LINEAR SYSTEMS ON THE BLOW-UP OF \((\mathbb{P}^1)^n\)

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Abstract. In this note we study linear systems on the blow-up of \((\mathbb{P}^1)^n\) at \(r\) points in very general position. We prove that the fibers of the projections \((\mathbb{P}^1)^n \to (\mathbb{P}^1)^s, 1 \leq s \leq n - 1\) can give contribution to the speciality of the linear system. This allows us to give a new definition of expected dimension of a linear system in \((\mathbb{P}^1)^n\) which we call fiber dimension. Finally, we state a conjecture about linear systems on \((\mathbb{P}^1)^3\).

Introduction

An open problem in algebraic geometry is that of determining the dimension of a linear system of hypersurfaces of \(\mathbb{P}^n\) of a given degree passing through finitely many points in very general position with prescribed multiplicities. This problem is related to polynomial interpolation in several variables to the Waring problem for polynomials and the classification of defective higher secant varieties of Veronese embeddings of projective spaces [Cil01, Sections 6 and 7]. In case \(n = 2\) the Segre-Harbourne-Gimigliano-Hirschowitz conjecture [Gim87, Har85, Hir89, Seg62] predicts the dimension of such linear systems. Several cases of this conjecture have been proved, see e.g. [Cil01, CHMR12, CM98, CM00, Lau99, Mig00]. In [LU06, LU12] an analogous conjecture is stated for \(n = 3\) and proved when the multiplicities of the points are \(\leq 5\) in [BB09, CBCS12]. There is no such a conjecture for higher values of \(n\), but the same there are partial results about the dimension of such linear systems [Dum09, Pau13] and in [BDP12] the authors determine the contribution to the dimension of linear systems given by linear subspaces.

Inspired by [BDP12] in this note we study linear systems of \((\mathbb{P}^1)^n\) through multiple points for \(n \geq 2\). Let \(\mathcal{L} := \mathcal{L}_{(d_1, \ldots, d_n)}(m_1, \ldots, m_r)\) be the linear system of hypersurfaces of degree \((d_1, \ldots, d_n)\) in \((\mathbb{P}^1)^n\) passing through a general union of \(r\) points with multiplicities respectively \(m_1, \ldots, m_r\). The virtual dimension of \(\mathcal{L}\) is

\[
\text{vdim}(\mathcal{L}) = \prod_{i=1}^{n} (d_i + 1) - \sum_{i=1}^{r} \left(\frac{n + m_i - 1}{n}\right) - 1
\]

the expected dimension of \(\mathcal{L}\) is \(\text{edim}(\mathcal{L}) = \max(\text{vdim}(\mathcal{L}), -1)\). The dimension of \(\mathcal{L}\) is minimum when the points are in very general position according to Remark 1.1. The inequality \(\dim(\mathcal{L}) \geq \text{edim}(\mathcal{L})\) always holds. The conditions imposed by the points are linearly dependent if and only if \(\dim(\mathcal{L}) > \text{edim}(\mathcal{L})\), in this case we say that \(\mathcal{L}\) is special. Otherwise we say that \(\mathcal{L}\) is non-special. Special linear systems have been classified when all the multiplicities are \(\leq 2\) in [CGG05b, LP13, Laf02, VT05].

We prove that a fiber of a projection map \((\mathbb{P}^1)^n \to (\mathbb{P}^1)^s, 1 \leq s \leq n - 1\), through a multiple point can contribute to the speciality of the linear system \(\mathcal{L}\).
We introduce the fiber dimension \( \text{fdim}(\mathcal{L}) \) which satisfies the inequalities
\[
\dim(\mathcal{L}) \geq \text{fdim}(\mathcal{L}) \geq \text{vdim}(\mathcal{L})
\]
and takes into account the speciality of \( \mathcal{L} \) coming from such fibers. We say that \( \mathcal{L} \) is fiber special if \( \dim(\mathcal{L}) > \text{fdim}(\mathcal{L}) \) and that it is fiber non-special otherwise. In Theorem 3.2 we show that a linear system through two multiple points is fiber non-special. If there are more than two multiple points then there are examples of fiber special systems (see Example 5.2). Linear systems through multiple points of \((\mathbb{P}^1)^n\) correspond to complete linear systems on the blow-up \( Y \) of \((\mathbb{P}^1)^n\) at those points. We recall a quadratic form on \( \text{Pic}(Y) \) introduced in [Muk04] and we study the action of its Weyl group, providing an algorithm for determining the element of minimal total degree in any orbit of an effective class. We say that such a class and the corresponding linear system are in standard form. For \( n = 3 \) we state a conjecture about special linear systems in standard form. Finally we recall a degeneration of \((\mathbb{P}^1)^n\) into two copies of \((\mathbb{P}^1)^3\) introduced in [LP13]. We relate the speciality of a linear system with that of the two linear systems arising from the degeneration.

The paper is organized as follows: In Section 1 we recall some definitions, notations and introduce a small modification between the blow-up of \( \mathbb{P}^n \) at \( n + r - 1 \) points and the blow-up of \((\mathbb{P}^1)^n\) at \( r \) points. In Section 2 we study the Weyl group of the Picard group of the blow-up of \((\mathbb{P}^1)^n\) at points in very general position. In Section 3 we prove that the fibers through the points can contribute to the speciality of linear systems and introduce the concept of fiber dimension. Section 4 deals with a degeneration of \((\mathbb{P}^1)^n\) and the related algorithm. Finally, in Section 5 we provide some examples and state a conjecture for \((\mathbb{P}^1)^3\).

Acknowledgements. It is a pleasure to thank Elisa Postinghel and Luca Ugaglia for several useful discussions.

1. Basic setup

In what follows we will denote by \( \mathbb{K} \) an algebraically closed field. Given an algebraic variety \( X \) we denote by \( h^i(X, D) \) the dimension of the \( i \)-th cohomology group of any line bundle whose class is \( D \in \text{Pic}(X) \).

In this section we recall some definitions, notations and results about linear systems on \((\mathbb{P}^1)^n\) and \( \mathbb{P}^n \). First of all we denote by \( \mathbb{K}[x_1, y_1, \ldots, x_n, y_n] \) the Cox ring of \((\mathbb{P}^1)^n\) and by \( \mathbb{K}[x_0, \ldots, x_n] \) the Cox ring of \( \mathbb{P}^n \). Let \( \pi_Y: Y \to (\mathbb{P}^1)^n \) (resp. \( \pi_X: X \to \mathbb{P}^n \)) be the blowing-up at \( r \) (resp. \( r+n-1 \)) points in very general position.

The Picard Group of \( Y \) is generated by the \( r + n \) classes of \( H_1, \ldots, H_n, E_1, \ldots, E_r \) where \( E_i \) is the exceptional divisor over the \( i \)-th point and \( H_i \) is the pull-back of the prime divisor of equation \( x_i = 0 \). The Picard Group of \( X \) is generated by the \( r + n \) classes of \( H, E_1, \ldots, E_{r+n-1} \) where \( E_i \) is the exceptional divisor over the \( i \)-th point and \( H \) is the pull-back of a hyperplane. We will call these bases tautological.

Remark 1.1. (Points in very general position). Let \( q_1, \ldots, q_r \) be distinct points of \((\mathbb{P}^1)^n\) and let \( m \in \mathbb{N}^r \). Consider the scheme \((\mathbb{P}^1)^{r}_{[m]}\) parametrizing \( r \)-tuples of points in \((\mathbb{P}^1)^n\) and let \( Q \in (\mathbb{P}^1)^{r}_{[m]} \) be the point corresponding to the \( q_i \). For \( d \in \mathbb{N}^n \) denote by \( \mathcal{H}(d, m, Q) \) the vector space of degree \( d \) homogeneous polynomials of \( \mathbb{K}[x_1, y_1, \ldots, x_n, y_n] \) with multiplicity at least \( m_i \) at each \( q_i \). Observe that
has already been considered in \( X \)-toric functor to the first one. Since the complementary of \( \varphi \) map whose cones are exactly the one-dimensional cones of \( \Sigma \), we say that the points \( q_1, \ldots, q_r \) are in very general position if the corresponding \( Q \) is in \( U \).

**Definition 1.2.** Given a birational map \( \phi : X \to Y \) of algebraic varieties we say that \( \phi \) is a small modification if there exist open subsets \( U \subseteq X \) and \( V \subseteq Y \) such that \( \varphi(U) \subseteq V \), the restriction \( \phi|_U \) is an isomorphism and both \( X - U \) and \( Y - V \) have codimension at least two. Note that any small modification induces mutually inverse isomorphisms of push-forward and pull-back

\[
\phi_* : \text{Pic}(X) \to \text{Pic}(Y) \quad \phi^* : \text{Pic}(Y) \to \text{Pic}(X).
\]

Moreover \( h^0(Y, \phi_*(D)) = h^0(X, D) \) for any \( D \in \text{Pic}(X) \) and \( h^0(X, \phi^*(D)) = h^0(Y, D) \) for any \( D \in \text{Pic}(Y) \).

**Definition 1.3.** Let \( X \) be an algebraic variety, \( D \in \text{Pic}(X) \) a divisor class and \( V \subset X \) a subvariety. We say that \( V \) is contained with multiplicity \( m \) in the base locus of \( D \) if the exceptional divisor \( E \) of the blow-up \( \pi : \tilde{X} \to X \) of \( X \) at \( V \) is contained with multiplicity \( m \) in the base locus of \( \pi^*(D) \).

**Remark 1.4.** Let \( \phi : \mathbb{P}^n \to (\mathbb{P}^1)^n \) be the birational map defined by \([x_0 : \cdots : x_n] \mapsto ([x_{n-1} : x_n] \cdots [x_0 : x_n])\). Let \( p_1, \ldots, p_{r+n-1} \) be points of \( \mathbb{P}^n \) in very general position such that the first \( n+1 \) are the fundamental ones and let \( q_1, \ldots, q_r \) be points of \( (\mathbb{P}^1)^n \) such that \( q_1 = ([0 : 1] \cdots [0 : 1]) \), \( q_2 = ([1 : 0] \cdots [1 : 0]) \) and \( q_{i+2} = \phi(p_{i+n}) \) for \( i \in \{1, \ldots, r-2\} \). This gives the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\pi_X \downarrow & & \pi_Y \downarrow \\
\mathbb{P}^n & \xrightarrow{\phi} & (\mathbb{P}^1)^n,
\end{array}
\]

where with abuse of notation we are denoting by the same symbol \( \phi \) and its lift.

To show that the above lift is a small modification it is enough to consider the case \( r = 2 \). In this case we have commutative diagrams

\[
\begin{array}{ccc}
\Sigma_{n+1} & \supseteq & \Sigma_2 \supseteq \Sigma_{1,n} \\
\Sigma & \supseteq & \Sigma_1 \supseteq \Sigma_{1,n} \\
\mathbb{P}^n & \supseteq & X_2 \\
\mathbb{P}^n & \supseteq & X(\Sigma) \subseteq Y_2 \supseteq \Sigma_{n+1} \\
\end{array}
\]

where the first diagram is obtained by completing in two different ways the fan \( \Sigma \) whose cones are exactly the one-dimensional cones of \( \mathbb{Z}^n \) generated by the vectors \( \{\pm e_1, \ldots, \pm e_n, \pm (e_1 + \cdots + e_n)\} \), while the second diagram is obtained applying the toric functor to the first one. Since the complementary of \( X(\Sigma) \) in both \( X(\Sigma_{n+1}) \) and \( X(\Sigma_n) \) is of codimension at least two, then the corresponding toric birational map \( \phi : X \to Y \) is small. We recall that the map \( \phi \) and its action on fat points has already been considered in [CGG05].
With the above notation the induced isomorphism \( \phi_* : \text{Pic}(X) \to \text{Pic}(Y) \) is given by

\[
\begin{align*}
    H &\mapsto \sum_{i=1}^{n} H_i - (n - 1)E_1 \\
    E_{n+1} &\mapsto E_2 \\
    E_i &\mapsto H_{n+1-i} - E_1 \quad \text{for } 1 \leq i \leq n \\
    E_i &\mapsto E_i - n+1 \quad \text{for } i > n+1.
\end{align*}
\]

\( (1.1) \)

\[ \begin{align*}
    H_i \cdot H_j &= 1 - \delta_{ij} \quad E_k \cdot E_s &= -\delta_{ks} \quad H_i \cdot E_k &= 0, \\
    \end{align*} \]
where \( i, j \in \{1, \ldots, n\} \) and \( k, s \in \{1, \ldots, r\} \). Observe that the lattice \( \text{Pic}(Y) \) equipped with the integer quadratic form induced by the above bilinear form has discriminant group isomorphic to \( \mathbb{Z}/(n-1)\mathbb{Z} \) and generated by the class \( \frac{1}{n-1}K_Y \).

Recall that given a non-degenerate lattice \( \Lambda \) and an element \( R \in \Lambda \) with \( R^2 = -2 \) one can define the \textit{Picard-Lefschetz reflection} defined by \( R \) as:

\[ \sigma_R : \Lambda \to \Lambda \quad D \mapsto D + (D \cdot R)R. \]

Observe that \( \sigma_R \) is the reflection in \( \Lambda \) with respect to the hyperplane orthogonal to \( R \). The \textit{Weyl group} of \( \Lambda \), denoted by \( W(\Lambda) \) is the subgroup of isometries of \( \Lambda \) generated by the Picard-Lefschetz reflections. For simplicity, given an algebraic variety \( X \) and a bilinear form on \( \text{Pic}(X) \) we denote the Weyl group of its Picard group by \( W(X) \).

The following is a particular case of [Muk04, Theorem 1]:

\textbf{Proposition 2.1.} For each transformation \( w : \text{Pic}(Y) \to \text{Pic}(Y) \) of \( W(Y) \), there is a small modification \( w : Y \to Y_w \) with the following property: \( Y_w \) is also a blow-up of \( (\mathbb{P}^1)^n \) in \( r \) points \( q_1, \ldots, q_r \) in general position and the pull-back of the tautological basis of \( \text{Pic}(Y_w) \) coincides with the transformation of the tautological basis of \( Y \) by \( w \).

In [CT06, Lemma 2.1] a set of generators for \( W(Y) \) consists of \( n+r-1 \) reflections with respect to the following roots:

\( H_1 - E_1 - E_2, \ H_1 - H_2, \ldots, H_{n-1} - H_n, \ E_1 - E_2, \ldots, E_{r-1} - E_r. \)

Let \( \omega \) be any element of the Weyl group \( W(Y) \) and let \( \varphi_{\omega} : Y \to Y_\omega \) be the corresponding small modification. We have that \( \varphi_{\omega} \) is the lift of a birational map \( \phi_{\omega} : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n \), moreover \( Y_\omega \) is the blow-up of \( (\mathbb{P}^1)^n \) at points \( q_1', \ldots, q_r' \) where \( q_1' = q_1, q_2' = q_2 \) and \( q_i' = \phi_{\omega}(q_i) \) for \( i \geq 3 \). The birational involution of \( (\mathbb{P}^1)^n \) associated to the root \( H_1 - E_1 - E_2 \) is the following [Muk04, pag. 128]:

\[ ([x_1 : y_1], \ldots, [x_n : y_n]) \mapsto \left( \begin{array}{c}
    \frac{1}{x_1} : \frac{1}{y_1} \\
    \frac{1}{x_2} : \frac{y_2}{y_1} \\
    \vdots \\
    \frac{x_n}{x_1} : \frac{y_n}{y_1}
\end{array} \right). \]
The birational involution of $(\mathbb{P}^1)^n$ associated to the root $H_i - H_{i+1}$ is the transformation of $(\mathbb{P}^1)^n$ which exchanges the $i$-coordinate with the $i+1$-coordinate, for $i \in \{1, \ldots, n-1\}$. Finally the birational map of $(\mathbb{P}^1)^n$ associated to the root $E_i - E_{i+1}$ is the identity map as we are just relabeling two points between the $q_i$’s.

Remark 2.2. Observe that the map $\phi_* : \text{Pic}(X) \to \text{Pic}(Y)$ is an isometry of lattices.

To see this it is enough to check that $\phi_*$ preserves the intersection matrix of the basis $(H, E_1, \ldots, E_r)$ of $\text{Pic}(X)$. This holds by (1.1) and the definition of the bilinear forms on the two lattices [CT06, 2.1]. We recall also that for any $\omega \in W(Y)$ and any $D, D' \in \text{Pic}(Y)$ we have $h(Y, D) = h(Y, \omega(D))$ by [CT06, Lemma 2.3] and $D$ is integral if and only if $\omega(D)$ is.

Definition 2.3. A class $D = \sum_{i=1}^n d_i H_i - \sum_{i}^r m_i E_i$ of $\text{Pic}(Y)$ is in pre-standard form if the following inequalities hold:

$$d_1 \geq d_2 \geq \cdots \geq d_n \geq 0 \quad m_1 \geq m_2 \geq \cdots \geq m_r \quad \sum_{i=2}^n d_i \geq m_1 + m_2.$$ 

If in addition $m_r \geq 0$, then $D$ is in standard form.

Remark 2.4. By Definition 2.3 [LU12, Definition 3.1] and the action of $\phi$ given above we have that a class $D$ in the Picard group of $X$ is in pre-standard form (resp. in standard form) if and only if $\phi_*(D)$ is in pre-standard form (resp. in standard form). In particular by [LU12, Proposition 3.2] we deduce that for any effective class $D \in \text{Pic}(Y)$ there exists a $w \in W(Y)$ such that $\omega(D)$ is in pre-standard form.

Remark 2.5. A $(-1)$-class of $\text{Pic}(Y)$ is the class of an irreducible and reduced divisor $E$ such that $E^2 = E \cdot K = -1$ where $K := \frac{1}{n-1}K_Y$. Observe that this definition coincides with the classical concept of $(-1)$-class when $n = 2$.

By Remark 2.2 and [LU12, Section 4] we conclude the following: The $(-1)$-classes form an orbit with respect to the action of the Weil group. Moreover if $D$ is a class in standard form, then $\omega(D) \cdot E \geq 0$ for any $(-1)$-class $E$ and any $D \in \text{Pic}(Y)$. Finally, some geometric properties of $(-1)$-curves on surfaces generalize to $(-1)$-classes: if $D$ is effective and $D \cdot E < 0$ for some $(-1)$-class $E$ then $E \subset Bs(D)$ and if $E, E'$ are two distinct $(-1)$-classes having negative product with $D$ then $E \cdot E' = 0$. 
The following program given a class $D \in \text{Pic}(Y)$ returns its standard form $D' \in \text{Pic}(Y)$ or return $0 \in \text{Pic}(Y)$ if the linear system induced by $D$ is empty.

**Input:** $(d, m) \in \mathbb{N}^n \times \mathbb{N}^r$, with $r \geq 2$.

**Output:** $(d, m) \in \mathbb{N}^n \times \mathbb{N}^r$ or $\emptyset$.

Sort both $d = (d_1, \ldots, d_n)$ and $m = (m_1, \ldots, m_r)$ in decreasing order;

**while** $k := \sum_{i=2}^n d_i - m_1 - m_2 < 0$ and $\min(d_1, \ldots, d_n) \geq 0$ **do**

$$
(d_1, m_1, m_2) := (d_1, m_1, m_2) + (k, k, k);
$$

Sort both $d$ and $m$ in decreasing order;

**if** $\min(d_1, \ldots, d_n) < 0$ **then**

**return** $\emptyset$

**else**

**return** $(d, m)$;

**end**

**Algorithm 1:** Standard form.

### 3. Fiber special systems

Recall that we denote by $\pi : Y \rightarrow (\mathbb{P}^1)^n$ the blow-up of $(\mathbb{P}^1)^n$ at $r$ points $q_1, \ldots, q_r$ in very general position. Given a subset $I \subset \{1, \ldots, n\}$ we denote by $P_I : (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^{|I|}$ the morphism defined by (if $I$ is empty $P_I$ is the constant morphism to a point)

$$
([x_1 : y_1], \ldots, [x_n : y_n]) \mapsto ([x_i : y_i] : i \in I).
$$

We denote by $F_{j,I}$ the fiber of $P_I$ through the point $q_j$ for any $j$. Given a vector $(d_1, \ldots, d_n) \in \mathbb{N}^n$ we will denote by

$$
(3.1) \quad s_I := \sum_{i \in I} d_i \quad \text{and} \quad S_I := 1 + |I| + s_I \quad \text{for any} \quad I \subset \{1, \ldots, n\}.
$$

Observe that by the assumption made on the points $F_{i,I} \cap F_{j,I} = \emptyset$ for any $i \neq j$. In Section 3 and Section 4 we use the notation $\mathcal{L} := \mathcal{L}_{[d_1, \ldots, d_n]}(m_1, \ldots, m_r)$ to denote a general linear system when no confusion arises. We denote by $\text{V}(<\mathcal{L}>)$ the subvector space of homogeneous polynomials of $K[x_1, y_1, \ldots, x_n, y_n]$ of degree $(d_1, \ldots, d_n)$ and multiplicity at least $m_1, \ldots, m_r$ at $q_1, \ldots, q_r$ respectively.

**Definition 3.1.** The fiber dimension of the linear system $\mathcal{L}$ is

$$
\text{fdim}(\mathcal{L}) := \prod_{i=1}^n (d_i + 1) + \sum_{m_i \geq S_I} (-1)^{|I|+1} \binom{m_i - S_I + n}{n} - 1
$$

and the fiber-expected dimension is $\text{efdim}(\mathcal{L}) := \max(-1, \text{fdim}(\mathcal{L}))$. Observe that $\dim(\mathcal{L}) \geq \text{efdim}(\mathcal{L})$. We say that $\mathcal{L}$ is fiber special if $\dim(\mathcal{L}) > \text{efdim}(\mathcal{L})$ and it is fiber non-special otherwise.

**Theorem 3.2.** A linear system $\mathcal{L}$ through two points is fiber non-special.

The proof of the following lemma is a direct consequence of the identity $\sum_{i=n}^{k} \binom{1}{n} = \binom{k+1}{n+1}$ which holds for any $k \geq n$. 

Lemma 3.3. Let $I$ be an ordered subset of $\{1, \ldots, n-1\}$, let $J := I \cup \{n\}$ and let $m$ be a non-negative integer. Given a vector $(d_1, \ldots, d_n) \in \mathbb{N}^n$ let $S_I$ be defined as in (3.1). Then the following holds

$$\sum_{j=0}^{d_n} \sum_{m-j \geq S_I} \left( \frac{m-j-S_I+n-1}{n-1} \right) = \sum_{m \geq S_I} \left( \frac{m-S_I+n}{n} \right) - \sum_{m \geq S_J} \left( \frac{m-S_J+n}{n} \right).$$

Proof of Theorem 3.2. Without loss of generality we can assume that $q_1 := (0 : 1), \ldots, [0 : 1])$, $q_2 := ([1 : 0], \ldots, [1 : 0])$. Hence a basis $\mathcal{B}(\mathcal{L})$ for $V(\mathcal{L})$ consists of the monomials of the form $\prod_{i=1}^{n} x_i^{a_i} y_i^{b_i}$ where $\sum_{i=1}^{n} a_i \geq m_1$, $\sum_{i=1}^{n} b_i \geq m_2$ and $a_i + b_i = d_i$ for any $i$. The statement follows by induction on $n$ using Lemma 3.3 and the equality

$$|\mathcal{B}(\mathcal{L})| = \sum_{j=0}^{d_n} |\mathcal{B}(\mathcal{L}(d_1, \ldots, d_{n-1}))(m_1 - j, m_2 - d_n + j)|.$$

\qed

Corollary 3.4. A linear system $\mathcal{L} := \mathcal{L}(d_1, \ldots, d_n)(m_1, m_2)$ is effective if and only if $\sum_{i=1}^{n} d_i \geq m_1 + m_2$.

Proof. If $\sum_{i=1}^{n} d_i < m_1 + m_2$ then, with the same notation of the proof of Theorem 3.2 either $\sum_{i=1}^{n} a_i < m_1$ or $\sum_{i=1}^{n} b_i < m_2$ so that there are no monomials in $V(\mathcal{L})$ and thus $\mathcal{L}$ is empty. On the other hand if $\sum_{i=1}^{n} d_i \geq m_1 + m_2$ then there are $a_i, b_i$ such that $\sum_{i=1}^{n} a_i \geq m_1$ and $\sum_{i=1}^{n} b_i \geq m_2$ and $a_i + b_i = d_i$ for any $i$. Thus $V(\mathcal{L})$ contains a monomial and hence $\mathcal{L}$ is not empty.

\qed

Proposition 3.5. Let $\mathcal{L}$ be a non-empty linear system. Then the fiber $F_{I,j}$ is contained in the base locus of $\mathcal{L}$ with multiplicity

$$\mu \geq \max\{m_j - s_{I,e}, 0\}$$

and the equality holds when $r \leq 2$.

Proof. Without loss of generality we can assume that $j = 1$ and $I = \{1, \ldots, i\}$. Let $\mathcal{M} := \mathcal{L}(d_1, \ldots, d_n)(m_1, m_2)$. The vector space $V(\mathcal{L})$ is a subspace of $V(\mathcal{M})$ which admits a monomial basis $\mathcal{B}(\mathcal{M})$ given in the proof of Theorem 3.2. The Cox ring of the blow-up $\pi_{1,I} : X_{1,I} \to (\mathbb{P}^1)^n$ of $(\mathbb{P}^1)^n$ at $F_{1,I}$ is isomorphic to $\mathbb{K}[z, x_1, y_1, \ldots, z x_i, y_i, x_{i+1}, y_{i+1}, \ldots, x_n, y_n]$, where $z$ corresponds to the exceptional divisor. Let $\mathcal{B}(\mathcal{M})$ be the pull-back of the basis $\mathcal{B}(\mathcal{M})$ via $\pi_{1,I}$. Then the basis $\mathcal{B}(\mathcal{M})$ consists of the following monomials

$$\prod_{j=1}^{i} (z x_j)^{a_j}(y_j)^{b_j} \prod_{j=i+1}^{n} x_j^{a_j} y_j^{b_j},$$

where $\sum_{i=1}^{n} a_i \geq m_1$, $\sum_{i=1}^{n} b_i \geq m_2$ and $a_i + b_i = d_i$ for each $i$. Observe that $\sum_{j=1}^{i} a_j \geq m_1 - \sum_{j=i+1}^{n} a_j \geq m_1 - s_{I,e}$, with equalities when $b_j = 0$ for any $j \in \{i + 1, \ldots, n\}$ and $\sum_{i=1}^{n} a_i = m_1$. Thus $z^{m_1 - s_{I,e}}$ divides any monomial in $\mathcal{B}(\mathcal{M})$ and this is the maximal power with this property when $\mathcal{M} = \mathcal{L}$, i.e. when $r \leq 2$.

\qed
Let \( \deg \) be a grading on \( P_d \) degree at most \( d \) of degree at most \( d \) the points. Let \( L \). Observe that \( V \) correspond to the integer points of the polytope \( \Delta(1) \). Assume that the following conditions hold:

1. \( L_1, L_2 \) are fiber non-special with \( (\text{fdim}(L_1) + 1)(\text{fdim}(L_2) + 1) \geq 0 \),
2. \( m_i \leq k \), for any \( i \in \{1, \ldots, s\} \),
3. \( m_j \leq d_n - k + 1 \) for any \( j \in \{s + 1, \ldots, n\} \).

Then the system \( \mathcal{L} := \mathcal{L}(d_1, \ldots, d_n)(m_1, \ldots, m_s) \) is fiber non-special.

Proof. Observe that we have an isomorphism of vector spaces

\[
\Phi: V \to \mathbb{K}^N
\]

be the function which maps \( f \) into the collection of all partial derivatives of \( f \), which correspond to the integer points of the polytope \( \Delta(m_i) \cap \prod_{i=1}^n [0, d_i] \), evaluated at \( p_i \) for each \( i \). Let \( L(\mathcal{L}) \) be the matrix of \( \Phi \) with respect to the monomial basis of \( V \) and the standard basis of \( \mathbb{K}^N \). The columns \( M(\mathcal{L}) \) are indexed by monomials of degree at most \( (d_1, \ldots, d_n) \), while rows are indexed by conditions imposed by the points. Let \( P = \mathbb{K}[p_1^1, \ldots, p_1^r, \ldots, p_s^1, \ldots, p_s^r] \), where \( p_k^j \) is the \( j \)-coordinate of the \( k \)-th point. Then the entries of \( M(\mathcal{L}) \) can be considered as polynomials in \( P \). Let \( \deg \) be a grading on \( P \) defined by \( \deg(p_k^j) = 1 \) if \( k = n \) and \( j \geq s + 1 \) and \( \deg(p_k^j) = 0 \) otherwise. In what follows we will adopt the following notation:

\[
(4.1) \quad L_1 := \mathcal{L}(d_1, \ldots, d_{n-1}, k)(m_1, \ldots, m_s) \quad L_2 := \mathcal{L}(d_1, \ldots, d_n, k)(m_{s+1}, \ldots, m_r).
\]

**Theorem 4.1.** Let \( L_1 \) and \( L_2 \) be defined as in (4.1). Assume that the following conditions hold:

1. \( L_1, L_2 \) are fiber non-special with \( (\text{fdim}(L_1) + 1)(\text{fdim}(L_2) + 1) \geq 0 \),
2. \( m_i \leq k \), for any \( i \in \{1, \ldots, s\} \),
3. \( m_j \leq d_n - k + 1 \) for any \( j \in \{s + 1, \ldots, n\} \).

Then the system \( \mathcal{L} := \mathcal{L}(d_1, \ldots, d_n)(m_1, \ldots, m_s) \) is fiber non-special.

Proof. Observe that we have an isomorphism of vector spaces

\[
\Psi: V(\mathcal{L}_2) \to V(\mathcal{L}(d_1, \ldots, d_{n-1}, k)(m_{s+1}, \ldots, m_r))
\]

where the multiplicities are imposed at the points \( p_{s+1}, \ldots, p_r \) respectively. After reordering the rows and the columns of the matrix \( M(\mathcal{L}) \) we can assume that its first \( \gamma \) columns are indexed by monomials of degree at most \( (d_1, \ldots, d_{n-1}, k - 1) \) and that its first \( \rho \) rows are indexed by conditions imposed at the points \( p_1, \ldots, p_s \). We write

\[
M(\mathcal{L}) = \begin{bmatrix}
M_1 & K_1 \\
K_2 & M_2
\end{bmatrix},
\]

where \( M_1 \) is a \( \gamma \times \rho \) matrix. Observe that \( M_1 = M(L_1) \) and \( M_2 \cong M(L_2) \) via the isomorphism \( \Psi \). Moreover by conditions (2), (3) and the fact that \( L_1, L_2 \) are fiber non-special, we deduce that both matrices have maximal rank. Assume now that \( \text{fdim}(L_1) \geq -1 \) and \( \text{fdim}(L_2) \geq -1 \) (the other case being analysed in a similar
way. Choose two submatrices $M_i'$ of $M_i$ of maximal rank, for $i \in \{1, 2\}$, and form the square submatrix of $M(L)$

$$M' = \begin{bmatrix} M_1' & K_1' \\ K_2' & M_2' \end{bmatrix}$$

where $K_1'$ is obtained from $K_1$ by deleting columns of $M_2$ and similarly for $K_2'$. By [Dum09, Lemma 2] we have that $\deg(\det(M'_2)) > \deg(\det(B))$ for any square submatrix $B$ of $[K_2' M_2']$. Thus, by the Laplace expansion with respect to the first $\rho$ rows we conclude that $\deg(\det(M')) = \deg(\det(M'_1) \cdot \det(M'_2)) > 0$ and the result follows. \hfill \square

The following program is a recursive program that uses Theorem 4.1 and Theorem 3.2 in order to conclude if the given linear system is non-special.

**Input**: $(d, m) \in \mathbb{N}^n \times \mathbb{N}^r$, with $r \geq 2$.

**Output**: $x \in \{\text{non-special, undecided, special}\}$.

if $\text{std}(d, m) = \emptyset$ then
    return non-special.
else if $\text{fdim}(\text{std}(d, m)) > \text{edim}(d, m)$ then
    return special;
else
    $(d, m) := \text{std}(d, m)$;
    if $r = 2$ then
        if $\text{fdim}(d, m) \geq \text{edim}(d, m)$ then
            return special;
        else
            return non-special;
    else
        for $k \in \{1, \ldots, d_1 - 1\}, s \in \{1, \ldots, r - 1\}$ do
            $d' := (k - 1, d_2, \ldots, d_n), m' := (m_1, \ldots, m_s)$;
            $d'' := (d_1 - k, d_2, \ldots, d_n), m'' := (m_{s+1}, \ldots, m_r)$;
            if $\text{sp}(d', m') = \text{non-special and sp}(d'', m'') = \text{non-special and}$
            $(\text{fdim}(d', m') + 1)(\text{fdim}(d'', m'') + 1) \geq 0$
            and $m_i \leq k$ for any $i \in \{1, \ldots, s\}$
            and $m_j \leq d_n - k$ for any $j \in \{s + 1, \ldots, r\}$
            then
                return non-special;
        end
    end
return undecided;
end

Algorithm 2: Speciality by degeneration.

5. Examples and conclusions

We have studied linear systems of $(\mathbb{P}^1)^n$ passing through points in very general position and concluded that the fibers of the projections $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^k$, for $1 \leq$
\( k < n \), can contribute to the speciality. The following is an example of a fiber special linear system \( \mathcal{L} \) whose standard form \( \mathcal{L}' \) is fiber non-special.

**Example 5.1.** The linear system \( \mathcal{L} := \mathcal{L}_{(13,9,5)}(11^2, 7^2, 3^2) \) of \((\mathbb{P}^1)^3\) is not in standard form with

\[
\text{vdim}(\mathcal{L}) = 12^2 \cdot 8^2 \cdot 4^2 - 2 \left( \binom{13}{3} + \binom{9}{3} + \binom{5}{3} \right) = 80 \quad \text{fdim}(\mathcal{L}) = 154.
\]

Using Algorithm 1 we obtain the following linear systems

\[
\mathcal{L}_{(13,9,5)}(11^2, 7^2, 3^2) \Rightarrow \mathcal{L}_{(5,9,5)}(7^2, 3^4) \Rightarrow \mathcal{L}_{(5,5,5)}(3^6) =: \mathcal{L}'
\]

where \( \mathcal{L}' \) is in standard form. Algorithm 2 degenerates \( \mathcal{L}' \) according to the following scheme:

\[
\begin{align*}
&\mathcal{L}_{(5,5,2)}(3^3) &\Rightarrow &\mathcal{L}_{(5,2,2)}(3^2) \\
&\mathcal{L}' &\Rightarrow &\mathcal{L}_{(5,2,2)}(3) \\
&\mathcal{L}_{(5,5,2)}(3^3) &\Rightarrow &\mathcal{L}_{(5,2,2)}(3^2) \\
&\mathcal{L}_{(5,5,2)}(3^3) &\Rightarrow &\mathcal{L}_{(5,2,2)}(3)
\end{align*}
\]

By Theorem 3.2 the last four linear systems are non-special, thus by repeated applications of Theorem 4.1 we conclude that \( \mathcal{L}' \) is non-special as well. In particular \( \dim(\mathcal{L}) = \dim(\mathcal{L}') = \text{vdim}(\mathcal{L}') = 156 \).

The following example show that there are other varieties giving contribution to the speciality of the linear system already when we blow-up three points in very general position.

**Example 5.2.** The linear system \( \mathcal{L} := \mathcal{L}_{(1,1,1,1,1,1)}(3^3) \) of \((\mathbb{P}^1)^7\) is in standard form with

\[
\text{vdim}(\mathcal{L}) = 2^7 - 3 \binom{9}{7} = 20 \quad \text{fdim}(\mathcal{L}) = \text{vdim}(\mathcal{L}) + 21 = 41,
\]

where the contribution on the right is given by the 21 one-dimensional fibers on the base locus, but we have that \( \dim(\mathcal{L}) = 42 \), then \( \mathcal{L} \) is fiber-special. Observe that Algorithm 2 returns undecided in this case since every degeneration gives a special linear system. The dimension of \( \mathcal{L} \) can be calculated by evaluating directly the rank of the matrix \( M(\mathcal{L}) \), appearing in the proof of Theorem 4.1, doing all the calculations in a finite field in order to get a quicker result. A computer calculation shows that the base locus of \( \mathcal{L} \) is the union of all the fibers through each of the three points plus an irreducible surface \( S \) through all of them. To describe \( \mathcal{S} \) we assume that the first two multiple points of \( \mathcal{L} \) are the canonical ones and let \( \mathcal{L}' = \mathcal{L}_{7,4}(3^9) \) be the linear system of \( \mathbb{P}^7 \) which is pull-back of \( \mathcal{L} \) via the birational map \( \phi^{-1} \) defined in Remark 1.4. The base locus of \( \mathcal{L}' \) consists of the 84 planes through three of the nine multiple points. The image via \( \phi \) of the plane through the three point outside the indeterminacy of \( \phi \) is \( S \). Observe that \( \mathcal{L}' \) is linearly non-special according to [BDP12, Definition 3.2] and moreover the varieties contributing to the speciality of \( \mathcal{L}' \) are just the lines through two points so that \( \phi^{-1}(S) \) does not contribute to the speciality of \( \mathcal{L}' \).
Denote, as before, by $Y$ the blow-up of $\mathbb{P}^3$ at $r$ points in very general position and by $\phi : \mathbb{P}^3 \to (\mathbb{P}^1)^3$ the birational map defined in Remark 1.4. Let $Q$ be a divisor in the strict transform of the linear system $L_{(1,1,1)}(\mathcal{L}^7)$ which is the image via $\phi^*$ of the class of the strict transform of the quadric through 9 points of $\mathbb{P}^3$. For any divisor $D$ in the strict transform of $L_{(d_1,d_2,d_3)}(m_1,\ldots,m_r)$ let

$$q(D) := \chi(D|_Q) = (d_1+1)(d_2+1)(d_3+1) - d_1d_2d_3 - \sum_{i=1}^7 m_i(m_i+1) \frac{1}{2}.$$

The following conjecture is equivalent to [LU12, Conjecture 6.3] via the small modification $\phi$.

**Conjecture 5.3.** Let $L := L_{(d_1,d_2,d_3)}(m_1,\ldots,m_r)$ be a linear system in standard form and let $D$ be a divisor in its strict transform.

- If $q(D) \leq 0$, then $h^0(D) = h^0(D - Q)$.
- If $q(D) > 0$, then $D$ is special if and only if $m_1 > d_n + 1$ and $D$ is fiber non-special.

**Example 5.4.** Let $L_n = L_{(n,n,n)}(n^7)$ be the linear system corresponding to the divisor class $nQ \in \text{Pic}(Y)$, where $n > 0$. This system has dimension 1 for any $n$ and it is non-special for $n = 1$. Its fiber dimension is

$$\text{fdim}(L_n) = \text{vdim}(L_n) = n^3 - 7\left(\frac{2n-1}{n}\right) < 0,$$

so that $L_n$ is fiber-special for $n > 1$. Its easy to check that $q(nQ) = 0$ for any $n$ and that in this case the conjecture holds.

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