3D reduction of the N-body Bethe-Salpeter equation.

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Abstract

We perform a 3D reduction of the two-fermion homogeneous Bethe-Salpeter equation, by series expansion around a positive-energy instantaneous approximation of the Bethe-Salpeter kernel, followed by another series expansion at the 3D level in order to get a manifestly hermitian 3D potential. It turns out that this potential does not depend on the choice of the starting approximation of the kernel anymore, and can be written in a very compact form. This result can also be obtained directly by starting with an approximation of the free propagator, based on integrals in the relative energies instead of the more usual δ-constraint. Furthermore, the method can be generalized to a system of N particles, consisting in any combination of bosons and fermions. As an example, we write the 3D equation for systems of two or three fermions exchanging photons, in Feynman or Coulomb’s gauge.

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1 Introduction.

The homogeneous Bethe-Salpeter equation \[1, 2\] is the usual tool for computing relativistic bound states. The first difficulty of this equation comes from the presence of N-1 (for N particles) unphysical degrees of freedom: the relative time-energy degrees of freedom. In the two-body problem, the relative energy is usually eliminated by replacing the free Green function by an expression combining a delta fixing the relative energy and a free 3D propagator\[3-17\]. The exact equivalence (in what concerns the physically measurable quantities of the pure two-fermion problem) with the original homogeneous Bethe-Salpeter equation can be obtained by recuperating the difference with the original free Green function in a series of correction terms to the 3D potential. The dependence in the relative energy of the 4D Bethe-Salpeter equation is thus transformed into a series of higher-order correction terms to a 3D equation. This series of "modified ladder terms" is to be combined with the series of "crossed terms" which gives the Bethe-Salpeter kernel as a sum of terms corresponding to irreducible graphs. We do not know if the result converges, and, if yes, which will be the level of approximation obtained with a given truncation. This will of course depend on the specific problem studied. The mutual cancellation between the higher-order "modified ladder terms" (generated by the 3D reduction) and the "crossed terms", which occurs at the one-body limit \[14, 15, 16\] is an
encouraging fact. The starting homogeneous Bethe-Salpeter equation (for the Bethe-Salpeter amplitude) is deduced from the inhomogeneous one (for the full propagator) by postulating the presence of a bound state pole, and is thus a priori valid only for bound states. However, the scattering amplitude built with the 3D free propagator and potential turns out to be correct on the mass shell.

This kind of 3D reduction of the two-fermion homogeneous Bethe-Salpeter equation has been performed by many authors [3-17]. The resulting equations are theoretically equivalent at the limit of all correction terms included. Of course, in practice, the series generated by the 3D reduction and the series giving the Bethe-Salpeter kernel itself will both have to be truncated and the estimation of the level of approximation obtained is not straightforward.

A less often used method is based on the replacement of the Bethe-Salpeter kernel by an "instantaneous" (i.e. independent of the relative energy) approximation. In this case, the resulting 3D potential is not manifestly hermitian (for each fixed value of the total energy on which its depends). In Phillips and Wallace’s method, the starting approximation is tuned, order by order, in such a way that it becomes the final 3D potential [18]. In a recent work, we also got an hermitian 3D potential by performing a supplementary series expansion at the 3D level and combining it with the first 3D reducing expansion [19].

In the three-fermion problem, a new difficulty comes from the unconnectedness of the two-body terms of the Bethe-Salpeter kernel, which are in fact the more important terms and often the only ones to be considered. In a constraining propagator-based reduction, the "approached" propagator would have to be a 3D propagator with two constraints (fixing the two relative energies). However, when an unconnected two-body term lies between two of these approached propagators, we get a constraint too much. In Gross’ spectator model [20, 21], the transition matrix element $T$ is approached by putting on their positive-energy mass shell the fermion lines which are not going to be interacting before and after. This means that each three-fermion free propagator will be "approached" in a way depending on its position in the expansion of $T$. This leads to a set of Faddeev-like equations, which is not equivalent to a single equation with a potential, unlike in the nonrelativistic case.

Recently, we succeeded in adapting our two-fermion kernel-based reduction to the three-fermion case. The 3D potential was a complicated sum of terms of various origins, although the first-order approximation was still manageable [19]. When working on an application of this 3D reduction method, we revisited our two-fermion kernel-based 3D reduction, and found an alternative to the second series expansion (the one performed at the 3D level to get an hermitian potential). After combining this series expansion with the first one (given by the 3D reduction itself), we found that the starting instantaneous approximation of the Bethe-Salpeter kernel disappears from the final 3D potential. The result is in fact a compact expression of the potential that Phillips and Wallace compute order by order [18]. Furthermore, it can more directly be obtained by a new integrating propagator-based reduction method, in which the relative energy
is integrated on, instead of being fixed by a $\delta$–fonction (or constraint). Here again, the physical scattering amplitude computed at the 3D level is also correct. In fact, the 3D full propagator turns out to be the retarded part of the 4D full propagator with the initial and final relative times put to zero, as proposed by Logunov and Tavkhelidze [22].

This integrating propagator-based reduction (and the Logunov and Tavkhelidze’s reduction [22]) can easily be adapted to the three-fermion problem, as the unconnectedness of the two-body parts of the Bethe-Salpeter kernel is not a difficulty anymore when integrals on the relative energies are used instead of $\delta$–fonctions. Furthermore, the method can be generalized to a system of N particles, consisting in any mixing of bosons and fermions. The 3D free propagator is given by the integral of the product of the N free Green functions with respect to the N-1 relative energies and consists in two terms: a term in which all particles have positive free energies, and a term in which they have all negative free energies (a slightly different 3D reduction, with simpler first terms, could be obtained by removing this part of the 3D propagator. We shall consider both options below). The absence of mixed free-energy signs preserves the 3D equation from continuum dissolution [25, 26, 27, 28]. The 3D scattering amplitudes are again correct when all particles are on their positive-energy mass shell (which has no measurable consequence when $N > 2$), but also in more complicated processes, like the scattering of a free particle by a bound state.

There are many ways of performing a constraining propagator-based 3D reduction of the two-body Bethe-Salpeter equation, according to the choice of the constraint and of the 3D free propagator. Our integrating propagator-based reduction approach, in contrast, leads to unexpectedly unifying results: independently of the choice of the initial instantaneous positive-energy kernel, the final 3D equation is always the same. Furthermore, it can also be obtained by Phillips and Wallace’s method [18] and by the quite different Logunov and Tavkhelidze’s method [22], while our integrating propagator-based reduction provides a more compact form for the 3D potential by recombining the series.

We do not pretend that this 3D reduction converges more rapidly than the more usual constraining propagator-based reductions (we try a comparison below). The important result consists in the fact that it can easily be generalized to more than two particles, giving a 3D equation for a wave function with a 3D potential, which must necessarily be truncated but can be indefinitely improved. By comparison, Gross’ spectator model for three particles [20, 21] (which can be considered as a constraining propagator-based reduction) leads to a set of coupled equations for partial transition operators, which can not be reduced to a single equation.

As a first test of our reduction method, we choose the problems of two and three fermions in QED, using Feynman’s gauge. We compare with a constraining propagator-based reduction method in the two-fermion case, and with Gross’ spectator model in the three-fermion case. In actual calculations, Coulomb’s gauge would be a better choice than Feynman’s gauge, but it is less suited to a
presentation of our method, as the corrections to the instantaneous approxima-
tion and the three-body terms are then of higher order, competing with a lot of
other contributions.

Two natural requirements in the writing of phenomenological equations for
systems of mutually interacting relativistic particles are Lorentz invariance and
cluster separability: if we split the system into clusters by "switching off" selec-
tively the interactions between the clusters or by by letting the distance between
them become infinite (this is not quite the same thing), we want to get inde-
pendent sets of correct covariant equations. It is not easy to satisfy Lorentz
invariance and cluster separability together, and, moreover, to include contin-
uum dissolution-preserving operators. In our 3D reductions of Bethe-Salpeter
equations the approach is different: the original inhomogeneous Bethe-Salpeter
equation (from which the homogeneous one is derived) is manifestly Lorentz co-
variant and cluster separable. The 3D reduction is performed in an unspecified
(at first) reference frame and the individual terms of the series giving the 3D
potential are not covariant, some of them being not cluster separable either. If
we choose to work in the global center of mass frame, all terms can be made
formally covariant, but this reference to the total momentum spoils the cluster
separability. However, if we see the 3D reduction only as a tool to calculate
the bound state spectrum, the exact results can in principle be indefinitely ap-
proached by including more and more higher-order contributions. Moreover, it
turns out that this same 3D equation is still valid for the various physical scatter-
ing amplitudes. The computation of electromagnetic form factors can also
be done by computing electromagnetic scatterings. What is then the practical
meaning of the cluster separability requirement? the selective "switching off"
is clearly a purely mathematical operation which does not correspond to some-
thing physical, while the problem of clusters going to infinity must be treated
as a scattering problem. A practical aspect of the Lorentz covariance / clus-
ter separability requirement could be the use of, for example, two-body bound
state or scattering data in a three-body context. When used directly, the two-
body data must always be extrapolated off the mass shell, in a model-dependent
way. In our 3D reductions, we could use two-body data indirectly, through the
adjustment of coupling parameters.

In section 2, we present the usual constraining propagator-based reduction of
the two-fermion Bethe-Salpeter equation, followed by our kernel-based reduc-
tion transformed into an integrating propagator-based reduction, and we com-
pare the first terms of the 3D potentials obtained by both methods. In section 3
we generalize this integrating propagator-based reduction to N-particle systems
(fermions, bosons or mixings of both). In section 4 we apply our method to sys-
tems of two or three fermions exchanging photons, in Feynman’s and Coulomb’s
gauges.
2 The two-fermion problem.

2.1 Constraining propagator-based reduction.

We shall write the Bethe-Salpeter equation for the bound states of two fermions as

\[ \Phi = G^0 K' \Phi, \]

where \( \Phi \) is the Bethe-Salpeter amplitude, function of the positions \( x_1, x_2 \) or of the momenta \( p_1, p_2 \) of the fermions, according to the representation chosen. The operator \( K' \) is the Bethe-Salpeter kernel, given as a factor of the kernel of an integral equation in momentum space by the sum of the irreducible two-fermion Feynman graphs. The operator \( G^0 \) is the product \( G_1^0 G_2^0 \) of the two individual fermion’s dressed propagators

\[ G_i^0 = \frac{1}{\gamma_i \cdot p_i - m_i - \Sigma_i + i\epsilon} \]

where \( m_i \) is the mass of the fermion \( i \) and \( \Sigma_i (p_i) \) the renormalized self-energy function, which becomes proportional to \( (\gamma_i \cdot p_i - m_i)^2 \) near the mass shell.

We shall need a Bethe-Salpeter equation in terms of the free propagator

\[ G^0 = G_1^0 G_2^0, \quad G_i^0 = \frac{1}{\gamma_i \cdot p_i - m_i + i\epsilon} = \frac{1}{p_{i0} - h_i + i\epsilon h_i} \beta_i \]

where the \( h_i \) are the Dirac free hamiltonians

\[ h_i = \vec{\alpha}_i \vec{p}_i + \beta_i m_i \quad (i = 1, 2). \]

In order to transfer the self-energy part of \( G^0 \) to the kernel, we shall write

\[ K = K' + \Sigma \]

with

\[ \Sigma = (G^0)^{-1} - (G^0)^{-1} = \Sigma_1 (G_2^0)^{-1} + \Sigma_2 (G_1^0)^{-1} - \Sigma_1 \Sigma_2 \]

and get the Bethe-Salpeter equation in the form

\[ \Phi = G^0 K' \Phi. \]

We shall define the total (or external, CM, global) and relative (or internal) variables:

\[ X = \frac{1}{2}(x_1 + x_2), \quad P = p_1 + p_2, \]

\[ x = x_1 - x_2, \quad p = \frac{1}{2}(p_1 - p_2). \]

and also

\[ S = h_1 + h_2, \quad E = E_1 + E_2, \quad E_i = \sqrt{h_i^2 + (p_i^2 + m_i^2)^2}. \]
We do not specify the reference frame in which we write noncovariant quantities like $h_i$ or $E_i$. Our 3D reduction will in fact be frame-dependent. Practically, we shall choose the global rest frame of the 2- (in this section), 3-, or N-particle system. In the N-particle problem we shall not try to boost the two-body kernels between the global rest frame and the (virtual) two-particle rest frames: these relativistic effects will be taken into account (if desired) by the inclusion of the higher-order terms of the series generated by the 3D reduction, since this series, when untruncated, leads to the same measurable quantities as the starting covariant Bethe-Salpeter equation.

If we consider the contributions of the poles of

$$G^0 = \frac{1}{\frac{1}{2}P_0 + p_0 - h_1 + i\epsilon h_1} \frac{1}{\frac{1}{2}P_0 - p_0 - h_2 + i\epsilon h_2} \beta_1 \beta_2$$

in an expression like $KG^0K$, we must perform an integration with respect to $p_0$. If $K$ is instantaneous, we get

$$\int dp_0 G^0(p_0) = -2i\pi \tau g^0 \beta_1 \beta_2, \quad g^0 = \frac{1}{P_0 - S + i\epsilon P_0}$$

where

$$\tau = \frac{1}{2}(\tau_1 + \tau_2), \quad \tau_i = \frac{h_i}{\sqrt{h_i^2}} = \frac{h_i}{E_i}$$

can also be written

$$\tau = \Lambda^{++} - \Lambda^{--}, \quad \Lambda^{ij} = \Lambda^i_1 \Lambda^j_2, \quad \Lambda^\pm_i = \frac{E_i \pm h_i}{2E_i}.$$ (14)

When $K$ is not instantaneous, we must add the contributions of its singularities. Furthermore, in the residues of the poles of $G^0$ we must take $K$ at $p_{10} = h_1$ (i.e. at $+E_1$ in the subspace built with the $h_1 = +E_1$ eigenstates of $h_1$, and at $-E_1$ in the complementary subspace) or at $p_{20} = h_2$, according to the chosen integration path and to the sign of $\tau$. We shall perform a 3D reduction based on the replacement of the free propagator $G^0$ by a carefully chosen expression

$$G^\delta(p_0) = -2i\pi \tau \delta(p_{10} - h_1) g^0 \beta_1 \beta_2$$

combining the 3D propagator $g^0$ with the constraint

$$\delta(p_{10} - h_1) \equiv \Lambda^+_1 \delta(p_{10} - E_1) + \Lambda^-_1 \delta(p_{10} + E_1) = \delta(p_0 - s_1), \quad s_1 = \frac{P_0}{2} + h_1.$$ (16)

The operator $\tau$ has a clear meaning in the basis built with the free solutions: it is $+1$ for $h_1, h_2 > 0$, $-1$ for $h_1, h_2 < 0$ and zero when they have opposite signs. It comes from the dependence of the $p_0$ integral on the signs of the $i\epsilon h_i$.

This choice of $G^\delta$ has three merits: a) It leads directly to a simple equation (Salpeter’s equation without higher-order correction terms to the potential)
with an instantaneous approximation of the kernel. The higher-order correction
terms (necessary to get the correct one-body limits) will then be the consequence
of the dependence of the kernel in the relative energy. b) In the two-fermion
plus potential problem and in the three-fermion problem, the operator \( \tau \)
presents the mixing of asymptotically free fermions with opposite energy signs,
which is the origin of the continuum dissolution problem [25, 24, 27, 28]. c) It
preserves a particle-antiparticle symmetry, which is a characteristic feature of
relativistic theories. We could also replace \( \tau \) by \( \Lambda^{++} \), as the \( \Lambda^{-+} \)
part does not contribute much in practice. One of the fermions would then be on its
positive-energy mass shell, as in Gross’ spectator model [14]. There is an infinity
of other possible choices, like fixing \( p_0 \) at its on mass shell value \( (h_1-h_2)/2 \),
instead of fixing \( p_{10} \) at its on mass shell value \( h_1 \), treating thus both fermions
in a symmetrical way. With Sazdjian’s propagator [17], one gets an explicitly
covariant equation (we recently combined this propagator with a kind of co-
variant positive-energy projector [29]). With Lepage [8], we get directly a 3D
equation written in Schrödinger’s form. An important feature is the presence or
absence of a continuum dissolution-preserving operator like \( \tau \) or \( \Lambda^{++} \). Without
such an operator, the propagator could not be used beyond the two-body prob-
lem, including the two-body in an external potential problem. Our constraining
operator contains thus the operator \( \tau \) (we shall also examine the \( \Lambda^{++} \) choice).
Besides that, the principal reason of our choice (15) is to allow for an easy term
by term comparison with our integrating propagator-based reduction below.

Let us now write the free propagator as the sum of the zero-order propagator,
plus a remainder:

\[
G^0 = G^\delta + GR.
\]  

(17)

The Bethe-Salpeter equation becomes then the inhomogeneous equation

\[
\Phi = G^0K\Phi = (G^\delta + GR)K\Phi = \Psi + GRK\Phi,
\]  

(18)

with

\[
\Psi = G^\delta K\Phi \quad (= G^\delta (G^0)^{-1}\Phi).
\]  

(19)

Solving (formally) the inhomogeneous equation (18) with respect to \( \Phi \) and
putting the result into (19), we get

\[
\Psi = G^\delta K(1 - GRK)^{-1}\Psi = G^\delta K^T\Psi
\]  

(20)

where

\[
K^T = K(1 - GRK)^{-1} = K + KG^R K + ... = (1 - KG^R)^{-1}K
\]  

(21)

obeys

\[
K^T = K + KG^R K^T = K + K^T GR K.
\]  

(22)

The reduction series (21) re-introduces in fact the reducible graphs into the
Bethe-Salpeter kernel, but with \( G^0 \) replaced by \( GR \). Equation (21) is a 3D
equivalent of the Bethe-Salpeter equation.
The relative energy dependence of eq. (20) can be easily eliminated:

\[ \Psi = \delta(p_0 - s_1) \psi \]  

and \( \psi \) obeys:

\[ \psi = -2i\pi g^0 \tau \int dp'_0 dp_0 \delta(p'_0 - s_1) \beta_1 \beta_2 K^T(p'_0, p_0) \delta(p_0 - s_1) \psi. \]  

Using the identity \( \psi = \tau^2 \psi \), we can write

\[ \psi = g^0 \tau V \psi, \quad V = -2i\pi \tau^2 \beta_1 \beta_2 K^T(s_1, s_1) \tau^2, \]  

\[ \beta_1 \beta_2 K^T(s_1, s_1) \equiv \int dp'_0 dp_0 \delta(p'_0 - s_1) \beta_1 \beta_2 K^T(p'_0, p_0) \delta(p_0 - s_1). \]  

Note that we write \( (p'_0, p_0) \) but \( (s_1, s_1) \), as we keep \( s_1 \) in operator form. This operator can be diagonalized in the spatial momentum space by using the \( \Lambda^{ij} \) projectors. The eigenvalue will depend on the position of \( s_1 \) in the formula: the eigenvalue of the first \( s_1 \) in (26) will be built with the final momenta and that of the last \( s_1 \) will be built with the initial momenta.

The inversion of the reduction is given by

\[ \Phi = (1 - G^R K)^{-1} \Psi = (1 + G^R K^T) \Psi = (1 + G^0 K^T - G^0 K^T) \Psi = G^0 K^T \Psi \]  

or, explicating the relative energy:

\[ \Phi(p'_0) = G^0(p'_0) K^T(p'_0, s_1) \psi. \]  

The splitting of \( G^0 \) into two terms containing a \( \delta \) is the origin of unphysical singularities in the terms of \( K^T \) when the argument of the delta vanishes on the singularities of \( K \). When the full \( K^T \) is computed, the singularities of the different terms cancel mutually. When \( K^T \) is truncated, some of the unphysical singularities have to be removed by hand [15, 20].

This 3D reduction can also be described in terms of transition operators. The 4D transition operator corresponding to our modified Bethe-Salpeter equation (7) is

\[ T = K + K G^0 K + \cdots \]  

and \( K^T \) can be obtained by keeping only the \( G^R \) part of \( G^0 \) in it. We have also

\[ T = K(1 - G^0 K)^{-1} = K(1 - G^R K - G^\delta K)^{-1} = K(1 - G^R K)^{-1}(1 - G^\delta K(1 - G^R K)^{-1})^{-1} \]

\[ = K^T(1 - G^\delta K^T)^{-1} = K^T + K^T G^\delta K^T + \cdots \]  

so that the 3D transition operator

\[ T^{3D} = V + V g^0 \tau V + \cdots \]
is also given by
\[ T^{3D} = -2i\pi \tau^2 \beta_1 \beta_2 T(s_1, s_1) \tau^2. \] (32)

The operator \( T \) and the operator \( T' \) corresponding to the original Bethe-Salpeter equation (1) are related by the common full propagator \( G \):
\[ G^0 + G^0 TG^0 = G = G^0 + G^0 T' G^0 \] (33)
with
\[ G^0 = G^0 (1 - \Sigma G^0)^{-1} = G^0_1 (1 - \Sigma_1 G^0_1)^{-1} G^0_2 (1 - \Sigma_2 G^0_2)^{-1}. \] (34)

We see that the 3D transition operator is a constrained form of that of field theory. When both fermions are on their positive-energy mass shells, the operators \( T(s_1, s_1), T, T' \), become equal and \( T^{3D} \) becomes proportional to the physical scattering amplitude. This was not guaranteed a priori, as our original two-fermion Bethe-Salpeter equation (3) was valid only for bound states. Our 3D equation (25) is a bound state equation too. To include the scattering states we should add an inhomogeneous term, or write the equation in the form
\[ (P_0 - S) \psi = \tau V \psi. \] (35)

2.2 Kernel-based reduction.

A few operators in this kernel-based reduction will be given the same name as similar operators used in the constraining propagator-based reduction, although being not identical to them. In order to avoid any confusion, we shall overwrite these operators with a \( \tilde{\ } \).

If the Bethe-Salpeter kernel were instantaneous (i.e. independent of \( p_0 \)), one would get Salpeter’s equation by integration with respect to \( p_0 \) [3]. In the realistic case of a non-instantaneous kernel, it is possible to compute the bound state energies at the 4D level by perturbations around an instantaneous approximation of the kernel [10, 31, 14]. Here, we want to build a 3D reduction around an approximation \( K^0 \) of the Bethe-Salpeter kernel. Let us write
\[ K = K^0 + K^R. \] (36)

The Bethe-Salpeter equation becomes
\[ \Phi = G^0 K^0 \Phi + G^0 K^R \Phi \] (37)
\[ \Phi = (1 - G^0 K^R)^{-1} G^0 K^0 \Phi = G^K K^0 \Phi \] (38)
with
\[ G^K = G^0 + G^0 K^R G^0 + \cdots \equiv G^0 + G^K R. \] (39)
If we now specialize $K^0$ to an instantaneous positive-energy kernel ($K^0 = \Lambda^{++} \beta_1 \beta_2 K^0 \Lambda^{++}$ and is independent of $p_0$), we get

$$\phi = (g^0 + g^{KR}) V^0 \phi$$  \hspace{1cm} (40)$$

with

$$\phi = \Lambda^{++} \int dp_0 \Phi(p_0), \quad V^0 = -2i\pi \beta_1 \beta_2 K^0,$$  \hspace{1cm} (41)$$

$$g^{KR} = \frac{-1}{2i\pi} \Lambda^{++} \int dp'_0 dp_0 G^{KR}(p'_0, p_0) \beta_1 \beta_2 \Lambda^{++}.$$  \hspace{1cm} (42)$$

The interaction term $(g^0)^{-1}(g^0 + g^{KR}) V^0$ is not hermitian (for a fixed value of the total energy on which it depends), so that we do not know if equation (10) will have a real energy spectrum. Let us write this equation in the form

$$\phi = (1 + g^0 T^{KR}) g^0 V^0 \phi, \quad T^{KR} = (g^0)^{-1} g^{KR}(g^0)^{-1}$$  \hspace{1cm} (43)$$
in order to perform the transformations

$$(1 + g^0 T^{KR})^{-1} \phi = g^0 V^0 \phi$$  \hspace{1cm} (44)$$

$$\phi = [g^0 V^0 + 1 - (1 + g^0 T^{KR})^{-1}] \phi = g^0 [V^0 + T^{KR}(1 + g^0 T^{KR})^{-1}] \phi$$

$$= g^0 [ < K^0 > + < K^R(1 - G^0 K^R)^{-1} > (1 + g^0 < K^R(1 - G^0 K^R)^{-1} >)^{-1} ] \phi$$  \hspace{1cm} (45)$$

with the definition

$$< A > = \frac{-1}{2i\pi} \Lambda^{++}(g^0)^{-1} \int dp'_0 dp_0 G^0(p'_0) A(p'_0, p_0) G^0(p_0) \beta_1 \beta_2 \Lambda^{++}(g^0)^{-1}.$$  \hspace{1cm} (46)$$

Let us expand $g^0$ into a 4D operator:

$$G^I = \phi$$  \hspace{1cm} (47)$$
defined as (writing the relative energy arguments, the other momentum arguments $P$ and $\vec{p}$ remaining local):

$$G^I(p'_0, p_0) = G^0(p'_0) \beta_1 \beta_2 \frac{\Lambda^{++}}{-2i\pi g^0} G^0(p_0)$$  \hspace{1cm} (48)$$

and write (45) in the form

$$\phi = g^0 < K^0 + K^R(1 - G^0 K^R)^{-1} > \phi$$

$$= g^0 < K^0 + K^R(1 - G^0 K^R + G^I K^R)^{-1} > \phi = g^0 < K^0 + K^R(1 - G^R K^R)^{-1} > \phi$$  \hspace{1cm} (49)$$

with

$$G^R(p'_0, p_0) = G^0(p_0) \delta(p'_0 - p_0) - G^I(p'_0, p_0).$$  \hspace{1cm} (50)$$
It is easy to see that $A \tilde{G}^R = \tilde{G} K^0 A = 0$ whenever $A$ is an instantaneous positive-energy operator like $K^0$. We can then recombine $K^0 + K^R = K$ in the 3D equation, which becomes

$$
\phi = g^0 \tilde{V} \phi, \quad \tilde{V} = < \tilde{K}^T >, \quad \tilde{K}^T = K(1 - \tilde{G}^R K)^{-1}.
$$

(51)

Computing the scattering matrix element as in the preceding subsection gives

$$
\tilde{T}^{3D} \equiv \tilde{V}(1 - g^0 \tilde{V})^{-1} = < T >.
$$

(52)

On the positive-energy mass shells $P_0 = E_1^0 + E_2^0 = E_1 + E_2$ the operators $(g^0)^{-1}$ of (46) vanish and we remain with the residues of the corresponding poles in the integrations with respect to $p_0'$ and $p_0$, so that

$$
<T> = -2i\pi \Lambda^{++} \beta_1 \beta_2 T (p_0' = E_i, p_{i0} = E_i) \Lambda^{++}
$$

(53)

and we have again the correct physical scattering amplitude ($T$ is written here with operators as arguments, as in (26)).

Equation (52) can be inverted to give $\tilde{V}$ in terms of $< T >$:

$$
\tilde{V} = < T > (1 + g^0 < T >)^{-1} = < \tilde{K}^T >.
$$

(54)

In fact, it is possible to use equation (52) to define the 3D reduction. The corresponding 3D full propagator

$$
g = g^0 \Lambda^{++} + g^0 \tilde{T}^{3D} g^0
$$

(55)

is then proportional to the integral of $G$ with respect to the initial and final relative energies, between two $\Lambda^{++}$ projectors. This manipulation preserves the position of the bound state poles and the physical scattering amplitudes. In configuration space, it means that we take the retarded part of $G$ at equal times. The transformation of the inhomogeneous Bethe-Salpeter equation into an equation for $g$ was the starting point of the 3D reduction of Logunov and Tavkhelidze [22].

Phillips and Wallace’s 3D reduction is a kernel-based reduction in the $\tau^2 = 1$ subspace. Their way of symmetrizing the interaction term consists in tuning $K^0$ in order to make $g^{KR}$ vanish, considering thus $g^{KR} = 0$ as an equation in $K^0$ to be solved order by order. If we do that here in the $\Lambda^{++} = 1$ subspace, we see that it implies that the operator at the left of (47) is the identical operator. The comparison with (51) leads to

$$
-2i\pi \beta_1 \beta_2 K^0 = < K(1 - \tilde{G}^R K)^{-1} >.
$$

(56)

Our 3D equation (51) is quite general, as it does not depend on the initial choice of $K^0$ anymore (at the 3D level, it is of course still possible to perform a perturbation calculation starting with $< K^0 >$, using (19)). This 3D reduction is also
identical to that of Phillips and Wallace, obtained by tuning an unspecified instantaneous kernel, and to that of Logunov and Tavkheldize, obtained by taking the retarded part of the full propagator \( G \) at equal times. Our own 3D reduction gives a compact expression of the 3D potential, by recombining the series.

The independence of our 3D equation on the choice of \( K^0 \) suggests that it could also be obtained by a new kind of propagator-based reduction, in which the relative energies are integrated on instead of being "constrained". We shall now present this new approach, and we shall see in section 3 that it can be easily generalized to N-body systems.

### 2.3 Integrating propagator-based reduction.

We shall now perform directly an integrating propagator-based reduction, inspired by our kernel-based reduction above, but we shall work in the larger \( \tau^2 = 1 \) subspace, defining

\[
G^I(p'_0, p_0) = G^0(p'_0) \beta_1 \beta_2 \frac{\tau}{-2i\pi g^0} G^0(p_0) \tag{57}
\]

\[
G^0 = G^I + \tilde{G}^R. \tag{58}
\]

\[
\Phi = G^0 K \Phi = (G^I + \tilde{G}^R) K \Phi = \tilde{\Psi} + \tilde{G}^R K \Phi, \tag{59}
\]

with

\[
\tilde{\Psi} = G^I K \Phi = G^I K (1 - \tilde{G}^R K)^{-1} \tilde{\Psi} = G^I \tilde{K}^T \tilde{\Psi} \tag{60}
\]

where

\[
\tilde{K}^T = K (1 - \tilde{G}^R K)^{-1} = K + K \tilde{G}^R K + ... = (1 - K \tilde{G}^R)^{-1} K. \tag{61}
\]

When we explicitate the relative energies in equation (60), we get

\[
\tilde{\Psi}(p''_0) = G^0(p''_0) \beta_1 \beta_2 \frac{\tau}{-2i\pi g^0} \int dp'_0 dp_0 G^0(p'_0) K(p'_0, p_0) \Phi(p_0)
\]

\[
= G^0(p''_0) \beta_1 \beta_2 \frac{\tau}{-2i\pi g^0} \int dp'_0 \Phi(p'_0)
\]

\[
= G^0(p''_0) \beta_1 \beta_2 \frac{\tau}{-2i\pi g^0} \int dp'_0 dp_0 G^0(p'_0) \tilde{K}^T(p'_0, p_0) \tilde{\Psi}(p_0). \tag{62}
\]

Defining now

\[
\phi = \tau \int dp_0 \Phi(p_0) \tag{63}
\]

\[
<A> = \frac{-1}{2i\pi} \tau^2 (g^0)^{-1} \int dp'_0 dp_0 G^0(p'_0) A(p'_0, p_0) G^0(p_0) \beta_1 \beta_2 \tau^2 (g^0)^{-1} \tag{64}
\]
so that
\[ G^l = > \tau g^0 < \]  
and writing (62) in terms of \( \phi \), we get
\[ \phi = g^0 \tau < \tilde{K}^T > \phi. \]  
If we perform again our kernel-based reduction of subsection 2.2, but now in the \( \tau^2 = 1 \) subspace and using the definitions (63-65), we get
\[ \chi = g^0 \tau \left[ < K^0 > - \tau < K^0 > \tau + \tau < \tilde{K}^T > \tau \right]\chi, \quad \chi = \tau \phi \]  
and \( K^0 \) will disappear if it does not connect the \((++)\) and the \((-\cdot)\) subspaces, leading to our equation (66). Phillips and Wallace’s potential, which must reappear after the symmetrizing transformation, will be \( < K^0 > = < \tilde{K}^T > \).

If we prefer to work in the \( \Lambda^{++} = 1 \) subspace, we can recover the results of subsection 2.2 by simply replacing \( \tau \) by \( \Lambda^{++} \).

2.4 Comparison of the lowest-order terms.

With the integrating propagator-based reduction, the final 3D potential will be
\[ \tilde{V} = < K > + < K \tilde{G}^R K > + \cdots \]
\[ = < K > + < K G^0 K > - < K > \tau g^0 < K > + \cdots \]  
At order 4 in the coupling constant, we should keep the ladder term and the first crossed term in \( < K > \) and only the ladder term in \( < K \tilde{G}^R K > \). The contributions of the higher-order terms should a priori decrease with the number of \( \tilde{G}^R \); once the leading contributions containing the poles in \( P_0 - (E_1 + E_2) \) have been removed, we remain with the smaller contributions coming from the residues of the poles of \( K \) or from its \((+-)\) and \((-+)\) components.

For comparison with the usual constraining propagator-based reduction of subsection 2.1, we shall split \( G^0 \) into \( G^\delta + G^R \) in the definition (46) of \( < A > \) (it must be noted that this splitting introduces unphysical singularities \([15, 20]\)). Keeping only the terms with zero or one \( G^R \), we get
\[ \tilde{V} \approx < K >^{\delta \delta} + < K G^R K >^{\delta \delta} \]
\[ + \left[ 1 - < K >^{\delta \delta} \tau g^0 \right] < K >^{R \delta} + < K >^{\delta R} \left[ 1 - \tau g^0 < K >^{\delta \delta} \right] \]
where the index \( R \) or \( \delta \) indicates which part of \( G^0 \) has been kept. The two first terms are also the two first terms of the constraining propagator-based reduction. The two other terms would not contribute at order 4 in a perturbation calculation beginning with \( < K >^{\delta \delta} \). We see the kind of rearrangements by which the constraining propagator-based reduction and the integrating propagator-based reduction will finally lead to the same energy spectrum.

At order 2 only in the coupling constant, we would get
\[ \tilde{V} \approx < K >^{\delta \delta} + < K >^{R \delta} + < K >^{\delta R} \]  
\[ (70) \]
and checking if the addition of the two last terms brings nearer to could help to choose between both approaches in a specific problem.

The 3D free propagator is in the constraining propagator as in the integrating propagator-based reductions. The 3D potentials and are both hermitian and both equations are equivalent. We must however not conclude that these potentials are equal: the dependence of the potential on the total energy allows for an infinity of equivalent equations.

3 N-body problem.

3.1 Generalization of the integrating propagator-based reduction.

The integrating propagator-based reduction is a good candidate to a double generalization: from two to N fermions and/or from N fermions to any system of f = 0,1,...N fermions with b = N-f bosons. The propagators of the fermion i and the boson j are respectively

\[ G^0_i = \frac{1}{p_{i0} - h_i + i\epsilon h_i} \beta_i, \]
\[ G^0_j = \frac{1}{p_{j0}^2 - E_j^2 + i\epsilon} = \frac{1}{2E_j} \sum_{|\sigma_j|} p_{j0} - \sigma_j E_j + i\epsilon \sigma_j \]  

with \( \sigma_j = \pm 1 \). The free propagator \( G^0 \) for a system of \( f = 0,1...N \) fermions and \( b = N-f \) bosons will be

\[ G^0 = G^0_1 ... G^0_f G^0_{f+1} ... G^0_N \]

and the corresponding 3D propagator will be proportional to

\[ \int dp_0 G^0(p_0) = \int \delta(P_0 - \sum_{i=1}^{N} p_{io}) dp_{10} ... dp_{N0} G^0(p_{10},...p_{N0}). \]  

We shall perform the integral in \( p_{10} \) by replacing it by \( (P_0 - \sum_{i=2}^{N} p_{i0}) \) in the first propagator. Each other \( p_{i0} \) will then appear twice: in \( G^0_i \) and in the corresponding \( G^0_j \), and the integral with respect to \( p_{i0} \) will be zero unless \( h_i \) (or \( \sigma_i \)) and \( h_1 \) (or \( \sigma_1 \)) have the same sign. The final result will be

\[ \int dp_0 G^0(p_0) = \frac{(-2i\pi)^{N-1}}{\omega} \frac{\tau}{g^0} \beta \]

with

\[ g^0 = \frac{1}{P_0 - S + i\epsilon P_0}, \quad S = E(\Lambda^+ - \Lambda^-), \quad \beta = \beta_1...\beta_f, \]
\[ \Lambda^\pm = \Lambda_1^\pm \ldots \Lambda_f^\pm, \quad \tau = \Lambda^+ + (-)^{f+1} \Lambda^-, \quad (77) \]

\[ E = \sum_{i=1}^{N} E_i, \quad \omega = 2^h E_{f+1} \ldots E_N \quad (78) \]

for \( f \neq 0 \), so that

\[ \tau g^0 = \frac{\Lambda^+}{P_0 - E + i\epsilon} + (-)^{f+1} \frac{\Lambda^-}{P_0 + E - i\epsilon} \quad (79) \]

When \( f = 0 \), (bosons only), we have no \( \tau \) and no \( \beta \) (we can replace them by 1 in (75)) and

\[ g^0 = \frac{1}{P_0 - E + i\epsilon} - \frac{1}{P_0 + E - i\epsilon} = \frac{2E}{P_0^2 - E^2 + i\epsilon}. \quad (80) \]

The Bethe-Salpeter equation for \( N \) particles can still be written

\[ \Phi = G^0 K \Phi \quad (81) \]

with

\[ K = K' + \Sigma, \quad \Sigma = (G^0)^{-1} - (G'^0)^{-1} \quad (82) \]

where \( K' \) will be given by a combination of irreducible \( 2 \leq n \leq N \)-body irreducible kernels.

\[ K' = K_{12}^' (G_3^0)^{-1} + K_{23}^' (G_1^0)^{-1} + K_{31}^' (G_2^0)^{-1} + K_{123}^'. \quad (83) \]

For \( N \geq 4 \) the writing of \( K' \) becomes more complicated, because of the presence of commutating kernels like \( K_{12}^' \) and \( K_{34}^' \) which would lead, without corrections, to an overcounting of some graphs in the expansion of \( G \). We examine this problem elsewhere [32].

Let us now define a \( N \)-body operator \( G^I \):

\[ G^I (p'_0, p_0) = G^0 (p'_0) \beta \frac{\tau \omega}{(-2i\pi)^{N-1}} \int dp_0 G^0 (p_0) \quad (84) \]

which is such that

\[ \int dp_0 G^I (p'_0, p_0) = G^0 (p'_0) \beta^2 = \tau^2 G^0 (p'_0), \]

\[ \int dp'_0 G^I (p'_0, p_0) = \tau^2 G^0 (p_0). \quad (85) \]

Performing a 3D reduction as in subsection 2.3, we get again

\[ \phi = g^0 \tau < \tilde{K}^T > \phi \quad (86) \]
with the definitions

\[ \phi = \tau \sqrt{\omega} \int dp_0 \Phi(p_0) \quad (87) \]

\[ \bar{K}^T = K(1 - \bar{G}^R K)^{-1}, \quad \bar{G}^R = G^0 - G^I \quad (88) \]

and

\[ < A > = \frac{1}{(-2\pi)^{N-1}} \frac{\tau^2 \sqrt{\omega}}{g^0} \int dp'_0 dp_0 G^0(p'_0) A(p'_0, p_0) G^0(p_0) \beta \frac{\tau^2 \sqrt{\omega}}{g^0} \quad (89) \]

so that

\[ G^I = > \tau g^0 < . \quad (90) \]

As in the two-fermion problem, we can choose to work in the \( \Lambda^+ = 1 \) subspace instead of the \( \tau^2 = 1 \) subspace, by simply replacing \( \tau \) by \( \Lambda^+ \). In this case also it is possible to start the 3D reduction by taking the retarded part of the full propagator at equal times \[2^{3+} \quad 24].

### 3.2 Scattering.

As in the two-fermion case, we have \( \bar{T}^{3D} = < T > \). This implies that \( \bar{T}^{3D} \) is equal to the physical scattering amplitude when all particles are on their positive-energy mass shells in their initial and final states. In actual scattering experiments, however, the initial state consists in two clusters only.

As shown already in \[2^{3}\], the 3D equation remains valid in these cases. The scattering of a bound state of fermions (23) by fermion 1 to a bound state of fermions (12) plus fermion 3, for example, can be described by a part of the full propagator \( G \):

\[ G = G^0_{12} T_{12,23} G_{23}^0 + \cdots \quad (91) \]

where \( T_{12,23} \) contains neither initial (23) nor final (12) interaction, these interactions being included in \( G_{23} \) and \( G_{12} \) respectively. The full (23) propagator can be expanded as \[3^{3} \quad 14\]

\[ G_{23}(p_{23}, p'_{23}, p_{23}) = \Phi_{23}(\vec{P}_{23}, p'_{23}) \frac{-i}{P_{230} - E_{23} + i\epsilon} \Phi_{23}(\vec{P}_{23}, p_{23}) + \cdots \quad (92) \]

where we isolated the pole at \( P_{230} = E_{23} = \sqrt{\vec{P}_{23}^2 + P_{23}^2} \), defining \( \Phi_{23} \) as \( \Phi_{23}^{+} \beta_{2} \beta_{3} \).

The full (12) propagator can be expanded similarly. The N-fermion 3D full propagator is, according to \[8^{3}\]:

\[ g = \frac{1}{(-2\pi)^{N-1}} \Lambda^+ \int dp'_0 dp_0 G(p'_0, p_0) \beta \Lambda^+. \quad (93) \]

Taking \( N = 3 \) and putting \[8^{1}\) into \(8^{3}\), we see that the integration of \( \Phi_{23}(\vec{P}_{23}, p_{23}) \Lambda_{23}^{+} \) with respect to \( p_{230} \) will give \( \phi_{23}(\vec{P}_{23}, p_{23}) \) while in the integration with respect to \( p_{10} \) we shall isolate the residue of the pole of \( G^0_{1} \) at
\( p_{10} = E_1 \) (and similarly for the final state). This leads to

\[
g = -\phi_{12} \frac{\Lambda_1^+}{P_0 - E_{12} - E_3 + i\epsilon} \left[ \mathbf{F}_{12} T_{12,23} \Phi_{23} \right]_{\delta \delta} \frac{\beta_1 \Lambda_1^+}{P_0 - E_{23} - E_1 + i\epsilon} \phi_{23}^+ + \cdots
\]

(94)

where

\[
\left[ \mathbf{F}_{12} T_{12,23} \Phi_{23} \right]_{\delta \delta} \equiv \int dp_0 dp_0' \delta(p_0' - E_3) T_{12,23} (p_0', p_0) \delta(p_{10} - E_1). \]

(95)

If we now compute \( g \) at the 3D level, using the free propagator \( \Lambda_+ g^0 \) and the potential \( \tilde{V} \), we get

\[
g = \Lambda_3^+ g_{12} \tilde{T}_{12,23}^3 \Lambda_1^+ \phi_{23} + \cdots
\]

(96)

while \( g_{23} \) is obtained from \( G_{23} \) through (93):

\[
g_{23} = \frac{1}{2\pi} \phi_{23} \frac{1}{P_0 - E_{23} - E_1 + i\epsilon} \phi_{23}^+ + \cdots
\]

(97)

and similarly for \( g_{12} \). If we identify (94) to (96) in momentum space, we get the following relation between the physical scattering amplitudes computed at the 3D and at the 4D levels:

\[
\phi_{12}^+ \tilde{T}_{12,23}^3 \phi_{23} = (-2i\pi)^2 \left[ \mathbf{F}_{12} T_{12,23} \Phi_{23} \right]_{\delta \delta} \beta_1
\]

(98)

the momenta being such that \( P_0 = E_{12}' + E_3' = E_{23} + E_1 \).

Here, \( \Phi_{23} \) depends on \( \vec{P}_{23}, \vec{p}_{23} \) while \( \phi_{23} \) depends on \( \vec{P}_{23}, \vec{p}_{23} \). We could compute \( \phi_{23} \) at \( \vec{P}_{23} = 0 \), then compute \( \Phi_{23} \) from it, boost \( \Phi_{23} \) to \( \vec{P}_{23} \neq 0 \), and use it in the right-hand side of (98), or we could try to compute directly \( \phi_{23} \) at \( \vec{P}_{23} \neq 0 \), and use it in the left-hand side of (98).

We shall stop here these considerations about the scattering, which is not the main subject of this work and would be better studied on a specific example. We only wanted to show how the 3D equation not only gives the bound state spectrum, but is also valid for the scattering amplitudes.

### 3.3 Reduction of the Dirac spinors.

Let us first consider a fermion in a positive-energy potential:

\[
(p_0 - h) \Psi = V \Psi, \quad V = \Lambda^+ V \Lambda^+.
\]

(99)

Using the \( \Lambda^+ \) projector:

\[
(p_0 - E) \Psi^+ = V \Psi^+, \quad \Psi^+ = \Lambda^+ \Psi.
\]

(100)
We shall apply a virtual boost on $\Psi^+$:

$$\Psi^+ = \frac{1 + (\gamma \cdot n)(\gamma \cdot u)}{\sqrt{(u + n)^2}} \Psi^0, \quad \Psi^0 = \frac{1 + (\gamma \cdot u)(\gamma \cdot n)}{\sqrt{(u + n)^2}} \Psi^+.$$ (101)

with

$$n = \frac{1}{m} (E, \vec{p}), \quad u = (1, \vec{0}).$$ (102)

In terms of $\Psi^0$, equation (100) becomes

$$(P_0 - E) \Psi^0 = \frac{1 + (\gamma \cdot u)(\gamma \cdot n)}{\sqrt{(u + n)^2}} V \frac{1 + (\gamma \cdot n)(\gamma \cdot u)}{\sqrt{(u + n)^2}} \Psi^0$$

$$= \frac{1 + \beta}{2} \sqrt{\frac{2m}{E + m}} V \sqrt{\frac{2E}{2m(E + m)}} \Psi^0.$$ (103)

Writing

$$\Psi^0 = \sqrt{\frac{m}{E}} \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$$ (104)

we get

$$(P_0 - E) \varphi = \sqrt{\frac{2E}{E + m}} \sqrt{\frac{2E}{E + m}} \varphi$$ (105)

where $\nu$ is the large-large part of $V$. Similarly, for $f$ fermions and a positive-energy integrating propagator-based reduction, we have

$$(P_0 - E) \varphi = \left[ \prod_{i=1}^{f} \sqrt{\frac{2E_i}{E_i + m}} \right] \nu \left[ \prod_{i=1}^{f} \sqrt{\frac{2E_i}{E_i + m}} \right] \varphi.$$ (106)

4 Example: two and three fermions in QED.

Our aim in this section is only to show how the integrating propagator-based 3D reduction method works for a system of $N > 2$ particles, since, as in any actual problem, specific complications are encountered (choice of the gauge, estimation of the order of the contributions for the exchange of zero-mass quanta, detection of the cancellations...). In Coulomb’s gauge the relative energy dependence of the Bethe-Salpeter kernel lies in the transverse part, the contributions of which are typically smaller by a factor $\alpha^2$. This makes the calculation of the first-order contributions (up to $\alpha^4$) more easy, but does not provide a good illustration of our 3D reduction method, the three-body contributions to the 3D potential being then of order $> \alpha^4$ and competing with a lot of other effects. We shall therefore work in Feynman’s gauge until subsection 4.4.
4.1 Two fermions.

For two fermions of charges $Z_1, Z_2$ the Bethe-Salpeter kernel is

$$K(p', p) = \frac{2i Z_1 Z_2}{(2\pi)^3} \frac{1}{k^2 + ie} (\gamma_1 \cdot \gamma_2) + \left[ \frac{2i Z_1 Z_2}{(2\pi)^3} \right]^2 \int \frac{dk}{k^2 + ie} \frac{1}{k^2 + ie} \gamma^{\mu \rho} \gamma^{\nu \mu} G_1^0(q_1) G_2^0(q_2^0) \gamma_{\mu \rho \nu} \gamma_{2 \mu} + \cdots$$  \hspace{1cm} (107)

We wrote explicitly the ladder term $K^L$ and the first crossed term $K^C$. The meaning of the variables can be read on figure 1, which represents $<K^L>, <K^C> \text{ and } <K^L G_R K^L > = <K^L G^0 K^L > - <K^L G^0 <K^L >$. The results of subsections 2.2 to 2.4 will be used, and specialized to the kernel [107]. If we choose to perform the 3D reduction in the $\Lambda^++=1$ subspace, we get

$$<K^L> (p', p) = \frac{2i Z_1 Z_2}{(2\pi)^3} \Lambda^{++} (p') (1 - \bar{a}_1 \cdot \bar{a}_2) \Lambda^{++} (p) I(p', p)$$  \hspace{1cm} (108)

$$I(p', p) = \frac{(P_0 - E'_1 - E'_2)(P_0 - E_1 - E_2)}{-2i\pi} \int dp_0 dp_0' \frac{1}{p_{10}' - E_1' + ie} \frac{1}{p_{20}' - E_2' + ie}$$

$$= \frac{-2i\pi}{(E_1' - E_1)^2 - \vec{k}^2 + ie} \left[ 1 + \frac{1}{2|\vec{k}|} R \right]$$  \hspace{1cm} (109)

$$R = \frac{(E_1' - E_1 - |\vec{k}|)}{P_0 - E_1' - E_2 - |\vec{k}| + ie} - \frac{(E_1' - E_1 + |\vec{k}|)}{P_0 - E_1' - E_2 + |\vec{k}| + ie}$$  \hspace{1cm} (110)

where the first part of $I$ corresponds to $<K^L>_{\delta \delta}$ and contributes in $\alpha^2$ to the energy. The two terms of $R$ correspond to $<K^L>_{\delta R}$ and $<K^L>_{R \delta}$ respectively. At lowest order

$$R \approx (P_0 - E_1 - E_2) + (P_0 - E_1' - E_2').$$  \hspace{1cm} (111)

By sheer power counting (pc), $R$ should contribute in $\alpha^3$ ( $|\vec{k}|^{-1}$ brings a factor $\alpha^{-1}$ and the difference of the energies a factor $\alpha^2$, to be combined with the $\alpha^2$ of Coulomb's potential). However, this dominant contribution, if isolated, is in $|\vec{k}|^{-3}$ and makes the future integrations with respect to $\vec{k}$ diverge at origin. A more careful estimation is thus necessary, and shows that $R$ contributes in fact in $\alpha^3 \log(\alpha)$. This divergence of the dominant contribution is a consequence of the zero mass of the photon. From now on, we shall estimate only the (pc) order of the contributions, keeping in mind that the real order could be lower.
The expression of \( I \) is apparently asymmetric for the permutation of the fermions, but it can be shown that it is in fact equal to the expression obtained after permutation or the fermions, on by directly closing the integration paths counterclockwise. The differences between \([-\vec{k}^2]\), \([(E'_1 - E_1)^2 - \vec{k}^2]\) and \([(E'_2 - E_2)^2 - \vec{k}^2]\) are in \( \alpha^4 \) (pc) (these three denominators are equal in the equal mass case if we work in the two-fermion rest frame).

A third way of computing \( I \) consists in writing the photon’s pole as

\[
\frac{1}{k^2 + i\epsilon} = \frac{-1}{2|k|} \left[ \frac{1}{p'_0 - p_0 + |k| - i\epsilon} - \frac{1}{p'_0 - p_0 - |k| + i\epsilon} \right]
\]

and closing for each part of (112) the integration path in order to leave the photon’s pole outside, with the result

\[
I(p',p) = 2i\pi \frac{2|k|}{2|k|} \left[ \frac{1}{E_1 + E_2 - P_0 + |k| - i\epsilon} + \frac{1}{E_1 + E'_2 - P_0 + |k| - i\epsilon} \right]
\]

which is the expression obtained in time-ordered perturbation theory [34], and again a rearrangement of (109).

If we include now the four-vertex graphs, we must compute

\[
\bar{V} \approx <K^L> + <K^C> + <K^L G^0 K^L> - <K^L> g^0 <K^L>,
\]

of which we already have \( <K^L> \). If we choose to split \( G^0 \) into \( G^5 + G^R \), we get, keeping only the terms with zero or one \( G^R \),

\[
\bar{V} \approx <K^L>^{\delta\delta} + <K^C>^{\delta\delta} + <K^L G^R K^L>^{\delta\delta} + [1 - <K^L>^{\delta\delta} g^0] <K^L>^{\delta\delta} + <K^L>^{\delta R} [1 - g^0 <K^L>^{\delta\delta}].
\]

The second line will not contribute to the first-order energy shift of a perturbation calculation starting with \( <K^L>^{\delta\delta} \). If we start with a Coulomb potential, then \( <K^L>^{\delta\delta} \) will provide perturbations in \( \alpha^4 \) (pc) and the second line perturbations in \( \alpha^3 \) (pc). The two last terms of the first line are in \( \alpha^3 \) (pc), but their higher-order contributions cancel mutually, leaving contributions in \( \alpha^4 \) (pc).

### 4.2 Three fermions.

For the three-fermion problem, we can write

\[
\phi = g^0 \bar{V} \phi, \quad \bar{V} = <K> + <K G^0 K> - <K> g^0 <K> + \cdots
\]

with the definitions

\[
<A> = <<G^0 A G^0>>
\]
\[
\frac{-1}{4\pi^2} \Lambda^{+++}(g^0)^{-1} \int dp_0 dp_0 G^0(p_0') A(p_0', p_0) G^0(p_0) \beta_1 \beta_2 \beta_3 \Lambda^{+++}(g^0)^{-1}
\]

\[
g^0 = \frac{1}{P_0 - E_1 - E_2 - E_3 + i\epsilon}, \quad G^0 = G^0_1 G^0_2 G^0_3
\]

\[
K = K_{12}(G^0_3)^{-1} + K_{23}(G^0_1)^{-1} + K_{31}(G^0_2)^{-1} + K_{123}
\]

\[
dp_0 = \delta(P_0 - p_{10} - p_{20} - p_{30}) dp_{10} dp_{20} dp_{30}.
\]

We introduced the notation \(<<>\) in order to exhibit the \(G^0_1\) operators below. The \(K_{ij}\) will be given by \(\text{(107)}\). We shall here neglect the three-fermion irreducible kernel which begins with 6-vertex interactions only. The two- and 4-vertex diagrams contributing to \(\tilde{V}\) are given in figure 2 (we draw only \(L_3, C_3, B_3\) and \(L_{31}\), as the other ones can be obtained by two-fermion permutations and/or three-fermion circular permutations). The computation of these diagrams is tedious but straightforward. Each external fermion line represents a propagator \((p_{10} - E_1 + i\epsilon)^{-1}\) and each internal fermion line a propagator \((p_{30} - \bar{h}_i + i\epsilon h_i)^{-1}\). We could integrate clockwise with respect to the energies on the lines 1 and 3 (for the diagrams drawn in the figure), replacing \(p_{20}\) by \(P_0 - p_{10} - p_{30}\). The leading contributions will come from the four residues of the positive-energy fermion poles on lines 1 and 3, while higher-order contributions will appear when one or several of these poles are replaced by a photon pole or by an internal fermion 2 pole (in \(C_3\) the residues on the internal 1 and 2 lines are both leading terms, if we close the integral on \(p_{10}\) clockwise, but they cancel mutually at leading order).

For the sum of the \((12)+3\) unconnected graphs, the residue of the pole in \((p_{30} - E_3 + i\epsilon)^{-1}\) will give the \((12)\) potential, with the \((12)\) energy \(P_{120}\) replaced by \(P_0 - E_3\) (if we close the \(p_{30}\) integration path clockwise). The sum of the residues of the propagator of the spectator fermion in the unconnected graphs will thus give the sum of the three two-fermion potentials:

\[
\tilde{V}_u = \langle K_{12}^T \rangle_{p_{20} = P_0 - E_3} + \langle \tilde{K}_{23}^T \rangle_{p_{230} = P_0 - E_1} + \langle \tilde{K}_{31}^T \rangle_{p_{310} = P_0 - E_2}.
\]

In graphs like \(C_3\) and \(B_3\), however, the internal fermion propagators will also provide poles inside the \(p_{30}\) integration path, unless we neglect the negative energy parts. These contributions (which can be neglected at order \(\alpha^4\)) apparently violate cluster separability, even if we do not choose the reference frame as being the global rest frame and left it unspecified. They do not vanish indeed when all interactions between particle 3 and the \((12)\) cluster are "switched off". However, such a "switching off" is of course not compatible with the existence of a global bound state, and we have shown above that our 3D equation gives anyway the correct cluster-particle scattering amplitudes.

The connected graph \(L_{13}\) is more specific of a three-body problem. It is given by

\[
L_{13} = \langle K_{23}^L G^0_2 K_{12}^L \rangle - \langle K_{23}^L (G^0_1)^{-1} \rangle g^0 \langle K_{12}^L (G^0_3)^{-1} \rangle
\]
\[ \delta G_L \]

If we perform the energy integrations of (124), we found, for the first and the last terms

\[ L^{(1+4)}_{13}(\bar{p}', \bar{p}) = -4 \pi^2 \frac{1}{2|k'_{23}|} \frac{1}{P_0 - E_1 - E_2 (\bar{q}_2) - E_3 - |k'_{23}|} \frac{1}{(E_1 - E_1)^2 - |k'_{12}|^2} \]

\[ I^{(1+4)}_{13}(\bar{p}', \bar{p}) = -4 \pi^2 \frac{1}{2} \frac{1}{|k'_{23}|} \left[ \frac{P_0 - E_1 - E_2 (\bar{q}_2) - E_3 - |k'_{23}|}{P_0 - E_1 - E_2 - E_3 - |k'_{23}|} + 1 \right] \]

where we neglected the \( \Lambda^+_2 (\bar{q}_2) \) part of the internal \( G^0_{2}(\bar{q}_2) \) propagator and its residue. The \( L^{(2+3)}_{13} \) part of \( L_{13} \) in obtained by the permutations

\[ |k'_{23}| \leftrightarrow |k'_{12}|, \quad E_1 \leftrightarrow E'_3, \quad E'_1 \leftrightarrow E_3, \quad E_2 \leftrightarrow E'_2. \]
4.3 Gross’ spectator model.

When the three-fermion irreducible Bethe-Salpeter kernel is neglected, the global propagator becomes a sequence of two-fermion interactions, for which the third fermion is a spectator. In Gross’ spectator model [20, 21], the spectator fermion is put on its positive-energy mass shell (we could say that we keep only the $G^0_i$ part of the corresponding propagator $G^0_i$). When a (12) interaction, for example, is followed by a (23) interaction (first part of figure 3), the fermions 1 and 3 between them are put on their positive-energy mass shell. The neglected $G^R_i$ parts and the neglected three-body irreducible parts of the kernel are then treated as higher-order corrections. The result is a set of 3D equations for partial transition operators, which can not be reduced to a single equation as the initial and final "spectator" fermions are not the same in all terms.

In the integrating propagator based reduction, all terms of the transition operator are treated in the same way. We expect of course that the most important contributions to the integrals in the relative energies will be the residues of these propagator’s poles which put the spectator fermions on their positive-energy mass shell. The 3D potential $\tilde{V}$ differs from the integrated transition operator $<T>$ by the presence of the counter-terms in $-\gamma^0$, as shown in figure 3. If we put the spectator fermions on their positive-energy mass shell in these terms too, then the two diagrams of figure 3 cancel mutually, and, finally, there remains only the unconnected contributions (12) to the potential. This is in fact the kind of potential we proposed in ref. [19] (in which there is also a comparison with Gross’ spectator model). The $G^R_i$ parts of the free propagators of the spectator fermions and the irreducible three-body terms provide corrections to it.

4.4 Coulomb gauge.

In Coulomb’s gauge, we would make the replacement
\[
\frac{1}{k^2 + i\epsilon} \beta_1 \beta_2 (\gamma_1 \cdot \gamma_2) \rightarrow \frac{1}{-k^2} - \frac{1}{k^2 + i\epsilon} (\tilde{\alpha}_1 \cdot \tilde{\alpha}_2)^T \tag{128}
\]
\[
(\tilde{\alpha}_1 \cdot \tilde{\alpha}_2)^T = (\tilde{\alpha}_1 \cdot \tilde{\alpha}_2) - \frac{(\tilde{\alpha}_1 \cdot \tilde{k}) (\tilde{\alpha}_2 \cdot \tilde{k})}{k^2}. \tag{129}
\]

In (128), the relative-energy dependence lies in the second (transverse) term. The contribution of this term to the energy begins at $\alpha^4$, as it connects the positive-energy components with the negative-energy components. At this order, we can replace $k^2$ by $-k^2$. In the two-fermion problem, we get
\[
I (p', \vec{p}) = \frac{2\pi}{k^2} \tag{130}
\]
\[
\tilde{V} (p', \vec{p}) = -\frac{1}{2\pi^2} \frac{Z_1 Z_2}{k^2} \Lambda^{++} (p') (1 - (\tilde{\alpha}_1 \cdot \tilde{\alpha}_2)^T) \Lambda^{++} (\vec{p}). \tag{131}
\]
In the three-fermion problem, up to order $\alpha^4$, we simply must add the three Coulomb and the three Breit potentials:

$$ \tilde{V}(\vec{p}', \vec{p}) = \tilde{V}_{12}(\vec{p}'_{12}, \vec{p}_{12}) \delta^3(\vec{p}'_3 - \vec{p}_3) $$

$$ + \tilde{V}_{23}(\vec{p}'_{23}, \vec{p}_{23}) \delta^3(\vec{p}'_1 - \vec{p}_1) + \tilde{V}_{31}(\vec{p}'_{31}, \vec{p}_{31}) \delta^3(\vec{p}'_2 - \vec{p}_2). $$

(132)

The contributions which are specific to the three-body problem will thus appear beyond $\alpha^4$.

5 Conclusions

Our integrating propagator-based reduction of the Bethe-Salpeter equation provides a quite straightforward way of writing the different terms of the 3D potential for any N-particle bound state equation. The continuum-dissolution problem is automatically avoided. Our reduction provides a link between the kernel-based reductions (especially that of Phillips and Wallace [18]) and the equal-time retarded full propagator method of Logunov and Tavkhelidze [22] and Kvinikhidze and Stoyanov [23, 24]. Although first developed for the two-fermion bound state problem, it has been naturally generalized to the N-particle bound state problem and is also valid for the 2-body and particle-cluster scattering problems (even if one may prefer another approach in this case). The 3D equation provides a (in principle) indefinitely improvable approach of the exact values of the measurable quantities, despite the fact that the individual terms of the series giving the 3D potential are neither Lorentz covariant nor all cluster separable.

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Figure 1. The lowest-order two-fermion graphs.

Figure 2. The lowest-order three-fermion graphs.

Figure 3. Gross' spectator model.