On the $P$- and $T$-non-invariant
two-component equation for the neutrino

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The relativistic two-component equation describing the free motion of particles with
zero mass and spin $\frac{1}{2}$, which is $P$- and $T$-non-invariant but $C$-invariant, is found. The
representation of the Poincaré group for zero mass and discrete spin is constructed.
The position operator for such a particle is defined.

1. Introduction

As is known, the Dirac equation for a particle with zero mass:

$$i \frac{\partial \Psi(t, x)}{\partial t} = \gamma_0 \gamma_k p_k \Psi(t, x), \quad k = 1, 2, 3, \quad (1.1)$$

is invariant with respect to the space-time reflections. If one chooses for the Dirac
matrices the Weyl representation eq. (1.1) decomposes into a system of two equations

$$i \frac{\partial \Psi_{\pm}(t, x)}{\partial t} = \pm \sigma_k p_k \Psi_{\pm}(t, x), \quad (1.2)$$

where $\sigma_k$ are the Pauli matrices and $\Psi_{\pm}$ is a two-component spinor. The Weyl
equation (1.2) for $\Psi_+$ (or $\Psi_-$) is not invariant under space reflection $P$ and charge
conjugation $C$ but is invariant under the $CP$- and $T$-operations.

Due to the fact that the space parity in the weak interactions is not conserved it
is usually assumed that neutrino is described, not by the four-component eq. (1.1),
but by a two-component one (1.2). Therefore, in papers [1] an hypothesis was put
forward that the weak interactions are invariant with respect to the $CP$ operation and
consequently to the $T$ operation, if the $CPT$ theorem is valid.

In this paper the two-component equation for a particle with zero mass and spin $\frac{1}{2}$,
which is non-invariant under the time reflection of $T$ and the $CP$ operation, is found.

2. Equation for a neutrino with “variable mass”

On the solutions of eq. (1.1) the generators of the Poincaré group $P(1, 3)$ have the form

$$P_0^\Psi = \mathcal{H}_\Psi = \gamma_0 \gamma_k p_k, \quad P_k^\Psi = p_k, \quad (2.1)$$

$$J_{kl}^\Psi = x_k p_l - x_l p_k + S_{kl}, \quad J_{0k}^\Psi = x_0 p_k - \frac{1}{2} [x_k, \mathcal{H}]_+, \quad (2.1')$$

$$S_{\mu \nu} = \frac{1}{4} i (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \quad S_{\mu 4} = \frac{1}{4} i (\gamma_\mu \gamma_4 - \gamma_4 \gamma_\mu),$$

$$S_{\mu 5} = \frac{1}{2} i \gamma_\mu, \quad S_{4 5} = \frac{1}{2} i \gamma_4, \quad \mu = 0, 1, 2, 3,$$
where $\gamma_\mu$ and $\gamma_4$ are the Dirac matrices.

If one performs a unitary transformation [2] over eq. (1.1)

$$U_1 = \exp \left\{ \frac{1}{2} i \pi S_{53} e_3 \right\}, \quad e_3 = \frac{p_3}{|p_3|}, \quad p_3 \neq 0,$$

(2.2)
or

$$U_1 = \frac{1}{\sqrt{2}} (1 + \gamma_3 e_3), \quad (2.2')$$

eq (1.1) has the form

$$i \frac{\partial \chi(t, x)}{\partial t} = (\gamma_0 \gamma_a p_a + \gamma_0 |p_3|) \chi(t, x), \quad a = 1, 2,$$

(2.3)

$$\chi = U_1 \Psi, \quad \chi \equiv \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}, \quad (2.4)$$

where $\chi^\pm$ is a two-component spinor.

The Poincaré group generators $P(1, 3)$ on $\{\chi\}$ being the solution of eq. (2.3) have the form

$$P_0^X = H^X = \gamma_0 \gamma_a p_a + \gamma_0 |p_3|, \quad P_k^X = p_k,$$

$$J_{ab}^X = x_a p_b - x_b p_a + S_{ab}, \quad J_{a3}^X = x_a p_3 - x_3 p_a - e_3 S_{a3},$$

$$J_{0k}^X = x_0 p_k - \frac{1}{2} [x_k, H^X]_+.$$

(2.5)

Choosing for the Dirac matrices somewhat unusual representation

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} i \sigma_a & 0 \\ 0 & -i \sigma_a \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

(2.6)

eq (2.3) decomposes into a system of two equations

$$i \frac{\partial \chi^\pm(t, x)}{\partial t} = \{i \sigma_3 \sigma_a p_a \pm \sigma_3 |p_3|\} \chi^\pm(t, x),$$

$$\chi^\pm = Q^\pm \chi, \quad Q^\pm = \frac{1}{2} \pm i S_{43} = \frac{1}{2} (1 \pm \gamma_3 \gamma_4).$$

(2.7)

Eq. (2.7) for the functions $\chi^+(t, x)$ (or $\chi^-(t, x)$) has quite the other properties relative to the discrete transformations than the Weyl equation (1.2).

We note the following:

(i) It is possible to arrive at eq. (2.3) (or (2.7)) in another way. If we “extract the square root” from the operator equation

$$(p_0^2 - p_a^2) \chi = p_3^2 \chi,$$

we obtain eqs. (2.3) (or eqs. (2.7)).

(ii) The fact that the Dirac equations for zero and non-zero mass are invariant under the $P_-$, $T$- and $C$-transformations is the consequence of the fact that they,
besides being invariant with respect to the group $P(1, 3)$, are invariant under the group $SU(2) \otimes SU(2) \sim O(4)$ (this question will be considered in detail in a following paper).

Eq. (2.3) coincides in form with a usual Dirac equation for zero mass if $|p_3|$ is considered as the mass of a particle. Therefore it is possible to say that eq. (2.3) describes a “flat neutrino” with variable mass $|p_3|$. Really the operator $|p_3|$ is the Casimir operator of the group $P(1, 2)$ but not of the group $P(1, 3)$.

Before passing to an investigation of the $P$-, $T$- and $C$-properties of eqs. (2.7) we shall construct the operator of the position in the space.

For eq. (2.3) the operator of the Foldy–Wouthuysen type has the form

$$ U_2 = \exp \left\{ \frac{S_{5a} p_a}{\sqrt{p_a^2 + |p_3|^2}} \arctg \sqrt{\frac{p_a^2}{|p_3|^2}} \right\}. \tag{2.8} $$

If the matrix $S_{5a}$ have the form of (2.1') then

$$ U_2 = \frac{E + |p_3| + \gamma_a p_a}{2E(E + |p_3|)}, \quad E = \sqrt{p_a^2 + p_3^2 + p_3^2}. \tag{2.9} $$

Eq. (2.3) after the transformation (2.9) transfers into

$$ i \frac{\partial \Phi(t, x)}{\partial t} = \mathcal{H}^\Phi(t, x) = \gamma_0 E \Phi(t, x), \quad \Phi(t, x) = U_2 \chi(t, x). \tag{2.10} $$

The generators of the group $P(1, 3)$ on $\{\Phi\}$ have the form

$$ P_0^\Phi = \mathcal{H}^\Phi = \gamma_0 E, \quad P_k^\Phi = p_k, $$

$$ J_{ab}^\Phi = x_a p_b - x_b p_a + S_{ab}, \quad J_{a3}^\Phi = x_a p_3 - x_3 p_a - \frac{e_3 S_{ab} p_b}{E + |p_3|}, \tag{2.11} $$

$$ J_{0a}^\Phi = x_0 p_a - \frac{1}{2} [x_a, \mathcal{H}^\Phi] + - \gamma_0 \frac{S_{ab} p_b}{E + |p_3|}, \quad J_{03}^\Phi = x_0 p_3 - \frac{1}{2} [x_3, \mathcal{H}^\Phi]_+ . $$

It must be noted that the operators (2.11), as it can be immediately verified, satisfy the algebra $P(1, 3)$ commutation relations not depending on the matrices $S_{ab}$ explicit form, i.e. the operators (2.11), if $\gamma_0$ is substituted for 1 (or $-1$) and realize irreducibly the algebra $P(1, 3)$ representation which is characterized by zero mass and discrete spin. The representation (2.11) differs from the corresponding Shirokov [3], Lomont–Moses [4] ones but is certainly equivalent to them.

The position operator on a set $\{\chi\}$ looks as

$$ X_a^\chi = U_2^{-1} x_a U_2 = x_a - \frac{S_{5a}}{E} + \frac{S_{5ac} p_c}{E^2 (E + |p_3|)} + \frac{S_{ac} p_c}{E(E + |p_3|)}, \tag{2.12} $$

$$ X_3^\chi = U_2^{-1} x_3 U_2 = x_3 + e_3 \frac{S_{ac} p_c}{E^2}, \quad S_{5c} = -\frac{1}{2} i \gamma_c. $$

The position operator on a set of solution $\{\Psi\}$ of eq. (1.1) looks as follows

$$ X_a^\Psi = U_1^{-1} X_a^\chi U_1 = x_a + e_3 \frac{\gamma_3 S_{5a}}{E} - e_3 \frac{S_{ac} p_c}{E^2 (E + |p_3|)} + \frac{S_{ac} p_c}{E(E + |p_3|)}, \tag{2.13} $$

$$ X_3^\Psi = U_1^{-1} X_3^\chi U_1 = x_3 - \frac{\gamma_3 S_{5c} p_c}{E^2}. $$
If one performs a transformation on eq. (1.1)
\[ \tilde{U}_1 = \frac{1}{\sqrt{2}} (1 + \gamma_3) \]  
(2.14)
and then a transformation
\[ \tilde{U}_2 = \frac{E + p_3 + \gamma_a p_a}{\sqrt{2E(E + |p_3|)}} \]  
(2.15)
it will transform into the equation
\[ i \frac{\partial \tilde{\Phi}(t, \mathbf{x})}{\partial t} = \gamma_0 \tilde{\Phi}(t, \mathbf{x}), \quad \tilde{\Phi} = \tilde{U}_2 \tilde{U}_1 \Psi. \]  
(2.16)

The generators of the group \( P(1, 3) \) on \( \{\tilde{\Phi}\} \) coincide with (2.11) where the substitution was made \( e_3 \rightarrow 1, |p_3| \rightarrow p_3 \).

3. \( P \)-, \( T \)- and \( C \)-properties of two-component equation

Here we shall study the properties of one of the two-component eqs. (2.7)
\[ i \frac{\partial \chi(t, \mathbf{x})}{\partial t} = (i \sigma_3 \sigma_a p_a + \sigma_3 |p_3|) \chi(t, \mathbf{x}), \]  
(3.1)
under the discrete transformations.

We shall denote through \( P^{(k)} \) \( (k = 1, 2, 3) \) the space inversion operator of one axis which is determined as
\[ P^{(1)} \chi(t, x_1, x_2, x_3) = r^{(1)} \chi(t, -x_1, x_2, x_3). \]  
(3.2)
Analogously \( P^{(2)} \) and \( P^{(3)} \) are determined.

As is well known, two non-equivalent definitions of the time-reflection operator exist. According to Wigner the time-inversion operator is
\[ T^{(1)} \chi(t, \mathbf{x}) = \tau^{(1)} \chi^*(-t, \mathbf{x}). \]  
(3.3)
According to Pauli it is:
\[ T^{(2)} \chi(t, \mathbf{x}) = \tau^{(2)} \chi(-t, \mathbf{x}). \]  
(3.4)
The operator of the charge conjugation can be defined as the product of the operators \( T^{(1)}, T^{(2)} \) or as
\[ C \chi(t, \mathbf{x}) = \tau^{(3)} \chi^*(t, \mathbf{x}), \]  
(3.5)
where \( r^{(k)}, \tau^{(k)} \) are the \( 2 \times 2 \) matrices.

The operators \( P, T, C \) with the group \( P(1, 3) \) generators satisfy the usual commutation relations.

The generators of the group \( P(1, 3) \) on the solutions \( \{\chi\} \) of eq. (3.1) have the form of eq. (2.5) where
\[ H^\chi \rightarrow i \sigma_3 \sigma_a p_a + \sigma_3 |p_3| = -\sigma_2 p_1 + \sigma_2 p_2 + \sigma_3 |p_3|, \]
\[ S_{ab} \rightarrow \frac{1}{4} i (\sigma_b \sigma_a - \sigma_a \sigma_b), \quad S_{a3} \gamma_3 \rightarrow -\frac{1}{2} \sigma_a, \]  
(3.6)
\[ ^1 \text{In what follows, under } \chi \text{ we shall understand the two-component spinor } \chi^+. \]
and the matrix $\gamma_0$ is substituted for the matrix $\sigma_3$.

Using the definitions (3.2)–(3.5) it is not difficult to verify that eq. (3.1) is $P(3)_-$, $C$-invariant but $P(1)_-\,, P(2)_-\,, T(1)_-\,, T(2)_-$-non-invariant.

Thus, eq. (3.1) is $P(3)_-\,, P(1)_-\,, P(2)_-\,, T(1)_-\,, T(2)_-$-invariant but $P(3)\,CT^{(a)}_-$ and $P^{(a)}\,C$-non-invariant.

We note the following:

(i) The result obtained is a consequence of the fact that the projection operators $Q_\pm$, with the operators of the discrete transformations, satisfy the following relations

$$P^{(a)}Q_\pm = Q_\mp P^{(a)}\,, \quad T^{(a)}Q_\pm = Q_\mp T^{(a)}\,, \quad P(3)_-Q_\pm = Q_\pm P(3)_-, \quad CQ_\pm = Q_\pm C.$$  \hspace{1cm} (3.7)

(ii) The two-component equations for the functions $\chi_+$ and $\chi_-$ are equivalent to the four-component one (2.3) with the subsidiary relativistic-invariant conditions

$$Q_-\chi = \left(\frac{1}{2} - iS_{43}\right)\chi = \frac{1}{2}(1 - \gamma_3\gamma_4)\chi = 0,$$

$$Q_+\chi = \left(\frac{1}{2} + iS_{43}\right)\chi = \frac{1}{2}(1 + \gamma_3\gamma_4)\chi = 0.$$  \hspace{1cm} (3.8)

respectively. For eq. (1.1) these conditions look like

$$\left(\frac{1}{2} + i\epsilon_{3}S_{45}\right)\Psi = \frac{1}{2}(1 - \epsilon_3\gamma_4)\Psi = 0,$$

$$\left(\frac{1}{2} - i\epsilon_{3}S_{45}\right)\Psi = \frac{1}{2}(1 + \epsilon_3\gamma_4)\Psi = 0.$$  \hspace{1cm} (3.8') (3.9')

Eq. (1.1) with the subsidiary conditions (3.8') and (3.9') can be joined and can be written in the form of two $P^{(a)}_-$ and $T^{(b)}_-$-non-invariant but $P(3)_-$ and $C$-invariant equations

$$\{\gamma_\mu p^\mu + \chi(1 + \epsilon_3\gamma_4)\} \Psi_1(t, x) = 0, \quad \{\gamma_\mu p^\mu + \chi(1 - \epsilon_3\gamma_4)\} \Psi_2(t, x) = 0,$$

where $\chi$ is some constant value. The four-component equations for the neutrino, which are the union of eq. (1.1) and the usual subsidiary condition, were recently considered in ref. [6]. These equations, as well as the Weyl equations (1.2), are $P$- and $C$-non-invariant but $T^{(1)}_-$-invariant.

The unitary operator of type $U_2$ for the two-component eq. (3.1) has the form

$$V_1 = \exp \left\{ \frac{S_a p_a}{\sqrt{p_a^2}} \arctg \sqrt{\frac{p_a^2}{|p_3|}} \right\}, \quad S_k = \frac{1}{2} \epsilon_{kln} S_{ln},$$  \hspace{1cm} (3.10)

or

$$V_1 = \frac{E + |p_3| + i\sigma_a p_a}{\{2E(E + |p_3|)\}^{1/2}}.$$  \hspace{1cm} (3.11)
The position operator on the set of solutions \( \{ \chi \} \) of eqs. (3.1) looks as follows

\[
X_a^{\chi^+} = V_1^{-1} x_a V_1 = x_a - \frac{\sigma_a}{2E} + \frac{\sigma_c p_a}{2E^2(E + |p_3|)} - i \frac{(\sigma_a \sigma_c - \sigma_c \sigma_a) p_c}{4E(E + |p_3|)}, \\
X_3^{\chi^+} = V_1^{-1} x_3 V_1 = x_3 + e_3 \frac{\sigma_5 p_6}{2E^2}.
\] (3.12)

To complete our treatment, we find the position operator for the neutrino which is described by the Weyl equation (1.2), for example for the function \( \Psi_+ \). This equation under a transformation

\[
V = \frac{E + |p_3| + i \sigma_k \xi_k}{2 \sqrt{\xi_k p_k}},
\] (3.13)

where the vector \( \xi \) has the following components

\[
\xi_k = \{ p_1, p_2 e_3, p_2 + e_3 p_1, e_3 (E + |p_3|) \},
\]
takes a canonical form

\[
i \frac{\partial \Phi_+(t, x)}{\partial t} = \sigma'_3 E \Phi_+(t, x), \quad \sigma'_3 = \sigma_3 e_3, \quad \Phi_+(t, x) = V \Psi_+(t, x).
\] (3.14)

The position operator for a neutrino which is described by the Weyl equation (1.2) (for \( \Psi_+ \)) looks like

\[
X_a^W = V^{-1} x_a V = x_a + i e_3 \frac{\sigma_3 \sigma_a}{2E} - i \frac{e_3 \sigma_3 \sigma_c p_a}{2E^2(E + |p_3|)} - i \frac{(\sigma_a \sigma_c - \sigma_c \sigma_a) p_c}{4E(E + |p_3|)},
\]

\[
X_3^W = V^{-1} x_3 V = x_3 - i \frac{\sigma_3 \sigma_5 p_6}{2E^2}.
\]

The other definitions of the operators \( X_k \) and \( V \) for the neutrino are given in ref. [5].

(iii) From Dirac eq. (1.1) one can, generally speaking, obtain three types of non-equivalent two-component equations. On the set of solutions of eq. (1.1) a direct sum of four irreducible representations \( D^\varepsilon(s) \) of the group \( P(1,3) \)

\[
D^{\varepsilon=1} \left( s = \frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = -\frac{1}{2} \right) \oplus D^{\varepsilon=1} \left( s = -\frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = \frac{1}{2} \right)
\] (3.15)

is realized, where \( \varepsilon \) is an energy sign, \( s \) is a helicity. Hence it follows that there exist three types of two-component equations on the set of which the following representation of the group \( P(1,3) \)

\[
D^{\varepsilon=1} \left( s = \frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = -\frac{1}{2} \right),
\]
or

\[
D^{\varepsilon=1} \left( s = -\frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = \frac{1}{2} \right), \quad D^{\varepsilon=1} \left( s = \frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = \frac{1}{2} \right).
\] (3.16)

or

\[
D^{\varepsilon=1} \left( s = -\frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = -\frac{1}{2} \right), \quad D^{\varepsilon=1} \left( s = \frac{1}{2} \right) \oplus D^{\varepsilon=-1} \left( s = -\frac{1}{2} \right).
\] (3.17)
\[ D^s=1 \left( s = \frac{1}{2} \right) \oplus D^{s=-1} \left( s = -\frac{1}{2} \right) \] (3.18)

are realized. If on the solutions of two-component equation there realizes the representation (3.16) then this equation will be \( T^{(1)} \)-invariant but \( C \)-, \( P \)-, \( T^{(2)} \)-non-invariant, if the representation (3.17) does then it will be \( T^{(1)} \), \( T^{(2)} \), \( C \)-invariant but \( P \)-non-invariant, and if the representation (3.18) it will be \( T^{(1)} \), \( P \)-invariant but \( C \), \( T^{(2)} \)-non-invariant. This problem will be considered in more detail in another paper.

4. Equation for a flat neutrino

The motion group in the Minkowski three-space is the \( P(1,2) \) group of rotations and translations conserving the form

\[ x^2 = x_0^2 - x_1^2 - x_2^2. \]

In this case the simplest spinor equation is

\[ i \frac{\partial \chi_\pm(t,x_1,x_2)}{\partial t} = (i\sigma_3\sigma_a p_a \pm \sigma_3 m) \chi_\pm(t,x_1,x_2), \] (4.1)

\( \chi_\pm \) is the two-component spinor and \( m \) is the eigenvalue of the operator \( \sqrt{p_2^2} \).

Eq. (4.1) for \( \chi_+ \) (or \( \chi_- \)) like eq. (3.1) is invariant under the \( P(1)P(2) \) and \( C \)-operations but non-invariant under the \( P^{(a)} \) and \( T^{(b)} \)-operations.

Thus, eq. (4.1) for the wave function \( \chi_+ \) (or \( \chi_- \)) is \( P(1)P(2)C \), \( T^{(a)}P^{(b)} \) and \( P^{(a)}CT^{(b)} \)-invariant but \( P^{(a)}C \)- and \( CT^{(a)} \)-non-invariant.

It should be noted that the equation being the “direct sum” of the equation for \( \chi_+(t,x_1,x_2) \) and \( \chi_-(t,x_1,x_2) \) is invariant under the \( P-, T- \) and \( C \)-transformations [7].

Finally, we quote one more example of the \( P- \) and \( C \)-non-invariant equation which is invariant with respect to the inhomogeneous De Sitter group. Such is the Dirac equation:

\[ i \frac{\partial \Psi(t,x,x_4)}{\partial t} = (\gamma_0\gamma_k p_k + \gamma_0 \kappa) \Psi(t,x,x_4), \quad k = 1, 2, 3, 4. \] (4.2)

This equation as is shown in refs. [2, 7] is \( T^{(1)} \), \( T^{(2)}C \)-invariant but \( P^{(k)} \), \( T^{(2)} \)- and \( C \)-non-invariant.

All the results obtained in this paper can be generalized for the arbitrary spin \( s \) case, if one uses for this the purpose the equation (ref. [2]):

\[ i \frac{\partial \Psi(t,x)}{\partial t} = \lambda S_0 p_l \Psi(t,x), \quad l = 1, 2, 3, \] (4.3)

where \( \lambda \) is some fixed parameter (for the Dirac equation \( \lambda = -2i \)), and \( S_{\mu\nu}, S_{\mu4}, S_{45} \) are the matrices (not \( 4 \times 4 \) ones) realizing the algebra \( O(1,5) \) representation.

(i) If we transform the usual Dirac equation describing the motion of the non-zero mass particle \( m \) with a spin \( \frac{1}{2} \) as

\[ V_2 = \frac{\gamma_3 p_3 + q_3 + m}{\sqrt{2q_3(q_3 + m)}}, \quad q_3 \equiv \sqrt{p_3^2 + m^2}, \] (4.4)
it has the form
\[ i \frac{\partial \Psi'(t, x)}{\partial t} = H' \Psi'(t, x), \quad (4.5) \]
\[ H' = \gamma_0 \gamma_a p_a + \gamma_0 q_3, \quad \Psi' = V_2 \Psi, \quad a = 1, 2. \quad (4.6) \]

Choosing the representation (2.6) for the Dirac matrices eq. (4.5) is decomposed into the set of two independent equations
\[ i \frac{\partial \Psi'_+(t, x)}{\partial t} = (-\sigma_2 p_1 + \sigma_1 p_2 + \sigma_3 q_3) \Psi'_+(t, x), \quad (4.7) \]
\[ i \frac{\partial \Psi'_-(t, x)}{\partial t} = (-\sigma_2 p_1 + \sigma_1 p_2 - \sigma_3 q_3) \Psi'_-(t, x), \quad (4.8) \]
where \( \Psi'_+ \) and \( \Psi'_- \) are two-component wave functions.

Eq. (4.7) or (4.8) describes a free motion of spinless particle and antiparticle with the mass \( m \). Thus besides of the Klein–Gordon equation there exist other equations of the type (4.7) and (4.8) which are also relativistically invariant and describe the spinless particle motion with non-zero mass. The two-component eq. (4.7) is equivalent to the four-component Dirac equation
\[ i \frac{\partial \Psi(t, x)}{\partial t} = (\gamma_0 \gamma_k p_k + \gamma_0 m) \Psi(t, x), \quad k = 1, 2, 3 \quad (4.9) \]
with such subsidiary condition
\[ \left(1 - \frac{\gamma_3 \gamma_4 m + \gamma_4 q_3}{q_3}\right) \Psi(t, x) = 0. \quad (4.10) \]

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