Random Walks on Strict Partitions

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Abstract

We consider a certain sequence of random walks. The state space of the nth random walk is the set of all strict partitions of n (that is, partitions without equal parts). We prove that, as n goes to infinity, these random walks converge to a continuous-time Markov process. The state space of this process is the infinite-dimensional simplex consisting of all nonincreasing infinite sequences of nonnegative numbers with sum less than or equal to one. The main result about the limit process is the expression of its the pre-generator as a formal second order differential operator in a polynomial algebra.

Of separate interest is the generalization of Kerov interlacing coordinates to the case of shifted Young diagrams.

1 Introduction

1.1 The Schur graph and its boundary

Let \( S_n, n \in \mathbb{Z}_{\geq 0} \), denote the finite set consisting of all strict partitions (that is, partitions without equal parts) of \( n \).\(^1\) Strict partitions are represented by shifted Young diagrams \(^2\), Ch. I, §1, Example 9]. The Schur graph is the graded graph consisting of all shifted Young diagrams (with edge multiplicities given by \(^2\) below).

As can be proved exactly as in \(^2\) using the results of \(^2\) the Martin boundary of the Schur graph can be identified with the infinite-dimensional ordered simplex

\[
\Omega_+ := \left\{ x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_i x_i \leq 1 \right\}.
\]

The simplex \( \Omega_+ \) viewed as a subspace of the infinite-dimensional cube \([0, 1]^\infty\) (which in turn is equipped with the product topology) is a compact, metrizable and separable space.

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\(^1\)The set \( S_0 \) consists of the empty partition \( \emptyset \).

\(^2\)Another proof can be found in the paper \(^2\).
1.2 Projective representations of symmetric groups

Each $S_n$, $n \geq 1$, may be regarded as a projective dual object to the symmetric group $S_n$ in the sense that $S_n$ parametrizes the irreducible projective representations of $S_n$ \[9, 24\]. The simplex $\Omega_+$ can be viewed as a kind of projective dual to the infinite symmetric group $S_\infty$. That is, the points of $\Omega_+$ parametrize the indecomposable normalized projective characters of the group $S_\infty$ \[18\].

The theory of projective representations of symmetric groups is in many aspects similar to the theory of ordinary representations. Let us indicate some of them:

- The (ordinary) dual object to $S_n$ is the set of all ordinary (i.e., not necessary strict) partitions of $n$;
- The role of $\Omega_+$ in the theory of ordinary representations is played by the Thoma simplex $\Omega$ that consists of couples $((\omega; \omega')) \in [0, 1]^\infty \times [0, 1]^\infty$ satisfying the following conditions:
\[
\omega_1 \geq \omega_2 \geq \cdots \geq 0, \quad \omega'_1 \geq \omega'_2 \geq \cdots \geq 0, \quad \sum_i \omega_i + \sum_j \omega'_j \leq 1.
\]

The points of $\Omega$ parametrize the indecomposable normalized characters of $S_\infty$ \[28\].
- The role that Schur’s $Q$-functions play in the theory of projective representations \[9\] is taken by ordinary Schur functions.

There is a natural embedding of $\Omega_+$ into $\Omega$ introduced in \[8\]. This map sends $x = (x_1, x_2, \ldots) \in \Omega_+$ to $(\omega, \omega') \in \Omega$ with $\omega = \omega' = (x_1/2, x_2/2, \ldots)$. In §8.5 below we discuss this embedding in more detail.

1.3 Multiplicative measures

In \[2\] A. Borodin introduced multiplicative coherent systems of measures (or central measures in the sense of \[15\]) on the Schur graph. This is a sequence of probability measures $M_n^\alpha$ on $S_n$, $n \in \mathbb{Z}_{\geq 0}$, depending on one real parameter $\alpha \in (0, +\infty)$. Below we call this object simply the multiplicative measures.

According to a general formalism (explained, e.g., in \[12\]) the multiplicative measures $\{M_n^\alpha\}$ give rise to a one-parameter family of probability measures $P^{(\alpha)}$ on $\Omega_+$. Namely, every set $S_n$ can be embedded into $\Omega_+$: a strict partition $\lambda \in S_n$ maps to a point $(\lambda_1/n, \lambda_2/n, \ldots) \in \Omega_+$, where $\lambda_i$ are the components of $\lambda$. As $\alpha$ remains fixed and $n$ goes to infinity, the images of $M_n^\alpha$ under these embeddings weakly converge to $P^{(\alpha)}$.

1.4 The model of random walks

To a coherent system of measures on a graded graph one can associate a sequence of random walks on the floors of the graph\[1\]. These random walks are called the
up-down Markov chains.

The up/down Markov chains first appeared in a paper by J. Fulman [6]. He was interested in such questions as their eigenstructure, eigenvectors and convergence rates. In the papers [6, 7] many examples of up/down Markov chains associated to various coherent systems on various graphs are studied including the Plancherel measures and the z-measures on the Young graph, the Ewens-Pitman’s partition structures on the Kingman graph [4] and the Plancherel measures on the Schur graph [5].

In [3, 21, 20] the limit behaviour of various up/down Markov chains is studied. The paper [3] deals with the limit behaviour of the chains associated to the z-measures on the Young graph. In [21] the chains corresponding to the Ewens-Pitman’s partition structure are studied, and in [20] the general case of the Young graph with Jack edge multiplicities is considered.

In this paper we consider a sequence of up/down Markov chains associated to the multiplicative measures on the Schur graph. These chains depend on the parameter $\alpha$. The nth chain lives on $S_n$ and preserves the probability measure $M_n^\alpha$. We study the limit behaviour of these random walks as $n \to \infty$.

1.5 The limit diffusion and its pre-generator

Assume that $\alpha \in (0, +\infty)$ is fixed. Let us embed each set $S_n$ into $\Omega_+$ as described above in [13] and let the discrete time of the nth up/down chain be scaled by the factor $n^{-2}$. We show that under these space and time scalings the up/down chains converge, as $n \to \infty$, to a diffusion process $\mathbf{X}_n(t), t \geq 0$, in the simplex $\Omega_+$. We also show that $\mathbf{X}_n(t)$ preserves the measure $\mathbf{P}^\alpha$, is reversible and ergodic with respect to it.

The main result of the present paper is the formula for the pre-generator of the process $\mathbf{X}_n(t)$. To formulate the result we need some notation.

By $C(\Omega_+)$ denote the Banach algebra of real-valued continuous functions on $\Omega_+$ with pointwise operations and the uniform norm. Let $\mathcal{F}$ be a dense subspace of $C(\Omega_+)$ freely generated (as a commutative unital algebra) by the algebraically independent continuous functions $q_{2k}(x) := \sum_{i=1}^{\infty} x_i^{2k+1}, k = 1, 2, \ldots$. Define an

as in [17, Ch. I, §1]). The z-measures originated from the problem of harmonic analysis for the infinite symmetric group $S_\infty$ [13, 14] and were studied in detail by A. Borodin and G. Olshanski, see the bibliography in [3].

The set of vertices of the Kingman graph is the same as of the Young graph, but the edge multiplicities are different. The Ewens-Pitman’s partition structure (this is a special name for the coherent system of measures on the Kingman graph, the term is due to J. F. C. Kingman [10]) was introduced in [5, 22]. It is closely related to the Poisson-Dirichlet measure (see, e.g., [23] and bibliography therein).

We give the definition of the Plancherel measures on the Schur graph in §2.3 below.

By a diffusion process we mean a strong Markov process with continuous sample paths.
operator $A: \mathcal{F} \to \mathcal{F}$ depending on the parameter $\alpha$:

$$
A = \sum_{i,j=1}^{\infty} (2i+1)(2j+1) (q_{2i+2j} - q_{2i}q_{2j}) \frac{\partial^2}{\partial q_{2i}\partial q_{2j}} + 2 \sum_{i,j=0}^{\infty} (2i + 2j + 3) q_{2i}q_{2j} \frac{\partial}{\partial q_{2i+2j+2}} - \sum_{i=1}^{\infty} (2i + 1) \left(2i + \frac{\alpha}{2}\right) q_{2i} \frac{\partial}{\partial q_{2i}},
$$

where, by agreement, $q_0 = 1$. This is a formal differential operator in the polynomial algebra $\mathcal{F} = \mathbb{R}[q_2,q_4,q_6,\ldots]$. We show that the operator $A$ is closable in $C(\Omega_+)$ and that the process $X_\alpha(t)$ is generated by the closure $\overline{A}$ of the operator $A$.

### 1.6 The method

The formulation of the results is given in probabilistic terms. However, the use of probabilistic technique in the proofs generally reduces to the application of certain results from the paper [29] and the book [4] concerning approximations of continuous semigroups by discrete ones.

The essential part of the paper consists of the computations in a polynomial algebra. To obtain the formula (1) for the pre-generator we use the methods similar to those of [20]. This involves the restatement of some of the results concerning ordinary Young diagrams to our situation. In particular, we introduce Kerov interlacing coordinates of shifted Young diagrams which are similar to interlacing coordinates of ordinary Young diagrams introduced and studied by S. Kerov in [11]. Kerov interlacing coordinates of shifted Young diagrams are of separate interest.

We also give an alternative expression for the pre-generator $A$. Namely, we compute the action of $A$ on Schur’s $Q$-functions (Proposition 8.7 (2) below). This is done in §4 exactly as in [3, §4] with ordinary Schur functions replaced by Schur’s $Q$-functions. In this argument we use the formula (21) for dimension of skew shifted Young diagrams which is due to V. Ivanov [10]. Note that the formula (47) for the action of $A$ on Schur’s $Q$-functions is not formally necessary for the rest of the results of the present paper (see Remark 8.11 below).

### 1.7 Organization of the paper

In §2.1–2.2 we recall the definition of the Schur graph. We also recall coherent systems associated to this graph and the corresponding up/down Markov chains. In §2.3 we recall the multiplicative measures on the Schur graph introduced by A. Borodin [2]. They depend on a parameter $\alpha \in (0, +\infty)$.

In §3 we introduce Kerov interlacing coordinates of shifted Young diagrams and study their properties. Here we restate some of the results of the paper [11] and apply them to our situation.

Note that in [2] the parameter $\alpha$ is denoted by $x$. 

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In §4.1 we consider the polynomial algebra \( \Gamma \) generated by the odd Newton power sums. The basis for \( \Gamma \) is formed by Schur’s \( Q \)-functions \( Q_\lambda \) (indexed by strict partitions). In §4.2–4.3 we prove a useful formula for the action of the \( n \)th up/down Markov chain transition operator (corresponding to the multiplicative measures on the Schur graph) on Schur’s \( Q \)-functions. The argument here is the same as in [3] §4.

In §5 we prove some facts concerning the algebra \( \Gamma \) that are used in §6–7.

In §6–§7 we compute the “differential” form of the \( n \)th up/down Markov chain transition operator corresponding to the multiplicative measures (see Theorem 7.1 below for the exact result).

In §8 we use the general results of the paper [29] and the book [4] to prove the convergence, as \( n \to \infty \), of our up/down Markov chains to a continuous time Markov process \( X_\alpha(t) \) in \( \Omega_+ \). We also prove the differential formula for the pre-generator of this process and study some other properties of \( X_\alpha(t) \).

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2 Multiplicative measures

2.1 The Schur graph

A partition is an infinite non-increasing sequence of nonnegative integers

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, 0, 0, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0, \quad \lambda_i \in \mathbb{Z}_{>0}, \]

having only finitely many nonzero members. Their number \( \ell(\lambda) \geq 0 \) is called the length of the partition. The weight of the partition is \( |\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i \). A partition \( \lambda \) is called strict if it does not contain similar terms: \( \lambda_1 > \lambda_2 > \cdots > \lambda_{\ell(\lambda)} > 0 \).

We denote strict partitions by \( \lambda, \mu, \nu, \kappa, \ldots \), and ordinary (i.e., not necessary strict) partitions by \( \sigma, \rho, \tau, \ldots \).

As explained in [17] Ch. I, §1, Example 9], to every strict partition corresponds a shifted Young diagram. The shifted Young diagram of the form \( \lambda \) consists of \( \ell(\lambda) \) rows. Each \( i \)th row \( (i = 1, \ldots, \ell(\lambda)) \) has \( \lambda_i \) boxes, and for \( j = 1, \ldots, \ell(\lambda) - 1 \) the first box of the \( (j+1) \)th row is right under the second box of the \( j \)th row. We identify strict partitions and corresponding shifted Young diagrams. For example, Figure I shows the shifted Young diagram of the form \( (6, 5, 3, 1) \).

If \( \lambda \) and \( \mu \) are shifted Young diagrams and \( \lambda \) is obtained from \( \mu \) by adding one box, then we write \( \lambda \searrow \mu \) (or, equivalently, \( \mu \nearrow \lambda \)). Denote this box (that distinguishes \( \lambda \) and \( \mu \)) by \( \lambda/\mu \).
For two shifted Young diagrams $\mu$ and $\lambda$ such that $|\lambda| = |\mu| + 1$ we set

$$\kappa(\mu, \lambda) := \begin{cases} 2, & \text{if } \mu \not\subset \lambda \text{ and } \ell(\lambda) = \ell(\mu); \\ 1, & \text{if } \mu \not\subset \lambda \text{ and } \ell(\lambda) = \ell(\mu) + 1; \\ 0, & \text{otherwise.} \end{cases}$$

(2)

All shifted Young diagrams are organized in a graded set $S = \bigsqcup_{n=0}^{\infty} S_n$, where $S_n := \{\lambda: |\lambda| = n\}$, $n \in \mathbb{Z}_{\geq 0}$, and $S_0 := \{\emptyset\}$. This set is equipped with the structure of a graded graph. It has edges only between consecutive floors $S_n$ and $S_{n+1}$, $n \in \mathbb{Z}_{\geq 0}$. If $\mu \in S_n$ and $\lambda \in S_{n+1}$, then we draw $\kappa(\mu, \lambda)$ edges between $\mu$ and $\lambda$. Let edges be oriented in the direction from $S_n$ to $S_{n+1}$. We call this oriented graded graph the Schur graph.

By $h(\mu, \lambda)$ denote the total number of (oriented) paths from $\mu$ to $\lambda$ in the graph $S$. Clearly, $h(\mu, \lambda)$ vanishes unless $\mu \subset \lambda$ (that is, unless the shifted Young diagram $\mu$ is a subset of the shifted Young diagram $\lambda$). Set $h(\lambda) := h(\emptyset, \lambda)$. This function has the form [17, Ch. III, §8, Example 12]:

$$h(\lambda) = 2^{|\lambda| - \ell(\lambda)} \cdot \frac{|\lambda|!}{\lambda_1! \lambda_2! \cdots \lambda_{\ell(\lambda)}!} \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \quad \lambda \in S. \quad (3)$$

Note that if $\lambda$ is not strict, then this formula reduces to $h(\lambda) = 0$. There is also an explicit formula for the function $h(\mu, \lambda)$, it was proved in [10]. We recall this result below, see (21).

### 2.2 Coherent systems and up/down Markov chains

Here we give definitions of a coherent system on the Schur graph and of up/down Markov chains associated to it. We follow [3, §1].

The down transition function for $\mu, \lambda \in S$ such that $|\lambda| = |\mu| + 1$ is

$$p^\downarrow(\lambda, \mu) := \frac{h(\mu)}{h(\lambda)^2} \kappa(\mu, \lambda).$$

(4)

**Footnotes:**

8 Sometimes (e.g., in [2]) the same graph with simple edges is called the Schur graph. These two graphs have the same down transition functions (see (22) below), hence for us the difference between them is inessential.

9 The factor $2^{|\lambda| - \ell(\lambda)}$ (that does not enter the corresponding formula in [17]) appears due to the edge multiplicities (2) in our version of the Schur graph.
It can be easily checked that

- \( p^↓(\lambda, \mu) \geq 0 \) for all \( \mu, \lambda \in \mathbb{S} \) such that \( |\lambda| = |\mu| + 1 \);
- \( p^↓(\lambda, \mu) \) vanishes unless \( \mu \not\succ \lambda \);
- if \( |\lambda| = n \geq 1 \), then \( \sum_{\mu : |\mu| = n-1} p^↓(\lambda, \mu) = 1 \).

**Definition 2.1.** A coherent system on \( \mathbb{S} \) is a system of probability measures \( M_n \) on \( \mathbb{S}_n \), \( n \in \mathbb{Z}_{\geq 0} \), consistent with the down transition function:

\[
M_n(\mu) = \sum_{\lambda : \lambda \prec_{\mathbb{S}} \mu} p^↓(\lambda, \mu)M_{n+1}(\lambda) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } \mu \in \mathbb{S}_n. \quad (5)
\]

Here by \( M_n(\mu) \) we denote the measure of a singleton \( \{\mu\} \).

Fix a coherent system \( \{M_n\} \). The up transition function for \( \lambda, \nu \in \mathbb{S} \) such that \( |\lambda| = n \), \( |\nu| = n+1 \), \( n \in \mathbb{Z}_{\geq 0} \), and \( M_n(\lambda) \neq 0 \) is

\[
p^↑(\lambda, \nu) := \frac{M_{n+1}(\nu)}{M_n(\lambda)}p^↓(\nu, \lambda).
\]

The up transition function depends on the choice of a coherent system. Moreover, \( \{M_n\} \) and \( p^↑ \) are consistent in a sense similar to (5):

\[
M_{n+1}(\nu) = \sum_{\lambda : \lambda \succ_{\mathbb{S}} \nu \land M_n(\lambda) \neq 0} p^↓(\lambda, \nu)M_n(\lambda) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } \nu \in \mathbb{S}_{n+1}. \quad (6)
\]

**Definition 2.2.** A system of measures \( \{M_n\} \), where \( M_n \) is a probability measure on \( \mathbb{S}_n \), \( n \in \mathbb{Z}_{\geq 0} \), is called nondegenerate, if \( M_n(\lambda) > 0 \) for all \( n \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{S}_n \).

Let \( \{M_n\} \) be a nondegenerate coherent system on \( \mathbb{S} \). For all \( n \in \mathbb{Z}_{\geq 0} \) we define a Markov chain \( T_n \) on the set \( \mathbb{S}_n \) with the following transition matrix:

\[
T_n(\lambda, \bar{\lambda}) := \sum_{\nu : |\nu| = n+1} p^↑(\lambda, \nu)p^↓(\nu, \bar{\lambda}), \quad \lambda, \bar{\lambda} \in \mathbb{S}_n.
\]

This is the composition of the up and down transition functions, from \( \mathbb{S}_n \) to \( \mathbb{S}_{n+1} \) and then back to \( \mathbb{S}_n \). From (5) and (6) it follows that \( M_n \) is a stationary distribution for \( T_n \). It can be readily shown that the matrix \( M_n(\lambda)T_n(\lambda, \bar{\lambda}) \) is symmetric with respect to the substitution \( \lambda \leftrightarrow \bar{\lambda} \). This means that the chain \( T_n \) is reversible with respect to \( M_n \).

**2.3 Multiplicative measures**

In this subsection we recall some definitions and results from [2] concerning multiplicative measures on the Schur graph.
**Definition 2.3.** For \( n \in \mathbb{Z}_{\geq 0} \) the *Plancherel measure* on the set \( S_n \) is defined as

\[
\text{Pl}_n(\lambda) := \frac{h(\lambda)^2}{n!} 2^{\ell(\lambda)} n^{-n}, \quad \lambda \in S_n,
\]

where \( h(\lambda) \) is given by (3).

The Plancherel measures form a nondegenerate coherent system \( \{\text{Pl}_n\} \) on \( S \).

**Definition 2.4.** Let \( M_n \) be a probability measure on \( S_n \) for all \( n \in \mathbb{Z}_{\geq 0} \). The system of measures \( \{M_n\} \) is called *multiplicative* if

\[
M_n(\lambda) = \text{Pl}_n(\lambda) \cdot \frac{1}{Z(n)} \cdot \prod_{\square \in \lambda} f(i(\square), j(\square)) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } \lambda \in S_n \tag{7}
\]

for some functions \( f: \mathbb{Z}_{\geq 0}^2 \to \mathbb{C} \) and \( Z: \mathbb{Z}_{\geq 0} \to \mathbb{C} \). Here the product is taken over all boxes in the shifted diagram \( \lambda \), and the numbers \( i(\square) \) and \( j(\square) \) are the row and column numbers of the box \( \square \), respectively.\(^\text{10}\)

**Theorem 2.5** (Borodin [2]). A nondegenerate multiplicative system of probability measures \( \{M_n\} \) is coherent if and only if the functions \( f(i, j) \) and \( Z(n) \) from (7) have the form

\[
f(i, j) = f_\alpha(i, j) := (j - i)(j - i + 1) + \alpha, \quad Z(n) = Z_\alpha(n) := \alpha(\alpha + 1)(\alpha + 2)(\alpha + 4)\ldots(\alpha + 2n - 2) \tag{8}
\]

for some parameter \( \alpha \in (0, +\infty] \).

We denote the multiplicative coherent system corresponding to \( \alpha \) by \( \{M_n^\alpha\} \). Below we call this object simply the *multiplicative measures*.\(^\text{11}\) The up transition function corresponding to \( \{M_n^\alpha\} \) can be written out explicitly:

\[
p_\alpha^1(\lambda, \nu) = \frac{c(\nu/\lambda)(c(\nu/\lambda) + 1) + \alpha}{2|\lambda| + \alpha} \cdot \frac{h(\nu)}{h(\lambda)(|\lambda| + 1)}, \tag{9}
\]

where \( \alpha \in (0, +\infty] \) and \( c(\square) := j(\square) - i(\square) \) is the *content* of the box \( \square \).

**Remark 2.6.** One can consider *degenerate multiplicative measures*. That is, for certain negative values of \( \alpha \) the formulas (7)–(8) define a system of measures \( \{M_n^\alpha\} \) not on the whole Schur graph \( S \), but on a certain finite subset of \( S \).

Namely, if \( \alpha = \alpha_N := -N(N + 1) \) for some \( N \in \mathbb{Z}_{>0} \), then \( M_n^{\alpha_N} \) is a probability measure on \( S_n \) for all \( n = 0, 1, \ldots, \frac{N(N + 1)}{2} \). The system \( \{M_n^{\alpha_N}\} \) satisfies (3) for \( n = 0, 1, \ldots, \frac{N(N + 1)}{2} \). It is clear from (7)–(8) that \( M_n^{\alpha_N}(\lambda) > 0 \) iff \( \lambda_1 \leq N \).

Thus, one can say that \( \{M_n^{\alpha_N}\}_{n=0,1,\ldots,\frac{N(N+1)}{2}} \) is a coherent system of measures on the finite graded graph \( S(N) := \{\lambda \in S: \lambda_1 \leq N\} \subset S \).

\(^\text{10}\) The row number is counted from up to down, and the column number is counted from left to right.

\(^\text{11}\) Note that for all \( \lambda \in S_n \) the ratio \( \prod_{\square \in \lambda} f_\alpha(i(\square), j(\square))/Z_\alpha(n) \) tends to one as \( \alpha \to +\infty \). Thus, one can say that \( \{M_n^\alpha\} \) coincides with the Plancherel coherent system.
The existence of degenerate multiplicative coherent systems is a useful observation, but in the present paper we concentrate on the case $\alpha \in (0, +\infty)$.

**Definition 2.7.** In the rest of the paper the parameter $\alpha$ takes values in $(0, +\infty)$. From now on by $T_n$ we denote the one-step transition operator of the $n$th up/down Markov chain corresponding to the multiplicative measures with parameter $\alpha$. This operator $T_n$ acts on functions on $\mathbb{S}_n$ (see §4.3 below for more detail).

### 3 Kerov interlacing coordinates of shifted Young diagrams

In this section we introduce Kerov interlacing coordinates of shifted Young diagrams and study their basic properties. These coordinates are similar to interlacing coordinates of ordinary Young diagrams introduced by S. Kerov, see [11]. In §3.2 and §3.3 we express the Schur graph’s Plancherel up transition function $p^1_\infty$ and the down transition function $p^1$, respectively, in terms of Kerov interlacing coordinates. This approach is similar to that explained in [11] and used in [20], but there are some significant differences.

#### 3.1 Definition and basic properties

Let $\lambda \in \mathbb{S}_n$, $n \geq 1$. Denote by $X(\lambda)$ the set of numbers $\{c(\nu/\lambda) : \nu \searrow \lambda\}$, that is, $X(\lambda)$ is the set of contents of all boxes that can be added to the shifted Young diagram $\lambda$. For every $x \in X(\lambda)$ there exists a unique shifted diagram $\nu \searrow \lambda$ such that $c(\nu/\lambda) = x$. Denote this diagram $\nu$ by $\lambda + \square(x)$. Similarly, let $Y(\lambda) := \{c(\lambda/\mu) : \mu \nearrow \lambda\}$ be the set of contents of all boxes that can be removed from the shifted Young diagram $\lambda$. For every $y \in Y(\lambda)$ there exists a unique shifted diagram $\mu \nearrow \lambda$ such that $c(\lambda/\mu) = y$. Denote this diagram $\mu$ by $\lambda - \square(y)$.

For $\lambda = \emptyset$ we set $X(\emptyset) := \{0\}$, $Y(\emptyset) := \emptyset$.

**Definition 3.1.** Let $\lambda \in \mathbb{S}$. Suppose that the sets $X(\lambda)$ and $Y(\lambda)$ are written in ascending order. The numbers $[X(\lambda); Y(\lambda)]$ are called **Kerov coordinates** of a shifted Young diagram $\lambda$.

Figure 2 shows Kerov coordinates of two different shifted Young diagrams. Namely, for $\mu = (6, 5, 1)$ Kerov coordinates are $X(\mu) = \{1, 6\}$ and $Y(\mu) = \{0, 4\}$ (Figure 2a-b); and for $\nu = (6, 5, 3)$ these coordinates are $X(\nu) = \{0, 3, 6\}$ and $Y(\nu) = \{2, 4\}$ (Figure 2c-d).

**Proposition 3.2** (The interlacing property). Let $\lambda$ be a shifted Young diagram.

(a) If $\lambda$ contains a one-box row (see, for example, Figure 2a-b), then for some integer $d \geq 1$ we have

$$X(\lambda) = \{x_1, \ldots, x_d\}, \quad Y(\lambda) = \{0, y_2, \ldots, y_d\}$$
and

\[ 0 = y_1 < x_1 < y_2 < x_2 < \cdots < y_d < x_d. \]

(b) If \( \lambda \) does not contain a one-box row (see, for example, Figure 2c-d), then for some integer \( d \geq 0 \) we have\(^{12}\)

\[ X(\lambda) = \{0, x_1, \ldots, x_d\}, \quad Y(\lambda) = \{y_1, \ldots, y_d\} \]

and

\[ 0 = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_d < x_d. \]

Proof. This can be simply proved by induction on the number of boxes of \( \lambda \), by consecutively adding a box to the diagram. During this procedure the change of the number of one-box rows in the diagram\(^{13}\) leads to transition from the case (a) to the case (b) and vice versa. \( \square \)

Remark 3.3. In the case of ordinary Young diagrams (see, e.g., [11, 20]) the number of elements in the set \( X(\lambda) \) is always greater by one than the number of elements in the set \( Y(\lambda) \). In our case it is not always true.

Let us define \( X'(\lambda) := X(\lambda) \setminus \{0\} \). It is clear that the numbers of elements in the sets \( X'(\lambda) \) and \( Y(\lambda) \) are equal for all shifted diagrams \( \lambda \). We will use this fact below.

Remark 3.4. As can be also proved by induction on the number of boxes (similarly to Proposition 3.2), a shifted Young diagram \( \lambda \) is uniquely determined by its Kerov coordinates \([X(\lambda); Y(\lambda)]\), or, equivalently, by the pair of sequences \( X'(\lambda) \) and \( Y(\lambda) \).

\(^{12}\)Note that \( d = 0 \) only for \( \lambda = \emptyset \).

\(^{13}\)This number is always zero or one.
Remark 3.5. Let $\lambda$ be a nonempty shifted Young diagram, and

$$X'(\lambda) = \{x_1, \ldots, x_d\}, \quad Y(\lambda) = \{y_1, \ldots, y_d\}$$

for some integer $d \geq 1$. It can be easily seen that $\lambda$ has the form

$$\lambda = (x_d, x_d - 1, \ldots, y_d + 1, x_d - 1, \ldots, y_d - 1, y_d + 1, \ldots, x_1, x_1 - 1, \ldots, y_1 + 1)$$

(see Figure 3). Here for all $j$ the numbers $x_j, x_j - 1, \ldots, y_j + 1$ are consecutive decreasing integers (for some $j$ it can happen that $x_j = y_j + 1$). Note that $y_1$ can be zero, this corresponds to the case (a) in Proposition 3.2.

![Figure 3](image)

Proposition 3.6. For every shifted Young diagram $\lambda$ we have

$$\sum_{x \in X(\lambda)} x(x + 1) - \sum_{y \in Y(\lambda)} y(y + 1) = 2|\lambda|.$$ 

Proof. If $\lambda = \emptyset$, the claim is obvious. Suppose $\lambda \neq 0$. We have

$$\sum_{x \in X(\lambda)} x(x + 1) - \sum_{y \in Y(\lambda)} y(y + 1) = \sum_{j=1}^{d} (x_j(x_j + 1) - y_j(y_j + 1)),$$

where the notation is as in Remark 3.5. For all $j$ the value $x_j(x_j + 1) - y_j(y_j + 1)$ clearly equals twice the area of the part of the shifted Young diagram $\lambda$ formed by the rows $x_j, x_j - 1, \ldots, y_j + 1$ (see Figure 3). This concludes the proof.

3.2 The Plancherel up transition function

The up transition function corresponding to the Plancherel coherent system on $S$ (see §2.3) can be written in terms of Kerov interlacing coordinates of shifted Young diagrams.
Let \( \lambda \) be an arbitrary shifted Young diagram and \( v \) be a complex variable. By definition, put

\[
\mathcal{R} \uparrow (v; \lambda) := \frac{\prod_{y \in Y(\lambda)} (v - y(y + 1))}{v \cdot \prod_{x \in X'(\lambda)} (v - x(x + 1))}.
\]  

(10)

It follows from Remark 3.3 that the degree of the denominator is always greater than the degree of the numerator by one. Next, from Proposition 3.2 it follows that if \( \lambda \) contains a one-box row, then the numerator and the denominator of \( \mathcal{R} \uparrow (v; \lambda) \) can both be divided by the factor \( v \), and if \( \lambda \) does not contain a one-box row, then the fraction in the RHS of (10) is irreducible. In either case, the denominator of the irreducible form of the fraction \( \mathcal{R} \uparrow (v; \lambda) \) is equal to

\[
\prod_{x \in X(\lambda)} (v - x(x + 1)).
\]

Let \( \theta^\uparrow_x (\lambda) \), \( x \in X(\lambda) \), be the following expansion coefficients of \( \mathcal{R} \uparrow (v; \lambda) \) as a sum of partial fractions:

\[
\mathcal{R} \uparrow (v; \lambda) = \sum_{x \in X(\lambda)} \frac{\theta^\uparrow_1 (\lambda)}{v - x(x + 1)}.
\]  

(11)

**Proposition 3.7.** For every shifted Young diagram \( \lambda \) and all \( x \in X(\lambda) \) we have

\[
\theta^\uparrow_x (\lambda) = p^\uparrow_\infty (\lambda, \lambda + \square(\hat{x})),
\]

where \( p^\uparrow_\infty (\cdot, \cdot) \) is the up transition function corresponding to the Plancherel coherent system on \( \mathbb{S} \) (see §2.3).

**Proof.** It follows from (11) by the residue formula that

\[
\theta^\uparrow_x (\lambda) = \begin{cases} 
\prod_{y \in Y(\lambda)} (\hat{x}(\hat{x} + 1) - y(y + 1)) & \text{if } \hat{x} \neq 0; \\
\hat{x}(\hat{x} + 1) \prod_{x \in X'(\lambda), \ x \neq \hat{x}} (\hat{x}(\hat{x} + 1) - x(x + 1)) & \text{if } \hat{x} = 0
\end{cases}
\]

(12)

for every \( \hat{x} \in X(\lambda) \).

Taking the limit as \( \alpha \to +\infty \) in (9), we obtain the following expression for the Plancherel up transition function:

\[
p^\uparrow_\infty (\lambda, \lambda + \square(\hat{x})) = \frac{h(\lambda + \square(\hat{x}))}{h(\lambda)(|\lambda| + 1)}, \quad \hat{x} \in X(\lambda),
\]

where \( h \) is given by (9).

Let us check that the two above expressions coincide. Assume first that \( \hat{x} \neq 0 \). Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) and \( \lambda + \square(\hat{x}) = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_\ell) \).
for some $1 \leq k \leq \ell$. Note that $\lambda_k = \hat{x}$. Using (3), we have

$$
p^\lambda_{\infty}(\lambda, \lambda + \Box(\hat{x})) = \frac{h(\lambda + \Box(\hat{x}))}{h(\lambda)(|\lambda| + 1)}
= \frac{2^{|\lambda|+1-\ell(\lambda+\Box(\hat{x}))}(|\lambda| + 1)!}{(\hat{x} + 1)! \cdot \prod_{i \neq k} \lambda_i!} \times
\prod_{i=1}^{k-1} \frac{\lambda_i - \hat{x} - 1}{\lambda_i + \hat{x} + 1} \cdot \prod_{j=k+1}^{\ell} \frac{\hat{x} + 1 - \lambda_j}{\hat{x} + 1 + \lambda_j} \cdot \prod_{1 \leq i < j \leq \ell, i,j \neq k} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \times
\frac{2^{\ell(\lambda)-\ell(\lambda+\Box(\hat{x}))}}{\hat{x} + 1} \cdot \prod_{1 \leq i \leq \ell, i \neq k} \frac{\hat{x}(\hat{x} + 1) - \lambda_i(\lambda_i - 1)}{\hat{x}(\hat{x} + 1) - \lambda_i(\lambda_i + 1)}.
$$

Using Remark 3.5 one can decompose the last product as follows:

$$
\prod_{1 \leq i \leq \ell, i \neq k} \frac{\hat{x}(\hat{x} + 1) - \lambda_i(\lambda_i - 1)}{\hat{x}(\hat{x} + 1) - \lambda_i(\lambda_i + 1)} = \prod_{m=1}^{d} \prod_{r=y_{m+1}}^{x_m} \frac{\hat{x}(\hat{x} + 1) - r(r - 1)}{\hat{x}(\hat{x} + 1) - r(r + 1)},
$$

where $X'(\lambda) = \{x_1, \ldots, x_d\}$ and $Y(\lambda) = \{y_1, \ldots, y_d\}$.

Fix $m = 1, \ldots, d$. It can be readily verified that

$$
\prod_{r=y_{m+1}}^{x_m} \frac{\hat{x}(\hat{x} + 1) - r(r - 1)}{\hat{x}(\hat{x} + 1) - r(r + 1)} = \begin{cases} 
\frac{\hat{x}(\hat{x} + 1) - y_m(y_m + 1)}{\hat{x}(\hat{x} + 1) - x_m(x_m + 1)}, & \hat{x} \neq x_m; \\
\frac{1}{2x}(\hat{x}(\hat{x} + 1) - y_m(y_m + 1)), & \hat{x} = x_m.
\end{cases}
$$

Observe that if $\hat{x} \neq 0$, then $\ell(\lambda) = \ell(\lambda + \Box(\hat{x}))$. It can be readily verified that the above expression (12) for $\theta^\lambda_{\hat{x}}(\lambda)$ coincides with

$$
p^\lambda_{\infty}(\lambda, \lambda + \Box(\hat{x})) = \frac{2}{\hat{x} + 1} \prod_{m=1}^{d} \prod_{r=y_{m+1}}^{x_m} \frac{\hat{x}(\hat{x} + 1) - r(r - 1)}{\hat{x}(\hat{x} + 1) - r(r + 1)}.
$$

The case $\hat{x} = 0$ can be considered similarly with the observation that in this case $\ell(\lambda + \Box(\hat{x})) = \ell(\lambda) + 1$. This concludes the proof.

**Remark 3.8.** If we take the limit transition as $v \to \infty$ in (11), we obtain

$$
\lim_{v \to \infty} vR^\lambda(v; \lambda) = 1 = \lim_{v \to \infty} \sum_{x \in X(\lambda)} \frac{v \cdot p^\lambda_{\infty}(\lambda, \lambda + \Box(x))}{v + x(x + 1)} = \sum_{x \in X(\lambda)} p^\lambda_{\infty}(\lambda, \lambda + \Box(x))
$$

for all $\lambda \in \mathbb{S}$ and $x \in X(\lambda)$, as it should be.
Note that now (12) can be rewritten as (here \( \alpha \in (0, +\infty) \))

\[
p_{\alpha}^1(\lambda, \lambda + \square(x)) = \frac{x(x + 1) + \alpha}{2|\lambda| + \alpha} \cdot \theta_{\alpha}^1(\lambda)
\]

for all \( \lambda \in S \) and \( x \in X(\lambda) \). \( (13) \)

### 3.3 The down transition function

The down transition function of the Schur graph can be written in terms of Kerov interlacing coordinates of shifted Young diagrams.

Let \( \lambda \) be an arbitrary nonempty shifted Young diagram and \( v \) be a complex variable. By definition, put

\[
R^\downarrow(v; \lambda) := \frac{1}{v R^\uparrow(v; \lambda)} = \prod_{x \in X'(\lambda)} (v - x(x + 1)) \prod_{y \in Y(\lambda)} (v - y(y + 1)).
\]

Observe that the numerator and the denominator both have \( v^d \) as the term of maximal degree in \( v \), where \( d \geq 0 \) is the number of elements in the set \( X'(\lambda) \) (or, equivalently, in \( Y(\lambda) \), see Remark 3.3).

Let \( \theta^\downarrow_y(\lambda), y \in Y(\lambda), \) be the following expansion coefficients of \( R^\downarrow(v; \lambda) \) as a sum of partial fractions:

\[
R^\downarrow(v; \lambda) = 1 - \sum_{y \in Y(\lambda)} \frac{\theta^\downarrow_y(\lambda)}{v - y(y + 1)}.
\] \( (14) \)

**Proposition 3.9.** For every nonempty shifted Young diagram \( \lambda \) we have

\[
\theta^\downarrow_y(\lambda) = 2|\lambda| \cdot p^\downarrow(\lambda, \lambda - \square(y)), \quad y \in Y(\lambda),
\]

where \( p^\downarrow(\cdot, \cdot) \) is the down transition function (see \( \S3 \) for the definition).

**Proof.** It follows from (14) by the residue formula that

\[
\theta^\downarrow_y(\lambda) = \frac{\prod_{x \in X'(\lambda)} (\hat{y}(\hat{y} + 1)) - x(x + 1)}{\prod_{y \in Y(\lambda), y \neq \hat{y}} (\hat{y}(\hat{y} + 1) - y(y + 1))}
\]

for every \( \hat{y} \in Y(\lambda) \).

Next, we can rewrite the definition of the down transition function (3) as

\[
p^\downarrow(\lambda, \lambda - \square(\hat{y})) = \frac{h(\lambda - \square(\hat{y}))}{h(\lambda)} \cdot 2^{1 - \delta(\hat{y})},
\]

where \( \delta(\cdot) \) is the Kronecker delta, and the function \( h \) is given by (3).

It can be shown exactly as in the proof of Proposition 3.7 (using Remark 3.5) that the two above expressions coincide. \( \square \)

### 4 The up/down Markov chains and doubly symmetric functions

In this section we compute the action of the operators \( T_n \) from Definition 2.7 on doubly symmetric functions (Theorem 4.8). We argue similarly to [3, §4].

14
4.1 Doubly symmetric functions

In this subsection we briefly recall the definitions of the algebra of doubly symmetric functions and some related objects. Exact definitions and proofs concerning this subject can be found, e.g., in the paper by V. Ivanov [10]. See also [17, Ch. III, §8] and [25].

Let \( \Lambda \) denote the algebra of real symmetric functions in \((n)\) variables \( y_1, y_2, \ldots \). This algebra is freely generated (as a commutative unital algebra) by Newton power sums \( p_k := \sum_{i=1}^{\infty} y_i^k, \quad k \in \mathbb{Z}_{>0} \). We write \( \Lambda = \mathbb{R}[p_1, p_2, p_3, \ldots] \).

By \( \Gamma \) we denote the subalgebra of \( \Lambda \) generated by the odd Newton power sums, \( \Gamma = \mathbb{R}[p_1, p_3, p_5, \ldots] \). We call \( \Gamma \) the algebra of doubly symmetric functions.

Remark 4.1. The subalgebra of \( \Lambda \) generated by the odd Newton power sums was studied by various authors. However, there is no common notation for it. For example, in [10] and [17] it is denoted by \( \Gamma \), in [9] — by \( \Delta \), in the papers [26, 27] — by \( \Omega \), and in a recent paper [11] — by \( D \). In [10] it is called the algebra of supersymmetric functions, and in [11] — the algebra of doubly symmetric functions. In the present paper we adopt the latter term and the notation \( \Gamma \) for this algebra.

We do not use the term “supersymmetric functions” because it was used by J. Stembridge [25] in a different sense. Namely, he studied the unital algebra generated by the following supersymmetric power sums in two sets of variables \( u_i \) and \( v_j \):

\[
p_k(u_1, u_2, \ldots; v_1, v_2, \ldots) = \sum_{i=1}^{\infty} u_i^k - \sum_{j=1}^{\infty} v_j^k, \quad k = 1, 2, \ldots.
\]

The algebra \( \Gamma \) defined above is generated by supersymmetric power sums in variables \( \{y_1, y_2, \ldots\} \) and \( \{-y_1, -y_2, \ldots\} \). Clearly, \( \Gamma \) can also be viewed as a subalgebra of that supersymmetric algebra.

The algebra \( \Gamma \) consists of all \( f \in \Lambda \) such that for every \( 1 \leq i < j \) the expression

\[
f(y_1, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_{j-1}, -z, y_{j+1}, \ldots)
\]

does not depend on \( z \) (here \( z \) is another independent formal variable).\(^{15}\) From [1] it follows that \( \Gamma \) can be viewed as the quotient of \( \Lambda \) by the ideal generated by all \( s_\sigma - s_{\sigma'} \), where \( s_\sigma \) is the ordinary Schur function, \( \sigma \) runs over all ordinary partitions and \( \sigma' \) denotes the conjugate of the partition \( \sigma \).

There is a natural filtration of the algebra \( \Lambda \) by degrees of polynomials in formal variables \( y_i \). This filtration is determined by setting \( \deg p_k = k, \quad k \in \mathbb{Z}_{>0} \).

The subalgebra \( \Gamma \subset \Lambda \) inherits this filtration from \( \Lambda \) and thus becomes a filtered

---

\(^{14}\) The definitions and related discussions can also be found in [17].

\(^{15}\) It is clear that the odd Newton power sums satisfy that property and the even do not. The fact that every \( f \in \Lambda \) satisfying that property is a polynomial in the odd Newton power sums follows from [25].
algebra with the filtration determined by setting \( \deg p_{2m-1} = 2m - 1, m \in \mathbb{Z}_{>0} \). More precisely,

\[
\Gamma = \bigcup_{m=0}^{\infty} \Gamma^{(m)}, \quad \Gamma^{(0)} \subset \Gamma^{(1)} \subset \Gamma^{(2)} \subset \ldots \subset \Gamma,
\]

where \( \Gamma^{(m)} \) is the finite-dimensional subspace of \( \Gamma \) consisting of elements of degree \( \leq m \):

\[\Gamma^{(0)} = \mathbb{R}1, \quad \Gamma^{(m)} = \text{span} \{p_{r_1}^{r_1} p_{r_2}^{r_2} \ldots : r_1 + 3r_2 + \cdots \leq m\}, \quad m = 1, 2, \ldots.\]

Finite products of the form \( p_{r_1}^{r_1} p_{r_2}^{r_2} \ldots \) constitute a linear basis for \( \Gamma \) as a vector space over \( \mathbb{R} \). Every element \( p_{r_1}^{r_1} p_{r_2}^{r_2} \ldots \) is homogeneous. We will need two more linear bases for \( \Gamma \), of which one is also homogeneous and the other is not.

**Definition 4.2** (Schur’s \( Q \)-functions). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, 0, 0, \ldots) \) be an arbitrary strict partition. For every \( n \geq \ell(\lambda) \) set

\[
R_{\lambda[n]}(y_1, \ldots, y_n) := y_1^{\lambda_1} \ldots y_n^{\lambda_{\ell(\lambda)}} \cdot \prod_{i \leq \ell(\lambda)} \frac{y_i + y_j}{y_i - y_j}. \quad (15)
\]

If \( n \geq \ell(\lambda) \), define\(^{10}\)

\[
Q_{\lambda}(y_1, \ldots, y_n, 0, \ldots) := \frac{2^{\ell(\lambda)}}{(n - \ell(\lambda))!} \sum_{w \in S_n} R_{\lambda[n]}(y_{w(1)}, \ldots, y_{w(n)}), \quad (16)
\]

and \( Q_{\lambda}(y_1, \ldots, y_n, 0, \ldots) := 0 \) otherwise. The expressions \( Q_{\lambda}(y_1, \ldots, y_n, 0, \ldots) \), \( n \in \mathbb{Z}_{>0} \), define a doubly symmetric function \( Q_{\lambda} \in \Gamma \)\(^{17}\). It is called Schur’s \( Q \)-function.

Each \( Q_{\lambda}, \lambda \in \mathcal{S} \), is a homogeneous element of degree \( |\lambda| \). The system \( \{Q_{\lambda}\}_{\lambda \in \mathcal{S}} \) is a linear basis for the algebra \( \Gamma \) over \( \mathbb{R} \).

**Definition 4.3** (Factorial Schur’s \( Q \)-functions). The factorial analogues of Schur’s \( Q \)-functions are defined as in \(^{15} \), \(^{10} \), with \( R_{\lambda[n]} \) replaced by

\[
R^{\lambda}_{\lambda[n]}(y_1, \ldots, y_n) := y_1^{1_{\lambda_1}} \ldots y_n^{1_{\lambda_{\ell(\lambda)}}} \cdot \prod_{i \leq \ell(\lambda)} \frac{y_i + y_j}{y_i - y_j}.
\]

Here \( y_i^{1_{\lambda_i}} \) is the decreasing factorial power defined as \( a^{1_k} := a(a-1) \ldots (a-k+1), k \in \mathbb{Z}_{>0}, a^{10} := 1 \). The functions \( Q^\lambda_{\lambda}, \lambda \in \mathcal{S} \), are called factorial Schur’s \( Q \)-functions.

For all \( \lambda \in \mathcal{S} \) we have \( Q^\lambda_{\lambda} = Q_{\lambda} + g \), where \( g \) is a doubly symmetric function with \( \deg g < |\lambda| = \deg Q_{\lambda} \). It follows that the system \( \{Q^\lambda_{\lambda}\}_{\lambda \in \mathcal{S}} \) is also a linear basis for \( \Gamma \) as a vector space over \( \mathbb{R} \).

\(^{16}\)Here \( \mathcal{S}_n \) is the symmetric group.

\(^{17}\)This follows from \(^{10} \). Note that that paper deals with Schur’s \( P \)-functions. They are linear multiples of the \( Q \)-functions: \( P_{\lambda} = 2^{-\ell(\lambda)}Q_{\lambda}, \lambda \in \mathcal{S} \).
4.2 A representation of $\mathfrak{sl}(2, \mathbb{C})$

By $\text{Fun}_0(\mathbb{S})$ denote the algebra of real finitely supported functions on the Schur graph $\mathbb{S}$ with pointwise operations. A natural basis for $\text{Fun}_0(\mathbb{S})$ is $\{\varepsilon_\mu\}_{\mu \in \mathbb{S}}$, where

$$\varepsilon_\mu(\lambda) = \begin{cases} 1, & \text{if } \lambda = \mu; \\ 0, & \text{otherwise}. \end{cases}$$

Let $E$, $F$, and $H$ be the following operators in $\text{Fun}_0(\mathbb{S})$ which are similar to Kerov’s operators (see [19] for definition):

$$E \varepsilon_\lambda := \sum_{x \in X(\lambda)} 2^{-\delta(x)} (x(x+1) + \alpha) \varepsilon_{\lambda+\Box(x)};$$

$$F \varepsilon_\lambda := -\sum_{y \in Y(\lambda)} \varepsilon_{\lambda-\Box(y)};$$

$$H \varepsilon_\lambda := \left(\frac{\alpha}{2} + 2|\lambda|\right) \varepsilon_\lambda.$$

Lemma 4.4. For all $\alpha \in \mathbb{R}$ these operators satisfy the commutation relations

$$[E,H] = -2E, \quad [F,H] = 2F, \quad [E,F] = H. \quad (17)$$

Proof. The proof uses the results of §3.1 and is similar to the proof of Lemma 4.2 of the paper [3]. \qed

Corollary 4.5. The correspondence

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to E, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to F, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to H$$

defines a representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in the space $\text{Fun}_0(\mathbb{S})$.

Lemma 4.6. Fix $N \in \mathbb{Z}_{>0}$. Let $V_N$ be the finite-dimensional subspace of $\text{Fun}_0(\mathbb{S})$ spanned by the basis vectors $\varepsilon_\lambda$ with $\lambda_1 \leq N$. If $\alpha = -N(N+1)$, then $V_N$ is invariant under the action of the operators $E$, $F$, and $H$, and the action of $\mathfrak{sl}(2, \mathbb{C})$ in $V_N$ defined in Corollary 4.5 lifts to a representation of the group $\text{SL}(2, \mathbb{C})$ in $V_N$.

Proof. This can be proved exactly as Lemma 4.3 of the paper [3]. \qed

4.3 The action of $T_n$ on factorial Schur’s $Q$-functions

For every set $\mathbb{X}$ by $\text{Fun}(\mathbb{X})$ denote the algebra of real-valued functions on $\mathbb{X}$ with pointwise operations.

Consider an embedding of the algebra $\Gamma$ described in §4.1 into the algebra $\text{Fun}(\mathbb{S})$. This embedding is defined on the generators of $\Gamma$:

$$p_k \mapsto p_k(\lambda) := \sum_{i=1}^{\ell(\lambda)} \lambda_i^k,$$ 

$k = 1, 3, 5, \ldots.$

---

18Here $\delta(\cdot)$ is the Kronecker delta.
19In fact, $\dim V_N = N(N+1)/2$. 

17
Thus, to every element \( f \in \Gamma \) corresponds a function from \( \text{Fun}(\mathbb{S}) \). Denote this function by \( f(\lambda) \). We identify the (abstract) algebra \( \Gamma \) with its image under this embedding, that is, with the algebra of functions \( \{ f(\cdot) \in \text{Fun}(\mathbb{S}) : f \in \Gamma \} \).

**Remark 4.7.** In [3] the role of \( \Gamma \) is played by the algebra generated by super-symmetric power sums (see Remark 4.1) in \( a_i \) and \(-b_j\), where \( a_i \) and \( b_j \) are the modified Frobenius coordinates of an ordinary Young diagram. The paper [20] deals with Jack deformations of these power sums.

For any \( f \in \Gamma \), by \( f_n \) denote the restriction of the function \( f(\cdot) \) to \( \mathbb{S}_n \subset \mathbb{S} \). It can be easily checked that the algebra \( \Gamma \subset \text{Fun}(\mathbb{S}) \) separates points of \( \mathbb{S} \). It follows that the functions of the form \( f_n \), with \( f \in \Gamma \), exhaust the (finite-dimensional) space \( \text{Fun}(\mathbb{S}_n) \), \( n \in \mathbb{Z}_{\geq 0} \).

Our aim in this section is to prove the following

**Theorem 4.8.** Let \( T_n : \text{Fun}(\mathbb{S}_n) \to \text{Fun}(\mathbb{S}_n) \), \( n \in \mathbb{Z}_{\geq 0} \), be the operator from Definition [2.7]. Its action on the functions \( (Q^*_\mu)_n \), \( \mu \in \mathbb{S} \), is as follows:

\[
(T_n - 1)(Q^*_\mu)_n = \frac{1}{(n+1)(n+\alpha/2)} \left[ -|\mu|(|\mu| + \alpha/2 - 1)(Q^*_\mu)_n + (n - |\mu| + 1) \sum_{y \in Y(\mu)} (y(y+1) + \alpha)(Q^*_{\mu-\Box(y)})_n \right],
\]

where \( 1 \) denotes the identity operator.

**Remark 4.9.** The above Theorem states that \( (T_n - 1)(Q^*_\mu)_n \) (for all \( \mu \in \mathbb{S} \)) is a linear combination of the function \( (Q^*_\mu)_n \) and the functions of the form \( (Q^*_\nu)_n \), where \( \nu \) runs over all shifted diagrams that can be obtained from \( \mu \) by deleting one box. Recall (3.1) that these diagrams are indexed by the set \( Y(\mu) \).

The proof of Theorem 4.8 uses the technique from [3 §4]. We do not want to repeat all details and in the rest of the section we give a scheme of the proof.

Fix arbitrary \( n \in \mathbb{Z}_{\geq 0} \) and \( \alpha \in (0, +\infty) \). We write \( T_n \) as the composition of “down” \( D_{n+1,n} : \text{Fun}(\mathbb{S}_n) \to \text{Fun}(\mathbb{S}_{n+1}) \) and “up” \( U_{n,n+1} : \text{Fun}(\mathbb{S}_{n+1}) \to \text{Fun}(\mathbb{S}_n) \) operators acting on functions:

\[
(D_{n+1,n}f_n)(\lambda) := \sum_{\mu : \mu \not\subseteq \lambda} p^\downarrow(\lambda, \mu)f_n(\mu), \quad \lambda \in \mathbb{S}_{n+1};
\]

\[
(U_{n,n+1}f_{n+1})(\nu) := \sum_{\kappa : \kappa \not\subseteq \nu} p^\uparrow(\nu, \kappa)f_{n+1}(\kappa), \quad \nu \in \mathbb{S}_n.
\]

The operator \( D_{n+1,n} \) is constructed using the down transition function \( p^\downarrow \) and does not depend on the parameter \( \alpha \). The operator \( U_{n,n+1} \) is constructed using the up transition function \( p^\uparrow \) and therefore depends on the parameter \( \alpha \).

**Remark 4.10.** These “down” and “up” operators act on functions. They are adjoint to the corresponding operators acting on measures. The latter act in accordance with their names, for example, the operator \( D^*_{n+1,n} \) maps \( M(\mathbb{S}_{n+1}) \) into \( M(\mathbb{S}_n) \), where \( M(X) \) denotes the space of measures on \( X \).
It clearly follows from the definition of the $n$th up/down Markov chain (§2.2) that $T_n = U_{n,n+1} \circ D_{n+1,n} : \text{Fun}(S_n) \to \text{Fun}(S_n)$. We deal with the operators $D_{n+1,n}$ and $U_{n,n+1}$ separately.

**Lemma 4.11** (The operator $D$). There exists a unique operator $D : \Gamma \to \Gamma$ such that

$$D_{n+1,n}f_n = \frac{1}{n+1}(Df)_{n+1}$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $f \in \Gamma$. In the basis $\{Q^*_\mu\}_{\mu \in S}$ for the algebra $\Gamma$ this operator has the form

$$DQ^*_\mu = (p_1 - |\mu|)Q^*_\mu.$$  \hfill (20)

**Proof.** The proof is exactly the same as the proof of Theorem 4.1 (1) of the paper [3], but instead of the facts about Frobenius-Schur functions we refer to the following formula which is due to V. Ivanov [10].

Let $|\lambda| = n$, $\mu \in S$ and $|\mu| \leq n$. Then

$$h(\mu, \lambda) = 2^{-|\mu|} \frac{(Q^*_\mu)_n(\lambda)}{n(n-1) \cdots (n-|\mu|+1)}.$$  \hfill (21)

We also use the recurrence relations for the function $h(\mu, \lambda)$ which directly follow from its definition (4.2):

$$h(\mu, \nu) = \sum_{\lambda : \lambda \vdash \nu} h(\mu, \lambda) \kappa(\lambda, \nu) \quad \text{for all } \mu, \nu \in S.$$  \hfill (4.2)

The rest of the proof repeats that of [3, Theorem 4.1 (1)]. \hfill \square

**Lemma 4.12** (The operator $U$). For every $\alpha \in (0, +\infty)$ there exists a unique operator $U : \Gamma \to \Gamma$ depending on $\alpha$ such that

$$U_{n,n+1}f_{n+1} = \frac{1}{n+\alpha/2}(Uf)_n$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $f \in \Gamma$. In the basis $\{Q^*_\mu\}_{\mu \in S}$ for the algebra $\Gamma$ this operator has the form

$$UQ^*_\mu = \left( p_1 + |\mu| + \frac{\alpha}{2} \right) Q^*_\mu + \sum_{y \in Y(\mu)} (y(y+1) + \alpha) Q^*_{\mu-\square(y)}.$$  \hfill (22)

**Proof.** The proof is similar to that of Theorem 4.1 (2) of the paper [3].

We must prove that

$$\left( n + \frac{\alpha}{2} \right) (U_{n,n+1}(Q^*_\mu)_{n+1})(\lambda) = \left( n + k + \frac{\alpha}{2} \right) (Q^*_\mu)_n(\lambda)$$

$$+ \sum_{y \in Y(\mu)} (y(y+1) + \alpha) (Q^*_{\mu-\square(y)})_n(\lambda)$$  \hfill (23)

for all $\mu, \lambda \in S$ such that $|\mu| = k$ and $|\lambda| = n \geq k$. 

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If \( |\mu| = 0 \), that is, \( \mu \) is an empty partition, then \( Q^*_\mu \equiv 1 \) and (23) clearly holds.

Now let \( |\mu| = k \geq 1 \). Using (9), (21) and the definition of \( U_{n,n+1} \) one can reduce (23) to the following equivalent combinatorial identity:

\[
\sum_{x \in X(\lambda)} (x(x+1) + \alpha) h(\mu, \lambda + \square(x)) = 2 \left( n + k + \frac{\alpha}{2} \right) (n - k + 1) h(\mu, \lambda) + \sum_{y \in Y(\mu)} \left( y(y+1) + \alpha \right) h(\mu - \square(y), \lambda),
\]

where \( |\lambda| = n \) and \( |\mu| = k \leq n \).

This combinatorial identity is verified exactly as the corresponding identity from the proof of [3, Theorem 4.1 (2)]. In our case one must use the results formulated in §4.2.

Theorem 4.8 now follows from Lemmas 4.11 and 4.12 and the fact that

\[(T_n - 1)f_n = -f_n + \frac{(UDf)_n}{(n+1)(n+\alpha/2)}, \quad f \in \Gamma. \quad (24)\]

5 Doubly symmetric functions on shifted Young diagrams

In this section we study the algebra \( \Gamma \subset \text{Fun}(S) \) (defined in §4) in more detail.

Let \( \lambda \) be an arbitrary shifted Young diagram and \( u \) be a complex variable. By definition, put

\[
\varphi(u; \lambda) := \prod_{i=1}^{\infty} \frac{u + \lambda_i}{u - \lambda_i}.
\]

Note that this product is actually finite, because any strict partition \( \lambda \) has only finitely many nonzero terms. Note also that \( \varphi(u; \lambda) \) is a rational function in \( u \) taking value 1 at \( u = \infty \).

Proposition 5.1. The algebra \( \Gamma \subset \text{Fun}(S) \) coincides with the commutative unital subalgebra of \( \text{Fun}(S) \) generated by the Taylor expansion coefficients of \( \varphi(u; \lambda) \) (or, equivalently, of \( \log \varphi(u; \lambda) \)) at \( u = \infty \) with respect to \( u^{-1} \).

Proof. The Taylor expansion of \( \log \varphi(u; \lambda) \) at \( u = \infty \) has the form

\[
\log \varphi(u; \lambda) = 2 \sum_{k \geq 1 \text{ odd}} \frac{p_k(\lambda)}{k} u^{-k}, \quad (25)
\]

where \( p_k(\lambda) = \sum_{i=1}^{\ell(\lambda)} \lambda_i^k \) are the Newton power sums. The algebra \( \Gamma \) is freely generated by the functions \( p_1, p_3, \ldots \in \text{Fun}(S) \), see §4.2. \( \square \)
By definition, put

\[ \Phi(v; \lambda) := \prod_{i=1}^{\infty} \frac{v - \lambda_i(\lambda_i - 1)}{v - \lambda_i(\lambda_i + 1)}. \]

The product here is also actually finite. Clearly, \( \Phi(v; \lambda) \) is a rational function in \( v \) taking value 1 at \( v = \infty \). It can be readily verified that \( \Phi(u^2 - u; \lambda) = \frac{\varphi(u - 1; \lambda)}{\varphi(u; \lambda)} \).

**Definition 5.2.** Let \( p_m(\cdot), g_m(\cdot), \hat{g}_m(\cdot) \in \text{Fun}(\mathbb{S}) \), \( m \in \mathbb{Z}_{>0} \), be the following Taylor expansion coefficients at \( v = \infty \) with respect to \( v^{-1} \):

\[
\log \Phi(v; \lambda) = \sum_{m=1}^{\infty} p_m(\lambda) v^{-m}; \\
\Phi(v; \lambda) = 1 + \sum_{m=1}^{\infty} g_m(\lambda) v^{-m}; \\
\frac{1}{\Phi(v; \lambda)} = 1 - \sum_{m=1}^{\infty} \hat{g}_m(\lambda) v^{-m}.
\]

Recall that the algebra \( \Gamma \) has a natural filtration (defined in §4.1) which is determined by setting

\[
\deg p_{2m-1} = 2m - 1, \quad m = 1, 2, \ldots . \tag{26}
\]

**Proposition 5.3.** The functions \( p_m(\lambda) \) belong to the algebra \( \Gamma \). More precisely,

\[
p_m(\lambda) = 2m \cdot p_{2m-1}(\lambda) + \ldots, \quad m \in \mathbb{Z}_{>0},
\]

where dots stand for lower degree terms in the algebra \( \Gamma \), which are a linear combination of \( p_{2l-1}(\lambda) \), where \( 1 \leq l \leq m - 1 \).

**Proof.** On one hand, by the definition of \( \Phi \) and by (25) we have

\[
\log \Phi(u^2 - u; \lambda) = \log \varphi(u - 1; \lambda) - \log \varphi(u; \lambda) \\
= 2 \sum_{k=1}^{\infty} \frac{p_{2k-1}(\lambda)}{2k - 1} \left( \frac{1}{(u - 1)^{2k-1}} - \frac{1}{u^{2k-1}} \right)
\]

for all \( \lambda \in \mathbb{S} \). Observe that

\[
\frac{1}{(u - 1)^{2k-1}} - \frac{1}{u^{2k-1}} = (2k - 1)u^{-2k} \left( 1 + \frac{k}{u} + \ldots \right),
\]

where \( k \in \mathbb{Z}_{>0} \) and dots stand for terms containing \( u^{-2}, u^{-3}, \ldots \).

---

\[\text{\[20\] Here } v \text{ is an independent complex variable.}\]
On the other hand, by Definition 5.2 we have

$$\log \Phi(u^2 - u; \lambda) = \sum_{m=1}^{\infty} \frac{p_m(\lambda)}{m} \frac{1}{(u^2 - u)^m}$$

for all \( \lambda \in \mathbb{S} \). Observe that

$$\frac{1}{(u^2 - u)^m} = u^{-2m} \left( 1 - \frac{m}{u} + \ldots \right),$$

where \( m \in \mathbb{Z}_{>0} \) and again dots stand for terms containing \( u^{-2}, u^{-3}, \ldots \).

Thus, we get the following identity:

$$2 \sum_{k=1}^{\infty} u^{-2k} p_{2k-1}(\lambda) \left( 1 - \frac{k}{u} + \ldots \right) = \sum_{m=1}^{\infty} u^{-2m} \frac{p_m(\lambda)}{m} \left( 1 - \frac{m}{u} + \ldots \right).$$

Comparing the coefficients of \( u^{-2m} \) in both sides, we get the claim. \( \Box \)

**Proposition 5.4.** We have \( g_1 = \hat{g}_1 = p_1 \)

and

$$kg_k = p_k + p_{k-1}g_1 + \cdots + p_1g_{k-1}, \quad \hat{g}_k = g_k - g_{k-1}\hat{g}_1 - \cdots - g_1\hat{g}_{k-1}$$

for all \( k = 2, 3, \ldots \).

**Proof.** The technique of this proof is similar to [17, Ch. I, §2]. Let \( w \) be an independent variable. Observe that

$$\sum_{m=1}^{\infty} \frac{p_m(\lambda)}{m} w^m = \log \left( 1 + \sum_{k=1}^{\infty} g_k(\lambda) w^k \right).$$

If we take \( d/dw \) of both sides and compare the coefficients by \( w^{k-1} \), we get the desired relation between \( p_k \)'s and \( g_k \)'s.

To prove the remaining relation between \( g_k \)'s and \( \hat{g}_k \)'s observe that

$$\left( 1 + \sum_{k=1}^{\infty} g_k(\lambda) w^k \right) \left( 1 - \sum_{k=1}^{\infty} \hat{g}_k(\lambda) w^k \right) = 1.$$

This concludes the proof. \( \Box \)

**Corollary 5.5.** Each of the three families \( \{p_1, p_2, p_3, \ldots \}, \{g_1, g_2, g_3, \ldots \} \) and \( \{\hat{g}_1, \hat{g}_2, \hat{g}_3, \ldots \} \) is a system of algebraically independent generators of the algebra \( \Gamma \). Under the identification of \( \Gamma \) with any of the algebras of polynomials

\[ \mathbb{R} [p_1, p_2, \ldots], \quad \mathbb{R} [g_1, g_2, \ldots] \quad \text{and} \quad \mathbb{R} [\hat{g}_1, \hat{g}_2, \ldots], \]

21Here and below we sometimes omit the argument \( \lambda \) to shorten the notation.
the natural filtration \([20]\) of \(\Gamma\) is determined by setting

\[
\begin{align*}
\deg p_m(\lambda) &= 2m - 1, \\
\deg g_m(\lambda) &= 2m - 1, \\
\deg \hat{g}_m(\lambda) &= 2m - 1,
\end{align*}
\]

\(m \in \mathbb{Z}_{>0}\), respectively.

**Proposition 5.6.** Let \(\lambda\) be an arbitrary shifted Young diagram with Kerov interlacing coordinates \([X(\lambda); X(\lambda)]\) (see §3.7). Then

\[
\Phi(v; \lambda) = \frac{\prod_{y \in Y(\lambda)} (v - y(y + 1))}{\prod_{x \in X'(\lambda)} (v - x(x + 1))} = v \cdot R^\uparrow(v; \lambda).
\]

Here the function \(R^\uparrow\) is defined by [17]. Recall that \(X'(\lambda) = X(\lambda) \setminus \{0\}\).

**Proof.** This can be proved exactly as Proposition 3.7 using Remark 3.5. \(\square\)

Using this Proposition one can express the functions \(p_m, g_m, \hat{g}_m, m \in \mathbb{Z}_{>0}\), through Kerov coordinates:

**Proposition 5.7.** Let \(\lambda \in S\) and \(m \in \mathbb{Z}_{>0}\). Then \(22\)

\[
\begin{align*}
p_m(\lambda) &= \sum_{x \in X(\lambda)} (x(x + 1))^m - \sum_{y \in Y(\lambda)} (y(y + 1))^m; \\
g_m(\lambda) &= \sum_{x \in X(\lambda)} \theta^1_x(\lambda) \cdot (x(x + 1))^m; \\
\hat{g}_m(\lambda) &= \sum_{y \in Y(\lambda)} \theta^1_y(\lambda) \cdot (y(y + 1))^{m-1}.
\end{align*}
\]

**Proof.** The first claim is a straightforward consequence of Proposition 5.6.

Let us prove the second claim. On one hand, from the definition of the numbers \(\{\theta^1_x(\lambda)\}\) (see §3.2) we have

\[
\begin{align*}
v \cdot R^\uparrow(v; \lambda) &= v \sum_{x \in X(\lambda)} \frac{\theta^1_x(\lambda)}{v - x(x + 1)} = \sum_{x \in X(\lambda)} \frac{\theta^1_x(\lambda)}{1 - \frac{x(x + 1)}{v}} \\
&= \sum_{x \in X(\lambda)} \theta^1_x(\lambda) \sum_{k=0}^{\infty} \left(\frac{x(x + 1)}{v}\right)^k \\
&= \sum_{k=0}^{\infty} v^{-k} \sum_{x \in X(\lambda)} \theta^1_x(\lambda) \cdot (x(x + 1))^k.
\end{align*}
\]

On the other hand, it follows from Proposition 5.6 that \(v \cdot R^\uparrow(v; \lambda) = \Phi(v; \lambda)\). Using the definition of the functions \(g_m\) (Definition 5.2) and comparing it to the above formula for \(v \cdot R^\uparrow(v; \lambda)\), we get the second claim.

The third claim can be verified similarly. \(\square\)

\(\text{22Recall that the numbers } \{\theta^1_x(\lambda)\}_{x \in X(\lambda)} \text{ and } \{\theta^1_y(\lambda)\}_{y \in Y(\lambda)} \text{ were introduced in [3]}\)
It follows from Propositions 3.6, 5.4 and 5.7 that
\[ p_1(\lambda) = g_1(\lambda) = \hat{g}_1(\lambda) = 2|\lambda|, \quad \lambda \in S. \] (27)

**Lemma 5.8.** Let \( \lambda \) be an arbitrary nonempty shifted Young diagram, \( x \in X(\lambda) \) and \( y \in Y(\lambda) \). Then
\[
\frac{\Phi(v; \lambda + \Box(x))}{\Phi(v; \lambda)} = \frac{(v - x(x + 1))^2}{(v - x(x + 1))^2 - 2(v + x(x + 1))}
\]
and
\[
\frac{\Phi(v; \lambda - \Box(y))}{\Phi(v; \lambda)} = \frac{(v - y(y + 1))^2 - 2(v + y(y + 1))}{(v - y(y + 1))^2}.
\]

**Proof.** This directly follows from Proposition 5.6 and the definitions of the diagrams \( \lambda + \Box(x) \) and \( \lambda - \Box(y) \) (§3.1).

6 The up and down operators in differential form

The aim of this section is to write the operators \( D \) and \( U \) in the algebra \( \Gamma \) (they were defined in Lemmas 4.11 and 4.12, respectively) in differential form. Here we use the results of §3 and §5. Our approach is inspired by the paper [20], but in our situation significant modifications are required.

6.1 Formulation of the theorem

We identify \( \Gamma \) with the polynomial algebra \( \mathbb{R}[g_1, g_2, \ldots] \). Recall that \( \Gamma \) is a filtered algebra, and under this identification the filtration is determined by setting (see Corollary 5.5) \( \deg g_m = 2m - 1, \ m \in \mathbb{Z}_{>0} \).

**Definition 6.1.** We say that an operator \( R: \Gamma \to \Gamma \) has degree \( \leq r \), where \( r \in \mathbb{Z} \), if \( \deg(Rf) \leq \deg f + r \) for any \( f \in \Gamma \).

**Remark 6.2.** Observe that any operator in the algebra of polynomials (in finitely or countably many variables) can be written as a differential operator with polynomial coefficients — a formal infinite sum of differential monomials. This fact is well known and can be readily proved. We do not need it but it is useful to keep it in mind while reading the formulation and the proof of Theorem 6.3.

**Theorem 6.3.** (1) The operator \( D: \Gamma \to \Gamma \) defined in Lemma 4.11 has degree 1 with respect to the filtration of \( \Gamma \) and looks as
\[
D = \frac{1}{2}g_1 + \sum_{r, s \geq 1} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s} - \sum_{r} (2r - 1)g_r \frac{\partial}{\partial g_r} + \sum_{r, s \geq 1} (r + s)g_r g_s \frac{\partial}{\partial g_{r+s}} + \text{operators of degree } \leq -2;
\]
For any fixed $\alpha \in (0, +\infty)$ the operator $U : \Gamma \to \Gamma$ defined in Lemma 4.12 has degree 1 with respect to the filtration of $\Gamma$ and looks as

$$U = \frac{1}{2}g_1 + \frac{1}{2}\alpha + a \frac{\partial}{\partial g_1} + \sum_{r,s \geq 1} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s}$$

$$+ \sum_{r \geq 1} (2r - 1)g_r \frac{\partial}{\partial g_r} + \sum_{r,s \geq 1} (r + s - 1)g_r g_s \frac{\partial}{\partial g_{r+s}}$$

+ operators of degree $\leq -2$.

**Scheme of proof.** The functions $g_k \in \Gamma$, $k \in \mathbb{Z}_{>0}$, generate the algebra $\Gamma$. However, we will not be dealing with actions of $D$ and $U$ on these generators. Instead, we consider the products of the form $\Phi(v_1; \lambda)\Phi(v_2; \lambda)\ldots$, where $v_1, v_2, \ldots$ are independent complex variables in a finite number (here $\Phi(v; \lambda)$ is defined in §5). It follows from Definition 5.2 that the products of the form $\Phi(v_1; \lambda)\Phi(v_2; \lambda)\ldots$ assemble various products of the generators, which in turn constitute a linear basis for $\Gamma$. Thus, we know the action of our operators if we know how they transform such products. It turns out that the transformation of the products $\Phi(v_1; \lambda)\Phi(v_2; \lambda)\ldots$ can be written down in a closed form. From this we can extract all the necessary information.

### 6.2 Action of $D$ and $U$ on generating series

It follows from Definition 5.2 that for any finite collection of independent complex variables $v_1, v_2, \ldots$ (we prefer not to indicate their number explicitly) we have

$$\Phi(v_1; \lambda)\Phi(v_2; \lambda)\ldots = \sum_{\rho} m_\rho(v_1^{-1}, v_2^{-1}, \ldots)g_\rho(\lambda), \quad (28)$$

where the sum is taken over all ordinary partitions $\rho$ such that $\ell(\rho)$ does not exceed the number of $v_i$’s, $m_\rho$ is the monomial symmetric function and $g_\rho = g_{\rho_1} \cdots g_{\rho_{\ell(\rho)}}$. It is convenient to set $g_0 := 1$.

**Remark 6.4.** Observe that $m_\rho(v_1^{-1}, v_2^{-1}, \ldots)$ vanishes if $\ell(\rho)$ is greater than the number of $v_i$’s. It follows that here and below in the sums similar to (28) we can let $\rho$ run over all ordinary partitions.

We regard the LHS of (28) as a generating series for the elements $g_\rho$ that constitute a linear basis for $\Gamma$. Thus, the action of an operator in $\Gamma$ on the LHS is determined by its action on the functions $g_\rho \in \Gamma$ in the RHS.

Occasionally, it will be convenient to omit the argument $\lambda$ in $\Phi(v; \lambda)$. Recall from §4.3 the notation $(\ldots)_n$ for the restriction of a function from $\Gamma \subset \text{Fun}(\mathfrak{S})$ to the subset $\mathfrak{S}_n \subset \mathfrak{S}$.

We start with the operators $U_{n,n+1}$ and $D_{n+1,n}$ defined by (19). By the very
definition of $U_{n,n+1}$ we have
\[
\left( U_{n,n+1} \left( \prod_l \Phi(v_l) \right) \right)_{n+1}(\lambda) = \sum_{x \in \mathcal{X}^{\lambda}} \frac{x(x+1)+\alpha}{2} \prod_l \frac{(v_l - x(x+1))^2}{(v_l - x(x+1))^2 - 2(v_l + x(x+1))} \theta_x^I(\lambda), \quad \lambda \in S_n. \]
Using (13) and Lemma 5.8 we get (note that $|\lambda| = n$):
\[
\left( (n+\alpha/2)U_{n,n+1} \left( \prod_l \Phi(v_l) \right) \right)_{n+1}(\lambda) = F^I(v_1,v_2,\ldots;\lambda) \cdot \prod_l \Phi(v_l;\lambda), \tag{29}
\]
where
\[
F^I(v_1,v_2,\ldots;\lambda) := \sum_{x \in \mathcal{X}^{\lambda}} \frac{x(x+1)+\alpha}{2} \prod_l \frac{(v_l - x(x+1))^2}{(v_l - x(x+1))^2 - 2(v_l + x(x+1))} \theta_x^I(\lambda). \tag{30}
\]
Likewise, for the operator $D_{n+1,n}$ we have
\[
\left( D_{n+1,n} \left( \prod_l \Phi(v_l) \right) \right)_{n}(\lambda) = \sum_{y \in \mathcal{Y}^{\lambda}} \frac{y(y+1)}{2} \prod_l \frac{(v_l - y(y+1))^2}{(v_l - y(y+1))^2 - 2(v_l + y(y+1))} \theta_y^I(\lambda), \quad \lambda \in S_{n+1}. \]
Using Proposition 3.9 and Lemma 5.8 we get (note that now $|\lambda| = n+1$):
\[
\left( (n+1)D_{n+1,n} \left( \prod_l \Phi(v_l) \right) \right)_{n}(\lambda) = F^I(v_1,v_2,\ldots;\lambda) \cdot \prod_l \Phi(v_l;\lambda), \tag{31}
\]
where
\[
F^I(v_1,v_2,\ldots;\lambda) := \sum_{y \in \mathcal{Y}^{\lambda}} \frac{y(y+1)}{2} \prod_l \frac{(v_l - y(y+1))^2}{(v_l - y(y+1))^2 - 2(v_l + y(y+1))} \theta_y^I(\lambda). \tag{32}
\]

**Lemma 6.5.** As functions in $\lambda$, both $F^I(v_1,v_2,\ldots;\lambda)$ and $F^I(v_1,v_2,\ldots;\lambda)$ are elements of the algebra $\Gamma$. More precisely, the both expressions can be viewed as elements of $\Gamma[[v_1^{-1},v_2^{-1},\ldots]]$.

**Proof.** Observe that the products on $l$ in (30) and (32) can be viewed as elements of $\mathbb{R}[x(x+1)][[v_1^{-1},v_2^{-1},\ldots]]$ and $\mathbb{R}[y(y+1)][[v_1^{-1},v_2^{-1},\ldots]]$, respectively\(^{23}\). Moreover, if $f(x(x+1))$ is a polynomial in $x(x+1)$, then the expression
\[
\sum_{x \in \mathcal{X}^{\lambda}} f(x(x+1)) \theta_x^I(\lambda)
\]
\(^{23}\)Here and below $\mathbb{R}[z(z+1)]$ denotes the algebra of polynomials in $z(z+1)$. 26
as function in \( \lambda \) belongs to \( \Gamma \) (this follows from Proposition \[5.7\]). Letting \( f \) be the corresponding formal power series in \( v_1^{-1}, v_2^{-1}, \dots \), we get the claim about \( F^1(v_1, v_2, \ldots; \lambda) \). The remaining claim about \( F^1(v_1, v_2, \ldots; \lambda) \) is verified similarly.

Now we proceed to the operators \( D \) and \( U \) in the algebra \( \Gamma \) defined in Lemmas \[4.11\] and \[4.12\] respectively. Using these Lemmas, we rewrite (29) and (31) as

\[
U(\Phi(v_1)\Phi(v_2)\ldots) = F^1(v_1, v_2, \ldots)\Phi(v_1)\Phi(v_2)\ldots;
D(\Phi(v_1)\Phi(v_2)\ldots) = F^1(v_1, v_2, \ldots)\Phi(v_1)\Phi(v_2)\ldots.
\]

These formulas contain in a compressed form all the information about the action of \( U \) and \( D \) on the basis elements \( g_\rho \), where \( \rho \) runs over all ordinary partitions. Our next step is to extract from (33) some explicit expressions for \( U g_\rho \) and \( D g_\rho \) using (28) and Proposition \[5.7\].

\[6.3\] \textbf{Action of \( U \) and \( D \) in the basis \( \{g_\rho\}\)}

Let us first introduce some extra notation. Let \( v \) and \( \xi \) be independent variables. Consider the following expansions at \( v = \infty \) with respect to \( v^{-1} \):

\[
\frac{(v - \xi)^2}{(v - \xi)^2 - 2(v + \xi)} = \sum_{s=0}^{\infty} a_s(\xi)v^{-s}, \quad a_s \in \mathbb{R}[\xi];
\]

\[
\frac{(v - \xi)^2 - 2(v + \xi)}{(v - \xi)^2} = \sum_{s=0}^{\infty} b_s(\xi)v^{-s}, \quad b_s \in \mathbb{R}[\xi].
\]

\[\text{Lemma 6.6.} \quad \text{We have}
\]

\[
a_0(\xi) = b_0(\xi) \equiv 1,
\]

and \( a_s(\xi) \) and \( b_s(\xi) \) have degree \( s - 1 \) for \( s \geq 1 \). More precisely,

\[
b_s(\xi) = -2(2s - 1)\xi^{s-1}, \quad s \geq 1,
\]

and \( a_s(\xi) \) has the form

\[
a_s(\xi) = 2(2s - 1)\xi^{s-1} + \text{terms of degree } (s - 2) \text{ in the variable } \xi, \quad s \geq 1.
\]

\[\text{Proof.} \quad \text{First, we compute explicitly } b_s(\xi):
\]

\[
\frac{(v - \xi)^2 - 2(v + \xi)}{(v - \xi)^2} = 1 - \frac{4\xi}{(v - \xi)^2} - \frac{2}{v - \xi} = 1 - 4 \sum_{s=1}^{\infty} s\xi^sv^{-s-1} - 2 \sum_{s=0}^{\infty} \xi^sv^{-s-1}
\]

\[
= 1 - \frac{2}{v} - \sum_{s=1}^{\infty} (4s + 2)\xi^sv^{-s-1} = 1 - \sum_{s=1}^{\infty} 2(2s - 1)\xi^{s-1}v^{-s}.
\]

Next, observe that \( (\sum_{s=0}^{\infty} a_s(\xi)v^{-s})(\sum_{s=0}^{\infty} b_s(\xi)v^{-s}) = 1 \), therefore, \( a_0(\xi) = 1 \) and for \( s \geq 1 \) the top degree term of \( a_s(\xi) \) is equal to \( 2(2s - 1)\xi^{s-1} \).
For an ordinary partition \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{\ell(\sigma)}) \) we set
\[
a_\sigma(\xi) := \prod_{i=1}^{\ell(\sigma)} a_{\sigma_i}(\xi), \quad b_\sigma(\xi) := \prod_{i=1}^{\ell(\sigma)} b_{\sigma_i}(\xi).
\]

Using (34) and the above definition, we get the following expressions for the products on \( l \) in (30) and (32):
\[
\prod_l \frac{(v_l - x(x+1))^2}{(v_l - x(x+1))^2 - 2(x + 1)} = \sum_{\sigma} a_\sigma(x(x+1)) m_\sigma(v_1^{-1}, v_2^{-1}, \ldots);
\]
\[
\prod_l \frac{(v_l - y(y+1))^2 - 2(v_l - y(y+1))}{(v_l - y(y+1))^2} = \sum_{\sigma} b_\sigma(y(y+1)) m_\sigma(v_1^{-1}, v_2^{-1}, \ldots).
\]

(35)

Here the sums in the right-hand sides are taken over all ordinary partitions.

Next, introduce two linear maps
\[
\mathbb{R}[x(x+1)] \to \Gamma, \quad f \mapsto \langle f \rangle^\uparrow; \\
\mathbb{R}[y(y+1)] \to \Gamma, \quad h \mapsto \langle h \rangle^\downarrow
\]
by setting
\[
\langle (x(x+1))^m \rangle^\uparrow := g_m, \quad \langle (y(y+1))^m \rangle^\downarrow := \hat{g}_{m+1}, \quad m \in \mathbb{Z}_{\geq 0}, \quad (36)
\]
where, by agreement, \( g_0 = 1 \). This definition is inspired by Proposition 5.7.

Finally, let \( c_{\sigma \tau}^\rho \) be the structure constants of the algebra \( \Lambda \) of all symmetric functions in the basis of monomial symmetric functions:
\[
m_\sigma m_\tau = \sum_{\rho} c_{\sigma \tau}^\rho m_\rho.
\]

Note that \( c_{\sigma \tau}^\rho \) can be nonzero only if \(|\rho| = |\sigma| + |\tau|\). Here \( \rho, \sigma, \) and \( \tau \) are ordinary partitions.

Now we are in a position to compute \( U g_\rho \) and \( D g_\rho \).

**Lemma 6.7.** With the notation introduced above we have
\[
U g_\rho = \sum_{\sigma, \tau: |\sigma| + |\tau| = |\rho|} \frac{1}{2} c_{\sigma \tau}^\rho \langle (x(x+1) + \alpha) \cdot a_\sigma(x(x+1)) \rangle^\uparrow g_\tau;
\]
\[
D g_\rho = \sum_{\sigma, \tau: |\sigma| + |\tau| = |\rho|} \frac{1}{2} c_{\sigma \tau}^\rho \langle b_\sigma(y(y+1)) \rangle^\downarrow g_\tau.
\]

**Proof.** Let us write
\[
F^\uparrow(v_1, v_2, \ldots) = \sum_{\sigma} F^\uparrow_{\sigma} m_\sigma(v_1^{-1}, v_2^{-1}, \ldots), \quad F^\uparrow_{\sigma} \in \Gamma;
\]
\[
F^\downarrow(v_1, v_2, \ldots) = \sum_{\sigma} F^\downarrow_{\sigma} m_\sigma(v_1^{-1}, v_2^{-1}, \ldots), \quad F^\downarrow_{\sigma} \in \Gamma,
\]

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where sums are taken over all ordinary partitions $\sigma$.

Using (28), we get
\[
\sum_{\rho} m_{\rho}(v_{1}^{-1}, v_{2}^{-1}, \ldots) U g_{\rho} = \left( \sum_{\sigma} F_{\sigma}^{\dagger} m_{\sigma}(v_{1}^{-1}, v_{2}^{-1}, \ldots) \right) \left( \sum_{\tau} m_{\tau}(v_{1}^{-1}, v_{2}^{-1}, \ldots) g_{\tau} \right),
\]
which implies
\[
U g_{\rho} = \sum_{\sigma, \tau : |\sigma|+|\tau|=|\rho|} c_{\sigma, \tau}^{\rho} F_{\sigma}^{\dagger} F_{\tau}^{\dagger} g_{\tau}.
\]
Similarly we obtain
\[
D g_{\rho} = \sum_{\sigma, \tau : |\sigma|+|\tau|=|\rho|} c_{\sigma, \tau}^{\rho} F_{\sigma}^{\dagger} F_{\tau}^{\dagger} g_{\tau}.
\]
The facts that
\[
F_{\sigma}^{\dagger} = \left\langle \frac{1}{2} (x(x+1) + \alpha) \cdot a_{\sigma} (x(x+1)) \right\rangle \quad \text{and} \quad F_{\sigma}^{\dagger} = \left\langle \frac{1}{2} b_{\sigma} (y(y+1)) \right\rangle
\]
follow directly from (30), (32) and (35).

6.4 The operator $D$ in differential form

In this subsection we prove claim (1) of Theorem 6.3, that is, compute the top degree terms of the operator $D : \Gamma \rightarrow \Gamma$.

By virtue of Lemma 6.7, we can write
\[
D = \sum_{\sigma} D_{\sigma} , \quad D_{\sigma} g_{\rho} := \sum_{\tau : |\tau|=|\rho|-|\sigma|} \frac{1}{2} \left\langle b_{\sigma} (y(y+1)) \right\rangle \downarrow c_{\sigma, \tau}^{\rho} g_{\tau}.
\]

Lemma 6.8. Let $\sigma$ be a nonempty ordinary partition. Then
\[
\deg D_{\sigma} \leq \max_{\rho, \tau} (\ell(\rho) - \ell(\tau) - 2\ell(\sigma) + 1),
\]
where the maximum is taken over all pairs $(\rho, \tau)$ such that $c_{\sigma, \tau}^{\rho} \neq 0$.

A more rough but simpler estimate is
\[
\deg D_{\sigma} \leq -\ell(\sigma) + 1.
\]

Proof. We have
\[
\deg D_{\sigma} \leq \max_{\rho, \tau} \left( \deg \left\langle b_{\rho} (y(y+1)) \right\rangle \downarrow + \deg g_{\tau} - \deg g_{\rho} \right) \\
= \max_{\rho, \tau} \left( \deg \left\langle b_{\tau} (y(y+1)) \right\rangle \downarrow + 2|\tau| - \ell(\tau) - 2|\rho| + \ell(\rho) \right) \\
\leq \max_{\rho, \tau} \left( \deg \left\langle b_{\tau} (y(y+1)) \right\rangle \downarrow - 2|\sigma| - \ell(\tau) + \ell(\rho) \right),
\]
Lemma 6.10. Consequently, \( \sigma \) is, Distribution to Lemma 6.8 shows that it must be \( \ell_b \) since Corollary 6.9.

Proof. Lemma 6.11. \( \hat{g}_m \) (Proposition 5.7). Note that \( \deg g_\rho \) for any ordinary partitions \( \rho \) and \( \tau \), and the third line holds because \( c_{\sigma \tau}^b \neq 0 \) only if \( |\rho| = |\sigma| + |\tau| \).

By assumption, \( \sigma \) is nonempty, therefore, \( \ell(\sigma) \geq 1 \). Set \( \sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)}) \), where \( \sigma_i \geq 1 \) for \( i = 1, \ldots, \ell(\sigma) \), and write \( \langle b_\sigma(y(y + 1)) \rangle^{\dagger} \) in more detail:

\[
\langle b_\sigma(y(y + 1)) \rangle^{\dagger} = \left( \prod_{i=1}^{\ell(\sigma)} b_{\sigma_i}(y(y + 1)) \right)^{\dagger} = \left( \prod_{i=1}^{\ell(\sigma)} -2(2\sigma_i - 1)(y(y + 1))^{\sigma_i-1} \right)^{\dagger}.
\]

We see that the polynomial \( b_\sigma(y(y + 1)) \) has degree \( |\sigma| - \ell(\sigma) \) in \( y(y + 1) \), therefore \( \langle b_\sigma(y(y + 1)) \rangle^{\dagger} \) is equal, within a nonzero scalar factor, to \( \hat{g}_{|\sigma| - \ell(\sigma) + 1} \) (Proposition 5.7). Note that \( \deg \hat{g}_{|\sigma| - \ell(\sigma) + 1} = 2|\sigma| - 2\ell(\sigma) + 1 \), therefore,

\[
\deg D_\sigma \leq \max_{\rho, \tau} \left( 2|\sigma| - 2\ell(\sigma) + 1 - 2|\sigma| - \ell(\tau) + \ell(\rho) \right),
\]

which is the first estimate. To prove the second one, observe that \( c_{\sigma \tau}^b \neq 0 \) implies \( \ell(\rho) \leq \ell(\sigma) + \ell(\tau) \).

\[
\hat{g}_1 = g_1. \quad \square
\]

From the second estimate of the above lemma follows a

**Corollary 6.9.** If \( \ell(\sigma) \geq 3 \), then \( \deg D_\sigma \leq -2 \).

By this corollary, it suffices to examine the operators \( D_\sigma \) with \( \ell(\sigma) = 0 \) (that is, \( \sigma = \emptyset \)), \( \ell(\sigma) = 2 \), and \( \ell(\sigma) = 1 \). The next three lemmas consider these cases consequently.

**Lemma 6.10.** \( D_\emptyset = \frac{1}{2} \hat{g}_1 \).

**Proof.** Suppose that \( \sigma = \emptyset \) in (37). Then \( \tau = \rho \) and \( c_{\sigma \tau}^b = 1 \). Moreover, since \( b_{\emptyset} = 1 \), then \( \langle b_\emptyset(y(y + 1)) \rangle^{\dagger} = \langle 1 \rangle^{\dagger} = \hat{g}_1 \). By virtue of Proposition 5.4 \( \hat{g}_1 = g_1 \). This concludes the proof. \( \square \)

**Lemma 6.11.**

\[
\sum_{\sigma : \ell(\sigma) = 2} D_\sigma = \sum_{r,s \geq 1} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s} + \text{operators of degree } \leq -2.
\]

**Proof.** Suppose that in (37) we have \( \ell(\sigma) = 2 \), that is, \( \sigma = (\sigma_1, \sigma_2), \sigma_1 \geq \sigma_2 \geq 1 \). Lemma 6.8 shows that it must be \( \ell(\rho) = \ell(\tau) + 2 \), otherwise the corresponding distribution to \( D_\sigma \) has degree \( \leq -2 \). This means that

\[
\sigma_1 = \rho_i, \quad \sigma_2 = \rho_j, \quad \tau = (\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_{\ell(\rho)})
\]
for some $1 \leq i < j \leq \ell(\rho)$. Using Lemma 6.6, we get
\[
\langle b_\sigma (y(y + 1)) \rangle^1 = 4(2\sigma_1 - 1)(2\sigma_2 - 1)\hat{g}_{\sigma_1 + \sigma_2 - 1}.
\]
It follows that
\[
\left( \sum_{\sigma: \ell(\sigma) = 2} D_\sigma \right) \hat{g}_\rho = \sum_{1 \leq i < j \leq \ell(\rho)} 2(2\rho_i - 1)(2\rho_j - 1)c_{\sigma \tau}^\rho \hat{g}_{\rho_i + \rho_j - 1} \hat{g}_{\rho \setminus \{\rho_i, \rho_j\}}.
\]
Note that for such $\rho, \sigma,$ and $\tau$ as described above we have
\[
c_{\sigma \tau}^\rho \hat{g}_{\rho \setminus \{\rho_i, \rho_j\}} = \begin{cases} 
\frac{\partial^2 \hat{g}_\rho}{\partial \hat{g}_{\rho_i} \partial \hat{g}_{\rho_j}}, & \text{if } \rho_i \neq \rho_j; \\
\frac{1}{2} \frac{\partial^2 \hat{g}_\rho}{\partial \hat{g}_{\rho_i}^2}, & \text{if } \rho_i = \rho_j.
\end{cases}
\] (38)
It follows that we can write
\[
\sum_{\sigma: \ell(\sigma) = 2} D_\sigma = \sum_{r_1 > r_2 \geq 1} 2(2r_1 - 1)(2r_2 - 1)\hat{g}_{r_1 + r_2 - 1} \frac{\partial^2}{\partial \hat{g}_{r_1} \partial \hat{g}_{r_2}} + \frac{1}{2} \sum_{r \geq 1} 2(2r - 1)^2 \hat{g}_{2r - 1} \frac{\partial^2}{\partial \hat{g}_{2r}^2}.
\]
Using Proposition 5.4, we can substitute each of $\hat{g}_k$’s above by $g_k$. Indeed, since $\hat{g}_k = g_k + \text{terms of degree } \leq 2k - 2$, $k = 1, 2, \ldots$, then replacing $\hat{g}_k$ by $g_k$ affects only negligible terms (that is, summands of degree $\leq -2$ in the expression for the operator $D$). The result of this substitution is the desired expression. \:

\textbf{Lemma 6.12.}
\[
\sum_{\sigma: \ell(\sigma) = 1} D_\sigma = - \sum_{r \geq 1} (2r - 1)g_r \frac{\partial}{\partial g_r} + \sum_{r,s \geq 1} (r + s)g_r g_s \frac{\partial}{\partial g_{r+s}} + \text{operators of degree } \leq -2.
\]
\textbf{Proof.} Suppose that in (37) we have $\ell(\sigma) = 1$, that is, $\sigma = (s)$ for some $s \in \mathbb{Z}_{>0}$.

It follows from Lemma 6.8 that either $\ell(\rho) = \ell(\tau) + 1$, or $\ell(\rho) = \ell(\tau)$. Let us examine these cases separately.

Assume first that $\ell(\rho) = \ell(\tau) + 1$. This means that
\[
s = \rho_i, \quad \tau = (\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{\ell(\rho)})
\]
for some $1 \leq i \leq \ell(\rho)$. Using Lemma 6.6 we get
\[
\langle b_\sigma (y(y + 1)) \rangle^1 = -2(2s - 1)\hat{g}_s.
\]
Therefore, the case $\ell(\rho) = \ell(\tau) + 1$ gives rise to the terms

$$-\sum_{i=1}^{\ell(\rho)} (2\rho_i - 1)c^\rho_{\sigma\tau};\hat{g}_{\rho_i\setminus\{\rho_i\}}.$$  

Note that in this case we have $c^\rho_{\sigma\tau};\hat{g}_{\rho_i\setminus\{\rho_i\}} = \partial g_{\rho_i}/\partial g_{\rho_i}$, and it follows that we obtain terms

$$-\sum_{r \geq 1} (2r - 1)\hat{g}_r \frac{\partial}{\partial g_r}$$

contributing to $\sum_{\sigma: \ell(\sigma) = 1} D_{\sigma}$.

Now assume that $\ell(\rho) = \ell(\tau)$. This means that $\tau$ is obtained from $\rho$ by subtracting $s$ from one of the parts $\rho_i$ of $\rho$; moreover, this part $\rho_i = r$ should be $\geq s + 1$. This gives rise to the terms

$$-\sum_{r \geq 2} \sum_{1 \leq s \leq r-1} (2s - 1)\hat{g}_s g_{r-s} \frac{\partial}{\partial g_r},$$

Finally, we get

$$\sum_{\sigma: \ell(\sigma) = 1} D_{\sigma} = -\sum_{r \geq 1} (2r - 1)\hat{g}_r \frac{\partial}{\partial g_r} - \sum_{r \geq 2} \sum_{1 \leq s \leq r-1} (2s - 1)\hat{g}_s g_{r-s} \frac{\partial}{\partial g_r},$$

It remains to express $\hat{g}_k$’s above in terms of $g_i$’s using Proposition 5.4 In the first sum we should do this as

$$\hat{g}_r \rightarrow g_r - g_{r-1} g_1 - \cdots - g_1 g_{r-1} + \text{terms of degree } \leq 2r - 2, \quad r = 1, 2, \ldots,$$

and in the second sum as

$$\hat{g}_s \rightarrow g_s + \text{terms of degree } \leq 2s - 1, \quad s = 1, 2, \ldots$$

It is clear that this substitution affects only negligible terms (that is, summands of degree $\leq -2$ in the expression for the operator $D$).

To conclude the proof it remains to perform a simple transformation. □

Theorem 6.3(1) now follows from Lemmas 6.10, 6.11 and 6.12.

### 6.5 The operator $U$ in differential form

Here we prove claim (2) of Theorem 6.3, that is, compute the top degree terms of the operator $U: \Gamma \rightarrow \Gamma$. The argument here is similar to that of the previous subsection. However, there are two differences which require us to go through the proof in full detail:

- There is a difference between the expressions for $Ug_{\rho}$ and $Dg_{\rho}$ (Lemma 6.7):
• The behaviour of the degree of \((x(x+1))^m)^1 = g_m\) differs from the behaviour of the degree of \((y(y+1))^m)^1 = \hat{g}_{m+1}\). Indeed, the expression \(\deg((y(y+1))^m)^1 = 2m+1\) is valid for all \(m \in \mathbb{Z}_{\geq 0}\), while the formula \(\deg((x(x+1))^m)^1 = 2m - 1\) is valid only for \(m > 0\).

It is convenient to decompose \(U\) using Lemma 6.7 as follows:

\[
U = \sum_\sigma (U^0_\sigma + U^1_\sigma),
\]

where the sum is taken over all ordinary partitions \(\sigma\),

\[
U^0_\sigma g_{\rho} := \frac{\alpha}{2} \sum_{\tau : |\tau| = |\rho| - |\sigma|} c^\rho_\sigma \langle a_\sigma (x(x+1)) \rangle^1 g_\tau \tag{39}
\]

and

\[
U^1_\sigma g_{\rho} := \frac{1}{2} \sum_{\tau : |\tau| = |\rho| - |\sigma|} c^\rho_\sigma \langle x(x+1) \cdot a_\sigma (x(x+1)) \rangle^1 g_\tau. \tag{40}
\]

Remark 6.13. This decomposition \(U = U^0 + U^1\) is similar to the decomposition of the corresponding operator \(U_{\theta,z,z'} = U^0_{\theta,z,z'} + U^1_{\theta,z,z'} + U^2_{\theta,z,z'}\) in the paper [20]. The operator \(U_{\theta,z,z'}\) is constructed using the up transition function for the \(z\)-measures on the Young graph with the Jack edge multiplicities (here \(\theta > 0\) is the Jack parameter), see [20, Thm. 6.1 (ii)].

Note that our \(U\) is expressed as the sum of two operators while the expression for \(U_{\theta,z,z'}\) has three terms. This is because the expression [20, (4.3)] for the up transition function for the \(z\)-measures involves terms of degrees zero, one and two in the (anisotropic) Kerov coordinates while the expression (9) for the up transition function for the multiplicative measures involves only terms of degrees zero and one in \(x(x+1)\).

Lemma 6.14 (Cf. Lemma 6.8). If \(\sigma\) is a nonempty ordinary partition, then

\[
\deg U^1_\sigma \leq \max_{\rho,\tau} (\ell(\rho) - \ell(\tau) - 2\ell(\sigma) + 1),
\]

where the maximum is taken over all pairs \((\rho, \tau)\) such that \(c^\rho_\sigma \neq 0\).

A more rough but simpler estimate is

\[
\deg U^1_\sigma \leq -\ell(\sigma) + 1.
\]

The case \(U^0_\sigma\) will be investigated separately, see Lemma 6.16 below.

Proof. Arguing as in Lemma 6.8, we get the estimate

\[
\deg U^1_\sigma \leq \max_{\rho,\tau} \left( \langle x(x+1) \cdot a_\sigma (x(x+1)) \rangle^1 - 2|\sigma| - \ell(\tau) + \ell(\rho) \right).
\]

24Our operator \(U\) is related to the multiplicative measures on the Schur graph in the same way, see Lemma 4.12.

25See also [20, Lemma 5.11] and Proposition 6.7. in the present paper, respectively.
It remains to compute $\deg \left\langle x(x+1) \cdot a_\sigma(x(x+1)) \right\rangle^\dagger$. Observe that the polynomial $x(x+1) \cdot a_\sigma(x(x+1))$ has degree $\geq 1$ in $x(x+1)$, therefore, by Lemma 6.6 and Proposition 5.7 we have

$$\deg \left\langle x(x+1) \cdot a_\sigma(x(x+1)) \right\rangle^\dagger = 2|\sigma| - 2\ell(\sigma) + 1,$$

this gives the first estimate. The second estimate is obtained as before, because if $c_{\rho\tau}^\sigma \neq 0$, then $\ell(\rho) \leq \ell(\sigma) + \ell(\tau)$.

From the second estimate of the above lemma follows a

**Corollary 6.15** (Cf. Corollary 6.9). If $\ell(\sigma) \geq 3$, then $\deg U^1_\sigma \leq -2$.

In the next Lemma we deal with the whole operator $\sum_\sigma U^0_\sigma$.

**Lemma 6.16.**

$$\sum_\sigma U^0_\sigma = \frac{\alpha}{2} + \alpha \frac{\partial}{\partial g_1} + \text{operators of degree } \leq -2.$$

**Proof.** If $\sigma$ is empty, then $\rho = \tau$ and $c_{\rho\tau}^\sigma = 1$ in (39), and we get

$$U^0_\sigma g_\rho = \frac{\alpha}{2} c_{\rho\sigma}^\sigma (1)^\dagger g_\rho = \frac{\alpha}{2} g_\rho$$

(compare this to Lemma 6.10).

Assume now that $\sigma$ is nonempty. Then $\sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)})$, and $\ell(\sigma) \geq 1$. Arguing as in Lemma 6.14 we get the estimate

$$\deg U^0_\sigma \leq \max_{\rho,\tau} \left( \deg \left\langle a_\sigma(x(x+1)) \right\rangle^\dagger + 2|\tau| - \ell(\tau) - 2|\rho| + \ell(\rho) \right) \leq \max_{\rho,\tau} \left( \deg \left\langle a_\sigma(x(x+1)) \right\rangle^\dagger - 2|\sigma| + \ell(\sigma) \right),$$

where the maximum is taken over all pairs $(\rho, \tau)$ such that $c_{\rho\tau}^\sigma \neq 0$. The second inequality holds because $\ell(\rho) - \ell(\tau) \leq \ell(\sigma)$.

If the polynomial $a_\sigma(x(x+1))$ has degree $\geq 1$ in $x(x+1)$, then using Lemma 6.6 and Proposition 5.7 we can write

$$\deg \left\langle a_\sigma(x(x+1)) \right\rangle^\dagger = \deg \left( g_{|\sigma| - \ell(\sigma)} \right) = 2|\sigma| - 2\ell(\sigma) - 1,$$

which implies

$$\deg U^0_\sigma \leq -\ell(\sigma) - 1 \leq -2.$$
then $\ell(\rho) - \ell(\tau) = 0$ is strictly smaller than $\ell(\sigma) = 1$, which again implies $\deg U_0^\sigma \leq -2$.

Thus, the only substantial contribution arises when $\sigma = (1)$, $\rho(\rho) = 1$ and $\tau = (\rho_1, \ldots, \rho(\rho) - 1)$ (that is, $\tau$ is obtained from $\rho$ by deleting a singleton). This gives rise to the term $U_0^\sigma g_\rho = \alpha \partial g_\rho / \partial g_1$. \hfill \Box

Lemmas 6.17, 6.18 and 6.19 below are similar to Lemmas 6.10, 6.11 and 6.12 respectively, and deal with the operator $\sum_\sigma U_1^\sigma$. \hfill \Box

**Lemma 6.17.** $U_1^\emptyset = \frac{1}{2} g_1$.

**Proof.** The proof is similar to that of Lemma 6.10. We should only note that $\langle x(x + 1) \cdot a_\emptyset (x(x + 1)) \rangle^1 = \langle x(x + 1) \rangle^1 = g_1$ in (10), therefore, $U_1^\emptyset$ reduces to multiplication by $g_1 / 2$. \hfill \Box

**Lemma 6.18.**

$$\sum_{\sigma: \ell(\sigma) = 2} U_1^\sigma = \sum_{r,s \geq 1} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s} + \text{operators of degree } \leq -2.$$  

**Proof.** The proof is similar to the proof of Lemma 6.11. Instead of Lemma 6.8, we refer to its analogue, Lemma 6.14.

Let $\ell(\sigma) = 2$ in (10), that is, $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 \geq \sigma_2 \geq 1$. Using Lemma 6.6 and Proposition 5.7, we get

$$\langle x(x + 1) \cdot a_\sigma (x(x + 1)) \rangle^1 = 4(2\sigma_1 - 1)(2\sigma_2 - 1)g_{\sigma_1 + \sigma_2 - 1} + \text{terms of degree } \leq 2|\sigma| - 5.$$  

Next,

$$\left( \sum_{\sigma: \ell(\sigma) = 2} U_1^\sigma \right) g_\rho = \sum_{1 \leq i < j \leq \ell(\rho)} 2(2\rho_i - 1)(2\rho_j - 1)c_{\rho_i \rho_j}^\rho g_{\rho_i + \rho_j - 1}g_{\rho \setminus \{\rho_i, \rho_j\}} + \text{terms of degree } \leq 2|\rho| - \ell(\rho) - 3,$$

and using (38), we conclude the proof. \hfill \Box

**Lemma 6.19.**

$$\sum_{\sigma: \ell(\sigma) = 1} U_1^\sigma = \sum_{r \geq 1} (2r - 1)g_r \frac{\partial}{\partial g_r} \sum_{r,s \geq 1} (r + s - 1)g_r g_s \frac{\partial}{\partial g_{r+s}} + \text{operators of degree } \leq -2.$$  

\footnote{It follows from Corollary 6.15 that it suffices to examine the cases when $\ell(\sigma) = 0$ (that is, $\sigma = \emptyset$), $\ell(\sigma) = 2$, and $\ell(\sigma) = 1$. We perform this consequently.}
Proof. The proof is similar to that of Lemma 6.12 (and we again use Lemma 6.14 instead of Lemma 6.8).

Let \( \ell(\sigma) = 1 \) in (40), that is, \( \sigma = (s) \) for some \( s \in \mathbb{Z}_{>0} \). Again, two cases are possible: either \( \ell(\rho) = \ell(\tau) + 1 \), or \( \ell(\rho) = \ell(\tau) \).

Assume first that \( \ell(\rho) = \ell(\tau) + 1 \). Using Lemma 6.6 and Proposition 5.7, we get
\[
\left\langle x(x + 1) \cdot a_\sigma(x(x + 1)) \right\rangle = 2(2s - 1)g_s + \text{terms of degree} \leq 2s - 3.
\]
This gives rise to the terms
\[
\sum_{r \geq 1} (2r - 1)g_r \frac{\partial}{\partial g_r} + \text{operators of degree} \leq -2.
\]
If \( \ell(\rho) = \ell(\tau) \), similarly to the proof of Lemma 6.12, we get the terms
\[
\sum_{\substack{r \geq 2 \ 1 \leq s \leq r-1}} (2s - 1)g_s g_{r-s} \frac{\partial}{\partial g_r} + \text{operators of order} \leq -2.
\]
To conclude the proof it remains to perform a simple transformation. \( \square \)

Theorem 6.3 (2) now follows from Lemmas 6.16, 6.17, 6.18 and 6.19.

7 The operator \( T_n \) in differential form

Let \( T_n : \text{Fun}(S_n) \to \text{Fun}(S_n) \), \( n \in \mathbb{Z}_{>0} \), be the operator from Definition 2.7. In [11] we have obtained a formula for the action of \( T_n \) on Schur’s \( Q \)-functions (Theorem 4.8). In this section using the results of [5] and Theorem 6.3 we prove another formula for \( T_n \):

Theorem 7.1. There exists a unique operator \( \tilde{B} : \Gamma \to \Gamma \) such that
\[
(T_n - 1)f_n = \frac{(\tilde{B}f)_n}{(n + \alpha/2)(n + 1)}
\]
for all \( f \in \Gamma \).

The operator \( \tilde{B} \) has zero degree. Under the identification of \( \Gamma \) with the polynomial algebra \( \mathbb{R}[p_1, p_3, p_5, \ldots] \), the zero-degree homogeneous component of \( \tilde{B} \), the operator \( B : \Gamma \to \Gamma \), has the form
\[
B = \sum_{i,j=2}^\infty (2i - 1)(2j - 1)(p_1 p_{2i+2j-3} - p_{2i-1} p_{2j-1}) \frac{\partial^2}{\partial p_{2i-1} \partial p_{2j-1}} + 2 \sum_{i,j=1}^\infty (2i + 2j - 1)p_1 p_{2i-1} p_{2j-1} \frac{\partial}{\partial p_{2i+2j-1}} \]
\[
- \sum_{i=2}^\infty (2i - 1) \left( 2i - 2 + \frac{\alpha}{2} \right) p_{2i-1} \frac{\partial}{\partial p_{2i-1}}.
\]
By the zero-degree homogeneous component of the operator \( \tilde{B} \) we mean the unique homogeneous operator \( B : \Gamma \to \Gamma \) of zero degree such that
\[
\tilde{B} = B + \text{operators of degree } \leq -1.
\]

First, we note an important corollary of Theorem 7.1:

**Corollary 7.2.** The operator \( B : \Gamma \to \Gamma \) commutes with the operator of multiplication by the element \( p_1 \in \Gamma \).

**Proof.** This follows from the fact that the expression (41) for \( B \) does not contain partial derivatives with respect to \( p_1 \).

In the rest of this section we prove Theorem 7.1.

First, (24) and (27) imply
\[
(T_n - 1)f_n = \frac{((UD - \frac{1}{4}(g_1 + \alpha))(g_1 + 2))f_n}{(n + \alpha/2)(n + 1)}
\]
for all \( f \in \Gamma \). Thus, \( \tilde{B} = UD - \frac{1}{4}(g_1 + \alpha)(g_1 + 2) \), and the uniqueness of \( \tilde{B} \) follows from the fact that the algebra \( \Gamma \subset \text{Fun}(S) \) separates points.

Now using Theorem 6.3 we write the operator \( \tilde{B} \) as a formal differential operator with respect to the generators \( g_k \), \( k \in \mathbb{Z}_{>0} \), of the algebra \( \Gamma \):

**Lemma 7.3.** Under the identification of the algebra \( \Gamma \) with the polynomial algebra \( \mathbb{R}[g_1, g_2, \ldots] \), the operator \( \tilde{B} = UD - \frac{1}{4}(g_1 + \alpha)(g_1 + 2) \) looks as follows:
\[
\tilde{B} = \sum_{r,s=2}^{\infty} (2r-1)(2s-1)(g_1 g_{r+s-1} - g_r g_s) \frac{\partial^2}{\partial g_r \partial g_s} + \sum_{r,s=1}^{\infty} (r + s - 1/2)g_1 g_r g_s \frac{\partial}{\partial g_{r+s}} - \sum_{r=2}^{\infty} (2r-1) \left( 2r - 2 + \frac{\alpha}{2} \right) g_r \frac{\partial}{\partial g_r} + \text{operators of degree } \leq -1.
\]

Note that the operators \( U \) and \( D \) both have degree 1 in the sense of Definition 6.1. However, it turns out that the operator \( \tilde{B} \) has zero degree instead of degree 2, because higher degree terms cancel out.

**Proof.** We write
\[
U = \frac{1}{2}g_1 + U_0 + U_{-1} + \ldots, \quad D = \frac{1}{2}g_1 + D_0 + D_{-1} + \ldots,
\]
where dots stand for operators of degree \( \leq -2 \),
\[
U_0 := \frac{1}{2} \alpha + \sum_{r=1}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r}, \quad D_0 := -\sum_{r=1}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r}
\]
(these are the zero degree parts),

\[
U_{-1} := \alpha \frac{\partial}{\partial g_1} + \sum_{r,s=1}^{\infty} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s} + \sum_{r,s=1}^{\infty} (r + s - 1)g_r g_s \frac{\partial}{\partial g_{r+s}}.
\]

\[
D_{-1} := \sum_{r,s=1}^{\infty} (2r - 1)(2s - 1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s} + \sum_{r,s=1}^{\infty} (r + s - 1)g_r g_s \frac{\partial}{\partial g_{r+s}}
\]

(these are the parts of degree $-1$).

We compute the top degree terms of the operator $\tilde{B} = UD - \frac{1}{4}(g_1 + \alpha)(g_1 + 2)$ consequently.

Terms of degree 2:

\[
\frac{1}{4} g_1^2 - \frac{1}{4} g_1^2 = 0.
\]

Terms of degree 1 are equal to

\[
\frac{1}{2} g_1 D_0 + \frac{1}{2} U_0 g_1 - \frac{1}{4} (\alpha + 2) g_1.
\]

Because the operator $\sum_{r=2}^{\infty} (2r - 1)g_r \frac{\partial}{\partial g_r}$ commutes with the multiplication by $g_1$, we have

\[
g_1 D_0 + U_0 g_1 = -g_1 \left( g_1 \frac{\partial}{\partial g_1} + \sum_{r=2}^{\infty} (2r - 1)g_r \frac{\partial}{\partial g_r} \right)
+ \left( \frac{\alpha}{2} + g_1 \frac{\partial}{\partial g_1} + \sum_{r=2}^{\infty} (2r - 1)g_r \frac{\partial}{\partial g_r} \right) g_1
= -g_1^2 \frac{\partial}{\partial g_1} + \frac{\alpha}{2} g_1 + g_1 \frac{\partial}{\partial g_1} g_1 = \frac{\alpha}{2} g_1 + g_1,
\]

and we see that the terms of degree 1 also cancel out.

It remains to compute terms of degree 0. They are equal to

\[
\frac{1}{2} g_1 D_{-1} + U_0 D_0 + \frac{1}{2} U_{-1} g_1 - \frac{\alpha}{2}.
\]

Observe that

\[
[U_{-1}, g_1] = \left[ \alpha \frac{\partial}{\partial g_1}, g_1 \right] + \left[ g_1 \frac{\partial^2}{\partial g_1^2}, g_1 \right] + 2 \sum_{r=2}^{\infty} (2r - 1)g_r \frac{\partial}{\partial g_r} \left[ \frac{\partial}{\partial g_1}, g_1 \right]
= \alpha + 2g_1 \frac{\partial}{\partial g_1} + 2 \sum_{r=2}^{\infty} (2r - 1)g_r \frac{\partial}{\partial g_r}.
\]
It can be readily verified that

\[
U_0D_0 = -\left( \frac{\alpha}{2} + g_1 \frac{\partial}{\partial g_1} + \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r} \right) \left( g_1 \frac{\partial}{\partial g_1} + \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r} \right)
\]

\[
= -\frac{1}{2} g_1 \frac{\partial}{\partial g_1} - g_1 \frac{\partial}{\partial g_1} - g_2 \frac{\partial^2}{\partial g_1^2} - 2 \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial^2}{\partial g_1 \partial g_r}
\]

\[
- \frac{\alpha}{2} \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r} - \sum_{r,s=2}^{\infty} (2r-1)(2s-1)g_r g_s \frac{\partial^2}{\partial g_r \partial g_s}
\]

\[
- \sum_{r=2}^{\infty} (2r-1)^2 g_r \frac{\partial}{\partial g_r}
\]

and

\[
\frac{1}{2} (D_0 + U_0) = \frac{1}{2} \frac{\partial}{\partial g_1} + \sum_{r,s=2}^{\infty} (2r-1)(2s-1)g_{r+s-1} \frac{\partial^2}{\partial g_r \partial g_s}
\]

\[
+ g_1 \frac{\partial^2}{\partial g_1^2} + 2 \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial^2}{\partial g_r \partial g_1} + \sum_{r,s=1}^{\infty} (r+s-1/2)g_r g_s \frac{\partial}{\partial g_{r+s}}
\]

Now we finally are able to compute the terms of degree 0:

\[
\frac{1}{2} (D_0) + U_0 = \frac{1}{2} g_1 (D_0 + U_0) + U_0 D_0 + \frac{\alpha}{2} g_1 \frac{\partial}{\partial g_1} + \sum_{r=2}^{\infty} (2r-1)g_r \frac{\partial}{\partial g_r}
\]

Combining three above formulas, we get the desired expression. □

To prove Theorem 7.1 it remains to substitute in the expression for \( \tilde{B} \) given by the previous Lemma the inhomogeneous generators \( g_k, k \in \mathbb{Z}_{>0} \), by the homogeneous generators \( p_{2m-1}, m \in \mathbb{Z}_{>0} \). This should be done according to the next Lemma:

**Lemma 7.4.** (1) \( g_k = 2p_{2k-1} \) + terms of degree \( \leq (2k-1) \), \( k \in \mathbb{Z}_{>0} \);

(2) Let \( f \in \Gamma \), then\(^{27} \)

\[
\frac{\partial f}{\partial g_k} = \frac{1}{2} \frac{\partial f}{\partial p_{2k-1}} + \text{terms of degree} \leq (\deg f - (2k-1)), \quad k \in \mathbb{Z}_{>0}.
\]

**Proof.** Claim (1) directly follows from Propositions 5.3 and 5.4.

To prove claim (2) observe that

\[
\frac{\partial f}{\partial g_k} = \sum_{i=k}^{\infty} \frac{\partial (2p_{2i-1})}{\partial g_k} \frac{\partial f}{\partial (2p_{2i-1})} = \frac{1}{2} \frac{\partial f}{\partial p_{2k-1}} + \sum_{i>k} \frac{\partial (2p_{2i-1})}{\partial g_k} \frac{\partial f}{\partial (2p_{2i-1})}
\]

\(^{27}\)Note that both \( \partial f/\partial g_k \) and \( \partial f/\partial p_{2k-1} \) have degree \( (\deg f - (2k-1)) \).
The $l$th summand in the last sum has degree
\[
\leq (2l - 3) - (2k - 1) + \deg f - (2l - 1) = \deg f - (2k - 1) - 2 < \deg f - (2k - 1).
\]
This concludes the proof.

Together Lemmas 7.3 and 7.4 imply Theorem 7.1.

8 The limit diffusion

In this section we prove that the Markov chains $T_n$ from Definition 2.7 converge (within a certain time scaling) to a continuous time Markov process $X_\alpha(t)$, $t \geq 0$, in the simplex $\Omega_+$. Using Theorem 7.1 we prove the expression (1) (from Introduction) for its the pre-generator. In §8.8 we study some further properties of $X_\alpha(t)$.

In §8.5 we discuss the embedding of the simplex $\Omega_+$ into the Thoma simplex $\Omega$ introduced in [8] (see also §1.2). This construction leads to another proof of one of the results from this section (namely, the first claim of Proposition 8.4) but it is also of separate interest.

8.1 An operator semigroup approximation theorem

We begin by stating a well-known general result on approximations of continuous contraction semigroups by discrete ones. We formulate it in a form (Theorem 8.3) best suitable for the application to our concrete situation. We refer to the paper [29] and the book [4]. In the book one can also find additional references.

Let $L$ and $L_n$, $n \in \mathbb{Z}_{>0}$, be real Banach spaces.\footnote{The norms of $L$ and of each $L_n$ are denoted by the same symbols $\| \cdot \|$.} Let $\pi_n : L \to L_n$, $n \in \mathbb{Z}_{>0}$, be bounded linear operators such that $\sup_n \| \pi_n \| < \infty$.

**Definition 8.1.** We say that a sequence of elements $\{f_n \in L_n\}$ converges to an element $f \in L$ if $\lim_{n \to \infty} \| \pi_n f - f_n \| = 0$. We write $f_n \to f$.

It our concrete situation described below in §8.2 the additional condition
\[
\lim_{n \to \infty} \| \pi_n f \| = \| f \| \quad \text{for all } f \in L
\] (42)
is satisfied. This condition implies that any sequence $\{f_n \in L_n\}$ may have at most one limit in $L$.

**Definition 8.2.** An operator $D$ in $L$ is called dissipative if $\|(s1 - D)f\| \geq s\|f\|$ for all $s \geq 0$ and all $f$ from the domain of $D$, where $1$ denotes the identity operator.
Now, assume that for all \(n \in \mathbb{Z}_{>0}\) we are given a contraction operator \(T_n\) in \(L_n\). Suppose that \(\{\varepsilon_n\}\) is a sequence of positive numbers converging to zero. Assume that there exists a dense subspace \(\mathcal{F} \subset L\) and an operator \(A: \mathcal{F} \to L\) such that in the sense of Definition 8.1
\[
\varepsilon_n^{-1}(T_n - 1)\pi_n f \to Af \quad \text{for all } f \in \mathcal{F}.
\]

**Theorem 8.3.** If
- The operator \(A: \mathcal{F} \to L\) is dissipative;
- For some \(s > 0\) the range of \((s1 - A)\) is dense in \(L\).

Then the operator \(A\) is closable in \(L\); its closure generates a strongly continuous contraction semigroup \(\{T(t)\}_{t \geq 0}\) in \(L\); and (again in the sense of Definition 8.1)
\[
T_n^{[\varepsilon_n^{-1}]} \pi_n f \to T(t)f \quad \text{for all } f \in L,
\]
for \(t \geq 0\) uniformly on bounded intervals.

**Proof.** Let \(\hat{A}: L \to L\) be the operator defined as \(f \mapsto \lim_{n \to \infty} \varepsilon_n^{-1}(T_n - 1)\pi_n f\) for all \(f \in L\) such that this limit exists in the sense of Definition 8.1. The domain of \(\hat{A}\) consists of such \(f \in L\). Clearly, \(\hat{A}|_F = A\).

Since each \(T_n\) is a contraction, each operator \(\varepsilon_n^{-1}(T_n - 1)\) is dissipative. Hence \(\hat{A}\) is dissipative, too.

The operator \(\hat{A}\) satisfies the conditions of [29, Thm. 5.3] with \(M = 1\) and \(K = 0\). Hence \(\hat{A}\) is closable, its closure \(\bar{A}\) generates a semigroup \(\{T(t)\}_{t \geq 0}\) in \(L\); and (again in the sense of Definition 8.1)
\[
T_n^{[\varepsilon_n^{-1}]} \pi_n f \to T(t)f \quad \text{for all } f \in L,
\]
for \(t \geq 0\) uniformly on bounded intervals.

Because each operator \(T_n\) is a contraction, the fact that each \(T(t)\) is a contraction follows from (42) and (43). By the Hille-Yosida theorem (see, e.g., [4, Ch. 1, Thm. 2.6]), the dissipativity of \(\hat{A}\) implies that the semigroup \(T(t)\) is strongly continuous.

Since \(\mathcal{F}\) is dense in \(L\) and for some \(s > 0\) the range of \((s1 - A)\) is dense in \(L\), the subspace \(\mathcal{F} \subset L\) is a core for \(\bar{A}\) in the sense of [4, Ch. 1, Sect. 3]. By [4, Ch. 1, Prop. 3.1], the operator \(A\) is closable in \(L\) and \(\bar{A} = \overline{A}\). This concludes the proof. \(\square\)

8.2 The simplex \(\Omega_+\)

We return to our concrete situation. As \(L_n, n \in \mathbb{Z}_{>0}\), we take the finite-dimensional vector space \(\text{Fun}(\mathbb{S}_n)\) of real-valued functions on \(\mathbb{S}_n\) with the supremum norm. As \(T_n\) we take the Markov transition operators from Definition 2.7. Clearly, each \(T_n\) is a contraction. As the scaling factors we take \(\varepsilon_n := 1/n^2\). To define the space \(L\) and the operators \(\pi_n: L \to L_n\) we need some extra notation.

Let \(\Omega_+\) be the subset of the infinite-dimensional cube \([0, 1]^\infty\) defined as
\[
\Omega_+ := \left\{ x = (x_1, x_2, \ldots) \in [0, 1]^\infty : x_1 \geq x_2 \geq \cdots \geq 0, \sum_i x_i \leq 1 \right\},
\]
We equip the cube $[0, 1]^\infty$ with the standard product topology. The subset $\Omega_+ \subset [0, 1]^\infty$ is a compact, metrizable and separable space. As $L$ we take the Banach space $C(\Omega_+)$ of all real continuous functions on $\Omega_+$ with pointwise operations and the supremum norm.

For $n \in \mathbb{Z}_{>0}$, we define an embedding $\iota_n$ of the set $S_n$ into the space $\Omega_+$:

$$
\iota_n : S_n \hookrightarrow \Omega_+, \quad \lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots) \mapsto \left(\frac{\lambda_1}{n}, \ldots, \frac{\lambda_n}{n}, 0, 0, \ldots\right) \in \Omega_+.
$$

Using $\iota_n$ we define the operators $\pi_n : L \to L_n$, that is, $\pi_n : C(\Omega_+) \to \text{Fun}(S_n)$:

$$(\pi_n f)(\lambda) := f(\iota_n(\lambda)), \quad \text{where } f \in C(\Omega_+) \text{ and } \lambda \in S_n.$$ (44)

Clearly, $\|\pi_n\| \leq 1$. Moreover, in our situation the condition (42) is satisfied because the space $\Omega_+$ is approximated by the sets $\iota_n(S_n) \subset \Omega_+$ in the sense that every open subset of $\Omega_+$ has a nonempty intersection with $\iota_n(S_n)$ for all $n$ large enough.

### 8.3 Moment coordinates

Here we define the dense subspace $\mathcal{F} \subset L = C(\Omega_+)$. To every point $x \in \Omega_+$ we assign a probability measure $
_x := \sum_{i=1}^{\infty} x_i \delta_{x_i} + \gamma \delta_0, \quad \gamma := 1 - \sum_{i=1}^{\infty} x_i$

on $[0, 1]$, where by $\delta_s$ we denote the Dirac measure at a point $s$. Denote by $q_k = q_k(x)$ the $k$th moment of the measure $\nu_x$:

$$q_k(x) := \int_0^1 u^k \nu_x(du) = \sum_{i=1}^{\infty} x_i^{k+1}, \quad k = 1, 2, \ldots.$$ 

Following [3], we call $q_1, q_2, \ldots$ the moment coordinates of the point $x \in \Omega_+$. They are continuous functions on $\Omega_+$. 29

Note that the functions $q_1, q_2, \ldots$ are algebraically independent as functions on $\Omega_+$. Clearly, any subcollection of $\{q_1, q_2, \ldots\}$ is also algebraically independent. As $\mathcal{F}$ we take the subalgebra of the Banach algebra $C(\Omega_+)$ freely generated by the even moment coordinates:

$$\mathcal{F} := \mathbb{R}[q_2, q_4, q_6, \ldots ] \subset C(\Omega_+).$$

#### Proposition 8.4

The functions $q_2, q_4, q_6, \ldots$ separate points of $\Omega_+$. Moreover, any infinite subcollection of $\{q_1, q_2, \ldots\}$ also possesses this property.

29Observe that the function $x \mapsto \sum_{i=1}^{\infty} x_i$ is not continuous in $x \in \Omega_+$. 

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Proof. Let \( \{q_{k_1}, q_{k_2}, \ldots \} \) be any infinite subcollection of \( \{q_1, q_2, \ldots \} \). It suffices to show that a point \( x \in \Omega_+ \) is uniquely determined by the sequence \( \{q_{k_1}(x), q_{k_2}(x), \ldots \} \). Observe that for every \( m = 0, 1, 2, \ldots \) we have
\[
\left( q_{k_n}(x) - \sum_{j=1}^{m} x_j^{k_n} \right)^{1/k_n} = x_{m+1} \cdot \left( 1 + \sum_{i=m+1}^{\infty} (x_i/x_{m+1})^{k_n} \right)^{1/k_n} \to x_{m+1}
\]
as \( n \to \infty \) (the convergence is pointwise). Here if \( m = 0 \), then by agreement there is no sum \( \sum_{j=1}^{m} \) in the LHS. Using this convergence, one can reconstruct the coordinates \( x_1, x_2, x_3, \ldots \) one after another using the sequence \( \{q_{k_1}(x), q_{k_2}(x), \ldots \} \). This concludes the proof.

Since the subalgebra \( \mathcal{F} \subset C(\Omega_+) \) separates points and contains the function 1, it is dense in \( C(\Omega_+) \).

Next, recall the algebra \( \Gamma = \mathbb{R}[p_1, p_3, p_5, \ldots] \) of doubly symmetric functions introduced in \( \S 4 \). Let \( I := (p_1 - 1)\Gamma \) be the principal ideal in \( \Gamma \) generated by \( p_1 - 1 \). Set \( \Gamma^\circ := \Gamma/I \). To every element \( f \in \Gamma \) corresponds an image in \( \Gamma^\circ \) denoted by \( f^\circ \). In particular, \( p_1^\circ = 1 \), and \( \Gamma^\circ \) is freely generated (as a commutative unital algebra) by the elements \( p_3^\circ, p_5^\circ, p_7^\circ, \ldots \).

Observe also that
\[
\Gamma = \mathbb{R}[p_1, p_3, p_5, p_7, \ldots] = I \oplus \mathbb{R}[p_3, p_5, p_7, \ldots],
\]
and hence \( \mathcal{F} \cong \mathbb{R}[p_3, p_5, p_7, \ldots] \). We will use this fact below.

The correspondence
\[
p_{2k+1}^\circ \leftrightarrow q_{2k}, \quad k = 1, 2, \ldots
\]
establishes an isomorphism between the algebras \( \Gamma^\circ \) and \( \mathcal{F} \). We will identify elements \( g \in \Gamma^\circ \) and the corresponding continuous functions \( g(x) \) on \( \Omega_+ \). Moreover, to every element \( \varphi \in \Gamma \) corresponds a continuous function on \( \Omega_+ \). Denote this function by \( \varphi^\circ(x) \). Equivalently, the map \( \varphi \to \varphi^\circ(\cdot) \) is determined by setting
\[
p_1^\circ(x) := 1, \quad p_{2k-1}^\circ(x) := \sum_{i=1}^{\infty} x_i^{2k-1}, \quad k = 2, 3, \ldots.
\]

### 8.4 The limit theorem for coherent systems

At this point it is convenient to formulate the following theorem about coherent systems on \( S \).

**Theorem 8.5.** Let \( \{M_n\} \) be a coherent system on \( S \) (see \( \S 2.2 \) for the definition). Then the push-forward of the measure \( M_n \) under the embedding \( \iota_n \) (defined in \( \S 8.2 \)) weakly converges, as \( n \to \infty \), to a probability measure \( P \) on \( \Omega_+ \). The measure \( P \) is called the boundary measure of the system \( \{M_n\} \).
Conversely, any coherent system on $S$ can be reconstructed from its boundary measure as follows:

$$M_n(\lambda) = 2^{-|\lambda|} h(\lambda) \int_{\Omega_+} \mathcal{Q}_\lambda^\alpha(x) \mathcal{P}(dx) \quad \text{for all } n \in \mathbb{Z}_{>0} \text{ and } \lambda \in S_n.$$  

Here $h(\lambda)$ is given by (3) and $\mathcal{Q}_\lambda^\alpha$ is the image in $\Gamma^\circ$ of doubly symmetric $Q$-Schur function defined in §4.1.

**Proof.** This theorem can be proved exactly as Theorem B of the paper [12] with the two following changes: instead of $\theta$-shifted Jack polynomials one should use factorial Schur’s $Q$-functions, and instead of [12, Theorem 6.1] one should refer to the relation (21) (proved in the paper by V. Ivanov [10]).

To the multiplicative coherent system $\{M_\alpha^n\}$ with parameter $\alpha \in (0, +\infty)$ corresponds a measure $\mathcal{P}^{(\alpha)}$ on $\Omega_+$. We may call it the **multiplicative boundary measure**.

### 8.5 Doubling of shifted Young diagrams and the Thoma simplex

In this subsection we discuss the embedding of $\Omega_+$ into the Thoma simplex $\Omega$ introduced in [8] (see also §1.2).

**8.5.1 Modified Frobenius coordinates and the Thoma simplex**

Here we recall some definitions from [3, §3.1 and §3.3].

Let $\sigma$ be an ordinary partition. Denote by $a_1, \ldots, a_k, b_1, \ldots, b_k$ its **modified Frobenius coordinates**. That is, $k$ is the number of diagonal boxes in $\sigma$, $a_i$ equals $\frac{1}{2}$ plus the number of boxes in the $i$th row to the right of the diagonal, and $b_j$ equals $\frac{1}{2}$ plus the number of boxes in the $j$th column below the diagonal. We write $\sigma = (a_1, \ldots, a_k \mid b_1, \ldots, b_k)$. Note that $\sum(a_i + b_i) = |\sigma|$, the number of boxes in the diagram $\sigma$. Note also that each of the sequences $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ is strictly decreasing. Recall that by $Y_n$ we denote the set of all ordinary partitions of weight $n$.

Let $\Omega$ be the **Thoma simplex**, that is, the space of couples $(\omega; \omega') \in [0, 1]^\infty \times [0, 1]^\infty$ satisfying the following conditions:

$$\omega_1 \geq \omega_2 \geq \cdots \geq 0, \quad \omega'_1 \geq \omega'_2 \geq \cdots \geq 0, \quad \sum_i \omega_i + \sum_j \omega'_j \leq 1.$$  

Here $[0, 1]^\infty$ is equipped with the product topology and hence the space $\Omega$ is a compact subset of $[0, 1]^\infty \times [0, 1]^\infty$.

Consider for all $n \in \mathbb{Z}_{>0}$ embeddings:

$$\hat{\iota}_n: Y_n \hookrightarrow \Omega, \quad \sigma \mapsto \left(\frac{a_1}{n}, \ldots, \frac{a_k}{n}, 0, \ldots; \frac{b_1}{n}, \ldots, \frac{b_k}{n}, 0, \ldots\right),$$

Ordinary partitions are identified with ordinary Young diagrams as in [17, Ch. I, §1]
where \( \sigma = (a_1, \ldots, a_k \mid b_1, \ldots, b_k) \) is written in terms of the modified Frobenius coordinates.

### 8.5.2 Doubling of shifted Young diagrams

Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a shifted Young diagram. By \( D\lambda \) denote the double of \( \lambda \), that is, the ordinary Young diagram that is written in the modified Frobenius coordinates as

\[
D\lambda = \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_\ell}{2} \mid \frac{\lambda_1 - 1}{2}, \ldots, \frac{\lambda_\ell - 1}{2} \right),
\]

see [17, Ch. I, §1, Example 9].\(^{31}\) In [9] this object is called the shift-symmetric diagram associated to a strict partition \( \lambda \). The number of boxes in \( D\lambda \) clearly equals twice the number of boxes in \( \lambda \). In this way we obtain embeddings \( S_n \hookrightarrow \mathcal{Y}_{2n} \) for all \( n \in \mathbb{Z}_{\geq 0} \) (and hence the whole Schur graph is embedded into the Young graph).

### 8.5.3 The embedding \( T: \Omega_+ \hookrightarrow \Omega \)

The sets \( \hat{i}_n(\mathcal{Y}_n) \) approximate the Thoma simplex \( \Omega \) in the same sense as the sets \( i_n(S_n) \) approximate \( \Omega_+ \) (see §8.2). Thus, it is natural to consider the following “limit” of the embeddings \( S_n \hookrightarrow \mathcal{Y}_{2n} \) as \( n \to \infty \):

\[
T x = (\omega; \omega') = \left( \frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_1}{2}, \frac{x_2}{2}, \ldots \right), \quad x = (x_1, x_2, \ldots) \in \Omega_+.
\]

The image of \( \Omega_+ \) is the whole diagonal subset \( \{(\omega, \omega'): \omega = \omega'\} \) of \( \Omega \). Moreover, \( T \) is a homeomorphism between \( \Omega_+ \) and this subset. The points \( x \in \Omega_+ \) such that \( \sum x_i = 1 \) map to the points \( (\omega; \omega) \) such that \( \sum (\omega_i + \omega_i) = 1 \). This embedding \( T \) was introduced in [8, §8.6].

The property that \( T: \Omega_+ \hookrightarrow \Omega \) is a limit in some sense of the embeddings \( S_n \hookrightarrow \mathcal{Y}_{2n} \) may be expressed as follows:

**Proposition 8.6.** Let \( \{\lambda(n)\}, \; n = 1, 2, \ldots, \), be a sequence of shifted Young diagrams, \( \lambda(n) \in \mathcal{S}_n \), such that, as \( n \to \infty \), the points \( i_n(\lambda(n)) \) tend to some point \( x \in \Omega_+ \). Then the points \( i_{2n}(D\lambda(n)) \) tend to \( T x \in \Omega \).

**Proof.** Clearly, \( T i_n(\lambda(n)) \to T x \) as \( n \to \infty \).

For any \( \mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{S}_n \) we have

\[
T i_n(\mu) = \left( \frac{\mu_1}{2n}, \ldots, \frac{\mu_\ell}{2n}, 0, \ldots; \frac{\mu_1}{2n}, \ldots, \frac{\mu_\ell}{2n}, 0, \ldots \right)
\]

and

\[
i_{2n}(D\mu) = \left( \frac{\mu_1 + 1/2}{2n}, \ldots, \frac{\mu_\ell + 1/2}{2n}, 0, \ldots; \frac{\mu_1 - 1/2}{2n}, \ldots, \frac{\mu_\ell - 1/2}{2n}, 0, \ldots \right).
\]

\(^{31}\)By agreement, \( D\emptyset = \emptyset \).
To conclude the proof observe that
\[
\left( \frac{1}{2n}, \ldots, \frac{1}{2n}, 0, \ldots, \frac{1}{2n}, \ldots \right) \to 0, \quad n \to \infty
\]
in the topology of \( \Omega \).

Informally, one can say that the previous Proposition states
\[
T_{\iota n}(\lambda) \approx \hat{\iota}_{2n}(\mathcal{D}\lambda), \quad \lambda \in S_n.
\]

8.5.4 Symmetric Thoma’s measures

Here we use the above construction to give another proof of the first claim of Proposition 8.4.

Let us recall the definition of the moment coordinates on the Thoma simplex [3, §3.4]. To every point \((\omega; \omega') \in \Omega\) one can assign the following probability measure on \([-1, 1]\):
\[
\hat{\nu}(\omega; \omega') := \sum_{i=1}^{\infty} \omega_i \delta_{\omega_i} + \sum_{i=1}^{\infty} \omega'_i \delta_{\omega'_i} + \hat{\gamma}(\omega; \omega') \delta_0,
\]
where \(\hat{\gamma}(\omega; \omega') = 1 - \sum_{i=1}^{\infty} (\omega_i + \omega'_i)\) and \(\delta_s\) denotes the Dirac measure at a point \(s\). This measure is called Thoma’s measure. The moments of Thoma’s measure \(\hat{\nu}(\omega; \omega')\) are called the moment coordinates of \((\omega; \omega') \in \Omega\):
\[
\hat{q}_m(\omega; \omega') = \sum_{i=1}^{\infty} \omega_i^{m+1} + (-1)^m \sum_{j=1}^{\infty} \omega'_j^{m+1}, \quad m = 1, 2, \ldots
\]
(compare these definitions of \(\hat{\nu}(\omega; \omega')\) and \(\hat{q}_m(\omega; \omega')\) to the definitions of \(\nu_x\) and \(q_m(x)\) from [335]).

If \(x \in \Omega_+\), then Thoma’s measure \(\hat{\nu}_x\) is symmetric with respect to the origin. Hence the odd moments of \(\hat{\nu}_x\) vanish. More precisely,
\[
q_m(Tx) = \begin{cases} 2^{-m}q_m(x), & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}
\]
Here \(q_m(x)\) are the moment coordinates on \(\Omega_+\).

A probability measure on \([-1, 1]\) is uniquely determined by its moments. Hence the functions \(q_1, q_2, \ldots\) separate points of \(\Omega\). It follows that a point \(Tx \in \Omega\) (where \(x \in \Omega_+)\) is uniquely determined by its moment coordinates \(q_2(Tx), q_4(Tx), q_6(Tx), \ldots\) (it suffices to take only even coordinates because the odd coordinates vanish). This is the same as to say that a point \(x \in \Omega_+\) is uniquely determined by its even moment coordinates \(q_2(x), q_4(x), q_6(x), \ldots\).

Hence the first claim of Proposition 8.4 holds.

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8.6 Convergence of generators

In this subsection we prove the convergence of the operators \(n^2(T_n - 1)\) to an operator \(A: \mathcal{F} \rightarrow \mathcal{F}\). In the next section using this convergence we apply the abstract Theorem 8.3 to our situation and prove the convergence of Markov chains corresponding to \(n^2(T_n - 1)\) to the Markov process \(X_\alpha(t)\) in \(\Omega_+\).

Proposition 8.7. In the sense of Definition 8.1 we have
\[
n^2(T_n - 1)\pi_n f \rightarrow Af \quad \text{for all } f \in \mathcal{F},
\]
where the operator \(A: \mathcal{F} \rightarrow \mathcal{F}\) can be written in one of the two following ways:

1. As a formal differential operator in the algebra \(\mathcal{F} = \mathbb{R}[q_2, q_4, q_6, \ldots]\):
   \[
   A = \sum_{i,j=1}^{\infty} (2i + 1)(2j + 1) (q_{2i+2j} - q_{2i}q_{2j}) \frac{\partial^2}{\partial q_{2i}\partial q_{2j}} + \sum_{i,j=0}^{\infty} (2i + 2j + 3) q_{2i}q_{2j} \frac{\partial}{\partial q_{2i+2j+2}} - \sum_{i=1}^{\infty} (2i + 1) \left(2i + \frac{\alpha}{2}\right) q_{2i} \frac{\partial}{\partial q_{2i}},
   \]

2. As an operator acting on functions \(Q_\mu \in \mathcal{F} \subset C(\Omega_+):\)
   \[
   AQ_\mu = -|\mu|(|\mu| + \alpha/2 - 1)Q_\mu + \sum_{y \in Y(\mu)} (y(y + 1) + \alpha)Q_\mu - \square(y);
   \]

where, by agreement, \(q_0 := 1\);

First, we prove two Lemmas.

Lemma 8.8. Let \(\varphi \in \Gamma\) and \(\deg \varphi \leq m - 1\) for some \(m \in \mathbb{Z}_{>0}\). Then
\[
\frac{1}{n^m} \varphi_n \rightarrow 0, \quad n \rightarrow \infty
\]
in the sense of Definition 8.1.34

Proof. The convergence to zero means that
\[
\sup_{\lambda \in \mathbb{S}_n} \frac{1}{n^m} |\varphi_n(\lambda)| \rightarrow 0, \quad n \rightarrow \infty.
\]

Observe that for all \(\lambda \in \mathbb{S}_n\) we have \(\lambda_i \leq n, \quad i = 1, \ldots, \ell(\lambda)\). Hence \(|\varphi_n(\lambda)| \leq \text{Const} \cdot n^{m-1}.

32See also Remark 6.2 about formal differential operators in polynomial algebras.
33Observe that the functions \(Q_\mu \in \Gamma^\circ \cong \mathcal{F}, \mu \in \mathbb{S}\), are not linearly independent. However, their linear span is \(\mathcal{F}\) because the system \(\{Q_\mu\}_{\mu \in \mathbb{S}}\) is a basis for \(\Gamma\). However, from the claim (1) it follows that the formula (47) for the action of \(A\) on \(Q_\mu\), \(\mu \in \mathbb{S}\), is consistent.
34Recall that \(\cdots)\) denotes the restriction of a function from the algebra \(\Gamma \subset \text{Fun}(\mathbb{S})\) to the subset \(\mathbb{S}_n \subset \mathbb{S}\).
Let $G$ denote the operator $\sum_{i=1}^{\infty} (2i-1)p_{2i-1} - \frac{\partial}{\partial p_{2i-1}}$ in the algebra $\Gamma$. In other words, on the homogeneous component of degree $m$ of the algebra $\Gamma$, $m \in \mathbb{Z}_{>0}$, the operator $G$ acts as multiplication by $m$. Let for all $s > 0$ the operator $s^G \: \Gamma \rightarrow \Gamma$ be the automorphism of $\Gamma$ which reduces to multiplication by $s^m$ on the $m$th homogeneous component of $\Gamma$.

Recall the maps $\pi_n \: C(\Omega_+) \rightarrow \text{Fun}(\mathbb{S}_n)$ defined in §8.2.

**Lemma 8.9.** Let $g \in \Gamma$ and $f = g^o \in \mathcal{F}$. For all $n \in \mathbb{Z}_{>0}$ we have

$$\pi_n f = (n^{-G}g)_n.$$

**Proof.** Fix $\lambda \in \mathbb{S}_n$. Consider the homomorphism $\Gamma \rightarrow \mathbb{R}$ defined on the generators as follows:

$$p_{2m-1} \mapsto \frac{1}{n^{2m-1}} \sum_{i=1}^{\ell(\lambda)} \lambda_i^{2m-1}, \quad m = 1, 2, 3, \ldots.$$

On one hand, this homomorphism is a composition of the automorphism $n^{-G}$ of $\Gamma$ and the map $\Gamma \rightarrow \mathbb{R}$, $\varphi \mapsto \varphi(n)$, hence $g \in \Gamma$ maps to $(n^{-G}g)_n(\lambda)$. On the other hand, since $p_i$ maps to 1, this homomorphism can be viewed as a composition of the canonical map $\Gamma \rightarrow \Gamma^o$ and the map $\Gamma^o \rightarrow \mathbb{R}$, $\psi \mapsto \psi(\iota_n(\lambda))$ (here $\iota_n$ is defined in §8.2). Hence $g \in \Gamma$ maps to $f(\iota_n(\lambda))$. This concludes the proof. \hfill $\Box$

**Proof of Proposition 8.4.** Fix arbitrary $f \in \mathcal{F}$. Let $g \in \Gamma$ be such that $f = g^o$. Theorem 7.4 and Lemma 8.9 imply

$$n^2(T_n - 1)\pi_n(f) = n^2(T_n - 1)(n^{-G}g)_n = \frac{n^2}{(n + \alpha/2)(n + 1)}(\tilde{B}n^{-G}g)_n, \quad (48)$$

where $\tilde{B}$ is a zero degree operator in the algebra $\Gamma$ with the top degree homogeneous part $B \: \Gamma \rightarrow \Gamma$ given by (41).

Because $B - \tilde{B}$ has degree $-1$, we can replace $\tilde{B}$ by $B$ in (48), this will affect only negligible terms.\footnote{One can argue as follows. Without loss of generality, assume that $g$ is homogeneous of degree $m \in \mathbb{Z}_{>0}$. Thus, $n^{-G}g = n^{-m}g$. Moreover, $\deg(B - \tilde{B})g \leq m - 1$ and hence by Lemma 8.8 $((B - \tilde{B})n^{-G}g)_n = \frac{1}{n}((B - \tilde{B})g)_n \rightarrow 0$, $n \rightarrow \infty$.}

We can also remove the factor $\frac{n^2}{(n + \alpha/2)(n + 1)}$. Thus, we have

$$n^2(T_n - 1)\pi_n(f) - (Bn^{-G}g)_n \rightarrow 0, \quad n \rightarrow \infty.$$ 

The operator $B \: \Gamma \rightarrow \Gamma$ is homogeneous, therefore, $Bn^{-G} = n^{-G}B$, and by Lemma 8.9 we have

$$(Bn^{-G}g)_n = (n^{-G}Bg)_n = \pi_n((Bg)^o)$$

Recall that the operator $B \: \Gamma \rightarrow \Gamma$ commutes with the multiplication by $p_1$ (Corollary 7.2), therefore, it induces an operator $A \: \mathcal{F} \rightarrow \mathcal{F}$, $f \mapsto (Bg)^o$, where $g \in \Gamma$ is such that $f = g^o$. Clearly, $Af$ does not depend on the choice of $g$. 

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Since $\mathcal{F} \cong \mathbb{R}[p_3, p_5, \ldots]$, we get (46) from (41) by replacing $p_1$ by 1 and each $p_{2m+1}$ by $q_{2m}$, $m \in \mathbb{Z}_{>0}$.

It remains to prove (47). Fix $\mu \in \mathbb{S}$. Multiply (18) by $n-|\mu| n^2$:

$$n^{-|\mu|} n^2 (T_n - 1)(Q_\mu)_n = \frac{n^2}{(n+\alpha/2)(n+1)} \left[ - |\mu|(|\mu| + \alpha/2 - 1) n^{-|\mu|}(Q_\mu)_n + \frac{n - |\mu| + 1}{n} \sum_{y \in Y(\mu)} (y(y+1) + \alpha) n^{-|\mu|+1}(Q_\mu - \square(y))_n \right].$$

Since $\deg(Q_\mu - Q_\lambda) \leq |\lambda| - 1$ (see §4.1), each function of the form $(Q_\mu)_n$ in the RHS can be replaced by $(Q_\lambda)_n$, this affects only negligible terms. We can also remove fractions containing $n$. Thus,

$$(n^{-|\mu|} n^2 (T_n - 1)(Q_\mu)_n + |\mu|(|\mu| + \alpha/2 - 1) n^{-|\mu|}(Q_\mu)_n - \sum_{y \in Y(\mu)} (y(y+1) + \alpha) n^{-|\mu|+1}(Q_\mu - \square(y))_n) \to 0, \quad n \to \infty$$

in the sense of Definition 8.1. To conclude the proof of Proposition 8.7 observe that $n^{-|\lambda|}(Q_\lambda)_n = (n^{-\lambda}Q_\lambda)_n = \pi_n(Q_\lambda)$.

### 8.7 Convergence of semigroups and the existence of the process

To apply Theorem 8.3 to our situation and finally get the existence of the process $X_\alpha(t)$ in $\Omega_+$ it remains to prove the following:

**Lemma 8.10.** (1) The operator $A: \mathcal{F} \to \mathcal{F}$ from Proposition 8.7 is dissipative.

(2) For all $s > 0$, the range of $sI - A$ is dense in $C(\Omega_+)$.  

**Proof.** (cf. [3, Proof of Proposition 1.4]) (1) Consider the filtration of the algebra $\mathcal{F} = \Gamma/(p_1 - 1)\Gamma$ inherited from the natural filtration (20) of $\Gamma$:

$$\mathcal{F} = \bigcup_{m=0}^\infty \mathcal{F}^m, \quad \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset \mathcal{F}.$$ 

It is clear that for fixed $m$ the operator $\pi_n: C(\Omega_+) \to \text{Fun}(\mathbb{S}_n)$ is injective on $\mathcal{F}^m$ for all $n$ large enough (this is true because each $\mathcal{F}^m$ is finite-dimensional and the spaces $\nu_n(\mathbb{S}_n)$ approximate the space $\Omega_+$, see §8.2). Thus, we can identify $\mathcal{F}^m$ with $\pi_n(\mathcal{F}^m)$ for such $n$.

It follows from (18) that the operator $T_n$ does not increase the degree of functions. Therefore, we can think of $T_n$ (and, clearly, of $n^2(T_n - 1)$) as an operator in $\mathcal{F}^m$. The convergence $n^2(T_n - 1) \to A$ established in Proposition 8.7 implies that $n^2(T_n - 1)$ converges to $A$ in every finite-dimensional space $\mathcal{F}^m$, $m \in \mathbb{Z}_{\geq 0}$.
Fix $m \in \mathbb{Z}_{\geq 0}$. For all $n$ large enough the operator $n^2(T_n - 1)$ (viewed as an operator in $\mathcal{F}^m$) is dissipative with respect to the norm of $\text{Fun}(S_n)$ because the operator $T_n$ is a transition operator of a Markov chain. Since the norms of $\text{Fun}(S_n)$ converge to the norm of $C(\Omega_+)$ (in the sense of (42)), we conclude that $A$ is dissipative.

(2) For every $m$ and $s > 0$ the operator $s \mathbf{1} - A$ maps $\mathcal{F}^m$ onto itself (this fact can be derived either from (47) or from the above proof of the claim (1) of the present Lemma). Thus, $(s \mathbf{1} - A) \mathcal{F} = \mathcal{F}$ and therefore $s \mathbf{1} - A$ has a dense range.

Now, from Theorem 8.3 it follows that the operator $A$ (given by Proposition 8.7) is closable in $C(\Omega_+)$ and its closure generates a strongly continuous contraction semigroup $\{T(t)\}_{t \geq 0}$.

We also have the convergence of semigroups $\{1, T_n, T_{n+1}, \ldots\}$ to $\{T(t)\}$ (43). Hence the semigroup $\{T(t)\}$ preserves the cone of nonnegative functions and the constant function 1 because each $T_n$ possesses this property.

From [4, Chapter 4, Theorem 2.7] follows that the semigroup $\{T(t)\}$ gives rise to a strong Markov process $X_\alpha(t)$ in $\Omega_+$. This process has càdlàg sample paths and can start from any point and any probability distribution on $\Omega_+$.

The operator $A$ is called the pre-generator of the process $X_\alpha(t)$, $t \geq 0$.

8.8 Some properties of the process $X_\alpha(t)$

Here we state some properties of the process $X_\alpha(t)$. They follow from the construction of $X_\alpha(t)$ and from the formulas (46) and (47) for its pre-generator. We do not give the proofs because they are similar to those from the paper [3].

Remark 8.11. The formula (47) for the action of the pre-generator $A$ on functions $Q^\mu \in \mathcal{F} \subset C(\Omega_+)$, $\mu \in S$, is not formally necessary for the convergence of the up/down Markov chains from Definition 2.7 to a continuous time Markov process in $\Omega_+$, as well as for the properties of the limit diffusion $X_\alpha(t)$ that are listed in this subsection. Indeed, from (46) one can easily obtain that

$$Af = -m(m - 1 + \alpha/2)f + g,$$

where $g \in \mathcal{F}^{m-1}$ (49)

for all $f \in \mathcal{F}^m$, $m \in \mathbb{Z}_{\geq 0}$. In other words, the action of $A$ on $\mathcal{F}^m$ up to lower degree terms (that is, terms from $\mathcal{F}^{m-1}$) is the multiplication by $-m(m - 1 + \alpha/2)$. It suffices to check (49) for $f = p^\sigma_{q_1} \ldots p^\sigma_{q_l}$, where $p^\sigma$ s are the Newton power sums and $\sigma = (\sigma_1, \ldots, \sigma_l)$ is an odd partition without parts equal to one. Indeed, for each $m \in \mathbb{Z}_{\geq 0}$ the functions of this form with $|\sigma| \leq m$ constitute a basis for $\mathcal{F}^m$. For such $f$ the relation (49) can be easily checked directly using (46) (note that $p^\sigma_{2i+1} = q_{2i}, i \geq 1$).

However, (47) allows to argue in a more straightforward way at some points. It can be also interesting to compare this formula with similar ones in other models (namely, [3, (5.1)] and [21, (14)]).
Continuity of sample paths. The process $X_\alpha(t)$ has continuous sample paths.

This is proved exactly as in [3, Coroll. 6.4 and Thm. 7.1], the proof uses the expression (46) for $A$.

The invariant symmetrizing measure. The multiplicative boundary measure $P^{(\alpha)}$ defined in Theorem 8.5 is an invariant measure for $X_\alpha(t)$. The process is reversible with respect to $P^{(\alpha)}$.

This follows from the facts that

• For all $n \in \mathbb{Z}_0$ the measure $M_n^\alpha$ is an invariant symmetrizing distribution for the Markov chain $T_n$ (see § 2);

• The measures $M_n^\alpha$ approximate the measure $P^{(\alpha)}$ in the sense of Theorem 8.5;

• The chains $T_n$ approximate the process $X_\alpha(t)$ (see § 8.4).

See also [3, Prop. 1.6, 1.7, Thm. 7.3 (2)].

Convergence of finite-dimensional distributions (Cf. [3, Prop. 1.8]). Let $X_\alpha(t)$ and all the chains $T_n$ are viewed in equilibrium (that is, starting from the invariant distribution). Then the finite-dimensional distributions for the $n$th Markov chain $T_n$ correspond to a small time interval $\Delta t \sim 1/n^2$.

The spectrum of the Markov generator in $L^2(\Omega_+, P^{(\alpha)})$. The space $\mathcal{F}$ viewed as the subspace of the Hilbert space $L^2(\Omega_+, P^{(\alpha)})$ is decomposed into the orthogonal direct sum of eigenspaces of the operator $A: \mathcal{F} \rightarrow \mathcal{F}$. The eigenvalues of $A$ are

$$\{0\} \cup \left\{-m \left(m - 1 + \frac{\alpha}{2}\right) : m = 2, 3, \ldots \right\}. \quad (50)$$

The eigenvalue 0 is simple, and the multiplicity of each $m$th eigenvalue is equal to the number of odd partitions of $m$ without parts equal to one, that is, to the number of solutions of the equation

$$3n_3 + 5n_5 + 7n_7 + \ldots = m$$

in nonnegative integers.

The reversibility property of the process $X_\alpha(t)$ implies that the operator $A$ is symmetric with respect to the inner product inherited from $L^2(\Omega_+, P^{(\alpha)})$. Moreover, $A$ preserves the filtration of $\mathcal{F}$ (defined in § 8.7). Indeed, this follows from the expression (47) for the pre-generator.

The fact that the eigenvalues of $A$ are described by (50) follows from (47). Indeed, $Q_\mu^\alpha \in \mathcal{F}^{[\mu]}$ for all $\mu \in \mathcal{S}$, and (47) is rewritten as $AQ_\mu^\alpha = -|\mu|(|\mu| - 1 + \alpha/2)Q_\mu^\alpha + g$, where $g \in \mathcal{F}^{[|\mu| - 1]}$. Thus, the eigenvalue 0 is simple, and the multiplicity of each $m$th eigenvalue is equal to the number of solutions of the equation $-m(m - 1 + \alpha/2) = 3n_3 + 5n_5 + 7n_7 + \ldots = m$.

Since $\mathcal{F} \cong \mathbb{R}[p_3, p_5, p_7, \ldots]$, finite products of the form $p_3^{\epsilon_1}p_5^{\epsilon_2}p_7^{\epsilon_3} \ldots$ constitute a linear basis for $\mathcal{F}$. This basis is compatible with the filtration $\{F^m\}$. Hence (dim $F^m$ – dim $F^{m-1}$) is equal to the number of basis vectors of degree

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The uniqueness of the invariant measure (Cf. [3] Thm. 7.3 (1)). The measure \( P^{(\alpha)} \) is a unique invariant measure for the process \( X_{\alpha}(t) \).

Ergodicity. The process \( X_{\alpha}(t) \) is ergodic with respect to the measure \( P^{(\alpha)} \).

This follows from the existence of a spectral gap of the process’ generator, see the eigenstructure above. See also [3] Thm. 7.3 (3)].

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