DK and BK-like spectra from Laplace sum rule at NLO

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Abstract

Encouraged by the agreement, with the recent LHCb data on the $D^-K^+$ invariant mass from $B \to D^+D^-K^+$ decay, of our results for the masses of the $0^+$ and $1^-$ open charm $(c\bar{d})(u$s) tetraquarks and molecules states from QCD spectral sum rules within stability criteria, which we review here, we extend our analysis to the $b$-quark channel. We find, in the $0^+$ case the lowest mass $M_{BK} = 5195(15)$ MeV with $f_{BK} = 8.3(2.4)$ keV and three (almost) degenerate states having respectively the masses $M_{S^0} = 5702(60)$ MeV, $M_{AA} = 5661(75)$ MeV and $M_{B-K^*} = 5720(71)$ MeV and couplings $f_{BS} = 22.2(2.3)$ keV, $f_{BA} = 30.1(3.1)$ keV and $f_{B-K^*} = 26.5(2.8)$ keV, from which we can associate a scalar tetramole with $M_{T_{sb}} = 5694(69)$ MeV and $f_{T_{sb}} = 26.5(2.7)$ keV. In the spin 1 case, we find four (almost) degenerate states associated with a tetramole having $M_{T_{sl}} = 5700(81)$ MeV and $f_{T_{sl}} = 16.2(2.6)$ keV. For the first radial excitation of the $BK$ molecule, we have $M_{(BK)_{0^+}} = 6265(146)$ MeV and $f_{(BK)_{0^+}} = 22.8(3.2)$ keV. For the remaining states, we associate a scalar and vector tetramoles having respectively $M_{(T_{sb})_{1^+}} = 7439(314)$ MeV, $f_{(T_{sb})_{1^+}} = 74.7(8.4)$ keV and $M_{(T_{sl})_{1^+}} = 7544(345)$ MeV, $f_{(T_{sl})_{1^+}} = 33.0(6.7)$ keV.

Keywords: QCD spectral Sum Rules, Perturbative and Non-perturbative QCD, Exotic hadrons, Masses and Decay constants.

1. Introduction

We have systematically calculated the masses and couplings of some possible configurations of the $0^+$ and $1^-$ molecules and tetraquarks open charm $(c\bar{d})(u$s) tetraquarks and molecules states from the inverse Laplace Transform (LSR) of QCD spectral sum rules (QSSR) \cite{25,26,27} within stability criteria in Ref.\cite{27} which we have used to explain the recent LHCb data on the $D^+K^-$ invariant mass \cite{1,2}.

We concluded that the 2400 MeV bump around $DK$ threshold can be due to $DK(2400)$ lowest mass molecule. The $0^+$ $X_0(2866)$ and $X_0(3150)$ (if it is a $0^+$ state) can e.g result from a mixing of the Tetramole with the $1^+$ radial excitation ($DK$) of the molecule state ($DK$) with a tiny mixing angle $\theta_0 = (5.2\pm1.9)^\circ$. The $1^-$ $X_1(2904)$ and $X_1(3350)$ (if it is a $1^-$ state) can result from a mixing of the Tetramole with its $1^+$ radial excitation with a tiny mixing angle $\theta_1 = (9.1\pm0.6)^\circ$. In this work, based on these coherent results, we extend our work to the $b$-channel, to estimate, from LSR, the masses and couplings of some $BK$-like spectra at NLO in the PT-series.

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2. The Laplace sum rule (LSR)

We shall work with the Finite Energy version of the QCD Inverse Laplace sum rules and their ratios:

\[ \mathcal{L}_n'(\tau, \mu) = \int_{M_b + m_\tau}^{\mu} dt \ e^{-\tau t} \frac{1}{\pi} \mathrm{Im} \Pi_{M,T}(t, \mu) \]

\[ \mathcal{R}_n'(\tau) = \frac{\mathcal{L}_{n+1}'(\tau)}{\mathcal{L}_n'(\tau)} \]

where \( M_b \) and \( m_\tau \) are the on-shell / pole bottom and running strange quark masses, \( \tau \) is the LSR variable, \( n = 0, 1 \) is the degree of moments, \( t_c \) is the threshold of the “QCD continuum” which parametrizes, from the discontinuity of the Feynman diagrams, the spectral function evaluated by the calculation of the correlator defined as:

\[ \Pi_{M,T}(t, \mu(q^2)) = \int d^4 x \ e^{-i q \cdot x} \langle 0| \mathcal{O}_{M,T}(x)\mathcal{O}_{M,T}^\dagger(0)|0 \rangle \]

where \( \mathcal{O}_{M,T}(x) \) are the interpolating currents for the tetraquarks \( T \) and molecules \( M \) states. The superscript \( J \) refers to the spin of the particles.

- \( \tau \) is an arbitrary parameter, then the mass and coupling have to be independent on him. We must get a stability w.r.t. \( \tau \).
- We find the beginning of the stability for different values of \( t_c \).
- The value of the mass (or coupling) was taken at the minimum or the inflexion point.

3. Interpolating currents

The interpolating currents for the molecules and for tetraquark states are given in Table 1.

| Tetraquarks | Molecules |
|-------------|-----------|
| 0^+ \( O_{SS} = \epsilon_{ijk} \epsilon_{mnk} (u_i^T C \gamma_5 d_j) (b_m \gamma_\mu C \bar{s}_n) \) | 0^+ \( O_{B-K} = (\bar{b} d) (\bar{s} u) \) |
| 1^+ \( O_{SA} = \epsilon_{ijk} \epsilon_{mnk} (u_i^T C \gamma_\mu d_j) (b_m \gamma_\mu C \bar{s}_n) \) | 1^+ \( O_{B-K} = (\bar{b} d) (\bar{s} u) \) |
| 0^+ \( O_{SV} = \epsilon_{ijk} \epsilon_{mnk} (u_i^T C \gamma_5 d_j) (b_m \gamma_\mu \gamma_5 C \bar{s}_n) \) | 1^+ \( O_{B-K} = (\bar{b} d) (\bar{s} u) \) |
| 1^- \( O_{PA} = \epsilon_{ijk} \epsilon_{mnk} (\bar{b}_m \gamma_\mu C \bar{s}_n) (u_i^T C \gamma_\mu d_j) \) | 1^- \( O_{B-K} = (\bar{b} d) (\bar{s} u) \) |

Table 1: Interpolating operators describing the scalar (0^+) and vector (1^-) molecules and tetraquark states used in this work

**Extracting the lowest ground state mass and coupling**

In [20], we have extracted the lowest ground state mass by using the minimal duality ansatz (MDA):

\[ \frac{1}{\pi} \mathrm{Im} \Pi_{M,T} = f_{M,T}^2 M_{M,T}^2 \delta(t - M_{M,T}^2) + \Theta(t - t_c) \]

The decay constant \( f_M \) (analogue to \( f_a = 132 \) MeV) for the molecule state is defined as:

\[ \langle 0| \mathcal{O}_{B_K}^\dagger | B_K \rangle = f_{B_K} M_{B_K}^2 \]

\[ \langle 0| \mathcal{O}_{B_K^*}^\dagger | B_K^* \rangle = e^\mu f_{B_K} M_{B_K}^2 \]

and analogously for the one \( f_T \) of the tetraquark state. Interpolating currents constructed from bilinear (pseudo)scalar currents are not renormalization group invariant such that the corresponding decay constants possess anomalous dimension:

\[ f_{B_K}(\mu) = f_{B_K}(\beta_{it}(a_t)) \left( 1 - k_f a_t \right) \]

\[ f_{B_K}(\mu) = f_{B_K}(\beta_{it}(a_t)) \left( 1 - k_f a_t / 2 \right) \]

where: \( f_M \) is the renormalization group invariant coupling and \( \beta_i = (1/21)(11 - 2t_f^2/3) \) is the first coefficient of the QCD \( \beta \)-function for \( n_f \) flavours. \( a_t \equiv (a_t/\pi) \) is the QCD coupling and \( k_f = 2.028(2.352) \) for \( n_f = 4(5) \) flavours. Within a such parametrization, one obtains:

\[ \mathcal{R}_n' \equiv R \approx M_{M,T}^2 \]

where \( M_{M,T} \) is the lowest ground state molecule or tetraquark mass.

**Higher orders PT corrections to the spectral functions**

We extract the NLO PT corrections by considering that the molecule/tetraquark two-point spectral function is the convolution of the two ones built from two quark bilinear currents (factorization). In this way, we obtain [21][22]:

\[ \frac{1}{\pi} \mathrm{Im} \Pi_{M,T}(t) = \theta \left( t - (M_b + m_\tau + m_d) \right) \left( \frac{k}{4\pi} \right)^2 \]

\[ \times \int_{(M_b + m_d)^2}^{\infty} dt_1 \int_{m_\tau^2}^{\infty} dt_2 A^{1/2} K(t_1, t_2) \]

where \( k \) is an appropriate normalization factor, \( M_b \) is the on-shell/pole perturbative heavy quark mass.

\[ K^{SS,PP} \equiv \left( \frac{t_1}{t} + \frac{t_2}{t} - 1 \right)^2 \times \frac{1}{\pi} \mathrm{Im} \phi^{S,P}(t_1) \frac{1}{\pi} \mathrm{Im} \phi^{S,P}(t_2) \]

\[ K^{VV,AA} \equiv \left( \left( \frac{t_1}{t} + \frac{t_2}{t} - 1 \right)^2 + 8 \left( \frac{t_1 t_2}{t^2} \right) \right) \]

\[ \times \frac{1}{\pi} \mathrm{Im} \phi^{V,A}(t_1) \frac{1}{\pi} \mathrm{Im} \phi^{V,A}(t_2) \]
for spin zero scalar state and:
\[ K_{VS,AP}^{AP} = 2\lambda \times \frac{1}{\pi} \text{Im} \psi^{VA}(t_1) \frac{1}{\pi} \text{Im} \psi^{V,AP}(t_2) \]  
for spin one vector state, with the phase space factor:
\[ A = \left( 1 - \frac{\sqrt{t_1} - \sqrt{t_2}}{t} \right)^2 \left( 1 - \frac{\sqrt{t_1} + \sqrt{t_2}}{t} \right)^2 \]  

For tetraquark, one interchanges s and d in the integrals of Eq.(7). We have taken \( m_0 = 0 \) for simplifying the expression but we shall also neglect \( m_0 \) in the numerical analysis.

- **QCD input parameters**

The PT QCD parameters which appear in this analysis are \( \alpha_s \), strange and bottom quark masses \( m_{s,b} \) (the light quark masses have been neglected). We also consider non-perturbative condensates which are the quark condensate \( \langle \bar{q}q \rangle \), the two-gluon condensate \( \langle g^2 G^2 \rangle \), the mixed condensate \( \langle \bar{q}Gq \rangle \), the three-gluon condensate \( \langle g^3 G^3 \rangle \), and the four-quark condensate \( \rho \langle \bar{q}q \rangle^2 \), where \( \rho \approx (3 - 4) \) indicates the deviation from the four-quark vacuum saturation. Their values are given in Table 2.

Table 2: QCD input parameters estimated from QSSR (Moments, LSR and ratios of sum rules).

| Parameters | Values         | Ref. |
|------------|----------------|------|
| \( \alpha_s(M_Z) \) | \( 0.1181(16) \) | \[23\,\[24\] \] |
| \( \bar{m}_s \) | \( 0.114 \pm 0.006 \) GeV | \[12\,\[25\,\[26\] \] |
| \( \bar{m}_b(m_b) \) | \( 420(8) \) MeV | \[27\] |
| \( \bar{\rho}_s \) | \( 253 \pm 6 \) MeV | \[12\,\[25\,\[26\] \] |
| \( M_0^2 \) | \( 0.8 \pm 0.2 \) GeV^2 | \[12\,\[19\,\[20\,\[33\] \] |
| \( \langle \bar{q}Gq \rangle \) | \( 6.35 \pm 0.35 \times 10^{-2} \) GeV^4 | \[23\] |
| \( \rho \langle \bar{q}q \rangle^2 \) | \( (8.2 \pm 0.2) \) GeV^2 \( \times \langle \bar{q}Gq \rangle \) | \[34\,\[35\] \] |

The Renormalization Group Invariant parameters are defined as \[11\,\[12\] : \]
\[ \hat{m}_I(\tau) = \hat{m}_I(-\beta_1 a_s)^{-2/\beta_1}, \quad \langle \bar{q}q \rangle(\tau) = \hat{m}_I^{-1}(-\beta_1 a_s)^{2/\beta_1}, \quad \langle \bar{q}Gq \rangle(\tau) = \hat{m}_I^{3/2}(-\beta_1 a_s)^{1/\beta_1} \]  
(11)

\( \hat{m}_I \) is the spontaneous RGI light quark condensate \[40\].

The running bottom mass \( m_b \) is related to the on-shell (pole) mass \( M_b \) used to compute the two-point correlator from the NLO relation \[41\,\[45\] :
\[ M_b(\mu) = \hat{m}_I(\mu) \left[ 1 + \frac{4}{3} \alpha_s(\mu) + \log \left( \frac{\mu}{M_b} \right) \right] a_s(\mu) + O(a_s^2) \]  
(12)

4. **Molecules and tetraquarks**

We will study the mass and coupling of some BK-like states. As the analysis will be performed using the same techniques, we shall illustrate the case of the SS \( (0^-) \) tetraquark.

- **\( f_{SS} \) and \( M_{SS} \)**

We study the behaviour of the coupling and mass in term of the LSR variable \( \tau \) for different values of \( t_c \) at NLO as shown in Fig 1. We consider as result the one corresponding to the beginning of the \( \tau \)-stability for \( t_c = 42 \) GeV^2 and \( \tau = 0.22 \) GeV^2 until the one where \( \tau \)-stability is reached for \( t_c \approx 50 \) GeV^2 and \( \tau = 0.28 \) GeV^2.

- **\( \mu \)-stability**

In Fig 2 we show the \( \mu \)-dependence of the results for given \( t_c = 50 \) GeV^2 and \( \tau = 0.29 \) GeV^2. One finds a common stability for \( \mu = (4.55 \pm 0.25) \) GeV.

![Figure 1: \( f_{SS} \) and \( M_{SS} \) at NLO as function of \( \tau \) for different values of \( t_c \) for \( \mu = 4.55 \) GeV](image1)

![Figure 2: \( f_{SS} \) and \( M_{SS} \) at NLO as function of \( \mu \) for \( t_c = 50 \) GeV^2 in the region of \( \tau \) stability](image2)
5. Tetramoles

Our results indicate that the molecules and tetraquark states leading to the same final states are almost degenerated in masses. Therefore, according to conclusion in [3], we expect that the "physical state" is a combination of almost degenerated molecules and tetraquark states with the same quantum numbers $j^{PC}$ which we shall call: Tetramole ($T_{M_1}$).

- Tetramole $0^+$

Taking our results in Table 3, one can see that we have three (almost) degenerate states:

$$M_{SS} = 5702(60) \text{ MeV, } M_{AA} = 5661(75) \text{ MeV} \text{ and } M_{B-K'} = 5720(71) \text{ MeV}$$

and their couplings to the corresponding currents are almost the same:

$$f_{SS} = 22.2(2.3) \text{ keV, } f_{AA} = 30.1(3.1) \text{ keV and } f_{B-K'} = 26.5(2.8) \text{ keV}$$

We assume that the physical state, hereafter called Tetramole ($T_{M_1}$), is a superposition of these nearly degenerated hypothetical states having the same quantum numbers. Taking its mass and coupling as (quadratic) means of the previous numbers, we obtain:

$$M_{T_{M_1}} = 5694(69) \text{ MeV and } f_{T_{M_1}} = 26.5(2.7) \text{ keV}$$

- Tetramole $1^-$

In Table 3, in the case of spin 1, we have four degenerate states with the same masses around 5700 GeV, taking the (quadratic) means, we obtain:

$$M_{T_{M_1}} = 5700(81) \text{ MeV and } f_{T_{M_1}} = 16.2(2.6) \text{ keV}$$

6. The first radial excitation

According to [3], we also extend our analysis by using a "Two resonances" $+ \theta(t - t_c)$ "QCD continuum" parametrization of the spectral function, we illustrate again the case of $SS$ tetraquark. To enhance the contribution of the first radial excitation, we will also work with the ratio of moments $\mathcal{R}_1$ for getting the mass of $(SS)_1$.

- $\tau$ and $t_c$-stability

We show in Fig. 4 the $\tau$ and $t_c$-behaviours of the coupling from $L_{\mathcal{R}_1}$ and the ones of the mass from $\mathcal{R}_1$ using as input the values of the lowest ground state mass and coupling. One can notice that the coupling from $L_{\mathcal{R}_1}$ stabilizes for $\tau \approx (0.15 \sim 0.26) \text{ GeV}^2$ which is slightly lower than the value $\tau = 0.29 \text{ GeV}^2$ corresponding to the one-resonance parametrization. The values of $t_c$ are 66 to 74 GeV$^{-2}$ compared to 42 to 50 GeV$^{-2}$ for the one resonance case. The result is given in Table 4.

- Tetramole $0^-$

One can also notice from Tables 3 that the radial excitations other than the one of $BK$ with a mass of 6265(146) MeV are almost degenerated around 7.4 GeV from which one can extract the masses and couplings (geometric mean) of the spin 0 excluding $(BK)_1$.

Then, for the $0^-$ case, we have three degenerate states having the mass and coupling (quadratic means):

$$M_{T_{M_0}} = 7439(314) \text{ MeV and } f_{T_{M_0}} = 74.7(8.4) \text{ keV}$$
• Tetramole $1^-$

For the spin $1^-$ case, in Table [3], we have four degenerate states around 7.5 GeV. Taking the (quadratic) means, we have:

\[ M_{(T_{2S})_{1^{-}}} = 7544(345) \text{ MeV and } f_{(T_{2S})_{1^{-}}} = 33.0(6.7) \text{ keV} \]

7. Conclusions

• We have presented improved predictions of QSSR for the masses and couplings of the $0^+$ and $1^-$ $BK$-like molecule and four-quark states at NLO of PT series and including up to dimension six non-perturbative condensates.

• Our analysis has been done within stability criteria with respect to the LSR variable $\tau$, the QCD continuum threshold $t_c$ and the subtraction constant $\mu$ which have provided successful predictions in different hadronic channels. The optimal values of the masses and couplings have been extracted at the same value of these parameters where the stability appears as an extremum and/or inflection points.

• The results for all the 4-quarks and molecule currents discussed here are below their thresholds.

• NLO radiative corrections are essential for giving a meaning on the input value of the charm and bottom quark masses which plays an important role in the analysis. We consider our results as improvement and a completion of the results obtained to LO from QCD spectral sum rules.

• We find as lowest mass $M_{BK} = 5195(15) \text{ MeV with } f_{BK} = 8.3(2.4) \text{ keV. We also obtain, in the } 0^+$ case, three (almost) degenerate states, to which we can associate a scalar tetramole with $M_{T_{2S}} = 5694(69) \text{ MeV and } f_{T_{2S}} = 26.5(2.7) \text{ keV. In the spin } 1^-$ case, we find four degenerate states leading to a tetramole having a mass $M_{T_{2P}} = 5700(81) \text{ MeV and and a coupling } f_{T_{2P}} = 16.2(2.6) \text{ keV.}$

• In an analogous way, we predict the first radial excitations of these states which give: $M_{(BK)_{1}} = 6265(146) \text{ MeV and } f_{(BK)_{1}} = 22.8(3.2) \text{ keV. We associate for the other states a scalar and vector tetramoles having respectively } M_{(T_{2S})_{1}} = 7439(314) \text{ MeV, } f_{(T_{2S})_{1}} = 74.7(8.4) \text{ keV and } M_{(T_{2P})_{1}} = 7544(345) \text{ MeV, } f_{(T_{2P})_{1}} = 33.0(6.7) \text{ keV.}$

• Our approach doesn’t discern, within the errors, the molecules and tetraquarks states.

References

[1] [LHCb Collab.], R. Aaij et al., Phys.Rev.Lett. 125, 242001 (2020).
[2] [LHCb Collab.], R. Aaij et al., Phys.Rev. D102, 112003 (2020).
[3] R.M. Albuquerque, S. Narison, D. Rabetiarivony and G. Randriananatraika, Nucl. Phys. A 1007 (2021) 122113.
[4] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B177 (1981) 218; Nucl. Phys. B187 (1981) 285.
[5] C. Becchi et al., Z. Phys. C 8, 335 (1981).
[6] R.A. Bertlmann, Acta Phys. Austriaca 53, 305 (1981) and references therein.
[7] R.A. Bertlmann and H. Neufeld, Z. Phys. C 27 (1985) 437.
[8] S. Narison and E. de Rafael, Phys. Lett. B 522 (2001) 266.
[9] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.
[10] V.I. Zakharov, Int. J. Mod. Phys. A 14, 4865 (1999).
[11] S. Narison, QCD as a theory of hadrons, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. 17 (2002) 1; [hep-ph/0205006].
[12] S. Narison, QCD spectral sum rules, World Sci. Lect. Notes Phys. 26 (1989) 1.
[13] S. Narison, Phys. Rept. 84 (1982) 263; Acta Phys. Pol. B 26 (1995) 687.
[14] E. de Rafael, hep-ph/9702448.
[15] F. J. Yndurain, The Theory of Quark and Gluon Interactions, 3rd ed. (Springer, New York, 1999).
[16] P. Fascal and R. Tarrach, QCD: Renormalization for Practitioner (Springer, New York, 1985).
[17] L. J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep. 127, (1985).
[18] B. L. Ioffe, Prog. Part. Nucl. Phys. 56, 232 (2006).
[19] H. G. Dosch, Non-Perturbative Methods, edited by S. Narison (World Scientific, Singapore, 1985).
[20] R. Albuquerque, S. Narison, A. Rabenmananjara and D. Rabetiarivony, Int. J. Mod. Phys. A 31, 1650009 (2016).
[21] A. Pich and E. de Rafael, Phys. Lett. B158 (1985) 477.
[22] S. Narison and A. Pivovarov, Phys. Lett. B327 (1994) 341.
[23] S. Narison, Int. J. Mod. Phys. A 33 (10) (2018) 1850045; S. Narison, (addendum) and references therein, Int. J. Mod. Phys. A 33 (10) (2018) 1850045.
[24] S. Narison, arXiv:1812.09360 [hep-ph], 2018.
[25] S. Narison, Int. J. Mod. Phys. A 30 (20) (2015) 1550116.
[26] S. Narison, Phys. Lett. B 738 (2014) 346.
[27] S. Narison, Phys. Lett. B 784, 261 (2018) ; Phys. Lett. B 802, 135221 (2020).
[28] B.L. Ioffe, Nucl. Phys. B 188 (1981) 317.
[29] Y. Chung et al., Z. Phys. C25 (1984) 151.
[30] H.G. Dosch, M. Jamin, S. Narison, Phys. Lett. B 220 (1989) 251.
[31] B.L. Ioffe, Nucl. Phys. B191 (1981) 591.
[32] A.A. Ovchinnikov, A.A. Pivovarov, Yad. Fiz. 48 (1988) 1135.
[33] S. Narison, Phys. Lett. B605 (2005) 319.
[34] S. Narison, Phys. Lett. B693 (2010) 559; Erratum ibid 705 (2011) 544.
[35] S. Narison, Phys. Lett. B706 (2011) 412.
[36] S. Narison, Phys. Lett. B707 (2012) 259.
[37] R.A. Bertlmann, G. Launer, E. de Rafael, Nucl. Phys. B 250 (1985) 61.
[38] S. Narison, Phys. Lett. B673 (2009) 30.
[39] G. Launer, S. Narison and R. Tarrach, Z. Phys. C26 (1984) 433.
[40] G. Floratos, S. Narison and E. de Rafael, Nucl. Phys. B155 (1979) 155.
[41] R. Tarrach, Nucl. Phys. B 183 (1981) 384.
[42] R. Coquereaux, Ann. Phys. 125 (1980) 401.
[43] P. Binetruy, T. Sucker, Nucl. Phys. B 178 (1981) 293.
[44] S. Narison, Phys. Lett. B 197 (1987) 405.
[45] S. Narison, Phys. Lett. B 216 (1989) 191.
Table 3: LSR predictions, at NLO, for the decay constants and masses of the ground state ($f_0, M_0$), for the molecules, tetraquark and the predicted tetramole states. The symbol "∗" indicates that BK did not contribute to the tetramole results.

| Observables | $M_0$ (MeV) | $f_0$ (keV) | $M_{T, M_0, 1}$ (MeV) | $M_{T, M_0, 1}$ (keV) |
|-------------|-------------|-------------|-----------------------|-----------------------|
| **0⁺ States** |             |             |                       |                       |
| **Molecule** |             |             |                       |                       |
| $B K$       | 5195(15)    | 08.3(2.4)   | *                     | *                     |
| $B' K^*$    | 5720(71)    | 26.5(2.8)   |                       |                       |
| $B K$       |             |             | 5694(69)              | 26.5(2.7)             |
| **Tetraquark** |             |             |                       |                       |
| $SS$        | 5702(60)    | 22.2(2.3)   |                       |                       |
| $AA$        | 5661(75)    | 30.1(3.1)   |                       |                       |
| **1⁻ States** |             |             |                       |                       |
| **Molecule** |             |             |                       |                       |
| $B K$       | 5714(66)    | 14.0(1.9)   |                       |                       |
| $B_0' K^*$  | 5676(92)    | 13.6(1.6)   |                       |                       |
| $B K$       |             |             | 5700(81)              | 16.2(2.6)             |
| **Tetraquark** |             |             |                       |                       |
| $PA$        | 5700(84)    | 19.1(3.9)   |                       |                       |
| $SV$        | 5711(79)    | 17.4(2.4)   |                       |                       |

Table 4: LSR predictions, at NLO, for the decay constants and masses of the first radial excitation ($f_1, M_1$), for the molecules, tetraquark and the predicted tetramole states. The symbol "∗" indicates that ($BK)_1$ did not contribute to the tetramole results.

| Observables | $M_1$ (MeV) | $f_1$ (keV) | $M_{T, M_1, 1}$ (MeV) | $f_{T, M_1, 1}$ (keV) |
|-------------|-------------|-------------|-----------------------|-----------------------|
| **0⁺ States** |             |             |                       |                       |
| **Molecule** |             |             |                       |                       |
| $(BK)_1$    | 6265(146)   | 22.8(3.2)   | *                     | *                     |
| $(B' K^*)_1$| 7494(232)   | 89.2(8.4)   |                       |                       |
| $(BK)_1$    |             |             | 7439(314)             | 74.7(8.4)             |
| **Tetraquark** |             |             |                       |                       |
| $(SS)_1$    | 7408(429)   | 51.5(6.8)   |                       |                       |
| $(AA)_1$    | 7415(240)   | 78.3(9.7)   |                       |                       |
| **1⁻ States** |             |             |                       |                       |
| **Molecule** |             |             |                       |                       |
| $(B K)_1$   | 7578(311)   | 27.3(6.7)   |                       |                       |
| $(B_0' K^*)_1$| 7459(256)  | 37.0(5.1)   |                       |                       |
| $(B K)_1$   |             |             | 7544(345)             | 33.0(6.7)             |
| **Tetraquark** |             |             |                       |                       |
| $(PA)_1$    | 7568(405)   | 34.4(7.8)   |                       |                       |
| $(SV)_1$    | 7569(388)   | 32.4(7.1)   |                       |                       |