Symmetrical Solutions of Backward Stochastic Volterra Integral Equations and their applications *

Tianxiao Wang and Yufeng Shi †
School of Mathematics, Shandong University, Jinan 250100, China

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Abstract

Backward stochastic Volterra integral equations (BSVIEs in short) are studied. We introduce the notion of adapted symmetrical solutions (S-solutions in short), which are different from the M-solutions introduced by Yong [16]. We also give some new results for them. At last a class of dynamic coherent risk measures were derived via certain BSVIEs.

Keywords: Backward stochastic Volterra integral equations, Adapted symmetrical solutions, Dynamic coherent risk measure

1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space on which a \(d\)-dimensional Brownian motion \(W(\cdot)\) is defined with \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) being its natural filtration augmented by all the \(\mathbb{P}\)-null sets. In this paper, we consider the following stochastic integral equations:

\[
Y(t) = \Psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]. \tag{1}
\]

where \(g : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^m\) and \(\Psi : [0, T] \times \Omega \rightarrow \mathbb{R}^m\) are some given measurable mappings with \(\Delta^c = \{(t, s) \in [0, T]^2 \mid t \leq s\}\). Such an equation is referred as a backward stochastic Volterra integral equation (BSVIE in short) (see [14] and [16]).

When \(\Psi(\cdot), g, Z(t, s)\) are independent of \(t, g\) is also independent of \(Z(s, t)\), BSVIE (1) is reduced to a nonlinear backward stochastic differential equation (BSDE in short)

\[
Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s),
\]

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†E-mail: xiaotian2008001@gmail.com yfshi@sdu.edu.cn
which was introduced by Pardoux and Peng in [8], where the existence and uniqueness of $\mathcal{F}$-adapted solutions are established under uniform Lipschitz conditions on $g$. Due to their important significance in many fields such as financial mathematics, optimal control, stochastic games and partial differential equations and so on, the theory of BSDEs has been extensively developed in the past two decades. The reader is referred to [3], [4], [6], [9], [12] and [11].

On the other hand, stochastic Volterra integral equations were firstly studied by Berger and Mizel in [1] and [2], then they were investigated by Protter in [13] and Pardoux and Protter in [10]. Lin [5] firstly introduced a kind of nonlinear backward stochastic Volterra integral equations of the form

$$Y(t) = \xi + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s).$$

But there is a gap in [5]. At the same time Yong [14] also investigated a more general version of BSVIEs, as the type of (1), and gave their applications to optimal control. In this paper, we give a further discussion to BSVIE (1).

Based on the martingale presentation theorem, especially for $Z(t, s), t \geq s$, Yong [16] introduced the concept of M-solutions for BSVIE (1). He gave some conditions that suffice BSVIE (1) is uniquely solvable. We realize that there should be other kinds of solutions for BSVIEs. In this paper, we introduce the notion of symmetrical solutions (called S-solutions) in the way of $Z(t, s) \equiv Z(s, t), t, s \in [0, T]$. It is worthy to point out that S-solutions should be solved in a more general Hilbert space, which is different from the one for M-solution, see more detailed accounts in Section 2. We prove the existence and uniqueness of S-solutions for BSVIEs. Some properties such as the continuity of $Y(t)$ are obtained. We then study the relations between S-solutions and other solutions. We give the notion of adapted solutions of (1) ($g$ is independent of $Z(s, t)$) and obtain the existence and uniqueness by virtue of the results of S-solution, which cover the ones in [5] and overcome its gap. Some relations between S-solutions and M-solutions are studied. By two examples we show that the two solutions usually are not equal, especially their values in $\Delta = \{(t, s) \in [0, T]^2 \mid t > s\}$. We also give a criteria for S-solutions of BSVIEs. At last by giving a comparison theorem for S-solutions of certain BSVIEs, we show a class of dynamic coherent risk measures by means of S-solutions for certain BSVIEs.

This paper is organized as follows: in the next section, we give some preliminary results. In Section 3, we prove the existence and uniqueness theorem of S-solutions of (1) and show some corollaries and some other new results on S-solution. In Section 4 we give a class of dynamic coherent risk measures by means of the S-solutions of a kind of BSVIEs.
2 Preliminary results

2.1 Notations and Definitions

In this subsection we give some notations and definitions that are needed in the following. For any $R, S \in [0, T]$, in the following we denote $\Delta^c[R, S] = \{(t, s) \in [R, S]^2; t \leq s\}$ and $\Delta[R, S] = \{(t, s) \in [R, S]^2; t > s\}$. Let $L^2_{\mathcal{F}_T}[0, T]$ be the set of $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$-measurable processes $X : [0, T] \times \Omega \to \mathbb{R}^m$ satisfying

$$E \int_0^T |X(t)|^2 dt < \infty.$$  

We denote

$$\mathbb{H}^2[R, S] = L^2_{\mathcal{F}}(\Omega; C[R, S]) \times L^2_{\mathcal{F}}(\Omega; L^2[R, S]).$$

which is a Banach space under the norm:

$$\|(y(\cdot), z(\cdot))\|_{\mathbb{H}^2[R, S]} = \left[ E \sup_{t \in [R, S]} |y(t)|^2 + E \int_R^S |z(t)|^2 dt \right]^{\frac{1}{2}}.$$  

Here $L^2_{\mathcal{F}}(\Omega; C[R, S])$ is the set of all continuous adapted processes $X : [R, S] \times \Omega \to \mathbb{R}^m$ satisfying

$$E \left[ \sup_{t \in [R, S]} |X(t)|^2 \right] < \infty.$$  

$L^2_{\mathcal{F}}(\Omega; L^2[R, S])$ is the set of all adapted processes $X : [R, S] \times \Omega \to \mathbb{R}^{m \times d}$ satisfying

$$E \int_R^S |X(t)|^2 dt < \infty.$$  

We denote

$$*\mathcal{H}^2[R, S] = L^2_{\mathcal{F}}(R, S) \times L^2_{\text{max, } \mathcal{F}}(R, S; L^2[R, S]),$$

$$\mathcal{H}^2[R, S] = L^2_{\mathcal{F}}(R, S) \times L^2(R, S; L^2_{\mathcal{F}}[R, S]).$$

Here $L^2_{\text{max, } \mathcal{F}}(R, S; L^2[R, S])$ is the set of all processes $z : [R, S]^2 \times \Omega \to \mathbb{R}^{m \times d}$ such that for almost all $t \in [R, S]$, $s \to z(t, s)$ is $\mathcal{F}_{t \vee s}$-measurable satisfying

$$E \int_R^S \int_R^S |z(t, s)|^2 dsdt < \infty.$$  

$L^2(R, S; L^2_{\mathcal{F}}[R, S])$ is the set of all processes $z : [R, S]^2 \times \Omega \to \mathbb{R}^{m \times d}$ such that for almost all $t \in [R, S]$, $z(t, \cdot) \in L^2_{\mathcal{F}}[R, S]$ satisfying

$$E \int_R^S \int_R^S |z(t, s)|^2 dsdt < \infty,$$
where $L^2_F[R, S] = L^2_F(\Omega; L^2[R, S])$.

We also define the norm of the elements in $H^2[R, S]$:

$$\|(y(\cdot), z(\cdot, \cdot))\|_{H^2[R, S]} = \left\{ E \int_R^S |y(t)|^2 \, dt + E \int_R^S \int_R^S |z(t, s)|^2 \, ds \, dt \right\}^{1/2}. $$

As to the norm of the element in $^\ast H^2[R, S]$,

$$\|(y(\cdot), z(\cdot, \cdot))\|_{^\ast H^2[R, S]} = \left\{ E \int_R^S |y(t)|^2 \, dt + E \int_R^S \int_R^S |z(t, s)|^2 \, ds \, dt \right\}^{1/2}. $$

From the definitions above, we know that $H^2[R, S]$ is a complete subspace of $^\ast H^2[R, S]$ under the norm above. Similarly we denote

$$H^2_1[R, S] = L^2_F[R, S] \times L^2(R, S; L^2_F[t, S]).$$

Here $L^2(R, S; L^2_F[t, S])$ is the set of all processes $z : \Delta^c[R, S] \times \Omega \to \mathbb{R}^{m \times d}$ such that for almost all $t \in [R, S]$, $z(t, \cdot) \in L^2_F[R, S]$ satisfying

$$E \int_R^S \int_t^S |z(t, s)|^2 \, ds \, dt < \infty.$$

We also define the norm of the elements in $H^2_1[R, S]$:

$$\|(y(\cdot), z(\cdot, \cdot))\|_{H^2_1[R, S]} = \left\{ E \int_R^S |y(t)|^2 \, dt + E \int_R^S \int_t^S |z(t, s)|^2 \, ds \, dt \right\}^{1/2}.$$

We now give the definition of M-solutions, introduced by Yong [16].

**Definition 2.1** Let $S \in [0, T]$. A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in H^2[S, T]$ is called an adapted M-solution of BSVIE (1) on $[S, T]$ if (1) holds in the usual Itô’s sense for almost all $t \in [S, T]$ and, in addition, the following holds:

$$Y(t) = E (Y(t) | F_S) + \int_S^t Z(t, s) \, dW(s).$$

In this paper, we introduce the concept of S-solutions as follows.

**Definition 2.2** Let $S \in [0, T]$. A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in ^\ast H^2[S, T]$ is called an adapted symmetrical solution (also called S-solution) of BSVIE (1) on $[S, T]$ if (1) holds in the usual Itô’s sense for almost all $t \in [S, T]$ and, in addition, the following holds:

$$Z(t, s) = Z(s, t), \quad t, s \in [S, T].$$
The following simple example shows the reason for considering the S-solution in $\mathcal{H}^2[0,T]$, rather than $\mathcal{H}^2[0,T]$. Let us consider the simple BSVIE

$$Y(t) = tW^2(T) - \int_t^T 1 ds - \int_t^T Z(t,s)dW(s).$$

It is easy to show that $\forall (t,s) \in \Delta$, $Y(t) = tW^2(t)$, $Z(t,s) = 2tW(s)$ is the adapted solution of the above equation. By the definition of S-solution, we have $\forall (t,s) \in \Delta, Z(t,s) = Z(s,t) = sW(t)$, obviously we get $(Y(\cdot),Z(\cdot,\cdot)) \in H^2[0,T]$, instead of $\mathcal{H}^2[0,T]$.

On the other hand, it is easy to see that $\forall (t,s) \in [0,T]^2$ and $(Y(\cdot),Z(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$, $Z(t,s)$ is $\mathcal{F}_0$-measurable, i.e., almost surely deterministic function if we consider the S-solution of BSVIE (1) in $\mathcal{H}^2[0,T]$. In fact, in this case, for any $(t,s) \in [0,T]^2$, $Z(t,s)$ (Z(s,t) respectively) is $\mathcal{F}_s$-measurable ($\mathcal{F}_t$-measurable respectively), then by $Z(t,s) = Z(s,t)$ we obtain that $Z$ should be $\mathcal{F}_0$-measurable.

Now we cite some definitions introduced in [15].

**Definition 2.3** A mapping $\rho : L^2_{\mathcal{F}_T}[0,T] \to L^2_{\mathcal{F}_T}[0,T]$ is called a dynamic risk measure if the following hold:

1) (Past independence) For any $\Psi(\cdot), \overline{\Psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, if $\Psi(s) = \overline{\Psi}(s)$, a.s. $\omega \in \Omega$, $s \in [t,T]$, for some $t \in [0,T]$, then $\rho(t;\Psi(\cdot)) = \rho(t;\overline{\Psi}(\cdot))$, a.s. $\omega \in \Omega$.

2) (Monotonicity) For any $\Psi(\cdot), \overline{\Psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, if $\Psi(s) \leq \overline{\Psi}(s)$, a.s. $\omega \in \Omega$, $s \in [t,T]$, for some $t \in [0,T]$, then $\rho(s;\Psi(\cdot)) \geq \rho(s;\overline{\Psi}(\cdot))$, a.s. $\omega \in \Omega$, $s \in [t,T]$.

**Definition 2.4** A dynamic risk measure $\rho : L^2_{\mathcal{F}_T}[0,T] \to L^2_{\mathcal{F}_T}[0,T]$ is called a coherent risk measure if the following hold: 1) There exists a deterministic integrable function $r(\cdot)$ such that for any $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$,

$$\rho(t;\Psi(\cdot) + c) = \rho(t;\Psi(\cdot)) - c e^{\int_t^T r(s)ds}, \ a.s. \omega \in \Omega, \ t \in [0,T].$$

2) For $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$ and $\lambda > 0$, $\rho(t;\lambda\Psi(\cdot)) = \lambda \rho(t;\Psi(\cdot))$ a.s. $\omega \in \Omega$, $t \in [0,T]$. 3) For any $\Psi(\cdot), \overline{\Psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$,

$$\rho(t;\Psi(\cdot) + \overline{\Psi}(\cdot)) \leq \rho(t;\Psi(\cdot)) + \rho(t;\overline{\Psi}(\cdot)), \ a.s. \omega \in \Omega, \ t \in [0,T].$$

**2.2 Some lemmas for S-solutions**

First we give some lemmas for S-solutions. For any $R,S \in [0,T]$, let us consider the following stochastic integral equation

$$\lambda(t,r) = \Psi(t) + \int_r^T h(t,s,\mu(t,s))ds - \int_r^T \mu(t,s)dW(s), \ r \in [S,T], \ t \in [R,T]. \ (2)$$

where $h : [R,T] \times [S,T] \times \mathbb{R}^{m \times d} \times \Omega \to \mathbb{R}^m$ is given. The unknown processes are $(\lambda(\cdot,\cdot),\mu(\cdot,\cdot))$, for which $(\lambda(t,\cdot),\mu(t,\cdot))$ are $\mathbb{F}$-adapted for all $t \in [R,T]$. We can regard
(2) as a family of BSDEs on $[S, T]$, parameterized by $t \in [R, T]$. Next we introduce the following assumption of $h$ in (2).

(H1) Let $R, S \in [0, T]$ and $h : [R, T] \times [S, T] \times R^{m \times d} \times \Omega \to R^m$ be $\mathcal{B}([R, T] \times [S, T] \times R^{m \times d}) \otimes \mathcal{F}_T$-measurable such that $s \mapsto h(t, s, z)$ is $\mathbb{F}$-progressively measurable for all $(t, z) \in [R, T] \times R^{m \times d}$ and

$$E \int_R^T \left( \int_S^T |h(t, s, 0)| ds \right)^2 dt < \infty. \tag{3}$$

Moreover, the following holds:

$$|h(t, s, z_1) - h(t, s, z_2)| \leq L(t, s)|z_1 - z_2|, \quad (t, s) \in [R, T] \times [S, T], z_1, z_2 \in R^{m \times d}, \tag{4}$$

where $L : [R, T] \times [S, T] \to [0, \infty)$ is a deterministic function such that for some $\varepsilon > 0$,

$$\sup_{t \in [R, T]} \int_S^T L(t, s)^{2+\varepsilon} ds < \infty.$$

Proposition 2.1 Let (H1) hold, then for any $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[R, T]$, (2) admits a unique adapted solution $(\lambda(t, \cdot), \mu(t, \cdot)) \in \mathbb{H}^2[S, T]$ for almost all $t \in [R, T]$, and the following estimate holds: $t \in [R, T]$

$$\| (\lambda(t, \cdot), \mu(t, \cdot)) \|^2_{\mathbb{H}^2[S, T]} \equiv E \left\{ \sup_{t \in [S, T]} |\lambda(t, r)|^2 + \int_S^T |\mu(t, s)|^2 ds \right\} \leq CE \left\{ |\Psi(t)|^2 + \left( \int_S^T |h(t, s, 0)|^2 ds \right)^2 \right\}. \tag{5}$$

Proof. The proof of Proposition 1 can be found in [16]. \(\square\)

Now we look at one special case of (2). Let $R = S$ and define

$$\begin{cases} Y(t) = \lambda(t, t), & t \in [S, T], \\ Z(t, s) = \mu(t, s), & (t, s) \in \Delta^c[S, T]. \end{cases}$$

Then the above (2) reads:

$$Y(t) = \Psi(t) + \int_t^T h(t, s, Z(t, s)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [S, T]. \tag{6}$$

Here we define $Z(t, s)$ for $(t, s) \in \Delta[S, T]$ by the following relation $Z(t, s) = Z(s, t)$, which is different from the way of defining M-solution, and that’s why we call it S-solution of (6). So we have the following lemma.
Lemma 2.1 Let (H1) hold, then for any $S \in [0, T]$, $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[S, T]$, (6) admits a unique adapted $S$-solution $(Y(\cdot), Z(\cdot, \cdot)) \in H^2[0, T]$, and the following estimate holds: $t \in [S, T],\]

$$E \left\{ |Y(t)|^2 + \int_t^T |Z(t, s)|^2 ds \right\} \leq CE \left\{ |\Psi(t)|^2 + \left( \int_t^T |h(t, s, 0)| ds \right)^2 \right\}. \quad (7)$$

Hereafter $C$ is a generic positive constant which may be different from line to line. If $h$ also satisfies (H1), $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[S, T]$, and $(\Psi(\cdot), \bar{Z}(\cdot, \cdot)) \in H^2[0, T]$ is the unique adapted $S$-solution of BSVIE (6) with $(h, \Psi)$ replaced by $(\eta, \bar{\Psi})$, then $\forall t \in [S, T],\]

$$E \left\{ |Y(t) - \Psi(t)|^2 + \int_t^T |Z(t, s) - \bar{Z}(t, s)|^2 ds \right\} \leq CE \left[ |\Psi(t) - \bar{\Psi}(t)|^2 + \left( \int_t^T |h(t, s, Z(t, s)) - h(t, s, Z(t, s))| ds \right)^2 \right]. \quad (8)$$

Furthermore, for any $t, \bar{t} \in [S, T],\]

$$E \left\{ |Y(t) - Y(\bar{t})|^2 + \int_{t \wedge \bar{t}}^T |Z(t, s) - Z(\bar{t}, s)|^2 ds \right\} \leq CE \left\{ |\Psi(t) - \Psi(\bar{t})|^2 + \left( \int_{t \wedge \bar{t}}^T |h(t \wedge \bar{t}, \bar{t}, s, Z(t \wedge \bar{t}, s))| ds \right)^2 \right\} + \int_{t \wedge \bar{t}}^T |Z(t \wedge \bar{t}, s)|^2 ds + \left( \int_{t \wedge \bar{t}}^T |h(t, s, Z(t, s)) - h(t, s, Z(t, s))| ds \right)^2 \right\}. \quad (9)$$

**Proof.** From Proposition 1 the existence and uniqueness of $S$-solution in $[S, T]$ is clear. As to the other estimates, the proof is the same as the one in [16].

Let’s give another special case. Let $r = S \in [R, T]$ be fixed. Define

$$\Psi^S(t) = \lambda(t, S), \quad Z(t, s) = \mu(t, s), \quad t \in [R, S], \quad s \in [S, T].$$

Then (2) becomes:

$$\Psi^S(t) = \Psi(t) + \int_0^T h(t, s, Z(t, s)) ds - \int_0^T Z(t, s) dW(s), \quad t \in [R, S], \quad (10)$$

and we have the following result.

**Lemma 2.2** Let (H1) hold, then for any $\Psi(\cdot) \in L^2_{\mathcal{F}_T}[R, S]$, (10) admits a unique adapted solution $(\Psi^S(\cdot), Z(\cdot, \cdot)) \in L^2_{\mathcal{F}_S}[R, S] \times L^2(R, S; L^2_S[R, T])$, and the following estimate holds: $t \in [R, S],\]

$$E \left\{ |\Psi^S(t)|^2 + \int_0^T |Z(t, s)|^2 ds \right\} \leq CE \left\{ |\Psi(t)|^2 + \left( \int_0^T |h(t, s, 0)| ds \right)^2 \right\}. \quad (11)$$

**Proof.** From Proposition 1 the result is obvious. \(\square\)
3 Well-posedness of S-solutions for BSVIEs

3.1 The existence and uniqueness of S-solutions

In this subsection we will give the existence and uniqueness of S-solutions. For it we need the following standing assumption.

(H2) Let \( g : \Delta^c \times R^m \times R^{m \times d} \times \Omega \rightarrow R^m \) be \( B(\Delta^c \times R^m \times R^{m \times d}) \otimes \mathcal{F}_T \)-measurable such that \( s \rightarrow g(t, s, y, z, \zeta) \) is \( \mathbb{F} \)-progressively measurable for all \((t, y, z, \zeta) \in [0, T] \times R^m \times R^{m \times d} \) and

\[
E \int_0^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt < \infty,
\]

where we denote \( g_0(t, s) \equiv g(t, s, 0, 0, 0) \). Moreover, it holds

\[
|g(t, s, y, z, \zeta) - g(t, s, \overline{y}, \overline{z}, \overline{\zeta})| \leq L(t, s) \left( |y - \overline{y}| + |z - \overline{z}| + |\zeta - \overline{\zeta}| \right), \quad \text{a.s.}
\]

\( \forall (t, s) \in \Delta^c, \ y, \overline{y} \in R^m, \ z, \overline{z}, \zeta, \overline{\zeta} \in R^{m \times d} \),

where \( L : \Delta^c \rightarrow R \) is a deterministic function such that the following holds: for some \( \epsilon > 0 \),

\[
\sup_{t \in [0, T]} \int_0^T L(t, s)^{2+\epsilon} ds < \infty.
\]

So we have:

**Theorem 3.1** Let (H2) hold, then for any \( \Psi(\cdot) \in L^2_{\mathbb{F}|[0,T]} \), (1) admits a unique adapted S-solution on \([0, T]\). Moreover, the following estimate holds: \( \forall S \in [0, T] \),

\[
\|(Y(\cdot), Z(\cdot, \cdot))\|_{\mathcal{H}^2[S,T]}^2 \equiv E \left\{ \int_S^T |Y(t)|^2 dt + \int_S^T \int_S^T |Z(t, s)|^2 ds dt \right\}
\leq CE \left\{ \int_S^T |\Psi(t)|^2 dt + \int_S^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt \right\}.
\]

**Proof.** We split the proof into two steps.

**Step 1** Here we consider the existence and uniqueness of the adapted S-solution of (1) on \([S,T]\) for some \( S \in [0, T] \). For all \( S \in [0, T] \), let \( \mathcal{S}^2[S,T] \) be the space of all \((y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^2[S,T]\) such that

\( z(t, s) = z(s, t), \text{a.s.}, \ t, s \in [S,T], \text{a.e.} \).

Clearly, \( \mathcal{S}^2[S,T] \) is a nontrivial closed subspace of \( \mathcal{H}^2[S,T] \). In fact, we assume that there is a series of elements \((y_n(\cdot), z_n(\cdot, \cdot)) \in \mathcal{S}^2[S,T]\), and the limit is \((y(\cdot), z(\cdot, \cdot))\), which belongs to \( \mathcal{H}^2[S,T] \). We easily know that the limit also belongs to \( \mathcal{S}^2[S,T] \).
Actually, we have the following:

\[
E \int_S^T \int_S^T |z(t, s) - z(s, t)|^2 dsdt 
\leq E \int_S^T \int_S^T |z(t, s) - z_n(t, s)|^2 dsdt 
+ E \int_S^T \int_S^T |z_n(s, t) - z(s, t)|^2 dsdt. \tag{14}
\]

As \( n \to \infty \), the limit of the right hand of (14) is zero, so we have

\[ z(t, s) = z(s, t), \text{a.s., } t, s \in [S, T], \text{a.e..} \]

Note that here the space \( S^2[0, T] \) is isomorphic to \( H_2^2[0, T] \) defined previously, i.e., there exists a bijection between them. For any \( (y(\cdot), z(\cdot, \cdot)) \in S^2[S, T] \), we have

\[
E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_T^t |z(t, s)|^2 dsdt \right\} 
\leq E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_S^t |z(t, s)|^2 dsdt \right\} 
= E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_s^t |z(t, s)|^2 dsdt + \int_S^T \int_t^T |z(t, s)|^2 dsdt \right\} 
= E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_s^T |z(s, t)|^2 dt ds + \int_S^T \int_t^T |z(t, s)|^2 dsdt \right\} 
= E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_t^T |z(t, s)|^2 dsdt + \int_S^T \int_T^t |z(t, s)|^2 dsdt \right\} 
\leq 2E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_t^T |z(t, s)|^2 dsdt \right\}.
\]

Hence, we can take a new norm for the elements of \( S^2[S, T] \) as follows:

\[
\|(y(\cdot), z(\cdot, \cdot))\|_{S^2[S,T]} = E \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_T^t |z(t, s)|^2 dsdt \right\}^{\frac{1}{2}}.
\]

Now we consider the following equation: \( t \in [S, T] \),

\[
Y(t) = \Psi(t) + \int_t^T g(t, s, y(s), Z(t, s), z(s, t)) ds - \int_t^T Z(t, s)dW(s), \tag{15}
\]

for any \( \Psi(\cdot) \in L_2^Z[F, S, T] \) and \( (y(\cdot), z(\cdot, \cdot)) \in S^2[S, T] \). By Lemma 2.5, (15) admits a unique adapted \( S \)-solution \( (Y(\cdot), Z(\cdot, \cdot)) \in S^2[S, T] \) and we can define a mapping \( \Theta: S^2[S, T] \to S^2[S, T] \) by

\[
\Theta(y(\cdot), z(\cdot, \cdot)) = (Y(\cdot), Z(\cdot, \cdot)), \text{ } \forall (y(\cdot), z(\cdot, \cdot)) \in S^2[S, T].
\]
Next we will prove $\Theta$ defined above is contracted when $T - S > 0$ is small enough. Let $(\overline{y}(\cdot), \overline{z}(\cdot, \cdot)) \in S^2[S, T]$ and $\Theta(\overline{y}(\cdot), \overline{z}(\cdot, \cdot)) = (\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot))$. From (8) we know that
\[
E|Y(t) - \overline{Y}(t)|^2 + E \int_t^T |Z(t, s) - \overline{Z}(t, s)|^2 ds 
\leq C E \left( \int_t^T |g(t, s, y(s), Z(t, s), \overline{z}(s, t)) - g(t, s, \overline{y}(s), Z(t, s), \overline{z}(s, t))|^2 ds \right)^{\frac{1}{2}} 
\leq C E \left\{ \int_t^T L(t, s)|y(s) - \overline{y}(s)| + |z(t, s) - \overline{z}(t, s)| ds \right\}^2 
\leq C(T - t)^{\frac{1}{1+\frac{1}{\lambda}}} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}} E \left\{ \int_T^T |y(s) - \overline{y}(s)|^2 ds \right\} 
+ C(T - t)^{\frac{1}{1+\frac{1}{\lambda}}} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}} E \left\{ \int_T^T |z(t, s) - \overline{z}(t, s)|^2 ds \right\}.
\]

The second inequality in (16) holds because of $z(t, s) \equiv z(s, t)$. Consequently,
\[
\| (Y(\cdot, Z(\cdot, \cdot)) - (\overline{Y}(\cdot, \overline{Z}(\cdot, \cdot))) \|^2_{S^2[S, T]} 
\equiv E \left\{ \int_S^T |Y(t) - \overline{Y}(t)|^2 dt + \int_t^T \int_t^T |Z(t, s) - \overline{Z}(t, s)|^2 ds dt \right\} 
\leq C(T - S)^{\frac{1}{1+\frac{1}{\lambda}}} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}} E \left\{ \int_S^T |y(t)|^2 dt \right\} 
+ C(T - S)^{\frac{1}{1+\frac{1}{\lambda}}} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}} E \left\{ \int_S^T \int_t^T |z(t, s) - \overline{z}(t, s)|^2 ds dt \right\}.
\]

Then we can choose $\eta$ so that $C' \max\{\eta^{\frac{1}{1+\frac{1}{\lambda}}}, \eta^{\frac{2}{2+\epsilon}}\} = \frac{1}{2}$, where
\[
C' = C \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}}.
\]

Hence (15) admits a unique fixed point $(Y(\cdot, Z(\cdot, \cdot)) \in S^2[S, T]$ which is the unique adapted $S$-solution of (1) in $[S, T]$ if $T - S \leq \eta$.

**Step 2:** Now we will prove the existence and uniqueness of the $S$-solution of (1) for $(t, s) \in [R, S] \times [R, S]$, for some $R \in [0, S]$. First we consider the following equation:
\[
t \in [R, S],
Y(t) = \Psi^S(t) + \int_t^S g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^S Z(t, s)dW(s),
\]
\[
\Psi^S(t) = \Psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s)dW(s).
\]
If we prove that $Ψ^S(t)$ is $\mathcal{F}_S$-measurable and $Ψ^S(t) \in L^2_{\mathcal{F}_S}[R, S]$, then we can use the same argument as Step 1 to show (1) is solvable on $[R,S]$ when $\eta \geq S - R > 0$ is small enough. Let $Z(t,s) \equiv Z(s,t)$ in (18), and we denote

$$h(t, s, Y(s), Z(t, s)) \equiv g(t, s, Y(s), Z(t, s), Z(s, t)).$$

From Step 1 we have known that $\{Y(s); s \in [S,T]\}$ is solved, then by Lemma 2.6, (18) admits a unique adapted solution $(Ψ^S(\cdot), Z(\cdot, \cdot)) \in L^2_{\mathcal{F}_S}[R, S] \times L^2(R, S; L^2_{\mathcal{F}}[S, T])$, and

$$E \left[ \int_R^S |Ψ^S(t)|^2 dt + \int_R^S \int_S^T |Z(t, s)|^2 ds dt \right] \leq CE \int_R^S |Ψ(t)|^2 dt + CE \int_R^S \left( \int_S^T |h(t, s, Y(s), 0)| ds \right)^2 dt$$

$$= CE \int_R^S |Ψ(t)|^2 dt + CE \int_R^S \left( \int_S^T |g(t, s, Y(s), 0, 0)| ds \right)^2 dt$$

$$\leq CE \int_R^S |Ψ(t)|^2 dt + CE \int_R^S \left( \int_S^T |g_0(t, s)| ds \right)^2 dt + CE \int_R^S \left( \int_S^T L(t, s)|Y(s)| ds \right)^2 dt.$$  \(19\)

Here $C$ is a constant depending on $\sup_{t \in [0,T]} \int_t^T L(t, s)^{2+\epsilon} ds$ and $T$. Then we have $E \int_R^S |Ψ^S(t)|^2 dt < \infty$. Thus we can repeat the argument in Step 2 to finish the proof of the existence and uniqueness of adapted $S$-solution of (1) on $[0,T]$.

Next we prove the estimate in the theorem. First we can choose $T_1 \in [0,T]$ so that it satisfies $\max \left\{ 8\theta^{2+\epsilon} A, 4\theta^{2+\epsilon} + 1 A \right\} = \frac{1}{2}$, where

$$\theta = T - T_1, \quad A = \sup_{t \in [0,T]} \left( \int_t^T L^{2+\epsilon}(t, s) ds \right)^{\frac{2}{2+\epsilon}},$$
then from BSVIE (1) we have, \( \forall u \in [T_1, T] \),
\[
E \int_u^T |Y(t)|^2 dt + E \int_u^T \int_t^T |Z(t, s)|^2 ds dt \\
\leq 2E \int_u^T |\Psi(t)|^2 dt + 4E \int_u^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt \\
+ 4E \int_u^T \left( \int_t^T |g(t, s, Y(s), Z(t, s), Z(s, t)) - g_0(t, s)| ds \right)^2 dt \\
\leq 2E \int_u^T |\Psi(t)|^2 dt + 4E \int_u^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt \\
+ 4(T - T_1)^{2+\epsilon} + 8(T - T_1)^{2+\epsilon} \sup_{t \in [0, T]} \left( \int_t^T L^{2+\epsilon}(t, s) ds \right)^\frac{4}{2+\epsilon} E \int_u^T |Y(s)|^2 ds \\
+ 8(T - T_1)^{2+\epsilon} \sup_{t \in [0, T]} \left( \int_t^T L^{2+\epsilon}(t, s) ds \right)^\frac{4}{2+\epsilon} E \int_u^T \int_t^T |Z(t, s)|^2 ds dt,
\]
thus by the way of choosing \( T_1 \) we know that
\[
E \int_u^T |Y(t)|^2 dt + E \int_u^T \int_t^T |Z(t, s)|^2 ds dt \\
\leq 4E \int_u^T |\Psi(t)|^2 dt + 8E \int_u^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt, \tag{20}
\]
furthermore we have, \( \forall u \in [T_1, T] \),
\[
E \int_u^T |Y(t)|^2 dt + E \int_u^T \int_t^T |Z(t, s)|^2 ds dt \\
= E \int_u^T |Y(t)|^2 dt + 2E \int_u^T \int_t^T |Z(t, s)|^2 ds dt \\
\leq CE \int_u^T |\Psi(t)|^2 dt + CE \int_u^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt. \tag{21}
\]
Similarly we can choose \( T_2 \in [0, T_1] \) satisfying \( T_1 - T_2 = \theta = T - T_1 \), so that \( u \in [T_2, T_1] \).
\[
E \left\{ \int_u^{T_1} |Y(t)|^2 dt + \int_u^{T_1} \int_u^{T_1} |Z(t, s)|^2 ds dt \right\} \\
\leq CE \left\{ \int_u^{T_1} |\Psi^{T_1}(t)|^2 dt + \int_u^{T_1} \left( \int_t^{T_1} |g_0(t, s)| ds \right)^2 dt \right\}, \tag{22}
\]
where
\[
\Psi^{T_1}(t) = \Psi(t) + \int_t^{T_1} g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^{T_1} Z(t, s) dW(s). \tag{23}
\]
By the definition of S-solution we have $Z(t, s) \equiv Z(s, t)$ in (23), then from the proof in Step 2 above we have, $\forall u \in [T_2, T_1]$,

$$E \left[ \int_u^{T_1} |\Psi_{T_1}(t)|^2 dt + \int_u^{T_1} \int_u^{T} |Z(t, s)|^2 ds dt \right]$$

$$\leq CE \int_u^{T_1} |\Psi(t)|^2 dt + CE \int_u^{T_1} \left( \int_t^{T} |g_0(t, s)| ds \right)^2 dt$$

$$+ CE \int_u^{T_1} \left( \int_t^{T} L(t, s)|Y(s)| ds \right)^2 dt$$

$$\leq CE \int_u^{T} |\Psi(t)|^2 dt + CE \int_u^{T} \left( \int_t^{T} |g_0(t, s)| ds \right)^2 dt, \quad (24)$$

then from (22) and (24), $\forall u \in [T_2, T_1]$,

$$E \left\{ \int_u^{T_1} |Y(t)|^2 dt + \int_u^{T_1} \int_u^{T_1} |Z(t, s)|^2 ds dt \right\}$$

$$\leq CE \int_u^{T} |\Psi(t)|^2 dt + CE \int_u^{T} \left( \int_t^{T} |g_0(t, s)| ds \right)^2 dt. \quad (25)$$

Here $C$ depends on $\sup_{t \in [0, T]} \int_t^{T} L(t, s)^2 ds$. Since we are considering the symmetrical form of $Z(\cdot, \cdot)$, by the stochastic Fubini Theorem we get:

$$E \int_u^{T_1} \int_u^{T_1} |Z(t, s)|^2 ds dt$$

$$= E \int_u^{T} \int_u^{T_1} |Z(t, s)|^2 ds dt$$

$$= E \int_u^{T_1} \int_u^{T} |Z(s, t)|^2 ds dt$$

$$= E \int_u^{T_1} \int_u^{T_1} |Z(t, s)|^2 ds dt. \quad (26)$$
From (21), (24), (25) and (26), we can estimate that, $\forall u \in [T_2, T_1]$,
\[
E \left\{ \int_u^T |Y(t)|^2 dt + \int_u^T \int_T^T |Z(t, s)|^2 ds dt \right\}
\]
\[= E \left\{ \int_{T_1}^T |Y(t)|^2 dt + \int_{T_1}^T \int_{T_1}^T |Z(t, s)|^2 ds dt \right\}
\]
\[+ E \left\{ \int_u^{T_1} |Y(t)|^2 dt + \int_u^{T_1} \int_u^{T_1} |Z(t, s)|^2 ds dt \right\}
\]
\[+ E \int_{T_1}^T \int_u^{T_1} |Z(t, s)|^2 ds dt + E \int_{T_1}^T \int_{T_1}^T |Z(t, s)|^2 ds dt
\]
\[\leq C \left\{ E \int_u^T |\Psi(t)|^2 dt + CE \int_u^T \left( \int_t^T |g_0(t, s)| ds \right)^2 dt \right\}.
\]
Thus we can repeat the argument above to obtain the estimate. □

### 3.2 Some corollaries for S-solutions

In this subsection we give some corollaries. Similar to [16], we easily claim the following results. We omit their proof.

**Corollary 3.1** Let $\overline{\Psi} : \Delta^c \times R^m \times R^{m \times d} \times R^{m \times d} \times \Omega \rightarrow R^m$ also satisfies (H2). Let $\overline{\Psi}(\cdot) \in L^2_{\mathcal{F}_T}[0, T]$ and $(\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot)) \in * \mathcal{H}^2[0, T]$ be the adapted S-solution of (1) with $g$ and $\Psi(\cdot)$ replaced by $\overline{g}$ and $\overline{\Psi}(\cdot)$, respectively, then we have the following: $\forall S \in [0, T]$,
\[
E \int_S^T |Y(t) - \overline{Y}(t)|^2 dt + E \int_S^T \int_S^T |Z(t, s) - \overline{Z}(t, s)|^2 ds dt
\]
\[\leq C E \int_S^T |\Psi(t) - \overline{\Psi}(t)|^2 dt + CE \int_S^T \left( \int_t^T |g - \overline{g}| ds \right)^2 dt,
\]
where $g = g(t, s, Y(s), Z(t, s), Z(s, t))$ and $\overline{g} = \overline{g}(t, s, Y(s), Z(t, s), Z(s, t))$.

**Corollary 3.2** Let (H2) hold and let $(Y(\cdot), Z(\cdot, \cdot)) \in * \mathcal{H}^2[0, T]$ be the adapted S-solution of (1). For any $t \in [0, T]$, let $(\lambda^i(\cdot), \mu^i(\cdot)) \in \mathbb{H}^2[t, T]$ be the adapted solution of the following BSDE:
\[
\lambda^i(r) = \Psi(t) + \int_r^T g(t, s, Y(s), \mu^i(s), Z(s, t)) ds - \int_r^T \mu^i(s) dW(s), \quad r \in [t, T]
\]
Let
\[
\begin{cases}
\overline{Y}(t) = \lambda^i(t), & t \in [0, T], \\
\overline{Z}(t, s) = \mu^i(s), & (t, s) \in \Delta^c.
\end{cases}
\]
and let the values $\overline{Z}(t, s)$ of $\overline{Z}(\cdot, \cdot)$ for $(t, s) \in \Delta$ be defined through:
\[
\overline{Z}(t, s) = \overline{Z}(s, t), \quad (t, s) \in \Delta
\]

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Then
\[
\begin{cases}
\overline{\mathbf{Y}}(t) = Y(t), & t \in [0, T], \\
\mathbf{Z}(t, s) = Z(t, s), & (t, s) \in [0, T]^2.
\end{cases}
\]

**Corollary 3.3** Let (H2) hold, and \( \Psi(\cdot) \in L^2_{\mathcal{F}^T} [0, T] \) and \( (Y(\cdot), Z(\cdot, \cdot)) \in^s \mathcal{H}^2[0, T] \) be the unique adapted S-solution of BSVIE (1) on \([0, T]\), then for all \( S \in [0, T] \),
\[
\Psi^S(t) = \Psi(t) + \int_S^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_S^T Z(t, s)dW(s)
\]
is \( \mathcal{F}_S \)-measurable for almost all \( t \in [0, S] \).

Next we will give an estimate to the S-solution of (1) which is stronger than (13). We assume (1) admits a unique S-solution \( (Y(\cdot), Z(\cdot, \cdot)) \in^s \mathcal{H}^2[0, T] \), and we will estimate:
\[
E \left( |Y(t)|^2 + \int_t^T |Z(t, s)|^2 ds \right).
\]
The author gave such an estimate of M-solution in [16], but there is a term \( E \int_t^T |Z(s, t)|^2 ds \) that cannot be estimated directly so he introduced the Malliavin calculus to treat this problem. But here we don’t have this kind of problem. In the following we assume \( (Y(\cdot), Z(\cdot, \cdot)) \) is S-solution of (1). From (1) we have:
\[
Y(t) + \int_t^T Z(t, s)dW(s) = \Psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds,
\]
thus
\[
E \left\{ |Y(t)|^2 + \int_t^T |Z(t, s)|^2 ds \right\}
\leq 2E|\Psi(t)|^2 + 4E \left( \int_t^T |g_0(t, s)|ds \right)^2
+ 4E \left( \int_t^T |g(t, s, Y(s), Z(t, s), Z(s, t)) - g_0(t, s)|ds \right)^2
\leq 2E|\Psi(t)|^2 + 4E \left( \int_t^T |g_0(t, s)|ds \right)^2
+ 4(T - t) \frac{2\pi}{T} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^<^{\frac{2}{2+\epsilon}} E \int_t^T |Y(s)|^2 ds \\
+ 8(T - t) \frac{2\pi}{T} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^<^{\frac{2}{2+\epsilon}} E \int_t^T |Z(t, s)|^2 ds.
\]
So there exists a constant \( \eta = T - T_1 \), such that \( \forall t \in [T_1, T] \),
\[
4(T - t) \frac{2\pi}{T} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^<^{\frac{2}{2+\epsilon}} \leq 4\eta \frac{2\pi}{T} \sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^<^{\frac{2}{2+\epsilon}} = \frac{1}{3}.\]
Then we have, $\forall t \in [T_1, T]$,

$$E|Y(t)|^2 \leq 2E|\Psi(t)|^2 + 4E \left( \int_t^T |g_0(t, s)| ds \right)^2 + \frac{1}{3} E \int_t^T |Y(s)|^2 ds \quad (27)$$

and

$$E \int_t^T |Z(t, s)|^2 ds$$

$$\leq 6E|\Psi(t)|^2 + 12E \left( \int_t^T |g_0(t, s)| ds \right)^2 + E \int_t^T |Y(s)|^2 ds,$$

thus from (27) and (28), we have:

$$E|Y(t)|^2 + E \int_t^T |Z(t, s)|^2 ds$$

$$\leq 8E|\Psi(t)|^2 + 16E \left( \int_t^T |g_0(t, s)| ds \right)^2 + \frac{4}{3} E \int_t^T |Y(s)|^2 ds$$

$$\leq 8E|\Psi(t)|^2 + \frac{4}{3} CE \int_t^T |\Psi(s)|^2 ds$$

$$+ 16E \left( \int_t^T |g_0(t, s)| ds \right)^2 + \frac{4}{3} CE \int_t^T \left( \int_s^T |g_0(s, u)| du \right)^2 ds. \quad (29)$$

The second inequality in (29) holds because of (13). $T$ is a finite constant, from the method of choice $T_1$, $T_1$ depends only on $\sup_{t \in [0, T]} \left( \int_t^T L(t, s)^{2+\epsilon} ds \right)^{\frac{2}{2+\epsilon}}$, thus there must exist finite partition on $[0, T]$, $T = T_0 \geq T_1 \geq T_2 \geq \cdots \geq T_k = 0$, such that $T_i - T_{i+1} \leq \eta$, and $\forall t \in [T_{i+1}, T_i]$ $(i = 0, 1, 2, \cdots k - 1)$, we have the following:

$$E|Y(t)|^2 + E \int_t^{T_i} |Z(t, s)|^2 ds$$

$$\leq 8E|\Psi^{T_i}(t)|^2 + 16E \left( \int_t^{T_i} |g_0(t, s)| ds \right)^2 + \frac{4}{3} E \int_t^{T_i} |Y(s)|^2 ds$$

$$\leq 8E|\Psi^{T_i}(t)|^2 + 16E \left( \int_t^{T_i} |g_0(t, s)| ds \right)^2 + \frac{4}{3} E \int_t^{T_i} |Y(s)|^2 ds$$

$$\leq 8E|\Psi^{T_i}(t)|^2 + \frac{4}{3} CE \int_t^{T_i} |\Psi(s)|^2 ds$$

$$+ 16E \left( \int_t^{T_i} |g_0(t, s)| ds \right)^2 + \frac{4}{3} CE \int_t^{T_i} \left( \int_s^{T_i} |g_0(s, u)| du \right)^2 ds. \quad (30)$$
Here

\[ Y(t) = \Psi^T_i(t) + \int_t^{T_i} g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^{T_i} Z(t, s) dW(s), \]

where

\[ \Psi^T_i(t) = \Psi(t) + \int_t^{T_i} g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^{T_i} Z(t, s) dW(s). \]  

(31)

Now we will give estimates for \( E|\Psi^T_i(t)|^2 \) and \( E \int_t^{T_i} |Z(t, s)|^2 ds, \) \( t \in [T_{i+1}, T_i], \) \( i = 1, 2, \ldots, k - 1. \) From Lemma 2.6 we have, \( \forall t \in [T_{i+1}, T_i], \)

\[
E|\Psi^T_i(t)|^2 + E \int_t^{T_i} |Z(t, s)|^2 ds \\
\leq CE|\Psi(t)|^2 + CE \left( \int_t^{T_i} |g(t, s, Y(s), 0, 0)| ds \right)^2 \\
\leq CE|\Psi(t)|^2 + CE \left( \int_t^{T_i} |g(t, s, 0, 0, 0)| ds \right)^2 \\
+ CE \left( \int_t^{T_i} L(t, s)|Y(s)| ds \right)^2 \\
\leq CE|\Psi(t)|^2 + CE \left( \int_t^{T_i} |g_0(t, s)| ds \right)^2 + CE \int_t^{T_i} |\Psi(s)|^2 ds \\
+ C \int_t^{T_i} \left( \int_s^{T_i} |g_0(s, u)| du \right)^2 ds. \]  

(32)

Hence we have for any \( t \in [0, T], \) there must exist one \( i, \) such that \( t \in [T_{i+1}, T_i]. \) From (30) and (32), we have,

\[
E|Y(t)|^2 + E \int_t^{T} |Z(t, s)|^2 ds \\
= E|Y(t)|^2 + E \int_t^{T_i} |Z(t, s)|^2 ds + E \int_t^{T_i} |Z(t, s)|^2 ds \\
\leq l_1 E \left( \int_t^{T_i} |g_0(t, s)| ds \right)^2 + l_2 E|\Psi(t)|^2 \\
+ l_3 E \int_t^{T} |\Psi(s)|^2 ds + l_4 E \left( \int_t^{T} \left( \int_s^{T_i} |g_0(s, u)| du \right)^2 ds, \right.
\]

where \( l_1, l_2, l_3, l_4 \) depend on \( T \) and \( \sup_{t \in [0, T]} L(t, s)^{2+\epsilon} ds. \)

To sum up the argument above, we give the estimate as follows:
Corollary 3.4 Let \((Y(\cdot), Z(\cdot, \cdot))\) be the \(S\)-solution of (1). Assume
\[
E\int_0^T \left( \int_s^T |g_0(s, u)| du \right)^2 ds < \infty, \quad E\int_0^T |\Psi(s)|^2 ds < \infty,
\]
and for any \(t \in [0, T]\), \(E|\Psi(t)|^2 < \infty\), then we have \(\forall t \in [0, T]\),
\[
E|Y(t)|^2 + E\int_t^T |Z(t, s)|^2 ds \\
\leq l_1 E\left( \int_t^T |g_0(t, s)| ds \right)^2 + l_2 E|\Psi(t)|^2 \\
+ l_3 E\int_t^T |\Psi(s)|^2 ds + l_4 E\int_t^T \left( \int_s^T |g_0(s, u)| du \right)^2 ds,
\]
(33)
where \(l_1, l_2, l_3, l_4\) depend on \(T\) and \(\sup_{t \in [0, T]} \int_t^T L(t, s)^{2+r} ds\).

We also have:

Corollary 3.5 Let \((Y(\cdot), Z(\cdot, \cdot))\) be the \(S\)-solution of (1). Assume
\[
\sup_{t \in [0, T]} E|\Psi(t)|^2 < \infty, \quad \sup_{t \in [0, T]} E\left( \int_t^T |g_0(t, s)| ds \right)^2 < \infty,
\]
then \(\sup_{t \in [0, T]} E|Y(t)|^2 < \infty\) and \(\sup_{t \in [0, T]} E\int_t^T |Z(t, s)|^2 ds < \infty\).

Next we show the continuity of \(Y(t)\) in \(t\). We have:

Corollary 3.6 Let \((Y(\cdot), Z(\cdot, \cdot))\) be the \(S\)-solution of (1), and assume
\[
\sup_{t \in [0, T]} E\left( \int_t^T |g_0(t, s)| ds \right)^2 < \infty, \quad \sup_{t \in [0, T]} E|\Psi(t)|^2 < \infty,
\]
(34)
then \(\forall t, \overline{t} \in [0, T]\), we have
\[
E \left\{ |Y(t) - Y(\overline{t})|^2 + \int_{t \wedge \overline{t}}^{t \vee \overline{t}} |Z(t, s) - Z(\overline{t}, s)|^2 ds \right\} \\
\leq CE \left\{ |\Psi(t) - \Psi(\overline{t})|^2 + \int_{t \wedge \overline{t}}^{t \vee \overline{t}} |Z(t \wedge \overline{t}, s)|^2 ds \\
+ \left( \int_{t \wedge \overline{t}}^{t \vee \overline{t}} |g(t, s, Y(s), Z(t, s), Z(s, t)) - g(\overline{t}, s, Y(s), Z(t, s), Z(s, t))| ds \right)^2 \\
+ \left( \int_{t \wedge \overline{t}}^{t \vee \overline{t}} |g(t \wedge \overline{t}, s, Y(s), Z(t \wedge \overline{t}, s), Z(s, t \wedge \overline{t}))| ds \right)^2 \right\}.
\]
(35)
Consequently, in the case that
\[
\lim_{|t-\bar{t}| \to 0} E|\Psi(t) - \Psi(\bar{t})|^2 = 0,
\]
and \( t \mapsto g(t, s, y, z, \zeta) \) is continuous in the sense that
\[
|g(t, s, y, z, \zeta) - g(\bar{t}, s, y, z, \zeta)| \leq C(1 + |y| + |z| + |\zeta|)|t - \bar{t}|,
\]
\( \forall t, \bar{t} \in [0, T] \), \( s \in [t \lor \bar{t}, T] \), \( y, z, \zeta \in \mathbb{R} \), for some modulus of continuity \( \rho(\cdot) \), then we have
\[
\lim_{|t-\bar{t}| \to 0} \left( E|Y(t) - Y(\bar{t})|^2 + E \int_{t \lor \bar{t}}^{T} |Z(t, s) - Z(\bar{t}, s)|^2 ds \right) = 0.
\]

**Proof.** We can easily obtain (35) by (9) with \( h(t, s, z) \equiv g(t, s, Y(s), z, z) \). We have
\[
E \left( \int_{t \lor \bar{t}}^{T} |g(t \land \bar{t}, s, Y(s), Z(t \land \bar{t}, s), Z(s, t \land \bar{t}))|^2 ds \right)^2
\leq CE \left( \int_{t \lor \bar{t}}^{T} |g_0(t \land \bar{t}, s)|^2 ds \right)^2 + C \int_{t \lor \bar{t}}^{T} |L(t \land \bar{t}, s)|^2 ds \cdot E \int_{t \lor \bar{t}}^{T} |Y(s)|^2 ds
+ C \int_{t \lor \bar{t}}^{T} |L(t \land \bar{t}, s)|^2 ds \cdot E \int_{t \lor \bar{t}}^{T} |Z(t \land \bar{t}, s)|^2 ds
+ C \int_{t \lor \bar{t}}^{T} |L(t \land \bar{t}, s)|^2 ds \cdot E \int_{t \lor \bar{t}}^{T} |Z(s, t \land \bar{t})|^2 ds.
\]
(39)
From Corollary 5 we have
\[
\sup_{t \in [0, T]} E \int_{t}^{T} |Z(t, s)|^2 ds < \infty.
\]
(40)
From (36), (37), (39) and (40), (38) is obtained. \( \square \)

### 3.3 The relations between S-solutions and other solutions

Let us consider the following BSVIE which is a generalization of BSVIE in [5].
\[
Y(t) = \Psi(t) + \int_{t}^{T} f(t, s, Y(s), Z(t, s)) ds - \int_{t}^{T} Z(t, s)dW(s).
\]
(41)
First we give a definition of the adapted solutions of BSVIEs.

**Definition 3.1** Let \( S \in [0, T] \). A pair of \( (Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2_{1}[S, T] \) is called an adapted solution of BSVIE (41) on \([S, T]\) if (41) holds in the usual Itô’s sense for almost all \( t \in [S, T] \).
There is a gap in [5]. Now we can easily prove the existence and uniqueness of adapted solution of (41) which is a generalization of the result in [5] and overcome the gap in [5]. We can claim:

**Theorem 3.2** Let \( f : \Delta^c \times R^m \times R^{m \times d} \times \Omega \to R^m \) be \( \mathcal{B}(\Delta^c \times R^m \times R^{m \times d}) \otimes \mathcal{F}_T \)-measurable such that \( s \mapsto f(t, s, y, z) \) is \( \mathbb{P} \)-progressively measurable for all \( (t, y, z) \in [0, T] \times R^m \times R^{m \times d} \) and

\[
E \int_0^T \left( \int_t^T |f_0(t, s)| ds \right)^2 dt < \infty,
\]

where \( f_0(t, s) \equiv f(t, s, 0, 0) \). Moreover, we assume \( \forall (t, s) \in \Delta^c, \ y, \overline{y} \in R^m, \ z, \overline{z} \in R^{m \times d}; \)

\[
|f(t, s, y, z) - f(t, s, \overline{y}, \overline{z})| \leq L(t, s)(|y - \overline{y}| + |z - \overline{z}|),
\]

where \( L : \Delta^c \to R \) is a deterministic function so that for some \( \epsilon > 0, \)

\[
\sup_{t \in [0, T]} \int_t^T L(t, s)^{2 + \epsilon} ds < \infty.
\]

If \( \Psi(\cdot) \in L^2_{\mathcal{F}_T}[0, T] \), then (41) admits a unique adapted solution.

**Proof.** Let \( g(t, s, Y(s), Z(t, s), Z(s, t)) \equiv f(t, s, Y(s), \frac{Z(t, s)}{2} + \frac{Z(s, t)}{2}) \). We consider the following BSVIE, \( t \in [0, T], \)

\[
Y(t) = \Psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s). \tag{42}
\]

\( \forall y, \overline{y}, z, \overline{z}, \zeta, \overline{\zeta} \in R \), we have

\[
|g(t, s, y, z, \zeta) - g(t, s, \overline{y}, \overline{z}, \overline{\zeta})| = \left| f(t, s, y, \frac{z}{2} + \frac{\zeta}{2}) - f(t, s, \overline{y}, \frac{\overline{z}}{2} + \frac{\overline{\zeta}}{2}) \right|
\leq L(t, s) \left( |y - \overline{y}| + \left| \frac{z}{2} + \frac{\zeta}{2} - \frac{\overline{z}}{2} - \frac{\overline{\zeta}}{2} \right| \right)
\leq L(t, s)(|y - \overline{y}| + |z - \overline{z}| + |\zeta - \overline{\zeta}|).
\]

So \( g \) satisfies (H2). Then BSVIE (42) admits a unique S-solution on \([0, T]\) by Theorem 3.1. Then we obtain the existence and uniqueness of the adapted solution of (41) on \([0, T]\). \( \square \)

**Remark 3.1** On the other hand, for the S-solution of (1), due to \( Z(t, s) \equiv Z(s, t) \), we can let

\[
f(t, s, Y(s), Z(t, s)) \equiv g(t, s, Y(s), Z(t, s), Z(s, t)),
\]

then (1) can be transformed into (41). If we have obtained the adapted solution \( (Y(\cdot), Z(\cdot, \cdot)) \in H^2_1[0, T] \) for (41), then we can get the S-solution of (1) in \( H^2[0, T] \) by defining the value for \( Z(t, s) \equiv Z(s, t) \). \( (t, s) \in \Delta \).
Therefore we obtain the M-solution of (43) as follows:

\[ Y(t) = tTW(T) - \int_t^T tY(s)/s^2\,ds - \int_t^T Z(t,s)dW(s), \quad t \in [T_1, T]. \tag{43} \]

Here \( T_1 > 0, \Psi(t) = tTW(T), g(t,s,Y(t),Z(t,s),Z(s,t)) = -tY(s)/s^2. \) It is easy to check that \( E \int_{T_1}^T t^2T^2W^2(T)\,dt = (T^6 - T^3T^3_1)/3 < \infty, \) and \( g \) satisfies the assumption (H2).

It is obvious that \( Y(t) = t^2W(t), Z(t,s) = ts \) satisfies (43), thus it is the unique S-solution of (43). We also know that BSVIE (43) has a unique M-solution (see [16]). But the M-solution is not equal to the S-solution of (43). In fact, if the unique S-solution of (43) also is the M-solution of (43), we have

\[ t^2W(t) = Y(t) = E(Y(t)|\mathcal{F}_{T_1}) + \int_{T_1}^T Z(t,s)dW(s) \]

\[ = t^2W(T_1) + \int_{T_1}^T tsdW(s) \]

\[ = t^2W(T_1) + t^2W(t) - tW(T_1)T_1 - t \int_{T_1}^T W(s)ds. \]

Thus

\[ tW(T_1)T_1 + t \int_{T_1}^T W(s)ds - t^2W(T_1) = 0, \quad \forall t \in [T_1, T], \]

then

\[ \frac{\int_{t_2}^{t_1} W(s)ds}{t_1 - t_2} = W(T_1), \quad \forall t_1, t_2 \in [T_1, T], \]

which means that \( W(t) = W(T_1) \) for any \( t \in [T_1, T]. \) Obviously it is a contradiction.

Now we give the explicit M-solution for (43). Let \( Z(t,s) = ts, T_1 \leq t \leq s \leq T, \) and \( Y(t) = t^2W(t), t \in [T_1, T]. \) Because

\[ E \int_{T_1}^T |D_sY(t)|^2\,ds = \int_{T_1}^T t^4I_{[0,t]}(s)ds < \infty, \]

then by Ocone-Clark formula (see [7]) and the definition of M-solution, we have

\[ Y(t) = E(Y(t)|\mathcal{F}_{T_1}) + \int_{T_1}^T E(D_sY(t)|\mathcal{F}_s)ds = E(Y(t)|\mathcal{F}_{T_1}) + \int_{T_1}^T Z(t,s)dW(s). \]

Thus

\[ Z(t,s) = E(D_s^2W(t)|\mathcal{F}_s) = t^2, \quad T_1 \leq s < t \leq T. \]

Therefore we obtain the M-solution of (43) as follows:

Remark 3.2 If BSVIE (1) degenerates to BSVIE (41), then the M-solution \((Y_1,Z_1)\) and S-solution \((Y_2,Z_2)\) of (1) are identical in \( \Delta^c, \) i.e., \( Y_1(t) \equiv Y_2(t), Z_1(t,s) \equiv Z_2(t,s), \quad 0 \leq t \leq s \leq T. \) But \( Z_1(t,s) \) and \( Z_2(t,s) \) may be different in \( \Delta. \) Now we give two examples to illustrate it. Let's consider the following BSVIE
\[
\begin{aligned}
Y(t) &= t^2 W(t), \quad t \in [T_1, T], \\
Z(t,s) &= ts, \quad t,s \in \Delta^c [T_1,T], \\
Z(t,s) &= t^2, \quad t,s \in \Delta [T_1,T].
\end{aligned}
\]

The above example is on \([T_1,T], \quad (T_1 > 0)\). Now we give an example on \([0,T]\).

Let's consider the following BSVIE, \(t \in [0,T]\),

\[
Y(t) = W(T)(T+1)(t+1) - \int_t^T \frac{(t+1)Y(s)}{(s+1)^2} ds - \int_t^T Z(t,s) dW(s), \quad (44)
\]

By the same method we can get the unique S-solution

\[
Y(t) = (t+1)^2 W(t), \quad Z(t,s) = (t+1)(s+1) \quad \text{of} \quad (44).
\]

From

\[
Z(t,s) = E(D_s(t+1)^2 W(t) | \mathcal{F}_s) = (t+1)^2, \quad 0 \leq s < t \leq T.
\]

we know the unique M-solution of (44) is as follows:

\[
\begin{aligned}
Y(t) &= (t+1)^2 W(t), \quad t \in [0,T], \\
Z(t,s) &= (t+1)(s+1), \quad t,s \in \Delta^c, \\
Z(t,s) &= (t+1)^2, \quad t,s \in \Delta.
\end{aligned}
\]

**Remark 3.3** From Remark 2 we know that when the generator of BSVIE is independent of \(Z(s,t)\) \(0 \leq s < t \leq T\), the M-solution and S-solution can be equal in \(\Delta^c\). However, the following example show that when the generator depends on \(Z(s,t)\) \(0 \leq s < t \leq T\), the M-solution and S-solution can also be equal. Let us consider the following BSVIE:

\[
Y(t) = \int_t^T g(t,s,Y(s),Z(s,t)) ds - \int_t^T Z(t,s) dW(s), \quad t \in [0,T].
\]

Here we assume (H2) holds and \(g(t,s,0,0) \equiv 0\). We can easily check that \(Y(t) \equiv 0, Z(t,s) \equiv 0, t,s \in [0,T]\) is not only the unique M-solution, but also the unique S-solution.

### 3.4 An interesting result for S-solutions

Now we give an interesting result for S-solutions. We consider the following BSVIE:

\[
Y(t) = \Psi(t) + \int_t^T g(t,s,Y(s),Z(s,t),Z(s,t)) ds - \int_t^T Z(s,t) dW(s). \quad (45)
\]

We denote

\[
\mathcal{H}^2[R,S] = L^2(R,S) \times \mathcal{T}^2([R,S]; L^2[R,S]).
\]
Here $\mathcal{T}^{H}_2([R,S]; L^2_{\mathbb{P}}(R,S))$ is the set of all processes $z : [R,S]^2 \times \Omega \to R^{m \times d}$ such that for almost all $t \in [R,S]$, $z(\cdot, t) \in L^2_{\mathbb{P}}(R,S)$ satisfying

$$E \int_R \int_R |z(s,t)|^2 ds dt < \infty.$$ 

We can define the norm of $\mathcal{T}^{H}_2([R,S])$ as the norm of $\mathcal{H}^2[R,S]$. We can define S-solution for (45). Obviously (45) has a unique S-solution which is the same as the one of (1). By the same method as in [16], we can also prove (45) admits a unique M’-solution defined as follows.

**Definition 3.2** Let $S \in [0,T]$. A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{T}^{H}_2[S,T]$ is called an adapted M’-solution of (45) on $[S,T]$, if (45) holds in the usual Itô’s sense for almost all $t \in [S,T]$ and, in addition, the following holds:

$$Y(t) = E[Y(t)|\mathcal{F}_S] + \int_S^t Z(s,t)dW(s), \text{ a.e. } t \in [S,T].$$

We have the following proposition.

**Proposition 3.1** We consider the following two equations: $t \in [0,T]$

$$Y_1(t) = \Psi_1(t) + \int_t^T g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)) ds - \int_t^T Z_1(t,s)dW(s). \quad (46)$$

$$Y_2(t) = \Psi_2(t) + \int_t^T g_2(t,s,Y_2(s),Z_2(t,s),Z_2(s,t)) ds - \int_t^T Z_2(t,s)dW(s). \quad (47)$$

We assume that for any $0 \leq t \leq s \leq T$ and $y \in R^m, z, \zeta \in R^{m \times d},$

$$\Psi_1(t) \equiv \Psi_2(t), \ g_1(t,s,y,z,\zeta) \equiv g_2(t,s,y,z,\zeta), \text{ a.s.}$$

then we have

$$Y_1(t) \equiv Y_2(t), \ Z_1(t,s) \equiv Z_2(t,s), \ \forall t, s \in [0,T],$$

if and only if $(Y_i(\cdot), Z_i(\cdot, \cdot)), \ (i = 1, 2)$, are the S-solutions of (46) and (47), respectively.

**Proof.** $\Leftarrow$: It is clear.

$\Rightarrow$: If $Y_1(t) \equiv Y_2(t), \ Z_1(t,s) \equiv Z_2(t,s), \ \forall t, s \in [0,T]$, thus

$$g_1(t,s,Y_1(s),Z_1(t,s),Z_1(s,t)) = g_2(t,s,Y_2(s),Z_2(t,s),Z_2(s,t)),$$

then $\int_t^T (Z_1(t,s) - Z_2(t,s)) dW(s) \equiv 0$, furthermore,

$$E \left( \int_t^T (Z_1(t,s) - Z_2(t,s)) dW(s) \right)^2 = E \int_t^T (Z_1(t,s) - Z_2(t,s))^2 ds \equiv 0, \ t \in [0,T].$$

So we have $Z_1(t,s) \equiv Z_2(t,s), \ t \leq s$. Because of the assumption of $Z_1(t,s) \equiv Z_2(t,s), \ t, s \in [0,T]$, we have $Z_2(t,s) \equiv Z_2(s,t), \ (t \leq s)$, and the solution $(Y_2, Z_2)$ is the S-solution of (47). By a similar method we can show $(Y_1, Z_1)$ is the S-solution of (46).
Remark 3.4 From the proposition above, we know that when two kinds of equations such as (46) and (47) have the same terminal condition and the same generator, they have the same solution if and only if both of the solutions are S-solution. Next we give an example to show this. We consider the following two BSVIEs

\[ Y_1(t) = tTW(T) - \int_t^T \frac{tY_1(s)}{s^2} \, ds - \int_t^T Z_1(t, s)dW(s), \quad t \in [T_1, T]. \quad (48) \]

\[ Y_2(t) = tTW(T) - \int_t^T \frac{tY_2(s)}{s^2} \, ds - \int_t^T Z_2(t, s)dW(s), \quad t \in [T_1, T]. \quad (49) \]

Obviously (48) and (49) have the same S-solution, for \( i = 1, 2 \)

\[ Y_i(t) = t^2W(t); \quad Z_i(t, s) = Z_i(s, t) = ts, \quad t, s \in [T_1, T], \]

however, the M-solution of (48) is not equal to the M'-solution of (49). In fact, from Remark 2 we know that \( Y_1(t) = Y_2(t) = t^2W(t) \) (\( t \in [T_1, T] \)), \( Z_1(t, s) = ts \) (\( t \leq s \)) and \( Z_2(s, t) = ts \) (\( t \leq s \)). We can also determine \( Z_1(t, s) \) (\( t > s \)) by

\[ Z_1(t, s) = E(D_s t^2 W(t) | F_s) = t^2, \quad T_1 \leq s < t \leq T. \quad (50) \]

and \( Z_2(s, t) \) (\( t > s \)) by

\[ Z_2(s, t) = E(D_s t^2 W(t) | F_s) = t^2, \quad T_1 \leq s < t \leq T. \quad (51) \]

So we have \( Z_1(t, s) = ts \neq s^2 = Z_2(t, s) \) for \( t < s \) and \( Z_1(t, s) = t^2 \neq ts = Z_2(t, s) \) for \( t > s \).

When neither the terminal conditions nor the generators are equal, the conclusion above can hold too. For example, we can choose \( \Psi_1(t) = \Psi(t) + c, \quad \Psi_2(t) = \Psi(t) - c, \quad g_1(t, s, y, z, \zeta) = g(t, s, y, z, \zeta) - c \) and \( g_2(t, s, y, z, \zeta) = g(t, s, y, z, \zeta) + c \), here \( c > 0 \) is a constant. We have \( \Psi_1(t) \neq \Psi_2(t) \) and \( g_1(t, s, y, z, \zeta) \neq g_2(t, s, y, z, \zeta) \), but the conclusion still holds.

4 Dynamic risk measures by special BSVIEs

In this section, we assume \( m = d = 1 \) and \( f \) is independent of \( \omega \). We know that the following BSVIE admits a unique adapted M-solution and a unique adapted S-solution when the generator and the terminal condition satisfy certain conditions:

\[ Y(t) = \psi(t) + \int_t^T f(t, s, Y(s), Z(t, s)) \, ds - \int_t^T Z(t, s)dW(s). \quad (52) \]

From the definition of the M-solution and S-solution, we know that both of them which solve (52) in the Itô sense have the same value in the following part

\( (Y(t), Z(t, s)), \quad 0 \leq t \leq s \leq T, \quad t \in [0, T], \)
and the only difference between the two kinds of solutions is the value of
\[ Z(t, s), \quad 0 \leq s < t \leq T. \]

Now we will give a comparison theorem on S-solution for the following BSVIE:
\[ Y(t) = -\psi(t) + \int_t^T (f(t, s, Y(s)) + r_1(s)Z(t, s) + r_2(s)Z(s, t))ds - \int_t^T Z(t, s)dW(s). \]

(53)

Here \( r_i(s) \) are two deterministic functions which satisfy \( e^{\frac{1}{2}} \int_0^T r_i^2(s)ds < \infty \). Thus we can determine the value \( Y(t), t \in [0, T] \) of S-solution to (53) by solving adapted solution of (54)
\[ Y(t) = -\psi(t) + \int_t^T (f(t, s, Y(s)) + (r_1(s) + r_2(s))Z(t, s))ds - \int_t^T Z(t, s)dW(s). \]

(54)

And we can use Girsanov theorem to rewrite (54)
\[ Y(t) = -\psi(t) + \int_t^T f(t, s, Y(s))ds - \int_t^T Z(t, s)d\tilde{W}(s), \]

(55)

where \( \tilde{W}(t) = W(t) + \int_0^t (r_1(s) + r_2(s))ds \) is a Brownian motion under new probability measure \( \tilde{P} \) defined by
\[ \frac{d\tilde{P}}{dP}(\omega) = \exp \left\{ \int_0^T r(s)dW(s) - \frac{1}{2} \int_0^T r^2(s)ds \right\}, \quad r(s) = r_1(s) + r_2(s). \]

Before proving the comparison theorem for S-solution, we need the following proposition in [15].

**Proposition 4.1** We consider the following BSVIE
\[ Y(t) = -\psi(t) + \int_t^T f(t, s, Y(s), Z(s, t))ds - \int_t^T Z(t, s)dW(s). \]

(56)

Let \( f, \overline{f} : \Delta^c \times R \times R^d \to R \) satisfy (H2) (here we assume \( L(t, s) \) is a bounded function), and let \( \psi(\cdot), \overline{\psi}(\cdot) \in L^2_{F_T}[0, T] \) such that
\[ f(t, s, y, z) \geq \overline{f}(t, s, y, z), \quad \forall (t, s, y, z) \in \Delta^c \times R \times R^d, \]
\[ \psi(t) \leq \overline{\psi}(t), \quad \text{a.s.} \quad t \in [0, T], \text{ a.e.} \]

(57)

Let \( (Y(\cdot), Z(\cdot, \cdot)) \) be the adapted M-solution of BSVIE (56), and \( (\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot)) \) be the adapted M-solution of BSVIE (56) with \( f \) and \( \psi(\cdot) \) replaced by \( \overline{f} \) and \( \overline{\psi}(\cdot) \), respectively. Then the following holds:
\[ Y(t) \geq \overline{Y}(t), \quad \text{a.s.} \quad t \in [0, T], \text{ a.e.} \]
We then have

**Lemma 4.1** Let $f, \overline{f} : \Delta^c \times R \mapsto R$ satisfy (H2) (here we assume $L(t, s)$ to be bounded), and let $\psi(\cdot), \overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}[0, T]$ such that

$$
\begin{align*}
  f(t, s, y) & \geq \overline{f}(t, s, y), \quad \forall (t, s, y) \in \Delta^c \times R, \\
  \psi(t) & \leq \overline{\psi}(t), \text{ a.s.} \quad t \in [0, T].
\end{align*}
$$

(58)

Let $(Y(\cdot), Z(\cdot, \cdot))$ be the adapted $S$-solution of BSVIE (53), and $(\overline{Y}(\cdot), \overline{Z}(\cdot, \cdot))$ be the adapted $S$-solution of BSVIE (53) with $f$ and $\psi(\cdot)$ replaced by $\overline{f}$ and $\overline{\psi}(\cdot)$, respectively. Then the following holds:

$$
Y(t) \geq \overline{Y}(t), \quad \text{a.s., } \forall t \in [0, T],
$$

(59)

**Proof.** We assume $(Y_1(\cdot), Z_1(\cdot, \cdot))$ is the unique M-solution of BSVIE (55), and $(\overline{Y}_1(\cdot), \overline{Z}_1(\cdot, \cdot))$ is the unique M-solution of (55) with $f$ and $\psi(\cdot)$ replaced by $\overline{f}$ and $\overline{\psi}(\cdot)$, respectively. Clearly, we have: $t \in [0, T],$

$$
\tilde{P}\{\omega; Y_1(t) = Y(t)\} = 1; \quad \tilde{P}\{\omega; \overline{Y}_1(t) = \overline{Y}(t)\} = 1.
$$

From Proposition 3, we know

$$
\tilde{P}\{\omega; Y_1(t) \geq \overline{Y}_1(t)\} = 1,
$$

then we have: $\forall t \in [0, T], \tilde{P}\{\omega; Y(t) \geq \overline{Y}(t)\} = 1,$ thus $P\{\omega; Y(t) \geq \overline{Y}(t)\} = 1. \quad \Box$

In what follows, we define

$$
\rho(t; \psi(\cdot)) = Y(t), \quad \forall t \in [0, T],
$$

(59)

where $(Y(\cdot), Z(\cdot, \cdot))$ is the unique adapted $S$-solution of BSVIE (53).

**Lemma 4.2** Let $f : \Delta^c \times R \mapsto R$ satisfy (H2).

1) Suppose $f$ is sub-additive, i.e.,

$$
\begin{align*}
  f(t, s, y_1 + y_2) & \leq f(t, s, y_1) + f(t, s, y_2), \quad (t, s) \in \Delta^c, \quad y_1, y_2 \in R, \text{ a.e.,}
\end{align*}
$$

then $\psi(\cdot) \mapsto \rho(t; \psi(\cdot))$ is sub-additive, i.e.,

$$
\rho(t; \psi_1(\cdot) + \psi_2(\cdot)) \leq \rho(t; \psi_1(\cdot)) + \rho(t; \psi_2(\cdot)), \text{ a.s., } \quad t \in [0, T]. \quad \text{a.e.}
$$

**Proof.** We can get the conclusion by Lemma 4.1. \quad \Box

**Lemma 4.3** 1) If the generator of (53) is: $f(t, s, y) = \eta(s)y$, with $\eta(\cdot)$ being a deterministic integrable function, then $\psi(\cdot) \mapsto \rho(t; \psi(\cdot))$ is translation invariant, i.e.,

$$
\rho(t; \psi(\cdot) + c) = \rho(t; \psi(\cdot)) - ce^{\int_t^T \eta(s)ds}, \quad \text{a.s., } \quad t \in [0, T], \quad \forall c \in R.
$$

In particular, if $\eta(\cdot) = 0$, then

$$
\rho(t; \psi(\cdot) + c) = \rho(t; \psi(\cdot)) - c, \quad \text{a.s., } \quad t \in [0, T], \quad \forall c \in R.
$$

2) If $f : \Delta^c \times R \mapsto R$ is positively homogeneous, i.e., $f(t, s, \lambda y) = \lambda f(t, s, y), \quad t, s \in \Delta^c, \text{ a.s., } \forall \lambda \in \mathbb{R}^+$, so is $\psi(\cdot) \mapsto \rho(t; \psi(\cdot)).$
Proof. The result is obvious. □ We then have

Theorem 4.1 Suppose \( f(t,s,y) = \eta(s)y \), with \( \eta(\cdot) \) being a deterministic bounded function, then \( \rho(\cdot) \) defined by (59) is a dynamic coherent risk measure.

Proof. It is not difficult to obtain the conclusion by the above lemmas. □

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