A Possible IIB Superstring Matrix Model with Euler Characteristic and a Double Scaling Limit

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Abstract

We show that a recently proposed Yang-Mills matrix model with an auxiliary field, which is a candidate for a non-perturbative description of type IIB superstrings, captures the Euler characteristic of moduli space of Riemann surfaces. This happens at the saddle point for the Yang-Mills field. It turns out that the large-$n$ limit in this matrix model corresponds to a double scaling limit in the Penner model.

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The conjecture by Banks, Fischler, Shenker, and Susskind \[1\] that M theory is an $n = \infty$ matrix model, has induced a large activity towards a non-perturbative formulation of type IIB superstrings by matrix models. It has been proposed by Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) \[2\] that the large-$n$ reduced model of ten-dimensional super Yang-Mills theory can be considered as such a formulation, provided the matrix size, $n$, is considered as a dynamical variable to be summed from $n = 0$ to $n = \infty$ in the partition function. It was recently proposed \[3\] that this sum over all $n$, which is somewhat foreign to the $n = \infty$ philosophy, should be replaced by an auxiliary matrix field $Y^i_j$, and the large-$n$ limit taken, thereby avoiding introducing small-$n$ matrices. The partition function is then given by \[3\]

$$Z_{\mathrm{NBI}} = \int \mathcal{D}A_\mu \mathcal{D}Y \; e^{-S_\epsilon} = C \int \mathcal{D}[A_\mu] e^{-S_{\mathrm{NBI}}},$$  \hspace{1cm} (1)$$

where $C$ is a constant, and the “dielectric” action $S_\epsilon$ is given by

$$S_\epsilon = -\frac{\bar{\alpha}}{4} \text{Tr} \left( Y^{-1} [A_\mu, A_\nu]^2 \right) + \beta \text{Tr} \; Y + \left( n - \frac{1}{2} \right) \text{Tr} \ln Y,$$  \hspace{1cm} (2)$$

and where the non-Abelian (strong field) Born-Infeld action $S_{\mathrm{NBI}}$ is given by

$$S_{\mathrm{NBI}} = -\sqrt{\bar{\alpha} \beta} \text{Tr} \sqrt{-[A_\mu, A_\nu]^2}.$$  \hspace{1cm} (3)$$

Also, the measure $[\mathcal{D}A_\mu]$ is given by

$$[\mathcal{D}A_\mu] = \mathcal{D}A_\mu \prod_{i<j} (z_i + z_j)^{-1},$$  \hspace{1cm} (4)$$

where the $z$’s are the eigenvalues of the commutator $-[A_\mu, A_\nu]^2$.

In writing these results we have ignored the fermionic parts of these actions (as well as the fermionic functional integrations), because they play no role in the following. The NBI action \[3\] allows one to pass to the Nambu-Goto form of superstrings, provided the $n \to \infty$ limit can be performed inside various sums ($\approx$ absence of tachyon singularities). We refer to \[3\] for the complete actions and a discussion of the limits involved in obtaining the Nambu-Goto string square root from $S_{\mathrm{NBI}}$.

It is worth emphasizing that the unconventional term $\text{Tr} \ln Y$ in eq. \[2\] is needed in order to get the square root action \[3\]. Likewise the coefficient $n - 1/2$ is crucial for the $Y$-integral to give eq. \[3\].

In this paper we address the question concerning the meaning of the auxiliary field $Y$ in \[2\]. We shall show that the physical interpretation of $Y$ is that it captures the Euler characteristic of moduli space of Riemann surfaces. It turns out, rather surprisingly, that the limit $n \to \infty$ in the $A_\mu$ saddle point action, arising from \[3\], automatically implies a double scaling limit \[4\], so that the partition function \[3\] contains the sum over all (virtual) Euler characteristic of moduli space of genus $g$. These results are closely related to works on the Penner model \[5, 6, 7, 8, 9\]. The appearance of the “automatic double scaling” limit is rather encouraging from the point of view that the partition function \[3\] should have something to do with superstrings.

We shall now look for a saddle point of the action \[2\] in $A_\mu$. As shown in ref. \[3\], this implies

$$[A_\mu, \{ Y^{-1}, [A_\mu, A_\nu] \}] = 0.$$  \hspace{1cm} (5)$$
Here \( \{a, b\} \) denotes the anti-commutator of \( a \) and \( b \). The equation of motion (5) differs from the one considered by IKKT \[2\] by the presence of the \( Y \)-field. Similarly to IKKT we can find a solution by using that the \( A_{\mu} \)-field commutes with the unit matrix, and hence

\[
\{Y^{-1}, [A_{\mu}, A_{\nu}]\}_j^i = 2i m_{\mu\nu} \delta_j^i, \quad \text{with} \quad m_{\mu\nu} = -m_{\nu\mu},
\]

where \( m_{\mu\nu} \) is a matrix with respect to the space indices, examples of which are given by IKKT \[2\] for D-strings. The solution of (6) is given by

\[
[A_{\mu}, A_{\nu}]_j^i = im_{\mu\nu} Y_j^i.
\]

From eq. (6) it follows that the action (2) at the saddle point is given by

\[
S_{\epsilon}^{\text{saddle}} = na \, \text{Tr} \, Y + (n - \frac{1}{2}) \, \text{Tr} \, \ln Y,
\]

where the parameter \( a \) is defined by

\[
a = (\bar{\beta} + \bar{\alpha} \, m_{\mu\nu}^2 / 4) / n.
\]

The parameters \( \bar{\alpha} \) and \( \bar{\beta} \) should be positive but are not otherwise fixed by the work in ref. \[3\]. In order to have a non-trivial large-\( n \) saddle point we need to assume that

\[
\bar{\alpha} = n \alpha \quad \text{and} \quad \bar{\beta} = n \beta,
\]

where the quantities without tildes are of order one. This is the reason for defining \( a \) as in (9). It should be noted that \( a \) is positive.

An important point is that taking \( \bar{\alpha} \) and \( \bar{\beta} \) of order \( n \) is not equivalent to taking the usual classical limit in string theory, i.e. \( g_s = 0 \). The limit \( g_s = 0 \) appears for \( \alpha, \beta \rightarrow \infty \) \[2\], and hence the term \( \text{Tr} \ln Y \) is ignored in this limit. Here we keep \( \alpha \) and \( \beta \) finite, and thus the string coupling does not vanish. Another way of seeing this is by noticing that at the saddle point the \( A_{\mu} \)-fields satisfy eqn. (6) where the auxiliary field \( Y \) is arbitrary, subject only to the constraints coming from integrating over \( Y \). More precisely, at the saddle point the commutator \([A_{\mu}, A_{\nu}]\) has a non-trivial distribution of eigenvalues, determined by the distribution of eigenvalues for the matrix \( Y \). The distribution of eigenvalues of the matrix \( Y \), on the other hand, is determined by the saddle point action (8). The matrix model defined by the action (8) is a well-understood model and, as we shall see, we are led to the conclusion that the field \( Y \) is a book-keeping field which allows us to distinguish between contributions from different genera.

The value of the partition function in the saddle point is given by the expression

\[
Z_{\text{NBI}}^{\text{saddle}} = \text{const.} \int \mathcal{D}Y \, \exp\left[-n \, \text{Tr}(a \, Y + t \, \ln Y)\right] \times G
\]

where \( G \) represents the terms resulting from the gaussian integration over the fluctuations of the \( A \)-fields. In the following we shall ignore the sub-dominant factor \( G \). It follows from eq. (2) that the parameter \( t \) is

\[
t = 1 - 1/2n.
\]

Now the crucial point is that the action (11) is of the Penner type \[4\]. We refer to refs. \[4, 5, 6, 7\] for a detailed discussion of this model. It should also be noticed that for the \( t \)-value (12) the \( Y \)-integral in (11) diverges\[4\]. Therefore we consider \( t \) first as a free parameter to be used in an analytic continuation. Ultimately \( t \) will reach the value given in (12).

\[1\]In doing the exact \( Y \)-integral in eq.(11) this problem does not occur \[4\], because of the term containing \( Y^{-1} \) in the action \( S_{\epsilon} \) in eq. \[1\]. This integral can be performed restricting the \( Y \)-integration to run over positive eigenvalues of the Hermitian matrix \( Y \).
The Penner model is given by the following matrix integral
\[
Z_{\text{Penner}} = e^F = \exp(-n^2 t) \int d\phi \exp\{-nt \ \text{Tr} \ (\ln \phi - \phi)\}.
\] (13)

To begin with this model is defined only for negative \( t \) (similarly to eq. (11)), but by analytical continuation one can extend its definition also to \( t > 0 \). There are two ways of evaluating the integral in (13): Either one can use the method of orthogonal polynomials with \( t \) less than zero [6] and obtain the result for all \( n \) and for positive \( t \) by analytic continuation, or one can evaluate it order by order in \( 1/n \) using a saddle point method [7, 8, 9]. In the last method the integration is deformed into the complex plane, and in general the matrix \( Y \) will have an eigenvalue distribution with complex eigenvalues, although the matrix \( Y \) originally is Hermitian [7, 9]. Of course, the non-real eigenvalues occur in complex conjugate pairs, so the resulting free energy is real.

Thus, in the language of the spectral density the region \( t > 0 \) is characterised by the eigenvalues of the matrix \( \phi \) not being restricted to the real axis but living on some curve in the complex plane. Examples of such curves for various negative values of \( t \) are shown in figs. 3–8 in ref. [9]. Using any of the methods in ref. [6] or [7, 8, 9], it turns out that the (analytically continued) Penner model has a critical point at \( t = t_c = 1 \). In the saddle point method, at this point the endpoints of the support of the eigenvalue distribution coalesce and the support of the eigenvalue distribution forms a closed loop in the complex plane [7, 9]. In the vicinity of the critical point \( t = t_c = 1 \) one can define a double scaling limit (in both approaches [6] and [7, 8, 9])
\[
t \to (t_c)_-, \ n \to \infty, \ and \ \mu = (t_c - t) n = \text{fixed},
\] (14)
so that the contribution to the free energy from surfaces of genus \( g \) reads
\[
F = F_0 + F_1 + \sum_{g=2}^{\infty} n^{2-2g} f_g,
\] (15)
with
\[
f_g = \frac{B_{2g}}{2g(2g - 2)} (t_c - t)^{2-2g}, \ g > 1,
\] (16)
where \( \{B_{2g}\} \) are the Bernoulli numbers and \( \chi_g = \frac{B_{2g}}{2g(2g - 2)} \) is the virtual Euler characteristic of moduli space of Riemann surfaces of genus \( g \). In the double scaling limit (14) the expansion (15) thus takes the form
\[
F(\mu) = F_0(\mu) + F_1(\mu) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g - 2)} \mu^{2-2g}.
\] (17)
As usual, the merit of the double scaling limit is that all genera are included.

In the double scaling limit defined in eq. (14), using the approach in ref. [6; 9], the endpoints of the support of the eigenvalue distribution \( x \) and \( y \) lie on the real axis and are separated by an infinitesimal distance of the order \( (1 - t)^{1/2} \sim O(1/\sqrt{n}) \). The support of the eigenvalue distribution is a curve, starting out at \( x \) and running along the real axis towards \( y \) but making a loop in the complex plane encircling the origin before actually reaching the point \( y \). In the approach of Distler and Vafa [4], the integral is anyhow evaluated exactly in the region where it converges (integration over only the positive eigenvalues of the hermitian matrix \( \phi \)), and then analytically continued.

\[2\] For a discussion of the genus zero free energy \( F_0 \), and \( F_1 \) see refs. [6] and [4; 7; 9].
Let us now return to the matrix model given by eq. (1), and consider the saddle point value (11), which is clearly of the Penner type. Here it should be remarked that if we change the coupling constant in front of the linear term in the Penner action in (13), (keeping its sign fixed), this only changes the partition function by an irrelevant constant and does not alter the position of the critical point or the behaviour (15) and (16). Therefore the results discussed above can be taken over without any changes. Due to eq. (12) we see that for this model

\[ t_c = 1, \quad t_c - t = 1/2n, \quad \text{i.e.} \quad \mu = (t_c - t)n = 1/2 = \text{fixed}. \]  

(18)

Therefore the matrix model (1) automatically satisfies the double scaling relations (14) in the limit \( n \to \infty \) with the “cosmological constant” \( \mu \) equal to 1/2. Thus, information about higher genera contributions is encoded in the “dielectric” matrix model of ref. [3] in the limit \( n \to \infty \). In this sense it is reasonable to assume that the auxiliary field \( Y \) is really a substitute for the summation over \( n \) in the IKKT approach [2].

The appearance of the Bernoulli numbers \( B_{2g} \) with positive coefficients shows that the genus expansion of the partition function is not Borel summable. It is well known that the same is true for the genus expansion in string theory [10].

It should be mentioned that the matrix model (1) also knows about the Euler characteristic \( \chi_{g,q} \) of the moduli space of Riemann surfaces of genus \( g \) with \( q \) punctures,

\[ \frac{\partial^n F}{\partial \mu^q} = q! \sum \chi_{g,q} \mu^{2g-q}. \]  

(19)

This was shown for the Penner model in ref. [6], and can be repeated without any change for saddle point value of the model (1) exhibited in eq. (11).

We end by a few remarks: The results obtained above show that the matrix model proposed in ref. [3] encodes some highly non-trivial information about Riemann surfaces, which is crucial for strings. In particular, information about higher genera surfaces is encoded already in the saddle point action, a feature which must be ascribed to the non-perturbative character of the model. It is interesting to note that the partition function (and the distribution of eigenvalues of the auxiliary field) at the saddle point can be evaluated exactly. This opens the possibility of studying the model also beyond the saddle point approximation.

Acknowledgements We thank Yu. Makeenko for interesting comments. One of us (P. O.) in addition thanks A. Fayyazuddin for interesting discussions.

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\( ^3 \)Since the critical point \( t_c = 1 \) appears, this indicates a connection between the “dielectric” matrix model [3] and \( c = 1 \) matter coupled to two-dimensional gravity.
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