DASH: Distributed Adaptive Sequencing Heuristic for Submodular Maximization

Tonmoy Dey  
Department of Computer Science  
Florida State University  
Tallahassee, Florida  
tdey@fsu.edu

Yixin Chen  
Department of Computer Science  
Texas A&M University  
College Station, Texas  
chen777@tamu.edu

Alan Kuhnle  
Department of Computer Science  
Texas A&M University  
College Station, Texas  
kuhnle@tamu.edu

Abstract

MapReduce (MR) model algorithms for maximizing monotone, submodular functions subject to a cardinality constraint (SMCC) are currently restricted to the use of the linear-adaptive (non-parallelizable) algorithm GREEDY. Low-adaptive algorithms do not satisfy the requirements of these distributed MR frameworks, thereby limiting their performance. We study the SMCC problem in a distributed setting and propose the first MR algorithms with sublinear adaptive complexity. Our algorithms, R-DASH, T-DASH and G-DASH provide \((0.316 - \varepsilon), (0.375 - \varepsilon)\) and \((0.632 - \varepsilon)\) approximation ratio respectively with near-optimal adaptive complexity. Additionally, we provide a memory-efficient framework MED that eliminates the memory limitations of all MR model algorithms resulting from large distributed setups or considerable cardinality constraints. Finally, we provide empirical evidence to demonstrate that our sublinear-adaptive distributed algorithms provide orders of magnitude faster speedup in runtime compared to current state-of-the-art distributed algorithms while producing near identical results.

1 Introduction

Submodular maximization has become an important problem in data mining and machine learning with real-world applications ranging from influence and revenue maximization in social networks to more complex tasks such as image and video summarization. A submodular function captures the diminishing gain property of adding an element to a set that decreases with increase in size of the set. Formally, a non-negative set function \(f : 2^\mathcal{N} \rightarrow \mathbb{R}_+\) is submodular iff for all sets \(A \subseteq B \subseteq \mathcal{N}\) and \(x \in \mathcal{N}\setminus B\), \(f(A \cup x) - f(A) \geq f(B \cup x) - f(B)\). The submodular function is monotone if \(f(A) \leq f(B)\) for all \(A \subseteq B\). In this paper, we study the following optimization problem for submodular optimization under cardinality constraint (SMCC) and address the subsequent issues:

\[
O \leftarrow \arg\max_{S \subseteq \mathcal{N}} f(S); \text{ subject to } |S| \leq k
\]

Issue 1. Big Data from different sources and modalities are growing at an unprecedented rate making them computationally intractable. Such massive data sets call for novel techniques that can work on the distributed data in order to remain computationally manageable.

Motivated by the growing data size, extensive effort has been made to develop distributed algorithms for SMCC using the MapReduce (MR) framework [29, 28, 26, 27, 1, 5, 8, 13, 14, 16, 22] (see

Preprint. Under review.
Table 1: Theoretical comparison to state-of-the-art MapReduce (MR) and Adaptive model algorithms.

| Reference                | Ratio               | MR Rounds | Adaptivity       | Queries       |
|-------------------------|---------------------|-----------|------------------|---------------|
| Barbosa et al. 2015 (RG)‡ | \(\frac{1}{2}(1 - \frac{1}{e})\) | 1         | \(O(k)\)        | \(O(nk)\)    |
| Barbosa et al. 2016 (PALG)‡ | \(1 - \frac{1}{e} - \varepsilon\) | \(O(\frac{1}{e^2})\) | \(O\left(\frac{k}{e^2}\right)\) | \(O\left(\frac{nk}{e^2}\right)\) |
| Kazemi et al. (DDIST)†   | \(1 - \frac{1}{e} - \varepsilon\) | \(O(\frac{1}{e})\) | \(O\left(\frac{k}{e^2}\right)\) | \(O\left(\frac{nk}{e^2}\right)\) |
| Epasto et al. (BCG)‡     | \(1 - \varepsilon\) | \(r\)     | \(O\left(\frac{nk}{e^2\log^n(\frac{1}{e^2})}\right)\) | \(O\left(\frac{nk}{e^2}\right)\) |

**Table 1.** In the MapReduce framework, defined formally in Appendix, the data is too large to fit on a single machine and is distributed across many machines. The proposed distributed algorithms of Barbosa et al. [5] and Barbosa et al. [6] provides a way to utilize an algorithm (say ALG) in a distributed setting where ALG is executed on the distributed data over a constant number of MR rounds to yield a constant approximation. However, in order for their theoretical guarantees to hold, these algorithms rely on a crucial property (formally defined in Property [4]) that must be satisfied by the initial algorithm ALG. GREEDY currently is the only known algorithm to satisfy this property and achieve constant approximation in a distributed setting.

**Issue 2.** GREEDY Algorithm achieves the optimal ratio of \(1 - 1/e\) for SMCC [37], but it requires \(O(kn)\) queries to the objective function \(f\) and linearly many adaptive rounds. In the era of big data, as the size of data continues to grow exponentially, GREEDY becomes impractical leading to a need for more scalable and parallelizable algorithms for SMCC.

Recently, parallelizable algorithms for SMCC in the adaptive complexity model have been extensively studied [4,15,17,11,12,12,24,25,21]. In this model, queries to the objective function \(f\) are organized into as few adaptive rounds as possible; within each adaptive round, the function queries are independent and thus may be arbitrarily parallelized. Several algorithms have been proposed that are nearly optimal in terms of ratio, adaptivity, and total query calls with \(O(\log(n))\) adaptivity and \(O(n\text{polylog}(n))\) query complexity. LS+PGB of Chen et al. [12] is the current state-of-the-art adaptive algorithm that obtains nearly the optimal \(1 - 1/e - \varepsilon\) approximation in \(O(\log(n))\) adaptive rounds and \(O(n)\) query complexity in expectation.

**Issue 3.** Size of Distributed Setups can influence the performance of distributed algorithms. Smaller number of machines lead to longer runtime during the primary MR round (map), while a larger setup could lead to: 1) slower overall runtime (Fig. 7A in Appendix) 2) memory overload; 3) smaller permissible cardinality constraint range.

The use of GREEDY (non-parallelizable) in MR algorithms currently restricts them from solving Issue [5]. As these MR algorithms cannot parallelize within a machine in the distributed setup, they cannot effectively utilize a setup consisting of a small number of machines without sacrificing performance in the preliminary MR round. Introducing a sublinear-adaptive algorithm would allow the use of smaller distributed setups that alleviates Issue [5], while yielding efficient performance in the preliminary MR round through optimum parallelism. In this work, we seek to address all three issues mentioned above by introducing sublinear-adaptivity (parallelism) within a distributed setting.
Table 2: Performance ratio of R-DASH to the RandGreedi (RG [5]) and BiCriteriaGreedy (BCG [16]) algorithm in terms of mean solution value and mean runtime across four applications;

| Application         | Centralized Data | Distributed Data |
|---------------------|------------------|------------------|
|                     | Mean Value \(^\dagger\) | Mean Runtime \(^\dagger\) | Mean Value \(^\ddagger\) | Mean Runtime \(^\ddagger\) |
|                     | R-DASH RG        | R-DASH BCG        | R-DASH RG        | R-DASH BCG        | R-DASH RG        | R-DASH BCG        | R-DASH RG        | R-DASH BCG        | R-DASH RG        | R-DASH BCG        | R-DASH RG        | R-DASH BCG        |
| ImageSumm           | 0.99            | 0.99             | 6.9 \times 10^4 | 4.4 \times 10^4 | 0.98            | 0.99             | 5.3 \times 10^3 | 1.3 \times 10^4 | DNC                | DNC                | 1.00            | 1.00             | 1.0 \times 10^3 | 1.0 \times 10^3 |
| InfluenceMax        | 1.00            | 1.00             | 2.6 \times 10^2 | 2.6 \times 10^3 | 1.00            | DNC               | DNC                | DNC                | DNC                | DNC                | DNC                |
| RevenueMax          | 0.97            | 0.97             | 2.3 \times 10^2 | 3.7 \times 10^3 | DNC              | DNC               | DNC                | DNC                | DNC                | DNC                | DNC                |
| MaxCover (BA)       | 1.00            | 1.00             | 1.5 \times 10^2 | 1.0 \times 10^3 | DNC              | DNC               | DNC                | DNC                | DNC                | DNC                | DNC                |

\(^\dagger\) Compares instances where RandGreedi (RG) / BiCriteriaGreedy (BCG) completes within timeout;
\(^\ddagger\) BiCriteriaGreedy (BCG) was evaluated with cardinality constraint \(k\) enforced.
\(^\ddagger\) No instance of the application was completed by RG/ BCG within the 24 hours timeout.

Technical Challenges  To our knowledge, no algorithm with low adaptive complexity works for distributed data in MapReduce setting: these algorithms require random access to the entire ground set on all machines. Moreover, none of these algorithms satisfy the consistency properties required by the distributed framework of Barbosa et al. [5, 6] stated as follows,

**Property 1** (Consistency). Let \(A\) and \(B\) be two disjoint subsets of \(\mathcal{N}\). Suppose that, for each element \(e \in B\), we have \(\text{ALG}(A \cup \{e\}) = \text{ALG}(A)\). Then \(\text{ALG}(A \cup B) = \text{ALG}(A)\).

In comparison to Greedy, adaptive algorithms involve random steps, such as random sampling or random sequencing that help reduce both the adaptive rounds and query calls but also make their output inconsistent while running the same algorithm multiple times. Hence, adaptive algorithms cannot be distributed directly. In fact, no distributed algorithm currently known uses sublinear number of adaptive rounds; which is useful for utilizing the multiple CPUs of the nodes in a cluster. Thus, the following question is posed:

**Question** Is it possible to design constant-factor approximation distributed algorithms with a constant number of MapReduce rounds and sublinear adaptive complexity (highly-parallelizable)?

Contributions.

- Our paper introduces a new randomness mechanism using a random vector \(\mathbf{q}\) that allows adaptive algorithms overcome the technical challenge of Property 1 and integrate with the distributed frameworks. This mechanism is the foundation of our theoretical contributions and provides a general framework that can be extended to other adaptive algorithms for monotone and non-monotone submodular objectives. In this paper, we extend this analysis to propose distributed adaptive sequencing heuristics (DASH) using two design approaches.

- First, using the Distributed-First-Adaptive (DFA) approach we propose two \(O(\log(k)\log(n))\)-adaptive algorithms R-DASH and G-DASH, that introduces sublinear adaptivity in distributed frameworks and provide \(\frac{1}{e}(1 - 1/e - \varepsilon) \approx 0.316\) and \(1 - \frac{1}{e} - \varepsilon \approx 0.632\) approximation guarantee respectively using one and \((1/\varepsilon)\) MapReduce rounds.

- Secondly, we present a novel single MR round algorithm T-DASH that achieves an improved approximation of 0.375-\(\varepsilon\) (among single MR round algorithms) with \(O(\log(n))\) adaptivity using the Adaptive-First-Distributed (AFD) approach. AFD algorithms like T-DASH are based on adaptive sequencing subroutines that provide constant factor approximation in a distributed setting while prioritising optimal \(O(\log(n))\) adaptivity.

- Additionally, we present MED, a general plug-in framework for distributed algorithms, which eliminates the cardinality and memory constraints (Issue 5) of distributed algorithms (say ALG) by running ALG with a smaller cardinality constraint \(k'\) over multiple MR rounds. When used in conjunction with a \(\gamma\)-approximation algorithm, MED provides an \((1 - e^{-\gamma})\)-approximate solution.

- Finally, we substantiate our theoretical claims with extensive empirical evaluations across a diverse set of four applications. We demonstrate that our distributed algorithms provide
orders of magnitude speedup in runtime compared to the existing state-of-the-art MR model algorithms while producing nearly identical solution values.

2 Consistent Low-Adaptive Procedures

In this section, we introduce a new randomness mechanism by conceptualizing a random vector \( q \) to deal with the inconsistency in the adaptive algorithms. We analyze two low-adaptive procedures, LAT (Alg. 5; Appendix) and LAG (Alg. 6 Appendix), variants of low-adaptive procedures proposed in Chen et al. [12]. This analysis enables their use in the distributed, MapReduce setting.

2.1 Random Vector \( q \)

Observe that the randomness of the low-adaptive algorithms only comes from the random permutations of \( V \). Consider an equivalent version of the algorithm in which the entire ground set \( \mathcal{N} \) is permuted randomly, from which the permutation of \( V \) is extracted. That is, if \( \sigma \) is the permutation of \( \mathcal{N} \), the permutation of \( V \) is given by \( v < w \iff \sigma(v) < \sigma(w) \), for \( v, w \in V \). The random vector \( q = (\sigma_1, \sigma_2, ...) \) is given as an input to the low-adaptive algorithms, which is a sequence of random permutation of \( \mathcal{N} \).

2.2 Low-Adaptive Threshold (LAT) Algorithm

This section presents the analysis of the low-adaptive threshold algorithm, LAT (Alg. 5) a variant of \textsc{ThresholdSeq} from Chen et al. [12], that incorporates the randomness \( q \) to provide consistency in a distributed setting.

**Lemma 1.** Let \( q \) be a fixed sequence of random bits. Suppose \( A \) and \( B \) are disjoint subsets of \( \mathcal{N} \), and that for each \( b \in B \), \( b \notin \text{LAT}(A \cup \{b\}, q) \). Also, suppose that \( \text{LAT}(A, q) \) terminates successfully. Then \( \text{LAT}(A \cup B, q) \) terminates successfully and \( \text{LAT}(A \cup B, q) = \text{LAT}(A, q) \).

**Proof.** Consider that the algorithm runs for all \( M + 1 \) iterations of the outer \textbf{for} loop; if the algorithm would have returned at iteration \( j \), the values \( S_i \) and \( V_i \), for \( i > j \) keep their values from when the algorithm would have returned. The proof relies upon the fact that every call to LAT(\( \cdot, q \)) uses the same sequence of permutations of \( \mathcal{N} \): \( (\sigma_1, \sigma_2, \ldots, \sigma_{M+1}) \). We refer to iterations of the outer \textbf{for} loop on Line 4 of Alg. 5 as simply as iterations.

We consider the runs of (1) LAT(\( A, q \)), (2) LAT(\( A \cup \{b\}, q \)), and (3) LAT(\( A \cup B, q \)) together. Variables of (1) are given the notation defined in the pseudocode; variables of (2) are given the superscript \( b \); and variables of (3) are given the superscript \( \prime \).

Let \( P(i) \) be the statement that

(i) \( S_i = S_i' \), and

(ii) for all \( b \in B \), \( S_i = S_i^b \), and

(iii) \( V_i = V_i' \setminus B \), and

(iv) for all \( b \in B \), \( V_i = V_i^b \setminus \{b\} \).

We prove in the Appendix that \( P(i) \) is true for every \( i \leq M + 1 \). Then, we consider the successful termination of LAT(\( A, q \)) in two cases. First, suppose LAT(\( A, q \)) terminates on iteration \( j \leq M + 1 \) with \( |S_{j-1}| = k \). Then by \( P(j) \), this termination condition also holds in each of the other runs, so LAT(\( A \cup B, q \)) succeeds and LAT(\( A \cup B, q \)) = LAT(\( A, q \)). Secondly, suppose LAT(\( A, q \)) terminates on iteration \( j \leq M + 1 \) with \( |V_j| = 0 \). Then on iteration \( j \) of LAT(\( A \cup B, q \)), \( V_j' = \emptyset \) as well, and also terminates successfully. Moreover, it returns \( S_{j-1}' \), which is equal to \( S_{j-1} \) by \( P(j - 1) \). \( \square \)
Another building block for our distributed algorithms is a simple, low-adaptive greedy algorithm Algorithm 1.

**Algorithm 1 RANDOMIZED-DASH (R-DASH)**

1. **Input:** Evaluation oracle \( f : 2^N \to \mathbb{R} \), constraint \( k \), error \( \varepsilon \), available machines \( M \)
2. for \( e \in N \) do
3. Assign \( e \) to machine chosen uniformly at random
4. for \( i \in M \) do
5. \( \triangleright \) On machine \( i \)
6. Let \( \mathcal{N}_i \) be the elements assigned to machine \( i \)
7. \( S_i \leftarrow \text{LAG}(f, \mathcal{N}_i, k, \varepsilon) \)
8. Send \( S_i \) to primary machine
9. \( \triangleright \) On primary machine
10. Gather \( S \leftarrow \bigcup_{i=1}^{M} S_i \)
11. return \( V \leftarrow \arg \max \{ f(T), f(S_1) \} \)

**2.3 Low-Adaptive Greedy (LAG) Algorithm**

Another building block for our distributed algorithms is a simple, low-adaptive greedy algorithm LAG (Alg. 6 in Appendix). This algorithm is an instantiation of the PARALLEL_GREEDYBOOST framework of Chen et al. [12], and it relies heavily on the low-adaptive procedure LAT (Alg. 5). The analysis for this algorithm can be found in Appendix.

**Modifications and Guarantees.** By Theorem 3 of Chen et al. [12], setting \( \Gamma = \max_{x \in X} f(\{x\}) \) and \( \alpha = 1/k \), LAG achieves ratio \( 1 - 1/e - \varepsilon \) in \( O(\log k \log n) \) adaptive rounds and \( O(n \log k) \) total queries.

**Consistency.** As in the analysis above for LAT, we will consider that the randomness of the algorithm is determined by a sequence \( q \) of random bits whose length depends on the size \( n \) of the ground set \( N \). We will consider the sequence \( q \) as an input to LAG.

**Lemma 2.** Let \( q \) be a fixed sequence of random bits. Suppose \( A \) and \( B \) are disjoint subsets of \( N \), and that for each \( b \in B \), LAG\((A \cup \{b\}, q) = LAG(A, q) \), and LAG\((A, q) \) terminates successfully. Then LAG\((A \cup B, q) \) terminates successfully and LAG\((A \cup B, q) = LAG(A, q) \).

**3 DISTRIBUTED-FIRST-ADAPTIVE Algorithms**

**Motivation** Is sublinear adaptivity possible for distributed frameworks?

In this section, we discuss two DFA algorithms, R-DASH and G-DASH that extends our consistency analysis to introduce sublinear adaptivity in a distributed setting. R-DASH and G-DASH generate \( \frac{1}{2}(1 - 1/e - \varepsilon) \) and \( 1 - 1/e - \varepsilon \) solution with \( O(\log(n) \log(k)) \) adaptivity.

**3.1 RANDOMIZED-DASH (R-DASH)**

**Description** The ground set is initially distributed at random by R-DASH across all machines \( M \). In its sole MR round, R-DASH runs LAG on every machine to obtain \( S_i \) in \( O(\log(k) \log(|\mathcal{N}_i|)) \) adaptive rounds. The solution from every machine is then returned to the primary machine, where LAG selects the output solution that guarantees \( \frac{1}{2}(1 - 1/e - \varepsilon) \) approximation in \( O(\log(k) \log(|S|)) \) adaptive rounds as stated in Theorem 1. The analysis for R-DASH is given in Appendix.

**Theorem 1.** R-DASH returns set \( V \) with a single MR round and \( O(\frac{1}{k} \log(k) \log(\frac{n}{\varepsilon})) \) adaptive rounds such that

\[
\mathbb{E}[f(V)] \geq \frac{1 - 1/e - \varepsilon}{2} \text{OPT}.
\]
where we replace standard greedy with LAG and get the following theorem. The detailed proof can be found in Appendix.

Algorithm 2 Greedy-DASH (G-DASH)

**Input:** evaluation oracle $f : 2^V \rightarrow \mathbb{R}$, constraint $k$, error $\varepsilon$, available machines $M$

1. $M \leftarrow \{1, 2, \ldots, |M|\}$, $S \leftarrow \emptyset$, $C_0 \leftarrow \emptyset$
2. for $r \leftarrow 1$ to $\left\lceil \frac{|M|}{2} \right\rceil$ do
3. 4. $X_r,i \leftarrow$ Elements assigned to machine $i$ chosen uniformly at random in round $r$
4. 5. $N_r,i \leftarrow X_r,i \cup C_{r-1}$
6. for $i \in M$ in parallel do
7. 8. $S_{r,i} \leftarrow$ LAG($f, N_{r,i}, k, \varepsilon$)
8. 9. $S \leftarrow \arg \max \{ f(S), f(S_{r,1}), \ldots, f(S_{r,|M|}) \}$
9. 10. $C_r \leftarrow \bigcup_{i=1}^{M} S_{r,i} \cup C_{r-1}$
10. \textbf{return} $S$

3.2 Greedy-DASH (G-DASH)

DistributedDistorted proposed by Kazemi et al. [22] is a distributed algorithm for regularized submodular maximization without loss of approximation ratio for an $\alpha$-approximation greedy algorithm. To the best of our knowledge, no algorithms besides the standard greedy algorithm of Fisher et al. [20] and the continuous greedy algorithm of Calinescu et al. [10], Feldman et al. [19] have been shown to work in this framework. In this section, we show that the algorithm LAG described in Algorithm 2 does fit into this framework. Therefore, this yields an algorithm that achieves the near optimal $(0.632-\varepsilon)$ ratio in $\left\lceil \frac{|M|}{2} \right\rceil$ MapReduce rounds, $O(\log(n) \log(k))$ adaptive rounds, and $O(n)$ total queries.

The Framework of Kazemi et al. [22]. DistributedDistorted has $\lceil 1/\varepsilon \rceil$ MR rounds where each round $r$ works as follows: First, it distributes the ground set into $m$ machines uniformly at random. Then, each machine $i$ runs Greedy (when modular term $f(\cdot) = 0$) on the data $N_{r,i}$ that combines the elements distributed before each round $X_{r,i}$ and the elements forwarded from the previous rounds $C_{r-1}$ to get the solution $S_{r,i}$. At the end, the final solution, which is the best among $S_{r,1}$ and all the previous solutions, is returned.

Fitting LAG into the Framework. To improve the adaptive rounds of DistributedDistorted, we replace standard greedy with LAG and get the following theorem. The detailed proof can be found in Appendix.

**Theorem 2.** G-DASH returns set $S$ with $O\left(1/\varepsilon^2\right)$ MR round and $O\left(\frac{1}{\varepsilon^2} \log(k) \log(n/\varepsilon)\right)$ adaptive rounds such that,

$$\mathbb{E}[f(S)] \geq \left(1 - 1/e - \varepsilon\right) \text{OPT}.$$  

4 Adaptive-First-Distributed Algorithms

Motivation Single MR round distributed frameworks employing an $\alpha$-approximate algorithm provides a ratio $\rho \leq \alpha/2$. Our DFA algorithm R-DASH provides a $(0.316-\varepsilon)$-approximate solution, the best possible solution using the framework. A natural question is: Does there exist a single MR round algorithm with better approximation and optimal adaptivity?

In this section, we present a novel $(0.375 - \varepsilon)$-approximate, single MR round algorithm Threshold-DASH (T-DASH), that uses adaptivity-first-distributed (AFD) design to achieve optimal $O \log(n)$ adaptivity.

4.1 Threshold-DASH (T-DASH)

Description The T-DASH algorithm described in Alg. 3 is a single MR round algorithm using the AFD approach that runs LAT concurrently on every machine for $\log_{1+\varepsilon}(k)$ different guesses of threshold $\tau_{i,j}$ in the range $\left[\frac{\Delta^*}{k}, \alpha \Delta^*\right]$; where $\alpha$ is the approximation of T-DASH and $\Delta^*$ is the maximum singleton in $N$. Every solution returned to the primary machine are placed into bins based on their corresponding threshold guess $\tau_{i,j}$ such that $\frac{\Delta}{k} \cdot (1 + \varepsilon)^x \leq \tau_{i,j} \leq \frac{\Delta^*}{k} \cdot (1 + \varepsilon)^{x+1}$; where $\Delta^*$ is the maximum singleton in $N$. Since there must exist a threshold $\tau^*$ that is close enough...
Algorithm 3 Threshold-DASH with no knowledge of OPT (T-DASH)

1. **Input:** Evaluation oracle $f : 2^N \rightarrow \mathbb{R}$, constraint $k$, error $\varepsilon$, available machines $M$
2. Initialize $\delta \leftarrow 1/(|M| + 1)$, $q \leftarrow$ a fixed sequence of random bits.
3. Set $\alpha \leftarrow \frac{3}{8}$
4. for $e \in \mathcal{N}$ do do
5. Assign $e$ to each machine independently with probability $1/|M|$.
6. for $i \in M$ do do
   7. $\triangleright$ On machine $i$
   8. Let $\mathcal{N}_i$ be the elements assigned to machine $i$
   9. Set $\Delta^+ \leftarrow \max\{f(e) : e \in \mathcal{N}_i\}$
10. for $j \leftarrow 0$ to $\log_{1+\varepsilon}(k)$ in parallel do
11. $\quad \tau_{i,j} \leftarrow \frac{\alpha \Delta^+}{j} (1 + \varepsilon)^j$
12. $\quad S_{i,j} \leftarrow \text{LAT}(f, \mathcal{N}_i, k, \delta, \varepsilon, \tau_j, q)$
13. $\quad$ Send $\Delta^+_i$ and all $(\tau_{i,j}, S_{i,j})$ to primary machine
14. $\triangleright$ On primary machine
15. Set $\Delta^* \leftarrow \max\{\Delta^+_i : 1 \leq i \leq |M|\}$
16. for $x \leftarrow 0$ to $\lceil \log_{1+\varepsilon}(k) \rceil + 1$ in parallel do
17. $\quad \tau_x \leftarrow \frac{\alpha \Delta^*}{x} (1 + \varepsilon)^x$
18. Let $S_x \leftarrow \{ \bigcup S_{i,j} : \tau_x \leq \tau_{i,j} \leq \tau_x + 1 \}$
19. Let $A_x \leftarrow \{ \text{Sample a solution } S_{i,j} : \tau_x \leq \tau_{i,j} \leq \tau_x + 1 \}$
20. Let $g_x(\cdot) \leftarrow f(A_x \cup \cdot) - f(A_x)$
21. $\quad T_x \leftarrow \text{LAT}(g_x, S_x, k - |A_x|, \delta, \varepsilon, \tau_x, q)$
22. $\quad T'_x \leftarrow A_x \cup T_x$
23. $T \leftarrow \arg \max\{f(T'_x) : 0 \leq x \leq \log_{1+\varepsilon}(k)\}$
24. return $T$

To $\alpha\text{OPT}/k$; running LAT (Line 21) on every bin in the range $[\frac{\alpha \Delta^*}{k}, \alpha \Delta^*]$ and selecting the best solution guarantees the $\alpha$-approximation of T-DASH in $O(\log(n))$ adaptive rounds.

**Theorem 3.** T-DASH returns set $T$ with a single MR round and $O\left(\frac{1}{\varepsilon} \log(n)\right)$ adaptive rounds such that,

$$
\mathbb{E}[f(T)] \geq \left(\frac{3}{8} - \varepsilon\right) \text{OPT}.
$$

**Overview of Proof.** Alg. 3 is inspired by Alg. 7 in Appendix. With $\Delta^* = \max\{f(e) : e \in \mathcal{N}\}$, there exists an $x_0$ such that $\tau_{x_0} \leq \alpha\text{OPT}(1 + \varepsilon)/k \leq \tau_{x_0 + 1}$. Then, on each machine $i$, we only consider the set $S_{i,j}$ such that $\tau_{x_0} \leq \tau_{i,j} \leq \tau_{x_0 + 1}$. If this $\tau_{i,j}$ does exist, $S_{i,j}$ works like $S_i$ in Alg. 7. If this $\tau_{i,j}$ does not exist, then for any $e \in \mathcal{N}_i$, it holds that $f(e) < \alpha\text{OPT}/k$, which means LAT($\mathcal{N}_i$) with $\tau = \alpha\text{OPT}/k$ will return an empty set.

**Proof.** First, for $x = 0$, $\tau_x = \alpha \Delta^* / k \leq \alpha\text{OPT} / k$; and for $x = \lceil \log_{1+\varepsilon}(k) \rceil + 1$,

$$
\tau_x \geq \alpha \Delta^*(1 + \varepsilon) \geq \alpha \sum_{o \in O} f(o)(1 + \varepsilon)/k \geq \alpha\text{OPT}(1 + \varepsilon)/k.
$$

Therefore, there exists an $x_0$ such that $\tau_{x_0} \leq \alpha\text{OPT}(1 + \varepsilon)/k \leq \tau_{x_0 + 1}$. Since $\tau_{x_0 + 1} = \tau_{x_0}(1 + \varepsilon)$, it holds that $\tau_{x_0} \geq \alpha\text{OPT}/k$ and $\tau_{x_0 + 1} \leq \alpha\text{OPT}(1 + \varepsilon)^2/k$.

Then, we only consider $T'_{x_0}$. If $|T'_{x_0}| = k$, by Theorem 2 of Chen et al. 12, it holds that,

$$
f(T) \geq f(T'_{x_0}) \geq \frac{1 - \varepsilon}{1 + \varepsilon} \tau_{x_0} k \geq \left(\frac{3}{8} - \varepsilon\right) \text{OPT}.
$$

Otherwise, in the case that $|T'_{x_0}| < k$, let $A_{x_0} = S_{i_0,j_0}$, $O_1 = \{o \in O : o \not\in \text{LAT}(\mathcal{N}_{i_0} \cup \{o\}, q)\}$.

Also, let $\tau_{i,j_0}$ be returned by machine $i$ that $\tau_{x_0} \leq \tau_{i,j_0} \leq \tau_{x_0 + 1}$, define

$$
S'_{i,j_0} = \begin{cases} S_{i,j_0} & \text{, if machine } i \text{ returned a } \tau_{i,j_0} \\ 0 & \text{, otherwise} \end{cases}
$$

7
Algorithm 4 MED($f, \mathcal{N}, k, \varepsilon, M, \text{ALG}$)

1: **Input:** evaluation oracle $f : 2^{\mathcal{N}} \to \mathbb{R}$, constraint $k$, error $\varepsilon$, available machines $M$, MR algorithm \text{ALG}
2: $n \leftarrow |\mathcal{N}|$, $\Psi \leftarrow$ Memory capacity (# elements) of primary machine
3: Choose $k' \leftarrow \max\{k' \in \mathbb{N} : k' \leq \frac{\Psi}{|M|}\}$
4: $m \leftarrow \lceil k'/k \rceil$
5: **for** $i \leftarrow 1$ **to** $m$ **do**
6: $A_i \leftarrow \text{ALG}(f, \mathcal{N}, \min\{k', k - |S_i|\}, \varepsilon)$
7: $S_{i+1} \leftarrow S_i \cup A_i$
8: **return** $S_m$

On the machine which does not return a $\tau_{i,(j_0)}$, we consider it runs LAT($\mathcal{N}_i, \tau_{x_0+1}$) and returns an empty set, and hence $\max\{f(e) : e \in \mathcal{N}_i\} < \tau_{x_0}$. Let $S_{\tau_0} = \bigcup_{i \in M} S'_{\tau_0}$, $O_2 = S_{\tau_0} \cap O$. Then, Lemma 5 in Appendix still holds in this case. We can calculate the approximation ratio as follows with $\varepsilon \leq 2/3$,

$$E[T] \geq E[T'_{\tau_0}] \geq E[f(O_1 \cup O_2)] - k \cdot \tau_{x_0+1} \geq \left(\frac{3}{8} - \varepsilon\right) \text{OPT}.$$

\[\square\]

5 Memory Efficient Distributed Framework (MED)

**Motivation** As the ground set $S$ accumulated on the primary machine after an MR round follows $|S| \propto k \cdot |M|$: large distributed setups or large $k$ values could lead to memory overload on the primary machine, limiting the practicality of MR algorithms. Thus we ask: Can MR algorithms eliminate this restriction with minimal loss in approximation and minimal change in their design?

In this section, we propose a general-purpose plug-in framework for distributed algorithms, \textsc{MemoryEfficientDistributed} (MED).

**Description** In MR Model, a dataset of size $n$ is distributed on $|M|$ machines, each with memory size of $\Psi$ elements. The total memory of the machines is constrained to be $\Psi \cdot |M| = O(n)$. After each round of computation, a machine may send $O(\Psi)$ amount of data to the primary machine. However, to avoid memory overload, MR model algorithms cannot process the objective for $k \geq \frac{\Psi}{|M|}$.

MED (Alg. 4) is a greedy plug-in framework for MR model algorithm that runs a $\gamma$-approximate MR algorithm \text{ALG} iteratively over $m$ iterations with a smaller cardinality constraint of $k' = k/m$. As a result, \text{ALG} can solve any instance of $k$ with an approximation of $(1 - e^{-\gamma})$. The analysis of MED is provided in the Appendix.

**Theorem 4.** Let $(f, k)$ be an instance of SMCC distributed over $m$ machines. Let $S_m$ be the set returned by MED. Then $f(S_m) \geq (1 - e^{-\gamma})$, where $\gamma$ is the approximation of \text{ALG}.

We provide evaluation results of MED in Appendix (Fig. 7) to compare the performance of MED to R-DASH and RANDGREEDI. The experiment set investigates whether there is a loss in solution value of MED in comparison to the algorithm it runs iteratively (R-DASH in this case).

6 Empirical Evaluation

This section presents empirical comparison of R-DASH, T-DASH and G-DASH to the state-of-the-art distributed algorithms on both centralized and distributed data. \textsc{Distributed} algorithms include RANDGREEDI (RG) of Barbosa et al. [5], PARALLEL\textsc{ALG} (PALG) of Barbosa et al. [6], \textsc{Distributed}DISTORTED (DDIST) of Kazemi et al. [22] and Bi\textsc{CriteriAGreedy} (BCG) of Epasto et al. [16]. Our results are summarized as follows:
Figure 1 and 2 presents performance comparison of distributed algorithms on centralized and distributed data of ImageSumm and InfluenceMax; other application results shown in Appendix; RAND-GREEDY (RG) is run with GREEDY as the algorithm ALG [5] to ensure the $\frac{1}{2} \left( 1 - 1/e \right)$ ratio. See Appendix for evaluation results of RG using LAG as the algorithm ALG; BiCRITERIAGREEDY (BCG) is evaluated for a single round as stated in Epasto et al. [16] and cardinality constraint is enforced during its post processing; DISTRIBUTEDDISTORTED (DDIST) is run with the modular term $\ell(\cdot) = 0$ (non-regularized; all applications are submodular) [22]. All GREEDY based algorithms use lazy calculations to improve the parallel runtime without compromising theoretical guarantees. Timeout for each application: centralized data = 6 hours; distributed data = 24 hours.

Experiment Set 1 shows R-DASH provides solutions that meet the value of GREEDY based distributed algorithms while being orders of magnitude faster. **Centralized Data (Fig. 1):** Across all instances of $k$ that RG and BCG completes within timeout, R-DASH is on average 300 and 700 times faster respectively; and maintains a 99% solution value of both algorithms. G-DASH provides marginally better solution than R-DASH in all applications but ImageSumm while remaining 14 and 35 times faster than RG and BCG; it is also 130 times faster than its approximation equivalent DDIST algorithm. The availability of only 4 threads per machine severely limits the parallelization of T-DASH, resulting in longer runtime; access to $\log_{1+\epsilon}(k)$ threads per machine should result in faster runtime than DASH. **Distributed Data (Fig. 2):** For large data ($n \geq 10^6$), RG completes only 1 instance of $k$ for InfluenceMax within 24 hours timeout and none for RevenueMax and MaxCover; on smaller groundsets, R-DASH is 190 times faster than RG with 98% of its solution value. R-DASH is 8000 times quicker than BCG on the only instance of ImageSumm it completes within timeout. Table 2 outlines performance ratio of R-DASH in comparison to RG and BCG. Figure 1 and 2 provides ImageSumm and InfluenceMax results; results on other applications in Appendix (Fig. 4, 5).

**Experiment Set 2** examines the large data ($n > 10^6$) scalability of R-DASH in terms of parallel runtime with increase in size of the distributed setup from 1 to 8 machines. Overall, R-DASH achieves a linear improvement in runtime with increase in cluster size without any loss of solution value. Figure 3 presents the scalability of R-DASH on InfluenceMax and RevMax for $k = k/100$. The availability of only 4 threads per machine severely limits the parallelization of T-DASH, resulting in longer runtime; access to $\log_{1+\epsilon}(k)$ threads per machine should result in faster runtime than DASH.
Figure 3: Performance of R-DASH vs. number of machines (M) in a distributed setup.

Additional Experiments in Appendix: 1) R-DASH vs. LS+PGB (Fig. 6); 2) MED vs. R-DASH (Fig. 7); 3) RG+LAG vs. R-DASH (Fig. 8); 4) Comparison of RANDGREEDI on 8 vs. 32 processing units (Fig. 10).

6.1 Experiments Setup

Parameters: Error $\varepsilon$ for all algorithms is set to 0.05.

Datasets are categorized into two groups: Centralized Data include datasets that are smaller than memory size of individual machines and range in size from $n=10,000$ to 100,000. Algorithms are given a $timeout$ of 6 hours to complete evaluation for $k=[100, 200, 300, 400, 500]$ on each application. All non-GREEDY based algorithms are repeated independently for five repetitions and the mean and standard deviation of the metrics are plotted. Distributed Data are large data files that are partitioned uniformly at random and assigned to $|M|$ machines before evaluation. Algorithms return rows of data files as solutions during MR rounds; thus, these experiments are limited to single MR round algorithms that require minimal communication (RG, BCG and R-DASH). Groundsets range from $n=50,000$ to 3,072,441 and $k=[1\%, 2\%, 5\%, 10\%]$ of $k'$ where $k' = n/|M|^2$. $timeout$ for each application is set to 24 hours. Additional dataset details are provided in Appendix.

Applications include image summarization (ImageSumm), influence maximization (InfMax), revenue maximization (RevMax) and maximum coverage (MaxCover). Details are provided in Appendix.

Implementation: Algorithms are implemented using the Message Passing Interface (MPI) and the runtime is measured using $MPI.Wtime()$ calls.

Environment: The experiments are conducted on 8 machines with 4 Intel(R) Core(TM) i5-2500 CPU @ 3.30GHz cores, 8 gigabytes of memory and running Ubuntu 18.64.2.

7 Conclusion

In this paper, we mitigate the practical limitations of MR model algorithms by introducing R-DASH, T-DASH, and G-DASH, the first MR model algorithms with sublinear adaptive complexity (highly parallelizable). Our algorithms provide state-of-the-art performance while maintaining theoretical guarantees for a distributed setting. In addition, we propose MED, a framework that eliminates existing memory and cardinality constraint limitations of MR algorithms. The limitations of this work lie in the data duplication required by T-DASH and G-DASH to achieve better theoretical guarantees, which we will try to address in the future.

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Algorithm 5 Low-Adaptive Threshold Algorithm (LAT)

1: procedure LAT($f, X, k, \delta, \varepsilon, \tau, q$)
2: \textbf{Input:} evaluation oracle $f : 2^N \rightarrow \mathbb{R}^+$, subset $X \subseteq N$, constraint $k$, confidence $\delta$, error $\varepsilon$, threshold $\tau$, fixed sequence of random bits $q$
3: Initialize $S_0 \leftarrow \emptyset$, $V_0 \leftarrow X$, $M \leftarrow \lfloor 4(1 + 2/\varepsilon) \log(n/\delta) \rceil$
4: \textbf{for} $j \leftarrow 1$ to $M + 1$ \textbf{do}
5: \hspace{1em} Update $V_j \leftarrow \{ x \in V_{j-1} : |\Delta (x \mid S_{j-1}) \geq \tau \}$ and filter out the rest
6: \hspace{1em} if $|V_j| = 0$ or $|S_{j-1}| = k$ then
7: \hspace{2em} \textbf{return} $S_{j-1}$, success
8: \hspace{1em} $V_j \leftarrow \text{random-permutation}(V_j, q_j)$.
9: \hspace{1em} $s \leftarrow \min\{k - |S_{j-1}|, |V_j|\}$
10: \hspace{1em} $\Lambda \leftarrow \{(1 + \varepsilon)^u : 1 \leq (1 + \varepsilon)^u \leq s, u \in \mathbb{N}\} \cup \{s\}$
11: $B \leftarrow \emptyset$
12: \textbf{for} $\lambda \in \Lambda$ in parallel \textbf{do}
13: \hspace{1em} $T_\lambda \leftarrow \{v_1, v_2, \ldots, v_\lambda\}$
14: \hspace{1em} if $\Delta (T_\lambda \mid S) / |T_\lambda| \geq (1 - \varepsilon)\tau$ then
15: \hspace{2em} $B \leftarrow B \cup \{\lambda\}$
16: \hspace{1em} $\lambda_\ast \leftarrow \min\{\lambda \in \Lambda : \lambda > b, \forall b \in B\}$
17: $S_j \leftarrow S_{j-1} \cup T_{\lambda_\ast}$
18: \textbf{return} $S_{M+1}$, failure

A MapReduce Model.

We adopt the following model: A dataset of size $n$ is distributed on $|M|$ machines, each with memory to hold at most $\Psi$ elements. The total memory of the machines is constrained to be $\Psi \cdot |M| = O(n)$. After each round of computation, a machine may send $O(\Psi)$ amount of data to other machines. We assume $|M| \leq n^{1-c}$ for some constant $c$.

B Lovász extension of submodular function.

Given submodular function $f$, the Lovász extension $F$ of $f$ is defined as follows: For $z \in [0, 1]^N$,

$$F(z) = (z_i)_{i \in N} = \mathbb{E}_{\lambda \sim U([0, 1])} [f(\{i : z_i \geq \lambda\})].$$

The Lovász extension satisfies the following properties: (1) $F$ is convex; (2) $F(cz) \geq cF(z)$ for any $c \in (0, 1)$. Moreover, we will require the following simple lemma:

Lemma 3 (5). Let $S$ be a random set, and suppose $\mathbb{E}[1_S] = c \cdot z$, for $c \in (0, 1)$. Then $\mathbb{E}[f(S)] \geq c \cdot F(z)$.

C Analysis of LAT

Proof of statement $P(i)$. Clearly, $P(0)$ holds. Let $i \leq M$. Suppose that, for all $j < i$ the statement $P(j)$ holds. If $|S_{i-1}| = k$, the variables do not update and $P(i)$ follows from $P(i-1)$. So, suppose that $|S_{i-1}| < k$. We will show that $P(i)$ holds. First, (iii) and (iv) clearly hold since by $P(i-1)$, the sets involved in updating $V_i$, $V'_i$, $V^b_i$ are equal (after $B$ is removed).

Next, suppose that $V_i = \emptyset$. We will show that $V'_i = \emptyset$ as well. By (iii), we have $V'_i \subseteq B$. If $V'_i \neq \emptyset$, let $b \in V'_i$. Then $b$ is not filtered at or before iteration $i$ in the run LAT($A \cup \{b\}$, $q_i$) either, so $V^b_i = \{b\}$. Since at least one element is added in any iteration of any run, this implies that $b \in S^1_i$, which contradicts the fact that $b \notin LAT(A \cup \{b\}, q_i)$. Therefore, $V'_i = \emptyset$ and $V^b_i = \emptyset$ for each $b \in B$, from which (i) and (ii) follow.

Finally, suppose $V_i \neq \emptyset$, let $T_{\lambda_i} \leftarrow \{v_1, v_2, \ldots, v_{\lambda_i}\}$ and let $b \in B$. By (iii) and (iv), $T_{\lambda_i} \subseteq V^b_i$ and $T_{\lambda_i} \subseteq V'_i$. Because $b \notin LAT(A \cup \{b\}, q_i)$, if $b \in V^b_i$, it must hold that $b \notin T_{\lambda_i}$. Since $V_i = V^b_i \setminus \{b\}$, it follows that $\lambda_{i}^{b} = \lambda_{i}$. This shows (ii). If $b \notin V^b_i$, then it must hold that for some $j < i$, $\Delta (b \mid S^b_j) < \tau$; this means that $\Delta (b \mid S^b_j) < \tau$, and therefore it also holds that $b \notin V'_i$. And
Algorithm 6 Low-Adaptive Greedy (LAG)

1: **Input:** Evaluation oracle $f : 2^N \rightarrow \mathbb{R}^+$, subset $C$, constraint $k$, accuracy parameter $\varepsilon$
2: $\Gamma \leftarrow \max_{x \in N} f(x)$
3: Initialize $\tau \leftarrow \Gamma$, $\delta \leftarrow 1/(\log_2(1/(3\varepsilon)) + 1)$, $S \leftarrow \emptyset$, fixed sequence of random bits $q$
4: while $\tau \geq \Gamma/(3\varepsilon)$ do
5: $\tau \leftarrow \tau (1-\varepsilon)$, $g(t) \leftarrow f(S \cup \cdot)$
6: $T \leftarrow \text{LAT}(g, C, k - |S|, \delta, \varepsilon/3, \tau, q_T)$
7: $S \leftarrow S \cup T$
8: if $|S| = k$ then
9: return $S$
10: return $S$

by above, each $b \in V_{\varepsilon}^b$ comes after every element of $T_{\varepsilon}$ in the permutation $\sigma_i$. It follows that $T_{\varepsilon}$ is a prefix of $V_i$, which shows (i).

**D Analysis of LAG**

**Lemma 2.** Let $q$ be a fixed sequence of random bits. Suppose $A$ and $B$ are disjoint subsets of $N$, and that for each $b \in B$, $\text{LAG}(A \cup \{b\}, q) = \text{LAG}(A, q)$, and $\text{LAG}(A, q)$ terminates successfully. Then $\text{LAG}(A \cup B, q)$ terminates successfully and $\text{LAG}(A \cup B, q) = \text{LAG}(A, q)$.

**Proof of Lemma 2.** Observe that the only randomness in LAG is from the calls to LAT. Since $\text{LAG}(A, q)$ succeeds, every call to LAT must succeed as well. Moreover, considering that $A$ is used to permute the underlying ground set $N$, changing the set argument $A$ of LAG does not change the sequence received by each call to LAT. Order the calls to LAT: LAT$(\cdot, q_1)$, LAT$(\cdot, q_2)$, ..., LAT$(\cdot, q_m)$. Since $b \notin \text{LAG}(A \cup \{b\}, q)$, it holds that $b \notin \text{LAG}(A \cup \{b\}, q_i)$ for each $b \in B$ and each $i$. By application of Lemma 1, LAT$(A \cup B, q)$ terminates successfully and LAT$(A \cup B, q_i) = \text{LAT}(A, q_i)$ for each $i$. Therefore, $\text{LAG}(A \cup B, q) = \text{LAG}(A, q)$ and the former call terminates successfully.

**E Analysis of R-DASH**

**Theorem 1.** R-DASH returns set $V$ with a single MR round and $O \left( \frac{1}{e^2} \log(k) \log \left( \frac{n}{\varepsilon} \right) \right)$ adaptive rounds such that

$$\mathbb{E}[f(V)] \geq \frac{1 - 1/e - \varepsilon}{2} \text{OPT}.$$ 

**Proof.** Let R-DASH be run with input $(f, k, \varepsilon, M)$. Since $|M| \leq n^{1-\varepsilon}$, $\mathbb{E}[|N_i|] = n/|M| \geq n^\varepsilon$. By an application of Chernoff’s bound to show that the size $|N_i|$ is concentrated, and with multiple, independent repetitions of LAG, we can ensure that R-DASH succeeds (that is, all calls to subroutines succeed) with probability $1 - (|M| + 1)/n \geq 1 - n^{-\varepsilon}$. For the remainder of the analysis, we condition on the event that all calls to LAG succeed.

Let $N(1/|M|)$ denote the random distribution over subsets of $N$ where each element is included independently with probability $1/|M|$. For $x \in N$, let

$$p_x = \begin{cases} 
\Pr_{x \sim N(1/|M|), q} \left[ \text{LAG}(X \cup \{x\}, q) \right], & \text{if } x \in O \\
\Pr_{x \sim N(1/|M|), q} \left[ \text{LAG}(X, q) \right], & \text{otherwise}
\end{cases}$$

Consider $S_1 = \text{LAG}(N, q)$ on machine $m_1$. Let $O_1 = \{o \in O : \text{LAG}(N_1 \cup \{o\}, q) = \text{LAG}(N_1, q)\}$. By Lemma 2, $\text{LAG}(N_1 \cup O_1, q)$ succeeds and $\text{LAG}(N_1 \cup O_1, q) = S_1$. Therefore, let $\alpha = 1 - 1/e - \varepsilon, f(S_1) \geq \alpha f(O_2)$. Next, let $O_2 = O \cap S$, where $S = \bigcup_{i=1}^M S_i$. It holds that $f(T) \geq \alpha f(O_2)$. Let $o \in O; o$ is assigned to $N_e$ on some machine $m_e$. It holds that,

$$\Pr[o \in O_2] = \Pr[o \in \text{LAG}(N_1)|o \in N_e] = \Pr[o \in \text{LAG}(N_e \cup \{o\})]$$

14
Therefore,

\[
E[f(V)] \geq \frac{1}{2} (E[f(S_1)] + E[f(T)]) \\
\geq \frac{\alpha}{2} (E[f(O_1)] + E[f(O_2)]) \\
\geq \frac{\alpha}{2} (F(p) + F(1o - p)) \\
\geq \frac{\alpha}{2} F(1o) = \frac{1-1/e - \varepsilon}{2} \text{OPT},
\]

where Inequality 2 follows from Lemma 3 and \( F \) is convex.

\( Q.E.D. \)

## F Analysis of GREEDY-DASH

**Theorem 2.** G-DASH returns set \( S \) with \( O(1/\varepsilon) \) MR round and \( O(\frac{1}{\varepsilon} \log(k) \log(n/\varepsilon)) \) adaptive rounds such that,

\[
E[f(S)] \geq (1 - 1/e - \varepsilon) \text{OPT}.
\]

**Proof.** Let \( O \) be the optimal solution. For any \( x \in \mathcal{N} \), define that,

\[
p_x^r = \begin{cases} 
Pr_{X \sim N(1/|M|),q} [x \notin C_{r-1} \text{ and } x \in \text{LAG}(X \cup C_{r-1} \cup \{x\})] & , \text{ if } x \in O \\
0 & , \text{ otherwise}
\end{cases}
\]

Then we provide the following lemma.

**Lemma 4.** For any \( x \in O \) and \( 1 \leq r \leq 1/\varepsilon \), \( Pr_x(x \in C_r) = \sum_{r=1}^r p_x^r \).

**Proof.**

\[
Pr_x(x \in C_r) = \sum_{r' = 1}^r Pr_x(x \in C_{r'} \setminus C_{r'-1}) \\
= \sum_{r' = 1}^r Pr_x \left( x \in \bigcup_{i=1}^{r'} S_{r',i} \setminus C_{r'-1} \right) \\
= \sum_{r' = 1}^r \frac{|M|}{|M|} Pr_x \left( x \notin S_{r',i} \setminus C_{r'-1} \right) \\
= \sum_{r' = 1}^r \frac{|M|}{|M|} Pr_x \left( x \notin C_{r'-1} \text{ and } x \in \text{LAG}(S_{r',i} \cup C_{r'-1} \cup \{x\}) \right) \\
= \sum_{r' = 1}^r p_x^r
\]

The rest of the proof bounds \( f(S_{r-1}) \) in the following two ways.

First, let \( O_{r,1} = \{ o \in O : o \notin \text{LAG}(N_{r,1} \cup C_{r-1} \cup \{o\}, q) \} \), and \( O_{r,2} = (C_{r-1} \cap O) \cup O_{r,1} \). By Lemma 2, it holds that \( \text{LAG}(N_{r,1} \cup C_{r-1}, q) = \text{LAG}(N_{r,1} \cup C_{r-1} \cup O_{r,1}, q) = S_{r,1} \). Since the approximation ratio for LAG is \( 1 - 1/e - \varepsilon, f(S_{r,1}) \geq (1 - 1/e - \varepsilon)f(O_{r,2}) \). And for any \( o \in O \),

\[
Pr(o \in O_{r,2}) = Pr(o \in C_{r-1} \text{ or } o \notin \text{LAG}(N_{r,1} \cup C_{r-1} \cup \{o\}, q))
\]

15
Table 3: Centralized and Distributed Data

| Application       | Centralized Data | Distributed Data |
|-------------------|------------------|------------------|
|                   | $n$ Edges        | $n$ Edges        |
| ImageSumm         | 10,000 $\approx 1.0 \times 10^8$ | 50,000 $\approx 2.5 \times 10^9$ |
| InfluenceMax      | 26,588 $\approx 1.0 \times 10^5$ | 1,134,890 $\approx 6.0 \times 10^6$ |
| RevenueMax        | 17,432 $\approx 1.8 \times 10^5$ | 3,072,441 $\approx 2.3 \times 10^8$ |
| MaxCover (BA)     | 100,000 $\approx 5.0 \times 10^5$ | 1,000,000 $\approx 1.0 \times 10^9$ |

$$= 1 - p^r_o.$$ Therefore,

$$\mathbb{E}[f(S_{r,1})] \geq (1 - 1/e - \varepsilon)\mathbb{E}[f(O_{r,2})] \geq (1 - 1/e - \varepsilon)F(1_O - p^r).$$

Second, let $O_{r,3} = C_{r-1} \cap O$. Similarly, it holds that $f(S_{r,1}) \geq (1 - 1/e - \varepsilon)f(O_{r,3})$. And for any $o \in O$, by Lemma 4, it holds that,

$$Pr(o \in O_{r,3}) = Pr(o \in C_{r-1}) = \sum_{r'=1}^{r-1} p_{x'}^r,$$

Therefore,

$$\mathbb{E}[f(S_{r,1})] \geq (1 - 1/e - \varepsilon)\mathbb{E}[f(O_{r,3})] \geq (1 - 1/e - \varepsilon)F(\sum_{r'=1}^{r-1} p_{x'}^r).$$

By Inequalities 3 and 5, we bound the approximation ratio of G-DASH by the following,

$$\mathbb{E}[f(S)] \geq \frac{1}{\varepsilon} \sum_{r=1}^{\frac{1}{\varepsilon}} \mathbb{E}[f(S_{r,1})] \geq \frac{1}{\varepsilon}(1 - 1/e - \varepsilon) \cdot \left( F \left( \sum_{r'=1}^{\frac{1}{\varepsilon}-1} p_{x'}^r \right) + \sum_{r=1}^{\frac{1}{\varepsilon}-1} F(1_O - p_{x'}^r) \right) \overset{(a)}{=} (1 - 1/e - \varepsilon')(1 - \varepsilon)F(1_O) = (1 - 1/e - \varepsilon')OPT,$$

where Inequality (a) follows from Lemma 4 and $F$ is convex.

G Threshold-DASH Knowing OPT

**Theorem 5.** T-DASH knowing OPT returns set $T'$ such that,

$$\mathbb{E}[f(T')] \geq \left( \frac{3}{8} - \varepsilon \right) OPT.$$  

**Proof.** In the case that $|T'| = k$, by Theorem 2 of Chen et al. [12], it holds that

$$f(T') \geq \frac{1 - \varepsilon}{1 + \varepsilon} \cdot k \geq \left( \frac{3}{8} - \varepsilon \right) OPT.$$ 

Otherwise, we consider the case that $|T'| < k$ in the following. First, we give a definition as follows.
Algorithm 7  THRESHOLD-DASH Knowing OPT

1: Input: Evaluation oracle $f : 2^N \rightarrow \mathbb{R}$, constraint $k$, error $\varepsilon$, available machines $M$, and OPT
2: Initialize $\delta \leftarrow 1/(|M| + 1)$, $q \leftarrow$ a fixed sequence of random bits.
3: Set $\alpha = \frac{3}{4}$
4: for $e \in N$ do do
5:  Assign $e$ to each machine independently with probability $1/|M|$
6: for $i \in M$ do
7:  $\triangleright$ On machine $i$
8:  Let $N_i$ be the elements assigned to machine $i$
9:  $S_i \leftarrow \text{LAT}(f, N_i, k, \delta, \varepsilon, \alpha \text{OPT}/k, q)$
10: if $|S_i| = k$ then return $T' \leftarrow S_i$
11: Send $S_i$ to primary machine
12: $\triangleright$ On primary machine
13: $S \leftarrow \bigcup_{i=1}^{|M|} S_i$
14: Let $g(\cdot) \leftarrow f(S \cup \cdot) - f(S)$
15: $T \leftarrow \text{LAT}(g, S, k - |S|, \delta, \varepsilon, \alpha \text{OPT}/k, q)$
16: $T' \leftarrow S \cup T$
17: return $T'$

Definition 1. For any $x \in N$, let

$$ p_x = \begin{cases} 
\Pr_{X \sim N(1/\ell), q} \left[ \text{LAT}(X \cup \{x\}, q) = \text{LAT}(X, q) \right], & \text{if } x \in O \\
0, & \text{otherwise}
\end{cases} $$

Let $O_1 = \{ o \in O : o \notin \text{LAT}(N_1 \cup \{o\}, q) \}$, $O_2 = S \cap O$. For any $o \in O_1$, $o$ is not selected in $S_1$. Since, $|S_1| < k$, by Theorem 2 of Chen et al. [12],

$$ \Delta (o | T') < \Delta (o | S_1) < \tau. $$

Also, for any $o \in O_2 \setminus T$, $o$ is not selected in $T$. Similarly,

$$ \Delta (o | T'') < \tau. $$

Then, we can get,

$$ f(O_1 \cup O_2) - f(T'') \leq f(O_1 \cup O_2 \cup T') - f(T'') $$

$$ \leq \sum_{o \in O_1 \cup O_2 \setminus T'} \Delta (o | T') $$

$$ \leq k \cdot \tau = 3 \text{OPT}/8. \quad (7) $$

Next, we provide the following lemma to complete the rest of the proof.

**Lemma 5.** For any $o \in O$, it holds that $\Pr (o \in O_1 \cup O_2) \geq 3/4$.

**Proof.** By Definition [1] it holds that $\Pr (o \in O_1) = p_o$. Since $o$ is assigned to each machine randomly with probability $1/|M|$,

$$ \Pr (o \in O_2) = \sum_{i=1}^{|M|} \Pr (o \in S_i) $$

$$ = |M| \cdot \Pr (o \in S_1 | o \in N_1) \cdot \Pr (o \in N_1) $$

$$ = \Pr (o \in \text{LAT}(N_1)) | o \in N_1 $$

$$ = \Pr (o \in \text{LAT}(N_1 \cup \{o\})) $$

$$ = 1 - p_o. $$

Moreover, we know that any two machines selects elements independently. So,

$$ \Pr (o \in O_2 | o \in O_1) = \Pr (o \in O_2 | o \notin \text{LAT}(N_1 \cup \{o\}, q)) $$
\[= 1 - Pr(o \not\in O_2 | o \not\in \text{LAT}(N_1 \cup \{o\}, q))
\]
\[= 1 - \prod_{i=1}^{[M]} Pr(o \not\in S_i | o \not\in \text{LAT}(N_1 \cup \{o\}, q)) \]
\[\stackrel{(a)}{=} 1 - \prod_{i=2}^{[M]} Pr(o \not\in S_i)
\]
\[\leq 1 - \Pr(o \not\in O_2)
\]
\[= Pr(o \in O_2) = 1 - p_o,
\]
where Inequality (a) follows from \(Pr(o \not\in \text{LAT}(N_1) | o \not\in \text{LAT}(N_1 \cup \{o\})) = 1\).

Thus, we can bound the probability by the following,
\[Pr(o \in O_1 \cup O_2) = Pr(o \in O_1) + Pr(o \in O_2)\]
\[\geq p_o + 1 - p_o - p_o(1 - p_o)
\]
\[\geq 3/4.
\]

Now, we know the probability that each element in \(O\) is also in \(O_1 \cup O_2\) is larger than 3/4. By Lemma 3,
\[\mathbb{E}[f(O_1 \cup O_2)] \geq 3/4 \cdot F(1) = 3/4 \cdot \text{OPT}.\] (8)

From Inequalities 7 and 8 we can bound the approximation ratio for Algorithm T-DASH knowing \(\text{OPT}\) by follows,
\[\mathbb{E}[f(T')] \geq 3\text{OPT}/8.
\]

\section{Analysis of MED}

\textbf{Theorem 4.} Let \((f, k)\) be an instance of SMCC distributed over \(m\) machines. Let \(S_m\) be the set returned by MED. Then \(f(S_m) \geq (1 - e^{-\gamma})), where \(\gamma\) is the approximation of \(\text{ALG}\).

\textit{Proof.} Let \(O\) be an optimal and \(O_1, O_2, \ldots, O_m\) be a partition of \(O\) into \(m\) pieces, each of size \(\leq k'\). Also \(\forall i, \hat{f}_i\) is monotone SMCC since \(f\) is monotone SMCC.

\[
f(S_{i+1}) - f(S_i) = f(S_i \cup A_i) - f(S_i)
\]
\[
= \hat{f}_i(A_i)
\]
\[
\geq \frac{1}{m} \sum_{j=1}^{m} \gamma \hat{f}_i(O_j)
\]
\[
\geq \frac{\gamma}{m} \hat{f}_i(O)
\]
\[
= \frac{\gamma}{m} (f(S_i \cup O) - f(S_i))
\]
\[
\geq \frac{\gamma}{m} (\text{OPT} - f(S_i)),
\]
where Inequality (a) follows from \(\hat{f}_i\) is SMCC; \(\text{ALG}\) is \(\gamma\)-approximation.

\[
\therefore \text{OPT} - f(S_{i+1}) \leq \text{OPT} - f(S_i) - \gamma/m (\text{OPT} - f(S_i))
\]
\[
= \left(1 - \frac{\gamma}{m}\right) \left[\text{OPT} - f(S_i)\right]
\]
\[
\leq \left(1 - \frac{\gamma}{m}\right)^i \left[\text{OPT} - f(\emptyset)\right]
\]

\[18\]
\[ = \left(1 - \frac{\gamma}{m}\right)^i \cdot OPT \]
\[ \therefore f(S_m) \geq \left[1 - (1 - \frac{\gamma}{m})^m\right] \cdot OPT \]
\[ \geq (1 - e^{-\gamma}) \cdot OPT \]

**Memory Requirement Justification** Assume each machine \( m_i \) can hold \( \Psi \) elements. Space needed by the primary machine for storing the sets returned from all machines is \(|M| \cdot k'\). Then
\[
|M| \cdot k' \leq \Psi
\]
\[ k' \leq \frac{\Psi}{|M|} \]; and

Iterations needed \( m = \frac{k}{k'} \).

Figure 4 and 5 presents performance comparison of distributed algorithms on centralized and distributed data of RevenueMax and MaxCover;

![Figure 4: Centralized Datasets](image)
![Figure 5: Distributed Datasets](image)

**I Experiment Setup**

**I.1 Implementation and Environment**

The experiments are conducted on 8 independent machines with 4 Intel(R) Core(TM) i5-2500 CPU @ 3.30GHz cores; 8 gigabytes of memory; running Ubuntu 18.64.2 with kernel version 5.8.0. MPICH version 3.3a2 (not OpenMPI) was installed on each machine; to implement and parallelize all algorithms using the Message Passing Interface (MPI); and run using the `mpirun` command. The \((LS+PGB)^H\) algorithm is a heuristic that executes in two stages, where the first stage runs...
LS+PGB on the assigned data $\mathcal{N}_i$ and the second stage selects the best set amongst all the solutions returned by the machines as the final solution. $(LS+PGB)_M$ does not hold theoretical guarantees for distributed data experiments. Centralized dataset evaluations for non-GREEDY based algorithms were independently run for five repetitions. No repetitions were performed for distributed dataset experiments. For each instance of $k$ value, the mean and standard deviation of the objective value and parallel (wall clock) runtime to the submodular function are observed.

I.2 Datasets

The dataset are categorized into two groups; Centralized and Distributed. The details are provided in Table 3. In Distributed Data experiments, large matrix files (size = $N$) are partitioned uniformly at random (such that $\bigcup_{i=1}^{\mathcal{M}} \mathcal{N}_i = \mathcal{N}$) and assigned to $|\mathcal{M}|$ machines. For data access during the experiments, each machine $\mathcal{M}_i$ loads the ($|\mathcal{N}_i| \times |\mathcal{N}|$) matrix into memory. By using MPI’s $\text{comm.send()}$, a matrix of size ($|\mathcal{S}_i| \times |\mathcal{N}|$) is sent to the primary machine based on the solution $\mathcal{S}_i$ obtained by the algorithm on machine $\mathcal{M}_i$. Using $\text{comm.recv()}$, the primary machine combines all the matrices sent from $|\mathcal{M}|$ machines and generates a new matrix of size ($|\mathcal{S}| \times \mathcal{N}$) for the second MapReduce round (where $\mathcal{S} \leftarrow \bigcup_{i=1}^{\mathcal{M}} \mathcal{S}_i$).

I.3 Applications

Given a constraint $k$, the objectives of the applications are defined as follows:

I.3.1 Max Cover

Maximize the number of nodes covered by choosing a set $\mathcal{S}$ of maximum size $k$, such that the number of nodes having at least one neighbour in the set $\mathcal{S}$. The application is run on synthetic random BA graphs of groundset size 100,000 and 1,000,000 generated using Barabási–Albert (BA) models for the centralized and distributed experiments respectively. For the centralized data, BA graphs were generated by adding $m = 5$ edges each iteration, while for the distributed data graphs were generated by adding $m = 500$ edges each iteration.

I.3.2 Image Summarization on CIFAR-10 data

Given large collection of images, find a subset of maximum size $k$ which is representative of the entire collection. The objective used for the experiments is a monotone variant of the image summarization from Fahrbach et al. [18]. For a groundset with $N$ images, it is defined as follows:

$$ f(\mathcal{S}) = \sum_{i \in \mathcal{N}} \max_{j \in \mathcal{S}} s_{i,j} $$

where $s_{i,j}$ is the cosine similarity of the pixel values between image $i$ and image $j$. The data for the image summarization experiments contains 10,000 and 50,000 CIFAR-10 [23] color images respectively for the centralized and distributed experiments.

I.3.3 Influence Maximization on a Social Network.

Maximise the aggregate influence to promote a topic by selecting a set of social network influencers of maximum size $k$. The probability that a random user $i$ will be influenced by the set of influencers in $\mathcal{S}$ is given by:

$$ f_i(\mathcal{S}) = 1 \quad \text{for} \ i \in \mathcal{S} $$

$$ f_i(\mathcal{S}) = 1 - (1 - p)^{|N_\mathcal{S}(i)|} \quad \text{for} \ i \not\in \mathcal{S} $$

where $|N_\mathcal{S}(i)|$ is the number of neighbors of node $i$ in $\mathcal{S}$. We use the Epinions data set consisting of 27,000 users from Rossi and Ahmed [32] for the centralized data experiments and the Youtube online social network data Yang and Leskovec [33] consisting more than 1 million users for distributed data experiments. The value of $p$ is set to 0.01.
Figure 6: Presents performance comparison of R-DASH Vs LS+PGB in terms of solution value on **Centralized Data** (top row); and R-DASH Vs (LS+PGB)$^H$ on **Distributed Data** (bottom row) across four applications; (LS+PGB)$^H$ is a heuristic that runs LS+PGB on each machine’s assigned data $N_i$ and returns the best solution amongst all machines as the output.

I.3.4 Revenue Maximization on YouTube.

Maximise revenue of a product by selecting set of users $S$ of maximum size $k$, where the network neighbors will be advertised a different product by the set of users $S$. It is based on the objective function from Mirzasoleiman et al. [30]. For a given set of users $X$ and $w_{i,j}$ as the influence between user $i$ and $j$, the objective function can be defined by:

$$ f(S) = \sum_{i \in X} V \left( \sum_{j \in S} w_{i,j} \right) $$

$$ V(y) = y^\alpha $$

where $V(S)$, the expected revenue from an user is a function of the sum of influences from neighbors who are in $S$ and $\alpha : 0 < \alpha < 1$ is a rate of diminishing returns parameter for increased cover.

We use the Youtube data set from Mirzasoleiman et al. [30] consisting of 18,000 users for centralized data experiments. For the distributed data experiments we perform empirical evaluation on the Orkut online social network data from Yang and Leskovec [33] consisting more than 3 million users. The value of $\alpha$ is set to 0.3

J Experiment Results

J.1 Additional Evaluation Results (continued from Experiment Section))

**Experiment Set 1:** The evaluation results on the RevenueMax and MaxCover objective for centralized and distributed data are shown in Fig. 4 and 5. **Centralized Data** Figure 4 indicates that R-DASH provides a 43 and 68 times faster speedup over the fastest GREEDY based distributed algorithm, RANDGREEDI, for RevenueMax and MaxCover objectives; with nearly matching solution values. For RevenueMax and MaxCover objectives, G-DASH remains two and three times faster than RANDGREEDI, with marginally better solution values than R-DASH. **Distributed Data** As
shown in Fig. 5 neither RANDGREEDI nor BiCRITERIAGREEDY complete a single instance of RevenueMax and MaxCover within the timeout of 24 hours.

J.2 Additional Experiments

J.2.1 Evaluation of LS+PGB: (Fig. 6)

This experiment compares R-DASH with the LS+PGB proposed by Chen et al. [12] on the smaller centralized data; while its parallel distributed heuristic (LS+PGB) is compared on the larger distributed data. Centralized Data R-DASH and LS+PGB provide almost identical solutions for all applications except RevenueMax; where R-DASH obtains 98% of the value obtained by LS+PGB. Distributed Data Using R-DASH for InfluenceMax, RevenueMax, and MaxCover provides a 170%, 80%, and 10% better solution value than (LS+PGB). According to the experiment, adaptive algorithms fail to deliver feasible solutions with parallel computation on distributed data; when they are executed concurrently on each machine’s local data.

J.2.2 Evaluation of MED

The purpose of this experiment set is to compare the performance of MED to the algorithm it runs iteratively over $m$ rounds, R-DASH and the most efficient GREEDY distributed algorithm, RANDGREEDI. We perform the comparison on centralized (small) datasets; to run all the algorithms on large constraints, $k = [k', 2k', 3k', 4k', 5k']$, where $k' = n/M^2$. Figure 7(a)-7(d) shows that MED is capable of maintaining or outperforming R-DASH in terms of solution value. As compared to RANDGREEDI, MED gets a solution 110 times faster on average (Fig. 7(e)-7(h)); while maintaining 99% of its solution value. This experiment set establishes that MED maintains a solution value similar to that R-DASH it runs iteratively.

![Figure 7: Empirical comparison of MED to R-DASH and RANDGREEDI on small centralized data. The plotted metrics are solution value (Fig. 7(a)-7(d)) and runtime (Fig. 7(e)-7(h)).](image-url)
Figure 8: Comparison of RANDGREEDI+LAG to RANDGREEDI and R-DASH on centralized data
J.2.3 Evaluation of RANDGREEDI+LAG (RANDGREEDI using LAG as ALG)

RANDGREEDI+LAG is a version of RANDGREEDI that uses LAG as the ALG component. This experiment investigates whether utilizing an optimal adaptive algorithm, LAG instead of GREEDY as the ALG component provides significant improvement in runtime. Centralized Data (Fig. 8) On average, RANDGREEDI+LAG is two times faster than RANDGREEDI; however, R-DASH provides a 160 time speedup over RANDGREEDI+LAG, with very similar solution values. Distributed Data (Fig. 9) The performance difference between RANDGREEDI+LAG and RANDGREEDI is negligible, due to the major contribution in runtime caused by GREEDY execution during the MapReduce round. This experiment shows that the GREEDY-based distributed algorithms cannot benefit from the use of efficient adaptive algorithms as their post-distributed processing algorithm.

J.2.4 Evaluation of RANDGREEDI on 8 vs. 32 independent processing units

Suppose that the size of the ground set is $n$, and the cardinality constraint is $k$. With only 1 thread on each machine, the total time of running RANDGREEDI on $M$ machines can be calculated by the following two parts: 1) The running time for one machine in the first MR round, which is proportional to $(n/M - (k - 1)/2)k$; 2) The running time for the primary machine in the second MR round (post-processing step), which is proportional to $(kM - (k - 1)/2)k$. The total running time would be proportional to $nk/M + Mk^2 - k(k - 1)$, which gets the optimal performance when $M = \sqrt{n/k}$. The issue was demonstrated by running RANDGREEDI on 32 processors and 8 processors with small constraints that $0.001 \leq \frac{k}{n} \leq 0.05$, and plotting the objective value and runtime. From Fig. 10 it is evident from the results that even when constraints are small ($\leq 5\%$), the overhead introduced in the second MR due to the increase in distributed setup can overshadow the performance gain during the first MR round. Therefore, with larger $k$ values, the performance of RANDGREEDI would be much worse with large distributed setups.
Figure 10: Comparison of RANDGREEDI when evaluated on 8 independent processing units (RG-8) and 32 independent processing units (RG-32) with no parallelism. R-DASH provides a reference for performance of sublinear adaptive distributed algorithm when evaluated on 8 independent processing units with 4 threads per processing unit.

K Replicating the Experiment Results

Our experiments can be replicated by running the following scripts:

- Install MPICH version 3.3a2 (DO NOT install OpenMPI and ensure `mpirun` utilizes mpich using the command `mpirun --version` (Ubuntu))
- Install pandas, mpi4py, scipy, networkx
- Set up an MPI cluster using the following tutorial: https://mpitutorial.com/tutorials/running-an-mpi-cluster-within-a-lan/
- Create and update the host file `../nodesFileIPnew` to store the ip addresses of all the connected MPI machines before running any experiments (First machine being the primary machine)
  - NOTE: Please place `nodesFileIPnew` inside the MPI shared repository; “cloud/” in this case (at the same level as the code base directory). DO NOT place it inside the code base “DASH-Distributed_SMCC-python/” directory.
- Clone the DASH-Distributed_SMCC-python repository inside the MPI shared repository (“cloud” in the case using the given tutorial)
  - NOTE: Please clone the “DASH-Distributed_SMCC-python” repository and execute the following commands on a machine with sufficient memory (RAM); capable of generating the large datasets. This repository NEED NOT be the primary repository (“cloud/DASH-Distributed_SMCC-python/”) on the shared memory of the cluster; that will be used for the experiments.
- Additional Datasets For Experiment 1: Please download the Image Similarity Matrix file "images_10K_mat.csv"(https://drive.google.com/file/d/1s9PzUhV-C5dW8LP4tZPvJSRX4PBhsrl/view?usp=sharing) and place it in the data/data_exp1/ directory.
• To generate the distributed data for Experiment 2 and 3: Please follow the below steps:
  
  – Execute bash GenerateDistributedData.bash nThreads nNodes
  
  – The previous command should generate nNodes directories in loading_data/ directory (with names machine<nodeNo> \Data)
  
  – Copy the data_exp2_split/ and data_exp3_split/ directories within each machine<\i> \Data directory to the corresponding machine Mi and place the directories outside /cloud (directory created after setting up an MPI cluster using the given tutorial)).

To run all experiments in the apper
Please read the README.md file in the "DASH-Distributed_SMCC-python" (Code/Data Appendix) for detailed information.