On the bulk block expansion
for a monodromy defect

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For a free–field flat monodromy defect, a formula for the finite part of the correlator is obtained as a double power series in $(1 - x)$ and $(1 - \bar{x})$ where $x$ and $\bar{x}$ are lightcone coordinates. It takes the particular form of a series in $(1 - x)$ with coefficients finite sums of hypergeometric functions of $1 - \bar{x}$ and is identified with a bulk block expansion. An expression for the coefficient of the $(1 - x)^n(1 - \bar{x})^m$ term is thereby found as an explicit function of the flux and dimension. Some typical examples are presented.

A transformation allows the bulk block expansion to be written as an Appell $F_3$ function which has simplifying consequences.
1. Introduction

This brief communication is a strictly technical addendum to two previous reports, [1], [2], concerning basic CFT quantities for the simplest codimension 2 monodromy defect. The set-up and basic structures have been detailed in these reports, and references therein, and so, to save space, I will assume that all the equations in [1] and [2] are available.

The question addressed here is the form of the bulk block expansion of the free-field correlator (Green function), $G$. The defect block expansion is relatively straightforward, amounting to the easily effected Fourier decomposition of $G$. The bulk block case is not so simple even for free fields. Its structure is outlined in the next section where the precise objective of the present calculation is explained.

2. The bulk block expansion

The relevant bulk block expansion for free fields has been given in a certain form in [3] where other references are given. It is algebraically convenient to use, as there, the (independent) lightcone Lorentzian coordinates $x$ and $\bar{x}$ on the two-dimensional space orthogonal to the planar defect and the bulk expansion reads, [3],

$$
G(x, \bar{x}) = \left( \frac{\sqrt{x \bar{x}}}{(1-x)(1-\bar{x})} \right)^\Delta \left( 1 + \sum_{l=0}^{\infty} c_l f_{2\Delta+l,l}(x, \bar{x}) \right).
$$

The bulk blocks take the form, in this case,

$$
((1-x)(1-\bar{x}))^{-\Delta} f_{2\Delta+l,l}(x, \bar{x}) \equiv \tilde{f}_{2\Delta+l,l}(x, \bar{x}) =
\sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{n,j}(\Delta, l)(1-x)^n(1-\bar{x})^{l+j} 2F1(\Delta+l+j, \Delta+l+j, 2(\Delta+l+j); 1-\bar{x}),
$$

with the constants $A_{n,j}$ separately determined by recursion. One objective is to compute the coefficients $c_l$ as functions of $\Delta$ and $\delta$, although I will not accomplish this here. Rather, I will recover the composition (1) just in the sense of being a double power series in $1-x$ and $1-\bar{x}$. This is the limited objective of the present note. The basic formulae now follow.

I remark that (1) has been written using the CFT correlator normalisation adopted in [3]. The normalisation used in [1,2], and here, is the standard QFT one,

\[\Delta\] here stands for $\Delta^\phi$ which equals $d/2 - 1$ for standard fields.
in terms of which (1) reads,

\[ G(x, \overline{x}, \delta) = G(x, \overline{x}, 0) \left(1 + \sum_{l=0}^{\infty} c_l f_{2\Delta+l,l}(x, \overline{x})\right), \]  

(3)

which can be rearranged to the form,

\[ G_{\text{sub}}(x, \overline{x}, \delta) = \mathcal{N} \sqrt{x \overline{x}} \Delta \sum_{l=0}^{\infty} c_l \tilde{f}_{2\Delta+l,l}(x, \overline{x}), \]  

(4)

where \( G_{\text{sub}} \) is the subtracted ‘Green function’ \((G(\delta) - G(0))\) derived in [2] in terms of the Appell \( F_1 \) function, \( \text{viz} \),

\[ G_{\text{sub}} = \mathcal{N} C x^\delta \sqrt{x \overline{x}} F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, 1 - x \overline{x}, 1 - x). \]  

(5)

\( \mathcal{N} \) is a convention dependent normalisation \(^3\), and cancels from (4). \( C \) is a constant that results from the calculation of \( G_{\text{sub}} \) as the integral of a cut discontinuity and equals, \(^4\)

\[ C = \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta) \sin \pi \delta}{\pi \Gamma(2\Delta + 1)}. \]  

(6)

I leave this factor understood and put it back at the end. Note that it is unchanged under \( \delta \to 1 - \delta \).

The \( \tilde{f} \) have the general power series expansion \((cf [3] \text{ equn.}(2.30))\),

\[ \tilde{f}_{2\Delta+l,l}(x, \overline{x}) = \sum_{n,m \geq 0} k_{n,m}(\Delta, l) \frac{(1-x)^n(1-\overline{x})^m}{n!m!}, \]

so that (4) is,

\[ G_{\text{sub}}(x, \overline{x}, \delta) = \mathcal{N}(\sqrt{x \overline{x}})\Delta \sum_{n,m \geq 0} C_{n,m} \frac{(1-x)^n(1-\overline{x})^m}{n!m!}, \]  

(7)

where,

\[ C_{n,m}(\Delta, \delta) = \sum_{l=0}^{\infty} c_l(\Delta, \delta) k_{n,m}(\Delta, l), \]  

(8)

\(^3\) It also contains a radial factor, \( r^{-\Delta} \).

\(^4\) This constant is denoted by \( -C_{\text{free}} \) in [3].
and the restricted aim here, as mentioned, is the determination of the ‘total’ coefficients, \( C_{n,m} \). From (7) and (5) this amounts to finding the double series expansion of the quantity,

\[
\tilde{F}(1 - x, 1 - \bar{x}) \equiv x^\delta F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, 1 - x\bar{x}, 1 - x) \\
\equiv x^\delta F_1(a, b, 1, c, 1 - x\bar{x}, 1 - x).
\]  \( (9) \)

First, the coefficient of \((1 - x)^n\) is sought, and then that of \((1 - \bar{x})^m\) in this is to be found.

Expanding (9) in powers of \((1 - \mu)\) reduces to the expansion of,

\[
(1 - \mu)^\delta F_1(a, b, 1, c; \mu + \bar{\mu} - \mu\bar{\mu}, \mu),
\]

in powers of \(\mu\) which is a slightly awkward process, as it stands.\(^5\)

However, the calculation is eased considerably on separating the \(\mu\) and \(\bar{\mu}\) dependencies in the \(F_1\) by applying the transformation, [4] p.30, equn.(54), \(^6\)

\[
F_1(a, b, b'; c; x, y) = (1 - y)^{-a} F_1(a, b, c - b - b', c; y - x y - 1, y y - 1),
\]

which yields a handier form for \(G_{sub}\) than (5) and gives for the right-hand side of (9),

\[
(1 - \mu)^\delta F_1(a, b, 1, c; \mu + \bar{\mu} - \mu\bar{\mu}, \mu) \\
= (1 - \mu)^{\delta - a} F_1(a, b, c - 1 - b, c; \frac{\bar{\mu}\mu - \bar{\mu}}{\mu - 1}, \frac{\mu}{\mu - 1}) \\
= (1 - \mu)^{-\Delta} F_1(a, b, c', c; \bar{\mu}, \frac{\mu}{\mu - 1}),
\]  \( (10) \)

which is simpler to expand in \(\mu\).

The outside factor can be treated by binomial expansion so I just concentrate on the \(r\)th derivative of the \(F_1\) in (10) with respect to \(\mu\).

The derivatives of any function of \(\frac{\mu}{\mu - 1}\) with respect to \(\mu\), evaluated at 0, can be expressed formally in terms of the generalised Laguerre polynomial, \(L_{n-1}^1(x)\), by,

\[
\left. \frac{1}{r!} \partial^r_{\mu} F \left( \frac{\mu}{\mu - 1} \right) \right|_{\mu = 0} = \left. L_{r-1}^1(\partial_{\mu}) F(\mu) \right|_{\mu = 0} \\
= \sum_{j=0}^{r} \frac{(-1)^j}{j!} \left( \frac{r - 1}{r - j} \right) \partial^j_{\mu} F(\mu) \bigg|_{\mu = 0} 
\]  \( (11) \)

\(^5\) But consult the Appendix.

\(^6\) This follows from a fractional coordinate transformation of the Picard integral representation of \(F_1\).
Since the \( j \)th derivative at 0 of the \( F_1 \) in (10) with respect to the second variable slot reduces to a hypergeometric function via, [4] p.15 and p.19 equn.(19),

\[
F_1^{(j)}(\mu,0) = \frac{(a_j)(c')_j}{(c)_j} F_1(a+j,b,c'+j,c+j;\mu,0) = \frac{(a_j)(c')_j}{(c)_j} 2F_1(a+j,b,c+j;\mu),
\]

the required \( r \)th derivative of \( F_1 \) is, from (11),

\[
\frac{\partial^r}{\partial \mu^r} F_1(a,b,c',c;\mu,\frac{\mu}{\mu-1}) \bigg|_{\mu=0} = r! \sum_{j=0}^{r} \frac{(-1)^j}{j!} \binom{r-1}{r-j} \frac{(a_j)(c')_j}{(c)_j} 2F_1(a+j,b,c+j;\mu).
\]

(12)

The next step is to combine (12) with the factor \((1 - \mu)^{-\Delta}\) in (10) to give the \( n \)th derivative, at \( \mu = 0 \). This produces,

\[
\frac{\partial^n}{\partial \mu^n} \tilde{F}(\mu,\mu) \bigg|_{\mu=0} = \sum_{r=0}^{n} r! \binom{n}{r} (\Lambda)_{n-r} \sum_{j=0}^{r} \frac{(-1)^j}{j!} \binom{r-1}{r-j} \frac{(a_j)(c')_j}{(c)_j} 2F_1(a+j,b,c+j;\mu).
\]

(13)

I note that the same equation holds when \( \mu \) and \( \overline{\mu} \) are interchanged, if, at the same time, \( \delta \) is replaced by \( 1 - \delta \).

Swapping round summations in (13) allows one to define ‘universal’ coefficients, polynomials in \( \Lambda \), by,

\[
U_{n,j}(\Lambda) \equiv \sum_{r=j}^{n} r! \binom{n}{r} \binom{r-1}{r-j} (\Lambda)_{n-r}, \quad (0 \leq j \leq n),
\]

and (13) takes the more compact form,

\[
\frac{\partial^n}{\partial \mu^n} \tilde{F}(\mu,\overline{\mu}) \bigg|_{\mu=0} = \sum_{j=0}^{n} (-1)^j U_{n,j}(\Lambda) \frac{(a_j)(c')_j}{j!(c)_j} 2F_1(a+j, b, c+j;\mu).
\]

(14)

To a factor of \( n! \), (14) is the coefficient of \((1 - x)^n\) in the bulk block expansion and one sees that it is given by a finite sum of hypergeometric functions of \((1 - \overline{x})\) which seems to be a different organisation compared to existing formulations.

The coefficient of \( \overline{\mu}^m \) in (14) can now be found using the definition of the hypergeometric function. This quickly leads to the coefficient of the term \( \mu^n\overline{\mu}^m/n!m! \) in the bulk block expansion which was my ultimate objective,

\[
C_{n,m}(\Lambda,\delta) \equiv C(\Lambda,\delta) \sum_{j=0}^{n} (-1)^j U_{n,j}(\Lambda) \frac{(a_j)(c')_j}{j!(c)_j} (a+j)_m(b)_m \frac{(c+j)_m}{(c+j)_m}
\]

\[
= C(\Lambda,\delta)(\Lambda)_m \sum_{j=0}^{n} (-1)^j U_{n,j}(\Lambda) \frac{(\Lambda + \delta)_{j+m}(\Lambda)_j}{j!(2\Lambda + 1)_{j+m}},
\]

(15)
where the constant $C$, (6), has been reinstated and the parameters $a = \Delta + \delta$, $b = \Delta$, $b' = 1$, $c' = \Delta$, $c = 2\Delta + 1$ inserted resulting in a final, easily calculable formula.

3. A few examples and properties of the coefficients

It is possible that further algebraic reductions could be found for (15) but, for the present, just some particular, lower order cases will be given in order to exhibit the general structure.

The ratio $C'_{n,m} = C_{n,m}/C$ is listed for a typical set of coefficients,

$$C'_{0,1} = \frac{\Delta(\delta + \Delta)}{1 + 2\Delta}, \quad C'_{0,2} = \frac{\Delta(\delta + \Delta)(1 + \delta + \Delta)}{2(1 + 2\Delta)},$$

$$C'_{4,0} = \frac{\Delta(3 + \Delta)(4 - \delta + \Delta)(3 - \delta + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)}{4(3 + 4\Delta(2 + \Delta))},$$

$$C'_{3,1} = \frac{\Delta^2(3 - \delta + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)(\delta + \Delta)}{4(3 + 4\Delta(2 + \Delta))},$$

$$C'_{2,2} = \frac{\Delta^2(1 + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)(\delta + \Delta)(1 + \delta + \Delta)}{4(2 + \Delta)(1 + 2\Delta)(3 + 2\Delta)}.$$(16)

The expressions simplify at the midpoint $\delta = 1/2$ ($\mathbb{Z}_2$ monodromy). For example, at this point,

$$C'_{6,0} = \frac{1}{512} \Delta(4 + \Delta)(5 + \Delta)(7 + 2\Delta)(9 + 2\Delta)(11 + 2\Delta),$$

$$C'_{5,1} = \frac{1}{512} \Delta^2(4 + \Delta)(1 + 2\Delta)(7 + 2\Delta)(9 + 2\Delta),$$

$$C'_{4,2} = \frac{1}{512} \Delta^2(1 + \Delta)(7 + 2\Delta)(3 + 4\Delta(2 + \Delta)), $$

$$C'_{3,3} = \frac{1}{512(3 + \Delta)} \Delta^2(1 + \Delta)(2 + \Delta)(1 + 2\Delta)(3 + 2\Delta)(5 + 2\Delta).$$

The coefficients satisfy the exchange requirement, $C_{n,m}(\Delta, \delta) = C_{m,n}(\Delta, 1 - \delta)$. This is not obvious from (15) and so provides a useful algebraic check.

The typical variation of a coefficient with flux and dimension is shown below in Fig.1. Physical quantities have to be made periodic in $\delta$ with period 1. Since there is some significance in the value at $\delta + 1$, the range of $\delta$ has been doubled.
As functions of $\Delta$, the coefficients vanish at $\Delta = 0$, i.e. in dimension $d = 2$, then rise to a maximum (if $0 < \delta < 1$) followed by a monotonic decrease to zero as $\Delta \to \infty$.

Fig1. Variation of the coefficient $C_{3,1}(\Delta, \delta)$ with flux, $\delta$, and dimension, $\Delta$. The range of $\delta$ has been extended to twice its unit cell.

4. Comments and conclusion

From existing formulae in the literature, e.g. [3], the coefficient of $\mu^n$, is, (see (2)),

$$\sum_{j=-n}^{n} A_{n,j}(\Delta, l) \sum_{l=0}^{\infty} c_l (1 - \overline{x})^{l+j} 2F_1(\Delta + l + j, \Delta + l + j, 2(\Delta + l + j); 1 - \overline{x}),$$

and it would seem a difficult task to reconcile this with (14) in general although individual coefficients could be compared. At the moment, I am not able to do this and there remains the problem of extracting the $c_l$ constants from (8).

The fact that the interchange symmetry, $C_{n,m}(\delta) = C_{m,n}(1 - \delta)$, is not immediately apparent from (15) suggests that there is a significant rearrangement that would make this so.\(^7\)

I note that physical quantities show a discontinuity in the $\delta$-derivative at the unit cell boundary.

\(^7\) Added note: This can be accomplished as described in the Appendix.
The use of the subtracted $G_{sub}$, rather than the complete $G$, which is the sum of two Appell $F_1$ functions, [1], [5], bypasses any divergence issues.

Appendix. Symmetry under $\delta \rightarrow 1 - \delta$. Reduction to an Appell $F_3$

The expression for the subtracted Green function, (5), can be transformed so as to make plain the symmetry under the combined conjugate (or exchange) operation $\mu \leftrightarrow \overline{\mu}$, $\delta \leftrightarrow 1 - \delta$, giving, as a consequence, a simpler form for the double power series.

The transformation in question is shown in [4] p.24 equn.(29) and is, for the variables in use here,

$$F_1(a, b, b'; c; \overline{\mu}, \frac{\mu}{\mu - 1}) = (1 - \mu)^\Delta F_3(a, c - a, b, b'; c; \overline{\mu}, \mu).$$

Hence the right–hand side of (9) is simply an Appell $F_3$ function, with no multiplying factors, yielding the elegant equality,

$$\tilde{F} = F_3(\Delta + \delta, \Delta + 1 - \delta; \Delta, \Delta, 2\Delta + 1; \overline{\mu}, \mu). \quad (18)$$

The requisite symmetry is now manifest because $F_3$ is invariant under the exchange of variables if, simultaneously, the first two parameters are interchanged and the third exchanged with the fourth. The last operation does nothing as this pair are equal and exchanging the first two is the same as $\delta \leftrightarrow 1 - \delta$. QED.

The double series definition of $F_3$ immediately delivers the sought for expansion without any of the earlier, slightly involved algebra which is thus rendered unnecessary (except as a check). The standard expansions are,

$$F_3(\Delta + \delta, \Delta + 1 - \delta; \Delta, \Delta, 2\Delta + 1; \overline{\mu}, \mu)$$

$$= \sum_{n,m=0}^{\infty} \frac{(\Delta + \delta)_m (\Delta + 1 - \delta)_n (\Delta)_m (\Delta)_n}{n! m! (2\Delta + 1)_m} \overline{\mu}^m \mu^n$$

$$= \sum_{m=0}^{\infty} \frac{(\Delta + \delta)_m \Delta_m}{m! (2\Delta + 1)_m} 2F_1(\Delta + 1 - \delta, \Delta, 2\Delta + 1 + m; \mu) \overline{\mu}^m \quad (19)$$

$$= \sum_{n=0}^{\infty} \frac{(\Delta + 1 - \delta)_n \Delta_n}{n! (2\Delta + 1)_n} 2F_1(\Delta + \delta, \Delta, 2\Delta + 1 + n; \overline{\mu}) \mu^n,$$

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8 It follows on applying a transformation to a hypergeometric function in the expansion of $F_1$. 7
from which an alternative, and much simpler, general expression for the coefficients, \( C_{n,m} \), can be read off. This supersedes that given earlier and will ease any comparison with existing expressions. It is, written out, putting \( C \) back, and simplifying,

\[
C_{n,m} = C(\Delta, \delta) \frac{(\Delta + \delta)_m (\Delta + 1 - \delta)_n (\Delta)_m (\Delta)_n}{(2\Delta + 1)_{m+n}}
\]

\[
= \frac{\sin \pi \delta}{\pi} (\Delta)_n (\Delta)_m B(\Delta + \delta + m, \Delta + 1 - \delta + n),
\]

in terms of the Beta function. Thankfully, computation brings agreement with the previous values e.g. (16). These are the building blocks of the bulk expansion.

Note that the coefficient of \( \mu^n \) in (19) is a single hypergeometric function, a significant algebraic improvement over (14). Furthermore, the derivatives with respect to \( x \) and \( \overline{x} \) can be found easily and the coincidence limit \((x \to 1, \overline{x} \to 1)\) will give a combination of Beta functions, as in [2].

Also useful to have visible, is the expression at \( \mu = 0 \),

\[
F_3(\Delta + \delta, \Delta + 1 - \delta; \Delta, 2\Delta + 1, \overline{\mu}, 0) = _2F_1(\Delta + \delta, 2\Delta + 1; \overline{\mu}, 0),
\]

which, as a check, is the result derived in [2] and agreeing with that conjectured in [3].

From all this, it is clear that \( F_3 \) provides the most fundamental realisation of the (subtracted) Green function. It could, no doubt, have been obtained without going through the intermediate \( F_1 \) form. The algebraic reason why \( F_3 \) does not appear earlier seems to be because, in the present scheme, the subtracted Green function occurred originally as a single integral while \( F_3 \) is expressible only as a double integral, in general. For the present parameters, this integral simplifies to,

\[
\frac{\Gamma(\Delta)^2}{\Gamma(2\Delta + 1)} F_3(\Delta + \delta, \Delta + 1 - \delta; \Delta, 2\Delta + 1, \overline{\mu}, \mu) = \int \int du dv (uv)^{\Delta-1} (1 - \overline{\mu} u)^{-\Delta-\delta} (1 - \mu v)^{\delta-1-\Delta},
\]

taken over the triangle, \( u \geq 0, v \geq 0, 1 - u - v \geq 0 \), showing, again, the exchange symmetry. Derivatives and coincidence limits can also be deduced readily from this form.

Presenting the subtracted Green function in terms of \( F_3 \) allows one easily to write down two, monodromy–dependent, second order partial differential equations satisfied by \( \tilde{F} \). One is, reverting to the \( x, \overline{x} \) coordinates,

\[
x(1 - \overline{x}) \frac{\partial^2 \tilde{F}}{\partial x^2} + (1 - \overline{x}) \frac{\partial^2 \tilde{F}}{\partial x \partial \overline{x}} + \delta(1 - x) \frac{\partial \tilde{F}}{\partial x} - \Delta(\Delta + \delta) \tilde{F} = 0,
\]
with the other obtained by conjugation. For the record, this system has rank 4 and
the singular locus is the union of the curves \( x = 0, x = 1, x = \infty, \overline{x} = 0, \overline{x} = 1, \overline{x} = \infty \) and \( x \overline{x} = 1 \) on the projective line product, \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Partial differential equations for \( G_{sub} = \mathcal{N} \sqrt{x \overline{x}} \tilde{F} \), follow trivially. I note, finally, the relation between \( F_3 \) and the Appell \( F_2 \),

\[
F_3(\Delta + \delta, \Delta + 1 - \delta; \Delta, \Delta, 2\Delta + 1; \mu, \mu) = (\mu \overline{\mu})^{-\Delta} F_2(0, \Delta, \Delta, 1 - \delta; \mu, \mu),
\]

which serves as an analytic continuation.

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