Classical Capacity of A Quantum Multiple Access Channel†

Minxin Huang†, Yongde Zhang§‡, Guang Hou‡

† Special Class for Gifted Young, University of Science and Technology of China, Hefei, 230026, P.R. China.
§ CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, P.R. China.
‡ Department of Modern Physics, University of Science and Technology of China, Hefei, 230027, P.R. China.

Abstract

We consider the transmission of classical information over a quantum channel by two senders. The channel capacity region is shown to be a convex hull bound by the Von Neumann entropy and the conditional Von Neumann entropy. We discuss some possible applications of our result. We also show that our scheme allows a reasonable distribution of channel capacity over two senders.

1 Introduction

In quantum information theory, a basic question often presented is how efficiently can one transmit classical information over a quantum channel. Because of the non-orthogonality of quantum states, the channel capacity is different from that of the classical channel discussed in . Recently, this question has attracted new attention due to the rapid progress in quantum information theory. In particular, a theorem is established that the maximum attainable rate of asymptotically error free transmission of classical information over a quantum channel is precisely the Holevo bound.

---

1 The project supported by National Natural Science Foundation of China
2 Email:minxin@ustc.edu
In this paper, we consider the transmission of classical information by two senders to a common receiver. The scheme can be viewed as a quantum multiple access channel, which is the quantum analogy of classical multiple access channel. Suppose two senders, Alice and Bob, are given an ensemble of normal letter states $|\Psi_{\alpha\beta}\rangle$, where $\alpha, \beta$ can be drawn from two alphabet sets: $\mathcal{H}_A = \{\alpha\}$ and $\mathcal{H}_B = \{\beta\}$, while Alice is allowed to choose the letter $\alpha$, Bob is allowed to choose the letter $\beta$. Then the letter state is sent to the receiver, Charlie, who subjects it to a measurement to determine which letters Alice and Bob have chosen. We should note that the sets $\{\alpha\}$, $\{\beta\}$ are purely classical alphabets in the hands of Alice and Bob, and $|\Psi_{\alpha\beta}\rangle$ does not imply a tensor product Hilbert space. For example, if Alice uses the two-symbol alphabet $\{A, B\}$, and Bob uses $\{C, D\}$, then it is still possible that the signal states live in a two-dimensional Hilbert space, that is $|\Psi_{AC}\rangle = |0\rangle$, $|\Psi_{AD}\rangle = |1\rangle$, $|\Psi_{BC}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\Psi_{BD}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. So entanglement is not at issue here at all. But we will see it coming up in the application in Sec 6.

Suppose $p_\alpha q_\beta$ is a product probability distribution of the letter state $|\Psi_{\alpha\beta}\rangle$, then denote the density matrix by

$$
\rho = \sum_{\alpha,\beta} p_\alpha q_\beta |\Psi_{\alpha\beta}\rangle \langle \Psi_{\alpha\beta}| \\
\rho_\alpha = \sum_\beta q_\beta |\Psi_{\alpha\beta}\rangle \langle \Psi_{\alpha\beta}| \\
\rho_\beta = \sum_\alpha p_\alpha |\Psi_{\alpha\beta}\rangle \langle \Psi_{\alpha\beta}| 
$$

We denote the conditional Von Neumann entropy by

$$
H_A = \sum_\beta q_\beta H(\rho_\beta) \\
H_B = \sum_\alpha p_\alpha H(\rho_\alpha) \tag{1}
$$

We can see $H_A, H_B \leq H(\rho)$ from the concavity of Von Neumann entropy. A code $((M,N), l)$ is defined to consist of $M$ $\alpha$-letter sequences and $N$ $\beta$-letter sequences of length $l$, and together they form $MN$ code words $\{|S_{ij}| i = 1, 2, \cdots, M; j = 1, 2, \cdots, N\}$, with each $\alpha$-letter sequence and $\beta$-letter sequence combined. We use the above example to make things clear.

Suppose $l = 2$, and Alice choose 2 $\alpha$-letter sequences $\{AA, AB\}$, Bob choose $\{CD, DD\}$, then $M = N = 2$, and we see that $|S_{11}| = |\Psi_{AC}\rangle \otimes |\Psi_{AD}\rangle$, $|S_{12}| = |\Psi_{AD}\rangle \otimes |\Psi_{AD}\rangle$, $|S_{21}| = |\Psi_{AC}\rangle \otimes |\Psi_{BD}\rangle$, and $|S_{22}| = |\Psi_{AD}\rangle \otimes |\Psi_{BD}\rangle$.  

2
We assume the MN code words have the same probability $\frac{1}{MN}$. A rate $(R_1, R_2)$ is said to be achievable if there exists a sequence of $\left(\left(2^{nR_1}, 2^{nR_2}\right), n\right)$ codes for which Charlie can decode the message by a measurement with error probability $P_E \to 0$ when $n$ tends to infinity. The capacity region of the multiple access channel is the closure of the set of achievable $(R_1, R_2)$ rate pairs. The following theorem is the main result of this paper.

**Theorem** The capacity region of a quantum multiple access channel is the closure of the convex hull of all $(R_1, R_2)$ satisfying

$$R_1 < H_A, R_2 < H_B, R_1 + R_2 < H(\rho)$$

for some product distribution $p_\alpha q_\beta$ on $\mathcal{H}_A \times \mathcal{H}_B$.

Thus $H_A, H_B$ are analogs of classical conditional mutual information. Our result provides an information-theoretical interpretation of the conditional Von Neumann entropy.

We should note that the problem addressed here have been considered and solved in previous papers. It was first raised by Allahverdyan and Saakian\[12\], who essentially discussed the converse theorem in this paper. Then Winter gave general formulation and solution for many users and noisy channels\[13\]. See also Winter’s Ph.D. Thesis\[14\]. However, Winter’s proof is highly abstract, and he himself spoke of desirability of a more direct proof. This is what is done in this paper, for the particular case of two users and noiseless channel.

The paper is organized as follows. In Section 2, we describe some useful properties which are necessary to the proof. In Section 3 and 4, we prove the achievability of this theorem. In Section 5, we prove the converse theorem. Finally, in Section 6, we discuss some applications of the theorem.

## 2 Some useful properties

We first prove a lemma about conditional Von Neumann entropy.

**Lemma 1** $p_\alpha q_\beta$ is a fixed product distribution, then

$$H_A + H_B \geq H(\rho)$$

[Proof] This lemma can be proved by the strong subadditivity of Von Neumann entropy\[9\]. If denote
\[
\rho^{RST} = \sum_{\alpha, \beta} p_{\alpha q_{\beta}} |\Psi_{\alpha \beta}\rangle \langle \Psi_{\alpha \beta}| \otimes |i_{\alpha}\rangle \otimes |j_{\beta}\rangle \langle j_{\beta}|
\]

here \(|i_{\alpha}\rangle, |j_{\beta}\rangle\) are orthogonal states in Hilbert space \(S\) and \(T\), then

\[
\begin{align*}
\rho^R &= \rho \\
\rho^{RS} &= \sum_{\alpha, \beta} p_{\alpha q_{\beta}} |\Psi_{\alpha \beta}\rangle \langle \Psi_{\alpha \beta}| \otimes |i_{\alpha}\rangle \langle i_{\alpha}|
\end{align*}
\]

\[
\begin{align*}
\rho^{RT} &= \sum_{\alpha, \beta} p_{\alpha q_{\beta}} |\Psi_{\alpha \beta}\rangle \langle \Psi_{\alpha \beta}| \otimes |j_{\beta}\rangle \langle j_{\beta}|
\end{align*}
\]

we use the strong subadditivity of Von Neumann entropy:

\[
H \left( \rho^{RST} \right) + H (\rho) \leq H \left( \rho^{RS} \right) + H \left( \rho^{RT} \right)
\]

It can be easily seen that

\[
\begin{align*}
H \left( \rho^{RST} \right) &= H (p_{\alpha q_{\beta}}) = H (p_{\alpha}) + H (q_{\beta}) \\
H \left( \rho^{RS} \right) &= H (p_{\alpha}) + H_{B} \\
H \left( \rho^{RT} \right) &= H (q_{\beta}) + H_{A}
\end{align*}
\]

where \(H (p_{\alpha q_{\beta}}), H (p_{\alpha})\) and \(H (q_{\beta})\) are Shannon entropies. Thus, we have

\[
H_{A} + H_{B} \geq H (\rho)
\]

Another property of the quantum multiple access channel is its convexity, i.e., if \((R_1, R_2)\) and \((R'_1, R'_2)\) are achievable rates, then \((\lambda R_1 + (1 - \lambda) R'_1, \lambda R_2 + (1 - \lambda) R'_2)\) is also achievable, for any \(0 \leq \lambda \leq 1\). The idea is the time sharing scheme. Given two sequences of codes at different rates \((R_1, R_2)\) and \((R'_1, R'_2)\), we can construct a third code book at the rate of \(\lambda (R_1, R_2) + (1 - \lambda) (R'_1, R'_2)\) by using the first code book for the first \(\lambda n\) symbols and the second for the remaining \((1 - \lambda) n\) symbols. Since the overall probability of error is less than the sum of the probability of error for each of the segments, the probability of error of the new code approaches zero, the rate is achievable.

For a fixed product distribution \(p_{\alpha q_{\beta}}\), since \(H_{A} + H_{B} \geq H (\rho) \geq H_{A}\) or \(H_{B}\), according to the convexity of capacity region, we need only to prove \((H (\rho) - H_{B}, H_{B})\) and \((H_{A}, H (\rho) - H_{A})\) are achievable rates in order to prove all rate pair satisfying Eq.\((2)\) is achievable. We will do this in Section 3 and 4.
There is a useful inequality in bounding the decoding error. Suppose $\langle 1|1 \rangle \leq 1$ and $\langle 2|2 \rangle \leq 1$, then

$$
|\langle 1|3 \rangle| \leq |\langle 2|3 \rangle| + |(\langle 1| - \langle 2|) |3 \rangle|
$$

$$
\leq |\langle 2|3 \rangle| + \sqrt{\langle 3|3 \rangle \sqrt{\langle 1|1 \rangle + \langle 2|2 \rangle - \langle 1|2 \rangle - \langle 2|1 \rangle}}
$$

therefore

$$
|\langle 2|3 \rangle| \geq |\langle 1|3 \rangle| - \sqrt{\langle 3|3 \rangle \sqrt{2 - \langle 1|2 \rangle - \langle 2|1 \rangle}}
$$

This inequality implies that if $\langle 1|3 \rangle$ and $\langle 1|2 \rangle$ are close to unity, then $\langle 2|3 \rangle$ is also close to unity.

### 3 Compound measurement

As noted in previous sections, we only need to prove $(H(\rho) - H_B, H_B)$ is achievable rate. We use latin character $a, b, \ldots$ to index Alice’s strings, and $a', b', \ldots$ to index Bob’s. Suppose the string length is $L$. Denote

$$
\rho_a = \sum_{allstrings a'} P_{a'} |S_{aa'} \rangle \langle S_{aa'} |
$$

Here $P_{a'}$ means the product probability (For example, if $a' = CD$, then $P_{a'} = q_C q_D$). The sum $a'$ is over all possible strings, so $\rho_a$ is a product state. For example, if $a = AB$, then $\rho_a = \rho_A \otimes \rho_B$.

$\rho_a$ has a complete orthonormal set of eigenstates, which we denote as $|t_{ak}\rangle$, and a corresponding set of eigenvalues $p_{k|a}$. Let $\epsilon, \delta > 0$. Then we can find a length $L$ long enough to enforce some typicality conditions. Noticing that the quantity $H(\rho) - H_B$ is the Holevo information of the ensemble $\{p_a, \rho_a\}$, it was proved in Ref.\[4\] that Alice can choose $M = 2^{L(H(\rho) - H_B - \delta)}$ strings, so that the decoder can distinguish the eigenstates of the mixed states $\rho_a$ by a POVM (Positive Operator Valued Measure). Suppose $|\tilde{u}_{ak}\rangle \langle \tilde{u}_{ak}|$ are the elements of the decoding POVM. Then for every string $a$ (in the $M$ strings), the probability of right guess is

$$
\sum_k p_{k|a} |\langle \tilde{u}_{ak}|t_{ak}\rangle|^2 > 1 - \epsilon
$$

(5)
Denote $\Pi_a$ as the projection onto the subspace of vectors $|t_{ak}\rangle$ satisfying

$$2^{-L(H_B+\delta)} < p_{k|a} < 2^{-L(H_B-\delta)} \quad (6)$$

Because the POVM element $|\tilde{u}_{ak}\rangle = 0$ when $p_{k|a}$ doesn’t satisfy Eq.\((6)\) (as noted in Ref.\([7]\)), and $\langle \tilde{u}_{ak}|u_{ak}\rangle \leq 1$, we have

$$tr (\Pi_a \rho_a \Pi_a) = \sum_k p_{k|a} \geq \sum_k p_{k|a} |\langle \tilde{u}_{ak}|t_{ak}\rangle|^2 > 1 - \varepsilon \quad (7)$$

$$tr (\Pi_a \rho_a^2 \Pi_a) \leq 2^{-L(H_B-3\delta)} \quad (8)$$

We choose the $M$ strings described above as Alice’s signal strings and we will use random code to select Bob’s signal strings.

The decoding process includes two measurements, the first can decode Alice’s signal and the second can decode Bob’s signal. Denote

$$A_a = \sum_k |t_{ak}\rangle \langle \tilde{u}_{ak}|$$

Since $\sum_a A_a^\dagger A_a = \sum_{ak} |\tilde{u}_{ak}\rangle \langle \tilde{u}_{ak}|$, $A_a$ is a decoding POVM element, which will be the decoder’s first measurement. Suppose the result of the first measurement is string $a$, then the decoder’s second measurement will be the so called ”pretty good measurement” used in Ref.\([6]\) to distinguish $N$ states $\{\Pi_a | S_{aa'}\}$ in order to determine $a'$. Suppose $|\tilde{\eta}_{a'|a}\rangle \langle \tilde{\eta}_{a'|a}|$ is the element of the POVM. Together these two measurements form a compound measurement. The probability of error is

$$P_E = 1 - \frac{1}{MN} \sum_{aa'} |\langle \tilde{\eta}_{a'|a}| A_a | S_{aa'}\rangle|^2$$

we denote

$$P_{Ea} = 1 - \frac{1}{N} \sum_{a'} |\langle \tilde{\eta}_{a'|a}| A_a | S_{aa'}\rangle|^2$$

then

$$P_E = \frac{1}{M} P_{Ea}$$

Denote the random code average by ”\langle \rangle_c ”. The random code is averaged over Bob’s codes. We will prove $\langle P_{Ea}\rangle_c < 8\varepsilon$ for every string $a$. 

6
4 Bob’s random codes

Using the Schwarz inequality and the inequality (4), we have

\[ \sqrt{1 - P_{Ea}} = \left( \frac{1}{N} \sum_{a'} \left| \langle \bar{\eta}_{a'}|a|S_{aa'} \rangle \right|^2 \right)^{\frac{1}{2}} \]
\[ \geq \frac{1}{N} \sum_{a'} \left| \langle \bar{\eta}_{a'}|a|S_{aa'} \rangle \right| \]
\[ \geq \frac{1}{N} \sum_{a'} \left| \langle \bar{\eta}_{a'}|a|S_{aa'} \rangle \right| \]
\[ - \sqrt{\frac{1}{N} \sum_{a'} \left( 2 - \langle S_{aa'}|A_a|S_{aa'} \rangle - \langle S_{aa'}|A_a^\dagger|S_{aa'} \rangle \right)} \]
\[ = \Omega_1 - \Omega_2 \]

here \( \Omega_1 = \frac{1}{N} \sum_{a'} \left| \langle \tilde{\eta}_{a'}|a|S_{aa'} \rangle \right| \),

\( \Omega_2 = \frac{1}{N} \sum_{a'} \sqrt{\langle \tilde{\eta}_{a'}|a|\tilde{\eta}_{a'}|a \rangle \left( 2 - \langle S_{aa'}|A_a|S_{aa'} \rangle - \langle S_{aa'}|A_a^\dagger|S_{aa'} \rangle \right)^{\frac{1}{2}}}. \)

According to Schwarz inequality,

\[ \sqrt{\langle 1 - P_{Ea} \rangle} \geq \langle \sqrt{1 - P_{Ea}} \rangle \]
\[ \geq \langle \Omega_1 \rangle - \langle \Omega_2 \rangle \]

We deal with \( \langle \Omega_1 \rangle \) and \( \langle \Omega_2 \rangle \), respectively. The calculation of \( \langle \Omega_1 \rangle \) is the same as in Ref. [6], it was proved by using Eqs. (3) that

\[ \langle \Omega_1 \rangle \geq 1 - \varepsilon - N \cdot 2^{-L(H_B - \beta)} \]

Next, we examine term \( \Omega_2 \). First, we notice that \( \langle \tilde{\eta}_{a'}|a|\tilde{\eta}_{a'}|a \rangle \) is the POVM element, so \( \langle \tilde{\eta}_{a'}|a|\tilde{\eta}_{a'}|a \rangle \leq 1 \). Then we have

\[ \Omega_2 \leq \left( \frac{1}{N} \sum_{a'} \langle \tilde{\eta}_{a'}|a|\tilde{\eta}_{a'}|a \rangle \right) \cdot \left( \frac{1}{N} \sum_{a'} \left( 2 - \langle S_{aa'}|A_a|S_{aa'} \rangle - \langle S_{aa'}|A_a^\dagger|S_{aa'} \rangle \right) \right) \]
\[ \leq 2 - \frac{1}{N} \sum_{a'} \left( \langle S_{aa'}|A_a + A_a^\dagger|S_{aa'} \rangle \right) \]

Averaged over Bob’s code, then

\[ \langle \Omega_2 \rangle \leq 2 - Tr \left[ \left( A_a + A_a^\dagger \right) \rho_a \right] \]
\[ = 2 - 2 \sum_k p_k |k| \langle \bar{u}_k|t_k \rangle \]
\[ \leq 2 \left( 1 - \sum_k p_k |k| \langle \bar{u}_k|t_k \rangle \right)^2 \]
\[ < 2 \varepsilon \]
So \( \sqrt{\langle 1 - P_{E_a} \rangle_c} > 1 - 3\varepsilon - N2^{-L(H_B - 3\delta)} \), then we choose \( N = 2^{L(H_B - 4\delta)} \). When \( L \) is large, we have \( \sqrt{\langle 1 - P_{E_a} \rangle_c} > 1 - 4\varepsilon \), so \( \langle P_{E_a} \rangle_c < 8\varepsilon \). And therefore

\[
\langle P_E \rangle_c = \frac{1}{M} \sum_a \langle P_{Ea} \rangle_c < 8\varepsilon
\]

The average probability of error is small, so Bob can find a particular code for which \( P_E < 8\varepsilon \), thus complete the proof of the achievability of the theorem.

5 About the converse theorem

Denote \( \mathcal{E} \) as the closure of the convex hull of all \((R_1, R_2)\) satisfying Eq.(2). Suppose Alice and Bob can send information to Charlie at the rate of \((R_1, R_2)\), then we shall prove that \((R_1, R_2) \in \mathcal{E}\). This is the converse of the theorem. Consider a \((2^{lR_1}, 2^{lR_2}, l)\) code in which the code words are \( |S_{aa'}\rangle \). Suppose Charlie can decode the signals asymptotically error free, then when \( l \) is sufficiently large we have the inequality

\[
R_1 + R_2 \leq \frac{1}{l} I(Charlie : Alice, Bob)
\]
\[
R_1 \leq \frac{1}{l} I(Charlie : Alice | Bob)
\]
\[
R_2 \leq \frac{1}{l} I(Charlie : Bob | Alice)
\]

Suppose \( M = 2^{lR_1}, N = 2^{lR_2} \). Denote

\[
\rho_{\text{code}} = \sum_{aa'} \frac{1}{MN} |S_{aa'}\rangle \langle S_{aa'}|
\]
\[
\rho_{\text{code}}^a = \sum_{a'} \frac{1}{N} |S_{aa'}\rangle \langle S_{aa'}|
\]
\[
\rho_{\text{code}}^{a'} = \sum_a \frac{1}{M} |S_{aa'}\rangle \langle S_{aa'}|
\]

and their Von Neumann entropies by \( H_{\text{code}}, H_{\text{code}}^a, H_{\text{code}}^{a'} \). According to Holevo theorem\(^3\), the mutual information is bounded by Von Neumann entropies:

\[
I(Charlie : Alice, Bob) \leq H_{\text{code}}
\]
\[
I(Charlie : Alice | Bob) \leq \sum_a \frac{1}{N} H_{\text{code}}^{a'}
\]
\[
I(Charlie : Bob | Alice) \leq \sum_a \frac{1}{M} H_{\text{code}}^a
\]

8
Let $\mathcal{E}$ be the ensemble of letter states that appear as first letters in the code words, we then have a product distribution, we can define an entropy $H^{(1)}$ and the conditional entropy $H^{(1)}_A$ and $H^{(1)}_B$. Similarly define $H^{(k)}$, $H^{(k)}_A$ and $H^{(k)}_B$ for each position $k = 1, 2, \ldots, l$ in the code words. According to the subadditivity of Von Neumann entropy \[9\], we have

$$H_{\text{code}} \leq H^{(1)} + \cdots + H^{(l)}$$

$$\sum_a \frac{1}{N} H^{a'}_{\text{code}} \leq H^{(1)}_A + \cdots + H^{(l)}_A$$

$$\sum_a \frac{1}{M} H^{a'}_{\text{code}} \leq H^{(1)}_B + \cdots + H^{(l)}_B$$

From Eqs.\((9,10,11)\) combined, we have

$$R_1 + R_2 \leq \frac{1}{7} \left( H^{(1)} + \cdots + H^{(l)} \right)$$

$$R_1 \leq \frac{1}{7} \left( H^{(1)}_A + \cdots + H^{(l)}_A \right)$$

$$R_2 \leq \frac{1}{7} \left( H^{(1)}_B + \cdots + H^{(l)}_B \right)$$

Denote $H = \frac{1}{l} \sum_{k=1}^{l} H^{(k)}$, $H_A = \frac{1}{l} \sum_{k=1}^{l} H^{(k)}_A$, $H_B = \frac{1}{l} \sum_{k=1}^{l} H^{(k)}_B$. Because $(H^{(k)} - H^{(k)}_B, H^{(k)}_B), (H^{(k)}_A, H^{(k)} - H^{(k)}_A) \in \mathcal{E}$, $k = 1, 2, \ldots, l$. According to the convexity of $\mathcal{E}$, we know $(H - H_B, H_B)$ and $(H_A, H - H_A)$ is also in $\mathcal{E}$. According to Eq.\((3)\), all $(R_1, R_2)$ satisfying Eq.\((12)\) form a rectangle with an angle cut off, of which $(H - H_B, H_B)$ and $(H_A, H - H_A)$ are the two outmost vertices. It follows that it must be $(R_1, R_2) \in \mathcal{E}$. Thus complete the proof of the converse of the theorem.

6 Some interpretations and applications of the theorem

The above theorem provides some intriguing quantum communication schemes, which can be viewed as a generalized superdense coding scheme. The superdense coding scheme proposed by Bennet and Wiesner\[10\] dealt with two-partite communication, but here we will deal with three-partite communication. Suppose Alice and Bob want to send classical information to Charlie by $N$-state quantum systems. If Alice and Bob send message independently, they can send $\log_2 N$ bits per system. But we suppose Alice and Bob initially share a considerable supply of $N$-state entanglement, how can they expand
their channel capacity? Note that Alice and Bob may be at two distant locations, so each must encode his/her messages independently of the other by a predetermined code.

Suppose the initial state Alice and Bob shared is $|\psi\rangle = \sum_{i=1}^{N} p_i |i\rangle_{Alice} |i\rangle_{Bob}$, $\rho_0 = |\psi\rangle \langle \psi|$, then Alice and Bob’s part of the density matrix is $\rho_A = tr_B (\rho_0)$, $\rho_B = tr_A (\rho_0)$. Denote $H_E = S (\rho^A) = S (\rho^B)$, which can be used to measure the entanglement between Alice and Bob’s systems. Alice and Bob can perform a unitary transformation on his/her systems, then they convey the systems to Charlie respectively.

Denote $\{T_A\}, \{T_B\}$ as Alice and Bob’s transformations. They correspond to the $\{\alpha\}, \{\beta\}$ discussed before. Then $\rho = \sum_{T_A T_B} p_{T_A} q_{T_B} (T_A T_B \rho_0 T_A^+ T_B^+)$ is in $N^2$ dimensional space, so $H (\rho) \leq 2 \log_2 N$. Next we examine the conditional entropy. We see $\rho_{T_A} = \sum_{T_B} q_{T_B} (T_A T_B \rho_0 T_A^+ T_B^+)$, because $T_B$ is Bob’s part of transformation, we have $tr_B (\rho_{T_A}) = T_A \rho_A T_A^+$, and $H (T_A \rho_A T_A^+) = H (\rho_A) = H_E$. According to subadditivity of Von Neumann entropy, we have $H (\rho_{T_A}) \leq H_E + H (tr_A (\rho_{T_A})) \leq H_E + \log_2 N$. We can similarly have $H (\rho_{TB}) \leq H_E + \log_2 N$.

So if Alice and Bob can send information at a rate $(R_1, R_2)$, then according to our theorem, it must be

$$R_1 + R_2 \leq 2 \log_2 N$$

$$R_1 \leq \log_2 N + H_E$$

$$R_2 \leq \log_2 N + H_E$$

(13)

Note that all $(R_1, R_2)$ satisfying Eq.(13) can be achieved with Alice and Bob’s ensembles of transformation including all permutations of Schmidt basis states of the initial state $|\psi\rangle$, rotations of the relative phases of these states, and the combination of the two cases (all with equal probability).

Although the total amount of information does not increase in the scheme, it is useful. Because the amount of information Alice and Bob want to send to Charlie may be different, if Alice has more information than Bob to send, we can adopt a code that increases Alice’s channel capacity at the sacrifice of Bob’s. This then allows us to distribute the channel capacity between two users properly, without the waste of entanglement. From Eq.(13) we see Alice can send information at the maximum rate of $(\log_2 N + H_E)$ bits
per system. In this case, our scheme reduced to the two-partite superdense coding between *Alice* and *Charlie*, while *Bob* can still send information to *Charlie* at the rate of \( \log_2 N - H_E \). This scheme can also be generalized to the case that *Alice*, *Bob* and *Charlie* share three-partite entanglement. In this case they can further expand their channel capacity.

## 7 Acknowledgement

We thank the referee and A.Winter for pointing out to us references [12] [13] [14].

**References**

[1] J.P. Gordon, in Quantum Electronics and Coherent Light, Proceeding of the International School of Physics "Enrico Fermi", Course XXXI, edited by P.A. Miles, (Academic, New York, 1964), pp. 165-181

[2] L.B. Levitin, Information, Complexity and Control in Quantum Physics, edited by A. Blaquiere, S. Diner and G. Lochak (Springer, Vienna, 1987), pp. 111-115

[3] A.S. Holevo, Probl. Peredachi Inf. 9, 177 (1973)

[4] C.E. Shannon, Bell Tech. J. 27, 379 (1948); 27, 623 (1948)

[5] T.M. Cover and J.A. Thomas, Elements of Information Theory (Wiley, New York, 1991)

[6] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland and W.K. Wooters, Phys.Rev.A 54, 1869 (1996)

[7] B. Schumacher and M. Westmoreland, Phys.Rev.A 56, 131 (1997)

[8] Holevo, quant-ph/9611023

[9] A. Wehrl, Rev.Mod.Phys. 50, 221 (1978)

[10] C.H. Bennett and S.J. Wiesner, Phys.Rev.Lett. 69, 2881 (1992)
[11] Please refer to a recent review: John Preskill’s Lecture Notes for Physics 229: Quantum Information and Computation, available from the WWW site of CIT (http://www.caltech.edu).

[12] A.E.Allahverdyan and D.B.Saakian, quant-ph/9712034

[13] A.Winter, quant-ph/9807019

[14] A.Winter, quant-ph/9907077