GENERATING THE GOERITZ GROUP

MARTIN SCHARLEMANN

Abstract. In 1980 J. Powell [Po] proposed that five specific elements sufficed to generate the Goeritz group for any Heegaard splitting of $S^3$. Here we prove

- an expansion of Powell’s proposed generators to include all eyeglass twists does generate the Goeritz group and, as a consequence,
- the Powell Conjecture is stably true.

1. Introduction

Suppose that $M$ is a compact orientable manifold and $M = A \cup_T B$ is a Heegaard splitting of $M$. Following [JM] the Goeritz group $G(M, T)$ is the group of isotopy classes of diffeomorphisms $(M, T) \to (M, T)$ for which the induced diffeomorphism $M \to M$ is isotopic to the identity. An element of the Goeritz group can be viewed as the final result of a (possibly non-unique) loop $T_\theta, 0 \leq \theta \leq 2\pi$ of embeddings of $T$ in $M$, that is an element in $\pi_1(\text{Diff}(M) / \text{Diff}(M, T))$ [JM Theorem 1]. That is the viewpoint we will take.

Little is known about the Goeritz group, even in the case that $M = S^3$ and the Heegaard splittings are fully described [Wa]. In [Po] J. Powell proposed (indeed he believed he had proven) that five specific isotopies generate the Goeritz group $G(S^3, T)$ of the 3-sphere. (In [Sc2] one is found to be redundant, and so the number is reduced to four.) Figures 3 to 6 give examples of Goeritz elements acting on the standard Heegaard splitting of $S^3$. The first three reflect Powell’s generators. Powell expected that restricting Figure 6 to the case where the red and blue disks are standard primitive meridians in their respective handlebodies would suffice; here we will need to include them all.

Date: November 24, 2020.

Michael Freedman has been an avid source of support for this project; his notion of a ‘cycle of weak reductions’ underpins it all. Aoife McCormick inspired the notion of ‘chamber complex’ - the method here for translating Freedman’s idea into a recipe for Heegaard surface reduction.
Powell’s conjecture has been verified for genus \( \leq 3 \) splittings of \( S^3 \) \cite{FS1}, but even the question of whether \( G(S^3, T) \) is finitely generated remains open when genus \( T \geq 4 \). Here we show:

- There is a generalization of Powell’s proposed generator \( D_\theta \) called an *eyeglass twist*. If \( D_\theta \) is replaced by the (infinite family) of eyeglass twists, the subgroup \( \mathcal{E} \subset G(S^3, T) \) thereby generated is all of \( G(S^3, T) \). See Theorem \[11.8\]. In particular, the Powell Conjecture is true if and only if each eyeglass twist is a consequence of the Powell generators.

- Suppose \( T \) is the standard genus \( g+1 \) Heegaard surface in \( S^3 \). Any element of \( G(S^3, T) \) that acts trivially on a standard genus 1 summand of \( T \) is a consequence of the Powell generators acting on \( T \). In other words, the Powell Conjecture is true, stably. See Corollary \[12.3\].

The strategy for the proof is rather simple: Suppose \( T \) is the standard genus \( g \) Heegaard splitting of \( S^3 \), and \( \tau \in G(S^3, T) \). In \cite{FS1} it is shown that there is a ‘cycle of weak reductions’ capturing \( \tau \). That is, given a loop of embeddings \( T_\theta, 0 \leq \theta \leq 2\pi \) of \( T \) in \( S^3 \) that represents \( \tau \), there is a natural topological way to extract, for generic \( \theta \), a pair of properly embedded disjoint essential disks \( a_\theta \subset A, b_\theta \subset B \) that thereby weakly reduce \( T_\theta \). Moreover, at the finite number of nongeneric points, the chosen weakly reducing pair does not change much. Since the method of choosing the pair \( (a_\theta, b_\theta) \) is topological, it follows relatively easily that \( (a_{2\pi}, b_{2\pi}) = (\tau(a_0), \tau(b_0)) \). A landmark result of Casson and Gordon \cite{CG} shows that a weakly reducing pair gives rise to a reducing sphere for \( T \); one might naturally hope that one could track such a reducing sphere \( K \) around \( \theta \), as was done for the weakly reducing pairs, and thereby be able to meaningfully compare \( K \) with \( \tau(K) \) and so understand the action of \( \tau \).

Sadly, the transition from a weakly reducing pair \( (a, b) \) to a reducing sphere involves much choice, so there is no naturally derived reducing sphere \( K \) for \( T \) as hoped for above. The program here is to find one, via this natural topological method: Let \( \mathcal{C} \subset S^3 \) be the surface obtained from \( T \) by weakly reducing along the pair \( (a, b) \). There is a natural and oft-used way to sweep out \( S^3 \) by level 2-spheres \( S_s \) and, given \( \mathcal{C} \), a natural value of \( s \) to pick: a value so that the genus of the part of \( \mathcal{C} \) lying below \( S_s \) matches the genus of the part above. Could \( S_s \) be turned into a viable, topologically defined reducing sphere for \( T \) that could play the role of \( K \) above?
It turns out that although $S$ itself may not play that role, it does give a recipe for weak reduction that is robust: it doesn’t change abruptly as $T$ moves through $S^3$. And as the weak reduction proceeds (as guided by $S$) and $C$ becomes more complicated, its complementary components are more likely to contain reducible components. Reducing spheres in these reducible components naturally cut off summands of the original Heegaard surface, and these (inductively) determine an isotopy to the standard picture that is unique, up to action by $\mathcal{E}$. If no complementary component becomes reducible, then at the end of the process we fall back on $S$ as the required reducing sphere for $T$.

Although the program is easy to describe, the technical difficulties encountered below are complex. Much of the machinery used is new, and then so is the terminology. We call the attention of the reader to the Index to this new terminology at the end of the paper. One surprise is that much of the argument can be carried out in an arbitrary compact orientable 3-manifold, as long as every closed surface separates. So perhaps the methodology can be useful in understanding stabilized Heegaard splittings of other 3-manifolds as well.

## 2. Chamber complexes

Suppose $M$ is a compact orientable 3-manifold in which each closed surface separates. Let $F \subset S^3$ be a (typically disconnected) closed surface in $S^3$. Call the pair $\mathcal{C} = (M, F)$ a chamber complex with underlying surface $F$; each component of $M - F$ is a chamber.

For $\mathcal{C}$ be a chamber complex as above, let $\mathcal{D}$ be a collection of disjoint properly embedded disks, some perhaps inessential, in the chambers of $\mathcal{C}$. We will call $\mathcal{D}$ a disk set in $\mathcal{C}$. Let $F_D \subset M$ be the surface obtained by doing surgery on $\mathcal{D}$, and denote $\hat{\mathcal{C}}_D$ the associated chamber complex. Each disk $D$ in $\mathcal{D}$ is the core of a 2-handle used in the surgery, and the boundary of that 2-handle contains two copies of $D$, called the scars of $D$, that lie in $F_D$. Let $C'$ be a chamber of $M - F_D$ and $D \in \mathcal{D}$ a disk which leaves a scar or two on $\partial C'$. If the disk $D$ lies outside the chamber $C'$ we call the scars external to $C'$; if it lies inside $C'$ we call them internal to $C'$. Clearly if $D$ leaves one internal scar on $\partial C'$ it will leave two, though the two may be on different components of $\partial C'$.

Let $F'$ be a component of $F_D$ and $W$ be one of the two components of $M - F'$. We say that $W$ is disky if $F \cap \text{int}(W)$ consists entirely of disks.

**Lemma 2.1.** If $W$ is disky then
- $W$ has no components of $F$ in its interior.
- Each component of $F_D$ that lies in the interior of $W$ is a sphere.
Let $C'$ be the chamber of $\hat{C}_D$ that is incident to $F'$ and lies in $W$, and suppose $D \in \mathcal{D}$ lies inside $C'$. Then the two scars it leaves must lie on different components of $\partial C'$.

**Proof.** Take the statements in order:

Each component of $F$ is a closed surface, so none can be a disk.

Suppose $F''$ is a component of $F_D$ that lies in the interior of $W$, and let $F''_\bullet$ be the compact surface obtained from $F''$ by deleting the interior of all its scars. Then $F''_\bullet$ lies in a component of $F \cap \text{int}(W)$ which, by assumption, is a disk. Hence $F''_\bullet$ has no genus, so neither can $F''$. See Figure 1

Suppose the two scars left by surgery on a disk $D \in \mathcal{D}_0$ were on the same component $F''$ of $\partial C'$. If that surface is $F''_\bullet$, then the annulus in $F$ running between the two scars of $D$ is a component of $F \cap \text{int}(W)$ that is not a disk, contradicting our assumption that $W$ is disky. Similarly, if the scars both lie on a sphere component of $\partial C'$ lying in the interior of $W$ then some component of $F \cap \text{int}(W)$ contains a punctured torus, and so cannot be a disk. \qed

![Figure 1](image_url)

**Lemma 2.2.** Suppose $W$ is disky, and $F''$ is a (sphere) component of $F_D$ lying in the interior of $W$. Then the component $W''$ of $\mathcal{M} - F''$ that lies in $W$ is disky.

**Proof.** By assumption, $F \cap \text{int}(W)$ consists of a collection $E$ of disks, each of them incident to the boundary of a scar in $F'$. Let $\mathcal{D}_0 \subset \mathcal{D}$ be the collection of disks lying in the interior of $W$, so $\partial \mathcal{D}_0 \subset E$. We induct on $n = |\mathcal{D}_0|$.

If $n = 0$ then $F''_\bullet$ is the union of a scar in $F'$ and a component of $E$, cutting off a component $W''$ whose interior must be disjoint from $F$ and so is disky by default.
Suppose \( n > 0 \) and let \( D_0 \in \mathcal{D}_0 \) be a disk whose boundary is innermost in \( E \) among the circles \( \partial \mathcal{D} \); let \( E_0 \) be the subdisk of \( E \) that it bounds. Inductively apply the theorem to \( \mathcal{D}' = \mathcal{D}_0 - D_0 \), creating a collection \( S' \) of spheres, each of which, by inductive assumption, cuts off a disky component of its complement in \( M \).

Final surgery on \( D_0 \) does not change the topology of \( F - S' \), so each sphere in \( S' \) remains disky, plus a new sphere \( S_0 = D_0 \cup E_0 \) is added and its complementary component lying in \( W \) has interior disjoint from \( F \) so it too is disky. The final surgery alters \( S' \), replacing \( E_0 \) by the disk \( D_0 \) and leaving \( E_0 \) as a new component of \( F - S' \). Since \( E_0 \) is a disk, this does not affect the fact that each component of \( S' \) (now altered) cuts off a disky component. See Figure 2. \( \square \)

**Figure 2.**

**Definition 2.3.** A chamber complex \( \mathcal{C} \) is tiny if either

- \( F = \emptyset \) or
- there is a chamber \( C \) of \( \mathcal{C} \) so that \( M - C \) consists of handlebody chambers (possibly balls).

Note that in the second case, \( C \) could itself be a handlebody. In other words, if \( F \) has a single component which divides \( M \) into two handlebodies it is considered a tiny chamber complex.

A Heegaard surface is a good example of a tiny chamber complex, and tiny chamber complexes are typically useless for our purposes. But a weak reduction of a Heegaard surface will typically give a chamber complex that is not tiny; that fact will be the entryway to our study of Heegaard splittings below. The word “tiny” is used because the tree dual to a chamber complex is particularly small for tiny chamber complexes: at its most complicated, a star graph with central vertex corresponding to the chamber \( C \).
Suppose $C$ is a chamber complex, $D$ is a disk set in $C$, and $\hat{C}_D$ is the chamber complex obtained by surgery on $D$. One can think of a component $\hat{C}$ of $\hat{C}_D$ as obtained in two stages: it starts as the complement $C_-$ in a chamber $C$ of $D$ after surgery on the set $D \cap C$. Then 2-handles are added along those disks in $D$ that are incident to $\partial C_- \cap \partial C$ but lie outside $C$. The component $\hat{C}$ is called a remnant of $C$, even though strictly speaking it doesn’t lie entirely inside of $C$.

We will be looking at possibly long sequences of decompositions, and at every stage the number of components will typically increase. Most components carry no useful information. In an effort to keep some control we will delete from the defining surface the boundary of some of the chambers likely to be of least interest, so the chamber no longer appears as a separate chamber in the chamber complex.

To that end, after a decomposition as above, remove from $\hat{C}_D$ some (perhaps not all) sphere components of $F_D$ that cut off disky ball components and call the resulting chamber complex $\hat{C}_D$. The balls themselves remain in $M$, but in the chamber complex they are absorbed into the adjacent chamber. $\hat{C}_D$ is said to be obtained from $C$ by decomposition along the disk set $D$. We will call the absorbed balls goneballs for the decomposition. A chamber in $\hat{C}_D$ that is obtained in this way from a remnant of $C$ in $\hat{C}_D$ will continue to be called a remnant of $C$.

**Proposition 2.4** (Tinyness pulls back). If $\hat{C}_D$ is tiny, and each of the handlebody chambers from Definition 2.3 is disky, then $C$ was tiny.

**Proof.** Suppose $\hat{C}_D$ is tiny, and $C'$ is the chamber of $\hat{C}_D$ so that $M - C'$ consists of a union $W'$ of handlebody chambers, each of which is disky (including perhaps disky balls that were not designated goneballs).

Let $W$ be the chambers in $C$ from which the $W'$ are remnants, and consider internal scars on $\partial W'$. Since $W'$ is disky, each internal scar on $\partial W'$ is parallel to a disk component of $F \cap W'$ lying inside $W'$. Between the scar and the disk is a ball, since $W'$ is irreducible, so, in order to understand the topology of $W$, we may as well isotope the disk through the scar, deleting both from $W'$. In other words, we may as well assume that there are no internal scars on $\partial W'$. (Disks in $D$ that are incident to the disks $F \cap W'$ make little difference - they just give rise to goneballs; see Figure 1.)

$W$ is obtained from $W'$ by attaching 1-handles dual to disks in $D$, and we have just seen that we can take all these 1-handles to lie in $C'$. Each end of a 1-handle lies either on $\partial W'$ or on a goneball in $C'$. Thus $W$ is obtained from $W'$ by attaching the neighborhood of a properly embedded graph lying in $C'$, with the valence 1 vertices incident on $W'$.
and other vertices corresponding to the goneballs in $C'$. But attaching the neighborhood of a graph to a collection of handlebodies creates a collection of handlebodies, so $W$ is indeed a collection of handlebodies. The chamber $C$ in $C$ that is the complement of the handlebodies $W$ is obtained from $C'$ by removing the neighborhood of a graph, so $C$ is a single chamber adjacent to each component of $W$. Thus $C$ is tiny.

The proof when $F_D$ is a single component, dividing $M$ into two disky handlebodies is easier: the argument above on $W'$ shows that $F$ is isotopic to $F_D$, so $C$ too is tiny. Similarly, if $F_D$ is empty then $W$ is obtained by attaching a number of 1-handles to the goneballs in $M$, in other words $W$ is a collection of handlebodies, and $M - W$ is a single chamber of $C$ that is adjacent to each such handlebody, so again $C$ is tiny. □

3. Heegaard splittings and chamber complexes

Chamber complexes are naturally relevant to weakly reducible Heegaard splittings, as we now describe. Suppose $M = A \cup_T B$ is a Heegaard splitting of the manifold $M$ and $A, B$ are disjoint non-empty families of essential disks, properly embedded in $A, B$ respectively. Such a pair of families is called weakly reducing, a notion with a long and important history in the study of Heegaard splittings (cf [CG]). Surger $T$ along $A \cup B$ and call the resulting surface $F$. Then $F$ defines a chamber complex $C$ in $M$. Call a chamber $C$ of $C$ an $A$-chamber if it contains a family $B_C \subset B$ of disks and call it a $B$ chamber if it contains a family $A_C \subset A$ of disks. Note that in any pair of adjacent chambers, one is an $A$-chamber and the other is a $B$-chamber. Each $A$-chamber $C$ inherits a natural Heegaard splitting $C = A_C \cup_{T_C} B_C$, where $B_C$ is the union of a collar of $\partial C$ and 2-handle neighborhoods of the disks $B_C$. (Note that $C$ may have spherical boundary components, so $B_C$ might be described as a punctured compression body.) We typically think of $B_C$ via its dual construction: attach to a collar of $\partial C$ the 1-handles whose cores are the cocores of the 2-handles around $B_C$. $A_C$ is the closed complement of $B_C$ in $C$. Everything is expressed symmetrically for $B$-chambers.

There is a natural inverse construction: Suppose we are given a chamber complex $C$ and alternately label the chambers $A$-chambers and $B$-chambers. Pick a Heegaard splitting $C = A_C \cup_{T_C} B_C$ for each $A$-chamber $C$ with the property that $\partial C = \partial - B_C$, and symmetrically for $B$-chambers. Notice that in each $A$-chamber $C$, $A_C$ is a handlebody, and symmetrically for each $B$-chamber. These Heegaard splittings induce a Heegaard splitting $M = A \cup_T B$ by amalgamating the Heegaard
splittings along \( F \), and the amalgamated Heegaard splitting is unique up to isotopy, cf \([La, \text{Proposition 3.1}]\).

We say in either case that the splitting \( M = A \cup_T B \) is supported by \( C \) and that \( C \) with such a choice of Heegaard splittings in each chamber is a **Heegaard split chamber complex**. Note that \( C \) may support many Heegaard splittings of \( M \), since Heegaard splittings of a chamber may not be unique. Similarly, \( C \) is not determined by \( T \), but also by the choice of weakly reducing disks.

Suppose \( C \) is a chamber in the chamber complex \( C \) obtained from \((T, A, B)\) as first described above. With no loss of generality, suppose that \( C \) is a \( B \)-chamber and \( A_C \) is the subset of \( A \) that lies in \( C \). The collection \( A_C \) then constitutes a complete collection of meridian disks for the compression body \( A_C \) in the induced Heegaard splitting \( C = A_C \cup_{T_C} B_C \). We denote this relation \( A \cap C = A_C \).

**Proposition 3.1.** Let \( A'_C \) be a possibly different complete collection of meridian disks for the compression body \( A_C \). Then there is a collection \( A' \) of meridian disks for \( A \) so that \( A' \)

- only differs from \( A \) by the choice of disks in \( C \), and
- \( A' \cap C = A'_C \)

**Proof.** This follows immediately from the proof of \([La, \text{Proposition 3.1}]\): a series of 1-handle slides on \( A_C \) in \( C \) will take \( A_C \) to \( A'_C \). These can be mimicked by handle slides in \( T \) that do not disturb \( A - A_C \). \( \Box \)

Note that the choice of handle slides on \( A \) that mimic those in \( A_C \) are not well-defined - most obviously because of the need to avoid the scars of \( B \) in adjacent chambers. This is an issue we will address later.

Suppose \( C \) is a Heegaard split chamber complex \( D \) is a disk set in \( C \). \( D \) is **aligned** with the Heegaard splitting if each disk in \( D \) is either disjoint from the Heegaard surface or intersects the Heegaard surface of the chamber in which it lies in a single essential circle. (In Heegaard language, the latter sort of disk is a \( \partial \)-reducing disk for the Heegaard splitting of the chamber.)

If \( D \) is aligned, it is straightforward to find disjoint complete collections of meridian disks \( A \) in \( A \) and \( B \) in \( B \) that are disjoint from \( D \). Similarly, if there are disjoint complete collections of meridian disks \( A \) in \( A \) and \( B \) in \( B \) that are disjoint from \( D \), then \( D \) will be aligned. In this case we will say that \( A \) and \( B \) miss \( D \).

If \( A \) and \( B \) miss \( D \) there is a natural Heegaard structure on the chamber complex \( C_D \) obtained by simply adding \( D \cap A \) to \( A \) and \( D \cap B \) to \( B \). Each chamber finds its Heegaard splitting \( \partial \)-reduced on the disks
in $\mathcal{D}$ contained in the chamber, and a 1-handle added that is dual to each incident disk in $\mathcal{D}$ that lies in an adjacent chamber. Finding an isotopy of $T$ so that afterwards $\mathcal{D}$ is aligned is not a problem:

**Proposition 3.2.** Suppose a Heegaard split chamber complex $\mathcal{C} \subset M$ supports a Heegaard splitting $A \cup_T B$ of $M$, and $\mathcal{D}$ is a disk set for $\mathcal{C}$. Then there is an isotopy of $T$, mimicking handleslides in the chambers of $\mathcal{C}$ so that afterward each disk in $\mathcal{D}$ is aligned.

**Proof.** This follows from [Sc1, Theorem 1.2] and the discussion in [La] about mimicking handleslides in the chambers by handleslides of $T$. □

For our purposes, Proposition 3.2 is not enough; we will also need to understand how well-defined the positioning of $T$ given by Proposition 3.2 is. It is on this question that the story really begins.

4. The eyeglass subgroup

In [FS1, Section 2] some consequences of the Powell generators are presented. These include the Powell generators themselves, but also include a collection of moves that make sense on any stabilized Heegaard splitting of any compact orientable manifold. Here is a brief description of those generalized moves; more detail can be found in [FS1]. The last move described (an eyeglass twist) is not known to be a consequence of the Powell generators, but rather significantly generalizes Powell’s move $D_\theta$.

For $M = A \cup_T B$ define a bubble for $T$ to be a 3-ball $b$ in $M$ whose boundary intersects $T$ in a single essential circle. Via Waldhausen [Wa] we know that $T$ intersects the ball in some number of standard stabilizing genus 1 summands; if there is only one such summand, we call $b$ a monobubble.

1. A bubble move is an isotopy of $b$ through some path in $T_g - b$ that returns $(b, b \cap T)$ to itself, see Figure 3.
(2) Let \( b \) be a monobubble. A \textit{flip} is the homeomorphism \((b, b \cap T) \to (b, b \cap T)\) called move \( D_\omega \) by Powell [Po]. See Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{flip.png}
\caption{A flip}
\end{figure}

(3) Let \( b_1, b_2 \) be disjoint monobubbles, and let \( v \subset T - (b_1 \cup b_2) \) be an arc connecting them. Let \( b \) be the genus 2 bubble obtained by tubing together the monobubbles along \( v \). A \textit{bubble exchange} exchanges the two monobubbles within \( B \), as shown in Powell’s move \( D_{\eta_2} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{switch.png}
\caption{A switch}
\end{figure}

(4) An \textit{eyeglass} is the union of two disks, \( \ell_a, \ell_b \) (the lenses) with an arc \( v \) (the bridge) connecting their boundaries. Suppose an eyeglass \( \eta \) is embedded in \( M \) so that the 1-skeleton of \( \eta \) (called the frame) lies in \( T \), one lens is properly embedded in \( A \), and the other lens is properly embedded in \( B \). The embedded \( \eta \) defines a natural automorphism \((M, T) \to (M, T)\), as illustrated in Figure 6 called an \textit{eyeglass twist}. Powell’s generator \( D_\theta \) is an example of this, but is restricted by Powell to the case in which the lenses are a standard meridian of one bubble and a standard longitude of another.

**Definition 4.1.** Suppose \( M = A \cup_T B \) is a Heegaard splitting. The subgroup of \( G(M, T) \) generated by the four types of moves just described
is called the eyeglass group $E \subset G(M,T)$. When $M = S^3$ the eyeglass group contains the subgroup of $G(S^3,T)$ generated by the Powell moves, which we will call the Powell group.

**Remarks:** The Powell Conjecture is that the Powell group is the entire Goeritz group $G(S^3,T)$; we will eventually show the weaker result that the eyeglass group is the entire Goeritz group. If one could show that each eyeglass twist on the standard Heegaard splitting of $S^3$ is in the Powell group, the Powell Conjecture would follow.

Suppose $M = A \cup_T B$ is a Heegaard splitting supported by chamber complex $C$, $C$ is a chamber of $C$ and $C = A_C \cup_{T_C} B_C$ is the induced Heegaard splitting. Let $E_C$ denote the eyeglass group on the latter Heegaard splitting. Since $E_C$ is defined by handleslides on $T_C$, one can mimic the action of any $\epsilon_C \in E_C$ by an element $\epsilon$ of the eyeglass group for $T$. Moreover, up to action by elements of $E$, the mimicking is well-defined: the path of a handleslide in $T_C$, when mimicked by a handleslide in $T$, is ambiguous only because of the need to miss the scars in $\partial C$ left by the 2-handles of $B$ that lie in adjacent chambers. But different paths around these scars differ by eyeglass moves.

Recall the following classical lemma ([La, Lemma 3.2], [Bo]):

**Lemma 4.2.** Any two complete collections of meridian disks for the compression body $A$ in a Heegaard splitting differ by a finite sequence of band moves, which can be viewed as handleslides.

The action in the next Lemma is confined to a chamber $C$ within a chamber complex. To simplify notation for the purposes of the proof, we drop the subscript $C$. That is, in statement and proof of Proposition 4.3 and the ensuing remarks, one should regard $A = A_C, B = B_C, T = T_C, E = E_C$ and $D = D \cap C$. 
Proposition 4.3. Suppose either the Heegaard splitting $C = A \cup_T B$ is not stabilized or $D$ is non-separating. Let $A$ and $A'$ be complete collection of meridian disks for $A$ that miss $D$, and $\tau_t : T \to C$ be an isotopy of $T$ in $C$ as in Lemma 4.2 for the sets $A, A'$. Then there is an isotopy $\epsilon_t \in E$ which takes the curves $\tau_t^{-1}(D)$ to $\tau_0^{-1}(D)$. After this isotopy, we can take the handleslides of $\tau$ to be disjoint from $D$.

Proof. When $C = A \cup_T B$ is not stabilized, all but the last claim follows from [FS2, Theorem 1.6]. The same is true when the splitting is stabilized and $D$ is non-separating: a required move, namely passing a bubble through a disk in $D$, can be realized by instead sliding the bubble around a loop in $T$ that intersects $D$ once, a bubble move in $E$.

So we focus on the last sentence. We can assume $\text{genus}(T) \geq 2$ since in a trivial splitting of the solid torus, $A = A' = \emptyset$.

The composition isotopy $\tau' = \epsilon \cdot \tau$, applied to the annuli $D \cap A$, is an isotopy of spanning annuli in the compression body $A$ during which the end of the annulus on $\partial_0 A = C$ is fixed. Each component of the space of embedded circles in a surface of genus at least 2 is contractible, so we can deform $\tau'$ so that the ends of these spanning annuli in $A$ are also fixed in $T$. (By which we mean $\tau_t^{-1}(D \cap T)$ is independent of $t$.) The isotopy of the spanning annuli then takes place entirely in the collar $A - \tau'(A) \cong \partial C \times I$ of $\partial C$. Since the space of diffeomorphisms of $\partial C \times I$ fixing its boundary is also contractible, we can take these spanning annuli also to be fixed in $A$ during the isotopy. Finally, the collection of essential disks $D \cap B \subset B$ may also be regarded as fixed during the isotopy ([Ha]), so we may take all of $D$ to be fixed during $\tau'$.

Put another way, in each component $C_-$ of the submanifold $C - D$ we can think of $\tau'$ as defining a series of handleslides for the Heegaard splitting $C_- = A_- \cup_T B_-$. These handleslides take $A \cap C'$ to the disks $\epsilon_1(A' \cap C')$. Applying Lemma 4.2 again (this time to $T_- \subset C_-$) there is a sequence of handleslides fixing $D$ that takes $\epsilon_1(A' \cap C')$ back to $A' \cap C'$. The composition then takes $A \cap C'$ to $A' \cap C'$. Each handleslide in each component of $C - D$ can be viewed as handleslides on $T$ and this defines the sequence of handleslides required.

A less formal way of describing these two propositions is this: if $D$ is a collection of essential disjoint disks in $C$ then $T$ can be isotoped so that $A$ intersects $D$ only in annuli. Moreover, up to the action of $E$, any two such positionings are isotopic by an isotopy that is fixed near $D$. 

□
Corollary 4.4. Under the hypotheses of Proposition 4.3, the Heegaard splitting \( C_- = A_- \cup_{T_-} B_- \) that \( T \) induces on any component \( C_- \) of \( C - D \) is well-defined.

Proof. Following the proof of Proposition 4.3, any two ways of isotoping \( A \) to intersect \( D \) only in a collar of \( \partial D \) differ by an element \( \epsilon \) of \( E \). Then \( \epsilon_1 \) provides a diffeomorphism from one version of \( C_- = A_- \cup_{T_-} B_- \) to the other. \( \square \)

Much of the argument in Proposition 4.3 remains intact even when \( D \) is separating and \( T \) is stabilized, with one exception. In the case that \( T \) is stabilized Theorem 1.6 of [FS2] requires in place of \( \epsilon \in E \) in the argument above a series of moves. Some are in \( E \) but the others pass a bubble through a disk in \( D \). The consequence is that the induced Heegaard splitting \( T_- \) on each component \( C_- \) of \( C - D \) is only known up to equivalence under stabilization by the maximum number of destabilizations possible for \( C_- = A_- \cup_{T_-} B_- \). We think of the Heegaard splittings induced on \( C_- \) as changeable by importing and/or exporting bubbles across the disks. The exact path by which a bubble is slid from one component to the other is unimportant, since sliding around a closed path in \( T \) is an element of \( E \).

5. Flagged chamber complexes

Our focus is on Heegaard splittings, but it is very difficult to describe and follow sequences of surgeries on disk sets using this language, particularly since, as noted at the end of Section 4, the exact Heegaard structure of each chamber is undetermined. Instead we introduce an elaborated form of chamber complex, which we use to carry forward the minimal information needed to ultimately apply the argument to Heegaard splittings.

Definition 5.1. A flagged chamber complex in \( M \) is a chamber complex in which some chambers are designated as empty. A chamber that is not empty is occupied. In order for a chamber \( C \) to be empty it must be a handlebody but not be a ball, whereas some handlebody chambers (including balls) may be occupied.

Suppose, as described above, \( \mathcal{C} \) is a flagged chamber complex, \( \mathcal{D} \) is an embedded collection of disks, and \( \hat{\mathcal{C}}_\mathcal{D} \) is the chamber complex that results from surgery on \( \mathcal{D} \). Remove some (but not necessarily all) disky balls to create \( \hat{\mathcal{C}}_\mathcal{D} \).

Definition 5.2. A flagged chamber complex structure on \( \mathcal{C}_\mathcal{D} \) is consistent with that on \( \mathcal{C} \) if it has the following properties:
1. Any chamber in $\mathcal{C}_D$ that is not a disky handlebody is occupied. A disky handlebody may be empty or occupied.

2. Any ball chamber in $\mathcal{C}_D$ is occupied. (This is anyway required in order for $\mathcal{C}_D$ to be a flagged chamber complex, but we call out that property here.)

3. At least one remnant of an occupied chamber in $\mathcal{C}$ is occupied in $\mathcal{C}_D$.

4. Unless every remnant of an occupied chamber is a disky handlebody, the occupied remnant is not a disky handlebody.

   In other words, if any occupied remnant of a chamber is a disky handlebody then every remnant is a disky handlebody.

5. Unless every remnant of an occupied chamber is a disky ball, all disky ball remnants are goneballs.

   In other words, if any occupied remnant of an occupied chamber is a disky ball then every remnant is a disky ball.

6. If every remnant of an occupied chamber is a disky ball, all but one disky ball is a goneball. Following the rules above, that remaining ball is occupied.

   In other words, if every remnant of an occupied chamber is a disky ball then there is only one remnant, an occupied ball.

Here is a helpful mental image that motivates the terminology: Suppose in $\mathcal{C}$ each occupied chamber possesses a flag. After surgery on $\mathcal{D}$, each such flag is passed on to the remnants of the chamber. The remnants have a rough priority for receiving the flag, with disky balls having lowest priority and disky handlebodies the next lowest. Any disky ball that doesn’t receive a flag is a goneball. The last set of rules describes this inheritance process; the first three direct how new flags must be issued to chambers in $\mathcal{C}_D$. The goal of the process, in terms of Heegaard splittings, is to show that, no matter what choices are made in aligning decomposing disks (in a decomposition sequence yet to be defined) eventually either one of the chambers is reducible, or some chamber is an occupied ball. Either of these outcomes carries important information about the Heegaard splitting.

**Definition 5.3.** A flagged chamber complex is tiny if it is a tiny chamber complex, as in Definition 2.3, and each of the handlebodies in $\mathcal{M} - \mathcal{C}$ is empty.

In particular, if $F$ is a single component dividing $\mathcal{M}$ into two handlebodies, so either of them can play the role of chamber $\mathcal{C}$, the flagged chamber complex is tiny unless both handlebodies are occupied.
Proposition 5.4 (Tinyness pulls back). Suppose $\mathcal{C}_D$ is a flagged chamber complex, whose flagging is consistent with that of $\mathcal{C}$. Then if $\mathcal{C}_D$ is tiny, so was $\mathcal{C}$.

Proof. Since $\mathcal{C}_D$ is tiny, each relevant handlebody is empty. It follows from Rule 11 of Definition 5.2 that each relevant handlebody is disky, so $\mathcal{C}$ is tiny as a chamber complex. It remains to show that each handlebody in $\mathcal{C}$ is empty. As shown in the proof of Proposition 2.4, the collection of remnants of the handlebodies in $\mathcal{C}$ consists of the handlebodies of $\mathcal{C}_D$ together with some goneballs. According to Definition 5.2 if one of the handlebodies in $\mathcal{C}$ is occupied, so would be one of its remnants. Hence one of the handlebodies (perhaps a remaining ball) of $\mathcal{C}_D$ would be occupied, violating the hypothesis. □

One point of both the flagging and the succession rules is this:

Corollary 5.5. Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_n$ is a sequence of flagged chamber complexes, each $\mathcal{C}_i$, $1 \leq i \leq n - 1$ has a disk set $D_i$, and $\mathcal{C}_{i+1}$ is the result of decomposing on $D_i$ with flagging consistent with that of $\mathcal{C}_i$. Then if $\mathcal{C}_1$ is not tiny, neither is $\mathcal{C}_n$. In particular, $\mathcal{C}_n$ is not a single chamber with $F_n = \emptyset$.

6. Heegaard split chamber trees

Given a Heegaard split chamber complex $\mathcal{C}$, establish a flag rubric by the following simple rule: a chamber is empty if and only if its splitting is trivial. Put another way, an $A$-chamber (resp $B$-chamber) $\mathcal{C}$ is empty if and only if $B_C$ (resp $A_C$) is a collar of $\partial C$ and symmetrically. This is consistent with Definition 5.1 that the only empty chambers are handlebodies, though a handlebody may be occupied.

Theorem 6.1. Let $\mathcal{C}$ be a flagged Heegaard split chamber complex containing a disk set $D$ and supporting a Heegaard splitting $M = A \cup_T B$ that is consistent with the flag rubric. $T$ can be isotoped so that $D$ is aligned with $T$ and a choice of goneballs in $\mathcal{C}_D$ made so that the flag rubric on the derived Heegaard split chamber complex $\mathcal{C}_D$ is consistent with that on $\mathcal{C}$.

Proof. It will be helpful to recall that the surface $F$ that defines the chamber complex $\mathcal{C}$ is (apart from some disks) a subsurface of the original Heegaard surface $T \subset M$, so information about how $F$ intersects a chamber $C'$ of $\mathcal{C}_D$ is directly relevant to the Heegaard splitting of $C'$ induced by $T$. Also, we will largely ignore the ambiguity up to $E_C$ in the positioning of $T_C$ in each chamber $C$, since this ambiguity can be
mimiced by action of $E$ on $T$. Finally, “condition (x)” below will refer to the 6 conditions of Definition 5.2.

Note that, so long as we do not declare a ball remnant with non-trivial splitting a goneball, condition (3) is always satisfied: If, in the Heegaard splitting $C = A_C \cup_{T_C} B_C$ the compression body $B_C$ is not a collar of the boundary, then at least one of the remnants will have the same property.

**Case 1:** The Heegaard splitting $T_C$ is not stabilized.

In this case the positioning of $T_C$ is well defined (up to action by $E$), and no ball remnant can have non-trivial splitting since such a splitting would, per Waldhausen, be stabilized. Declare a ball remnant to be a goneball if and only if it is disky.

Condition (6) is always satisfied: if every remnant of $C$ is a disky ball, then the argument of Proposition 2.1 shows that $C$ is a handlebody. But any non-trivial splitting of a handlebody is stabilized, so we conclude the splitting of $C$ must be trivial and so $C$ is empty.

Similarly, no remnant that the flag rubric declares to be occupied can be a handlebody, since a non-trivial splitting of a handlebody is stabilized, contradicting the hypothesis of this case. Thus (4) and (5) are also vacuously satisfied.

It remains to check conditions (1) and (2).

If a remnant $C'$ is not disky, then there is a component of $F \cap C'$ which either has positive genus or more than one boundary component (perhaps both). In either case, this component guarantees that the Heegaard splitting on $C'$ induced by the original Heegaard surface $T$ is non-trivial. In particular, the flag rubric calls $C'$ occupied, so (1) and (2) are satisfied.

So suppose $C'$ is disky. If $C'$ is not a handlebody then its Heegaard splitting cannot be trivial, so the flag rubric says it is occupied and again (1) and (2) are satisfied.

Finally, suppose $C'$ is a disky handlebody, but not a ball, since such a ball has been declared a goneball. Then conditions (1) and (2) are vacuously satisfied.

**Case 2:** The Heegaard splitting $T_C$ is stabilized.

Suppose first that some remnant $C'$ is not a disky handlebody. In this case, again declare that a ball remnant is a goneball if and only it is disky. As in case 1, the flag rubric declares that $C'$ is occupied. Position $T_C$ so that any bubble lies in $C'$, so no other remnant is stabilized. This means that no other remnant is an occupied handlebody. Then (4)-(6) are again vacuously satisfied and, for $C''$ any other remnant, the argument proceeds for $C''$ as it did for $C'$ in the previous case.
Suppose lastly that all remnants are disky handlebodies. Then immediately (1) and (4) are satisfied.

If any remnant $C'$ is not a ball, then (6) is satisfied. Again position $T_C$ so that any bubble lies in $C'$ and declare any ball remnant, necessarily now having trivial splitting, to be a goneball. Then (2) and (5) are satisfied.

If every remnant is a ball, then pick an arbitrary one $C'$ and proceed as above: position $T_C$ so that all stabilizing bubbles lie in $C'$ and declare any other ball remnant, necessarily now having trivial splitting, to be a goneball. Then (2), (5) and (6) are satisfied.

A Heegaard split chamber complex $C'$ obtained as in the proof of Theorem 6.1 is said to be derived from $C, D$.

**Corollary 6.2.** Suppose, as above, $C$ is a Heegaard split chamber complex with the flag rubric; that $D$ is a disk set in $C$; and $C'$ is a derived Heegaard split chamber complex. Then

1. If no Heegaard splitting of a chamber in $C$ is stabilized, then $C'$ and the Heegaard splittings of its chambers is fully determined.
2. If no chamber in $C$ is an occupied handlebody, then $C'$ is fully determined as a flagged chamber complex, but the Heegaard structure of each chamber in $C'$ may not be well defined.

**Proof.** Both statements follow easily from the proof of Theorem 6.1. For the second claim, note that no choice is made of which remnants of a chamber $C$ are goneballs or which remnants are occupied unless all remnants of $C$ are disky handlebodies and at least one is occupied. But in this case the argument of Proposition 2.4 shows that $C$ is an occupied handlebody. □

7. **Adding another disk to $D$**

Suppose $C_1 \xrightarrow{D} C_2$ is a flagged chamber complex decomposition, and $E$ is a properly embedded disk disjoint from $D$ in a chamber $C$ of $C_1$. Suppose we add $E$ to $D$. What is the effect on $C_2$? Clearly if $E$ is parallel in $C$ to a disk in $D$, there is no effect: the ball between them becomes a goneball. More generally, the same is true if $E$ is inessential in a chamber of $C_2$ and the ball between them is disky. But even here there can be circumstances in Definition 5.2 where what appears to be a goneball might instead be occupied.

The general situation can be complicated to describe, but also important to understand. As $T$ moves through a loop of embeddings in $S^3$, the chamber complex that our process gives rise to will change, with disks being added to or deleted from the disk set defining the
associated chamber complex. The philosophy will be to try to delay the addition of a disk to a disk set as long as possible, perhaps to the last stage of the process, where the picture is very concrete. We will find that if delaying the change in the disk set doesn’t work well, the process may instead identify a reducing sphere that provides continuity as the disk set changes - a very useful outcome. Or, under special circumstances, neither occurs and instead one of the complexes has a “bullseye” inserted, as we now describe.

Definition 7.1. A bullseye is a chamber complex in $B^3$ consisting of a torus $T$ surrounded by a nested sequence of $k \geq 0$ spheres. A blank bullseye is a bullseye with the torus deleted. See Figure 7.

Figure 7. Bullseyes, each $k = 2$

Lemma 7.2. Suppose a flagged chamber complex $C$ is obtained from $C'$ by inserting a bullseye into a chamber $C'$ of $C'$. Suppose $D$ is a disk set in $C$ and $D' \subset D$ consists of those disks which are not incident to the inserted bullseye. Then either:

1. the chamber $C'$ is reducible (that is, a reducible 3-manifold) or
2. the chamber complex $C_D$ is obtained from $C'_D$ also by inserting a bullseye, with at least as many spheres.

Remarks: The inserted bullseye in the second outcome can be blank even when the original bullseye was not. This happens when some disk $E \in D$ has essential boundary on the torus $T$. In this case the number of spheres can rise because surgery on $E$ alone turns $T$ into a sphere bounding an occupied ball, the second example in Figure 7.
Let a dotted arrow represent the construction “insert a bullseye.” Then outcome (2) can be expressed as this commutative square:

$$
\begin{array}{ccc}
\mathcal{C} & \overset{\mathcal{D}}{\longrightarrow} & \mathcal{C}_D \\
\downarrow & & \downarrow \\
\mathcal{C}' & \overset{\mathcal{D}'}{\longrightarrow} & \mathcal{C}'_D
\end{array}
$$

The direction of the dotted arrow can be confusing. Think of $\mathcal{C}'$ being duplicated exactly in $\mathcal{C}$, where it is supplemented by a bullseye.

That the number of spheres in the bullseye does not decrease does not seem to be important to the ongoing proof, but is included here because its proof is easy.

**Proof.** One way of expressing the hypothesis is that we start off with a bullseye $\odot$ in $\mathcal{C}'$, but don’t regard it as defining part of the chamber complex, only as a surface that happens to lie in a chamber of $\mathcal{C}'$. In $\mathcal{C}$, in contrast, the bullseye $\odot$ is regarded as part of the underlying surface $F$ defining the chamber complex.

Let $\mathcal{D}_B \subset \mathcal{D}$ be the set of disks incident to $\odot$, and $Q \subset \mathcal{C}'$ be the surface obtained from $\odot$ by surgery simply on $\mathcal{D}_B$. Since surgery does not raise genus, $Q$ is a collection of spheres, plus a torus if and only if no surgery disk has essential boundary on the bullseye’s torus $T$. If any sphere in $Q$ is essential in $\mathcal{C}'$ (see Figure 8) we have outcome (1), so assume all spheres in $Q$ bound balls in $\mathcal{C}'$.

**Figure 8.** Outcome (1): $\mathcal{C}'$ reducible
First consider the case in which $D_B$ does not contain a disk whose boundary is essential in $T$, possibly because $\ominus$ is blank. In that case, each ball in $C'$ bounded by a sphere component of $Q$ becomes a goneball in $C_D$ if and only if it does not contain any component of $\ominus$ in its interior, since to be disky it cannot have such a component inside. Hence the sphere components of $Q$ that remain in $C_D$ just add spheres to the nested sequence in $\ominus$, giving a new bullseye $\ominus_+$. The disks $D - D_B$ are disjoint from both $D_B$ and $\ominus$, so they are disjoint from $\ominus_+$. Thus, except for the count of spheres, we have outcome (2). (Note that there may be other goneballs in $C'_D$ that have $\ominus_+$ in their interior. But their boundaries just add to the number of spheres in $\ominus_+$.)

If $D_B$ does contain a disk whose boundary is essential in $T$, a little deeper analysis is needed. Let $Y$ be the chamber in the bullseye whose boundary is $T$, either a solid torus or a knot complement; see Figure 9. First assume that the only disks in $D_B$ that are incident to $T$ have essential boundaries on $T$. Surgery on a pair of these yields a goneball in between, so we may as well assume that there is exactly one such disk $E$. If $E$ lies in $Y$ then surgery on $E$ yields a ball, which is a goneball if $Y$ was empty and an occupied ball if it was not. If $E$ lies outside $Y$, as illustrated in the second panel of Figure 9 then the sphere created by surgery has non-disky interior (namely the rest of $T$) so the ball is occupied. This is outcome (2), in which the new inserted bullseye is blank (but has more spheres).

![Figure 9. E and Y in the bullseyes](image)

The argument is little changed if there is a disk in $D_B$ that has inessential boundary on $T$: Of all such disks, let the disk $D_0 \in D_B$ have boundary innermost on $T$, cutting off a subdisk $D_T$ of $T - \partial D_B$. 

If $D_0$ lies in $Y$ then $D_0 \cup D_T$ is a goneball, so $D_0$ may as well have been discarded from $\mathcal{D}$ originally. Thus we can assume that $D_0$ lies in the chamber adjacent to $Y$; if it is inessential there it could also have been discarded. If it is essential there, then its ball interior is not disky (since it contains the rest of $T$) so in $\mathcal{C}_\mathcal{D}$ it becomes a sphere $Q_0 \subset Q$ bounding an occupied ball. See Figure 10. Components of $Q$ that lie outside this occupied ball remain in $\mathcal{C}_\mathcal{D}$ if and only if they contain $Q_0$ and so these form a nested sequence of spheres. Inside the ball bounded by $Q_0$ all remaining disks of $\mathcal{D}_B$ lie between $Q_0 - D_T$ and $T - D_T$ so they are inessential and could have been discarded originally.

To count the number of spheres in outcome (2), choose a point in a chamber other than $C'$ and draw an arc to a point in $Y$. Orient the components in $\circ$ consistently, say towards $Y$. Then the arc has algebraic intersection $k + 1$ with $\circ$. Surgery on disks won’t change the algebraic intersection, nor will removing goneballs. So we conclude that the arc has algebraic intersection $k + 1$ with $\circ_+$. Hence it must have at least $k + 1$ components.

Return then to our discussion of the flagged chamber complex decomposition $\mathcal{C}_1 \xrightarrow{\mathcal{D}} \mathcal{C}_2$ and $E$ a properly embedded disk disjoint from $D_1$ in a chamber $C$ of $\mathcal{C}_1$. What is the effect on $\mathcal{C}_2$ of adding $E$ to $\mathcal{D}$?

For our purposes, it will suffice to ask a related question: what is the effect of delaying the addition of $E$ to the disk set? To that end, thinking of $E$ as a disk set in itself, let $\mathcal{C}_{ED}$ be a flagged chamber complex derived from $\mathcal{C}_1$ by first using $E$ and then $\mathcal{D}$, and let $\mathcal{C}_{DE}$ be one derived by first using $\mathcal{D}$ and then $E$.

**Question:** Will $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$ be the same flagged chamber complexes? If not, how do they differ?
Of course, if we did not remove goneballs, both chamber complexes would be the same, namely the chamber complex obtained by simply doing surgery on $\mathcal{D} \cup E$, so the question is really about the sequencing of goneballs and how that sequencing changes the outcome.

Here’s an easy warm-up

**Proposition 7.3.** Suppose $E$ is parallel to a disk $E' \subset F$ and no disk in $\mathcal{D}$ is incident to $E'$. Then $\mathcal{C}_{DE} = \mathcal{C}_{ED}$.

**Proof.** The union of $E$ and $E'$ is a goneball both for $\mathcal{C}_{DE}$ and $\mathcal{C}_E$, so $\mathcal{C}_E = \mathcal{C}$ and $\mathcal{C}_{ED} = \mathcal{C}_{DE} = \mathcal{C}_{D}$.

In the general situation, more complicated outcomes are possible:

**Proposition 7.4.** Following surgeries on $\mathcal{D}$ and $E$ either

1. $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$ are the same flagged chamber complexes
2. a chamber of $\mathcal{C}_E$ contains a reducing sphere that lies entirely in a chamber of $\mathcal{C}$
3. $\mathcal{C}_{\mathcal{D}}$ is obtained from $\mathcal{C}_{ED}$ by inserting a bullseye, with $E$ the meridian of the torus component. If there are no spheres in the bullseye, then the chamber $Y$ that the torus $T$ bounds in $\mathcal{C}_{\mathcal{D}}$ does not contain the meridian $E$.
4. $\mathcal{C}_{ED}$ is obtained from $\mathcal{C}_{DE}$ by inserting a blank bullseye. This only occurs when $E$ is parallel to a subdisk of $F_{\mathcal{D}}$ and the ball between them is an occupied ball in $\mathcal{C}_{ED}$.

The last two outcomes in diagrams:

$$
\begin{array}{cc}
\mathcal{C} \xrightarrow{\mathcal{D}} \mathcal{C}_D & \mathcal{C} \xrightarrow{\mathcal{D}} \mathcal{C}_{\mathcal{D}} \\
\downarrow \mathcal{E} & \downarrow \mathcal{E} \\
\mathcal{C}_E \xrightarrow{\mathcal{D}} \mathcal{C}_{ED} & \mathcal{C}_E \xrightarrow{\mathcal{D}} \mathcal{C}_{DE}
\end{array}
$$

**Proof.** **Case 1a:** $C$ is an empty solid torus, $E$ is a meridian of $C$, and there is a ball in $M$ that contains the chamber $C$ in its interior but is otherwise disjoint from $F$.

In this case, $C$ can be regarded as a bullseye with the number of spheres $k = 0$, and $\mathcal{C}_E$ is the chamber complex obtained from $\mathcal{C}$ by deleting the torus $\partial C$ from $F$. Put another way, $\mathcal{C}$ is obtained from $\mathcal{C}_E$ by inserting the bullseye $C$. Then, in Lemma 7.2, we can consider $\mathcal{D} \cup E$ here as playing the role of $\mathcal{D}$ there, and $\mathcal{D}$ here playing the role
of $\mathcal{D}'$ there. Outcome (1) of Lemma 7.2 then translates to outcome (2) here, and outcome (2) of Lemma 7.2 translates to outcome (3) here.

**Case 1b:** The chamber $C$ is an empty solid torus, $E$ is a meridian of $C$, but $C$ does not lie in a ball as described in the previous case. In this case, suppose some disks in $\mathcal{D}$ have inessential boundary on the torus $\partial C$ and let $D_0$ be one whose boundary is innermost on the torus $T = \partial C$, cutting off a disk $D_C$ from $\partial C$. If $D_0 \cup D_C$ is essential in a chamber of $\mathcal{C}_E$ we have outcome (2). Otherwise, since $D_0 \cup D_C$ can’t bound a ball containing $T$, it bounds a goneball, and we may as well have discarded $D_0$ from the start.

If all disks in $\mathcal{D}$ incident to $\partial C$ have boundary essential on $\partial C$ then they are parallel to $E$ and we have outcome (1).

**Case 2:** $C$ is not an empty solid torus with $E$ a meridian. In this case, decomposing along $E$ does not create a goneball, so $\mathcal{D}' = \mathcal{D}$. We consider then $\hat{\mathcal{C}}_{D+e}$ obtained from surgering along $\mathcal{D} \cup E$, and examine how goneballs of this surgery interact with the decomposition trajectories $\mathcal{C} \rightarrow \mathcal{C}_{ED}$ vs $\mathcal{C} \rightarrow \mathcal{C}_{DE}$. To this end, let $B_G$ be a ball of $\hat{\mathcal{C}}_{D+e}$ that is bounded by a sphere $G$.

**Case 2a:** $E$ lies outside $B$ and has no scars on $G$.

Then $B_G$ is a goneball in both $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$ if and only if $B_G$ is diskly for $\hat{\mathcal{C}}_{D+e}$ (that is, the underlying surface $F$ of $\mathcal{C}$ intersects $B_G$ only in disks). Thus $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$ do not differ here, consistent with outcome (1).

**Case 2b:** $E$ lies inside $B_G$ and neither scar of $E$ lies on $G$.

Then $G$ is also a sphere in $\hat{\mathcal{C}}_D$, before surgery on $E$ and before elimination of goneballs.

If $F$ intersects $B_G$ only in disks, then $E$ is parallel to a subdisk of $F$ in one of these disks, and $B_G$ is a goneball both in $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$.

If $\text{int}(B_G) \cap F$ contains a component that is not a disk then $G$ is an occupied ball in $\mathcal{C}_D$ and so persists into $\mathcal{C}_{DE}$. If a component of $\text{int}(B_G) \cap F$ remains not a disk after surgery on $E$, then $G$ also persists in $\mathcal{C}_{ED}$, consistent with outcome (1).

The remaining possibility is that surgery on $E$ turns the unique non-disk component of $\text{int}(B_G) \cap F$ into a disk, so that component is a once-punctured torus with $E$ as a meridian. See Figure 11. Then $G$ remains a sphere in $\mathcal{C}_D$ and surrounds an empty torus whose meridian is $E$, whereas in $\mathcal{C}_{ED}$, $B_G$ becomes a goneball. This is outcome (3), with $k \geq 1$.

**Case 2c:** $E$ lies inside $B_G$ and leaves 2 scars on $G$.

The outcome is similar: Let $A$ denote the annulus in $B_G$ whose meridian is $E$. If any component of $\text{int}(B_G) \cap F$ other than $A$ is not a
Figure 11. Some of Case 2b

disk, $G$ remains a sphere in $C_{ED}$. In $C_D$, $G$ is a torus whose meridian is $E$, so in $C_{DE}$ $B_G$ is not disky and $G$ remains there too, consistent with outcome (1).

If each component of $\text{int}(B_G) \cap F$ other than $A$ is a disk, $B_G$ becomes a goneball in $C_{ED}$, but in $C_D$ the union of $A$ and the subannulus of $G$ lying between the components of $\partial A$ persists as a torus in $C_D$ with meridian $E$, and the region between them is the component $Y$ whose boundary is $T$. This is outcome (3) with perhaps $k = 0$.

**Case 2d:** $E$ lies inside $B_G$ and leaves one scar on $G$.

Again let $A$ denote the annulus whose meridian is $E$. In $C_{D+E}$, $A$ tubes the sphere $G$ to a component $K$ of $C_{D+E}$.
First consider the case that in $\mathcal{C}_D$, $K$ becomes a sphere $K_D$ with disky interior. Then $K_D$ bounds a goneball in $\mathcal{C}_D$. $B_G$ will also be a goneball in $\mathcal{C}_D$ if and only if its interior is disky, which requires that $K$ itself be a sphere and every other component of $\text{int}(B_G) \cap F$ be a disk. Whereas in $\mathcal{C}_{ED}$, $K_D$ will still bound a goneball but $G$ may not, if $K$ is not a sphere or if there are other components of $\text{int}(B_G) \cap F$ that are not disks. So, to summarize in this case, if $K$ is not a sphere but all other components of $\text{int}(B_G) \cap F$ are disks, $G$ is a goneball in $\mathcal{C}_D$ (and so in $\mathcal{C}_{DE}$) but not in $\mathcal{C}_{ED}$. This is outcome (4); see Figure 12. In the other cases mentioned, $B_G$ is a goneball in both $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$, or in neither, consistent with outcome (1).
On the other hand, if \( K_D \) is not a sphere or if the ball that it bounds is not disky, then \( K \) persists in \( C_D \) and so into \( C_{DE} \), and \( G \) does also. Since \( K_D \) does not bound a disky ball in \( C_D \), \( K \) is not a sphere in \( C_E \). So it cannot bound a goneball in \( C_E \), and persists into and does not bound a disky ball in \( C_{ED} \). So \( G \) also persists into \( C_{ED} \). That is \( G \) persists into both \( C_{ED} \) and \( C_{DE} \), consistent with outcome (1).

Case 2e: \( E \) lies outside \( B_G \) and leaves two scars on \( G \). Then in \( C_D \) the union of \( B_G \) and the cocore of \( E \) is a solid torus \( W \) with \( E \) as meridian. Suppose there is a non-disk component of \( \text{int}(B_G) \cap F \). Then \( G \) persists as an occupied ball in \( C_{ED} \), \( W \) is an
occupied torus in $\mathcal{C}_D$ and so $G$ becomes an occupied ball in $\mathcal{C}_{DE}$, consistent with outcome $\text{II}$.

On the other hand, if all components of $\text{int}(B_G) \cap F$ are disks, then $W$ is an empty torus, and $G$ is a goneball in both $\mathcal{C}_{ED}$ and $\mathcal{C}_{DE}$, again consistent with outcome $\text{II}$.

**Case 2f:** $E$ lies outside $B_G$ and leaves one scar on $G$

Again let $A$ denote the annulus whose meridian is $E$. Suppose $\text{int}(B_G) \cap F$ consists of disks. Then $B_G$ becomes a goneball in both $\mathcal{C}_{DE}$ and $\mathcal{C}_{ED}$, consistent with outcome $\text{II}$.

Suppose $\text{int}(B_G) \cap F$ contains a component not a disk. Then $G$ persists as a sphere in $\mathcal{C}_{ED}$, bounding an occupied ball. However, in $\mathcal{C}_{DE}$, the components of $\text{int}(B_G) \cap F$ may themselves become goneballs, in which case $E$ is just parallel, via $B_G$, to a disk in $F_D$, so $B_G$ acts as a goneball in the decomposition $\mathcal{C}_D \to \mathcal{C}_{DE}$. See Figure 14. This is outcome $\text{IV}$.

$$\square$$

**Corollary 7.5.** Let $\mathcal{C}_{D+E}$ be the flagged chamber complex obtained by surgery on $D \cup E$. Then either:

1. $\mathcal{C}_{ED}$ and $\mathcal{C}_{D+E}$ are the same flagged chamber complexes.
2. A chamber of $\mathcal{C}_E$ contains a reducing sphere that lies entirely in a chamber of $\mathcal{C}$.
3. $\mathcal{C}_{D+E}$ is obtained from $\mathcal{C}_{ED}$ by inserting a blank bullseye.

**Proof.** The proof is identical, with the added observation that the circumstances in the proof of Proposition 7.4 which led to outcome $\text{IV}$ do not arise if $E$ and $D$ are surgered simultaneously. $\square$

8. **Certificates and guiding spheres**

**Definition 8.1.** A flagged chamber complex $\mathcal{C}$ certifies if some chamber is reducible or an occupied ball. A reducing sphere in such a chamber (or the boundary of an occupied ball chamber) is a certificate issued by $\mathcal{C}$.

**Definition 8.2.** A flagged chamber complex decomposition sequence

$$\vec{C} : \mathcal{C}_1 \xrightarrow{\mathcal{D}_1} \mathcal{C}_2 \xrightarrow{\mathcal{D}_2} \ldots \xrightarrow{\mathcal{D}_{n-1}} \mathcal{C}_n$$

is a sequence in which each $\mathcal{C}_i$ is a flagged chamber complex, each $\mathcal{D}_i$ is a disk set in $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ is a flagged disk complex consistent with surgery on $\mathcal{D}_i$ as defined in Section 2.

A flagged chamber complex decomposition sequence certifies if some $\mathcal{C}_i$ certifies.
Two flagged chamber complex decomposition sequences

\[ \tilde{C} : \ C_1 \overset{D_1}{\rightarrow} C_2 \overset{D_2}{\rightarrow} \ldots \overset{D_{n-1}}{\rightarrow} C_n \]

and

\[ \tilde{C}' : \ C'_1 \overset{D'_1}{\rightarrow} C'_2 \overset{D'_2}{\rightarrow} \ldots \overset{D'_{n-1}}{\rightarrow} C'_m \]

interact if there are chamber complexes \( C_i, C'_j \) containing chambers \( C, C' \) so that one of the two chambers contains a certificate for both \( C_i \) and \( C'_j \). In particular both sequences certify.
Two flagged chamber complex decomposition sequences $\vec{C}, \vec{C}'$ issue equivalent certificates if there is a sequence

$$\vec{C} = \vec{C}_1, \vec{C}_2, \ldots, \vec{C}_n = \vec{C}'.$$

so that for each $1 \leq i \leq n - 1$, $\vec{C}_i$ interacts with $\vec{C}_{i+1}$. We write $\vec{C} \sim \vec{C}'$

An aside to motivate and explain: There are now several dimensions of construction at play.

- **The flagged chamber complex itself**: We look for a reducing sphere (a ‘certificate’) in one of the chambers. If $\mathcal{M}$ is irreducible (e. g. $S^3$) such a reducing sphere will determine (up to action by $\mathcal{E}$) a reducing sphere for the original Heegaard splitting. This is our principal goal.

- **A decomposition sequence of flagged chamber complexes, to be guided by a sphere**: The hope is to identify in this decomposition sequence a chamber complex that must certify. Failing that, we end up with a concrete picture of the last stage of the decomposition sequence, and this will suffice.

- **A sequence of decomposition sequences of flagged chamber complexes**: These arise as the original Heegaard surface moves through a loop in the space of its embeddings in $\mathcal{M}$ determined by an element of the Goeritz group. If we can transition from each one to the next via a common certificate, then when the Heegaard surface completes the loop we will have found reducing spheres in $\mathcal{T}$ before and after the action of the Goeritz element, along with a natural way to transition between them. This will suffice to show that the Goeritz element does not change the eyeglass equivalence class of the Heegaard splitting.

Return now to the argument:

Suppose $\mathcal{C}$ is a flagged chamber complex in $\mathcal{M}$ with underlying surface $\mathcal{F}$, and $\mathcal{S} \subset \mathcal{M}$ is a sphere transverse to each chamber in $\mathcal{C}$.

**Definition 8.3.** A decomposition $\mathcal{C} \xrightarrow{\mathcal{D}} \mathcal{C}'$ is guided by $\mathcal{S}$ if the disk set $\mathcal{D}$ consists of the disk components of $\mathcal{S} - \mathcal{F}$.

More generally, a decomposition sequence

$$\mathcal{C}_1 \xrightarrow{\mathcal{D}_1} \mathcal{C}_2 \xrightarrow{\mathcal{D}_2} \ldots \xrightarrow{\mathcal{D}_{n-1}} \mathcal{C}_n$$

guided by $\mathcal{S}$ is one in which each decomposition is guided by $\mathcal{S}$. It is complete if $\mathcal{S}$ lies entirely in a chamber of $\mathcal{C}_n$.

We denote this decomposition sequence $(\mathcal{C}_1, \mathcal{S})$. 
Theorem 8.4. Suppose $C_1$ is a flagged chamber complex and the sphere $S \subset M$ is transverse to $C_1$. Let $E$ be a properly embedded disk in a chamber of $C_1$ so that $E$ and $S$ are disjoint. Let $C'_1$ be the flagged chamber complex obtained by decomposing $C_1$ by disk set $\{E\}$ alone. Let

$$
\vec{C} : C_1 \xrightarrow{D_1} C_2 \xrightarrow{D_2} \ldots \xrightarrow{D_{k-1}} C_k
$$

and

$$
\vec{C}' : C'_1 \xrightarrow{D'_1} C'_2 \xrightarrow{D'_2} \ldots \xrightarrow{D'_{k-1}} C'_k
$$

be the respective flagged chamber complex decomposition sequences guided by $S$. Then either:

- Each $C'_i$ is obtained from $C_i$ by decomposition on the disk set $E$ alone, written $\vec{C} \xrightarrow{E} \vec{C}'$. In particular $C_k \xrightarrow{E} C'_k$.

- $C_k$ is obtained from $C'_k$ by inserting a bullseye, written $\vec{C} \leftarrow \vec{C}'$, with $E$ the meridian of the torus component.

- $C'_k$ is obtained from $C_k$ by inserting a blank bullseye, written $\vec{C} \rightarrow \vec{C}'$. In this case, $E$ is parallel to a disk in $C_k$.

- The two sequences interact, so $\vec{C} \sim \vec{C}'$.

Proof. This is a consequence of Proposition 7.4. We begin with the diagram

$$
\begin{array}{cccc}
C_1 & \xrightarrow{D_1} & C_2 & \xrightarrow{D_2} \ldots & \xrightarrow{D_{k-1}} & C_k \\
\downarrow E & \quad & \downarrow E & \quad & \downarrow E & \\
C'_1 & \xrightarrow{D'_1} & C'_2 & \xrightarrow{D'_2} \ldots & \xrightarrow{D'_{k-1}} & C'_k
\end{array}
$$

extending the downward arrows to the right for as long as the diagram commutes. We then will turn to the diagram in the proof of Proposition 7.4.

If all the squares commute, then $\vec{C} \xrightarrow{E} \vec{C}'$.

If not all the squares commute, we may as well assume that the first square does not commute. If outcome (2) of Proposition 7.4 occurs we have $\vec{C} \sim \vec{C}'$.

If instead outcome (3) of Proposition 7.4 occurs, we have $C_2 \leftrightarrow C'_2$. We then turn to Lemma 7.2. If outcome (2) of Lemma 7.2 occurs for all remaining squares we have $\vec{C} \leftrightarrow \vec{C}'$. If outcome (1) of Lemma 7.2 ever occurs we have $\vec{C} \sim \vec{C}'$.

If outcome (4) of Proposition 7.4 occurs, the same argument applies, with the roles of $\vec{C}$ and $\vec{C}'$ reversed. $\square$
The outcomes in Theorem 8.4 are not meant to be mutually exclusive.

**Corollary 8.5.** With the hypotheses of Theorem 8.4

- If both $C_k$ and $C'_k$ certify then $\vec{C} \sim \vec{C}'$.
- If $C'_k$ certifies then either $\vec{C} \sim \vec{C}'$ or $C_k \rightarrow C'_k$. If $C_k \rightarrow C'_k$ then $E$ is parallel to a disk in $C_k$.
- If $C_k$ certifies then either $\vec{C} \sim \vec{C}'$ or $C_k \leftarrow C'_k$. If $C_k \leftarrow C'_k$ then $E$ is parallel to a disk in $C_k$.

**Proof.** Suppose first that both $C_k$ and $C'_k$ certify

**Case 1: $C_k \rightarrow C'_k$.**

Since $C_k$ certifies, there is a chamber $C$ of $C_k$ that contains a reducing sphere. By standard innermost disk arguments, a reducing sphere $Z \subset C$ can be found that is disjoint from $E$ and so persists (perhaps as a goneball) into $C'_k$. By hypothesis, there is also a reducing sphere in a chamber of $C'_k$. Again, by standard arguments, we may pick $Z' \subset C'$, a chamber of $C'_k$, so that $Z'$ and $Z$ are disjoint.

If $Z$ does not bound a goneball in $C'_k$ it certifies in both $C_k$ and $C'_k$, so $\vec{C} \sim \vec{C}'$. If $Z$ does bound a goneball, then $E$ must have been inside that goneball and $Z'$ outside of it, so $Z'$ certifies both $C_k$ and $C'_k$ and again $\vec{C} \sim \vec{C}'$.

**Case 2:** It is not true that $C_k \rightarrow C'_k$.

Then there is a point where one of the squares in Theorem 8.4 does not commute, and we may as well assume it’s the first square. We then turn to Proposition 7.4. If the second outcome there occurs, $\vec{C} \sim \vec{C}'$. If the third occurs then appeal to Lemma 7.2 at every succeeding stage. Either we get outcome (1) of that Lemma at some stage, which implies $\vec{C} \sim \vec{C}'$, or we have $C'_k \rightarrow C_k$. But $C_k$ and $C'_k$ then differ only by an extra blank bullseye in a chamber of $C_k$, so the certificate for $C'_k$ is also a certificate for $C_k$. Thus $\vec{C} \sim \vec{C}'$.

If the fourth occurs then the same argument using Lemma 7.2 applies again, switching the roles of $C_k$ and $C'_k$, so $C_k \rightarrow C'_k$.

For the second part of Corollary 8.5 suppose that we are only given that $C'_k$ certifies. Again we turn to Theorem 8.4. If the second outcome occurs, so $C_k \leftarrow C'_k$, then $C_k$ also certifies and we are done by the previous part. The other two alternatives are what is claimed.

For the third part of Corollary 8.5 suppose that we are only given that $C_k$ certifies. If the third outcome of Theorem 8.4 occurs, so $C_k \rightarrow C'_k$, then the blank bullseye in $C'_k$ certifies, so $C_k \sim C'_k$. The other two alternatives are what is claimed. □
Definition 8.6. Suppose $S \subset M$ is a sphere transverse to each chamber in flagged chamber complex $C$, with $X, Y$ the complementary components of $S$, and $F$ the underlying surface for $C$. Then $S$ is balanced in $C$ if the compact surfaces $F \cap X$ and $F \cap Y$ are both non-empty and they are either both planar (a planar balance) or both non-planar (a non-planar balance).

Theorem 8.7. Suppose $C_1 \xrightarrow{D_1} C_2 \xrightarrow{D_2} \cdots \xrightarrow{D_{n-1}} C_n$ is a flagged chamber complex decomposition sequence guided by $S \subset M$. Then

- If $S$ is planar (resp non-planar) balanced in any chamber complex in the sequence, then it is in every chamber complex in the sequence. In this case, call $S$ planar (resp non-planar) balanced in the sequence.
- If $S$ is balanced in the sequence, the sequence is complete, and $C_1$ is not tiny, then the sequence certifies.

Proof. For the first statement, note that the effect on $F \cap X$, say, of decomposing along $D_i$ is 2-fold: first cap off some of boundary components of $F \cap X$ with disks, then delete some spheres from $F$ (the boundaries of the goneballs). Neither step affects the genus of $F \cap X$ or $F \cap Y$.

For the second statement, note that since $C_1$ is not tiny, $C_n$ is not empty (Corollary 5.5). Let $F_n \neq \emptyset$ be the underlying surface for $C_n$. When the sequence is non-planar balanced, there are (non-planar) closed components of $F_n$ in both $X$ and $Y$. This implies that $S$ is a reducing sphere for the chamber of $C_n$ in which it lies, so $S$ itself is a certificate. If the sequence is planar balanced, then each component of $F_n$ is a closed planar surface, i.e. a sphere. Since there are no empty balls in a flagged chamber complex (5.2), an innermost sphere in $F_n$ is a certificate.

9. Isotopies of guiding spheres

Theorem 9.1. Suppose $C$ is a flagged chamber complex in $M$ that is not tiny. Suppose further that $S$ and $S'$ are balanced spheres for $C$ that are isotopic in $M$ through balanced spheres. Then $(C, S) \sim (C, S')$.

Proof. It suffices to consider the case in which $S$ and $S'$ pass through a single tangency point with the underlying surface $F$ of $C$. If the tangency is a max or min the conclusion is obvious, since the decompositions differ by just an initial goneball. (See Proposition 7.3.) So
we focus on a saddle tangency point \( v \). Note that in this case (and so the general case) \( S \) and \( S' \) are either both planar balanced or both non-planar balanced, since moving across a saddle point from below to above, say, cannot increase the genus above, nor decrease the genus below.

Let \( S_0 \) be the sphere, lying between \( S \) and \( S' \), that meets \( F \) at the saddle tangency. A neighborhood in \( F \) of \( S_0 \cap F \) (a figure eight) is a pair of pants with three disjoint circles: \( c \), whose parallel in \( S_0 \) is incident to \( v \) in two points, and \( c_1, c_2 \) each of whose parallels in \( S_0 \) is incident to \( v \) in a single point. For the purposes of the proof, call the spheres \( S_a(bove) \) lying above \( S_0 \) and containing two of the three circles, and \( S_b(elow) \) lying below \( S_0 \), containing the remaining circle.

\[ \text{Figure 15.} \]

During the decomposition guided by \( S_a \) or \( S_b \) the disks that the circles bound in the sphere will disappear; the order of disappearance determines the viewpoint from which we will argue. Note that \( c_1, c_2 \) lie on the same side of \( c \) in \( S_0 \). If \( c \) disappears after \( c_1 \) and \( c_2 \) (Case 1 below) we picture the side of \( c \) containing \( c_1, c_2 \) and note that the circles \( c_i \) will not be nested in and will both in \( S_a \). If, say, \( c_2 \) disappears after \( c \) (Case 2 below) then we will observe the side of \( c_2 \) that contains \( c_1 \), where the \( c_i \) appear nested. In this case, \( c \) and \( c_1 \) lie in \( S_a \) and \( c_2 \) lies in \( S_b \). If all circles disappear simultaneously (so \( S_0 \cap F \) is exactly the figure 8) we take the non-nested point of view (Case 1).

**Case 1:** The circles \( c_1 \) and \( c_2 \) lie in \( S_a \) and are not nested there.
There is no difference between the guidance of $S_a$ and the guidance of $S_b$ until at least one of the circles $c, c_1, c_2$ bounds a disk in $S_a$ or $S_b$. In Case 1, we are assuming it is one or both of the non-nested $c_i$. Say $c_1$ bounds a disk $E_a \subset S_a - F$. Surger $F$ along $E_a$ to get a new chamber complex $C'$ with underlying surface $F'$. Note that this does not change the fact that $S_a$ is balanced for $C'$, so $C'$ guided by $S_a$ certifies. Note also that $S_a \cap F' = S_b \cap F'$, so $S_b$ is also balanced for $C'$ and they issue the same certificate: \( (C', S_a) = (C', S_b) \). Furthermore, since both $C$ and $C'$ certify for both $S_a$ and $S_b$, Corollary 8.5 says \( (C, S_a) \sim (C', S_a) \) and \( (C, S_b) \sim (C', S_b) \). It follows that

\[
(C, S_a) \sim (C', S_a) = (C', S_b) \sim (C, S_b)
\]
as required. See Figure 16.

Case 2: The circles $c$ and $c_1$ lie in $S_a$, with $c$ nested inside $c_1$; $c_2$ lies in $S_b$.

The proof is essentially the same as in Case 1 if the first circle to bound a disk in the spheres is $c_1 \subset S_a$. So we henceforth assume that $c_2 \subset S_b$ is the first to bound a disk.

Consider a special situation: Suppose the next tangency encountered if $S_b$ were to move to a lower level is a minimum that caps off $c$. We compare the guidance given by $S_a$ and $S_b$, see Figure 17. Any difference in guidance does not arise until $c_1$ bounds a disk $E_a$ in $S_a$ (second panel of Figure 17). At that point, let $C'$ be the chamber complex obtained by surgery on $E_a$ in $S_a$. The guidance given by $S_a$ to $C'$ is the same as the guidance that $S_b$ gives to $C'$, so, as in Case 1, $S_a$ and $S_b$ both remain balanced for $C'$ and \( (C', S_a) = (C', S_b) \). Invoking Corollary 8.5 again we
have
$$(C, S_a) \sim (C', S_a) = (C', S_b) \sim (C, S_b).$$
See Figure 17.

Now return to the proof of Case 2: Suppose that $c_2 \subset S_b$ bounds a disk $E_b \subset S_b$ and let $C_b$ be the chamber complex obtained by surgering along $E_b$ at a level just below $S_b$. $C_b$ remains balanced for $S_b$. The same is true for $S_a$ since compressing $E_b$ will not change the genus of $F$ above $S_a$ and visibly does not change it below. Invoking Corollary 8.5 we have $(C_b, S_b) \sim (C, S_b)$ and $(C_b, S_a) \sim (C, S_a)$. But $C_b$ is now as in the special situation above, so $(C_b, S_a) \sim (C_b, S_b)$. We conclude
$$(C, S_a) \sim (C_b, S_a) \sim (C_b, S_b) \sim (C, S_b)$$
as required.

**Proposition 9.2.** Suppose $C$ is a flagged chamber complex in $M$ that is not tiny, $S$ is a planar balanced sphere for $C$, and $S'$ differs from $S$ simply by passing through one saddle tangency with the surface $F$ underlying $C$. Then either

- $(C, S) \sim (C, S')$ or
- $M$ is $S^3$, $(C, S')$ does not certify and at its terminal stage, $C$ is a single torus, with both complementary components occupied, and the spheres $S$ and $S'$ differ by passing through a single essential tangency on the torus.

**Proof.** If $S'$ is also balanced the first outcome follows from Theorem 9.1. So we will assume that $S'$ is not balanced and adopt the same picture of the saddle tangency as in the proof of Theorem 9.1. The only way in which genus can change from balanced to unbalanced as we pass through the tangency is if the genus of $F$ above $S_b$ is 1 and the genus of $F$ above $S_a$ is 0. Since $S$ is planar balanced $S = S_a$ and
$S' = S_b$, so in the proof of Theorem 9.1 the information we no longer have available is that $S_b$ is balanced. We explore how that changes the proof and conclusion.

Since $(C, S_a)$ certifies, but $(C, S_b)$ may not we could find ourselves, for example, only having $(C, S_b) \xrightarrow{E} (C', S_b)$ or $(C, S_b) \rightarrow (C', S_b)$ for the relevant chamber complex $\hat{C}$, following the second part of Corollary 8.5. In particular, the first displayed formula may become

$$\begin{align*}
(C, S_a) \sim (C', S_a) &= (C', S_b) \xleftarrow{E_a} (C, S_b) \\
(C, S_a) \sim (C', S_a) &= (C', S_b) \leftarrow (C, S_b)
\end{align*}
$$

But a closer look shows that the latter can’t arise in this case: According to Corollary 8.5 the outcome $(C', S_b) \leftarrow (C, S_b)$, harkening back to outcome 1 of Proposition 7.4 requires that $E_a$ (a disk in $S_a$ bounded by $c_1$) be parallel to a disk in the stage of the decomposition of $C$ at which $E_a$ is compressed. In other words, $E_a$ is coplanar with some subset of those disks which the sphere $S_b$ is also guiding us to compress along. But if that’s the case, compressing on $E_a$ can’t change the genus above $S_b$, so $(C, S_b)$ would be balanced planar, contradicting hypothesis.

So we are left with the case $(C', S_b) \xleftarrow{E_a} (C, S_b)$. This means that we can delay surgery on $E_a$ until the final stage, after $C$ has become a single torus $T$ with meridian $E_a$ and possibly some sphere components, and at the end stage the tangency point is an essential saddle singularity in $T$. If $M$ is not $S^3$ then surgery on $E_a$ creates an occupied ball lying in $M$, whose complement is then a reducible chamber, so $(C, S) \sim (C, S')$. On the other hand, if $M$ is $S^3$, the complement of the occupied ball is just a ball, whose boundary is compressible in the complement of $T$, so it doesn’t certify. We return to this case in a moment.

For the second formula, note that in the argument for Case 2 there, we might have to replace $(C', S_b) \sim (C, S_b)$ with merely $(C', S_b) \xleftarrow{E_a} (C, S_b)$ so in the third formula we have merely

$$\begin{align*}
(C, S_a) \sim (C_b, S_a) \xleftarrow{E_a} (C_b, S_b) &= (C, S_b)
\end{align*}
$$

(The right equality follows because when $E_b$ is used to compress in the decomposition of $C$ guided by $S_b$, $C_b$ will differ exactly by a goneball.)

But again, the relation $(C_b, S_a) \xleftarrow{E_a} (C_b, S_b)$ implies that at the terminal stage of the decomposition by $S_b$ we have a torus $T$ with meridian
$E_a$ above $S_b$, and possibly also some spheres, and at the end stage the tangency point is an essential saddle singularity in $T$.

If there are any spheres at all (or some are added via $\langle C_b, S_a \rangle \xleftarrow{E_a} \langle C_b, S_b \rangle$) then this terminal stage certifies for $\langle C, S_b \rangle$ as well as for $\langle C, S_a \rangle$. Thus $\langle C, S \rangle \sim \langle C, S' \rangle$, as desired. If there are no spheres and so only $T$, each complementary component must be occupied, since $C$ is not tiny. This is the second outcome. □

**Proposition 9.3.** Suppose $C$ is a flagged chamber complex in $M$ that is not tiny, $S$ is a non-planar balanced sphere for $C$, and $S'$ differs from $S$ simply by passing through one saddle tangency with the surface $F$ underlying $C$. Then either

- $\langle C, S \rangle \sim \langle C, S' \rangle$
- $M$ is $S^3$, $\langle C, S' \rangle$ does not certify and at its terminal stage, $\langle C, S \rangle$ consists of two unlinked tori separated by $k \geq 0$ spheres. At least one of the tori bounds an empty solid torus, and the terminal stage of $\langle C, S' \rangle$ is obtained by compressing away this empty torus and removing the spheres.

**Proof.** The proof begins just as the proof of the previous Proposition 9.2 with the standard picture, except in this case the balanced sphere $S$ is identified with $S_b$ and $S'$ with $S_a$, so $S_b$ is non-planar balanced and $S_a$ is not balanced, with the part of $F$ above $S_a$ of genus 0 and the part of $F$ below of genus 1. We again explore how that changes the proof of Theorem 9.1 and the conclusion.

**Step 1:** First focus on the relation $\langle C, S_a \rangle \sim \langle C_b, S_a \rangle$ occurring in the last formula in the proof of Theorem 9.1 which relied on $S_a$ being balanced and therefore certifying. Here instead we refer back to the proof of Theorem 8.4 as applied to the disk $E_b$. The compression along $E_b$ just below $S_b$ can first be made when the disk becomes disjoint from $F$. As described in the proof of Theorem 8.4 delay making that compression so long as the diagram there commutes. If at any stage we have the last outcome, $\langle C, S_a \rangle \sim \langle C_b, S_a \rangle$ we have what we need for these two sequence, so we turn to the other three possibilities.

If surgery on $E_b$ can be pushed back until after surgery on $E_a$, then notice $c_2$ has become parallel with $c$ in $F$, so after that point $\langle C, S_a \rangle = \langle C_b, S_a \rangle$, which suffices. So we need not consider the first possibility $\langle C, S_a \rangle \xleftarrow{E_a} \langle C_b, S_a \rangle$. 


Suppose at some point between the creation of $E_b$ and surgery on $E_a$ the diagram fails to commute. The third outcome there, $(C, S_a) \rightarrow (C_b, S_b)$ will, per Proposition 7.4, only occur if $E_b$ is parallel, via $F$ to other disks being surgered along simultaneously by the decomposition sequence. This can’t happen above $E_b$ since this would imply that $c_1$ is inessential in $F$ at that stage, contradicting the fact that later surgery on it will reduce the genus of $F$ above $E_a$. The parallelism also can’t happen below, since the surface below remains genus 1 throughout the decomposition. We conclude that this outcome does not occur.

For the second outcome of Theorem 8.4 to occur, per Proposition 7.4, $E$ must be the meridian of a torus at the time of compression. But this would again imply that $c_1$ is inessential in $F$ at that point, and that would again contradict that the later surgery on $E_a$ reduces the genus. Hence the second outcome also is impossible.

**Step 2:** The remaining issue is the use of $(C', S_b) \sim (C, S_b)$ in the proof of Theorem 9.1. (Recall that here $C'$ is obtained from $C$ by compressing along $E_a$.) We no longer know this when $S_a$ is not balanced, so $(C', S_b)$ is no longer known to certify. On the other hand, since $S_b$ is balanced for $C$ we do know that $(C, S_b)$ certifies. So, per the third part of 8.5 we know that either $(C', S_b) \sim (C, S_b)$, and there is no further reason for concern, or we have one of

$$(C', S_b) \rightarrow (C, S_b) \quad (C', S_b) \xleftarrow{E_a} (C, S_b)$$

Consider the last stage of the decomposition, and consider first $(C, S_b)$. We know at this stage $F$ has been divided by $S_b$ into two tori components and possibly some spheres. Unless $M = S^3$ and all the sphere components come from an inserted bullseye, $S_b$ or another reducing sphere will persist into the last stage of $(C', S_b)$, implying that $(C', S_b)$ certifies and so $(C', S_b) \sim (C, S_b)$ as required.

So suppose $M = S^3$ and all sphere components of the terminal stage of $(C, S_b)$, if any, come from an inserted bullseye. Unless surgery on $E$ in the center of this bullseye creates a goneball, the resulting occupied ball in the final stage of $(C', S_b)$ will again certify, and we are done. If it is empty, we have the concrete description given in the second part of the Proposition.

The remaining possibility is that $(C', S_b) \xleftarrow{E_a} (C, S_b)$. Again consider the last stage. If compressing the torus component by $E_a$ gives an occupied ball, we are done as before. We deduce that the torus is
empty, and again we have the second concrete description, this time with no spheres.

Aside: One might observe that the second, highly specialized outcome of Propositions 9.2 could be avoided if we regarded an occupied solid torus as certifying, based on the fact that it issues an ‘implicit’ certificate: the occupied ball one obtains by $\partial$-reducing the solid torus along its meridian. When we return in our discussion to Heegaard splittings we will see that this is a justifiable approach, since any non-trivial Heegaard splitting of a solid torus is a stabilization of the trivial splitting. But introducing this very special case in Definition 8.2 and carrying it through the entire argument up to this point would also be problematic, so we have elected to deal with this case when needed as we will do here.

10. Simultaneous saddle tangencies

Suppose $S_0$ is a sphere in $M$ that meets the underlying surface $F$ of a flagged chamber complex $C$ in two saddle tangency points $v_1, v_2$ and is otherwise transverse to it. There are four ways to resolve the tangencies, two for each tangency point, to obtain generic intersections. One way to visualize the possibilities is to let $I \times I$ parameterize a small push-off of $S_0$ near $v_1$ in one $I$ factor, and a small push-off near $v_1$ with the other. It is customary to view this parameterization in a way that is itself generic, by rotating it $\pi/4$, and that is how it typically appears in sweep-out graphics (to be discussed further below).

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (-1,-1) -- (1,1);
\draw (-1,1) -- (1,-1);
\node at (0,0) {\text{N}};
\node at (1,0) {\text{Q}};
\node at (0,1) {\text{P}};
\node at (0,-1) {\text{R}};
\end{tikzpicture}
\caption{Figure 18.}
\end{figure}

In that context, suppose the 4 quadrants of resolution are as shown in Figure 18. Some notation will be useful: We label quadrant $X \frac{0}{\text{T}}$ (resp $\frac{1}{\text{T}}$) if the part of $F$ above $S_X$ is planar and the part below is non-planar (resp vice versa). Similarly $\frac{0}{\text{P}}$ and $\frac{1}{\text{P}}$ denote planar balanced and non-planar balanced respectively. The spheres that result from resolving the tangencies into the respective quadrants will be labeled $S_P, S_Q, S_N, S_R$. 
Lemma 10.1. Suppose \( P \) and \( Q \) represent balanced positions \( S_P, S_Q \) for \( S \). Then either \( (C, S_P) \sim (C, S_Q) \) or the quadrant \( N \) is labeled \( \frac{0}{1} \) and \( R \) is labeled \( \frac{1}{0} \), or vice versa.

Proof. If either quadrant \( S_N \) or \( S_R \) is balanced, Theorem 9.1 implies \( (C, S_P) \sim (C, S_Q) \). On the other hand, if neither is balanced, we may assume without loss of generality that \( N \) is labeled \( \frac{1}{0} \). If \( Q \) is labeled \( \frac{1}{1} \) the change in numerators between \( Q \) and \( N \) implies that \( S_N \) lies above \( S_Q \); if \( Q \) is labeled \( \frac{0}{1} \), so does the change in denominators.

Since \( S_N \) lies above \( S_Q \), the sphere \( S_P \) lies above \( S_R \). If \( P \) is labeled \( \frac{0}{1} \) his implies that \( R \) is not labeled \( \frac{0}{1} \), since that would be inconsistent with the denominator; if \( P \) is labeled \( \frac{1}{1} \) this implies that \( R \) is not labeled \( \frac{0}{1} \), since that would be inconsistent with the numerator. We conclude that \( R \) is labeled \( \frac{1}{0} \). \( \square \)

Figure 19 illustrates what we know so far.

\[
\begin{array}{c}
\text{N} \quad \frac{0}{1} \\
\text{P} & \text{Q} \\
\text{R} \quad \frac{1}{0}
\end{array}
\]

Figure 19.

Lemma 10.2. Suppose quadrants \( P \) and \( Q \) represent non-planar balanced positions \( S_P, S_Q \) for \( S \), then \( S_0 \cap F \) and its pair of tangency points, is as shown in Figure 20.

\[
\begin{array}{c}
1 & 2 & 3 & 4 \\
\text{v}_1 & & & \\
& \text{v}_2
\end{array}
\]

Figure 20.
Proof. There are six possibilities, as seen in Figure 21, if one makes use of the symmetries available in $S_0$. In the connected diagrams (the bottom row) the first two have in common that there is a resolution of the two tangencies with three components; the other three resolutions (and so the other three quadrants) are obtained by various ways of fusing these three components together. We will show that this is inconsistent with the Lemma’s hypothesis.

The first observation is that for a surface $F$, fusing together two circle boundary components may increase, but it cannot decrease the genus. Indeed, if the circles are fused together by adding a band between them, the original surface is contained within the new surface; if they are fused together by cutting along an arc in $F$ between the boundary components, this increases the Euler characteristic by 1 and decreases the number of boundary components by 1, leaving genus unchanged.

Now apply this to the situation at hand: let $S$ be a sphere for which the two tangencies have been resolved so as to yield 3 circle components in $F \cap S$, as above. Then, since the other resolutions involve fusing some of these three together, the genus of $F_a$ for $S$ is minimal among all four quadrants, which by hypothesis is 0. Similarly for $F_b$. Hence $S$ is balanced planar, and this contradicts the hypothesis of the lemma.

A similar argument leading to the same contradiction, with 4 circles instead of 3, applies to all three configurations in the top row of Figure 21. The only remaining possibility is the bottom right configuration, which is what was to be proved. 

Proposition 10.3. Suppose quadrants $P$ and $Q$ represent non-planar balanced positions $S_P, S_Q$ for $S$. Then $(\overrightarrow{C}, S_P) \sim (\overrightarrow{C}, S_Q)$

Proof. Following Lemma 10.2 we examine the region above and below the sphere $S_0$, where $S_0 \cap F$ contains the two tangency points in the
component $L$ shown in Figure 20. Typically there are other circles, frequently nested, in each of the complementary regions of $S_0 - L$ labeled 1-4 in Figure 20. As before, let $S_N, S_P, S_Q, S_R$ be the spheres near $S_0$ that resolve the singularities. The figure is very symmetric, so without loss of generality, we suppose that $S_N \cap F$ resolves the singularities in $L$ into the circles bounding regions 2 and 4, and $S_R \cap F$ resolves the singularities into circles bounding regions 1 and 3. The other two resolutions each create a single circle; one appears in $S_P$ and one in $S_Q$.

As the decomposition of $\vec{C}$ progresses for each of the 4 spheres, nothing changes until one of the regions 1-4 is devoid of circles. Without loss, assume that region 2 is (one of) the first to empty. At that point the ‘difference’ between $S_P \cap F$ and $S_Q \cap F$ is as shown in Figure 22 - namely, there is no difference! So we conclude $\overrightarrow{(C, S_P)} \sim \overrightarrow{(C, S_Q)}$.$\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure22.png}
\caption{Figure 22.}
\end{figure}

**Proposition 10.4.** Suppose quadrants $P$ and $Q$ represent planar balanced positions $S_P, S_Q$ for $S$. Then $\overrightarrow{(C, S_P)} \sim \overrightarrow{(C, S_Q)}$.

**Proof.** Here is a summary of the situation:

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure23.png}
\caption{Figure 23.}
\end{figure}

Following Proposition 9.2 either adjacent quadrants issue equivalent certificates, or the picture of the transition at the final stage is concrete: a single torus $T$ in $N$ or $Q$ for which both sides are occupied is
compressed to a sphere that bounds occupied balls on both sides. Furthermore, similar analysis to Lemma 10.2 shows that $S_0 \cap T$ is again as shown in Figure 20 but this time the underlying surface component of $C$ is the torus $T$, and it is unknotted, with quadrants $P$ and $Q$ yielding resolutions of the tangencies that correspond to pairs of meridians and longitudes respectively.

Combining these two facts, we examine the two transitions from quadrant $N$, say. If both are equivalences we are done. If not, the end stage of $\rightarrow (C, S_P)$ is the sphere obtained from $T$ by compressing meridians, and the end stage of $\rightarrow (C, S_Q)$ is the sphere obtained from $T$ by compressing longitudes. But these are the same sphere, and so $\rightarrow (C, S_P) \sim (C, S_Q)$.

**Proposition 10.5.** Suppose quadrant $P$ represents a planar balanced position $S_P$ for $S$ and $Q$ represents a non-planar balanced positions. Then $(C, S_P) \sim (C, S_Q)$.

**Proof.** Here is a summary of the situation:

![Figure 24](image)

If the quadrants $N$ and $P$ issue equivalent certificates, that is $(C, S_P) \sim (C, S_N)$, then in particular $(C, S_N)$ certifies. It follows, essentially from Corollary 8.5 that also $(C, S_Q) \sim (C, S_N)$, and consequently $(C, S_P) \sim (C, S_Q)$ as required. Similarly if the transition from quadrant $R$ to quadrant $P$ is an equivalence. We are left with the possibility that neither is true, so at the last stage of the transitions of both quadrant $N$ and quadrant $R$ to quadrant $P$, we have the concrete second outcome of Proposition 9.2.

Let us consider then the transition from $Q$ to $N$, for which we can call on Proposition 9.3. If $(C, S_N) \sim (C, S_Q)$ then in particular $(C, S_N)$ certifies so $(C, S_N) \sim (C, S_P)$ and we are done. The other possibility is that at the terminal stage of the decompositions, the chamber
complexes $C_N$ and $C_Q$ satisfy the concrete description given second in Proposition 9.3. In particular, the torus in $(C, S_Q)$ above $S_b$ bounds an empty solid torus. The symmetric argument in the transition from quadrant $R$ to $N$ would show that the the torus in $(C, S_Q)$ below $S_b$ bounds an empty solid torus. But this is the torus that appears in $C_N$. Since $C_N$ is not tiny, this torus cannot be the only defining surface in $C_N$, so there must also be spheres. In particular $C_N$ certify, so $(C, S_P) \sim (C, S_N) \sim (C, S_Q)$ as required. □

11. The Goeritz group is the eyeglass group

Let $C \subset S^3$ be a flagged chamber complex that is not tiny (so it is not just a Heegaard splitting) and whose underlying surface $F$ has positive genus and is in general position with respect to the standard height function on $S^3$. Use the standard height function on $S^3$ to define a sweepout $S_s, 0 < s < 1$ of $S^3$ by level spheres.

When $s$ is near 1, the part of $F$ above $S_s$ will be planar (in fact empty) and the part below will be non-planar, since $\text{genus}(F) \geq 1$. For $s$ near 0 exactly the opposite will be true. The genus above $S_s$ will not increase as $s$ ascends, nor will the genus below $S_s$ decrease, and passing through a saddle point will change the genus on at most one side. It follows that there is an $S_{s*}$ which is balanced for $C$, thus $(C, S_{s*})$ certifies. We next will argue that the certificate issued is, up to equivalence, independent of our choice of $S_{s*}$ and indeed invariant under isotopy of $C$.

**Proposition 11.1.** Let $C_t, 0 \leq t \leq 1$ be an ambient isotopy of $C$ in $S^3$ and $S_{s_0}, S_{s_1}$ be balanced level spheres for $C_0$ and $C_1$ respectively. Then $(C_0, S_{s_0}) \sim (C_1, S_{s_1})$.

**Proof.** Consider the graphic on the square $I \times I$ determined by $C_t$ and $S_s$ (see [FS]). Label a region in the graphic much as in the proofs in Section 10: pick a point $(t, s) \in (I \times I)$ that lies in the region and label it $\frac{0}{0}$ if $S_s$ is planar balanced for $C_t$; $\frac{1}{1}$ if $S_s$ is non-planar balanced for $C_t$; by $\frac{1}{0}$ if the part of $F$ above $S_s$ is non-planar, and the part below is planar; and by $\frac{0}{1}$ if the part of $F$ above $S_s$ is planar, and the part below is not. Let $B(\text{balanced})$ represent all regions labeled $\frac{0}{0}$ or $\frac{1}{1}$.

**Claim:** The closure $\overline{B}$ in $I \times I$ of $B$ is connected.

Pick any generic $t \in I$ and consider how the interval $\{t\} \times I$ intersects $\overline{B}$. We noted above that $s$ rises it must pass through some region in $B$; suppose it passes through two successive such regions $B_a$ and $B_b$, with $B_a$ above $B_b$ and no intersection with a $B$ region in between.
A look at $B_b$ shows that the unbalanced region in between must be labeled $\frac{0}{1}$: If $B_b$ is labeled $\frac{0}{0}$ increasing $s$ can’t increase the genus of the part of $F$ lying above $S_s$, so the numerator must remain 0; if $B_b$ is labeled $\frac{1}{0}$ increasing $s$ can’t decrease the genus of the part of $F$ lying below $S_s$, so the denominator must remain 1. But a similar analysis of the label just below $B_a$ leads to the opposite conclusion: the region between $B_b$ and $B_a$ must be labeled $\frac{1}{0}$. The contradiction shows that for any generic $t$, $\{t\} \times I$ intersects $B$ in a single closed interval and, similarly, even at a non-generic value of $t$, the intersection is either a single point or a single closed interval. This verifies the claim.

It follows that there is a path in $\overline{B}$ from the point $(0, s_0)$ to $(1, s_1)$. See Figure 25. The path may pass through vertices in the graphic satisfying the hypotheses of one of the Propositions of Section 10. No matter: following those propositions and Theorem 9.1 the path shows $(C_0, S_{s_0}) \sim (C_1, S_{s_1})$, as required. $\square$

Following Proposition 11.1 we can define the certificate class issued by $C$ as that issued by any level balanced sphere for $C$, no matter how $C$ has been isotoped in $S^3$.

**Definition 11.2.** For $C$, $C'$ flagged chamber complexes in $S^3$ say $C \sim C'$ if for some balanced sphere decomposition $\bar{C}$ of $C$ and $\bar{C}'$ of $C'$, $\bar{C} \sim \bar{C}'$.

The proof of Proposition 11.1 suggests a Goeritz-like corollary:

Following [JM] let $\text{Img}(S^3, C) = \text{Diff}(S^3)/\text{Diff}(S^3, C)$, the space of images of $C$ in $S^3$. Consider the group $\pi_1(\text{Img}(S^3, C))$, in analogy to the Goeritz group for Heegaard splittings of $S^3$. For $\tau \in \pi_1(\text{Img}(S^3, C))$ let $\tau_\theta : C \to S^3, 0 \leq \theta \leq 2\pi$ be a representative, and define $\hat{\tau} = \tau_{2\pi} : (S^3, C) \to (S^3, C)$. Then $\hat{\tau}$ is well-defined up to isotopy of pairs; indeed
\( \hat{\tau} \in \text{Diff}(S^3, C) \) is the image of \( \tau \) in the fibration sequence \( \text{Diff}(S^3, C) \to \text{Diff}(S^3) \to \text{Img}(S^3, C) \).

**Figure 26.**

**Corollary 11.3.** Suppose \( Z \) is a certificate for \( C \). Then \( \hat{\tau}(Z) \) is an equivalent certificate for \( C \).

*Proof.* Apply the same argument as in Proposition 11.1 to the graphic on the annulus \( I \times S^1 \) determined by \( S_s \) and \( \tau(\theta) \). See Figure 26. There is an essential circle in \( I \times S^1 \) that lies between the regions labeled \( \frac{1}{T} \) and those labeled \( \frac{0}{T} \), a circle through balanced regions of the graphic. Via Theorem 9.1 and the Propositions of Section 10 the circle defines an equivalence of \( Z \) to its image under \( \hat{\tau} \), as required. \( \square \)

**Proposition 11.4.** Suppose \( C \) is a flagged chamber complex in \( S^3 \) that is not tiny and does not consist only of spheres. Suppose, for some level sphere \( S_e \), \( E \) is a disk component of \( S_e - F \). Let \( C' \) be the flagged chamber complex obtained by surgery on \( E \). Then \( C \sim C' \).

*Proof.* We may as well take \( S_e \) to be part of a generic sweep-out \( S_s \) of \( S^3 \) by level spheres. For some \( 0 < s_0 < s_1 < 1 \), each \( S_s, s_0 < s < s_1 \) is a balanced sphere for \( C \). With no loss of generality we may assume \( e < s_1 \). Let \( S_b \) be a sphere just below \( S_{s_1} \), so \( S_b \) is balanced, but as the level spheres rise to \( S_a \), just above \( S_{s_1} \), they pass through a single saddle singularity at level \( s_1 \) and are no longer balanced. In the usual way this implies that the part of the defining surface \( F \) for \( C \) lying above \( S_a \) is planar, and the part below is non-planar, summarized as \( S_a \) is labeled \( \frac{\partial}{T} \).

Consider the effect of surgering on \( E \). Since \( S_e \) lies below \( S_b \), the surgery cannot increase the genus of the part of \( F \) below \( S_b \). Thus if \( S_b \)
is planar balanced before the surgery, it will be planar balanced after. And even if \( S_b \) is non-planar balanced before the surgery it may remain non-planar balanced afterwards. In either case, the result follows from Corollary 8.5.

The remaining possibility is that \( S_b \) is non-planar balanced for \( C \) but is no longer balanced after surgery on \( E \), that is for \( C' \). In this case observe that because the part of \( F \) above \( S_b \) goes from non-planar to planar by moving through a single tangency point to \( S_a \), it must have genus 1. Similarly, because a single surgery on the part of \( F \) below \( S_b \) can be made planar by surgery on \( E \), it too must have genus 1. Figure 27 shows what we have derived, where all non-planar surfaces are of genus 1.

This chart is a dead-ringer for the graphic shown in Figure 24 and it is not a coincidence. The proof proceeds as it did in Proposition 10.5 as we now briefly summarize:

If either \( (C', S_b) \) or \( (C, S_a) \) certify, we are done, essentially by Corollary 8.5 and this is what we aim to show. The final stage of the decomposition \( (C, S_b) \) consists of a single torus above \( S_b \), a single torus below \( S_b \) and some number of spheres. If either of the tori bounds an empty solid torus chamber, say the torus above \( S_b \) does, then in \( (C', S_b) \) the final stage must contain also a sphere component, since \( (C', S_b) \) is not tiny. That sphere certifies, and so we are done. On the other hand, if the torus above \( S_b \) does not bound an empty solid torus, then in \( (C, S_a) \) the now compressed torus component becomes an occupied ball, which certifies, and again we are done.

Suppose \((S^3, T)\) is a Heegaard splitting with \( \text{genus}(T) \geq 3 \) and \( \tau \) is an element of the Goeritz group \( G(S^3, T) \), as described in [JM]. Let
$T_\theta \subset S^3, 0 \leq \theta \leq 2\pi$ be a representative of $\tau$ in $\pi_1(\text{Img}(S^3, T))$. We briefly review some of the results of [FS1].

There are values $0 < \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$ so that for each $\theta \notin \{\theta_i, 1 \leq i \leq n\}$ there is a pair of weakly reducing disks $(a_\theta, b_\theta)$ associated to $T_\theta$ so that:

- The isotopy class of the pair $(a_\theta, b_\theta)$ does not change throughout each interval in $S^1 - \{\theta_i, 1 \leq i \leq n\}$.
- $(a_{2\pi}, b_{2\pi}) = (\tau(a_0), \tau(b_0))$
- For each $1 \leq i \leq n$ either $a_{\theta - \epsilon} = a_{\theta + \epsilon}$ and this disk, together with the disks $b_{\theta - \epsilon}, b_{\theta + \epsilon}$ are all disjoint, or symmetrically.
- For each $1 \leq i \leq n$ these three disks $a_{\theta \pm \epsilon}, b_{\theta - \epsilon}, b_{\theta + \epsilon}$ (or symmetrically) all lie near a level sphere.
- The flagged chamber complex $C_\theta$ obtained from $T$ by weak reduction on $(a_\theta, b_\theta)$ is not tiny, nor does it consist only of spheres.

The last two properties, not explicit in [FS1], follows from the construction described in [FS1, Appendix]. In particular, weak reduction ensures that two adjacent chambers of $C_\theta$ are occupied, so $C_\theta$ is not tiny. And since the weak reduction involves compressing only two disks, the genus of the underlying surface $F$ for $C_\theta$ is at least $\text{genus}(T) - 2 \geq 1$.

Now define, for each $\theta_i$, two flagged chamber complexes $C_i$ and $C'_i$ that support the Heegaard splitting $(S^3, T_{\theta_i})$ as follows. Without loss of generality, assume $a_{\theta_i - \epsilon} = a_{\theta_i + \epsilon}$.

- $C_i$ is obtained from $T_{\theta_i}$ by weakly reducing along the two disks $a_{\theta_i \pm \epsilon}, b_{\theta_i - \epsilon}$.
- $C'_i$ is obtained from $T_{\theta_i}$ by weakly reducing along the three disks $a_{\theta_i \pm \epsilon}, b_{\theta_i - \epsilon}, b_{\theta_i + \epsilon}$.

We are given that the isotopy from $T_{\theta_i + \epsilon}$ to $T_{\theta_i - \epsilon}$ carries the pair $(a_{\theta_i + \epsilon}, b_{\theta_i + \epsilon})$ to $(a_{\theta_i - \epsilon}, b_{\theta_i - \epsilon})$. So in a similar spirit, we define $C_{n+1}$ as the flagged chamber complex obtained from $T_{\theta_n + \epsilon}$ by decomposing along $(a_{\theta_n + \epsilon}, b_{\theta_n + \epsilon})$. See Figure 28.

More generally, because the isotopy from $T_{\theta_i + \epsilon}$ to $T_{(\theta_i+1) - \epsilon}$ carries the pair $(a_{\theta_i + \epsilon}, b_{\theta_i + \epsilon})$ to $(a_{(\theta_i+1) - \epsilon}, b_{(\theta_i+1) - \epsilon})$, it follows that $C_i$ can be gotten both by compressing $C_i$ along $b_{\theta_i + \epsilon}$ and compressing $C_{i+1}$ along $b_{(\theta_i+1) - \epsilon}$. Hence we have, from Proposition 11.4 that $C_i \sim C'_i \sim C_{i+1}$. Applied at each $C_i, 1 \leq i \leq n$ we obtain $C_1 \sim C_{n+1}$.

**Proposition 11.5.** Given a Heegaard splitting $(S^3, T)$ and $\tau \in G(S^3, T)$, there are flagged chamber complexes $C, C' \subset S^3$ such that:

- Neither $C$ nor $C'$ is tiny, nor does either have underlying surface consisting only of spheres.
- Both $C$ and $C'$ support $T$. 


There are balanced spheres \( S, S' \) for \( C \) and \( C' \) respectively, so that \((C, S)\) and \((C', S')\) issue equivalent certificates.

For any certificate \( Z \) issued by \((C, S)\), \( \tau(Z) \) is a certificate issued by \((C', S')\).

**Proof.** In the discussion above, choose \( C = C_1 \) and \( C' = C_{n+1} \).

The next proposition does not require that \( M \) be \( S^3 \), only that \( M \) be irreducible (as well as being a compact orientable manifold in which every closed surface separates). Let \( M = A \cup_T B \) be a Heegaard splitting of \( M \).

**Proposition 11.6.** Suppose \( C \subset M \) is a flagged chamber complex in \( M \) that supports the Heegaard splitting \( T \), and \( Z \) is a certificate for \( C \). Then \( Z \) corresponds to a bubble (a stabilizing summand) \( K(Z) \) for \( T \) that is unique up to eyeglass moves.

**Proof.** If \( Z \) is the boundary of an occupied ball chamber \( C \), this is immediate: since \( C \) is occupied the corresponding Heegaard surface \( T_C \) is a non-trivial splitting of the 3-ball, hence a bubble in \( T \) by Waldhausen [Wa].

So suppose \( Z \) is a reducing sphere for the chamber \( C \) in \( C \), and \( C = A_C \cup_{T_C} B_C \) is the Heegaard splitting determined by \( T \). By [FS2 Theorem 1.6], \( T_C \) can be isotoped in \( C \) so that \( Z \) becomes a reducing sphere for \( T_C \) (that is, \( Z \) intersects \( T_C \) in a single essential circle) and the isotopy of \( T_C \) is unique up to eyeglass moves. Amalgamate the Heegaard splittings of all the chambers into the Heegaard splitting \( T \) and observe that \( Z \) remains a reducing sphere for the Heegaard surface \( T \). But since \( M \) is irreducible, \( Z \) bounds a ball in \( M \). Since \( Z \) is a reducing sphere for the Heegaard splitting \( T \), \( T \) restricts to a non-trivial
Heegaard splitting of this ball. The induced Heegaard splitting of the ball is necessarily a bubble, via Waldhausen again.

**Proposition 11.7.** Suppose $C$ and $C'$ are flagged chamber complexes in $M$, each supporting the Heegaard splitting $M = A \cup_T B$. Suppose $C$ and $C'$ belong to certifying flagged chamber complex decomposition sequences $\vec{C}$ and $\vec{C'}$ respectively. Suppose $Z$ is a certificate issued by $\vec{C}$ and $Z'$ is a certificate issued by $\vec{C'}$. If $\vec{C} \sim \vec{C'}$, there is a sequence of reducing spheres $K(Z) = K_1, ..., K_m = K(Z')$ for $T$, each bounding a non-trivial bubble in $T$, so that for each $1 \leq i < m$, $K_i$ is disjoint from $K_{i+1}$.

**Proof.** It suffices to consider the elementary situation in which $\vec{C}$ interacts with $\vec{C'}$. In that case, let $C_j$ be a flagged chamber complex in the sequence $\vec{C}$, $C'_k$ a flagged chamber complex in the sequence $\vec{C'}$, each containing chambers $C, C'$ respectively, so that, say $C$ contains certifying spheres for both $C_j$ and $C'_k$.

If $C$ itself is an occupied ball, so its certificate is its boundary, the certificate for $C'_k$ lying within it is disjoint. Suppose $C$ is not an occupied ball, so its certificate $Z_C$ is a reducing sphere for $C$. A standard innermost (in $Z_C'$) disk argument shows that there is a reducing sphere in $C$ (hence a certificate for $\vec{C}$) that is disjoint from the certificate $Z'_C$ for $C'_k$. So in either case, there are corresponding reducing spheres $K_C, K'_C$ for $T$ that are disjoint, lie in this chamber, and certify for $\vec{C}$ and $\vec{C'}$ respectively.

It suffices now to show there is a sequence of spheres as in the Proposition, beginning with $K(Z)$ and ending with $K_C$. Then, symmetrically, there is such a sequence from $K'_C$ to $K(Z')$.

We may as well assume that the original certificate $Z$ lies in a chamber $C''$ of some $C_i$ with $i \leq j$, since we can argue symmetrically, switching the roles of $Z$ and $Z_C$ if $i > j$. Suppose $i = j$, so $K(Z)$ and $K_C$ lie in the same chamber complex. If they lie in separate chambers then of course they are disjoint. If they both lie in the chamber $C$, and so are reducing spheres there, apply a classic innermost disk argument to show that there is a sequence of reducing spheres in $C$, each disjoint from the next, beginning with $K(Z)$ and ending with $K_C$.

Finally, suppose $i < j$ and induct on $j - i$. It is again a classic innermost disk argument that there is a reducing sphere $K$ in $C$ that is disjoint from decomposing disks $D$ in $C_i \xrightarrow{D} C_{i+1}$ and so a sequence of reducing spheres for $C''$ beginning with $K(Z)$ and ending with $K$. Then we may as well substitute $K$ for $K(Z)$, and note that $K$ survives
intact as a sphere in a chamber $\hat{C}$ of $C_{i+1}$. If $\hat{K}$ is a reducing sphere for $\hat{C}$, this completes the inductive step, and we are done.

It is possible that $\hat{K}$ is not a reducing sphere for $\hat{C}$, though. For example, the ball that $\hat{K}$ bounds in $M$ could have contained a handlebody chamber adjacent to $C''$ in $C_i$, and the handlebody chamber disappears in $C_{i+1}$ because it is decomposed by $D$ into a goneball. The bubble in $T$ that it bounds then already appears as a bubble in the induced Heegaard splitting of $C_{i+1}$. In this case, simply make the bubble very small in $\hat{C}$, avoiding all further decomposing disks, until the bubble arrives intact in $C_j$ disjoint from the sphere $K$. There is ambiguity as to which chamber the bubble lies in as the decomposition sequence proceeds, but recall from the comments following Corollary 4.4 that how decomposing disks intersect the Heegaard surface are anyway ambiguous, partly by action of the eyeglass group, but also because bubbles may slide across decomposing disks. So just ensuring that later decomposing disks miss the bubble adds no new ambiguity. □

**Theorem 11.8.** For any Heegaard splitting $S^3 = A \cup_T B$ of $S^3$, the Goeritz group $G(S^3, T)$ is generated by the eyeglass group.

**Proof.** Let $g = \text{genus}(T)$ and suppose inductively that the theorem has been proven for all splittings of $S^3$ of lower genus. Per [FS1] we may as well take $g \geq 4$, though $g \geq 3$ is all that’s used. Given $\tau \in G(S^3, T)$, Proposition 11.5 provides certifying flagged chamber complex decomposition sequences $\vec{C}, \vec{C}'$ in $S^3$ so that $\vec{C} \sim \vec{C}'$, and for any certificate $Z$ issued by $\vec{C}$, $\tau(Z)$ is a certificate issued by $\vec{C}'$.

Choose one such certificate $Z$, defining (up to action of the eyeglass group) a reducing sphere $K$ for $T$, which naturally divides $T$ into two non-trivial bubbles. According to Proposition 11.7 there is then a sequence of non-trivial bubbles $b = b_1, \ldots, b_m = \tau(b)$ so that for each $1 \leq i < m$, the bubbles $b_i, b_{i+1}$ are disjoint, and $b$ is a bubble bounded by $K$.

Motivated by [FS1] Section 3] let $T_g$ be a fixed genus $g$ Heegaard splitting of $S^3$, with orthogonal meridian disks in $A$ and in $B$, labeled $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ respectively. (By orthogonal disks we mean $|a_i \cap b_j| = \delta_{ij}$.) Two homeomorphisms $h_1, h_2 : (S^3, T) \to (S^3, T_g)$ are **eyeglass equivalent** if there is $\alpha \in \mathcal{E}$ so that $h_2$ is isotopic to $h_1\alpha$.

**Claim 1:** Under our inductive assumption, a non-trivial bubble $b$ in $T$ defines an eyeglass equivalence class of homeomorphisms $(S^3, T) \to (S^3, T_g)$.

Let $a \subset A$ be a **primitive** meridian disk, meaning that there is a meridian disk $b \subset B$ so that $|a \cap b| = 1$. It is shown in [FS1]. Corollary
3.6] that, under our inductive assumption, a choice of such a meridian disk \( a \) determines a homeomorphism \((S^3, T, a) \to (S^3, T_g, a_1)\) that is unique up to eyeglass equivalence. (In [FS1] it was shown to be true up to Powell equivalence, but this stricter outcome relied on the stricter inductive assumption that the Powell conjecture was known for splittings of lower genus.)

Let \((b, T_b)\) denote the induced Heegaard splitting in the bubble, of genus \( \leq g - 1 \). Suppose \( a, a' \) are primitive meridians for \( A \) lying in \( b \). There is a homeomorphism \((b, T_b) \to (b, T_b)\) that carries \( a \) to \( a' \); by inductive assumption the homeomorphism is in the eyeglass group. So a choice of any primitive meridian for \( A \) in \( b \) determines the same eyeglass equivalence class of homeomorphisms \((S^3, T) \to (S^3, T_g)\), verifying the claim.

**Claim 2:** Suppose \( b \) and \( b' \) are disjoint bubbles for \( T \) and \( h, h' : (S^3, T) \to (S^3, T_g) \) are homeomorphisms determined by the bubbles, as described in Claim 1. Then \( h, h' \) are eyeglass equivalent.

Pick primitive meridian disks \( a, a' \) in \( b \) and \( b' \) respectively. Consider a homeomorphism \( \hat{h} : (S^3, T) \to (S^3, T_g) \) so that \( \hat{h}(a) = a_1, \hat{h}(a') = a_g \). By construction \( \hat{h} \) is in the eyeglass equivalence class defined by \( b \); if we post-compose \( \hat{h} \) with the homeomorphism \( \sigma : (S^3, T_g) \to (S^3, T_g) \) that exchanges the monobubbles \((a_1, b_1)\) and \((a_g, b_g)\) we get a homeomorphism in the eyeglass equivalence class defined by \( b' \). But a monobubble exchange is in the eyeglass group, so the two are eyeglass equivalent, verifying the claim.

Having now verified the claims, select a primitive \( a \) from the bubble \( b \) and consider the homeomorphisms \( h : (S^3, T, a) \to (S^3, T_g, a_1) \) and \( h\tau^{-1} : (S^3, T, \tau(a)) \to (S^3, T_g, a_1) \). The former is a homeomorphism determined by \( b \) and the latter by \( \tau(b) \). Following the claims, the sequence of bubbles \( b = b_1, ..., b_m = \tau(b) \) shows that the two are eyeglass equivalent. Hence \( \tau \) must be in the eyeglass group. \( \square \)

**12. Eyeglass twists are stably Powell**

We briefly recall the conventions and definitions from [FS1], filling out and reviewing the conventions introduced in the proof of Theorem 11.8. Let \( T_g \subset S^3 \) be the standard genus \( g \) Heegaard surface in \( S^3 \), dividing \( S^3 \) into the genus \( g \) handlebodies \( A \) and \( B \). Let \( \{a_1, ..., a_g\} \) and \( \{b_1, ..., b_g\} \) be orthogonal sets of disks in \( A \) and \( B \) respectively. We can view each pair \( \partial a_i, \partial b_i \) as respectively the meridian and longitude of one of the standard summands of \( T_g \). Let \( \{c_1, ..., c_{g-1}\} \) be the disjoint separating circles on \( T_g \) shown in [FS1, Figure 2], with each \( c_i \) separating the first \( i \) standard summands from the last \( g - i \) standard
summands. Note that each \( c_i \) bounds a disk in both \( A \) and \( B \) and so defines a reducing sphere \( S_i \) for \( T_g \).

**Definition 12.1.** Any finite composition of Powell generators, illustrated in [FS1] Fig. 1, will be called a Powell move.

**Proposition 12.2.** Suppose \( \tau \) is an eyeglass twist whose frame \( \eta \) is disjoint from \( c_1 \). Then \( \tau \) is a Powell move.

**Proof.** Let \( \{\ell_a, \ell_b, v\} \) be the frame of \( \eta \). Let \( u \) be an arc such that

- the ends of \( u \) lie on \( a_1 \) near the point \( a_1 \cap b_1 \) and on \( \ell_a \) near the point \( \ell_a \cap v \),
- \( u \) crosses \( c_1 \) once and
- \( u \) is otherwise disjoint from \( a_1, b_1 \), and \( \eta \).

See Figure 29.

![Figure 29.](image)

Let \( \eta' \) be the eyeglass twist whose frame is \( \{a_1, \ell_b, u \cup v\} \). We know from [FS1] Lemma 3.4 that an eyeglass twist \( \tau' \) along \( \eta' \) is a Powell move. Let \( \ell'_a \) be the band sum of \( \ell_a \) and \( a_1 \) along \( u \) and observe, by watching the motion of \( \ell_b \), that the composition \( \tau \tau' \) is an eyeglass twist \( \tau_+ \) whose frame \( \eta_+ \) is \( \{\ell'_a, \ell_b, v\} \).

Now let \( \eta'_1 \) be the eyeglass whose frame is \( \{\ell_a, b_1, u\} \) and \( \tau_1 \) be the eyeglass twist along \( \eta'_1 \). See Figure 30. Again, from [FS1] Lemma 3.4, \( \tau_1 \) is a Powell move. Further observe that \( \tau_1^{-1}(\eta_+) \) is a frame \( \eta'_+ \) with lenses \( a_1 \) and \( \ell_b \) and a bridge that intersects \( c_1 \) in a single point, so again an eyeglass twist \( \tau'_+ \) along \( \eta'_+ \) is a Powell move. See Figure 31.

Since \( \tau_1^{-1}(\eta_+) = \eta'_+ \) it follows that \( \tau_+ = \tau_1 \tau'_+ \tau_1^{-1} \). As a composition of Powell moves, \( \tau_+ \) is a Powell move. But we have earlier shown that \( \tau = \tau_+ \tau'^{-1} \) so, as a composition of Powell moves, \( \tau \) is a Powell move. \( \square \)
Corollary 12.3. Suppose $\tau \in G(S, T)$ and $\tau : (S^3, T) \rightarrow (S^3, T)$ leaves the curve $c_1$ invariant. Then $\tau$ is a Powell move.

References

[Bo] F Bonahon, Cobordism of automorphisms of surfaces, Ann. Sci. École Norm. Sup.(4) 16 1983) 237–270
[CG] A. Casson and C. Gordon, Reducing Heegaard splittings, Topology and its applications, 27 (1987) 275-283.
[FS1] M. Freedman and M. Scharlemann, Powell moves and the Goeritz group, Arxiv preprint 1804.05909
[FS2] M. Freedman and M. Scharlemann, Uniqueness in Haken’s Theorem, Arxiv preprint 2004.07385
[Ha] A Hatcher, Homeomorphisms of sufficiently large $P^2$-irreducible 3-manifolds, Topology 15 (1976), 343–347.
[JM] J. Johnson and D. McCullough, The space of Heegaard splittings J. Reine Angew. Math 679 (2013) 155-179
[La] M. Lackenby, An algorithm to determine the Heegaard genus of simple 3-manifolds with nonempty boundary Alg. Geom. Top. 8 (2008) 911-934.
[Po] J. Powell, Homeomorphisms of $S^3$ leaving a Heegaard surface invariant, Trans. Amer. Math. Soc. 257 (1980) 193–216.
[Sc1] M. Scharlemann, A Strong Haken’s Theorem , Arxiv preprint 2003.08523
[Sc2] M. Scharlemann, One Powell generator is redundant, to appear Proc. Amer. Math. Soc., arxiv preprint 1908.00479.
[Wa] F. Waldhausen, Heegaard-Zerlegungen der 3-Sphäre, Topology 7 (1968) 195–203.
Index

Aligned disk set, 8
Balanced sphere, 32
Blank bullseye, 18
Bubble, 9
Bubble exchange, 10
Bubble move, 9
Bullseye, 18
Certify; certificate, 27
Chamber complex, 3
Decomposition of chamber complex, 6
Disk set, 3
Disky, 8
Empty chamber, 13
Eyeglass, 10
Eyeglass group, 11
Eyeglass twist, 10
Flag rubric, 15
Flagged chamber complex, 13
Flip, 10
Goneball, 6
Guiding sphere, 29
Heegaard split chamber complex, 8
interacting flagged chamber complex decompositions, 28
Occupied chamber, 13
Powell group, 11
Remnant of chamber, 6
Scars, 3
Tiny chamber complex, 5

Martin Scharlemann, Mathematics Department, University of California, Santa Barbara, CA 93106-3080 USA
Email address: mgscharl@math.ucsb.edu