Serre-Taubes duality for pseudoholomorphic curves

Ivan Smith*

Abstract

According to Taubes, the Gromov invariants of a symplectic four-manifold $X$ with $b_+ > 1$ satisfy the duality $Gr(\alpha) = \pm Gr(\kappa - \alpha)$, where $\kappa$ is Poincaré dual to the canonical class. Extending joint work with Simon Donaldson, we interpret this result in terms of Serre duality on the fibres of a Lefschetz pencil on $X$, by proving an analogous symmetry for invariants counting sections of associated bundles of symmetric products. Using similar methods, we give a new proof of an existence theorem for symplectic surfaces in four-manifolds with $b_+ = 1$ and $b_1 = 0$. This reproves another theorem due to Taubes: two symplectic homology projective planes with negative canonical class and equal volume are symplectomorphic.

1 Introduction

Many questions concerning the topology of symplectic manifolds can be formulated in terms of the existence (or otherwise) of appropriate symplectic submanifolds. In four dimensions, this viewpoint has been especially fruitful given rather general existence theorems for symplectic surfaces coming from Taubes’ theory relating Seiberg-Witten invariants to pseudoholomorphic curves [Tau95]. Taubes’ results imply in particular a curious and striking symmetry for counting holomorphic curves in symplectic four-manifolds with $b_+ > 1$: the Gromov invariants for the classes $\alpha$ and $\kappa - \alpha$ are equal up to sign, where $\alpha, \kappa \in H_2(X; \mathbb{Z})$ and $\kappa$ is Poincaré dual to the canonical class of the symplectic structure. (As we recall in the next section, this is a consequence of Serre duality when $X$ is Kähler, and can be seen as a symplectic shadow of the Serre duality theorem.) Given Taubes’ amazing theorem on the equivalence “$\text{SW} = \text{Gr}$” this is rather trivial: it is the translation of a routine symmetry in Seiberg-Witten theory arising from flipping a spin$^c$-structure $L$ to its dual $L^*$. Nonetheless, one might hope that there was a direct proof of the symmetry in the context of Gromov invariants and holomorphic curves, which are in a real sense the more geometric objects of interest in the symplectic setting. This paper is designed, amongst other things, to outline one route to such a geometric interpretation of the duality.

In [DS], in joint work with Simon Donaldson, we explained how to obtain symplectic surfaces in four-manifolds with rational symplectic form from Lefschetz pencils. These arise as sections of associated fibrations of symmetric products down the fibres of the Lefschetz pencil; using this viewpoint, we were able in particular to give a new proof of Taubes’ theorem that for “most” symplectic four-manifolds the class $\kappa$ can be represented by an embedded symplectic

*New College, Oxford, OX1 3BN, England: smithi@maths.ox.ac.uk
1 INTRODUCTION

surface. More precisely (a fuller review will be given in the third section) we defined an
invariant - the “standard surface count” - \( I_{(X,f)}(\alpha) \) as follows. Be given a Lefschetz pencil
\( f : X \rightarrow \mathbb{S}^2 \) of genus \( g \) curves and construct a fibre bundle \( F : X_r(f) \rightarrow \mathbb{S}^2 \) with fibres the
\( r = (2g-2) \)-th symmetric products of fibres of \( f \). There is a natural injection \( \phi \) from the set
of sections of \( F \) to \( H_2(X;\mathbb{Z}) \). Then \( I_{(X,f)}(\alpha) \) counted the holomorphic sections of
\( F \) in the unique class \( \tilde{\alpha} \) with image \( \phi(\tilde{\alpha}) = \alpha \), and was defined to be zero for \( \alpha \notin \text{im}(\phi) \). The main
theorems of [DS], for manifolds with rational symplectic form, were then:

- if \( I_{(X,f)}(\alpha) \neq 0 \) then \( \alpha \) may be represented by an embedded symplectic surface in \( X \);
- for any \( X \) with \( b_+(X) > 1 + b_1(X) \) and any Lefschetz pencil \( f \) on \( X \) (of sufficiently
  high degree) we have \( I_{(X,f)}(\kappa) = \pm 1 \).

The main theorems of this sequel paper are, in these terms, the following: again \( I \) will denote
an invariant which counts sections of a compactified bundle of \( r \)-th symmetric products, for
arbitrary possible \( r \).

(1.1) THEOREM: Fix a symplectic four-manifold \( X \) and a Lefschetz pencil \( f \) of sufficiently
high degree on \( X \):

- if \( b_+(X) > 1 + b_1(X) \) we have an equality \( I_{(X,f)}(\alpha) = \pm I_{(X,f)}(\kappa - \alpha) \).
- if \( b_+(X) = 1 \) and \( b_1(X) = 0 \) and \( \alpha \cdot [\omega] > 0 \), \( \alpha^2 > K_X \cdot \alpha \) then \( I_{(X,f)}(\alpha) = \pm 1 \).

The broad strategy is as in [DS]. First we set up enough theory to define the relevant Gromov
invariants. Then we use almost complex structures adapted to diagonal loci in symmetric
products to obtain symplectic surfaces, and lastly we use almost complex structures adapted
to the geometry of the Abel-Jacobi map to perform explicit computations. (The third stage
in particular is conceptually as well as technically more involved here.) These results should
be considered together with the following

(1.2) THEOREM: Suppose \((X,\omega)\) contains no embedded symplectic torus of square zero. Then
there is an equality \( I_{(X,f)} = Gr_X : H_2(X;\mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2 \) of the mod two reductions of
the standard surface count and the Gromov invariant.

The hypothesis on \( X \) is satisfied, for instance, if \( K_X = \lambda[\omega_X] \) for any \( \lambda \in \mathbb{R}^* \). Here the
invariant \( Gr_X \) is that introduced by Taubes, counting embedded but not necessarily connected
holomorphic curves in \( X \). It would be natural to extend the theorem by taking account of
the signs (to lift to an equivalence of integer-valued invariants) and by including the delicate
situation for tori; we leave those extensions for elsewhere. Up to certain universal weights in
the definition of the \( I \)-invariants when tori are present, we expect:

(1.3) CONJECTURE: Once \( f \) is of sufficiently high degree, \( I_{(X,f)} \) is independent of \( f \) and
defines a symplectic invariant of \((X,\omega)\). Moreover there is an equality \( I_{(X,f)} = Gr_X : H_2(X;\mathbb{Z}) \rightarrow \mathbb{Z} \).
1 INTRODUCTION

It seems likely, given the stabilisation procedure for Lefschetz pencils, that the first half of this can be proven directly without identifying $\mathcal{I}$ and $\text{Gr}_X$. We make a few remarks on this at the end of the fifth section. Of course the conjecture, combined with Taubes’ results, would imply the first theorem. Rather, the point is that one should be able to prove the conjecture independently of Taubes’ results, and then (1.1, part (i)) yields a new proof of the symmetry of the Gromov invariants. In this paper, we shall concern ourselves with the properties of the $\mathcal{I}$-invariants, but let us point out that along with (1.2) we can obtain holomorphic curves in any $X$:

\section*{(1.4) Theorem:} Let $X$ be a symplectic four-manifold and fix a taming almost complex structure $J$ on $X$. If $\mathcal{I}(X,f)(\alpha) \neq 0$ for all sufficiently high degree pencils $f$ on $X$ then $X$ contains $J$-holomorphic curves in the class $\alpha$.

As a technical remark, note that this enables one to find symplectic surfaces in $X$ even if the symplectic form is not rational. One might hope that the equivalence $\mathcal{I} = \text{Gr}$, together with the conjectural routes to establishing $\text{SW} = \mathcal{I}$ as proposed by Dietmar Salamon, would give a more intuitive framework for Taubes identification $\text{SW} = \text{Gr}$. In our picture, the symmetry (1.1, (i)) will arise entirely naturally from Serre duality on the fibres of the Lefschetz fibration; the key geometric ingredient is the fact that the $(2g - 2 - r)$-th symmetric product of a Riemann surface fibres over the Jacobian with exceptional fibres precisely over an image of the $r$-th symmetric product, when $r < g - 1$. For technical reasons, we will make use of a strengthening of this observation which gives us enough control to work in families (6.2). This stronger result will follow easily from results, due to Eisenbud and Harris, in the Brill-Noether theory of Riemann surfaces; one appeal of the current proof is that such results become of relevance in four-dimensional symplectic geometry.

The symmetry of the Gromov invariants is false for symplectic manifolds with $b^+ = 1$. Indeed, if $X$ is minimal and has $b_+ = 1$, and if in addition $\text{Gr}(\alpha)$ and $\text{Gr}(\kappa - \alpha)$ are both non-trivial, then necessarily $K^2_X = 0$ and $K_X = n\alpha$ for some integer $n$. This is an easy consequence of properties of the intersection form on such four-manifolds; for Kähler surfaces, if both $\text{Gr}(\alpha)$ and $\text{Gr}(\kappa - \alpha)$ are non-trivial then the holomorphic curves in the two homology classes come from sections of line bundles which may be tensored to produce a non-trivial element of $H^0(K_X)$, forcing $b_+ > 1$. Taubes in fact proves the theorem under the weakest possible constraint $b_+ > 1$. We shall assume throughout the bulk of the paper that $b_+ > 1 + b_1$; this weaker assumption simplifies the arguments, and keeps the geometry to the fore. At the end of the paper we shall sketch how to improve the arguments to hold in case $b_+ > 2$. This still falls short of Taubes, and superficially at least the Hard Lefschetz theorem plays a complicating role. Presumably sufficient ingenuity would cover the missing case $b_+ = 2$, but the author could not find an argument. In any case, rather than being sidetracked we hope to emphasise the key geometric ingredients of the new proof. After all, the theorem itself already has one beautiful and very detailed exposition, thanks to Taubes, and our intention is to supplement and not supplant the gauge theory\footnote{“Il n’y a point de secte en géométrie”, Voltaire (Dictionnaire Philosophique)}.

The non-vanishing result (1.1, part (ii)) - which is described at the end of the paper - implies in particular an existence theorem for symplectic surfaces in four-manifolds with $b_+ = 1$.\footnote{“Il n’y a point de secte en géométrie”, Voltaire (Dictionnaire Philosophique)}
Such results are well-known, and go back to McDuff \cite{McD98}; similar work has been done by T.J.Li and A.K.Liu \cite{LL}. Each of these earlier proofs has relied on wall-crossing formulae for Seiberg-Witten invariants; our arguments are “more symplectic” and may cast a new light on the relevant geometry. We remark that here it is important to use a definition of $I(\alpha)$ in which we cut down a positive dimensional moduli space by intersecting the image of an evaluation map with appropriate divisors (corresponding to forcing holomorphic curves to pass through points in the four-manifold $X$); we will explain this more properly below.

The proofs of both parts of (1.1) run along similar lines to the proof of the main theorems of \cite{DS}, and much of the technical material is already present in that paper. As before, monotonicity of the fibres of $X_r(f)$ enables us to use elementary machinery from the theory of pseudoholomorphic curves, so the proofs are not too hard. Using the results of the two papers together, it now becomes possible to re-derive some of the standard structure theorems for symplectic four-manifolds from the perspective of the existence of Lefschetz pencils. Here are some sample results, whose proofs are well-known: for completeness we briefly recall the arguments in the last section of the paper.

\begin{enumerate}
\item \textbf{(Taubes)} If $X$ is minimal with $b_+ > 1$ then $2\sigma(X) + 3\sigma(X) \geq 0$. In particular, manifolds such as $K3^\sharp K3^\sharp K3$ admit no symplectic structure.
\item \textbf{(Taubes)} A homology symplectic projective plane with $K_X \cdot [\omega] < 0$ is symplectomorphic to $(\mathbb{C}P^2, \mu \omega_{FS})$ for some $\mu > 0$.
\item \textbf{(Ohta-Ono)} More generally, if $c_1(X) = \lambda[\omega]$ for some $\lambda \in \mathbb{R}_{>0}$ then $X$ is diffeomorphic to a del Pezzo surface.
\item \textbf{(Li-Liu)} If $X$ is minimal with $b_+ = 1$ and $K_X^2 > 0$, $K_X \cdot \omega > 0$ then the canonical class contains symplectic forms.
\end{enumerate}

The examples, though not exhaustive, serve also to highlight some of the profound successes of the gauge theory which remain mysterious from the perspective of symplectic linear systems: one such is the role of positive scalar curvature as an obstruction to the existence of holomorphic curves.

Let us remark on three further directions suggested by \cite{DS} and this paper. The first concerns non-symplectic four-manifolds. Work of Presas \cite{Pre} suggests that symplectic manifolds with contact boundary should also admit pencils of sections, and one could hope to complement Taubes’ theorems on the Seiberg-Witten equations on manifolds with self-dual forms \cite{Tau99} with existence statements for holomorphic curves with boundary. A second concerns higher dimensional symplectic manifolds. For complex three-folds fibred smoothly over curves, the relative Hilbert schemes are smooth and one can approach the Gromov invariants of the three-folds through sections of the associated bundles. Counting such sections gives invariants of loops of symplectomorphisms for complex surfaces which refine those of \cite{Sei97}. More generally, after finitely many blow-ups any symplectic six-manifold $Z$ admits a map to the projective plane; a complex curve $\Sigma$ in $Z$ projects to a complex curve $C$ in $\mathbb{C}P^2$, and
generically at least $\Sigma$ lies inside the total space of a Lefschetz fibration over $C$. This suggests an inductive approach to the Gromov invariants of $Z$, similar in flavour to work of Seidel [Sei00]. In a third direction, both [DS] and this paper concern solutions of the Seiberg-Witten equations on the total space of a Lefschetz fibration. An analogous story for the instanton equations is the subject of work in progress by the author and will be the topic of a sequel paper.

Outline of the paper:

1. Remark: although we will not repeat all details of the local constructions of [DS], we will give a (more) coherent development of the global theory that we require, so the paper should be accessible in its own right.

2. In the next section, we explain how the symmetry $Gr_X(\alpha) = \pm Gr_X(\kappa - \alpha)$ follows from Serre duality if $X$ is a Kähler surface with $b_1 = 0$. (This motivates various later constructions.)

3. In the third section, we recall the basics of Lefschetz fibrations, and prove that given $(X, f)$ the relative Hilbert scheme $X_r(f)$ provides a smooth symplectic compactification of the family of $r$-th symmetric products of the fibres. (We have tried to illuminate the structure of this space.)

4. In the fourth section, we compute the virtual dimensions of moduli spaces of sections of $X_r(f)$ and define an invariant $I$ which counts sections in a fixed homotopy class (this requires a compactness theorem). We also give a simple “blow-up” formula.

5. In the fifth section, using a natural almost complex structure on $X_r(f)$ and Gromov compactness, we show that if $I(\alpha) \neq 0$ then the moduli space of holomorphic curves representing $\alpha$ is non-empty for any taming almost complex structure on $X$. We also sketch how to obtain the equivalence $I = Gr_X$ (mod 2) for manifolds containing no symplectic square zero tori.

6. In the sixth section, we prove the main result (1.1): this involves a short detour into Brill-Noether theory and some obstruction computations modelled on those of [DS]. We assume, for simplicity, that $b_+ > 1 + b_1$ or $b_+ = 1, b_1 = 0$ in this section.

7. In the final section, we give the proofs of the applications listed above and explain how to extend the arguments of [DS] (and in principle Section 6) to the case where $b_+ > 2$.

Acknowledgements: Many of the ideas here arose from conversations with Simon Donaldson, whose influence has been accordingly pervasive. Thanks also to Denis Auroux, Eyal Markman, Paul Seidel and Bernd Siebert for helpful remarks.

2 Digression on algebraic surfaces

We shall begin (semantic sensibilities regardless!) with a digression. If $X$ is Kähler then one can often compute the Gromov invariants of $X$ directly; we will review this, and explain how
the symmetry $Gr_X(\alpha) = \pm Gr_X(\kappa - \alpha)$ emerges in this framework. Suppose for simplicity that $b_1(X) = 0$.

The key point is that a holomorphic curve, for the integrable complex structure, is exactly a divisor and as such gives rise to a section of a line bundle. Moreover, generically at least, the locus of holomorphic sections of a given line bundle yielding singular complex curves will have positive codimension, whilst those yielding curves with worse than nodal singularities will have complex codimension at least two. It follows that the linear system in which the divisor moves defines a suitable compactification of the space of smooth holomorphic curves for computing Gromov invariants. Again for conceptual clarity, and since we shall not use the results of this section later on, we will suppose we are always in this situation. The desired invariant itself can be computed as the Euler class of an obstruction bundle over the moduli space (cf. [Sal], Prop. 11.29).

\begin{equation}
\text{(2.1) Proposition:} \quad \text{Let } X \text{ be a Kähler surface with } b_+ > 1 \text{ and } b_1 = 0. \text{ The Gromov invariants of } X \text{ manifest the symmetry } Gr(\alpha) = \pm Gr(\kappa - \alpha).
\end{equation}

\begin{proof}
Fix a suitable class $\alpha \in H_2(X; \mathbb{Z})$ which we suppose to be Poincaré dual to a class $D$. We will blur $D$ with the unique holomorphic line bundle $\mathcal{O}(D)$ with first Chern class $D$. Our assumptions imply that we have $H^1(X, \mathcal{O}_X) = 0$; then $\mathbb{P} = \mathbb{P}(H^0(D))$ is the moduli space of pseudoholomorphic curves in the homology class $\alpha$ - there are no other line bundles with the same first Chern class. Write $\Delta \subset \mathbb{P} \times X$ for the universal divisor with projections $\pi : \Delta \to \mathbb{P}$ and $p : \Delta \to X$. The obstruction bundle by definition is given by $R^1\pi_*\mathcal{O}_{\Delta}(\Delta)$.

Suppose first that the virtual dimension is zero; $\alpha^2 = K_X \cdot \alpha$. Then the invariant is the Euler class of the obstruction bundle. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P} \times X} \to \mathcal{O}_{\mathbb{P} \times X}(\Delta) \to \mathcal{O}_{\mathbb{P} \times X}(\Delta)|_\Delta \to 0$$

where the last non-zero term is $\mathcal{O}_{\Delta}(\Delta)$ by definition of notation. It follows that

$$\pi_!\mathcal{O}_{\mathbb{P} \times X} + \pi_!\mathcal{O}_{\Delta}(\Delta) = \pi_!\mathcal{O}_{\mathbb{P} \times X}(\Delta)$$

in K-theory, and hence taking total Chern classes that

$$c(\pi_!\mathcal{O}_{\Delta}(\Delta)) = c(\pi_!\mathcal{O}_{\mathbb{P} \times X}(\Delta)). \quad (2.2)$$

Now $\pi : \Delta \to \mathbb{P}$ has one-dimensional fibres and hence

$$\pi_!\mathcal{O}_{\Delta}(\Delta) = R^0\pi_*\mathcal{O}_{\Delta}(\Delta) - R^1\pi_*\mathcal{O}_{\Delta}(\Delta).$$

Note that the first of these is by deformation theory just the tangent space to the moduli space $\mathbb{P}$, whilst the latter is the obstruction space we require. It follows, also using (2.2), that

$$c(\text{Obs}) = c(\pi_!\mathcal{O}_{\mathbb{P} \times X}(\Delta))^{-1} \cdot c(T\mathbb{P}).$$

Of course the term $c(T\mathbb{P})$ is just $(1 + H)^{h^0 - 1}$ where $H$ is the generator of the cohomology of the projective space and $h^0$ is the rank of $H^0(X, D)$. We must therefore understand the other term in the last expression. Observe
\[ \pi_! \mathcal{O}_{P \times X}(\Delta) = R^0 \pi_* - R^1 \pi_* + R^2 \pi_* \]
in an obvious notation. But we also have
\[ \mathcal{O}_{P \times X}(\Delta) = \pi^* \mathcal{O}_P(1) \otimes p^* D \]
and hence \( R^i \pi_* = \mathcal{O}_P(1)^{h^i(D)} \). Taking total Chern classes one last time, we deduce that
\[ c(\text{Obs}) = c(\mathcal{O}_P(1))^{-\text{Index}} \cdot c(T\mathbb{P}) = (1 + H)^{h^1(D) - h^2(D)}. \]
Here \( \text{ind} \) denotes the index of the \( \bar{\partial} \)-operator on the bundle \( D \), equivalently the alternating sum \( \sum (-1)^i h^i(D) \). Now we use the assumption that \( b_1(X) = 0 \), and hence \( H^1(X; K_X) \cong H^1(X; \mathcal{O})^* = 0 \) again. For this implies that
\[ H^0(X, D) \otimes H^1(X; K_X - D) \to H^1(X; K_X) \]
is a trivial map, which in turn implies that whenever \( H^0(X, D) \neq \{0\} \) we have \( H^1(X, K_X - D) = \{0\} \) and hence \( H^1(X, D) = \{0\} \). Here we use an old result of Hopf which asserts that whenever \( V \otimes V' \to W \) is a map of complex vector spaces linear on each factor, then the dimension of the image exceeds \( rk(V) + rk(V') - 1 \). The upshot is that Serre duality implies that the Gromov invariant we require is exactly
\[ Gr_X(\alpha) = \left( \frac{-h^2(D)}{h^0(D) - 1} \right); \]
for by the binomial theorem, this is just the Euler class of the obstruction bundle over the projective space which is the moduli space of holomorphic curves representing \( D \). Recall the definition of a binomial coefficient with a negative numerator:
\[ (-n \choose k) = (-1)^k \left( \frac{n + k - 1}{k} \right). \]
This shows that
\[ Gr_X(\alpha) = \pm \left( \frac{h^0(D) + h^2(D) - 2}{h^0(D) - 1} \right) \pm \left( \frac{h^0(K - D) + h^2(K - D) - 2}{h^0(K - D) - 1} \right) = \pm Gr_X(\kappa - \alpha). \]
Here the identity in the middle is given by the Riemann-Roch theorem and Serre duality for the bundles \( \mathcal{O}(D) \) and \( \mathcal{O}(K_X - D) \) on \( X \). If the virtual dimension \( \alpha^2 - K_X \cdot \alpha = r \) is in fact positive, the argument is similar; the relevant equality in this instance is
\[ \left( \frac{-[h^2(D) - r]}{[h^0(D) - 1] - r} \right) = \left( \frac{-[h^2(K_X - D) - r]}{[h^0(K_X - D) - 1] - r} \right). \]
To see this, note that each incidence condition on the holomorphic curves defines a hyperplane on the projective space of sections. The final result again follows from the definition of the binomial coefficients. ■
One can use arguments analogous to those above to perform explicit computations. Setting \( \alpha = \kappa \) above, we see that if \( b_+ > 1 \) then the Gromov invariant of the canonical class on the surface is \( \pm 1 \). This is the “holomorphic” case of the main theorem of [DS]. Here is a more substantial assertion. Suppose \( X \) is spin and \( c_2^2(X) = 0 \); then Rokhlin’s theorem shows that \( b_+(X) = 4n - 1 \) for some \( n \geq 1 \).

(2.3) Lemma: For \( X \) as above, the Gromov invariant of \( D = K_X/2 \) is \( (2n-2) \).

To see this, argue as follows. The multiplication map

\[
H^0(X; D) \otimes H^0(X; D) \rightarrow H^0(X; K_X)
\]

shows that \([b_+ + 1]/2 \geq 2h^0(D)\), by the old result of Hopf alluded to previously. But from the Riemann-Roch theorem, and the duality \( H^2(X; D) \cong H^0(X; D)^* \) we see that \( 2h^0(X; D) \geq \text{Index}_{\mathcal{O}(D)}(\mathcal{O}) = [b_+ + 1]/2 \). Hence \( h^0(X; D) = [b_+ + 1]/4 \). Now the arguments as above show that the obstruction bundle over the projective space \( \mathbb{P}H^0(X; D) \) satisfies

\[
c(\text{Obs}) = (1 + h)^{-\text{Index}} c(\mathcal{P}) = (1 + h)^{-[b_+ + 1]/4}.
\]

The Gromov invariant is given by the coefficient of \( h^{[b_+ + 1]} \) in this expression, which is exactly as claimed by the binomial expansion. (If \( X \) is spin but \( c_1^2(X) \neq 0 \) then the moduli space of sections in the class \( K_X/2 \) is not zero-dimensional and the invariant will vanish as soon as \( b_+ > 1 \).) We will return to this result in a symplectic setting later, to illustrate a weakness of our current theory.

3 Symplectic surfaces and symmetric products

In this section we shall review some of the basics of Lefschetz pencils, and establish the core ideas on which the rest of the paper is founded. We include various extensions of results from [DS] that will be important later, and have tried to make the discussion essentially self-contained wherever the statements of [DS] are inadequate for our applications. We have also included an elementary proof of smoothness of the relative Hilbert scheme for families of curves with only nodal singularities. Using this, one can build all the spaces we need - and verify the properties we need - directly, but at the cost of losing the naturality which comes for free in the algebraic geometry. For foundations on Hilbert schemes and compactified Jacobians, we defer to the papers [AK80] and [OS79].

3.1 Lefschetz pencils

Let \( X \) be an integral symplectic four-manifold: \( X \) is equipped with a symplectic form \( \omega \) whose cohomology class satisfies \( [\omega]/2\pi \in H^2(X; \mathbb{Z}) \subset H^2(X; \mathbb{R}) \). Then there is a line bundle \( \mathcal{L} \) with connexion with curvature \( \omega \), and Donaldson [Don90] has shown that appropriate sections of high tensor powers \( \mathcal{L}^k \) of this line bundle give symplectic surfaces in \( X \) Poincaré dual to \( k[\omega]/2\pi \). Extending this in [Don91], Donaldson shows that integral symplectic manifolds admit complex Morse functions or Lefschetz pencils: we can find a map \( f : X \rightarrow S^2 \) defined
on the complement of finitely many points $q_j$ in $X$, with finitely many critical points $p_i$, all with distinct image under $f$, such that:

1. $f$ has the local model $(z_1, z_2) \mapsto z_1 z_2$ at each of the $p_i$, 
2. $f$ has the local model $(z_1, z_2) \mapsto z_1/z_2$ at each of the $q_j$,

where all local complex co-ordinates are compatible with fixed global orientations. The fibres are again symplectic submanifolds Poincaré dual to the class $k[\omega]/2\pi$, for some large $k$. We can assume the symplectic form is positive of type $(1, 1)$ in a fixed almost complex structure at each $p_i, q_j$. The topology of such a situation is described in [Sm99] or [GS99] for instance; after blowing up at each $q_j$ we have a manifold $X'$ fibred over $S^2$ with smooth two-dimensional surfaces of some fixed genus $g$ as fibres over the points of $S^2 \setminus \{f(p_i)\}$ and with critical fibres surfaces with a single ordinary double point. The exceptional spheres $E_i$ of the blow-ups form distinct sections of the fibration of $X' = X\#CP^2$, where $r = \sharp\{q_j\}$. We remark that one can always ensure that the singular fibres are irreducible, in the sense that the vanishing cycles which have collapsed to the node are homologically essential, or equivalently that removing the node does not separate the singular fibre [Sm01]. We will tacitly assume that all Lefschetz fibrations satisfy this hypothesis henceforth. The following definition is from [DS]:

(3.1) **Definition:** Let $f : X' \to S^2$ be a Lefschetz fibration arising from a Lefschetz pencil $(X, f)$. A standard surface in $X'$ is an embedded surface $\Sigma \subset (X\setminus\{p_i\})$ for which the restriction $f|\Sigma : \Sigma \to S^2$ is a branched covering of positive degree with simple branch points.

Here we assume that $f$ is of positive degree on each component of $\Sigma$ in the case when the surface is disconnected. As explained in [DS], a symplectic structure on $X$ induces a family of symplectic forms $(\omega_{(N)} = p^*(k\omega_X) + N f^*\omega_{S^2})$ on $X'$ which are symplectomorphic, under the obvious identification away from a small neighbourhood of the exceptional sections, to the original form on $X$ up to scaling by $k(1 + N)$. Moreover, a standard surface is necessarily symplectic with respect to $\omega_{(N)}$ for large enough $N$, and hence a standard surface disjoint from the $E_i$ gives rise to a symplectic surface in $X$. We will sometimes refer to a finite collection of standard surfaces with locally positive transverse intersections as a **positive symplectic divisor**.

Donaldson’s construction of Lefschetz pencils $(X, f)$ involves a subsidiary choice of almost complex structure $J$ on $X$, compatible with $\omega_X$; after perturbing $J$ by a $C^k$-small amount (for any given $k$), we can assume that the fibration $f$ is pseudoholomorphic and all the fibres are almost complex surfaces in $X'$. Let us also remark that given any (not necessarily integral) symplectic four-manifold, arbitrarily small perturbations of the symplectic form give rise to a rational form and hence to Lefschetz pencils with fibres dual to a multiple of the form. Since the $J$-holomorphic surfaces remain symplectic for small perturbations of $\omega$, this means that arbitrary symplectic manifolds admit topological Lefschetz pencils: that is, if we drop the integrality hypothesis on the form, we only lose the explicit identification of the homology class of the fibre.
3.2 Relative Hilbert schemes

The choice of \( J \) on the total space of the Lefschetz fibration also defines a smooth map from the base \( \phi_f : S^2 \to \overline{M}_g \) to the Deligne-Mumford moduli space of stable curves, where the extension over the critical values of \( f \) follows precisely from our requirements on the local normal forms of the singularities. The map is defined up to “admissible isotopy”, that is, isotopies which do not change the geometric intersection number with the divisors of nodal curves; these intersections are locally positive. By choosing \( J \) generically we can assume that the map \( \phi_f \) has image disjoint from the orbifold singular loci of moduli space, and hence lies inside the fine moduli space of curves without automorphisms. Hence we have various universal families over \( \phi_f(S^2) \), including a universal curve - which just defines \( X' \) - and universal families of symmetric products and Picard varieties of curves.

Recall that associated to any smooth Riemann surface \( \Sigma \) we have a complex torus parameterising line bundles of some fixed degree \( r \) and a smooth complex variety parameterising effective divisors of degree \( r \) on \( \Sigma \). Thinking of these as moduli spaces, for coherent torsion-free sheaves and zero-dimensional subschemes of fixed length respectively, machinery from geometric invariant theory ([HL97], [AK80] etc) provides relative moduli schemes which fibre over \( \overline{M}_g \) - or at least over the locus of irreducible curves with at most one node - with fibre at a smooth point \( \tau \in \overline{M}_g \) just the Picard variety or symmetric product of \( \Sigma_\tau \). These universal fibre bundles give rise to fibre bundles on the base \( S^2 \) of a Lefschetz pencil, once we have made a choice of complex structures on the fibres. More explicitly, each of these fibre bundles can also be defined by local charts, called restricted charts in [DS]. A restricted chart is a diffeomorphism \( \chi : D_1 \times D_2 \to X' \) which is a smooth family, indexed by \( \tau \in D_1 \subset S^2 \), of holomorphic diffeomorphisms \( \chi_\tau : \{\tau\} \times D_2 \to U_\tau \subset f^{-1}(\tau) \) onto open subsets of the fibres of \( X' \). Here each \( D_i \) denotes the unit disc in the complex plane, with its standard integrable structure. The existences of atlases of restricted charts is implied by the Riemann mapping theorem with smooth dependence on parameters. At any rate, with this background we can make the following:

**3.2 Definition:** Let \( (X, f) \) be a Lefschetz pencil inducing \( \phi_f : S^2 \to \overline{M}_g \).

- Denote by \( F : X_r(f) \to S^2 \) the pullback by \( \phi_f \) of the universal relative Hilbert scheme for zero-dimensional length \( r \) subschemes of fibres of the universal curve \( \pi : C_g \to \overline{M}_g \).

- Denote by \( G : P_r(f) \to S^2 \) the pullback by \( \phi_f \) of the universal relative Picard scheme for degree \( r \) torsion-free sheaves on the fibres of \( \pi : C_g \to \overline{M}_g \).

The natural smooth map \( u : X_r(f) \to P_r(f) \) will be referred to as the Abel-Jacobi map.

The existence of the map \( u \) is proven in [AK80] or can be deduced (in the smooth category) from our constructions. Here is a set-theoretic description of the singular fibres of \( G \) and \( F \).

1. The degree \( r \) torsion-free sheaves on an irreducible nodal curve \( C_0 \) are of two forms: locally free, or push-forwards of locally free sheaves from the normalisation \( \pi : \tilde{C}_0 \to C_0 \). A locally free sheaf is completely determined by a pair \( (L, \lambda) \) where \( L \to \tilde{C}_0 \) is a degree \( r \) line bundle on the normalisation and \( \lambda \in \text{Iso}(L_\alpha, L_\beta) \cong \mathbb{C}^* \) is a gluing parameter that identifies the fibres \( L_\alpha \) and \( L_\beta \) of \( L \) over the preimages of the node of \( C_0 \). This
gives a $\mathbb{C}^*$-bundle over $\text{Pic}_r(\tilde{C}_0)$. The non-locally free sheaves are of the form $\pi_*L'$ where $L' \to \tilde{C}_0$ is locally free of degree $r - 1$. These arise by compactifying $\mathbb{C}^*$ to $\mathbb{P}^1$ and identifying the two degenerate gluings - the 0 and $\infty$ sections of the resulting $\mathbb{P}^1$-bundle - over a translation by the action of $\mathcal{O}(\alpha - \beta)$ in $\text{Pic}_r(\tilde{C}_0)$.

2. The fibre of $F$ can be completely described by giving the fibre of $G$, as above, and the fibres of the map $u$. The latter are projective spaces. At a point $(L, \lambda)$ the fibre of $u$ is the subspace of the linear system $\mathbb{P}H^0(L)$ comprising those sections $s \in H^0(L)$ for which $s(\alpha) = \lambda s(\beta)$. At a point $\pi_*L'$ the fibre of $u$ is just the entire linear system $\mathbb{P}H^0(L')$.

The sets $X_r(f)$ and $P_r(f)$ obtained above carry obvious topologies: given a sequence $(D_n = p_n + D)_{n \in \mathbb{N}}$ of distinct $r$-tuples of points in the singular fibre of $f$, with $D$ a fixed $(r - 1)$-tuple and $p_n \to \text{Node}$, then the points of the Hilbert scheme converge to the obvious point of $\text{Sym}^{-1}(C_0)$ which is determined by the divisor $D$ and the associated line bundle $L' = \mathcal{O}(D)$ of rank $r - 1$ on $\tilde{C}_0$. The general behaviour is analogous. We can put smooth structures on the spaces using explicit local charts. The following result - which may be known to algebraic geometers but does not appear in the literature - is central for this paper.

**Theorem (3.3):** For any $(X, f)$ and each $r$, the total spaces of $X_r(f)$ and $P_r(f)$ are smooth compact symplectic manifolds.

**Proof:** Up to diffeomorphism, the total space of the relative Picard variety for a fibration with a section is independent of $r$; hence the proof given for $r = 2g - 2$ in [DS] is sufficient. We recall the main point: any torsion-free sheaf of degree $r$ on an irreducible nodal curve is either locally free or is the push-forward of a locally free sheaf of degree $r - 1$ on the normalisation. This follows from the existence of a short exact sequence

$$0 \to \mathcal{O}_{\tilde{C}} \to \pi_*\mathcal{O}_{\tilde{C}} \to \mathbb{C}_{(p)} \to 0$$

where $\pi : \tilde{C} \to C$ is the normalisation and the skyscraper sheaf $\mathbb{C}_{(p)}$ is supported at the node. The locally free sheaves on $C$ come from degree $r$ locally free sheaves on $\tilde{C}$ with a gluing parameter $\lambda \in \mathbb{C}^*$ to identify the fibres of the line bundle at the two preimages of the node. The compactification by adding torsion free sheaves arises from compactifying $\mathbb{C}^*$ (as a $\mathbb{C}^* \times \mathbb{C}^*$-space) to $\mathbb{C}\mathbb{P}^1$ and gluing together the 0- and $\infty$-sections over a translation in the base, cf. the Appendix to [DS]. The resulting variety has normal crossings and the total space of the relative Picard scheme is easily checked to be smooth, modelled transverse to the singularities of the central fibre on a family of semistable elliptic curves.

For the relative Hilbert scheme, we can argue as follows. Clearly we have smoothness away from the singular fibres, and more generally when the subscheme is supported away from the nodes of fibres of $f$. Moreover, given a zero-dimensional subscheme of a nodal fibre $\Sigma_0$ of $f$ supported at a collection of points $x_1, \ldots, x_s$ we can take product charts around each of the $x_i$ and reduce to the situation at which all of the points lie at the node of $\Sigma_0$. Hence it will be sufficient to prove smoothness for the local model: that is, for the relative Hilbert scheme $\mathbb{C}_r(f)$ of the map $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $(z, w) \mapsto zw$. 
According to Nakajima [Nak99], the Hilbert scheme $\text{Hilb}^{[r]}(\mathbb{C}^2)$ is globally smooth and may be described explicitly as follows. Let $\tilde{\mathcal{H}}$ denote the space

$$\{(B_1, B_2, v) \in \mathcal{M}_r(\mathbb{C})^2 \times \mathbb{C}^r \mid [B_1, B_2] = 0, (\ast)\}$$

where the stability condition $(\ast)$ asserts that for any $S \subseteq \mathbb{C}^r$ invariant under both $B_1$ we have $v \not\in S$. There is a $GL_r(\mathbb{C})$ action on $\tilde{\mathcal{H}}$ defined by

$$g \cdot (B_1, B_2, v) \mapsto (g B_1 g^{-1}, g B_2 g^{-1}, g v)$$

and the stability condition implies that this is free. Moreover, the cokernel of the map

$$\tilde{\mathcal{H}} \to \mathcal{M}_r(\mathbb{C}); \quad (B_1, B_2, v) \mapsto [B_1, B_2]$$

has constant rank $r$ (the stability condition shows that the map $\xi \mapsto \xi(v)$ is an isomorphism from the cokernel to $\mathbb{C}^r$). It follows that $\tilde{\mathcal{H}}$ is smooth, and the freeness of the action gives a smooth structure on the quotient $\mathcal{H} = \tilde{\mathcal{H}}/GL_r(\mathbb{C})$; but this quotient is exactly $\text{Hilb}^{[r]}(\mathbb{C}^2)$.

To see this, note that an ideal $\mathcal{I} \subset \mathbb{C}[z, w]$ of length $r$ defines a quotient vector space $V = \mathbb{C}[z, w]/\mathcal{I}$ of dimension $r$; now define endomorphisms $B_1, B_2$ via multiplication by $z, w$ respectively, and $v$ by the image of 1. Conversely, given $(B_1, B_2, v)$ we set $\mathcal{I} = \ker(\phi)$ for $\phi : \mathbb{C}[z, w] \to \mathbb{C}^r$ defined by $f \mapsto f(B_1, B_2)v$; this is the inverse map. There is an obvious family of zero-dimensional subschemes over $\mathcal{H}$; given any other such family $\pi : Z \to W$, there is a locally free sheaf of rank $r$ over $W$, given by $\pi_* \mathcal{O}_Z$. Taking a cover of $W$ and trivialising locally, we can define multiplication maps $B_i$ by the co-ordinate functions $z_i$ and hence show this family is indeed a pullback by a map $W \to \mathcal{H}$, which gives the required universal property.

This gives a simple description of the relative Hilbert scheme $\mathcal{H}(f)$ as the subscheme of those ideals containing $zw - \lambda$ for some $\lambda \in \mathbb{C}$. Explicitly, setting $I_r$ for the identity element of $\text{End}(\mathbb{C}^r)$, we have:

$$\mathcal{H}(f) = \{(B_1, B_2, \lambda, v) \in \mathcal{M}_r(\mathbb{C})^2 \times \mathbb{C} \times \mathbb{C}^r \mid B_1B_2 = \lambda I_r, B_2B_1 = \lambda I_r, (\ast)\}$$

this still carries a free $GL_r(\mathbb{C})$ action (trivial on the $\lambda$-component) and the quotient is the relative Hilbert scheme $\mathcal{H}(f)$. Note that if $\lambda \neq 0$ then each of the $B_i$ is invertible, the first two equations are equivalent, and the two matrices are simultaneously diagonalisable; then the fibre over $\lambda \in \mathbb{C}^*$ is just the obvious copy of $\mathbb{C}^r = \text{Sym}^r(\mathbb{C})$. Smoothness of the total space now follows from the fact that the map

$$\mathcal{M}_r(\mathbb{C})^2 \oplus \mathbb{C} \to \mathcal{M}_r(\mathbb{C})^2; \quad (C_1, C_2, \mu) \mapsto (C_1B_2 + B_1C_2 - \mu I_r, B_2C_1 + C_2B_1 - \mu I_r)$$

(which is the differential of the defining equations) has constant dimensional kernel $r^2 + 1$, independent of $(B_1, B_2, \lambda, v) \in \mathcal{H}(f)$. We can see this with a routine if tedious computation. First, if $\lambda \neq 0$, then the equation $B_1B_2 = \lambda I_r$ shows that each $B_i$ is invertible; the two defining equations are then equivalent, and at any point of the kernel, $C_1$ is uniquely determined by $C_2$ and $\mu$ which can be prescribed arbitrarily:
Here $B_1 = \sum h_i B_i^1 v$, similarly $B_2 = \sum H_i B_i^2 v$, using the fact that the subspaces $\langle v, B_i v, B_i^2 v, \ldots \rangle$ are invariant under $B_j$, $j = 1, 2$. It is now easy to check that $h_0 = 0 = H_0$, from the condition that $B_1 B_2 = B_2 B_1 = 0$ and neither $B_1 \equiv 0$. Let us then write, in obvious notation, an element $(C_1, C_2, \mu)$ in the kernel of the differential as follows, in block matrix form for the $C_i$:

$$C_1 = \begin{pmatrix} \beta & \tau & \chi \\ \tau & \mu_{1,j} & \nu_{1,j} \\ \chi & \phi_{1,j} & \psi_{1,j} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \alpha & v & u \\ \bar{v} & m_{i,j} & n_{i,j} \\ \bar{u} & p_{i,j} & q_{i,j} \end{pmatrix}$$

Here $\tau, v$ are $(1 \times n)$-vectors, $\chi, u$ are $(1 \times m)$-vectors (and similarly for the $\bar{\cdot}$-entries); whilst $m_{i,j}, \mu_{i,j}$ are in $M_n(\mathbb{C})$, $q_{i,j}, \psi_{i,j} \in M_m(\mathbb{C})$ and the other entries are $(n \times m)$ or $(m \times n)$ blocks in the obvious way. In this schematic, the linearisation of the defining equations for our relative Hilbert scheme become:
\[
\mu I_r = B_2 C_1 + C_2 B_1 = \begin{pmatrix}
v_{1,1} & \Phi v \\
m_{i,1} & (m_{i,j}) \Phi \\
\beta + \Psi \chi + p_{j,1} & \tau + \Psi (\phi_{i,j}) + (p_{i,j}) \Phi \\
\end{pmatrix}
\]

and simultaneously the equation:

\[
\mu I_r = B_1 C_2 + C_1 B_2 = \begin{pmatrix}
\chi_1 & 0 \\
\alpha + v_{1,1} + (\bar{v})' & 0 \\
\psi_{k,1} & \chi' \\
\end{pmatrix}
\]

Here we have written \(\Phi\) and \(\Psi\) for the non-trivial \((n \times n)\), resp. \((m \times m)\), blocks in the matrices \(B_1\) and \(B_2\). In addition, \(\chi'\) is the vector \((\chi_2, \ldots, \chi_{m-1}, \sum H_{i+1} \chi_i)\) and \((\bar{v})'\) is similarly a linear combination of the entries of \(\bar{v}\) and the \(h_i\) (the precise formula is not really important, and easily worked out by the curious). First of all, we claim that necessarily \(\mu = 0\); i.e. the matrices above must vanish. From the first equation, we see from the left hand column that \(m_{i,1}\) vanishes for each \(i\), and hence the matrix \((m_{i,j})\) has trivial determinant; but then we cannot have \((m_{i,j}) \Phi = \mu I_n\) unless \(\mu = 0\). Hence the equations for our differential amount to the vanishing of the equations above. Clearly we can freely choose the parameters \(q_{i,j}, \mu_{i,j}, \tilde{u}\) and \(\tilde{\tau}\), which do not even appear on the RHS. A moment’s inspection shows that one can also prescribe \(\alpha, \beta, u, \tau\) freely, and then \(p_{i,j}\) and \(n_{i,j}\); then all the other data is determined. This is clear if \(\Phi\) and \(\Psi\) are invertible, just by manipulating the above. In this case, an element of the kernel is completely determined by the choice of

\[
\alpha, \beta, u, \tau, \tilde{u}, \tilde{\tau}, q_{i,j}, \mu_{i,j}, p_{i,j}, n_{i,j}
\]

of respective dimensions:

\[
1, 1, m, n, n, m, n^2, mn, mn.
\]

Hence the dimension of the kernel is \((m + n + 1)^2 + 1 = r^2 + 1\), as required. We claim this is still the case even if \(\Phi\) and \(\Psi\) are not necessarily invertible. For instance, we already know that the matrix \(m_{i,j}\) is of the form

\[
m_{i,j} = \begin{pmatrix} 0_{n,1} & m'_{i,j} \end{pmatrix}
\]

for some \((n \times (n-1))\)-matrix \(m'_{i,j}\). Then if in fact \(\Phi\) has trivial determinant, we can see that

\[
\Phi = \begin{pmatrix} 0_{n-1,1} & 0 \\ I_{n-1} & (h_j) \end{pmatrix}
\]

from which it easily follows that \((m_{i,j}) \Phi = 0 \Rightarrow (m_{i,j}) \equiv 0\). Then the rest of the data can be determined successively.

This leaves only the case where (without loss of generality) \(B_1 \equiv 0\). In this case, it is no longer true that \(\mu\) necessarily vanishes; but it is easy to check the kernel is now \((r^2 + 1)\)-dimensional. For instance, if \(\mu \neq 0\) then \(\mu\) determines \(C_2 = \mu B_1^{-1}\) and \(C_1\) can be chosen freely. It follows that the space \(\mathcal{H}(f)\) is determined everywhere by a map to a vector space of
constant rank, hence is smooth, and then the quotient by the free $GL_r(\mathbb{C})$ action is smooth by Luna’s slice theorem. Hence we have smooth compact manifolds. To put symplectic structures on the spaces $X_r(f)$ follows from a theorem of Gompf [Gom01], adapting an old argument of Thurston, and is discussed below. ■

(3.4) Remark: One can use the above to define the relative Hilbert scheme, as a smooth manifold via local charts, for readers wary of sheaf quotients. Note also that the above shows that we have a smooth compactification of the symmetric product fibration even when there are reducible fibres present. □

In fact the spaces $X_r(f)$ all have normal crossing singularities (they are locally of the form $\{zw = 0\} \times \{\text{Smooth}\}$). For large $r$ this follows from the result for the relative Picard fibration and the existence of the Abel-Jacobi map, whose total space is a projective bundle over the base once $r > 2g - 2$. Since the condition is local, the result for large $r$ can be used to deduce it in general. The $r$-th symmetric product of $\{zw = 0\}$ comprises the spaces $\mathbb{C}^r \times \mathbb{C}^s$ glued together along various affine hyperplanes, and the Hilbert scheme is obtained by successively blowing up the strata; the projective co-ordinates in the normalisation are described in [Nak99] and the Appendix to [DS]. A “conceptual” proof of smoothness from this point of view is also given there, though the computations above apply more easily to the case of small $r$.

Since we have integrable complex structures near the singular fibres, we can extend (for instance via a connexion) to obtain almost complex structures on the total spaces of the $X_r(f)$. For us, the holomorphic structure on the fibres is always standard. The existence of (a canonical deformation equivalence class of) symplectic structures on $X_r(f)$ runs as follows.

The obvious integrable Kähler forms on the fibres patch to give a global two-form $\Omega_0$; then if $\omega_{st}$ is the area form on the sphere, with total area 1, the forms $\Omega_t = \Omega_0 + tF^*\omega_{st}$ are symplectic for all sufficiently large $t$.

Write $\mathcal{J} = \mathcal{J}_r$ for the class of almost complex structures $J$ on $X_r(f)$ which agree with the standard integrable structures on all the fibres and which agree with an integrable structure induced from an almost complex structure on $X'$ in some tubular neighbourhood of each critical fibre. Usually we will deal with smooth almost complex structures, though in the next section it shall be important to allow ones that are only Hölder continuous. In any case, all our almost complex structures shall be of this form.

(3.5) Lemma: The space $\mathcal{J}$ is non-empty and contractible. A given $J \in \mathcal{J}$ is tamed by the symplectic forms $\Omega_t$ for all sufficiently large $t > t(J) \in \mathbb{R}_+$.

This is just as in [DS]; if we fix the neighbourhoods of the critical fibres where the structure is induced by an integrable structure on $X$ then the remaining choice is of a section of a fibre bundle with contractible (affine) fibres. Note that the complex structures $J$ will not be compatible with $\Omega_t$; the class of compatible structures is not large enough to achieve regularity for spaces of holomorphic sections.
Let us recall various facts concerning the geometry of the Abel-Jacobi map. As we observed at the start of the section, the fibres of $u$ are projective spaces (linear systems). We can check their generic dimension at the singular fibre. A locally free sheaf $L$ on $C_0$ is given by a line bundle $\tilde{L}$ on $\tilde{C}_0$ with a gluing map $\lambda: \mathbb{C} \to \mathbb{C}$ of the fibres over the two preimages $p, q$ of the node. Since $\tilde{C}_0$ is of genus $g - 1$ when $C_0$ has (arithmetic) genus $g$, if $L$ has degree $d$ then by the Riemann-Roch theorem we find that $\text{Index} \tilde{L}(\partial) = d - (g - 1) + 1$. Generically this is the dimension of $H^0(\tilde{L})$ and then $H^0(L) = \{ s \in H^0(\tilde{L}) \mid s(p) = \lambda s(q) \}$.

Provided $\lambda \in \mathbb{C}^*$ and not every section vanishes at $p, q$ this is a hyperplane; hence $h^0(L) = d - g + 1$. Thus the generic dimension of the fibres of the Picard fibration is the same over the singular fibre as over the smooth fibres. There are two different behaviours we should emphasise:

- Along the normal crossing divisor of the singular fibre of $P_r(f)$ we have $\lambda \in \{0, \infty\}$ (these two cases are glued together, cf. [DS]). Here all sections represent Weil divisors which are not Cartier and do not arise from locally free sheaves; for instance, any subscheme supported at the node to order exactly one is of this form.

- We also have an embedding (of varieties of locally free sheaves) $\text{Pic}_d(C_0) \to \text{Pic}_d(C_0)$ induced by the embedding $\text{Pic}_{d-2}(\tilde{C}_0) \to \text{Pic}_d(\tilde{C}_0)$ which takes a line bundle $\tilde{L} \to \tilde{L} \otimes O(p + q)$. At these points, the space of sections of $H^0(L)$ is the entirety of the space of sections of $H^0(\tilde{L} \otimes O(-p - q))$ - the parameter $\lambda$ plays no role now - which although not a hyperplane again has the right dimension, since the bundle upstairs has a different degree.

In particular, surfaces in the four-manifold $X'$ which pass through a node of a singular fibre can arise from smooth sections of the symmetric product fibration (necessarily disjoint from the singular fibre of the Abel-Jacobi map as over the smooth fibres). There are two different behaviours we should emphasise:

- Along the normal crossing divisor of the singular fibre of $P_r(f)$ we have $\lambda \in \{0, \infty\}$ (these two cases are glued together, cf. [DS]). Here all sections represent Weil divisors which are not Cartier and do not arise from locally free sheaves; for instance, any subscheme supported at the node to order exactly one is of this form.

- We also have an embedding (of varieties of locally free sheaves) $\text{Pic}_d(C_0) \to \text{Pic}_d(C_0)$ induced by the embedding $\text{Pic}_{d-2}(\tilde{C}_0) \to \text{Pic}_d(\tilde{C}_0)$ which takes a line bundle $\tilde{L} \to \tilde{L} \otimes O(p + q)$. At these points, the space of sections of $H^0(L)$ is the entirety of the space of sections of $H^0(\tilde{L} \otimes O(-p - q))$ - the parameter $\lambda$ plays no role now - which although not a hyperplane again has the right dimension, since the bundle upstairs has a different degree.

In particular, surfaces in the four-manifold $X'$ which pass through a node of a singular fibre can arise from smooth sections of the symmetric product fibration (necessarily disjoint from the locus of critical values for the projection $F$). If this was not the case, we could not hope to obtain a compactness theorem for spaces of sections.

In [DS] we didn’t identify the monodromy of these associated families of Jacobians or symmetric products, so let us do that here. Remark for the readers’ convenience that this shall not be used later on and is included just for the sake of intuition. The author learned this general material from unpublished notes of Paul Seidel [Sei97].

**Definition:** Let $(X^{2n}, \omega_X)$ be a symplectic manifold. Be given a symplectic manifold $(Y^{2n-2}, \omega_Y)$ and a principal circle bundle $S^1 \to W \xrightarrow{\pi} Y$, together with an embedding $\iota: W \hookrightarrow X$ which satisfies $\iota^*\omega_X = \pi^*\omega_Y$. Then a generalised Dehn twist along $W$ is a symplectomorphism $\phi: X \to X$ with the following two properties:

- $\phi$ is the identity outside a tubular neighbourhood of $W$;

- on each circle fibre $\phi$ acts as the antipodal map.

(If $n = 1$ and $Y$ is a point then this is a Dehn twist along a curve in a real surface in the classical sense.)
It is not hard to show that such a symplectomorphism \( \phi \) always exists in this situation (and indeed for more general coisotropic embeddings of sphere bundles with orthogonal structure group). An important result, due to Seidel, is that the data distinguishes a unique Hamiltonian isotopy class of symplectomorphism containing generalised Dehn twists. With this terminology established, we then have:

**Proposition:** Let \( X \) be a family of genus \( g \) curves over the disc \( D \) with a single irreducible nodal fibre \( \Sigma_0 \) over 0. Write \( \tilde{\Sigma}_0 \) for the normalisation of this nodal fibre.

- The monodromy of the Picard fibration \( P_r(f) \to D \) around \( \partial D \) is a generalised Dehn twist along a circle bundle over an embedded copy of \( \text{Pic}_{r-1}(\tilde{\Sigma}_0) \).
- The monodromy of the relative Hilbert scheme \( X_r(f) \to X \) around \( \partial D \) is a generalised Dehn twist along a circle bundle over an embedded copy of \( \text{Sym}^{r-1}(\tilde{\Sigma}_0) \).

In each case, the base of the circle bundle is isomorphic to the singular locus of the fibre over 0, and the singularity arises as the circle fibres shrink to zero size as we approach the origin of the disc.

**Proof:** [Sketch] The point is that for any holomorphic fibration with a normal crossing fibre over \( 0 \in D \), the monodromy is a generalised Dehn twist of this form. This is because the normal crossing data defines a unique local model in a neighbourhood of the singular locus: the circle bundle is just the bundle of vanishing cycles defined with respect to some symplectic connexion arising from a local Kähler form. Seidel’s notes give a detailed construction of the symplectomorphism from this data. It follows that it is enough to identify the singular locus of the singular fibres. (Of course this discussion holds for \( X \) itself. If we fix a smooth fibre over a point \( t \in D^* \), an embedded ray \( \gamma \) from \( t \) to 0 in \( D \) and a symplectic form on the total space of \( X \), then the fibre \( X_t \) contains a distinguished real circle, which is the vanishing cycle associated to the critical point of \( f \) by the path \( \gamma \). The monodromy is a Dehn twist about this circle.)

- The Picard fibration \( W \), up to diffeomorphism (for instance on choosing a local section), is just the total space of the quotient \( R^1f_*(\mathcal{O})/R^1f_*(\mathbb{Z}) \). The singular locus of \( W_0 \) is just a copy of the Jacobian of the normalisation of the singular fibre \( X_0 \):

\[
\text{Sing}(W_0) \cong \text{Jac}(\tilde{X}_0) \cong \mathbb{T}^{2g-2}.
\]

This follows from the explicit construction of the generalised Jacobian given in the Appendix to \([DS]\). In fact we can see clearly why the monodromy of the fibration is a generalised Dehn twist. From our smooth description, the total space is diffeomorphic to \( \text{Jac}(\tilde{X}_0) \times E \), with \( E \) a fibration of elliptic curves with a unique semistable nodal fibre. This is because the integral homology lattices give a flat connexion on a real codimension two subbundle of the homology bundles. The monodromy is just \( \text{id} \times \tau \), where \( \tau \) is the diffeomorphism induced by (the homological action of) the Dehn twist about \( \gamma \) on the subspace of \( H_2(X_t) \) generated by \( \gamma \) and a transverse longitudinal curve. Invariantly, the singular locus is the degree \( r - 1 \) Picard variety of the normalisation since the natural map \( \mathcal{O}_{X_0} \to \pi_*\mathcal{O}_{\tilde{X}_0} \) has cokernel of length one.
4 COUNTING STANDARD SURFACES

• Consider now the associated family $Z = \text{Hilb}^{[r]}(f)$. When $r$ is large (say $r > 2g - 2$), we can use the above and the Abel-Jacobi map. For then the Hilbert scheme is a projective bundle over the Picard variety, and the singular locus of the zero fibre is a projective bundle over $\text{Pic}_{r-1}(\tilde{\Sigma}_0)$. The total space of this bundle is isomorphic to a copy of $\text{Sym}^{r-1}(\tilde{\Sigma}_0)$. Naively this description breaks down for small $r$, although the result is still true; it is a consequence, for $r = 2$, of the examples described in the Appendix to [DS], and more generally of the fact that the singular locus represents precisely Weil non-Cartier divisors. Now the fact that every torsion-free sheaf is either locally free or the push-forward from the normalisation of something locally free implies the Proposition; sections of sheaves $\pi_* L$ correspond to the obvious linear systems on the normalisation. ■

4 Counting standard surfaces

In this section, we present the holomorphic curve theory for our associated fibrations. We shall compute the virtual dimensions of moduli spaces, prove that in fact they are a priori compact for regular almost complex structures, and define the appropriate Gromov invariant counting sections.

4.1 Sections, cycles and index theory

From a Lefschetz pencil $(X, f)$ we build the fibration $X'$ and the associated fibrations $F = F_r : X_r(f) \to S^2$. Suppose $s : S^2 \to X_r(f)$ is a smooth section. At every point $t \in S^2$ we have a collection of $r$ points on the fibre $\Sigma_t$ of $f : X' \to S^2$. As $t$ varies these trace out some cycle $C_s \subset X'$ and the association $s \mapsto [C_s]$ defines a map $\phi : \Gamma(F) \to H_2(X'; \mathbb{Z})$ from homotopy classes of sections of $F$ to homology classes of cycles in $X'$.

(4.1) Lemma: The map $\phi : \Pi_t \Gamma(F_r) \to H_2(X'; \mathbb{Z})$ is injective.

Proof: Clearly the image homology class determines $r$, as the algebraic intersection number with the fibre of $f$. So we can restrict to a single $F = F_r$ henceforth. We will construct a “partial inverse”. Fix $A \in H_2(X'; \mathbb{Z})$; this defines a complex line bundle $L_A \to X'$. A fixed symplectic form on $X'$ defines a connexion (field of horizontal subspaces) on the smooth locus of $f : X' \to S^2$. As $t$ varies these trace out some cycle $C_s \subset X'$ and the association $s \mapsto [C_s]$ defines a map $\phi : \Pi_t \Gamma(F) \to H_2(X'; \mathbb{Z})$ from homotopy classes of sections of $F$ to homology classes of cycles in $X'$.

$\phi : \Gamma(F_r) \to H_2(X'; \mathbb{Z})$ is injective.

Proof: Clearly the image homology class determines $r$, as the algebraic intersection number with the fibre of $f$. So we can restrict to a single $F = F_r$ henceforth. We will construct a “partial inverse”. Fix $A \in H_2(X'; \mathbb{Z})$; this defines a complex line bundle $L_A \to X'$. A fixed symplectic form on $X'$ defines a connexion (field of horizontal subspaces) on the smooth locus of $f : X' \to S^2$; also pick a connexion $\nabla$ on the total space of the principal circle bundle $P_A \to X'$. Now consider the circle bundle $P_t$ given by restricting $P_A$ to a smooth fibre $C_t = f^{-1}(t)$ of the Lefschetz fibration. At each point $u \in P_t$ we have a natural decomposition of the tangent space of $(P_A)_u$ given by

$$T_u(P_A) = T_u(S^1) \oplus T_u(C_t) \oplus f^* T_f(u)(S^2)$$

using first the connexion $\nabla$ and then the symplectic form. In particular, the tangent space $T_u(P_t)$ is naturally split and this splitting is obviously invariant under the action of $S^1$; hence we induce a natural connexion on each of the complex line bundles $L_t = L_A|_{C_t}$. But every connexion induces an integrable holomorphic structure over a one-dimensional complex manifold (i.e. for each $L_t$ we have $\bar{\partial}^2 = 0$). Thus the (contractible) choice of
connexion $\nabla$ gives rise to a distinguished section of the bundle $P_r(f)$ which is just the class of the holomorphic line bundle $L_t \to C_t$ as $t$ varies over $S^2$. Riemann’s removable singularities theorem, with holomorphic data near the singular fibres, takes care of the extension even where the connexion becomes singular. In other words, $H_2(X'; Z)$ is in one-to-one correspondence with the set of different homotopy classes of sections of bundles $P_r(f)$ as $r$ varies. (One can also see this by thinking of connexions on $L_t$ as an infinite dimensional affine space; any fibre bundle of such spaces admits a section.)

Now a section of $P_r(f)$ may not lift to any section of $X_r(f)$, but if it does lift then the different homotopy classes of lifts differ by at most an action of $Z$ which acts by adding a generator $h$ of $\pi_2(\mathbb{P}^N)$ to a given section: here $\mathbb{P}^N$ is a linear system fibre of the Abel-Jacobi map. (These projective spaces may be empty, or have varying dimension as we move the section by homotopy, but they contribute at most a single integral class to $H_2(X'; Z)$.) Now it is easy to check directly that if a section $s$ of $X_r(f)$ defines a homology class $A_s \in H_2(X'; Z)$ then $s + h$ defines a homology class $A_s + [\text{Fibre}]$, and these of course differ. It follows that at most a single homotopy class of section can yield any given $A$ under the above map $\phi$. $
$
We will also need to introduce the twisting map $\iota : H_2(X; Z) \to H_2(X'; Z)$ which takes a cycle $C \subset X$ to the cycle $C \cup E$ where $E$ is the union of the exceptional sections of $f$, each taken with multiplicity one. Clearly this is an embedding of $H_2(X; Z)$ onto a direct summand of $H_2(X'; Z)$. We will sometimes identify homology classes on $X$ with elements in the homology of $X'$ under the natural map $i$ on $H_2$ induced by the blow-down map; that is, use Poincaré duality and the pullback on $H^2$, or just choose a cycle not passing through the points we blow-up and take its preimage. Note this “obvious” embedding $i : H_2(X; Z) \to H_2(X'; Z)$ is different from the twisting map: $\iota(\alpha) = i(\alpha) + \sum E_i$. With this convention, for any $\alpha \in H_2(X; Z)$ there is an equality

$$\alpha^2 - \alpha \cdot K_X = \iota(\alpha)^2 - i(\alpha) \cdot K_{X'}.$$  \hfill (4.2)

where $\cdot$ denotes intersection product and we identify the canonical classes with their Poincaré duals. This reflects the basic “blow-up” formula for Gromov invariants in four-manifolds: the virtual dimension of the space of holomorphic curves in a four-manifold $W$ in a class $A$ is the same as the dimension of curves on $W'$ in the class $A + E$, where $E$ is the exceptional divisor of a blow-up $W' \to W$. Indeed the actual invariants co-incide; the analogue of this for the spaces $X_r(f)$ will be important later. To end the section, we shall give an index result.

\textbf{(4.3) Proposition:} Let $\phi(s) = \alpha \in H_2(X'; Z)$ where $s \in \Gamma(F)$. Then the complex virtual dimension of the space of pseudoholomorphic sections of $F : X_r(f) \to S^2$ in the homotopy class $s$ is given by $[\alpha^2 - K_X \cdot \alpha]/2$.

\textbf{Proof:} By standard arguments, the virtual dimension is given by the sum of the rank of the vertical tangent bundle and its first Chern class evaluated on the homology class of the section. We can fix a smooth section which passes through the open dense set $\text{Sym}^r(\Sigma_0 \setminus \{\text{Node}\})$ at each critical fibre, and then we are just working with the vertical tangent bundle of a family of symmetric products of curves. It will be helpful to adopt the universal viewpoint: via a
4 COUNTING STANDARD SURFACES

generic choice of fibrewise metrics, regard $X' \to S^2$ as smoothly embedded inside the total space of the universal curve $C_g \to \overline{M}_g$. Indeed, we can form the fibre product $Z = C \times_{\mu} \text{Sym}^{r}(C)$ of the universal curve and the universal relative Hilbert scheme. This contains a universal divisor $\Delta$, which is just the closure of the obvious fibrewise divisor in $\Sigma \times \text{Sym}^{r}(\Sigma)$ where $\Sigma \in M_g \subset \overline{M}_g$. Write $\pi$ for the projection $Z \to \text{Sym}^{r}(C)$ to the universal Hilbert scheme. Then the vertical tangent bundle to $\mu : \text{Sym}^{r}(C) \to \overline{M}_g$ is exactly $\pi^*O_\Delta(\Delta)$. This follows from the naturality of the construction of \cite[pages 171-3]{A}. From the exact sequence

$$0 \to O \to O(\Delta) \to O_\Delta(\Delta) \to 0$$

of bundles on $Z$, and the associated long exact sequence in cohomology, it follows that

$$ch(T^v(\mu)) = ch(T^v(Pic)) + ch(\pi^*O(\Delta)) - 1.$$

Here the constant term 1 comes from $ch(\pi^*O)$. If we take the degree zero terms in the above, we get an equation in the ranks of the bundles:

$$r = g + (r - g + 1) - 1$$

using Riemann-Roch on each fibre; this checks! We have also used the fact that $R^1\pi_*O_\Delta(\Delta) = \{0\}$, which holds since skyscraper sheaves on curves have no higher cohomology. The term $R^1\pi_*O$ has fibre at $(D \in \Sigma) \in \text{Sym}^{r}(C)$ the cohomology group $H^1(\Sigma; O)$, independent of $D$. This is just the pullback from $\overline{M}_g$ of the vertical tangent bundle to the Picard bundle.

We can apply the Grothendieck-Riemann-Roch theorem to the term $ch(\pi^*O(\Delta))$. This yields

$$ch(\pi^*O(\Delta)) = \pi_*[O(\Delta)]\text{Todd}(T^v(\pi)).$$

We need only work up to degree four on the RHS since we will evaluate the push-forward on a sphere $[S^2]$. The relevant terms of the Chern character of the line bundle $O(\Delta)$ are then $1 + [\Delta] + [\Delta]^2/2$. The vertical tangent bundle to $\pi$ is just (the pullback of) the vertical tangent bundle to the universal curve. Hence, the degree four term on the RHS is given by

$$[\Delta]^2/2 - \omega_{C/M_g} \cdot [\Delta]/2 + \text{Todd}_{deg(4)}(T^v(\pi)). \quad (4.4)$$

Now evaluate on the sphere arising from the Lefschetz fibration $X' \to S^2$; by construction, the class $\Delta$ gives $\ell(\alpha)$ whilst the first Chern class of the relative dualising sheaf gives the canonical class of $X'$ twisted by the canonical class of the base $\mathbb{P}^1$: $c_1(\omega) = K_{X'} - f^*K_{\mathbb{P}^1}$. This twist means that $c_1(\omega) \cdot [\Delta]$ is just $\alpha \cdot K_{X'} + 2r$, where $r = \alpha \cdot [\text{Fibre}]$ inside $X'$. The third term in (4.4) cancels with the term coming from $T^v(Pic)$ above; indeed the relative dualising sheaf is dual to the vertical tangent bundle to the Picard bundle. When we put these pieces together, we find that

$$\text{virdim}_{C}(\mathcal{M}_J(s)) = r + \alpha^2/2 - K_{X'} \cdot \alpha/2 - r$$

where the first $r$ comes from the rank of the vertical tangent bundle and the second from the discrepancy between $K_{X'}$ and $K_{X'} - f^*K_{\mathbb{P}^1}$. This gives the required answer. \qed
4.2 Defining the invariants

We need one last ingredient which was not relevant in the discussions of [DS]; a method for cutting down the dimensions of moduli spaces. Any point \( z \in X \) disjoint from the base-points and critical points of the Lefschetz pencil gives rise to a smooth divisor \( D(z) \) in the fibre over \( f(z) \) of \( X_r(f) \). To define this divisor, note that there is a holomorphic map

\[
\text{Sym}^{r-1}(\Sigma) \to \text{Sym}^r(\Sigma); \quad D \mapsto D + z
\]

whenever \( z \in \Sigma \) is a point of a fixed Riemann surface \( \Sigma \). The image of the map is a smoothly embedded copy of \( \text{Sym}^{r-1}(\Sigma) \) characterised as precisely those points whose support contains the point \( z \). If we take \( \Sigma = f^{-1}(f(z)) \) then the image of the map above is exactly \( D(z) \subset \text{Sym}^r(\Sigma) \), where we suppress the index \( r \) for clarity. Whenever we discuss these divisors \( D(z_i) \), for points \( z_i \in X \), we will assume that the points have been chosen generically and in particular lie in \( X \setminus (\{p_i\} \cup \{q_j\}) \). Given this background, we can now define the invariants which play a fundamental role in this paper. Recall the maps \( i \) and \( \iota \) taking \( H_2(X) \) to \( H_2(X') \).

(4.5) Definition: Let \( X \) be any symplectic four-manifold and choose a Lefschetz pencil \( f \) on \( X \) (always supposed to be of high degree). Fix \( \alpha \in H_2(X;\mathbb{Z}) \). The standard surface count \( I(X,f)(\alpha) \) is defined as follows: for each \( r \), fix some generic \( J \in J_r \).

1. If \( i(\alpha) \notin \text{im} (\phi) \) then \( I(X,f)(\alpha) = 0 \);
2. If \( \alpha^2 - K \cdot \alpha < 0 \) then \( I(X,f)(\alpha) = 0 \);
3. If \( i(\alpha) = \phi(s) \), with \( s \in \Gamma(F_r) \) and \( [\alpha^2 - K \cdot \alpha]/2 = m \geq 0 \), then \( I(X,f)(\alpha) \) is the Gromov invariant \( \text{Gr}_{X_r(f)}(s;z_1,\ldots,z_m) \) which counts \( J \)-holomorphic sections of \( F \) in the class \( s \) passing through the fibre-divisors \( D(z_i), 1 \leq i \leq m \).

Note that it is not \textit{a priori} clear that this is independent of the choice of Lefschetz pencil; for applications coming from non-vanishing results that’s not actually important. The next result explains why the invariant is indeed well-defined. Our treatment is formal, since the required Sobolev machinery is standard; note that we can do without the virtual fundamental class machinery of Li and Tian, via a compactness theorem. The second part of the Theorem is a “blow-up” formula to help with later computations.

(4.6) Theorem: Let \( (X,f) \) be a symplectic Lefschetz pencil (with integral symplectic form) and fix a compatible almost complex structure \( J \) on \( X \). Use this to define the associated fibre bundles as above, and equip these with almost complex and symplectic structures. Fix \( \alpha \in H_2(X;\mathbb{Z}) \).

- The invariant \( I(\alpha) \) (when not defined to be zero) can be computed as the signed count of the points of a compact zero-dimensional moduli space in which each point has a uniquely attached sign \( \pm 1 \).
- \( i(\alpha) \in \text{im} (\phi) \iff \iota(\alpha) \in \text{im} (\phi) \) and in obvious notation \( I(\alpha) = I(\iota(\alpha)) \).
Proof: Most of this is implied by the arguments of [DS]. Let us review the principal parts.

- For generic $J$ on $X_r(f)$, compatible with the fibration (i.e. $J \in \mathcal{J}$, inducing the given integrable structures on the fibres), the moduli space of holomorphic sections in a given homotopy class $s$ will be a smooth open manifold of real dimension

$$d(s) = C_{\phi(s)} \cdot C_{\phi(s)} - K_{X'} \cdot C_{\phi(s)}.$$

According to general theory, we can compactify the space by adding two kinds of element. First, the cusp sections: that is, holomorphic sections which may have a number of bubbles, necessarily lying in fibres of $F : X_r(f) \to \mathbb{S}^2$. Second, we add curves which are not actually sections in the sense that we may add curves passing through the critical values of $F$. We need to know that the points we add do not have excess dimension. For the cusp curves, argue as follows. Any bubble in any fibre projects to a sphere in the Picard fibration $P_r(f)$. This either lies in the smooth part of a fibre or lifts to the normalisation of a singular fibre. Arguing as in Lemma (8.11) of [DS] - which applies unchanged to the case of general $r$ and not just $r = 2g - 2$ - the latter situation cannot occur and hence any bubble represents a multiple of the generator of a projective space fibre for some linear system. If we split off $n$ bubbles, the remaining section component $s'$ maps under $\phi$ to the homology class $C_{\phi(s)} - n[Fibre]$. Provided at least the degree of the pencil is large, so $C \cdot [Fibre] \gg 0$, the virtual dimension for holomorphic sections in this new class will be very negative. Then, by the assumed regularity of $J$, when the degree of the pencil is large enough the total dimension of the space of cusp curves is negative and hence there are no bubbles in any moduli spaces.

To see that the curves passing through the critical values cannot be of excess dimension, the argument is even stronger. Namely, we claim that no curve passing through the critical loci can arise as the limit of a sequence of smooth sections for any $J$. For although the limit curve is (in the appropriate Sobolev setting) only a $L^{2,2}$-map, the section component $s$ is a holomorphic curve for a smooth almost complex structure on an almost complex manifold and hence is smooth by elliptic regularity. The composite map $F \circ s : \mathbb{S}^2 \to \mathbb{S}^2$ is then holomorphic and degree one away from the isolated point of intersection with the normal crossing divisor in the singular fibre. Hence the map is a degree one diffeomorphism everywhere, and hence $s$ is indeed a section; so it cannot pass through any point of the total space where $dF = 0$. We have already ruled out bubble components, and hence there are no curves passing through the singular loci.

We deduce that the moduli space of holomorphic sections carries a fundamental class of the correct (virtual) dimension. Global orientability of the moduli space follows since the $\overline{\partial}$-operator of a non-integrable $J$ is just a zeroth order perturbation of the usual $\overline{\partial}$-operator. The standard surface count, as we have defined it, is given by cutting down the moduli space to be zero-dimensional. Note that cutting down the moduli spaces introduces no analytic problems since we are only dealing with spheres, in which case evaluation maps are always submersions by a result of [MS94]. At this stage, orientability is exactly the assignment of a sign to each point, giving the first part of the above theorem. That we really have an invariant, independent of the choice of almost complex structure, follows from the usual cobordism argument for one-parameter families together with the observation (3.5).

- To see that it is equivalent to count holomorphic sections yielding surfaces in a class $i(\alpha)$
or $i(\alpha)$ in $H_2(X';\mathbb{Z})$, we use the analogue of the divisors $D(z)$ coming from the exceptional sections. That is, notice an exceptional curve $E$ defines a point $e_t \in \Sigma_t$ for each $t \in \mathbb{S}^2$ and hence a smooth submanifold $X_{t-1}^E(f) \to X_r(f)$. As in [DS], there are almost complex structures on $X_r(f)$ which give rise to smooth almost complex structures on each $X_{t-1}^E(f)$ and for which the natural inclusions are holomorphic. Choose $J$ on $X_r(f)$ regular amongst almost complex structures in $J$ with this property; then the compactified space of $J$-sections yielding curves in the class $i(\alpha)$ will be a smooth compact manifold where none of the curves contain bubbles. We claim that in fact all of these sections lie inside the intersections of the images of all the $X_{t-1}^E(f)$, and that this defines an isomorphism of moduli spaces $\mathcal{M}_J(i(\alpha)) \sim \mathcal{M}_J(i(\alpha))$. Suppose otherwise; then some element $s \in \mathcal{M}_J(i(\alpha))$ does not lie inside $X_{t-1}^E(f)$, and hence meets it with locally positive intersection. Hence the cycle $C_s$ in $X'$ meets $E$ with locally positive intersection; but the algebraic intersection number is just $E \cdot [C_s] = E \cdot [\alpha + \sum E_i] = -1$, and this yields a contradiction.

So all the sections lie inside all the loci defined by exceptional sections, and hence give rise to canonical holomorphic sections of the intersection $\cap E X_{t-1}^E(f) \sim X_{r-N(E)}(f)$, where $N(E)$ is the number of exceptional curves. But then $i(\alpha) \cdot [\text{Fibre}] = i(\alpha) \cdot [\text{Fibre}] - N(E)$, and it is easy to check this yields the required isomorphism of moduli spaces. The virtual dimensions for the classes coincide, so we cut down with a fixed set of $D(z_i)$ to obtain isomorphic compact zero-dimensional moduli spaces. The reader can easily check that the signs associated to points agree; the result follows. ■

Let us draw attention to a part of the above, a “fibred monotonicity” property:

(4.7) Corollary: For a pencil $f$ of high degree on $X$, and any generic almost complex structure $J \in \mathcal{J}$ on $F : X_r(f) \to \mathbb{S}^2$, all moduli spaces of smooth holomorphic sections of $F$ are already compact.

The above arguments show that whenever we have a non-zero standard surface count for a class $i(\alpha)$ we can find symplectic surfaces in $X'$ disjoint from the $E_i$ and hence push them down to surfaces in $X$. The importance of this is that it allows us to stabilise the intersection number $r$ to be large, and hence take advantage of the geometry of the Abel-Jacobi map. On the other hand, the blow-up identity is also important; for instance, $i(\alpha)$ and $i(K_X - \alpha)$ meet the fibres of a Lefschetz pencil respectively $r$ and $2g - 2 - r$ times, and this is where Serre duality will enter. Indeed, the explicit computation of the standard surface count for the canonical class in [DS] involved dealing with a class of almost complex structures - those compatible with the zero-sections - which were tailored to the Abel-Jacobi map, and it is a strict generalisation of this latter class of structures which will underlie Theorem (1.1).

5 Standard surfaces and holomorphic curves

In the previous section we defined new invariants of Lefschetz pencils, but did not relate them to symplectic submanifolds in $X$. The point is that, for appropriate complex structures, the sections of $X_r(f)$ provided by the non-triviality of a standard surface count yield standard symplectic surfaces. In [DS] we reached this conclusion via an intermediate stage:
we constructed unions of such surfaces with positive local intersections and then applied a smoothing lemma. This has the advantage of simplicity, but the disadvantage of remaining bound within the realm of integral symplectic manifolds: we always assumed the ray $\mathbb{R}_+ \langle \omega_X \rangle$ defined a point of $\mathbb{P}H^2(X; \mathbb{Q}) \subset \mathbb{P}H^2(X; \mathbb{R})$. To avoid this assumption, here we will build holomorphic curves in $X$ from non-triviality of the $I$-invariants. Additionally, we will sketch a proof of the result “$I = Gr \ (mod\ 2)$”, at least in the absence of square zero tori. This result is intended to be motivational, and to make the present paper coherent, more than to make serious headway on a full proof of (1.3).

5.1 Almost complex diagonals

To obtain symplectic surfaces, we need to use special almost complex structures on $X_r(f)$, and for this we need to introduce the diagonal strata (cf. [DS], Section 6). For each partition $\pi$ of the form $r = \sum a_i n_i$ and each smooth fibre, we have a map

$$\text{Sym}^{n_1}(\Sigma) \times \cdots \times \text{Sym}^{n_s}(\Sigma) \to \text{Sym}^r(\Sigma); \quad (D_1, \ldots, D_s) \mapsto \sum a_i D_i.$$ We induce a smooth map of fibre bundles

$$Y_\pi = X_{n_1}(f) \times_f \cdots \times_f X_{n_s}(f) \to X_r(f)$$

which is finite and generically a homeomorphism onto its image. An almost complex structure $J$ on $X_r(f)$ is compatible with the strata if there are almost complex structures $j_\pi$ on $Y_\pi$, for every partition $\pi$, making the above maps $(j_\pi, J)$-holomorphic. That these exist follows from a local computation with restricted charts: the point is that almost complex structures on each of $X_r(f), Y_\pi$ arise by patching local canonically defined structures via partitions of unity. Since a smooth partition of unity on $X_r(f)$ defines smooth partitions of unity on all the $Y_\pi$, pulling back under the canonical smooth maps above, we deduce existence. In fact, we obtain the following, given as Propositions (6.3) and (7.4) in [DS]:

\textbf{(5.1) Lemma:} Let $J_\Delta \subset J$ denote the class of smooth almost complex structures on $X_r(f)$ which are compatible with the strata. Then this is non-empty, and there is an open dense set $\mathcal{U}$ in $J_\Delta$ with the following property. If $J \in \mathcal{U}$ then all moduli spaces of smooth holomorphic sections of all fibrations $Y_\pi$, including $X_r(f)$ itself, have the expected dimension. Moreover, a dense set of points of each moduli space corresponds to sections transverse to all diagonal strata in which they are not contained.

The regularity again follows standard lines: locally, perturbations of almost complex structures compatible with the strata generate the whole tangent space to the space of sections (i.e. all vector fields tangent to the fibres along the image $\mathbb{S}^2$). Note that we can immediately assert:

\textbf{(5.2) Proposition:} If $s : \mathbb{S}^2 \to X_r(f)$ is a section which meets the diagonal strata transversely at embedded points and with locally positive intersections, then the cycle $C_s \subset X'$ is a (not necessarily connected) smooth standard surface.
5 STANDARD SURFACES AND HOLOMORPHIC CURVES

This is just because transverse intersections with the top stratum of the diagonals (and no other intersections) give tangency points of $C_s$ to the fibres of $f$, whereas singularities of $C_s$ arise from non-transverse intersections with the diagonals; for instance, nodes of $C_s$ arise from tangencies to the diagonals. Before continuing, we need further discussion of almost complex structures on fibre bundles and their associated symmetric product bundles.

For any fibration of manifolds $\pi : Z \to B$ we can form the fibre product $Z \times_{\pi} \cdots \times_{\pi} Z \to B$ whose fibre over $b \in B$ is just $\pi^{-1}(b) \times \cdots \times \pi^{-1}(b)$. That this is smooth follows immediately from the surjectivity of $d(\pi \times \cdots \times \pi)$ viewed as a map $Z \times \cdots \times Z \to B \times \cdots \times B$.

If the original manifolds $Z$ and $B$ carry almost complex structures $J, \eta$ for which $\pi$ is a holomorphic projection, then the product complex structure $J \times \cdots \times J$ induces an almost complex structure on the total space of the fibre product. Now if we remove all the diagonals from the fibre product, then there is a free and holomorphic action of the symmetric group $S_r$, and the quotient inherits a natural smooth almost complex structure: in other words, for any pseudoholomorphic fibration, the space $Z_r(\pi) \backslash \{\text{Diagonals}\}$ carries a natural smooth almost complex structure $\mathcal{J}$.

In general we cannot say any more that this; the total space of the relative symmetric product $Z_r(\pi)$ will not carry the structure of a smooth manifold, even, and we cannot make sense of the tangent space at the points of the diagonal. If however the fibres of $\pi$ are two-real-dimensional surfaces, then using families of restricted charts, we find that the fixed $J$ on $Z$ does induce on $Z_r(\pi)$ the structure of a smooth manifold. In this case, we can ask about the induced almost complex structure $\mathcal{J}$ in a neighbourhood of the diagonals; from the topological set-up it is clear that it extends continuously, but higher regularity is not obvious. Let $J$ be given in local complex co-ordinates $z, w$ on $X'$ by a matrix

$$
\begin{pmatrix}
i & \mu \\
0 & i
\end{pmatrix}
$$

for a complex anti-linear homomorphism $\mu : f^*T_bS^2 \to T_pX'$, so $\mu \cdot i + i \cdot \mu = 0$. Here $z$ is the fibre co-ordinate and $w$ a co-ordinate in the base. Taking product charts, reduce to the case of a point $p + \cdots + p$ in the small diagonal of the symmetric product. There are induced co-ordinates $\sigma_1, \ldots, \sigma_r, w$ for $\sigma_i$ the $i$-th elementary symmetric function of $r$ copies $z_i, 1 \leq i \leq r$, of the local co-ordinate $z$ near $p$. Then $J$ is given near this point as

$$
\begin{pmatrix}
\cdots & \mu_1 \\
\text{Sym}^i(i) & \vdots \\
\cdots & \mu_r \\
0 & i
\end{pmatrix}.
$$

The functions $\mu_i$ are obtained from smooth functions of $\mu$ which are invariant under the action of $S_r$ on the fibre product. The elementary symmetric functions $\sigma_i$ are polynomial in given co-ordinate functions $x_i, y_i$ upstairs, and hence the smooth function $\mu$ of the real co-ordinates is a priori just a function of the fractional powers $\sigma^{1/i}$ (and their complex conjugates) for $i \leq r$. Nonetheless, some regularity does persist. The following was observed independently by the author and, in a mildly different context, by Siebert and Tian.  

(5.3) Lemma: The extension of the almost complex structure $\mathcal{J}$ on the relative symmetric
product $Z_r(\pi)$ from the complement of the diagonals to the total space is Hölder continuous, of Hölder exponent $C/r$ for some constant $C$. Along the top open stratum where at most two points co-incide the extension is Lipschitz.

Proof: The key computation, due to Barlet, is the following. Be given a holomorphic branched covering $D \to D'$ of complex domains of maximal ramification order $n$ and with branch locus having normal crossing singularities. Let $\phi$ be a smooth function on $D$ and let $\overline{\phi}$ be the “trace” of $\phi$ on $D'$, that is $\overline{\phi}(z) = \int_{\phi^{-1}(z)} \phi$. Then $\overline{\phi}$ is Hölder continuous of exponent $2/n$. As a special case, if a group $G$ acts holomorphically on a complex manifold $D$ and $D/G$ is smooth, then a smooth $G$-invariant function $\phi$ on $D$ defines a Hölder continuous function on the quotient. Since any holomorphic branched covering can be resolved to have normal crossing branch locus, it follows that traces of smooth functions are always Hölder continuous of exponent depending only on the ramification. In our situation this applies to the branched covering from the fibre product to the symmetric product, ramified along the diagonals. If only two points co-incide, then the ramification locus is already smooth: there is no need to resolve, and the extension is Hölder continuous for every $\alpha < 1$, in other words Lipschitz continuous. The local computation - and much besides - is given in Barlet’s paper [Bar82] (this reference is due to Siebert and Tian).

Notice that the above implies a tautological correspondence between smooth $J$-holomorphic curves in $X$ and Hölder $J$-holomorphic sections of $X_r(f)$ (in the case where the sections lie inside a diagonal stratum, one can check this by pulling back to the minimal such stratum, where the complex structure $J$ is now generically smooth). Such a correspondence presumably lies at the heart of (1.3).

5.2 Relation to the Gromov invariant

Basic theory of, and crucially the compactness theorem for, pseudoholomorphic curves has been proven with rather weak assumptions on the regularity of the almost complex structure: the best reference is [AL94], in particular the article by J.C. Sikorav.

(5.4) Proposition: Let $X$ be a symplectic four-manifold and fix a taming almost complex structure $J$ on $X$. Suppose for every Lefschetz pencil $f$ of sufficiently high degree the standard surface count $I_{(X,f)}(\alpha) \neq 0$. Then there is a $J$-holomorphic curve in $X$ in the homology class $\alpha$.

Proof: We can choose a sequence $J_n$ of almost complex structures on $X$ which converge in (say) $C^2$-norm to $J$ and such that for each $n$, there is a Lefschetz pencil of $J_n$-holomorphic curves on $X$. (The degrees of these pencils may have to increase with $n$, hence the wording in the hypothesis of the Proposition.) This follows from the main theorem of [Don96]: for any fixed $\varepsilon > 0$ we know $X$ admits Lefschetz pencils of surfaces whose tangent spaces deviate from integrability by at most $\varepsilon$ measured in any given $C^k$-norm. Hence it will be enough to show that for an almost complex structure $j$ on $X$ for which a holomorphic Lefschetz pencil (of high enough degree) exists, there are $j$-holomorphic curves in the class $\alpha$. Then we can finish the proof using Gromov compactness for a sequence of $J_n$-holomorphic curves.
Now fix such a $j$ on $X$ and form the fibre bundle $X_r(f) \to S^2$, where $r = \alpha \cdot [\text{Fibre}]$ in the usual way. The almost complex structure $j$ on $X$ induces a canonical smooth almost complex structure on $X^* = X_r^*(f) \setminus \Delta$, that is on the complement of the diagonals in $X_r^*(f) = X_r(f) \setminus \{F^{-1}(\text{Crit}(f))\}$. To see this, note that $X' \to S^2$ is a smooth fibre bundle away from the critical fibres; now the above discussion yields such a canonical $\mathcal{J}$ on $X^*$, at least Hölder continuous at the diagonals. Near the critical fibres, the original fibration was actually holomorphic, and the relative Hilbert scheme carries a smooth integrable complex structure. The data patches, by naturality, and so we obtain a Hölder continuous almost complex structure $\mathcal{J}$ on $X_r(f)$ which is smooth on a dense set.

The smooth sections of any vector bundle are always dense in the Hölder sections. It follows that we can choose a sequence of smooth almost complex structures $\mathcal{J}_n$ on $X_r(f)$ which converge to the canonical structure, in $C^{0,\alpha}$-norm say. Since the structures on $X_r(f)$ compatible with the strata form a dense Baire set, we can assume that each $\mathcal{J}_n$ lies in $\mathcal{J}_{\mathcal{S}}$. For each $n$ we then have a section $s_n$ of $X_r(f)$ in the homotopy class $\phi(\alpha)$, and the results of [DS] as in the previous section, assert that this defines a positive symplectic divisor $C'_n$ in $X^7$ which contains the exceptional curves and descends to a symplectic divisor $C_n$ in $X$ in the class $\alpha$. The symplectic condition controls the genera and area of all the surfaces uniformly.

For each surface $C_n$ we can find an almost complex structure $j_n$ on $X$ for which $C_n$ is $j_n$-holomorphic. Since we have convergence of the $\mathcal{J}_n$ in $C^{0,\alpha}$ on $X_r(f)$, we can choose the $j_n$ so as to converge in $C^{0,\alpha}$-norm to the given almost complex structure $j$. In this case, we can complete the proof using the Gromov compactness theorem; this holds for sequences of almost complex structures converging only in $C^{0,\alpha}$ by the results of Sikorav and Pansu in [AL94]. The compactness will yield a $j$-holomorphic curve $C$ in the class $\alpha$ (for the proof of Pansu shows that no area is lost in the limit), and since the almost complex structure $j$ is smooth, elliptic regularity asserts that the curve $C$ is the image of a smooth map.

Let’s add in the following well-known regularity result, due to Ruan (and developed by Taubes):

**(5.5) Proposition:** [Ruan] Let $X$ be a symplectic four-manifold and $\alpha \in H_2(X,\mathbb{Z})$. There is an open dense set $\mathcal{J}_{\text{reg}}$ in the space of compatible almost complex structures on $X$ for which the following hold for $j \in \mathcal{J}_{\text{reg}}$:

- if $|\alpha^2 - K_X \cdot \alpha|/2 = d \geq 0$ then the space of $j$-holomorphic curves representing $\alpha$ and passing through $d$ generic points of $X$ is finite;

- each such holomorphic curve is embedded or an unramified cover of a square zero torus.

An immediate consequence of this, and the preceding existence result, is the following

**(5.6) Corollary:** Let $(X,\omega)$ be a symplectic four-manifold. Let $\omega'$ be a rational perturbation of $\omega$ and construct Lefschetz pencils $f$ on $X$ with fibres dual to $k[\omega']/2\pi$. If $\mathcal{J}_{(X,\omega',k)}(\alpha) \neq 0$ for all $k \gg 0$ then $\alpha$ may be represented by embedded $\omega$-symplectic surfaces in $X$. 
Since there is no rationality hypothesis on the original symplectic form on $X$, this statement - even for the canonical class - is stronger than that obtained in [DS]. Unfortunately, Hölder regularity of the almost complex structure $J$ does not seem strong enough to directly apply the usual implicit function theorem. Hence, it seems non-trivial to prove that the map from holomorphic sections of $F$ to holomorphic curves in $X$ outlined above is onto. This would prevent any naive comparison of the $I$ and $Gr$ invariants. However, in principle we can get around this, using the fact that along the top stratum of the diagonal we actually have Lipschitz control. First we need to strengthen Ruan’s regularity result to a fibred situation. An almost complex structure on a four-manifold is compatible with a Lefschetz pencil $f$ if locally near each base-point there is a $j$-holomorphic diffeomorphism to a family of complex discs through the origin in $C^2$ and if all the fibres of $f$ are $j$-holomorphic curves in $X$. Considering the sphere in moduli space defined by the pencil, for instance, we find that such $j$ always exist and form a connected subspace $J_f$ of the space of all $\omega$-taming forms.

(5.7) **Lemma:** Let $X$ be a symplectic four-manifold and fix $\alpha \in H_2(X; \mathbb{Z})$. For Lefschetz pencils $f$ of sufficiently high degree on $X$ and for almost complex structures $j$ on $X$ generic amongst those compatible with $f$, the moduli spaces of $j$-holomorphic curves in the class $\alpha$ are smooth and of the correct dimension.

**Proof:** Fix some pencil $f$ of symplectic surfaces of area greater than $\alpha \cdot [\omega]$. If $j$ varied in the class of all taming almost complex structures on $X$ this would be exactly Ruan’s theorem, but we are restricting to almost complex structures for which the fibres of the pencil $f$ are holomorphic. In particular, such a $j$ cannot be regular for curves in the class of the fibre. Nonetheless, by positivity of intersections of holomorphic curves, any $j$-curve $C$ in the class $\alpha$ will be some multisection of the fibration (by consideration of area no such curve can contain any fibre components). In local co-ordinates near some point $p \in X$ disjoint from the base locus of $f$, all perturbations of $j$ in horizontal directions (i.e. in the term $\mu$ in the description before 5.3) yield almost complex structures which are compatible with $f$. Hence the class of compatible $j$ is large enough for deformations $(C', j')$ of $(C, j)$ (through curves holomorphic for some $j' \in J_f$) to generate the entire tangent space to $X$, at least over an open subset where the given curve is a multisection and not tangent to any fibre. By an argument with Aronszajn’s Theorem, this is enough to achieve transversality, as in [MS94] or the related result ([DS] Lemma 7.5). ■

For generic almost complex structures on a four-manifold with $b_+ > 1$, the Gromov invariant used by Taubes counts all curves that are not square zero tori with weight $\pm 1$, whilst the Gromov invariant counting somewhere injective spheres used to define the standard surface count - cf. [MS94] for instance - again counts each point of the moduli space with weight $\pm 1$. Hence a bijection of moduli spaces will give (5.3) modulo two, at least for $X$ containing no embedded symplectic tori of square zero and for zero-dimensional moduli spaces. (In fact, when $b_+ > 2$, we know that the only non-trivial standard surface counts do occur in zero-dimensional moduli spaces; we will deduce this from (1.1) later in the paper, and the same property is standard for the Gromov invariants. But it is no harder to establish the bijection
even if we first have to cut down dimensions.) The previous lemma shows, in essence, that there are \( f \)-compatible \( j \) which are sufficiently regular for the computation of the Gromov invariant of the class \( \alpha \). By perturbing \( j \) amongst \( f \)-compatible structures again if necessary, we see:

\[ (5.8) \text{ Corollary: Suppose } X \text{ contains no symplectic square zero tori. Fix } \alpha \in H_2(X; \mathbb{Z}) \text{ and a Lefschetz pencil } f \text{ on } X \text{ as in (5.7). For generic } j \text{ compatible with } f, \text{ the invariant } \text{Gr}_X(\alpha) \text{ is given by a signed count of finitely many embedded holomorphic curves, each of which is transverse to the fibres of } f \text{ away from finitely many tangencies all at smooth points.} \]

For this \( j \), the associated \( J \) on the relative Hilbert scheme has - by the tautological correspondence - a moduli space of sections in the class \( \phi(\alpha) \) comprising finitely many points, each of which is a section which meets the diagonal strata transversely (and hence only in the top stratum). We need a key technical result, due to Siebert and Tian ([ST], Theorem II): we have phrased their result in terminology appropriate to our situation. Fix a restricted chart \( D \times D \to X \) and work, by pulling back, locally in the domain. Hence we start with a smooth family \( \pi : D \times D \to D \) of complex discs over the disc, with a smoothly varying family of holomorphic structures on the fibres. As usual, there is a fibration \( C_r(\pi) \to D \) with fibres the \( r \)-th symmetric products of fibres of \( \pi \), and this carries an induced almost complex structure \( J \) with regularity as in (5.3). Fix some finite \( p > 2 \).

\[ (5.9) \text{ Theorem: [Siebert, Tian] If } C \text{ is a } J \text{-holomorphic section of } C_r(\pi) \to D \text{ with finitely many transverse intersections with the diagonals, then the moduli space of } J \text{-holomorphic sections near } C \text{ is naturally a smooth Banach manifold, modelled on the Banach space } L^{1,p}_{\text{hol}}(D; C^r). \]

In other words, we can parametrise local pseudoholomorphic curves provided the almost complex structure is at worst Lipschitz and smooth off a finite set. (Indeed one can relax the last condition.) In [AL94] Sikorav observes that the \( \bar{\partial} \)-equation can be linearised at any Hölder complex structure, and that solutions \( f \) of \( \bar{\partial}_J(f) = 0 \) - with \( J \) Hölder of exponent \( \alpha \) - will necessarily be of class \( C^{1,\alpha} \) and hence differentiable everywhere. The key in the result above is to show that, in the Lipschitz setting, an approximate right-inverse to the linearisation (provided as usual by the Cauchy integral operator) satisfies strong enough bounds to apply a contraction mapping and parametrise local \( J \)-curves by perturbations of the given curve \( C \).

Now suppose we have a \( J \)-holomorphic section \( s : S^2 \to X_r(f) \) which satisfies the transversality properties coming from (5.8). Combining the local perturbations above, near its intersections with the diagonals, with the usual perturbation arguments for holomorphic curves in smooth almost complex manifolds, we can see that globally over the base of the Lefschetz pencil the linearisation of the \( \bar{\partial}_J \)-operator will be surjective at \( s \). For the argument of [MS94] (page 35, cf. the footnote) works here even without \textit{a priori} \( L^{2,2} \)-regularity on the maps and Aronszajn’s Theorem. That is, an element in the kernel of the adjoint of the linearisation must be orthogonal to all vertical vector fields (since a given section of \( X_r(f) \) is never tangent to the fibres), and hence trivial. In this context, on the open subset
$X^0(f)$ of $X_r(f)$ given by removing a neighbourhood of the lower strata of the diagonals, we can use the implicit function theorem for the $\overline{\partial}$-equation. This is viewed as a section of the bundle over the space $C^{1,\alpha}(S^2; X^0_r(f)) \times J^{Lip}$ with fibre the one-form valued sections $C^{0,\alpha}(S^2, u^*(TX^0_r(f))^{\text{et}} \otimes K_{S^2})$. The implicit function theorem in particular shows that the moduli space of holomorphic sections of $X^0_r(f)$ is independent, up to diffeomorphism, of sufficiently small Lipschitz perturbations of the almost complex structure. On the other hand, we know that in the space of all Hölder almost complex structures on $X_r(f)$ the smooth structures are dense, and that in this dense subspace the structures for which all points of zero-dimensional moduli spaces represent sections having only transverse intersections with the diagonals are also dense. Hence, approximating $J$ by smooth structures $J_n$ (convergent in $C^{0,C/r}$) which satisfy this genericity, which $a$ priori admit no holomorphic sections lying inside the diagonals, and which Lipschitz approximate $J$ elsewhere, yields the following.

**Proposition (5.10):** Fix a generic $j$ on $X$ compatible with a sufficiently high degree pencil $f$. The moduli space of $J$-holomorphic sections of $X_r(f)$ in a homotopy class $\phi(\alpha)$ is a smooth Banach manifold. Moreover, this manifold is diffeomorphic to the moduli space of $J_n$-holomorphic sections for any sufficiently large $n$, where $(J_n)_{n \in \mathbb{N}}$ is a $C^{0,C/r}$-approximating sequence of smooth almost complex structures on $X_r(f)$ as above. Here, by cutting down with point conditions in $X$ or with the $D(z)$ in $X_r(f)$, we are implicitly working only with zero-dimensional moduli spaces. In this setting, the smoothness of the moduli space amounts to the regularity of the isolated points (they carry no obstruction information). The partial result (1.2) is almost a consequence; by our general theory, the “mod two” Gromov invariant on $X$ is the parity of the number of points in the moduli space of holomorphic sections of $X_r(f)$ for the canonical Hölder almost complex structure. This is now identified with the sections for a nearby smooth almost complex structure. The signed count of the second of these finite sets is by definition the standard surface count of the class $\alpha$. It’s not immediate from the discussion so far that any fixed pencil $f$ is of sufficiently high degree to obtain equality of $I$ and $Gr$ modulo two for all classes simultaneously. But in fact each of the invariants can be non-zero for at most finitely many classes, and so if we choose $f$ as in (5.7) varying $\alpha$ over a suitable finite set we obtain the theorem.

**Remark (5.11):** Recall that for any fibre bundle $Z \to B$ which has holomorphic total space and projection, the fibrewise symmetric product is naturally almost complex away from the diagonals. Suppose again we have two-real-dimensional fibres and a global smooth structure does exist. Fix a connexion on $f : Z \to B$, so at each point $p$ we have a splitting

$$T_pZ = T_pF_{f(p)} \oplus (\text{Hor}_p \cong f^*T_{f(p)}B).$$

If $Z_r(f) \to B$ is smooth, then it carries a distinguished connexion: at a tuple $p_1 + \cdots + p_r$ of points in one fibre $F_b$ we have the tangent space to the fibre and a splitting given by the diagonal

$$T_pB \cong \Delta \subset \text{Hor}_{p_1} \oplus \cdots \oplus \text{Hor}_{p_r}.$$

Now the connexion induces an almost complex structure $\mathcal{J}$ on $Z_r(f)$ by taking the product
structure in charts, so the holomorphic sections are just the flat sections; if the original almost complex structure \( j \) on \( Z \) is itself split, then this serves to extend the canonical structure on \( Z^* \), as described above, over the diagonals. (More simply, if \( \mu = 0 \) before (5.3) then all the \( \mu_i = 0 \), and these are smooth!)

For a Lefschetz fibration, there is a distinguished connexion defined by the symplectic form, but this does not extend over the critical points; equivalently, the flat almost complex structure \( j \) on \( X^* \to \mathbb{P}^1 \) defined on the complement of the critical fibres does not extend to a smooth structure on \( X' \). However, for smooth fibred four-manifolds there is no problem, and one can use the above point of view to prove (1.3) in this special case; the almost complex structures are smooth but we do not have regularity, so instead we need to analyse the obstruction theory. □

Proving the equivalence “\( I = \text{Gr} \)” seems particularly appealing considering the fact that, to date, there is no (non-homotopy) invariant of a general symplectic manifold which is defined from one Lefschetz pencil and proved independent of the choice of pencil. Note also that this would give a version of the Gromov-Taubes invariant of the four-manifold in which one only had to count spheres, for which the analysis is often simpler than for counting higher genus curves.

5.3 Heuristic interlude

In this subsection we give three digressions to related ideas. First, with future higher-dimensional applications in mind, we sketch how to obtain holomorphic curves from geometric measure theory. Second, we outline one approach to proving the standard surface counts are independent of the choice of Lefschetz pencil. Lastly, we discuss some ideas relating the conjectural equivalence of these invariants and the Seiberg-Witten invariants.

- For any symplectic manifold, an almost complex structure \( J \) is strictly compatible if it tames the symplectic form and if \( \omega(Ju, Jv) = \omega(u, v) \) for any tangent vectors \( u, v \). The space of these is contractible. For constructions of surfaces in fibred six-manifolds (where all the fibres are Hilbert schemes and Hölder regularity is less clear) another strategy may be important, so we sketch the argument; it also appears in Taubes [Tauf9].

Recall that, by approximating the canonical almost complex structure on the relative Hilbert scheme by smooth structures compatible with the strata, we had a sequence of symplectic surfaces \( C_n \) of uniformly bounded genus and fixed homology class. There is therefore a fixed upper bound \( K \), independent of \( n \), on the number of points of \( C_n \) which lie in one of the critical fibres of \( f \) or which are tangent to one of the fibres. Reparametrising, we can assume we are in the following situation: we have a fixed surface \( \Sigma \) and a sequence \( h_n : \Sigma \to X \) of symplectic immersions of \( \Sigma \) in a fixed homology class \( \alpha \). Moreover there is a finite set of \( K \) points on \( \Sigma \) such that the surfaces \( h_n(\Sigma \setminus K) \) become pointwise \( j \)-holomorphic as \( n \to \infty \). But now \( j \) is smooth on \( X \), which itself is compact. It follows that if \( \|\partial_j(h_n)\| \leq 1/n \) on \( D^* \) and \( h_n \) is a priori smooth on the whole disc, then this bound holds at the origin. Each \( C_n \) defines a (much better than) rectifiable integral current. We assume that in fact \( j \) was strictly compatible with \( \omega_X \) and write \( |\cdot| \) for the associated metric. Then the pointwise formula
\[ |dh|^2 = h^*\omega + \overline{\partial h}|^2 \] (5.12)

for any map \( h : \Sigma \to X \), together with the convergence to zero of the \( |\overline{\partial}_j(h_n)| \) on compacta of \( \Sigma \setminus K \) and the smoothness of \( j \), shows that the sequence of currents has bounded area. We now use three results:

1. (Federer [Fed69]) The compactness theorem for bounded area currents guarantees a limit current \( C \) which will be area-minimizing by (5.12). Such a current is necessarily a union \( A \cup B \) where \( B \) is the singular set and \( A \) is locally a union of finitely many submanifolds.

2. (Almgren, [Alm00]) An area-minimizing integral current is rectifiable; its singular set \( B \) has Hausdorff codimension at least two.

3. (Chang, [Cha88]) An area-minimizing rectifiable integral two-dimensional current is a classical minimal surface: it is the image of a smooth map \( h : \Sigma \to X \) with only finitely many singular points.

But now, using (5.12) again, an area-minimizing surface is necessarily \( j \)-holomorphic, and this completes the argument. The caveat is that Almgren’s theorem is harder than “SW = Gr^+” so even in higher dimensions there should be a better strategy.

- According to Donaldson [Don99] the Lefschetz pencils from pairs of approximately holomorphic sections become symplectic invariants when the degree \( k \) of the pencils is (arbitrarily) large. There is a stabilization procedure for Lefschetz pencils, described in [Smi01] and [AK], which has the following consequence. Fix a degree \( k \) Lefschetz pencil \( \{ s_0 + \lambda s_1 \}_{\lambda \in \mathbb{P}^1} \) on \( X \). We can find a family of degree \( 2k \) Lefschetz pencils, parametrised by the open complex disc \( D \subset \mathbb{C} \), such that for \( t \in D^* \) the pencil is a smooth degree \( 2k \) Lefschetz pencil, whilst at \( t = 0 \) we have the degenerate pencil \( \{ s_0^2 + \lambda s_0 s_1 \}_{\lambda \in \mathbb{P}^1} \). Results of Auroux [Aur01] suggest that the pencils over \( D^* \) do satisfy the approximate holomorphicity constraints of [Don99]. If we blow up the total space of \( X \times D \) at the section of degree \( 4k^2|\omega|^2 \) given by the base-points of the pencils away from 0, and at the base curve \( \{ s_0 = 0 \} \subset X \) inside \( X \times \{ 0 \} \), the resulting space carries a smooth family of Lefschetz fibrations over \( D^* \) but has singular total space. However, further blow-ups give a desingularisation, and one eventually obtains a smooth space with a globally defined map to \( \mathbb{P}^1 \times D \). Moreover, the fibres of this map are nodal Riemann surfaces (which may be reducible and have many nodes each) at least away from the point \( (\lambda = 0, 0) \).

Now the relative Hilbert scheme can be defined for this larger family of curves, and at least where the singularities are nodal and in codimension one in the base, the total space will be smooth. One can now hope to identify the standard surface counts for the degree \( k \) and \( 2k \) Lefschetz pencils with a fixed Gromov invariant on this larger dimensional manifold (which can be compactified or replaced with its symplectization at infinity, for analytical purposes). Suppose we again work with almost complex structures making the projection to \( \mathbb{P}^1 \times D \) holomorphic. Then for a generic \( t \in D \), the holomorphic curves in some class \( \alpha \) will lie in a fibre which is just the total space of a Lefschetz fibration \( X' \) given by blowing up the base-points of a degree \( 2k \) pencil; whilst at \( t = 0 \), the holomorphic curves will all lie
in an irreducible component of the relative Hilbert scheme formally obtained from a partial

gluing of the space \( X_r(f) \) arising from the degree \( k \) pencil and a trivial bundle of symmetric

products \( \text{Sym}^r((s_0 = 0)) \times \mathbb{P}^1 \). Hence the moduli spaces of holomorphic sections in the total

space of the larger symmetric product fibration, for differing \( J \), should recover the spaces of

sections arising from the degree \( 2k \) and degree \( k \) pencils, setting up an equivalence between

\( \mathcal{I}(X,f_k) \) and \( \mathcal{I}(X,f_{2k}) \). Note that, in principle, the technical difficulties here are no greater

than understanding relative Hilbert schemes for certain degenerate but integrable families of
curves over a disc. We hope to return to this programme elsewhere.

• To finish this section, let us remark that the conjecture (1.3) implies that we have a triangle

of equal invariants: \( SW = \mathcal{I} = \text{Gr} \). Indeed, we believe that the equivalence of Seiberg-Witten

and Gromov invariants should factor naturally through the \( \mathcal{I} \)-invariants. The link comes

from the vortex equations, which from this point of view are just the equations for solutions

\( (A, \Phi) \) to the \( SW \) equations on \( \Sigma \times \mathbb{R}^2 \) which are translation invariant in the two Euclidean

variables. It is well known that the moduli space of finite energy solutions of these is a copy

of the symmetric product of \( \Sigma \).

In [Sal99] Dietmar Salamon shows that, formally at least, the adiabatic limit of the Seiberg-

Witten equations on a fibred four-manifold, as the metric on the fibres shrinks to zero, yields

the Cauchy-Riemann equations for holomorphic maps of the base into a universal family of

solutions to the vortex equations. The almost complex structure on this family arises from

a canonical connexion in the universal vortex family. The upshot is, formally at least, that

Seiberg-Witten solutions on \( X' \) should be determined by finite energy solutions on \( X^* \), and

these should come from the punctured holomorphic sections of \( X^* \) that arose above.

The vortex equations depend on a real stability parameter \( \tau \): for each fixed value, the moduli

space is a smooth symplectic manifold canonically diffeomorphic to the geometric invariant

theory quotient which is just the usual symmetric product. Although the complex structure is

fixed, under the adiabatic limit \( \tau \to \infty \) the symplectic structure degenerates (its cohomology

class varies linearly with \( \tau \)). Work of Hong, Jost and Struwe [HJS96] shows that in this

limit, the solutions of the vortex equations converge to Dirac delta solutions at the zero-

sets of the Higgs field \( \Phi \). However, the symplectic structure on the moduli space contains

a term formally of the shape \( \omega(\Phi_1, \Phi_2) = \int_{\Sigma} \langle \Phi_1, \Phi_2 \rangle \omega_{\Sigma} \). Thus, as Michael Hutchings and

Michael Thaddeus pointed out to the author, in the limit one expects to obtain the degenerate

symplectic structure on \( \text{Sym}^r(\Sigma) \) given by pushing forward \( \omega \times \cdots \times \omega \) from \( \Sigma \times \cdots \times \Sigma \).

This is analogous to degenerating almost complex structures on \( X_r(f) \) to ones coming from the

fibre product.

6 Serre Duality for symplectic surfaces

In this section we shall prove Theorem (1.1). As we observed in the Introduction, the

geometric input for the first part arises from Serre duality on the fibres of a Lefschetz fibration.

In order to compute Gromov invariants, however, we need almost complex structures which

behave well when we pass to one-parameter families of curves. The key technical ingredient

allowing this is a result from Brill-Noether theory, due to Eisenbud and Harris, which we

introduce at once.
6.1 Brill-Noether theory

Let $C$ be a Riemann surface of genus $g$. Let $W^s_r(C)$ denote the locus of linear systems $g^s_r$ of degree $r$ and dimension $s$ on $C$, viewed as a subscheme of the Picard variety $\text{Pic}^r(C)$. This has virtual dimension given by the Brill-Noether number

$$\rho = g - (s + 1)(g - r + s).$$

The famous “existence theorem” \([85]\) asserts that if $\rho \geq 0$ then $W^s_r(C)$ is not empty, for any $C \in M_g$. On the other hand, the “dimension theorem” (op. cit.) asserts that if $\rho < 0$ then $W^s_r(C)$ is empty for a general curve of genus $g$; in other words, it is non-empty only on a subvariety of positive codimension in $M_g$. One could hope to sharpen this statement, at least in good cases, to estimate the codimension of this subvariety in terms of the deficiency $-|\rho|$ of the (negative) Brill-Noether number. In general the naive estimates fail (as shown in many cases in \([85]\)), but eliminating codimension one components does turn out to be possible \([85]\).

(6.1) Theorem: [Eisenbud-Harris] Suppose $\rho < -1$. Then the locus $W^s_r \subset M_g$ comprising curves $C$ for which $W^s_r(C) \neq \emptyset$ has codimension greater than one in the moduli space.

In \([85]\), a central ingredient in the computation of the Gromov invariant was the simple form of the Abel-Jacobi map $X_{2g-2}(f) \to P_{2g-2}(f)$, with a unique fibre of excess dimension. This holds for any choice of fibrewise metrics on the Lefschetz pencil. Such a simple description cannot exist for all Abel-Jacobi maps $X_r(f) \to P_r(f)$; if $r = g - 1$ then the geometry of the theta-divisor is well known to be subtly dependent on the curve in moduli space. However, using the above proposition and the adjunction formula, we can obtain a pretty description for the cases of interest to us. Fix a symplectic four-manifold $X$ and let $\alpha \in H_2(X; \mathbb{Z})$. Under the map $i : H_2(X; \mathbb{Z}) \to H_2(X'; \mathbb{Z})$ we map $\alpha$ to a class $\alpha + \sum_{i=1}^N E_i$ whose intersection with the fibre of the Lefschetz fibration $f : X' \to S^2$ is at least as large as the number $N$ of exceptional curves. But by adjunction on $X$, if the fibres of the pencil represent a class $W$ in homology,

$$2g(\text{Fibre}) - 2 = K_X \cdot W + W^2,$$

where of course $N = W^2$. As $W = k[\omega]$ then by increasing the degree of the pencil, the second term on the right grows quadratically with $k$ and the first term on the right only linearly; hence the ratio $(2g - 2)/N$ tends to $1$. In particular, for any $\alpha$, the intersection number $r$ of $i(\alpha)$ and the fibre of the Lefschetz fibration is such that $(2g - 2)/r \to 1$ as the degree increases.

(6.2) Proposition: For sufficiently large $k$ and $r(k)$, there is an embedding of the residual fibration $T \cong X_{2g-2-r}(f) \hookrightarrow P_r(f)$. The natural map $u : X_r(f) \to P_r(f)$ has (projective space) fibres of dimension $r - g$ away from $T$ and of dimension $r - g + 1$ over $T$.

Proof: By the above remarks, we can certainly assume that $r > g - 1$. Fix a smooth curve $C$. The fibre of $u : \text{Sym}^r(C) \to \text{Pic}^r(C)$ has dimension $r - g + 1$ for line bundles with no
higher cohomology. Conversely, the line bundle $L$ has higher cohomology if and only if $K - L$ admits sections; such a section defines an effective divisor of degree $2g - 2 - r$, and hence the space $\text{Sym}^{2g-2-r}(C)$ parametrises all of the residual linear systems at which points the map $u$ may have vanishing differential. Now for $2g - 2 \geq r \gg g$, we see $\rho$ is negative: setting $d = 2g - 2 - r$ to be the degree of the residual divisors, we have

$$\rho(W^s_d(C)) = g - 2(g - 1 + d) = 2 - g - 2d \leq 2 - g$$

which will be substantially smaller than $-1$ for high degree pencils. The virtual dimensions of the $W^s_d(C)$ with $s > 1$ are still smaller; since $d < g - 1$ we expect the line bundle $K - L$ to have no sections, and it to be increasingly unlikely that linear systems of dimension $s$ exist as $s \mapsto s + 1$. Hence for a generic $C$, meaning a point $C \in M_g$ in the complement of a subvariety of codimension at least 2, the Abel-Jacobi map $\text{Sym}^r(C) \to \text{Pic}^r(C)$ has excess fibres of dimension at most one greater than the generic dimension, and moreover the locus of points in $\text{Pic}^r(C)$ where this happens is exactly a copy of the dual symmetric product. This embeds in $\text{Pic}^r(C)$ under the obvious map $\sum z_i \mapsto (K_C - O(\sum z_i))$.

It follows, by the Eisenbud-Harris result, that for a generic choice of fibrewise metrics the sphere $\phi_f(S^2) \subset M_g$ will be disjoint from the locus $W^s_d$ for every $s$, at least over the smooth locus. We now need to argue for the behaviour over the singular fibres. Recall from [AK80] that there is always a natural morphism $\text{Hilb}^d(\Sigma_0) \to \text{Pic}_d(\Sigma_0)$, for every $d$, with projective space fibres; indeed, a generic point of $\text{Pic}_d$ gives a locally free sheaf on the normalisation and the projective space is just the linear system of divisors of this line bundle. On the moduli space $M_{g-1}$ there is a family of complex surfaces given by the fibre product of the universal curve with itself, and a natural (gluing) morphism from this onto the divisor in $M_g$ of irreducible curves with one node. The Eisenbud-Harris result, applied for genus $g - 1$ curves, now shows that the embedding we have constructed away from the singular fibres will extend to embed $T \mapsto P_r(f)$. The rest of the result is clear; for every degree $r$ torsion-free sheaf $L$ on each (not necessarily smooth) curve $C$ in the Lefschetz fibration, the dimension of the linear system $K - L$ is at most zero, meaning that the effective divisors of degree $2g - 2 - r$ never move in non-trivial systems. ■

\textbf{(6.3) Example:} As in the Appendix to [DS] the situation is most clear for the case of divisors of degree two. Let $\Sigma_0$ be a nodal curve of high genus; the Hilbert scheme $\text{Hilb}^2(\Sigma_0)$ is given by blowing up the second symmetric product at the point (Node, Node). $\text{Pic}_2(\Sigma_0)$ is given by gluing two sections of a $\mathbb{P}^1$-bundle over $\text{Pic}_2(\Sigma_0)$ over a translation in the base. The embedding $\text{Hilb} \to \text{Pic}$ defines a map from $\text{Hilb}^2$ to the normalisation, which maps the exceptional sphere to a fibre. Inside $\text{Pic}_2$ the sphere remains embedded, and meets the singular locus at two points, which are just the two points of the exceptional curve singular inside $\text{Hilb}^2$. □
6.2 Adapted complex structures

We now have a very simple picture in which to compute the standard surface counts of homology classes. Suppose we have a fibre bundle \( \pi : Z \to B \) with almost complex fibres \( F_b \) and almost complex base \( B \). Any choice of connexion (horizontal splitting) for \( \pi \) defines an almost complex structure on \( Z \); we use the given \( J_F \) on the tangent spaces vertically and use the connexion to lift \( j_B \) to the horizontal planes. Now connexions may always be extended from closed subsets - their obstruction theory is trivial. (As usual: given some connexion on the fibration, a particular connexion on a subfibration is given in each point by the graph of a linear map from the horizontal to the vertical space at that point. Since the space of such linear homomorphisms is contractible, the given section over the closed subspace may be extended.) It follows that we may find almost complex structures on the total space making any subfibration by almost complex subspaces of the fibres an almost complex submanifold of the total space.

(6.4) Definition: In the situation of the previous proposition, there are almost complex structures on \( \mathcal{P}_r(f) \) and \( \mathcal{X}_r(f) \) extending the obvious integrable structures on the fibres and for which

1. the submanifold \( \mathcal{X}_{2g-2-r}(f) \to \mathcal{P}_r(f) \) is an almost complex subspace;
2. the Abel-Jacobi map \( u : \mathcal{X}_r(f) \to \mathcal{P}_r(f) \) is holomorphic;
3. the almost complex structure on the restriction \( u^{-1}(\mathcal{X}_{2g-2-r}(f)) \) is induced by a linear connexion on the total space of a vector bundle over \( \mathcal{X}_{2g-2-r}(f) \).

We will call such almost complex structures \( J \in J \) on \( \mathcal{X}_r(f) \) compatible with duality. If \( r = 2g - 2 \), as in \([DS]\), we will say \( J \) is compatible with the zero-sections.

This is an immediate consequence of the discussion. Choose an almost complex structure on \( \mathcal{X}_{2g-2-r}(f) \) and then a connexion on a vector bundle over this space with fibre at a divisor \( D \) the sections \( H^0(K - O(D)) \). That these complex vector spaces, by assumption now of constant rank, fit together to yield a vector bundle is essentially standard elliptic regularity (cf. the next proposition). More easily, the vector bundle is just the pullback by \( \phi_f : S^2 \to \overline{M}_g \) of a universal vector bundle over the moduli space. This is the push-forward of the bundle over \( \mathcal{S}^{2g-2-r}(C) = \text{Hilb}^{[2g-2-r]}(C_g/\overline{M}_g) \) with fibre at a subscheme \( Z \subset C \) the line bundle \( K_C - O(Z) \). The ambiguity arising from the fact that the relative dualising sheaf of a fibration is not the same as the restriction of the canonical bundle of the total space to each fibre is eliminated since it is only the projectivisation of the vector bundle which embeds into \( \mathcal{X}_r(f) \). Now given a complex structure on the total space of the projective bundle, viewed as a closed subset inside \( \mathcal{X}_r(f) \), we can extend arbitrarily by choosing an extension of the connexion to the whole of \( u : \mathcal{X}_r(f) \to \mathcal{P}_r(f) \).

Since these vector bundles of sections on the fibres of a Lefschetz fibration appear as holomorphic subspaces of the \( \mathcal{X}_r(f) \), it will be helpful to understand their spaces of holomorphic sections. Here is a general result in this line, formulated in the framework and notation of Lemma (4.1). Let \( \alpha \in H_2(X; \mathbb{Z}) \) and bear in mind (4.2).
6 SERRE DUALITY FOR SYMPLECTIC SURFACES

(6.5) Proposition: Suppose, for each \( j \in \{0, 1\} \), the value \( h^j(L_\alpha|_{\Sigma_t}) \) is constant for \( t \in S^2 \). Then the vector spaces \( H^0(L_\alpha|_{\Sigma_t}) \) and \( H^1(L_\alpha|_{\Sigma_t}) \) define vector bundles over \( S^2 \). The element \( V_\alpha = \{H^0 - H^1\}_{t \in P^1} \) of the K-theory of \( \mathbb{C}P^1 \) satisfies

\[
\text{Index}(V_\alpha) = \left[\alpha^2 - K_X \cdot \alpha \right]/2 + (b_+ + 1 - b_1)/2.
\]

Here the index \( \text{Index}(\cdot) \) is defined as the sum of the rank and first Chern class of the virtual bundle.

Proof: The existence of the vector bundles is an application of standard results in index theory for families of elliptic operators. Away from the critical fibres we have a smoothly varying family of \( \bar{\partial} \)-operators. Over small discs around each critical value of the Lefschetz fibration, we can assume both \( X \) and the families of operators vary holomorphically, and the index virtual bundle is just the push-forward \( f_!(L_\alpha) \). There is a canonical identification of these two objects over annuli around each critical value, coming from the canonical holomorphic structure on the index bundle for a family of Hermitian operators on the fibres of a holomorphic submersion, as explained in Freed’s survey [Fre87].

The first Chern class of this index bundle could be computed using either the Atiyah-Singer theorem, were the family smooth (no critical fibres), or the Grothendieck-Riemann-Roch theorem were it globally holomorphic (even with critical fibres). Although neither apply here, we can get around the deficiency. One approach (which we shall not develop) uses excision, and removes a neighbourhood of the critical fibres. Alternatively, our family of \( \bar{\partial} \)-operators is classified by a smooth map \( \psi : S^2 \to \text{Pic} \) to a universal family over the universal curve. The homotopy invariance property for indices of families of operators asserts that the index is determined by the homotopy class of this map, which is encoded by the homology class \( \alpha \). The index bundle \( V_\alpha \) is just the pullback \( \psi^*(\pi_! \mathcal{F}) \) of the push-forward of a holomorphic sheaf \( \mathcal{F} \to \text{Pic} \). We can now argue precisely as in the proof of (4.3) to obtain the formula.

\[ \blacksquare \]

Note that in the proof of (4.3) and above, the universal perspective is used to reduce the computation to the specific case of the first Chern class of the relative dualising sheaf, which is computed separately in [Smi99]. It would be interesting to develop a direct argument for giving the indices of smooth families of operators with locally holomorphic singularities.

(6.6) Remark: Let us stress again the two key geometric features of both the index formulae we have given: the same geometry, in a mildly more complicated scenario, will play a role in the next subsection. For complex structures compatible with the zero-sections, the geometry of the situation is reflected in maps:

\[ \mathbb{P}(V) \to X_r(f) \xrightarrow{E} P_r(f). \]

These induce:

\[
0 \to T \mathbb{P}(V) \to T^{\text{rel}}X_r(f) \to \text{im}(dF) \to 0;
\]

\[
0 \to \text{im}(dF) \to T^{\text{rel}}P_r(f) \to \text{cok}(dF) \to 0.
\]
If we take the two long exact sequences in cohomology, we obtain a sequence:

\[ 0 \to H^0(W) \to \text{Obs}_u \to C(b, -1)/2 \to H^1(W) \to 0. \]

Here we have used two identifications:

1. At any point \( \sum p_i \in \text{Sym}^r(\Sigma) \) the cokernel of the Abel-Jacobi map is canonically isomorphic to \( H^1(\Sigma, \mathcal{O}(\sum p_i)) \) and this globalises to the identity \( \text{cok}(dF)_u \sim W \), where \( W \) denotes the appropriate push-forward sheaf. Now the first and last terms in the sequence are just \( H^i(\text{cok}(dF)_u) \) for \( i \in \{0, 1\} \).

2. The vertical tangent bundle to \( P_r(f) \) near any section, by the remarks above, is isomorphic to the bundle with fibre \( H^1(\Sigma, \mathcal{O}) \). Relative duality - a parametrised form of Serre duality which holds for any flat family of curves, and hence for the universal curve from which our bundles are pulled back [Har66] - asserts that

\[ T^{vt}P_r(f)|_{u(\Sigma^2)} \cong (f_*\omega_{X'/\mathbb{P}^1})^*. \]

These comments on vertical tangent bundles implicitly assume that all our sections are smooth, in the sense that there are no bubbles. We have proved that fact for regular almost complex structures, and shown that structures compatible with the diagonal strata can be regular, but we are now working with almost complex structures which fall into neither of these classes. Hence we must provide a fresh argument.

**Proposition (6.7):** Let \( J \in \mathcal{J} \) on \( X_r(f) \) be generic amongst structures which are compatible with duality. Then for any holomorphic section \( w \) of \( X_{2g-2-r}(f) \), the holomorphic sections of the projective subbundle of \( X_r(f) \) lying over \( w \) arise from constant sections of a vector bundle and contain no bubbles.

**Proof:** Fix the section \( w \) of \( X_{2g-2-r}(f) \). We will work in the preimage of this fixed section. The holomorphic sections of a projective bundle \( \mathbb{P}(V) \) coming from sections of \( V \) have bubbles iff the sections of \( V \) vanish at certain points. For us, once we have fixed a homology class \( \alpha \in H_2(X; \mathbb{Z}) \), the relevant vector bundle \( V^0 \) is given by taking the holomorphic sections of the bundles \( L_\lambda = L_\alpha|_{\Sigma_\lambda} \) on the fibres of the Lefschetz fibration. We have already given the index of the \( \partial \)-operator on the element \( V^0 - V^1 \) in K-theory, where \( V^1 \) is the bundle of first cohomology groups on the fibres. For high degree pencils the rank of \( V^0 \) is very large, growing with \( k \), whilst \( J \) being compatible with duality precisely asserts that the bundle \( V^1 \) has rank one. Hence it is enough to show that the first Chern class of \( V^1 \) is small, and then the constant value of \( \text{Index}(V^0 - V^1) \) will force the first Chern class of \( V^0 \) to be negative for high degree pencils. In this case, by stability, the bundle will generically be of the form

\[ V^0 = \mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-1); \]

the only holomorphic sections of such a bundle over \( \mathbb{P}^1 \) are constant sections, and these have no isolated zeroes and give no bubbles.
In fact, we claim that the first Chern class of the line bundle $V^1$ is necessarily zero, in other words the line bundle is topologically trivial. (In the situation of this amounts to saying that the push-forward sheaf $R^1f_*\mathcal{O}$, down the fibres of a Lefschetz fibration, is trivial.) The result is an application of relative duality. Fibrewise, $H^1(\Sigma_t; L_\alpha|_{\Sigma_t})$ is dual to $W_t = H^0(K_{\Sigma_t} - L_\alpha|_{\Sigma_t})$; hence it is enough to understand the Chern class of the bundle with this as fibre. But a section $u$ of $W \rightarrow \mathbb{P}^1$ gives, at each value $t$, a section $u_t$ of the residual bundle and hence a divisor in $\text{Sym}^{2g-2-r}(\Sigma_t)$ - which is necessarily the unique point $w(t)$ in the linear system, under our assumption that $\text{Sym}^{2g-2-r} \hookrightarrow \text{Pic}_r$. There is a subtlety here; the section of the projective bundle is only a divisor on each fibre up to scale, and so one might think this scalar indeterminacy precisely hid the first Chern class. In fact, if the section $u$ has a zero, then its zero-set will contain certain fibres, but the collection of points of the symmetric products defined by the original section $w = P(u)$ does not.

More invariantly, argue as follows. In general, we can’t assume that an almost complex structure $J$ compatible with duality also respects the stratification coming from exceptional sections (i.e. makes the subfibrations $X^E_{r-1}(f)$ all holomorphic). However, if we distinguish one fixed $E$, we can also assume $X^E_{r-1}(f)$ is an almost complex subvariety, since one condition on a vector space is always linearly independent and cuts out a hyperplane. By the usual arguments that enable us to push cycles from $X'$ down to $X$, the original section $w$ of $X_{2g-2-r}(f)$ must in fact be disjoint from $X^E_{r-1}(f)$. It follows that, by scaling to make the evaluation of the section $u$ at the unique point of $E$ in any given fibre equal to 1, we can explicitly smoothly trivialise $W$. This completes the argument.

A consequence of the above result, and the preceding remarks, is the following: the arguments are similar so we leave them to the reader.

\textbf{(6.8) Corollary:} The normal bundle of the embedding of $X_{2g-2-r}(f)$ inside $P_r(f)$ has negative first Chern class.

With these two facts in hand, we can assemble all the pieces for our version of Taubes’ duality theorem.

\textit{6.3 Proof of the duality}

The proof will be completed as follows. We shall fix an almost complex structure on $X_r(f)$ which is compatible with duality, and explicitly determine the moduli space of smooth holomorphic sections. Despite the non-genericity, this moduli space will already be compact, and hence we will be able to compute the obstruction bundle (by using elementary arguments to understand its fibre at any given smooth section). For the moment, let’s work with an easier linear constraint:

\textbf{(6.9) Theorem:} Let $X$ be a symplectic four-manifold with $b_+(X) > 1 + b_1(X)$. For any class $\alpha \in H_2(X; \mathbb{Z})$ the standard surface counts for $\alpha$ and $\kappa - \alpha$ co-incide up to sign.

\textbf{Proof:} By the blow-up formula \textbf{(4.6)} given before, it is enough to compare the counts of
holomorphic sections giving curves in classes \(\iota(\alpha)\) and \(\kappa_{X'} - \iota(\alpha) = i(\kappa_X - \alpha)\).

For \(\iota(\alpha)\), the algebraic intersection number with the fibre of \(f : X' \to \mathbb{S}^2\) is \(r \gg g - 1\) and we can assume that the fibration \(X_r(f)\) maps to \(P_r(f)\) with the standard topological format described above. Fix an almost complex structure \(J \in \mathcal{J}\) which is generic on \(X_{2g-2-r}(f)\) and which is generic amongst those compatible with duality in the sense of the preceding Proposition. Since the Abel-Jacobi map is assumed holomorphic, every \(J\)-holomorphic section of \(F\) maps to a holomorphic section of \(P_r(f)\). We claim that, for our particular \(J\), this must lie inside \(X_{2g-2-r}(f) \subset P_r(f)\). To see this, recall that the index of the \(\partial\)-operator on the vector bundle \(T^\mathfrak{h} P_r(f) \to \mathbb{S}^2\) is negative, more precisely \(1 + b_1 - b_+\). This follows since for any fixed section \(\phi\) of \(P_r(f)\) there is a diffeomorphism of pairs

\[(P_r(f), \phi) \to (P_{g-2}(f), \phi_K)\]

where \(\phi_K\) is the section coming from the canonical bundles on the fibres. This identifies the vertical tangent bundle near \(\phi\) with that near \(\phi_K\), and the first Chern class of this bundle determines the index: but then the index for the section \(\phi_K\) is determined by the signature formula of \(\text{Sm}^99\), as explained earlier in the paper. (This is the \(\alpha = 0\) instance of 6.3.)

It follows that for any holomorphic section \(\phi\) of \(P_r(f)\) passing through some (previously) fixed open subset \(U \subset P_r(f) \setminus X_{2g-2-r}(f)\), the moduli space of holomorphic sections in the class \([\phi]\) is regular near \(\phi\) and hence of the correct (virtual) dimension, which is negative. We observed above that the normal bundle \(N\) to the embedding of \(X_{2g-2-r}(f)\) inside \(P_r(f)\) has negative Chern class. Therefore if \(J\) is generic on a tubular neighbourhood of the image of the embedding, and vertically generic on its preimage under the Abel-Jacobi map, we can deduce that in fact \(N\) has no sections either. Hence sections of \(X_{2g-2-r}(f)\) do not deform infinitesimally in the total space unless they remain inside this subfibration. This means that all holomorphic sections for the almost complex structure \(J\) have image inside the embedded copy of \(X_{2g-2-r}(f)\). By the definition of the embedding

\[X_{2g-2-r}(f) \to P_r(f); \quad \sum z_i \mapsto K - O(\sum z_i)\]

the section \(u \circ \phi : \mathbb{S}^2 \to X_{2g-2-r}(f)\) gives rise to a cycle in the homology class \(i(\kappa - \alpha)\) if \(\phi\) gives a cycle in the class \(i(\alpha)\). Conversely, for any holomorphic section of \(X_{2g-2-r}(f)\) in the former class, the restriction of \(u\) to the preimage of this section is just the projection map of the projectivisation of a vector bundle \(V\), by the result \([2]\). Hence for some \(a = \text{Index}_V(\mathcal{E} - 1)\), we have exhibited the moduli space of \(J\)-holomorphic sections of \(X_r(f)\) in the class \(\phi^{-1}(i(\alpha))\) as a \(\mathbb{P}^a\)-bundle over the moduli space of \(J\)-holomorphic sections of \(X_{2g-2-r}(f)\) in the class \(\phi^{-1}(i(\kappa - \alpha))\). It remains to compare the obstruction bundles on the two moduli spaces.

First, since we have a generic almost complex structure downstairs, we can choose \(m = [\alpha^2 - K \cdot \alpha]/2\) points on \(X\) and cut down the moduli space of sections of \(X_{2g-2-r}(f)\) to be zero-dimensional, and hence a signed set of points. Their total number is exactly the standard surface count for \(\kappa - \alpha\). Since the rank of the linear system over a point of \(P_r(f)\) is large relative to \(m\), each condition cuts down each linear system by a hyperplane (the conditions are independent). Thus the new geometric situation is again that of a moduli space which is a projective bundle but now over finitely many points. The obstruction theories behave compatibly, so we reduce to studying the \(m = 0\) problem for a smaller family.
of symmetric products Sym^ rm; so assume m = 0 from the start. It will be enough to show that the obstruction bundle on each \( \mathbb{P}^a \) is just the quotient bundle (that is, the cokernel of the embedding of the tautological line bundle into the trivial bundle of rank \( a + 1 \)). Then each \( \mathbb{P}^a \) will contribute ±1 to the moduli space of sections in the class \( \iota(\alpha) \), since the quotient bundle has Euler class ±1 depending only on the dimension \( a \).

Write \( Z \) for the total preimage of \( X_{2g−2−r}(f) \) inside \( X_r(f) \) under the Abel-Jacobi map. Recall that the sections we are interested in stay away from all the critical loci of the projection maps, and so we can consider the cokernel \( W \) of the natural projection \( π \):

\[
0 \to T(X_r(f))/TZ \xrightarrow{\pi} T(P_r(f))/T(X_{2g−2−r}(f)) \to W \to 0. \tag{6.10}
\]

We also have, for a given section \( s \in \mathbb{P}^a \) in one component of our moduli space, a sequence

\[
0 \to ν_{s/Z} \to ν_{s/X_r(f)} \to ν_{X_r(f)|s} \to 0
\]

of normal bundles. Here we identify \( s \) with its image, and the sequence is induced by the inclusions \( s \subset Z \subset X_r(f) \). The long exact sequence in cohomology for this second sequence shows

\[
0 = H^1(ν_{s/Z}) \to H^1(ν_{s/X_r(f)}) = \text{Obs}(s) \to H^1(ν_{X_r(f)|s}) \to 0. \tag{6.11}
\]

The first identity holds since the almost complex structure \( J \) is generic over the whole of \( Z \); the identification in the middle is just the definition of the obstruction space at \( s \). Now we use the sequence (6.10):

\[
0 \to H^0(W) \to \text{Obs}(s) \to H^1(T(P_r(f))/T(X_{2g−2−r}(f)))|_s \to H^1(W) \to 0. \tag{6.12}
\]

The penultimate term depends only on the image of \( s \) in \( P_r(f) \) so is constant as we vary over the fixed projective space; i.e. it varies to give a trivial bundle over \( \mathbb{P}^a \). We must identify the cokernel bundle \( W \). The fibre of \( W \) at a divisor \( D = \sum p_i \) is exactly (canonically) \( H^1(Σ; D) = H^0(Σ; K − D)^{\vee} \). On the other hand, \( H^0(K − D) \) is generated by the unique effective divisor which is the point \( u(D) \in X_{2g−2−r}(f) \), where \( u \) is the reciprocity map. It follows, as in [DS], that we can identify \( W \cong K_{Σ^1} \), where the twist comes from the relative duality isomorphism, and then the fact that \( u(D) \) generates \( H^0(K − D) \) identifies the sequence with:

\[
0 \to \mathcal{O}(−1) \to \mathbb{C}^{a+1} \to \text{Obs} \to 0.
\]

The first term is the tautological bundle, and hence the (quotient) obstruction bundle is the quotient bundle on projective space as claimed.

The above applies to each component \( \mathbb{P}^a \) of the moduli space of sections representing \( \iota(\alpha) \). But the Euler class of the quotient bundle is ±1 where the sign is determined by \( a \). This is fixed once and for all by \( \alpha \) and \( X \), so we indeed find that the standard surface counts for \( \alpha \) and \( \kappa − \alpha \) can only differ by a single overall sign. \( \blacksquare \)
As an immediate application, here is the “simple type” result that we alluded to earlier.

(6.13) Proposition: Let $X$ be a symplectic four-manifold with $b_+(X) > 1 + b_1(X)$. Then the standard surface counts vanish for any class $\alpha$ with $\alpha^2 \neq \kappa \cdot \alpha$.

Proof: Fix a compatible almost complex structure $J$ on $X$. Suppose the standard surface count for a class $\alpha \in H_2(X; \mathbb{Z})$ is non-zero. Then, by the symmetry proven above and the results of the previous section, both $\alpha$ and $\kappa - \alpha$ can be represented by embedded (or smoothly multiply covered) $J$-holomorphic curves in $X$. It is well known that $J$-holomorphic curves have locally positive intersections in a four-manifold; hence $\alpha$ and $\kappa - \alpha$ must have non-negative intersection unless they have common components. Suppose $C$ is such a component. Then $\alpha \cdot C = C^2 = \kappa - \alpha \cdot C$, which means that $\kappa \cdot C = 2C^2$. On the other hand, by adjunction, $\kappa \cdot C + C^2 = 2g(C) - 2$ and hence $C^2 \geq 0$. It follows that in all cases $\alpha \cdot (\kappa - \alpha) \geq 0$. Contrastingly, the (real) dimension of the moduli space of holomorphic sections of $X_r(f)$ in the class corresponding to $\alpha$ is exactly $\alpha^2 - \kappa \cdot \alpha$, also proven above; this must be non-negative if the invariant is non-zero. Hence

$$\alpha^2 \leq \kappa \cdot \alpha; \quad \alpha^2 \geq \kappa \cdot \alpha.$$  \hfill (6.14)

This gives $\alpha^2 = \kappa \cdot \alpha$, as required. ■

It’s now easy to check the following, which gives one familiar “basic class” type obstruction to symplectic manifolds being Kähler.

(6.15) Corollary: Suppose $b_+(X) > 1 + b_1(X)$. If $I_{(X,f)}(\alpha) \neq 0$ then $0 \leq \alpha \cdot [\omega] \leq K_X \cdot [\omega]$, with equality iff $\alpha \in \{0, K_X\}$. If in fact $(X; \omega)$ is a minimal Kähler surface of general type, then the standard surface counts are non-zero iff $\alpha \in \{0, K_X\}$.

Proof: The first half is immediate from the above. The result on the invariants for general type surfaces follows from the fact that $K_X$ is in the closure of the ample cone. Indeed, using results on pluricanonical linear systems [BPV84], it’s easy to see that in fact $K_X$ contains symplectic forms deformation equivalent to the given $\omega$ (just take an exact perturbation near any contracted $(-2)$-spheres). So then if $I_{(X,f)}(\alpha) \neq 0$ then

$$\alpha \cdot [\omega] \leq K_X \cdot [\omega] \Rightarrow \alpha \cdot K_X \leq K_X^2.$$  

Now the Hodge index theorem and Cauchy-Schwartz assert that $(\alpha^2) \cdot (K_X^2) \leq (\alpha \cdot K_X)^2$, which forces either $\alpha = 0$ or an equality $\alpha^2 = K_X^2$. The result is a simple consequence. ■

6.4 More symplectic surfaces

In this subsection, we prove the second part of Theorem (1.1). When $b_+ = 1$ the virtual dimension for sections of the Picard fibration is never negative, and so the arguments used above to control the geometry of the moduli spaces must be refined somewhat.
(6.16) Lemma: Let \((X, f)\) be any symplectic Lefschetz pencil on a manifold with \(b_+ = 1\). Form the Picard bundle \(P_r(f) \to \Sigma^2\) and fix a section \(s\) of \(P_r(f)\). Then there is an almost complex structure \(j\) on \(P_r(f)\) for which the moduli space of sections in the homotopy class defined by \(s\) is exactly a complex torus \(\mathbb{T}^{b_1(X)/2}\).

Proof: Note first the statement is coherent; if \(b_+ = 1\) then \(b_1\) is even. Decompose the base \(\Sigma^2\) into a union \(D\) of discs around the critical values of \(f\) over which the fibration is holomorphic, annuli \(A\) surrounding \(D\) and the complement \(R\) of slightly smaller annuli. Over \(R\) the degree zero Picard fibration (that is, the Jacobian fibration) carries a canonical flat connexion defined by the integral lattices in each Picard torus fibre, and this defines an almost complex structure on the total space which is linear in the sense that it lifts to a linear connexion on the complex vector bundle \(V|_R \to R\) where \(V \to \mathbb{P}^1\) has fibre \(H^1(\Sigma, \mathcal{O})\) at \(t \in \mathbb{P}^1\). Over the discs \(D\) the total space of the Picard bundle is exactly isomorphic to the sheaf quotient \(\mathcal{R}^1 f_* \mathcal{O}/\mathcal{R}^1 f_* \mathbb{Z}\) and hence the induced integrable complex structure on \(P_r(f)|_D\) lifts to the given linear complex structure on the same vector bundle \(V\). Now over the interpolating annuli \(A\), we can interpolate the two connexions on \(V\) equivariantly with respect to the action of the lattice \(\mathbb{Z}^{2g}\) (fix a homotopy on a fundamental domain and then extend). This defines a complex structure on \(P_0(f)\).

Now be given some \(P_r(f)\) and a fixed section \(s\). We use the section to define a diffeomorphism of \(P_r(f)\) and \(P_0(f)\) which takes \(s\) to the zero-section, and equip \(P_r(f)\) with the pullback of the above complex structure. Clearly the given section is now holomorphic. Given any other holomorphic section \(s'\) of \(P_r(f)\) which is homotopic to \(s\), the differences \(y(t) = s(t)^{-1} \otimes s'(t)\) define a section of the bundle \(P_0(f)\) which is also holomorphic. Since we have worked with integrable structures near the singular fibres, and the group action is holomorphic in the algebraic setting, this is globally well-defined. By the construction of the complex structure from a connexion, this section \(y\) lifts (non-uniquely!) to a section of the vector bundle \(V \to \mathbb{P}^1\), which for a generic interpolating choice of connexions over the annuli \(A\) carries its most stable complex structure. But the index of the \(\bar{\partial}\)-operator on \(V\) is just \(b_1(X)/2\) and the rank of the bundle is \(g \gg 0\). Hence the most stable complex structure is of the shape

\[ V \cong \mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-1) \]

and the moduli space of sections of \(V\) in the required homotopy class is just \(\mathcal{C}^{b_1(X)/2}\). To compute the moduli space of sections of \(P_r(f)\) in the projected homotopy class, we must divide out the non-uniqueness of the lift, and this is just the action of the sublattice that preserves the holomorphically trivial subbundle of \(V\). This yields a torus of the claimed dimension. ■

Let us put this observation in context. Suppose for a moment that we are looking for symplectic surfaces in a class \(\alpha\) which arise from a section of the bundle of symmetric products \(X_r(f)\) with \(r > 2g - 2\). This is always the case, by adjunction, for a class of the form \(\iota(\alpha)\) when \(K_X \cdot \omega < 0\). In this case the geometry simplifies considerably; the total space of \(X_r(f)\) is a projective bundle over \(P_r(f)\). The map \(F : X_r(f) \to P_r(f)\) is a submersion away from the critical fibres, and the exact sequences of (6.4) and (6.5) disappear. We find that the obstruction bundle is topologically trivial. This has two consequences:
If \( b_+ > 1 \) then the obstruction bundle has trivial Euler class, and hence the standard surface counts cannot be non-trivial for any class \( \alpha \) with \( \alpha \cdot \omega > K_X \cdot \omega \);

- If \( b_+ = 1 \) then the obstruction bundle is of rank zero, and hence moduli spaces of symplectic surfaces are unobstructed.

The first statement above we knew already, from the duality \( I(\alpha) = \pm I(\kappa - \alpha) \). The second statement should be interpreted as follows. View the invariant \( I(\alpha) \) as a homology class in \( H_{\text{vir dim}}(\Gamma(S^2; X_r(f))) \), before cutting down dimensions; then this invariant is realised by the fundamental class of the space of \( J \)-holomorphic curves for any \( J \in \mathcal{J} \). Now return to the main theme:

(6.17) Theorem: Let \( X \) be a symplectic four-manifold with \( b_+ = 1 \) and \( b_1 = 0 \). Fix \( \alpha \in H_2(X; \mathbb{Z}) \) satisfying \( \alpha^2 > K_X \cdot \alpha \) and \( \alpha \cdot \omega > 0 \). Then \( \alpha \) contains embedded symplectic surfaces in \( X \).

Proof: It is enough to prove that for a high degree pencil \( f \) on \( X \), the invariant \( I(X,f)(\alpha) \neq 0 \). After twisting by all the exceptional curves, there is certainly a homotopy class of sections which yields surfaces in the class \( \alpha + \sum E_i \); just choose a family of \( \overline{\partial} \)-operators on the restrictions of the line bundle with this first Chern class to the fibres of \( f \), and observe that an associated projective bundle down the fibres embeds in some \( X_r(f) \). By an earlier result, this homotopy class of sections is unique.

Fix an almost complex structure \( j \) on \( P_r(f) \) for which the moduli space of sections is just a point (a zero-dimensional torus). Extend to \( J \) on \( X_r(f) \) for which the Abel-Jacobi map is holomorphic. Then all holomorphic sections of \( X_r(f) \) lie over this unique section of \( P_r(f) \), and hence the moduli space is just the space of sections of some projective bundle. The conditions on \( \alpha \) show that this projective bundle does indeed have sections \( \overline{\partial} \). In the usual way, this shows that the (compact) space of sections is a projective space. Each point condition \( D(z_i) \) defines a hyperplane in this space, and so the standard surface count is \( \pm 1 \). The result follows.

This is just an instance of the “wall-crossing formula”. In general, although the symmetry \( I(\alpha) = \pm I(\kappa - \alpha) \) breaks down when \( b_+ = 1 \), the difference \( |I(\alpha) - I(\kappa - \alpha)| \) is given by the Euler class of an obstruction bundle over \( T^{b_1(X)/2} \). Given (6.16), one can prove this using the techniques of this paper, thereby removing the hypothesis \( b_1 = 0 \) from the last two applications in (1.5).

7 Applications and refinements

In this section, we shall give the proofs of the applications listed in the Introduction. These are due to other authors, and are included for completeness. We also give a weak result on homotopy \( K3 \) surfaces. At the end of the section, we shall sketch how to improve the linear constraint \( b_+ > 1 + b_1 \) to \( b_+ > 2 \), which mildly improves the scope of the main theorems of this paper and [DS].
7 APPLICATIONS AND REFINEMENTS

7.1 Symplectic consequences

The first result is an immediate consequence of the main result of [DS], and is originally due to Taubes [Tau95].

(7.1) Corollary: Let $X$ be a symplectic four-manifold with $b_+ > 1 + b_1$. If $X$ is minimal then $2e(X) + 3\sigma(X) \geq 0$.

Proof: We may represent $K_X$ by an embedded symplectic surface; by adjunction, the only components of negative square are $-1$-spheres, which we exclude by assumption. So $c_1^2(X) \geq 0$, and this is just the inequality given. ■

If $b_1(X) = 0$, then the above also shows that $K_X \cdot [\omega_X] < 0 \Rightarrow b_+ = 1$. If in fact $b_2(X) = 1$ then every symplectic form is rational. Hence, even without proving that one obtains holomorphic curves from the $Z$-invariants and not just symplectic surfaces, we recover the theorem of Taubes [Tau96].

(7.2) Corollary: If $X$ is a symplectic homology projective plane with $K_X \cdot [\omega] < 0$ then $(X, \omega) \cong (\mathbb{C}P^2, \mu \omega_{FS})$ for some $\mu > 0$.

Proof: According to (6.17) the generator for the homology contains smooth symplectic surfaces. These must be connected, by irreducibility of the homology class, and then the adjunction formula shows that such a surface must in fact be a sphere. Since the virtual dimension of the moduli space was 2, there is in fact such a sphere through any two generic points of $X$. Then, as in Gromov [Gro85], one uses such a family of spheres to construct an explicit diffeomorphism from $X$ to $\mathbb{C}P^2$ which pulls back the standard symplectic form. ■

This was the original motivation for the section. We can extend that result as follows: the following is due to Ohta and Ono [OO96].

(7.3) Corollary: Let $X$ be a symplectic four-manifold with $b_1 = 0$ and with $K_X \cdot [\omega] < 0$. Then $X$ is diffeomorphic to $S^2 \times S^2$ or to $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ with $n \leq 8$.

Proof: Again it is enough to construct a symplectic sphere of non-negative square. Then work of McDuff provides a diffeomorphism to one of the listed del Pezzo surfaces (and in fact work of Lalonde-McDuff on the classification of symplectic forms on rational manifolds pins down the precise symplectic structure). Note that their work does not rely on gauge theory! Now if $K_X \cdot \omega < 0$ then for any $\alpha$ with $\alpha \cdot \omega > 0$ we have that $\nu(\alpha) \cdot [\text{Fibre}] > 2g - 2$; this is a trivial consequence of adjunction. It follows, by the remarks before Theorem (6.17), that if $\alpha$ satisfies $\alpha^2 > K \cdot \alpha$ then $\alpha$ will have embedded symplectic representatives through any point. At least one component of each will be a sphere provided $\alpha^2 + K_X \cdot \alpha < 0$, and this underlies the result.

Since $K_X = -\lambda[\omega]$ we know $c_1^2(X) > 0$. However, since $b_+ = 1$ and $b_1 = 0$, then
\[ 2e(X) + 3\sigma(X) = 4 + 2(1 + b_-) + 3(1 - b_-) = 9 - b_-. \]

It follows that \( b_- \leq 8 \). The Hasse-Minkowski classification of intersection forms shows that the only possible intersection forms are \( \mathbb{Z}(1) \oplus b_- \mathbb{Z}(-1) \), in the odd case, and the rank two hyperbolic form in the even case. In both situations, the generator of the positive summand \( \alpha = (1, 0) \) is quickly checked to satisfy all the required conditions, and this homology class contains symplectic spheres. \( \blacksquare \)

Finally, we note that these methods give rise to constructions of symplectic forms on four-manifolds with \( b_+ = 1 \). There are two standard ways of building symplectic forms on four-manifolds:

- **By integration**: find a large family \( S \) of irreducible homologous surfaces which cover \( X \) and all have pairwise locally positive strictly positive intersections. Define a current by taking a two-form \( \chi \) to the number \( \int_{u \in S} \int_{S_u} \chi \); under fairly general circumstances, this defines a symplectic form (cf. Gompf’s work in \cite{Gom95} and \cite{Gom01}) in the class Poincaré dual to the homology class of any surface.

- **By inflation**: be given one symplectic form \( \omega \) on \( X \), and a connected symplectic surface \( S \subset X \), and then deform \( \omega \) by adding positive forms supported in the normal bundle of \( S \) (cf. McDuff’s \cite{McD98} and Biran’s \cite{Bir99}). One obtains forms in the class \([\omega] + tPD[S]\) for all \( t \geq 0 \).

The second method is just a special case of the first: the normal bundle of the surface \( S \) is filled by homologous symplectic surfaces. We can use either method here; the latter is mildly simpler. This result, and generalisations which also follow from our work, was noted by Li and Liu \cite{LL}.

**Corollary:** Suppose that \( X \) is minimal and \( b_+ = 1 \), \( b_1 = 0 \). If \( K^2 > 0 \) and \( K \cdot \omega > 0 \) then the canonical class contains symplectic forms.

**Proof:** For large enough \( N \), and arguments as above, we find that the homology class \( NK - [\omega] \) contains symplectic surfaces in \( X \). An easy argument with the intersection form (and the minimality assumption) ensures that these surfaces are connected. Then inflate! \( \blacksquare \)

This finishes our treatment of the applications \( \text{(1.5)} \); but we will end by discussing an application that we cannot yet complete. In the proof of \( \text{(1.1)} \), we used the Brill-Noether theory to get control on the geometry of the map \( X_r(f) \rightarrow P_r(f) \). For \( r \) neither very large nor small compared to \( g \), say \( r = g - 1 \), we have no such control. Hence we cheat and make a definition. Every Riemann surface of genus \( g \) has an associated \( \Theta \)-divisor, the image of \( \text{Sym}^{g-1}(C) \rightarrow \text{Pic}_{g-1}(C) \). The image is some subvariety of the Picard torus, and for a Zariski open set \( \mathcal{U} \) in moduli space the deformation type of the pair \( (\text{Pic}_{g-1}, \text{Sym}^{g-1}) \) will vary locally constantly. It is easy to check that the complement \( \mathcal{Q} \) of this open set has divisorial components.
Definition: A symplectic four-manifold $X$ is $\Theta$-positive if for all Lefschetz pencils $f$ of high enough degree, the associated sphere in moduli space $\mathcal{M}_g$ meets $Q$ locally positively.

Algebraic positivity is proved in [Smi01], but our techniques do not yet permit “Whitney moves” in $\mathcal{M}_g$ to cancel excess intersections. At least for the class of $\Theta$-positive manifolds, we can reprove a nice result of Morgan and Szabo, which follows from [MS97]. We shall just sketch the idea of the proof.

Proposition: Suppose $X$ is a $\Theta$-positive, simply-connected symplectic four-manifold with $c_1(X) = 0$. Then $X$ is a homotopy $K3$ surface.

Proof: [Sketch] It would be enough to prove the following (which is Morgan and Szabo’s theorem): for $X$ spin symplectic with $b_1 = 0$ and $c_1^2 = 0$ then $Gr_X(K_X/2)$ is odd iff $b_+ = 3$. This is the mod two version of the computation given at the end of section two. (There are examples which show that in the symplectic category, the Gromov invariant of $K_X/2$ can be any even or odd number for each fixed homotopy type, depending on whether $b_+ > 3$ or $b_+ = 3$.) Let us re-cast the last stage of that earlier argument. The map

$$H^0(K_X/2) \times H^0(K_X/2) \to H^0(K_X)$$

induces a map on projective spaces $\mathbb{P}^a \times \mathbb{P}^a \to \mathbb{P}^{2a}$ which can be described as follows: $\mathbb{P}^a = \text{Sym}^a(\mathbb{P}^1)$ and the map is just addition of divisors. The Kähler surfaces of the given homotopy type are elliptic; each (half-)canonical divisor is a collection of fibres, determined by a set of points on the base $\mathbb{P}^1$ of the elliptic fibration. Now the Gromov invariant of $K_X/2$ is just the degree of the map $+: \mathbb{P}^a \times \mathbb{P}^a \to \mathbb{P}^{2a}$, which is even provided the image has positive dimension, i.e. if $b_+ > 3$.

Be given a spin symplectic manifold. If $c_1^2 = 0$ we know that $b_+ \geq 3$, and if also $b_1 = 0$ then the index of the $\partial$-operator on the Picard bundle is necessarily negative. Split the exceptional sections into two collections of equal numbers of spheres $E_A$ and $E_B$. We obtain two sections of $P_{g-1}(f)$, corresponding to $K_X/2 + E_A$ and $K_{X'} - (K_X/2 + E_A) = K_X/2 + E_B$, and by generically extending some $J_{\text{Pic}}$ away from these we can assume these are the only holomorphic sections of the Picard bundle. If $X$ is $\Theta$-positive then we can find a $J$ on $X_{g-1}(f)$ which maps holomorphically to the Picard bundle with structure $J_{\text{Pic}}$. Then the moduli space of holomorphic sections of $X_{g-1}(f)$ in the class $s_A$ is a projective space $\mathbb{P}^a$ and in the class $s_B$ it’s a projective space $\mathbb{P}^{a+1}$ of the same dimension. Moreover, we have a map

$$+: \mathbb{P}^a \times \mathbb{P}^a \to \mathbb{P}^{N}; \quad N = [b_+ - 3]/2.$$

This multiplies holomorphic curves in the obvious way to give a section of the canonical bundle on the fibre: the projective space on the right is just the moduli space of holomorphic sections of $X_{2g-2}(f)$ for an almost complex structure standard near the zero-sections. On each fibre, then, the $+$ map is given by the natural map $\text{Sym}^{g-1} \times \text{Sym}^{g-1} \to \text{Sym}^{2g-2}$. But, if $a > 0$, this last map has even degree onto its image; it factors through $\text{Sym}^{2}(\mathbb{P}^a)$ for instance. The standard surface count for the class $K_X/2$ can be obtained from the obstruction bundle over $\mathbb{P}^a$ which can be described in terms of the obstruction bundle over its image in $\mathbb{P}^N$ via pullback by the map $+$. The evenness of the degree of this map translates to saying that the
Euler class is even, and we finally obtain that when \( b_+ > 3 \) the standard surface count for \( K_X/2 \) is even. But this, together with \([DS]\), implies that \( K_X \neq 0 \); hence \( \Theta \)-positivity implies the Morgan-Szabo theorem. ■

It would be interesting to see if one could use the geometry of the theta-divisor, or of harmonic maps to the moduli space of curves, to prove the positivity always holds.

### 7.2 A better linear constraint

Throughout the last sections, we have worked with complex structures on bundles and Jacobians generic away from certain specified loci. The factor \( b_1(X) \) in all the various inequalities occurs because (6.16 aside) we have ignored the topological structure - in the form of trivial subrepresentations of the homological monodromy - coming from line bundles on the four-manifold. We shall now indicate one naive route to taking account of the extra structure; in doing so we will improve the linear constraint in the main theorem of [DS] from \( b_+ > 1 + b_1 \) to the slightly improved constraint \( b_+ > 2 \). A similar analysis would yield the same improvement for the main theorem of this paper. This still falls short of Taubes, however, so the discussion should be regarded as somewhat parenthetical. We include it only to indicate that the problems arising here are related to the failure of the Hard Lefschetz theorem for symplectic manifolds. Indeed, if \( b_1(X) \) is odd, then we can’t “share out” the homology equally between \( H^{1,0}(\Sigma) \) and \( H^{0,1}(\Sigma) \) for a hyperplane \( \Sigma \). But this introduces an asymmetry into the construction: the projectivisation of the first vector space is the fibre of the bundle of fibrewise canonical forms, and the second vector space is the fibre of the tangent bundle to the Picard fibration at the zero-section. Presumably a different adaptation will avoid this hiccup, but it sheds some light on the different role of the first homology of the four-manifold in our treatment and that of Taubes. The argument is very closely related to (6.16).

As piece of notation, write \( R \) for \( b_1(X)/2 \) when \( b_1 \) is even and for \( (b_1(X) - 1)/2 \) when \( b_1 \) is odd. We adopt the notations of \([DS]\) wherever not already defined. In particular, \( f_*K \) denotes the vector bundle of fibrewise canonical forms. That is, if the bundle \( W \) has fibre canonically \( H^1,0(X) = H^0(X, \Omega^1_X) \). The subbundle arises from the obvious restriction map on holomorphic differentials. Moreover there is a decomposition \( H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1} \). The vector space on the LHS carries a trivial monodromy representation for any (Lefschetz) fibration, and hence we have a holomorphically trivial subbundle of \( f_*\omega_{X/B} \) over \( \mathbb{P}^1 \). The rank of the subbundle is \( R \). The proposition relies on the analogue of this for a general symplectic pencil.

### (7.7) Proposition:

If \( b_1(X) \) is even then there is a holomorphic structure on \( f_*K \) for which the space of holomorphic sections has dimension \( (b_+ - 1)/2 \). If \( b_1(X) \) is odd there is a holomorphic structure for which the space of holomorphic sections has dimension \( (b_+ - 2)/2 \).

### Proof:

For a Kähler surface there is a subbundle of \( f_*\omega_{X/B} \) with fibre at a point \( t \) given by \( H^{1,0}(X) = H^0(X, \Omega^1_X) \). The subbundle arises from the obvious restriction map on holomorphic differentials. Moreover there is a decomposition \( H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1} \). The vector space on the LHS carries a trivial monodromy representation for any (Lefschetz) fibration, and hence we have a holomorphically trivial subbundle of \( f_*\omega_{X/B} \) over \( \mathbb{P}^1 \). The rank of the subbundle is \( R \). The proposition relies on the analogue of this for a general symplectic pencil.

Using a family of metrics on the fibres we can write, at any point \( t \in \mathbb{S}^2 \), the cohomology group
7 APPLICATIONS AND REFINEMENTS

\[ H^1(\Sigma_t, \mathbb{C}) = H^{1,0}(\Sigma_t) \oplus H^{0,1}(\Sigma_t) = H^0(\Sigma_t, K_{\Sigma_t}) \oplus H^0(\Sigma_t, K_{\Sigma_t})^*. \]

We have an inclusion \( H^1(X, \mathbb{C}) \subset H^1(\Sigma_t, \mathbb{C}) \) as the subgroup of monodromy invariants, but now there may be no non-trivial intersection of \( H^1(X, \mathbb{C}) \) with \( H^{1,0}(\Sigma_t) \). We can change the choice of almost complex structures on the fibres to avoid this. Recall that a complex structure on the real vector space \( H^1(\Sigma, \mathbb{R}) \) is precisely a decomposition of the complexification \( H^1(\Sigma, \mathbb{C}) = H^1(\Sigma, \mathbb{R}) \otimes \mathbb{C} \) into two half-dimensional subspaces which are conjugate under the action of the conjugation on the tensor product factor. Given one reference complex structure \( H^1(\Sigma, \mathbb{C}) = E \oplus \mathbf{E} \), for instance a splitting into holomorphic and antiholomorphic forms, any other is given by a complex linear homomorphism \( E \to \mathbf{E} \) whose graph defines one of the two new decomposing subspaces. Now the subspace \( H^1(X, \mathbb{C}) \subset H^1(\Sigma, \mathbb{C}) \) is preserved by the complex conjugation, though no non-trivial complex subspace is fixed pointwise. It follows that we can choose a homomorphism in \( \text{Hom}(E, \mathbf{E}) \) for which an \( R \)-dimensional subspace of \( H^1(X, \mathbb{C}) \) lies inside the new space of holomorphic forms. Equivalently, we can choose a fibrewise family of complex structures on the fibres of a Lefschetz pencil to ensure that the intersection of the trivial complex bundle with fibre \( H^1(X, \mathbb{C}) \) and the bundle \( f_* \omega_{X/\mathbb{C}^2} \) has rank \( R \).

It follows that there is a sequence of topological bundles

\[ 0 \to \mathbb{C}^R \to f_* \omega_{X/\mathbb{C}^2} \to Q \to 0 \quad (7.8) \]

where \( Q \) is defined as the cokernel: this gives a sequence

\[ 0 \to (K_{\mathbb{P}^1})^\otimes R \to f_* K \to Q' \to 0. \]

Suppose we choose a complex structure on the vector bundle \( f_* K \) which makes the subbundle of the sequence holomorphic and which extends that connexion generically to \( Q' \). Then from the long exact sequence in cohomology we find that \( H^0(f_* K) = \text{Index}_{\mathbb{Z}^R}(\overline{\partial}) \) which is easily computed to be as claimed in the proposition. \( \blacksquare \)

Given this, one can find holomorphic sections of the projective bundle whenever \( b_+ > 2 \).

To run the rest of the computation of \( I(\kappa) \) requires one or two further modifications. The principal of these is a new definition for the complex structure \( J_{\text{ext}} \) compatible with the zero-sections (or more generally compatible with duality). We start with the linear complex structure on \( \mathbb{P}(f_* K) \) provided by (7.7). Inside the fibration of degree zero Jacobian varieties we have a subfibration \( \text{Pic}^X \) of tori of dimension \( R \), using the sequence (7.8) and the duality given at the end of Remark (7.6). These fibres are Picard varieties for line bundles on the symplectic manifold \( X \). There is an analogous subfibration inside all the higher degree Picard fibrations, defined up to translation. We can choose a complex structure on \( \text{Sym}^{(2g-2)} \) to make not only \( \mathbb{P}(f_* K) \) a holomorphic subset but also the preimage of this entire subfibration of \( P_{2g-2}(f) \). Choose a generic such almost complex structure.

(7.9) Lemma: Let \( X \) be a symplectic four-manifold with \( 2 < b_+(X) \leq 1 + b_1(X) \). For the \( J_{\text{ext}} \) described above, the whole moduli space of sections of \( \text{Sym}^{(2g-2)} \) is still the projective space of sections of \( \mathbb{P}(f_* K) \).
Proof: Suppose as before we have two sections $u_1, u_2$ of the fibration of symmetric products. As in (6.16), the difference $\tau(u_1)(\tau(u_2))^{-1}$ is well-defined as a section of the bundle $R^1 f_* \mathcal{O} \to S^2$. The sections of this are dual to the constant sections of the trivial subbundle of $f_\omega$. It follows that although not all sections of $\text{Pic}_f$ coincide with the image under $\tau$ of the projective bundle $\mathbb{P}(f_* K)$, all sections are constant translates of this section still lying inside the subfibration $\text{Pic}_f X$. Hence all the holomorphic sections of $\text{Sym}_f^{(2g-2)}$ are in fact holomorphic sections of $\mathbb{P}(f_* K)$ or of a projective bundle of rank $g-2$ over $S^2$. But the index of the $\overline{\partial}$-operator on all of these other projective bundles is still negative, by the assumption on the Betti numbers of $X$. Hence for a generic extension of the almost complex structure from $\mathbb{P}(f_* K)$ to the rest of the fibration $\tau^{-1}(\text{Pic}_f X)$, none of these other projective bundles have any sections; not all sections of $P_r(f)$ lift to the fibration of symmetric products. The result follows. ■

Note that it is easier to “fill in” the remaining four-manifolds with $2 < b_+ \leq 1 + b_1$ than to give a treatment for all four-manifolds with $b_+ > 1$ in one step. There is one last alteration required in the proof. In the long exact sequence in cohomology which underlies the obstruction computation (6.11), the first $H^1$-term no longer vanishes, since there is a non-trivial obstruction bundle for sections when viewed as lying inside $\mathbb{P}(f_* K)$. However, the preceding map in the exact sequence with image this obstruction space is an isomorphism, and the rest of the argument proceeds much as before.

References

[A+85] E. Arbarello et al. Geometry of Algebraic Curves, volume 1. Springer, 1985.

[AK] D. Auroux and L. Katzarkov. The degree doubling formula for braid monodromies and Lefschetz pencils. Preprint, 2000.

[AK80] A. Altman and S. Kleiman. Compactifying the Picard scheme. Advances in Math., 35:50–112, 1980.

[AL94] M. Audin and J. Lafontaine, editors. Holomorphic curves in symplectic geometry. Birkhäuser, 1994.

[Alm00] F. Almgren. Q-valued functions minimizing Dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension two. World Scientific, 2000.

[Aur01] D. Auroux. Symplectic maps to projective spaces and symplectic invariants. Turkish J. Math., 25:1–42, 2001.

[Bar82] D. Barlet. Developpement asymptotique des fonctions obtenues par integration sur les fibres. Invent. Math., 68:129–174, 1982.

[Bir99] P. Biran. A stability property of symplectic packing. Invent. Math., 136:123–55, 1999.

[BPV84] W. Barth, C. Peters, and A. Van de Ven. Compact Complex Surfaces. Springer, 1984.
REFERENCES

[Cha88] S. Chang. Two-dimensional area-minimizing integral currents are classical minimal surfaces. *J. Amer. Math. Soc.*, 1:699–788, 1988.

[Don96] S.K. Donaldson. Symplectic submanifolds and almost complex geometry. *J. Diff. Geom.*, 44:666–705, 1996.

[Don99] S.K. Donaldson. Lefschetz pencils on symplectic manifolds. *J. Diff. Geom.*, 53:205–36, 1999.

[DS] S. Donaldson and I. Smith. Lefschetz pencils and the canonical class for symplectic 4-manifolds. Preprint, 2000, available at math.SG/0012067.

[EH89] D. Eisenbud and J. Harris. Irreducibility of some families of linear series with Brill-Noether number 1. I. *Ann. Sci. Ecole Norm. Sup.*, 22:33–53, 1989.

[Fed69] H. Federer. *Geometric measure theory*. Springer, 1969.

[Fre87] D. Freed. On determinant line bundles. In S.T. Yau, editor, *Mathematical Aspects of String Theory I*. World Scientific, 1987.

[Gom95] R. Gompf. A new construction of symplectic manifolds. *Annals of Math.*, 142:527–595, 1995.

[Gom01] R. Gompf. The topology of symplectic manifolds. *Turkish J. Math.*, 25:43–59, 2001.

[Gro85] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82:307–47, 1985.

[GS99] R. Gompf and A. Stipsicz. *4-manifolds and Kirby calculus*. American Mathematical Society, 1999.

[Har66] R. Hartshorne. *Residues and Duality*. Springer, 1966.

[HJS96] M.C. Hong, J. Jost, and M. Struwe. Asymptotic limits of a Ginzburg-Landau type functional. In J. Jost, editor, *Geometric analysis and the calculus of variations for Stefan Hildebrandt*, pages 99–123. International Press, 1996.

[HL97] D. Huybrechts and M. Lehn. *The Geometry of Moduli Spaces of Shaves*. Number 31 in Aspects of Math. Max Plank-Institut, 1997.

[LL] T-J. Li and A-K. Liu. Uniqueness of symplectic canonical class, surface cone and symplectic cone of four-manifolds with $b_+ = 1$. Preprint, 2000.

[McD98] D. McDuff. From symplectic deformation to isotopy. In *Topics in symplectic 4-manifolds (Irvine, 1996)*. International Press, 1998.

[MS94] D. McDuff and D. Salamon. *J-holomorphic curves and quantum cohomology*. Amer. Math. Soc., 1994.

[MS97] J. Morgan and Z. Szabo. Homotopy $K3$ surfaces and mod 2 Seiberg-Witten invariants. *Math. Res. Lett.*, 4:17–21, 1997.
REFERENCES

[Nak99] H. Nakajima. Lectures on Hilbert schemes of points on surfaces. Amer. Math. Soc. Publ., 1999.

[OO96] H. Ohta and K. Ono. Notes on symplectic four-manifolds with $b_+ = 1$. II. Internat. J. Math., 7:755–760, 1996.

[OS79] T. Oda and C.S. Seshadri. Compactifications of the generalised Jacobian variety. Trans. Amer. Math. Soc., 253:1–90, 1979.

[Pre] F. Presas. Submanifolds of symplectic manifolds with contact border. Preprint, 2000, available at math.SG/0007037.

[Sal] D. Salamon. Spin geometry and Seiberg-Witten invariants. Unpublished book.

[Sal99] D. Salamon. Seiberg-Witten invariants of mapping tori, symplectic fixed points, and Lefschetz numbers. Turkish J. Math., 23:117–143, 1999.

[Sei97a] P. Seidel. $\pi_1$ of symplectic automorphism groups and invertibles in quantum homology rings. GAFA, 7:1046–95, 1997.

[Sei97b] P. Seidel. Thesis draft. Unpublished notes, 1997.

[Sei00] P. Seidel. Vanishing cycles and mutations. Preprint, available at math.SG/0007115, to appear in Proceedings of the European Congress of Mathematicians, Birkhauser, 2000.

[Smi99] I. Smith. Lefschetz fibrations and the Hodge bundle. Geometry and Topology, 3:211–33, 1999.

[Smi01] I. Smith. Lefschetz pencils and divisors in moduli space. Geometry and Topology, 5:579–608, 2001.

[ST] B. Siebert and G. Tian. Weierstrass polynomials and plane pseudoholomorphic curves. Preprint, 2001.

[Tau95] C.H. Taubes. The Seiberg-Witten and the Gromov invariants. Math. Res. Letters, 2:221–38, 1995.

[Tau96] C.H. Taubes. $SW \Rightarrow Gr$: From the Seiberg-Witten equations to pseudoholomorphic curves. Jour. Amer. Math. Soc., 9:845–918, 1996.

[Tau99] C.H. Taubes. Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual harmonic two-forms. Geometry and Topology, 3:167–210, 1999.