On flat connections with non-diagonalizable holonomies

K. Selivanov
ITEP, Moscow, 117259, B.Cheryomushkinskaya 25

Abstract

Recently the long-standing puzzle about counting the Witten index in $N = 1$ supersymmetric gauge theories was resolved. The resolution was based on existence (for higher orthogonal $SO(N)$, $N \geq 7$ and exceptional gauge groups) of flat connections on $T^3$ which have commuting holonomies but cannot be gauged to a Cartan torus. A number of papers has been published which studied moduli spaces and some topological characteristics of those flat connections. In the present letter an explicit description of such flat connection for the basic case of $Spin(7)$ is given.

Recently the long standing paradox with counting the Witten index in $N = 1$ supersymmetric gauge theory has been resolved [1]. The essence of the paradox was that different ways of computing the Witten index gave different results for the higher orthogonal ($SO(N)$, $N \geq 7$) and the exceptional gauge groups.

The first way was to put the gauge theory into a finite spacial box and to count the number of supersymmetric vacuum states [2] which resulted in $\text{Tr}(-1)^F = r + 1$ where $r$ is the rank of the gauge group. For higher orthogonal and exceptional groups, this result disagrees with the one based on counting of gluino zero modes in the instanton background and also on the analysis of weakly coupled theories with additional matter super-multiplets [2, 3].

$$\text{Tr}(-1)^F = h^\vee,$$  \hspace{1cm} (1)

where $h^\vee$ is the dual Coxeter number of the group (see e.g. [3], Chapt. 6; it coincides with the Casimir $T_\alpha T^\alpha$ in the adjoint representation when a proper
normalization is chosen.). For $SO(N \geq 7)$, $h^V = N - 2 > r + 1$. Also for exceptional groups $G_2, F_4, E_{6,7,8}$, the index (1) is larger than the Witten’s original estimate.

In [1] Witten has found a flaw in his original arguments and shown that, for $SO(N \geq 7)$, vacuum moduli space is richer than it was thought before so that the total number of quantum vacua is $N - 2$ in accordance with the result (1). The derivation in [2] was based on the assumption that a flat connection on 3d torus $T^3$ can be gauged to a Cartan sub-algebra of the corresponding Lie algebra. This assumption seems to be quite natural since for a connected, simply connected gauge group a connection on $T^3$ is flat when and only when it has commuting holonomies $\Omega_j, j = 1, 2, 3$ over three independent nontrivial cycles of $T^3$, and it is very natural to expect that commuting holonomies are representable as exponentials of commuting Lie algebra elements. Nevertheless, it is not true. For higher orthogonal and exceptional gauge groups there are triples of commuting holonomies which cannot be represented as exponentials of a Cartan sub-algebra elements. This fact is in the heart of resolution of the paradox in [1].

Interestingly, existence of such triples has been known to topologists for a long (see, e.g. [3, 4], where examples of such triples were constructed).

After the Witten’s work [1] (see also its interpretation for pedestrians in [7]) a new interest to such triples has arisen. Moduli spaces of such triples (that is, additional components of moduli spaces of flat connections on $T^3$ for exceptional gauge group) have been described in [8] and in [9]. Later these results have been re-derived and extended in some respects in [10].

The purpose of this letter is to explicitely describe flat connections corresponding to such triples (actually, only the basic case of $Spin(7)$ is considered here; the description is likely generalizable to other cases). Although the explicit description is not needed in the problem of computing the Witten index, it may be useful elsewhere, in particular, in answering the question whether the existence of the new components of vacua moduli space affects only the Witten index or also some other observables in supersymmetric gauge theories with orthogonal or exceptional gauge groups. It may also be interesting per se.

For $Spin(7)$ group there is a unique (up to conjugation) nontrivial triple. It can be chosen in the form [6]. [1]
\[\begin{align*}
\Omega_1 &= \gamma_{1234} \\
\Omega_2 &= \gamma_{1256} \\
\Omega_3 &= \gamma_{1357},
\end{align*}\]  

where and in what follows we use the notation

\[\gamma_{ijkl...} = \gamma_i \gamma_j \gamma_k \gamma_l ...\]  

and \(\gamma_i, i = 1, \ldots 7\) stand for the gamma-matrices. \(\Omega\)'s in Eq.(2) mutually commute and cannot be conjugated to the Cartan torus (see, e.g. [4]).

Let \(x, y, z\) be coordinates on the cube in \(\mathbb{R}^3\) which gives \(T^3\) upon identification \(x \sim x + 1, y \sim y + 1, z \sim z + 1\). We would like to explicitly describe a flat \(Spin(7)\) connection \(A_i, i = 1, 2, 3\) on \(T^3\) which has holonomies \(\Omega_i, i = 1, 2, 3\) from Eq.(2), that is

\[\begin{align*}
\Omega_1 &= \text{P exp} \left( \int_0^1 A_1(x, 0, 0) dx \right) \\
\Omega_2 &= \text{P exp} \left( \int_0^1 A_2(0, y, 0) dy \right) \\
\Omega_3 &= \text{P exp} \left( \int_0^1 A_3(0, 0, z) dz \right).
\end{align*}\]  

It will be represented in the form

\[A_i(x, y, z) = g(x, y, z)^{-1} \frac{\partial}{\partial x^i} g(x, y, z)\]  

where \(g(x, y, z)\) takes values in the group and has the following properties:

\[\begin{align*}
g(1, y, z) &= \Omega_1 g(0, y, z) \\
g(x, 1, z) &= \Omega_2 g(x, 0, z) \\
g(x, y, 1) &= \Omega_3 g(x, y, 0).
\end{align*}\]  

with \(\Omega\)'s from Eq.(2).

Obviously, with \(g(x, y, z)\) obeying Eq.(4) the flat connection \(A_i\) from Eq.(5) is periodical on \(T^3\) and has the appropriate holonomies.

Now introduce the following definitions:

\[\begin{align*}
g_1 &= e^{\frac{\pi}{2} x(\gamma_{12} + \gamma_{34})} \\
g_2 &= e^{\frac{\pi}{2} y(\gamma_{15} - \gamma_{26})} \\
g_3 &= e^{\frac{\pi}{2} z(\gamma_{14} + \gamma_{57})}.
\end{align*}\]
$g_i$ Eq.(7) produces the monodromy $\Omega_i$ when the corresponding coordinate changes by 1. The specific choice of $g_i$’s is related to the following property:

$$[g_i, \Omega_{i+1}] = 0 \quad (8)$$

By explicit computation one verifies that

$$g_i \Omega_{i-1} = \Omega_{i-1} g_i^{-1} \quad (9)$$

Introduce also $\tilde{g}_3$ such that it commutes with $g_2$ (and, consequently, with $\Omega_2$),

$$\tilde{g}_3 = \exp \left( \frac{\pi}{2} z(\gamma_{15} - \gamma_{37}) \right). \quad (10)$$

$\tilde{g}_3$ as well as $g_3$ produces $\Omega_3$ when the corresponding coordinate ($z$) changes by 1.

One can also verify that

$$\tilde{g}_3 \Omega_1 = \Omega_1 (\tilde{g}_3)^{-1}. \quad (11)$$

Using all these properties, one can see that the following $g(x, y, z)$

$$g(x, y, z) = g_1 g_2 g_1^{-1} g_3 \tilde{g}_3^{-1} g_1 g_2^{-1} g_1^{-1} \tilde{g}_3 g_1 g_2 \quad (12)$$

obeys the key property Eq.(3), and hence the corresponding flat connection, Eq.(5), is periodic and has the monodromies $\Omega$’s.

I would like to thank A.Gorsky and A.Rosly for discussions and for convincing me to publish this letter. This work was partially supported by INTAS-96-0482.

References

[1] E. Witten, *J. High En. Phys.* **9802** (1998) 6

[2] E. Witten, *Nucl. Phys.* **B202** (1982) 253

[3] S.F. Cordes and M. Dine, *Nucl. Phys.* **B273** (1986) 581; A. Morozov, M. Ol’shanetsky, and M. Shifman, *Nucl. Phys.* **B304** (1988) 291

[4] V.G. Kac, *Infinite–dimensional Lie algebras*, 3d edition, Cambridge U. Press, 1990
[5] A. Borel and J.-P. Serre, *Comment. Math. Helv.* **27** (1953) 128

[6] A. Borel, *Tohoku Math. J.* **13** (1961) 216

[7] A. Keurentjes, A. Rosly, and A.V. Smilga, *Phys. Rev.* **D58** (1998) 081701

[8] V.G. Kac and A. Smilga, NANTES-PHT-99-3, [hep-th/9902029](http://arxiv.org/abs/hep-th/9902029)

[9] A.Keurentjes, JHEP 9905:001,1999, JHEP 9905:014,1999

[10] A.Borel, R.Friedman, and J.Morgan, [math.gr/9907007](http://arxiv.org/abs/math.gr/9907007)