Online enrichment is the extension of an existing reduced solution space based on the solution of the reduced model. It is used in many localized model order reduction techniques: ArbiLoMod, introduced in Buhr et al. (2017), employs online enrichment. For the Localized Reduced Basis Multiscale Method (LRBMS) which was introduced in Albrecht et al. (2012), online enrichment was presented in Albrecht and Ohlberger (2013) and in Ohlberger and Schindler (2015). For the Generalized Multiscale Finite Element Method (GMsFEM) which was introduced in Efendiev et al. (2013), online enrichment was discussed in Chung et al. (2015). For the Constraint Energy Minimizing Generalized Multiscale Finite Element Method (CEM-GMsFEM) which was introduced in Chung et al. (2017a), online enrichment was discussed in Chung et al. (2017b). However, no a priori convergence analysis is available for the ArbiLoMod and the LRBMS. For the GMsFEM and CEM-GMsFEM, exponential convergence was shown in Chung et al. (2015) and Chung et al. (2017b), but only for the case where the reduced space already has good properties. The proofs given here do not require any properties of the reduced space.

However, no a priori convergence analysis is available for the ArbiLoMod and the LRBMS, exponential convergence was shown for the GMsFEM and CEM-GMsFEM, but only for the case where the reduced space already has good properties. The proofs given here do not require any properties of the reduced space.

The online enrichment presented here is usually applied within a localized model order reduction method like the methods mentioned above. To analyze the convergence behavior and obtain a priori estimates, we isolated the online enrichment.

While the setting presented in the following resembles overlapping domain decomposition methods, it is different as it generates reduced solution spaces in each iteration. The approximate solution in each step is the solution of the reduced problem, in contrast to domain decomposition methods, were the approximate solution is calculated as the sum of the previous approximation and a correction term.

2. SETTING

On the domain $\Omega$ with $\text{dim}(\Omega) = d \in \{2, 3\}$ we approximate the solution of the stationary heat equation

$$-\nabla(\kappa \nabla u) = f$$

with homogeneous Dirichlet boundary conditions. Non-homogeneous Dirichlet boundary conditions could be handled with the usual shifting technique. The methods presented below could easily be extended for Neumann boundary conditions. $\kappa$ is the heat conductivity. In the space $V := H_0^1(\Omega)$, $u \in V$ is the unique solution of

$$a(u, \varphi) = f(\varphi) \quad \forall \varphi \in V$$

with the coercive bilinear form

$$a(u, \varphi) := \int_\Omega \kappa \nabla u \nabla \varphi dx$$

and the linear form

$$f(\varphi) := \int_\Omega f \varphi dx.$$
Each reduced space $V_n$ is constructed by enriching the previous reduced space with an additional basis function $\tilde{u}_n$, which lies in one of the localized spaces $O_i$.

By using an overlapping domain decomposition, all reduced local spaces and local spaces are subspaces of $V$ (assuming an extension with zero to the whole domain). A non-overlapping domain decomposition would require local spaces with non-zero boundary conditions which are not subspaces of $V$. This would require a completely different treatment.

Residual Based Enrichment

The residual based enrichment algorithm (given as Algorithm 1) first selects the local enrichment space from which the enrichment function $\tilde{u}_n$ is taken. The local space $O_k$ which maximizes the dual norm of the residual $\|R_n\|_{O_k}^* = \sup_{\varphi \in O_k \setminus \{0\}} \frac{R_n(\varphi)}{\|\varphi\|}$ is chosen, i.e.

$$k := \arg \max_{i=1,\ldots,N_D} \|R_n\|_{O_i}. \quad (5)$$

The residual $R_n \in V' \setminus \{0\}$ is defined as $R_n(\cdot) := f(\cdot) - a(\tilde{u}_n, \cdot)$. Then a localized problem is formed by a Galerkin projection of the original problem onto this local space $O_k$, and replacing the right hand side $f$ by the last residual $R_n$. The solution of the localized problem is the enrichment function $\tilde{u}_n$.

Globally Coupled Local Enrichment

The globally coupled local enrichment algorithm (given as Algorithm 2) couples the global reduced space with the full local space. First it iterates over all local spaces $O_i$ and solves the coupled problem: It solves the original problem projected on the space $\tilde{V}_n \oplus O_i$, the solution of this coupled problem is called $\tilde{u}_{n,i}$. Then the local space $O_i$ is selected which maximizes the change in the solution $\|\tilde{u}_n - \tilde{u}_{n,i}\|_a$, i.e.

$$k := \arg \max_{i=1,\ldots,N_D} \|\tilde{u}_n - \tilde{u}_{n,i}\|_a. \quad (6)$$

The function $\tilde{u}_{n,k}$ is used to enrich the space $\tilde{V}_n$. Note that this is an enrichment in $O_k$, even though $\tilde{u}_{n,k}$ has global support.

4. A PRIORI CONVERGENCE

First we prove exponential convergence for the residual based enrichment.

**Theorem 1.** (Exponential convergence). For the reduced solutions $\tilde{u}_{n+1}$ in Algorithm 1 it holds that

$$\|u - \tilde{u}_{n+1}\|_a \leq c \cdot \|u - \tilde{u}_n\|_a$$

with

$$c := \sqrt{1 - \frac{1}{N_D c_{pu}^2}}. \quad (7)$$

The constant $c_{pu}$ is explained and defined later in this section.

**Proof.** As we use the energy norm, the solution is the best approximation

$$\|u - \tilde{u}_{n+1} - u\|_a \leq \|\varphi - u\|_a \quad \forall \varphi \in \tilde{V}_{n+1}. \quad (8)$$

This holds that for $\varphi = \tilde{u}_n + \alpha \tilde{u}_{n,i}$ for all $\alpha \in \mathbb{R}$. Because of the symmetry of $a$ it holds that

$$\|\tilde{u}_n + \alpha \tilde{u}_{n,i} - u\|_a^2 = \|\tilde{u}_n - u\|_a^2 - 2\alpha R_n(\tilde{u}_n) + \alpha^2 \|\tilde{u}_{n,i}\|_a^2.$$

This term is minimized by choosing $\alpha = R_n(\tilde{u}_n)/\|\tilde{u}_{n,i}\|_a^2$. We use this $\alpha$ and realize that $R_n(\tilde{u}_n) = \|\tilde{u}_{n,i}\|_a^2 = \|R_n\|_{O_k}^2$.
because \( \tilde{u}_n \) is the Riesz representative of \( R_n \) in the energy norm. It follows that

\[
\| \tilde{u}_{n+1} - u \|_a^2 \leq \| \tilde{u}_n - u \|_a^2 - \| R_n \|_{O_k}^2. \tag{10}
\]

Till this point, the proof followed the structure given in (Chung et al., 2015, Section 4). We defined \( k \) to select the largest local residual, so it holds that

\[
\| R_n \|_{O_k}^2 \geq \frac{1}{N_D} \sum_{i=1}^{N_D} \| R_n \|_{O_i}^2. \tag{11}
\]

Furthermore, from (Buhr et al., 2017, Proposition 5.1) we know

\[
\| R_n \|_{V_k}^2 \leq c_{pu}^2 \sum_{i=1}^{N_D} \| R_n \|_{O_i}^2, \tag{12}
\]

with a constant \( c_{pu} \) which is a stability constant for the partition of unity (pu). Magnitude and scaling behavior of \( c_{pu} \) depend on the choice of the partition of unity and the norm of the spaces. See Buhr et al. (2017) for more details. As we use the energy norm, we have

\[
\| R_n \|_{V_k}^2 = \| \tilde{u}_n - u \|_a^2. \tag{13}
\]

Combining (11), (12), and (13) we obtain

\[
\| R_n \|_{O_k}^2 \geq \frac{1}{N_D} \frac{1}{c_{pu}^2} \| \tilde{u}_n - u \|_a^2. \tag{14}
\]

Combining (14) with (10) yields

\[
\| \tilde{u}_{n+1} - u \|_a^2 \leq \left( 1 - \frac{1}{N_D} \frac{1}{c_{pu}^2} \right) \| \tilde{u}_n - u \|_a^2
\]

and thus the claim. \( \square \)

**Corollary 2.** For the reduced solutions \( \tilde{u}_n \) in Algorithm 1 it holds that

\[
\| \tilde{u}_n - u \|_a \leq c^n \cdot \| u \|_a \tag{16}
\]

with \( c \) as defined in (8).

**Proof.** This follows from Theorem 1, because \( \tilde{u}_0 = 0 \). \( \square \)

**Concerning \( c_{pu} \)**

The constant \( c_{pu} \) is defined to be

\[
c_{pu}^2 := \sup_{\varphi \in V \setminus \{0\}} \| \varphi \|_a^2. \tag{17}
\]

With this constant, it holds that

\[
\| \zeta \|_{V_k}^2 \leq c_{pu}^2 \sum_{i=1}^{N_D} \| \zeta \|_{O_i}^2 \tag{18}
\]

for any element \( \zeta \) of the dual space \( V' \), especially for the residual \( R_n \) (see (Buhr et al., 2017, Proposition 5.1) for details). An upper bound for \( c_{pu} \) is devised in the following proposition.

**Proposition 3.** (Upper bound of \( c_{pu} \)). With \( J \) being the maximum number of functions \( \varphi_i \) having support in any given point \( x \) in \( \Omega \), \( \| \varphi \|_{\inf} \) being the contrast of the problem, \( c_f \) the constant of the Friedrich’s inequality on \( \Omega \) and \( \| \cdot \|_\infty \) the infinity norm, it holds that

\[
c_{pu}^2 \leq 2J \left( c_f \frac{K_{\max}}{K_{\min}} \max_i \| \nabla \varphi_i \|_\infty + \max_i \| \varphi_i \|_\infty \right). \tag{19}
\]

**Proof.** Starting from (17), we estimate \( \sum_{i=1}^{N_D} \| \varphi_i \|_a^2 \). It holds that

\[
\sum_{i=1}^{N_D} \| \varphi_i \|_a^2 = \sum_{i=1}^{N_D} \int_{\omega_i} \kappa |\nabla (\varphi_i \varphi)|^2 \, dx \tag{20}
\]

\[
\leq \sum_{i=1}^{N_D} \int_{\omega_i} \kappa [2|\nabla \varphi_i|^2 + 2 |\varphi_i (\nabla \varphi)|^2] \, dx. \tag{21}
\]

For the second term in (21) it holds that

\[
\sum_{i=1}^{N_D} \int_{\omega_i} 2|\nabla \varphi_i|^2 \, dx \leq 2J \max_i \| \varphi_i \|_\infty \| \varphi_i \|_a^2 \tag{22}
\]

and for the first term we have

\[
\sum_{i=1}^{N_D} \int_{\omega_i} 2 |\varphi_i (\nabla \varphi)|^2 \, dx \leq 2J c_f \frac{K_{\max}}{K_{\min}} \max_i \| \nabla \varphi_i \|_\infty \| \varphi_i \|_a^2 \tag{23}
\]

Combining these yields the claim. \( \square \)

**Globally coupled enrichment**

The globally coupled enrichment given in Algorithm 2 is the optimal enrichment: Among all enrichment functions from all local spaces, it selects the one which minimizes the resulting error in the energy norm.

**Theorem 4.** (Optimality of Algorithm 2). For the reduced solutions \( \tilde{u}_{n+1} \) in Algorithm 2 it holds that

\[
\| u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a
\tag{24}
\]

\[
\{ \| u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

\[
\{ u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

\[
\{ u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

\[
\{ u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

\[
\{ u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

\[
\{ u - \tilde{u}_{n+1} \|_a = \min_{i=1,\ldots,N_D} \| u - u_{\psi_i} \|_a \}
\]

**Proof.** First we realize that the solution \( \tilde{u}_{n+1} \) is identical to \( \tilde{u}_{n,k} \), because \( \tilde{u}_{n,k} \) solves \( a(\tilde{u}_{n,k}, \varphi) = f(\varphi) \) in \( \tilde{V}_{n+1} \), which is a subspace of \( \tilde{V}_n \oplus O_k \) and the solution is unique:

\[
\tilde{u}_{n+1} = \tilde{u}_{n,k}. \tag{25}
\]

Second, since \( u - \tilde{u}_{n,i} \) is \( a \)-orthogonal to \( \tilde{u}_{n,i} - \tilde{u}_n \), it holds that

\[
\| u - \tilde{u}_n \|_a^2 = \| u - \tilde{u}_{n,i} \|_a^2 + \| \tilde{u}_{n,i} - \tilde{u}_n \|_a^2 \tag{26}
\]

and thus

\[
k = \arg \max_{i=1,\ldots,N_D} \| \tilde{u}_{n,i} - \tilde{u}_n \|_a \tag{27}
\]

implies

\[
k = \arg \min_{i=1,\ldots,N_D} \| u - \tilde{u}_{n,i} \|_a \tag{28}
\]

So \( \tilde{u}_{n,k} \) is closest to \( u \) among all \( \tilde{u}_{n,i} \).

Third, \( \tilde{u}_{n,i} \) is the best approximation in \( \tilde{V}_n \oplus O_i \). It is not possible to get closer to \( u \) with any other enrichment in \( O_i \). \( \square \)

**Corollary 5.** The results in Theorem 1 and Corollary 2 also hold for Algorithm 2.

**Proof.** The enrichment in Algorithm 2 is optimal, so it is not worse than the enrichment of Algorithm 1. Any bound for Algorithm 1 holds also for Algorithm 2. \( \square \)

5. NUMERICAL EXPERIMENTS

The experiments were carried out using the software package pyMOR (Milk et al. (2016)). The source code to
reproduce the results presented here can be obtained at Zenodo (Buhr (2017)).

While the method and the proofs work both in two and three dimensions, experiments were conducted for the two dimensional case only. We define \( \Omega \) to be the unit square \((0,1)^2 \). We discretize the problem using P1 finite elements on a regular grid of \( 200 \times 200 \) squares, each divided into four triangles, resulting in \( 80,401 \) degrees of freedom. We use a coefficient field \( \kappa \) with high contrast \((\kappa_{\text{max}}/\kappa_{\text{min}} = 10^5) \) and high conductivity channels to get interesting behavior (see Fig. 1 and 2). As domain decomposition \( \omega_i \) we use domains of size \( 0.2 \times 0.2 \) with overlap 0.1, resulting in 81 subdomains (Fig. 3). The resulting error decay is shown in Fig. 4, Fig. 5, and 6.

To compare with the theory, we calculate an upper bound for \( c_{pu} \) using \( J = 4, c_f = 1/(\sqrt{2\pi}), \kappa_{\text{max}}/\kappa_{\text{min}} = 10^5, \max_i \| \nabla \varrho_i \|_\infty = 2H^{-2} = 200, \max_i \| \varrho_i \|_\infty = 1 \) and obtain 
\[
c^2_{pu} \leq 3.6013 \cdot 10^7.
\]
This results in an estimate of \( 1 - c \geq 1.714 \cdot 10^{-10} \). The rate of convergence observed in the experiment is several orders of magnitude better than the rate guaranteed by the a priori theory and is close to the optimal convergence rate (Fig. 6).

To investigate the reason for this, we plot the quotient of the larger part and the smaller part of the estimates (10), (11), and (12) in Fig. 7. It can be observed that the estimate (10) is rather sharp, except when the error drops after a plateau. In estimate (11), around one order of magnitude is lost. This could be improved by not enriching only one space but using a marking strategy instead. However, the main reason for the a priori theory to be so pessimistic seems to be in estimate (12).

6. CONCLUSION AND OUTLOOK

We have shown that residual based online enrichment converges exponentially. The observed rate of convergence is far better than the rate guaranteed by theory and close to the rate of optimal convergence. However, these results do not transfer immediately to methods like ArbiLoMod or LRBMS, because these methods do not enrich with a local solution, but they apply a subspace projection to the local solution before adding it to the reduced basis. The convergence behavior with an additional subspace projection is subject to future work. Additionally, the results presented in this publication are restricted to not parameterized...
Fig. 6. Convergence. 1 would be convergence within one iteration, 0 would be stagnation.

Fig. 7. Sharpness of inequalities in Algorithm 1: For equation (10), \( \frac{\|\tilde{u}_n - u\|_a^2 - \|R_n\|_{O_k}^2}{\|\tilde{u}_{n+1} - u\|_a^2} \) is plotted. For equation (11), \( \frac{\|R_n\|_{O_k}^2}{\left(\frac{1}{ND} \sum_{i=1}^{ND} \|R_n\|_{O_i}^2\right)} \) is plotted. For equation (12), \( \frac{\left(\sum_{i=1}^{ND} \|R_n\|_{O_i}^2\right)}{\left(\|R_n\|_{V_o}^2\right)} \) is plotted. Problems. Also the extension to parameterized problems is an interesting question which remains to be answered.

REFERENCES

Albrecht, F., Haasdonk, B., Ohlberger, M., and Kaulmann, S. (2012). The localized reduced basis multi-scale method. Proceedings of Algoritmy 2012, Conference on Scientific Computing, Vysoke Tatry, Podbanske, September 9-14, 2012, 393–403.

Albrecht, F. and Ohlberger, M. (2013). The localized reduced basis multi-scale method with online enrichment. Oberwolfach Rep., 7, 406–409. doi:10.4171/OWR/2013/07.

Buhr, A. (2017). Source Code to “Exponential Convergence of Online Enrichment in Localized Reduced Basis Methods”. doi:10.5281/zenodo.1002767.

Buhr, A., Engwer, C., Ohlberger, M., and Rave, S. (2017). ArbiLoMod, a Simulation Technique Designed for Arbitrary Local Modifications. SIAM J. Sci. Comput., 39(4), A1435–A1465. doi:10.1137/15M1054213.

Chung, E.T., Efendiev, Y., and Leung, W.T. (2015). Residual-driven online generalized multiscale finite element methods. J. Comput. Phys., 302, 176–190. doi:10.1016/j.jcp.2015.07.068.

Chung, E.T., Efendiev, Y., and Leung, W.T. (2017a). Constraint energy minimizing generalized multiscale finite element method. arXiv preprint arXiv:1704.03193.

Chung, E.T., Efendiev, Y., and Leung, W.T. (2017b). Fast online generalized multiscale finite element method using constraint energy minimization. arXiv preprint arXiv:1706.07093.

Efendiev, Y., Galvis, J., and Hou, T.Y. (2013). Generalized multiscale finite element methods (GMsFEM). J. Comput. Phys., 251, 116–135. doi:10.1016/j.jcp.2013.04.045.

Milk, R., Rave, S., and Schindler, F. (2016). pyMOR – Generic Algorithms and Interfaces for Model Order Reduction. SIAM J. Sci. Comput., 38(5), S194–S216. doi:10.1137/15M1026614.

Ohlberger, M. and Schindler, F. (2015). Error control for the localized reduced basis multi-scale method with adaptive on-line enrichment. SIAM J. Sci. Comput., 37(6), A2865–A2895. doi:10.1137/151003660.