Scaling of geometric phase versus band structure in cluster-Ising models

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Introduction.— Gapped ground states define quantum phases of matter. Yet, they can be of very different nature. Some of them exhibit approximate orders on a local scale and they can be characterized by their symmetries. Others possess subtler orders that can only be captured by highly non-local observables. One of the major challenging themes in modern condensed matter physics, with applications to quantum technology, is to devise a unified understanding of all the possible quantum phases of matter \[1,5]. Indeed, the scientific community has been applying integrated methods by combining quantum information, foundational notions of quantum mechanics and many-body physics to study the problem \[6,9]. Recent outcomes in topological matter e.g., topological insulators, Weyl semi-metals \[10,12] and superconductors have demonstrated how the topology of the energy bands of the system can be useful in a novel way for analyzing the quantum phases of matter \[12,13]. Here, we study a specific many-body system that can display exotic orders by exploiting Berry phase and winding numbers \[14,16].

We focus on a one dimensional spin system whose ground state can be tuned to be an ordered state with local order parameter or to be a state with an exotic order of topological nature (so-called symmetry protected topologically ordered phases). Quantum phase transitions with dynamical critical exponents \(z = 1\) or \(z = 2\) are found. Quantum phase transitions are analyzed through finite-size scaling of the geometric phase accumulated when the spins of the lattice perform an adiabatic precession. In particular, we quantify the scaling behavior of the geometric phase in relation with the topology and low energy properties of the band structure of the system.

We study the phase diagram of a class of models in which a generalized cluster interaction can be quenched by Ising exchange interaction and external magnetic field. We characterize the various phases through winding numbers. They may be ordinary phases with local order parameter or exotic ones, known as symmetry protected topologically ordered phases. Quantum phase transitions with quantum information, foundational notions of quantum mechanics. Others possess subtler orders that can only be captured by highly non-local observables. One of the major challenging themes in modern condensed matter physics, with applications to quantum technology, is to devise a unified understanding of all the possible quantum phases of matter \[1,5]. Indeed, the scientific community has been applying integrated methods by combining quantum information, foundational notions of quantum mechanics and many-body physics to study the problem \[6,9]. Recent outcomes in topological matter e.g., topological insulators, Weyl semi-metals \[10,12] and superconductors have demonstrated how the topology of the energy bands of the system can be useful in a novel way for analyzing the quantum phases of matter \[12,13]. Here, we study a specific many-body system that can display exotic orders by exploiting Berry phase and winding numbers \[14,16].

Finite-size scaling analysis of geometric phase to reveal order-disorder quantum phase transition was studied in the XY\textsuperscript{+} spin chain \[23\].

In this paper, we consider a set of localized spins in one dimensional lattice enjoying a specific higher order (multispin) interaction; at the same time, such interaction competes with Ising exchange; finally the chain is placed in a transverse magnetic field. Indeed, the systems under scrutiny are generalizations of the cluster-Ising model that was formulated in the cross-fertilization area between many-body physics, quantum correlations and ultracold atoms \[26,28\]. The cluster-Ising model displays a second order quantum phase transition between a phase with local order parameter and a symmetry protected topological quantum phase \[29,30\]. Here, we study the phase diagrams of the generalized cluster-Ising models by looking at the winding numbers of the ground states (for a specific subclass of models studied here, winding numbers were recently studied in \[31\]). We explore the criticality of the system by studying the finite-size scaling of the geometric phase. In particular, we find that critical points with nonlinear low-energy dispersions are characterized by an anomalous logarithmic scaling of the geometric phase.

Model.—We consider a class of models describing interactions between \(l + 1\) spins competing with exchange interaction, in external field. The Hamiltonian reads \[20,21,29,30\]

\[
H^{(l)} = \sum_{j=-L}^{M} -\lambda \sigma_j^z Z_{j,l} \sigma_{j+1}^x + a \sigma_j^y \sigma_{j+1}^y + g \sigma_j^z, \quad (1)
\]

with \(Z_{j,l} = \sigma_{j+1}^z \cdots \sigma_{j+l-1}^z\) and \(M = (L - 1)/2\) for odd \(L\). The operators \(\sigma_n^\alpha (\alpha = x, y, z)\) are the Pauli matri-
FIG. 1: (Color online). Phase diagrams of cluster-Ising models. (a)-(f) are phase diagrams of the system for $l$ from 1 to 6. We choose $\lambda = 1$. The horizontal and vertical axes are represented by $g$ and $a$, respectively. The abbreviations mean different phases: paramagnetic (P), ferromagnetic (F), antiferromagnetic (AF), cluster (C); the superscripts X, Y specify the directions of the orders. (a)-(f) are phase diagrams of the system for $l = 2$ or generalized cluster phases ($l > 2$). $C^*_l$ mean dual (generalized) cluster phases (see the text for the definition).

TABLE I: Phase and winding number for specific part in the cluster-Ising models.

| Interaction | Phase | Winding number |
|-------------|-------|----------------|
| $\pm \sum_j \sigma_j^y$ | AFM$^y$, FM$^y$ | +1 |
| $\pm \sum_j \sigma_j^y \sigma_{j+1}^y$ | AFM$^x$, FM$^x$ | -1 |
| $\pm \sum_j \sigma_j^z Z_{j,l} \sigma_{j+1}^z$ | $C^*_l$, $C_l$ | $-l$ |
the Zeeman field is the control parameter. When \( g > 0 \), phases with even integer winding numbers are generated. The Ising interaction \( a \) tunes the ferromagnetic or antiferromagnetic phase. The roles of \( g \) and \( a \) are exchanged for odd \( l \) (Figs. 1(a), 1(c) and 1(e)). The \( C_2 \) and \( C_2^* \) (the so-called dual cluster phase) display a string order of the cluster state with two Majorana modes at the edges of the system; such two phases are characterized by string order parameters with different spin polarizations at the edges [21]. Similarly, \( C_3 \) phases are cluster phases with three Majorana modes at the edges of the system. \( C_3 \) and \( C_3^* \) phases in Fig. 1(d) are distinguished from each other by the negative and positive Ising interaction \( a \). The \( C_m \) and \( C_m^* \) phases with \( m > 2 \) in Figs. 1(c), 1(f) are defined with a similar logic. The different phases \( C_m \) with fixed \( l \) in the different panels of Fig. 1 can be connected adiabatically: a fixed phase \( C_m \) of a given Hamiltonian \( H^{(l)} \) evolves into \( C_m^* \) of \( H^{(l+1)} \) under \( H^{(l,l+1)} = (t - 1)H^{(l)} + tH^{(l+1)} \), \( t \in [0, 1] \).

Phases boundaries are obtained as the combination of the Hamiltonian parameters for which a specific low-energy mode emerges (for a specific value of critical momentum \( k_c \)) in the band structure. The critical momenta of the phase boundaries Figs. 1(b) and 1(c) are shown in Figs. 2(a) and 2(b), respectively. The green-solid lines in Figs. 1 are in the XX universality class. Along there, the critical momentum depends on the parameters \( a \) and \( g \). For the blue-dotted (red-dashed) straight lines indicating Ising phase transitions in Fig. 1 there is one Dirac point at \( k_c = 0 (\pi) \). The XX and Ising transitions have a topological difference. The two phases separated by the XX line have winding number difference equals to 2. However, for ground states separated by the Ising type transition, the winding number difference is 1. Elaborating on the findings for \( l = 2 \) [29], Lahtinen and Ardonne demonstrated that the multicritical points of the system may be indeed characterized by the \( \text{so}(N)_1 \) conformal field theory. For \( M_0 \), with two Ising criticalities, the XX gets to the \( \text{so}(2)_1 \) universality class. If the cluster type of order is involved, more branches (\( \geq 3 \)) with linear dispersions show up at the criticality (Fig. 2). The multicritical points are combined in the \( \text{so}(l)_1 \) and \( \text{so}(l + 1)_1 \) universality classes [20]. Specifically, in \( M_3 \) there are two degenerate points in Brillouin zone. As for \( M_4 \), there are three degenerate points in the band structure. Therefore, \( M_4 \) enjoys a \( \text{so}(3)_1 \) criticality rather than the XX one. It is interesting to note that there is quadratic band touching at \( M_1 \) and \( M_4 \). Indeed, a quadratic band touching may lead to interesting non-Fermi liquid interaction effects [37] [38].

**Scaling of geometric phase.**—Berry phase arises when the spin variables localized in the lattice points along the chain are rotated adiabatically [22]. The rotating system can be described by \( H_R^{(l)} = \mathcal{R}_l H^{(l)} \mathcal{R}_l^\dagger \), with \( \mathcal{R}_l = e^{i \sum_{j=1}^M \phi_j} \). For our model Eq. (1), the ground state of \( H_R^{(l)} \) is the vacuum of free fermionic modes: \( |gs\rangle^{(l)} = \prod_{k=1}^M |gs\rangle_k^{(l)} \) with \( |gs\rangle_k^{(l)} = |cos \frac{g}{2(l+1)}|0\rangle_k |0\rangle_{-k} - i e^{-i \phi} \sin \frac{g}{2(l+1)} |1\rangle_k |1\rangle_{-k} \) where \( |0\rangle_k, |1\rangle_k \) are the vacuum and single excitation of the \( k \)-th mode, \( c_k \), respectively. Adiabatically varying the angle \( \phi \) from 0 to \( \pi \), the geometric phase of \( |gs\rangle \) results [22] [23]:

\[
\varphi_{gs}^{(l)} = \frac{i}{M} \int_0^{\pi} \langle gs \rangle \langle \partial_\phi |gs\rangle \rangle \langle d\phi = \frac{\pi}{M} \sum_k (1 - \cos \theta_k^{(l)}). \quad (3)
\]

In Fig. 1(a) the horizontal line (green-solid) at \( a = -1 \), \( -2 < g < 2 \) defines a XX critical state with quasi-long-range order. In such a state, the Berry phase is identically vanishing. If a non-trivial cluster state order is involved the universality class of the transition changes: multicritical points \( M_2 \) and \( M_4 \) appear as shown in Fig. 1(b). The Berry phase near the criticality (green-solid) is non-vanishing. We explore the criticality via \( d\varphi/dg \).

(I) Scaling close to quantum phase transitions with the critical exponent \( z = 1 \). The scaling ansatz for (derivative of) the geometric phase is [23]:

\[
\frac{d\varphi_{gs}^{(l)}}{dg} |_{g_m} \approx k_1 \ln N + \text{const}, \quad (4)
\]

\[
\frac{d\varphi_{gs}^{(l)}}{dg} \approx k_2 \ln |g - g_c| + \text{const}, \quad (5)
\]
where \( g_c \) is the critical value of \( g \) for infinite long spin chain, and \( g_m \) marks the anomaly for the finite-size system. According to the scaling ansatz, in the case of logarithmic singularities, the ratio \(|\kappa_2/\kappa_1|\) is the exponent \( \nu \) that governs the divergence of correlation length. We note that the scaling behavior is related to the band structure at low energy.

As for topological quantum phase transitions, we first consider \( l = 2 \). The critical properties are found symmetric about \( a = 0 \) as shown in Fig. 1(b). We discuss the phase boundaries with \( 0 \leq a \leq 2 \). For \( a = 0 \), phase transitions occur at \(|g| = 1\) which separates a paramagnetic and a cluster phase. As expected by looking at the dispersion curves, \( M_3 \) and \( M_5 \) share the same criticality. Similarly, the quantum multicritical points \( M_2 \) and \( M_4 \) involving the cluster phase enjoy the same scaling behavior. In Figs. 3(a) and 3(b) we present the scaling behaviors characterized by Eqs. (4) and (5). For critical regime with critical exponent \( z = 1 \) and linear low-energy dispersions, the ratio \(|\kappa_2/\kappa_1|\sim 1\). The scaling parameter \( \kappa_1 \) for \( l = 2 \) is represented in (c). We find that for multicritical points with multiple degeneracies in energy bands, the scaling coefficients are discontinuously connected to the neighboring critical points which share the same topologies of band structures. The discontinuity (sudden change) of the scaling parameter \( \kappa_1 \) renders the topological change of band structure. The smooth variation of \( \kappa_1 \) in the XX criticality arises from the fact that slopes of linear dispersions change depending on Ising exchange interaction and transverse magnetic field. In Fig. 3(d), we show \( \kappa_1 \) for \( l = 3 \) and also observe the sudden change at multicritical points. Similar behaviors also exist for \( \kappa_2 \).

(II) Scaling close to quantum phase transitions with \( z = 2 \). Quantum phase transitions implied in Eq. (1) may be characterised by a low-energy dispersions \( \sim k^2 \) (e.g., \( M_1 \) and \( M_5 \) for \( l = 2 \)). Consistently with the scaling theory, the dynamical critical index for such phase transition is \( z = 2 \). The scaling behavior of this band topology at critical regime are shown in Fig. 4. For \( M_1 \) and \( M_5 \), we found that the scaling ansatz Eqs. (4) and (5) should be modified to

\[
\ln \frac{d^2 \phi_{gs}}{dg^2} \bigg|_{g_m} \simeq \tilde{\kappa}_1 \ln N + \text{const},
\]

and

\[
\ln \frac{d^2 \phi_{gs}}{dg^2} \bigg|_{g_m} \simeq \tilde{\kappa}_2 \ln |g - g_c| + \text{const}.
\]

\( \tilde{\kappa}_1 \) and \( \tilde{\kappa}_2 \) in Figs. 4(a) and 4(b) are found 1.999
and $-0.492$, respectively. Close to critical points with quadratic dispersions for $I > 2$, we find a similar log scaling behavior.

Conclusions.— We have studied the criticality of generalized cluster-Ising models through the scaling properties of the geometric phase. The criticality with parameter-dependent critical momentum is generically found of the $XX$ type. At the multicritical points with linear gapless modes the quantum phase transitions are in the $so(N)_{1}$ universality classes. We have found that the critical points with linear and quadratic low-energy dispersions obey different scaling ansatz. Specifically, the critical point with critical exponent $z = 2$ shows anomalous logarithmic scaling behavior which is markedly different from that one with $z = 1$, with linear dispersions. We also employed the scaling parameters to study the band topology in critical regime. We have also found that the scaling parameters change smoothly along the phase boundary with $z = 1$. In contrast, the scaling parameters are found very sensitive to topological change (Figs. 3(c) and 3(d)). In this paper, we observed that there is a close connection between topological phase transition, quantum criticality, energy band structure and geometric phase.

Our approach may be generalized to other spin chain with multispin interactions, like the Wen-plaquette model, simulated in nuclear magnetic resonance systems, or Baxter-Wu models. Recently, multispin interactions were demonstrated to arise in Floquet driven lattices. We finally observe that the multicritical points in the generalized cluster-Ising models can be used to investigate nonequilibrium dynamics in many-body physics. It is interesting to study the nonadiabatic driving scheme across the multicritical points in cluster-Ising spin chain and the interplay between geometric phase and dynamics in the near future.

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