UNIQUENESS AND STABILITY FOR THE SOLUTION OF A NONLINEAR LEAST SQUARES PROBLEM

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Abstract. In this paper, we focus on the nonlinear least squares: \( \min_{x \in \mathbb{H}^d} \| |Ax| - b \| \) where \( A \in \mathbb{H}^{m \times d}, b \in \mathbb{R}^m \) with \( \mathbb{H} \in \{ \mathbb{R}, \mathbb{C} \} \) and consider the uniqueness and stability of solutions. Such problem arises, for instance, in phase retrieval and absolute value rectification neural networks. For the case where \( b = |Ax_0| \) for some \( x_0 \in \mathbb{H}^d \), many results have been developed to characterize the uniqueness and stability of solutions. However, for the case where \( b \neq |Ax_0| \) for any \( x_0 \in \mathbb{H}^d \), there is no existing result for it to the best of our knowledge. In this paper, we first focus on the uniqueness of solutions and show for any matrix \( A \in \mathbb{H}^{m \times d} \) there always exists a vector \( b \in \mathbb{R}^m \) such that the solution is not unique. But, in real case, such “bad” vectors \( b \) are negligible, namely, if \( b \in \mathbb{R}^m_+ \) does not lie in some measure zero set, then the solution is unique. We also present some conditions under which the solution is unique. For the stability of solutions, we prove that the solution is never uniformly stable. But if we restrict the vectors \( b \) to any convex set then it is stable.

1. Introduction

1.1. Problem setup. Assume that \( A := [a_1, \ldots, a_m]^* \in \mathbb{H}^{m \times d} \) and \( b := [b_1, \ldots, b_m]^* \in \mathbb{R}^m \) where \( \mathbb{H} \in \{ \mathbb{R}, \mathbb{C} \} \). We are interested in the following program

\[
\Phi_A(b) := \arg\min_{x \in \mathbb{H}^d} \| |Ax| - b \|^2,
\]

where \(|\cdot|\) is understood to act entrywise. Such model has a rich history in statistics and is widely used in phase retrieval (see \([34, 37, 12, 28, 14]\)) and deep learning \([22, 16]\). Although one has developed many algorithms to solve (1.1), especially in the randomized setting (meaning that the matrix \( A \) is drawn at random), there are very few results about the properties of the program, such as the uniqueness and stability of the solution.

For convenience, we set

\[
\mathcal{K}_A := \{ y = |Ax| \in \mathbb{R}^m : x \in \mathbb{H}^d \}
\]

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and call $\mathcal{K}_A$ as a phaseless surface corresponding to $A$. When $b \in \mathcal{K}_A$, i.e., $b = |Ax_0|$ for some $x_0 \in \mathbb{H}^d$, the recovery of $x_0$ from the phaseless observation vector $b$ is known as phase retrieval. For this case where $b \in \mathcal{K}_A$, there are many results for the uniqueness and stability of the solution to (1.1). For instance, if $m \geq 2d - 1$ (resp. $m \geq 4d - 4$) then the generic matrix $A \in \mathbb{R}^{m \times d}$ (resp. $A \in \mathbb{C}^{m \times d}$) suffices to guarantee the uniqueness of the solution to (1.1) (see e.g. [2, 8, 35]); moreover, for $b \in \mathcal{K}_A$, the solution to (1.1) is always stable for any fixed $d$ (i.e., finite-dimensional Hilbert space) [3, 17] while it is unstable in any infinite-dimensional Hilbert space [1, 6]. However, in the noisy phase retrieval, we often encounter the case where $b \notin \mathcal{K}_A$. In this setting, to our knowledge, there is no result concerning the uniqueness and stability of solutions. Naturally, one may be interested in whether the solution to (1.1) is unique or stable for any $b \in \mathbb{R}^m$, which is the topic of this paper.

As said before, the aim of this paper is to address the uniqueness and stability of solutions of the nonlinear least squares problem (1.1). Particularly, we are interested in the following questions:

Question I (Uniqueness of solutions) Does there exist a matrix $A \in \mathbb{H}^{m \times d}$ so that the solution to (1.1) is unique up to a unimodular constant for all the vectors $b \in \mathbb{R}^m$?

Question II (Conditions for uniqueness) For which vector $b \in \mathbb{R}^m$, the solution to (1.1) is unique?

Question III (Stability of solutions) Is there a matrix $A \in \mathbb{H}^{m \times d}$ and a constant $c$ only depending on $A$ so that

$$\min_{x \in \Phi_A(b_1), y \in \Phi_A(b_2)} \|x - y\| \leq c \|b_1 - b_2\|$$

holds for all $b_1, b_2 \in \mathbb{R}^m$?

Note that if $x \in \mathbb{H}^d$ is a solution to (1.1) then $cx$ is also a solution to (1.1) for any unimodular constant $c$. Thus, we say $x \sim y$ if $x = cy$ for some unimodular constant $c$. Let $\mathbb{H}^d := \mathbb{H}^d/\sim$. We shall use $[x]$ to denote the equivalent class containing $x$. We say that the solution to (1.1) is unique if $\Phi_A(b)$ only contains one element in $\mathbb{H}^d$. The distance between $\bar{x}$ and $\bar{y}$ is defined as $\|\bar{x} - \bar{y}\| := \min_{c \in \mathbb{H}, |c| = 1} \|x - cy\|$.

1.2. Related work.
1.2.1. **Phase retrieval.** The most related example to (1.1) is *phase retrieval*, which aims to recover the signals from the magnitudes of measurements. The phase retrieval problem arises in many areas, such as X-ray crystallography [19, 27], optics [33], astronomical imaging [11], diffraction imaging [4], and microscopy [26]. In these areas, the phase information of an object is lost due to physical limitations of scientific instruments. More specifically, suppose that a signal \( x_0 \in \mathbb{H}^d \) is measured via measurement vectors \( a_i \in \mathbb{H}^d \) to obtain \( b_i = |\langle a_i, x_0 \rangle|, \ i = 1, \ldots, m \). The phase retrieval problem aims to recover the signal \( x_0 \) based on measurement matrix \( A := [a_1, \ldots, a_m]^* \in \mathbb{H}^{m \times d} \) and vector \( b := [b_1, \ldots, b_m]^* \in \mathbb{R}^m \). A natural approach to reconstruct \( x_0 \) is to employ (1.1). Many efficient algorithms have been proposed for solving (1.1) with the proviso that \( A \) is a Gaussian random matrix, such as Truncated Amplitude Flow [34], Reshaped Wirtinger Flow [37], Perturbed Amplitude Flow [13] and Smoothed Amplitude Flow [5].

We say a matrix \( A \in \mathbb{H}^{m \times d} \) has *phase retrieval property* if one can recover any \( x_0 \in \mathbb{H}^d \) from \( b = |Ax_0| \in K_A \). For the real case, the matrix \( A \in \mathbb{R}^{m \times d} \) has phase retrieval property if and only if \( A \) satisfies the complement property [3], which implies \( m \geq 2d - 1 \) generic vectors of \( \mathbb{R}^d \) are sufficient to have phase retrieval property. For the complex case, Balan, Casazza and Edidin in [2] show that \( A \in \mathbb{C}^{m \times d} \) has phase retrieval property if \( m \geq 4d - 2 \) and \( a_1, \ldots, a_m \) are generic vectors in \( \mathbb{C}^d \). Lately, Bandeira, Cahill, Mixon and Nelson improve this result to \( m \geq 4d - 4 \) generic vectors [8].

Recently, the phase retrieval problem under a generative prior is studied in [18, 30] and is termed as *deep phase retrieval*. In such setting, the signal of interest is the output of a generative model which is a \( n \)-layer, fully-connected, feed forward neural network with Rectifying Linear Unit (ReLU) activation functions and no bias terms. To recover the signal, they consider the empirical risk minimization problem:

\[
\min_{x \in \mathbb{R}^d} \| |AG(x)| - |AG(x_0)| \|^2,
\]

where \( G(x) := \text{ReLU}(W_n \cdots \text{ReLU}(W_2(\text{ReLU}(W_1 x)))) \) with the weights \( W_i \in \mathbb{R}^{k_i \times k_{i-1}} \) and \( \text{ReLU}(z) = \max(0, z) \). The results of [18, 30] show that the objective function of (1.2) exhibits favorable geometry landscape and does not have any spurious local minima away from neighborhoods of the true solutions provided \( A \in \mathbb{R}^{m \times k_n} \) is Gaussian random matrix and \( m = \Omega(dn \log(k_1 \cdots k_n)) \). A simple observation is that (1.2) has a unique solution up
to a unimodular constant if $A$ has phase retrieval property. That is another example of program (1.1) which combine phase retrieval and deep learning.

1.2.2. Shallow neural networks. Another example related to the program (1.1) is shallow neural networks with absolute value rectification. More specifically, given training data \( \{(a_i, b_i)\}_{i=1}^m \in \mathbb{R}^d \times \mathbb{R} \), we consider a neural network with zero hidden unit and a single output with absolute value activation to fit the data. A natural approach is to minimize the least squares misfit aggregated over the data, which is in the form of (1.1) exactly. Absolute value rectification $g(z) = |z|$ is a generalization of ReLU units. Since the slope is non-zero when $z$ is negative, it can be used to avoid the dead ReLU problem. Fitting the data with absolute value rectification has several advantages over others activation functions when taking into account the (sign) symmetry of features [36]. For example, for the object recognition from images, it makes sense to use absolute value rectification to seek features that are invariant under a polarity reversal of the input illumination [16]. We would like to point out that there is an interesting growing literature [20, 23, 24, 25, 31] on learning shallow neural networks with zero hidden unit and a single output, most of which focus on geometric landscape analysis and the convergence of gradient-based methods to the global optimum under various assumptions. To our knowledge, there is little works considering the uniqueness and stability of solutions. Since neural networks have achieved remarkable empirical success [7, 15, 29, 10] while still lack of theoretical guarantees, we believe that the results in our paper are useful in reducing the gap.

1.3. Our Contribution. The aim of this paper is trying to answer Question I, Question II and Question III. For Question I, we prove that for arbitrary matrix $A \in \mathbb{H}^{m \times d}$ there always exists $b \in \mathbb{R}^m$ such that the solution to (1.1) is not unique, which gives a negative answer for it. We then turn to Question II in the real case. First, we show that the set of nonuniqueness vectors $b$ is negligible in the nonnegative orthant, i.e., for all vectors $b \in \mathbb{R}_+^m$ except a measure zero set the solution to (1.1) is unique. Recall that we use $\Phi_A(b)$ to denote the solutions set to (1.1). We next prove that $\# \Phi_A(b)$ is finite provided $A$ satisfies the phase retrieval property. Finally, we present a sufficient condition, the vector $b$ is very close to the set $K_A$, under which the solution to (1.1) is unique. These explain the reason why the solution to (1.1) is often unique in many numerical experiments. Although the results only
hold in the real case, it sheds light on the relationship of the vector $b$ to uniqueness of the solution.

Finally, we consider Question III, i.e., the stability of solutions. We prove that for any $\epsilon > 0$ there always exist $b_1, b_2 \in \mathbb{R}^m$ so that $\text{dist}(\Phi_A(b_1), \Phi_A(b_2)) \geq \|b_1 - b_2\|/\epsilon$, which means the solution to (1.1) is never uniformly stable. But if we restrict the vector $b$ to some convex sets, then the solution to (1.1) is stable.

1.4. Organization. The paper is organized as follows. In Section 2, we introduce some notations and lemmas which are useful in this paper. In Section 3, we present a negative result for Question I and show the solution to (1.1) is not unique for some vectors $b$. Section 4 is devoted to establishing several uniqueness results under some appropriate conditions, which gives a positive answer to Question II. Finally, Section 5 is concerned with the stability of solutions to (1.1), which gives the answers to Question III.

2. Preliminaries

In this section, we introduce a few notations and lemmas that will be used in our paper.

2.1. The best approximation and Chebyshev sets. Assume that $K \subset \mathbb{R}^m$ is nonempty. For any fixed $b \in \mathbb{R}^m$, if $y^\# \in \mathbb{R}^m$ satisfies

$$\|y^\# - b\| = \min_{y \in K} \|y - b\|$$

then $y^\#$ is called a best approximation to $b$ from $K$ and $d(K, b) := \|y^\# - b\|$ is called the distance from $b$ to $K$. We use $P_K(b)$ to denote the set of all best approximations to $b$ from $K$. In the context of the best approximation theory, $K$ is called a Chebyshev set if each $b \in \mathbb{R}^m$ has a unique best approximation in $K$ (see [9]). The following lemma presents a characterization of Chebyshev set in finite-dimensional Hilbert space.

**Lemma 2.1.** ([9, Theorem 12.7]) Assume that $K$ is a nonempty subset of $\mathbb{R}^m$. Then $K$ is a Chebyshev set if and only if $K$ is closed and convex.

The next lemma states that the distance function is nonexpansive for any nonempty set.
Lemma 2.2. [9, Theorem 5.3] Assume that $K \subset \mathbb{R}^m$ is a nonempty set. Then for every pair $b, b' \in \mathbb{R}^m$, 
\[ |d(K, b) - d(K, b')| \leq \|b - b'\|. \]

The following lemma shows that the projection operator onto a Chebyshev set is also nonexpansive.

Lemma 2.3. [9, Theorem 12.3] Assume that $K \subset \mathbb{R}^m$ is closed and convex. Then 
\[ \|P_K(b) - P_K(b')\| \leq \|b - b'\| \text{ for all } b, b' \in \mathbb{R}^m. \]

2.2. Some results about phase retrieval. As stated before, we say a matrix $A \in \mathbb{H}^{m \times d}$ has phase retrieval property if any $x_0 \in \mathbb{H}^d$ can be recovered from $|Ax_0| \in \mathbb{R}^m$.

The following lemma presents a relationship between the solution to (1.1) and the best approximation to $b$ from $K_A := \{y = |Ax| \in \mathbb{R}^m : x \in \mathbb{H}^d\}$.

Lemma 2.4. Assume that $A \in \mathbb{H}^{m \times d}$ has phase retrieval property. For any vector $b \in \mathbb{R}^m$, the program (1.1) has a unique solution if and only if the best approximation to $b$ from the $K_A$ has exactly one element, i.e., $\#P_{K_A}(b) = 1$.

Proof. We assume that the best approximation to $b$ from $K_A$ has exactly one element. Then there exists a vector $b_1 \in K_A$ such that
\[ (2.1) \quad \|b - b_1\| < \|b - b_2\| \quad \text{for any } b_2 \in K_A \setminus \{b_1\}. \]

Since $A$ has phase retrieval property, there exists a unique $x_1 \in \mathbb{H}^d$ such that $b_1 = |Ax_1|$. According to (2.1), we have
\[ ||Ax_1| - b|| < ||Ax_2| - b|| \quad \text{for any } x_2 \in \mathbb{H}^d \setminus \{x_1\}, \]
which implies the solution to (1.1) is unique.

We next assume that (1.1) has a unique solution. We will show that the best approximation to $b$ from the phaseless surface $K_A$ contains only one element. For the aim of contradiction, we assume there exist two best approximations to $b$ for $K_A$, say $b_1$ and $b_2$. Then there exist two vectors $x_1, x_2 \in \mathbb{H}^d$ with $x_1 \neq x_2$ such that $b_1 = |Ax_1|$, $b_2 = |Ax_2|$ and
\[ ||Ax_1| - b|| = ||Ax_2| - b|| < ||Ax| - b|| \quad \text{for any } x \in \mathbb{H}^d \setminus \{x_1, x_2\}. \]
which implies (1.1) has two solutions $x_1$ and $x_2$. This contradicts to the assumption.

□

For the case where $\mathbb{H} = \mathbb{R}$, the matrix $A \in \mathbb{R}^{m \times d}$ has phase retrieval property if and only if $A$ satisfies the complement property:

**Lemma 2.5.** [3] The matrix $A := [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times d}$ has phase retrieval property in $\mathbb{R}^d$ if and only if for every $I \subset \{1, \ldots, m\}$, either $\text{span}\{a_j : j \in I\} = \mathbb{R}^d$ or $\text{span}\{a_j : j \in I^c\} = \mathbb{R}^d$.

The following lemma shows that, for the real case, any solution to (1.1) satisfies a fixed-point equation.

**Lemma 2.6.** [21] Suppose that $A := [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. Assume that $\hat{x}$ is a solution to (1.1). Then $\hat{x}$ satisfies the following fixed-point equation:

$$\hat{x} = (A^\top A)^{-1} A^\top (b \odot s(A\hat{x})), $$

where $\odot$ denotes the Hadamard product and $s(A\hat{x}) := \left(\frac{\langle a_1, \hat{x} \rangle}{|\langle a_1, \hat{x} \rangle|}, \ldots, \frac{\langle a_m, \hat{x} \rangle}{|\langle a_m, \hat{x} \rangle|}\right)$ for any $\hat{x} \in \mathbb{R}^d$.

Here, $\frac{\langle a_j, \hat{x} \rangle}{|\langle a_j, \hat{x} \rangle|} = 1$ is adopted if $\langle a_j, \hat{x} \rangle = 0$.

3. The non-uniqueness of solutions to (1.1)

The aim of this section is to answer Question I by showing that the solution to (1.1) is nonunique for some vectors $b \in \mathbb{R}^m$. We state the main result of this section as follows.

**Theorem 3.1.** Assume that $m, d$ are positive integers. For arbitrary matrix $A \in \mathbb{H}^{m \times d}$, there exists $b \in \mathbb{R}^m$ so that the solution to (1.1) is not unique where $\mathbb{H} \in \{\mathbb{R}, \mathbb{C}\}$.

To prove this theorem, according to Lemma 2.4, it is enough to show the set $\mathcal{K}_A$ is not a Chebyshev set. From Lemma 2.1, we can do it by showing the set $\mathcal{K}_A$ is not a convex set.

**Lemma 3.2.** Assume that $A \in \mathbb{H}^{m \times d}$ has phase retrieval property. Then the set $\mathcal{K}_A \subset \mathbb{R}^m$ is non-convex.

**Proof.** We first prove it in the real case where $\mathbb{H} = \mathbb{R}$. For the aim of contradiction, we assume that $\mathcal{K}_A$ is convex. Let $A := [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times d}$ with $a_j = (a_{j,1}, \ldots, a_{j,d})^\top \in \mathbb{R}^d$. 


Without loss of generality, we assume that \( a_j = e_j, j = 1, \ldots, d \). Since \( A \) has phase retrieval property, there exists \( j_0 \in \{d + 1, \ldots, m\} \) so that \( \|a_{j_0}\|_0 \geq 2 \). Without loss of generality, we assume that \( j_0 = d + 1 \) and the \( d \)-th component of \( a_{d+1} \) is positive, i.e., \( a_{d+1,d} > 0 \). For any \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we have \( |\langle a_j, x \rangle| = |x_j|, \; j = 1, \ldots, d \). Then there exist \( \epsilon > 0 \) and \( r_0 > 0 \) so that \( \langle a_{d+1}, x \rangle > 0 \) provided \( x \in [-\epsilon, \epsilon]^d \times [r_0, \infty] \). Let \( x' := (-x_1, \ldots, -x_{d-1}, x_d) \). Since \( \mathcal{K}_A \) is convex, we have

\[
\frac{1}{2} |Ax| + \frac{1}{2} |Ax'| \in \mathcal{K}_A,
\]

where \( x, x' \in [-\epsilon, \epsilon]^d \times [r_0, \infty] \). Note that the first \( d + 1 \) entries of \( \frac{1}{2} |Ax| + \frac{1}{2} |Ax'| \) is \( |x_1|, \ldots, |x_d|, a_{d+1,d}x_d \) provided \( x, x' \in [-\epsilon, \epsilon]^d \times [r_0, \infty] \). Here, we use \( \langle a_{d+1}, x \rangle > 0 \) and \( \langle a_{d+1}, x' \rangle > 0 \) if \( x, x' \in [-\epsilon, \epsilon]^d \times [r_0, \infty] \). According to (3.1), for any \( x \in [-\epsilon, \epsilon]^d \times [r_0, \infty] \), there exists \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_d) \) with \( \hat{x}_d > 0 \) so that

\[
|\hat{x}_j| = |x_j|, j = 1, \ldots, d, \quad |\langle a_{d+1}, \hat{x} \rangle| = a_{d+1,d}x_d.
\]

Note that \( x_d > r_0 > 0 \). Combining \( |x_d| = |\hat{x}_d| \) and \( \hat{x}_d > 0 \), we have \( \hat{x}_d = x_d \). Since \( |\hat{x}_j| = |x_j| \leq \epsilon \) for all \( j = 1, \ldots, d - 1 \), then the choice of \( r_0 \) implies \( \langle a_{d+1}, \hat{x} \rangle > 0 \). According to (3.2), we have

\[
\langle a_{d+1}, \hat{x} \rangle = a_{d+1,d}x_d,
\]

which implies that

\[
a_{d+1,1}\hat{x}_1 + \cdots + a_{d+1,d-1}\hat{x}_{d-1} = 0
\]

holds for any \((x_1, \ldots, x_{d-1}) \in [-\epsilon, \epsilon]^{d-1}\). Combining (3.3) and \( |\hat{x}_j| = |x_j| \), we obtain that \( a_{d+1,1} = \cdots = a_{d+1,d-1} = 0 \), which contradicts to \( \|a_{d+1}\|_0 \geq 2 \).

We next turn to the complex case where \( \mathbb{H} = \mathbb{C} \). For the aim of contradiction, we assume that \( \mathcal{K}_A \) is convex. Without loss of generality, we assume that \( a_j = e_j, j = 1, \ldots, d \). Since \( A \in \mathbb{C}^{m \times d} \) has phase retrieval property, there exist distinct \( j_0, k_0 \in \{d + 1, \ldots, m\} \) so that \( |a_{j_0,d-1}|^2 + |a_{j_0,d}|^2 \neq 0 \) and \( |a_{k_0,d-1}|^2 + |a_{k_0,d}|^2 \neq 0 \). Otherwise, one can not recover the vector in the form of \((0, \ldots, 0, x_{d-1}, x_d) \in \mathbb{C}^d \). Without loss of generality, we assume that \( j_0 = d + 1, k_0 = d + 2 \) and \( a_{d+1,d} = a_{d+2,d} = 1 \). We assume that \( x = (0, \ldots, 0, r'e^{-i\theta_1}, x_d) \) and \( x' = (0, \ldots, 0, r'e^{-i\theta_2}, x_d) \). Here, \( \theta_1, \theta_2 \in [0, 2\pi) \) and \( r' > 0, x_d > 0 \) are fixed constants. We have

\[
\frac{1}{2} |Ax| + \frac{1}{2} |Ax'| \in \mathcal{K}_A.
\]
A simple calculation shows that the first \(d+2\) entries of \(\frac{1}{2}|Ax| + \frac{1}{2}|Ax'|\) are \((0, \ldots, 0, r', x_d, s_1, s_2)^\top\)

where \(s_1 := s_1(\theta_1, \theta_2) = \frac{1}{2}(|a_{d+1,d-1}r'e^{-i\theta_1} + x_d| + |a_{d+1,d-1}r'e^{-i\theta_2} + x_d|)\) and \(s_2 := s_2(\theta_1, \theta_2) = \frac{1}{2}(|a_{d+2,d-1}r'e^{-i\theta_1} + x_d| + |a_{d+2,d-1}r'e^{-i\theta_2} + x_d|).\) According to (3.4), there exists \(\hat{x} = (0, \ldots, 0, \hat{x}_{d-1}, \hat{x}_d)\) with \(\hat{x}_d > 0\) so that

\[
|\hat{x}_{d-1}| = |x_{d-1}| = r', \quad |\hat{x}_d| = x_d, \quad |a_{d+1,d-1}\hat{x}_{d-1} + \hat{x}_d| = s_1, \quad |a_{d+2,d-1}\hat{x}_{d-1} + \hat{x}_d| = s_2.
\]

Since \(\hat{x}_d > 0\) and \(|\hat{x}_d| = x_d\), we have \(\hat{x}_d = x_d\). The \(|\hat{x}_{d-1}| = r'\) implies \(\hat{x}_{d-1} = r'e^{i\theta'}\) for some \(\theta' \in \mathbb{R}\). So, \(|a_{d+1,d-1}\hat{x}_{d-1} + \hat{x}_d|, |a_{d+2,d-1}\hat{x}_{d-1} + \hat{x}_d|\) is a one dimensional manifold with respect to \(\theta'\) while \((s_1, s_2)\) is two dimensional manifold with respect to \(\theta_1\) and \(\theta_2\), which is a contradiction. \(\square\)

We next present the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We divide the proof into two cases:

**Case 1:** The matrix \(A\) does not have phase retrieval property. From the definition of phase retrievable, there exist two vectors \(x_1, x_2 \in \mathbb{H}^d\) with \(x_1 \neq x_2\) such that \(|Ax_1| = |Ax_2|\). Let \(b := |Ax_1|\). Then (1.1) has two solutions \(x_1, x_2\) for such vector \(b\). The conclusion holds.

**Case 2:** The matrix \(A\) has phase retrieval property. According to Lemma 3.2, \(\mathcal{K}_A\) is non-convex, which means the set \(\mathcal{K}_A\) is not a Chebyshev set by Lemma 2.1. Thus, the conclusion immediately follows from Lemma 2.4. \(\square\)

The next result shows that, in the real case, \(#\Phi_A(b)\) is finite, i.e., the solutions to (1.1) are finite.

**Theorem 3.3.** Assume that \(A \in \mathbb{R}^{m \times d}\) has phase retrieval property. Then for any vector \(b \in \mathbb{R}^m\), the number of solutions to (1.1) is finite.

**Proof.** Since \(A\) is phase retrievable, it suffices to show the number of the best approximations to \(b\) from \(\mathcal{K}_A\) is finite. Note that

\[
\mathcal{K}_A = \left\{ y : y = |Ax|, x \in \mathbb{R}^d \right\} = \cup_{S \subseteq \{1, \ldots, m\}} \left\{ y \geq 0 : y = A_S x, x \in \mathbb{R}^d \right\}.
\]

Here, the \(i\)th row of \(A_S \in \mathbb{R}^{m \times d}\) is defined as follows

\[
(A_S)i := \begin{cases} a_1^\top, & i \in S, \\ -a_1^\top, & \text{otherwise.} \end{cases}
\]
Since the set \( \{ y \geq 0 : y = A_S x, x \in \mathbb{R}^d \} \) is convex, it means the best approximation to \( b \) from \( \{ y \geq 0 : y = A_S x, x \in \mathbb{R}^d \} \) is unique. Note that the number \(|S| \leq 2^m\). Hence, the number of the best approximations to \( b \) from \( \mathcal{K}_A \) is at most \( 2^m \), which is finite. This completes the proof. \( \square \)

4. The conditions of uniqueness of solutions to (1.1)

In this section, we focus on Question II: For which vector \( b \in \mathbb{R}^m \) the solution to (1.1) is unique? We will present several sufficient conditions for it. Throughout this section, we assume that \( H = \mathbb{R} \).

4.1. Almost all the vectors \( b \in \mathbb{R}_+^m \). The following theorem shows that for almost all the vectors \( b \in \mathbb{R}_+^m \) the solution to (1.1) is unique provided \( A \in \mathbb{R}^{m \times d} \) has phase retrieval property.

**Theorem 4.1.** Suppose that \( A \in \mathbb{R}^{m \times d} \) has phase retrieval property in \( \mathbb{R}^d \). Then for all vectors \( b \in \mathbb{R}_+^m \) except for a measure zero set, the program (1.1) has a unique solution.

Theorem 4.1 only considers the real case. We conjecture a similar result holds for complex case:

**Conjecture 4.2.** Suppose that \( A \in \mathbb{C}^{m \times d} \) has phase retrieval property in \( \mathbb{C}^d \). Then for all vectors \( b \in \mathbb{R}^m \) except for a measure zero set, the program (1.1) has a unique solution.

Before presenting the proof of Theorem 4.1, we introduce the following lemma, which will be used in the proof of Theorem 4.1.

**Lemma 4.3.** Assume that \( A := [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times d} \) has phase retrieval property. Set

\[
P_{\epsilon, \epsilon'}(z) := \|A^\top D_\epsilon z\|^2 - \|A^\top D_{\epsilon'} z\|^2,
\]

where \( z \in \mathbb{R}^m \), \( \epsilon, \epsilon' \in \{1, -1\}^m \), \( D_\epsilon = \text{Diag}(\epsilon_1, \ldots, \epsilon_m) \) and \( D_{\epsilon'} = \text{Diag}(\epsilon'_1, \ldots, \epsilon'_m) \). Then \( P_{\epsilon, \epsilon'}(z) \neq 0 \) provided \( \epsilon \neq \pm \epsilon' \).

**Proof.** A simple observation is that \( P_{\epsilon, \epsilon'} \) is a polynomial with respect to \( z \in \mathbb{R}^m \). To obtain the conclusion, it is enough to show \( P_{\epsilon, \epsilon'}(z_0) \neq 0 \) for some \( z_0 \in \mathbb{R}^m \). Without loss of
generality, we assume that $\epsilon' = e := (1, \ldots, 1)$. Hence, $\epsilon \neq \pm e$. For convenience, we set $P_\epsilon(z) := P_{e,\epsilon}(z)$. A simple calculation leads to

$$P_\epsilon(z) = \sum_{j \neq k} z_j z_k \langle a_j, a_k \rangle - \sum_{j \neq k} \epsilon_j \epsilon_k z_j z_k \langle a_j, a_k \rangle,$$

where $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$. Set $I := \{j : \epsilon_j = 1\}$ and $I^c = \{1, \ldots, m\} \setminus I$. Then $I \neq \emptyset$ and $I^c \neq \emptyset$ due to $(\epsilon_1, \ldots, \epsilon_m) \neq (1, \ldots, 1)$. Since the matrix $A$ has phase retrieval property, according to Lemma 2.5, we have $\text{span}\{a_j : j \in I\} = \mathbb{R}^d$ or $\text{span}\{a_j : j \in I^c\} = \mathbb{R}^d$.

Without loss of generality, we assume that $\text{span}\{a_j : j \in I^c\} = \mathbb{R}^d$ and that $\epsilon_1 = 1$. Since $\text{span}\{a_j : j \in I^c\} = \mathbb{R}^d$, there exists $j_0 \in I^c$ such that $\langle a_1, a_{j_0} \rangle \neq 0$. Noting that $j_0 \in I^c$, we have $\epsilon_1 \epsilon_{j_0} = -1$. Set $z_0 := (1, 0, \ldots, 0, -1, 0, \ldots, 0) \in \mathbb{R}^m$. Then $|P_\epsilon(z_0)| = 2|\langle a_1, a_{j_0} \rangle| \neq 0$ which implies $P_\epsilon(z) \neq 0$. More precisely, $P_\epsilon(z)$ is a nonzero homogeneous polynomial with respect to $z \in \mathbb{R}^m$. \hfill \Box

We are now ready to prove the main result in this subsection.

**Proof of Theorem 4.1.** Since $A$ has phase retrieval property, it then follows from Lemma 2.5 that $\text{rank}(A) = d$. We assume that the singular decomposition of $A$ is $A = UDV^\top$ where $D \in \mathbb{R}^{d \times d}$ is an invertible diagonal matrix. Set $\tilde{A} := A V D^{-1}$ and $\tilde{x} := D V^\top x$. Note that

$$||Ax - b|| = ||\tilde{A}x - b||.$$

Hence, the program (1.1) has a unique solution iff $\arg\min_{x \in \mathbb{R}^d} ||\tilde{A}x - b||$ has a unique solution. Observe that $\tilde{A}^\top \tilde{A} = I$. So, to this end, it is enough to consider the case where $A^\top A = I$.

Assume that $b_0 \in \mathbb{R}^m$ so that $\Phi_A(b_0)$ contains at least two elements, i.e., the program (1.1) has at least two solutions for the vector $b_0$. We claim that $b_0$ satisfies $P_{\epsilon',\epsilon}(b_0) = 0$ for some $\epsilon', \epsilon \in \{-1, 1\}^m$ with $\epsilon \neq \pm \epsilon'$, where

$$P_{\epsilon',\epsilon}(z) := ||A^\top D_{\epsilon'} z||^2 - ||A^\top D_{\epsilon} z||^2.$$

According to Lemma 4.3, $P_{\epsilon',\epsilon}(b_0)$ is a nonzero homogeneous polynomial with respect to $b_0 \in \mathbb{R}^m$. It means that the Lebesgue measure of the set $\bigcup_{\epsilon \neq \pm \epsilon'} \{b_0 \in \mathbb{R}^d : P_{\epsilon',\epsilon}(b_0) = 0\}$ is 0. This establishes the conclusion.
It remains to prove the claim that \( P_{\epsilon',\epsilon}(b_0) = 0 \). Assume that \( x_1 \) and \( x_2 \) are two global solutions to (1.1) with \( x_1 \neq x_2 \), namely, \( x_1 \neq \pm x_2 \). Set

\[
S_1 := \left\{ i : a_i^\top x_1 \geq 0 \right\} \quad \text{and} \quad S_2 := \left\{ i : a_i^\top x_2 \geq 0 \right\}.
\]

According to Lemma 2.6, we obtain

\[
x_1 = A^\top D_{S_1} b_0 \quad \text{and} \quad x_2 = A^\top D_{S_2} b_0.
\]

Here, \( D_S := \text{Diag}(\delta_S(1), \ldots, \delta_S(m)) \in \mathbb{R}^{m \times m} \),

\[
\delta_S(j) := \begin{cases} 
1, & j \in S, \\
-1, & \text{otherwise}.
\end{cases}
\]

Using the notations above, we have

\[
|Ax_1| = D_{S_1}(AA^\top D_{S_1} b_0) \quad \text{and} \quad |Ax_2| = D_{S_2}(AA^\top D_{S_2} b_0).
\]

Since \( x_1 \) and \( x_2 \) are two global solutions to (1.1), it gives

\[
\|D_{S_1}(AA^\top D_{S_1} b_S) - b_0\|^2 = \|D_{S_2}(AA^\top D_{S_2} b_S) - b_0\|^2,
\]

which is equivalent to

\[
(4.1) \quad \|AA^\top D_{S_1} b_0 - D_{S_1} b_0\|^2 = \|AA^\top D_{S_2} b_0 - D_{S_2} b_0\|^2.
\]

A simple calculation shows that

\[
(4.2) \quad \|AA^\top D_{S_1} b_0 - D_{S_1} b_0\|^2 = \langle AA^\top D_{S_1} b_0, AA^\top D_{S_1} b_0 \rangle - 2 \langle AA^\top D_{S_1} b_0, D_{S_1} b_0 \rangle + \|D_{S_1} b_0\|^2
\]

\[
= \langle A^\top AA^\top D_{S_1} b_0, A^\top D_{S_1} b_0 \rangle - 2 \langle A^\top D_{S_1} b_0, A^\top D_{S_1} b_0 \rangle + \|D_{S_1} b_0\|^2
\]

\[
= -\|A^\top D_{S_1} b_0\|^2 + \|b_0\|^2,
\]

where we use the fact \( A^\top A = I \). Combining (4.1) and (4.2), we obtain

\[
(4.3) \quad \|A^\top D_{S_1} b_0\|^2 = \|A^\top D_{S_2} b_0\|^2.
\]

Take \( \epsilon' = (\delta_{S_1}(1), \ldots, \delta_{S_1}(m)) \) and \( \epsilon = (\delta_{S_2}(1), \ldots, \delta_{S_2}(m)) \). Since \( x_1 \neq \pm x_2 \), we know \( \epsilon \neq \pm \epsilon' \). From (4.3), we have \( P_{\epsilon',\epsilon}(b_0) = 0 \), as claimed.  \( \square \)

4.2. The \( b \in \mathbb{R}^m \) is close to the set \( \mathcal{K}_A \). In this subsection, we show if the vector \( b \) is close to the set \( \mathcal{K}_A = \{ y = |Ax| : x \in \mathbb{R}^d \} \) then the solution to (1.1) is unique. For convenience, we introduce the definition of strong complement property which was firstly introduced in [3] (see also [32]).
Definition 4.4. [3, Definition 17] We say the matrix $A \in \mathbb{R}^{m \times d}$ satisfies the $\sigma$-strong complement property if
\[
\max \left\{ \lambda_{\min}(A^\top \Gamma A_{\Gamma}), \lambda_{\min}(A^\top \Gamma c A_{\Gamma c}) \right\} \geq \sigma^2
\]
for every $\Gamma \subset \{1, \ldots, m\}$, where $A_{\Gamma} := [a_j : j \in \Gamma]^\top$ denotes the sub-matrix of $A$.

We next present a sufficient condition under which the solution to (1.1) is unique.

Theorem 4.5. Assume that $A \in \mathbb{R}^{m \times d}$ has $\beta\sigma_{\max}(A)$-strong complement property for some $\beta \in (0, 1)$, where $\sigma_{\max}(A)$ is the largest singular value of $A$. Suppose $b = |Ax_0| + \eta \in \mathbb{R}^m$ for some $x_0 \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^m$. If $\|\eta\| \leq \beta^2 \lambda$ then the program (1.1) has a unique solution where $\lambda := \min_i \{|Ax_0|_i : i = 1, \ldots, m\}$.

Proof. We first consider the case where $A^\top A = I$. Note that $\sigma_{\max}(A) = 1$. Thus the matrix $A$ has $\beta$-strong complement property. It then follows from Lemma 2.5 that $A$ has phase retrievable property. To obtain the conclusion, according to Lemma 2.4, it is enough to show the best approximation to $b$ from $K_A$ is unique.

Recognize that $b = |Ax_0| + \eta$. Let $S^* := \{i : a_i^\top x_0 \geq 0\}$. Then
\[(4.4) \quad b = D_{S^*} Ax_0 + \eta.
\]
Here, $D_{S^*} := \text{Diag}(\delta_{S^*}(1), \ldots, \delta_{S^*}(m)) \in \mathbb{R}^{m \times m}$ with
\[
\delta_{S^*}(j) := \begin{cases} 1, & j \in S^*, \\ -1, & \text{otherwise}. \end{cases}
\]
For convenience, we set $K_S := \{y \in \mathbb{R}^m : y = D_S Ax, x \in \mathbb{R}^d\} \cap \mathbb{R}^m_+$. A simple observation is that
\[
K_A = \left\{ y : y = |Ax|, x \in \mathbb{R}^d \right\} = \bigcup_{S \subset \{1, \ldots, m\}} K_S.
\]
We claim that, for any $S \subset \{1, \ldots, m\}$ with $S \neq S^*$, we have
\[(4.5) \quad d(K_{S^*}, b) < d(K_S, b),
\]
which implies that the best approximation to $b$ from $K_A$ is unique. We arrive at the conclusion.

We next prove the claim (4.5). Let $P_S(b) \in \mathbb{R}^m$ be the projection of vector $b \in \mathbb{R}^m$ onto the subspace $\{y \in \mathbb{R}^m : y = D_S Ax, x \in \mathbb{R}^d\}$. Note that $A^\top A = I$. A simple calculation
leads to

\[ P_S(b) = D_S A (D_S A)^\top b. \]

For the set \( S^* \), we have

\[ P_{S^*}(b) = D_{S^*} A (D_{S^*} A)^\top b = D_{S^*} A (D_{S^*} A)^\top (D_{S^*} Ax_0 + \eta) = D_{S^*} Ax_0 + D_{S^*} A (D_{S^*} A)^\top \eta. \]

A simple observation is that the subspace \( \{ y \in \mathbb{R}^m : y = D_{S^*} Ax, x \in \mathbb{R}^d \} \) contains the point \( D_{S^*} Ax_0 \in \mathbb{R}^m_+ \), which implies the set \( \{ y \in \mathbb{R}^m : y = D_{S^*} Ax \} \cap \mathbb{R}^m_+ \) is non-empty. We next prove \( P_{S^*}(b) \in K_{S^*}. \) To this end, we only need to show \( P_{S^*}(b) \in \mathbb{R}^m_+ \). Note that

\[ (4.6) \quad P_{S^*}(b) = D_{S^*} Ax_0 + D_{S^*} A (D_{S^*} A)^\top \eta = |Ax_0| + D_{S^*} A (D_{S^*} A)^\top \eta. \]

It is sufficient to prove \( \|D_{S^*} A (D_{S^*} A)^\top \eta\|_\infty \leq \lambda := \min_i \{|Ax_0_i|\} \). Note that

\[ \|D_{S^*} A (D_{S^*} A)^\top \eta\|_\infty \leq \max_{1 \leq i \leq m} \|(D_{S^*} A (D_{S^*} A)^\top) \eta\|_i \|\eta\| \]

\[ = \|\eta\| \cdot \max_{1 \leq i \leq m} \left( \sum_{k=1}^m |\langle a_k, a_i \rangle|^2 \right)^{1/2} = \max_{1 \leq i \leq m} \|a_i\| \|\eta\|, \]

where \( (D_{S^*} A (D_{S^*} A)^\top) \eta \) denotes the \( i \)-th row of \( D_{S^*} A (D_{S^*} A)^\top \) and the last equality follows from \( A^\top A = I \). Since the columns of \( A \) are orthonormal, we have \( \|a_i\|^2 \leq 1 \) for all \( i \). Thus,

\[ \|D_{S^*} A (D_{S^*} A)^\top \eta\|_\infty \leq \|\eta\| \leq \lambda, \]

where we use the condition \( \|\eta\| \leq \beta^2 \lambda \) with \( \beta \leq 1 \) in the last inequality. This immediately gives \( P_{S^*}(b) \in \mathbb{R}^m_+ \).

Combining (4.4), (4.6) and \( P_{S^*}(b) \in K_{S^*} \), we have

\[ (4.7) \quad d(K_{S^*}, b) = \|b - P_{S^*}(b)\| = \|(I - D_{S^*} A (D_{S^*} A)^\top) \eta\| \leq \|\eta\|, \]

where the inequality comes from the fact that \( I - D_{S^*} A (D_{S^*} A)^\top \) is an orthogonal projection matrix.

Next, we turn to evaluate \( d(K_{S^*}, b) \). For any fixed \( S \subset \{1, \ldots, m\} \), define \( \Gamma := (S \backslash S^*) \cup (S^* \backslash S) \). If \( S \neq S^* \) then \( \Gamma \neq \emptyset \). Since \( A \) has \( \beta \)-strong complement property, we have

\[ (4.8) \quad \max \left\{ \lambda_{\min}(A_{[\Gamma]}^\top A_{[\Gamma]}), \lambda_{\min}(A_{[\Gamma \backslash \Gamma]}^\top A_{[\Gamma \backslash \Gamma]}) \right\} \geq \beta^2. \]

Without loss of generality, we assume \( \lambda_{\min}(A_{[\Gamma]}^\top A_{[\Gamma]}) \geq \beta^2 \). Note that the subspaces spanned by \( D_S A \) and \( D_{S^*} A \) are the same. Hence, if \( \lambda_{\min}(A_{[\Gamma]}^\top A_{[\Gamma]}) \geq \beta^2 \) then we only need
to replace the subset $S$ and $\Gamma$ by $S^c$ and $\Gamma^c := (S^c \setminus S^* ) \cup (S^* \setminus S^c)$, respectively. Recall that
\[
P_S(b) = D_S A (D_S A)^\top (D_S A x_0 + \eta).
\]
A simple observation is
\[
(D_S A)^\top D_S A = I - 2A_{[\Gamma]}^\top A_{[\Gamma]},
\]
where $(A_{[\Gamma]})_i = a_i^\top$ if $i \in \Gamma$ and $(A_{[\Gamma]})_i = 0$ if $i \notin \Gamma$. Then
\[
P_S(b) = D_S A (D_S A)^\top (D_S^* A x_0 + \eta) = D_S A (I - 2A_{[\Gamma]}^\top A_{[\Gamma]}) x_0 + D_S A (D_S A)^\top \eta
\]
\[
= D_S A x_0 - 2D_S A A_{[\Gamma]}^\top A_{[\Gamma]} x_0 + D_S A (D_S A)^\top \eta
\]
\[
= D_S^* A x_0 - 2D_S^* A_{[\Gamma]} x_0 - 2D_S A A_{[\Gamma]}^\top A_{[\Gamma]} x_0 + D_S A (D_S A)^\top \eta,
\]
which implies
\[
d(\mathcal{K}_S, b) \geq \|b - P_S(b)\| = \|(I - D_S A (D_S A)^\top) \eta + 2D_S^* A_{[\Gamma]} x_0 + 2D_S A A_{[\Gamma]}^\top A_{[\Gamma]} x_0\|
\]
\[
\geq 2\|D_S^* A_{[\Gamma]} x_0 + D_S A A_{[\Gamma]}^\top A_{[\Gamma]} x_0\| - \|(I - D_S A (D_S A)^\top) \eta\|
\]
\[
= 2\|D_S^* A_{[\Gamma]} x_0 + D_S A (D_S^* A_{[\Gamma]}^\top D_S^* A_{[\Gamma]} x_0\| - \|(I - D_S A (D_S A)^\top) \eta\|
\]
(4.9)
\[
\|D_S^* A_{[\Gamma]} x_0 + D_S A (D_S^* A_{[\Gamma]}^\top D_S^* A_{[\Gamma]} x_0\| \geq \|\hat{b}_{[\Gamma]} - D_S^* A_{[\Gamma]} (D_S^* A_{[\Gamma]}^\top \hat{b}_{[\Gamma]}\|
\]
(4.10)
\[
\|\hat{b}_{[\Gamma]} - D_S^* A_{[\Gamma]} (D_S^* A_{[\Gamma]}^\top \hat{b}_{[\Gamma]}\| \geq \beta^2 \|\hat{b}_{[\Gamma]}\|.
\]
From strong complement property (4.8), we know
\[
\|D_S^* A_{[\Gamma]} (D_S^* A_{[\Gamma]}^\top D_S^* A_{[\Gamma]}\| = \|D_S^* A_{[\Gamma]}^\top D_S^* A_{[\Gamma]}\| = \|A_{[\Gamma]}^\top A_{[\Gamma]}\| = \|I - A_{[\Gamma]}^\top A_{[\Gamma]}\| \leq 1 - \beta^2,
\]
which implies
\[
\|\hat{b}_{[\Gamma]} - D_S^* A_{[\Gamma]} (D_S^* A_{[\Gamma]}^\top \hat{b}_{[\Gamma]}\| \geq \beta^2 \|\hat{b}_{[\Gamma]}\|.
\]
(4.11)
Putting (4.10) and (4.11) into (4.9), we have
\[
d(\mathcal{K}_S, b) \geq 2\beta^2 \|\hat{b}_{[\Gamma]}\| - \|(I - D_S A (D_S A)^\top) \eta\| \geq 2\beta^2 \lambda - \|\eta\|.
\]
Combining the above estimator with (4.7) and noting $\|\eta\| < \beta^2 \lambda$, we obtain
\[
d(\mathcal{K}_S, b) > \|\eta\| \geq d(\mathcal{K}_{S^*}, b)
\]
for any $S \neq S^*$. Thus we complete the proof for the case where $A^\top A = I$.

Finally, for general matrix $A \in \mathbb{R}^{m \times d}$, we assume the singular decomposition of $A$ is $A = U D V^\top$ where $U := [u_1, \ldots, u_m]^\top \in \mathbb{R}^{m \times d}$ and $D \in \mathbb{R}^{d \times d}$ is an invertible diagonal matrix. Let $\tilde{A} = U$, $\tilde{x} = D V^\top x$ and $\tilde{x}_0 = D V^\top x_0$. Then we have
\[
\|Ax - b\| = \|\tilde{A}\tilde{x} - b\| \quad \text{and} \quad b = |Ax_0| + \eta = |\tilde{A}\tilde{x}_0| + \eta.
\]
Note that the program (1.1) has a unique solution iff \( \arg\min_x \|A\tilde{x} - b\| \) has a unique solution. Hence, if \( \tilde{A} \) has \( \beta \)-strong complement property then we arrive at the conclusion. Indeed, for any \( \Gamma \subset \{1, \ldots, m\} \)
\[
\tilde{A}^\top \Gamma \tilde{A} \Gamma = \sum_{i \in \Gamma} u_i u_i^\top \geq \frac{1}{\sigma_2^2(A)} \cdot \sum_{i \in \Gamma} \sigma_i^2 u_i u_i^\top = \frac{1}{\sigma_{\max}(A)} \cdot A_{\Gamma} A_{\Gamma}^\top,
\]
where \( \sigma_i \) are the singular values of \( A \). Since \( A \in \mathbb{R}^{m \times d} \) has \( \beta \sigma_{\max}(A) \)-strong complement property, it immediately gives that
\[
\lambda_{\min}(\tilde{A}^\top \Gamma \tilde{A} \Gamma), \lambda_{\min}(\tilde{A}^\top \Gamma_c \tilde{A} \Gamma_c) \geq \beta^2.
\]
Hence, \( \tilde{A} \) has \( \beta \)-strong complement property. This completes the proof. \( \square \)

Theorem 4.5 requires the matrix \( A \) has strong complement property. The next lemma shows that the Gaussian random matrix satisfies such property with high probability.

**Lemma 4.6.** [3, Theorem 20] Assume that \( A \in \mathbb{R}^{m \times d} \) is a Gaussian random matrix with independent standard normal entries. If \( m > 2d \) then for every \( \epsilon > 0 \) the matrix \( A \) has \( \sigma \)-strong complement property with probability at least \( 1 - \exp(-\epsilon m) \) where \( R = m/d \) and
\[
\sigma = \frac{1}{\sqrt{2e^{1 + \epsilon/(R - 2)}}} \cdot \frac{m - 2d + 2}{2^{R/(R - 2)} \sqrt{m}}.
\]

Combining Theorem 4.5 and Lemma 4.6, we have the following corollary.

**Corollary 4.7.** Assume \( m \geq Cd \). Let \( A \in \mathbb{R}^{m \times d} \) be a Gaussian random matrix with independent standard normal entries. Suppose \( b = |Ax_0| + \eta \in \mathbb{R}_+^m \) for some \( x_0 \in \mathbb{R}^d \) and \( \eta \in \mathbb{R}^m \). Let \( \lambda := \min_i \{|Ax_0|_i\} \). If \( \|\eta\| \leq c\lambda \) then with probability at least \( 1 - \exp(-c_0 m) \) the program (1.1) has a unique solution, where \( C, c \) and \( c_0 \) are universal constants.

**Proof.** Picking \( \epsilon = 1 \) in Lemma 4.6, we obtain for \( m > 2d \), with probability at least \( 1 - \exp(-m) \), the matrix \( A \) has \( \sigma \)-strong complement property with
\[
\sigma = \frac{1}{\sqrt{2e^{1 + 1/(R - 2)}}} \cdot \frac{m - 2d + 2}{2^{R/(R - 2)} \sqrt{m}},
\]
where \( R = m/d \). A simple observation is that if \( m \geq 4d \) then
\[
\frac{1}{8\sqrt{2e^{3/2}}} \cdot \sqrt{m} \leq \sigma \leq \frac{1}{2\sqrt{2e}} \cdot \sqrt{m}.
\]
On the other hand, since $A$ is a Gaussian random matrix, with probability at least $1 - \exp(-cm)$, we have
\[
\sqrt{m} \leq \sigma_{\max}(A) \leq 3\sqrt{m}
\]
provided $m \geq Cd$, where $C$ and $c$ are universal constants.

Combining (4.12) with (4.13), we obtain that for $m \geq Cd$, with probability at least $1 - \exp(-c_0m)$, the matrix $A$ has $\beta \sigma_{\max}(A)$-strong complement property for some constant $0 < \beta < 1$, where $c_0$ is a universal constant. Then Theorem 4.5 implies the conclusion holds.

\[
\square
\]

5. Stability of solutions to (1.1)

In this section, we focus on Question III, i.e., the stability of solutions to (1.1). For convenience, we set
\[
U_A := \{ b \in \mathbb{R}^m : \#P_{K_A}(b) = 1 \},
\]
where $P_{K_A}(b)$ is the set of the best approximation to $b$ from $K_A$ and $\#P_{K_A}(b)$ is the cardinality of $P_{K_A}(b)$. Recall that the program (1.1) is
\[
\Phi_A(b) := \arg\min_{x \in \mathbb{H}^d} \| A x - b \|^2.
\]
The following lemma states $\Phi_A(b)$ is bilipschitz if $b \in K_A$.

**Lemma 5.1.** [17] Assume that $A \in \mathbb{H}^{m \times d}$ has phase retrieval property. There exist constants $\alpha > 0, \beta > 0$ which only depend on $A$ so that
\[
\alpha \cdot \text{dist}(\Phi_A(b), \Phi_A(b')) \leq \| b - b' \| \leq \beta \cdot \text{dist}(\Phi_A(b), \Phi_A(b'))
\]
holds for all $b, b' \in K_A$.

Recall that $b \in K_A$ means $b = |A x_0|$ for some $x_0$. Hence, Lemma 5.1 shows that the phase retrieval problem in the absence of noise is stable.

5.1. Instability of solutions to (1.1). Lemma 5.1 only consider the case where $b \in K_A$. In this subsection, we focus on the general case where $b \notin K_A$ showing the program (1.1) is unstable without the assumption of $b \in K_A$. To begin with, we need the following lemma.
Lemma 5.2. Assume $A \in \mathbb{H}^{m \times d}$ has phase retrieval property. For arbitrary constant $\epsilon > 0$ there exist $b_1, b_2 \in U_A \subset \mathbb{R}^m$ so that

$$\|b_1 - b_2\| < \epsilon \cdot \|P_{\mathcal{K}_A}(b_1) - P_{\mathcal{K}_A}(b_2)\|.$$ 

Proof. According to Theorem 3.1, there exists $b_0 \in \mathbb{R}^m$ so that $\#P_{\mathcal{K}_A}(b_0) \geq 2$. Assume $y_1, y_2 \in P_{\mathcal{K}_A}(b_0)$ with $y_1 \neq y_2$. We claim $P_{\mathcal{K}_A}(\lambda y_1 + (1 - \lambda)b_0) = \{y_1\}$ and $P_{\mathcal{K}_A}(\lambda y_2 + (1 - \lambda)b_0) = \{y_2\}$ for any $\lambda \in (0, 1)$. Indeed, if $P_{\mathcal{K}_A}(\lambda_0 y_1 + (1 - \lambda_0)b_0)$ contains some vector $y_0 \in \mathcal{K}_A$ with $y_0 \neq y_1$ for some $\lambda_0 \in (0, 1)$ then

$$\|b_0 - y_0\| \leq \|b_0 - (\lambda_0 y_1 + (1 - \lambda_0)b_0)\| + \|((\lambda_0 y_1 + (1 - \lambda_0)b_0) - y_0\|,$$

which contradicts to the fact that $y_1$ is a best approximation to $b_0$ from $\mathcal{K}_A$. Hence, we immediately obtain $P_{\mathcal{K}_A}(\lambda y_1 + (1 - \lambda)b_0) = \{y_1\}$. Similarly, we can show $P_{\mathcal{K}_A}(\lambda y_2 + (1 - \lambda)b_0) = \{y_2\}$. Taking $b_1 = \frac{\epsilon}{2} y_1 + (1 - \frac{\epsilon}{2})b_0 \in U_A$ and $b_2 = \frac{\epsilon}{2} y_2 + (1 - \frac{\epsilon}{2})b_0 \in U_A$, we arrive at the conclusion that

$$\|b_1 - b_2\| = \frac{\epsilon}{2} \|y_1 - y_2\| < \epsilon \|y_1 - y_2\| = \epsilon \cdot \|P_{\mathcal{K}_A}(b_1) - P_{\mathcal{K}_A}(b_2)\|.$$ 

Combining Lemma 5.1 and Lemma 5.2, we have the following instability result.

Theorem 5.3. Assume that $A \in \mathbb{H}^{m \times d}$ has phase retrieval property. Then for any $\epsilon > 0$ there exist $b_1, b_2 \in U_A$ so that

$$\|b_1 - b_2\| \leq \epsilon \cdot \text{dist}(\Phi_A(b_1), \Phi_A(b_2)).$$

Proof. According to Lemma 5.2, there exist $b_1, b_2 \in U_A$ so that

(5.2) $$\|b_1 - b_2\| < \frac{\epsilon}{|P_{\mathcal{K}_A}(b_1) - P_{\mathcal{K}_A}(b_2)|},$$

where $\beta$ is the constant given in Lemma 5.1. Note that $P_{\mathcal{K}_A}(b_j) \in \mathcal{K}_A$ and $\Phi_A(P_{\mathcal{K}_A}(b_j)) = \Phi_A(b_j), j = 1, 2$. Combining the right hand sides of (5.1) and (5.2), we arrive at the conclusion.  \[\square\]
The above theorem shows that the program (1.1) is not uniformly stable. However, we next show that if we restrict the vector \( \mathbf{b} \) to any convex set \( \Omega \subset U_A \) then it is stable. To show that, we introduce a lemma first.

**Lemma 5.4.** Assume that \( \Omega \subset U_A \) is a convex domain. For any matrix \( A \in \mathbb{R}^{m \times d} \), the \( P_{\mathcal{K}_A}(\cdot) \) is continuous on \( \Omega \).

**Proof.** For the aim of contradiction, we assume that there is a point \( \mathbf{x} \in \Omega \) and a sequence \( \{\mathbf{x}_j\}_{j=1}^\infty \subset \Omega \) with \( \lim_{j \to \infty} \mathbf{x}_j = \mathbf{x} \) such that for every \( j \) it holds

\[
(5.3) \quad \|P_{\mathcal{K}_A}(\mathbf{x}_j) - P_{\mathcal{K}_A}(\mathbf{x})\| \geq \epsilon > 0.
\]

From the definition, we have

\[
(5.4) \quad d(\mathcal{K}_A, \mathbf{x}) \leq \|\mathbf{x} - P_{\mathcal{K}_A}(\mathbf{x}_j)\| \leq \|\mathbf{x} - \mathbf{x}_j\| + \|\mathbf{x}_j - P_{\mathcal{K}_A}(\mathbf{x}_j)\| = \|\mathbf{x} - \mathbf{x}_j\| + d(\mathcal{K}_A, \mathbf{x}_j).
\]

According to Lemma 2.2, we know \( d(\mathcal{K}_A, \mathbf{x}_j) \) converges to \( d(\mathcal{K}_A, \mathbf{x}) \). By squeeze theorem, (5.4) implies \( \|\mathbf{x} - P_{\mathcal{K}_A}(\mathbf{x}_j)\| \) converges to \( d(\mathcal{K}_A, \mathbf{x}) \). Note that \( \{P_{\mathcal{K}_A}(\mathbf{x}_j)\}_{j=1}^\infty \) is a bounded sequence. There exists a subsequence which is convergent. We assume \( P_{\mathcal{K}_A}(\mathbf{x}_{j_l}) \) converges to \( \mathbf{y} \in \mathbb{R}^m \). Then

\[
\|\mathbf{x} - \mathbf{y}\| = \lim_{l \to \infty} \|\mathbf{x} - P_{\mathcal{K}_A}(\mathbf{x}_{j_l})\| = d(\mathcal{K}_A, \mathbf{x}).
\]

It implies \( P_{\mathcal{K}_A}(\mathbf{x}) = \{\mathbf{y}\} \). Hence,

\[
lim_{l \to \infty} \|P_{\mathcal{K}_A}(\mathbf{x}_{j_l}) - P_{\mathcal{K}_A}(\mathbf{x})\| = lim_{l \to \infty} \|P_{\mathcal{K}_A}(\mathbf{x}_{j_l}) - \mathbf{y}\| = 0,
\]

which contradicts to (5.3). \( \square \)

Now, we could extend the stability result in Lemma 5.1 from \( \mathbf{b} \in \mathcal{K}_A \) to any convex set \( \Omega \subset U_A \) in the real case. We also conjecture a similar result holds for the complex case.

**Theorem 5.5.** Assume that \( A \in \mathbb{R}^{m \times d} \) has phase retrieval property in \( \mathbb{R}^d \). Let \( \Omega \subset U_A \) be a convex domain. Then there exists a constant \( \alpha > 0 \) which only depends on \( A \) so that

\[
(5.5) \quad \alpha \cdot \text{dist}(\Phi_A(\mathbf{b}), \Phi_A(\mathbf{b'})) \leq \|\mathbf{b} - \mathbf{b'}\| \quad \text{for all} \quad \mathbf{b}, \mathbf{b'} \in \Omega.
\]

**Proof.** Assume that \( \mathbf{b}, \mathbf{b'} \in \Omega \). If \( \mathbf{b} = \mathbf{b'} \), then the conclusion (5.5) holds. Thus, we just need to consider the case where \( \mathbf{b} \neq \mathbf{b'} \). Note that \( P_{\mathcal{K}_A}(\mathbf{b}), P_{\mathcal{K}_A}(\mathbf{b'}) \in \mathcal{K}_A \). According to Lemma 5.1, there exists a constant \( \alpha > 0 \) which only depends on \( A \) so that

\[
\alpha \cdot \text{dist}(\Phi_A(\mathbf{b}), \Phi_A(\mathbf{b'})) = \alpha \cdot \text{dist}(\Phi_A(P_{\mathcal{K}_A}(\mathbf{b})), \Phi_A(P_{\mathcal{K}_A}(\mathbf{b'}))) \leq \|P_{\mathcal{K}_A}(\mathbf{b}) - P_{\mathcal{K}_A}(\mathbf{b'})\|.
\]
So, to prove the conclusion, it is enough to show that
\[ \| P_{K_A}(b) - P_{K_A}(b') \| \leq \| b - b' \|. \]

Set
\[ [b, b'] := \{(1 - \lambda)b + \lambda b' : \lambda \in [0, 1]\}. \]

Since \( \Omega \) is convex, we have \([b, b'] \subset \Omega \subset U_A\). According to Lemma 5.4, \( P_{K_A} \) is continuous on \([b, b']\). We use \( D_\epsilon \in \mathbb{R}^{m \times m} \) to denote a diagonal matrix whose diagonal is \( \epsilon \in \{1, -1\}^m \).

Set
\[ H_\epsilon := \{ D_\epsilon A x \geq 0 : x \in \mathbb{R}^d \}. \]

A simple observation is that \( H_\epsilon \subset K_A \) is convex for any fixed \( \epsilon \in \{1, -1\}^m \). We assume that \((b_0, b_1, \ldots, b_k)\) is a partition of \([b, b']\) such that \( b = b_0 < b_1 < \cdots < b_k = b' \) and \( P_{K_A}([b_{t-1}, b_t]) \subset H_{\epsilon_t} \) where \( \epsilon_t \in \{1, -1\}^m, t \in \{1, \ldots, k\} \). Since \( P_{K_A} \) is continuous on \([b, b']\), it means \( P_{K_A}(b_t) \in H_{\epsilon_t} \cap H_{\epsilon_{t+1}} \) for \( t = 1, 2, \ldots, k - 1 \). According to \( P_{K_A}([b_{t-1}, b_t]) \subset H_{\epsilon_t} \), we have \( P_{K_A}([b_{t-1}, b_t]) = P_{H_{\epsilon_t}}([b_{t-1}, b_t]) \). Note that \( H_{\epsilon_t} \) is convex. It then follows from Lemma 2.3 that
\[ \| P_{K_A}(b_t) - P_{K_A}(b_{t-1}) \| = \| P_{H_{\epsilon_t}}(b_t) - P_{H_{\epsilon_t}}(b_{t-1}) \| \leq \| b_t - b_{t-1} \|. \]

Thus we have
\[ \| P_{K_A}(b) - P_{K_A}(b') \| \leq \sum_{t=1}^{k} \| P_{K_A}(b_t) - P_{K_A}(b_{t-1}) \| \leq \sum_{t=1}^{k} \| b_t - b_{t-1} \| = \| b - b' \|, \]

where the last equation follows from \( b_t, t = 0, 1, \ldots, k, \) are collinear points. We arrive at the conclusion.

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