Comment on "Conditional Decoupling of Quantum Information"

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Abstract

Berta et al [Phys. Rev. Lett., 121, 040504 (2018)] claim that their result provides a conceptually new extension of the decoupling approach to quantum information theory. We provide an alternate proof using the plain-vanilla decoupling approach for the achievable rates of their main result and hence, their claim is unwarranted, and the title can be misleading when taken in conjunction with the claim.

Berta et al [1] analyze two protocols namely deconstruction and conditional erasure, differing only in error constraints, closely related to the one in [2], for which the decoupling approach works [3], and make the following claims:

• Our models for deconstruction and conditional erasure extend the decoupling approach to quantum information theory.

• Our result can alternatively be read as a conditional decoupling theorem and hence provides a conceptually new extension of the decoupling approach to quantum information theory.

These claims are made without qualifications and one could at least weakly interpret them as saying that their result is not merely to show that an approach powered by the Quantum State Redistribution (QSR) [4, 5] works for their protocols, moderately interpret them as saying that their result is beyond what a mere decoupling approach can provide, and strongly interpret them as saying that there is something fundamental about "conditional decoupling". The strong interpretation looks quite plausible considering the audacity and scope of the claims when taken along with the title.

Clearly, the possibility of a "conceptually new extension" of a method by considering two example protocols arises only if the method cannot address the protocols. There

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is no proof provided in [1] that it is impossible to analyze their protocols by using the decoupling approach. Furthermore, the exploitation of the QSR in [1] for their achievable rates is by itself a ratification of the decoupling approach since it can be used for QSR as shown in [3] rendering the claims inscrutable since [3] pre-dates [1].

The authors of [1] harp on the subtle differences between the two protocols. Notice these sentences from [1]:

*We would like to emphasize again that deconstruction and conditional erasure protocols are more delicate than standard decoupling, the latter sometimes described as having the relatively indiscriminate goal of destruction [38]. That is, a straightforward application of the decoupling method is too blunt of a tool to apply in a state deconstruction protocol.*

Ultimately, the true test of any claims or insights is the proof itself, and in its absence, one cannot fathom where this lack of faith in decoupling and a leap of faith in something beyond decoupling are coming from.

The definition of deconstruction requires the notion of the optimal permissible recovery map as solution to the problem: for a tripartite quantum state $\rho^{ABR}$, find a map $\mathcal{R}^{B\rightarrow AB}$ such that $\mathcal{R}(\rho^{BR})$ is closest to $\rho^{ABR}$.

Even if the benefit of doubt is given to the authors in the second sentence above on the deconstruction protocol that the recovery map one gets by leveraging the conditional erasure protocol is trivial giving rise to a tensor product state, the fact remains that this trivial map forms the basis in [1] to show the achievable rates only for the conditional erasure and the converse takes care of the fact that it is also optimal for the deconstruction. If the authors had shown that the optimal rates for the deconstruction and the conditional erasure are different perhaps due to certain vagaries of the optimal recovery map or, even less, that the trivial map one gets by the leveraging the conditional erasure was insufficient for providing the optimal rate for the deconstruction and this can only be rectified by something beyond the "blunt" decoupling, then the second sentence may be valid. (Clearly, for the vagaries of the optimal recovery maps to kick in, the problem statement would have to change, but we digress.) No such thing was proved and a slight issue here is that the aforesaid optimal rates being exactly the same is the main result of [1]!

In general, mankind knows little to nothing about the optimal recovery maps and in this realm of ungloomy ignorance where we somehow survive, to make statements that seemingly pertain to the other unknown realm to pulverize a method when the proofs, including those in [1], elucidate no such possibility is verbal voodoo, quantum or not!

We provide the achievable rates for the protocols by the good-old decoupling approach. This implies that their aforesaid claims, no matter which of the weak to strong interpretations one clings to, and statements are uncalled-for.

Perhaps there is an exciting new concept of "conditional decoupling" out there waiting to be discovered, but [1] fails to deliver!

The proof is similar to the one in Section 11 in [3]. $|A|$ is the dimension of $A$, $\pi^A$ is the maximally mixed state on $A$, for $\rho^{AR}$, $H_\alpha(A|R)_\rho$ is the Rényi quantum conditional entropy of $A$ given $R$ and since there are several options, one could pick one’s favorite
— see [3], the von Neumann entropies for \( \alpha = 1 \) are denoted by \( H(A|R)_\rho \), for \( \rho^{ABR} \), the conditional mutual information between \( A \) and \( R \) given \( B \) is given by \( I(A; R|B)_\rho = H(A|B)_\rho - H(A|RB)_\rho \). \( || \cdot ||_1 \) is the trace norm, and \( \Xi(\varepsilon) \equiv \sqrt{\varepsilon(2 + \varepsilon + 2\sqrt{1+\varepsilon})} \) for \( \varepsilon \geq 0 \). For \( \Psi_{ABRE} \), pure, there are duality relations such that \( H_\alpha(A|RE)_\Psi = -H_\tilde{\alpha}(A|B)_\Psi \) and the relationships between \( \alpha \) and \( \tilde{\alpha} \) are provided in [6] and references therein. Define

\[
T_{\mathcal{W}}^{A\rightarrow B}(\sigma^A) \equiv \frac{|A|}{|B|} (W^{A\rightarrow B} \cdot \sigma^A),
\]

(1)

where \( W^{A\rightarrow B}, |A| \geq |B| \), is a full-rank partial isometry.

**Definition 1** (Deconstruction and Conditional erasure). Consider \( n \) copies of a tripartite state \( \rho^{ABR} \), with \( M \) Unitaries \( U_i, i = 1, ..., M \), applied over \( A^n B^n \) such that

\[
\Upsilon^{A^n B^n R^n} \equiv \frac{1}{M} \sum_{i=1}^{M} U_i \cdot [(\rho^{ABR})^\otimes n],
\]

(2)

A \((\rho, \text{error}, n)\) deconstruction protocol ensures

\[
\sup_{\mathcal{R}^{B^n\rightarrow A^n B^n}} \left\| \Upsilon^{A^n B^n R^n} - \mathcal{R}^{B^n\rightarrow A^n B^n} (\Upsilon^{B^n R^n}) \right\|_1 \leq \text{error} \quad \text{and} \quad \left\| \Upsilon^{B^n R^n} - (\rho^{BR})^\otimes n \right\|_1 \leq \text{error},
\]

(3)

where the supremum is taken over all the completely positive and trace preserving (cptp) recovery maps \( \mathcal{R}^{B^n\rightarrow A^n B^n} \).

A \((\rho, \text{error}, n)\) conditional erasure protocol ensures

\[
\left\| \Upsilon^{A^n B^n R^n} - \tau^A \otimes \Upsilon^{B^n R^n} \right\|_1 \leq \text{error} \quad \text{and} \quad \left\| \Upsilon^{B^n R^n} - (\rho^{BR})^\otimes n \right\|_1 \leq \text{error},
\]

(4)

where \( \tau^A \) is a density matrix and we make no apriori restrictions on the choice of \( \tau^A \).

The number \( (\log M)/n \) is called the rate of the protocol. A real number \( r \) is called an achievable rate if protocols exist for \( n \rightarrow \infty \) with rate approaching \( r \) and the error approaching 0.

**Theorem 1** (Berta et al., 2018 [1]). The smallest achievable rate for both the protocols is \( I(A : R|B)_\rho \).

We prove the following theorem.

**Theorem 2.** For any \( n \in \mathbb{N} \), there exist \((\rho, \text{error}, n)\) deconstruction and conditional erasure protocols such that for any \( \delta > 0, \alpha \in (1, 2] \) and \( |\Psi\rangle_{ABRE} \) a purification of \( \rho^{ABR} \),

\[
\frac{\log M}{n} = H_\alpha(A|B)_\rho - H_\alpha(A|BR)_\rho + (|E| + |B|) ||R|| \frac{\log(n+1)}{n} + \delta,
\]

(5)

and the error approaches 0 exponentially in \( n \).
Consider now the following Unitaries over $A^n$. For $M \leq |F|^2$, choose $M$ Heisenberg-Weyl Unitaries $V_i^F$, and let $\mathcal{V}_M^F$ be a ctp map given by

$$\mathcal{V}_M(\sigma^F) \equiv \frac{1}{M} \sum_{i=1}^{M} V_i^F \cdot \sigma^F.$$  \hfill (6)

Then, from Corollary 2 in [3], for any $\alpha \in (1, 2]$, there exists a Unitary $U$ over $A^n$ such that

$$\left\| \text{Tr}_F \circ \mathcal{T}_W \left[ U \cdot (\rho^{ABR})_{\otimes n} \right] - (\rho^{BR})_{\otimes n} \right\|_1$$

$$\leq 8 \exp \left\{ \frac{\alpha - 1}{2\alpha} \left[ |R| \log(n + 1) - nH_\alpha(A|RE)_\rho - \log |F| \right] \right\}$$

$$= 8 \exp \left\{ \frac{\alpha - 1}{2\alpha} \left[ |R| \log(n + 1) + nH_\alpha(A|BR)_\rho - \log |F| \right] \right\} \equiv \varepsilon_n, \hfill (7)$$

and

$$\left\| \mathcal{V}_M \circ \mathcal{T}_W \left[ U \cdot (\rho^{ABR})_{\otimes n} \right] - \pi^F \otimes (\rho^{BR})_{\otimes n} \right\|_1$$

$$\leq 8 \exp \left\{ \frac{\alpha - 1}{2\alpha} \left[ |R| \log(n + 1) - nH_\alpha(A|BR)_\rho - \log M + \log |F| \right] \right\} \equiv \vartheta_n, \hfill (8)$$

where in (8), we have also used Lemma 23 in [3]. From (7) and Lemma 31 in [3], we claim that there exists a Unitary $V_U^{A^nB^n}$ over $A^nB^n$ such that

$$\left\| W^\dagger \cdot \mathcal{T}_W \left[ U \cdot (\Psi^{ABRE})_{\otimes n} \right] - V_U \cdot (\Psi^{ABRE})_{\otimes n} \right\|_1 \leq \Xi(\varepsilon_n). \hfill (9)$$

Consider now the following Unitaries over $A^n$ constructed from $V_i^F$ as $V_i^{A^n} = W^\dagger \cdot V_i^F + (1^A - W^\dagger W)$. Note that $V_i^{A^n}W^\dagger = W^\dagger V_i^F$. We now claim that $V_i^{A^n}V_U$, $i = 1, ..., M$, are precisely the $M$ Unitaries we need. For

$$\Gamma^{A^nB^nR^n} \equiv \frac{1}{M} \sum_{i=1}^{M} (V_i^{A^n}V_U) \cdot (\rho^{ABR})_{\otimes n}, \hfill (10)$$

we have

$$\left\| \Gamma^{A^nB^nR^n} - (W^\dagger \cdot \pi^F) \otimes (\rho^{BR})_{\otimes n} \right\|_1$$

$$\leq \left\| \frac{1}{M} \sum_{i=1}^{M} (V_i^{A^n}V_U) \cdot (\rho^{ABR})_{\otimes n} - \frac{1}{M} \sum_{i=1}^{M} (V_i^{A^n}W^\dagger) \cdot \mathcal{T}_W \left[ U \cdot (\rho^{ABR})_{\otimes n} \right] \right\|_1 +$$

$$\left\| \frac{1}{M} \sum_{i=1}^{M} (V_i^{A^n}W^\dagger) \cdot \mathcal{T}_W \left[ U \cdot (\rho^{ABR})_{\otimes n} \right] - (W^\dagger \cdot \pi^F) \otimes (\rho^{BR})_{\otimes n} \right\|_1$$

$$\leq \frac{1}{M} \sum_{i=1}^{M} \left\| (V_i^{A^n}V_U) \cdot (\rho^{ABR})_{\otimes n} - (V_i^{A^n}W^\dagger) \cdot \mathcal{T}_W \left[ U \cdot (\rho^{ABR})_{\otimes n} \right] \right\|_1 +$$
where the first inequality follows from the triangle inequality, in the second inequality, the first term follows from the convexity of the trace norm and the second term follows by invoking $V_i^A W^\dagger = W^\dagger V_i^B$, in the third inequality, the first term follows by invoking the Unitary invariance and monotonicity of the trace norm and the second term from monotonicity, in the fourth inequality, the first term is upper bounded using (9) and the second term is upper bounded using (8).

Using monotonicity and (11), we get
\[
\left\| \Upsilon^{B^n R^n} - (\rho^{BR})^{\otimes n} \right\|_1 \leq \Xi(\varepsilon_n) + \vartheta_n,
\]
and using (11), (12), and triangle inequality, we get
\[
\left\| \Upsilon^{A^n B^n R^n} - (W^\dagger \cdot \pi^F) \otimes \Upsilon^{B^n R^n} \right\|_1 \leq 2 \Xi(\varepsilon_n) + 2\vartheta_n,
\]
which along with (12) proves the claim for the conditional erasure protocol. By substituting the recovery map (given in [1]) in (13) as
\[
R^{B^n \to A^n B^n}(\Upsilon^{B^n R^n}) = W^\dagger \cdot \pi^F \otimes \Upsilon^{B^n R^n},
\]
and from (12), the claim also follows for the deconstruction protocol.

\[\blacksquare\]

**Remarks:**

- Unlike the proof in [1], we do not need any ancilla.
- Unsurprisingly, with similar approach as [7, 8] that use QSR, one could instead derive lower bounds to $I(A; R|E)_\rho$ using (11) since for $V_{u}^{A^n B^n \to A^n B^n X}$ as the Stinespring dilation isometry mocking up the application of $M$ Unitaries over $A^n B^n$ with $|X| = M$, (11) implies the existence of a recovery operation $\mathcal{E}_{X E^n \to \tilde{A}^n E^n}$ such that
\[
\mathcal{E}_{X E^n \to \tilde{A}^n E^n} \circ \text{Tr}_{A^n B^n} \circ V_{u}^{A^n B^n \to A^n B^n X}[\psi_{ABR}^{\otimes n}] \approx \rho^{\tilde{A}^n R^n E^n}
\]
and noting that
\[
\text{Tr}_{A^n B^n X} \circ V_{u}^{A^n B^n \to A^n B^n X}[\psi_{ABR}^{\otimes n}] = (\rho^{RE})^{\otimes n}.
\]
- The converse can be proved by standard arguments such as continuity and hence, contains no surprises that go against this comment.
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