GIBBS STATES ON RANDOM CONFIGURATIONS

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Abstract. Gibbs states of a spin system with the single-spin space $S = \mathbb{R}^m$ and unbounded pair interactions is studied. The spins are attached to the points of a realization $\gamma$ of a random point process in $\mathbb{R}^n$. Under certain conditions on the model parameters we prove that, for almost all $\gamma$, the set $\mathcal{G}(S^\gamma)$ of all Gibbs states is nonempty and its elements have support properties, explicitly described in the paper. We also show the existence of measurable selections $\gamma \mapsto \nu_\gamma \in \mathcal{G}(S^\gamma)$ (random Gibbs measures) and derive the corresponding averaged moment estimates.

1. Introduction

The aim of this paper is to study Gibbs states (states of thermal equilibrium) of the following system of interacting particles. The underlying set is a countable collection of point particles chaotically distributed over a Euclidean space $X$, modeled by a random point process in $X$. Each particle $x$ in the collection possesses an internal structure described by a mark (spin) $\sigma(x)$ taking values in a single-spin space $S_x$ and characterized by a single-spin measure $\chi_x$. The system as a whole is characterized by the law of the underlying point process $\mu$, by the spin-spin pair interaction dependent on the location of the particles, and by the family of single-spin measures $\{\chi_x\}_{x \in \mathbb{R}^d}$. For a fixed realization of the point process $\gamma$, a Gibbs state is a probability measure on the product space $S^\gamma = \prod_{x \in \gamma} S_x$ constructed in the following way. First we equip $S^\gamma$ with the usual product topology and the corresponding Borel $\sigma$-algebra $\mathcal{B}(S^\gamma)$. Then we introduce the set $\mathcal{P}(S^\gamma)$ of all probability measures on $(S^\gamma, \mathcal{B}(S^\gamma))$. If the pair interaction is absent, i.e., the spins are independent, the unique Gibbs state is just the product $\otimes_{x \in \gamma} \chi_x \in \mathcal{P}(S^\gamma)$ of the single-spin measures. The states of the system with interacting spins are constructed as perturbations of the mentioned product measure by the “densities”

\begin{equation}
\exp \left( - \sum_{\{x,y\} \subset \gamma} W_{xy}(\sigma(x), \sigma(y)) \right),
\end{equation}

where $W_{xy} : S \times S \to \mathbb{R}$ are measurable functions – interaction potentials. Clearly, (1.1) is just a heuristic expression – the rigorous definition is based on the Gibbs specification constructed by means of the potentials $W_{xy}$. Then the Gibbs states $\nu_\gamma$ are defined as elements of $\mathcal{P}(S^\gamma)$ that solve the Dobrushin-Lanford-Ruelle (DLR) equation corresponding to the Gibbs specification (see e.g. [9, 25] and Introduction in [14]).

If the underlying set is fixed and reasonably regular, the only problem which one faces in constructing Gibbs states of models with interactions as
in (1.1) is the possible unboundedness of the potentials $W_{xy}$. Works in this direction were originated in seminal papers [20, 28] where the underlying set is a cubic lattice $\mathbb{Z}^d$ and the potentials are unbounded functions on $\mathbb{R} \times \mathbb{R}$, see also [18] for a more recent results. The case where the underlying set is a fixed unbounded degree graph was studied in [13].

In the present paper, we study Gibbs states of a spin system of this kind with the underlying set chosen at random from the collection of all locally finite subsets of a Euclidean space $X = \mathbb{R}^n$, $n \geq 1$ (called 'simple configurations' in $X$); that is, from the set

\begin{equation}
\Gamma(X) = \{ \gamma \subset X : N(\gamma \cap \Lambda) < \infty, \ \Lambda \in \mathcal{B}_0(X) \},
\end{equation}

where $N(A)$ stands for the cardinality of $A$ and $\mathcal{B}_0(X)$ is the collection of all compact subsets of $X$. The set $\Gamma(X)$ is endowed with a Polish space structure (see e.g. [11, Section 15.7.7] and [26, Proposition 3.17]), by means of which we introduce the Borel $\sigma$-algebra $\mathcal{B}(\Gamma(X))$. Then we fix a probability measure $\mu$ on $\mathcal{B}(\Gamma(X))$ and interpret the underlying set $\gamma$ as a realization of a random point process. A typical choice of $\mu$ is a Poisson measure. However, our results are valid for a wide class of probability measures on $\Gamma(X)$ introduced below. Our goal is to study the set $G(S_{\gamma})$ of all Gibbs measures associated with the collections of $W_{xy}$ and $\chi_x$, for $\mu$-almost all configurations $\gamma$. In the physical terminology, cf. [4], the elements of $G(S_{\gamma})$ are quenched Gibbs states of an amorphous magnet. Here we discuss the questions of existence of such states and their measurable dependence on $\gamma$, while the complementary paper [6] is devoted to the problem of phase transitions in a more specific (ferromagnetic) version of this model.

We assume that the interaction potentials have finite range, that is, they satisfy condition $W_{xy} \equiv 0$ whenever $|x - y| > R$ for some fixed $R > 0$. This allows for introducing a graph structure on $\gamma$ in the following way: the vertex set of the graph is $\gamma$ itself, whereas the edge set is defined as

\begin{equation}
E_\gamma = \{ \{x, y\} \subset \gamma : |x - y| \leq R \}.
\end{equation}

Correspondingly, vertices $x$ and $y$ are called neighbors or adjacent in $\gamma$, if $|x - y| \leq R$. Let $n_{\gamma,R}(x)$ denote the degree of vertex $x \in \gamma$, i.e., the number of neighbors of $x$ in $\gamma$. In our model the function $n_{\gamma,R}$ appears to be unbounded for $\mu$-a.a. $\gamma \in \Gamma(X)$. For unbounded degree graphs and unbounded spins, the question of existence of Gibbs measures was first studied in [13], where certain growth conditions on the degrees and stability conditions on $\chi$ and $W_{xy}$ were imposed. Observe that in the case of a compact spin space $S$ the answer to the existence question is always positive, see e.g. [25, Proposition 5.3]. For a comprehensive review of the theory of Gibbs measures on graphs see [10] and references therein.

The structure of this paper is as follows. In Section 2 we describe certain properties of the graph defined in (1.3), which hold for $\mu$-almost all $\gamma$. The only assumption is that the measure $\mu$ has correlation functions up to a certain order, which are essentially bounded. In particular, we obtain bounds on the growth of $n_{\gamma,R}(x)$ and show that, for $\mu$-a.a. $\gamma \in \Gamma(X)$, the graph $(\gamma, E_\gamma)$ satisfies the corresponding conditions of work [13]. In Section 3 by means of the results of [13] we show that the set $G(S_{\gamma})$ is non-empty. In addition, we describe the support of the elements of $G(S_{\gamma})$ and obtain uniform
estimates on their exponential moments. By means of these estimates, we prove that \( G(S^\gamma) \) contains elements with a priori prescribed support properties. These are tempered Gibbs measures. The use of such measures is typical for systems of unbounded spins. The proof of these results is based on exponential moment bounds for the local Gibbs specification of our model and its weak dependence on the boundary conditions. Such a technique is effective in dealing with spatially irregular systems, see [13, 14, 17]. The two fundamental tools – Ruelle’s (super-)stability technique [27, 28] and general Dobrushin’s existence and uniqueness criteria [3] – are not directly applicable to our model (due to the unboundedness of the degree function \( n_{\gamma,R} \) and the lack of the spatial transitivity of \( \mu \)-almost all \( \gamma \)). At the same time, for our model the uniqueness problem remains open. Thus, the map \( \Gamma(X) \ni \gamma \mapsto \nu_\gamma \in G(S^\gamma) \) is in general set-valued.

The results of Section 3, however, do not answer the following important question: is it possible to select \( \nu_\gamma \in G(S^\gamma) \) in such a way that the resulting map \( \gamma \mapsto \nu_\gamma \) is measurable (existence of measurable selections)? This measurability is a key property that allows one to define averages of the type of \( \int_{\Gamma(X)} \Phi(E_{\nu_\gamma} F) \mu(d\gamma) \). Similar problems appear in the theory of Gibbs fields with random components, e.g., random interactions. The measurable maps \( \gamma \mapsto \nu_\gamma \) are then called random Gibbs measures, see [14] and Section 6.2 in [4]. In Section 4, we prove the existence of random Gibbs measures for our model. For unbounded spins with random interactions on a lattice, a similar result was obtained in [14]. The novelty of the present situation is that the measures \( \nu_\gamma \) live (for different \( \gamma \)) on different spaces, and a priori it is not clear in what sense the mentioned measurability can be understood. In Section 4, we develop a constructive procedure of obtaining measurable selections \( \gamma \mapsto \nu_\gamma \). For this, we identify the spaces \( S^\gamma, \gamma \in \Gamma(X) \), with the fibres of a natural bundle over \( \Gamma(X) \). It turns out that its total space \( X \) has the structure of the marked configuration space \( \Gamma(X, S) \). For definitions and main facts on marked configuration spaces we refer to [7, 1, 5, 19].

Using the appropriate moment bounds, we construct an auxiliary measure \( \hat{\nu} \) on \( \Gamma(X, S) \) and define its conditional distribution (i.e. disintegration) \( (\nu_{\gamma})_{\gamma \in \Gamma(X)} \subset \mathcal{P}(S^\gamma) \) with respect to \( \mu \), so that the measurability required holds. Then we prove that \( \nu_\gamma \in G(S^\gamma) \) and that each \( \nu_\gamma \) is a tempered measure. Note that \( \nu_\gamma \) need not in general coincide with the element of \( G(S^\gamma) \) constructed in the proof of Theorem 3.1 and be represented as the limit of a sequence of local Gibbs measures. However, by means of Komlós’ theorem, we show the existence (and hence measurability) of limiting Gibbs measures \( \nu_\gamma \) obtained from sequences of the Cesàro means of local Gibbs measures. It resembles the Newman–Stein approach [22, 23] in the theory of disordered spin systems, in which the so called ‘chaotic size dependence’ is tamed by means of a space averaging, see also [14].

2. Estimates for a typical configuration.

Let \( C_0(X) \) denote the set of all continuous functions on \( f : X \to \mathbb{R} \) which have compact support. The configuration space \( \Gamma(X) \) defined in (1.2) is equipped with the vague topology – the weakest topology that makes
continuous all the mappings
\[ \Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X). \]

It is known that this topology is completely metrizable, which makes \( \Gamma(X) \) a Polish space (see, e.g., [11, Section 15.7.7] or [26, Proposition 3.17]); an explicit construction of the appropriate metric can be found in [16]. By \( P(\Gamma(X)) \) we denote the space of all probability measures on the corresponding Borel \( \sigma \)-algebra \( B(\Gamma(X)) \). We will also use the algebra \( B_0(\Gamma(X)) \) of local sets, \( B_0(\Gamma(X)) := \cup_{\Lambda \in B_0(X)} B(\Gamma(\Lambda)). \) The space of \( B(\Gamma(X)) \) (resp. \( B_0(\Gamma(X)) \)) measurable bounded functions \( f : \Gamma(X) \to \mathbb{R} \) will be denoted by \( L_\infty(\Gamma(X)) \) (resp. \( L_0^\infty(\Gamma(X)) \)).

For a given \( \mu \in P(\Gamma(X)) \), a measurable symmetric (w.r.t. permutations of its arguments) function \( 0 \leq k_m : X^m \to \mathbb{R}, \quad m \in \mathbb{N}, \) is called the \( m \)-th order correlation function of \( \mu \) if for any non-negative measurable symmetric function \( g : X^m \to \mathbb{R} \) the following holds
\[
\int_{\Gamma(X)} \sum_{\{x_1, \ldots, x_m\} \subset \gamma} g(x_1, \ldots, x_m) \mu(d\gamma) = \frac{1}{m!} \int_{X^m} g(x_1, \ldots, x_m) k_m(x_1, \ldots, x_m) dx_1 \ldots dx_m.
\]

From now on we assume that \( \mu \) is fixed and that it has all correlation functions up to some order \( M \in \mathbb{N} \), which are essentially bounded, i.e.,
\[
||k_m||_\infty := \text{ess sup}_{X^m} k_m(x_1, \ldots, x_m) < \infty, \quad 1 \leq m \leq M.
\]

Remark 2.1. In the theory of random point processes, correlation functions \( k_m \) appear as densities (w.r.t. \( dx_1 \ldots dx_m \)) of the so-called \( m \)-th factorial moment measures corresponding to \( \mu \) (see e.g. [7, Section 5.4]). The boundedness as in (2.2) holds for a wide class of measures on \( \Gamma(X) \) and implies the finiteness of local moments, i.e.,
\[
\int_{\Gamma(X)} |\langle f, \gamma \rangle|^m \mu(d\gamma) < \infty, \quad f \in C_0(X), \quad m \leq M.
\]

For the standard Poisson point process \( \mu = \pi_z \) with the activity parameter \( z > 0 \) and Lebesgue intensity measure \( zdx \), the correlation functions \( k_m(x_1, \ldots, x_m) \) are just constants \( z^m, m \in \mathbb{N} \). If there exists \( \zeta > 0 \) such that \( ||k_m||_\infty \leq \zeta^m \) for all \( m \in \mathbb{N} \), we say that the correlations functions \( k_m \) are sub-Poissonian or satisfy Ruelle’s bound. Such measures typically arise in classical statistical mechanics as Gibbs modifications of the Poisson measure \( \pi_z \) by means of stable interactions, see [27, 28]. Note that any \( \mu \in P(\Gamma(X)) \) such that \( (k_m)_{m \in \mathbb{N}} \leq \zeta^m \) for all \( m \in \mathbb{N} \), is uniquely determined by its correlation functions. General criteria allowing for reconstructing a state \( \mu \in P(\Gamma(X)) \) from a given system of functions \( (k_m)_{m \in \mathbb{N}} \) were established in [11, 15, 21].
Now let us turn to the graph $(\gamma, E_\gamma)$ defined in (1.3). For $x \in \gamma$, its degree in this graph is

$$n_{\gamma,R}(x) := N(\{y \in \gamma : y \sim x\}) \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\},$$

where $x \sim y$ means that $\{x, y\} \in E_\gamma$. For $\alpha, r > 0$, we introduce weights

$$w_\alpha(x) := e^{-\alpha|x|}, \quad x \in X,$$

and consider the following functions on $\Gamma(X)$:

$$a_{\alpha,r}(\gamma) := \sum_{\{x,y\} \in E_\gamma} w_\alpha(x) [n_{\gamma,R}(x)n_{\gamma,R}(y)]^r, \quad r \geq 0,$$

(2.3)

$$b_\alpha(\gamma) := \sum_{x \in \gamma} w_\alpha(x) = \langle w_\alpha, \gamma \rangle.$$

Standard arguments (based on $n$-particle expansions) show that $a_{\alpha,r}$ and $b_\alpha$ are $B(\Gamma(X))$-measurable.

**Proposition 2.2.** Let $\mu$ be such that (2.2) holds with some integer $M \geq 2$. Then, for any $\alpha > 0$ and $0 \leq r \leq M/2 - 1$, we have inclusions $a_{\alpha,r}, b_\alpha \in L^1(\Gamma(X), \mu)$.

**Proof.**

1) Applying (2.1) to $w_\alpha \in L^1(X)$ we obtain

$$\int_{\Gamma(X)} b_\alpha(\gamma) \mu(d\gamma) = \int_{\Gamma(X)} \sum_{x \in \gamma} w_\alpha(x) \mu(d\gamma)$$

$$= \int_X w_\alpha(x)k_1(x)dx \leq \||k_1||_\infty \int_X e^{-\alpha|x|}dx < \infty.$$

2) Since $n_{\gamma,R}(x)n_{\gamma,R}(y)$ is either 0 or $\geq 1$, we have $a_{\alpha,r}(\gamma) \leq a_{\alpha,r}(\gamma)$ whenever $r \leq r'$. Thus it is sufficient to prove the inclusion $a_{\alpha,r} \in L^1(\Gamma(X), \mu)$ just for $r = M/2 - 1$.

Let us fix some $x \in \gamma$. Clearly, for any $y \in \gamma$ such that $|x - y| \leq R$, we have

$$n_{\gamma,R}(y) \leq n_{\gamma,2R}(x),$$

which yields

$$\sum_{y \in \gamma \setminus \{x\}} [n_{\gamma,R}(x)n_{\gamma,R}(y)]^r \leq n_{\gamma,R}(x) [n_{\gamma,R}(x)n_{\gamma,2R}(x)]^r$$

(2.4)

$$\leq n_{\gamma,2R}(x)^{2r+1} = n_{\gamma,2R}(x)^{M-1}.$$

Observe that

$$n_{\gamma,2R}(x) = N(\{y \in \gamma : 0 < |x - y| \leq 2R\}) = \sum_{\substack{y \in \gamma \setminus \gamma \setminus \{x\}} \chi_{B_{2R}}(y - x),}$$

where $B_{2R}$ is the closed ball of radius $2R$ centred at the origin and $\chi_{B_{2R}}$ is the corresponding indicator function. Thus, we have the multinomial
expansion

\[ n_{\gamma,2R}(x)^{M-1} = \left( \sum_{y \in \gamma \backslash \{x\}} 1_{B_{2R}}(y-x) \right)^{M-1} \]

\[ = \sum_{y_1, \ldots, y_{M-1} \in \gamma \backslash \{x\}} \prod_{k=1}^{M-1} 1_{B_{2R}}(y_k-x) \]

\[ = \sum_{j=1}^{M-1} c_j \sum_{\{y_1, \ldots, y_j\} \in \gamma \backslash \{x\}} \prod_{k=1}^{j} 1_{B_{2R}}(y_k-x) \]

with the coefficients

\[ c_j := \sum_{i_1, \ldots, i_j \in \mathbb{N}} \frac{(M-1)!}{i_1! \cdots i_j!}, \quad 1 \leq j \leq M-1. \]

Let us introduce notations \( \bar{y}_j := (y_0, y_1, \ldots, y_j) \in \gamma^{j+1} \) and \( \{ \bar{y}_j \} := \{ y_0, y_1, \ldots, y_j \} \subset \gamma \) for the vector and configuration with components \( y_0, y_1, \ldots, y_j \in \gamma \), respectively, and consider functions

\[ f_j(\bar{y}_j) = w_\alpha(y_0) \prod_{k=1}^{j} 1_{B_{2R}}(y_k-y_0) \]

and

\[ \hat{f}_j(\bar{y}_j) = \sum_{s \in S_{j+1}} f_j(s(\bar{y}_j)), \]

where \( S_m \) is the symmetric group of order \( m \). Inequality (2.4) implies that

\[ a_{\alpha,r}(\gamma) \leq \sum_{x \in \gamma} w_\alpha(x)n_{\gamma,2R}(x)^{M-1} = \sum_{j=1}^{M-1} c_j \sum_{\{\bar{y}_j\} \subset \gamma} \hat{f}_j(\bar{y}_j). \]

The application of (2.1) to the right-hand side of (2.5) shows that

\[ \int_{\Gamma(X)} a_{\alpha,r}(\gamma) \mu(d\gamma) \leq \sum_{j=1}^{M-1} \frac{c_j}{(j+1)!} \int_{\mathcal{X}^{j+1}} \hat{f}_j(\bar{y}_j) k_{j+1}(\bar{y}_j) d\bar{y}_j. \]

Thus we obtain the estimate

\[ \int_{\Gamma(X)} a_{\alpha,r}(\gamma) \mu(d\gamma) \leq \sum_{j=1}^{M-1} \frac{c_j |S_{j+1}|}{(j+1)!} \int_{\mathcal{X}^{j+1}} w_\alpha(y_0) \prod_{k=1}^{j} 1_{B_{2R}}(y_k-y_0) k_{j+1}(\bar{y}_j) d\bar{y}_j \]

\[ \leq ||k||_\infty \sum_{j=1}^{M-1} c_j \text{Vol}(B_{2R})^j \int_X e^{-\alpha|x|} dx < \infty, \]

where \( \text{Vol}(B_{2R}) \) is the volume of the ball \( B_{2R} \) and \( ||k||_\infty := \max_{1 \leq m \leq M} ||k_m||_\infty \), which completes the proof. \( \square \)
3. Construction of Gibbs measures

In the standard Dobrushin-Lanford-Ruelle approach in statistical mechanics [9, 25] which we follow in this work, Gibbs states are constructed by means of their local conditional distributions (constituting the so-called Gibbsian specification). The main technical problem in realizing this approach is to control the spatial irregularity of the configuration $\gamma$ and the unboundedness of the interaction potentials $W_{xy}$.

In what follows, we write $| \cdot |$ for the corresponding Euclidean norms in both $X$ and $S$. Let $W_{xy} : S \times S \to \mathbb{R}$, $x, y \in X$, be measurable functions satisfying the polynomial growth estimate

$$|W_{xy}(u, v)| \leq I_W (|u|^r + |v|^r) + J_W, \quad u, v \in S,$$

and the finite range condition $W_{xy} \equiv 0$ if $|x - y| \leq R$ for all $x, y \in X$ and some constants $I_W, J_W, R, r \geq 0$. We assume also that $W_{xy}(u, v)$ is symmetric with respect to the permutation of $(x, u)$ and $(y, v)$. A typical example is given by the bilinear form

$$W_{xy}(u, v) = A(x - y)u \cdot v, \quad u, v \in S,$$

where $\cdot$ denotes the Euclidean inner product in $S$ and $A$ is a uniformly bounded measurable mapping with values in the space of symmetric $m \times m$ matrices such that $\text{supp } A \subset B_R = \{ x \in X : |x| \leq R \}$.

In the sequel, we take the single-spin measures in the following form

$$\chi_x(du) := e^{-V(u)}du,$$

where $V : S \to \mathbb{R}$ is a measurable functions satisfying

$$V(u) \geq a_V |u|^q - b_V, \quad u \in S,$$

for some constants $a_V, b_V > 0$, and $q > 2$. Note that $\chi_x(S) < \infty$ in view of (3.3), which is aimed to compensate the destabilizing effects of the unbounded interactions potential $W_{xy}$. Note also that the case of $q = 2$ cannot be covered by our scheme due to the lack of uniform bounds on vertex degrees $n_{\gamma, R}(x)$ in the underlying graph $(\gamma, \xi_\gamma)$.

For a fixed $\gamma \in \Gamma(X)$, we will denote by $\sigma_{\gamma}, \xi_\gamma$, etc. elements of the space $S^\gamma$, and omit the subscript $\gamma$ whenever possible. Let $\mathcal{F}(\gamma)$ be the collection of all finite subsets of $\gamma$. For any $\eta \in \mathcal{F}(\gamma)$, $\sigma_\eta = (\sigma(x))_{x \in \eta} \in S^\eta$ and $\xi_\gamma = (\xi(y))_{y \in \gamma} \in S^\gamma$ define the relative local interaction energy

$$E_\eta(\sigma | \xi) = \sum_{\{x, y\} \in \eta} W_{xy}(\sigma(x), \sigma(y)) + \sum_{\substack{x \in \eta \\cap \\gamma \\backslash \\eta \\backslash \\eta \\gamma}} W_{xy}(\sigma(x), \xi(y)).$$

The corresponding specification kernel $\Pi_\eta(\sigma | \xi) \in \mathcal{P}(S^\gamma)$ is given by the formula

$$\int_{S^\gamma} f(\sigma)\Pi_\eta(\sigma | \xi) = Z(\xi)^{-1} \int_{S^\gamma} f(\sigma_\eta \times \xi_{\gamma \backslash \eta}) \exp \{-E_\eta(\sigma | \xi)\} \chi_{\eta}(d\sigma_\eta),$$

where

$$\chi_{\eta}(d\sigma_\eta) := \bigotimes_{x \in \eta} \chi_x(\sigma_\eta(x)),$$

and
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\[ f \in L^\infty(S^\gamma) \quad (=: \text{the set of bounded Borel function on } S^\gamma) \] and

\[ Z(\xi) = \int_{S^\gamma} \exp \left[ -E_\eta(\sigma|\xi) \right] \chi_\eta(d\sigma) \]

is a normalizing factor. Observe that the integral in the right-hand side of (3.4) is well-defined in view of (3.3). For each fixed \( \xi \in S^\gamma \), \( \Pi_\eta(d\sigma|\xi) \) is a probability measure on \( S^\gamma \) and, for each fixed \( B \in \mathcal{B}(S^\gamma) \), the map \( S^\gamma \ni \xi \mapsto \Pi_\eta(\sigma|\xi) \in [0,1] \) is measurable. The family \( \Pi_\gamma := \{ \Pi_\eta(d\sigma|\xi) \}_{\eta \in \mathcal{F}(\gamma), \xi \in S^\gamma} \) is called the Gibbsian specification (see e.g. [9, 25]). By construction, it satisfies the consistency property

\[ \int_{S^\gamma} \Pi_\eta(B|\sigma) \Pi_\eta(d\sigma|\xi) = \Pi_\eta(B|\xi), \]

which holds for any \( B \in \mathcal{B}(S^\gamma) \), \( \xi \in S^\gamma \) and \( \eta_1, \eta_2 \in \mathcal{F}(\gamma) \) such that \( \eta_1 \subset \eta_2 \).

A probability measure \( \nu \) on \( S^\gamma \) is said to be a Gibbs measure associated with the potentials \( W \) and \( V \) if it satisfies the DLR equation

\[ \nu(B) = \int_{S^\gamma} \Pi_\eta(B|\xi) \nu(d\xi), \quad B \in \mathcal{B}(S^\gamma), \]

for all \( \eta \in \mathcal{F}(\gamma) \). Equivalently, one can fix an exhausting sequence \( (\Lambda_N) \) of compact sets in \( X \) and require (3.7) only for \( \eta = \gamma \cap \Lambda_N, N \in \mathbb{N} \). For a given \( \gamma \in \Gamma(X) \), by \( \mathcal{G}(S^\gamma) \) we denote the set of all such measures.

Our next goal is to prove the existence of Gibbs measures supported on certain sets of tempered sequences from \( S^\gamma \) for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \). Let us assume that the measure \( \mu \) satisfies (2.2) with an integer \( M \) (cf. Proposition 2.2) such that

\[ M > \frac{2q}{q-2} > 2, \]

where \( q \) is the same as in (3.3). Fix a parameter

\[ p \in \left[ \frac{2M}{M-2}, q \right] \]

and set

\[ p' := 2(p-2)^{-1}, \]

so that

\[ \frac{2}{q-2} \leq p' \leq M/2 - 1. \]

Thus, according to Proposition 2.2, \( a_{\alpha, p'}, b_\alpha \in L^1(\Gamma(X), \mu) \) for any \( \alpha > 0 \), and thus

\[ a_{\alpha, p'}(\gamma), \ b_\alpha(\gamma) < \infty \]

for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \).

For \( \sigma \in S^\gamma \), we define the norm

\[ \|\sigma\|_{\alpha, p} := \left( \sum_{x \in \gamma} |\sigma(x)|^p w_\alpha(x) \right)^{1/p} \]

and consider the Banach space

\[ l^p_\alpha(\gamma, S) := \left\{ \sigma \in S^\gamma : \|\sigma\|_{\alpha, p} < \infty \right\}. \]
By $\mathcal{G}_{\alpha,p}(S^\gamma) \subset \mathcal{G}(S^\gamma)$ we denote the set of all Gibbs measures on $\gamma$ associated with $W$ and $V$, which are supported on $I_{\alpha}^p(\gamma, S)$. These measures are called tempered.

**Theorem 3.1.** Assume that conditions (3.8) and (3.9) are satisfied. Then the following statements hold for $\mu$-a.a. $\gamma \in \Gamma(X)$:
1) the set $\mathcal{G}_{\alpha,p}(S^\gamma)$ is not empty;
2) for any $\lambda \in \mathbb{R}_+$, there exists a constant $\Xi_\gamma(\lambda) > 0$ such that every $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$ satisfies the estimate

$$\int_{S^\gamma} \exp \left\{ \lambda \|\sigma\|_{\alpha,p}^p \right\} \nu(d\sigma) \leq \exp \Xi_\gamma(\lambda).$$

**Proof.** For $\gamma \in \Gamma(X)$ satisfying (3.10), statements 1) and 2) follow by the direct application of Theorem 1 of [13] to the graph $(\gamma, \mathcal{E}_\gamma)$. The key technical step is to establish the exponential moment bound

$$\sup_{\eta \in \mathcal{F}(\gamma)} \int_{S^\gamma} e^{\lambda \|\sigma\|_{\alpha,p}^p} \Pi_{\eta}(d\sigma | \xi) < \infty, \xi \in \ell_{\alpha}^p(\gamma, S),$$

which holds uniformly in $\eta \in \mathcal{F}(\gamma)$ and implies the local equicontinuity of the family

$$\{\Pi_{\eta}(d\sigma | \xi)\}_{\eta \in \mathcal{F}(\gamma)}$$

for any $\xi \in \ell_{\alpha}^p(\gamma, S)$ (cf. Definition 4.6 in [9]) and hence its relative compactness in the topology of set-wise convergence on the algebra $\mathcal{B}_0(S^\gamma) := \cup_{\eta \in \mathcal{F}(\gamma)} \mathcal{B}_\eta(S^\gamma)$ of local subsets of $S^\gamma$. Here $\mathcal{B}_\eta(S^\gamma)$ is the $\sigma$-algebra of sets $C_A = \{\sigma \in \mathcal{B}(S^\gamma) : \sigma_\eta \in A\}, A \in \mathcal{B}(S^\eta)$, which is isomorphic to $\mathcal{B}(S^\eta)$.

This ensures the existence of accumulation points $\nu^\xi \in \mathcal{P}(S^\gamma)$. Standard limit transition arguments show that $\nu^\xi \in \mathcal{G}_{\alpha,p}(S^\gamma)$, and that estimate (3.12) holds for all $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$.

The only additional features of the present framework (in comparison to [13]) are the multidimensionality of the spin space $S$ and the dependence of the potentials $W_{xy}$ on $x, y \in \gamma$, which however does not affect the proof in view of the uniformity of the estimates in (3.11). □

The next important auxiliary statement is a byproduct of the proof of Lemma 1 in [13].

**Proposition 3.2.** For each $\lambda > 0$, $\beta \in (0, e^{\alpha R} \lambda/2)$ and $p' = 2(p - 2)^{-1}$, there exist constants $C_1, C_2, C_3 \geq 0$ such that the following estimate holds:

$$\int_{S^\gamma} \|\sigma\|_{\alpha,p}^p \Pi_{\eta}(d\sigma | \xi) \leq C_1 b_\alpha(\gamma) + C_2 a_{\alpha,p'}(\gamma) + C_3 \|\xi_\gamma\|_{\alpha,p}',$$

uniformly for all $\eta \in \mathcal{F}(\gamma)$. Moreover,

$$\int_{S^\gamma} \|\sigma\|_{\alpha,p}^p \nu(d\sigma) \leq C_1 b_\alpha(\gamma) + C_2 a_{\alpha,p'}(\gamma),$$

holding for any $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$.

**Proof.** The application of Jensen’s inequality to the right-hand side of formula (3.6) of [13] together with (3.18) of the same work implies (3.13). Bound (3.14) can be proved by limit transition arguments combined with the DLR equation, similar to the proof of (3.12). □
Remark 3.3. Condition (\ref{3.8}) establishes a relation between the growth rate \( q \) of \( V \) and the number \( M \) related to the correlation functions \( k_m \), \( 1 \leq m \leq M \), of the underlying random point process \( \mu \). In the case where \( \mu \) has bounded correlation functions of arbitrary order, condition (\ref{3.8}) holds for any \( q > 2 \) and \( p \in (2, q] \). Observe that higher values of \( p \) guarantee the existence of Gibbs measures with the smaller support set \( \mathcal{P}_\alpha(\gamma, S) \). Unfortunately, our method does not allow us to control the case of \( q = 2 \), even when the underlying particle configuration \( \gamma \) is distributed according to the homogeneous Poisson random field \( \mu = \pi_x \) on \( \Gamma(X) \). In particular, the existence problem is still open for the important class of ferromagnetic harmonic systems on \( S^\gamma \) with the pair interactions of the form (\ref{3.2}) with \( A(x − y) ≤ 0 \) and \( V(u) = a_V |u|^2 \), \( a_V > 0 \).

Remark 3.4. As already mentioned in the Introduction, in this paper we do not touch the question of uniqueness of \( \nu \in \mathcal{G}(S^\gamma) \). This is a highly nontrivial problem and general conditions that guarantee that \( N(\mathcal{G}(S^\gamma)) = 1 \) are not known (even for small interaction strength). On the other hand, in \cite{6} we studied a class of models with ferromagnetic pair interaction living on Poisson random graphs and showed the existence of multiple Gibbs states, that is, that \( N(\mathcal{G}(S^\gamma)) > 1 \) (and therefore \( = \infty \)) for a.a. \( \gamma \in \Gamma(X) \).

4. Measurable dependence on \( \gamma \).

In the proof of Theorem 3.1 a measure \( \nu_\xi^\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma) \) has been constructed for each tempered \( \xi \in \Gamma(X) \) as a limit of a sequence of ‘finite volume’ measures \( \Pi_{m_n}(d\sigma|\xi), n \in \mathbb{N} \). However, the measurability of the map \( \Gamma(X) \ni \gamma \mapsto \nu_\xi^\gamma \) is far from being clear. Indeed, the sequence \( \eta = (\eta_n)_{n \in \mathbb{N}} \subset \mathcal{F}(\gamma) \) can depend on the random parameter \( \gamma \) in some uncontrollable way (the so-called chaotic size dependence, see the discussion in \cite{22, 23}). In this section we address this problem. A difficulty here is that, for different \( \gamma \), the measures \( \nu_\alpha \in \mathcal{G}_{\alpha,p}(S^\gamma) \) are defined on different spaces, and it is not clear in what sense this measurability can be understood. To overcome this difficulty we will identify the spaces \( S^\gamma \) that support measures \( \nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma) \) with the measurable subspaces of the marked configuration space

\begin{equation}
\Gamma(X, S) := \{ \widehat{\gamma} \in \Gamma(X \times S) : p_X(\widehat{\gamma}) \in \Gamma(X) \},
\end{equation}

where \( p_X \) is the natural extension to \( \Gamma(X \times S) \) of the canonical projection \( X \times S → S \). For basic definitions and properties of marked configuration spaces we refer to e.g. \cite{1, 0, 2, 9}.

In order to proceed, we endow the space \( \Gamma(X, S) \) with a (completely metrizable) topology defined as the weakest topology that makes the map \( \Gamma(X, S) \ni \widehat{\gamma} \mapsto \langle f, \widehat{\gamma} \rangle \) continuous for any \( f \in C_{b,0}(X \times S) \) (\( =: \) the set of continuous bounded functions on \( X \times S \) with support \( S_\Lambda := \Lambda \times S, \Lambda \in \mathcal{B}_0(X) \)). Let \( \mathcal{B}(\Gamma(X, S)) \) be the corresponding Borel \( \sigma \)-algebra. The space of \( \mathcal{B}(\Gamma(X, S)) \) measurable bounded functions \( f : \Gamma(X, S) → \mathbb{R} \) will be denoted by \( L^\infty(\Gamma(X, S)) \).

The space \( \Gamma(X, S) \) has the structure of a fibre bundle over \( \Gamma(X) \), with fibres \( p_X^{-1}(\gamma) \), which can be identified with the product spaces \( S^\gamma \). As before, elements of \( S^\gamma \) will be denoted by \( \sigma \equiv \sigma_\gamma = (\sigma(x))_{x \in \gamma} \), with the subscript
\( \gamma \) omitted when possible. Thus each \( \hat{\gamma} \in \Gamma(X, S) \) can be represented by the pair

\[
\hat{\gamma} = (\gamma, \sigma), \quad \text{where} \quad \gamma = p_X(\hat{\gamma}) \in \Gamma(X), \quad \sigma \in S^\gamma.
\]

It follows directly from the definition of the corresponding topologies that the map \( p_X: \Gamma(X, S) \to \Gamma(X) \) is continuous, which implies that the space \( S^\gamma \) is a Borel subset of \( \Gamma(X, S) \) for any configuration \( \gamma \in \Gamma(X) \). Moreover, \( S^\gamma \) is a Polish space embedded into \( \Gamma(X, S) \), which is a Polish space as well. By the Kuratowski theorem \([24,\ page\ 21]\), the latter implies that the Borel \( \sigma \)-algebras \( \mathcal{B}(S^\gamma) \)

\[
\mathcal{A}(\mathbb{R}^\gamma) := \{ A \in \mathcal{B}(\Gamma(X, S)) : A \subset S^\gamma \}
\]

are measurably isomorphic. Thus, any probability measure \( \mu \) on \( \mathcal{B}(\Gamma(X, S)) \) with the property \( \mu(S^\gamma) = 1 \) can be redefined as a measure on \( \mathcal{B}(S^\gamma) \), for which we will you the same notation.

Let \( \mathcal{P}(\Gamma(X, S)) \) stand for the space of all Borel probability measures on \( \Gamma(X, S) \). We say that a map \( \Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{P}(\Gamma(X, S)) \) is measurable if the map \( \Gamma(X) \ni \gamma \mapsto \nu_\gamma(A) \in \mathbb{R} \) is measurable for all \( A \in \mathcal{B}(S^\gamma) \), which in turn is equivalent to the measurability of the map \( \Gamma(X) \ni \gamma \mapsto \int_{\Gamma(X, S)} f(\hat{\gamma}) \nu_\gamma(d\hat{\gamma}) \in \mathbb{R} \) for all \( f \in L^\infty(\Gamma(X, S)) \).

The following theorem is the main result of this section.

**Theorem 4.1.** There exists a measurable mapping

\[
\Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{P}(\Gamma(X, S))
\]

such that \( \nu_\gamma(S^\gamma) = 1 \) and \( \nu_\gamma \in \mathcal{G}_{\alpha, p}(S^\gamma) \) for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \).

The proof will go along the following lines. First, using moment bounds \([5, 13]\), we will construct an auxiliary measure \( \hat{\nu} \) on \( \Gamma(X, S) \) and define its conditional distribution (i.e. disintegration) \( (\nu_\gamma)_{\gamma \in \Gamma(X)} \subset \mathcal{P}(S^\gamma) \) with respect to \( \mu \), so that the measurability required in \((4.2)\) holds. Then we will prove the inclusion \( \nu_\gamma \in \mathcal{G}_{\alpha, p}(S^\gamma) \).

Let us fix a measurable mapping \( u : X \to S \) satisfying the bound \( |u(x)| \leq c e^{\beta|x|} \) for some \( c, \beta \in \mathbb{R} \) and define the map

\[
\Gamma(X) \ni \gamma \mapsto \xi_\gamma = (u(\gamma))_{\gamma \in \gamma} \in S^\gamma.
\]

Obviously, we have the estimate

\[
\|\xi_\gamma\|_{\alpha, p}^\beta \leq b_{\alpha'}(\gamma), \quad \text{for any} \quad \alpha' > p\beta,
\]

so that \( \xi_\gamma \in l^\beta_{\alpha'}(\gamma, S) \) for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \).

**Proposition 4.2.** Let \( \xi_\gamma \) be as in \((4.3)\). Then the map

\[
\Gamma(X) \ni \gamma \mapsto (\gamma, \xi_\gamma) \in \Gamma(X, S)
\]

is measurable.

**Proof.** By the definition of the measurable structure of \( \Gamma(X, S) \), the claim is equivalent to the measurability of the maps \( \Gamma(x) \ni \gamma \mapsto F(\gamma) := \langle f, \hat{\gamma} \rangle \), \( \hat{\gamma} = (\gamma, \xi_\gamma) \), for all \( f \in C_{b, 0}(X \times S) \). It is clear that \( F(\gamma) = \langle g, \gamma \rangle \), where \( g(x) = f(x, u(x)) \), so that \( g \) is measurable and has compact support. The assertion follows now from the definition of the measurable structure of \( \Gamma(X) \). \( \square \)
Let us fix $\Delta \in B_0(X)$ (e.g. a closed ball or cube) and define $\gamma_\Delta := \Delta \cap \gamma$, $\gamma \in \Gamma(X)$. Obviously, $\gamma_\Delta \in \mathcal{F}(\gamma)$. Consider the measure $\nu_\Delta^\xi$ on $\Gamma(X, S)$ defined by the formula

$$\nu_\Delta^\xi(d\gamma) = \Pi_{\gamma_\Delta}(d\sigma | \xi_\Delta)\mu(d\gamma), \quad \gamma = p_X(\hat{\gamma}),$$

or, equivalently,

$$\int_{\Gamma(X, S)} F(\hat{\gamma})\nu_\Delta^\xi(d\hat{\gamma}) = \int_{\Gamma(X)} \Phi_F(\gamma)\mu(d\gamma)$$

holding for each $F \in C_b(\Gamma(X, S))$, where

$$\Phi_F(\gamma) := \int_{S^\gamma} F(\gamma, \sigma) \Pi_{\gamma_\Delta}(d\sigma | \xi_\Delta).$$

The measure $\nu_\Delta^\xi$ is well-defined because of the next result.

**Proposition 4.3.** The function $\Phi_F : \Gamma(X) \to \mathbb{R}$ is measurable.

**Proof.** For any $A, B \subset X$, introduce the function $W_{A,B}(x \times u, y \times v) = 1_A(x)1_B(y)W_{xy}(u, v)$, $x, y \in X, u, v \in S$, and set $\hat{W}_\Delta(\hat{\gamma}) := \sum_{\{\hat{x}, \hat{y}\} \subset \hat{\gamma}} (W_{\Delta \times \Delta} + W_{\Delta \times c\Delta}) (\hat{x}, \hat{y})$. Observe that $\hat{W}_\Delta : \Gamma(X, S) \to \mathbb{R}$ is measurable. For $\hat{\gamma} = (\gamma, \sigma_\Delta \times \xi_\Delta)$ we have the equality $E_{\gamma_\Delta}(\sigma | \xi_\Delta) = \hat{W}_\Delta(\hat{\gamma})$, which implies the measurability of the map $\gamma \mapsto E_{\gamma_\Delta}(\sigma | \xi_\Delta)$. It follows from (3.3) and (3.4) that

$$\Phi_F(\gamma) = Z^{-1} \int F(\hat{\gamma}) \exp \left( -E_{\gamma_\Delta}(\sigma | \xi_\Delta) \right) \chi_{\gamma_\Delta}(d\sigma | \xi_\Delta),$$

where $Z = \int \exp \left( -E_{\gamma_\Delta}(\sigma | \xi_\Delta) \right) \chi_{\gamma_\Delta}(d\sigma | \xi_\Delta)$. Without loss of generality we can assume that $\chi_\sigma$ is a probability measure. It was proved in [3] (2.18) and Appendix A] that the map

$$\Gamma(X) \ni \gamma \mapsto \int G(\gamma, \sigma) \chi_{\gamma}(d\sigma) \in \mathbb{R}$$

is measurable for any measurable function $G : \Gamma(X, S) \to \mathbb{R}$. The result follows now from the measurability of the map $\Gamma(X) \ni \gamma \mapsto \gamma_\Delta \in \Gamma(\Delta)$.

Let us consider the algebra $B_0(\Gamma(X, S)) := \cup_{A \in B_0(X)} B_A(\Gamma(X, S))$ of local subsets of $\Gamma(X, S)$. Here $B_A(\Gamma(X, S))$ is the $\sigma$-algebra of sets $C_A := \{ \hat{\gamma} \in \Gamma(X, S) : \hat{\gamma} \cap S_A \in A \}, A \in B(\Gamma(\Lambda, S))$, which is isomorphic to $B(\Gamma(\Lambda, S))$. The space of $B_0(\Gamma(X, S))$ measurable bounded functions $f : \Gamma(X, S) \to \mathbb{R}$ will be denoted by $L_{0 \infty}^\sigma(\Gamma(X, S))$.

Our next goal is to show that the family $\{ \nu_\Delta^\xi, \Delta \in B_0(X) \} \subset \mathcal{P}(\Gamma(X, S))$ has an accumulation point. For this, we equip the space $\mathcal{P}(\Gamma(X, S))$ with the topology $\tau_{loc}$ of local setwise convergence (cf. [9, Sec. 4.1, Prop. 4.9]). This is the weakest topology that makes the maps $\mathcal{P}(\Gamma(X, S)) \ni \mu \mapsto \mu(B) \in \mathbb{R}$ (resp. $\mathcal{P}(\Gamma(X, S)) \ni \mu \mapsto \int f(\hat{\gamma})\mu(d\hat{\gamma}) \in \mathbb{R}$) continuous for all $B \in B_0(\Gamma(X, S))$ (resp. all $B_0(\Gamma(X, S))$-measurable bounded functions $f : \Gamma(X, S) \to \mathbb{R}$), so that $\mathcal{P}(\Gamma(X, S)) \ni \mu_n \xrightarrow{\tau_{loc}} \mu \in \mathcal{P}(\Gamma(X, S))$, $n \to \infty$, if $\mu_n(B) \to \mu(B)$ for all $B \in B_0(\Gamma(X, S))$. 

Definition 4.4. We say that a family of probability measures \( \{ \mu_m \}_{m \in \mathbb{N}} \) on \( \Gamma(X, S) \) is locally equicontinuous (LEC) if for any \( \Delta \in \mathcal{B}_0(X) \) and any sequence \( \{ \mathcal{B}_n \}_{n \in \mathbb{N}} \in \mathcal{B}(\Gamma(\Delta, S)) \), \( \mathcal{B}_n \downarrow \emptyset, n \to \infty \), we have
\[
\lim_{n \to \infty} \limsup_{m \in \mathbb{N}} \mu_m(\mathcal{B}_n) = 0.
\] (4.6)

Observe that the local setwise convergence is equivalent to convergence in the space \([0, 1]^{\mathcal{B}_0} \), where \( \mathcal{B}_0 := \mathcal{B}_0(\Gamma(X, S)) \). The following fact is essentially well-known, see [9, Prop. 4.9].

Proposition 4.5. Let \( \{ \mu_n \}_{n \in \mathbb{N}} \) be a LEC family of probability measures on \( \Gamma(X, S) \). Then it has a \( \tau_{loc} \)-cluster point, which is also a probability measure on \( \Gamma(X, S) \).

Proof. We give here a sketch of the proof from [9, Prop. 4.9] adapted to our setting. It is straightforward that the family \( \{ \mu_n \}_{n \in \mathbb{N}} \) contains a cluster point \( \mu \) as an element of the compact space \([0, 1]^{\mathcal{B}_0} \), and \( \mu \) is an additive function on \( \mathcal{B}_0 \). The LEC property (1.10) implies that the projection \( \mu_\Lambda \) of \( \mu \) onto \( \mathcal{B}(\Gamma(\Lambda, S)) \) is \( \sigma \)-additive for each \( \Lambda \in \mathcal{B}_0(X) \). Thus \( \{ \mu_\Lambda \}_{\Lambda \in \mathcal{B}_0(X)} \) forms a consistent (w.r.t. projective maps \( p_{\Lambda_2, \Lambda_1} : \Gamma(\Lambda_2, S) \ni \gamma_1 \mapsto \gamma_{\Lambda_1} := (\gamma_{\Lambda_1}, \sigma_{\Lambda_1}) \in \Gamma(\Lambda_1, S), \Lambda_1 \subset \Lambda_2 \) family of measures and by the corresponding version of the Kolmogorov theorem (see [24, Theorem V.3.2]) generates a probability measure on \( \Gamma(X, S) \) (which obviously coincides with \( \mu \)). \( \Box \)

Corollary 4.6. There exists a subsequence \( \{ \mu_{n_k} \}_{k \in \mathbb{N}} \) such that \( \mu_{n_k} \to \mu \), \( k \to \infty \).

Let us consider the function
\[
\phi(\hat{\gamma}) := b_\alpha(\gamma) + \|\sigma_\gamma\|_{\alpha,p}^p,
\]
with \( b_\alpha(\gamma) \) and \( \|\sigma_\gamma\|_{\alpha,p} \) given by formulae (2.3) and (3.11), respectively. Using estimate (3.13) we obtain the inequality
\[
\int_{\Gamma(X,S)} \phi(\hat{\gamma}) \mathcal{A}_\Delta^p(d\hat{\gamma}) \leq \int_{\Gamma(X)} \left[ (C_1 + 1)b_\alpha(\gamma) + C_2 a_{\alpha,p}(\gamma) + C_3 \|\xi_\Delta\|_{\alpha,p}^p \right] \mu(d\gamma).
\]
It follows now from estimate (4.4) and Proposition 2.2 that
\[
\int_{\Gamma(X,S)} \phi(\hat{\gamma}) \mathcal{A}_\Delta^p(d\hat{\gamma}) \leq C
\]
for some constant \( C \in \mathbb{R} \) and all \( \Delta \in \mathcal{B}_0(X) \).

Define the set
\[
\hat{\Gamma}_T := \{ \hat{\gamma} \in \Gamma(X, S) : \phi(\hat{\gamma}) \leq T \}, \ T > 0.
\]
and observe that for any set \( \Lambda \in \mathcal{B}_0(X) \), there exists a constant \( c_\Lambda \) such that
\[
N(\hat{\gamma}_\Lambda) \leq c_\Lambda T, \ \hat{\gamma} \in \hat{\Gamma}_T, \ T > 0.
\]

Consider the family of measures \( \hat{\mathcal{P}}_m(d\hat{\gamma}) := \mathcal{A}_m^\xi(d\hat{\gamma}), \ m \in \mathbb{N}, \) where \( \{ \Lambda_m \}_{m \in \mathbb{N}} \subset \mathcal{B}_0(X) \) is an increasing sequence exhausting \( X \) and \( \mathcal{A}_m^\xi(d\hat{\gamma}) \) are defined by formula (4.5).

Proposition 4.7. The family \( \{ \hat{\mathcal{P}}_m \}_{m \in \mathbb{N}} \) is LEC.
First, we fix $T > (4.8)$ holding for each $m$ we obtain the bounds

$$\sum_{\{x,y\} \subset \gamma} \sum_{\phi(x,y) \in \Gamma} (|\sigma(x)|r + |\sigma(y)|r) + J_W \frac{N(\gamma \Delta} \leq \frac{N(\gamma \Delta) - 1}{2}$$

for any $m \geq m_0$ and $n \geq n_0$. The following estimate follows from (4.1) by an easy calculation:

$$E_{\gamma \Delta} (|\sigma| \zeta) = \sum_{\{x,y\} \subset \gamma} W_{x,y} (\sigma(x), \sigma(y)) + \sum_{x \in \gamma \Delta} \sum_{y \in \gamma \Delta'} (|\sigma(x)|r + |\sigma(y)|r) + J_W N(\gamma \Delta \cap \Delta_{R})$$

$$\leq I_W \sum_{\{x,y\} \subset \gamma} \sum_{\phi(x,y) \in \Gamma} (|\sigma(x)|r + |\sigma(y)|r) + J_W N(\gamma \Delta) N(\gamma \Delta' \cap \Delta_R)$$

$$\leq I_W \left( (2N(\gamma \Delta) + N(\gamma \Delta' \cap \Delta_R)) \sum_{x \in \gamma \Delta} |\sigma(x)|r + N(\gamma \Delta) \sum_{y \in \gamma \Delta' \cap \Delta_R} |\zeta(y)|r \right)$$

$$+ J_W (N(\gamma \Delta)^2 + N(\gamma \Delta) N(\gamma \Delta' \cap \Delta_R))$$

First, we fix $T > 0$ and estimate the corresponding measures of the sets $B_n \cap \Gamma_T$ and $B_n \cap \left( \hat{\Gamma}_T \right)^c$. Taking into account that $|u|^r \leq |u|^p + 1$ for $r < p$, we obtain the bounds:

$$\sum_{x \in \gamma \Delta} |\sigma(x)|r \leq \phi(\hat{\gamma}) + N(\gamma \Delta), \sum_{y \in \gamma \Delta' \cap \Delta_R} |\zeta(y)|r \leq \varphi(\hat{\gamma}) + N(\gamma \Delta' \cap \Delta_R),$$

where $\hat{\gamma} = (\gamma, \sigma_{\gamma \Delta} \times \zeta_{\gamma \Delta'})$. Thus for $\hat{\gamma} \in \hat{\Gamma}_T$ we have:

$$\sum_{x \in \gamma \Delta} |\sigma(x)|r \leq cT, \sum_{y \in \gamma \Delta' \cap \Delta_R} |\zeta(y)|r \leq cT$$

for some constant $c$, which in turn implies that

$$1_{B_n \cap \Gamma_T} (\hat{\gamma}) |E_{\gamma \Delta} (|\sigma| \zeta|) \leq I_W (3cT^2 + cT^2) + 2T^2 J_W.$$

Thus there exists a constant $a(T)$ such that

$$1_{B_n \cap \Gamma_T} (\hat{\gamma}) \exp (-E_{\gamma \Delta} (|\sigma| \zeta)) \leq a(T)$$

and

$$1_{B_n \cap \Gamma_T} (\hat{\gamma}) Z_{\gamma \Delta}^{-1} (\zeta) \leq \exp \left\{ \int_{\gamma \Delta} 1_{B_n \cap \Gamma_T} (\hat{\gamma}) E_{\gamma \Delta} (|\sigma| \zeta) \chi_{\gamma \Delta} (d\sigma) \right\}$$

$$\leq a(T)$$

for all $\hat{\gamma} \in \Gamma(X,S)$ and $n \in \mathbb{N}$.

By Chebyshev’s inequality applied to measure $\hat{\Pi}_m$ on $\Gamma(X,S)$ we have

$$\hat{\Pi}_m \left( \{ \hat{\gamma} \in \Gamma(X,S) : \varphi(\hat{\gamma}) \geq T \} \right) \leq T^{-2} \int_{\Gamma(X,S)} |\varphi(\hat{\gamma})|^2 \hat{\Pi}_m (d\hat{\gamma})$$

holding for each $T > 0$, which together with (4.7) shows yields

(4.8) \hspace{1cm} \hat{\Pi}_m \left( \left( \hat{\Gamma}_T \right)^c \right) \leq \varepsilon,
holding for each \( \varepsilon > 0 \) and \( T \) bigger than some \( T(\varepsilon, \hat{\zeta}) \). On the other hand,

\[
\hat{\Pi}_m \left( B_n \cap \hat{\Gamma}_T \right) = \int_{\Gamma(X,S)} 1_{B_n \cap \hat{\Gamma}_T} (\hat{\gamma}) \hat{\Pi}_m (d\hat{\gamma}) = \int_{\Gamma(X)} I(\gamma) \mu(d\gamma),
\]

where

\[
I(\gamma) := \int_{S^7} 1_{B_n \cap \hat{\Gamma}_T} (\gamma, \sigma) \Pi_{\gamma \Lambda_m} (d\sigma | \xi_\gamma).
\]

Observe that there exists \( m_0 \) such that \( \Lambda_m \supset \Delta \) for \( m \geq m_0 \). For all such \( m \), it follows from consistency property (3.6) that

\[
\hat{\Pi}_m \left( B_n \cap \hat{\Gamma}_T \right) \leq a(\Delta, T)^2 \int_{\Gamma(X)} \int_{S^7} 1_{B_n \cap \hat{\Gamma}_T} (\gamma, \sigma \gamma \Delta \times \xi_{\gamma \Delta} \varepsilon) \chi_{\gamma \Delta} (d\sigma) \mu(d\hat{\gamma}) < \varepsilon
\]

for \( n \) greater than some \( n(\varepsilon, T) \) since \( B_n \to \emptyset, n \to \infty \). Combining this with estimate (11.8) we can see that \( \forall \varepsilon > 0 \) and \( m \geq m_0, n \geq n_0 = n(\varepsilon/2, T(\varepsilon/2)) \) we have

\[
\hat{\Pi}_m (B_n) = \hat{\Pi}_m \left( B_n \cap \left( \hat{\Gamma}_T \right)^{\varepsilon} \right) + \hat{\Pi}_m \left( B_n \cap \hat{\Gamma}_T \right)
\]

\[
\leq \hat{\Pi}_m \left( \left( \hat{\Gamma}_T \right)^{\varepsilon} \right) + \int_{\Gamma(X,S)} 1_{B_n \cap \hat{\Gamma}_T} (\hat{\gamma}) \hat{\Pi}_m (d\hat{\gamma}) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

The proof is complete. \( \square \)

**Corollary 4.8.** The family of measures \( \left\{ \hat{\nu}_{\Delta_n}^\varepsilon, \Delta \in \mathcal{B}_0(X) \right\} \) contains a sequence \( \hat{\nu}_{\Delta_n}^\varepsilon \), \( n \in \mathbb{N} \), which \( \tau_{\text{loc}} \)-converges to a probability measure \( \hat{\nu}^\varepsilon \) on \( \Gamma(X,S) \). Without loss of generality we can assume that the sequence of sets \( \Delta_n \) is increasing and exhausts \( X \).

Let \( \nu_X^\varepsilon \) be the projection of the measure \( \hat{\nu}^\varepsilon \) onto \( \Gamma(X) \) and \( F \in L_0^\infty(\Gamma(X)) \). For the function \( \hat{F} := F \circ p_X \in L_0^\infty(\Gamma(X,S)) \) we have

\[
\int_{\Gamma(X,S)} \hat{F}(\hat{\gamma}) \hat{\nu}_{\Delta_n}^\varepsilon (d\hat{\gamma}) \to \int_{\Gamma(X)} F(\gamma) \nu_X^\varepsilon (d\gamma), n \to \infty.
\]

On the other hand, taking the limit in

\[
\int_{\Gamma(X,S)} \hat{F}(\hat{\gamma}) \hat{\nu}_{\Delta_n}^\varepsilon (d\hat{\gamma}) = \int_{\Gamma(X)} F(\gamma) \left( \int_{S^7} 1_{\gamma \Delta_n} (d\sigma | \xi_\gamma) \right) \mu(d\gamma) = \int_{\Gamma(X)} F(\gamma) \mu(d\gamma)
\]
we get
\[ \int_{\Gamma(X,S)} \hat{F}(\gamma) \nu^\xi(d\gamma) = \int_{\Gamma(X)} F(\gamma) \mu(d\gamma), \]
which in turn implies
\[ \mu = \nu_X. \]

The application of Theorem 8.1 of [24] to the measurable map \( p_X : \Gamma(X,S) \to \Gamma(X) \) yields the existence of the corresponding regular conditional probability distribution \( \nu^\xi_\gamma, \gamma \in \Gamma(X) \), that is, a family of probability measures \( \nu^\xi_\gamma \) on \( \Gamma(X,S) \) such that for any measurable set \( A \subset \Gamma(X,S) \), it follows that
\[ \hat{\nu}^\xi(A) = \int_{\Gamma(X)} \nu^\xi_\gamma(A) \mu(d\gamma), \]
and the map
\[ \Gamma(X) \ni \gamma \mapsto \nu^\xi_\gamma(A) \]
is measurable. Moreover, \( \nu^\xi_\gamma[\Gamma(X,S) \setminus p_X(\gamma)] = 0 \) for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \). Thus \( \nu^\xi_\gamma \) generates (for \( \mu \)-a.a. \( \gamma \in \Gamma(X) \)) a probability measure on \( p_X(\gamma) = S^\gamma \) (for which we preserve the notation \( \nu^\xi_\gamma \)) such that the map
\[ \Gamma(X) : \gamma \mapsto \int_{S^\gamma} F(\gamma,\sigma) \nu^\xi_\gamma(d\sigma) \]
is measurable for any \( F \in L^\infty_0(\Gamma(X,S)) \) and
\[ \int_{\Gamma(X,S)} F(\gamma) \hat{\nu}^\xi(d\gamma) = \int_{\Gamma(X)} \left( \int_{S^\gamma} F(\gamma,\sigma) \nu^\xi_\gamma(d\sigma) \right) \mu(d\gamma). \]

**Proposition 4.9.** For \( \mu \)-a.a. \( \gamma \in \Gamma(X) \) and any \( \xi \) of the form (4.3), we have that
\[ \nu^\xi_\gamma \in \mathcal{G}_{a,p}(S^\gamma). \]

**Proof.** We first prove that \( \nu^\xi_\gamma \) satisfies the DLR equation (3.7). Fix \( \Lambda \subset \mathcal{B}_0(X) \) and \( \Delta_n \) such that \( \Lambda \subset \Delta_n \). According to consistency property (3.6) we have
\[ \int_{S^\gamma} \Pi_{\gamma \Delta_n}(d\sigma | \zeta) \Pi_{\gamma \Delta_n}(d\zeta | \xi) = \Pi_{\gamma \Delta_n}(d\sigma | \xi), \]
so that
\[ \int_{\Gamma(X,S)} g(\gamma,\zeta) \Pi_{\gamma \Delta_n}(d\zeta | \xi) \mu(d\gamma) = \int_{\Gamma(X,S)} G(\gamma,\sigma) \Pi_{\gamma \Delta_n}(d\sigma | \xi) \mu(d\gamma), \]
where \( G \in L^\infty_0(\Gamma(X,S)) \), \( g(\gamma,\zeta) := \int_{S^\gamma} G(\gamma,\sigma) \Pi_{\gamma \Lambda}(d\sigma | \zeta) \) and \( \xi \) is given by (4.3). Observe that \( g \in L^\infty_0(\Gamma(X,S)) \), so that we can pass to the limit as \( n \to \infty \) and obtain the equality
\[ \int_{\Gamma(X,S)} g(\gamma) \hat{\nu}^\xi(d\gamma) = \int_{\Gamma(X,S)} G(\gamma) \hat{\nu}^\xi(d\gamma), \]
or
\[ \int_{\Gamma(X)} \int_{S^\gamma} G(\gamma,\sigma) \Pi_{\gamma \Lambda}(d\sigma | \zeta) \nu^\xi_\gamma(d\zeta) \mu(d\gamma) = \int_{\Gamma(X)} \int_{S^\gamma} G(\gamma,\sigma) \nu^\xi_\gamma(d\sigma) \mu(d\gamma), \]
which in turn implies that
\[ \int_{S^\gamma} \Pi_\gamma (d\sigma | \zeta) \nu_\gamma^\xi (d\zeta) = \nu_\gamma^\xi (d\sigma) \]
for a.a. \( \gamma \). Thus, (3.7) does hold.

In order to prove that \( \nu_\gamma^\xi \) is supported on \( l_p^\alpha (\gamma, S) \) for a.a. \( \gamma \in \Gamma (X) \), we introduce the cut-off
\[ \phi_{L,K} (\hat{\gamma}) := \sum_{|k| \leq K} e^{-\alpha |k|} (N(\gamma_k) \wedge L) + \left( \| \sigma_\gamma \|_{\alpha, p} \wedge L \right), \]

\( K, L \in \mathbb{N} \), and observe that
\[ \int_{\Gamma (X, S)} \phi_{L,K} (\hat{\gamma}) \nu_\gamma^\xi (d\hat{\gamma}) = \lim_{K \to \infty} \lim_{L \to \infty} \int_{\Gamma (X, S)} \phi_{L,K} (\hat{\gamma}) \nu_\gamma^\xi (d\hat{\gamma}). \]
Moreover, \( \phi_{L,K} \in L_0^\infty (\Gamma (X, S)) \), and the limit transition as \( n \to \infty \) together with the estimate (4.7) show that
\[ \int_{\Gamma (X, S)} \phi (\hat{\gamma}) \nu_\gamma^\xi (d\hat{\gamma}) < \infty. \]
Thus
\[ \int_{\Gamma (X)} \int_{S^\gamma} \| \sigma_\gamma \|_{\alpha, p} \nu_\gamma^\xi (d\sigma_\gamma) \mu (d\gamma) \leq \int_{\Gamma (X, S)} \phi (\hat{\gamma}) \nu_\gamma^\xi (d\hat{\gamma}) < \infty, \]
so that \( \int_{S^\gamma} \| \sigma_\gamma \|_{\alpha, p} \nu_\gamma^\xi (d\sigma_\gamma) < \infty \) for a.a. \( \gamma \in \Gamma (X) \), which in turn implies that
\( \nu_\gamma^\xi (l_p^\alpha (\gamma, S)) = 1 \).

**Proof of Theorem 4.1**. The result follows directly from formula (4.9) and Proposition 4.9. \( \square \)

**Remark 4.10**. Let \( \nu_\gamma \in \mathcal{G}_{\alpha, p}(S^\gamma), \gamma \in \Gamma (X) \), be a family of Gibbs measures satisfying the measurability condition (4.2). For \( \mu \)-a.a. \( \gamma \in \Gamma (X) \) the measure \( \nu_\gamma \) obeys the moment estimate (3.11). Integrating both sides of this inequality we obtain
\[ \int_{\Gamma (X)} \left( \int_{S^\gamma} \| \sigma_\gamma \|_{\alpha, p} \nu_\gamma (d\sigma) \right) \mu (d\gamma) \leq C_1 \int_{\Gamma (X)} h_\alpha (\gamma) \mu (d\gamma) + C_2 \int_{\Gamma (X)} a_{\alpha, p'} (\gamma) \mu (d\gamma) < \infty \]
because of Proposition 2.2.

Let us note that the convergence of the measures \( \hat{\nu}_n^\xi, n \in \mathbb{N} \), to \( \hat{\nu}_n^\xi \) does not in general imply the convergence of their conditional distributions \( \Pi_{\gamma_n} (d\sigma | \xi_n) \) to \( \nu_\gamma^\xi (d\sigma) \) for \( \mu \)-a.a. \( \gamma \). However, we can make use of Komlós’ theorem (see e.g. [2]) and prove the following result.

**Proposition 4.11**. There exists a sequence \( (n_j, j \in \mathbb{N}) \subset \mathbb{N} \) such that for \( \mu \)-a.a. \( \gamma \in \Gamma (X) \) and \( \gamma_j := \gamma_{n_j} \) we have the local setwise convergence of measures
\[ \frac{1}{N} \sum_{j=1}^N \Pi_{\gamma_j} (\cdot | \xi_{\gamma_j}) \to \nu_\gamma^\xi, \quad N \to \infty. \]
Diagonal procedure (as in the proof of Theorem 3.6 in [14]) on can show that there exists a sequence $(n, m)$ all the proof.

It is clear that $g^{(m)}_{\Delta_n}(\gamma) := \int_{S^\gamma} f_m(\gamma, \sigma) \Pi_{\Delta_n}(d\sigma|\xi_\gamma), \ n, m \in \mathbb{N}.$

Proof. It has been shown in [6] that there exists a countable family of functions $\{f_m, m \in \mathbb{N}\} \subset L^\infty_0(\Gamma(X, S)),$ which form a separating class for $\mathcal{P}(\Gamma(X, S)).$ That is, for any two measures $\mu, \nu \in \mathcal{P}(\Gamma(X, S))$ the equality $\int f_m d\mu = \int f_m d\nu, \ m \in \mathbb{N},$ implies that $\mu = \nu.$ Consider the family of functions

$$g^{(m)}_{\Delta_n}(\gamma) := \int_{S^\gamma} f_m(\gamma, \sigma) \Pi_{\Delta_n}(d\sigma|\xi_\gamma), \ n, m \in \mathbb{N}.$$ 

It is clear that $g^{(m)}_{\Delta_n} \in L^1(\Gamma(X, \mu)).$ Applying Komlós’ theorem and the diagonal procedure (as in the proof of Theorem 3.6 in [14]) one can show that there exists a sequence $(n_j, j \in \mathbb{N}) \subset \mathbb{N}$ such that for $\mu$-a.a. $\gamma \in \Gamma(X),$ all $m \in \mathbb{N}$ and any subsequence $(n_{j_k}, k \in \mathbb{N})$ we have the Cesaro means convergence

$$\frac{1}{N} \sum_{k=1}^{N} g^{(m)}_{n_{j_k}}(\gamma) \rightarrow g^{(m)}(\gamma), \ N \rightarrow \infty,$$

where $g^{(m)}_{n_{j_k}} := g^{(m)}_{\Delta_{n_{j_k}}}$ and $g^{(m)} \in L^1(\Gamma(X, \mu)).$ Observe that $g^{(m)}$ is independent of the choie of the subsequence $(n_{j_k}, k \in \mathbb{N}).$

Moreover, for $\mu$-a.a. $\gamma \in \Gamma(X)$ the family of measures $\Pi_{\gamma_j}(d\sigma|\xi_\gamma) := \Pi_{\gamma_{\Delta_{n_j}}}(d\sigma|\xi_\gamma), \ j \in \mathbb{N},$ is relatively compact in $\tau_{loc}$ topology (see proof of Theorem 3.1) and thus contains a sequence $\Pi_{\gamma_{j_k}}(d\sigma|\xi_\gamma)$ converging to some measure $\eta_\gamma^\xi(d\sigma)$ on $S^\gamma$. This together with (4.11) implies the equality

$$g^{(m)}(\gamma) = \int_{S^\gamma} f_m(\gamma, \sigma) \eta_\gamma^\xi(d\sigma)$$

for all $m \in \mathbb{N}$ and $\mu$-a.a. $\gamma \in \Gamma(X).$ Integration of both sides of (4.12) shows that

$$\int_{\Gamma(X)} g^{(m)}(\gamma) \mu(d\gamma) = \int_{\Gamma(X, S)} f_m(\gamma, \sigma) \eta_\gamma^\xi(d\sigma) \mu(d\gamma).$$

On the other hand, the convergence $\nu_\gamma^\xi_{\Delta_n} \rightarrow \nu_\gamma^\xi$ together with (4.10) implies that

$$\int_{\Gamma(X, S)} f_m(\gamma, \sigma) \frac{1}{N} \sum_{j=1}^{N} \Pi_{\gamma_j}(d\sigma|\xi_\gamma) \mu(d\gamma) \rightarrow \int_{\Gamma(X, S)} f_m(\gamma, \sigma) \nu_\gamma^\xi(d\sigma) \mu(d\gamma),$$

$N \rightarrow \infty,$ so that

$$\int_{\Gamma(X)} g^{(m)}(\gamma) \mu(d\gamma) = \int_{\Gamma(X, S)} f_m(\gamma, \sigma) \nu_\gamma^\xi(d\sigma) \mu(d\gamma)$$

for all $m \in \mathbb{N}$ and $\mu$-a.a. $\gamma \in \Gamma(X).$ The combination of equalities (4.13) and (4.14) together with the measure separating property of the family $\{f_m, m \in \mathbb{N}\}$ shows that $\nu_\gamma^\xi = \eta_\gamma$ for $\mu$-a.a. $\gamma \in \Gamma(X),$ which completes the proof. □
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