Existence of Peregrine Solitons in fractional reaction-diffusion equations*

Agustín Besteiro† Diego Rial ‡

June 2018

Abstract

In this article, we will analyze the existence of Peregrine type solutions for the fractional diffusion reaction equation by applying Splitting-type methods. These functions that have two main characteristics, they are direct sum of functions of periodic type and functions that tend to zero at infinity. Global existence results are obtained for each particular characteristic, for then finally combining both results.

Keywords — Fractional diffusion, global existence, Lie–Trotter method.
AMS code — 35K55; 35K57; 35R11; 35Q92; 92D25

1 Introduction

We consider the non autonomous system

$$\partial_t u + \sigma(-\Delta)\beta u = F(t,u), \quad (1.1)$$

where $u(x,t) \in Z$ for $x \in \mathbb{R}^n$, $t > 0$, $\sigma \geq 0$ and $0 < \beta \leq 1$, $F : \mathbb{R} \times Z \to Z$ a continuous map and $Z$ a Banach space. We consider the initial problem $u(x,0) = u_0(x)$.

The aim of this paper is to prove the existence of Peregrine type of solutions for the fractional reaction diffusion equation, using recent numerical splitting techniques ([5], [12], and [4]) introduced for other purposes. Peregrine solitons were studied in ([20]), and has multiple applications (See for example, [3], [11], [15], [14] and [24]). "Peregrine solitons" are functions with two main characteristics: These are direct sum of periodic functions and functions that tend to zero when the spatial variable tends to infinity.

Fractional reaction-diffusion equations are frequently used on many different topics of applied mathematics such as biological models, population dynamics models, nuclear reactor models, just to name a few (see [2], [7], [8] and references therein).

The fractional model captures the faster spreading rates and power law invasion profiles observed in many applications compared to the classical model ($\beta = 1$). The main reason for this behavior is given by the fractional Laplacian, that is described

---

*Received date, and accepted date (The correct dates will be entered by the editor).
†Instituto de Matemática Luis Santaló, CONICET–UBA, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina, (abesteiro@dm.uba.ar).
‡Dpto. de Matemática, FCEyN–UBA, Instituto de Matemática Luis Santaló, CONICET–UBA, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina, (drial@dm.uba.ar).
by standard theories of fractional calculus (for a complete survey see [19]). There are many different equivalent definitions of the fractional Laplacian and its behavior is well understood (see [6], [13], [16], [18], [25], [21] and [17]).

The non-autonomous nonlinear reaction diffusion equation dynamics were studied by [22] and others, analyzing the stability and evolution of the problem.

The paper is organized as follows: In Section 2 we set notations and preliminary results and in Section 3 we present main results, focusing first on each characteristic of the direct sum separately, for finally joining both results to reach the existence of Peregrine Solitons.

2 Notations and Preliminaries.

We are interested in continuous functions to vectorial values, that is to say, whose evaluations take values in Banach Spaces.

Let $Z$ be a Banach space, we define $C_u(\mathbb{R}^d,Z)$ as the set of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in $Z$. Taking the norm

$$\|u\|_{\infty,Z} = \sup_{x \in \mathbb{R}^d} |u(x)|_Z,$$

$C_u(\mathbb{R}^d,Z)$ is a Banach space. It is easy to see that if $g \in L^1(\mathbb{R}^d)$ and $u \in C_u(\mathbb{R}^d,Z)$ the Bochner integral is defined in the following way,

$$(g * u)(x) = \int_{\mathbb{R}^d} g(y) u(x - y) dy$$

This defines an element of $C_u(\mathbb{R}^d,Z)$ and the linear operator $u \mapsto g * u$ is continuous (see [10]).

The following results show that the operator $-(\Delta)^{\beta}$ defines a continuous contraction semigroup in the Banach space $C_u(\mathbb{R}^d,Z)$. The following lemma is a consequence of Lévy–Khintchine formula for infinitely divisible distributions and the properties of the Fourier transform.

Lemma 1. Let $0 < \beta \leq 1$ and $g_\beta \in C_0(\mathbb{R}^d)$ such that $\hat{g}_\beta(\xi) = e^{-|\xi|^2\beta}$, it holds $g_\beta$ is positive, invariant under rotations of $\mathbb{R}^d$, integrable and

$$\int_{\mathbb{R}^d} g_\beta(x) dx = 1.$$

Proof. The first statement follows from Theorem 14.14 of [23], the remaining claims are immediate from the definition of $\hat{g}_\beta$. 

Based on the previous lemma, we study Green’s function associated to the linear operator $\partial_t + \sigma(-\Delta)^{\beta}$.

Proposition 2. Let $\sigma > 0$ and $0 < \beta \leq 1$, the function $G_{\alpha,\beta}$ given by

$$G_{\alpha,\beta}(t,x) = (\sigma t)^{-\frac{d-\alpha}{\alpha}} g_\beta((\sigma t)^{-\frac{1}{\alpha}} x),$$

verifies

i. $G_{\alpha,\beta}(\cdot, t) > 0;$
ii. $G_{\sigma,\beta}(\cdot,t) \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} G_{\sigma,\beta}(t,x)dx = 1;$$

iii. $G_{\sigma,\beta}(\cdot,t) * G_{\sigma,\beta}(\cdot,t') = G_{\sigma,\beta}(\cdot,t + t'),$ for $t, t' > 0$;

iv. $\partial_t G_{\sigma,\beta} + \sigma(-\Delta)^\beta G_{\sigma,\beta} = 0$ for $t > 0$.

**Proof.** The first and second statements are a consequence of the definition of $\hat{g}_\beta$. The third and fourth statements are immediate applying Fourier transform.

In the following proposition, we have that the linear operator $-\sigma(-\Delta)^\beta$ defines a contraction continuous semigroup in the set $C_u(\mathbb{R}^d, Z)$.

**Proposition 3.** For any $\sigma > 0$ and $0 < \beta \leq 1$, the map $S: \mathbb{R}_+ \to \mathcal{B}(C_u(\mathbb{R}^d, Z))$ defined by $S(t)u = G_{\sigma,\beta}(\cdot,t) * u$ is a contraction continuous semigroup.

**Proof.** The proof can be found in [4] Proposition 2.2.

**Remark 4.** If $u \in C_u(\mathbb{R}^d, Z)$ is a constant, then $S(t)u = u$.

In this paper, we consider integral solutions of the problem (1.1). We say that $u \in C([0,T], C_u(\mathbb{R}^d, Z))$ is a mild solution of (1.1) if, for $u$ verifies

$$u(t) = S(t)u_0 + \int_0^t S(t-t')F(t', u(t'))dt'.$$  \hspace{1cm} (2.1)

Since our method to build solutions of (2.1) is based on the application of the Lie-Trotter method, it is necessary to study the non-linear problem associated with $F$. We remark that some regularity condition is necessary for convergence, as it is shown in the counterexample given in [9].

Let $F: \mathbb{R}_+ \times Z \to Z$ be a continuous map, we say that is locally Lipschitz in the second variable if, given $R, T > 0$ there exists $L = L(R, T) > 0$ such that if $t \in [0,T]$ and $z, \tilde{z} \in Z$ with $|z|, |\tilde{z}| \leq R$, then

$$|F(t, z) - F(t, \tilde{z})|_Z \leq L|z - \tilde{z}|_Z.$$  \hspace{1cm} (2.2)

In this case, for any $z_0 \in Z$ there exists a unique (maximal) solution of the Cauchy problem

$$z(t) = z_0 + \int_{t_0}^t F(t', z(t'))dt'$$
defined on $[t_0, t_0 + T^*(t_0, z_0))$, with $T^*(t_0, z_0)$ is the maximal time of existence. It is easy to see that there exists a nonincreasing function $\mathcal{T}: \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\mathcal{T}(T, R) \leq \inf \{T^*(t_0, z_0) : 0 \leq t_0 \leq T, |z_0|_Z \leq R\}.$$  \hspace{1cm} (2.2)

Also, one of the following alternatives holds:

- $T^*(t_0, z_0) = \infty$;
- $T^*(t_0, z_0) < \infty$ and $|z(t)|_Z \to \infty$ when $t \uparrow t_0 + T^*(t_0, z_0)$.
We can see that $F: \mathbb{R}_+ \times C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z)$, given by $F(t, u)(x) = F(t, u(x))$ is continuous and locally Lipschitz in the second variable. Therefore, we can consider problem (2.2) in $C_u(\mathbb{R}^d, Z)$. We denote by $N: \mathbb{R} \times \mathbb{R} \times C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z)$ the flow generated by the integral equation (2.2) as $u(t) = N(t, t_0, u_0)$, defined for $t_0 \leq t < t_0 + T^*(t_0, u_0)$.

The following result relates the solutions of (2.2) with the problem (2.1) in the case of having constant initial data.

**Proposition 5.** If $u_0$ is a constant function, then $u(t) = N(t, t_0, u_0)$ is a solution of (2.1).

**Proof.** Since $u_0$ is a constant function, from the uniqueness of the problem (2.2), we have $u(t)$ is a constant function for any $t > 0$ where the solution is defined. Therefore,

$$u(t) = u_0 + \int_0^t F(t', u(t'))dt' = S(t)u_0 + \int_0^t S(t-t')F(t', u(t'))dt',$$

which proves our assertion. \hfill \Box

**Theorem 6.** There exists a function $T^*: C_u(\mathbb{R}^d, Z) \to \mathbb{R}_+$ such that for $u_0 \in C_u(\mathbb{R}^d, Z)$, exists a unique $u \in C([0, T^*(u_0)), C_u(\mathbb{R}^d, Z))$ mild solution of (1.1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:

- $T^*(u_0) = \infty$;
- $T^*(u_0) < \infty$ and $\lim_{t \uparrow T^*(u_0)} \|u(t)\|_{C_u(\mathbb{R}^d, Z)} = \infty$.

**Proof.** See Theorem 4.3.4 in [10]. \hfill \Box

**Proposition 7.** Under conditions of theorem above, then

1. $T^*: C_u(\mathbb{R}^d, Z) \to \mathbb{R}_+$ is lower semi-continuous;

2. If $u_{0,n} \to u_0$ in $C_u(\mathbb{R}^d, Z)$ and $0 < T < T^*(u_0)$, then $u_{n} \to u$ in the Banach space $C([0, T], C_u(\mathbb{R}^d, Z))$.

**Proof.** See Proposition 4.3.7 in [10]. \hfill \Box

### 3 Periodic solutions

In this section, we will analyze the existence of solutions for the fractional reaction diffusion equation by applying Splitting methods to functions that have two main characteristics: these are direct sum of functions of periodic type and functions that tend to zero at infinity. This type of solution is also studied in the non-linear Schroedinger equation, under the name of "Peregrine solitons" [20]. Well posedness results are obtained for each particular characteristic, to then combine both results. In addition, we will observe that the evolution of the periodic part is independent of the part that tends to zero at infinity.

For instance, suppose that the non-linearity is of polynomial type (as in the Fitzhugh-Nagumo equation, see [1]), in this case we use $F(u) = u^2$. If $u(t) = v(t) + w(t)$, where $v(t)$ is a periodic function and $w(t)$ is a function that tends to zero when the spatial variable tends to infinity, then we have that

$$F(u) = F(v + w) = (v + w)^2 = v^2 + 2vw + w^2$$
where, \( v^2 \) is periodic and \( 2vw + w^2 \) tends to zero. In this specific case we can appreciate the absorption of the part that tends to zero, in the crossed terms. As \( v^2 = F(v) \), we expect that the periodic part of the initial data evolve independently from the part that tends to zero for the non linear equation. In this section we obtain general results to which this example refers.

Let \( \{ \gamma_1, \ldots, \gamma_q \} \) be \( q \) linearly independent vectors of \( \mathbb{R}^d \) and let \( \Gamma \) be the lattice generated, i.e., \( \Gamma = \{ n_1 \gamma_1 + \cdots + n_q \gamma_q : n_j \in \mathbb{Z} \} \). A function \( u \in C_u(\mathbb{R}^d, Z) \) is \( \Gamma \)-periodic if 

\[
(u + \gamma) = u(x) \text{ for any } \gamma \in \Gamma.
\]

We denote the set of \( \Gamma \)-periodic functions of \( C_u(\mathbb{R}^d, Z) \) by \( C_u(\mathbb{R}^d/\Gamma, Z) \).

We consider the space \( C_0(\mathbb{R}^d, Z) \) of functions which converge to 0 when \( |x| \to \infty \). It is easy to prove the following result.

**Proposition 8.** \( C_u(\mathbb{R}^d/\Gamma, Z), C_0(\mathbb{R}^d, Z) \subset C_u(\mathbb{R}^d, Z) \) are closed subspaces. Moreover, \( C_u(\mathbb{R}^d/\Gamma, Z) \cap C_0(\mathbb{R}^d, Z) = \{ 0 \} \).

**Proof.** Let \( u \in C_u(\mathbb{R}^d/\Gamma, Z) \), we set \( x \in \mathbb{R}^d \)

\[
\lim_{|\gamma| \to \infty} u(x + \gamma) = 0.
\]

Then \( u(x) = 0 \) for any \( x \in \mathbb{R}^d \).

**Lemma 9.** Let \( X \) be a Banach space and let \( X_1, X_2 \subset X \) be closed subspaces such that \( X_1 \cap X_2 = \{ 0 \} \), the following statement are equivalent

i. \( X_1 \oplus X_2 \) is closed.

ii. The projector \( P: X_1 \oplus X_2 \to X_1 \) is continuous.

**Proof.** Since \( X_1 \oplus X_2 \) is a Banach space, the linear map \( \phi: X_1 \times X_2 \to X_1 \oplus X_2 \) given by \( \phi(x, y) = x + y \) is bijective, and from the closed graph theorem we have \( \phi \) and \( \phi^{-1} \) are continuous operator. We can write \( P = \pi_1 \phi^{-1} \) and then \( P \) is continuous. On the other hand, \( X_1 \oplus X_2 = P^{-1} X_1 \), since \( P \) continuous and \( X_1 \) a closed subspace, \( X_1 \oplus X_2 \) is closed.

**Lemma 10.** The projector \( P: C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d/\Gamma, Z) \) is continuous.

**Proof.** Let \( u = v + w \in C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z) \), \( v \in C_u(\mathbb{R}^d/\Gamma, Z) \) and \( w \in C_0(\mathbb{R}^d, Z) \). For any \( x \in \mathbb{R}^d \), we can see that

\[
v(x) = \lim_{|\gamma| \to \infty} v(x + \gamma) = \lim_{|\gamma| \to \infty} u(x + \gamma),
\]

then \( |v(x)| \leq \|u\|_{C_u(\mathbb{R}^d, Z)} \), which implies \( \|v\|_{C_u(\mathbb{R}^d, Z)} = \|Pu\|_{C_u(\mathbb{R}^d, Z)} \leq \|u\|_{C_u(\mathbb{R}^d, Z)}. \)

**Corollary 11.** The direct sum \( X_{\Gamma, Z} = C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z) \) is a closed subspace of \( C_u(\mathbb{R}^d, Z) \).

To obtain the existence of solutions in the space \( X_{\Gamma, Z} \), we first study each case separately. We analyze the existence of solutions for the case of \( \Gamma \) periodic functions using the translation function.

Given \( \gamma \in \mathbb{R}^d \) define \( T_\gamma : C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z) \) as \( (T_\gamma u)(x) = u(x + \gamma) \). Since \( S(t) \) is a convolution operator, it is easy to see that \( T_\gamma S(t) = S(t)T_\gamma \). Using that \( T_\gamma F(t, u) = F(t, T_\gamma u) \) we obtain

\[
T_\gamma u(t) = S(t)T_\gamma u_0 + \int_0^t S(t-t')F(t, T_\gamma u(t')) dt'.
\]

Therefore, \( T_\gamma u \) is the solution of (2.1) with initial data \( T_\gamma u_0 \).
Proposition 12. If \( u_0 \in C_u(\mathbb{R}^d/\Gamma, Z) \), then the solution \( u \) of the equation \( (2.1) \) verifies \( u(t) \in C_u(\mathbb{R}^d/\Gamma, Z) \) for \( 0 \leq t < T^*(u_0) \).

Proof. Since \( T_\gamma u_0 = u_0 \) for any \( \gamma \in \Gamma \), \( T_\gamma u, u \) are solutions with the same initial data. From uniqueness, we have \( T_\gamma u = u \). Therefore, \( u(t) \in C_u(\mathbb{R}^d/\Gamma, Z) \).

We now analyze the existence of solutions of functions that tend to zero when the spatial variable tends to infinity.

Lemma 13. If \( u \in C_0(\mathbb{R}^d, Z) \), then \( S(t)u \in C_0(\mathbb{R}^d, Z) \) for \( t \in \mathbb{R}_+ \).

Proof. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence verifying \( |x_n| \to \infty \), we have
\[
| \langle S(t)u(x_n) \rangle |Z \leq \int_{\mathbb{R}^d} G_{\sigma,\beta}(t,y)|u(x_n - y)|Z dy.
\]

As \( G_{\sigma,\beta}(t,.)|u(x_n - .)|Z \leq G_{\sigma,\beta}(t,.)\|u\|_{\infty,Z} \) and \( G_{\sigma,\beta}(t,y)|u(x_n - y)|Z \to 0 \), from dominated convergence theorem we obtain that \( \lim_{n \to \infty}|\langle S(t)u(x_n) \rangle |Z = 0 \). Since \( \{x_n\}_{n \in \mathbb{N}} \) an arbitrary sequence, we have that \( S(t)u \in C_0(\mathbb{R}^d, Z) \).

Lemma 14. Let \( u_0, \tilde{u}_0 \in C_u(\mathbb{R}^d, Z) \), if \( u_0 - \tilde{u}_0 \in C_0(\mathbb{R}^d, Z) \), then \( N(t,t_0,u_0) - N(t,t_0,\tilde{u}_0) \in C_0(\mathbb{R}^d, Z) \) for \( 0 \leq t < \min\{T^*(u_0), T^*(\tilde{u}_0)\} \).

Proof. Let \( u(t) = N(t,t_0,u_0) \) and \( \tilde{u}(t) = N(t,t_0,\tilde{u}_0) \), for any \( x \in \mathbb{R}^d \) we have
\[
|u(x,t) - \tilde{u}(x,t)|Z \leq |u_0(x) - \tilde{u}_0(x)|Z + \int_{t_0}^t |F(t',u(x,t')) - F(t',\tilde{u}(x,t'))|Z dt' \leq |u_0(x) - \tilde{u}_0(x)|Z + L \int_{t_0}^t |u(x,t') - \tilde{u}(x,t')|Z dt'.
\]

From Gronwall’s lemma, we get \( |u(x,t) - \tilde{u}(x,t)|Z \leq e^{L(t-t_0)}|u_0(x) - \tilde{u}_0(x)|Z \). Given \( \varepsilon > 0 \), there exists \( r > 0 \) such that \( |u_0(x) - \tilde{u}_0(x)|Z < \varepsilon e^{-rL(t-t_0)} \) for \( |x| > r \), then \( |u(x,t) - \tilde{u}(x,t)|Z < \varepsilon \), which implies \( u(t) - \tilde{u}(t) \in C_0(\mathbb{R}^d, Z) \).

Proposition 15. Let \( u_0, \tilde{u}_0 \in C_u(\mathbb{R}^d, Z) \), such that \( u_0 - \tilde{u}_0 \in C_0(\mathbb{R}^d, Z) \) and let \( u, \tilde{u} \) be the respective solutions of \( (2.1) \). For any \( 0 \leq t < \min\{T^*(u_0), T^*(\tilde{u}_0)\} \), it is verified \( u(t) - \tilde{u}(t) \in C_0(\mathbb{R}^d, Z) \).

Proof. For \( t \in [0, \min\{T^*(u_0), T^*(\tilde{u}_0)\}] \), let \( n \in \mathbb{N} \), \( h = t/n \) and \( \{U_{h,k}\}_{0 \leq j \leq n}, \{\tilde{U}_{h,k}\}_{0 \leq j \leq n} \) sequences defined in terms of a recurrence, in the following way:

Let \( \{U_{h,k}\}_{0 \leq k \leq n}, \{V_{h,k}\}_{1 \leq k \leq n} \) be the sequences given by \( U_{h,0} = u_0 \),
\[
\begin{align*}
V_{h,k+1} &= S(h)U_{h,k}, \\
U_{h,k+1} &= N(kh+h,kh+h/2,V_{h,k+1}), \quad k = 0, \ldots, n-1.
\end{align*}
\]

We claim that \( U_{h,k} - \tilde{U}_{h,k} \in C_0(\mathbb{R}^d, Z) \) for \( k = 0, \ldots, n \). Clearly, the assertion is true for \( k = 0 \). If \( U_{h,k-1} - \tilde{U}_{h,k-1} \in C_0(\mathbb{R}^d, Z) \), from Lemma 14, we have \( N(kh,kh-h/2,V_{h,k-1}) - N(kh,kh-h/2,\tilde{V}_{h,k-1}) \in C_0(\mathbb{R}^d, Z) \). Using lemma 13, we can see that
\[
V_{h,k} - \tilde{V}_{h,k} = S(h)(N(kh,kh-h/2,V_{h,k-1}) - N(kh,kh-h/2,\tilde{V}_{h,k-1})) \in C_0(\mathbb{R}^d, Z).
\]
As \( C_0(\mathbb{R}^d, Z) \) is closed and \( U_{h,n} - \tilde{U}_{h,n} \to u(t) - \tilde{u}(t) \), we obtain the result.
Theorem 16. Let $u_0 \in X_{\Gamma, Z}$, the solution $u$ of the equation (2.1) verifies $u(t) \in X_{\Gamma, Z}$ for $0 \leq t < T^*(u_0)$. Moreover, if $u_0 = v_0 + w_0$ with $v_0 \in C_0(\mathbb{R}^d / \Gamma, Z)$ and $w_0 \in C_0(\mathbb{R}^d, Z)$, then $u(t) = v(t) + w(t)$, where $v$ is the solution of (2.1) with initial data $v_0$ and $w$ is the solution of

$$w(t) = S(t)w_0 + \int_0^t S(t-t')(F(v(t')) + w(t')) dt'.$$

Proof. As $u_0 \in X_{\Gamma, Z} \subseteq C_0(\mathbb{R}^d, Z)$, by theorem 6 we have that $u(t) \in C_0(\mathbb{R}^d, Z)$ with maximal time of existence $T^*(u_0)$. We observe that as $v_0 \in C_0(\mathbb{R}^d / \Gamma, Z)$ then by proposition 12 we know that $v(t) \in C_0(\mathbb{R}^d / \Gamma, Z)$ with maximal time of existence $T^*(v_0)$. We define $w(t) = u(t) - v(t)$. By hypothesis, we have that $w_0 = w(0) = u(0) - v(0) = u_0 - v_0 \in C_0(\mathbb{R}^d, Z)$ therefore, by proposition 15 we know that $w(t) \in C_0(\mathbb{R}^d, Z)$. Then, we obtain that $u(t) = v(t) + w(t) \in X_{\Gamma, Z}$, where $v(t) \in C_0(\mathbb{R}^d / \Gamma, Z)$ and $w(t) \in C_0(\mathbb{R}^d, Z)$ in the interval $[0, T_{\min})$ donde $T_{\min} = \min\{T(u_0), T(v_0)\}$. If it were that $T^*(v_0) = T^*(u_0)$ then we have the result.

Suppose that $T^*(v_0) < T^*(u_0)$.

Let $T \in (0, T^*(u_0))$, $M = \max_{0 \leq t \leq T} \|u(t)\|$. We define $\mathcal{T} = \{t \in [0, T] : u(t) \notin X_{\Gamma, Z}\}$, that is, the times for which we have that $u(t)$ is not a direct sum. Suppose that $\mathcal{T} \neq \emptyset$. Then there exists $t_1 = \inf \mathcal{T}$. We analyze if the infimum can be equal to zero or greater than zero.

The case in which $t_1 = 0$ is not possible because we have already seen that $u(t) \in X_{\Gamma, Z}$, in the interval $[0, T^*(v_0))$. In the same way, if $t_1 > 0$ and additionally $t_1 < T^*(v_0)$ we have that $u(t) \in X_{\Gamma, Z}$. We analyze the remaining case, $t_1 > 0$ and $T > t_1 > T^*(v_0)$.

We observe that, by theorem 6 we obtain that $\lim_{t \to T^*(v_0)} \|v(t)\| = +\infty$ but on the other hand, by lemma 10 we have that $\|v(t)\| \leq \|P\| \|u(t)\| \leq \|P\| M$ that is, the norm $v(t)$ is bounded for $t \in [0, T^*(v_0)) \subset [0, T]$, which is a contradiction.

So we finally have that $u(t) \in X_{\Gamma, Z}$ for $t \in [0, T^*(u_0))$.

\[ \square \]

Acknowledgement

This work was partially supported by CONICET–Argentina, PIP 11220130100006.

References

[1] Z. Asgari, M. Ghaemi and M. G. Mahjani Pattern Formation of the FitzHugh-Nagumo Model: Cellular Automata Approach, Iran. J. Chem. Chem. Eng., Vol. 30, Issue 1, p. 135-142, 2011.

[2] B. Baernmer, M. Kovács and M. M. Meerschaert Fractional Reproduction-Dispersal Equations and Heavy Tail Dispersal Kernels, Bull. Math. Biol, Vol. 69, p. 2281-2297, 2007.

[3] H. Bailung, S. K. Sharma, and Y. Nakamura, Observation of Peregrine solitons in a multicomponent plasma with negative ions, Bull. Math. Biol, Vol. 107, No.25, p. 255005, 2011.

[4] A.T. Besteiro and D.F. Rial Global existence for vector valued fractional reaction-diffusion equations, arXiv preprint arXiv:1805.09985, 2018.
[5] J. P. Borgna, M. De Leo, D. Rial and C. Sanchez de la Vega, *General Splitting methods for abstract semilinear evolution equations*, Commun. Math. Sci, Int. Press Boston, Inc., Vol. 13 p. 83-101, 2015.

[6] C. Bucur, E. Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, Springer, 2016.

[7] A. Bueno-Orovio, D. Kay and K. Burrage, *Fourier spectral methods for fractional-in-space reaction-diffusion equations*, BIT Numer. Math., Springer, Vol. 54, No.4 p. 937–954, 2014.

[8] K. Burrage, N. Hale and D. Kay, *An efficient implicit FEM scheme for fractional-in-space reaction-diffusion equations*, SIAM J. Sci. Comput., Vol. 34, No.4 p. A2145–A2172, 2012.

[9] C. Canzi and G. Guerra, *A simple counterexample related to the Lie–Trotter product formula* Semigroup Forum, Vol. 84, p. 499-504, 2012.

[10] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Ser. Math. Appl., Clarendon Press, Rev ed. edition, 1999.

[11] A. Chabchoub, N.P. Hoffmann and N. Akhmediev, *Rogue wave observation in a water wave tank*, Phys. Rev. Lett., Vol. 106, No.20, p. 204502, 2011.

[12] M. De Leo, D. Rial and C. Sanchez de la Vega, *High-order time-splitting methods for irreversible equations*, IMA J. Numer. Anal., p. 1842-1866, 2015.

[13] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., Vol. 136, No. 5, p. 521-573, 2012.

[14] K. Hammani, B. Kibler, C. Finot, P. Morin, J. Fatome, J.M. Dudley and G. Millot, *Peregrine soliton generation and breakup in standard telecommunications fiber*, Optics Letters, Vol. 36, No. 2, p. 112-114, 2011.

[15] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev and J.M. Dudley, *The Peregrine soliton in nonlinear fibre optics*, Nature Physics, Vol. 6, No. 10, p. 790, 2010.

[16] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal., Vol. 20, p. 7-51, 2017.

[17] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York-Heidelberg, 1972.

[18] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai and M. Meerschaert, M. Ainsworth, and G. Karniadakis, *What is the Fractional Laplacian?*, preprint, arXiv:1801.09767, 2018.

[19] J. T. Machado and V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul., Elsevier, Vol. 16, No.3, p.1140–1153, 2011.

[20] D.H. Peregrine, *Water waves, nonlinear Schrödinger equations and their solutions*, J. Aust. Math. Soc., Vol. 25, No.1, p. 16-43, 1983.

[21] C. Pozrikidis, *The fractional Laplacian*, CRC Press, Boca Raton, FL, 2016.
[22] J.C. Robinson, A. Rodríguez-Bernal, and A. Vidal-Lopez, *Pullback attractors and extremal complete trajectories for non-autonomous reaction-diffusion problems*, J. Differential Equations, Vol. 238, p. 289-337, 2007.

[23] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Stud. Adv. Math., Cambridge Univer. Press, 1999.

[24] V.I. Shrira and V.V. Geogjaev, *What makes the Peregrine soliton so special as a prototype of freak waves?*, J. Eng. Math., Vol. 67, No.1-2, p.11-22, 2009.

[25] L. E. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Thesis (Ph.D.)-The University of Texas at Austin, 2005.