Article

Sharp Coefficient Problems of Functions with Bounded Turnings Subordinated by Sigmoid Function

Muhammad Arif 1,*, Safa Marwa 1, Qin Xin 2,*, Fairouz Tchier 3,*, Muhammad Ayaz 1, and Sarfraz Nawaz Malik 4,*

1 Faculty of Physical and Numerical Sciences, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan
2 Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark
3 Department of Mathematics, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia
4 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
* Correspondence: smalik110@ciitwah.edu.pk or smalik110@yahoo.com

Abstract: This study deals with analytic functions with bounded turnings, defined in the disk
$O_d = \{z : |z| < 1\}$. These functions are subordinated by sigmoid function $\frac{1}{1+e^{-z}}$ and their class
is denoted by $\mathcal{B}_{f_0}$. Sharp coefficient inequalities, including the third Hankel determinant for class
$\mathcal{B}_{f_0}$, were investigated here. The same was also included for the logarithmic coefficients related to
functions of the class $\mathcal{B}_{f_0}$.

Keywords: bounded turning function; logarithmic coefficients; Hankel determinant; sigmoid function

MSC: 30C45; 30C50

1. Introduction and Definitions

Let the class of analytic functions in disk $O_d = \{z \in \mathbb{C} : |z| < 1\}$ be denoted by the
notation $H(D)$ and suppose that $\mathfrak{A}$ is the sub-family of $H(D)$ defined as follows.

$$\mathfrak{A} := \left\{ f \in H(D) : f(z) = \sum_{k=1}^{\infty} a_k z^k, \text{ with } a_1 = 1 \right\}. \quad (1)$$

Moreover, all univalent functions from class $\mathfrak{A}$ are composed in a class, named as $S$. For two functions $g_1, g_2 \in H(D)$, the function $g_1$ is said to be subordinated by $g_2$, mathematically denoted as $g_1 \prec g_2$, if a regular function $v$ defined in $O_d$ exists with the property that
$v(0) = 0$ and $|v(z)| < 1$ such that $f(z) = g(v(z))$. Moreover, if $g_2$ is univalent in $O_d$, then
the relation $g_1 \prec g_2$ with $g_1(0) = g_2(0)$ implies that $g_1(O_d) \subset g_2(O_d)$. For details, see [1–3]
and the references therein.

Although the univalent function theory was initiated in 1851, the coefficient conjecture, proposed by Bieberbach [4] in 1916 and laterally proved by de-Branges [5] in 1985, turned the theory into one of the emerging areas of potential research. During the era between 1916 and 1985, several researchers attempted to prove or disprove this conjecture, which resulted in the formation of many subclasses of the class $S$ that are based on the geometry of image domains. The most studied and fundamental subclasses of $S$ are $S^*$ and $K$, which contain starlike and convex functions, respectively. Ma and Minda [6], in 1992, introduced the following general form of the class $S^*$:

$$S^*(\phi) := \left\{ f \in \mathfrak{A} : \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\},$$

where $\phi$ is an analytic function with $\Re \phi(z) > 0$, $z \in O_d$. In addition, the function $f$ maps
$O_d$ onto a star-shaped domain with respect to $\phi(0) = 1$ and is symmetric about the real
axis. With the variation in function $\phi$, the class $S^*(\phi)$ generates several sub-families of $S^*$, which include some of the ones listed below.

(i). If we choose $\phi(z) = \frac{1+iz}{1+Mz}$ with $-1 \leq M < L \leq 1$, then we obtain $S^*[L, M] \equiv S^*(\frac{1+iz}{1+Mz})$, which is described as the class of Janowski starlike functions studied in [7]. Moreover, $S^*(\xi) := S^*[1 - 2\xi, -1]$ is the famous starlike functions’ class of order $\xi$ with $0 \leq \xi < 1$.

(ii). The family $S^*(\phi)$ with $\phi(z) = \sqrt{1+z}$ was developed by Sokól and Stankiewicz in [8], which maps the symmetric disk $D$ onto the region bounded by $|\alpha^2 - 1| < 1$.

(iii). By choosing the function $\phi(z) = 1 + \sin^{-1}z$, we obtain $S^*_p := S^*(1 + \sin^{-1}z)$, which was recently introduced by Kumar and Arora [9]. In 2021, Barukab et al. [10] found the sharp upper bound of the third Hankel determinant for functions of the following class:

$$\mathcal{R}_s = \left\{ f \in \mathbb{A} : f'(z) < 1 + \sin^{-1}z, \quad z \in O_d \right\}.$$ 

Later, in 2022, Shi et al. [11] determined the sharp second-order Hankel determinant for the above class, but with logarithmic coefficients.

(iv). By choosing $\phi(z) = 1 + \frac{1}{2}z + \frac{1}{2}z^2$, the class $S^*(\phi)$ reduces to class $S^*_M$, which was studied by Gandhi et al. [12]. In 2022, Arif et al. [13] determined the sharp third-order Hankel determinant for functions of class $S^*_M$. Later, Shi et al. [14] determined the sharp second Hankel determinant of the same class with logarithmic coefficients.

(v). Raza and Bano [15] and Alotaibi et al. [16] contributed the families $S^*_{\cos} := S^*(\cos(z))$ and $S^*_{\cosh} := S^*(\cosh(z))$, respectively. These researchers investigated some geometric characteristics of the functions of these families.

(vi). We obtain the family $S^*_\sin$ by choosing $\phi(z) = 1 + \sin z$, which was established in [17].

In this paper, the authors determined radii problems for the defined class $S^*_\sin$.

For functions $f \in S$ of the series form as given in (1), the Hankel determinant $\mathcal{H}_{q,n}(f)$ (with $q, n \in \mathbb{N} = \{1, 2, \ldots \}$ and $a_1 = 1$) was given by Pommerenke [18,19] and is defined as follows:

$$\mathcal{H}_{q,n}(f) := \det \begin{bmatrix} a_2 & a_3 \cdots a_{n+q-1} \\ a_3 & a_4 & \cdots a_{n+q-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{bmatrix}.$$ 

With certain variations in $q$ and $n$, we have the following second and third-order Hankel determinants, respectively:

$$\mathcal{H}_{2,1}(f) = \begin{bmatrix} 1 \\ a_2 \\ a_3 \end{bmatrix} = a_3 - a_2^2,$$

$$\mathcal{H}_{2,2}(f) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix} = a_2 a_4 - a_3^2,$$

$$\mathcal{H}_{3,1}(f) = \begin{bmatrix} 1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \left(a_2 a_4 - a_3^2\right) a_3 - \left(a_4 - a_2 a_3\right) a_4 + \left(a_3 - a_2^2\right) a_5. \quad (2)$$

For the functions $f \in S$, the best established sharp inequality is $|\mathcal{H}_{2,n}(f)| \leq \lambda \sqrt{n}$, where $\lambda$ is an absolute constant and this inequality is due to Hayman [20]. Furthermore, the following results for the class $S$ can be accessed from [21].

$$|\mathcal{H}_{2,2}(f)| \leq \lambda, \text{ for } 1 \leq \lambda \leq \frac{11}{3},$$

$$|\mathcal{H}_{3,1}(f)| \leq \mu, \text{ for } \frac{4}{9} \leq \mu \leq \frac{32 + \sqrt{285}}{15}.$$
The following sharp bound of $|H_{2,2}(f)|$ was given by Janteng et al. [22,23].

$$|H_{2,2}(f)| \leq \begin{cases} 
\frac{1}{8}, & \text{for } f \in K, \\
1, & \text{for } f \in S^*, \\
\frac{4}{9}, & \text{for } f \in \mathcal{R}, 
\end{cases}$$

where $\mathcal{R}$ denotes the class of functions with bounded turnings which is defined as

$$\mathcal{R} = \left\{ f \in S : f'(z) < \frac{1+z}{1-z}, \quad z \in \mathbb{O}_d \right\}.$$

To date, a number of researchers have contributed to the work on Hankel determinants and have achieved remarkable milestones. Some of the recent developments can be accessed from [24–34] and the references therein. The logarithmic function associated to function $f \in S$ is defined as

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k. \quad (3)$$

If $f \in S$ assumes the series of the form given in (1), then (3) gives the following relations:

$$\begin{align*}
\gamma_1 &= \frac{1}{2} a_2, \\
\gamma_2 &= \frac{1}{2} \left( a_3 - \frac{1}{2} a_2^2 \right), \\
\gamma_3 &= \frac{1}{2} \left( a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right), \\
\gamma_4 &= \frac{1}{2} \left( a_5 - a_2 a_4 + a_2^2 a_3 - \frac{1}{2} a_2^2 - \frac{1}{4} a_2^3 \right). 
\end{align*} \quad (4) \quad (5) \quad (6) \quad (7)$$

Recently, in [35,36], Kowalczyk and Lecko introduced the following $q$th-order Hankel determinant $H_{q,n} \left( \frac{Ff}{2} \right)$ containing the logarithmic coefficients of $f$.

$$H_{q,n} \left( \frac{Ff}{2} \right) = \begin{vmatrix} 
\gamma_1 & \gamma_2 & \cdots & \gamma_{n-q} & \\
\gamma_2 & \gamma_3 & \cdots & \gamma_{n-q+1} & \\
\vdots & \vdots & \ddots & \vdots & \\
\gamma_{n-q} & \gamma_{n-q+1} & \cdots & \gamma_{n+2q-2} 
\end{vmatrix}. \quad (8)$$

From above, one can easily deduce that

$$H_{2,1} \left( \frac{Ff}{2} \right) = \gamma_1 \gamma_3 - \gamma_2^2, \quad (9)$$
$$H_{2,2} \left( \frac{Ff}{2} \right) = \gamma_2 \gamma_4 - \gamma_3^2. \quad (10)$$

We now define the class of functions with bounded turnings and associated with the sigmoid function $\phi(z) = \frac{2}{1+e^{-z}}$ with $\phi(0) = 1$ and $\Re \phi(z) > 0$ as follows:

$$\mathbb{BS}_\phi := \left\{ f \in S : f'(z) < \frac{2}{1+e^{-z}}, \quad z \in \mathbb{O}_d \right\}. \quad (11)$$

We intended to find the sharp bound of $|H_{3,1}(f)|$ and $|a_3 a_5 - a_4^2|$ for the class $\mathbb{BS}_\phi$. In addition, we investigated the sharp bounds of $|H_{2,1} \left( \frac{Ff}{2} \right)|$ and $|H_{2,2} \left( \frac{Ff}{2} \right)|$ for the class $\mathbb{BS}_\phi$. 

2. A Set of Lemmas

Definition 1. A function $p \in \mathcal{P}$ if and only if, it has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \Omega_{\alpha})$$

along with the $\Re p(z) \geq 0$ for $z \in \Omega_{\alpha}$.

Lemma 1. If the function $p \in \mathcal{P}$ has the series representation as given in (12), then

$$|c_{n+k} - \lambda c_n c_k| \leq 2 \max\{1, |2 \lambda - 1|\} = \begin{cases} 2 & \text{for } 0 \leq \lambda \leq 1; \\ 2|2 \lambda - 1| & \text{otherwise.} \end{cases}$$

and

$$|c_n| \leq 2 \quad \text{for } n \geq 1. \quad (14)$$

The inequalities (13) and (14) are sharp and their proofs can be accessed from [37] and [38], respectively.

Lemma 2 ([39]). If the function $p \in \mathcal{P}$ has the series representation as given in (12) and if $B \in [0, 1]$ with $B(2B - 1) \leq D \leq B$, then, we have

$$|c_3 - 2Bc_1c_2 + Dc_1^2| \leq 2. \quad (15)$$

Lemma 3. If the function $p \in \mathcal{P}$ has the series representation as given in (12), then, for $x, \delta, \rho \in \Omega_{\alpha} = \{z \in \mathbb{C} : |z| \leq 1\}$, we have

$$2c_2 = x(4 - c_1^2) + c_1^2, \quad \text{(16)}$$

$$4c_3 = -x^2c_4(4 - c_1^2) + 2xc_1(4 - c_1^2) + 2(1 - |x|^2)(4 - c_1^2)\delta + c_1^3, \quad \text{(17)}$$

$$8c_4 = \left[4x + \left(x^2 - 3x + 3\right)c_1^2\right]x(4 - c_1^2) - 4(1 - |x|^2)(4 - c_1^2)$$
$$- \rho(1 - |\delta|^2) + (x - 1)\delta c + \delta^2 x + c_1^4. \quad \text{(18)}$$

Here, the readers can refer to the formula for $c_2$ given in [37]. The formula for $c_3$ is due to Libera and Zlotkiewicz [40], and the formula for $c_4$ is proved in [41].

Lemma 4 ([42]). If the function $p \in \mathcal{P}$ has the series representation as given in (12) and if $\eta, a, \alpha$ and $\beta$ satisfy the inequalities $0 < \alpha < 1, 0 < a < 1$, and

$$8\left(\frac{(a(a + \alpha) - \beta)^2 + (a\beta - 2\eta)^2}{(1 - a)a + a(\beta - 2aa)^2(1 - \alpha)}\right) \leq 4aa^2(1 - a)^2(1 - a),$$

then,

$$|\eta c_1^4 + ac_2^2 + 2ac_1c_3 - \frac{3}{2} \beta c_1^2c_2 - c_4| \leq 2. \quad (19)$$

3. Third Hankel Determinant for the Class $\mathcal{B} \Sigma_{\alpha} \mathcal{E}_\theta$

Theorem 1. If the function $f \in \mathcal{B} \Sigma_{\alpha} \mathcal{E}_\theta$ assumes the series representation as given in (1), then

$$|\mathcal{H}_{3,1}(f)| \leq \frac{1}{64}.$$

The inequality is sharp and sharpness can be achieved from

$$f(z) = \int_{0}^{1} \left(\frac{2}{1 + e^{-r^2}}\right) dt = z + \frac{z^4}{8} - \frac{z^{10}}{240} + \cdots.$$
Proof. Let \( f \in \mathcal{H}_c \). Then,
\[
f'(z) = \frac{2}{1 + e^{-w(z)}}. \tag{20}\]

If \( p \in \mathcal{P} \), then
\[
p(z) = \frac{1 + (w(z))}{1 - (w(z))} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots . \]

This implies that
\[
w(z) = \left( \frac{1}{2}c_1 \right)z + \left( -\frac{1}{4}c_1^2 + \frac{1}{2}c_2 \right)z^2 + \left( -\frac{1}{8}c_1c_2 + \frac{1}{8}c_1^3 + \frac{1}{2}c_3 \right)z^3
+ \left( \frac{3}{8}c_1^2c_2 + \frac{1}{16}c_4 - \frac{3}{4}c_2c_3 - \frac{1}{2}c_1c_3 - \frac{1}{16}c_4 \right)z^4 + \cdots . \tag{21}\]

Using (21) in (20), we obtain
\[
f'(z) = 1 + \frac{1}{4}c_1z + \left( -\frac{1}{8}c_1^2 + \frac{1}{4}c_2 \right)z^2 + \left( \frac{1}{4}c_3 - \frac{1}{4}c_2c_1 + \frac{11}{192}c_4 \right)z^3
+ \left( \frac{11}{64}c_1^2c_2 - \frac{1}{8}c_2^2 - \frac{1}{4}c_1c_3 - \frac{3}{128}c_2^4 + \frac{1}{4}c_4 \right)z^4 + \cdots . \tag{22}\]

From the series defined in (1), it follows that
\[
f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots . \tag{23}\]

By comparing (22) and (23), one may have
\[
a_2 = \frac{c_1}{8}, \tag{24}\]
\[
a_3 = -\frac{c_1^2}{24} + \frac{c_2}{12}, \tag{25}\]
\[
a_4 = \frac{11c_3^3}{768} + \frac{c_3}{16} - \frac{c_1c_2}{16}, \tag{26}\]
\[
a_5 = -\frac{3c_4}{640} + \frac{11c_1^2c_2}{320} + \frac{c_4}{20} - \frac{c_1c_3}{20} - \frac{c_2^2}{40}. \tag{27}\]

Substituting (24)–(27) in (2) and setting \( c_1 = c \), we obtain
\[
\mathcal{H}_{3,1}(f) = \frac{1}{L} \left( -25344c^2c_4 - 384c^2c_2^2 - 23552c_3^2 + 3744c^3c_3 - 119c_6^2
- 480c^4c_2 + 43776cc_2c_3 - 34560c_3^2 + 36864c_2c_4 \right), \tag{28}\]

where \( L = 8,847,360 \). Now, assuming \( c_1 = c \) and \( m = (4 - c_1^2) \) in (16)–(18), we obtain the following:
\[
c_2 = \frac{1}{2}\left(mx + c^2\right),
\]
\[
c_3 = \frac{1}{4}\left(-mx^2c + 2mxc + 2m(1 - |x|^2)\delta + c^3\right),
\]
\[
c_4 = \frac{1}{8}\left[(4x + (x^2 - 3x + 3)c^2)m - 4m(1 - |x|^2)
- \rho(1 - |\delta|^2) + (x - 1)\delta c + \delta^2 x + c^4\right].
\]
By inserting the above expressions into (28), one may have
\[
\mathcal{H}_{3,1}(f) = \frac{1}{L} \left\{ -3456 \left( 1 - \left| \delta^2 \right| \right) \left( 1 - |x|^2 \right) c^2 m \rho - 15 c^6 + 720 c^3 m \left( 1 - |x|^2 \right) \delta \\
+ 2880 \left( 1 - |x|^2 \right) c x m \delta + 9216 \left( 1 - \left| \delta^2 \right| \right) \left( 1 - |x|^2 \right) m^2 x \rho - 2944 m^3 x^3 \\
+ 3456 \left( 1 - |x|^2 \right) c^3 x m \delta + 504 c^4 x^2 m + 3456 \left( 1 - |x|^2 \right) c^2 m \delta^2 \right\}.
\]

Since \( m = (4 - c^2) \), it follows that
\[
\mathcal{H}_{3,1}(f) = \frac{1}{L} \left\{ m_0(c, x) + m_1(c, x) \delta + m_2(c, x) \delta^2 + \Pi(c, x, \delta) \rho \right\},
\]
where \( \rho, x, \delta \in \mathbb{C} \), and
\[
m_0(c, x) = -15 c^6 - 8 x \left[ 2 x \left( 50 x c^2 - 9 x^2 c^2 + 160 x - 18 c^2 \right) (4 - c^2) \\
+ 432 x c^2 - 12 c^4 + 108 c^4 x^2 - 63 c^4 x \right] (4 - c^2),
\]
\[
m_1(c, x) = -144 c \left[ (4 - c^2) \left( -20 x + 4 x^2 \right) - 5 c^2 - 24 x c^2 \right] (1 - |x|^2) (4 - c^2),
\]
\[
m_2(c, x) = -576 \left[ (4 - c^2) \left( 15 + |x|^2 \right) - 6 c^2 x \right] (1 - |x|^2) (4 - c^2),
\]
\[
\Pi(c, x, \delta) = 1152 \left[ (4 - c^2) (1 - |x|^2) (1 - \left| \delta^2 \right|) 8 x (4 - c^2) - 3 c^2 \right].
\]

By replacing \( |\delta| \) with \( y \) and \( |x| \) with \( x \), if we apply the statement \( |\rho| \leq 1 \), it follows that
\[
|\mathcal{H}_{3,1}(f)| \leq \frac{1}{L} \left( |m_0(c, x)| + |m_1(c, x)| y + |m_2(c, x)| y^2 + |\Pi(c, x, \delta)| \right),
\]
\[
\leq \frac{1}{L} \left( f(c, x, y) \right). \tag{29}
\]

where
\[
f(c, x, y) = n_0(c, x) + n_1(c, x) y + n_2(c, x) y^2 + n_3(c, x) \left( 1 - y^2 \right). \tag{30}
\]

with
\[
n_0(c, x) = 15 c^6 + 8 x \left[ 2 x \left( 50 x c^2 + 9 x^2 c^2 + 160 x + 18 c^2 \right) (4 - c^2) \\
+ 432 x c^2 + 12 c^4 + 108 c^4 x^2 + 63 c^4 x \right] (4 - c^2),
\]
\[
n_1(c, x) = 144 c \left[ (4 - c^2) \left( 20 x + 4 x^2 \right) + 5 c^2 + 24 x c^2 \right] (1 - x^2) (4 - c^2),
\]
\[
n_2(c, x) = 576 \left[ (4 - c^2) \left( 15 + x^2 \right) + 6 c^2 x \right] (1 - x^2) (4 - c^2),
\]
\[
n_3(c, x) = 1152 \left[ 3 c^2 + (4 - c^2) 8 x \right] (1 - x^2) (4 - c^2).
\]

Now, we have to maximize \( f(c, x, y) \) in the closed cuboid \( \Delta : [0, 2] \times [0, 1] \times [0, 1] \). For this purpose, we need to find \( \max f(c, x, y) \) in the interior of \( \Delta \), in the interior of all of its six faces, and on the twelve edges of cuboid \( \Delta \).
(I). Initially, we will look for the maxima of $f(c, x, y)$ in the interior of $\Delta$. Let $(c, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$. Then, on differentiating (30) partially with respect to the parameter $y$, it yields

$$\frac{\partial f}{\partial y} = (1 - x^2)(4 - c^2)144\left[8(x - 15)(4 - c^2) + 48c^2\right]y(x - 1) + c\left(4x(4 - c^2)(15 + x) + 24c^2(x + 5)\right).$$

Taking $\frac{\partial f}{\partial y} = 0$, we obtain

$$y = \frac{c(4x(4 - c^2)(15 + x) + 24c^2(x + 5))}{(8(15 - x)(4 - c^2) - 48c^2)(x - 1)} := y_1.$$

For $y_1$ to belong to $(0, 1)$, it is possible only if

$$24c^3(x + 5) + 4cx(x + 15)(4 - c^2) + 8(1 - x)(15 - x)(4 - c^2) < 48(1 - x)c^2 \quad (31)$$

and

$$c^2 > \frac{4(120 - 8x)}{168 - 8x}. \quad (32)$$

Now, in order to find the solutions that meet both inequalities (31) and (32), we consider $\nu(x) = \frac{4(120 - 8x)}{168 - 8x}$. It follows that $\nu'(x) < 0$ in $(0, 1)$; therefore, $\nu(x)$ is decreasing over $(0, 1)$. Hence, $c^2 > \frac{14}{9}$ and an easy calculation illustrates that (31) does not hold for $x \in (0, 1)$. This implies that $f$ does not have critical point in $(0, 2) \times (0, 1) \times (0, 1)$.

(II). Next, we will look for the maxima of $f(c, x, y)$ in the interior of all six faces of $\Delta$. Choosing $c = 0$, we achieve

$$p_1(x, y) = f(0, x, y) = 40960x^3 + \left(9216(x - 1)(x - 15)y^2 + 147456x\left(1 - x^2\right)\right).$$

Hence, we have found no maxima for $f(0, x, y)$ in $(0, 1) \times (0, 1)$. Taking $c = 2$, we obtain

$$f(2, x, y) = 960. \quad (33)$$

Setting $x = 0$, we have

$$p_2(c, y) = f(c, 0, y) = 720c^3y(4 - c^2) + 15c^6 - 3456c^4 + 12096c^4y^2 + 13824c^2 - 82944c^2y^2 + 138240y^2.$$

By solving $\frac{\partial p_2}{\partial y} = 0$ and $\frac{\partial p_2}{\partial c} = 0$, we obtain the critical points. Setting $\frac{\partial p_2}{\partial y} = 0$, we achieve

$$y = \frac{5c^3}{24(7c^2 - 20)} := y_0. \quad (34)$$

For the provided range of $y$, $y_0$ would belong to $(0, 1)$ if $c > c_0$ for $c_0 \approx 1.73573$. Further, $\frac{\partial p_2}{\partial c} = 0$ gives

$$2160c^2y(4 - c^2) - 1440c^4y + 90c^5 - 13824c^3 + 48384c^3y^2 + 27648c - 165888cy^2 = 0.$$

By inserting (34) into the above equation, we obtain

$$1260c^6 - 682176c^7 + 5225472c^5 - 13271040c^3 + 11059200c = 0.$$
Now, solving for $c \in (0, 2)$, we achieve $c \approx 1.402523$. Hence, no optimal solution is achieved for $f(c, 0, y)$ in $(0, 2) \times (0, 1)$. If we choose $x = 1$, we find that

$$p_3(c, y) = f(c, 1, y) = -217c^6 - 4896c^4 + 13056c^2 + 40960.$$ (35)

For the critical point, $\frac{\partial p_3}{\partial c} = 0$ gives $c_0 \approx 1.110119$, at which, $p_3$ attains its maxima, which is

$$p_3(c, y) \leq 49207.94534.$$  

Taking $y = 0$, we obtain

$$f(c, x, 0) = p_4(c, x) = 144c^6x^4 + 15c^6 - 64c^6x^3 - 1152c^4x^4 - 216c^6x^2$$
$$+ 2304c^2x^4 - 96c^6x - 288c^4x^2 + 9600c^4x - 9600c^4x^3$$
$$- 3456c^4 + 4608c^2x^2 + 66048c^2x^3 - 73278c^2x$$
$$+ 13824c^2 - 106496x^3 + 147456x.$$  

A computation indicates that the solution for the system of equations

$$\frac{\partial p_4}{\partial c} = 0 \quad \text{&} \quad \frac{\partial p_4}{\partial x} = 0$$  

in $(0, 2) \times (0, 1)$ does not exist. Considering $y = 1$, we have

$$p_5(c, x) = f(c, x, 1) = -64c^6x^3 + 15c^6 - 576c^5x^4 + 144c^6x^4 + 576c^5x^3$$
$$- 1728c^4x^4 - 216c^6x^2 + 6912c^2x^4 + 9216c^3x^3 + 82944c^2x^2$$
$$+ 1296c^3x^2 - 96c^6x + 3072c^4x^3 - 11808c^4x^2 - 576c^5x + 4608c^4x$$
$$- 3072c^4x - 21504c^2x^3 - 7488c^3x^2 + 8640c^4 - 9216c^4x - 720c^5$$
$$- 9216c^3x - 69120c^2 + 46080cx - 129024x^2 + 138240 - 9216x^4$$
$$- 46080cx^3 + 2880x^3 + 13824x^2 + 9216cx^2 + 40960x^3.$$  

From the computation, we conclude that the solution for the system of equations

$$\frac{\partial p_5}{\partial x} = 0 \quad \text{&} \quad \frac{\partial p_5}{\partial c} = 0$$  

in $(0, 2) \times (0, 1)$ does not exist.  

(III). Finally, we look forward to the maximum of $f(c, x, y)$ at the twelve edges of $\Delta$. By substituting $y = 0$ and $x = 0$, it yields

$$p_6(c) = f(c, 0, 0) = 15c^6 - 3456c^4 + 13824c^2.$$  

For the critical point, the equation $\frac{\partial p_6}{\partial c} = 0$ gives $c_0 \approx 1.4236371$, at which, the maximum value is achieved for $p_6(c)$. That is,

$$f(c, 0, 0) \leq 13946.406.$$  

By selecting $y = 1$ and $x = 0$, we have

$$p_7(c) = f(c, 0, 1) = -720c^5 + 15c^6 + 2880c^3 + 8640c^4 + 138240 - 69120c^2.$$  

As $\frac{\partial p_7}{\partial c} < 0$ for $[0, 2]$, $p_7(c)$ is decreasing and achieves its maximum at $c = 0$. Thus,

$$f(c, 0, 1) \leq 138240.$$
By taking \( x = 0 \) and \( c = 0 \), we obtain
\[
p_8(y) = f(0, 0, y) = 138240y^2.
\]
Clearly, \( \frac{\partial p_8}{\partial y} > 0 \) for \([0, 1]\), which indicates that \( p_8(y) \) is increasing over \([0, 1]\) and that its maximum value is attained at \( y = 1 \). Thus, one may have
\[
f(0, 0, y) \leq 138240.
\]
We note that \((35)\) is free of \( y \). It follows that
\[
p_9(c) = f(c, 1, 0) = f(c, 1, 1).
\]
\[
p_9(c) = -217c^6 - 4896c^4 + 13056c^2 + 40960.
\]
For the critical point, the equation \( \frac{\partial p_9}{\partial c} = 0 \) gives \( c_0 = 1.110119 \), at which, the maximum value of \( p_9(c) \) is attained. Thus, one may conclude that
\[
p_9(c) \leq 49207.94534.
\]
For \( x = 1 \) and \( c = 0 \), we obtain
\[
p_{10}(y) = f(0, 1, y) = 40960.
\]
By choosing \( c = 2 \), we see that \((33)\) is free of \( y \), \( x \), and \( c \). It follows that
\[
f(2, 0, y) = f(2, 1, y) = f(2, x, 1) = f(2, x, 0) = 960.
\]
By substituting \( y = 0 \) and \( c = 0 \), we have
\[
p_{11}(x) = f(0, x, 0) = -106496x^3 + 147456x.
\]
For the critical point, the equation \( \frac{\partial p_{11}}{\partial x} = 0 \) gives \( x_0 = 0.679366 \), at which, the maximum value of \( p_{11}(x) \) is attained. Hence, we have
\[
f(0, x, 0) \leq \frac{98304}{13} \sqrt{6} \sqrt[3]{13}.
\]
By setting \( y = 1 \) and \( c = 0 \), we obtain
\[
p_{12}(x) = f(0, x, 1) = -9216x^4 + 40960x^3 - 129024x^2 + 138240.
\]
By simple computation, we see that \( p_{12}(x) \) obtains its maximum value at 0, so we have
\[
f(0, x, 1) \leq 138240.
\]
Hence, from the above situations, we conclude that
\[
f(c, x, y) \leq 138240 \text{ on } \Delta : [0, 2] \times [0, 1] \times [0, 1].
\]
By using Equation \((29)\), it follows that
\[
\mathcal{H}_{3,1}(f) \leq \frac{1}{L}(f(c, x, y)) \leq \frac{1}{64}.
\]
Thus, the required result is accomplished. \(\square\)
Theorem 2. If the function \( f \in \mathcal{F}_{c_0} \) assumes the series representation as given in (1), then

\[
|a_{3a5} - a_{43}^2| \leq \frac{1}{64}.
\]

The result is the best possible and equality is attained from the function

\[
f(z) = \int_0^z \left( \frac{2}{1+e^{-t^2}} \right) dt = z + \frac{z^3}{8} - \frac{z^{10}}{240} + \cdots.
\]

Proof. Substituting (25)–(27) with \( c_1 = c \), we have

\[
a_{3a5} - a_{43}^2 = \frac{1}{2949120}(-29c^6 - 96c^4c_2 + 864c^3c_3 - 6144c^2c_4 + 10752c_2c_3 - 6144c_2^2 + 12288c_2c_4 - 11520c_2^2).
\]

Using the resulting form of (16)–(18) with \( m = (4 - c^2) \), we obtain

\[
10752c_2c_3 = -1344c^2m^3x^3 - 1344c^4mx^2 + 2688c^2m(1 - |x|^2)\delta + 2688c^2m^2x^2 + 1344c^6 + 4032c^4mx,
\]

\[
96c^4c_2 = 48c^4mx + 48c^6,
\]

\[
6144c_2^2 = 6144c_3 = 6144c_4 = 6144c_5 = \frac{1}{2949120}(4(1 - |x|^2)\delta - 3c^2mx^2 - 4c^3m^2x^2).
\]

\[
64c^3c_2 = 64c^3m^2x^2 + 4c^3m^2x^2 + c^6 + 4c^4mx + 4c^3m^2x^2 - 4(1 - |x|^2)\delta - 3c^2mx^2 - 4c^3m^2x^2 - 4c^3m^2x^2.
\]

\[
864c^3c_3 = 432c^4mx + 432(1 - |x|^2)\delta + 4c^4mx.
\]

\[
11520c_2^3 = 720(c^2m^2x^4 - 4c^2m^2x^3(1 - |x|^2)\delta - 4c^2m^2x^3 - 2c^4mx^2 + c^6 + 4c^4mx + 8c^2x^3(1 - |x|^2)\delta + 4c^3m(1 - |x|^2)\delta + 4c^2m^2x^2 + 4m^2x(1 - |x|^2)^2\delta).
\]

By inserting the above expressions into (36), one may have

\[
a_{3a5} - a_{43}^2 = \frac{1}{2949120}\left\{-5c^6 + 240c^3m(1 - |x|^2)\delta - 120c^4x^2m - 192c^2m^2x^2 - 2880m^2(1 - |x|^2)\delta^2 - 768m^3x^3 + 3072x^3m^2 + 48c^2m^2x^4 - 192c^2x^2m^2(1 - |x|^2)\delta + 3072m^2x(1 - |x|^2)(1 - |\delta|^2)\rho - 3072m^2x(1 - |x|^2)\delta^2 - 768c^2m^2x^3\right\}.
\]

Since \( m = 4 - c^2 \), it follows that

\[
a_{3a5} - a_{43}^2 = \frac{1}{2949120}\left\{b_0(c, x) + b_1(c, x)\delta + b_2(c, x)\delta^2 + b_3(c, x, \delta)\rho\right\}.
\]
where \( \delta, \rho, x \in \mathbb{O}_0 \), and

\[
\begin{align*}
    b_0(c, x) &= -5c^6 - 24x^2 \left[ 5c^4 + 2 \left( 4c^2 - x^2c^2 \right) \left( 4 - c^2 \right) \right] \left( 4 - c^2 \right), \\
    b_1(c, x) &= -48 \left[ -5c^2 + 4x^2 \left( 4 - c^2 \right) \right] \left( 1 - |x|^2 \right) \left( 4 - c^2 \right), \\
    b_2(c, x) &= -192 \left( 15 + |x|^2 \right) \left( 1 - |x|^2 \right) \left( 4 - c^2 \right)^2, \\
    b_3(c, x, \delta) &= 3072 \left( 1 - |\delta|^2 \right) \left( 4 - c^2 \right) \left( 1 - |x|^2 \right). 
\end{align*}
\]

By replacing \(|\delta|\) by \(y\) and \(|x|\) by \(x\), if we apply the relation \(|\rho| \leq 1\), it follows that

\[
|a_{3\alpha 5} - a_{3\alpha}^2| \leq \frac{1}{2949120} \left( |b_0(c, x)| + |b_1(c, x)|y + |b_2(c, x)|y^2 + |b_3(c, x, \delta)| \right), \\
\leq \frac{1}{2949120} \left( f(c, x, y) \right), \tag{37}
\]

where

\[
\begin{align*}
    f(c, x, y) &= 3_0(c, x) + 3_1(c, x)y + 3_2(c, x)y^2 + 3_3(c, x) \left( 1 - y^2 \right) \tag{38} \\
\end{align*}
\]

with

\[
\begin{align*}
    3_0(c, x) &= 5c^6 + 24x^2 \left[ 5c^4 + 2 \left( 4c^2 - x^2c^2 \right) \left( 4 - c^2 \right) \right] \left( 4 - c^2 \right), \\
    3_1(c, x) &= 48c \left[ 5c^2 + 4x^2 \left( 4 - c^2 \right) \right] \left( 1 - x^2 \right) \left( 4 - c^2 \right), \\
    3_2(c, x) &= 192 \left( 15 + x^2 \right) \left( 1 - x^2 \right) \left( 4 - c^2 \right)^2, \\
    3_3(c, x) &= 3072 \left( 1 - x^2 \right) \left( 4 - c^2 \right)^2 x. 
\end{align*}
\]

Now, we have to maximize \(f(c, x, y)\) in the closed cuboid \(\Delta : [0,2] \times [0,1] \times [0,1]\). For this purpose, we have to find the maximum of \(f(c, x, y)\) in the interior of \(\Delta\), in the interior of all its six faces, and on the twelve edges of \(\Delta\).

(i). Initially, we will look for the maxima of \(f(c, x, y)\) in the interior of \(\Delta\).

Let \((c, x, y) \in (0,2) \times (0,1) \times (0,1)\). Then, on differentiating (38) partially with respect to parameter \(y\), one may obtain

\[
\frac{\partial f}{\partial y} = 48 \left( 4 - c^2 \right) \left( 1 - x^2 \right) \left[ 8(x - 15) \left( 4 - c^2 \right)(x - 1)y + \left( 4 \left( 4 - c^2 \right)x^2 + 5c^2 \right) \right].
\]

Taking \(\frac{\partial f}{\partial y} = 0\), we obtain

\[
y = \frac{(4x^2(4 - c^2) + 5c^2)c}{8(15 - x)(4 - c^2)(x - 1)} := y_0.
\]

For \(y_0\) to belong to \((0,1)\), we must have

\[
5c^3 + 4cx^2(4 - c^2) < 8(x - 1)(4 - c^2)(15 - x) \tag{39}
\]

and

\[
c^2 > 4. \tag{40}
\]

Now, in order to find the solutions that meet both inequalities (39) and (40), we see that \(c^2 > 4\), and an easy calculation shows that (39) does not hold for all \(x \in (0,1)\). This implies that we found no optimal point of \(f\) in \((0,2) \times (0,1) \times (0,1)\).
(ii). Next, we look forward to the maximum of \( f(c, x, y) \) in the interior of all six faces of \( \Delta \). If we choose \( c = 0 \), then we obtain
\[
j_1(x, y) = f(0, x, y) = 3072 \left( (x - 15)(x - 1)y^2 + 16x \right) (1 - x^2),
\]
which shows that there does not exist any point of extrema for \( f(0, x, y) \) in \((0, 1) \times (0, 1)\).
Taking \( c = 2 \), we obtain
\[
f(2, x, y) = 320. \tag{41}
\]
Setting \( x = 0 \), we have
\[
j_2(c, y) = 5c^6 + \left( 4 - c^2 \right) \left( 11520y^2 + 240c^3y - 2880c^2y^2 \right) = f(c, 0, y).
\]
We found no solution for the following system of equations
\[
\frac{\partial j_2}{\partial c} = 0, \quad \frac{\partial j_2}{\partial y} = 0,
\]
in the interval \((0, 2) \times (0, 1)\). Choosing \( x = 1 \), we obtain
\[
j_3(c, y) = f(c, 1, y) = 125c^6 - 1440c^4 + 3840c^2. \tag{42}
\]
For the critical point, \( \frac{\partial j_3}{\partial x} = 0 \) gives \( c_0 \approx 1.3104808 \), at which, \( j_3 \) attains its maximum. That is,
\[
j_3(c, y) \leq 2980.765545.
\]
Selecting \( y = 0 \), we have
\[
j_4(c, x) = f(c, x, 0) = 48c^6x^4 + 5c^6 - 384c^4x^4 + 72c^6x^2 - 1056c^4x^2 - 3072c^4x^3 + 768c^2x^4 - 49152x + 3072c^2x^2 + 24576c^2x^3 - 24576c^2x - 49152x^3 + 3072c^4x.
\]
A computation indicates that the solution does not exist for the following equations:
\[
\frac{\partial j_4}{\partial c} = 0, \quad \frac{\partial j_4}{\partial x} = 0
\]
in \((0, 2) \times (0, 1)\). Substituting \( y = 1 \), we obtain
\[
j_5(c, x) = f(c, x, 1) = 5c^6 + 48c^6x^4 - 192c^5x^4 + 72c^6x^2 - 576c^4x^4 + 432c^2x^2 + 1536c^3x^4 - 3744c^4x^2 - 3072c^4x^2 + 46080 + 2304c^2x^4 + 960c^3 - 240c^5 - 2496c^3x^2 - 3072c^4x^4 + 2880c^4 - 23040c^2 - 43008x^2 + 24576c^2x^2,
\]
which shows that the following system of equations
\[
\frac{\partial j_5}{\partial c} = 0, \quad \frac{\partial j_5}{\partial x} = 0.
\]
has no optimal solution in \((0, 2) \times (0, 1)\).
(iii). Finally, we now look forward to the maximum of \( f(c, x, y) \) at the edges of \( \Delta \). By selecting \( y = 0 \) and \( x = 0 \), we obtain
\[
j_6(c) = f(c, 0, 0) = 5c^6,
\]
which gives that
\[
f(c, 0, 0) \leq 320.
By choosing $y = 1$ and $x = 0$, we obtain

$$j_7(c) = f(c, 0, 1) = 5c^6 - 240c^5 + 2880c^4 + 960c^3 - 23040c^2 + 46080.$$  

Clearly, $\frac{\partial j_7}{\partial c} < 0$ for $[0, 2]$, which indicates that $j_7(c)$ is decreasing and achieve its maximum at $c = 0$. Thus,

$$j_7(c) \leq 46080.$$

By taking $x = 0$ and $c = 0$, we obtain

$$j_8(y) = f(0, 0, y) = 46080y^2,$$

which shows that

$$f(0, 0, y) \leq 46080.$$

We note that (42) is free of $y$. It follows that

$$j_9(c) = f(c, 1, 0) = f(c, 1, 1) = 125c^6 - 1440c^4 + 3840c^2.$$

For the critical point, $\frac{\partial j_9}{\partial c} = 0$ gives $c_0 = 1.310480$, at which, the maximum value is attained for $j_9(c)$. Thus,

$$j_9(c) \leq 2980.76554.$$

By setting $x = 1$ and $c = 0$, we obtain

$$f(0, 1, y) = 0.$$  

By substituting $c = 2$, we see that (41) is free from $y, x,$ and $c$. It follows that

$$f(2, 0, y) = f(2, 1, y) = f(2, x, 1) = f(2, x, 0) = 320.$$  

By choosing $y = 0$ and $c = 0$, the edge $f(c, x, y)$ yields

$$j_{10}(x) = f(0, x, 0) = -49152x^3 + 49152x.$$  

For the critical point, $\frac{\partial j_{10}}{\partial x} = 0$ gives $x_0 = 0.577350$, at which, the maximum value is achieved for $j_{10}(x)$. Thus,

$$j_{10}(x) \leq \frac{32768}{3\sqrt{3}}.$$  

By taking $y = 1$ and $c = 0$, we obtain

$$j_{11}(x) = f(0, x, 1) = -3072(1 - x^2)(x^2 + 15).$$  

By simple computation, we see that $j_{11}(x)$ obtains its maximum value at 0, so we have

$$f(0, x, 1) \leq 46080.$$  

Hence, from the above situations, we conclude that

$$f(c, x, y) \leq 46080 \text{ on } \Delta : [0, 2] \times [0, 1] \times [0, 1].$$  

By using Equation (37), it follows that

$$\left|a_3a_5 - a_4^2\right| \leq \frac{1}{2949120}(f(c, x, y)) \leq \frac{1}{64}.$$  

The required inequality is accomplished. □
4. Logarithmic Coefficient Inequalities for the Class $\mathcal{BT}_{S_g}$

**Theorem 3.** If the function $f \in \mathcal{BT}_{S_g}$ assumes the series representation as given in (1), then

$$|\gamma_1| \leq \frac{1}{8}, \quad |\gamma_2| \leq \frac{1}{12}, \quad |\gamma_3| \leq \frac{1}{16}, \quad |\gamma_4| \leq \frac{1}{20}.$$ 

These inequalities are sharp, which can be seen from (4)—(7) and

$$f_0(z) = \int_0^z \left( \frac{2}{1 + e^{-t}} \right) dt = z + \frac{z^2}{4} - \frac{z^4}{96} + \cdots,$$

$$f_1(z) = \int_0^z \left( \frac{2}{1 + e^{-t^2}} \right) dt = z + \frac{z^3}{6} - \frac{z^7}{168} + \cdots,$$

$$f_2(z) = \int_0^z \left( \frac{2}{1 + e^{-t^3}} \right) dt = z + \frac{z^4}{8} - \frac{z^{10}}{240} + \cdots,$$

$$f_3(z) = \int_0^z \left( \frac{2}{1 + e^{-t^4}} \right) dt = z + \frac{z^5}{10} - \frac{z^{13}}{312} + \cdots.$$ 

**Proof.** Substituting (24)—(27) into (4)—(7), we obtain

$$\gamma_1 = \left( \frac{1}{16} \right) c_1,$$  \hspace{1cm}  (43)

$$\gamma_2 = -\frac{19}{768} c_1^2 + \frac{1}{24} c_2,$$  \hspace{1cm}  (44)

$$\gamma_3 = -\frac{7}{192} c_1 c_2 + \frac{31}{3072} c_1^3 + \frac{1}{32} c_3,$$  \hspace{1cm}  (45)

$$\gamma_4 = \frac{541}{23040} c_1^2 c_2 - \frac{37}{1280} c_1 c_3 - \frac{5941}{1474560} c_1^4 + \frac{1}{40} c_4 - \frac{41}{2880} c_2^2.$$  \hspace{1cm}  (46)

Applying (14) in (43), we obtain

$$|\gamma_1| \leq \frac{1}{8}.$$ 

Now, from (44), we can write

$$\gamma_2 = \frac{1}{24} \left( c_2 - \frac{19}{32} c_1^2 \right).$$

Applying (13), we obtain

$$|\gamma_2| \leq \frac{1}{12}.$$ 

For (45), we deduce that

$$|\gamma_3| = \frac{1}{32} \left| c_3 - 2 \left( \frac{7}{12} \right) c_1 c_2 + \left( \frac{31}{96} \right) c_1^3 \right|.$$ 

From (15), let

$$B = \frac{7}{12} \quad \text{and} \quad D = \frac{31}{96}.$$ 

It is clear that $0 \leq B \leq 1, B \geq D$ and

$$B(2B - 1) = \frac{7}{72} \leq D.$$
Thus, all of the conditions of Lemma (2) are satisfied. Thus, we have

$$|\gamma_3| \leq \frac{1}{16}.$$  

From (46), we conclude that

$$\gamma_4 = -\frac{1}{40} \left( \frac{5941}{36864} c_4^4 + \left( \frac{41}{72} \right) c_2^2 + 2 \left( \frac{37}{64} \right) c_1 c_3 - \frac{3}{2} \left( \frac{541}{864} \right) c_1^2 c_2 - c_4 \right)$$

(47)

$$\gamma_4 = -\frac{1}{40} \left( \eta c_4^4 + ac_2^2 + 2ac_1 c_3 - \frac{3}{2} \beta c_1^2 c_2 - c_4 \right),$$

(48)

where

$$\eta = \frac{5941}{36864}, \quad a = \frac{41}{72}, \quad \alpha = \frac{37}{64}, \quad \beta = \frac{541}{864}.$$  

We see that 0 < a < 1, 0 < \alpha < 1, and

$$8a(1-a) \left( (\alpha \beta - 2\eta)^2 + (\alpha(a+\alpha) - \beta)^2 \right) + a(1-a)(\beta - 2aa)^2 = 0.006067,$$

and

$$4aa^2 (1-a)^2 (1-a) = 0.05833.$$  

Thus, the conditions of Lemma 4 are satisfied. Thus, we have

$$|\gamma_4| \leq \frac{1}{20}.$$  

This completes the proof. \(\square\)

**Theorem 4.** Let the function \(f \in B \Sigma \mathcal{E}_q\) be of the form given in (1). Then,

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| \leq \max \left\{ \frac{1}{12}, \left| \frac{3 + 3|\lambda|}{192} \right| \right\}, \text{ for } \lambda \in \mathbb{C}.$$  

The result is sharp, which can be obtained from (4), (5), and

$$f_1(z) = \int_0^z \left( \frac{2}{1 + e^{(-z)}} \right) dt = z + \frac{z^3}{6} - \frac{z^7}{168} + \cdots.$$  

**Proof.** From (43) and (44), one may write

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| = \left| \frac{1}{24} c_2 - \frac{19}{768} c_1^2 - \frac{\lambda}{256} c_1^2 \right|.$$  

The application of (13) gives that

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| \leq \frac{2}{24} \max \left\{ 1, \left| \frac{19 + 3|\lambda|}{16} \right| \right\}.$$

After simplification, we obtain

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| \leq \max \left\{ \frac{1}{12}, \left| \frac{3 + 3|\lambda|}{192} \right| \right\},$$

which gives the required result. \(\square\)

**Theorem 5.** Let the function \(f \in B \Sigma \mathcal{E}_q\) be of the form given in (1). Then,

$$|\gamma_1 \gamma_2 - \gamma_3| \leq \frac{1}{16}.$$  

The inequality is sharp for (4)–(6) and
\[ f_2(z) = \int_0^z \left( \frac{2}{1 + e^{-ct}} \right) dt = z + \frac{z^4}{8} - \frac{z^{10}}{240} + \cdots. \]

**Proof.** Using (43–45), we have
\[ |\gamma_1 \gamma_2 - \gamma_3| = \frac{1}{32} \left| c_3 - 2 \left( \frac{5}{8} \right) c_1 c_2 + \left( \frac{143}{384} \right) c_4^3 \right|. \]

From (15), let \( B = \frac{5}{8} \) and \( D = \frac{143}{384} \).

Applying Lemma 2, we obtain
\[ |\gamma_1 \gamma_2 - \gamma_3| \leq \frac{1}{16}. \]

\( \square \)

**Theorem 6.** Let the function \( f \in B_{\Sigma \in \mathfrak{g}} \) be of the form given in (1). Then,
\[ |\gamma_4 - \gamma_2^2| \leq \frac{1}{20}. \]

The result is sharp for (5), (7) and
\[ f_3(z) = \int_0^z \left( \frac{2}{1 + e^{-ct}} \right) dt = z + \frac{z^5}{10} - \frac{z^{13}}{312} + \cdots. \]

**Proof.** By using (44) and (46), we obtain
\[
\begin{align*}
|\gamma_4 - \gamma_2^2| &= \left| -\frac{23}{1440} c_1^2 + \frac{1177}{46080} c_1^2 c_2 - \frac{13687}{2949120} c_1^4 - \frac{37}{1280} c_1 c_3 + \frac{1}{40} c_4^4 \right| \\
&= \frac{1}{40} \left| \frac{13687}{73728} c_1^4 + \frac{23}{36} c_2^2 + 2 \left( \frac{37}{64} \right) c_1 c_3 - \frac{3}{2} \left( \frac{1177}{1728} \right) c_2^2 c_4 - c_4 \right| \\
&= \frac{1}{40} \left| \eta c_1^4 + ac_2^2 + 2ac_1 c_3 - \frac{3}{2} \beta c_2^2 c_4 - c_4 \right|,
\end{align*}
\]
where
\[ \eta = \frac{13687}{73728}, \quad a = \frac{23}{36}, \quad \alpha = \frac{37}{64}, \quad \beta = \frac{1177}{1728}. \]

We see that \( 0 < a < 1, 0 < \alpha < 1, \) and
\[ 8a(1-a) \left( (\alpha \beta - 2\eta)^2 + (\alpha(\alpha + \alpha) - \beta)^2 \right) + a(1-a)(\beta - 2\alpha)^2 = 0.002673, \]
and
\[ 4\alpha^2 a^2 (1-a)^2 (1-a) = 0.05489. \]

Thus, all of the conditions of Lemma 4 are satisfied. Hence, we have
\[ |\gamma_4 - \gamma_2^2| \leq \frac{1}{20}. \]

\( \square \)
5. Hankel Determinant with Logarithmic Coefficients for the Class $\mathcal{B}\mathcal{T}_{\mathfrak{e}_0}$

**Theorem 7.** If the function $f \in \mathcal{B}\mathcal{T}_{\mathfrak{e}_0}$ assumes the series representation given in (1), then

$$\left| \mathbf{H}_{2,1}(F_f/2) \right| \leq \frac{1}{144}.$$  

The result is sharp and equality can be achieved from (4)–(6) and

$$f_1(z) = \int_0^z \left( \frac{2}{1 + e^{-|z|}} \right) dt = z + \frac{z^3}{6} - \frac{z^7}{168} + \cdots.$$ 

**Proof.** Substituting (43)–(45) into (9), we have

$$\mathbf{H}_{2,1}(F_f/2) = -\frac{1}{4608} c_1 c_2 + \frac{11}{589824} c_1^4 + \frac{1}{512} c_1 c_3 - \frac{1}{576} c_2^2.$$  

Using (16) and (17) to express $c_2$ and $c_3$ in terms of $c_1$ and setting $c_1 = \delta$, one may have

$$\left| \mathbf{H}_{2,1}(F_f/2) \right| = \left| -\frac{1}{2304} (4 - c_1)^2 x^2 - \frac{1}{2048} (4 - c_1)^2 x^2 - \frac{7}{196608} c_1^4 \right.$$  

$$+ \frac{1}{1024} \left( 1 - |x|^2 \right) (4 - c_1) c_1 \left|.$$  

Setting $|x| = u$, $|^| \leq 1$ with $u \leq 1$ and applying the triangle inequality, we have

$$\left| \mathbf{H}_{2,1}(F_f/2) \right| \leq \frac{1}{2304} (4 - c_1)^2 u^2 + \frac{1}{2048} (4 - c_1)^2 u^2 + \frac{7}{196608} c_1^4$$  

$$+ \frac{1}{1024} \left( 1 - u^2 \right) (4 - c_1) c_1 := \Psi(c, u).$$  

Now, differentiating $\Psi(c, u)$ with respect to parameter $u$, we have

$$\frac{\partial \Psi(c, u)}{\partial u} = \left( \frac{1}{9216} c_1^2 + \frac{1}{288} - \frac{1}{512} c_1 \right) (4 - c_1) u.$$  

We see that $\frac{\partial \Psi(c, u)}{\partial u} \geq 0$ on $[0, 1]$, which shows that $\Psi(c, u) \leq \Psi(c, 1)$. Thus,

$$\left| \mathbf{H}_{2,1}(F_f/2) \right| \leq \frac{1}{2304} (4 - c_1)^2 + \frac{c_1^2}{2048} (4 - c_1)^2 + \frac{7c_1^4}{196608} := L(c).$$  

By simple computation, it follows that $L(c)$ obtains its maximum value at $c = 0$. Hence,

$$\left| \mathbf{H}_{2,1}(F_f/2) \right| \leq \frac{1}{144}.$$  

□

**Theorem 8.** If the function $f \in \mathcal{B}\mathcal{T}_{\mathfrak{e}_0}$ assumes the series representation given in (1), then

$$\left| \mathbf{H}_{2,2}(F_f/2) \right| \leq \frac{1}{256}.$$  

The result is sharp. Equality can be achieved from (5)–(7) and

$$f_2(z) = \int_0^z \left( \frac{2}{1 + e^{-|z|}} \right) dt = z + \frac{z^4}{6} - \frac{z^{10}}{240} + \cdots.$$  

**Proof.** The determinant $\mathbf{H}_{2,2}(F_f/2)$ is described as follows.

$$\mathbf{H}_{2,2}(F_f/2) = \gamma_2 \gamma_4 - \gamma_5^2.$$
By virtue of (44)–(46), along with \( c_1 = c \in [0, 2] \), it can be written that

\[
\mathcal{H}_{2,2}(F_f/2) = \frac{1}{T}\left(95616c^3c_3 + 1179648c_2c_4 - 14888c^4c_2 - 671744c_3^2 - 1105920c_3^3 + 1216512c_2^2c_3 - 2441c^6 - 700416c^2c_4 + 1536c^2c_2^2\right),
\]

where \( T = 1132462080 \) and by substituting \( m = 4 - c^2 \) in (16)–(18). Now, applying these lemmas, we have

\[
\begin{align*}
95616c^3c_3 &= 23904 \left(c^6 - c^4mx^2\right) + 47808 \left(c^4xm + c^3m \left(1 - |x|^2\right)\delta\right), \\
14688c^4c_2 &= 7344c^4mx + 7344c^6, \\
671744c_3^2 &= 251904 \left(c^4mx + c^2m^2x^2\right) + 83968 \left(m^3x^3 + c^6\right), \\
1216512c_2c_3 &= -152064c^3m^2c^2 - 152064c^4mx^2 + 304128cxm^2 \left(1 - |x|^2\right)\delta \\& + 304128cx^2m^2 + 304128c^3m \left(1 - |x|^2\right)\delta + 456192c^4xm + 152064c^6, \\
1179648c_2c_4 &= -294912 \left(1 - |x|^2\right)\delta m^2c^2 + 73728c^4mx^3 + 294912c^4xm + 294912m^3c^2x^2 + 73728c^4mx^2 + 221184c^2x^2m^2 - 221184c^4mx^2 + 73728c^6 \\
&+ 294912 \left(1 - |x|^2\right)\delta m^2x + 294912 \left(1 - |x|^2\right)^2 \left(1 - |x|^2\right)c^2pm \\
&- 294912 \left(1 - |x|^2\right)^2\delta m - 221184c^3m^2c^2 + 294912c^3m^2 \\
&- 294912c^2x^2m \left(1 - |x|^2\right)c^6 - 294912c^2x^2m \left(1 - |x|^2\right)\delta - 294912\delta m^2c^2 + 305208 \left(1 - |x|^2\right)\delta c^2 + 552012c^4xm + 294912m^3x \left(1 - |x|^2\right)\delta + 26265c^6, \\
700416c^2c_4 &= 87552c^4mx^3 - 350208 \left(1 - |x|^2\right)\delta^3m^2c^2 + 87552 \delta^6 + 26265c^4xm + 350208 \left(1 - |x|^2\right)\delta^2m^2 + 350208m^2c^2x^2 - 26265c^4mx^2 + 350208 \left(1 - |x|^2\right)\delta^2m^2 \\
&- 350208 \left(1 - |x|^2\right)^2\delta m - 350208 \left(1 - |x|^2\right)c^2m^2x, \\
1105920c_3^3 &= -276480x^2m^2 \left(1 - |x|^2\right)\delta + 69120x^2m^2c^2 - 138240c^4mx^2 \\
&- 276480x^2m^2c^2 + 276480c^2x^2m^2 + 552960cxm^2 \left(1 - |x|^2\right)\delta \\
&+ 276480m^2 \left(1 - |x|^2\right)^2\delta^2 + 276480c^4xm + 276480c^3m \left(1 - |x|^2\right)\delta + 69120c^6, \\
1536c^2c_2^2 &= 768c^4mx + 384 \left(c^6 + c^2m^2x^2\right).
\end{align*}
\]

Inserting the above expression into (49), we achieve

\[
\mathcal{H}_{2,2}(F_f/2) = \frac{1}{T}\left(55296c^3m \left(1 - |x|^2\right)x\delta + 55296c^2m \left(1 - |x|^2\right)\delta^2m^2 + 55296c^2m \left(1 - |x|^2\right)\delta^2m^2 + 46080c^4xm \left(1 - |x|^2\right)\delta \\
- 18432x^2m^2 \left(1 - |x|^2\right)c^6 - 18432x^2m^2 \left(1 - |x|^2\right)c^6 - 18432x^2m^2 \left(1 - |x|^2\right)c^6 - 18432c^6m \left(1 - |x|^2\right)c^6x^2m^2 \\
+ 276480m^2 \left(1 - |x|^2\right)\delta^2 + 20160c^3m \left(1 - |x|^2\right)\delta - 83968x^3m^3 + 294912c^2\delta^2m^2 - 345c^6\right).
\]
Since \( m = 4 - c^2 \), it follows that
\[
\mathcal{H}_{2,2}(F_{f}/2) = \frac{1}{T} \left( b_1(c,x) + b_2(c,x)\delta + b_3(c,x)\delta^2 + (c,x,\delta)\rho \right),
\]
where \( \delta, x, \rho \in \mathbb{R} \), and
\[
b_1(c,x) = -345c^6 + \left( 4 - c^2 \right) \left( 4608c^2x^4 - 40960cx^3 - 12800cx^3 + 3744c^2x^2 \right),
\]
\[
b_2(c,x) = \left( 4 - c^2 \right) \left( 1 - |x|^2 \right) \left( 4 - c^2 \right) \left( -18432c^3x^2 + 46080cx \right) + 55296c^3x + 20160c^3,
\]
\[
b_3(c,x) = \left( 4 - c^2 \right) \left( 1 - |x|^2 \right) \left( 4 - c^2 \right) \left( -276480 - 18432|x|^2 \right) + 55296c^2x,
\]
\[
(c,x,\delta) = \left( 4 - c^2 \right) \left( 1 - |x|^2 \right) \left( 1 - |\delta|^2 \right) \left[ -55296c^2 + 294912x \left( 4 - c^2 \right) \right].
\]

By replacing \( |\delta| \) with \( y \) and \( |x| \) with \( x \), if we apply the inequality \( |\rho| \leq 1 \), it follows that
\[
\left| \mathcal{H}_{2,2}(F_{f}/2) \right| \leq \frac{1}{T} \left( |b_1(c,x)| + y|b_2(c,x)| + y^2|b_3(c,x)| + |(c,x,\delta)| \right),
\]
\[
\leq \frac{1}{T}(W(c,x,y)), \quad (50)
\]
where
\[
W(c,x,y) = t_0(c,x) + t_1(c,x)y + t_2(c,x)y^2 + t_3(c,x) \left( 1 - y^2 \right),
\]
with
\[
t_0(c,x) = 345c^6 + \left( 4 - c^2 \right) \left( 40960x^3 + 4608c^2x^4 + 12800cx^3 + 3744c^2x^2 \right),
\]
\[
+ 3744c^2x^2 + 13824c^4x^3 + 55296c^3x^2 + 1296c^4x \left( 4 - c^2 \right),
\]
\[
t_1(c,x) = \left( 4 - c^2 \right) \left( 1 - x^2 \right) \left( 4 - c^2 \right) \left( 18432c^2x + 46080cx \right) + 55296c^3x + 20160c^3,
\]
\[
t_2(c,x) = \left( 4 - c^2 \right) \left( 1 - x^2 \right) \left( 4 - c^2 \right) \left( 276480 + 18432x^2 \right) + 55296c^2x,
\]
\[
t_3(c,x) = \left( 4 - c^2 \right) \left( 1 - x^2 \right) \left[ -55296c^2 + 294912x \left( 4 - c^2 \right) \right].
\]

Now, we have to maximize \( W(c,x,y) \) in the closed cuboid \( \Delta : [0,2] \times [0,1] \times [0,1] \). For this purpose, we have to find the maxima of \( W(c,x,y) \) in the interior of \( \Delta \), in the interior of its six faces, and on the twelve edges of \( \Delta \).

(i) Initially, we find the maximum of \( W(c,x,y) \) in the interior of \( \Delta \). Let \( (c,x,y) \in (0,2) \times (0,1) \times (0,1) \). Then, when differentiating \( W(c,x,y) \) partially with respect to parameter \( y \), it implies that
\[
\frac{\partial W}{\partial y} = 576(1 - x^2) \left( 4 - c^2 \right) \left[ \left( 192c^2 + 64 \left( 4 - c^2 \right)(x - 15) \right)(x - 1)y 
\right.
\]
\[
+ \left( c^2(35 + 96x) + x \left( 4 - c^2 \right)(80 + 32x) \right)c \right].
\]

The equation \( \frac{\partial W}{\partial y} = 0 \) gives
\[
y = \frac{\left( c^2(35 + 96x) + x \left( 4 - c^2 \right)(80 + 32x) \right)c}{\left( -192c^2 + 64 \left( 4 - c^2 \right)(15 - x) \right)(x - 1)} := y_3.
\]

For \( y_3 \) to belong to \( (0,1) \), it must follow that
\[
c^3(35 + 96x) + cx(80 + 32x) \left( 4 - c^2 \right) + 64(1 - x)(15 - x) \left( 4 - c^2 \right) < 192c^2(1 - x). \quad (51)
\]
and

\[ c^2 > \frac{3840 - 256x}{1152 - 64x}. \]  (52)

Now, in order to find the solutions that meet both inequalities (51) and (52), we see that \( v(x) = \frac{3840 - 256x}{1152 - 64x} \) gives that \( v'(x) < 0 \) for \((0, 1)\), which shows that \( v(x) \) is decreasing over \((0, 1)\). Thus, \( c^2 > \frac{36}{17} \) and an easy calculation shows that (51) does not hold for all \( x \in (0, 1) \). Therefore, there are no critical points of \( W \) in the interval \((0, 2) \times (0, 1) \times (0, 1)\).

(ii) Next, we look forward to find the maximum of \( W(c, x, y) \) in the interior of all six faces of \( \Delta \).

Choosing \( c = 0 \), we obtain

\[
t_1(x,y) = W(0, x, y) = -4063232x^3 + 294912(x - 1)(x - 15)(1 - x^2)y^2 + 4718592x,
\]

which shows that there does not exist any max \( W(0, x, y) \) in \((0, 1) \times (0, 1)\). Taking \( c = 2 \), we have

\[
W(2, x, y) = 22080. \quad (53)
\]

Selecting \( x = 0 \), we find that

\[
t_2(c, y) = W(c, 0, y) = 345c^6 + (4 - c^2)\left(20160c^3y - 331776c^2y^2 + 55296c^2 + 1105920y^2\right).
\]

For the critical points, we see that \( \frac{\partial t_2}{\partial y} = 0 \) gives

\[
y = \frac{35c^3}{384(3c^2 - 10)} := y_0. \quad (54)
\]

For the concerned range of \( y, y_0 \) would belong to \((0, 1)\) only if \( c > 1.880236 \). Furthermore,

\[
\frac{\partial t_2}{\partial c} = \left(4 - c^2\right)\left(60480c^2y - 663552c^2y^2 + 110592c\right) + 2070c^5 - 40320c^4y + 663552c^2y^2 - 110592c^3 - 2211840cy^2.
\]

Inserting (54) into the above expression and setting \( \frac{\partial t_2}{\partial c} = 0 \), we obtain

\[
\frac{\partial t_2}{\partial c} = 4185c^6 - 3994512c^7 + 34477704c^5 - 97320960c^3 + 88473600c = 0.
\]

Now, solving for \( c \in (0, 2) \), we obtain \( c \approx 1.4072104 \). Thus, no optimal solution is achieved for \( W(c, 0, y) \) in \((0, 2) \times (0, 1)\). If we choose \( x = 1 \), we obtain

\[
t_3(c, y) = W(c, 1, y) = 1577c^6 - 99648c^4 + 215040c^2 + 655360. \quad (55)
\]

For the critical point, \( \frac{\partial t_3}{\partial c} = 0 \) gives \( c_0 \approx 1.052686 \), at which, \( t_3 \) attains its maximum value; that is,

\[ W(c, 1, y) \leq 773435.177. \]

Setting \( y = 0 \), we have

\[
t_4(c, x) = W(c, x, 0) = -1024c^6x^3 + 4608c^6x^4 - 36864c^4x^4 - 1056c^6x^2 + 2236416c^2x^3 - 301056c^4x^3 - 6528c^4x^2 + 30096c^4x + 73728c^2x^4 - 1296c^6x + 43008c^2x^2 - 55296c^4 + 345c^6 - 2359296c^2x + 221184c^2 + 4718592x - 4063232x^3.
\]
A computation shows that a solution for a system of the following equations
\[
\frac{\partial t_4}{\partial c} = 0 \quad \text{and} \quad \frac{\partial t_4}{\partial x} = 0
\]
in \((0,2) \times (0,1)\) does not exist. Substituting \(y = 1\), we find that
\[
t_5(c, x) = W(c, x, 1) = -1024c^6x^3 + 4608c^6x^4 - 1056c^6x^2 - 18432c^5x^4 + 9216c^5x^3 + 345c^6
\]
\[-1296c^6x - 55296c^4x^4 + 49152c^4x^3 + 38592c^5x^2 + 655360x^3
\]-\(9216c^5x + 147456c^3x^4 + 147456c^3x^3 - 319872c^4x^2 + 294912c^2x^2
\]-\(20160c^5 + 221184c^2x^4 - 228096c^3x^2 - 50112c^4x - 221184c^2
\]-\(344064c^2x^2 - 294912c^3x^3 - 344064c^2x^3 + 2328576c^3x^2 - 294912x^4 - 737280c^3x^3 + 221184c^2x + 80640c^3 + 737280cx
\]+\(4423680 + 276480c^4 - 4128768x^2 - 147456c^3x.\)

In this situation, we came to the same conclusion as for \(t_4\); that is, the system
\[
\frac{\partial t_5}{\partial c} = 0, \quad \frac{\partial t_5}{\partial x} = 0
\]
has no solution in \((0,2) \times (0,1)\).

(iii) Finally, we now intend to find the maximum value of \(W(c, x, y)\) at the twelve edges of \(\Delta\). For this, we proceed as follows.

By choosing \(y = 0\) and \(x = 0\), it yields that
\[
t_6(c) = W(c, 0, 0) = 345c^6 - 55296c^4 + 221184c^2.
\]

For the critical point, \(\frac{\partial t_6}{\partial c} = 0\) gives \(c_0 \approx 1.4279024\), at which, the maximum value is achieved for \(t_6(c)\). That is,
\[
W(c, 0, 0) \leq \frac{22402453}{100}.
\]

By selecting \(y = 1\) and \(x = 0\), we obtain
\[
t_7(c) = W(c, 0, 1) = -20160c^5 + 345c^6 + 80640c^3 + 276480c^4 + 276480c^4 + 4423680 + 276480c^4.
\]

By simple computation, we see that \(t_7(c)\) obtains its maximum value at 0, so we have
\[
W(c, 0, 1) \leq 4423680.
\]

By setting \(x = 0\) and \(c = 0\), we have
\[
t_8(y) = W(0, 0, y) = 4423680y^2.
\]

It follows that \(\frac{\partial t_8}{\partial y} > 0\) for \([0, 1]\) shows that \(t_8(y)\) is decreasing and its maxima is achieved at 1. Hence, we have
\[
W(0, 0, y) \leq 4423680.
\]

We note that (55) is free from \(y\). It follows that
\[
t_9(c) = W(c, 1, 1) = W(c, 1, 0)
\]
\[
= 1577c^6 - 99648c^4 + 215040c^2 + 655360.
\]
For the critical point, \( \frac{\partial}{\partial c} 9 \frac{\partial}{\partial c} = 0 \) gives \( c_0 \approx 1.052686 \), at which, the maximum value is achieved for \( t_0(c) \). We conclude that
\[
W(c, 1, 0) \leq 773435.177.
\]
By substituting \( x = 1 \) and \( c = 0 \), we obtain
\[
W(0, 1, y) = 655360.
\]
By selecting \( c = 2 \), we see that \((53)\) is free from \( y, x, \) and \( c \). It follows that
\[
W(2, 1, y) = W(2, 0, y) = W(2, x, 1) = W(2, x, 0) = 22080.
\]
By taking \( y = 1 \) and \( c = 0 \), we obtain
\[
t_{10}(x) = W(0, x, 1) = -294912x^4 + 655360x^3 - 4128768x^2 + 4423680.
\]
By simple computation, we see that \( t_{10}(x) \) obtains its maximum value at 0, so we have
\[
t_{10}(x) \leq 4423680.
\]
By choosing \( y = 0 \) and \( c = 0 \), we have
\[
t_{11}(x) = W(0, x, 0) = -4063232x^3 + 4718592x.
\]
For the critical point, \( \frac{\partial t_{11}}{\partial x} = 0 \) gives \( x_0 \approx 0.6221710 \), at which, the maximum value is achieved for \( t_{11}(x) \). Therefore, we have
\[
W(0, x, 0) \leq \frac{6291456}{31} \sqrt{3} \sqrt{31}.
\]
Hence, from the above situations, we conclude that
\[
W(c, x, y) \leq 4423680 \text{ on } \Delta : [0, 2] \times [0, 1] \times [0, 1].
\]
By using Equation \((50)\), it follows that
\[
\left| H_{2,2} \left( F_j / 2 \right) \right| \leq \frac{1}{T} \left( W(c, x, y) \right) \leq \frac{1}{256},
\]
which completes the proof. \( \square \)

6. Conclusions
We have obtained the sharp bounds of Hankel determinants of order three for the class of functions with bounding turning that are associated with the sigmoid function. All of the bounds that we found here were sharp. Moreover, we investigated the sharp bounds of logarithmic coefficients linked with the functions of bounded turnings. This also includes the third-order Hankel determinant for these logarithmic coefficients. This work will help in finding the fourth-order Hankel determinants for the same types of analytic functions that have been considered in this study.

Author Contributions: Conceptualization, M.A. (Muhammad Arif) and Q.X.; data curation, M.A. (Muhammad Ayaz); formal analysis, S.N.M.; funding acquisition, M.A. (Muhammad Arif) and F.T.; investigation, S.M. and Q.X.; methodology, S.M. and Q.X.; project administration, M.A. (Muhammad Ayaz); resources, M.A. (Muhammad Ayaz); software, S.N.M.; supervision, M.A. (Muhammad Arif); validation, F.T.; visualization, F.T.; writing—original draft, S.N.M.; writing—review and editing, S.N.M. All authors contributed equally and approved the final manuscript.

Funding: This research received no external funding.
Data Availability Statement: Not applicable.

Acknowledgments: This research was supported by the researchers Supporting Project Number [RSP-2021/401], King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Salii, A.; Noor, K.I.; Hussain, S.; Darus, M. On Quantum Differential Subordination Related with Certain Family of Analytic Functions. *J. Math.* 2020, 2020, 6675732. [CrossRef]

2. Salii, A.; Jabeen, K.; Al-shbeil, I.; Oladejo, S.O.; Cătăș, A. Radius and Differential Subordination Results for Starlikeness Associated with Limaçon Class. *J. Funct. Spaces* 2022, 2022, 8264693. [CrossRef]

3. Al-Shbeil, I.; Salii, A.; Cătăș, A.; Malik, S.N.; Oladejo, S.O. Some Geometrical Results Associated with Secant Hyperbolic Functions. *Mathematics* 2022, 10, 2697. [CrossRef]

4. Bieberbach, L. Über die koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitzungsberichte Preuss. Akad. Der Wiss.* 1916, 138, 940–955.

5. De-Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* 1970, 154, 137–152. [CrossRef]

6. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press: New York, NY, USA, 1992; pp. 157–169.

7. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* 1970, 23, 159–177. [CrossRef]

8. Sokół, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* 1996, 19, 101–105.

9. Kumar, S.S.; Arora, K. Starlike functions associated with a petal shaped domain. *arXiv* 2020; arXiv:2010.10072.

10. Barukab, O.; Arif, M.; Abbas, M.; Khan, S.K. Sharp bounds of the coefficient results for the family of bounded turning functions associated with petal shaped domain. *J. Funct. Spaces* 2021, 2021, 5535629. [CrossRef]

11. Shi, L.; Arif, M.; Rafiq, A.; Abbas, M.; Iqbal, J. Sharp Bounds of Hankel Determinant on Logarithmic Coefficients for Functions of Bounded Turning Associated with Petal-Shaped Domain. *Mathematics* 2022, 10, 1939. [CrossRef]

12. Gandhi, S. Radius estimates for three leaf function and convex combination of starlike functions. In *Proceedings of the International Conference on Recent Advances in Pure and Applied Mathematics, Gorakhpur, India, 12–13 April 2018*; pp. 173–184.

13. Arif, M.; Barukab, O.M.; Afzal Khan, S.; Abbas, M. The Sharp Bounds of Hankel Determinants for the Families of Three-Leaf-Type Functions. *Fractal Fract.* 2022, 6, 291. [CrossRef]

14. Shi, L.; Arif, M.; Raza, M.; Abbas, M. Hankel Determinant Containing Logarithmic Coefficients for Bounded Turning Functions Connected to a Three-Leaf-Shaped Domain. *Mathematics* 2022, 10, 2924. [CrossRef]

15. Bano, K.; Raza, M. Starlike functions associated with cosine function. *Bull. Iran. Math. Soc.* 2021, 47, 1513–1532. [CrossRef]

16. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikeness Associated with Cosine Hyperbolic Function. *Mathematics* 2020, 8, 1118. [CrossRef]

17. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* 2019, 45, 213–232. [CrossRef]

18. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* 1966, 1, 111–122. [CrossRef]

19. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* 1967, 14, 108–112. [CrossRef]

20. Hayman, W.K. On second Hankel determinant of mean univalent functions. *Proc. Lond. Math. Soc.* 1968, 3, 77–794. [CrossRef]

21. Obradović, M.; Tuneski, N. Hankel determinants of second and third order for the class S of univalent functions. *Mathematica Slovaca* 2021, 71, 649–654. [CrossRef]

22. Janteng, A.; Halim, S.A.; Darus, M. Coefficient inequality for a function whose derivative has a positive real part. *J. Inequalities Pure Appl. Math.* 2006, 7, 1–5.

23. Janteng, A.; Halim, S.A.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math.* 2007, 1, 619–625.

24. Khan, B.; Aldawish, I.; Araci, S.; Khan, M.G. Third Hankel Determinant for the Logarithmic Coefficients of Starlike Functions Associated with Sine Function. *Fractal Fract.* 2022, 6, 261. [CrossRef]

25. Zhang, H.-Y.; Huo, T. A study of fourth-order Hankel determinants for starlike functions connected with the sine function. *J. Funct. Spaces* 2021, 2021, 9991460. [CrossRef]

26. Shafiq, M.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Darus, M.; Kiran, S. An Upper Bound of the Third Hankel Determinant for a Subclass of k-Starlike Functions Associated with k-Fibonacci Numbers. *Symmetry* 2020, 12, 1043. [CrossRef]

27. Murugusundaramoorthy, G.; Bulboaca, T. Hankel Determinants for New Subclasses of Analytic Functions Related to a Shell Shaped Region. *Mathematics* 2020, 8, 1041. [CrossRef]

28. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper Bound of the Third Hankel Determinant for a Subclass of q-Starlike Functions. *Symmetry* 2019, 11, 347. [CrossRef]
29. Srivastava, H.M.; Ahmad, Q.Z.; Darus, M.; Khan, N.; Khan, B.; Zaman, N.; Shah, H.H. Upper Bound of the Third Hankel Determinant for a Subclass of Close-to-Convex Functions Associated with the Lemniscate of Bernoulli. *Mathematics* 2019, 7, 848. [CrossRef]
30. Raza, M.; Riaz, A.; Xin, Q.; Malik, S.N. Hankel Determinants and Coefficient Estimates for Starlike Functions Related to Symmetric Booth Lemniscate. *Symmetry* 2022, 14, 1366. [CrossRef]
31. Arif, M.; Raza, M.; Tang, H.; Hussain, S.; Khan, H. Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Math.* 2019, 17, 1615–1630. [CrossRef]
32. Riaz, A.; Raza, M.; Thomas, D.K. The Third Hankel determinant for starlike functions associated with sigmoid functions. *Forum Math.* 2022, 34, 137–156. [CrossRef]
33. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz Determinants for a Subclass of $q$-Starlike Functions Associated with a General Conic Domain. *Mathematics* 2019, 7, 181. [CrossRef]
34. Saliu, A.; Noor, K.I. On Coefficients Problems for Certain Classes of Analytic Functions. *J. Math. Anal.* 2021, 12, 13–22.
35. Kowalczyk, B.; Lecko, A. Second Hankel determinant of logarithmic coefficients of convex and starlike functions. *Bull. Aust. Math. Soc.* 2022, 105, 458–467. [CrossRef]
36. Kowalczyk, B.; Lecko, A. Second Hankel Determinant of logarithmic coefficients of convex and starlike functions of order alpha. *Bull. Malays. Math. Sci. Soc.* 2022, 45, 727–740. [CrossRef]
37. Pommerenke, C. *Univalent Function*; Vanderhoeck & Ruprecht: Göttingen, Germany, 1975.
38. Carathéodory, C. Über den Variabilitätsbereich der Fourier’schen Konstanten von position harmonischen Funktionen. *Rendiconti Del Circolo Matematico di Palermo* 1911, 32, 193–217. [CrossRef]
39. Libera, R.J.; Złotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivative in $P$. *Proc. Am. Math. Soc.* 1983, 87, 251–257. [CrossRef]
40. Libera, R.J.; Złotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Am. Math. Soc.* 1982, 85, 225–230. [CrossRef]
41. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* 2018, 18, 307–314. [CrossRef]
42. Ravichandran, V.; Verma, S. Bound for the fifth coefficient of certain starlike functions. *C. R. Math. Acad. Sci. Paris* 2015, 353, 505–510. [CrossRef]