I argue that the complete partition function of 3D quantum gravity is given by a path integral over gauge-inequivalent manifolds times the Chern-Simons partition function. In a discrete version, it gives a sum over simplicial complexes weighted with the Turaev-Viro invariant. Then, I discuss how this invariant can be included in the general framework of lattice gauge theory (qQCD$_3$). To make sense of it, one needs a quantum analog of the Peter-Weyl theorem and an invariant measure, which are introduced explicitly. The consideration here is limited to the simplest and most interesting case of $SL_q(2)$, $q = e^{i\pi k/2}$. At the end, I dwell on 3D generalizations of matrix models.
1 Introduction

During the last few years, considerable progress has been made in our understanding of 2D quantum gravity and string theory (see review [1] and references therein). What helped greatly to fight the problem was the fortunate interplay of the methods of conformal field theory and the computational power of matrix models. For those who tries to think of quantum gravity seriously the next step has naturally been the path integral over 3D manifolds. It is not a priori doomed-to-fail enterprise. Indeed, although the problem is really a hard one, some interesting results have already been obtained. As well as in the 2D case, there are essentially two approaches. The first starts with a continuous formulation trying to make sense of a path integral over metrics. The main achievement on this way has been the connection with the Chern-Simons theory established by E.Witten [2]. The second approach is based completely on lattice experience. Here the path integral is substituted by a sum over all simplicial (or another kind of) complexes. The gained advantage is the finiteness of all involved quantities and relative simplicity, which allows for numerical investigations. However, the main problem, native to all lattice models, is what kind of continuum limit (if any) can be reached in every particular case?

In the present paper, I try to establish a connection between these two approaches, paying more attention to the second one, however.

In Section 2 I remind a reader the basic notions of 3D general relativity and describe its connection with the Chern-Simons theory.

Section 3 is devoted to 3D simplicial gravity. I formulate the model and review some results of numerical investigations.

In Section 4 I define qQCD$_3$ and show that its weak-coupling limit is related to the Turaev-Viro invariant.

In Section 5 a model which can be regarded as a 3D generalization of the one-matrix model are introduced.

Section 6 contains some general remarks.
2 3D gravity and Chern-Simons interpretation

The partition function in Euclidian quantum gravity is intuitively defined as a sum (path integral) over all manifolds weighted with the exponential of a reparametrization invariant action

\[ \mathcal{P} = \sum_{\mathcal{M}} e^{S} \]  

By definition, the manifold is a topological space which can be globally covered with local coordinate systems. In other words, every its point has an open vicinity allowing for a continuous one-to-one map into \( \mathbb{R}^3 \). On a manifold, one can define functions, vector fields, forms and tensors. To make sense of the partition function (1), a metric tensor, a volume form and an affine connection are needed. The metric tensor is a scalar bi-linear symmetric function on vectors, i.e., a second-rank contra-variant tensor \( g_{ij} \). The matrix \( g_{ij} \) has to be invertible: \( g_{ij}g^{jk} = \delta^k_i \). If \( g_{ij} \equiv 0 \) on some sub-manifold, it should be regarded as non-compactness. Without metric one cannot define the functional integral measure.

The volume form is some fixed 3-form \( \tilde{V} \). It is always convenient to make it compatible with the metric. Then, in coordinates,

\[ \tilde{V} = \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3 \]  

To choose a coordinate basis means to fix 3 mutually commutative vector fields \( \hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3 : [\hat{\partial}_i, \hat{\partial}_j] = 0 \). Sometimes, it is convenient to have a non-coordinate basis \( \{ \hat{e}_a \} \):

\[ [\hat{e}_a, \hat{e}_b] = C^c_{ab} \hat{e}_c \]  

Let me choose it such that

\[ g_{ij} e^i_a e^j_b = \delta_{ab} \]  

where \( e^i_a \) are the components: \( \hat{e}_a = e^i_a \hat{\partial}_i \). I shall refer to them as the dreibein.

\(^{a}\)In what follows, for convenience, I denote forms with the tilde and vector fields with the hat, e.g., \( \hat{\partial}_i \equiv \frac{\partial}{\partial x^i} \).
To introduce the Riemann tensor one needs the notion of an affine connection, which defines rules of a parallel transport of vectors: \( \nabla \hat{e}_a \hat{e}_b = \omega^c_{ab} \hat{e}_c \).

If one uses forms \( \tilde{e}^a = e^a_i dx^i \) and \( \tilde{\omega}^a_b = \omega^a_{bi} dx^i \), one can introduce the Riemann tensor \( \tilde{R}^a_b = \frac{1}{2} R^a_{bc} dx^c \wedge dx^b \) and the torsion \( \tilde{T}^a = \frac{1}{2} T^a_{ij} dx^i \wedge dx^j \) in the most elegant way (Cartan’s structural equations)

\[
\tilde{R}^a_b = \frac{1}{2} \left( \tilde{\omega}^c_b - \tilde{\omega}^c_a \right) \wedge \tilde{e}_c \tag{5}
\]

\[
\tilde{T}^a = \frac{1}{2} \left( \tilde{\omega}^a_b \right) \wedge \tilde{e}^b \tag{6}
\]

If the torsion vanishes, the connection is said to be symmetric. In this case, it is determined by the commutator \( [\tilde{\omega}_{ab}, \tilde{\omega}_{cd}] = 0 \)

\[
\omega^{c}_{ab} = \frac{1}{2} \left( C_{cb}^a + C_{ca}^b - C_{ab}^c \right) \tag{7}
\]

The Einstein-Hilbert action can be written in the form

\[
S = \lambda \int \epsilon_{abc} \tilde{R}^a_b \wedge \tilde{e}^c + \beta \int \tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \tag{8}
\]

Witten suggested to consider the dreibein and the Levi-Civita connection as the gauge variable:

\[
A_i = e^a_i P_a + \omega^a_{bi} J_{ab} \tag{9}
\]

taking values in the \( ISO(3) \) Lie algebra (if the signature is Euclidian and the cosmological constant is zero):

\[
[J_{ab}, J_{cd}] = \delta_{ac} J_{bd} + \delta_{bd} J_{ac} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc}
\]

\[
[J_{ab}, P_c] = P_a \delta_{bc} - P_b \delta_{ac} \tag{10}
\]

\[
[P_a, P_b] = 0
\]

with the invariant metric on the algebra: \( \langle P_a, P_b \rangle = \langle J_{ab}, J_{cd} \rangle = 0, \langle P_a, J_{bc} \rangle = \epsilon_{abc} \).

The obvious problem here is: what meaning do we give to the generators? If \( P_a \)’s are to represent vector fields, \( \tilde{P}_a \), forming a coordinate basis (they commute), then \( [\epsilon^a_i \tilde{P}_a, \epsilon^b_j \tilde{P}_b] \) is not zero except for the case when the space is

\[b\text{The groups of indices } abc \text{ and } ijk \text{ belong to different bases!} \]
flat and the dreibein appears as a coordinate transformation: $e_i^a = \frac{\partial y^a}{\partial x^i}$. In a curved space, the Lie algebra generators have an indefinite meaning.

However, the construction is not so restrictive as might seem from this consideration.

Let $\tilde{v} = v^i \hat{\partial}_i = \gamma^a \hat{e}_a$ be an infinitesimal vector field. The variation of the basis $\hat{\partial}_i$ under the diffeomorphism generated by $\tilde{v}$ is given by the Lie derivative

$$\mathcal{L}_{\tilde{v}} \hat{\partial}_i \equiv [\tilde{v}, \hat{\partial}_i] = \nabla_{\tilde{v}} \hat{\partial}_i - \nabla_{\hat{\partial}_i} \tilde{v} = (-\nabla_i \gamma^a + \omega^a_{bj} v^j e^b_i + v^j e^a_{i,j}) \hat{e}_a$$

where the comma means the derivative with respect to $x^j$. The first equality holds if the torsion tensor identically vanishes. So, from the viewpoint of the fixed non-coordinate basis $\hat{e}_a$, the variation of the basis $\hat{\partial}_i$ consists of

(i) a “gauge transformation” $\nabla_i \gamma^a$,
(ii) a “Lorentz rotation” $\omega^a_{bj} v^j e^b_i = \tau^a_{jb} e_i^j$ and
(iii) a coordinate shift $v^j e^a_{i,j} \approx e^a_i (x + v) - e^a_i (x)$. The last term can be removed by “pulling back” $e^a_i$ to its initial point in the $x$-frame as a scalar function.

Of course, it just repeats the famous Witten’s argument \[2\] that diffeomorphisms generated by vector fields can be regarded on-shell as gauge transformations of the field \[9\]. Maybe, it should be stressed here that one is restricted to reparametrizations, which are not the most general transformations. In particular, they do not affect the commutators \[3\].

The complete algebra of vector fields is infinite dimensional, since the structure constants $C_{bc}^a$ are arbitrary functions of coordinates. Hence, it cannot as a whole be reduced to any finite dimensional symmetry, if one insists on the interpretation of its generators as vector fields. However, we have seen that reparametrizations can be regarded as the gauge transformations. One can, in principle, get rid of them by fixing a gauge and pulling out of the path integral a volume they produce. For compact manifolds, this volume gives a topological invariant (up to some trivial (but maybe infinite) factor). Indeed, one can choose an arbitrary background metric, and the most convenient choice is a solution to the Einstein equation (classical vacuum). In this case, one finds an integral over flat connections. Witten has noticed that, if one considers the dreibein and connection as independent variables, the Riemann-Hilbert action \[8\] takes the form of the Chern-Simons one for the group $SO(4)$. In this case, the vanishing torsion and the Einstein equation
are implied by the equations of motion and one finds that the gauge volume should be given by the Chern-Simons partition function. Off-shell, of course, any equivalence between diffeomorphisms and the gauge transformations disappears.

The non-renormalizability of 3D gravity means that one should work within a regularization scheme, the choice of which can be crucial (i.e., answers will vary from scheme to scheme drastically).

3 Quantum Regge calculus

The heuristic consideration of the previous section serves to support the following substitution for the path integral over all 3D geometries:

\[
P = \sum_{\text{topologies}} I_{TV} \sum_C e^S
\]  

(12)

where the first sum goes over all topologies; \( \sum_C \) is the sum over all simplicial complexes of a given topology; \( I_{TV} \) is the Turaev-Viro invariant \( [3] \); \( S \) is a lattice action, which can be taken in the form

\[
S = \alpha N_1 - \beta N_3
\]  

(13)

\( N_k \) is the number of \( k \)-dimensional simplexes in a complex.

The Turaev-Viro invariant is the most reasonable substitution for the gauge volume. For a negative cosmological constant, one finds \( SO(4) \) Chern-Simons theory and, as \( SO(4) = SU(2) \times SU(2) \), \( I_{TV} \) seems to be the most appropriate candidate \( [4] \). Its “classical” limit was investigated long ago by Ponzano and Regge \( [5] \) in the framework of the Regge calculus \( [6] \). Provided a triangulation is fixed, it describes an integral over lengths of all links with a weight equal to an exponential of the discretized Einstein-Hilbert action. The Turaev-Viro invariant in this context may be regarded simply as a regularization of the Ponzano-Regge construction.

As all lengths are included in \( I_{TV} \), we can choose every tetrahedron in \( \sum_C \) to be equilateral. This sum serves as a natural regularization of the path integral over classes of gauge-inequivalent manifolds.

Using reparametrizations, one can make lengths of the dreibein vectors equal to unity, three remaining local degrees of freedom being angles between them. As usual, on a lattice, one should work with a group rather than an
algebra. It means that, instead of dreibeins, their integral curves have to be considered. In a discrete version, one fixes a finite number of the curves going from every vertex and associate them with links of a lattice. It is convenient to make lengths of all links equal to one another. In simplicial complexes, angles between them are quantized, which leads to a quantized total curvature. The Regge calculus gives the expression for it

\[ \int d^3x \sqrt{g}R = a \left( 2\pi N_1 - 6N_3 \arccos \frac{1}{3} \right) \]  

(14)

\( a \) is a lattice spacing.

For manifolds, the Euler character vanishes \( \chi = N_0 - N_1 + N_2 - N_3 = 0 \). Together with the constraint \( N_2 = 2N_3 \), it implies that a natural action (linear in the numbers of simplexes) depends on two free parameters which should be related to bare cosmological and Newton constants.

To simulate all geometries, one has to sum over all possible complexes. Indeed, if a triangulation is fixed, commutators of lattice shifts (analogous of the structure constants in Eq. (3)) are fixed. Fluctuating geometry assumes a fluctuating lattice.

If a topology is fixed, the sum over simplicial complexes can be investigated numerically. Any two complexes of the same topology can be connected by a sequence of moves shown in Figure 1. The first move is called the triangle-link exchange: the common triangle of two tetrahedra on the left of Figure 1(a) is removed and three new triangles sharing the new link appear on the right. It increases (the inverse one decreases) the number of tetrahedra by 1. The second move consists in the subdivision of a tetrahedron: 4 new tetrahedra fill an old one. The inverse move is seldom possible. However, to perform it, one can always decrease the coordination number of a vertex by applying the triangle-link exchange.

Monte-Carlo simulations using these moves as basic "infinitesimal steps" appear to be quite efficient.

I do not intend to give a review of the numerical results here. An interested reader is referred to the original papers [7, 8, 9]. However, a few words should be said. All simulations so far have been carried out for the spherical topology of complexes. It appears that the number of spherical complexes of a given volume, \( N_3 \), is exponentially bounded as a function of \( N_3 \) for an arbitrary value of \( \alpha \). It means that the definition (12) is reasonable and \( P \) hopefully has an appropriate continuum limit.
One of the most interesting observations is the resolution of the problem of the unboundedness of the Riemann-Hilbert action within the discrete model. As Eq. (13) is linear in $N_1$, it seems that most probable configurations should be those having the maximum mean curvature, but they are surely lattice artifacts. However, it happened that, at $N_3$ and $\alpha$ fixed, the probability distribution for $N_1$ has roughly speaking the gaussian shape 

$$P_{N_3,\alpha}(N_1) \approx e^{-\frac{(N_1-\langle N_1 \rangle)^2}{2\sigma^2}}$$  

It means that, varying $\alpha$, one just shifts a position of the maximum. Moreover, in Refs. [8], a first order phase transition was found at some critical value, $\alpha_c$. In the “hot” phase ($\alpha < \alpha_c$), crumpled manifolds dominate the partition function. This phase is clearly unphysical. In the “cold” phase ($\alpha > \alpha_c$), it happens that 

$$\langle N_1 \rangle(N_3) = c_1 N_3 + c_2$$ 

is a linear function; $c_1$ is a constant smoothly depending on $\alpha$. Hence, the mean curvature per unit volume makes sense in the large volume limit. However, after the naive rescaling, one finds that its value tends to the infinity in the continuum limit. But, the total curvature can not be regarded as an
observable in quantum gravity. In the Einstein-Hilbert action, there are two terms with dimensionful coupling constants in fronts of them. After a regularization, one finds the action (13) (or similar) where the total curvature has lost its individuality and is mixed with the volume. Let us imagine a kind of renormalization group procedure: one increases a cut-off and integrate over fluctuations inside the blocks. The additive nature of the total curvature means that it should undergo an additive renormalization as well as a multiplicative one. It is natural to kill the first by shifting the cosmological constant. Therefore, the mean value of the total curvature is scheme-dependent and only fluctuations make sense.

Four dimensional numerical simulations show a similar picture [10]. The phase transition there is, presumably, of the second order which might be an evidence for graviton-like (i.e., long-range) excitations in the system.

Here, the following comment is in order. One can easily obtain a continuous manifold from a simplicial complex by using a piece-wise linear approximation and then to smooth it. In three dimensions, any continuous manifold allows for a unique differentiable structure and vice versa. It means that the continuous and discrete models are hopefully equivalent. In four dimensions, the situation is much more complicated [11] and it is unclear whether an entropy of smooth manifolds can be correctly estimated within a lattice approximation. However, simplicial gravity is interesting in its own rights. One can simply say that, at the quantum level, the notion of the continuous manifold is more fundamental than of the smooth one.

4 q-deformed lattice gauge theory (qQCD$_3$)

In this section, I would like to show that the Turaev-Viro invariant can be interpreted as a lattice gauge model (although a not quite standard one). Let me start with reminding basic facts about lattice QCD [12].

Given a $d$-dimensional lattice, a gauge variable $g_{\ell}$ taking values in a compact group $G$ is attached to each 1-dimensional link, $\ell$, and the Boltzmann weight,

$$w_{\beta}(x_f) = \sum_R d_R \chi_R(x_f)e^{-\beta C_R},$$

(17)

to each 2-dimensional face, $f$. The argument is a holonomy along the face,
i.e., the ordered product of gauge variables along a boundary, $\partial f$, of the face $f$:

$$x_f = \prod_{k \in \partial f} g_k$$  \hspace{1cm} (18)

In Eq. (18), every factor is taken respecting an orientation of links and faces. The change of the orientation corresponds to the conjugation $g_k \rightarrow g_k^+$ (or $x_f \rightarrow x_f^+$).

By a lattice I mean a cell (polyhedral) decomposition of a $d$-dimensional manifold such that any cell can enter in a boundary of another one only once, and every two cells can border upon each other along only one less dimensional cell. Simplicial complexes and their duals obey this restriction by definition. In eq. (17), $\sum_R$ is the sum over all irreps of the gauge group $G$; $\chi_R(x_f)$ is the character of an irrep $R$; $d_R = \chi_R(I)$ is its dimension; $C_R$ is a second Casimir and $\beta$ is a number. The construction makes sense for compact groups when unitary finite dimensional irreps span the regular representation. Therefore, $R$ is always a discrete index. The choice (17) provides that $w_\beta(x_f)$ becomes the group $\delta$-function when $\beta \rightarrow 0$:

$$w_0(x_f) = \delta(x_f, I)$$  \hspace{1cm} (19)

The partition function is defined as the integral over all field configurations:

$$Z_\beta = \int_G \prod_\ell dg_\ell \prod_f w_\beta(\prod_{k \in \partial f} g_k)$$  \hspace{1cm} (20)

where $dg$ is the Haar measure on the group $G$.

Now, we would like to make the gauge group quantum\footnote{In this context, the word “quantum” may be misleading, but it has already become standard having actually supplanted the term “q-deformed.”}. The simplest example of quantum group is $GL_q(2)$ elements of which can be defined as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$  \hspace{1cm} (21)

where

$$ab = qba \quad bd = qdb \quad bc = cb$$

$$ac = qca \quad cd = qdc \quad ad - da = (q - q^{-1})bc$$  \hspace{1cm} (22)
The matrices can be multiplied. If elements of both $g_1$ and $g_2$ obey Eq. (22) and are mutually commutative, the elements of the product obey (22) as well [13]. Therefore, matrices on different links of a lattice have to commute with one another in the tensor product (as well as with matrices performing gauge transformations).

The determinant

$$\text{Det}_q g = ad - qbc$$

is central, therefore, one can put it equal to 1. In this way, one arrives at $SL_q(2)$, which has two real forms: $SU_q(2)$, for real $q$, and $SL_q(2, R)$, for $|q| = 1$ [13].

The relations (22) imply the existence of the $R$-matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and the $RTT = TTR$ equation

$$R g_1 \otimes g_2 = g_2 \otimes g_1 R$$

$R$ itself obeys the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

Indices show at which positions in the tensor cube $V \otimes V \otimes V$ acts the $R$-matrix.

For classical gauge groups the self-consistency of the model follows from the Peter-Weyl theorem stating that the algebra of regular functions on a compact group is isomorphic to the algebra of matrix elements of finite dimensional representations. The quantum version of this theorem was proven for real $q$’s in Refs. [14]. In this case there is the one-to-one correspondence between representations of $SU_q(N)$ and $SU(N)$, and the notion of the matrix element is naturally generalized.

Therefore, by the space of functions, one may mean a vector space spanned by matrix elements of irreps, $T_{j,\alpha\beta}(g)$. Eq. (21) can be regarded as the fun-
damental representation. Matrix elements always obey the $RTT = TTR$ equation (25) and, by definition,

$$T_{j,\alpha\beta}(gh) = \sum_{\gamma=-j}^{j} T_{j,\alpha\gamma}(g) T_{j,\gamma\beta}(h)$$

(27)

The next ingredient is the integral, whose existence is postulated. It is defined simply as

$$\int dg ~ T_j(g) = \delta_{j,0}$$

(28)

i.e., whatever is integrated the answer is always zero except for the trivial representation, which is just a constant.

If one has a product of functions, one can always re-expand products of matrix elements by using Clebsch-Gordan coefficients:

$$T_{j_1,\alpha\beta}(g) T_{j_2,\gamma\delta}(g) = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} \sum_{\sigma,\varepsilon=-j_3}^{j_3} \langle j_1\alpha, j_2\gamma|j_3\sigma\rangle T_{j_3,\sigma\varepsilon}(g) \langle j_3\varepsilon|j_1\beta, j_2\delta\rangle$$

(29)

Applying this equation successively one can, in principle, reduce an arbitrary integral to the basic one (28).

The last ingredient is the character entering the definition of the weight (17). It should be said that this notion is missing for $SL_q(2)$. If one naively defines it as the quantum trace of a matrix element,

$$\chi_j(g) \overset{?}{=} \text{Tr}_q T_j(g) = \sum_{\alpha=-j}^{j} q^\alpha T_{j,\alpha\alpha}(g)$$

(30)

then one finds that $[\chi_j(gh), \chi_i(gf)] \neq 0$. It seems to be impossible to $q$-deform the partition function (20) in a self-consistent way simply starting with this definition! It is a manifestation of the fact that qQCD$_D$ does not exist at arbitrary $D$.

On the other hand, the quantum dimension is equal to the quantum trace of the identity operator:

$$[2j + 1] \equiv \sum_{\alpha=-j}^{j} q^\alpha = \frac{q^{j+\frac{1}{2}} - q^{-j-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

(31)
However, to define the partition function (20) in the quantum case, one does not actually need the notion of the character!

The profound correspondence between quantum groups and links of knots\footnote{I use the term “link” to denote 1-dimensional simplexes as well hoping it should not lead to misunderstanding.} suggests that the most adequate way to define $q$QCD would be to connect all involved notions with certain geometric objects. After a projection onto a plane, the partition function can be given a meaning by putting into correspondence quantum-group quantities to all geometrical elements. If one takes another plane, quantum-group symmetries should provide the independence of the construction from a way of projection. I shall follow closely Ref. [15]. The basic notion is the tangle, which is defined as follows. One takes a spherical ball inside which there are a number of oriented loops and segments whose ends lie on the boundary of the ball. They are all colored with $SL_q(2)$ representations. One puts into correspondence to every tangle an operator $O$ acting in the tensor product of representation spaces $V_j \otimes \ldots \otimes V_{j_n}$, (if there are $n$ segments colored $j_1, \ldots, j_n$; their orientations show the direction of the action of $O$):

\[ O_{j_1, \alpha_1; \ldots; j_n, \alpha_n} \equiv O_{j_1, \alpha_1; \ldots; j_n, \alpha_n} \]  

For example, if there is only one segment and no loops, one finds the $\delta$-function:

\[ \delta_{\alpha, \beta} \equiv \delta_{\alpha, \beta} \]
The $R$-matrix distinguishes between under- and over-crossings:

\[ \equiv R = \sum_i a_i \otimes b_i \quad (34) \]

\[ \equiv R^{-1} = \sum_i b_i \otimes s(a_i) \quad (35) \]

where $s$ is the antipod in the $\mathcal{U}_q(sl(2))$ Hopf algebra. The Clebsch-Gordan coefficients are represented as the 3-valent vertices

\[ \equiv \langle j_3 \gamma | j_1 \alpha, j_2 \beta \rangle \quad (36) \]

\[ \equiv \langle j_1 \alpha, j_2 \beta | j_3 \gamma \rangle \quad (37) \]

Matrix elements can be drawn as

\[ \equiv T_{j,\alpha\beta}(g) \quad (38) \]
The Yang-Baxter and $RTT = TTR$ equations take the familiar graphical forms

\[ \begin{array}{c|c|c} \hline & & \\
\hline & & \\
\hline \end{array} = \begin{array}{c|c|c} \hline & & \\
\hline & & \\
\hline \end{array} \quad (39) \]

and

\[ \begin{array}{c|c|c} \hline & & \\
\hline & & \\
\hline \end{array} = \begin{array}{c|c|c} \hline & & \\
\hline & & \\
\hline \end{array} \quad (40) \]

One needs also the quantum trace of an operator, which is equivalent to the closure of a tangle

\[
\begin{array}{c|c|c} \hline & & \\
\hline & & \\
\hline \end{array} \equiv \sum_{\alpha_1 = -j_1}^{j_1} \cdots \sum_{\alpha_n = -j_n}^{j_n} \prod_{i=1}^{n} q^{\alpha_i} O_{j_1,\alpha_1;\ldots;j_n,\alpha_n \alpha_n} \quad (41)
\]

To each link of the lattice, one puts into correspondence an integral of a product of matrix elements, the number of which is equal to the number of faces incident to the link. One can associate a tangle with every such integral. It means a cell decomposition of the manifold. The partition function can be constructed by connecting these tangles together or, equivalently, by gluing up the 3-cells. There appears an index loop going along a boundary of every face. In three dimensions, there is a natural cyclic order of faces sharing the same link. The index loops have to be ordered according to it. After that the partition function can be unambiguously defined.
If \( q = e^{i \frac{2\pi}{k+2}} \), one has to restrict all indices to the fusion ring: \( j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2} \). In this case the following tangle can serve as the definition of the matrix element

\[
T_{j,\alpha\beta}(g) \equiv \begin{array}{c}
\ \ \ j,\alpha \\
\downarrow \downarrow \hspace{1cm} \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ j,\beta \\
\ \ \ j,\beta \\
\end{array}
\]

The tensor product of matrix elements looks as

\[
T_{j_1,\alpha_1\beta_1}(g)T_{j_2,\alpha_2\beta_2}(g) \equiv \begin{array}{c}
\ \ \ j_1,\alpha_1 \\
\downarrow \\
\ \ \ j_2,\alpha_2 \\
\downarrow \\
\ \ \ j_2,\beta_2 \\
\downarrow \\
\ \ \ j_1,\beta_1 \\
\end{array}
\]

and the integral takes the form of the finite sum

\[
\int dg \ T_{j,\alpha\beta}(g) \equiv d_0 \sum_{i=0}^{k/2} d_i \left( \begin{array}{c}
\ \ \ j,\alpha \\
\downarrow \\
\ \ \ j,\beta \\
\end{array} \right)^i = \\
\frac{2 \sin \frac{\pi}{k+2}}{k+2} \sum_{i=0}^{k/2} \sin \frac{\pi(2i+1)}{k+2} \frac{\sin \frac{\pi(2i+1)(2j+1)}{k+2}}{\sin \frac{\pi(2j+1)}{k+2}} \delta_{\alpha,\beta} = \delta_{j,0} \delta_{\alpha,0} \delta_{\beta,0}
\]

where \( d_j \) is the quantum dimension conveniently normalized:

\[
d_j = \sqrt{\frac{2}{k+2} \sin \frac{\pi(2j+1)}{k+2}}
\]

To prove Eq. (44), I used results of Reshetikhin and Turaev [15]. My claim is that it can be regarded as the definition of the integral on the fusion ring of \( SL_q(2) \), \( q = e^{i \frac{2\pi}{k+2}} \).

In addition to the fusion ring irreps \( \{V_j\}, \ j = 0, \frac{1}{2}, \ldots, \frac{k}{2}; \ U_q(sl(2)) \) has a number of representations having the vanishing quantum dimension [16]: \( \{I_p\}, \ p = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots, \frac{k+1}{2} \). Representations \( I_p \) for \( 0 \leq p \leq \frac{k+1}{2} \) although not irreducible are indecomposable.
The tensor product of two irreps from the fusion ring has the following decomposition

\[ V_i \otimes V_j = \left( \bigoplus_{m=|i-j|} V_m \right) \oplus \left( \bigoplus_{\frac{1}{2} \leq p \leq \frac{k+2}{2}} \oplus_{\frac{m}{m+\frac{k-j}{m+1}}} I_p \right) \]  

(46)

The set of representations \( \{ I_p \} \) forms an ideal

\[ \{ V_j \} \otimes \{ I_p \} \subset \{ I_p \} \quad \{ I_p \} \otimes \{ I_p \} \subset \{ I_p \} \]  

(47)

As was proven by Reshetikhin and Turaev \([15]\), the closure of any tangle vanishes if at least one representation of this type appears in it. More precisely, they have shown that any \( \mathcal{U}_q(sl(2)) \)-linear operator acting in \( \{ I_p \} \) has the vanishing quantum trace. Obviously, it holds for operators obtained by cutting an internal line in an arbitrary closed tangle. As it takes place for any line, colors from the set \( \{ I_p \} \) never appear. Therefore, when all index loops are closed, these representations can be simply ignored. Thus, one has the following orthogonality property

\[ \int dg \, T_{j_1, \alpha_1 \beta_1}(g) T_{j_2, \alpha_2 \beta_2}(g) = d_0 \sum_{i=0}^{k/2} d_i \]  

(48)

which allows for a Fourier decomposition of an arbitrary function spanned by matrix elements from the fusion ring. It gives an analog of the Peter-Weyl theorem. A reader must realize that Eqs. (44) and (48) do not hold for \( \{ I_p \} \) representations. For self-consistency, all greek indices have to be summed over to form a link of 3-valent graphs and loops. All equalities between tangles have to be understood as taking place after closing with an arbitrary tangle.
So, we arrive at the following definition of qQCD$_3$ partition function on a 3-manifold $\mathcal{M}$ [17]:

$$Z_{\beta}(\mathcal{M}) = d_0^{N_2+N_0-2} \prod_{f=1}^{N_2} \prod_{j_f} \{d_{j_f} c_{j_f}(\beta)\} \sum_{\ell=1}^{N_1} d_{j_{\ell}} J_{\{j_{\ell}\},\{j_f\}}(\mathcal{L})$$

(49)

where $J_{\{j_{\ell}\},\{j_f\}}(\mathcal{L})$ is the Jones polynomial for a link $\mathcal{L}$ defined by a cell decomposition of $\mathcal{M}$. This link consists of $N_1+N_2$ unframed loops colored with sets of representations $\{j_{\ell}\}$ and $\{j_f\}$. Loops from the first set, $\{j_{\ell}\}$, go around 1-cells (links) pinching bunches of loops from the second set, $\{j_f\}$, which go along boundaries of 2-cells (faces); $c_{j_f}(\beta)$’s are numbers (weights). If all $c_j \equiv 1$, $Z_0$ is a topological invariant. In this case, the partition function (49) is obviously self-dual with respect to the Poincaré duality of complexes.

Eq. (49) is just a particular implementation of the general Reshetikhin-Turaev construction [15]. However, the link $\mathcal{L}$ here is not related to a surgery representation of the manifold.

In order to establish a connection with the Turaev-Viro invariant, let us consider lattices dual to simplicial complexes. Their 1-skeletons are 4-valent graphs and exactly 3 faces are incident to each link giving the integral of 3 matrix elements for every triangle in a simplicial complex:

$$\int dg \ T_{j_1,\alpha_1\beta_1}(g) T_{j_2,\alpha_2\beta_2}(g) T_{j_3,\alpha_3\beta_3}(g) = \frac{d_0}{d_{j_3}} \sum_{\gamma_1,\gamma_2,\gamma_3} \{j_{\ell_1} j_{\ell_2} j_{\ell_3} \}$$

(50)

The right hand side of eq. (50) is the product of two 3-$j$ symbols. Summing over lower indices one gets a Racah-Wigner 6-$j$ symbol

$$\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \} = \frac{d_0^2}{d_{j_6} d_{j_2} d_{j_5}}$$

(51)
inside each tetrahedron of a simplicial complex. Representation indices, \(j_f\), are attached to its 1-simplexes, \(f\), (i.e. faces of the dual lattice). The partition function \(Z_0\) can be written then in the Turaev-Viro form

\[
Z_0 = d_0^{N_1-N_2-2} \sum_{\{j_f\}} \prod_{f=1}^{N_2} d_{j_f} \prod_{t=1}^{N_0} \left\{ \begin{array}{ccc}
    j_{t_1} & j_{t_2} & j_{t_3} \\
    j_{t_4} & j_{t_5} & j_{t_6}
\end{array} \right\}
\]

where the indices \(t_1, \ldots, t_6\) denote six edges of a \(t\)th tetrahedron.

To prove that \(Z_0\) is indeed a topological invariant it is sufficient to show that it is unchanged under the moves shown in Figure 1. However, the link representation (59) is more convenient in this respect. By using the analog of the group measure invariance

\[
\int dg f(gh) \equiv d_0^{k/2} \sum_{i=0} d_i = d_0^{k/2} \sum_{i=0} d_i = \int dg f(g)
\]

one can reduce the number of loops. The corresponding operations have a nice interpretation as topology preserving transformations of complexes \[47\].

For lattices, as they have been defined above, all loops are unframed. However, transforming complexes, one can obtain non-trivial framings and, in real calculations, has to follow them carefully. Practically, it is convenient to use the ribbon graph representation \[48\]. The framing of a ribbon loop is defined as a linking number of its edges. It is fixed by the condition that one of two sides of the ribbon is always turned toward the inside of a 2-cell which it encircles (or toward a 1-cell which it wraps).

The invariant is multiplicative with respect to the connected sum of complexes

\[
Z_0(C_1 \# C_2) = Z_0(C_1)Z_0(C_2)
\]

because for the sphere

\[
Z_0(S^3) = 1
\]

Every oriented complex can be transformed into the canonical form, when there are single 0- and 3-dimensional cells and the equal number, \(\nu\), of 1- and
2-cells:

\[ C = \sigma^0 \cup \left( \bigcup_{i=1}^{\nu} \sigma_1^i \right) \cup \left( \bigcup_{j=1}^{\nu} \sigma_2^j \right) \cup \sigma^3 \]  

(56)

One can put into correspondence with each 1-cell, \( \sigma_1^i \), a generator of the fundamental group \( \gamma_i \in \pi_1(C) \). Each 2-cell, \( \sigma_2^j \), gives a defining relation for \( \pi_1(C) \):

\[ \Gamma_j = \prod_{\sigma_1^k \in \partial \sigma_2^j} \gamma_k = I \]  

(57)

If the gauge group is a classical finite group \( G \), the partition function \( Z_0 \) is well defined (after substituting the sum \( \sum_{g \in G} \) for \( \int dg \)):

\[ Z_0^{(G)} = \sum \prod_{\{g_i\} \in \nu} \delta(\prod_{\sigma_1^k \in \partial \sigma_2^j} g_k, I) \]  

(58)

This expression equals the number of representations of the fundamental group by elements of the gauge one: \( \pi_1(C) \to G \). Hence, it is an integer. In the quantum case, a similar interpretation exists. One have to consider an action of the fundamental group on the universal covering of a complex. It acts permuting cells of the covering, which can be regarded as a \( \pi(C) \)-module. The invariant can be said to be the “quantum analog” of Eq. (58), where the \( \pi(C) \)-action on the universal covering is represented by elements of a quantum gauge group. It is a real number.

As was proven by Turaev [18], the Turaev-Viro invariant is equal to the Reshetikhin-Turaev-Witten one modulo squared: \( Z_0(\mathcal{M}) = |I(\mathcal{M})|^2 \). Kohno [19] has shown that it is bounded from above as

\[ Z_0(\mathcal{M}) \leq \left( \frac{1}{d_0} \right)^{2h} \]  

(59)

where \( h \) is a Heegaar genus of \( \mathcal{M} \), i.e., the minimum genus of handlebodies appearing in Heegaar splittings of \( \mathcal{M} \).
5 Generating function for simplicial complexes.

In Refs. [20, 17] the zero-dimensional field model generating all possible simplicial complexes weighted with the partition function (52) was suggested. Let \( \phi(x, y, z) \) be a function on \( G \otimes G \otimes G \) invariant under right shifts

\[
\phi(x, y, z) = \phi(xu, yu, zu) \quad \forall x, y, z, u \in G
\]

and symmetric under even permutations. Odd ones are equivalent to the complex conjugation:

\[
\phi(x, y, z) = \phi(y, z, x) = \phi(z, x, y) = \overline{\phi(y, x, z)}
\]

It can be represented in terms of matrix elements as

\[
\phi(x, y, z) = \sum_{\{j_1 j_2 j_3\}} \sqrt{d_{j_1} d_{j_2} j_{a_1} a_3} T_{j_1, a_1 b_1}(x) T_{j_2, a_2 b_2}(y) T_{j_3, a_3 b_3}(z) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ b_1 & b_2 & b_3 \end{array} \right)
\]

\[
= \sum_{\{j_1 a_1 b_1\}} \sqrt{d_{j_1} d_{j_2} d_{j_3}} \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \right)
\]

where \( \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ b_1 & b_2 & b_3 \end{array} \right) \) is the 3-\( j \) symbol; \( \varphi_{a_1 a_2 a_3}^{j_1 j_2 j_3} = \varphi_{a_2 a_1 a_3}^{j_2 j_1 j_3} \) and symmetric under cyclic permutations. This equation is a general Fourier decomposition of a function obeying (60).

The partition function is defined as the integral

\[
P = \int D\phi \ e^{-S}
\]

where the action is taken in the form

\[
S = \frac{1}{2} \int dx dy dz \ |\phi(x, y, z)|^2 - \frac{\lambda}{12} \int dx dy dz du dv dw \ \phi(x, y, z) \phi(x, u, v) \phi(y, v, w) \phi(z, w, u)
\]

20
The first term in eq. (64) can be imagined as two glued triangles and the second, as four triangles forming a tetrahedron. It is not surprising that, after the Fourier transformation, one finds a 6-\(j\) symbol associated with it:

\[
S = \frac{1}{2} \sum_{j_{1},j_{2},j_{3}} \frac{1}{d_{j_{3}}} - \frac{\lambda}{12} \sum_{j_{1}...j_{6}} \frac{1}{d_{j_{1}}^{2}d_{j_{2}}d_{j_{3}}}
\]

The measure can be written in terms of Fourier coefficients

\[
\mathcal{D}\phi = \prod_{j_{1}j_{2}j_{3}}^{a_{1}a_{2}a_{3}} d\varphi^{j_{1}j_{2}j_{3}}
\]

If \(q = e^{i \frac{2\pi}{k+2}}\), the product in eq. (66) runs over irreps from the fusion ring and, hence, is finite.

Practically, the partition function (63) has a meaning within the perturbation expansion in \(\lambda\). Performing all possible Wick pairings, one gets in every order in \(\lambda\) all oriented simplicial complexes. For every 1-simplex in a simplicial complex, one has a loop carrying a representation index. It gives a corresponding quantum dimension. A 6-\(j\) symbol inside each tetrahedron has already appeared in eq. (65). Summing over all representations on links, one reproduces the Turaev-Viro partition function for a given simplicial complex.

Therefore, \(\log P\) is a generating function of 3D simplicial complexes weighted with the Turaev-Viro invariant. Of course, \(P\) is only formally defined. However, this construction gives a framework for the strong coupling expansion in simplicial gravity, which can be carried out by iterating the Schwinger-Dyson equation for the partition function (63).

A more down-to-earth model can be obtained by taking classical finite gauge group. Repeating all steps, one finds the sum over all simplicial complexes weighted with the invariant (58) times a volume dependent factor. To make a contact with the discrete action (13), one has to introduce a fugacity \(\mu\) for the number of links as well. It can be done by adding three indices to \(\phi\):
\[
\log P^{(G)} = \int \prod_{\{x_i \in G\}} \prod_{k_i = 1}^{\mu} d\phi^1_{x_1x_2x_3} \exp \left\{ - \frac{1}{2} \sum_{\{x_i \in G\}} \sum_{k_i = 1}^{\mu} |\phi^1_{x_1x_2x_3}|^2 + \sum_{\{x_i \in G\}} \sum_{k_i = 1}^{\mu} \phi^1_{x_1x_2x_3} \phi^1_{x_4x_5x_6} \phi^1_{x_7x_8x_9} \phi^1_{x_{10}x_{11}x_{12}} = \right.
\]
\[
\sum_{\{C\}} |G|^{N_0 - N_1} \mu^{N_1} Z_0^{(G)} (C) \tag{67}
\]
where \( |G| = \sum_{G} 1 \) is the rank of the group.

It can be easily seen that simplicial complexes have non-negative Euler characters

\[
\chi = \sum_{i=1}^{N_0} p_i \geq 0
\]

where the sum runs over all vertices. Tetrahedra touching the \( i \)'th vertex form a 3D ball; \( p_i \) is the genus of its 2D boundary. By definition, a complex is a manifold iff \( p_i = 0 \ \forall i \); i.e., the vicinity of every point is a spherical ball.

After the rescaling, \( \lambda = |G| \bar{\lambda}, \mu = \frac{1}{|G|} \bar{\mu} \), one obtains

\[
|G| \log P^{(G)} = \sum_{\{C\}} \bar{\lambda}^{N_0} \bar{\mu}^{N_1} |G|^\chi Z_0^{(G)} (C) \tag{69}
\]
and in the formal limit \( |G| \to 0 \) only manifolds for which \( Z_0^{(G)} \) is finite contribute.

Any finite group can be embedded in the permutation group, \( S_n \), for sufficiently large \( n \); \( |G| = n! \) in this case. It suggests that, at \( \bar{\lambda} \) and \( \bar{\mu} \) fixed, one should take \( n \) much bigger than the maximum rank of the fundamental group for typical complexes and try to continue analytically to \( n! = 0 \) (But how to do it practically?!). If the Poincaré hypothesis is true, only spheres should survive in this limit. Technically, it could mean a kind of double scaling.

Unfortunately, the model seems to be too complicated to be investigated analytically.
6 Conclusion

My aim in the present paper has been to draw attention to the quite promising problem of 3D quantum gravity. What one could learn from it concerns fundamental properties of the quantum vacuum. Non-renormalizability of gravity, non-boundedness of the Einstein-Hilbert action, topology changing processes, cosmological constant problem can be addressed within this simplified (comparing to 4D gravity) framework. Three dimensional geometry and topology possess a lot of beautiful mathematical structures. Many fundamental and long standing problems have not yet been solved. It is still a field of intensive research, which create an exciting atmosphere of a parallel rise of mathematical results and physical understanding.

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