On the last zero process of a spectrally negative Lévy process

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Abstract
Let $X$ be a spectrally negative Lévy process and consider $g_t$ the last time $X$ is below the level zero before time $t \geq 0$. We derive an Itô formula for the three dimensional process \{(g_t, t, X_t), t \geq 0\} and its infinitesimal generator using a perturbation method for Lévy processes. We also find an explicit formula for calculating functionals that include the whole path of the length of current positive excursion at time $t \geq 0$, $q U_t := t - g_t$. These results are applied to optimal prediction problems for the last zero $g = \lim_{t \to \infty} g_t$, when $X$ drifts to infinity. Moreover, the joint Laplace transform of $(U_{e^q}, X_{e^q})$ where $e^q$ is an independent exponential time is found and a formula for a density of the $q$-potential measure of the process \{(U_t, X_t), t \geq 0\} is derived.

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1 Introduction

Last passage times have received considerable attention in the recent literature. For instance, in the classic ruin theory (which describes the capital of an insurance company), the moment of ruin is considered as the first time the process is below the level zero. However, in more recent literature the last passage time below zero is treated as the moment of ruin and the Cramér–Lundberg has been generalised to spectrally negative Lévy processes (see e.g. Nok Chiu and Yin (2005)). Moreover, in Paroissin and Rabehasaina (2015) spectrally positive Lévy processes are considered as degradation models and the last passage time above a certain fixed boundary is considered as the failure time.

Assume that $X$ is a spectrally negative Lévy process that drifts to infinity. Let $g$ the last zero of the process, i.e.

$$g = \sup\{t > 0 : X_t \leq 0\}.$$ 

Note that $g$ is a random variable that is not $\mathcal{F}_t$ measurable for all $t \geq 0$. That means, to know the value of $g$ it is necessary to have information of the whole trajectory of $X$. Suppose that at any time $t \geq 0$ we need to know the value of $g$ so then some actions can be taken. Then an alternative is to approximate $g$ with random variables that its realisation can be determined at any time $t \geq 0$. That is, we want to find the stopping time that minimises its “distance” from $g$.

We then introduce an optimal prediction problem for $g$. Consider a non-decreasing convex function $d : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and denote $d'_+$ as its right derivative. We have the optimal prediction problem

$$V_\tau = \inf_{\tau \in \mathcal{T}} \mathbb{E}(d(|\tau - g|)),$$

where $\mathcal{T}$ is the set of all stopping times of $X$. It can be shown that, under the assumption that $\mathbb{E}(d(g)) < \infty$, solving the optimal prediction problem is equivalent to solve an optimal stopping problem. Namely, for every

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\[ (u, x) \in E := \{(u, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : u > 0 \text{ and } x > 0\} \cup \{(0, x) \in \mathbb{R}^2 : x \leq 0\}, \]

consider the optimal stopping problem given by

\[
V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x} \left[ \int_0^\tau G(U_s, X_s)ds \right],
\]

(2)

where \( U_t = t - g_t \) and \( g_t = \sup\{0 \leq s \leq t : X_s \leq 0\} \) for all \( t \geq 0 \), the measure \( \mathbb{P}_{u,x} \) is given in equation (12) and the function \( G \) is given by

\[
G(u, x) = d'_+(u)\mathbb{P}_x(g = 0) - \mathbb{E}_x(d'_+(g)\mathbb{1}_{g>0}).
\]

We have that \( V_* = V(0, 0) + \mathbb{E}(d(g)) \).

This paper is organised as follows. In Section 2 we introduce some notation and some fluctuation identities of spectrally negative Lévy processes. Section 3 is dedicated to the definition of the last zero process in which basic properties of this process are shown. Moreover, a derivation of Itô formula, infinitesimal generator and formula for the expectation of a functional of \( U_t \) are the main results of this section (see Theorems 3.3 and 3.6 and Corollary 3.5). Then the aforementioned results are applied to sketch the solutions to optimal prediction problems and formulas for the joint Laplace transform of \((U, X)\) before an exponential time and the \( q \)-potential measures are found. Lastly, Section 4 is exclusively dedicated to introduce a perturbated Lévy process and Theorems 3.3 and 3.6 are proven.

2 Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a filtered probability space where \( \mathbb{F} = \{\mathcal{F}_t, t \geq 0\} \) is a filtration which is naturally enlarged (see Definition 1.3.38 of Bichteler (2002)). A Lévy process \( X = \{X_t, t \geq 0\} \) is an almost surely càdlàg process that has independent and stationary increments such that \( \mathbb{P}(X_0 = 0) = 1 \). From the stationary and independent increments property the law of \( X \) is characterised by the distribution of \( X_1 \). We hence define the characteristic exponent of \( X \), \( \Psi(\theta) := - \log(\mathbb{E}(e^{i\theta X_1})) \). The Lévy–Khintchine formula guarantees the existence of constants, \( \mu \in \mathbb{R}, \sigma \geq 0 \) and a measure \( \Pi \) concentrated in \( \mathbb{R} \setminus \{0\} \) with the property that \( \int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty \) (called the Lévy measure) such that

\[
\Psi(\theta) = i\mu \theta + \frac{1}{2} \sigma^2 \theta^2 - \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y\mathbb{1}_{|y|<1})\Pi(dy).
\]

Moreover, from the Lévy–Itô decomposition we can write

\[
X_t = \sigma B_t - \mu t + \int_{[0,t]} \int_{(-\infty,-1)} x N(ds \times dx) + \int_{[0,t]} \int_{(-1,0)} x(N(ds \times dx) - ds\Pi(dx)),
\]

where \( N \) is a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity \( dt \times \Pi(dx) \). We state now some properties and facts about Lévy processes. The reader can refer, for example, to Bertoin (1996), Sato (1999) and Kyprianou (2014) for more details. Every Lévy process \( X \) is also a strong Markov \( \mathbb{F} \)-adapted process. For all \( x \in \mathbb{R} \), denote \( \mathbb{P}_x \) as the law of \( X \) when started at the point \( x \in \mathbb{R} \), that is, \( \mathbb{E}_x(\cdot) = \mathbb{E}(\cdot | X_0 = x) \). Due to the spatial homogeneity of Lévy processes, the law of \( X \) under \( \mathbb{P}_x \) is the same as that of \( X + x \) under \( \mathbb{P} \).

The process \( X \) is a spectrally negative Lévy process if it has no negative jumps (\( \Pi(0, \infty) = 0 \)) with no monotone paths. We state now some important properties and fluctuation identities of spectrally negative Lévy processes which will be useful in latter sections, see Bertoin (1996), Chapter VII or Chapter 8 in Kyprianou (2014) for details.

Due to the absence of positive jumps, we can define the Laplace transform of \( X_1 \). We denote \( \psi(\beta) \) as the Laplace exponent of the process, that is, \( \psi(\beta) = \log(\mathbb{E}(e^{\beta X_1})) \). Then for all \( \beta \geq 0 \),

\[
\psi(\beta) = -\mu \beta + \frac{1}{2} \sigma^2 \beta^2 + \int_{(-\infty,0)} (e^{\beta y} - 1 - \beta y\mathbb{1}_{y<0})\Pi(dy).
\]
It can be shown that $\psi$ is infinitely differentiable and strictly convex function on $(0, \infty)$ such that tends to infinity at infinity. In particular, $\psi'(0+) = \mathbb{E}(X_1) \in [-\infty, \infty)$ and determines the value of $X$ at infinity. When $\psi'(0+) > 0$ the process $X$ drifts to infinity, i.e., $\lim_{t \to \infty} X_t = \infty$; when $\psi'(0+) < 0$, $X$ drifts to minus infinity and the condition $\psi'(0+) = 0$ implies that $X$ oscillates, that is, $\limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty$. We also define the right-inverse of $\psi$,

$$\Phi(q) = \sup\{\beta \geq 0 : \psi(\beta) = q\}, \quad q \geq 0.$$  

The process $X$ has paths of finite variation if and only if $\sigma = 0$ and $\int_{(-1,1)} |x| \Pi(dx) < \infty$, otherwise $X$ has paths of infinite variation. Define $\tau_a^+$ as the first passage time above the level $a > 0$,

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}.$$  

Then for any $a > 0$ the Laplace transform of $\tau_a^+$ is given by

$$\mathbb{E}(e^{-a\tau_a^+}) = e^{-\Phi(a)}.$$  

A very important family of functions for spectrally negative Lévy processes are the scale functions, $W^{(q)}$. For all $q \geq 0$, the scale function $W^{(q)} : \mathbb{R} \mapsto \mathbb{R}_+$ is such that $W^{(q)}(x) = 0$ for all $x < 0$ and it is characterised on the interval $[0, \infty)$ as a strictly and continuous function with Laplace transform given by

$$\int_0^\infty e^{-\beta x}W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}, \quad \text{for } \beta > \Phi(q).$$  

For the case $q = 0$ we simply denote $W = W^{(0)}$. When $X$ has paths of infinite variation, $W^{(q)}$ is continuous on $\mathbb{R}$ and $W^{(q)}(0) = 0$ for all $q \geq 0$, otherwise for all $q \geq 0$, we have $W^{(q)}(0) = 1/d$, where

$$d = -\mu - \int_{(-1,0)} x \Pi(dx) > 0.$$  

The behaviour of $W^{(q)}$ at infinity is the following. For $q \geq 0$ we have, $\lim_{x \to \infty} e^{-\Phi(q)x}W^{(q)}(x) = \Phi'(q)$.

There are some important fluctuation identities of Lévy processes in terms of the scale functions. In particular we list some that will be useful in later sections. Denote by $\tau_x^-$ as the first $X$ is strictly below the level $x \leq 0$, i.e.,

$$\tau_x^- = \inf\{t > 0 : X_t < x\}.$$  

The joint Laplace transform of $\tau_0^-$ and $X_0^-$ is

$$\mathbb{E}_x(e^{-q\tau_0^- + \beta X_0^- 1_{\tau_0^- < \infty}}) = e^{\beta x \mathcal{I}^{(q, \beta)}(x)}$$  

for all $x \in \mathbb{R}$ and $q > \psi(\beta) \vee 0$, where the function $\mathcal{I}^{(q, \beta)}$ is given by

$$\mathcal{I}^{(q, \beta)}(x) := 1 + (q - \psi(\beta)) \int_0^x e^{-\beta y}W^{(q)}(y)dy - \frac{q - \psi(\beta)}{\Phi(q) - \beta}e^{-\beta x}W^{(q)}(x) \quad x \in \mathbb{R}.$$  

When $\beta = \Phi(q)$ for $q \geq 0$ we understand the equation above in the limiting sense, i.e.

$$\mathcal{I}^{(q, \Phi(q))}(x) = 1 - \psi'(\Phi(q)) e^{-\Phi(q)x}W^{(q)}(x) \quad x \in \mathbb{R}.$$  

Denote by $\sigma_x^-$ the first time the process $X$ is below or equal to the level $x$, i.e.

$$\sigma_x^- = \inf\{t > 0 : X_t \leq x\}.$$  

It is easy to show that $\sigma_x^-$ and $\tau_x^-$ have the same distribution for all $x < 0$. When $X$ is of infinite variation, $X$ enters instantly to the set $(\infty, 0)$ whilst in the finite variation case there is a positive time before the process enters it. That implies that in the infinite variation case $\tau_0^- = \sigma_0^- = 0$ a.s. whereas in the finite
variation case, $\sigma_0^- = 0$ and $\tau_0^- > 0$.

Let $q > 0$ and $a \in \mathbb{R}$. The $q$-potential measure of $X$ killed on exiting $(-\infty, a]$ is absolutely continuous with respect to Lebesgue measure with a density given by
\[
e^{-\Phi(q)(a-x)}W(q)(a-y) - W(q)(x-y), \quad x, y \leq a,
\]
Similarly, the $q$-potential measure of $X$ killed on exiting $[0, \infty)$
\[
\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy, t < \tau_0^-)dt
\]
is absolutely continuous with respect to Lebesgue measure and it has a density given by
\[
e^{-\Phi(q)y}W(q)(x) - W(q)(x-y), \quad x, y \geq 0.
\]
Let $\beta \geq 0$, the process given by $\{e^{\beta X_t - \psi(\beta)t}, t \geq 0\}$ is a martingale. Then for each such $\beta$, we can define a change of measure given by
\[
\frac{d\mathbb{P}^\beta}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{\beta X_t - \psi(\beta)t}.
\]
Under the measure $\mathbb{P}^\beta$, $X$ is a Lévy process with Laplace exponent given by $\psi_\beta(\lambda) = \psi(\lambda + \beta) - \psi(\beta)$ for $\lambda \geq -\beta$ and hence $\Phi_\beta(q) := \sup\{\lambda \geq -\beta : \psi_\beta(\lambda) = q\} = \Phi(q + \psi(\beta)) - \beta$ for $q \geq -\psi(\beta)$. In the particular case when $\beta = \Phi(q)$ for $q \geq 0$ we have that $\psi_{\Phi(q)}(\lambda) = \psi(\lambda + \Phi(q)) - q$. That implies that for any $q > 0$, $\psi_{\Phi(q)}(0^+) = \psi'(\Phi(q)) \geq 0$ and then the process $X$ drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$. Moreover, denote $W^{\Phi(q)}_{\Phi(q)}$ the $0$-scale function under the measure $\mathbb{P}^{\Phi(q)}$, we have that $W^{\Phi(q)}_{\Phi(q)}(x) = e^{\Phi(q)x}W^{\Phi(q)}_{\Phi(q)}(x)$ for all $x \in \mathbb{R}$ and $q > 0$.

Another family of martingales is the following. Let $q \geq 0$, then the process
\[
\{e^{-q(t \wedge \tau_0^-)}W(q)(X_{t \wedge \tau_0^-}), t \geq 0\}
\]
is a $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}$.

### 3 The last zero process

Let $X$ be a spectrally negative Lévy process. For any $t \geq 0$ and $x \in \mathbb{R}$, we define as $g_t^{(x)}$ as the last time that the process is below $x$ before time $t$, i.e.,
\[
g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},
\]
with the convention $\sup\emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. A similar version of this random time is studied in Revuz and Yor (2004) (see Chapter XII.3), namely the last hitting time at zero, before any time $t \geq 0$, to describe excursions straddling at a given time. It is also shown that this random time at time $t = 1$ follows the arcsine distribution. The last hitting time to zero has some applications in the study of Azéma’s martingale (see Azéma and Yor (1989)). In Salminen (1988) the distribution of the the last hitting time of a moving boundary is found.

For any stopping time $\tau$, the random variable $g_{\tau}^{(x)}$ is $\mathcal{F}_{\tau}$ measurable. In particular we get that $\{g_{\tau}^{(x)}, t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$. Moreover, It is easy to show that for a fixed $x \in \mathbb{R}$, the stochastic process $\{g_t, t \geq 0\}$ is non-decreasing, right-continuous with left limits. Similarly, for a fixed $t \geq 0$ the mapping $x \mapsto g_t^{(x)}$ is non-decreasing and almost surely right-continuous with left limits.

It turns out that for all $x \in \mathbb{R}$ the process $\{g_t^{(x)}, t \geq 0\}$ is not a Markov process, in particular not Lévy process. However, the strong Markov property holds for the three dimensional process $\{(g_t, t, X_t), t \geq 0\}$.
Proposition 3.1. The process \(\{(g_t, t, X_t), t \geq 0\}\) is a strong Markov process with respect to the filtration \(\{\mathcal{F}_t, t \geq 0\}\) with state space given by \(E_g = \{(\gamma, t, x) : 0 \leq \gamma < t \text{ and } x > 0\} \cup \{(\gamma, t, x) : 0 \leq \gamma = t \text{ and } x \leq 0\}\).

Proof. From the definition of \(g_t\) it easy to note that for all \(t \geq 0\) we have that \(X_t \leq 0\) if and only if \(g_t = t\) from which we obtain that \((g_t, t, X_t)\) can take only values in \(E_g\). Now we proceed to show the strong Markov property. Consider a measurable positive function \(h: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}\). Then we have for any stopping time \(\tau\) and \(s \geq 0\),

\[
\mathbb{E}(h(g_{t+s}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) = \mathbb{E}(h(g_t \vee \sup\{r \in [\tau, s + \tau] : X_r \leq 0\}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) \leq \mathbb{E}(h(g_t \vee \sup\{r \in [\tau, s + \tau] : \tilde{X}_{r-\tau} + X_r \leq 0\}, \tau + s, X_{\tau+s} + \tilde{X}_{\tau} + X_\tau)|\mathcal{F}_\tau),
\]

where \(\tilde{X}_\tau = X_{\tau+\tau} - X_\tau\) and \(a \vee b := \max(a, b)\) for any \(a, b \in \mathbb{R}\). Using the strong Markov property of Lévy processes and the fact that \(g_t\) and \(X_t\) are \(\mathcal{F}_t\) measurable we obtain that

\[
\mathbb{E}(h(g_{t+s}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) = f_s(g_t, \tau, X_t),
\]

where for any \(x \in \mathbb{R}\) and \(0 \leq \gamma \leq t\), the function \(f_s\) is given by

\[
f_s(\gamma, t, x) = \mathbb{E}_x(h(\gamma, t + s, X_s)\mathbb{I}_{[\sigma_0^- > s]}) + \mathbb{E}_x(h(g_s + t + s, X_s)\mathbb{I}_{[\sigma_0^- \leq s]}). \tag{10}
\]

On the other hand, similar calculations lead us to

\[
\mathbb{E}(h(g_{t+s}, \tau + s, X_{\tau+s})|\sigma(g_t, \tau, X_t)) = f_s(g_t, \tau, X_t).
\]

Hence, for any measurable positive function \(h\) we obtain

\[
\mathbb{E}(h(g_{t+s}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) = \mathbb{E}(h(g_{t+s}, \tau + s, X_{\tau+s})|\sigma(g_t, \tau, X_t)).
\]

Therefore the process \(\{(g_t, t, X_t), t \geq 0\}\) is a strong Markov process. \(\square\)

In the spirit of the above Proposition we define for all \((\gamma, t, x) \in E_g\) the probability measure \(\mathbb{P}_{\gamma,t,x}\) in the following way: for every measurable and positive function \(h\) we define

\[
\mathbb{E}_{\gamma,t,x}(h(g_{t+s}, t + s, X_{t+s})) := \mathbb{E}(h(g_{t+s}, t + s, X_{t+s})| (g_t, t, X_t) = (\gamma, t, x)) = f_s(\gamma, t, x),
\]

where \(f_s\) is given in (10). Then we can write \(\mathbb{P}_{\gamma,t,x}\) in terms of \(\mathbb{P}_x\) by

\[
\mathbb{E}_{\gamma,t,x}(h(g_{t+s}, t + s, X_{t+s})) = \mathbb{E}_x(h(\gamma, t + s, X_s)\mathbb{I}_{[\sigma_0^- > s]}) + \mathbb{E}_x(h(g_s + t + s, X_s)\mathbb{I}_{[\sigma_0^- \leq s]}). \tag{11}
\]

Define \(U_t = t - g_t\) as the length of the current excursion above the level zero. As a direct consequence we have that the process \(\{(U_t, X_t), t \geq 0\}\) is also a strong Markov process with state space given by \(E = \{(u, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : u > 0 \text{ and } x > 0\} \cup \{(0, x) \in \mathbb{R}^2 : x \leq 0\}\). We hence can define a probability measure \(\mathbb{P}_{u,x}\), for all \((u, x) \in E\), by

\[
\mathbb{E}_{u,x}(f(U_t, X_t)) = \mathbb{E}_x(f(u + s, X_s)\mathbb{I}_{[\sigma_0^- > s]}) + \mathbb{E}_x(h(U_s, X_s)\mathbb{I}_{[\sigma_0^- \leq s]}). \tag{12}
\]

for any positive and measurable function \(f\).

Remark 3.2. We know that for any \(x \in \mathbb{R}\), \(g^{(x)}_t\) is a non-decreasing process. That directly implies that \(g^{(x)}_t\) is a process of finite variation and then it has a countable number of jumps. Moreover, with a close inspection to the definition of \(g^{(x)}_t\) we notice that \(g^{(x)}_t = t\) on the set \(\{t \geq 0 : X_t \leq x\}\), it is flat when \(X\) is in the set \((x, \infty)\) and it has a jump when \(X\) enters the set \((\infty, x]\). Moreover, if \(X\) is a process of infinite variation we know that the set of times in which \(X\) visits the level \(x\) from above is infinite. That implies that when \(X\) is of infinite variation, \(t \mapsto g^{(x)}_t\) has an infinite number of arbitrary small jumps.

We develop a version of Itô formula and derive the infinitesimal generator of the process \(\{(g_t, t, X_t), t \geq 0\}\). The proof can be found in Section 4.2.
**Theorem 3.3** (Itô formula). Let \( X \) be any spectrally negative and \( F \) a \( C^{1,1,2}(E_g) \) real-valued function. In addition, in the case that \( \sigma > 0 \) assume that \( \lim_{h\to 0} F(\gamma,t,h) = F(t,t,0) \) for all \( \gamma \leq t \). Then we have the following version of Itô formula for the three dimensional process \( \{g(t,t,X_t), t \geq 0\} \).

\[
F(g_t,t,X_t) = F(g_0,0,X_0) + \int_0^t \frac{\partial}{\partial \gamma} F(g_s,s,X_s)I_{\{X_s \leq 0\}} ds + \int_0^t \frac{\partial}{\partial t} F(g_s,s,X_s) ds \\
+ \int_0^t \frac{\partial}{\partial x} F(g_{s-},s,X_{s-}) dX_s + \frac{1}{2} \sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(g_s,s,X_s) ds \\
+ \int_{[0,t]} \int_{(-\infty,0)} \left( F(g_s,s,X_{s-} + y) - F(g_{s-},s,X_{s-}) - y \frac{\partial}{\partial x} F(g_{s-},s,X_{s-}) \right) N(ds \times dy)
\]

**Remark 3.4.** When \( \sigma > 0 \), the Brownian motion part of \( X \) implies that \( X \) can visit the interval \( (-\infty,0) \) by creeping. That implies that \( t \mapsto g_t \) has two types of jumps: those as a consequence of \( X \) jumping from the positive half line to \( (-\infty,0) \) which is finite (since \( \Pi(-\infty,-\varepsilon) < \infty \) for all \( \varepsilon > 0 \)) and those as a consequence of creeping. The limit condition imposed for \( F \) (when \( \sigma > 0 \)) ensures that the jumps due to the Brownian component vanish, otherwise a more careful analysis involving the local needs to be done.

Now that we have an Itô’s formula for the three dimensional process \( (g_t,t,X_t) \) we are ready to state its infinitesimal generator. Denote by \( C^{1,1,2}_b(E_g) \) the set of bounded \( C^{1,1,2}(E_g) \) functions with bounded derivatives. We have the following Corollary which proof follows directly from equation (20) and using standard arguments so it is omitted.

**Corollary 3.5.** Let \( X \) be any spectrally negative Lévy process and \( F \) a \( C^{1,1,2}_b(E_g) \) function such that when \( \sigma > 0 \), \( \lim_{h\to 0} F(\gamma,t,h) = F(t,t,0) \) for all \( \gamma \leq t \). Then the infinitesimal generator \( A_Z \) of the process \( Z_t = (g_t,t,X_t) \) satisfies:

\[
A_Z F(\gamma,t,x) = \frac{\partial}{\partial \gamma} F(\gamma,t,x)I_{\{x \leq 0\}} + \frac{\partial}{\partial t} F(\gamma,t,x) - \mu \frac{\partial}{\partial x} F(\gamma,t,x) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} F(\gamma,t,x) \\
+ \int_{(-\infty,0)} \left( F(\gamma,t,x+y) - F(\gamma,t,x) - y\Pi_{\{y>0\}} \frac{\partial}{\partial x} F(\gamma,t,x) \right) I_{\{x+y>0\}} \Pi(dy) \\
+ \int_{(-\infty,0)} \left( F(t,t,x+y) - F(t,t,x) - y\Pi_{\{y>0\}} \frac{\partial}{\partial x} F(t,t,x) \right) I_{\{x \leq 0\}} \Pi(dy) \\
+ \int_{(-\infty,0)} \left( F(t,t,x+y) - F(\gamma,t,x) - y\Pi_{\{y>0\}} \frac{\partial}{\partial x} F(\gamma,t,x) \right) I_{\{x > 0\}} I_{\{x+y \leq 0\}} \Pi(dy) 
\]

(13)

Recall from Remark 3.2 that the behaviour of \( g_t \) and then \( U_t \) can be determined from the excursions of \( X \) away from zero. Then, using that fact, we are able to derive a formula for a functional that involves the whole trajectory of the process \( U_t \). The next theorem provides a formula to calculate an integral involving the process \( \{(U_t,X_t), t \geq 0\} \) with respect of time in terms of the excursions of \( X \) above and below zero.

**Theorem 3.6.** Let \( q \geq 0 \) and \( X \) be a spectrally negative Lévy process and \( K : E \to \mathbb{R} \) a left-continuous function in each argument. Assume that there exists a non-negative function \( C : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) such that \( u \mapsto C(u,x) \) is a non-decreasing function for all \( x \in \mathbb{R} \), \( |K(u,s)| \leq C(u,x) \) and \( \mathbb{E}_x \left( \int_0^\infty e^{-\psi u} C(r,x_r) dr \right) < \infty \) for all \( x \in \mathbb{R} \). Then we have that for any \( (u,x) \in E \) that

\[
\mathbb{E}_{u,x} \left( \int_0^\infty e^{-\psi u} K(U_r,X_r) dr \right) = K^+(u,x) + \int_0^\infty K(0,y) \left[ e^{\psi(q)(x-y)} \Phi'(q) - W(q)(x-y) \right] dy \\
+ e^{\Phi(q)x} \left[ 1 - \psi' (\Phi(q) + e^{-\Phi(q)x} W(q)(x)) \right] \lim_{\varepsilon \downarrow 0} \frac{K^+(0,\varepsilon)}{\psi' (\Phi(q) + e^{-\Phi(q)x} W(q)(\varepsilon))},
\]

(14)

where \( K^+ \) is given by

\[
K^+(u,x) = \mathbb{E}_x \left( \int_0^x e^{-\psi u} K(u+r,X_r) dr \right), \quad (u,x) \in E.
\]
Remark 3.7. Note that from the proof of Theorem 3.6 we can find an alternative representation for formula (14) as a limit in terms of excursions of X above and below zero divided by a normalisation term,

\[
\mathbb{E}_{u,x} \left( \int_0^\infty e^{-qr} K(U_r, X_r)dr \right) = K^+(u, x) + \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-qr_0^-} K^-(X_{r_0^-} - \varepsilon) \right)
+ e^{\Phi(q)x} \left[ 1 - \psi'(\Phi(q) + ) e^{-\Phi(q)x} W(q)(x) \right] \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-qr_0^-} K^-(X_{r_0^-} - \varepsilon) \right) + K^+(0, \varepsilon),
\]
where \( K^- \) is given by

\[
K^-(x) = \mathbb{E}_x \left( \int_0^{\tau_0^+} e^{-qr} K(0, X_r)dr \right),
\]
for all \( x \in \mathbb{R} \).

3.1 Applications

In this section we consider an application of Theorems (3.3) and (3.6) and Corollary (3.5). Consider the optimal stopping problem given in (2). Under some integrability conditions imposed to \( X \) it can be shown that a stopping time that minimises (2) exists and can be taken of the form

\[
\tau_D = \inf \{ t \geq 0 : (U_t, X_t) \in D \},
\]
where \( D = \{ (u, x) \in E : V(u, x) = 0 \} \subset \mathbb{R}_+ \times \mathbb{R}_+ \). Moreover, it is well known that if \( V \) is a smooth enough function it solves the Dirichlet/Poisson problem (see Peskir and Shiryaev (2006) section III.7.2). That is, \( V \) solves

\[
A_Z V = -G \quad \text{in } C,
V|_{\partial C} = 0,
\]
where \( C = E \setminus D \) and \( A_Z \) is the infinitesimal generator of \( Z = \{ (U_t, X_t), t \geq 0 \} \) (see Corollary 3.5). Therefore, from Itô formula (if \( V \) satisfies the conditions of Theorem 3.3) we have that for any \((u, x) \in E\) that

\[
V(u, x) = \mathbb{E}_{u,x} \left( \int_0^\infty G(U_r, X_r) \mathbb{I}_{\{(U_r, X_r) \in C\}} dr \right)
- \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty, 0)} V(U_r, X_r + y) \mathbb{I}_{\{X_r + y > 0\}} \mathbb{I}_{\{(U_r, X_r) \in D\}} dy dr \right)
- \mathbb{E}_{u,x} \left( \int_0^\infty \int_{(-\infty, 0)} V(0, X_r + y) \mathbb{I}_{\{X_r + y \leq 0\}} \mathbb{I}_{\{(U_r, X_r) \in D\}} dy dr \right).
\]

Note that, in the representation above of \( V \), the stopping time on the upper integration limit disappeared and it became an indicator function depending on the stopping set \( D \). Then with the help of Theorem 3.6 further calculations can be done.

We also apply the result derived in Theorem (3.6) to calculate the joint Laplace transform of \((U_{e_q}, X_{e_q})\) where \( e_q \) is an exponential time with parameter \( q > 0 \) independent of \( X \).

Corollary 3.8. Let \( X \) be a spectrally negative Lévy process. Let \( q > 0 \) and \( \alpha \in \mathbb{R}, \beta \geq 0 \) such that \( q > \psi(\beta) \lor (\psi(\beta) - \alpha) \). We have that for all \((u, x) \in E\),

\[
\mathbb{E}_{u,x} \left( e^{-\alpha U_{e_q} + \beta X_{e_q}} \right)
= \frac{qe^{\beta x}}{q - \psi(\beta)} + e^{\Phi(q)x} \Phi'(q) \left[ \frac{q}{\Phi(q + \alpha) - \beta} - \frac{q}{\Phi(q) - \beta} \right] + e^{\beta x} q \int_0^x e^{-\beta y} [W(q)(y) - e^{-\alpha u} W(q + \alpha)(y)] dy
+ \frac{q}{\Phi(q + \alpha) - \beta} \left[ e^{-\alpha u} W(q + \alpha)(x) - W(q)(x) \right].
\]
Proof. Consider the function $K(u, x) = e^{-\alpha u + \beta x}$ for all $(u, x) \in E$. We have that $K$ is a continuous function and $K(u, x) \leq e^{-(\alpha \wedge 0) u + \beta x}$ for all $(u, x) \in E$. Moreover we have for all $q > 0$ such that $q > \psi(\beta) \lor (\psi(\beta) - \alpha) = \psi(\beta) - (\alpha \wedge 0)$ that

$$
\mathbb{E}_x \left( \int_0^\infty e^{-q r} e^{-\alpha(0 r) + \beta X_r} \, dr \right) = e^{\beta x} \int_0^\infty e^{-(\alpha + \beta) r} - \psi(\beta) < \infty
$$

for all $x \in \mathbb{R}$. Then for all $u > 0$ and $x > 0$ we have, by Fubini’s theorem and from equation (8), that

$$
K^+(u, x) = \mathbb{E}_x \left( \int_0^{\tau^-} e^{-q r} e^{-\alpha(u + r) + \beta X_r} \, dr \right)
= e^{-\alpha u} \int_{(0, \infty)} e^{\beta y} \int_0^\infty e^{-(\alpha + \beta) r} \mathbb{P}_x \left( X_r, r < \tau^-_0 \right) \, dr
= e^{-\alpha u} \int_{(0, \infty)} e^{\beta y} \left[ e^{-\Phi(q \alpha)} W(q \alpha)(x) - W(q \alpha)(x - y) \right] \, dy
= e^{-\alpha u} W(q \alpha)(x) / \Phi(q + \alpha) - \beta - e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy,
$$

Similarly, we calculate for any $x \in \mathbb{R}$

$$
\int_{-\infty}^0 e^{\beta y} \left[ e^{\Phi(q \alpha)} - W(q \alpha)(x - y) \right] \, dy = \Phi(q) e^{\Phi(q \alpha)} \int_0^\infty e^{-(\beta - \Phi(q)) y} \, dy - e^{\beta x} \int_0^\infty e^{-\beta y} W(q \alpha)(y) \, dy
= \frac{\Phi(q) e^{\Phi(q \alpha)}}{\beta - \Phi(q) - q} + e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy,
$$

where the last equality follows from equation (8) and the last integral is understood like 0 when $x < 0$. Then from (14) we get that for all $(u, x) \in E$,

$$
\mathbb{E}_{u, x} \left( \int_0^\infty e^{-q r} e^{-\alpha U_r + \beta X_r} \, dr \right)
= e^{-\alpha u} W(q \alpha)(x) / \Phi(q + \alpha) - \beta - e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy + \frac{\Phi(q) \mathbb{I}(q \alpha)}{\beta - \Phi(q) - q}
+ e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy

= e^{-\alpha u} W(q \alpha)(x) / \Phi(q + \alpha) - \beta - e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy + \frac{\Phi(q) \mathbb{I}(q \alpha)}{\beta - \Phi(q) - q}
+ e^{\beta x} \int_0^x e^{-\beta y} W(q \alpha)(y) \, dy

+ e^{\beta x} \left[ 1 - \psi(\Phi(q) \alpha) e^{-\Phi(q \alpha) W(q \alpha)(x)} \right] \frac{\Phi(q)}{\Phi(q + \alpha) - \beta},
$$

where in the last equality we used the fact that $\Phi(q) = 1 / \psi(\Phi(q) \alpha)$, $W(q \alpha)(x)$ is non-negative and strictly increasing on $[0, \infty)$ for all $q \geq 0$ and that

$$
\lim_{\varepsilon \downarrow 0} W^* \left( \frac{q \alpha}{2} \varepsilon \right) = 1
$$

for all $\alpha, q \geq 0$. The latter fact follows from the representation $W(q \alpha)(x) = \sum_{k=0}^\infty q^k W^* (k+1)(x)$ and the estimate $W^* (k+1)(x) \leq x^k / k! W(x)^{k+1}$ (see equations (8.28) and (8.29) in Kyprianou (2014), pp 241-242).

Rearranging the terms and using the fact that

$$
\mathbb{E}_{u, x} \left( e^{-\alpha U_r + \beta X_r} \right) = q \mathbb{E}_{u, x} \left( \int_0^\infty e^{-q r} e^{-\alpha U_r + \beta X_r} \, dr \right)
$$

for all $(u, x) \in E$, we obtain the desired result.
Remark 3.9. Note that from formula (15) we can recover some known expressions for Lévy processes. If we take \( \alpha = 0 \) we obtain for all \( \beta \geq 0 \) and \( q > \psi(\beta) \) \( \forall 0 \) and \( x \in \mathbb{R} \),

\[
\mathbb{E}_x(e^{\beta X_t}) = \frac{q e^{\beta x}}{q - \psi(\beta)}.
\]

On the other for any \( \theta \geq 0, q \geq 0 \) and \( x \in \mathbb{R} \) we have that

\[
\mathbb{E}_x(e^{-\theta q x}) = \int_0^\infty q e^{-q t} \mathbb{E}_x(e^{-\theta q t}) dt = \int_0^\infty q e^{-(q + \theta)t} \mathbb{E}_x(e^{\theta U_t}) dt = \frac{q}{q + \theta} \mathbb{E}_x(e^{\theta U_t})
\]

where \( e_q \) is an exponential random variable with parameter \( q + \theta \). The result coincides with the one found in Baurdoux (2009) (see Theorem 2).

Let \( q > 0 \) we consider the \( q \)-potential measure of \((U, X)\) given by

\[
\int_0^\infty e^{-q t} \mathbb{P}_{u,v}(U_t \in dv, X_t \in dy) dt
\]

for \((u, x), (v, y) \in E\). From the fact that for all \( t > 0, U_t = 0 \) if and only if \( X_t \leq 0 \) we have that for any \((u, x) \in E\) and \( y \leq 0 \)

\[
\int_0^\infty e^{-q t} \mathbb{P}_{u,v}(U_t = 0, X_t \in dy) dt = \int_0^\infty e^{-q t} \mathbb{P}_x(X_t \in dy) dt
\]

In the next corollary we find the an expression for a density when \( v, x > 0 \).

Corollary 3.10. Let \( q > 0 \). The \( q \)-potential measure of \((U, X)\) has a density given by

\[
\int_0^\infty e^{-q t} \mathbb{P}_{u,v}(U_t \in dv, X_t \in dy) dt = e^{-q(v-u)} \mathbb{P}_x(X_{v-u} \in dy, v-u < \tau_0^{-}) \mathbb{I}_{\{v-u\}}(y) dv
\]

\[
+ \left[ e^{\Phi(q) x} \Phi'(q) - W'(q)(x) \right] \frac{y}{v} e^{-q v} \mathbb{P}(X_v \in dy) dv
\]

for all \((u, x) \in E\) and \( v, y > 0 \).

Proof. Let \( 0 < u_1 < u_2 \) and \( 0 < x_1 < x_2 \) and define the sets \( A = (u_1, u_2] \) and \( Y = (x_1, x_2] \). Then the function \( K(u, x) = \mathbb{I}_{\{u \in A, x \in Y\}} \) is left-continuous and bounded by above by \( C(x) = \mathbb{I}_{\{x \in Y\}} \). Moreover, we have that for all \( q > 0 \) and \( x \in \mathbb{R} \),

\[
\mathbb{E}_x \left( \int_0^\infty e^{-q t} \mathbb{I}_{\{X_t \in Y\}} dt \right) < \infty.
\]

First we calculate for all \( u, x > 0 \) such that \( u < u_1 \),

\[
K^+(u, x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_0^-} e^{-q t} \mathbb{I}_{\{U_t \in A, X_t \in Y\}} dt \right) = \int_A \int_Y e^{-q(r-u)} \mathbb{P}_x(X_{r-u} \in dy, r-u < \tau_0^-) dr
\]

and for every \( x \leq 0 \) we have that

\[
K^-(x) = \mathbb{E}_{u,x} \left( \int_0^{\tau_0^+} e^{-q t} \mathbb{I}_{\{U_t \in A, X_t \in Y\}} dt \right) = 0
\]

Hence, for all \((u, x) \in E\) we obtain that

\[
\mathbb{E}_{u,x} \left( \int_0^\infty e^{-q t} \mathbb{I}_{\{U_t \in A, X_t \in Y\}} dt \right)
\]

\[
= \int_A \int_Y e^{-q(r-u)} \mathbb{P}_x(X_{r-u} \in dy, r-u < \tau_0^-) dr
\]

\[
+ e^{\Phi(q) x} \left[ 1 - \psi'(\Phi(q) +) e^{-\Phi(q)x} W'(q)(x) \right] \int_A \int_Y \lim_{\varepsilon \to 0} \frac{e^{-q r} \mathbb{P}_x(X_r \in dy, r < \tau_0^-)}{\psi'(\Phi(q) +) W'(q)(\varepsilon)} dr.
\]
We calculate the limit on the right-hand side of the equation above. Denote \( P^\epsilon_x \) as the law of \( X \) starting from \( \epsilon \) conditioned to stay positive. We have that for all \( x \in \mathbb{R} \) and \( y > 0 \) that
\[
\lim_{\epsilon \downarrow 0} \frac{P_x(X_{r} \in dy, r < \tau_0^-)}{W^{(q)}(\epsilon)} = \lim_{\epsilon \downarrow 0} \frac{P_x(X_{r} \in dy, r < \tau_0^-)}{W(\epsilon)} = \frac{1}{W(y)} \mathbb{P}(X_{r} \in dy) \]
where the second equality follows from the definition of \( \mathbb{P}^\epsilon \) (see e.g. Bertoin (1996) section VII.3 equation (6)) and the last follows since \( \lim_{\epsilon \downarrow 0} W(\epsilon)/W^{(q)}(\epsilon) = 1 \) and \( \mathbb{P}^\epsilon \) converges to \( \mathbb{P}^\epsilon \) in the sense of finite-dimensional distributions (see Proposition VII.3.14 in Bertoin (1996)). Moreover we have that for all \( y, r > 0 \), \( \mathbb{P}^\epsilon(X_{t} \in dy) = yW(y)\mathbb{P}(X_{r} \in dy)/r \) (see Corollary VII.3.16 in Bertoin (1996)) so that
\[
\lim_{\epsilon \downarrow 0} \frac{P_x(X_{r} \in dy, r < \tau_0^-)}{W^{(q)}(\epsilon)} = \frac{y}{r} \mathbb{P}(X_{r} \in dy).
\]
Therefore we have that for all \((u, x) \in E\),
\[
E_{u,x} \left( \int_0^\infty e^{-qr} \mathbb{1}_{(U_r,\epsilon A, X_r \in Y)} dr \right) = \int_A \int_Y e^{-q(r-u)} P_x(X_{r-u} \in dy, r-u < \tau_0^-) dr + \Phi'(q)e^{\Phi(q)x} - W^{(q)}(x) \int_A \int_Y \frac{y}{r} e^{-qr} \mathbb{P}(X_r \in dy) dr.
\]

**Remark 3.11.** Bingham (1975) showed that the q-potential measure of \( X \) has a density that is absolutely continuous with respect to the Lebesgue measure. This can be shown moving the killing barrier on the q-potential measure killed on entering the set \((-\infty, 0] \) (see (8)) and taking limits. Alternatively, it can deduced taking limits on (14). Moreover, Corollary 3.10 provides an alternative method for finding the aforementioned density as a solution of an ordinary differential equation. Let \( u, y > 0 \) and \( x \in \mathbb{R} \), integrating (16) with respect to \( v \) we obtain that
\[
\int_0^\infty e^{-qv} P_x(X_r \in dy) dr = \int_{(0, \infty)} \int_0^\infty e^{-qv} P_{u,x}(U_r \in dv, X_r \in dy) dr
\]
where the last equality follows from (8) . On the other hand, from the strong Markov property applied applied to time \( \tau_0^+ \) and equation (3) we obtain that for any \( x < 0 \),
\[
\int_0^\infty e^{-qv} P_x(X_r \in dy) dr = e^{\Phi(q)x} f_y(q) dy,
\]
where for all \( q > 0 \), \( f_y(q) := \int_0^\infty e^{-qv} \mathbb{P}(X_r \in dy) \). Then from the two equations above we obtain the integral equation
\[
f_y(q) = \Phi'(q)y \int_q^\infty f_y(p) dp.
\]
Solving the system we obtain that \( f_y(q) = \Phi'(q)e^{-\Phi(q)y}C \) for some constant \( C > 0 \). Therefore for all \( x \in \mathbb{R} \),
\[
\int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in dy)dr = \left( e^{\Phi(q)(x-y)} \Phi'(q)C - W(q)(x-y) + e^{-\Phi(q)y}W'(q)(x)[C-1] \right) dy.
\]
Using the space homogeneity of Lévy processes we have that for all \( y > 0 \),
\[
\int_0^\infty e^{-qr} \mathbb{P}_y(X_r \in dy)dr = \lim_{y \downarrow 0} \int_0^\infty e^{-qr} \mathbb{P}_y(X_r \in dy)dr = \lim_{y \downarrow 0} f_y(q)dy. \]
Comparing the latter with the equation above we conclude that \( C = 1 \). Therefore for all \( x \in \mathbb{R} \) and \( y > 0 \),
\[
\int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in dy)dr = \left( e^{\Phi(q)(x-y)} \Phi'(q) - W'(q)(x-y) \right) dy.
\]
Moreover, from the space homogeneity of Lévy processes we conclude that the equation above holds for all \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \).

4 Main proofs

Suppose that \( X \) is a spectrally negative Lévy process of finite variation. Then with positive probability it takes a positive amount of time to cross below 0, i.e. \( \tau_0^- > 0 \) a.s. Hence, stopping at the consecutive times in which \( X \) is below zero and the ideas mentioned in Remark 3.2 we can fully describe the behaviour of \( g_t \) and then derive the results mentioned in Theorems 3.3 and 3.6. However, in the case \( X \) is of infinite variation it is well known that the closed zero set of \( X \) is perfect and nowhere dense and the mentioned approach proves to be no longer useful (since we have that \( \tau_0^- = 0 \) a.s). Therefore, in order to exploit the idea applicable for finite variation processes we make use of a perturbation method. This method is mainly based on the work of Dassios and Wu (2011) and Revuz and Yor (2004) (see Theorem VI.1.10) which consists in construct a new “perturbed” process \( X^{(\varepsilon)} \) (for \( \varepsilon \) sufficiently small) that approximates \( X \) with the property that \( X^{(\varepsilon)} \) visits the level zero a finite number of times before any time \( t \geq 0 \). Then we approximate \( g_t \) by the corresponding last zero process of \( X^{(\varepsilon)} \).

4.1 Perturbed Lévy process

We describe formally the construction of the “perturbed” process \( X^{(\varepsilon)} \). Let \( \varepsilon > 0 \), define the stopping times \( \sigma_{1,\varepsilon} = 0 \) and for any \( k \geq 1 \),
\[
\sigma_{k,\varepsilon}^- := \inf\{t > \sigma_{k-1,\varepsilon}^- : X_t \geq \varepsilon\} \quad \sigma_{k+1,\varepsilon}^- := \inf\{t > \sigma_{k,\varepsilon}^+ : X_t < 0\}
\]
and define the auxiliary process \( X^{(\varepsilon)} = \{X_t^{(\varepsilon)}, t \geq 0\} \) where
\[
X_t^{(\varepsilon)} = \begin{cases} X_t - \varepsilon & \text{if } \sigma_{k,\varepsilon}^- \leq t < \sigma_{k,\varepsilon}^+ \\ X_t & \text{if } \sigma_{k,\varepsilon}^+ \leq t < \sigma_{k+1,\varepsilon}^- \end{cases}.
\]
It is straightforward from the definition of \( X^{(\varepsilon)} \) that \( X_t - \varepsilon \leq X_t^{(\varepsilon)} \leq X_t \) and that \( X^{(\varepsilon)} \uparrow X \) uniformly when \( \varepsilon \downarrow 0 \), i.e.
\[
\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} |X_t^{(\varepsilon)} - X_t| = 0
\]
In addition we define the last zero process \( g_{\varepsilon,t} \) associated to the process \( X^{(\varepsilon)} \), i.e.
\[
g_{\varepsilon,t} = \sup\{0 \leq s \leq t : X_s^{(\varepsilon)} \leq 0\}.
\]
The inequality \( g_t \leq g_{c,t} \leq g_t^{(c)} \) holds for all \( t \geq 0 \). Taking \( \varepsilon \downarrow 0 \) and by right continuity of \( x \mapsto g_t^\varepsilon \) we obtain that \( g_{c,t} \downarrow g_t \) when \( \varepsilon \downarrow 0 \) for all \( t \geq 0 \). Therefore we have that \( t-U_{c,t} =: U_{c,t} \uparrow U_t \) when \( \varepsilon \downarrow 0 \) for all \( t \geq 0 \).

Recall that the local time at zero, \( L = \{ L_t, t \geq 0 \} \), is a continuous process defined in terms of the Itô–Tanaka formula (see Protter (2005) Chapter IV) and its measure \( dL_t \) is carried by the set \( \{ s \geq 0 : X_{s-} = X_s = 0 \} \).

Denote \( M_t^{(c)} \) as the number of downcrossings of the level zero at time \( t \geq 0 \) of the process \( X_t^{(c)} \), i.e.

\[
M_t^{(c)} = \sum_{k=1}^{\infty} \mathbb{1}\{\sigma_{c,-}^{k} < t\}.
\]

We simply denote \( M^{(c)} = \lim_{t \to \infty} M_t^{(c)} \) for all \( \varepsilon > 0 \). It turns out that \( M_t^{(c)} \) works as an approximation of the local time at zero in some sense. We have the following lemma. The proof follows an analogous argument than Revuz and Yor (2004) (see Exercise VI.1.19) and the proof is omitted.

**Lemma 4.1.** Suppose that \( X \) is a spectrally negative Lévy process. Then for all \( t \geq 0 \),

\[
\lim_{\varepsilon \downarrow 0} \varepsilon M_t^{(c)} = \frac{1}{2} L_t \quad \text{a.s.}
\]

In the next Lemma we calculate explicitly the probability mass function of the random variable \( M_{e_p} \).

**Lemma 4.2.** Let \( e_p \) an independent exponential random variable with parameter \( p \geq 0 \). For all \( \varepsilon > 0 \) we have that the probability mass function of the random variable \( M_{e_p}^{(c)} \) is given by

\[
\mathbb{P}_x(M_{e_p}^{(c)} = n) = \begin{cases} 1 & \text{if } n = 1 \\ \mathcal{I}(p,0)(\varepsilon) e^{-\Phi(p)(\varepsilon-x)} & \text{if } n = 2 \\ \mathcal{I}(p,0)(\varepsilon) e^{-\Phi(p)(\varepsilon-x)} \mathcal{T}(p,\Phi(p))(\varepsilon)^{n-2} [1 - \mathcal{T}(p,\Phi(p))(\varepsilon)] & \text{if } n \geq 2 \end{cases}
\]

for all \( x \leq \varepsilon \).

**Proof.** We calculate the probability of the event \( \{ M_{e_p}^{(c)} \geq n \} \) for \( n \geq 2 \) which happens if and only if \( \{ \sigma_{c,-}^{n} < e_p \} \). First, for any \( x < \varepsilon \) we calculate

\[
\mathbb{P}_x(M_{e_p}^{(c)} \geq 2) = \mathbb{P}_x(\sigma_{c,-}^{x} < e_p) = \mathbb{E}_x(\mathbb{P}_x(\sigma_{c,-}^{x} < e_p, \sigma_{c,-}^{1} < e_p | \mathcal{F}_{\sigma_{c,-}^{1}}))
\]

\[
= \mathbb{E}_x(\mathbb{E}_x(e^{-p\tau_{\varepsilon}^{x}} \mathbb{1}_{\{\tau_{\varepsilon}^{x} < \infty\}} | \mathcal{F}_{\sigma_{c,-}^{1}}) e^{-p\tau_{\varepsilon}^{x}} \mathbb{1}_{\{\tau_{\varepsilon}^{x} < \infty\}})
\]

\[
= \mathcal{I}(p,0)(\varepsilon) e^{-\Phi(p)(\varepsilon-x)},
\]

where the second last equality follows from the strong Markov property and the lack of memory property of the exponential distribution, the last by equations (3) and (5). Next assume that \( n \geq 3 \), we have that for any \( x < \varepsilon \)

\[
\mathbb{P}_x(M_{e_p}^{(c)} \geq n) = \mathbb{P}_x(\sigma_{n,-}^{x} < e_p)
\]

\[
= \mathbb{E}_x(\mathbb{P}_x(\sigma_{n,-}^{x} < e_p, \sigma_{n-1,-}^{x} < e_p | \mathcal{F}_{\sigma_{n-1,-}^{x}}))
\]

\[
= \mathbb{E}_x(e^{-p\tau_{\varepsilon}^{x}} \mathbb{1}_{\{\tau_{\varepsilon}^{x} < \infty\}} | \mathcal{F}_{\sigma_{n-1,-}^{x}}) \mathbb{E}_x(e^{-p\tau_{\varepsilon}^{x}} \mathbb{1}_{\{\tau_{\varepsilon}^{x} < \infty\}})
\]

\[
= \mathcal{I}(p,0)(\varepsilon) \mathbb{E}_x(e^{-p\tau_{\varepsilon}^{x}} \mathbb{1}_{\{\sigma_{n-1,-}^{x} < \infty\}}),
\]

where the third equality follows from the strong Markov property and the lack of memory property of the exponential distribution and the last equality by equation (5). Applying the strong Markov property at the stopping time \( \sigma_{n-1,-}^{x} \) we get
where the last equality follows from equation (3). We apply the strong Markov property at \( \sigma_{n-2,\varepsilon}^+ \) and we use the fact that for all \( k \geq 2 \), \( X_{\sigma_{k,\varepsilon}^+} = \varepsilon \) on the event \( \{0 < \sigma_{k,\varepsilon}^+ < \infty\} \) to deduce for all \( n \geq 3 \) that

\[
\mathbb{P}_x(M_{\varepsilon}^{(c)} \geq n) = \mathcal{T}^{(p,0)}(\varepsilon) \mathbb{E}_x(e^{-pR_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma_{n-1,\varepsilon}^+ < \infty\}}) = \mathcal{T}^{(p,0)}(\varepsilon) \mathbb{E}_x(e^{-pR_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma_{n-1,\varepsilon}^+ < \infty\}}) \mathbb{E}_\varepsilon (e^{-pR_{\sigma_{n-1,\varepsilon}^+} + \Phi(p)X_{\sigma_{n-1,\varepsilon}^+}^\varepsilon} \mathbb{I}_{\{\sigma_{n-1,\varepsilon}^+ < \infty\}}),
\]

where last equality follows from equations (5) and (18). Then by an induction argument we get that for all \( n \geq 2 \) and \( x < \varepsilon \)

\[
\mathbb{P}_x(M_{\varepsilon}^{(c)} \geq n) = \mathcal{T}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-x)}[\mathcal{T}^{(p,\Phi(p))}(\varepsilon)]^{n-2}.
\]  

(19)

When \( x \geq \varepsilon \) we get from the Markov property at time \( \sigma_{2,\varepsilon}^- \) that for all \( n \geq 2 \),

\[
\mathbb{P}_x(M_{\varepsilon}^{(c)} \geq n) = \mathbb{E}_x(\mathbb{P}_x(\sigma_{n,\varepsilon}^- < e_p | \sigma_{2,\varepsilon}^-)) = \mathbb{E}_x(e^{-pR_{\sigma_{2,\varepsilon}^-} \mathbb{P}_{\varepsilon} M_{\varepsilon}^{(c)} \geq n-1} | \sigma_{2,\varepsilon}^- < \infty)).
\]

The result follows from equations (5) and (19) and the fact that \( \sigma_{2,\varepsilon}^- = \tau_{0}^- \) when \( x = \varepsilon \). \( \square \)

**Remark 4.3.** For all \( \varepsilon > 0 \) we can describe the paths of the process \( \{g_{\varepsilon,t}, t \geq 0\} \) in terms of the stopping times \( \{\sigma_{k,\varepsilon}^- , \sigma_{k,\varepsilon}^+, k \geq 1\} \). When \( X_{t}^{(\varepsilon)} \leq 0 \) we have that \( \sigma_{k,\varepsilon}^- \leq t < \sigma_{k+1,\varepsilon}^- \) for some \( k \geq 1 \) and then \( g_{\varepsilon,t} = t \).

Similarly, when \( X_{t}^{(\varepsilon)} > 0 \) there exists \( k \geq 1 \) such that \( \sigma_{k,\varepsilon}^+ \leq t < \sigma_{k+1,\varepsilon}^- \) and hence \( g_{\varepsilon,t} = \sigma_{k,\varepsilon}^+ \).

**4.2 Proof of Theorem 3.3**

Suppose that \( X_t > 0 \) and choose \( \varepsilon < X_t \). For any function \( F \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}) \) we have that using a telescopic sum

\[
F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)}) = F(g_{\varepsilon,0}, 0, X_0^{(\varepsilon)}) + \sum_{k=1}^{M_t^{(\varepsilon)}} [F(g_{\varepsilon,\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-}, X_t^{(\varepsilon)}) - F(g_{\varepsilon,\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-}, X_t^{(\varepsilon)})] + \sum_{k=1}^{M_t^{(\varepsilon)}} [F(g_{\varepsilon,\sigma_{k+1,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-}, X_t^{(\varepsilon)}) - F(g_{\varepsilon,\sigma_{k+1,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-}, X_t^{(\varepsilon)})] + [F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)}) - F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)})] + [F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)}) - F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)})].
\]

Note that \( g_{\varepsilon,\sigma_{k+1,\varepsilon}^-} = \sigma_{k,\varepsilon}^+ , g_{\varepsilon,\sigma_{k,\varepsilon}^-} = \sigma_{k,\varepsilon}^- \) and \( g_{\varepsilon,\sigma_{k,\varepsilon}^+} = \sigma_{k,\varepsilon}^+ \) for all \( k \geq 1 \). Thus,
\[
F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)}) = F(g_{\varepsilon,0}, 0, X_0^{(\varepsilon)}) + \sum_{k=1}^{M(\varepsilon)} [F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^+, X_{\sigma_{k,\varepsilon}^+}^{(\varepsilon)} - \varepsilon) - F(\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon)] \\
+ \sum_{k=1}^{M(\varepsilon)} [F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)}) - F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon)] \\
+ \sum_{k=1}^{M(\varepsilon)-1} [F(\sigma_{k,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-, X_{(\sigma_{k+1,\varepsilon})^-}^{(\varepsilon)}) - F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon)] \\
+ \sum_{k=1}^{M(\varepsilon)-1} [F(\sigma_{k+1,\varepsilon}^+, \sigma_{k,\varepsilon}^- X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon) - F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon)] \\
+ [F(\sigma_{M(\varepsilon)}^+, \sigma_{M(\varepsilon)}^-, (\sigma_{M(\varepsilon)}^-)) - F(\sigma_{M(\varepsilon)}^+, \sigma_{M(\varepsilon)}^-, X_{\sigma_{M(\varepsilon)}^-}^{(\varepsilon)})],
\]

where we also used that $X_s^{(\varepsilon)} = X_s$ when $s \in [\sigma_{k,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-]$ and $X_s^{(\varepsilon)} = X_s - \varepsilon$ when $s \in [\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^+]$ for all $k \geq 1$. Applying Itô formula on intervals of the form $(\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^+)$ for $k \geq 1$ we have that

\[
\sum_{k=1}^{M(\varepsilon)} F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon) - F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^-, X_{\sigma_{k,\varepsilon}^-}^{(\varepsilon)} - \varepsilon)
\]

\[
= \sum_{k=1}^{M(\varepsilon)} \left[ \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \frac{\partial}{\partial \gamma} F(s, s, X_s - \varepsilon) ds + \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \frac{\partial}{\partial \tau} F(s, s, X_s - \varepsilon) ds \right] \\
+ \sum_{k=1}^{M(\varepsilon)} \left[ \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \frac{1}{2} \sigma^2(s, s, X_s - \varepsilon) ds + \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \frac{\partial}{\partial \gamma} F(s, s, X_s - \varepsilon) ds + \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \frac{\partial}{\partial \tau} F(s, s, X_s - \varepsilon) ds \right] \\
+ \sum_{k=1}^{M(\varepsilon)} \left[ \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} \int_{-\infty,0} \left( F(s, s, X_s - y - \varepsilon) - F(s, s, X_s - \varepsilon) - y \frac{\partial}{\partial \gamma} F(s, s, X_s - \varepsilon) \right) N(ds \times dy) \right]
\]

\[
= \int_0^t \frac{\partial}{\partial \gamma} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) I_{\{X_s^{(\varepsilon)} \leq 0\}} ds \\
+ \int_0^t \frac{\partial}{\partial \tau} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) I_{\{X_s^{(\varepsilon)} \leq 0\}} ds + \frac{1}{2} \sigma^2 \int_0^t \frac{\partial^2}{\partial \gamma^2} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) I_{\{X_s^{(\varepsilon)} \leq 0\}} ds \\
+ \int_{[0,t]} \frac{\partial}{\partial \gamma} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) I_{\{X_s^{(\varepsilon)} \leq 0\}} ds + \int_{[0,t]} \frac{\partial}{\partial \gamma} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) I_{\{X_s^{(\varepsilon)} > 0\}} I_{\{X_s^{(\varepsilon)} + y < 0\}} N(ds \times dy) \\
+ \int_{[0,t]} \int_{-\infty,0} \left( F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)} + y) - F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) - y \frac{\partial}{\partial \gamma} F(g_{\varepsilon,s}, s, X_s^{(\varepsilon)}) \right) I_{\{X_s^{(\varepsilon)} \leq 0\}} N(ds \times dy),
\]

where the last equality follows since $X_s^{(\varepsilon)} \leq 0$ if and only if $s \in [\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^+]$ for some $k \geq 1$ (and hence $g_{\varepsilon,s} = s$), $X$ has a jump at time $s$ on the event $\{X_s^{(\varepsilon)} > 0\} \cap \{X_s^{(\varepsilon)} < 0\}$ and there are no jumps at time $\sigma_{k,\varepsilon}^+$ for all $k \geq 1$. Similarly, applying Itô formula on intervals of the form $(\sigma_{k,\varepsilon}^-, \sigma_{k+1,\varepsilon}^-)$ for $k \geq 1$, there are no jumps at time $\sigma_{k,\varepsilon}^-$ for all $k \geq 1$ and the fact that $X_s^{(\varepsilon)} > 0$ if and only if $s \in [\sigma_{k,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-]$ for some $k \geq 1$ (and hence $g_{\varepsilon,s} = s, g_{\varepsilon,s}^+ = \sigma_{k,\varepsilon}^+$) we have that
\[
\sum_{k=1}^{M_{(s)}-1} [F(\sigma_{k,+}, \sigma_{k-1,+}, X_{(\sigma_{k-1,+}^-)}) - F(\sigma_{k,+}^+, \sigma_{k-1,+}^+, X_{\sigma_{k-1,+}^+})] + [F(\sigma_{M_{(s)}-1,+}, \sigma_{M_{(s)}-1,-}, X_{\sigma_{M_{(s)}-1}^-})]
\]

\[
= \int_0^t \frac{\partial}{\partial t} F(g_{\varepsilon,s}, s, X_{(s)}^+) \mathbb{I}_{\{X_{(s)}^+ > 0\}} ds
\]

\[
+ \frac{1}{2} \varepsilon^2 \int_0^t \frac{\partial^2}{\partial x^2} F(g_{\varepsilon,s}, s, X_{s}^+) \mathbb{I}_{\{X_{s}^+ > 0\}} ds + \int_{[0,t]} \frac{\partial}{\partial x} F(g_{\varepsilon,s-}, s, X_{s-}^+) \mathbb{I}_{\{X_{s-}^+ > 0\}} dX_s
\]

\[
+ \int_{[0,t]} \int_{(-\infty,0)} \left( F(g_{\varepsilon,s-}, s, X_{s-}^+) + y - F(g_{\varepsilon,s-}, s, X_{s-}^+) - y \frac{\partial}{\partial x} F(g_{\varepsilon,s-}, s, X_{s-}^+) \right) \mathbb{I}_{\{X_{s-}^+ > 0\}} N(ds \times dy).
\]

Hence, we obtain that

\[
F(g_{\varepsilon,t}, t, X_t^+) = F(g_{\varepsilon,0}, 0, X_0^+)
\]

\[
+ \sum_{k=1}^{M_{(s)}-1} [F(\sigma_{k,+}, \sigma_{k-1,+}, X_{\sigma_{k-1,+}^+}) - F(\sigma_{k,+}^+, \sigma_{k-1,+}^+, X_{\sigma_{k-1,+}^+})]
\]

Since \( X_{\sigma_{k-1,+}^-} = \varepsilon \) and that \( X \) can cross below 0 either by creeping or by a jump we have that the last two terms in the expression above become

\[
\sum_{k=1}^{M_{(s)}-1} [F(\sigma_{k,+}, \sigma_{k-1,+}, X_{\sigma_{k-1,+}^-}) - F(\sigma_{k,+}^+, \sigma_{k-1,+}^+, X_{\sigma_{k-1,+}^-})]
\]

\[
= \sum_{k=1}^{M_{(s)}-1} [F(\sigma_{k+1,+}, \sigma_{k-1,+}^+), \varepsilon) - F(\sigma_{k+1,+}, \sigma_{k-1,+}^+), 0)]
\]

\[
+ \sum_{k=1}^{M_{(s)}-1} [F(\sigma_{k-1,+}^-), \sigma_{k-1,+}^-, \varepsilon) - F(\sigma_{k-1,+}^-), \sigma_{k-1,+}^-, 0)]\mathbb{I}_{\{X_{\sigma_{k-1,+}^-} = 0\}}
\]

\[
+ \int_{[0,t]} \int_{(-\infty,0)} F(s, s, X_{s-}^+ + y - \varepsilon) - F(g_{\varepsilon,s-}, s, X_{s-}^+) \mathbb{I}_{\{X_{s-}^+ > 0\}} \mathbb{I}_{\{X_{s-}^+ + y < 0\}} N(ds \times dy),
\]

where we used the fact that when \( \sigma > 0, \lim_{t\to0} F(\gamma, t, h) = F(t, t, 0) \) for all \( 0 \leq \gamma \leq t \) by assumption, \( F \) is continuous and that \( X_{(\sigma_{k-1,+}^-)}^- = 0 \) on the event of creeping. By the mean value theorem we have that for each \( k \geq 1 \) there exist \( c_{1,k} \in (0, \varepsilon) \) and \( c_{2,k} \in (-\varepsilon, 0) \) such that
\[
\sum_{k=1}^{M^{(e)}-1} [F(\sigma_{k+1,\varepsilon}^+, \sigma_{k+1,\varepsilon}^+), \varepsilon) - F(\sigma_{k+1,\varepsilon}^+, \sigma_{k+1,\varepsilon}^+, 0)] \\
+ \sum_{k=1}^{M^{(e)}-1} [F(\sigma_{k+1,\varepsilon}^-, \sigma_{k+1,\varepsilon}^-), -\varepsilon) - F(\sigma_{k+1,\varepsilon}^-, \sigma_{k+1,\varepsilon}^-, 0)] I_{\{X_{k+1,\varepsilon}^- = 0\}} \\
\leq \sum_{k=1}^{M^{(e)}-1} \left| \frac{\partial}{\partial x} F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^+, s) - \frac{\partial}{\partial x} F(\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^-, s) \right| I_{\{X_{k+1,\varepsilon}^- = 0\}} \\
\leq 2K_1 \varepsilon (M^{(e)} - 1),
\]

where we used the fact that \( F \) is \( C^{1,1,2} \) and then \( \frac{\partial}{\partial x} F \) is bounded in the set \([0, t] \times [0, t] \times [-\varepsilon, \varepsilon] \) by a constant, namely \( K_1 > 0 \). Moreover, we know that \( \varepsilon M^{(e)}_t \to L_t/2 \) a.s. when \( \varepsilon \downarrow 0 \) (see Lemma 4.1). Hence using the dominated convergence theorem and continuity of \( F \) we deduce that

\[
\lim_{\varepsilon \to 0} \left| \sum_{k=1}^{M^{(e)}-1} [F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^+, \varepsilon) - F(\sigma_{k,\varepsilon}^+, \sigma_{k,\varepsilon}^+, 0)] + [F(\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^-, \varepsilon) - F(\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^-, 0)] I_{\{X_{k+1,\varepsilon}^- = 0\}} \right| = 0.
\]

Therefore, by the dominated convergence theorem for stochastic integrals, we deduce that

\[
F(g_t, t, x_t) = F(g_0, 0, x_0) + \int_0^t \frac{\partial}{\partial y} F(g_s, s, x_s) I_{\{x_s \leq 0\}} ds + \int_0^t \frac{\partial}{\partial t} F(g_s, s, x_s) ds \\
+ \int_0^t \frac{\partial}{\partial x} F(g_s, s, x_s) dx_s + \frac{1}{2} \sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(g_s, s, x_s) ds \\
+ \int_{[0, t]} \int_{(-\infty, 0)} \left( F(g_s, s, x_s + y) - F(g_s, s, x_s) - y \frac{\partial}{\partial x} F(g_s, s, x_s) \right) I_{\{x_s > 0\}} N(ds \times dy) \\
+ \int_{[0, t]} \int_{(-\infty, 0)} \left( F(g_s, s, x_s + y) - F(g_s, s, x_s) - y \frac{\partial}{\partial x} F(g_s, s, x_s) \right) I_{\{x_s \leq 0\}} N(ds \times dy) \\
+ \int_{[0, t]} \int_{(-\infty, 0)} \left( F(s, s, x_s + y) - F(s, s, x_s) - y \frac{\partial}{\partial x} F(s, s, x_s) \right) I_{\{0 < x_s < \varepsilon \}} N(ds \times dy).
\]

From the fact that \( g_t \) is continuous in the set \( \{t \geq 0 : x_t > 0 \text{ or } x_{t-} \leq 0\} \) we obtain the desired result. The case when \( x_t \leq 0 \) is similar and proof is omitted.

### 4.3 Proof of Theorem 3.6

First note that, since \( |K(U_s, X_s)| \leq C(s, X_s) \) for all \( s \geq 0 \) and \( \mathbb{E}_x \left( \int_0^\infty e^{-u t} C(X_t) dr \right) < \infty \) for all \( (u, x) \in E \), we have that \( K^+ \) and \( K^- \) are finite. Moreover, we have that for all \( r \geq 0 \) and \( \varepsilon > 0 \),

\[
|K(U_{r, x}, X_{r}^{(e)})| \leq C(r, X_{r}^{(e)}) \leq C(r, X_r) + C(r, X_r - \varepsilon),
\]

where the last inequality follows since \( C \) is non-negative. It follows from integrability of \( e^{-u t} C(X_t) \) with respect to the product measure \( \mathbb{P}_x \times dr \) for all \( x \in \mathbb{R} \), by dominated convergence theorem and left-continuity in each argument of \( K \) that for all \( (u, x) \in E \),

\[
\mathbb{E}_{u, x} \left( \int_0^\infty e^{-u t} K(U_r, X_r) dr \right) = \lim_{\varepsilon \downarrow 0} \mathbb{E}_{u, x} \left( \int_0^\infty e^{-u t} K(U_{r, x}, X_{r}^{(e)}) dr \right).
\]
Then we calculate the right-hand side of the equation above. Fix $\varepsilon > 0$, using the fact that \( \{ M^{(\varepsilon)} = n \} \cap \{ \sigma_{n,\varepsilon}^+ < \infty \} \cap \{ \sigma_{n+1,\varepsilon}^- = \infty \} \) for $n = 1, 2, \ldots$, we have for any $x \leq 0$ that

\[
E_x \left( \int_0^\infty e^{-qr} K(U_{\varepsilon,r}, X_r^{(\varepsilon)}) \, dr \right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( \sum_{r \in \{ \sigma_{n,\varepsilon}^+ < \infty \}} \int_{\sigma_{n,\varepsilon}^+}^{\sigma_{n+1,\varepsilon}^-} e^{-qr} K(0, X_r^{(\varepsilon)}) \, dr \right)
\]

where the last equality follows from the fact that \( g_{\varepsilon,r} = r \) when \( r \in [\sigma_{k,\varepsilon}^- \sigma_{k,\varepsilon}^+ ] \) and \( g_{\varepsilon,r} = \sigma_{k,\varepsilon}^+ \) when \( r \in [\sigma_{k,\varepsilon}^-, \sigma_{k+1,\varepsilon}^+] \) for some $k \geq 1$. We first analyse the first double sum on the right-hand side of the expression above. Conditioning with respect to the filtration at the stopping time \( \sigma_{k,\varepsilon}^+ \), the strong Markov property and the fact that $X$ creeps upwards we get

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( \sum_{r \in \{ \sigma_{n,\varepsilon}^+ < \infty \}} \int_{\sigma_{n,\varepsilon}^+}^{\sigma_{n+1,\varepsilon}^-} e^{-qr} K(0, X_r^{(\varepsilon)}) \, dr \right)
\]

where the first term in the last equality corresponds to the first excursion of $X^{(\varepsilon)}$ below zero (case $k = 1$) and we used the fact that $X_r^{(\varepsilon)} = X_r - \varepsilon$ for $r \in [\sigma_{k,\varepsilon}^-, \sigma_{k+1,\varepsilon}^+]$ for any $k \geq 1$. We define the auxiliary functions

\[
K^-(x) := E_x \left( \int_0^{\tau_0^+} e^{-qr} K(0, X_r) \, dr \right),
\]

\[
K_1^-(x) := E_x \left( \int_{\tau_0^-}^{\tau_0^+} e^{-qr} K(0, X_r) \, dr \right),
\]

\[
K_2^-(x) := E_x \left( \int_{\tau_0^-}^{\tau_0^+} e^{-qr} K(0, X_r) \, dr \right)
\]

for all $x \in \mathbb{R}$. Then we have that $K^-(x) = K_1^-(x) + K_2^-(x)$ for all $x \in \mathbb{R}$. Conditioning again with respect to the filtration at time $\sigma_{k,\varepsilon}$ (resp. $\sigma_{n,\varepsilon}$) we have
\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) \\
= K^-(x - \varepsilon) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = n - k + 1) \\
+ \sum_{n=2}^{\infty} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} e^{-qr} K(X_{\sigma_{n,\varepsilon} - \varepsilon}) \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = 1) \\
= K^-(x - \varepsilon) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} e^{-qr} K(X_{\sigma_{n,\varepsilon} - \varepsilon}) \right) \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = n - k + 1) \\
+ \sum_{n=2}^{\infty} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} e^{-qr} K(X_{\sigma_{n,\varepsilon} - \varepsilon}) \right) \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = 1) \\
= K^-(x - \varepsilon) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} e^{-qr} K(X_{\sigma_{n,\varepsilon} - \varepsilon}) \right) \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = n - k + 1) \\
+ \sum_{n=2}^{\infty} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} e^{-qr} K(X_{\sigma_{n,\varepsilon} - \varepsilon}) \right) \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) P_x(M(\varepsilon) = 1),
\]
where the second equality follows from conditioning with respect to time \(\sigma_{k-1,\varepsilon}\) (resp. \(\sigma_{n-1,\varepsilon}\)) and the Markov property of \(X\) and the last from equation (18). From Lemma 4.2 and solving the corresponding series we get

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(0, X_r(\varepsilon)) \, d\tau \right) \\
= K^-(x - \varepsilon) + \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0 < \infty\}} e^{-qr} K(X_{\tau_0 - \varepsilon}) \right) \frac{e^{-\Phi(q) (\varepsilon - x)}}{1 - \mathbb{I}(q, \Phi(q)) (\varepsilon)}. 
\]

Using similar arguments we have, from the strong Markov property, the fact that \(X\) creeps upwards, equation (5) and Lemma 4.2, that

\[
\sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(r - \sigma_{k,\varepsilon}, X_r(\varepsilon)) \, d\tau \right) \\
+ \sum_{n=1}^{\infty} \mathbb{E}_x \left( \mathbb{I}_{\{\sigma_{n,\varepsilon} < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon} = \infty\}} \int_{\sigma_{k,\varepsilon}}^{\sigma_{k+1,\varepsilon}} e^{-qr} K(r - \sigma_{n,\varepsilon}, X_r(\varepsilon)) \, d\tau \right) \\
= K^+(0, \varepsilon) \frac{e^{-\Phi(q) (\varepsilon - x)}}{1 - \mathbb{I}(q, \Phi(q)) (\varepsilon)}. 
\]

Therefore, by the dominated convergence theorem we have that for all \(x \leq 0\)

\[
\mathbb{E}_x \left( \int_0^{\infty} e^{-qr} K(U_r, X_r) \, d\tau \right) \\
= \lim_{\varepsilon \downarrow 0} \left\{ K^-(x - \varepsilon) + \frac{e^{-\Phi(q) (\varepsilon - x)}}{1 - \mathbb{I}(q, \Phi(q)) (\varepsilon)} \left[ \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0 < \infty\}} e^{-qr} K(X_{\tau_0 - \varepsilon}) \right) + K^+(0, \varepsilon) \right] \right\}. 
\]
When $u, x > 0$ we have that
\[
\mathbb{E}_{u,x} \left( \int_0^\infty e^{-qr} K(U_r, X_r)dr \right)
\]
\[
= \mathbb{E}_x \left( \int_0^{\tau_0^-} e^{-qr} K(u + r, X_r)dr \right) + \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} \int_{\tau_0^-}^\infty e^{-qr} K(U_r, X_r)dr \right)
\]
\[
= K^+(u, x) + \lim_{\varepsilon \downarrow 0} \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} K^- (X_{\tau_0^-} - \varepsilon) \right)
\]
\[
+ e^{\Phi(q)x} T(q, \Phi(q)) (x) \lim_{\varepsilon \downarrow 0} \frac{e^{-\Phi(q)x}}{1 - T(q, \Phi(q)) (\varepsilon)} \left[ \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} K^- (X_{\tau_0^-} - \varepsilon) \right) + K^+(0, \varepsilon) \right],
\] (21)
where the last equality follows from conditioning at time $\tau_0^-$ and the strong Markov property. Using Fubini’s theorem and equation (7) we have that for all $x < 0$,
\[
K^-(x) = \int_{(-\infty, 0)} K(0, y) \int_0^\infty e^{-qr} P_x (X_r \in dy, r < \tau_0^+) dr = \int_{-\infty}^0 K(0, y) [e^{\Phi(q)x} W(q)(-y) - W(q)(x-y)] dy
\]
Then for any $x, \varepsilon > 0$ we have that by Fubini’s theorem and equation (5),
\[
\mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} K^- (X_{\tau_0^-} - \varepsilon) \right)
\]
\[
= e^{\Phi(q)(x-\varepsilon)} T(q, \Phi(q)) (x) \int_{-\varepsilon}^0 K(0, y) W(q)(-y) dy
\]
\[
+ \int_{-\infty}^{-\varepsilon} K(0, y) \left[ e^{\Phi(q)(x-\varepsilon)} T(q, \Phi(q)) (x) W(q)(-y) - \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} W(q)(X_{\tau_0^-} - \varepsilon - y) \right) \right] dy.
\]
Let $x, \varepsilon > 0$ and $y < -\varepsilon$, using a change of measure (see equation (9)) we obtain that
\[
\mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- = \infty\}} e^{-q\tau_0^-} W(q)(X_{\tau_0^-} - \varepsilon - y) \right)
\]
\[
= e^{\Phi(q)(x-\varepsilon-y)} \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- = \infty\}} e^{-\Phi(q)(X_{\tau_0^-} - \varepsilon - y)} W(q)(X_{\tau_0^-} - \varepsilon - y) \right)
\]
\[
= e^{\Phi(q)(x-\varepsilon-y)} \Phi'(q) \mathbb{E}_x (\tau_0^- = \infty)
\]
\[
= e^{-\Phi(q)(\varepsilon + y)} W(q)(x),
\]
where in the second equality we used that fact that $X$ drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$ and the last follows from equation (5). Then from the fact that $e^{-q(t \wedge \tau_0^-)} W(q)(X_{t \wedge \tau_0^-})$ is a martingale and since $\tau_0^- - y < \tau_0^-$ we have that
\[
\mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} W(q)(X_{\tau_0^-} - \varepsilon - y) \right)
\]
\[
= \mathbb{E}_x \left( e^{-q\tau_0^-} W(q)(X_{\tau_0^-} - \varepsilon - y) \right) - \mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- = \infty\}} e^{-q\tau_0^-} W(q)(X_{\tau_0^-} - \varepsilon - y) \right)
\]
\[
= W(q)(x - \varepsilon - y) - e^{-\Phi(q)(\varepsilon + y)} W(q)(x).
\]
Hence we obtain that for any $x > 0$,
\[
\mathbb{E}_x \left( \mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} K^- (X_{\tau_0^-} - \varepsilon) \right)
\]
\[
= e^{\Phi(q)(x-\varepsilon)} T(q, \Phi(q)) (x) \int_{-\varepsilon}^0 K(0, y) W(q)(-y) dy
\]
\[
+ \int_{-\infty}^{-\varepsilon} K(0, y) \left[ e^{\Phi(q)(x-\varepsilon)} T(q, \Phi(q)) (x) W(q)(-y) - W(q)(x - \varepsilon - y) + e^{-\Phi(q)(\varepsilon + y)} W(q)(x) \right] dy.
\]
Substituting the expression above in (21) and taking limits we obtain that for all $(u, x) \in E$,
\[
\mathbb{E}_{u,x} \left( \int_0^\infty e^{-qr} K(U_r, X_r)dr \right) = K^+(u, x) + \int_{-\infty}^0 K(0, y) \left[ e^{\Phi(q)(x-\varepsilon)} \Phi'(q) - W(q)(x-y) \right] dy
\]
\[
+ e^{\Phi(q)x} \left[ 1 - \Phi'(q) + e^{-\Phi(q)x} W(q)(x) \right] \lim_{\varepsilon \downarrow 0} \frac{K^+(0, \varepsilon)}{\Phi'(q)(+)} W(q)(\varepsilon).
\]

19
The result follows from equation (5).

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