FACTORIZATIONS OF RELATIVE EXTREMAL PROJECTORS

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Abstract. We survey earlier results on factorizations of extremal projectors and relative extremal projectors and present preliminary results on non-commutative factorizations of relative extremal projectors: we deduce the existence of such factorizations for \(\mathfrak{sl}_4\) and \(\mathfrak{sl}_5\).

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This article is dedicated to V. S. Varadarajan on the occasion of his retirement.

1. Introduction

Extremal projectors were first investigated in the 1960's and 1970's by Asherova, Smirnov, and Tolstoi; their results are summarized in [AST79]. Zhelobenko wrote a series of articles on projectors in the 1980's and 1990's, including the survey monograph [Zh90]. In this note we will focus on relative extremal projectors, but we begin with a brief account of extremal projectors. We will take the liberty of attributing to [AST79] and [Zh90] some results which were in fact first discovered in earlier works of the same authors: see the references of those papers.

Given any Lie algebra \(\mathfrak{g}\), we have the universal enveloping algebra \(U(\mathfrak{g})\) and the augmentation ideal \(U^+(\mathfrak{g}) := \mathfrak{g}U(\mathfrak{g})\). If \(W\) is a \(\mathfrak{g}\)-module, we write \(W^\mathfrak{g}\) for the space of \(\mathfrak{g}\)-invariants in \(W\). Throughout this article we write \(\mathbb{N}\) for the non-negative integers and \(\mathbb{Z}^+\) for the positive integers.

Let \(\mathfrak{g}\) be a finite dimensional complex reductive Lie algebra, \(\mathfrak{h}\) a Cartan subalgebra, and \(\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\) a triangular decomposition of \(\mathfrak{g}\). We denote the associated positive and simple root systems by \(\Delta(\mathfrak{n}^+)\) and \(\Pi(\mathfrak{n}^+)\), respectively. More generally, given any \(\mathfrak{h}\)-module \(V\) we write \(\Delta(V)\) for its weights in \(\mathfrak{h}^*\) and \(V_\mu\) for its \(\mu\)-weight space. Throughout the article, all \(\mathfrak{h}\)-invariant subalgebras of \(\mathfrak{g}\) will be endowed with the positive and negative systems inherited from \(\mathfrak{g}\). We sometimes write \(\mathfrak{g}_{ss}\) for the semisimple part of \(\mathfrak{g}\) and \(\mathfrak{z}(\mathfrak{g})\) for its center, so that \(\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})\).

One may ask the following naive question: is there an element of \(U(\mathfrak{g})\) which projects any representation \(V\) of \(\mathfrak{g}\) in the category \(O(\mathfrak{g})\) to its highest weight space \(V^\mathfrak{n}^+\) along the sum of its lower weight spaces \(\mathfrak{n}^- V\)? The answer is no, but there is such an element in a certain extension \(\mathfrak{F}(\mathfrak{g})\) of \(U(\mathfrak{g})\), the extremal projector \(P(\mathfrak{g})\). Its action is defined on all weight spaces \(V_\mu\) such that

\[
V^\mathfrak{n}^+ \cap \mathfrak{n}^- V = 0.
\]

In order to define \(\mathfrak{F}(\mathfrak{g})\), fix an \(\mathfrak{sl}_2\)-triple \(\{E_\alpha, F_\alpha, H_\alpha\}\) for each positive root \(\alpha\) in \(\Delta(\mathfrak{n}^+)\). Thus \(E_\alpha\) spans \(\mathfrak{n}^+_\alpha\), \(F_\alpha\) spans \(\mathfrak{n}^-\alpha\), \(H_\alpha\) is the element \([E_\alpha, F_\alpha]\) of \(\mathfrak{h}\), and

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\( \alpha(H_a) = 2 \). We index the positive roots and use multinomial notation:

\[
\Delta(n^+) := \{ \alpha_1, \ldots, \alpha_m \}, \quad E^I := E_{\alpha_1} \cdots E_{\alpha_m}, \quad F^I := F_{\alpha_1} \cdots F_{\alpha_m},
\]

where \( I \in \mathbb{N}^m \). For any \( K \in \mathbb{Z}^m \), let \( |K| \) denote the weight \( \sum_{r=1}^m K_r \alpha_r \).

**Definition.** \( \mathfrak{F}(g) \) is the algebraic direct sum \( \bigoplus_{\gamma \in \Delta(\mathfrak{u}(g))} \mathfrak{F}(g)_\gamma \) of its weight spaces, where \( \mathfrak{F}(g)_\gamma \) is the space of formal series in the monomials \( F^I E^J \) of weight \( \gamma \) with coefficients in the fraction field \( \text{Frac} \mathfrak{u}(h) \) of \( \mathfrak{u}(h) \):

\[
\mathfrak{F}(g)_\gamma := \left\{ \sum_{|J-I| = \gamma} F^I E^J h_{IJ} : h_{IJ} \in \text{Frac} \mathfrak{u}(h) \right\}.
\]

The commutation relations of \( \mathfrak{u}(g) \) extend to an algebra structure on \( \mathfrak{F}(g) \). Recall that the Cartan involution \( \theta \) of \( g \) exchanges \( E_\alpha \) and \( -F_\alpha \) for \( \alpha \) simple and is \(-1\) on \( \mathfrak{h} \). Let \( \Omega \mapsto \Omega^* \) be the Hermitian anti-involution of \( \mathfrak{F}(g) \), which is \(-\theta\) on \( g \). Elements of \( \mathfrak{F}(g) \) fixed by this anti-involution are called Hermitian. For reference, note that \( g = \mathfrak{h} \) gives

\[
\mathfrak{F}(h) = \text{Frac} \mathfrak{u}(h).
\]

**Theorem 1.1.** [AST179] There is a unique non-zero Hermitian idempotent \( P(g) \) in \( \mathfrak{F}(g)_0 \), the extremal projector, such that

\[
n^+ P(g) = 0 = P(g)n^-.
\]

The universal Verma module \( M(g) \) is \( \mathfrak{F}(g)/\mathfrak{F}(g)n^+ \). It is a two-sided \( \mathfrak{F}(h) \)-module, spanned freely by the image of \( \mathfrak{u}(n^-) \) under both the right and left actions. Under the adjoint action of \( \mathfrak{h} \), \( F^I \) has weight \(-|I|\) and \( M(g) \) is the direct sum of its weight spaces:

\[
M(g) = \bigoplus_{\nu \in \Delta(\mathfrak{u}(n^+))} M(g)_{-\nu}, \quad M(g)_{-\nu} = \text{Span}_{\mathfrak{F}(h)} \{ F^I : |I| = \nu \}.
\]

Write \( \text{End}_{\rho(h)} M(g) \) for the endomorphisms of \( M(g) \) commuting with the right action \( \rho \) of \( \mathfrak{h} \). Since \( \mathfrak{F}(g) \) acts on \( M(g) \) from the left, there is a natural homomorphism from \( \mathfrak{F}(g) \) to \( \text{End}_{\rho(h)} M(g) \).

The Shapovalov form \( \langle \cdot, \cdot \rangle \) on \( M(g) \) is \( \mathfrak{F}(h) \)-valued: for \( \Theta \) and \( \Theta' \) in the space \( \mathfrak{u}(n^-) \mathfrak{F}(h) \), \( \langle \Theta, \Theta' \rangle \) is the projection of the weight zero component of \( \Theta^* \Theta' \) to \( \mathfrak{F}(h) \) along \( (n^- \mathfrak{F}(g)n^+)_0 \). It is non-degenerate, symmetric, and right \( \mathfrak{h} \)-bilinear, and Hermitian elements of \( \mathfrak{F}(g) \) act on \( M(g) \) by Hermitian operators.

The universal Verma module is the Hermitian direct sum of its highest weight space \( M(g)^{n^+} \) and the sum of its lower weight spaces \( n^- M(g) \):

\[
M(g)^{n^+} = M(g)_0, \quad n^- M(g) = \bigoplus_{\nu \in \Delta(\mathfrak{u}(n^+))} M(g)_{-\nu},
\]

and

\[
M(g) = M(g)^{n^+} \oplus n^- M(g).
\]

**Theorem 1.2.** [Zh90] The homomorphism from \( \mathfrak{F}(g) \) to \( \text{End}_{\rho(h)} M(g) \) is an isomorphism. It carries \( P(g) \) to the projection from \( M(g) \) to \( M(g)^{n^+} \) along \( n^- M(g) \).
2. Factorizations of the extremal projector

2.1. Non-commutative factorizations. As usual, write \( w_0 \) for the longest element of the Weyl group \( W(\mathfrak{g}) \), \( \rho_\theta \) for the half-sum \( \frac{1}{2} \sum_r \Delta(\mathfrak{n}^+) \alpha \) of the positive roots, and \( s_\alpha \) for the reflection in a root \( \alpha \).

Definition. An ordering \( (\alpha_1, \ldots, \alpha_m) \) of \( \Delta(\mathfrak{n}^+) \) is normal if whenever \( \alpha_r + \alpha_s \) is a root \( \alpha_t \), then either \( r < t < s \) or \( s < t < r \).

Proposition 2.1. [Zh87] Normal orders are in bijection with reduced expressions for \( w_0 \): if \( \alpha_1, \ldots, \alpha_m \) are elements of \( \Pi(\mathfrak{n}^+) \) (usually not distinct) such that \( w_0 = s_{\alpha_1'} \cdots s_{\alpha_m'} \), then \( (\alpha_1, \ldots, \alpha_m) \) is a normal order of \( \Delta(\mathfrak{n}^+) \) for \( \alpha_r = s_{\alpha_1'} \cdots s_{\alpha_{r-1}'} \alpha_r \).

Conversely, if \( (\alpha_1, \ldots, \alpha_m) \) is a normal order of \( \Delta(\mathfrak{n}^+) \), then the roots \( \alpha_r := s_{\alpha_1} \cdots s_{\alpha_{r-1}} \alpha_r \) are in \( \Pi(\mathfrak{n}^+) \) and \( w_0 = s_{\alpha_1'} \cdots s_{\alpha_m'} \).

The main result of [AST79] is a beautiful class of explicit formulas for \( P(\mathfrak{g}) \), non-commutative factorizations along normal orders of \( \Delta(\mathfrak{n}^+) \). For \( \mathfrak{sl}_2 \), it is an enjoyable exercise to prove

\[ P(\mathfrak{sl}_2) = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k F^k E^k \prod_{i=1}^{k} (H + 1 + i)^{-1}. \]

The AST factors of \( P(\mathfrak{g}) \) generalize \( P(\mathfrak{sl}_2) \). For \( t \in \mathbb{C} \), define

\[ Q_t(\mathfrak{sl}_2) := \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k F^k E^k \prod_{i=1}^{k} (H + t + i)^{-1} \in \mathfrak{g}(\mathfrak{sl}_2). \]

Note that \( Q_1(\mathfrak{sl}_2) = P(\mathfrak{sl}_2) \). For \( \alpha \in D(\mathfrak{n}^+) \), let \( \mathfrak{a}_\alpha \) be the copy of \( \mathfrak{sl}_2 \) in \( \mathfrak{g} \) corresponding to \( \alpha \), and let \( t_\alpha \) be its augmentation by \( \mathfrak{h} \):

\[ a_\alpha := \text{Span}_\mathbb{C} \{ E_\alpha, F_\alpha, H_\alpha \}, \quad t_\alpha := \mathfrak{h} + a_\alpha. \]

We remark that \( P(t_\alpha) \) and \( P(a_\alpha) \) are the same. Indeed, \( P(\mathfrak{g}) = P(\mathfrak{g}_\infty) \) by definition.

Definition. Fix a normal order \( (\alpha_1, \ldots, \alpha_m) \) of \( \Delta(\mathfrak{n}^+) \). For \( \tau \in \mathfrak{h}^* \), set

\[ Q_\tau(\mathfrak{g}) := \prod_{r=1}^{m} Q_{\tau(H_{\alpha_r})}(a_{\alpha_r}). \]

Theorem 2.2. [AST79] For any normal ordering of \( \Delta(\mathfrak{n}^+) \), \( P(\mathfrak{g}) = Q_{\rho_\theta}(\mathfrak{g}) \).

Observe that \( \rho_\theta(H_{\alpha}) \in \mathbb{Z}^+ \) for \( \alpha \in \Delta(\mathfrak{n}^+) \), so those \( Q_t \) occurring as AST factors of \( P(\mathfrak{g}) \) have \( t \in \mathbb{Z}^+ \). Theorem 2.2 is in fact a corollary of the following more general result, discovered later by Zhelobenko.

Theorem 2.3. [Zh90] For all \( \tau \in \mathfrak{h}^* \), \( Q_\tau(\mathfrak{g}) \) is independent of the choice of normal order of \( \Delta(\mathfrak{n}^+) \).
2.2. Denominators. In an obvious sense, the total denominator of each of the factorizations of \( P(\mathfrak{g}) \) in Theorem 2.2 is the commutative formal product

\[
D(\mathfrak{g}) := \prod_{i=1}^{\infty} \prod_{\alpha \in \Delta(n^+)} (H_\alpha + \rho_0(H_\alpha) + i).
\]

This has the following implication: if \( V \) is any representation in \( \mathcal{O}(\mathfrak{g}) \) and \( \mu \) is any weight on which no factor of \( D(\mathfrak{g}) \) is zero, then the formula of Theorem 2.2 defines an action of \( P(\mathfrak{g}) \) on \( V_\mu \).

Proposition 2.4. \( D(\mathfrak{g}) \) divides the total denominator of any formula for \( P(\mathfrak{g}) \).

Idea of proof. Suppose that \( \mu \) is a weight annihilating some factor \( (H_\alpha + \rho_0(H_\alpha) + i) \) of \( D(\mathfrak{g}) \). By (4), it suffices to find an object \( V \) of \( \mathcal{O}(\mathfrak{g}) \) with \( V_\mu^{n^+} \cap n^- V \neq 0 \).

Let \( \cdot \) denote the affine dot action

\[
w \cdot \mu := w(\mu + \rho_0) - \rho_0
\]

of \( W(\mathfrak{g}) \) on \( \mathfrak{h}^* \). Then \( s_\alpha \cdot \mu - \mu = i\alpha \), so by a well known result of Bernstein, Gel'fand, and Gel'fand, the Verma module \( M(\mathfrak{g}, s_\alpha \cdot \mu) \) of \( \mathfrak{g} \) with highest weight \( s_\alpha \cdot \mu \) satisfies

\[
M(\mathfrak{g}, s_\alpha \cdot \mu)^{n^+} \cap n^- M(\mathfrak{g}, s_\alpha \cdot \mu) \neq 0.
\]

2.3. Infinite commutative factorizations. Let \( \mathfrak{z}(\mathfrak{g}) \) be the center of \( \mathfrak{u}(\mathfrak{g}) \). Gel'fand discovered an infinite commutative factorization of \( P(\mathfrak{g}) \) built from the Casimir element \( \Omega_2 \) of \( \mathfrak{z}(\mathfrak{g}) \). In order to describe it we must extend the dot action of \( W(\mathfrak{g}) \) to \( \mathfrak{u}(\mathfrak{h}) \) and define the shift action of \( \mathfrak{h}^* \) on \( \mathfrak{u}(\mathfrak{h}) \). Regard elements \( h \) of \( \mathfrak{u}(\mathfrak{h}) \) as polynomials on \( \mathfrak{h}^* \), and for \( w \in W(\mathfrak{g}) \) and \( \nu, \mu \in \mathfrak{h}^* \), set

\[
(wh)(\mu) := h(w^{-1} \mu), \quad (w \cdot h)(\mu) := h(w^{-1} \cdot \mu), \quad h^\nu(\mu) := h(\nu + \mu).
\]

Write \( \mathfrak{u}(\mathfrak{h})^{W(\mathfrak{g})^*} \) for the subalgebra of \( \mathfrak{u}(\mathfrak{h}) \) invariant under the dot action.

Recall that the subalgebra \( \mathfrak{u}(\mathfrak{g})^b = \mathfrak{u}(\mathfrak{g})_0 \) of \( \mathfrak{u}(\mathfrak{g}) \) decomposes as \( (n^- \mathfrak{u}(\mathfrak{g}) n^+)_0 \oplus \mathfrak{u}(\mathfrak{h}) \), where \( (n^- \mathfrak{u}(\mathfrak{g}) n^+)_0 \) is a two-sided ideal. The Harish-Chandra homomorphism \( HC_\rho \) is the associated projection from \( \mathfrak{u}(\mathfrak{g})^b \) to \( \mathfrak{u}(\mathfrak{h}) \). By a well known result of Harish-Chandra, it restricts to an isomorphism

\[
HC_\rho : \mathfrak{z}(\mathfrak{g}) \rightarrow \mathfrak{u}(\mathfrak{h})^{W(\mathfrak{g})^*}.
\]

Theorem 2.5. \( [Zh93] \) As an element of \( \text{End}_{\mathfrak{u}(\mathfrak{h})} M(\mathfrak{g}) \),

\[
P(\mathfrak{g}) = \prod_{\nu \in \Delta(\mathfrak{u}(n^+))} \frac{\Omega_2 - (HC_\rho \Omega_2)^\nu}{(HC_\rho \Omega_2 - HC_\rho \Omega_2)^\nu}.
\]

Idea of proof. The infinite product is interpreted as follows: if the factors are applied successively to any \( \Theta \in M(\mathfrak{g}) \), the resulting sequence eventually stabilizes. To prove that it stabilizes at \( P(\mathfrak{g})\Theta \), recall (4) and check that \( \Omega_2 \) acts on \( M(\mathfrak{g})_{-\nu} \) by \( (HC_\rho \Omega_2)^\nu \) for all \( \nu \in \mathfrak{u}(n^+) \). Therefore the \( \nu \)-factor of the infinite product acts by 0 on \( M(\mathfrak{g})_{-\nu} \) and by 1 on \( M(\mathfrak{g})_0 \), so the entire product acts by 1 on \( M(\mathfrak{g})_0 \) and by 0 on all other weight spaces. Now apply (4) and Theorem 1.2 \( \square \)
It was observed in [CS05] that for \( g \) simple, Theorem 2.3 holds for any non-constant element \( \Omega \) of \( \mathfrak{Z}(g) \) replacing \( \Omega_2 \). For \( \mathfrak{sl}_2 \), it may be rewritten as

\[
P(\mathfrak{sl}_2) = \prod_{i=1}^{\infty} \left( 1 - \frac{FE}{i(H + 1 + i)} \right).
\]

It is an intriguing fact that the AST factors \( Q_t \) occurring in Theorem 2.2 have themselves a similar infinite commutative factorization. In a natural telescopic sense explained in Theorem 15 of [CS05], for \( t \in \mathbb{Z}^+ \) we have

\[
(10) \quad Q_t(\mathfrak{sl}_2) = \prod_{i=t}^{\infty} \left( 1 - \frac{FE}{i(H + 1 + i)} \right).
\]

In particular, \( Q_t(\mathfrak{sl}_2) \) annihilates all but the highest \( t \) weight spaces of \( M(\mathfrak{sl}_2) \): its image is \( \bigoplus_{i=0}^{t-1} M(\mathfrak{sl}_2)_{-2i} \). However, for \( t > 1 \) it is not the Hermitian projection operator onto this sum.

### 3. The relative extremal projector

Let \( \mathfrak{l} \) be a standard reductive subalgebra of \( g \), i.e., the Levi subalgebra of a standard parabolic subalgebra. Thus \( \mathfrak{l} \) contains \( \mathfrak{h} \) and has triangular decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}^+ \), where \( \mathfrak{l}^\pm := \mathfrak{l} \cap \mathfrak{n}^\pm \), and its positive root system \( \Delta(\mathfrak{l}^+) \) has simple system

\[
\Pi(\mathfrak{l}^+) = \Delta(\mathfrak{l}^+) \cap \Pi(\mathfrak{n}^+).
\]

Let \( \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+ \) be the \( \mathfrak{l} \)-invariant decomposition of \( g \) such that \( \mathfrak{u}^\pm \subset \mathfrak{n}^\pm \).

The relative extremal projector is introduced in the next two theorems, which are parallel to Theorems 1.1 and 1.2.

**Theorem 3.1.** [CS03] There is a unique non-zero Hermitian idempotent \( P(\mathfrak{g}, \mathfrak{l}) \) in \( \mathfrak{S}(\mathfrak{g})_0 \), the relative extremal projector, which commutes with \( \mathfrak{l} \) and satisfies

\[
\mathfrak{u}^+ P(\mathfrak{g}, \mathfrak{l}) = 0 = P(\mathfrak{g}, \mathfrak{l})\mathfrak{u}^-.
\]

Recall that \( \mathfrak{S}(\mathfrak{g}) \) is identified with \( \text{End}_{\rho(\mathfrak{h})} M(\mathfrak{g}) \). The relative analog of (1) was observed in Lemma 1 of [CS03]. In order to state it, we must explain how \( M(\mathfrak{g}) \) decomposes as a direct sum of copies of \( M(\mathfrak{l}) \).

Note that those monomials \( F^I \) from (2) which are contained in \( \mathfrak{U}(\mathfrak{u}^-) \) form a basis of it. The extremal projector \( P(\mathfrak{l}) \) of \( \mathfrak{l} \) maps each of them to a non-zero highest weight vector \( P(\mathfrak{l})(F^I) \) in \( M(\mathfrak{g}) \). Applying \( \mathfrak{U}(\mathfrak{g}^-) \mathfrak{S}(\mathfrak{h}) \) to this highest weight vector gives an \( \mathfrak{l} \)-submodule of \( M(\mathfrak{g}) \) isomorphic to \( M(\mathfrak{l}) \), and \( M(\mathfrak{g}) \) is the Hermitian direct sum of these submodules:

\[
(11) \quad M(\mathfrak{g}) = \bigoplus_{F^I \in \mathfrak{U}(\mathfrak{u}^-)} \mathfrak{U}(\mathfrak{g}^-) \mathfrak{S}(\mathfrak{h}) P(\mathfrak{l})(F^I).
\]

The highest \( \mathfrak{l} \)-submodule \( M(\mathfrak{g})\mathfrak{u}^+ \) of \( M(\mathfrak{g}) \) is the copy of \( M(\mathfrak{l}) \) generated by 1, and the lower submodules are the other copies:

\[
M(\mathfrak{g})\mathfrak{u}^+ = \mathfrak{U}(\mathfrak{g}^-) \mathfrak{S}(\mathfrak{h}),
\]

\[
(12) \quad \mathfrak{u}^- M(\mathfrak{g}) = \bigoplus_{F^I \in \mathfrak{U}^+(\mathfrak{u}^-)} \mathfrak{U}(\mathfrak{g}^-) \mathfrak{S}(\mathfrak{h}) P(\mathfrak{l})(F^I),
\]

\[
M(\mathfrak{g}) = M(\mathfrak{g})\mathfrak{u}^+ \oplus \mathfrak{u}^- M(\mathfrak{g}).
\]
Theorem 3.2. [CS03] As an element of \( \text{End}_{\rho(b)} M(g) \), the relative extremal projector \( P(g,l) \) is the projection from \( M(g) \) to \( M(g)^{u^+} \) with kernel \( u^- M(g) \).

Because \( P(g,l) \) commutes with \( l \), we seek formulas for it whose terms lie in \( \mathfrak{Z}(g)^l \), the commutant of \( l \) in \( \mathfrak{Z}(g) \). In the relative case, the natural analogs of the factorizations of \( P(g) \) discussed above have factors with numerators in \( \Omega(l)^l \) and denominators in the center \( \mathfrak{Z}(l) \) of \( \mathfrak{M}(l) \). Such denominators are allowed because by Lemma 2 of [CS03], non-zero elements of \( \mathfrak{Z}(l) \) are invertible in \( \mathfrak{Z}(g) \).

Remark. By Theorem 6 of [CS03], if \( l' \) is a standard reductive subalgebra of \( l \) then
\[
P(g, l') = P(g, l)P(l, l') = P(l, l')P(g, l).
\]
In particular, \( l' = h \) gives \( P(g) = P(g, l)P(l) = P(l)P(g, l) \), as \( P(g, h) = P(g) \).

Compare this factorization of \( P(g) \) to the AST factorizations from Theorem 2.2

Any normal order of \( \Delta(l^+) \) can be extended to a normal order of \( \Delta(n^+) \) with \( \Delta(l^+) \) all to the left or all to the right. For such normal orders of \( \Delta(n^+) \), the product of those AST factors \( Q_{\varrho_\alpha(\mathfrak{h})}(a_\alpha) \) of \( P(g) \) with \( \alpha \in \Delta(l^+) \) is equal to \( P(l) \). However, one cannot “cancel” this factor \( P(l) \) of \( P(g) \) with the one in (13) for \( l' = h \); \( P(g, l) \) is not in general the product of the non-\( \Delta(l^+) \) AST factors of \( P(g) \).

For example, in obvious notation, let \( \{\alpha_{12}, \alpha_{13}, \alpha_{23}\} \) be a normal order of the positive roots of \( \mathfrak{s}_l \), and write \( a_{ij} \) and \( l_{ij} \) for the subalgebras in (7). Then (13) and Theorem 2.2 give
\[
P(\mathfrak{s}_{13}) = P(\mathfrak{s}_{12}, \mathfrak{s}_{23})P(l_{23}) = P(l_{12})Q_2(a_{13})P(l_{23}),
\]
but \( P(\mathfrak{s}_{12}, \mathfrak{s}_{23}) \) is not equal to \( P(l_{12})Q_2(a_{13}) \). Indeed, \( P(\mathfrak{s}_{12}, \mathfrak{s}_{23}) \) projects \( M(\mathfrak{s}_{13}) \) to \( \bigoplus_{n=0}^{\infty} M(\mathfrak{s}_{13})^{-n\alpha_{23}} \), so it annihilates \( F_{\alpha_{13}} \). On the other hand, \( P(l_{12})Q_2(a_{13}) \) does not:
\[
Q_2(a_{13})(F_{\alpha_{13}}) \text{ is a non-zero } \mathfrak{Z}(h) \text{-multiple of } F_{\alpha_{13}}, \text{ and } P(l_{12})(F_{\alpha_{13}}) \neq 0.
\]

The following lemma is in a sense a converse of (13).

Lemma 3.3. \( P(g, l) \) is the unique element of \( \mathfrak{Z}(g)^l \) such that \( P(\mathfrak{g}, l)P(l) = P(g) \).

Proof. We saw in (11) that as an \( l \)-module, \( M(g) \) is a direct sum of copies of \( M(l) \). Each copy of \( M(l) \) is generated under \( l \) by its \( l \)-highest weight vector, and \( P(l)M(g) \) is the space of all such highest weight vectors. Thus any element \( \pi \) of \( \mathfrak{Z}(g)^l \) is determined by its action on \( P(l)M(g) \). In particular, if \( \pi P(l) = P(g) \), then \( \pi \) must be \( P(g, l) \) because \( P(\mathfrak{g}, l)P(l) = P(g) \). \( \square \)

3.1. Infinite commutative factorizations. In [CS02] and [CS05] we give infinite commutative factorizations of \( P(g, l) \). Theorem 7 of [CS03] is the relative analog of Theorem 2.5, a factorization built from the Casimir element \( \Omega_2 \). Theorem 4 (3) of [CS05] shows that it holds with almost any element of \( \mathfrak{Z}(g) \) replacing \( \Omega_2 \).

Following Section 2.3, write \( \mathfrak{U}(h)^{W(l)} \) for the subalgebra of \( \mathfrak{U}(h) \) invariant under the \( l \)-dot action of the Weyl group \( W(l) \) of \( l \). The Harish-Chandra isomorphism \( \text{HC}_1 : \mathfrak{Z}(l) \rightarrow \mathfrak{U}(h)^{W(l)} \) extends to an isomorphism from \( \text{Frac}(\mathfrak{Z}(l)) \) to \( \mathfrak{Z}(h)^{W(l)} \).

We extend it further by \( t \rightarrow t \) to an isomorphism
\[
\text{HC}_1 : \text{Frac}(\mathfrak{Z}(l))|l| \rightarrow \mathfrak{Z}(h)^{W(l)}|l|.
\]

It is an elementary but crucial observation that although in general \( \rho_l \neq \rho_g \), the \( l \)- and \( g \)-dot actions of \( W(l) \) are the same, because \( W(l) \) stabilizes \( \rho_g - \rho_l \). Therefore we may speak unambiguously of the dot action of \( l \). Taking \( w \in W(g) \), \( \nu \in h^* \), and \( h \in \mathfrak{Z}(h) \), let us observe that
\[
w \cdot (h^\nu) = (w \cdot h)^{\nu} = (wh)^{\nu - \rho_g + \rho_h}, \quad \mathfrak{U}(h)^{W(g)} \subseteq \mathfrak{U}(h)^{W(l)}.
\]
Theorem 3.4. [CS05] Let $\Omega$ be an element of $\mathfrak{z}(\mathfrak{g})$ that has a non-constant component over every simple summand of $\mathfrak{g}$. Then as elements of $\text{End}_{\mathfrak{h}(\mathfrak{g})} M(\mathfrak{g})$,

$$P(\mathfrak{g}, l) = \left( \frac{\text{HC}_{1}^{-1}}{l} \prod_{\nu \in \Delta(\mathfrak{u}^+(\mathfrak{u}^{+}))} \frac{t - (\text{HC}_{g} \Omega)^{\nu}}{(\text{HC}_{g} \Omega) - (\text{HC}_{g} \Omega)^{\nu}} \right)_{t=\Omega}.$$  

Idea of proof. We first explain the expression. $\text{HC}_{g} \Omega$ is dot-invariant, so (14) gives

$$w \cdot \left( \frac{t - (\text{HC}_{g} \Omega)^{\nu}}{(\text{HC}_{g} \Omega) - (\text{HC}_{g} \Omega)^{\nu}} \right) = \frac{t - (\text{HC}_{g} \Omega)^{w\nu}}{(\text{HC}_{g} \Omega) - (\text{HC}_{g} \Omega)^{w\nu}}.$$  

Since $W(l)$ leaves $\Delta(\mathfrak{u}^+(\mathfrak{u}^{+}))$ invariant and partitions it into finite orbits, the product may be written as an infinite product of finite products over these orbits. Each finite product is $W(l)$ dot-invariant, so $\text{HC}_{1}^{-1}$ may be applied to it to give an element of $\text{Frac}(\mathfrak{z}(l))[t]$. Then substituting $\Omega$ for $t$ gives an element of $\mathfrak{z}(\mathfrak{g}) \text{Frac} \mathfrak{z}(l)$. The product of all these elements is the right hand side.

The proof that this infinite product is $P(\mathfrak{g}, l)$ is similar to the proof of Theorem [2,5]. Its factors commute with $l$, so by (11) and (12) it suffices to prove that it acts by 1 on 1 and by 0 on $P(1)(F^l)$ for all $F^l$ in $\mathfrak{u}^+(\mathfrak{u}^{+})$. It is not hard to check that on $P(1)(F^l)$, $\Omega$ acts by $(\text{HC}_{g} \Omega)^{|l|}$ and the elements of $\mathfrak{z}(l)$ act by their images under $\text{HC}_{l}$, so the $\nu$-factor contributes action

$$\frac{(\text{HC}_{g} \Omega)^{|l|} - (\text{HC}_{g} \Omega)^{\nu}}{(\text{HC}_{g} \Omega) - (\text{HC}_{g} \Omega)^{\nu}}.$$  

This numerator is 0 for $\nu = |l|$, so it only remains to prove that none of the denominators are 0. This follows from the non-constancy condition on $\Omega$. □

The total denominator $\text{HC}_{1}^{-1} \prod_{\nu \in \Delta(\mathfrak{u}^+(\mathfrak{u}^{+}))} (\text{HC}_{g} \Omega - (\text{HC}_{g} \Omega)^{\nu})$ of the formula for $P(\mathfrak{g}, l)$ given in Theorem [3,4] is a formal product of elements of $\mathfrak{z}(l)$. The formula is not efficient: its denominator is larger than necessary. Theorem 3 of [CS05] gives a general scheme for constructing infinite factorizations of $P(\mathfrak{g}, l)$ with factors in $\mathfrak{z}(\mathfrak{g}) \text{Frac} \mathfrak{z}(l)$, as opposed to simply $\mathbb{C}[\Omega] \text{Frac} \mathfrak{z}(l)$ for some $\Omega$ in $\mathfrak{z}(\mathfrak{g})$. It is applied in Theorems 4 (1) and 4 (2) to give factorizations with smaller denominators. These two parts of Theorem 4 are identical for $l$ maximal, the most important case in the context of finding a relative version of Theorem [22]. We now recall Theorem 4 (1). The center $\mathfrak{z}(l)$ of $l$ is of course a subalgebra of $\mathfrak{h}$. Define

$$\mathfrak{z}^+(l) := \{ T \in \mathfrak{z}(l) : \text{Real Part}(\alpha(T)) > 0 \ \forall \ \alpha \in \Delta(\mathfrak{u}^{+}) \}.$$  

For $T \in \mathfrak{h}$, write $W(\mathfrak{g})^T$ for the $W(\mathfrak{g})$-stabilizer of $T$. The stabilizer and dot-stabilizer of $T$ are the same, so the following polynomial is in $\mathfrak{u}(\mathfrak{h}) W(\mathfrak{g})^T[t]$:  

$$p_T(t) := \prod_{w \in W(\mathfrak{g})/W(\mathfrak{g})^T} (t - w \cdot T).$$

Theorem 3.5. [CS05] For all $T \in \mathfrak{z}^+(l)$,

$$P(\mathfrak{g}, l) = \prod_{c \in \Delta(\mathfrak{u}^+(\mathfrak{u}^{+}))((T))} \left. \left( \text{HC}_{1}^{-1} p_T(t) \right) \right|_{t=T+c}.$$  

Idea of proof. As in the proof of Theorem [3,3] the factors commute with $l$, so it suffices to prove that the product acts by 1 on 1 and by 0 on $P(1)(F^l)$ for all $F^l$ in
Check that the action of the numerators and denominators on $P(l)(F^I)$ is multiplication by the following quantities:

$$
(H_{g^{-1}} p_T(t))_{t=T-c} \sim p_T^{[l]}(T + c) = \prod_{w \in W(g)/W(g)^T} (T + c - (w \cdot T)^{[l]}),
$$

$$
H_{i^{-1}} p_T(T + c) \sim p_T(T + c) = \prod_{w \in W(g)/W(g)^T} (T + c - w \cdot T).
$$

These multipliers are equal for $I = 0$, and $T \in z^+(l)$ implies that the denominators never act by zero. For $I > 0$, the factor of the numerator’s multiplier with $c = |I|(T)$ and $w = e$ acts by zero. □

3.2. Denominators. As discussed in Section 2.2 for $P(g)$, formulas for $P(g,l)$ with smaller denominators are better, as they have larger domains of definition. We now recall Theorem 8 and Conjecture 1 of [CS05]. The theorem gives a lower bound for the denominator of $P(g,l)$. It generalizes Proposition 2.4, and its proof is again a BGG argument. The conjecture predicts that the lower bound can be achieved.

**Proposition 3.6.** The total denominator of any formula for $P(g,l)$ is divisible by

$$
D(g,l) := \prod_{i=1}^{\infty} H_{i^{-1}} \left( \prod_{\alpha \in \Delta(u^+)} (H_\alpha + i)^{\rho_\alpha} \right).
$$

**Conjecture 1.** There is a formula for $P(g,l)$ with total denominator $D(g,l)$.

For $l = \mathfrak{h}$, this follows from Theorem 2.2. In Theorem 13 of [CS05] we use Theorem 5.5 to prove it in a few additional cases:

**Theorem 3.7.** Conjecture 1 holds if $g$ is of type $A_n$ or $B_n$ and the simple roots of $l$ form a “ray”: a connected segment of the Dynkin diagram of $g$ including an end root, short in the case of $B_n$.

**Idea of proof.** Check that $T - w \cdot T = (T - wT)^{\rho_\alpha}$, so the denominator of (15) is

$$
D(g,l,T) := \prod_{c \in \Delta(\Delta(u^+))(T)} H_{i^{-1}} \left( \prod_{w \in W(g)/W(g)^T} (T - wT + c)^{\rho_\alpha} \right).
$$

If $I$ is maximal, $u^+$ is irreducible under $l$, and $|\Delta(u^+)| = |W(g)/W(l)| - 1$, then $D(g,l,T)$ is proportional to $D(g,l)$ for any $T \in z^+(l)$. However, these conditions hold if and only if $g$ is of type $A_n$ or $B_n$ and the simple root missing from $l$ is an end root, long in the case of $B_n$. An inductive argument based on Theorem 5 of [CS05] now gives the result. □

**Remark.** In order to understand some of the obstacles to further progress, it may be helpful to consider the case that $g$ is $\mathfrak{o}_5$ and $l$ is a long $\mathfrak{g}l_2$. Let $\pm \epsilon_1, \pm \epsilon_2$, and $\pm \epsilon_1 \pm \epsilon_2$ be the roots of $\mathfrak{o}_5$. Take simple roots $\epsilon_1 - \epsilon_2$ and $\epsilon_2$ and let $l$ have simple root $\epsilon_1 - \epsilon_2$. We may identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via

$$
H_{\pm \epsilon_1 \pm \epsilon_2} \equiv \pm \epsilon_1 \pm \epsilon_2.
$$

When $l$ is maximal, $g(l)$ is 1-dimensional, so there is essentially only one choice of $T$ in Theorem 5.5. Here that choice is $H_{\epsilon_1 + \epsilon_2}$. Up to proportionality, the “extra factors” of $D(\mathfrak{o}_5,l,H_{\epsilon_1 + \epsilon_2})$, those not occurring in $D(\mathfrak{o}_5,l)$, are

$$
(H_{\epsilon_1 + \epsilon_2} + \frac{3}{2})^{\rho_\alpha}, (H_{\epsilon_1 + \epsilon_2} + \frac{5}{2})^{\rho_\alpha}, \ldots
$$
We expect that in this example, no factorization of \( P(\mathfrak{o}_5, l) \) with factors drawn from \( \mathfrak{sl}_2 \) Frac \( \mathfrak{sl}(I) \) achieves the minimal denominator \( D(\mathfrak{o}_5, l) \); our guess is that it can only be attained by factorizations over \( \mathfrak{u}(\mathfrak{o}_5)^l \) Frac \( \mathfrak{sl}(I) \).

3.3. Non-commutative factorizations. We now give some new results in the case that \( \mathfrak{g} \) is \( \mathfrak{sl}_4 \) or \( \mathfrak{sl}_5 \): for any standard reductive subalgebra \( I \) of \( \mathfrak{sl}_4 \) and for all but one such subalgebra of \( \mathfrak{sl}_5 \), we prove that \( P(\mathfrak{g}, l) \) has non-commutative factorizations analogous to the factorizations of \( P(\mathfrak{g}) \) given in Theorem 2.2. In these factorizations, the factors of \( P(\mathfrak{g}, l) \) are indexed by certain reductive subalgebras \( \mathfrak{m} \) of \( \mathfrak{g} \), not in general standard in \( \mathfrak{g} \), which themselves contain \( I \) as a maximal standard reductive subalgebra. The factor \( Q(\mathfrak{m}, l) \) corresponding to \( \mathfrak{m} \) is an element of \( \mathfrak{g}(\mathfrak{m}) \).

Unlike the method of [AST79], the method we will present is non-constructive: it only shows that the factors \( Q(\mathfrak{m}, l) \) exist. They are relative analogs of the AST factors \( Q_I(\mathfrak{a}_\mathfrak{m}) \), and it would be interesting to have explicit formulas for them such as \((\mathfrak{g})\) and \((\mathfrak{I})\). Formulas for their total denominators as formal products in \( \mathfrak{g}(I) \) would allow a resolution of Conjecture 1.

If \( I \) is a maximal standard subalgebra of \( \mathfrak{g} \), our method gives no non-trivial factorizations of \( P(\mathfrak{g}, I) \). At the other extreme, for \( I = \mathfrak{h} \) it gives only a weaker version of Theorem 2.2. Thus the interest lies in the cases \( 1 \leq |\Pi(I^+) | \leq |\Pi(n^+) | - 2 \), that is, \( 1 \leq \text{rank}(l_\mathfrak{m}) \leq \text{rank}(\mathfrak{g}_\mathfrak{m}) - 2 \).

Let us recall some standard notation for \( \mathfrak{sl}_n \). Take \( n^+ \) and \( n^- \) to be the upper and lower triangular matrices, respectively, and \( \mathfrak{h} \) to be the diagonal matrices. Writing \( e_{ij} \) for the usual elementary \( n \times n \) matrix, the positive and negative root vectors and corresponding elements of \( \mathfrak{h} \) are

\[
E_{ij} := e_{ij}, \quad F_{ij} := e_{ji}, \quad H_{ij} := e_{ii} - e_{jj}; \quad 1 \leq i < j \leq n.
\]

Let \( \epsilon_i \) be the \( i \)th standard basis vector of \( \mathbb{R}^n \), so that \( E_{ij} \) has root \( \alpha_{ij} := \epsilon_i - \epsilon_j \).

For \( 1 \leq i < i_2 < \cdots < i_r \leq n \), define subalgebras \( \mathfrak{a}_{i_1, \ldots, i_r} \) and \( \mathfrak{l}_{i_1, \ldots, i_r} \) of \( \mathfrak{sl}_n \) by

\[
\mathfrak{a}_{i_1, \ldots, i_r} := \text{Span}(E_{i_a i_b}, F_{i_a i_b}, H_{i_a i_b} : 1 \leq a < b \leq r), \quad \mathfrak{l}_{i_1, \ldots, i_r} := \mathfrak{h} + \mathfrak{a}_{i_1, \ldots, i_r}.
\]

Note that \( \mathfrak{l}_{i_1, \ldots, i_r} \) is standard if and only if \( i_1, \ldots, i_r \) are consecutive.

If \( \{ j_1 < \cdots < j_s \} \subseteq \{ i_1 < \cdots < i_r \} \), then \( \mathfrak{l}_{j_1, \ldots, j_s} \subseteq \mathfrak{l}_{i_1, \ldots, i_r} \). In this case we write

\[
P^{j_1, \ldots, j_s}_{i_1, \ldots, i_r} := P(\mathfrak{l}_{i_1, \ldots, i_r}, \mathfrak{l}_{j_1, \ldots, j_s}), \quad P^{j_1, \ldots, j_s}_{1 \ldots n} := P(\mathfrak{sl}_n, \mathfrak{l}_{j_1, \ldots, j_s}).
\]

At the other extreme, if \( \{ i_1 < \cdots < i_r \} \) and \( \{ j_1 < \cdots < j_s \} \) are disjoint, then \( \mathfrak{l}_{i_1, \ldots, i_r} \) and \( \mathfrak{l}_{j_1, \ldots, j_s} \) commute. In this case \( \mathfrak{sl}_n \) has the reductive subalgebra

\[
\mathfrak{l}_{i_1, \ldots, i_r, j_1, \ldots, j_s} := \mathfrak{l}_{i_1, \ldots, i_r} + \mathfrak{l}_{j_1, \ldots, j_s},
\]

and \( P(\mathfrak{l}_{i_1, \ldots, i_r, j_1, \ldots, j_s}) \) is simply \( P_{i_1, \ldots, i_r} \).

We now state our results; their proofs are given in Section 4. Keep in mind that in these factorizations of \( P(\mathfrak{g}, l) \), the factors commute with \( I \) but not always with each other. Some of them do commute, and the reader will note that their possible orders are closely related to normal orders of \( D(n^+) \). Some of the factors coincide with the AST operators \( Q_I(\mathfrak{sl}_2) \) in (\(\mathfrak{g}\)), and so for \( 1 \leq a < b \leq n \) we define

\[
Q_{ab} := Q_{b-a}(\mathfrak{a}_{ab}).
\]

For \( \mathfrak{sl}_4 \) we are concerned only with the case \( |\Pi(I^+) | = 1 \), and for \( \mathfrak{sl}_5 \) only with the cases \( |\Pi(I^+) | = 1 \) or \( 2 \). For both \( \mathfrak{sl}_4 \) and \( \mathfrak{sl}_5 \), up to isomorphism the only choices of \( I \) with \( |\Pi(I^+) | = 1 \) are \( l_{12} \) and \( l_{13} \). For \( \mathfrak{sl}_5 \) there are four choices with \( |\Pi(I^+) | = 2 \):

\[
l_{12}, l_{13}, l_{12,34}, l_{12,45}.
\]
We will use obvious notation such as $P_{12,34}^{12,34}$ for $P(\mathfrak{sl}_5, l_{12,34})$. The one case we will not treat is that of $P_{12,45}^{12,45}$; the reason for this is explained in Section 5.

**Theorem 3.8.** For $|\Pi(l^+)| = 1$, $P(\mathfrak{sl}_5, l)$ has the following factorizations:

(i) For $l = l_{12}$, there is a unique element $Q_{12}^{12}$ of $\mathfrak{f}(l_{12})^{l_{12}}$ such that
\[ P_{12}^{12} = P_{12}^{12} Q_{12}^{12} P_{34}. \]

(ii) For $l = l_{23}$, $Q_{14}$ is the unique element of $\mathfrak{f}(l_{14})^b$ such that
\[ P_{23}^{12} = P_{12}^{12} Q_{14} P_{23}^{12}. \]

**Theorem 3.9.** For $|\Pi(l^+)| = 1$, $P(\mathfrak{sl}_5, l)$ has the following factorizations:

(i) For $l = l_{12}$, there is a unique element $Q_{12}^{12}$ of $\mathfrak{f}(l_{12})^{l_{12}}$ such that
\[ P_{12}^{12} = P_{12}^{12} Q_{12}^{12} P_{345} = P_{12}^{12} Q_{12}^{12} P_{34} Q_{12}^{12} Q_{35} P_{45}. \]

(ii) Let $Q_{12}^{12} \in \mathfrak{f}(l_{12})^{l_{12}}$ be $Q_{12}^{12}$ with all indices shifted up one. For $l = l_{23}$, $Q_{15}$ is the unique element of $\mathfrak{f}(l_{15})^b$ such that
\[ P_{23}^{12} = P_{23}^{12} Q_{15} P_{23}^{12} = P_{23}^{12} Q_{14} P_{23}^{12} Q_{15} P_{23}^{12} P_{45}. \]

**Theorem 3.10.** For $|\Pi(l^+)| = 2$, $P(\mathfrak{sl}_5, l)$ has the following factorizations:

(i) For $l = l_{123}$, there is a unique element $Q_{123}^{123}$ of $\mathfrak{f}(l_{123})^{l_{123}}$ such that
\[ P_{123}^{123} = P_{123}^{123} Q_{123}^{123} P_{345}. \]

(ii) For $l = l_{234}$, $Q_{15}$ is the unique element of $\mathfrak{f}(l_{15})^b$ such that
\[ P_{234}^{123} = P_{123}^{123} Q_{15} P_{234}^{123} = P_{123}^{123} Q_{14} P_{234}^{123} Q_{15} P_{234}^{123} P_{45}. \]

(iii) For $l = l_{12,34}$, $Q_{12}^{12}$ above is the unique element of $\mathfrak{f}(l_{12})^{l_{12}}$ such that
\[ P_{12,34}^{12,34} = P_{12}^{12} Q_{12}^{12} P_{12,34}^{12}. \]

4. Proofs

In this section we prove Theorems 3.8, 3.9, and 3.10. We will need the generalization of Lemma 1 of [CS03] given above as (11), to reductive subalgebras $\mathfrak{m}$ of $\mathfrak{g}$ which contain $\mathfrak{h}$ but are not necessarily standard. Such $\mathfrak{m}$ have triangular decomposition $\mathfrak{m}^- \oplus \mathfrak{h} \oplus \mathfrak{m}^+$, where $\mathfrak{m}^\pm := \mathfrak{m} \cap \mathfrak{n}^\pm$, but $\Delta(\mathfrak{m}^+) \cap \Pi(n^+)$ is not necessarily a simple system of the positive system $\Delta(\mathfrak{m}^+)$. Throughout we will work over the field $\mathfrak{f}(\mathfrak{h})$, and we abbreviate the phrase “highest weight vector” to HWV. Remembering (2), define
\[ \mathcal{I}(\mathfrak{g}, \mathfrak{m}) := \{ l \in \mathbb{N}^m : I_r = 0 \forall \alpha_r \in \Delta(\mathfrak{m}^+) \}. \]

**Lemma 4.1.** The set $\{ P(\mathfrak{m})(F^I) : I \in \mathcal{I}(\mathfrak{g}, \mathfrak{m}) \}$ is an $\mathfrak{f}(\mathfrak{h})$-basis of $M(\mathfrak{g})^m$. For $I \in \mathcal{I}(\mathfrak{g}, \mathfrak{m})$, the space $\mathcal{U}(\mathfrak{m}^-)\mathfrak{f}(\mathfrak{h})P(\mathfrak{m})(F^I)$ is an $\mathfrak{m}$-submodule of $M(\mathfrak{g})$ isomorphic to $M(\mathfrak{m})$. Moreover, $M(\mathfrak{g})$ is the direct sum of these submodules:
\[ M(\mathfrak{g}) = \bigoplus_{I \in \mathcal{I}(\mathfrak{g}, \mathfrak{m})} \mathcal{U}(\mathfrak{m}^-)\mathfrak{f}(\mathfrak{h})P(\mathfrak{m})(F^I). \]

**Proof.** If $P(\mathfrak{m})(F^I)$ is non-zero then it is an $\mathfrak{m}$-HWV, so $\mathcal{U}(\mathfrak{m}^-)\mathfrak{f}(\mathfrak{h})P(\mathfrak{m})(F^I)$ is $\mathfrak{m}$-isomorphic to $M(\mathfrak{m})$ because $\mathfrak{n}^-$ acts freely on $M(\mathfrak{g})$.

A PBW argument shows that the weight space dimensions on the right side of (16) are no bigger than those on the left, with equality only if the sum is direct. Conversely, if the right side contains $F^I$ for all $I \in \mathcal{I}(\mathfrak{g}, \mathfrak{m})$, then it is $M(\mathfrak{g})$. To
and the
Projector We have just seen that it is possible to define the relative extremal
Remark. Therefore it suffices to choose ˜
modulo the sum of those
Projector by (11), all
Projector by (11), both of these spaces are non-zero. By the second of the two equations, we
\begin{equation}
\nu \notin \text{Span}_\mathbb{N}\{\alpha_{13}, \alpha_{23}\} \cap \text{Span}_\mathbb{N}\{\alpha_{12}, \alpha_{13}\} = \mathbb{N}\alpha_{13}.
\end{equation}
Therefore it suffices to choose ˜Q_{13} so that P_{12}˜Q_{13}P_{23} maps 1 to 1 and annihilates
\begin{equation}
M(\mathfrak{sl}_3)_{-\alpha_{13}} \text{ for } n > 0.
\end{equation}
Step 2. By the PBW theorem, \(F_{12}^j F_{23}^j F_{13}^{n-j}\) and \(F_{23}^j F_{12}^j F_{13}^{n-j}\) are both bases
\begin{equation}
P_{12}(M(\mathfrak{sl}_3)_{-\alpha_{13}}) = \mathfrak{F}(\mathfrak{h})P_{12}(F_{13}^n), \quad P_{23}(M(\mathfrak{sl}_3)_{-\alpha_{13}}) = \mathfrak{F}(\mathfrak{h})P_{23}(F_{13}^n).
\end{equation}
By (11), both of these spaces are non-zero. By the second of the two equations, we are done if we prove that there is a unique choice of ˜Q_{13} such that P_{12}˜Q_{13}P_{23} maps 1 to 1 and P_{23}(F_{13}^n) to 0 for n > 0.
Step 3. By a weight argument, Lemma (11) implies that
\begin{equation}
\bigoplus_{j=0}^{\infty} M(\mathfrak{sl}_3)_{-j\alpha_{13}} = \bigoplus_{j=0}^{\infty} \mathfrak{U}(l_{13})\mathfrak{F}(\mathfrak{h})P_{13}(F_{12}^j F_{23}^j).
\end{equation}
In particular, for unique elements h_0, \ldots, h_n of \mathfrak{F}(\mathfrak{h}),
\begin{equation}
P_{23}(F_{13}^n) = \sum_{i=0}^{n} h_i F_{13}^i P_{13}(F_{12}^{n-i} F_{23}^{n-i}).
\end{equation}
Step 4. The operator P(l_{13}, \mathfrak{h}, F_{13}^k) in \mathfrak{F}(l_{13}) defined after Lemma (11) projects
\begin{equation}
Q_{13} = \sum_{k=0}^{\infty} q_k P_{13}[k].
\end{equation}
It is a crucial point that \( P_{13}[k](F_{13}^i P_{13}(F_{12}^j F_{23}^n)) = \delta_{i,k} P_{13}(F_{12}^j F_{23}^n) \), because elements of \( \mathfrak{S}(l_{13}) \) such as \( P_{13}[k] \) see \( l_{13} \)-HWVs such as \( P_{13}(F_{12}^j F_{23}^n) \) as 1. Therefore

\[
\hat{Q}_{13} P_{23}(F_{13}^n) = \sum_{i=0}^{n} q_i h_i F_{13}^i P_{13}(F_{12}^{n-i} F_{23}^{n-i}).
\]

**Step 5.** Now apply \( P_{12} \): we must choose \( q_0, q_1, q_2, \ldots \) so that

\[
P_{12} \hat{Q}_{13} P_{23}(F_{13}^n) = \sum_{i=0}^{n} q_i h_i P_{12} \left( F_{13}^i P_{13}(F_{12}^{n-i} F_{23}^{n-i}) \right)
\]

is 1 for \( n = 0 \) and 0 for all \( n > 0 \). We choose them successively. Clearly \( q_0 \) must be 1. Suppose that \( q_1, \ldots, q_{n-1} \) have been determined. By the first equation in (17), every summand of (20) is a multiple of \( P_{12}(F_{13}^n) \). Therefore there is a unique choice of \( q_n \) such that the right side is zero if and only if the coefficient of \( q_n \) is non-zero. Since \( P_{12}(F_{13}^n) \neq 0 \), we reduce to proving \( h_n \neq 0 \).

**Step 6.** Apply \( E_{13}^n \) to (18): since \( E_{13} P_{13} = 0 \), we obtain \( E_{13}^n P_{23}(F_{13}^n) = E_{13}^n h_n F_{13}^n \). Thus we reduce to proving \( E_{13}^n P_{23}(F_{13}^n) \neq 0 \).

**Step 7.** For any \( H \) in \( \mathfrak{h} \), define \( d_k(H) := \prod_{i=1}^{k} (H + 1 + i) \). By (5),

\[
E_{13}^n P_{23}(F_{13}^n) = \sum_{k=0}^{n} \frac{(-1)^{k}/k!}{d_k(H_{23})^{-n_{13}}} E_{13}^n F_{23}^k E_{23}^k F_{13}^n,
\]

where \( d_k(H_{23})^{-n_{13}} \) is the \( -n_{13} \)-shift of \( d_k(H_{23}) \) defined in (9). We are working in \( M(\mathfrak{sl}_3) \), so (21) is in \( M(\mathfrak{sl}_3)_0 \), which is \( \mathfrak{S}(\mathfrak{h}) \).

We can conclude the proof efficiently with the following trick. All of the denominators \( d_k(H_{23})^{-n_{13}} \) with \( k < n \) are strict divisors of \( d_k(H_{23})^{-n_{13}} \), so if there is no cancellation between \( d_k(H_{23})^{-n_{13}} \) and \( E_{13}^n F_{23}^k E_{23}^k F_{13}^n \), then (21) is non-zero.

Simplifying \( E_{13}^n F_{23}^k E_{13}^n P_{13} \) in \( M(\mathfrak{sl}_3) \), \( E_{13}^n F_{23}^k \) becomes a \( \mathbb{C} \)-multiple of \( F_{12}^n \) and \( E_{13}^n F_{23}^k \) becomes a \( \mathbb{C} \)-multiple of \( E_{13}^n \). Hence the whole expression is a polynomial in \( H_{23} \), which admits no cancellation with any polynomial in \( H_{23} \). \( \square \)

### 4.2. \( \mathfrak{sl}_4 \): Proof of Theorem 3.8

**Proof of Part (i).** By Lemma 3.3 the equation holds if and only if multiplying its right side by \( P_{12} \) gives \( P_{1234} \). Since \( P_{12} \) is idempotent and commutes with the factors, we must prove that there is a unique \( Q_{12}^{12} \) in \( \mathfrak{S}(l_{24})^{12} \) satisfying

\[
P_{1234} = (P_{123}^{12} P_{12}) Q_{124}^{12} (P_{34} P_{12}) = P_{125} Q_{124}^{12} P_{12,34}.
\]

**Step 1.** By (14), all \( l_{24} \)-HWVs in \( M(\mathfrak{sl}_4) \) have weights in \( -\text{Span}_{\mathfrak{h}} \{\alpha_{14}, \alpha_{24}, \alpha_{34}\} \), and all \( l_{12,34} \)-HWVs in \( M(\mathfrak{sl}_4) \) have weights in \( -\text{Span}_{\mathfrak{h}} \{\alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}\} \). It follows that for any \( Q_{12}^{12} \) in \( \mathfrak{S}(l_{24})^{12} \), \( P_{123} Q_{12}^{12} P_{12,34} \) annihilates \( M(\mathfrak{sl}_4)_{-\nu} \) for

\[
\nu \notin \text{Span}_{\mathfrak{h}} \{\alpha_{14}, \alpha_{24}, \alpha_{34}\} \cap \text{Span}_{\mathfrak{h}} \{\alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}\} = \text{Span}_{\mathfrak{h}} \{\alpha_{14}, \alpha_{24}\}.
\]

During this proof, write \( n \) for an ordered pair \((n_{14}, n_{24})\) in \( \mathbb{N}^2 \) and set

\[
F^n := F_{14}^{n_{14}} F_{24}^{n_{24}}, \quad E^n := E_{14}^{n_{14}} E_{24}^{n_{24}}, \quad n \cdot \alpha := n_{14} \alpha_{14} + n_{24} \alpha_{24}, \quad |n| := n_{14} + n_{24}.
\]

Equip \( \mathbb{N}^2 \) with the usual partial order. The preceding paragraph shows that it will suffice to choose \( Q_{12}^{12} \) so that \( P_{123} Q_{124}^{12} P_{12,34} \) maps 1 to 1 and annihilates \( M(\mathfrak{sl}_4)_{-n,\alpha} \) for \( n > 0 \).
Step 2. Using two PBW bases of $\mathfrak{u}(n^-)$, one with $F_{12}$ and $F_{34}$ to the left and the other with $F_{12}$, $F_{23}$, and $F_{13}$ to the left, we find that

\begin{equation}
P_{123}(M(\mathfrak{sl}_4)_{-\nu}) = \mathfrak{F}(h)P_{123}(F^n), \quad P_{12,34}(M(\mathfrak{sl}_4)_{-\nu}) = \mathfrak{F}(h)P_{12,34}(F^n).
\end{equation}

By (11), both of these spaces are non-zero. By the second of the two equations, we are done if we prove that there is a unique choice of $Q_{124}^{12}$ such that $P_{123}Q_{124}^{12}$ maps 1 to 1 and $P_{12,34}(F^n)$ to 0 for $n > 0$.

Step 3. Note that $\Delta(\mathfrak{u}(l_{124})) = -\text{Span}_q\{\alpha_1, \alpha_2\}$. Lemma (11) implies that the sum of the weight spaces of $M(\mathfrak{sl}_4)$ with weights in this set is $l_{124}$-invariant and is a sum of copies of $M(l_{124})$, as follows:

\[ \bigoplus_{\nu \in \Delta(\mathfrak{u}(l_{124}))} M(\mathfrak{sl}_4)_{-\nu} = \bigoplus_{j \in \mathbb{N}^2} \mathfrak{u}(l_{124})\mathfrak{F}(h)v_j, \quad v_{(j_1, j_{24})} := P_{124}(F_{13}^{j_1}F_{23}^{j_{24}}P_{124}^{j_1}P_{34}^{j_{24}}).
\]

Applying $P_{12}$ to this equation, we find that for $n \in \mathbb{N}^2$ the space of $l_{124}$-HWVs in $M(\mathfrak{sl}_4)$ of weight $-n \cdot \alpha$ is

\[ P_{12}(M(\mathfrak{sl}_4)_{-\nu}) = \text{Span}_{\mathfrak{F}(h)}\{P_{12}(F^iv_{n-i}) : i \in \mathbb{N}^2, \ 0 \leq i \leq n\}. \]

In particular, for unique elements $h_i$ of $\mathfrak{F}(h)$ with $0 \leq i \leq n$,

\begin{equation}
P_{12,34}(F^n) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} h_i P_{12}(F^iv_{n-i}).
\end{equation}

Step 4. By (11), $M(l_{124})$ is the direct sum of the copies of $M(l_{12})$ with HWVs $P_{12}(F^k)$, where $k = (k_{14}, k_{24})$ runs over $\mathbb{N}^2$. Write $P_{124}^{12}[k]$ for the projection operator $P(l_{124}, l_{12}, F^k)$ projecting $M(l_{124})$ to $\mathfrak{u}(l_{124})\mathfrak{F}(h)P_{12}(F^k)$ along the other summands of (11). Note that $P_{124}^{12}[k]$ is in $\mathfrak{F}(l_{124})$ and, for any element $Q_{124}^{12}$ of $\mathfrak{F}(l_{124})$ there exist unique elements $q_k$ of $\mathfrak{F}(h)$ such that

\begin{equation}
Q_{124}^{12} = \sum_{k \in \mathbb{N}^2} q_k P_{124}^{12}[k].
\end{equation}

Now $P_{124}^{12}[k]$ acts on $P_{12}(F^i v_{n-i})$ by 1 if $i = k$ and by 0 otherwise. Therefore

\[ Q_{124}^{12} P_{12,34}(F^n) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} q_i h_i P_{12}(F^i v_{n-i}). \]

Step 5. Apply $P_{123}$: because $P_{123}P_{12} = P_{123}$, we must choose the $q_k$ so that

\begin{equation}
P_{123}Q_{124}^{12} P_{12,34}(F^n) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} q_i h_i P_{123}(F^i v_{n-i})
\end{equation}

is 1 for $n = 0$ and 0 for all $n > 0$. We choose them by induction on the partial order on $\mathbb{N}^2$. Clearly $q_0$ must be 1. Suppose that $q_i$ has been determined for $i < n$. By the first equation in (22), every summand of (24) is a multiple of $P_{123}(F^n)$. Therefore there is a unique choice of $q_n$ such that the right side is zero if and only if the coefficient of $q_n$ is non-zero. Since $P_{123}(F^n) \neq 0$, we reduce to proving $h_n \neq 0$.

Step 6. Apply $E^n$ to (25). On the right side, the summand $E^n h_i P_{124}(F^i v_{n-i})$ is in the copy $\mathfrak{u}(l_{124})\mathfrak{F}(h)v_{n-i}$ of $M(l_{124})$ generated by $v_{n-i}$. But the weights of this space are all $\leq -(n-i) \cdot \alpha$, so only the summand at $i = n$ can be non-zero. Thus $E^n P_{12,34}(F^n)$ is $E^n h_n P_{124}(F^n)$, so we need only prove $E^n P_{12,34}(F^n) \neq 0$. 


Step 7. We use the same denominator trick used for $\mathfrak{sl}_3$. Disregarding $\mathbb{C}$-scalars, the summands of $P_{12,34} = P_{12}P_{34}$ may be written as $F_{12}^a F_{34}^b E_{12}^n$, with denominators $d_{\alpha}(H_{12})d_{\alpha}(H_{34})$. Consider

$$E^n F_{12}^a F_{34}^b E_{12}^n = E_{24}^{n+1} F_{12}^a F_{34}^b E_{24}^{n+1} F_{12}^a F_{34}^b E_{12}^n F_{24}^{n+1}$$

in $M(\mathfrak{sl}_4)_0 = \mathfrak{g}(h)$. Because $E_{12}$ commutes with $F_{24}$, this term is zero for either $a > n_{14}$ or $b > |n|$. Conversely, the denominators of the non-zero terms with either $a < n_{14}$ or $b < |n|$ are strict divisors of $d_{n_{14}}(H_{12})d_{|n|}(H_{34})$. Thus if (20) is non-zero at $a = n_{14}$ and $b = |n|$ and admits no $\mathfrak{g}(h)$-cancellation against $d_{n_{14}}(H_{12})d_{|n|}(H_{34})$, we are done.

Using $[E_{14}, F_{12}] = -E_{24}$ and $[E_{12}, F_{14}] = -F_{24}$, check that (20) at $a = n_{14}$ and $b = |n|$ is $C$-proportional to $E_{24} F_{34}^a E_{12} F_{24}^b$. This in turn is $C$-proportional to $E_{23}^{n_1} F_{23}^b$, which is a polynomial in $H_{23}$. □

Proof of Part (ii). By Lemma 3.3, the equation holds if and only if multiplying the right side by $P_{23}$ gives $P_{1234}$. Since $P_{23}$ is idempotent and commutes with the factors, we must prove that $P_{1234}$ is

$$(P_{23}P_{23})Q_{14} = (P_{23}Q_{14}P_{23}) = (P_{12}Q_{13}P_{23})Q_{14} = (P_{23}Q_{24}P_{34}).$$

This simplifies to $P_{12}Q_{13}P_{23}Q_{24}P_{34}$, which is $P_{1234}$ by Theorem 2.2.

We do not have a short proof of the uniqueness of $Q_{14}$. An argument parallel to the one used to prove Part (i) shows that there exists a unique element $Q_{14}$ of $\mathfrak{g}(l_{14})_0$ such that $P_{1234}^2 = P_{12}Q_{13}P_{23}Q_{24}P_{34}$: we will not give the details. By the preceding paragraph, $Q_{14}$ must be $Q_{14}$. □

4.3. $\mathfrak{sl}_5$: Proofs of Theorems 3.9 and 3.10.

Proof of Theorem 3.9 (i). Most of the arguments are similar to those in the proof of Theorem 3.8 (i). Multiplying by $P_{12}$ and applying Lemma 3.3 we come down to proving that there is a unique $Q_{12}^{12}$ in $\mathfrak{g}(l_{12})^{12}$ satisfying

$$P_{1234} = P_{1234}Q_{12}^{12}P_{123}.45.$$

Step 1. By (11), the weights of $P_{1234}(M(\mathfrak{sl}_5))$ comprise $-\text{Span}_\mathbb{H}\{\alpha_r : r < 5\}$ and the weights of $P_{1234}(M(\mathfrak{sl}_5))$ comprise $-\text{Span}_\mathbb{H}\{\alpha_1, \alpha_2, \alpha_3 : r > 2\}$. Conclude that for any $Q_{12}^{12}$ in $\mathfrak{g}(l_{12})^{12}$, $P_{1234}Q_{12}^{12}P_{1234}$ annihilates $M(\mathfrak{sl}_5)_{\nu}$ for

$$\nu \notin \text{Span}_\mathbb{H}\{\alpha_5, \alpha_6\}.$$
Step 3. By Lemma 4.1, $\bigoplus_{\nu \in \Delta(\mathfrak{sl}(1_{25}))} M(\mathfrak{sl}_5)_{-\nu}$ is $I_{25}$-invariant and is a sum of copies of $M(I_{125})$. As in the case of $\mathfrak{sl}_4$, the highest weights of these copies of $M(I_{125})$ are all in $-\text{Span}_F\{\alpha_5, \alpha_{25}\}$, but here there is more than one copy for each highest weight. Writing temporarily $\mathcal{I}_{125}$ for the set of $I$ in $\mathcal{I}(\mathfrak{sl}_5, I_{125})$ such that $|I|$ is in $\text{Span}_F\{\alpha_5, \alpha_{25}\}$, Lemma 4.1 leads to

$$\bigoplus_{\nu \in \Delta(\mathfrak{sl}(1_{25}))} M(\mathfrak{sl}_5)_{-\nu} = \bigoplus_{I \in \mathcal{I}_{125}} \mathcal{M}(I_{125}) \mathfrak{s}(h) P_{125}(F^I).$$

Because $P_{12}(M(I_{125}))$ has 1-dimensional weight spaces, we find that for $0 \leq j \leq n$ in $\mathbb{N}$ there are unique $I_{125}$-HWVs $v_j$ of weight $-j \cdot \alpha$ in $M(\mathfrak{sl}_5)$ such that

$$P_{12, 345}(F^n) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} P_{12}(F^i v_{n-i}).$$

In this notation, the coefficients $h_i$ in (23) have been absorbed by the $v_{n-i}$. The key point is that $v_0$ is in $\mathfrak{s}(h)$; we will not be concerned with the other $v_j$.

Step 4. The first paragraph remains as for $\mathfrak{sl}_4$ except that 4 is replaced by 5:

$$Q_{125}^{12} = \sum_{k \in \mathbb{N}^2} q_k P_{12}^{12}(k).$$

Since $P_{12}^{12}(k)(P_{12}(F^i v_{n-i})) = \delta_{i,k} P_{12}(F^i v_{n-i})$, we have

$$Q_{125}^{12} P_{12, 345}(F^n) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} q_k P_{12}(F^i v_{n-i}).$$

Step 5. Apply $P_{1234}$: because $P_{1234} P_{12} = P_{1234}$, we must choose the $q_k$ so that

$$P_{1234}(Q_{125}^{12} P_{12, 345}(F^n)) = \sum_{0 \leq i \leq n \in \mathbb{N}^2} q_k P_{1234}(F^i v_{n-i}) = \delta_{0,n}.$$

Proceed as for $\mathfrak{sl}_4$: there is a unique solution for $q_n$ if and only if $v_0 \neq 0$.

Step 6. Apply $E^n$ to (23). by the weight argument used before, the right side becomes $E^n P_{12}(F^n) v_0$, so we need only prove $E^n P_{12, 345}(F^n) \neq 0$.

Step 7. Recall that $P_{12, 345} = P_{12} P_{345}$ and $P_{345} = P_{15} P_{35} P_{34}$. Since $P_{34}(F^n) = F^n$, the denominators of the summands of $E^n P_{12, 345}(F^n)$ are polynomials in $H_{12}$, $H_{35}$, and $H_{45}$, and the numerators are terms like

$$E_{25}^{115} E_{12}^{15} F_{12}^{5} F_{35}^{5} F_{35}^{5} F_{12}^{15} F_{15}^{15} F_{25}.$$
5. Remarks

5.1. Our primary goal at this point is to characterize those \( g \) and \( l \) for which our method gives a non-commutative factorization of \( P(\mathfrak{g}, l) \). It does not apply in multiply laced cases, even for \( l = \mathfrak{h} \): it is instructive to examine its failure for \( P(\mathfrak{a}_5) \). It can also break down when \( l \) is not simple; this type of failure first occurs for \( g = \mathfrak{sl}_5 \) and \( l = l_{12,45} \), the “missing case” in Theorem 3.10. Here the natural conjecture is that there is a unique element \( Q_{12,45} \) of \( \mathfrak{g}(l_{12,45}) \) such that

\[
P_{12,45}^{12,45} = P_{12,45}^{12,45} Q_{12,45}^{12,45} P_{345}^{12,45}.
\]

Multiplying by \( P_{12,45} \) and applying Lemma 3.3 we may replace this equation by

\[
P_{12345} = P_{12345}^{12,45} Q_{1245}^{12,45} P_{12,345}^{12,45}.
\]

Carrying out Step 1, we find that we need only consider \( M(\mathfrak{sl}_5)_{-\nu} \) for

\[
\nu \in \text{Span}_\mathbb{N} \{ \alpha_{14}, \alpha_{24}, \alpha_{15}, \alpha_{25} \}.
\]

However, when we apply \( P_{12345} \) and \( P_{12,345} \) to these \( M(\mathfrak{sl}_5)_{-\nu} \) in Step 2, we do not in general obtain 1-dimensional images over \( \mathfrak{g}(\mathfrak{h}) \). For example, \( F_{14} F_{25} \) and \( F_{15} F_{24} \) have the same weight but, by Lemma 3.3, independent images. This causes Step 5 to fail. Maybe the failure can be repaired using the fact that here the projections \( P_{1245}^{12,45}[k] \) appearing in Step 4 have more than one \( k \) corresponding to each weight, but we have not yet overcome the difficulty.

5.2. We would also like to have some description of the factors \( Q(\mathfrak{m}, l) \) of \( P(\mathfrak{g}, l) \). An explicit formula would be best, but short of that one could try to prove that they have certain properties possessed by the factors \( Q_{\mathfrak{g}e(\mu_a)}(\mathfrak{a}_a) \) of \( P(\mathfrak{g}) \).

For example, consider (19). As mentioned above (10), in fact only \( q_0 \) and \( q_1 \) are non-zero, and in the analogous expression for \( Q(\mathfrak{sl}_2) \), only \( q_0, \ldots, q_{l-1} \) are non-zero. It is natural to predict that this phenomenon occurs also in the relative case. The first instance of this is (24), where we expect that only the first few \( q_k \) are non-zero: probably only \( q(0,0) \), \( q(1,0) \), and \( q(0,1) \). Relative versions of (10) giving infinite commutative factorizations of the \( Q(\mathfrak{m}, l) \) would resolve the situation.

To give another example, recall that for \( l = \mathfrak{h} \), the nonstandard subalgebras \( \mathfrak{m} \) corresponding to the AST factors of \( P(\mathfrak{g}) \) are \( \mathfrak{l}_a = \mathfrak{a}_a + \mathfrak{h} \). As pointed out in Section 3.1 in this setting our method only proves the existence of factors \( Q(\mathfrak{l}_a, \mathfrak{h}) \) in \( \mathfrak{g}(\mathfrak{l}_a)^{\mathfrak{h}} \), although in fact they are in \( \mathfrak{g}(\mathfrak{a}_a)^{\mathfrak{h}} \). Does this generalize to all \( l \)? That is, are the factors \( Q(\mathfrak{m}, l) \) in \( \mathfrak{g}(\mathfrak{m}_{ss})^{\mathfrak{h}} \)? In the context of the preceding paragraph, affirming this amounts to proving that the coefficients \( q_k \) are in \( \mathfrak{g}(\mathfrak{h} \cap \mathfrak{m}_{ss}) \).

5.3. We conclude by explaining some implications of our results for Conjecture 1. In all the cases we treat, the subalgebras \( \mathfrak{m} \) and the factors \( Q(\mathfrak{m}, l) \) have the following properties:

(i) \( \Delta(\mathfrak{m}^+) \setminus \Delta(\mathfrak{l}^+) \) is the disjoint union \( \bigsqcup_{\mathfrak{m}} \Delta(\mathfrak{m}^+) \setminus \Delta(\mathfrak{l}^+) \).

(ii) \( Q(\mathfrak{m}, l) \) is the projector \( P(\mathfrak{m}, l) \) if and only if \( \mathfrak{m} \) is standard in \( \mathfrak{g} \).
We expect that these properties hold in general. When they do, Conjecture \[\text{1}\] would follow immediately if one could prove that there is a formula for \[Q(m, l)\] with total denominator
\[
D(m, l) := \prod_{i=1}^{\infty} \mathcal{H}C_{l}^{-1} \left( \prod_{\alpha \in \Delta(m^+, 0)} (H_{\alpha} + i)^{a_{\alpha}} \right).
\]

In several cases in Theorems 3.8, 3.9, and 3.10, \(m = l + a_{\alpha}\) for some \(\alpha\) in \(\Delta(n^+)\) such that \(a_{\alpha}\) commutes with \(l\). In all of these cases the factor \(Q(m, l)\) is simply \(Q_{\rho_{g}(H_{\alpha})}(a_{\alpha})\) from (6). For such \(m\), \(H_{\alpha}\) is in \(\mathfrak{z}(l)\), so (29) does indeed reduce to the denominator of \(Q(m, l)\). Coupling these observations with Theorem 3.7 proves Conjecture \[\text{1}\] for \(P(\mathfrak{sl}_4, l_{23})\) and \(P(\mathfrak{sl}_5, l_{234})\). More generally, using the proof of Theorem 3.7 (ii) one easily obtains:

**Lemma 5.1.** For \(1 < a < b < n\), \(P_{1, a a+1 \cdots b} = P_{1, a a+1 \cdots b} Q_{1, a a+1 \cdots b} P_{2, a a+1 \cdots b}^r = P_{1, a a+1 \cdots b} Q_{1, a a+1 \cdots b} P_{2, a a+1 \cdots b}^r\).

Hence Theorem 3.7 and induction on \(n\) give:

**Proposition 5.2.** For \(1 < a < b < n\), Conjecture \[\text{1}\] holds for \(P(\mathfrak{sl}_n, l_{a a+1 \cdots b})\). Put differently, the conjecture holds if \(\mathfrak{g}\) is of type \(A_n\) and the simple roots of \(l\) form a connected segment of the Dynkin diagram of \(\mathfrak{g}\).

**REFERENCES**

[AST79] R. Asherova, Y. Smirnov, V. Tolstoi, Description of a class of projection operators for semisimple complex Lie algebras, *Math. Notes* 26 (1979), 499–504.

[CS03] C. H. Conley, M. R. Sepanski, Relative extremal projectors, *Adv. Math.* 174 (2003), no. 2, 155–166.

[CS05] C. H. Conley, M. R. Sepanski, Infinite commutative product formulas for relative extremal projectors, *Adv. Math.* 196 (2005), 52–77.

[TV00] P. Etingof, A. Varchenko, Dynamical Weyl groups and applications, *Adv. Math.* 167 (2002), no. 1, 74–127.

[Zh90] A. Molev, Yangians and Classical Lie Algebras, *Mathematical Surveys and Monographs*, 143, American Mathematical Society, Providence, 2007.

[Tar90] V. Tarasov, A. Varchenko, Difference equations compatible with trigonometric KZ differential equations, *Internat. Math. Res. Notices* 2000, no. 15, 801–829.

[Zh87] D. P. Zhelobenko, Extremal cocycles of Weyl groups, *Trans. Amer. Math. Soc.* 21 (1987), no. 3, 11–21.

[Zh93] D. P. Zhelobenko, S-algebras and Harish-Chandra modules over symmetric Lie algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* 54 (1990), no. 4, 659–675.

[Zh93] D. P. Zhelobenko, Constructive modules and extremal projectors over Chevalley algebras, *Funct. Anal. Appl.* 27 (1993), no. 3, 158–165.