CLEBSCH-GORDAN AND RACAH-WIGNER COEFFICIENTS FOR $U_q(SU(1,1))$

N.A. LISKOVA and A.N. KIRILLOV
Steklov Mathematical Institute
Fontanka 27, Leningrad 191011, USSR

Received October 21, 1991

ABSTRACT
The Clebsch-Gordan and Racah-Wigner coefficients for the positive (or negative) discrete series of irreducible representations for the noncompact form $U_q(SU(1,1))$ of the algebra $U_q(sl(2))$ are computed.

0. Introduction
Now it is well known the great significance which the Clebsch-Gordan and Racah-Wigner coefficients for the algebra $U_q(SU(2))$ has in the conformal field theory, topological field theory, low-dimensional topology, in the theory of a $q$-special functions. In this note we compute the Clebsch-Gordan and Racah-Wigner coefficients for the non-compact form $U_q(SU(1,1))$ of the Hopf algebra $U_q(sl(2))$ in the case corresponding to the tensor product of the irreducible representations of positive (or negative) discrete series. Our main result consists of two parts. At first we obtain the formula for the Clebsch-Gordan and Racah-Wigner coefficients in the case mentioned above as an analytical continuation of corresponding formula for the algebra $U_q(sl(2))$ in the region of negative values of parameters. At the second we find the simple substitutions which transforms the corresponding formula for $U_q(sl(2))$ into ones for $U_q(SU(1,1))$ and vice versa.

Acknowledgements. The authors thank F.A. Smirnov, L.A. Takhtajan and L.L. Vaksman for interesting discussions and remarks. We would like to express gratitude to the organizers of the RIMS 91 Project “Infinite Analysis” for the invitation to take participation in the workshop of this Project and the secretaries of RIMS for the various assistance and the help in preparing the manuscript to publication.
n°1. Algebra $U_q(sl(2))$ and its compact forms.

The algebra $U_q(sl(2))$, [1,2], is generated by elements $\{K, K^{-1}, X_\pm\}$ with the commutation relations:

\[
K \cdot K^{-1} = K^{-1} \cdot K = 1, \quad KX_\pm K^{-1} = q^{\pm 1}X_\pm;
\]

\[
X_+X_- - X_-X_+ = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}.
\]

(1)

The following formula for the comultiplication [3], the antipode and counit on the generators define the structure of a Hopf algebra on $U_q(sl(2))$:

\[
\Delta(X_\pm) = X_\pm \otimes K^{1/2} + K^{-1/2} \otimes X_\pm, \quad \Delta(K) = K \otimes K;
\]

\[
S(X_\pm) = -q^{\pm 1/2}X_\pm, \quad S(K) = K^{-1};
\]

\[
\varepsilon(K) = 1, \quad \varepsilon(X_\pm) = 0.
\]

(2)

(3)

(4)

We denote this Hopf algebra by $U_q := (U_q(sl(2)), \Delta, S, \varepsilon)$. The maps $\Delta' = \sigma \circ \Delta$, $S' = S^{-1}$, where $\sigma$ is the permutation in $U_q(sl(2))^\otimes 2$, i.e. $\sigma(a \otimes b) = b \otimes a$, also define the structure of Hopf algebra on $U_q(sl(2))$. From (2) and (3) it follows that

\[
U_{q^{-1}} := (U_q(sl(2)), \Delta', S', \varepsilon) \simeq (U_{q^{-1}}(sl(2)), \Delta, S, \varepsilon)
\]

(5)

as the Hopf algebras. It is well known (e.g. [6]) that $U_q(sl(2))$ and $U_{q^{-1}}(sl(2))$ are isomorphism as a Hopf algebras if $q_1 = q_2$ or $q_1q_2 = 1$. Remark that the square of the antipode $S^2(K) = K$, $S^2(X_\pm) = q^{\pm 1}X_\pm$ does not coincide with identity map.

Comultiplications $\Delta$ and $\Delta'$ are connected in $U_q(sl(2))^\otimes 2$ by the following automorphism [5]

\[
\Delta'(a) = R\Delta(a)R^{-1}, \quad a \in U_q,
\]

(6)

where $R \in U_q(sl(2))^\otimes 2$. The element $R$ is called the universal $R$-matrix. It satisfies the relations

\[
(\Delta \otimes \text{id})R = R_{13}R_{23}
\]

\[
(\text{id} \otimes \Delta)R = R_{13}R_{12}
\]

\[
(s \otimes \text{id})R = R^{-1}
\]

(7)

where the indices show the embeddings $R$ into $U_q(sl(2))^\otimes 3$. The center of the algebra $U_q(sl(2))$ (if $q$ is not equal to a root of unity) is generated by the $q$-analog of Casimir's element [2,3]

\[
2c = \left(q^{1/2} + q^{-1/2}\right) \left(\frac{K^{1/2} - K^{-1/2}}{q^{1/2} - q^{-1/2}}\right)^2 + X_+X_- + X_-X_+.
\]

(8)

Let us give now some useful formulas

\[
\Delta(X_\pm^m) = \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] X_\pm^l K^{l-m} \otimes X_\pm^{m-l} K^{1/2};
\]

\[
[X_\pm^n, X_\pm^m] = \sum_{l=1}^{\min(m,n)} \left[ \begin{array}{c} m \\ l \end{array} \right] \left[ \begin{array}{c} n \\ l \end{array} \right] X_\pm^{m-l} X_\pm^{n-l} \prod_{j=1}^{l} \frac{q^{-1/2}(m-n+l-j)}{q^{1/2} - q^{-1/2}} \frac{K - q^{1/2}(m-n+l-j)}{q^{1/2} - q^{-1/2}} K^{-1}.
\]

(9)
where we use the following notations
\[ [m] = \frac{q^{m} - q^{-m}}{q^{1/2} - q^{-1/2}}, \quad [m]! = \prod_{j=1}^{m} [j], \quad [0]! = 1, \]
\[ \left[ \frac{m}{l} \right] = \frac{[m]!}{[l]![m-l]!}, \quad \text{if} \quad 0 \leq l \leq m. \]

\textbf{n°2. The real forms of a Hopf algebra} \( A = (A, m, \Delta, S, \varepsilon) \).

First, let us recall the definition of \( * \)-antiinvolution of Hopf algebra \( A \) (e.g. \([6,7]\)). It is a map \( * : A \to A \) such that the following diagrams are commutative

1) \( A \otimes A \xrightarrow{\mu} A \xrightarrow{\sigma} \quad \xrightarrow{*} \quad A \xrightarrow{\varepsilon} A \) (antiautomorphism of algebra);

2) \( A \xrightarrow{\Delta} A \xrightarrow{\ast \otimes \ast} \quad \xrightarrow{\ast} \quad A \xrightarrow{\ast \otimes \ast} A \otimes A \) (automorphism of co-algebra);

3) \( \ast^2 = id_A \) (involution);

4) \( \xrightarrow{s} \quad A \xrightarrow{S} A, \quad \text{i.e.} \quad (\ast \circ S)^2 = id_A; \)

5) \( \varepsilon(a^*) = \overline{\varepsilon(a)}, \quad a \in A. \)

Two antiinvolutions \( \ast_1 \) and \( \ast_2 \) are called to be equivalent if there exists automorphism \( \varphi \) of the Hopf algebra \( A \) such that the diagram

\[ \xymatrix{ A \ar[r]^{\ast_1} & A \ar[d]^{\varphi} \ar[r]^{\ast_2} & A \ar[d]^{\varphi} } \]

is commutative. The real form of the Hopf algebra \( A \) is by definition the pair \( (A, \ast) \) consisting of the Hopf algebra \( A \) and the class of antiinvolutions, which are equivalent to \( \ast \). The real forms of \( U_q(sl(n)) \) are classified in \([4]\) and for the case \( U_q(sl(2)) \) in \([6,7]\).

\textbf{Proposition 1} \([4,6,7]\). \textbf{The real forms of} \( U_q(sl(2)) \) \textbf{are exhausted by the following types:}

a) \( U_q(SU(2)), -1 < q < 1, q \neq 0 \) \quad (a compact real form),
\[ K^* = K, \quad X^*_\pm = X_\pm; \]
b) \( U_q(SU(1,1)), -1 < q < 1, q \neq 0 \) (a non compact real form),
\[ K^* = K, \quad X_\pm^* = -X_\mp; \]

c) \( U_q(sl(2,\mathbb{R})), |q| = 1 \) (a non compact real form),
\[ K^* = K, \quad X_\mp^* = -X_\pm. \]

We note that the real Lie algebras \( SU(1,1) \) and \( sl(2,\mathbb{R}) \) are equivalent (via the Cayley transformation) in the classical case \((q = 1)\), but in the quantum case these two real forms are not equivalent. It is an interesting problem to quantize the irreducible unitary representations of the Lie algebras \( sl(2,\mathbb{R}) \) and \( sl(2,\mathbb{C}) \) (see e.g. [12]).

3. Irreducible unitary representations of \( U_q(SU(1,1)) \), \( 0 < q < 1 \).

Let us remind that the left \( U_q(SU(1,1)) \)-module \( V \) is called unitary if there exists a positive definite Hermitian scalar product \((\cdot,\cdot)\) on \( V \) such that
\[ (ax, y) = (x, a^* y), \quad x, y \in V, \quad a \in U_q(SU(1,1)), \]
where the antinvolution \(*\) defines the real form \( U_q(SU(1,1)) \). The Casimir operator (see the formula (8)) acts on an irreducible unitary representation \( V \) of \( U_q(SU(1,1)) \) as a scalar: \( C|_V = c_V \cdot Id_V \) and \((-c_V) \in \mathbb{R}_+\). Before to formulate the result (e.g. [8],[6]) concerning the classification of unitary irreducible representations of the algebra \( U_q(SU(1,1)) \) let us introduce some notations. Let us fix \( q = \exp(-h), \) \( h \in \mathbb{R}^*_+ \) and take \( \varepsilon = 0, 1/2 \). Let \( \mathcal{H}_\varepsilon \) be a complex Hilbert space with orthonormal bases
\[ \{e_m \mid m = \varepsilon + n, n \in \mathbb{Z}\}. \]

For any complex number \( j \) consider the following representation \( V^j_\varepsilon \) of the algebra \( U_q(SU(1,1)) \) in the space \( \mathcal{H}_\varepsilon \)
\[ X_\pm e_m^j = \pm ([m \mp j][m \mp j + 1])^{1/2} e_{m \pm 1}^j, \]
\[ Ke_m^j = q^m e_m^j, \]
where we use notations \( e_m^j \) for the bases of \( \mathcal{H}_\varepsilon \) instead of \( e_m \).

The irreducible unitary representations of \( U_q(SU(1,1)), 0 < q < 1 \), are classified (up to the unitary equivalence) by the following types:

I. Continuous (or principal) series
\[ V^j_\varepsilon, \quad j = \frac{1}{2} - i\sigma, \quad 0 < \sigma < \frac{\pi}{h}. \]

II. Strange series
\[ V^j_\varepsilon, \quad j = \frac{1}{2} - \frac{\pi i}{h} - s, \quad s > 0. \]

III. Complementary series
\[ V^j_0, \quad 0 < j < \frac{1}{2}. \]
IV. Discrete series
a) Positive \( j \in \mathbb{Z}^+ \) or \( j \in \frac{1}{2} + \mathbb{Z}^+ \),
\[
V^j_+ = \{ e^j_m \mid m - j \in \mathbb{Z}^+ \}.
\]
b) Negative \( j \in \mathbb{Z}^+ \) or \( j \in \frac{1}{2} + \mathbb{Z}^+ \),
\[
V^j_- = \{ e^j_m \mid m + j \in \mathbb{Z}^- \}.
\]

V. Exceptional representations \( j = \frac{1}{2} \),
\[
V^{\frac{1}{2}+} = \{ e^j_m \mid m \in -\frac{1}{2} + \mathbb{Z}^+ \}, \quad V^{\frac{1}{2}-} = \{ e^j_m \mid m \in \frac{1}{2} + \mathbb{Z}^- \}.
\]

The action of the generators of \( U_q(SU(1,1)) \) in the cases IV and V are given by (12).

Note that the continuous series we have
\[
-c_j = \left( \frac{1}{q^{1/4} + q^{-1/4}} \right)^2 + \left( \frac{2 \sin \frac{\sigma h}{2}}{q^{1/2} - q^{-1/2}} \right)^2 > 0,
\]
\[
[m \pm j][m \mp j \pm 1] = \left[ m \pm \frac{1}{2} \right]^2 + \left( \frac{2 \sin \frac{\sigma h}{2}}{q^{1/2} - q^{-1/2}} \right)^2 > 0,
\]
and for the strange series
\[
-c_j = \left( \frac{1}{q^{1/4} - q^{-1/4}} \right)^2 + [s]^2 > 0,
\]
\[
[m \pm j][m \mp j \pm 1] = \left[ m \pm \frac{1}{2} \right]^2 + \left( \frac{q^{s/2} + q^{-s/2}}{q^{1/2} - q^{-1/2}} \right)^2 > 0.
\]
The same inequalities are correct in all other cases.

\textbf{n°4. Quantum Clebsch-Gordan coefficients for } U_q(SU(1,1)).

We study the decomposition of the tensor product of two irreducible representations of positive (or negative) discrete series for the algebra \( U_q(SU(1,1)) \) and the corresponding quantum \( q - 3j \) symbols. Our approach follows to the papers [9,10,11]. In the sequel we use notation \( V^j := V^j_+ \).

\textbf{Theorem 1} (Clebsch-Gordan series for \( U_q(SU(1,1)) \)). We have the following decomposition
\[
V^{j_1} \otimes V^{j_2} = \bigoplus_{j \geq j_1 + j_2} V^j, \quad j - j_1 - j_2 \in \mathbb{Z}^+.
\]
Proof. We will construct the lowest vectors in every irreducible component \( V^j \rightarrow V^{j_1} \otimes V^{j_2} \). For this aim let us consider a vector
\[
e_j^{j_1 j_2} = \sum_{m_1 + m_2 = j} a_{m_1, m_2} e_{m_1}^{j_1} \otimes e_{m_2}^{j_2} \in V^{j_1} \otimes V^{j_2}. \tag{13}
\]

It is easy to see that \( e_j^{j_1 j_2} \in V^j \). This vector is a lowest vector in the component \( V^j \) if \( \Delta(X_-) e_j^{j_1 j_2} = 0 \). So we obtain the recurrence relation on the coefficients \( a_{m_1, m_2} \), namely
\[
a_{m_1 + 1, m_2} (m_1 - j_1 + 1)(m_1 + j_1) \frac{1}{2} q^{-m_1} + a_{m_1, m_2 + 1} (m_2 - j_2 + 1)(m_2 + j_2) \frac{1}{2} q^{-m_2} = 0, \tag{14}
\]
where \( j_1 \leq m_1 \leq j - j_2 \) and \( m_1 - j_1 \in \mathbb{Z} \). It is easy to find the solution of (14). We have
\[
a_{j_1 + k, j - j_2 - k} = a_0 (-1)^k q^{-\frac{k}{2}(j - 1)} \left\{ \frac{[j - j_1 - j_2][j - j_1 + j_2 - 1][2j_1 - 1]}{[k][j - j_1 - j_2 - k][j - j_1 + j_2 - k - 1][2j_1 + k - 1]} \right\}^{1/2}. \tag{15}
\]

The initial constant \( a_0 \) may be found from the condition that vector (13) have the norm equals to 1
\[
\|e_j^{j_1 j_2}\|^2 = a_0^2 \sum_{k=0}^{j-j_1-j_2} q^{-k(j - 1)} q^{-k(j - 1)} \frac{[j - j_1 - j_2][j - j_1 + j_2 - 1][2j_1 - 1]}{[k][j - j_1 - j_2 - k][j - j_1 + j_2 - k - 1][2j_1 + k - 1]}.
\]

Now we use the identity (e.g. [11])
\[
\sum_{k \geq 0} q^{-\frac{2k}{2}} \frac{1}{[k][b - k][c - k][a - b - c + k]} = q^{-bc} \frac{[a]!}{[b]![c]![a - b]![a - c]!}.
\]

In our case we have \( a = 2j - 2, b = j - j_1 - j_2, c = j - j_1 + j_2 - 1 \). Consequently
\[
a_0 = q^{\frac{1}{2}(j-j_1-j_2)(j-j_1+j_2-1)} \left\{ \frac{[j - j_2 + j_1 - 1][j + j_1 + j_2 - 2][2j_1 - 1]}{[2j - 2][2j_1 - 1]} \right\}^{1/2}.
\]

After substitution this expression into (15) and (13) we finally obtain the exact formula for (13)
\[
e_j^{j_1 j_2} = \sum_{m_1, m_2} \left[ \begin{array}{c} j_1 \\ m_1 \\ j_2 \\ m_2 \\ j \\ j \end{array} \right]_{SU(1,1)} e_{m_1}^{j_1} \otimes e_{m_2}^{j_2},
\]
From the formula (9) and (17), (18) we deduce that all factorials in the denominator are nonnegative. On the other side, from Theorem 2 it is easy to see that there exists the following relation between the Clebsch-Gordan and Racah-Wigner coefficients for $U_q(SU(1, 1))$ from the decomposition

$$e^{j_1 j_2}_{m} = \sum_{m_1, m_2} \left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_{q}^{SU(1, 1)} e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}. \tag{18}$$

From the formula (9) and (17), (18) we deduce

**Theorem 2** (Formula for the Clebsch-Gordan coefficients).

$$\left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_{q}^{SU(1, 1)} = \delta_{m_1 + m_2, m} \cdot (-1)^{j_1 - m_1} q^{2(c_j + c_{j_1} - c_{j_2}) - \frac{m + m_1 - 1}{2}} \tilde{\Delta}(j_1 j_2 j) \cdot \left\{ \frac{[2j - 1][m - j][m_1 - j_1][m_1 + j_1 - 1][m_2 - j_2][m_2 + j_2 - 1]}{[m + j - 1]} \right\}^{1/2} \cdot \sum_{r \geq 0} (-1)^{r} q^{\frac{r}{2}} \cdot \frac{1}{[r][m - j - r][m_1 - j_1 - r][m_1 + j_1 - r - 1]} \cdot \frac{1}{[j_2 - m_1 + r][j_2 - m_1 + r - 1]} \cdot \left\{ \frac{[2j - 1][m - j][m_1 - j][m_1 + j_1 - 1][m_2 - j_2][m_2 + j_2 - 1]}{[m + j - 1]} \right\}^{1/2} \cdot \tilde{\Delta}(j_1 j_2 j) = \left[ \begin{array}{ccc} j_1 - j_2 & j_2 - j_1 & j \\ m_1 & m_2 & m \end{array} \right]_{q}^{SU(1, 1)}.$$ 

where

$$c_j = j(j - 1),$$

$$\tilde{\Delta}(j_1 j_2 j) = \left[ \begin{array}{ccc} j - j_1 & j - j_2 & j \\ m_1 & m_2 & m \end{array} \right]_{q}^{SU(1, 1)} = \left[ \begin{array}{ccc} j_1 - j_2 & j_2 - j_1 & j \\ m_1 & m_2 & m \end{array} \right]_{q}^{SU(1, 1)}.$$ 

Note that formula (19) may be obtained from [10], formula (3.4), by the formal replacements $m \rightarrow -m, j \rightarrow -j, j_\alpha \rightarrow j_\alpha, m_\alpha \rightarrow -m_\alpha$ ($\alpha = 1, 2$), $q \rightarrow q^{-1}$ and $[-n]! \rightarrow \frac{1}{[n - 1]!}$ if $n \geq 0$. Let us recall that the summation in (19) is taken only over such $r$ that all factorials in the denominator are nonnegative. On the other side, from Theorem 2 it is easy to see that there exists the following relation between the Clebsch-Gordan coefficients for $U_q(SU(2))$ and $U_q(SU(1, 1))$. 

---

**Clebsch-Gordan and Racah-Wigner coefficients for $U_q(SU(1, 1))**
Theorem 3. We have
\[
\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}^{SU(2)}_q = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi \\ n_1 & n_2 & n \end{bmatrix}^{SU(1,1)}_{q^{-1}},
\]
where
\[
\begin{align*}
j_1 &= \frac{1}{2}(n_1 + n_2 + \varphi_2 - \varphi_1 - 1), & n_1 &= \frac{1}{2}(j_1 + j_2 - m_1 + m_2 + 1), \\
j_2 &= \frac{1}{2}(n_1 + n_2 + \varphi_1 - \varphi_2 - 1), & n_2 &= \frac{1}{2}(j_1 + j_2 + m_1 - m_2 + 1), \\
m_1 &= \frac{1}{2}(n_2 - n_1 + \varphi_1 + \varphi_2 - 1), & \varphi_1 &= \frac{1}{2}(j_2 - j_1 + m_1 + m_2 + 1), \\
m_2 &= \frac{1}{2}(n_1 - n_2 + \varphi_1 + \varphi_2 - 1), & \varphi_2 &= \frac{1}{2}(j_1 - j_2 + m_1 + m_2 + 1), \\
j &= \varphi - 1, & m &= \varphi_1 + \varphi_2 - 1, & \varphi &= j + 1, & n &= j_1 + j_2 + 1.
\end{align*}
\]
The theorem 3 is the quantum analog of the corresponding classical result (for \( q = 1 \), see e.g. [13]).

The symmetry’s properties for the Clebsch-Gordan coefficients of the algebra \( U_q(SU(1,1)) \) follows according to the Theorem 3 from the corresponding ones for the algebra \( U_q(SU(2)) \) (e.g. [11]). Here we mention only one.

Corollary 4. We have
\[
\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}^{SU(1,1)}_q = (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}^{SU(1,1)}_{q^{-1}}.
\]

Similarly, it is possible to define the decomposition into irreducible component of the tensor product of the irreducible representations for the negative discrete series and to compute the corresponding Clebsch-Gordan coefficients. We give only the answer.

Theorem 5. Assume that \( V^{j_1} \) and \( V^{j_2} \) lies in the negative discrete series for \( U_q(SU(1,1)) \). Then
\begin{itemize}
\item[a)] \( V^{j_1} \otimes V^{j_2} = \bigoplus_{j \leq j_1+j_2} V^j, j - j_1 - j_2 \in \mathbb{Z}^-; \)
\item[b)] \[
\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}^{SU(1,1)}_q = \delta_{m_1+m_2,m}(-1)^{j_1-m_1} \Delta(-j_1,-j_2,-j)q^{\frac{1}{2}(c_j+c_{j_1}-c_{j_2})+m_1(m_1+1)}
\cdot \left\{ \left[ \sum \frac{(-1)^r q^{\frac{r}{2}((m+j+1)} \left[ r!(j-m-r)!(j_1-m_1-r)!(m_1-j_1-r-1)! \right]^{1/2} \right]^{1/2} \right\}^{1/2} \left[ \sum \frac{(-1)^r q^{\frac{r}{2}((m+j+1)} \left[ r!(j-m-r)!(j_1-m_1-r)!(m_1-j_1-r-1)! \right]^{1/2} \right]^{1/2} \right\}^{1/2}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\cdot \frac{1}{[j_2-j+m_1+r]![j_1-j_2+m_1+r-1]!}
\end{bmatrix}^{SU(1,1)}_q = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi \\ n_1 & n_2 & n \end{bmatrix}^{SU(1,1)}_{q^{-1}},
\]
\end{itemize}

Remark. The formula (21) may be obtained from (19) by the replacements \( m_\alpha, j_\alpha, m, j, q \) on \( -m_\alpha, -j_\alpha, -m, -j, q^{-1} \) (\( \alpha = 1, 2 \)).
n°5. Quantum Racah-Wigner coefficients for $U_q(SU(1,1))$.

In this section we consider the $q$-analog of a $6j$-symbols for the tensor product $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ of three irreducible representations of the positive discrete series for $U_q(SU(1,1))$. As in the case of the algebra $U_q(SU(2))$, there are two ways to obtain an irreducible components in this tensor product. One is to decompose first $V^{j_1} \otimes V^{j_2} = \oplus_{j_{12}} V^{j_{12}}$ and then to take an irreducible submodule in $V^{j_{12}} \otimes V^{j_3}$. The other is to decompose first $V^{j_2} \otimes V^{j_3} = \oplus_{j_{23}} V^{j_{23}}$ and then $V^{j_1} \otimes V^{j_{23}}$. These two ways give two complete orthogonal bases in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$:

$$
e^{j_{12}j}_{m}(j_1 j_2 | j_3) = \sum_{m_{1},m_{2},m_{3}} \left[ \begin{array}{ccc} j_{12} & j_3 & j \\ m_{12} & m_3 & m \\ j_1 & j_2 & j_12 \end{array} \right]_{q}^{SU(1,1)} \left[ \begin{array}{ccc} j_1 & j_2 & j_12 \\ m_1 & m_2 & m_{12} \end{array} \right]_{q}^{SU(1,1)} \cdot e^{j_{1}j}_{m_{1}} \otimes e^{j_{2}j}_{m_{2}} \otimes e^{j_{3}j}_{m_{3}}; \quad (22)$$

$$
e^{j_{23}j}_{m}(j_1 | j_2 j_3) = \sum_{m_{1},m_{2},m_{3}} \left[ \begin{array}{ccc} j_1 & j_23 & j \\ m_{1} & m_{23} & m \\ j_1 & j_3 & j_23 \end{array} \right]_{q}^{SU(1,1)} \left[ \begin{array}{ccc} j_2 & j_3 & j_23 \\ m_2 & m_3 & m_{23} \end{array} \right]_{q}^{SU(1,1)} \cdot e^{j_{1}j}_{m_{1}} \otimes e^{j_{2}j}_{m_{2}} \otimes e^{j_{3}j}_{m_{3}}. \quad (23)$$

The matrix elements of the matrix, connecting these bases will be called $SU(1,1) q$-$6j$-symbols:

$$
e^{j_{23}j}_{m}(j_1 j_2 | j_3) = \sum_{j_{23}} \left[ \begin{array}{ccc} j_1 & j_2 & j_12 \\ j_3 & j & j_23 \end{array} \right]_{q}^{SU(1,1)} \cdot e^{j_{23}j}_{m}(j_1 | j_2 j_3). \quad (24)$$

Using the graphical technique (e.g. [10]) we may rewrite the definition (24) of $q$-$6j$-symbols in the form

$$\begin{array}{cccccc}
\begin{array}{ccc}
j_1 & j_2 & j_3 \\
j_1 & j_2 & j_3 \\
\end{array}
&=& \sum_{j_{23}} \left[ \begin{array}{ccc} j_1 & j_2 & j_12 \\ j_3 & j & j_23 \end{array} \right]_{q}^{SU(1,1)} \\
\end{array} \quad j$$

Acting by the same way as in the case of Hopf algebra $U_q(sl(2))$ (e.g. [10]), we may find the formula for $q$-$6j$-symbols for an irreducible representations of the positive discrete series for $U_q(SU(1,1))$. The answer may be obtained from [10], formula (5.7) by the replacements all $j$’s on $-j$’s and $|n|!$ on $\frac{1}{(n-n)!}$, if $n > 0$. However, it is possible to receive for $q$-$6j$-symbols the result of the type (20).

Theorem 6.

$$\left\{ \begin{array}{ccc} a & b & c \\
d & e & f \end{array} \right\}_{q}^{SU(1,1)} = \left\{ \begin{array}{ccc} \alpha & \beta & \varepsilon \\
\delta & \gamma & \varphi \end{array} \right\}_{q}^{SU(2)}.$$
where
\[ \alpha = \frac{a + b + c + d}{2} - 1; \quad a = \frac{\alpha - \beta + \gamma - \delta + 1}{2}; \]
\[ \beta = \frac{c - a - b - d}{2}; \quad b = \frac{\alpha - \beta - \gamma + \delta}{2}; \]
\[ \gamma = \frac{a + c + d - b}{2} - 1; \quad c = \frac{\alpha + \beta + \gamma + \delta + 1}{2}; \]
\[ \delta = \frac{b + c + d - a}{2} - 1; \quad d = \frac{\gamma + \delta - \alpha - \beta}{2}; \]
\[ \varepsilon = e - 1; \quad e = \varepsilon + 1; \]
\[ \varphi = f - 1; \quad f = \varphi + 1; \]

So, it is easy to see that \( q \)-6j-symbols \( \{ a \ b \ e \ \choose \ d \ c \ f \}_q \) satisfies the orthogonality relation, the Racah identity, the Biedenharn-Elliot identity and the face variant of quantum Yang-Baxter equation, e.g. see identities (6.16)-(6.19) from [10].

After the completion of this note the authors were known about the work of Y. Shibukawa [14] which also contains the calculation of the Clebsch-Gordan coefficients for the positive discrete series of the algebra \( U_q(SU(1,1)) \).

References
1. Kulish, P.P., Reshetikhin N.Yu, Quantum linear problem for the Sine-Gordon equation and higher representations, Zap. Nauch. Semin. LOMI 101 (1980) 101-110 (in Russian).
2. Jimbo, M., A \( q \)-difference analog of \( U(\mathfrak{g}) \) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63–69.
3. Sklyanin E.K., Uspehi Mat. Nauk, 40 (1985), N2, 214 (in Russian).
4. Faddeev, L., Reshetikhin, N.Yu., and Takhtajan, L.A., Quantization of Lie algebras and Lie groups, Algebra i Analiz 1 (1989), N1 (in Russian).
5. Drinfeld V.G., Quantum groups, Proc. ICM 1 (Berkeley Academic Press, 1986), 798–820.
6. Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., Saburi, Y., and Ueno, K., Unitary representations of the quantum group \( SU_q(1,1) \) I, II, Lett. Math. Phys. 19 (1990), 187–204.
7. Soybelman, Y.L, and Vaksman, L.L., Algebra functions on quantum group \( SU(2) \), Funk Anal. i appl. 22 (1988) N3, 1-14 (in Russian).
8. Vaksman, L.L., and Korogodsky, L.I., Harmonic analysis on quantum hyperbolids, Preprint, 1990 (in Russian).
9. Kirillov, A.N., and Reshetikhin, N.Yu., Representations of the algebra \( U_q(sl(2)) \), \( q \)-orthogonal polynomials and invariants of links, LOMI Preprint E-9-88, Leningrad, 1988.
10. Kirillov, A.N., Quantum Clebsch-Gordon coefficients, Zap. Nauch. Semin. LOMI, 168 (1988), 67-84 (in Russian).
12. Podleˇc, P., Complex quantum groups and their real representations, *RIMS Preprint*, Kyoto Univ., 754, May 1991.

13. Barut, A., and Raczk, R., *Theory of group representations and applications*, (Warszawa PWN-Polish Scientific Publishers, 1980), 717p.

14. Shibukawa Y., Clebsch-Gordan coefficients for $U_q(SU(1,1))$ and $U_q(sl(2))$, and linearization formula of matrix elements. Preprint 1991.