Higgs-Electroweak Chiral Lagrangian: One-Loop Renormalization Group Equations

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Starting from the one-loop divergences we obtained previously, we work out the renormalization of the Higgs-Electroweak Chiral Lagrangian explicitly and in detail. This includes the renormalization of the lowest-order Lagrangian, as well as the decomposition of the remaining divergences into a complete basis of next-to-leading-order counterterms. We provide the list of the corresponding beta functions. We show how our results match the one-loop renormalization of some of the dimension-6 operators in SMEFT. We further point out differences with related work in the literature and discuss them. As an application of the obtained results, we evaluate the divergences of the vacuum expectation value of the Higgs field at one loop and show that they can be appropriately removed by the corresponding renormalization. We also work out the finite renormalization required to keep the no-tadpole condition on the Higgs field at one loop.
I. INTRODUCTION

Exploring the Higgs sector at the Large Hadron Collider (LHC) requires a parametrization of Higgs-boson properties, able to account for effects beyond the Standard Model (SM) in a well-defined way. This is frequently done using the $\kappa$-framework, where SM couplings of the Higgs boson are rescaled by phenomenological factors [1]. The electroweak chiral Lagrangian including a light Higgs boson (EWChL for short) provides us with a systematic effective field theory (EFT) formulation of this idea [2] (see also [3] for a detailed discussion of the precise relationship between the $\kappa$ parameters and EWChL coefficients). The power counting of the EWChL is governed by chiral dimensions, equivalent to an expansion in loop orders. The leading-order Lagrangian $\mathcal{L}_2$, of chiral dimension 2, naturally contains the dominant anomalous Higgs couplings.

Several authors have contributed to the development of the chiral Lagrangian with a light Higgs boson as an EFT of electroweak symmetry breaking [4–13], which includes the complete classification of the next-to-leading order terms [14] and a systematic review of its power counting [15]. Motivated by the importance of the EWChL as an EFT of anomalous Higgs couplings, we have computed its complete one-loop renormalization in a previous article [16]. Similar work was reported shortly thereafter in [17]. The consideration of one-loop corrections is, in particular, needed when treating subleading effects, which are of interest for their impact on decay distributions.

Since the leading-order Lagrangian $\mathcal{L}_2$ of the EWChL is non-renormalizable, the one-loop renormalization, besides affecting $\mathcal{L}_2$ itself, requires the addition of counterterms with chiral dimension 4. In the present paper, we investigate the one-loop divergences of the electroweak chiral Lagrangian in detail. This includes their decomposition into a basis of counterterms, their renormalization, and the derivation of renormalization group equations (RGEs) for all parameters in explicit terms. This work parallels similar calculations done in the past in the context of chiral perturbation theory for pions and kaons coupled to photons [18] and light fermions [19], extending the original results of [20].

The outline of the present paper is as follows. We set up our notation in Section II. In Section III we decompose the one-loop divergences into the various classes of basis operators. We work out the renormalization of the leading-order Lagrangian and the next-to-leading order counterterms in Sections IV and V, respectively, deriving also the one-loop RGEs for all parameters of the theory. Section VI presents a comparison with the one-loop renormalization of SM Effective Field Theory (SMEFT), which provides us with additional cross-checks of our results. A brief survey of the literature on the renormalization of the EWChL is given in Section VII. As an application, we treat the one-loop corrections to the vacuum expectation value (vev) of the Higgs field in Section VIII, showing how it is renormalized affecting $\mathcal{L}_2 \equiv \mathcal{L}_2(h)$, normalized as $\langle \frac{\partial_{\mu} h \partial^{\mu} h}{v^2} \rangle = 1$.

II. NOTATION AND LEADING-ORDER LAGRANGIAN

In this paper we discuss the one-loop renormalization of the EWChL at lowest order, which can be written as [14, 15]

$$\mathcal{L}_2 = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2} (W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{v^2}{4} (L_{\mu} L^{\mu}) F \left( \frac{h}{v} \right) + \frac{1}{2} \partial_{\mu} h \partial^{\mu} h - V \left( \frac{h}{v} \right)$$

starting from the one-loop divergences obtained in Ref. [16]. In this expression, $G_{\mu\nu}^a$, $W_a$ and $B_\mu$ are the gauge fields of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, respectively, $h$ denotes the Higgs field and $v = 246$ GeV is the electroweak scale. The $SU(2)$ trace is denoted by $\langle \ldots \rangle$.

The SM fermions are collected in the field $\psi = (u_i, d_i, \nu_i, e_i)^T$, where $u_i, d_i, \nu_i$ and $e_i$ are Dirac spinors and $i$ is the generation index. The covariant derivative on the fermion field reads

$$D_\mu \psi = (\partial_\mu + ig_\mu G_\mu + ig W_\mu P_L + ig' B_\mu (Y_L P_L + Y_R P_R)) \psi,$$

where $P_L, P_R$ are the left and right chiral projectors and the weak hypercharge is described by the diagonal matrices

$$Y_L = \text{diag}(1/6, 1/6, -1/2, -1/2), \quad Y_R = \text{diag}(2/3, -1/3, 0, -1).$$

(2.3)

We will also use the notation $\psi_L \equiv P_L \psi, \psi_R \equiv P_R \psi$ and $q = (u_i, d_i)^T, l = (\nu_i, e_i)^T$.

The electroweak Goldstone bosons are parametrized as $U = \exp(2i\varphi/v)$, where $\varphi = \varphi^a T^a$ and $T^a$ denote the generators of $SU(2)$, normalized as $(T^a T^b) = \delta^{ab}/2$. It is convenient to define their covariant derivative as

$$L_\mu = iU D_\mu U^\dagger, \quad D_\mu U = \partial_\mu U + ig W_\mu U - ig' B_\mu \tau_L U, \quad \tau_L = UT^3 U^\dagger.$$

(2.4)
The Yukawa term has been compactly expressed through
\[ m(\eta, U) \equiv U M(\eta) P_R + M^\dagger(\eta) U^\dagger P_L, \tag{2.5} \]
where \( \eta \equiv h/v \) and \( M \) is the block-diagonal mass matrix
\[ M = \text{diag}(M_u, M_d, M_\nu, M_e) \tag{2.6} \]
acting on \( \psi \). The entries \( M_f \equiv M_f(\eta) \) are \( h \)-dependent matrices in generation space. It is understood that the right-handed neutrinos are absent when we assume SM particle content. Accordingly, we will take \( M_\nu = 0 \) in our calculation.

The Higgs-dependent functions can be expanded as
\[ F(\eta) = 1 + \sum_{n=1}^{\infty} F_n \eta^n, \quad V(\eta) = v^4 \sum_{n=2}^{\infty} V_n \eta^n, \quad M(\eta) = \sum_{n=0}^{\infty} M_n \eta^n, \tag{2.7} \]
where the potential is defined such that \( V'(0) = 0 \).

In order to express our results in a compact way we will use the combinations
\[ \kappa = \frac{1}{2} F' F^{-1/2}, \quad B = -F^{-1/2} \kappa' = \frac{F''}{4F^2} - \frac{F'''}{2F}. \tag{2.8} \]
Here and in the following, a prime on \( \eta \)-dependent functions denotes differentiation with respect to this variable. Of particular interest are the values of these functions for \( \eta = 0 \),
\[ F(0) = 1, \quad \kappa(0) = F_1/2, \quad B(0) = \frac{F_2^2}{4} - F_2, \quad V(0) = V'(0) = 0, \quad V''(0) = 2v^4 V_2 = v^2 m_h^2, \quad M(0) = M_0, \tag{2.9} \]
and the deviations from these values,
\[ \bar{F}(\eta) = F(\eta) - 1, \quad \bar{\kappa}(\eta) = \kappa(\eta) - F_1/2, \quad \bar{B} = B - \frac{F_1^2}{4} + F_2, \quad \bar{M} = M - M_0. \tag{2.10} \]

III. DECOMPOSITION OF THE ONE-LOOP DIVERGENCES ON A BASIS OF COUNTERTERMS

The first step will be to decompose the one-loop divergences worked out in Ref. [16] and project them onto the basis of one-loop counterterms derived in Ref. [14]. Some of the divergences displayed in Ref. [16] actually have the same structure as the lowest-order effective Lagrangian \( L_2 \), and are thus dealt with by means of a set of counterterms \( \Delta L_2 \), involving only the structures already present in \( L_2 \).

It is convenient to decompose the one-loop divergent parts according to the presence of spin-one field strengths, scalar fields or fermion fields as
\[ L_{\text{div}} = L_{\text{div}}^{(1)} + L_{\text{div}}^{(0)} + L_{\text{div}}^{(1/2)}, \tag{3.1} \]
where the explicit expressions for each term on the right-hand side can be found in Eqs. (C.1), (C.2), and (C.11), respectively, in Appendix C. We proceed with each term in turn.

In manipulating the one-loop divergent pieces, one is entitled to make use of the equations of motion at leading order, which are collected in Appendix A. One reason for this is that the divergences are expressed in terms of the classical background fields. One may thus use the classical, lowest-order, equations of motion as long as the corresponding counterterms are inserted into tree-level diagrams only, which is the case for the computation of amplitudes at next-to-leading accuracy. More generally, the use of the equations of motion can also be implemented as a field redefinition, which does not change the S-matrix elements. For a general discussion on this issue, see e.g. Ref. [21, 22]. As a practical consequence, it is therefore not necessary to introduce next-to-leading counterterms corresponding to the field renormalizations for fermions and the Higgs boson.

In the results collected in Eqs. (C.1), (C.2), and (C.3), which are taken from Ref. [16], the equations of motion had already been used for the scalar fields \( U \) and \( h \). In Appendix C we extend this to fermions.
A. Decomposition of $\mathcal{L}_{\text{div}}^{(1)}$

From Eq. (C.1), one has ($N_c$ stands for the number of colors, $N_f$ for the number of quark flavors, and $N_g$ for the number of generations)

$$
\mathcal{L}_{\text{div}}^{(1)} = \frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{g^2}{2} G^\alpha_{\mu\nu} G^\alpha_{\mu\nu} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) + \frac{g^2}{2} \langle W^\mu W^\nu \rangle \left[ \frac{43}{3} - \frac{2}{3} (N_c + 1) N_g \right] - \frac{F^2}{24} \right\} + \frac{g^2}{2} B^\mu B^\nu \left[ - \left( \frac{11}{27} N_c + 1 \right) N_g - \frac{1}{6} \right] - \frac{1}{24} \tilde{\kappa} (\tilde{\kappa} + F_1) \left[ g^2 B^\mu B^\nu + 2g^2 \langle W^\mu W^\nu \rangle \right] - i \frac{F_1^2}{48} \left[ g^2 B^\mu B^\nu + 2g^2 \langle W^\mu W^\nu \rangle \right] - \frac{1}{6} \partial^\mu (\kappa^2) \left( g \langle W^\mu W^\nu \rangle - g^2 B^\mu \langle \tau_L L^\nu \rangle \right),
$$

(3.2)

where we have separated explicitly terms with and without Higgs field dependence.

It is tempting to read off the one-loop beta functions for the gauge fields directly from the coefficients of their kinetic terms, as given by the first three terms above. This would however be incorrect in the case of the $SU(2)$ and $U(1)$ couplings, due to the presence of the terms in the fourth line. In order to reach the right result, one may use, in the first two terms of the fourth line, the relations

$$
\langle W^\mu [L_\mu, L_\nu] \rangle = ig \langle W^\mu W^\nu \rangle - ig B^\mu \langle \tau_L W^\nu \rangle + ig \bar{\psi}_L \gamma_\nu \psi_L - ig \frac{v^2}{2} \frac{2}{F(\eta)} \langle L^\nu L_\nu \rangle,
$$

(3.3)

$$
B^\mu \langle \tau_L [L_\mu, L_\nu] \rangle = \frac{ig}{2} B^\mu B_\mu - ig B^\mu \langle \tau_L W^\nu \rangle - 2ig \tilde{\kappa} \langle \tau_L L^\nu \rangle \left( \bar{\psi}_\nu (Y_L P_L + Y_R P_R) \psi + \frac{v^2}{2} F(\eta) \langle L^\nu \tau_L \rangle \right),
$$

(3.4)

which follow from the identities [14]

$$
D_L L_\nu - D_\nu L_\mu = g W^\mu - g^2 B^\mu \tau_L + i[L_\mu, L_\nu], \\
D_\mu \tau_L = i[L_\mu, \tau_L],
$$

(3.5)

$$
\langle T^a L^\nu \rangle \langle T^a L_\nu \rangle = \frac{1}{2} \langle L^\nu L_\nu \rangle, \\
\langle L^\nu T^a \rangle \bar{\psi}_\nu \psi_L = \frac{1}{2} \bar{\psi}_L \psi_L,
$$

from the equations of motion given in Appendix A, and from integrating by parts. Because of the last point, they can only be used in the form given here if the operators on the left-hand side are not multiplied by $h$-dependent functions. The expression of $\mathcal{L}_{\text{div}}^{(1)}$ then can be rewritten as

$$
\mathcal{L}_{\text{div}}^{(1)} = \frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{g^2}{2} G^\alpha_{\mu\nu} G^\alpha_{\mu\nu} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) + \frac{g^2}{2} \langle W^\mu W^\nu \rangle \left[ \frac{43}{3} - \frac{2}{3} (N_c + 1) N_g \right] - \frac{F^2}{24} \right\} + \frac{g^2}{2} B^\mu B_\mu \left[ - \left( \frac{11}{27} N_c + 1 \right) N_g - \frac{1}{6} \right] - \frac{1}{24} \tilde{\kappa} (\tilde{\kappa} + F_1) \left[ g^2 B^\mu B_\mu + 2g^2 \langle W^\mu W^\nu \rangle \right] - i \frac{F_1^2}{48} \left[ g^2 B^\mu B_\mu + 2g^2 \langle W^\mu W^\nu \rangle \right] - \frac{1}{6} \partial^\mu (\kappa^2) \left( g \langle W^\mu W^\nu \rangle - g^2 B^\mu \langle \tau_L L^\nu \rangle \right),
$$

(3.6)

The one-loop beta functions of the gauge fields are now correctly given by the first three terms between the curly brackets, and they coincide with the expressions obtained in the SM. We next consider the two terms proportional to $\partial^\mu (\kappa^2)$ on the last line of the above expression. At this stage, we will need the equations of motion of the electroweak gauge fields, given in Eqs. (A.1) and (A.2), as well as the identities of Eq. (3.5). Up to total-derivative terms, which are discarded, we obtain

$$
\partial^\mu (\kappa^2) \langle W^\mu W^\nu \rangle = - \frac{\tilde{\kappa} (\tilde{\kappa} + F_1)}{2} \left( \frac{v^2}{2} \frac{2}{F(\eta)} \langle L^\nu L_\nu \rangle + \bar{\psi}_L \psi_L \right).
$$
\[ -\frac{\bar{\kappa}(\kappa + F_1)}{2} \left[ g(W_{\mu\nu}W_{\mu\nu}) - g' B^{\mu\nu} (W_{\mu\nu}\tau_L) + i(W_{\mu\nu} [L^\mu, L^\nu]) \right], \quad (3.7) \]

and

\[ \partial^\mu (\kappa^2) B_{\mu\nu} \langle \tau_L L^\nu \rangle = -\bar{\kappa}(\kappa + F_1) g' \left[ \frac{v^2}{2} (L_\nu \tau_L) F(\eta) + \bar{\psi}_\gamma \gamma_\nu (Y_L P_L + Y_R P_R) \psi \right] \langle \tau_L L^\nu \rangle \\
+ \frac{\bar{\kappa}(\kappa + F_1)}{2} B_{\mu\nu} \langle i \tau_L [L^\mu, L^\nu] - g \tau_L W_{\mu\nu} + g' \tau^2_2 B^{\mu\nu} \rangle. \quad (3.8) \]

Making the corresponding substitutions in the above expression of \( \mathcal{L}^{(1)}_{\text{div}} \), we find

\[ \mathcal{L}^{(1)}_{\text{div}} = -\frac{1}{16\pi^2} \frac{1}{d - 4} \left\{ -\frac{g^2}{2} G^\mu\nu G_\mu^\nu \left[ \frac{11}{3} N_c - \frac{2}{3} N_f \right] - \frac{g^2}{2} \langle W_{\mu\nu} W_{\mu\nu} \rangle \left[ \frac{43}{3} - \frac{2}{3} (N_c + 1) N_g \right] \\
+ \frac{g^2}{2} B^{\mu\nu} B_{\mu\nu} \left[ -\frac{11}{27} N_c + 1 \right] N_g - \frac{1}{6} - g^2 \frac{v^2}{24} (\kappa^2 - 1) F(\eta) \langle L^\nu L^\nu \rangle \\
+ \frac{\kappa^2 - 1}{12} g^2 \bar{\psi}_L \psi_L - \frac{\kappa^2 - 1}{6} g^2 \frac{v^2}{2} (L_\nu \tau_L) F(\eta) + \bar{\psi}_\gamma \gamma_\nu (Y_L P_L + Y_R P_R) \psi \langle \tau_L L^\nu \rangle \right\} \]

\[ \equiv \Delta^{(1)} \mathcal{L}_2 + \Delta^{(1)} \mathcal{L}_{\beta_1} + \Delta^{(1)} \mathcal{L}_{\psi^2 U h D}. \quad (3.9) \]

The term

\[ \Delta^{(1)} \mathcal{L}_2 = -\frac{1}{16\pi^2} \frac{1}{d - 4} \left\{ -\frac{g^2}{4} G^\mu\nu G_\mu^\nu \left[ -\frac{22}{3} N_c + \frac{4}{3} N_f \right] - \frac{g^2}{2} \langle W_{\mu\nu} W_{\mu\nu} \rangle \left[ -\frac{44}{3} + \frac{2}{3} (N_c + 1) N_g + \frac{1}{3} \right] \\
- \frac{g^2}{4} B^{\mu\nu} B_{\mu\nu} \left[ 2 \left( \frac{11}{27} N_c + 1 \right) N_g + \frac{1}{3} \right] - g^2 \frac{v^2}{24} (\kappa^2 - 1) F(\eta) \langle L^\nu L^\nu \rangle \right\} \quad (3.10) \]

renormalizes the gauge-field and Goldstone-boson kinetic terms in \( \mathcal{L}_2 \). The two last terms of \( \mathcal{L}^{(1)}_{\text{div}} \), to which we will return later, read

\[ \Delta^{(1)} \mathcal{L}_{\beta_1} = -\frac{1}{16\pi^2} \frac{1}{d - 4} \times g^2 \frac{v^2}{2} (\kappa^2 - 1) \frac{1}{6} \langle L^\nu \tau_L \rangle \langle L_\nu \tau_L \rangle \]

and

\[ \Delta^{(1)} \mathcal{L}_{\psi^2 U h D} = -\frac{1}{16\pi^2} \frac{1}{d - 4} \times \frac{\kappa^2 - 1}{12} \left\{ g^2 \bar{\psi}_L \psi_L - 2 g^2 \langle \tau_L L^\nu \rangle \bar{\psi}_\gamma \gamma_\nu (Y_L P_L + Y_R P_R) \psi \right\}. \quad (3.11) \]

\( \Delta^{(1)} \mathcal{L}_{\beta_1} \) renormalizes the custodial-symmetry breaking operator \( \mathcal{L}_{\beta_1} \), while \( \Delta^{(1)} \mathcal{L}_{\psi^2 U h D} \) renormalizes some of the counterterms of the class \( \psi^2 U h D \), respectively. We refer to Appendix B for the detailed definition of operators and their classes. We employ the nomenclature of Ref. [14], which will be used throughout. Incidentally, we note that the divergences involving the next-to-leading order operators of the classes \( X^2 U h \) and \( X U h D^2 \) have canceled.

Finally, in order to bring all the terms in correspondence to the operator basis retained in Ref. [14], the quantities of the type \( \bar{\psi}_L O_1 \bar{\psi}_2 \psi_L \) can be transformed as follows (\( P_{12} = T^1 + i T^2 \), \( P_{21} = T^1 - i T^2 \)):

\[ \bar{\psi}_L O_1 \bar{\psi}_2 \psi_L = 2 \bar{\psi}_L \gamma^\mu O_1 U T^a U^\dagger O_2 \psi_L (U T^a U^\dagger L_\mu) = \bar{\psi}_L \gamma^\mu O_1 U P_{12} U^\dagger \bar{\psi}_2 \psi_L (U P_{21} U^\dagger L_\mu) + \bar{\psi}_L \gamma^\mu O_1 U P_{21} U^\dagger O_2 \psi_L (U P_{12} U^\dagger L_\mu) + 2 \bar{\psi}_L \gamma^\mu O_1 \tau_L \bar{\psi}_2 \psi_L (\tau_L L_\mu). \]

Using the previous relation, we find

\[ \Delta^{(1)} \mathcal{L}_{\psi^2 U h D} = -\frac{1}{16\pi^2} \frac{1}{d - 4} \times \frac{\kappa^2 - 1}{12} \left\{ 2 g^2 \bar{\psi}_L \gamma^\mu \psi_L (\tau_L L_\mu) \\
+ g^2 \bar{\psi}_L U P_{21} \gamma^\mu \psi_L (U P_{12} U^\dagger L_\mu) + g^2 \bar{\psi}_L U P_{12} \gamma^\mu \psi_L (U P_{21} U^\dagger L_\mu) \\
- 2 g^2 (\langle \tau_L L^\nu \rangle \bar{\psi}_L \gamma_\nu Y_L \psi_L - \frac{4}{3} g^2 (\langle \tau_L L^\nu \rangle \bar{\psi}_R \gamma_\nu Y_R \psi_L + \frac{2}{3} g^2 (\langle \tau_L L^\nu \rangle \bar{\psi}_R \gamma_\nu \psi_L + 2 g^2 (\langle \tau_L L^\nu \rangle \bar{e}_R \gamma_\nu e_R \right). \quad (3.14) \]
B. Decomposition of $\mathcal{L}^{(0)}_{\text{div}}$

We consider next the divergent pieces involving spin-zero fields given in Eq. (C.2), which we rewrite as

$$\mathcal{L}^{(0)}_{\text{div}} = \Delta^{(0)}L_2 + \Delta^{(0)}L_{U\bar{h}D^4} + \Delta^{(0)}L_{\beta_i},$$

(3.15)

where $\langle \langle \ldots \rangle \rangle$ denotes a trace over isospin, as well as generations and color.

$$\Delta^{(0)}L_2 = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{v^2}{4} \left[ -\frac{g^2}{8}(F_1^2 + 12) + \frac{3g^2}{4}(F_1^2 + 4) + V_2(F_1^2 - 4F_2) + \frac{4}{v^4} \langle \bar{M}_0^\dagger M_0 \rangle \right] - \frac{g^2}{8}(F_1^2 + 12)F - \frac{g^2}{2}\bar{\kappa}(\bar{\kappa} + F_1)F - \frac{3g^2}{4}(F_1^2 + 4)\bar{F} - 3g^2\bar{\kappa}(\bar{\kappa} + F_1)F \\
- 2(\kappa^2 - 1)\frac{F'V'}{v^4} + 2\frac{(V'' - 2v^4 V_2)}{v^4} FB + 4V_2(\bar{F}\bar{B}) + \frac{4}{v^2} \langle \bar{M}_1^\dagger M_0 + \bar{M}_0^\dagger \bar{M} + \bar{M}_1^\dagger \bar{M} \rangle \right\} (L^\mu L_\mu)$$

$$+ \frac{1}{2} \left[ -\frac{3F^2_1 + 4F_2}{8}(3g^2 + g'^2) + \frac{4}{v^2} \langle \bar{M}_1^\dagger M_1 \rangle \right] \partial^\mu \partial^\nu h_{\mu\nu}$$

$$+ \frac{3}{2}(3g^2 + 2g'^2 + g'^4)v^4F + \frac{3}{8}g^2 + \frac{3}{8}F'V' + \frac{3}{8} \left( \frac{F'V'}{v^2} \right)^2 + \frac{1}{2} \left( \frac{V''}{v^2} \right)^2 - 2\langle \langle M_1^\dagger M_1 \rangle \rangle$$

$$+ 4i(\tau_L L_\mu) \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle,$$

(3.16)

$$\Delta^{(0)}L_{U\bar{h}D^4} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{(\kappa^2 - 1)^2}{6} \langle L_\mu L_\nu \rangle \langle L^\mu L_\nu \rangle + \frac{(\kappa^2 - 1)^2}{12} + \frac{F^2B^2}{8} \right\} (L^\mu L_\mu)^2 - \frac{2}{3}\kappa^2 \langle L_\mu L_\nu \rangle \partial^\mu \eta \partial^\nu \eta$$

$$- \frac{(\kappa^2 - 1)B + \kappa^2}{6} \langle L_\mu L_\nu \rangle \partial^\mu \eta \partial^\nu \eta + \frac{3}{2}B^2 \langle \partial^\mu \eta \partial^\nu \eta \rangle \right\},$$

(3.17)

$$\Delta^{(0)}L_{\beta_i} = -\frac{1}{16\pi^2} \frac{1}{d-4} \times \frac{3}{4}g^2v^2(1 - \kappa^2)F \langle \tau_L L_\mu \rangle \langle \tau_L L_\mu \rangle.$$

(3.18)

The two last terms renormalize the counterterms of the class $U\bar{h}D^4$ and the custodial-symmetry breaking operator $L_{\beta_i}$, respectively. $\Delta^{(0)}L_{U\bar{h}D^4}$ comes already fully expressed in terms of the basis elements displayed in Ref. [14].

The last term of Eq. (3.16) does not naturally appear as a renormalization of $L_2$, but can be shown to renormalize the Yukawa term. Using the equation of motion for $B_\mu$ in (A.1), one may write

$$4i(\tau_L L_\mu) \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle = \frac{8i}{v^2 F} \left[ \frac{1}{g^2} \partial^\nu B_{\nu\mu} - \bar{\psi} \gamma_\mu (Y_L P_L + Y_R P_R) \psi \right] \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle,$$

(3.19)

The first term between square brackets on the right-hand side of this relation leads to a total derivative,

$$F^{-1}\partial^\nu B_{\nu\mu} \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle = \partial^\nu \left[ F^{-1} B_{\nu\mu} \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle \right] - B_{\nu\mu} \partial^\nu \left[ F^{-1} \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle \right],$$

(3.20)

since the second term in the above relation vanishes, being proportional either to $B_{\mu\nu}(\partial^\mu \eta)(\partial^\nu \eta)$ or to $B_{\mu\nu}(\partial^\mu \eta')(\partial^\nu \eta')$. One thus ends up with

$$4i(\tau_L L_\mu) \langle \langle (\partial^\mu M_1^\dagger M - M_1^\dagger \partial^\nu M)T^3 \rangle \rangle = \frac{8i}{v^2} \left[ \langle \langle (F^{-1/2} M_1^\dagger F^{-1/2} M) - \partial^\mu (F^{-1/2} M_1^\dagger F^{-1/2} M) \rangle \rangle \right]$$

$$\times \bar{\psi} \gamma_\mu (Y_L P_L + Y_R P_R) \psi$$

$$= \frac{8i}{v^2} \langle \langle (F^{-1/2} M_1^\dagger F^{-1/2} M) - (F^{-1/2} M_1^\dagger F^{-1/2} M) \rangle \rangle$$

$$\times \langle \langle (\partial^\mu \eta')(\bar{\psi} \gamma_\mu (Y_L P_L + Y_R P_R) \psi) \rangle \rangle$$

$$= \frac{8i}{v^2} \partial^\mu \int_0^1 ds \langle \langle (F^{-1/2} M_1^\dagger F^{-1/2} M) - (F^{-1/2} M_1^\dagger F^{-1/2} M) \rangle \rangle$$

$$\times \langle \langle (\partial^\mu \eta) \rangle \rangle.$$
Here the integration variable \( s \) has been introduced, which denotes the dependence of the integrands on \( h/v = s \). In the last step, an integration by parts has been performed, the resulting total-derivative term has been dropped, and the equations of motion for the fermionic fields have been used. Notice that \( Y_L - Y_L = T^3 \). Objects like

\[
\int_0^\eta ds \left[ F^{-1/2} \mathcal{M}^\dagger (F^{-1/2} \mathcal{M})' - (F^{-1/2} \mathcal{M})' F^{-1/2} \mathcal{M} \right]
\]

are perfectly well defined as formal power series in \( \eta \), obtained upon multiplication of the formal series defining, e.g., \( F^{-1/2}(s) \mathcal{M}^\dagger(s) \) and \( (F^{-1/2}(s) \mathcal{M}(s))' \), and term-by-term integration.

### C. Decomposition of \( \mathcal{L}_{\text{div}}^{(1/2)} \)

Turning finally to the divergences involving also fermionic fields, Eq. (C.11) gives

\[
\mathcal{L}_{\text{div}}^{(1/2)} = -\frac{1}{16\pi^2} \frac{1}{d-4} \{ \bar{\psi}_L \left( \begin{array}{cc} 3 & 2g^2 + 2g^2 Y_L^2 \\ 2g^2 \end{array} \right) i\mathcal{D}\psi_L + \bar{\psi}_R 2g^2 Y_R^2 i\mathcal{D}\psi_R \\
+ 2g^2 C_F \bar{q} (i\mathcal{D} - 4(U\mathcal{M} q P_R + \mathcal{M} q U P_L)) q \\
+ V'' (\bar{\psi}_L U M'' \psi_R + \text{h.c.}) - 8g^2 (\bar{\psi}_L Y_L U M_{Y_R} \psi_R + \text{h.c.}) \\
+ \left( \left( 3g^2 + g^2 \frac{v^2}{4} + 3 \frac{F'}{F} \frac{v^2}{2} \right) - \frac{1}{v^2} \left( \bar{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right) \right. \\
\left. + \frac{3}{v^2} F^{-1} \bar{\psi}_L U \left( M_i i \partial M \right) \psi_R + \frac{1}{2v^2} \bar{\psi}_R \left( M_{i\partial i}^T \psi_R + \text{h.c.} \right) \right. \\
\left. + \frac{2}{\sqrt{v^2}} (\bar{\psi}_L U M M^\dagger M' \psi_R + \text{h.c.}) + \frac{1}{2v^2} (\bar{\psi}_L U M M^\dagger M' \psi_R + \text{h.c.}) + \frac{1}{2v^2} (\bar{\psi}_L U M M^\dagger M' \psi_R + \text{h.c.}) \right. \\
\left. + \frac{3F^{-1}}{\sqrt{v^2}} \bar{\psi}_R \left( M_{i\partial i}^T \psi_R + \text{h.c.} \right) - \frac{1}{2v^2} \bar{\psi}_R \left( M_{i\partial i}^T \psi_R + \text{h.c.} \right) \\
\left. + \frac{1}{2v^2} \bar{\psi}_L \left( U M M^\dagger U^\dagger + U M M^\dagger U \psi_L \right) U L \\
\left. + \Delta^{(1/2)} \mathcal{L}_{\psi^2 U hD^2} + \Delta^{(1/2)} \mathcal{L}_{\psi^4 U h} \right\}
\]

with

\[
\Delta^{(1/2)} \mathcal{L}_{\psi^2 U hD^2} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ L^\mu L^\mu \left[ \begin{array}{cc} F B & \bar{\psi}_L U M'' \psi_R - \frac{\kappa^2}{F_v^2} \bar{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right] \\
+ \frac{2\kappa'}{v^2} \partial^\mu \eta \left[ \bar{\psi}_L U (F^{-1/2} \mathcal{M})' \psi_R + \text{h.c.} \right] + \frac{3B}{F_v^2} \partial^\mu \eta \partial_\mu \eta \left[ \bar{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right] \right\},
\]

\[
\Delta^{(1/2)} \mathcal{L}_{\psi^4 U h} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ 3F^{-2} \left[ \bar{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right]^2 + \frac{1}{4v^2} \left[ \bar{\psi}_L U M'' \psi_R + \text{h.c.} \right]^2 \left\{ \bar{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right\} \right\}.
\]
The first six lines in the above expression of $L_{\text{div}}^{(1/2)}$ correspond to structures already present in the lowest-order Lagrangian $L_2$. The five lines that follow are of the type $\psi^2 U_h D$, but are not present in the basis considered in Ref. [14]. They need therefore to be transformed, proceeding as for the last term in $\Delta^{(0)} L_2$. Considering the first term of this type, one may write

$$
\frac{3F^{-1}}{2v^2} \bar{\psi}_R (M^\dagger i \partial M - i \partial M^\dagger M) \psi_R = \frac{3}{2v^2} \bar{\psi}_R [F^{-1/2} M^\dagger i \partial (F^{-1/2} M) - i \partial (F^{-1/2} M^\dagger) F^{-1/2} M] \psi_R \\
= \frac{3}{2v^2} \bar{\psi}_R \sum_{i} \partial_i \bar{\psi}_R^n [F^{-1/2} M^\dagger (F^{-1/2} M)^i - (F^{-1/2} M)^i (F^{-1/2} M)'] \psi_R \\
= \frac{3}{2v^2} \bar{\psi}_R \sum_{i} \int_0^n ds [F^{-1/2} M^\dagger (F^{-1/2} M)^i - (F^{-1/2} M)^i (F^{-1/2} M)] \psi_R \\
= - \frac{3}{2v^2} \bar{\psi}_R \sum_{i} \int_0^n ds [F^{-1/2} M^\dagger (F^{-1/2} M)^i - (F^{-1/2} M)^i (F^{-1/2} M)] M^\dagger U^\dagger \psi_L \\
+ \frac{3}{2v^2} \bar{\psi}_L U M \int_0^n ds [F^{-1/2} M^\dagger (F^{-1/2} M)^i - (F^{-1/2} M)^i (F^{-1/2} M)] \psi_R.
$$

(3.26)

In the last step, an integration by parts has been performed, the resulting total-derivative term has been dropped, and the lowest-order equations of motion for the fermionic fields have been used. The other structures of this type can be handled in a similar manner to obtain:

$$
\frac{1}{2v^2} \bar{\psi}_R (M^\dagger i \partial M' - i \partial M^\dagger M') \psi_R = \frac{1}{2v^2} \bar{\psi}_R \sum_{i} \int_0^n ds [M^\dagger M' - M' M^\dagger] M^\dagger U^\dagger \psi_L \\
+ \frac{1}{2v^2} \bar{\psi}_L U M \int_0^n ds [M^\dagger M' - M' M^\dagger] \psi_R,
$$

$$
\frac{1}{2v^2} \bar{\psi}_L U (M^\dagger i \partial M' - i \partial M^\dagger M') U^\dagger \psi_L = \frac{1}{2v^2} \bar{\psi}_R M^\dagger \int_0^n ds [F^{-1/2} M (F^{-1/2} M')' - (F^{-1/2} M')' F^{-1/2} M] U^\dagger \psi_L \\
+ \frac{1}{2v^2} \bar{\psi}_L U \int_0^n ds [F^{-1/2} M' F^{-1/2} M' - F^{-1/2} M (F^{-1/2} M')] U^\dagger \psi_L \\
+ \frac{1}{2v^2} \bar{\psi}_L U \int_0^n ds [F^{-1/2} M (F^{-1/2} M')' - (F^{-1/2} M')' F^{-1/2} M] U^\dagger \psi_L,
$$

(3.27)

which reduce to structures already present in $L_2$. The structures of the form $\psi_I O barslash O_I \psi_L$, where $I = L, R$, can be handled upon using the identity (3.13), such that the whole structure can be expressed in terms of the basis of Ref. [14]. One thus obtains the decomposition

$$
L_{\text{div}}^{(1/2)} = \Delta^{(1/2)} L_2 + \Delta^{(1/2)} L_{\psi^2 U_h D} + \Delta^{(1/2)} L_{\psi^3 U_h D^2} + \Delta^{(1/2)} L_{\psi^4 U_h},
$$

(3.28)
with (the lowest-order equations of motion of the fermion fields have been applied)

\[
\Delta^{(1/2)} \mathcal{L}_2 = - \frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{3}{4} g^2 \tilde{\psi}_L U \mathcal{M} \psi_R + g^2 \tilde{\psi}_L Y^2 U \mathcal{M} \psi_R + g^2 \tilde{\psi}_L U \mathcal{M} Y^2 \psi_R \\
-6 g^2 C_F \bar{q} U \mathcal{M} q + \frac{V''}{v^4} \tilde{\psi}_L U \mathcal{M}' \psi_R - 8 g^2 \tilde{\psi}_L Y U \mathcal{M} Y \psi_R \\
+ \left( 3g^2 + g'^2 \right) \frac{v^2}{4} F + \frac{3}{2} F' \tilde{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R \\
+ \frac{3}{v^2} F^{-1} \tilde{\psi}_L U \mathcal{M} \mathcal{M}' \psi_R - \frac{3}{v^2} F^{-1} \tilde{\psi}_L U \left( \mathcal{M} \mathcal{M}' \right) \psi_R \\
+ \frac{2}{v^2} \tilde{\psi}_L U \mathcal{M} \mathcal{M}' \psi_R + \frac{1}{2v^2} \tilde{\psi}_L U \mathcal{M} \mathcal{M}' \mathcal{M}' \psi_R + \frac{1}{2v^2} \tilde{\psi}_L U \mathcal{M} \mathcal{M}' \mathcal{M}' \psi_R \\
+ \frac{3}{2v^2} \tilde{\psi}_L U \mathcal{M} \int_0^\eta ds \left( F^{-1/2} \mathcal{M}' \left( F^{-1/2} \mathcal{M} \right)' - \left( F^{-1/2} \mathcal{M} \right)' F^{-1/2} \mathcal{M} \right) \psi_R \\
- \frac{1}{2v^2} \tilde{\psi}_L U \int_0^\eta ds \left[ \left( F^{-1/2} \mathcal{M} \right)' F^{-1/2} \mathcal{M}' - F^{-1/2} \mathcal{M} \left( F^{-1/2} \mathcal{M} \right)' \right] \psi_R \\
+ \frac{1}{2v^2} \tilde{\psi}_L U \int_0^\eta ds \left[ \mathcal{M} \mathcal{M}'' - \mathcal{M} \mathcal{M}'' \right] \psi_R + \frac{1}{2v^2} \tilde{\psi}_L U \int_0^\eta ds \left[ \mathcal{M} \mathcal{M}'' - \mathcal{M} \mathcal{M}'' \right] \psi_R \\
- \frac{1}{v^2} \tilde{\psi}_L U \int_0^\eta ds \left[ F^{-1/2} \mathcal{M} \left( F^{-1/2} \mathcal{M} \right)' - \left( F^{-1/2} \mathcal{M} \right)' F^{-1/2} \mathcal{M} \right] \psi_R + h.c. \right\}, (3.29)
\]

and

\[
\Delta^{(1/2)} \mathcal{L}_{\psi^2 u,h,d} = - \frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{F^{-1}}{v^2} \tilde{\psi}_R \mathcal{M} \mathcal{M} U \psi_R - \frac{1}{v^2} \tilde{\psi}_R \mathcal{M} \mathcal{M} U \psi_R \\
+ \frac{1}{2v^2} \frac{F^{-1/2}}{v^2} \left( \tilde{\psi}_R \mathcal{M} \mathcal{M} U \psi_R + h.c. \right) + \frac{F^{-1/2}}{v^2} \left( \tilde{\psi}_R \mathcal{M} \mathcal{M} \mathcal{M} U \psi_R + h.c. \right) \\
+ \frac{1}{2v^2} \left( \tilde{\psi}_R \mathcal{M} \mathcal{M} \mathcal{M} U \psi_R + h.c. \right) - \frac{1}{v^2} \tilde{\psi}_L U \int_0^\eta ds \left[ \left( F^{-1/2} \mathcal{M} \right)' F^{-1/2} \mathcal{M}' - F^{-1/2} \mathcal{M} \left( F^{-1/2} \mathcal{M} \right)' \right] U \psi_R + h.c. \right\}. (3.30)
\]

D. Summary

In this section, the divergences at next-to-leading order have been decomposed into the basis of counterterms given in Ref. [14]. The result of this procedure is summarized by

\[
\mathcal{L}_{\text{div}} = \left[ \Delta^{(0)} \mathcal{L}_2 + \Delta^{(1)} \mathcal{L}_2 + \Delta^{(1/2)} \mathcal{L}_2 \right] + \left[ \Delta^{(0)} \mathcal{L}_\beta_1 + \Delta^{(1)} \mathcal{L}_\beta_1 \right] + \Delta^{(0)} \mathcal{L}_{u,h,d} + \left[ \Delta^{(1)} \mathcal{L}_{\psi^2 u,h,d} + \Delta^{(1/2)} \mathcal{L}_{\psi^2 u,h,d} \right] + \Delta^{(1/2)} \mathcal{L}_{\psi^2 u,h,d} + \Delta^{(1/2)} \mathcal{L}_{\psi^2 u,h}, (3.31)
\]

where the expressions for each term are given above.

IV. RENORMALIZATION OF THE LEADING-ORDER LAGRANGIAN

The renormalization of the parameters in \( \mathcal{L}_2 \) is derived from the first bracket in (3.31),

\[
\Delta \mathcal{L}_2 \equiv \Delta^{(0)} \mathcal{L}_2 + \Delta^{(1)} \mathcal{L}_2 + \Delta^{(1/2)} \mathcal{L}_2 , (4.1)
\]

The expressions of \( \Delta^{(1)} \mathcal{L}_2 \), \( \Delta^{(0)} \mathcal{L}_2 \), and \( \Delta^{(1/2)} \mathcal{L}_2 \) are given in Eqs. (3.10), (3.16), and (3.29), respectively. We rewrite (4.1) as

\[
\Delta \mathcal{L}_2 = \Delta \mathcal{L}_{2,\text{gauge}} + \Delta \mathcal{L}_{2,\text{scalar}} . (4.2)
\]
where
\[ 32\pi^2 \varepsilon \Delta L_{2,\text{gauge}} = -\frac{g^2}{4} G^\mu\nu G^{\mu\nu} \left( -\frac{22}{3} N_c + \frac{4}{3} N_f \right) - \frac{g^2}{2} (W^{\mu\nu} W_{\mu\nu}) \left[ -\frac{44}{3} + \frac{2}{3} (N_c + 1) N_g + \frac{1}{3} \right] \]
\[ -\frac{g^2}{4} B^{\mu\nu} B_{\mu\nu} \left[ 2 \left( \frac{11}{27} N_c + 1 \right) N_g + \frac{1}{3} \right] \]
\[ (4.3) \]

\[ 32\pi^2 \varepsilon \Delta L_{2,\text{scalar}} = \frac{v^2}{4} \langle L_\mu L^\mu \rangle A_F + \partial_\mu h \partial^\mu h A_h + A_V - (\bar{\psi} U A_M P_R \psi + \text{h.c.}) . \]
\[ (4.4) \]

Here and in the following, we define for the dimension of space-time
\[ d \equiv 4 - 2\varepsilon. \]
\[ (4.5) \]

The various functions introduced on the right-hand side of (4.4) read
\[ A_F(\eta) = -\frac{g^2}{2} (\kappa^2 + 3) F - \frac{g^2}{6} (19\kappa^2 + 17) F - 2(\kappa^2 - 1) \frac{F' V'}{Fv^4} + 2 \frac{V''}{v^4} BF + \frac{4}{v^2} \langle \mathcal{M}^4 \rangle, \]
\[ A_h(\eta) = \frac{1}{4} \left( (3g^2 + g'^2)(FB - 4\kappa^2) + 3 \frac{F' V'}{Fv^4} B + \frac{2}{v^2} \langle \mathcal{M}^4 \rangle \right), \]
\[ A_V(\eta) = \frac{3}{2} (3g^2 + 2g^2 g^2 + g'^2) \frac{v^4}{16} F^2 + \frac{3g^2 + g'^2}{2} F' V' + \frac{3}{8} \left( \frac{F' V'}{Fv^2} \right)^2 + \frac{1}{2} \left( \frac{V''}{v^2} \right)^2 - 2 \langle \langle \mathcal{M}^4 \rangle^2 \rangle, \]
\[ A_M(\eta) = \frac{3}{4} g^2 F - \frac{g}{2} (\kappa Y_L^2 + \kappa Y_R^2) + 6g^2 C_F M_q + 8g^2 Y_L M_Y R
\[ - \left[ (3g^2 + g'^2) \frac{v^2}{4} F + 3 \frac{F V'}{2 Fv^2} F^{-1} \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M}' \right) - \frac{V''}{v^4} \mathcal{M}' \right]
\[ - \frac{3}{2v^4} \int_0^\eta ds \left[ \mathcal{M}^{-1} \mathcal{M}^{-1} - \frac{1}{2v^2} \mathcal{M}' \mathcal{M}' \right]
\[ + \frac{1}{2v^2} \int_0^\eta ds \left( \mathcal{M}^{-1} \mathcal{M}' - \mathcal{M}' \mathcal{M}' \right) \mathcal{M}
\[ - \frac{1}{2v^2} \int_0^\eta ds \left[ \mathcal{M}' \mathcal{M}' - \mathcal{M}' \mathcal{M}' \right] = - \frac{1}{2v^2} \int_0^\eta ds \left[ \mathcal{M}' \mathcal{M}' - \mathcal{M}' \mathcal{M}' \right] \mathcal{M}
\[ + \frac{1}{2v^2} \int_0^\eta ds \left[ \mathcal{M}^{-1} \mathcal{M}' - \mathcal{M}' \mathcal{M}' \right] \mathcal{M}
\[ + \frac{8}{v^2} MT^3 \int_0^\eta ds \left( \mathcal{M}^{-1} \mathcal{M}' - \mathcal{M}' \mathcal{M}' \right) \mathcal{M}\right]. \]
\[ (4.9) \]

In the expression for \( A_M \) it is understood that the contribution proportional to \( g'^2 \) only affects the quark fields.

The renormalization that cancels \( \Delta L_{2,\text{gauge}} \) is standard and will be summarized below. In order to treat \( \Delta L_{2,\text{scalar}} \), it is convenient to first apply suitable redefinitions of the Higgs field \( h \), which bring \( \Delta L_{2,\text{scalar}} \) to a canonical form. The renormalization of \( \mathcal{L}_2 \) then proceeds in three steps. First, we eliminate the term \( \partial_\mu h \partial^\mu h A_h \) from \( \Delta L_{2,\text{scalar}} \) using integration by parts and the lowest-order equation of motion for \( h \). Second, we shift the Higgs field, \( \eta \rightarrow \eta + \delta \), and fix \( \delta \) such that the minimum of the potential remains at \( \eta = 0 \). In a third step, we renormalize the fields and parameters in \( \mathcal{L}_2 \). The resulting counterterms are then determined in the usual way from the requirement that they cancel the divergences of \( \Delta L_2 \).

We remark that for the field shift \( \delta \), only the divergent part, relevant for the UV renormalization, is considered in the present section. However, \( \delta \) also includes a finite piece, which must be chosen such as to preserve the condition \( V'(0) = 0 \). Minimal subtraction is therefore not sufficient here. The finite counterterm that is required in this case is computed in Section VIII.

### A. Redefinitions of the Higgs field

In a first step, we eliminate \( \partial_\mu h \partial^\mu h A_h \) from (4.4) by writing
\[ \partial_\mu h \partial^\mu h A_h(\eta) = -v \partial^2 h \int_0^\eta ds A_h(s) = \left[ -F' \frac{v^2}{4} \langle L_\mu L^\mu \rangle + V' + \bar{\psi} m'(\eta, U) \psi \right] \int_0^\eta ds A_h(s), \]
\[ (4.10) \]
where we used an integration by parts and the equation of motion of the Higgs field (A.4). Inserting (4.10) in (4.4) we obtain

\[
32\pi^2\varepsilon \Delta L_{2,\text{scalar}} = \frac{v^2}{4} \langle L_\mu L^\mu \rangle \left[ A_F - F' \int_0^n ds A_h(s) \right] + A_V + V' \int_0^n ds A_h(s)
- \left( \bar{\psi} U \left[ A_M - M' \int_0^n ds A_h(s) \right] P_R \psi + \text{h.c.} \right).
\] (4.11)

This step is equivalent to a field redefinition for \( h \). The form of (4.11) has the advantage that the Higgs kinetic term \( \partial h \partial h \) no longer receives a (divergent) one-loop correction. Consequently, there is no need to renormalize \( h \). This is fully analogous to our treatment of the fermions, where we have also used the equations of motion to dispense with their field renormalization.

In a second step, we shift \( \eta \to \eta + \delta \) to ensure a minimum of the potential at \( \eta = 0 \). Since \( V'(0) = 0 \) at leading order, \( \delta \) is of the order of a one-loop contribution. The shift then leads to an additional one-loop term in the Lagrangian:

\[
\mathcal{L}_2 \to \mathcal{L}_2 + \frac{v^2}{4} \langle L_\mu L^\mu \rangle F' \delta - V' \delta - \left( \bar{\psi} U M' \delta P_R \psi + \text{h.c.} \right).
\] (4.12)

The divergent Lagrangian \( \Delta \mathcal{L}_{2,\text{scalar}} \) then becomes

\[
32\pi^2\varepsilon \Delta L_{2,\text{scalar}} = \frac{v^2}{4} \langle L_\mu L^\mu \rangle \left[ A_F - F' \int_0^n ds A_h(s) + F' \delta \right] + A_V + V' \int_0^n ds A_h(s) - V' \delta
- \left( \bar{\psi} U \left[ A_M - M' \int_0^n ds A_h(s) + M' \delta \right] P_R \psi + \text{h.c.} \right),
\] (4.13)

where

\[
\tilde{\delta} \equiv 32\pi^2\varepsilon \delta
\] (4.14)

The condition that the minimum of the effective potential remains at \( \eta = 0 \) implies for the divergent part of \( \delta \)

\[
\left[ \left( A_V + V' \int_0^n ds A_h(s) \right)' - V'' \delta \right]_{\eta=0} = 0,
\] (4.15)

and accordingly

\[
\tilde{\delta} = \frac{3F_1}{32V_2} \left( 3g^4 + 2g^2g' + g'^4 \right) + \frac{3g^2 + g'^2}{8} F_1 + 6V_3 - \frac{2}{v^4V_2} \langle \langle M_1^4 M_0 + M_0^4 M_1 \rangle \rangle.
\] (4.16)

**B. Renormalization of \( \mathcal{L}_2 \)**

We start from the leading order Lagrangian (2.1), where we consider all fields and parameters \( X \) as unrenormalized, denoted by \( \tilde{X} \). As discussed above, no renormalization is needed for the Higgs field and the fermions, and we thus take \( \tilde{h} = h, \tilde{\psi} = \psi \). For the remaining quantities we introduce renormalization constants in the form

\[
\tilde{G}_\mu^a = Z_G^{1/2} G_\mu^a, \quad \tilde{W}_\mu^a = Z_W^{1/2} W_\mu^a, \quad \tilde{B}_\mu = Z_B^{1/2} B_\mu,
\]

\[
\tilde{g}_s = Z_G^{-1/2} g_s \mu^s, \quad \tilde{g} = Z_W^{-1/2} g \mu^e, \quad \tilde{g}' = Z_B^{-1/2} g' \mu^e,
\]

\[
\tilde{\varphi}^a = Z_{\varphi}^{1/2} \varphi^a, \quad \tilde{v} = Z_v^{1/2} v \mu^{-e},
\]

\[
\tilde{F}_n = Z_{F_n} F_n, \quad \tilde{V}_n = Z_{V_n} V_n \mu^{2e}, \quad \tilde{\mathcal{M}}_n = \mathcal{M}_n + \Delta \mathcal{M}_n.
\] (4.17)

Notice that the renormalization of the gauge fields and of the corresponding gauge couplings involve the same renormalization constants. This is because we are using the background field gauge, with the effect that the Slavnov-Taylor identities boil down to the simple QED-type Ward-Takahashi identities [23].
we observe that the Goldstone kinetic term and the gauge-boson mass term receive the same divergent one-loop corrections. In addition, the term \( gW_\mu - g'B_\mu \delta^{a3} \) is not renormalized within our scheme. It follows that the renormalization factors for \( \varphi^n \) and \( v \) are identical, a fact which we have already used in (4.17). As a consequence, \( \varphi^n/v \) and the Goldstone matrix \( U \) are not renormalized.

It will be convenient to write the one-loop \( Z \)-factor for a quantity \( X \) in the minimal subtraction (MS) scheme as

\[
Z_X = 1 + \Delta Z_X = 1 + \frac{\Delta \tilde{Z}_X}{32\pi^2\varepsilon}, \quad \Delta \tilde{Z}_X = 32\pi^2\varepsilon \Delta Z_X, \quad \Delta \tilde{M}_n = 32\pi^2\varepsilon \Delta M_n. \tag{4.19}
\]

In the MS scheme, the pole term is replaced by

\[
\frac{1}{\varepsilon} = \frac{2}{4-d} \to \frac{2}{4-d} - \gamma_E + \ln 4\pi. \tag{4.20}
\]

Employing (4.17) in the unrenormalized version of (2.1), and subtracting the renormalized Lagrangian, we obtain the counterterm \( \mathcal{L}_{2,\text{CT}} = \tilde{\mathcal{L}}_2 - \mathcal{L}_2 \), where

\[
\mathcal{L}_{2,\text{CT}} = \frac{1}{4}(Z_G - 1) G^{a\mu} G^{a\nu} - \frac{1}{2}(Z_W - 1) (W^{\mu \nu} W^{\mu \nu}) - \frac{1}{4}(Z_B - 1) B^{\mu \nu} B^{\mu \nu}
\]

\[
\quad + \frac{v^2}{4} \langle L_\mu L^\mu \rangle \sum_{n=0}^\infty \left( Z_\varphi^{1-n/2} F_n - 1 \right) F_n \eta^n - \sum_{n=2}^\infty \left( Z^{2-n/2}_\varphi V_n - 1 \right) \eta^n (\sum_{\alpha} p^{\alpha}_W + \text{h.c.})
\]

\[
\quad - \left( \bar{\psi} U \sum_{n=0}^\infty \left( - \frac{n}{2} \Delta Z_\varphi M_n + \Delta \tilde{M}_n \right) \eta^n P_R \psi + \text{h.c.} \right). \tag{4.21}
\]

The approximation in the second equality holds to one-loop order.

The renormalization constants are fixed by requiring \(-\mathcal{L}_{2,\text{CT}} = \Delta \mathcal{L}_{2,\text{gauge}} + \Delta \mathcal{L}_{2,\text{scalar}} \), with \( \Delta \mathcal{L}_{2,\text{gauge}} \) in (4.3) and \( \Delta \mathcal{L}_{2,\text{scalar}} \) in the redefined form of (4.13). This implies

\[
\Delta \tilde{Z}_G = - \left( -\frac{22}{3} N_c + \frac{4}{3} N_f \right) g_s^2,
\]

\[
\Delta \tilde{Z}_W = - \left[ -\frac{44}{3} + \frac{2}{3} (N_c + 1) N_g + \frac{1}{3} \right] g^2,
\]

\[
\Delta \tilde{Z}_B = - \left[ 2 \left( \frac{11}{27} N_c + 1 \right) N_g + \frac{1}{3} \right] g^2. \tag{4.22}
\]

and

\[
\sum_{n=0}^\infty \left( \left( 1 - \frac{n}{2} \right) \Delta Z_\varphi + \Delta \tilde{Z}_{F_n} \right) F_n \eta^n = -A_F + F' \int_0^\eta ds A_h(s) - F' \tilde{\delta}, \tag{4.23}
\]

\[
\sum_{n=2}^\infty \left( \left( 2 - \frac{n}{2} \right) \Delta \tilde{Z}_\varphi + \Delta \tilde{Z}_{V_n} \right) \eta^n (\sum_{\alpha} p^{\alpha}_W + \text{h.c.}) = A_V + V' \int_0^\eta ds A_h(s) - V' \tilde{\delta}, \tag{4.24}
\]

\[
\sum_{n=0}^\infty \left( - \frac{n}{2} \Delta Z_\varphi M_n + \Delta \tilde{M}_n \right) \eta^n = -A_M + M' \int_0^\eta ds A_h(s) - M' \tilde{\delta}. \tag{4.25}
\]
\[ \Delta \hat{Z}_\varphi \] follows from the \( n = 0 \) term in (4.23), noting that \( F_0 = 1 \) and \( \Delta \hat{Z}_{F_0} = 0 \). We find
\[
\Delta \hat{Z}_\varphi = \left( \frac{F_1^2}{4} + \frac{17}{6} \right) \frac{g^2}{3} - \frac{3}{2} g'^2 - V_2 F_1^2 + 4 V_2 F_2 - 6 F_1 V_3 \frac{4}{\mu^2} \langle (\mathcal{M}_{\mu}^i \mathcal{M}_0) \rangle \\
- \frac{3 F_1^2}{32 V_2} (3 g^4 + 2 g'^2 g'^2 + g'^4) + \frac{2 F_1}{\mu^4 V_2} \langle (\mathcal{M}_{\mu}^i \mathcal{M}_0 + \mathcal{M}_{\mu}^i \mathcal{M}_1) \mathcal{M}_{\mu}^i \mathcal{M}_0 \rangle.
\] (4.26)

Eqs. (4.22) – (4.25), together with (4.26), fix the complete set of one-loop counterterms that renormalize the parameters contained in the leading-order Lagrangian (2.1).

C. Renormalization group equations for the coefficients of \( L_2 \)

Knowing the renormalization constants, it is straightforward to derive the renormalization-group beta functions. For any parameter \( X \) we define
\[
\beta_X = 16 \pi^2 \frac{dX}{d \ln \mu}.
\] (4.27)

Recalling (4.19), we have
\[
\dot{X} = Z_X X \mu^\epsilon \Rightarrow \beta_X = - \frac{1}{2 \epsilon} \frac{d \Delta \hat{Z}_X}{d \ln \mu} X = \Delta \hat{Z}_X X.
\] (4.28)

In the last step, we have used
\[
\frac{d \Delta \hat{Z}_X}{d \ln \mu} = -2 \epsilon \Delta \hat{Z}_X,
\] (4.29)

which holds for \( X = g_\mu, g', v^2, F_n, V_n \), and also for \( \Delta \hat{Z}_X \rightarrow \Delta \mathcal{M}_n \). In all these cases, \( \Delta \hat{Z}_X \) is a homogeneous function of degree 2 in the weak couplings \( k = g_\mu, g', \sqrt{V_n}, \mathcal{M}_n / v \). This implies (4.29), since at tree level in \( d = 4 - 2 \epsilon \) dimensions
\[
\frac{dk}{d \ln \mu} = - \epsilon k, \text{ whereas } \frac{dF_n}{d \ln \mu} = \frac{d \mathcal{M}_n}{d \ln \mu} = 0.
\] (4.30)

Using (4.28) and the results of section IV B, we finally obtain
\[
\beta_{g_\mu} = - \frac{1}{2} \Delta \hat{Z}_G g_\mu, \quad \beta_g = - \frac{1}{2} \Delta \hat{Z}_W g, \quad \beta_{g'} = - \frac{1}{2} \Delta \hat{Z}_B g',
\] (4.31)
\[
\sum_{n=0}^{\infty} \beta_{F_n} \eta^n = - A_F + F' \int_0^{\eta} ds A_h(s) - F' \delta \left( F - \frac{\eta}{2} F' \right) \Delta \hat{Z}_\varphi,
\] (4.32)
\[
v^4 \sum_{n=2}^{\infty} \beta_{V_n} \eta^n = - A_V + V' \int_0^{\eta} ds A_h(s) - V' \delta \left( 2V - \frac{\eta}{2} V' \right) \Delta \hat{Z}_\varphi,
\] (4.33)
\[
\sum_{n=0}^{\infty} \beta_{\mathcal{M}_n} \eta^n = - A_{\mathcal{M}} + \mathcal{M}' \int_0^{\eta} ds A_h(s) - \mathcal{M}' \delta + \frac{\eta}{2} \mathcal{M}' \Delta \hat{Z}_\varphi.
\] (4.34)
\[
\beta_{v^2} = \Delta \hat{Z}_\varphi v^2.
\] (4.35)

The functions \( A_F, A_h, A_V \) and \( A_{\mathcal{M}} \) are defined in (4.7) – (4.9).

Eqs. (4.31) – (4.35) summarize the one-loop beta functions in the MS (or \( \overline{\text{MS}} \)) scheme for all the parameters of the leading-order electroweak chiral Lagrangian (2.1). We emphasize that the beta functions for the gauge couplings (4.31) are identical with their SM expressions, i.e., the contributions to the gauge-beta functions from the scalar sector only depend on the Goldstone modes and are independent of the Higgs couplings to gauge bosons. We have checked that (4.32) – (4.35) are in agreement with the SM results, when the corresponding limit is taken. In particular, the function on the right-hand side of (4.32) vanishes in this limit, as it should.
D. Running of the $hVV$ coupling $F_1$

As an example for the RGE running of anomalous Higgs couplings within the EWChL we consider $F_1$, the coupling of $h$ to a pair of vector bosons. From (4.32) we find

$$
\beta_{F_1} = \frac{3}{64 V_2} F_1 \left( F_1^2 - 4 F_2 \right) \left( 3 g^4 + 2 g^2 g'^2 + g'^4 \right) - \frac{9}{12} F_1 \left[ \frac{37}{4} \left( F_1^2 - 4 F_2 \right) + 17 (F_2 - 1) \right] - \frac{3}{16} g^2 F_1 (F_1^2 - 4) \\
+ V_2 \left[ F_1 \left( \frac{5}{2} \left( F_1^2 - 4 F_2 \right) + 4 (F_2 - 1) \right) + 12 F_3 \right] - \frac{F_2^2 - 4 F_2}{v^2 V_2} \langle \left\langle M_0^4 M_0 \left( M_0^4 + M_1^4 + M_2^4 \right) \right\rangle \rangle \\
+ 2 \frac{F_1 - 2}{v^2} \langle \left\langle \left( M_0^4 + M_1^4 \right) \right\rangle \rangle + 4 \frac{4}{v^2} \langle \left\langle \left( M_1^4 - M_0^4 \right) \left( M_1 - M_0 \right) \right\rangle \rangle
$$

(4.36)

In the limit of large top mass and Yukawa coupling this becomes

$$
\beta_{F_1} \approx (4 F_2 - F_1^2) 4 N_c \frac{m_t^4}{v^2 m_h^2}.
$$

(4.37)

Therefore

$$
\frac{F_1(\mu)}{2} \approx \frac{F_1(v)}{2} + \frac{\beta_{F_1}}{32 \pi^2} \ln \frac{\mu}{v} \approx \frac{F_1(v)}{2} + 0.036 (4 F_2 - F_1^2) \ln \frac{\mu}{v} \approx \frac{F_1(v)}{2} + 0.125 (4 F_2 - F_1^2),
$$

(4.38)

where in the last expression we have taken $\mu = 8$ TeV as a representative cut-off scale for the EWChL. Experimentally $F_1/2$ is close to 1 (within 10%), but $F_2$ may still deviate significantly from its SM value $F_2 = 1$. Eq. (4.38) indicates that the difference between $F_1$ at the electroweak scale $v$ and at the high scale $\mu$ may be appreciable.

V. RENORMALIZATION OF THE NEXT-TO-LEADING-ORDER COUNTERTERMS

Since $L_2$ is not renormalizable, one also needs to introduce a set $L_4$ of new operators in order to absorb all the divergences generated at one loop. The structure of $L_4$ is entirely dictated by the symmetries of $L_2$ and by power counting. A complete set of counterterms at the one-loop level is already available [14] and we will stick to the operator basis displayed there.

In the previous Section we dealt with the divergences that can be absorbed through renormalization of the leading-order Lagrangian $L_2$. The remaining divergences have to be eliminated through the renormalization of the counterterms at next-to-leading order. These remaining divergences are given by

$$
L_{\text{div}} - \Delta L_2 = \Delta L_{\beta_1} + \Delta L_{U_{H_4}} + \Delta L_{\psi^2_{U_{H_4}}} + \Delta L_{\psi^2_{U_{H_2}}} + \Delta L_{\psi^2_{U_{H_1}}},
$$

(5.1)

with

$$
\Delta L_{\beta_1} \equiv \Delta^{(0)} L_{\beta_1} + \Delta^{(1)} L_{\beta_1} = -\frac{1}{16 \pi^2} \frac{1}{d-4} \left( \frac{5}{6} \right) g^2 v^2 (\kappa^2 - 1) F \langle \tau_L L^\mu \rangle \langle \tau_L L^\mu \rangle
$$

(5.2)

$$
\Delta L_{U_{H_4}} \equiv \Delta^{(0)} L_{U_{H_4}} = -\frac{1}{16 \pi^2} \frac{1}{d-4} \left\{ \left( \frac{(\kappa^2 - 1)^2}{12} + \frac{F^2 B^2}{8} \right) \langle L^\mu L_\mu \rangle^2 + \frac{2}{3} \kappa^2 \langle L^\mu L_\nu \rangle \partial^\mu \eta ^2 \partial^\nu \eta + \frac{3}{2} B^2 \langle \partial^\mu \eta \partial^\nu \eta \rangle^2 \right\},
$$

(5.3)

$$
\Delta L_{\psi^2_{U_{H_2}}} \equiv \Delta^{(1/2)} L_{\psi^2_{U_{H_2}}} + \Delta^{(1)} L_{\psi^2_{U_{H_2}}} =
$$

$$
= \frac{1}{16 \pi^2} \frac{1}{d-4} \left\{ \bar{\psi}_L U \bar{\psi}_{L^1} (M, M^1)^{U^1} \bar{\psi}_L + \bar{\psi}_L U \bar{\psi}_{L^1} (M, M^1)^{U^1} \bar{\psi}_L \\
+ \frac{\kappa^2 - 1}{12} \left[ 2 g^2 \langle \tau_L L^\nu \rangle \bar{e}_R \gamma_\nu e_R - 2 g^2 \langle \tau_L L^\nu \rangle \bar{\psi}_L \gamma_\nu Y_{\psi} \gamma_\nu \psi \rangle \psi \rangle - \frac{4}{3} F^2 \langle \tau_L L^\nu \rangle \bar{u}_R \gamma_\nu u_R + \frac{2}{3} g^2 \langle \tau_L L^\nu \rangle \bar{d}_R \gamma_\nu d_R \right] \\
- \frac{F^{-1}}{v^2} \bar{\psi}_R M^1 U^\dagger \bar{\psi} U M \psi_R - \frac{1}{v^2} \bar{\psi}_R M^1 U^\dagger \bar{\psi} U M' \psi_R + \frac{\kappa}{v^2} F^{-1/2} \langle \bar{\psi}_R M^1 U^\dagger \bar{\psi} U M' \psi_R + \text{h.c.} \rangle \right\},
$$

(5.4)
\[
\Delta \mathcal{L}_{\psi^2 UhD^4} = \Delta^{(1/2)} \mathcal{L}_{\psi^2 UhD^4} \\
= -\frac{1}{16 \pi^2} \frac{1}{d-4} \left\{ \langle L^\mu L_\mu \rangle \left[ \frac{F B}{2 v^2} \bar{\psi}_L U \mathcal{M}' \psi_R - \frac{\kappa^2 - 1}{2} \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R + \text{h.c.} \right] \\
+ 2 \kappa' \frac{\partial^\mu \eta}{v} \left( i \bar{\psi}_L L_\mu U \left( F^{-1/2} \mathcal{M}' \right) \psi_R + \text{h.c.} \right) + \frac{3 B}{2 v^2} \partial^\mu \eta \partial_\mu \eta \left( \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R + \text{h.c.} \right) \right\} ,
\]

\[
\Delta \mathcal{L}_{\psi^4 Uh} = \Delta^{(1/2)} \mathcal{L}_{\psi^4 Uh} \\
= -\frac{1}{16 \pi^2} \frac{1}{d-4} \left\{ 3 \frac{F-2}{2 v^4} \left( \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R + \text{h.c.} \right)^2 + \frac{1}{2 v^4} \left( \bar{\psi}_L U \mathcal{M}'' \psi_R + \text{h.c.} \right)^2 \\
+ \frac{4}{v^4} \left( i \bar{\psi}_L U T^a \left( F^{-1/2} \mathcal{M}' \right) \psi_R + \text{h.c.} \right)^2 \right\} .
\]

In the expression of \( \Delta \mathcal{L}_{\psi^2 UhD^4} \), the quantity
\[
\mathcal{L}(\mathcal{M}, \mathcal{M}') = g^2 \kappa^2 - \frac{1}{24} + \frac{1}{2 v^2} \left[ \mathcal{M}' \mathcal{M}' + F^{-1} \mathcal{M} \mathcal{M}' \right] - \frac{\kappa}{v^2} F^{-1/2} \mathcal{M}' \mathcal{M}' \\
- \frac{1}{2 v^2} \int_0^\eta ds \left[ \mathcal{M}' \mathcal{M}' + \mathcal{M} \mathcal{M}' + F^{-1} \mathcal{M'} \mathcal{M} + F^{-1} \mathcal{M} \mathcal{M}' \right]
\]
has been introduced.

**A. Renormalization of the counterterm \( \beta_1 \)**

Let us start with the elimination of the divergent term \( \Delta \mathcal{L}_{\beta_1} \). It requires the counterterm \( \mathcal{L}_{\beta_1} \) given in [14], namely (note the slight change in notation as compared to this reference)
\[
\mathcal{L}_{\beta_1} = -\hat{v}^2 \left( \tau_L L^\mu \right) \left[ \beta_1 + F_{\beta_1}(\eta) \right] , \quad F_{\beta_1}(\eta) = \sum_{n \geq 1} f_{\beta_1,n} \eta^n .
\]

In order to perform the renormalization (for instance, in the \( \overline{\text{MS}} \) scheme), one interprets the coefficients as unrenormalized ones and writes
\[
\hat{f}_{\beta_1,n} = f_{\beta_1,n}(\mu) + \frac{\beta_{\beta_1,n}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \beta_{\beta_1} = \beta_1(\mu) + \frac{\gamma_{\beta_1}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{F}_{\beta_1}(\eta) = F_{\beta_1}(\eta; \mu) + \frac{\Gamma_{\beta_1}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\gamma_{\beta_1} = \frac{5}{24} g^2 (F_1^2 - 4) , \quad \Gamma_{\beta_1}(\eta) = \frac{5}{6} g^2 \left[ \tilde{\kappa}(\tilde{\kappa} + F_1) + \bar{F}(\kappa^2 - 1) \right] .
\]

**B. Renormalization of the counterterms in the class \( UhD^4 \)**

The elimination of the divergences contained in \( \Delta^{(0)} \mathcal{L}_{UhD^4} \) requires five counterterms of the class \( UhD^4 \),
\[
\mathcal{L}_{UhD^4} = \sum_{i=1}^{15} \mathcal{O}_{D_i} \left[ c_{D_i} + F_{D_i}(\eta) \right] , \quad F_{D_i}(\eta) = \sum_{n \geq 1} f_{D_i,n} \eta^n ,
\]
\[
\mathcal{O}_{D_i} \text{ for } i = 1, 2, 7, 8, 11 . \quad \text{Notice that in contrast to Ref. [14], we have not written the overall factor } v^2 / \Lambda^2 ,
\]
and we have introduced the specific couplings \( c_{D_i} \) in a different way. The renormalization proceeds as previously,
\[
\hat{c}_{D_1} = c_{D_1}(\mu) + \frac{\gamma_{D_1}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_1}(\eta) = F_{D_1}(\eta; \mu) + \frac{\Gamma_{D_1}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{c}_{D_2} = c_{D_2}(\mu) + \frac{\gamma_{D_2}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_2}(\eta) = F_{D_2}(\eta; \mu) + \frac{\Gamma_{D_2}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{c}_{D_3} = c_{D_3}(\mu) + \frac{\gamma_{D_3}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_3}(\eta) = F_{D_3}(\eta; \mu) + \frac{\Gamma_{D_3}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{c}_{D_4} = c_{D_4}(\mu) + \frac{\gamma_{D_4}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_4}(\eta) = F_{D_4}(\eta; \mu) + \frac{\Gamma_{D_4}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{c}_{D_5} = c_{D_5}(\mu) + \frac{\gamma_{D_5}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_5}(\eta) = F_{D_5}(\eta; \mu) + \frac{\Gamma_{D_5}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] ,
\]
\[
\hat{c}_{D_6} = c_{D_6}(\mu) + \frac{\gamma_{D_6}}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] , \quad \hat{F}_{D_6}(\eta) = F_{D_6}(\eta; \mu) + \frac{\Gamma_{D_6}(\eta)}{16 \pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4 \pi - \gamma_E \right) \right] .
\]
with
\[ \gamma_{D1} = \frac{1}{12} \left( \frac{F_1^2}{4} - 1 \right)^2 + \frac{1}{8} \left( \frac{F_1^2}{4} - F_2 \right)^2, \quad \gamma_{D2} = \frac{1}{6} \left( \frac{F_1^2}{4} - 1 \right)^2, \]
\[ \gamma_{D7} = \frac{1}{6} \left( \frac{F_1^2}{4} - F_2 \right) \left( \frac{7}{4} F_1^2 - F_2 - 6 \right), \quad \gamma_{D8} = \frac{2}{3} \left( \frac{F_1^2}{4} - F_2 \right)^2, \quad \gamma_{D11} = \frac{3}{2} \left( \frac{F_1^2}{4} - F_2 \right)^2, \]
and
\[ \Gamma_{D1} = \frac{1}{12} \bar{\kappa}(\bar{\kappa} + F_1) \left[ \bar{\kappa}(\bar{\kappa} + F_1) + \frac{F_1^2}{2} - 2 \right] - \frac{1}{8} \left( FB - \frac{F_1^2}{4} + F_2 \right) \left( FB + \frac{F_1^2}{4} - F_2 \right), \]
\[ \Gamma_{D2} = \frac{1}{6} \bar{\kappa}(\bar{\kappa} + F_1) \left[ \bar{\kappa}(\bar{\kappa} + F_1) + \frac{F_1^2}{2} - 2 \right], \]
\[ \Gamma_{D7} = - \bar{B} \left( \frac{F_2}{4} - 1 \right) - B \bar{\kappa}(\bar{\kappa} + F_1) - \frac{F_2}{6} \left( \frac{F_1^2}{4} - F_2 \right)^2 - \frac{F_1^2}{6} \bar{B} \left( \bar{B} + \frac{F_1^2}{2} - 2F_2 \right), \]
\[ \Gamma_{D8} = \frac{2}{3} dB \left( \bar{B} + \frac{F_1^2}{2} - 2F_2 \right) + \frac{2}{3} F \left( \frac{F_1^2}{4} - F_2 \right)^2, \]
\[ \Gamma_{D11} = \frac{3}{2} \bar{B} \left( \bar{B} + \frac{F_1^2}{2} - 2F_2 \right). \]

C. Renormalization of the counterterms in the class \( \psi^2 U h D \)

In order to proceed with the renormalization of the operators of the class \( \psi^2 U h D \), we first need to express \( \mathcal{L}_{\psi^2 U h D} \) in terms of the basis operators given in Ref. [14]. Recalling that \( \mathcal{M} \) and therefore also \( \mathcal{E} \) of (5.7) are diagonal in \( SU(2) \) space, one has
\[ \mathcal{E} = \frac{1}{2} (\mathcal{E}) + 2T^3(T^3\mathcal{E}), \]
so that
\[ \psi_L U \mathcal{U}^\dagger \psi_L = \frac{1}{2} \bar{\psi}_L (\mathcal{E}) \psi_L + 2 \bar{\psi}_L (\mathcal{E} T^3) \bar{T} L \psi_L. \]

Note that traces are over \( SU(2) \) only, i.e., the expressions on the right-hand side of (5.16) are still matrices in flavor space.

Upon using the identity (3.13), (5.17) becomes \( (P_L = \frac{1}{2} \pm T^3, P_{12} = T^1 + iT^2, P_{21} = T^1 - iT^2) \)
\[ \bar{\psi}_L U \mathcal{U}^\dagger \psi_L = \bar{\psi}_L T^\mu (\mathcal{E} T^\mu) \bar{T} L \psi_L \left( \tau_L L_{\mu} \right) + \bar{\psi}_L T^\mu (\mathcal{E}) U P_{12} U^\dagger \psi_L \left( U P_{21} U^\dagger L_{\mu} \right) + \bar{\psi}_L T^\mu (\mathcal{E}) U P_{21} U^\dagger \psi_L \left( U P_{12} U^\dagger L_{\mu} \right) \]
\[ + \bar{\psi}_L T^\mu (\mathcal{E} T^3) \psi_L \left( \tau_L L_{\mu} \right) + \bar{\psi}_L T^\mu (\mathcal{E} T^3) U P_{12} U^\dagger \psi_L \left( U P_{21} U^\dagger L_{\mu} \right) - \bar{\psi}_L T^\mu (\mathcal{E} T^3) U P_{21} U^\dagger \psi_L \left( U P_{12} U^\dagger L_{\mu} \right) \]
\[ = \bar{\psi}_L T^\mu (\mathcal{E} T^3) \psi_L \left( \tau_L L_{\mu} \right) + \bar{\psi}_L T^\mu (\mathcal{E}) T^3 \psi_L \left( \tau_L L_{\mu} \right) \]
\[ + \bar{\psi}_L T^\mu (\mathcal{E} P_{\pm}) U P_{12} U^\dagger \psi_L \left( U P_{21} U^\dagger L_{\mu} \right) + \bar{\psi}_L T^\mu (\mathcal{E} P_{\pm}) U P_{21} U^\dagger \psi_L \left( U P_{12} U^\dagger L_{\mu} \right), \]
due to the relations \( T^3 P_{12} = +P_{12}/2 \) and \( T^3 P_{21} = -P_{21}/2 \).

The terms involving the right-handed fermion fields can essentially be handled along similar lines. One first establishes an identity similar to (3.13),
\[ \bar{\psi}_R O_1 U^\dagger \mathcal{L} O_2 \psi_R = 2 \bar{\psi}_R T^\mu O_1 T^3 O_2 \psi_R \left( U T^3 U^\dagger L_{\mu} \right) \]
\[ = \bar{\psi}_R T^\mu O_1 O_2 \psi_R \left( U P_{21} U^\dagger L_{\mu} \right) + \bar{\psi}_R T^\mu O_1 O_2 \psi_R \left( U P_{12} U^\dagger L_{\mu} \right) \]
\[ + 2 \bar{\psi}_R T^3 O_1 O_2 \psi_R \left( \tau_L L_{\mu} \right). \]

Then one obtains, for instance,
\[ \bar{\psi}_R \mathcal{M} U^\dagger \mathcal{L} U \mathcal{M} \psi_R = 2 \bar{\psi}_R T^\mu \mathcal{M} T^3 \mathcal{M} \psi_R \left( \tau_L L_{\mu} \right) + \bar{\psi}_R T^\mu \mathcal{M} \left( P_{12} \mathcal{M} \psi_R \left( U P_{21} U^\dagger L_{\mu} \right) + \bar{\psi}_R T^\mu \mathcal{M} \left( P_{21} \mathcal{M} \psi_R \left( U P_{12} U^\dagger L_{\mu} \right) \right. \right. \]
\[ \begin{align*}
\Delta = \bar{u}_R \gamma^\mu M_1^\mu M_u u_R \langle \tau_L L_\mu \rangle - \bar{d}_R \gamma^\mu M_1^\mu d_R \langle \tau_L L_\mu \rangle - \bar{e}_R \gamma^\mu M_1^\mu e_R \langle \tau_L L_\mu \rangle \\
+ \bar{u}_R \gamma^\mu M_1^\mu M_d u_R \langle U P_{21} U^\dagger L_\mu \rangle + \bar{d}_R \gamma^\mu M_1^\mu M_u u_R \langle U P_{12} U^\dagger L_\mu \rangle.
\end{align*} \] (5.20)

These divergences can now be removed through the renormalization of the operators \( \mathcal{O}_{\psi V} \) and \( \mathcal{O}_{\psi V}^\dagger \) of \( \mathcal{L}_{\psi^2 U h D} \).

Explicitly, one has (notice that the functions \( F_{\psi V} \) are actually matrices in generation space)

\[ F_{\psi V} (\eta) = F_{\psi V} (\eta; \mu) + \frac{\Gamma_{\psi V} (\eta)}{16 \pi^2} \mu^{d-4} \left[ 1 - \frac{1}{2} (\ln 4 \pi - \gamma_E) \right], \] (5.21)

with

\[ \begin{align*}
\Gamma_{\psi V 1} &= - \frac{1}{2} (\mathcal{L}_u - \mathcal{L}_d + \text{h.c.}) + g^2 \frac{\kappa^2 - 1}{36}, \\
\Gamma_{\psi V 2} &= - \mathcal{L}_u - \mathcal{L}_d + \text{h.c.}, \\
\Gamma_{\psi V 3} &= - \mathcal{L}_u - \mathcal{L}_d^\dagger, \\
\Gamma_{\psi V 4} &= + \frac{1}{v^2} \left[ F^{-1} M_{1 u}^\dagger M_u + M_{1 u}^\dagger M_u' - \kappa F^{-1/2} (M_{1 u}^\dagger M_u' + M_{1 u}^\dagger M_u) \right] + g^2 \frac{\kappa^2 - 1}{9}, \\
\Gamma_{\psi V 5} &= - \frac{1}{v^2} \left[ F^{-1} M_{1 d}^\dagger M_d + M_{1 d}^\dagger M_d' - \kappa F^{-1/2} (M_{1 d}^\dagger M_d' + M_{1 d}^\dagger M_d) \right] - g^2 \frac{\kappa^2 - 1}{18}, \\
\Gamma_{\psi V 6} &= + \frac{1}{v^2} \left[ F^{-1} M_{1 u}^\dagger M_u + M_{1 u}^\dagger M_u' - \kappa F^{-1/2} (M_{1 u}^\dagger M_u' + M_{1 u}^\dagger M_u) \right], \\
\Gamma_{\psi V 7} &= + \frac{1}{2} (\mathcal{L}_e + \text{h.c.}) - g^2 \frac{\kappa^2 - 1}{12}, \\
\Gamma_{\psi V 8} &= - \mathcal{L}_e + \text{h.c.}, \\
\Gamma_{\psi V 9} &= - \mathcal{L}_e^\dagger, \\
\Gamma_{\psi V 10} &= - \frac{1}{v^2} \left[ F^{-1} M_{1 e}^\dagger M_e + M_{1 e}^\dagger M_e' - \kappa F^{-1/2} (M_{1 e}^\dagger M_e' + M_{1 e}^\dagger M_e) \right] - g^2 \frac{\kappa^2 - 1}{6}.
\end{align*} \] (5.22)

In these expressions, \( \mathcal{L}_j \equiv \mathcal{L}(M_j, M_j^\dagger) \) with \( j = u, d, e \).

**D. Renormalization of the counterterms in the class \( \psi^2 U h D^2 \)**

Recalling that the matrix \( \mathcal{M} \) is \( SU(2) \)-diagonal, one has the following relations:

\[ \begin{align*}
\bar{\psi}_L U F(\mathcal{M}) \psi_R &= \bar{q}_L U P_{12} F(\mathcal{M}) q_R + \bar{q}_L U P_{21} F(\mathcal{M}) q_R + \bar{l}_L U P_{1} F(\mathcal{M}) l_R, \\
\bar{\psi}_L L u U F(\mathcal{M}) \psi_R &= 2 \bar{\psi}_L U T^a F(\mathcal{M}) \psi_R \langle U T^a U^\dagger L_\mu \rangle \\
&= \bar{\psi}_L U P_{12} F(\mathcal{M}) \psi_R \langle U P_{21} U^\dagger L_\mu \rangle + \bar{\psi}_L U P_{21} F(\mathcal{M}) \psi_R \langle U P_{12} U^\dagger L_\mu \rangle + 2 \bar{\psi}_L U T^3 F(\mathcal{M}) \psi_R \langle \tau_L L_\mu \rangle \\
&= \bar{\psi}_L U P_{12} F(\mathcal{M}) \psi_R \langle U P_{21} U^\dagger L_\mu \rangle + \bar{\psi}_L U P_{21} F(\mathcal{M}) \psi_R \langle U P_{12} U^\dagger L_\mu \rangle + \bar{\psi}_L U P_{1} F(\mathcal{M}) \psi_R \langle \tau_L L_\mu \rangle - \bar{\psi}_L U P_{1} F(\mathcal{M}) \psi_R \langle \tau_L L_\mu \rangle \\
&= \bar{q}_L U P_{12} F(\mathcal{M}) q_R \langle U P_{21} U^\dagger L_\mu \rangle + \bar{q}_L U P_{21} F(\mathcal{M}) q_R \langle U P_{12} U^\dagger L_\mu \rangle + \bar{l}_L U P_{1} F(\mathcal{M}) l_R \langle U P_{12} U^\dagger L_\mu \rangle + \bar{l}_L U P_{1} F(\mathcal{M}) l_R \langle \tau_L L_\mu \rangle - \bar{l}_L U P_{1} F(\mathcal{M}) l_R \langle \tau_L L_\mu \rangle.
\end{align*} \] (5.24)

Using them one can rewrite \( \Delta^{(1/2)} \mathcal{L}_{\psi^2 U h D^2} \) as

\[ \begin{align*}
\Delta^{(1/2)} \mathcal{L}_{\psi^2 U h D^2} = - \frac{1}{16 \pi^2} \frac{1}{d-4} \left( \bar{q}_L U P_+ \left[ \frac{F B}{v^2} \mathcal{M}_u'' - \frac{\kappa^2 - 1}{v^2} \left( \frac{F'}{2} \mathcal{M}_u' - \mathcal{M}_u \right) \right] \right) q_R \langle L^\mu L_\mu \rangle \\
+ \bar{q}_L U P_+ \left[ \frac{F B}{v^2} \mathcal{M}_d'' - \frac{\kappa^2 - 1}{v^2} \left( \frac{F'}{2} \mathcal{M}_d' - \mathcal{M}_d \right) \right] q_R \langle L^\mu L_\mu \rangle \\
+ \bar{l}_L U P_+ \left[ \frac{F B}{v^2} \mathcal{M}_e'' - \frac{\kappa^2 - 1}{v^2} \left( \frac{F'}{2} \mathcal{M}_e' - \mathcal{M}_e \right) \right] l_R \langle L^\mu L_\mu \rangle \\
+ 2 \frac{1}{v^2} \delta^\mu_\eta \left( \bar{q}_L U P_{12} (F^{-1/2} \mathcal{M}_d) \right) q_R \langle U P_{21} U^\dagger L_\mu \rangle + \bar{q}_L U P_{21} (F^{-1/2} \mathcal{M}_u) q_R \langle U P_{12} U^\dagger L_\mu \rangle
\end{align*} \]
These divergences are removed through the renormalization of the operators $O_{\psi S_i}$ (and $O_{\psi S_i}^\dagger$) of $\mathcal{L}_{\psi^3U^4D^2}$, with (in order of appearance in the previous formula) $i = 1, 2, 7, 12, 13, 17, 10, 11, 16, 14, 15, 18$. Explicitly, one has (notice that the functions $F_{\psi S_i}$ are actually matrices in generation space)

$$
\tilde{F}_{\psi S_i}(\eta) = F_{\psi S_i}(\eta; \mu) + \frac{\Gamma_{\psi S_i}(\eta)}{16\pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2}(\ln 4\pi - \gamma_E) \right],
$$

with

\begin{align*}
\Gamma_{\psi S_1} &= \frac{FB}{2v^2} \mathcal{M}_u' - \frac{\kappa^2 - 1}{F v^2} \left( \frac{F'}{2} \mathcal{M}_u - \mathcal{M}_u \right), \\
\Gamma_{\psi S_2} &= \frac{FB}{2v^2} \mathcal{M}_d' - \frac{\kappa^2 - 1}{F v^2} \left( \frac{F'}{2} \mathcal{M}_d - \mathcal{M}_d \right), \\
\Gamma_{\psi S_7} &= \frac{FB}{2v^2} \mathcal{M}_e' - \frac{\kappa^2 - 1}{F v^2} \left( \frac{F'}{2} \mathcal{M}_e - \mathcal{M}_e \right), \\
\Gamma_{\psi S_{10}} &= \frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_u)', \\
\Gamma_{\psi S_{11}} &= -\frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_d)', \\
\Gamma_{\psi S_{12}} &= \frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_d)', \\
\Gamma_{\psi S_{13}} &= \frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_u)', \\
\Gamma_{\psi S_{14}} &= \frac{3B}{F v^2} \left( \frac{F'}{2} \mathcal{M}_u - \mathcal{M}_u \right), \\
\Gamma_{\psi S_{15}} &= \frac{3B}{F v^2} \left( \frac{F'}{2} \mathcal{M}_d - \mathcal{M}_d \right), \\
\Gamma_{\psi S_{16}} &= \frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_e)', \\
\Gamma_{\psi S_{17}} &= \frac{2in'}{v^2} (F^{-1/2} \mathcal{M}_e)', \\
\Gamma_{\psi S_{18}} &= \frac{3B}{F v^2} \left( \frac{F'}{2} \mathcal{M}_e - \mathcal{M}_e \right).
\end{align*}

E. Renormalization of the counterterms in the class $\psi^4U\bar{h}$

The expression to start with reads

$$
\Delta \mathcal{L}_{\psi^4U\bar{h}} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{3F^{-2}}{2v^4} \left( \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R + \hc \right)^2 + \frac{1}{2v^4} \left( \bar{\psi}_L U \mathcal{M}'' \psi_R + \hc \right)^2 + \frac{4}{v^4} \left( i\bar{\psi}_L U T^a \left( \mathcal{M}^{-1/2} \right) \psi_R + \hc \right)^2 \right\}
$$

$$
= -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \frac{3F^{-2}}{2v^4} \left( \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R \right)^2 + \frac{3F^{-2}}{2v^4} \left( \bar{\psi}_R \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) U^\dagger \psi_L \right)^2 \right\}
$$
\[ + \frac{3F^{-2}}{v^4} \left( \bar{\psi}_L U \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_R \right) \left( \bar{\psi}_R U^\dagger \left( \frac{F'}{2} \mathcal{M}' - \mathcal{M} \right) \psi_L \right) \]
\[ + \frac{1}{2v^4} \left( \bar{\psi}_L U \mathcal{M}' \psi_R \right)^2 + \frac{1}{2v^4} \left( \bar{\psi}_R \mathcal{M}' U^\dagger \psi_R \right)^2 + \frac{1}{v^4} \left( \bar{\psi}_L U \mathcal{M}' \psi_R \right) \left( \bar{\psi}_R \mathcal{M}' U^\dagger \psi_L \right) \]
\[ - \frac{4}{v^4} \left( \bar{\psi}_L U T^a \left( F^{-1/2} \mathcal{M} \right)' \psi_R \right) \left( \bar{\psi}_L U T^a \left( F^{-1/2} \mathcal{M} \right)' \psi_R \right) \]
\[ - \frac{4}{v^4} \left( \bar{\psi}_R \left( F^{-1/2} \mathcal{M} \right)' T^a U^\dagger \psi_R \right) \left( \bar{\psi}_R \left( F^{-1/2} \mathcal{M} \right)' T^a U^\dagger \psi_R \right) \]
\[ + \frac{8}{v^4} \left( \bar{\psi}_L U T^a \left( F^{-1/2} \mathcal{M} \right)' \psi_R \right) \left( \bar{\psi}_R \left( F^{-1/2} \mathcal{M} \right)' T^a U^\dagger \psi_R \right) \].

(5.28)

We have to consider the following structures:

\[ (\bar{\psi}_L U \mathcal{F}(\mathcal{M}) \psi_R)^2 + \text{h.c.}, \quad (\bar{\psi}_L U \mathcal{F}(\mathcal{M}) \psi_R)(\bar{\psi}_R \mathcal{F}(\mathcal{M}) U^\dagger \psi_L) \]

(5.29)

and

\[ (\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R)(\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R) + \text{h.c.}, \quad (\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R)(\bar{\psi}_R \mathcal{F}(\mathcal{M}) U^\dagger T^a U^\dagger \psi_L) \]

(5.30)

and decompose them onto the operator basis of Ref. [14]. Notice that the coefficients of the four-fermion operators are actually rank-four tensors in generation space. We use the notation

\[ \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \mathcal{O} \equiv \mathcal{F}^{(1)}_{i_j} \mathcal{F}^{(2)}_{k_l} \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \]

\[ \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \mathcal{O} \equiv \mathcal{F}^{(1)}_{i_l} \mathcal{F}^{(2)}_{k_j} \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \]

(5.31)

for an operator \( \mathcal{O} \equiv \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \), where \( i, j, k, l \) are generation indices. For the basis operators \( \mathcal{O}_{AB} \) to be used below, we follow the notation of [24].

Using Eq. (5.23), one obtains

\[ (\bar{\psi}_L U \mathcal{F}(\mathcal{M}) \psi_R)^2 = \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right)^2 + \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right)^2 + \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{e}) \psi_R \right)^2 \]

\[ + \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{e}) \psi_R \right)^2 + \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \]

\[ + \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \]

\[ + 2 \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \]

\[ + 2 \left( \bar{\psi}_L U P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \]

\[ = \mathcal{F}(\mathcal{M}_{u}) \otimes \mathcal{F}(\mathcal{M}_{u}) \mathcal{O}_{FY1} + \mathcal{F}(\mathcal{M}_{d}) \otimes \mathcal{F}(\mathcal{M}_{d}) \mathcal{O}_{FY3} + \mathcal{F}(\mathcal{M}_{e}) \otimes \mathcal{F}(\mathcal{M}_{e}) \mathcal{O}_{FY10} \]

\[ + 2 \mathcal{F}(\mathcal{M}_{u}) \otimes \mathcal{F}(\mathcal{M}_{d}) \mathcal{O}_{ST5} + 2 \mathcal{F}(\mathcal{M}_{u}) \otimes \mathcal{F}(\mathcal{M}_{e}) \mathcal{O}_{ST3} + 2 \mathcal{F}(\mathcal{M}_{d}) \otimes \mathcal{F}(\mathcal{M}_{e}) \mathcal{O}_{FY7}, \]

(5.32)

and

\[ (\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R)(\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R) = \left( \bar{\psi}_L U T^a P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \left( \bar{\psi}_L U T^a P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \]

\[ + \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \]

\[ + \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{e}) \psi_R \right) \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{e}) \psi_R \right) \]

\[ + 2 \left( \bar{\psi}_L U T^a P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \]

\[ + 2 \left( \bar{\psi}_L U T^a P_{-} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \left( \bar{\psi}_L U T^a P_{+} \mathcal{F}(\mathcal{M}_{d}) \psi_R \right) \]

\[ = \mathcal{F}(\mathcal{M}_{u}) \otimes \mathcal{F}(\mathcal{M}_{u}) \mathcal{O}_{FY1} + \mathcal{F}(\mathcal{M}_{d}) \otimes \mathcal{F}(\mathcal{M}_{d}) \mathcal{O}_{FY3} + \mathcal{F}(\mathcal{M}_{e}) \otimes \mathcal{F}(\mathcal{M}_{e}) \mathcal{O}_{FY10} \]

(5.33)

Next, since

\[ T^a \otimes T^a = \frac{1}{2} P_{12} \otimes P_{21} + \frac{1}{2} P_{21} \otimes P_{12} + \frac{1}{4} P_{+} \otimes P_{+} + \frac{1}{4} P_{-} \otimes P_{+} - \frac{1}{4} P_{+} \otimes P_{-} - \frac{1}{4} P_{-} \otimes P_{+}, \]

(5.34)

one has

\[ T^a P_{+} \otimes T^a P_{+} = \frac{1}{4} P_{+} \otimes P_{+}, \quad T^a P_{+} \otimes T^a P_{-} = \frac{1}{2} P_{21} \otimes P_{12} - \frac{1}{4} P_{+} \otimes P_{-}, \quad T^a P_{-} \otimes T^a P_{+} = \frac{1}{2} P_{12} \otimes P_{21} - \frac{1}{4} P_{-} \otimes P_{+}, \]

(5.35)

so that

\[ (\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R)(\bar{\psi}_L U T^a \mathcal{F}(\mathcal{M}) \psi_R) = \frac{1}{4} \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \left( \bar{\psi}_L U P_{+} \mathcal{F}(\mathcal{M}_{u}) \psi_R \right) \]

(5.36)
so that
\[
\psi_{\mathcal{L}} U P - \mathcal{F} (M_d) q_R (\bar{q}_L U P - \mathcal{F} (M_d) q_R) \\
+ \frac{1}{4} (\bar{q}_L U P - \mathcal{F} (M_u) l_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
+ \frac{1}{4} (\bar{q}_L U P - \mathcal{F} (M_u) q_R) (\bar{q}_L U P_2 \mathcal{F} (M_d) q_R) \\
+ \frac{1}{2} (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{q}_L U P - \mathcal{F} (M_d) q_R) \\
+ \frac{1}{2} (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
- \frac{1}{2} (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
+ \frac{1}{4} (\bar{q}_L U P - \mathcal{F} (M_d) q_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
= \frac{1}{4} \mathcal{F} (M_u) \otimes \mathcal{F} (M_d) O_{FY1} + \frac{1}{4} \mathcal{F} (M_d) \otimes \mathcal{F} (M_d) O_{FY3} \\
+ \frac{1}{4} \mathcal{F} (M_u) \otimes \mathcal{F} (M_d) O_{FY10} + \mathcal{F} (M_u) \otimes \mathcal{F} (M_d) O_{ST6} \\
- \frac{3}{2} \mathcal{F} (M_u) \otimes \mathcal{F} (M_d) O_{ST5} + \mathcal{F} (M_u) \otimes \mathcal{F} (M_c) O_{ST10} \\
- \frac{1}{2} \mathcal{F} (M_u) \otimes \mathcal{F} (M_c) O_{ST9} + \frac{1}{2} \mathcal{F} (M_d) \otimes \mathcal{F} (M_c) O_{FY7}. \tag{5.36}
\]

In turn, the structure \((\bar{\psi}_L \mathcal{U} \mathcal{F} (\mathcal{M}) \psi_R) (\bar{\psi}_R \mathcal{F} (\mathcal{M})^\dagger U^\dagger \psi_L)\) can be simplified using Eq. (5.23):
\[
(\bar{\psi}_L \mathcal{U} \mathcal{F} (\mathcal{M}) \psi_R) (\bar{\psi}_R \mathcal{F} (\mathcal{M})^\dagger U^\dagger \psi_L) = \left[ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
\times (\bar{q}_L U P - \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{I}_L U P - \mathcal{F} (M_e) l_R) (\bar{I}_R \mathcal{F} (M_e))^\dagger P_U^\dagger l_L \\
+ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{q}_L U P - \mathcal{F} (M_d) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
+ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_R \mathcal{F} (M_e))^\dagger P_U^\dagger l_L) \right] \\
= (\bar{q}_L U P - \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{I}_L U P - \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{q}_L U P - \mathcal{F} (M_d) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{I}_L U P - \mathcal{F} (M_u) q_R) (\bar{q}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L \\
+ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_L U P - \mathcal{F} (M_e) l_R) \\
+ (\bar{q}_L U P + \mathcal{F} (M_u) q_R) (\bar{I}_R \mathcal{F} (M_e))^\dagger P_U^\dagger l_L) \right]. \tag{5.37}
\]

The terms obtained this way can be decomposed on the operator basis through the use of the Fierz identities
\[
(\bar{q}_L U P + \mathcal{F} (M_u) q_R)(\bar{q}_R \mathcal{F} (M_d))^\dagger P_L U^\dagger q_L) = - \mathcal{F} (M_u) \otimes \mathcal{F} (M_d)^\dagger \left[ \frac{1}{12} O_{LR1} + \frac{1}{2} O_{LR2} + \frac{1}{6} O_{LR10} - O_{LR11} \right], \tag{5.38}
\]
\[
(\bar{q}_L U P - \mathcal{F} (M_u) q_R)(\bar{q}_R \mathcal{F} (M_d))^\dagger P_L U^\dagger q_L) = - \mathcal{F} (M_d) \otimes \mathcal{F} (M_d)^\dagger \left[ \frac{1}{12} O_{LR3} + \frac{1}{2} O_{LR4} - \frac{1}{12} O_{LR12} + O_{LR13} \right],
\]
\[
(\bar{I}_L U P + \mathcal{F} (M_u) l_R)(\bar{I}_R \mathcal{F} (M_c))^\dagger P_L U^\dagger q_L) = - \mathcal{F} (M_c) \otimes \mathcal{F} (M_c)^\dagger \left[ \frac{1}{4} O_{LR8} - \frac{1}{2} O_{LR17} \right],
\]
\[
(\bar{q}_L U P - \mathcal{F} (M_d) q_R)(\bar{I}_R \mathcal{F} (M_e))^\dagger P_L U^\dagger q_L) = - \mathcal{F} (M_e) \otimes \mathcal{F} (M_e)^\dagger \left[ \frac{1}{4} O_{LR9} - \frac{1}{2} O_{LR18} \right].
\]

so that
\[
(\bar{\psi}_L \mathcal{U} \mathcal{F} (\mathcal{M}) \psi_R) (\bar{\psi}_R \mathcal{F} (\mathcal{M})^\dagger U^\dagger \psi_L) = - \mathcal{F} (M_u) \otimes \mathcal{F} (M_u)^\dagger \left[ \frac{1}{12} O_{LR1} + \frac{1}{2} O_{LR2} + \frac{1}{6} O_{LR10} + O_{LR11} \right],
\]
\[
- \mathcal{F} (M_d) \otimes \mathcal{F} (M_d)^\dagger \left[ \frac{1}{12} O_{LR3} + \frac{1}{2} O_{LR4} - \frac{1}{12} O_{LR12} - O_{LR13} \right],
\]
\[
- \mathcal{F} (M_e) \otimes \mathcal{F} (M_e)^\dagger \left[ \frac{1}{4} O_{LR8} - \frac{1}{2} O_{LR17} \right].
\]
+ \mathcal{F}(M_d) \otimes \mathcal{F}(M_u)^\dagger \mathcal{O}_{FY5} + \mathcal{F}(M_u) \otimes \mathcal{F}(M_d)^\dagger \mathcal{O}_{FY5} \\
+ \mathcal{F}(M_e) \otimes \mathcal{F}(M_u)^\dagger \mathcal{O}_{FY9} + \mathcal{F}(M_u) \otimes \mathcal{F}(M_e)^\dagger \mathcal{O}_{FY9} \\
- \mathcal{F}(M_d) \otimes \mathcal{F}(M_e)^\dagger \left[ \frac{1}{4} \mathcal{O}_{LR9} - \frac{1}{2} \mathcal{O}_{LR18} \right] \\
- \mathcal{F}(M_e) \otimes \mathcal{F}(M_d)^\dagger \left[ \frac{1}{4} \mathcal{O}_{LR9} - \frac{1}{2} \mathcal{O}_{LR18} \right]. $$

Finally, one has

$$(\bar{\psi}_L U T^a \mathcal{F}(M) \psi_R)(\bar{\psi}_R \mathcal{F}(M)^\dagger T^a U^\dagger \psi_L) = \left[ (\bar{q}_L U T^a P_+ \mathcal{F}(M_u) q_R) + (\bar{q}_L U T^a P_- \mathcal{F}(M_d) q_R) + (\bar{q}_L U T^a P_- \mathcal{F}(M_u) l_R) \right] \times \left[ (\bar{q}_R \mathcal{F}(M_d)^\dagger P_+ T^a U^\dagger q_L) + (\bar{q}_R \mathcal{F}(M_e)^\dagger P_- T^a U^\dagger l_L) \right]$$

Using now

$$T^a P_+ \otimes P_+ T^a = \frac{1}{2} P_{21} \otimes P_{12} + \frac{1}{4} P_+ \otimes P_+ , \quad T^a P_- \otimes P_- T^a = \frac{1}{2} P_{12} \otimes P_{21} + \frac{1}{4} P_- \otimes P_- , \quad T^a P_\pm \otimes P_\mp T^a = - \frac{1}{4} P_\pm \otimes P_\mp ,$$

together with the previous Fierz identities and the following ones:

$$(\bar{q}_L U P_{12} \mathcal{F}(M_d) q_R)(\bar{q}_R \mathcal{F}(M_d)^\dagger P_{21} U^\dagger q_L) = - \mathcal{F}(M_d) \otimes \mathcal{F}(M_d)^\dagger \left[ \frac{1}{12} \mathcal{O}_{LR3} + \frac{1}{2} \mathcal{O}_{LR4} + \frac{1}{6} \mathcal{O}_{LR12} + \mathcal{O}_{LR13} \right] ,$$

$$(\bar{q}_L U P_{21} \mathcal{F}(M_u) q_R)(\bar{q}_R \mathcal{F}(M_u)^\dagger P_{12} U^\dagger q_L) = - \mathcal{F}(M_u) \otimes \mathcal{F}(M_u)^\dagger \left[ \frac{1}{12} \mathcal{O}_{LR1} + \frac{1}{2} \mathcal{O}_{LR2} - \frac{1}{6} \mathcal{O}_{LR10} - \mathcal{O}_{LR11} \right] ,$$

$$(\bar{I}_L U P_{12} \mathcal{F}(M_e) l_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_{21} U^\dagger l_L) = - \mathcal{F}(M_e) \otimes \mathcal{F}(M_e)^\dagger \left[ \frac{1}{4} \mathcal{O}_{LR8} + \frac{1}{2} \mathcal{O}_{LR17} \right] ,$$

$$(\bar{q}_L U P_{12} \mathcal{F}(M_d) q_R)(\bar{I}_R \mathcal{F}(M_d)^\dagger P_{21} U^\dagger l_L) = - \mathcal{F}(M_d) \otimes \mathcal{F}(M_d)^\dagger \left[ \frac{1}{4} \mathcal{O}_{LR9} + \frac{1}{2} \mathcal{O}_{LR18} \right] ,$$

one obtains

$$(\bar{\psi}_L U T^a \mathcal{F}(M) \psi_R)(\bar{\psi}_R \mathcal{F}(M)^\dagger T^a U^\dagger \psi_L) =$$

$$\frac{1}{2} (\bar{q}_L U P_{12} \mathcal{F}(M_u) q_R)(\bar{q}_R \mathcal{F}(M_u)^\dagger P_{21} U^\dagger q_L) + \frac{1}{4} (\bar{q}_L U P_+ \mathcal{F}(M_u) q_R)(\bar{q}_R \mathcal{F}(M_u)^\dagger P_+ U^\dagger q_L)
+ \frac{1}{2} (\bar{q}_L U P_{12} \mathcal{F}(M_d) q_R)(\bar{q}_R \mathcal{F}(M_d)^\dagger P_{21} U^\dagger q_L) + \frac{1}{4} (\bar{q}_L U P_- \mathcal{F}(M_d) q_R)(\bar{q}_R \mathcal{F}(M_d)^\dagger P_- U^\dagger q_L)
+ \frac{1}{2} (\bar{I}_L U P_{12} \mathcal{F}(M_e) l_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_{21} U^\dagger l_L) + \frac{1}{4} (\bar{I}_L U P_+ \mathcal{F}(M_e) l_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_+ U^\dagger l_L)
- \frac{1}{2} (\bar{q}_L U P_+ \mathcal{F}(M_u) q_R)(\bar{q}_R \mathcal{F}(M_u)^\dagger P_+ U^\dagger q_L) - \frac{1}{4} (\bar{q}_L U P_- \mathcal{F}(M_u) q_R)(\bar{q}_R \mathcal{F}(M_u)^\dagger P_- U^\dagger q_L)
- \frac{1}{2} (\bar{q}_L U P_+ \mathcal{F}(M_d) q_R)(\bar{q}_R \mathcal{F}(M_d)^\dagger P_+ U^\dagger q_L) - \frac{1}{4} (\bar{q}_L U P_- \mathcal{F}(M_d) q_R)(\bar{q}_R \mathcal{F}(M_d)^\dagger P_- U^\dagger q_L)
+ \frac{1}{2} (\bar{I}_L U P_{12} \mathcal{F}(M_e) q_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_{21} U^\dagger l_L) + \frac{1}{4} (\bar{I}_L U P_+ \mathcal{F}(M_e) q_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_+ U^\dagger l_L)
+ \frac{1}{2} (\bar{I}_L U P_{12} \mathcal{F}(M_e) q_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_{21} U^\dagger l_L) + \frac{1}{4} (\bar{I}_L U P_+ \mathcal{F}(M_e) q_R)(\bar{I}_R \mathcal{F}(M_e)^\dagger P_+ U^\dagger l_L).$$
\[
\begin{align*}
&= -\mathcal{F}(\mathcal{M}_u) \hat{\otimes} \mathcal{F}(\mathcal{M}_u)^\dagger \left[ \frac{1}{24} \mathcal{O}_{LR1} + \frac{1}{4} \mathcal{O}_{LR2} - \frac{1}{12} \mathcal{O}_{LR10} - \frac{1}{2} \mathcal{O}_{LR11} \right] \\
&- \mathcal{F}(\mathcal{M}_u) \hat{\otimes} \mathcal{F}(\mathcal{M}_u)^\dagger \left[ \frac{1}{48} \mathcal{O}_{LR1} + \frac{1}{8} \mathcal{O}_{LR2} + \frac{1}{24} \mathcal{O}_{LR10} + \frac{1}{4} \mathcal{O}_{LR11} \right] \\
&- \mathcal{F}(\mathcal{M}_d) \hat{\otimes} \mathcal{F}(\mathcal{M}_d)^\dagger \left[ \frac{1}{48} \mathcal{O}_{LR3} + \frac{1}{4} \mathcal{O}_{LR4} + \frac{1}{24} \mathcal{O}_{LR12} + \frac{1}{2} \mathcal{O}_{LR13} \right] \\
&- \mathcal{F}(\mathcal{M}_d) \hat{\otimes} \mathcal{F}(\mathcal{M}_d)^\dagger \left[ \frac{1}{48} \mathcal{O}_{LR3} + \frac{1}{8} \mathcal{O}_{LR4} - \frac{1}{24} \mathcal{O}_{LR12} - \frac{1}{4} \mathcal{O}_{LR13} \right] \\
&- \mathcal{F}(\mathcal{M}_e) \hat{\otimes} \mathcal{F}(\mathcal{M}_e)^\dagger \left[ \frac{1}{8} \mathcal{O}_{LR8} + \frac{1}{4} \mathcal{O}_{LR17} \right] - \mathcal{F}(\mathcal{M}_e) \hat{\otimes} \mathcal{F}(\mathcal{M}_e)^\dagger \left[ \frac{1}{16} \mathcal{O}_{LR8} - \frac{1}{8} \mathcal{O}_{LR17} \right] \\
&- \mathcal{F}(\mathcal{M}_d) \hat{\otimes} \mathcal{F}(\mathcal{M}_d)^\dagger \left[ \frac{1}{8} \mathcal{O}_{LR9} + \frac{1}{4} \mathcal{O}_{LR18} \right] - \frac{1}{4} \mathcal{F}(\mathcal{M}_d) \otimes \mathcal{F}(\mathcal{M}_d)^\dagger \mathcal{O}_{FY5} \\
&- \mathcal{F}(\mathcal{M}_d) \hat{\otimes} \mathcal{F}(\mathcal{M}_d)^\dagger \left[ \frac{1}{16} \mathcal{O}_{LR9} - \frac{1}{8} \mathcal{O}_{LR18} \right] - \frac{1}{4} \mathcal{F}(\mathcal{M}_d) \otimes \mathcal{F}(\mathcal{M}_d)^\dagger \mathcal{O}_{FY5}' \\
&- \mathcal{F}(\mathcal{M}_e) \hat{\otimes} \mathcal{F}(\mathcal{M}_e)^\dagger \left[ \frac{1}{8} \mathcal{O}_{LR9} + \frac{1}{4} \mathcal{O}_{LR18} \right] - \frac{1}{4} \mathcal{F}(\mathcal{M}_e) \otimes \mathcal{F}(\mathcal{M}_e)^\dagger \mathcal{O}_{FY9} \\
&- \mathcal{F}(\mathcal{M}_e) \hat{\otimes} \mathcal{F}(\mathcal{M}_e)^\dagger \left[ \frac{1}{16} \mathcal{O}_{LR9} - \frac{1}{8} \mathcal{O}_{LR18} \right] - \frac{1}{4} \mathcal{F}(\mathcal{M}_e) \otimes \mathcal{F}(\mathcal{M}_e)^\dagger \mathcal{O}_{FY9}'.
\end{align*}
\]

(5.43)

These divergences are removed through the renormalization of the operators \(\mathcal{O}_{FYi}, \mathcal{O}_{STi},\) and \(\mathcal{O}_{LRi}\) of \(\mathcal{L}_{\psi^4 U Y}\). Explicitly, one has (recall that the functions \(\mathcal{F}_{FYi}, \mathcal{F}_{STi},\) and \(\mathcal{F}_{LRi}\) are actually rank-four tensors in generation space)

\[
\begin{align*}
\mathcal{F}_{FY1}(\eta) &= \mathcal{F}_{FY1}(\eta; \mu) + \frac{\Gamma_{FY1}(\eta)}{16\pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi - \gamma_E) \right], \\
\mathcal{F}_{ST1}(\eta) &= \mathcal{F}_{ST1}(\eta; \mu) + \frac{\Gamma_{ST1}(\eta)}{16\pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi - \gamma_E) \right], \\
\mathcal{F}_{LR1}(\eta) &= \mathcal{F}_{LR1}(\eta; \mu) + \frac{\Gamma_{LR1}(\eta)}{16\pi^2} \mu^{d-4} \left[ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi - \gamma_E) \right],
\end{align*}
\]

(5.44)

with

\[
\begin{align*}
\Gamma_{FY1} &= \frac{3F^{-2}}{2v^4} \left( \frac{F'}{2} M_u' - M_u \right) \otimes \left( \frac{F'}{2} M_u' - M_u \right) + \frac{1}{2v^4} M_d'' \otimes M_u'' - \frac{1}{v^4} \left( F^{-1/2} M_u \right)' \otimes \left( F^{-1/2} M_u \right)', \\
\Gamma_{FY3} &= \frac{3F^{-2}}{2v^4} \left( \frac{F'}{2} M_d' - M_d \right) \otimes \left( \frac{F'}{2} M_d' - M_d \right) + \frac{1}{2v^4} M_d'' \otimes M_d'' - \frac{1}{v^4} \left( F^{-1/2} M_d \right)' \otimes \left( F^{-1/2} M_d \right)', \\
\Gamma_{FY5} &= \frac{3F^{-2}}{v^4} \left( \frac{F'}{2} M_d' - M_d \right) \otimes \left( \frac{F'}{2} M_u' - M_u \right) + \frac{1}{v^4} M_d'' \otimes M_u'' - \frac{2}{v^4} \left( F^{-1/2} M_d \right)' \otimes \left( F^{-1/2} M_u \right)', \\
\Gamma_{FY7} &= \frac{3F^{-2}}{v^4} \left( \frac{F'}{2} M_d' - M_d \right) \otimes \left( \frac{F'}{2} M_e' - M_e \right) + \frac{1}{v^4} M_d'' \otimes M_e'' - \frac{2}{v^4} \left( F^{-1/2} M_d \right)' \otimes \left( F^{-1/2} M_e \right)', \\
\Gamma_{FY9} &= \frac{3F^{-2}}{v^4} \left( \frac{F'}{2} M_e' - M_e \right) \otimes \left( \frac{F'}{2} M_u' - M_u \right) + \frac{1}{v^4} M_e'' \otimes M_u'' - \frac{2}{v^4} \left( F^{-1/2} M_e \right)' \otimes \left( F^{-1/2} M_u \right)', \\
\Gamma_{FY10} &= \frac{3F^{-2}}{2v^4} \left( \frac{F'}{2} M_e' - M_e \right) \otimes \left( \frac{F'}{2} M_e' - M_e \right) + \frac{1}{2v^4} M_e'' \otimes M_e'' - \frac{1}{v^4} \left( F^{-1/2} M_e \right)' \otimes \left( F^{-1/2} M_e \right)',
\end{align*}
\]

(5.45)

\[
\begin{align*}
\Gamma_{ST5} &= \frac{3F^{-2}}{v^4} \left( \frac{F'}{2} M_u' - M_u \right) \otimes \left( \frac{F'}{2} M_d' - M_d \right) + \frac{1}{v^4} M_u'' \otimes M_d'' + \frac{2}{v^4} \left( F^{-1/2} M_u \right)' \otimes \left( F^{-1/2} M_d \right)', \\
\Gamma_{ST6} &= -\frac{4}{v^4} \left( F^{-1/2} M_u \right)' \otimes \left( F^{-1/2} M_d \right)',
\end{align*}
\]
\[ \Gamma_{ST9} = \frac{3F^2}{4v^4} \left( \frac{F'}{2} \mathcal{M}'_u - \mathcal{M}_u \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) + \frac{1}{v^2} \mathcal{M}'_u \otimes \mathcal{M}''_c + \frac{2}{v^4} \left( F^{-1/2} \mathcal{M}_u \right)' \otimes \left( F^{-1/2} \mathcal{M}_c \right)', \]
\[ \Gamma_{ST10} = -\frac{4}{v^4} (F^{-1/2} \mathcal{M}_u)' \otimes (F^{-1/2} \mathcal{M}_c)', \]

(5.46)

\[ \Gamma_{LR1} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_u - \mathcal{M}_u \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) - \frac{1}{12v^4} \mathcal{M}'_u \otimes \mathcal{M}''_d - \frac{1}{2v^4} \left( F^{-1/2} \mathcal{M}_u \right)' \otimes \left( F^{-1/2} \mathcal{M}_d \right)', \]
\[ \Gamma_{LR2} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_u - \mathcal{M}_u \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_l - \mathcal{M}_l \right) - \frac{1}{2v^4} \mathcal{M}'_u \otimes \mathcal{M}''_l - \frac{3}{v^4} \left( F^{-1/2} \mathcal{M}_u \right)' \otimes \left( F^{-1/2} \mathcal{M}_l \right)', \]
\[ \Gamma_{LR3} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) - \frac{1}{12v^4} \mathcal{M}'_d \otimes \mathcal{M}''_d - \frac{1}{2v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_d \right)', \]
\[ \Gamma_{LR4} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) - \frac{1}{2v^4} \mathcal{M}'_d \otimes \mathcal{M}''_c - \frac{3}{v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_c \right)', \]
\[ \Gamma_{LR8} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_e - \mathcal{M}_e \right) - \frac{1}{4v^4} \mathcal{M}'_d \otimes \mathcal{M}''_e - \frac{3}{2v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_e \right)', \]
\[ \Gamma_{LR9} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) - \frac{1}{4v^4} \mathcal{M}'_d \otimes \mathcal{M}''_c - \frac{3}{2v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_c \right)', \]
\[ \Gamma_{LR10} = -\frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) - \frac{1}{6v^4} \mathcal{M}'_d \otimes \mathcal{M}''_d + \frac{1}{3v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_d \right)', \]
\[ \Gamma_{LR11} = -\frac{3F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_u - \mathcal{M}_u \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_u - \mathcal{M}_u \right) - \frac{1}{v^2} \mathcal{M}'_u \otimes \mathcal{M}''_u + \frac{2}{v^4} \left( F^{-1/2} \mathcal{M}_u \right)' \otimes \left( F^{-1/2} \mathcal{M}_u \right)', \]
\[ \Gamma_{LR12} = \frac{F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) + \frac{1}{6v^4} \mathcal{M}'_d \otimes \mathcal{M}''_d - \frac{1}{3v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_d \right)', \]
\[ \Gamma_{LR13} = \frac{3F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) + \frac{1}{v^2} \mathcal{M}'_d \otimes \mathcal{M}''_d - \frac{2}{v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_d \right)', \]
\[ \Gamma_{LR17} = \frac{3F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) + \frac{2}{v^2} \mathcal{M}'_c \otimes \mathcal{M}''_c - \frac{1}{v^4} \left( F^{-1/2} \mathcal{M}_c \right)' \otimes \left( F^{-1/2} \mathcal{M}_c \right)', \]
\[ \Gamma_{LR18} = \frac{3F^2}{2v^4} \left( \frac{F'}{2} \mathcal{M}'_d - \mathcal{M}_d \right) \otimes \left( \frac{F'}{2} \mathcal{M}'_c - \mathcal{M}_c \right) + \frac{1}{v^2} \mathcal{M}'_d \otimes \mathcal{M}''_c - \frac{1}{v^4} \left( F^{-1/2} \mathcal{M}_d \right)' \otimes \left( F^{-1/2} \mathcal{M}_c \right)' . \] (5.47)

F. Renormalization group equations for the coefficients of \( \mathcal{L}_4 \)

In the preceding subsections we obtained the counterterms of the next-to-leading order Lagrangian \( \mathcal{L}_4 \), employing a systematic decomposition into basis operators. We are now in a position to derive the renormalization group equations for the operator coefficients.

Summarizing the previous results, we may write

\[ \mathcal{L}_4 = \sum_i \mathcal{O}_i \mathcal{F}_i = \sum_i \mathcal{O}_i \left( F_i + \frac{\Gamma_i}{16\pi^2} \frac{1}{d-4} \right) \mu^{d-4}, \]

(5.48)

where the sum extends over all the NLO terms in our basis, comprising the classes \( \beta_1, UhD^4, \psi^2UhD, \psi^2UhD^2, \) and \( \psi^4Uh \). The quantities \( F_i \) and \( \Gamma_i \) are functions of \( \eta = h/v \) and, for the fermionic terms, tensors in generation space.

The second equality in (5.48) expresses the unrenormalized coefficients \( \mathcal{F}_i \) through their renormalized version plus counterterms \( \sim \Gamma_i \), written here in the MS scheme. The counterterms are equal and opposite in sign to the one-loop divergences, displayed in (5.6). Inspecting the latter, we note that all terms \( \mathcal{O}_i \Gamma_i \) have canonical dimension exactly 4, in \( d = 4-2\varepsilon \) space-time dimensions, once in this context we take \( g, g' \) and \( v \) to mean \( g_y \varepsilon, g' \mu^\varepsilon \) and \( v \mu^{-\varepsilon} \), respectively. Then their dimensions are \( [g] = [g'] = \varepsilon, [v] = 1 - \varepsilon \), and \( g, g' \) and \( v \) become \( \mu \)-independent at tree level. It follows that also \( d\Gamma_i/d\ln \mu = 0 \) at tree level. Since \([\mathcal{L}_4] = d, [\mathcal{O}_i \Gamma_i] = 4 \) implies the factor \( \mu^{d-4} \) shown on the right-hand
side of (5.48). From the $\mu$-independence of $\tilde{F}_i$ we then infer the renormalization group equations (in 4 space-time dimensions)

$$16\pi^2 \frac{d}{d \ln \mu} F_i = - \Gamma_i. \quad (5.49)$$

The various $\Gamma_i$ are given in the previous sections, for $i = \beta_j$ in (5.11), $D1, D2, D7, D8, D11$ in (5.14), (5.15), $\psi V k$ in (5.22), $\psi S k$ in (5.27), $FY k$ in (5.45), $ST k$ in (5.46), and $LR k$ in (5.47). The one-loop beta functions vanish for all couplings not present in the preceding list.

### VI. COMPARISON WITH SMEFT

SMEFT [25, 26] is the effective field theory formulation of the electroweak and strong interactions, where the operators are organized as an expansion according to their canonical dimension. To lowest order (dimension 4), it coincides with the SM. Excluding lepton-number violating effects, the leading corrections are given by operators of dimension 6.

Even though SMEFT is organized differently than the EWChL, there is an overlap [27], which can be used as a cross-check of our calculation. The one-loop renormalization of the SM at dimension 4 has already been shown in [16] to follow as a special case from the renormalization of the EWChL. Beyond that, the one-loop divergences of the EWChL discussed here also contain the renormalization of those dimension-6 terms in SMEFT that have chiral dimension 2 and are thus contained in (2.1). In the SMEFT basis of [26], these terms can be expressed as

$$\Delta L_{2, d=6} = \frac{1}{\Lambda^2} \left( C_{\phi Q} Q_{\phi Q} + C_{\phi Q} Q_{\phi Q} + C_{\psi \phi Q}^* Q_{\psi \phi}^* \right), \quad (6.1)$$

where

$$Q_{\phi \phi} = \phi^\dagger \phi \Box \phi^\dagger \phi \quad (6.2)$$

$$Q_{\phi} = (\phi^\dagger \phi)^3 \quad (6.3)$$

$$Q_{\psi \phi}^* = \phi^\dagger \phi (\bar{\psi}_L (\hat{\phi}, \phi))^\dagger \psi_R^*. \quad (6.4)$$

Here $r, s$ are fermion flavor indices and $\phi$ is the Higgs doublet.

By extracting the corresponding terms from the one-loop divergences of the EWChL, one should obtain the known one-loop renormalization of SMEFT [28–31], coming from the single insertion of the dimension-6 operators (6.2) – (6.4). In order to do this, we need the relations between the non-linear and the linear representation of the Higgs sector, in particular,

$$\langle \tilde{\phi}, \phi \rangle = \frac{v}{\sqrt{2}} (1 + \eta) U, \quad \phi^\dagger \phi = \frac{v^2}{2} (1 + \eta)^2, \quad D^\mu \phi^\dagger D_\mu \phi = \frac{1}{2} \partial^\mu h \partial_\mu h + \frac{v^2}{4} \langle L^\mu L_\mu \rangle (1 + \eta)^2, \quad (6.5)$$

where $\phi$ is the complex Higgs doublet in the conventional linear representation, and $\tilde{\phi}_i = e_i j \phi^*_j$.

The operators in (6.2) – (6.4) modify the $\eta$-dependent functions $F$, $V$ and $M$ from their SM form

$$F_{SM} = (1 + \eta)^2, \quad V_{SM} = -\frac{m^2 v^2}{2} (1 + \eta)^2 + \frac{\lambda v^4}{8} (1 + \eta)^4, \quad M_{SM} = \frac{v}{\sqrt{2}} \gamma (1 + \eta), \quad (6.6)$$

where $\gamma = \text{diag}(\gamma_u, \gamma_d, \gamma_u, \gamma_e)$ collects the Yukawa matrices of the SM.

The simplest case is the renormalization of $C_{\phi Q}$. Here $F$ and $M$ take their SM values, while the potential becomes $V = V_{SM} + \Delta V$, with

$$\Delta V = - \frac{C_{\phi}}{\Lambda^2} \frac{v^6}{8} (1 + \eta)^6. \quad (6.7)$$

This term only affects the contribution $\mathcal{L}_{\text{div}}^{(0)}$ in (C.2), from which we extract the term of first order in $C_{\phi}$,

$$32\pi^2 \varepsilon \Delta \mathcal{L}_{\text{div}}^{(0)} = - \frac{C_{\phi}}{\Lambda^2} (\phi^\dagger \phi)^3 \left[ \frac{3}{2} (3g^2 + g'^2) + 54 \lambda \right] + 24 \frac{m^2}{\Lambda^2} C_{\phi} (\phi^\dagger \phi)^2. \quad (6.8)$$
As discussed e.g. in [32], we still need to subtract \( K_\phi = 6(3g^2 + g'^2 - \langle \mathcal{Y} \mathcal{Y} \rangle) \) from the term in square brackets, due to the renormalization of \( \phi \) inside \( Q_\phi \). We then find

\[
\beta_\phi \geq \left[ -\frac{9}{2}(3g^2 + g'^2) + 54\lambda + 6\langle \mathcal{Y} \mathcal{Y} \rangle \right] C_\phi, \quad \beta_\lambda \geq 48 \frac{m^2}{\Lambda^2} C_\phi \tag{6.9}
\]

for the contribution of \( C_\phi \) to the beta functions of SMEFT at dimension 4 and 6, in agreement with [28–31] and the compilation in [33]. We recall that the beta functions of coefficient \( C_i \) are defined as \( \beta_i = 16\pi^2 dC_i / d\ln \mu \) and the operator multiplying \( \lambda \) is \( Q_\lambda^{(4)} = -(\phi^3 \phi)/2 \) [32].

We next consider the renormalization of the modified Yukawa term \( C_{\psi \phi}^* Q_{\psi \phi}^* \). In this case, \( F \) and \( V \) are the same as in the SM, but \( \mathcal{M} = \mathcal{M}_{\text{SM}} + \Delta \mathcal{M} \), with

\[
\Delta \mathcal{M} = \frac{-v^3}{2\sqrt{2}\Lambda^2} (1 + \eta)^3 C_{\psi \phi}. \tag{6.10}
\]

Here \( C_{\psi \phi} = \text{diag}(C_{u \phi}, C_{d \phi}, C_{\nu \phi}, C_{e \phi}) \), where the entries are matrices in generation space. Working out the terms to first order in \( \Delta \mathcal{M} \) from \( \mathcal{L}_{\text{div}} \) in (3.1), and expressing the result through the Higgs-doublet \( \phi \), we obtain

\[
32\pi^2 \varepsilon \Delta C_{\psi \phi} = 4\eta_1 m^2 (\phi \phi)^2 + 4(-\eta_1 \lambda + \langle C_\psi^\dagger \mathcal{Y} \mathcal{Y}^\dagger \mathcal{Y} + \text{h.c.} \rangle) (\phi \phi)^3
\]
\[
+ \left\{ 6m^2 \bar{\psi}_L \gamma^\mu (\phi) C_{\psi \phi} \psi_R - 2\eta_1 \phi \phi \bar{\psi}_L (\phi, \phi) \psi_R + 4i\eta_5 \phi \phi \bar{\psi}_L (\phi, \phi) \lambda T_3 \psi_R \right. \\
\left. - \phi \phi \bar{\psi}_L (\phi, \phi) \left[ -6C_F g_s^2 C_{\psi \phi} + \left( \frac{3}{4} (3g^2 + g'^2) - 6Y_L Y_R g'^2 + 12\lambda \right) C_{\psi \phi} \right] \psi_R \\
\right. \\
- \phi \phi \bar{\psi}_L (\phi, \phi) \left[ 7\mathcal{Y} \mathcal{Y} C_{\psi \phi} + 6C_F \phi \phi \mathcal{Y} \mathcal{Y} + 2Y \phi \phi \mathcal{Y} \mathcal{Y} - (C_{\psi \phi} \mathcal{Y} \mathcal{Y}) \mathcal{Y} - 2 \langle Y \rangle C_{\psi \phi} \mathcal{Y} \mathcal{Y} - \frac{3}{2} (\mathcal{Y} \mathcal{Y}) C_{\psi \phi} \right] \psi_R \\
+ \text{h.c.}, \tag{6.11}
\]

where we defined \( C_{q \phi} = \text{diag}(C_{u \phi}, C_{d \phi}, 0, 0) \) and [33]

\[
\eta_1 \equiv \frac{1}{2} \langle C_{\psi \phi} \mathcal{Y} \mathcal{Y} \mathcal{Y} \mathcal{Y} \rangle, \quad i\eta_5 \equiv \langle (C_{\psi \phi} \mathcal{Y} \mathcal{Y} \mathcal{Y} \mathcal{Y}) T_3 \rangle. \tag{6.12}
\]

Since the EWChL is formulated explicitly in the broken phase, in contrast to SMEFT, terms vanishing as \( v \to 0 \) after expressing all scalar fields through the doublet \( \phi \) have to be omitted in deriving (6.11). The beta-function contributions proportional to \( C_{\psi \phi} \) can be read off from (6.11), once the field renormalization factor \( K_{\psi \phi} = 3(3g^2 + g'^2 - \langle \mathcal{Y} \mathcal{Y} \rangle) \) has been subtracted from the coefficient of \( -C_{\psi \phi}^* Q_{\psi \phi}^* \). We find the entries

\[
\beta_\phi \geq 8\eta_1 \frac{m^2}{\Lambda^2}, \quad \beta_\lambda \geq 6\frac{m^2}{\Lambda^2} C_{\psi \phi}, \quad \beta_\mu \geq 4\eta_1 \lambda - 4 \langle C_{\psi \phi} \mathcal{Y} \mathcal{Y} \mathcal{Y} \mathcal{Y} + \text{h.c.} \rangle, \tag{6.13}
\]
\[
\beta_{q \phi} \geq 2\eta_1 \mathcal{Y} \mathcal{Y} T_3 - 6C_F g_s^2 C_{\psi \phi} + \left( -\frac{9}{4} (3g^2 + g'^2) - 6Y_L Y_R g'^2 + 12\lambda + 3 \langle \mathcal{Y} \mathcal{Y} \rangle \right) C_{\psi \phi}
\]
\[
+ 7\mathcal{Y} \mathcal{Y} C_{\psi \phi} + 6C_F \phi \phi \mathcal{Y} \mathcal{Y} + 2Y \phi \phi \mathcal{Y} \mathcal{Y} - (C_{\psi \phi} \mathcal{Y} \mathcal{Y}) \mathcal{Y} - 2 \langle Y \rangle C_{\psi \phi} \mathcal{Y} \mathcal{Y} - \frac{3}{2} (\mathcal{Y} \mathcal{Y}) C_{\psi \phi}. \tag{6.14}
\]

Finally, we extract the one-loop divergences induced by a single insertion of the operator \( Q_{q \phi} \). This case is more complicated than the previous ones, since \( Q_{q \phi} \) does not match the canonical form of the chiral Lagrangian \( \mathcal{L}_2 \) in (2.1). Employing a suitable field redefinition the desired information can nevertheless be obtained.

The operator \( Q_{q \phi} \) modifies the SM Lagrangian such that the kinetic term of \( h = v \eta \) becomes

\[
\mathcal{L}_h = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{C_{q \phi}}{\Lambda^2} Q_{q \phi} = \left( 1 - 2 \frac{C_{q \phi}}{\Lambda^2} v^2 (1 + \eta)^2 \right) \frac{1}{2} \partial_\mu h \partial^\mu h = \frac{1}{2} \partial_\mu \tilde{h} \partial^\mu \tilde{h}, \tag{6.15}
\]

with all other terms unchanged. In the second step, (6.2), (6.5) and an integration by parts have been used. In the last step, the canonical form of the kinetic term is recovered by means of the field redefinition

\[
\eta \equiv \tilde{\eta} + \frac{C_{q \phi}}{\Lambda^2} v^2 (1 + \eta)^2 \equiv \tilde{\eta} + \Delta \eta, \tag{6.16}
\]
to first order in $C_{\phi^2}$. The functions in (6.6) can next be expressed as

$$F_{SM}(\eta) = F_{SM}(\tilde{\eta} + \Delta \eta) \equiv F_{SM}(\tilde{\eta}) + \frac{C_{\phi^2} v^2}{\Lambda^2} \left( 1 + \eta \right)^3 F'_{SM}(\eta) \equiv F_{SM}(\tilde{\eta}) + \Delta F$$

(6.17)

and similarly for $V_{SM}$ and $M_{SM}$. The SM Lagrangian including the $Q_{\phi^2}$ term can thus be brought to the form of (2.1), now written with $\tilde{\eta}$ replacing $\eta$, and with the functions

$$F(\tilde{\eta}) = F_{SM}(\tilde{\eta}) + \Delta F, \quad V(\tilde{\eta}) = V_{SM}(\tilde{\eta}) + \Delta V, \quad M(\tilde{\eta}) = M_{SM}(\tilde{\eta}) + \Delta M.$$  

(6.18)

In order to find the one-loop divergences proportional to $C_{\phi^2}$, we extract from (3.1)

$$L_{\text{div}}(F_{SM}(\tilde{\eta}) + \Delta F, V_{SM}(\tilde{\eta}) + \Delta V, M_{SM}(\tilde{\eta}) + \Delta M)$$

(6.19)

the terms to first order in $\Delta F$, $\Delta V$ and $\Delta M$. In addition, we need to re-express the SM limit of the divergences through the original field $\eta$, which introduces a further contribution

$$L_{\text{div}}^{SM}(\tilde{\eta}) = L_{\text{div}}^{SM}(\eta - \Delta \eta) \equiv L_{\text{div}}^{SM}(\eta) + \hat{L}_{\text{div}}^{\Delta \eta}.$$  

(6.20)

Adding all four contributions and expressing the scalar fields in terms of $\phi$, we obtain

$$32\pi^2 \varepsilon \frac{A^2}{C_{\phi^2}} \Delta L^{\phi}_{\text{div}} = 8m^4 \phi^4 + m^2 \left( \frac{20}{3} g^2 - 32 \lambda \right) (\phi^4)^2 - 2m^2 (\bar{\psi}_L \phi \gamma \psi_R + \text{h.c.})$$

$$+ \left( \frac{40 \lambda^2}{\lambda} - \frac{20}{3} g^2 \lambda \right) (\phi^4)^3 - \left( \frac{8g^2 + 8}{3} g^2 + 12 \lambda \right) Q_{\phi^2} - \frac{20}{3} g^2 Q_{\phi D}$$

$$+ \phi^4 \left( \bar{\psi}_L \phi \gamma \psi_R \right) \left( \phi^4 \right)^2 - 2 \left( \bar{\phi}^4 D_{\mu} \phi \bar{\psi}_R \gamma \phi \gamma \psi_R + \text{h.c.} \right)$$

$$+ \phi^4 i D_{\mu} \phi \bar{\psi}_L \left( \frac{g^2}{3} Y + \frac{1}{2} \left( \phi^4 \gamma \psi_R \right) \right)$$

$$\beta_{\phi^2} \geq \left( \frac{20}{3} g^2 \lambda - 40 \lambda^2 \right) C_{\phi^2}, \quad \beta_{\phi^2} \geq \left( -4g^2 - \frac{4}{3} g^2 + 12 \lambda + 4 \langle \phi^4 \gamma \psi_R \rangle \right) C_{\phi^2}, \quad \beta_{\phi D} \geq \frac{20}{3} g^2 C_{\phi^2}$$

(6.23)

$$\beta_{\psi \phi} \geq \left( \frac{10}{3} g^2 - 2 \lambda \right) \gamma - 6 \langle \phi^4 \gamma \psi_R \rangle C_{\phi^2}.$$  

(6.24)

$$\beta_{\phi \psi} \geq \left( \frac{g^2}{3} Y - \langle \phi^4 \gamma \psi_R \rangle \right) C_{\phi^2}, \quad \beta_{\phi u d} \geq 2 \langle \phi^4 \gamma \psi_R \rangle.$$  

(6.25)

$$\beta_{\phi^4}^{(1)} \geq \left( \frac{g^2}{3} Y + \frac{1}{2} \langle \phi^4 \gamma \psi_R \rangle \right) C_{\phi^2}, \quad \beta_{\phi^4}^{(3)} \geq \left( \frac{g^2}{6} - \frac{1}{2} \langle \phi^4 \gamma \psi_R \rangle \right) C_{\phi^2}.$$  

(6.26)

All terms are in agreement with the known SMEFT results, as summarized e.g. in [33].
VII. BRIEF SURVEY OF RELATED LITERATURE

Several groups computed subsets of the one-loop renormalization and RGEs of the EWChL in the past [34–39]. In our previous paper [16], we compared and found agreement with the one-loop divergences reported in [38], which treated the subset of divergences that arise from scalar loops and their corresponding beta functions. Previously, [36] had computed the scalar 1, 2, 3, and 4-point functions at one loop. They focused on the scalar sector, not only regarding the particles running in the loop, but also regarding the set of operators. Since this set is not closed under the application of the equations of motion, there are ambiguities that make the RGEs listed there hard to compare with our result. In addition, the operator $\partial h\partial h F_h(h)$ was considered, which is always redundant [14]. However, [36] compared their results with [34] and claimed agreement in the corresponding limit.

Ref. [34] and its follow-ups [35, 37] considered explicit $VV$ scattering processes and derived the RGEs for the couplings involved. These results were soon thereafter corroborated by [38]. In a different approach, [39] considered a geometric formulation of the scalar sector. They also computed the divergence structure of scalar loops and found full agreement with [38]. This means that all results previous to [16] have been cross-checked against each other, either directly or indirectly. Note that [38] also compared to the result in the Higgsless limit [40–44] and found agreement.

Shortly after we published [16], work on the same subject was described in [17]. The authors claim in the journal version of [17]: "Gauge bosons have the SM renormalization plus an extra contribution”. We find that, after basis projection (see discussion at the beginning of our section III A), this is not the case and the gauge couplings just have the one-loop beta functions of the SM. This result can be confirmed by looking at the SMEFT RGE (through dimension 6). Here the only BSM contribution to the gauge-beta functions comes from $C_{\phi X} Q_{\phi X}$, an operator structure that is absent in the leading-order EWChL. The absence of new contributions to the one-loop gauge-beta functions can also be seen by comparing the SM limit with the Higgsless limit of our result. They both agree, indicating that only Goldstone-boson and gauge loops contribute to the one-loop gauge-beta functions, while the Higgs couplings (modified or not) do not. One can actually check that the contribution of Higgs one-loop diagrams does not lead to divergences, only to finite pieces.

VIII. VACUUM EXPECTATION VALUE OF $h$ AND TADPOLE COUNTERTERM

The potential $V$ for the scalar field $h$ is chosen such that $V'(0) = 0$, and hence the vev of the scalar field $h$ vanishes, $\langle h \rangle = 0$ at lowest order. This property is not maintained when loop corrections are considered, and one needs to perform a finite renormalization of the potential in order to enforce it. This finite renormalization is computed here to illustrate the application of the EWChL at one loop.

Employing the background field method, we separate the various fields into a classical or background part, and a quantum, fluctuating, part. The terms involving only the background fields provide the tree-level amplitudes. This is the case, in particular for all the terms in $L_4$. The part linear in the quantum fields does not matter, it vanishes when the classical equations of motion are enforced on the background fields, and terms with three or more quantum fields are not relevant for the computation of one-loop amplitudes. For general one-loop calculations, therefore, only terms exactly quadratic in the quantum fields, but with arbitrary powers of background fields, are required. As we are only interested in the Higgs potential here, all background fields, except for the Higgs, may be dropped in the following. Specialising to the tadpole contribution, eventually only the terms linear in the background Higgs field need to be kept.

Hereafter, quantum fields are written with a caret, the fields without carets being background fields (except for the ghosts, which are always quantum fields). The decomposition in terms of background and quantum fields is a linear one, with the exception of the Goldstone fields, where the split is multiplicative. Omitting background fields, the Goldstone field reduces to its quantum part, for which we use

$$\hat{U} = e^{2\hat{\varphi} F^{-1/2}/v}, \quad \hat{\varphi} = \phi^a T^a.$$  \hfill (8.1)

Finally, one also needs to fix the gauge and add the corresponding contributions from the ghost fields. For the former, a background-gauge invariant choice [45] is made, namely \(^1\)

$$L_{g.f.} = -\frac{1}{2} \left( \partial_\mu \hat{B}_\mu + \frac{g' v}{2} F^{1/2} \hat{\varphi}^3 \right)^2 - \left( D^\mu \hat{W}_\mu - \frac{g v}{2} F^{1/2} \hat{\varphi} \right)^2.$$  \hfill (8.2)

\(^1\) At this stage, the QCD part does not matter, and is therefore omitted for the time being.
This means that the Faddeev-Popov ghost Lagrangian reads

$$\mathcal{L}_{\text{ghosts}} = \partial^\mu \bar{c} \partial_\mu c + \left(\frac{g'v}{2}\right) F \bar{c} \frac{g}{2} \left[ c^3 - \frac{g'v}{2} c \right] + \partial^\mu \bar{c} \partial_\mu c^a - \left(\frac{g}{2}\right)^2 F (c^1 c^1 + c^2 c^2) - \frac{g}{2} F \bar{c}^3 \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right). \tag{8.3}$$

In order to compute the Higgs tadpole at one loop, we first extract from $\mathcal{L}_2 + \mathcal{L}_{g.t.} + \mathcal{L}_{\text{ghosts}}$ the contributions quadratic in all quantum fields, linear in $h$, and without any other classical field. This gives

$$\mathcal{L}_2 + \mathcal{L}_{g.t.} + \mathcal{L}_{\text{ghosts}} =$$

$$= - \frac{1}{4} (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu \tilde{W}^a_\nu - \partial^\nu \tilde{W}^a_\mu) - \frac{1}{4} (\partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu)(\partial^\mu \hat{B}^\nu - \partial^\nu \hat{B}^\mu)$$

$$+ \frac{g'^2}{4} F_i \left(\frac{g}{2} \partial^\mu (\psi F^{-1/2}) + g \tilde{W}^a - g' \hat{B}^a T^3 \right) \left(\frac{2}{v} \partial_\mu (\psi F^{-1/2}) + g \tilde{W}_\mu - g' \hat{B}_\mu T^3 \right)$$

$$+ \frac{1}{2} \partial^\mu \tilde{h} \delta_{\mu h} - v^2 V_2 h^2 - 3v V_3 h^2 + i \psi \bar{h} \tilde{\psi} - \tilde{\psi}(\mathcal{M} P_R + \mathcal{M} P_L) \tilde{\psi}$$

$$- \frac{1}{2} \left(\partial^\mu \hat{B}_\mu + \frac{g'v}{2} F^{1/2} \partial^a \right) \left. \frac{1}{2} \left(\partial^\mu \tilde{W}_\mu^a - \frac{g}{2} F^{1/2} \partial^a \right) \right| \left(\partial^\mu \tilde{W}_\mu^a - \frac{g}{2} F^{1/2} \partial^a \right)$$

$$+ \partial^\mu \bar{c} \partial_\mu c + \frac{g'v}{2} F \bar{c} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \partial^\mu \bar{c} \partial_\mu c^a - \left(\frac{g}{2}\right)^2 F (c^1 c^1 + c^2 c^2) - \frac{g}{2} F \bar{c}^3 \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \cdots$$

$$= - \frac{1}{2} (\partial_\mu W^a_\nu)(\partial^\mu W^{a \nu}) + \frac{F}{2} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \frac{F}{2} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right)$$

$$+ \frac{1}{2} \partial^\mu \tilde{h} \delta_{\mu h} - v^2 V_2 h^2 - 3v V_3 h^2$$

$$+ \frac{1}{2} \left(\partial^\mu \tilde{W}_\mu^a \right) \left(\partial^\mu \tilde{W}_\mu^a \right) + \frac{1}{4} F \left(\tilde{\psi} \bar{h} \tilde{\psi} - \tilde{\psi}(\mathcal{M} P_R + \mathcal{M} P_L) \tilde{\psi} \right)$$

$$+ \partial^\mu \bar{c} \partial_\mu c + \frac{g'v}{2} F \bar{c} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \partial^\mu \bar{c} \partial_\mu c^a - \left(\frac{g}{2}\right)^2 F (c^1 c^1 + c^2 c^2) - \frac{g}{2} F \bar{c}^3 \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \cdots$$

$$= - \frac{1}{2} (\partial_\mu W_1)(\partial^\mu W_1) - \frac{1}{2} (\partial_\mu W_2)(\partial^\mu W_2) + \frac{F}{2} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right)$$

$$- \frac{1}{2} (\partial_\mu \tilde{Z}_\nu)(\partial^\mu \tilde{Z}_\nu) + \frac{F}{2} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right)$$

$$+ \frac{1}{2} \partial^\mu \tilde{h} \delta_{\mu h} - v^2 V_2 h^2 - 3v V_3 h^2$$

$$+ \frac{1}{2} \left(\partial^\mu \tilde{W}_\mu^a \right) \left(\partial^\mu \tilde{W}_\mu^a \right) + \frac{1}{4} F \left(\tilde{\psi} \bar{h} \tilde{\psi} - \tilde{\psi}(\mathcal{M} P_R + \mathcal{M} P_L) \tilde{\psi} \right)$$

$$+ \partial^\mu \bar{c} \partial_\mu c + \partial^\mu \bar{c} \partial_\mu c^a - \left(\frac{g}{2}\right)^2 F (c^1 c^1 + c^2 c^2) - \frac{g}{2} F \bar{c}^3 \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \cdots$$

$$= - \frac{1}{2} (\partial_\mu \hat{B}_\nu)(\partial^\mu \hat{B}_\nu) - \frac{1}{2} (\partial_\mu \hat{B}_\nu)(\partial^\mu \hat{B}_\nu) + \frac{F}{2} \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right)$$

$$- \frac{1}{2} \partial^\mu \tilde{h} \delta_{\mu h} - v^2 V_2 h^2 - 3v V_3 h^2$$

$$+ \frac{1}{2} \left(\partial^\mu \tilde{W}_\mu^a \right) \left(\partial^\mu \tilde{W}_\mu^a \right) + \frac{1}{4} F \left(\tilde{\psi} \bar{h} \tilde{\psi} - \tilde{\psi}(\mathcal{M} P_R + \mathcal{M} P_L) \tilde{\psi} \right)$$

$$+ \partial^\mu \bar{c} \partial_\mu c + \partial^\mu \bar{c} \partial_\mu c^a - \left(\frac{g}{2}\right)^2 F (c^1 c^1 + c^2 c^2) - \frac{g}{2} F \bar{c}^3 \left(\frac{g}{2} c^3 - \frac{g'v}{2} c \right) + \cdots$$

$$(8.4)$$

In the first equality, the terms of the fourth line represent the contributions from the gauge-fixing term and the fifth line the contributions from the corresponding Faddeev-Popov ghosts. Total derivatives have been dropped. In the last step, we have introduced mass eigenstates in the gauge sector in the usual way by defining the combinations

$$A_\mu = \frac{g}{\sqrt{g^2 + g^2}} B_\mu + \frac{g'}{\sqrt{g^2 + g^2}} W_\mu^a, \quad Z_\mu = \frac{g}{\sqrt{g^2 + g^2}} W_\mu^3 - \frac{g'}{\sqrt{g^2 + g^2}} B_\mu,$$

$$c_a = \frac{g}{\sqrt{g^2 + g^2}} c + \frac{g'}{\sqrt{g^2 + g^2}} c^3, \quad c_Z = \frac{g}{\sqrt{g^2 + g^2}} c - \frac{g'}{\sqrt{g^2 + g^2}} c. \tag{8.5}$$

In addition, the last expression involves $\Box h$, where $h$ is the classical field, so that one may use its equation of motion:

$$\Box h = -m_h^2 h + \cdots, \quad m_h^2 = 2v^2 V_2. \tag{8.6}$$

Taking the background field $h$ to zero in (8.4), we obtain the kinetic terms for all the quantum fields. Those terms fix the propagators entering the calculation. In order to compute the vev of $h$ at one loop, we further need to extract
from (8.4) all the cubic terms that are both linear in the background field \( h \) and quadratic in the quantum fields, i.e.

\[
\mathcal{L}_2 + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghosts}} = \cdots + \frac{h}{v} F_1 \left[ - \frac{g^2}{2} V_2 (\dot{\varphi}^a \dot{\varphi}^a) - \frac{1}{2} \left( \frac{g v^2}{2} \right)^2 (\dot{\varphi}^1 \dot{\varphi}^1 + \dot{\varphi}^2 \dot{\varphi}^2) - \frac{1}{2} \left( \frac{v}{2} \right)^2 (g^2 + g'^2) \dot{\varphi}^3 \dot{\varphi}^3 - 3 v^2 V_3 \dot{h}^2 + \frac{1}{2} \left( \frac{g v^2}{2} \right)^2 (W_\mu^1 \dot{W}^\mu_1 + W_\mu^2 \dot{W}^\mu_2) + \frac{1}{2} \left( \frac{v}{2} \right)^2 (g^2 + g'^2) \dot{Z}_\mu \dot{Z}^\mu - \frac{g' v}{2} \partial^\mu (B_\mu \dot{\varphi}^3) + \frac{g v}{2} \partial^\mu (W_\mu^a \dot{\varphi}^a) \right] - \frac{1}{F_1} \bar{\psi} (\mathcal{M}_1 P_R + \mathcal{M}_1^* P_L) \psi - \left( \frac{g v^2}{2} \right)^2 (\dot{c}^1 \dot{c}^1 + \dot{c}^2 \dot{c}^2) - \left( \frac{v}{2} \right)^2 (g^2 + g'^2) \dot{c} \dot{c} Z \dot{Z} \right] + \cdots. \tag{8.7}
\]

Then one finds

\[
< h >_{\text{loop}} = \frac{1}{2 v^2 V_2} \times \frac{i F_1}{2 v} \int \frac{d^d q}{(2 \pi)^d} \left[ \frac{-v^2 V_2 - (g v/2)^2}{q^2 - (g v/2)^2} + \frac{-v^2 V_2 - v^2 (g^2 + g'^2)/4}{q^2 - v^2 (g^2 + g'^2)/4} - 6 v^2 V_3 \right] \frac{1}{F_1} \frac{1}{q^2 - 2 v^2 V_2}
\]

\[
+ d \times 2 \left( \frac{g v^2}{2} \right)^2 \frac{1}{q^2 - (g v/2)^2} + d \times \left( \frac{v}{2} \right)^2 \frac{1}{q^2 - (g^2 + g'^2)/4} - 4 (-1) \left( \frac{g v^2}{2} \right)^2 \frac{1}{q^2 - (g v/2)^2 - 2 (-1) \left( \frac{v}{2} \right)^2 \frac{1}{q^2 - v^2 (g^2 + g'^2)/4}
\]

\[
= \frac{2}{F_1} (-1) \left[ \left( M_1 P_R + M_1^* P_L \right) \left( g + M_0 P_L + M_0^* P_R \right) \left( \frac{1}{q^2 - M_0 M_0^*} P_R + \frac{1}{q^2 - M_0^* M_0} P_L \right) \right]
\]

\[
= \frac{i}{2 v^2 V_2} \times \frac{F_1}{2 v} \left[ 2 (1 - d) \left( \frac{g v^2}{2} \right)^2 A \left( \frac{g^2 v^2}{4} \right) + (1 - d) \left( \frac{v}{2} \right)^2 (g^2 + g'^2) A \left( \frac{v^2}{4} (g^2 + g'^2) \right)
\]

\[
- 2 v^2 V_2 A \left( \frac{g v^2}{4} \right) - v^2 V_2 A \left( \frac{v}{4} (g^2 + g'^2) \right) - 6 v^2 V_3 A \left( 2 v^2 V_2 \right)
\]

\[
+ \frac{4}{F_1} \left[ \left\langle M_1 M_0^* A (M_0 M_0^* + M_0^* M_0) \right\rangle \right]. \tag{8.8}
\]

The first equality shows, successively, the contributions from the scalar fields (first line), from the gauge fields (second line), from the ghosts (third line), and from the fermion fields (fourth line). The second equality involves the dimensionally regularized one-point loop function

\[
A(m^2) = \int \frac{d^d q}{(2 \pi)^d} \frac{1}{q^2 - m^2} \rightarrow \mu^{-d} \int \frac{d^d q}{(2 \pi)^d} \frac{1}{q^2 - m^2} = -i m^2 \frac{1}{16 \pi^2} \left[ \frac{2}{d - 4} - \ln 4 \pi + \gamma_E + \frac{m^2}{\mu^2} - 1 \right]. \tag{8.9}
\]

On the other hand, in the \( \overline{\text{nMS}} \) scheme, the counterterm \( \Delta V = V' \delta \) in (4.12) contributes to \( < h > \) as

\[
< h >_{\Delta V} = \frac{1}{2 v^2 V_2} \delta \left( \frac{-\Delta V}{h} \right) \bigg|_{h=0}
\]

\[
= \left[ \left( \frac{1}{2 v^2 V_2} \right) \left( \frac{1}{16 \pi^2} \right) \left[ \frac{3}{2} \left( 3 g^4 + 2 g^2 g'^2 + g'^4 \right) \frac{v^3}{8} F_1 + (3 g^2 + g'^2) \frac{v^3}{4} F_1 V_2 \right]
\]

\[
+ 12 v^3 V_3 - \frac{4}{v} \left\langle M_1^* M_0 (M_1 M_0 + M_0^* M_1) \right\rangle \right]. \tag{8.10}
\]

One thus obtains the finite, but scale-dependent, result

\[
< h >_{\text{loop}} + < h >_{\Delta V} = \frac{1}{2 v^2 V_2} \times \left[ \frac{3 v^3 F_1}{16} \left( \frac{g v^2}{2} \right)^2 \left( \frac{4 \mu^2}{4 \mu^2} - 1 \right) \left( \frac{3 v^2 F_1}{32} (g^2 + g'^2) \left( \frac{g^2 + g'^2}{4 \mu^2} - 1 \right) \right) \right]
\]

\[
\frac{-v^3}{8} F_1 V_2 \left( \frac{g^2}{4 \mu^2} - 1 \right) \left( \frac{4 \mu^2}{4 \mu^2} - 1 \right) - 6 v^3 V_3 \left( \frac{2 v^2 V_2}{\mu^2} - 1 \right)
\]

\[
+ \frac{2}{v} \left\langle M_1 M_0^* M_0 M_0^* \right\rangle \left( \frac{M_0 M_0^*}{\mu^2} - 1 \right) + M_1^* M_0 M_0^* M_0 \left( \frac{1}{\mu^2} - 1 \right) \right], \tag{8.11}
\]

where \((d - 1) A(m^2) = 3A(m^2) - im^2/(8\pi^2)\) has been used.
In order to ensure that $V'(0)$ vanishes at the one-loop level, one thus must add to $\Delta V$ a tadpole part, $\Delta V \rightarrow \Delta V + hT(\mu)$, with $T(\mu)$ chosen such that $<h>_{\text{loop}} + <h>_{\Delta V} + <h>_{T} = 0$. This requires

$$T(\mu) = -\frac{1}{16\pi^2} \left[ \frac{3v^3F_1}{16} g^4 \left( \ln \frac{g^2v^2}{4\mu^2} - \frac{1}{3} \right) + \frac{3v^3F_1}{32} (g^2 + g'^2)^2 \left( \ln \frac{(g^2 + g'^2)v^2}{4\mu^2} - \frac{1}{3} \right) ight. $$

$$ + \frac{g^2}{8} F_1 V_2 \left( 2g^2 \left( \ln \frac{g^2v^2}{4\mu^2} - 1 \right) + (g^2 + g'^2) \left( \ln \frac{(g^2 + g'^2)v^2}{4\mu^2} - 1 \right) \right) + 6v^3 V_3 \left( \ln \frac{2v^2V_2}{\mu^2} - 1 \right) $$

$$ - 2 \left[ \mathcal{M}_1^{\mu_0} \mathcal{M}_0^{\mu_0} \mathcal{M}_1^{\mu_0} \mathcal{M}_0^{\mu_0} \right] \left( \ln \frac{\mathcal{M}_1^{\mu_0} \mathcal{M}_0^{\mu_0}}{\mu^2} - 1 \right) \mathcal{M}_1^{\mu_0} \mathcal{M}_0^{\mu_0} \mathcal{M}_0^{\mu_0} \mathcal{M}_0^{\mu_0} \left( \ln \frac{\mathcal{M}_1^{\mu_0} \mathcal{M}_0^{\mu_0}}{\mu^2} - 1 \right) \right].$$

(8.12)

We note that the tadpole term $T(\mu)$ corresponds to a finite shift of the Higgs field, which can be expressed as a finite contribution $\delta_{\text{fin}}$ to the divergent parameter $\delta$ introduced in (4.12),

$$\delta \rightarrow \delta + \delta_{\text{fin}}, \quad \delta_{\text{fin}} = \frac{1}{2v^3V_2} T(\mu).$$

(8.13)

According to (4.12) this leads to finite corrections in the effective Lagrangian, which cancel the Higgs tadpole in all amplitudes to one-loop order.

**IX. CONCLUSIONS**

In this paper we have worked out the one-loop renormalization group equations of the EWChL, taking as starting point the one-loop divergences given in [16]. The transition between the divergent structures of the theory and its beta functions is most conveniently done if the results are projected onto a complete basis. In this paper we have used the conventions adopted in [14] and worked out in detail the one-loop renormalization of both the leading order and next-to-leading order EWChL. Besides the complete list of the beta functions, we provide, for completeness, the explicit calculation of the finite piece needed to enforce the no-tadpole condition at one loop.

One of the most interesting results of our analysis is that the one-loop beta functions of the gauge couplings happen to be universal, i.e. they are not affected by potential deviations of the Higgs couplings with respect to their SM values. In order to reach this conclusion in a transparent way, it is crucial to reduce the divergent operator set to a minimal, non-redundant basis. Only after this step is done it is possible to directly read off the gauge-beta functions from the divergent piece in front of the kinetic terms without doing an additional calculation.

Our results have been crosschecked in a number of ways. All the renormalization group equations correctly reduce to the SM ones in the appropriate limit. The one-loop renormalization of SMEFT has also been used for comparison. The EWChL and SMEFT are different electroweak EFTs but their one-loop divergences partly overlap. We have explicitly shown that our results are consistent in this overlapping sector. Finally, the fact that the computation of the finite tadpole contribution yields actually a finite result is yet another meaningful cross-check.

The computation presented in this paper is of relevance for the analyses of Higgs interactions at the LHC. The EWChL is the right tool to implement consistently the $\kappa$ formalism into an EFT language. The one-loop renormalization presented in this paper is necessary if one wants to extend the $\kappa$ formalism to study differential distributions in Higgs processes. Various processes of interest will be considered in the future. The framework is now available to extend their treatment to one loop.

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of Energy, Office of Science, Office of High Energy Physics. For cross-checks of our calculations, the programs FeynCalc [46, 47] and Mathematica [48] proved useful, as well as the compilation of formulas in [49].

Appendix A: Equations of motion

The equations of motion (eom), derived from the leading-order Lagrangian in (2.1), are needed in particular to reduce NLO terms to a set of basis operators (see Appendix B). Here we collect the eom for the gauge fields $B_{\mu}$, $W_{\mu}$, $G_{\mu}$, the scalars $\eta = h/v$, $\varphi$, and the fermions $\psi_{L,R}$:

$$\partial^\mu B_{\mu\nu} = g^\prime \left[ \bar{\psi}_L \gamma_\nu Y_L \psi_L + \bar{\psi}_R \gamma_\nu Y_R \psi_R + \frac{v^2}{2} F(\tau L L) \right]$$  \quad (A.1)

$$D^\mu W^a_{\mu\nu} = g \left[ \bar{\psi}_L \gamma_\nu T^a \psi_L - \frac{v^2}{2} F(\tau L L) \right]$$  \quad (A.2)

$$D^\mu G^A_{\mu\nu} = g_s \bar{q} \gamma_\nu T^A q$$  \quad (A.3)

$$v^2 \Box \eta = -V^I + \frac{v^2}{4} \langle L_L L \rangle F^I - \bar{\psi} m \psi$$  \quad (A.4)

$$FD^\mu L_{\mu} = -F^I \partial^\mu \eta L_{\mu} - \frac{4}{v^2} UT^a U^I \left( \bar{\psi}_L U i T^a \mathcal{M} \psi_R + \text{h.c.} \right)$$  \quad (A.5)

$$i D \psi_L = U \mathcal{M} \psi_R$$  \quad (A.6)

$$i D \psi_R = \mathcal{M}^U U^I \psi_L$$  \quad (A.7)

Appendix B: Basis of NLO operators

In this section we list a basis of NLO operators for the EWChL, following [14, 15]. The NLO terms are the independent operators of chiral dimension 4, with the field content and the symmetries of the Lagrangian in (2.1).

In the following we will assume that custodial symmetry breaking takes place at the electroweak scale. This means that the spurions of custodial symmetry breaking $\sim \tau_L$ carry chiral dimension. Accordingly, terms of chiral dimension 4 with extra factors of $\tau_L$, which were kept in the general analysis of NLO operators in [14], will be omitted. We further assume that tensor currents, e.g. $\bar{q} \sigma_{\mu\nu} U r$, only arise with a chiral dimension larger than 2. This eliminates operators with tensor currents in [14] from the list of NLO terms to be considered here.

The NLO operators can then be divided into the classes $UhD^2$, $UhD^4$, $X^2Uh$, $XUhD^2$, $\psi^2UhD$, $\psi^2UhD^2$ and $\psi^4Uh$. A well-defined subset of these operators represents the counterterms necessary to renormalize the one-loop divergences of the EWChL calculated in this paper.

All operators arising at chiral dimension 4 in the classes $UhD^2$, $UhD^4$, $X^2Uh$, $XUhD^2$, $\psi^2UhD$ and $\psi^2UhD^2$ are listed below. All the operators in the classes $UhD^2$, $UhD^4$, $\psi^2UhD$ and $\psi^2UhD^2$ are needed as counterterms. On the other hand, no operator in class $X^2Uh$ or $XUhD^2$ is required to absorb one-loop divergences. Within the class $\psi^4Uh$, we only list the operators that actually appear as (divergent) counterterms.

Class $UhD^2$:

$$\mathcal{O}_{\beta_i} = -v^2 \langle \tau L L_{\mu} \rangle \langle \tau L L^{\mu} \rangle.$$  \quad (B.1)

This operator has only two derivatives, but its coefficient comes with two powers of the weak coupling $g^\prime$, related to custodial symmetry breaking. In total, the term has chiral dimension 4 and enters at NLO in the EFT.
Class $U h D^4$:

\[ O_{D1} = \langle L_\mu L^\mu \rangle^2, \quad O_{D2} = \langle L_\mu L_\nu \rangle \langle L_\mu L^\nu \rangle \]
\[ O_{D7} = \langle L_\mu L^\mu \rangle \partial_\nu \eta \partial^\nu \eta, \quad O_{D8} = \langle L_\mu L_\nu \rangle \partial^\mu \eta \partial^\nu \eta, \quad O_{D11} = (\partial_\mu \eta \partial^\mu \eta)^2. \] (B.2)

All these operators are CP even.

Class $X^2Uh$:

The CP-even operators are

\[ O_{Xh1} = g^2 B_{\mu \nu} B^{\mu \nu} F_{Xh1}(h) \]
\[ O_{Xh2} = g^2 \langle W_{\mu \nu} W^{\mu \nu} \rangle F_{Xh2}(h) \]
\[ O_{Xh3} = \frac{g^2}{2} G_\alpha^{\mu \nu} G_\alpha^{\mu \nu} F_{Xh3}(h). \] (B.3)

The CP-odd operators read

\[ O_{Xh4} = g^2 \varepsilon_{\mu \nu \lambda \rho} B^{\mu \nu} B^{\lambda \rho} F_{Xh4}(h) \]
\[ O_{Xh5} = g^2 \varepsilon_{\mu \nu \lambda \rho} \langle W^{\mu \nu} W^{\lambda \rho} \rangle F_{Xh5}(h) \]
\[ O_{Xh6} = \frac{g^2}{2} \varepsilon^{\mu \nu \lambda \rho} C_\mu^{\alpha} G_\alpha^{\mu \nu} G_\lambda^{\alpha \rho} F_{Xh6}(h). \] (B.4)

Here

\[ F_{Xi}(h) = \sum_{n=1}^{\infty} f_{Xi,n} \left( \frac{h}{u} \right)^n. \] (B.5)

Class $XUhD^2$:

CP-even operators:

\[ O_{XU1} = g' g B_{\mu \nu} \langle W^{\mu \nu} \tau_L \rangle (1 + F_{XU1}(h)) \]
\[ O_{XU7} = ig' B_{\mu \nu} \langle \tau_L [L_\mu, L_\nu] \rangle F_{XU7}(h) \]
\[ O_{XU8} = ig \langle W_{\mu \nu} [L_\mu, L_\nu] \rangle F_{XU8}(h). \] (B.6)

CP-odd operators:

\[ O_{XU4} = g' g \varepsilon_{\mu \nu \lambda \rho} \langle \tau_L W^{\mu \nu} \rangle B^{\lambda \rho} (1 + F_{XU4}(h)) \]
\[ O_{XU10} = ig' \varepsilon_{\mu \nu \lambda \rho} B^{\mu \nu} \langle \tau_L [L^\lambda, L^\rho] \rangle F_{XU10}(h) \]
\[ O_{XU11} = ig \varepsilon_{\mu \nu \lambda \rho} \langle W^{\mu \nu} [L^\lambda, L^\rho] \rangle F_{XU11}(h), \] (B.7)

with $F_{Xi}(h)$ as in (B.5).
Class $\psi^2 UhD$:

\begin{align*}
\mathcal{O}_{\psi V1} &= -\bar{q}_L \gamma^\mu q_L \langle \tau_L L_\mu \rangle & \mathcal{O}_{\psi V2} &= -\bar{q}_L \gamma^\mu \tau_L q_L \langle \tau_L L_\mu \rangle \\
\mathcal{O}_{\psi V3} &= -\bar{q}_L \gamma^\mu U P_{12} U^\dagger q_L \langle L_\mu U P_{21} U^\dagger \rangle & \mathcal{O}_{\psi V4} &= -\bar{u}_R \gamma^\mu u_R \langle \tau_L L_\mu \rangle \\
\mathcal{O}_{\psi V5} &= -\bar{d}_R \gamma^\mu d_R \langle \tau_L L_\mu \rangle & \mathcal{O}_{\psi V6} &= -\bar{u}_R \gamma^\mu d_R \langle L_\mu U P_{21} U^\dagger \rangle \\
\mathcal{O}_{\psi V7} &= -\bar{l}_L \gamma^\mu l_R \langle \tau_L L_\mu \rangle & \mathcal{O}_{\psi V8} &= -\bar{l}_L \gamma^\mu \tau_L l_R \langle \tau_L L_\mu \rangle \\
\mathcal{O}_{\psi V9} &= -\bar{l}_L \gamma^\mu U P_{12} U^\dagger l_L \langle L_\mu U P_{21} U^\dagger \rangle & \mathcal{O}_{\psi V10} &= -\bar{e}_R \gamma^\mu e_R \langle \tau_L L_\mu \rangle,
\end{align*}

_together with the hermitian conjugates $\mathcal{O}_{\psi V3}^\dagger$, $\mathcal{O}_{\psi V6}^\dagger$ and $\mathcal{O}_{\psi V9}^\dagger$._

Class $\psi^2 UhD^2$:

\begin{align*}
\mathcal{O}_{\psi S1} &= \bar{q}_L U P_+ \bar{q}_R \langle L_\mu L^\mu \rangle & \mathcal{O}_{\psi S2} &= \bar{q}_L U P_- \bar{q}_R \langle L_\mu L^\mu \rangle & \mathcal{O}_{\psi S7} &= \bar{l}_L U P_+ \bar{l}_R \langle L_\mu L^\mu \rangle \\
\mathcal{O}_{\psi S10} &= \bar{q}_L U P_+ \bar{q}_R \langle \tau_L L_\mu \rangle \partial^\mu \eta & \mathcal{O}_{\psi S11} &= \bar{q}_L U P_- \bar{q}_R \langle \tau_L L_\mu \rangle \partial^\mu \eta & \mathcal{O}_{\psi S12} &= \bar{q}_L U P_{12} \bar{q}_R \langle U P_{21} U^\dagger L_\mu \rangle \partial^\mu \eta \\
\mathcal{O}_{\psi S13} &= \bar{q}_L U P_{21} \bar{q}_R \langle U P_{12} U^\dagger L_\mu \rangle \partial^\mu \eta & \mathcal{O}_{\psi S14} &= \bar{q}_L U P_- \bar{q}_R \partial _\mu \eta \partial^\mu \eta & \mathcal{O}_{\psi S15} &= \bar{q}_L U P_- \bar{q}_R \partial _\mu \eta \partial^\mu \eta \\
\mathcal{O}_{\psi S16} &= \bar{l}_L U P_- \bar{l}_R \langle \tau_L L_\mu \rangle \partial^\mu \eta & \mathcal{O}_{\psi S17} &= \bar{l}_L U P_{21} \bar{l}_R \langle U P_{21} U^\dagger L_\mu \rangle \partial^\mu \eta & \mathcal{O}_{\psi S18} &= \bar{l}_L U P_- \bar{l}_R \partial _\mu \eta \partial^\mu \eta.
\end{align*}

_(B.8)_

For this class, hermitean conjugate versions have not been listed separately.

Class $\psi^4 Uh$:

Here we list the four-fermion operators that are generated as one-loop counterterms of the EWChL. The complete basis can be found in [24]. Note that some of the $ST$-type operators originally listed there are redundant, as pointed out in Appendix A.4 of [50]. This redundancy doesn’t affect the terms ST5, ST6, ST9, ST10 appearing below. Generation indices are suppressed. $T^A$ denotes the generators of color $SU(3)$.

\begin{align*}
\mathcal{O}_{LR1} &= \bar{q}_L \gamma^\mu q_L \bar{u}_R \gamma^\mu u_R & \mathcal{O}_{LR2} &= \bar{q}_L \gamma^\mu T^A q_L \bar{u}_R \gamma^\mu T^A u_R \\
\mathcal{O}_{LR3} &= \bar{q}_L \gamma^\mu q_L \bar{d}_R \gamma^\mu d_R & \mathcal{O}_{LR4} &= \bar{q}_L \gamma^\mu T^A q_L \bar{d}_R \gamma^\mu T^A d_R \\
\mathcal{O}_{LR8} &= \bar{l}_L \gamma^\mu l_R \bar{e}_R \gamma^\mu e_R & \mathcal{O}_{LR9} &= \bar{l}_L \gamma^\mu l_R \bar{e}_R \gamma^\mu e_R \\
\mathcal{O}_{LR10} &= \bar{q}_L \gamma^\mu UT_3 U^\dagger q_L \bar{u}_R \gamma^\mu u_R & \mathcal{O}_{LR11} &= \bar{q}_L \gamma^\mu T^A U T_3 U^\dagger q_L \bar{u}_R \gamma^\mu T^A u_R \\
\mathcal{O}_{LR12} &= \bar{q}_L \gamma^\mu UT_3 U^\dagger q_L \bar{d}_R \gamma^\mu d_R & \mathcal{O}_{LR13} &= \bar{q}_L \gamma^\mu T^A U T_3 U^\dagger q_L \bar{d}_R \gamma^\mu T^A d_R \\
\mathcal{O}_{LR17} &= \bar{l}_L \gamma^\mu UT_3 U^\dagger l_R \bar{e}_R \gamma^\mu e_R & \mathcal{O}_{LR18} &= \bar{q}_L \gamma^\mu UT_3 U^\dagger l_R \bar{e}_R \gamma^\mu e_R \\
\mathcal{O}_{ST5} &= \bar{q}_L U P_+ \bar{q}_R \bar{q}_L U P_- \bar{q}_R & \mathcal{O}_{ST6} &= \bar{q}_L U P_{21} \bar{q}_R \bar{q}_L U P_{12} \bar{q}_R \\
\mathcal{O}_{ST9} &= \bar{q}_L U P_+ \bar{q}_R \bar{l}_L U P_- \bar{l}_R & \mathcal{O}_{ST10} &= \bar{q}_L U P_{21} \bar{q}_R \bar{l}_L U P_{12} \bar{l}_R \\
\mathcal{O}_{FY1} &= \bar{q}_L U P_+ \bar{q}_R \bar{q}_L U P_- \bar{q}_R & \mathcal{O}_{FY3} &= \bar{q}_L U P_- \bar{q}_R \bar{q}_L U P_- \bar{q}_R \\
\mathcal{O}_{FY5} &= \bar{q}_L U P_- \bar{q}_R \bar{q}_R P_+ U^\dagger q_L & \mathcal{O}_{FY7} &= \bar{q}_L U P_- \bar{q}_R \bar{l}_L U P_- \bar{l}_R \\
\mathcal{O}_{FY9} &= \bar{l}_L U P_- \bar{l}_R \bar{q}_R P_+ U^\dagger q_L & \mathcal{O}_{FY10} &= \bar{l}_L U P_- \bar{l}_R \bar{l}_L U P_- \bar{l}_R.
\end{align*}

_(B.9)_

_(B.10)_

_(B.11)_

_(B.12)_
Appendix C: One-loop divergences

This appendix gathers the explicit expressions for the complete one-loop divergences of the EWChL obtained in Ref. [16]. In terms of the decomposition introduced in Eq. (3.1), they read:

\[ \mathcal{L}^{(1)}_{\text{div}} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ -\frac{g^2}{4} \left( \frac{22}{27} N_c + 2 \right) N_g + \frac{g^2}{3} \left( \frac{22}{27} N_c + 2 \right) N_g + \frac{\kappa^2 + 1}{6} \right\} B_{\mu\nu} B_{\mu\nu} \\
+ \frac{\kappa^2 - 1}{6} g g' \langle \tau L W_{\mu\nu} B_{\mu\nu} \rangle - \frac{\kappa^2 - 1}{12} \left( i g \langle W_{\mu\nu} [L, L_\nu] \rangle + i g' B_{\mu\nu} \langle \tau L [L, L_\nu] \rangle \right) \\
- \frac{\kappa^2}{4} \langle \psi \langle W_{\mu\nu} \rangle - g' B_{\mu\nu} \langle \tau L \rangle \rangle \right\}, \tag{C.1} \]

\[ \mathcal{L}^{(0)}_{\text{div}} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ -\frac{1}{2} \left[ g^2 (\kappa^2 + 3) \frac{v^2 F}{4} + 3g^2 (\kappa^2 + 1) \frac{v^2 F}{2} + (\kappa^2 - 1) \frac{F'V'}{F v^2} - \frac{V'' F}{v^2} B - 2\langle (\mathcal{M}^\dagger \mathcal{M}) \rangle \right] \langle L_\mu L_\mu \rangle \\
+ \frac{1}{4} \left[ (3g^2 + g^2) v^2 (FB - \kappa^2) + 6 \frac{F'V'}{F v^2} B \right] \partial^\mu \eta \partial_\mu \eta + 2 \langle (\partial^\mu \mathcal{M}^\dagger \partial_\mu \mathcal{M}) \rangle \\
+ \frac{3}{2} (3g^2 + 2g^2 v^2 + g^4) \frac{v^2}{16 F} + \frac{3g^2 + 2g^2}{8} \frac{F'V'}{F v^2} + \frac{3}{8} \left( \frac{F'V'}{F v^2} \right)^2 + \frac{1}{2} \left( \frac{V''}{v^2} \right)^2 - 2 \langle (\mathcal{M}^\dagger \mathcal{M}) \rangle^2 \right\} \langle L_\mu L_\mu \rangle \\
+ \left( \frac{\kappa^2 - 1}{6} \right) \langle L_\mu L_\mu \rangle \langle L_\mu L_\mu \rangle + \left( \frac{(\kappa^2 - 1)^2}{12} + \frac{F^2 B^2}{8} \right) \langle L_\mu L_\mu \rangle^2 + \frac{2}{3} \kappa (\kappa^2 - 1) \langle L_\mu \rangle \partial^\mu \eta \partial_\mu \eta \\
- \left( (\kappa^2 - 1) B - \frac{\kappa^2}{6} \right) \langle L_\mu L_\mu \rangle \partial^\mu \eta \partial_\mu \eta + \frac{3}{2} \langle B^2 (\partial^\mu \eta \partial_\mu \eta)^2 + \frac{3}{4} g^2 v^2 (1 - \kappa^2) F \langle \tau L L_\mu \rangle \langle \tau L L_\mu \rangle \\
+ 4i \langle (\partial^\mu \mathcal{M}^\dagger \mathcal{M} - \mathcal{M}^\dagger \partial^\mu \mathcal{M}) \rangle^2 \rangle \langle \tau L L_\mu \rangle \rangle \right\}. \tag{C.2} \]

Here \( \langle \ldots \rangle \) denotes the trace over isospin, as well as generation and color indices, in distinction to \( \langle \ldots \rangle \), which refers to the trace over isospin indices only.

\[ \mathcal{L}^{(1/2)}_{\text{div}} = -\frac{1}{16\pi^2} \frac{1}{d-4} \left\{ \tilde{\psi}_L \left( \frac{3}{2} g^2 + 2g^2 Y_L^2 \right) i \mathcal{D} \psi_L + \tilde{\psi}_R 2g^2 Y_R^2 i \mathcal{D} \psi_R \\
+ 2g^2 C_F \tilde{q} (i \mathcal{D} - 4 (U \mathcal{M}_F P_R + \mathcal{M}_F U^\dagger P_L)) \tilde{q} \\
+ \frac{V''}{v^4} \left( \tilde{\psi}_L U^\dagger M^\mu \psi_R + \text{h.c.} \right) - 8g^2 \left( \tilde{\psi}_L Y_L U^\dagger M^\mu \psi_R + \text{h.c.} \right) \right\} \\
+ \left( (3g^2 + g^2) \frac{v^2}{4} \frac{F}{v^2} + 3 \frac{F'V'}{2v^2} + \frac{F^{-1}}{v^2} \left( \tilde{\psi}_L U \left( \frac{F'}{2} M^\mu - \mathcal{M} \right) \psi_R + \text{h.c.} \right) \right) \\
- \frac{8}{v^2} \frac{F}{v^2} \left( \tilde{\psi}_L U T^a M^\mu \psi_R + \text{h.c.} \right) + \frac{2}{v^2} \left( \tilde{\psi}_L U M^\mu \psi_R + \text{h.c.} \right) \right\} \\
+ \langle L_\mu L_\mu \rangle \left[ \frac{F B}{2v^2} \tilde{\psi}_L U M^\mu \psi_R - \frac{\kappa^2 - 1}{v^2} \tilde{\psi}_L U \left( \frac{F'}{2} M^\mu - \mathcal{M} \right) \psi_R + \text{h.c.} \right] \\
+ \frac{2\kappa^2}{v^2} \theta^\mu \eta \left( i \tilde{\psi}_L U (F^{-1/2} M) \psi_R + \text{h.c.} \right) \\
+ \frac{3B}{F v^2} \eta \theta^\mu \eta \left( \tilde{\psi}_L U \left( \frac{F'}{2} M^\mu - \mathcal{M} \right) \psi_R + \text{h.c.} \right) \\
+ \frac{3 F^{-2}}{2v^2} \left( \tilde{\psi}_L U \left( \frac{F'}{2} M^\mu - \mathcal{M} \right) \psi_R + \text{h.c.} \right)^2 \right\} \\
+ \frac{4}{v^4} \left( i \tilde{\psi}_L U T^a (F^{-1/2} M) \psi_R + \text{h.c.} \right) \\
+ \frac{4}{v^2} \tilde{\psi}_L U T^a M F^{-1/2} i \phi \left( M^\mu F^{-1/2} \right) T^a U^\dagger \psi_L \right\}. \tag{C.3} \]
\[ UT^a \mathcal{M} \mathcal{M}^\dagger T^a U^\dagger = \frac{1}{2} \langle \mathcal{M} \mathcal{M} \dagger \rangle - \frac{1}{4} U \mathcal{M} \mathcal{M}^\dagger U^\dagger. \]  

\[ i \partial \psi_L = U \mathcal{M} \psi_R, \quad i \partial \psi_R = \mathcal{M}^\dagger U^\dagger \psi_L, \]  

one establishes the identities

\[ \frac{3}{v^2} \bar{\psi}_R \mathcal{M}^\dagger F^{-1/2} i \partial \left( M F^{-1/2} \psi_R \right) = \frac{3i}{2v^2} \partial_\mu \left( F^{-1} \bar{\psi}_R \gamma^\mu \mathcal{M} \mathcal{M}^\dagger \psi_R \right) + \frac{3F^{-1}}{2v^2} \bar{\psi}_R \left( \mathcal{M}^\dagger \bar{\partial} \mathcal{M} - i \bar{\partial} \mathcal{M} \mathcal{M}^\dagger \right) \psi_R + \frac{F^{-1}}{2v^2} \bar{\psi}_R \mathcal{M} \mathcal{M}^\dagger U^\dagger \psi_L + \bar{\psi}_L U \mathcal{M} \mathcal{M}^\dagger \mathcal{M}^\dagger \psi_R, \]  

\[ \frac{1}{v^2} \bar{\psi}_L \mathcal{M} (i \bar{\partial} \mathcal{M}^\dagger) U^\dagger \psi_L + \frac{1}{v^2} \bar{\psi}_L \mathcal{M} \mathcal{M}^\dagger U^\dagger (i \partial + \slashed{E}) \psi_L = \frac{i}{2v^2} \partial_\mu \left( \bar{\psi}_L U \mathcal{M} \mathcal{M}^\dagger U^\dagger \gamma^\mu \psi_L \right) + \frac{1}{2v^2} \bar{\psi}_L U \left( \mathcal{M} \mathcal{M}^\dagger - i \bar{\partial} \mathcal{M} \mathcal{M}^\dagger \right) U^\dagger \psi_L + \frac{1}{2v^2} \bar{\psi}_L \left( U \mathcal{M} \mathcal{M}^\dagger U^\dagger \slashed{E} + \slashed{E} \mathcal{M} \mathcal{M}^\dagger U^\dagger \right) \psi_L + \frac{1}{2v^2} \left( \bar{\psi}_L U \mathcal{M} \mathcal{M}^\dagger \mathcal{M}^\dagger \psi_R + \bar{\psi}_R \mathcal{M} \mathcal{M}^\dagger U^\dagger \psi_L \right), \]  

\[ \frac{2F^{-1/2}}{v^2} \bar{\psi}_L (\mathcal{M} \bar{\partial} (\mathcal{M}^\dagger F^{-1/2})) \psi_L + \frac{2F^{-1}}{v^2} \bar{\psi}_L \mathcal{M} (\mathcal{M}^\dagger \partial \psi_L) = \frac{i}{2v^2} \partial_\mu \left( \bar{\psi}_L U \mathcal{M} \mathcal{M}^\dagger U^\dagger \gamma^\mu \psi_L \right) + \frac{F^{-1}}{v^2} \bar{\psi}_L \mathcal{M} \mathcal{M}^\dagger \psi_R + \frac{F^{-1}}{v^2} \bar{\psi}_R \mathcal{M} (\mathcal{M}^\dagger \mathcal{M}^\dagger) U^\dagger \psi_L, \]  

\[ - \frac{F^{-1/2}}{v^2} \bar{\psi}_L \mathcal{M} \mathcal{M}^\dagger U^\dagger \psi_L - \frac{F^{-1}}{v^2} \bar{\psi}_L U \mathcal{M} \mathcal{M}^\dagger U^\dagger \psi_L + \frac{F^{-1}}{v^2} \bar{\psi}_L \slashed{E} \mathcal{M} \mathcal{M}^\dagger U^\dagger \psi_L = 0. \]
so that, upon dropping the total derivatives, \( L_{\text{div}}^{(1/2)} \) may be rewritten as

\[
L_{\text{div}}^{(1/2)} = -\frac{1}{16\pi^2} \left( \frac{1}{d-4} \left( \tilde{\psi}_L \left( \frac{3}{2} g^2 + 2g'^2Y_L^2 \right) i \slashed{D} \psi_L + \tilde{\psi}_R 2g'^2Y_R^2 i \slashed{D} \psi_R \\
+ 2g'^2C_F \bar{q} \left( i \slashed{D} - 4(U \tau_{\alpha} P_R + M^\dagger v \tau_{\alpha} P_L) \right) q \right) \\
+ \frac{V'}{v^2} (\tilde{\psi}_L U M' \psi_R + \text{h.c.}) - 8g^2 (\tilde{\psi}_L Y_L U M \psi_R + \text{h.c.}) \\
+ \left( \frac{3g^2 + g'^2}{v^2} F + \frac{3F'V}{2F} \right) \frac{F^{-1}}{v^2} \left( \tilde{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right) \\
+ \frac{3}{v^2} F^{-1} (\tilde{\psi}_L U M^\dagger M \psi_R + \text{h.c.}) - \frac{3}{v^2} F^{-1} (\tilde{\psi}_L U M M^\dagger \psi_R + \text{h.c.}) \\
+ \frac{2}{v^2} (\tilde{\psi}_L U M^\dagger M' \psi_R + \text{h.c.}) + \frac{1}{2v^2} (\tilde{\psi}_L U M^\dagger M' \psi_R + \text{h.c.}) + \frac{1}{2v^2} (\tilde{\psi}_L U M^\dagger M' \psi_R + \text{h.c.}) \\
+ \frac{3F^{-1}}{2v^2} \tilde{\psi}_R (M^\dagger i \slashed{D} M - i \slashed{D} M^\dagger M) \psi_R + \frac{1}{2v^2} \tilde{\psi}_R (M^\dagger i \slashed{D} M' - i \slashed{D} M^\dagger M') \psi_R \\
- \frac{F^{-1}}{2v^2} \tilde{\psi}_L (M^\dagger i \slashed{D} M - i \slashed{D} M^\dagger M) \psi_R + \frac{1}{2v^2} \tilde{\psi}_L (M^\dagger i \slashed{D} M' - i \slashed{D} M^\dagger M') \psi_R \\
+ \frac{F^{-1}}{v^2} \tilde{\psi}_L (M^\dagger i \slashed{D} M - i \slashed{D} M^\dagger M) \psi_R - \frac{F^{-1}}{v^2} \tilde{\psi}_R M^\dagger M U M \psi_R - \frac{1}{v^2} \tilde{\psi}_R M^\dagger M U \bar{L} U M' \psi_R \\
+ \frac{\kappa}{v^2} F^{-1/2} (\tilde{\psi}_R M^\dagger M U \bar{L} U M' \psi_R + \text{h.c.}) - \frac{\kappa}{v^2} F^{-1/2} (\tilde{\psi}_L U M^\dagger M' U \bar{L} \psi_L + \text{h.c.}) \\
+ \frac{1}{2v^2} \tilde{\psi}_L (U M^\dagger M U U + U M \bar{L} M^\dagger U \bar{L}) \psi_L + \frac{F^{-1}}{2v^2} \tilde{\psi}_L (U U M^\dagger M' U \bar{L} \psi_L + \text{h.c.}) \\
+ \langle L \mu \Lambda \rangle \left[ \frac{F \beta}{2v^2} \tilde{\psi}_L U M^\dagger \psi_R - \frac{\kappa^2 - 1}{F v^2} \tilde{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right] \\
+ \frac{2\mu'}{v^2} \partial'^\mu \eta \left( i \tilde{\psi}_L U \rho (\psi_L U (F^{-1/2}M') \psi_R + \text{h.c.}) \right) \\
+ \frac{3B}{F v^2} \partial'^\mu \eta \tilde{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right) \\
+ \frac{3F^{-2}}{2v^2} \left( \tilde{\psi}_L U \left( \frac{F'}{2} M' - M \right) \psi_R + \text{h.c.} \right)^2 + \frac{1}{2v^2} (\tilde{\psi}_L U M^\dagger M \psi_R + \text{h.c.})^2 \\
+ \frac{4}{v^2} \left( i \tilde{\psi}_L U T^a \left( \frac{F^{-1/2}M'}{2} \right) \psi_R + \text{h.c.} \right)^2 \right].
\]

(C.10)
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