NONLOCAL MINIMAL LAWSON CONES

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Abstract. We prove the existence of the analog of Lawson’s minimal cones for a notion of nonlocal minimal surface introduced by Caffarelli, Roquejoffre and Savin, and establish their stability/instability in low dimensions. In particular we find that there are nonlocal stable minimal cones in dimension 7, in contrast with the case of classical minimal surfaces.

1. Introduction

In [4], Caffarelli, Roquejoffre and Savin introduced a nonlocal notion of perimeter of a set, which generalizes the \((N-1)\)-dimensional surface area of \(\partial E\). For \(0 < s < 1\), the \(s\)-perimeter of \(E \subset \mathbb{R}^N\) is defined (formally) as

\[
\text{Per}_s(E) = \int_E \int_{\mathbb{R}^N \setminus E} \frac{dx \, dy}{|x-y|^{N+s}}.
\]

This notion is localized to a bounded open set \(\Omega\) by setting

\[
\text{Per}_s(E, \Omega) = \int_E \int_{\mathbb{R}^N \setminus E} \frac{dx \, dy}{|x-y|^{N+s}} - \int_{E \cap \Omega} \int_{\mathbb{R}^N \setminus (E \cup \Omega)} \frac{dx \, dy}{|x-y|^{N+s}}.
\]

This quantity makes sense, even if the last two terms above are infinite, by rewriting it in the form

\[
\text{Per}_s(E, \Omega) = \int_{E \cap \Omega} \int_{\mathbb{R}^N \setminus E} \frac{dx \, dy}{|x-y|^{N+s}} + \int_{E \setminus \Omega} \int_{\mathbb{R}^N \setminus E} \frac{dx \, dy}{|x-y|^{N+s}}.
\]

Let us assume that \(E\) is an open set set with \(\partial E \cap \Omega\) smooth. The usual notion of perimeter is recovered by the formula

\[
\lim_{s \to 1} (1-s) \text{Per}_s(E, \Omega) = \text{Per}(E, \Omega) = c_N \mathcal{H}^{N-1}(\partial E \cap \Omega),
\]

see [13]. Let us consider a unit normal vector field \(\nu\) of \(\Sigma = \partial E\) pointing to the exterior of \(E\), and consider functions \(h \in C_0^\infty(\Omega \cap \Sigma)\). For a number \(t\) sufficiently small, we let \(E_{th}\) be the set whose boundary \(\partial E_{th}\) is parametrized as

\[
\partial E_{th} = \{ x + th(x)\nu(x) / x \in \partial E \},
\]

with exterior normal vector close to \(\nu\). The first variation of the perimeter along these normal perturbations yields

\[
\frac{d}{dt} \text{Per}_s(E_{th}, \Omega) \bigg|_{t=0} = - \int_\Sigma H^s_\Sigma h,
\]

where

\[
H^s_\Sigma(p) := \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{\mathbb{R}^N \setminus E}(x)}{|x-p|^{N+s}} \, dx \quad \text{for } p \in \Sigma.
\]
This integral is well-defined in the principal value sense provided that $\Sigma$ is regular near $p$. We say that the set $\Sigma = \partial E$ is a nonlocal minimal surface in $\Omega$ if the surface $\Sigma \cap \Omega$ is sufficiently regular, and it satisfies the nonlocal minimal surface equation

$$H^s_\Sigma(p) = 0 \quad \text{for all } p \in \Sigma \cap \Omega.$$ 

We may naturally call $H^s_\Sigma(p)$ the nonlocal mean curvature of $\Sigma$ at $p$.

Let $\Sigma = \partial E$ be a nonlocal minimal surface. As we will prove in Section 4, the second variation of the $s$-perimeter in $\Omega$ can be computed for functions $h$ smooth and compactly supported in $\Sigma \cap \Omega$ as

$$\frac{d^2}{dt^2} \text{Per}_s(E_{th}, \Omega) \bigg|_{t=0} = -2 \int_\Sigma J^s_\Sigma[h] h \quad (1.3)$$

where $J^s_\Sigma[h]$ is the nonlocal Jacobi operator given by

$$J^s_\Sigma[h](p) = \text{p.v.} \int_\Sigma \frac{h(x) - h(p)}{|p - x|^{N+s}} dx + h(p) \int_\Sigma \langle \frac{\nu(p) - \nu(x), \nu(p)}{|p - x|^{N+s}} \rangle \frac{dx}{|p - x|^{N+s}}, \quad p \in \Sigma. \quad (1.4)$$

In agreement with formula (1.3), we say that an $s$-minimal surface $\Sigma$ is stable in $\Omega$ if

$$-\int_\Sigma J^s_\Sigma[h] h \geq 0 \quad \text{for all } h \in C^\infty_0(\Sigma \cap \Omega).$$

A basic example of a stable nonlocal minimal surface is a nonlocal area minimizing surface. We say that $\Sigma = \partial E$ is nonlocal area minimizing in $\Omega$ if

$$\text{Per}_s(E, \Omega) \leq \text{Per}_s(F, \Omega) \quad (1.5)$$

for all $F$ such that $(E \setminus F) \cup (F \setminus E)$ is compactly contained in $\Omega$. In [4], Caffarelli, Roquejoffre and Savin proved that if $\Omega$ and $E_0 \subset \mathbb{R}^N \setminus \Omega$ are given, and sufficiently regular, then there exists a set $E$ with $E \cap (\mathbb{R}^N \setminus \Omega) = E_0$ which satisfies (1.5). They proved that $\Sigma = \partial E$ is a surface of class $C^{1,\alpha}$ outside a closed set of Hausdorff dimension $N - 2$.

In this paper we will focus our attention on nonlocal minimal cones. By a (solid) cone in $\mathbb{R}^N$, we mean a set of the form

$$E = \{tx / t > 0, \ x \in \mathcal{O}\}$$

where $\mathcal{O}$ is a regular open subset of the sphere $S^{N-1}$. The cone (mantus) $\Sigma = \partial E$ is an $(N - 1)$-dimensional surface which is regular, except at the origin.

Existence or non-existence of area minimizing cones for a given dimension is a crucial element in the classical regularity theory of minimal surfaces. Simons [15] proved that no stable minimal cone exists in dimension $N \leq 7$, except for hyperplanes. This result is a main ingredient in regularity theory: it implies that area minimizing surfaces must be smooth outside a closed set of Hausdorff dimension $N - 8$.

Savin and Valdinoci [13], by proving the nonexistence of a nonlocal minimizing cone in $\mathbb{R}^2$, established the regularity of any nonlocal minimizing surface outside a set of Hausdorff dimension $N - 3$, thus improving the original result in [4].

In [5], Caffarelli and Valdinoci proved that regularity of non-local minimizers holds up to a $(N - 8)$-dimensional set, provided that $s$ is sufficiently close to 1.
The purpose of this paper is to analyze a specific class of nonlocal minimal cones. Let \( n, m \geq 1 \), \( n + m = N \) and \( \alpha > 0 \). Let us call
\[
C_\alpha = \{ x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^n / |z| = \alpha |y| \}.
\]
(1.6)
It is a well-known fact that \( C_\alpha \) is a minimal surface in \( \mathbb{R}^N \setminus \{0\} \) (its mean curvature equals zero) if and only
\[
n \geq 2, \ m \geq 2, \ \alpha = \sqrt{\frac{n-1}{m-1}}.
\]
We call this minimal Lawson cone \( C_{n,m}(\alpha) \). As for the stability-minimizing character of these cones, the result of Simons [15] tells us that they are all unstable for \( n + m = 7 \). Simons also proved that the cone \( C_{1,1}^4 \) is stable and conjectured that it was minimizing. Bombieri, De Giorgi and Giusti in [3] found a family of disjoint minimal surfaces asymptotic to the cone, foliating \( \mathbb{R}^4 \times \mathbb{R}^4 \). This implies \( \gamma = C_{1,1}^4 \) is area minimizing. For \( N > 8 \) the cones \( C_{n,m}^n \) are all area minimizing. For \( N = 8 \) they are area minimizing if and only if \( |m-n| \leq 2 \). These facts were established by Lawson [11] and Simoes [14], see also [12, 6, 1, 8].

For the non-local scenario we find the existence of analogs of the cones \( C_{n,m}^n \).

**Theorem 1.** For any given \( m \geq 1 \), \( n \geq 1 \), \( 0 < s < 1 \), there is a unique \( \alpha = \alpha(s, m, n) > 0 \) such that \( C_\alpha = \{ x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^n / |z| = \alpha |y| \} \) is a nonlocal minimal cone. We call this cone \( C_{n,m}^n(s) \).

The above result includes the existence of a minimal cone \( C_{1,m}^n(s) \), \( m \geq 1 \). Such an object does not exist in the classical setting for \( C_{1,m}^n \) is defined only if \( n, m \geq 2 \).

We have found a (computable) criterion to decide whether or not \( C_{m,n}^n(s) \) is stable. As a consequence we find the following result for \( s \) close to 0 which shows a sharp contrast with the classical case.

**Theorem 2.** There is a \( s_0 > 0 \) such that for each \( s \in (0, s_0) \), all minimal cones \( C_{m,n}^n(s) \) are unstable if \( N = m + n \leq 6 \) and stable if \( N = 7 \).

We recall that in the classical case \( C_{m,n}^n \) is unstable for \( N = 7 \). It is natural to conjecture that the above cones for \( N = 7 \) are minimizers of perimeter. Being that the case, the best regularity possible for small \( s \) would be up to an \( (N-7) \)-dimensional set.

As far as we know, at this moment, there are no examples of regular nontrivial nonlocal minimal surfaces ([14]). Formula (1.1) suggests that for \( s \) close to 1 there may be nontrivial nonlocal minimal surfaces close to the classical ones. In a forthcoming paper [7] we prove that this is indeed the case. We construct nonlocal catenoids as well as nonlocal Costa surfaces for \( s \) close to 1 by interpolating the classical minimal surfaces in compact regions with the nonlocal Lawson’s cones \( C_{1,m}^1 \) far away. Thus these nonlocal catenoids can be considered as foliations of the nonlocal Lawson’s cones \( C_{1,m}^1 \). A natural question, as in the classical minimal cones case ([8]), is the existence of foliations for general nonlocal Lawson’s cones \( C_{m,n}^n \).

In section [2] we prove theorem [1] and in section [3] we show that also for \( s = 0 \) there is a unique minimal cone. In section [4] we obtain formula (1.4) for the nonlocal Jacobi operator and section [5] is devoted to the proof of theorem [2].
2. Existence and Uniqueness

Let us write

\[ E_\alpha = \{ x = (y, z) : y \in \mathbb{R}^m, z \in \mathbb{R}^n, |z| > \alpha |y| \}, \]

so that \( C_\alpha = \partial E_\alpha \) is the cone defined in (1.3).

**Proof of theorem 1.**

**Existence.** We fix \( N, m, n \) with \( N = m + n, n \leq m \) and also fix \( 0 < s < 1 \). If \( m = n \) then \( C_1 \) is a minimal cone, since (1.2) is satisfied by symmetry. So we concentrate next on the case \( n < m \).

Before proceeding we remark that for a cone \( C_\alpha \) the quantity appearing in (1.2) has a fixed sign for all \( p \in C_\alpha, p \neq 0 \), since by rotation we can always assume that \( p = r\alpha \) for some \( r > 0 \) where

\[ p_\alpha = \frac{1}{\sqrt{1 + \alpha^2}}(\alpha e_1^{(m)}, \alpha e_1^{(n)}) \]

and similarly for \( e_1^{(n)} \). Then we observe that

\[ \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha}(x)}{|x - r\alpha|^{N+s}} \, dx = \frac{1}{r^s} \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha}(x)}{|x - \alpha|^{N+s}} \, dx. \]

Let us define

\[ H(\alpha) = \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha}(x)}{|x - \alpha|^{N+s}} \, dx \]

and note that it is a continuous function of \( \alpha \in (0, \infty) \).

**Claim 1.** We have

\[ H(1) \leq 0. \]

Indeed, write \( y \in \mathbb{R}^m \) as \( y = (y_1, y_2) \) with \( y_1 \in \mathbb{R}^n \) and \( y_2 \in \mathbb{R}^{m-n} \). Abbreviating \( e_1 = e_1^{(n)} = (1, 0, \ldots, 0) \in \mathbb{R}^n \) we rewrite

\[
H(1) = \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B(p_1, \delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha}(x)}{|x - p_1|^{N+s}} \, dx \\
= \lim_{\delta \to 0} \int_{A_\delta} \left( |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 \right)^{N+s/2} \\
- \lim_{\delta \to 0} \int_{B_\delta} \left( |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 \right)^{N+s/2},
\]

where

\[
A_\delta = \{ |z|^2 > |y_1|^2 + |y_2|^2, |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 > \delta^2 \} \\
B_\delta = \{ |z|^2 < |y_1|^2 + |y_2|^2, |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 > \delta^2 \}.
\]
But the first integral can be rewritten as
\[
\int_{A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2)^{\frac{s-N}{2}}} \, dx
= \int_{\tilde{A}_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2)^{\frac{s-N}{2}}} \, dx
\]
where
\[
\tilde{A}_\delta = \{|y_1|^2 > |z|^2 + |y_2|^2, \ |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 > \delta^2\}
\]
(we just have exchanged \(y_1\) by \(z\) and noted that the integrand is symmetric in these variables). But \(\tilde{A}_\delta \subset B_\delta\) and so
\[
\int_{\mathbb{R}^N \setminus B(p_1, \delta)} \frac{\chi_{E_1}(x) - \chi_{E_1^c}(x)}{|x - p_1|^{N+s}} \, dx
= - \int_{B_\delta \setminus \tilde{A}_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2)^{\frac{s-N}{2}}} \leq 0.
\]
This shows the validity of (2.4).

Claim 2. We have
\[
H(\alpha) \to +\infty \quad \text{as} \quad \alpha \to 0. \quad (2.5)
\]
Let \(0 < \delta < 1/2\) be fixed and write
\[
H(\alpha) = I_\alpha + J_\alpha
\]
where
\[
I_\alpha = \int_{\mathbb{R}^N \setminus B(p_\alpha, \delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}} \, dx
\]
\[
J_\alpha = \text{p.v.} \int_{B(p_\alpha, \delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}} \, dx.
\]
With \(\delta\) fixed
\[
\lim_{\alpha \to 0} I_\alpha = \int_{\mathbb{R}^N \setminus B(p_\alpha, \delta)} \frac{1}{|x - p_0|^{N+s}} \, dx > 0. \quad (2.6)
\]
For \(J_\alpha\) we make a change of variables \(x = \alpha \tilde{x} + p_\alpha\) and obtain
\[
J_\alpha = \text{p.v.} \int_{B(p_\alpha, \delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}} \, dx = \frac{1}{\alpha^s} \text{p.v.} \int_{B(0, \delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde{x}) - \chi_{F_\alpha^c}(\tilde{x})}{|\tilde{x}|^{N+s}} \, d\tilde{x} \quad (2.7)
\]
where \(F_\alpha = \frac{1}{\alpha}(E_\alpha - p_\alpha)\). But
\[
\text{p.v.} \int_{B(0, \delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde{x}) - \chi_{F_\alpha^c}(\tilde{x})}{|\tilde{x}|^{N+s}} \, d\tilde{x} \to \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{F_\alpha}(x) - \chi_{F_\alpha^c}(x)}{|x|^{N+s}} \, dx
\]
as $\alpha \to 0$ where $F_0 = \{ x = (y, z) : y \in \mathbb{R}^m, z \in \mathbb{R}^n, \, |z + c_1^{(n)}| > 1 \}$. But writing
$z = (z_1, \ldots, z_n)$ we see that
$$
p.v. \int_{\mathbb{R}^n} \frac{\chi_{E_0}(x) - \chi_{F_0}(x)}{|x|^{N+s}} \, dx \geq p.v. \int_{\mathbb{R}^n} \frac{\chi_{|z|>2 \text{ or } z_1<0}}{|x|^{N+s}} \, dx
$$
and this number is positive. This and (2.7) show that $J_\alpha \to +\infty$ as $\alpha \to 0$ and combined with (2.6) we obtain the desired conclusion.

By (2.4), (2.5) and continuity we obtain the existence of $\alpha \in (0,1]$ such that $H(\alpha) = 0$.

**Uniqueness.** Consider 2 cones $C_{\alpha_1}$, $C_{\alpha_2}$ with $\alpha_1 > \alpha_2 > 0$, associated to solid cones $E_{\alpha_1}$ and $E_{\alpha_2}$. We claim that there is a rotation $R$ so that $R(E_{\alpha_1}) \subset E_{\alpha_2}$ (strictly) and that
$$H(\alpha_1) = p.v. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}} \, dx.
$$
Note that the denominator in the integrand is the same that appears in (2.3) for $\alpha_2$ and then
$$H(\alpha_1) = p.v. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}} \, dx < p.v. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{E_{\alpha_2}}(x) - \chi_{E_{\alpha_2}^c}(x)}{|x - p_{\alpha_2}|^{N+s}} \, dx = H(\alpha_2).
$$
This shows that $H(\alpha)$ is decreasing in $\alpha$ and hence the uniqueness. To construct the rotation let us write as before $x = (y, z) \in \mathbb{R}^N$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, and $y = (y_1, y_2)$ with $y_1 \in \mathbb{R}^n$, $y_2 \in \mathbb{R}^{m-n}$ (we assume always $n \leq m$). Let us write the vector $(y_1, z)$ in spherical coordinates of $\mathbb{R}^{2n}$ as follows
$$y_1 = \rho \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \cos(\varphi_2) \\ \sin(\varphi_1) \sin(\varphi_2) \cos(\varphi_3) \\ \vdots \\ \sin(\varphi_1) \sin(\varphi_2) \sin(\varphi_3) \ldots \sin(\varphi_{n-1}) \cos(\varphi_n) \end{bmatrix}
$$
$$z = \rho \begin{bmatrix} \sin(\varphi_1) \sin(\varphi_2) \sin(\varphi_3) \ldots \sin(\varphi_n) \cos(\varphi_{n+1}) \\ \sin(\varphi_1) \sin(\varphi_2) \sin(\varphi_3) \ldots \sin(\varphi_{2n-2}) \cos(\varphi_{2n-1}) \\ \sin(\varphi_1) \sin(\varphi_2) \sin(\varphi_3) \ldots \sin(\varphi_{2n-2}) \sin(\varphi_{2n-1}) \end{bmatrix}
$$
where $\rho > 0$, $\varphi_{2n-1} \in [0, 2\pi)$, $\varphi_j \in [0, \pi]$ for $j = 1, \ldots, 2n-2$. Then
$$|y|^2 = \rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2, \quad |y_1|^2 + |z|^2 = \rho^2.
$$
The equation for the solid cone $E_{\alpha_1}$, namely $|z| > \alpha_1 |y|$, can be rewritten as
$$\rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2 > \alpha_1^2 (|y_1|^2 + |y_2|^2).
$$
Adding $\alpha_1^2 |z|^2$ to both sides this is equivalent to
$$\sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2 > \sin(\beta_1)^2 (1 + \frac{|y_2|^2}{\rho^2}).$$
where $\beta_i = \arctan(\alpha_i)$. We let $\theta = \beta_1 - \beta_2 \in (0, \pi/2)$, and define the rotated cone $R_\theta(E_{\alpha_1})$ by the equation

$$\sin(\varphi_1 + \theta) \sin(\varphi_2) \ldots \sin(\varphi_n) > \sin(\beta_1)(1 + \frac{|y_2|^2}{\rho^2}).$$

We want to show that $R_\theta(E_{\alpha_1}) \subset E_{\alpha_2}$. To do so, it suffices to prove that for any given $t \geq 1$, if $\varphi$ satisfies the inequality $|\sin(\varphi + \theta)| > \sin(\beta_1)t$ then it also satisfies $|\sin(\varphi)| > \sin(\beta_2)t$. This in turn can be proved from the inequality

$$\arccos(\sin(\beta_1)t) + \theta < \arccos(\sin(\beta_2)t)$$

for $1 < t \leq \frac{1}{\sin(\beta_1)}$. For $t = 1$ we have equality by definition of $\theta$. The inequality for $1 < t \leq \frac{1}{\sin(\beta_1)}$ can be checked by computing a derivative with respect to $t$. The strict inequality in $\mathbb{R}$ is because $R(E_{\alpha_1}) \subset E_{\alpha_2}$ strictly. \hfill \qed

3. MINIMAL CONES FOR S = 0

In this section we derive the limiting value $\alpha_0 = \lim_{s \to 0} \alpha_s$ where $\alpha_s$ is such that $C_{\alpha_s}$ is an $s$-minimal cone.

**Proposition 3.1.** Assume that $n \leq m$ in $\mathbb{R}^n$, $N = m + n$. The number $\alpha_0$ is the unique solution to

$$\int_{\alpha}^{\infty} \frac{t^{n-1}}{(1 + t^2)^{\frac{3}{2}}} \, dt - \int_{0}^{\alpha} \frac{t^{n-1}}{(1 + t^2)^{\frac{3}{2}}} \, dt = 0.$$

**Proof.** We write $x = (y, z) \in \mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. Let us assume in the rest of the proof that $n \geq 2$. The case $n = 1$ is similar. We evaluate the integral in $\int_{\alpha}^{\infty}$ for the point $p = (e_1^{(m)}, \alpha e_1^{(n)})$ using spherical coordinates for $y = r\omega_1$ and $z = \rho\omega_2$ where $r, \rho > 0$ and

$$\omega_1 = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \cos(\theta_2) \\ \vdots \\ \sin(\theta_1) \sin(\theta_2) \ldots \sin(\theta_{m-2}) \cos(\theta_{m-1}) \\ \sin(\theta_1) \sin(\theta_2) \ldots \sin(\theta_{m-2}) \sin(\theta_{m-1}) \end{bmatrix},$$

$$\omega_2 = \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \cos(\varphi_2) \\ \vdots \\ \sin(\varphi_1) \sin(\varphi_2) \ldots \sin(\varphi_{n-2}) \cos(\varphi_{n-1}) \\ \sin(\varphi_1) \sin(\varphi_2) \ldots \sin(\varphi_{n-2}) \sin(\varphi_{n-1}) \end{bmatrix},$$

where $\theta_j \in [0, \pi]$ for $j = 1, \ldots, m-2$, $\theta_{m-1} \in [0, 2\pi]$, $\varphi_j \in [0, \pi]$ for $j = 1, \ldots, n-2$, $\varphi_{n-1} \in [0, 2\pi]$. Then

$$|(y, z) - (e_1^{(m)}, \alpha e_1^{(n)})|^2 = r^2 + 1 - 2r \cos(\theta_1) + \rho^2 + \alpha^2 - 2\rho \alpha \cos(\varphi_1).$$

Assuming that $\alpha = \alpha_s > 0$ is such that $C_{\alpha_s}$ is an $s$-minimal cone, $\int_{\alpha}^{\infty}$ yields the following equation for $\alpha$

$$p.v. \int_{0}^{\infty} r^{n-1}(A_{\alpha,s}(r) - B_{\alpha,s}(r)) \, dr = 0 \quad (3.3)$$
where

\[ A_{\alpha,s}(r) = \int_0^\infty \int_0^\infty \int_0^\pi \frac{\rho^{n-2} \sin(\theta_1)^m \sin(\varphi_1)^{n-2}}{(1 + \frac{1}{\rho^2} - \frac{2}{\rho} \cos(\theta_1) + t^2)} \, \rho^m \, t^n \, d\theta_1 \, d\varphi_1 \, d\rho \]

\[ B_{\alpha,s}(r) = \int_0^\infty \int_0^\infty \int_0^\pi \frac{\rho^{n-1} \sin(\theta_1)^{m-2} \sin(\varphi_1)^{n-2}}{(1 + \frac{1}{\rho^2} - \frac{2}{\rho} \cos(\theta_1) + t^2)} \, \rho^m \, t^n \, d\theta_1 \, d\varphi_1 \, d\rho, \]

which are well defined for \( r \neq 1 \). Setting \( \rho = rt \) we get

\[ A_{\alpha,s}(r) = r^{-m-s} \int_0^\infty \int_0^\pi \int_0^\pi \frac{t^{n-1} \sin(\varphi_1)^{m-2} \sin(\theta_1)^{n-2}}{(1 + t^2)^{\frac{n+s}{2}}} \, d\theta_1 \, d\varphi_1 \, dt \]

\[ = C_{m,n} r^{-m-s} \int_0^\infty \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt + O(r^{-m-s-1}) \]

as \( r \to \infty \) and this is uniform in \( s \) for \( s > 0 \) small. Here \( C_{m,n} > 0 \) is some constant. Similarly

\[ B_{\alpha,s}(r) = C_{m,n} r^{-m-s} \int_0^\infty \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt + O(r^{-m-s-1}) \]

Then (3.3) takes the form

\[ 0 = \int_0^\infty \int_0^\alpha \int_0^\pi \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt - \int_0^\infty \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt \]

where

\[ C_s(\alpha) = \int_0^\alpha \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt - \int_0^\infty \frac{t^{n-1}}{(1 + t^2)^{\frac{n+s}{2}}} \, dt \]

and \( O(1) \) is uniform as \( s \to 0 \), because \( 0 < \alpha_s \leq 1 \) by theorem (1) and the only singularity in (3.3) occurs at \( r = 1 \). This implies that \( \alpha_0 = \lim_{s \to 0} \alpha_s \) has to satisfy \( C_0(\alpha_0) = 0 \).

### 4. The Jacobi operator

In this section we prove formula (1.3) and derive the formula for the nonlocal Jacobi operator (1.4).

Let \( E \subset \mathbb{R}^N \) be an open set with smooth boundary and \( \Omega \) be a bounded open set. Let \( \nu \) be the unit normal vector field of \( \Sigma = \partial E \) pointing to the exterior of \( E \). Given \( h \in C_0^\infty(\Omega \cap \Sigma) \) and \( t \) small, let \( E_{th} \) be the set whose boundary \( \partial E_{th} \) is parametrized as

\[ \partial E_{th} = \{ x + th(x)\nu(x) \mid x \in \partial E \}, \]

with exterior normal vector close to \( \nu \).

**Proposition 4.1.** For \( h \in C_0^\infty(\Omega \cap \Sigma) \)

\[ \frac{d^2}{dt^2} \Per_s(E_{th}, \Omega) \bigg|_{t=0} = -2 \int_\Sigma J_2^s[h] h - \int_\Sigma h^2 \HH^s, \quad (4.1) \]

where \( J_2^s \) is the nonlocal Jacobi operator defined in (1.4), \( \HH^s \) is the classical mean curvature of \( \Sigma \) and \( H_2^s \) is the nonlocal mean curvature defined in (1.2).

In case that \( \Sigma \) is a nonlocal minimal surface in \( \Omega \) we obtain formula (1.3). Another related formula is the following.
Proposition 4.2. Let $\Sigma_{th} = \partial E_{th}$. For $p \in \Sigma$ fixed let $p_t = p + th(p)\nu(p) \in \Sigma_{th}$. Then for $h \in C^\infty(\Sigma) \cap L^\infty(\Sigma)$

$$\frac{d}{dt}H^s_{\Sigma_{th}}(p_t)\bigg|_{t=0} = 2J^s_\Sigma[h](p).$$ (4.2)

A consequence of proposition 4.2 is that entire nonlocal minimal graphs are stable.

Corollary 4.1. Suppose that $\Sigma = \partial E$ with

$$E = \{(x', F(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}\}$$

is a nonlocal minimal surface. Then

$$-\int_\Sigma J^s_\Sigma[h] h \geq 0 \quad \text{for all} \quad h \in C^\infty_0(\Sigma).$$ (4.3)

Proof of proposition 4.1

Let

$$K_\delta(z) = \frac{1}{|z|^{N+\delta}} \eta_\delta(z)$$

where $\eta_\delta(x) = \eta(x/\delta)$ ($\delta > 0$) and $\eta \in C^\infty(\mathbb{R}^N)$ is a radially symmetric cut-off function with $\eta(x) = 1$ for $|x| \geq 2$, $\eta(x) = 0$ for $|x| \leq 1$.

Consider

$$Per_{s,\delta}(E_{th}, \Omega) = \int_{E_{th} \cap \Omega} \int_{\Omega \setminus E_{th}} K_\delta(x - y) dy dx + \int_{E_{th} \setminus \Omega} \int_{\Omega \setminus E_{th}} K_\delta(x - y) dy dx.$$ (4.4)

We will show that $\frac{d^2}{dt}Per_{s,\delta}(E_{th}, \Omega)$ approaches a certain limit $D_2(t)$ as $\delta \to 0$, uniformly for $t$ in a neighborhood of 0 and that

$$D_2(0) = -2\int_\Sigma J^s_\Sigma[h] h - \int_\Sigma h^2 HH^s_\Sigma.$$ (4.3)

First we need some extensions of $\nu$ and $h$ to $\mathbb{R}^N$. To define them, let $K \subset \Sigma$ be the support of $h$ and $U_0$ be an open bounded neighborhood of $K$ such that for any $x \in U_0$, the closest point $\hat{x} \in \Sigma$ to $x$ is unique and defines a smooth function of $x$. We also take $U_0$ smaller if necessary as to have $\overline{U_0} \subset \Omega$. Let $\nu : \mathbb{R}^N \to \mathbb{R}^N$ be a globally defined smooth unit vector field such that $\nu(x) = \nu(\hat{x})$ for $x \in U_0$. We also extend $h$ to $\tilde{h} : \mathbb{R}^N \to \mathbb{R}$ such that it is smooth with compact support contained in $\Omega$ and $\tilde{h}(x) = h(\hat{x})$ for $x \in U_0$. From now one we omit the tildes (‘) in the definitions of the extensions of $\nu$ and $h$. For $t$ small $\bar{x} \mapsto \bar{x} + th(\bar{x})\nu(\bar{x})$ is a global diffeomorphism in $\mathbb{R}^N$. Let us write

$$u(\bar{x}) = h(\bar{x})\nu(\bar{x}) \quad \text{for} \quad \bar{x} \in \mathbb{R}^N,$$

$$\nu = (\nu^1, \ldots, \nu^N), \quad u = (u^1, \ldots, u^N)$$

and let

$$J_t(\bar{x}) = J_{id+tu}(\bar{x})$$

be the Jacobian determinant of $id + tu$.

We change variables

$$x = \bar{x} + tu(\bar{x}), \quad y = \bar{y} + tu(\bar{y}),$$

where $\bar{y} = \bar{y}(\bar{x})$.
in \([4.4]\)

\[
\begin{align*}
\text{Per}_{s,\delta}(E_{th}, \Omega) &= \int_{E \cap \phi_t(\Omega)} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) J_t(x) J_t(y) dy dx,
+ \int_{E \cap \phi_t(\Omega)} \int_{\phi_t(\Omega) \setminus E} K_\delta(x - y) J_t(y) dy dx,
\end{align*}
\]

where \(\phi_t\) is the inverse of the map \(\bar{x} \mapsto \bar{x} + tu(\bar{x})\).

Differentiating with respect to \(t\):

\[
\frac{d}{dt} \text{Per}_{s,\delta}(E_{th}, \Omega) = \int_{E \cap \phi_t(\Omega)} \int_{\mathbb{R}^N \setminus E} \left[ \nabla K_\delta(x - y)(u(\bar{x}) - u(\bar{y})) J_t(x) J_t(y) 
+ K_\delta(x - y)(J_t'(x) J_t(y) + J_t(x) J_t'(y)) \right] dy dx.
\]

where

\[
J_t'(x) = \frac{d}{dt} J_t(x).
\]

Note that there are no integrals on \(\partial \phi_t(\Omega)\) for \(t\) small because \(u\) vanishes in a neighborhood of \(\partial \Omega\).

Since the integrands in \(\frac{d}{dt} \text{Per}_{s,\delta}(E_{th}, \Omega)\) have compact support contained in \(\phi_t(\Omega)\) (\(t\) small), we can write

\[
\frac{d}{dt} \text{Per}_{s,\delta}(E_{th}, \Omega) = \int_{E} \int_{\mathbb{R}^N \setminus E} \left[ \nabla K_\delta(x - y)(u(\bar{x}) - u(\bar{y})) J_t(x) J_t(y) 
+ K_\delta(x - y)(J_t'(x) J_t(y) + J_t(x) J_t'(y)) \right] dy dx.
\]

Differentiating once more

\[
\frac{d^2}{dt^2} \text{Per}_{s,\delta}(E_{th}, \Omega) = A(\delta, t) + B(\delta, t) + C(\delta, t)
\]

where

\[
A(\delta, t) = \int_{E} \int_{\mathbb{R}^N \setminus E} D^2 K_\delta(x - y)(u(\bar{x}) - u(\bar{y}))(u(\bar{x}) - u(\bar{y})) J_t(x) J_t(y) dy dx,
\]

\[
B(\delta, t) = 2 \int_{E} \int_{\mathbb{R}^N \setminus E} \nabla K_\delta(x - y)(u(\bar{x}) - u(\bar{y}))(J_t'(x) J_t(y) + J_t(x) J_t'(y)) dy dx
\]

\[
C(\delta, t) = \int_{E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y)(J_t''(x) J_t(y) + 2 J_t'(x) J_t'(y) + J_t(x) J_t''(y)) dy dx.
\]

We claim that \(A(\delta, t), B(\delta, t)\) and \(C(\delta, t)\) converge as \(\delta \to 0\) for uniformly for \(t\) near 0, to limit expressions \(A(0, t), B(0, t)\) and \(C(0, t)\), which are the same as above replacing \(\delta\) by 0, and that the integrals appearing in \(A(0, t), B(0, t)\) and \(C(0, t)\) are well defined. Indeed, we can estimate

\[
|A(\delta, t) - A(0, t)| \leq C \int_{x \in E \cap K_\delta} \int_{y \in E^c, |x - y| \leq 2\delta} \frac{1}{|x - y|^{N+\alpha}} dy dx,
\]
where $K_0$ is a fixed bounded set. For $x \in E \cap K_0$ we see that
\[
\int_{y \in E^c, |x-y| \leq 2\delta} \frac{1}{|x-y|^{|N+s|}} dy \leq \frac{C}{\text{dist}(x,E^c)^s}
\]
and therefore
\[
|A(\delta,t) - A(0,t)| \leq C \int_{x \in E \cap K_0, \text{dist}(x,E^c) \leq 2\delta} \frac{1}{\text{dist}(x,E^c)^s} dx \leq C\delta^{1-s}.
\]
The differences $B(\delta,t) - B(0,t)$, $C(\delta,t) - C(0,t)$ can be estimated similarly. This shows that
\[
\frac{d}{dt}^2 \text{Per}_s(E_{t\delta}, \Omega) \bigg|_{t=0} = \lim_{\delta \to 0} \frac{d}{dt}^2 \text{Per}_s(x_{t\delta}, \Omega) \bigg|_{t=0} = \lim_{\delta \to 0} A(\delta,0) + B(\delta,0) + C(\delta,0).
\]
In what follows we will evaluate $A(\delta,0) + B(\delta,0) + C(\delta,0)$. At $t = 0$ we have
\[
A(\delta,0) = \int_E \int_{\mathbb{R}^N \setminus E} D_{x,x_j} K_\delta(x-y)(u^j(x) - u^i(y))(u^i(x) - u^j(y)) dy
dx = A_{11} + A_{12} + A_{21} + A_{22}
\]
where
\[
A_{11} = \int_E \int_{\mathbb{R}^N \setminus E} D_{x,x_j} K_\delta(x-y)u^i(x)u^j(x) dy
dx
\]
\[
A_{12} = -\int_E \int_{\mathbb{R}^N \setminus E} D_{x,x_j} K_\delta(x-y)u^i(x)u^j(y) dy
dx
\]
\[
A_{21} = -\int_E \int_{\mathbb{R}^N \setminus E} D_{x,x_j} K_\delta(x-y)u^i(y)u^j(x) dy
dx
\]
\[
A_{22} = \int_E \int_{\mathbb{R}^N \setminus E} D_{x,x_j} K_\delta(x-y)u^i(y)u^j(y) dy
dx.
\]
Let us also write
\[
B(\delta,0) = 2 \int_E \int_{\mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y)(u^j(x) - u^i(y))(\text{div}(u)(x) + \text{div}(u)(y)) dy
dx = B_{11} + B_{12} + B_{21} + B_{22},
\]
where
\[
B_{11} = 2 \int_E \int_{\mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y)u^i(x)\text{div}(u)(x) dy
dx
\]
\[
B_{12} = 2 \int_E \int_{\mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y)u^i(x)\text{div}(u)(y) dy
dx
\]
\[
B_{21} = -2 \int_E \int_{\mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y)u^i(y)\text{div}(u)(x) dy
dx
\]
\[
B_{22} = 2 \int_E \int_{\mathbb{R}^N \setminus E} D_{y_j} K_\delta(x-y)u^i(y)\text{div}(u)(y) dy
dx,
\]
and
\[
C(\delta,0) = C_1 + C_2 + C_3,
\]
where
\[ C_1 = \int_E \int_{E \cap \mathbb{R}^N \setminus E} K_\delta(x-y) \left( \text{div}(u)(x) - \text{div}(D(x)) \right) \, dy \, dx \]
\[ C_2 = \int_E \int_{E \cap \mathbb{R}^N \setminus E} \left( \frac{\text{div}(u)(y)}{2} - \text{div}(D(y)) \right) \, dy \, dx \]
\[ C_3 = 2 \int_E \int_{E \cap \mathbb{R}^N \setminus E} K_\delta(x-y) \text{div}(u)(x) \text{div}(u)(y) \, dy \, dx. \]

We compute
\[
A_{11} = \int_E \int_{E \cap \mathbb{R}^N \setminus E} D_{x_i} \left[ D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \right] \, dy \, dx
\]
\[ - \int_E \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) D_{x_i} \left[ u^i(x) u^j(x) \right] \, dy \, dx \]
\[ = \int_{\partial E} \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^j(x) \, dy \, dx
\]
\[ - \int_E \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) \left[ D_{x_i} u^i(x) u^j(x) + u^i(x) D_{x_j} u^j(x) \right] \, dy \, dx. \]

Therefore
\[
A_{11} + B_{11} = \int_{\partial E} \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^j(x) \, dy \, dx
\]
\[ + \int_E \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) \left[ D_{x_i} u^i(x) u^j(x) - u^i(x) D_{x_j} u^j(x) \right] \, dy \, dx. \]

We express the first term as
\[
\int_{\partial E} \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^j(x) \, dy \, dx
\]
\[ = - \int_{\partial E} \int_{E \cap \mathbb{R}^N \setminus E} D_{y_j} K_\delta(x-y) u^i(x) u^j(x) \nu^j(x) \, dy \, dx \]
\[ = \int_{\partial E} \int_{\partial E} K_\delta(x-y) u^i(x) u^j(x) \nu^j(x) \nu^j(y) \, dy \, dx \]
\[ = \int_{\partial E} \int_{\partial E} K_\delta(x-y) h(x)^2 \nu(x) \nu(y) \, dy \, dx. \]

For the second term of \( A_{11} + B_{11} \) let us write
\[
\int_E \int_{E \cap \mathbb{R}^N \setminus E} D_{x_j} K_\delta(x-y) D_{x_i} u^i(x) u^j(x) \, dy \, dx
\]
\[ = \int_E \int_{E \cap \mathbb{R}^N \setminus E} \left[ D_{x_j} K_\delta(x-y) D_{x_i} u^i(x) u^j(x) \right] \, dy \, dx \]
\[ - \int_E \int_{E \cap \mathbb{R}^N \setminus E} K_\delta(x-y) D_{x_j} \left[ D_{x_i} u^i(x) u^j(x) \right] \, dy \, dx \]
\[ = \int_{\partial E} \int_{E \cap \mathbb{R}^N \setminus E} K_\delta(x-y) D_{x_i} u^i(x) u^j(x) \nu^j(x) \, dy \, dx
\]
\[ - \int_E \int_{E \cap \mathbb{R}^N \setminus E} K_\delta(x-y) \left[ D_{x_i} u^i(x) u^j(x) + \text{div}(u)(x)^2 \right] \, dy \, dx. \]
The third term of $A_{11} + B_{11}$ is

\[-\int_E \int_{\mathbb{R}^N\setminus E} D_x \frac{\partial}{\partial x} K_\delta(x-y)u^i(x)D_x u^j(x) \, dy \, dx\]

\[= -\int_E \int_{\mathbb{R}^N\setminus E} D_x \left[K_\delta(x-y)u^i(x)D_x u^j(x) \right] \, dy \, dx \]

\[+ \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y)D_x \left[u^i(x)D_x u^j(x) \right] \, dy \, dx \]

\[= -\int_{\partial E} \int_{\mathbb{R}^N\setminus E} K_\delta(x-y)u^i(x)D_x u^j(x) \, dy \, dx \]

\[+ \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \left[ D_x u^i(x)D_x u^j(x) + u^i(x)D_{x,x} u^j(x) \right] \, dy \, dx. \]

Therefore

\[A_{11} + B_{11} = \int_{\partial E} \int_{\partial E} K_\delta(x-y)h(x)^2 \nu(x)\nu(y) \, dy \, dx \]

\[+ \int_{\partial E} \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \left[ D_x u^i(x)u^j(x)\nu^j(x) - u^i(x)D_x u^j(x)\nu^j(x) \right] \, dy \, dx \]

\[+ \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \left[ D_x u^i(x)D_x u^j(x) - \text{div}(u)(x)^2 \right] \, dy \, dx, \]

so that

\[A_{11} + B_{11} + C_1 = \int_{\partial E} \int_{\partial E} K_\delta(x-y)h(x)^2 \nu(x)\nu(y) \, dy \, dx \]

\[+ \int_{\partial E} \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \left[ D_x u^i(x)u^j(x)\nu^j(x) - u^i(x)D_x u^j(x)\nu^j(x) \right] \, dy \, dx. \]

But using $u = \nu h$ and $\text{div}(\nu) = H$ where $H$ is the mean curvature of $\partial E$ we have

\[D_x u^i(x)u^j(x)\nu^j(x) - u^i(x)D_x u^j(x)\nu^j(x) = h(x)^2 H(x)\]

and therefore

\[A_{11} + B_{11} + C_1 = \int_{\partial E} \int_{\partial E} K_\delta(x-y)h(x)^2 \nu(x)\nu(y) \, dy \, dx + \int_{\partial E} \int_{\mathbb{R}^N\setminus E} K_\delta(x-y)h(x)^2 H(x). \]

In a similar way, we have

\[A_{22} + B_{22} + C_2 = \int_{\partial E} \int_{\partial E} K_\delta(x-y)h(y)^2 \nu(x)\nu(y) \, dy \, dx \]

\[- \int_E \int_{\partial E} K_\delta(x-y) \left[ D_y u^i(y)u^j(y)\nu^j(y) - u^i(y)D_y u^j(y)\nu^j(y) \right] \, dy \, dx \]

\[= \int_{\partial E} \int_{\partial E} K_\delta(x-y)h(y)^2 \nu(x)\nu(y) \, dy \, dx - \int_E \int_{\partial E} K_\delta(x-y)h(y)^2 H(y) \, dy \, dx. \]
Further calculations show that

\[ A_{12} = -\int_{\partial E} \int_{\partial E} K_\delta(x - y) h(x)h(y) \, dy \, dx \]
\[ - \int_{\partial E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(y)u^i(x)) \nu^i(x) \, dy \, dx \]
\[ + \int_{E} \int_{\partial E} K_\delta(x - y) \text{div}(u(x)) \nu^j(y) \, dy \, dx \]
\[ + \int_{E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(x)) \text{div}(u(y)) \, dy \, dx, \]

and

\[ A_{21} = -\int_{\partial E} \int_{\partial E} K_\delta(x - y) h(x)h(y) \, dy \, dx \]
\[ - \int_{\partial E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(y)u^i(x)) \nu^j(x) \, dy \, dx \]
\[ + \int_{E} \int_{\partial E} K_\delta(x - y) \text{div}(u(x)) \nu^j(y) \, dy \, dx \]
\[ + \int_{E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(x)) \text{div}(u(y)) \, dy \, dx, \]

and

\[ B_{12} + B_{21} = 2 \int_{\partial E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(y)u^j(x)) \nu^j(x) \, dy \, dx \]
\[ - 2 \int_{E} \int_{\partial E} K_\delta(x - y) \text{div}(u(x)) \nu^j(y) \, dy \, dx \]
\[ - 4 \int_{E} \int_{\mathbb{R}^N \setminus E} K_\delta(x - y) \text{div}(u(x)) \text{div}(u(y)) \, dy \, dx, \]

so that

\[ A_{12} + A_{21} + B_{12} + B_{21} + C_3 = -2 \int_{\partial E} \int_{\partial E} K_\delta(x - y) h(x)h(y) \, dy \, dx. \]

Therefore

\[ \frac{d^2}{dt^2} \text{Per}_{s,\delta}(E_{th}, \Omega)|_{t=0} = 2 \int_{\partial E} \int_{\partial E} K_\delta(x - y) h(x)^2 (\nu(x) \nu(y) - 1) \, dy \, dx \]
\[ - 2 \int_{\partial E} h(x) \int_{\partial E} K_\delta(x - y) (h(y) - h(x)) \, dy \, dx \]
\[ - \int_{\partial E} h(x)^2 H(x) \int_{\mathbb{R}^N} \chi_E(y) \chi_{E^c}(y) K_\delta(x - y) \, dy \, dx. \]

Taking the limit as \( \delta \to 0 \) we find (4.3). □

**Proof of proposition 4.2.** Let \( \nu_t(x) \) denote the unit normal vector to \( \partial E_t \) at \( x \in \partial E_t \) pointing out of \( E_t \). Note that \( \nu(x) = \nu_0(x) \). Let \( L_t \) be the half space defined by \( L_t = \{ x : \langle x - p_t, \nu_t(p_t) \rangle > 0 \} \). Then

\[ H^s_{\text{th}}(p_t) = \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_t}(x) - \chi_{E^c}(x) + \chi_{L_t^c}(x)}{|x - p_t|^{N+s}} \, dx \quad (4.5) \]
We integrate the third term by parts

$$H^s_{2;1}(p_t) = 2 \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}} \, dx.$$ 

For $\delta > 0$ let $\eta \in C^\infty(\mathbb{R}^N)$ be a radially symmetric cut-off function with $\eta(x) = 1$ for $|x| \geq 2$, $\eta(x) = 0$ for $|x| \leq 1$. Define $\eta_\delta(x) = \eta(x/\delta)$ and write

$$\int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}} \eta_\delta(x - p_t) \, dx = f_\delta(t) + g_\delta(t)$$

where

$$f_\delta(t) = \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}} \eta_\delta(x - p_t) \, dx$$

and $g_\delta(t)$ is the rest. Then it is direct that $f_\delta$ is differentiable and

$$f'_\delta(0) = \int_{\partial E} \frac{h(x)}{|x - p|^{N+s}} \eta_\delta(x - p) \, d\nu - \int_{\partial L_0} \frac{h(p)(\nu(p), \nu(p))}{|x - p|^{N+s}} \eta_\delta(x - p) \, d\nu$$

$$+ (N + s)h(p) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x - p|^{N+s+2}} \langle x - p, \nu(p) \rangle \eta_\delta(x - p) \, dx$$

$$- h(p) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x - p|^{N+s}} \langle \nabla \eta_\delta(x - p), \nu(p) \rangle \, dx.$$ 

We integrate the third term by parts

$$(N + s) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x - p|^{N+s+2}} \langle x - p, \nu(p) \rangle \eta_\delta(x - p) \, dx$$

$$= - \int_{\mathbb{R}^N} \langle \chi_E(x) - \chi_{L_0}(x) \rangle \langle \nabla \frac{1}{|x - p|^{N+s}}, \nu(p) \rangle \eta_\delta(x - p) \, dx$$

$$= - \int_{\partial E} \langle \nu(x), \nu(p) \rangle \eta_\delta(x - p) + \int_{\partial L_0} \langle \nu(p), \nu(p) \rangle \eta_\delta(x - p)$$

$$+ \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x - p|^{N+s}} \langle \nabla \eta_\delta(x - p), \nu(p) \rangle \, dx.$$ 

Since $\eta_\delta$ is radially symmetric,

$$\int_{\partial L_0} \langle \frac{x - p, \nu(p)}{|x - p|^{N+s}} \rangle \eta_\delta(x - p) \, d\nu = 0$$

and then

$$f'_\delta(0) = \int_{\partial E} \frac{h(x)}{|x - p|^{N+s}} \eta_\delta(x - p) \, d\nu - h(p) \int_{\partial E} \langle \nu(x), \nu(p) \rangle \eta_\delta(x - p) \, dx$$

$$+ h(p) \int_{\partial E} \frac{1 - \langle \nu(x), \nu(p) \rangle}{|x - p|^{N+s}} \eta_\delta(x - p) \, dx.$$ 

We claim that $g'_\delta(t) \to 0$ as $\delta \to 0$, uniformly for $t$ in a neighborhood of 0. Indeed, in a neighborhood of $p_t$ we can represent $\partial E_t$ as a graph of a function $G_t$ over $L_t$ ∩
Therefore the same argument as in the proof of proposition 4.2 shows that if \( F \) and smooth in all its variables (we write \( \delta \to 0 \) so that
\[
B(p_t, 2\delta), \quad \text{with } G_t \text{ defined in a neighborhood of 0 in } \mathbb{R}^{N-1}, \quad G_t(0) = 0, \quad \nabla_y G_t(0) = 0
\]
and smooth in all its variables (we write \( \delta \to 0 \) so that
\[
\frac{1}{|y'|^2 + G_t(y')^2} \frac{\partial G_t}{\partial t}(1 - \eta_0(y', y_N))dy_Ndy'.
\]
so that
\[
g_s(t) = \int_{|y'| < \delta} \int_0^{G_t(y')} \frac{1}{(|y'|^2 + G_t(y')^2) \frac{\partial G_t}{\partial t}(1 - \eta_0(y', y_N))dy_Ndy'.
\]
But \( |G_t(y')| \leq K|y'|^2 \) and \( \frac{1}{|y'|^2 + G_t(y')^2} \leq K|y'|^2 \), so
\[
g_s(t) \leq C\delta^{1-s}.
\]
Therefore
\[
\frac{d}{dt} H_{E_{th}}^S(p_t) \bigg|_{t=0} = 2 \lim_{\delta \to 0} \left[ \int_{\partial E} \frac{h(x) - h(p)}{|x - p|^{N+s}} \eta_0(x - p)dx + \frac{1 - \langle \nu(x), \nu(p) \rangle}{|x - p|^{N+s}} \eta_0(x - p)dx \right].
\]
Letting \( \delta \to 0 \) we find (4.2). \( \square \)

**Proof of corollary 4.1.** The same argument as in the proof of proposition 4.2 shows that if \( F : \Sigma \to \mathbb{R}^N \) is a smooth bounded vector field and we let \( E_t \) be the set whose boundary \( \Sigma_t = \partial E_t \) is parametrized as
\[
\partial E_{th} = \{ x + tF(x) / x \in \partial E \},
\]
with exterior normal vector close to \( \nu \), then
\[
\frac{d}{dt} H_{E_{th}}^S(p_t) \bigg|_{t=0} = 2\mathcal{J}_E^S[(F, \nu)](p),
\]
where \( p_t = p + tF(p) \). Taking as \( F(x) = e_N = (0, \ldots, 0, 1) \) we conclude that
\[
\mathcal{J}_E^S[w](x) = 0 \quad \text{for all } x \in \Sigma.
\]
More explicitly
\[
p.v. \int_{\Sigma} \frac{w(y) - w(x)}{|y - x|^{N+s}} dy + w(x)A(x) = 0 \quad \text{for all } x \in \Sigma,
\]
where
\[
A(x) = \int_{\Sigma} \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^{N+s}} dy.
\]

As in the classical setting we can show that \( \Sigma \) is stable in the sense that (4.3) holds. Let \( \phi \in C^\infty_0(\Sigma) \) and observe that
\[
\frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{N+s}} dxdy = \int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))\phi(x)}{|x - y|^{N+s}} dxdy.
\]
Write \( \phi = w\psi \) with \( \psi \in C^\infty_0(\Sigma) \). Then
\[
\int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))\phi(x)}{|x - y|^{N+s}} dxdy = \int_{\Sigma} \int_{\Sigma} \frac{(w(x) - w(y))w(x)\psi(x)^2}{|x - y|^{N+s}} dxdy
\]
\[+ \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))w(x)w(y)\psi(x)}{|x - y|^{N+s}} dxdy. \quad (4.7)
\]
Multiplying (4.6) by $w\psi^2$ and integrating we get
\[
\int_{\Sigma} \int_{\Sigma} \frac{(w(x) - w(y))w(x)w(y)\psi(x)^2}{|x-y|^{N+s}} \, dx \, dy = \int_{\Sigma} A(x)w(x)^2 \psi(x)^2 \, dx = \int_{\Sigma} A(x)\phi(x)^2 \, dx.
\]
(4.8)

For the second term in (4.7) we observe that
\[
\int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))w(x)w(y)\psi(x)}{|x-y|^{N+s}} \, dx \, dy = \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy.
\]
(4.9)

Therefore, combining (4.7), (4.8), (4.9) we obtain
\[
\frac{1}{2} \int_{\Sigma} \int_{\Sigma} (\phi(x) - \phi(y))^2 \, dx \, dy = \int_{\Sigma} A(x)\phi(x)^2 \, dx
\]
\[
+ \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy.
\]
and this shows (4.3).

\[
\frac{1}{2} \int_{\Sigma} \int_{\Sigma} (\phi(x) - \phi(y))^2 \, dx \, dy = \int_{\Sigma} A(x)\phi(x)^2 \, dx
\]

\[
+ \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy.
\]

5. Stability and Instability

We consider the nonlocal minimal cone $C^m_n(s) = \partial E_\alpha$ where $E_\alpha$ is defined in (2.1) and $\alpha$ is the one of theorem 1. For $0 \leq s < 1$ we obtain a characterization of their stability in terms of constants that depend on $m$, $n$ and $s$. For the case $s = 0$ we consider the limiting cone with parameter $\alpha_0$ given in proposition 3.1.

Note that in the case $s = 0$ the limiting Jacobi operator $J^0_{E_{\alpha_0}}$ is well defined for smooth functions with compact support.

For brevity, in this section we write $\Sigma = C^m_n(s)$.

Recall that
\[
J_\Sigma^2[\phi](x) = \text{p.v.} \int_{\Sigma} \frac{\phi(y) - \phi(x)}{|y-x|^{N+s}} \, dy + \phi(x) \int_{\Sigma} \frac{1 - \langle \nu(x), \nu(y) \rangle}{|x-y|^{N+s}} \, dy
\]
for $\phi \in C^\infty_0(\Sigma \setminus \{0\})$. Let us rewrite this operator in the form
\[
J_\Sigma^2[\phi](x) = \text{p.v.} \int_{\Sigma} \frac{\phi(y) - \phi(x)}{|x-y|^{N+s}} \, dy + \frac{A_0(m,n,s)^2}{|x|^{1+s}} \phi(x)
\]
where
\[
A_0(m,n,s)^2 = \int_{\Sigma} \frac{\langle \nu(\hat{\rho}) - \nu(x), \nu(\hat{\rho}) \rangle}{|\hat{\rho} - x|^{N+s}} \, dx \geq 0
\]
and this integral is evaluated at any $\hat{\rho} \in \Sigma$ with $|\hat{\rho}| = 1$. We can think of $J_\Sigma^2$ as analogous to the fractional Hardy operator
\[
-(-\Delta)^{\frac{1+s}{2}} \phi + \frac{C}{|x|^{1+s}} \phi \quad \text{in } \mathbb{R}^{N-1},
\]
for which positivity is related to a fractional Hardy inequality with best constant, see Herbst [10]. This suggests that the positivity of $J_\Sigma^2$ is related to the existence of $\beta$ in an appropriate range such that $J_{\Sigma}^2[|x|^{-\beta}] \leq 0$, and it turns out that the best choice of $\beta$ is $\beta = \frac{N-2-s}{2}$. This motivates the definition
\[
H(m,n,s) = \text{p.v.} \int_{\Sigma} \frac{1 - |y|^{-\frac{N-2-s}{2}}}{|\hat{\rho} - y|^{N+s}} \, dy
\]
where \( \hat{p} \in \Sigma \) is any point with \( |\hat{p}| = 1 \).

We have then the following Hardy inequality with best constant:

**Proposition 5.1.** For any \( \phi \in C_0^{\infty}(\Sigma \setminus \{0\}) \) we have

\[
H(m, n, s) \int_{\Sigma} \frac{(\phi(x))^2}{|x|^{1+s}} \, dx \leq \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} \, dx \, dy \tag{5.1}
\]

and \( H(m, n, s) \) is the best possible constant in this inequality.

As a result we have:

**Corollary 5.1.** The cone \( C^m_n(s) \) is stable if and only if \( H(m, n, s) \geq A_0(m, n, s)^2 \).

Other related fractional Hardy inequalities have appeared in the literature, see for instance [2, 9].

**Proof of proposition 5.1.** Let us write \( H = H(m, n, s) \) for simplicity. To prove the validity of (5.1) let \( w(x) = |x|^\beta \) with \( \beta = \frac{N-2-s}{2} \) so that from the definition of \( H \) and homogeneity we have

\[
p.v. \int_{\Sigma} \frac{w(y) - w(x)}{|x-y|^{N+s}} \, dy + \frac{H}{|x|^{1+s}} w(x) = 0 \quad \text{for all } x \in \Sigma \setminus \{0\}.
\]

Now the same argument as in the proof of corollary [11] shows that

\[
\frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} \, dx \, dy = \int_{\Sigma} \frac{H}{|x|^{1+s}} \phi(x)^2 \, dx \tag{5.2}
\]

\[
+ \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy.
\]

for all \( \phi \in C_0^{\infty}(\Sigma \setminus \{0\}) \) with \( \psi = \frac{\phi}{\tilde{w}} \in C_0^{\infty}(\Sigma \setminus \{0\}) \).

Now let us show that \( H \) is the best possible constant in (5.1). Assume that

\[
\tilde{H} \int_{\Sigma} \frac{\phi(x)^2}{|x|^{1+s}} \, dx \leq \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} \, dx \, dy
\]

for all \( \phi \in C_0^{\infty}(\Sigma \setminus \{0\}) \). Using (5.2) and letting \( \phi = w\psi \) with \( \psi \in C_0^{\infty}(\Sigma \setminus \{0\}) \) we then have

\[
\tilde{H} \int_{\Sigma} \frac{w(x)^2\psi(x)^2}{|x|^{1+s}} \, dx \leq H \int_{\Sigma} \frac{w(x)^2\psi(x)^2}{|x|^{1+s}} \, dx
\]

\[
+ \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(\psi(x) - \psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy.
\]

For \( R > 3 \) let \( \psi_R : \Sigma \to [0, 1] \) be a radial function such that \( \psi_R(x) = 0 \) for \( |x| \leq 1 \), \( \psi_R(x) = 1 \) for \( 2 \leq |x| \leq 2R \), \( \psi_R(x) = 0 \) for \( |x| \geq 3R \). We also require \( |\nabla \psi_R(x)| \leq C \) for \( |x| \leq 3 \), \( |\nabla \psi_R(x)| \leq C/R \) for \( 2R \leq |x| \leq 3R \). We claim that

\[
a_0 \log(R) - C \leq \int_{\Sigma} \frac{w(x)^2\psi_R(x)^2}{|x|^{1+s}} \, dx \leq a_0 \log(R) + C \tag{5.3}
\]

where \( a_0 > 0 \), \( C > 0 \) are independent of \( R \), while

\[
\left| \int_{\Sigma} \int_{\Sigma} \frac{(\psi_R(x) - \psi_R(y))^2 w(x)w(y)}{|x-y|^{N+s}} \, dx \, dy \right| \leq C. \tag{5.4}
\]

Letting then \( R \to \infty \) we deduce that \( \tilde{H} \leq H \).

To prove the upper bound in (5.3) let us write points in \( \Sigma \) as \( x = (y, z) \), with \( y \in \mathbb{R}^m \), \( z \in \mathbb{R}^n \). Let us write \( y = r\omega_1 \), \( z = r\omega_2 \), with \( r > 0 \), \( \omega_1 \in S^{m-1} \), \( \omega_2 \in S^{n-1} \).
and use spherical coordinates \((\theta_1, \ldots, \theta_{m-1})\) and \((\varphi_1, \ldots, \varphi_{n-1})\) for \(\omega_1\) and \(\omega_2\) as in \((3.1)\) and \((3.2)\). We assume here that \(m \geq n \geq 2\). In the remaining cases the computations are similar. Then we have

\[
\int_{\Sigma} \frac{w(x)^2 \psi_R(x)^2}{|x|^{1+s}} \, dx \leq a_0 \int_1^{4R} \frac{1}{r^{N-2-s}} \frac{1}{y^{1+s}} \, r^{N-2} \, dr \leq a_0 \log(R) + C
\]

where

\[
a_0 = \sqrt{1 + \alpha^2 A_{m-1} A_{n-1}}
\]

and \(A_k\) denotes the area of the sphere \(S^k \subseteq \mathbb{R}^{k+1}\) and is given by

\[
A_k = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}.
\]

The lower bound in \((5.3)\) is similar.

To obtain \((5.4)\) we split \(\Sigma\) into the regions \(R_1 = \{x : |x| \leq 3\}, R_2 = \{x : 3 \leq x \leq R\}, R_3 = \{x : R \leq |x| \leq 4R\}\) and \(R_4 = \{x : |x| \geq 4R\}\) and let

\[
I_{i,j} = \int_{x \in R_i} \int_{y \in R_j} \frac{(\psi_R(x) - \psi_R(y))^2}{|x-y|^{N+s}} w(x) w(y) \, dx \, dy.
\]

Then \(I_{i,j} = I_{j,i}\) and \(I_{i,j} = 0\) for \(j = 2, 4\). Moreover \(I_{1,1} = O(1)\) since the region of integration is bounded and \(\psi_R\) is uniformly Lipschitz.

Estimate of \(I_{1,2}\): We bound \(w(x) \leq C\) for \(|x| \geq 1\) and then

\[
|I_{1,2}| \leq C \int_{y \in R_2} \frac{w(y)}{|y|^{N+s}} \, dy \leq C \int_2^R \frac{1}{r^{x-\frac{1}{2}}} \frac{1}{y^{N+s}} \, y^{N-2} \, dy \leq C,
\]

where \(p \in \Sigma\) is fixed with \(|p| = 2\).

By the same argument \(I_{1.3} = O(1)\) and \(I_{1,4} = O(1)\) as \(R \to \infty\).

Estimate of \(I_{2,3}\): for \(y \in R_3\), \(w(y) \leq CR^{-\frac{N-2-s}{2}}\), so

\[
|I_{2,3}| \leq CR^{-\frac{N-2-s}{2}} \int_{x \in R_2} \frac{1}{|x|^{N-\frac{1}{2}}} \int_{y \in R_3} \frac{(\psi_R(x) - \psi_R(y))^2}{|x-y|^{N+s}} \, dy \, dx
\]

\[
\leq CR^{-\frac{N-2-s}{2}} \frac{Vol(R_3)}{R^{N+s}} \int_{x \in R_2} \frac{1}{|x|^{\frac{N-1}{2}}} \, dx \leq C.
\]

Estimate of \(I_{2,4}\):

\[
|I_{2,4}| \leq C \int_{x \in R_2} \frac{1}{|x|^{\frac{N-1}{2}}} \int_{y \in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-1}{2}}} \, dy \, dx.
\]

By scaling

\[
\int_{y \in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-1}{2}}} \, dy \leq CR^{-\frac{N-1}{2}} \quad \text{for } x \in R_2,
\]

so that

\[
|I_{2,4}| \leq CR^{-\frac{N-1}{2}} \int_{x \in R_2} \frac{1}{|x|^{\frac{N-1}{2}}} \, dx \leq C.
\]

To estimate \(I_{3,3}\) we use \(|\psi_R(x) - \psi_R(y)| \leq C|x-y|\) for \(x, y \in R_3\), which yields

\[
|I_{3,3}| \leq \frac{C}{R^2} \int_{x, y \in R_3} \frac{1}{|x-y|^{N+s-2}} \, dy \, dx.
\]
The integral is finite and by scaling we see that is bounded by $C R^{N-s}$, so that

$$|I_{3,3}| \leq C.$$ 

Estimate of $I_{3,4}$:

$$|I_{3,4}| \leq C R^{-\frac{N-2-s}{2}} \int_{x \in R_3} \int_{y \in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{n+2}} dy dx.$$ 

By scaling

$$\int_{y \in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{n+2}} dy \leq \frac{C}{|x|^{n+2}}$$

for $x \in R_3$. Therefore

$$|I_{3,4}| \leq C R^{-\frac{N-2-s}{2}} \int_{x \in R_3} \frac{1}{|x|^{n+2}} dx \leq C.$$ 

This concludes the proof of (5.4).

**Proof of Theorem 2.** In what follows we will obtain expressions for $H(m, n, s)$ and $A_0(m, n, s)^2$ for $m \geq 2$, $n \geq 1$, $0 \leq s < 1$. We always assume $m \geq n$. For the sake of generality, we will compute

$$C(m, n, s, \beta) = \text{p.v.} \int_{\Sigma} \frac{1 - |\tilde{p} - x|^{-\beta}}{|\tilde{p} - x|^{N+s}} dx$$

where $\tilde{p} \in \Sigma$, $|\tilde{p}| = 1$, and $\beta \in (0, N-2-s)$, so that $H(m, n, s) = C(m, n, s, \frac{N-2-s}{2})$.

Let $x = (y, z) \in \Sigma$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. For simplicity in the next formulas we take $p = (e^{(m)}_1, \alpha e^{(n)}_k)$ (see the notation in (2.1)), and $h(y, z) = |y|^{-\beta}$, so that

$$C(m, n, s, \beta) = (1 + \alpha^2)^{\frac{m+4}{4}} \text{p.v.} \int_{\Sigma} \frac{h(p) - h(x)}{|p - x|^{N+s}} dx.$$ 

**Computation of $C(m, 1, s, \beta)$**. Write $y = r\omega_1$, $z = \pm \alpha r\omega_1$, with $r > 0$, $\omega_1 \in S^{m-1}$. Let us use the notation $\Sigma^+_\alpha = \Sigma \cap \{z > 0\}$, $\Sigma^-_\alpha = \Sigma \cap \{z < 0\}$. Using polar coordinates $(\theta_1, \ldots, \theta_{m-1})$ for $\omega_1$ as in (5.1) we have

$$|x - p|^2 = |r \theta_1 - e^{(m)}_1|^2 + \alpha^2 |r \theta_1 - e^{(m)}_1|^2 = r^2 + 1 - 2r \cos(\theta_1) + \alpha^2 (r+1)^2,$$

for $x \in \Sigma^+_\alpha$ and

$$|x - p|^2 = |r \theta_1 - e^{(m)}_1|^2 + \alpha^2 |r \theta_1 - e^{(m)}_1|^2 = r^2 + 1 - 2r \cos(\theta_1) + \alpha^2 (r+1)^2,$$

for $x \in \Sigma^-_\alpha$. Hence, with $h(y, z) = |y|^{-\beta}$

$$\text{p.v.} \int_{\Sigma} \frac{h(p) - h(x)}{|x - p|^{N+s}} dx = \sqrt{1 + \alpha^2} A_{m-2} \text{p.v.} \int_0^\infty (1 - r^{-\beta})(I_+(r) + I_-(r)) r^{N-2} dr$$

(5.6)

where

$$I_+(r) = \int_0^\pi \frac{\sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r \cos(\theta_1) + \alpha^2 (r+1)^2)^{\frac{n+2}{2}}} d\theta_1$$

$$I_-(r) = \frac{\sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r \cos(\theta_1) + \alpha^2 (r+1)^2)^{\frac{n+2}{2}}} d\theta_1,$$
and $A_{m-2}$ is defined in (5.5) for $m \geq 2$. From (5.6) we obtain

$$C(m, 1, s, \beta) = (1 + \alpha^2) \frac{2\pi}{A_{m-2}} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s})(I_+(r) + I_-(r))dr.$$  

(5.7)

**Computation of $A_0(m, 1, s)^2$.** Let $x = (r\theta_1, \pm\alpha r), p = (e_1^{(n)}, \alpha)$ so that

$$\nu(x) = \frac{(-\alpha\omega_1, \pm 1)}{\sqrt{1 + \alpha^2}}, \quad \nu(p) = \frac{(-\alpha e_1^{(n)}, 1)}{\sqrt{1 + \alpha^2}},$$

and hence

$$\int_{\Sigma} \frac{1 - \langle \nu(x), \nu(p) \rangle}{|p - x|^{N+s}} dx = \sqrt{1 + \alpha^2} A_{m-2} \int_0^\infty (J_+(r) + J_-(r)) r^{N-2} dr$$

$$= \sqrt{1 + \alpha^2} A_{m-2} \int_0^1 (r^{N-2} + r^s)(J_+(r) + J_-(r))dr,$$

where

$$J_+(r) = \frac{\alpha^2}{1 + \alpha^2} \int_0^\pi \frac{(1 - \cos(\theta_1)) \sin(\theta_1)^{m-2}}{(r^2 + 1 - 2\cos(\theta_1) + \alpha^2(r - 1)^2)^{N+2}} d\theta_1,$$

$$J_-(r) = \frac{1}{1 + \alpha^2} \int_0^\pi \frac{2 \alpha^2 - \alpha^2 \cos(\theta_1) \sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r \cos(\theta_1) + \alpha^2(r + 1)^2)^{N+2}} d\theta_1.$$

Therefore we find

$$A_0(m, 1, s)^2 = (1 + \alpha^2) \frac{2\pi}{A_{m-2}} \int_0^1 (r^{N-2} + r^s)(J_+(r) + J_-(r))dr.$$  

**Computation of $C(m, n, s, \beta)$ for $n \geq 2$.** Write $y = r\omega_1, z = r\omega_2, w > 0, \omega_1 \in S^{m-1}, \omega_2 \in S^{n-1}$ and let us use spherical coordinates $(\theta_1, \ldots, \theta_{m-1})$ and $(\varphi_1, \ldots, \varphi_{n-1})$ for $\omega_1$ and $\omega_2$ as in (5.1) and (5.2). Recalling that $p = (e_1^{(m)}, \alpha e_2^{(n)})$, we have

$$|x - p|^2 = |r\theta_1 - e_1^{(m)}|^2 + |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r \cos(\theta_1) + \alpha^2(r^2 + 1 - 2r \cos(\varphi_1)).$$

Hence, with $h(y, z) = |y|^{-\beta}$

$$\text{p.v.} \int_{\Sigma} \frac{h(p) - h(x)}{|x - p|^N} dx = \sqrt{1 + \alpha^2} A_{m-2} A_{n-2} \text{p.v.} \int_0^\infty (1 - r^{-\beta}) I(r) r^{N-2} dr$$

$$= \sqrt{1 + \alpha^2} A_{m-2} A_{n-2} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s}) I(r) dr$$

where

$$I(r) = \int_0^\pi \int_0^\pi \frac{\sin(\theta_1)^{m-2} \sin(\varphi_1)^{n-2}}{(r^2 + 1 - 2r \cos(\theta_1) + \alpha^2(r^2 + 1 - 2r \cos(\varphi_1)))^{N+2}} d\theta_1 d\varphi_1.$$  

We find then that

$$C(m, n, s, \beta) = (1 + \alpha) \frac{2\pi}{A_{m-2} A_{n-2}} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s}) I(r) dr.$$  

(5.8)

**Computation of $A_0(m, n, s)^2$ for $n \geq 2$.** Similarly as before we have, for $x =
Table 1. Values of \( H(m, n, 0) \) and \( A_0(m, n, 0)^2 \) divided by \((1 + \alpha^2)^{\frac{3s}{2}} A_{m-2} A_{n-2}\)

\[
\begin{array}{cccccccc}
m & n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & H & 0.8140 & 1.0679 & 0.8140 & 1.0679 & 0.8140 & 1.0679 & 0.8140 \\
A_0^2 & 3.2669 & 2.3015 & 3.2669 & 2.3015 & 3.2669 & 2.3015 & 3.2669 \\
3 & H & 1.1978 & 1.2346 & 0.3926 & 0.3926 & 0.3926 & 0.3926 & 0.3926 \\
A_0^2 & 2.5984 & 1.7918 & 0.4463 & 0.4463 & 0.4463 & 0.4463 & 0.4463 \\
4 & H & 1.3968 & 1.3649 & 0.4477 & 0.4477 & 0.4477 & 0.4477 & 0.4477 \\
A_0^2 & 2.0413 & 1.5534 & 0.4288 & 0.4288 & 0.4288 & 0.4288 & 0.4288 \\
5 & H & 1.5117 & 1.4570 & 0.4895 & 0.4895 & 0.4895 & 0.4895 & 0.4895 \\
A_0^2 & 1.7332 & 1.3981 & 0.4118 & 0.4118 & 0.4118 & 0.4118 & 0.4118 \\
6 & H & 1.5833 & 1.5231 & 0.5215 & 0.5215 & 0.5215 & 0.5215 & 0.5215 \\
A_0^2 & 1.5318 & 1.2841 & 0.3955 & 0.3955 & 0.3955 & 0.3955 & 0.3955 \\
7 & H & 1.6303 & 1.5719 & 0.5465 & 0.5465 & 0.5465 & 0.5465 & 0.5465 \\
A_0^2 & 1.3872 & 1.1951 & 0.3802 & 0.3802 & 0.3802 & 0.3802 & 0.3802 \\
\end{array}
\]

\((\omega_1, \alpha \omega_2) \in \Sigma, \) and \( p = (e_1^{(m)}, e_2^{(n)}) \):

\[
\nu(x) = \frac{(-\alpha \omega_1, \omega_2)}{\sqrt{1 + \alpha^2}}, \quad \nu(p) = \frac{(-\alpha e_1^{(n)}, 1)}{\sqrt{1 + \alpha^2}}.
\]

Hence

\[
\int_{\Sigma} \frac{1 - (\nu(x), \nu(p))}{|p - x|^{N+s}} \, dx = \sqrt{1 + \alpha^2} \int_0^{\infty} r^{N-2} J(r) \, dr
\]

\[
= \sqrt{1 + \alpha^2} \int_0^{\infty} (r^{N-2} + r^s) J(r) \, dr
\]

where

\[
J(r) = \frac{1}{1 + \alpha^2} \int_0^\pi \int_0^\pi \frac{(1 + \alpha^2 - \alpha^2 \cos(\theta_1) - \cos(\varphi_1)) \sin(\theta_1) \sin(\varphi_1)^{n-2}}{\cos(\varphi_1)^{n-2} + 1 + 2r \cos(\varphi_1)} \, d\theta_1 d\varphi_1.
\]

We finally obtain

\[
A_0(m, n, s)^2 = (1 + \alpha^2)^{\frac{3s}{2}} A_{m-2} A_{n-2} \int_0^1 (r^{N-2} + r^s) J(r) \, dr.
\]

In table 1 we show the values obtained for \( H(m, n, 0) \) and \( A_0(m, n, 0)^2 \), divided by \((1 + \alpha^2)^{\frac{3s}{2}} A_{m-2} A_{n-2}\), from numerical approximation of the integrals. From these results we can say that for \( s = 0 \), \( \Sigma \) is stable if \( n + m = 7 \) and unstable if \( n + m \leq 6 \). The same holds for \( s > 0 \) close to zero by continuity of the values with respect to \( s \).

\[\Box\]

**Remark 5.1.** We see from formulas [5.7] and [5.8] that \( C(m, n, s, \beta) \) is symmetric with respect to \( \frac{N-2-s}{2} \) and is maximized for \( \beta = \frac{N-2-s}{2} \).

**Remark 5.2.** In table 2 we give some numerical values of \( \alpha \), \( H(m, n, s) \) and \( A_0(m, n, s)^2 \) divided by \((1 + \alpha^2)^{\frac{3s}{2}} A_{m-2} A_{n-2}\) for \( m = 4, n = 3 \), which show how in this dimension stability depends on \( s \). One may conjecture that there is \( s_0 \) such that the cone is stable for \( 0 \leq s \leq s_0 \) and unstable for \( s_0 < s < 1 \).
Table 2. Values of $H(m,n,s)$ and $A_0(m,n,s)^2$ divided by $(1 + \alpha^2)^{1/2}A_{m-2}A_{n-2}$ for $m = 4$, $n = 3$.

| $s$  | 0.1    | 0.2    | 0.3    | 0.4    |
|------|--------|--------|--------|--------|
| $\alpha$ | 0.8379 | 0.8361 | 0.8341 | 0.8319 |
| $H(4,3,s)$ | 0.4113 | 0.3856 | 0.3699 | 0.3639 |
| $A_0(4,3,s)^2$ | 0.4007 | 0.3830 | 0.3756 | 0.3786 |

References

[1] D. Benarros, M. Miranda, Lawson cones and the Bernstein theorem. Advances in geometric analysis and continuum mechanics (Stanford, CA, 1993), 44–56, Int. Press, Cambridge, MA, 1995.

[2] K. Bogdan, B. Dyda, The best constant in a fractional Hardy inequality. Math. Nachr. 284 (2011), no. 5-6, 629–638.

[3] E. Bombieri E. de Giorgi, E. Giusti, Minimal cones and the Bernstein problem, Invent. Math., 7 (1969), pp. 243-268.

[4] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces. Comm. pure Appl. Math. 63 (2010), no. 9, 1111–1144.

[5] L. Caffarelli, E. Valdinoci, Uniform estimates and limiting arguments for nonlocal minimal surfaces. Calc. Var. partial Differential Equations 41 (2011), no. 1-2, 203–240.

[6] P. Conczi, M. Miranda, MACSYMA and minimal surfaces. Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), 163–169, Proc. Sympos. Pure Math., 44, Amer. Math. Soc., Providence, RI, 1986.

[7] J. Davila, M. del Pino and J. Wei, Examples of nonlocal minimal surfaces, preprint.

[8] A. Davini, On calibrations for Lawson’s cones. Rend. Sem. Mat. Univ. Padova 111 (2004), 55–70.

[9] B. Dyda, R.L. Frank, Fractional Hardy-Sobolev-Maz’ya inequality for domains. Studia Math. 208 (2012), no. 2, 151–166.

[10] I.W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. Comm. Math. Phys. 53 (1977), no. 3, 285–294.

[11] H. B. Lawson Jr., The equivariant Plateau problem and interior regularity. Trans. Amer. Math. Soc., 173 (1972), pp. 231–249.

[12] M. Miranda, Grafici minimi completi. Ann. Univ. Ferrara Sez. VII (N.S.) 23 (1977), 269–272 (1978).

[13] O. Savin, E. Valdinoci, Regularity of nonlocal minimal cones in dimension 2. preprint 2012.

[14] P. Simoes, A class of minimal cones in $R^N$, $N \geq 8$, that minimize area, Ph. D. Thesis, University of California, (Berkeley, CA, 1973).

[15] J. Simons, Minimal varieties in riemannian manifolds. Ann. of Math. (2) 88 (1968), 62–105.

[16] E. Valdinoci, A fractional framework for perimeters and phase transitions, arXiv:1210.5612v1.

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