Phase transition at exceptional point in Hermitian systems

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Exceptional point (EP) is a spectral singularity in non-Hermitian systems. The passing over the EP leads to a phase transition, which endows the system with unconventional features that find a wide range of applications. However, the need of using the dissipation and amplification limits the possible applications of systems with the EP. In this work, the concept of phase transitions at the EP is expanded to Hermitian systems that are free from dissipation and amplification. It is considered a composite Hermitian system including both two coupled subsystems and their environment consisting only of several tens degrees of freedom such that the energy can return from the environment to the subsystems. It is shown that the dynamics of such a Hermitian system demonstrates a clear phase transition. It occurs at the critical coupling strength between subsystems corresponding to the EP in the non-Hermitian system. This phase transition manifests itself even in the non-Markovian regime of the system dynamics in which collapses and revivals of the energy occur. A phonic circuit is proposed for observing the EP phase transition in systems free from dissipation and amplification. The obtained results extend the range of practical applications of the EP phenomena to Hermitian systems.

Non-Hermitian systems are actively investigated in last decades [1]-[3]. Unlike the Hermitian systems, the eigenstates of non-Hermitian systems are no mutually orthogonal [1, 2]. The point in the space of system parameters, at which some of eigenstates coalesce and corresponding eigenvalues coincide, is called an exceptional point (EP) of non-Hermitian system [1, 2]. By changing parameters, the system can pass over the EP. This passing is accompanied by qualitative change in the eigenstates [4, 5], which is referred to as an exceptional point phase transition [2],[6]-[11]. An example of such a transition is the spontaneous symmetry breaking in PT-symmetry systems [5],[12]-[14]. The EP phase transitions also take place in strongly coupled cavity-atom [4, 15, 16], polaritonic [17]-[19], optomechanical [20, 21], and parametrically driven [22, 23] systems. In the vicinity of the phase transition

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point the non-Hermitian systems demonstrate unusual properties, which find a num-
erous applications [4, 5, 13, 14]. In particular, they are used to enhance the sensitivity of
laser gyroscopes [24] and sensors [25]-[28], to control light propagation [29]-[35], to control
the laser characteristics [36]-[39] and even to achieve lasing without population inversion
[23, 40].

In spite of the significant advances in the applications of the systems with the EP, the
necessity of using the gain and loss limits the utility of these devices [41]-[49]. For instance,
the dissipation as well as amplification leads to increase of noises near the EP that prevents
to sensor operation [41]-[47]. Moreover, the dissipation (amplification) impedes a stable
device operation due to the decay (growth) of energy in the systems [48, 49]. Therefore,
overcoming the negative influence of dissipation and amplification on the characteristics of
the devices with the EP is an important problem.

In this paper, we extend the concept of the EP phase transition to the Hermitian
systems. To this end, we consider a Hermitian system together with its environment. The
interaction of system with environment leads to energy exchange between them. In the
case of environment with infinite number of degrees of freedom, the energy irreversibly
flows from the system to the environment, which leads to energy dissipation in the system.
In the case of environment with large but finite number of degrees of freedom, situation
qualitatively changes. At small times, an exponential decay of the amplitude oscillations in
this system occurs, which is due to the energy flows from the system into the environment.
However, at large times, the inverse energy flow from the environment leads to the revivals
of oscillations in the system [50, 51]. The time of the first appearance of the revival can
be called as a return time [52]. Usually, it is assumed that the environment is so large
that the return time is much longer than system observation time, and revivals can be
excluded from consideration. In this case, the Hermitian systems interacting with the
environment can be described by non-Hermitian equations. However, despite the fact that
the system demonstrates the non-Hermitian dynamics at times smaller than the return
time, the eigenstates of the Hermitian system including an environment are always mutually
orthogonal so EPs cannot exist [2]. Therefore, it seems that the EP phase transition is
distinctive feature of the non-Hermitian systems and can manifest itself only at time smaller
than the return time.

Here, we demonstrate that the EP phase transition takes place in Hermitian systems
including an environment with finite number of degrees of freedom. As an example, we
consider a Hermitian system including both two coupled oscillators and the environment
consisting only of several tens of degrees of freedom. We demonstrate that the dynamics of
such a Hermitian system exhibits a phase transition at the change in the coupling strength
between the oscillators. This phase transition manifests itself even at times much greater
than the return time, when the system displays the non-Markovian dynamics including
revivals of the energy of the oscillators. We show that there is a critical coupling strength
between the oscillators. Below the critical coupling strength, the system evolves from any
initial state into a state with a strictly specified ratio of the oscillators’ amplitudes. Above
the critical coupling strength, the ratio of the oscillators’ amplitudes in the final state
depends on the initial state. To illustrate this fact, we calculate the dependence of the
variance of the amplitudes’ ratio by varying the initial states on the coupling strength.
We demonstrate that this variance is about zero below the critical coupling strength and
sharply increases when the coupling strength exceeds the critical value. Therefore, the
variance of the amplitudes’ ratio can serve as an order parameter of the phase transition.
We show that the critical coupling strength in the Hermitian system coincides with the
one corresponding to the EP in the non-Hermitian system. We propose a photonic circuit
Figure 1: A scheme of Hermitian system under consideration consisting of two coupled oscillators and two reservoirs of \( N_1, N_2 \) oscillators. Each of the oscillators interacts with its reservoir.

based on two coupled microcavities interacting with finite-length waveguide for observation of the EP phase transition in a Hermitian dissipative-free system.

The obtained results can help to overcome the limitations arising from amplification and dissipation for the practical utilization of systems with the EP. This opens the way for creation of a new type of devices for laser and sensoric applications.

1 System under consideration

We consider a Hermitian system which includes two coupled oscillators and two reservoirs, each of which interacts with one of oscillators (Figure 1). The frequencies of two coupled oscillators are equal to \( \omega_0 \). The first and second reservoirs consist of sets of \( N_1, N_2 \) oscillators with frequencies \( \omega_k^{(1,2)} = \omega_0 + \delta \omega_{1,2} (k - N_1,2/2) \), respectively. Here \( \delta \omega_1 \) and \( \delta \omega_2 \) are steps between the oscillator frequencies in the first and second reservoirs.

To describe this system, we use the Jaynes-Cummings Hamiltonian in the rotating-wave approximation [52]:

\[
\hat{H} = \omega_0 \hat{a}_1^{\dagger} \hat{a}_1 + \omega_0 \hat{a}_2^{\dagger} \hat{a}_2 + \Omega (\hat{a}_1^{\dagger} \hat{a}_2 + \hat{a}_2^{\dagger} \hat{a}_1) + \sum_{k=1}^{N_1} \omega_k^{(1)} \hat{b}_k^{\dagger} \hat{b}_k + \sum_{k=1}^{N_2} \omega_k^{(2)} \hat{c}_k^{\dagger} \hat{c}_k \\
+ \sum_{k=1}^{N_1} g_1 (\hat{b}_k^{\dagger} \hat{a}_1 + \hat{a}_1^{\dagger} \hat{b}_k) + \sum_{k=1}^{N_2} g_2 (\hat{c}_k^{\dagger} \hat{a}_2 + \hat{a}_2^{\dagger} \hat{c}_k) \tag{1}
\]

First, second and third terms are the Hamiltonians of the first and second oscillators and the interaction between them, respectively. \( \hat{a}_{1,2} \) and \( \hat{a}_{1,2}^{\dagger} \) are the annihilation and creation operators for the first and second oscillators, obeying the boson commutation relations [52]. \( \Omega \) is a coupling strength between the oscillators. Fourth and fifth terms are the Hamiltonians of the reservoirs interacting with the first and second oscillators, respectively. \( \hat{b}_k, \hat{c}_k \) and \( \hat{b}_k^{\dagger}, \hat{c}_k^{\dagger} \) are the annihilation and creation operators for the oscillators with the
frequency $\omega_k^{(1), (2)}$ in the first and second reservoirs, respectively. $\sum_{k=1}^{N_1} g_1 (\hat{b}_k^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{b}_k)$ and $\sum_{k=1}^{N_2} g_2 (\hat{c}_k^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{c}_k)$ are the Hamiltonians of the interaction between the oscillators and their reservoirs. $g_{1,2}$ are the coupling strengths between the respective oscillator and the oscillators in the reservoirs. Such a model describes a wide class of systems interacting with reservoirs [52, 53]. We propose a photonic realization of this system in Section 3.

Using the Heisenberg equation for operators [53, 54], we obtain the closed system of equations for operators $\hat{a}_1, \hat{a}_2, \hat{b}_k, \hat{c}_k$. Moving from the operators to their averages, we obtain the linear system of equations

$$\frac{da_1}{dt} = -i\omega_0 a_1 - i \Omega a_2 - i \sum_{k=1}^{N_1} g_1 b_k$$

$$\frac{da_2}{dt} = -i\omega_0 a_2 - i \Omega a_1 - i \sum_{k=1}^{N_2} g_2 c_k$$

$$\frac{db_k}{dt} = -i\omega_k^{(1)} b_k - i g_1 a_1$$

$$\frac{dc_k}{dt} = -i\omega_k^{(2)} c_k - i g_2 a_2$$

where $a_1 = \langle \hat{a}_1 \rangle$, $a_2 = \langle \hat{a}_2 \rangle$, $b_k = \langle \hat{b}_k \rangle$ and $c_k = \langle \hat{c}_k \rangle$. Underline that Eqns. (2)–(5) are a closed system for the operator averages and thus can be solved without additional approximations.

2 Results

2.1 Crossover between Hermitian and non-Hermitian systems

The interaction of the oscillators with their reservoirs results in the energy exchange between them. When the number of degrees of freedom in the reservoirs tends to infinity ($N_{1,2} \to \infty$ and $\delta\omega_{1,2} \to 0$), the interaction leads to an exponential decay of the oscillators amplitudes (Figure 2a). In this case, the system dynamics can be described by the effective non-Hermitian equation [52, 55]:

$$\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -i\omega_0 - \gamma_1 & -i\Omega \\ -i\Omega & -i\omega_0 - \gamma_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

where $\gamma_{1,2}$ are effective decay rates, which, in the case of infinite reservoirs, are given as $\gamma_{1,2} = \sum_{k=1}^{N_{1,2}} \pi g_{1,2}^2 \delta \left( \omega_0 - \omega_k^{(1), (2)} \right)$ [52]. This system has an EP, at which the eigenstates coalesce and their eigenvalues coincide with each other. It occurs at

$$\Omega = \pm \Omega_{EP} = \pm \frac{|\gamma_1 - \gamma_2|}{2}$$

The passing through the EP is accompanied by a spontaneous symmetry breaking of eigenstates,

$$h_{1,2} = \left( i \left( \frac{\gamma_1 - \gamma_2}{2} \pm \sqrt{\left( \frac{\gamma_1 - \gamma_2}{2} \right)^2 - \Omega^2} \right), \ \Omega \right)^T$$

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Figure 2: Dependencies of the absolute values of the amplitudes of the first (blue line) and second (red line) oscillators on time calculated by simulating Eqns. (2)–(5). The black dashed lines show temporal dependencies of the amplitudes of the first and second oscillators calculated by Eq. (6) with decay rates (10). \( \Omega < \Omega_{EP} = \frac{\omega_0}{2}; T_{R1} = \frac{2\pi}{\delta \omega_1} \) is the return time of the first reservoir; the return time of the second reservoir \( T_{R2} = \sqrt{2}T_{R1} \). Here the following parameters are used: \( N_1 = N_2 = 40; \delta \omega_1 = 5 \times 10^{-3} \omega_0; \delta \omega_2 = 5 \times 10^{-3} \omega_0 / \sqrt{2}; \Omega = 10^{-4} \omega_0; g_{1,2} = 1.5 \times 10^{-3} \omega_0 \). The initial conditions are \( a_1 (0) = 1; a_2 (0) = 1; b_k (0) = 0; c_k (0) = 0 \).

and by a change of the eigenvalues \( \lambda_{1,2} = -21 \pi \gamma_{1,2} \pm \sqrt{\left(2(1-\gamma_{1,2})^2 - \Omega^2 \right)^2} \). Below the EP (\( |\Omega| < \Omega_{EP} \)), the eigenstates are non-PT-symmetrical and have different decay rates (\( \text{Re}\lambda_1 \neq \text{Re}\lambda_2 \)). Above the EP (\( |\Omega| \geq \Omega_{EP} \)), the eigenstates are PT-symmetrical and their decay rates become equal to each other (\( \text{Re}\lambda_1 = \text{Re}\lambda_2 \)). The spontaneous symmetry breaking occurring at the EP is referred to as an EP phase transition [2],[6]-[11]. The EP phase transition becomes apparent in the system dynamics [1, 7, 56], which is determined by as follows

\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = c_1 h_1 e^{\lambda_1 t} + c_2 h_2 e^{\lambda_2 t}
\]

where \( c_1 \) and \( c_2 \) are amplitudes of eigenstates which are determined from the initial conditions. The dynamics is sensitive to the changes in the eigenstates and the eigenvalues. Often, it is the system dynamics that is the object of study when detecting the EP phase transition [32, 34, 35].

At finite sizes of reservoirs, i.e., when \( N_{1,2} \) are finite and \( \delta \omega_{1,2} / \omega_0 \) are small but nonzero, from Eqns. (2)–(5) it follows that the exponential decay takes place only when time is much smaller than the return times of both reservoirs, i.e., when \( t << T_{R1,R2} = 2\pi / \delta \omega_{1,2} \). At this stage the decay rates can be evaluated as

\[
\gamma_{1,2} = \frac{\pi g_{1,2}^2}{\delta \omega_{1,2}}
\]

At \( t << T_{R1,R2} \) the temporal behavior of the oscillators interacting with the reservoirs of the finite number of degrees of freedom coincides with the behavior predicted by the non-Hermitian Eq. (6) with the decay rates (10) (cf. the blue and red lines with the dashed black line in Figure 2).

At \( t > T_{R1,R2} \), the temporal dynamics of the oscillators becomes more complex (Figure 2). The revivals of the oscillations in the system occur at times \( t \sim n T_{R1,R2}, n \) is a natural number. When revival occurs, the energy flows from the reservoirs to the system. This behavior does not take place in the non-Hermitian systems describing by Eq. (6).
Thus, the dynamics of Hermitian system interacting with reservoirs of finite sizes can be described by the non-Hermitian Eq. (6) only at times much smaller than the return time. For this reason, it is expected that an analog of the EP phase transition can be manifested in the behavior of Hermitian system only at small times. However, the question arises "is the EP phase transition manifested at time much greater than the return time?" Below, we show that this is so. We provide a new approach to describe the EP phase transition and introduce an order parameter that is suitable for both the non-Hermitian and Hermitian systems. Then, we demonstrate an existence of the EP phase transition in entirely Hermitian (i.e. dissipation-free) system including the reservoirs with only several tens degrees of freedom.

2.2 Criterion of phase transition at the EP

To proceed, we consider dynamics of the non-Hermitian system (6) both below and above the EP. The EP phase transition is accompanied by the change in the relaxation rates of the eigenstates. Below the EP, the eigenstates of the non-Hermitian system (6) have different decay rates (to be specific, we consider $\Re \lambda_1 > \Re \lambda_2$). In this case, at $t \gg |\Re \lambda_1 - \Re \lambda_2|^{-1}$ the contribution of the first eigenstate to the system state becomes prevailing (see Eq. (9)). As a result, at $t \gg |\Re \lambda_1 - \Re \lambda_2|^{-1}$ the ratio of amplitudes of the first and the second oscillators, $a_1/a_2$, coincides with the one for the first eigenstate (see Eq. (8)). Note that though the oscillators' amplitudes decay over time, for a given eigenstate they decay with the same rates. Thus, the ratio $a_1/a_2$ becomes fixed. As a result, the non-Hermitian system evolves from any initial state to a final state with a fixed ratio of $a_1/a_2$ determined by the ratio of oscillators' amplitudes in the eigenstate with the smallest decay rate.

Above the EP, the eigenstates of the non-Hermitian system (6) have the same decay rates ($\Re \lambda_1 = \Re \lambda_2$). In this case, the ratio of $a_1/a_2$ in a final state depends on the initial state and is determined by the contributions of both eigenstates to the system initial state.

Thus, we conclude that the passing through the EP can be detected by the dependence of $a_1(t \to \infty)/a_2(t \to \infty)$ on the initial state.

For quantitative description of the EP phase transition, using Eq. (6) we derive the equation for the quantity $A = a_1/a_2$:

$$\frac{dA}{dt} = (\gamma_2 - \gamma_1) A + i\Omega A^2 - i\Omega$$

Eq. (11) can be rewritten as two first-order differential equations for the real and imaginary parts of $A = a_1/a_2$:

$$\frac{d\Re A}{dt} = (\gamma_2 - \gamma_1) \Re A - 2\Omega \Re A \cdot \Im A$$

$$\frac{d\Im A}{dt} = (\gamma_2 - \gamma_1) \Im A + \Omega \left( (\Re A)^2 - (\Im A)^2 \right) - \Omega$$

The phase portraits of Eq. (12), i.e., the trajectories of the system dynamics in the space $\Re A$ and $\Im A$ are shown in Figure 3.

There are two fixed points of Eq. (12) in the coordinate plane of $\Re A$ and $\Im A$. These points are defined by following formulas:

$$(\Re A, \Im A) = \left( 0, \frac{\gamma_1 - \gamma_2 \pm \sqrt{(\gamma_1 - \gamma_2)^2 - 4\Omega^2}}{2\Omega} \right), \quad \Omega < \Omega_{EP}$$

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The points correspond to the eigenstates of the non-Hermitian system of Eq. (6).

It is seen that below the EP ($\Omega < \Omega_{EP}$), one of the fixed point is an attractive point, while other point is repulsion point (Figure 3a). As a result, the system evolves from arbitrary initial state to the specified final state. Note that Eq. (12) and the phase portraits are symmetrical relative to the change $\text{Re}A \rightarrow -\text{Re}A$. Below the EP, the final state is also symmetrical relative to this transformation of variables (in the final state $\text{Re}A = 0$).

Above the EP ($\Omega > \Omega_{EP}$), the system evolves along closed trajectories (Figure 3c). The ratio of the oscillators’ amplitudes depends on time and the initial state. It is important that, though Eq. (12) and the phase portraits are symmetrical relative to the change $\text{Re}A \rightarrow -\text{Re}A$, the system trajectories are not. Depending on the initial condition, the system evolution occurs either at positive or at negative values of $\text{Re}A$ (Figure 3c).

Thus, the trajectories along which the system evolves lose the symmetry. This behavior is manifestation of a spontaneous symmetry breaking in the non-Hermitian system and corresponds to the EP phase transition.

### 2.3 Order parameter for the EP phase transition

To characterize the EP phase transition, we introduce an order parameter based on the change in the system dynamics. To this end, we calculate the integral $I_{12}(T) = \frac{1}{T} \int_{0}^{T} A(t) \, dt$ for the different initial conditions; $T$ is the observation time. We consider the case $T >> T_{R1,R2}$. Then we find the variance $D_{12}(T) = \langle I_{12}(T)^2 \rangle - \langle I_{12}(T) \rangle^2$ over the initial states of the system. Below the EP, the system state tends to the eigenstate with the lowest relaxation rate. In this case, the ratio of $a_1/a_2$ in the final state does not depend on the initial state and, therefore, the value of $\lim_{T \rightarrow \infty} I_{12}(T)$ also does not depend on the initial state and $\lim_{T \rightarrow \infty} (D_{12}(T)) = 0$ (Figure 4). Above the EP, the eigenstates have the same relaxation rates and the ratio of $a_1/a_2$ changes over time and the system evolution depends on the initial state (see previous section). In this case, the value of $\lim_{T \rightarrow \infty} I_{12}(T)$ depends on the initial state and $\lim_{T \rightarrow \infty} \langle D_{12}(T) \rangle$ is not zero (Figure 4).

It is seen that the variance $\langle D_{12}(T \rightarrow \infty) \rangle$ experiences qualitative change at the EP
2.4 EP phase transition in a Hermitian system

In this section, we expand the concept of EP phase transitions to Hermitian systems, which include reservoirs of finite sizes. For this purpose, we study the time evolution of the Hermitian system, which is described by Eqns. (2)–(5) and consists of two coupled oscillators interacting with their reservoirs. The reservoirs consist of set of $N_1, 2$ oscillators ($N_1 = N_2 = 40$).

Similar to the case of non-Hermitian system, we calculate the variance $D_{12} (T) = \langle I_{12} (T)^2 \rangle - \langle I_{12} (T) \rangle^2$ over the initial states of the Hermitian system. At time much smaller than the return times ($t << T_{R1,R2}$) the evolution of Hermitian system approximately coincides with the one of the non-Hermitian system (see Figure 2). Therefore, it is not surprising that the variance $D_{12} (T)$ calculated at time $T << T_{R1,R2}$ demonstrates the same behavior as the one in the non-Hermitian system (cf. Figure 4 and Figure 5a). Thus, at $t << T_{R1,R2}$, the EP phase transition is clearly visible in the behavior of Hermitian system and the variance continues to play the role of the order parameter.
At time greater than the return times \( t > T_{R1,R2} \) the dynamics of Hermitian system qualitatively differs from the one of the non-Hermitian system (Figure 2). Instead of the exponential decay, the collapses and revivals of the oscillations take place in the system. At larger times \( t >> T_{R1,R2} \), the collapses and revivals are mixed and the system dynamics becomes complex (Figure 6).

Since at \( t >> T_{R1,R2} \) the behavior of Hermitian system qualitatively differs from the one of the non-Hermitian system, it is difficult to expect that the EP phase transition manifests itself in the dynamics of Hermitian system in this case. However, our calculations show that the EP phase transition takes place even at \( t >> T_{R1,R2} \). To demonstrate this, we calculate the variance \( D_{12} (T) \) for times much greater than the return times (Figure 5b). It is seen that when passing through the critical coupling strength the variance \( D_{12} (T >> T_{R1,R2}) \) changes similar to the one in the non-Hermitian system (Figure 5b). This critical coupling strength equals \( \Omega_{EP} \), i.e., corresponds to the EP in the non-Hermitian system (Eq. (7)), where the relaxation rates are determined by Eq. (10). Below, we call this critical coupling strength as an EP for Hermitian system.

Thus, the variance \( D_{12} (T >> T_{R1,R2}) \) sharply increases at the coupling strength corresponding to the EP. This behavior indicates that there is a phase transition manifesting in the dynamics of Hermitian system that is similar to the EP phase transition in the non-Hermitian system. The change occurring at the EP phase transition is clearly visible in the dynamics of Hermitian system, in which the reservoirs have different frequency steps \( \delta \omega_1 \neq \delta \omega_2 \). In this system, the return times of the first and second reservoirs are different from each other. When the coupling strength between the oscillators is smaller than \( \Omega_{EP} \), the revivals of oscillations in the first and second oscillators occur at different times (Figure 6a). Thus, in this regime the oscillators’ states are not synchronized with each other. When the coupling strength between the oscillators is greater than \( \Omega_{EP} \), the dynamics of oscillators becomes locked-in (Figure 6b).

Thus, we conclude that the EP phase transition manifests itself in the dynamics of Hermitian system at times both smaller and greater than the return times. This transition leads to synchronization of the oscillations in the coupled subsystems.

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**Figure 6**: Dependencies of the absolute values of the amplitudes of the first (blue line) and second (red line) oscillators on time calculated by Eqns. (2)–(5) when the coupling strength \( \Omega = 10^{-4} \omega_0 \approx 0.01 \Omega_{EP} \) (a); \( \Omega = 2 \cdot 10^{-2} \omega_0 \approx 2.12 \Omega_{EP} \) (b). Here \( \Omega_{EP} = 3\pi \cdot 10^{-3} \omega_0 \) and \( T_{R1} = 2\pi/\delta \omega_1 \) is the return time of the first reservoir. The following parameters are used: \( N_1 = N_2 = 40 \); \( \delta \omega_1 = 5 \times 10^{-3} \omega_0 \); \( g_1 = 2\sqrt{10} \times 10^{-3} \omega_0 \); \( g_2 = \sqrt{10} \times 10^{-3} \omega_0 \). The initial state is \( a_1 (0) = 1; a_2 (0) = 1; b_k (0) = 0; c_k (0) = 0 \).
2.5 The EP phase in a Hermitian system with small number of degrees of freedom

In the previous section, we demonstrated that the EP phase transition takes place in the Hermitian system. In this section, we study manifestations of the EP in the Hermitian systems with reservoirs with small number of degrees of freedom. To this end, we consider the Hermitian system of two coupled oscillators (2)–(5) interacting with two reservoirs with $N_1 = N_2 = 4$ degrees of freedom. This Hermitian system has only ten degrees of freedom.

The dynamics of this system differs qualitatively from the one of the non-Hermitian system at all times (cf. Figure 2 and Figure 7a). It is complex and resembles random fluctuations (Figure 7a). However, the EP phase transition manifests itself even in the dynamics of such a system (Figure 7b). The fact that the phase transition occurs even in the Hermitian system with only ten degrees of freedom allows us to argue about a Hermitian nature of the EP phase transition. Moreover, the small number of degrees of freedom that is sufficient for observing the EP phase transition can simplify the experimental implementation of such systems.

3 Experimental setup for observation of the EP phase transition in a Hermitian system

In the previous sections, we demonstrated that the EP phase transition manifests itself even in the Hermitian systems with several tens of degrees of freedom. Here we discuss possible experimental schemes for observing the EP phase transition in dissipative-free systems. Such systems should consist of two coupled bosonic subsystems, at least one of which interacts with a multimode system simulating the reservoir with finite number of degrees of freedom. High-Q optical cavities [57]-[61], superconducting qubits [62]-[69] can play the role of such bosonic subsystems. In turn, the finite length waveguides or multimode cavities can serve as reservoirs with a finite number of degrees of freedom. The length of waveguide or cavity determines the step between the frequencies in such a reservoir.

As an example of dissipative-free system suitable for observing the EP phase transition,
Figure 8: (a) Scheme of a photonic system for observing the EP phase transition. (b) Dependence of the absolute value of the variance $D_{12}(T)$ on the coupling strength between the optical cavities $\Omega$. $N = 40; \delta \omega = 5 \cdot 10^{-3} \omega_0; g = 2\sqrt{10} \times 10^{-3} \omega_0$.

we propose optical circuit consisting of two coupled optical cavities, one of which interacts with a finite length waveguide (Figure 8a). In such a system, the interaction of cavity with the finite length waveguide leads to the collapses and the revivals of the energy in the cavities. The return time $T_R = 2\pi/\delta \omega$ is a characteristic time of these processes. Therefore, we can assume that the system is a dissipative-free, when the relaxation times of the electromagnetic field in the cavities and the waveguide are much greater than the return time. At this condition, the collapses and revivals have time to occur.

To illustrate the EP phase transition in this system, based on Eqns. (2)–(5), we calculate the time dependence of light amplitudes in the cavities. Our calculations show that the variance $D_{12}(T)$ demonstrates the behavior that is typical for the EP phase transition (Figure 8b). Therefore, this optical scheme enables to observe the EP phase transition in dissipative-free system. The variance $D_{12}(T)$ plays the role of the order parameter for such a phase transition.

Note that besides the proposed optical scheme, there are a number of other circuits that are suitable for observation of the EP phase transition. For example, in the work [69], the authors investigated the system of Xmon qubits strongly coupled to a slow-light phonon waveguide. They demonstrated revivals and collapses in the system dynamics [69]. Such systems based on superconducting qubits are also suitable for observing the EP phase transition.

Thus, we conclude that the EP phase transition can be observed in the different types of systems including photonics and qubit systems for which losses are a critical problem.

4 Conclusions

To conclude, we consider the Hermitian system consisting of the two coupled oscillators and the reservoirs interacting with the oscillators. Usually, the non-Hermitian equations are used to describe the behavior of this system. The non-Hermitian equations are obtained by the elimination of the degrees of freedom of the reservoirs, which are usually considered with infinite number degrees of freedom [52, 53]. The resultant non-Hermitian system has an exceptional point (EP), at which two eigenstates coalesce and their eigenvalues coincide. The passing through the EP leads to the spontaneous symmetry breaking of eigenstates, which is associated with the EP phase transition. In the case of reservoirs with a finite number of degrees of freedom, the reverse energy flow from the environment into the system appears that leads to collapses and revivals of the amplitude oscillations in the system. As
a result, the dynamics of Hermitian system, which includes both the oscillators and the reservoirs, cannot be described by the non-Hermitian equations. The manifestation of the EP phase transitions in such Hermitian systems is a controversial question.

To solve this problem, we study the behavior of the Hermitian system of two coupled oscillators interacting with the reservoirs with only few tens of degrees of freedom. We show that the dynamics of this Hermitian system demonstrates a clear phase transition occurring at the coupling strength between the oscillators corresponding to the EP in the non-Hermitian system. This transition is manifested in the change in the system dynamics, which observes even at time much greater than the return time when the dynamics of Hermitian system differs qualitatively from the one of non-Hermitian system. The existence of the EP phase transition at times that much greater than the return time indicates that the EP phase transition is property of the Hermitian system. We find an order parameter characterizing the EP phase transition in the Hermitian system and demonstrate the existence of spontaneous symmetry breaking in the phase trajectories of this system. Our results demonstrate the existence of a new class of the EP phase transitions that occur in Hermitian systems and establish a connection between the phase transitions in the Hermitian and non-Hermitian systems.

We propose the photonic circuit for observing the EP phase transition in Hermitian system. In this way, we extend an area of the practical applications of the EP phenomena to Hermitian systems including photonic and qubit circuits for which losses are a critical problem.

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