UNIFORM RATES OF CONVERGENCE OF SOME REPRESENTATIONS OF EXTREMES: A FIRST APPROACH

Abstract. Uniform convergence rates are provided for asymptotic representations of sample extremes. These bounds which are universal in the sense that they do not depend on the extreme value index are meant to be extended to arbitrary sample extremes in coming papers.

Résumé. Des vitesses de convergence uniforme, universelles en ce sens qu’elles ne dépendent pas de l’index d’attraction, sont fournies pour des représentions asymptotiques des extrêmes d’Achantaillons. Ces bornes sont attendues dans le cas général des extrêmes dans des articles à venir.

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1. Introduction

This paper presents a view on the rate of convergence of the univariate extremes of samples in a simple form. Rather than trying to handle the general problem for a distribution function in the extreme domain of attraction, we focus here on the simplest representations for which we give the most precise rates. The results are expected to serve as tools in general
for the univariate case and later for the multivariate frame.

In a few words, the extreme value theory started with the univariate case, especially with independent data. Given a sequence of independent and identically distributed random variables \((X_n)_{n \geq 0}\) with common cumulative distribution function \((cdf)\) and defined on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\), the max-stability problem consists of finding possible limits in distribution of the sequences of partial maxima

\[ M_n = \max(X_1, ..., X_n), \ n \geq 1, \]

when appropriately centered and normalized. Precisely, we want to find non-random sequences \((a_n > 0)_{n \geq 1} \subset \mathbb{R}\) and \((b_n)_{n \geq 1} \subset \mathbb{R}\) such that \((M_n - b_n)/a_n\) converges in distribution to a random variable \(Z\) as \(n \to +\infty\), denoted as

\[ \frac{M_n - b_n}{a_n} \xrightarrow{D} Z. \quad (M) \]

This problem was solved around the middle of 20-th century with the contributions of many people, from whom we can cite Gnedenko (1943), Fisher and Tippet (1928), FrÃ¶chet (1927), etc. The following result is usually quoted as the Gnedenko (1943) result since he had the chance to close the characterization theorem. If \(Z\) is non-degenerate, that is: \(Z\) takes at least two different values, Formula (M) can hold only if \(Z\) is one of the three types in terms of its \(cdf\):

- the FrÃ¶chet type with parameter \(\gamma \in \Gamma_1 = \{x \in \mathbb{R}, \ x > 0\}\):
  \[ H_\gamma(x) \equiv \phi_\gamma(x) = \exp(-x^{-1/\gamma})1_{(x \geq 0)}, \]
- the Weibull type with parameter \(\gamma \in \Gamma_2 = \{x \in \mathbb{R}, \ x < 0\}\):
  \[ H_\gamma(x) \equiv \psi_\gamma(x) = \exp((-x)^{-1/\gamma})1_{(x < 0)} + 1_{(x \geq 0)} \]
- and the Gumbel type with parameter \(\gamma \in \Gamma_0 = \{0\}\):
  \[ H_\gamma(x) \equiv \Lambda(x) = \exp(-e^{-x}), \ x \in \mathbb{R}. \]

By type of distribution, we mean that any non-degenerate \(Z\) in Formula (M) have a \(cdf\) \(F_Z\) which is of the form \(F_Z(x) = H_\gamma(Ax + B)\), where \(0 < A \in \mathbb{R},\)
A modern account of the theory, including statistical estimation, can be found in: Galambos (1958), de Haan (1970), de Haan and Ferreira (2006), Embrechts et al. (1997), Resnick (1987), Beirlant et al. (2014), etc. This theory is part of the general weak convergence which is thoroughly treated in Billingsley (1968) and van der Vaart and Wellner (1996). But all our needs in extreme value theory and in weak convergence are already gathered in Lo et al. (2016) and Lo (2016), which are currently cited in the paper.

We denote by $Z_1 = Fr(\gamma)$, $Z_2 = Wei(\gamma)$ and $Z_0 = \Lambda$ random variables admitting the above cdf’s respectively. We remark that the cdf’s $H_\gamma$ are continuous, and the cdf of $(M_n - b_n)/a_n$ is $F((a_n x + b_n)n$, $x \in \mathbb{R}$. By Theorem 3 in Chapter 2 and Point (5) in Chapter 4, Section 1 in Lo et al. (2016), (M) is equivalent to

$$R_n = \sup_{x \in \mathbb{R}} |F(a_n x + b_n)n - H_\gamma(x)| \to 0 \text{ as } n \to +\infty.$$  

The general problem of the rate of convergence here is to find $R_n$ or to have universal bounds for it.

As we already stressed, here we propose results to be used in the general case. To introduce this, suppose that we have a sequence of independent and uniformly distributed random variables $(U_n)_{n \geq 0}$ on $(0,1)$ and let us consider for each the $U_{1,n} = \min(U_1,\ldots,U_n)$ for each $n \geq 1$ and define

$$Z_n(1, \gamma) = (n U_{1,n})^{-\gamma}, \gamma \in \Gamma_1; \quad Z_n(2, \gamma) = -(n U_{1,n})^{-\gamma}, \gamma \in \Gamma_2$$

and

$$Z_n(0, \gamma) = -\log(n U_{1,n}), \gamma \in \Gamma_0.$$  

It is straightforward to show that for $\gamma \in \Gamma_i$, $Z_n(i, \gamma)$ weakly converges to $Fr(\gamma)$, to $Wei(\gamma)$ and to $\Lambda$ accordingly to $i = 0$, $i = 1$ and $i = 2$. But the most important is that, based on Section 3.2 in Chapter 1 in Lo (2016) for example, if (M) holds with $Z$ non-degenerate, we may always get the following asymptotic representation in distribution
\[
\frac{M_n - b_n}{a_n} = Z_n(i, \gamma) + r_n, \quad \text{with } r_n \to 0 \text{ as } n \to +\infty, \quad (MG)
\]

for some \(i \in \{0, 1, 2\}\) with suitable sequences \((a_n > 0)_{n \geq 1} \subset \mathbb{R}\) and \((b_n)_{n \geq 1} \subset \mathbb{R}\).

Because of the general formula \((MG)\), we think it is important to have a special study of the rate of convergence of the \(Z_n(i, \gamma)\)'s, which paves the way, further, for the handling of the general rate \(r_n\).

This is what we are going to do exactly in Section 2.

## 2. Rate of convergence of \(Z_n(i, \gamma)\)

We are going to see that a universal bound will be guided by the following rate of convergence
\[
\left( e\left(1 - \frac{1}{n}\right)^{-1} - 1 \right)
\]
which is estimated as follows.

**Lemma 1.** For all \(n \geq 2\)
\[
0 \leq \left( e\left(1 - \frac{1}{n}\right)^{-1} - 1 \right) \leq C_0 \left( \frac{1}{2n} + \frac{1}{n^2 \log n} \right).
\]

where \(C_0 = \exp(0.6106739) = 1.841673\).

**Proof.** It is given in the Appendix.

The main result of the paper is the following.

**Theorem 1.** We have for all \(n \geq 2\), for \(C_1 = 2 + C_0\),
\[
\sup_{i \in \{0,1,2\}} \sup_{\gamma \in \Gamma} \sup_{x \in \mathbb{R}} |F_{Z_n(i, \gamma)}(x) - F_{Z_i}(x)| \leq \frac{C_1}{4n} + \frac{C_0}{2n^2 \log n}.
\]

**Proof.** We first deal with the \(Z_n(1, \gamma), \gamma > 0\). Set \(\alpha = 1/\gamma\). We have
\[
\forall (x \in \mathbb{R}), \quad F_n(x) = F_{Z_n(1, \gamma)}(x) = \begin{cases} (1 - \frac{1}{nx^\alpha})^n & \text{if } nx^\alpha > 1 \\ 0 & \text{otherwise} \end{cases}
\]
\( \forall (x \in \mathbb{R}), \ f_n(x) = F_{Z_n(1, \gamma)}(x) = \begin{cases} \alpha x^{-\alpha - 1} \left(1 - \frac{1}{nx^\alpha}\right)^{n-1} & \text{if } nx^\alpha > 1 \\ 0 & \text{otherwise.} \end{cases} \)

Since, as \( n \to +\infty \),
\[ Z_n(1, \gamma) \Rightarrow Fr(1/\gamma), \]
we have, by Scheffé’s Theorem (see for example Theorem 4, Chapter 3 in Lo et al. (2016)) :
\[
\sup_{x \in \mathbb{R}} |F_{Z_n(1, \gamma)}(x) - \phi_{\gamma}(x)| \leq \sup_{B \text{ measurable}} \int_B |f_n(x) - f(x)| \, dx
\]
\[
= \frac{1}{2} \int |f_n(x) - f(x)| \, dx,
\]
with
\[
\forall (x \in \mathbb{R}), \ f(x) = F_{Fr(1/\gamma)}(x) = \begin{cases} \alpha x^{-\alpha - 1} e^{-x^\alpha} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

For short, we also write \( F_{Fr(1/\gamma)} = F \). Now, we have to compute

\[
(2.1) \quad 2a_n(\alpha) = \int |f_n(x) - f(x)| \, dx.
\]

We have
\[
\int |f_n(x) - f(x)| \, dx = \int_{[0, n^{-1/\alpha}]} |f_n(x) - f(x)| \, dx + \int_{n^{-1/\alpha}, +\infty[} |f_n(x) - f(x)| \, dx
\]
\[
= F(n^{-1/\alpha}) + \int_{n^{-1/\alpha}, +\infty[} |f_n(x) - f(x)| \, dx.
\]

We also have
\[
\int_{n^{-1/\alpha}, +\infty[} |f_n(x) - f(x)| \, dx = \int_{n^{-1/\alpha}, +\infty[} \alpha x^{-\alpha - 1} e^{-x^\alpha} \left| 1 - e^{x^\alpha} \left(1 - \frac{1}{nx^\alpha}\right)^{n-1} \right| \, dx.
\]
Now, let us study
\[ g_{n,\alpha}(x) = e^{x^{-\alpha}} \left(1 - \frac{1}{nx^\alpha}\right)^{n-1} \]
on \left[n^{-1/\alpha}, +\infty\right]. Differentiating \( g_{n,\alpha}(x) \) gives
\[ g'_{n,\alpha}(x) = \frac{\alpha}{n} e^{x^{-\alpha}} x^{-\alpha-1} \left(1 - \frac{1}{nx^\alpha}\right)^{n-2} (x^{-\alpha} - 1) \text{ on } \left[n^{-1/\alpha}, +\infty\right]. \]

For \( n \geq 2 \), \( g'_{n,\alpha}(x) \) is positive on \( \left[n^{-1/\alpha}, 1\right] \) and negative on \( [1, +\infty[ \) so that \( g_{n,\alpha}(x) \) is increasing \( \left[n^{-1/\alpha}, 1\right] \) and decreasing on \( [1, +\infty[ \). Since for \( n \geq 2 \),
\[ g_{n,\alpha}(1) = e(1 - 1/n)^{n-1} > 1 \]
and, as \( n \to +\infty \),
\[ g_{n,\alpha}(1) \to 1^+ \text{ (i.e., by from above 1)}, \]
we have a unique number \( x_{n,\alpha} \in [n^{-1/\alpha}, 1] \) such that \( g_{n,\alpha}(x_{n,\alpha}) = 1 \) and
\[ 0 \leq g_{n,\alpha}(x) \leq 1 \text{ on } \left[n^{-1/\alpha}, x_{n,\alpha}\right] \]
and for all \( x \geq n^{-1/\alpha}, \)
\[ 1 \leq g_{n,\alpha}(x) \leq g_{n,\alpha}(1) = e(1 - 1/n)^{n-1}. \]

By exploiting the sign of \( 1 - g_{n,\alpha}(x) \) on \( \left[n^{-1/\alpha}, 1\right] \) and \( [1, +\infty[ \) respectively, we have
\[
\int_{\left[n^{-1/\alpha}, +\infty[}^{x_{n,\alpha}} \alpha x^{-\alpha-1} e^{-x^\alpha} \left|1 - e^{x^{-\alpha}} \left(1 - \frac{1}{nx^\alpha}\right)^{n-1}\right| \, dx \\
\leq \int_{n^{-1/\alpha}}^{x_{n,\alpha}} \alpha x^{-\alpha-1} e^{-x^\alpha} (1 - g_{n,\alpha}(x)) \, dx + \int_{x_{n,\alpha}}^{\infty} \alpha x^{-\alpha-1} e^{-x^\alpha} (g_{n,\alpha}(x) - 1) \, dx \\
[2.3] \quad a_n(\alpha, 1) + a_n(\alpha, 2). \]
Next, we have

$$a_n(\alpha, 1) = \int_{x_{n, \alpha}^{1-\alpha}}^{x_{n, \alpha}} f(x) \, dx - \int_{x_{n, \alpha}^{1-\alpha}}^{x_{n, \alpha}} \left( \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right)^n \right)' \, dx$$

$$= F(x_{n, \alpha}) - F(n^{-1/\alpha}) - \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right)^n. \quad (2.4)$$

Combining all, this leads to

$$a_n(\alpha, 2) \leq (g_{n, \alpha}(1) - 1)(1 - F(x_{n, \alpha}))$$

$$\leq \left((e(1 - 1/n)^{n-1} - 1) \equiv \alpha_n(3) \right), \quad (2.5)$$

In total, by putting Formulas 2.1-2.5, we get

$$2\alpha_n(\alpha) \leq \exp(-x_{n, \alpha}^{-\alpha}) - \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right)^n + \alpha_n(3)$$

$$\leq \exp(-x_{n, \alpha}^{-\alpha}) \left(1 - \exp(x_{n, \alpha}^{-\alpha}) \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right)^{n-1} \right) \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right) + \alpha_n(3)$$

$$\leq \exp(-x_{n, \alpha}^{-\alpha}) \left(1 - g_{n, \alpha}(x_{n, \alpha}) \left(1 - \frac{1}{n x_{n, \alpha}^{\alpha}} \right) \right) + \alpha_n(3) \quad (L2)$$

$$\leq \exp(-x_{n, \alpha}^{-\alpha}) \frac{1}{n x_{n, \alpha}^{\alpha}} + \alpha_n(3),$$

$$\leq \frac{1}{n} \left(x_{n, \alpha}^{-\alpha} \exp(-x_{n, \alpha}^{-\alpha}) \right) + \alpha_n(3),$$

where we identified $g_{n, \alpha}(x_{n, \alpha})$, which is equal to one, in Line 2 in the above group of formulas. Now, the term between the big parentheses is bounded by the supremum of the function $\ell(x) = x^{-\alpha} e^{-x^{-\alpha}}$ on $(0, 1)$ whose derivative, $\alpha x^{-\alpha-1} e^{-x^{-\alpha}} (x^{-\alpha} - 1)$, is non-negative on $(0, 1)$. Hence our bound is $\ell(1) = 1/e$. We conclude that for $n \geq 2$

$$2\alpha_n(\alpha) \leq \frac{1}{n} + \left(e(1 - 1/n)^{n-1} - 1 \right).$$

By using Lemma 1, we finally get
\[ \alpha_n(\alpha) \leq \frac{1}{2n} + C_0 \left( \frac{1}{4n} + \frac{2}{n^2 \log n} \right). \]

From there, we use Lemma 1 to conclude the proof for \( Z_n(1, \gamma) \). To finish for the other cases, it is enough to see that we have the following two formulas:

\[
\sup_{x \in \mathbb{R}} |F_{Z_n(2, \gamma)}(x) - F_{Z_2}(x)| = \sup_{x \in \mathbb{R}} |F_{Z_n(1, -1/\gamma)}(-x^{-1}) - F_{Z_1}(-x^{-1})|
\]

and

\[
\sup_{x \in \mathbb{R}} |F_{Z_n(0, \gamma)}(x) - F_{Z_0}(x)| = \sup_{x \in \mathbb{R}} |F_{Z_n(1, 1)}(e^x) - F_{Z_1}(e^x)|,
\]

which puts an end to the proof.

**Remark.** The bounds provided here are to be used in the general theory of a coming paper.

3. **Appendix**

**Proof.** Fix \( n \geq 2 \). We have

\[
(n - 1) \log(1 - 1/n) = -1 + \sum_{k \geq 1} \left( \frac{1}{k} - \frac{1}{k + 1} \right) \frac{1}{n^k},
\]

which in turn implies

\[
1 \leq g_{n, \alpha}(1) = e \left( 1 - \frac{1}{n} \right)^{n-1} = \exp \left( \sum_{k \geq 1} \frac{1}{k(k + 1)n^k} \right). \ (F1)
\]

By the comparison methods between series with monotonic terms, say \( a_n = a(n) \) with the appropriate improper Riemann integral of \( a(x) \), we have

\[
\sum_{k \geq 1} \frac{1}{k(k + 1)n^k} \leq \frac{1}{2n} + \int_2^{+\infty} \frac{dx}{n^x} \leq \frac{1}{n^2 \log n}. \ (F2)
\]
By combining (F1) and (F2), we have, for \(0 \leq \theta \leq 1\),

\[
0 \leq \left( e \left( 1 - \frac{1}{n} \right)^{n-1} - 1 \right) \leq \exp \left( \frac{1}{2n} + \frac{1}{n^2 \log n} \right) - 1
\]

\[
\leq \left( \frac{1}{2n} + \frac{1}{n^2 \log n} \right) \exp \left( \theta \left( \frac{1}{2n} + \frac{1}{n^2 \log n} \right) \right)
\]

\[
\leq \exp(0.6106739) \left( \frac{1}{2n} + \frac{1}{n^2 \log n} \right).
\]

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