Quandles and Lefschetz Fibrations

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Abstract: We show that isotopy classes of simple closed curves in any oriented surface admit a quandle structure with operations induced by Dehn twists, the Dehn quandle of the surface. We further show that the monodromy of a Lefschetz fibration can be conveniently encoded as a quandle homomorphism from the knot quandle of the base as a manifold with a codimension 2 subspace (the set of singular values) to the Dehn quandle of the generic fibre and discuss prospects for construction of invariants arising naturally from this description of monodromy.

1 Quandles, Fundamental Quandles and Knot Quandles

Quandles were originally introduced by Joyce [Joy79, Joy82] as an algebraic invariant of classical knots and links. They may be regarded as an abstraction from groups in as much as some of the most important examples arise by considering a group with left and right conjugation as operations.

Definition 1 A quandle is a set $Q$ equipped with two binary operations $\triangleright$ and $\triangleright =$ satisfying

\[
\forall x \in Q \quad x \triangleright x = x
\]

\[
\forall x, y \in Q \quad (x \triangleright y) \triangleright = x = (x \triangleright y) \triangleright y
\]

\[
\forall x, y, z \in Q \quad (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)
\]

Algebraic structures satisfying the second and third axioms only have been studied under the name “racks” by Fenn and Rourke [FR92], structures satisfying the third axiom only are called “right distributive semigroups” by universal algebraists.

Examples abound:

Example 2 If $G$ is any group, we can make $G$ into a quandle by letting $x \triangleright y = y^{-1}xy$ and $x \triangleright = y = yxy^{-1}$. Likewise any union of conjugacy classes in a group $G$ forms a subquandle.

This example is of particular importance for the theory of quandles, as a representation theorem due to Joyce [Joy79] shows that all quandles admit representations as a union of conjugacy classes in some group.

Example 3 Given a linear automorphism $T$ of a vectorspace $V$, $V$ becomes a quandle with $x \triangleright y = T(x - y) + y$ and $x \triangleright = y = T^{-1}(x - y) + y$.

The following examples are of particular interest to us, as we will see a topological application later:

Example 4 Let $R$ be any commutative ring, and $X$ be a free $R$-module equipped with an antisymmetric bilinear form $\langle - , - \rangle : X \times X \to R$. (If $(R, +)$ has any two-torsion, we actually need alternating rather than just antisymmetric.) Then $X$ is a quandle when equipped with the operations

\[
(x \triangleright y) \triangleright z = (x \triangleright (y \triangleright z)) \triangleright = (x \triangleright = z) \triangleright = (y \triangleright = z)
\]
\[ x \triangleright y = x + (x, y)y \]
\[ x \trianglerighteq y = x - (x, y)y \]

The proof that this last example satisfies the quandle axioms is routine, but we indicated it to give the reader unfamiliar with quandles the flavor of such things:

Observe that since \((-,-)\) is alternating, we have \(x \triangleright x = x\). Likewise by bilinearity and alternating-ness, it follows that \((x + ay, y) = (x, y)\) from which the second quandle axiom follows.

For the third, we calculate

\[
(x \triangleright y) \triangleright z = (x + (x, y)y) \triangleright z \\
= x + (x, y)y + (x + (x, y)y, z)z \\
= x + (x, y)y + (x, z)z + (x, y)(y, z)z
\]

while

\[
(x \triangleright z) \triangleright (y \triangleright z) = (x + (x, z)z) \triangleright (y + (y, z)) \\
= x + (x, z)z + (x + (x, z)z, y + (y, z)z(y + (y, z)z) \\
= x + (x, z)z + \\
\quad [(x, y)y + (x, z)(y, z) + (y, z)(x, z) + (x, z)(y, z)(z, z)(y + (y, z)z) \\
= x + (x, y)y + (x, z)z + (x, y)(y, z)z
\]

where the equations follow from the definitions (twice), bilinearity and alternating-ness respectively.

We will call a quandle arising in this way an alternating quandle.

We will not directly apply alternating quandles, but rather a particular quotient quandle which exists for any alternating quandle:

**Example 5** Given an alternating bilinear form \((-,-)\) on an \(R\)-module \(V\), the alternating quandle structure on \(V\) induces a quandle structure on the space of orbits of the action of the multiplicative group \(\{1, -1\}\) on \(V\) by scalar multiplication (note: if \(1 = -1\) in \(R\) the action is trivial):

Negating \(x\) negates \(x \triangleright y\) and \(x \trianglerighteq y\), while negating \(y\) leaves them unchanged (since the negation of the instance of \(y\) in the bilinear form cancels the negation of \(y\) outside).

We will call a quandle arising in this way a reduced alternating quandle.

Joyce’s principal motivation in considering this structure was to provide an algebro-topological invariant of classical knots more sensitive that the fundamental group of the complement.

As we will need the corresponding notion down a dimension, we recall his constructions, but state them in arbitrary dimensions. We consider pairs of a space and a subspace, equipped with a point in the complement of the subspace \((X, S, p)\). In particular we consider the “noose” or “lollipop”: \((N, \{0\}, 2)\) where \(N\) is the subspace of \(\mathbb{C}\) consisting of union of the unit disk and the line segment \([1, 2] in the real axis.

By a map of pointed pairs we mean a continuous map which preserves the base point and both the subspace and its complement. We can then make
Definition 6 The fundamental quandle $\Pi(X, S, p)$ of a pointed pair $(X, S, p)$ is the set of homotopy classes of maps of pointed pairs (where homotopies are through maps of pointed pairs), equipped with the operations $x \cdot y$ (resp. $x \cdot' y$) induced by appending the path from the base point obtained by traversing $(S^1)$ oriented counterclockwise (resp. clockwise), followed by traversing $(S^1)$ in the opposite direction to the path $x([1,2])$ and reparametrizing.

For the proof that this gives a quandle structure, see [Joyce79, Joyce82].

In the case where both the space and its subspace are smooth oriented manifolds and the subspace is of codimension 2, it is possible to identify a particularly interesting subquandle of the fundamental quandle.

Definition 7 The knot quandle $Q(M, K, p)$ of a pointed pair $(M, K, p)$, where $M$ is a smooth manifold, $K$ a smooth embedded submanifold of codimension 2 is the subquandle of $\Pi(M, K, p)$ consisting of all maps of the noose such that the bounding $S^1$ has linking number 1 with $K$. (Note: this is in the signed sense.)

Joyce [Joyce79, Joyce82] showed that the knot quandle of a classical knot determined the knot up to orientation.

We, however, will be concerned here with the knot quandle of set of (positively oriented) points in a surface: Given a (path connected) oriented surface $\Sigma$, equipped with a finite set of points $S$, and a point $p$ not lying in $S$, the quandle $Q(\Sigma, S, p)$ has as elements all isotopy classes of maps of pointed pairs from the noose to $(\Sigma, S, p)$ which map the boundary of the disk with winding number 1.

It is easy to see that there is a relationship between $Q(\Sigma, S, p)$ and $\pi_1(\Sigma \setminus S, p)$: an action of the fundamental group $\pi_1(\Sigma \setminus S, p)$ on $Q(\Sigma, S, p)$ by quandle homomorphisms is given by appending a loop representing an element of $\pi_1$ to the initial path of the noose and rescaling.

There is, however, an more intimate relationship between $Q(\Sigma, S, p)$ and $\pi_1(\Sigma \setminus S, p)$:

Definition 8 [Joyce79, Joyce82] An augmented quandle is a quadruple $(Q, G, \ell : Q \to G, \cdot : Q \times G \to Q)$ where $Q$ is a quandle, $G$ is a group, $\cdot$ is a right-action of $G$ on $Q$ by quandle homomorphisms, and the set-map $\ell$ (called the augmentation) satisfies

$$q \cdot \ell(q) = q$$
$$\ell(q \cdot \gamma) = \gamma^{-1}\ell(q)\gamma$$

Proposition 9 For any oriented manifold $M$ with an oriented, properly embedded codimension 2 submanifold $K$ and a point $p \in M \setminus K$, the quadruple

$$(Q(M, K, p), \pi_1(M \setminus K, p), \ell, \cdot)$$

where $\cdot$ is the action described above, and $\ell(q)$ is the homotopy class of the loop at $p$ which traverses the arc, then the boundary of the disk counterclockwise, then the arc back to $p$, is an augmented quandle. We call the loop at $p$ just described as a representative for $\ell(q)$ the canonical loop of the noose $q$.

proof: Having noted that the action of $\pi_1(M \setminus K, p)$ is by quandle homomorphisms (a fact which follows essentially by conjugation in the fundamental groupoid–the reader may fill in the details),
it remains only to verify that the map \( \ell \) satisfies the two conditions specified in the definition of augmented quandles.

The first reduces to the idempotence of the quandle operation. The second follows from the fact that the appended loop occurs twice in the specification of \( \ell(q \cdot \gamma) \), initially in the outgoing arc from \( p \) with positive orientation, and again in the incoming arc to \( p \) with reversed orientation. \( \square \)

In the case where \( M \) is simply connected we have

**Proposition 10** The image of the augmentation, \( \ell(Q(M,K,p)) \) generates \( \pi_1(M \setminus K, p) \).

**proof:** This follows from van Kampen’s Theorem: killing all of the noose boundaries kills the fundamental group, and thus the noose boundaries generate. \( \square \)

### 2 Dehn Quandles

We now consider a rather different geometric construction of quandles, related to mapping class groups of surfaces in a way weakly analogous to the relationship between knot quandles and fundamental groups.

From [Bir75] we recall

**Definition 11** If \( \Sigma \) is a surface, the mapping class group of \( \Sigma \) is the group \( M(\Sigma) = \pi_0(F\Sigma) \), where \( F\Sigma \) is the group of all orientation-preserving self-diffeomorphisms of \( \Sigma \), endowed with the compact-open topology.

Birman [Bir75] actually defines more general objects depending on a set of distinguished points lying in \( \Sigma \). Following the usual convention, if \( \Sigma \) is of genus \( g \), we denote its mapping class group by \( M(g,0) \), the 0 indicating the lack of distinguished points.

It is easy to verify that \( M(0,0) \) is trivial. It is also well-known that \( M(1,0) \cong SL(2,\mathbb{Z}) \).

Birman and Hilden [BH71] gave a finite presentation for \( M(2,0) \). Building on work of McCool [McC73] and Hatcher and Thurston [HT80], Harer [Har83] gave finite presentations for the higher genus case, which were improved by Wajnryb [Waj83].

The key to approaching presentations of mapping class groups, and to our related quandles, however, predates these developments, and is due to Dehn [Deh38]. It depends upon a particular construction of self-diffeomorphisms from an embedded curve:

**Definition 12** Let \( \Sigma \) be an oriented surface, and \( c \) a simple closed curve lying in \( \Sigma \). \( c \) then admits a bicollar neighborhood \( U \). If we identify this bicollar neighborhood with the annulus

\[
A = \{ z | 1 < |z| < 2 \}
\]

in \( \mathbb{C} \) by an orientation preserving diffeomorphism, \( \phi: U \to A \) which maps \( c \) to \( \{ z| |z| = \frac{3}{2} \} \) given in polar coordinates by \( \phi = (r_\phi, \theta_\phi) \), the self-homeomorphism \( t^+_c : \Sigma \to \Sigma \) given by

\[
t^+_c(x) = \begin{cases} 
x & \text{if } x \in \Sigma \setminus U \\
\phi^{-1}(r_\phi(x), \theta_\phi(x) + 2\pi r_\phi(x)) & \text{if } x \in U
\end{cases}
\]

or any self-diffeomorphism obtained by smoothing \( t^+_c \) is called a positive (or right-handed) Dehn twist about \( c \).

Negative (or left-handed) Dehn twists are defined similarly using

\[
t^-_c(x) = \begin{cases} 
x & \text{if } x \in \Sigma \setminus U \\
\phi^{-1}(r_\phi(x), \theta_\phi(x) - 2\pi r_\phi(x)) & \text{if } x \in U
\end{cases}
\]
It is easy to see that the positive and negative Dehn twists about a curve \( c \) are inverse to each other (in the smoothed case up to isotopy).

It is well-known that the positive Dehn twists along isotopic simple closed curves are isotopic as diffeomorphisms. Thus each isotopy class of simple closed curves determines an element of the mapping class group. Similarly the images of simple closed curves under isotopic diffeomorphisms will be isotopic.

We may thus make the following definition

**Definition 13** The Dehn quandle \( D(\Sigma) \) of an oriented surface \( \Sigma \) is the set of isotopy classes of simple closed curves in \( \Sigma \) equipped with the operations

\[
\begin{align*}
  x \searrow y &= t_y^+(x) \\
  x \nearrow y &= t_y^-(x)
\end{align*}
\]

where by abuse of notation we use the same symbol to denote the isotopy class and a representative curve.

By the discussion above, it is clear that the operations described are independent of the choice of representing curve and define a well-defined isotopy class of curves. We now establish

**Proposition 14** The operations of Definition 13 satisfy the quandle axioms.

**proof:** It is clear by the discussion above that the second quandle axiom is satisfied. Likewise observe that \( t_y^+ \) fixes the curve \( x \) up to isotopy. Thus the first quandle axiom is satisfied. It thus remains only to verify the third axiom. This may be seen from the fact that any self-diffeomorphism of \( \Sigma \) induces an automorphism of the algebraic structure with operations \( \searrow \) and \( \nearrow \), in particular \( - \searrow y = t_y^+(-) \) is such an automorphism. \( \square \)

We consider now the case of a genus one surface, where the structure of the Dehn quandle can be completely determined. As noted above \( M(1,0) \cong SL(2,\mathbb{Z}) \). Recall also that isotopy classes of essential simple closed curve are given by slopes \( \frac{y}{x} \) with \( x \) and \( y \) relatively prime integers (and 0 is allowed in either place).

As noted in Casson and Bleiler [CB88], elements of \( SL(2,\mathbb{Z}) \) corresponding to powers of Dehn twists are the integer matrices of trace 2 and determinant 1. A fairly routine calculation shows that the positive Dehn twist along a curve of slope \( \frac{y}{x} \) (\( \gcd(x,y)=1 \)) is given by the matrix

\[
M_{\frac{y}{x}} = \begin{bmatrix} xy + 1 & -x^2 \\ y^2 & 1 - xy \end{bmatrix}
\]

Observe also that this transformation from slopes to matrices is well-defined, being independent of the choice of signs for \( x \) and \( y \), and one-to-one: given a matrix of the given form, \( x \) and \( y \) may be recovered up to sign from the off-diagonal entries, while the diagonal entries determine the sign of the product \( xy \), and thus the sign of the slope \( \frac{y}{x} \).

We can thus determine a formula for the operations of the Dehn quandle from the observation that

\[
t_{h(c)}^+ = h(t_c^+(h^{-1})) \quad (*)
\]

Applying this fact in the case where \( h \) itself is a positive Dehn twist gives us
Example 15 The Dehn quandle of the torus $D(T^2)$ has underlying set
\[
\{ \frac{y}{x} | x, y \in \mathbb{Z}, \gcd(x, y) = 1 \} \cup \{ I \}
\]
where $I$ represents the (unique) isotopy class of contractible simple closed curves and $\frac{y}{x}$ represents the isotopy class of essential simple closed curves of slope $\frac{y}{x}$.

The quandle operations on $D(T^2)$ are given by
\[
\begin{align*}
\frac{v}{u} \triangleright \frac{y}{x} &= \frac{v + vxy - uy^2}{u - uxy + vx^2} \\
I \triangleright q &= I \\
qu \triangleright I &= q \\
\frac{v}{u} \triangleright \frac{y}{x} &= \frac{v - vxy + uy^2}{u + uxy - vx^2} \\
I \triangleright q &= I \\
qu \triangleright I &= q
\end{align*}
\]
where $q$ is any element of the quandle and $x$, $y$, $u$, and $v$ are integers with $\gcd(x, y) = \gcd(u, v) = 1$.

It is easy to see that Dehn twist on contractible curves are isotopic to the identity, and likewise that the isotopy class of contractible curves is fixed by any Dehn twist. The first of the remaining two relations may be obtained by using the equation $(\ast)$ above in the case where $h = t^\frac{1}{x}$ and $c$ has slope $\frac{u}{v}$, computing the conjugate $M_{\frac{1}{x}}^{-1} M_{\frac{u}{v}} M_{\frac{x}{y}}$ and identifying the numerator and denominator which give rise to the resulting matrix. The last remaining relation may be verified by observing that it provides the inverse operation to that just computed.

As in the case of the fundamental and knot quandles, the Dehn quandle admits an augmentation in the obvious related group:

Proposition 16 There is an obvious right action of $M(\Sigma)$ on $D(\Sigma)$ by quandle homomorphisms given by \([q] \cdot [h] = [h(q)]\). Let $\ell : D(\Sigma) \rightarrow M(\Sigma)$ be given by mapping an isotopy class of simple closed curve in $\Sigma$ to the isotopy class of the positive Dehn twist about any of its representatives. Then
\[
(D(\Sigma), M(\Sigma), \ell, \cdot)
\]
is an augmented quandle. We call it the augmented Dehn quandle of $\Sigma$.

proof: The proof is routine.

As was the case with the augmented knot quandle for a simply connected underlying manifold, so with augmented Dehn quandles we have

Proposition 17 The image of the augmentation $\ell(D(\Sigma))$ generates $M(\Sigma)$

proof: This is simply a restatement of the classical theorem that the mapping class group is generated by (positive) Dehn twists \cite{Deh38, Lic64}.

As of this writing, the structure of the Dehn quandle for higher genus surfaces has yet to be determined. One thing which can be read off from the well-known presentation for the mapping class group for a surface $\Sigma_2$ of genus two is
**Proposition 18** The Dehn quandle $D(\Sigma_2)$ of a surface of genus two admits a quotient to a seventeen element quandle, two of whose elements act trivially and the other fifteen of which form the quandle of all transpositions in $S_6$.

**proof:** First pass to the subquandle of $M(\Sigma_2)$ under conjugation by the augmentation map, then to the subquandle of $\mathbb{Z}/10 \times S_6$ under the quandle map induced by the group homomorphism which maps the generator $\zeta_i$ to $(1, (i \ i + 1))$. The image is then the subset $\{(0, e), (2, e)\} \cup \{(1, (a \ b)) | 1 \leq a < b \leq 6\}$. The element $(2, e)$ is the image of (any of) the product(s) of twelve Dehn twists about non-separating curves which give a Dehn twist about a separating curve, all of which become trivial in the quotient to $S_6$ and map to 2 in the quotient to $\mathbb{Z}/10$. This set is readily verified to be closed under conjugation, which induces the quandle structure describe in the proposition. $\blacksquare$

It is also possible in general to find interesting quotients of a Dehn quandle $D(\Sigma)$ by considering the reduced alternating quandle associated to $H_1(\Sigma, R)$ with the intersection form, where $R$ is any quotient of $\mathbb{Z}$. We call the alternating quandle associated to the intersection form the $R$-homology quandle of $\Sigma$ and denote it by $HQ_R(\Sigma)$, omitting the $R$ when $R = \mathbb{Z}$. (As an aside, by the same construction, we can put a quandle structure on $H_2^{n+1}(X, R)$ for any $4n + 2$ manifold.)

Since the Dehn quandle has as elements isotopy classes of unoriented simple closed curves, they can be more naturally related to the reduced alternating quandle associated to the intersection form, which we call the $R$-homology Dehn quandle of $\Sigma$ and denote by $HD_R(\Sigma)$, as before omitting the subscript $R$ when $R = \mathbb{Z}$.

Any unoriented simple closed curve represents an element of $HD_R(\Sigma)$, with isotopic simple closed curves representing the same element. We thus have a map $D(\Sigma) \to HD_R(\Sigma)$ for any surface $\Sigma$.

To see that this map is a quandle map we must relate the geometric construction of the operations in $D(\Sigma)$ to the algebraic construction of the operation on $HD_R(\Sigma)$ from the intersection form. Consider a pair $a, b$ of unoriented simple closed curves in an oriented surface $\Sigma$. Depending on how they are oriented, their intersection number (if it is non-zero) may be given either sign. Since we are really concerned with isotopy classes of curves, we may assume the curves intersect transversely.

Now choose an orientation on $a$. We may induce an orientation on $b$ as follows: orient $b$ so that at each intersection point, the “turn right” rule defining a right-handed Dehn twist about $b$ causes the curve representing $a \triangleright b$ in $D(\Sigma)$ to traverse $b$ with the same sign as the intersection point.

The curve representing $a \triangleright b$ in $D(\Sigma)$, oriented to agree with the orientation on $a$, then represents the homology class $a + \langle a, b \rangle b$ in $H_1(\Sigma, R)$. Passing to the quotient $HD_R(\Sigma)$ then removes any dependence on orientation, and we see that the map carrying a simple closed curve to the $\{\pm 1\}$-orbit of its homology class is a quandle homomorphism.

One thing which should be observed is that for genus 1, $D(\Sigma^2) \cong HD(\Sigma^2)$ since each homology class is represented by a unique isotopy class of oriented simple closed curve. In higher genus, $HD(\Sigma)$ will be a proper quotient of $D(\Sigma)$, as different isotopy classes of curves can represent the same homology class. For example, the boundary of a disk and a curve which separates a surface of genus two into two surfaces with boundary each of genus one are both null-homologous, but represent different isotopy classes.
3 Lefschetz Fibrations

We briefly recall the relevant facts about Lefschetz fibrations, following Gompf and Stipsicz [GS99]:

**Definition 19** A Lefschetz fibration of a smooth, compact oriented 4-manifold $X$ (possibly with boundary) is a smooth map $f : X \to \Sigma$, where $\Sigma$ is a compact connected oriented surface, $f^{-1}(\partial \Sigma) = \partial X$ and such that each critical point of $f$ lies in the interior of $X$ and has a local coordinate chart modelled (in complex coordinates) by $f(z, w) = z^2 + w^2$.

We moreover require that each singular fiber have a unique singular point.

Now the generic fiber $F$ of $f$ is a compact, canonically oriented surface. The genus of $F$ is called the genus of the fibration $f$.

As is pointed out in [GS99], the choice of any regular point of the fibration $p \in \Sigma$ and an identification of the fiber over $p$ with a standard surface $F$ of the appropriate genus gives rise to a group homomorphism $\Psi : \pi_1(\Sigma \setminus S, p) \to M(F)$, where $S$ is the set of critical values of $f$, called the monodromy representation of $f$.

In the case of genus $g \geq 2$, this group homomorphism completely determines the structure of the Lefschetz fibration by a theorem of Matsumoto [Mat96]. There are, however, restrictions on which group homomorphisms can occur. In particular the image of any loop linking exactly one critical value with linking number one must be a positive Dehn twist about the vanishing cycle of the singularity—the simple closed curve which collapses to a point at the singular point [GS99].

Due to the awkwardness of imposing such a condition while trying to work in a group theoretic context, when discussing Lefschetz fibrations over the disk $D^2$ and the sphere $S^2$, Gompf and Stipsicz [GS99] work instead with the monodromy of the fibration: the $|S|$-tuple of Dehn twists given by a family of generating loops each of which links a single critical value with linking number one.

This, then, has the drawback that the $|S|$-tuple is determined only up to an overall conjugation by an element of $M(F)$, cyclic permutation, and combinatorial moves given by swapping two of the Dehn twists while conjugating one of them by its partner in a suitable sense.

Both drawbacks—the use of geometric side-conditions in what would otherwise be the purely group theoretic setting monodromy representations and the ambiguity of definition inherent in the notion of the monodromy are removed by considering

**Definition 20** The quandle monodromy of a Lefschetz fibration $f : X \to \Sigma$ with critical set $S \subset \Sigma$, relative to a regular point $p$ is the quandle homomorphism

$$\mu : Q(\Sigma, S, p) \to D(F)$$

given by mapping each element of $Q(\Sigma, S, p)$ to the monodromy around the canonical loop of any representing noose.

The augmented quandle monodromy of a Lefschetz fibration $f : X \to \Sigma$ relative to $p$ is the map of augmented quandles $(\mu, \Psi)$, where $\mu$ is the quandle monodromy and $\Psi$ is the monodromy representation.

We then have

**Theorem 21** The isomorphism type of the augmented quandle monodromy determines the isomorphism class of any Lefschetz fibration of genus $g \geq 2$. Moreover, if $g \geq 2$ and the base $\Sigma$ is $D^2$ or $S^2$, the isomorphism class of the quandle monodromy determines the isomorphism class of the Lefschetz fibration.
proof: The first statement follows a fortiori from the theorem of Matsumoto [Mat96]. The second statement follows from the first, Propositions 10 and 17, and the fact that in either case $\pi_1(\Sigma \setminus S, p)$ is free. □

Observe that the first statement of this formulation includes the restriction on which homomorphisms $\Psi : \pi_1(\Sigma \setminus S, p) \to M(F)$ actually occur as an algebraic rather than combinatorial condition.

In the case of $S^2$, the second statement has an analogous deficiency to the classical formulations: not all quandle homomorphisms extend to augmented quandle homomorphisms, a suitably ordered product of the Dehn twists (images of curves under the augmentation) must be the identity in $M(F)$.

4 Prospects for Quandle Invariants of Lefschetz Fibrations

Although the purpose of this note was to introduce Dehn quandles and their use to describe the monodromy of Lefschetz fibrations, we conclude with a brief consideration of prospects for using this approach to study Lefschetz fibrations.

Having reduced the description of monodromy to quandle theory, a number of approaches to the algebraic construction of invariants of Lefschetz fibrations present themselves:

- Simple counting invariants: count the number of homomorphisms of (augmented) quandle maps (that is commuting squares of (augmented) quandle maps) from the (augmented) quandle monodromy to a fixed (augmented) quandle map between finite (augmented) quandles. Variants of this include counting factorizations of a fixed quandle map from $Q(\Sigma, S, p)$ to a finite quandle through $D(F)$.

- Quandles map valued invariants: Joyce [Joy79] considered quandles satisfying additional axioms (e.g. involutory quandles where $P = \overline{P}$, and abelian quandles which satisfy $(w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z)$). We may consider the induced map between (universal) quotient quandles as an invariant of the Lefschetz fibration.

Similarly the map $\eta : Q(\Sigma, S, p) \to HD(F, R)$, the “$R$-homology quandle monodromy” is plainly an invariant of the Lefschetz fibration. This particular invariant, being constructed out of homology and intersection theory seems likely to have some geometric significance.

- Invariants based on the quandle (co)homology of Carter, Jelsovsky, Kamada, Langford and Saito [CJK+99, CJKS]: this structure may be considered in two ways–first as a variant of counting invariants in their guise as counting “colorings” and second homologically: the quandle monodromy giving rise to a (co)chain map between the quandle (co)chain complexes, the (co)homology of whose cone is then an invariant of the Lefschetz fibration.

Geometric interpretation of this latter invariant would then depend upon understanding the geometric significance of the quandle (co)homology of knot quandles and Dehn quandles.
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