A note on the Turán number for the traces of hypergraphs

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Abstract
Let $H$ be an $r$-uniform hypergraph and $F$ be a graph. We say $H$ contains $F$ as a trace if there exists some set $S \subseteq V(H)$ such that $H|_S := \{E \cap S : E \in E(H)\}$ contains a subgraph isomorphic to $F$. Let $ex_r(n, Tr(F))$ denote the maximum number of edges of an $n$-vertex $r$-uniform hypergraph $H$ which does not contain $F$ as a trace. In this paper, we improve the lower bounds of $ex_r(n, Tr(F))$ when $F$ is a star, and give some optimal cases. We also improve the upper bound for the case when $H$ is $3$-uniform and $F$ is $K_{2,t}$ when $t$ is small.

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1 Introduction

A hypergraph $H$ is $r$-uniform if it is a family of $r$-element subsets of a finite vertex set. We usually denote its vertex set and edge set by $V(H)$ and $E(H)$, respectively.

Definition 1.1. For a graph $F$ with vertex set $\{v_1, \ldots, v_p\}$ and edge set $\{e_1, \ldots, e_q\}$, a hypergraph $H$ contains $F$ as a trace if there exists a set of distinct vertices $W := \{w_1, \ldots, w_p\} \subseteq V(H)$ and distinct edges $\{f_1, \ldots, f_q\} \subseteq E(H)$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} = f_i \cap W$. The vertices $\{w_1, \ldots, w_p\}$ are called the base vertices of $F$.

For $r \geq 2$, let $ex_r(n, Tr(F))$ be the maximum number of edges of an $n$-vertex $r$-uniform hypergraph that does not contain $F$ as a trace.

In [5], Mubayi and Zhao determined the asymptotic value of $ex_r(n, Tr(K_s))$ for all $r$ when $s \in \{3, 4\}$ and conjectured that for $s \geq 5$, $ex_r(n, Tr(K_s)) \sim \left(\frac{n}{s}\right)^{s-1}$. Later, Sali and Spiro [6] determined the order of magnitude of $ex_r(n, Tr(K_{s,t}))$ when $t \geq (s-1)! + 1$, $s \geq 2r - 1$. Recently, Füredi and Luo [3] deduced the order of magnitude of $ex_r(n, Tr(F))$ for all graphs $F$ in terms of their generalized Turán numbers. In particular, they showed

$$ex_r(n, Tr(F)) = \Theta \left( \max_{2 \leq s \leq r} ex(n, K_s, F) \right),$$

where $ex(n, K_s, F)$ denotes the maximum number of copies of $K_s$ in an $F$-free graph on $n$ vertices. In the case where $F$ is outerplanar, they gave the following theorem.

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Theorem 1.2 (3). If $F$ is an $m$-vertex outerplanar graph, then

$$ex(n-r+2,F) \leq ex_r(n,Tr(F)) \leq \frac{1}{2}r^r(m-2)^{(r-2)}ex(n,F).$$

In [3], Füredi and Luo also investigated the case when $F$ is a star and gave both lower and upper bounds. Later, Luo and Spiro [4] gave an upper bound on the number of edges in a 3-uniform hypergraph which is $Tr(K_{2,t})$-free for $t \geq 14$.

In this note, we first improve the lower bounds for $ex_r(n,Tr(K_{1,t}))$ and show that our lower bounds are optimal for infinitely many cases. Secondly, by adapting the method used in [4], we improve the upper bound for $ex_3(n,Tr(K_{2,t}))$ when $t$ is small.

The paper is organised as follows. In Sections 2 and 3, we study the lower bounds for $ex_r(n,K_{1,t})$ and the upper bounds for $ex_3(n,Tr(K_{2,t}))$, respectively. We conclude in Section 4.

2 Improved lower bounds for $ex_r(n,Tr(K_{1,t}))$

We first give the definition of the well-known structure called disjointly representable family.

Definition 2.1. The sets $E_1, E_2, \ldots, E_k \subseteq X$ are said to be disjointly representable if there exist $x_1, x_2, \ldots, x_k \in X$ such that

$$x_i \in E_i \Leftrightarrow i = j \text{ (} 1 \leq i, j \leq k \text{)},$$

in other words, no $E_i$ is contained in the union of the others.

Let $f(r,k) = \max |\mathcal{H}|$, where the maximum is taken over all $r$-uniform set-systems $\mathcal{H}$ containing no $k$ disjointly representable members. In the literature, the best upper and lower bounds for $f(r,k)$ were given by Frankl and Pach [2].

Theorem 2.2 (2). We have

$$f(r,k) \leq \binom{r+k-1}{k-1}, \text{where } r \geq 1, k \geq 2,$$

$$f(r,3) = \left\lfloor \frac{r+2}{2} \right\rfloor \frac{r+2}{2}, \text{where } r \geq 1,$$

and (up to isomorphism) the only $r$-uniform set-system $\mathcal{H}_{r,3}$ without 3 disjointly representable members, satisfying $\mathcal{H}_{r,3} = f(r,3)$, can be constructed as follows. Let $A$ and $B$ be disjoint set with $|A| = \left\lfloor \frac{r+2}{2} \right\rfloor$ and $|B| = \left\lceil \frac{r+2}{2} \right\rceil$, and let $\mathcal{H}_{r,3} := \{ E \subseteq A \cup B \mid |E| = r \text{ and } |A \setminus E| = |B \setminus E| = 1 \}.$

$$f(2,k) = \left\lfloor \frac{k+1}{2} \right\rfloor \frac{k+1}{2}, \text{where } k \geq 2,$$

and the only graph $\mathcal{H}_{2,k}$ with $|\mathcal{H}_{2,k}| = f(2,k)$, which contains no $k$ disjointly representable edges, can be obtained from the complete graph $K_{k+1}$ by deleting $\lfloor \frac{k+1}{2} \rfloor$ edges as disjoint as possible.

In [3], Füredi and Luo gave a connection between $f(r,k)$ and the upper bound of $ex_r(n,Tr(K_{1,t}))$, and gave the following theorem.

Theorem 2.3 (3). For any $r \geq 2, t \geq 2$, if $n = a(r+t-2) + b$ with $b \leq r + t - 3$ then

$$a \binom{r+t-2}{r} + \binom{b}{r} \leq ex_r(n,Tr(K_{1,t})) \leq \frac{n}{r} f(r-1,t) \leq \frac{n}{r} \binom{r+t-2}{r-1}.$$

In particular, if $n$ is divisible by $r+t-2$, the lower bound is $\frac{n}{r} \binom{r+t-3}{r-1}$. 

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Before we improve the lower bound, we first give the following definition.

**Definition 2.4.** Let $v \geq k \geq t$ and let $X$ be a set of $v$ elements (points). A $t(v, k, \lambda)$ covering is a collection of $k$-subsets (blocks) of $X$, denoted by $C$, such that every $t$-subset of points occurs in at least $\lambda$ blocks in $C$. The size of a covering is just the size of $C$.

Now we improve the lower bound.

**Theorem 2.5.** For any $r \geq 2, t \geq 2$, if $n = a(r + t - 1) + b$ with $b \leq r + t - 2$, and there exists a minimal $(r-1)-(r+t-1, r, 1)$ covering with size $c$, then

\[
a \left( \frac{r + t - 1}{r} - c \right) + \binom{b}{r} \leq ex_r (n, Tr(K_{1, t})) .
\]

**Proof.** Let each component of $H$ be a clique of size $r + t - 1$ such that there are as many cliques as possible. For each clique, we delete a minimal $(r-1)-(r+t-1, r, 1)$ covering. And the left vertices just form a clique. If $n = a(r + t - 1) + b$ with $b \leq r + t - 2$, then $e(H) = a \left( \binom{r + t - 1}{r} - c \right) + \binom{b}{r}$.

It only needs to show that $H$ is $Tr(K_{1, t})$-free.

1. If the component with vertex size $b \leq r + t - 2$ contains a $Tr(K_{1, t})$ with base vertices \{$v_1, \ldots, v_{t+1}$\}, then each edge must contain at least $3$ base vertices, a contradiction.

2. If the component with vertex size $r + t - 1$ contains a $Tr(K_{1, t})$ with vertices \{$v_1, \ldots, v_{t+1}$\} and edges \{$v_1v_2, v_1v_3, \ldots, v_{t+1}$\}. By the definition of $Tr(K_{1, t})$, the corresponding hyperedges of $H$ must share the left common $r - 2$ vertices \{$v_{t+2}, \ldots, v_{r+t-1}$\}. And these $r$ hyperedges are all the edges containing \{$v_1, v_{t+2}, \ldots, v_{r+t-1}$\} in the clique, which is a contradiction since we have deleted an $(r-1)-(r+t-1, r, 1)$ covering.

Now, we give some exact results for the values of $ex_r (n, Tr(K_{1, t}))$.

**Theorem 2.6.** For the case $r = 3$, the following holds.

1. If $t + 2 \equiv 0 \mod 6$ and $t + 2 \mid n$, then $ex_3 (n, Tr(K_{1, t})) = \frac{n}{t}(t^2 - 2)$.

2. If $t + 2 \equiv 1$ or $3 \mod 6$ and $t + 2 \mid n$, then $ex_3 (n, Tr(K_{1, t})) = \frac{n}{t}(t^2 - 1)$.

**Proof.** We first prove the upper bounds. By Theorem 2.1 and Theorem 2.3, we know that $ex_3 (n, Tr(K_{1, t})) \leq \frac{n}{t} f(2, t) = \frac{n}{t} \left( \binom{t+1}{2} - \lceil \frac{t+1}{2} \rceil \right)$.

1. If $t + 2 \equiv 0 \mod 6$, then $t$ is even, and $ex_3 (n, Tr(K_{1, t})) \leq \frac{n}{t} \left( \binom{t+1}{2} - \lceil \frac{t+1}{2} \rceil \right) = \frac{n}{t} \left( \binom{t+1}{2} - \frac{t+1}{2} \right) = \frac{n}{t} \left( \frac{t^2 - 1}{2} \right)$.

2. If $t + 2 \equiv 1$ or $3 \mod 6$, then $t$ is odd, and $ex_3 (n, Tr(K_{1, t})) \leq \frac{n}{t} \left( \binom{t+1}{2} - \lceil \frac{t+1}{2} \rceil \right) = \frac{n}{t} \left( \binom{t+1}{2} - \frac{t}{2} \right) = \frac{n}{t} \left( t^2 - 1 \right)$.

Now we prove the lower bounds. If $t + 2 \equiv 0 \mod 6$, it is known (see [1]) that there exists a $2-(t + 2, 3, 1)$ covering of size $\frac{t+2}{3}$, so when $t + 2 \mid n$, the lower bound follows from Theorem 2.5. If $t + 2 \equiv 1$ or $3 \mod 6$, the well-known Steiner Triple System $(t + 2, 3, 1)$ exists, thus if $t + 2 \mid n$, then the lower bound also follows from Theorem 2.5.

**Theorem 2.7.** When $r = 2k$ and $2k(k + 1) \mid n$, we have that $ex_r (n, Tr(K_{1, 3})) = \frac{n(k+1)}{2}$.

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Lemma 3.3. For any analysis, and obtain an improved upper bound for $ex(n, Tr(K_{1,3})) \leq \frac{n(k+1)}{2}$ follows from Theorem 2.2 and Theorem 2.3.

Turán’s theorem (see [1]) says that suppose $q = \lfloor \frac{n'}{n} \rfloor$, then there exists an $h$-$(n', n'-2, 1)$ covering of size $q(n') - (9^{k+1})(n' - h - 1)$. In this case $n' = r + 2, h = r - 1, q = \lfloor \frac{r+2}{r} \rfloor = k + 1$, and there exists an $(r-1)-(r+2, r, 1)$ covering of size $(k+1)(2k+2) - 2(k^{k+2}) = k(k+1)$. Thus $ex(n, Tr(K_{1,3})) \geq \frac{n(k+1)}{2}$ follows from Theorem 2.3.

3 Improved upper bounds for $ex_3(n, K_{2,t})$ when $t$ is small

In this section, we focus on 3-uniform hypergraphs. In [4], Luo and Spiro gave the following theorem.

Theorem 3.1 ([4]). For $t \geq 14,

$$ex_3(n, Tr(K_{2,t})) \leq \frac{1}{6} \left( 2^{3/2} + 55t \sqrt{\log t} \right) n^{3/2} + o(n^{3/2}).$$

In this section, we generalise the result of Luo and Spiro for $ex_3(n, Tr(C_4))$ in [4] by a similar analysis, and obtain an improved upper bound for $ex_3(n, Tr(K_{2,t}))$ when $t$ is small.

Given a hypergraph $H$, we define $d_H(x, y)$ to be the number of edges of $H$ containing $\{x, y\}$, and we call this number the co-degree of $\{x, y\}$. We denote $A$ the set of edges containing at least one pair of vertices with co-degree 1. We will often identify hypergraphs by their set of edges and write e.g. $H \setminus A$ to denote the hypergraph $H$ after deleting some set of edges $A$ from $E(H)$.

Lemma 3.2 ([4]). Let $t \geq 2$ and let $H$ be a $Tr_3(K_{2,t})$-free 3-uniform hypergraph on $[n]$. For any pair $\{x, y\}$, we have $d_{H \setminus A}(x, y) \leq 3t - 3$.

Following the notations in [4], for $v \in V(H)$, we define the 1 and 2-neighborhood of $v$ as

$$N_1(v) = \{x \in V(H) : \exists e \in E(H), \{v, x\} \subseteq e\},$$

$$N_2(v) = \{x \in V(H) \setminus (N_1(v) \cup \{v\}) : \exists e \in E(H), x \in e, e \cap N_1(v) \neq \emptyset\}.$$

That is, $N_i(v)$ is the set of vertices that are at distance $i$ from $v$.

When the co-degrees of all pairs of vertices in $V(H)$ are at most $k$, it is observed in [4] that if $E$ is a set of edges containing some vertex $v$ and $V$ is the set of vertices $u \neq v$ with $u \in e$ for some $e \in E$, then

$$|V| \geq \frac{2}{k} |E|, \tag{1}$$

as each vertex in $V$ is contained in at most $k$ edges with $v$.

Let $H$ be an $n$-vertex 3-uniform hypergraph with no $K_{2,t}$ trace for $t \geq 3$. Let $A$ be the edges with at least one pair of co-degree 1, and $B = H \setminus A$. It was shown in [4] that

$$|A| \leq ex(n, K_{2,t}) \leq \frac{\sqrt{t-1}}{2} n^{3/2} + o(n^{3/2}).$$

Now we focus on the size of $B$. From now on, we write $d_B(v), d_B(u, v)$ as $d(v), d(u, v)$. By Lemma 3.2 $d(x, y) \leq 3t - 3$ for all $\{x, y\} \subseteq V(H)$. For any vertex $v$ we also let $N_1(v)$ and $V_2(v)$ denote the 1 and 2-neighborhoods of $v$ in $B$, respectively.

Lemma 3.3. For any $x, y \in V(H)$, $|N_1(x) \cap N_1(y)| \leq (t - 1)(6t - 2)$. 

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Proof. Suppose that there exist \(x, y \in V(H)\) and some set \(\{u_1, u_2, \ldots, u_{(t-1)(6t-2)+1}\} \subseteq N_1(x) \cap N_1(y)\). At most \(3t-3\) \(u_i\)'s, say \(\{u_{(t-1)(6t-5)+2}, u_{(t-1)(6t-5)+3}, \ldots, u_{(t-1)(6t-2)+1}\}\), are in edges of the form \(\{x, y, u_i\} \in B\). Let \(G\) be a graph on \([(t-1)(6t-5)+1]\) where \(ij \in E(G)\) if and only if either \(\{x, y, u_i\} \in B\) or \(\{y, u_i, u_j\} \in B\). Because pairs in \(B\) have co-degree at most \(3t-3\), we have that \(\Delta(G) \leq 6t-6\). By greedy algorithm, we can always find a \(K_t\), say \(\{u_1, u_2, \ldots, u_t\}\). Let \(e(x, i)\) and \(e(y, i)\) be the edges of \(B\) containing \(\{x, u_i\}\) and \(\{y, u_i\}\) for \(1 \leq i \leq t\), respectively. Note that both \(y, u_j \notin e(x, u_i)\) and \(x, u_j \notin e(y, u_i)\) hold for \(i \neq j\). Thus, these \(2t\) edges form a \(K_{2t}\) trace in \(B\), which is a contradiction. \(\Box\)

Fix any vertex \(v \in V(H)\). We define \(E_u = \{e \in B : e \cap N_1(v) = \{u\}\}\) and \(V_u = \{w \in N_2(v) : \exists e \in E_u, w \in e\}\). Since \(V_u \subseteq N_1(u)\) for all \(u\), we have the following corollaries.

**Corollary 3.4.** Let \(e = \{v, u, w\} \in B\) be any edge containing \(v\). Then \(|V_u \cap V_w| \leq (t-1)(6t-2)\).

**Proof.** It follows directly from Lemma 3.3. \(\Box\)

**Corollary 3.5.** For all \(u \in N_1(v)\),

\[
|V_u| \geq \frac{2}{3t-3} (d(u) - (3t - 3) - 3(t - 1)^2(6t - 2)).
\]

**Proof.** Note that \(E_u\) consists of every edge containing \(u\) except for at most \(3t - 3\) edges which also contain \(v\) and the edges \(\{e \in B : u \in e, |e \cap N_1(v)| \geq 2\}\). We claim that the latter set has cardinality at most \(3(t-1)^2(6t - 2)\). Indeed, any such edge would contribute a vertex to \(N_1(u) \cap N_1(v)\), of which there are at most \((t-1)(6t-2)\) vertices by the previous lemma. Each of such vertices can be contained in at most \(3t - 3\) edges together with \(u\) because \(B\) has maximum co-degree at most \(3t-3\). Therefore, we have that \(|E_u| \geq d(u) - (3t - 3) - 3(t - 1)^2(6t - 2)\).

Since \(B\) has maximum co-degree at most \(3t - 3\), and by \(\Box\), we have that \(|V_u| \geq \frac{2}{3t-3}|E_u|\). This completes the proof. \(\Box\)

**Lemma 3.6.** \(\sum_{u \in N_1(v)} |V_u| \leq (t-1)(6t-2)n\).

**Proof.** If \(|N_1(v)| \leq (t-1)(6t-2)\), then the conclusion follows directly since \(|V_u| \leq n\). Now we assume that \(|N_1(v)| > (t-1)(6t-2)\), if for any distinct \((t-1)(6t-2) + 1\) vertices \(u_1, \ldots, u_{(t-1)(6t-2)+1} \in N_1(v)\), there exists a vertex \(w \in V_u \cap \cdots \cap V_{u_{(t-1)(6t-2)+1}}\), then we have that \(|N_1(v) \cap N_1(w)| > (t-1)(6t-2) + 1\), which contradicts to Lemma 3.3. Thus, any vertex can be contained in at most \((t-1)(6t-2)\) different \(V_u\)s, the conclusion follows by double counting. \(\Box\)

By the discussions above, we have that

\[
\sum_{u \in N_1(v)} d(u) \leq \sum_{u \in N_1(v)} \left(\frac{3t-3}{2}|V_u| + f(t)\right) \leq 3(t-1)^2(3t-1)n + 2d(v)f(t),
\]

where \(f(t) = 3t - 3 + (3t - 1)^2(6t-2)\), and the last inequality comes from the fact that \(|N_1(v)| \leq 2d(v)\).

Let \(d = 3e(H)/n\) be the average degree of \(H\). We have

\[
d^2n \leq \sum_{u \in V(H)} d(u)^2 \leq \sum_{u \in V(H)} \sum_{v \in N_1(u)} d(u) = \sum_{v \in V(H)} \sum_{u \in N_1(v)} d(u) \leq 3(t-1)^2(3t-1)n^2 + 2f(t)d(n).
\]

Therefore, \(d \leq \sqrt{3(3t-1)(t-1)n} + O(t)\), where \(O(t)\) means that it is a constant depending only on \(t\). Thus, \(|B| = dn/3 \leq \sqrt{\frac{18(t-1)(t-1)n}{3}} + O(n)\).

Now the following theorem can be derived directly.
Theorem 3.7. Let \( t \geq 3 \) be an integer. We have that
\[
\text{ex}_3(n, \text{Tr}(K_{2,t})) \leq \left( \frac{\sqrt{3(3t-1)(t-1)}}{3} + \frac{\sqrt{t-1}}{2} \right) n^{3/2} + o(n^{3/2}).
\]

Remark 3.8. 1. By combining the conclusions in [4] directly, one can also derive upper bounds for \( \text{ex}_3(n, \text{Tr}(K_{2,t})) \) when \( t \) is small, but our result, i.e., Theorem 3.7 is better.

2. It was proved in [4] that
\[
\lim_{t \to \infty} \lim_{n \to \infty} \frac{\text{ex}_3(n, \text{Tr}(K_{2,t}))}{t^{3/2}n^{3/2}} = \frac{1}{6},
\]
so, Theorem 3.7 works well only when \( t \) is small.

4 Concluding remarks
In this note, we give a new lower bound for \( \text{ex}_r(n, \text{Tr}(K_{1,t})) \) by combinatorial coverings, and obtain optimal constructions for some cases. We also derive an improved upper bound for \( \text{ex}_3(n, \text{Tr}(K_{2,t})) \) when \( t \) is small. However, when \( t \) is fixed, we still don’t know the limit (if it exists) \( \lim_{n \to \infty} \frac{\text{ex}_3(n, \text{Tr}(K_{2,t}))}{n^{3/2}} \).

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