1. Introduction

Augmented algebraic vector bundles often have moduli spaces which depend not only on the topological type of the augmented bundle, but also on an additional parameter. The result is that the moduli spaces occur in discrete families. First exploited by Thaddeus in a proof of the Verlinde formula, this phenomenon has been responsible for several interesting applications (cf. \cite{BeDW}, \cite{BGG}, \cite{BG2}). In this paper we examine the augmented bundles known as coherent systems and discuss the use of their moduli spaces as a tool in Brill-Noether theory.

By a coherent system on an algebraic variety (or scheme) we mean an algebraic vector bundle together with a linear subspace of prescribed dimension of its space of sections. As such it is an example of an augmented bundle. Introduced in \cite{KN}, \cite{RV} and \cite{LeP}, there is a notion of stability which permits the construction of moduli spaces. This notion depends on a real parameter, and thus leads to a family of moduli spaces.

That these moduli spaces are related to Brill-Noether loci follows almost immediately from the definitions. The Brill-Noether loci are natural subvarieties within the moduli spaces of stable bundles over an algebraic curve, defined by the condition that the bundles have at least a prescribed number of linearly independent sections. A similar condition defines Brill-Noether loci in the moduli spaces of (S-equivalence classes of) semistable bundles. But any bundle which occurs as part of a coherent
system must evidently have at least a prescribed number of linearly independent sections. Conversely, a bundle with a prescribed number of linearly independent sections determines, in a natural way, a coherent system.

In order to convert these observations into a precise relationship between the coherent systems moduli spaces and the Brill-Noether loci, one extra tool is needed, namely a precise relationship between bundle stability and coherent system stability. In general, for an arbitrary choice of the coherent system stability parameter, no such relationship exists. However, for values of the parameter close to 0, the required relationship holds and there is a map from the coherent systems moduli space to the semistable Brill-Noether locus. While this map is not necessarily surjective, it does include the entire stable Brill-Noether locus in its image. It is via this map that information about the coherent systems moduli spaces can be applied to answer questions about higher rank Brill-Noether theory.

In [BG2] the first two authors initiated a programme to do just this, i.e. to use coherent systems moduli spaces to study higher rank Brill-Noether theory. There the goals were limited to explaining results of [BGN] from the perspective of coherent systems. This turned out to require only a limited understanding of the coherent systems moduli spaces. In this paper we build on the foundation laid in [BG2].

We study the moduli spaces for coherent systems \((E, V)\) consisting of an algebraic vector bundle \(E\) together with a linear subspace \(V\) of its space of sections. While not required by the definitions, we consider only the case of bundles over a smooth irreducible projective algebraic curve \(X\) of genus \(g\). The type of the coherent system is defined by a triple of integers \((n, d, k)\) giving the rank of \(E\), the degree of \(E\), and the dimension of the subspace \(V\).

The infinitesimal study of coherent systems follows a standard pattern and is summarised in section 3. This allows us in particular to identify the Zariski tangent space to each moduli space at any point, and to show that every irreducible component of every moduli space has dimension at least equal to a certain number \(\beta(n, d, k)\), called the Brill-Noether number and often referred to as the expected dimension.

For each type \((n, d, k)\), there is a family of moduli spaces. While each such family of moduli spaces has some properties which depend on \((n, d, k)\), there are some features that are common to all types. In particular:

- **The families have only a finite number of distinct members.** The different members in the family correspond to different values for the real parameter \(\alpha\) in the definition of stability. As \(\alpha\) varies, the stability condition changes only as \(\alpha\) passes through one of a discrete set of points in the real line. In some cases (if \(k < n\)) the range for the parameter is a finite interval, in which case it follows automatically that the family has only finitely many distinct members. However, even in the cases for which the range of the parameter is infinite, it turns out that there can be only a finite number of distinct moduli spaces. This is a consequence of the stabilisation theorem Proposition 4.6.

- **The families are ordered, and the coherent systems in the terminal member are as simple as possible.** The ordering comes from the fact that the moduli spaces are labelled by intervals on the real line. By the terminal member of the family we
mean the moduli space corresponding to the last of these intervals (in the natural ordering of \( \mathbb{R} \)). In section 3 we analyse the coherent systems corresponding to the points in this terminal moduli space. While the specifics depend on the type of the coherent system, in all cases we find that the structure of these coherent systems is (in a suitable sense) the best possible.

The most obvious type-dependent feature is the description of the terminal moduli space. This divides naturally into distinct cases, according to whether \( k < n \), \( k = n \) or \( k > n \). The case \( k < n \) was discussed in some detail in \([BG2]\). The results are summarized in section 5.1. When \( k \geq n \), we show (in section 5.2) how to relate the terminal moduli space to a Grothendieck Quot scheme of quotients of the trivial bundle of rank \( k \). Denoting the terminal moduli space of stable (respectively semistable) objects by \( G_L \) (respectively \( \tilde{G}_L \)), we prove

**Theorem 1.1.** [Theorem 5.6] Let \( k = n \geq 2 \). If \( \tilde{G}_L \) is non-empty then \( d = 0 \) or \( d \geq n \). For \( d > n \), \( \tilde{G}_L \) is irreducible and \( G_L \) is smooth of the expected dimension \( dn - n^2 + 1 \). For \( d = n \), \( G_L \) is empty and \( \tilde{G}_L \) is irreducible of dimension \( n \) (not of the expected dimension). For \( d = 0 \), \( G_L \) is empty and \( \tilde{G}_L \) consists of a single point.

The non-emptiness of \( G_L \) for the case \( k > n \) is not so obvious and is related to the non-emptiness of Quot schemes. However, in this case there is a duality construction that relates coherent systems of types \( (n, d, k) \) and \( (k - n, d, k) \). If the parameter is large these “dual” moduli spaces are birationally equivalent. A similar idea has been considered for small \( \alpha \) by Butler \([Bu2]\), but the construction seems to be more natural for large \( \alpha \) and turns out to be an important tool to prove non-emptiness for large values of the parameter. For instance if \( k = n + 1 \), the non-emptiness is given by the classical rank 1 Brill-Noether theory, i.e. we get

**Theorem 1.2.** [Theorem 5.11] Suppose that the curve \( X \) is generic and that \( k = n + 1 \). Then \( G_L \) is non-empty if and only if \( \beta = g - (n + 1)(n - d + g) \geq 0 \). Moreover \( G_L \) has dimension \( \beta \) and it is irreducible whenever \( \beta > 0 \).

In addition to these ‘absolute’ results about the terminal moduli spaces, we also give ‘relative’ results which characterize the differences between moduli spaces within a given family. We compare the moduli spaces and identify subvarieties within which the differences are localised. In section 6 we give some general results estimating the codimensions of these subvarieties.

With a view to applications, we examine a number of special cases in which either \( k \) or \( n \) (or both) are small. In these cases, discussed in sections 7 - 10, we can get more detailed results, especially for the codimension estimates on the difference loci between moduli spaces within a family.

Having amassed all this information about the coherent systems moduli spaces, we end with some applications to Brill-Noether theory in section 11. In all cases the strategy is the same: starting with information about \( G_L \), and using our results about the relation between different moduli spaces within a given family, we deduce properties of the moduli space corresponding to the smallest values of the stability parameter. This is then passed down to the Brill-Noether loci using the morphism
from the coherent systems moduli space to the moduli space of semistable bundles. This allows us, for example,

- **[Theorems 11.7 and 11.11]** to prove the irreducibility of the Brill-Noether loci for \(k = 1, 2, 3\), and
- **[Theorem 11.13]** to compute the Picard group of the smooth part of the Brill-Noether locus for \(k = 1\).

While the irreducibility result was previously known for \(k = 1\) and any \(d\), and for \(k = 2, 3, k < n \) and \(d < \min\{2n, n + g\}\), our theorems have no such restrictions. These results should be regarded as a sample of what can be done. Our methods are certainly applicable more widely and we propose to pursue this in future papers.

Throughout the paper \(X\) denotes a fixed smooth irreducible projective algebraic curve of genus \(g\) defined over the complex numbers. Unless otherwise stated, we make no assumption about \(g\). For simplicity we shall write \(O\) for \(O_X\) and \(H^0(E)\) for \(H^0(X, E)\). We shall consistently denote the ranks of bundles \(E, E', E_1\ldots\) by \(n, n', n_1\ldots\), their degrees by \(d, d', d_1\ldots\) and the dimensions of subspaces \(V, V', V_1\ldots\) of their spaces of sections by \(k, k', k_1\ldots\).

2. Definitions and basic facts

2.1. Coherent systems and their moduli spaces. Recall (cf. [LeP], [KN]) that a coherent system \((E, V)\) on \(X\) of type \((n, d, k)\) consists of an algebraic vector bundle \(E\) over \(X\) of rank \(n\) and degree \(d\), and a linear subspace \(V \subseteq H^0(E)\) of dimension \(k\). Strictly speaking, it is better to consider triples \((E, V, \phi)\) where \(V\) is a dimension \(k\) vector space and \(\phi : V \otimes O \to E\) is a sheaf map such that the induced map \(H^0(\phi) : V \to H^0(E)\) is injective. The linear space \(V \subseteq H^0(E)\) is then the image \(H^0(\phi)(V)\).

Under the natural concepts of isomorphism, isomorphism classes of such triples are in bijective correspondence with isomorphism classes of coherent systems. We will usually use the simpler notation \((E, V)\), but occasionally it is helpful to use the longer one. For a summary of basic results about coherent systems (and other related augmented bundles) we refer the reader to [BDGW].

By introducing a suitable definition of stability, one can construct moduli spaces of coherent systems. The correct notion (i.e. the one dictated by Geometric Invariant Theory) depends on a real parameter \(\alpha\), which a posteriori must be non-negative (cf. [KN]). In the situation under consideration (i.e. where \(E\) is a vector bundle over a smooth algebraic curve), the definition may be given as follows.

**Definition 2.1.** Fix \(\alpha \in \mathbb{R}\). Let \((E, V)\) be a coherent system of type \((n, d, k)\). The \(\alpha\)-slope \(\mu_\alpha(E, V)\) is defined by

\[
\mu_\alpha(E, V) = \frac{d}{n} + \alpha \frac{k}{n}.
\]

We say \((E, V)\) is \(\alpha\)-stable if

\[
\mu_\alpha(E', V') < \mu_\alpha(E, V)
\]

for all proper subsystems \((E', V')\) (i.e. for every non-zero subbundle \(E'\) of \(E\) and every subspace \(V' \subseteq V \cap H^0(E')\) with \((E', V') \neq (E, V))\). We define \(\alpha\)-semistability
by replacing the above strict inequality with a weak inequality. A coherent system is called \( \alpha \)-polystable if it is the direct sum of \( \alpha \)-stable coherent systems of the same \( \alpha \)-slope.

Sometimes it is necessary to consider a larger class of objects than coherent systems in which one replaces \( E \) by a general coherent sheaf and \( H^0(\phi) : \mathcal{V} \to H^0(E) \) is not necessarily injective. By doing so one obtains an abelian category \([KN]\). One can easily extend the definition of \( \alpha \)-stability to this category. It turns out, however, that \( \alpha \)-semistability forces \( E \) to be locally free and \( H^0(\phi) \) to be injective, and hence \( \alpha \)-semistable objects in this category can be identified with \( \alpha \)-semistable coherent systems up to an appropriate definition of isomorphism. One has the following result.

**Proposition 2.2.** ([KN, Corollary 2.5.1]) The \( \alpha \)-semistable coherent systems of any fixed \( \alpha \)-slope form a noetherian and artinian abelian category in which the simple objects are precisely the \( \alpha \)-stable systems. In particular the following statements hold.

(i) (Jordan-H"older Theorem) For any \( \alpha \)-semistable coherent system \( (E,V) \), there exists a filtration by \( \alpha \)-semistable coherent systems \( (E_j,V_j) \),

\[
0 = (E_0,V_0) \subset (E_1,V_1) \subset \ldots \subset (E_m,V_m) = (E,V),
\]

with \( (E_j,V_j)/(E_{j-1},V_{j-1}) \) an \( \alpha \)-stable coherent system and

\[
\mu_\alpha((E_j,V_j)/(E_{j-1},V_{j-1})) = \mu_\alpha(E,V) \quad \text{for} \quad 1 \leq j \leq m.
\]

(ii) If \( (E,V) \) is an \( \alpha \)-stable coherent system, then \( \text{End}(E,V) \cong \mathbb{C} \).

Any filtration as in (i) is called a Jordan-H"older filtration of \( (E,V) \). It is not necessarily unique, but the associated graded object is uniquely determined by \( (E,V) \).

**Definition 2.3.** We define the graduation of \( (E,V) \) to be the \( \alpha \)-polystable coherent system

\[
gr(E,V) = \bigoplus_j (E_j,V_j)/(E_{j-1},V_{j-1}).
\]

Two \( \alpha \)-semistable coherent systems \( (E,V) \) and \( (E',V') \) are said to be S-equivalent if \( gr(E,V) \cong gr(E',V') \).

We shall denote the moduli space of \( \alpha \)-stable coherent systems of type \((n,d,k)\) by \( G(\alpha) = G(\alpha; n,d,k) \), and the moduli space of S-equivalence classes of \( \alpha \)-semistable coherent systems of type \((n,d,k)\) by \( \tilde{G}(\alpha) = \tilde{G}(\alpha; n,d,k) \). The moduli space \( \tilde{G}(\alpha) \) is a projective variety which contains \( G(\alpha) \) as an open set.

Now suppose that \( k \geq 1 \). By applying the \( \alpha \)-semistability condition for \( (E,V) \) to the subsystem \( (E,0) \) one obtains that \( \alpha \geq 0 \). This means that there are no semistable coherent systems for negative values of \( \alpha \). For \( \alpha = 0 \), \( (E,V) \) is 0-semistable if and only if \( E \) is semistable. For \( k \geq 1 \) there are no 0-stable coherent systems.

**Definition 2.4.** We say that \( \alpha > 0 \) is a virtual critical value if it is numerically possible to have a proper subsystem \( (E',V') \) such that \( \frac{k'}{n'} \neq \frac{k}{n} \) but \( \mu_\alpha(E',V') = \mu_\alpha(E,V) \). We also regard 0 as a virtual critical value. If there is a coherent system \( (E,V) \) and a subsystem \( (E',V') \) such that this actually holds, we say that \( \alpha \) is an actual critical value.
It follows from this (cf. [BDGW]) that, for coherent systems of type \((n,d,k)\), the non-zero virtual critical values of \(\alpha\) all lie in the set

\[
\left\{ \frac{nd' - n'd}{n'k - nk'} \mid 0 \leq k' \leq k, \ 0 < n' < n, \ n'k \neq nk' \right\} \cap (0, \infty).
\]

We say that \(\alpha\) is generic if it is not critical. Note that, if \(\text{GCD}(n,d,k) = 1\) and \(\alpha\) is generic, then \(\alpha\)-semistability is equivalent to \(\alpha\)-stability. If we label the critical values of \(\alpha\) by \(\alpha_i\), starting with \(\alpha_0 = 0\), we get a partition of the \(\alpha\)-range into a set of intervals \((\alpha_i, \alpha_{i+1})\). For numerical reasons it is clear that within the interval \((\alpha_i, \alpha_{i+1})\) the property of \(\alpha\)-stability is independent of \(\alpha\), that is if \(\alpha, \alpha' \in (\alpha_i, \alpha_{i+1})\), \(G(\alpha) = G(\alpha')\). We shall denote this moduli space by \(G_i = G_i(n,d,k)\). The construction of moduli spaces thus yields one moduli space \(G_i\) for the interval \((\alpha_i, \alpha_{i+1})\). If \(\text{GCD}(n,d,k) \neq 1\), one can define similarly the moduli spaces \(\tilde{G}_i\) of semistable coherent systems. The GIT construction of these moduli spaces has been given in [LeP] and [KN]. A previous construction for \(G_0\) had been given in [RV] and in [Be] for big degrees. When \(k = 1\) the moduli space of coherent systems is equivalent to the moduli space of vortex pairs studied in [B, BD1, BD2, G, HL1, HL2, Th].

The relationship between the semistability of a coherent system and the underlying vector bundle is given by the following (cf. [BDGW], [KN]).

**Proposition 2.5.** Let \(\alpha_1\) be the first critical value after 0 and let \(0 < \alpha < \alpha_1\). Then

(i) \((E, V)\) \(\alpha\)-stable implies \(E\) semistable;
(ii) \(E\) stable implies \((E, V)\) \(\alpha\)-stable.

2.2. Brill-Noether loci.

**Definition 2.6.** Let \(X\) be an algebraic curve, and let \(M(n,d)\) be the moduli space of stable bundles of rank \(n\) and degree \(d\). Let \(k \geq 0\). The Brill-Noether loci of stable bundles are defined by

\[
B(n,d,k) := \{ E \in M(n,d) \mid \dim H^0(E) \geq k \}.
\]

Similarly one defines the Brill-Noether loci of semistable bundles

\[
\tilde{B}(n,d,k) := \{ [E] \in \tilde{M}(n,d) \mid \dim H^0(\text{gr}(E)) \geq k \},
\]

where \(\tilde{M}(n,d)\) is the moduli space of S-equivalence classes of semistable bundles, \([E]\) is the S-equivalence class of \(E\) and \(\text{gr}(E)\) is the polystable bundle defined by a Jordan-Hölder filtration of \(E\).

The spaces \(B(n,d,k)\) and \(\tilde{B}(n,d,k)\) have previously been denoted by \(W^{k-1}_{n,d}\) and \(\tilde{W}^{k-1}_{n,d}\), but we have chosen to change the classical notation in an attempt to get rid of the \(k-1\), which in the arbitrary rank case does not make much sense. (In fact the same loci have also been denoted by \(W^k_{n,d}\) and \(\tilde{W}^k_{n,d}\), but, while logical, this seems a little confusing!)

By semicontinuity, the Brill-Noether loci are closed subschemes of the appropriate moduli spaces. The main object of Brill-Noether theory is the study of these subschemes, in particular questions related to their non-emptiness, connectedness, irreducibility, dimension, and topological and geometric structure. It is in particular not
difficult to describe them as determinantal loci, from which one obtains the following general result. We begin with a definition.

**Definition 2.7.** For any \((n, d, k)\), the Brill-Noether number \(\beta(n, d, k)\) is defined by
\[
\beta(n, d, k) = n^2(g - 1) + 1 - k(k - d + n(g - 1)).
\]

**Theorem 2.8.** If \(B(n, d, k)\) is non-empty and \(B(n, d, k) \neq M(n, d)\), then
- every irreducible component \(B\) of \(B(n, d, k)\) has dimension
  \[
  \dim B \geq \beta(n, d, k),
  \]
- \(B(n, d, k + 1) \subset \text{Sing}B(n, d, k)\),
- the tangent space of \(B(n, d, k)\) at a point \(E\) with \(\dim H^0(E) = k\) can be identified with the dual of the cokernel of the Petri map
  \[
  H^0(E) \otimes H^0(E^* \otimes K) \longrightarrow H^0(\text{End} E \otimes K)
  \]
  (given by multiplication of sections),
- \(B(n, d, k)\) is smooth of dimension \(\beta(n, d, k)\) at \(E\) if and only if the Petri map is injective.

For details, see for example [M3].

When \(n = 1\), \(M(n, d)\) is just \(J^d\), the Jacobian of \(X\) consisting of line bundles of degree \(d\), and the Brill-Noether loci are the classical ones for which a thorough modern presentation is given in [ACGH]. In particular we have the following results.

- If \(\beta(1, d, k) \geq 0\), then \(B(1, d, k)\) is non-empty.
- If \(\beta(1, d, k) > 0\), then \(B(1, d, k)\) is connected.
- For a generic curve \(X\) and \(n = 1\), the Petri map is always injective. Hence
  - \(B(1, d, k)\) is smooth outside \(B(1, d, k + 1)\).
  - \(B(1, d, k)\) has dimension \(\beta(1, d, k)\) whenever it is non-empty and not equal to \(M(n, d)\).
  - \(B(1, d, k)\) is irreducible if \(\beta(1, d, k) > 0\).

None of these statements is true for \(n \geq 2\) (see, for example, [T1, T2, BGN, BeF, Mu]).

Rather than referring repeatedly to a generic curve, we prefer to use the following more precise term.

**Definition 2.9.** A curve \(X\) is called a Petri curve if the Petri map
\[
H^0(L) \otimes H^0(L^* \otimes K) \longrightarrow H^0(K)
\]
is injective for every line bundle \(L\) over \(X\).

One may note that any curve of genus \(g \leq 2\) is Petri, the simplest examples of non-Petri curves being hyperelliptic curves with \(g \geq 3\). There is currently no sensible generalisation of Definition 2.3 to higher rank. Indeed, at least for \(g \geq 6\), there exist stable bundles \(E\) on Petri curves for which the Petri map (2) is not injective (see [T1, §5]). Moreover the condition of Definition 2.3 is not sufficient to determine even the non-emptiness of Brill-Noether loci in higher rank (see [M3, Mu, V]).
2.3. Relationship between \( B(n, d, k) \) and \( G_0 \). The relevance of the moduli spaces of coherent systems in relation to Brill-Noether theory is given by Proposition 2.5. The assignment \((E, V) \mapsto E\) defines a map

\[
G_0(n, d, k) \rightarrow \tilde{B}(n, d, k),
\]

which is one-to-one over \( B(n, d, k) - B(n, d, k+1) \) and whose image contains \( B(n, d, k) \).
When \( \gcd(n, d, k) \neq 1 \), this map can be extended to

\[
\tilde{G}_0(n, d, k) \rightarrow \tilde{B}(n, d, k).
\]

Even (4) may fail to be surjective. This happens for example (as observed in \([BG2]\)) when \( d = 0, 0 < k < n \). In this case \( \tilde{G}_0 = \emptyset \) but \( \tilde{B} \) is non-empty. On the other hand, if \( (n, d) = 1 \), the loci \( B \) and \( \tilde{B} \) coincide and (3) is surjective.

Even when \((n, d) \neq 1\), we may be able to obtain information about \( B \) from properties of \( G_0 \). For example, if \( G_0(n, d, k) \) is non-empty, then certainly \( \tilde{B}(n, d, k) \) is non-empty. Moreover, if \( B(n, d, k) \) is non-empty and \( G_0(n, d, k) \) is irreducible, then \( \tilde{B}(n, d, k) \) is also irreducible.

We are therefore interested in studying \( G_0 = G_0(n, d, k) \). Our approach to this consists of having

- a detailed description of at least one (usually large \( \alpha \)) moduli space,
- a thorough understanding of the “flips” to go from \( G_i \) to \( G_{i-1} \), until we get to \( G_0 \).

The meaning of ‘thorough’ can vary, depending on the application. For instance, for non-emptiness questions all we require are the codimensions of the flip loci, or at least sufficiently good estimates thereof.

In the case \( n = 1 \), everything is much simpler. The concept of stability is vacuous and independent of \( \alpha \in (0, \infty) \). We shall therefore denote the moduli space of coherent systems by \( G(1, d, k) = G(\alpha; 1, d, k) \). It consists of coherent systems \((L, V)\) such that \( L \) is a line bundle of degree \( d \) and \( V \subset H^0(L) \) is any subspace of dimension \( k \). These spaces have been studied classically (see for example \([ACGH]\), where \( G(1, d, k) \) is denoted by \( G^d_{k-1} \)). The map (3) becomes

\[
G(1, d, k) \rightarrow B(1, d, k)
\]

and is always surjective. The fibre of (3) over \( L \) can be identified with the Grassmannian \( \text{Gr}(k, h^0(L)) \).

When \( X \) is a Petri curve, we have

- \( G(1, d, k) \) is non-empty if and only if \( \beta = g - k(k - d + g - 1) \geq 0 \),
- If \( \beta \geq 0 \), \( G(1, d, k) \) is smooth of dimension \( \beta \),
- If \( \beta > 0 \), \( G(1, d, k) \) is irreducible.

Comparing this with section 2.2, we see that, for \( X \) a Petri curve, \( G(1, d, k) \) provides a desingularisation of \( B(1, d, k) \) whenever \( B(1, d, k) \neq J^d \). In higher rank, if \( \gcd(n, d, k) = 1 \), \( \beta(n, d, k) \leq n^2(g - 1) \) and \( G_0(n, d, k) \) is smooth and irreducible, and \( B(n, d, k) \) is non-empty, then the map (3) is a desingularisation of the closure of \( B(n, d, k) \). We shall see that, for \( k = 1 \), all these conditions hold (see section
Note that it was proved in [RV] that $G_0(n, n(g - 1), 1)$ is a desingularisation of $\tilde{B}(n, n(g - 1), 1)$, which coincides with the generalised theta-divisor in $M(n, n(g - 1))$.

3. Infinitesimal study and extensions

The infinitesimal study of the moduli space of coherent systems as well as the study of extensions of coherent systems is carried out in [HI, LeP] (see also [II, RV]). We review here the main results and refer to these papers, in particular for omitted proofs.

Given two coherent systems $(E, V)$ and $(E', V')$ one defines the groups

$$\text{Ext}^q((E', V'), (E, V)),$$

and considers the long exact sequence ([HI, Corollaire 1.6])

$$
\begin{align*}
0 & \longrightarrow \text{Hom}((E', V'), (E, V)) \longrightarrow \text{Hom}(E', E) \longrightarrow \text{Hom}(V', H^0(E)/V) \\
& \longrightarrow \text{Ext}^1((E', V'), (E, V)) \longrightarrow \text{Ext}^1(E', E) \longrightarrow \text{Hom}(V', H^1(E)) \\
& \longrightarrow \text{Ext}^2((E', V'), (E, V)) \longrightarrow 0.
\end{align*}
$$

(6)

Notice that since we are on a curve $\text{Ext}^2(E', E) = 0$. Also, since $E'$ is a vector bundle,

$$\text{Ext}^1(E', E) = H^1(E'^* \otimes E).$$

We can now apply this to the study of infinitesimal deformations of the moduli space of coherent systems as well as to the study of extensions of coherent systems.

3.1. Extensions. We will have to deal later with extensions of coherent systems arising from the one-step Jordan-Hölder filtration of a semistable coherent system. By standard results on abelian categories, we have

**Proposition 3.1.** Let $(E_1, V_1)$ and $(E_2, V_2)$ be two coherent systems on $X$. The space of equivalence classes of extensions

$$0 \longrightarrow (E_1, V_1) \longrightarrow (E, V) \longrightarrow (E_2, V_2) \longrightarrow 0$$

is isomorphic to $\text{Ext}^1((E_2, V_2), (E_1, V_1))$. Hence the quotient of the space of non-trivial extensions by the natural action of $\mathbb{C}^*$ can be identified with the projective space $\mathbb{P}(\text{Ext}^1((E_2, V_2), (E_1, V_1)))$.

**Proposition 3.2.** Let $(E_1, V_1)$ and $(E_2, V_2)$ be two coherent systems on $X$ of types $(n_1, d_1, k_1)$ and $(n_2, d_2, k_2)$ respectively. Let $\mathbb{H}^0_{21} := \text{Hom}((E_2, V_2), (E_1, V_1))$ and $\mathbb{H}^2_{21} := \text{Ext}^2((E_2, V_2), (E_1, V_1))$. Then

$$\dim \text{Ext}^1((E_2, V_2), (E_1, V_1)) = C_{21} + \dim \mathbb{H}^0_{21} + \dim \mathbb{H}^2_{21},$$

where

$$C_{21} := k_2 \chi(E_1) - \chi(E'_2 \otimes E_1) - k_1 k_2$$

(7)

$$= n_1 n_2 (g - 1) - d_1 n_2 + d_2 n_1 + k_2 d_1 - k_2 n_1 (g - 1) - k_1 k_2.$$

Moreover,

$$\mathbb{H}^2_{21} = \text{Ker}(H^0(E_1^* \otimes K) \otimes V_2 \to H^0(E_1^* \otimes E_2 \otimes K))^*.$$

(9)
Finally, if $N_2$ is the kernel of the natural map $V_2 \otimes \mathcal{O} \to E_2$ then
\begin{equation}
H^2_{21} = H^0(E_1^* \otimes N_2 \otimes K)^*.
\end{equation}

**Proof.** This follows from (6) applied to $(E_1, V_1) = (E, V)$ and $(E_2, V_2) = (E', V')$, together with Serre duality for the last part.

In order to use this result we will need to be able to estimate the dimension of $H^2_{21}$.

**Lemma 3.3.** Suppose that $k_2 > 0$ and $h^0(E_1^* \otimes K) \neq 0$. Then the dimension of $H^2_{21}$ is bounded above by $(k_2 - 1)(h^0(E_1^* \otimes K) - 1)$.

**Proof.** Use the Hopf lemma which states that, if $\phi: A \otimes B \to C$ is a bilinear map between finite-dimensional complex vector spaces such that, for any $a \in A \setminus \{0\}$, $\phi(a, \cdot)$ is injective and, for any $b \in B \setminus \{0\}$, $\phi(\cdot, b)$ is injective, then the image of $\phi$ has dimension at least $\dim A + \dim B - 1$. The result follows from this and (9).

\section*{3.2. Infinitesimal deformations.}

By standard arguments in deformation theory we have (see [He, Théorème 3.12])

**Proposition 3.4.** Let $(E, V)$ be an $\alpha$-stable coherent system.

(i) If $\text{Ext}^2((E, V), (E, V)) = 0$, then the moduli space of $\alpha$-stable coherent systems is smooth in a neighbourhood of the point defined by $(E, V)$. This condition is satisfied if and only if the homomorphism $\text{Ext}^1(E, E) \to \text{Hom}(V, H^1(E))$ is surjective.

(ii) The Zariski tangent space to the moduli space at the point defined by $(E, V)$ is isomorphic to $\text{Ext}^1((E, V), (E, V))$.

**Lemma 3.5.** Let $(E, V)$ be an $\alpha$-stable coherent system of type $(n, d, k)$. Then
\[ \dim \text{Ext}^1((E, V), (E, V)) = \beta(n, d, k) + \dim \text{Ext}^2((E, V), (E, V)), \]
where $\beta(n, d, k)$ is the Brill-Noether number defined in Definition 2.7.

**Proof.** By considering the long exact sequence (8) for $(E', V') = (E, V)$, we see that
\begin{align*}
\dim \text{Ext}^1((E, V), (E, V)) &= k \chi(E) - \chi(\text{End} E) - k^2 + \dim \text{End}(E, V) \\
&\quad + \dim \text{Ext}^2((E, V), (E, V)) \\
&= k(d + n(1 - g)) - n^2(1 - g) - k^2 + 1 \\
&\quad + \dim \text{Ext}^2((E, V), (E, V)) \\
&= n^2(g - 1) + 1 - k(k - d + n(g - 1)) \\
&\quad + \dim \text{Ext}^2((E, V), (E, V)),
\end{align*}
(11)
since $\text{End}(E, V) \cong \mathbb{C}$ by Proposition 2.2(ii).

**Corollary 3.6.** Every irreducible component $G$ of every moduli space $G_i(n, d, k)$ has dimension
\[ \dim G \geq \beta(n, d, k). \]

**Proof.** See [He, Corollaire 3.14].

The following further corollary of Lemma 3.3 will be useful.
Corollary 3.7. Let $C_{21}$ be defined by (8) and $C_{12}$ by interchanging indices in (8). Then
\[ \beta(n, d, k) = \beta(n_1, d_1, k_1) + \beta(n_2, d_2, k_2) + C_{12} + C_{21} - 1. \]

Proof. This follows from (8) and (11) using the facts that $\chi(E) = \chi(E_1) + \chi(E_2)$ and $\chi(\operatorname{End} E) = \chi(\operatorname{End} E_1) + \chi(\operatorname{End} E_2) + \chi(E_1^* \otimes E_2) + \chi(E_2^* \otimes E_1)$.

Remark 3.8. Notice that if $k > n$ then $\beta(n, d, k) = \beta(k - n, d, k)$. This is easily seen by writing
\[ \beta(n, d, k) = n(g - 1)(n - k) - k(k - d) + 1. \]
We will come back to this symmetry later when studying the dual span of a coherent system (see section 5.4).

We are now ready to extend to coherent systems the standard fact about smoothness of Brill-Noether loci. First we extend the definition of Petri map.

Definition 3.9. Let $(E, V)$ be a coherent system. The Petri map of $(E, V)$ is the map
\[ V \otimes H^0(E^* \otimes K) \to H^0(\operatorname{End} E \otimes K) \]
given by multiplication of sections.

Proposition 3.10. Let $(E, V)$ be an $\alpha$-stable coherent system of type $(n, d, k)$. Then the moduli space $G(\alpha; n, d, k)$ is smooth of dimension $\beta(n, d, k)$ at the point corresponding to $(E, V)$ if and only if the Petri map is injective.

Proof. By Proposition 3.4 and Lemma 3.3, the moduli space is smooth of the correct dimension at $(E, V)$ if and only if $\operatorname{Ext}^2((E, V), (E, V)) = 0$. The result is now a special case of (9).

Remark 3.11. This is a strengthening of the result for Brill-Noether loci (Theorem 2.8), and it justifies the idea that the spaces of coherent systems provide partial desingularisations of the Brill-Noether loci (see sections 2.3 and 11). In view of Proposition 3.10 and Corollary 3.6, we often refer to $\beta(n, d, k)$ as the expected dimension of $G_i(n, d, k)$.

There is a special case in which it is easy to check the injectivity of the Petri map.

Proposition 3.12. Let $(E, V)$ be an $\alpha$-stable coherent system such that $k \leq n$. If $V \otimes \mathcal{O} \to E$ is injective then the moduli space is smooth of dimension $\beta(n, d, k)$ at the point corresponding to $(E, V)$. This happens in particular when $k = 1$.

Proof. We have an exact sequence
\[ 0 \to V \otimes \mathcal{O} \to E \to F \to 0, \]
where $F$ is a coherent sheaf. Tensoring with $E^* \otimes K$ gives
\[ 0 \to V \otimes E^* \otimes K \to \operatorname{End} E \otimes K \to F \otimes E^* \otimes K \to 0. \]
Taking sections, we see that the Petri map is injective.

When the hypotheses of Proposition 3.12 fail, we will need to estimate the dimension of the kernel of the Petri map. In fact Lemma 3.3 gives us such an estimate.
4. Range for $\alpha$

**Lemma 4.1.** If $k < n$ then the moduli space of $\alpha$-semistable coherent systems of type $(n, d, k)$ is empty for $\alpha > \frac{d}{n-k}$. In particular, we must have $d \geq 0$ in order for $\alpha$-semistable coherent systems to exist. Also we must have $d > 0$ in order for $\alpha$-stable coherent systems to exist.

**Proof.** Suppose that $(E, V)$ is an $\alpha$-semistable coherent system of type $(n, d, k)$. By applying the $\alpha$-semistability condition to the subsystem $(E', V)$, where $E' = \text{Im}(V \otimes \mathcal{O} \to E)$, one obtains the upper bound $\alpha \leq \frac{d}{n-k}$, which in particular implies that $d \geq 0$ in order to have non-empty moduli spaces. The final assertion is similar. \hfill \Box

From this lemma and the considerations of section 2.1, we deduce at once the following proposition.

**Proposition 4.2.** Let $k < n$ and let $\alpha_L$ be the biggest critical value smaller than $\frac{d}{n-k}$. The $\alpha$-range is then divided in a finite set of intervals determined by

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L < \frac{d}{n-k}.$$  

Moreover, if $\alpha_i$ and $\alpha_{i+1}$ are two consecutive critical values, the moduli spaces for two different values of $\alpha$ in the interval $(\alpha_i, \alpha_{i+1})$ coincide, and if $\alpha > \frac{d}{n-k}$ the moduli spaces are empty.

**Lemma 4.3.** Let $k \geq n$. We must have $d \geq 0$ in order for $\alpha$-semistable coherent systems of type $(n, d, k)$ to exist. Also we must have $d > 0$ in order for $\alpha$-stable coherent systems to exist except in the case $(n, d, k) = (1, 0, 1)$.

**Proof.** The first assertion is clear if the map $\mathcal{O} \otimes V \to E$ is generically surjective, otherwise one has to apply the $\alpha$-semistability condition to the subsystem $(E', V)$, where $E' = \text{Im}(\mathcal{O} \otimes V \to E)$.

For the second assertion, suppose $d = 0$ and apply the $\alpha$-stability condition to $(E', V)$, where $E' = \text{Im}(\mathcal{O} \otimes V \to E)$, to get that $\mathcal{O} \otimes V \to E$ is generically surjective. Therefore $E \cong \mathcal{O}^n$ and $\alpha$-stability forces $k = n = 1$. \hfill \Box

Although in the case $k \geq n$ the stability condition does not provide us with a bound for $\alpha$, we will show that in fact after a certain finite value of $\alpha$ the moduli spaces do not change. We show first that for $\alpha$ big enough the vector bundle $E$ for an $\alpha$-semistable coherent system is generically generated by the sections in $V$. More precisely

**Proposition 4.4.** Suppose $k \geq n$. Then there exists $\alpha_g > 0$ such that for $\alpha \geq \alpha_g$ if $(E, V)$ is $\alpha$-semistable then the map $\phi : V \otimes \mathcal{O} \to E$ is generically surjective, i.e. we have an exact sequence

$$0 \to N \to \mathcal{O}^\oplus k \to E \to T \to 0$$

where

(i) $T$ is a torsion sheaf (possibly 0),

(ii) $\text{rk} N = k - n$,

(iii) $H^0(N) = 0$. 

In fact, \[ \alpha_g \leq \frac{d(n-1)}{k}. \]

**Proof.** Let \( N = \text{Ker } \phi \) and \( I = \text{Im } \phi \), and suppose \( \text{rk } I = n - l < n \). One has the exact sequence

\[ 0 \to N \to \mathcal{O}^{\otimes k} \to E \to E/I \to 0. \]

Consider the subsystem \((I, V)\). One has \( d_I := \deg I \geq 0 \) since \( I \) is generated by global sections. Now \( \alpha \)-semistability implies that \( \mu_\alpha(I, V) \leq \mu_\alpha(E, V) \), which means that

\[ \frac{d_I}{n-l} + \frac{\alpha}{n-l} \leq \frac{d}{n} + \frac{\alpha}{n}, \]

and hence

\[ \alpha \leq \frac{d(n-l)}{kl} \leq \frac{d(n-1)}{k}. \]

We conclude that if \( \alpha > \frac{d(n-1)}{k} \) then \( \text{rk } I = n \) so that \( E/I = T \) is pure torsion. Finally \( H^0(N) = 0 \) since \( H^0(\phi) \) is injective by definition of coherent system. \( \square \)

Our next object is to show that the \( \alpha \)-stability condition is independent of \( \alpha \) for \( \alpha > d(n-1) \). More precisely

**Proposition 4.5.** (i) If there exists a subsystem \((E', V')\) of \((E, V)\) with \( \frac{k'}{n'} > \frac{k}{n} \), then \((E, V)\) is not \( \alpha \)-semistable for \( \alpha > d(n-1) \).

(ii) If there exists a subsystem \((E', V')\) of \((E, V)\) with \( \frac{k'}{n'} = \frac{k}{n} \) and \( \frac{d'}{n'} \geq \frac{d}{n} \), then \((E, V)\) is not \( \alpha \)-stable for any \( \alpha \).

(iii) If neither (i) nor (ii) holds, and \( E \) is generically generated by its sections, then \((E, V)\) is \( \alpha \)-stable for \( \alpha > d(n-1) \).

**Proof.** (i) Suppose \((E, V)\) is \( \alpha \)-semistable. Replacing \( E' \) by a subbundle if necessary, we can suppose that \( E' \) is generically generated by its sections and hence \( d' \geq 0 \). Then we have

\[ \alpha \frac{k'}{n'} \leq \frac{d}{n} + \frac{k}{n}, \]

i.e.

\[ \alpha \leq \frac{n'd}{nk' - n'k} \leq d(n-1). \]

(ii) is obvious.

(iii) If neither (i) nor (ii) holds and \((E', V')\) contradicts the \( \alpha \)-stability of \((E, V)\), then we must have \( \frac{n'}{n'} < \frac{k'}{k} \). If \( E \) is generically generated by its sections, then so is \( E/E' \); hence \( \deg(E/E') \geq 0 \) and \( d' = \deg E' \leq d \). Thus we have

\[ \frac{d}{n} + \frac{k}{n} \leq \frac{d'}{n'} + \frac{k'}{n'}, \]

i.e.

\[ \alpha \leq \frac{d(n-n')}{n'k - nk'} \leq d(n-1). \]

\( \square \)
We have thus proved the following.

**Proposition 4.6.** Let \( k \geq n \). Then there is a critical value, denoted by \( \alpha_L \), after which the moduli spaces stabilise, i.e. \( G(\alpha) = G_L \) if \( \alpha > \alpha_L \). The \( \alpha \)-range is thus divided into a finite set of intervals bounded by critical values

\[
0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L < \infty
\]

and such that

(i) if \( \alpha_i \) and \( \alpha_{i+1} \) are two consecutive critical values, the moduli spaces for any two different values of \( \alpha \) in the interval \( (\alpha_i, \alpha_{i+1}) \) coincide,

(ii) for any two different values of \( \alpha \) in the range \( (\alpha_L, \infty) \), the moduli spaces coincide.

5. **Moduli for \( \alpha \) large**

5.1. **The moduli space \( G_L \) for \( k < n \).** Recall that, when \( k < n \), \( G_L \) denotes the moduli space of coherent systems for \( \alpha \) large, i.e. \( \alpha_L < \alpha < \frac{d}{n-k} \). The description of \( G_L \) in this case has been carried out in [BG2], where we refer for details. We summarise here the main results.

**Definition 5.1.** A BGN extension ([BGN]) is an extension of vector bundles

\[
0 \rightarrow \mathcal{O}^{\oplus k} \rightarrow E \rightarrow F \rightarrow 0
\]

which satisfies the following conditions:

- \( \text{rk} \ E = n > k \),
- \( \text{deg} \ E = d > 0 \),
- \( H^0(F^*) = 0 \),
- if \( \bar{e} = (e_1, \ldots, e_k) \in H^1(F^* \otimes \mathcal{O}^{\oplus k}) = H^1(F^*)^{\oplus k} \) denotes the class of the extension, then \( e_1, \ldots, e_k \) are linearly independent as vectors in \( H^1(F^*) \).

The BGN extensions which differ only by an automorphism of \( \mathcal{O}^{\oplus k} \), i.e. by the action of an element in \( \text{GL}(k) \), comprise a BGN extension class of type \((n, d, k)\).

**Proposition 5.2.** Suppose that \( 0 < k < n \) and \( d > 0 \). Let \( \alpha_L < \alpha < \frac{d}{n-k} \). Let \((E, V)\) be an \( \alpha \)-semistable coherent system of type \((n, d, k)\). Then \((E, V)\) defines a BGN extension class represented by an extension

\[
0 \rightarrow \mathcal{O}^{\oplus k} \rightarrow E \rightarrow F \rightarrow 0, \tag{12}
\]

with \( F \) semistable. In the converse direction, any BGN extension of type \((n, d, k)\) in which the quotient \( F \) is stable gives rise to an \( \alpha \)-stable coherent system of the same type.

**Remark 5.3.** In the last part of Proposition 5.2, it is essential to have \( F \) stable. If \( F \) is only semistable, the coherent system can fail to be \( \alpha \)-semistable.

**Theorem 5.4.** Let \( 0 < k < n \) and \( d > 0 \). If \( g \geq 2 \), the moduli space \( G_L(n, d, k) \) of \( \alpha \)-stable coherent systems of type \((n, d, k)\) is birationally equivalent to a fibration over the moduli space \( M(n-k, d) \) of stable bundles of rank \( n-k \) and degree \( d \) with fibre the Grassmannian \( \text{Gr}(k, d + (n-k)(g-1)) \). In particular \( G_L \) is non-empty if and only
Proposition 5.2 remains true when preceded by Remark 5.5.

Proof. This follows directly from Proposition 5.2 and the remark immediately preceding it. □

Remark 5.5. Proposition 5.2 remains true when \( g = 0 \) or 1, but Theorem 5.4 can fail because \( M(n-k,d) \) may be empty. In fact, if \( g = 0 \), \( M(n-k,d) = \emptyset \) unless \( n-k = 1 \). Furthermore, if \( n-k \) does not divide \( d \), then \( \tilde{M}(n-k,d) = \emptyset \), and it follows from Proposition 5.2 that \( G_L(n,d,k) = \emptyset \). If \( d = (n-k)a \) with \( a \in \mathbb{Z} \), then \( \tilde{M}(n-k,d) \) consists of a single point corresponding to the bundle

\[ F = \mathcal{O}(a) \oplus \ldots \oplus \mathcal{O}(a). \]

It is not clear from the results of [BG2] whether there exist \( \alpha \)-stable coherent systems as in (12). Thus we conclude, for \( g = 0 \),

- \( G_L(n,d,n-1) \neq \emptyset \) if and only if \( d \geq n \), and it is then isomorphic to the Grassmannian \( \text{Gr}(n-1,d-1) \),
- \( G_L(n,d,k) = \emptyset \) if \( k \leq n-2 \) and \( d \) is not divisible by \( n-k \),
- if \( k \leq n-2 \) and \( d = (n-k)a \) with \( a \in \mathbb{Z} \), then \( G_L(n,d,k) = \emptyset \) if \( d < n \); if \( d \geq n \), a more detailed analysis is required.

Turning now to \( g = 1 \), we know that \( \tilde{M}(n-k,d) \) is always non-empty and that \( M(n-k,d) \neq \emptyset \) if and only if \( (n-k,d) = 1 \); moreover in this case \( M(n-k,d) \) is isomorphic to the curve \( X \). (All this follows essentially from [At].) We conclude, for \( g = 1 \),

- if \( (n-k,d) = 1 \), then \( G_L(n,d,k) \neq \emptyset \) if and only if \( d \geq k \), and it is then isomorphic to a fibration over \( X \) with fibre \( \text{Gr}(k,d) \),
- if \( (n-k,d) \neq 1 \), a more detailed analysis is required.

5.2. Quot schemes. When \( k \geq n \), we can follow [BeDW] and relate \( G_L \) to a Grothendieck Quot scheme. In fact, by Proposition 4.4, any element of \( \tilde{G}_L \) can be represented in the form

\[ 0 \rightarrow N \rightarrow V \otimes \mathcal{O} \xrightarrow{\phi} E, \]

where \( \phi \) is generically surjective. Dualising (13), we obtain

\[ 0 \rightarrow E^* \rightarrow V^* \otimes \mathcal{O} \rightarrow F \rightarrow 0, \]

where \( F \) is a coherent sheaf but is not torsion-free (unless \( \phi \) is surjective). Conversely, given (14), one can recover (13) (in fact \( N \cong F^* \)). It follows that there is a bijective correspondence between triples \((E,V,\phi)\) and points of \( Q = \text{Quot}_{k-n,d}(\mathcal{O}^{\oplus k}) \), the Quot scheme of quotients of \( \mathcal{O}^{\oplus k} \) of rank \( k-n \) and degree \( d \). In order to obtain \( \tilde{G}_L \), we therefore need to construct a GIT quotient of \( Q \) by the natural action of \( \text{GL}(k) \) with respect to a stability condition corresponding to the \( \alpha \)-stability of coherent systems for large \( \alpha \). This situation requires detailed analysis, but even if we complete the construction, it may still be difficult to obtain information about \( G_L \), since even basic information about \( Q \) is often lacking, e.g., when it is non-empty, irreducible etc. However, potentially this would be a useful source of information about \( G_L \).
In sections 5.3 and 5.4, we shall use the sequences (13) and (14) to obtain information about $G_L$ in the cases $k = n$ and $k > n$.

5.3. The moduli space $G_L$ for $k = n$. We are now able to prove Theorem 1.1 in a stronger form which covers $G_L$ as well as $G_L$.

**Theorem 5.6.** Let $k = n \geq 2$. If $\widetilde{G}_L$ is non-empty then $d = 0$ or $d \geq n$. For $d > n$, $\widetilde{G}_L$ is irreducible and $G_L$ is smooth of the expected dimension $dn - n^2 + 1$. For $d = n$, $G_L$ is empty and $\widetilde{G}_L$ is irreducible of dimension $n$ (not of the expected dimension). For $d = 0$, $G_L$ is empty and $\widetilde{G}_L$ consists of a single point.

**Proof.** Let $(E, V)$ be an $\alpha$-semistable coherent system for any $\alpha$, represented by $\phi : \mathcal{O}^\oplus n \to E$. If $\phi$ is an isomorphism, then $d = 0$ and $(E, V) \cong (\mathcal{O}^\oplus n, \mathbb{C}^n)$, which is clearly $\alpha$-semistable but not $\alpha$-stable. Otherwise there exists $\mathcal{O} \subset \mathcal{O}^\oplus n$ which defines a section of $E$ with a zero. It follows that this section is contained in a subbundle of $E$ of rank 1 with degree $> 0$. This subbundle together with the section defines a subsystem which contradicts $\alpha$-semistability if $d < n$ and $\alpha$-stability if $d = n$.

Now suppose $d \geq n$. Let $(E, V)$ be any $\alpha$-semistable coherent system for $\alpha$ large. By Proposition 4.4 with $k = n$, we have an extension

$$0 \to \mathcal{O}^\oplus n \to E \to T \to 0,$$

where $T$ is torsion. The generic torsion sheaf is of the form $T = \mathcal{O}_D$ for a divisor $D$ consisting of $d$ distinct points. For such $T$, $E$ is given by an extension class $\xi \in \text{Ext}^1(T, \mathcal{O}^\oplus n) \cong \text{Hom}(T, \mathbb{C}^n)$, which is equivalent to a collection of $d$ vectors $\xi_i \in \mathbb{C}^n$, one for each point $P_i$ in the support of $D$.

We claim that all the coherent systems defined by extensions in

$$U = \{(\mathcal{O}_D, \xi) | \text{any subset of } n \text{ vectors of } \xi_1, \ldots, \xi_d \text{ is linearly independent}\}$$

are $\alpha$-stable (for $d > n$) or $\alpha$-semistable (for $d = n$).

Suppose for the moment that the claim holds. Let

$$U^{ss} = \{(T, \xi) \mid \xi \in \text{Ext}^1(T, \mathcal{O}^\oplus n) \text{ and determines an } \alpha\text{-semistable coherent system}\}.$$

Then $U$ is dense and open in $U^{ss}$. Also $U^{ss}$ dominates the moduli space of $\alpha$-semistable coherent systems. Thus $\widetilde{G}_L$ is irreducible. The fact that $G_L$ is smooth of the expected dimension follows at once from Proposition 3.12. We can also compute directly the dimension of the space of coherent systems determined by $U$. The space $\text{Ext}^1(T, \mathcal{O}^\oplus n)$ has dimension $dn$, and we have to quotient out by the automorphisms $\text{Aut}_T = \text{GL}(1)^d$, for $T = \mathcal{O}_D$, and by $\text{Aut}_T \mathcal{O}^\oplus n = \text{GL}(n)$. For $d > n$, the centraliser of the action of the product on $(\mathcal{O}_D, \xi) \in U$ is $\mathbb{C}^*$. So the dimension of the space of coherent systems determined by $U$ is $d + dn - d - n^2 + 1$, which is in agreement with our previous answer.

For $d = n$, $\text{GL}(n)$ acts freely on any collections of $n$ linearly independent vectors in $\mathbb{C}^n$. So the dimension of the space of coherent systems determined by $U$ is $d + dn - n^2 = n$. It is possible that different elements of $U$ give rise to $S$-equivalent systems, thus reducing $\dim \widetilde{G}_L$. However, if $g \geq 1$, the coherent systems

$$\left( \mathcal{O}(P_1), H^0(\mathcal{O}(P_1)) \right) \oplus \ldots \oplus \left( \mathcal{O}(P_n), H^0(\mathcal{O}(P_n)) \right),$$

...
where \( P_1, \ldots, P_n \in X \), are clearly \( \alpha \)-semistable and no two of them are S-equivalent, so \( \dim G_L \geq n \), which completes the computation. If \( g = 0 \), there is a unique line bundle \( O(1) \) of degree 1, and \( h^0(O(1)) = 2 \); in this case the coherent systems
\[
(O(1), V_i) \oplus \ldots \oplus (O(1), V_n),
\]
where \( V_1, \ldots, V_n \) are subspaces of dimension 1 of \( H^0(O(1)) \), are \( \alpha \)-semistable and form a family of dimension \( n \). This gives the same conclusion.

Now we prove the claim, i.e. every \((E, V)\) in the image of \( U \) is \( \alpha \)-stable for \( \alpha \) large. Let \((E_1, V_1)\) be a coherent subsystem of \((E, V)\). As \( E_1 \subset E \) we must have \( k_1 \leq n_1 \). If \( k_1 < n_1 \) then \((E_1, V_1)\) cannot violate \( \alpha \)-stability. If \( k_1 = n_1 \) then we have a diagram
\[
\begin{array}{ccc}
O^{\otimes n_1} & \rightarrow & E_1 \\
\downarrow & & \downarrow \\
O^{\otimes m} & \rightarrow & E
\end{array}
\]
where \( d_1 = \deg T_1 \). Then the image of \( \xi \in \text{Ext}^1(T, O^{\otimes n}) \) in \( \text{Ext}^1(T_1, O^{\otimes n_1}) \) lies in the subspace \( \text{Ext}^1(T_1, O^{\otimes n_1}) \). This is equivalent to \( \xi_i \in \mathbb{C}^{n_1} \) for any \( P_i \) in the support of \( T_1 \). \( E \) is \( \alpha \)-semistable if \( d_1/n_1 \leq d/n \) for all possible choices of diagrams as above.

Now any subcollection of \( n \) vectors of the \( \xi_i \) is linearly independent, so for \( d_1 \geq n \) we have \( n_1 = n \) and \( E_1 = E \). For \( d_1 < n \) we have \( n_1 \geq d_1 \) and \( d_1/n_1 \leq 1 \). Hence, for \( d > n \) the coherent systems are \( \alpha \)-stable, while for \( d = n \) they are \( \alpha \)-semistable. \( \square \)

**Remark 5.7.** For \( d \leq n \), the proof shows that \((E, V)\) cannot be \( \alpha \)-stable for any \( \alpha \); moreover, if \( 0 < d < n \), \((E, V)\) cannot be \( \alpha \)-semistable. For \( d = 0 \), any \( \alpha \)-semistable coherent system is isomorphic to \((O^{\otimes n}, \mathbb{C}^n)\). For \( d = n \), one can show that \( \tilde{G}(\alpha) \) is independent of \( \alpha \) and that \( \tilde{G}(\alpha) \cong S^nX \) (see [BGN, Theorem 8.3] for the case \( \alpha = 0 \)).

### 5.4. The moduli space \( G_L \) for \( k > n \). The dual span construction
We can represent a coherent system by a sequence \([13]\), where we now suppose that \( k > n \) and that \( \phi \) is surjective; so we have
\[
0 \rightarrow N \rightarrow V \otimes O \xrightarrow{\psi} E \rightarrow 0
\]
and
\[
0 \rightarrow E^* \rightarrow V^* \otimes O \xrightarrow{\phi} N^* \rightarrow 0.
\]
In the case where \( V = H^0(E) \) and \( E \) is generated by its sections, this construction has been used by a number of authors (see for example [Bu1, Bu2, Fl, Mil, Pr]), the main question being to determine conditions under which the stability of \( E \) implies that of \( N \). Recently Butler noted that the construction belongs more naturally to the theory of coherent systems and began to investigate it using \( \alpha \)-stability. However he restricted attention to small \( \alpha \). Our purpose in this section is to show that the construction works better if we consider large \( \alpha \).

It is convenient here to make partial use of the wider notion of coherent system, introduced in [KN] and mentioned in section 2.7, by dropping the assumption that \( H^0(\phi) \) is injective. This makes no essential difference as \((E, V)\) cannot be \( \alpha \)-semistable unless \( H^0(\phi) \) is injective. It does however mean that \((N^*, V^*, \psi)\) always determines a coherent system, which we may call the dual span of \((E, V, \phi)\) and denote by \( D(E, V, \phi) \) (or \( D(E, V) \) in the case where \( H^0(\phi) = 0 \)).
Definition 5.8. \((E, \mathbb{V}, \phi)\) is strongly unstable if there exists a proper coherent subsystem \((E', \mathbb{V}', \phi')\) such that
\[
\frac{k'}{n'} > \frac{k}{n}.
\]

Proposition 5.9. Suppose that \(E\) is generated by \(\mathbb{V}\). Then \((E, \mathbb{V}, \phi)\) is strongly unstable if and only if \(D(E, \mathbb{V}, \phi)\) is strongly unstable.

Proof. Suppose that \((E, \mathbb{V}, \phi)\) is strongly unstable and that \((E', \mathbb{V}', \phi')\) is a subsystem as in the definition above. Replacing \(E'\) by the (sheaf) image of \(\mathbb{V}'\) in \(E\) if necessary, we have a short exact sequence
\[
0 \rightarrow N' \rightarrow \mathbb{V}' \otimes O \rightarrow E' \rightarrow 0.
\]

\(D(E', \mathbb{V}', \phi')\) is then a quotient system of \(D(E, \mathbb{V}, \phi)\). The corresponding subsystem has rank \((k - n) - (k' - n')\) and dimension \(k - k'\) and
\[
(k - n)(k' - n) - ((k - n) - (k' - n'))k = nk' - n'k > 0.
\]

So \(D(E, \mathbb{V}, \phi)\) is strongly unstable. The converse is similar, which completes the proof. \(\square\)

By Proposition 4.5, any strongly unstable coherent system fails to be \(\alpha\)-semistable for \(\alpha > d(n - 1)\). The converse may fail because we have to take account of subsystems with \(\frac{k'}{n'} = \frac{k}{n}\). However, if \((n, k) = 1\), there are no such subsystems and we have

Corollary 5.10. Suppose that \(E\) is generated by \(V\). If \((n, k) = 1\), then \((E, V)\) is \(\alpha\)-stable for large \(\alpha\) if and only if \(D(E, V)\) is \(\alpha\)-stable for large \(\alpha\).

By Proposition 4.5 it is sufficient to take \(\alpha > \max\{d(n - 1), d(k - n - 1)\}\).

Theorem 5.11. Suppose that \(X\) is a Petri curve and that \(k = n + 1\). Then \(G_L\) is non-empty if and only if \(\beta = g - (n + 1)(n - d + g) \geq 0\). Moreover \(G_L\) has dimension \(\beta\) and it is irreducible whenever \(\beta > 0\).

Proof. If \((E, V) \in G_L\), then by Proposition 4.4, \(E\) is generically generated by \(V\). If we suppose further that \(E\) is generated by \(V\), then \(D(E, V) \in G(1, d, n + 1)\). Since \(X\) is Petri, \(G(1, d, n + 1)\) is non-empty if and only if the Brill-Noether number
\[
\beta = \beta(1, d, n + 1) = g - (n + 1)(n - d + g) \geq 0.
\]

Moreover, if this holds, \(G(1, d, n + 1)\) has dimension \(\beta\), and it is irreducible whenever \(\beta > 0\). Note also that the dimension of the subvariety consisting of systems \((L, W)\) for which \(L\) is not generated by \(W\) has dimension at most
\[
g - (n + 1)(n - (d - 1) + g) + 1 < \beta.
\]

So \(G(1, d, n + 1)\) has a dense open subset in which \(L\) is generated by \(W\). The Brill-Noether number \(\beta(n, d, n + 1) = \beta\) by Remark 3.8 so the systems \((E, V)\) which are \(\alpha\)-stable for large \(\alpha\) and for which \(V\) generates \(E\) are parametrised by a variety of the expected dimension. If \(E\) is only generically generated by \(V\) and \(E'\) is the subsheaf generated by \(V\), we can put \(\deg E' = d - t\) with \(t > 0\). Then, by the argument above,
the variety parametrising the systems \((E', V)\) has the expected dimension, which is 
\[ \beta - (n + 1)t. \]
On the other hand, the variety parametrising the extensions
\[ 0 \to E' \to E \to T \to 0, \]
where \(T\) is a torsion sheaf of length \(t\), has dimension \(nt\) (after factoring out by the action of \(\text{Aut } T\)). So the variety parametrising all the corresponding systems \((E, V)\) has dimension \(< \beta\). Since every component of \(G_L\) has dimension \(\geq \beta\), this completes the proof.

\[ \square \]

6. Crossing critical values

In this section we analyse the differences between consecutive moduli spaces in the family \(\{G_0, G_1, \ldots, G_L\}\). Recall that \(G_i\) denotes the moduli space of \(\alpha\)-stable coherent systems, where \(\alpha\) is (anywhere) in the interval bounded by the critical values \(\alpha_i\) and \(\alpha_{i+1}\). The differences between \(G_{i-1}\) and \(G_i\) are thus due to the differences between the \(\alpha\)-stability conditions for \(\alpha < \alpha_i\) and \(\alpha > \alpha_i\).

6.1. The basic mechanism. The following lemma describes the basic mechanism responsible for a change in the stability property of a coherent system.

**Lemma 6.1.** Let \((E, V)\) be a coherent system of type \((n, d, k)\) and let \((E', V')\) be a subsystem of type \((n', d', k')\). Then \(\mu_\alpha(E', V') - \mu_\alpha(E, V)\) is a linear function of \(\alpha\) which is

- monotonically increasing if \(\frac{k'}{n'} - \frac{k}{n} > 0\),
- monotonically decreasing if \(\frac{k'}{n'} - \frac{k}{n} < 0\),
- constant if \(\frac{k'}{n'} - \frac{k}{n} = 0\).

In particular, if \(\alpha_i\) is a critical value and \(\mu_{\alpha_i}(E', V') = \mu_{\alpha_i}(E, V)\), then

- \((\mu_\alpha(E', V') - \mu_\alpha(E, V))(\alpha - \alpha_i) > 0\), for all \(\alpha \neq \alpha_i\), if \(\frac{k'}{n'} - \frac{k}{n} > 0\),
- \((\mu_\alpha(E', V') - \mu_\alpha(E, V))(\alpha - \alpha_i) < 0\), for all \(\alpha \neq \alpha_i\), if \(\frac{k'}{n'} - \frac{k}{n} < 0\),
- \(\mu_\alpha(E', V') - \mu_\alpha(E, V) = 0\) for all \(\alpha\) if \(\frac{k'}{n'} - \frac{k}{n} = 0\).

**Proof.** This follows easily from
\[ \mu_\alpha(E', V') - \mu_\alpha(E, V) = \frac{d'}{n'} - \frac{d}{n} + \alpha \left( \frac{k'}{n'} - \frac{k}{n} \right). \]

\[ \square \]

In particular, we have

**Lemma 6.2.** Let \((E, V)\) be a coherent system of type \((n, d, k)\). Suppose that it is \(\alpha\)-stable for \(\alpha > \alpha_i\), but is strictly \(\alpha\)-semistable for \(\alpha = \alpha_i\). Then \((E, V)\) is unstable for all \(\alpha < \alpha_i\).

**Proof.** Any such coherent system must have a subsystem, say \((E', V')\), for which \(\mu_{\alpha_i}(E', V') = \mu_{\alpha_i}(E, V)\) but such that \(\mu_\alpha(E', V') < \mu_\alpha(E, V)\) if \(\alpha > \alpha_i\). It follows from the previous lemma that \(\mu_\alpha(E', V') > \mu_\alpha(E, V)\) for all \(\alpha < \alpha_i\), i.e. the subsystem \((E', V')\) is destabilising for all \(\alpha < \alpha_i\).

\[ \square \]
Remark 6.3. Thus, if we study the effect on $G_L(n,d,k)$ of monotonically reducing $\alpha$, we see that “once a coherent system is removed it can never return”. In contrast to this, it can happen that “a coherent system once added may have to be later removed” [BG2].

Definition 6.4. We define $G^+_i \subseteq G_i = G_i(n,d,k)$ to be the set of all $(E,V)$ in $G_i$ which are not $\alpha$-stable if $\alpha < \alpha_i$. Similarly, we define $G^-_i \subseteq G_{i-1}$ to be the set of all $(E,V)$ in $G_{i-1}$ which are not $\alpha$-stable if $\alpha > \alpha_i$.

We can identify the sets $G_i - G_i^+ = G_{i-1} - G_i^-$ and hence (set theoretically) we get

- $G_{i+1} = G_i - G_{i+1}^- + G^+_i$,
- $G_{i-1} = G_i - G^+_i + G_i^-$.\[10pt]

In fact, we can be more precise. The subset $G_i^+$ consists of the points in $G_i$ corresponding to coherent systems which are not $\alpha_i$-stable; they therefore form a closed subscheme of $G_i$. Similarly $G_i^-$ is a closed subscheme of $G_{i-1}$. Hence $G_i^+ - G_i^-$ and $G_{i-1} - G_i^-$ have natural scheme structures, and as such are isomorphic.

6.2. Destabilising patterns. The following lemma allows us to describe the sets $G_i^+$ and $G_i^-$, and also to estimate their codimensions in the moduli spaces $G_i$. It is important to note that, unlike the Jordan-Hölder filtrations for semistable objects, the descriptions we obtain are always as extensions, i.e. $1$-step filtrations. This simplification results from a careful exploitation of the stability parameter. For convenience, we denote values of $\alpha$ in the intervals on either side of $\alpha_i$ by $\alpha_i^-$ and $\alpha_i^+$ respectively.

Lemma 6.5. Let $\alpha_i$ be a critical value of $\alpha$ with $1 \leq i \leq L$. Let $(E,V)$ be a coherent system of type $(n,d,k)$.

(i) Suppose that $(E,V)$ is $\alpha_i^+$-stable but $\alpha_i^-$-unstable. Then $(E,V)$ appears as the middle term in an extension\[10pt]

$$0 \to (E_1,V_1) \to (E,V) \to (E_2,V_2) \to 0 \tag{15}$$

in which

- (a) $(E_1,V_1)$ and $(E_2,V_2)$ are both $\alpha_i^+$-stable, with $\mu_{\alpha_i^+}(E_1,V_1) < \mu_{\alpha_i^+}(E_2,V_2)$,
- (b) $(E_1,V_1)$ and $(E_2,V_2)$ are both $\alpha_i$-semistable, with $\mu_{\alpha_i}(E_1,V_1) = \mu_{\alpha_i}(E_2,V_2)$,
- (c) $\frac{k}{n_1}$ is a maximum among all proper subsystems $(E_1,V_1) \subset (E,V)$ which satisfy (b),
- (d) $n_1$ is a minimum among all subsystems which satisfy (c).

(ii) Similarly, if $(E,V)$ is $\alpha_i^-$-stable but $\alpha_i^+$-unstable, then $(E,V)$ appears as the middle term in an extension \([\text{II}]\) in which

- (a) $(E_1,V_1)$ and $(E_2,V_2)$ are both $\alpha_i^-$-stable, with $\mu_{\alpha_i^-}(E_1,V_1) < \mu_{\alpha_i^-}(E_2,V_2)$,
- (b) $(E_1,V_1)$ and $(E_2,V_2)$ are both $\alpha_i$-semistable, with $\mu_{\alpha_i}(E_1,V_1) = \mu_{\alpha_i}(E_2,V_2)$,
- (c) $\frac{k}{n_1}$ is a minimum among all proper subsystems $(E_1,V_1) \subset (E,V)$ which satisfy (b),
- (d) $n_1$ is a minimum among all subsystems which satisfy (c).
**Proof.** Since its stability property changes at $\alpha_i$, the coherent system $(E, V)$ must be strictly $\alpha_i$-semistable, i.e. it must have a proper subsystem $(E', V')$ with $\mu_{\alpha_i}(E', V') = \mu_{\alpha_i}(E, V)$. Consider the (non-empty) set

$$\mathcal{F}_1 = \{ (E_1, V_1) \subset (E, V) \mid \mu_{\alpha_i}(E_1, V_1) = \mu_{\alpha_i}(E, V) \}.$$  

Any such subsystem $(E_1, V_1)$ must have $n_1 < n$ and $V_1 = V \cap H^0(E_1)$ (otherwise replacing $V_1$ by $V \cap H^0(E_1)$ would contradict the $\alpha_i$-semistability of $(E, V)$).

**Proof of (i).** Suppose first that $(E, V)$ is $\alpha_i^+$-stable but $\alpha_i^-$-unstable. We observe that if $(E_1, V_1) \in \mathcal{F}_1$, then $\frac{k_1}{n_1} < \frac{k}{n}$, since otherwise $(E, V)$ could not be $\alpha_i^+$-stable. But the allowed values for $\frac{k_1}{n_1}$ are limited by the constraints $0 < n_1 < n$ and $0 \leq k_1 \leq k$. We can thus define

$$\lambda_0 = \max \left\{ \frac{k_1}{n_1} \mid (E_1, V_1) \in \mathcal{F}_1 \right\}$$

and set

$$\mathcal{F}_2 = \left\{ (E_1, V_1) \subset \mathcal{F}_1 \mid \frac{k_1}{n_1} = \lambda_0 \right\}.$$  

Let $(E_1, V_1)$ be any coherent system in $\mathcal{F}_2$. Since $V_1 = V \cap H^0(E_1)$, we can write

$$0 \to (E_1, V_1) \to (E, V) \to (E_2, V_2) \to 0$$

for some coherent system $(E_2, V_2)$. Since $\mu_{\alpha_i}(E_1, V_1) = \mu_{\alpha_i}(E, V) = \mu_{\alpha_i}(E_2, V_2)$ and $(E, V)$ is $\alpha_i$-semistable, it follows that both $(E_1, V_1)$ and $(E_2, V_2)$ are $\alpha_i$-semistable.

We now show that $(E_2, V_2)$ is $\alpha_i^+$-stable. Suppose not. Then there is a proper subsystem $(E_2', V_2') \subset (E_2, V_2)$ with

- $\mu_{\alpha_i}(E_2', V_2') = \mu_{\alpha_i}(E_2, V_2)$,
- $\frac{k_1'}{n_1'} \geq \frac{k_2}{n_2}$.

Consider now the subsystem $(E', V') \subset (E, V)$ defined by the pull-back diagram

$$0 \to (E_1, V_1) \to (E', V') \to (E_2', V_2') \to 0.$$  

This has $\mu_{\alpha_i}(E', V') = \mu_{\alpha_i}(E, V)$ and thus satisfies $\frac{k_1'}{n_1'} + \frac{k_1}{n_1} \leq \frac{k_1}{n_1}$. It follows that

$$\frac{k_1'}{n_1'} \leq \frac{k_1}{n_1} < \frac{k_2}{n_2},$$

which is a contradiction.

Now consider $(E_1, V_1) \in \mathcal{F}_2$ with minimum rank in $\mathcal{F}_2$. If $(E_1, V_1)$ is not $\alpha_i^+$-stable, then it must have a proper subsystem $(E_1', V_1')$ with

- $\mu_{\alpha_i}(E_1', V_1') = \mu_{\alpha_i}(E_1, V_1)$,
- $\frac{k_1'}{n_1'} \geq \frac{k_1}{n_1}$.

But then $n_1' < n_1$, which contradicts the minimality of $n_1$. Finally, notice that since $(E, V)$ is $\alpha_i^+$-stable, we must have $\mu_{\alpha_i^+}(E_1, V_1) < \mu_{\alpha_i^+}(E, V) < \mu_{\alpha_i^+}(E_2, V_2)$.
Proof of (ii). If \((E,V)\) is \(\alpha_i^-\)-stable but \(\alpha_i^+\)-unstable, then \(\frac{k_1}{n_1} > \frac{\ell}{n}\) for all \((E_1,V_1) \in \mathcal{F}_1\). The proof of (i) must thus be modified as follows. With
\[
\lambda_0 = \min \left\{ \frac{k_1}{n_1} \mid (E_1,V_1) \in \mathcal{F}_1 \right\}
\]
we can define
\[
\mathcal{F}_2 = \left\{ (E_1,V_1) \subset \mathcal{F}_1 \mid \frac{k_1}{n_1} = \lambda_0 \right\}
\]
and select \((E_1,V_1) \in \mathcal{F}_2\) such that \(E_1\) has minimal rank in \(\mathcal{F}_2\). It follows in a similar fashion to that above that \((E,V)\) has a description as
\[
0 \to (E_1,V_1) \to (E,V) \to (E_2,V_2) \to 0
\]
in which both \((E_1,V_1)\) and \((E_2,V_2)\) are \(\alpha_i^-\)-stable.

We refer to the extensions of the form (15) with the properties of Lemma 6.5 as the *destabilising patterns* of the coherent systems.

6.3. Codimension estimates for \(G_i^-\) and \(G_i^+\).

**Definition 6.6.** Let \(W^+(\alpha, \lambda, n_1; n, d, k)\) (abbreviated to \(W^+_i(\lambda, n_1)\) whenever possible) denote the set of all destabilising patterns
\[
0 \to (E_1,V_1) \to (E,V) \to (E_2,V_2) \to 0
\]
in which
- \((E,V)\) is \(\alpha_i^+\)-stable and of type \((n,d,k)\),
- \(\text{rk}(E_1) = n_1\) and \(\dim(V_1) = \lambda n_1\),
- \(\mu_{\alpha_i}(E_1,V_1) = \mu_{\alpha_i}(E_2,V_2) = \mu_{\alpha_i}(E,V)\),
- \((E_1,V_1)\) and \((E_2,V_2)\) are both \(\alpha_i^+\)-stable,
- \(\dim(V_1)\) and \(\text{rk}(E_1)\) satisfy the min-max criteria given in (c) and (d) of Lemma 6.5(i).

Similarly, let \(W^-(\alpha, \lambda, n_1; n, d, k)\) (abbreviated to \(W^-_i(\lambda, n_1)\) whenever possible) denote the set of all destabilising patterns
\[
0 \to (E_1,V_1) \to (E,V) \to (E_2,V_2) \to 0
\]
in which
- \((E,V)\) is \(\alpha_i^-\)-stable and of type \((n,d,k)\),
- \(\text{rk}(E_1) = n_1\) and \(\dim(V_1) = \lambda n_1\),
- \(\mu_{\alpha_i}(E_1,V_1) = \mu_{\alpha_i}(E_2,V_2) = \mu_{\alpha_i}(E,V)\),
- \((E_1,V_1)\) and \((E_2,V_2)\) are both \(\alpha_i^-\)-stable,
- \(\dim(V_1)\) and \(\text{rk}(E_1)\) satisfy the min-min criteria given in (c) and (d) of Lemma 6.5(ii).

Define
\[
W^+(\alpha, n, d, k) = \bigsqcup_{\lambda < \frac{\ell}{n}, n_1 < n} W^+(\alpha, \lambda, n_1; n, d, k),
\]
\[ W^- (\alpha_i, n, d, k) = \bigcup_{\lambda > \frac{k}{n}, n_1 < n} W^- (\alpha, \lambda, n_1; n, d, k). \]

We abbreviate these to \( W^+_i \) and \( W^-_i \) whenever possible.

**Lemma 6.7.** Fix \((n, d, k)\) and also \(\alpha_i\). Then each set \( W^\pm_i (\lambda, n_1) \) is contained in a family of dimension bounded above by

\[
\begin{align*}
 w^\pm_i (\lambda, n_1) &= \dim G(\alpha^\pm_i; n_1, d_1, k_1) + \dim G(\alpha^\pm_i; n_2, d_2, k_2) \\
 &\quad + \max \dim \text{Ext}^1((E_2, V_2), (E_1, V_1)) - 1.
\end{align*}
\]

Here \( n = n_1 + n_2, d = d_1 + d_2 \) and \( k = k_1 + k_2 \), and the maximum is taken over all \((E_1, V_1), (E_2, V_2)\) which satisfy the relevant part of Definition 6.6. Thus the set \( W^+_i \) is contained in a family whose dimension is bounded above by the maximum of \( w^+_i (\lambda, n_1) \) for all \( \lambda < \frac{k}{n} \) and \( n_1 < n \). Similarly, the set \( W^-_i \) is contained in a family whose dimension is bounded above by the maximum of \( w^-_i (\lambda, n_1) \) for all \( \lambda > \frac{k}{n} \) and \( n_1 < n \).

**Proof.** In general the coherent systems moduli spaces do not support universal objects. In order to obtain families in the strict sense of the term, it is necessary to lift back from the moduli spaces to a level (for example, that of Quot schemes) on which families can be constructed. One can then do a dimensional calculation. In fact this gives the same answer if we simply assumed that the moduli spaces support genuine families (for a similar calculation, see, for example, [BGN, Lemma 4.1]). Given this, the lemma follows at once from the definitions and Lemma 6.5. \(\square\)

Note that \( G(\alpha^+_i; n_1, d_1, k_1) = G_i(n_1, d_1, k_1) \) and \( G(\alpha^-_i; n_1, d_1, k_1) = G_{i-1}(n_1, d_1, k_1) \); the version used in the lemma appears more natural in this context.

There are clearly surjective maps

\[ W^\pm_i \twoheadrightarrow G^\pm_i. \]

The maps may fail to be injective because a coherent system in \( G^\pm_i \) may have more than one subsystem which satisfies the criteria on \((E_1, V_1)\) in Lemma 6.5. Nevertheless, we can use the dimension estimates on \( W^\pm_i \) to estimate the codimension of \( G^\pm_i \) in \( G(\alpha^\pm_i; n, d, k) \) by

\[
\begin{align*}
\text{codim } G^+_i &\geq \dim G(\alpha^+_i; n, d, k) - \max \left\{ w^+_i (\lambda, n_1) \bigg| \lambda < \frac{k}{n}, \ n_1 < n \right\} \\
\text{and} \\
\text{codim } G^-_i &\geq \dim G(\alpha^-_i; n, d, k) - \max \left\{ w^-_i (\lambda, n_1) \bigg| \lambda > \frac{k}{n}, \ n_1 < n \right\}.
\end{align*}
\]

It follows from (15) and Proposition 2.2(ii) that in our situation

\[ \mathbb{H}^0_{21} = \text{Hom}((E_2, V_2), (E_1, V_1)) = 0. \]
When $G(\alpha_i^+; n, d, k)$, $G(\alpha_i^-; n_1, d_1, k_1)$ and $G(\alpha_i^+; n_2, d_2, k_2)$ have their expected dimensions, and $H^2_{21}$ is zero for all relevant $(E_1, V_1)$ and $(E_2, V_2)$, we have

\[
\text{codim } G_i^+ \geq \beta(n, d, k) - \max \left\{ (\beta(n_1, d_1, k_1) + \beta(n_2, d_2, k_2) + C_{21} - 1) \left| \frac{k_1}{n_1} < \frac{k}{n}, \ n_1 < n \right\} \quad (16)
\]

\[
(16) = \min \left\{ C_{12} \left| \frac{k_1}{n_1} < \frac{k}{n}, \ n_1 < n \right\}
\]

by Corollary 3.7. Similarly

\[
\text{codim } G_i^- \geq \min \left\{ C_{12} \left| \frac{k_1}{n_1} > \frac{k}{n}, \ n_1 < n \right\}. \quad (17)
\]

Of course in general we have to allow for the fact that the moduli spaces may have dimensions greater than the expected ones and take into account the contribution from $H^2$ in the computations of the actual dimensions. For later use, we state a very general result and then we particularise to a result that covers the cases considered in this paper.

In general, we shall describe the process of going from $G(\alpha_i^+; n, d, k)$ to $G(\alpha_i^-; n, d, k)$ (or vice versa) as a flip, although it is not necessarily a flip in any technical sense. For all allowable values of $(\lambda, n_1)$, we denote the image of $W_i^+(\lambda, n_1)$ in $G_i^+$ by $G_i^+(\lambda, n_1)$. For any irreducible component $G$ of $G(\alpha_i^+; n, d, k)$, we shall say that the flip is $(\lambda, n_1)$-good on $G$ if $G_i^+(\lambda, n_1) \cap G$ has positive codimension in $G$. A similar definition applies to irreducible components of $G(\alpha_i^-; n, d, k)$. If a flip is $(\lambda, n_1)$-good on all irreducible components of both $G(\alpha_i^+; n, d, k)$ and $G(\alpha_i^-; n, d, k)$ and for all allowable values of $(n_1, \lambda)$, we shall call it a good flip.

**Lemma 6.8.** Let $\alpha_i$ be a critical value and suppose that

- $n_1 + n_2 = n$, $d_1 + d_2 = d$, $k_1 + k_2 = k$,
- $\frac{d_1}{n_1} + \alpha_i \frac{k_1}{n_1} = \frac{d_2}{n_2} + \alpha_i \frac{k_2}{n_2} = \frac{d}{n} + \alpha_i \frac{k}{n}$,
- $\lambda = \frac{k}{n_1} < \frac{k}{n}$.

Let $G$ be an irreducible component of $G(\alpha_i^+; n, d, k)$ of excess dimension $e \geq 0$. Let $\{S_t\}$ be a stratification of

\[
G(\alpha_i^+; n_1, d_1, k_1) \times G(\alpha_i^+; n_2, d_2, k_2)
\]

such that $\dim H^2_{21}$ is constant on each $S_t$. Write $e_1$, $e_2$ for the excess dimensions of irreducible components $G^1$ of $G(\alpha_i^+; n_1, d_1, k_1)$ and $G^2$ of $G(\alpha_i^+; n_2, d_2, k_2)$. Then the flip at $\alpha_i$ is $(\lambda, n_1)$-good if

\[
C_{12} > \dim H^2_{21} + e_1 + e_2 - e - \text{codim}_{G^1 \times G^2}(S_t \cap (G^1 \times G^2)) \quad (18)
\]

for all $G^1$, $G^2$ and all $S_t$ such that there exist extensions (13) satisfying the conditions of Lemma 6.3(ii) with $(E, V) \in G$ and $((E_1, V_1), (E_2, V_2)) \in S_t \cap (G^1 \times G^2)$.

A similar result holds for $G(\alpha_i^-; n, d, k)$ if we replace the condition $\lambda < \frac{k}{n}$ by $\lambda > \frac{k}{n}$ and Lemma 6.3(i) by Lemma 6.3(ii).
\textbf{Proof.} We need to adjust the formulae (16) and (17) by allowing for all the obstructions. For this we use (7) and recall that we have already noted that $\mathbb{H}^0_{21} = 0$. \hfill \Box

\textbf{Corollary 6.9.} Suppose that, for every allowable choice of $(n_1, d_1, k_1)$ with $\frac{k_1}{n_1} < \frac{k}{n}$, $G(\alpha_i^\pm; n_1, d_1, k_1)$ and $G(\alpha_i^\pm; n_2, d_2, k_2)$ have the expected dimensions, and that stratifications $\{S^+_i\}, \{S^-_i\}$ of

\[ G(\alpha_i^+; n_1, d_1, k_1) \times G(\alpha_i^+; n_2, d_2, k_2), \quad G(\alpha_i^-; n_1, d_1, k_1) \times G(\alpha_i^-; n_2, d_2, k_2) \]

exist such that $\dim \mathbb{H}^2_{21}$ is constant on every stratum $S^+_i$ and $\dim \mathbb{H}^2_{12}$ is constant on every stratum $S^-_i$. Suppose further that

\[ C_{12} > \dim \mathbb{H}^2_{21} - \operatorname{codim} S^+_i, \quad \text{and} \quad C_{21} > \dim \mathbb{H}^2_{12} - \operatorname{codim} S^-_i \]

for every $(n_1, d_1, k_1)$ and every stratum $S^+_i$. Then the flip at $\alpha_i$ is good.

\textbf{Proof.} The hypotheses give $e_1 = e_2 = 0$ for every choice of $G^1, G^2$. The flip is therefore $(\lambda, n_1)$-good for $\lambda < \frac{k}{n}$ by Lemma 6.8.\hfill \Box

Now note that interchanging the indices 12 changes a destabilising pattern with $\lambda = \frac{k_1}{n_1} < \frac{k}{n}$ into one with $\lambda = \frac{k_2}{n_2} > \frac{k}{n}$ and vice-versa. So the second inequality in the statement shows that the flip is good for $\lambda > \frac{k}{n}$. \hfill \Box

Of course, one needs to prove (16) only for non-empty strata. Moreover, if the extension (13) is trivial, $(E, V)$ cannot be $\alpha$-stable for any $\alpha$. So, for proving the first inequality, we may also assume that $\dim \operatorname{Ext}^1((E_2, V_2), (E_1, V_1)) > 0$, i.e. by (7)

\[ C_{21} + \dim \mathbb{H}^2_{21} > 0. \]

Similarly, for the second inequality, we may assume

\[ C_{12} + \dim \mathbb{H}^2_{12} > 0. \]

\textbf{7. Coherent systems with $k = 1$}

We want to deal with applications of the theory developed so far to the case of coherent systems with few sections and also to the case of small rank. We devote the following sections to this task.

We start by analysing the case $k = 1$ and $n \geq 2$. The moduli space of coherent systems in this case coincides with the moduli space of pairs $(E, \phi)$ which are $\alpha$-stable (see [BG1]). The particular case $n = 2, k = 1, d > 0$ has been studied thoroughly by Thaddeus [11], showing in particular that the spaces $G(\alpha; 2, d, 1)$ are irreducible and of the expected dimension $2g + d - 2$. We assume that $g \geq 2$ partly because of the complications of Remark 5.5 and partly because the proof fails for $g = 0$.

\textbf{Theorem 7.1 (BD1, BD2, C, BDW, BDGW).} Let $g \geq 2$. For $n > 1$, the moduli spaces $G_i(n, d, 1)$ are non-empty, smooth, irreducible and of the expected dimension $\beta = (n^2 - n)(g - 1) + d$. They are birationally equivalent for different values of $i$. The critical values are all of the form $\frac{s}{m} \in (0, \frac{d}{n-1})$ with $0 < m < n$ and $0 < s < d$.

\textbf{Proof.} The smoothness property follows from Proposition 3.12. Theorem 7.4 shows that the large $\alpha$ moduli space $G_L$ is irreducible and of the expected dimension. So it only remains to prove that all the moduli spaces are birationally equivalent for different
values of $\alpha$. This follows at once when we check that the flips are good. By Corollary 6.9 we need only to verify the inequalities (19) for $k_1 = 0$, $k_2 = 1$, but we do need to know that all non-empty $G(\alpha_i^\pm; n_1, d_1, k_1)$ with $n_1 < n$ and $k_1 = 0, 1$ have the expected dimensions. For $k_1 = 0$, these spaces are the full moduli spaces, for which we know the result to be true. We can therefore proceed by induction on $n$.

For the base case, we take the equivalent theorem for $n = 1$, namely that $G(1, d, 1)$ has dimension $d$. This is clear since $G(1, d, 1) = S^d X$.

We can therefore proceed to the inductive step. Note first that $H^2 V_1 = 0$ by Lemma 3.3 and $H^2_1 = 0$ since $V_1 = 0$. The critical value $\alpha_i$ is given by

$$\frac{d_1}{n_1} = \frac{d}{n} + \frac{\alpha_i}{n_2} = \frac{d_2}{n_2} + \frac{\alpha_i}{n_2},$$

i.e.

$$\alpha_i = \frac{1}{n_1} (d_1 n_2 - d_2 n_1)$$

(20)

We have by (8)

$$C_{12} = n_1 n_2 (g - 1) - d_2 n_1 + d_1 n_2 = n_1 n_2 (g - 1) + n_1 \alpha_i > 0.$$  

On the other hand

$$C_{21} = n_1 n_2 (g - 1) - d_1 n_2 + d_2 n_1 + d_1 - n_1 (g - 1).$$

Now

$$d_1 n_2 - d_2 n_1 = n_1 \alpha_i < \frac{n_1 d}{n - 1},$$

which gives $d_1 n_2 - d_2 n_1 < d_1$. So

$$C_{21} > n_1 (n_2 - 1)(g - 1) \geq 0.$$  

$\square$

8. Coherent systems with $k = 2$

We look next at the case $k = 2$.

**Theorem 8.1.** Let $X$ be a Petri curve of genus $g \geq 2$. Then we have

- For $n = 2$ the moduli spaces $G_i(2, d, 2)$ are non-empty if and only if $d > 2$. They are irreducible and of the expected dimension $2d - 3$.
- For $n > 2$ the moduli spaces $G_i(n, d, 2)$ are non-empty if and only if $d > 0$. They are always irreducible and of the expected dimension $(n^2 - 2n)(g - 1) + 2d - 3$.

**Proof.** We start by considering the moduli space $G_L$. Here the result follows from Theorem 7.4 when $n = 2$ and from Theorem 5.4 when $n > 2$.

It remains to prove that all the flips are good. Again we proceed by induction on $n$, noting that we already know that the moduli spaces for $k = 0, 1$ do have the expected dimensions. For the base case, we take the statement that the moduli spaces $G(1, d, 2)$ have the expected dimensions. This is true by section 2.3 since we are assuming that the curve is Petri. Note incidentally that these spaces are not necessarily irreducible, but irreducibility is not needed for the argument.
We now proceed to the inductive step. According to Corollary 6.9, we can restrict attention to the two cases \( k_1 = 0, k_2 = 2 \) and \( k_1 = k_2 = 1, n_1 > n_2 \). In each case we need to prove the inequalities (19).

(i) \( k_1 = 0, k_2 = 2 \). The critical value is given by

\[
\frac{d_1}{n_1} = \frac{d_2}{n_2} + \frac{2\alpha_i}{n_2},
\]

i.e.

\[
\alpha_i = \frac{1}{2n_1}(d_1n_2 - d_2n_1).
\]

So

\[
C_{12} = n_1n_2(g-1) - d_2n_1 + d_1n_2 = n_1n_2(g-1) + 2n_1\alpha_i > 0.
\]

By Proposition 3.2, \( \mathbb{H}^2_{21} = \text{Ext}^2((E_2, V_2), (E_1, 0)) = H^0(E_1^* \otimes N_2 \otimes K)^* \), where \( N_2 \) is the kernel of \( \mathcal{O}_2 \to E_2 \). If \( N_2 = 0 \) we have finished as we have already proved that \( C_{12} > 0 \). When \( N_2 \) is non-zero we have an exact sequence \( N_2 \to \mathcal{O}_2 \to L \) onto some line bundle \( L \) with at least two sections. Therefore \( \deg N_2 = -\deg L \leq -\frac{g+2}{2} \), by section 2.2, since the curve is Petri. So

\[
\deg(E_1^* \otimes N_2 \otimes K) \leq -d_1 + n_1(2g - 2 - \frac{g+2}{2}) < n_1(2g - 2).
\]

Then by Clifford’s theorem \[\text{[BCN]}\] applied to the semistable bundle \( E_1^* \otimes N_2 \otimes K \), if \( h^0(E_1^* \otimes N_2 \otimes K) > 0 \) then

\[
dim \mathbb{H}^2_{21} \leq \frac{-d_1 + n_1(2g - 2 - \frac{g+2}{2})}{2} + n_1 = -\frac{d_1}{2} + \frac{n_1}{4}(3g - 2)
\]

\[
< \frac{3}{4}n_1(g - 2) + n_1 < n_1n_2(g - 2) + n_1n_2 + 2n_1\alpha_i = C_{12}.
\]

On the other hand, \( \mathbb{H}^2_{12} = 0 \) since \( k_1 = 0 \). Therefore we only need to prove that \( C_{21} > 0 \). Now

\[
C_{21} = n_1n_2(g-1) + n_1d_2 - n_2d_1 - 2n_1(g-1) + 2d_1.
\]

If \( n_2 > 2 \) then we use the bound on the \( \alpha \)-range given by \( \alpha_i < \frac{d_2}{n_2-2} \). Hence

\[
\alpha_i < \frac{d_2}{n_2} + \frac{2\alpha_i}{n_2} \quad \text{and} \quad C_{21} = n_1(n_2-2)(g-1) + 2d_1 - 2n_1\alpha_i > 0.
\]

If \( n_2 = 2 \) then \( d_2 > 2 \) by induction hypothesis, and so \( \frac{d_2}{n_1} = \frac{d_2}{2} + \alpha_i > \alpha_i \), whence

\[
C_{21} = 2d_1 - 2n_1\alpha_i > 0.
\]

If \( n_1 = 1 \) then \( d_2 \geq \frac{g+2}{2} \) since \( E_2 \) is a line bundle with at least two sections on a Petri curve. As \( \frac{d_1}{n_1} > d_2 \), we have

\[
C_{21} = -n_1(g-1) + n_1d_2 + d_1 > 2n_1d_2 - n_1(g-1)
\]

\[
\geq n_1(g + 2 - g + 1) > 0.
\]

So in all the cases \( C_{21} > 0 \), as required.

(ii) \( k_1 = k_2 = 1, n_1 > n_2 \). The critical value is given by

\[
\frac{d_1}{n_1} + \alpha_i = \frac{d_2}{n_2} + \alpha_i = \frac{d}{n} + \frac{2}{n} \alpha_i.
\]
\[ \alpha_i = \frac{1}{n_1 - n_2}(d_1 n_2 - n_1 d_2). \]

By Lemma 3.3, we have \( H^0_{21} = 0 \) and \( H^2_{12} = 0 \). We compute

\[
\begin{align*}
C_{12} &= n_1 n_2 (g - 1) - n_1 d_2 + n_2 d_1 - n_2 (g - 1) + d_2 - 1 = \\
&= (n_1 - 1)n_2 (g - 1) + \alpha_i (n_1 - n_2) + d_2 - 1 > 0, \\
C_{21} &= n_1 n_2 (g - 1) + n_1 d_2 - n_2 d_1 - n_1 (g - 1) + d_1 - 1 = \\
&= (n_2 - 1)n_1 (g - 1) + d_1 - \alpha_i (n_1 - n_2) - 1.
\end{align*}
\]

For \( n_2 > 1 \) we use the \( \alpha \)-range condition to get \( \alpha_i < \frac{d_1}{n_1 - 1} \) and so \( d_1 - \alpha_i (n_1 - n_2) > d_1 - \frac{d_1}{n_1 - 1} (n_1 - n_2) \geq 0 \) and thus \( C_{21} > 0 \). In the case \( n_2 = 1 \), we have \( C_{21} = n_1 d_2 - 1 > 0 \).

\[ \square \]

**Remark 8.2.** In the case \( n = d = k = 2 \), \( \tilde{G}_L \) consists only of reducible coherent systems and it is irreducible and of dimension 2 by Theorem 5.6. It is easy to see that in this case there are no flips.

## 9. Coherent systems with \( n = 2 \)

Now we are going to deal with coherent systems of rank 2. Our results in this case are partial. This is due to two reasons. On the one hand our understanding of the moduli space \( G_L \) of coherent systems for large values of the parameter \( \alpha \) for \( k \geq 4 \) is very limited, in particular we do not know whether these spaces are irreducible and of the expected dimension. On the other hand we only manage to check that the flips are good for \( k \leq 4 \). We need a preliminary result on rank 1 coherent systems.

**Lemma 9.1.** Let \( X \) be a Petri curve of genus \( g \geq 2 \). Consider in \( G(1, d, k) \) the stratification given by the sets \( S_t = \{(L, V) \in G(1, d, k) \mid h^0(L) = t\} \). Then

- If \( d \leq g - 1 + k \) then the number of sections \( h^0(L) \) of a generic \((L, V) \in G(1, d, k)\) is \( k \), and \( \operatorname{codim} S_{k+j} = j(g - d - 1 + k + j) \), when non-empty.
- If \( d \geq g - 1 + k \) then the number of sections \( h^0(L) \) of a generic \((L, V) \in G(1, d, k)\) is \( p = d - g + 1 \), and \( \operatorname{codim} S_{p+j} = j(d - g + 1 - k + j) \), when non-empty.

**Proof.** Let \( p \) be the number of sections of a generic \((L, V) \in G(1, d, k)\). Then it must be \( \dim G(1, d, k) = \dim G(1, d, p) + \dim \operatorname{Gr}(k, p) \). By an easy computation it follows that either \( p = d - g + 1 \) or \( p = k \). If \( d < g - 1 + k \) then it must be \( p = k \) and \( \operatorname{codim} S_{k+j} = \dim G(1, d, k) - \dim G(1, d, k + j) - \dim \operatorname{Gr}(k, k + j) \). If \( d \geq g - 1 + k \) then \( \operatorname{codim} S_{d-g+1} = 0 \) so \( p = d - g + 1 \). The computation of \( \operatorname{codim} S_{p+j} \) is left to the reader. \[ \square \]

Now we focus on the study of \( G_i(2, d, k) \) for \( k > 0 \). The expected dimension is \( \beta(2, d, k) = (4 - 2k)g + kd - k^2 + 2k - 3 \). For \( k = 1 \) this has been treated in section 4 and for \( k = 2 \) in section 8. So we may restrict to the case \( k > 2 \). By Lemma 1.3 it must be \( d > 0 \) for stable objects to exist.

**Theorem 9.2.** Let \( X \) be a Petri curve of genus \( g \geq 2 \). Then
• For $k = 2$ the moduli spaces $G_i(2, d, 2)$ are non-empty if and only if $d > 2$. They are irreducible and of the expected dimension $\beta = 2d - 3$.

• For $k = 3$ the moduli spaces $G_i(2, d, 3)$ are non-empty if and only if $d \geq \frac{2g + 6}{3}$. They are always of the expected dimension $\beta = 3d - 2g - 6$ and irreducible when $\beta > 0$.

• For $k = 4$ the moduli spaces $G_i(2, d, 4)$ are birational to each other.

**Proof.** We start by considering the moduli space $G_L$. Here the result follows from Theorem 5.6 for $k = 2$ and from Theorem 5.11 for $k = 3$.

Let now $k = 2, 3$ or 4 and we will prove that the flips are good. By Corollary 6.3 we have to prove the inequalities (19) for $n_1 = n_2 = 1$ and $k_1 < \frac{k}{2}$, since the moduli spaces of coherent systems of type $(1, d', k')$ have the expected dimension for a Petri curve, by section 2.3. As $k \leq 4$ we have that $k_1 = 0$ or 1.

More in general, let $k \geq 2$ be an integer, and consider extensions as in (13) of the form $(L_1, V_1) \to (E, V) \to (L_2, V_2)$ where $n_1 = n_2 = 1$ and $k_1 < \frac{k}{2}$ satisfying $k_1 \leq 1$. Then we are going to prove that the inequalities (19) are satisfied. By Lemma 6.8 this implies that the flip is $(\lambda, 1)$-good on $G(\alpha_i^+; 2, d, k)$ for $\lambda = 0, 1$ and $(\lambda, 1)$-good on $G(\alpha_i^-; 2, d, k)$ for $\lambda = k, k - 1$.

The critical value $\alpha_i$ is given by

$$d_1 + k_1 \alpha_i = \frac{d}{2} + \frac{k}{2} \alpha_i = d_2 + k_2 \alpha_i,$$

i.e.

$$\alpha_i = \frac{d_1 - d_2}{k_2 - k_1}.$$

We start by proving the second inequality in (19). In this case Lemma 3.3 implies that $\mathbb{H}^2_{12} = 0$ since $k_1 \leq 1$. By Theorem 2.8 in order for coherent systems of type $(1, d_2, k_2)$ to exist we must have

$$d_1 > d_2 \geq \frac{k_2 - 1}{k_2} g + k_2 - 1. \quad (21)$$

We compute

$$C_{21} = g - 1 + d_2 - d_1 + k_2(d_1 - g + 1 - k_1)$$

$$= d_2 + (k_2 - 1)(d_1 - g + 1 - k_1) - k_1$$

$$\geq k_2 \left( \frac{k_2 - 1}{k_2} g + k_2 - 1 \right) + (k_2 - 1)(-g + 2 - k_1) - k_1$$

$$= (k_2 - k_1 - 1)k_2 + 2(k_2 - 1) \geq 2k_1 > 0.$$

Now we prove the first inequality in (19). We have

$$C_{12} = g - 1 + d_1 - d_2 + k_1(d_2 - g + 1 - k_2)$$

$$\geq g - 1 + 1 + k_1 \left( \frac{k_2 - 1}{k_2} g - g \right) = \frac{k_2 - k_1}{k_2} g > 0.$$
We stratify $G(1,d_1,k_1)$ by using the subsets defined in Lemma 9.1. Let $S_t$ be the subspace of those $(L_1, V_1) \in G(1,d_1,k_1)$ with $h^0(L_1) = t$. It only remains to check that $C_{12} > \dim \mathbb{H}^{21}_2 - \text{codim } S_t$ at the points in $S_t$.

Suppose first that $d_1 > g - 1 + k_1$. Lemma 9.1 says that the generic number of sections $h^0(L_1)$ of an element $(L_1, V_1) \in G(1,d_1,k_1)$ is $p = d_1 - g + 1$ and that $\text{codim } S_{p+t} = t(d_1 - g + 1 - k_1 + t)$. Also $h^0(L_1^* \otimes K) = t$ at a point in $S_{p+t}$. Suppose that $\dim \mathbb{H}^{21}_2 - \text{codim } S_{p+t} > 0$ since otherwise there is nothing to prove. So

\[
\dim \mathbb{H}^{21}_2 - \text{codim } S_{p+t} \leq (k_2 - 1)(t - 1) - t(d_1 - g + 1 - k_1 + t)
\]

(22)

\[
\leq t(g - d_1 - 2 + k - t)
\]

\[
\leq (k_2 - 1)(g - d_1 + k - 3),
\]

since it must be $1 \leq t \leq k_2 - 1$ for the second line to be non-negative. In the other case, $d_1 \leq g - 1 + k_1$, the generic number of sections of $L_1$ is $p = k_1$ and $\text{codim } S_{p+t} = t(g - 1 - d_1 + k_1 + t)$. Since $h^0(L_1^* \otimes K) = g - 1 - d_1 + k_1 + t$ at a point in $S_{p+t}$, we have

\[
\dim \mathbb{H}^{21}_2 - \text{codim } S_{p+t} \leq (k_2 - 1)(g - 1 - d_1 + k_1 + t - 1) - t(g - 1 - d_1 + k_1 + t)
\]

(23)

\[
\leq (k_2 - 1 - t)(g - 1 - d_1 + k_1 + t)
\]

\[
\leq (k_2 - 1)(g - d_1 + k - 3).
\]

So using either (22) or (23) it only remains to prove that

\[
g - 1 + d_1 - d_2 + k_1(d_2 - g + 1 - k_1k_2) > (k_2 - 1)(g - d_1 + k - 3).
\]

Rearranging terms this is equivalent to

\[
k_2d_1 + (k_1 - 1)d_2 > (k - 2)g + (k - 2)(k_2 - 2) + k_1k_2.
\]

Using (24) it suffices to show that

\[
k_2 + (k - 1)\left(\frac{k_2 - 1}{k_2}g + k_2 - 1\right) > (k - 2)g + (k - 2)(k_2 - 2) + k_1k_2.
\]

This holds for $k_1 = 0$ or 1 and $k_2 = k - k_1$. \hfill \Box

**Remark 9.3.** In order to have any flips, (24) imposes the condition

\[
d \geq 2\left(\frac{k_2 - 1}{k_2}g + k_2 - 1\right) + 1,
\]

for some $k_2 > \frac{k}{2}$. This implies that $d \geq 2(\frac{k}{2} + 1) + 1 = g + 3$. So when $d \leq g + 2$ there are no flips for $G(\alpha; d, k)$.

Checking whether the flips are good when $k_1 > 1$ is difficult in general. Nonetheless we have the following positive result for the case $k_1 = 2$.

**Theorem 9.4.** Let $X$ be a Petri curve of genus $g \geq 2$. Consider the moduli spaces of coherent systems of type $(2,d,k)$ with $k > 4$, and let $\alpha_i$ be a critical value corresponding to coherent subsystems with $n_1 = 1$ and $k_1 = 2$. Then the flip at $\alpha_i$ is $(\lambda = k - 2, n_1 = 1)$-good on $G(\alpha_i; 2, d, k)$. In particular, when $k = 5$ or $k = 6$, if the moduli space $G_0(2,d,k)$ is non-empty then $G_+(2,d,k)$ is non-empty also.
Proof. By Lemma 6.8 we need to check that for \( n_1 = n_2 = 1 \) and \( k_1 = 2 \) we have the inequality \( C_{21} > \dim \mathbb{H}^2_{12} - \operatorname{codim} S_t \) at the points of \( S_t \), for a suitable stratification \( \{ S_t \} \) of \( G(\alpha_1^\circ; 1, d_1, 2) \times G(\alpha_2^\circ; 1, d_2, k - 2) \). By the proof of Theorem 9.2, we already know that \( C_{21} \geq 2k_1 = 4 > 0 \).

We distinguish two cases. First suppose that \( d_2 \geq g + k - 3 \). We consider the stratification of \( G(1, d_2, k - 2) \) given by \( S_t = \{(L_2, V_2) \mid h^0(L_2) = t\} \). By Lemma 3.3 we know that \( \dim \mathbb{H}^2_{12} \leq h^0(L_2^* \otimes K) - 1 \). The generic number of sections of \( L_2 \) for an element \((L_2, V_2) \in G(1, d_2, k - 2)\) is \( p = d_2 - g + 1 \). Using Lemma 6.1 we have that for any \( t \geq 0 \), at a point in \( S_{p+t} \),

\[
\dim \mathbb{H}^2_{12} - \operatorname{codim} S_{p+t} \leq t - 1 - t(d_2 - g + 1 - k + 2 + t) \leq 0 < C_{21},
\]
as required.

The other case is \( d_2 < g + k - 3 \). Then the generic number of sections of \( L_2 \) for an element \((L_2, V_2) \in S_{k-2} \subset G(1, d_2, k - 2) \). A coherent system \((L_2, V_2) \in S_{k-2}\) is determined by its underlying line bundle \( L_2 \). Now consider the exact sequence \( N_1 \rightarrow \mathcal{O}^2 \rightarrow L_1 \), where \( N_1 \) is the kernel. Then \( N_1 \) is a line bundle of degree \(-l\), say. One clearly has \( l \leq d_1 \). Define the stratification of \( S_{k-2} \) given by the subsets

\[
T_l = \{(L_2, V_2) \in S_{k-2} \mid h^0(N_1^* \otimes L_2) = t\}.
\]

Clearly \( \dim T_l \leq \dim G(1, d_2 + l, t) \). Also we stratify \( G(1, d_1, 2) \) by the subsets \( W_i \) of those coherent systems \((L_1, V_1)\) such that the image of the map \( \mathcal{O}^2 \rightarrow L_1 \) is a line bundle of degree \( l \). Generically this map is surjective, so \( W_{d_1} \) is an open dense subset.

We start by considering the stratum \( W_{d_1} \subset G(1, d_1, 2) \). An easy calculation using that \( d_1 > d_2 \geq \frac{k-3}{k-2}g + k - 3 \) (see [21]) and \( k \geq 5 \) shows that

\[
\dim G(1, d, d - g + 4) < \dim G(1, d_2, k - 2).
\]

Therefore the generic number of sections of the line bundle \( N_1^* \otimes L_2 \), for \((L_2, V_2) \in S_{k-2}\) and \((L_1, V_1) \in W_{d_1}\), is \( p \leq d - g + 3 \). Note that in particular \( d - g + 3 \geq 0 \). At a point of \( T_i \subset S_{k-2} \) with \( t \leq d - g + 3 \) we have

\[
\dim \mathbb{H}^2_{12} = \dim H^0(L_2^* \otimes N_1 \otimes K) = g - 1 - d + t \leq 2 < C_{21}.
\]

At a point of \( T_{d-g+4+t} \) with \( t \geq 0 \) we have,

\[
\dim \mathbb{H}^2_{12} - \operatorname{codim} T_{d-g+4+t} = g - 1 - d - g + 4 + t - \operatorname{codim} T_{d-g+4+t} \\
\leq 3 + t - \dim G(1, d_2, k - 2) + \dim G(1, d, d - g + 4 + t) \\
< 3 + t - (d - g + 7 + t) \leq 3 \leq C_{21}.
\]

For the stratum \( W_{d_1-1} \) we use that \( \dim G(1, d - 1, d - g + 3) < \dim G(1, d_2, k - 2) \) to prove that the generic number of sections of \( N_1^* \otimes L_2 \), for \((L_2, V_2) \in S_{k-2}\) and \((L_1, V_1) \in W_{d_1-1}\), is \( p \leq d - g + 2 \). Working as before we get that

\[
\dim \mathbb{H}^2_{12} - \operatorname{codim} T_{d-g+2+t} \leq 2 < C_{21},
\]

for \( t \geq 0 \).
Finally consider the strata $W_l \subset G(1, d, 2)$ where $l \leq d_1 - 2$. It is easy to check that $\text{codim} W_l = d_1 - l \geq 2$. We have that $l \geq \frac{g+2}{2}$, since $N^*_1$ has two sections and $X$ is a Petri curve. Now an easy calculation shows that

$$\dim G(1, d_2 + d, d_2 + l - g + 7) < \dim G(1, d, k - 2).$$

Therefore the generic number of sections of $N^*_1 \otimes L_2$ is $p \leq d_2 + l - g + 6$. At a point $((L_1, V_1), (L_2, V_2)) \in W_l \times T_l \subset G(1, d_1, 2) \times S_{k-2}$ with $t \leq d_2 + l - g + 6$ we have

$$\dim \mathbb{H}^2_{12} - \text{codim} W_l \leq g - 1 - (d_2 + l) + t - 2 \leq 3 < C_{21}.$$  

At a point of $W_t \times T_{d_2 + l - g + 7 + t}$ with $t \geq 0$, we have

$$\dim \mathbb{H}^2_{12} - \text{codim} W_l - \text{codim} T_{d_2 + l - g + 7 + t} < 6 + t - t(d_2 + l - g + 13 + t) - 2 \leq 4 \leq C_{21},$$

concluding that in all cases the flip is $(k - 2, 1)$-good. \hfill \square

10. Coherent systems with $k = 3$

Now we shall work out the case of the moduli spaces $G_3(n, d, 3)$ of coherent systems with $k = 3$ sections and rank $n > 1$. Note that the case $n = 2$ follows from section 9. We need a preliminary result, similar in spirit to Lemma 9.1 but for the case of bundles of higher rank. This result is somewhat restricted as the only input is information on coherent systems with at most 2 sections.

**Lemma 10.1.** Let $d \leq n(g - 1)$. Stratify the moduli space $M(n, d)$ by $S_t = \{F \in M(n, d) \mid h^0(F) = t\}$. Then $2h^0(F^* \otimes K) - \text{codim} S_t \leq 2(n(g - 1) - d) + 1$ at a point in $S_t$.

**Proof.** For $F \in S_0$ we have $2h^0(F^* \otimes K) = 2(n(g - 1) - d)$. For $t = 1$ we have, by Proposition 2.5, $\dim S_1 \leq \dim G(\alpha; n, d, 1) = (n^2 - n)(g - 1) + d$, where $\alpha > 0$ is a small number. Hence $\dim S_1 \geq n^2(g - 1) + 1 - (n^2 - n)(g - 1) + d = n(g - 1) - d + 1$ and $2h^0(F^* \otimes K) - \text{codim} S_1 \leq 2(n(g - 1) - d + 1) - n(g - 1) - d + 1 = n(g - 1) - d + 1$.

For $t \geq 2$ we have that $\dim S_t + \dim \text{Gr}(2, t) \leq \dim G(\alpha; n, d, 2) = (n^2 - 2n)(g - 1) + 2d - 3$, using Theorem 8.1. So we deduce that

$$2h^0(F^* \otimes K) - \text{codim} S_t \leq$$

$$\leq 2(n(g - 1) - d + t) - n^2(g - 1) - 1 + (n^2 - 2n)(g - 1) + 2d - 3 - 2(t - 2) = 0.$$

The statement follows. \hfill \square

Now we obtain Clifford bounds type results for coherent systems. The following results are not sharp, but they are good enough for our purposes in this section. In the next two Lemmas, $X$ is any curve of genus $g \geq 2$.

**Lemma 10.2.** Suppose $(E, V)$ is an $\alpha$-semistable coherent system with $\mu(E) \geq 2g - 2$ and $h^1(E) > 0$. Then

$$h^0(E) \leq \frac{d}{2} + n + (n - 1)k\alpha.$$
Proof. We want to bound $h^0(E) = h^1(E)+d+n(1-g)$. Put $N = h^1(E) = h^0(E^* \otimes K)$. Then there are $N$ linearly independent maps $E \to K$. For any divisor $D$ on $X$ of degree $\left[ \frac{N-1}{n} \right]$ we may find a non-zero map $E \to K(-D)$. The $\alpha$-semistability implies then
\[
\frac{d}{n} + \alpha \frac{k}{n} \leq 2g - 2 - \deg D + \alpha k;
\]
\[
\left[ \frac{N-1}{n} \right] \leq 2g - 2 - \frac{d}{n} + \alpha k \frac{n-1}{n},
\]
\[
N \leq n(2g - 2) - d + \alpha k(n-1) + n,
\]
\[
h^0(E) \leq n(g - 1) + n + \alpha k(n-1) \leq \frac{d}{2} + n + \alpha k(n-1).
\]

Lemma 10.3. Let $(E, V)$ be an $\alpha$-semistable coherent system with $0 \leq \mu(E) < 2g-2$. Then
\[
h^0(E) \leq \frac{d}{2} + n + (n-1)\alpha k.
\]

Proof. For $n = 1$ the last term is dropped and the result is the usual Clifford theorem for line bundles. Also for $\alpha > 0$ very small, $E$ is a semistable bundle and the result follows by the Clifford theorem in \cite{BGN}. We also may suppose that $k > 0$. Note that the bound weakens as we increase $\alpha$, so it is enough to check what happens when we cross a critical value $\alpha_i$ to the coherent systems $(E, V)$ that are $\alpha_i$-semistable but not $\alpha_i$-semistable. Then there is a pattern
\[
0 \to (E_1, V_1) \to (E, V) \to (E_2, V_2) \to 0,
\]
with $k_1/n_1 < k/n < k_2/n_2$, $\mu_{\alpha_i}(E_1, V_1) = \mu_{\alpha_i}(E_2, V_2) = \mu_{\alpha_i}(E, V)$ and where $(E_1, V_1)$, $(E_2, V_2)$ are $\alpha_i$-semistable. Therefore $k_2 > 0$, and by Lemma 4.3 we have $d_2 \geq 0$. Hence $0 \leq d_2/n_2 < d/n < 2g-2$, and by induction,
\[
h^0(E_2) \leq \frac{d_2}{2} + n_2 + (n_2-1)k_2\alpha_i.
\]

There are three cases to consider:

- $d_1/n_1 < 2g-2$. As $d_1/n_1 > d/n \geq 0$, we apply induction to get
\[
h^0(E_1) \leq \frac{d_1}{2} + n_1 + (n_1-1)k_1\alpha_i,
\]
which together with (24) gives the result using that $h^0(E) \leq h^0(E_1) + h^0(E_2)$. Note that $(n_1-1)k_1 + (n_2-1)k_2 \leq (n-1)k$, whenever $n = n_1 + n_2$, $0 < n_1, n_2 < n$ and $k = k_1 + k_2$, $k_1, k_2 \geq 0$.

- $d_1/n_1 \geq 2g-2$ and $h^1(E_1) \neq 0$. We use Lemma 10.2 to conclude
\[
h^0(E_1) \leq \frac{d_1}{2} + n_1 + (n_1-1)k_1\alpha_i,
\]
and the result follows as in the previous case.

- $d_1/n_1 \geq 2g-2$ and $h^1(E_1) = 0$. Then $h^0(E_1) = d_1 + n_1(1-g)$. We have
\[
h^0(E_1) = \frac{d_1}{2} + \frac{n_1}{2} \left[ \frac{d}{n} + \alpha_i \left( \frac{k}{n} - \frac{k_1}{n_1} \right) \right] + n_1(1-g) <
\]
\[ < \frac{d_1}{2} + \frac{n_1}{2} (2g - 2) + n_1 (1 - g) + \alpha_i \frac{n_1 k}{2n} < \frac{d_1}{2} + n_1 + \alpha_i \frac{(n - 1)k}{2n}, \]

from which we get again the result since \( \frac{(n-1)k}{2n} + (n_2 - 1)k_2 \leq (n - 1)k. \)

\[ \square \]

**Theorem 10.4.** Let \( X \) be a Petri curve of genus \( g \geq 2 \). Then we have

- For \( n = 2 \) the moduli spaces \( G_i(2, d, 3) \) are non-empty if and only if \( d \geq \frac{2g + 6}{3} \). They are always of the expected dimension \( \beta = 3d - 2g - 6 \) and irreducible when \( \beta > 0 \).
- For \( n = 3 \) the moduli spaces \( G_i(3, d, 3) \) are non-empty if and only if \( d > 3 \). They are irreducible and of the expected dimension \( \beta = 3d - 8 \).
- For \( n > 3 \) the moduli spaces \( G_i(n, d, 3) \) are non-empty if and only if \( d > 0 \) and \( d \geq n - (n - 3)g \). They are always irreducible and of the expected dimension \( \beta = (n^2 - 3n)g + 3d - 8 \).

**Proof.** The case \( n = 2 \) follows from Theorem 9.2, so we may restrict to the case \( n \geq 3 \). The moduli space \( G_L \) for the largest possible values of the parameter satisfies the statement of the Theorem, using Theorem 5.0 for the case \( k = n = 3 \) and Theorem 5.4 for the case \( n > k = 3 \).

It remains to check that the flips are good. We proceed by induction on \( n \), noting that we already know that the moduli spaces for \( k = 0, 1, 2 \) have the expected dimensions for a Petri curve. According to Corollary 8.9, we have two cases: \( k_1 = 0, k_2 = 3 \) and \( k_1 = 1, k_2 = 2 \).

(i) \( k_1 = 0, k_2 = 3 \). The critical value \( \alpha_i \) is given by

\[ \frac{d_1}{n_1} = \frac{d_2}{n_2} + \frac{3}{n_2} \alpha_i = \frac{d}{n} + \frac{3}{n} \alpha_i. \]

i.e.

\[ \alpha_i = \frac{d_1 n_2 - d_2 n_1}{3n_1}. \]

We start by proving the first inequality in (13). We have

\[ C_{12} = n_1 n_2 (g - 1) - n_1 d_2 + n_2 d_1 = n_1 n_2 (g - 1) + 3n_1 \alpha_i > 0. \]

Now Lemma 5.3 implies \( \dim \mathbb{H}^3_{21} \leq 2(h^0(E_1^* \otimes K) - 1) \) or else \( \mathbb{H}^2_{21} = 0 \). There are two cases:

(a) If \( d_1 \leq n_1 (g - 1) \) then we use Lemma 10.1. Define the stratification given by

\[ S_i = \{ E_1 \in M(n_1, d_1) \mid h^0(E_1) = i \}. \]

Then

\[ 2h^0(E_1^* \otimes K) - \text{codim} S_i \leq 2(n_1 (g - 1) - d_1) + 1. \]

Hence \( C_{12} > \dim \mathbb{H}^3_{21} - \text{codim} S_i \) is implied by

\[ n_1 (n_2 - 2) (g - 1) + 2d_1 + 3n_1 \alpha_i > -1. \]

For \( n_2 \geq 2 \) this obviously holds. For \( n_2 = 1 \) we have that

\[ -n_1 (g - 1) + 2d_1 + 3n_1 \alpha_i = -n_1 (g - 1) + 2d_1 + d_1 - n_1 d_2 = \]

\[ = 3d_1 - n_1 (g - 1 + d_2) \geq n_1 (2d_2 - g + 1) > -1, \]
using that $\frac{d_1}{n_1} > d_2 \geq \frac{2g+6}{3}$, the last inequality being necessary for the existence of coherent systems of type $(1, d_2, 3)$ on a Petri curve.

(b) If $d_1 > n_1(g - 1)$ then we use Clifford theorem for the stable bundle $E_1^* \otimes K$. So either $h^0(E_1^* \otimes K) = 0$ in which case there is nothing to prove, or

$$h^0(E_1^* \otimes K) \leq \frac{n_1(2g - 2) - d_1}{2} + n_1,$$

whence $\dim \mathbb{H}_{21}^2 \leq 2n_1g - d_1 - 2$. The inequality $C_{12} > 2n_1g - d_1 - 2$ is equivalent to

$$n_1(n_2 - 2)(g - 2) + n_1(n_2 - 4) + d_1 + 3n_1\alpha_i > -2.$$

For $n_2 \geq 4$ this is obviously true. For $n_2 = 3$ it must be $d_2 > 3$ by induction hypothesis, so $\frac{d_1}{n_1} > 1 + \alpha_i$ and $d_1 - n_1 > 0$, which yields the result. For $n_2 = 2$ we have $\frac{d_1}{n_1} > \frac{d_2}{2} \geq 2$ as $d_2 \geq \frac{2g+6}{3}$, by induction hypothesis. So $-2n_1 + d_1 > 0$ and we are done. For $n_2 = 1$ and $g \leq 5$ we have that $\frac{d_1}{n_1} > d_2 \geq \frac{2g+6}{3}$ implies $\frac{d_1}{n_1} > d_2 \geq g + 1$ and hence

$$-n_1(g - 2) - 3n_1 + d_1 + 3n_1\alpha_i > d_1 - n_1(g + 1) > -2,$$

as required. The same argument covers the case $n_2 = 1$ and $d_1 \geq n_1(g + 1)$. Finally the case $n_2 = 1$, $n_1(g - 1) < d_1 < n_1(g + 1)$ and $g \geq 6$ requires a special treatment. We use the improvement of Clifford theorem given in [M4, Theorem 1]. Since $2 + \frac{2}{g-4} \leq g - 3 < 2g - 2 - \frac{d_1}{n_1} < g - 1$ and the curve is Petri, we have

$$h^0(E_1^* \otimes K) \leq \frac{n_1(2g - 2) - d_1}{2},$$

which gives $\dim \mathbb{H}_{21}^2 \leq 2n_1(g - 1) - d_1 - 2$, and hence

$$C_{12} = n_1(g - 1) + 3n_1\alpha_i > \dim \mathbb{H}_{21}^2.$$

Now we pass on to prove the second inequality in (19). In this case $\mathbb{H}_{12}^2 = 0$. We compute

$$C_{21} = n_1n_2(g - 1) - 3n_1\alpha_i + 3(d_1 - n_1(g - 1)) = n_1(n_2 - 3)(g - 1) + 3d_1 - 3n_1\alpha_i.$$

We have the following cases:

(a) If $n_2 > 3$ then $\alpha_i < \frac{d_2}{n_2-3}$. Computing we obtain that $\alpha_i < \frac{d_2}{n_2} + \frac{3}{n_2} \alpha_i = \frac{d_1}{n_1}$ and thus $C_{21} > 0$.

(b) If $n_2 = 3$ then $C_{12} = 3d_1 - 3n_1\alpha_i = d_2n_1 > 0$.

(c) If $n_2 = 2$ then $\frac{d_1}{n_1} > \frac{d_2}{2}$. As $d_2 \geq \frac{2g+6}{3}$ by induction hypothesis, we have

$$C_{21} = -n_1(g - 1) + 3d_1 - 2d_1 + d_2n_1 = n_1(d_2 - g + 1) + d_1$$

$$> n_1 \left( \frac{3}{2}d_2 - g + 1 \right) \geq n_1(g + 3 - g + 1) > 0.$$
(d) If \( n_2 = 1 \) then \( d_2 \geq \frac{2g+6}{3} \) in order to have stable coherent systems of type (1, \( d_2, 3 \)) on a Petri curve. Also \( \frac{d_1}{n_1} > d_2 \), so

\[
C_{21} = -2n_1(g - 1) + 3d_1 - d_1 + d_2n_1 = n_1(d_2 - 2g + 2) + 2d_1 > n_1(3d_2 - 2g + 2) \geq n_1(2g + 6 - 2g + 2) > 0.
\]

(ii) \( k_1 = 1, k_2 = 2 \). The critical value is given by

\[
\frac{d_1}{n_1} + \frac{1}{n_1} \alpha_i = \frac{d_2}{n_2} + \frac{2}{n_2} \alpha_i = \frac{d}{n} + \frac{3}{n} \alpha_i,
\]

i.e.

\[
\alpha_i = \frac{d_2n_1 - d_1n_2}{n_2 - 2n_1}.
\]

It must be \( n_2 - 2n_1 \neq 0 \). We start proving the second inequality in (19). We have \( \mathbb{H}^2_{l_2} = 0 \) and

\[
C_{21} = n_1n_2(g - 1) + n_1d_2 - n_2d_1 + 2(d_1 - n_1(g - 1) - 1).
\]

We have the following cases:

(a) \( n_2 - 2n_1 > 0 \). Then \( C_{21} = n_1(n_2 - 2)(g - 1) + \alpha_i(n_2 - 2n_1) + 2d_1 - 2 > 0 \) since \( d_1 > 0 \) and \( n_2 > 2 \).

(b) \( n_2 - 2n_1 < 0 \) and \( n_2 \geq 2 \). Then \( \alpha_i = \frac{d_1}{n_1} < \frac{d_1}{n_1 - 1} \) implies that \( \alpha_i(2n_1 - n_2) < 2d_1 - d_2 \). So

\[
C_{21} = n_1(n_2 - 2)(g - 1) - \alpha_i(2n_1 - n_2) + 2d_1 - 2 > n_1(n_2 - 2)(g - 1) + d_2 - 2 \geq 0.
\]

(Recall that in the particular case \( n_2 = 2 \) we must have \( d_2 > 2 \).)

(c) \( n_2 - 2n_1 < 0 \) and \( n_2 = 1 \). Using that \( \frac{d_1}{n_1} > d_2 \) and \( d_2 \geq \frac{g+2}{2} \), we have

\[
C_{21} = n_1(d_2 - g + 1) + d_1 - 2 > n_1(2d_2 - g + 1) - 2 \geq n_1(g + 2 - g + 1) - 2 = 3n_1 - 2 > 0.
\]

Now we pass on to prove the first inequality in (19). We have

\[
C_{12} = n_1n_2(g - 1) - n_1d_2 + n_2d_1 + (d_2 - n_2(g - 1) - 2).
\]

On the other hand, either \( \mathbb{H}^2_{l_1} = 0 \) or else Lemma [3.3] and Lemma [10.2] or Lemma [10.3] imply that

\[
\dim \mathbb{H}^2_{l_1} \leq h^0(E^*_1 \otimes K) - 1 \leq n_1g - \frac{d_1}{2} + (n_1 - 1)\alpha_i - 1.
\]

We have the following cases:

- \( 2n_1 - n_2 > 0 \). As we are supposing \( n \geq 3 \) it follows that \( n_1 > 1 \). Then

\[
C_{12} = (n_1 - 1)n_2(g - 1) + \alpha_i(2n_1 - n_2) + d_2 - 2 \geq 1 + 1 + 1 - 2 > 0.
\]

We need to prove that \( C_{12} > \dim \mathbb{H}^2_{l_1} \) using (25).

(a) \( n_2 = 1, n_1 \geq 2 \). Then \( C_{12} > n_1g - \frac{d_1}{2} + (n_1 - 1)\alpha_i - 1 \) is equivalent to \( n_1\alpha_i + d_2 + \frac{d_1}{2} > n_1 + g \). In order for coherent systems of type (1, \( d_2, 2 \)) to exist on a Petri curve, it is necessary that \( d_2 \geq \frac{g+2}{2} \). Also

\[
d_1 > n_1d_2 \geq n_1\frac{g + 2}{2} \geq g + 2n_1 - 2.
\]
Easily we get the result.

(b) \( n_2 = 2, \ n_1 \geq 2 \). Then \( C_{12} > n_1 g - \frac{d_1}{2} + (n_1 - 1)\alpha_i - 1 \) is equivalent to

\[
(n_1 - 2)(g - 2) + \alpha_i(n_1 - 1) + d_2 + \frac{d_1}{2} > 3.
\]

This holds since \( d_2 \geq 3 \), for a stable coherent system of type \((2, d_2, 2)\) to exist.

(c) \( n_2 > 2, \ n_1 \geq 2 \). Then generically the map \( \mathcal{O}^2 \to E_2 \) has no kernel, for \((E_2, V_2) \in G_i(n_2, d_2, 2)\). This happens since in \( G_i(n_2, d_2, 2) \) all coherent systems have this property by Proposition 5.2, and because all \( G_i(n_2, d_2, 2) \) are birational to each other, by induction hypothesis. Therefore the subset \( S \subset G_i(n_2, d_2, 2) \) of those coherent systems \((E_2, V_2)\) such that \( \mathcal{O}^2 \to E_2 \) is not injective is of positive codimension. For \((E_2, V_2) \not\in S\) we have \( \mathbb{H}^2_{21} = 0 \) by (I). So it is enough to prove \( C_{12} > \dim \mathbb{H}^2_{21} - 1 \), i.e.

\[
(n_1 - 2)(g - 2) + (n_2 - 2)(n_1 - 1)(g - 1) + (d_2 - \alpha_i(n_2 - 2)) + \alpha_i(n_1 - 1) + \frac{d_1}{2} > 2.
\]

This holds clearly. The only case to be considered separately is \( g = 2, \ n_1 = 2, \ n_2 = 3, \ d_1 = 1 \). But in this case \( \alpha_i \in \mathbb{Z} \), hence \( \alpha_i \geq 1 \) and the result follows easily.

• \( 2n_1 - n_2 < 0 \). Then

\[
C_{12} = (n_1 - 1)n_2(g - 1) - \alpha_i(n_2 - 2n_1) + d_2 - 2.
\]

Now \( \alpha_i = \frac{d_2n_1 - d_1n_2}{n_2 - 2n_1} < \frac{d_2}{n_2 - 2} \) gives \( \alpha_i(n_2 - 2n_1) < d_2 - 2d_1 \). Thus

\[
C_{12} > (n_1 - 1)n_2(g - 1) + 2d_1 - 2 \geq 0.
\]

We have the following cases:

(a) \( n_1 \geq 2 \). We use the bound (25). Then \( C_{12} > n_1 g - \frac{d_1}{2} + (n_1 - 1)\alpha_i - 1 \) is equivalent to

\[
(n_1 - 2)(g - 2) + (n_2 - 2)(n_1 - 1)(g - 1) + (d_2 - \alpha_i(n_2 - 2n_1)) - \alpha_i(n_1 - 1) + \frac{d_1}{2} > 3.
\]

Use that \( \alpha_i(n_1 - 1) < d_1 \) and \( d_2 - \alpha_i(n_2 - 2n_1) > 2d_1 \) to get that the left hand side is bigger or equal than \( 3 + 2d_1 - d_1 + \frac{d_1}{2} > 3 \).

(b) \( n_1 = 1, \ n_2 > 2 \). By Proposition 3.2, \( \mathbb{H}^2_{21} = H^0(E_1^* \otimes N_2 \otimes K)^* \), where \( N_2 \to \mathcal{O}^2 \to E_2 \). Hence if \( N_2 = 0 \) then \( \mathbb{H}^2_{21} = 0 \) and we have finished. So we may suppose that \( N_2 \not\equiv 0 \). By (25) it is enough to prove that \( C_{12} = d_1 n_2 - 2 > g - \frac{d_1}{2} - 1 \). Let \( L \) be the image of \( \mathcal{O}^2 \to E_2 \), which is a line bundle of degree \( l \geq \frac{g+2}{2} \). We may write an inclusion of coherent systems \((L, V_2) \subset (E_2, V_2)\). By \( \alpha_i \)-semistability, \( l + 2\alpha_i \leq \frac{d_2 + 2\alpha_i}{n_2} = d_1 + \alpha_i \). So \( d_1 \geq \alpha_i + \frac{g+2}{2} \) and then

\[
d_1 n_2 - 2 > n_2 g + 2 - 2 > g - \frac{d_1}{2} - 1.
\]

\[\square\]
11. Applications of coherent systems to Brill-Noether theory

In this section, we shall describe in more detail the relationship between $G_0(n,d,k)$ and $B(n,d,k)$ introduced in section 2.3, and give some applications of our results to Brill-Noether theory. Although a good deal is known about non-emptiness of Brill-Noether loci, even quite simple geometrical properties (for example, irreducibility) have been established only in a very few cases. The results given here begin to fill these gaps in our knowledge, and should be regarded as a sample of what is possible. We plan to return to these questions in future papers and obtain more extensive and comprehensive results.

Although many of the proofs are valid for all $g$, one may as well assume in this section that $g \geq 2$, since Brill-Noether theory itself is trivial for $g = 0, 1$.

11.1. General remarks.

Lemma 11.1. If $\beta(n,d,k) \geq n^2(g - 1) + 1$, then $B(n,d,k) = M(n,d)$.

Proof. By (1),

\[ \beta(n,d,k) \geq n^2(g - 1) + 1 \iff d - n(g - 1) \geq k. \]

(26)

When these equivalent conditions hold, it follows from the Riemann-Roch Theorem that, for any $E \in M(n,d)$,

\[ \dim H^0(E) \geq k. \]

So $B(n,d,k) = M(n,d)$. □

On the other hand, we have

Lemma 11.2. If $\beta(n,d,k) \leq n^2(g - 1)$, then every irreducible component $B$ of $B(n,d,k)$ contains a point outside $B(n,d,k + 1)$.

Proof. (This is [Lau, Lemma 2.6]; for the convenience of the reader, we include a proof.) The content of the statement is that there exists $E \in B$ such that $\dim H^0(E) = k$. To see this, note first that, if $\dim H^0(E') \geq 1$ and $P$ is a point of $X$ such that the sections of $E'$ generate a non-zero subspace of the fibre $E'_P$, we can find an extension

\[ 0 \to F \to E' \to O_P \to 0 \]

such that the map $H^0(E') \to O_P$ is non-zero and hence $\dim H^0(F) = \dim H^0(E') - 1$.

Now let $E'$ be a point of $B$ not contained in any other irreducible component of $B(n,d,k)$ and suppose that $H^0(E') = k + r$ with $r \geq 1$. By iterating the above construction, we can find points $P_1, \ldots, P_r$ of $X$ and an exact sequence

\[ 0 \to F \to E' \to O_{P_1} \oplus \ldots \oplus O_{P_r} \to 0 \]

such that $\dim H^0(F) = \dim H^0(E') - r = k$. Now consider the extensions

\[ 0 \to F \to E \to O_{Q_1} \oplus \ldots \oplus O_{Q_r} \to 0, \]

(27)
where \( Q_1, \ldots, Q_r \in X \). These form an irreducible family of bundles with \( \dim H^0(E) \geq k \), whose generic member is stable (since \( E' \) is stable). It follows that the generic extension (27) belongs to \( B \). Moreover, by the Riemann-Roch Theorem and (27),

\[
\dim H^1(F) = \dim H^0(F) - (d - r) + n(g - 1) > k - k + r = r.
\]

By considering the dual sequence

\[
0 \to E^* \otimes K \to F^* \otimes K \to O_{Q_1} \oplus \ldots \oplus O_{Q_r} \to 0,
\]

in which \( \dim H^0(F^* \otimes K) > r \), we can choose \( Q_1, \ldots, Q_r \) and \( E \) so that

\[
\dim H^0(E^* \otimes K) = \dim H^0(F^* \otimes K) - r;
\]

hence (again by Riemann-Roch)

\[
\dim H^0(E) = \dim H^0(F).
\]

It follows that the generic extension (27) satisfies \( \dim H^0(E) = k \). Since we already know that \( E \in B \), this completes the proof. \( \square \)

As envisaged at the end of section 2.3, we introduce

**Conditions 11.3.**

- \( \beta(n, d, k) \leq n^2(g - 1) \),
- \( G_0(n, d, k) \) is irreducible,
- \( B(n, d, k) \neq \emptyset \).

For the moment we do not assume that \( \gcd(n, d, k) = 1 \) or that \( G_0(n, d, k) \) is smooth. We denote by

\[
\psi : G_0(n, d, k) \to \tilde{B}(n, d, k)
\]

the map given by assigning to every \( (E, V) \in G_0(n, d, k) \) the underlying bundle \( E \) (see (3)).

**Theorem 11.4.** Suppose Conditions 11.3 hold. Then

(i) \( B(n, d, k) \) is irreducible,

(ii) \( \psi \) is one-to-one over \( B(n, d, k) - B(n, d, k + 1) \),

(iii) \( \dim B(n, d, k) = \dim G_0(n, d, k) \),

(iv) for any \( E \in B(n, d, k) - B(n, d, k + 1) \), the linear map

\[
\text{d} \psi : T_{(E, H^0(E))} G_0(n, d, k) \longrightarrow T_E B(n, d, k)
\]

of Zariski tangent spaces is an isomorphism.

**Proof.** (i) If \( E \in B(n, d, k) \), then \( (E, V) \in G_0(n, d, k) \) for any \( k \)-dimensional subspace of \( H^0(E) \). It follows that the image of \( \psi \) contains \( B(n, d, k) \) as a non-empty Zariski-open subset. Since \( G_0(n, d, k) \) is irreducible, it follows that \( B(n, d, k) \) is irreducible.

(ii) If \( E \in B(n, d, k) - B(n, d, k + 1) \), then \( \psi^{-1}(E) = \{(E, H^0(E))\} \).

(iii) follows from (i), (ii) and Lemma 11.2.

(iv) Taking \( (E', V') = (E, V) \) in (3) and putting \( V = H^0(E) \), we get a map

\[
\text{Ext}^1((E, H^0(V)), (E, H^0(V))) \to \text{Ext}^1(E, E)
\]
which can be identified with the map

\[ T_{(E,H^0(E))} G_{0}(n,d,k) \rightarrow T_{E}M(n,d) \]

induced by \( \psi \). By (6) this map is injective and its image is

\[ \text{Ker}(\text{Ext}^1(E,E) \rightarrow \text{Hom}(H^0(E),H^1(E))). \]

By standard Brill-Noether theory, this image becomes identified with the subspace \( T_{E}B(n,d,k) \) of \( T_{E}M(n,d) \).

\[ \blacksquare \]

Corollary 11.5. Suppose Conditions 11.3 hold and \( G_{0}(n,d,k) \) is smooth. Then \( \psi \) is an isomorphism over \( B(n,d,k) - B(n,d,k+1) \). Moreover, if \( \text{GCD}(n,d,k) = 1 \), then \( G_{0}(n,d,k) \) is a desingularisation of the closure \( \overline{B(n,d,k)} \) of \( B(n,d,k) \) in the projective variety \( M(n,d) \).

Proof. The first part follows from (ii) and (iv). For the second part, recall that, when \( \text{GCD}(n,d,k) = 1 \), \( G_{0}(n,d,k) \) is projective; hence the image of \( \psi \) is precisely \( \overline{B(n,d,k)} \).

\[ \blacksquare \]

Corollary 11.6. Suppose Conditions 11.3 hold, \( G_{0}(n,d,k) \) is smooth and \( (n,d) = 1 \). Then \( B(n,d,k) \) is projective and \( G_{0}(n,d,k) \) is a desingularisation of \( B(n,d,k) \).

Proof. In this case \( M(n,d) = \widetilde{M}(n,d) \).

\[ \blacksquare \]

11.2. Irreducibility and dimension of Brill-Noether loci. In many cases our methods yield information about the irreducibility and dimension of \( B(n,d,k) \), and more precisely about its birational structure. We illustrate this with results for \( k = 1,2,3 \), where we have good estimates for the codimensions of the flips. The main respect in which our results improve those previously known is that they impose no restriction on \( d \) other than that required for the Brill-Noether locus to be non-empty and not equal to \( M(n,d) \).

We begin with \( k = 1 \).

Theorem 11.7. Suppose \( 0 < d \leq n(g - 1) \). Then

(i) \( G_{0}(n,d,1) \) is a desingularisation of \( \overline{B(n,d,1)} \),
(ii) \( B(n,d,1) \) is irreducible of dimension \( \beta(n,d,1) \), smooth outside \( B(n,d,2) \),
(iii) \( B(n,d,1) \) is birationally equivalent to a fibration over \( M(n-1,d) \) with fibre \( \mathbb{P}^{d+(n-1)(g-1)-1} \),
(iv) if \( (n-1,d) = 1 \), \( B(n,d,1) \) is birationally equivalent to

\[ M(n-1,d) \times \mathbb{P}^{d+(n-1)(g-1)-1}. \]

Remark 11.8. In the case \( d = n(g - 1) \), a stronger form of (i) is proved in [RV]. Part (ii) is proved in [Su]. Parts (iii) and (iv) are implicit in [Su]. We have chosen to prove the complete theorem to illustrate our methods.

Proof. We first check Conditions 11.3. The first follows at once from (26), the second from Theorem 7.1 and the third is elementary and well known (see for example [Su]). Moreover \( G_{0}(n,d,1) \) is smooth of dimension \( \beta(n,d,1) \) by Theorem 7.1 (or Proposition 3.12).
Parts (i) and (ii) now follow from Theorem 11.4 and Corollary 11.5. Part (iii) follows from Theorem 5.4 and Theorem 7.1, as does part (iv) if we note that in this case the existence of a universal bundle over \( X \times M(n - 1, d) \) implies that the fibration of Theorem 7.4 is locally trivial in the Zariski topology.

For \( k = 2, 3 \), we need a lemma.

**Lemma 11.9.** Suppose \( k = 2 \) or \( 3 \), \( n \geq 2 \) and that \( X \) is a Petri curve of genus \( g \geq 2 \). Then \( B(n, d, k) \) is non-empty precisely in the following cases:

(a) \( k = 2, n = 2, d \geq 3 \),
(b) \( k = 2, n \geq 3, d \geq 1 \),
(c) \( k = 3, n = 2, d \geq \frac{2g+6}{3} \),
(d) \( k = 3, n = 3, d \geq 4 \),
(e) \( k = 3, n = 4, \) either \( g = 2 \) and \( d \geq 2 \) or \( g \geq 3 \) and \( d \geq 1 \),
(f) \( k = 3, n \geq 5, d \geq 1 \).

**Remark 11.10.** The Petri condition is required only for case (c).

**Proof.** All parts except (c) follow from [BGN] and either [T1] or [M2]. For (c), see [Bu2] or [Tan]. \( \Box \)

**Theorem 11.11.** Let \( X \) be a Petri curve of genus \( g \geq 2 \), \( k = 2 \) or \( 3 \), \( d < n(g-1)+k \). Suppose further that one of the conditions of Lemma 11.3 holds. Then

(i) \( B(n, d, k) \) is irreducible of dimension \( \beta(n, d, k) \).

If in addition \( k < n \) (i.e. in cases (b), (e), (f) of Lemma 11.3), then

(ii) \( B(n, d, k) \) is birationally equivalent to a fibration over \( M(n-k, d) \) with fibre \( \text{Gr}(k, d+(n-k)(g-1)) \);

(iii) if \( (n-k, d) = 1 \), \( B(n, d, k) \) is birationally equivalent to \( M(n-k, d) \times \text{Gr}(k, d+(n-k)(g-1)) \).

**Proof.** (i) Note that the conditions of Lemma 11.3 for the non-emptiness of \( B(n, d, k) \) are exactly the same as those of Theorems 8.1 and 10.4 for the non-emptiness of \( G_0(n, d, k) \). Conditions 11.3 follow from (26) and Theorems 8.1 and 10.4. The result now follows from Theorem 11.4.

(ii) and (iii) follow from Theorem 5.4 in the same way as the corresponding parts of Theorem 11.4. \( \Box \)

**Remark 11.12.** When \( k < n \) (cases (b), (e), (f)), the irreducibility of \( B(n, d, k) \) for \( d < \min\{2n, n+g\} \) and the fact that \( B(n, d, k) \) has the expected dimension for \( d \leq 2n \) have been proved previously [BGN, M1]. For \( k \leq n \) (all cases except (c)), it was proved in [T1] that \( B(n, d, k) \) has a component of the expected dimension. Parts (ii) and (iii) are known for \( d < \min\{2n, n+g\} \) [M1].

11.3. Picard group. Our methods become potentially even more useful in computing cohomological information about Brill-Noether loci. In general, the calculations will be complicated and we restrict attention here to computing the Picard group in the case \( k = 1 \).
**Theorem 11.13.** Let $X$ be a Petri curve of genus $g \geq 2$. Suppose $0 < d \leq n(g - 1)$, $n \geq 3$, $(n - 1, d) = 1$ and $(n, d) = 1$. Then

$$\text{Pic}(B(n, d, 1) - B(n, d, 2)) \cong \text{Pic}(M(n - 1, d)) \times \mathbb{Z}.$$  

**Proof.** Note first that, by Theorem 5.4, $G_L(n, d, 1)$ is a projective bundle over $M(n - 1, d)$, so

$$\text{Pic}(G_L(n, d, 1)) = \text{Pic}(M(n - 1, d)) \times \mathbb{Z}.$$  

From the proof of Theorem 7.1, we see that the codimensions $C_{12}$, $C_{21}$ are both at least 2 (we need $n \geq 3$ here since otherwise we could have $n_1 = n_2 = 1$, $d_2 = 1$, giving $C_{21} = 1$). Hence

$$\text{Pic}(G_0(n, d, 1)) = \text{Pic}(M(n - 1, d)) \times \mathbb{Z}.$$  

To complete the proof, we need to show that $\psi^{-1}(B(n, d, 2))$ has codimension at least 2 in $G_0(n, d, 1)$. Now the fibre of $\psi$ over a point of $B(n, d, k) - B(n, d, k + 1)$ is a projective space of dimension $k - 1$. It is therefore sufficient to prove that $B(n, d, k)$ has codimension at least $k + 1$ in $B(n, d, 1)$ for all $k \geq 2$. In view of Lemma 11.2, it is enough to prove this for $k = 2$, i.e. to prove

$$\text{codim}_{B(n, d, 1)} B(n, d, 2) \geq 3.$$  

For this, note that

$$\beta(n, d, 2) = \beta(n, d, 1) - n(g - 1) + d - 3$$

$$\leq \beta(n, d, 1) - 3$$  

since $d \leq n(g - 1)$. The result now follows from Theorem 11.11.

**Remark 11.14.** Note that we need Theorem 11.11 here to show that $B(n, d, 2)$ always has the expected dimension. This is the only point in the proof where the Petri condition is used. It may be that this condition is not essential.

**References**

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Vol. 1, Springer-Verlag, New York, 1985.

[At] M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc. (3)* 7 (1957) 414–452.

[Be] A. Bertram, Stable pairs and stable parabolic pairs, *J. Alg. Geom.* 3 (1994) 703–724.

[BeDW] A. Bertram, G. Daskalopoulos and R. Wentworth, Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians, *J. Amer. Math. Soc.* 9 (1996) 529–571.

[BeF] A. Bertram and B. Feinberg, On stable rank 2 bundles with canonical determinant and many sections, *Algebraic Geometry: papers presented for the EUROPROJ conferences in Catania and Barcelona*, ed. P. E. Newstead, Lecture Notes in Pure and Applied Mathematics 200, 259–269, Marcel Dekker, New York, 1998.

[B] S.B. Bradlow, Special metrics and stability for holomorphic bundles with global sections, *J. Diff. Geom.* 33 (1991) 169–214.

[BD1] S.B. Bradlow and G. Daskalopoulos, Moduli of stable pairs for holomorphic bundles over Riemann surfaces, *Internat. J. Math.* 2 (1991) 477–513.

[BD2] S.B. Bradlow and G. Daskalopoulos, Moduli of stable pairs for holomorphic bundles over Riemann surfaces II, *Internat. J. Math.* 4 (1993) 903–925.

[BDW] S.B. Bradlow, G. Daskalopoulos and R. Wentworth, Birational equivalences of vortex moduli, *Topology*, 35 (1996) 731–748.
[BDGW] S.B. Bradlow, G. Daskalopoulos, O. García-Prada and R. Wentworth, Stable augmented bundles over Riemann surfaces. *Vector Bundles in Algebraic Geometry, Durham 1993*, ed. N.J. Hitchin, P.E. Newstead and W.M. Oxbury, LMS Lecture Notes Series 208, 15–67, Cambridge University Press, 1995.

[BG1] S.B. Bradlow and O. García-Prada, A Hitchin-Kobayashi correspondence for coherent systems on Riemann surfaces, *J. London Math. Soc. (2) 60* (1999) 155–170.

[BG2] S.B. Bradlow and O. García-Prada, An application of coherent systems to a Brill-Noether problem, *J. Reine Angew. Math.*, to appear.

[BGG] S.B. Bradlow, O. García-Prada and P. Gothen, Representations of the fundamental group of a surface in $PU(p,q)$ and holomorphic triples, *C. R. Acad. Sci. Paris, 333*, Série I (2001), 347–352.

[BGN] L. Brambila-Paz, I. Grzegorczyk and P. Newstead, Geography of Brill-Noether loci for small slope. *J. Alg. Geom. 6* (1997) 645–669.

[BMNO] L. Brambila-Paz, V. Mercat, P. Newstead and F. Ongay, Nonemptiness of Brill-Noether loci, *Internat. J. Math. 11* (2000) 737–760.

[Bu1] D. Butler, Normal generation of vector bundles over a curve, *J. Diff. Geom. 39* (1994) 1–34.

[Bu2] D. Butler, Birational maps of moduli of Brill-Noether pairs, preprint.

[EL] L. Ein and R.K. Lazarsfeld, Stability and restrictions of Picard bundles with an application to the normal bundles of elliptic curves, *Complex Projective Geometry*, ed. G. Ellingsrud, C. Peskine, G. Sacchiero and S. A. Stromme, LMS Lecture Notes Series 179, Cambridge University Press, 1992.

[G] O. García-Prada, Dimensional reduction of stable bundles, vortices and stable pairs, *Internat. J. Math. 5* (1994) 1–52.

[He] M. He, Espaces de modules de systèmes cohérents, *Internat. J. Math. 9* (1998) 545–598.

[HL1] D. Huybrechts and M. Lehn, Stable pairs on curves and surfaces, *J. Alg. Geom. 4* (1995) 67–104.

[HL2] D. Huybrechts and M. Lehn, Framed modules and their moduli, *Internat. J. Math. 6* (1995) 297–324.

[KN] A. King and P. Newstead, Moduli of Brill-Noether pairs on algebraic curves, *Internat. J. Math. 6* (1995) 733–748.

[Lau] G. Laumon, Fibrés vectoriels spéciaux, *Bull. Soc. Math. France 119* (1991) 97–119.

[LeP] J. Le Potier, Faisceaux semi-stables et systèmes cohérents, *Vector Bundles in Algebraic Geometry, Durham 1993*, ed. N.J. Hitchin, P.E. Newstead and W.M. Oxbury, LMS Lecture Notes Series 208, 179–239, Cambridge University Press, 1995.

[M1] V. Mercat, Le problème de Brill-Noether pour des fibrés stables de petite pente, *J. reine angew. Math. 56* (1999) 1–41.

[M2] V. Mercat, Le problème de Brill-Noether et le théorème de Teixidor, *Manuscripta Math. 98* (1999) 75–85.

[M3] V. Mercat, Le problème de Brill-Noether: Présentation, http://www.bnt.math.jussieu.fr/

[M4] V. Mercat, Clifford’s Theorem and higher rank vector bundles, *Internat. J. Math.*, to appear.

[Mu] S. Mukai, Vector bundles and Brill-Noether theory, *Current Topics in Algebraic Geometry*, Math. Sci. res. Inst. Publ. 28, 145–158, Cambridge University Press, 1995.

[PR] K. Paranjape and S. Ramanan, On the canonical ring of a curve, *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata* (1987) 503–516.

[RV] N. Raghavendra and P.A. Vishwanath, Moduli of pairs and generalized theta divisors, *Tohoku Math. J. 46* (1994) 321–340.

[Su] N. Sundaram, Special divisors and vector bundles, *Tohoku Math. J. 39* (1987) 175–213.

[Tan] Tan X.-J., Some results on the existence of rank 2 special stable bundles, *Manuscripta Math. 75* (1992) 365–373.

[T1] M. Teixidor i Bigas, Brill-Noether theory for stable vector bundles, *Duke Math. J. 62* (1991) 385–400.

[T2] M. Teixidor i Bigas, On the Gieseker-Petri map for rank 2 vector bundles, *Manuscripta Math. 75* (1992) 375–382.
[Th] M. Thaddeus, Stable pairs, linear systems and the Verlinde formula, *Invent. Math.* **117** (1994) 317–353.

[V] C. Voisin, Sur l’application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri, *Acta Math.* **168** (1992) 249–272.

**Department of Mathematics, University of Illinois, Urbana, IL 61801, USA**

*E-mail address: bradlow@math.uiuc.edu*

**Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain**

*E-mail address: oscar.garcia-prada@uam.es*

**Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain**

*E-mail address: vicente.munoz@uam.es*

**Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, UK**

*E-mail address: newstead@liverpool.ac.uk*