A nonextensive view of the stellar braking indices

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Abstract – The present work is based on the effects of the magnetic braking for the angular-momentum loss evolution and, consequently, the relationship between stellar rotation and age. In general, this loss rate denoted by dJ/dt depends on the angular velocity Ω in the form dJ/dt ∝ Ω^q, where q is a parameter from nonextensive statistical mechanics. In the context of the stellar rotation, this parameter is directly related to the braking index. For q equal to unity, the scenario of the saturation of the magnetic field is recovered. Such an approach was proposed and investigated by de Freitas and De Medeiros for unsaturated field stars. We propose a new nonextensive approach for the stellar rotational evolution based on the Reiners and Mohanty model. We developed a nonextensive version of Reiners and Mohanty torque, and compare it with the model proposed in de Freitas and De Medeiros, by using a sample of velocity v sin i for ~16000 field F- and G- stars. As a result, we show that the Kawaler and Reiners-Mohanty models exhibit strong discrepancies in relation to the domain of validity of the entropic index q. These discrepancies are mainly due to sensitivity on the stellar radius. Our results also showed that the modified Kawaler prescription is consistent in a wider mass ranges, while the Reiners and Mohanty model is restricted to masses less than from G6 stars.

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Introduction. – It is generally accepted that the magnetic braking is the fundamental mechanism responsible for angular-momentum loss through magnetic stellar winds in several classes of stars, such as main-sequence field stars and cluster stars. The magnetic braking was initially suggested by Schatzman [1] in 1962. Later, Kraft [2] showed that the behavior of mean rotation velocity of low-mass main-sequence stars below 1.5M⊙ is preferentially due to magnetic wind. As mentioned by Kawaler [3], main-sequence stars (hereafter MS) with mass less than 1.5M⊙ contain their deep surface convection zones which support magnetic field as well as acoustically controlled stellar winds.

Skumanich’s [4] pioneering work showed that the angular-momentum loss of G-type main-sequence stars for Hyades and Pleiades obeys the relationship dJ/dt ∝ Ω^3, where Ω denotes the angular velocity. Thus, the projected rotation velocity v sin i decreasing roughly with t^{-1/2}, is valid only within the equatorial plane. Here, t is the stellar age and the exponent −1/2 is related to the exponent from dJ/dt by the simple relationship t^{-1/2} = t^1/(1−3), where dJ/dt is the angular-momentum loss rate. In addition, unsaturated G stars are in agreement with this relationship. As quoted by Schrijver and Zwaan [5], the empirical Skumanich relationship, revisited by several authors, has stood the test of time, albeit the equations fitted to the data do differ somewhat from author to author. In contrast, Chaboyer et al. [6] elaborated a general parametrization developed by Kawaler [3] and showed that stars rotating substantially faster are subject to magnetic-field saturation at a critical rotation rate ωsat. Krishnamurthi et al. [7] proposed the inclusion of a Rossby scaling at saturation velocity for stars more massive than 0.5M⊙. However, Sills, Pinsonneault and Terndrup [8] argue that the Rossby number is inadequate for stars with mass less than 0.6M⊙.

Recently, Reiners and Mohanty [9] proposed a new braking approach based on the strong dependence on the

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stellar radius which arises from the definition of the surface magnetic strength \((B_0 \propto \Omega^3)\). In contrast, Kawaler [3] assumes that rotation is related to the magnetic-field strength \((B_0 R^2 \propto \Omega^3)\) reducing the dependence on the radius. According to [9], the factor of \(R^2\) implies deep changes in the angular-momentum loss law. Another important difference between the modified Kawaler and Reiners-Mohanty torques is in the saturation threshold \(\omega_{\text{sat}}\). Reiners and Mohanty [9] assume that \(\omega_{\text{sat}}\) does not depend on the mass and it only affects the torque in unsaturated stars. Their work showed that the angular-momentum loss law for low-mass stars obeys the relationship \(dJ/dt \propto \Omega^5\). For this torque, the rotational velocity decreases with stellar age according to \(t^{-1/4}\).

More recently, de Freitas and De Medeiros [10] revisited the modified Kawaler parametrization proposed by Chaboyer et al. [6] in the light of the nonextensive statistical mechanics [11]. This new statistical theory was introduced by Tsallis [11], and further developed by Curado and Tsallis [12] and Tsallis, Mendes and Plastino [13], with the goal of extending the regime of the applicability of Boltzmann-Gibbs (B-G) statistical mechanics that at present just plainly fail when applied in out-of-equilibrium systems. de Freitas and De Medeiros [10] analyzed the rotational evolution of unsaturated F- and G-type main-sequence stars in the solar neighborhood limited in age and mass. They considered the entropic index \(q\) from the Tsallis formalism as a parameter describing the magnetic braking level, namely the “braking index”. They also associated such a parameter to the exponent of the dynamo theory \((a)\) and to the magnetic-field topology \((N)\) by writing \(q = 1 + 4aN/3\). As a result, they showed that the saturated regime can be recovered in the nonextensive context at the limit \(q \rightarrow 1\). This limit is particularly important because it represents the thermodynamic equilibrium valid in the Boltzmannian regime. Indeed, the torque in the nonextensive version is given by \(dJ/dt \propto \Omega^5\) revealing that the rotational velocity of F- and G-type main-sequence stars decreases with age according to \(t^{3(1-\alpha)}\). The values of \(q\) obtained by de Freitas and De Medeiros [10] point out that this parameter is strongly stellar-mass dependent.

Some applications have been made to test the nonextensive approach proposed by de Freitas and De Medeiros [10], among them Silva et al. [14] and de Freitas et al. [15] using this model for open cluster stars. In Silva et al. [14], the authors define the index \(q\) extracted from the modified Kawaler parametrization as \(q_K\), where the subscript \(K\) is an abbreviation for Kawaler. In this scenario, Silva et al. [14] suggest that the \(q_K\) is a possible parameter that control the degree of anti-correlation between \(q\) from empirical rotational distribution and the cluster age \(t\) given by \(q \approx q_0(1 - \Delta t/q_K), \text{ where } \Delta t = t - t_0\) is the difference between the logarithms of the cluster ages, and \(q_0\) is the index \(q\) for the cluster with age \(t_0\). In general, this behavior indicates that stars lose the memory of past angular-momentum history when their rotational distribution becomes extensive, i.e., \(q = 1\). However, we wonder if this result set prevails when invoking another parametrization model of the angular-momentum loss, in our specific case, the Reiners and Mohanty torque.

Epstein and Pinsonneault [16] compare the prescriptions for the strength of the stellar wind of the modified Kawaler and Reiners and Mohanty torque with and without the effect of circumstellar disk for stars with mass between 0.5 and 1.0\(M_\odot\). They conclude that the largest differences between two prescriptions appear among low-mass stars. As a result, they claim that the Reiners and Mohanty torque preserves rapid rotators of low-mass stars. This effect is related to the Reiners and Mohanty torque, which predicts a slower spin-down than Kawaler does for the first few Myr. Epstein and Pinsonneault [16] still claim that the Kawaler parametrization is indistinguishable from the Skumanich law beyond the zero age main sequence (ZAMS). Indeed, this result is expected because the values adopted for \(a\) and \(N\) imply that angular velocity decays with time asymptotically as \(\Omega \sim t^{-1/2}\). In contrast, for the Reiners and Mohanty torque using the same values of \(a = 1.5\) and \(N = 2\), the spin-down rate decreases to a weaker power law, i.e., \(\Omega \sim t^{-1/4}\). In fact, the Kawaler (standard) framework of the angular-momentum loss rate described above is limited to the stars later than late-F stars. For stars earlier than early-F a simple scaling of the solar torque by \(\omega^3\) does not explain the fast rotators in young stars; in this case, it is necessary a saturation in the angular-momentum loss. In both loss laws, the rotation rate from saturated stars diminishes exponentially to a critical saturation velocity. In general, according to de Freitas and De Medeiros [10], as the exponent of \(\omega\) is sensitive to stellar mass, we obtain fluctuations of this exponent in a mass range narrower than those we defined above.

In this study we compare the Kawaler and Reiners-Mohanty torques in the context of the Tsallis’ nonextensive statistical mechanics. In the next section, we revisit and generalize the parametric Reiners and Mohanty model for angular-momentum loss by magnetic stellar wind, and compares these models. In the last section, we present our conclusions.

**Generalized Reiners and Mohanty torque.** Reiners and Mohanty [9] propose a new approach for the stellar magnetic braking scenario. Their approach considers two major deviations from the modified Kawaler prescription (see [10] for a review). As mentioned by Epstein and Pinsonneault [16], first Reiners and Mohanty [9] argue that rotation is related to the magnetic-field strength rather than to the surface magnetic flux as adopted by Kawaler in [6]. This dependence on \(R^2\) modifies the structure of the angular-momentum loss law, indicating that the Kawaler model presents a weak dependence on the stellar radius. Second, these authors assume that the saturation angular velocity \(\omega_{\text{sat}}\) does not depend on the stellar mass. In this case, \(\omega_{\text{sat}}\) is insensitive to the structural differences in the stellar interior. In this context, Reiners and
Mohanty [9] restrict the rate of the angular-momentum loss to the radial fields and a dynamo law \( a = 1.5 \) for unsaturated regime, as given by the following equations:

\[
\frac{dJ}{dt} = -C \left[ \Omega \left( \frac{R^{16}}{M^2} \right)^{1/3} \right]
\]  

(1)

for \( \Omega \geq \omega_{sat} \). For \( \Omega < \omega_{sat} \), we have

\[
\frac{dJ}{dt} = -C' \left[ \left( \frac{\Omega}{\omega_{sat}} \right)^4 \Omega \left( \frac{R^{16}}{M^2} \right)^{1/3} \right]
\]  

(2)

with \( C = \frac{2}{3} \left( \frac{R_{sat}^8}{G M \Omega K_{V}} \right)^{1/3} \).

Here, \( G \) is the gravitational constant, \( \omega_{sat} \) is the threshold angular velocity beyond which the saturation occurs, \( B_{sat} \) denotes the saturation field strength, \( K_{V} \) is a calibration constant and \( R, M \) and \( \dot{M} \) represent the stellar radius, mass and loss-mass rate in units of \( 10^{-14} M_{\odot} \) yr\(^{-1} \), respectively.

For a future comparison with the parameters obtained in the nonextensive Kawaler model given in de Freitas and De Medeiros [10], we eliminated the restrictions in \( a \) and in the magnetic-field geometry \( N \) adopted by Reiners and Mohanty [9]. After a rather tedious algebraic manipulation, the generalized Reiners and Mohanty torque, for the different saturation regimes, where the parameters \( a \) and \( N \) are free, is given by

\[
\frac{dJ}{dt} = -C' \left[ \Omega \left( \frac{R^{16-8N}}{M^2} \right)^{1/3} \right]
\]  

(3)

for \( \Omega \geq \omega_{sat} \). In this regime, the equation above does not depend on the parameter \( a \). For \( \Omega < \omega_{sat} \), we have

\[
\frac{dJ}{dt} = -C' \left[ \left( \frac{\Omega}{\omega_{sat}} \right)^{8a/(3-4N)} \Omega \left( \frac{R^{16-8N}}{M^2} \right)^{1/3} \right]
\]  

(4)

with \( C' = \left[ \left( \frac{3}{4} \right)^a \right]^{1/(3-4N)} \). If \( a = 1.5 \) and \( N = 0 \) (later we will replace \( N = 2 \)), we recover the Reiners and Mohanty torque.

In the present study, we assume that since the moment of inertia \( I \) and stellar radius \( R \) change slowly during the main-sequence evolution, the angular-momentum loss law can be simplified by considering the limit of \( dJ \)/\( dt \) for constants \( I \) and \( R \), that is, this loss law is fully specified by the rotational velocity, while the star is braked by the stellar wind [17]. We also consider that in the absence of angular-momentum loss, the equatorial rotational velocity \( v \) can be determined by the simple condition of angular-momentum conservation, denoted by

\[
J = \left( \frac{I}{R} \right)_{const.} v;
\]  

(5)

furthermore, we assume that the stars spin down as a solid body.

Two pairs of equations were combined to obtain the time dependence of \( v \). First, from eqs. (3) and the derivative of (5), we have

\[
v(t) = v_0 \exp \left[ -\frac{(t-t_0)}{\tau_1} \right] \quad (t_0 \leq t < t_{sat})
\]

(6)

with

\[
\tau_1 = \frac{C'}{T} \left( \frac{R^{16-8N}}{M^2} \right)^{1/3} \]  

(7)

for the saturated domain.

On the other hand, by combining eqs. (4) and the derivative of (5), we obtain the equation

\[
v(t) = v_{sat} \left[ 1 + \frac{(t-t_{sat})}{\tau} \right]^{-\left(3-4N\right)/8a} \quad (t \geq t_{sat})
\]

(8)

with

\[
\tau = \left[ \frac{8a}{3-4N} \frac{C'}{T} \left( \frac{R^{16-8N}}{M^2} \right)^{1/3} \right]^{-1}
\]

(9)

In this respect, the \( \tau_1 \) and \( \tau \) are the MS spin-down timescales in the saturated and unsaturated regimes, respectively. The other parameters, \( t_0, v_0 \) and \( t_{sat} \), are the age at which the star arrives at the MS, the rotational velocity at that time, and subsequent age at which the star slows down to the values below the critical velocity \( v_{sat} \), respectively [9]. Time \( t_{sat} \) can be determined by setting \( v(t) = v_{sat} \) in eq. (6). Thus, we have

\[
t_{sat} = t_0 + \tau_1 \ln \left( \frac{v_0}{v_{sat}} \right).
\]

(10)

Equation (6) reveals that in the saturated regime the star slows down exponentially from time \( t_0 \) up to velocity \( v_{sat} \) in time \( t_{sat} \). On the other hand, eq. (8) shows that from this time onward, the spin-down rate decreases to a power law given by \(-\left(3-4N\right)/8a\). Thus, the star remains within a factor of a few \( v_{sat} \) for the remainder of its MS lifetime.

**Nonextensive approach for stellar rotation-age relation for the unsaturated regime.** - By applying Tsallis statistics [18], we can assume that eq. (6) follows a simple linear differential equation in the form

\[
\frac{d}{dt} \left( \frac{v}{v_0} \right) = -\lambda_1 \left( \frac{v}{v_0} \right),
\]

(11)

from which the solution is given by

\[
v(t) = v_0 \exp \left[ -\lambda_1 (t-t_0) \right].
\]

(12)

According to Lyra et al. [19], in contrast to the exponential behavior presented in eq. (12), eq. (11) must be replaced with a slower power law at critical points where long correlations develop. In this context, a similar expression can be created to characterize the possible behavior of
the stellar-rotation distribution in an unsaturated regime. As proposed by de Freitas and De Medeiros \cite{10}, the spin-down law of a group of the stars limited by mass is usually given in the generalized form

$$\frac{d}{dt} \left( \frac{v}{v_{\text{sat}}} \right) = -\lambda_q \left( \frac{v}{v_{\text{sat}}} \right)^q \quad (\lambda_q \geq 0; q \geq 1),$$  \hspace{1cm} (13)

for $q = 3$, the spin down is determined by a torque generated by magnetically controlled stellar winds.

Integrating the equation above, we have

$$v(t) = v_{\text{sat}} \left[ 1 + (q - 1)\lambda_q (t - t_{\text{sat}}) \right]^{1/q},$$  \hspace{1cm} (14)

where $[1 + (1 - q)x]^{1/(1-q)} \equiv \exp_q(x)$ is known in a nonextensive scenario as $q$-exponential (see \cite{20} for a review of the $q$-calculus). In the present context, $\lambda_q$ indicates the braking strength.

Thus, in the nonextensive scenario, eq. (8) is well described as a nonlinear differential equation in the form (13) with solution (14). When $q = 1$, eq. (12) is recovered, indicating that a system governed by a saturated regime is in thermodynamic equilibrium. By contrast, when $q$ differs from unity, a system controlled by unsaturation is out of equilibrium.

In agreement with our unsaturated-regime model and replacing $N$ by $N - 2$, we find that the “braking index” $q_{RM}$ can be described by a pair $(a, N)$ given by

$$q_{RM} = 1 + \frac{8a}{4N - 5},$$  \hspace{1cm} (15)

whereas for the saturated regime, $q_{RM} = 1$ due to the usual exponential function. As shown in table 1, the parameters $a$ and $N$ are a function of spectral type, \textit{i.e.}, the braking index $q$ is related to the stellar mass and evolution of the stellar magnetic field. In fig. 1, we have the plot of index $q$ against the field topology $N$. Straight lines represent the Kawaler entropic index given by the identity $q_K \equiv 1 + 4aN/3$, while the curves denote the Reiners and Mohanty prescription given by the term (15). For such a model, the plot shows lines and curves as an iso-$a$ scenario (specifically $a = 1$, $a = 1.5$ and $a = 2$). The vertical dash-dotted line shows that there is only one value for $N$ for which $q_K$ and $q_{RM}$ are equal, \textit{i.e.}, $N = 2$. From the referred figure, the point $A$ represents the possible lowest value of $q_{RM}$. The point $B$ denotes the exponent adopted by Reiners and Mohanty [\cite{9}, \textit{i.e.}, $q_{RM} = 5$. The point $C$ is the maximum possible value of $q$ among the models. The plot shows also that the Skumanich law cannot be represented on the generalized Reiners and Mohanty torque.

In general, the $q$-parameter is related to the degree of nonextensivity that emerges within the thermodynamic formalism proposed by Tsallis in [\cite{11}]. As revealed in eq. (15), $q_{RM}$ is a function of the magnetic-field topology.

Table 1: Best parameter values of our unsaturated model using eq. (14). The values of $N$ using eq. (15) are also shown.

| Stars $(M/M_\odot)$ | $q_{RM}$ | $N_{a=2}; N_{a=1}$ | $a_{N=0}; a_{N=2}$ |
|---------------------|----------|----------------------|----------------------|
| F0–F5               | 1.36     | 1.80 ± 0.1           | 6.25; 3.75           |
| F6–F9               | 1.22     | 1.96 ± 0.2           | 5.42; 3.33           |
| G0–G5               | 1.11     | 2.18 ± 0.2           | 4.64; 2.95           |
| G6–G9               | 0.98     | 4.38 ± 0.6           | 2.43; 1.84           |
| All F               | 1.91 ± 0.1 | 5.65; 3.45        | 0.57; 0.34           |
| All G               | 2.18 ± 0.4 | 4.64; 2.95        | 0.74; 0.44           |

Fig. 1: Plot for $q$ from the model against the field topology $N$. Straight lines represent the Kawaler entropic index, while curves denote the Reiners and Mohanty one. The vertical dash-dotted line shows that there exists only one value for $N$ for which the $q$’s are equal, \textit{i.e.}, when $N = 2$. The point $A$ represents the lowest value of $q_{RM}$. The point $B$ denotes the Reiners and Mohanty [\cite{9}] exponent, \textit{i.e.}, $t^{-1/4}$. Point $C$ is the maximum possible value of $q$ among the models. The plot shows also that the Skumanich law cannot be represented on the generalized Reiners and Mohanty torque.
and the dynamo law $a$, which depend on the stellar evolution. According to Kawaler [3] and de Freitas and De Medeiros [10], small $N$ values result in a weak wind that acts on the MS phase of evolution, while winds with large $N$ values remove significant amounts of angular momentum early in the pre-main sequence (PMS). In this phase, for a given mass, the maximum rotational velocity $v$ decreases as $N$ increases. This result is important because, within a thermostatistical framework in which eq. (8) naturally emerges, and for a given value of $a$ and $N$, we obtain the scale laws found in the literature, such as those proposed by Skumanich [4], Soderblom et al. [21] and Reiners and Mohanty [9].

Conclusions. – In summary, our study presents a new approach to the angular-momentum loss law found in the literature. The present analysis shows that the generalized torques of Kawaler and Reiners-Mohanty are clearly antagonistic, except for $N = 2$. Table 1 demonstrates that the modified Reiners and Mohanty model is valid only for spectral types beyond G6. According to fig. 1, the lowest value of $q_{RM}$ is 3.67. As a result, the Skumanich law cannot be represented on the generalized Reiners and Mohanty relationship, i.e., this model does not explain the rotational behavior of solar-type stars. This limitation is mainly due to the sensitivity on the stellar radius given by term $R^{16/3}$, i.e., the angular-momentum loss rate would have become entirely dependent on the stellar radius. In the Kawaler model, this dependence is given by the maximum term $R^2$, where $\frac{dJ}{dt}$ depends entirely on the stellar velocity for unsaturated stars. However, our results reveal which Reiners and Mohanty model is consistent for the low-mass stars, specifically, M-type stars. In other words, the Reiners and Mohanty model is a good approximation for fully convective stars.

The entropic index $q$ can be interpreted as an indicator of the memory of the system or, in our case, the memory of the primordial angular momentum (see de Freitas and De Medeiros [10] and Silva et al. [14] for further details on the link between $q$ and the memory of the system). As $q$ increases from F to M, very-low-mass stars maintain this memory. In fact, the memory grows at same scale of $q$. In contrast, because of the much shorter interaction of the young star with its circumstellar disc, higher-mass stars do not maintain the memory of the initial angular momentum for a long time. According to de Freitas and De Medeiros [10] and present results, both nonextensive models show that the Kawaler framework is compatible with a wider mass range, while the Reiners and Mohanty model is restricted to low-mass stars (less than G6 masses). This result suggests that a decrease in the mass would lead to a weaker power law than the Skumanich one. Finally, our results point out that a strong dependence of the angular-momentum loss rate on the stellar radius is limited to very-low-mass stars encountering discrepancies when applied to F and G stars.

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