Loop quantization as a continuum limit

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Abstract
We present an implementation of Wilson’s renormalization group and a continuum limit tailored for loop quantization. The dynamics of loop-quantized theories is constructed as a continuum limit of the dynamics of effective theories. After presenting the general formalism we show as a first explicit example the 2D Ising field theory, an interacting relativistic quantum field theory with local degrees of freedom quantized by loop quantization techniques.

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1. Introduction and summary of results
Loop quantization was originally motivated by its application to theories free of a background metric like gravity and gravity coupled to matter. However, an extended version of the formalism, sometimes referred to as a ‘polymer representation’, has been developed for gauge theories with compact gauge group, sigma models with compact group and scalar fields and it is equally applicable to theories free of a background metric and to theories that use a background metric. Much of the interest in this extended formalism is the possibility of comparing with well-established physics. The kinematics of such quantum field theories is very well understood. They are characterized by a space of generalized ‘polymer-like’ field configurations (connections in the case of gauge theories) $\mathcal{A}_M$ that has been extensively studied. In this work, we will loosely refer to theories in the continuum whose kinematical basis is a ‘polymer representation’ on the space $\mathcal{A}_M$ as loop-quantized theories. In contrast with the kinematical side, the dynamics of quantum theories constructed by loop quantization is regarded as well understood only for topological theories [1–3] and for two-dimensional Yang–Mills theories [4].
It has been argued that it would be natural to construct the dynamics of loop-quantized theories as a continuum limit of the dynamics of effective theories following Wilson’s renormalization group ideas, as is done in lattice gauge theories [3, 5]. However, the concept of scale that lies at the centre of the renormalization group is background metric-dependent.

In our work, we overcome this obstacle by postulating an ‘extended notion of scale’; then we implement Wilson’s renormalization group and the continuum limit based on this notion. When such a continuum limit exists, it defines the dynamics of a loop-quantized theory. If this ‘extended notion of scale’ is used in metric-dependent theories, it can be reduced to standard regular lattices. Structures of the type presented in this paper should be able to bring a lot of the work on metric theories done by ordinary lattice gauge theory (analytical and numerical) to the extended loop quantization framework.

In discrete approaches to 4D quantum gravity, the intention to implement Wilson’s renormalization group goes back to Regge calculus [6] and reappears in the context of spin foam models [7, 8]. Also in this context, the issue of the theory defined in a continuum (‘ultra-violet’) limit has always been of interest. In this work and in [9], the ‘extended notion of scale’ mentioned above places a cut-off in the continuum of (a spacetime version of) loop quantization that yields effective theories that are discrete models of the type used in [8].

Loop quantized theories live in the continuum, but many of its constructions rely on auxiliary discrete structures. These constructions include a procedure in which the regulator is removed to end with a continuum free of any foreign discrete structure. For example, discrete structures are used (i) to define/regularize the scalar constraint in loop quantum gravity [10], (ii) to construct kinematical states that approximate given classical geometries [11], (iii) to assign a meaning to continuum path integrals by means of a ‘summation prescription’ for spin foams [5], (iv) in 3D gravity to make the Reisenberger–Rovelli construction of the ‘generalized projector’ concrete and realize it as a continuum path integral in the sense of (iii). This regularization by Perez and Noui defines the space of physical states solving the constraints of canonical 3D gravity as a sum over spin foams therefore constructing a clear link between spin foams and canonical loop quantum gravity [2].

By contrast, in this work the discrete models have the physical interpretation of effective theories at a given ‘scale’. Even the continuum limit theory should be thought of in terms of effective theories at a given ‘scale’ and renormalization. In our context, a loop-quantized theory in the continuum has the physical interpretation of a bookkeeping device organizing the information contained in completely renormalized theories at any given ‘scale’. Moreover, this interpretation comes together with a powerful method for constructing non-trivial loop-quantized theories.

We conclude with an explicit example; using our implementation of Wilson’s renormalization group and our continuum limit, we construct the dynamics of a loop-quantized quantum field theory. The example is a very well-studied quantum field theory: 2D Ising field theory. It is an interacting relativistic quantum field theory with local degrees of freedom quantized by loop quantization techniques. We remark that although this example is background dependent, the formalism that we present in this paper does not use the background directly. The same formalism can be applied successfully to topological theories. Thus, our formalism is applicable to background-independent theories and can produce quantum field theories with local degrees of freedom. We have not yet succeeded in constructing a background-independent quantum field theory with local degrees of freedom.

The paper is organized as follows. In section 2, we review basic notions and notation of cellular decompositions and loop quantization. Our implementation of Wilson’s renormalization group and the continuum limit are contained in section 3. The example of the 2D Ising field theory is presented in the closing section.
2. Basic notions

2.1. Cellular decompositions

We start with a review of a few preliminary notions and notation on cellular decompositions of manifolds. In this work by a cellular decomposition $C$ of a manifold $M$ we mean a presentation of it as a locally finite union of disjoint cells

$$M = \bigcup_{c_\alpha \in C} c_\alpha, \quad c_\alpha \cap c_\beta = \emptyset \quad \text{if} \quad \alpha \neq \beta.$$ 

Each cell $c_\alpha$ is the embedding of an open convex polyhedron of dimension between zero and $\dim M$. A typical example of a cellular decomposition is the minimal triangulation of the sphere whose cells are: four triangles, six edges and four vertices.

We will be concerned with the calculation of $n$-point functions. Thus, we will have a manifold $M$ with a set of $n$ marked points $\{p_1, \ldots, p_n\}$. A generic cellular decomposition of a manifold $M$ with $n$ marked points is a cellular decomposition of $M$ such that each of the marked points is contained in a cell of the same dimension as $M$.

The families of cellular decompositions that we use enjoy three properties which let them play the role of ‘cut-off scales’:

(i) Partial order relation. Two cellular decompositions are related $C_1 \leq C_2$ if any cell in the coarser decomposition $C_1$ is a finite union of cells of the finer decomposition $C_2$.

(ii) Common refinement. Given any two cellular decompositions $C_1, C_2$ there is a common refinement $C_3$ such that $C_1 \leq C_3$ and $C_2 \leq C_3$. This property makes the family of cellular decompositions a partially ordered and directed set. (Directed towards refinement.)

(iii) Infinite refinement. Given any open set $U$ of $M$ there is a fine enough cellular decomposition $C_0$ in the family with a cell $c_\alpha \in C_0$ of dimension $\dim(c_\alpha) = \dim(M)$ that is completely contained in the open set, $c_\alpha \subset U$.

There are many families of cellular decompositions enjoying these properties.

For a given cellular decomposition $C$, the set of its cells will be denoted by $L(C)$; if $c_\alpha$ is a cell in $C$ we will refer to the corresponding element of $L(C)$ simply by its label $\alpha \in L(C)$. The elements of this abstract set can be thought of as points, but are not a priori related to specific points in $M$. However, the structure of the cellular decomposition has a preferred type of map from $L(C)$ to $M$ that we will call representative embeddings. These maps $\text{Emb}_{L(C)} : L(C) \to M$ must obey $\text{Emb}_{L(C)}(\alpha) \in c_\alpha \subset M$ for every cell of the cellular decomposition. Clearly specifying one such representative embedding implies a choice.

Given two cellular decompositions $C_1 \leq C_2$ there is a natural map $r_{2,1} : L(C_2) \to L(C_1)$ which sets $r_{2,1}(\alpha^{(2)}) = \alpha^{(1)}$ if and only if $c_\alpha^{(2)} \subseteq c_\alpha^{(1)}$ ($c_\alpha^{(2)}$ is a cell in $C_2$ and $c_\alpha^{(1)}$ is a cell in $C_1$). Again one can embed $L(C_1)$ into $L(C_2)$ at will, but we call an embedding $\text{Emb}_{1,2} : L(C_1) \to L(C_2)$ representative if $r_{2,1} \circ \text{Emb}_{1,2} = \text{id}$. With this we close our review of preliminary notions of cellular decompositions.

2.2. Kinematics of the continuum: kinematics of loop-quantized Euclidean quantum field theories

Here we will work in the Euclidean (imaginary time) description of quantum field theory, but a similar structure also exists in a Hamiltonian description. In this work, we will give the general prescription for spin systems, sigma models and scalar fields; for gauge systems the different aspects are treated carefully in [9].
The space of Euclidean histories will be denoted by $\bar{A}_M$ and its elements $s \in \bar{A}_M$ assign an element of a compact group\(^3\) to any point of spacetime $s(p) \in G$ without any continuity requirement. Thus, the algebra of fundamental observables is considered to be the so-called ‘cylindrical functions’ $f \in \text{Cyl}(\bar{A}_M)$ which are functions that depend on the histories restricted to finitely many points of spacetime, $f = f((s(p)))$. A very important particular case is the product of functions that depend on a single point of spacetime; the expectation values of such products are the familiar $n$-point functions.

2.3. Dynamics of the continuum: physical measure

A physical measure $\mu_M$ on $\bar{A}_M$ lets us calculate expectation values and correlations among physical observables; $\langle f \rangle = \int_{\bar{A}_M} f \, d\mu_M$ gives a precise meaning to expressions of the type $\langle f \rangle = \frac{1}{2} \int \mathcal{D}\phi \exp(-S(\phi)) f(\phi)$. We stress that this measure encodes the dynamics of the theory in contrast to the auxiliary measure used to define the inner product in the usual kinematical Hilbert space of canonical loop-quantized theories. The construction of physical measures is the primary goal of our work.

3. Implementation of Wilson’s renormalization group and the continuum limit

We define a space of effective Euclidean histories at scale $C$ as the space of $C$-constant field configurations, $A_C$. By definition $s \in A_C \subset \bar{A}_M$ if and only if for any $p, q \in M$ contained in the same cell of $C$ $s(p) = s(q)$. Clearly the space of $C$-constant functions on $M$ is in one-to-one correspondence with the space of functions on $L(C)$. Thus, there is a natural identification between $A_C$ and the space $L(C)$ consisting of functions from $L(C)$ to $G$. The algebra of effective observables at scale $C$ is denoted by $\text{Cyl}(A_C)$ and consists of functions depending on the configurations of finitely many cells.

The effective theory at scale $C$ is defined after we specify a measure in the space of effective histories that lets us calculate the expectation value of physical observables, $\langle f \rangle_C = \int_{A_C} f \, d\mu_C$. Thus, an effective theory at scale $C$ is a pair $(A_C, \mu_C)$. As mentioned above, in this work we define effective theories for spin systems, sigma models and scalar fields; for gauge systems the different aspects are treated carefully in [9].

Given two scales $C_1 \ll C_2$ there are two ways to relate the corresponding effective theories. One map $i_{C_1} \to i_{C_2}$ in the direction of refinement (which will be shown to induce regularization) and a coarse-graining map $\pi_{C_2} \to \pi_{C_1}$.

$$(A_{C_1}, \mu_{C_1}) \xrightarrow{i_{C_1}, \pi_{C_2}} (A_{C_2}, \mu_{C_2}).$$

For these maps $i_{C_1} \circ \pi_{C_2} = \text{id}$. First let us describe the map in the direction of refinement. Since $A_{C_1} \subset A_{C_2}$, the needed map is the inclusion map $i_{C_2} : A_{C_2} \to A_{C_1}$. It can be useful to note that in terms of the spaces $A_{L(C)}$ the map in the direction of refinement is $i_{C_1} \circ i_{L(C)} = r_{1^*} : A_{L(C)} \to A_{L(C)}$. Clearly also $A_C \subset \bar{A}_M$ and we also have the inclusion map $i_C : A_C \to \bar{A}_M$. This map induces a regularization map that brings any observable of the continuum to scale $C$, $i_C^* : \text{Cyl}(\bar{A}_M) \to \text{Cyl}(A_C)$. These regularization maps link all the effective theories to the theory in the continuum. There are many observables of the continuum that get regularized to the same observable at scale $C$. However, due to the infinite refinement property (iii) of our families of cellular decompositions, two cylindrical

\(^3\) The loop quantization (polymer representation) of the scalar field is based on a compact group, the Bohr compactification of $\mathbb{R}$ [12].
functions of the continuum \( f, g \in \text{Cyl}(\mathcal{A}_M) \) are different if and only if there is a sufficiently fine scale \( C_0 \) such that \( l_{C_0}^f \neq l_{C_0}^g \). This property of regularization on our families of effective configurations says that the subset of \( \mathcal{A}_M \) consisting of elements that are eventually in \( \mathcal{A}_C \) (with respect to the directed partial order of the family) is a dense subset of \( \mathcal{A}_M \). This result can be written symbolically as

\[
\lim_{C \to M} \mathcal{A}_C = \mathcal{A}_M,
\]

and its main significance for the rest of this work is that it will allow us to construct interesting measures in \( \mathcal{A}_M \) as a continuum limit of effective measures.

The coarse-graining map \( \pi_{C \to C'} \) models an ‘averaging procedure’ by means of decimation. For each cell \( c_\alpha \) in \( C_1 \) we choose one ‘preferred cell’ \( c'_\alpha \) in \( C_2 \) among the cells that are contained in \( c_\alpha \) and define the coarse-graining map based on this choice. If we are given \( s \in \mathcal{A}_{C_2} \), the configuration \( \pi_{c_\alpha \to C_1} s \) is the \( C_1 \)-constant configuration whose value at \( p \in c_\alpha \) is \( s(p') \) for any point \( p' \in c'_\alpha \). We could say that \( \pi_{c_\alpha \to C_1} \) forgets everything about the configuration \( s \in \mathcal{A}_{C_2} \) except for its values at the ‘preferred’ cells. It is convenient to formulate this definition in terms of the spaces \( \mathcal{A}_{L(C)} \). Given a choice of representative embedding \( \text{Emb}_{1,2} : L(C_1) \to L(C_2) \), our definition is \( \pi_{c_\alpha \to C_1} := \text{Emb}_{1,2} : \mathcal{A}_{L(C_1)} \to \mathcal{A}_{L(C_1)} \). Coarse graining from the continuum is done similarly; after the choice of \( \text{Emb}_{L(C)} : L(C) \to M \), we define \( \pi_C := \text{Emb}_{L(C)} : \mathcal{A}_M \to \mathcal{A}_{L(C)} \).

Coarse graining maps naturally act on measures letting us calculate expectation values of functions of coarse observables according to the effective theory defined at the fine scale. As in any decimation, we simply integrate out the degrees of freedom that do not have an impact at the coarser scale,

\[
\langle f \rangle_{C_1(C)} := \int_{\mathcal{A}_{C_1}} f \, d(\pi_{C \to C_1}, \mu_{C_1}) = \int_{\mathcal{A}_{C_2}} (f \circ \pi_{C \to C_1}) \, d\mu_{C_1}.
\]

By this formula we define the measure \( \pi_{C \to C_1}, \mu_{C_1} \) in \( \mathcal{A}_{C_1} \). Similarly, measures \( \mu_M \) on \( \mathcal{A}_M \) can be coarse-grained to measures \( \pi_{C \to C_1}, \mu_{C_1} \) on \( \mathcal{A}_C \) defined by \( \langle f \rangle_{C(M)} := \int_{\mathcal{A}_M} f \, d(\pi_{C}, \mu_M) = \int_{\mathcal{A}_M} (f \circ \pi_{C}) \, d\mu_M \).

We have defined effective theories, regularization and coarse graining. With these elements at hand, we now present our implementation of Wilson’s renormalization group and the continuum limit.

Consider a sequence of increasingly finer scales \( \{ C_i \} \). Any two effective theories of the sequence are related by a coarse-graining map, different coarse-graining maps being restrained by the compatibility condition \( \pi_{C_1 \to C_2} \circ \pi_{C_2 \to C_3} = \pi_{C_1 \to C_3} \) whenever \( C_1 \leq C_2 \leq C_3 \). Now given any observable \( f \in \text{Cyl}(\mathcal{A}_C) \) at a fixed scale \( C_0 \), we can use a coarse-grained measure \( \pi_{C \to C_0}, \mu_{C_0} \) to calculate its expectation value \( \langle f \rangle_{C_0(C)} := \int_{\mathcal{A}_{C_0}} (f \circ \pi_{C \to C_0}) \, d\mu_{C_0} \) for any \( C_i \geq C_0 \). In this sense, the physics of a system at a given scale can be described by measures at any finer scale. Observe that the maps between measures

\[
\mu_{C_i} = \pi_{C_1 \to C_i}, \mu_{C_1}
\]

go from the finer to the coarser scale. We will refer to them as exact renormalization group transformations.

However, these transformations do not directly define a continuum limit for the measure given that this limit is to be taken in the opposite direction, i.e., towards finer scales. Furthermore, measures related by exact renormalization group transformations are generically of a very complicated functional form. In the following, we will therefore rather consider measures restricted to a certain functional form when written in terms of Boltzmann weights. Such measures will be denoted as \( \mu_{\beta(C)} \) and are parametrized by a small set of coupling.
constants $\beta(C_i)$. To fix these coupling constants at any scale, so-called renormalization conditions are imposed on the measures. They usually consist in prescribing the expectation values of certain key observables at some scale, $(f_{i})_{C_{0}}$; that is, the equations
\[
\langle f_{i} \rangle_{C_{0}} := \int_{A_{C_{0}}} f_{i} \, d\mu_{\beta}(C_{0}) = \int_{A_{C_{i}}} (f_{i} \circ \pi_{C_{i},C_{0}}) \, d\mu_{\beta}(C_{i})
\]
are considered as conditions that fix measures $\{\mu_{\beta}(C_{i})\}$ for any $C_{i} \geq C_{0}$.

Note that we also consider situations in which the measure $\mu_{\beta}(C)$ is not ‘homogeneous’ in any way and to completely specify it we need local coupling constants [8]; that is coupling constants that can take different values at different cells of our cellular decompositions. We then also need local renormalization conditions to specify the measure. Such a situation arises for example in inhomogeneous materials and in systems that do not depend on a metric background. Thus, allowing such systems is essential to include quantum gravity in our framework.

We are now in a position to consider the continuum limit $C_{i} \to M$ of ever finer scales $\{C_{i}\}$, where the corresponding measures are determined by an appropriate set of renormalization conditions. The relation (2) will not be satisfied at any scale $C_{i}$, but it may be satisfied asymptotically in the following sense. We will say that a theory ‘possesses a continuum limit’ when for an appropriate set of renormalization conditions the expectation values $\langle f \rangle_{C_{0}(C_{i})}$ of all observables $f \in \text{Cyl}(A_{C_{i}})$ at any given scale $C_{0}$ converge in the limit $C_{i} \to M$. In other words, the completely renormalized measures $\mu_{C_{0}}^{\text{ren}}$ defined by
\[
\langle f \rangle_{C_{0}}^{\text{ren}} := \lim_{C_{i} \to M} \langle f \rangle_{C_{0}(C_{i})} = \lim_{C_{i} \to M} \int_{A_{C_{i}}} (f \circ \pi_{C_{i},C_{0}}) \, d\mu_{\beta}(C_{i})
\]
should exist at any given scale $C_{0}$. We clarify that the meaning of the limit written above is that given any fixed $f \in \text{Cyl}(A_{C_{0}})$ and $\epsilon > 0$ there is a sufficiently fine cellular decomposition $C$ such that $\left| \langle f \rangle_{C_{0}}^{\text{ren}} - \langle f \rangle_{C_{0}(C_{i})} \right| \leq \epsilon$ for any $C_{i} \geq C$.

Note that when the above limits exist, the completely renormalized measures in the sequence $\{\mu_{C_{0}}^{\text{ren}}\}$ are compatible with coarse graining in the sense that
\[
\mu_{C_{i}}^{\text{ren}} = \pi_{C_{i},C_{0}}^{\text{ren}} \mu_{C_{0}}^{\text{ren}}. \tag{3}
\]
Observe also that any macroscopical observable at some scale $f \in \text{Cyl}(A_{C_{0}})$ naturally induces a sequence of macroscopical observables to any finer scale $C_{i} \geq C_{0}$, $\{f \circ \pi_{C_{i},C_{0}}\}$. Due to the compatibility of completely renormalized measures (3), we can define
\[
\left\langle \left\{f \circ \pi_{C_{i},C_{0}}\right\} \right\rangle_{C}^{\text{ren}} := \left\langle f \circ \pi_{C_{0}}^{\text{ren}}\right\rangle_{C}
\]
for any scale $C \geq C_{0}$. Thus, there is a continuum limit which describes observables that are macroscopic at some scale.

Note, however, that a sequence of compatible coarse-graining maps $\{\pi_{C_{i},C_{0}}\}$ with $C_{i} \to M$ induces a coarse-graining map from the continuum $\pi_{C_{0}} : \mathcal{M} \to \mathcal{A}_{C_{0}}$ and vice versa. Then the sequence $\{f \circ \pi_{C_{i},C_{0}}\}$ can be seen as $f \circ \pi_{C_{0}} \in \text{Cyl}(\mathcal{A}_{M})$. In this form, the macroscopical character of these observables is lost because any observable $g \in \text{Cyl}(\mathcal{A}_{M})$ would be of the form $f \circ \pi_{C_{0}}$ for an appropriate choice of $\pi_{C_{0}}$, or what is the same, for an appropriate choice of compatible coarse-graining maps $\{\pi_{C_{i},C_{0}}\}$.\footnote{This happens because our $\pi_{C_{i},C_{0}}$ are brute decimation maps; observables of the type $f \circ \pi_{C_{0}}$ would be smoothly smeared if our coarse-graining maps were constructed by a true averaging procedure.} From this point of view, it is natural to ask for a continuum limit measure that can act on any observable in $\text{Cyl}(\mathcal{A}_{M})$. This is where our regularization maps enter the continuum limit; given any $f \in \text{Cyl}(\mathcal{A}_{M})$, we calculate the expectation value of its regularization to scale $C$, $\langle f \rangle_{C}$, and look for convergence as the
regularization scale gets ever finer. We say that the family of measures \( \{ \mu_C \} \) has \( \mu_M \) as a continuum limit if for any \( f \in \text{Cyl}(\mathcal{A}_M) \) the following limit exists:

\[
(f)_{M} = \lim_{C \to M} (i_C^* f)_C.
\] (4)

To complete the formal definition of this continuum limit, we only have to say that it is taken in the subfamily of cellular decompositions that are generic with respect to \( f \). If the cylindrical function \( f \) is sensitive to the collection of \( G \)-configurations \( \{ s(p_1), \ldots, s(p_n) \} \), the condition on the cellular decompositions is that the points \( p_i \) lie on cells of maximal dimension (see section 2). If the limit in equation (4) exists for every cylindrical function, we have given a constructive definition of a functional \( \mu_M \) in \( \mathcal{A}_M \). The proof of linearity is trivial and the positivity of \( \mu_C \) implies non-negativity of \( \mu_M \).

The physical relevance of the continuum limit measure \( \mu_M \) rests on the fact that when it exists the completely renormalized measures \( \mu_\text{ren} \) also exist and they satisfy

\[
\pi_{C,\cdot}^{-1} \mu_M = \mu_\text{ren}_{\cdot},
\]

which means that \( \mu_M \) acts on any observable in \( \text{Cyl}(\mathcal{A}_M) \) while inducing the correct completely renormalized measures; thus, (4) extends the continuum limit to \( \mathcal{A}_M \).

**Remark.** If we have chosen to work with a family of cellular decompositions that remains invariant under the transformations of the group of spacetime symmetries of our system, then no background structure foreign to the symmetry group was used in the construction of the measure (4). Alternatively, we can choose a small, more economic, family of cellular decompositions that is not invariant under the group of spacetime symmetries of our system. In this case, our choice of family is completely arbitrary and it is necessary to check whether the resulting measure depends on it before the results are trusted. For example, if the system has the rotation group as a symmetry and the economic family breaks rotational invariance by the introduction of preferred coordinate axes, one must check that in the continuum limit rotational symmetry is restored.

### 4. Explicit example: 2D Ising field theory

Here, we simply present 2D Ising field theory as an example of our construction of measures in the continuum. The physics of the scaling limit in critical phenomena is at the core of our proposal and it is amply discussed in the literature. During the course of our work, we found Kadanoff’s book [13] particularly enlightening.

In this case, the space of Euclidean histories is the space of spin fields on \( \mathbb{R}^2, \mathcal{A}_\mathbb{R}^2 \). We assume that the system that we are describing has physical correlation length \( \xi_\text{phys} \) and that this datum is given to us.

Our approach will use the economic families of cellular decompositions of \( \mathbb{R}^2 \) composed of **lattice-type cellular decompositions**. Here, we describe them in detail. \( C_{m,t} \) is a Cartesian lattice-type cellular decomposition of size \( a_m = 1/2^m \) (with the directions of coordinate axis fixed). The two-dimensional cells of \( C_{m,t} \) are squares, its one-dimensional cells are horizontal or vertical open segments and its zero-dimensional cells are points (that we will call vertices).

We can describe \( C_{m,t} \) by the position of its vertices. First, label the vertices using a pair of integers, say \( I \in \mathbb{Z} \) for the \( x \) direction and \( J \in \mathbb{Z} \) for the \( y \) direction, \( v^{IJ} \). Their position is given by \( (v^I_x, v^I_y) = (a_m I, a_m J) + (t_x, t_y) \). (The parameter \( t \) ‘slightly shifts’ the whole lattice-type cellular decomposition, translating it rigidly by \( t \).)

For any fixed value of the parameter \( t \) and letting \( m \) run through the naturals, \( \{ C_{m,t} \}_{m \in \mathbb{N}} \) is a family of cellular decompositions that satisfies the properties (i)–(iii) of section 2. Also,
given any collection of marked points \( \{ p_1, \ldots, p_n \} \) there are values of \( t \) such that \( \{ C_{m,t} \}_{m \in \mathbb{N}} \) is generic with respect to them. We recall that this means that for any cellular decomposition of the family all the marked points fall inside two-dimensional cells, \( \{ \alpha^{(m)}_1, \ldots, \alpha^{(m)}_n \} \).

The ‘\( n \)-point’ functions completely characterize the dynamics of an effective theory and later we will use \( n \)-point functions to characterize the measure in the continuum. At scale \( C_{m,t} \), the ‘\( n \)-point functions’ are

\[
\langle s(\alpha_1) \cdots s(\alpha_n) \rangle_{C_{m,t}} = \frac{1}{Z_{C_{m,t}}} \sum s(\alpha_1) \cdots s(\alpha_n) \exp \left[-\beta_{C_{m,t}} \sum_{(\alpha_j,\alpha_k)} s(\alpha_j) s(\alpha_k) \right]
\]

where the sum runs over pairs \((\alpha_j,\alpha_k)\) of neighbouring 2D cells, \( s(\alpha_j) = s(p) \) for any point \( p \) in the 2D cell \( \alpha_j \) and \( M(\beta_{C_{m,t}}) = |1 - \left( \sinh^{-4}(2\beta_{C_{m,t}}) \right)|^{1/8} \).

The regularization of the product of \( n \) spins at different points of the continuum is simply

\[
i^s_{C_{m,t}} s(p_1) \cdots s(p_n) = s(\alpha^{(m)}_1) \cdots s(\alpha^{(m)}_n) \]

where \( p_j \in \alpha^{(m)}_j \). Thus, after regularization to scale \( C_{m,t} \), the \( n \)-point functions are exactly of the type written above.

Coarse graining is done by a decimation of half of the rows and half of the columns (in one coarse-graining step our choice could be say keeping the even rows and the even columns). Thus, a single step coarse-graining map is defined as \( \pi_{m+1,1} = \text{Emb}_{m,m+1} \) with the representative embedding \( \text{Emb}_{m,m+1} : L(C_{m,t}) \rightarrow L(C_{m+1,t}) \) defined below. To specify the embedding, we label each two-dimensional cell in \( C_{m,t} \) by a pair of integers, \( \alpha^{(m)}(X,Y) \) corresponding to their \( x \) and \( y \) coordinates. Then, \( \text{Emb}_{m,m+1}(\alpha^{(m)}(X,Y)) := \alpha^{(m+1)}(2X,2Y) \).

To complete the prescription of a representative embedding of \( L(C_{m,t}) \), we also prescribe the embedding of the one-dimensional cells. There are only two choices to embed a one-dimensional cell in a representative way: the even and the odd. We again choose the even option. Since we will calculate the continuum limit of \( n \)-point functions using only generic cellular decompositions, the choice of embedding of one-dimensional cells turns out to be irrelevant.

Then, the coarse graining of \( n \)-point functions is simply

\[
\langle s(\alpha^{(m)}_1) \cdots s(\alpha^{(m)}_n) \rangle_{C_{m+1,t}} = \langle s(\text{Emb}_{m,m+1}(\alpha^{(m)}_1)) \cdots s(\text{Emb}_{m,m+1}(\alpha^{(m)}_n)) \rangle_{C_{m+1,t}}.
\]

(5)

Now we will construct a measure in the continuum by calculating the continuum limit of the \( n \)-point functions calculated using the coupling constants that solve an appropriate renormalization prescription.

Let \( \{ p_1, \ldots, p_n \} \) be a set of marked points in \( \mathbb{R}^2 \). At the scale \( C_{m,t} \), they induce a set of marked 2-cells \( \{ \alpha^{(m)}_1, \ldots, \alpha^{(m)}_n \} \). The relative positions of these 2-cells are described in terms of the differences between their \( x \) and \( y \) coordinates. The \( x \) coordinate of cell \( \alpha^{(m)}_j \) will be denoted by \( X_j(m) \) and a similar notation will be used for the \( y \) coordinate. Then the integers \( X_{jk}(m) := X_j(m) - X_k(m), Y_{jk}(m) := Y_j(m) - Y_k(m) \) measure the relative position of cells \( \alpha^{(m)}_j, \alpha^{(m)}_k \) in the lattice of 2-cells. Clearly, as we refine the scale \( C_{m,t} \rightarrow \mathbb{R}^2, X_{jk}(m) \rightarrow \infty, Y_{jk}(m) \rightarrow \infty \) and the size of the cells shrinks to zero, \( a_m \rightarrow 0 \). However, since the \( n \) physical points \( \{ p_1, \ldots, p_n \} \) are fixed, \( a_m X_{jk}(m) \) and \( a_m Y_{jk}(m) \) have a well-defined limit. At the same time if the coupling constant is properly adjusted, the correlation length calculated in lattice units \( \xi_m \) diverges in a way that should make \( a_m \xi_m \) converge to the physical correlation length of the system \( \xi_{\text{phys}} \) as \( m \rightarrow \infty \). Recall that by definition the correlation length \( \xi \) measures the asymptotic behaviour of the correlation function, \( \langle s(\alpha(0,0)) \cdot s(\alpha(R,0)) \rangle \sim R^{-p} \exp(-R/\xi) \) for large \( R \). We will postulate as a renormalization prescription that

\[
a_m \xi_m = a_m \xi_{m+1}(m+1)
\]

(6)

where \( \xi_{m+1}(m+1) \) is the correlation length calculated using the 2-point function \( \langle s(\alpha^{(m)}_1) \cdot s(\alpha^{(m)}_2) \rangle_{C_{m+1,t}} \). Note that, since the correlation length is sensitive only to asymptotic
behave, we have \( a_m \xi_{m+1} = a_{m+1} \xi_m \). Thus the renormalization prescription can be written as \( a_m \xi_m = a_{m+1} \xi_{m+1} \) or more precisely \( a_m \xi_m = a_m [z_m^2 + 2z_m - 1]^{-1/2} [z_m(1 - z_m^2)]^{1/2} = \xi_{phys} \) where \( z_m = \tanh (\beta C_{m,t}) \).

In fact, McCoy, Tracy and Wu find convergence in this scaling limit of \( n \)-point functions of the two-dimensional Ising model. In the language of this work, they prove the following theorem [14].

**Theorem 1** (McCoy, Tracy, Wu). Choose \( \beta_{C_{m,t}} \) so as to satisfy the renormalization prescription (6). Then a measure \( \mu_{\mathbb{R}^2} \) in \( \mathcal{A}_{\mathbb{R}^2} \) is defined by its \( n \)-point functions that are calculated as a continuum limit

\[
\lim_{C_{m,t} \to \mathbb{R}^2} \langle s(p_1) \cdots s(p_n) \rangle_{C_{m,t}} = \langle s(p_1) \cdots s(p_n) \rangle_{\mathbb{R}^2}.
\]

Moreover, explicit expressions for these \( n \)-point functions are given below.

In the case that the coupling constants involved in the renormalization group flow are below the critical point, \( \beta_{C_{m,t}} < \beta_{critical} = \tanh^{-1}(\sqrt{2} - 1) \), they show

\[
\lim_{C_{m,t} \to \mathbb{R}^2} \langle s(p_1) \cdots s(p_n) \rangle_{C_{m,t}} = \exp \left( \sum_{k=2}^{\infty} f^{(k)}_n \right)
\]

where

\[
f^{(k)}_n = \frac{-1}{2k(2\pi^2)^k} \int_{-\infty}^{\infty} dv_1 \cdots dv_k du_1 \cdots du_k \prod_{i=1}^{k} (1 + v_i^2 + u_i^2)^{-1} \frac{u_i + u_{i+1}}{v_i - v_{i+1} + i\epsilon} \text{Tr} \left[ \prod_{r=1}^{k} A(r) \right]
\]

and the entries of the \( n \times n \) matrix \( A(r) \) are \( A(r)_{ij} = \text{sgn}(\xi_{m} x_{ij}) \exp(-ix_{ij} y_{r} - iy_{ij} x_{r}) \), where \( x_{ij} = X_{ij} / \xi_{m} \) and \( y_{ij} = Y_{ij} / \xi_{m} \). The diagonal matrix elements of \( a(r) \) vanish. The explicit form of the correlation length is \( \xi_m = |z_m^2 + 2z_m - 1|^{-1/2} [z_m(1 - z_m^2)]^{1/2} \) for \( z_m = \tanh (\beta_{C_{m,t}}) \).

In the case that the coupling constants involved satisfy \( \beta_{C_{m,t}} > \beta_{critical} \), they show

\[
\lim_{C_{m,t} \to \mathbb{R}^2} \langle s(p_1) \cdots s(p_n) \rangle_{C_{m,t}} = \left| \det \left( \sum_{k=1}^{\infty} g^{(k)}_{(n/j)} \right) \right|^{1/2} \exp \left( \sum_{k=2}^{\infty} f^{(k)}_n \right)
\]

where

\[
g^{(k)}_{(n/j)} = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dy_1 \cdots dy_k dx_1 \cdots dx_k \prod_{i=1}^{k} (1 + x_i^2 + y_i^2)^{-1} \prod_{i=1}^{k-1} \frac{y_i + y_{i+1}}{x_i - x_{i+1} + i\epsilon} \left[ \prod_{r=1}^{k} A(r) \right]_{ij}
\]

Once we have constructed the theory in the continuum, we can go back to scale \( C_{m,t} \), and calculate corrections to the effective theory. The completely renormalized theory at scale \( C_{m,t} \) is determined by the coarse graining of \( n \)-point functions from the continuum

\[
\langle s(p_1) \cdots s(p_n) \rangle^{\text{ren}}_{C_{m,t}} = \langle s(p_1) \cdots s(p_n) \rangle_{\mathbb{R}^2}
\]

with \( p_j = \text{Emb}_{L(C_{m,t})} \alpha_j \).

The scaling limit of the 2D Ising model has been extensively studied (for a review see [15]). Of particular importance is the proof [16] showing that the \( n \)-point functions in the continuum defined above satisfy the Osterwalder–Schrader axioms [17]. In particular, the rotational invariance that was broken by working with an economic family is known to be restored in the continuum limit. Regarding the implied relativistic quantum field theory, we can

\[5\] Another natural renormalization prescription is to ask that a given 2-point function stays fixed after coarse graining. The resulting continuum limit is the same for both renormalization prescriptions.
explicitly give a covariant Hamiltonian quantum theory following an Osterwalder–Schrader-type construction on the space $\hat{A}_M$ like the one proposed in [18]. Explicit knowledge of the relation between this quantum field theory and the standard realization of the 2D Ising field theory ($\phi^4$ Landau–Ginzburg model) in terms of linear scalar fields would certainly be desirable. If we knew such a relation, we could import very important concepts into the loop-quantized world. For example, for the standard realization a basis of quasi-particles has been found and in that basis the S-matrix is explicitly known [19].

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