\textbf{$L^2$-harmonic forms on complete Vaisman manifold}

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Abstract

In this article, we first consider the $L^2$ Morse-Novikov cohomology on a complete Riemannian manifold equipped with a parallel 1-form. Based on a vanishing theorem of $L^2$ Morse-Novikov cohomology, we prove that the $L^2$-harmonic forms on a complete, simply connected, Vaisman manifold $(M, J, g, \omega, \theta)$ are identically zero.

Keywords. Vaisman manifold, Morse-Novikov cohomology, Vanishing theorem

1 Introduction

Let $M$ be a complete $n$-dimensional Riemannian manifold. A basic question, pertaining both the function theory and topology on $M$, is: when are there non-trivial harmonic $k$-forms on $M$? When $X$ is not compact, a growth condition on the harmonic forms at infinity must be imposed, in order that the answer to this question would be useful. A natural growth condition is square-integrable, if $\Omega^k_{(2)}(X)$ denotes the $L^2$-forms of degree $k$ on $M$ and $\mathcal{H}^k_{(2)}(X)$ the harmonic forms in $\Omega^k_{(2)}(X)$. One version of this basic question is: what is the structure of $\mathcal{H}^k_{(2)}(X)$?

The Hodge theorem for compact manifolds states that every real cohomology class of a compact manifold $M$ is represented by a unique harmonic form. That is, the space of solutions to the differential equation $(d + d^*)\alpha = 0$ on $L^2$-forms over $M$ is a space that depends on the metric on $M$. This space is canonically isomorphic to the purely topological real cohomology space of $M$. The study of $\mathcal{H}^k_{(2)}(M)$, a question of so-called $L^2$-cohomology of $M$, is rooted in the attempt extending Hodge theory to non-compact manifolds. No such result holds in general for complete non-compact manifolds, but there are numerous partial result about the $L^2$-cohomology of non-compact manifold. The study of the $L^2$-harmonic forms on a complete Riemannian manifold is a very fascinating and important subject. There has been some recent interest in the study of $L^2$ harmonic forms on certain non-compact moduli spaces occurring in gauge theories [12].

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Suppose that \((M, \omega)\) is a complete, Kähler manifold of complex dimension \(n\) and \(\omega\) is the Kähler form on \(M\). The metric induced by Kähler form allows one to define the class of square-integrable forms of all bi-degrees, \(\Omega^{p,q}_{(2)}(M)\). We denote by \(H^{p,q}_{(2)}(M)\) the space of \(L^2\)-harmonic \((p, q)\)-forms. There are many articles study the Kähler case [1, 6, 7, 8, 10, 17]. In many situations, e.g., \((M, \omega) = \text{hyperbolic upper half plane in } \mathbb{C}^n\), it happens that \(H^{p,q}_{(2)}(M) = \{0\}\) unless \(p + q = n\). The middle dimension, when \(p + q = n\), is always a special case. For example, there are no results in [17] about \(L^2\) harmonic forms in these dimensions.

The main object of the present paper is the following notion. Let \((M, J, g)\) be a connected complex Hermitian manifold of complex dimension at least 2. Denote by \(\omega\) its fundamental Hermitian two-form, with the convention \(\omega(X, Y) = g(X, JY)\). A locally conformally Kähler (LCK) manifold is a complex Hermitian manifold, with a Hermitian form \(\omega\) satisfying \(d\omega = \theta \wedge \omega\), where \(\theta\) is a closed 1-form, called the Lee form of \(M\). A compact LCK manifold never admits a Kähler structure, unless the cohomology class \([\theta] \in H^1_{dR}(M)\) vanishes [23]. LCK manifolds form an interesting class of complex non-Kähler manifolds, including all non-Kähler surfaces which are not class VII. In many situations, the LCK structure becomes useful for the study of topology and complex geometry of an LCK-manifold. An LCK manifold \((M, J, g, \omega, \theta)\) is called Vaisman if \(\nabla \theta = 0\), where \(\nabla\) is the Levi-Civita connection of metric \(g\). The Vaisman manifold is a distinguished class among the LCK manifolds. In this article, we focus on the Vaisman case. We prove a vanishing theorem as follows.

**Theorem 1.1.** Let \((M, J, g, \omega, \theta)\) be a complete, simply-connected, Vaisman manifold of complex dimension \(n\). Then for any \(k \geq 0\), the space \(H^k_{(2)}(M)\) of \(L^2\)-harmonic \(k\)-forms is trivial.

### 2 Preliminaries

#### 2.1 \(L^2\)-harmonic forms

Let \((M, g)\) be a complete Riemannian manifold. Let \(\Omega^k(M)\) and \(\Omega^k_0(M)\) denote the smooth \(k\)-forms on \(M\) and the smooth \(k\)-forms with compact support on \(M\). Let \(\langle \cdot, \cdot \rangle\) denote the pointwise inner product on \(\Omega^*(M)\) given by \(g\). The global inner product is defined by

\[
(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle d\text{Vol}.
\]

We also write \(|\alpha|^2 = \langle \alpha, \alpha \rangle\), \(\|\alpha\|^2_{L^2(M, g)} = \int_M |\alpha|^2 d\text{Vol}\) and let

\[
\Omega^k_{(2)}(M, g) = \{\alpha \in \Omega^k(M) : \|\alpha\|_{L^2(M, g)} < \infty\}.
\]

The operator of exterior differentiation is \(d : \Omega^k_0(M) \to \Omega^{k+1}_0(M)\), and it satisfies \(d^2 = 0\); its formal adjoint is \(\delta : \Omega^{k+1}_0(M) \to \Omega^k_0(M)\); we have

\[
\forall \alpha \in \Omega^k_0(M), \forall \beta \in \Omega^{k+1}_0(M), \int_M \langle d\alpha, \beta \rangle = \int_M \langle \alpha, \delta \beta \rangle.
\]
We can define
\[
\mathcal{H}^k_{(2)}(M) = \{ \alpha \in \Omega^k_{(2)}(M) : d\alpha = 0, \ \delta\alpha = 0 \}.
\]
Because the operator \(d + \delta\) is elliptic, we have by elliptic regularity: \(\mathcal{H}^k_{(2)}(M) \subset \Omega^k(M)\). The space \(\Omega^k(M)\) has the following Hodge-de Rham-Kodaira orthogonal decomposition
\[
\Omega^k_{(2)}(M) = \mathcal{H}^k_{(2)}(M) \oplus d(\Omega^{k-1}_0(M)) \oplus \delta(\Omega^{k+1}_0(M)),
\]
where the closure is taken with respect to the \(L^2\) topology [2].

### 2.2 Morse-Novikov cohomology

Let \(M\) be a differential manifold and \(\eta\) a closed 1-form on \(M\). The Morse-Novikov cohomology of a manifold \(M\) refers to the cohomology of complex of smooth real form \(\Omega^*(M)\), with the differential operator defined as follow
\[
d_{\eta} = d + e(\eta),
\]
d being the exterior differential and \(e(\eta)\) the operator given by
\[
e(\eta)(\alpha) = \eta \wedge \alpha, \ \forall \alpha \in \Omega^*(M).
\]
Recall that the Morse-Novikov cohomology, also known as Lichnerowicz cohomology (defined independently by Novikov and Lichnerowicz in [16] and [18]) is the cohomology of the complex \((\Omega^*(M), d_{\eta})\)
\[
\Omega^0(M) \xrightarrow{d_{\eta}} \Omega^1(M) \xrightarrow{d_{\eta}} \Omega^2(M) \rightarrow \cdots
\]
Denote by \(H^*_\eta(M)\) the cohomology of the complex \((\Omega^*(M), d_{\eta})\). In fact, the sequence above is an acyclic resolution for \(\ker d_{\eta}\), as each \(\Omega^*(M)\) is soft, [5 Proposition 2.1.6 and Theorem 2.1.9]. Thus, by taking global sections in (2.3), we compute the cohomology groups of \(M\) with values in the sheaf \(\ker d_{\eta}, H^i(M, \ker d_{\eta})\). What we obtain is actually the Morse-Novikov cohomology.

**Proposition 2.1.** ([3 Proposition 4.4]) Let \(M\) be a differentiable manifold and \(\eta\) a closed 1-form on \(M\). Then,
(i) The differential complex \((\Omega^*(M), d_{\eta})\) is elliptic. Thus, if \(M\) is compact the cohomology groups \(H^k_{\eta}\) have finite dimension.
(ii) If \(\eta\) is exact the \(H^*_\eta(M) \cong H^*_d\), (M).

If the 1-form \(\eta\) is not exact then, in general, \(H^k_{\eta}(M) \not\cong H^k_d\). We recall some results proved by Guédira-Lichnerowicz [11] which will be useful in the sequel. Suppose that \(M\) is a differential manifold of dimensional \(n\), that \(\eta\) is a closed 1-form on \(M\) and that \(g\) is a Riemannian metric. Consider the vector field \(U\) on \(M\) characterized by the connection \(\eta(X) = g(X, U)\), for all vector field \(X\) on \(M\). Denote by \(i_U\) the contraction by the vector field \(U\), that is
\[
i_U(\alpha) = (-1)^{nk+n}(\ast \circ e(\eta) \circ \ast)(\alpha), \ \text{for} \ \alpha \in \Omega^k(M),
\]

being the Hodge star isomorphism. Then, we define the operator \( \delta_\eta : \Omega^k_0(M) \to \Omega^{k-1}_0(M) \) by
\[
\delta_\eta = \delta + i_U.
\] (2.5)

Then, it is easy to prove that \( (d_\eta \alpha, \beta) = (\alpha, \delta_\eta \beta) \), for all \( \alpha \in \Omega^{k-1}_0(M) \) and \( \beta \in \Omega^k_0(M) \). If \( M \) is compact, since the complex \( (\Omega^*(M), d_\eta) \) is elliptic, we obtain an orthogonal decomposition of the space \( \Omega^k(M) \) as follows
\[
\Omega^k(M) = \mathcal{H}^k_\eta(M) \oplus d_\eta(\Omega^{k-1}(M)) \oplus \delta_\eta(\Omega^{k+1}(M)),
\] (2.6)
where
\[
\mathcal{H}^k_\eta(M) = \{ \alpha \in \Omega^k(M) : d_\eta(\alpha) = 0, \delta_\eta(\alpha) = 0 \}.
\]

From (2.6), it follows that \( \mathcal{H}^k_\eta(M) \cong \mathcal{H}^k_\eta(M) \). Unlike de Rham cohomology, Morse-Novikov cohomology \( H^i_\eta(M) \), is not a topological invariant, it depends on \([\eta] \in H^1_{dR}(M)\). Also, Riemannian properties involving this one-form can be important. For instance, it was shown in [3] that if on a compact manifold \( M \) there exists a Riemannian metric \( g \) and a closed one-form \( \eta \) such that \( \eta \) is parallel with respect to \( g \), then for any \( i \geq 0 \), \( H^i_\eta(M) = 0 \).

We now define the spaces of generalized \( L^2 \)-harmonic forms as follows:
\[
\mathcal{H}^k_{(2),\eta}(M) = \{ \alpha \in \Omega^k_{(2)}(M) : d_\eta(\alpha) = 0, \delta_\eta(\alpha) = 0 \}.
\]

Following the idea in [3], we have

**Theorem 2.2.** Let \( M \) be a complete manifold and \( \eta \) a closed 1-form on \( M \), \( \eta \neq 0 \). Suppose that \( g \) is a Riemannian metric on \( M \) such that \( \eta \) is parallel with respect to \( g \). Then, \( \mathcal{H}^*_{(2),\eta} \) is trivial.

**Proof.** Since \( \eta \) is a parallel and non-null it follows that \( |\eta| = c \), with \( c \) constant, \( c > 0 \). Assume, without the loss of generality, that \( c = 1 \). Note that if \( c \neq 1 \), we can consider the Riemannian metric \( g' = c^2 g \) and it is clear that the module of \( \eta \) with respect to \( g' \) is 1 and that \( \eta \) is also parallel with respect to \( g' \). Under the hypothesis is \( c = 1 \), we have that
\[
\eta(U) = 1.
\] (2.7)

Using that \( \eta \) is parallel and that \( U \) is Killing, we obtain that (see (2.4))
\[
\mathcal{L}_U = -\delta \circ e(\eta) - e(\eta) \circ \delta,
\] (2.8)
\[
\delta \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta.
\] (2.9)

From (2.1–2.5), (2.7) and (2.9), we deduce the following relations:
\[
d_\eta \circ i_U = -i_U \circ d_\eta + \mathcal{L}_U + Id, \quad \delta_\eta \circ i_U = -i_U \circ \delta_\eta,
\] (2.10)
\[
d_\eta \circ \mathcal{L}_U = \mathcal{L}_U \circ d_\eta, \quad \delta_\eta \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta_\eta.
\] (2.11)
where $Id$ denotes the identity transformation. Let $\xi : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \xi \leq 1$,

\[ \xi(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t \geq 1 \end{cases} \]

and consider the compactly supported function

\[ f_j(x) = \xi(\rho(x_0, x) - j), \]

where $j$ is a positive integer and $\rho(x_0, x)$ stands for the Riemannian distance between $x$ and a base point $x_0$. On the other hand, (2.8) implies that

\[ \langle L_U \alpha, f_j \alpha \rangle = -\langle \alpha, df_j \alpha \rangle - \langle \alpha, i_U (df_j) \wedge \alpha \rangle \]

for all $\alpha \in \Omega^k(M)$. Here we use the identity

\[ L_U (\alpha \wedge \beta) = (L_U \alpha) \wedge \beta + \alpha \wedge (L_U \beta). \]

Noting that $0 \leq f_j \leq 1$ and $\lim_{i \to \infty} f_j(x) = \alpha(x)$, it follows from the dominated convergence theorem that

\[ \lim_{i \to \infty} \langle f_j, \alpha \rangle = \| \alpha \|^2. \]  (2.15)

Since $|U| = 1$ and $supp(df_j) \subset B_{j_i + 1} \setminus B_{j_i}$, one obtains

\[ |(i_U \alpha, *(df_j \wedge \alpha))| \leq C \int_{B_{j_i + 1} \setminus B_{j_i}} |\alpha(x)|^2 \to 0, \text{ as } i \to \infty, \]  (2.16)

where $C$ is a constant independent of $j_i$. It now follows from (2.12) and (2.14)–(2.16) that $\alpha = 0$. This proves that $\mathcal{H}^{(2)}_{(2), \eta}(M) = \{0\}$. 

\[ \Box \]
2.3 Locally conformally Kähler manifolds

In this section we will give the necessary definitions and properties of locally conformally Kähler (LCK) manifolds. In what follows, $M$ will denotes a connected, smooth manifold of complex dimension $n$; $J$ will be an integrable complex structure. For a Hermitian metric $g$, we denote by $\nabla$ the Levi-Civita connection and by $\omega$ the fundamental two-form defined as $\omega(X,Y) = g(JX,Y)$.

A Locally conformally Kähler manifolds is a complex manifold $X$ covered by a system of open subsets $U_\alpha$ endowed with local Kähler metrics $g_\alpha$, conformal on overlaps $U_\alpha \cap U_\beta$: $g_\alpha = c_{\alpha\beta} g_\beta$. The metrics $e^{f_\alpha} g_\alpha$ glue to a global metric whose associated 2-form satisfies the integrability condition $d\omega = \theta \wedge \omega$, thus being locally conformal with the Kähler metrics $g_\alpha$. Here $\theta|_{U_\alpha} = df_\alpha$. The closed 1-form $\theta$ is called the Lee form. This gives another definition of an LCK structure, which will be used in this paper.

Definition 2.3. Let $(M, g, \omega)$ be a complex Hermitian manifold, $\dim_{\mathbb{C}} M > 1$, with

$$d\omega = \theta \wedge \omega,$$

where $\theta$ is a closed 1-form. Then $M$ is called a locally conformally Kähler (LCK) manifold.

If one performs a conformal change, $\omega_1 = e^{f}\omega$, the Lee form $\theta$ changes to $\theta_1 = \theta + df$. The cohomology class $[\theta] \in H^1_{dR}(M)$ is an important invariant of an LCK-manifold. Clearly, $[\theta] \in H^1_{dR}(M)$ vanishes if and only if $\omega$ is conformally equivalent to a Kähler structure. In this case $(M, \omega)$ is called globally conformally Kähler.

An LCK-form $\omega$ on an LCK-manifold satisfies $d\omega = \omega \wedge \theta$, therefore it is $(d - \theta)$-closed. The cohomology class $[\omega] \in H^2(M)$ is called the Morse-Novikov class of the LCK-manifold. It is an invariant of the LCK-manifold, roughly analogous to the Kähler class on a Kähler manifold. In [19], the author defined three cohomology invariants, the Lee class, the Morse-Novikov class, and the Bott-Chern class, of an LCK-structure. These invariants play together the same role as the Kähler class in Kähler geometry [20, 21, 22].

Among the LCK manifolds, a distinguished class is the following:

Definition 2.4. Let $(M, g, \omega, \theta)$ be an LCK manifold and $\nabla$ its Levi-Civita connection. We say that $M$ is an LCK manifold with parallel Lee form $\theta$, or Vaisman manifold, if $\nabla \theta = 0$. If $\theta \neq 0$, then after rescaling, we may always assume that $|\theta| = 1$. Unless otherwise stated, we shall assume implicitly that $|\theta| = 1$ for all Vaisman manifolds we consider.

We can constructed a Kähler potential on a Vaisman manifold with exact Lee form [25, 27].

Proposition 2.5. Let $M$ be an LCK manifold with parallel Lee form $\theta$, and $\theta^\sharp$ be the dual vector field of $\theta$. Consider a diffeomorphism flow $\psi_t$ associated with $\theta^\sharp$. Then $\psi_t$ acts on $M$ preserving the LCK structure.

Proof. For a more detailed proof see [9] or [26, Proposition 4.1].
The Lee form $\theta$ is by definition closed. Passing to a covering if necessary, we may assume that it is exact: $\theta = dt$. Write $r = e^{-t}$.

**Definition 2.6.** Let $M$ be an LCK manifold with exact Lee form $\theta = dt$. The function $r = e^{-t}$ is called the potential of $M$. Clearly, $r$ is defined uniquely up to a positive constant multiplier.

Let $(M, g, \omega)$ be an LCK manifold with exact Lee form $\theta$, $r$ its potential and $\omega \in \Omega^{1,1}(M)$ the Hermitian form of $(M, g)$. One can see that $r\omega$ is positive definite since $r$ is a positive function. Then, we have

$$d(r\omega) = rd\omega - e^{-t}dt \wedge \omega = r(\theta \wedge \omega) - \theta \wedge (r\omega) = 0,$$

i.e., $r\omega$ is a Kähler form.

**Proposition 2.7.** ([26, Proposition 4.4]) Let $(M, J, g)$ be a Vaisman manifold. Assume that the Lee form $\theta$ is exact, and let $r$ be the corresponding potential function. Then $r$ is the Kähler potential for the Kähler form $r\omega$.

**Proof.** Let $L_{\theta}$ be the operator of Lie derivative along the vector field $\theta^\sharp$ dual to $\theta$. Then $L_{\theta}\omega = 0$ by Proposition 2.5. Similarly,

$$L_{\theta^\sharp}r = i_{\theta^\sharp}dr = *(\theta \wedge *dr) = -* (\theta \wedge *(r\theta)) = -r.$$

Therefore, $L_{\theta^\sharp}(r\omega) = -r\omega$. On the other hand, $r\omega$ is closed, i.e., $d(r\omega) = 0$. We obtain

$$r\omega = -L_{\theta^\sharp}(r\omega) = -di_{\theta^\sharp}(r\omega) = -d*(\theta \wedge *(r\omega)).$$

Let $d^c = -J \circ d \circ J$ be the twisted de Rham differential. We notice that

$$\theta \wedge *(r\omega) = r\theta \wedge \frac{\omega^{n-1}}{(n-1)!} = r\left(\frac{1}{(n-1)!}L^{n-1}\theta\right) = *J(r\theta),$$

and $dr = -r\theta$. Therefore,

$$r\omega = dJ(r\theta) = -dJdr = dd^cr.$$

Hence the function $r$ is a Kähler potential for the form $r\omega$. 

The Kähler potential on the universal covering space $\tilde{M}$ was first noted by Verbitsky [26]. As a consequence, Ornea-Verbitsky [20, 21, 22] introduced and started the study of the more general notion of a LCK metric with (positive) potential.

**Remark 2.8.** Suppose that the Kähler form $\omega$ on a Kähler manifold $M$ is given by a global potential, $\omega = i\partial\bar{\partial}\lambda$ for a smooth $\lambda \in C^2(M)$ with $\lambda \geq 1$. Suppose that for all $x \in M$, there exists a constants $A, B < \infty$ such that

$$|\partial\lambda(x)|^2_\omega \leq (A + B\lambda(x)), \tag{2.17}$$

where \(| \cdot |_\omega\) is the norm induced by the Kähler form \(\omega\). If \(M\) is complete, then \((M, \omega)\) was called Kähler convex [17].

We denote by \(r_g\) the metric on a complete, simply-connected, Vaisman manifold \((M, J, g, \omega, \theta)\) induced by Kähler form \(r_\omega = dd^c r = i\partial\bar{\partial}(2r)\). Noting that \(|\partial t|^2_g = |\partial \theta|^2_g = \frac{1}{2}\) since \(\theta = dt\) and \(|\theta|^2_g = 1\). We then have
\[
|\partial (2r)|^2_g = |2e^{-f} \partial t|^2_g = 4r |\partial t|^2_g = 2r.
\]
In particular, the Kähler potential \(2r\) satisfies (2.17). It is true that a complete, simply-connected Vaisman manifold become Kähler after a conformal change of the metric, but this metric is never complete.

Let \((M, J, \theta)\) be a Vaisman manifold. Since the Lee form \(\theta\) is parallel, \(|\theta| = 2c\) for some \(c \in \mathbb{R}\), \(c \neq 0\). We adopt the notations
\[
u = |\theta|^{-1}\theta, \ U = u^\sharp, \ v = -u \circ J, \ V = -JU.
\]
We recall that given a real \((2n - 1)\)-dimensional \(C^\infty\) differentiable manifold \(N\) and \(c \in \mathbb{R}\), \(c \neq 0\), a \(c\)-Sasakian structure on \(N\) is a synthetic object \((\psi, \xi, \eta, \gamma)\) consisting of a \((1, 1)\)-tensor field \(\psi\), a vector field \(\xi \in \mathcal{X}(N)\), a 1-form \(\eta\), and a Riemannian metric \(\gamma\), satisfying the following identities:
\[
\begin{align*}
\psi^2 &= -I + \eta \otimes \xi \\
\eta \circ \psi &= 0, \ \eta(\xi) = 1 \\
\gamma(\psi X, \psi Y) &= \gamma(X, Y) - \eta(X)\eta(Y) \\
[\psi, \psi] + 2(d\eta) \otimes \xi &= 0 \\
d\eta &= c\phi,
\end{align*}
\]
where the 2-form \(\phi\) is given by \(\phi(X, Y) = \gamma(X, \psi Y)\). A \((2n - 1)\)-dimensional manifold \(N\) carrying a \(c\)-Sasakian structure is a \(c\)-Sasakian manifold. Of course, one may always go back to a usual Sasakian structure by a transformation:
\[
\hat{\psi} = \psi, \ \hat{\xi} = \frac{1}{c}\xi, \ \hat{\eta} = c\eta, \ \hat{\gamma} = c^2\gamma.
\]
We then have

**Proposition 2.9.** ([24] and [9] Proposition 5.1) Let \(M\) be a Vaisman manifold. Let \(S\) be a leaf of \(\mathcal{F}_0\) and \(i : S \hookrightarrow M\) the inclusion. Let \((\psi, \xi, \eta, \gamma)\) on \(S\) be given by
\[
\psi = J \circ (di) + (i^* v) \otimes (U \circ i) \\
\xi = V \circ i, \ \eta = i^* v \\
\gamma = i^* g,
\]
Then \((\psi, \xi, \eta, \gamma)\) is a \(c\)-Sasakian structure on \(S\).
Using 
\[ g = \gamma + u \otimes u \]
and the De Rham decomposition theorem, we obtain

**Proposition 2.10.** ([27] and [9, Proposition 5.2]) The universal Riemannian covering manifold \( M \) of a complete Vaisman manifold is the Riemannian product of a simply connected \( c \)-Sasakian manifold \( N \), which is the universal covering space of a leaf \( N \) of \( F_0 \) and the real line.

## 3 Vanishing theorems

### 3.1 Riemannian manifold with parallel 1-form

In this section, we recall some notations and definitions on differential geometry [27]. Let \( M \) be a \( C^\infty \)-manifold. We denote by \( \Omega^\ast(X) \) the smooth forms on \( M \). Given an odd or even from \( \alpha \in \Omega^\ast(M) \), we denote by \( \tilde{\alpha} \) its parity, which is equal to 0 for even forms, and 1 for odd forms. An operator \( f \in \text{End}(\Lambda^\ast(M)) \) preserving parity is called even, and one exchanging odd and even forms is odd, \( \tilde{f} \) is equal to 0 for even forms and 1 for odd ones. Given a \( C^\infty \)-linear map \( \Omega^1(M) \to \Omega^{\text{even}}(M) \) or \( \Omega^1(M) \to \Omega^{\text{odd}}(M) \), \( p \) can be uniquely extended to a \( C^\infty \)-linear derivation \( \rho \) on \( \Omega^\ast(M) \), using the rule

\[ \rho|_{\Omega^0(M)} = 0, \quad \rho|_{\Omega^1(M)} = p, \quad \rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}\tilde{\beta}} \alpha \wedge \rho(\beta). \]

Then, \( \rho \) is an even (or odd) differentiation of the graded commutative algebra \( \Omega^\ast(M) \). Verbitsky gave a definition of the structure operator of \( (M, \omega) \), see [27] Definition 2.1.

**Definition 3.1.** Let \( M \) be a Riemannian manifold equipped with a parallel differential \( k \)-form \( \eta \). Consider an operator \( C : \Omega^1(M) \to \Omega^{k-1}(M) \) mapping \( \alpha \in \Omega^1(M) \) to \(*(*\omega \wedge \alpha) \). The corresponding differentiation

\[ C : \Omega^\ast(M) \to \Omega^{\ast+k-2}(M) \]

is called the structure operator of \( (M, \eta) \).

**Definition 3.2.** ([27 Definition 2.3]) Let \( M \) be a Riemannian manifold, and \( \omega \in \Omega^k(M) \) a differential form, which is parallel with respect to the Levi-Civita connection. Denote by \( d_C \) the supercommutator

\[ \{d, C\} := dC - (-1)^{\tilde{C}d}Cd. \]

This operator is called the twisted de Rham operator of \( (M, \omega) \). Being a graded commutator of two graded differentiations, \( d_C \) is also a graded differentiation of \( \Omega^\ast(M) \).

**Lemma 3.3.** ([27 Proposition 2.5]) Let \( M \) be a Riemannian manifold equipped with a parallel differential \( k \)-form \( \eta \), and \( L_\eta \) the operator \( \alpha \mapsto \alpha \wedge \eta \). Then

\[ d_C = \{L_\eta, d^\ast\}, \]

where \( \{\cdot, \cdot\} \) denotes the supercommutator, and \( d^\ast \) is the adjoint to \( d \).
Remark 3.4. If \( \eta \) is a parallel one form on \( M \), then in fact the structure operator of \( M \) is \( C = i_\eta \). The operator \( d_C \) is the Lie derivative \( \mathcal{L}_\eta \) since \( \{d, i_\eta\} = \mathcal{L}_\eta \).

We recall some results which proved by Verbitsky (See [27, Proposition 2.5 and Corollary 2.9]).

**Proposition 3.5.** Let \( M \) be a Riemannian manifold equipped with a parallel differential \( k \)-form \( \eta \), \( d_C \) the twisted de Rham operator constructed above and \( \ast_C \) its Hermitian adjoint. Then,

(i) The following supercommutators vanish:

\[
\{d, d_C\} = 0, \quad \{d, d_C^*\} = 0, \quad \{d^*, d_C\} = 0, \quad \{d^*, d_C^*\} = 0.
\]

(ii) The Laplacian \( \Delta = \{d, d^*\} \) commutes with \( L_\eta : \alpha \mapsto \alpha \wedge \eta \) and its adjoint operator \( \Lambda_\eta \) which is denoted as \( \Lambda_\eta : \Omega^i(M) \to \Omega^{i-k}(M) \).

Following Proposition 3.5, if \( \alpha \) is a harmonic form on \( M \), then \( \alpha \wedge \eta \) is harmonic.

**Corollary 3.6.** Let \( M \) be a complete manifold and \( \eta \) a closed 1-form on \( M \), \( \eta \neq 0 \). Suppose that \( g \) is a Riemannian metric on \( M \) such that \( \eta \) is parallel with respect to \( g \). If \( \alpha \) is a \( L^2 \)-harmonic \( k \)-form on \( M \), then \( \eta \wedge \alpha \) is a \( L^2 \)-harmonic \((k+1)\)-form.

**Proof.** Since the 1-form \( \eta \) is a parallel, \( |\eta| = \text{const} \). Therefore, \( \eta \wedge \alpha \in \Omega^{k+1}(M) \). Following Proposition 3.5, it implies that \( \Delta(\eta \wedge \alpha) = 0 \). Then, we have \( d(\eta \wedge \alpha) = 0 \) and \( d^*(\eta \wedge \alpha) = 0 \).

Let \((M, g)\) be a complete Riemannian manifold. A differential form \( \alpha \) is called \( d \)-bounded if there exists a form \( \beta \) on \( M \) such that \( \alpha = d\beta \) and

\[
\|\beta\|_{L^\infty(M,g)} = \sup_{x \in M} |\beta(x)|_g < \infty.
\]

It is obvious that if \( M \) is compact, then every exact form is \( d \)-bounded. However, when \( M \) is not compact, there exist smooth differential forms which are exact but not \( d \)-bounded. For instance, on \( \mathbb{R}^n \), \( \alpha = dx^1 \wedge \cdots \wedge dx^n \) is exact, but it is not \( d \)-bounded.

Let us recall some concepts introduced by Cao-Xavier in [1]. A differential form \( \alpha \) on a complete non-compact Riemannian manifold \((M, g)\) is called \( d \)-sublinear if there exist a differential form \( \beta \) and a number \( c > 0 \) such that \( \alpha = d\beta \) and

\[
|\alpha(x)|_g \leq c, \quad |\beta(x)|_g \leq c(1 + \rho(x, x_0)),
\]

where \( \rho(x, x_0) \) stands for the Riemannian distance between \( x \) and a base point \( x_0 \) with respect to \( g \).

In [13], the author extended the idea of Cao-Xavier’s [1] to the case of Riemannian manifold equipped with a parallel differential form. We then have a result as follows. Here, we give a proof in detail for the readers convenience.

**Theorem 3.7.** [13 Theorem 2.9] Let \((M, \eta)\) be a Riemannian manifold equipped with a parallel differential \( k \)-form \( \eta \). If \( \eta = d\beta \) is \( d \)-sublinear, then for any \( \alpha \in \mathcal{H}^p(\mathcal{X}) \), we have

\[
\eta \wedge \alpha = 0.
\]
Proof. Let \( \{f_j\}_{j=0,1,2,...} \) be the compactly supported functions which are the same as the functions in the proof of Theorem 2.2. Let \( \alpha \) be a harmonic \( p \)-form in \( L^2 \), and consider the form \( \nu = \beta \wedge \alpha \). Observing that \( d^*(\eta \wedge \alpha) = 0 \) since \( \eta \wedge \alpha \in H^{p+k}_c(X) \) and noticing that \( f_j \nu \) has compact support, one has

\[
0 = (d^*(\eta \wedge \alpha), f_j \nu) = (\eta \wedge \alpha, d(f_j \nu)) = (\eta \wedge \alpha, f_j \eta \wedge \alpha) + (\eta \wedge \alpha, df_j \wedge \beta \wedge \alpha). \tag{3.1}
\]

Since \( 0 \leq f_j \leq 1 \) and \( \lim_{j \to \infty} f_j(x)(\eta \wedge \alpha)(x) = (\eta \wedge \alpha)(x) \), it follows from the dominated convergence theorem that

\[
\lim_{j \to \infty} (\eta \wedge \alpha, f_j \eta \wedge \alpha) = ||\eta \wedge \alpha||^2. \tag{3.2}
\]

Since \( \eta \) is bounded, \( \text{supp}(df_j) \subset B_{j+1} \setminus B_j \) and \( |\beta(x)| = O(\rho(x_0, x)) \), one obtains

\[
| (\eta \wedge \alpha, df_j \wedge \beta \wedge \alpha) | \leq (j + 1)C \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 d\text{Vol}, \tag{3.3}
\]

where \( C \) is a constant independent of \( j \).

We claim that there exists a subsequence \( \{j_i\}_{i \geq 1} \) such that

\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 d\text{Vol} = 0. \tag{3.4}
\]

If not, there would exist a positive constant \( a \) such that

\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 d\text{Vol} \geq a > 0, \quad j \geq 1.
\]

This inequality implies

\[
\int_M |\alpha(x)|^2 d\text{Vol} = \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 d\text{Vol} \geq a \sum_{j=0}^{\infty} \frac{1}{j+1} = +\infty
\]

a contradiction to the assumption \( \int_M |\alpha(x)|^2 d\text{Vol} < \infty \). Hence, there exists a subsequence \( \{j_i\}_{i \geq 1} \) for which (3.4) holds. Using (3.3) and (3.4), one obtains

\[
\lim_{i \to \infty} (\eta \wedge \alpha, df_j \wedge \beta \wedge \alpha) = 0 \tag{3.5}
\]

It now follows from (3.1), (3.2) and (3.5) that \( \eta \wedge \alpha = 0 \). 

There is a very known result. Let \( M \) be a compact Riemannian manifold, \( \alpha \) be a closed 1-form and \( \pi : \tilde{M} \to M \) be the universal covering. Then the pull back form \( \pi^*(\alpha) \) is \( d \) (sublinear) (see [14, Proposition 1]). We observe a useful lemma as follows.

Lemma 3.8. Let \( M \) be a complete, non-compact Riemannian manifold. If the \( C^\infty \)-function \( f \) on \( M \) satisfies \( \nabla^2 f = 0 \), then for any \( x \in M \),

\[
|f(x)| \leq c(\rho(x, x_0) + 1),
\]

where \( c \) is a uniform positive constant.
Proof. Since $\nabla^2 f = 0$, $|\nabla f| = \text{const.}$ Let $x_0$ be a fix point on $M$. For any point $x$ in $M$, there exists a geodesic $s : [0, 1] \to M$ such that $s(0) = x_0$ and $s(1) = x$. Thus

$$|f(s(1)) - f(s(0))| \leq c\rho|\nabla f|,$$

where $\rho$ is the Riemannian distance between $x_0$ and $x$, $c$ is a positive constant independent on $x \in M$.

We then have

Lemma 3.9. Let $M$ be a complete, simply-connected manifold and $\eta$ a closed 1-form on $M$, $\eta \neq 0$. Suppose that $g$ is a Riemannian metric on $M$ such that $\eta$ is parallel with respect to $g$. Then for any $L^2$-harmonic $k$-form $\alpha$ on $M$, we have

$$e(\eta)(\alpha) = 0, \ i_U(\alpha) = 0. \quad (3.6)$$

In particular, for any $k \geq 0$,

$$\mathcal{H}^k(M) \subset \mathcal{H}^k_{(2),\eta}(M).$$

Proof. Since $M$ is simply-connected, there is a function $f$ on $M$ such that $\eta = df$. Therefore, $\nabla^2 f = 0$ since $\nabla \eta = 0$. By the Lemma 3.8, $\eta$ is $d$(sublinear). Let $\alpha$ be a $L^2$-harmonic $k$-form. Then, $\ast \alpha$ is also a $L^2$-harmonic $(n - k)$-form. Following Theorem 3.7 and Corollary 3.6, it implies that

$$e(\eta)(\alpha) = \eta \wedge \alpha = 0, \ i_U(\alpha) = (-1)^{nk+n} \ast e(\eta)(\ast \alpha) = 0.$$

Therefore, $d_\eta \alpha = d\alpha + e(\eta)\alpha = 0$ and $\delta_\eta \alpha = \delta \alpha + i_U \alpha = 0$, i.e., $\alpha \in \mathcal{H}^k_{(2),\eta}(M)$.

Noting that the Lee form from $\theta$ on a complete simply connected Vaisman manifold is parallel. Following Proposition 3.9, we then have

Corollary 3.10. Let $(M, J, g, \omega, \theta)$ be a complete, simply-connected Vaisman manifold. If $\alpha$ is a $L^2$-harmonic $k$-form on $M$, then

$$e(\theta)(\alpha) = 0, \ i_{\theta^\sharp}(\alpha) = 0. \quad (3.7)$$

where $\theta$ is the Lee form and $\theta^\sharp$ is the Lee filed. In particular, for any $k \geq 0$,

$$\mathcal{H}^k_{(2)}(M) \subset \mathcal{H}^k_{(2),\theta}(M).$$

Theorem 3.11. Let $M$ be a complete, simply-connected manifold and $\eta$ a closed 1-form on $M$, $\eta \neq 0$. Suppose that $g$ is a Riemannian metric on $M$ such that $\eta$ is parallel with respect to $g$. Then for any $k \geq 0$, the spaces $\mathcal{H}^k_{(2)}(M)$ of $L^2$-harmonic $k$-forms for all $k$ are trivial.

Proof. The conclusion follows from Theorem 2.2 and Lemma 3.9.

Proof of Theorem 1.1. The conclusion follows from Theorem 3.11 and Corollary 3.10.
Remark 3.12. If $(M, J, g, \omega, \theta)$ is a closed, smooth Vaisman manifold and $\pi : (\tilde{M}, \tilde{J}, \tilde{g}, \tilde{\omega}, \tilde{\theta}) \to (M, J, g, \omega, \theta)$ its universal covering. Then $(\tilde{M}, \tilde{J}, \tilde{g}, \tilde{\omega}, \tilde{\theta})$ is a complete, simply-connected Vaisman manifold. Therefore, $\mathcal{H}^n_{(2)}(\tilde{M}, \tilde{g}) = \{0\}$.

We recall a well-known property of the space of $L^2$ harmonic $n$-forms on a complete $2n$-dimensional manifold under conformal metric (see [2, Proposition 5.2]).

**Proposition 3.13.** If $g_1 = e^f g_2$ are two conformally equivalent Riemannian metric on a smooth $2n$-dimensional manifold $M$, then $$\mathcal{H}^n_{(2)}(M, g_1) = \mathcal{H}^n_{(2)}(M, g_2).$$

**Corollary 3.14.** Let $(M, J, g)$ be a Vaisman manifold. Assume that the Lee form $\theta$ is exact, and let $r$ be the corresponding potential function. Then $\mathcal{H}^n_{(2)}(M, rg) = 0$.

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