Entanglement of formation for qubit-qudit system using partition of qudit system into a set of qubit system

Wen-Chao Qiang and W. B. Cardoso

Faculty of Science, Xi’an University of Architecture and Technology, Xi’an, 710055, China
Instituto de Física, Universidade Federal de Goiás, 74.001-970, Goiânia - GO, Brazil

Gerjuoy [Phys. Rev. A 67, 052308 (2003)] has derived a closed-form lower bound for the entanglement of formation of a mixed qubit-qudit system (qudit system has $d$ levels with $d \geq 3$). In this paper, inspired by Gerjuoy’s method, we propose a scheme that partitions a qubit-qudit system into $d(d−1)/2$ qubit-qudit systems, which can be treated by all known methods pertinent to qubit-qudit system. The method is demonstrated by a qubit-qudit system (The levels of qudit are $d = 3$ and $d = 5$, respectively).

Keywords: entanglement of formation, qubit-qudit system, partition

I. INTRODUCTION

Entanglement of quantum systems is an important physical resource to realize quantum information tasks and quantum computation [1]. The quantitative measure of entanglement is one of the main research areas in quantum information theory and quantum computation [2], which have attracted much attention of many researchers. In this sense, many useful measures were developed, such as: concurrence [3–11], entanglement of formation (EOF) [12–17], geometric measure [18–22], entanglement witness [23–25], quantum discord [24, 25], three-tangle [26], etc. These entanglement measurements are usually defined first for pure states and then are extended to mixed states via the convex roof construction. Because it requires complicated optimization procedure, generally speaking, computing an entanglement measurement for a given mixed quantum state is difficult. To the best of our knowledge, only a few analytic expressions of some entanglement measurements for some special quantum systems, for example, the EOF and geometric measure of qubit-qubit states and isotropic states, are obtained [20, 26]. Some numerical algorithms for computing some entanglement measures were also developed [27, 28].

The concurrence and the EOF among all entanglement measurements play an essential role due to some of other entanglement measurements can be expressed by concurrence and the method employed to derive analytic expressions of the EOF can be used to derive analytic expressions of other entanglement measurements. Wootters have obtained an elegant formula for qubit-qubit system [3]. Wei and Goldbart have also derived an analytic expression of the geometric measure for two-qubit mixed states [18]. Due to the fact of the concurrence, as defined originally, is only suitable for qubit-qudit systems and the optimization process to get analytic expressions for the EOF of a general entangled state in higher-dimensional space to be complicated, several schemes were proposed to find the lower bounds of the concurrence and the EOF of general entanglement mixed states [7, 8, 18]. Using the Schmidt decomposition theorem [1], Gerjuoy derived the lower bounds of the concurrence and the EOF of any qubit-qudit system, and easily obtained Wootters’ formula. The method employs a set of $(2 \times d) \times (2 \times d)$ matrices $S_{ij}^y$, that are constructed via $\sigma_y^{(2)} \otimes \sigma_y^{(d)}$, where $\sigma_y^{(2)} (\sigma_y^{(d)})$ is the usual Pauli matrices for the qubit (qudit) case. In this sense, in the present paper we propose a scheme that partitions a qubit-qudit system into $d(d−1)/2$ qubit-qudit systems, which can be treated by all known methods pertinent to qubit-qudit system.

This paper is organized as follows. In the next section, we analyze Gerjuoy’s method and propose our scheme, which simplify Gerjuoy’s procedure. In the section II, two examples, qubit-qutrit ($d = 3$) and qubit-qudit ($d = 5$) systems, are given to illustrate our scheme. The summary and discussion are given in section III.

II. ANALYSES OF GEJUOY’S SCHEME AND PARTITION OF QUBIT-QUDIT SYSTEM INTO QUBIT-QUBIT SYSTEM

To find the lower bound of entanglement of formation (EOF) of the qubit-qudit system, E. Gerjuoy first defined $d(d−1)/2$ symmetric square matrices $S_{ij}^y, 0 \leq i \leq d − 2$ and $j > i$, whose elements all are zero, except for

\[
S^y_{i,j+d} = S^y_{j+d,i} = 1, \quad (1a)
\]

\[
S^y_{i,j+d} = S^y_{i+d,j} = -1. \quad (1b)
\]

Second, he defined

\[
C_{ij}(\rho) = \max(0, \lambda_1^y - \lambda_2^y - \lambda_3^y - \lambda_4^y), \quad (2)
\]

where the $\lambda^y$, ordered decreasingly, are the square roots of the four largest eigenvalues of the matrix $\rho S^y S^y \rho^*$. $\rho$ is the density matrix of the qubit-qudit system and $\rho^*$ is its conjugate. Thirdly, he denoted the lower bound of the concurrence $C(\rho)$ of the qubit-qudit system by $C_{dB}(\rho)$.

\[
C_{dB} = \left[ \sum_{j>i}^{d-2} \sum_{i=0}^{d-2} C_{ij}^2(\rho) \right]^{1/2} \leq C(\rho). \quad (3)
\]

\[\text{EOF}\] of general entanglement mixed states $\rho$. Using the Schmidt decomposition theorem [1], Gerjuoy derived the lower bounds of the concurrence and the EOF of any qubit-qudit system, and easily obtained Wootters’ formula. The method employs a set of $(2 \times d) \times (2 \times d)$ matrices $S_{ij}^y$, that are constructed via $\sigma_y^{(2)} \otimes \sigma_y^{(d)}$, where $\sigma_y^{(2)} (\sigma_y^{(d)})$ is the usual Pauli matrices for the qubit (qudit) case. In this sense, in the present paper we propose a scheme that partitions a qubit-qudit system into $d(d−1)/2$ qubit-qudit systems, which can be treated by all known methods pertinent to qubit-qudit system.

This paper is organized as follows. In the next section, we analyze Gerjuoy’s method and propose our scheme, which simplify Gerjuoy’s procedure. In the section II, two examples, qubit-qutrit ($d = 3$) and qubit-qudit ($d = 5$) systems, are given to illustrate our scheme. The summary and discussion are given in section III.

II. ANALYSES OF GEJUOY’S SCHEME AND PARTITION OF QUBIT-QUDIT SYSTEM INTO QUBIT-QUBIT SYSTEM

To find the lower bound of entanglement of formation (EOF) of the qubit-qudit system, E. Gerjuoy first defined $d(d−1)/2$ symmetric square matrices $S_{ij}^y, 0 \leq i \leq d − 2$ and $j > i$, whose elements all are zero, except for

$$S^y_{i,j+d} = S^y_{j+d,i} = 1, \quad (1a)$$

$$S^y_{i,j+d} = S^y_{i+d,j} = -1. \quad (1b)$$

Second, he defined

$$C_{ij}(\rho) = \max(0, \lambda_1^y - \lambda_2^y - \lambda_3^y - \lambda_4^y), \quad (2)$$

where the $\lambda^y$, ordered decreasingly, are the square roots of the four largest eigenvalues of the matrix $\rho S^y S^y \rho^*$. $\rho$ is the density matrix of the qubit-qudit system and $\rho^*$ is its conjugate. Thirdly, he denoted the lower bound of the concurrence $C(\rho)$ of the qubit-qudit system by $C_{dB}(\rho)$.

$$C_{dB} = \left[ \sum_{j>i}^{d-2} \sum_{i=0}^{d-2} C_{ij}^2(\rho) \right]^{1/2} \leq C(\rho). \quad (3)$$

* Corresponding author.
E-mail address: qwqj@163.com (Wen-Chao Qiang).
The desired lower bound on the qubit-qudit EOF is $\varepsilon[C_{dB}(\rho)]$.

For a qubit-qudit mixed state system, there are only three $S^{ij}$ expressed as $S_x$, $S_y$ and $S_z$,

$$S_x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$ (4)

$$S_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$ (5)

$$S_z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$ (6)

respectively. For the qubit-qudit mixed state, there is only one $S^{ij}$ denoted as

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$ (7)

Note that the Eq. (7) is a matrix constructed via $\sigma_x^A \otimes \sigma_y^B$, with $A$ and $B$ denoting the indexes of qubit $A$ and qubit $B$, respectively. Considering a mixed qubit-qudit quantum system with levels $\{0\}$ and $\{1\}$ one can construct the Pauli operator $\sigma_y$ for the subsystems $A$ and $B$ given by

$$\sigma_y = i (|1_x⟩⟨0_x| - |0_x⟩⟨1_x|), \quad (x = A, B).$$ (8)

Then, if we consider a mixed qubit-qudit system composed of subsystem $A$ with two levels $\{0_A\}$ and $\{1_A\}$ and a subsystem $B$ with three levels $\{0_B\}$, $\{1_B\}$, $\{2_B\}$, the Pauli operator $\sigma_y$ can be now expressed by Dirac notation as:

$$\sigma_y = i (|j_x⟩⟨i_x| - |i_x⟩⟨j_x|), \quad (x = A, B),$$ (9)

where $\{i, j\} = \{0, 1\}$ for $x = A$ and $\{i, j\} = \{0, 1\}$ and $\{1, 2\}$, respectively, for $x = B$. It is easy to test the matrix forms of $\sigma_{y,01}^A \otimes \sigma_{y,01}^B$, $\sigma_{y,01}^A \otimes \sigma_{y,02}^B$ and $\sigma_{y,01}^A \otimes \sigma_{y,12}^B$ in $\{|0_1, 0_2\}$, $\{|0_1, 1_2\}$, $\{|0_1, 2_2\}$, $\{|1_1, 0_2\}$, $\{|1_1, 1_2\}$, $\{|1_1, 2_2\}$ are $-S_x$, $-S_y$ and $-S_z$, respectively. Generally speaking, for a qubit-qudit mixed state system the matrix forms of $\sigma_{y,01}^A \otimes \sigma_{y,01}^B (i < j)$ are $S^{ij}$.

It is notable that the difference between the matrix expression of $\sigma_{y,01}^A \otimes \sigma_{y,01}^B$ and $S^{ij}$ is only a minus. It doesn’t affect the eigenvalues of $\rho S^{ij} \rho^*, S^{ij}$ if we replace $S^{ij}$ by $\sigma_{y,01}^A \otimes \sigma_{y,01}^B$.

It is now clear that Gerjuoy’s approach, in fact, is to treat the qubit-qudit mixed system as a set of $d!/(2!d-2)!$ qubit-qudit system. Therefore, for such a system, we can take any two levels of qubit subsystem to combine with the qubit subsystem as a qubit-qudit. Then, all methods to solving the qubit-qudit problem can be used. In our present case, we only need to use matrix $S$ instead of $S^{ij}$ (or $S_x$, $S_y$ and $S_z$) to calculate the concurrence $C^{ij}(C_x, C_y$ and $C_z$) for those qubit-qudit subsystem. This will greatly simplify the calculation. In the next section, we shall demonstrate our proposal by a concrete example.

### III. Illustrative Examples

To illustrate our method let us consider two atoms (A and B), each of them interacting resonantly with a single quantized mode of a cavity field (system C) in a Fock state. This physical situation is described by the two-atom Tavis-Cummings (TC) Hamiltonian: $H = \hbar g [(\sigma_A + \sigma_B)a^*_C + (\sigma_A^+ + \sigma_B^+)a_C]$, where $\sigma_j$ and $\sigma_j^+$ are the Pauli ladder operators for the $j$th atom, $a(a^\dagger)$ is the annihilation (creation) operator for photons in cavity $C$, and $g$ is the coupling constant. We assume the system is initially in the state $|\psi(0)\rangle = \alpha|0_0, 0_1, n_c\rangle + \beta|1_1, 1_1, n_c\rangle$. Since the TC Hamiltonian preserves the total number of excitations, the cavity mode will evolve within a five-dimensional Hilbert space spanned by $\{|(n-2)_c\rangle, |(n-1)_c\rangle, |n_c\rangle, |(n+1)_c\rangle, |(n+2)_c\rangle\}$ for $n \geq 2$. When $n = 0, 1$ the dimension will be 3 and 4, respectively. On the other hand, the atomic system will evolve within the subspace $\{|0_0, 0_1\rangle, |+\rangle, |1_1, 1_1\rangle\}$ with $|+\rangle = |1_1, 0_1\rangle + |0_1, 1_1\rangle)/\sqrt{2}$ independently of $n$. By solving the Schrödinger equation, the system at time $t$ is described by the state

$$|\psi(t)\rangle = c_1(t)|0_0, 0_1\rangle|(n+2)_c\rangle + c_2(t)|+\rangle|(n+1)_c\rangle$$

$$+ c_3(t)|1_1, 1_1\rangle|n_c\rangle + c_4(t)|0_0, 0_1\rangle|n-1_c\rangle + c_5(t)|+\rangle|(n-1)_c\rangle + c_6(t)|1_1, 1_1\rangle|n-2_c\rangle,$$ (10)

where the probability amplitudes are

$$c_1(t) = -\beta\sqrt{(n+1)(n+2)}/[2n+3][1 - \cos(\sqrt{2}(2n+3)gt)],$$ (11)

$$c_2(t) = -\frac{i\beta\sqrt{n+1}}{\sqrt{2n+3}}\sin(\sqrt{2(2n+3)gt}),$$ (12)

$$c_3(t) = \beta\left[1 - \frac{n+1}{2n+3}[1 - \cos(\sqrt{2}(2n+3)gt)]\right],$$ (13)

$$c_4(t) = a\left[1 - \frac{n}{2n-1}[1 - \cos(\sqrt{2}(2n-1)gt)]\right].$$ (14)
\[ c_5(t) = -\frac{i\alpha \sqrt{n}}{\sqrt{2n-1}} \sin(\sqrt{2(2n-1)}gt), \] (15)

\[ c_6(t) = -\frac{\alpha \sqrt{n(n-1)}}{2n-1} [1 - \cos(\sqrt{2(2n-1)}gt)]. \] (16)

Now, we take trace of density operator \( \rho = |\psi(t)\rangle \langle \psi(t)| \) over atom \( B \) resulting in the reduced density operator of the qubit-qudit system \( \rho_{AC} \).

### A. Qubit-Qutrit case

When \( n = 0 \), atom \( A \) and cavity \( C \) compose a qubit-qutrit system. As described in the above section, we delete terms not containing \(|i_A, 0c\rangle \langle j_A, 0c|, |i_A, 0c\rangle \langle j_A, 1c|, |i_A, 1c\rangle \langle j_A, 0c| \) and \(|i_A, 1c\rangle \langle j_A, 1c| \) \( (i, j = 0, 1) \) in \( \rho_{AC} \) to form

\[ \rho_{1A}^{01} = c_3^2 |1_A, 0c\rangle \langle 1_A, 0c| + c_4^2 |0_A, 0c\rangle \langle 0_A, 0c| \] 
\[ + \frac{1}{2} c_2^2 |0_A, 1c\rangle \langle 0_A, 1c| + |1_A, 1c\rangle \langle 1_A, 1c| \] 
\[ + \frac{1}{\sqrt{2}} (c_2 c_1^2 |1_A, 1c\rangle \langle 0_A, 0c| + c_2 c_4 |0_A, 0c\rangle \langle 1_A, 1c|) \] 
\[ + \frac{1}{\sqrt{2}} (c_2 c_5 |1_A, 0c\rangle \langle 0_A, 1c| + c_2 c_3 |0_A, 1c\rangle \langle 1_A, 0c|), \] (17)

The matrix form of which is

\[ \rho_{1A}^{01} = \begin{pmatrix} c_3^2 & 0 & 0 & c_4^2 c_2^2 / \sqrt{2} \\ 0 & c_2^2 c_1^2 / 2 & c_2 c_4 c_2^3 / \sqrt{2} & 0 \\ 0 & 0 & c_2 c_4 c_2^5 / \sqrt{2} & 0 \\ c_2 c_3^2 / \sqrt{2} & 0 & 0 & c_2 c_4 c_2^7 / 2 \end{pmatrix}. \] (18)

Similarly we obtain matrices \( \rho_{1A}^{02} \) and \( \rho_{1A}^{12} \) respectively

\[ \rho_{1A}^{02} = \begin{pmatrix} c_4 c_1^2 & 0 & 0 & 0 \\ c_2 c_1^2 c_4 & 0 & 0 & 0 \\ 0 & c_2 c_1^2 c_4 & 0 & 0 \\ 0 & 0 & c_2 c_1^2 c_4 & 0 \end{pmatrix}, \] (19)

\[ \rho_{1A}^{12} = \begin{pmatrix} c_2 c_4^2 / 2 & 0 & 0 & 0 \\ 0 & c_1 c_4^2 & c_2 c_4^2 / \sqrt{2} & 0 \\ 0 & 0 & c_2 c_4^2 & 0 \end{pmatrix}. \] (20)

The matrices \( \rho_{1A}^{01} \) and \( \rho_{1A}^{12} \) are X form \([28]\). The corresponding concurrences can be read out

\[ C_x = C_{01} = \sqrt{2} |c_2 c_4| = |c_2 c_3|, \] (21)

\[ C_2 = C_{12} = \sqrt{2} |c_2 c_4| = |c_2 c_3|. \] (22)

Unfortunately, The matrix \( \rho_{1A}^{02} \) is not the X form. The square roots of four eigenvalues of \( \rho_{1A}^{02} S \rho_{1A}^{02} S \) are

\[ \{0, 0, |c_1 c_3|, |c_1 c_3|\}, \] therefore, \( C_y = C_{02} = 0 \). Consequently, the lower bound of concurrence \( C(\rho) \) of this qubit-qutrit system is \( C_{AC} = \sqrt{2} |c_2 c_4|^2 + (|c_2 c_4| - |c_2 c_3|)^2 \).

\[ \rho_{AC}^{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_3 c_2 c_4 & 0 & 0 \\ 0 & 0 & c_3 c_2 c_4 & 0 \\ 0 & 0 & 0 & c_3 c_2 c_4 \end{pmatrix}, \] (23)

\[ \rho_{AC}^{02} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_3 c_2 c_4 & 0 & 0 \\ 0 & 0 & c_3 c_2 c_4 & 0 \\ 0 & 0 & 0 & c_3 c_2 c_4 \end{pmatrix}, \] (24)

\[ \rho_{AC}^{03} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_3 c_2 c_4 & 0 & 0 \\ 0 & 0 & c_3 c_2 c_4 & 0 \\ 0 & 0 & 0 & c_3 c_2 c_4 \end{pmatrix}, \] (25)

\[ \rho_{AC}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_3 c_2 c_4 & 0 & 0 \\ 0 & 0 & c_3 c_2 c_4 & 0 \\ 0 & 0 & 0 & c_3 c_2 c_4 \end{pmatrix}. \]
We plot the evolution of $E_{AC}$ with the dimensionless time $\tau = \sqrt{14\gamma t}/(6\pi)$ in Fig. 2. $E_{AC}$ is also computed according to the lower bound of $C_{AC}$ given by Ref. [9] and plotted in the same figure for comparison. Though two lines in Fig. 2 have different trends in some interval of $\tau$, their global behaviors of evolution with time are basically the same.

### IV. SUMMARY

We have analyzed Gerjuoy’s approach on calculating the lower bound on entanglement of formation for qubit-qudit system and we found that his method, in fact, is to treat qubit-qudit system as a set of qubit-qubit system. Therefore, we proposed a simple scheme to solve qubit-qudit problem. The scheme consists of three steps: (1) partition the qudit system into a set of qubit system; (2) compose the original qubit and partitioned qubit into a set of qubit-qbdt systems and treat them by all methods suitable to qubit-qubit system. Find the measurements you want for every qubit-qubit system; (3) obtain the measurement of whole qubit-qudit system. For the case discussed in the present paper, we calculated the concurrences for every qubit-qubit system and the lower bound of the concurrence of the qubit-qudit or qubit-qudit system. Our method has the advantage of avoiding finding many matrices $S^{ij}$ and only using one matrix $S = \sigma_y \otimes \sigma_y$. This method greatly simplified the calculation about the measurement of qubit-qudit system. We hope this method can be extended to treat other problems of the qubit-qudit system.

### Acknowledgments

This work is supported by the Special Funds for Theoretical Physics of the National Natural Science Foundation of China (Grant No.11147161). The partial support by the CNPq and INCT-IQ (WBC) are also acknowledged.
FIG. 2: (Color online) The evolution of entanglement of formation between the atom $A$ and the cavity mode $C$ for the initial state $|\psi(0)\rangle = (\alpha |0_0\rangle + \beta |1_1\rangle) |2_C\rangle$ with $\alpha = \beta = 1/\sqrt{2}$. The red solid line corresponds to the present expression of $C_{AC}$ and the blue dashed line to the lower bound of $C_{AC}$ given by Ref. [9]. The dimensionless time $\tau = \sqrt{\frac{14}{6\pi}}gt$.

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[3] W. K. Wootters, Phys. Rev. Lett. 88, 2245 (1998).
[4] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[5] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A 64, 042315 (2001).
[6] E. Gerjuoy, Phys. Rev. A 67, 052308 (2003).
[7] F. Mintert, M. Kuś and A. Buchleitner, Phys. Rev. Lett. 92, 167902 (2004).
[8] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. 95, 210501 (2005).
[9] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. 95, 040504 (2005).
[10] A. Uhlmann, Phys. Rev. A 62, 032307 (2000).
[11] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[12] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[13] F. Lastra, C. E. López, L. Roa, and J. C. Retamal, Phys. Rev. A 85, 022320 (2012).
[14] B. M. Terhal and KarlGerdt H. Vollbrecht, Phys. Rev. Lett. 85, 2625 (2000).
[15] P. Rungta and C. M. Caves, Phys. Rev. A 67, 012307 (2003).
[16] Ming-Jing Zhao, Teng Ma, Shao-Ming Fei, and Zhi-Xi Wang, Phys. Rev. A 83, 052120 (2011).
[17] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A 64, 062307 (2001).
[18] T.-C. Wei and P. M. Goldbart, Phys. Rev. A 68, 042307 (2003).
[19] L. Tamaryan, D. K. Park, and S. Tamaryan, Phys. Rev. A 77, 022325 (2008).
[20] A. Streltsov, H. Kampermann, and Dagmar Bruß, Phys. Rev. A, 84, 022323 (2011).
[21] B. Lari, P. Durganandini, and P. S. Joag, Phys. Rev. A 82, 062302 (2010).
[22] K. Chen and L. A. Wu, Phys. Rev. A 69, 022312 (2004).
[23] S.-S. B. Lee and H.-S. Sim, Phys. Rev. A 85, 022325 (2012).
[24] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
[25] M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. A 81, 042105 (2010).
[26] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[27] Guifrè Vidal, Phys. Rev. A, 62, 062315 (2000).
[28] Zhi-Jian Li, Jun-Qi Li, Yan-Hong Jin and Yi-Hang Nie, J. Phys. B: At. Mol. Opt. Phys. 40, 3401 (2007).