THE LITTLEWOOD CONJECTURE FOR A RESTRICTED CLASS OF PAIRS OF REAL NUMBERS

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ABSTRACT. We prove that the Littlewood conjecture is satisfied for a restricted class of pairs $(\alpha, \beta)$ of badly approximable numbers. We use the localization of the roots of a cubic equation with coefficients depending on the diophantine properties for the considered pair $(\alpha, \beta)$. The estimates of the roots rely on the properties of the denominators of the convergents of the continued fraction expansion of $\alpha$ and $\beta$.

1. INTRODUCTION

The Littlewood conjecture. Around 1930, J.E. Littlewood asked the question whether given any two distinct real numbers $\alpha$ and $\beta$, the sequence $(n|\sin(n\alpha)\sin(n\beta)|)_{n \geq 0}$ can take arbitrary small values ([10] Problem 5, p.19). The values of such sequences strongly depend on the diophantine properties of the pair $(\alpha, \beta)$. If we denote by $\| \cdot \|$ the function distance to the nearest integer, $\| x \| = d(x, \mathbb{Z})$. The problem of Littlewood amounts to solving the following conjecture which is called the Littlewood conjecture in the literature.

Conjecture 1.1 (Littlewood). For any $(\alpha, \beta) \in \mathbb{R}^2$, one has

$$\liminf_{n \to \infty} n\|n\alpha\|\|n\beta\| = 0.$$ 

The conjecture is obviously true if $\alpha$ or $\beta$ have unbounded partial quotients in their continued fraction expansion or if $1, \alpha$ and $\beta$ are linearly dependent over the rationals. Thus, it remains to prove the conjecture when $\alpha, \beta \in B$ where $B$ is the set of real numbers which have uniformly bounded partial quotients and when $1, \alpha$ and $\beta$ are linearly independent over the rationals.

This problem remains unsolved but it has been proved for some specific classes of pairs $(\alpha, \beta)$.

- For the pairs $(\alpha, \beta)$, where $\alpha$ and $\beta$ belongs to the same cubic field. (Cassels-Swninerton-Dyer, 1955, [5]). It is not known whether this result is relevant or not, indeed it is believed that cubic irrationals are not badly approximable numbers.
- For pairs $(\alpha, \beta)$ of badly approximable numbers, such that $\beta \in B(\alpha)$ where $B(\alpha)$ is a subset of $\mathbb{B}$ with $\dim_H B(\alpha) = 1$. (Pollington-Velani, 2000, [16]). This is the first result proving the existence of badly approximable solutions. An effective version of this result has been proved by Bugeaud [3] (2014).
- The first construction of explicit pairs of badly approximable numbers have been discovered by De Mathan [6] (2003) and Adamczewski-Bugeaud [1] (2006). Both results gives examples in terms of their continued fraction expansion.
- The set of exceptional pairs which do not satisfy the conjecture has been proved to be of Hausdorff dimension zero (Einsiedler, Katok, Lindenstrauss, 2006, [7]). This is the strongest evidence towards the conjecture. The full conjecture is implied by a deep conjecture of Margulis on the distribution of orbits under diagonal flows acting on $\text{SL}_3(\mathbb{R})/\text{SL}_3(\mathbb{Z})$.
- Lindenstrauss ([12]) suggested to exploit the result of (Einsiedler, Katok, Lindenstrauss, 2006) in order to find a criterion on a real number $\alpha$ such that any pair $(\alpha, \beta)$ satisfies the
conjecture. This leads to the notion of combinatorial entropy of a real number. Lindenstrausz gives the following criterion, if the entropy of a real number \( \alpha \) is positive then any pair \((\alpha, \beta)\) satisfies the Littlewood conjecture. Similarly to the dynamical case, the set of reals with null entropy has Hausdorff dimension zero.

For more precise information about the conjecture one can look at the references ([17], [19]) and ([20]) for the dynamical point of view.

1.1. **The main result.** We give sufficient conditions which implies the validity of the conjecture for a pair of badly approximable numbers. The conditions are given in terms of the denominators of the convergents of the continued fraction expansion of \( \alpha \) and \( \beta \).

**Theorem 1.2.** Let \((\alpha, \beta) \in \mathbb{R}^2\). If one can find a sequence \((\eta_n)_{n \geq 1}\) of real numbers with \(0 \leq \eta_n < 1/3\) for every \(n\), such that the two following conditions hold for \(n\) large enough

1. \[ q_{2n}(\beta)^{11/12+\eta_n/4} \leq q_{2n}(\alpha) \leq q_{2n}(\beta). \]
2. \[ \text{lcm}(q_{2n}(\alpha), q_{2n}(\beta)) \leq q_{2n}(\beta)^{1+\eta_n}. \]

Then the pair \((\alpha, \beta)\) satisfies the Littlewood conjecture.

**Remarks.**

1. The proof relies on the properties of an auxiliary cubic form \(F_n(t)\) whose coefficients depend on the diophantine properties of \(\alpha\) and \(\beta\). The conditions (1) and (2) of the theorem have been set in order to ensure, given an \(\varepsilon > 0\), the existence of an integer \(t_n\) which is a multiple of \(\text{lcm}(q_{2n}(\alpha), q_{2n}(\beta))\) and such that \(|F_n(t_n)| \leq \varepsilon\).

2. It is quite easy to find a pair of badly approximable numbers which satisfies the first condition. However the second condition is less flexible due to the lack of information on the divisibility properties of the denominators of the convergents. We discuss the special case of quadratic irrationals in section §5. An interesting question is to prove whether or not the Hausdorff dimension of the set of pairs satisfying the theorem is positive and, in the latter case, to estimate its value.

3. We expect that this result can be improved by refining the method for finding a multiple of \(\text{lcm}(q_{2n}(\alpha), q_{2n}(\beta))\). Indeed, we have used the fact the length of the interval \(I_j\) is larger than \(\text{lcm}(q_{2n}(\alpha), q_{2n}(\beta))\) which is asking too much. A better estimation of the roots of \(F_n(t) = \varepsilon\) may lead to the discarding of the constrains. This can be done by using the algebraic expression of the roots of a cubic instead of the trigonometric one.

4. A similar problem concerns the values of indefinite quadratic forms at integers instead of the product of linear forms. The analog of the Littlewood conjecture is the Oppenheim conjecture which is now a theorem due to G.A. Margulis [14],[15]. Recently using the same method we gave a proof of Margulis’ Theorem for indefinite irrational diagonal forms in dimension 3 ([12]).
2. Rational Lines of Approximations and Cubic Equations

2.1. Continued fractions expansion of a real number. Let us recall some of the main properties of the theory of continued fractions we are going to use, see for instance Chapter 3 [8] and [19]. The error in the approximation of $\beta$ by $c_n(\beta)$ is denoted by $e_n(\beta)$, i.e. $e_n(\beta) = \beta - c_n(\beta)$, it is exactly given by

$$e_n(\beta) = \frac{(-1)^n}{\beta_{n+1} q_n(\beta) + q_{n+1}(\beta)}$$

where $\beta_{n+1}$ is such that $\beta = [b_0; b_1, \ldots, b_n, \beta_{n+1}]$ (see Lemma 3 E, [19]). In particular, $e_{2n}(\beta)$ are all positive and one has ( [8] ex. 3.1.5 p. 76 )

$$\frac{1}{2q_{2n+1}(\beta)^2} \leq e_{2n}(\beta) \leq \frac{1}{q_{2n}(\beta)^2}.$$  \hfill (1)

Badly approximable numbers. The elements of $\mathbb{B}$ are the real numbers with bounded partial quotient in their continued fractions representation. It is not difficult to show that

$$\mathbb{B} := \{ \beta \in \mathbb{R} \mid \inf_{q \geq 1} q\|q\beta\| > 0 \text{ for every } q \in \mathbb{N} \}.$$

Dirichlet’s theorem applied to a badly approximable number $\beta$ is illustrated by the following estimate

$$\|q\beta\| \asymp \frac{1}{q}.$$ 

One has for any $\beta \in \mathbb{B}$ that $q_{2n+1}(\beta) \ll q_{2n}(\beta)$ for every $n$ thus using (1) we get the important estimate where the constant involved depends on the maximum value $M$ among the $b_i$ for all $i \geq 0$,

$$e_{2n}(\beta) \asymp \frac{1}{q_{2n}(\beta)^2}.$$ \hfill (2)

2.2. Small values of the Littlewood cubic form. Let us fix a pair $(\alpha, \beta)$ of positive real numbers and let us introduce the ternary cubic form deduced from the conjecture

$$f(x, y, z) := x(\alpha x - y)(\beta x - z).$$ \hfill (3)

This cubic is degenerate in the sense that it is the product of three linear forms. The set of zeros of $f$ is the union of the three planes of respective equations $x = 0$, $y = \alpha x$ and $z = \beta x$. If we assume that $\alpha$ and $\beta$ are two irrational numbers, then no nonzero integral vector could lie in the set $f = 0$. The Littlewood conjecture amounts to proving that

$$m(f) = \inf_{v \in \mathbb{Z}^3, v \neq 0} |f(v)| = 0.$$ \hfill (4)

This is equivalent to the assertion that for any real $\varepsilon > 0$, there exists $(x, y, z) \in \mathbb{Z}^3$, $x \neq 0$ such that

$$|f(x, y, z)| \leq \varepsilon.$$

Let us fix an arbitrary small real number $\varepsilon > 0$ and set

$$\mathcal{D}(\varepsilon) := \{ (x, y, z) \in \mathbb{R}^3 : |f(x, y, z)| \leq \varepsilon \}.$$

If $x$ is outside the planes $x = 0$, $y = \alpha x$ and $z = \beta x$, then $\mathcal{D}(\varepsilon)$ is bounded by the two parametric surfaces

$$\alpha x - \frac{\varepsilon}{x(\beta x - z)} \leq y \leq \alpha x + \frac{\varepsilon}{x(\beta x - z)}.$$
\[ \beta x - \frac{\varepsilon}{x(\alpha x - y)} \leq z \leq \beta x + \frac{\varepsilon}{x(\alpha x - y)}. \]

One can check that \( D(\varepsilon) \) contains the slant asymptote \( \mathbb{R}(1, \alpha, \beta) \) intersection of the two planes \( y = \alpha x \) and \( z = \beta x \). The set \( D(\varepsilon) \) is quite complicated since it consists in 2^3 connected components; each component is contained in one of the intersection of the half-spaces bounded by the three planes \( x = 0, y = \alpha x \) and \( z = \beta \). Finding one lattice point in any of these 8 connected component will suffices to ensure the existence of a lattice point in all other components by performing a change of sign on each coordinate. Usual methods of the geometry of numbers fails to apply in this situation due to the irregular shape of the domain \( D(\varepsilon) \). We are going to look for lattice points lying in rational line segments in \( D(\varepsilon) \).

### 2.3. Dirichlet lattice points near \( D(\varepsilon) \)

For any arbitrary \( \varepsilon > 0 \), the line \( \mathbb{R}(1, \alpha, \beta) \) lies in the domain \( D(\varepsilon) \), and it may happen that one could find a lattice point on it and solving the problem. But a non-trivial pair must satisfy that \( 1, \alpha \) and \( \beta \) are algebraically independent over \( \mathbb{Q} \), this breaks the hope of finding a lattice point on \( \mathbb{R}(1, \alpha, \beta) \). Nevertheless, Dirichlet’s approximation theorem tells us that it is always possible to find a lattice point lying arbitrarily near the line in question with a certain level of precision. The crucial issue is that the speed of convergence is not strong enough in order to ensure that such integral vectors are in \( D(\varepsilon) \). More precisely, if we consider a sequence of integers given by \( (N_n)_{1 \leq i \leq n} \). The two-dimensional version of Dirichlet’s approximation Theorem tells us that for every \( n \geq 1 \) there exists an integral vector \( M_n = (x_n, y_n, z_n) \in \mathbb{Z}^3 \) with \( 1 \leq x_n \leq N_n \) such that

\[ \begin{align*}
|\alpha x_n - y_n| &\leq N_n^{-1/2} \\
|\beta x_n - z_n| &\leq N_n^{-1/2}.
\end{align*} \tag{5} \]

Geometrically, this says that the lattice point \( M_n \) approaches the line \( \mathbb{R}(1, \alpha, \beta) \) as \( n \) increases with a rate of convergence given by \( 1/\sqrt{N_n} \). Applying this to the cubic form \( f \) we get that

\[ |f(M_n)| = |x_n||\alpha x_n - y_n||\beta x_n - z_n| \leq \frac{x_n}{N_n} \leq 1. \]

Not surprisingly, Dirichlet’s Theorem is not enough for proving the Littlewood conjecture. It can happen that \( |f(M_n)| \leq \varepsilon \) for some positive integer \( n \), in this case the theorem is proved. Thus from now on, we assume that it is not the case, meaning that for every positive integer \( n \),

\[ |f(M_n)| \geq \varepsilon. \]

Using (5), we obtain

\[ \varepsilon \leq |f(M_n)| = |x_n||\alpha x_n - y_n||\beta x_n - z_n| \leq \frac{x_n}{N_n}. \]

This yields the crucial bound on \( x_n \),

\[ \varepsilon N_n \leq x_n \leq N_n. \tag{6} \]

Recall that we are assuming that \( \alpha \) and \( \beta \) are badly approximable numbers, which amounts to saying that

\[ C := \min\{\inf_{q \geq 1} q\|q\alpha\|, \inf_{q \geq 1} q\|q\beta\|\} > 0. \]

In particular, this information together with (5) give the following inequalities
We have the following obvious bound on growth, we can assume that there exists an integer $n$

since the denominators for the convergents of a badly approximable number have a geometrical every $n$

$$2.4.$$ Line approximation. Using the lattice points $(M_n)_{n \geq 1}$ arising from Dirichlet’s theorem, we construct a family of rational lines which are nearly parallel to the line $\mathbb{R}(1, \alpha, \beta)$ in $\mathbb{R}^3$ and which converges towards it. To do so, we introduce the line passing through $P_n$ and directed by the vector given by $(1, c_{2n}(\alpha), c_{2n}(\beta))$, that is,

$$L^n_{\alpha, \beta} = M_n + \mathbb{R}(1, c_{2n}(\alpha), c_{2n}(\beta))$$

where $c_{2n}(\alpha)$ (resp. $c_{2n}(\beta)$) is the $2n$th convergent of $\alpha$ (resp. $\beta$).

A parametrization of the line segment given by $v_n : \mathbb{R} \to \mathbb{R}^3$ is defined as

$$v_n(t) := M_n - t(1, c_{2n}(\alpha), c_{2n}(\beta)) = (x_n(t), y_n(t), z_n(t))$$

where

$$\left( L^n_{\alpha, \beta} : \begin{array}{l}
x_n(t) = x_n - t \\
y_n(t) = y_n - tc_{2n}(\alpha) \\
z_n(t) = z_n - tc_{2n}(\beta).
\end{array} \right.$$ 

The strategy for proving the main theorem consists in showing the existence of an integer $t_n$ satisfying the three conditions

1. (Non vanishing condition) $x_n(t_n) \neq 0$.
2. (Geometric condition) For every $\varepsilon > 0$, $v_n(t_n) \in \mathcal{D}(\varepsilon)$ i.e. $0 < |f(v_n(t_n))| \leq \varepsilon$.
3. (Arithmetic condition) $t_n$ is an integral multiple of $l_n := \text{lcm}(q_{2n}(\alpha), q_{2n}(\beta)).$

We have the following obvious bound on $l_n$

$$\max\{q_{2n}(\alpha), q_{2n}(\beta)\} \leq l_n \leq q_{2n}(\alpha)q_{2n}(\beta).$$ \hspace{1cm} (8)

Since the denominators for the convergents of a badly approximable number have a geometrical growth, we can assume that there exists an integer $n_0$ large enough such that $q_{2n}(\alpha) \leq q_{2n}(\beta)$ for every $n \geq n_0$. In other words, for $n \geq n_0$ (8) is just

$$q_{2n}(\beta) \leq l_n \leq q_{2n}(\alpha)q_{2n}(\beta).$$ \hspace{1cm} (9)

Let us introduce the logarithmic ratio between $q_{2n}(\beta)$ and $q_{2n}(\alpha)$ as

$$\gamma_n := \frac{\ln q_{2n}(\alpha)}{\ln q_{2n}(\beta)}.$$ 

Thus one has $0 < \gamma_n \leq 1$ for every $n$ greater than $n_0$, and we can now write

$$q_{2n}(\alpha) = q_{2n}(\beta)^{\gamma_n}.\hspace{1cm} (10)$$

For $n$ large enough, the bounds (9) can be rewritten

$$q_{2n}(\beta) \leq l_n \leq q_{2n}(\beta)^{1+\gamma_n}.\hspace{1cm} (11)$$

The assumptions of Theorem 1.2 tell us that we have the bounds

$$\frac{11}{12} + \frac{\eta_n}{4} < \gamma_n \leq 1\hspace{1cm} (12)$$
under the condition $0 \leq \eta_n < 1/3$ when $n \geq n_0$.

2.5. **The cubic polynomial $F_n(t)$ associated to the Dirichlet lattice points $(M_n)_{n \geq 1}$.** In order to study the existence of lattice points in $D(\varepsilon)$, we will restrict to study their eventual presence on the lines $(L^a_{\alpha, \beta}) = \{v_n(t) : t \in \mathbb{R}\}$. First, we evaluate the cubic $f$ on $(L^a_{\alpha, \beta})$. One has, for any real $t$,

$$f(v_n(t)) = (x_n - t) \{\alpha(x_n - t) - (y_n - te2n(\alpha))\} \{\beta(x_n - t) - (z_n - te2n(\beta))\}$$

$$= (x_n - t) \{(\alpha x_n - y_n) - te2n(\alpha)\} \{(\beta x_n - z_n) - te2n(\beta)\}$$

Let us denote $\delta_n(\alpha) := \alpha x_n - y_n$ and $\delta_n(\beta) := \beta x_n - z_n$, without loss of generality and changing signs of $y_n$ and $z_n$ or of $f$ if necessary, we can assume that $\delta_n(\alpha)$ and $\delta_n(\beta)$ are both positive. Thus, we get

$$f(v_n(t)) = (x_n - t) \{\delta_n(\alpha) - te2n(\alpha)\} \{\delta_n(\beta) - te2n(\beta)\}$$

Also we set $t_n(\alpha) = \delta_n(\alpha)/e2n(\alpha)$, $t_n(\beta) = \delta_n(\beta)/e2n(\beta)$ and $A_n = e2n(\alpha)e2n(\beta)$. Hence,

$$f(v_n(t)) = -A_n(t - x_n)(t - t_n(\alpha))(t - t_n(\beta)).$$

The expression of $f(v_n(t))$ defines a cubic polynomial in $t$ which we denote $F_n(t)$. It assumes three real (positive)\(^1\) roots, namely $x_n$, $t_n(\alpha)$ and $t_n(\beta)$.

2.6. **Estimates of the coefficients of $F_n$.** Let us introduce the elementary symmetric polynomials of the roots $x_n$, $t_n(\alpha)$ and $t_n(\beta)$ of $F_n$.

$$\begin{align*}
\Sigma_1 & = x_n + t_n(\alpha) + t_n(\beta) \\
\Sigma_2 & = x_n t_n(\alpha) + x_n t_n(\beta) + t_n(\alpha) t_n(\beta) \\
\Sigma_3 & = x_n t_n(\alpha) t_n(\beta).
\end{align*}$$

We define a sequence of real numbers $(\delta_n)_{n \geq 1}$ such that

$$N_n = q2n(\beta)^{\delta_n}. \tag{14}$$

In order words $\delta_n$ is just the quotient $\log N_n/ \log q2n(\beta)$. The following proposition gives asymptotic bounds depending on $\varepsilon$ for each $\Sigma_i$ ($i = 1, 2, 3$).

**Proposition 2.1.** For $n$ large enough, one has

1. $q2n(\beta)^{\delta_n} + q2n(\beta)^{2 - \delta_n} \ll \varepsilon \Sigma_1 \ll \varepsilon q2n(\beta)^{\delta_n} + q2n(\beta)^{2 - \delta_n}/2$.

2. $q2n(\beta)^2 + q2n(\beta)^{2(1 + \gamma_n) - \delta_n} \ll \varepsilon \Sigma_2 \ll \varepsilon q2n(\beta)^{2(1 + \gamma_n) - \delta_n} + q2n(\beta)^{2 + \delta_n}/2$.

3. $\Sigma_3 \ll \varepsilon q2n(\beta)^{2(1 + \gamma_n)}$.

**Proof of the proposition.** (1) From 1, 6 and 7 we get

$$\varepsilon N_n < x_n \leq N_n, \tag{15}$$

\(^1\) The positivity of roots is not important but it allows us to work without taking care of the sign in the estimates and inequalities. At the end we are going to solve $|f(v)| < \varepsilon$ with the absolute value.
\[ y = \alpha x + \frac{\varepsilon}{\beta x^2} \]

\[ y = \alpha x - \frac{\varepsilon}{\beta x^2} \]

\[ (L_{\alpha,\beta}^n) \]

\[ (x(t_n(\varepsilon)), y_n, 0) \]

\[ (N_n, x_n, 0) \]

\[ (N_n, \alpha N_n, 0) \]

\[ x(t_n(\varepsilon)) \]

\[ x_n N_n \]

\[ y = \alpha x \]

**Figure 1.** A view of the picture in the plane \( z = 0 \). The red line \( (L_{\alpha,\beta}^n) \) passes through the Dirichlet lattice vector and cuts \( D(\varepsilon) \) at \( x(t_n(\varepsilon)) \). The boundary of \( D(\varepsilon) \) is colored in blue.

An estimate of \( \Sigma_1 \) is thus given by,

\[ \frac{q_{2n}(\beta)^2}{x_n} \ll t_n(\beta) \ll \frac{q_{2n}(\beta)^2}{\sqrt{N_n}}, \quad (16) \]

\[ \frac{q_{2n}(\alpha)^2}{x_n} \ll t_n(\alpha) \ll \frac{q_{2n}(\alpha)^2}{\sqrt{N_n}}. \quad (17) \]

Using the notation 14,

\[ q_{2n}(\beta)^{\delta_n} + q_{2n}(\beta)^{2\gamma_n-\delta_n} + q_{2n}(\beta)^{2-\delta_n} \ll \varepsilon \Sigma_1 \ll \varepsilon \left( q_{2n}(\alpha)^2 + q_{2n}(\beta)^2 \right). \quad (18) \]

Since \( 0 < \gamma_n \leq 1 \) then, we are reduced to

\[ q_{2n}(\beta)^{\delta_n} + q_{2n}(\beta)^{2-\delta_n} \ll \varepsilon \Sigma_1 \ll \varepsilon \left( q_{2n}(\beta)^{\delta_n} + q_{2n}(\beta)^{2-\delta_n} \right). \quad (19) \]
(2) Let us consider $\Sigma_2$. The asymptotic estimates (16), (17) allow us to obtain

$$q_{2n}(\alpha)^2 \ll x_n t_n(\alpha) \ll \sqrt{N_n} q_{2n}(\alpha)^2,$$  \hspace{1cm} (21)

$$q_{2n}(\beta)^2 \ll x_n t_n(\beta) \ll \sqrt{N_n} q_{2n}(\beta)^2.$$  \hspace{1cm} (22)

For the product $t_n(\alpha)t_n(\beta)$, the estimates (16) and (17) do not provide optimal bounds. Instead we use the assumption that $|f(x_n, y_n, z_n)| > \varepsilon$, which we assume to hold otherwise the conjecture is proved. This condition is equivalent to

$$\varepsilon < x_n \delta_n(\alpha) \delta_n(\beta)$$  \hspace{1cm} (23)

or also,

$$\frac{\varepsilon}{A_n} < x_n t_n(\alpha)t_n(\beta).$$  \hspace{1cm} (24)

Using (2) again, we arrive to

$$\frac{\varepsilon q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{x_n} \ll t_n(\alpha)t_n(\beta).$$  \hspace{1cm} (25)

Thus, we obtain the following refined estimate

$$\frac{q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{x_n} \ll \varepsilon t_n(\alpha)t_n(\beta) \ll \varepsilon \frac{q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{N_n}.$$  \hspace{1cm} (26)

From (21), (22), (26),

$$\frac{q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{N_n} + (q_{2n}(\alpha)^2 + q_{2n}(\beta)^2) \ll \varepsilon \Sigma_2 \ll \varepsilon \frac{1}{N_n} q_{2n}(\alpha)^2 q_{2n}(\beta)^2 + \sqrt{N_n} (q_{2n}(\alpha)^2 + q_{2n}(\beta)^2).$$  \hspace{1cm} (27)

As $N_n = q_{2n}(\beta)^{\delta_n}$, we arrive to

$$q_{2n}(\beta)^2(1+\gamma_n) - t_n (\alpha)\beta_n(\beta)^2 + q_{2n}(\beta)^2 \ll \varepsilon \Sigma_2 \ll \varepsilon \frac{q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{N_n} + \frac{q_{2n}(\alpha)^2}{N_n}.$$  \hspace{1cm} (28)

Thus,

$$q_{2n}(\beta)^2 + q_{2n}(\beta)^2(1+\gamma_n) - t_n (\alpha)\beta_n(\beta)^2 \ll \varepsilon \Sigma_2 \ll \varepsilon \frac{q_{2n}(\alpha)^2 q_{2n}(\beta)^2}{N_n} + q_{2n}(\beta)^2 + \delta_n.$$  \hspace{1cm} (29)

and this shows (2).

(3) For $\Sigma_3$, we already have from (26) that

$$\frac{\varepsilon}{A_n} < x_n t_n(\alpha) t_n(\beta) \ll q_{2n}(\alpha)^2 q_{2n}(\beta)^2.$$  \hspace{1cm} (30)

That is,

$$x_n t_n(\alpha) t_n(\beta) \ll q_{2n}(\alpha)^2 q_{2n}(\beta)^2.$$  \hspace{1cm} (31)

Hence,

$$\Sigma_3 \ll \varepsilon q_{2n}(\beta)^2(1+\gamma_n).$$  \hspace{1cm} (32)
2.7. **Asymptotic shape of the graph** $y = F_n(t)$. The derivative of $F_n(t)$ with respect to the real variable is

$$F'_n(t) = -A_n \left\{3t^2 - 2\Sigma_1 t + \Sigma_2 \right\}.$$ 

To find the local extrema of $F_n$, we are then reduced to compute the zeroes of $3t^2 - 2\Sigma_1 t + \Sigma_2$. The fact that we have three roots for $F_n$, a graphical argument shows one has two extrema so that the discriminant of $F'_n$ is nonnegative for every integer $n \geq 1$,

$$\Delta_n = 4\Sigma_1^2 - 12\Sigma_2 = 4(\Sigma_1^2 - 3\Sigma_2).$$

The roots of the derivative $F'_n(t)$ are then given by

$$\tau_n^\pm = \Sigma_1 \pm \frac{1}{2}\sqrt{\Delta_n}. \quad \text{(33)}$$

The values of the local maximum and minimum are respectively given by

$$F_n(\tau_n^\pm) = -A_n(\tau_n^\pm - x_n)(\tau_n^\pm - t_n(\alpha))(\tau_n^\pm - t_n(\beta)).$$

In fact we have much more precise than this, it requires the use of a result due to Laguerre-Cesáro (see e.g. Theorem 6.5.1 [18]) which says in our case that $\tau_n^\pm$ is in the middle third of the interval bounded by two the critical numbers bounding $\tau_n^\pm$. In concrete words, if the root are ordered as $\tau_n^{(1)} < \tau_n^{(2)} < \tau_n^{(3)}$ so that we have the following configuration

$$\tau_n^{(1)} < \tau_n^- < \tau_n^{(2)} < \tau_n^+ < \tau_n^{(3)}.$$ 

$$\tau_n^- \in \left[\tau_n^{(1)} + \frac{\tau_n^{(2)} - \tau_n^{(1)}}{3}, \tau_n^{(2)} - \frac{\tau_n^{(2)} - \tau_n^{(1)}}{3}\right]$$

and

![Figure 2. The shape of $y = F_n(t)$](image-url)
\[ \tau^+_n \in \left[ \tau^{(2)}_n + \frac{\tau^{(3)}_n - \tau^{(2)}_n}{3}, \tau^{(3)}_n - \frac{\tau^{(3)}_n - \tau^{(2)}_n}{1} \right]. \]

Also one has the equivalent
\[ |F_n(\tau^+_n)| \asymp \frac{1}{q_2(n)(\alpha)^2 q_2(n)(\beta)^2} |x_n - \tau^+_n| |\tau^+_n - t_n(\alpha)| |\tau^+_n - t_n(\beta)|. \]

Given \( \varepsilon > 0 \) arbitrary, replacing \( F_n \) by \(-F_n\) if necessary we are reduced to the three possible scenarios

1. \( \max\{|F_n(\tau^-_n)|, |F_n(\tau^+_n)|\} < \varepsilon \) then \( J_n \) is a single interval.
2. \( F_n(\tau^-_n) < -\varepsilon \) and \( F_n(\tau^+_n) \leq \varepsilon \) \( J_n \) is the union of two intervals.
3. \(-\varepsilon < F_n(\tau^-_n) \) and \( \varepsilon < F_n(\tau^+_n) \) \( J_n \) is the union of two intervals.
4. \( \min\{|F_n(\tau^-_n)|, |F_n(\tau^+_n)|\} \geq \varepsilon \) \( J_n \) is the union of three intervals.

Unfortunately, we are not able to estimate the factors in \( |F_n(\tau^+_n)| \) due to the lack of a 2-dimensional analog of the notion of continued fraction for real numbers. We are going to treat the problem of the intervals by solving directly the equation \( F_n(t) = \varepsilon \) in §3. We finish the study of the shape of the graph of \( F_n \) by giving an upper bound on the measure of \( t \) such that \( |F_n(t)| \leq \varepsilon \). This will follow from the study of the set of the small values of the cubic polynomial \( F_n(t) \). A key result due to H. Cartan [4] given in the cubic case shows that the measure of set of small values cannot be too large.

**Theorem 2.2** (Theorem 3.1, VIII, §3 [11]). Let \( f(t) = (t-t_1)(t-t_2)(t-t_3) \) be a monic polynomial in \( \mathbb{C}[t] \) and \( \varepsilon \) be a positive real. Then, the set of \( t \) such that
\[ |f(t)| \leq \varepsilon \]
is contained in the union of a most 3 intervals such that the sum of the lengths is bounded by \( 6\varepsilon \varepsilon^{1/3} \).

Applying this result to the monic polynomial \( p_n(t) = -A_n^{-1}F_n(t) \) tells us that there exists a covering of \( \mathcal{D}(\varepsilon) \) consisting in at most three intervals \( I_1, I_2, I_3 \) respectively with length \( l_1, l_2, l_3 \) so that their sums is less or equal to \( 6\varepsilon|A_n|\varepsilon^{1/3} \). In particular, we have
\[ \lambda(t \in \mathbb{R} : 0 < |F_n(t)| \leq \varepsilon) \leq \lambda(I^n_1) + \lambda(I^n_2) + \lambda(I^n_3) \leq 6\varepsilon(|A_n^{-1}|\varepsilon)^{1/3} \]
and
\[ \lambda(t \in \mathbb{R} : 0 < |F_n(t)| \leq \varepsilon) \ll (q_2(n)(\alpha)^2 q_2(n)(\beta)^2)\varepsilon^{1/3}. \]

We have obtained the following asymptotic estimate of the total length of the interval(s)

**Proposition 2.3.** For every positive real \( \varepsilon \), there exists an integer \( n_0 \) such that for all \( n \geq n_0 \),
\[ \lambda(t \in \mathbb{R} : 0 < |F_n(t)| \leq \varepsilon) \ll (q_2(n)(\alpha) q_2(n)(\beta))^2/3. \]

3. INTervals FOR THE SOLUTIONS OF \( F_n(t) = \pm \varepsilon \)

**Real solutions of the cubic equation** \( F_n(t) = \pm \varepsilon \). Now we need to find all the possible intersection points which correspond to the values of \( t \) such that \( F_n(t) = \pm \varepsilon \). We have the expanded form
\[ F_n(t) = -A_n \left\{ t^3 - \sum_1 t^2 + \sum_2 t - \sum_3 \right\}. \]
The equation \( F_n(t) = \pm \varepsilon \) takes the following form
\[ -A_n \left\{ t^3 - \sum_1 t^2 + \sum_2 t - \sum_3 \right\} \pm \varepsilon = 0. \]
We rather consider the equivalent equation

\[ G_n(\pm \varepsilon)(t) := t^3 - \Sigma_1 t^2 + \Sigma_2 t - \left( \Sigma_3 \pm \frac{\varepsilon}{A_n} \right) = 0. \tag{35} \]

If we specialize \( \varepsilon \) to be zero, we recover the solution of the cubic equation \( F_n(t) = 0 \). In particular we see that \( F_n(t) \) and \( G_n(\pm \varepsilon)(t) \) share the same symmetric functions of the roots \( \Sigma_1 \) and \( \Sigma_2 \), only the third differs by a factor \( \pm \frac{\varepsilon}{A_n} \).

We choose the trigonometric parametrization of the solution of this cubic equation \( G_n(\pm \varepsilon)(t) = 0 \) following the presentation of [2] (Appendix of Chapter 4). The first step consists to perform the Vieta transform \( t = y + \frac{\Sigma_1}{3} \) to get the following cubic in the reduced form (the \( t^2 \) term has vanished)

\[ y^3 + P_n y = Q_n(\pm \varepsilon) \tag{36} \]

where

\[ P_n = \frac{3 \Sigma_2 - \Sigma_1^2}{3} \quad \text{and} \quad Q_n(\pm \varepsilon) = -\frac{9 \Sigma_1 \Sigma_2 + 2 \Sigma_3^3}{27} + \left( \Sigma_3 \pm \frac{\varepsilon}{A_n} \right). \]

The discriminant of \( G_n(\pm \varepsilon) \) is given by

\[ D_n(\pm \varepsilon) := -4P_n^3 - 27Q_n(\pm \varepsilon)^2. \]

Let us solve (36), to do so we set \( y = h_n z \) where \( h_n = 2 \sqrt{|P_n|}/3 \) and replace in (36)

\[ h_n^3z^3 + P_n h_n z = \left( \frac{4|P_n|}{3} \right)^{3/2} z^3 + P_n \left( \frac{4|P_n|}{3} \right) z = Q_n(\pm \varepsilon). \]

Then we divide by \( \left( \frac{4|P_n|}{3} \right)^{3/2} \)

\[ z^3 + \frac{3P_n}{4|P_n|} z = \left( \frac{4|P_n|}{3} \right)^{-3/2} Q_n(\pm \varepsilon). \]

We are therefore reduced to solve the following equation,

\[ 4z^3 + 3 \sgn(P_n) z = C_n(\pm \varepsilon) \tag{37} \]

where

\[ C_n(\pm \varepsilon) = \frac{1}{2} \left( \frac{3}{|P_n|} \right)^{3/2} Q_n(\pm \varepsilon). \]

If we replace \( \varepsilon \) by zero, we recover the solution of the cubic equation \( F_n(t) = 0 \). The latter equation has 3 real solutions \( x_n, t_n(\alpha) \) and \( t_n(\beta) \), thus \( P_n < 0 \) (\( P_n \) does not depend on \( \varepsilon \)). Therefore we are left with two cases.

**Case 1.** \( \sgn(P_n) < 0 \), \( |C_n(\pm \varepsilon)| \geq 1 \) in this case, one real root one has to solve

\[ 4z^3 + 3z = C_n(\pm \varepsilon). \]

Using the formula \( \cosh 3\theta = 4 \cosh^3 \theta - 3 \cosh^2 \theta \)

\[ z = \pm \cosh \left( \frac{1}{3} \arccosh(C_n(\pm \varepsilon)) \right) \]
where the sign + is involved when \( C_n(\pm \epsilon) \geq 1 \) and the sign - when \( C_n(\pm \epsilon) \leq -1 \). If we assume that, say \( C_n(\pm \epsilon) \geq 1 \) then we have a unique real root for equation (36)

\[
y = 2\sqrt{-P_n/3} \cosh \left( \frac{1}{3} \Phi_n(\pm \epsilon) \right)
\]

where \( \Phi_n(\pm \epsilon) = \arccosh(C_n(\pm \epsilon)) \). Finally, we get the solutions of the equation \( F_n(t) = \pm \epsilon \)

\[
t(\pm \epsilon) = 2\sqrt{-P_n/3} \cosh \left( \frac{1}{3} \Phi_n(\pm \epsilon) \right) + \frac{\Sigma_1}{3}.
\]

(38)

Thus \( \mathcal{D}(\epsilon) = \{ t \in \mathbb{R}, |F_n(t)| \leq \epsilon \} \) is exactly the interval \( I_n = [t(\epsilon), t(-\epsilon)] \).

**Case 2.** \( \text{sgn}(P_n) < 0, |C_n(\pm \epsilon)| < 1 \).

One is reduced to solve

\[
4z^3 - 3z = C_n(\pm \epsilon).
\]

Taking advantage of the relation \( \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \) we deduce the solutions

\[
z_k = \cos \left( \frac{1}{3} \arccos(C_n(\pm \epsilon)) + \frac{2(k-1)\pi}{3} \right) \quad (k = 1, 2, 3).
\]

Thus in the case when \( |C_n(\pm \epsilon)| \leq 1 \), one has three real distinct roots for equation (36)

\[
y_k = 2\sqrt{-P_n/3} \cos \left( \frac{1}{3} \Phi_n(\pm \epsilon) + \frac{2(k-1)\pi}{3} \right) \quad (k = 1, 2, 3)
\]

where \( \Phi_n(\pm \epsilon) = \arccos(C_n(\pm \epsilon)) \) lies in the open interval \((0, \pi)\). Finally, we get the solutions of the equation \( F_n(t) = \pm \epsilon \)

\[
t_k(\pm \epsilon) = 2\sqrt{-P_n/3} \cos \left( \frac{1}{3} \Phi_n(\pm \epsilon) + \frac{2(k-1)\pi}{3} \right) + \frac{\Sigma_1}{3} \quad (k = 1, 2, 3).
\]

(39)

Thus \( \mathcal{D}(\epsilon) = \{ t \in \mathbb{R}, |F_n(t)| \leq \epsilon \} \) is the union of three the interval \( I_1^* = [t_1(\epsilon), t_1(-\epsilon)] \), \( I_2^* = [t_2(-\epsilon), t_2(\epsilon)] \), \( I_3^* = [t_3(\epsilon), t_3(-\epsilon)] \). Hence we are in the worst case where \( \mathcal{D}(\epsilon) \) is the union of three intervals.
Asymptotic estimate for $-P_n$. The next lemma is crucial it gives an asymptotic estimate for $-P_n$ with constants depending on $\varepsilon$.

Lemma 3.1. Assume $\delta_n > 4/3$ for every $n$, then as $n$ gets large, one has

$$-P_n \asymp_{\varepsilon} q_2n(\beta)^{2\delta_n}.$$ 

Proof. We have from Proposition 2.1

$$q_2n(\beta)^{\delta_n} + q_2n(\beta)^{2-\delta_n} \ll_{\varepsilon} \Sigma_1 \ll_{\varepsilon} q_2n(\beta)^{\delta_n} + q_2n(\beta)^{2-\delta_n/2}.$$ 

First, one has

$$\delta_n - (2 - \frac{\delta_n}{2}) = \delta_n - 2 + \frac{\delta_n}{2} = \frac{3\delta_n}{2} - 2 > 0.$$ 

Also,

$$\delta_n - (2 - \delta_n) = 2\delta_n - 2 = 2(\delta_n - 1) > 0.$$ 

Thus,

$$q_2n(\beta)^{\delta_n} \ll_{\varepsilon} \Sigma_1 \ll_{\varepsilon} q_2n(\beta)^{\delta_n}.$$ 

By definition $-P_n = \frac{1}{3}\Sigma_1^2 - \Sigma_2$ and it is positive because $F_n(t)$ has three real roots. Therefore $\Sigma_2^2$ is the dominant term of $-P_n$, then

$$-P_n \asymp_{\varepsilon} \Sigma_1^2 \asymp_{\varepsilon} q_2n(\beta)^{2\delta_n}.$$ 

4. Proof of Theorem 1.2

Let us fix $\varepsilon > 0$, and $N_n = q_2n(\beta)^{\delta_n}$ where $\delta_n > 4/3$ is chosen such that

$$\delta_n \in \left[\frac{4}{3}, \frac{1}{5}(3 + 4\gamma_n - \eta_n)\right] \neq \emptyset$$ 

for $n$ large enough. The nonemptyness in (40) is ensured by condition (12), indeed the length of

$$\left[\frac{4}{3}, \frac{1}{5}(3 + 4\gamma_n - \eta_n)\right]$$ 

is given by

$$\frac{1}{5}(3 + 4\gamma_n - \eta_n) - \frac{4}{3} = \frac{4}{5}\gamma_n - \frac{1}{5}\eta_n - \frac{11}{15} > \frac{4}{5}\left(\frac{11}{12} + \frac{\eta_n}{4}\right) - \frac{1}{5}\eta_n - \frac{11}{15} = 0.$$ 

![Figure 4](image-url)

**Figure 4.** One the left the worst scenario $|C_n(\pm\varepsilon)| < 1$ (three intervals) and one the right the best scenario $|C_n(\pm\varepsilon)| \geq 1$ (one interval).
We assume we are in the worst case\footnote{The shape of $\mathcal{D}(\varepsilon)$ can convince the reader that the best case is when the line cuts the boundary of $\mathcal{D}(\varepsilon)$ only twice.}, namely the case $2$ when $|C_{n}(\pm\varepsilon)| < 1$. Thus $F_{n}(t) = \pm\varepsilon$ has three real roots. In that case we have seen in \cite{9} that the three (trigonometric) solutions are given by

$$t_{k}(\pm\varepsilon) = 2\sqrt{-P_{n}/3} \cos \left( \frac{1}{3} \Phi_{n}(\pm\varepsilon) + \frac{2(k - 1)\pi}{3} \right) + \frac{\Sigma_{1}}{3} \quad (k = 1, 2, 3).$$

Thus, the set of times $t$ such that $|F_{n}(t)| \leq \varepsilon$ is the union of three intervals $I^{n}_{1}$, $I^{n}_{2}$ and $I^{n}_{3}$ bounded respectively by the roots $t_{1}(\pm\varepsilon)$, $t_{2}(\pm\varepsilon)$ and $t_{3}(\pm\varepsilon)$.

Since the lengths of each interval $I^{n}_{1}$, $I^{n}_{2}$ and $I^{n}_{3}$ are all equal, we only need to focus on the length of $I^{n}_{1}$. One has

$$t_{1}(\pm\varepsilon) = 2\sqrt{-P_{n}/3} \cos \left( \frac{1}{3} \Phi_{n}(\pm\varepsilon) \right) + \frac{\Sigma_{1}}{3}.$$

The length of $I^{n}_{1}$ is given by

$$\lambda(I^{n}_{1}) = \text{length}(I^{n}_{1}) = |t_{1}(\varepsilon) - t_{1}(-\varepsilon)| = 2\sqrt{-P_{n}/3} \left| \cos \frac{\Phi_{n}(\varepsilon)}{3} - \cos \frac{\Phi_{n}(-\varepsilon)}{3} \right|.$$

A lower bound for the length of $I^{n}_{1}$ is given by,

**Proposition 4.1.** For $n$ large enough,

$$\lambda(I^{n}_{1}) \gg \frac{q_{2n}(\beta)^{4(1+\gamma_{n})}}{(-P_{n})^{5/2}}.$$

**Proof of the Proposition.** Using the identity $\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)$, we get

$$\lambda(I^{n}_{1}) = 4\sqrt{-P_{n}/3} \left| \sin \left( \frac{\Phi_{n}(\varepsilon) + \Phi_{n}(-\varepsilon)}{6} \right) \sin \left( \frac{\Phi_{n}(\varepsilon) - \Phi_{n}(-\varepsilon)}{6} \right) \right|.$$

Remark that, one has $|(\Phi_{n}(\varepsilon) \pm \Phi_{n}(-\varepsilon))/6| \in [0, \pi/3]$, and thus one can apply the inequality $\sin x \geq x/2$ which is valid for all $x \in [0, \pi/2]$. We obtain

$$\lambda(I^{n}_{1}) \geq \frac{1}{36} \sqrt{-P_{n}/3} \left| \Phi_{n}(\varepsilon)^{2} - \Phi_{n}(-\varepsilon)^{2} \right|.$$

or more precisely,

$$\lambda(I^{n}_{1}) \geq \frac{1}{36} \sqrt{-P_{n}/3} \left| \arccos^{2}(C_{n}(\varepsilon)) - \arccos^{2}(C_{n}(-\varepsilon)) \right|.$$

**Lemma 4.2.** For all $x, y \in [-1, 1]$,

$$|\arccos^{2}x - \arccos^{2}y| \geq |x - y|^{2}.$$

**Proof.** One remarks that

$$|\arccos^{2}x - \arccos^{2}y| = |\arccos x - \arccos y| \cdot |\arccos x + \arccos y|.$$

Noting the fact that the function $\arccos$ assumes positive values on the open interval $(-1, 1)$, we can write that

$$|\arccos x + \arccos y| = \arccos x + \arccos y \geq |\arccos x - \arccos y|.$$

Thus,

$$|\arccos^{2}x - \arccos^{2}y| \geq |\arccos x - \arccos y|^{2}.$$
Moreover, the Lipschitz property for the cosine yields
\[ |\cos u - \cos v| \leq |u - v|. \]
The latter applied to \( u = \arccos x \) and \( v = \arccos y \) we get
\[ |x - y| \leq |\arccos x - \arccos y|. \] (41)
Hence,
\[ |\arccos^2 x - \arccos^2 y| \geq |x - y|^2. \]

Remembering that \( \Phi_n(\pm \varepsilon) = \arccos(C_n(\pm \varepsilon)) \) the previous lemma gives us
\[ |C_n(\varepsilon) - C_n(-\varepsilon)|^2 \leq |\Phi_n(\varepsilon)^2 - \Phi_n(-\varepsilon)^2|. \]

Therefore,
\[ \lambda(I_1^n) \geq \frac{1}{36\sqrt{3}} \sqrt{-P_n} |\Phi_n(\varepsilon)^2 - \Phi_n(-\varepsilon)^2| \gg \sqrt{-P_n} |C_n(\varepsilon) - C_n(-\varepsilon)|^2. \]
Thus, we obtain
\[ \lambda(I_1^n) \gg \sqrt{-P_n} |C_n(\varepsilon) - C_n(-\varepsilon)|^2. \]

Concerning the right hand side, one has
\[ C_n(\varepsilon) - C_n(-\varepsilon) = \frac{1}{2} \left( \frac{3}{-P_n} \right)^{3/2} (Q_n(\varepsilon) - Q_n(-\varepsilon)) = \frac{1}{2} \left( \frac{3}{-P_n} \right)^{3/2} \frac{2\varepsilon}{A_n} = \frac{3\sqrt{3}\varepsilon}{(-P_n)^{3/2} A_n}. \]
Thus, for \( n \) large enough
\[ |C_n(\varepsilon) - C_n(-\varepsilon)| \asymp \frac{q_{2n}(\beta)^{2(1+\gamma_n)}}{(-P_n)^{3/2}} \]
since \( A_n = (e_{2n}(\alpha)e_{2n}(\alpha))^{-1} \sim q_{2n}(\alpha)^2 q_{2n}(\beta)^2 = q_{2n}(\beta)^2(1+\gamma_n) \) by (2). Hence, we get the lower bound for the length
\[ \lambda(I_1^n) \gg \varepsilon \sqrt{-P_n} |C_n(\varepsilon) - C_n(-\varepsilon)|^2 \gg \varepsilon \frac{q_{2n}(\beta)^{4(1+\gamma_n)}}{(-P_n)^{3/2}}. \] (42)

The previous proposition combined with Lemma 3.1 yields,
\[ \lambda(I_1^n) \gg \varepsilon q_{2n}(\beta)^{4(1+\gamma_n)-5\delta_n}. \] (43)

We are ready to complete the proof of the theorem. One hypothesis of the theorem tells us that the least common multiple \( l_n \) of \( q_{2n}(\alpha) \) and \( q_{2n}(\beta) \) satisfies
\[ l_n \leq q_{2n}(\beta)^{1+\eta_n} \]
with some \( 0 \leq \eta_n \leq 1/3 \). Using the latter with Lemma 4.1,
\[ \frac{\lambda(I_1^n)}{l_n} \gg \varepsilon q_{2n}(\beta)^{4(1+\gamma_n)-5\delta_n} \]
which reduces to
\[ \frac{\lambda(I_1^n)}{l_n} \gg \varepsilon q_{2n}(\beta)^{3+4\gamma_n-\eta_n-5\delta_n}. \] (45)
For each \( n \), let us consider \( j_n \in \{1, 2, 3\} \) to be the index of one of two intervals \( I^n_{j_n} \) which do not contain \( x_n \). The initial choice of the sequence \( \delta_n \) in (40) implies that \( 3 + 4\gamma_n - \eta_n - 5\delta_n \) is positive for \( n \) large enough. In particular,

\[
\lim_{n} \frac{\lambda(I^n_1)}{l_n} = \infty.
\]

(46)

The same holds for \( \lambda(I^n_{j_n}) = \lambda(I^n_1) \) and therefore \( I^n_{j_n} \) always contains at least one multiple of \( l_n \).

For each \( n \) large enough, let \( t_n = k_n l_n \in I_{j_n} \) be a multiple of \( l_n \), and by definition of \( j_n \), \( x_n \neq t_n \). Moreover, if we set \( u_n = v_n(t_n) \), we have

\[
0 < |f(u_n)| \leq \varepsilon.
\]

In other words, we have found \( u_n \in \mathbb{Z}^3 \) with nonzero first component which satisfies

\[
\inf_{u \in \mathbb{Z}^3, u_1 \neq 0} |f(u)| = 0.
\]

Hence \((\alpha, \beta)\) satisfies the Littlewood conjecture. The theorem 1.2 is proved.

\[\Box\]

5. Comments on Theorem 1.2

Our main theorem gives sufficient conditions which ensure the validity of the Littlewood conjecture up to some restrictions involving the denominators of the convergents. The natural question is whether or not, there exists a pair \((\alpha, \beta)\) of badly approximable numbers which satisfies both conditions and such that \(1, \alpha\) and \(\beta\) are \(\mathbb{Q}\)-algebraically independent.

Even if it is not the aim of our method, finding an explicit pair \((\alpha, \beta)\) of quadratic irrationals seems extremely difficult. In our case the main difficulty is to get an explicit of the denominators of the convergents of an irrational quadratic number. However, for a special case of quadratic real numbers with continued fraction expansion of the form \([b,a]\), with \(a\) dividing \(b\), there is an explicit formula for the denominators of the convergents (e.g. [9] §10.13 Theorem 178). We will restrict to the case when \(a = b\). Those numbers form a family of irrational quadratic numbers which generalizes the golden ratio \(\varphi = (1 + \sqrt{5})/2 = [1; 1]\).

5.1. Denominators of the convergents of quadratic irrational numbers. Let \(b \geq 1\) be an integer, we define the real number \(\beta = [b; b]\), for \(b = 1\) we recover the golden ratio. That is not difficult to see that \(\beta\) is a irrational quadratic number satisfying the equation \(\beta^2 - b\beta - 1 = 0\), thus

\[
\beta = [b; b] = \frac{b + \sqrt{b^2 + 4}}{2}.
\]

The denominators of the convergents are given by the recurrence relation

\[
q_{n+1}(\beta) = bq_n(\beta) + q_{n-1}(\beta).
\]

The general expression of \(q_n(\beta)\) is

\[
q_n(\beta) = \frac{\beta^{n+1} - \beta^{n+1}}{\beta - \beta'}.
\]

where \(\beta' = \frac{b - \sqrt{b^2 + 4}}{2}\), that is,

\[
q_n(\beta) = \frac{1}{2^{n+2}} \frac{(b + \sqrt{b^2 + 4})^{n+1} - (b - \sqrt{b^2 + 4})^{n+1}}{\sqrt{b^2 + 4}}.
\]
Using binomial expansion, one can prove that
\[ q_n(\beta) = \sum_{k=0}^\lfloor n/2 \rfloor \binom{n-k}{k} b^{n-2k}. \]
For even indices we get,
\[ q_{2n}(\beta) = \sum_{k=0}^n \binom{2n-k}{k} b^{2(n-k)}. \]

An important fact is that \( q_{2n}(\beta) \) can be seen as the value of a monic polynomial of degree \( 2n \) with integral coefficients at \( b \). Analogously, if we consider \( \alpha = [a; \bar{a}] \) with \( b/a \notin \mathbb{Z} \) then \( \alpha \in \mathbb{Q} [\sqrt{a^2 + 4}] \) and \( \beta \in \mathbb{Q} [\sqrt{b^2 + 4}] \) with the property that \( \{1, \alpha, \beta\} \) are linearly independent over \( \mathbb{Q} \).

5.2. The condition on the size of \( q_{2n}(\alpha) \) relatively to \( q_{2n}(\alpha) \) in Theorem 1.2.

Lemma 5.1. Given a real number \( 0 \leq \eta < 1/3 \), there exists a positive real number \( b_c(\eta) \), such that the interval \([b^{11/12+\eta/4}, b]\) contains at least one positive integer as soon as \( b > b_c(\eta) \).

Proof. Let us consider the fonction \( \psi(x) := \frac{\log(x-1)}{\log x} \) for \( x > 1 \). We claim that \([b^{11/12+\eta/4}, b] \cap \mathbb{N} \neq \emptyset \) if and only if \( \psi(b) > \frac{11}{12} + \frac{\eta}{4} \). Indeed, saying that \([b^{11/12+\eta/4}, b] \cap \mathbb{N} \neq \emptyset \) is equivalent to \( b - b^{11/12+\eta/4} > 1 \).

The latter holds if and only if
\[ \log(b - 1) > \left( \frac{11}{12} + \frac{\eta}{4} \right) \log b. \]
Thus the claim. Moreover, for \( x \geq 2 \)
\[ \psi'(x) := \frac{x \log x - (x-1) \log(x)}{x(x-1)(\log x)^2} > 0, \]
the function \( \psi \) is increasing, tends to 1 as \( x \to \infty \) and thus \( \psi \) realizes a bijection \([2, \infty) \) onto \([0, 1) \).

In particular, there exists a unique value \( b_c(\eta) > 1 \) such that \( \psi(b_c(\eta)) = \frac{11}{12} + \frac{\eta}{4} \). Hence, the claim tells us that \([b^{11/12+\eta/4}, b] \cap \mathbb{N} \neq \emptyset \) as soon as \( b > b_c(\eta) \). The lemma is proved.

Remark. It is possible to compute numerically the values of \( b_c(\eta) \). For instance one has,
- For \( \eta = 0 \), one has \( b_c(0) = 6.78199... \)
- For \( \eta = 1/100 \), one has \( b_c(10^{-2}) = 6.912... \)
- For \( \eta = 1/10 \), one has \( b_c(10^{-1}) = 8.514... \)
- For \( \eta = 1/4 \), one has \( b_c(1/4) = 17.332... \)
  
  ...  
- For \( \eta = 1/3 \), one has \( b_c(1/3) = \infty \).

The smallest possible integer \( b \) for which \([b^{11/12+\eta/4}, b] \cap \mathbb{N} \neq \emptyset \) is \( b = 7 \). Let us assess the values of \( \eta \) which makes \([7^{11/12+\eta/4}, 7] \cap \mathbb{N} \neq \emptyset \).

For \( \eta = 0 \), one can check that
\[ 7^{11/12} = 5.95... < 6 < 7. \]
Thus the interval $[7^{11/12}, 7]$ contains 6. The solution $b = 7$ works not only for $\eta = 0$, but also for $\eta = \frac{4}{250} = \frac{2}{125}$, indeed for this value one has

$$7^{11/12+1/250} = 5.998... < 6 < 7$$

thus the interval $[7^{11/12+1/250}, 7]$ contains 6. However, for $\eta = 1/50$ we have that $7^{11/12+1/200} = 6.01...$ and then the interval $[7^{11/12+1/50}, 7]$ contains no integer. Thus for $\eta \geq 1/50$ there are no chance that $[7^{11/12+\eta/4}, 7]$ contains 6. The critical value $\eta_0$ for $b = 7$ is somewhere between $\frac{2}{125} = 0.016$ and $\frac{1}{50} = 0.020$.

**Lemma 5.2.** For a fixed real $0 < \eta < 1/3$, one can find a couple of positive integers $(a, b)$ such that for any integer $n$ large enough we have

$$q_{2n}(\beta)^{11+\eta \over 4} \leq q_{2n}(\alpha) \leq q_{2n}(\beta)$$

where $\alpha = [a; \overline{\pi}]$ and $\beta = [b; \overline{\pi}]$.

**Proof.** The previous lemma ensures that there exists a couple of integers $(a, b)$ such that

$$b^{11+\eta \over 4} < a < b$$

as soon as $b > b_\alpha(\eta)$. We define the corresponding numbers $\alpha = [a; \overline{\pi}]$ and $\beta = [b; \overline{\pi}]$. We have seen earlier that $q_{2n}(\beta)$ (resp. $q_{2n}(\alpha)$) is a monic polynomial of degree $2n$ in $b$ (resp. $a$). Therefore we have the following asymptotic comparisons,

$$\frac{q_{2n}(\beta)^{11+\eta \over 4}}{q_{2n}(\alpha)} = \left(\frac{b^{11+\eta \over 4}}{a^{2n}}\right)^{2n} \left(1 + o(1)\right) \sim_{n \to \infty} \left(\frac{b^{11+\eta \over 4}}{a}\right)^{2n} \leq 1$$

since $b^{11+\eta \over 4} < a$.

In a similar way, using the fact that $a < b$ one gets

$$\frac{q_{2n}(\beta)}{q_{2n}(\alpha)} = \frac{b^{2n}(1 + o(1))}{a^{2n}(1 + o(1))} \sim_{n \to \infty} \left(\frac{b}{a}\right)^{2n} \geq 1.$$  

To sum up, we obtain the following for $n \geq n_0$

$$q_{2n}(\beta)^{11+\eta \over 4} \leq q_{2n}(\alpha) \leq q_{2n}(\beta).$$

\[\square\]

Now given $0 < \eta < 1/3$, we have found a couple of badly approximable numbers $(\alpha, \beta)$ of the form $\alpha = [a; \overline{\pi}]$ and $\beta = [b; \overline{\pi}]$ which satisfies the first assumption of Theorem 1.2 as proved in Lemma 5.2.

\[\square\]

**Remark.** The couple $(\alpha, \beta)$ obtained in the Lemma satisfies the linear condition $\{1, \alpha, \beta\}$ is a linearly independent family over $\mathbb{Z}$ if $a$ does not divide $b$.

5.3. **The condition on the least common multiple of $q_{2n}(\alpha)$ and $q_{2n}(\beta)$**. This condition is unfortunately very difficult to check in practice. As we have noticed earlier it is almost impossible to find the general expression of the denominators of the convergents of an irrational real number. For the simplest case, when $\beta = [b; \overline{\pi}]$, one has

$$q_{2n}(\beta) = \sum_{k=0}^{n} \binom{2n - k}{k} b^{2(n-k)}.$$
Already in this case, it seems difficult to find the prime decomposition of \(q_{2n}(\beta)\) even if \(b\) is known. Let us consider the pair \((\alpha, \beta)\) obtained in Lemma 5.2 and let us write their respective prime decomposition for \(n \geq n_0\)

\[
q_{2n}(\alpha) = p_1^{a_{1,n}} \ldots p_{r_n}^{a_{r_n,n}} \text{ and } q_{2n}(\beta) = p_1^{b_{1,n}} \ldots p_{r_n}^{b_{r_n,n}}
\]

where \(a_{i,n}, \ b_{i,n}\) are nonnegative integers not both zero for every \(i = 1, \ldots, r_n\) which we denote simply \(\alpha_i, \beta_i\). The prime numbers \(p_1 < \ldots < p_{r_n}\) are given in increasing order. In the one hand, after applying the logarithm in Lemma 5.2 we infer that

\[
\mu_n \sum_{i=1}^{r_n} \beta_i \log p_i \leq \sum_{i=1}^{r_n} \alpha_i \log p_i \leq \sum_{i=1}^{r_n} \beta_i \log p_i
\]

where \(\mu_n = 11/12 + \eta_n/4\). In the other hand, the lcm condition for \(q_{2n}(\alpha)\) and \(q_{2n}(\beta)\) reads

\[
\sum_{i=1}^{r_n} \beta_i \log p_i \leq \sum_{i=1}^{r_n} \max(\alpha_i, \beta_i) \log p_i \leq (1 + \eta_n) \sum_{i=1}^{r_n} \beta_i \log p_i.
\] (48)

If the latter condition is satisfied therefore the pair \((\alpha, \beta)\) of Lemma 5.2 fulfills the hypothesis of Theorem 1.2. Let us precise the condition 48. Set \(I_n = \{1 \leq i \leq r_n, \ \alpha_i > \beta_i\}\) which is non empty otherwise \(q_{2n}(\alpha)\) divides \(q_{2n}(\beta)\). Therefore,

\[
\sum_{i=1}^{r_n} \max(\alpha_i, \beta_i) \log p_i = \sum_{i \in I_n} \alpha_i \log p_i + \sum_{i \notin I_n} \beta_i \log p_i.
\]

Thus,

\[
\sum_{i \in I_n} \beta_i \log p_i \leq \sum_{i \in I_n} \alpha_i \log p_i \leq \sum_{i \in I_n} \beta_i \log p_i + \eta_n \sum_{i=1}^{r_n} \beta_i \log p_i.
\]

(49)

Or also,

\[
0 \leq \sum_{i \in I_n} (\alpha_i - \beta_i) \log p_i \leq \eta_n \left( \sum_{i=1}^{r_n} \beta_i \log p_i \right).
\]

(50)

A sufficient condition which implies 50,

\[
|I_n| \max_{i \in I_n} [(\alpha_i - \beta_i) \log p_i] \leq \eta_n r_n \min_{1 \leq i \leq r_n} \beta_i \log p_i.
\]

(51)

also,

\[
|I_n| \max_{i \in I_n} (\alpha_i - \beta_i) \log p_{r_n} \leq \eta_n r_n \log p_1 \min_{1 \leq i \leq r_n} \beta_i.
\]

(52)

Therefore a sufficient condition which implies 49 is given by

\[
|I_n| \max_{i \in I_n} (\alpha_i - \beta_i) \leq \eta_n \frac{r_n}{\log p_{r_n}} \min_{1 \leq i \leq r_n} \beta_i.
\]

(53)

A very important feature of irrational quadratic numbers (also true for irrational cubics) is that the greatest prime factor of the denominator of its convergents tends to infinity, this result is due to K. Mahler [13]. Thus, the sequence \(p_{r_n}\) tends to infinity as \(n\) gets large, and in particular \(r_{n}\) too. To sum up, the pair \((\alpha, \beta)\) satisfies the assumptions of the theorem if it fulfills Lemma 5.2 and the inequality 53 for the prime decomposition. The fact that \(\alpha\) and \(\beta\) lies in two distinct quadratic fields which are not equivalent implies that \(\{1, \alpha, \beta\}\) are independent over \(\mathbb{Q}\), thus providing a nontrivial example.
6. Final Comments

The main theorem gives a class of pairs of reals which satisfies the Littlewood conjecture. We ignore if this class is empty, in the latter case it means that an eventual counterexample of the conjecture has to be found in this class. The first interesting question concerns the size of the class of pairs satisfying the main theorem. More precisely, let $\Omega_\eta$ be the set of pairs satisfying the hypothesis of the previous theorem and $\Omega$ be the union of all $\Omega_\eta$ over all the sequences $\eta$ valued in $[0, 1/3]$. A natural question is whether the Hausdorff dimension $\dim_H(\Omega)$ is positive or not? In the positive case, eventually finding bounds for $\dim_H(\Omega)$.

The method of the paper can be refined in order to go beyond the restricted class $\Omega$. Indeed, requiring that the measure of $I_j^n$ tends to infinity faster than $\text{lcm}(q_{2n}(\alpha), q_{2n}(\beta))$ is a strong assumption. By a finer analysis of the coefficients using for instance, algebraic formula for the roots instead of the trigonometric method, might lead to a better localization of the roots and hopefully allowing us to relax the two assumptions of the theorem. From a far reaching point of view we expect that any improvement of this method for instance using analytic or combinatorial number theory could help to solve the problem of the existence of a multiple of $l_n$ in one of the $I_j$’s without requiring that their lengths have to diverge rapidly. The latter point is the heart of the problem. Presumably, this method can also be applied to some variations around the conjecture like the $p$-adic analog for instance.

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