Universal bound states of one-dimensional bosons with two- and three-body attractions

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(Dated: March 2017)

When quantum particles are confined into lower dimensions, an effective three-body interaction inevitably arises and may cause significant consequences. Here we study bosons in one dimension with weak two-body and three-body interactions, predict the existence of two three-body bound states when both interactions are attractive, and determine their binding energies as universal functions of the two-body and three-body scattering lengths. We also show that an infinitesimal three-body attraction induces an excited bound state only for 3, 39, or more bosons. Our findings here have direct relevance to a broad range of quasi-one-dimensional systems realized with ultracold atoms.

I. INTRODUCTION

Effective three- and higher-body interactions are ubiquitous and play important roles in various subfields of physics [1,3]. One such example is provided by quantum particles confined into lower dimensions even when their interaction in free space is purely pairwise. As far as low-energy physics relative to the transverse excitation energy is concerned, the system admits an effective low-dimensional description where multibody interactions inevitably arise from virtual transverse excitations. In particular, the three-body interaction in onedimensional systems may cause significant consequences because it breaks the integrability [4–8] and is marginally relevant when attractive [9,10]. The purpose of this work is to elucidate possible consequences of the three-body interaction for bound states of bosons in one dimension.

A. Model and universality

Bosons in one dimension with two-body and three-body interactions are described by

\[ H = \int dx \left[ \frac{1}{2m} \frac{d^2 \phi(x)}{dx^2} + \frac{u_2}{2m} |\phi(x)|^4 + \frac{u_3}{6m} |\phi(x)|^6 \right], \tag{1} \]

where we set \( \hbar = 1 \) and \( |\phi(x)|^{2n} = |\phi_1(x)|^n |\phi(x)|^n \). When this system is realized by confining weakly interacting bosons with a two-dimensional harmonic potential [11], the two-body and three-body couplings are provided by

\[ u_2 = 2 \frac{a_{3D}}{l_\perp^2} \quad \text{and} \quad u_3 = -12 \ln(4/3) \frac{a_{3D}^2}{l_\perp}, \tag{2} \]

respectively, for \( |a_{3D}| \ll l_\perp \), where \( a_{3D} \) is the s-wave scattering length in free space and \( l_\perp \equiv 1/\sqrt{ma_{3D}} \) is the harmonic oscillator length [12,13]. While the two-body interaction can be either attractive or repulsive depending on the sign of \( a_{3D} \), the three-body interaction is always attractive (\( u_3 < 0 \)) because it arises from the second-order perturbation theory [8]. We note that four- and higher-body interactions also exist but are irrelevant to low-energy physics.

It is more convenient to parametrize the two-body and three-body couplings in terms of the scattering lengths. The two-body scattering length is introduced as \( a_2 = -2/u_2 \). With this definition, the binding energy of a two-body bound state (dimer) is provided by \( E_2 = -1/(ma_2^2) \) for \( a_2 \gg l_\perp \) [11]. Similarly, the three-body scattering length is introduced so that the binding energy of a three-body bound state (trimer) is provided by \( E_3 = -1/(ma_3^3) \) for \( a_3 \gg l_\perp \) when the two-body interaction is assumed to be absent [9]. This definition leads to \( a_3 \sim e^{-\sqrt{3} \pi/u_3 l_\perp} \) as we will see later in Eq. (4). While \( a_3 \gg |a_2| \gg l_\perp \) is naturally realized for weakly interacting bosons with \( |a_{3D}| \ll l_\perp \), we study the system with an arbitrary \( -\infty < a_3/a_2 < +\infty \) because the two-body and three-body interactions are independently tunable in principle with ultracold atoms [14,17]. As far as both interactions are weak in the sense of \( |a_2|, a_3 \gg l_\perp \), low-energy physics of the system at \( |E| \ll 1/(ma_2^2) \) is universal, i.e., depends only on the two scattering lengths.

II. THREE-BOSON SYSTEM

A. Formulation

We now focus on the system of three bosons whose Schrödinger equation reads

\[ \left[ \frac{1}{2m} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \frac{u_2}{m} \sum_{1 \leq i < j \leq 3} \delta(x_{ij}) + \frac{u_3}{m} \delta(x_{12}) \delta(x_{23}) \right] \times \Psi(x_1, x_2, x_3) = E \Psi(x_1, x_2, x_3), \tag{3} \]

where \( x_{ij} \equiv x_i - x_j \) is the interparticle separation. For a bound state with its binding energy \( E \equiv -\kappa^2/m < 0 \), the Schrödinger equation is formally solved in Fourier space.
by

\[ \tilde{\Psi}(p_1, p_2, p_3) = -\sum_{i=1}^{3} \tilde{\Psi}_2(P_{123} - p_i; p_i) + \tilde{\Psi}_3(P_{123}), \]

\[ \kappa^2 + \sum_{i=1}^{3} \frac{p_i^2}{2}, \]

(4)

where \( P_{123} \equiv p_1 + p_2 + p_3 \) is the center-of-mass momentum and

\[ \tilde{\Psi}_2(P; p) = u_2 \int \frac{dq}{2\pi} \tilde{\Psi}(P - q, q, p), \]

(5a)

\[ \tilde{\Psi}_3(P) = u_3 \int \frac{dq}{(2\pi)^2} \tilde{\Psi}(P - q, q, r), \]

(5b)

are the Fourier transforms of \( u_2 \Psi(X, X, x) \) and \( u_3 \Psi(X, X, X) \), respectively. After rewriting \( p_1 \to P - p - q, p_2 \to p, \) and \( p_3 \to q \) in Eq. (4), the integration over \( q \) leads to

\[ \frac{1}{u_2} \tilde{\Psi}_2(P - p; p) = -\int \frac{dq}{(2\pi)^2} \frac{2\tilde{\Psi}_2(P - q; q)}{2\kappa^2 + \frac{(P - p - q)^2}{4} + p^2 + q^2} \]

\[ - \frac{\tilde{\Psi}_2(P - p; p) + \tilde{\Psi}_3(P)}{2\kappa^2 + \frac{(P - p)^2}{4} + p^2}, \]

(6a)

while the integration over \( p \) and \( q \) leads to

\[ \frac{1}{u_3} \tilde{\Psi}_3(P) = -\int \frac{dq}{(2\pi)^2} \frac{3\tilde{\Psi}_2(P - q; q)}{2\sqrt{\kappa^2 + \frac{(P - q)^2}{4} + q^2}} \]

\[ - \frac{1}{\sqrt{3\pi}} \ln \left( \frac{\Lambda}{\kappa^2 + \frac{q^2}{4}} \right) \tilde{\Psi}_3(P), \]

(6b)

where \( \Lambda \sim \frac{1}{\kappa} \) is the momentum cutoff and Eqs. are used on the left-hand sides. Finally, by substituting the ansatz of \( \tilde{\Psi}_2(P - p; p) \equiv 2\pi \delta(P) \tilde{\psi}_2(p) \) and \( \tilde{\Psi}_3(P) \equiv 2\pi \delta(P) \tilde{\psi}_3 \) (i.e., zero center-of-mass momentum) into Eqs. as well as the two-body and three-body couplings parametrized as

\[ u_2 = -\frac{2}{a_2} \quad \text{and} \quad u_3 = -\frac{\sqrt{3\pi}}{\ln(a_3)} \]

(7)

we obtain

\[ \left( \frac{a_2}{2} - \frac{1}{2\sqrt{\kappa^2 + 3\pi^2}} \right) \tilde{\psi}_2(p) \]

\[ = \int \frac{dq}{2\pi} \frac{2\tilde{\psi}_2(q)}{\kappa^2 + p^2 + q^2 + pq} + \frac{\tilde{\psi}_3}{2\sqrt{\kappa^2 + 3\pi^2}} \]

(8a)

and

\[ \frac{\ln(a_3 \kappa)}{\sqrt{3\pi}} \tilde{\psi}_3 = \int \frac{dq}{2\pi} \frac{3\tilde{\psi}_2(q)}{2\sqrt{\kappa^2 + 3\pi^2}} \]

(8b)

Equation \( 5b \) with \( \tilde{\psi}_3 \) eliminated by Eq. \( 5a \) provides the closed one-dimensional integral equation for \( \tilde{\psi}_3(p) \), which is to be solved numerically. We note that nontrivial solutions exist only in the even-parity channel where \( \tilde{\psi}_2(p) = \tilde{\psi}_2(-p) \).

As we can see in Eq. (7), the positive (negative) two-body scattering length corresponds to the attractive (repulsive) two-body interaction. The two-body attraction increases with increasing \( 1/a_2 \) from the strong repulsion \( 1/a_2 \to -\infty \) via no interaction \( 1/a_2 = 0 \) to the strong attraction \( 1/a_2 \to +\infty \). On the other hand, the three-body scattering length is positive definite and the three-body attraction increases with increasing \( 1/a_2 \) from the weak attraction \( 1/a_3 \to 0 \) to the strong attraction \( 1/a_3 \to +\infty \). For later discussion, we identify the prefactor of \( \tilde{\psi}_3 \) in Eq. (8b) as \( -1/\tilde{u}_3(\kappa) \), where

\[ \tilde{u}_3(\kappa) \equiv -\frac{\sqrt{3\pi}}{\ln(a_3 \kappa)} \]

(9)

is the renormalized three-body coupling with logarithmic energy dependence \( 6 \).

B. Binding energies

The numerical solutions for \( \kappa > \theta(a_2)/a_2 \) are plotted as functions of \( a_3/a_2 \) in Fig. 1 with the different normalizations \( 3 \). Here we find that the ground state trimer appears at \( a_3/a_2 \approx -0.149218 \). Its binding energy is \( \kappa = 1/a_3 \) at \( a_3/a_2 = 0 \) by the definition of \( a_3 \) and asymptotically approaches \( \kappa = 2/a_2 \) as

\[ \kappa \to \frac{2}{a_2} + \frac{2}{\sqrt{3} a_2 \ln(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +\infty. \]

(10)

On the other hand, we find that the excited state trimer appears right at \( a_3/a_2 = 0 \) where the dimer state also appears. Its binding energy asymptotically approaches \( \kappa = 2/a_2 \) as

\[ \kappa \to \frac{2}{a_2} + \frac{2}{\sqrt{3} a_2 \ln(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +0, \]

(11)

while it asymptotically approaches \( \kappa = 1/a_2 \) as

\[ \kappa \to \frac{1}{a_2} + \frac{\pi^2}{18 a_2 \ln^2(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +\infty. \]

(12)

The subleading term in Eq. (12) indicates that the atom-dimer term in the scattering length is provided

\( ^2 \) Their analytical expressions were recently obtained in Ref. \( 22 \).
vanishingly small toward the three-boson threshold \( a_{3K} \to +0 \). Indeed, the subleading terms in Eqs. (10) and (11) for \( \ln(a_{3}/a_{2}) \to \pm\infty \) can both be obtained from the expectation value of the renormalized three-body interaction energy \( V_{3} = \langle \bar{u}_{3}(\kappa)/m \rangle \delta(x_{12})\delta(x_{23}) \) with respect to the wave function of the McGuire trimer; \( \Psi(x_{1}, x_{2}, x_{3}) = \sqrt{\frac{3}{3a_{2}L}} e^{\sum_{1 \leq i < j \leq 3} |x_{ij}|/a_{2}} \).}

### III. N-BOSON SYSTEM

While we have so far focused on the system of three bosons, it is straightforward to generalize our formulation and some results to an arbitrary \( N \) number of bosons. In particular, when the three-body interaction is assumed to be absent, McGuire also predicted a single \( N \)-body bound state for every \( N \) with its binding energy \( E_{N}^{(MG)} \equiv -N(N^{2}-1)/(6ma_{2}^{2}) \). Its wave function in the domain of \( x_{1} < x_{2} \cdots < x_{N} \) is provided by

\[
\Psi_{N}(x) = \sqrt{\frac{(N-1)!}{NL}} \frac{2}{a_{2}} \frac{N-1}{a_{2}} \exp\left(\frac{\sum_{i=1}^{N} N + 1 - 2i}{a_{2}} x_{i}\right),
\]

(13)

where \( x \equiv (x_{1}, x_{2}, \ldots, x_{N}) \). Then, the expectation value of the renormalized three-body interaction energy \( V_{3} = \langle \bar{u}_{3}(\kappa)/m \rangle \sum_{1 \leq i < j < k \leq N} \delta(x_{ij})\delta(x_{jk}) \) with respect to the wave function in Eq. (13) leads to the binding-energy shift induced by an infinitesimal three-body attraction, which is found to be

\[
\Delta E_{N} \equiv E_{N} - E_{N}^{(MG)} \to -\sqrt{3}\pi N(N^{2}-1)(N^{2}-4)/45ma_{2}^{2} \ln(a_{3}/a_{2})
\]

(14)

for \( \ln(a_{3}/a_{2}) \to +\infty \).

Similarly, regarding the scattering state consisting of an atom with momentum \( k \) and an \( (N-1) \)-body bound state at rest, its wave function in the domain of \( x_{1} < x_{2} \cdots < x_{N} \) is provided by

\[
\Psi_{1,N-1}(x) = \frac{N}{a_{2}} \frac{2}{a_{2}} \exp\left(-\frac{\sum_{i=1}^{N} N + 1 - 2i}{a_{2}} x_{i}\right) \times \frac{e^{ikx_1}}{\sqrt{NL}} \Psi_{N-1}(x \setminus \{x_{j}\}),
\]

(15)

where \( x \setminus \{x_{j}\} \) refers to \( x \) with \( x_{j} \) excluded. Because the wave function factorizes as \( \Psi_{1,N-1}(x) \to \sqrt{\frac{e^{ikx}}{NL}} \Psi_{N-1}(x \setminus \{x_{j}\}) \) at a large separation \( x_{j} \ll x \setminus \{x_{j}\} \), the scattering length between the atom and the \( (N-1) \)-body bound state is divergent, i.e., noninteracting. Then, the expectation value of the renormalized three-body interaction energy \( V_{3} = \langle \bar{u}_{3}(\kappa)/m \rangle \sum_{1 \leq i < j < k \leq N} \delta(x_{ij})\delta(x_{jk}) \) with respect to the wave function in Eq. (15) at \( k \to 0 \) is found to be

\[
\lim_{k \to 0} V_{3}(1,N-1) = \Delta E_{N-1} - \frac{N}{(N-1)ma_{1,N-1}L}.
\]

(16)
TABLE I. Values of $\beta_{1,N-1}$ for some selected boson numbers $N$. 

| $N$ | $\beta_{1,N-1}$ | $N$ | $\beta_{1,N-1}$ |
|-----|-----------------|-----|-----------------|
| 3   | 2/9             | 20  | $-2.32241 \times 10^2$ |
| 4   | $-3$            | 30  | $-4.54773 \times 10^3$ |
| 5   | $-184/15$       | 40  | $2.94072 \times 10^5$ |
| 6   | $-275/9$        | 50  | $4.06680 \times 10^4$ |
| 7   | $-19162/315$    | 100 | $2.32605 \times 10^6$ |
| 8   | $-1589/15$      | 200 | $6.36300 \times 10^7$ |
| 9   | $-22744/135$    | 300 | $3.99017 \times 10^7$ |
| 10  | $-6269/25$      | 400 | $1.43180 \times 10^8$ |

where the leading term is just the binding-energy shift in Eq. (14) but the subleading term reflects the interaction between the atom and the $(N - 1)$-body bound state induced by an infinitesimal three-body attraction. The extracted scattering length $a_{1,N-1} \equiv a_2 \ln(a_3/a_2)/[3\pi \alpha_{1,N-1}]$ is plotted in Fig. 2 and turns out to be positive for $N = 3$ and $N = 39$ but negative for $4 \leq N \leq 38$, which correspond to the attractive and repulsive interactions between the atom and the $(N - 1)$-body bound state, respectively. Therefore, they in the former case with $a_{1,N-1} \gg a_2$ constitute another $N$-body bound state induced by the infinitesimal three-body attraction, whose binding energy measured from the threshold at $E = E_{N-1}$ reads

$$-\frac{N}{2(N-1)m\alpha_{1,N-1}^2} = -\frac{3\pi^2 N \beta_{1,N-1}^2}{2(N-1)m\alpha_{2}^2 \ln^2(a_3/a_2)}$$

for $\ln(a_3/a_2) \rightarrow +\infty$. The values of $\beta_{1,N-1}$ for some selected $N$ are presented in Table I.

Beyond the limit of infinitesimal three-body attraction, the binding energies of $N$ bosons are to be determined by generalizing Eqs. (18) as

$$\frac{1}{\sqrt{3\pi}} \ln \left( a_3 \sqrt{\kappa^2 + \frac{1}{6} \left( \sum_{i=4}^{N} p_i \right)^2 + \sum_{i=2}^{N} \frac{p_i^2}{2} } \right) \tilde{\psi}_3(p \setminus \{p_1, p_2\})$$

and

$$\frac{1}{\sqrt{3\pi}} \ln \left( a_3 \sqrt{\kappa^2 + \frac{1}{6} \left( \sum_{i=4}^{N} p_i \right)^2 + \sum_{i=2}^{N} \frac{p_i^2}{2} } \right) \tilde{\psi}_3(p \setminus \{p_1, p_2, p_3\})$$

While elaborate analyses of these coupled integral equations are deferred to a future work, we note that Eq. (18b)
without $\psi_2$ was solved numerically for $N = 4$ in the absence of the two-body interaction $a_3/a_2 = 0$ [9]. Here three four-body bound states (tetramers) were found with their binding energies provided by $\kappa = 873.456/a_4$, 11.7181/$a_3$, and 1.45739/$a_3$. On the other hand, in the opposite limit $a_3/a_2 \to +\infty$ where the three-body attraction is infinitesimal, we find above that there exists only one tetramer state with its binding energy $\kappa \to \sqrt{10}/a_2$. Therefore, the bound-state spectrum of four or more bosons as a function of $a_3/a_2$ is rather nontrivial and should be elucidated in the future work.

IV. CONCLUSION

In this work, we studied bosons in one dimension with weak two-body and three-body interactions, predicted the existence of two trimer states when both interactions are attractive, and determined their binding energies as universal functions of the two-body and three-body scattering lengths. We also showed that an infinitesimal three-body attraction induces an excited bound state only for 3, 39, or more bosons. Because the effective three-body attraction inevitably arises by confining weakly interacting bosons into lower dimensions, our findings herein have direct relevance to a broad range of quasi-one-dimensional systems realized with ultracold atoms [11, 22–24]. In particular, when $a_{3D} < 0$ and $|a_{3D}| \ll l_\perp$, the $N$-body to dimer binding-energy ratios predicted from Eqs. (2), (7), (14), and (17) read

$$\frac{E_N}{E_2} = \frac{E_N^{(MG)}}{E_2} + \frac{4N(N^2 - 1)(N^2 - 4) \ln(4/3)}{15} \left( \frac{a_{3D}}{l_\perp} \right)^2$$

for the ground state and

$$\frac{E_N}{E_2} = \frac{E_{N-1}}{E_2} + \frac{72N^2 \beta_{1,N-1}^2 \ln^2(4/3)}{N - 1} \left( \frac{a_{3D}}{l_\perp} \right)^4$$

for the excited state with $N = 3$ or $N \geq 39$ which may be observable in ultracold atom experiments.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grants No. JP15K17727 and No. JP15H05855.

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