A determination of the mass gap 
in the O($n$) $\sigma$ model

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Abstract

We calculate the finite volume mass gap $M(L)$ at 3-loop level in the non-linear O($n$) $\sigma$-model in two dimensions in small volumes. By applying the Monte Carlo measurements of the running coupling $\bar{g}(L) = 2nM(L)L/(n-1)$ by Lüscher, Weisz and Wolff measured in units of the physical mass gap $m$, the result is used to determine $m$ in units of the $\Lambda$-parameter in the O(3) and O(4) models. Our determinations show good agreement with those by Hasenfratz, Maggiore and Niedermayer in both models. We note that this manuscript has been revised in our paper hep-lat/9810025 by using the corrected four-loop $\beta$-function on the lattice.
1 Introduction

The non-linear $O(n)$ $\sigma$-model which describes an $n$-component spin field has thus far found many applications in theoretical physics. In condensed matter physics it has been investigated to study, e.g. ferromagnets. In elementary particle physics, on the other hand, the model in two dimensions shares with QCD many common properties; on the quantum level it is like QCD renormalizable [4], asymptotically free according to perturbation theory [5, 6, 7] and thought to have a mass gap $m$. In addition, the O(3) model has instanton solutions. Further, the $\sigma$-model has an additional advantage: because of low dimensionality it is simple to simulate numerically. Therefore, in many cases the model can be used for tests of new ideas in lattice theories.

One of the most interesting properties of the $\sigma$-model is the existence of an infinite set of conserved non-local quantum charges [8, 9, 10, 11]. From these, one can exclude particle production and also derive the “factorization equations” directly. Using these properties, one can then determine the $S$-matrix exactly up to CDD-ambiguity [12], provided the particle spectrum of the model is known [13, 14]. Furthermore, the CDD-ambiguity can also be restricted with help of the assumption that there are no bound states. The absence of the bound states was shown in the $1/n$-expansion to the order of $1/n$ [13]. Further, the exact $S$-matrix constructed in this way was confirmed by the same method to the second order in $1/n$ [15].

Since there is only one scale in the model, one can, in principle, express all physical quantities with help of this scale. In particular, the mass gap of the theory is proportional to the $\Lambda$-parameter. This situation is similar to what happens in QCD with massless quarks where the $\Lambda$-parameter sets the scale for scale violations, for example, in deep inelastic scattering.

Using the exact $S$-matrix, a few years ago Hasenfratz, Maggiore and Niedermayer succeeded in an analytic determination of the ratio $m/\Lambda$ in the O($n$) $\sigma$-model [2, 3]. In this derivation, however, various assumptions including the thermodynamical Bethe ansatz are made, which are plausible, for which, however, the rigorous proofs are still lacking. An independent calculation of $m/\Lambda$ is therefore desirable.

In the literature [1], Lüscher, Weisz and Wolff estimated the ratio $m/\Lambda$ by measuring the coupling $\bar{g}^2(L) = 2mM(L)L/(n - 1)$ running with the volume $L$ of the system in units of the physical mass gap $m[=M(\infty)]$ by means of Monte Carlo simulations and applying the measurements to the perturbative expression of $\Lambda$-parameter in this scheme. Although their determination of $m/\Lambda$ where $\Lambda$ was evaluated up to three loops in perturbation theory reproduces that of Hasenfratz et. al quite well, the agreement of two results was still not completely undisputable. In order to determine $m/\Lambda$ more precisely, we decided to evaluate the $\Lambda$-parameter to one loop order higher. This requires the calculation of the finite volume mass gap $M(L)$ at 3-loop level.

Our work is also interesting in another respect. It is generally accepted that the O($n$) $\sigma$-model is asymptotically free. This assumption was, however, questioned by
Patrascioiu and Seiler \cite{16, 17, 18}. Since in the determination of \( m/\Lambda \) by Hasenfratz et al. the validity of the asymptotic freedom was required, the confirmation of their result through our independent determination will, at the same time, support the correctness of asymptotic freedom.

The work is arranged as follows. In the next chapter, we present our 3-loop calculations of the finite volume mass gap \( M(L) \). By computing the spin-spin correlation function, we first evaluate the mass gap on a lattice. We then take the continuum limit and finally convert it to the MS-scheme. In chapter 3, we apply the result to the determination of mass gap \( m \) in units of the \( \Lambda \)-parameter in the \( \text{O}(3) \) and \( \text{O}(4) \) models. Two approaches will be considered. First, we introduce the mass shift \( \delta_0 = [M(L) - m]/m \) which is known for \( L \to \infty \) and match this with the perturbative \( M(L) \) \cite{19, 20, 21}. As a second approach, we determine \( m \) by applying the Monte Carlo measurements of the running coupling \( \bar{g}^2(L) \) to the perturbative \( \Lambda \)-parameter \cite{4}. Finally, the Feynman diagrams and some technical details are presented in the appendices.

2 Calculation of mass gap at 3-loop level

2.1 Mass gap on a lattice

We consider an \( n \)-component spin field \( q^i(x) \) \((i = 1, \cdots, n)\) with unit length \( q(x)^2 = 1 \) on a two-dimensional finite lattice

\[
\Lambda = \left\{ x \in \mathbb{Z}^2 \mid -T \leq x_0 \leq T; \ x_1 \equiv x_1 + L \right\},
\]

where \( T \) and \( L \) are fixed integers with \( T, L \geq 2 \). We work with periodic boundary conditions in space direction and, for reasons which will become clear below, free boundary conditions in time direction.

The action is given by

\[
S = \frac{n}{2f_0} \sum_{x_1=1}^{L} \left\{ \sum_{x_0=-T}^{T-1} \partial_0 q(x) \cdot \partial_0 q(x) + \sum_{x_0=-T}^{T} \partial_1 q(x) \cdot \partial_1 q(x) \right\},
\]

where \( f_0 \) denotes the bare coupling constant. The forward and backward lattice derivatives are defined by

\[
\partial_\mu q(x) = q(x + \hat{\mu}) - q(x)
\]

\[
\partial^\mu q(x) = q(x) - q(x - \hat{\mu}).
\]

\( \hat{\mu} \) is the unit vector to the positive \( \mu \)-direction \((\mu = 0, 1)\). The expectation values of the observables can be calculated by the formula

\[
\langle O \rangle = \frac{1}{Z} \int \prod_{x \in \Lambda} \left[ d^n q(x) \delta(q(x)^2 - 1) \right] O e^{-S},
\]

\footnote{We set the lattice constant \( a \) to 1.}
where the partition function $Z$ is such that $\langle 1 \rangle = 1$.

Concerning the energy spectrum, there is a following possibility for the calculation of the mass gap $M(L)$ in finite volume for $f_0 \to 0$. We calculate the expectation value of the spatially averaged spin-spin correlation function in the limit $T \to \infty$ ($\tau > 0$):

$$C(\tau) = \lim_{T \to \infty} \frac{1}{L^2} \sum_{x_1=1}^{L} \sum_{y_1=1}^{L} (q(x) \cdot q(y)) \mid_{x_0=-y_0=\tau}. \tag{2.4}$$

If the bare coupling $f_0$ is fixed, the correlation function $C(\tau)$ converges to the vacuum expectation value of the 2-point function for spin field operators at large $T$ independently of whether one chooses free or periodic boundary conditions in the time direction. In perturbation theory, we are, however, first expanding in powers of the coupling $f_0$ and then let $T$ go to infinity. Therefore, we can not, in general, have such a converging behavior, and one requires a proper choice of boundary conditions to get the desired vacuum expectation value.

In free boundary conditions, we are projecting on states which are $O(n)$ invariant at large times. The energies $\varepsilon(f_0)$ in these states, on the other hand, have the property

$$\lim_{f_0 \to 0} \left\{ \varepsilon(f_0) - \varepsilon_0(f_0) \right\} > 0, \tag{2.5}$$

where $\varepsilon_0(f_0)$ denotes the ground state energy. Except for the ground state, all these states are therefore exponentially suppressed at large $T$ and we arrive at the desired vacuum expectation value even though we are first expanding in powers of $f_0$.

After having taken $T$ to infinity so that $C(\tau)$ been reduced to the vacuum expectation value of the product of two spin field operators, only the vector intermediate states contribute. The energy $\varepsilon_1(f_0)$ of the first excited state of this system has the property

$$\varepsilon_1(f_0) - \varepsilon_0(f_0) = O(f_0) \quad \text{for} \quad f_0 \to 0. \tag{2.6}$$

For the energies of the other higher excited states, eq.(2.5) is valid and their contribution to the spin correlation function hence vanishes exponentially at large $\tau$.

From this consideration, the mass gap $M(L)$ which is defined by the l.h.s of eq.(2.6) can be calculated by

$$M(L) = - \lim_{\tau \to \infty} \frac{1}{2} \frac{\partial}{\partial \tau} \ln C(\tau). \tag{2.7}$$

If one expands $C(\tau)$ in perturbation theory for $f_0 \to 0$, the mass gap has the general form

$$M(L) = 2L \sum_{\nu=1}^{\infty} f_0^{\nu} \Delta^{(\nu)}. \tag{2.8}$$

$\Delta^{(\nu)}$ can thus be determined by calculating the 2-point function [eq.(2.4)] with the mentioned boundary conditions in perturbation theory.
The calculations up to the third order in $f_0$ were already done by Lüsenher and Weisz \[22\]. Their results read

\[
\Delta^{(1)} = \Delta_0^{(1)}
\]

\[
\Delta^{(2)} = \Delta_0^{(2)} + \Delta_1^{(2)}[\ln L]
\]

\[
\Delta^{(3)} = \Delta_0^{(3)} + \Delta_1^{(3)}[\ln L] + \Delta_2^{(3)}[\ln L]^2
\]

where the coefficients are given by

\[
\Delta_0^{(1)} = \frac{n - 1}{n}
\]

\[
\Delta_0^{(2)} = -\frac{(n - 1)^2}{2\pi n^2} \left\{ \ln \frac{\pi}{\sqrt{2}} - \gamma \right\} + \frac{n - 1}{2\pi n^2} \left\{ \ln \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} - \gamma \right\}
\]

\[
\Delta_1^{(2)} = \frac{(n - 1)^2}{2\pi n^2} - \frac{n - 1}{2\pi n^2}
\]

\[
\Delta_0^{(3)} = \frac{n - 1}{n^3} \left\{ k_0 + (n - 1)k_1 + (n - 1)^2k_2 \right\}
\]

\[
\Delta_1^{(3)} = -\frac{(n - 1)^3}{(2\pi)^2 n^3} \left\{ 2\ln \frac{\pi}{\sqrt{2}} - 2\gamma \right\}
\]

\[
+ \frac{(n - 1)^2}{(2\pi)^2 n^3} \left\{ 4\ln \frac{\pi}{\sqrt{2}} + \pi - 4\gamma + 1 \right\}
\]

\[
- \frac{n - 1}{(2\pi)^2 n^3} \left\{ 2\ln \frac{\pi}{\sqrt{2}} + \pi - 2\gamma + 1 \right\}
\]

\[
\Delta_2^{(3)} = \frac{(n - 1)^3}{(2\pi)^2 n^3} - 2\frac{(n - 1)^2}{(2\pi)^2 n^3} + \frac{n - 1}{(2\pi)^2 n^3}
\]

with

\[
k_0 = 0.11141943681128(1)
\]

\[
k_1 = 0.00192740414869(1)
\]

\[
k_2 = \frac{1}{(2\pi)^2} \left( \ln \frac{\pi}{\sqrt{2}} - \gamma \right)^2
\]

\[\gamma\] denotes the Euler constant (\(\gamma = 0.577216 \cdots\)).

In order to determine the mass gap in the fourth order, we first evaluate $\Delta^{(4)}$ on finite lattices and then extrapolate to continuum limit, i.e., $L/a \to \infty$. The 2- and 3-loop diagrams contributing to $\Delta^{(4)}$ are illustrated in figures (8) and (9) which are presented in appendix A due to the very large number of the diagrams. In carrying out Wick’s contractions from them, we used the symbolic language MATHEMATICA.

We calculated the generated terms numerically. In the numerical computations, we take $T$ and $\tau$ large, but finite. The corrections to the limit $T, \tau \to \infty$ are here exponentially suppressed by the order of $e^{-\frac{\pi}{4\tau}}$ and $e^{-\frac{\pi}{4}(T-\tau)}$. We therefore achieve the best approximation to the limit $T, \tau \to \infty$ by keeping $T \simeq 3\tau$.\[\]
The numerical work, however, turned out to be very complicated due to the problems caused by rounding errors and running time which increases with $L$ very quickly (in the order of $L^6$). In addition, problems appear from the fact that on the 3-loop level on which we are working there are extremely many terms to treat. One needs therefore a very efficient computer program which requires, among others, the free propagator running fast without significant loss of precision. In this way, we succeeded in evaluating all diagrams up to $L = 20$ with good enough precision and reasonable running time (for $L = 20$ we needed the CPU time of around 7 days on the IBM RISC 6000/32H), so that we could extrapolate to the continuum limit to get our desired results.

With regard to the extrapolation to the continuum limit, we note that in this limit $\Delta^{(4)}$ has the general form

$$\Delta^{(4)} = \Delta_0^{(4)} + \Delta_1^{(4)}(\ln L) + \Delta_2^{(4)}(\ln L)^2 + \Delta_3^{(4)}(\ln L)^3$$

up to terms of the order $L^{-2}(\ln L)^3$ where the coefficients $\Delta_0^{(4)}, \ldots, \Delta_3^{(4)}$ are independent of $L$. All of these coefficients except $\Delta_0^{(4)}$ can be derived with help of the renormalization group equation and the results from the calculations of the mass gap at lower orders:

$$\Delta_1^{(4)} = \frac{3(n-1)^4}{(2\pi)n^4}k_2$$

$$+ \frac{(n-1)^3}{(2\pi)^3n^4}\left\{h_1 - 2\left(\ln \frac{\pi}{\sqrt{2}} - \gamma\right) + 3(2\pi)^2(k_1 - k_2)\right\}$$

$$+ \frac{(n-1)^2}{(2\pi)^3n^4}\left\{-2h_1 + h_2 + 4\left(\ln \frac{\pi}{\sqrt{2}} - \gamma\right) + \pi + 3(2\pi)^2(k_0 - k_1)\right\}$$

$$+ \frac{n-1}{(2\pi)^3n^4}\left\{h_1 - h_2 - 2\left(\ln \frac{\pi}{\sqrt{2}} - \gamma\right) - \pi - 3(2\pi)^2k_0\right\}$$

$$\Delta_2^{(4)} = -\frac{3(n-1)^4}{(2\pi)^3n^4}\left\{\ln \frac{\pi}{\sqrt{2}} - \gamma\right\}$$

$$+ \frac{(n-1)^3}{(2\pi)^3n^4}\left\{9\left(\ln \frac{\pi}{\sqrt{2}} - \gamma\right) + \frac{1}{2}(3\pi + 5)\right\}$$

$$- \frac{(n-1)^2}{(2\pi)^3n^4}\left\{3\left(3\ln \frac{\pi}{\sqrt{2}} + \pi - 3\gamma + 1\right) + 2\right\}$$

$$+ \frac{n-1}{(2\pi)^3n^4}\left\{1 + \frac{3}{2}\left(2\ln \frac{\pi}{\sqrt{2}} + \pi - 2\gamma + 1\right)\right\}$$

$$\Delta_3^{(4)} = \frac{(n-1)^4}{(2\pi)^3n^4} - \frac{3(n-1)^3}{(2\pi)^3n^4} + \frac{3(n-1)^2}{(2\pi)^3n^4} - \frac{n-1}{(2\pi)^3n^4}$$

In order to determine the unknown coefficient $\Delta_0^{(4)}$, we decompose it into $n$-independent constants:

$$\Delta_0^{(4)} = \frac{n-1}{n^4}\left\{s_0 + (n-1)s_1 + (n-1)^2s_2 + (n-1)^3s_3\right\}.$$
The coefficients $s_0, \cdots, s_3$ can then be determined by inserting $\Delta^{(4)}$ which was evaluated for different $L$’s in eq. (2.21) and extrapolating to continuum limit.

The extrapolation of $s_k$ ($k = 0, \cdots, 3$) to $L = \infty$ can be done very efficiently with help of the procedure by Lüscher and Weisz described in detail in [23]. We note that in our case the constants $s_k$ have corrections of the form

$$s_k^0(L) = s_k + \sum_{p=1}^{\infty} \left\{ a_p + b_p \ln L + c_p (\ln L)^2 + d_p (\ln L)^3 \right\}/L^{2p}. \quad (2.26)$$

Through the extrapolation in the region $5 \leq L \leq 20$, we obtain

$$s_0 = 0.03954(1) \quad (2.27)$$
$$s_1 = 0.02903(1) \quad (2.28)$$
$$s_2 = 0.000756(1) \quad (2.29)$$
$$s_3 = -0.000649(1) \quad (2.30)$$

At this point, we would like to emphasize that for all four constants $s_k$ the limit $L \to \infty$ exists. This is, of course, only the case if $\Delta^{(4)}$ calculated by us contains the terms diverging logarithmically with $L$, so that they cancel exactly with the other divergent factors of eq. (2.21) in a non-trivial way. This correct logarithmic behavior of $\Delta^{(4)}$ is a strong consistency check on our evaluations of the diagrams.

### 2.2 Conversion of the mass gap to the MS-scheme

The $\beta$-function in the MS-scheme of dimensional regularization is defined by

$$\beta(f) \equiv \mu \frac{\partial f}{\partial \mu} \bigg|_{f_0} = -f \sum_{\nu=1}^{\infty} b_{\nu} f^{\nu}, \quad (2.31)$$

where $f$ denotes the renormalized coupling and $\mu$ the normalization mass. The first two coefficients in (2.31) are scheme independent:

$$b_1 = \frac{n-2}{2\pi n} \quad (2.32)$$
$$b_2 = \frac{n-2}{(2\pi n)^2} \quad (2.33)$$

The remaining coefficients are known to four loops [24, 25, 26, 27]:

$$b_3 = \frac{n-2}{(2\pi n)^3} \left[ (n-2) \frac{1}{4} + 1 \right] \quad (2.34)$$
$$b_4 = \frac{n-2}{(2\pi n)^4} \left[ - (n-2)^2 \frac{1}{12} + (n-2)u_1 + u_2 \right] \quad (2.35)$$
with

\[ u_1 = \frac{3}{2}(v + 1) \]  
\[ u_2 = -\frac{1}{2}(3v - 1) \]

(2.36)  
(2.37)

where the constant \( v \) is given by \( v \approx 1.2020569 \).

The corresponding \( \Lambda \)-parameter is

\[ \Lambda_{MS} = \mu(b_1f)^{-b_2/b_1^2}e^{-1/(b_1f)} \cdot \lambda(f), \]

(2.38)

\[ \lambda(f) = \exp \left[ -\int_0^f dx \left( \frac{1}{\beta(x)} + \frac{1}{b_1x^2} - \frac{b_2}{b_1^2x} \right) \right]. \]

(2.39)

The function \( \lambda(f) \) is well-defined at \( f = 0 \) and can therefore be expanded as a power series in \( f \). Up to 4-loop, we find

\[ \Lambda_{MS} = \mu(b_1f)^{-b_2/b_1^2}e^{-1/(b_1f)} \left\{ 1 + \frac{b_3}{b_1^3}f + \frac{b_4}{2b_1^6}f^2 + \mathcal{O}(f^3) \right\}. \]

(2.40)

On the other hand, the \( \beta \) function on the lattice is defined by

\[ \hat{\beta}(f_0) \equiv -a \frac{\partial f_0}{\partial a} \bigg|_f = -f_0 \sum_{\nu=1}^\infty \hat{b}_\nu f_0^\nu. \]

(2.41)

The coefficients here are also known to four loops [28, 29, 30]:

\[ \hat{b}_1 = b_1 \]

(2.42)

\[ \hat{b}_2 = b_2 \]

(2.43)

\[ \hat{b}_3 = \frac{n-2}{(2\pi n)^3}[(n-2)h_1 + h_2] \]

(2.44)

\[ \hat{b}_4 = \frac{n-2}{(2\pi n)^4}[(n-2)^2t_1 + (n-2)t_2 + t_3] \]

(2.45)

with

\[ h_1 = -0.088766484(1) \]

(2.46)

\[ h_2 = 1 + \pi/2 - 5\pi^2/24 \]

(2.47)

\[ t_1 = -1.015(1) \]

(2.48)

\[ t_2 = -5.44(1) \]

(2.49)

\[ t_3 = -9.093756(4) \]

(2.50)

\[ ^2 \text{In this section, we introduce the lattice constant } a. \]
where the difficult 4-loop computation was performed recently by Caracciolo and Pelissetto [30].

The \( \Lambda \)-parameter on the lattice \( \Lambda_L \) can be obtained from \( \Lambda_{MS} \) of eq.(2.38) through the replacements \( \mu \rightarrow a^{-1} \), \( f \rightarrow f_0 \) and \( \beta(x) \rightarrow \hat{\beta}(x) \). The two \( \Lambda \)-parameters are related by the formula:

\[
\frac{\Lambda_L}{\Lambda_{MS}} = \exp \left\{ \frac{1}{2} \left( \ln \frac{\pi}{8} - \gamma - \frac{\pi}{n-2} \right) \right\}. \tag{2.51}
\]

From this, it follows the relation between the coupling constants in both schemes. Up to order \( \mathcal{O}(f^4) \), we find

\[
f_0 = f \left\{ 1 + \sum_{\nu=1}^{\infty} X^{(\nu)} f^{\nu} \right\} \tag{2.52}
\]

\[
X^{(1)} = \frac{n-2}{4\pi n} \left[ \ln \left( \frac{\pi}{8} a^2 \mu^2 \right) - \gamma \right] - \frac{1}{4n} \tag{2.53}
\]

\[
X^{(2)} = [X^{(1)}]^2 + \frac{1}{2\pi n} X^{(1)} + \frac{1}{(2\pi n)^2} \left[ (n-2) \left( h_1 - \frac{1}{4} \right) + h_2 - 1 \right] \tag{2.54}
\]

\[
X^{(3)} = [X^{(1)}]^3 + \frac{5}{4\pi n} [X^{(1)}]^2 + \frac{1}{(2\pi n)^2} \left[ (n-2) \left( 3h_1 - \frac{1}{2} \right) + 3h_2 - 2 \right] X^{(1)} + \frac{1}{(2\pi n)^3} \left[ (n-2)^2 \left( t_1 + \frac{1}{12} \right) + (n-2)(t_2 - u_1) + (t_3 - u_2) \right] \tag{2.55}
\]

Finally, we use this relation to express the mass gap [eqs. (2.9)-(2.11) and (2.21)] in terms of the renormalized coupling in MS-scheme. After a lengthy, but straightforward calculation we get

\[
M(L) = \frac{n-1}{2nL} f \left\{ 1 + \sum_{\nu=1}^{\infty} \kappa^{(\nu)} f^{\nu} \right\} \tag{2.56}
\]

\[
\kappa^{(1)} = \frac{n-2}{4\pi n} \left\{ \ln \frac{\mu^2 L^2}{4\pi} + \gamma \right\} \tag{2.57}
\]

\[
\kappa^{(2)} = [\kappa^{(1)}]^2 + \frac{1}{2\pi n} \kappa^{(1)} + 3 \frac{n-2}{(4\pi n)^2} \tag{2.58}
\]

\[
\kappa^{(3)} = [\kappa^{(1)}]^3 + \rho_1 \frac{(n-2)^3}{4\pi n} + \frac{5}{4\pi n} [\kappa^{(1)}]^2 + \rho_2 \frac{2(n-2)}{(4\pi n)^2} \kappa^{(1)}
+ \rho_3 \frac{(n-2)^2}{3 (4\pi n)^3} + \rho_4 \frac{1}{(4\pi n)^2} \kappa^{(1)} + \rho_5 \frac{n-2}{(4\pi n)^3} + \rho_6 \frac{1}{(4\pi n)^3} \tag{2.59}
\]

The expressions for the coefficients \( \rho_1, \ldots, \rho_6 \) of \( \kappa^{(3)} \) are rather long; we therefore write them in appendix B. Here, the coefficients \( \kappa^{(1)} \) and \( \kappa^{(2)} \) are the result of L"uscher and Weisz [22]. By using dimensional regularization, \( \kappa^{(2)} \) was first calculated by Floratos and Petcher [21] with less numerical precision. Our result for \( \kappa^{(3)} \) is the extension of their 1- and 2-loop computations to 3-loop.
With dimensional regularization, \( \kappa^{(n)} = 0 \) for \( n = 2 \) since we formally have a free theory in this case. \( \kappa^{(1)} \) and \( \kappa^{(2)} \) already satisfy this condition. We checked that \( \kappa^{(3)} \) is also zero within the errors, which shows a further check on the correctness of our final results in the lattice calculations, \( s_0, s_1, s_2 \) and \( s_3 \) [eqs. (2.27) - (2.30)]. This is, however, not only a check on our computations, but also that on the 4-loop coefficient of the \( \beta \) function on the lattice \( \hat{b}_4 \) which was introduced in our calculations by the conversion of the result on the lattice to the MS-scheme.

### 2.3 Expansion in \( z \)

In view of the evaluation of the finite volume mass gap \( M(L) \) in the infinite volume limit, we introduce the variable

\[
z = M(L)L
\]

and express the ratio \( M(L)/\Lambda_{\overline{\text{MS}}} \) in terms of this dimensionless and renormalization group invariant parameter in perturbation theory for \( z \to 0 \), as suggested by L"uscher [13]. Here, \( \Lambda_{\overline{\text{MS}}} \) is related with \( \Lambda_{\text{MS}} \) of eq. (2.38) through

\[
\Lambda_{\overline{\text{MS}}} = \Lambda_{\text{MS}} \exp \left[ \frac{1}{2} (\ln 4\pi - \gamma) \right].
\]

(2.61)

For that purpose, we invert at first the \( f \)-expansion of eq. (2.56) in the \( z \)-expansion:

\[
f = d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \mathcal{O}(z^5), \tag{2.62}
\]

where

\[
d_1 = \frac{2n}{n - 1} \tag{2.63}
\]

\[
d_2 = -d_1^2 \kappa^{(1)} \tag{2.64}
\]

\[
d_3 = -[2d_1 d_2 \kappa^{(1)} + d_1^3 \kappa^{(2)}] \tag{2.65}
\]

\[
d_4 = -[(d_2^2 + 2d_1 d_3) \kappa^{(1)} + 3d_1^2 d_2 \kappa^{(2)} + d_1^4 \kappa^{(3)}] \tag{2.66}
\]

The insertion of this series in eq. (2.38), together with eq. (2.61), yields finally

\[
\frac{M(L)}{\Lambda_{\overline{\text{MS}}}} = e^\gamma \left( \frac{n - 2}{n - 1} \right)^{\frac{1}{n - 2}} \left( \frac{\pi e}{n - 1} \right)^{\frac{n - 1}{n - 2}} \left\{ 1 + \sum_{\nu=1}^{\infty} a_\nu z^\nu \right\} \tag{2.67}
\]

with

\[
a_1 = \frac{1}{\pi (n - 1)} \tag{2.68}
\]

\[
a_2 = \frac{1}{24\pi^2 (n - 1)^2 (n - 2)} (a_1 n^3 + a_2 n^2 + a_3 n + a_4) \tag{2.69}
\]
The expressions for the constants $\alpha_1, \ldots, \alpha_4$ are given in appendix B. We note that the coefficients $a_\nu$ in eq. (2.67) are independent of $\mu$ and $L$ since $M(L)$, $\Lambda_{\text{MS}}$ and $z$ are renormalization group invariants. In the above result, the leading term and the first coefficient $a_1$ were calculated in ref. [22], and our determination of $a_2$ is the 3-loop correction to their calculations.

2.4 Definition of running coupling $\bar{g}^2(L)$

Observing that the mass gap $M(L)$ in finite volume [eq.(2.56)] is proportional to $f$ in the leading order, Lüscher, Weisz and Wolff [1] defined a coupling running with $L$ by

$$\bar{g}^2(L) = 2nM(L)L/(n - 1).$$

We can then expand it, for $\mu = 1/L$, in the coupling of the MS-scheme:

$$\bar{g}^2(L) = f + c_1 f^2 + c_2 f^3 + c_3 f^4 + \mathcal{O}(f^5),$$

where

$$c_1 = -\frac{n - 2}{4\pi n} [\ln(4\pi) - \gamma] \quad (2.72)$$

$$c_2 = c_1^2 + \frac{c_1}{2\pi n} + \frac{3(n - 2)}{(4\pi n)^2} \quad (2.73)$$

$$c_3 = c_1^3 + c_2 \left( \frac{n - 2}{4\pi n} \right)^3 + \frac{5}{4\pi n} c_1^2 + c_2 \left( \frac{2(n - 2)}{(4\pi n)^2} \right) c_1$$

$$+ \rho_3 \left( \frac{1(n - 2)^2}{3(4\pi n)^3} + \rho_1 \frac{1}{(4\pi n)^2} c_1 + \rho_2 \frac{n - 2}{(4\pi n)^3} \right) \quad (2.74)$$

For the coefficients of $\beta$-function

$$\bar{\beta}(g^2) \equiv -L \frac{\partial g^2}{\partial L} = -g^2 \sum_{l=1}^{\infty} \bar{b}_l (\bar{g}^2)^l \quad (2.75)$$

we find

$$\bar{b}_1 = b_1 \quad (2.76)$$

$$\bar{b}_2 = b_2 \quad (2.77)$$

$$\bar{b}_3 = \frac{(n - 1)(n - 2)}{(2\pi n)^3} \quad (2.78)$$

$$\bar{b}_4 = \frac{1}{4} \frac{n - 2}{(2\pi n)^4} [(n - 2)^3 \chi_1 + (n - 2)^2 \chi_2 + (n - 2) \chi_3 + \chi_4] \quad (2.79)$$

where we have used the $\beta$-function in MS-scheme [eqs.(2.31)-(2.35)]. The expressions for the constants $\chi_1, \ldots, \chi_4$ are listed in appendix B.
The corresponding $\Lambda$-parameter is given by

\[
\Lambda_{FV} = \frac{1}{L} (b_1 \bar{g}^2)^{-b_2/b_1^2} e^{-1/(b_1 \bar{g}^2)} \cdot \tilde{\lambda}(\bar{g}^2),
\]
(2.80)

\[
\tilde{\lambda}(\bar{g}^2) = \exp\left[ -\int_0^{\bar{g}^2} dx \left( \frac{1}{\beta(x)} + \frac{1}{b_1 x^2} - \frac{b_2}{b_1^2 x^2} \right) \right].
\]
(2.81)

From the relation between two couplings in eq.(2.71), we can derive the ratio of $\Lambda$-parameters in both schemes:

\[
\frac{\Lambda_{FV}}{\Lambda_{MS}} = \frac{e^{\frac{1}{2} \gamma}}{2 \sqrt{\pi}}.
\]
(2.82)

3 Results and Discussions

In this chapter, we determine the mass gap $m$ in units of the $\Lambda$-parameter by applying our results in the last chapter. For this, the following two approaches will be considered:

- Determination of the ratio $m/\Lambda_{MS}$ by matching the behavior of the finite volume mass gap $M(L)$ evaluated at small $L$ with that of the mass shift $\delta_0 = [M(L) - m]/m$ known at large $L$
- Determination of the ratio $\Lambda_{FV}/m$ by means of running coupling $\bar{g}^2(L)$

We discuss these two methods in detail and give the results.

After that, it is interesting to compare them with the analytic determination performed by Hasenfratz, Maggiore and Niedermayer a few years ago [2, 3]. Their result in the general $O(n)$ model has the following simple form:

\[
m = \left( \frac{8}{e} \right)^{1/(n-2)} \frac{1}{\Gamma[1 + 1/(n-2)]} \Lambda_{MS}.
\]
(3.1)

In our work, we will need the values in the $O(3)$ and $O(4)$ models:

\[
m = \left( \frac{8}{e} \right) \Lambda_{MS} \quad \text{for} \quad O(3)
\]
(3.2)

\[
m = \left( \frac{32}{\pi e} \right)^{1/2} \Lambda_{MS} \quad \text{for} \quad O(4)
\]
(3.3)

By using the equations (2.82) and (2.61), these formulas can be rewritten in terms of $\Lambda_{FV}$ defined in eq.(2.80) in connection with the running coupling $\bar{g}^2(L)$:

\[
\frac{\Lambda_{FV}}{m} = \frac{e^{\gamma+1}}{32 \pi} \quad \text{for} \quad O(3)
\]
(3.4)

\[
\frac{\Lambda_{FV}}{m} = \frac{1}{16} \sqrt{e} \cdot e^\gamma \quad \text{for} \quad O(4)
\]
(3.5)
3.1 Determination of $m/\Lambda_{\overline{MS}}$ by applying $\delta_0$

Our aim is to determine the mass gap $m$ in infinite volume by applying our result of the mass gap $M(L)$ in finite volume [eq.(2.67)]. For this purpose, we introduce two quantities $C_0(z)$ and $c_0$ by

$$M(L) = C_0(z) \Lambda_{\overline{MS}}$$

(3.6)

$$m = c_0 \Lambda_{\overline{MS}}$$

(3.7)

where $m$ is obtained from $M(L)$ by $m = \lim_{L\to\infty} M(L)$.

The problem is then the determination of $c_0$ which may be done by attempting to extrapolate $C_0(z)$ to the infinite volume limit, i.e. to $z \to \infty$. Inspired by the successful application of the method in the $O(n)$ model at $n = \infty$ where one can solve the given problem exactly, Lüscher [19] tried, with his 1-loop calculation of $C_0(z)$, to estimate $c_0$ in the $O(3)$ model as the value of $C_0(z)$ at some intermediate region of $z$ (at around $3 < z < 4$) with the hope that in this region the perturbative $C_0(z)$ would be a good approximation for the infinite volume. Here, we extend his 1-loop calculation to up to 3-loop which is plotted in fig. 1 as a function of $z$ in the $O(3)$ model. The lowest curve is the 1-loop result by Lüscher while the middle one comes from the 2-loop computation by Floratos and Petcher [21]. Finally, the uppermost curve shows our 3-loop result. The expected behavior of the curves, supported by the study in the $O(\infty)$ model, is that they rapidly decrease at small $z$ and then quickly become flat. From the figure we, however, see that it is difficult to make a clear estimation for $C_0(\infty)$ in this way.

For a better estimation of $c_0$, we make use of the mass shift defined by

$$\delta_0 = \frac{M(L) - m}{m}$$

(3.8)

as done in ref. [21]. $\delta_0$ was first introduced by Lüscher [20] and can be calculated at large $\zeta = mL$ exactly up to an exponentially small correction term if the elastic forward scattering amplitude is known. In the $O(n)$ $\sigma$-model, the exact scattering matrix was determined by brothers Zamolodchikov [13, 14]. In particular, the forward scattering amplitude is known and hence one can calculate $\delta_0$ in this model at large $\zeta$.

By using the definition of $\delta_0$ together with eq.(3.7), we obtain the following relation between $C_0(z)$ and $c_0$:

$$C_0 = (1 + \delta_0)c_0.$$ 

(3.9)

From this, a formula expressing $c_0$ as a function of $\zeta$ follows:

$$c_0(\zeta) = [1 + \delta_0(\zeta)]^{-1} C_0[z(\zeta)].$$

(3.10)

$^3\zeta$ is related with $z$ by $z = (1 + \delta_0)\zeta$ so that $z \simeq \zeta$ at large $\zeta$. 

Actually, $c_0$ is independent of $\zeta$. Therefore, it must be valid for all $\zeta$: $C_0[z(\zeta)] \sim 1 + \delta_0(\zeta)$ if $C_0$ and $\delta_0$ are known exactly. However, we know the two quantities only approximately; $C_0$ for small $z$ and $\delta_0$, on the other hand, for large $\zeta$. Although in this approximate relation $c_0$ is in general dependent on $\zeta$, there may exist the possibility that one finds somewhere an overlapping region where both approximations are good and thus $c_0$ becomes independent of $\zeta$.

This idea works very well in the model for $n = \infty$, as shown in ref. \cite{21}. We therefore apply it to the determination of $c_0$ in the O($n$) model, explicitly in the O(3) and O(4) models. In the O(3) model, $\delta_0$ was calculated by Lüscher \cite{20}:

$$
\delta_0 = 4\pi \int_{-\infty}^{\infty} dt \, e^{-\zeta \cosh t} \frac{\cosh t}{t^2 + \frac{\kappa}{4}\pi^2} + O(e^{-\kappa \zeta}), \quad (3.11)
$$

where $\kappa$ is a constant which is not smaller than $\sqrt{\frac{3}{2}}$ and may become as large as 3. If we now insert this quantity together with $C_0(z)$ at $n = 3$ in eq.(3.10), we obtain a functional relation of $c_0$ in terms of $\zeta$ which is illustrated in fig. 2.

The 1-loop result displays a region where $c_0 \simeq 1.6$. This value was the earlier estimation by Lüscher by this method \cite{19,20}. There, Lüscher gave an optimistic error of 20 % which he apparently underestimated. The 2-loop result corrects this
by about 30% to \( c_0 \simeq 2.15 \). This is in agreement with the estimation by Floratos and Petcher who give \( c_0 \simeq 2.1 \) \cite{24}. Finally, we see from our 3-loop result that the curve rises rather monotonically instead of showing a wider flat area. If this method should work well, we would expect that the flat area at the 3-loop becomes wider than at the 2-loop. We attribute the bad applicability of this method to the fact that the convergence radius of \( C_0(z) \) is apparently too small. Nevertheless, we conservatively estimate the 3-loop approximation to \( c_0 \) by \( c_0 \simeq 2.3 \). We note that it is very difficult to give a systematic error on our estimation.

Our determination is to be compared with that by Hasenfratz et al. which is \( c_0 = 2.94304 \cdot \cdot \cdot \) [see eq. (3.2)]. The value of \( c_0 \simeq 2.3 \) at the 3-loop still deviates noticeably from that by Hasenfratz et al.. One does not, however, need to be disturbed because, from the convergence of the 1-, 2- and 3-loop approximations, one sees a definite indication that the results in higher orders would arrive at that of Hasenfratz et al..

Now, we investigate the O(4) model and see how the picture changes in this model. The mass shift here has the form

\[
\delta_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \cosh t e^{-\zeta \cosh t} \left\{ 4 - \left[ \frac{2\pi t}{t^2 + \frac{\pi}{4}} B\left( \frac{3}{4} \right) - \frac{it}{2\pi} \right] \right\} + O(e^{-\lambda \zeta}), \tag{3.12}
\]
Figure 3: $c_0$ as a function of $\zeta$ in the O(4) model. The lowest curve contains for $C_0(z)$ only the 1-loop contribution, the middle one the contributions up to 2-loop and the uppermost curve up to 3-loop.

where $B$ denotes the beta function and $\lambda$ is a constant whose value lies in the same region as $\kappa$ in eq.(3.11). We again insert this quantity together with $C_0(z)$ at $n = 4$ in eq.(3.10) to get the functional relation of $c_0$ in terms of $\zeta$. This relation is plotted in fig. 3.

The 1-loop result shows a stable region in the neighborhood $\zeta = 3$, from which one may estimate $c_0 \simeq 1.5$. If one, however, looks at the curves of the higher loops, they become steeper and rise rather monotonically, which makes it difficult to estimate the desired value. We again see, as in the O(3) model, this method does not work very well. However, a remarkable information is contained in the estimations of $c_0$ from the curves; by going to the higher loops, they are tending to converge to the value by Hasenfratz et al. of $c_0 = 1.93577 \cdots$ [see eq.(3.3)].

From our determinations of the mass gap $m$, we conclude that in both O(3) and O(4) models our results support the correctness of those by Hasenfratz et al..

### 3.2 Determination of $\Lambda_{FV}/m$ by applying $\bar{g}^2(L)$

In chapter 2, we introduced a coupling $\bar{g}^2(L)$ running with $L$. There, the coefficients of the corresponding $\beta$-function $\tilde{\beta}(\bar{g}^2)$ were calculated up to 4-loop order [eqs.(2.76)–
Table 1: $\bar{g}^2(L)$ in the O(3) model as a function of $L$ in units of $m$.

| $mL$   | $\bar{g}^2(L)$ |
|--------|---------------|
| 0.0019(3) | 0.5372 |
| 0.0038(3) | 0.5747 |
| 0.0063(4) | 0.6060 |
| 0.0127(4) | 0.6553 |
| 0.0211(5) | 0.7383 |
| 0.0327(4) | 0.7646 |
| 0.0422(4) | 0.8166 |
| 0.05557(13) | 1.2680 |

We make use of these perturbative coefficients to evaluate the $\Lambda$-parameter $\Lambda_{FV}$ in this scheme in perturbation theory.

In order to obtain the perturbative expressions for $\Lambda_{FV}$, we expand $\tilde{\lambda}(\bar{g}^2)$ of eq.(2.81) in $\bar{g}^2$. Up to order $O(\bar{g}^4)$, we find

$$\Lambda_{FV}^{(2)} = \frac{1}{L} (b_1 \bar{g}^2 - b_2/b_1^2 e^{-1/(b_1 \bar{g}^2)})$$

(3.13)

$$\Lambda_{FV}^{(3)} = \frac{1}{L} (b_1 \bar{g}^2 - b_2/b_1^2 e^{-1/(b_1 \bar{g}^2)}) \left\{ 1 + \frac{b_3^2}{b_1^3 \bar{g}^2} \right\}$$

(3.14)

$$\Lambda_{FV}^{(4)} = \frac{1}{L} (b_1 \bar{g}^2 - b_2/b_1^2 e^{-1/(b_1 \bar{g}^2)}) \left\{ 1 + \frac{b_3^2}{b_1^3 \bar{g}^2} \right\} + \frac{b_4^2}{2b_1^3} - \frac{b_2^2}{b_1^2} + 2b_1 b_2 \bar{b}_3 - 2b_1 b_2 \bar{b}_3 + b_1^2 \bar{b}_3^2 - b_1^4 \bar{b}_4^2$$

(3.15)

where $\Lambda_{FV}^{(2)}$, $\Lambda_{FV}^{(3)}$, and $\Lambda_{FV}^{(4)}$ represent the 2-, 3- and 4-loop approximation to $\Lambda_{FV}$ respectively. In the limit $L \to 0$ where the coupling $\bar{g}^2(L)$ also goes to zero, they converge to the exact

$$\Lambda_{FV} = \lim_{L \to 0} \frac{1}{L} (b_1 \bar{g}^2 - b_2/b_1^2 e^{-1/(b_1 \bar{g}^2)})$$

(3.16)

with the rates $\bar{g}^2(L)$, $\bar{g}^4(L)$ and $\bar{g}^6(L)$ respectively.

For determination of these perturbative $\Lambda$-parameters, the values of the running coupling $\bar{g}^2(L)$ are needed. In ref. [1], they were measured numerically by Lüscher, Weisz and Wolff through a finite size technique for given $L$ in units of the mass gap $m$ in the O(3) model. Furthermore, Hasenbusch made available to me his new, more precise data [31] which contain the measurements in the smaller region of the coupling. Their results are listed in table 1. In deriving the numerical values
of the table, the extrapolation to the continuum limit was done by fitting with a polynomial in \((a/L)^2\) which was supposed by Symanzik \[32\].

If we insert the numbers of table 1 in eqs.\((3.13)-(3.15)\), we get \(\Lambda_{FV}/m\) in 2-, 3- and 4-loop approximations. The results are listed in table 2 and also illustrated in fig. 4. In the figure, we also draw the analytic value of Hasenfratz et al. for comparison. We see in the figure the difference between the 3- and 4-loop results is very small. Nevertheless, our 4-loop result lies between the 2- and 3-loop ones, so that by consideration of the higher order for \(\Lambda_{FV}\) the ratio \(\Lambda_{FV}/m\) lies nearer to the analytic value. The remarkable fact, however, is that the 3- and 4-loop curves which, with the old data by Lüscher et al. alone, seemed to fall down continuously again rise together to the compared value if one adds the new data for smaller coupling by Hasenbusch beginning from \(\bar{g}^2(L) = 0.6970\). We can thus expect very well that in the limit \(\bar{g}^2(L) \to 0\) the perturbative \(\Lambda_{FV}/m\) would converge to the result of Hasenfratz et al..

To give a further impression on the systematic errors, we consider an alternative definition of \(n\)-loop approximations to \(\Lambda_{FV}\); instead of expanding \(\tilde{\lambda}(\bar{g}^2)\) in \(\bar{g}^2\), we insert the perturbative coefficients of \(\tilde{\beta}(x)\) in \(\tilde{\lambda}(\bar{g}^2)\) and integrate this exactly. If we apply the coefficients calculated up to 4-loop order, we obtain the following approximations to \(\Lambda_{FV}\) at 2-, 3- and 4-loop levels which are a little modified from eqs.\((3.13)-(3.15)\):

\[
\bar{\Lambda}_{FV}^{(2)} = \frac{1}{L}(b_1 \bar{g}^2)^{-b_2/b_1} e^{-1/(b_1 \bar{g}^2)} \cdot \exp \left[ - \int_0^{\bar{g}^2} dx \left( \frac{1}{x^2(b_1 + b_2 x)} + \frac{1}{b_1 x^2} - \frac{b_2}{b_1^2 x} \right) \right] \tag{3.17}
\]

\[
\bar{\Lambda}_{FV}^{(3)} = \frac{1}{L}(b_1 \bar{g}^2)^{-b_2/b_1} e^{-1/(b_1 \bar{g}^2)} \cdot \exp \left[ - \int_0^{\bar{g}^2} dx \left( \frac{1}{x^2(b_1 + b_2 x + b_3 x^2)} + \frac{1}{b_1 x^2} - \frac{b_2}{b_1^2 x} \right) \right] \tag{3.18}
\]

\[
\bar{\Lambda}_{FV}^{(4)} = \frac{1}{L}(b_1 \bar{g}^2)^{-b_2/b_1} e^{-1/(b_1 \bar{g}^2)} \cdot \exp \left[ - \int_0^{\bar{g}^2} dx \left( \frac{1}{x^2(b_1 + b_2 x + b_3 x^2 + b_4 x^3)} + \frac{1}{b_1 x^2} - \frac{b_2}{b_1^2 x} \right) \right] \tag{3.19}
\]

Insertion of the data of table 1 in these equations gives the numbers in table 3 which are also plotted in fig. 4. Apart from the fact that the difference between the 3- and 4-loop results is a little larger, we have here almost the same picture as in fig. 4. Although the 4-loop result lies below the 3-loop one so that it deviates a little more from the value of Hasenfratz et al., there is no cause for alarm since both curves approach the analytic value together. From our study through the alternative definition of the perturbative \(\Lambda_{FV}\), we therefore conclude that our determination of \(\Lambda_{FV}/m\), as in the first case, agrees with that by Hasenfratz et al. very well.

In ref. \[33\], Hasenbusch and Horgan measured the running coupling \(\bar{g}^2(L)\) as a function of \(L\) in units of the mass gap \(m\) in the O(4) model by using the finite-size
| $\bar{g}^2(L)$ | $\Lambda_{FV}^{(2)}/m$ | $\Lambda_{FV}^{(3)}/m$ | $\Lambda_{FV}^{(4)}/m$ |
|--------------|-------------------|-------------------|-------------------|
| 0.5372       | 0.0511(8)         | 0.0467(6)         | 0.0467(17)        |
| 0.5747       | 0.0512(7)         | 0.0465(6)         | 0.0466(17)        |
| 0.6060       | 0.0514(7)         | 0.0464(6)         | 0.0465(15)        |
| 0.6553       | 0.0518(6)         | 0.0464(5)         | 0.0464(15)        |
| 0.6970       | 0.0520(6)         | 0.0462(5)         | 0.0463(15)        |
| 0.7383       | 0.0525(6)         | 0.0463(5)         | 0.0463(13)        |
| 0.7646       | 0.0526(6)         | 0.0462(5)         | 0.0463(13)        |
| 0.8166       | 0.0532(6)         | 0.0463(5)         | 0.0463(11)        |
| 0.9176       | 0.0552(5)         | 0.0471(5)         | 0.0472(10)        |
| 1.0595       | 0.0574(4)         | 0.0478(3)         | 0.0479(8)         |
| 1.2680       | 0.0628(1)         | 0.0502(2)         | 0.0503(8)         |

Table 2: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(3) model

Figure 4: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(3) model from table 2. The triangular points show the 2-loop approximation to the $\Lambda$-parameter, the hexagonal points the 3-loop and the circular points the 4-loop approximation. The dotted line is the value of Hasenfratz et al. ($\Lambda_{FV}/m = 0.04815 \cdots$).
Table 3: Alternative 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(3) model

| $\bar{g}^2(L)$ | $\Lambda_{FV}^{(2)}/m$ | $\Lambda_{FV}^{(3)}/m$ | $\Lambda_{FV}^{(4)}/m$ |
|-----------------|------------------------|------------------------|------------------------|
| 0.5372          | 0.0555(8)              | 0.0474(6)              | 0.0468(17)             |
| 0.5747          | 0.0559(7)              | 0.0473(6)              | 0.0466(17)             |
| 0.6060          | 0.0563(7)              | 0.0473(6)              | 0.0466(15)             |
| 0.6553          | 0.0572(6)              | 0.0474(5)              | 0.0465(15)             |
| 0.6970          | 0.0578(6)              | 0.0474(5)              | 0.0464(15)             |
| 0.7383          | 0.0586(6)              | 0.0476(5)              | 0.0465(23)             |
| 0.7646          | 0.0590(6)              | 0.0476(5)              | 0.0465(13)             |
| 0.8166          | 0.0601(6)              | 0.0479(5)              | 0.0466(11)             |
| 0.9176          | 0.0633(5)              | 0.0492(5)              | 0.0476(10)             |
| 1.0595          | 0.0671(4)              | 0.0505(3)              | 0.0484(8)              |
| 1.2680          | 0.0755(1)              | 0.0544(2)              | 0.0513(8)              |

Figure 5: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(3) model from table 3. The arrangement of the points and dotted line is same as in fig. [4].
Table 4: $\bar{g}^2(L)$ in the O(4) model as a function of $L$ in units of $m$

| $mL$ | $\bar{g}^2(L)$ |
|------|----------|
| 1/8  | 0.863(2) |
| 1/4  | 1.011(2) |
| 1/2  | 1.228(2) |
| 1    | 1.584(4) |
| 2    | 2.309(10) |
| 4    | 4.132(10) |

scaling analysis of Lüscher et al. [1]. We make use of the measurements to determine the ratio $\Lambda_{FV}/m$ in this model. Their data for the running coupling are given in table 4. If we apply the numbers in the table to the eqs. (3.13)-(3.15), we obtain $\Lambda_{FV}/m$ in 2-, 3- and 4-loop approximations. They are written in table 5 and also illustrated in fig. 6. The figure shows very good agreement between our result and the one by Hasenfratz et al.. Already at $\bar{g}^2(L) = 1.584$, $\Lambda_{FV}/m$ at 4-loop level is almost the same as the analytic value.

As in the O(3) model, by inserting the data of table 4 in the eqs. (3.17)-(3.19), we also determine $\Lambda_{FV}/m$ by means of the modified definition of the perturbative $\Lambda_{FV}$. The results are written in table 6 and also plotted in fig. 7. Apart from the fact that the curves for the 3- and 4-loop approximations in the figure changed their places, we see here the same picture as in fig. 6. The agreement of two values at a little smaller $\bar{g}^2(L)$ is irrelevant from the fact that we are at the end interested in the determination of $\Lambda_{FV}/m$ at the limit $\bar{g}^2(L) \to 0$.

As a whole, we conclude that not only in the O(3) but also in the O(4) model our determinations of $\Lambda_{FV}/m$ agree with those by Hasenfratz et al. very well.

**Acknowledgement**

I thank Peter Weisz for suggesting this work and many useful discussions. He also read through this manuscript, which is greatly acknowledged as well. I express my thanks also to Martin Hasenbusch for having made available to me his unpublished numerical data.
Table 5: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(4) model

| $g^2(L)$ | $\Lambda_{FV}^{(2)}/m$ | $\Lambda_{FV}^{(3)}/m$ | $\Lambda_{FV}^{(4)}/m$ |
|----------|------------------------|------------------------|------------------------|
| 0.863    | 0.0795(7)              | 0.0722(7)              | 0.0725(16)             |
| 1.011    | 0.0817(7)              | 0.0729(6)              | 0.0733(15)             |
| 1.228    | 0.0844(5)              | 0.0734(6)              | 0.0740(12)             |
| 1.584    | 0.0880(5)              | 0.0733(6)              | 0.0742(10)             |
| 2.309    | 0.0928(4)              | 0.0701(4)              | 0.0723(9)              |
| 4.132    | 0.0853(4)              | 0.0479(4)              | 0.0544(9)              |

Figure 6: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(4) model from table 5. The arrangements of the points is same as in fig. 4. The dotted line is the value of Hasenfratz et al. ($\Lambda_{FV}/m = 0.07321 \ldots$).
Table 6: Alternative 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(4) model

| $\bar{g}^2(L)$ | $\Lambda^{(2)}_{FV}/m$ | $\Lambda^{(3)}_{FV}/m$ | $\Lambda^{(4)}_{FV}/m$ |
|----------------|------------------------|------------------------|------------------------|
| 0.863          | 0.0830(7)              | 0.0733(7)              | 0.0726(16)             |
| 1.011          | 0.0859(7)              | 0.0744(6)              | 0.0734(15)             |
| 1.228          | 0.0898(5)              | 0.0757(6)              | 0.0743(12)             |
| 1.584          | 0.0952(5)              | 0.0771(6)              | 0.0749(10)             |
| 2.309          | 0.1035(4)              | 0.0780(4)              | 0.0741(9)              |
| 4.132          | 0.1024(4)              | 0.0678(4)              | 0.0609(9)              |

Figure 7: 2-, 3- and 4-loop approximations to $\Lambda_{FV}/m$ in the O(4) model from table 6. The arrangement of the points and dotted line is same as in fig. 6.
A Feynman diagrams

We list here the 2- and 3-loop Feynman diagrams contributing to the mass gap in fourth order. In the diagrams, the internal dot denotes the vertices while the cross means those coming from “$\pi^2$ insertion.”

Figure 8: 2-loop diagrams
Figure 9: 3-loop diagrams
Figure 9: 3-loop diagrams (continued)
Figure 9: 3-loop diagrams (continued)
B Expressions of Coefficients

In this appendix, we show the expressions of the coefficients defined in the main text.

B.1 Expressions of the coefficients appearing in eq. (2.59)

\[
\rho_1 = \left( \ln \frac{\pi^2}{2} - 2\gamma \right)^3 + (4\pi)^3 s_3 \\
\rho_2 = 3 \left( \ln \frac{\pi^2}{2} \right)^2 + 3(\pi - 4\gamma + 1) \ln \frac{\pi^2}{2} - 6\gamma(\pi + 1) \\
\quad + 24k_1\pi^2 + 6h_1 + 12\gamma^2 - 1 \\
\rho_3 = 18 \left( \ln \frac{\pi^2}{2} \right)^3 + 3(3\pi - 36\gamma + 1) \left( \ln \frac{\pi^2}{2} \right) \left( \ln \frac{\pi^2}{2} \right)^2 \\
\quad + 12 \left[ 12k_1\pi^2 + h_1 + 18\gamma^2 - \gamma(3\pi + 1) \right] \ln \frac{\pi^2}{2} \\
\quad + 192\pi^3(s_2 + 3s_3) - 144\gamma^3 + 12\gamma^2(3\pi + 1) \\
\quad - 24\gamma(12\pi^2k_1 + h_1) + 12t_1 + 1 \\
\rho_4 = 3 \left( \ln \frac{\pi^2}{2} - 2\gamma \right)^2 + 3(16k_0 + 16k_1 - 1)\pi^2 + 12h_2 - 6\pi - 8
\]
\[
\rho_5 = 3 \left( \ln \frac{\pi^2}{2} \right)^3 - 6(3\gamma + \pi) \left( \ln \frac{\pi^2}{2} \right)^2 \\
+ 2 \left[ 3(8k_0 + 8k_1 - 1)\pi^2 + (12\gamma - 1)\pi + 2h_2 + 18\gamma^2 \right] \ln \frac{\pi^2}{2} \\
+ 16\pi^3(4s_1 + 8s_2 + 12s_3 - 3k_1) - 12\gamma\pi^2(8k_0 + 8k_1 - 1) \\
- 4\pi (h_1 + 6\gamma^2 - \gamma) + 4(t_2 - u_1 - 2\gamma h_2 - 6\gamma^3)
\]
\[
\rho_6 = (4\pi^3(s_0 + s_1 + s_2 + s_3) - 48\pi^3(k_0 + k_1 + k_2) \\
+ 2\pi^3 + \pi^2 - 4\pi h_2 + 4(t_3 - u_2)
\]

**B.2 Expressions of the coefficients appearing in eq. (2.69)**

\[
\alpha_1 = 3 \left[ \left( \ln \frac{\pi^2}{2} - 2\gamma \right)^3 + (4\pi)^3 s_3 \right]
\]
\[
\alpha_2 = -3 \left[ 6 \left( \ln \frac{\pi^2}{2} \right)^3 + (3\pi - 36\gamma + 5) \left( \ln \frac{\pi^2}{2} \right)^2 \\
+ 4 \left( 18\gamma^2 + 2h_1 - 3\gamma\pi - 5\gamma - 3 \right) \ln \frac{\pi^2}{2} \\
+ 4 \left( 3\gamma^2\pi - 12\gamma^3 + 5\gamma^2 - 4\gamma h_1 + 6\gamma - 16\pi^3 s_3 + 48\pi^3 s_3 - t_1 \right) \right]
\]
\[
\alpha_3 = 2 \left[ 18 \left( \ln \frac{\pi^2}{2} \right)^3 + 6(3\pi - 18\gamma + 5) \left( \ln \frac{\pi^2}{2} \right)^2 \\
+ \left( 7\pi^2 - 9\pi(8\gamma - 1) + 216\gamma^2 - 120\gamma + 48h_1 - 66 \right) \ln \frac{\pi^2}{2} \\
+ 96\pi^3(s_1 - 2s_2 + 3s_3) - 14\gamma\pi^2 + 6\pi(12\gamma^2 - 3\gamma + 2h_1 - 3) \\
- 6(16\gamma h_1 + 24\gamma^3 - 20\gamma^2 - 22\gamma + 4t_1 - t_2 + 1) \right]
\]
\[
\alpha_4 = -4 \left[ 6 \left( \ln \frac{\pi^2}{2} \right)^3 + 3(3\pi - 12\gamma + 5) \left( \ln \frac{\pi^2}{2} \right)^2 \\
+ \left( 7\pi^2 - 9\pi(4\gamma - 1) + 72\gamma^2 - 60\gamma + 24h_1 - 30 \right) \ln \frac{\pi^2}{2} \\
+ 96\pi^3(s_1 + s_3) - 14\gamma\pi^2 + 6\pi(6\gamma^2 - 3\gamma + 2h_1 - 3) \\
- 3(16\gamma h_1 + 16\gamma^3 - 20\gamma^2 - 20\gamma + 4t_1 - 2t_2 + u_2 + 1) \right]
\]

**B.3 Expressions of the coefficients appearing in eq. (2.79)**

\[
\chi_1 = \left( \ln \frac{\pi^2}{2} - 2\gamma \right)^3 + (4\pi)^3 s_3
\]
\[
\chi_2 = 6 \left( \ln \frac{\pi^2}{2} \right)^3 + (3\pi - 36\gamma + 1) \left( \ln \frac{\pi^2}{2} \right)^2
\]
\[ +4[12k_1\pi^2 + 18\gamma^2 - \gamma(3\pi + 1) + h_1] \ln \frac{\pi^2}{2} + (4\pi)^3(s_2 + 3s_3) \]

\[ -2[\ln(4\pi) - \gamma] \cdot \left[ 3\left( \ln \frac{\pi^2}{2} \right)^2 + 3(\pi - 4\gamma + 1) \ln \frac{\pi^2}{2} \right] \]

\[ +12\gamma^2 - 6\gamma(\pi + 1) + 24\pi^2k_1 + 6h_1 - 1 \] + 10 \ln(4\pi)

\[ -48\gamma^3 + 4\pi^2(3\pi + 1) - 2\gamma(48k_1\pi^2 + 4h_1 + 5) + 4t_1 \]

\[ \chi_3 = 3\left( \ln \frac{\pi^2}{2} \right)^3 - 6(3\gamma + \pi)\left( \ln \frac{\pi^2}{2} \right)^2 \]

\[ +2[3\pi^2(8k_0 + 8k_1 - 1) + \pi(12\gamma - 1) + 18\gamma^2 + 2h_2] \ln \frac{\pi^2}{2} \]

\[ -[\ln(4\pi) - \gamma] \cdot \left[ 3\left( \ln \frac{\pi^2}{2} - 2\gamma \right)^2 \right] \]

\[ +3\pi^2(16k_0 + 16k_1 - 1) - 6\pi + 12h_2 - 8 \] + 4 \ln(4\pi)

\[ +16\pi^3(4s_1 + 8s_2 + 12s_3 - 3k_1) - 12\gamma\pi^2(8k_0 + 8k_1 - 1) \]

\[ -4\pi(6\gamma^2 - \gamma + h_1) - 24\gamma^3 - 4\gamma(2h_2 + 1) + 4t_2 \]

\[ \chi_4 = 4u_2 \]

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