Uncertain time series forecasting method for the water demand prediction in Beijing

Haiyan Li³, Xiaosheng Wang⁴,* and Haiying Guo³
³School of Economics, Nankai University, Tianjin 300071, China
⁴School of Mathematics and Physics, Hebei University of Engineering, Handan 056038, China
*Corresponding author. E-mail: xswang@hebeu.edu.cn

ABSTRACT

Water demand prediction is crucial for effectively planning and management of water supply systems to handle the problem of water scarcity. Taking into account the uncertainties and imprecisions within the framework of water demand forecasting, the uncertain time series prediction method is introduced for the water demand prediction. Uncertain time series is a sequence of imprecisely observed values that are characterized by uncertain variables and the corresponding uncertain autoregressive model is employed to describe it for predicting the future values. The main contributions of this paper are shown as follows. Firstly, by defining the auto-similarity of uncertain time series, the identification algorithm of uncertain autoregressive model order is proposed. Secondly, a new parameter estimation method based on the uncertain programming is developed. Thirdly, the imprecisely observed values are assumed as the linear uncertain variables and a ratio-based method is presented for constructing the uncertain time series. Finally, the proposed methodologies are applied to model and forecast the Beijing’s water demand under different confidence levels and compared with the traditional time series, i.e., ARIMA method. The experimental results are evaluated on the basis of performance criteria, which shows that the proposed method outperforms over the ARIMA method for water demand prediction.

Key words: uncertainty theory, uncertain time series, water demand prediction

HIGHLIGHT

Taking into account the uncertainties and imprecisions within the framework of water demand forecasting, the uncertain time series prediction method is introduced for the water demand prediction. The proposed methodologies are applied to model and forecast the Beijing’s water demand under different confidence levels and compared with the traditional time series, i.e., ARIMA method.

1. INTRODUCTION

Water is an indispensable natural resource on our planet, which plays an important role in man’s life and activity. Apart from drinking and personal hygiene, water is still a necessary resource for agricultural and industrial production, economic and ecological development (Deng et al. 2015; Liu et al. 2015). However, due to climate change, socio-economic development and population growth, water consumption (especially for freshwater) is growing rapidly, and water supply is facing with many challenges (Choksi et al. 2015), especially the problem of water scarcity (Frederick 1997; Pahl-Wostl 2007; Arnell & Lloyd-Hughes 2014; Wang et al. 2015). This has led to the need for effectively planning, managing and operating the finite water resources (Odouro-Kwarteng et al. 2009; Wang et al. 2018). Therefore, water demand forecasting is a fundamental phase for optimal allocation of water resources and aims to provide the simulated view of future demand, which can assist decision makers in devising appropriate management scheme to relieve the conflict between growing demand and limited supply of water resources. To this end, many researchers have proposed different methods to model and forecast water demand.

Time series analysis is one of the commonly used methods for the water demand prediction. It was proposed by Box and Jenkins that considered the dependence among the data. The model, namely, ARIMA, is regarded as classical forecasting technique, describing a predicted value as a linear function of previous data and random errors and including a cyclical or seasonal component. For example, Maidment et al. (1985) applied the time series model of daily municipal water use as a function of rainfall and air temperature for short-term forecasting of daily water use in Austin, Texas. Smith (1988) developed an autoregressive process with randomly varying mean to forecast the daily municipal water use, which captured the
seasonality and day-week effects in the model through the unit demand function. Aly & Wanakule (2004) utilized deterministic smoothing algorithm that considered level, trend and seasonality components of time series to estimate monthly water use. Zhai et al. (2012) employed the time series forecasting method for predicting the future needs of water in Beijing by analyzing the driving mechanism of changes of water consumption and water consumed structure.

In time series modelling application, the determination of the model order is the fundamental step towards describing any dynamic process which has been considerable interest for a long time. Primarily, the determination method of the order of time series model is based on the properties of sample autocorrelation coefficient and partial autocorrelation coefficient. Following that, several order selection approaches based on information theoretic criteria, such as Akaike’s information criterion (AIC) (Akaike 1974), Akaike’s final prediction error (FPE) (Akaike 1970), minimum description length (MDL) (Rissanen 1978; Liang et al. 1993) and so on, have been developed. The another common method, namely, the linear algebraic method based upon the determinant and rank testing algorithms, was proposed in (Fuchs 1987; Sadabadi et al. 2007). Apart from two methods above, many other methods like bayesian information criterion (BIC) (Schwarz 1978), edge detection-based approach (Al-Smadi & Al-Zaben 2005), optimal instrumental variable (IV) algorithm (Sadabadi et al. 2009) and so on were investigated to estimate the order of the time series model. Another estimation problem have also considerably investigated in the aspect of coefficients determination of time series model. Commonly used methods for estimation of unknown coefficients are least-squares (LS) estimator and maximum likelihood estimator (MLE) methods.

Based on the mentioned methods of model order identification and parameters estimation, the time series models can be formulated to forecast water demand. It is worth noting that the aforementioned models provided a single valued forecast of water demand disregarding the uncertainty inherent in some situations where the influential factors that affect water demand are uncertain, which lead to the uncertainty of water demand. This would limit the usefulness of these deterministic models. One classical way to handle the uncertainty is to use a probabilistic model (Almutaz et al. 2012; Haque et al. 2014) based on the Monte Carlo Simulations (MCS) to obtain the distribution of water demand and provide an estimate of the overall uncertainty in the predictions connected to uncertainty of influential factors.

Unfortunately, the distribution function obtained in most practical problems is not close enough to the actual frequency, especially in the case of emergencies and lack of history data. In addition, the water demand data possess uncertain characteristics caused by inaccuracies in measurements that need to be given by experts. This motivates us to apply a new mathematical tool to deal with a range of uncertainties inherent in certain water demand data. Recently, uncertainty theory was proposed by Liu (2009) in 2007, which is an effective way to solve previous problem for imprecisely observed values. Based on the uncertainty theory, many researchers have done a lot of work including the determination of uncertain distribution (Wang et al. 2012a, 2012b; Wang & Peng 2014), hypothesis test (Guo et al. 2017; Ye & Liu 2020a), and uncertain regression analysis (Wang et al. 2012a, 2012b; Lio & Liu 2018; Yao & Liu 2018; Ye & Liu 2020b). Furthermore, the concept of uncertain time series was firstly proposed by Yang & Liu (2019) based on uncertain theory in 2019. Like the traditional time series analysis, there might be more than one approach to model time series. But in their study, to describe uncertain time series, the uncertain autoregressive model was employed to predict the future values based on previously imprecisely observed values that are characterized in terms of uncertain variables. Based on the imprecisely observed values, Yang & Liu (2019) presented the least-squares method to estimate the coefficients of the uncertain autoregressive model for predicting the carbon emission.

However, there are still many important issues that have not been touched. Firstly, the identification of uncertain autoregressive model order is one of these, because it is the first step in estimating the model parameters. In the work of Yang & Liu (2019), the 2-order uncertain autoregressive model was directly employed to forecast the future values. This method is too subjective and lacks a certain theoretical foundation, which may reduce the prediction accuracy of the model. So, in this paper, by defining the auto-similarity of uncertain time series, an algorithm for determining the optimal order of autoregressive model is designed. Secondly, its novel parameter estimation approach is developed based on the uncertain programming. Within the proposed method, the original problem including uncertain measure is transformed to the equivalent crisp mathematical programming. Thirdly, in our daily life, most information are uncertain in nature. For example, the water demand naturally takes different values with minimum water demand and maximum water demand, which are inherently imprecisely observed values at times \( t, t = 1, 2, \ldots, n \), respectively, so the linear uncertain variables are selected for this purpose. Hence, it is a critical issue for us to determine the lower and upper bounds for the actual data belonging to a range. That is, how to construct an uncertain time series based on observed historical point data. Referring to the work of Huarng (2006), we introduced a novel ratio-based approach to determine the effective uncertain time series. Furthermore, the proposed uncertain
time series forecasting approach is used to predict the urban water demand. As the second-largest city of China, Beijing’s rapid development has attracted many immigrants in recent years, water consumption is growing rapidly, which led to the increasingly sharp conflict between demand and supply of water resources. This situation has become the important constraint on the sustainable development of Beijing. Therefore, the water demand prediction of Beijing is a fundamental stage for water resources planning and utilize, which contributes to harmonious development between the socio-economy and resources environment in Beijing. To further verify the accuracy of the proposed methodologies, traditional time series method is selected as a competitor. The results are judged on the basis of presented criteria, i.e., the prediction reliability and accuracy compared to ARIMA method.

The organization of this paper is as follows. Section 2 briefly presents some fundamental concepts properties and theorems in uncertainty theory. Section 3 introduces the forecasting procedure of uncertain time series analysis. Section 4 provides an experimental analysis to validate the effectiveness of the proposed method and access its performance by comparing with the conventional time series (ARIMA) method. Finally, some conclusions are drawn.

2. PRELIMINARIES

In this section, we will present some fundamental definitions and theorems on uncertainty theory.

**Definition 1. (Liu 2007)** Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) be a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in \mathcal{L} \) is called an event. A number \( M(\Lambda) \) indicates the belief degree that \( \Lambda \) will occur. Then \( M \) is called an uncertain measure if it satisfies the following axioms:

**Axiom 1:** (Normality Axiom) \( M(\Gamma) = 1 \) for the nonempty set \( \Gamma \).

**Axiom 2:** (Duality Axiom) \( M(\Lambda) + M(\Lambda^c) = 1 \) for any event \( \Lambda \).

**Axiom 3:** (Subadditivity Axiom) For every countable sequence of events \( \Lambda_i, \ i = 1, 2, \ldots \), we have

\[
M\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i)
\]

In this case, the triplet \( (\Gamma, \mathcal{L}, M) \) is called an uncertainty space.

Then the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) was defined by Liu (2009), producing the fourth axiom of uncertain measure.

**Axiom 4:** (Product Axiom) Let \( (\Gamma_k, \mathcal{L}_k, M_k) \) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( M \) is an uncertain measure satisfying

\[
M\left( \prod_{k=1}^{\infty} \Lambda_k \right) = \prod_{k=1}^{\infty} M_k(\Lambda_k)
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L} \) for \( k = 1, 2, \ldots \), respectively.

The concept of uncertain variable \( \xi \) was introduced by Liu as a measurable function from an uncertainty space \( (\Gamma, \mathcal{L}, M) \) to the set of real numbers.

**Definition 2. (Liu 2007)** An uncertain variable is a measure function \( \xi \) from an uncertain space \( (\Gamma, \mathcal{L}, M) \) to the set of real number. That is, for any Borel set \( B \), the set

\[
\{\xi \in B\} = \{\gamma \in \mathcal{L} | \xi(\gamma) \in B\}
\]

is an event.

**Definition 3. (Liu 2007)** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be uncertain variables, and let \( f \) be a real-valued measurable function. Then \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) is an uncertain variable defined by

\[
\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \ \forall \gamma \in \Gamma
\]
**Definition 4.** (Liu 2007) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is

$$\Phi(x) = \mathcal{M}(\xi \leq x), \quad \forall x \in \mathbb{R}$$  \hspace{1cm} (5)

**Definition 5.** (Liu 2007) An uncertain variable is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \frac{(x - a)}{(b - a)}, & \text{if} \quad a \leq x \leq b \\ 1, & \text{if} \quad x \geq b \end{cases}$$  \hspace{1cm} (6)

denoted by $L(a, b)$ where $a$ and $b$ are real numbers with $a < b$.

**Definition 6.** (Liu 2010a) An uncertainty distribution $\Phi(x)$ is said to be regular if it is continuous and strictly increasing function with respect to $x$ at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to \infty} \Phi(x) = 1.$$  \hspace{1cm} (7)

**Definition 7.** (Liu 2010a) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(a)$ is called the inverse uncertainty distribution of $\xi$.

**Example 1.** The inverse uncertainty distribution of linear uncertain variable $L(a, b)$ is

$$\Phi^{-1}(a) = (1 - \alpha)a + \alpha b.$$  \hspace{1cm} (8)

**Definition 8.** (Liu 2007) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{-\infty}^{+\infty} \mathcal{M}(\xi \geq r)dr - \int_{-\infty}^{0} \mathcal{M}(\xi \leq r)dr$$  \hspace{1cm} (9)

provided that at least one of the two integral is finite.

**Theorem 1.** (Liu 2010a) Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then

$$E[\xi] = \int_{0}^{+\infty} \Phi^{-1}(\alpha)d\alpha$$  \hspace{1cm} (10)

**Theorem 2.** (Liu 2010a) Let $\xi$ and $\eta$ be independent uncertain variables with finite expected values. Then for any real number $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$  \hspace{1cm} (11)

**Theorem 3.** (Liu 2010b) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is a strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$  \hspace{1cm} (12)

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \cdots, \Phi_n^{-1}(1 - \alpha))$$  \hspace{1cm} (13)

**Theorem 4.** (Liu 2010b) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If constraint function $f(x, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and
strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n \), then

\[
M(\xi, \xi_1, \xi_2, \ldots, \xi_n) \leq 0 \quad \alpha
\]

holds if and only if

\[
f(\xi, \Phi^{-1}_1(\alpha), \ldots, \Phi^{-1}_m(\alpha), \Phi^{-1}_{m+1}(1-\alpha), \ldots, \Phi^{-1}_n(1-\alpha)) \leq 0
\]

3. UNCERTAIN TIME SERIES FORECASTING METHOD

Uncertain time series was proposed by Yang & Liu (2019) in 2019 so as to predict the future values based on previously imprecisely observed values that are described by uncertain variables. The basic definition of uncertain time series is as follows.

**Definition 10.** (Yang & Liu 2019) An uncertain time series is a sequence of imprecisely observed values that are characterized in terms of uncertain variables. Mathematically, an uncertain time series is represented by

\[
X = \{X_1, X_2, \ldots, X_n\}
\]

where \( X_t \) are imprecisely observed values (i.e., uncertain variables) at times \( t, t = 1, 2, \ldots, n \), respectively.

After giving the uncertain time series, it is necessary to formulate function relations between the observations at time \( t \) and those at previous times to describe uncertain time series. Generally, the relationship between uncertain variables can be expressed by the following function

\[
X_t = f(X_{t-1}, X_{t-2}, \ldots, X_{t-p}, \epsilon_t) + \epsilon_t, \quad t = p + 1, p + 2, \ldots, n
\]

According to the research (Yang & Liu 2019), the method for modelling uncertain time series is the autoregressive model,

\[
X_t = \varphi_0 + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \cdots + \varphi_p X_{t-p} + \epsilon_t, \quad t = p + 1, p + 2, \ldots, n
\]

where \( \varphi_0, \varphi_1, \ldots, \varphi_p \) are unknown parameters, \( \epsilon_t \) is an uncertain variable, and \( p \) is called the order of the autoregressive model. In order to recognize a good uncertain autoregressive model as a forecast tool for the given data, the following three problems need to be solved which include: the determination of order of uncertain autoregressive model, parameter estimation and the construction of uncertain time series. In the following subsections, all the detailed procedure of using our methodology to make predictions is presented, which can be divided into four stages clearly differentiated as Figure 1.

3.1. The determination of the order of uncertain autoregressive model

During the modeling process of uncertain time series, a fundamental phase is the identification of order of uncertain autoregressive model. Generally, we make predictions about the future to make strategies, which should not only get information from the data before but also get information from the near past, although they might not have the same effect strength. Therefore, it is crucial that finding an appropriate order to determine the lagged variables existing in the model and establish the truly effective model. If the model order is not recognized efficiently, the accuracy of the predictions produced by the model will be compromised. Just like traditional time series analysis, correlation is a very important concept used in analyzing data in the time series. We often identify the model based on the trailing or truncating properties of the autocorrelation coefficient and the partial correlation coefficient. However, for the imprecisely observed values represented by uncertain variables, the traditional statistical method above is problematic. In this subsection, we introduce an order determination method based on the notion of similarity, which is generated from the distance measure defined by Li & Liu (2015).

It can be represented by

\[
D_p(\xi, \eta) = (E[|\xi - \eta|^p])^{1/p}, \quad p > 0
\]
where \( \xi \) and \( \eta \) are uncertain variables. It is easy to understand that the distance between uncertain variables essentially reflects their difference. The greater the distance is, the smaller the similarity is, and vice versa. Here, we just set out one, i.e. \( p = 1 \), as the application for the following definitions.

### 3.1.1. Basic definitions

Based on the above distance measure, the average distance between two uncertain variables in the uncertain time series is defined as below.

**Definition 9.** *Average distance*
Let \( \{X_t\}(t = 1, 2, \cdots, n) \) be an uncertain time series and \( X_1, X_2, \cdots, X_n \) be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertain distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \) respectively, \( m \) denote the experimental order. Then the average distance between uncertain variables \( X_t \) and \( X_{t-k} \) \( (k = 1, 2, \cdots m) \) is defined as

\[
AD_k = \frac{1}{n-k} \sum_{t=k+1}^{n} D_1(X_t, X_{t-k})
= \frac{1}{n-k} \sum_{t=k+1}^{n} |E[X_t - X_{t-k}]|^2, \quad k = 1, 2, \cdots, m
\]  

(20)

To further provide ease of use, the study applies the following definition to make some adjustments.

**Definition 10.** Auto-similarity of uncertain time series

Let \( AD_k(k = 1, 2, \cdots, m) \) be average distance between uncertain variables \( X_t \) and \( X_{t-k} \) \( (k = 1, 2, \cdots, m) \) of uncertain time series \( \{X_t\}(t = 1, 2, \cdots, n) \), \( m \) denote the experimental order. Then the auto-similarity of uncertain time series is defined as

\[
AS_k = 1 - \frac{\text{min}_{1 \leq k \leq m} \{AD_k\}}{\text{max}_{1 \leq k \leq m} \{AD_k\}}, \quad k = 1, 2, \cdots, m
\]  

(21)

It is clear that \( AS_k \in [0, 1], k = 1, 2, \cdots, m \). The Definition 10 shows that the greater \( AS_k \) is, the higher the similarity between the uncertain variables \( X_t \) and \( X_{t-k} \) is.

### 3.1.2. Model order selection algorithm

According to the above definitions, an algorithm for determining the appropriate order of uncertain autoregressive model is presented as follows.

1. **Step 1.** We set \( k = 1 \) as an alternative order.
2. **Step 2.** Determine the confidence level \( \alpha \).
3. **Step 3.** If \( |AS_k - AS_{k+1}| \leq \alpha \), then select the \((k + 1)\)th order to add the set of alternative order numbers. That is, enter into the Step 4.
   - If \( |AS_k - AS_{k+1}| > \alpha \), then choose the \( k \)th order as the optimal order.
4. **Step 4.** Set \( k = k + 1 \) and return to the Step 2.
5. **Step 5.** Optimal order obtained from Step 1 to Step 4 is regarded as the order of uncertain autoregressive model.
6. **Step 6.** If we cannot find the effective order until the predetermined experimental order \( m \) is reached, then let \( k = 1 \).

We want to note again that the proposed algorithm is also presented in Figure 2.

### 3.2. A new parameter estimation method based on uncertain programming

Once the order of the model is determined, we need to estimate the parameters of uncertain autoregressive model to make predictions. Based on the imprecisely observed values, Yang & Liu (2019) investigated the least squares approach to estimate coefficients of uncertain autoregressive model. Different from the previous research, in this subsection, we will propose a new method of parameter estimation based on uncertain programming, which can give more flexibility to the uncertain autoregressive model to make predictions. In general, we hope that the given parameters should make the differences between the predicted values \( \hat{X}_t \) and observed values \( X_t \) as small as possible. In uncertain time series model, let \( X_1, X_2, \cdots, X_n \) be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertain distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Then, the estimation of unknown parameters \( \varphi_0, \varphi_1, \cdots, \varphi_p \) in the uncertain autoregressive model

\[
\hat{X}_t = \varphi_0 + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \cdots + \varphi_p X_{t-p} + e_t, \quad t = p + 1, p + 2, \cdots, n
\]  

(22)
can solve the following programming model

$$\begin{align*}
\min \sum_{t=p+1}^{n} a_t \\
\text{subject to} \\
M((X_t - \hat{X}_t)^2 \leq a_t) \geq \beta_t \\
t = p + 1, p + 2, \ldots, n,
\end{align*}$$

(23)

where $a_t$ is the target variable and $\beta_t \in (0, 1)$ is a given level by the domain experts according to their experience knowledge.
In order to obtain the optimal solution, we need to transform it into an equivalent deterministic model. The following theorem will address this problem.

**Theorem 5.** The model (23) is equivalent to the crisp mathematical programming

\[
\begin{align*}
\min & \sum_{t=p+1}^{n} a_t \\
\text{subject to} & \left( \Phi_t^{-1}(\beta_t) - \varphi_0 - \sum_{i=1}^{p} \varphi_i Y_{t-1}^{-1}(\beta_i) \right)^2 - a_t \leq 0 \\
& t = p + 1, p + 2, \ldots, n,
\end{align*}
\]

where

\[ Y_{t-1}^{-1}(\beta_t, \varphi_i) = \begin{cases} 
\Phi_t^{-1}(1 - \beta_t), & \text{if } \varphi_i \geq 0 \\
\Phi_t^{-1}(\beta_t), & \text{if } \varphi_i < 0 
\end{cases} \]  

for \( i = 1, 2, \ldots, p \).

Proof: It follows from Theorem 4 immediately.

### 3.3. The construction of uncertain time series

Uncertain time series was proposed so as to deal with such forecasting problems where the historical data are not crisp numbers but are imprecisely observed values. Besides, in practical cases, most traditional point data possess uncertainty characteristics due to the measurement errors. For instance, the water demand variables that naturally take a finite set of numerical values varying between a lower and upper bound are regarded as interval-valued variables, which stand for the inaccuracies in measurements. For doing so, each interval-valued variable is assumed as the linear uncertain variable. However, for the same uncertain time series model, the difference of constructed intervals by adopting different ways can result in different forecasting performance. So, how to use an efficient way to choose effective length of interval is especially critical to improve uncertain time series forecasting performance. A key point in determining the proper length of interval is that they should not too large or small. When an effective length of interval is too wide, the prediction results will be meaningless in the uncertain time series. While the length is too small, the uncertain time series will become very close to the traditional time series and the result is not intended. On the other hand, many traditional time series have the momentum to vibrate in a certain period of time. Therefore, in the process of constructing uncertain time series, we should consider the trend information of data of time series itself, which makes the determined interval series more reasonable and really reflects the variation tendency of data of time series. By following the two requirements, in this subsection, we propose a new ratio-based approach to determine the length of interval to obtain the high forecasting accuracy. The step of the algorithm of the method presented can be given as follows:

Step 1. Take the first order of differences between any two consecutive observations \( y_t - y_{t-1} \) for any \( y_t \) and \( y_{t-1} \), \( t = 2, 3, \ldots, n \).

Step 2. Calculate relative differences \( \frac{y_t - y_{t-1}}{y_{t-1}} \) for all \( t = 2, 3, \ldots, n \).

Step 3. Determine the lower and upper bounds of the initial value.

Let

\[ y_1^U - y_1^L = \frac{1}{n} \sum_{t=1}^{n} y_t - \frac{1}{n} \sum_{t=1}^{n} y_t \]  

and

\[ y_1^U + y_1^L = 2y_1 \]

Step 4. Determine the lower and upper bounds on the interval series.
Let
\[ y_i^L = y_{i-1}^L \times (1 + r_t), \]  \hspace{1cm} (28)
and
\[ y_i^U = y_{i-1}^U \times (1 + r_t), \]  \hspace{1cm} (29)
t = 2, 3, \ldots, n,
be the lower and upper bounds for observation \( y_t \) at time \( t, t = 2, 3, \ldots, n \) respectively.

Step 5. Construct the uncertain time series. Following steps above, the interval-valued variables, \( X_t = \{ [y_t^L, y_t^U] : y_t^L, y_t^U \in R, y_t^L < y_t^U \}, t = 1, 2, \ldots, n \), are obtained. We can consider each interval variable \( X_t \) at time \( t \) as the linear uncertain variable with linear uncertain distribution \( L(y_t^L, y_t^U) \). Then, an uncertain time series can be represented as
\[ X = \{ X_1, X_2, \ldots, X_n \}. \]  \hspace{1cm} (30)

4. CASE STUDY

This section presents the application of the proposed methods for water demand forecast in Beijing. In Section 4.1, the location and dataset used in model development are given. Section 4.2 provides the implementations of the proposed uncertain time series model. For the purpose of comparison, the ARIMA model is selected to contrast the forecasting performance, and the classical measure methods are adopted to evaluate the forecast accuracy of the models in Section 4.3.

4.1. Location and dataset

In this work, the study area is located in Beijing. As the capital of China, Beijing is China’s political, cultural, and international communication center and located at the interlaced terrace of North China Plain and Mongolian Plateau. In Beijing, drinking water is mainly supplied by the Yongdinghe and Chaobaihe rivers. As a result of China’s rapid development and dense population, the Beijing’s water demand consumption is increasing rapidly and Beijing is experiencing a shortage of water resources. According to the Beijing Water Authority (BWA), the annual water resources per capita is less than 300 m\(^3\), which is only 12.5% of the national average and far below the internationally recognized minimum standard of 1,000 m\(^3\) per year. This situation of increasing water demands and limited water resource supplies has also become the vital restrictive factor affecting the socio-economic development and environmental health of Beijing for a long time into the future. Therefore, it is particularly important to apply the proposed method to forecast the water demand in Beijing, which contributes to the water resources planning and management in the near future.

All the research dataset of uncertain time series are obtained from Beijing Water Resources Bulletin during the time period between 1988 and 2016. A total of 29 data points are collected and shown in the Table 1. In order to illustrate the effectiveness of the proposed uncertain time series method, the data from 1988 to 2013 are used as an estimation sample to determine the coefficients of the estimation model, while the rest of data are reserved as the hold-out sample, used to test model and access the performance of prediction.

4.2. Methodologies implementations

4.2.1. Construction of uncertain time series

As mentioned previously, water demand data are not crisp numbers but imprecisely observed values, so the data pre-processing is essential. For the purpose of implementation, we utilize the linear uncertain variables to describe the water demand observations and construct the uncertain time series by following the algorithm in the previous subsection.

(1) Take the first order of differences between any two consecutive observations, which are listed in the fourth column of Table 1.
(2) Calculate the relative differences between any two consecutive observations. All the relative differences are listed in the fifth column of Table 1.
(3) Determine the lower and upper bounds of the interval-valued series. Firstly, the initial interval is calculated as follows.
By

\[ y_1^{U} - y_1^{L} = \frac{1}{n} \sum_{t=1}^{n} y_t - \frac{1}{n} \sum_{t=1}^{n} y_t \]

\[ = \frac{1}{26} \sum_{t=1}^{26} y_t - \frac{1}{26} \sum_{t=1}^{26} y_t \]

\[ = \frac{1}{26} \sum_{t=1}^{26} [y_t - 391] \]

\[ = 36 \] (31)

and

\[ y_1^{U} + y_1^{L} = 2 \times y_1 \]

\[ = 848 \] (32)
we have
\[ y_L^1 = 406, \quad y_U^1 = 442 \]  \hspace{1cm} (33)

Then, it is obvious that \( X_1 = [406, 442] \). We consider the initial interval-valued variable \( X_1 \) as the linear uncertain variable with linear uncertain distribution \( L(y_L^1, y_U^1) \). Secondly, by following the Step 3 in Section 3.3, the corresponding interval series can be constructed as the Table 1. Thus, we can obtain an uncertain time series that are characterized in terms of linear uncertain variables, i.e.,
\[ X = \{X_1, X_2, \ldots, X_{26}\} \]  \hspace{1cm} (34)

### 4.2.2. The determination of the model order

In the model order selection phase, different experimental orders are examined based on the definition of auto-similarity of uncertain time series and the best one among them is selected. Generally, we set the maximum lagging order \( m = 5 \). The details of the calculation can be showed in the following paragraphs.

Firstly, we assume that linear uncertain variables \( X_1, X_2, \ldots, X_{26} \) are independent. According to the Definition 9, the average distance of the first order lag of all uncertain variables is calculated as follows
\[
AD_1 = \frac{1}{25} \sum_{t=2}^{26} D_1(X_t, X_{t-1})
\]
\[
= \frac{1}{25} \sum_{t=2}^{26} \left| E\{X_t - X_{t-1}\}\right|^\frac{1}{2}
\]
\[
= \frac{1}{25} \sum_{t=2}^{26} \left[ \int_0^1 (\Phi_{t-1}(\alpha) - \Phi_{t-1}(1 - \alpha)) d\alpha \right]^{\frac{1}{2}}
\]
\[
= 3.2075.
\]  \hspace{1cm} (35)

Similar, we have,
\[
AD_2 = \frac{1}{24} \sum_{t=3}^{26} D_1(X_t, X_{t-2}) = 3.5781,
\]  \hspace{1cm} (36)

\[
AD_3 = \frac{1}{23} \sum_{t=4}^{26} D_1(X_t, X_{t-3}) = 4.2078,
\]  \hspace{1cm} (37)

\[
AD_4 = \frac{1}{22} \sum_{t=5}^{26} D_1(X_t, X_{t-4}) = 4.7491,
\]  \hspace{1cm} (38)

and
\[
AD_5 = \frac{1}{21} \sum_{t=6}^{26} D_1(X_t, X_{t-5}) = 5.1953.
\]  \hspace{1cm} (39)
Then, \( \min_{1 \leq k \leq 5} \{ AD_k \} = AD_1, \max_{1 \leq k \leq 5} \{ AD_k \} = AD_5 \) It follows from Definition 10 that
\[
AS_1 = 1, AS_2 = 0.8136, AS_3 = 0.4968, AS_4 = 0.2245, AS_5 = 0. \tag{40}
\]

By using the algorithm in Section 3.1, we can find the appropriate order.

Let \( \alpha = 0.2 \). Because \( |AS_1 - AS_2| = 0.1864 < 0.2 = \alpha \) and \( |AS_2 - AS_3| = 0.3186 > 0.2 = \alpha \). At the same time \( AS_2 = 0.1836 > 0.4968 = AS_5 \), thus the order of uncertain autoregressive model is \( k = 2 \).

### 4.2.3. The parameter estimation for the uncertain autoregressive model

According to the Section 4.2.2, we obtain the 2-order uncertain autoregressive model
\[
\hat{X}_t = \varphi_0 + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \epsilon_t, \quad t = 3, 4, \ldots, 26. \tag{41}
\]

By using the Theorem 5, we get the following mathematical programming model to estimate the unknown coefficients \( \varphi_i (i = 0, 1, 2) \).

\[
\begin{align*}
\min & \sum_{t=3}^{26} a_t \\
\text{subject to} & \\
& \left((31 + 30 \times \varphi_1 + 31 \times \varphi_2) \times \beta_{26} + 349 - \varphi_0 - 374 \times \varphi_1 - 375 \times \varphi_2\right)^2 \leq a_{26} \\
& \vdots \\
& \left((35 + 38 \times \varphi_1 + 36 \times \varphi_2) \times \beta_{3} + 394 - \varphi_0 - 465 \times \varphi_1 - 442 \times \varphi_2\right)^2 \leq a_{3} \\
& \varphi_0 > 0, \varphi_1 > 0, \varphi_2 > 0.
\end{align*} \tag{42}
\]

Let \( \beta_i = 0.75, (i = 3, 4, \ldots, 26) \). Then we obtain the optimal solution
\[
(\varphi_0, \varphi_1, \varphi_2) = (44.90, 0.81, 0.10), \tag{43}
\]

and the corresponding autoregressive model
\[
\hat{X}_t = 44.90 + 0.81X_{t-1} + 0.10X_{t-2}, \quad t = 3, 4, \ldots, 26. \tag{44}
\]

According to different confidence levels (i.e., 0.75, 0.85, 0.90), different regression models can be established to compare with the existing traditional time series model.

### 4.3. Comparison with the existing method

In this subsection, we apply the proposed uncertain autoregressive model to forecast the water demand of Beijing from 2014 to 2016. Following the above forecasting model (44), the forecasting result is an uncertain variable. However, in most cases, the results we require are often crisp values, so the research uses the expected value of the uncertain variable as the predicted value. The prediction results under different confidence levels (i.e., 0.75, 0.85, 0.90) are presented in the Table 2. In order to further verify the effectiveness of the proposed methodologies, the traditional ARIMA time series method is selected as a competitor to contrast the forecasting performance. According to the historical data of water demand from 1988 to 2015, the result of ARIMA model is represented as follows
\[
\hat{X}_t = X_{t-1} + \epsilon_t \tag{45}
\]

The performances of the models are evaluated based on the classical measure methods, i.e., average relative error (ARE) and the total absolute error (AE). Obviously, lower ARE and AE values lead to better performance. The definitions of all these
performance criteria are represented by

$$ARE = \frac{1}{s} \sum_{j=1}^{s} \left| \frac{\hat{y}_j - y_j}{y_j} \right|$$  \hspace{1cm} (46)$$

and

$$AE = \sum_{j=1}^{s} |\hat{y}_j - y_j|$$  \hspace{1cm} (47)$$

where $s$ is the total number of data needed to predict, $\hat{y}_j$ and $y_j$ denote the predicted value and the actual observed value, respectively.

Therefore, the prediction performance are shown in Table 3 and Figure 3. From the experimental results obtained, it can be concluded that the proposed uncertain time series forecasting method has the better forecasting performance than the ARIMA method under the considered levels.

As far as the comparison between the proposed method under the 0.75 and 0.85 confidence levels and the ARIMA method is concerned, the former outperforms the latter in all cases. The total prediction error is reduced by 78.15 and 17.26% respectively, and the average relative error by 78.36 and 17.38% respectively. In addition, for the prediction error of each observation, the proposed method under the 0.75 and 0.85 levels is smaller than the ARIMA method. Especially for the associated 0.75 confidence level, the improvements of forecasting performance are more obvious.

When considering the comparison between the proposed method under the 0.90 confidence level and the ARIMA method in all cases, we can see that, the maximum prediction error of the proposed method is a little higher than the ARIMA method. Overall, the former almost wins. The total prediction error is reduced by 0.59% and the average relative error by 0.68%. This reduction is crucial in the planning and management of water supply systems.

Apart from the statistical criteria discussed above, we implement the forecasting trend to evaluate the performance of the above methods. According to the current trend, the Beijing’s total water demand has been increasing from 2014 to 2016. However, the prediction trend of the ARIMA method was declining. This is not in accordance with the reality of the Beijing’s
total water demand during 2014 and 2016. So, this predicted results may not provide support to the water resource management in the near future. It is worthy to note that the predicted results of the proposed method could reflect the realistic water demand trend in the short term and contributes to the decision makers devise reasonable management scheme.

5. CONCLUSION

In this study, we presented a modified time series method for demand estimation of water resources in Beijing. Considering that the uncertainty of water demand in real life, we attempted to combine the uncertainty theory with time series model, called uncertain time series, to handle the above problems. In the presented method, we employed the uncertain autoregressive model to describe uncertain time series for predicting the future values. First, the auto-similarity of uncertain time series, as a principle of justifiable recognition, is defined and the identification algorithm of determining the optimal model order is proposed, which enables estimate the correct parameters of the model. Second, we propose an uncertain programming approach for estimating the parameters of model. Then, the imprecisely observed values are assumed as the linear uncertain variables and a ratio-based method is presented for constructing the uncertain time series. Finally, we tested the performance of the prosed model and the traditional time series model (ARIMA) based on the statistical criteria. The results demonstrated that the proposed model provided much better accuracy over the traditional model mentioned above for water demand predictions. The possible reason is that the traditional model cannot effectively handle the imprecisely observed values, this allows the possibility of the loss of effective information, which leads to the reduction of prediction accuracy.

Although the proposed uncertain autoregressive model has greatly improved the traditional water demand time series method, there are still some limitations needed to improved such as the determination of the model order and the construction of the uncertain time series. In the future, we will further study the algorithm optimization of model order, and provide
better solutions for the uncertain autoregressive model applications which improve the accuracy of forecast. On the other hand, we only investigate the construction of liner uncertain time series due to the interval-valued data are encountered frequently in multiple situations. Further study may attempt to construct the normal uncertain time series to expand the application field of the model.

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DATA AVAILABILITY STATEMENT
All relevant data are included in the paper or its Supplementary Information.

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