Glueball spectrum and hadronic processes in low-energy QCD

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Abstract

Low-energy limit of quantum chromodynamics (QCD) is obtained using a mapping theorem recently proved. This theorem states that, classically, solutions of a massless quartic scalar field theory are approximate solutions of Yang-Mills equations in the limit of the gauge coupling going to infinity. Low-energy QCD is described by a Yukawa theory further reducible to a Nambu-Jona-Lasinio model. At the leading order one can compute glue-quark interactions and one is able to calculate the properties of the $\sigma$ and $\eta - \eta'$ mesons. Finally, it is seen that all the physics of strong interactions, both in the infrared and ultraviolet limit, is described by a single constant $\Lambda$ arising from dimensional transmutation and in the infrared as an integration constant.

Keywords:

1. Introduction

Understanding low-energy QCD is a key to uncover the structure of light unflavored mesons. To paraphrase Luciano Maiani: “If these are all tetraquark states, where are exotic states?”[1]. We will give a first answer to this question.

KLOE-2 measurements[2] seem to rule out quark contributions to the structure of lighter mesons. Similarly, their results support a glue component in $\eta'$ meson that, having a glue component in the structure, is seen to possibly emit a glue state that finally decays in two pions. Consistency is only obtained if the emitted glue state is a $\sigma$ meson.

Structure of $\sigma$ resonance is hotly debated. Common view is that this meson should be a tetraquark state, member of a low-lying nonet. This state has not been seen yet in lattice computations either quenched or not. Recent analysis on $\gamma \gamma$ decay and data from NA48/2[3] seem to support the idea that this is a glue state rather than a quark composite particle.

Similarly, $f_0(980)$ appears to be a possible glue state and can be seen as an excited state of $\sigma$. Mass of this state should take into account its KK decay.

All properties of hadronic processes must be consistent with a single parameter of the theory, $\Lambda$, representing a constant arising in ultraviolet limit by dimensional transmutation and in the infrared limit as an integration constant of the theory. Theory must be working with a single $\Lambda$ parameter, even if some authors admit a possible dependency on the energy scale[4]. Our aim here is to fix this constant from a low-energy theoretical analysis.

2. Mapping theorem and low-energy QCD

In order to manage QCD, it appears essential to find a way to reduce this theory to a simpler one. With this aim in mind the following theorem has been proved: MAPPING THEOREM: An extremum of the action

$$S = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 - \frac{\lambda}{4} \phi^4 \right]$$

is also an extremum of the SU(N) Yang-Mills Lagrangian when one properly chooses $A_{\mu}^{a}$ with some components being zero and all others being equal, and $\lambda = Ng^2$, being $g$ the coupling constant of the Yang-Mills field, when only time dependence is retained. In the most general case the following mapping holds

$$A_{\mu}^{a}(x) = \eta_{\mu}^{a} \phi(x) + O(1/\sqrt{Ng})$$

being $\eta_{\mu}^{a}$ constant, that becomes exact for the Lorenz gauge.

A first proof of this theorem was given in [5] and, after a criticism by Terence Tao, a final proof was presented in [6] also agreed with Tao[7].
Applying the above mapping theorem, holding in the large coupling limit, QCD generating functional becomes:

\[
Z(\eta, j_\phi) = \int \prod_n [d\eta] [dQ]\ e^{i\int \sqrt{\frac{g}{4\pi}} \int d^4x \left( \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \partial_{\mu} \phi \partial_{\nu} \bar{\phi} F_{\rho\sigma}^a \right)^2 - \frac{1}{2} g \bar{\phi} \eta \phi - \frac{1}{2} g^2 \bar{\phi} \sigma^{\mu\nu} \phi \partial_{\mu} \phi \partial_{\nu} \phi}.
\]

At this order ghost field just decouples so we can safely ignore it. QCD is so reduced to a Yukawa model by the use of the mapping theorem, at the leading order of a development in the inverse of the \( ' \) t Hooft coupling. All the parameters of the model, at this order, are fixed by QCD. In the limit \( A \to \infty \), the scalar field term takes a Gaussian form \([8]\). This theory is trivial in this limit and the beta function goes like \( \beta(A) = 4\lambda \) in four dimensions at lower momenta.

The generating functional for the Yukawa model can be cast into a Gaussian form in the strong coupling limit \( g \to \infty \), knowing the gluon propagator. Indeed, one can write

\[
Z(\eta, j_\phi) = \exp\left\{\int d^4x \left[ \frac{g}{2} \bar{\phi} \phi \eta \right] \right\} \exp\left\{\int d^4x \left[ \frac{g^2}{4} \bar{\phi} \sigma^{\mu\nu} \phi \partial_{\mu} \phi \partial_{\nu} \phi \right] \right\},
\]

provided \( j_\phi = j \cdot \eta \) and the gluon propagator

\[
\Delta(p) = \sum_{n=1}^{\infty} \frac{B_n}{p^2 - m_n^2 + i\epsilon},
\]

being

\[
B_n = (2n + 1) \frac{n!^2}{K(i)} \frac{(-1)^n e^{-\pi i / 2} \pi}{1 + e^{-2\pi n + 1}}.
\]

We get a first key formula and this is the spectrum of the theory, in the strong coupling limit, given by

\[
m_n = \left( n + \frac{1}{2} \right) \frac{\pi}{K(i)} \left( \frac{N_c^2}{2} \right)^{1/2} \Lambda.
\]

From the mass spectrum we can identify a string tension as

\[
\sqrt{\sigma} = \left( \frac{N_c^2}{2} \right)^{1/2} \Lambda = (2\pi N_c \Lambda)^{1/2} \Lambda.
\]

Presently, the parameter \( \Lambda \) appears rather arbitrary. Being an integration constant, it should be obtained from experiment. We just note from this that \( \sigma_{SU(3)} / \sigma_{SU(2)} = \sqrt{2}/3 \), as seen on lattice\([8]\). We recognize that, at lower energies, strong interactions are mediated by a kind of bosons that can be seen as due to Yang-Mills field self-interaction. These are the physical states in a strong coupling limit.

We realize that the low-energy limit of QCD can be further reduced to a Nambu-Jona-Lasinio model. In the gluon propagator we just take the low momenta limit producing the contact interaction\([10]\)

\[
A(x - y) \approx \frac{3.76}{\sigma} \delta^3(x - y).
\]

A first analysis of simple interactions may be accomplished by neglecting quark loops or decays involving quarks, just quark-glue vertexes. So, for a first understanding we just consider quark-glue interaction. In this case, QCD is exactly integrable producing a non-trivial Gaussian generating functional. So, the generating functional can be finally integrated producing the final result

\[
Z(\eta^2, j_\phi) = \exp\left\{\int d^4x \left[ \frac{3}{2} \eta \phi \bar{\phi} + \frac{1}{2} \eta^2 \phi^2 \right] \right\} \exp\left\{\int d^4x \left[ \frac{3}{2} \eta \phi \bar{\phi} + \frac{1}{2} \eta^2 \phi^2 \right] \right\}.
\]

The quark propagator, considering that a gradient expansion corresponds to a strong coupling expansion\([11]\), is

\[
S[f_\alpha, x - y] = \theta(t_x - t_y) \delta^3(x - y) \times e^{i \int d^4x \left[ \frac{3}{2} \eta \phi \bar{\phi} + \frac{1}{2} \eta^2 \phi^2 \right]}
\]

We have a defined leading order term for a strong coupling expansion in QCD. Higher order terms can be obtained from fields. QCD at the leading order in a strong coupling expansion \( g \to \infty \) appears a confining and yet renormalizable theory.

### 3. \( \sigma \) and \( \eta - \eta' \) Mesons

We can identify the \( \sigma \) meson as the lowest state in Yang-Mills theory. Being massive, this theory shows up a mass gap. The mass gap, that is also the mass of the lowest glue excitation, is given by

\[
m_\sigma = \frac{\pi}{2K(i)} \sqrt{\sigma} = \frac{\pi}{2K(i)} \frac{6\pi \Lambda}{\sigma^2} \Lambda.
\]

Width is given by\([12]\) \( (G_{N_c}^\sigma = \frac{32\pi^3}{3}g = \frac{3.76}{\sigma}) \)

\[
\Gamma_\sigma = \frac{2}{\pi} G_{N_c}^2 m_\sigma f_\sigma^4 \int \frac{1 - \frac{4m_\pi^2}{m_\sigma^2}}{m_\sigma^2}.
\]

Decay constants for all glue excitations can be straightforwardly obtained. Using the mapping theorem, one has a Fourier series for SU(3)

\[
A_\mu(0, y) = f_\mu \int \frac{1}{2\pi} \int \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{i(n+1/2)x} \frac{e^{i\phi(x)}}{1 + e^{2\pi n - x}} + e.c.c.
\]
From this series we can easily read the decay constants for the glue excitations

\[ f_{\sigma} = \frac{\sqrt{\sigma}}{6\pi\alpha_s} \frac{2\pi}{k(i)} (-1)^n \frac{e^{-(\pi^2 + \pi)} \pi}{1 + e^{-2\pi^2 + 1\pi}}. \]  

Finally, for the \( \sigma \), setting \( n = 0 \), we have

\[ f_{\sigma} = \frac{\sqrt{\sigma}}{6\pi\alpha_s} \frac{2\pi}{k(i)} + e^{-\pi}. \]  

It is interesting to note that, using again the mapping theorem, the correlator is given by

\[ \langle A_i^\sigma(0,t)A_j^\sigma(0,0) \rangle = \eta \eta^\pi(\phi(0,t)\phi(0,0)) = \sum_{n=0}^\infty B_n e^{-im_\tau t} e^{-i\theta} + c.c. + O(1/\sqrt{N}) \]

and so, eq. (5) is indeed the spectrum of the theory in a strong coupling limit.

\( \eta' \) decay is one of the key processes to understand glue role in hadronic physics. This is mainly \( \eta' \to \eta \pi \pi \). Measures at DAoNE proved that \( \eta' \) has a significant glue component \([2]\). So, we consider this decay as a two step process and essentially due to glue emission: \( \eta' \to \eta \pi \). This will give the following width

\[ \Gamma_{\eta'} = \frac{1}{2\pi} G_{NJL}(m_{\eta'}) f_{\eta'} f_{\pi}^2 \times \sqrt{\pi \pi \pi \pi \pi \pi}, \]

The opposite process \( \sigma \eta \to \eta' \) can be also easily computed by time reversal symmetry of QCD. Glue production by \( \eta' \) is a threshold process and so it can be used to fix \( \sigma \) mass as

\[ m_\sigma = m_{\eta'} - m_\eta. \]  

We can use the measured width \([13]\) to determine \( f_{\eta} \). Taking for \( \Gamma_{\eta'} \) the same value observed for the process \( \eta' \to \eta \pi \pi \) and \( f_{\sigma} \) the one computed above, one has for \( m_{\sigma} \) at the threshold

\[ |f_{\sigma}| \approx 0.019 \text{ GeV}. \]  

Now, we assume \( \eta \) and \( \eta' \) to mix with an angle \( \theta \approx -14^\circ \) and also \( f_{\eta} \approx 0.13 \text{ GeV}, f_{\sigma} \approx -0.45 f_{\eta} \) and \( f_{\eta'} \approx 1.2 f_{\eta} \), so one has

\[ f_{\eta'} = f_{\eta} \cos \theta + f_{\sigma} \sin \theta \approx -0.019 \text{ GeV}. \]  

while \( f_{\eta'} \approx 0.16 \text{ GeV} \). This computation does not depend on the value of \( \alpha_s \).

4. QCD and \( \Lambda \) constant

Now, we check above scenario against a proper value of \( \alpha_s \). One has for the running coupling with six flavors \([13]\)

\[ \alpha_s(q^2, \Lambda) = \frac{1}{\frac{2\pi}{\sqrt{\ln \ln (q^2/\Lambda^2)}}} - \frac{7}{2\pi} \ln \left( \frac{q^2}{\Lambda^2} \right) + \ldots \]  

and we stop at this order to avoid dependencies from a renormalization scheme. Our aim is to fix \( \Lambda \) in order to get a consistent scheme for low-energy QCD. This is the same parameter both for high and low-energy physics. In our case, a key quantity is string tension. We will have

\[ \sigma(q^2, \Lambda) = \sqrt{\frac{6\pi\alpha_s(q^2, \Lambda)}{\Lambda^2}}. \]  

So, the mass of the first glue state is obtained by solving the equation

\[ m_{\sigma} = \pi \frac{2K(i)}{2K(i)} \sigma(m_{\sigma}^2, \Lambda). \]  

The next excited glue state will be given by

\[ m_{1} = \frac{3\pi}{2K(i)} \sigma(m_{1}^2, \Lambda). \]  

We can write down the corresponding decay widths as

\[ \Gamma_{\sigma} = \frac{2}{\pi} G_{NJL}(m_{\sigma}^2, \Lambda) m_{\sigma} f_{\pi}^2 \sqrt{1 - \frac{4m_{\sigma}^2}{m_{\sigma}^2}} \]  

and for \( f(0\pi) \)

\[ \Gamma_{1\pi} = \frac{2}{\pi} G_{NJL}(m_{\pi}^2, \Lambda) m_{1} f_{\pi}^2 \sqrt{1 - \frac{4m_{\pi}^2}{m_{\pi}^2}}. \]  

For the decay constants of glue states we will have

\[ f_{\sigma} = \frac{\sqrt{\sigma(m_{\sigma}^2, \Lambda)^2 \pi^2}}{6\pi\alpha_s(m_{\sigma}^2, \Lambda) K(i)} (-1)^n \frac{e^{-(\pi^2 + \pi)}}{1 + e^{-2\pi^2 + 1\pi}}. \]  

On a similar ground we can write, as observed for \( f(0\pi) \) from its KK decay,

\[ \Gamma_{1\pi K} = \frac{2}{\pi} G_{NJL}(m_{1}^2, \Lambda) m_{1} f_{K}^2 \sqrt{1 - \frac{4m_{K}^2}{m_{1}^2}}. \]  

We can compute the ratio \( r_{f/K} = |g_{f/K}|/|g_{f/\pi}| \) predicted to be 2.59(1.34) by Mennessier, Narison, and Wang \([3]\). These authors also agree with a content of \( f(0\pi) \) being mostly glue. We have

\[ g_{f/K} = 8 \sqrt{6m_{f} f_{\pi}^2 \sqrt{\pi\alpha_s}}. \]  

\[ (24) \]
and similarly for decay to Ks. This implies a sizable
coupling of the $\sigma$ with K mesons. This computation will
represent a leading order approximation as we neglect
mixing effects that should be anyhow present. Finally,
we will check our computation of $\Lambda$ against the corre-
sponding value of $\alpha_s$ as obtained from other sources and
given by PDG[13].

So our estimation is

$$\Lambda = 0.171 \pm 0.001 \text{ GeV}$$ (25)

that yields the following results for $\sigma$ mass

$$m_\sigma = 0.410 \pm 0.007 \text{ GeV},$$ (26)

f0(980) mass

$$m_{f0(980)} = 1.023 \pm 0.002 \text{ GeV},$$ (27)

mass ratio

$$m_{f0(980)}/m_\sigma = 2.49 \pm 0.05, $$ (28)

very near the theoretical value 3 but here we are consid-
ering running coupling and string tension. This may be
relevant for understanding lattice results. $\sigma$ width is

$$\Gamma_\sigma = 0.260 \pm 0.001 \text{ GeV}. $$ (29)

It is interesting to note that these values for width and
mass are very near those of a recent analysis[14]. Coupling
ratio for f0(980)

$$r_{f0K} = 1.42 \pm 0.02$$ (30)
in close agreement with Mennessier, Narison, and
Wang [3]. $\sigma$ decay constant is $f_\sigma = 0.139 \pm 0.008 \text{ GeV}.$
Finally, we just note that the scenario given through this
analysis matches rather well that given in a pioneering
work of Narison and Veneziano [15, 16]. With their
choice of $\Lambda$, they get a width for $\sigma$ very near to the
correct one obtainable from our formulas and a corre-
sponding decay constant increasing with mass.

5. Conclusions

Low-energy limit of QCD can be obtained by an expan-
sion at very large coupling and remapping Yang-Mills
field on a quartic massless scalar field theory. This map-
ing is a proved mathematical theorem. A mass gap
arises due to strong self-interaction of the fields already
at classical level. QCD infrared limit is a renormaliz-
able Yukawa model that, in the proper approximation,
reduces to a Nambu-Jona-Lasinio model. This is possible
for the gluon propagator in close agreement with lattice
data. A lot of hadronic processes can be described by
a generating functional obtained in closed form from
QCD in the above limit but neglecting quark loops and
quark-quark interaction and having only quark-glue in-
teractions. From this, one can compute decay constants,
width and several other experimental meaningful ob-
servables. $\sigma$ and f0(980) appear to be glue excitations
entering into a lot of hadronic processes and describ-
ing the true physical states of glue for low-energy QCD.
It is worthwhile to note that decay constants for glue
excitations are strongly damped for higher excitations
making observable just the first few states. $\eta'$ appears
mostly to emit a glue state that, due to a threshold con-
dition, should properly fix the value of the mass of $\sigma$
resonance. In agreement with KLOE-2 evidences, this
gives a sound explanation of the main decay of this me-
sion resembling the case of QED of an excited atom de-
caying into an atom plus the force carrier, i.e. a photon.
This decay can also be used to determine the decay con-
stant of $\eta$. All observables are properly fixed with a sin-
gle constant $\Lambda = 171 \pm 1 \text{ MeV}$ that recovers the correct
value of the measured running coupling at $M_Z.$

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