On the Structure of Cohomology of Hamiltonian $p$-Algebras

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Abstract. We demonstrate advantages of non-standard grading for computing cohomology of restricted Hamiltonian and Poisson algebras. These algebras contain the inner grading element in the properly defined symmetric grading compatible with the symplectic structure. Using modulo $p$ analog of the theorem on the structure of cohomology of Lie algebra with inner grading element, we show that all nontrivial cohomology classes are located in the grades which are the multiples of the characteristic $p$. Besides, this grading implies another symmetries in the structure of cohomology. These symmetries are based on the Poincaré duality and symmetry with respect to transpositions of conjugate variables of the symplectic space. Some results obtained by computer program utilizing these peculiarities in the cohomology structure are presented.

1 Introduction

The ground field is $F$. The Hamiltonian algebra $\mathfrak{h}(n)$ is the Lie algebra of vector fields on $F^n$ annihilating the 2-form

$$\omega = \sum_{i=1}^{m} dx_i \wedge dx_{i+m}, \quad \text{where } n = 2m. \quad (1)$$

Since this algebra plays a key role in both classical and quantum physics, it has been widely studied by different methods, in particular, using the tools of algebraic topology. Its cohomologies are important invariants and for the past four decades I. Gelfand and his collaborators tried to compute them. All the computations performed so far are partial; for an account, see [1, 2] (the latest result, with Kontsevich, concerned a version of $\mathfrak{h}(n)$ whose elements were vector fields with Laurent polynomials as coefficients).

One of the main approaches to deal with the cohomology of a given infinite-dimensional algebra is to extract some finite-dimensional subcomplexes based usually on an expertly introduced grading. Nevertheless, in many important cases, e.g., in investigation of deformations via cohomology with coefficients in the adjoint module, this trick does not work.

Another approach is based on construction of finite-dimensional models of infinite-dimensional algebras.

Any Lie algebra of polynomial vector fields has finite-dimensional analogs defined over fields of positive characteristic $p$. These analogs sometimes possess a special structure; they are called restricted Lie algebras or Lie $p$-algebras. These algebras were first systematically studied by Jacobson in [3]. The corresponding cohomology theory was first considered by Hochschild in [4]. For most general and mathematically rigorous information on the subject, see [5–12]. Here we consider only few elementary constructions sufficient for our purposes.

We shall use in what follows the construction called the algebra of divided powers. Let the characteristic of $F$ be $p > 0$ and $x = (x_1, \ldots, x_n)$ be a set of indeterminates. For a multiindex $r = (r_1, \ldots, r_n)$, set

$$x^{(r)} = \prod_{i=1}^{n} x_i^{(r_i)} = \prod_{i=1}^{n} \frac{x_i^{r_i}}{r_i!}. \quad (2)$$

Let $F[x]_p = \text{span} \{x^{(r)}|r_i < p\}$, then one can see that $F[x]_p$ is a subalgebra of algebra $F[x]$ if char$F = p$, since the multiplication for basis elements of form (2) is given by the formula

$$x^{(r)}x^{(s)} = \binom{r + s}{r} x^{(r+s)}$$

and $\binom{r}{i} = 0 \mod p$ for integer $i$ and $j$ such that $0 \leq i, j < p$ and $i + j \geq p$.

The main result of this paper, presented in Section 4, is demonstration that the structure of cohomology of Lie $p$-algebra with the Poisson bracket becomes more clear and easier for computation if we use the grading compatible with symplectic structure (1).
2 The Restricted Hamiltonian Algebras

The elements of \( h(n) \) — the Hamiltonian vector fields — can be expressed in terms of generating functions. The elements of the central extension of the Hamiltonian algebra, the Poisson algebra \( \mathfrak{po}(n) \), can also be naturally described by means of generating function.

If we express the generating function in terms of divided power monomials (2), then, in the characteristic \( p \), we obtain the truncated Hamiltonian and Poisson algebras denoted in what follows by \( h(n)_p \) and \( \mathfrak{po}(n)_p \), respectively.

Observe that although, for the elements of \( F[x]_p \), the degrees of indeterminates are \(< p\), the monomials of generating functions for \( h(n)_p \) and \( \mathfrak{po}(n)_p \) may contain degrees equal to \( p \) since the mapping of the space of generating functions into the space of vector fields involves differentiations.

We see that

\[
\mathfrak{po}(n)_p = \text{span} \left\{ x_1^{(r_1)} \cdots x_n^{(r_n)} | 0 \leq r_i < p \right\} \oplus \text{span} \left\{ x_1^{(p)}, \ldots, x_n^{(p)} \right\},
\]

\[
h(n)_p = \text{span} \left\{ x_1^{(r_1)} \cdots x_n^{(r_n)} | 0 \leq r_i < p, 0 < \sum_{i=0}^{n} r_i \right\} \oplus \text{span} \left\{ x_1^{(p)}, \ldots, x_n^{(p)} \right\},
\]

\[
\dim \mathfrak{po}(n)_p = p^n + n, \quad \dim h(n)_p = p^n + n - 1.
\]

In what follows, we will mainly restrict our attention to the Hamiltonian case. The algebra \( h(n)_p \) is not semisimple, in other words, its first cohomology \( H^1(h(n)_p) = (h(n)_p \otimes h(n)_p)' \) is non-trivial. The ideal

\[
h^{(1)}(n)_p = [h(n)_p, h(n)_p] = \text{span} \left\{ x_1^{(r_1)} \cdots x_n^{(r_n)} | 0 \leq r_i < p, 0 < \sum_{i=0}^{n} r_i \right\},
\]

\[
\dim h^{(1)}(n)_p = p^n - 1
\]

still remains non-semisimple, but the next ideal

\[
h^{(2)}(n)_p = \left[ h^{(1)}(n)_p, h^{(1)}(n)_p \right] = \text{span} \left\{ x_1^{(r_1)} \cdots x_n^{(r_n)} | 0 \leq r_i < p, 0 < \sum_{i=0}^{n} r_i < n(p - 1) \right\},
\]

\[
\dim h^{(2)}(n)_p = p^n - 2
\]

is a simple Lie \( p \)-algebra for \( p \neq 2 \). The algebra \( h^{(2)}(n)_p \) is identified with the Lie \( p \)-algebra denoted by \( H_m \) \( (m = n/2) \) in the well known list\(^1\) of restricted simple Lie algebras of Cartan type.

3 A Preliminary Example

To give some idea of how non-standard grading may be useful, let us look at an example. We consider here the cohomologies \( H^k_g(h^{(2)}_3) \) \(^2\) computed by the author’s program described in [13]. In Table 1 the dimensions of \( H^k_g(h^{(2)}_3) \) in the standard grading (\( \deg x_i = 1 \) for all \( i \)) are presented. In this and subsequent tables an empty box means that \( \dim C^k_g = 0 \), a dot means that \( \dim C^k_g \neq 0 \) but \( \dim H^k_g = 0 \).

The picture in Table 1 does not look instructive at all, so let us introduce another grading: \( \deg x_i = -1 \), \( \deg x_{i+m} = 1 \) for \( i \leq m \). We shall call this (of non-Weisfeiler’s type) grading symmetric. Repeating computation with symmetric grading we obtain results presented in Table 2.

Now the picture is much more attractive. The reasons are explained in the next section.

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\(^1\) This list contains four series of algebras: \( W_n, S_n, H_n, K_n \), see [6].

\(^2\) It might seem more natural to consider as an example some simple ideal instead of \( h^{(2)}_3 \), but \( h^{(2)}(2)_3 \) is too small to be illustrative, whereas \( h^{(2)}(2)_5 \) is too large.
4 The Symmetric Grading

Let $a(n)_p$ denote any of the algebras $h(n)_p$, $h^{(2)}(n)_p$ or the corresponding Poisson algebras. Let $N = \dim a(n)_p$.

The standard $\mathbb{Z}$-grading of $a(n)_p$ is induced by prescribing the grades $\deg x_i = 1$ to all $n$ indeterminates. This grading divides the algebra $a(n)_p$ into the homogeneous degree $i$ subspaces $L_i$:

$$a(n)_p = (L_{-2} \oplus) L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r,$$

(3)

$L_{-2} = 0$ for Hamiltonian algebras, $r = (p - 1)n - 3$ for $h^{(2)}(n)_p$.

Imposing grading like (3) is, perhaps, the only possibility to extract finite-dimensional subcomplexes when computing cohomology of infinite-dimensional algebra.

But for finite-dimensional restricted Lie algebra of vector fields with the Poisson brackets we can use the symmetric grading $\deg x_i = -1$, $\deg x_{i+m} = 1$ reflecting the fact that $x_i$ and $x_{i+m}$ are conjugate indeterminates for symplectic structure (1). This grading induces the following decomposition:

$$a(n)_p = L_{-r} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r,$$

(4)

$r = \frac{(p-1)n}{2}$ for $h^{(2)}(n)_p$. The advantages of grading (4) are clear from the following propositions 1–3.

**Proposition 1.** Cohomologies in subcomplexes with opposite grades are isomorphic:

$$H^k_{\sigma}(a(n)_p) \cong H^{-k}_{-\sigma}(a(n)_p).$$

(5)

**Proof:** The isomorphism follows from the symmetry of all constructions with respect to transpositions of the conjugate indeterminates $x_i$ and $x_{i+m}$.

$$\square$$
Comment: Turning to Table 2, we see that with Proposition 1 it suffices to compute only one half (upper or lower) of the table.

**Proposition 2.** Cohomology of degree \( k \) in a given grade \( g \) is isomorphic to the homology of degree \( N - k \) in the same grade \( g \):

\[
H^k_g(\alpha(n)_p) \cong (H^{N-k}_g(\alpha(n)_p))' = H_{N-k,g}(\alpha(n)_p),
\]

Proof: The proposition follows immediately from the Poincaré duality. The algebra \( \alpha(n)_p \) is unitary [1], i.e., \( H_N(\alpha(n)_p) \neq 0 \). Since in the symmetric grading \( C_N(\alpha(n)_p) \) has zero grade, the Poincaré dual to the \( H^k_g(\alpha(n)_p) \) is \( H_{N-k,g}(\alpha(n)_p) \), and using (5) to change sign of grade we obtain (6).

Comment: Proposition 2 also reduces the computation to one half (left or right) of the table. Note also, that in the standard grading we have (see Table 1) only 60 non-empty \((g,k)\)-subcomplexes, and using the above propositions it suffices to process only 13 of them.

**Proposition 3.** For the symmetric grading all non-trivial cohomologies \( H^k_g(\alpha(n)_p) \) lie in the grades \( g = \pm pj \) for \( j \) integer.

Proof: This proposition is, in fact, “modulo \( p \)” version of standard theorem [1] on the inner grading element. One should only reformulate slightly the theorem and repeat the proof replacing the arithmetic in zero characteristic by the modular arithmetic. The inner grading element for \( \alpha(n)_p \) in symmetric grading is represented by the generating function \( \sum_{i=1}^m x_i x_{i+m} \).

Comment: Proposition 3 manifests in Table 2 in the fact that non-trivial cohomologies are located in the rows \(-6, -3, 0, 3, 6\) only.

From the computational point of view, the combination of Propositions 1–3 reduces the computation of the cohomology roughly by the factor \( 1/4p \). Turning to Table 2 we see that there is 108 \((g,k)\)-subcomplexes with non-empty cochain spaces, and using the above propositions it suffices to process only 13 of them. Note also, that in the standard grading we have (see Table 1) only 60 non-empty \((g,k)\)-subcomplexes, i.e., the average size of subcomplex in the symmetric grading is smaller than in the standard grading (and, hence, on the average, the subcomplexes should be easier for computation).

5 Computation of \( H^k_0(\mathfrak{h}(2)(2)_5) \)

Since \( \dim \mathfrak{h}(2)(2)_5 = 23 \), the total (i.e., for all degrees and grades) dimension of the cochain space is \( \dim C^*_0(\mathfrak{h}(2)(2)_5) = 8388608 \). Thus, the full computation of the cohomology is a rather difficult task. Nevertheless we have completed the task. Using the above listed symmetries in the structure of cohomology it suffices to carry out computations for degrees \( 1 \leq k \leq 11 \) in the grades \( 0, 5, 10, 15, 20 \). The data presented in Table 3 were obtained on a 1133MHz Pentium III PC with 512Mb under the Windows XP. The computation of \( H^k_0(\mathfrak{h}(2)(2)_5) \) took 22 h 29 min. It is the most difficult subtask. All other \((g,k)\)-boxes of the table take much smaller time of calculations.

| \( g \) \ \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 5 | 1 | 1 | 4 | 4 | 8 | 12 | 9 | 18 | 14 | 30 | 1 |
| \pm 10 | 1 | 1 | 3 | 2 | 6 | 9 | 8 | 15 | 14 | 25 | 1 |
| \pm 15 | 2 | 1 | 1 | 4 | 3 | 1 | 3 | 6 | 4 | 9 | 7 |
| \pm 20 | 2 | 1 | 1 | 4 | 3 | 1 | 3 | 6 | 4 | 9 | 7 |

6 Conclusions

As is clear, the full computations for \( p > 5 \) are hardly possible on the present-day computers, e.g., \( \dim \mathfrak{h}(2)(2)_7 = 47 \) and \( \dim C^*_0(\mathfrak{h}(2)(2)_7) = 140737488355328 \). Applying different tricks and efforts we can, probably, reduce the last value by several orders, but this is quite insufficient.
On the other hand, we can try to derive some hints about the structure of cohomology of classical Hamiltonian algebra analyzing the small degree cohomologies of $p$-analogs of the classical algebra. We have performed some computations for $p = 7, 11, 13$ and it seems that the classical cohomology classes are among the zero grade cocycles of $p$-algebras. Of course, for any finite $p$, zero grade series of cocycles contain multiplicative combinations of cocycles in grades $\pm p, \pm 2p, \ldots$, etc. and these should be separated from the classical cocycles.

It seems also that the sequences of dimensions in the grades $jp$ with growing $p$ tend to some stable sequences depending only on the number $j$ (not on $p$) but these observations are too preliminary to be discussed seriously at present.

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References

1. Fuks, D.B.: Cohomology of Infinite Dimensional Lie Algebras. Consultants Bureau, New York (1987)
2. Fuks, D.B.: Finite-dimensionality of the homology of the Lie algebra of Hamiltonian vector fields on the plane. Funktsional. Anal. i Prilozhen. 19 (1985), no. 4, 68–73 (Russian); Functional Anal. Appl. 19 (1985), no. 4, 305–310 (English).
3. Jacobson, N.: Restricted Lie Algebras of Characteristic $p$. Trans. Amer. Math. Soc. 50 (1941) 15–25
4. Hochschild, G.: Cohomology of Restricted Lie Algebras. Amer. J. Math. 76 (1954) 555–580
5. Kostrikin, A. I., Shafarevich, I. R.: Cartan pseudogroups and Lie $p$-algebras. Dokl. Akad. Nauk SSSR 168 (1966) 740–742 (Russian); Soviet Math. Dokl. 7 (1966) 715–718 (English)
6. Kostrikin, A. I., Shafarevich, I. R.: Graded Lie algebras of finite characteristic. Izv. Akad. Nauk. SSSR Ser. Mat. 33 (1969) 251–322 (Russian); Math.USSR-Izv. 3 (1969) 237–304 (English)
7. Strade, H.: The Classification of the Simple Lie Algebras over Fields with Positive Characteristic. Hamburger Beitrage zur Mathematik aus dem Mathematischen Seminar, Heft 31 (1997), Heft 87 (2000).
8. Strade, H.: The classification of the simple modular Lie algebras: VI. Solving the final case. Trans. Amer. Math. Soc. 250 (1998) 2553–2628
9. Strade, H., Farnsteiner, R.: Modular Lie Algebras and their Representations. Marcel Dekker Textbooks and monographs, Marcel Dekker Inc., New Yourk, 1988; Vol 116.
10. Premet, A.A., Strade, H.: Simple Lie algebras of small characteristic: I. Sandwich elements. Journal of Algebra 189 (1997) 419–480
11. Premet, A.A., Strade, H.: Simple Lie algebras of small characteristic: II. Exceptional roots. Journal of Algebra 216 (1999) 190–301
12. Premet, A.A., Strade, H.: Simple Lie algebras of small characteristic: III. Toral Rank 2 Case. Journal of Algebra 242 (2001) 236–337
13. Kornyak, V.V.: Modular Algorithm for Computing Cohomology: Lie Superalgebra of Special Vector Fields on (2|2)-dimensional Odd-Symplectic Superspace. In: Computer Algebra in Scientific Computing, CASC 2003, V.G.Ganzha, E.W.Mayr and E.V.Vorozhtsov (Eds.), TUM, München (2003) 227–240; URL: http://arXiv.org/abs/math.RT/0305155