Quasi-Isotropic Expansion for a Two-Fluid Cosmological Model Containing Radiation and String Gas

I. M. Khalatnikov*, A. Yu. Kamenshchik*a,b,***, and A. A. Starobinsky*a,c,***

* Landau Institute for Theoretical Physics, Russian Academy of Sciences, Moscow, 119334 Russia
b Dipartimento di Fisica e Astronomia, Università di Bologna and INFN, Via Irnerio 46, Bologna, 40126 Italy
c Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow oblast, 141980 Russia

* e-mail: khalat@itp.ac.ru
** e-mail: kamenshchik@bo.infn.it
*** e-mail: alstar@landau.ac.ru

Received April 15, 2019; revised April 15, 2019; accepted April 16, 2019

Abstract—The quasi-isotropic expansion for a simple two-fluid cosmological model consisting of radiation and string gas is constructed. Expressions for the first-order gradient-dependent corrections to the space-time metric, energy densities, and velocities are presented explicitly. Their small- and large-time behavior is investigated. It is found that the anisotropic component of the metric grows faster than the isotropic (trace-proportional) one at late times. As a result, the behavior of the metric tensor does not have the standard quasi-isotropic form corresponding to the string gas equation of state.

DOI: 10.1134/S1063776119100066

1. INTRODUCTION

The quasi-isotropic solution of the Einstein equations near a cosmological singularity was first found by Lifshitz and Khalatnikov [1] for the Universe filled by radiation with the equation of state \( p = \frac{\varepsilon}{3} \) in the early 60th. In the paper [2], we presented its generalization to the case of an arbitrary one-fluid cosmological model. Then this solution was further generalized to the case of the Universe filled by two ideal barotropic fluids [3].

As is well known, modern cosmology deals with many very different types of matter. In comparison with the old standard model of the hot radiation dominated Universe (dubbed the Big Bang), the situation has been dramatically changed, first, with the development of inflationary cosmological models which contain an inflaton effective scalar field or/and other exotic types of matter as an important ingredient [4–9], and second, with the understanding that the main part of the non-relativistic matter in the present Universe is nonbaryonic—cold dark matter (CDM). Furthermore, the appearance of brane and M-theory cosmological models [10, 11] and the discovery of the cosmic acceleration [12, 13] (see also [14–16] for a review) suggests that matter playing an essential role at different stages of cosmological evolution is multi-component generically, and these components may obey very different equations of state. Moreover, the very notion of the equation of state appears to be not fundamental; it has only a limited range of validity as compared to a more fundamental field-theoretical description. From this general point of view, the generalization of the quasi-isotropic solution to the case of two ideal barotropic fluids with constant but different \( p/\varepsilon \) ratios seems to be a natural and important next logical step.

To explain the physical sense of the quasi-isotropic solution, let us remind that it represents the most generic spatially inhomogeneous generalization of the Friedmann–Lemaître–Robertson–Walker (FLRW) model in which the space-time is locally FLRW-like near the cosmological singularity \( t = 0 \) (in particular, its Weyl tensor is much less than its Riemann tensor). On the other hand, generically it is very inhomogeneous globally and may have a very complicated spatial topology. As was shown in [2, 17] (see also [18, 19]), such a solution contains 3 arbitrary functions of spatial coordinates. From the FLRW point of view, these degrees of freedom represent the growing (non-decreasing in terms of metric perturbations) mode of adiabatic perturbations and the non-decreasing mode of gravitational waves (with two polarizations) in the case when deviations of a space-time metric from the FLRW one are not small. So, the quasi-isotropic solution is not a generic solution of the Einstein equations
with a barotropic fluid. Therefore, one should not expect this solution to arise in the course of generic gravitational collapse (in particular, inside a black hole event horizon). The generic solution near a space-like curvature singularity (for \( p < e \)) has a completely different structure consisting of the infinite sequence of anisotropic vacuum Kasner-like eras with space-dependent Kasner exponents [20–22].

For this reason, the quasi-isotropic solution had not attracted much interest for about twenty years. Its new life began after the development of successful inflationary models (i.e., with “graceful exit” from inflation) and the theory of generation of perturbations during inflation, because it had immediately become clear that generically (without fine tuning of initial conditions) scalar metric perturbations after the end of inflation remain small in a finite (though still very large compared to the presently observable part of the Universe) region of space which is much less than the whole causally connected space volume produced by inflation. It appears that the quasi-isotropic solution can be used for a global description of a part of space-time after inflation which belongs to “one post-inflationary universe.” The latter is defined as a connected part of space-time after inflation which cannot be expressed as a function of the cosmic time \( t \) describing the moment when inflation ends is space-like and, therefore, can be made the surface of constant (zero) synchronous time \( t \) by a coordinate transformation. This directly follows from the derivation of perturbations generated during inflation given in [23] (see Eq. (17) of that paper) which is valid in case of large perturbations, too. Thus, when used in this context, the quasi-isotropic solution represents an intermediate asymptotic regime during expansion of the Universe after inflation. The synchronous time \( t \) appearing in it is the proper time since the end of inflation, and the region of validity of the solution is from \( t = 0 \) up to a moment in future when spatial gradients become important. For sufficiently large scales, the latter moment may be rather late, even of the order or larger than the present age of the Universe. Note also analogues of the quasi-isotropic solution related to the Universe behavior before the end of inflation which are produced by generic globally inhomogeneous late-time asymptotic solutions of the Einstein equations either with a cosmological constant [24] or with a scalar field having the exponential potential (power-law inflation) [25]. These solutions can be smoothly matched across the hypersurface of the end of inflation to a post-inflationary quasi-isotropic solution of the type we are studying (of course, the matter content has to be changed beyond this hypersurface, too).

A slightly different versions of the quasi-isotropic expansion were independently developed during last decades which are known under the names of long-wave expansion, gradient expansion, or the separate universe approach [26–32]. However, the specific of our approach is that we consider only solutions having the local FLRW behavior near singularity at \( t = 0 \).

Originally the quasi-isotropic expansion was developed as a technique of generation of some kind of perturbative expansion in the vicinity of cosmological singularity at \( t = 0 \), where the synchronous time \( t \) serves as a small parameter. However, the more general treatment of the quasi-isotropic expansion is possible if one notices that the next order of the quasi-isotropic expansion contains higher orders of spatial derivatives of metric coefficients. Thus, it is possible to construct a natural generalization of the quasi-isotropic solution of the Einstein equations which would be valid not only in the vicinity of cosmological singularity, but in the full time range. In this case simple algebraic equations, which one resolves to find higher orders of the quasi-isotropic approximation in the vicinity of singularity are substituted by differential equations, where the time dependence of the space-time metric can be rather complicated in contrast to the power-law behavior of its coefficients of the original quasi-isotropic expansion.

In the present paper we construct this expansion for a relatively simple two-fluid cosmological model containing radiation and the cosmic string gas (see e.g. [33]). Such a model has a technical advantage: the corresponding Friedmann equation is exactly solvable in terms of the synchronous time \( t \) and, hence, \( t \) is a natural parameter for constructing the quasi-isotropic solution. In the second section of the paper we explicitly construct next order terms of the quasi-isotropic expansion for the metric tensor in the synchronous reference frame, energy densities and velocities of two fluids, and determine their asymptotic behavior at early and late times. The last section contains some concluding remarks. In the Appendix we apply the developed formalism to the case of a one-fluid cosmological model. In this case the solutions valid in the full time range coincide with those valid in the vicinity of singularity [2].

\[ p_R = \frac{1}{3} \varepsilon_R \quad (1) \]

and the string gas with the equation of state

\[ p_S = -\frac{1}{3} \varepsilon_S. \quad (2) \]

The solution of the Friedmann equation for such a flat universe filled with a mixture of these two fluids can be expressed as a function of the cosmic time \( t \) as

\[ a(t) = \sqrt{At^2 + Bt}, \quad (3) \]
where \(a(t)\) is the cosmological radius. Thus, we can take as a zero-level for the quasi-isotropic approximation the following spatial metric in the synchronous reference system:

\[
ds^2 = dt^2 - \gamma_{ab} dx^a dx^b,
\]

(4)

\[
\gamma_{ab} = a_{ab}(x)(t + b(x)t^2),
\]

(5)

where the Greek indices are spatial and \(x\) stays also for spatial coordinates. As usually for a two-fluid model, the zero-order in spatial gradients term of the quasi-isotropic solution contains the following arbitrary functions of spatial coordinates: a symmetric tensor \(a_{ab}(x)\) and a scalar \(b(x)\) [3]. They describe the growing adiabatic mode and the constant isocurvature mode of scalar perturbations, and the constant mode of tensor perturbations (primordial gravitational waves) in the fully nonlinear regime. We look for the next order of the quasi-isotropic expansion which is proportional to the square of spatial derivatives of these functions:

\[
\gamma_{ab} = a_{ab}(x)(t + b(x)t^2) + c_{ab}(x,t).
\]

(6)

In what follows we omit the argument \(x\) from the corresponding functions. We shall need the following expressions: the inverse metric

\[
v^{ab} = \frac{a^{ab}}{b^2 + t} - \frac{c_{ab}}{(b^2 + t)^2},
\]

(7)

Here the matrix \(a^{ab}\) is defined by relation

\[
a^{ad}a_{db} = \delta_{ab}^d
\]

(8)

and the indices in the metric \(c\) are raised and lowered by matrices \(a^{ab}\) and \(a_{ab}\). However, the indices in the matrix \(\gamma_{ab}\) and in the extrinsic curvature tensors are raised and lowered by the whole matrices \(\gamma^{ab}\) and \(\gamma_{ab}\). The extrinsic curvature in the first approximation is

\[
K_{ab} = (2bt + 1)a_{ab} + \dot{c}_{ab},
\]

(9)

where “dot” as usual means the differentiation with respect to the time parameter \(t\). Then we have

\[
K^\alpha_0 = \frac{(2bt + 1)b^\alpha_0}{b^2 + t} + \frac{\dot{c}_0^\alpha}{b^2 + t} - \frac{2(2bt + 1)c_0^\alpha}{(b^2 + t)^2},
\]

(10)

\[
K = K^\alpha_0 = \frac{3(2bt + 1)}{b^2 + t} + \frac{\dot{c}_0}{b^2 + t} - \frac{2(2bt + 1)c_0}{(b^2 + t)^2},
\]

(11)

\[
\frac{\partial K^\alpha_0}{\partial t} = \frac{2b\dot{c}_0^\alpha}{b^2 + t} - \frac{(2bt + 1)c_0^\alpha}{b^2 + t} + \frac{\dot{c}_0^\alpha}{b^2 + t} - \frac{2(2bt + 1)c_0^\alpha}{(b^2 + t)^2}
\]

(12)

\[
\frac{\partial K^\alpha}{\partial t} = \frac{6b}{b^2 + t} - \frac{3(2bt + 1)}{b^2 + t} + \frac{\dot{c}}{b^2 + t} - \frac{2(2bt + 1)c}{(b^2 + t)^2} + \frac{2(2bt + 1)c}{(b^2 + t)^3},
\]

(13)

\[
K^\beta_0K^\alpha_0 = \frac{3(2bt + 1)^2}{(b^2 + t)^2} + \frac{2(2bt + 1)c}{(b^2 + t)^2} - \frac{2(2bt + 1)^2c}{(b^2 + t)^3}.
\]

(14)

Now we can write down the expression for \(R_0\) component of the Ricci tensor using the known formula [1, 34]:

\[
R_0 = -\frac{1}{2} \frac{\partial K^\alpha_0}{\partial t} - \frac{1}{4} K^\alpha_0 K^\alpha_0.
\]

(15)

Thus,

\[
R_0 = \frac{3(2bt + 1)^2}{4(b^2 + t)^2} - \frac{3b}{b^2 + t} - \frac{\dot{c}}{2(b^2 + t)}
\]

(16)

\[
+ \frac{(2bt + 1)c}{2(b^2 + t)^2} + \frac{bc}{2(b^2 + t)^2} - \frac{2(2bt + 1)^2c}{2(b^2 + t)^3}.
\]

The energy-momentum tensor for a perfect fluid has the form

\[
T^i_j = (\varepsilon + p)\mu^i \mu^j - p\delta^i_j,
\]

(17)

where \(u_i\) is a four-velocity normalized as usual as

\[
u_i u^i = 1,
\]

(18)

or, in other words, as

\[
u_0^2 - \gamma^{ab} u_a u_b = 1.
\]

(19)

In our approximation the spatial components of the four-velocities will be proportional to the spatial gradients of the scalar function \(b\):

\[
u_a = v b_a.
\]

(20)

The trace of the energy-momentum tensor is equal to

\[
T = T^i_i = \varepsilon - 3p.
\]

(21)

For our mixture of two fluids with the equations of state (1) and (2) one has

\[
T = 2\varepsilon_S.
\]

(22)

We shall choose the Newton constant in such a way that the Einstein equations look as

\[
G^i_j = T^i_j - \frac{1}{2} \delta^i_j T.
\]

(23)

Then, the temporal–temporal component of the system of the Einstein equations is

\[
R^0_0 = T^0_0 - \frac{1}{2} T.
\]

(24)

Using expressions (17)–(21) one gets

\[
T^0_0 - \frac{1}{2} T = \varepsilon_R + \frac{4}{3} \varepsilon_R \varepsilon_S^2 b_{0\alpha} b_{0\beta} \gamma^{0\alpha} + \frac{2}{3} \varepsilon_S \varepsilon_S^2 b_{0\alpha} b_{0\beta} \gamma^{0\alpha}. \]

(25)

Substituting Eqs. (25) and (16) into Eq. (24) and taking into consideration only the zero-order terms, we get

\[
\varepsilon_R^{(0)} = \frac{3}{4(b^2 + t)^2}.
\]

(26)
The spatial-temporal components of the Ricci tensor are given by the formula [34]:

$$R^b_{\alpha} = -P^b_{\alpha} - \frac{1}{2} \frac{\partial K^b_{\alpha}}{\partial t} - \frac{1}{4} K^b_{\gamma} K^\gamma_{\alpha},$$  \hspace{1cm} (27)$$

where $P^b_{\alpha}$ is the Ricci tensor constructed from the spatial metric $\gamma_{\mu\nu}$. The Ricci scalar is

$$R = R^0_0 + R^\alpha_{\alpha} = -P - \frac{\partial K^\alpha_{\alpha}}{\partial t} - \frac{1}{2} K^\alpha_{\beta} K^\beta_{\alpha}. \hspace{1cm} (28)$$

Now, using the contracted Einstein equation

$$R = -\mathcal{T}$$  \hspace{1cm} (29)$$

together with Eq. (22) and taking into account only the zero-order contributions, one obtains

$$\varepsilon^{(0)}_S = \frac{3b}{b^2 + t}.$$

We shall need also the expression for the mixed, spatial-temporal components of the Ricci tensor:

$$R^0_\alpha = \frac{1}{2} (K^0_{\alpha b} - K^b_{\alpha b}),$$  \hspace{1cm} (31)$$

where the covariant derivative is taken by using the Christoffel symbols constructed from the metric $\gamma_{\mu\nu}$. In the first non-vanishing order

$$R^0_\alpha = -\frac{\mathcal{T}}{(b^2 + t)} b^\alpha.$$  \hspace{1cm} (32)$$

The mixed component of the energy-momentum tensor is in the same order

$$T^0_\alpha = \left(\frac{4}{3} \varepsilon^{(0)}_R v_R + \frac{2}{3} \varepsilon^{(0)}_S v_S\right) b^\alpha.$$  \hspace{1cm} (33)$$

Comparing Eqs. (32) and (33) and using expressions (26) and (30) we get

$$v_R + 2b(b^2 + t)v_S = -\frac{\mathcal{T}}{b^2 + t}.$$  \hspace{1cm} (34)$$

Equation (34) is not sufficient to find the velocities $v_R$ and $v_S$ and we should use the energy-momentum tensor conservation laws. We suppose that our fluids do not interact and hence the corresponding energy-momentum tensors should be conserved separately. The spatial component of the energy-momentum tensor conservation law can be written down as

$$T^0_\alpha + \frac{1}{2} K^0_{\alpha b} T^0_b - p^\alpha = 0.$$  \hspace{1cm} (35)$$

This equation implies that radiation and string gas velocity functions, respectively, satisfy the following first-order differential equations:

$$\frac{d\varepsilon_R}{dt} = \frac{1}{6} K^0_{\alpha b} b^\alpha,$$  \hspace{1cm} (36)$$

$$\frac{d\varepsilon_S}{dt} = \frac{1}{6} K^0_{\alpha b} b^\alpha.$$  \hspace{1cm} (37)$$

Solving these equations we obtain

$$v_R = \frac{-\sqrt{bt^2 + t}}{2b^{5/2}} \text{Arccosh}(2bt + 1) + \frac{t}{b^2 + t},$$  \hspace{1cm} (38)$$

$$v_S = \frac{-1}{2b^2 + 1} \text{Arccosh}(2bt + 1).$$  \hspace{1cm} (39)$$

It is easy to check that these velocities satisfy Eq. (34).

We shall need also the 0 component of the energy-momentum conservation law, which looks like

$$T^0_0 + T^0_{0,\alpha} - \Gamma^0_{0\alpha} T^\alpha_0 + \Gamma^0_{0\alpha} T^\alpha_0 = 0.$$  \hspace{1cm} (40)$$

Up to first order the relevant quantities are written as

$$T^0_0 = \varepsilon^{(0)} + \varepsilon^{(1)} + \frac{(\varepsilon + p)^{(0)} v^2}{b^2 + t} b^\alpha,$$  \hspace{1cm} (41)$$

$$T^0_{0,\alpha} = -\frac{(\varepsilon + p)^{(0)} v^2 b^\alpha}{b^2 + t},$$  \hspace{1cm} (42)$$

$$T^\alpha_0 = -\frac{(\varepsilon + p)^{(0)} v b^\alpha}{b^2 + t},$$  \hspace{1cm} (43)$$

$$\Gamma^0_{0\alpha} = \frac{1}{2} K^0_{\alpha b} + \frac{1}{2} K^b_{\alpha b}.$$  \hspace{1cm} (44)$$

Using Eqs. (40)–(44) one can find the following equations for the first corrections to the energy densities $\varepsilon^{(1)}$:

$$\frac{d\varepsilon^{(1)}}{dt} + \frac{1}{2} K^{(0)}_{\alpha b} \left(\frac{\frac{d\varepsilon^{(0)}}{dt} v^2 b^\alpha}{b^2 + t} + \frac{4(\varepsilon + p)^{(0)} v^2 b^\alpha b^\alpha}{3(b^2 + t)}\right)$$  \hspace{1cm} (45)$$

$$+ \frac{1}{2} K^{(0)}_{\alpha b} \left(\frac{(\varepsilon + p)^{(0)} v^2}{b^2 + t}\right) = 0.$$  \hspace{1cm} (45)$$

Equations (45) could be explicitly integrated for both the energy densities $\varepsilon^{(1)}_R$ and $\varepsilon^{(1)}_S$, expressing their relation to other unknown quantity: the trace of the first correction to the metric $c$:

$$\varepsilon^{(1)}_R = -\frac{c}{2(b^2 + t)^3} - \frac{4(\varepsilon + p)^{(0)} v^2 b^\alpha b^\alpha}{3(b^2 + t)^3} + \varepsilon^{(1)}_R,$$  \hspace{1cm} (46)$$

where

$$\varepsilon^{(1)}_R = -\frac{1}{4} \text{Arccosh}^2(2bt + 1) - \ln(b + bt + 1) \frac{b^\alpha b^\alpha}{b^2 + t}$$  \hspace{1cm} (47)$$

$$+ \frac{1}{b^2 (b^2 + t)^2} \left(-2\ln(1 + bt) + \frac{3b^2 t}{b^2 + t}\right) + \text{Arccosh}^2(2bt + 1) - \frac{5bt \text{Arccosh}(2bt + 1)}{2b^2 (b^2 + t)} b^\alpha b^\alpha,$$  \hspace{1cm} (47)$$

$$\varepsilon^{(1)}_S = -\frac{b c}{(b^2 + t)^3} - \frac{2(\varepsilon + p)^{(0)} v^2 b^\alpha b^\alpha}{3(b^2 + t)^3} + \varepsilon^{(1)}_S,$$  \hspace{1cm} (48)$$

JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS  Vol. 129  No. 4  2019
\[ \Xi_{S}^{(3)} = \frac{b_{\alpha}^{\mu}}{b t (2 b t^2 + t)} \times \left( 2 \ln(b t + 1) + 2 - \frac{(2 b t + 1) \text{Arccosh}(2 b t + 1)}{b t^2 + t} \right) + \frac{b_{\alpha} b_{\beta}^{\mu}}{4 b^2 (b t^2 + t)} \left( \text{Arccosh}^2(2 b t + 1) - 2 b t + b t^2 + t \right) + \frac{(4 b t - 1) \text{Arccosh}(2 b t + 1) + 6 - 48 b t - b^2 t^2}{b t^2 + t} \right). \]  

Now combining the Einstein equations in such a way to exclude from them the term including the second time derivatives of \( c \) we have the equation

\[ 2 \varepsilon_{R}^{(i)} + 2 \varepsilon_{S}^{(i)} - P + \frac{1}{4} \mathbf{K}_{a}^{\mu} \mathbf{K}^{\mu}_{a} - \frac{1}{4} (\mathbf{K}_{a}^{\mu})^2, \]

\[ + \frac{8 \epsilon_{R}^{(i)} \gamma^{\mu} \gamma^{\nu} b_{\alpha} a_{\beta}}{b t^2 + t} + \frac{4 \epsilon_{R}^{(i)} \gamma^{\mu} \gamma^{\nu} b_{\alpha} a_{\beta}}{3 b t + t} = 0. \]  

We shall need also the following expression for the spatial curvature scalar

\[ P = \frac{\mathbf{P} \mathbf{P}}{b t^2 + t} - \frac{2 b^2 t}{(b t^2 + t)^2} b_{\alpha} a_{\beta} + \frac{3 b^4}{2 b t^2 + t} b_{\alpha} b_{\beta}. \]  

where \( \mathbf{P} \) denotes the spatial curvature scalar constructed from the metric \( a_{\alpha \beta} \).

Then

\[ c = (2 b t + 1) \int dt \left( \frac{b t^2 + t}{(2 b t + t)^2} \right) \left( - P + 2 \varepsilon_{R}^{(i)} + 2 \varepsilon_{S}^{(i)} \right). \]

The expression for the trace of the correction to the metric is

\[ c = - \frac{\mathbf{P} \mathbf{P} - 11 b_{\alpha} a_{\beta}}{2} + \frac{9 b_{\alpha} a_{\beta} b_{\gamma}}{10 b^2}. \]

At large values of the time parameter one has

\[ c = - \frac{\mathbf{P} \mathbf{P}}{2} + \frac{b_{\alpha} a_{\beta} b_{\gamma}}{b^2}, \]

Thus, at large values of time \( t \) the corrections to the metric represents some tensor depending only on special coordinates, multiplied by \( t^2 \).

Now, substituting into the formulas (46) and (48) the expressions (53), (47), (49) and the velocities (38) and (39) we obtain

\[ \varepsilon_{R}^{(i)} = \frac{\mathbf{P} \mathbf{P}}{4 b t^2 + t} + \frac{b_{\alpha} b_{\gamma}}{2 b t^2 + t} \times \left( - b t^2 + b t - b t(1 + b t) \ln(1 + b t) - (1 + b t) \ln(1 + b t) \right) - \frac{1}{4} \left[ (2 b t^2 + 2 b t + 1) \text{Arccosh}^2(2 b t + 1) \right] \]

\[ + \frac{b_{\alpha} b_{\beta}^{\mu}}{b t^2 + t} \left[ \frac{15 b_t b^2 + 8 b t}{48} + \frac{b t + 1}{4} \ln(1 + b t) \right] \]

\[ + \frac{b_{\alpha} b_{\beta}^{\mu}}{b t^2 + t} \left[ \frac{11 b^2 + 8 b t}{48} + \frac{5 b t + 1}{4} \ln(1 + b t) \right] \]

\[ - \frac{5 b t}{2} \text{Arccosh}^2(2 b t + 1) \ln \left( \frac{b t}{1 + b t} \right) \]

\[ - \frac{1}{4} \text{Li}_2(e^{-2 \text{Arccosh}(2 b t + 1)}) \]

\[ - \frac{1}{4} \text{Li}_2(e^{-2 \text{Arccosh}(2 b t + 1)}) \]
we use the Einstein equation for the spatial components of the Ricci and the energy-momentum tensor, taking their traceless parts:
\[ \tilde{R}^\alpha_\beta = \tilde{T}^\alpha_\beta. \]

From Eqs. (27) and (11) one finds that the traceless part of the spatial–spatial components of the Ricci tensor is
\[ \tilde{R}^\alpha_\alpha = -\frac{1}{2} \frac{\partial \tilde{K}^\alpha_\beta}{\partial t} - \frac{3(2b^2 + 1)}{4(b^2 + 1)} \tilde{K}^\beta_\alpha, \]

while from Eq. (43) one finds that
\[ \tilde{T}^\beta_\alpha = -\frac{1}{2b^2 + t}. \]

We shall need also the expression for the spatial Ricci tensor
\[ P^\beta_\alpha = \frac{\tilde{P}^\beta_\alpha}{b^2 + t} - \frac{2t^2}{(b^2 + t)^2} b^\alpha_\beta + \frac{3t^4}{2(b^2 + t^2)} b^\alpha_\beta. \]

Now combining Eqs. (66) and (67) we obtain the following equation for the traceless part of the extrinsic curvature:
\[ (b^2 + t)^{-3/2} \partial_t (\tilde{K}^\beta_\alpha(b^2 + t)^{3/2}) = -2\tilde{P}^\beta_\alpha - 2\tilde{T}^\beta_\alpha, \]

which immediately gives
\[ \tilde{K}^\beta_\alpha = -2(b^2 + t)^{-3/2} \int (b^2 + t)^{3/2} \tilde{P}^\beta_\alpha + \tilde{T}^\beta_\alpha dt. \]

Substituting into Eq. (71) expressions (68), (69), (26), (30), (38), (39) one finds the following expression for the traceless part of the extrinsic curvature:
\[ \tilde{K}^\beta_\alpha = -\left( -\frac{2b^2 + 1}{2b^2 + t} \frac{\partial \tilde{K}^\beta_\alpha}{\partial t} - \frac{3(2b^2 + 1)}{4(b^2 + t)^{3/2}} \tilde{K}^\beta_\alpha \right) \tilde{P}^\beta_\alpha + \tilde{T}^\beta_\alpha dt. \]

To find the traceless part of the first correction to the metric
\[ \tilde{c}_{\alpha\beta} \equiv c_{\alpha\beta} - \frac{1}{3} a_{\alpha\beta} e. \]

Finally for the traceless part of \( c_{\alpha\beta} \) we obtain
At small $t$ the expression (74) behaves as

$$\tilde{c}_{\alpha\beta} = \frac{4}{3} t^2 \tilde{P}_{\alpha\beta} + \frac{4}{5} t^3 \left(b_{\alpha\beta} - \frac{1}{3} a_{\alpha\beta} b_{\gamma\gamma}\right) + \frac{4}{45} b t^3 \left(b_{\alpha\beta} b_{\gamma\gamma} - \frac{1}{3} a_{\alpha\beta} b_{\gamma\gamma}\right).$$

(75)

At large values of $t$ it looks like

$$\tilde{c}_{\alpha\beta} = t^2 \ln b \left(-\tilde{P}_{\alpha\beta} + \frac{2}{b} \left(b_{\alpha\beta} - \frac{1}{3} a_{\alpha\beta} b_{\gamma\gamma}\right) - \frac{3}{2b^2} \left(b_{\alpha\beta} b_{\gamma\gamma} - \frac{1}{3} a_{\alpha\beta} b_{\gamma\gamma}\right)\right).$$

(76)

Thus, the large-time asymptotic behavior of the traceless “anisotropic” part of the metric is very different from the asymptotic behavior of the trace of the metric $c$ (58). The anisotropic part of the metric grows faster (by logarithm $\ln bt$).

To understand better such a behavior of the anisotropic part of the metric, let us remember that at large values of $t$ the string gas dominate radiation. One can make transition from the two-fluid case to one fluid-one, where only the string gas is present, considering the limit

$$b \to \infty, \quad b a_{\alpha\beta} = \text{const.}$$

(77)

Substituting (77) into the expressions (53) and (74) one sees that while the limiting value of the trace part of the metric $c$ is regular, the coefficient $c_{\alpha\beta}$ diverges as $\ln bt$. It is quite natural because we know that first correction to the traceless part of the metric in the quasi-isotropic expansion diverges for the barotropic fluid with $w = -\frac{1}{3}$ (see [2] and Appendix).

3. CONCLUDING REMARKS

We have calculated explicitly the next order terms in the quasi-isotropic solution for the metric tensor (53), (74) and for energy-densities (59), (60) and velocities (38), (39) of two fluids. Also we have found their asymptotic behavior for small and large values of the cosmic synchronous time $t$. As was easily predictable, the structure of the solution for small $t$ is determined only by the radiation component, and it coincides with that found in the original paper [1]. However, the late time behavior of the metric tensor reveals an unusual feature: the anisotropic part of the metric (76) grows essentially faster than the isotropic one (75), and their ratio is $\propto \ln bt$. It seems that this effect is due to the two-fluid character of the model considered in this paper and to the specific property of the string gas equation of state which leads to the singular character of the quasi-isotropic expansion for the string gas alone (see [2] and the Appendix of the present paper). From the mathematical point of view, this is reflected in the fact that right-hand sides of Eqs. (58) and (76) do not have a regular limit for $b \to 0$, i.e. when the isocurvature mode becomes arbitrary small.

It is not clear if the appearance of terms having non-power-law behavior is present in two-fluid quasi-isotropic models with other equations of state. To answer this question it is necessary to develop the formalism of building the quasi-isotropic expansion valid for the full range of time for arbitrary two-fluid models that is much more complicated technically.

APPENDIX

In this Appendix we consider the quasi-isotropic expansion for one-fluid cosmological model, which is valid for the full time range. The method of calculation
is the same as in Section 2, but in one-fluid case they are much simpler. We shall see that form of the expression for the first correction to the metric coefficients valid for the full time range coincides this that valid in the vicinity of the singularity [2], obtained by the method first proposed in [1].

We consider the universe with the fluid with the equation of state \( p = \omega \epsilon \). The spatial metric now is

\[
\gamma_{\alpha\beta} = a_{\alpha\beta}t^\kappa + c_{\alpha\beta},
\]

where

\[
\kappa = \frac{4}{3(1 + \omega)}.
\]

Inverse metric is

\[
\gamma^{\alpha\beta} = a^{\alpha\beta}t^{-\kappa} - c^{\alpha\beta}t^{-2\kappa}.
\]

Then we have the following formulae for the extrinsic curvature:

\[
K_{\alpha\beta} = a_{\alpha\beta} \kappa t^{-\kappa} + \dot{c}_{\alpha\beta},
\]

\[
K^\beta_\alpha = \frac{\partial^\beta}{\partial t} \kappa + \dot{c}t^{-\kappa} - \frac{\kappa - 1}{\kappa} \kappa t^{-\kappa - 1},
\]

\[
K = \frac{3\kappa}{t} + \dot{c}t^{-\kappa} - \kappa t^{-\kappa - 1},
\]

\[
\frac{\partial K}{\partial t} = -\frac{3\kappa}{t^2} + \dot{c}t^{-\kappa} - 2\kappa t^{-\kappa + 1} + c\kappa(\kappa + 1)t^{-\kappa - 2},
\]

\[
K^\beta_\alpha K^\alpha_\beta = \frac{3\kappa^2}{t^2} + 2\dot{c} \kappa t^{-\kappa - 1} - 2\kappa^2 t^{-\kappa - 2}.
\]

Substituting formulas (84), (85) into (15) we have

\[
R^\beta_\alpha = \frac{3\kappa(2 - \kappa)}{4t^2} - \frac{\dot{c}t^{-\kappa} + \dot{\kappa} t^{-\kappa + 1} - \frac{\kappa - 1}{\kappa} \kappa t^{-\kappa - 2}}{2}.
\]

Using now the Einstein equation (24) in the lowest order of the approximation we obtain for the energy density of the fluid under consideration

\[
\epsilon^{(0)} = \frac{4}{3(1 + \omega)^2 t^2}.
\]

Using the 0 component of the energy-momentum conservation law (40) we can find the relation between the first correction to the energy density \( \epsilon^{(1)} \) and the trace of the first correction to the metric \( c \):

\[
\epsilon^{(1)} = \frac{-c \kappa t^{-\kappa - 2}}{2}.
\]

Now, using the expression for the scalar curvature \( R \) (28) and the 00 component of the Einstein equation in the form \( R^0_0 = \frac{1}{2} (R + T^0_0) \) for in the first quasi-isotropic order the following equation:

\[
\frac{P}{2} + \frac{1}{4} K^{(0)} K^{(1)} - \frac{1}{8} (K^\beta_\alpha K^\alpha_\beta)^{(1)} = \epsilon^{(0)},
\]

Combining (89) and (88) we obtain the following differential equation for \( c \):

\[
\dot{c} = \frac{c(\kappa - 1)}{t} - \frac{\overline{P}}{\kappa}.
\]

Integrating (90) we obtain

\[
c = \frac{\overline{P}^2}{\kappa(\kappa - 3)} = -\frac{9(1 + \omega)^2 \overline{\Omega}^2}{4(5 + 9\omega)}.
\]

Now to find the traceless part of the first correction to the metric \( \tilde{c}_{\alpha\beta} \) we shall use traceless part of the spatial-spatial component of the Einstein equations, which in the case of one fluid and in the first order approximation has a particularly simple form:

\[
\tilde{R}^\beta_\alpha = -\frac{\overline{P}^0_\alpha}{2} - \frac{1}{2t} \frac{\partial K^\beta_\alpha}{\partial t} - \frac{3\kappa}{4t} \tilde{K}^\beta_\alpha = 0.
\]

(Notice that the traceless part of the extrinsic curvature \( \tilde{K}^\beta_\alpha \) does not have zero-order terms.) Equation (92) can be rewritten as

\[
\frac{\partial \tilde{K}^\beta_\alpha}{\partial t} + \frac{3\kappa}{2t} \tilde{K}^\beta_\alpha = -\frac{2\overline{P}^0_\alpha t^{-\kappa}}{\kappa + 1}.
\]

Integrating (93) one obtains

\[
\tilde{R}^\beta_\alpha = -\frac{4}{\kappa + 2} \overline{P}^0_\alpha t^{-\kappa + 1}.
\]

Using relation

\[
\tilde{c}^\beta_\alpha = \frac{1}{t} \int \tilde{K}^\beta_\alpha dt,
\]

we come to

\[
\tilde{c}^\beta_\alpha = \frac{4}{\kappa^2 - 4} \overline{P}^0_\alpha t^2 = -\frac{9(1 + \omega)^2 \overline{\Omega}^2}{(3\omega + 1)(3\omega + 5)} \overline{P}^0_\alpha t^2.
\]

One can see that the results (91) and (96) valid in the full range of time coincide with those valid in the vicinity of the initial cosmological singularity (\( t = 0 \)) [2] obtained by the algebraic method [1]. The general expression for the first correction to the metric for one-fluid case is given in the formula (37) in [2]. The metric \( b_{\alpha\beta} \) in [2] corresponds to \( c_{\alpha\beta} \) in the present paper, while for the equation of state parameter the symbol \( k \) is used instead of \( w \). The formula (37) contains a misprint: in front of the second term in the brackets in the right-hand side of this equation should stay the factor 1/4. At first glance the first correction to the metric (37) contains a pole at 3\( k + 1 = 0 \), however calculating the trace of this metric, one sees that this pole is canceled and is present only in its anisotropic part.

Thus, for the case of the universe filled with the string gas \( w = -\frac{1}{3} \) the quasi-isotropic expansion does not work because the expression for \( \tilde{c}_{\alpha\beta} \) becomes singular. As we have seen before the quasi-isotropic expansion for the universe filled with the mixture of
string gas and radiation does work, but at large values of the time parameter \( t \), when the influence of the string gas becomes dominant, the metric coefficient \( c_{ab} \) grows rapidly as \( t \ln bt \) (76).

In the conclusion let us consider a special case when the metric \( a_{\alpha\beta} \) has a conformally flat form:

\[
a_{\alpha\beta} = e^{\phi(x)}\delta_{\alpha\beta}.
\]  

(97)

In this case the spatial Ricci tensor is

\[
\overline{P}_{\alpha\beta} = \frac{1}{4}(\rho_{\alpha\rho}^\beta - 2\rho^\beta_{\alpha\rho}) - \frac{1}{4}\delta_{\alpha\beta}(\rho^\mu\rho^\mu + 2\rho^\mu_{\mu})
\]  

(98)

or

\[
\overline{P}^\beta_{\alpha} = \frac{1}{4}(\rho_{\alpha\rho}^\beta - 2\rho^\beta_{\alpha\rho}) - \frac{1}{4}\delta^\beta_{\alpha}(\rho^\mu\rho^\mu + 2\rho^\mu_{\mu}).
\]  

(99)

Correspondingly

\[
\overline{P} = -2\rho^\mu_{\mu} - \frac{1}{2}\rho^- (100)
\]

and the traceless part of the Ricci tensor is

\[
\overline{P}_{\alpha\beta} = \frac{1}{4}(\rho_{\alpha\rho}^\beta - 2\rho^\beta_{\alpha\rho}) + \frac{1}{12}\delta^\beta_{\alpha}(2\rho^\mu\rho^\mu - \rho^\mu_{\mu})^2.
\]  

(101)

If

\[
\rho = A_{\nu\lambda}x^\nu x^\lambda
\]  

(102)

then

\[
\overline{P} = -4A^\mu_{\mu} - 2A^\mu_{\mu}A^\nu^{\nu\alpha}x^\alpha,
\]  

(103)

\[
\overline{P}_{\alpha\beta} = \frac{1}{3}x^\mu x^\nu (3A_{\nu\mu}A^\beta_{\mu} - \delta^\beta_{\alpha}A_{\mu\nu}A^\mu_{\nu}).
\]  

(104)

Thus, it is easy to see that if the metric in the lowest order of the quasi-isotropic expansion has the Gaussian form determined by Eqs. (97) and (102) already its first correction \( c_{ab} \) determined by the curvature tensors (103) and (104) has non-Gaussian form due to the presence of the quadratic in \( x^\mu \) terms in front of the Gaussian exponential.

ACKNOWLEDGMENTS

IMK, AYK, and AAS were partially supported by the Russian Foundation for Basic Research, grant 17-02-01008.

REFERENCES

1. E. M. Lifshitz and I. M. Khalatnikov, Sov. Phys. JETP 12, 108 (1960).
2. I. M. Khalatnikov, A. Yu. Kamenshchik, and A. A. Starobinsky, Class. Quantum Grav. 19, 3845 (2002).
3. I. M. Khalatnikov, A. Yu. Kamenshchik, M. Martellini, and A. A. Starobinsky, J. Cosmol. Astropart. Phys. 0303, 001 (2003).
4. A. A. Starobinsky, JETP Lett. 30, 682 (1979).
5. A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).
6. K. Sato, Mon. Not. R. Astron. Soc. 195, 467 (1981).
7. A. H. Guth, Phys. Rev. D 23, 347 (1981).
8. A. D. Linde, Phys. Lett. B 108, 389 (1982).
9. A. D. Linde, Phys. Lett. B 129, 177 (1983).
10. T. Banks and W. Fischler, hep-th/0102077 (2001).
11. T. Banks and W. Fischler, hep-th/0111142 (2001).
12. S. J. Perlmutter et al., Astroph. J. 517, 565 (1999).
13. A. Riess et al., Astron. J. 116, 1009 (1998).
14. V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000).
15. P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003).
16. T. Padmanabhan, Phys. Rep. 380, 235 (2003).
17. E. M. Lifshitz and I. M. Khalatnikov, Sov. Phys. Usp. 6, 495 (1964).
18. G. L. Comer, N. Deruelle, D. Langlois and J. Perry, Phys. Rev. D 49, 2759 (1994).
19. N. Deruelle and D. Langlois, Phys. Rev. D 52, 2007 (1995).
20. V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. 19, 525 (1970).
21. V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. 31, 639 (1982).
22. C. W. Misner, Phys. Rev. Lett. 22, 1071 (1969).
23. A. A. Starobinsky, Phys. Lett. B 117, 175 (1982).
24. A. A. Starobinsky, JETP Lett. 37, 66 (1983).
25. V. Müller, H.-J. Schmidt, and A. A. Starobinsky, Class. Quantum Grav. 7, 1163 (1990).
26. D. S. Salopek and J. R. Bond, Phys. Rev. D 42, 3936 (1990).
27. D. S. Salopek, Phys. Rev. D 43, 3214 (1991).
28. D. S. Salopek and J. M. Stewart, Class. Quantum Grav. 9, 1943 (1992).
29. D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, Phys. Rev. D 62, 043527 (2000).
30. Y. Nambu and A. Taruya, Class. Quantum Grav. 13, 705 (1996).
31. Y. Tanaka and M. Sasaki, Prog. Theor. Phys. 117, 633 (2007).
32. S. Cotsakis, S. Kadry, and D. Trachilis, Int. J. Mod. Phys. A 31, 1650130 (2016).
33. A. Y. Kamenshchik and I. M. Khalatnikov, Int. J. Mod. Phys. D 21, 1250004 (2012).
34. L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields (Nauka, Moscow, 1988; Pergamon, Oxford, 1975).