Hamiltonian quantization of General Relativity with the change of signature

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Abstract

We show in this article how the usual hamiltonian formalism of General Relativity should be modified in order to allow the inclusion of the Euclidean classical solutions of Einstein’s equations. We study the effect that the dynamical change of signature has on the superspace and we prove that it induces a passage of the signature of the supermetric from (−+++++) to (+−−−−−−). Next, all these features are more particularly studied on the example of minisuperspaces.

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Finally, we consider the problem of quantization of the Euclidean solutions. The consequences of different choices of boundary conditions are examined.
1 Introduction

One of the most important results in quantum cosmology has been the suggestion that the very early Universe might be described by an Euclidean manifold. This idea arises from the interpretation of the behaviour of wave function of the Universe which is very similar to the behaviour of the wave function of a particle in a classically forbidden region. Thus, Euclidean Universes should be a natural feature of quantum cosmology. However, recent works have shown that solutions of Einstein’s equations with change of signature might be possible also as a part of classical theory. The Hamiltonian formulation of General Relativity (ADM formalism) is particularly well adapted to the study of the signature change. Indeed, it is conceived in the very beginning as a slicing of the four-dimensional spacetime into three-dimensional spacelike hypersurfaces. The geometry of these hypersurfaces contains the true degrees of freedom of the theory, whereas the way how the hypersurfaces are stacked together is essentially arbitrary and depends on the dynamics.

One of the aims of this article is to show how we can construct a slightly modified usual Hamiltonian formalism in order to incorporate classical Euclidean solutions. Then, we study the consequences of the dynamical change of signature on the metric of the superspace. We prove that Euclidean solutions correspond to a region in the superspace where signature of the metric is (+ − − − −) instead of (− + + + +). Finally, we consider the quantization of this model of gravitation. Since the classical Euclidean solutions do exist, it seems interesting to find out how the corresponding wave function behaves and to compare its behaviour with the behaviour of the wave functions which are interpreted as describing Euclidean regions of the Uni-
verse. Another motivation is that the change of signature could occur during the Planck epoch \[1\]. In that case the quantum theoretical approach would become necessary.

2 Hamiltonian formalism

In this section we show how classical Einstein’s solutions with change of signature can be obtained by means of a slight modification of the ADM formalism of General Relativity. We consider a four-dimensional manifold \((M, g_{\mu\nu})\) whose structure can be symbolized by:

\[ M = M^+ \cup \Sigma \cup M^- \quad (1) \]

where \(M^+\) is a submanifold endowed with an Euclidean metric (signature \(++++)\) whereas \(M^-\) is endowed with a Lorentzian metric (signature \(-+++)\). These two submanifolds are matched together at \(\Sigma\), the surface of change of the signature on which \(g_{\mu\nu}\) is degenerate. An additional difficulty arises while considering the ADM formalism since the basic idea of the Hamiltonian formulation of General Relativity consists in slicing along the timelike coordinate the four-dimensional manifold into three-dimensional spacelike hypersurfaces and in following the evolution of these hypersurfaces in time. In the Euclidean region, space and time are completely equivalent and there is no a priori direction which could be considered as being the coordinate with respect to which the slicing should be performed. However, the time coordinate of the Lorentzian region induces in the vicinity of the separating surface \(\Sigma\) a privileged direction in the Euclidean region which will

\[1\] In this article, Greek indices run from 0 to 3 whereas Latin indices run from 1 to 3
be chosen in order to slice the submanifold $M^+$. Following the notations adopted by Ellis et al. [1], let us define the symbol $\epsilon$ to be equal to $-1$ on $M^+$ and $+1$ on $M^-:
\begin{align*}
\epsilon &= -1 \quad \text{on } M^+ \\
\epsilon &= +1 \quad \text{on } M^-
\end{align*}
(2)

The three-dimensional hypersurfaces are endowed with the metric $h_{ij}$. Then, the metric $g_{\mu\nu}$ can be written as:

\begin{align*}
\mathrm{ds}^2 &= g_{\mu\nu} \, dx^\mu \, dx^\nu \\
&= -\epsilon(\sqrt{\epsilon N} \, dt)^2 + h_{ij}(dx^i + N_i \, dt)(dx^j + N_j \, dt)
\end{align*}
(4)

where $N(x^\mu)$ and $N^i(x^\mu)$ are respectively the usual lapse and shift functions of the ADM formalism [5]. The term $\epsilon$ in front of the first term takes into account the fact that the distance between two neighbour points (nothing else than the Pythagoras' theorem!) is calculated differently according to the signature. We note that the proper time elapsed between two points with the same spacelike coordinates is now given by the formula:

\[ d\tau = \sqrt{\epsilon N} \, dt \]
(5)

Then, the metric of the manifold $M$ can be written as:

\[ \mathrm{ds}^2 = (N^i N_i - N) dt^2 + 2N_i \, dx^i \, dt + h_{ij} \, dx^i \, dx^j \]
(6)

with $N$ allowed to take on any real values, which leads to the possibility of including the signature changes. The inverse metric is:

\[
\begin{pmatrix}
-\frac{1}{N} & \frac{N^i}{N} \\
\frac{N^i}{N} & h^{ij} - \frac{N^i N^j}{N}
\end{pmatrix}
\]
We have to examine now how the extrinsic curvature can be expressed in such a manifold. Let $n$ be the vector perpendicular to the congruence of three-dimensional hypersurfaces, $\partial_t$ the vector associated to the timelike coordinate $t$ and $\partial_i$ three basic (coordinate) vectors of the spacelike hypersurfaces. The obvious relation holds:

$$\partial_t = \sqrt{\epsilon N} n + N^i \partial_i,$$

from which follows that the contravariant and covariant components of the vector $n$ are:

$$n^\mu = \frac{1}{\sqrt{\epsilon N}} (1, -N^i)$$

$$n_\mu = (-\sqrt{\epsilon N}, 0, 0, 0)$$

To compute the extrinsic curvature, we use the following coordinate-independent formula:

$$K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to the vector field $n$. In terms of lapse and shift functions, the above relation can be written as:

$$K_{ij} = \epsilon \sqrt{\epsilon N} \Gamma^0_{ij}$$

$$= -\frac{1}{2 \sqrt{\epsilon N}} ((3)\nabla_i N_j) - \partial_i h_{ij}$$

where $(3)\nabla$ is the intrinsic covariant derivative on the three-dimensional spacelike hypersurfaces. Using the expression (6) of the metric, we can now compute the components of the Ricci tensor. They are given by:

$$R_{00} = -\sqrt{\epsilon N} h^{ki} \hat{K}_{ki} + \frac{1}{2} (3)\nabla_k (\partial^k N) + 2 \sqrt{\epsilon N} K^i_{\ j} (3)\nabla_i N^j + 2 \sqrt{\epsilon N} N^i_{\ j} (3)\nabla_k K^{kj} - N_j \sqrt{\epsilon N} (3)\nabla_j K^k_k + \epsilon NK_{kj} K^{kj}$$
\[ R_{00} = \frac{\epsilon}{\sqrt{\epsilon N}} N^k \dot{K}_{ki} + N^j(3) R_{ij} - \frac{N^k}{2N} \nabla_k (\partial_i N) - \sqrt{\epsilon N} \partial_i K^j_j - \epsilon N^k K_{ik} K^j_j - \frac{\epsilon}{\sqrt{\epsilon N}} N^j N^k K_{ik} \nabla_j N^m - 2\epsilon N^j K_{ik} K^j_j - \epsilon N^k \nabla_i N \partial_k N \]

We note that the last two terms in the expression of \( R_{00} \) are not present in the analogous formula without change of signature (see the appendix). The other terms are, in general, slightly modified in comparison with the usual formula (typically, \( N \) is replaced by \( \sqrt{\epsilon N} \)). However, the term \( \epsilon \) appears sometimes outside the square root of the lapse function and we shall see that the consequences of this simple fact will be very important. The expression giving the \((0i)\)-components of the Ricci tensor is:

\[ R_{0i} = \frac{\epsilon}{\sqrt{\epsilon N}} N^k \dot{K}_{ki} + N^j(3) R_{ij} - \frac{N^k}{2N} \nabla_k (\partial_i N) - \sqrt{\epsilon N} \partial_i K^j_j - \epsilon N^k K_{ik} K^j_j - \frac{\epsilon}{\sqrt{\epsilon N}} N^j N^k K_{ik} \nabla_j N^m - 2\epsilon N^j K_{ik} K^j_j - \epsilon N^k \nabla_i N \partial_k N \]

As in the previous case, we note that the last term is absent in the usual expression. The space-space components of the Ricci tensor can be written as:

\[ R_{ij} = (3) R_{ij} + \frac{\epsilon}{\sqrt{\epsilon N}} \dot{K}_{ij} - 2\epsilon K_{ki} K^j_j + \epsilon K^k_k K_{ij} - \epsilon N^k \nabla_k K_{ij} - \frac{1}{2N} \nabla_j (\partial_i N) - \frac{\epsilon}{\sqrt{\epsilon N}} K_{ki} \nabla_j N^k \]
Here again, an extra term (the last one) is present. Then, using the expression of the inverse metric, we can compute the Ricci scalar; after quite straightforward calculations, we find the following expression:

\[
\begin{align*}
R &= (3)R - 3\epsilon K_{k_i}K^{k_i} + \epsilon K^2 - 2\frac{\epsilon}{\sqrt{\epsilon}N}N^i \partial_i K^j \partial^j \\
&\quad + 2\frac{\epsilon}{\sqrt{\epsilon}N}h^{k_i} \dot{K}_{k_i} - \frac{1}{N} (3)\nabla_k (\partial^k N) - \frac{4\epsilon}{\sqrt{\epsilon}N}K_k (3)\nabla_j N^k \\
&\quad + \frac{1}{2N^2} \partial_k N \partial^k N
\end{align*}
\]

Some of the additional terms have been eliminated, but an extra term still remains. Noting that:

\[
h^{k_i} \dot{K}_{k_i} = \dot{K} + 2\sqrt{\epsilon}N K^{k_i} K_{k_i} + 2K_{k_i} (3)\nabla^k N^i
\]

we can re-express the formula for the Ricci scalar curvature; it takes the following form:

\[
\begin{align*}
R &= (3)R + \epsilon \dot{K}_{k_i} K^{k_i} + \epsilon K^2 + \frac{2\epsilon}{\sqrt{\epsilon}N} \dot{K} - \frac{2\epsilon}{\sqrt{\epsilon}N}N^i \partial_i K \\
&\quad - \frac{1}{N} (3)\nabla_k (\partial^k N) + \frac{1}{2N^2} \partial_k N \partial^k N
\end{align*}
\]

Now, we can compute the Einstein-Hilbert Lagrangian with change of signature, which is

\[
\mathcal{L}_G = \sqrt{-g}R = h^{\frac{1}{2}} \sqrt{\epsilon N} R
\]

Using the previous expression for R, we obtain the following definition of the Lagrangian of the gravitational field:

\[
\mathcal{L}_G = h^{\frac{1}{2}} \sqrt{\epsilon N} ((3)R + \epsilon \dot{K}_{k_i} K^{k_i} - \epsilon K^2) \\
+ 2\epsilon \frac{d}{dt} (h^{\frac{1}{2}} K) - 2\partial_i (\epsilon h^{\frac{1}{2}} K N^i + h^{\frac{1}{2}} h^{k_i} \partial_k \sqrt{\epsilon N})
\]
This equation looks like the usual one since it contains three parts, one being composed by the kinetic and potential terms, the two others representing surface terms. However, we note the presence of the factor $\epsilon$ in the dynamical term. The extra terms have disappeared or have been incorporated in the surface terms which are slightly modified in comparison with the case without the change of signature. In what follows, we shall drop out the surface terms and shall consider only the Lagrangian $L_G$ defined by:

$$L_G[N, N^i, h_{ij}] = \frac{\hbar}{2}\sqrt{\epsilon N}(3)R + \epsilon K_{ij}K^{ij} - \epsilon K^2)$$ \hspace{1cm} (21)

Starting from the previous equation, we can perform now full Hamiltonian analysis. The conjugate momenta are given by:

$$\pi_\mu = \frac{\delta L_G}{\delta (\partial_t N^\mu)} = 0 \hspace{1cm} (22)$$

$$\pi^{ij} = \epsilon h^{\frac{1}{2}}(K^{ij} - h_{ij}K) \hspace{1cm} (23)$$

where $K = h^{ij}K_{ij}$. Equation (22) shows that $L_G$ is a constrained Lagrangian and therefore requires the use of the Dirac formalism developed especially to treat this kind of systems [6, 7, 8, 9]. Equation (23) can be inversed to provide the following expression for $K^{ij}$:

$$K^{ij} = \epsilon h^{-\frac{1}{2}}(\pi^{ij} - \frac{\pi}{2}h_{ij}) \hspace{1cm} (24)$$

where $\pi = h_{ij}\pi_{ij}$. The canonical hamiltonian can be written as:

$$H_G = \pi_\mu \partial_t N^\mu + \pi^{ij} \partial_t h_{ij} - \mathcal{L}_G \hspace{1cm} (25)$$

$$= 2\pi^{ij(3)}\nabla_i N_j - h^{\frac{1}{2}}\sqrt{\epsilon N}(\frac{\epsilon}{2}\pi^2 - \epsilon \pi^{ij}\pi_{ij} + h^{(3)}R) \hspace{1cm} (26)$$

Integrating by parts and dropping out the surface term, we obtain:

$$H_c = \int d^3x(\sqrt{\epsilon N}H + N_jH^j) \hspace{1cm} (27)$$
where the expressions of $\mathcal{H}$ and $\mathcal{H}^j$ are:

$$\mathcal{H} = h^{-\frac{1}{2}}(\epsilon \pi^{ij} \pi_{ij} - \frac{\epsilon}{2} \pi^2) - h^{\frac{1}{2}}(3) \tilde{R}$$  \hspace{1cm} (28)

$$\mathcal{H}^j = -2^{(3)} \nabla_i \pi^{ij}$$  \hspace{1cm} (29)

$\mathcal{H}$ can be written in such a way that the metric of the superspace appears explicitly:

$$\mathcal{H} = G_{ijkl} \pi^{ij} \pi^{kl} - h^{\frac{1}{2}}(3) \tilde{R}$$  \hspace{1cm} (30)

where $G_{ijkl}$ is defined by:

$$G_{ijkl} = \epsilon h^{-\frac{1}{2}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$$  \hspace{1cm} (31)

We see that the only effect of the change of signature is to modify the respective sign between the kinetic part and the potential part in the definition of the Hamiltonian (30). In other words, the change from an Euclidean signature to a Lorentzian signature in the physical manifold $M$ induces a change in the superspace from the signature ($-++++)$ to the signature ($++----$). We note that the superspace metric remains globally hyperbolic even if the metric of $M$ happens to be Euclidean. This becomes more explicit if we define the coordinate $\xi$ in the superspace following [10]:

$$\xi = 4 \sqrt{\frac{2}{3} h^{\frac{1}{4}}}$$  \hspace{1cm} (32)

and five coordinates $\xi^A (A = 1 \ldots 5)$ orthogonal (in the sense of the supermetric) to $\xi$. Then, the interval in the superspace takes on the form:

$$ds^2 = -\epsilon d\xi^2 + \frac{3}{8} \epsilon \tilde{G}_{AB} d\xi^A d\xi^B$$  \hspace{1cm} (33)

where $\tilde{G}_{AB}$ is the metric of a five-dimensional subspace of the superspace whose signature is ($++++)$. Using previous relations, we can compute
the curvature invariants \((R_{ABCD}R^{ABCD})\) for example) and the geodesics in the superspace \([10]\) and show that all these quantities are well-behavin when the change of signature from \((-+++++)\) to \((+-------)\) occurs. This suggests another argument in favor of considering the solutions of Einstein’s equations with change of signature, since there exist a priori no reason for studying only a limited region of the superspace.

In the next section, we are going to consider minisuperspaces and to show how the equations of motion can be found using the Dirac formalism for constrained systems.

3 Applications to minisuperspaces

In this section, we restrict our considerations to minisuperspaces \([11]\). This means that we impose very strong symmetry conditions on the solutions so that almost all degrees of freedom are frozen, except a few ones, which leads to the superspace of finite dimension in which the analytic computations and finite integrations can be performed. The coordinates \(q^\alpha\) in the minisuperspace are the components of the metric tensor and eventually the fields describing the matter. In the case of change of the signature, the Lagrangian can be written as:

\[
L = \sqrt{\epsilon N} \left( \frac{1}{2\epsilon N} f_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V(q^\alpha) \right) \tag{1}
\]

where \(f_{\alpha\beta}\) is the metric \(G_{ijkl}\) restricted to the minisuperspace, and the canonical Hamiltonian deduced from the above expression of the Lagrangian is given by:

\[
H_c = \sqrt{\epsilon N} \left( \frac{1}{2} f^{\alpha\beta} \pi_\alpha \pi_\beta + V(q^\alpha) \right) \tag{2}
\]
In what follows, we shall suppose that the manifold has the geometry of a Friedmann-Robertson-Walker (FRW) Universe:

\[
 ds^2 = -N(t)dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right)
\]  

(3)

where \( k \) can take on the values 0, ±1 according to the case where the three-dimensional spacelike hypersurfaces are flat, open or closed. The matter will be described by a unique scalar field. Putting equation (3) in the expression (21) of the Lagrangian, we find that the action for the gravitational field can be written as:

\[
 S_G = \alpha \int dt(k\sqrt{\epsilon Na} - \epsilon \frac{\dot{a}^2}{\sqrt{\epsilon N}})
\]  

(4)

where \( \alpha \) is a dimensional constant. With the assumption that the scalar field is spatially homogeneous, that is to say \( \phi = \phi(t) \), the action of the matter takes on the form:

\[
 S_M = -\int d^4x\sqrt{\epsilon Nh^\frac{1}{2}}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + V(\phi)\right)
\]

(5)

\[
 = \frac{\alpha}{6} \int da^3\left(\frac{\epsilon \dot{\phi}^2}{2\sqrt{\epsilon N}} - \sqrt{\epsilon N}V(\phi)\right)
\]

(6)

The full Lagrangian of the system (gravitational field and scalar field), \( L = L_G + L_M \) is given by:

\[
 L = \sqrt{\epsilon N}\left(\frac{1}{2\epsilon N}(-2\epsilon a \dot{a}^2 + \epsilon \frac{a^3}{6} \dot{\phi}^2) - U(a, \phi)\right)
\]

(7)

where the potential \( U(a, \phi) \) is defined by:

\[
 U(a, \phi) = \frac{a^3}{6}V(\phi) - ka
\]

(8)

The metric \( f_{\alpha\beta} \) of the minisuperspace can readily be obtained from the expression of \( L \):

\[
 f_{\alpha\beta} = \begin{pmatrix}
 -\frac{4}{2a} & 0 \\
 0 & \frac{6}{a^2}
\end{pmatrix}
\]
As announced, the signature of the metric of the minisuperspace changes from 
\((-+\)) to (++) when the signature of the FRW metric becomes Lorentzian.
We can now compute the conjugate momenta:

\[
\pi_a = \frac{\partial L}{\partial \dot{a}} = -2 \epsilon \frac{\dot{a} \sqrt{\epsilon N}}{a} \tag{9}
\]
\[
\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\epsilon a^3}{6 \sqrt{\epsilon N}} \dot{\phi} \tag{10}
\]
\[
\pi_N = 0 \tag{11}
\]

The last equality is a primary constraint and is a consequence of the gauge
invariance of General Relativity (Thus, following Dirac’s classification, it is
also a first class constraint). The canonical Hamiltonian is given by:

\[
H_c = \sqrt{\epsilon N} \left( -\frac{\epsilon}{4a} \pi_a^2 + \frac{3\epsilon}{a^3} \pi_\phi^2 + U(a, \phi) \right) \tag{12}
\]
\[
= \sqrt{\epsilon N} H \tag{13}
\]

One can easily verify that \(H_c\) has the form given by the equation (2). Then,
the total Hamiltonian can be written as:

\[
H_T = \sqrt{\epsilon N} H + \lambda \pi_N \tag{14}
\]

where \(\lambda\) is a Lagrange multiplier. According to the Dirac algorithm for
constrained systems, we have to check that the primary constraint \(\pi_N \approx 0\)
\((\approx\) is the Dirac weak equality) is preserved in time:

\[
\{\pi_N, H_T\}_P \approx 0 \tag{15}
\]
where \( \{ , \}_P \) is the Poisson bracket in the minisuperspace defined by

\[
\{ f, g \}_P = \frac{\partial f}{\partial N} \frac{\partial g}{\partial \pi_N} + \frac{\partial f}{\partial a} \frac{\partial g}{\partial \pi_a} - \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \pi_\phi} - \frac{\partial f}{\partial \pi_N} \frac{\partial g}{\partial N} - \frac{\partial f}{\partial \pi_a} \frac{\partial g}{\partial a} - \frac{\partial f}{\partial \pi_\phi} \frac{\partial g}{\partial \phi} \] (16)

This leads to the secondary constraint:

\[
- \frac{\epsilon}{2\sqrt{\epsilon}N} H \approx 0 \tag{17}
\]

We note the difference in comparison with the usual case where we obtain
\( H \approx 0 \). One have to be very careful since equation (17) has the form
\( f(p, q)g(p, q) \approx 0 \). Earliest works \cite{12} have shown that if we assume that
the last relation implies \( f(p, q) \approx 0 \) or \( g(p, q) \approx 0 \), this can lead to a contra-
diction with the Dirac conjecture which states that all secondary first class
constraints must generate symmetries. Therefore, at this stage, we are not
allowed to deduce \( H \approx 0 \) from equation (17). On the other hand, if the re-
lation \( h(p, q) \approx 0 \) holds, it implies that for all functions \( f(p, q) \), the equation
\( f(p, q)h(p, q) \approx 0 \) must hold, too. For consistency, the secondary constraints
must be preserved in time:

\[
\{- \frac{\epsilon}{2\sqrt{\epsilon}N} H, H_T \}_P = - \frac{\epsilon\lambda}{2} \{ \frac{1}{\sqrt{\epsilon}N}, \pi_N \}_P H \tag{18}
\]

\[
= \frac{\lambda}{2\epsilon N^2} \left( - \frac{\epsilon}{2\sqrt{\epsilon}N} H \right) \tag{19}
\]

\[
\approx 0 \tag{20}
\]

according to the previous discussion. Thus, there are no further constraints.

The two constraints are first class ones since their Poisson bracket vanishes.

\footnote{This the restricted version to the minisuperspace of the general formula:}

\[
\{ f, g \}_P = \sum_{i, j} \int d^3 x \left( \frac{\delta f(z)}{\delta h_{ij}(x)} \frac{\delta g(z)}{\delta \pi_{ij}(x)} - \frac{\delta g(z)}{\delta \pi_{ij}(x)} \frac{\delta f(z)}{\delta h_{ij}(x)} \right) \]

14
The extended Hamiltonian can be written now as:

\[ H_E = \lambda \pi_N + H(\sqrt{\epsilon N} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}}) \]  

(21)

where \( \eta \) is a Lagrange multiplier. We can note that this expression is different from the usual one (i.e. \( H_E = \lambda \pi_N + NH \)). Finally, we want to show how the equations of motion can be deduced from the extended Hamiltonian. The Hamilton equations are:

\[ \dot{N} \approx \{N, H_E\}_P = \lambda \]  

(22)

This equation shows that \( N \) is completely arbitrary since its derivative is equal to the Lagrange multiplier. Next,

\[ \dot{\pi}_N \approx \{\pi_N, H_E\}_P \]  

\[ \approx -\frac{\epsilon}{2\sqrt{\epsilon N}} H + \frac{H \eta}{2N}\left(-\frac{\epsilon}{2\sqrt{\epsilon N}} H\right) \]  

(23)

\[ = 0 \]  

(24)

\[ \approx -\frac{1}{\sqrt{\epsilon N}}(N - \frac{\eta}{2} \frac{\pi_a}{a}) \]  

(25)

where we have used the constraints to transform the weak equality into a strong one.

\[ \dot{a} \approx \{a, H_E\}_P \]  

(26)

\[ \dot{a} \approx (\sqrt{\epsilon N} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}})\{a, H\}_P \]  

(27)

\[ \approx -\frac{1}{\sqrt{\epsilon N}}(N - \frac{\eta}{2} \frac{\pi_a}{2a}) \]  

(28)

But as we have just seen with expression (22), \( N \) is arbitrary. Therefore, the last equation can be written as:

\[ \dot{a} \approx -\frac{N}{\sqrt{\epsilon N} 2a} \]  

(29)

\[ \approx -\frac{\epsilon \sqrt{\epsilon N} \pi_a}{2a} \]  

(30)
The equation of motion for $\pi_a$ is given by:

$$\dot{\pi}_a \approx \{\pi_a, H_E\}_P$$

$$\approx \left(\sqrt{\epsilon N} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}}\right)\{\pi_a, H\}_P$$

$$\approx \sqrt{\epsilon N}\left(-\frac{\epsilon}{4a^2}\pi_a^2 + \frac{9\epsilon}{a^4}\pi_a^2 + k - \frac{a^2}{2}V(\phi)\right)$$

For the scalar field, we obtain:

$$\dot{\phi} \approx \{\phi, H_E\}_P$$

$$\approx \left(\sqrt{\epsilon N} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}}\right)\{\phi, \frac{3\epsilon}{a^3}\pi_\phi^2\}_P$$

$$\approx \epsilon\sqrt{\epsilon N}\frac{6}{a^3}\pi_\phi$$

The dynamical equation for the conjugate momentum of the scalar field is:

$$\dot{\pi}_\phi \approx \{\pi_\phi, H_E\}_P$$

$$\approx -\sqrt{\epsilon N}\frac{a^3 dV}{6 d\phi}$$

It is now straightforward to show that the previous equations are equivalent to the equations already found in earlier papers [1, 2] describing the change of signature of a FRW Universe filled with a scalar field:

$$\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{N}}{2aN} + \frac{\dot{\phi}^2}{8} - \frac{N}{6}V = 0$$

$$\frac{\dddot{\phi}}{2N} - \frac{\dot{N}\dot{\phi}}{2N} + 3\frac{\ddot{\phi}}{a} + \frac{N}{a^3}dV = 0$$

$$\frac{kN}{a^2} + \frac{a^2}{a^2} = \frac{N}{6}\left(\frac{\dot{\phi}^2}{2N} + V(\phi)\right)$$

Thus, we have verified that the hamiltonian formalism adapted to the case including the change of signature, used with care, provides the right equations.
of motion. Obviously, the choice of $N$ remains to be made in order to obtain a good junction condition. This point has been discussed with more details in Refs. [1, 2, 13, 14].

4 Canonical quantization

Following the suggestion of Hartle and Hawking [15], we shall suppose that an Euclidean region could have existed in the very early Universe. In this regime, the behaviour of the wave function of the Universe corresponds to the behaviour of a wave function of usual quantum mechanics in a classically forbidden region. However, we have seen that classical Euclidean solutions of Einstein’s equations can also exist, so that the problem of quantization of such solutions and of the behaviour of the wave function in the Euclidean regime naturally arises. Let us study this problem now.

We want to quantize a theory with the constraints of first class described by the extended Hamiltonian [16]:

$$H_E = \lambda \pi_N + (\sqrt{\epsilon N} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}})H$$

(1)

According to the prescriptions of Dirac, this leads to the relations:

$$\hat{\pi}_N \Psi(a, \phi, N) = 0 \implies \Psi = \Psi(a, \phi)$$

(2)

$$\left(\sqrt{\epsilon \hat{N}} - \frac{\eta \epsilon}{2\sqrt{\epsilon N}}\right)\hat{H} \Psi(a, \phi) = 0 \implies \hat{H} \Psi(a, \phi) = 0,$$

(3)

the last equation resulting from the fact that the wave function $\Psi(a, \phi)$ does not depend on $N$ (see the equation (2)). Consequently, the Wheeler-De Witt equation is given by [14]:

$$\left(-\frac{\epsilon}{4a} \pi_a^2 + \frac{3\epsilon}{a^3} \pi_\phi^2 - ka + \frac{a^3 V(\phi)}{6}\right) \Psi(a, \phi) = 0$$

(4)
where the hat on the symbols denoting the operators has been omitted. If we multiply by \(-\epsilon a\), we obtain:
\[
\left( \frac{1}{4} \pi^2_a - \frac{3}{a^2} \pi^2_\phi - \epsilon a^2 \frac{V(\phi)}{6} - k \right) \Psi(a, \phi) = 0
\] (5)

Now, we can interpret the change of signature in a different way: the last equation has the usual Wheeler-De Witt form (no change from \((-+\)) to \((+-)\)), but the potential is modified by the factor \(\epsilon\) which takes into account the difference of signs between the Euclidean and Lorentzian regions:
\[
\tilde{U}(a, \phi) = \epsilon a^2 \left( a^2 \frac{V(\phi)}{6} - k \right)
\] (6)

In other words, all the effects of the signature change have been incorporated into the potential term only. Because of the presence of factor \(\epsilon\), we have to check if \(\tilde{U}(a, \phi)\) is well-behaving at the surface of change \(\Sigma\). If we assume that \(N(t) = \epsilon\), then the equation (41) becomes:
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\dot{\phi}^2}{12} + \epsilon \left( \frac{V}{6} - \frac{k}{a^2} \right)
\] (7)

If we require the continuity of the scale factor and of its first derivative, this provides a criterion (cf. [1], [2]) for determining the value of \(V\) at which the change of signature does occur:
\[
V = \frac{6k}{a^2} \quad \text{on } \Sigma
\] (8)
\[
V > \frac{6k}{a^2} \quad \text{if } \epsilon = 1
\] (9)
\[
V < \frac{6k}{a^2} \quad \text{if } \epsilon = -1
\] (10)

This criterion assures that the potential \(\tilde{U}(a, \phi)\) is continuous (but not its derivative) when the change of signature occurs. Equation (5) leads to a
differential equation if we apply the correspondence principle, that is to say when we replace the square of the conjugate momentum according to the usual rule \(^{17}\):

\[
\pi^2 \rightarrow -q^{-p} \frac{\partial}{\partial q}(q^p \frac{\partial}{\partial q})
\]  

(11)

where the exponent \(p\) takes into account the operator-ordering problem. In what follows, we shall assume that \(p = -1\) in order to obtain a differential equation which can be solved analytically. We shall also assume that the scalar field (and therefore the potential \(V(\phi)\)) is constant and plays the rôle of a cosmological term in Einstein’s equations. This will enable us to find a connection with the classical solutions displaying a change of signature already found in \(^{1}\) and \(^{2}\). With these assumptions, the Wheeler-De Witt equation can be written as:

\[
\frac{d^2}{da^2} \Psi(a) - \frac{1}{a} \frac{d}{da} \Psi(a) + 4\epsilon a^2 (H^2 a^2 - 1) \Psi(a) = 0
\]  

(12)

where \(H^2 = \frac{V}{6}\) and \(k = 1\). By defining \(z = -\epsilon H^{-\frac{2}{3}}(H^2 a^2 - 1)\), the above equation becomes:

\[
\frac{d^2}{dz^2} \Psi(z) - z \Psi(z) = 0
\]  

(13)

which is the differential equation defining the Airy functions \(^{18,19}\). The general solution of the Wheeler-De Witt equation is given in this case by:

\[
\Psi(z) = \lambda(\phi) Ai(z) + \eta(\phi) Bi(z)
\]  

(14)

where \(Ai(z)\) and \(Bi(z)\) are the Airy functions of first kind and second kind. \(\lambda\) and \(\eta\) are arbitrary functions of the parameter \(\phi\). When the scale factor \(a(t)\) comes close to zero, the term \(a^4\) can be neglected and the Wheeler-De Witt equation can be written as:

\[
\frac{d^2 \Psi}{da^2} - \frac{1}{a} \frac{d \Psi}{da} + 4a^2 \Psi = 0
\]  

(15)
The condition $a(t) \rightarrow 0$ implies that $\epsilon$ must be equal to $-1$. Then, $\Psi(a = 0, \phi)$ is a constant and does not depend on $\phi$. Since $z(a = 0) = -H^{-\frac{4}{3}}$, the general solution (14) is (in what follows we shall put a “CS” superscript meaning “Change of Signature” to distinguish the wave functions obtained here from the wave functions of usual quantum cosmology):

$$\Psi^{(CS)}(z) = \frac{\alpha Ai\left(-eH^{-\frac{4}{3}}(H^2a^2 - 1)\right) + \beta Bi\left(-eH^{-\frac{4}{3}}(H^2a^2 - 1)\right)}{\alpha Ai(-H^{-\frac{4}{3}}) + \beta Bi(-H^{-\frac{4}{3}})}$$

(16)

In order to determine the value of $\alpha$ and $\beta$, we have to solve the problem of boundary conditions of quantum cosmology.

Let us analyze the difference between the wave functions (16) and those of usual quantum cosmology in each region separately (i.e. for $\epsilon = \pm 1$). In the region where $a < \frac{1}{H}(\epsilon = -1)$, the WKB solutions of the Wheeler-De Witt equation take on the form:

$$\Psi^{(CS)} \propto e^{\pm \left(\frac{2}{3H^2(1-H^2a^2)^2 + \frac{\pi}{4}}\right)}$$

(17)

In the simplified model considered here, the wave function of the Universe is formally equivalent to a wave function of a particle with energy equal to zero [11]. Since for $a < \frac{1}{H}$, the potential of the usual Wheeler-De Witt equation is positive, this region is classically forbidden and we obtain an exponential behaviour $\Psi \sim e^{-I}$, where $I$ denotes the Euclidean action: this phenomenon is the well-known tunnel effect. The fact that the wave function $\Psi^{(CS)}$ adopts an oscillatory behaviour seems quite natural here since, due to the value $-1$ of the term $\epsilon$, the potential is now negative and the energy of the fictitious particle is above this value, so that the Euclidean region is no longer forbidden. Using Wigner’s functions [20, 21, 22, 23, 24] or canonical
transformations [23] it is easy to show that oscillatory behaviour means that 
\( \Psi^{(CS)} \) carries quantum correlations. Obviously, these correlations correspond 
to the classical solutions of Einstein’s equations with change of signature.

In the region \( a > \frac{1}{H}(\epsilon = 1) \), the wave function is supposed to describe 
our Universe and we can use it for cosmological interpretations. The predictions coming from \( \Psi^{(CS)} \) should be compared with those coming from the 
wave functions of ordinary quantum cosmology (written without “CS” super-
script). To fix the wave function, we have to solve the problem of boundary 
conditions. Many propositions have been made, one of the best known is the 
“no boundary” proposition of Hawking and Hartle [15]. The wave function 
on the three-dimensional hypersurface \( B \) is constructed by performing the 
Euclidean path integral:

\[
\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B] = \sum_M \int Dg_{\mu\nu} D\Phi \exp(-I[g_{\mu\nu}, \Phi]) \quad (18)
\]

where the sum is taken over all manifolds \( M \) having \( B \) as part of their 
boundary and over all metrics \( g_{\mu\nu} \) and matter fields \( \Phi \) which induce \( \tilde{h}_{ij} \) and \( \tilde{\Phi} \) on \( B \); \( I \) denotes the Euclidean action. The proposal of Hawking and 
Hartle consists in restricting the sum to compact four-dimensional manifolds 
\( M \) whose only boundary is \( B \). In the case of the minisuperspace considered, 
this would lead to a wave function which takes on the form \( (\alpha = 1, \beta = 0) \):

\[
\Psi_{HH}(z) = \frac{Ai\left(H^{-\frac{3}{4}}(1 - H^2a^2)\right)}{Ai(H^{-\frac{1}{4}})} \quad (19)
\]

We don’t know how to compute \( \Psi^{(CS)}_{HH} \). Another well-known choice is the 
tunneling boundary condition of Vilenkin [24]. The wave function is assumed 
to have only a WKB component \( e^{-iS} \) in the semi-classical regime. This 
choice is made in order to describe the birth of the Universe that might be
interpreted as the tunnel effect of common quantum mechanics. This leads to the following wave functions ($\alpha = 1$, $\beta = i$):

$$
\Psi_V(z) = \frac{Ai\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right) + iBi\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right)}{Ai(H^{-\frac{4}{3}}) + iBi(H^{-\frac{4}{3}})}
$$

(20)

$$
\Psi_{V}^{(CS)}(z) = \frac{Ai\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right) + iBi\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right)}{Ai(-H^{-\frac{4}{3}}) + iBi(-H^{-\frac{4}{3}})}
$$

(21)

$\Psi_V(z)$ and $\Psi_{V}^{(CS)}(z)$ do not coincide, in the region where $\epsilon = 1$, only because of the presence of $-H^{-\frac{4}{3}}$ instead of $H^{-\frac{4}{3}}$ in the argument of the Airy functions of the denominator. Clearly, this is true for all boundary conditions. It comes from the fact that while computing the denominator, we used the condition $\Psi_{V}^{(CS)}(a = 0, \phi) = 0$ fixed in the region $\epsilon = -1$ where the classical theories are not identical. As we shall see, this will have important consequences.

Another boundary condition seems also quite natural. It consists in assuming the continuity of the first derivative of the wave function at points at which the change of signature occurs:

$$
\lim_{z \to 0^+} \Psi_\epsilon' = -1(z) = \lim_{z \to 0^-} \Psi_\epsilon' = +1(z)
$$

(22)

We remind that it happens at $a = \frac{1}{H}$ or $z = 0$. This condition leads to:

$$
-\alpha Ai'(0) - \beta Bi'(0) = \alpha Ai'(0) + \beta Bi'(0)
$$

(23)

Recalling one of the properties of the Airy functions which is [16, 17]:

$$
Bi'(0) = -\sqrt{3}Ai'(0),
$$

(24)

we find that the wave function is now given by ($\alpha = \sqrt{3}, \beta = 1$):

$$
\Psi_{V}^{(CS)}(z) = \frac{\sqrt{3}Ai\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right) + Bi\left(H^{-\frac{4}{3}}(1 - H^2a^2)\right)}{\sqrt{3}Ai(-H^{-\frac{4}{3}}) + Bi(-H^{-\frac{4}{3}})}
$$

(25)
\( \Psi_C^{(CS)} \) is the only wave function which is of class \( C^1 \) when the change of signature occurs. This boundary condition makes no sense in ordinary quantum cosmology where \( \Psi_C \) cannot appear (more precisely all the wave functions of ordinary quantum cosmology are \( C^1 \)-continuous). We can write the WKB approximation \( \Psi = Ae^{iS} \) for these four wave functions using the asymptotic form of Airy’s functions \([18, 19]\):

\[
\lim_{z \to \infty} Ai(-z) = \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) \tag{26}
\]

\[
\lim_{z \to \infty} Bi(-z) = \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}} \cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) \tag{27}
\]

This asymptotic form can be used in the numerator of the expression (14) since \( a \to \infty \) in the region \( \epsilon = 1 \) but also in its denominator because \( H^{-\frac{1}{2}} = \left(\frac{\dot{\phi}}{\phi}\right)^\frac{3}{2} \gg 1 \) since \( |V| << 1 \) in the semiclassical regime. We obtain (for \( \Psi_C^{(CS)} \) we have just considered the part of the wave function corresponding to the Universe in expansion. One can show that the two components \( e^{iS} \) and \( e^{-iS} \) does not interfere [29]):

\[
\Psi_{HH}(a, \phi) \propto e^{\frac{4}{V} \left( H^2a^2 - 1 \right)} \left( e^{-\frac{1}{4} e^{-i\frac{2}{3H^3}(H^2a^2-1)^\frac{3}{2}}} \right) \tag{28}
\]

\[
\Psi_C^{(CS)}(a, \phi) \propto \frac{V^{-\frac{1}{2}}}{\cos\left(\frac{1}{V} - \frac{1}{12}\right)} \left( H^2a^2 - 1 \right)^{-\frac{1}{4}} e^{-i\frac{2}{3H^2}(H^2a^2-1)^\frac{3}{2}} \tag{29}
\]

\[
\Psi_V^{(CS)}(a, \phi) \propto e^{im\left(\frac{4}{V} + \frac{1}{2}\right) \left( H^2a^2 - 1 \right)} \left( e^{-\frac{1}{4} e^{-i\frac{2}{3H^3}(H^2a^2-1)^\frac{3}{2}}} \right) \tag{30}
\]

\[
\Psi_V(a, \phi) \propto e^{-\frac{1}{4} (H^2a^2 - 1)^{-\frac{1}{4}} e^{-i\frac{2}{3H^2}(H^2a^2-1)^\frac{3}{2}}} \tag{31}
\]

Following Grishchuk and Gibbons [28, 29], we can display \( \Psi_{HH}, \Psi_C^{(CS)}, \Psi_V^{(CS)} \) and \( \Psi_V \) as seen in the space of all the wave functions defined on our minisuperspace. Introducing the notation \( \alpha = |A|e^{i\gamma_1}, \beta = |B|e^{i\gamma_2} \), the choice of boundary conditions is equivalent to fixing the complex number \( \xi \) defined
by:
\[ \xi = xe^{i\gamma} \tag{32} \]
where \( x = \frac{|B|}{|A|} \) and \( \gamma = \gamma_2 - \gamma_1 \). If we identify \( x \) with \( \theta \) via the relation \( x = \cotan \frac{\theta}{2} \), the numerator of the wave function is represented by a point \((\theta, \gamma)\) on a sphere of unit radius. The denominator can be found by identifying its modulus with the distance \( \rho \) from the center of the sphere. Then a wave function is characterized by a point \((\rho, \theta, \gamma)\) of a three-dimensional parameter space (cf. the figure). For example, \( \Psi^{(CS)}_C \) corresponds to \( \alpha = \sqrt{3} \) and \( \beta = 1 \), then \( \gamma = 0 \) and \( \theta = 2 \tan \sqrt{3} = \frac{2\pi}{3} \). Consequently \( \Psi^{(CS)}_C \) will be represented on the radius \((\theta = \frac{2\pi}{3}, \gamma = 0)\). \( \Psi^{(CS)}_V \) and \( \Psi_V \) will be plotted in the same direction \((\theta = \frac{\pi}{2}, \gamma = \frac{\pi}{2})\) but as their moduli differ, they will be represented at different points. These wave functions are peaked around the classical solutions:
\[
\begin{align*}
a(t) &= \frac{1}{H} \cosh(Ht + t_0) \tag{33} \\
\phi &= \phi_i \tag{34}
\end{align*}
\]
where the constant \( \phi_i \) is the initial value of the scalar field and \( t_0 \) a constant. Among these solutions there is the continuous solution, corresponding to \( t_0 = 0 \). All the solutions display inflationary behaviour, but the rate of inflation depends on the value of \( \phi_i \) \cite{11,30}. To find what is the most probable value of \( \phi_i \) given by the wave function, we have to use the prefactor of the WKB solutions. This prefactor allows us to define a conserved current given by the expression:
\[
J^a = |A|^2 \nabla^a S \tag{35}
\]
Then, we can define the probability measure:
\[
dP = J^a d\phi \tag{36}
\]
24
where \( J^a \) is the component of the current associated with the coordinate \( a \) in the minisuperspace (for the complete discussion, see e.g. [11, 26, 31]).

Inserting (28), (29), (30) and (31) in (36), we obtain:

\[
\begin{align*}
    dP_{HH} &\propto e^{\frac{8}{V}}d\phi \\
    dP_{C}^{(CS)} &\propto \frac{V^{-\frac{2}{3}}}{\cos^2(\frac{1}{V} - \frac{\pi}{12})}d\phi \\
    dP_{V}^{(CS)} &\propto d\phi \\
    dP_{V} &\propto e^{-\frac{8}{V}}d\phi
\end{align*}
\]

The measure \( P_C^{(CS)} \) diverges when \( V_k = \frac{48}{\pi(12k+7)} \). In what follows, we restrict our considerations to the values of the scalar field such that \(|V| << 1\) in order to be sure of the validity of the semiclassical approximation (if \( V(\phi) = m^2\phi^2 \) then \( \phi << 10^4 \) in Planck units [11, 32]). But all the singularities of \( P_C^{(CS)} \) are contained in the range \( V < 2.5 \). Then, in spite of its nice form, the choice leading to \( \Psi_C^{(CS)} \) must not be considered. We would like to emphasize that this problem would occur for all the wave functions with classical change of signature except for \( \Psi_V^{(CS)} \); this is a direct consequence of the Hamiltonian quantization with the change of signature. Indeed, the denominator in the expression \( (16) \) leads, when we take its asymptotic form, to trigonometric functions which possess lot of zeros and therefore produces divergencies. The probability that \( \phi_i \) must be greater than \( \phi_{suf} \) (the value of the scalar field sufficient to produce inflation in agreement with present observations, \( \phi_{suf} \approx 4.4 \) [11, 32]) knowing that \( \phi_i \) is smaller than \( 10^4 \), is given by:

\[
P_V^{(CS)}(\phi_i > \phi_{suf} | 0 < \phi_i < 10^4) = \frac{\int_{\phi_{suf}}^{10^4} d\phi}{\int_0^{10^4} d\phi} \approx 1
\]

Sufficient inflation is predicted indeed by \( \Psi_V^{(CS)} \). It is easy to show that
sufficient inflation is also predicted by $\Psi_V$ but not by $\Psi_{HH}$. We have seen that if we try to quantize General Relativity with the classical change of signature, a unique boundary condition that is acceptable is the tunneling boundary condition (contrary to usual quantum cosmology where many choices are possible). The wave function satisfying this condition predicts sufficient inflation. However, its first derivative is not continuous when the change of signature occurs.

5 Conclusion

The aim of this article was to construct a modified version of the Hamiltonian formulation of General Relativity in order to incorporate classical solutions of Einstein’s equations displaying the change of signature. We have shown that Euclidean solutions correspond to a region in the superspace with a signature of the supermetric being $(+-----)$ instead of $(-+++++)$. The case of the minisuperspace describing the Robertson-Walker cosmological solution with a scalar field has been studied in particular. Next, we have quantized the theory in this minisuperspace. This has allowed us to compute the wave function corresponding to classical Euclidean solutions. We have also considered different boundary conditions. Only the boundary condition introduced by Vilenkin leads to an acceptable behaviour of the wave function. However, the derivative of the wave function is not continuous when the change of signature occurs.
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A Appendix

In this appendix, we remind the basic formulae of the Hamiltonian formalism of General Relativity in order to be able to compare them with the relations including the change of signature considered at the first section of this article. The metric is decomposed according to the expression:

\[ ds^2 = (N^i N_i - N^2) dt^2 + 2 N_i dx^i dt + h_{ij} dx^i dx^j \] (1)

and the extrinsic curvature can be written as:

\[ K_{ij} = -\frac{1}{2N} (\nabla_i (N j) - \partial_t h_{ij}) \] (2)

Then, using the previous equations, the computation of the Christoffel symbols yields [33]:

\[
\begin{align*}
\Gamma^0_{00} &= \frac{\dot{N}}{N} + \frac{N^i N_j}{N} + \frac{N^i N^j}{N} K_{ij} \\
\Gamma^0_{0i} &= \frac{N_i}{N} + \frac{N^k}{N} K_{ik} \\
\Gamma^0_{ij} &= \frac{1}{N} K_{ij} \\
\Gamma^i_{00} &= -\frac{1}{2} h^{ik} (N_j N^j - N^2)_{,k} + N h^{ik} \partial_t \left( \frac{N_k}{N} \right) - \frac{N^i N^k}{N} N_{,k}
\end{align*}
\]
\[ -\frac{N^i N^k N^m}{N} K_{km} \] (6)

\[ \Gamma^i_{j0} = N(K^i_j + (3) \nabla(\frac{N^i}{N}) - \frac{N^i N^k}{N^2} K_{jk}) \] (7)

\[ \Gamma^k_{ij} = (3) \Gamma^i_{jk} - \frac{N^k}{N} K_{ij} \] (8)

The components of the Ricci tensor are given by [21]:

\[ R_{00} = -Nh^{ki} \dot{K}_{ki} + (3)\nabla_k(\partial^k N) + 2NK^i_j (3)\nabla_i N^j \]
\[ + 2N N^j (3)\nabla K_j + N N^j (3)\nabla_j K_k + N^2 K_{kj} K^{kj} \]
\[ + N^l N^m K_{lj} K_{mj} - 2N^i N^k K_{kj} K^j_i \]
\[ + \frac{N^i N^k}{N} K_{ki} - \frac{N^i N^k}{N} (3)\nabla_k(\partial_i N) - \frac{2}{N} N^i N^k K_{kj} (3)\nabla_i N^j \]
\[ - \frac{1}{N} N^i N^k N^j (3)\nabla_i N^j \] (9)

\[ R_{0i} = \frac{N^k}{N} \dot{K}_{ki} + N^j (3) R_{ij} - \frac{N^k}{N} (3)\nabla_k(\partial_i N) - N \partial_i K_j^j \]
\[ + N^j (3)\nabla_j (K^j_i) + N^k N_{kh} K^j_k - \frac{N^j N^m}{N} (3)\nabla_j K_{im} \]
\[ - \frac{N^k}{N} K_{jk} (3)\nabla_i N^j - \frac{N^j}{N} K_{im} (3)\nabla_j N^m - 2N^j K^m_i K_{mj} \] (10)

\[ R_{ij} = (3) R_{ij} + \frac{1}{N} K_{ij} - 2K_{ki} K^k_j + K^k_k K_{ij} \]
\[ - \frac{N^k}{N} (3)\nabla K_{ij} - \frac{1}{N} (3)\nabla_j(\partial_i N) - \frac{1}{N} K_{ki} (3)\nabla_j N^k \]
\[ - \frac{1}{N} K_{jk} (3)\nabla_i N^k \] (11)

The Ricci scalar deduced from the above equations takes on the form:

\[ R = (3) R - 3K_{ki} K^{ki} + K^2 - \frac{2}{N} N^i \partial_i K^j_j \]
\[ + \frac{2}{N} h^{ki} \dot{K}_{ki} - \frac{2}{N} (3)\nabla_k(\partial^k N) - \frac{4}{N} K^j_k (3)\nabla_j N^k \] (12)
One can also write $R$ as:

$$R = (3)R + K_{ki}K^{ki} + K^2 + \frac{2}{N}\dot{K} - \frac{2N^i}{N}N^i\partial_iK$$

$$- \frac{2}{N}(3)\nabla_k(\partial^kN)$$

Finally, we obtain the well-known formula giving the Lagrangian of the gravitational field including the surface term:

$$\mathcal{L}_G = \frac{1}{2N}(3)R + K_{ki}K^{ki} - K^2$$

$$+ 2\frac{d}{dt}(h^{\frac{1}{2}}K) - 2\partial_i(h^{\frac{1}{2}}KN^i + h^{\frac{1}{2}}h^{ki}\partial_kN)$$

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Caption of the figure:

$\Psi_{HH}, \Psi_V, \Psi^{(CS)}_C$ and $\Psi^{(CS)}_C$ in the space of Wave functions.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9402028v1