On the Vertex Operator Representation of Lie Algebras of Matrices

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Abstract

The polynomial ring $B_r := \mathbb{Q}[e_1, \ldots, e_r]$ in $r$ indeterminates is a representation of the Lie algebra of all the endomorphism of $\mathbb{Q}[X]$ vanishing at powers $X^j$ for all but finitely many $j$. We determine a $B_r$-valued formal power series in $r + 2$ indeterminates which encode the images of all the basis elements of $B_r$ under the action of the generating function of elementary endomorphisms of $\mathbb{Q}[X]$, which we call the structural series of the representation. The obtained expression implies (and improves) a formula by Gatto & Salehyan, which only computes, for one chosen basis element, the generating function of its images. For sake of completeness we construct in the last section the $B = B_\infty$-valued structural formal power series which consists in the evaluation of the vertex operator describing the bosonic representation of $\mathfrak{gl}_\infty(\mathbb{Q})$ against the generating function of the standard Schur basis of $B$. This provide an alternative description of the bosonic representation of $\mathfrak{gl}_\infty$ due to Date, Jimbo, Kashiwara and Miwa which does not involve explicitly exponential of differential operators.

1 Introduction

1.1 This paper is concerned with the following general and rather elementary fact. There is a natural way to multiply polynomials by matrices of infinite size (with all zero entries but finitely many) which is compatible with the matrix commutator, i.e. $M(Np) - N(Mp) = [M, N]p$ for all matrix pair $(M, N)$ and for all polynomials $p$. This observation is basically due to Date, Jimbo, Kashiwara and Miwa (DJKM) [6] who, more than that, determine a vertex operator representation of the generating function of the elementary matrices acting on the ring $B$ (the bosonic Fock space) of polynomials in infinitely many indeterminates (see also [17, Section 5.1] for an elementary account).

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Our main contribution consists in the study of the \textit{finite type} version of the DJKM description of the bosonic representation of matrices, improving the output of the approach taken in [12]. To be more precise we need to introduce two main actors.

The former is the vector space \( V := \mathbb{Q}[X] \) of polynomials in the indeterminate \( X \), with basis \( (X^i)_{i \geq 0} \), to which one attaches the Lie algebra \( \mathfrak{gl}(V) := \bigoplus_{i,j \geq 0} \mathbb{Q} \cdot X^i \otimes \partial^j \), where \( \partial^j \) denotes the unique linear form such that \( \partial^j(X^i) = \delta^{ij} \).

The latter is the polynomial ring \( B_r := \mathbb{Q}[e_1, \ldots, e_r] \), in the \( r \) indeterminates \( (e_1, \ldots, e_r) \). Denote by \( H_r \) the sequence \( (h_n)_{n \in \mathbb{Z}} \) in \( B_r \), where \( h_n := \det(e_{i-j+1})_{1 \leq i,j \leq n} \), setting by convention, \( e_0 = 1 \) and \( e_j = 0 \) for \( j < 0 \). Let \( \mathcal{P}_r \) be the set of all partitions of length at most \( r \). It turns out that the set of all \( \Delta_\lambda(H_r) = \det(h_{\lambda_{j-i+1}}) \) provide a \( \mathbb{Q} \)-basis of \( B_r \) parametrized by \( \mathcal{P}_r \), i.e. \( B_r := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot \Delta_\lambda(H_r) \). The main point is that \( B_r \) can be made into an irreducible representation of \( \mathfrak{gl}(V) \) (in particular: polynomials in \( B_r \) can be multiplied by matrices compatibly with the Lie algebra structure of \( \mathfrak{gl}(V) \)), by pulling back the natural one on \( \bigwedge^r V \) through the vector space isomorphism \( B_r \to \bigwedge^r V \) as in [18, Main Theorem], the \textit{finite type boson-fermion correspondence}.

Our Theorem 4.11 determines an expression for the formal power series \( E_r(z, w^{-1}, t_r) \in B_r[z, w^{-1}, t_r] \) resulting from the evaluation of the generating function
\[
E(z, w^{-1}) = \sum_{i,j \geq 0} X^i \otimes \partial^j \cdot z^i w^{-j}
\]
of the elementary endomorphisms \( X^i \otimes \partial^j \in \text{End}_\mathbb{Q}(V) \) against a suitable generating function of the basis \( (\Delta_\lambda(H_r))_{\lambda \in \mathcal{P}_r} \) of \( B_r \).

For sake of a dutiful comparison, we apply the same procedure to the genuine DJKM-representation of \( \mathfrak{gl}_\infty(\mathbb{Q}) \) in Section 5, by merely evaluating the bosonic vertex operator against the generating function of the natural Schur basis of \( B_r \), which can be determined via Cauchy-type formulas. One so obtains a description of the \( \mathfrak{gl}_\infty(\mathbb{Q}) \)-module structure of \( B = B_\infty \) (Theorem 5.11) which is equivalent to the DJKM one, but with no explicit occurrence of exponentials of differential operators. The rest of the introduction will be devoted to state more precisely our results and to say a few words about the history of the subject and motivations.

1.2 Precise Statement of the Main Results. The most natural candidate to be a generating series for the basis \( (\Delta_\lambda(H_r)) \) is
\[
\Delta(H_r; t_r) := \sum_{\lambda \in \mathcal{P}_r} \Delta_\lambda(H_r) s_\lambda(t_r)
\]
where \( t_r := (t_1, \ldots, t_r) \) is an \( r \)-tuple of indeterminates and \( s_\lambda(t_r) \) are the usual Schur symmetric polynomials as defined, e.g., in [8, Formula 3.1]. The finite type version of the \textit{boson fermion correspondence} says that the \( r \)-th exterior power \( \bigwedge^r V \) is a free \( B_r \)-module of rank \( 1 \) generated by \( X^r(0) := X^{r-1} \wedge \cdots \wedge X^0 \) such that
\[
X^r(\lambda) := X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r} = \Delta_\lambda(H_r) \cdot X^r(0).
\]
One then defines a \( \ast \)-product, to make \( B_r \) into a \( \mathfrak{gl}(V) \)-module, as follows:
\[
[(X^i \otimes \partial^j) \ast \Delta_\lambda(H_r)]X^r(0) := X^i \wedge (\partial^j \wedge X^r(\lambda))
\]
where $\partial^{j}: \bigwedge V \to \bigwedge V$ is the natural contraction operator. Define

$$E_r(w) = 1 - e_1 w + \cdots + (-1)^r e_r w^r \in B_r[w] \quad (1)$$

and

$$E_r\left( t_r; \frac{1}{w} \right) = 1 - e_1(t_r)w^{-1} + \cdots + (-1)^r e_r(t_r)w^{-r} = \prod_{j=1}^{r} (1 - t_j w^{-1}) \in \mathbb{Q}[t_r, w^{-1}] \quad (2)$$

in such a way that $e_j(t_r)$ is the elementary symmetric polynomial of degree $j$ in the variable $(t_1, \ldots, t_r)$. The polynomials $e_j(t_r)$ should not be confused with the indeterminates $e_j$ of the ring $B_r$, which could be rather thought of as the elementary symmetric polynomials of the universal roots of the generic monic polynomial $X^r - e_1X^{r-1} + \cdots + (-1)^r e_r$. Our main result consists in computing the action of the generating function $E(z, w^{-1})$ against the generating function $\Delta(H_r; t_r)$ of the basis of $B_r$.

**Theorem A** (Theorem 4.11). Let

$$E_r(z, w^{-1}, t_r) := \sum_{\{(i, j), \lambda\} \in \mathbb{N}^2 \times \mathcal{P}} \left[ X^i \otimes \partial^j \ast \Delta_{\lambda}(H_r) \right] z^i w^{-j} s_\lambda(t_r)$$

be the generating function of the structure constant of $B_r$ as a representation of $\mathfrak{gl}(V)$. Then:

$$E_r(z, w, t_r) =$$

$$\frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n(t_r) \left( \frac{1}{w^n} - \frac{1}{z^n} \right) + x_n p_n(z, t_r) \right) \left( E_r(w) + (-1)^{r+1} e_r w^r E_r(t_r, \frac{1}{w}) \right).$$

The proof of Theorem A is based on the formalism of Schubert derivations, in the same vein of [2, 3, 4].

It is then natural, beside being dutiful, to see how the same procedure can be applied to rephrase the DJKM-description of the $\mathfrak{gl}_\infty(\mathbb{Q})$-structure of $B = B_\infty$. In this case we identify $B$ with the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots]$, where the variables $x$ are related to the $e_i$ through the equality

$$\exp(\sum_{i \geq 1} x_i z^i) = \frac{1}{E_\infty(z)} = (1 - e_1 z + e_2 z^2 + \cdots)^{-1} \in B[z].$$

To be more adherent with the standard notation in the case of $B_\infty$, the Schur basis will be denoted by $S_\lambda(x) = \det(S_{\lambda_{\vdash j+i}}(x))$ where $S_\kappa(x) = \det(e_{i-j+1})_{1 \leq i,j \leq k}$. The generating function of the basis elements of $B$ is then $\sum_{\lambda \in \mathcal{P}} S_\lambda(x)s_\lambda(t)$, where $t := t_\infty$. Then our second main result is:

**Theorem B** (Theorem 5.11). Let

$$E(z, w^{-1}, t_r) = \sum_{\{(i, j), \lambda\} \in \mathbb{Z} \times \mathbb{Z} \times \mathcal{P}} \left( X^i \otimes \partial^j \ast S_\lambda(x) \right) z^i w^{-j} s_\lambda(t) \in B[z, w, t_r, z^{-1}, w^{-1}]$$
where \( t := (t_1, t_2, \ldots) \) is a sequence of infinitely many indeterminates. Then

\[
E(z, w^{-1}, t_r) = \exp \left( \sum_{n \geq 1} \frac{1}{n} \left( \frac{w^n}{z^n} - \frac{p_n(t_r)}{z^n} + \frac{p_n(t_r)}{w^n} \right) + x_n (z^n - w^n + p_n(t_r)) \right) \tag{3}
\]

i.e. the image of \( S_{\lambda}(x) \in B \) through the multiplication by the elementary endomorphism \( X^i \otimes \partial^j \) is the coefficient of \( z^i w^{-j} \) of (3).

1.3 History and motivations. The bosonic vertex operator representation of the Lie algebra \( \mathfrak{gl}_\infty(\mathbb{C}) \) was determined by Date, Jimbo, Kashiwara and Miwa within the framework of algebraic analysis and mathematical physics related to the KP hierarchy, see e.g. [16]. The KP hierarchy is a system of infinitely many nonlinear PDEs whose polynomial solutions are parametrized by the points of the orbit of \( 1 \in B \otimes_\mathbb{Q} \mathbb{C} \) through a natural action of the group

\[ GL_\infty(\mathbb{C}) := \{ \text{invertible } A \in \text{End}_\mathbb{C}(\mathbb{C}[X^{-1}, X]) \mid AX^i = X^i \text{ for all but finitely many } j \} \]

The \( GL_\infty(\mathbb{C}) \) orbit of 1 does correspond to the locus of the decomposable tensors (the Sato Universal Grassmann Manifold, see [19] and also [17, p. 73]) in infinite wedge power, roughly speaking \( \bigwedge^\infty \mathbb{C}^\infty \), the Fermionic Fock space of charge 0, to which B is isomorphic via the so called boson-fermion correspondence. It turns out that the DJKM representation is the linearization of this natural action of \( GL_\infty(\mathbb{C}) \) on B.

In [12, 13] is recognized that the DJKM description is a natural consequence of the well known basic linear algebraic fact, namely that each vector space is a representation of its Lie algebra of endomorphisms. More generally, it turns out that the DJKM representation is a particular (extremal) case of a more general picture which in [12] was summarized by the slogan “the cohomology of the Grassmannian is a \( \mathfrak{gl}_n \)-module”, which entitles the paper.

This can be quickly explained as follows. Each \( \mathbb{Q} \)-vector space \( V \) of finite dimension \( n \) is isomorphic to \( \mathbb{Q}[X]/(X^n) = H^*(\mathbb{P}^{n-1}, \mathbb{Q}) \) and one already sees that the singular cohomology of the projective space \( \mathbb{P}^{n-1} \) is a module over the Lie algebra of \( \mathbb{Q} \)-valued square \( n \times n \) matrices. The trivial claim that \( \mathbb{Q}[X] \) is a \( \mathfrak{gl}(\mathbb{Q}[X]) \)-module generalizes to the fact that \( B_r \) is a \( \mathfrak{gl}(\mathbb{Q}[X]) \)-module because \( \bigwedge^r \mathbb{Q}[X] \) is naturally a representation of \( \mathfrak{gl}(\mathbb{Q}[X]) \) and because of the isomorphism \( B_r \to \bigwedge^r \mathbb{Q}[X] \). The composition of maps \( B_r \to \bigwedge^r \mathbb{Q}[X] \to \bigwedge^r \mathbb{Q}[X]/(X^n) \) factorizes through a ring \( B_{r,n} \) which turns out to be the the singular cohomology ring of the complex Grassmannian \( G(r, n) \). The case \( n = \infty \) corresponds to the situation coped with in our Theorem 4.11.

One main point is that we do not know any direct way to infer our Theorem 4.11 from the DJKM expression, also because in the finite type case (i.e. in the \( B_r \)-representation rather than the \( B_\infty \)-one) many technical issues arise, at the point that in [12], unlike in the DJKM case, the authors are not able to provide a formula for the generating function of the elementary endomorphisms tout-court, but only the generating function of all the images of a specified basis element through the elementary endomorphisms. This causes an unpleasant dependence of their formula from the partition parametrizing the basis element whose image is computed. Our idea is to remove the explicit dependence from the partition,
by evaluating the image of the generating function of the Schur basis of $B_r$. Again, we would have not able to achieve the goal without heavily using the formalism of Schubert derivations as in [3, 4, 12, 13].

We should finally remark that the way Schubert derivations remove the necessity to deal with partial derivatives, carries a big potential to extend our main results to tropical situations as indicated e.g. in [15]. As a matter of fact, in [15] a Grassmann semi-algebra is constructed within the framework of systems and the constructions lends itself to extend the Schubert derivations. In addition, in the last section of the recent preprint [5], a semi-algebra version of our Theorem 3.2 is also provided, essentially due to the fact that the techniques exposed in [12] work in that more constrained situation. It seems likely that a sharpening of [5, Theorem 7.24] can be naturally achieved using Cauchy type formulas for polynomial semi-algebras, a task which we temporarily postpone to further investigations.

1.4 Structure of the paper. We collect in Section 2 the minimal background to follow the proofs of the main result, all based on the manipulation with Schubert derivations. Section 3 is just a reformulation of the main formula in [12]. Section 4 contains the proof of the main theorem as well as the proof of many technical lemmas which on one hand are interesting in their own and, on the other hand, can be read within the classical theory of symmetric functions as in the classical reference [8]. Section 5 is finally devoted to rephrase the DJKM representation with the purpose to present it within a unified perspective together with Theorem 4.11.

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2 Preliminaries and Notation

2.1 Let $X$ be an indeterminate over $\mathbb{Q}$. We denote by $V := \mathbb{Q}[X^{-1}, X]$ the vector space of Laurent polynomials, with basis $(X^i)_{i \in \mathbb{Z}}$ and for all $j \in \mathbb{Z}$ we write $\partial^j$ for the unique linear form on $V$ such that $\partial^j(X^i) = \delta^i_j$. Let $V := \mathbb{Q}[X]$ be the vector space of polynomials. It is a vector subspace of $V$. The vector spaces $V^* := \bigoplus_{j \in \mathbb{Z}} \mathbb{Q} \partial^j$ and $V^* := \bigoplus_{j \geq 0} \mathbb{Q} \partial^j$ are the restricted duals of $V$ and $V$ respectively. Let $gl(V)$ and $gl(V)$ be respectively $V \otimes V^*$ and $V \otimes V^*$.

We denote by $X(z)$ and $\partial(w^{-1})$ the generating series of the basis elements of $V$ and of $V^*$ respectively, i.e.:

$$X(z) := \sum_{i \geq 0} X^i z^i \quad \text{and} \quad \partial(w^{-1}) := \sum_{j \geq 0} \partial^j w^{-j}.$$

2.2 Partitions and exterior algebras. We denote by $P$ the set of all partitions. This is the set of all non-increasing sequences $\lambda := (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)$ of integers whose terms
are all zero but finitely many. The terms of $\lambda$ are its parts. The length $\ell(\lambda)$ is the number of non zero parts and $P_r$ stands for the set of all partitions with length at most $r$. Let $\wedge V = \bigoplus_{i \geq 0} \wedge^i V$ be the exterior algebra of $V$. It is a graded algebra where $\wedge^0 V = \mathbb{Q}$ and, for all $r \geq 1$:

$$
\wedge^r V := \bigoplus_{\lambda \in P_r} \mathbb{Q} \cdot X^r(\lambda),
$$

where we have used the notation:

$$
X^r(\lambda) := X^{r-1+\lambda_1} \wedge X^{r-2+\lambda_2} \wedge \ldots \wedge X^{\lambda_r}.
$$

In particular $\wedge^1 V = \bigoplus \mathbb{Q} \cdot X(\lambda) = \bigoplus \mathbb{Q} \cdot X^1 = V$.

2.3 The ring $B_r$. For $r \geq 1$, let $B_r := \mathbb{Q}[e_1, \ldots, e_r]$ be the polynomial ring in the $r$ indeterminates $(e_1, \ldots, e_r)$. By convention one sets $B_0 = \mathbb{Q}$. Consider the generic polynomial $E_r(z) := 1 - e_1 z + \cdots + (-1)^r e_r z^r \in B_r[z]$, and the sequences $H_r := (h_j)_{j \in \mathbb{Z}}$ and $x_r = (x_i)_{i \in \mathbb{Z}}$ defined through the equality:

$$
\sum_{n \in \mathbb{Z}} h_n z^n := \frac{1}{E_r(z)} = \exp \left( \sum_{i \geq 1} x_i z^i \right),
$$

holding in $B_r[z]$. In particular $h_j = 0$ if $j < 0$ and $h_0 = 1$. Moreover for $j \geq 0$, $h_j$ is an explicit isobaric polynomial of degree $j$ in $(e_1, \ldots, e_r)$, once one gives the weight of $e_j$ to be $j$.

2.4 It is well known that the Schur determinants

$$
\Delta_\lambda(H_r) := \det(h_{\lambda_j-j+i})_{1 \leq i, j \leq r} =
\begin{vmatrix}
  h_{\lambda_1} & h_{\lambda_2-1} & \cdots & h_{\lambda_r-r+1} \\
  h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r-r+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r}
\end{vmatrix},
$$

form a $\mathbb{Q}$-basis of $B_r$ parametrized by the partitions of length at most $r$, i.e.:

$$
B_r := \bigoplus_{\lambda \in P_r} \mathbb{Q} \cdot \Delta_\lambda(H_r).
$$

Thus the linear extension of the sets map

$$
\Delta_\lambda(H_r) \mapsto X^r(\lambda).
$$

gives a natural $\mathbb{Q}$-vector space isomorphism $B_r \to \bigwedge^r V$.

If $r = \infty$, one sets

$$
B = B_\infty = \mathbb{Q}[x_1, x_2, \ldots]
$$

and denotes by $S_\lambda(x) := \det(S_{\lambda_j-j+i}(x))$ the $\mathbb{Q}$-basis element of $B$ corresponding to the partition $\lambda$, where the sequence $(S_1(x), S_2(x), \ldots)$ is obtained through the equation

$$
\sum_{j \in \mathbb{Z}} S_j(x) z^j = \exp \left( \sum_{i \geq 1} x_i z^i \right).
$$
2.5 Schur Polynomials. Let \( z_k := (z_1, \ldots, z_k) \) be an ordered finite sequence of formal variables and consider the \( \wedge^k V \)-valued formal power series

\[
X(z_k) \wedge \cdots \wedge X(z_1).
\]

It vanishes along all the diagonals \( z_i - z_j = 0 \) (\( i \neq j \)). Therefore it is divisible by the Vandermonde determinant \( \Delta_0(z_k) = \prod_{1 \leq i < j \leq k} (z_j - z_i) \). The equality

\[
\sum_{\mu \in \mathcal{P}_k} X^k(\mu) s_\lambda(z_k) \Delta_0(z_k) = X(z_k) \wedge \cdots \wedge X(z_1),
\]

define the Schur symmetric polynomial \( s_\lambda(x_k) \). This definition coincides with the usual one as in [8, formula (3.1)].

2.6 The \( \text{End}_Q(V) \)-module structure of \( \wedge V \). Given \( \phi \in \text{End}_Q(V) \) let us denote by \( \delta(\phi) \) the unique derivation of \( \wedge V \) such that \( \delta(\phi)|_V = \phi \) (see [2, Section 3.1]). In other words,

\[
\delta(\phi)(u \wedge v) = \delta(\phi)u \wedge v + u \wedge \delta(\phi)v \quad \forall u, v \in \wedge V
\]

and \( \delta(\phi)w = \phi(w) \) for all \( w \in V = \wedge^1 V \).

2.7 Proposition The plethystic exponential of \( \delta(\phi) \in \text{End}_Q(\wedge V) \)

\[
D^\phi(z) := \exp(\delta(\phi)z) := \exp\left( \sum_{i \geq 1} \frac{1}{i} \delta(\phi)^i z^i \right) : \wedge V \to \wedge V[z]
\]

is the unique Hasse-Schmidt derivation on \( \wedge V \) such that

\[
D^\phi(z)|_V = \sum_{i \geq 0} \phi^i z^i \in \text{End}_Q(V).
\]

See [1] for the proof. Recall by e.g. [9] or [10] that to say \( D^\phi(z) \) is a Hasse-Schmidt (HS) derivation means that

\[
D^\phi(z)(u \wedge v) = D^\phi(z)u \wedge D^\phi(z)v \quad \forall u, v \in \wedge V
\]

In particular, putting \( \overline{D}^\phi(z) = \exp(-\delta(\phi)z) \), which is easily seen to be a HS derivation as well, the integration by parts formula holds:

\[
\overline{D}^\phi(z)u \wedge v = D^\phi(z)(u \wedge \overline{D}^\phi(z)v) \tag{4}
\]

2.8 By abuse of notation we denote by \( X \) and \( X^{-1} \) the \( \mathbb{Q} \)-linear maps \( V \to V \) given by multiplication by \( X \) and \( X^{-1} \) respectively, where for all \( i > 0 \)

\[
X^{-i}X^i = \begin{cases} X^{i-i} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}
\]
Consider

\[
\sigma_+(z) = \sum_{i \geq 0} \sigma_i z^i := \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(X^i) z^i \right) : \wedge V \to \wedge V[z], \quad (5)
\]

\[
\sigma_-(z) = \sum_{i \geq 0} \sigma_{-i} z^{-i} := \exp \left( - \sum_{i \geq 1} \frac{1}{i} \delta(X^{-i}) z^{-i} \right) : \wedge V \to \wedge V[z^{-1}], \quad (6)
\]

and their inverses as elements of \( \text{End}_Q(\wedge V)[z^{\pm 1}] \)

\[
\overline{\sigma}_+(z) = \sum_{i \geq 0} (-1)^i \overline{\sigma}_i z^i := \exp \left( - \sum_{i \geq 1} \frac{1}{i} \delta(X^i) z^i \right) : \wedge V \to \wedge V[z], \quad (7)
\]

\[
\overline{\sigma}_-(z) = \sum_{i \geq 0} (-1)^i \overline{\sigma}_{-i} z^{-i} := \exp \left( - \sum_{i \geq 1} \frac{1}{i} \delta(X^{-i}) z^{-i} \right) : \wedge V \to \wedge V[z^{-1}]. \quad (8)
\]

2.9 Proposition ([11, Proposition 2.2]). The maps \( \sigma_{\pm}(z) \) and \( \overline{\sigma}_{\pm}(z) \) are the unique (HS) derivations on the exterior algebra \( \wedge V \) such that

\[
\sigma_+(z)X^i = \sum_{i \geq 0} X^{i+i} z^i, \quad \overline{\sigma}_+(z)X^i = X^i - X^{i+1} z , \quad (9)
\]

and

\[
\sigma_-(z)X^i = \sum_{i \geq 0} X^{i-i} z^{-i}, \quad \overline{\sigma}_-(z)X^i = X^i - X^{i-1} z^{-1}, \quad (10)
\]

putting \( X^i = 0 \) for \( i < 0 \). They are called Schubert derivations.

2.10 Transposition. The transpose \( \sigma_{\pm}(z)^T : \wedge V^* \to \wedge V^*[z^{\pm 1}] \) of the Schubert derivation \( \sigma_{\pm}(z) \) is defined via its action on homogeneous elements. If \( \eta \in \wedge V^* \), then one stipulates that \( \sigma_{\pm}(z)^T \eta(u) = \eta(\sigma_{\pm}(z) u) \), for all \( u \in \wedge V \). By [11, Proposition 2.8] \( \sigma_{\pm}(z)^T \) is a HS–derivation of \( \wedge V^* \). In the sequel we will need the fact that

\[
\sigma_-(z)^T X^i = \sum_{i \geq 0} X^{i+i} z^{-i}.
\]

2.11 The \( B_r \)-module structure of \( \wedge^r V \). For all \( u \in \wedge^r V \), define

\[
e_i u = \overline{\sigma}_i u \quad \text{or, equivalently,} \quad h_i u = \sigma_i u. \quad (11)
\]

In particular:

\[
\overline{\sigma}_+(z) u = E_r(z) \cdot u \quad \text{and} \quad \sigma_+(z) u := \frac{1}{E_r(z)} u, \quad \forall u \in \wedge^r V.
\]

2.12 Proposition . Equations (11) make \( \wedge^r V \) into a free \( B_r \)-module generated by \( X^r(0) := X^{r-1} \wedge \cdots \wedge X^0 \).
Proof. It is a consequence of the fact, explained in [9], see also [2, Proposition 3.5], that Giambelli’s formula for the Schubert derivation \( \sigma_+(z) \) holds:

\[
X^r(\lambda) = \Delta_\lambda(\sigma_+(z)) \cdot X^r(0) := (\det(\sigma_{\lambda_{i-j+i}})_{1 \leq i,j \leq r}) \cdot X^r(0).
\]  

(12)

Therefore

\[
h_i \cdot X^r(\lambda) = \sigma_i X^r(\lambda) = \sum_{\mu \in \mathcal{P}_r} X^r(\mu) = \sum_{\mu \in \mathcal{P}_r} \Delta_\mu(H_r) X^r(0) = (h_i \Delta_\lambda(H_r)) X^r(0)
\]

Hence \( \bigwedge^r V \) is a free \( B_r \)-module of rank 1 generated by \( X^r(0) \).

The fact that \( \bigwedge^r V \) is a free \( B_r \)-module of rank 1 generated by \( X^r(0) \), as prescribed by equality (12), shows that the Schubert derivations \( \sigma_-(z), \overline{\sigma}_-(z) \) induce maps \( B_r \to B_r[z^{-1}] \), that, abusing notation, will be denoted in the same way. Their action on a basis element \( \Delta_\lambda(H_r) \) of \( B_r \) is defined through its action on \( \bigwedge^r V \):

\[
(\overline{\sigma}_-(z) \Delta_\lambda(H_r)) X^r(0) := \overline{\sigma}_-(z) X^r(\lambda),
\]  

(13)

\[
(\sigma_-(z) \Delta_\lambda(H_r)) X^r(0) := \sigma_-(z) X^r(\lambda).
\]  

(14)

Denote by \( \overline{\sigma}_-(z)H_r \) (respectively \( \sigma_-(z)H_r \)) the sequence \( (\overline{\sigma}_-(z)h_j)_{j \in \mathbb{Z}} \) (respectively \( (\sigma_-(z)h_j)_{j \in \mathbb{Z}} \)). By using [11, Theorem 5.7], and exploiting the Laksov & Thorup determinantal formula as in [18, Main Theorem 0.1], one obtains the following statement, which gives a practical way to evaluate the image of \( \Delta_\lambda(H_r) \) through the maps \( \overline{\sigma}_-(z) \) and \( \sigma_-(z) \) defined by (14) and (13).

2.13 Proposition ([11, Proposition 5.3]). For all \( r \geq 0 \) and all \( \lambda \in \mathcal{P}_r \)

\[
\sigma_-(z)h_j = \sum_{i \geq 0} h_{j-i} z^{-i} \quad \text{and} \quad \overline{\sigma}_-(z)h_j = h_j - h_{j-1} z^{-1}.
\]

Moreover,

\[
\sigma_-(z) \Delta_\lambda(H_r) = \Delta_\lambda(\sigma_-(z)H_r) \quad \text{and} \quad \overline{\sigma}_-(z) \Delta_\lambda(H_r) = \Delta_\lambda(\overline{\sigma}_-(z)H_r).
\]

3 The \( \mathfrak{gl}(V) \)-structure of \( B_r \) revisited 1

In this section we revisit [12, Theorems 5.7 and 6.4] and [13, Section 9], getting a more explicitly and elegant expression to realize the \( \mathfrak{gl}(V) := V \otimes V^* \)-module structure of the ring \( B_r \).

Let \( \beta \in V^* \). In the paper [2, Section 5.1 and Lemma 5.3] one learns to phrase the usual contraction endomorphism

\[
\beta_J : \bigwedge V \to \bigwedge V
\]
via the following diagram

\[
\begin{pmatrix}
\beta(X^{r-1+\lambda_1}) & \beta(X^{r-2+\lambda_2}) & \cdots & \beta(X^{\lambda_r}) \\
X^{r-1+\lambda_1} & X^{r-2+\lambda_2} & \cdots & X^{\lambda_r}
\end{pmatrix},
\]

which means that the scalar \((-1)^{i+1}\beta(X^{r-i+\lambda_i})\) is the coefficient of the element of \(\wedge^{r-1} V\) obtained by removing the \(i\)-th exterior factor of the wedge product of the elements in the second row, namely \(X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r} = X^r(\lambda)\).

For example, it follows by the very definition \((15)\) that

\[
\partial^0_\lambda X^r(0) = (-1)^{r-1} X^{r-1}(1^{r-1}).
\]

3.1 The Lie algebra \(\mathfrak{gl}(V)\) acts on the exterior algebra via the map \(\delta\) as in Section 2.6.

In particular

\[
\delta(X^i \otimes \partial^j)(u) = X^i \wedge \partial^j u \quad \forall u \in \wedge V.
\]

Using the definition of the Schubert derivation \(\sigma_+ (z)\) as in \((9)\), the generating function \(E(z, w^{-1})\) of the basis \((X^i \otimes \partial^j)_{i,j \geq 0}\) of \(\mathfrak{gl}(V)\) can be written as

\[
E(z, w^{-1}) = X(z) \otimes \partial(w^{-1}) = \sigma_+(z)X^0 \otimes \partial(w^{-1})
\]

and acts on \(\wedge V\) as

\[
E(z, w^{-1})(X^r(\lambda)) = \sigma_+(z)X^0 \wedge (\partial(w^{-1})_\lambda X^r(\lambda)).
\]

The following rephrases in a more elegant and transparent way the description of the \(\mathfrak{gl}(V)\) structure of \(\wedge^r V\) proposed in \([12, \text{Theorem 4.3}]\).

3.2 **Proposition.** The action of \(E(z, w^{-1})\) on the basis element \(X^r(\lambda)\) is given by:

\[
E(z, w^{-1})X^r(\lambda) = \frac{z^{-r}}{w^{r-1}} \sigma_+(z) \overline{\sigma}_-(z) \begin{pmatrix}
w^{-\lambda_1} & w^{-\lambda_2+1} & \cdots & w^{r-1-\lambda_r} & 0 \\
X^{r-1+\lambda_1} & X^{r-1+\lambda_2} & \cdots & X^{r+\lambda_r} & X^0
\end{pmatrix}.
\]

**Proof.** By applying equation \((16)\) and diagram \((15)\), we obtain

\[
E(z, w^{-1})X^r(\lambda) = \sigma_+(z)X^0 \wedge (\partial(w^{-1})_\lambda X^r(\lambda))
\]

\[
= \sigma_+(z)X^0 \wedge \begin{pmatrix}
\partial(w^{-1})X^{r-1+\lambda_1} & \cdots & \partial(w^{-1})X^{\lambda_r} \\
X^{r-1+\lambda_1} & \cdots & X^{\lambda_r}
\end{pmatrix}
\]

\[
= \sigma_+(z)X^0 \wedge \begin{pmatrix}
w^{-r+1-\lambda_1} & \cdots & w^{-\lambda_r} \\
X^{r-1+\lambda_1} & \cdots & X^{\lambda_r}
\end{pmatrix}.
\]
Using the integration by parts (4) for the Schubert derivation $\sigma_+(z)$, one obtains:

$$
\mathcal{E}(z, w^{-1})X^r(\lambda) = \sigma_+(z) \left( X^0 \wedge \underline{\sigma}_+(z) \begin{vmatrix}
X^r \ldots X^0 \\
X^r \ldots X^0 \\
\underline{\sigma}_+(z)X^r \ldots \underline{\sigma}_+(z)X^0 \\
\wedge X^0
\end{vmatrix}
\right)
$$

$$
= \sigma_+(z) \left( (-1)^{r-1} \begin{vmatrix}
w^{-r+1-\lambda_1} \ldots w^{-\lambda_r} \\
w^{-r+1-\lambda_1} \ldots w^{-\lambda_r} \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\wedge X^0
\end{vmatrix}
\right)
$$

By (9) and (10) one has $\underline{\sigma}_+(z)X^j = -z\underline{\sigma}_-(z)X^{l+1}$. Hence (17) can be rewritten as

$$
\sigma_+(z) \left( \begin{vmatrix}
w^{-r+1-\lambda_1} \ldots w^{-\lambda_r} \\
w^{-r+1-\lambda_1} \ldots w^{-\lambda_r} \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\wedge X^0
\end{vmatrix}
\right)
$$

$$
= \sigma_+(z) \left( \begin{vmatrix}
w^{-\lambda_1} \ldots w^{-\lambda_2+1} \ldots w^{r-1-\lambda_r} \ldots 0 \\
w^{-\lambda_1} \ldots w^{-\lambda_2+1} \ldots w^{r-1-\lambda_r} \ldots 0 \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\wedge X^0
\end{vmatrix}
\right)
$$

which is the desired expression.

### 3.3 Example

Applying Proposition 3.2 to the particular case $r = 2$, and using the properties of Schubert derivations in equations (9) and (10), one gets

$$
\mathcal{E}(z, w^{-1})X^2(\lambda) = \frac{z}{w} \sigma_+(z)\underline{\sigma}_-(z) \left( \begin{vmatrix}
w^{-\lambda_1} \ldots w^{-\lambda_2+1} \ldots w^{r-1-\lambda_r} \ldots 0 \\
w^{-\lambda_1} \ldots w^{-\lambda_2+1} \ldots w^{r-1-\lambda_r} \ldots 0 \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\underline{\sigma}^{-}(z)X^{r+\lambda_1} \ldots \underline{\sigma}^{-}(z)X^{l+\lambda_r} \\
\wedge X^0
\end{vmatrix}
\right)
$$

$$
= \frac{z}{w} \sigma_+(z)\underline{\sigma}_-(z) \left( w^{-\lambda_1}(X^{1+\lambda_2} \wedge X^0) - w^{-\lambda_2+1}(X^{2+\lambda_1} \wedge X^0) \right)
$$

$$
= \frac{z}{w^{1+\lambda_2}}(\sigma_+(z)(X^{1+\lambda_2} - X^{\lambda_2}z^{-1}) \wedge \sigma_+(z)X^0)
$$

$$
- \frac{z}{w^{\lambda_2}}(\sigma_+(z)(X^{2+\lambda_1} - X^{1+\lambda_1}z^{-1}) \wedge \sigma_+(z)X^0)
$$

$$
= \frac{-1}{w^{1+\lambda_1}}(X^{\lambda_2} \wedge \sigma_+(z)X^0) + \frac{1}{w^{\lambda_2}}(X^{1+\lambda_1} \wedge \sigma_+(z)X^0)
$$

$$
= \frac{-1}{w^{1+\lambda_1}}(X^{\lambda_2} \wedge \sigma_+(z)X^0) + \frac{1}{w^{\lambda_2}}(X^{1+\lambda_1} \wedge \sigma_+(z)X^0)
$$

$$
= (\frac{X^{1+\lambda_1}}{w^{1+\lambda_1}} - \frac{X^{\lambda_2}}{w^{1+\lambda_1}}) \wedge \sigma_+(z)X^0
$$

$$
= \left( \frac{X^{1+\lambda_1}}{w^{\lambda_2}} - \frac{X^{\lambda_2}}{w^{1+\lambda_1}} \right) \wedge X(z).
$$
3.4 Remark. Besides the formula displayed in Proposition 3.2 being very manageable and explicit, there is still a dependence on $\lambda$ that is desirable to remove. The idea to do this is to sum all the above expression for $\lambda$ ranging over all the partitions. Better, we will apply the generating series $E(z,w)$ to a the generating function $\sum_{\lambda \in P} X^r(\lambda)s_\lambda(t_r)$. By [2, Lemma 8.2] (which is essentially an exponential way to write the Cauchy formula as in [7, Formula (3), p. 52]) that is

$$\exp\left(\sum_{i \geq 1} x_i p_i(t_r)\right) X^r(0),$$

where $p_i(t_r) := t_1^i + \cdots + t_r^i$.

4 The $gl(V)$ structure of $B_\tau$ revisited 2

Recall that the $B_\tau$-module structure of $\bigwedge^r V$, established in Proposition 2.12, maps $B_\tau$-valued formal power series to $\bigwedge^r V$-valued formal power series via the map $\phi \mapsto \phi \cdot X^r(0)$.

4.1 Definition. We denote by $E_r(z,w^{-1},t_r)$ the unique $B_\tau$-valued formal power series in the indeterminates $(z,w^{-1},t_r)$ such that

$$E_r(z,w^{-1},t_r)X^r(0) = \sum_{((i,j),\lambda) \in N^2 \times P_r} [X^i \otimes \partial^j \Delta_\lambda(H_r)] z^i w^{-j} s_\lambda(t_r) X^r(0)$$

$$= \sum_{\lambda \in P_r} X(z) \land (\partial(w^{-1}) \lambda X^r(\lambda)s_\lambda(t_r)).$$

Clearly $E(z,w^{-1},t_r) \in B_\tau[z,w^{-1},t_r]$ and the purpose of this section is to determine an exponential-like expression for

$$E_r(z,w^{-1},t_r) := \sum_{((i,j),\lambda) \in N^2 \times P_r} [X^i \otimes \partial^j \Delta_\lambda(H_r)] z^i w^{-j} s_\lambda(t_r) \in B_\tau[z,w^{-1},t_r].$$

To this purpose we first invoke [2, Lemma 8.2] according which

$$\sum_{\lambda \in P_r} X^r(\lambda)s_\lambda(t_r) = \sigma_+(t_r)X^r(0). \quad (18)$$

Recall that the symbols $\sigma_\pm(t_r)$ and $\overline{\sigma}_\pm(t_r)$ stand for multivariables Schubert derivations (see [2, Section 4.1]):

$$\sigma_\pm(t_r) := \sigma_\pm(t_1) \cdots \sigma_\pm(t_r) \quad \text{and} \quad \overline{\sigma}_\pm(t_r) := \overline{\sigma}_\pm(t_1) \cdots \overline{\sigma}_\pm(t_r).$$

They are HS derivations on $\bigwedge V$, in the sense that $\sigma_\pm(t_r)(u \land v) = \sigma_\pm(t_r)(u) \land \sigma_\pm(t_r)(v)$, holding the same to their inverses $\overline{\sigma}_\pm(t_r)$.

Next we need some preliminaries results.

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4.2 Lemma. Let $t$ be one formal variable over $\mathbb{Q}$. The following equality holds for all $r \geq 2$:

$$\sigma_+(t)X^r(0) = \sigma_+(t)X^{r-1} \land X^{r-1}(0).$$

Proof. One argues by induction on $r$. The case $r = 1$ is ruled by the equality

$$\sigma_+(t)X^i = X^i + t\sigma_+(t)X^{i+1}. \quad (19)$$

Now suppose that the property holds for $r - 1 \geq 1$, then:

$$\begin{align*}
\sigma_+(t)X^r(0) &= \sigma_+(t)X^{r-1} \land \sigma_+(t)X^{r-1}(0) \quad \text{(because } \sigma_+(t) \in HS(\land V)) \\
&= \sigma_+(t)X^{r-1} \land (\sigma_+(t)X^{r-2} \land X^{r-2}(0)) \quad \text{(induction hypothesis)} \\
&= \sigma_+(t)X^{r-1} \land (\sigma_+(t)X^{r-2} \land X^{r-3} \land \ldots \land X^0) \\
&= \sigma_+(t)X^{r-1} \land (X^{r-2} + t\sigma_+(t)X^{r-1}) \land X^{r-3} \land \ldots \land X^0 \quad \text{(eq. (19))} \\
&= \sigma_+(t)X^{r-1} \land X^{r-1}(0)
\end{align*}$$

as desired. ■

4.3 Proposition. The multivariable Schubert derivations satisfy the following equalities:

$$\begin{align*}
\overline{\sigma}_+(t_r)X^0 &= X^0 + \sum_{i=1}^{r} (-1)^i e_i(t_r)X^i \\
\sigma_+(t_r)X^0 &= X^0 + \sum_{i \geq 1} h_i(t_r)X^i,
\end{align*} \quad (20, 21)$$

where $e_i(t_r)$ and $h_i(t_r)$ are, respectively, the elementary and complete symmetric polynomial of degree $i$ in the indeterminates $t_1, \ldots, t_r$.

Proof. For $r = 1$ is the definition of $\sigma_+(t)$ and $\overline{\sigma}_+(t)$. The general case follows by induction on $r$.

Indeed, if (20) holds for $r - 1 \geq 1$, then

$$\begin{align*}
\overline{\sigma}_+(t_r)X^0 &= \overline{\sigma}_+(t_r)\overline{\sigma}_+(t_{r-1})X^0 = \sum_{i=0}^{r-1} (-1)^i(X^i - t_rX^{i+1})e_i(t_{r-1}) \\
&= \sum_{i=0}^{r-1} (-1)^i e_i(t_{r-1})X^i + \sum_{i=0}^{r-1} (-1)^{i+1}t_r e_i(t_{r-1})X^{i+1} \\
&= \sum_{i=0}^{r-1} (-1)^i e_i(t_{r-1})X^i + \sum_{i=1}^{r} (-1)^i t_r e_{i-1}(t_{r-1})X^i \\
&= X^0 + \sum_{i=1}^{r-1} (-1)^i (e_i(t_{r-1}) + t_r e_{i-1}(t_{r-1}))X^i + (-1)^r t_r e_{r-1}(t_{r-1})X^r \\
&= \sum_{i=0}^{r} (-1)^i X^i e_i(t_r).
\end{align*}$$
Equality (21) is a consequence of (20), because $\sigma_+ (t_r)$ and $\overline{\sigma}_+ (t_r)$ are mutually inverses in $\text{End}_\mathbb{Q} (\wedge^r V)[t_r]$. 

4.4 Lemma. In the ring of formal power series $\mathbb{Q}[u]$ the following equality holds:

$$1 - u = \exp \left( - \sum_{i \geq 1} \frac{u^i}{i} \right).$$

Proof. This is well known. Just write

$$1 - u = \exp (\log(1 - u)) = \exp \left( -(1 + u + u^2 + \cdots) du \right) = \exp \left( - \sum_{i \geq 1} \frac{u^i}{i} \right).$$

4.5 Proposition. The following equalities hold:

$$\sigma_-(w) \sigma_+ (t_r) X^r(0) = \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+ (t_r) X^r(0), \quad (22)$$

$$\overline{\sigma}_-(w) \sigma_+ (t_r) X^r(0) = \exp \left( - \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+ (t_r) X^r(0). \quad (23)$$

Proof. We first prove that for a single formal variable $t$ one has:

$$\sigma_-(w) \sigma_+ (t) X^0 = \exp \left( \sum_{i \geq 1} \frac{t^i}{w^i} \right) \sigma_+ (t) X^0$$

Indeed by the very definition of $\sigma_+ (t)$:

$$\sigma_-(w) \sigma_+ (t) X^0 = \sigma_-(w) \left( \sum_{i \geq 0} X^i t^i \right) = X^0 + \sigma_-(w) \left( \sum_{i \geq 0} X^i+1 t^{i+1} \right)$$

$$= \sum_{i \geq 0} \sum_{j=0}^i X^{i-j} \frac{t^i}{w^i} = \left( \sum_{i \geq 0} \frac{t^i}{w^i} \right) \sigma_+ (t) X^0$$

$$= \frac{1}{1 - \frac{t}{w}} \sigma_+ (t) X^0 = \exp \left( \sum_{i \geq 1} \frac{t^i}{w^i} \right) \sigma_+ (t) X^0$$

where in the last equality we used Lemma 4.4. For $r > 1$, denote by $\Delta_0 (t_r)$ the Vandermonde determinant

$$\Delta_0 (t_r) = \prod_{1 \leq i < j \leq r} (t_j - t_i), \quad (24)$$

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we have:

\[ \Delta_0(t_r)\sigma_-(w)\sigma_+(t_r)X^r(0) = \sigma_-(w)(\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r)X^0) \]

\[ = \sigma_-(w)\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_-(w)\sigma_+(t_r)X^0 \]

and because \( \sigma_-(w) \) is a HS-derivation we have

\[ \Delta_0(t_r)\sigma_-(w)\sigma_+(t_r)X^r(0) = \sigma_-(w)\left(\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r)X^0\right) \]

\[ = \sigma_-(w)\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_-(w)\sigma_+(t_r)X^0. \]

Now by virtue of Lemma 4.4, the last above expression can be rewritten as

\[ = \exp\left(\sum_{n \geq 1} \frac{1}{n} \frac{t^n}{w^n}\right) \cdots \exp\left(\sum_{n \geq 1} \frac{1}{n} \frac{t^n}{w^n}\right) \left(\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r)X^0\right) \]

\[ = \Delta_0(t_r)\exp\left(\sum_{n \geq 1} \frac{1}{n} p_n\left(\frac{t_r}{w}\right)\right) \sigma_+(t_r)X^r(0). \]

whence (22) because \( \Delta_0(t_r) \) is not a zero divisor in \( \mathbb{Q}[t_r] \). The proof of (23) is similar and we omit it.

\[ \blacksquare \]

4.6 Lemma. Let \( \ell \) be a non-negative integer and set \((\ell^r)\) to be the partition with \( r \) parts equal to \( \ell \). For each \( r \geq 1 \) we have the following identities:

1. \( \overline{\sigma}_+(z)X^r(\ell^r) = X^r(\ell^r) + \sum_{i=1}^r (-1)^i z^i X^r((\ell + 1)^i \ell^{r-i}) \) and

2. \( e_i X^r(\ell^r) := \overline{\sigma}_i X^r(\ell^r) = X^r((\ell + 1)^i \ell^{r-i}), \) for \( 0 \leq i \leq r \).

Proof. The second item is a consequence of the first one. Assuming \( r = 1 \), item 1) can be seen as follows

\[ \overline{\sigma}_+(z)X^1(\ell) = \overline{\sigma}_+(z)X^\ell = X^\ell - zX^{\ell+1} = X^1(\ell) - zX^1(\ell + 1). \]

Assuming that the statement holds true for some \( r \geq 1 \), the proof follows by induction on \( r \).
\[ \sigma_+(z)X^{r+1}(l^{r+1}) \]
\[ = \sigma_+(z)(X^{l+r} \land X^r(l^r)) \]
\[ = (X^{l+r} - zX^{l+r+1}) \land \sigma_+(z)X^r(l^r) \]  
\[(\sigma_+(z) \in HS(\land V) \text{ and eq. (9))} \]
\[ = (X^{l+r} - zX^{l+r+1}) \land \left( X^r(l^r) + \sum_{i=1}^{r} (-1)^i z^i X^r((l + 1)^i l^{r-i}) \right) \]  
\[(\text{induction hypothesis}) \]
\[ = X^{l+r} \land X^r(l^r) - zX^{l+r+1} \land X^r(l^r) + \sum_{i=1}^{r} (-1)^i z^i \cdot X^{l+r+1} \land X^r((l + 1)^i l^{r-i}) \]
\[ = X^{r+1}(l^{r+1}) - zX^{r+1}((l + 1)^{l^r}) + \sum_{i=2}^{r+1} (-1)^i z^i \cdot X^{l+r+1} \land X^r((l + 1)^i l^{r-(i-1)}) \]
\[ = X^{r+1}(l^{r+1}) - zX^{r+1}((l + 1)^{l^r}) + \sum_{i=2}^{r+1} (-1)^i z^i \cdot X^{r+1}(l^{r+1-i}) \]
\[ = X^{r+1}(l^{r+1}) + \sum_{i=1}^{r+1} (-1)^i z^i \cdot X^{r+1}((l + 1)^i l^{r+1-i}) \]

4.7 Lemma. The following commutations rule hold

1. \( \partial^0 \cdot \sigma_+(t)X^r(\lambda) = \sigma_+(t)(\partial^0 \cdot X^r(\lambda)) \);
2. \( \partial^0 \cdot \sigma_+(t)X^r(\lambda) = \sigma_+(t)(\partial^0 \cdot X^r(\lambda)) \).

Proof. If the length of the partition is exactly \( r \), \( l(\lambda) = r \), then both sides of (1) are equal to zero. If \( l(\lambda) < r \), since \( X^r(\lambda) = X^{r-1}(\lambda + (1^r)) \land X^0 \), then the left hand side of (1) is
\[ \partial^0 \cdot \sigma_+(t)(X^{r-1}(\lambda + (1^r)) \land X^0) = \partial^0(\sigma_+(t)X^{r-1}(\lambda + (1^r)) \land \sigma_+(t)X^0) \]
\[ = (-1)^{r-1} \sigma_+(t)X^{r-1}(\lambda + (1^r)) \]
\[ = \sigma_+(t)(\partial^0 \cdot X^r(\lambda)). \]

The proof of item (2) is completely analogous.

4.8 Lemma. It follows from [2, Lemma 5.5] that
\[ \partial(w^{-1}) \cdot \sigma_+(t)X^r(\lambda) = \sigma_+(w)(\partial^0 \cdot \sigma_-(w)X^r(\lambda)), \]
and in particular we have
\[ \partial(w^{-1}) \cdot \sigma_+(t_r)X^r(0) = \sigma_+(w)(\partial^0 \cdot \sigma_-(w)\sigma_+(t_r)X^r(0)). \]
As defined in the introduction, cf. equations (1) and (2), the following two polynomials are required for the next Lemma.

\[ E_r(w) = 1 - e_1w + \cdots + (-1)^{r}e_r w^r \in B_r[w] \]

and

\[ E_r\left(t_r; \frac{1}{w}\right) = 1 - e_1(t_r)w^{-1} + \cdots + (-1)^{r}e_r(t_r)w^{-r} = \prod_{j=1}^{r}(1 - t_j w^{-1}) \in Q[t_r, w^{-1}] \]

4.9 Lemma. The following identity holds:

\[ \overline{\sigma}_+ (w)\overline{\sigma}_{r-1}X^{r-1}(0) \land \overline{\sigma}_+(t_r)X^0 = \left( E_r(w) + (-1)^{r+1}e_r w^r E_r\left(t_r, \frac{1}{w}\right) \right) X^r(0). \]

Proof. We first note that \( \overline{\sigma}_{r-1}X^{r-1}(0) = X^{r-1}(1^{r-1}) \), and by virtue of (1) of Lemma 4.6, where \( \ell = 1 \), we obtain

\[ \overline{\sigma}_+(w)\overline{\sigma}_{r-1}X^{r-1}(0) = X^{r-1}(1^{r-1}) + \sum_{i=1}^{r-1} (-1)^i w^i X^{r-1}(2^i 1^{r-1-i}), \]

where for \( k \geq \ell \), \((k^i0^{j})\) stands for the partition having \( i \) parts equal to \( k \) and \( j \) parts equal to \( \ell \). The Proposition 4.3 assures that \( \overline{\sigma}_+(t_r)X^0 = \sum_{i=0}^{r} (-1)^{i} e_r(t_r)X^i \), and then

\[ X^{r-1}(1^{r-1}) \land \overline{\sigma}_+(t_r)X^0 = X^{r-1}(1^{r-1}) \land X^0 + (-1)^{r}e_r(t_r)X^{r-1}(1^{r-1}) \land X^r \]

\[ = X^r(0) + (-1)^r(-1)^{r-1}e_r(t_r)e_rX^r(0) \]

\[ = (e_0 + (-1)^{r}e_r(-1)^{r-1}e_r(t_r))X^r(0). \]

Now, for each \( 1 \leq i \leq r - 1 \) we can write

\[ X^{r-1}(2^i 1^{r-1-i}) = X^r \land \cdots \land X^{r+1-i} \land X^{r-1-i} \land X^{r-2-i} \land \cdots \land X^1, \]

and so

\[ X^{r-1}(2^i 1^{r-1-i}) \land X^0 = X^r(1^i 0^{r-i}). \]

Therefore,

\[ (-1)^i w^i X^{r-1}(2^i 1^{r-1-i}) \land \overline{\sigma}_+(t_r)X^0 \]

\[ = (-1)^i w^i X^{r-1}(2^i 1^{r-1-i}) \land (X^0 + (-1)^{r-i} e_{r-i}(t_r)X^{r-i}) \]

\[ = (-1)^i w^i X^r(1^i 0^{r-i}) + (-1)^i(-1)^{r-i}(-1)^{r-i-1}w^i e_{r-i}(t_r)X^r \land \cdots \land X^1 \]

\[ = ((-1)^i e_i w^i + (-1)^r e_r \cdot (-1)^{r-i-1} e_{r-i}(t_r)w^i) X^r(0). \]
Adding up all the summands above, we get the desired result.

The last required preliminary result right before state and prove the main theorem of this section is the following one.

4.10 Proposition. ([14, Proposition 4.2]) For all \( u \in \wedge^r V \) the following equality holds:

\[
\sigma_+(z)X^0 \wedge u = z^r \sigma_+(z)\overline{\sigma}_-(z)(\overline{\sigma}_r u \wedge X^0).
\]

4.11 Theorem. For each \( r \geq 2 \),

\[
\mathcal{E}_r(z, w^{-1}, t_r) = \frac{z^r}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{p_n(t_r)}{n} \left( \frac{1}{w^n} - \frac{1}{z^n} \right) + x_n p_n(z, t_r) \right) \cdot \left( \mathcal{E}_r(w) + (-1)^{r+1} e_r w^r \mathcal{E}_r \left( t_r, \frac{1}{w} \right) \right).
\]

4.12 Remark. We call \( \mathcal{E}(z, w^{-1}, t_r) \) the structural formal power series because it determines, and is determined, by the structural constants of the representation. If \( g \) is an \( R \) Lie algebra with basis \( (\gamma_a)_{a \in A} \) and \( M \) is a \( g \)-module with \( R \)-basis \( (m_b)_{b \in B} \), the structural constants of the representation are the scalars \( R^c_{ab} \) defined by the equality

\[
\gamma_a \ast m_b = \sum_{c \in B} R^c_{ab} m_c \in M.
\]

Proof of Theorem 4.11. It amounts to work out the definition 4.1 of \( \mathcal{E}(z, w^{-1}, t_r) \). One has:

\[
\mathcal{E}_r(z, w^{-1}, t_r)X^r(0) = \sum_{\lambda \in \Sigma_r} X(z) \wedge (\partial(w^{-1})\lambda X^r(\lambda) s_\lambda(t_r))
\]

\[
= X(z) \wedge (\partial(w^{-1})\sigma_+(t_r) X^r(0)) \quad (\text{eq. (18)})
\]

\[
= \sigma_+(z)X^0 \wedge \overline{\sigma}_-(w) \left( \partial^0 \sigma_-(w) \sigma_+(t_r) X^r(0) \right) \quad (\text{Lemma 4.8})
\]

Now, Proposition 4.5, equation (22), assures that the expression

\[
\sigma_+(z)X^0 \wedge \overline{\sigma}_-(w) \left( \partial^0 \sigma_-(w) \sigma_+(t_r) X^r(0) \right)
\]

can be written as

\[
\exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z)X^0 \wedge \overline{\sigma}_-(w) \left( \partial^0 \sigma_+(t_r) X^r(0) \right),
\]

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Now by Lemma 4.7 this last expression becomes

\[
\exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z) X^0 \wedge \overline{\sigma}_-(w) \sigma_+(t_r)(\delta^0 \wedge X^r(0))
\]

\[
= (-1)^{r-1} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z) X^0 \wedge \overline{\sigma}_-(w) \sigma_+(t_r)X^{r-1}(1^{r-1}).
\]  

(25)

One can see that \(\sigma_+(t_r)\) commutes with \(\overline{\sigma}_-(w)\) when applied to any \(u \in \wedge V\) such that \(X^0 \wedge u \neq 0 \in \wedge^{r+1} V\). We have to avoid \(X^0\) just because it is fixed by \(\overline{\sigma}_-(w)\). So in particular, \(\overline{\sigma}_-(w)\sigma_+(t_r)X^{r-1}(1^{r-1}) = \sigma_+(t_r)\overline{\sigma}_-(w)X^{r-1}(1^{r-1})\). It is also known that for any \(r \geq 1\) one has \(\sigma_+(t_r)\overline{\sigma}_-(w)X^r(1^r) = \frac{(-1)^r}{w^r} \overline{\sigma}_+(w) \sigma_+(t_r)X^r(0)\). Hence

\[
\overline{\sigma}_-(w)\sigma_+(t_r)X^{r-1}(1^{r-1}) = \frac{(-1)^{r-1}}{w^{r-1}} \overline{\sigma}_+(w) \sigma_+(t_r)X^{r-1}(0).
\]

The above identity, together Proposition 4.10, imply that the right hand side of equation (25) becomes

\[
\frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z) \overline{\sigma}_-(z) \left( \overline{\sigma}_+(w) \sigma_+(t_r) \overline{\sigma}_r(0) \wedge X^0 \right).
\]

Using the fact that \(\overline{\sigma}_+(w)\) and \(\sigma_+(t_r)\) commute and integrating by parts (4), the previous expression can be written as

\[
\frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z) \overline{\sigma}_-(z) \sigma_+(t_r) \left( \overline{\sigma}_+(w) \overline{\sigma}_r(0) \wedge \sigma_+(t_r)X^0 \right).
\]  

(26)

To make notation more compact we set

\[
\mathcal{E}(w, t_r) := \left( E_r(w) + (-1)^{r+1} e_r w^r E_r \left( t_r, \frac{1}{w} \right) \right).
\]

By virtue of Lemma 4.9 and also by equation (23) of Proposition 4.5, the expression in above equation (26) is equals to

\[
\frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \sigma_+(z) \overline{\sigma}_-(z) \sigma_+(t_r) \mathcal{E}(w, t_r) X^r(0)
\]

\[
= \frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{w} \right) \right) \exp \left( - \sum_{n \geq 1} \frac{1}{n} p_n \left( \frac{t_r}{z} \right) \right) \sigma_+(z) \sigma_+(t_r) \mathcal{E}(w, t_r) X^r(0)
\]

(27)

\[
= \frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n(t_r) \left( \frac{1}{w^n} - \frac{1}{z^n} \right) \right) \sigma_+(z) \sigma_+(t_r) \mathcal{E}(w, t_r) X^r(0).
\]
Since $X^r(0)$ is eigenvector of $\sigma_+(z)$ with eigenvalue $\frac{1}{E_r(z)}$, we introduce new formal variables $(\chi_n)_{n \geq 1}$ through the equality
\[
\exp \left( \sum_{n \geq 1} \chi_n z^n \right) = \frac{1}{E_r(z)}.
\]
Hence, substituting
\[
\frac{1}{E_r(z)} \frac{1}{E_r(t_1)} \cdots \frac{1}{E_r(t_r)} = \exp \left( \sum_{n \geq 1} \chi_n p_n(z, t_r) \right),
\]
where $p_n(z, t_r) = z^n + p_n(t_r)$, in last expression of equation (27) gives
\[
\frac{z^{r-1}}{w^{r-1}} \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n(t_r) \left( \frac{1}{w^n} - \frac{1}{z^n} \right) + \chi_n p_n(z, t_r) \right) \mathcal{E}(w, t_r) X^r(0),
\]
that concludes the proof of the theorem. 

5 DJKM Representation

5.1 In this section we shall work on the vector space $V := \mathbb{Q}[X^{-1}, X]$. We use the following notation:

\[ [X]^0 = X^0 \wedge X^{-1} \wedge \cdots = X^0 \wedge [X]^{-1} = X^0 \wedge X^{-1} \wedge [X]^{-2} = \cdots. \]

and if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{P}_r$, we set

\[ [X]^{\lambda} = X^{\lambda_1} \wedge X^{-1+\lambda_2} \wedge \cdots \wedge X^{-r+1+\lambda_r} \wedge [X]^{-r}. \]

5.2 Definition. The Fermionic Fock space of charge 0 associated to $V$ is:
\[ \mathcal{F} := \mathcal{F}(V) := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot [X]^\lambda. \]

5.3 Definition. The Schubert derivations extend to $\mathcal{F}$ as follows (see [13] for details). First one declare that
\[
\sigma_+(z)[X]^m = \sigma_+(z)X^m \wedge [X]^{m-1},
\]
\[
\overline{\sigma}_+(z)[X]^m = [X]^m - z[X]^{m+1} + z^2[X]^{m+(1^2)} - z^3[X]^{m+(1^3)} + \cdots,
\]
\[
\sigma_-(z)[X]^m = [X]^m \quad \text{and} \quad \overline{\sigma}_-(z)[X]^m = [X]^m
\]
Then one sets:
\[
\sigma_\pm(z)X^\lambda = \sigma_\pm(z) \left( X^{\lambda_1} \land X^{-1+\lambda_2} \land \cdots \land X^{-r+1+\lambda_r} \right) \land \sigma_\pm(z)X^{-r}
\]
\[
\overline{\sigma}_\pm(z)X^\lambda = \overline{\sigma}_\pm(z) \left( X^{\lambda_1} \land X^{-1+\lambda_2} \land \cdots \land X^{-r+1+\lambda_r} \right) \land \overline{\sigma}_\pm(z)X^{-r}.
\]

5.4 Lemma. The following equation holds:
\[
\sigma_+(t_1)X^0 \land \cdots \land \sigma_+(t_r)X^0 \land [X]^{-r} = \Delta_0(t_r)\sigma_+(t_r)X^0.
\]

Proof. Starting on the left hand side, we have
\[
\sigma_+(t_1)X^0 \land \cdots \land \sigma_+(t_r)X^0 \land [X]^{-r}
\]
\[
= \left( \sum_{i_1, \ldots, i_r \geq 0} X^{i_1} \land \cdots \land X^{i_r} \cdot t_1^{i_1} \cdots t_r^{i_r} \right) \land [X]^{-r}
\]
\[
= \left( \sum_{(\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_r} X^{\lambda_1} \land \cdots \land X^{-r+1+\lambda_r} s_\lambda(t_r)\Delta_0(t_r) \right) \land [X]^{-r}
\]
\[
= \Delta_0(t_r) \cdot \sigma_+(t_1, \ldots, t_r) \left( X^0 \land X^{-1} \land \cdots \land X^{-r+1} \right) \land [X]^{-r}
\]
\[
= \Delta_0(t_r)\sigma_+(t_r) \left( X^0 \land X^{-1} \land \cdots \land X^{-r+1} \right) \land [X]^{-r}
\]
\[
= \Delta_0(t_r)\sigma_+(t_r)X^0
\]
as desired.

5.5 Lemma. We have:
\[
\sum_{\mu \in \mathcal{P}_k} [X]^\lambda s_\lambda(t_r) \land [X]^{-r} = \sigma_+(t_r)X^0.
\]

Proof. By [13, Proposition 5.12] the product \( \sigma_+(t_1) \cdots \sigma_+(t_r) \) of \( r \) Schubert derivations acts only on the first \( r \) exterior factors of \( [X]^0 \), namely
\[
\sigma_+(t_r)[X]^0 = \sigma_+(t_r) \left( X^0 \land \cdots \land X^{-r+1} \right) \land [X]^{-r}.
\]

5.6 Lemma. The following commutation rule holds:
\[
\sigma_-(z)\sigma_+(t)[X]^0 = \exp \left( \sum_{n \geq 1} \frac{1}{nz^n} \right) \sigma_+(t)\sigma_-(z)[X]^0 = \exp \left( \sum_{n \geq 1} \frac{1}{nz^n} \right) \sigma_+(t)[X]^0.
\]

Proof. It amounts to straightforward mechanical manipulation which we report below with details just for sake of completeness. Using Definition 5.3 for the extension of Schubert derivations to the Fermionic Fock space
\[
\sigma_-(z) \left[ \sigma_+(t)[X]^0 \right] = \sigma_-(z) \left( \sigma_+(t)X^0 \land [X]^{-1} \right)
\]

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\[
\sigma_-(z)\sigma_+(t)X^0 \land [X]^{-1} = \sigma_-(z) \left( \sum_{i \geq 0} X^i t^i \right) \land [X]^{-1}
\]

\[
\sigma_+(z)X^0 \left( 1 + \frac{t}{z} + \frac{t^2}{z^2} + \cdots \right) \land [X]^{-1}
\]

\[
= \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(z)X^0 \land [X]^{-1}
\]

\[
= \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(z)X^0
\]

\[
5.7 \text{ Lemma. The following commutation rule holds}
\]

\[
\sigma_-(z)\sigma_+(t)[X]^0 = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(t)\sigma_-(-z)[X]^0 = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(t)[X]^0
\]

\textbf{Proof.} First one writes:

\[
\sigma_+(t)[X]^0 = \sigma_-(z)\sigma_+(t)[X]^0
\]

Now we use Lemma 5.6, to commute \(\sigma_-(z)\) and \(\sigma_+(t)\), so obtaining:

\[
\sigma_+(t)[X]^0 = \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(z)\sigma_+(t)[X]^0,
\]

from which

\[
\sigma_-(z)\sigma_+(t)[X]^0 = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{t^n}{z^n} \right) \sigma_+(t)[X]^0.
\]

Recall the notation for Vandermonde determinant in the variables \(t_r\) as in (24).
5.8 Corollary. For all $r \geq 1$,

$$\sigma_-(z)\sigma_+(t_r)[X]^0 = \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{z^n} \right) \sigma_+(t_r)[X]^0.$$  

Proof. One has:

$$\Delta_0(t_r)\sigma_-(z)\sigma_+(t_r)[X]^0 = \sigma_-(z) \left( \sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_+(t_r)X^0 \right) \wedge [X]^{-r}$$

$$= \Delta_0(t_r)\sigma_-(z)\sigma_+(t_1)X^0 \wedge \cdots \wedge \sigma_-(z)\sigma_+(t_r)X^0 \wedge [X]^{-r}$$

$$= \Delta_0(t_r) \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{t_1^n}{z^n} \right) \cdots \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{t_r^n}{z^n} \right) \sigma_+(t_r)[X]^0$$

$$= \Delta_0(t_r) \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{z^n} \right) \sigma_+(t_r)[X]^0.$$

The proof ends by dividing the very first and the very last member by $\Delta_0(t_r)$.

5.9 Corollary. For all $r \geq 1$,

$$\overline{\sigma}_-(z)\sigma_+(t_r)[X]^0 = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{z^n} \right) \sigma_+(t_r)[X]^0.$$

Proof. It is an obvious consequence of Corollary 5.8, playing with the inverse of the Schubert derivations.

The Boson–Fermion correspondence implies that $B = \mathbb{Q}[x_1, x_2, \ldots]$ is naturally isomorphic to $\mathcal{F}_0$ via the $\mathbb{Q}$-linear extension of the sets map

$$S_{\lambda}(x) \mapsto [X]^\lambda.$$

5.10 Gatto & Salehyan in [13] show that

$$\left( \exp \left( \sum_{n \geq 1} \left( z^n - w^n \right) \right) S_{\lambda}(x) \right) [X]^0 = \sigma_+(z)\overline{\sigma}_+(w)[X]^\lambda,$$

and that

$$\left( \exp \left( - \sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{z^n} - \frac{1}{w^n} \right) \frac{\partial}{\partial x_n} \right) S_{\lambda}(x) \right) [X]^0 = \overline{\sigma}_-(z)\sigma_-(w)[X]^\lambda.$$

As in the case for finite $r$, $\mathfrak{gl}(\mathcal{V}) = \mathcal{V} \otimes \mathcal{V}^*$ acts on $B$ as follows:

$$[(X^i \otimes \partial^j) S_{\lambda}(x)] [X]^0 = X^i \wedge \partial^j [X]^\lambda.$$
Putting as usual
\[ \mathcal{E}(z, w) = \sum_{i, j \in \mathbb{Z}} (\chi^i \land \partial^j) z^i w^{-j}, \]
The DJKM result [6] (see also [17, Theorem 5.1]) says that:
\[ \mathcal{E}(z, w) = \frac{1}{1 - \frac{w}{z}} \cdot \Gamma(z, w), \tag{28} \]
where
\[ \Gamma(z, w) = \exp \left( \sum_{n \geq 1} \left( z^n - w^n \right) \right) \exp \left( - \sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{z^n} - \frac{1}{w^n} \right) \frac{\partial}{\partial x_n} \right). \]

5.11 Theorem. Let
\[ \mathcal{E}(z, w^{-1}, t_r) := \sum_{\lambda \in \mathcal{P}_r} \Gamma(z, w) \mathcal{S}_\lambda(x)s_\lambda(t_r) \in B[z, w, t_r, z^{-1}, w^{-1}] \]
be the structural formal power series of the B-representation of \( \mathfrak{gl}(\mathcal{V}) \) (Cf. Remark 4.12). Then
\[ \mathcal{E}(z, w^{-1}, t_r) = \exp \left( \sum_{n \geq 1} \frac{1}{n} \left( \frac{w^n}{z^n} - \frac{p_n(t_r)}{w^n} + \frac{p_n(t_r)}{w^n} \right) + x_n (z^n - w^n + p_n(t_r)) \right). \]

Proof. We first use the exponential expression of the geometric series:
\[ \left( 1 - \frac{w}{z} \right)^{-1} = \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{w^n}{z^n} \right), \]
to put in (28). Now we observe that:
\[ \left( \sum_{\lambda \in \mathcal{P}_r} \Gamma(z, w) \mathcal{S}_\lambda(x)s_\lambda(t_r) \right) [X]^0 = \sigma_+(z) \overline{\sigma}_+(w) \overline{\sigma}_-(z) \sigma_-(w) \sum_{\lambda \in \mathcal{P}_r} [X]^\lambda \mathcal{S}_\lambda(t_r) = \sigma_+(z) \overline{\sigma}_+(w) \overline{\sigma}_-(z) \sigma_-(w) \sigma_+(t_r) [X]^0. \]

Applying Lemma 5.6 and Lemma 5.7:
\[ = \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{w^n} \right) \sigma_+(z) \overline{\sigma}_+(w) \overline{\sigma}_-(z) \sigma_+(t_r) [X]^0 \]
\[ = \exp \left( \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{w^n} \right) \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{p_n(t_r)}{z^n} \right) \sigma_+(z) \overline{\sigma}_+(w) \sigma_+(t_r) [X]^0. \tag{29} \]

By recalling that \([X]^0\) is eigenvalue of \( \sigma_+(z), \overline{\sigma}_+(w) \) and \( \sigma_+(t_r) \), we obtain
\[ \sigma_+(z) \overline{\sigma}_+(w) \sigma_+(t_r) [X]^0 = \exp \left( \sum_{n \geq 1} x_n z^n \right) \exp \left( - \sum_{n \geq 1} x_n w^n \right) \exp \left( \sum_{n \geq 1} x_n p_n(t_r) \right) [X]^0. \]
Substituting the previous equation on (29) and simplifying, we may conclude
\[
\mathcal{E}(z, w^{-1}, t_r) = \sum_{\lambda \in \mathcal{P}_r} \Gamma(z, w) S_\lambda(x) s_\lambda(t_r)
\]
\[
= \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n(t_r) \left( \frac{1}{z^n} - \frac{1}{w^n} \right) + x_n \left( z^n - w^n + p_n(t_r) \right) \right),
\]
as claimed.

References

[1] O. Behzad, *Hasse-Schmidt Derivation and Vertex Operators on Exterior Algebras*, Ph.D. Thesis, 2021, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran.

[2] O. Behzad, A. Contiero, L. Gatto, and R. Vidal Martins, *Polynomial representations of endomorphisms of exterior powers*, Collect. Math. (2021) to appear, https://doi.org/10.1007/s13348-020-00310-5.

[3] O. Behzad and L. Gatto, *Bosonic and Fermionic Representations of Endomorphisms of Exterior Algebras*, Fundamenta Mathematica (2021) to appear, arXiv:2009.00479.

[4] O. Behzad and A. Nasrollah-Nejad, *Universal Decomposition Algebras Represent Endomorphism*, J. Algebra and its Applications (2021) to appear, https://doi.org/10.1142/S0219498822500724.

[5] A. Chapman, L. Gatto, and L. Rowen, *Clifford Semialgebras*, arXiv:2108.03617.pdf, 2021.

[6] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *Transformation groups for soliton equations. III. Operator approach to the Kadomtsev-Petviashvili equation*, J. Phys. Soc. Japan 50 (1981) 3806–3812.

[7] W. Fulton, *Young tableaux. With applications to representation theory and geometry*, London Mathematical Society Student Texts 35, Cambridge University Press, 1997.

[8] I. G-Macdonald, *Symmetric functions and Hall polynomials*, Oxford Classic Texts in the Physical Sciences, Oxford University Press, New York, 2015.

[9] L. Gatto, *Schubert calculus via Hasse-Schmidt derivations*, Asian J. Math. 9 (2005) 315–321.

[10] L. Gatto and P. Salehyan, *Hasse-Schmidt derivations on Grassmann algebras. With Applications to Vertex Operators*, IMPA Monographs, vol. 4, Springer Verlage, 2016.

[11] ______, *On Plücker equations characterizing Grassmann cones*, Schubert varieties, equivariant cohomology and characteristic classes — IMPANGA 15, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2018, pp. 97–125, arXiv:1603.00510.
[12] The cohomology of the Grassmannian is a $\mathfrak{gl}_n$-module, Comm. Algebra 48 (2020) 274–290.

[13] Schubert derivations on the infinite exterior power, Bull. Braz. Math. Soc., New Series 52 (2021) 149–174.

[14] L. Gatto and I. Scherbak, Hasse-Schmidt derivations and Cayley-Hamilton theorem for exterior algebras, Contemp. Math. 733 (2019) 149–165.

[15] Letterio Gatto and Louis Rowen, Grassmann semialgebras and the Cayley-Hamilton theorem, Proc. Amer. Math. Soc. 7 (2020) 183–201.

[16] M. Jimbo and T. Miwa, Solitons and infinite-dimensional Lie algebras, Publ. Res. Inst. Math. Sci. 19 (1983) 943–1001.

[17] V. G. Kac, A. K. Raina, and N. Rozhkovskaya, Bombay lectures on highest weight representations of infinite dimensional Lie algebras, Advanced Series in Mathematical Physics, vol. 29, World Scientific Publishing, 2013.

[18] D. Laksov and A. Thorup, A determinantal formula for the exterior powers of the polynomial ring, Indiana Univ. Math. J. 56 (2007) 825–845.

[19] M. Sato, The KP hierarchy and infinite-dimensional Grassmann manifolds, Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc. (1989) 51–66.

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