Exact paraxial diffraction theory for polygonal apertures under Gaussian illumination

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Abstract
Paraxial diffraction of monochromatic Gaussian beams by arbitrarily shaped polygonal apertures is analytically explored within the boundary diffraction wave theory framework. Exact closed-form expressions of the diffracted wavefield are obtained, as well as an interesting connection between classical optics and probability theory.

1 Introduction
Fresnel’s diffraction is a cornerstone of classical optics since more than two century. Gaussian beams are the basic model to describe the light emitted by real laser sources. The present paper is aimed at providing exact analytical expressions for the optical wavefield produced, within Fresnel’s approximation, when a Gaussian beam is diffracted by an arbitrarily shaped sharp-edge polygonal aperture. Although analytical descriptions for plane-wave Fraunhofer diffraction by polygonal apertures are available since many years [1, 3, 2], the same cannot be said when the diffracted field has to be estimated in the near field, apart from a few notable exceptions [4, 5, 6].

We believe what is contained in the present paper could represent a useful mathematical tool for further improving our theoretical understanding of diffraction. At the same time, the availability of a complete and exact knowledge of diffraction under an incomparably more realistic model of illumination than a plane wave should be viewed, from a practical point of view, as an important achievement. The whole theoretical analysis described below is within the context of a “genuinely paraxial” formulation of the Young-Maggi-Rubinowicz boundary diffraction wave (BDW) theory [7] and its generalization by Miyamoto and Wolf [8], recently developed [9, 10, 11, 12, 13]. In what follows only the main ideas and results will be highlighted, with several mathematical steps and details having been dropped. We encourage the interested readers to go through the above quoted papers to get a more complete idea of the general context.
2 Theoretical analysis

The geometry of the problem is sketched in Fig. 1: a monochromatic Gaussian beam (wavenumber \( k \), waist size \( w_0 \)) orthogonally impinges on an opaque transverse plane placed at a distance \( D \) from its waist plane having a sharp-edge aperture \( \mathcal{A} \) arbitrarily shaped as a \( N \)-side polygon. The problem to find

![Diagram of the problem](image)

the disturbance of the diffracted wavefield at the observation point \( P \equiv (r; z) \) on a transverse plane at a distance \( z > 0 \) from the aperture plane has been addressed from a rather general point of view in [13], whose main aspects will now be briefly resumed. A key role is played by the following decomposition formula:

\[
-\frac{ik}{2\pi z} \int_{\rho \in \mathcal{A}} d^2\rho \exp \left( \frac{ik}{2z} |r - \rho|^2 \right) = \psi_G + \psi_{BDW}, \tag{1}
\]

where the complex quantities \( \psi_G \) and \( \psi_{BDW} \) are both expressed via suitable one-dimensional contour integrals defined onto the aperture boundary \( \Gamma = \partial \mathcal{A} \).

On denoting the position of a typical point on \( \Gamma \) by \( Q \), let \( Q = Q(t) \) be a suitable parametrization of \( \Gamma \), with \( t \) being a real parameter ranging within a given interval.

Then, on introducing the transverse vector \( \mathbf{R} = \overrightarrow{PQ} \), quantities \( \psi_G \) and \( \psi_{BDW} \) can be evaluated through [13]

\[
\psi_G = \frac{1}{2\pi} \oint_{\Gamma} dt \frac{\mathbf{R} \times \dot{\mathbf{R}}}{\mathbf{R} \cdot \mathbf{R}}, \tag{2}
\]
2 Theoretical analysis

\begin{equation}
\psi_{\text{BDW}} = -\frac{1}{2\pi} \oint_{\Gamma} dt \frac{R \times \dot{R}}{R \cdot R} \exp \left( \frac{ik}{2z} R \cdot R \right),
\end{equation}

respectively. Here \( \dot{R} \) denotes the derivative of \( R \) with respect to the parameter \( t \), while the cross product must be intended as the sole \( z \)-component, being both vectors \( R \) and \( \dot{R} \) purely transverse. In \cite{13} it was conjectured the above integral representations to be valid, in principle, also for complex values of the Cartesian coordinates of the observation point \( P \). Accordingly, the following recipe to retrieve the diffracted wavefield produced by the Gaussian beam in Fig. 1 was then proposed \cite{13}: (i) multiply all Cartesian coordinates of \( P \), i.e., the position vector \((r; z)\), by the following dimensionless complex factor

\begin{equation}
\frac{1 + iD/L}{1 + i(z + D)/L},
\end{equation}

with \( L = kw_0^2/2 \) being the incident Gaussian beam Rayleigh length; (ii) evaluate both \( \psi_G \) and \( \psi_{\text{BDW}} \) for the above complex values of coordinates through Eqs. (2) and (3); (iii) multiply \( \psi_G + \psi_{\text{BDW}} \) by the wavefield obtained by letting the Gaussian beam to freely propagate from the waist plane up to the real observation point \( P \equiv (r; z) \) to retrieve the diffracted wavefield. That is all.

Now we are going to show how, when the aperture boundary \( \Gamma \) is an arbitrarily shaped polygon, Eqs. (2) and (3) can be exactly expressed through analytical closed forms. The idea is simple: both \( \psi_G \) and \( \psi_{\text{BDW}} \) can be written as the sum of a finite number of contributions, each of them coming from one polygon side. Accordingly, In this way the main problem reduces to evaluate Eqs. (2) and (3) for a typical segment, say \( AB \), placed at the transverse plane at \( z \), as sketched in Fig. 2.

We start with the contribution to the geometrical component, say \( \psi_G(AB) \). To this aim, a natural parametrization of the segment is (see Fig. 2a)

\begin{equation}
R(t) = \overrightarrow{PA} + \overrightarrow{AQ} = \overrightarrow{PA} + \overrightarrow{AB}t, \quad t \in [0, 1],
\end{equation}

so that \( \dot{R} = \overrightarrow{AB} \). On substituting from Eq. (5) into Eq. (2) we obtain at once

\begin{equation}
\psi_G(AB) = \frac{1}{2\pi} \frac{\overrightarrow{PA} \times \overrightarrow{AB}}{AB^2} \int_0^1 \frac{dt}{t^2 + 2\eta t + \chi^2},
\end{equation}

where

\begin{equation}
\eta = \frac{\overrightarrow{PA} \cdot \overrightarrow{AB}}{AB^2}, \quad \chi^2 = \frac{\overrightarrow{PA} \cdot \overrightarrow{PA}}{AB^2}.
\end{equation}

Despite its apparent simplicity, the evaluation of the integral in Eq. (6) requires some care since it has to be computed, for what it was said above, when \( P \) attains complex values of its coordinates, say \((r \exp(i\phi); z \exp(i\phi))\). Without detailing all mathematical steps, the following closed-form formula can be established:

\begin{equation}
\int_0^1 \frac{dt}{t^2 + 2\eta t + \chi^2} = \frac{1}{2\sqrt{\eta^2 - \chi^2}} \sum_{\sigma = \pm 1} \sigma \log \left( 1 + \frac{\eta + \sigma \sqrt{\eta^2 - \chi^2}}{\chi^2} \right),
\end{equation}

(8)
2 Theoretical analysis

Fig. 2: Parametrizations of the segment $AB$ for the evaluation of the contributions to the geometrical component $\psi_G(AB)$ (a) and of the BDW component $\psi_{BDW}(AB)$ (b) at the observation point $P$.

which, together with Eqs. (6) and (7), gives the exact representation of the geometrical wavefield contribution due to the segment $AB$. As an easy check, it is an academic exercise to prove that, for real values of $\eta$ and $\chi$, corresponding to plane-wave illumination, Eqs. (6)-(8) give at once

$$\psi_G(AB) = \frac{\beta - \alpha}{2\pi},$$

where the sign of angles $\alpha$ and $\beta$ is positive for counterclockwise rotations (see Fig. 2b, for the meaning of $\alpha$ and $\beta$).

The evaluation of the contribution to the BDW component of the diffracted wavefield coming from the segment $AB$, say $\psi_{BDW}(AB)$, is more cumbersome, since involves the use of a special function which is well known in statistics but still unknown in optics. To evaluate this contribution, the integration parameter into Eq. (3) has to be changed from $t$ (see Fig. 2a) to the polar angle $\varphi$ shown in Fig. 2b. Let $D$ denote the distance between $P$ and $AB$. On introducing the Cartesian axis $x$ whose origin coincides with $P$ and which is orthogonal to the $AB$ direction, the polar representation of the vector $R$ is given by $R(\varphi) = D/\cos \varphi$, with $\varphi \in [\alpha, \beta]$. Accordingly, the contribution of $AB$ to the BDW wavefield turns out to be

$$\psi_{BDW}(AB) = -\frac{1}{2\pi} \int_{\alpha}^{\beta} \exp \left( \frac{iu/2}{\cos^2 \varphi} \right) d\varphi,$$

where the dimensionless parameter $u = kD^2/z$ has been introduced. Note that
the distance $D$ in Fig. 2b can formally be analytically expressed as

$$D = \frac{\overrightarrow{PA} \times \overrightarrow{AB}}{AB},$$

(11)

where use has been made of $\overrightarrow{PB} = \overrightarrow{PA} + \overrightarrow{AB}$. To evaluate the integral in Eq. (10), it is then sufficient to make the variable change $\tau = \tan \varphi$, which yields

$$\psi_{BDW}(AB) = T\left(\sqrt{-iu}, \tan \alpha\right) - T\left(\sqrt{-iu}, \tan \beta\right),$$

(12)

where the symbol $T(a, b)$ denotes the so-called Owen $T$-function [14], defined as follows:

$$T(a, b) = \frac{1}{2\pi} \int_0^b \frac{\exp\left[\frac{-a^2}{2(1 + \tau)^2}\right]}{1 + \tau^2} \, d\tau.$$  

(13)

For real values of both parameters, the quantity $T(a, b)$ gives the probability that $\{X > a \text{ and } 0 < Y < bX\}$, with $X$ and $Y$ being two statistically independent normal random variables [14]. For our purposes it is important to note that Owen’s function $T(a, b)$ can be analytically continued to complex values of both arguments. A precious work about the main properties of the Owen function (including integral and series representations, connections with other special functions, evaluation of integrals and series containing the $T$-function) has recently been published [15]. Readers are encouraged to go through this nice work to familiarize with the Owen function. Analytical expressions of both $\tan \alpha$ and $\tan \beta$ in terms of the positions of points $A$, $B$, and $P$ can also be found through elementary geometry, which yields

$$\begin{align*}
\tan \alpha &= \frac{\overrightarrow{PA} \cdot \overrightarrow{AB}}{PA \times AB}, \\
\tan \beta &= \frac{\overrightarrow{PB} \cdot \overrightarrow{AB}}{PA \times AB}.
\end{align*}$$

(14)

Equations (6) - (8), together with Eqs. (11) - (14), are the main result of the present paper. Through them, paraxial diffraction of Gaussian beams by polygonal sharp-edges apertures finds an exact solution, as promised at the beginning.

In the rest of the paper a single but significant example of application of the theory previously exposed will be illustrated. Such example received a considerable attention, even recently [5, 16, 17, 18, 19]. Figure 3 depicts the geometry of the problem: a sharp-edge aperture shaped in the form of an equilateral triangle is orthogonally illuminated by a Gaussian beam whose mean propagation axis passes through the triangle centre. It is worth introducing a Cartesian reference frame $Oxyz$, in which the coordinates of the triangle vertices are $A \equiv (-\sqrt{3}, -1)$, $B \equiv (\sqrt{3}, -1)$, and $C \equiv (0, 2)$, where for simplicity the length unit has been chosen in such way that each triangle side measures $2\sqrt{3}$. Then, on denoting $(x, y)$ the Cartesian representation of the observation point $P$, Eqs. (7), (11),
and \[14\] allows all parameters $\eta$, $\chi^2$, $D$, $\tan\alpha$, and $\tan\beta$ to be easily evaluated for all triplets $(A, B, P)$, $(B, C, P)$, and $(C, A, P)$. For example, the parameters related to the triplet $(A, B, P)$ take on the following expressions:

\[
\begin{align*}
\eta &= -\frac{3 + x\sqrt{3}}{6}, \\
\chi^2 &= \frac{x^2 + 2x\sqrt{3} + y(2 + y) + 4}{12}, \\
D &= 1 + y, \\
\tan\alpha &= -\frac{\sqrt{3} + x}{1 + y}, \\
\tan\beta &= \frac{\sqrt{3} - x}{1 + y}.
\end{align*}
\]

Similar expressions follow for the other two triplets $(B, C, P)$ and $(C, A, P)$ but will not be given here.

The geometrical component $\psi_G$ is first investigated. Several years ago Otis \[20\] claimed a Gaussian beam impinging on a sharp-edge aperture would produce a geometrical “Gaussian shadow” having the same shape as the aperture and scaled by the modulus of the factor in Eq. \[4\]. Otis’ conjecture implies the geometrical field defined by Eq. \[2\] once evaluated at complex transverse positions...
$r \exp(i\phi)$ must satisfy the following relation:

$$
\psi_G = \begin{cases} 
1 & r \in A, \\
0 & r \notin A,
\end{cases}
$$

(16)

irrespective the value of $\phi$. Equation (16) was rigorously proved for circular apertures sharing the axial symmetry with the impinging Gaussian beam [13]. In the same work it was also argued such a case to be the sole for which the Otis conjecture holds. The fascinating problem about such “Gaussian shadows” will surely be the subject of future investigations. In the present paper we shall limit to provide a single check about the conjectures of [13].

Figure 4 shows the geometry: the observation point $P$ belongs to the $x$-axis. For a given value of $x$, the geometrical wavefield $\psi_G$ is then computed at $x \exp(i\phi)$, with $\phi \in [-\pi, \pi]$. The result is shown in Fig. 5: it turns out that $\psi_G = 1$ within the white region, $\psi_G = 0$ within the black region, and $\psi_G = 1/2$ within the grey region.

It is thus seen that Eq. (16) is not satisfied, except when the observation points is inside the inscribed circle $\gamma_m$, for which $\psi_G \equiv 1$, or when it is outside the circumscribed circle $\gamma_M$, for which $\psi_G \equiv 0$. Several others numerical trials, made at different observation points but not shown here, gave results in perfect agreement with what has been conjectured in [13].

In order to offer at least one example of computation of $\psi_{BDW}$, in Fig. 6 two-dimensional maps of the optical intensity produced by the diffraction of
3 Summary and conclusions

An analytical exact derivation of the paraxial wavefield generated by arbitrarily shaped polygonal sharp-edge apertures under Gaussian beam illumination has been presented. No further approximations beyond Fresnel diffraction have been invoked. We believe our achievement could help in further improving the theoretical understanding of paraxial diffraction. Moreover, the analytical and complete knowledge of the diffracted field under such a realistic model of illumination should be acknowledged as an effective computational tool of considerable usefulness for a wide range of applications.
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Fig. 6: Two-dimensional maps of the optical intensity produced by the diffraction of a Gaussian beam with $\lambda = 0.5 \, \mu m$ and $D = 0$ at the observation plane $z = 100 \, mm$. The radius of the circumscribed circle $\gamma_M$ is set to 1 mm. The spot-size has been set to $w_0 = 500 \, \mu m$ (a), $w_0 = 1 \, mm$ (b), $w_0 = 200 \, \mu m$ (c), and $w_0 = 10 \, mm$ (d).

References

[1] R. C. Smith and J. S. Marsh, “Diffraction patterns of simple apertures,” J. Opt. Soc. Am. 64, 798 - 803 (1974).

[2] J. Komrska, “Simple derivation of formulas for Fraunhofer diffraction at polygonal apertures,” J. Opt. Soc. Am. 72, 1382 - 1384 (1982).

[3] S. Ganci, “Simple derivation of formulas for Fraunhofer diffraction at polygonal apertures from Maggi - Rubinowicz transformation,” J. Opt. Soc. Am. A 1, 559 - 561 (1984).

[4] G. W. Forbes and A. A. Asatryan, “Reducing canonical diffraction problems to singularity-free one-dimensional integrals,” J. Opt. Soc. Am. A 15, 1320 (1998).

[5] J. G. Huang, J. M. Christian, and G. S. McDonald, “Fresnel diffraction
and fractal patterns from polygonal apertures," J. Opt. Soc. Am. A 23, 2768 - 2774 (2006).

[6] J. Naraga and N. Hermosa, “Diffraction of polygonal slits using catastrophe optics,” J. Appl. Phys. 124, 034902 (2018).

[7] M. Born and E. Wolf, Principles of Optics (Cambridge University Press, Cambridge, 1999).

[8] K. Miyamoto and E. Wolf, “Generalization of the Maggi-Rubinowicz theory of the boundary-diffraction wave,” J. Opt. Soc. Am. 52, 615 - 625 (part I) and 626-637 (part II) (1962).

[9] R. Borghi, “Uniform asymptotic of paraxial boundary-diffraction waves,” J. Opt. Soc. Am. A 32, 685 - 696 (2015).

[10] R. Borghi, “Catastrophe optics of sharp-edge diffraction,” Opt. Lett. 41, 3114 - 3117 (2016).

[11] R. Borghi, “Heart diffraction,” Opt. Lett. 42, 2070 - 2073 (2017).

[12] R. Borghi, “Tailoring axial intensity of laser beams with a heart-shaped hole, by Wang et al.: Comment,” Opt. Lett. 43, 3240 (2018).

[13] R. Borghi, “Sharp-edge diffraction under Gaussian illumination: a paraxial revisitation of Miyamoto-Wolf’s theory,” J. Opt. Soc. Am. A, 36, 1048 (2019).

[14] D. B. Owen, Ann. Math. Statist. 27, 1075 (1956).

[15] Yu. A. Brychkov and N. V. Savischenko, “Some properties of the Owen T-function,” Integral Transforms and Special Functions 27, 163 (2016)

[16] C. Stahl and G. Gbur, “Analytic calculation of vortex diffraction by a triangular aperture,” J. Opt. Soc. Am. A 33,1175 (2016).

[17] J. A. Rivera, T. C. Galvin, A. W. Steinforth, and J. G. Eden “Fractal modes and multi-beam generation from hybrid microlaser resonators,” Nature Commun. 9, 2594 (2018).

[18] H. Sroor, D. Naidoo, S. W. Miller, J. Nelson, J. Courtial, and A. Forbes, “Fractal light from lasers,” Phys. Rev. A 99, 013848 (2019).

[19] J. C. A. Rocha, J. P. Amaral, E. J. S. Fonseca, and A. J. Jesus-Silva “Study of the conservation of the topological charge strength in diffraction by apertures ,” J. Opt. Soc. Am. B 36, 2114 (2019).

[20] G. Otis, “Application of the boundary-diffraction-wave theory to Gaussian beams,” J. Opt. Soc. Am. 64, 1545 - 1550 (1974).