On gravitational fluctuations and the semi-classical limit in minisuperspace models

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Abstract

An attempt is made to go beyond the semi-classical approximation for gravity in the Born-Oppenheimer decomposition of the wave-function in minisuperspace. New terms are included which correspond to quantum gravitational fluctuations on the background metric. They induce a back-reaction on the semi-classical background and can lead to the avoidance of the singularities the classical theory predicts in cosmology and in the gravitational collapse of compact objects.

1 Introduction

The canonical quantization of highly symmetrical general relativistic systems carried out in suitably chosen variables leads to the dynamics being determined by the (super)Hamiltonian constraint \[ \mathcal{H} \] of the Arnowitt, Deser and Misner (ADM) construction \cite{3}. Such an approach is particularly useful to investigate self-gravitating quantized matter with gravity in the semi-classical regime. One performs a Born-Oppenheimer (BO) decomposition of the wave-function satisfying the Wheeler-DeWitt (WDW) equation into two parts \[ \psi \]. The first one represents a collective degree of freedom associated with gravity (slow component) and, in the semi-classical approximation, leads to an Hamilton-Jacobi (HJ) equation for the gravitational degree of freedom; the second part describes microscopic matter (fast component) and satisfies a Schrödinger equation in the time defined by semi-classical gravity.

Alternative approaches have been attempted, such as the one in Ref. \cite{6}, where, however, relative phases of matter and gravity were incorrectly identified \[ \psi \], or which involve an expansion in the Planck mass (see, e.g., Ref. \cite{7,8}). The latter expansion is potentially dangerous, since it has been shown that it can lead to violation of unitarity within the framework of canonical quantization \[ \mathcal{H} \] and to incorrect identification of the background as an empty solution of Einstein equations \[ \mathcal{H} \]. In the BO approach \[ \psi \] the collective degree of freedom evolves slowly because it is associated with the total mass of the system which is (many) times the mass of each constituting matter quantum (regardless of the latter being smaller than the Planck mass).

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For these reasons we shall appeal to the approach introduced in Ref. [5] as the best suited for the purpose of analyzing the semi-classical limit and shall not attempt at expanding the total wave-function in the Planck mass.

The BO approach was subsequently applied to two physical models of general interest: the gravitational collapse of a sphere of homogeneous dust in empty space [11] and spatially homogeneous Universes [9, 12] (see also Ref. [13] for collapsing shells). For the former system the novel effect of non-adiabatic production of matter has been studied with the analytical method of the (adiabatic) invariants for time dependent Hamiltonians [14, 9] in Ref. [15]. The same technique, supplemented by numerical simulations, has shown the possibility of having an inflationary phase in the primordial Universe which is driven by purely quantum fluctuations of the inflaton and has finite duration [16]. In both cases there are one degree of freedom for gravity, \( R \) (related to the external radius of the sphere or the scale factor of the Universe), and one degree of freedom for matter, \( \phi \) (homogeneous scalar field). The phase space is then the usual Friedmann-Robertson-Walker (FRW) minisuperspace of the space-time metric

\[
ds^2 = R \left[ -d\eta^2 + \frac{d\rho^2}{1 - \epsilon \rho^2} + \rho^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]
\]

(\( \epsilon = 0, \pm 1 \) respectively for flat, spherical and hyperbolic space and \( \rho \geq 0 \) for \( \epsilon = 0, -1; 0 \leq \rho \leq 1 \) for \( \epsilon = 1 \)) and homogeneous scalar matter. The coordinates \((\eta, \rho, \theta, \phi)\) define a comoving reference frame and for the sphere of dust \( \rho \leq \rho_s \), where \( \rho_s \) is the (constant) comoving radius of the sphere.

One must be careful in modeling dust with a scalar field, since the latter indeed describes a perfect fluid with pressure equal to the Lagrangian density, \( p \sim \frac{1}{2} \left( \dot{\phi}^2 - \ell_\phi^{-2} \dot{\phi}^2 \right) \) [17]. If the scalar field has mass \( m_\phi = \hbar/\ell_\phi \), then \( p \) oscillates with frequency \( \sim 2/\ell_\phi \), e.g., for \( m_\phi \sim 10^{-27} \) kg (the proton mass) this means a period \( T \sim 10^{-23} \) s. It is thus reasonable to approximate the actual pressure with its time average over one period (that is, set \( p = 0 \) provided the radius \( R \) does not change appreciably on the time scale \( T \) (quantum adiabatic approximation for the state of the scalar field). Moreover, this adiabatic approximation becomes exact in the classical limit for \( \phi \), as can be seen by taking \( \hbar \to 0 \) with \( m_\phi \) held fixed \( (\ell_\phi \to 0 \text{ and } T \sim \ell_\phi \text{ vanishes}) \), and a mode of the homogeneous massive scalar field can be identified with dust. In fact in Ref. [11] it was verified that in this approximation one recovers the classical Oppenheimer-Snyder (OS) model [18].

A major restriction in Refs. [3, 4, 11, 12, 13, 15, 16] is that quantum fluctuations of the gravitational degree of freedom were suppressed a priori and \( R \) was approximated by a classical trajectory \( R_c(\eta_c) \) (\( \eta_c \) being the conformal time associated with that trajectory) which, in turn, was determined solely by the matter content. The aim of the present notes is to allow the variable \( R \) to have quantum fluctuations around the classical trajectory and modify the expressions of the general formalism [3] accordingly. Of course, it would be much more interesting to allow for inhomogeneous fluctuations, but this would inevitably render the system intractable analytically and is left for future developments. As a by-product, we will see that one can have a significant back-reaction of the gravitational quantum dynamics on the semi-classical trajectory. This affects the singularity classical General Relativity generically predicts in cosmology and as the final state of a collapsing body (see [13] and Refs. therein).

The possibility of avoiding space-time singularities in a quantum theory has been studied for a long time and the literature on this topic is wide. Here we only refer to two approaches:
1. In quantum field theory in curved space-time (see, e.g., Ref. [20]) gravity is described by a classical background on which quantum matter fields propagate. In Ref. [21] it was found that there are states of matter for which the Universe admits a minimum non-zero scale factor, provided the number of particles is not conserved.

2. It was suggested that canonical quantization of the gravitational degrees of freedom could bypass the cosmological singularity [22]. In Ref. [23] the constraints were implemented before quantizing and one ended up with quantized gravitational degrees of freedom only. In this case no significant change in the classical behaviour was found.

It is a trivial observation that in a quantum theory a point-like singularity is meaningless since it would violate Heisenberg’s principle. What we shall show in the proposed approach is that one expects the singularity is avoided under a broad assignment of initial conditions. In fact the semi-classical approximation breaks down before the point-like singularity is reached (but within the adiabatic approximation for the gravitational degree of freedom) and the very concept of a trajectory loses its meaning at a value of \( R \) which can be appreciably big (in a sense that will be specified later).

The plan of the paper is as follows. In the next Section quantum gravitational fluctuations are treated in the standard BO formalism for the FRW minisuperspace and it is shown that their energy cannot always be neglected with respect to the energy of matter. In Section 3 the energy of such fluctuations is incorporated in a modified semi-classical HJ equation which is solved under certain approximations. Such approximations are then analyzed to determine the range of validity of the solutions. In Section 4 some conclusions are drawn for cosmological models and for the collapse of homogeneous spheres of dust. Finally in Section 5 the results are summarized and commented. We shall use units in which \( c = 1, \kappa = 8\pi G_N, \ell_p = \sqrt{\hbar\kappa} \) is the Planck length.

2 Quantum gravitational fluctuations in the BO approach

Let us start directly from the WDW equation in the minisuperspace of the two variables \( R \) and \( \phi \) (for a derivation from first principles see [1]) with a convenient operator ordering in the gravitational kinetic term [11]:

\[
\left[ \hat{H}_G + \hat{H}_M \right] \Psi = \frac{1}{2} \left[ \kappa h^2 \frac{\partial^2}{\partial R^2} R - \frac{\epsilon}{\kappa} R - \frac{h^2}{R^3} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\ell^2} \phi^2 R^3 \right] \Psi(R, \phi) = 0 .
\] (2.1)

The wave-function \( \Psi \) can be expressed in the factorized form \( \Psi(R, \phi) = R \psi(R) \chi(\phi, R) \) which, after multiplying on the LHS of Eq. (2.1) by \( \chi^* \) and integrating over the matter degrees of freedom, leads to the equation for the gravitational part [9]

\[
\frac{1}{2} \left[ \left( \kappa h^2 \frac{\partial^2}{\partial R^2} - \frac{\epsilon}{\kappa} R^2 \right) + \frac{1}{\langle \chi | \chi \rangle} \langle \chi | \frac{\partial^2}{\partial \phi^2} + \frac{1}{\ell^2} \phi^2 R^4 | \chi \rangle \right] \tilde{\psi} = \left[ \hat{H}_G R + R \langle \hat{H}_M \rangle \right] \tilde{\psi} = \frac{\kappa h^2}{2 \langle \chi | \chi \rangle} \langle \chi | \frac{\partial}{\partial R} \left( 1 - \frac{\langle \chi | \chi \rangle}{\langle \chi | \chi \rangle} \right) \frac{\partial}{\partial R} | \chi \rangle \tilde{\psi} .
\] (2.2)

The scalar product \( \langle \chi | \chi \rangle \equiv \int d\phi \chi^*(\phi, R) \chi(\phi, R) \) and

\[
\tilde{\psi} = e^{-i} \int R A(R') dR' \tilde{\psi} \quad \chi = e^{+i} \int R A(R') dR' \tilde{\chi} ,
\] (2.3)
with $A \equiv -i\langle \chi | \chi \rangle^{-1} \langle \chi | \partial_R | \chi \rangle \equiv -i \langle \partial_R \rangle$. If we now multiply Eq. (2.2) by $\tilde{\chi}$ and subtract it from Eq. (2.1) we obtain the equation for the matter function $\tilde{\chi}$:

$$\tilde{\psi} R \left[ \hat{H}_M - \langle \hat{H}_M \rangle \right] \tilde{\chi} + \kappa h^2 \left( \frac{\partial \tilde{\psi}}{\partial R} \right) \frac{\partial \tilde{\chi}}{\partial R} = \frac{\kappa h^2}{2} \tilde{\psi} \left[ \left( \frac{\partial^2}{\partial R^2} \right) - \frac{\partial^2}{\partial R^2} \right] \tilde{\chi}.$$  (2.4)

The Eqs. (2.2) and (2.4), as well as the WDW equation, are exact, in the sense that no approximation has been assumed yet for the wave-functions $\tilde{\chi}$ and $\tilde{\psi}$, and contain no time variable.

A way one can introduce the time is by taking the semi-classical limit for gravity [24, 25, 5]. In order to do so, first one needs to neglect the RHS’s of Eqs. (2.2) and (2.4) which are related to quantum transitions among different semi-classical trajectories [9]. As usual [5, 9], we shall check the consistency of all approximations once the solutions to the semi-classical equations have been obtained (see Section 3.2). In fact, it is not necessary (nor possible, in general) to prove that the RHS’s are small from the onset, but it is sufficient to show a posteriori that they are negligible for the cases considered. Then one writes a semi-classical (WKB) approximation for the wave function $\tilde{\psi}$

$$\tilde{\psi}_c = \frac{1}{\sqrt{-P_c}} e^{\frac{i}{\hbar} \int P_c dR_c},$$  (2.5)

where

$$P_c = -\frac{1}{\kappa} \frac{\partial R_c}{\partial \eta_c} = -\frac{1}{\kappa} \sqrt{2 \kappa R_c \langle \hat{H}_M \rangle - \epsilon R_c^2}$$  (2.6)

is the canonical momentum conjugated to $R$ in the classically allowed region $\langle \hat{H}_M \rangle > \epsilon R^2 / 2 \kappa$ and the integral in the exponent is computed along the (so far unspecified) semi-classical trajectory $R = R_c(\eta_c)$ with momentum $P = P_c(\eta_c)$. Moreover, the derivatives with respect to the conformal time $\eta_c$ are defined according to Eq. (2.4) as

$$\frac{\partial}{\partial \eta_c} \equiv -\kappa \psi_c P \left. \frac{\partial}{\partial R} \right|_{R_c} = -\kappa P_c \frac{\partial}{\partial R} \bigg|_{R_c},$$  (2.7)

where the last step follows from $\psi_c$ having support only for $R \sim R_c$, $\eta \sim \eta_c$.

Upon substituting $\tilde{\psi} = \psi_c$ into Eq. (2.2), the gravitational equation finally reduces to the semi-classical HJ equation

$$\psi_c \left[ -\frac{1}{2\kappa} \left( \frac{dR_c}{d\eta_c} \right)^2 - \frac{\epsilon}{2\kappa} R_c^2 + R \langle \hat{H}_M \rangle \right] = -\frac{1}{2\kappa} \left( \frac{dR_c}{d\eta_c} \right)^2 - \frac{\epsilon}{2\kappa} R_c^2 + R_c \langle \hat{H}_M \rangle = 0,$$  (2.8)

which can now be used to determine $R_c$ explicitly once $\langle \hat{H}_M \rangle$ is given.

It is important to note that the semi-classical regime is not defined simply as the limit $\hbar \to 0$, but rather by a specific choice of the wave-function $\psi_c$. For instance, with $P_c$ given by Eq. (2.4), $\langle \hat{H}_M \rangle = N_\phi m_\phi$ and a constant radial number density of scalar quanta $N_\phi > 0$ (in practice this is the statement of the quantum adiabatic approximation), one has (see, e.g., Ref. [26])

$$R_c = N_\phi \frac{\ell_p^2}{\ell_\phi} \times \begin{cases} (\cosh \eta_c - 1) & \epsilon = -1 \\ \eta_c^2/2 & \epsilon = 0 \\ (1 - \cos \eta_c) & \epsilon = +1 \end{cases}.$$  (2.9)
that is the usual FRW cosmological models for increasing \( \eta_c \) or the OS model of gravitational collapse for decreasing \( \eta_c \) (the classical singularity occurs at \( \eta_c = 0 \) in both cases).

Substituting \( \tilde{\psi} = \psi_c \) in Eq. (2.4) gives the Schrödinger equation

\[
i \hbar \frac{\partial \chi_s}{\partial \eta_c} = \frac{1}{2} \left[ -\frac{\hbar^2}{R_c^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\ell^2} R_c^4 \phi^2 \right] \chi_s
\]

for the rescaled matter function \( \tilde{\chi} = \chi_s \exp\{ (i/\hbar) \int_0^\eta R_c \, d\eta' \langle \hat{H}_M \rangle \} \). We note in passing that the difference between \( \chi_s \) and the original \( \chi \) amounts exactly to the phase factor between eigenvalues of an hermitian invariant for the Hamiltonian in Eq. (2.10) \cite{14} and exact solutions of Eq. (2.10) \cite{9}. The above Schrödinger equation together with the HJ equation (2.8) was the starting point for the results found in Refs. \cite{11, 12, 13, 15, 16} and led to the conclusions briefly mentioned in the introduction.

So far matter and gravity are determined by two equations of clearly different types. Suppose instead one defines

\[
\tilde{\psi} = \psi_c f ,
\]

where \( f = f(R) \) will encode quantum fluctuations around the trajectory \( R_c \) swept by \( \psi_c \). Then Eq. (2.2), again neglecting the RHS’s (see Section 3.2 for a detailed description of the difference with the previous case), becomes an equation for \( f \) (‘\( \equiv \partial/\partial R \))

\[
\psi_c \left[ \frac{k \hbar^2}{2} \left( \frac{3}{4} \frac{P''^2}{P^2} - \frac{1}{2 \kappa^2} \frac{P''}{P} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial R^2} \right) + i \kappa \hbar P \frac{\partial}{\partial R} \right] f
\]

\[
\quad = \psi_c \left[ \frac{k}{2} P^2 + \frac{\epsilon}{2 \kappa} R^2 - R \langle \hat{H}_M \rangle \right] f .
\]

Upon using the definition (2.7), the above takes the form of a time-dependent Schrödinger equation (‘\( \equiv \partial/\partial \eta \))

\[
i \hbar \frac{\partial f}{\partial \eta_c} = \psi_c \left[ \frac{k \hbar^2}{2} \left( \frac{5}{4 \kappa^2} \frac{P''^2}{P^4} - \frac{1}{2 \kappa^2} \frac{P''}{P} \frac{P}{\partial R} + \frac{P}{\partial R} + \frac{\partial^2}{\partial R^2} \right) \right] f
\]

\[
\quad + \psi_c \left[ -\frac{\kappa}{2} P^2 - \frac{\epsilon}{2 \kappa} R^2 + R \langle \hat{H}_M \rangle \right] f .
\]

In the RHS, due to the factor \( \psi_c \), the quantity \( P \) is still evaluated at the classical momentum and \( \eta = \eta_c \). Therefore one can simplify Eq. (2.13) by making use of the HJ equation (2.8) and obtains

\[
i \hbar \frac{\partial f}{\partial \eta_c} = \frac{k \hbar^2}{2} \left[ \left( \frac{5}{4 \kappa^2} \frac{P''^2}{P^4} - \frac{1}{2 \kappa^2} \frac{P''}{P} \frac{P}{P_c^2} \right) f(R_c) + \frac{1}{\kappa} \frac{P_c}{P_c^2} \frac{\partial}{\partial R} f(R_c) + \frac{\partial^2}{\partial R^2} f(R_c) \right]
\]

which is now the analogue of Eq. (2.10) for gravity.

Since Eq. (2.14) is involved, let us take the \textit{classical adiabatic limit} for \( R_c \), to wit \( \left| \dot{R}_c \right| \ll R_c \Rightarrow \left| \frac{\partial^{n+1} R_c}{\partial \eta_c^{n+1}} \right| \ll \left| \dot{R}_c \right| \) for any integer \( n > 0 \). This approximation is not to be confused with the previously mentioned quantum adiabatic limit on the state of \( \phi \) and will be further
discussed in Section 3.2. From the definitions (2.6) and (2.7) it then follows that one can neglect terms containing \( \dot{P}_c \) and \( \ddot{P}_c \). Hence, only the last term survives in the RHS above and one finds

\[
i \hbar \frac{\partial f}{\partial \eta_c} = \kappa \hbar^2 \frac{\partial^2 f}{\partial R^2} \bigg|_{R_c},
\]

which resembles the non-relativistic equation for a free particle of “mass” \( 1/\kappa \) and negative kinetic energy.

**Plane waves**

Eq. (2.15) admits solutions in the form of plane waves,

\[
f(\lambda) = \exp \left\{ i \frac{\ell_p^2 \eta_c}{2 \lambda^2} \frac{R_c}{\lambda} \right\},
\]

where the \( \lambda \)'s are real numbers (\( \lambda > 0 \) for left movers and \( \lambda < 0 \) for right movers in \( R \) space). One observes that, due to the “wrong” sign mentioned above, the energy (conjugated to the proper time \( d\tau = R_c d\eta_c \)) associated with each mode of wavelength \( \lambda \),

\[
E(\lambda) = -\frac{\hbar^2}{2 R_c \lambda^2},
\]

is negative. Although disturbing at first sight, this is in agreement with gravity contributing negative amounts to the total (super)Hamiltonian \([27]\).

**Exponential waves**

For imaginary \( \lambda = il \), from Eq. (2.16) one obtains a new set of solutions given by

\[
f(l) = \exp \left\{ -i \frac{\ell_p^2 \eta_c}{2 l^2} \frac{R_c}{l} \right\},
\]

whose energies are positive,

\[
E_l = \frac{\hbar}{2 R_c} \frac{\ell_p^2}{l^2}.
\]

Of course the amplitude of the above solutions increase with \( R_c \) for \( l > 0 \) and decrease with \( R_c \) for \( l < 0 \), thus signalling an instability. In fact, these modes can be related to the tunneling of \( R \) across classically forbidden regions, since \( i P (\ln f_l)' \) is real if \( P \) is imaginary. For this reason, one cannot superpose solutions from the two sets \( \{f_\lambda\} \) and \( \{f_l\} \) if the classical limit for \( R \) has to make sense \([28]\) (due to the factor \( \psi_c \) either \( R \) is in a classically allowed region or it is not).

In the following we will only consider real values for \( \lambda \).
Full solutions

The full gravitational state corresponding to the modes $f_\lambda$ found above are given by

$$\tilde{\psi}_\lambda \equiv \psi_c f_\lambda = \psi_c \exp \left\{ i \frac{\ell_p^2 \eta_c}{2 \lambda^2} + i \frac{R_c}{\lambda} \right\},$$

(2.20)

where the weight $\psi_c$ ensures that $\int dR \tilde{\psi}_\lambda^* \hat{O} \tilde{\psi}_\lambda = (\hat{O} f_\lambda)(R_c)$ for every operator $\hat{O}(R, \partial/\partial R)$. Then the general solution to Eq. (2.15) is a superposition of the form $\sum c_\lambda f_\lambda(R_c)$ and the total energy associated to quantum gravitational fluctuations is given by

$$E_f = -\frac{\hbar \ell_p^2}{2 R_c} \sum_\lambda |c_\lambda|^2 \frac{\lambda^2}{\chi^2},$$

(2.21)

where the $c_\lambda$ are normalization coefficients.

It is clear from Eq. (2.21) that $|E_f|$ can be very large, depending on the modes $f_\lambda$ which are included. In particular, $|E_\lambda| \ll \langle \hat{H}_M \rangle$ only for

$$\lambda^2 \gg \lambda^2_c \equiv \frac{\hbar}{2 R_c \langle \hat{H}_M \rangle}.$$

(2.22)

An interesting observation is that $\lambda_c$ is time-dependent (via $R_c$) and, in the quantum adiabatic approximation for $\chi_s$, the term $\langle \hat{H}_M \rangle = N_\phi m_\phi$ is constant [11] and one has

$$\chi^2_c = \frac{\ell_p^2 \ell_\phi}{2 R_c N_\phi}.$$

(2.23)

For an expanding universe in which $R_c$ increases in time without bounds, $\lambda_c$ will eventually vanish after it had been as big as possible in the far past. On the other hand, for the case of a collapsing sphere of dust with monotonically decreasing $R_c$, $\lambda_c$ will diverge and, no matter how long are the wavelengths of the initial gravitational fluctuations, $|E_f|$ will overcome $\langle \hat{H}_M \rangle$ before the sphere reaches the classical singularity $R_c = 0$.

We thus arrive at the following paradoxical conclusion. Our equations show that there are quantum gravitational fluctuations which can be generally associated to the classical solutions $R_c$ in Eq. (2.9). The energy of such fluctuations becomes inevitably larger than the matter energy at certain times but, since it does not appear in the HJ equation, the presence of gravitational fluctuations does not affect the semi-classical motion in any way. In the next Section we shall show that this paradox is due to an incorrect identification of the semi-classical limit and propose an approach to include the back-reaction of gravitational fluctuations.

3 Improved BO approach

The aim of this Section is to propose a redefinition of the semi-classical limit for gravity in minisuperspace which includes the (negative) energy of the gravitational fluctuations found in the previous Section into the HJ equation. This amounts to treat the gravitational fluctuations as an extra “matter” contribution, in much the same fashion as is usually done in perturbation theory around a fixed background in order to compute the back-reaction on the metric (see, e.g.,
The main advantage of the BO decomposition with respect to the latter approach is that we now derive such a description from a (supposedly more fundamental) unitary quantum theory (the WDW equation) together with explicit conditions for the semi-classical approximation which should otherwise be deduced from external principles. In fact, this will give us (semi)classical trajectories \( R_f \) corresponding to the matter content \( \langle \hat{H}_M \rangle \) and the gravitational state \( f \) together with the consistency conditions discussed in Section 3.2.

In order to simplify the analysis from now on we shall consider one gravitational mode at a time and set \( \tilde{\psi} = \psi_{\lambda} \) so that the energy of the gravitational fluctuations is given by \( E_\lambda \) in Eq. (2.17). With the above restriction, Eq. (2.12) in the classical adiabatic approximation \( \dot{R} \ll R \) becomes

\[
- i \hbar \psi_c P \frac{\partial f_\lambda}{\partial R} = \psi_c \left[ -\frac{\kappa}{2} P^2 - \frac{\epsilon}{2\kappa} R^2 + R \langle \hat{H}_M \rangle - \frac{\kappa \hbar^2}{2 \lambda^2} \right] f_\lambda .
\] (3.1)

The term of order \((\hbar/\lambda)^2\) survives in the semi-classical limit only provided one allows for very short wavelengths, such that \( \ell_p/|\lambda| \) does not vanish for \( \hbar \to 0 \). This is just the analogue of what is required for the expectation value of the matter Hamiltonian \( \sim \hbar/\ell_\phi \), to wit \( \ell_\phi \sim \hbar \). The condition \( |\lambda| \sim \ell_p \), in turn, would refer to a fully quantum theory of gravity, if \( \lambda \) is interpreted as a spatial wavelength, and one might prefer to place a ultra-violet cut-off \( \Lambda \geq \ell_p \) for the values of \( \lambda \). We prefer to stick to a more heuristic attitude and assign a physical meaning only to the energy \( E_\lambda \), keeping it finite (and mostly small) throughout the computations. Of course one can always consider \( \lambda \sim \ell_p \) as a (limiting) case of particular interest. Indeed, we will see in the next Section that one can obtain significant corrections induced by such modes in a way which is phenomenologically acceptable within the semi-classical treatment.

One observes that the factorization of the wave-function \( \tilde{\psi}_\lambda \) (gravitational state) into \( f_\lambda \) (fluctuations) and a specific \( \psi_c \) (classical part) is not forced by Eq. (3.1) or any other equation following from the WDW equation (2.3). In Refs. 25, 5, 9 it was rather determined by the implicit assumption that \( \langle \hat{H}_M \rangle \) is the dominant contribution in the semi-classical limit. This physical assumption takes mathematical form in the condition (2.22) which leads to the definition of the classical momentum \( P_c \) in Eq. (2.6). However, since in the last Section we concluded that there are times at which \( E_\lambda \sim \langle \hat{H}_M \rangle \) for every \( \lambda \), this is clearly contradictory and one should instead treat \( E_\lambda \) as a source for the dynamics of \( R \) on the same footing as \( \langle \hat{H}_M \rangle \). This can be achieved straightforwardly by introducing the modified momentum

\[
P_\lambda = - \frac{1}{\kappa} \frac{\partial R_\lambda}{\partial \eta_\lambda} = - \frac{1}{\kappa} \sqrt{2 \kappa R_\lambda \langle \hat{H}_M \rangle} - \epsilon R_\lambda^2 - \frac{\ell_\phi^2}{\lambda^2} .
\] (3.2)

A more formal way to derive this result is by defining a new WKB wave-function \( \psi_\lambda \) peaked on a modified trajectory \( R_\lambda \), parameterized by a time variable \( \eta_\lambda \) \( (d\eta_\lambda = \eta_c d\eta_c) \), such that

\[
\tilde{\psi}_\lambda = \frac{1}{\sqrt{-P_\lambda}} e^{i \int P_\lambda dR_\lambda} e^{i \frac{\kappa}{2}} \equiv \psi_\lambda \tilde{f}_\lambda .
\] (3.3)

Then, upon substituting into Eq. (2.22), in the classical adiabatic approximation \( \dot{R} \ll R \) and neglecting the RHS, one obtains

\[
- i \hbar \psi_\lambda P \frac{\partial \tilde{f}_\lambda}{\partial R} = \psi_\lambda \left[ -\frac{\kappa}{2} P^2 - \frac{\epsilon}{2\kappa} R^2 + R \langle \hat{H}_M \rangle - \frac{\kappa \hbar^2}{2 \lambda^2} \right] \tilde{f}_\lambda .
\] (3.4)
The LHS vanishes identically, since \(-\kappa \psi \lambda \partial \bar{f}_\lambda / \partial R \equiv \partial \bar{f}_\lambda / \partial \eta_\lambda = 0\). Therefore the RHS gives the modified HJ equation

\[- \frac{1}{2\kappa} \left( \frac{dR_\lambda}{d\eta_\lambda} \right)^2 - \frac{\epsilon}{2\kappa} R_\lambda^2 + R_\lambda \langle \bar{H}_M \rangle - \frac{\hbar \ell_p^2}{2 \lambda^2} = 0\, , \tag{3.5}\]

which is Eq. (3.2).

### 3.1 Examples

The above Eq. (3.5) will now be solved in the quantum adiabatic approximation \(\langle \bar{H}_M \rangle = N_\phi m_\phi\) constant and for the three values taken by the parameter \(\epsilon\) for the purpose of showing explicit results. However, we emphasize that the latter approximation is not essential for the general formalism and is assumed just because it allows to carry on the computation analytically.

**Negative curvature**

For \(\epsilon = -1\) the velocity \(\dot{R}_\lambda\) vanishes at

\[R_\lambda^\pm = -N_\phi \frac{\ell_p^2}{\ell_\phi} \left[ 1 \pm \sqrt{1 + \frac{1}{N_\phi^2 \lambda^2}} \right] , \tag{3.6}\]

thus one expects a turning point at \(R_\lambda = R_\lambda^-\) (\(R_\lambda^+ < 0\) is unphysical). The latter reduces to the turning point \(R_c^- = 0\) for \(\ell_\phi / |\lambda| \to 0\). Upon setting \(R_\lambda(0) = R_\lambda^-\), the modified trajectory is given by \((\eta \equiv \eta_\lambda)\)

\[R_\lambda = N_\phi \frac{\ell_p^2}{\ell_\phi} \left[ \sqrt{1 + \frac{1}{N_\phi^2 \lambda^2}} \cosh \eta - 1 \right] , \tag{3.7}\]

and the solution \(R_c\) given in Eq. (2.9) is recovered as \(R_\infty\) in the limit \(\ell_\phi / |\lambda| \to 0\) (\(R_\lambda^- \to 0\)). In the opposite limit, \(|\lambda| / \ell_p \to 0\), \(R_\lambda^-\) diverges and the trajectory eventually reduces to a point.

**Flat space**

For \(\epsilon = 0\) the modified trajectory is given by

\[R_\lambda = \frac{\ell_p^2}{2 \lambda^2} \frac{\ell_\phi}{N_\phi} + \frac{N_\phi \ell_p^2}{2 \ell_\phi} \eta^2 , \tag{3.8}\]

with a turning point at \(R_\lambda(0)\). The solution \(R_c\) in Eq. (2.9) is recovered in the limit \(\ell_\phi / |\lambda| \to 0\) \((R_\lambda(0) \to 0)\) as for \(\epsilon = -1\). Also, the opposite limit behaves the same as for negative curvature.

**Positive curvature**

For \(\epsilon = +1\) there are two turning points at

\[R_\lambda^\pm = N_\phi \frac{\ell_p^2}{\ell_\phi} \left[ 1 \pm \sqrt{1 - \frac{1}{N_\phi^2 \lambda^2}} \right] , \tag{3.9}\]
provided the square root is real, that is $|\lambda| > \ell_\phi/N_\phi$. As before, for $\ell_\phi/|\lambda| \to 0$ the minimum $R_\lambda^- \to R_\lambda^- = 0$ and the maximum $R_\lambda^+ \to R_\lambda^+ = 2 N_\phi \ell_\phi^2/\ell_\phi$. Again, upon setting $R_\lambda(0) = R_\lambda^-$, the modified solution is

$$R_\lambda = N_\phi \frac{\ell_\phi^2}{\ell_p} \left[ 1 - \sqrt{1 - \frac{1}{N_\phi^2 \lambda^2} \ell_\phi^2 \cos \eta} \right].$$

(3.10)

Thus, the effect of the extra term in the HJ equation is to make $R_\lambda$ oscillate between a minimum value $R_\lambda^-$ which is shifted above zero and a maximum value which is below the turning point $R_\lambda^{+}$. The shifts vanish and the solution $R_\lambda$ given in Eq. (2.9) is recovered in the limit $\ell_\phi/|\lambda| \to 0$. At the opposite limit stands the case $|\lambda| = \ell_\phi/N_\phi$ for which the amplitude of the oscillation $R_\lambda^+ - R_\lambda^- = 0$.

### 3.2 Consistency of the approximations

The trajectories $R_\lambda$ displayed above are consistent only when three different approximations hold simultaneously:

1. The RHS’s of Eqs. (2.2) and (2.4) must be negligible. This is the only relevant approximation which must hold for the general formalism as developed in this Section to apply. Since the RHS of Eq. (2.2) amounts to the expectation value of a gravitational operator on the matter state $\tilde{\chi}$, it is not substantially modified by the new definition of $\tilde{\psi}$ in Eq. (2.11) with respect to the state (2.5) (one only expects corrections because $\tilde{\chi}$ will evolve differently with the new scale factor $R_\lambda$). Hence, we refer the reader to Refs. [11, 12, 13, 15, 16], where this approximation has been studied at best for the case in Eq. (2.5).

However, in order to ensure that the corrections we have computed are significant, we now need to check that the RHS of Eq. (2.2) is negligible with respect to the new term $|E_\lambda|$. Taking the estimate for the RHS as given in Appendix B of Ref. [11],

$$\text{RHS} \sim \frac{\hbar \ell_\phi^2}{R^3} N_\phi^2,$$

(3.11)

we obtain the condition

$$\text{RHS} \ll |E_\lambda| \Rightarrow R \gg N_\phi \lambda \sim N_\phi \ell_p,$$

(3.12)

where we keep on singling out the particular value $\lambda \sim \ell_p$. We can now take the trajectories given in the previous Section and find that, for $\epsilon = 0, -1$, the turning point $R_{\lambda=\epsilon \ell_p}(0)$ satisfies Eq. (3.12) provided

$$N_\phi^2 \ll \frac{\ell_\phi^2 \ell_p}{\lambda^3} \sim \frac{\ell_\phi}{\ell_p},$$

(3.13)

while for $\epsilon = +1$ such a condition would be unphysical and is never met. This means that, for $\epsilon = +1$, the semi-classical approximation breaks down before any rebound occurs and quantum fluctuations prevail at small $R$ so that the superposition among different semi-classical trajectories cannot be avoided. Strictly speaking, the latter conclusion extends to all values of $\epsilon$, since the WKB approximation breaks down near the classical turning points.
This actually turns out to be a blessing in disguise, since, in the case of the collapse, the rebound of the sphere at a finite radius inside the Schwarzschild radius would eventually violate causality [29].

A more complete analysis of the dynamics when the RHS’s are not negligible is currently under study [28]. It is expected that one can still determine the evolution of the system in this non-classical range (at least numerically), although the geometrical interpretation is then lost (see, e.g., Ref. [30]).

2. The quantum adiabatic approximation $\langle \hat{H}_M \rangle$ constant. As we mentioned previously, this approximation is not essential for the development of the general formalism and was actually relaxed in Refs. [13, 15, 16]. However, it is only for $\langle \hat{H}_M \rangle$ constant that solutions can be computed analytically. Again we refer to Refs. [11, 12, 13, 15, 16] for a detailed analysis. For the case of the gravitational collapse, one obtains the condition [11]

$$ r - r(0) \gg \ell_\phi \ , $$

(3.14)

where $r \equiv \rho_s R_\lambda(\eta)$ is the areal radius of the sphere and $r(0)$ is the turning point $\rho_s R_\lambda(0)$.

3. The classical adiabatic approximation $\dot{R}_\lambda \ll R_\lambda$. Despite the terminology, this approximation is not related to the previous one, although, to some extent, it is inessential for the general formalism developed here. However, it is necessary for Eq. (2.14) for $f$ to simplify to the tractable form (2.15) and leads to a new condition which turns out to be not really restrictive for the example worked out here. In fact, it is easy to see that for the proposed solutions the ratio

$$ \frac{\dot{R}_\lambda}{R_\lambda} \to 0 \ , $$

(3.15)

for $\lambda \to 0$ ($\epsilon = -1, 0$) and $\lambda \to \ell_\phi/N_{\phi}$ (the minimum allowed value for $\epsilon = +1$).

From the above considerations one thus concludes that the break-down of the semi-classical approximation might occur at relatively large values $\sim R_\lambda(0)$. In the following Section, such turning/breaking points in the gravitational collapse will be regarded as significantly different from the point-like singularity only if they are bigger than the Compton wave-length $\ell_\phi$ of the particles.

4 Applications

As described in Section 2, the contribution of gravitational fluctuations incorporated in the theory is of quantum origin. It is generally taken for granted that gravity in the world we can test is classical, which leads one to assume the energy stored in quantum gravitational fluctuations is negligible at the time $\eta_0$ when measurements take place. To be more precise, let us introduce the ratio $\alpha^2(\eta) \equiv |E_\lambda|/\langle \hat{H}_M \rangle$ between quantum gravitational fluctuation energy and matter energy and assume $\alpha_0 \equiv \alpha(\eta_0) \ll 1$. From Eq. (2.17) this definition can be used to express $\lambda$ as

$$ \lambda^2 = \frac{1}{2 \alpha_0^2} \frac{\ell_\phi^2}{R_0 N_{\phi}} \ , $$

(4.1)
where $R_0 = R_\lambda(\eta_0)$.

The second important issue is whether the inclusion of $E_\lambda$ in the HJ equation leads to observable effects, that is, one will have to check when (if ever) deviations from the standard trajectories $R_c$ are physically significant in magnitude.

In order to clarify the above points, we now specialize the very simple analytic solutions found so far to two models. In so doing we do not expect our results to be definite answers to any basic physical question in either cases, however, we believe they give hints as to the possible relevance of the predicted effects.

### 4.1 Cosmology

In the cosmological case one takes $\eta_0$ equal to the (conformal) age of the Universe, so that the energy stored in the gravitational fluctuations is totally negligible today. However, this does not prevent $\alpha^2 \sim 1/R_\lambda$ to be comparable with one or bigger at very early stages. The key observation is precisely that the present scale factor of the universe, $R_0$, is related to the initial (minimum) scale factor $R(0)$ by

$$\alpha_0^2 = \alpha^2(0) \frac{R(0)}{R_0} .$$

One also recalls that in the RW metric the spatial distance between two points arbitrarily set at $\rho = 0$ and at $\rho = \rho_d$ is given by ($\rho_d \leq 1$ for $\epsilon = 0$)

$$s = R \int_0^{\rho_d} \frac{d\rho}{\sqrt{1 - \epsilon \rho^2}} .$$

#### Open Universe

When the spatial curvature $\epsilon = -1$, from Eqs. (3.6) and (4.1) one obtains

$$R(0) = R^-_\lambda = N_\phi \ell_\phi^2 \left[ \sqrt{1 + 2 \alpha_0^2 \frac{R_0 \ell_\phi}{\ell_p^2 N_\phi} - 1} \right] .$$

On using Eq. (4.2) one finds

$$R(0) = 2 N_\phi \ell_\phi^2 \left( \alpha^2(0) - 1 \right) ,$$

so that $R(0) > 0$ implies $\alpha^2(0) > 1$ and quantum gravitational fluctuations must dominate the early stages in order to have a start at non-zero scale factor (this is due to the gravitational potential contributing with the same sign as $\langle \hat{H}_M \rangle$ in the HJ equation for $\epsilon = -1$).

It is now interesting to relate $\alpha_0$ to physically meaningful quantities. For instance, the relative difference $\Delta s = s_f - s_c$ between spatial distances measured when $R = R_\lambda(\eta_0)$ and $R = R_c(\eta_0)$ is

$$\frac{\Delta s_0}{s_0} \simeq \sqrt{1 + 2 \alpha_0^2 \frac{R_0 \ell_\phi}{\ell_p^2 N_\phi} - 1} ,$$

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where we used Eqs. (2.9) and (3.7) in the limit \( \cosh \eta_0 \gg 1 \) (\( \Delta s_0 > 0 \) since \( R_\lambda \sim R_c + R_\lambda^- \)). Hence Eq. (4.4) becomes

\[
R(0) \simeq N_\phi \frac{\ell_\phi^2}{\ell_\phi} \frac{\Delta s_0}{s_0} .
\] (4.7)

If one takes for \( \Delta s_0/s_0 \) the accuracy with which distances are measured in the present Universe, the above equation gives us a (very rough indeed) estimate of the maximum value for the initial scale factor which cannot be ruled out by present measurements.

A further consequence of having \( R(0) > 0 \) is that two points in space were causally disconnected at \( \eta = 0 \) if their distance \( s(0) \gg \ell_\phi \), that is, from Eq. (4.3),

\[
\rho_d \gg \rho_c = \sinh \left( \frac{\ell_\phi}{N_\phi} \frac{s_0}{\ell_\phi} \Delta s_0 \right) .
\] (4.8)

In the flat case \( \epsilon = 0 \), \( \alpha^2(0) = 1 \), as follows from a quick inspection of Eq. (3.5), and the expression of the initial scale factor simplifies to

\[
R(0) = \alpha^2_0 R_0 = \frac{\Delta s_0}{s_0} R_0 .
\] (4.9)

Then the causal comoving radius \( \rho_c = \frac{\ell_\phi}{R_0} \frac{s_0}{\Delta s_0} \).

**Closed Universe**

For \( \epsilon = +1 \) we have shown in Section 3.2 that our corrections are less relevant, however, we also consider this case for completeness. On taking \( R_0 \sim R_\Lambda^+ \), one has

\[
R_0 \simeq 2 N_\phi \frac{\ell_\phi^2}{\ell_\phi} \left( 1 - \alpha^2_0 \right) ,
\] (4.10)

which holds for \( \alpha^2_0 < 1 \) (the gravitational potential in the HJ equation is now opposite in sign to \( \langle \hat{H}_M \rangle \)). The relative difference in distances is

\[
\frac{\Delta s_0}{s_0} \simeq -\alpha^2_0 ,
\] (4.11)

where we used Eqs. (2.9) and (3.10) in the limit \( \cos \eta_0 \simeq -1 \) (\( \Delta s_0 < 0 \) because the maximum of \( R_\lambda \) is shifted down with respect to the maximum of \( R_c \)). Putting the pieces together gives

\[
R(0) = \left| \frac{\Delta s_0}{s_0} \right| R_0 ,
\] (4.12)

and \( s(0) \gg \ell_\phi \) if

\[
\rho \gg \rho_c = \sin \left( \frac{\ell_\phi}{R_0} \frac{s_0}{\Delta s_0} \right) .
\] (4.13)

Since \( \rho < 1 \), the latter condition can be satisfied only if

\[
\left| \frac{\Delta s_0}{s_0} \right| \gg \frac{2 \ell_\phi}{\pi R_0} .
\] (4.14)
No need to say the present model is too simplified to take the expression for the comoving radius $\rho_c$ seriously as a prediction for the scale of Cosmic Background Fluctuations or related cosmological quantities. In fact, there is no inflationary stage and regions outside of $\rho_c$ will eventually come into the causal cone after a finite (short) time due to the slow expansion of the scale factor in the FRW models. The situation might change in case one considers a more realistic description. Further, we recall that the existence of a minimum scale factor $R(0) > 0$ is a basic ingredient of Pre-Big-Bang Cosmology (see, e.g. [31] and Refs. therein), in which case $\phi$ should be identified with the homogeneous mode of the dilaton predicted by the low energy limit of string theory [32].

4.2 Gravitational collapse

For the case of the collapsing sphere of dust the above framework is inverted since now $|E_\lambda|$ increases along the classical trajectory. Thus, although one starts with $\alpha_0 \ll 1$ so that the energy of the quantum gravitational fluctuations is totally negligible, when the singularity is approached the gravitational fluctuations induce an effective quantum pressure which slows down the collapse and causes the break-down of the semi-classical approximation at a finite radius of the sphere $r(0)$. A very important observation, already mentioned at the end of Section 3, is that $r(0)$ is physically distinguished from the singularity $\rho_s R_c = 0$ only if it is bigger than the Compton wavelength $\ell_\phi$.

Before proceeding, it is useful to recall that the Schwarzschild radius of the sphere $r_H = 2M$ where the ADM mass parameter is

$$M = \rho_s^3 N_\phi \frac{\ell_\phi^2}{\nabla},$$

regardless of the value of $\epsilon$. We will assume $M/\kappa$ is the mass that is measured for astronomical objects, although it generally differs from the proper mass $N_\phi (\hbar/\ell_\phi) \int_0^{\rho_s} \rho^2 d\rho/\sqrt{1 - \epsilon \rho^2}$. Further, $\rho_s$ is related to the geodesic energy parameter $\mathcal{E}$ of the trajectory $r_s = \rho_s R$ of the radius of the sphere in the outer Schwarzschild space-time with mass parameter $M$ by $\mathcal{E}^2 = 1 - \epsilon \rho_s^2$ ($-1 < \mathcal{E} < 1$ for bound orbits, $\mathcal{E} \geq 1$ for unbound orbits) [29].

Scattering orbits

For $\mathcal{E} > 1$ one can choose the starting radius of the sphere $r_0 = \rho_s R_0$ is any value greater than $r_H$. Then the sphere will bounce in correspondence to $R_\lambda$ at

$$r(0) = \frac{M}{\mathcal{E}^2 - 1} \left[ \sqrt{1 + 2 (\mathcal{E}^2 - 1) \alpha_0^2 \frac{r_0}{M}} - 1 \right].$$

As mentioned above, $r(0)$ must be greater than $\ell_\phi$ to be physical, that is

$$\alpha_0^2 \gg \frac{\ell_\phi [r_H + (\mathcal{E}^2 - 1) \ell_\phi]}{r_0 r_H}. \quad (4.17)$$

For $\mathcal{E}^2 - 1$ small one can expand the square root in Eq. (4.16) and obtains

$$r(0) \simeq \alpha_0^2 r_0, \quad (4.18)$$
with the condition $\alpha_0^2 \gg \ell_\phi/r_0$. The above result is exact for $\mathcal{E} = 1$, in which case $\rho_s$ is arbitrary and $r = (\alpha_0^2 r_0) + \rho_s R_c$. In the opposite limit, $\mathcal{E} \gg 1$, one obtains

$$r(0) \sim \frac{\alpha_0}{\mathcal{E}} \sqrt{r_0 r_H},$$  \hfill (4.19)$$

with $\alpha_0^2 \gg \mathcal{E}^2 \ell_\phi^2/r_H r_0$.

In all cases ($\epsilon = -1, 0$), $r(0)$ is bigger than $r_H$ provided $\alpha_0^2 > r_H \mathcal{E}^2/r_0$, or, using Eq. (4.13) and (4.15), the comoving radius is given by

$$\rho_s^2 \sqrt{1 - \epsilon \rho_s^2} \sim \frac{2 M}{\ell_p},$$  \hfill (4.20)$$

for the limiting case $\lambda \sim \ell_p$.

**Bound orbits**

For $\mathcal{E}^2 < 1$, on setting $r_0 = \rho_s R_c^+$, the choice (4.1) and the trajectory (3.10) give

$$r_0 = \frac{r_H}{1 - \mathcal{E}^2} \left(1 - \alpha_0^2\right).$$  \hfill (4.21)$$

Then, the radius at which the sphere bounces is given by

$$r(0) = \frac{M}{1 - \mathcal{E}^2} \left[1 - \sqrt{1 - 4 \alpha_0^2 (1 - \alpha_0^2)}\right] \approx \alpha_0^2 r_0,$$  \hfill (4.22)$$

and one concludes that Eq. (4.18) holds for $-1 < \mathcal{E} \leq 1$, in which cases $r(0) \gg \ell_\phi$ provided $\ell_\phi/r_0 \ll \alpha_0^2 \ll 1$ and $r(0) > r_H$ for $r_\ell/r_0 < \alpha_0^2 \ll 1$. For the limiting value $\lambda \sim \ell_p$ this means

$$\rho_s^2 \approx \frac{2 M}{\ell_p} \gg 1,$$  \hfill (4.23)$$

which lies outside the allowed range of $\rho_s$ and the breaking point cannot thus be bigger than $r_H$ for $\epsilon = +1$.

It is worth noting that the minima obtained above are fairly generic in that they do not depend on the detailed structure of the sphere nor on the specific form of the quantum gravitational fluctuations ($\ell_\phi$, $\ell_p$ and $\lambda$ do not appear explicitly). Furthermore, the turning point at which the semi-classical approximation breaks can be rather big and, from Eq. (4.18), one cannot exclude it occurs at a radius comparable with the error with which one measures $r_0$. Of course, to model a star with a sphere of dust ignores (among the rest) the crucial role played by the pressure in keeping the star in equilibrium and contrasting the collapse itself. Hence, it is clear that the actual value of $r(0)$ could be significantly different from the one estimated here.

We conclude by mentioning an independent argument which supports the possibility of having the kind of pressure emerging from the quantum fluctuations discussed in this Section. In fact, besides the WDW equation, one has the conservation of the total energy of the system, namely its ADM mass $\tilde{\mathcal{M}}$

$$M = \rho_s^3 \left(\langle \hat{H}_M \rangle - E_f\right) \approx \rho_s^3 \langle \hat{H}_M(\eta_0)\rangle.$$

(4.24)
Since $\langle \hat{H}_M \rangle$ increases in time, due to non-adiabatic production of matter particles \cite{13,15} (an effect totally ignored in the present notes), Eq. (4.24) requires that either $\rho_s$ decreases or $E_f$ increases (or both effects take place). The first case amounts to quantum jumps to classical trajectories with geodesic energy closer and closer to $E = 1$ which would act as a semi-classical attractor \cite{34}. The second case would imply that, although one can start with a state in which only matter modes are present (as appears sensible for a sphere of large initial radius), the price to pay for preserving the classical dynamics is the generation of gravitational perturbations in an amount such that the total energy (along with $E$) is conserved. We remark that the spherical symmetry assumed for the model would prevent these perturbations from propagating in the external vacuum as gravitational waves (Birkhoff’s theorem).

5 Conclusions

We have generalized the BO approach to the WDW equation in FRW minisuperspace \cite{9} in order to include homogeneous quantum gravitational fluctuations around the WKB trajectory. In a standard approach, such fluctuations are treated as perturbations of a fixed background and satisfy a Schrödinger equation, whose solutions in (double) adiabatic approximation were displayed both in Lorentzian and Euclidean space. One then realizes that these solutions signal a possible instability, since their energy can grow without bound for small $R$. The latter result suggested that the semi-classical limit had been incorrectly identified. Then a second approach was proposed in which the semi-classical limit includes the back-reaction of quantum gravitational fluctuations on the metric from the start, in much the same fashion as in quantum field theory on curved background one replaces the classical energy-momentum tensor of matter with the expectation value of the corresponding quantum operator (for matter and gravitational waves). This in fact led to different classical trajectories with a possible non-vanishing minimum size for a FRW Universe or a collapsing body at which the semi-classical approximation breaks down and the metric does not admit a description in terms of classical variables.

Although the FRW minisuperspace model is too simple to be realistic, one learns that the role played by quantum gravitational fluctuations, at least, should not be overlooked when considering self-gravitating matter. The guiding analogy is the treatment of infrared divergences in quantum field theory. In that context one encounters diverging quantities when studying, \textit{e.g.}, the Bremsstrahlung from an electron moving in the external electric field of a nucleus. At the classical level (the tree level of the quantum theory) the transition amplitude diverges for vanishing energy of the emitted photon. This problem is cured by adding the (diverging) one-loop contribution and noting that experimental measurements would not distinguish the final state of the electron with energy $E$ from any state with energy $E - \Delta E$ if $\Delta E$ is smaller than the precision $\Lambda$ of the apparatus. Then one has to sum over all the (tree level) emissions of (soft) photons of energy $\Delta E < \Lambda$ and obtains the counter-term which precisely cancels against the one-loop diverging term. Perhaps one can rephrase the results obtained in these notes by saying that the inclusion of (soft) quantum gravitational fluctuations with energy smaller than the precision with which we measure the energy of matter seems to cure the singularity in the density distribution of matter which develops at the classical level according to General Relativity.
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