Numerical Implementation of the Multisymplectic Preissman Scheme and Its Equivalent Schemes

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Abstract

We analyze the multisymplectic Preissman scheme for the KdV equation with the periodic boundary condition and show that the un-convergence of the widely-used iterative methods to solve the resulting nonlinear algebra system of the Preissman scheme is due to the introduced potential function. A artificial numerical condition is added to the periodic boundary condition. The added boundary condition makes the numerical implementation of the multisymplectic Preissman scheme practical and is proved not to change the numerical solutions of the KdV equation. Based on our analysis, we derive some new schemes which are not restricted by the artificial boundary condition and more efficient than the Preissman scheme because of less computing cost and less computer storages. By eliminating the auxiliary variables, we also derive two schemes for the KdV equation, one is a 12-point scheme and the other is an 8-point scheme. As the byproducts, we present two new explicit schemes which are not multisymplectic but still have remarkable numerical stable property. Numerical experiments on soliton collisions are also provided to confirm our conclusion and to show the benefits of the multisymplectic schemes with comparison of the spectral method and Zabusky-Kruskal scheme.

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1 Introduction

The Korteweg-de Veris equation has been used to describe various phenomena such as acoustic waves in an anharmonic crystal, waves in bubble-liquid mixtures, magnetohydrodynamic waves in warm plasma, and ion acoustic waves. This equation has two fascinating and significant features. One is the existence of permanent wave solutions, including solitary wave solutions, and the other is the recurrence of the initial state of the wave form. In 1965 Zabusky and Kruskal used a finite difference method, i.e. the famous Zabusky-Kruskal scheme, to show the existence of solitons which propagate with their own velocities, exerting essentially no influence on each other. They also discussed the recurrence of an initial state and guessed that the KdV equation led to the recurrence. Since then, various methods including the finite difference method, the Fourier expansion method and the finite element method have been proposed to solve the KdV equation. Unfortunately, the difference solutions often exhibit nonlinear instabilities when a long time integration is performed. In the 1990s, the symplectic schemes were introduced and systematically developed for the Hamiltonian systems within the framework of symplectic geometry. Numerical results show that symplectic schemes have superior performance, especially in long time simulations. The symplectic schemes can be applied to the KdV equation which may be transformed into the form of Hamiltonian system.

Recently, J. E. Marsden etc. and T. J. Bridges etc. proposed the concept of multisymplectic PDEs and multisymplectic schemes which can be viewed as the generalization of symplectic schemes. Many soliton equations such as the KdV equation, the Kadomtsev-Petviashvili equation, the Zabolotskaya-Khokhlov equation and the sine-Gordon equation can be reformulated into the multisymplectic PDEs and can be solved numerically by the multisymplectic schemes. The simplest and basic multisymplectic scheme is the Preissman scheme, which has been hot in the last two years. However, sometimes the direct numerical implementation of the Preissman scheme for the multisymplectic equation has puzzled researchers all along. When it is applied to solve the periodic boundary problem of the soliton equations with degenerate lagrangian like the KdV, K-P equation and the water waves equation, the general widely-used iterative method referred as the simple iterative method is not convergent, so are the other iterative method such as Newton method and conjugate gradient method. Why?

To settle the problem, S. Reich eliminated the auxiliary variables to get a equivalent scheme for the KdV equation, but he did not consider the influence of the boundary condition.
In the present paper, taking the KdV equation as an example, we analyze the practical computation of the Preissman scheme and find that the unconvergence is due to the indeterminacy of the potential function. We add a condition on the potential function to fix it up and prove that the added condition will not change the numerical solutions of the KdV equation. This condition can be stated as a restriction on numerical periodic boundary condition. Based on our analysis, we present some new multisymplectic implicit schemes that are equivalent to, but more efficient than the Preissman scheme because of less computing cost and less computer storages. By converting the implicit term in the multisymplectic schemes to an explicit one, we obtain two stable, efficient, explicit schemes for the KdV equation. Of course, they are not multisymplectic any more. An elementary but useful method to eliminate the auxiliary variables of the multisymplectic schemes is also presented and two new multisymplectic schemes for the KdV equation are derived. One is a 12-point scheme, the other is an 8-point scheme.

The main purpose of this paper is to develop a method to analyze the multisymplectic scheme for the Hamiltonian PDEs and to show how to choose the proper numerical boundary condition for the multisymplectic schemes and how to derive the new schemes for the PDEs. The method presented in the paper can be applied to the Preissman scheme for other PDEs and to other multisymplectic schemes. Another aim is to compare the performance of the multisymplectic schemes with other kind numerical method to see if the multisymplectic schemes benefit the finite difference approximations of the PDEs. Thus a series of numerical experiments on soliton collisions are presented. Compared with the Zabusky-Kruskal scheme and the spectral method, the multisymplectic schemes are shown to have superior stability, excellent ability to preserve the conservation laws and remarkable capacity of long time computing.

This paper is organized as follows. In section 2 we take a brief review of multisymplectic structure of the KdV equation and the multisymplectic Preissman scheme. The Preissman scheme is analyzed in section 3, where we present an artificial numerical boundary condition for the Preissman scheme and verify its rationality. In section 4, some new multisymplectic schemes for the KdV equation are derived. Section 5 is for numerical experiments and we finish the paper with concluding remarks in section 6.
2 Multisymplectic structure of the KdV equation and the Preissman scheme

The general form of the KdV equation with the initial value and the periodic boundary condition is

\[
\frac{\partial u}{\partial t} + \eta u \frac{\partial u}{\partial x} + \delta^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, \quad (2.1)
\]

\[u(t = 0, x) = u_0(x), \quad u(t, x + a) = u(t, x + b),\]

where \( \eta \) and \( \delta \) are two real numbers.

Introducing the potential \( \phi_x = u \), momenta \( v = \delta u_x \) and variable \( w = \frac{1}{2} \phi_t + \delta v_x + V'(u) \), \( V(u) = \eta u^3 / 6 \), the KdV equation (2.1) can be rewritten as the following Hamiltonian PDEs.

\[M z_t + K z_x = \nabla_z S(z), \quad (2.2)\]

where

\[M = \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -\delta & 0 \\
0 & \delta & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad z = \begin{pmatrix}
\phi \\
u \\
v \\
w
\end{pmatrix}\]

and \( S(z) = \frac{1}{2} v^2 - uw + V(u) \).

Each of the two skew-symmetric matrices \( M \) and \( K \) can be identified with a closed two forms.

\[\omega^1(u, v) = < M u, v >, \quad \omega^2(u, v) = < K u, v >,\]

where \( u, v \) are any vectors on \( \mathbb{R}^4 \) and \( < \cdot, \cdot > \) is the standard Euclidean inner product on \( \mathbb{R}^4 \).

Both forms \( \omega^i \) \( i = 1, 2 \) are closed and therefore pre-symplectic on \( \mathbb{R}^4 \), and on subspaces where they are non-degenerated, they are symplectic forms. In other words, \( (\mathbb{R}^2, \omega^1) \), and \( (\mathbb{R}^4, \omega^2) \) are two distinct symplectic manifolds. Moreover, each two forms is associated with a different direction. \( \omega^1 \) is associated with time and \( \omega^2 \) is associated with space. In this sense the first order PDEs (2.2) is called multisymplectic PDEs or Hamiltonian PDEs. The KdV equation is completely characterized by the function \( S(z) \), and the two skew-symmetric operators \( M \) and \( K \). They are all defined on a finite dimensional space.
The multisymplectic Hamiltonian equation (2.2) satisfies the important multisymplectic conservation law
\[ \partial_t [dz \wedge Mdz] + \partial_x [dz \wedge Kdz] = 0, \] (2.3)
which, for the KdV equation (2.1), is equivalent to
\[ \partial_t [d\phi \wedge du] + 2\partial_x [d\phi \wedge dw + \delta dv \wedge du] = 0, \] (2.4)
where \( \wedge \) is the standard exterior product operator of the differential forms.

The conservation law (2.4) is a strictly local conservation concept that does not depend on a specific boundary condition. That is to say, in the arbitrary domain of the space-time plane, changes in the wedge product \( d\phi \wedge du \) in time are exactly compensated for by changes in the wedge product \( -2(d\phi \wedge dw + \delta dv \wedge du) \) in space.

Bridges and Reich [8] showed that the Preissman scheme for (2.2) is a multisymplectic scheme which preserves the discrete form of (2.3). The Preissman scheme for (2.2) is
\[ \frac{1}{\tau} M(z_{n+\frac{1}{2}}^{m+1} - z_{n+\frac{1}{2}}^m) + \frac{1}{h} K(z_{n+1}^{m+\frac{1}{2}} - z_n^{m+\frac{1}{2}}) = \nabla_z S(z_n^{m+\frac{1}{2}}), \] (2.5)

where \( \tau \) is the time step, \( h \) is the space step, \( x_n, n = 1, 2, \ldots, N; t_m, m = 1, 2, \ldots \) is the regular grids of the integral domain, \( z_n^m \) is an approximation to \( z(x_n, t_m), z_{n+\frac{1}{2}}^m = \frac{1}{2}(z_{n+1}^m + z_n^m), z_n^{m+\frac{1}{2}} = \frac{1}{2}(z_{n+1}^{m+1} + z_n^{m+1}), z_n^{m+1} + z_n^{m+1} + z_n^{m+1}, \phi = (\phi, u, v, w)^T, \) and the corresponding discretized multisymplectic conservation law is
\[
\frac{d\phi_{n+\frac{1}{2}}^{m+\frac{1}{2}} \wedge du_{n+\frac{1}{2}}^{m+\frac{1}{2}} - d\phi_{n+\frac{1}{2}}^{m} \wedge du_{n+\frac{1}{2}}^{m} - 2d\phi_{n+\frac{1}{2}}^{m+\frac{1}{2}} \wedge dw_{n+\frac{1}{2}}^{m+\frac{1}{2}} - \delta dv_{n+\frac{1}{2}}^{m+\frac{1}{2}} \wedge du_{n+\frac{1}{2}}^{m+\frac{1}{2}} - \delta dv_{n+\frac{1}{2}}^{m+\frac{1}{2}} \wedge du_{n+\frac{1}{2}}^{m+\frac{1}{2}}}{h} = 0.
\]

### 3 Analysis of the Preissman Scheme

The Preissman scheme (2.5) is an implicit scheme that involves solving a nonlinear equations for \( z^{m+1} \) at each time step. The widely-used iterative method of this nonlinear equations is as follows. (The analysis and the comparison between this iterative technique with other iterative methods...
such as Newton’s Method can be found in [2].

\[
\frac{1}{2}(u_i^{j+1} + u_{i+1}^{j+1}) - r(u_i^{j+1} - u_{i+1}^{j+1}) = \frac{1}{2}(u_i^j + u_{i+1}^j) + r(u_i^j - u_{i+1}^j),
\]

\[
\frac{\tau}{2}(w_i^{j+1} + w_{i+1}^{j+1}) - \frac{1}{2}(\phi_i^{j+1} + \phi_{i+1}^{j+1}) + \delta r(v_i^{j+1} - v_{i+1}^{j+1}) =
\]

\[-\frac{\tau}{2}(w_i^j + w_{i+1}^j) - \frac{1}{2}(\phi_i^j + \phi_{i+1}^j) - \delta r(v_i^j - v_{i+1}^j) + 2\tau V'(\bar{u}_i^j), \quad (3.1)\]

\[
\frac{h}{2}(u_i^{j+1} + u_{i+1}^{j+1}) + (\phi_i^{j+1} - \phi_{i+1}^{j+1}) = -\frac{h}{2}(u_i^j + u_{i+1}^j) - (\phi_i^j - \phi_{i+1}^j),
\]

where \( i = 1, 2, \ldots, n \), \( \bar{u}_i^j = \frac{1}{4}(u_i^{j+1} + u_{i+1}^{j+1} + u_i^j + u_{i+1}^j) \), \( r = \frac{\tau}{h} \) is the ratio between temporal and spatial steps.

Here we discuss the numerical boundary conditions.

By \( \phi_x = u \),

\[
\phi(b, t) = \phi(a, t) + \int_a^b u(x, t)dx,
\]

Set \( c = \int_a^b u(x, t)dx \), then

\[
\frac{dc}{dt} = \int_a^b u_t dx
\]

\[
= -\int_a^b (cu_x + \delta^2 u_{xxx})dx
\]

\[
= -\int_a^b \left( \frac{c}{2} u_x^2 + \delta^2 u_{xx} \right) dx
\]

\[= 0.\]

We obtain \( c = \int_a^b u_0(x)dx = \text{a constant.} \)

Thus the periodic numerical boundary conditions are

\[
\begin{align*}
    u_1^{j+1} &= u_{n+1}^{j+1}, & v_1^{j+1} &= v_{n+1}^{j+1}, & \phi_1^{j+1} &= \phi_{n+1}^{j+1} + c, & w_1^{j+1} &= w_{n+1}^{j+1}. \quad (3.2)
\end{align*}
\]

Let

\[
A = \begin{bmatrix}
    1 & 1 & 0 & \cdots & 0 & 0 \\
    0 & 1 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 1 & 1 \\
    1 & 0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix}_{n \times n},
\]

\[
B = \begin{bmatrix}
    -1 & 1 & 0 & \cdots & 0 & 0 \\
    0 & -1 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & -1 & 1 \\
    1 & 0 & 0 & \cdots & 0 & -1 \\
\end{bmatrix}_{n \times n},
\]

6
where $P$ is the unit lower triangular matrix, $R$ is an upper triangular matrix.

Unfortunately, this iteration is not convergent because the coefficient matrix is degenerated. Note that $B$ is singular, and its rank is $n - 1$. $A$ is a nonsingular matrix only if $n$ is an odd number. To take the row operation of the coefficient matrix $D$, we suppose $n$ be an odd number and $A^{-1}$ exist. Then there is a permutation matrix $P$, so that

$$PD = R,$$

The initial guess is generally chosen as the value of the previous time step, i.e.

$$X^{(l)} = (u^{j}, v^{j}, w^{j}, \phi^{j})^T.$$
The left multiplying (3.4) by the permutation matrix $P$ yields an equivalent system

$$\tilde{D}X^{(l+1)} = \tilde{b}(X^{(l)}), \ l = 0, 1, \cdots ,$$

where the augmented coefficient matrix now becomes $(\tilde{D}, \tilde{b}) = P \cdot (D, b) =

$$\begin{pmatrix}
\frac{h}{2}A & 0 & 0 & -B & -\frac{h}{2}Au^j + B\Phi^j + c \\
0 & \frac{h}{2}A & 0 & -\frac{2h}{h}BA^{-1}B & \frac{2h}{h}BA^{-1}(B\Phi^j + c) - \frac{h}{2}Av^j \\
0 & 0 & \frac{h}{2} + \frac{8h^2}{h^2}(BA^{-1})^3 & -\frac{A}{2} - r(\frac{2h}{h}BA^{-1})^2 B & r(\frac{2h}{h}BA^{-1})^2(B\Phi^j + c) - \frac{A}{2}Aw^j - \frac{4}{2}\Phi^j + 2rV \\
0 & 0 & 0 & \frac{2B}{h} + \frac{8h^2}{h^2}(BA^{-1})^3 B & Au^j - \frac{8h^2}{h^2}(BA^{-1})^3(B\Phi^j + c) - 4rBA^{-1}V - \frac{c}{h}
\end{pmatrix}$$

(3.6)

**Proposition 3.1.** The coefficient matrix $D$ of the iteration (3.4) is rank 1 deficient and the deficiency is relative to the potential function $\Phi$.

**Proof.** By (3.6)

$$\text{rank}(D) = \text{rank}(\tilde{D}) = 3\text{rank}(A) + \text{rank}([I_{n \times n} + 4h^2 \frac{r}{h^2}(BA^{-1})^3]B)$$

Note that $A$ is a full rank matrix and $B$ is a rank 1 deficient matrix, we may take a proper $r$ ratio, so that

$$\det(I_{n \times n} + 4h^2 \frac{r}{h^2}(BA^{-1})^3) \neq 0,$$

$$\text{rank}([I_{n \times n} + 4h^2 \frac{r}{h^2}(BA^{-1})^3]B) = \text{rank}(B).$$

Thus $D$ is a rank 1 deficient matrix and it is obvious by the (3.5) that the deficiency is relative to the potential function $\Phi$.

This proposition implies that we need one and only one more condition to fix up the system (3.4) and the condition is related to the variable $\Phi$. For example,

$$\phi_i^{j+1} = \text{a constant} \quad \text{some } i \in [1, n].$$

(3.7)

If we add this condition to the (3.3), the rank of the coefficient matrix $D$ is just full and the iteration is convergent. It will be clarified by the following discussion and the numerical experiments in section 5.

What we concern about is the solution of the KdV equation (2.1), i.e. the variable $u^{j+1}$ in system (3.3). We hope that the added condition (3.7) will not change the value of the variable $u^{j+1}$ in system (3.3), which will be proved to be true in the following proposition.

**Proposition 3.2.** The variables $u^{j+1}, v^{j+1}$, and $B\Phi^{j+1}$ in system (3.3) are independent on the added condition (3.7), i.e., the value of $\phi_i^{j+1}$. They
are only dependent on $B\Phi^j, u^j, v^j$.

Proof. The four iterative equations in the system (3.5) may be written as

$$\frac{h}{2} A u^{(l+1)} - B \Phi^{(l+1)} = - \frac{h}{2} A u^j + B \Phi^j + c, \quad (3.8)$$

$$\frac{h}{2} A v^{(l+1)} - \frac{2\delta}{h} B A^{-1} B \Phi^{(l+1)} = - \frac{h}{2} A v^j + \frac{2\delta}{h} B A^{-1} (B \Phi^j + c), \quad (3.9)$$

$$\frac{\tau}{2} A w^{(l+1)} - \left(\frac{A}{2} + r(\frac{2\delta}{h} B A^{-1})^2 B\right) \Phi^{(l+1)} = r\left(\frac{2\delta}{h} B A^{-1}\right)^2 (B \Phi^j + c)$$

$$- \frac{\tau}{2} A w^j - \frac{A}{2} \Phi^j + 2\tau V(u^j, u^l), \quad (3.10)$$

$$\left(\frac{2}{h} I_{n \times n} + \frac{8\delta^2 r}{h^3} (BA^{-1})^3 B\right) B \Phi^{(l+1)} = A u^j - \frac{8\delta^2 r}{h^3} (BA^{-1})^3 (B \Phi^j + c)$$

$$- 4r B A^{-1} V(u^j, u^l) - \frac{c}{h}. \quad (3.11)$$

We now check the process of the iteration. Recall that the values of $u^j, v^j, \text{ and } B \Phi^j$ is given and $u^{(0)} = u^j$. By (3.11), $B \Phi^{(1)}$ is determined. It is independent upon the value of $\phi^{j+1}$. The determined $B \Phi^{(1)}$ and (3.8) fix up $A u^{(1)}$. Because $A$ is invertible, $u^{(1)}$ is fixed up. For the same reason, $v^{(1)}$ is also determined by (3.9). They are all independent upon the value of $\phi^{j+1}$. Substituting $u^{(1)}$ into (3.11) yields independence of $B \Phi^{(2)}$. As the iteration goes on, we get three sequences $u^{(l+1)}, v^{(l+1)}, \text{ and } B \Phi^{(l+1)}, l = 0, 1, \cdots$, which are all independent on the value of $\phi^{j+1}$. Thus the convergent point also has this property, namely, $u^{j+1}, v^{j+1}, \text{ and } B \Phi^{j+1}$ are independent on the value of $\phi^{j+1}$.

This proposition assures the reliability of adding the condition (3.7) to the system (3.3). Actually, the values of $u, v \text{ and } B \Phi$ are fixed up by the Preissman scheme, they have nothing to do with the added condition. When to compute the value of the variable $\Phi$, we need a condition like (3.7) because the rank of the matrix $B$ is $n - 1$. For convenience, we may take the condition (3.7) as a boundary condition in practical computation. That is $\phi^{j+1}_1 = 0$ (or $\phi^{j+1}_{n+1} = 0$). The periodic numerical boundary condition (3.2) now becomes

$$u^{j+1}_1 = u^{j+1}_{n+1}, \quad v^{j+1}_1 = v^{j+1}_{n+1}, \quad w^{j+1}_1 = w^{j+1}_{n+1}, \quad \phi^{j+1}_1 = \phi^{j+1}_{n+1} + C = 0, \quad (3.12)$$

which makes the direct numerical implementation of the Preissman scheme practical without changing the numerical solution of the KdV equation. The corresponding numerical results on soliton collisions will be presented in the next section. Remark:
• The above analysis is based on the condition that the number of spatial grid points \( n \) is an odd number. How to deal with the case with an even number? In reference [7], the numerical experiments presented by Marsden et. al. also imply the question: why sometimes the numerical results supported on odd spatial grid points are quite different from that on even grid points?

• By (3.6), we know that the variable \( w^{j+1} \) will change if we change the value of \( \phi^{j+1} \) in the added condition (3.7). This means that the variable \( w = \frac{1}{2} \phi + \delta V + V'(u) \) in the Preissman scheme is not fixed up. Does the variable \( w \) have the important physical meaning? If it does, how to fix it up?

4 Some new multisymplectic schemes for the KdV equation

In this section, we present several new multisymplectic schemes for the KdV equation.

Inspired by (3.6), we obtain a new scheme for the KdV equation (2.1)

\[
\begin{align*}
\mathbf{p}^{j+1} & = M_1(q^j - \frac{1}{h}c) + M_2(p^j + c) + M_3\left(\frac{q^j + q^{j+1}}{4}\right)^2, \\
q^{j+1} & = -q^j + \frac{2}{h}(p^j + p^{j+1} + c).
\end{align*}
\]

(4.1)

where \( \mathbf{p} = B\Phi \), \( \mathbf{q} = Au \), \( M_1 \), \( M_2 \), \( M_3 \) are three constant matrices, \( M_1 = \left[ \frac{2}{h}I_n + \frac{8\delta^2r}{h^3}(BA^{-1})^3 \right]^{-1}, M_2 = -\frac{4\delta^2r}{h^3}(BA^{-1}) - I_n, M_3 = -2\eta r M_1 BA^{-1}, q^2 = (q_1^2, q_2^2, \ldots, q_n^2)^T \).

This scheme is equivalent to the multisymplectic Preissman scheme, so it is also a multisymplectic scheme and has an excellent stability. Actually, it is composed of the first and the forth line in (3.6). This scheme is more efficient than the Preissman scheme because we need not to compute the variable \( w, v \) and \( \phi \). we may take \( \mathbf{p} = B\Phi \) and \( \mathbf{q} = Au \) as new variables, furthermore the condition (3.7) does not need. It enhances the conclusion that \( u^{j+1} \) is independent on the additional numerical boundary value of \( \phi \).

Moreover, Scheme (4.1) implies a natural iterative form

\[
\begin{align*}
\mathbf{p}^{(l+1)} & = M_1q^j + M_2(p^j + c) + M_3\left(\frac{q^j + q^{(l)}}{4}\right)^2 - M_1 \frac{c}{h}, \\
q^{(l+1)} & = -q^j + \frac{2}{h}(p^j + p^{(l+1)} + c).
\end{align*}
\]
The computations of the iteration only involves multiplication of matrices and vectors. It avoids from solving the algebra equations which is the main part of computation in other general implicit scheme such as the Preissman scheme. After the convergent point \((p^{k+1}, q^{k+1})\) is obtained, solving the equations \(A u^{k+1} = q^{k+1}\) yields the numerical solution at the \(k+1\)th time step of the KdV equation (2.1). If we want to solve the system \(B \Phi^{k+1} = p^{k+1}\) to get the numerical results of the potential \(\Phi^{k+1}\), the additional condition of \(\phi\) like (3.7) is also needed, for the coefficient matrix \(B\) is rank 1 deficient.

Eliminating the variable \(p\) in the scheme (4.1), we have
\[
\frac{h}{2}(q^{j+1} + q^j) = (M_1 + \frac{h}{2}M_2)(q^j + q^{j-1}) + M_3 \left[ \frac{(q^{j+1} + q^j)^2 + (q^j + q^{j-1})^2}{4} \right] + (I + M_2 - \frac{1}{h}M_1)c. \tag{4.2}
\]

Set \(z^{j+1} = q^{j+1} + q^j\), then
\[
\frac{h}{2}z^{j+1} = (M_1 + \frac{h}{2}M_2)z^j + M_3 \left[ \frac{(z^{j+1})^2 + (z^j)^2}{4} \right] + (I + M_2 - \frac{1}{h}M_1)c. \tag{4.3}
\]
Here the matrices \(M_1\), \(M_2\) and \(M_3\) are defined in (4.1).

This scheme is equivalent to the multisymplectic scheme (4.1). It contains only the variable \(u\), thus it can be viewed as a multisymplectic scheme for the original KdV equation (2.1). Its stability and capacity of long-time simulation are the same with the multisymplectic Preissman scheme and scheme (4.1). Furthermore this scheme has more benefits such as simple form, to practice easily and less computations.

It is worth mention that even if we modify the scheme (4.1) into a real explicit scheme which don’t need iterations when being applied, the resulting scheme still have very nice numerical performance which will be shown in next section. The explicit scheme is
\[
\begin{align*}
p^{j+1} &= M_1 q^j + M_2 p^j + M_3 \frac{q^j}{2} + (M_2 - \frac{1}{h}M_1)c, \\
q^{j+1} &= -q^j + \frac{2}{h}(p^j + p^{j+1} + c). \tag{4.4}
\end{align*}
\]
Eliminating the variable \(p\) yields an explicit scheme for the original KdV
equation \(2.1\)

\[
\frac{h}{2}(q^{j+1} + q^j) = (M_1 + \frac{h}{2}M_2)(q^j + q^{j-1}) + M_3 \left[ \left( \frac{q^j}{2} \right)^2 + \left( \frac{q^{j-1}}{2} \right)^2 \right] + (I + M_2 - \frac{1}{h}M_1)c.
\]

If we modify the implicit term in scheme \(4.2\) into an explicit one, we obtain another explicit scheme for the KdV equation

\[
\frac{h}{2}(q^{j+1} + q^j) = (M_1 + \frac{h}{2}M_2)(q^j + q^{j-1}) + M_3 \left[ \left( \frac{q^j}{2} \right)^2 + \left( \frac{q^{j-1}}{2} \right)^2 \right] + (I + M_2 - \frac{1}{h}M_1)c.
\]

But numerical results show that this is an unstable scheme.

All the schemes above are invalid provided the number \(n\) of the spatial grid points is even. Next we introduce another method to eliminate the auxiliary variables of the multisymplectic Preissman scheme to get two multisymplectic schemes for the KdV equation. Both schemes are valid whether \(n\) is odd or even.

Let us state the multisymplectic Preissman scheme for equation \(2.2\) in the form

\[
\frac{1}{2\Delta t}(u_{i-1}^{j+1} + u_i^{j+1} - u_i^{j-1} - u_{i-1}^{j-1}) + \frac{1}{\Delta x}(w_i^{j-1} + w_i^j - w_{i-1}^{j-1} - w_{i-1}^j) = 0,
\]

\[
\frac{\delta}{\Delta x}(u_{i-1}^{j-1} + u_i^j - u_i^{j-1} - u_{i-1}^j) - \frac{1}{2}(v_{i-1}^{j-1} + v_i^j + v_i^{j-1} + v_{i-1}^j) = 0,
\]

\[
\frac{1}{2}(w_{i-1}^{j-1} + w_i^j + w_i^{j-1} + w_{i-1}^j) - \frac{1}{2\Delta x}(\phi_{i-1}^{j-1} + \phi_i^j - \phi_i^{j-1} - \phi_{i-1}^j) = 0,
\]

\[
\frac{1}{2}(w_{i-1}^{j-1} + w_i^j + w_i^{j-1} + w_{i-1}^j) - \frac{1}{2\Delta t}(\phi_{i-1}^{j-1} + \phi_i^j - \phi_i^{j-1} - \phi_{i-1}^j)
- \frac{\delta}{\Delta x}(v_{i-1}^{j-1} + v_i^j - v_i^{j-1} - v_{i-1}^j) = 2V'(\frac{1}{4}(u_{i-1}^{j-1} + u_i^j + u_i^{j-1} + u_{i-1}^j)),
\]

\(4.10\)
Eliminating the variable $w$ by (4.7) and (4.10), we obtain
\[
\frac{\Delta x}{4\Delta t}[u^j_{i+1} - u^j_{i+1} + 2(u^j_i - u^j_{i-1}) + u^j_{i-1} - u^j_{i-1}] + \frac{1}{2\Delta t}[(\phi^j_{i+1} - \phi^j_{i+1}) - (\phi^j_{i-1} - \phi^j_{i-1})] + \frac{\delta}{\Delta x}[v^j_{i+1} + v^j_{i+1} - 2(v^j_i + v^j_{i-1}) + v^j_{i-1} + v^j_{i-1}] = -2[V'(\frac{1}{4}(u^j_{i+1} + u^j_{i+1} + u^j_{i-1} + u^j_i)) - V'(\frac{1}{4}(u^j_{i-1} + u^j_{i-1} + u^j_{i-2} + u^j_{i-2}))].
\]

In the same manner, we may eliminate the variable $v$ by combining (4.11) and (4.8) to obtain
\[
\frac{\Delta x}{4\Delta t}[(u^j_{i+1} - u^j_{i+1}) + 3(u^j_i - u^j_{i-1}) + 3(u^j_{i-1} - u^j_{i-1}) + u^j_{i-2} - u^j_{i-2}] + \frac{1}{2\Delta t}[(\phi^j_{i+1} - \phi^j_{i+1}) + (\phi^j_i - \phi^j_{i-1}) - (\phi^j_{i-1} - \phi^j_{i-1}) - (\phi^j_{i-2} - \phi^j_{i-2})] + \frac{2\delta^2}{\Delta x^2}[(u^j_{i+1} + u^j_{i+1} - 3(u^j_i + u^j_{i-1}) + 3(u^j_{i-1} + u^j_{i-1}) - (u^j_{i-2} + u^j_{i-2})] = -2[V'(\frac{1}{4}(u^j_{i+1} + u^j_{i+1} + u^j_{i-1} + u^j_i)) - V'(\frac{1}{4}(u^j_{i-1} + u^j_{i-1} + u^j_{i-2} + u^j_{i-2}))],
\]

which together with (4.9) yields a new 8-points scheme in only variable $u$, by eliminating the variable $\phi$,
\[
\frac{1}{4\Delta t}[(u^j_{i+1} + 3u^j_i + 3u^j_{i-1} + u^j_{i-2}) - (u^j_{i+1} + 3u^j_i + 3u^j_{i-1} + u^j_{i-2})] + \frac{\delta^2}{\Delta x^3}[(u^j_{i+1} - 3u^j_i + 3u^j_{i-1} - u^j_{i-2}) + (u^j_{i+1} - 3u^j_i + 3u^j_{i-1} - u^j_{i-2})] + \frac{1}{\Delta x}[V'(\frac{1}{4}(u^j_{i+1} + u^j_{i+1} + u^j_{i-1} + u^j_i)) - V'(\frac{1}{4}(u^j_{i-1} + u^j_{i-1} + u^j_{i-2} + u^j_{i-2}))] = 0.
\]

Converting the implicit term in above scheme to a explicit one, we obtain a new explicit scheme for the KdV equation whose remarkable numerical
property will be shown in next section.

\[
\frac{1}{4\Delta t}[(u_{i+1}^{j+1} + 3u_i^{j+1} + 3u_{i-1}^{j+1} + u_{i-2}^{j+1}) - (u_{i+1}^{j-1} + 3u_i^{j-1} + 3u_{i-1}^{j-1} + u_{i-2}^{j-1})]
+ \frac{\delta^2}{\Delta x^2}[(u_{i+1}^{j-1} - 3u_i^{j-1} - u_{i-2}^{j-1}) + (u_{i+1}^{j-1} - 3u_i^{j-1} - u_{i-2}^{j-1})]
+ \frac{1}{\Delta x}[V''(\frac{1}{2}(u_{i+1}^{j-1} + u_i^{j-1})) - V''(\frac{1}{2}(u_{i-1}^{j-1} + u_{i-2}^{j-1}))] = 0.
\] 

(4.14)

In the appendix of this paper we present another process of eliminating the auxiliary variables to derive a 12-points scheme

\[
\frac{1}{16\Delta t}[(u_{i+1}^{j+1} - u_{i-1}^{j-1} + 3u_i^{j+1} - 3u_i^{j-1} + 3u_{i+1}^{j+1} - 3u_{i-1}^{j-1} - u_{i-2}^{j-1})]
+ \frac{\delta^2}{4\Delta x^2}[(u_{i+1}^{j+1} - 3u_i^{j+1} - u_{i-1}^{j+1}) + 2u_{i+1}^{j+1} - 6u_i]
+ 6u_i - 2u_i^{j+1} + u_{i+1}^{j+1} - 3u_i^{j-1} + 3u_{i+1}^{j-1} - u_{i-2}^{j-1}]
+ \frac{1}{4\Delta x}[V''(\frac{1}{4}(u_i + u_i^{j+1} + u_i^{j+1} + u_{i+1}^{j+1})) - V''(\frac{1}{4}(u_i^{j-1} + u_i^{j+1} + u_i^{j-1} + u_{i+1}^{j+1}))]
+ V''(\frac{1}{4}(u_i^{j-1} + u_i^{j+1} + u_i^{j-1} + u_{i+1}^{j+1})) - V''(\frac{1}{4}(u_i^{j-1} + u_i^{j+1} + u_i^{j-1} + u_{i+1}^{j+1})) = 0.
\] 

(4.15)

Both schemes (4.13) and (4.15) are derived from the the Preissman scheme (2.5), thus they should be equivalent to each other. Actually they can be derived from each other.

**Remark:** The method introduced above can be applied to the Preissman scheme, as well as other multisymplectic scheme, for other Hamiltonian PEDs to obtain new schemes. For example, we can get a nine-point scheme for the sine-Gordon equation, a six-point scheme for the Schrödinger equation, a forty-five-point for the Kadomtsev-Petviashvili equation and so on. All these schemes, except for the nine-point scheme, which was discussed by Marsden et. al. in [7], are new and expected to have excellent numerical stability and capacity of long-time simulation.

## 5 Concluding Remarks

We analyze the multisymplectic scheme for the KdV equation and find that the unconvergence of the widely-used iteration method to solve the resulting nonlinear algebra system is due to the introduced potential function \( \phi \). We
add a artificial numerical boundary condition on the original periodic numerical boundary condition. It leads to a new numerical boundary condition (3.12) which makes the implementation of the Preissman scheme practical without changing the numerical solution of the KdV equation. The numerical results obtained with the presented numerical boundary condition show the correctness of the condition and the merits of the multisymplectic schemes. This method for analysis can be easily generalized to other multisymplectic schemes and other PDEs.

We obtain two new multisymplectic schemes for the KdV equation, which are equivalent to, but more efficient than the Preissman scheme. We also develop a method to eliminate the auxiliary variables of the Preissman scheme and get two equivalent multisymplectic schemes in only \( u \) for the KdV equation. One is a 12-point scheme and the other is an 8-point scheme. Compared with the Zabusky-Kruskal scheme and the spectral method, the new multisymplectic schemes are used to simulate the solitary waves. Numerical results show that the multisymplecticity do bring the finite differential schemes some benefits such as stability, capacity for long time computation, and ability to preserve the conservational laws.

At last we like point out the explicit schemes (4.5) and (4.14) is also an excellent schemes for the KdV equation. they can give the most accurate waveforms, which catch well up with those in Figure 6. Furthermore, its the stability, capacity for long time computation and efficiency are much better than that of the Zabusky-Kruskal scheme, as presented in subsection 5.5. We are currently analyzing theoretically the stability, conservation and other proprieties of the explicit schemes.

**Remark:** During the preparation of this paper, Prof. R. MacLachan has also derived the same results on the eight-points for the KdV equation. We thank him for the discussions and many important suggestions.

**References**

[1] N.J.Zabusky & M.D.Kruskal, Interaction of "soliton" in a Collisionless Plasma and Recurrence if Initial States, Phys. Rev. Letters, 15, 240-243, 1965.

[2] S.B.Wineberg, J.F.Megrath, E.F.Gabl, ect., Implicit Spectral Methods for Wave Propagation Problems, Jour. Comp. Phys. 97, 311-336, 1991.

[3] R. Winther, A conservative finite element method for the Korteweg-de Vries equation, Math. Compu. 34, 23-43, 1980.
[4] Ernst Hairer, Christian Lubich, Invariant tori of dissipatively perturbed Hamiltonian systems under symplectic discretization, Appl. Numer. Math. 29, 57-71, 1999.

[5] Sanz-Serna J M, Calvo M P. Numerical hamiltonian problem. Chapman and Hall, London, 1994

[6] Feng K, Qin M Z, The symplectic methods for computation of Hamiltonian systems, In Zhu Y L, Guo Ben-Yu, ed, Proc Conf on Numerical Methods for PDEs, Berlin: Springer, 1987, 1-37, Lecture notes in Math, 1297.

[7] J.E.Marsden, G.P.Patrick and S.Shkoller, Multisymplectic geometry, variational integrators, and Nonlinear PDEs, Comm. Math. Phys, 199, 351-395(1998).

[8] T.J.Bridges, S.Reich, Multi-symplectic Integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity, Physics letter A, 2001, 284(4-5):184-193.

[9] P.F. Zhao, M.Z. Qin, multisymplectic Geometry and Multisymplectic Preissman Scheme for the KdV Equation, J. Phys. A: Math. Gen, 33, 3613-1626, 2000.

[10] Jing-Bo Chen, New schemes for the nonlinear Schroedinger equation, Applied Mathematics and Computation, 124(3), 371-379, 2001.

[11] S. Reich, Notes on Numerical Methods for Hamiltonian PDEs, Reading Material of International Workshop on structure-Preserving Algorithms, Vol. 6(3), 153-174, 2001.

[12] Yushun Wang, Mengzhao Qin, Multisymplectic Geometry and Multisymplectic Scheme for the Nonlinear Klein Gordon Equation. Journal of the Physical Society of Japan. Vol.70, No.3, 653-661, March 2001.
Appendix

A The detail process of produce the 12-point scheme for the KdV equation

Let us state the multisymplectic Preissman scheme for equation (2.2) in the form
\[
\frac{1}{2\Delta t}(u_{i-1}^j + u_i^j - u_{i-1}^{j-1} - u_i^{j-1}) + \frac{1}{\Delta x}(w_i^j - w_i^{j-1} - w_{i-1}^j - w_{i-1}^{j-1}) = 0, 
\] (A.1)
\[
\frac{\delta}{\Delta x}(u_i^j - u_i^{j-1} - u_{i-1}^j + u_{i-1}^{j-1}) - \frac{1}{2}(v_i^j - v_i^{j-1} + v_{i-1}^j + v_{i-1}^{j-1}) = 0, 
\] (A.2)
\[
\frac{1}{2}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j) - \frac{1}{\Delta x}(\varphi_i^j - \varphi_i^{j-1} - \varphi_{i+1}^j + \varphi_{i+1}^{j-1}) = 0, 
\] (A.3)
\[
\frac{1}{2}(w_i^{j-1} + w_i^j + w_{i+1}^{j-1} + w_{i+1}^j) - \frac{1}{2\Delta t}(\varphi_i^j + \varphi_i^{j-1} - \varphi_{i+1}^j + \varphi_{i+1}^{j-1}) 
- \frac{\delta}{\Delta x}(v_i^{j-1} + v_i^j - v_{i-1}^{j-1} - v_{i-1}^j) = 2V'(\frac{1}{4}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j)). 
\] (A.4)

Taking \(i = i + 1\) in (A.4), we obtain
\[
\frac{1}{2}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j) - \frac{1}{2\Delta t}(\varphi_i^j + \varphi_i^{j-1} - \varphi_{i+1}^j - \varphi_{i+1}^{j-1}) 
- \frac{\delta}{\Delta x}(v_{i+1}^{j-1} + v_{i+1}^j - v_i^{j-1} - v_i^j) = 2V'(\frac{1}{4}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j)). 
\] (A.5)
\[
\frac{1}{2\Delta x}(w_{i+1}^{j-1} + w_{i+1}^j - w_i^{j-1} - w_i^j) - \frac{1}{2\Delta t}(\varphi_{i+1}^j - \varphi_{i+1}^{j-1} - \varphi_i^j + \varphi_i^{j-1}) 
- \frac{\delta}{\Delta x^2}(v_{i+1}^{j-1} + v_{i+1}^j - 2v_i^{j-1} - 2v_i^j + v_{i-1}^{j-1} + v_{i-1}^j) = 
\frac{2}{\Delta x}(V'\left(\frac{1}{4}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j)\right) - V'\left(\frac{1}{4}(u_i^{j-1} + u_i^j + u_{i+1}^{j-1} + u_{i+1}^j)\right)). 
\] (A.6)
Let $i = i - 1$ in (A.6), then

\[
\frac{1}{2\Delta x}(u_i^{j-1} + w_i^j - w_{i-1}^{j-1} - w_{i-2}^j) - \frac{1}{2\Delta t\Delta x}(\varphi_i^j - \varphi_i^{j-1} - \varphi_{i-1}^j + \varphi_{i-2}^{j-1})
- \frac{\delta}{\Delta x^2}(v_i^{j-1} + v_i^j - 2v_{i-1}^j - 2v_{i-1}^{j-1} + v_{i-2}^j + v_{i-2}^{j-1})
= \frac{2}{\Delta x} \left( V' \left( \frac{1}{4}(u_{i-1}^{j-1} + u_i^{j-1} + u_i^{j-1}) \right) - V' \left( \frac{1}{4}(u_{i-2}^{j-1} + u_{i-1}^{j-1} + u_i^{j-1}) \right) \right).
\] (A.7)

(A.3) and (A.4) yields

\[
\frac{1}{4\Delta x}(u_{i+1}^{j-1} + u_{i+1}^j - w_{i+1}^{j-1} - w_{i+1}^j + w_i^{j-1} - w_i^j - w_{i-1}^{j-1} - w_{i-1}^j)
- \frac{1}{4\Delta t\Delta x}(\varphi_{i+1}^j - \varphi_{i+1}^{j-1} - \varphi_{i-1}^j + \varphi_{i-1}^{j-1} + \varphi_i^{j-1} - \varphi_i^j - \varphi_{i-2}^j + \varphi_{i-2}^{j-1})
- \frac{\delta}{2\Delta x^2}(v_{i+1}^{j-1} + v_{i+1}^j - v_i^{j-1} - v_i^j - v_{i-1}^{j-1} - v_{i-1}^j + v_{i-2}^{j-1} + v_{i-2}^j)
= \frac{1}{\Delta x} \left( V' \left( \frac{1}{4}(u_{i+1}^{j-1} + u_i^{j-1} + u_i^{j-1} + u_i^{j+1}) \right) - V' \left( \frac{1}{4}(u_{i+2}^{j-1} + u_{i+1}^{j-1} + u_i^{j-1} + u_i^{j+1}) \right) \right).
\] (A.8)

and, similarly, taking $i = i + 1$ in (A.2), we have

\[
\frac{\delta}{\Delta x}(u_{i+1}^{j-1} + u_{i+1}^j - u_i^{j-1} - u_i^j) - \frac{1}{2}(v_{i+1}^{j-1} + v_i^j + v_{i+1}^{j-1} + v_{i+1}^j) = 0. \tag{A.9}
\]

calculation of \(\frac{\Delta}{\Delta x}(A.9) - (A.2)\) leads to

\[
\frac{\delta}{\Delta x^2}(u_{i+1}^{j-1} + u_{i+1}^j - 2u_i^j - 2u_i^{j-1} + u_{i-1}^{j-1} + u_{i-1}^j)
- \frac{1}{2\Delta x}(v_{i+1}^{j-1} + v_i^j + v_{i+1}^{j-1} + v_{i+1}^j) = 0.
\] (A.10)

which implies \((i = i - 1)\)

\[
\frac{\delta}{\Delta x^2}(u_i^{j-1} + u_i^j - 2u_{i-1}^j - 2u_{i-1}^{j-1} + u_{i-2}^{j-1} + u_{i-2}^j)
- \frac{1}{2\Delta x}(v_i^{j-1} + v_i^j - v_{i-2}^{j-1} - v_{i-2}^j) = 0. \tag{A.11}
\]
\( A.10 \) \( \rightarrow A.11 \) yields

\[
\frac{\delta}{\triangle x^3} (u_{i+1}^{j+1} + u_{i-1}^{j} - 3u_i^j - 3u_{i-1}^j + 3u_{i-1}^{j-1} + 3u_{i-2}^j - u_{i-2}^j - u_{i-2}^{j-1}) - \frac{1}{2\triangle x^2} (v_{i+1}^{j+1} + v_{i}^{j+1} - v_{i-1}^j - v_{i-1}^j - v_{i-1}^{j-1} + v_{i-2}^j + v_{i-2}^{j-1}) = 0.
\]

(A.12)

In the same manner, taking \( i = i + 1 \) in (A.11), we have

\[
\frac{1}{2\triangle t} (u_{i+1}^j + u_{i+1}^{j-1} - u_{i}^{j-1}) + \frac{1}{\triangle x} (w_{i+1}^j + w_{i+1}^{j-1} - w_{i}^{j-1} - w_{i}^{j}) = 0. \quad (A.13)
\]

and the sum of the above formula and (1) is

\[
\frac{1}{2\triangle t} (w_{i+1}^j - w_{i+1}^{j-1} + 2u_i^j - 2w_i^j + u_{i-1}^j - w_{i-1}^j) + \frac{1}{\triangle x} (w_{i+1}^{j-1} + w_{i+1}^j - w_{i}^{j-1} - w_{i}^j) = 0. \quad (A.14)
\]

Combining (A.8), (A.12), (A.14), we obtain

\[
\frac{1}{8\triangle t} (u_{i+1}^j - u_{i+1}^{j-1} + 3u_i^j - 3u_{i-1}^j + 3u_{i-1}^{j-1} - 3u_{i-2}^j + u_{i-2}^{j-1}) - \frac{1}{4\triangle t \triangle x} (\varphi_{i+1}^{j+1} - \varphi_{i+1}^{j-1} - \varphi_{i-1}^j + \varphi_{i-1}^{j-1} + \varphi_i^j - \varphi_i^{j-1} + \varphi_{i+1}^{j+1} + \varphi_{i+1}^{j-1})
\]

\[
- \frac{\delta^2}{\triangle x} (u_{i+1}^j + u_{i+1}^{j-1} - 3u_i^j - 3u_{i-1}^j + 3u_{i-1}^{j-1} + 3u_{i-2}^j - u_{i-2}^{j-1})
\]

\[
= \frac{1}{\triangle x} \left( V' \left( \frac{1}{4} (u_i^j + u_{i+1}^j + u_{i+1}^{j-1} + u_{i+1}^j) \right) - V' \left( \frac{1}{4} (u_{i-2}^j + u_{i-2}^j + u_{i-1}^{j-1} + u_{i-1}^{j-1}) \right) \right).
\]

(A.15)

This leads, if \( j \) is replaced with \( j + 1 \),

\[
\frac{1}{8\triangle t} (u_{i+1}^{j+1} - u_{i+1}^j + 3u_i^{j+1} - 3u_i^j + 3u_{i+1}^{j+1} - 3u_{i+1}^j + u_{i+1}^{j+1} + u_{i+2}^j - u_{i+2}^j)
\]

\[
- \frac{1}{4\triangle t \triangle x} (\varphi_{i+1}^{j+1} - \varphi_{i+1}^j - \varphi_{i-1}^{j+1} + \varphi_{i-1}^j + \varphi_i^{j+1} - \varphi_i^j + \varphi_{i+1}^{j+1} + \varphi_{i+1}^j)
\]

\[
- \frac{\delta^2}{\triangle x^2} (u_{i+1}^{j+1} + u_{i+1}^j - 3u_i^{j+1} - 3u_i^j + 3u_{i+1}^{j+1} + 3u_{i+1}^j - u_{i+1}^{j+1} - u_{i+1}^j)
\]

\[
= \frac{1}{\triangle x} \left( V' \left( \frac{1}{4} (u_i^{j+1} + u_{i+1}^{j+1} + u_{i+1}^j + u_{i+1}^{j+1}) \right) - V' \left( \frac{1}{4} (u_{i-2}^{j+1} + u_{i-2}^{j+1} + u_{i-1}^{j+1} + u_{i-1}^{j+1}) \right) \right).
\]

(A.16)
The sum of the above two formulas is

\[
-\frac{1}{8\Delta t}(u_{i+1}^{j+1} - u_{i+1}^{j-1} + 3u_i^{j+1} - 3u_i^{j-1} - 3u_{i-1}^{j+1} + 3u_{i-1}^{j-1} + u_{i-2}^{j+1} - u_{i-2}^{j-1})
\]

\[
-\frac{1}{4\Delta t\Delta x}(\varphi_{i+1}^{j+1} - \varphi_{i+1}^{j-1} + \varphi_i^{j+1} - \varphi_i^{j-1} - \varphi_{i-1}^{j+1} + \varphi_{i-1}^{j-1} + \varphi_{i-2}^{j+1} - \varphi_{i-2}^{j-1})
\]

\[
-\frac{\delta^2}{\Delta x^2}(u_{i+1}^{j+1} + 2u_i^{j+1} + u_{i-1}^{j+1} - 3u_i^{j+1} - 6u_i^{j-1} - 3u_i^{j-1})
\]

\[
+ 6u_{i-1}^{j-1} + 3u_{i-1}^{j+1} - 2u_{i-2}^{j+1} - u_{i-2}^{j-1}
\]

\[
= \frac{1}{\Delta x} \left(V'(\frac{1}{4}(u_i^{j+1} + u_i^{j+1} + u_{i+1}^{j+1} + u_{i+1}^{j+1})) - V'(\frac{1}{4}(u_{i-2}^{j+1} + u_{i-2}^{j+1} + u_{i-1}^{j+1} + u_{i-1}^{j+1}))
\]

\[
+ V'(\frac{1}{4}(u_i^{j-1} + u_i^{j-1} + u_{i+1}^{j-1} + u_{i+1}^{j-1})) - V'(\frac{1}{4}(u_{i-2}^{j-1} + u_{i-2}^{j-1} + u_{i-1}^{j-1} + u_{i-1}^{j-1})) \right).
\]

(A.17)

Meanwhile, we take \(j = j + 1\) in (A.3)

\[
\frac{1}{2}(u_i^{j+1} + u_i^{j+1} + u_i^{j+1} + u_i^{j+1}) - \frac{1}{\Delta x}(\varphi_i^j + \varphi_i^j + \varphi_i^j - \varphi_i^j) = 0.
\]

(A.18)

(A.18 \text{ - A.3}) yields

\[
\frac{1}{2\Delta t}(u_{i+1}^{j+1} + u_{i+1}^{j+1} - u_i^{j+1} - u_i^{j-1}) - \frac{1}{\Delta x\Delta t}(\varphi_i^{j+1} - \varphi_i^{j-1} - \varphi_i^{j-1} + \varphi_i^{j-1}) = 0.
\]

(A.19)

Taking \(i = i + 1\) in (A.19), we obtain

\[
\frac{1}{2\Delta t}(u_{i+1}^{j+1} + u_{i+1}^{j+1} - u_i^{j+1} - u_i^{j-1}) - \frac{1}{\Delta x\Delta t}(\varphi_i^{j+1} - \varphi_i^{j+1} - \varphi_i^{j-1} + \varphi_i^{j-1}) = 0.
\]

(A.20)

By (A.19) + (A.20), we have

\[
\frac{1}{2\Delta t}(u_i^{j+1} + 2u_i^{j+1} + u_i^{j+1} - 2u_i^{j+1} - u_i^{j+1} - u_i^{j+1})
\]

\[
- \frac{1}{\Delta x\Delta t}(\varphi_i^{j+1} - \varphi_i^{j+1} - \varphi_i^{j+1} + \varphi_i^{j+1}) = 0.
\]

(A.21)

which is

\[
\frac{1}{2\Delta t}(u_i^{j+1} + 2u_i^{j+1} + u_i^{j+1} - 2u_i^{j+1} - u_i^{j+1} - u_i^{j+1})
\]

\[
- \frac{1}{\Delta x\Delta t}(\varphi_i^{j+1} - \varphi_i^{j+1} - \varphi_i^{j+1} + \varphi_i^{j+1}) = 0.
\]

(A.22)
if $i$ is replaced with $i - 1$.

Combining (A.21), (A.22) and (A.17), we obtain a new multisymplectic twelve points scheme for the KdV equation (2.1)

$$
\frac{1}{16\Delta t}(u_{i+1}^{j+1} - u_{i+1}^{j-1} + 3u_{i}^{j+1} - 3u_{i-1}^{j-1} + 3u_{i+1}^{j+1} + u_{i-2}^{j+1} - u_{i-2}^{j-1})
+ \frac{\delta^2}{4\Delta x^3}(u_{i+1}^{j+1} - 3u_{i}^{j+1} + 3u_{i-1}^{j+1} - u_{i-2}^{j+1} + 2u_{i+1}^{j} - 6u_i^j)
+ 6u_{i-1}^j - 2u_{i-2}^j + u_{i+1}^{j-1} - 3u_{i-1}^{j-1} + 3u_{i+1}^{j-1} - u_{i-2}^{j-1})
+ \frac{1}{4\Delta x}[V'(\frac{1}{4}(u_{i}^{j+1} + u_{i-1}^{j+1} + u_{i+1}^{j+1})) - V'(\frac{1}{4}(u_{i+1}^{j+1} + u_{i-1}^{j+1} + u_{i+1}^{j+1}))]
+ V'(\frac{1}{4}(u_{i}^{j-1} + u_{i+1}^{j-1} + u_{i+1}^{j+1})) - V'(\frac{1}{4}(u_{i-2}^{j-1} + u_{i-2}^{j-1} + u_{i-1}^{j-1} + u_{i-1}^{j-1}))]
= 0.
\quad (A.23)
$$