Energy distribution of charged dilaton black holes

S. S. Xulu*

Abstract

Chamorro and Virbhadra studied, using the energy-momentum complex of Einstein, the energy distribution associated with static spherically symmetric charged dilaton black holes for an arbitrary value of the coupling parameter \( \gamma \) which controls the strength of the dilaton to the Maxwell field. We study the same in Tolman’s prescription and get the same result as obtained by Chamorro and Virbhadra. The energy distribution of charged dilaton black holes depends on the value of \( \gamma \) and the total energy is independent of this parameter.

04.70.Bw,04.20.Cv

*Dept of Applied Mathematics, University of Zululand, P/Bag X1001, 3886 Kwa-D Langezwa, SOUTH AFRICA  E-mail: ssxulu@pan.uzulu.ac.za
The subject of energy-momentum localization is associated with some degree of controversy. There are different opinions on this subject. Contradicting the viewpoint of Misner et al. [1] that the energy is localizable only for spherical systems, Cooperstock and Sarracino [2] argued that if the energy localization is meaningful for spherical systems then it is meaningful for all systems. Bondi [3] expressed that a non-localizable form of energy is inadmissible in relativity and its location can in principle be found. Einstein energy-momentum complex was followed by many definitions of energy, momentum, and angular momentum proposed for a general relativistic system (see in [4] and references therein). There lies a dispute with the importance of nontensorial energy-momentum complexes whose physical interpretation has been questioned by a number of physicists, including Weyl, Pauli and Eddington (see in [5]). There is a suspicion that, in a given spacetime, different energy distributions would be obtained from different energy-momentum complexes. Several examples of particular spacetimes (the Kerr-Newman, the Einstein-Rosen, and the Bonnor-Vaidya) have been investigated and different energy-momentum complexes are known to give the same energy distribution for a given spacetime [6]. Recently, Aguirregabiria et al. [4] showed that several energy-momentum complexes coincide for any Kerr-Schild class metric.

Virbhadra and Parikh [7], using the energy-momentum complex of Einstein, calculated the energy distribution with stringy charged black holes and found that the entire energy is confined to interior of the holes. We [8] found the same result using the Tolman definition. Chamorro and Virbhadra [9], using the energy-momentum complex of Einstein, studied energy distribution for the Garfinkle-Horowitz-Strominger (GHS) charged dilaton black holes [10]. In this paper we investigate the same using Tolman energy-momentum complex to see whether or not one gets the same result. This is the purpose of our present investigation. We use the convention that Latin indices take values from 0 to 3 and Greek indices values from 1 to 3, and take $G = 1$ and $c = 1$ units.

The Garfinkle-Horowitz-Strominger [10] static spherically symmetric asymptotically flat black hole is described by the line element
\[ ds^2 = (1 - \frac{r^+}{r})(1 - \frac{r^-}{r})^\sigma dt^2 - (1 - \frac{r^+}{r})^{-1}(1 - \frac{r^-}{r})^{-\sigma} dr^2 - (1 - \frac{r^-}{r})^{1-\sigma} r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

(1)

and the dilaton field \( \Phi \) is given by

\[ e^{2\Phi} = \left(1 - \frac{r^-}{r}\right)^{\frac{1-\sigma}{\gamma}}, \]

(2)

where

\[ \sigma = \frac{1 - \gamma^2}{1 + \gamma^2}. \]

(3)

\( r_- \) and \( r_+ \) are related to mass \( M \) and charge \( Q \) parameters as follows:

\[ M = \frac{r_+ + \sigma r_-}{2}, \]

\[ Q^2 = \frac{r_+ r_-}{(1 + \gamma^2)}. \]

(4)

Virbhadra \[11\] proved that for \( Q = 0 \) the GHS solution yields the Janis-Newman-Winicour solution \[12\] to the Einstein-Massless Scalar equations. Virbhadra et al. \[13\] showed that the Janis-Newman-Winicour solution has a globally naked strong curvature singularity. However, the GHS solution \((Q \neq 0)\) is a black hole solution.

Charged dilaton black holes have been a subject of study in many recent investigations \[14\]- \[16\]. A number of interesting properties of charged dilaton black holes critically depend on a dimensionless parameter \( \gamma \) which controls the coupling between the dilaton and the Maxwell fields. The maximum charge, for a given mass, that can be carried by a charged dilaton depends on \( \gamma \) \[14\]. When \( \gamma \neq 0 \), the surface \( r = r_- \) is a curvature singularity while at \( \gamma = 0 \) the surface \( r = r_- \) is a nonsingular inner horizon \[15\]. Both the entropy and temperature of these black holes depend on \( \gamma \) \[14\]. The gyromagnetic ratio for charged slowly rotating dilaton black holes depends on parameter \( \gamma \) \[14\]. Chamorro and Virbhadra \[8\] showed, using Einstein’s prescription, that the energy distribution of charged dilaton black holes depends on the value of \( \gamma \).

We start by transforming the line element (1) to quasi-Cartesian coordinates:
\[ ds^2 = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r})^\sigma dt^2 - (1 - \frac{r_+}{r})^{-\sigma}(dx^2 + dy^2 + dz^2) \]
\[ - \frac{(1 - \frac{r_+}{r})^{-1}(1 - \frac{r_-}{r})^\sigma - (1 - \frac{r_-}{r})^{-1}}{r^2} (x dx + y dy + z dz)^2, \tag{5} \]

according to
\[ r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \quad \phi = \tan^{-1}(y/x). \tag{6} \]

Tolman’s \cite{17} energy-momentum complex is
\[ \mathcal{T}_k^i = \frac{1}{8\pi} U_{k,j}^{ij}, \tag{7} \]

where
\[ U_{k}^{ij} = \sqrt{-g} \left[ -g^{pi}\left(-\Gamma_{kp}^j + \frac{1}{2}g_k^j\Gamma_{ap}^a + \frac{1}{2}g_p^j\Gamma_{ak}^a\right) + \frac{1}{2}g_k^jg_{pm}^{ij}\left(-\Gamma_{pm}^j + \frac{1}{2}g_p^j\Gamma_{am}^a + \frac{1}{2}g_m^j\Gamma_{ap}^a\right) \right], \tag{8} \]
\( \mathcal{T}_0^0 \) is the energy density, \( \mathcal{T}_0^\alpha \) are the components of energy current density, \( \mathcal{T}_\alpha^0 \) are the momentum density components. Therefore, energy \( E \) for a stationary metric is given by the expression
\[ E = \frac{1}{8\pi} \int \int \int U_{0,\alpha}^{\alpha} \, dx dy dz \tag{9} \]

After applying the Gauss theorem one has
\[ E = \frac{1}{8\pi} \int \int U_{0}^{\alpha} \mu_\alpha \, dS, \tag{10} \]

where \( \mu_\alpha = (x/r, y/r, z/r) \) are the three components of a normal vector over an infinitesimal surface element \( dS = r^2 \sin\theta d\theta d\phi \).

The determinant of the metric tensor and its non-vanishing contravariant components are obtained by Chamorro and Virbhadra \cite{9}. To compute energy using Eq. \cite{8} we require the following list of nonvanishing components of the Christoffel symbol of the second kind.

\[ \Gamma_{11}^1 = x(c_1 + c_2x^2), \quad \Gamma_{22}^2 = y(c_1 + c_2y^2), \]
\[ \Gamma_{33}^3 = z(c_1 + c_2z^2), \quad \Gamma_{12}^1 = x(c_3 + c_2y^2), \]
\[ \Gamma_{11}^2 = y(c_3 + c_2x^2), \quad \Gamma_{33}^1 = x(c_3 + c_2z^2), \]
Now using (13) with (4) in (10) we get

\[ \Gamma_{11}^3 = z(c_3 + c_2x^2), \quad \Gamma_{33}^2 = y(c_3 + c_2z^2), \]
\[ \Gamma_{22}^3 = z(c_3 + c_2y^2), \quad \Gamma_{12}^1 = y(c_4 + c_2x^2), \]
\[ \Gamma_{13}^4 = z(c_4 + c_2x^2), \quad \Gamma_{21}^2 = x(c_4 + c_2y^2), \]
\[ \Gamma_{23}^2 = z(c_4 + c_2y^2), \quad \Gamma_{31}^3 = x(c_4 + c_2z^2), \]
\[ \Gamma_{32}^3 = y(c_4 + c_2z^2), \quad \Gamma_{00}^1 = xc_5, \]
\[ \Gamma_{00}^2 = yc_5, \quad \Gamma_{01}^3 = zc_5, \]
\[ \Gamma_{02}^0 = yc_6, \quad \Gamma_{03}^0 = zc_6, \]
\[ \Gamma_{23}^1 = \Gamma_{13}^1 = \Gamma_{12}^3 = c_2xyz. \] (11)

where

\[ c_1 = \frac{1}{2r^4(r - r_-)}[2r_-r^2 + 3r_-r^2 - 3r_-r_+r + r_-^2r_+ - r_-^2r + (r_-r_+ - r_-r_- + r_-r_+ - r_-^2)r_-\sigma], \]
\[ c_2 = \frac{1}{2r^4} \left[ \frac{2r_-^2r + 6r_-r_+r - 3r_-r_-^2 - 3r_-r_+^2 - 3r_-r_+^2}{r - r_+} + \frac{(2r_- - r_-\sigma)(r_-r_+ - r_-r_+ + r_-^2)}{r - r_-} \right], \]
\[ c_3 = \frac{1}{2r^4} [r_-r + 2r_+r - r_-r_+ + (r - r_+)r_-\sigma], \]
\[ c_4 = \frac{1}{2r^2} \left[ \frac{r_- - r_-\sigma}{r - r_-} \right], \]
\[ c_5 = \frac{(r - r_-)^{2\sigma - 1}}{2r^{2\sigma + 4}}(r - r_-)[(r - r_-)r_+ + (r - r_+)r_-\sigma], \]
\[ c_6 = \frac{1}{2r^2} \left[ \frac{r_+}{r - r_+} + \frac{r_-\sigma}{r - r_-} \right]. \] (12)

Using Eqs. (8) and (11) we obtain required components of \( U_{ij}^k \). These are

\[ U_{01}^{01} = \frac{x}{r^4}[r(\sigma r_- + r_+) - \sigma r_-r_+], \]
\[ U_{02}^{02} = \frac{y}{r^4}[r(\sigma r_- + r_+) - \sigma r_-r_+], \] (13)
\[ U_{03}^{03} = \frac{z}{r^4}[r(\sigma r_- + r_+) - \sigma r_-r_+]. \]

Now using (13) with (4) in (11) we get

\[ E(r) = M - \frac{Q^2}{2r}(1 - \gamma^2). \] (14)
Thus, we get the same result as Chamorro and Virbhadra \cite{9} obtained using the Einstein energy-momentum complex. This is against the “folklore” that different energy-momentum complexes could give different and hence unacceptable energy distribution in a given spacetime. For the Reissner-Nordström metric one gets \( E = M - Q^2/2r \), which is the same as obtained by using several other energy-momentum complexes \cite{4} and definitions of Penrose as well as Hayward \cite{18}. \( E(r) \), given by (14), can be interpreted as the “effective gravitational mass” that a neutral test particle “feels” in the GHS spacetime. The “effective gravitational mass” becomes negative at radial distances less than \( Q^2(1 - \gamma^2)/2M \).

ACKNOWLEDGMENTS

Thanks are due to T. A. Dube and K. S. Virbhadra for their guidance.
REFERENCES

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., NY, 1973) p.603.

[2] F. I. Cooperstock and R. S. Sarracino, *J. Phys.* A11, 877 (1978).

[3] H. Bondi, *Proc. R. Soc. London* A427, 249 (1990).

[4] J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, *Gen. Relativ. Gravit.* 28, 1393 (1996).

[5] S. Chandrasekhar and V. Ferrari, *Proc. R. Soc. London* A435, 645 (1991).

[6] K. S. Virbhadra, *Phys. Rev.* D41, 1086 (1990); *Phys. Rev.* D42, 2919 (1990); F. I. Cooperstock and S. A. Richardson, in *Proc. 4th Canadian Conf. on General Relativity and Relativistic Astrophysics* (World Scientific, Singapore, 1991); N. Rosen and K. S. Virbhadra, *Gen. Relativ. Gravit.* 25, 429 (1993); K. S. Virbhadra, *Pramana - J. Phys.* 45, 215 (1995); A. Chamorro and K. S. Virbhadra, Pramana-J. Phys. 45, 181 (1995).

[7] K. S. Virbhadra and J. C. Parikh, *Phys. Lett.* B317, 312 (1993).

[8] S. S. Xulu, gr-qc/9712100, *Int. J. Theor. Phys.*, to appear.

[9] A. Chamorro and K. S. Virbhadra, *Int. J. Mod. Phys.* D5, 251 (1996).

[10] D. Garfinkle, G. T. Horowitz and A. Strominger, *Phys. Rev.* D43, 3140 (1991).

[11] K. S. Virbhadra, *Int. J. Mod. Phys.* A12, 4831 (1997).

[12] A. I. Janis, E. T. Newman and J. Winicour, *Phys. Rev. Lett.* 20, 878 (1968).

[13] K. S. Virbhadra, S. Jhingan and P. S. Joshi, *Int. J. Mod. Phys.* D6, 357 (1997).

[14] C. F. E. Holzhey and F. Wilczek, *Nucl. Phys.* B380, 447 (1992).

[15] J. H. Horne and G. T. Horowitz, *Phys. Rev.* D46, 1340 (1992).
[16] A. Shapere, S. Trivedi and F. Wilczek, *Mod. Phys. Lett.* **A6**, 2677 (1991); K. Shiraishi, *Phys. Lett.* **A166**, 298 (1992); J. A. Harvey and A. Strominger, Quantum aspects of black holes, hep-th/9209055; F. Dowker, J. P. Gauntlett, D. A. Kastor and J. Traschen, *Phys. Rev.* **D49**, 2909 (1994).

[17] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, London, 1934) p. 227.

[18] K. P. Tod, *Proc. Roy. Soc. London* **A388**, 467 (1983); S. A. Hayward, *Phys. Rev.* **D49**, 831 (1994).