A semi-parametric model for target localization in distributed systems

Rohit K. Patra
University of Florida
Moulinath Banerjee
University of Michigan
George Michailidis
University of Florida

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Abstract

Distributed systems serve as a key technological infrastructure for monitoring diverse systems across space and time. Examples of their widespread applications include: precision agriculture, surveillance, ecosystem and physical infrastructure monitoring, animal behavior and tracking, disaster response and recovery to name a few. Such systems comprise of a large number of sensor devices at fixed locations, wherein each individual sensor obtains measurements that are subsequently fused and processed at a central processing node. A key problem for such systems is to detect targets and identify their locations, for which a large body of literature has been developed focusing primarily on employing parametric models for signal attenuation from target to device. In this paper, we adopt a nonparametric approach that only assumes that the signal is nonincreasing as function of the distance between the sensor and the target. We propose a simple tuning parameter free estimator for the target location, namely, the simple score estimator (SSCE). We show that the SSCE is \(\sqrt{n}\) consistent and has a Gaussian limit distribution which can be used to construct asymptotic confidence regions for the location of the target. We study the performance of the SSCE through extensive simulations, and finally demonstrate an application to target detection in a video surveillance data set.

1 Introduction

Target detection and localization represents a canonical problem in distributed systems, wherein information is obtained from sensing devices and then appropriately fused to identify the presence and location of target(s). Sensing technologies have evolved over time from phased arrays in radar systems (see Niu et al. (2012) and references therein), to wireless sensor networks involving many inexpensive sensors (see survey paper Akyildiz et al. (2002)), to highly sophisticated surveillance/monitoring systems integrating video and other sensor data Joshi and Thakore (2012). Examples of this canonical problem based on such diverse technologies abound and include precision agriculture Cardell-Oliver et al. (2005), surveillance Estrin (2007), animal behavior Mainwaring et al. (2002), drone tracking, emergent disaster response and recovery Blatt and Hero (2006), fire hazards Son et al. (2006), structural integrity of critical infrastructure Chen and Jahanshahi (2017).

Such distributed systems comprise a large number of sensors (acoustic, image/video, chemical, environmental) deployed at various (fixed or random) locations, wherein each individual sensor acquires

\*Corresponding author. E-mail: rohitpatra@ufl.edu
signals from the surrounding area at fixed time intervals. The task of a central location is to integrate or fuse the data recorded by the sensors to locate or track an object or other quantity of interest, such as a crack in a bridge, or a chemical spill in the environment. In what follows, we formally describe the problem for a single fixed time.

Consider $n$ identical sensors deployed at locations $\{X_i\}_{i=1}^n \in \mathbb{R}^d$ over a $d$-dimensional region $\chi \in \mathbb{R}^d$, where $d$ is 1, 2, or 3. Our object of interest is the location of a target that emits a signal; e.g., infrared, acoustic, temperature, etc. Let $\theta_0 := (\theta_{0,1}, \ldots, \theta_{0,d}) \in \Theta$ denote the position of the target, where $\Theta \subset \mathbb{R}^d$ is called the ’monitoring region’. We assume that the energy or intensity of the signal attenuates with distance (from the target) according to a nonincreasing function $\eta_0 : \mathbb{R}^d \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive real line, i.e., the true energy/intensity of the signal at sensor located at $X$ is $\eta_0(|\theta_0 - X|^2)$, where $|\cdot|$ denotes the Euclidean norm and $\eta_0(0)$ is the energy of the signal at the target. However, since the sensor measurements are error prone, the observed energy at a sensor at $X$ is

$$Y = \eta_0(|\theta_0 - X|^2) + \epsilon,$$  \hspace{1cm} (1)

where $\epsilon$ is the unobserved measurement error. We will assume that $\mathbb{E}(\epsilon|X) = 0$ and $\mathbb{E}(\epsilon^2|X) < \infty$ for almost every $X$. The goal here is to estimate the unknown function $\eta_0$ and $\theta_0 \in \Theta$ based on an i.i.d. sample $(X_i, Y_i)$ under minimal assumptions on $\eta_0$.

Given the importance and wide applicability of this canonical problem, a large body of work has emerged; for a comprehensive review see the book Varshney (2012). The majority of existing work imposes a parametric functional form on the signal attenuation function $\eta$ leveraging information about the nature of the signal obtained by the sensors. This is a justified approach when dealing with thermal or acoustic signals that exhibit exponential and polynomial rates of decay, respectively see e.g., Blatt and Hero (2006); Clouqueur et al. (2001); Li et al. (2002); Sheng and Hu (2005). However, in many real-life scenarios, such parametric assumptions fall short. For example, the above assumptions are hard to justify for image/video acquiring sensors, or fail to hold in non-ideal environments, like the presence of dense vegetation or at high altitude, where the signal attenuation deviates from such nice parametric forms Watanabe and Yamada (1996). To address such issues, some approaches quantize the signal (record a value 0 if the signal is below a certain threshold and 1 otherwise) – see Katenka et al. (2007, 2008) and references therein. However, this strategy can lead to significant loss of information thereby negatively impacting target detection and localization capabilities of distributed systems. To that end, we propose to address the problem of estimating $\theta_0$ and $\eta_0$ in (1) without any parametric assumptions on $\eta_0$.

Note that the nature of the problem under consideration corresponds to semiparametric estimation with bundled parameters, wherein the parametric and nonparametric components are intertwined (see Huang and Wellner (1997)). Such problems have been studied in the literature based on estimators that require tuning parameters, see e.g., Powell et al. (1989); Li and Duan (1989); Ichimura (1993); Härdle et al. (1993); Hristache et al. (2001); Delecroix et al. (2006); Wang and Yang (2009); Cui et al. (2011) and references therein. However, observe that in our setting, the nonparametric component $\eta_0$ is governed by a natural shape constraint: monotonicity. It is by now very well known that the use of shape constraints like monotonicity, convexity, log-concavity, etc., lead to elegant tuning parameter free estimates in a wide repertoire of nonparametric problems involving function estimation that, at least in one dimension (i.e., shape constrained functions of one variable), produce minimax optimal rates under minimal smoothness assumptions, see e.g., Groeneboom et al. (2001); Zhang (2002); Guntuboyina and Sen (2015); Groeneboom and Jongbloed (2014); Kim and Samworth (2016); Balabdaoui et al. (2009); Han and Wellner (2019); Kuchibhotla and Patra (2019); Gao et al. (2020) and references therein. Bypassing the tuning parameter selection step provides estimates that are truly data-driven: in fact, shape-constrained procedures have an adaptive data-driven bandwidth choice built into the algorithms for their computation, and therefore extraneous stipulations of bandwidth via cross-validation or other techniques are not necessary.
Hence, we adopt the shape-constrained approach to leverage its advantages for the problem at hand. This is rendered feasible in our version of the bundled parameters problem by the recent developments in Groeneboom and Hendrickx (2018); Balabdaoui et al. (2019a,b); Kuchibhotla and Patra (2019) in the related single index model (where $E(Y|X) = \eta_0(\theta_0, X)$) with shape constrained link function $\eta$, which demonstrated how the parameter of interest $\theta_0$ can be estimated at the optimal $\sqrt{n}$ rate while using a tuning parameter free approach for the estimation of the nuisance parameter $\eta$.

The main difficulty in studying the asymptotics for the estimator of $\theta_0$ in the shape constrained framework of (1) comes from the fact that the monotonically constrained estimator of $\eta_0$ is piecewise constant and thus lies on the “boundary” of the space of monotone functions. When the parametric ($\theta_0$) and non-parametric ($\eta_0$) components are not bundled (e.g., Cox proportional hazards model or partial linear regression model), the discontinuity (and boundary problem) of the estimator of $\eta_0$ can be overcome using traditional techniques because the asymptotics of the estimators for $\theta_0$ do not involve $\eta_0$, see e.g., Van der Vaart (2002); Huang (1996, 2002). This is, however, not true when $\theta_0$ and $\eta_0$ are intertwined, see e.g., Kuchibhotla et al. (2017). To overcome the above difficulties, we adapt the powerful and elegant techniques developed in Groeneboom and Hendrickx (2018) and Balabdaoui et al. (2019b) to develop and study tuning parameter free estimators for (1). As opposed to the single index model studied in the above works, the index in (1) $|(\theta_0 - X)|$ is not a linear function of the parameter. This creates a number of new technical challenges that require careful handling; see e.g., Section G.1. Our work shows that the tools developed in Groeneboom and Hendrickx (2018) can be used for general bundled problems (where the index is not linear in $\theta$ or $X$), provided the index is only locally linear, and therefore expands the scope of these techniques to a broader set of problems/models. Finally, Balabdaoui et al. (2019b) assumes that the errors have all moments, while we relax this assumption significantly and establish $\sqrt{n}$-consistency for the estimator of $\theta_0$ for heavier tailed errors; we require errors to have only finite sixth moment (conditional on $X$). This relaxation is important in many applications, due to the nature of the operating environment Liu et al. (2009) or possible adversarial signal contamination (Swami and Sadler, 2002; Dai and So, 2017). A summary of the key technical contributions of this work is provided next:

1. We find simple conditions on $\chi$ the support of $X$ and $\eta_0$ for the parameter in (1) ($\eta_0$ and $\theta_0$) to be identifiable.

2. We provide two tuning parameter free estimators for $\theta_0$; namely the simple score estimator and least squares estimator, see Section 2. Furthermore, each of the estimators is associated with a tuning parameter free estimator for $\eta_0$.

3. In contrast to most works in the shape constrained literature, we find the rate of convergence of the location estimators under heavy-tailed errors. We allow the errors to be arbitrarily dependent on $X$. We show that the simple score estimator is $\sqrt{n}$ consistent and asymptotically normal as long as $\mathbb{E}(\epsilon^6|X) < \infty$ almost every $X$.

4. We study the performance of both the simple score estimator and least squares estimator through extensive simulations, and analyze a real life dataset in Section 5. We use the $m$-out-of-$n$ bootstrap to provide tuning parameter free inference for the simple score estimator.

Organization. The remainder of the paper is organized as follows. In Section 2, we provide simple conditions for identifiability of the model, followed by two parameter free estimators for the target location $\theta_0$. In Section 3, we provide asymptotic analyses for the two estimators. Section 3.1 finds rate upper bounds for the least squares estimators of $\theta_0$ and $\eta_0$. Section 3.2 shows that the simple score estimator for $\theta_0$ is $\sqrt{n}$ consistent and asymptotically normal. In Section 4, we study the finite sample performance of both estimators through extensive simulations. We also illustrate that the $m$-out-of-$n$ bootstrap can be used for
valid inference for the simple score estimator. In Section 5, we use the proposed estimators in a surveillance application to locate an individual. Specifically, we use video footage from a wide angle CCTV camera for the entrance lobby of the INRIA Labs at Grenoble, France (Fisher et al., 2005). Section 6 summarizes the contribution of the paper and provides some concluding remarks, and in particular thoughts about extending this approach to the time-series case [i.e. a series of data on sensor readings that arrive at consecutive time points] which is relevant to tracking a moving target. The proofs of all the results in the Appendix.

2 Model identifiability and estimation

We start by addressing identifiability issues for the model posited in (1) based on the following two assumptions.

(A1) The support $\chi$ of $X$ is a bounded convex set with at least one interior point. The covariate $X$ has a bounded density with respect to the Lebesgue measure on $\chi$. The parameter set $\Theta$ is bounded with non empty interior and $\theta_0$ belongs to the interior of $\Theta$. Further, let $T \in \mathbb{R}$ be some finite number such that $\sup_{x \in \chi} |x| \leq T$ and $\sup_{\theta \in \Theta} |\theta| \leq T$.

(A2) The function $t \mapsto \eta_0(t)$ is nonconstant, continuously differentiable, and nonincreasing on $\mathbb{R}^+$. The boundedness assumptions on $\chi$ can be replaced by a sub-Gaussianity or heavy-tail moment assumption. In that case, the rate upper bound derived for the estimators of $\eta$ will suffer. On the other hand, as long as the elements of $X$ have enough moments, the score estimator proposed later in the paper will still be $\sqrt{n}$ consistent; also see Remark 3 of Balabdaoui et al. (2019b). In the following result (proved in Section A), we establish the identifiability of (1). The boundedness assumption on $\Theta$ is natural as in practice the monitoring regions are well known in advance. The continuity, non-constancy, and monotonicity assumptions on the attenuation function is very natural is justified by the physics of signal attenuation.

Lemma 2.1. Suppose assumptions (A1) and (A2) hold. Then, $\theta_0$ and $\eta_0$ are unique. In other words, if there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta \in \mathbb{R}^d$ such that

$$\eta_0(|\theta_0 - x|^2) = g(|\beta - x|^2) \quad \text{for all} \ x \in \chi,$$

then $\theta_0 = \beta$ and $\eta_0 = g$ on $\{|\theta_0 - x|^2 : x \in \chi\}$.

In this paper, we suppose that we have $n$ i.i.d. observations $\{(X_i, Y_i) \in \chi \times \mathbb{R}, 1 \leq i \leq n\}$ from (1). Before introducing the estimators for the location parameter $\theta_0$, we propose the following simple profile least squares estimator for the attenuation function for any location $\theta \in \Theta$

$$\tilde{\eta}_\theta := \arg\min_{\eta \in \mathcal{M}} Q_n(\eta, \theta),$$

where

$$Q_n(\eta, \theta) := \sum_{i=1}^{n} (Y_i - \eta(|\theta - X_i|^2))^2$$

and

$$\mathcal{M} := \{ \eta : [0, 4T^2] \to \mathbb{R}^+ : \eta \text{ is a non-increasing function} \}.$$

For every fixed $\theta$, the optimization problem in (2) can be shown to be convex. However, note that $\tilde{\eta}_\theta$ is well defined only at $\{|\theta - X_i|^2\}_{i=1}^{n}$. In this paper, we consider the canonical extension of $\tilde{\eta}_\theta$, and define $t \mapsto \tilde{\eta}_\theta(t)$ to be the unique right continuous piecewise constant function on $[0, 4T^2]$ with potential jumps.
at \(\{\|\theta - X_i\|\}^n_{i=1}\). Further, it is well known that, when there are no ties in \(\{\|\theta - X_i\|\}^n_{i=1}\), the profiled estimator \(\tilde{\eta}_0\) is the left derivative of the least concave majorant of the cumulative sum diagram
\[
\left\{ (0, 0), (1, Y_{(1, \theta)}), \ldots, \left(k, \sum_{j=1}^{k} Y_{(j, \theta)}\right), \ldots, \left(n, \sum_{j=1}^{n} Y_{(j, \theta)}\right) \right\},
\]
where \(Y_{(k, \theta)}\) is the measurement corresponding to the sensor that is \(k\)th closest to \(\theta\); see for example (Robertson et al., 1988, Theorem 1.2.1) or (Barlow et al., 1972, Theorem 1.1). For ease of presentation, we assume that there are no ties in \(\{\|\theta - X_i\|\}^n_{i=1}\). The case of ties can be easily handled by “merging” tied data points and considering a weighted least squares problem, see e.g., Balabdaoui et al. (2019a,b); Kuchibhotla and Patra (2020). From (4), the profile least squares estimator can be easily with a complexity of \(O(n)\) via the pool adjacent violators algorithm (PAVA); see (Robertson et al., 1988, Section 1.2) and Grotzinger and Witzgall (1984).

For any \(\theta \in \Theta\), we define the “population” version of \(\tilde{\eta}_0\) as follows:
\[
\eta_0(t) := \mathbb{E}\left(\eta_0(\|\theta_0 - X\|^2) | \|\theta - X\|^2 = t\right) \quad \text{for all } t \in \mathbb{R}^+.
\] (5)

The following lemma states a useful characterization for \(\eta_0(t)\). It shows that for some fixed \(\theta\), \(\eta_0(\cdot)\) can be thought of as the minimizer of the population version squared error loss in (3).

**Lemma 2.2.** Suppose \(\eta_0\) is a strictly decreasing function. Then, there exists \(\delta_0 > 0\) such that for every \(\theta \in B(\theta_0, \delta_0)\), \(\eta_0\) uniquely minimizes \(\eta \mapsto \mathbb{E}(\|\theta - X\|^2) - \eta_0(\|\theta_0 - X\|^2)\) over the class of nonincreasing functions \(\mathcal{M}\).

Observe that \(\eta_{\theta_0} = \eta_0\). Thus, if \(\theta_0\) were known, then \(\tilde{\eta}_{\theta_0}\) would be the estimator for \(\eta_0\). In Theorem 3.1, we will study the asymptotic properties of \(\tilde{\eta}_{\theta_0}\) as \(\theta\) varies in a small neighborhood of \(\theta_0\). Below, we use the profile least squares estimator to propose two tuning parameter free estimators for \(\theta_0\).

**Least Squares Estimator (LSE).** The LSE for \(\theta_0\) defined as
\[
\hat{\theta} := \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_0(\|\theta - X_i\|^2))^2,
\] (6)
where the profile least squares estimator \(\tilde{\eta}_0\) is defined via (2). Note that the above minimization problem is free of tuning parameters. However, unlike (2), the optimization problem in (6) is typically non-convex. Recall that the cumulative sum diagram in (4) depends only on the ordering of \(\{\|\theta - X_i\|\}^n_{i=1}\). Thus, for every \(\theta\) in the interior of the parameter space, and \(\beta\) in some small neighborhood of \(\theta\), we have that
\[
\sum_{i=1}^{n} (Y_i - \tilde{\eta}_0(\|\theta - X_i\|^2))^2 = \sum_{i=1}^{n} (Y_i - \tilde{\eta}_\beta(\|\beta - X_i\|^2))^2.
\] (7)

Further, for every \(t \in [0, 4T^2]\) the function \(\theta \mapsto \tilde{\eta}_0(t)\) is piecewise constant, as changes in \(\theta\) may lead to different ordering for \(\{\|\theta - X_i\|\}^n_{i=1}\). As a consequence, \(\theta \mapsto \sum_{i=1}^{n}(Y_i - \tilde{\eta}_0(\|\theta - X_i\|^2))^2\) is piecewise constant with multiple global minimizers. The results that follow hold true for any global minimizer \(\hat{\theta}\). Once we have the LSE for \(\theta_0\), we define \(\tilde{\eta}\), the LSE for \(\eta_0\), as
\[
\tilde{\eta} := \arg\min_{\eta \in \mathcal{M}} Q_n(\eta, \hat{\theta}).
\]
We study the asymptotic behavior of LSE \((\tilde{\eta}, \hat{\theta})\) in Theorem 3.2.
Figure 1: Plot of \( \theta \mapsto \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_\theta(|\theta - X_i|^2))X_i \) (left panel) and \( \theta \mapsto \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_\theta(|\theta - X_i|^2))^2 \) (right panel) for sample sizes 500 (solid) and 5000 (dotted). In both cases, we have taken \( \theta = 0, X \sim \text{Uniform}[-3, 3], \epsilon \sim \text{t}_7/4, \) and \( Y = 1/(1 + .1|X - \theta|^2)^2 + \epsilon. \)

**Simple Score Estimator (SSCE).** To motivate this estimator, assume for the moment that \( \tilde{\eta}_\theta \) is differentiable; then, \( \hat{\theta} \) can be defined as the solution to the following score equation:

\[
\sum_{i=1}^{n} (Y_i - \tilde{\eta}_\theta(|\theta - X_i|^2)) \tilde{\eta}'_\theta(|\theta - X_i|^2)(X_i - \theta) = \mathbf{0}_d, \tag{8}
\]

where \( \mathbf{0}_d \in \mathbb{R}^d \) is a vector comprising of zeros. The above score equation is equivalent to a semiparametric efficient score equation, and hence it is reasonable to expect that its solution would give rise to a semiparametrically efficient estimator for \( \theta_0 \). However, since \( \tilde{\eta}_\theta \) is a piecewise constant function, \( \tilde{\eta}_\theta' \) does not exist. In light of this, we propose a new estimator based on the following simple modification of the above efficient score equation Balabdaoui et al. (2019b); Groeneboom and Hendrickx (2018). We define an SSCE by a zero of

\[
\theta \mapsto n^{-1} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_\theta(|\theta - X_i|^2))(X_i - \theta).
\]

Further, by the property of the isotonic estimator \( \tilde{\eta}_\theta \), we have that \( \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \tilde{\eta}_\theta(|\theta - X_i|^2) \). Thus, the SSCE is the zero of \( \theta \mapsto \mathbb{M}_n(\theta) \), where

\[
\mathbb{M}_n(\theta) := \sum_{i=1}^{n} (Y_i - \tilde{\eta}_\theta(|\theta - X_i|^2))X_i. \tag{9}
\]

Note that in the definition of \( \mathbb{M}_n(\cdot) \), we have ignored the non-differentiability of \( \tilde{\eta}_\theta \) and replaced \( \tilde{\eta}_\theta' \) in (8) with 1. The motivation for (9) is that in the absence of \( \tilde{\eta}_\theta' \), \( \mathbb{M}_n(\cdot) \) can be seen as a “rough approximation” of the efficient score equation Balabdaoui et al. (2019b); Groeneboom and Hendrickx (2018). Further, \( \phi(x, y) = (y - \tilde{\eta}_\theta(|\theta - x|^2))(x - \theta) \) is a valid influence function in the sense of (Van der Vaart, 2002, Chapter 1.2). Another motivation for SSCE stems from observing that \( \mathbb{E}[X|Y - \eta_0(\theta_0 - X^2)] \), the “population” version of \( \mathbb{M}_n(\theta_0) \), is \( \mathbf{0}_d \). In Section 3.2, we discuss assumptions (see assumptions (A4) and (A5)) under which \( \theta_0 \) is the unique zero of \( \theta \mapsto \mathbb{E}[X|Y - \eta_0(\theta - X^2)] \).

After obtaining the SSCE for \( \theta_0 \), define \( \hat{\eta} \) (the SSCE for \( \eta_0 \)) as

\[
\hat{\eta} := \arg\min_{\eta \in \mathcal{M}} Q_n(\eta, \hat{\theta}). \tag{10}
\]
Now recall that $\theta \mapsto \tilde{\eta}_\theta(t)$ is piecewise constant with discontinuities (for any $t \in [0, 4T^2]$). This and (7) imply that $\theta \mapsto M_n(\theta)$ will have discontinuities and exact zeros in (9) may not always exist. Instead, we define the SSCE $\hat{\theta}$ as a “zero crossing” of $\theta \mapsto M_n(\theta)$. The following definition is from Groeneboom and Hendrickx (2018).

**Definition 2.1 (Zero Crossing, Groeneboom and Hendrickx (2018)).** We say that $\beta^*$ is a zero crossing of a real-valued function $\beta \mapsto \zeta(\beta)$ on a set $A$ if each open neighborhood of $\beta^*$ contains points $\beta_1, \beta_2 \in A$ such that $\zeta(\beta_1)\zeta(\beta_2) \leq 0$. We say that an $m$-dimensional function $\beta \mapsto \zeta(\beta) = (\zeta_1(\beta), \ldots, \zeta_m(\beta))'$ has a crossing of zero at a point $\beta^*$, if $\beta^*$ is a crossing of zero of each component $\beta \mapsto \zeta_j(\beta)$ for every $j \in [m]$.

We study the asymptotic behavior of SSCE ($\tilde{\eta}, \hat{\theta}$) in Theorem 3.3 and find the asymptotic distribution of $\hat{\theta}$ in Theorem 3.4. Following the work of Balabdaoui and Groeneboom (2020) in a single index model, one can show that the SSCE is asymptotically equivalent to the following minimizer $\theta^\dagger := \arg\min_{\theta \in \Theta} |M_n(\theta)|$. However, we do not pursue this extension here.

### 3 Asymptotic analysis of the estimators

We start our analysis by establishing properties of the simple profile least squares estimator for $\tilde{\eta}_\theta$. Henceforth, we also require the following assumption on the distribution of $\epsilon$.

(A3) The error $\epsilon$ in model (1) has finite $q$-th moment, i.e., $K_q := \mathbb{E}(|\epsilon|^q)^{1/q} < \infty$ where $q \geq 2$. Further, $\mathbb{E}(\epsilon|X) = 0$, $P_X$ a.e. and $\sigma^2(x) := \mathbb{E}(\epsilon^2|X = x) \leq \sigma^2 < \infty$ for all $x \in \chi$.

The above assumption on $\epsilon$ is fairly general and allows for heteroscedastic errors. Further, contrary to most existing work for similar models that require sub-Gaussian or sub-exponential errors (see, e.g., Balabdaoui et al. (2019a,b); Balabdaoui and Groeneboom (2020)), we allow the error distribution to have only finitely many moments. The following result, established in Section B, shows that $\tilde{\eta}_\theta$ (defined in (2)) converges to $\eta_\theta$ (defined in (3)) uniformly in $\theta$ in a neighborhood of $\theta_0$.

**Theorem 3.1.** Suppose assumptions (A1)–(A3) hold, then

$$\sup_{\theta \in \Theta} \|\tilde{\eta}_\theta\|_\infty = O_p(n^{1/q}).$$

Moreover, let $P_X$ denote the distribution of $X$, then there exists a fixed $\delta_0 > 0$, such that

$$\sup_{\theta \in B(\theta_0, \delta_0)} \int \left\{ \tilde{\eta}_\theta(|\theta - x|^2) - \eta_\theta(|\theta - x|^2) \right\}^2 dP_X(x) = O_p(n^{-2/3}n^{2/q}).$$

The profiled estimator $\tilde{\eta}_\theta$ of $\eta_\theta$ plays a crucial role in the definition and analysis of both the LSE and SSCE. The first part of Theorem 3.1 shows that even though $\mathcal{M}$ is an unbounded class, $\|\tilde{\eta}_\theta\|_\infty$ is not too large. Moreover, the above uniform convergence result helps us study the behavior of the criterion/loss function around $\theta_0$ for both the LSE and SSCE.

#### 3.1 Asymptotic analysis of the LSE

In this section, we compute upper bounds on the rate of convergence of $\tilde{\eta}(|\tilde{\theta} - x|^2)$ and $\tilde{\theta}$, to $\eta_\theta(|\theta_0 - x|^2)$ and $\theta_0$, respectively. Since $\tilde{\theta}$ is a minimizer $\theta \mapsto Q_n(\tilde{\eta}_\theta, \theta)$ over $\chi$ and $\tilde{\eta}_\theta \in \mathcal{M}$ for all $\theta \in \chi$, the next step in characterizing the asymptotic behavior of $\tilde{\theta}$ is to calculate the metric entropy of the class of functions
\{η(|θ−·|^2) : η ∈ \mathcal{M}, θ ∈ Θ\}. However, by (11), we have that for large enough n, \(P(\sup_{θ ∈ B(θ_0, δ_0)} \|\tilde{η}_θ\|_∞ > Cn^{1/q}) \leq ε\) for some C depending only on ε. Thus, we will study the following class of functions,

\[\mathcal{F}_K := \{x \mapsto η(|θ − x|^2) : η ∈ \mathcal{M}, θ ∈ Θ, \|η\|_∞ ≤ K\} .\]

Let \(N_\|\| ε, \mathcal{F}_K, L_2(P_X)\) denote the \(ε\)-bracketing number of \(\mathcal{F}_K\) in the \(L_2(P_X)\) metric (see Section 2.1.1 of van der Vaart and Wellner (1996) for a formal definition). The following lemma, proved in Section C, computes the bracketing entropy of \(\mathcal{F}_K\).

**Lemma 3.1.** Let \(ε > 0\) and \(K > ε\). Then, there exists a constant \(A_1 > 0\) depending only on \(d, Θ, \) and \(χ\) such that \(\log N_\|\| ε, \mathcal{F}_K, L_2(P_X)\) \(≤ A_1 K / ε\).

In Section D, we use the above result to establish the following upper bounds on the rate of convergence for \(x \mapsto \tilde{η}(|\tilde{θ} − x|^2)\). To show that \(\tilde{θ}\) inherits the rate of convergence of the (joint) regression function, we will need the following assumption.

**(A4‘)** There exists an open set \(A \subset \{\|θ_0 − x\|^2 : x ∈ χ\}\), such that \(t \mapsto η_0(t)\) is continuously differentiable on \(A\), \(\inf_{t ∈ A} |η_0'(t)| > 0\), and \(P(\|θ_0 − X\|^2 ∈ A) > 0\).

**Theorem 3.2.** If assumptions (A1)–(A3) hold and \(q ≥ 5\), then we have

\[\int \left\{\tilde{η}(|\tilde{θ} − x|^2) − η_0(|θ_0 − x|^2)\right\}^2 dP_X(x) = O_p(\sqrt{n}^{-2/3} n^{2/q}) .\]

Moreover, if assumption (A4‘) also holds, then

\[|\tilde{θ} − θ_0| = O_p(\sqrt{n}^{-1/3} n^{1/q}) .\]

Assumption (A4‘) is inspired by (Balabdaoui et al., 2019a, Assumption (A5)) and allows us get the rate of convergence of \(\tilde{θ}\) from the rate of convergence of \(\tilde{η}(|\tilde{θ} − ·|^2)\). If assumption (A4‘) doesn’t hold then, the second part of proof of Theorem 3.2 can be easily modified to show that there exists a positive semi-definite matrix \(I\) such that \(|I_{dx,d}(\tilde{θ} − θ_0)| = O_p(n^{-1/3} n^{1/q})\). Assumption (A4‘) essentially says that \(η_0\) must be smooth and non-constant on a region (in \(χ\)) of positive mass.

The LSE discussed above is a natural tuning parameter free estimator for \(θ_0\). The sub-\(\sqrt{n}\) upper bound on rate of convergence, however, raises the question whether the rate bound above is tight or the LSE actually converges at the much faster \(\sqrt{n}\) rate. To investigate this, we have done an extensive simulation study in Section 4 of the paper. The simulations suggests that, indeed the above rate upper bound is not tight. However, it is still unknown whether the LSE is \(\sqrt{n}\) consistent, let alone its asymptotic distribution. The rate of convergence of the LSE from the various simulation settings considered in Section 4 is inconclusive, e.g., in Table 1, the \(\sqrt{n} × \text{Var}(\tilde{θ})\) appears to decrease, while Figures 3 and 4 display an almost opposite trend. We believe that the difficulties in finding the true rate of convergence of the LSE for \(θ_0\) stems from the fact that \(\tilde{η}_θ\) is continuous and \(η_0\) and \(θ_0\) are intertwined. A similar phenomenon is observed for the LSE in the monotone single index models, where faster than \(n^{1/3}\) rate (under sub-exponential errors) is conjectured and observed but not proved (Tanaka, 2008; Balabdaoui et al., 2019a). Lastly, the dependence of \(q\) in the rate upper bound above in Theorem 3.2 can be improved by using the techniques developed in (Kuchibhotla and Patra, 2019, Theorem 3.1 and Corollary 3.1) and (Balabdaoui et al., 2019a, Theorems 4.1 and 7.3). However we do not pursue this marginal improvement in the current paper. Instead, we focus on studying a \(\sqrt{n}\) consistent estimator with a tractable limit distribution that is practically useful, namely the SSCE.
3.2 Asymptotic analysis of the SSCE

In this section, we study the asymptotics of the SSCE defined in (9). We will first prove its existence and consistency. Before stating the main results of this section, let us define $M : \mathcal{X} \to \mathbb{R}$, the population version of $M_n(\cdot)$:

$$M(\theta) := \mathbb{E}\left(\left[Y - \eta_0(\theta - X)\right] (X - \theta)\right).$$

It is easy to see that $M(\theta_0) = 0_d$. However, it is not clear whether $\theta_0$ is the unique zero of $\theta \mapsto M(\theta)$ and/or “well-separated” in the sense of (Van der Vaart, 1998, Theorem 5.9). In Lemmas G.2 and G.3 (stated and proved in Section G.1), we will use the following two assumptions to show that $\theta \mapsto M(\theta)$ has a unique zero at $\theta_0$, is differentiable at $\theta_0$, and $M'(\theta_0)$ is non-singular.

(A4) $\mathbb{E}\left(\eta_0^2(\theta_0 - X)^2 \text{Cov}(X | \theta_0 - X) \right)$ is a positive definite matrix.

(A5) There exists a $\delta_0 > 0$ such that for all $\theta \in B(\theta_0, \delta_0)$ and $\theta \neq \theta_0$, the random variable

$$\text{Cov}\left((\theta - \theta_0)^\top X, \eta_0(\theta_0 - X) \right) \neq 0 \text{ almost everywhere.}$$

Note that (A4') is a sufficient condition for both (A4) and (A5). Assumptions (A4) and (A5) are similar to (Kuchibhotla et al., 2017, Assumption A4), (Balabdaoui et al., 2019b, Assumption A5), and (Groeneboom and Hendrickx, 2018, Theorem 4.1) among others. The following result, proved in Section E, shows that $\hat{\theta}$ exists with probability approaching one and is consistent.

Theorem 3.3. Suppose assumptions (A1)–(A5) hold, then the SSCE exists with probability approaching one and $\hat{\theta}$ is consistent for $\theta_0$, i.e., $\hat{\theta} \overset{P}{\to} \theta_0$.

If assumption (A4) does not hold, then just as in the case of Theorem 3.2, we can show that $A^\top (\hat{\theta} - \theta_0) = o_p(1)$, where $A := \mathbb{E}(\eta_0^2(\theta_0 - X)^2 \text{Cov}(X | \theta_0 - X))$. To prove the asymptotic normality of $\hat{\theta}$, the SSCE, we will require the following smoothness assumption on the conditional expectation of $X$ given $|\theta_0 - X|$.

(A6) The function $u \mapsto \mathbb{E}|X| |\theta - X| = u$ is twice continuously differentiable, except possibly at a finite number of points, and there exists a finite constant $\tilde{M} > 0$ such that for every $\theta_1, \theta_2 \in \Theta$,

$$\sup_{u \in [0,4 \tilde{M}]} \left| \mathbb{E}|X| |\theta_1 - X| = u - \mathbb{E}|X| |\theta_2 - X| = u \right| \leq \tilde{M}|\theta_1 - \theta_2|.$$ 

The assumption (A6) is standard and widely used for semiparametric regression models of similar nature. Assumption (A6) is similar to those in (Murphy et al., 1999, Theorem 3.2), (Groeneboom and Hendrickx, 2018, Assumption A5), (Balabdaoui et al., 2019b, Assumption A5), and (Kuchibhotla et al., 2017, Assumption B3); also see (Song, 2014, Assumption G2 (ii)). The above papers, discuss many distributions of $X$ such that (A6) holds.

Based on the discussion preceding (9) in Section 2, it is intuitively apparent that $\hat{\theta}$ is not semiparametrically efficient as (8) is not the efficient score equation. The following result (proved in Section F) shows that $\hat{\theta}$ is nonetheless asymptotically normal and finds its asymptotic distribution.

Theorem 3.4. Suppose assumptions (A1)–(A6) hold and $t \mapsto \eta_0(t)$ is strictly decreasing. Furthermore, suppose $q \geq 6^2$ and let

$$A := \mathbb{E}\left(\eta_0^2(\theta_0 - X)^2 \text{Cov}(X | \theta_0 - X) \right).$$

1In Lemma G.3, we show that $M'(\theta_0) = \mathbb{E}(\eta_0^2(\theta_0 - X)^2 \text{Cov}(X | \theta_0 - X))$.

2Using Theorem 5.1 and Corollary 3.1 of Kuchibhotla and Patra (2019) and techniques used in the proof of Theorems 4.1 and 7.3 of Balabdaoui et al. (2019a) one can improve the assumptions that $q \geq 6$. However we do not pursue this marginal improvement in the current paper.
and
\[ \Sigma := \mathbb{E}\left[ \sigma^2(X) (X - \mathbb{E}(X \mid \theta_0 - X)^2) \right]. \]

Then
\[ \sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{D} N(0, A^{-1}\Sigma A^{-1}). \]

**Remark 3.1 (Efficient estimation of \( \theta_0 \)).** As discussed above, the SSCE is not semiparametrically efficient. We can improve upon it by considering the following estimator:
\[ \hat{\theta}_h := \operatorname{argmin}_{\theta \in \Theta} \hat{m}_n(\theta) \quad \text{where} \quad \hat{m}_n(\theta) = \left\| \sum_{i=1}^{n} (Y_i - \hat{\eta}_0(|\theta - X_i|)) \hat{\eta}_{\theta,h}(|\theta - X_i|) X_i \right\|, \]
where for every \( u \in [0, 4T^2] \) and \( h > 0 \), we define
\[ \hat{\eta}_{\theta,h}(u) := \frac{1}{h} \int_{0}^{4T^2} K\left( \frac{u - x}{h} \right) d\hat{\eta}_0(x), \]
where \( x \to K(x) \) is a twice differentiable kernel with support \([0, 1]\). Using the techniques developed in this paper and Balabdaoui et al. (2019b), one can show that if \( h \asymp n^{-1/7} \), then \( \hat{\theta}_h \) is an efficient estimator for \( \theta_0 \) under some additional smoothness assumptions on \( \eta_0 \). However, we do not consider this estimator any further, since it involves a tuning parameter and the finite sample performance of \( \hat{\theta}_h \) can depend heavily on the choice of the bandwidth \( h \). Finally, it is important to note that \( \hat{\theta}_h \) will be efficient only when the errors are homoscedastic. In case of heteroscedastic errors, it may be the case that the SSCE has lower asymptotic variance than the above estimator. In fact, in a closely related model Balabdaoui and Groeneboom (2020) give an example where the efficient estimator (under homoscedastic error) has worse finite sample (and asymptotic) variance than a non-efficient estimator.

## 4 Performance evaluation

To investigate the performance of the LSE and SSCE, we carry out several simulation experiments. The R codes developed to implement these estimators and the scripts replicating the numerical results are available at [http://stat.ufl.edu/rohitpatra/](http://stat.ufl.edu/rohitpatra/). We consider i.i.d. observations from
\[ Y = \eta_0(|\theta_0 - X|^2) + \epsilon, \]  
where the distribution of the covariates, the attenuation function, and distribution of the errors vary across a wide range of options. In the following subsections, we study the behavior of \( \hat{\theta}_1 \) and \( \hat{\theta}_1 \), where \( \hat{\theta} := (\hat{\theta}_1, \ldots, \hat{\theta}_d) \) and \( \hat{\theta} := (\hat{\theta}_1, \ldots, \hat{\theta}_d) \). However, before undertaking a comprehensive comparison across different settings, we focus on the potential statistical inefficiency of the LSE, shown in the next example.

### 4.1 A simple example showcasing the inefficiency of the LSE

We consider the following setup for the model (13):
\[ \eta_0(t) = 1/(1 + 0.1t), \quad X \sim \text{Uniform}[-3, 3]^2, \quad \text{and} \quad \epsilon \mid X \sim \text{Normal}(0, (0.1)^2). \]

In Table 1, we list the sample variance of (centered and scaled) LSE and SSCE for the above setting. It can be seen that the variance of \( \sqrt{n}(\hat{\theta}_1 - \theta_{0,1}) \) stabilizes and converges to its asymptotic limit for even relatively small sample size (≥ 500). On the other hand, the variance of \( \sqrt{n}(\hat{\theta}_1 - \theta_{0,1}) \) appears to grow.\(^5\)

\(^5\)Not that here we have chosen \( \theta_0 = \theta_{0,1} \) without loss of generality.
with \( n \). This finding together with similar results presented in the sequel suggest that the LSE might not be \( \sqrt{n} \) consistent. Note that analogous inconclusive behavior is observed for the LSE estimator in the closely related monotone single index models in Tanaka (2008); Balabdaoui et al. (2019a).\textsuperscript{4} It is remarkable, though, that the estimated sample variance of SSCE is significantly lower than that of LSE (even under homoscedastic error). Thus, even though it is unknown whether the LSE is \( \sqrt{n} \) consistent, it is safe to conclude that the LSE is not efficient for the estimation of \( \theta_0 \).

Table 1: Scaled sample variance for the LSE (\( \hat{\theta}_1 \)) and SSCE (\( \hat{\theta}_1 \)) for \( \theta_{0,1} \) over 500 replications as the sample size grows.

| Sample size | 500   | 1000  | 3000  | 5000  | 10000 | 15000 |
|-------------|-------|-------|-------|-------|-------|-------|
| \( n \times \text{var}(\hat{\theta}_1) \) | 1.43  | 1.50  | 1.57  | 1.46  | 1.75  | 1.82  |
| \( n \times \text{var}(\hat{\theta}_1) \) | 0.74  | 0.75  | 0.74  | 0.73  | 0.74  | 0.74  |

Table 2: Table showing choices for the attenuation function and the distribution of the covariates and error for the simulation considered in Section 4.2.

| Choice of attenuation function | Choices of covariate distribution |
|-------------------------------|----------------------------------|
| \( 5 + (1 + t/5)^{-3} \)     | \( X_1 \sim \text{Unif}[-3, 3] \) and \( X_2 | X_1 \sim 0.2X_1 + .8 \text{Unif}[-3, 3] \) |
| \( 5 + \exp(-t/4) \)        | \( X \sim \text{Unif}[-3, 3]^2 \) |
| \( 5 + (1 - t/10) \times \mathbb{1}_{[0,10]}(t) \) | \( X \sim \text{Normal}(0, I_{2 \times 2}) \) |

| Homoscedastic error distributions | Heteroscedastic error distributions |
|-----------------------------------|-------------------------------------|
| \( \epsilon | X \sim \text{Normal}(0, 1) \) | \( \epsilon | X \sim \log \left( 2 + \| \theta_0 - X \|^2 \right) \times \text{Normal}(0, 1) \) |
| \( \epsilon | X \sim (-1)^{\text{Ber}(1)} \times \text{Beta}(2, 3) \) | \( \epsilon | X \sim \log \left( 2 + \| \theta_0 - X \|^2 \right) \times (-1)^{\text{Ber}(1)} \times \text{Beta}(2, 3) \) |
| \( \epsilon | X \sim t_3 \) | \( \epsilon | X \sim \log \left( 2 + \| \theta_0 - X \|^2 \right) \times t_3 \) |
| \( \epsilon | X \sim t_7 \) | \( \epsilon | X \sim \log \left( 2 + \| \theta_0 - X \|^2 \right) \times t_7 \) |

### 4.2 Extensive comparison of LSE and SSCE

We consider a grid of settings for the attenuation function, the distribution of both the covariates, and the errors as described in Table 2. The dimension is fixed as \( d = 2 \) and \( \theta_0 = (0, 0) \). In figures 2–4, we illustrate the finite sample performance of the LSE and SSCE by plotting the sample variance (scaled by \( n \)) of the two estimators of \( \theta_{0,1} \), the first co-ordinate of \( \theta_0 \). In each figure, we fix the choice of the attenuation function and vary across the choices for the distribution of covariates and errors. The figures illustrate that in almost all settings considered, the empirical variance of the SSCE is significantly lower than that of the LSE. Further, the sample biases (not shown here) of both the estimators are close to zero. We see that LSE has a smaller finite sample variance than the SSCE in only seven out of 74 simulation settings considered in Figures 2–4. Indeed, a similar behavior of the SSCE was observed in the monotone single index models by Balabdaoui et al. (2019b) and Balabdaoui and Groeneboom (2020).

The analysis in Section 3.1 established that \( |\hat{\theta} - \theta_0| = O_p(n^{-1/3}n^{1/d}) \). However, simulations suggest that the above rate is not tight. We conjecture that \( \hat{\theta} \) converges to \( \theta_0 \) at a rate faster than \( n^{1/3} \), but is not \( \sqrt{n} \) consistent. This is in line with the extensive work on the monotone current status and single index models (see Balabdaoui et al. (2019a); Groeneboom and Hendrickx (2018); Balabdaoui et al. (2019b); Balabdaoui

\textsuperscript{4}Under the stronger assumption of convexity, Kuchibhotla et al. (2017) show that a minor variant of the LSE is not only \( \sqrt{n} \) consistent, but also semiparametrically efficient.
**Figure 2:** Plot of scaled sample variance \((n \times \text{var}(\theta^1))\) for the LSE (dashed) and SSE (solid) for the first co-ordinate of \(\theta_0\) when \(\eta_0(t) = 5 + (1 + t/5)^{-3}\). The sample variance is calculated by using 200 replications.
Covariate: Dependent Uniform  Covariate: Standard Normal  Covariate: Independent Uniform $[-3,3]$

Heteroscedastic Standard Normal  Homoscedastic Standard Normal

Heteroscedastic Symmetric Beta  Homoscedastic Symmetric Beta

Heteroscedastic Student t$_3$  Homoscedastic Student t$_3$

Heteroscedastic Student t$_7$  Homoscedastic Student t$_7$

Figure 3: Plot of scaled variance ($n \times \text{var}({\hat{\theta}}^1)$) for the LSE (dashed) and SSE (solid) for the first co-ordinate of $\theta_0$ when $\eta_0(t) = 5 + \exp(-t/4)$.

and Groeneboom (2020)). Another interesting empirical observation is that $\hat{\theta}$, the estimator based on a simple score equation, thoroughly outperforms the LSE.

4.3 Confidence intervals for the SSCE based on the Bootstrap

The goal of this subsection is to compute a confidence interval for $\theta_0$ based on the SSCE. Theorem 3.4 establishes that the asymptotic distribution of the SSCE depends on nuisance parameters such as the $\sigma^2(X)$,
Figure 4: Plot of scaled variance ($n \times \text{var}(\theta^\dagger)$) for the LSE (dashed) and SSE (solid) for the first co-ordinate of $\theta_0$ when $\eta_0(t) = 5 + (1 - t/10) \times 1(0 \leq t \leq 10)$.

$E(X||X - \theta_0|^2)$, and $\eta'_0$. One can use consistent estimates of these quantities to estimate the asymptotic variance of the SSCE and create an asymptotic confidence interval for $\theta_0$. However, such estimators often involve tuning parameters. Following the theme of this paper for a tuning parameter free approach, we will use the standard $m$-out-of-$n$ bootstrap procedure on the data $\{(X_i, Y_i)\}_{i=1}^n$ to compute a confidence interval for $\theta_0$. Figure 5 shows the empirical bootstrap distribution of $\hat{\theta}_1$ for the wild bootstrap Mammen (1993) (left panel) and $m$-out-of-$n$ bootstrap Bickel et al. (2012) (right panel). Extensive simulation results suggest that the wild bootstrap with Mammen’s two-point distribution Mammen (1993) is inconsistent for
the SSCE. However, for all the settings considered in the paper, the $m$-out-of-$n$ bootstrap performs well for most (valid) choices of $m$. Figure 6 depicts the empirical coverage of both bootstrap procedures for sample sizes ranging from 300 to 1200 for a number of the settings described in Table 2. The results suggest that coverage for the $m$-out-of-$n$ bootstrap is close to the nominal level (90%) for all settings involving an exponential attenuation function, but about 5% below for the polynomial attenuation one. On the other hand, the wild bootstrap’s coverage falls significantly short of the nominal level, even for very large sample sizes.

5 Locating an individual from video surveillance footage

Next, we use the proposed estimators in a surveillance application. Specifically, video footage is available from a wide angle CCTV camera for the entrance lobby of the INRIA Labs at Grenoble, France. Individuals walk in and out of the lobby and the objective is to determine their locations. The video frames can be downloaded from the Context Aware Vision using Image-based Active Recognition (CAVIAR) project Fisher et al. (2005). The specific data set (Walk1) employed in this paper can be downloaded from http://groups.inf.ed.ac.uk/vision/CAVIAR/CAVIARDATA1/Walk1/Walk1.jpg.tar.gz.

In the video under consideration, there were two time windows (frames 1002–1019 and 1182–1230) where there was no movement in the lobby. Starting at frame 1235 and ending at frame 1525 a person walks across the lobby. Given an image, the objective is to locate the person in the lobby. The frames are stored in an RGB format. Thus, each frame consists of three channels: Red, Green, and Blue; for each channel, we have a gray scale matrix of dimension $384 \times 288$. Thus, a single frame can be represented as a $384 \times 288 \times 3$ tensor. To leverage our model, we analyze the three channels of the frame independently. For each channel, the person in the frame is considered to be the target and each pixel corresponds to a sensor measurement. In this section, we analyze frame number 1380. To adapt the data into our framework, we convert the data in the frame of interest (# 1380) into a long vector of size $384 \times 288$ independent observations from the model

$$Z = f(X) + \epsilon,$$

(14)
Figure 6: Empirical coverage for two 90% nonparametric bootstrap confidence intervals for $\theta_{0,1}$ based on the SSCE under model (1) for a number of settings for the attenuation functions and distributions of the error listed in Table 2. For wild bootstrap, we use Mammen’s two-point distribution Mammen (1993) and for $m$-out-of-$n$ bootstrap we used $m = \lfloor n^{7/8} \rfloor$. The computed empirical coverage is based on 500 bootstrap replications.
Figure 7: Intensity plot for the frame 1380 from the “Walk1” benchmark data set of the CAVIAR project for the red, green, and blue channels.

Figure 8: Intensity plot of the estimate of $b(X)$ (i.e., the average of 66 frames where the lobby is empty) for the red, green, and blue channels.

where $X$ is a two dimensional vector denoting the location of the pixel, $Y$ is the measurement at the corresponding pixel, and $\epsilon$ is a noise term. We further assume that the function $f$ can be modeled via an additive structure as

$$f(X) = b_0(X) + s_0(X),$$

where $b_0(X)$ denotes the unknown “background” and $s_0(X)$ denotes the unknown “signal”. The function $X \mapsto b_0(X)$ is unknown. However, each of the 66 frames (wherein the lobby is empty) provides independent and identically distributed observations from $Z = b_0(X) + \epsilon$ for every $X$. Thus, we can estimate $X \mapsto b_0(X)$ consistently for each $X$ via a simple sample average. Thus, for the remainder of the section, we treat $X \mapsto b_0(X)$ as a known function and assume we have measurements from the following model:

$$Y = Z - b_0(X) = s_0(X) + \epsilon. \tag{15}$$

Since in the current setup the target corresponds to a person walking through the lobby, in an ideal
Figure 9: Intensity plot of all pixels of re-centered image (see (15)) for the frame number 1380.

world (one without light bleeding),\(^5\) \(x \mapsto s_0(x)\) would a step function of the form \(c_1 \times 1(x \in A)\) for some constant \(c_1\) and subset \(A \subset \chi\). The presence of the target at a pixel location will elevate (or deprecate) the true gray scale intensity, while the gray scale intensity at the other locations is zero. Observe that the true gray scale intensity can be taken to be zero, because we assume that \(b_0(X)\) has been estimated well from all preceding frames. Further, in this paper, we will assume that the target (set \(A\)) can be well approximated by a disk. This implies that, in the ideal world, we can assume that \(s_0(X)\) is a step function of the form \(c_1 \times 1(|\theta_0 - x| \leq c_2)\) for some constants \(c_1\) and \(c_2\). However, to accommodate for light bleeding in the data and to allow for the developed theory to be applicable to this dataset, we posit that \(s_0(\cdot)\) can be approximated by a rapidly (strictly) decreasing function around the target. For the rest of the section, we assume that

\[
s_0(X) = \eta_0(|X - \theta_0|^2),
\]

where \(\theta_0\) can be interpreted as the location of the target and \(\eta_0\) is a monotone function. For an example of \(\eta_0\) see Figure 11 where we plot the estimated functions for the three color bands. Combining (15) and (16), the data are converted to our posited framework with the following exception: the design points \(\{X_i\}_{i=1}^n\) are fixed grid locations in this application, while in the technical developments we assume a random design setup. To remedy this, we sample the grid locations uniformly at random.

Remark 5.1. A natural question is, how does the practitioner know whether to assume that \(t \mapsto \eta(t)\) is decreasing or increasing? This is an important question, because the intensity (values of \(Y\)) of the person relative to the background can be higher or lower depending on factors such as lighting and colors in the picture. A simple way to address this, is to fit both an increasing and a decreasing function for (15) and choose the fit that has the smaller squared error loss \((\sum_{i=1}^n (Y_i - \eta(|X_i - \hat{\theta}|^2)))\) at the SSCE. In fact, this is how we decided to fit a decreasing function for \(\eta_0\) in (16).

The image for frame 1380 from the "Walk1" benchmark data set is shown in Figure 7. As mentioned earlier, we use frames 1002–1019 and 1182–1230 to estimate the background levels (i.e., the function \(b(\cdot)\)). As the above 66 frames do not record any movement, we assume that the data in the model follows model (14) with \(f(x) = b(x)\) for all \(x\) and estimate it by the sample mean of the 66 frames. In Figure 8, we plot the estimates of \(x \mapsto b_0(X)\) for each of the three channels. We now treat \(b_0(\cdot)\) as known, and assume that we have observations \((X, Y)\) from (15). Figure 9 corresponds to a heat map of the centered frame; i.e., we treat the image as being generated by (15) with \(Y\) being the intensity and \(X\) being the location of the pixel.

\(^5\)Light bleeding is the phenomenon, where photo-charge from an pixel bleeds/leaks into other nearby pixels, thus affecting
To replicate the random design scenario, we sample (uniformly) a grid of size $100 \times 100$ from the images in Figure 9. The top row of Figure 10 depicts this “observed” data set. The next step is to compute the SSCE. The second row of Figure 10 depicts the heat map of $\theta \mapsto |M_\theta(\theta)|$ as $\theta$ varies over the location of the sensors. In each of the heat maps, the location of the minimum is marked with solid dots that correspond to the estimated location of the individual. Finally, in the left panel of Figure 11, we overlay the original image with the detected location from each of the channels. The right panel plots the $t \mapsto \hat{\eta}(t)$ corresponding to the three channels - recall that $\hat{\eta}$ is defined as $\tilde{\eta}_\theta$ as in (10). Finally, in Figure 12, we plot the ellipsoid confidence regions based on a normal approximation, with the dispersion matrix based on an $m$-out-of-$n$ bootstrap with $m = \lfloor n^{7/8} \rfloor$. It is worth noting that the confidence ellipsoids contain the target for all three of the channels.

6 Concluding remarks

This paper proposed two tuning parameter free estimators (the LSE and SSCE) for the location of a target based on measurements acquired from distributed sensors using an index type regression model. We proved that the SSCE for the unknown location is $\sqrt{n}$ consistent and asymptotically normal under heavy-tailed and heteroscedastic errors. A numerical comparison between the SSCE and LSE reveals that proposed score estimator performs well in wide variety of settings. Unlike most work in the target detection literature, we do not assume a parametric model for the signal strength attenuation function. Further, the estimation procedure is completely automated and doesn’t require any tuning parameters. These additional detected sensitivity.
Figure 11: Left panel: input image overlaid with estimated location (SSCE, $\hat{\theta}$) of person based on the three color channels. Right Panel: isotonic estimates of the attenuation function at the estimated location ($\hat{\eta} = \hat{\eta}$) for each of the three color channels.

Figure 12: Ellipsoid confidence regions (90% and 95%) based on normal approximation with the variance estimated via 200 $m$-out-of-$n$ bootstrap replicates with $m = \lfloor n^{7/8} \rfloor$.

Advantages make the proposed methodology applicable to a wide variety of problems that leverage sensing infrastructure in distributed systems. We conclude by outlining some exciting future research directions. The rate of convergence of the LSE and limiting distribution of the monotone LSE is an open problem in the wider field of semi-parametric inference. Further, the current paper focuses only on locating a target for a fixed time point. Extending the current methodology to tracking one or multiple targets is a challenging but important problem. When tracking a moving target, one can potentially “combine” the estimates of the attenuation function from different time points to provide an accurate and tuning parameter free estimate of the current location.
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A Proof of Lemma 2.1

If we could prove that \( \theta_0 = \beta \), this would imply that \( \eta_0(\cdot) \equiv g(\cdot) \) on \( \{|x - \theta_0|^2 : x \in \chi\} \). Hence, it suffices to show that \( \theta_0 = \beta \). To show that \( \theta_0 = \beta \), we first notice that because of the convexity of \( \chi \), for small enough \( L > 0 \) we can find an open ball \( B \) with radius \( L \) included in \( \chi \) on which \( x \mapsto \eta_0(|x - \theta_0|^2) \) is not constant and

\[
\eta_0(|\theta_0 - x|^2) = g(|\beta - x|^2) \quad \text{for all } x \in B. \tag{17}
\]

Since, \( x \mapsto \eta_0(|x - \theta_0|^2) \) is not constant on \( B \), there exists a point \( b \in \{|x - \theta_0|^2 : x \in B\} \) and \( \epsilon > \delta > 0 \) such that \( b + \epsilon \in \{|x - \theta_0|^2 : x \in B\} \) and

\[
t_0 =: \eta_0(b + \epsilon) < \eta_0(b + \delta) := t_1 < \eta_0(b) := t_2.
\]

Thus

\[
\phi \subseteq \{x \in \chi : \eta_0(|x - \theta_0|^2) \leq t_0\} \cap B \subseteq \{x \in \chi : \eta_0(|x - \theta_0|^2) \leq t_1\} \cap B \subseteq B \tag{18}
\]

Observe that \( \{x \in \mathbb{R}^d : \eta_0(|x - \theta_0|^2) \leq t_0\} \) and \( \{x \in \mathbb{R}^d : \eta_0(|x - \theta_0|^2) \leq t_1\} \) are two distinct (by (18)) concentric discs centered at \( \theta_0 \). Similarly, by (17), we have that \( \{x \in \mathbb{R}^d : g(|x - \beta|^2) \leq t_0\} \) and \( \{x \in \mathbb{R}^d : g(|x - \beta|^2) \leq t_1\} \) are two (by (18)) concentric discs centered at \( \beta \). They are distinct because

\[
\{x \in \mathbb{R}^d : \eta_0(|x - \theta_0|^2) \leq t_0\} \cap B = \{x \in \mathbb{R}^d : g(|x - \beta|^2) \leq t_0\} \cap B \n\]

and \( \{x \in \chi : \eta_0(|x - \theta_0|^2) \leq t_1\} \cap B \neq B \). Thus \( \theta_0 = \beta \). And the proof is complete.
Proof of Theorem 3.1

We will first prove (11). For each $\theta \in \Theta$, recall that

$$\tilde{\eta}_\theta := \arg\min_{\eta \in \mathcal{M}} Q_n(\eta, \theta).$$

From Robertson et al. (1988, Theorem 1.3.4), for any $q \leq j \leq n$, we have

$$\min_{1 \leq i \leq n} Y_i \leq \tilde{\eta}_\theta(|\theta - X_j|^2) \leq \max_{1 \leq i \leq n} Y_i.$$

Thus the proof of (11) will be complete if we can show that $\max_{1 \leq i \leq n} |Y_i| = O_p \left( n^{1/q} \right)$. In this regard observe that for any $C > 0$,

$$P \left( \max_{1 \leq i \leq n} |Y_i| \geq C n^{1/q} \right) \leq \sum_{i=1}^n P \left( |Y_i| \geq C n^{1/q} \right) \leq \sum_{i=1}^n P \left( |\epsilon_i| \geq C n^{1/q}/2 \right),$$

for all $n$ such that $n^{1/q} \geq \|\eta_0\|_{\infty}$. By assumption (A3) and Markov’s inequality, we have

$$P \left( \max_{1 \leq i \leq n} |Y_i| \geq C n^{1/q} \right) \leq \sum_{i=1}^n P \left( |\epsilon_i| \geq C n^{1/q}/2 \right) \leq n \frac{2^q K_q^q}{C_q n} = 2^q K^q C^{-q}.$$

The above upper bound converges to zero as $C \uparrow \infty$ for all large enough $n$ (independent of $C$). Thus $\max_{1 \leq i \leq n} |Y_i| = O_p \left( n^{1/q} \right)$.

In the following we prove (12). We will use the following newly defined quantities in the proof:

1. $\mathcal{M}^K := \{\eta : D \rightarrow \mathbb{R} : \eta \text{ is non-decreasing and } \|\eta\|_{\infty} \leq K\}$
2. $M_0 := \|\eta_0\|_{\infty}$
3. $M_1 := \sup_{\theta \in B(\theta_0, \rho)} \|\eta_\theta\|_{\infty}$
4. $C_\epsilon := 8 \mathbb{E} \left( \max_{1 \leq i \leq n} |\epsilon_i| \right) \leq 8 \left[ \mathbb{E} \left( \max_{1 \leq i \leq n} |\epsilon_i|^q \right) \right]^{1/q} \leq 8 \left( n \mathbb{E} |\epsilon|^q \right)^{1/q} \leq 8 \|\epsilon\|_{q n^{1/q}}.$

We will use arguments similar to Theorem 3.2.5 of van der Vaart and Wellner (2000) and Balabdaoui et al. (2019a). Recall that

$$\tilde{\eta}_\theta := \arg\min_{\eta \in \mathcal{M}} Q_n(\eta, \theta) = \arg\max_{\eta \in \mathcal{M}} \sum_{i=1}^n \left\{ Y_i \eta \circ \theta(x_i) - \frac{1}{2} \eta^2 \circ \theta(x_i) \right\}$$

and

$$\eta_\theta := \arg\min_{\eta \in \mathcal{M}} Q(\eta, \theta) = \arg\max_{\eta \in \mathcal{M}} \mathbb{E} \left\{ Y \eta(|\theta - X|^2) - \frac{1}{2} \eta^2(|\theta - X|^2) \right\}.$$
Observe that
\[ Q_n(\eta, \theta) - Q_n(\eta_0, \theta) = \sum_{i=1}^{n} \left\{ Y_i \eta \circ \theta(x_i) - \frac{1}{2} \eta^2 \circ \theta(x_i) - Y_i \eta_0 \circ \theta(x_i) + \frac{1}{2} \eta_0^2 \circ \theta(x_i) \right\} \]
\[ = \sum_{i=1}^{n} \left\{ \epsilon_i (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i)) + \eta_0 \circ \theta_0(x_i) (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i)) - \frac{1}{2} \eta^2 \circ \theta(x_i) + \frac{1}{2} \eta_0^2 \circ \theta(x_i) \right\} \]
\[ = \sum_{i=1}^{n} \epsilon_i (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i)) + \sum_{i=1}^{n} (\eta_0 \circ \theta_0(x_i) - \eta_0 \circ \theta(x_i)) (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i)) \]
\[ + \sum_{i=1}^{n} \frac{1}{2} (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i))^2. \]

We first will show that for each \( \theta \)
\[ Q(\eta, \theta) - Q(\eta_0, \theta) = -\frac{1}{2} d^2_\theta(\eta, \eta_0) \leq -d^2_\theta(\eta, \eta_0), \] (19)

where for any \( \eta_1, \eta_2 \in \mathcal{M} \),
\[ d^2_\theta(\eta_1, \eta_2) = \int (\eta_1(|\theta - X|^2) - \eta_2(|\theta - X|^2))^2 dP_0(X) = \int (\eta(t) - \eta_2(t))^2 f_{|\theta - X|^2}(t) dt. \]

Observe that
\[ Q(\eta, \theta) - Q(\eta_0, \theta) = \mathbb{E} \left( Y \eta(|\theta - X|^2) - \frac{1}{2} \eta^2(|\theta - X|^2) - Y \eta_0(|\theta - X|^2) + \frac{1}{2} \eta_0^2(|\theta - X|^2) \right) \]
\[ = \mathbb{E} \left( \mathbb{E} \left[ Y \eta(|\theta - X|^2) - \frac{1}{2} \eta^2(|\theta - X|^2) - Y \eta_0(|\theta - X|^2) + \frac{1}{2} \eta_0^2(|\theta - X|^2) | \theta - X|^2 \right] \right) \]
\[ = \mathbb{E} \left( \mathbb{E} \left[ \eta(|\theta - X|^2) - \eta_0(|\theta - X|^2) \right] - \frac{1}{2} \eta^2(|\theta - X|^2) + \frac{1}{2} \eta_0^2(|\theta - X|^2) \right) \]
\[ = \mathbb{E} \left( \eta_0(|\theta - X|^2) - \eta_0(|\theta - X|^2) \right) - \frac{1}{2} \eta^2(|\theta - X|^2) + \frac{1}{2} \eta_0^2(|\theta - X|^2) \]
\[ = \frac{1}{2} d^2_\theta(\eta, \eta_0). \]

Let us fix \( r > 0 \) and let \( \phi_n : \mathbb{R}^+ \to \mathbb{R}^+ \), be a function such that
\[ \sqrt{n} \mathbb{E}^* \sup_{\theta \in B(\theta_0, r)} \sup_{\eta \in \mathcal{M}_0^K(\delta)} \left| (Q_n - Q)(\eta, \theta) - (Q_n - Q)(\eta_0, \theta) \right| \lesssim \phi_n(\delta), \]
where
\[ \mathcal{M}_0^K(\delta) := \{ \eta : \|\eta\|_{\infty} < K \text{ and } d_\theta(\eta, \eta_0) \leq \delta \} \]
and there exists an \( \alpha < 2 \) such that \( \delta \mapsto \phi_n(\delta)/\delta^\alpha \) is decreasing and \( r_n^2 \phi(1/r_n) \leq \sqrt{n} \). Our goal is to show that
\[ \sup_{\theta \in B(\theta_0, r)} d_\theta(\eta_0, \eta_0) = O_p \left( n^{-1/3} n^{1/q} \right). \]
Note that
\[
\begin{align*}
&P\left(r_n \sup_{\theta \in B(\theta_0, r)} d_\theta(\tilde{\eta}_\theta, \eta_\theta) > \delta\right) \\
&\leq P\left(\sup_{\theta \in B(\theta_0, r)} \|\tilde{\eta}_\theta\| \geq K\right) + \sum_{j > M} P\left(2^{j-1}\delta < r_n \sup_{\theta \in B(\theta_0, r)} d_\theta(\tilde{\eta}_\theta, \eta_\theta) \leq 2^j\delta \text{ and } \sup_{\theta \in B(\theta_0, r)} \|\tilde{\eta}_\theta\| \leq K\right).
\end{align*}
\] (20)

By (11), we can make the first probability on the right hand side of the equation small by choosing \(K = Cn^{1/\theta}\) for an appropriate choice of \(C\). We will now try to bound the second probability. Now observe that \(2^{j-1}\delta < r_n \sup\theta \in B(\theta_0, r) d_\theta(\tilde{\eta}_\theta, \eta_\theta) \leq 2^j\delta\) implies that there exists \(\theta_a \in B(\theta_0, r)\) such that \(2^{j-1}\delta < r_n d_\theta(\tilde{\eta}_{\theta_a}, \eta_{\theta_a}) \leq 2^j\delta\). Now let us define a set
\[
\mathcal{A}_{n,j} := \{(\eta, \theta) : \theta \in B(\theta_0, r), \|\eta\|_{\infty} \leq K, \text{ and } 2^{j-1}\delta < r_n d_\theta(\eta, \eta_\theta) \leq 2^j\delta\}.
\]

As \((\theta_a, \tilde{\eta}_{\theta_a}) \in \mathcal{A}_{n,j}\), we have that
\[
2^{j-1}\delta < r_n \sup_{\theta \in B(\theta_0, r)} d_\theta(\tilde{\eta}_\theta, \eta_\theta) \leq 2^j\delta \Rightarrow \sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} (\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta)) \geq \mathbb{Q}_n(\tilde{\eta}_{\theta_a}, \theta_a) - \mathbb{Q}_n(\tilde{\eta}_{\theta_a}, \theta_a) \geq 0.
\] (21)

Moreover, by (19) we have that
\[
\sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} (\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta)) = \sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} -\frac{1}{2} d^2_\theta(\eta, \eta_\theta) = \frac{-2^{2j-2}\delta^2}{2r_n^2}.
\] (22)

Combining (21) and (22), we have that if \(2^{j-1}\delta < r_n \sup\theta \in B(\theta_0, r) d_\theta(\tilde{\eta}_\theta, \eta_\theta) \leq 2^j\delta\) then
\[
0 \leq \sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} (\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta)) - \frac{1}{2} d^2_\theta(\eta, \eta_\theta) \geq \frac{2^{2j-2}\delta^2}{2r_n^2}.
\] (23)

Now combining (20) and (23), we have that
\[
\begin{align*}
P\left(r_n \sup_{\theta \in B(\theta_0, r)} d_\theta(\tilde{\eta}_\theta, \eta_\theta) > \delta\right) &= P\left(\exists j \geq 1 \sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} (\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta)) \geq \frac{2^{2j-2}\delta^2}{2r_n^2}\right) \\
&\leq \sum_{j \geq 1} \frac{2r_n^2}{\sqrt{n}2^{2j-2}\delta^2} \mathbb{P}\left(\frac{1}{\sqrt{n}} \sup_{(\eta, \theta) \in \mathcal{A}_{n,j}} \sqrt{n} |(\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta)|\right) \\
&\leq \sum_{j \geq 1} \frac{2r_n^2}{\sqrt{n}2^{2j-2}\delta^2} \mathbb{E}\left(\sup_{(\eta, \theta) \in F_j} \sqrt{n} |(\mathbb{Q}_n(\eta, \theta) - \mathbb{Q}_n(\eta_\theta, \theta))|\right),
\end{align*}
\]
where
\[
F_j := \bigcup_{i=1}^j \mathcal{A}_{n,i} = \{(\eta, \theta) : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}_0^K(2^j\delta/r_n)\}.
\]
We will now compute an upper bound for the expectation in the display. Observe that
\[
\sqrt{n}\left|\mathbb{Q}_n - \mathbb{Q}\right| = \sqrt{n}\left|\mathbb{Q}_n - \mathbb{Q}\right| (\eta, \theta) + \sqrt{n}\left|\mathbb{Q}_n - \mathbb{Q}\right| (\eta, \theta) \\
\leq \left|\mathbb{G}_n \varepsilon (\eta \circ \theta - \eta \circ \theta)\right| + \left|\mathbb{G}_n \left[\left(\eta_0 \circ \theta_0(x_i) - \eta_0 \circ \theta(x_i)\right) (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i))\right]\right| + \frac{1}{2} \left|\mathbb{G}_n (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i))^2\right|
\]

Thus we have that
\[
\mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \sqrt{n}\left|\mathbb{Q}_n - \mathbb{Q}\right| (\eta, \theta) - (\mathbb{Q}_n - \mathbb{Q})(\eta, \theta) \right) \\
\leq \mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \left|\mathbb{G}_n \varepsilon (\eta \circ \theta - \eta \circ \theta)\right| \right) + \mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \left|\mathbb{G}_n \left[\left(\eta_0 \circ \theta_0(x_i) - \eta_0 \circ \theta(x_i)\right) (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i))\right]\right| \right) \\
+ \frac{1}{2} \mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \left|\mathbb{G}_n (\eta \circ \theta(x_i) - \eta_0 \circ \theta(x_i))^2\right| \right). \tag{24}
\]

Next we give upper bounds for each of the terms in the right of (24). As
\[
\sup_{(\eta, \theta) \in \mathcal{F}_j} \left\|\left(\eta \circ \theta - \eta_0 \circ \theta\right)\right\|_{\infty} \leq M_1 + K,
\]
by Lemma S.5.1 of Kuchibhotla et al. (2017), we have that
\[
\mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \left|\mathbb{G}_n \varepsilon (\eta \circ \theta - \eta \circ \theta)\right| \right) \leq \mathbb{E} \left( \sup_{(\eta, \theta) \in \mathcal{F}_j} \left|\mathbb{G}_n \varepsilon (\eta \circ \theta - \eta \circ \theta)\right| \right) + 2 \frac{(M_1 + K)\mathbb{C}_\varepsilon}{\sqrt{n}},
\]
where for \(i = 1, \ldots, n\)
\[
\bar{\varepsilon}_i := \varepsilon \mathbf{1}_{|i| \leq \mathbb{C}_\varepsilon} \quad \text{and} \quad \varepsilon^*_i := \varepsilon - \bar{\varepsilon}_i.
\]

We will bound the last expectation on the right of (24) via symmetrization and contraction (Theorem 3.1.21 and Corollary 3.2.2 of Giné and Nickl (2016)). Note that
\[
\mathbb{E} \left( \sup_{\mathcal{F}_j} \left|\mathbb{G}_n (\eta \circ \theta - \eta_0 \circ \theta)^2\right| \right) \leq 2\mathbb{E} \left( \sup_{\mathcal{F}_j} \left|\mathbb{P}_n R(\eta \circ \theta - \eta_0 \circ \theta)^2\right| \right) \\
\leq 8 \sup_{\mathcal{F}_j} \|\eta \circ \theta - \eta_0 \circ \theta\|_{\infty} \mathbb{E} \left( \sup_{\mathcal{F}_j} \left|\mathbb{P}_n R(\eta \circ \theta - \eta_0 \circ \theta)^2\right| \right) \tag{25}
\]
\[
= 8 \sup_{\mathcal{F}_j} \|\eta \circ \theta - \eta_0 \circ \theta\|_{\infty} \mathbb{E} \left( \sup_{\mathcal{F}_j} \left|\mathbb{G}_n R(\eta \circ \theta - \eta_0 \circ \theta)\right| \right),
\]
here \(R_1, \ldots, R_n\) are i.i.d. Rademacher random variables (i.e., \(\mathbb{P}(R = 1) = \mathbb{P}(R = 0) = 1/2\)) independent of \((X_i, \varepsilon_i)_{i=1}^n\).

Let \(\mathcal{F}(\gamma) := \{(\eta, \theta) : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}_\theta^K(\gamma)\}\).
By combining (24) and (25) and Lemma B.1, we have that

\[
\mathbb{E}
\left(\sup_{(\eta, \theta) \in \mathcal{F}(\gamma)} \sqrt{n} \left| (Q_n - Q)(\eta, \theta) - (Q_n - Q)(\eta_\theta, \theta) \right|\right)
\lesssim \sigma [K\gamma]^{1/2} + \frac{\sigma^2 \gamma^{-1} K^2 C_e}{\sqrt{n}} + \frac{KC_e}{\sqrt{n}} + \frac{K^3 \gamma^{-1}}{\sqrt{n}} + \frac{K_\gamma^{1/2}}{\sqrt{n}}
\lesssim \sigma K^{3/2} \gamma^{-1/2} + \frac{K^2 (K + C_e) \gamma^{-1}}{\sqrt{n}} + \frac{KC_e}{\sqrt{n}}
\]

Let \(\gamma_j = 2^j \delta / r_n, C_e = C n^{1/q}\), and \(K = C n^{1/q}\), then

\[
P\left( r_n \sup_{\theta \in B(\theta_0, r)} d_{\theta}(\tilde{\eta}_\theta, \eta_\theta) > \delta \right)
\lesssim \frac{8r_n^2}{\sqrt{n} \delta^2} \sum_{j \geq 1} \left( \sup_{(\eta, \theta) \in \mathcal{F}(\gamma_j)} \sqrt{n} \left| (Q_n - Q)(\eta, \theta) - (Q_n - Q)(\eta_\theta, \theta) \right| \right)
\lesssim \frac{8r_n^2}{\sqrt{n} \delta^2} \left[ \frac{\sigma K^{3/2}}{n^2} \sum_{j \geq 1} \frac{1}{2^{2j}} \left[ 2^j \delta / r_n \right]^{1/2} + \frac{K^2 (K + C_e)}{\sqrt{n}} \sum_{j \geq 1} \frac{1}{2^{2j}} \left[ 2^j \delta / r_n \right]^{-1} + \frac{KC_e}{\sqrt{n}} \sum_{j \geq 1} \frac{1}{2^{2j}} \right]
\lesssim \frac{8r_n^2}{\sqrt{n} \delta^2} \left[ \frac{\sigma K^{3/2}}{n^2} \sum_{j \geq 1} \frac{1}{2^{2j}} \left[ 2^j \right]^{1/2} + \frac{K^2 (K + C_e) r_n}{\delta} \sum_{j \geq 1} \frac{1}{2^{2j}} \left[ 2^j \right]^{-1} + \frac{KC_e}{\sqrt{n}} \sum_{j \geq 1} \frac{1}{2^{2j}} \right]
\lesssim \frac{8r_n^2}{\sqrt{n} \delta^2} \left[ \frac{\sigma K^{3/2}}{n^2} \left( \frac{\delta}{r_n} \right)^{1/2} + \frac{K^2 (K + C_e) r_n}{\delta} + \frac{KC_e}{\sqrt{n}} \right]
\lesssim \frac{8r_n^2}{\sqrt{n} \delta^2} + \frac{8r_n^3}{n^3 \delta^3} K^2 (K + C_e) + \frac{8KC_e^2 r_n}{n^2 \delta^2}
\lesssim \sigma K^{3/2} \frac{8r_n^2}{\sqrt{n} \delta^2} + \frac{8r_n^3}{n^3 \delta^3} n^{-3/q} + \frac{8n^2 / q r_n^2}{n^2 \delta^2}
\]

Thus if \(r_n = n^{1/3 - 1/q}\), then

\[
P\left( r_n \sup_{\theta \in B(\theta_0, r)} d_{\theta}(\tilde{\eta}_\theta, \eta_\theta) > \delta \right) \lesssim \frac{8\sigma}{\delta^{3/2}} + \frac{8}{\delta^3} + \frac{8n^{-1/3}}{\delta^2}.
\]

**Lemma B.1.** Let

\[
\mathcal{F}(\gamma) := \{ (\eta, \theta) : \theta \in B(\theta_0, r) and \eta \in \mathcal{M}_{\theta}^K (\gamma) \},
\]

then

\[
\mathbb{E}
\left( \sup_{\mathcal{F}(\gamma)} | G_n (\eta \circ \theta - \eta_\theta \circ \theta) | \right) \lesssim \sigma [K\gamma]^{1/2} + \frac{\sigma^2 \gamma^{-1} K^2 C_e}{\sqrt{n}},
\]

\[
\mathbb{E}
\left( \sup_{\mathcal{F}(\gamma)} | G_n R (\eta \circ \theta - \eta_\theta \circ \theta) | \right) \lesssim [K\gamma]^{1/2} + \frac{K^2 \gamma^{-1}}{\sqrt{n}},
\]

\[
\mathbb{E}
\left( \sup_{\mathcal{F}(\gamma)} | G_n (\eta_0 \circ \theta_0 - \eta_\theta \circ \theta) (\eta \circ \theta - \eta_\theta \circ \theta) | \right) \lesssim [K\gamma]^{1/2} + \frac{K^2 \gamma^{-1}}{\sqrt{n}}.
\]
Proof. Note that \( \epsilon(\theta - \eta \circ \theta), R(\theta - \eta \circ \theta), (\eta_0 \circ \theta_0 - \eta \circ \theta)(\eta \circ \theta - \eta_0 \circ \theta) \) are uniformly bounded by \( C_\epsilon(M_1 + K), C_\epsilon(M_1 + K), \) and \((M_1 + K)(M_1 + M_0),\) respectively. In Lemma C.1, we show that
\[
\log N_{[1]}(\nu, \{\eta_0 \circ \theta : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) \leq \frac{2A(M_1 + K)}{\nu}
\]
and
\[
\log N_{[1]}(\nu, \{\eta_0 \circ \theta : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) = \log N_{[1]}(\nu, \{\eta_0 \circ \theta_0 - \eta_0 \circ \theta : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) \leq \frac{AM_1}{\nu}.
\]
Thus it is clear that
\[
\log N_{[1]}(\nu, \{\epsilon(\theta - \eta \circ \theta) : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) \leq \frac{2A(M_1 + K)\sigma}{\nu},
\]
as \( \|\epsilon(X - \cdot)\|_\infty \leq \sigma \) and
\[
\log N_{[1]}(\nu, \{R(\eta \circ \theta - \eta_0 \circ \theta) : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) \leq \frac{2A(M_1 + K)}{\nu}.
\]
Finally, in Lemma C.1, we show that
\[
\log N_{[1]}(\nu, \{(\eta_0 \circ \theta_0 - \eta_0 \circ \theta)(\eta \circ \theta - \eta_0 \circ \theta) : (\eta, \theta) \in \mathcal{F}(\gamma), \| \cdot \| \}) \leq \frac{8A(M_1 + K)M_1}{\nu},
\]
Then by Lemma 3.4.2 of van der Vaart and Wellner (1996) we have that
\[
\mathbb{E}\left( \sup_{\mathcal{F}(\gamma)} \left| \mathbb{G}_n \epsilon(\eta \circ \theta - \eta_0 \circ \theta) \right| \right) \lesssim \sigma\sqrt{A(M_1 + K)\gamma^{1/2}} \left( 1 + \frac{\sigma\sqrt{A(M_1 + K)\gamma^{1/2}C_\epsilon(M_1 + K)}}{\gamma^2 \sqrt{n}} \right)
\]
\[
= \sigma\sqrt{A(M_1 + K)\gamma^{1/2}} + \sigma^2 A(M_1 + K) \frac{\gamma^{-1}C_\epsilon(M_1 + K)}{\sqrt{n}}
\]
\[
\lesssim \sigma [K\gamma]^{1/2} + \frac{\sigma^2 \gamma^{-1}K^2 C_\epsilon}{\sqrt{n}}
\]
and
\[
\mathbb{E}\left( \sup_{\mathcal{F}(\gamma)} \left| \mathbb{G}_n R(\eta \circ \theta - \eta_0 \circ \theta) \right| \right) \lesssim \left[ \sqrt{A(M_1 + K)\gamma} \right]^{1/2} \left( 1 + \frac{\sqrt{A(M_1 + K)\gamma^{1/2}C_\epsilon(M_1 + K)}}{\gamma^2 \sqrt{n}} \right)
\]
\[
\lesssim [K\gamma]^{1/2} + \frac{K^2 \gamma^{-1}}{\sqrt{n}}
\]
and
\[
\mathbb{E}\left( \sup_{\mathcal{F}(\gamma)} \left| \mathbb{G}_n (\eta_0 \circ \theta_0 - \eta_0 \circ \theta)(\eta \circ \theta - \eta_0 \circ \theta) \right| \right)
\]
\[
\lesssim \sqrt{A(M_1 + K)M_1 [\gamma(M_0 + M_1)]^{1/2}} \left( 1 + \frac{[\gamma(M_0 + M_1)]^{1/2} \sqrt{A(M_1 + K)M_1(M_1 + K)(M_0 + M_1)}}{[\gamma(M_0 + M_1)]^{2} \sqrt{n}} \right)
\]
\[
= \sqrt{A(M_1 + K)M_1 [\gamma(M_0 + M_1)]^{1/2}} + \frac{[\gamma(M_0 + M_1)]^{-1} A(M_1 + K)M_1(M_1 + K)(M_0 + M_1)}{\sqrt{n}}
\]
\[
= \sqrt{A(M_1 + K)M_1 [\gamma(M_0 + M_1)]^{1/2}} + \frac{[\gamma(M_0 + M_1)]^{-1} A(M_1 + K)M_1(M_1 + K)(M_0 + M_1)}{\sqrt{n}}
\]
\[
\lesssim [K\gamma]^{1/2} + \frac{K^2 \gamma^{-1}}{\sqrt{n}}.
\]
\[\square\]
C Entropy Calculations

The following lemma, proves Lemma 3.1 and finds metric entropies of other related function classes.

Lemma C.1. If \( \nu > 0 \), then

\[
\log N_{[]}^{[]} (\nu, \{ \eta \circ \theta : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|) \leq \frac{AK}{\nu} \tag{27}
\]

\[
\log N_{[]}^{[]} (\nu, \{ \eta_0 \circ \theta : \theta \in B(\theta_0, r) \}, \| \cdot \|) \leq \frac{AM_1}{\nu},
\]

\[
\log N_{[]}^{[]} (\nu, \{ \eta \circ \theta \circ \theta : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|) \leq \frac{2A(M_1 + K)}{\nu}, \tag{28}
\]

where \( A \) is a universal constant depending only on \( D \) and \( d \) (the dimension of \( X \)). Further, let

\[
\mathcal{A}^K := \{ (\eta \circ \theta_0 - \eta \circ \theta) (\eta \circ \theta - \eta_0 \circ \theta) : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \},
\]

then

\[
\log N_{[]}^{[]} (\nu, \mathcal{A}^K, \| \cdot \|) \leq \frac{8A(M_1 + K)M_1}{\nu}. \tag{29}
\]

Proof. By triangle inequality, we have that

\[
| | \theta_2 - x | - | \theta_1 - x || \leq | \theta_1 - \theta_2 |.
\]

Thus the proof of (27)–(28) follows directly from Lemma K.1 of Kuchibhotla et al. (2017); also see Lemma 4.9 of Balabdaoui et al. (2019a). We will now prove (29). Consider the following two classes of functions:

\[
\mathcal{A}_1^K := \{ (\eta \circ \theta - \eta \circ \theta_0 ) \eta_0 \circ \theta_0 : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}
\]

\[
\mathcal{A}_2^K := \{ (\eta \circ \theta - \eta_0 \circ \theta) \eta_0 \circ \theta : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}
\]

Note that \( \| \eta_0 \circ \theta_0 \| \infty \leq M_0 \). Thus by (28), we have that

\[
\log N_{[]}^{[]} (\nu, \mathcal{A}_1^K, \| \cdot \|) \leq \frac{2A(M_1 + K)M_0}{\nu}.
\]

Now we will compute the entropy of \( \mathcal{A}_2^K \). Note that for any \( \theta \in B(\theta_0, r) \) and \( \eta \in \mathcal{M}_K \), \( f(t) := \eta(t) \eta_0(t) \) is a monotone function on \( D \rightarrow \mathbb{R} \) and bounded by \( KM_1 \). Thus by Lemma C.1, we have that

\[
\log N_{[]}^{[]} (\nu, \{ \eta_0 \circ \theta \circ \theta : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|) \leq \frac{AM_1 K}{\nu}.
\]

Similarly, we have that

\[
\log N_{[]}^{[]} (\nu, \{ (\eta_0 \circ \theta)^2 : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|) \leq \frac{AM_1^2}{\nu}.
\]

Thus

\[
\log N_{[]}^{[]} (\nu, \mathcal{A}_2^K, \| \cdot \|) \leq \log N_{[]}^{[]} (\nu/2, \{ (\eta_0 \circ \theta)^2 : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|)
\]

\[
\quad + \log N_{[]}^{[]} (\nu/2, \{ (\eta_0 \circ \theta) : \theta \in B(\theta_0, r) \text{ and } \eta \in \mathcal{M}^K \}, \| \cdot \|)
\]

\[
\leq \frac{2AM_1 K}{\nu} + \frac{2AM_1^2}{\nu} = \frac{2A(M_1 + K)M_1}{\nu}
\]

Finally as \( \mathcal{A}^K \subset \mathcal{A}_1^K + \mathcal{A}_2^K \), we have

\[
\log N_{[]}^{[]} (\nu, \mathcal{A}^K, \| \cdot \|) \leq \log N_{[]}^{[]} (\nu/2, \mathcal{A}_1^K, \| \cdot \|) + \log N_{[]}^{[]} (\nu/2, \mathcal{A}_2^K, \| \cdot \|)
\]

\[
\leq \frac{4A(M_1 + K)M_0}{\nu} + \frac{4A(M_1 + K)M_1}{\nu}
\]

\[
\leq \frac{8A(M_1 + K)M_1}{\nu} \tag*{□}
\]
D  Proof of Theorem 3.2

Proof of joint rate of the LSE  We will first show that for every $q \geq 5$, we have
\[
\|\tilde{\eta}(\hat{\theta} - X^2) - \eta_0(|\theta_0 - X^2|)\| = O_p\left(n^{-1/3}n^{1/q}\right).
\]  \hspace{1cm} (30)

We will prove (30) via an application of Theorem 2.1 of Kuchibhotla and Patra (2019). Fix $C > 0$, define $\tilde{F}_n := F_{Cn^{1/q}}$, where for any $K > 0$, $F_K$ is defined as in Lemma 3.1. Now define, $f^\dagger := \arg\min_{f \in \tilde{F}_n} \sum_{i=1}^n (Y_i - f(X_i))^2$. Observe that by Theorem 3.1, we have that $\mathbb{P}(\tilde{\eta}(\hat{\theta} - \cdot^2) \not\in \tilde{F}_n) = o(1)$. Thus $\mathbb{P}(f^\dagger(\cdot) \equiv \tilde{\eta}(\hat{\theta} - \cdot^2)) = 1 - o(1)$ and the rate of convergence of $f^\dagger(\cdot)$ and $\tilde{\eta}(\hat{\theta} - \cdot^2)$ coincide. To complete the proof of (30), we will find the rate of convergence of $f^\dagger$ by applying Theorem 2.1 of Kuchibhotla and Patra (2019). Note that $\tilde{F}_n$ satisfies the assumption of Theorem 2.1 of Kuchibhotla and Patra (2019) with $A = Cn^{1/q}$, $\alpha = 1$, $s = 0$ and $\Phi = \max\{Kq, Cn^{1/q}\}$. Thus we have that
\[
\|f^\dagger(X) - \eta_0(|\theta_0 - X^2|)\| = O_p\left(\max\left\{n^{-1/3}n^{1/q}, n^{-(q-3)/2q}, n^{-(q-1-(3q-1)/(2q-1))}\right\}\right).
\]

Now observe that $n^{-1-(3q-1)/(2q-1)} \leq n^{-1/3}n^{1/q}$ when $q \geq 5$ and $n^{-(q-3)/2q} \leq n^{-1/3}n^{1/q}$ when $q \geq 5$.

Consistency of the separated parameters:  We will now use the above result to prove that $|\hat{\theta} - \theta_0| = o_p(1)$. The following argument is similar to the proof of Theorem 5.2 of Balabdaoui et al. (2019a). To show the dependence of $n$ in the definition of $\tilde{\eta}$ and $\hat{\theta}$, we will use $\tilde{\eta}_n$ and $\hat{\theta}_n$, respectively. By (30), we have that for every subsequence $\{n_k\}$, there exists a further subsequence $\{n_k\}$ such that $\|\tilde{\eta}_{n_k}(\hat{\theta}_{n_k} - X^2) - \eta_0(|\theta_0 - X^2|)\|$ converges to 0 almost surely; see Theorem 2.3.2 of Durrett (2010).

Now fix $\omega$. Since we will argue along the subsequences, to avoid this messy subsequence notation, in what follows, we will assume without loss of generality that $\|\tilde{\eta}_n(\hat{\theta}_n - X^2) - \eta_0(|\theta_0 - X^2|)\|$ converges to 0 almost surely. We will use compactness based arguments to show that $\hat{\theta}_n$ is consistent. Define $\tilde{\eta}_n : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:
\[
\tilde{\eta}_n(t) := \begin{cases} \tilde{\eta}_n(t) & \text{if } \tilde{\eta}_n(t) \in [0, \|\eta_0\|_{\infty}] \\ \|\eta_0\|_{\infty} & \text{otherwise.} \end{cases}
\]

Now recall that the space of bounded, monotone, left-continuous functions are compact under pointwise convergence. Moreover, since $\Theta$ is compact. Let $m_0$ and $\alpha_0$ be a limit points of $\{\tilde{\eta}_n\}$ and $\{\hat{\theta}_n\}$, respectively. If we can show that
\[
\|m_0(\alpha_0 - X^2) - \eta_0(|\theta_0 - X^2|)\| = 0,
\]  \hspace{1cm} (31)
identifiability (Lemma 2.1) and monotonicity of $\eta_0$ and $m_0$ will imply that all limit points of $\{\tilde{\eta}_n\}$ and $\{\hat{\theta}_n\}$ are $\eta_0$ and $\theta_0$, respectively. We will now prove (31). By triangle inequality, we conclude that
\[
\|m_0(\alpha_0 - X^2) - \eta_0(|\theta_0 - X^2|)\| \leq \|m_0(\alpha_0 - X^2) - m_0(\hat{\theta}_n - X^2)\| + \|m_0(\hat{\theta}_n - X^2) - \eta_0(|\theta_0 - X^2|)\| + \|\eta_0(|\theta_0 - X^2|) - \eta_0(|\theta_0 - X^2|)\|.
\]

We now provide a bound for the first term. Since $X$ has density with respect to the Lebesgue measure (by (A1)) and $m_0$ has only countably many discontinuities, we have that $\|m_0(\alpha_0 - X^2) - m_0(\hat{\theta}_n - X^2)\| \to 0$ along a subsequence as $n \to \infty$. For the third term, observe that
\[
\|\tilde{\eta}_n(\hat{\theta}_n - X^2) - \eta_0(|\theta_0 - X^2|)\| \leq \|\tilde{\eta}_n(\hat{\theta}_n - X^2) - \eta_0(|\theta_0 - X^2|)\|,
\]

\footnote{All the subsequences used in the following arguments depend on $\omega$.}
by definition of $\tilde{\eta}_n$. Thus $\|\tilde{\eta}_n(\tilde{\theta}_n - X)^2 - \eta_0(\theta_0 - X)^2\| \to 0$ along a subsequence as $n \to \infty$. For the second term, observe that $\tilde{\eta}_n$ converges pointwise (along a subsequence) to $m_0$ at each continuity point of $m_0$ and both functions are bounded. Moreover, since $|\tilde{\theta}_n - X|^2$ has density wrt Lebesgue measure, we have that the points of discontinuities of $m_0$ wrt to density of $|\tilde{\theta}_n - X|^2$ is measure zero. Thus by the Dominated convergence theorem, we have that $\|m_0(|\tilde{\theta}_n - X|^2) - \tilde{\eta}_n(\theta_0 - X)^2\| \to 0$ along a subsequence as $n \to \infty$. Thus, we have (31). Thus, we have that all limit points of $\theta_n$ are identical to $\theta_0$. Thus $|\tilde{\theta} - \theta_0| = o_p(1)$.

**Proof of rate of convergence of $\tilde{\theta}$:** We will use the above two results and assumption $(A4')$ to show that the $\tilde{\theta}$ inherits the rate of convergence of $x \mapsto \tilde{\eta}(|\tilde{\theta} - x|^2)$. The proof borrows from (Murphy et al., 1999, Lemma 5.7), (Kuchibhotla et al., 2017, Theorem 3.8), and (Balabdaoui et al., 2019a, Corollary 5.3). Let $g_1(x) := [\tilde{\eta}(|\tilde{\theta} - x|^2) - \eta_0(|\tilde{\theta} - x|^2)]$ and $g_2(x) := [\eta_0(|\tilde{\theta} - x|^2) - \eta_0(|\theta_0 - x|^2)]$. For any set $B \subset \chi$ with nonempty interior, let $X_B$ a random variable such that $\mathbb{P}(X \in A) = \mathbb{P}(X \in A \cap B)/\mathbb{P}(X \in B)$. By the Cauchy-Schwarz inequality, we have

$$
(P_{X_B}[g_1 g_2])^2 = (P_{X_B}[[\eta(|\tilde{\theta} - |x|^2)| \eta_0(|\tilde{\theta} - |x|^2)]g_2(\cdot)])^2
= \left(\frac{P_{X_B}[g_1(\cdot)P_{X_B}[g_2(X_B)||\tilde{\theta} - X_B)]}{P_{X_B}[g_2(X_B)||\tilde{\theta} - X_B]}\right)^2
\leq P_{X_B}[g_1^2] P_{X_B}\left[P_{X_B}[g_2(X_B)||\tilde{\theta} - X_B]\right]
= c_n P_{X_B}g_1^2 P_{X_B}g_2^2,
$$

where

$$
c_n := \frac{P_{X_B}\left[P_{X_B}[g_2(X_B)||\tilde{\theta} - X_B]\right]}{P_{X_B}g_2^2} = \frac{P\left([\eta_0(|\tilde{\theta} - X_B|^2) - P_{X_B}[\eta_0(|\theta_0 - X_B|^2)||\tilde{\theta} - X_B)]^2\right)}{P\left([\eta_0(|\tilde{\theta} - X_B|^2) - \eta_0(|\theta_0 - X_B|^2)]^2\right)}.
$$

If $c_n < 1$, then by Lemma 5.7 Murphy et al. (1999), we can infer that

$$
P_{X_B}g_1^2 + P_{X_B}g_2^2 \leq \frac{1}{1 - \sqrt{c_n}} P_{X_B}(g_1 + g_2)^2 = \frac{1}{1 - \sqrt{c_n}} P_X(X \in B) \int_B \left\{\tilde{\eta}(|\tilde{\theta} - x|^2) - \eta_0(|\theta_0 - x|^2)\right\}^2 dP_X(x)
$$

Now fix $\varepsilon > 0$. By consistency of $\tilde{\theta}$, we can easily find $B \subset \chi$ such that the interior of $B$ is not empty, $\mathbb{P}(X \in B) > 0$, $\{||\theta_0 - x||^2 : x \in B\} \subset A$, and $\mathbb{P}(\{||\theta - x||^2 : x \in B\} \subset A) \geq 1 - \varepsilon$ for all $n > n_0$. If we can show that $c_n < 1$ with probability tending to 1 for the above choice of $B$. Then previous two parts of the proof, we can find large constants $M_1, M_2, M_3$, and $n_0$ such that for any $n > n_0$, the following three inequalities hold:

$$
\mathbb{P}\left(\int \left\{\tilde{\eta}(|\tilde{\theta} - x|^2) - \eta_0(|\theta_0 - x|^2)\right\}^2 dP_X(x) > M_1 n^{-2/3} n^{2/9}\right) \leq \varepsilon,
\mathbb{P}(||\tilde{\theta} - \theta_0| \geq 1/M_2) \leq \varepsilon,
\text{and} \quad \mathbb{P}(c_n \geq 1 - 1/M_3) \leq \varepsilon.
$$

Combining the above two displays for the above choice of $B$ above, for all $n > n_0$, we get

$$
\mathbb{P}\left(P_{X_B}g_2^2 \lesssim n^{-2/3} n^{2/9}\right) \geq 1 - 4\varepsilon.
$$

(32)

---

5We will choose an appropriate $B$ later.
We will now show that \((32)\), implies that \(\mathbb{P}((\bar{\theta} - \theta_0) \leq n^{-2/3} n^{2/q}) \leq 4\varepsilon\).

\[
1 - 4\varepsilon \leq \mathbb{P}\left(P_{X_B} g_2^2 \leq n^{-2/3} n^{2/q}\right)
\]
\[
= \mathbb{P}\left(\int_B \left\{\eta_0(|\bar{\theta} - x|) - \eta_0(|\theta_0 - x|)\right\}^2 dP_X(x) \leq \mathbb{P}(X \in B) n^{-2/3} n^{2/q}\right)
\]
\[
\leq \mathbb{P}\left(k_0 \int_B \left\{|\bar{\theta} - x| - |\theta_0 - x|\right\}^2 dP_X(x) \leq \mathbb{P}(X \in B) n^{-2/3} n^{2/q}\right)
\]
\[
\leq \mathbb{P}\left(k_0|\bar{\theta} - \theta_0|^2 \inf_{\beta \in S_{d-1}} \int_B \left\{\beta^\top (\bar{\theta} + \theta_0 - 2x)\right\}^2 dP_X(x) \leq \mathbb{P}(X \in B) n^{-2/3} n^{2/q}\right),
\]

where \(k_0 := \min_{t \in A} |\eta'_0(t)|^2\). Recall that by continuity of \(t \mapsto \eta'_0(t)\) (by \((A4')\)), we have that \(k_0 > 0\). Since \(B\) does not depend on particular value of \(\bar{\theta}(\omega)\) and \(dP_X\) is positive everywhere on \(B\), we obtain

\[
1 - 4\varepsilon \leq \mathbb{P}\left(k_0|\bar{\theta} - \theta_0|^2 \inf_{\beta \in S_{d-1}} \int_B \left\{\beta^\top (\bar{\theta} + \theta_0 - 2x)\right\}^2 dP_X(x) \leq \mathbb{P}(X \in B) n^{-2/3} n^{2/q}\right)
\]
\[
\leq \mathbb{P}\left(|\bar{\theta} - \theta_0|^2 \leq n^{-2/3} n^{2/q}\right).
\]

Since we can do this for every \(\varepsilon > 0\), we have that \(\|\bar{\theta} - \theta_0\| = O_p\left(n^{-1/3} n^{1/q}\right)\).

We will now complete the proof by showing that \(c_n < 1\) with probability tending to 1. By continuous differentiability of \(\eta_0\) on \(||\theta_0 - x|\| : x \in B\), for all \(x \in B\), we have

\[
\eta_0(|\theta_0 - x|) = \eta_0(|\bar{\theta} - x|) + \eta'_0(|\bar{\theta} - x|)(|\theta_0 - x| - |\bar{\theta} - x|) + o(|\theta_0 - x| - |\bar{\theta} - x|).
\]

(33)

Since \(|\theta_0 - x| - |\bar{\theta} - x| = (\theta_0 - \bar{\theta})^\top (\theta_0 + \bar{\theta} - 2x)\), and both \(\Theta\) and \(\chi\) are bounded, we conclude that \(|\theta_0 - x| - |\bar{\theta} - x|\| \leq 3T|\theta_0 - \bar{\theta}|\). Letting \(r_n := |\theta_0 - \bar{\theta}|\) and by Taylor series expansion (in (33)), we conclude

\[
P\left(\left[\eta_0(|\bar{\theta} - X_B|)^2 - P\left(\eta_0(|\theta_0 - X_B|) - \eta_0(|\bar{\theta} - X_B|)^2\right)\right]^2\right)
\]
\[
= P\left(P\left(\left[\eta'_0(|\bar{\theta} - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2) + o(r_n)\right) - \eta_0(|\bar{\theta} - X_B|)^2\right)\right] + o(r_n^2)
\]
\[
= P\left(\left[P\left(\eta'_0(|\bar{\theta} - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2)\right) - \eta_0(|\bar{\theta} - X_B|)^2\right] + o(r_n^2)\right)
\]
\[
+ o(r_n^2)\left(P\left(\eta'_0(|\bar{\theta} - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2)\right)\right).
\]

Another Taylor expansion implies that

\[
P\left(\left[\eta_0(|\bar{\theta} - X_B|^2) - \eta_0(|\theta_0 - X_B|^2)\right]^2\right) = P\left(\left[P\left(\eta'_0(|\bar{\theta} - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2)\right) + o(r_n^2)\right)
\]
\[
+ o(r_n)\left(P\left(\eta'_0(|\bar{\theta} - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2)\right)\right).
\]

Recall that by \((A4')\), \(t \mapsto \eta'_0(t)\) is continuous on \(A\) and \(0 < \inf_{t \in A} |\eta'_0(t)| \leq \sup_{t \in \{|\theta_0 - x|: x \in B\}} |\eta'_0(t)| < \infty\).

Hence we deduce that \(r_n \leq P\left(P\left(\eta'_0(|\theta_0 - X_B|^2)(|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2)\right)\right) \leq r_n\). Thus, we have that

\[
c_n = \frac{P\left(\left[\eta'_0(|\theta_0 - X_B|^2)\right]^2\right) + o(r_n^2)}{P\left(\eta'_0(|\theta_0 - X_B|^2)\right)^2}.
\]

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We will now simplify parts of the numerator and denominator of \( c_n \). Observe that

\[
\left[ P\left( |\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2 \right) |\bar{\theta} - X_B| \right]^2 = 4(\bar{\theta} - \theta_0)^\top P(X_B | \bar{\theta} - X_B)P(X_B^\top | \bar{\theta} - X_B)(\bar{\theta} - \theta_0) + (|\bar{\theta}|^2 - |\theta_0|^2)^2 + 4(|\bar{\theta}|^2 - |\theta_0|^2)(\bar{\theta} - \theta_0)^\top P(X_B | \bar{\theta} - X_B),
\]

and

\[
\left[ |\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2 \right]^2 = 4(\bar{\theta} - \theta_0)^\top X_B X_B^\top (\bar{\theta} - \theta_0) + (|\bar{\theta}|^2 - |\theta_0|^2)^2 + 4(|\bar{\theta}|^2 - |\theta_0|^2)(\bar{\theta} - \theta_0)^\top X_B.
\]

Substituting these in \( c_n \), we derive

\[
c_n = \frac{4(\bar{\theta} - \theta_0)^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] P(X_B | \bar{\theta} - X_B)P(X_B^\top | \bar{\theta} - X_B) \right)(\bar{\theta} - \theta_0) + b_n + o(r_n^2)}{4(\bar{\theta} - \theta_0)^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right)(\bar{\theta} - \theta_0) + b_n + o(r_n^2)},
\]

where

\[
b_n := (|\bar{\theta}|^2 - |\theta_0|^2)^2 P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] \right) + 4(|\bar{\theta}|^2 - |\theta_0|^2) P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] \right)^2 (\bar{\theta} - \theta_0)^\top X.
\]

Thus

\[
c_n = d_n + \frac{(1 - d_n)(b_n + o(r_n^2))}{P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right)^2} + o(r_n^2),
\]

where

\[
d_n := \frac{(\bar{\theta} - \theta_0)^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right)(\bar{\theta} - \theta_0)}{(\bar{\theta} - \theta_0)^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right)(\bar{\theta} - \theta_0)},
\]

Note that \(|(b_n + o(r_n^2))/P(\eta_0^\top(\bar{\theta} - X_B)^2)[|\theta_0 - X_B|^2 - |\bar{\theta} - X_B|^2^2] + o(r_n^2)\)| \(\leq 1\), thus if we can show that \(d_n(\omega) < 1\), then we have that \(c_n(\omega) < 1\). Define \(\gamma := (\bar{\theta} - \theta_0)/|\bar{\theta} - \theta_0|\) and \(S_{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \}\). Since \(t \mapsto \eta_0(t)\) is continuous on \(A\), we have that

\[
1 - d_n = \frac{\gamma^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B - P(X_B | \bar{\theta} - X_B) \right) \left( X_B^\top - P(X_B^\top | \bar{\theta} - X_B) \right) \gamma + \gamma^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right) \gamma}{\gamma^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right) \gamma}
\]

\[
\geq \inf_{\gamma \in S_{d-1}} \frac{\gamma^\top P\left( [X_B - P(X_B | \bar{\theta} - X_B)] \left( X_B^\top - P(X_B^\top | \bar{\theta} - X_B) \right) \right) \gamma}{\gamma^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right) \gamma}
\]

\[
\geq k_0 \inf_{\gamma \in S_{d-1}} \frac{\gamma^\top P\left( \left( X_B - P(X_B | \bar{\theta} - X_B) \right) \left( X_B^\top - P(X_B^\top | \bar{\theta} - X_B) \right) \right) \gamma}{\gamma^\top P\left( [\eta_0^\top(\bar{\theta} - X_B)^2] X_B X_B^\top \right) \gamma}
\]

\[
= k_0 \frac{\inf_{\gamma \in S_{d-1}} P\left( [\gamma^\top X_B - P(\gamma^\top X_B | \bar{\theta} - X_B)] \right)^2}{\sup_{\gamma \in S_{d-1}} P \left( [\eta_0^\top(\bar{\theta} - X_B)^2] (\gamma^\top X_B)^2 \right)}
\]

= k_0.
where \( k_0 := \min_{t \in A} [\eta_0(t)]^2 \). We will now show that the numerator is strictly positive. Let \( \{e_1, \ldots, e_{d-1}, e_d\} \) be orthonormal basis of \( \mathbb{R}^d \). Then
\[
\inf_{\gamma \in \mathbb{S}_{d-1}} P \left( \left[ \gamma^\top X_B - P(\gamma^\top X_B|\tilde{\theta} - X_B) \right]^2 \right) = \min_{1 \leq i \leq d} P \left( [e_i^\top X_B - P(e_i^\top X_B|\tilde{\theta} - X_B)]^2 \right).
\]
Since \( B \) has a nonempty interior, \( \mathbb{P}(X \in B) > 0 \), and \( X \) has a Lebesgue density, we have that \( P \left( [e_i^\top X_B - P(e_i^\top X_B|\tilde{\theta} - X_B)]^2 \right) > 0 \) for all \( i \). Thus we have that \( d_n < 1 \) with probability tending to 1. And the proof is now complete.

E  Proof of Theorem 3.3

Recall that
\[
M_n(\theta) := \sum_{i=1}^n (Y_i - \hat{\eta}_\theta(|\theta - X_i|^2)) (X_i - \theta).
\]
and \( M : \Theta \mapsto \mathbb{R}^d \) is defined as
\[
M(\theta) := \int_D \left[ Y - \eta_\theta(|\theta - X|^2) \right] (X - \theta) dP_X,
\]
where
\[
\eta_\theta(u^2) := \arg\min_{\eta \in \mathcal{M}} \mathbb{Q}(\eta, \theta) = \arg\max_{\eta \in \mathcal{M}} \mathbb{E} \left( Y\eta(|\theta - X|^2) - \frac{1}{2} \eta^2(|\theta - X|^2) \right) = \mathbb{E}(\eta_\theta(|\theta_0 - X|^2)|\theta - X = u).
\]
Since \( \theta \mapsto M_n(\theta) \) is a piecewise constant function with finitely many jumps, it is not clear if \( \hat{\theta} \) (a zero crossing) exists. The proof of the theorem is split into two parts: (1) existence of a zero crossing (Section E.1) and (2) consistency of the zero crossing (Section E.2).

E.1  Existence of a zero crossing

**Theorem E.1.** Suppose the conditions of Theorem 3.3 hold. Then for all \( \epsilon > 0 \), there exists a \( N(\epsilon) \) such that
\[
\mathbb{P} \left( M_n(\cdot) \text{ has a zero crossing in } B(\theta_0, r) \right) \geq 1 - \epsilon, \quad \forall n > N(\epsilon).
\]

**Proof.** In Lemma G.3, we show that \( M(\theta) \) is a differentiable function. Hence, multivariate Taylor’s theorem implies that
\[
M(\theta) = M(\theta_0) + M'(\theta_0)(\theta - \theta_0) + o(|\theta - \theta_0|)
\]
Thus by Lemma G.1, we have
\[
M_n(\theta) = M'(\theta_0)(\theta - \theta_0) + o(|\theta - \theta_0|) + B_n(\theta), \quad (34)
\]
where \( B_n(\theta) \) is such that \( \sup_{\theta \in B(\theta_0, r)} |B_n(\theta)| = o_p(1) \). Let us define
\[
R_n(\theta) := M_n(\theta) - M'(\theta_0)(\theta - \theta_0), \quad (35)
\]
and for any \( h > 0 \), \( K_h(\cdot) \) be any (fixed) \( d \)-dimensional product differentiable kernel with finite support. Let us now define
\[
\tilde{R}_{nh}(\theta) := \int_{\mathbb{R}^d} R_n(v)K_h(v - \theta)dv,
\]
33
and
\[ \tilde{M}_{n,h}(\theta) := \int M_n(v)K_h(v - \theta)dv = M'(\theta_0)(\theta - \theta_0) + \tilde{R}_{n,h}(\theta). \]

We will first show that for every large \( n \), \( \tilde{M}_{n,h}(\theta) \) has a zero in \( B(\theta_0, r) \) with probability tending to one. We will then show that, this in turn implies that for every large \( n \), \( M_n(\theta) \) has a zero crossing in \( B(\theta_0, r) \) with probability tending to one. Consider the following reparameterization of \( \theta, \gamma := M'(\theta_0)\theta \) and \( \gamma_0 := M'(\theta_0)\theta_0 \). Now consider the function
\[ k_{n,h}(\gamma) := \gamma_0 - \tilde{R}_{n,h}([M'(\theta_0)]^{-1}\gamma). \]

Note that \( M'(\theta_0) \) is invertible and \( \tilde{R}_{n,h} \) is a continuous map. Thus by (34) and (35), we have for each small enough \( \delta > 0 \), \( h \) small enough such that
\[ \mathbb{P}(k_{n,h}(B(\gamma_0, \delta)) \subset B(\gamma_0, \delta)) \geq 1 - \epsilon, \]
for all large enough \( n \). If \( k_{n,h}(B(\gamma_0, \delta)) \subset B(\gamma_0, \delta) \), then by Brouwer’s fixed point theorem, we have that there exists a \( \gamma_{n,h} \) such that \( k_{n,h}(\gamma_{n,h}) = \gamma_{n,h} \), i.e., \( \gamma_{n,h} = \gamma_0 - \tilde{R}_{n,h}([M'(\theta_0)]^{-1}\gamma_{n,h}) \). Defining \( \theta_{n,h} := ([M'(\theta_0)]^{-1}\gamma_{n,h}) \), we get that
\[ \tilde{M}_{n,h}(\theta_{n,h}) = M'(\theta_0)(\theta_{n,h} - \theta_0) + \tilde{R}_{n,h}(\theta_{n,h}) = \gamma_0 - \gamma_{n,h} - \tilde{R}_{n,h}([M'(\theta_0)]^{-1}\gamma_{n,h}) = 0. \] (36)

For each fixed \( n \), consider the sequence of \( \theta_{n,h_i} \) as \( h_i \downarrow 0 \). By compactness of \( B(\gamma_0, \delta) \), we have that \( \theta_{n,h_i}'s \) have a limit point. Let us denote this point by \( \theta_n \).

In the following, we will show that \( M_{n,j}(\theta) \) (the \( j \)th component of \( M_n(\theta) \)) has a zero crossing at \( \theta_n \) for all \( j \leq d \). Suppose \( M_{n,j}(\theta) \) does not have a zero crossing at \( \theta_n \). Then there exists a \( \delta > 0 \), such that \( M_{n,j}(\theta) \) must have the same sign for all \( \theta \in B(\theta_n, \delta) \). Let \( M_{n,j}(\theta) > 0 \) for all \( \theta \in B(\theta_n, \delta) \). Since \( M_{n,j}(\theta) \) takes only finitely many values, there exists a \( c > 0 \) such that \( M_{n,j}(\theta) \geq c > 0 \) for all \( \theta \in B(\theta_n, \delta) \). Observe that \( \tilde{M}_{n,h_j}(\theta) = \int M_{n,j}(u)K_h(u - \theta)du > c/2 \), for all \( \theta \in B(\theta_n, \delta) \). A contradiction to (36), since \( \theta_{n,h} \in B(\theta_n, \delta) \) for large \( h \).

### E.2 Consistency of the SSCE

**Theorem E.2.** Suppose the conditions of Theorem 3.3 hold, then \( |\hat{\theta} - \theta_0| = o_p(1) \).

**Proof.** By Theorem E.1, Lemma G.1, and the Borel-Cantelli Lemma, there exists a sequence \( \{n_k\} \) such that
\[ \sum_{k=1}^{\infty} \mathbb{P}(M_{n_k} \text{ does not have a zero crossing}) < \infty \text{ and } \sup_{\theta \in B(\theta_0, r)} |M_{n_k}(\theta) - M(\theta)| \xrightarrow{a.s.} 0. \] (37)

If it exists, let \( \hat{\theta}_n \) be any zero crossing of \( M_n \). Thus for almost every \( \omega \), \( \hat{\theta}_n \) exists for all but finitely many \( k \)'s. Hence we can find a subsequence \( \{n_{k_j}\} \) (can depend on \( \omega \)) such that \( \hat{\theta}_{n_{k_j}}(\omega) \rightarrow \theta_* \) for some \( \theta_* \in B(\theta_0, r) \).

The fact that \( M(\cdot) \) is a continuous function and (37) imply that \( |M_{n_k}(\hat{\theta}_{n_{k_j}}(\omega)) - M(\theta_*)| \rightarrow 0 \) for almost all \( \omega \). By the fact that limit of zero crossing become roots of the limit, we will have that, we have that \( M(\theta_*) = 0 \). We will next show that this implies that \( \theta_* = \theta_0 \). Recall that (see (75))
\[ 0 = (\theta_0 - \theta_*)M(\theta_*) = E \left[ \text{Cov}((\theta_0 - \theta_*)^T(X - \theta), \eta_0(|\theta_0 - X|)(|\theta - X|)^2) \right] \] (38)

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In the proof of Lemma G.2, we have shown that \( \text{Cov} \left( (\theta - \theta_0)^\top (X - \theta), \eta_0(|\theta_0 - X|)^2 | \theta - X |^2 \right) \geq 0. \) Further by assumption (A5), we have that \( \text{Cov} \left( (\theta - \theta_0)^\top (X - \theta), \eta_0(|\theta_0 - X|)^2 | \theta - X |^2 \right) \) is not equal to 0 almost surely for all \( \theta \neq \theta_0. \) Thus (38) implies \( \theta_0 = \theta^*. \) As for almost all \( \omega, \{ \tilde{\theta}_{nkj} (\omega) \} \) converges to \( \theta_0, \) by Theorem 2.3.2 of Durrett (2010), we have that \( \hat{\theta}_n \) converges to \( \theta_0 \) in probability.

\[ \text{F Proof of Theorem 3.4} \]

To avoid messy and technical details, in the proof we will assume that \( \hat{\theta}_n \) exists for each \( n. \) We will first show that

\[
\mathbb{M}_n(\hat{\theta}) = \int_D \left( \eta_{\hat{\theta}_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) (X - \theta_0 - h_{\hat{\theta}_0}(|\theta_0 - X|)) dP_X
\]

\[ + \int_D (Y - \eta_{\hat{\theta}_0}(|\theta_0 - X|^2)) (X - \theta_0 - h_{\hat{\theta}_0}(|\theta_0 - X|)) d(\mathbb{P}_n - P_0) + o_p(n^{-1/2} + \hat{\theta} - \theta_0), \]

where

\[
h_{\hat{\theta}}(u) := \mathbb{E}(X||X - \theta| = u) - \theta. \quad (40)
\]

By Lemma G.1 and the fact that \( \hat{\theta} \xrightarrow{d} \theta_0, \) we have that

\[
\int_D (X - \mathbb{E}(X|X - \theta)) \left( \eta_{\hat{\theta}_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) dP_X
\]

\[ = M'(\theta_0)(\hat{\theta} - \theta_0) + o_p(\hat{\theta} - \theta_0). \]

Recall that in Lemma G.3, we show that \( M'(\theta_0) = \mathbb{E} \left( \eta_{\hat{\theta}_0}(|\theta_0 - X|^2) \text{Cov}(X|\theta_0 - X|^2) \right). \) As \( \mathbb{M}_n(\hat{\theta}) = 0 \) and \( M'(\theta_0) \) is invertible (see (A4)), we have that

\[
\sqrt{n}(\hat{\theta} - \theta_0) = -\left\{ M'(\theta_0) \right\}^{-1} \sqrt{n} \int_D (Y - \eta_{\hat{\theta}_0}(|\theta_0 - X|^2)) (X - \theta_0 - h_{\hat{\theta}_0}(|\theta_0 - X|)) d(\mathbb{P}_n - P_0)
\]

\[ + o_p(1 + \sqrt{n}(\hat{\theta} - \theta_0)), \]

We can then conclude that

\[
\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}_d(0, (M'(\theta_0))^{-1} \Sigma(M'(\theta_0))^{-1}), \]

where

\[
\Sigma = \mathbb{E} \left( \epsilon^2 [X - \mathbb{E}(X||X - \theta_0||)] [X - \mathbb{E}(X||X - \theta_0||)]^\top \right).
\]

\[ \text{Proof of (39):} \] Observe that

\[
\mathbb{M}_n(\hat{\theta}) = \int \left( Y - \tilde{\eta}_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) [X - \hat{\theta}] d\mathbb{P}_n
\]

\[ = \int \left( Y - \tilde{\eta}_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_{\hat{\theta}}(|\hat{\theta} - X|)] d\mathbb{P}_n
\]

\[ + \int \left( Y - \tilde{\eta}_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) h_{\hat{\theta}}(|\hat{\theta} - X|) - \tilde{h}_{n,\hat{\theta}}(|\hat{\theta} - X|) d\mathbb{P}_n
\]

\[ + \int \left( Y - \tilde{\eta}_{\hat{\theta}}(|\hat{\theta} - X|^2) \right) \tilde{h}_{n,\hat{\theta}}(|\hat{\theta} - X|) d\mathbb{P}_n \]

\[ \text{(41)} \]
where \( \bar{h}_{n,\theta} \) be a piecewise constant function defined as follows:

\[
\bar{h}_{n,\theta} := \begin{cases} 
    h_\theta(\tau_{i,\theta}) & \text{if } \eta_\theta(u) > \bar{\eta}_\theta(\tau_{i,\theta}) \text{ for all } u \in (\tau_{i,\theta}, \tau_{i+1,\theta}) \\
    h_\theta(s) & \text{if } \eta_\theta(s) = \bar{\eta}_\theta(\tau_{i,\theta}) \text{ for some } s \in (\tau_{i,\theta}, \tau_{i+1,\theta}) \\
    h_\theta(\tau_{i+1,\theta}) & \text{if } \eta_\theta(u) < \bar{\eta}_\theta(\tau_{i,\theta}) \text{ for all } u \in (\tau_{i,\theta}, \tau_{i+1,\theta}),
\end{cases}
\]

where \( \{\tau_{i,\theta}\}_{i=1}^n \) is the values \( \{|X_i - \theta|\}_{i=1}^n \) in increasing order. By definition of \( \bar{h}_{n,\theta} \) and the fact that \( \bar{\eta}_\theta \) is the minimizer of \( \sum_{i=1}^n (Y_i - \bar{\eta}_\theta(|\theta - X_i|^2))^2 \), we have that

\[
\int \left( Y - \bar{\eta}_\theta(|\hat{\theta} - X|^2) \right) \bar{h}_{n,\hat{\theta}}(|\hat{\theta} - X|) d\mathbb{P}_n = 0^8.
\]

In Lemma F.1, we show that

\[
\left| \int \left( Y - \bar{\eta}_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \bar{h}_{n,\theta}(|\hat{\theta} - X|)] d\mathbb{P}_n \right| = o_p(n^{-1/2} + |\hat{\theta} - \theta_0|). \tag{42}
\]

Thus by (41) and (42), we have that

\[
\mathbb{M}_n(\hat{\theta}) = \int \left( Y - \bar{\eta}_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d\mathbb{P}_n + o_p(n^{-1/2} + |\hat{\theta} - \theta_0|)
= \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d\mathbb{P}_n \tag{43} + \int \left( \eta_\theta(|\hat{\theta} - X|^2) - \bar{\eta}_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d\mathbb{P}_n + o_p(n^{-1/2} + |\hat{\theta} - \theta_0|)
\]

In Lemma F.4, we show that

\[
\int \left( \eta_\theta(|\hat{\theta} - X|^2) - \bar{\eta}_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d\mathbb{P}_n = o_p(n^{-1/2} + |\hat{\theta} - \theta_0|). \tag{44}
\]

Thus by (43) and (44), we have that

\[
\mathbb{M}_n(\hat{\theta}) = \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d\mathbb{P}_n + o_p(n^{-1/2} + |\hat{\theta} - \theta_0|)
= \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \tag{45} + \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] dP_0 + o_p(n^{-1/2} + |\hat{\theta} - \theta_0|)
\]

Observe that

\[
\int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] dP_0
= \int \left( \eta_{\theta_0}(|\theta_0 - X|^2) - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] dP_X \tag{46}
= \int \left( \eta_{\theta_0}(|\theta_0 - X|^2 - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \theta_0 - h_{\theta_0}(|\hat{\theta} - X|)] dP_X + o_p(|\hat{\theta} - \theta_0|).
\]

---

\(^8\)See (Groeneboom and Jongbloed, 2014, Page 332) for this property of the isotonic estimator; also see Groeneboom and Hendrickx (2018).
Here the last step is due to assumption (A6) and Theorem 3.1. Moreover

\[
\int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \\
= \int \left( Y - \eta_{\theta_0}(|\theta_0 - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \\
+ \int \left( \eta_{\theta_0}(|\theta_0 - X|^2) - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \\
= \int \left( Y - \eta_{\theta_0}(|\theta_0 - X|^2) \right) [X - \theta_0 - h_{\theta_0}(|\theta_0 - X|)] d(P_n - P_0) \\
+ \int \left( Y - \eta_{\theta_0}(|\theta_0 - X|^2) \right) [\theta_0 + h_{\theta_0}(|\theta_0 - X|) - \theta - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \\
+ \int \left( \eta_{\theta_0}(|\theta_0 - X|^2) - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0)
\]

By (47), (48), and (49), we have that

\[
\left| \int \left( Y - \eta_{\theta_0}(|\theta_0 - X|^2) \right) [\theta_0 + h_{\theta_0}(|\theta_0 - X|) - \theta - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \right| = o_p(n^{-1/2})
\]

and

\[
\left| \int \left( \eta_{\theta_0}(|\theta_0 - X|^2) - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0) \right| = o_p(n^{-1/2})
\]

Thus by (47), (48), and (49), we have that

\[
\int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)] d(P_n - P_0)
\]

\[
= \int_D \left( Y - \eta_{\theta_0}(|\theta_0 - X|^2) \right) (X - \theta_0 - h_{\theta_0}(|\theta_0 - X|)) d(P_n - P_0) + o_p(n^{-1/2}).
\]

Combining (45), (46), and (50), we have that (39). Thus completing the proof.

**Lemma F.1.** Suppose assumptions (A1)–(A5) hold. If \( q \geq 6 \), then

\[
\left| \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \tilde{h}_{n,\theta}(|\hat{\theta} - X|)] dP_n \right| = o_p(n^{-1/2} + |\hat{\theta} - \theta_0|)
\]

**Proof.** Let us split the quantity of interest into three parts,

\[
\int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \tilde{h}_{n,\theta}(|\hat{\theta} - X|)] dP_n
\]

\[
= \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \tilde{h}_{n,\theta}(|\hat{\theta} - X|)] d(P_n - P_0)
\]

\[
+ \int \left( Y - \eta_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \tilde{h}_{n,\theta}(|\hat{\theta} - X|)] dP_0
\]

\[
+ \int \left( \eta_\theta(|\hat{\theta} - X|^2) - \eta_\theta(|\hat{\theta} - X|^2) \right) [h_\theta(|\hat{\theta} - X|) - \tilde{h}_{n,\theta}(|\hat{\theta} - X|)] dP_X
\]

\[
= \text{I} + \text{II} + \text{III}
\]

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Let us start by providing an upper bound for $\mathbf{II}$. Observe that Lemma F.2, the Dominated Convergence Theorem, Lemma G.3, and consistency of $\hat{\theta}$, imply that

$$
\left| \int \left( Y - \eta_\theta (|\hat{\theta} - X|^2) \right) \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] dP_0 \right|
= \left| \int \left( \eta_\theta (|\theta_0 - X|^2) - \eta_\theta (|\hat{\theta} - X|^2) \right) \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] dP_X \right|
= \left| (\hat{\theta} - \theta_0) (1 + o(|\hat{\theta} - \theta_0|)) \int \eta_\theta (|\theta_0 - X|^2) \left[ X - \theta - [\eta_\theta (|\theta_0 - X|)] \right] h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) dP_X \right|
\leq \left| (\hat{\theta} - \theta_0) (1 + o(|\hat{\theta} - \theta_0|)) \int \eta_\theta (|\theta_0 - X|^2) \left[ X - \theta - [\eta_\theta (|\theta_0 - X|)] \right] h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) dP_X \right|
\leq dR \| \eta_\theta \|_\infty |(\hat{\theta} - \theta_0) (1 + o(|\hat{\theta} - \theta_0|))| \int \left| \eta_\theta (|\hat{\theta} - X|^2) - \bar{\eta}_\theta (|\hat{\theta} - X|^2) \right| dP_X
= o_p (|\hat{\theta} - \theta_0|).
$$

We will now bound $\mathbf{III}$. By Lemma F.2 and Theorem 3.1, we have that

$$
\mathbf{III} = \left| \int \left( \eta_\theta (|\hat{\theta} - X|^2) - \bar{\eta}_\theta (|\hat{\theta} - X|^2) \right) \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] dP_X \right|
\leq M_h^* \sqrt{d} \int \left( \eta_\theta (|\hat{\theta} - X|^2) - \bar{\eta}_\theta (|\hat{\theta} - X|^2) \right)^2 dP_X
= O_p (n^{-2/3} n^{2/3}).
$$

The proof will be complete if we can show that $I = o_p (n^{-1/2})$. Observe that

$$
\sqrt{n} I = \left| G_n \left[ \epsilon \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] \right] \right|
+ \left| G_n \left[ \eta_\theta (|\theta_0 - X|^2) - \bar{\eta}_\theta (|\hat{\theta} - X|^2) \right] \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] \right|.
$$

In Lemma F.2, we show that that for any $\delta > 0$

$$
\mathbb{P} \left( \left| G_n \left[ \epsilon \left[ h_\theta (|\hat{\theta} - X|) - \bar{h}_{n,\hat{\theta}} (|\hat{\theta} - X|) \right] \right] \right| \geq \delta \right) = o(1).
$$

In Lemma F.3, we show that the second term on the right side of (51) is also $o_p (1)$. □

**Lemma F.2.** Suppose assumptions (A1)–(A5) hold. Let $\{h_{\theta,i}\}_{i=1}^d$ denote the $d$ components of $d$-dimensional function $h_\theta$, i.e., $h_\theta(\cdot) := (h_{\theta,1}(\cdot), \ldots, h_{\theta,d}(\cdot))$. There exists constant $M_h$ and $M_h^*$ such that

$$
\sup_{\theta \in B(\theta_0, r)} \| h_\theta \|_{2,\infty} \leq M_h, \quad \sup_{\theta \in B(\theta_0, r)} \sum_{i=1}^d TV (h_{\theta,i}) \leq M_h,
$$

and

$$
|h_\theta (u) - \bar{h}_{n,\hat{\theta}} (u)| \leq M_h^* \sqrt{d} |\eta_\theta (u) - \bar{\eta}_\theta (u)| \quad \forall \theta \in B(\theta_0, r) \text{ and } u \in \mathcal{D}.
$$

Moreover, let

$$
\mathcal{H} := \{ h(|\theta - x|) | h : \mathcal{D} \to \mathbb{R}^d, h_i = f_{i,1} - f_{i,2}, f_{i,1}, f_{i,2} \in \mathcal{M}^4 M_h, \text{ and } \theta \in B(\theta_0, r) \}.
$$
Then
\[
\log N_\|\nu, \bar{H}, \|_{2,P_X} \leq \frac{8dAM_h}{\nu}, \quad (54)
\]

Finally
\[
P \left( \left\| G_n \left[ e \left[ h_{\hat{\theta}}((\hat{\theta} - X)) - \bar{h}_{n,\hat{\theta}}((\hat{\theta} - X)) \right] \right] \right\| \geq \delta \right) = o(1) \quad (55)
\]

Proof. Recall that by Lemma C.1, we have that
\[
\log N_\|\nu, \bar{H}, \|_{2,P_X} \leq \frac{8dAM_h}{\nu}.
\]
Thus by stability property of Donsker classes, we have that
\[
\log N_\|\nu, \bar{H}, \|_{2,P_X} \leq \frac{8dAM_h}{\nu}.
\]
As \( \sup_{x \in X} |x| \leq R \), it is easy to see that \( \sup_{\theta \in B(\theta_0, r)} \| h_\theta \|_{2,\infty} \leq R + \| \theta \| + r \). The proof of finite total variation follows from the proof of Lemma G.3 and Lemma F.4 of Balabdaoui et al. (2019b). Recall that \( t \mapsto \eta_0(t) \) is strictly decreasing and continuously differentiable, then by Lemma 2.2, we can conclude that \( \eta_0(\cdot) \) is bounded away from zero for all \( \theta \in B(\theta_0, r) \). We have also assumed that \( u \mapsto h_\theta(u) \) has a totally bounded derivative (see (A6)). Thus techniques used in (10.64) of Groeneboom and Jongbloed (2014) imply that there exist constant \( M_h^* \) such that

\[
|h_\theta(u) - \bar{h}_{n,\hat{\theta}}(u)| \leq M_h^* \sqrt{\delta} \eta_\theta(u) - \bar{\eta}_\theta(u) \quad \forall \theta \in B(\theta_0, r) \text{ and } u \in D.
\]

Observe that \( h_\theta - \bar{h}_{\hat{n},\theta} \in \bar{H} \), as both \( h_\theta \) and \( \bar{h}_{n,\theta} \) satisfy (52) and by Lemma F.5 of Balabdaoui et al. (2019b), we have that \( h_\theta - \bar{h}_{n,\theta} = (f_{1,1} - f_{1,2}, \ldots, f_{d,1} - f_{d,2}) \), for some functions \( f_{i,1}, f_{i,2} \in \mathcal{M}_h^* \) for all \( i \in \{1, \ldots, d\} \). Recall that by Theorem 3.1 and (53), we have that
\[
\sup_{\theta \in B(\theta_0, r)} \| h_\theta(|\theta - X|) - \bar{h}_{n,\theta}(|\theta - X|) \|_{2,P_X} = O_p(dn^{-1/3}n^{1/4}).
\]

Let us define
\[
\hat{H}_n := \{ h(x) : h \in \bar{H}, \text{ and } \| h(X) \|_{2,P_X} \leq \zeta_n \}, \quad \text{where} \quad \zeta_n := d \log nn^{-1/3}n^{1/4}. \quad (56)
\]
Then by Chebyshev’s inequality
\[
\begin{align*}
P \left( \left\| G_n \left[ e \left[ h_{\hat{\theta}}((\hat{\theta} - X)) - \bar{h}_{n,\hat{\theta}}((\hat{\theta} - X)) \right] \right] \right\| \geq \delta \right) \\
\leq P \left( \sup_{\theta \in B(\theta_0, r)} \left\| G_n \left[ e \left[ h_{\theta}(|\theta - X|) - \bar{h}_{n,\theta}(|\theta - X|) \right] \right] \right\| \geq \delta \right) \\
\leq P \left( \sup_{h(\cdot) \in \hat{H}_n} \| G_n \epsilon h \| \geq \delta \right) + P \left( \sup_{\theta \in B(\theta_0, r)} \| h_\theta(|\theta - X|) - \bar{h}_{n,\theta}(|\theta - X|) \|_{2,P_X} \geq \zeta_n \right) \\
\leq 2\delta^{-1}\sqrt{d} \sum_{i=1}^d E \left( \sup_{h(\cdot) \in \hat{H}_n} |G_n \epsilon h_i| \right) + o(1).
\end{align*}
\]

\footnote{A real-valued uniformly bounded function of bounded variation can be written as difference of two bounded and monotone functions.}
By definition of \( \bar{\mathcal{H}} \), we have that \( \sup_{h \in \bar{\mathcal{H}}} \| h \|_{2, \infty} \leq 2M_h \). Thus by arguments similar to those in the proof of Theorem 3.1 and Lemma F.4 of Kuchibhotla et al. (2017), we have that for every \( i \in \{1, \ldots, d\} \)

\[
\mathbb{E}\left( \sup_{h(\cdot) \in \mathcal{H}_n} |G_n \epsilon h_i| \right) \leq \mathbb{E}\left( \sup_{h(\cdot) \in \mathcal{H}_n} |G_n \epsilon h_i| \right) + 2\frac{M_h C_\epsilon}{\sqrt{n}},
\]

(58)

where \( \epsilon_i := \epsilon_i \mathbb{1}_{\{|\epsilon_i| \leq C_\epsilon\}} \) and \( C_\epsilon \leq n^{1/q} \). Thus by Lemma F.7 of Kuchibhotla et al. (2017), we have that

\[
\mathbb{E}\left( \sup_{h(\cdot) \in \mathcal{H}_n} |G_n \epsilon h_i| \right) \leq \sigma \sqrt{4dAM\zeta_n^{1/2}} \left( 1 + \frac{\sigma \sqrt{4dAM\zeta_n^{1/2} C_\epsilon 2M_h}}{\zeta_n^{1/2}} \right) = o(1).
\]

(59)

Thus combining (57), (58), and (59), we have (55).

\[\square\]

**Lemma F.3.** Suppose the assumptions of Theorem 3.4 hold. If \( q \geq 6 \),

\[
\left| G_n \left[ \eta_{\theta_0}(|\theta_0 - X|^2) - \bar{\eta}_\theta(\hat{\theta} - X|^2) \right] \left[ h_\theta(\hat{\theta} - X) - \bar{h}_{\theta, \hat{\theta}}(\hat{\theta} - X) \right] \right| = o_p(1).
\]

Proof. Let \( \mathcal{F}_n^\theta := \{ \eta_0 \circ \theta_0 - f \circ \theta : \theta \in \mathcal{B}(\theta_0, r), f \in \mathcal{M}^{B_n}, \text{ and } \| \eta_0 \circ \theta_0 - f \circ \theta \|_{P_X} \leq \zeta_n \} \),

(60)

where \( \zeta_n \) is defined in (56) and

\[
B_n := n^{1/q} \log n.
\]

(61)

Then by Lemma C.1 we have that

\[
\log N(\nu, \mathcal{F}_n^\theta, \| \cdot \|_{P_X}) \leq \frac{A(B_n + M_1)}{\nu},
\]

Let \( \mathcal{F}_n^\theta \times \bar{\mathcal{H}} := \{ \text{all } f \in \mathcal{F}_n^\theta \} \). By (54) and Lemma 9.25 of Kosorok (2008), we have that

\[
\log N(\nu, \mathcal{F}_n^\theta \times \bar{\mathcal{H}}, \| \cdot \|_{2, P_X}) \leq \frac{A(4dM_h + M_1 + B_n)}{\nu}.
\]

Define \( A_n := \sup_{f \in \mathcal{F}_n^\theta \times \bar{\mathcal{H}}} \| f \|_{2, P_X} \). As \( \bar{\mathcal{H}} \) is uniformly bounded by \( 2M_h \), by definition of \( \mathcal{F}_n^\theta \), we have that

\[
A_n = \sup_{f \in \mathcal{F}_n^\theta \times \bar{\mathcal{H}}} \| f \|_{2, P_X} \leq 2\sqrt{dM_h} \sup_{f \in \mathcal{F}_n^\theta} \| f \|_{P_X} \leq 2\sqrt{d}M_h \zeta_n.
\]

Recall that by (60), Theorem 3.1, and definition of \( \bar{\mathcal{H}} \), we have that

\[
\mathbb{P}\left( \eta_{\theta_0}(|\theta_0 - X|^2) - \bar{\eta}_\theta(\hat{\theta} - X|^2) \in \mathcal{F}_n^\theta \right) = o(1) \text{ and } \mathbb{P}\left( [h_\theta(|\hat{\theta} - X|) - \bar{h}_{\theta, \hat{\theta}}(|\hat{\theta} - X|)] \in \bar{\mathcal{H}} \right) = 1.
\]

Thus

\[
\mathbb{P}\left( \left| G_n \left[ \eta_{\theta_0}(|\theta_0 - X|^2) - \bar{\eta}_\theta(\hat{\theta} - X|^2) \right] \left[ h_\theta(|\hat{\theta} - X|) - \bar{h}_{\theta, \hat{\theta}}(|\hat{\theta} - X|) \right] \right| > \delta \right)
\]

\[
\leq \mathbb{P}\left( \sup_{f \in \mathcal{F}_n^\theta \times \bar{\mathcal{H}}} |G_n f| > \delta \right) + \mathbb{P}\left( \left[ \eta_{\theta_0}(|\theta_0 - X|^2) - \bar{\eta}_\theta(\hat{\theta} - X|^2) \right] \left[ h_\theta(|\hat{\theta} - X|) - \bar{h}_{\theta, \hat{\theta}}(|\hat{\theta} - X|) \right] \notin \mathcal{F}_n^\theta \times \bar{\mathcal{H}} \right)
\]

\[
= \mathbb{P}\left( \sup_{f \in \mathcal{F}_n^\theta \times \bar{\mathcal{H}}} |G_n f| > \delta \text{ and } \right) + o(1).
\]

(62)
The proof is now complete, as by Lemma 3.4.2 of van der Vaart and Wellner (1996) (for uniformly bounded function classes) and (62) imply that

\[
\mathbb{P} \left( \left| \mathcal{G}_n \left[ \eta_0(\theta_0 - X^2) - \eta_\hat{\theta}(\hat{\theta} - X^2) \right] [h_\hat{\theta}(\hat{\theta} - X) - \hat{h}_{n, \hat{\theta}}(\hat{\theta} - X)] \right| > \delta \right) \\
\leq \delta^{-1} \sqrt{\mathbb{E}} \sum_{i=1}^{n} \left( \sup_{f \in \mathcal{F}_n} \left| \mathcal{G}_n \left[ f_i \right] \right| \right) + o(1) \\
= \left( (4dM_h + M_1 + B_n)A_n \right)^{1/2} \left( 1 + \frac{((4dM_h + M_1 + B_n)A_n)^{1/2}}{\sqrt{n}A_n} \right) + o(1) \\
\lesssim \left( B_nA_n \right)^{1/2} + \frac{B_n^2}{\sqrt{n}A_n} + o(1) \\
= O \left( n^{-1/6} \log n + (\log n)^2 n^{1/4} n^{-1/6} \right) + o(1) = o(1). \quad \square
\]

**Lemma F.4.** Suppose the assumptions of Theorem 3.4 hold. If \( q \geq 6 \), we have that

\[
\int \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) [X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X)] d\mathbb{P}_n = o_p(n^{-1/2} + |\hat{\theta} - \theta_0|).
\]

**Proof.** First note that

\[
\int \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) [X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X)] d\mathbb{P}_0 \\
= \mathbb{E} \left[ \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) \left( X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X) \right) \right] \\
= \mathbb{E} \left[ \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) \mathbb{E} \left[ \left( X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X) \right) \left( X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X) \right) \right] \right] = 0.
\]

Thus, we have

\[
\int \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) [X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X)] d\mathbb{P}_n \\
= \int \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) [X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X)] d(\mathbb{P}_n - P_0) \quad (63)
\]

Let us define

\[
\mathcal{G}_n^* := \left\{ \left( \eta_\theta(\theta - X^2) - f(|\theta - X|) \right) (X - \theta - h_\theta(|\theta - X|)) : f \in \mathcal{M}_{B_n}, \right\}
\]

where \( B_n \) and \( \zeta_n \) are defined as in (61). By Theorem 3.1, we have that

\[
\mathbb{P} \left( \left( \eta_\hat{\theta}(\hat{\theta} - X^2) - \tilde{\eta}_\hat{\theta}(\hat{\theta} - X^2) \right) (X - \hat{\theta} - h_\hat{\theta}(\hat{\theta} - X)) \notin \mathcal{G}_n^* \right) = o(1). \quad (64)
\]

Then by (27), (26), (54), and Lemma 9.25 of Kosorok (2008), we have that

\[
\log N_{\nu}(\nu, \mathcal{G}_n^*, \| \cdot \|_{2, P_X}) \leq \frac{4A M_h}{\nu} \frac{A(M_1 + B_n)}{\nu} \leq \frac{B_n}{\nu}. \quad (65)
\]
Combining (63), (64), (65), and Theorem 3.4.2 of van der Vaart and Wellner (1996), we get
\[
\mathbb{P} \left( \mathcal{G}_n \left[ (\eta_0(|\hat{\theta} - X|) - \tilde{\eta}_0(|\hat{\theta} - X|)^2) (X - \hat{\theta} - h_\theta(|\hat{\theta} - X|)) \right] \geq \delta \right) \\
\leq \delta^{-1} \sqrt{d} \sum_{i=1}^{d} \mathbb{E} \left( \sup_{\varphi \in \mathcal{G}_n^*} \mathbb{E}_{G_{n}^{i}} \right) + o(1) \\
\lesssim \delta^{-1} \sqrt{d} B^{1/2}_{n} \zeta^{1/2} \left( 1 + \frac{B^{1/2}_{n} \zeta^{1/2}}{\sqrt{n} \zeta^{2}} (B_n + M_1)R \right) + o(1).
\]

(66)

The proof is now complete, as it is very easy to see that the first term is \( o(1) \) if \( q \geq 6 \) as
\[
B_n \zeta_n = n^{-1/3} n^{2/q} (\log n)^{2} \text{ and } \frac{B_n^{2}}{\sqrt{n} \zeta_n^{2}} = \log n \log^2 n.
\]

\[ \square \]

**Lemma F.5.** Suppose the assumptions of Theorem 3.4 hold, then
\[
\left| \mathbb{G}_n (Y - \eta_0(|\theta_0 - X|^2)) \left[ \hat{\theta} + h_{\theta_0}(|\theta_0 - X|) - \hat{\theta} + h_\theta(|\hat{\theta} - X|) \right] \right| = o_p(1).
\]

**Proof.** For the proof of this lemma, let us define
\[
g_\theta(\cdot) := \mathbb{E} \left( X \mid X - \theta = \cdot \right).
\]

By definition of (40), we have that
\[
\theta_0 + h_{\theta_0}(|\theta_0 - x|) - [\theta + h_\theta(|\hat{\theta} - x|)] = g_\theta_0(|\theta_0 - x|) - g_\theta(|\hat{\theta} - x|).
\]

By Lemma F.2 and the fact that \( \chi \) and \( \Theta \) is bounded, we have that
\[
\sup_{\theta \in B(\theta_0, r)} \| g_\theta \|_{2, \infty} \leq T, \quad \sup_{\theta \in B(\theta_0, r)} \sum_{i=1}^{n} TV(g_{\theta, i}) \leq M_h,
\]
and
\[
\| g_\theta - g_{\theta_1} \|_{2, \infty} \leq M_h |\theta - \theta_1| \quad \forall \theta, \theta_1 \in B(\theta_0, r).
\]

By arguments similar to the proof of (54) or proof of (26) in Lemma C.1 and the fact that a bounded function with finite total variation can be written as a difference of two bounded and monotone functions (also see Lemma F.5 of Balabdaoui et al. (2019b)), we can show that
\[
\log N_0(\nu, \{ g_\theta(|X - \theta|) - g_{\theta_0}(|X - \theta_0|) : \theta \in B(\theta_0, r) \}, \| \cdot \|_{2, P_X}) \lesssim \frac{(M_h + T)}{\eta}.
\]

Let \( \{a_n\} \) be a sequence such that \( a_n \to 0, a_n \log n \to \infty, \) and \( |\hat{\theta} - \theta_0| = o_p(a_n) \). Thus for every \( \zeta > 0 \), there exist \( N_\zeta \) and \( C_\zeta \) such that \( \mathbb{P}(|\hat{\theta} - \theta_0| \geq C_\zeta a_n) \leq \zeta/2 \) for all \( n > N_\zeta \). Now for any \( n > N_\zeta \), let us define
\[
B_n^\zeta := \left\{ g_\theta(|\theta - x|) - g_{\theta_0}(|\theta_0 - x|) : \theta \in B(\theta_0, C_\zeta a_n) \right\}
\]
\[ \text{Note that Theorem E.2 guarantees the existence of such a sequence.} \]
For all $n > N_\zeta$, by Chebyshev’s inequality, we have
\[
\mathbb{P}\left(\left|G_n \varepsilon (g_{\theta_0}(|\theta_0 - X|)) - g_\hat{\theta}(|\hat{\theta} - X|)\right| \geq \delta\right)
\leq \mathbb{P}\left(\left|G_n \varepsilon (g_{\theta_0}(|\theta_0 - X|)) - g_\hat{\theta}(|\hat{\theta} - X|)\right| \geq \delta \text{ and } \hat{\theta} \in B(\theta_0, C_\zeta a_n)\right) + \mathbb{P}\left(\hat{\theta} \notin B(\theta_0, C_\zeta a_n)\right)
\leq \mathbb{P}\left(\sup_{g \in \mathcal{B}_n} |G_n \varepsilon g| \geq \delta\right) + \frac{\zeta}{2}.
\] (67)

Observe that $\sup_{g \in \mathcal{B}_n} ||g||_{2, \infty} \leq 4T$ as $\sup_{x \in \chi} |x| \leq T$. Thus by arguments similar to those in the proof of Theorem 3.1 and Lemma F.4 of Kuchibhotla et al. (2017), we have that for every $i \in \{1, \ldots, d\}$
\[
\mathbb{E}\left(\sup_{g \in \mathcal{B}_n} |G_n \varepsilon g_i|\right) \leq \mathbb{E}\left(\sup_{g \in \mathcal{B}_n} |G_n \varepsilon g_i|\right) + \frac{4RC_\varepsilon}{\sqrt{n}},
\] (69)
where $\bar{\varepsilon}_i := \varepsilon_i \mathbb{1}_{\{|\varepsilon_i| \leq C_\varepsilon\}}$ and $C_\varepsilon \lesssim n^{1/4}$. We will now use Lemma F.7 of Kuchibhotla et al. (2017) to bound the first term on the right of (69). Observe that for every $i \in \{1, \ldots, d\}$
\[
\sup_{g \in \mathcal{B}_n} ||g||_{2, P_X} \leq M^*_h \sup_{\theta \in B(\theta_0, C_\zeta a_n)} |\theta - \theta_0| = M^*_h C_\zeta a_n.
\]
Thus by Lemma F.7 of Kuchibhotla et al. (2017), we have that
\[
\mathbb{E}\left(\sup_{g \in \mathcal{B}_n} |G_n \varepsilon g_i|\right) \leq C \left(\left(M_h + T\right)M^*_h C_\zeta a_n\right)^{1/2} \left(1 + \frac{\sigma \left(\left(M_h + T\right)M^*_h C_\zeta a_n\right)^{1/2}}{\sqrt{n}}\right)
\]
\[
\lesssim C_n^{1/2} + \frac{C_\varepsilon}{\sqrt{n}a_n}.
\]

The proof is complete by combining (67), (68), (69), and the facts that $C_n \lesssim n^{1/4}$, $a_n \to 0$, and $\log na_n \to \infty$.

**Lemma F.6.** Suppose the assumptions of Theorem 3.4 hold, then
\[
\left|G_n \left[\eta_{\theta_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2)\right] \left[X - \hat{\theta} - h_{\hat{\theta}}(|\hat{\theta} - X|)\right]\right| = o_p(1).
\]

**Proof.** Observe that $\|X - \hat{\theta} - h_{\hat{\theta}}(|\hat{\theta} - X|)\|_{2, \infty} \leq 4\sqrt{d}T$. Thus
\[
\left\|\left[\eta_{\theta_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2)\right] \left[X - \hat{\theta} - h_{\hat{\theta}}(|\hat{\theta} - X|)\right]\right\|_{2, \infty} \leq \sqrt{d}TM_1.
\] (70)

Using arguments similar to Lemma F.3 of Balabdaoui et al. (2019b) and Lemma G.3, we have that there exists a constant $C$ such that
\[
\sup_{x \in \chi} |\eta_{\theta_0}(|\theta_0 - x|) - \eta_{\theta}(|\theta - x|)| \leq C|\theta_0 - \theta| \quad \forall \theta \in B(\theta_0, r).
\]

Thus, we have that
\[
\left\|\left[\eta_{\theta_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2)\right] \left[X - \hat{\theta} - h_{\hat{\theta}}(|\hat{\theta} - X|)\right]\right\|_{2, P_X}
\leq \sqrt{d}2R \left\|\eta_{\theta_0}(|\theta_0 - X|^2) - \eta_{\hat{\theta}}(|\hat{\theta} - X|^2)\right\|_{P_X}
\leq 2RC|\hat{\theta} - \theta_0|.
\] (71)
Let us define
\[ \mathcal{K} := \left\{ f(\|\theta - x\|)(x - \theta - g_1(\|\theta - x\|) + g_2(\|\theta - x\|)) : f \in \mathcal{M}^{M_1}, g_1, g_2 : \mathcal{D} \to \mathbb{R}^d, \right. \]
\[ \left. g_{i,j} \in \mathcal{M}^{2M_h} \text{ for } i \in \{1, 2\} \text{ and } j \in \{1, \ldots, d\}, \text{ and } \theta \in B(\theta_0, R) \right\} \]

Using arguments similar to the proof of (54), we can show that
\[ [\eta_0(\|\theta_0 - X\|^2) - \eta_0(\|\hat{\theta} - X\|^2)] [X - \hat{\theta} - h_\hat{\theta}(\|\hat{\theta} - X\|)] \in \mathcal{K}. \]

Furthermore, by stability of Donsker classes, we have that
\[ \log N_\|\| (\nu, \mathcal{K}, \|\cdot\|_2, P_X) \lesssim \frac{M_1 + dM_h}{\nu}. \]

Let us now define \( \mathcal{K}_n := \{ f \in \mathcal{K} : \|f\|_2, P_X \leq 2RCa_n \} \), where \( \{a_n\} \) be a sequence such that \( a_n \to 0 \), \( a_n \log n \to \infty \), and \( |\hat{\theta} - \theta_0| = o_p(a_n) \). By arguments similar to (66), we have that
\[ \mathbb{P} \left( \left| \mathbb{E} \left( \sup_{g \in \mathcal{K}_n} |G_ng_i| \right) + \mathbb{P} \left( [\eta_0(\|\theta_0 - X\|^2) - \eta_0(\|\hat{\theta} - X\|^2)] [X - \hat{\theta} - h_\hat{\theta}(\|\hat{\theta} - X\|)] \notin \mathcal{K}_n \right) \right) \leq \delta^{-1} \sqrt{d} \sum_{i=1}^{d} \mathbb{P} \left( \left| \mathbb{E} \left( \sup_{g \in \mathcal{K}_n} |G_ng_i| \right) + \mathbb{P} \left( [\eta_0(\|\theta_0 - X\|^2) - \eta_0(\|\hat{\theta} - X\|^2)] [X - \hat{\theta} - h_\hat{\theta}(\|\hat{\theta} - X\|)] \notin \mathcal{K}_n \right) \right) \]

However, by (71) and definition of \( a_n \), we have that
\[ \mathbb{P} \left( [\eta_0(\|\theta_0 - X\|^2) - \eta_0(\|\hat{\theta} - X\|^2)] [X - \hat{\theta} - h_\hat{\theta}(\|\hat{\theta} - X\|)] \notin \mathcal{K}_n \right) = o(1). \]

We will now bound the expectation at the right of (72). By Lemma 3.4.2 of van der Vaart and Wellner (1996), (70), and (71), we have that
\[ \mathbb{E} \left( \sup_{g \in \mathcal{K}_n} |G_ng_i| \right) \leq ((M_1 + dM_h)Ra_n)^{1/2} \left( 1 + \frac{(M_1 + dM_h)Ra_n}{{\sqrt{n}}(Ra_n)^2} \sqrt{dRM_1} \right) = o(1), \]

where the last inequality follows from the definition of \( a_n \).

\[ \square \]

G Auxiliary lemmas for Appendices E and F

Lemma G.1. Suppose assumptions (A1)–(A3) hold, then
\[ \sup_{\theta \in B(\theta_0, r)} \left| \mathbb{M}_n(\theta) - M(\theta) \right| = o_p(1). \]

Proof. For any \( \theta \in B(\theta_0, r) \), we have
\[ \mathbb{M}_n(\theta) = n^{-1} \sum_{i=1}^{n} [Y_i - \eta_0(\|\theta - X_i\|^2)] (X_i - \theta) + n^{-1} \sum_{i=1}^{n} [\eta_0(\|\theta - X_i\|^2) - \bar{\eta}_0(\|\theta - X_i\|^2)] (X_i - \theta) \]
\[ = M(\theta) + \int [y - \eta_0 \circ \theta(x)] (x - \theta) d(\mathbb{P}_n - P_0)(x, y) \]
\[ + \int [\eta_0 \circ \theta(x) - \bar{\eta}_0 \circ \theta(x)] (x - \theta) d(\mathbb{P}_n - P_0)(x, y) \]
\[ + \int [\eta_0 \circ \theta(x) - \bar{\eta}_0 \circ \theta(x)] (x - \theta) dP_0(x, y) \]
By (12) of Theorem 3.1, we have

\[
\sup_{\theta \in B(\theta_0, \delta_0)} \int [\eta_0 \circ \theta(x) - \tilde{\eta}_0 \circ \theta(x)] (x - \theta) dP_X = O_p(n^{-1/3} n^{1/q})
\]

In the following we will now prove that

\[
\sup_{\theta \in B(\theta_0, r)} \left| \int [y - \eta_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| = O_p(n^{-1/2}) \tag{73}
\]

\[
\sup_{\theta \in B(\theta_0, r)} \left| \int [\eta_0 \circ \theta(x) - \tilde{\eta}_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| = O_p(n^{-3/6} n^{1/q}) \tag{74}
\]

First, observe that

\[
\sup_{\theta \in B(\theta_0, r)} \left| \int [y - \eta_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| \leq \sup_{\theta \in B(\theta_0, r)} \left| \int [\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| + \sup_{\theta \in B(\theta_0, r)} \left| \int \epsilon(x - \theta) d(P_n - P_0)(x, y) \right|
\]

\[
\leq \sup_{\theta \in B(\theta_0, r)} \left| \int [\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| + |P_n \epsilon| \sup_{\theta \in B(\theta_0, r)} |\theta|
\]

\[
\leq \sup_{\theta \in B(\theta_0, r)} \left| \int [\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)] (x - \theta) d(P_n - P_0)(x, y) \right| + O_p(n^{-1})
\]

\[
= \frac{1}{\sqrt{n}} \sup_{\theta \in B(\theta_0, r)} \left| G_n \left[ (\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)) (x - \theta) \right] \right| + O_p(n^{-1}).
\]

The proof of (73), will be complete if we can show that \(\sup_{\theta \in B(\theta_0, r)} \left| G_n \left[ (\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)) (x - \theta) \right] \right| = O_p(1)\). We show this next. Note that

\[
\sup_{\theta \in B(\theta_0, r)} \| \eta_0 \circ \theta_0 - \eta_0 \circ \theta \| \leq \sup_{\theta \in B(\theta_0, r)} \| \eta_0 \circ \theta_0 - \eta_0 \circ \theta \|_\infty \leq M_1,
\]

and by Lemma C.1, we have that

\[
\log N_1(\nu, \{\eta_0 \circ \theta_0 - \eta_0 \circ \theta : \theta \in B(\theta_0, r)\}, \| \cdot \|) \leq \frac{A_0 M_1}{\nu},
\]

where \(A_0\) is a constant depending only on \(\chi\). Now by Theorem 3.4.2 of van der Vaart and Wellner (1996), we have

\[
E \left( \sup_{\theta \in B(\theta_0, r)} \left| G_n \left[ (\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)) (x - \theta) \right] \right| \right)
\]

\[
\leq d \sum_{j=1}^d E \left( \sup_{\theta \in B(\theta_0, r)} \left| G_n \left[ (\eta_0 \circ \theta_0(x) - \eta_0 \circ \theta(x)) (x_j - \theta_j) \right] \right| \right)
\]

\[
\leq d \sqrt{A_0 M_1^{3/2}} \left( 1 + \sqrt{\frac{A_0 M_1^{3/2} M_1}{M_1^2 n}} \right) \leq d \sqrt{A_0 M_1^2}
\]
We will now establish (74). Fix any \( \delta > 0 \). Let \( C_1 \) and \( C_2 \) be constants such that

\[
P \left( \sup_{\theta \in B(\theta_0, \delta_0)} \| \tilde{\eta}_\theta \|_\infty \geq C n^{1/q} \right) \leq \delta / 4.
\]

and

\[
P \left( \sup_{\theta \in B(\theta_0, \delta_0)} \int \{ \tilde{\eta}_\theta(|\theta - x|^2) - \eta_\theta(|\theta - x|^2) \}^2 dP(x) \geq C_2 n^{-2/3} n^{2/3} \right) \leq \delta / 4.
\]

Note that such constants exist by Theorem 3.1. Now consider the following class of functions

\[
B_1(\gamma) := \left\{ \eta \circ \theta - \eta_\theta \circ \theta : \eta \in \mathcal{M}^{C_1 n^{1/q}}, \ \theta \in B(\theta_0, r), \right. \\
\left. \text{and } \int \{ \tilde{\eta}_\theta(|\theta - x|^2) - \eta_\theta(|\theta - x|^2) \}^2 dG(x) \leq \gamma \right\}.
\]

Then by Lemma C.1, we have that

\[
\log N_{\eta}(\nu, B_1(\nu), \| \cdot \|) \leq \frac{2A(M_1 + C_1 n^{1/q})}{\nu}.
\]

Observe that

\[
P \left( \frac{1}{n^{1/2}} \sup_{\theta \in B(\theta_0, r)} \left| \int [\eta_\theta \circ \theta(x) - \tilde{\eta}_\theta \circ \theta(x)] (x - \theta) d(\nu_n - P_\theta)(x, y) \right| \geq C_3 n^{1/q} n^{-1/6} \right)
\]

\[
\leq P \left( \sup_{\theta \in B(\theta_0, r)} \left| \mathbb{G}_n \left[ (\eta_\theta \circ \theta(x) - \tilde{\eta}_\theta \circ \theta(x)) (x - \theta) \right] \right| \geq C_3 n^{1/q} n^{-1/6}, \sup_{\theta \in B(\theta_0, \delta_0)} \| \tilde{\eta}_\theta \|_\infty \leq C n^{1/q}, \right.
\]

\[
\left. \text{and } \sup_{\theta \in B(\theta_0, \delta_0)} \int \{ \tilde{\eta}_\theta(|\theta - x|^2) - \eta_\theta(|\theta - x|^2) \}^2 dG(x) \leq C_2 n^{-2/3} n^{1/q} \right) + \delta / 2
\]

\[
\leq P \left( \sup_{f_1 \in B_1(n^{2/3} n^{1/q})} \left| \mathbb{G}_n \left[ (x - \theta) f \right] \right| \geq C_3 n^{1/q} n^{-1/6} \right) + \delta / 2
\]

\[
\leq \frac{1}{C_3 n^{1/q} n^{-1/6}} \mathbb{E} \left( \sup_{f_1 \in B_1(n^{2/3} n^{1/q})} \left| \mathbb{G}_n \left[ (x - \theta) f \right] \right| \right) + \delta / 2
\]

\[
\leq \frac{1}{C_3 n^{1/q} n^{-1/6}} \sum_{j=1}^d \mathbb{E} \left( \sup_{f_1 \in B_1(n^{2/3} n^{1/q})} \left| \mathbb{G}_n \left[ (x_j - \theta_j) \right] \right| \right) + \delta / 2
\]

Moreover, by Theorem 3.4.2 of van der Vaart and Wellner (1996), we have that

\[
\mathbb{E} \left( \sup_{f_1 \in B_1(n^{2/3} n^{1/q})} \left| \mathbb{G}_n \left[ (x_j - \theta_j) \right] \right| \right)
\]

\[
\leq J_{\eta}(C_2 n^{-2/3} n^{1/q}, B_1, \| \cdot \|) \left( 1 + \frac{J_{\eta}(C_2 n^{-2/3} n^{1/q}, B_1, \| \cdot \|)}{\sqrt{n} [C_2 n^{-2/3} n^{1/q}]^2} C_1 n^{1/q} \right)
\]

\[
\lesssim \sqrt{n^{2/q} n^{-1/3}} \left( 1 + \frac{\sqrt{n^{2/q} n^{-1/3} n^{1/q}}}{\sqrt{n^{2/q} n^{-2/3}}} \right)
\]

\[
\lesssim \sqrt{n^{2/q} n^{-1/3}} + \frac{n^{3/q} n^{-1/3}}{\sqrt{n^{2/q} n^{-2/3}}}
\]

\[
\lesssim n^{1/q} n^{-1/6}.
\]
We have now proved (74), as
\[
P\left( \sup_{\theta \in B(\theta_0, \delta)} \left| \int [\eta_\theta \circ \theta(x) - \tilde{\eta}_\theta \circ \theta(x)] (x - \theta) d(\mathbb{P}_n - P_0) (x, y) \right| \geq C_3 n^{1/4} n^{-4/6} \right) \\
\leq \frac{1}{C_3 n^{1/4} n^{-1/6}} n^{1/4} n^{-1/6} \leq \frac{1}{C_3}.
\]

\[\square\]

G.1 Property of \( M(\theta) \)

The following two lemmas establish some properties of \( \theta \mapsto M(\theta) \). Recall that
\[
M(\theta) := \int_{\mathcal{D}} [Y - \eta_\theta(|\theta - X|^2)] (X - \theta) dP_X,
\]
is the population version of \( \mathbb{M}_n(\cdot) \) and
\[
\eta_\theta(u) := \mathbb{E}(\eta_\theta(|\theta_0 - X|^2) | \theta - X|^2 = u).
\]

Lemma G.2. It is easy to see that \( |M(\theta_0)| = 0 \). Suppose assumptions (A1)–(A3), and (A5) hold, then

1. For all \( \theta \in B(\theta_0, \delta) \), we have that \( (\theta - \theta_0)^\top M(\theta) \geq 0 \).

2. There does not exist \( \theta_1 \neq \theta_0 \) such that \( (\theta - \theta_1)^\top M(\theta) \geq 0 \) for all \( \theta \in B(\theta_0, \delta) \).

Proof. Proof of (1): The proof here is similar to Proof of Lemma F.2 Balabdaoui et al. (2019b). By definition of \( \eta_\theta(\cdot) \) (see (5)), we have
\[
M(\theta) = \int_{\mathcal{D}} (x - \theta) \left[ \eta_\theta(|\theta_0 - x|^2) - \eta_\theta(|\theta - x|^2) \right] dP_X(x)
= \int_{\mathcal{D}} (x - \theta) \left[ \eta_\theta(|\theta_0 - x|^2) - \mathbb{E}(\eta_\theta(|\theta_0 - X|^2) | \theta - X|^2 = |\theta - x|^2) \right] dP_X(x)
= \mathbb{E} \left[ \text{Cov}(X - \theta, \eta_\theta(|\theta_0 - X|^2) | \theta - X|^2) \right].
\]

Thus
\[
(\theta - \theta_0)^\top M(\theta) = \mathbb{E} \left[ \text{Cov}( (\theta - \theta_0)^\top (X - \theta), \eta_\theta(|\theta_0 - X|^2) | \theta - X|^2) \right]
= \mathbb{E} \left[ \text{Cov}( (\theta - \theta_0)^\top (X - \theta), \eta_\theta(|X - \theta|^2 + |\theta - \theta_0|^2 + 2(\theta - \theta_0)^\top (X - \theta)) | \theta - X|^2) \right].
\]

We will next show that \( \text{Cov}( (\theta - \theta_0)^\top (X - \theta), \eta_\theta(|X - \theta|^2 + |\theta - \theta_0|^2 + 2(\theta - \theta_0)^\top (X - \theta)) | \theta - X|^2) \) is positive because \( \eta_\theta \) is increasing. Define \( Z_1 = (\theta - \theta_0)^\top (X - \theta) \) and \( Z_2 = \eta_\theta(u + Z_1) \). Let \( \tilde{z}_2 := \eta_\theta^{-1}(z_2) - u \), then by monotonicity of \( \eta_\theta \), we have that
\[
P(Z_1 \geq z_1, Z_2 \geq z_2) = P(Z_1 \geq \max (z_1, \tilde{z}_2)) \geq P(Z_1 \geq \max (z_1, \tilde{z}_2)) P(Z_1 \geq \min (z_1, \tilde{z}_2))
= P(Z_1 \geq z_1) P(Z_2 \geq z_2).
\]
Thus for all $\theta \in B(\theta_0, \delta)$, we have
\[
\text{Cov}\left(\left(\theta - \theta_0\right)^\top (X - \theta), \eta_0(|X - \theta|^2 + |\theta - \theta_0|^2 + 2(\theta - \theta_0)^\top (X - \theta)) \big| \theta - X|^2 = u\right)
\]
\[
= \int \left[\mathbb{P}(Z_1 \geq z_1, Z_2 \geq z_2) - \mathbb{P}(Z_1 \geq z_1)\mathbb{P}(Z_2 \geq z_2)\right] dz_1 dz_2
\]
\[
\geq 0
\]

**Proof of (2):** Suppose there exists $\theta_1$ such that $(\theta - \theta_1)^\top M(\theta) \geq 0$ for all $\theta \in B(\theta_0, \delta)$. Take $\theta' = (\theta_1 + \theta_0)/2$, then $(\theta' - \theta_0)^\top M(\theta') = -(\theta' - \theta_1)^\top M(\theta')$. A contradiction since by Assumption (A5), we have that
\[
\text{Cov}\left(\left(\theta - \theta_0\right)^\top (X - \theta), \eta_0(|\theta_0 - X|^2) \big| |\theta - X|^2 \right) \neq 0 \quad \text{almost everywhere.}
\]

**Lemma G.3.** Under assumptions (A1), (A2), (A3), and (A5), we have
\[
\frac{\partial \eta_0(|\theta - x|^2)}{\partial \theta} \bigg|_{\theta = \theta_0} = \left(\mathbb{E}(X||\theta_0 - X|^2 = |\theta_0 - x|^2) - X\right)\eta_0'(||\theta_0 - x|^2)
\]

and
\[
M'(\theta_0) = \mathbb{E}\left(\eta_0'(|\theta_0 - X|^2)\text{Cov}(X||\theta_0 - X|^2)\right)
\]

**Proof.** Let $h_+^\theta(\cdot|u)$ be the conditional density of $(X_2, \ldots, X_d)$ when $|X - \theta| = u$ and $\text{sign}(X_1 - \theta_1) = 1$ and $h_-^\theta(\cdot|u)$ be the conditional density of $(X_2, \ldots, X_d)$ when $|X - \theta| = u$ and $\text{sign}(X_1 - \theta_1) = -1$. Further, let $p_+^\theta = \mathbb{P}(\text{sign}(X_1 - \theta_1) = 1)$ and $p_-^\theta = 1 - p_+^\theta$. In this proof, we use the following notation $\theta = (\theta_1, \ldots, \theta_d)$ and $\theta_0 = (\theta_{01}, \ldots, \theta_{0d})$. Then
\[
\eta_0(|\theta - x|^2) = \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2)
\]
\[
= p_+^\theta \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1) \\
+ p_-^\theta \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = -1)
\]

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and

\[
\frac{\partial}{\partial \theta_j} \eta_0(|\theta - x|^2) |_{\theta = \theta_0} = \frac{\partial}{\partial \theta_j} p_0^+ |_{\theta = \theta_0} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta_0 = |\theta_0 - x|^2, \text{sign}(X_1 - \theta_01) = 1)
\]

\[
+ \frac{\partial}{\partial \theta_j} p_0^- |_{\theta = \theta_0} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta_0 = |\theta_0 - x|^2, \text{sign}(X_1 - \theta_01) = -1)
\]

\[
+ p_0^+ \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1) |_{\theta = \theta_0}
\]

\[
+ p_0^- \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = -1) |_{\theta = \theta_0}
\]

\[
= \frac{\partial}{\partial \theta_j} p_0^+ |_{\theta = \theta_0} \eta_0(|\theta_0 - x|^2) + \frac{\partial}{\partial \theta_j} \eta_0(|\theta_0 - x|^2)
\]

\[
+ p_0^+ \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1) |_{\theta = \theta_0}
\]

\[
+ p_0^- \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = -1) |_{\theta = \theta_0}
\]

\[
= p_0^+ \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1) |_{\theta = \theta_0}
\]

\[
+ p_0^- \frac{\partial}{\partial \theta_j} \mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = -1) |_{\theta = \theta_0}
\]

\[
, \quad (77)
\]

for all \(j = 1, \ldots, d\). Note that

\[
|\theta_0 - X|^2 = (X_1 - \theta_01)^2 + \sum_{k=2}^d (X_k - \theta_{0k})^2
\]

\[
= |X_1 - \theta|^2 + (\theta_1 - \theta_01)^2 + 2(\theta_1 - \theta_1)(\theta_1 - \theta_01) + \sum_{k=2}^d (X_k - \theta_{0k})^2
\]

\[
= |X - \theta|^2 - \sum_{k=2}^d (X_k - \theta_k)^2 + 2 \text{sign}(X_1 - \theta_1)(\theta_1 - \theta_01) \sqrt{|X - \theta|^2 - \sum_{k=2}^d (X_k - \theta_k)^2}
\]

\[
+ (\theta_1 - \theta_01)^2 + \sum_{k=2}^d (X_k - \theta_{0k})^2.
\]

Thus

\[
\mathbb{E}(\eta_0(|\theta_0 - X|^2) | \theta = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1)
\]

\[
= \int_{\text{sign}(\tilde{x}_1 - \theta_1) = 1} \eta_0(|\theta_0 - X|^2) h^\theta_+ (\tilde{x}_2, \ldots, \tilde{x}_d | \theta - x) d\tilde{x}_2 \ldots d\tilde{x}_d \quad (78)
\]

\[
= \int_{\text{sign}(\tilde{x}_1 - \theta_1) = 1} \eta_0 \left( |x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2 + 2(\theta_1 - \theta_01) \sqrt{|x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2}
\]

\[
+ (\theta_1 - \theta_01)^2 + \sum_{k=2}^d (\tilde{x}_k - \theta_{0k})^2 \right) h^\theta_+ (\tilde{x}_2, \ldots, \tilde{x}_d | \theta - x) d\tilde{x}_2 \ldots d\tilde{x}_d.
\]
Define,
\[ h(x, \theta, \{\tilde{x}\}_2^d) := |x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2 + 2(\theta_{1} - \theta_0) \sqrt{|x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2 + (\theta_{1} - \theta_0)^2 + \sum_{k=2}^d (\tilde{x}_k - \theta_{0k})^2}. \]

Note that for \( j \geq 2 \), we have
\[ \frac{\partial}{\partial \theta_j} h(x, \theta, \{\tilde{x}\}_2^d) = 2(\tilde{x}_j - x_j)\left\{1 + \frac{\theta_1 - \theta_0}{\sqrt{|x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2}}\right\}. \]

For \( j = 2, \ldots, d \), we have that
\[ \frac{\partial \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1)}{\partial \theta_j} \]
\[ = \int_{\text{sign}(\tilde{x}_1 - \theta_1) = 1} \eta_0(\theta)(h(x, \theta, \{\tilde{x}\}_2^d)) \frac{\partial}{\partial \theta_j} h_+(\tilde{x}_2, \ldots, \tilde{x}_d||\theta - x||)d\tilde{x}_2 \ldots d\tilde{x}_d \]
\[ + \int_{\text{sign}(\tilde{x}_1 - \theta_1) = 1} \eta_0'(h(x, \theta, \{\tilde{x}\}_2^d)) 2(\tilde{x}_j - x_j)\left\{1 + \frac{\theta_1 - \theta_0}{\sqrt{|x - \theta|^2 - \sum_{k=2}^d (\tilde{x}_k - \theta_k)^2}}\right\} h_+(\tilde{x}_2, \ldots, \tilde{x}_d||\theta - x||)d\tilde{x}_2 \ldots d\tilde{x}_d \]

Thus
\[ \frac{\partial \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1)}{\partial \theta_j} \bigg|_{\theta = \theta_0} \]
\[ = \int_{\text{sign}(\tilde{x}_1 - \theta_1) = 1} \eta_0'(|x - \theta_0|^2) 2(\tilde{x}_j - x_j) h_+(\tilde{x}_2, \ldots, \tilde{x}_d||\theta_0 - x||)d\tilde{x}_2 \ldots d\tilde{x}_d \]
\[ = \eta_0'(|x - \theta_0|^2) 2\left(\mathbb{E}(X_j|\text{sign}(X_1 - \theta_0) = 1, |\theta_0 - X| = |\theta - x|) - x_j\right) \quad (79) \]

Similarly, we can show that
\[ \frac{\partial \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = -1)}{\partial \theta_j} \bigg|_{\theta = \theta_0} \]
\[ = \eta_0'(|x - \theta_0|^2) 2\left(\mathbb{E}(X_j|\text{sign}(X_1 - \theta_0) = -1, |\theta_0 - X| = |\theta - x|) - x_j\right) \quad (80) \]

Combining (77) with (79) and (80), we get that
\[ \frac{\partial}{\partial \theta_j} \eta_0(|\theta - x|^2) \bigg|_{\theta = \theta_0} = \eta_0'(|x - \theta_0|^2) 2\left(\mathbb{E}(X_j|\theta_0 - X| = |\theta - x|) - x_j\right), \]
for all \( j \geq 2 \). We will now compute (77) for \( j = 1 \). By (78), we have that
\[
\frac{\partial \mathbb{E}(\eta_0(|\theta_0 - X|^2)||\theta - X|^2 = |\theta - x|^2, \text{sign}(X_1 - \theta_1) = 1)}{\partial \theta_1} \bigg|_{\theta = \theta_0}
= \int \eta_0^0 \left(h(x, \theta, \{\tilde{x}\}^d) \frac{2}{2^{d}} \sum_{k=2}^d (\tilde{x}_k - \theta_0k)^2 - 2x_1 + 2\theta_01 \right) h_+^0(\tilde{x}_2, \ldots, \tilde{x}_d||\theta_0 - x|) d\tilde{x}_2 \ldots d\tilde{x}_d
\]

The other case can be solved similarly. Combining the above results, we have (76) and

\[
M'(\theta_0) = \mathbb{E} \left( - \frac{\partial \eta_0(|\theta - x|^2)}{\partial \theta} \bigg|_{\theta = \theta_0} (X - \theta_0) \right) = \mathbb{E} \left( \eta_0^0(|\theta_0 - X|^2) \text{Cov}(X||\theta_0 - X|^2) \right). \]

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