Two New Families of Finsler Connections on Even-Dimensional Manifolds

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ABSTRACT

Let \( F^{2n} = (M, M', F^*) \) be an even-dimensional pseudo-Finsler manifold. We construct an almost hypercomplex structure on any chart domain of a certain atlas of \( M' \) by using a considered non-linear connection. Then by using the almost hypercomplex structure we define two new families of Finsler connections. Also we show that for any Finsler connection \( \nabla \) there exists a linear connection \( D \) such that the local almost hypercomplex structure is parallel with respect to it.

Keywords: pseudo-Finsler manifold; Finsler connection; almost hypercomplex structure.

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1. Introduction

A. Bejancu and H.R. Farran, in [1] and [2], for a pseudo-Finsler manifold \( F^m = (M, M', F^*) \) with a non-linear connection \( H M' \) and any two skew-symmetric Finsler tensor fields of type \((1, 2)\) on \( F^m \), introduced a notion of Finsler connections which named "\((H M', S, T)\)-Cartan connections". After them in [3] we reconstruct the same Finsler connections by using almost complex structures. On the other hand almost hypercomplex and hypercomplex structures which are important in differential geometry have many interesting and effective applications in theoretical physics. For example the background objects of HKT-geometry are hypercomplex manifolds. These spaces appear in \( N = 4 \) supersymmetric model (see [4, 5]). Applications of Riemannian metrics on these spaces persuade us to study the geometry of Finsler metrics on them (see [6, 7]). In the present paper we study the relations between Finsler structures and almost hypercomplex structures in a different viewpoint. We use almost hypercomplex structures to construct new Finsler connections on even-dimensional pseudo-Finsler manifolds. For this purpose we construct a local almost hypercomplex structure by using a considered non-linear connection. Then by using the almost hypercomplex structure we define two new families of Finsler connections. Also we show that for any Finsler connection \( \nabla \) there exists a linear connection \( D \) such that the local almost hypercomplex structure is parallel with respect to it.

2. Preliminaries and notations

Assume that \( M \) is a real \( m \)-dimensional smooth manifold and \( TM \) is the tangent bundle of \( M \). Let \( M' \) be a nonempty open submanifold of \( TM \) such that \( \pi(M') = M \) and \( \theta(M) \cap M' = \emptyset \), where \( \theta \) is the zero section of \( TM \). Suppose that \( F^m = (M, M', F^*) \) is a pseudo-Finsler manifold where \( F^* : M' \rightarrow \mathbb{R} \) is a smooth function which satisfies the following conditions in any coordinate system \( \{(U', \Phi') : x^i, y^j \} \) in \( M' \),

- \( F^* \) is positively homogeneous of degree two with respect to \((y^1, \ldots, y^m)\), i.e., we have
  \[ F^*(x^1, \ldots, x^m, ky^1, \ldots, ky^m) = k^2 F^*(x^1, \ldots, x^m, y^1, \ldots, y^m) \]
  for any point \((x, y) \in (U', \Phi')\) and \( k > 0 \).

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• At any point \((x, y) \in (U', \Phi')\), \(g_{ij}\) are the components of a quadratic form on \(\mathbb{R}^m\) with \(q\) negative eigenvalues and \(m - q\) positive eigenvalues, \(0 < q < m\) (see [1]).

Consider the tangent mapping \(\pi_\ast : TM' \rightarrow TM\) of the projection map \(\pi : M' \rightarrow M\) and define the vector bundle \(VM' = \ker \pi_\ast\). A complementary distribution \(HM'\) to \(VM'\) in \(TM'\) is called a non-linear connection or a horizontal distribution on \(M'\) \(TM' = HM' \oplus VM'. \quad (3.1)\)

In local coordinates let \(\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} - N_i^j(x, y) \frac{\partial}{\partial y^j}\), \((i, j = 1 \cdots m)\) be the canonical basis for \(TM'\). Let \(F^m = (M, M', F^\ast)\) be a pseudo-Finsler manifold. Then a Finsler connection on \(F^m\) is a pair \(FC = (HM', \nabla)\) where \(HM'\) is a non-linear connection on \(M'\) and \(\nabla\) is a linear connection on the vertical vector bundle \(VM'\) (see [1]).

An almost hypercomplex manifold is a \(4n\)-dimensional manifold \(M\) with three globally-defined almost complex structures \(J_i, i = 1, 2, 3\), satisfying the quaternion identities
\[
J_1 J_2 = -J_2 J_1 = J_3, \quad J_i^2 = -I_{TM}, \quad i = 1, 2, 3. \quad (2.2)
\]
\[
J_i^2 = -I_{TM}, \quad i = 1, 2, 3. \quad (2.3)
\]

**Remark 2.1.** From now on we suppose that \(F^{2n} = (M, M', F^\ast)\) is an even-dimensional pseudo-Finsler manifold. We use \(h\) and \(v\) for the projections on \(HM'\) and \(VM'\), respectively. Also we use \(\delta_i\) and \(\partial_i\) as \(\frac{\partial}{\partial x_i}\) and \(\frac{\partial}{\partial y^i}\), respectively.

Throughout the article, we use the following rules for the indices.

- The indicant \(a\) is equal to 1 and 3, only.
- Latin indices (except the alphabets \(a\) and \(n\)) run from 1 to 2n.
- Greek indices \(\alpha, \beta, \gamma\) and \(\theta\) run from 1 to \(n\).
- Any repeated pair of indices (except the alphabets \(a\) and \(n\)), provided that one is up and the other is down, is automatically summed.
- The matrix \((g^{ij})\) is the matrix inverse of \((g_{ij})\).

## 3. New Finsler connections on even-dimensional manifolds

In this section we construct two new families of Finsler connections on \(M'\). For this reason we consider an atlas on \(M'\), also we suppose that \(HM'\) is a non-linear connection on \(M'\). Then by using the non-linear connection we define an almost hypercomplex structure on any chart. After this step we define new Finsler connections on any chart by using the almost hypercomplex structure. Finally it is enough to paste the local connections by a partition of unity to have a connection on \(M'\).

Let \(A\) be an atlas on \(M'\) and \((U', \Phi') \in A\). We construct the following almost hypercomplex structure on \(U'\) by using non-linear connection \(HM'\).
\[
J_1 (\delta_\alpha) = \partial_\alpha, \quad J_2 (\delta_\alpha) = \delta_{n+\alpha}, \quad J_3 (\delta_\alpha) = -\delta_{n+\alpha},
\]
\[
J_1 (\partial_{n+\alpha}) = -\partial_{n+\alpha}, \quad J_2 (\partial_{n+\alpha}) = -\delta_\alpha, \quad J_3 (\partial_{n+\alpha}) = \delta_\alpha
\]
\[
J_1 (\partial_\alpha) = -\delta_\alpha, \quad J_2 (\partial_\alpha) = \partial_{n+\alpha}, \quad J_3 (\partial_\alpha) = \delta_{n+\alpha}
\]
\[
J_1 (\partial_{n+\alpha}) = \delta_{n+\alpha}, \quad J_2 (\partial_{n+\alpha}) = -\delta_{n+\alpha}, \quad J_3 (\partial_{n+\alpha}) = -\partial_\alpha
\]

\( (3.1) \)

It is easy to show
\[
J_1^2 = J_2^2 = J_3^2 = -I_{TU'} \quad (3.2)
\]
\[
J_3 = J_1 \circ J_2 = -J_2 \circ J_1 \quad (3.3)
\]

Now by using \(J_1\) and \(J_3\) we construct two linear connections on \(U'\) as follows.

**Theorem 3.1.** Let \(\nabla\) be a Finsler connection on \(U'\). The differential operators \(D^a, a = 1, 3\), which are defined by
\[
D^a_X Y := \nabla_X^a v^a Y - J_a \nabla_X J_a b Y \quad \forall X, Y \in \Gamma(TM')
\]

\( (3.4) \)

are two linear connections on \(U'\). Also \(J_1\) and \(J_3\) are parallel with respect to \(D^1\) and \(D^3\), respectively.

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Proof. The proof is easy so we omit it.

Let \( T^{D^a}, a = 1, 3 \), denotes the torsion tensor of \( D^a \). It is simple to see,

\[
T^{D^a}(X, Y) = (\nabla_X vY - \nabla_Y vX - v[X, Y]) - J_a(\nabla_X J_a hY - \nabla_Y J_a hX - J_a h[X, Y]) \quad \forall X, Y \in \Gamma(TU').
\] (3.5)

Now we can introduce two families of new Finsler connections by using \( J_1 \) and \( J_3 \).

**Theorem 3.2.** Suppose that \( HM' \) is a non-linear connection on \( M' \) and \( S \) and \( T \) are two arbitrary skew-symmetric Finsler tensor fields of type \((1, 2)\) on \( F^{2n} \). Then there exists a unique linear connection \( \nabla^a \) on \( MU' \) satisfying the conditions:

1. \( \nabla^a \) is a metric connection.
2. \( T^{D^a} \), \( S \) and \( T \) satisfy,
   i. \( T^{D^a}(vX, vY) = S(vX, vY) \),
   ii. \( hT^{D^a}(hX, hY) = J_a T(J_a hY, J_a hX) \).

**Proof.** For any \( X, Y, Z \in \Gamma(TU') \) we define a linear connection \( \nabla^a \) on \( MU' \) by the following equations.

\[
2g(\nabla^a_{vX} vY, vZ) = vX(g(vY, vZ)) + vY(g(vZ, vX)) - vZ(g(vX, vY))
+ g(vY, vZ) + g(vZ, vX) - g(vX, vY)
\] (3.6)

and

\[
2g(\nabla^a_{J_a hY} J_a hZ) = hX(g(J_a hY, J_a hZ)) + hY(g(J_a hZ, J_a hX))
- hZ(g(J_a hX, J_a hY)) + g(J_a hY, J_a hZ)
i
+ g(J_a hZ, J_a hX) + g(J_a hZ, J_a hX)
i
- g(J_a hY, T(J_a hZ, J_a hX)) + g(J_a hX, T(J_a hY, J_a hY))
\] (3.7)

Now we show that \( g \) is parallel with respect to \( \nabla^a \). It is easy to see \( J_a \circ v = h \circ J_a \) and \( v \circ J_a = J_a \circ h \). After performing some computations for any \( X, Y, Z \in \Gamma(TU') \) we have,

\[
(\nabla^a_X g)(vY, vZ) = (\nabla^a_{vX+hX} g)(vY, vZ)
\]

\[
= vX(g(vY, vZ)) - g(\nabla^a_{vX} vY, vZ) - g(vY, \nabla^a_{vX} vZ)
+ hX(g(vY, vZ)) - g(\nabla^a_{J_a hY} J_a hZ, vZ) - g(vY, \nabla^a_{J_a hY} J_a hZ) = 0.
\]

So \( \nabla^a \) is a metric connection.

Locally we set

\[
\nabla^1_{\partial_j} \partial_i = F^i_{jk} \partial_k \quad \nabla^1_{\partial_j} \partial_i = \tilde{F}^i_{jk} \partial_k
\] (3.8)

\[
\nabla^1_{\partial_j} \partial_i = C^i_{jk} \partial_k \quad \nabla^1_{\partial_j} \partial_i = \tilde{C}^i_{jk} \partial_k
\] (3.9)

\[
S(\partial_j, \partial_i) = S^i_{jk} \partial_k \quad T(\partial_j, \partial_i) = T^i_{jk} \partial_k.
\] (3.10)

In the relation 3.6 let \( X = \partial_j, Y = \partial_i \) and \( Z = \partial_k \), then we can obtain the coefficients \( C^i_{jk} \) and \( \tilde{C}^i_{jk} \) as follows:

\[
\tilde{C}^i_{jk} = C^i_{jk} = \frac{1}{2} g^{ik} \{ \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ij} + S^i_{jl} g_{kh} + S^i_{jl} g_{kh} - S^i_{lj} g_{kh} \}.
\] (3.11)

By attention to the relation 3.1, for computing the coefficients \( F^i_{jk} \) and \( \tilde{F}^i_{jk} \) we must consider four cases for any connection \( \nabla^a \) as follows:

\[
F^k_{r\alpha \beta}, F^k_{\alpha n+\beta}, F^k_{n+\alpha \beta}, F^k_{n+\alpha n+\beta}
\] (3.12)

\[
\tilde{F}^k_{r\alpha \beta}, \tilde{F}^k_{\alpha n+\beta}, \tilde{F}^k_{n+\alpha \beta}, \tilde{F}^k_{n+\alpha n+\beta}
\] (3.13)

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The Computation of $F_{a\beta}^k$ and $F_{n+\alpha\beta}^k$.
In the equation 3.7 let $X = \delta_{\beta\alpha}, Y = \delta_{\alpha\beta}$ and $Z = \delta_{\beta\theta}$. Then we have,
\begin{align*}
2F_{a\beta}^k g_{\theta} & = \delta_{\beta}\alpha g_{\alpha\beta} + \delta_{\alpha}\beta g_{\beta\alpha} + \delta_{\beta}\alpha g_{\alpha\beta} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} - T_{a\beta}^h g_{\alpha h} - T_{a\alpha}^h g_{\beta h} \\
2F_{n+\alpha\beta}^k g_{\theta} & = \delta_{\beta}\alpha g_{\alpha\beta} + \delta_{\alpha}\beta g_{\beta\alpha} + \delta_{\beta}\alpha g_{\alpha\beta} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} - T_{a\beta}^h g_{\alpha h} - T_{a\alpha}^h g_{\beta h}.
\end{align*}
(3.14)

Now for the same $X$ and $Y$ let $Z = \delta_{n+\beta\theta}$, after some computations we have
\begin{align*}
2F_{a\beta}^k g_{\theta} & = \delta_{\beta}\alpha g_{\alpha\beta} + \delta_{\alpha}\beta g_{\beta\alpha} + \delta_{\beta}\alpha g_{\alpha\beta} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} - T_{a\beta}^h g_{\alpha h} - T_{a\alpha}^h g_{\beta h} \\
2F_{n+\alpha\beta}^k g_{\theta} & = \delta_{\beta}\alpha g_{\alpha\beta} + \delta_{\alpha}\beta g_{\beta\alpha} + \delta_{\beta}\alpha g_{\alpha\beta} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} - T_{a\beta}^h g_{\alpha h} - T_{a\alpha}^h g_{\beta h}.
\end{align*}
(3.16)

Now the relations (3.14,3.16) and the relations (3.15,3.17) respectively show that
\begin{align*}
F_{a\beta}^k & = \frac{1}{2} g_{kl} \{\delta_{\beta}\alpha g_{kl} + \delta_{\alpha}\beta g_{kl} + T_{a\beta}^h g_{kl} + T_{a\alpha}^h g_{kl} - T_{a\beta}^h g_{kl} - T_{a\alpha}^h g_{kl}\} \\
& \quad - \frac{1}{2} (g_{k\gamma} \delta_{\alpha\gamma} - g_{k\gamma} n+\theta) g_{\delta\alpha} \\
\bar{F}_{n+\alpha\beta}^k & = \frac{1}{2} g_{kl} \{\delta_{\beta}\alpha g_{kl} + \delta_{\alpha}\beta g_{kl} + T_{a\beta}^h g_{kl} + T_{a\alpha}^h g_{kl} - T_{a\beta}^h g_{kl} - T_{a\alpha}^h g_{kl}\} \\
& \quad + T_{n+\alpha\beta}^h g_{kl} + T_{n+\alpha\beta}^h g_{kl} - T_{n+\alpha\beta}^h g_{kl}.
\end{align*}
(3.18)

The Computation of $F_{a\beta}^k$ and $F_{n+\alpha\beta}^k$.
Similar to the pervious case in the equation 3.7, let $X = \delta_{n+\beta\alpha}, Y = \delta_{\alpha\beta}$ and $Z = \delta_{\beta\theta}$. Then we have,
\begin{align*}
2F_{a\beta}^k g_{\theta} & = \delta_{n+\beta\alpha} g_{\alpha\beta} - \delta_{\beta\alpha} g_{\alpha\beta} + \delta_{n+\beta\alpha} g_{\alpha\beta} - T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} \\
2\bar{F}_{n+\alpha\beta}^k g_{\theta} & = \delta_{n+\beta\alpha} g_{\alpha\beta} - \delta_{\beta\alpha} g_{\alpha\beta} + \delta_{n+\beta\alpha} g_{\alpha\beta} - T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h}.
\end{align*}
(3.20)

Now for $X = \delta_{n+\beta\alpha}, Y = \delta_{\alpha\beta}$ let $Z = \delta_{n+\theta\alpha}$, then we have
\begin{align*}
2F_{a\beta}^k g_{\theta} & = \delta_{n+\beta\alpha} g_{\alpha\beta} - \delta_{\beta\alpha} g_{\alpha\beta} + \delta_{n+\beta\alpha} g_{\alpha\beta} - T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} \\
2\bar{F}_{n+\alpha\beta}^k g_{\theta} & = \delta_{n+\beta\alpha} g_{\alpha\beta} - \delta_{\beta\alpha} g_{\alpha\beta} + \delta_{n+\beta\alpha} g_{\alpha\beta} - T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h} + T_{a\beta}^h g_{\alpha h} + T_{a\alpha}^h g_{\beta h}.
\end{align*}
(3.22)

Now the relations (3.20,3.22) and the relations (3.21,3.23) respectively show that
\begin{align*}
F_{a\beta}^k & = \frac{1}{2} g_{kl} \{\delta_{n+\beta\alpha} g_{kl} - \delta_{\beta\alpha} g_{kl} - T_{n+\beta\alpha} g_{kl} - T_{n+\beta\alpha} g_{kl} + T_{n+\beta\alpha} g_{kl} + T_{n+\beta\alpha} g_{kl}\} \\
& \quad + \frac{1}{2} (g_{k\gamma} \delta_{\beta\gamma} - g_{k\gamma} n+\theta) g_{\alpha\beta} \\
\bar{F}_{n+\alpha\beta}^k & = \frac{1}{2} g_{kl} \{\delta_{n+\beta\alpha} g_{kl} - \delta_{\beta\alpha} g_{kl} - T_{n+\beta\alpha} g_{kl} - T_{n+\beta\alpha} g_{kl} + T_{n+\beta\alpha} g_{kl} + T_{n+\beta\alpha} g_{kl}\} \\
& \quad + T_{n+\alpha\beta} g_{kl} + T_{n+\alpha\beta} g_{kl} - T_{n+\alpha\beta} g_{kl}.
\end{align*}
(3.24)

The Computation of $F_{n+\alpha\beta}^k$ and $\bar{F}_{n+\beta}^k$.
In 3.7 consider $X = \delta_{\beta\alpha}, Y = \delta_{n+\alpha}$ and $Z = \delta_{\beta\theta}$. Then we have
\begin{align*}
2F_{n+\alpha\beta} g_{\theta} & = \delta_{\beta\alpha} g_{\alpha\beta} - \delta_{n+\alpha\beta} g_{\beta\alpha} + \delta_{n+\alpha\beta} g_{\beta\alpha} - T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h} + T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h} \\
2\bar{F}_{n+\alpha\beta} g_{\theta} & = \delta_{\beta\alpha} g_{\alpha\beta} - \delta_{n+\alpha\beta} g_{\beta\alpha} + \delta_{n+\alpha\beta} g_{\beta\alpha} - T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h} + T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h}.
\end{align*}
(3.26)

\begin{align*}
2\bar{F}_{n+\alpha\beta} g_{\theta} & = \delta_{\beta\alpha} g_{\alpha\beta} - \delta_{n+\alpha\beta} g_{\beta\alpha} + \delta_{n+\alpha\beta} g_{\beta\alpha} - T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h} + T_{n+\beta\alpha} g_{\alpha h} + T_{n+\beta\alpha} g_{\beta h}.
\end{align*}
(3.27)
Now for the same $X$ and $Y$ let $Z = \delta_{n+\theta}$, then we have

\[
2F_{n+\alpha}^{k} g_{n+\theta} = -\delta \gamma g_{n+\alpha} n+\theta - \delta_{n+\alpha} g_{n+\theta} + \delta n+\alpha\beta n+\alpha
\]

(3.28)

\[
2F_{n+\alpha}^{k} g_{n+\theta} = \delta_{n+\alpha} g_{n+\theta} + \delta_{n+\alpha} g_{n+\theta} n+\beta - \delta_{n+\alpha} g_{n+\beta} n+\alpha
\]

(3.29)

Equations (3.26,3.28) and equations (3.27,3.29) respectively show that

\[
F_{n+\alpha}^{k} = \frac{1}{2} g^{kl} \{ \delta_{n+\alpha} g + \delta_{n+\alpha} g n+\alpha + \delta_{n+\alpha} g n+\alpha n+\alpha
\]

(3.30)

\[
F_{n+\alpha}^{k} = \frac{1}{2} g^{kl} \{ \delta_{n+\alpha} g + \delta_{n+\alpha} g n+\alpha - \delta_{n+\alpha} g n+\alpha n+\alpha
\]

(3.31)

The Computation of $F_{n+\alpha}^{k}$ and $F_{n+\alpha}^{k}$

Now for the last time in equation 3.7, let $X = \delta_{n+\theta}, Y = \delta_{n+\alpha}$ and $Z = \delta_{\theta}$. Then we have,

\[
2F_{n+\alpha}^{k} g_{n+\theta} = -\delta_{n+\alpha} g_{n+\theta} + \delta_{n+\alpha} g_{n+\theta} n+\alpha
\]

(3.32)

\[
2F_{n+\alpha}^{k} g_{n+\theta} = \delta_{n+\alpha} g_{n+\theta} + \delta_{n+\alpha} g_{n+\theta} n+\beta - \delta_{n+\alpha} g_{n+\beta} n+\alpha
\]

(3.33)

Similar to the previous cases for the same $X$ and $Y$, let $Z = \delta_{n+\theta},$ after some computations we have

\[
2F_{n+\alpha}^{k} g_{n+\theta} = -\delta_{n+\alpha} g_{n+\theta} + \delta_{n+\alpha} g_{n+\theta} n+\alpha
\]

(3.34)

\[
2F_{n+\alpha}^{k} g_{n+\theta} = \delta_{n+\alpha} g_{n+\theta} + \delta_{n+\alpha} g_{n+\theta} n+\beta - \delta_{n+\alpha} g_{n+\beta} n+\alpha
\]

(3.35)

Now by the equations (3.32,3.34) and (3.33,3.35) we respectively have

\[
F_{n+\alpha}^{k} = \frac{1}{2} g^{kl} \{ \delta_{n+\alpha} g + \delta_{n+\alpha} g n+\alpha + \delta_{n+\alpha} g n+\alpha n+\alpha
\]

(3.36)

\[
F_{n+\alpha}^{k} = \frac{1}{2} g^{kl} \{ \delta_{n+\theta} g + \delta_{n+\theta} g n+\theta - \delta_{n+\theta} g n+\theta n+\alpha
\]

(3.37)

Now by using the relations $J_{a} \circ v = h \circ J_{a}, v \circ J_{a} = J_{a} \circ h$ and 3.5, we have

\[
T^{D^{*}}(vX, vY) = \nabla_{v}XvY - \nabla_{v}YvX - [vX, vY]
\]

(3.38)

\[
hT^{D^{*}}(hX, hY) = J_{a}(\nabla_{hY}J_{a}hX - \nabla_{hX}J_{a}hY) + J_{a}h[1[hX, hY]]
\]

(3.39)

Suppose that $X, Y \in \Gamma(TU')$ are two arbitrary vector fields on $U'$ which have the following representations in local coordinates:

\[
X = z^{\alpha} \partial_{\alpha} + z^{n+\alpha} \partial_{n+\alpha} + \partial_{\alpha} + \partial_{n+\alpha}
\]

(3.40)

\[
Y = z^{\alpha} \partial_{\alpha} + z^{n+\alpha} \partial_{n+\alpha} + \partial_{\alpha} + \partial_{n+\alpha}
\]

(3.41)
After performing some computations we have:

\[ T^{D_\alpha}(vX, vY) = S(vX, vY) \]  
\[ hT^{D_\alpha}(hX, hY) = J_\alpha T(J_\alpha hY, J_\alpha hX). \]

The last equations show that \( \nabla^\alpha \) satisfies the second condition of the theorem. 

In this step we show that \( \nabla^\alpha \) is unique. Let \( \tilde{\nabla}^\alpha \) be another linear connection on \( \mathcal{U}' \) satisfying conditions (1) and (2).

By the equation \((\tilde{\nabla}_{vX} g)(vY, vZ) = 0 \) we have

\[ vX(g(vY, vZ)) = g(\tilde{\nabla}_{vX} vY, vZ) + g(vY, \tilde{\nabla}_{vX} vZ), \]

and so,

\[ vX(g(vY, vZ)) + vY(g(vZ, vX)) - vZ(g(vX, vY)) = g(2\tilde{\nabla}_{vX} vX + T^{\tilde{D}_\alpha}(vX, vY) + [vX, vY], vZ) + g(T^{D_\alpha}(vX, vZ) + [vX, vZ], vY) + g(T^{D_\alpha}(vZ, vY) + [vZ, vY], vX), \]

on the other hand we have,

\[ (\tilde{\nabla}_{vX} g)(vJ_a Y, vJ_a Z) = 0, \]

which shows that,

\[ hX(g(vJ_a Y, vJ_a Z)) + hY(g(vJ_a Z, vJ_a X)) - hZ(g(vJ_a X, vJ_a Y)) = g(J_a hT^{\tilde{D}_\alpha}(hX, hY) + 2\tilde{\nabla}_{hX}^a hX + J_a h[hX, hY], J_a hZ) + g(J_a hT^{D_\alpha}(hX, hZ) + J_a h[hX, hZ], J_a hY) + g(-J_a hT^{\tilde{D}_\alpha}(hZ, hY) - hX h[hY, hZ], J_a hX), \]

where \( \tilde{D}_\alpha \) is the linear connection induced by \( \tilde{\nabla}^\alpha \) and theorem 3.1.

The relations 3.45 and 3.46 show that \( \tilde{\nabla}^\alpha \) satisfies 3.6 and 3.7, respectively. Therefore \( \nabla^\alpha = \tilde{\nabla}^\alpha \).

It is a natural question whether we can use \( J_2 \) to introduce a new connection similar to \( J_1 \) and \( J_3 \)? The answer is negative because \( J_2 \) dose not involve with the non-linear connection and we can not introduce a connection \( D^2 \) in a similar way to \( D^1 \) and \( D^3 \) but we can have the following theorem.

**Theorem 3.3.** Suppose that \( \nabla \) is a Finsler connection on \( \mathcal{U}' \). Then the differential operator \( D \) which is defined by

\[ D_{X+Y} Z = \frac{1}{2} \{ \nabla_X vY - J_1 \nabla_X hY - J_2 \nabla_X J_2 vY - J_3 \nabla_X J_3 hY \}, \]

where \( X, Y, Z \in \Gamma(T\mathcal{U}') \), is a linear connection on \( \mathcal{U}' \). Also \( J_1, J_2 \) and \( J_3 \) are parallel with respect to \( D \).

**Proof.** For any \( X, Y, Z \in \Gamma(T\mathcal{U}') \) and \( f \in C^\infty(\mathcal{U'}) \) we have

\[ D_{fX+Y} Z = \frac{1}{2} \{ f\nabla_X vZ + \nabla_Y vZ - J_1(f\nabla_X hZ + \nabla_Y J_1 hZ) - J_2(f\nabla_X J_2 hZ + \nabla_Y J_2 vZ) - J_3(f\nabla_X J_3 hZ + \nabla_Y J_3 hZ) \} = fD_{X} Z + D_{Y} Z \]

\[ D_{X} (fY + Z) = \frac{1}{2} \{ Xf(vY) + f\nabla_X vY + \nabla_X vZ - J_1(Xf(J_1 hY) + f\nabla_X J_1 hY + \nabla_X J_1 hZ - J_2(Xf(J_2 vY) + f\nabla_X J_2 vY + \nabla_X J_2 vZ) - J_3(Xf(J_3 hY) + f\nabla_X J_3 hY + \nabla_X J_3 hZ) \} = \frac{1}{2} (2Xf(vY + hY)) + fD_{X} Y + D_{X} Z = Xf(Y) + fD_{X} Y + D_{X} Z. \]

Therefore \( D \) is a linear connection on \( \mathcal{U}' \).

Now we show that the almost hypercomplex structure is parallel with respect to \( D \).
Let $Y$ be a vector field on $\mathcal{U}'$ which have the representation $Y = x^\alpha \delta_\alpha + y^\alpha \delta_{n+\alpha} + z^\alpha \partial_\alpha + w^\alpha \partial_{n+\alpha}$ in local coordinates. Then by a simple computation we have

\[
(D_X J_1)(Y) = \frac{1}{2} \left\{ \nabla_X v J_1 Y - J_1 \nabla_X J_1 h J_1 Y - J_2 \nabla_X J_2 v J_1 Y - J_3 \nabla_X J_3 h J_1 Y \right\} + \frac{1}{2} \left\{ \nabla_X v J_1 Y - J_1 \nabla_X J_1 h Y - J_2 \nabla_X J_2 v Y - J_3 \nabla_X J_3 h Y \right\}
\]

(3.50)

In a similar way we can show $(D_X J_2)(Y) = (D_X J_3)(Y) = 0$. Hence $J_1$, $J_2$ and $J_3$ are parallel with respect to $D$. 

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