Leapover lengths and first passage time statistics for Lévy flights

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Exact results for the first passage time and leapover statistics of symmetric and one-sided Lévy flights (LFs) are derived. LFs with stable index \( \alpha \) are shown to have leapover lengths, that are asymptotically power-law distributed with index \( \alpha \) for one-sided LFs and, surprisingly, with index \( \alpha/2 \) for symmetric LFs. The first passage time distribution scales like a power-law with index 1/2 as required by the Sparre Andersen theorem for symmetric LFs, whereas one-sided LFs have a narrow distribution of first passage times. The exact analytic results are confirmed by extensive simulations.

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The statistics of first passage times is a classical concept to quantify processes, in which it is of interest when the dynamic variable crosses a given threshold value for the first time, e.g., when a tracer in some aquifer reaches a certain probe position, two molecules meet to form a chemical bond, animals search for sparse food locations, or a share at the stock market crosses a preset market value \( \$1 \). Here, we revisit the first passage time problem for processes with non-trivial jump length distributions, namely, Lévy flights (LFs) and derive exact asymptotic expressions for the first passage time density \( p_f(\tau) \) of symmetric and one-sided LFs. For the former, we obtain the Sparre Andersen universality \( p_f(\tau) \approx \tau^{-3/2} \), while a narrow behavior is found for one-sided LFs. Apart from calculating the first passage times, we investigate the behavior of the first passage leapovers, that is, the distance the random walker overshoots the threshold value \( d \) in a single jump (see Fig. 1). Surprisingly, for symmetric LFs with jump length distribution \( \lambda(x) \approx |x|^{-\alpha} \ (0 < \alpha < 2) \) the distribution of leapover lengths across \( x = d \) is distributed like \( p_l(\ell) \approx \ell^{-1-\alpha/2} \), i.e., it is much broader than the original jump length distribution. In contrast, for one-sided LFs the scaling of \( p_l(\ell) \) bears the same index \( \alpha \).

For processes subject to a narrow jump length distribution with finite second moment \( \int_{-\infty}^{\infty} x^2 \lambda(x) dx \) the crossing of a given threshold value \( d \) is identical to the first arrival at \( x = d \) \( [2] \). This is no longer true for LFs: Intuitively, a particle, whose jump lengths are distributed according to the symmetric long-tailed distribution \( \lambda(x) \approx |x|^{-\alpha} \ (0 < \alpha < 2) \) is likely to criss-cross the point \( x = d \) multiple times before it eventually hits it, causing the first arrival at \( d \) to be slower than its first passage across \( d \) \( [4] \). A measure for the ability to criss-cross \( d \) is the distribution of leapover lengths, \( p_l(\ell) \).

Information on the leapover behavior is therefore important to the understanding of how far proteins searching for their specific binding site along DNA overshoot their target \( [5] \), climatic forcing visible in ice core records exceeds a given value \( [6] \), or to define better stock market strategies determining when to buy or sell a certain share instead of fixing a threshold price \( [7] \). The quantification of leapovers is vital to estimate how far diseases would spread once a carrier of that disease crosses a certain border \( [8] \). Leapover statistics of one-sided LFs provide an interesting alternative interpretation of the distribution of the first waiting time in ageing continuous time random walks \( [9] \), just to name a few examples.

The master equation for a Markovian diffusion process,

\[
\frac{\partial P(x,t)}{\partial t} = \frac{1}{\tau} \int_{-\infty}^{\infty} \left[ \lambda(x-x') P(x',t) - \lambda(x-x) P(x,t) \right] dx' \tag{1}
\]

accounts for the influx of probability to position \( x \), and the outflux away from \( x \), where \( \lambda(x) \) is a general, normalized jump length distribution. The time scale for single jumps is \( \tau \). The solution to Eq. (1) in Fourier space is \( P(k,t) = e^{-[1-\lambda(k)]t/\tau} \), denoting the Fourier transform \( f(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) dx \) by explicit dependence on the wave number \( k \). For instance, for the symmetric jump length distribution \( \lambda(x) \approx \sigma^\alpha |x|^{-1-\alpha} \), one finds

\[
P(k,t) = e^{-K(\alpha)|k|^{\alpha} t} \tag{2}
\]

with \( K(\alpha) = \sigma^\alpha/\tau \), the characteristic function of a symmetric Lévy stable law as obtained from continuous time random walk theory in the diffusion limit or from the equivalent space fractional diffusion equation \( [10] \).

In the following we study processes with the long-tailed composite jump length distribution

\[
\lambda(x)/\tau = \Theta(|x| - \varepsilon) [c_1 \Theta(-x) + c_2 \Theta(x)] / |x|^{1+\alpha}, \tag{3}
\]
where $\Theta(x)$ is the Heaviside function. For $c_1 = c_2$, $\lambda(x)$ defines a symmetric LF, and for $c_1 = 0$ and $c_2 > 0$ a completely asymmetric (one-sided) LF permitting exclusively forward jumps. The cutoff $\varepsilon$ excludes the singularity at $x = 0$, but can be taken to be small, $\varepsilon \to 0$ [11].

In the theory of homogeneous random processes with independent jumps there exists a theorem, which provides an exact expression for the joint PDF $p(\tau, \ell)$ of first passage time $\tau$ and leapover length $\ell$ ($\ell \geq 0$) across $x = d$ for a particle initially seeded at $x = 0$ [12,13]. We here evaluate this theorem, that appears to have been widely overlooked, and derive a number of new analytic results for $p_f(\tau)$ and $p_l(\ell)$ of symmetric and one-sided LFs. With the probability to jump longer than $x$, the theorem states that the double Laplace transform of the joint PDF [12,13]

$$M(x) = \int_x^\infty \lambda(x')dx', \quad x > 0,$$

(4)

the theorem states that the double Laplace transform of $\lambda$ is given in terms of the multiple integral

$$p(u, \mu) = \int_0^\infty \int_0^\infty e^{-u\tau - \mu\ell} p(\tau, \ell) d\tau d\ell$$

(5)

given in terms of the multiple integral

$$p(u, \mu) = 1 - q_+(u, d) - \frac{\mu}{u} \int_0^d \frac{\partial q_+(u, s)}{\partial s} ds \times \int_{-\infty}^0 \frac{\partial q_+(u, s')}{\partial s'} ds' \int_0^\infty e^{-\mu s''} x \cdot M(d + s'' - s') ds''.$$  

(6)

Here, we use the two auxiliary measures $q_{\pm}(u, x)$ defined through Fourier transforms

$$\tilde{q}_\pm(u, k) = \int_{-\infty}^\infty e^{ikx} \frac{\partial q_{\pm}(u, x)}{\partial x} dx = \exp \left\{ \pm \int_0^\infty e^{-ut} t_0^\infty (e^{ikx} - 1) P(x, t)dx dt \right\},$$

(7)

and the condition $q_{\pm}(u, 0) = 0$. They are related to the cumulative distributions of the maximum, $Q_+(t, d) = \Pr \{ \max_{0 \leq \tau \leq t} x(\tau) < d \}$, and minimum, $Q_-(t, d) = \Pr \{ \min_{0 \leq \tau \leq t} x(\tau) < d \}$, of the position $x(t)$ such that $q_{\pm}(u, d) = u \int_0^\infty e^{-ut} Q_{\pm}(t, d) dt$. The complicated integrals above reduce to elegant results for symmetric and one-sided LFs, as we show now.

For symmetric LFs ($c_1 = c_2 = c$), the propagator is defined by the characteristic function [2] with generalized diffusion coefficient $K(\alpha) = 2\alpha \Gamma(1 - \alpha) \cos(\pi \alpha/2)/\alpha$. In the limit $u \to 0$ (long time limit), we obtain from Eq. (7)

$$q_+(u, k) \sim \frac{u^{1/2}}{\sqrt{K(\alpha) k^{1/2}}} \exp \left\{ \frac{\text{sign}(k) \pi \alpha}{4} \right\}.$$  

(8)

Inverse Fourier transform yields

$$q_+(u, d) \sim \frac{2u^{1/2}}{\alpha \sqrt{K(\alpha) \Gamma(\alpha/2)}} d^{\alpha/2}, \quad d > 0$$  

(9)

such that from $p_f(u) = 1 - q_+(u, d)$ we find

$$p_f(\tau) \sim \frac{d^{\alpha/2}}{\alpha \sqrt{\pi K(\alpha) \Gamma(\alpha/2)}} \tau^{-3/2}.$$  

(10)

This is the exact asymptotic first passage time PDF of symmetric LFs. Fig. 2 demonstrates good agreement with simulations results, for which the algorithm from Ref. [14] was used to obtain random numbers distributed according to Lévy stable laws. We note that previously only the $\tau^{-3/2}$ scaling was known from simulations and application of the Sparre Andersen theorem [4].

For symmetric LFs, for $0 < \alpha < 2$ we obtain that

$$M(x) = \frac{K(\alpha)}{2\Gamma(1 - \alpha) \cos(\pi \alpha/2)} x^{-\alpha}, \quad x > 0.$$  

(11)

Using that for symmetric LFs $q_-(\tau, x) = q_+(\tau, -x)$ it turns out after some transformations from Eq. (9) that

$$p_l(\mu) = \int_0^\infty e^{-\mu t} \frac{\sin(\pi \alpha/2)}{\pi} \frac{d^{\alpha/2}}{\ell^{\alpha/2}(d + \ell)} dt,$$  

(12)

from which it follows immediately that

$$p_l(\ell) = \frac{\sin(\pi \alpha/2)}{\pi} \frac{d^{\alpha/2}}{\ell^{\alpha/2}(d + \ell)},$$  

(13)

see Fig. 3. Note that $p_l$ is normalized. In the limit $\alpha \to 2$, $p_l(\ell)$ tends to zero if $\ell \neq 0$ and to infinity at $\ell = 0$ corresponding to the absence of leapovers in the Gaussian continuum limit. However, for $0 < \alpha < 2$ the leapover PDF follows an asymptotic power-law with index $\alpha/2$, and is thus broader than the original jump length PDF $\lambda(x)$ with index $\alpha$. This is a remarkable finding: while $\lambda$ for $1 < \alpha < 2$ has a finite characteristic length $\langle |x| \rangle$, the mean leapover length diverges.
whose series expansion and asymptotic behavior are [10]. Here, we used the definition of the Mittag-Leffler function as
\[E_\alpha(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(1 + \alpha n)},\]
and we change the variable \(ik \rightarrow -s\) to find
\[E_\alpha(-z) = E_\alpha\left[-\frac{u}{K(\alpha)} \cos \left(\frac{\pi \alpha}{2}\right) \right].\]

The series representation and asymptotic behavior with exponential decay
\[M_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(1 - \alpha - \alpha n)} \sim \frac{(\alpha z)^{\alpha - 1/2}}{\sqrt{2\pi(1 - \alpha)}} \exp \left[\frac{1 - \alpha}{\alpha} (\alpha z)^{1/2} \right].\]
The moments of the \(M_\alpha\)-function are obtained through
\[\int_0^\infty z^n M_\alpha(z) dz = \lim_{s \rightarrow 0} (-1)^n \frac{d^n}{ds^n} E_\alpha(-s) = \frac{\Gamma(n + 1)}{\Gamma(1 + \alpha n)}.\]

The first passage PDF \(p_f(\tau)\) is displayed in Fig. 4 in nice agreement with the simulations. Note that for \(\alpha \leq 1/2\) the tail of \(M(x)\) is so long that it is most likely to cross \(x = d\) in the first jump, while for \(\alpha > 1/2\), \(p_f(\tau)\) has a maximum at finite \(\tau > 0\). To obtain the leaper statistics for the one-sided LF, we first note that since \(P(x < 0, t) = 0\) (only forward steps are permitted) we have \(q_-(u, k) = 1\), and thus \(\partial q_-(u, x)/\partial x = \delta(x)\). Combining Eqs. (16) and (17),
\[\langle e^{-\mu \tau} \rangle = 1 - \lim_{u \rightarrow 0} \frac{\mu}{u} \int_0^d e^{-\mu s'} M(d + s' - s) \frac{\partial q_+(u, s)}{\partial s} ds' ds.\]
With the small $u$ expansion of the Mittag-Leffler function, Eqs. (17) and (19) produce
\[ \frac{\partial u_+ (u, x)}{\partial x} = \frac{u \cos(\pi \alpha/2)}{K^{(\alpha)}G^{(\alpha)}} x^{\alpha-1}. \] (28)

Eqs. (15) and (28) inserted into Eq. (27) then yield
\[ p_1(\mu) = (e^{-\mu t}) = \frac{\sin(\pi \alpha)}{\pi} \int_0^\infty e^{-\mu t} \frac{d^\alpha}{t^\alpha (d + t)}, \] (29)
leading to the leapover PDF
\[ p_1(\ell) = \frac{\sin(\pi \alpha)}{\pi} \frac{d^\alpha}{\ell^\alpha (d + \ell)}, \] (30)
which corresponds to the result obtained in Ref. [17] from a different method. Thus, for the one-sided LF, the scaling of the leapover is exactly the same as for the jump length distribution, namely, with exponent $\alpha$.

The leapover distribution (30) also provides a new aspect to the first waiting time in a renewal process with broad waiting time distribution $\psi(t) \sim t^{-1-\beta}$ ($0 < \beta < 1$). Interpret the position $x$ as time and the jump lengths drawn from the one-sided $\lambda(x)$ as waiting times $t$. Consider an experiment, starting at time $t_0$, on a system prepared at time 0 (corresponding to position $x = 0$). Then the first recorded waiting time $t_1$ of the system will be distributed like $p_1(t_1) = \pi^{-1} \sin(\pi \alpha) \ell_0^\alpha / [\ell_0^\alpha (t_0 + t_1)]$, as obtained from a different reasoning in Ref. [9]. We note that the first passage time $\tau$ in this analogy corresponds to the number of waiting events.

While for symmetric LFs it was previously established that the first passage time distribution follows the universal Sparre Andersen asymptotics $p_1(\tau) \sim \tau^{-3/2}$, here we derived the prefactor of this law, in particular, its dependence on the generalized diffusion coefficient $K^{(\alpha)}$. For the same case, we derived the leapover distribution $p_1(\ell)$, that is surprising for two reasons: first, $p_1(\ell)$ is independent of $K^{(\alpha)}$, synonymous to the noise strength; and second, its power-law exponent is $\alpha/2$, and thus $p_1(\ell)$ is broader than the original jump length distribution.

For one-sided LFs, we recovered the previously reported leapover distribution and derived the so far unknown first passage time distribution. While the leapovers follow the same asymptotic scaling $p_1(\ell) \sim \ell^{-1-\beta}$ as the jump lengths $\lambda(x)$, once more independent of $K^{(\alpha)}$, the first passage times are narrowly distributed. We also drew an analogy between the leapovers and the first waiting time in a subdiffusive renewal process. For both symmetric and one-sided LFs, extensive simulations showed nice agreement with the theoretical results, without adjustable parameters.

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