymers;
- probabilities of events correspond to the absolute values of activities of polymers;
- probability of the good event to be positive corresponds to require that the partition function evaluated at \(-|w|\) (which is the worst case, see e.g. [44] or [11]) to be strictly positive.

1.4 Moser-Tardos algorithmic version of the LLL

The LLL is very general, in particular in its statement and proof there is no need to specify anything about the probability space. Of course, in the applications the probability space \( \Omega \) has to be specified and it is natural to wonder, once condition (1.1) is satisfied, if it is possible to find a polynomial algorithm in this specified probability space able to find a configuration in \( \Omega \) which realizes the event \( \bigcap_{e \in \mathcal{E}} e \). During many years researchers have tried to find methods to devise general algorithms able to find such a configuration for as many as possible applications covered by the LLL. These efforts have been only partially successful in the sense that the class of example for which an efficient algorithm could be found was limited and the condition (1.1) got worse, see for example [4], [9]. Such situation changed radically in 2009 when in a breakthrough paper [37] Moser and Tardos presented a fully algorithmic version of the LLL which covered the vast majority of LLL applications.

The scheme proposed by Moser and Tardos is called nowadays the variable setting. It starts from the assumption that the probability space \( \Omega \) is a product space generated by a collection of mutually independent random variables \( \{\psi_x\}_{x \in \Lambda} \), where \( \Lambda \) is a finite set whose elements will be called atoms hereafter. In general, each random variable \( \psi_x \) takes values in its own space \( \Psi_x \) according to its own distribution, but in a vast majority of the applications these variables take values in a common finite space \( \Psi \) (e.g. the set of colors). Therefore \( \Omega = \prod_{x \in \Lambda} \Psi_x \equiv \Psi_\Lambda \) and an element \( \omega \in \Psi_\Lambda \) is called a configuration. An event \( \epsilon \) in such a probability space \( \Omega \) (also called a flaw) is a subset of \( \Psi_\Lambda \) and it depends in general on all the random variables \( \{\psi_x\}_{x \in \Lambda} \) generating \( \Omega \). When \( \epsilon \) depends only on variables of a proper subset \( A \subset \Lambda \) (i.e. \( \epsilon \) is fully determined by variables \( \Psi_A = \{\psi_x\}_{x \in A} \)), we say that the event \( \epsilon \) is tempered and we write \( \text{supp}(\epsilon) = A \) and \( \text{supp}(\epsilon) = A \subset \Lambda \) is sometimes called the scope of \( \epsilon \). A tempered event \( \epsilon \) is hereafter called elementary if it is formed only by one single configuration \( \omega \in \Psi_{\text{supp}(\epsilon)} \).

Moser and Tardos considered situations in which all of the bad events constituting the family \( \mathcal{F} \) are tempered. With this assumption any two events \( \epsilon, \epsilon' \) of \( \mathcal{F} \) such that \( \text{supp}(\epsilon) \cap \text{supp}(\epsilon') = \emptyset \) are necessarily independent. This implies that the graph \( \mathcal{G} \) with vertex set \( \mathcal{F} \) and edge set...
constituted by the pairs \{e, e'\} such that \text{supp}(e) \cap \text{supp}(e') \neq \emptyset is a natural dependency graph for the family \mathcal{F}.

In this setting Moser and Tardos defined the following algorithm.

| Resampling. |
|-------------|
| 1. Take a random evaluation \( \omega_0 \in \Psi_\Lambda \). |
| 2. While there is a bad event belonging to \( \mathcal{F} \) occurring, select an event \( e \) and re-sample all variables \( \{\psi_x\}_{x \in \text{supp}(e)} \). |
| 3. End while. |
| 4. Output current evaluation. |

Moser and Tardos proved that if condition (1.1) of Theorem 1.1 holds, then Resampling terminates rapidly finding a configuration \( \omega \in \Psi_\Lambda \) such that none of the bad events of the family \( \mathcal{F} \) occurs. Later, inspired by the paper by Bissacot et al. [12], Pegden [40] improved the Moser-Tardos result replacing condition (1.1) with condition (1.7).

**Theorem 1.4 (Pegden)** Given a finite set \( \Lambda \) and its associated family of mutually independent random variables \( \psi_\Lambda \), let \( \mathcal{F} \) be a family of tempered bad events with natural dependency graph \( \mathcal{G} \). Let \( \mu = \{\mu_e\}_{e \in \mathcal{F}} \) be non-negative numbers. If, for each \( e \in \mathcal{F} \),

\[
\text{Prob}(e) \leq \frac{\mu_e}{\Xi_e(\mu, \mathcal{G})}
\]

with \( \Xi_e(\mu, \mathcal{G}) \) defined in (1.8), then

\[
\text{Prob}\left( \bigcap_{e \in \mathcal{F}} \bar{e} \right) > 0
\]

and algorithm Resampling finds \( \omega \in \bigcap_{e \in \mathcal{F}} \bar{e} \) in an expected total number of steps less than or equal to \( \sum_{e \in \mathcal{F}} \mu_e \).

It is also worth to mention that Kolipaka and Szegedy showed in [35] that algorithm Resampling is successful in polynomial time also if Shearer conditions hold (see also [9] for a similar result).

**Remark.** In the Moser-Tardos setting above described the function \( \Xi_e(\mu, \mathcal{G}) \), defined in (1.8), admits a somehow natural upper bound so that condition (1.9) can be greatly simplified (paying some price as we will see in a moment). Just observe that, for any \( y \in \Lambda \), the set \( \mathcal{F}(y) = \{e \in \mathcal{F} : y \in \text{supp}(e)\} \) is a clique of the natural dependency graph \( \mathcal{G} \) of the family \( \mathcal{F} \): any pair \( \{e, e'\} \subset \mathcal{F}(y) \) is such that \( \text{supp}(e) \cap \text{supp}(e') \supset \{y\} \neq \emptyset \), i.e., any pair \( \{e, e'\} \subset \mathcal{F}(y) \) is an edge of \( \mathcal{G} \). Thus, the neighborhood \( \Gamma^*_\mathcal{G}(e) \) of any event \( e \in \mathcal{F} \) is the union of cliques \( \{\mathcal{F}(y)\}_{y \in \text{supp}(e)} \), and then we can write \( \Gamma^*_\mathcal{G}(e) = \bigcup_{y \in \text{supp}(e)} \mathcal{F}(y) \). Therefore

\[
\Xi_e(\mu, \mathcal{G}) \leq \Xi^{\text{clique}}_e(\mu, \mathcal{G}) \equiv \prod_{y \in \text{supp}(e)} \left[ 1 + \sum_{e' \in \mathcal{F}(y)} \mu_{e'} \right].
\]  

Using the expression \( \Xi^{\text{clique}}_e(\mu, \mathcal{G}) \), which is much simpler to evaluate, in place of \( \Xi_e(\mu, \mathcal{G}) \) in condition (1.7), it was possible to improve estimate for latin transversal in [12] and improve
bounds for several chromatic indices in \cite{38} and \cite{13}. It is important however to stress that the estimate (1.10) is efficient only if the cliques \( \{ \mathcal{F}(y) \} \) are not too overlapped. In general this union is not disjoint, but when cliques \( \{ \mathcal{F}(y) \} \) are too overlapped, the replacement \( \Xi_e(\mu, G) \) with \( \Xi_{\text{clique}} (\mu, G) \) tends to be a too crude estimate. This situation occurs for example in the case of perfect and separating hash families (see \cite{11}).

1.5 The subset gas

In the context of the connection between the LLL and the APS it is natural to ask where the Moser-Tardos setting defined above does fit. We explain below that the APS counterpart of the Moser-Tardos setting is the so-called subset gas.

The subset gas, originally proposed by Gruber and Kunz \cite{27}, is a particular realization of the abstract polymer gas which appears in many physical situations and it is defined as follows.

Given a countable set \( V \), the space of polymers \( P \) is defined as the collection of all finite subsets of \( V \), namely \( P = \{ \gamma \subset V : |\gamma| < +\infty \} \). The Boltzmann factor is then defined as \( W(\gamma, \gamma') = 0 \) if \( \gamma \cap \gamma' \neq \emptyset \) and \( W(\gamma, \gamma') = 1 \) if \( \gamma \cap \gamma' = \emptyset \). Thus, in such a realization of the APS, polymers have a cardinality, so that one can speak about large polymers and small polymers. Of course, as before, each polymer \( \gamma \) has an associated activity \( w_\gamma \). In most of the physical realizations, \( V \) is the vertex set of a (possibly infinite) graph, typically the cubic lattice \( \mathbb{Z}^d \) with the edge set being the set of nearest neighbor in \( \mathbb{Z}^d \).

Actually, the subset gases appearing in the framework of statistical mechanics and specifically spin systems on \( \mathbb{Z}^d \) have in general a further characteristic. Typically, in each site \( x \in \mathbb{Z}^d \) is defined a random variable \( s_x \) (the spin at \( x \)) taking values in some space \( S_x \) and frequently this space \( S_x \) is the same for all \( x \in V \). Then the space of the polymers \( P \) is formed by the pairs \( \gamma = (\text{supp}(\gamma), s_\gamma) \), where \( \text{supp}(\gamma) \) is, as before, a finite subset of \( V \) and \( s_\gamma \) is the spin configuration of \( \gamma \), i.e. a function from \( \text{supp}(\gamma) \) to \( \prod_{x \in \text{supp}(\gamma)} S_x \).

The reader can see at this point the evident parallel with the variable setting of the LLL. Namely, polymers \( \gamma = (\text{supp}(\gamma), s_\gamma) \) in the subset gas correspond to (elementary) events \( e \) in the Moser-Tardos variable setting.

Generally, as far as the subset gas is concerned, the bound (1.10) is always used. So, for the subset gas, the condition (1.3) can be written as

\[
|w_\gamma| \leq \frac{\mu_\gamma}{\prod_{x \in \text{supp}(\gamma)} \left[ 1 + \sum_{\gamma' \in P} \mu_{\gamma'} \right]}, \quad \forall \gamma \in P.
\]

Note that the above criterion is constituted by many inequalities, i.e. as many inequalities as the number of total polymers, so if \( P \) is infinite this number can be infinite. However, in the specific case of the subset gas this set of inequalities can be replaced (and usually is!) by a unique global inequality to which the set of activities must obey. Indeed, since by (1.11) we necessarily have that \( \mu_\gamma > |w_\gamma| \), a typical choice is to set

\[
\mu_\gamma = |w_\gamma|e^{a|\text{supp}(\gamma)|}
\]

with \( a > 0 \). Such a choice permits to resume the set of conditions (1.11) in terms of a simple “global” conditions on the set of activities \( w \). Namely, the thesis of Theorem (1.3) holds, if

\[
\sup_{x \in V} \sum_{\gamma \in P} |w_\gamma|e^{a|\text{supp}(\gamma)|} \leq e^a - 1 \quad \text{for some } a > 0.
\]
The above discussion on the connection between the LLL and the APS leads to conclude that the set of conditions of the CELL criterion can be reexpressed in terms of a global unique condition of the probabilities \( \text{Prob}(\epsilon) \) of the events \( \epsilon \) as far as we are in the Moser-Tardos variable setting.

**Lemma 1.5** Given a finite set \( \Lambda \) and a family of mutually independent random variables \( \psi_\Lambda \), let \( \mathcal{F} \) be a family of tempered events and for \( \epsilon \in \mathcal{F} \) let \( \text{Prob}(\epsilon) \) be its probability in the product space generated by the variables \( \psi_\Lambda \). If it is possible to find \( a > 0 \) such that

\[
\sup_{x \in \Lambda} \sum_{\epsilon \in \mathcal{F} \in \text{ supp}(\epsilon)} \text{Prob}(\epsilon)e^{a|\text{ supp}(\epsilon)|} \leq e^a - 1, \tag{1.13}
\]

then

\[
\text{Prob}\left( \bigcap_{\epsilon \in \mathcal{F}} \bar{\epsilon} \right) > 0
\]

and algorithm \textsc{Resampling} finds a configuration \( \omega \in \bigcap_{\epsilon \in \mathcal{F}} \bar{\epsilon} \) in an expected total number of steps less than or equal to \( \sum_{\epsilon \in \mathcal{F}} \text{Prob}(\epsilon)e^{a|\text{ supp}(\epsilon)|} \).

The latter global “subset gas condition” \eqref{1.13} is able to reproduce all improvements obtained via CELL \cite{38,13,12} with the exception of \cite{11}, where the general condition \eqref{1.7} has been used. It is worth to mention that similar (but less effective) global conditions deduced from the original LLL in the variable setting have been already formulated in the literature and used in specific examples (see e.g. Lemma 3 in \cite{34} and reference therein).

### 1.6 The entropy compression method

The Moser-Tardos algorithmic version of the LLL, since its appearance, has been the subject of a very intense study by several researchers in the areas of computer science, combinatorics and probability. In this regard two main directions can be pointed out. The first one concerns the (successful) efforts made to extend the validity of the algorithmic version of the LLL beyond the variable setting (see e.g. \cite{31,1,30,3,33,29} and references therein) with the objective to include important applications of the non constructive LLL (such as latin transversal) which does not fit in the independent variable setting.

The second direction was motivated by the fact that the algorithm \textsc{Resampling} proposed by Moser and Tardos was extremely simple, so it makes sense to try to modify/refine it in order to improve the final criterion beyond \eqref{1.7}. These ideas have been originally developed in \cite{17,28,20} where backtracking algorithms have been implemented for specific graph coloring problems to obtain bounds which are better than those obtainable by LLL or CELL.

In particular, Esperet and Parreau devised in \cite{20} an algorithm able to obtain a new upper bound for the chromatic index of the acyclic edge coloring of a graph with maximum degree \( \Delta \) which sensibly improves on the bound obtained by Ndreca et al. \cite{38} just an year before via the CELL. The algorithm proposed by Esperet and Parreau presents evident differences from the algorithm \textsc{Resampling}: instead of sampling all variables at once and then resampling some of them until all flaws are avoided, the Esperet-Parreau algorithm starts with the empty configuration in which no variable has an assigned value. Then, step by step a (random) value is attributed to a currently unassigned variable; when this leads to the appearance of one or more flaws, the algorithm backtracks to a partial non-violating configuration by retracting some set of variables.
Esperet and Parreau suggested, through further examples and applications, that their algorithm could be adapted to treat most of the applications in graph coloring problems covered by the LLL. Indeed, this was confirmed in several successive papers \cite{26,42,39,43,15,14,45,24,8}, where the Esperet-Parreau scheme has been applied to various graph coloring problems and beyond, generally improving previous results obtained via the LLL/CELL (sometimes the improvement is more sensible, sometimes less). However, in all papers mentioned above the Esperet-Parreau algorithmic scheme, usually called \textit{entropy compression method} (the name is probably due to Tao \cite{48}), has been commonly utilized as a set of \textit{ad hoc} instructions to be implemented on a case-by-case basis. A systematization of the entropy compression method providing a general criterion similar to those given by the LLL and CELL has been an open question since the beginning \cite{20,26} and it is still demanded even in the very recent paper by Achlioptas and Iliopoulos \cite{2}. In this regard we mention a non-algorithmic general criterion proposed by Bernshteyn \cite{10} which is able to reproduce several results obtained by the entropy compression method. The systematization of the Esperet-Parreau backtrack algorithm can actually be found \cite{7} in which the setting where entropy compression can be used is clearly outlined, a general \textit{entropy compression criterion} is proposed and a connection between bounds obtained via this method and LLL conditions is elucidated. According to \cite{7}, the entropy compression method can be implemented in any application that can be analyzed through a variable setting \textit{a la} Moser-Tardos with the further restriction that, for all \(x \in \Lambda\), variables \(\psi_x\) take values in a common space \(\Psi_x = [k]\), where \(k \in \mathbb{N}\) (or, more generally, in possibly distinct spaces \(\Psi_x\) but all with the same cardinality \(k\)) according to the uniform distribution. We will refer to this particular realization of the variable setting as the \textit{uniform variable setting}. Let us review rapidly for later comparison the entropy compression criterion obtained in \cite{7}.

### 1.6.1 The entropy compression setting

As said above, the entropy compression method can be applied in the so-called \textit{uniform variable setting} where all random variables \(\{\psi_x\}_{x \in \Lambda}\) take values in the common space \(\Psi = [k] \equiv \{1, 2, \ldots, k\}\) according to the uniform distribution. We sometimes refer to \([k]\) as the set of “colors” and we set \([k]_0 = [k] \cup \{0\}\). So the entropy compression setting is determined by the pair \((\Lambda, k)\), where \(\Lambda\) is a finite set and \(k\) is a positive integer. A configuration \(w\) of \(X \subseteq \Lambda\) is a function \(w : X \to [k]\), and a partial configuration \(w\) of \(X \subseteq \Lambda\) is a function \(w : X \to [k]_0\) and when \(w(x) = 0\) we say that the variable \(\psi_x\) is unassigned (or uncolored). For any non-empty \(X \subseteq \Lambda\), let \([k]^X\) and \([k]_0^X\) denote the sets of configuration and partial configuration in \(X\) respectively. Given \(Y \subseteq X\) and \(w \in [k]^X\) we denote by \(w|_Y\) the restriction of the configuration \(w\) to \(Y\). Of course \(w|_Y \in [k]^Y\).

\textbf{Remark.} More generally, one can also suppose that variables \(\{\psi_x\}_{x \in \Lambda}\) take values in possibly different spaces \(\{\Psi_x\}_{x \in \Lambda}\) but all having a common cardinality \(k\), i.e. such that \(|\Psi_x| = k\) for all \(x \in \Lambda\) and in each space \(\Psi_x\) the random variable \(\psi_x\) takes values according to the uniform distribution. The Example 2 in Section 4 below falls in this more general setting.

Given the pair \((\Lambda, k)\) and \(A \subseteq \Lambda\), a \textit{flaw of} \(A\) is a subset \(\epsilon \subseteq [k]^A\), i.e, it is a tempered bad event in the Moser-Tardos scheme. As before, we write \(\text{supp}(\epsilon) = A\) and we call \(|\text{supp}(\epsilon)|\) \textit{the size of} \(\epsilon\) while \(|\epsilon|\) is the number of configurations forming \(\epsilon\). An event is \textit{elementary} if \(|\epsilon| = 1\). As usual, \(\tilde{\epsilon}\) will denote the complement of \(\epsilon\) in \([k]^A\), i.e. \(\tilde{\epsilon} = [k]^A \setminus \epsilon\).

Given a family \(\mathcal{F}\) of tempered events, a \textit{a good configuration w.r.t. to} \(\mathcal{F}\) is a configuration \(\omega \in [k]^\Lambda\).
Remark. We stress once again that virtually all applications of the entropy compression method available in the literature fall in this uniform variable setting, with the sole exception, as far as we know, of the acyclic edge coloring of bounded degree graphs (see [7]).

Definition 1.6 Given an event $e$, a non empty subset $X \subseteq \text{supp}(e)$ is called a seed of $e$ if $e|_X = [k]^X$ and $|e| = k^{|X|}$. An event $e$ is said tidy if either it is elementary, or it is such that for all $y \in \text{supp}(e)$, there exists a non-empty set $X \subseteq \{\text{supp}(e) \setminus \{y\}\}$ which is a seed of $e$.

Clearly, by definition, all seeds of a tidy event $e$ must have all the same cardinality which we denote by $\kappa(e)$. If $e$ is elementary, we set $\kappa(e) = 0$. We further set

$$\|e\| = |\text{supp}(e)| - \kappa(e)$$

and refer to $\|e\|$ as the power of the event $e$. Note that $\|e\| = |\text{supp}(e)|$ if and only if $e$ is elementary. Moreover, the properties listed here below follow immediately from Definition 1.6:

1. If $e$ is tidy and $X$ is a seed of $e$, then any configuration $w \in e$ is uniquely determined by its restriction to $X$ and no $Y \subseteq X$ has this property.

2. If $e$ is tidy, then

$$\text{Prob}(e) = \frac{1}{k^{\|\text{supp}(e)|-\kappa(e)}}.$$ 

3. Let $\mathcal{F}$ be a family of tempered and tidy events. Suppose that an event $e$ is occurring in a given configuration and $X \subseteq \text{supp}(e)$ is a seed of $e$. If we resample all the variables in $\text{supp}(e) \setminus X$ leaving all other variables unchanged, then in the so obtained new configuration the probability for an event $e' \in \mathcal{F}$ to occur is $\text{Prob}(e')$.

Remark 1. In [7] the definition of seed is slightly more general. Namely it coincides with item 1 of the above list: a seed $X \subseteq \{\text{supp}(e) \setminus \{y\}\}$ is such that any coloring $w \in e$ is uniquely determined by its restriction to $X$ and no $Y \subseteq X$ has this property. The difference is subtle. As an example, suppose that the edges of a graph $G = (V,E)$ are colored at random using $k$ colors, uniformly and independently and consider a cycle $C$ of $G$ with an even number of edges and then let the event $e_C$ be “$C$ is properly bichromatic”, i.e. its edges are colored with two colors and no pair of adjacent edges are monochromatic. Then any cherry (i.e. two incident edges) of $C$ is a seed of $e_C$ according to the definition given in [7], while $e_C$ is not tidy according to the definition 1.6 since the restriction of $e_C$ to any cherry $c \in C$ is not in $[k]^c$, once the monochromatic configurations of $c$ are not allowed if $C$ is properly bichromatic. On the other hand given a path $p$ of $G$ constituted by an even number of edges, let $e_p$ be the event “the second half of $p$ is colored in the same way as the first half”, then $e_p$ is tidy according to the Definition 1.6. For example, we can set as a seed the set of edges constituting the first half of the path or the set of edges constituting the second half of the path.

Remark 2. By definition, a tidy flaw has the empty set as its unique seed if and only if is elementary. Note that if $e$ is not tidy, then it can be seen as the disjoint union of tidy (in the worst case elementary) flaws. Therefore there is no loss of generality in considering only families in which all flaws are tidy.
We introduce the following notations. 

\[ \mathcal{F}_s(y) = \{ \epsilon \in \mathcal{F} : y \in \text{supp}(\epsilon) \text{ and } \|\epsilon\| = s \} \]

and 

\[ d_s = \max_{y \in \Lambda} |\mathcal{F}_s(y)|. \] (1.15)

Namely, \( d_s \) is an upper bound for the number of events with power \( s \) whose support contains a common element of \( \Lambda \). We finally define 

\[ E'_\delta = \{ s \in \mathbb{N} : \exists \epsilon \in \mathcal{F} \text{ such that } \|\epsilon\| = s \}. \] (1.16)

### 1.6.2 The entropy compression algorithm and the entropy compression Lemma

We assume that a total order has been chosen in the sets \( \Lambda \) and \( \mathcal{F} \). We choose, for each \( y \in \Lambda \) and \( \epsilon \in \mathcal{F}(y) \), a unique subset \( G(\epsilon, y) \subset \text{supp}(\epsilon) \setminus \{y\} \) such that \( G(\epsilon, y) \) is a seed of \( \epsilon \). We also denote shortly \( G^c(\epsilon, y) = \text{supp}(\epsilon) \setminus G(\epsilon, y) \). Note that \( y \in G^c(\epsilon, y) \). Given a partial coloring \( w \in [k]^\Lambda_0 \), given \( X \subset \Lambda \) and given \( s \in [k] \cup \{0\} \) we denote by \( w^s_X \) the partial coloring which coincides with \( w \) in the set \( \Lambda \setminus X \) and it takes the value \( s \) at every \( x \in X \). If \( X = \{x\} \) we set shortly \( \omega^s_X = \omega^s_{x} \).

Let \( t \) be an arbitrary natural number (which can be taken as large as we please) and let \( V_t \) be an element of \( [k]^t \), i.e. \( V_t \) is a vector with \( t \) entries such that each entry takes values in the set \( [k] \).

The algorithm **ENTROPY COMPRESSION** has input \( V_t \), performs (at most) \( t \) steps and, at each step \( i \in [t] \), produces a partial coloring \( w_i \) as described below.

| ENTROPY COMPRESSION (with input \( V_t \)) |
|-----------------------------------------|
| - Step 0. Set \( w_0 = 0 \), i.e. in the beginning no element \( \psi_x \), with \( x \in \Lambda \), is colored. |
| - Step \( i \) (for \( i \geq 1 \)). |
| \( i^0 \) If \( w_{i-1}^{-1}(0) \neq \emptyset \), let \( y \) be the smallest element of \( \Lambda \) (in the total order chosen) such that \( \psi_y \) is uncolored in the partial coloring \( w_{i-1} \). Take the \( i^{th} \) entry of the vector \( V_t \) and let \( s \in [k] \) be this entry. Color \( \psi_y \) with the color \( s \) and consider the partial coloring \( w_{i-1}^s \) obtained from \( w_{i-1} \) by coloring \( \psi_y \) with the color \( s \). |
| \( i^1 \) If no flaw occurs in \( w_{i-1}^s \), set \( \omega_i = w_{i-1}^s \) and go to the step \( i + 1 \). |
| \( i^2 \) Conversely, if some flaw occur in \( w_{i-1}^s \), select the smallest, say \( \epsilon \), which by construction belongs to the set \( \mathcal{F}(y) \). Set \( w_i = w_{i-1}^{0}_{G^c(\epsilon,y)} \) and go to the step \( i + 1 \). In words, \( w_i \) is obtained from \( w_{i-1} \) by discoloring all \( \psi_x \) such that \( x \in \text{supp}(\epsilon) \setminus G(\epsilon, y) \). |
| \( i^* \) If \( w_{i-1}^{-1}(0) = \emptyset \), stop the algorithm discarding all entries \( v_i, v_{i+1}, \ldots, v_t \) of \( V_t \). |

Note that the partial coloring \( w_i \) returned by the algorithm at the end of each step \( i \) necessarily avoids all flaws in \( \mathcal{F} \). **ENTROPY COMPRESSION** performs at most \( t \) steps but it can stop earlier, i.e. after having performed \( m < t \) steps and \( \omega_m^{-1}(0) = \emptyset \). In this case only the first \( m \) entries of
the vector \( V_t \) are used. ENTROPY COMPRESSION is successful if it stops after \( m < t \) steps, or it lasts \( t \) steps and after the last step \( t \) we have \( \omega_t^{-1}(0) = \emptyset \). Conversely, ENTROPY COMPRESSION fails if it performs all \( t \) steps and \( \omega_t^{-1}(0) \neq \emptyset \). Clearly when ENTROPY COMPRESSION is successful \( \omega_t \) is a good configuration. Observe that ENTROPY COMPRESSION can be either deterministic, if \( V_t \) is a given prefixed vector, or random, if the entries of \( V_t \) are uniformly sampled from the set \([k]\) sequentially and independently. In [7] the following theorem is proved.

**Theorem 1.7 (Entropy compression lemma)** Assume that a pair \((\Lambda, k)\) is given together with a family \( \mathcal{F} \) of tempered and tidy flaws. If there is \( \alpha > 0 \) such that

\[
1 + \sum_{s \in E'_t} d_s \alpha^s e^a < k, \tag{1.17}
\]

then

\[
\bigcap_{e \in \mathcal{F}} \bar{e} \neq \emptyset.
\]

Moreover, ENTROPY COMPRESSION finds a configuration \( \omega \in \bigcap_{e \in \mathcal{F}} \bar{e} \) in an expected number of steps linear in \(|\Lambda|\).

We will refer to the inequality (1.17) as the *entropy compression criterion*. We stress that this theorem is able to reproduce all results obtained in these last years via the entropy compression method.

It is now simple to compare the above entropy compression criterion with the global CELL criterion (1.13). Since we are in the entropy compression setting determined by the pair \((\Lambda, k)\), the probability space is generated by \(|\Lambda|\) i.i.d. uniformly distributed random variables taking values in \([k]\). As recalled above, in the restricted variable setting covered by the entropy compression method, we have that \( \text{Prob}(\epsilon) = \frac{1}{k^{\|\epsilon\|}} \equiv p_{\|\epsilon\|} \). Setting

\[
q = \max_{e \in \mathcal{F}} \left\{ \frac{\|\supp(\epsilon)\|}{\|\epsilon\|} \right\}, \tag{1.18}
\]

and grouping events in terms of their powers we get

\[
\sup_{x \in \Lambda} \sum_{\epsilon \in \mathcal{F}} \sum_{x \in \supp(\epsilon)} \text{Prob}(\epsilon) e^{a \|\supp(\epsilon)\|} \leq \sum_{s \in E'_t} d_s p_s e^{aq s}. \]

So condition (1.13) is fulfilled if

\[
\sum_{s \in E'_t} d_s p_s e^{aq s} \leq e^a - 1, \tag{1.19}
\]

or, setting \( \alpha = \frac{e^{aq}}{k} \), if there is \( \alpha > 0 \) such that

\[
\left( 1 + \sum_{s \in E'_t} d_s \alpha^s \right)^q \leq k. \tag{1.20}
\]

The reader can immediately compare (1.20) with the entropy compression condition (1.17). The presence of the exponent \( q \) defined in (1.18) in inequality (1.20) is the only reason why entropy compression condition (1.17) can give better bounds than LLL. It must however be stressed that we are excluding here the case of the acyclic edge coloring of a graph with maximal degree.
In this pretty singular case, the entropy compression scheme is fruitfully combined with the crucial observation that it is possible to properly color the edges of \( G \) using just \( 2\Delta - 1 \) color in such a way to avoid bichromatic cycles of length 4 (see Lemma 4 in \cite{7}). This fruitful strategy leaded Esperet and Parreau to a very sensible improvement of the upper bound of the acyclic edge chromatic index of \( G \) with respect to the bound obtained via CELL. Due to its specificity, the case of the acyclic edge coloring must be treated separately (see comments below and see also the remark in Section 4.2.5 of \cite{7}).

### 1.7 Motivations and plan of this paper

Concluding this introduction, we need to mention two recent papers, \cite{32} and \cite{25}, proposing a variant of the Moser-Tardos resampling algorithm, which has actually motivated the present paper. In particular, in \cite{25} Giotis et al. are able to slightly improve just the specific case of the acyclic edge chromatic index of a graph with maximum degree \( \Delta \). The intriguing fact is that Giotis et al. use in \cite{25} the Moser-Tardos resampling algorithm with the unique variant that the successive resampled bad events must be chosen, when possible, in the neighbor of the previous bad event. Their result is somehow surprising considering that, as mentioned above, the CELL criterion applied to acyclic edge coloring gives a much worse bound than entropy compression method.

In the present paper we manage to combine the ideas of \cite{25} (also foreshadowed in \cite{32}) with the observation explained above that the power of an event (possibly tidy) can be considered in place of the cardinality of its support and we show that the criterion (1.17) based on the backtracking algorithm ENTROPY COMPRESSION can be reobtained in the usual Moser-Tardos scheme by doing the slight modification of the algorithm RESAMPLING illustrated in \cite{32} and \cite{25} (which give rise to forests instead of tree as a register of the steps of the algorithm) jointly with the prescription proposed by Esperet-Parreau to not resample certain variables of the bad events (the previously seen “seeds” of the events).

In the very specific and singular case of acyclic edge coloring, the Moser-Tardos modified algorithm presented in this paper is able to further slightly improve the bound obtained in \cite{25}, and this latter issue is the subject of a separate paper \cite{22}.

The rest of the paper is organized as follows. In Section 2 we describe the variant of the Moser-Tardos algorithm and state our main result, i.e. Theorem 2.2. Section 3 is devoted to the proof of Theorem 2.2. Finally in Section 4 we present some examples.

### 2 A variant of the Moser-Tardos Algorithm

Let us consider the general Moser-Tardos framework. Given a finite set \( \Lambda \) with cardinality \( m = |\Lambda| \), let \( \psi_\Lambda \equiv \{\psi_x\}_{x \in \Lambda} \) be a set of \( m \) mutually independent random variables such that each \( \psi_x \) takes values in \( \Psi_x \) and let \( \Omega_\Lambda = \prod_{x \in \Lambda} \Psi_x \) be the product probability space generated by these variables. Let \( \mathcal{F} \) be a finite collection of tempered events in \( \Omega \). We recall that for each event \( \varepsilon \in \mathcal{F} \) there exists a subset \( \text{supp}(\varepsilon) \subset \Lambda \) such that \( \varepsilon \) depends on variables \( \{\psi_x\}_{x \in \text{supp}(\varepsilon)} \).

As usual, if \( U \subset \Lambda \), we set \( \psi_U = \{\psi_x\}_{x \in U} \) and \( \Omega_U = \prod_{x \in U} \Psi_x \). Moreover, given a random configuration \( \omega \in \Omega_\Lambda \), \( \text{Prob}(\varepsilon) \) denotes the probability of the event \( \varepsilon \in \mathcal{F} \) to occur. We also recall that if \( \varepsilon \) is a tempered event such that there is a unique configuration \( \omega \in \Omega_{\text{supp}(\varepsilon)} \) which realizes \( \varepsilon \) we say that \( \varepsilon \) is an elementary event.
Definition 2.1 (Seed) Let $e \in \mathcal{F}$. A proper subset $U \subsetneq \text{supp}(e)$ is called a “seed” of $e$ if it is such that, given a configuration $\omega \in \Omega$ such that the event $e$ is occurring, if we resample all the variables in $\text{supp}(e) \setminus U$ leaving unchanged the values of all the other variables, then in the new configuration $\omega'$ so obtained all the events $e' \in \mathcal{F}$ have at most $\text{Prob}(e')$ to happen and any $U' \supset U$ has not this property. We denote by $S_e$ the set of all seeds of $e$. An event $e \in \mathcal{F}$ is tidy if it is such that all seeds of $e$ have the same non-zero cardinality $\kappa(e)$ and for all $x \in \text{supp}(e)$ there exists $U \in S_e$ such that $x /\in U$.

Remark. Of course, in the uniform variable setting, where the variables $\{\psi_x\}_{x \in \Lambda}$ beside being independents are also identically and uniformly distributed and taking values in the common set $[k] = \{1, \ldots, k\}$, Definition 2.1 and Definition 1.6 are equivalent. It is also important to stress that Definition 2.1 says that once we reach a configuration $\omega$ in which the event $e$ occurs and $U$ is a non-empty seed of $e$, passing to a new configuration $\omega'$ obtained from $\omega$ by resampling only variables $\{\psi\}_{\text{supp}(e) \setminus U}$ does not help any event to happen. The transition $\omega \rightarrow \omega'$ reminds the definition of resampling oracle given in [31].

We define the power of the event $e$ as the number

$$\|e\| = \begin{cases} |	ext{supp}(e)| - \kappa(e) & \text{if } e \text{ is tidy}, \\ |	ext{supp}(e)| & \text{otherwise}. \end{cases} \quad (2.1)$$

Moreover, for any tidy event $e \in \mathcal{F}$ and any $x \in \text{supp}(e)$, we fix a rule to choose uniquely a seed $S_x(e)$ of $e$ such that $x /\in S_x(e)$. If $e$ is either elementary or not tidy we set $S_x(e) = \emptyset$. Note that in any case

$$|	ext{supp}(e)| - |S_x(e)| = \|e\|. \quad (2.2)$$

We will classify the events $e \in \mathcal{F}$ according to their power $\|e\|$. Let $E'_{\mathcal{F}} \subset \mathbb{N}$ be defined as

$$E'_{\mathcal{F}} = \{s \in \mathbb{N} : \exists e \in \mathcal{F} \text{ s.t. } \|e\| = s\}. \quad (2.3)$$

For $s \in E'_{\mathcal{F}}$, we set

$$\mathcal{F}_s = \{e \in \mathcal{F} : \|e\| = s\}. \quad (2.4)$$

Finally, for $x \in \Lambda$ and $s \in E'_{\mathcal{F}}$, we define $d_s(x)$ as

$$d_s(x) = |\{e \in \mathcal{F}_s : x \in \text{supp}(e)\}|, \quad (2.5)$$

and set

$$d_s = \max_{x \in \Lambda} d_s(x). \quad (2.6)$$

Hereafter we will assume that a total order is fixed in the set $\Lambda$ as well as on the set of events $\mathcal{F}$. Following [25], we now describe a procedure, called FOREST-ALGORITHM which samples (and eventually resamples) the variables $\psi_\Lambda$. Given an evaluation $\omega$ of all variables $\psi_\Lambda$, we say shortly that the atom $x \in \Lambda$ is bad if some $e \in \mathcal{F}$ occurs in the evaluation $\omega$ and $x \in \text{supp}(e)$. Otherwise we say that $x$ is good.
Forest-Algorithm.

1. Sample all variables $\psi_A$.

2. While there is a bad atom, select the pair $(x, \epsilon)$ where $x$ is the smallest bad atom and where $\epsilon$ is the smallest event occurring such that $x \in \text{supp}(\epsilon)$, and do

3. Resample $(x, \epsilon)$.

4. End while.

5. Output current evaluation.

Resample $(x, \epsilon)$

1. Resample all variables $\psi_y$ such that $y \in \text{supp}(\epsilon) \setminus S_x(\epsilon)$.

2. While there is a bad atom in $\text{supp}(\epsilon) \setminus S_x(\epsilon)$, let $x'$ be the smallest of these atoms and let $\epsilon'$ be the smallest event occurring such that $x' \in \text{supp}(\epsilon')$ and do

3. Resample $(x', \epsilon')$.

4. End while.

A step of Forest-Algorithm is the procedure described in Line 2 of Resample $(x, \epsilon)$. Observe that, since $x \notin S_x(\epsilon)$ for any $\epsilon$ such that $x \in \text{supp}(\epsilon)$, in Resample $(x, \epsilon)$ the variable $\psi_x$ is always resampled. A phase of Forest-Algorithm is the collection of steps made by Forest-Algorithm during a call of Resample $(x, \epsilon)$ in Line 3 of Forest-Algorithm. Note that during a phase many steps occur, the first step of the $i$-th phase will be called the root of the phase $i$. The record of the algorithm is the list

$$L = ((x_1, \epsilon_1), (x_2, \epsilon_2), \ldots)$$

constituted by the steps done by the algorithm during its execution. We will denote by atom label (resp. event label) any atom (resp. event) listed in the record $L$. According to the prescriptions described above, $L$ is a random variable determined by the random samplings performed by the algorithm in each step. If $L$ is finite, i.e. if $|L| = n$ for some $n \in \mathbb{N}$, then the algorithm terminates having performed $n$ steps and produces an evaluation $\omega \in \bigcap_{\epsilon \in \bar{\Gamma}} \epsilon$. Let us define

$$P_n = \text{Prob}(|L| = n). \quad (2.7)$$

In other words $P_n$ is the probability that Forest-Algorithm runs $n$ steps.

We are now in the position to state the main result of this paper. To do this we introduce the following notations. For $s \in E'_\delta$ and $\xi > 0$, define

$$p_s = \max_{\epsilon \in \delta_s} \text{Prob}(\epsilon), \quad (2.8)$$

$$\phi_\delta(\xi) = \sum_{s \in E'_\delta} p_s d_s (\xi + 1)^s. \quad (2.9)$$
**Theorem 2.2** Given a finite set $\Lambda$ and a family of mutually independent random variables $\psi_\Lambda$, let $\mathcal{F}$ be a family of tempered events depending on $\psi_\Lambda$. Suppose that
\[
\min_{\xi>0} \frac{\phi_3(\xi)}{\xi} < 1,
\]
then there is an evaluation of the variables $\psi_\Lambda$ such that none of the events in the family $\mathcal{F}$ occur. Moreover, FOREST-ALGORITHM finds a configuration $w \in \bigcap_{e \in \mathcal{F}} \bar{e}$ in an expected number of steps polynomial in $m = |\Lambda|$.

**Remark.** Note that (2.10) is completely equivalent to the condition (1.17) of entropy compression lemma (Theorem 1.7). Indeed, in the uniform variable setting defined by the pair $(\Lambda, k)$, we have that $p_s = \frac{1}{k^\alpha}$ and therefore, posing $\alpha = (\xi + 1)/k$, condition (2.10) is rewritten in the form (1.17).

3 Proof of Theorem 2.2

Let us start by proving some important properties of FOREST-ALGORITHM.

**Lemma 3.1** Consider any call of RESAMPLE($x, \epsilon$) and let $Y$ be the set of all good atoms at the beginning of this call. If this call finishes, then all the atoms in $Y \cup \{\supp(\epsilon) \setminus S_x(\epsilon)\}$ are good.

**Proof:** According to the algorithm if RESAMPLE($x, \epsilon$) finishes then $\supp(\epsilon) \setminus S_x(\epsilon)$ are good atoms, so we just need to prove that the atoms in $Y$ continue to be good in the end of RESAMPLE($x, \epsilon$). Let $y \in Y$, and assume that RESAMPLE($x, \epsilon$) finishes and performs $n$ steps. Suppose by contradiction that after these $n$ steps performed by RESAMPLE($x, \epsilon$) $y$ is bad. Then there exists a last step $t \leq n$ of RESAMPLE($x, \epsilon$) such that $y$ was good at step $t - 1$, became bad at step $t$ and stayed bad during the remaining $n - t$ steps of RESAMPLE($x, \epsilon$). This means that there is an event $\epsilon'$ and an atom $z \in \supp(\epsilon')$ such that RESAMPLE($z, \epsilon'$) was called at step $t - 1$ and $y \in \supp(\epsilon') \setminus S_z(\epsilon')$ became bad as soon as the variables $\psi_{\supp(\epsilon') \setminus S_z(\epsilon')}$ were resampled. But RESAMPLE($z, \epsilon'$) must end at a step $t' > t$ and at this step all variables of $\supp(\epsilon') \setminus S_z(\epsilon')$ must be good and thus $y$, which belongs to $\supp(\epsilon') \setminus S_z(\epsilon')$, is good at step $t' > t$ in contradiction with the assumption. 

**Lemma 3.2** FOREST-ALGORITHM performs at most $m = |\Lambda|$ phases.

**Proof.** Consider two phases $l$ and $s$, with $l < s$, generated by an execution of FOREST-ALGORITHM and let $(x_l, \epsilon_l)$ and $(x_s, \epsilon_s)$ be the pairs resampled at their initial steps respectively, i.e, the roots of phase $l$ and $s$ respectively. By Lemma 3.1 all atoms in $\supp(\epsilon_l) \setminus S_{x_l}(\epsilon_l)$ are good when phase $l$ ends and at the beginning of any successive phase. In particular, since $x_l \in \{\supp(\epsilon_l) \setminus S_{x_l}(\epsilon_l)\}$, $x_l$ is good and thus $x_l \notin \supp(\epsilon_s)$. In conclusion $x_l \neq x_s$.

3.1 Witness Forest

We will associate to an execution of FOREST-ALGORITHM a labeled forest formed by plane rooted trees whose vertices are labeled with pairs $(x, \epsilon)$ belonging to $\mathcal{L}$.
Suppose that the algorithm performs \( r \) phases and during the phase \( s \), \( s \in \{1, \ldots, r\} \), the algorithm performs \( n_s \) steps, in such a way that the record of the algorithm is

\[
\mathcal{L} = (\langle x_1^1, \epsilon_1^1 \rangle, \ldots, \langle x_{n_1}^1, \epsilon_{n_1}^1 \rangle, \langle x_1^2, \epsilon_1^2 \rangle, \ldots, \langle x_{n_2}^2, \epsilon_{n_2}^2 \rangle, \ldots, \langle x_1^r, \epsilon_1^r \rangle, \ldots, \langle x_{n_r}^r, \epsilon_{n_r}^r \rangle). \tag{3.11}
\]

At each phase \( s \), \( 1 \leq s \leq r \), we will associate a tree \( \tau'_s \). Let

\[
\langle x_1^s, \epsilon_1^s \rangle, \ldots, \langle x_{n_s}^s, \epsilon_{n_s}^s \rangle, \tag{3.12}
\]

be the pairs resampled at phase \( s \). We construct the tree \( \tau'_s \) in the following way.

a) The root of \( \tau'_s \) has label \( \langle x_1^s, \epsilon_1^s \rangle \).

b) For \( i > 1 \), we proceed by checking if \( \langle x_i^s, \epsilon_i^s \rangle \) is such that \( x_i^s \in (\text{supp}(\epsilon_{i-1}^s) \setminus S_{x_i^{s-1}}(\epsilon_{i-1}^s)) \),

- if yes, we add \( \langle x_i^s, \epsilon_i^s \rangle \) as a child of \( \langle x_{i-1}^s, \epsilon_{i-1}^s \rangle \),

- if no, we go back in (3.12) checking the ancestors of the vertex labeled by \( \langle x_{i-1}^s, \epsilon_{i-1}^s \rangle \) until we find a pair \( \langle x_j^s, \epsilon_j^s \rangle \), with \( j < i \), such that \( x_i^s \in (\text{supp}(\epsilon_j^s) \setminus S_{x_j^{s-1}}(\epsilon_j^s)) \), and we add \( \langle x_i^s, \epsilon_i^s \rangle \) as a child of \( \langle x_j^s, \epsilon_j^s \rangle \).

Observe that by the construction of \textsc{Forest-Algorithm}, all pairs \( \langle x_i^s, \epsilon_i^s \rangle \) can be added to \( \tau'_s \) in this way, then \( \tau'_s \) has \( n_s \) vertices (leaves included) with labels \( \langle x_i^s, \epsilon_i^s \rangle \) with \( i = 1, \ldots, n_s \). By Lemma 3.1 the pair \( \langle x_i^{s+1}, \epsilon_i^{s+1} \rangle \) is the first pair in (3.11) that can not be added to \( \tau'_s \) in this way, so we build a new tree \( \tau'_{s+1} \) with root \( \langle x_i^{s+1}, \epsilon_i^{s+1} \rangle \) following the same rule described to build \( \tau'_s \).

Note that the vertices of the forest defined above are naturally ordered according to the natural order of the steps made by the algorithm. The forest \( F' = \{\tau'_1, \ldots, \tau'_r\} \) so obtained uniquely associated to the record \( \mathcal{L} \) is such that, for each \( s \in [r] \), \( \tau'_s \) is a rooted plane tree with \( n_s \) vertices and each vertex of \( \tau'_s \) has label \( \langle x, \epsilon \rangle \) where \( x \in \text{supp}(\epsilon) \) and \( \epsilon \in \mathfrak{E} \).

Note that, by Lemma 3.2 we have that \( r \leq m \) and thus the forest \( F' \) contains at most \( m \) trees.

Note also that in each tree \( \tau'_s \) of \( F' \) the list of labels of the vertices of \( \tau'_s \) ordered according to the depth-first search, coincides with the list (3.12).

Note finally that, by construction, the correspondence \( \mathcal{L} \mapsto F' \) is an injection.

\textbf{Lemma 3.3} Consider a tree \( \tau' \in F' \), and let \( v_i \) and \( v_j \) be two vertices in \( \tau' \) with labels \( \langle x_i, \epsilon_i \rangle \) and \( \langle x_j, \epsilon_j \rangle \) respectively. We have that

a) If \( v_i \) is a child of \( v_j \), then \( x_i \in \text{supp}(\epsilon_j) \setminus S_{x_j}(\epsilon_j) \).

b) If \( v_i \) and \( v_j \) are siblings in \( \tau' \), then \( x_i \neq x_j \).

c) Any vertex \( v \in \tau' \) with label \( \langle x, \epsilon \rangle \) has at most \( ||\epsilon|| \) children, where \( ||\epsilon|| \) is defined in (2.7).

\textbf{Proof.}\n
a) It is trivial by construction of the algorithm.

b) As \( v_i \) and \( v_j \) are siblings, suppose that \( v_i \) and \( v_j \) are the \( i \)-th and the \( j \)-th children of a vertex in \( \tau' \), with \( i < j \) in the natural order of the vertices of \( \tau' \) induced by the steps of the algorithm. For \( q \) such that \( i \leq q < j \), let \( \langle x_q, \epsilon_q \rangle \) be the label of the \( q \)-th sibling. By Lemma 3.1 when \textsc{Resample}(\( x_q, \epsilon_q \)) ends all the atoms \( x_1, \ldots, x_q \) are good as well the atoms in \( \langle \text{supp}(\epsilon_1) \setminus S_{x_1}(\epsilon_1) \rangle \cup \ldots \cup \langle \text{supp}(\epsilon_q) \setminus S_{x_q}(\epsilon_q) \rangle \). Therefore \( x_j \) can not be in the set \( \{x_i, x_{i+1}, \ldots, x_{j-1}\} \).

c) Follows trivially from items a) and b).
Given a forest $F'$ produced by the algorithm, we let $X_{F'}$ be the set of atoms which label the roots of the trees of the forest, i.e.,

$$X_{F'} = \{ x \in \Lambda : \exists e \in F' \text{ such that } (x, e) \text{ is the root label of some } \tau' \in F' \}.$$  

Lemma 3.2 implies that atoms in $X_{F'}$ are all distinct.

**Definition 3.4 (Witness forest)** Given the record $L$ of Forest-Algorithm and the forest $F'$ associate to $L$, we construct a new forest $F$ by adding to $F'$ new vertices in the following way:

1) Add to the forest $F'$ as many isolated vertices as the atoms which are in $\Lambda \setminus X_{F'}$, and give to these isolated vertices the label $(x, \emptyset)$ for all $x \in \Lambda \setminus X_{F'}$.

2) For each vertex $v$ of the forest $F'$ with label $(x, e)$ with less than $\|e\|_c$ children, do the following: let $H_v$ be the set of atoms in $\text{supp}(e) \setminus S_x(e)$ which are not atoms labels of the children of $v$.

   For each $y \in H_v$ we add to $v$ a leaf with label $(y, \emptyset)$ in such a way that $v$ has now exactly $\|e\|_c$ children.

The new labeled forest $F$, so obtained uniquely associated to the random variable $L$ by the prescriptions described above, is called the *witness forest* produced by Forest-Algorithm. This witness forest $F$ has, by construction, the following properties.

**Properties of the witness forest $F$.**

1. $F$ is constituted by exactly $|\Lambda| = m$ labeled rooted trees $\tau_1, \ldots, \tau_m$ (some of which are just isolated vertices).

2. Let the vertex $u$ be a child of the vertex $v$ in $\tau \in F$ and let $(x_u, e_u)$ and $(x_v, e_v)$ be their labels respectively. Then $x_u \in \text{supp}(e_v)$.

3. Each internal vertex $v$ of $\tau \in F$ carries a label $(x_v, e_v)$ where $x_v \in \text{supp}(e_v)$ and $e_v \in \mathcal{F}$, while each leaf $\ell$ of $\tau$ carries a label $(x_\ell, \emptyset)$ and $x_\ell \in \text{supp}(e_\ell)$, where $w$ is the vertex parent of $\ell$.

4. Let the vertices $v$ and $v'$ be the $i^{th}$ and the $j^{th}$ siblings in $\tau \in F$, with $i < j$ in the depth-first search order of $\tau$, and let $(x_i, e_i)$ and $(x_j, e_j)$ be their labels respectively, then $x_i \neq x_j$.

5. Let $v$ be an internal vertex of $\tau \in F$ and let $(x, e)$ be its label, then the vertex $v$ has exactly $\|e\|_c$ children.

Let $F_n$ be the set of labeled forests satisfying properties 1-5 above, that contains $n$ internal vertices in total and let $F = \bigcup_{n \geq 0} F_n$.

It is important to stress that the map $L \mapsto F$ is an injection. Therefore, since Forest-Algorithm lasts $n$ steps if and only if the witness forest associated to the record $L$ of Forest-Algorithm has $n$ internal vertices, the probability $P_n$ defined in (2.7) can be written as

$$P_n = \text{Prob}($the witness forest associated to $L$ has $n$ internal vertices$).$$  

Then the next goal is to estimate the probability that Forest-Algorithm produces a witness forest $F$ with $n$ internal vertices.
3.2 The validation algorithm

**Definition 3.5 (Admissible sequence)** We say that a sequence \( S = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \) is admissible if \( x_i \in \text{supp}(\varepsilon_i) \), for all \( i = 1, \ldots, n \).

Given a witness forest \( F \) with \( n \) internal vertices, we can associate to \( F \), in a natural way, the admissible sequence \( S_F = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \) formed by the labels of its internal vertices. Namely, the sequence \( S_F \) coincides with (3.11).

We now describe a validation algorithm, called S-CHECK, whose input is an admissible sequence \( S = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \). S-CHECK first samples all variables in \( \psi \Lambda \) and then resamples some of the variables in \( \psi \).

\[
\text{S-CHECK.}
\]

Given the admissible sequence \( S = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \)

1. Sample all variables in \( \psi \Lambda \).
2. For \( i = 1, \ldots, n \), do
3. If \( \varepsilon_i \) occurs, resample all the variables \( \psi_y \) with \( y \in \text{supp}(\varepsilon_i) \setminus S_{x_1}(\varepsilon_i) \). If the event \( \varepsilon_i \) does not occur, return failure.
4. End for.

The procedure described at line 3 of S-CHECK is called a step. Of course, if \( S = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \) is the input for S-CHECK, its execution will perform exactly \( n \) steps if it does not return failure. Observe that S-CHECK does not return failure (i.e. passes) if, and only if, in each step \( i \) the event \( \varepsilon_i \) occurs under the current evaluation of the variables.

**Lemma 3.6** Let \( S = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\} \) be an admissible sequence. Then

\[
\text{Prob}(\text{S-check with input S passes}) \leq \prod_{i=1}^{n} \text{Prob}(\varepsilon_i). \tag{3.14}
\]

**Proof.** Consider the first step of S-CHECK: we sample all variables \( \psi \Lambda \) reaching a configuration \( \omega_0 \) and we have to check if the event \( \varepsilon_1 \) happens, if \( \varepsilon_1 \) does not happen we stop, otherwise we resample the variables in \( \text{supp}(\varepsilon_1) \setminus S_{x_1}(\varepsilon_1) \). As \( S_{x_1}(\varepsilon_1) \) is a seed, we have that the new configuration \( \omega_1 \) is such that any event \( \varepsilon \) has probability to occur at most \( \text{Prob}(\varepsilon) \). Therefore, by induction, at each step \( i \) the probability of \( \varepsilon_i \) to occur is at most \( \text{Prob}(\varepsilon_i) \). As S-CHECK is successful if and only if all events \( \varepsilon_i \) occur, then

\[
\text{Prob}(\text{S-check with input S passes}) \leq \prod_{i=1}^{n} \text{Prob}(\varepsilon_i). \tag{3.15}
\]

**Lemma 3.7** Given a witness forest \( F \in \mathcal{F}_n \) whose internal vertices carry labels

\[
S_F = \{(x_1, \varepsilon_1), \ldots, (x_n, \varepsilon_n)\},
\]

we have that

\[
\text{Prob}(\text{Forest-Algorithm produces F}) \leq \prod_{i=1}^{n} \text{Prob}(\varepsilon_i). \tag{3.16}
\]
**Proof.** Observe that if all the random choices made by an execution of **Forest-Algorithm** that produces \( F \) as witness forest are also made by the Algorithm **S-Check** with input \( S_F \), then in each step \( i \) the event \( \epsilon_i \) occurs and so **S-Check** does not return failure. Then

\[
\text{Prob(the Forest-Algorithm produces } F) \leq \text{Prob(S-Check with input } S_F \text{ passes).} \tag{3.17}
\]

Now (3.16) follows from Lemma 3.6 □

**Remark.** In the \( i \)-th phase of the **S-Check** just the values of the variables \( \psi_{\supp(e_i) \setminus S_{\epsilon_i}} \) are resampled. So, at the beginning of phase \( i+1 \) the distribution of the evaluation of the variables is the same as the Line 1. This is not the case for **Forest-Algorithm**, once it would mean that this algorithm is not making any progress in the search of a configuration such that any event in \( \mathcal{S} \) occurs.

### 3.3 The unlabeled Forest

The strategy to prove Theorem 2.2 is to show that the probability that **Forest-Algorithm** lasts at least \( n \) steps decays exponentially in \( n \), which implies that **Forest-Algorithm** terminates almost surely, returning an evaluation of \( \psi_{\Lambda} \) such that all events in \( \mathcal{F} \) do not occur.

If **Forest-Algorithm** lasts \( n \) steps then it produces a witness forest with \( n \) internal nodes. Recall that, if the internal vertex \( v \) of the witness forest has event label \( \epsilon_v \) then this vertex has exactly \( s_v = \|\epsilon_v\| \) children.

Let \( \mathcal{F}_n \) be the set of all unlabeled forests constituted by \( |\Lambda| = m \) plane trees having in total \( n \) internal vertices and such that each internal vertex \( v \) has a number of children in the set \( E_v' \) defined in (2.3). Given the record \( L \) of **Forest-Algorithm** such that \( |L| = n \) and given the witness forest \( F \in \mathcal{F}_n \) associated to \( L \), we define the function

\[
f : \mathcal{F}_n \rightarrow \mathcal{F}_n^* \\
f(F) = \Phi
\]

that removes all the labels of \( F \) obtaining an unlabeled witness forest \( \Phi \in \mathcal{F}_n^* \). We call \( \Phi \) the **associated unlabeled witness forest** produced by **Forest-Algorithm**. Given an internal vertex \( v \) of an unlabeled forest \( \Phi \in \mathcal{F}_n^* \), we let \( s_v \) be the number of children on \( v \).

For \( \Phi \in \mathcal{F}_n^* \) let us define

\[
P_\Phi = \text{Prob}(\Phi \text{ is the associated unlabeled witness forest produced by **Forest-Algorithm**}). \tag{3.18}
\]

\[
P_\Phi = \sum_{F \in \mathcal{F}_n, f(F) = \Phi} \text{Prob(the Forest-Algorithm produces the witness forest } F) \tag{3.19}
\]

\[
\leq \sum_{F \in \mathcal{F}_n, f(F) = \Phi} \prod_{v \in F} \text{Prob}(\epsilon_v) \tag{3.20}
\]

\[
\leq \prod_{v \in \Phi} p_{s_v} \sum_{F \in \mathcal{F}_n, f(F) = \Phi} 1, \tag{3.21}
\]

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where $p_s$ is defined in (2.8) and inequality (3.20) is due to Lemma 3.7. Now observe that

$$
\sum_{\mathcal{F} \in \mathcal{F}_n} 1 \leq \prod_{v \in \Phi} d_{s_v},
$$

(3.22)

since for each vertex $v \in \Phi$ with $s_v$ children, we have $d_{s_v}$ options for its event label, and fixed the atom label and the event label of the parent of $v$, we can determine uniquely the atom label of $v$. Indeed, suppose that the parent of $v$ is the vertex $u$, which has $s_u$ children, and $x_u$ is its atom label, then for the event label of $u$ we have $d_{s_u}$ options, and once fixed the event label of $u$, suppose $e_u$, we know the atom labels of all children of $u$, namely they are, in order, the atoms in $\text{supp}(e_u) \setminus S_{x_u}(e_u)$. So, if $v$ is the $i$-th child of $u$, then its atom label is the $i$-th atom in $\text{supp}(e_u) \setminus S_{x_u}(e_u)$. Proceeding recursively, observe that what we need to know is the atom label of the roots of each tree in $\Phi$, however this information is easily obtained by the construction of a witness forest, as the trees are organized by the atom labels of their roots.

Then,

$$
P_{\Phi} \leq \prod_{v \in \Phi} d_{s_v}p_{s_v}.
$$

(3.23)

We now can bound the probability $P_n$ (see (3.13)) that FOREST-ALGORITHM lasts $n$ steps as

$$
P_n \leq \sum_{\Phi \in \mathcal{F}_n^*} P_{\Phi}.
$$

To estimate $\sum_{\mathcal{F} \in \mathcal{F}_n} P_{\Phi}$, observe that every forest $\Phi \in \mathcal{F}_n^*$ is constituted by $m$ trees $\tau_1, \ldots, \tau_m$ with $n_1, \ldots, n_m$ internal vertices respectively. The numbers $n_1, \ldots, n_m$ are such that $n_i \geq 0$ for all $i = 1, \ldots, m$ and $n_1 + n_2 + \ldots + n_m = n$. Recall also that the number of children of the internal vertices of any $\tau_i$ takes values in the set $E'_F$. Let us denote by $\mathcal{T}$ the set of plane trees with number of children of the internal vertices taking values in the set $E'_F$ and let $\mathcal{T}_n$ be the subset of $\mathcal{T}$ formed by the trees with exactly $n$ internal vertices.

Let us denote shortly, for $s \in E'_F$,

$$
w_s = d_s p_s.
$$

(3.24)

For a tree $\tau \in \mathcal{T}$, let $V_{\tau}$ be the set of its internal vertices. Then define the weight of $\tau$ as

$$
\omega(\tau) = \prod_{v \in V_{\tau}} w_{s_v}
$$

where we recall that $s_v$ is the number of children of the vertex $v$.

For a given $n \in \mathbb{N}$, let

$$
Q_n = \sum_{\tau \in \mathcal{T}_n} \omega(\tau).
$$

Therefore, the probability that the FOREST-ALGORITHM lasts $n$ steps is bounded by

$$
P_n \leq \sum_{n_1 + \ldots + n_m = n, \ n_i \geq 0} Q_{n_1} \cdots Q_{n_m}.
$$

(3.25)

It is now easy to check that $Q_n$ is defined by the recurrence relation

$$
Q_n = \sum_{s \in E'_F} w_s \sum_{n_1 + \ldots + n_k = n-1, \ n_1 \geq 0, \ldots, n_k \geq 0} Q_{n_1} \cdots Q_{n_{k-1}},
$$

(3.26)
with \( Q_0 = 1 \). Now let

\[
W(z) = \sum_{n=1}^{\infty} Q_n z^n,
\]

be the generating function encoding the sequence \( \{Q_n\}_{n \geq 1} \). Then we have from (3.26)

\[
W(z) = z \sum_{n=1}^{\infty} \sum_{s \in E'_{\bar{\delta}}} w_s \prod_{i=1}^{s} Q_{n_i} z^{n_i}
\]

\[
= z \sum_{s \in E'_{\bar{\delta}}} w_s \prod_{i=1}^{s} \sum_{n_i \geq 0} Q_{n_i} z^{n_i}
\]

\[
= z \sum_{s \in E'_{\bar{\delta}}} w_s \prod_{i=1}^{s} [1 + \sum_{n_i \geq 1} Q_{n_i} z^{n_i}]
\]

\[
= z \sum_{s \in E'_{\bar{\delta}}} w_s (1 + W(z))^s,
\]

i.e. denoting, for \( \xi > 0 \)

\[
\phi_{\bar{\delta}}(\xi) = \sum_{s \in E'_{\bar{\delta}}} w_s (1 + \xi)^s
\]

we have

\[
W(z) = z \phi_{\bar{\delta}}(W(z))
\]

and thus, by a well known result in analytic combinatorics (see e.g. Proposition IV.5 of [23] or also Theorem 5 in [18]) we have that the coefficients of the generating function \( W(z) \) are bounded as follows.

\[
Q_n \leq \rho^n
\]

where

\[
\rho = \min_{\xi > 0} \frac{\phi_{\bar{\delta}}(\xi)}{\xi}.
\]

Hence,

\[
P_n \leq \sum_{n_1 + \ldots + n_m = n \atop n_i \geq 0} Q_{n_1} \cdots Q_{n_m} \leq \rho^n \sum_{n_1 + \ldots + n_m = n \atop n_i \geq 0} 1 = \rho^n \binom{n + m - 1}{m - 1}
\]

(3.35)

Now, if condition (2.10) holds, we have that the probability that the Forest-Algorithm runs at least \( n \) steps decays exponentially in \( n \) if \( n \) is sufficiently large. In particular it is easy to check that

\[
\rho^n \binom{n + m - 1}{m - 1} \leq \rho^2
\]

as soon as

\[
\frac{n}{\ln n} \geq \frac{2m}{|\ln(\rho)|}
\]

i.e. as soon as

\[
n \geq \frac{2m}{|\ln(\rho)|} \ln^2 \left( \frac{2m}{|\ln(\rho)|} \right) \equiv N.
\]
Thus, if we estimate $P_n = 1$ if $n \leq N$ and $P_n \leq \rho^n/2$ if $n > N$, the expected number of steps $T$ of Forest-Algorithm is given by

$$T \leq \frac{N(N+1)}{2} + \sum_{n=N+1}^{\infty} n\rho^n.$$ 

### 4 Examples

In what follows $G = (V,E)$ is a graph with maximum degree $\Delta$ and $k \in \mathbb{N}$. A coloring of the vertices (resp. edges) of $G$ is a function $c : V \to [k]$ (resp. $c' : E \to [k]$).

**Example 1: Nonrepetitive vertex coloring of a graph**

A coloring of the vertices of $G$ is nonrepetitive if, for any $n \geq 1$, no path $p = \{v_1,v_2,\ldots,v_{2n}\}$ is colored repetitively, i.e. such that $c(v_i) = c(v_{i+n})$ for all $i = 1,2,\ldots,n$. The minimum number of colors needed such that $G$ has a non-repetitive vertex coloring is called the non-repetitive chromatic index of $G$ and it is denoted by $\pi(G)$. Here we are in the uniform variable setting, where the set of atoms $\Lambda$ coincides with $V$ and to each atom/vertex $v \in V$ we associate a random variable $\psi_v$, the color of $v$, that takes values in $[k]$ according to the uniform distribution. Let $P_n$ be the set of all paths with $2n$ vertices and set $P = \bigcup_{n \geq 1} P_n$. The family $\mathcal{F}$ of bad events is the set $\mathcal{F} = \{e_p\}_{p \in P}$, where $e_p$ is the event “the path $p$ is colored repetitively”. For any $p \in P$, given a vertex $v$ a seed of $e_p$ not containing $v$ is the half of $p$ that does not contain $v$. Thus, if $p$ is a path with $2n$ vertices, then $e_p$ is tidy with seeds of size $n$ and therefore $\|e_p\| = n$ and $\text{Prob}(e_p) = \frac{1}{k^n}$. So in this case $E_{\mathcal{F}}^c = \{1,2,3,\ldots\}$. In order to apply Theorem 2.2 we have to estimate $d_s$, the maximum number of events of power $s$ containing a fixed vertex. In the present case $d_s$ coincides with the maximum number of paths in $G$ of size $2s$ containing a fixed vertex. We have

$$d_s \leq s\Delta^{2s-1}.$$ 

Therefore the function $\phi_{\mathcal{F}}(\xi)$ defined in (2.9) is in the present case

$$\phi_{\mathcal{F}}(\xi) = \sum_{s \geq 1} s\Delta^{2s-1} \frac{1}{k^s}(\xi + 1)^s$$

$$= \frac{1}{\Delta} \sum_{s \geq 1} s \left( \frac{\Delta^2}{k}(\xi + 1) \right)^s$$

$$= \frac{1}{\Delta} \frac{\Delta^2(\xi + 1)}{1 - \frac{\Delta^2}{k}(\xi + 1)^2}$$

$$= \frac{1}{\Delta} \frac{(b+1)(\xi + 1)}{(b - \xi^2)}.$$ 

where in the last line we have set $k = (1+b)\Delta^2$.

Thus condition (2.10) is in this case

$$\min_{\xi > 0} \left( \frac{1}{\Delta} \frac{(b+1)(\xi + 1)}{\xi(b - \xi^2)} \right) < 1.$$
Observe that the minimum occurs at
$$
\xi_0 = \frac{\sqrt{9 + 8b} - 3}{4},
$$
and
$$
\phi_{\Psi}(\xi_0) = \frac{1}{\Delta} \frac{\sqrt{(8b + 9)^3 + 8b^2 + 36b + 27}}{8b^3},
$$
and thus if we let $b_0(\Delta)$ be the solution of the equation
$$
\frac{\sqrt{(8b + 9)^3 + 8b^2 + 36b + 27}}{8b^3} = \Delta,
$$
we have that the non repetitive chromatic index $\pi(G)$ of a graph with maximum degree $\Delta$ is such that
$$
\pi(G) \leq (1 + b_0(\Delta))\Delta^2. \tag{4.36}
$$
Comparing our bound with Theorem 8 in [26], which states that
$$
\pi(G) \leq \Delta^2 + \Delta^3 \left[ \frac{3}{2^{2/3}} + \frac{2^{2/3}}{\Delta^{1/3} - 2^{1/3}} \right], \tag{4.37}
$$
we observe that bound (4.36) is better than (4.37) for low values of $\Delta$ while becomes asymptotically equivalent for large values of $\Delta$.

**Example 2: Facial Thue Choice Index of planar graphs**

We suppose here that the graph $G = (V, E)$ is planar. Suppose moreover that for all edge $e \in E$, a list $L_e$ of $k$ colors is given. A facial path of $G$ is a path of $G$ which is part of the boundary of a face of $G$. The least integer $k$ such that for every collection of lists $\{L_e\}_{e \in E}$ with $|L_e| = k$ there is an edge coloring of $G$ such that every facial path of $G$ is nonrepetitive is called the facial Thue choice index of $G$ and is denoted by $\pi_{fl}(G)$. Observe that the set of independent random variables is in this case $\Psi = \{L_e\}_{e \in E}$.

Let $P$ denotes the set of all facial paths with even number of edges. For all $p \in P$ let $\epsilon_p$ be the event “$p$ is repetitive”, i.e., if $p = \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ we have $c'(e_i) = c'(e_{i+n})$ for all $i \in [n]$ where $c'(e)$ is the color chosen in the list $L_e$ via the random experiment. The family of bad events is thus $\mathcal{F} = \{\epsilon_p\}_{p \in P}$. Observe that, analogously to the previous example, any event $\epsilon_p$ with $p \in P$ is tidy and we can take as a seed of $\epsilon_p$ the first or the second half of the path $p$. This implies that as before $E_{\mathcal{F}}' = \{1, 2, 3, \ldots, \}$. Moreover, if $|p| = 2n$, we have that $\text{Prob}(\epsilon_p) \leq \frac{1}{k^n}$, and since every edge of a planar graph is contained in at most $4n$ facial paths of $G$ of size $2n$, for $s \in E_{\mathcal{F}}'$ we have that $d_s \leq 4s$, and therefore

$$
\phi_{\Psi}(\xi) \leq \sum_{s \geq 1} \frac{1}{k^s} 4s(\xi + 1)^s
\leq \frac{4(\xi + 1)}{k} \frac{1}{(1 - \frac{\xi + 1}{k})^2}
= \frac{4k(\xi + 1)}{(k - \xi - 1)^2}.
$$
Then, we have

\[
\min_{\xi > 0} \frac{\phi_\beta(\xi)}{\xi} < 1
\]

as soon as \(k \geq 12\), which is the same bound obtained in [42] via entropy compression method.

**Example 3: Coloring graphs frugally**

A proper vertex coloring of a graph \(G\) is said \(\beta\)-frugal if any vertex has at most \(\beta\) members of any color class in its neighborhood. The minimum number of colors required such that a graph \(G\) has at least one \(\beta\)-frugal proper vertex coloring is called the \(\beta\)-frugal chromatic number of \(G\) and will be denoted by \(\chi_\beta(G)\). Analogously to the Example 1, we are in the entropy compression setting where \((\Lambda, k) \equiv (V, k)\) and to each \(v \in V\) we associate a random variable \(\psi_v\) (the color of \(v\)) that takes values in \([k]\) according to the uniform distribution.

Observe that in the present case we have only two kind of bad events. First the coloring has to be proper. So, for each edge \(e = \{u, v\}\) of \(G\) we must avoid the event \(e\) that “\(u\) and \(v\) have the same color”, and let \(F_1 = \{e\} e \in E\).

We say that a set \(\sigma\) formed by \(\beta + 1\) vertices of \(G\) is a \(\beta\)-star of \(G\) if all members of \(\sigma\) are neighbors of a common vertex \(v \in V\), in other words, if there is \(v \in V\) such that \(\sigma \subset \Gamma_G(v)\). Let \(S_\beta\) denote the set of all \(\beta\)-stars of \(G\). Given \(\sigma \in S_\beta\), let \(e_\sigma\) be the event “all the \(\beta + 1\) vertices forming \(\sigma\) receive the same color”, i.e., \(\sigma\) is monochromatic. We thus have a second family of bad events \(F_\beta = \{e_\sigma\} \sigma \in S_\beta\). Clearly the events of the family \(F_1\) are tidy with seeds of size 1 and power equal to 1, while all events of the family \(F_\beta\) are tidy with seeds of size 1 and power equal to \(\beta\). A \(\beta\)-frugal coloring of the vertices of \(G\) occurs if none of the events of the family \(F = F_1 \cup F_\beta\) occurs.

In the present case, \(E'_\beta = \{1, \beta\}\), and for every \(e \in E\) and \(\sigma \in S_\beta\), we have that \(\text{Prob}(e_\sigma) = \frac{1}{k}\) and \(\text{Prob}(\sigma) \leq \frac{1}{\beta^\beta}\) respectively.

To check condition (2.10) we just need to estimate \(d_\beta\). Observe that \(d_1 = \Delta\) and

\[
d_\beta \leq \Delta \left(\frac{\Delta}{\beta}\right) \leq \frac{\Delta^{1+\beta}}{\beta!}.
\]

Then, in the present case the function \(\phi_\beta(\xi)\) defined in [24] takes the form

\[
\phi_\beta(\xi) \leq \frac{\Delta}{k} (\xi + 1) + \frac{1}{k^\beta} \frac{\Delta^{1+\beta}}{\beta!} (\xi + 1)^\beta.
\]

And hence, with some calculation, we obtain the upper bound

\[
\chi_\beta(G) \leq \frac{\Delta^{1+\beta}}{\beta! 1/\beta} \beta(\beta - 1)^{\frac{1}{\beta} - 1} + \Delta,
\]

which, of course, is the same bound obtained in [7] via entropy compression method.

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