Delocalization and ergodicity of the Anderson model on Bethe lattices

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In this paper we review the state of the art on the delocalized non-ergodic regime of the Anderson model on Bethe lattices. We also present new results using Belief Propagation, which consists in solving the self-consistent recursion relations for the Green’s functions directly on a given sample. This allows us to numerically study very large system sizes and to directly access observables related to the eigenfunctions and energy level statistics, such as level compressibility and eigenstates correlation functions. In agreement with recent works, we establish the existence of a delocalized non-ergodic phase on Cayley trees. On random regular graphs instead our results indicate that ergodicity is recovered when the system size is larger than a cross-over scale $N_c(W)$, which diverges exponentially fast approaching the localization transition. This scale corresponds to the size at which the mean-level spacing becomes smaller than the Thouless energy $E_{\text{Th}}(W)$. Such energy scale, which vanishes exponentially fast approaching the localization transition, is the one below which ergodicity in the level statistics is restored in the thermodynamic limit. Remarkably, the behavior of random regular graphs below $N_c(W)$ coincides with the one found close to the root of loop-less infinite Cayley trees, \textit{i.e.} only above $N_c(W)$ the effects of loops emerge and random regular graphs behave differently from Cayley trees.

All in all, our results indicate that ergodicity is recovered in the thermodynamic limit on random regular graph. This notwithstanding, all observables probing volumes smaller than $N_c(W)$ and times smaller than $\hbar/E_{\text{Th}}(W)$ are expected to behave as if there were an intermediate phase. Given the very fast divergence of $N_c(W)$ and $\hbar/E_{\text{Th}}(W)$ these non-ergodic effects are very pronounced in a large region preceding the localization transition, and they can be related to the intermediate phase present on Cayley trees.

I. INTRODUCTION

After more than a half century, the subject of Anderson localization is still very much alive\textsuperscript{11} as proved by the recent observations of Anderson localization of atomic gases in one dimension\textsuperscript{2} and of classical sound elastic waves in three dimensions\textsuperscript{3}. On the theoretical side several questions remain open: Although there is by now a good understanding of the localization transition in low dimensional systems, culminating in a functional renormalization group analysis by a $2+\epsilon$ expansion\textsuperscript{4}, the behavior in high dimensions\textsuperscript{5}, in particular the existence of an upper critical dimension and the relationship with Bethe lattice analysis\textsuperscript{6}, is still an issue. Recently, there has been a renewal of interest on this problem because of its relationship with Many-Body localization (MBL)\textsuperscript{7}, a fascinating new kind of phase transition between a low temperature non-ergodic phase—a purely quantum glass—and a high temperature ergodic phase. This phenomenon has been argued to take place for several disordered isolated interacting quantum systems, in particular disordered electrons\textsuperscript{7}, and was also independently investigated in\textsuperscript{8} to explain the quantum ergodicity transition of complex molecules. MBL can be thought of as localization in the Fock space of Slater determinants, which play the role of lattice sites in a disordered Anderson tight-binding model. A paradigmatic representation of this transition\textsuperscript{7,11} is indeed (single-particle) Anderson localization on a very high dimensional lattice, which for spinless electrons consists in an $N$-dimensional hyper-cube (where $N \gg 1$ is the number of sites of the lattice system). Anderson localization on Cayley trees and Bethe lattices is a drastic simplification of this problem. It is very useful to obtain a qualitative understanding but neglect correlations between energies and rare loops.

Localization had an impact on several fields, in particular Random Matrices and Quantum Chaos. As a matter of fact, in the delocalized phase the level statistics is described by random matrix theory and generally corresponds to the Gaussian Orthogonal Ensemble (GOE), whereas instead in the localized phase is determined by Poisson statistics because wave-functions close in energy are exponentially localized on very distant sites and hence do not overlap; thus, contrary to the GOE case, there is no level-repulsion and eigen-energies are distributed similarly to random points thrown on a line.

The relationship with quantum chaos goes back to the Bohigas-Giannoni-Schmidt conjecture, which states that the level statistics of chaotic (or ergodic) systems is given by random matrix theory, whereas integrable systems instead are characterized by Poisson statistics\textsuperscript{12}. This result can be fully worked out and understood in the semi-classical limit\textsuperscript{13,14} for a quantum chaotic system, in the $\hbar \to 0$ limit, wave-functions at a given energy become uniformly spread over the micro-canonical hypersurface of the configuration space. They are fully delocalized as expected for an ergodic classical system that covers regions with same energy uniformly. Instead, quantum non-ergodic models, such as integrable systems, are characterized by Poisson statistics and localized wave-
functions. All those results support a general relationship between delocalization—GOE statistics—ergodicity (similarly between localization—Poisson statistics—lack of ergodicity).

However, recent numerical studies [15–20] of the Anderson model on a Random-Regular Graph (RRG)—a random lattice that has locally a tree-like structure but does not have a boundary, see below for a precise definition—seem to indicate the possibility of the existence of a novel intermediate delocalized but non-ergodic phase in a broad disorder range, as first suggested in [9]. Such phase should be characterized by multifractal eigenfunctions (with the fractal dimensions depending on the disorder strength), anomalous (sub-diffusive) transport along rare, ramified, paths, and, possibly, non-universal level statistics on a scale larger than the mean-level spacing (while the level statistics on the scale of the mean-level spacing is expected to be described by the GOE ensemble). The arguments in favour of this scenario rely mostly on numerical results obtained from Exact Diagonalization (ED) of large but finite samples [15–18] and on an analytic approximation scheme based on Replica Symmetry Breaking (RSB) and “inflationary population dynamics” developed ad hoc to deal with non-ergodic states [18–20].

The possibility of a multifractal delocalized phase in a disordered system is clearly very intriguing, especially due to its relationship with MBL. In fact, this scenario is explicitly realized in suitable models which possess critical states such as the Rosenzweig-Porter random matrix [21, 22] and the power-law random banded matrix [29] models, and also occurs in the tight-binding Anderson model on the (loop-less) Cayley tree, as recently shown in [21–25]. However, it appears to be in explicit conflict with the analytical predictions based on the supersymmetric approach for the Anderson model on sparse random graphs [26–27]. In fact the supersymmetric analysis indicates that the Inverse Participation Ratio (IPR) defined as $\gamma_{\rho} = \left( \sum_{i=1}^{N} |\langle i|m \rangle|^4 \right)$ (where $\langle i|m \rangle$ is the amplitude of the wave-function $|m\rangle$ on site $i$), scales as $\gamma_{\rho} \sim C/N$ (where the prefactor $C$ depends on the disorder strength, approaching its Gaussian-ensemble value 3 deeply in the metallic phase and diverging as $\ln C \sim (W_L - W)^{-1/2}$ at the localization transition).

Moreover, recent numerical investigations based on the finite-size scaling of energy levels and wave-functions statistics on the delocalized side of the Anderson model on RRG [28] and similar sparse random lattices [29, 30] provided new indications against the existence of a truly intermediate non-ergodic extended phase. Such indications rely on the observation of a non-monotonous behavior of the observables as a function of the system size on the delocalized side of the transition, which can be explained in terms of (i) the presence of a characteristic scale which diverges exponentially fast approaching the transition and is already very large far from it; (ii) the localized nature of the critical point in the limit of infinite dimension [3] [27] [31] [32]. The combination of these two elements are argued to produce dramatic and highly non-trivial finite size effects even very far from the critical point, and give rise to a strong non-ergodic behavior in a crossover region where the correlation volume $N_c(W)$ is larger than the accessible system sizes. Still, important questions remain answered. Probably, the most puzzling feature is the fact that the non-ergodic crossover region observed when the system size is smaller than the correlation volume exhibits non-trivial disorder-dependent (apparent) fractal exponents associated to the spectral statistics, which are independent on $N$ in a broad range of system sizes smaller than $N_c$. Note that, strictly speaking, these exponents are not rigorously defined since the system is ergodic in the thermodynamic limit. However, since $N_c(W)$ is so large, an effective non-ergodic behavior, that one can describe with effective exponents on several decades, is observed. The main questions are then: What gives rise to this effective non-ergodic behavior? Why do the effective exponents change with $W$ (usual finite size scaling would imply a behavior independent of $W$ when $N \ll N_c(W)$)? How can one explain theoretically these phenomena?

The existence of this controversy, and the fact that several questions remain open in, after-all, a very old model, is somewhat surprising, especially if one thinks that the Anderson transition on tree-like lattices allows, in principle, for an exact solution [6, 26, 27, 31–35]. This can be obtained in terms of the self-consistent equations for the Green’s functions, which allow to establish the transition point and the corresponding critical behavior. Nevertheless, such exact solution is obtained in the limit of infinite system size, and by introducing an infinitesimal imaginary regulator $\eta$ which gives an infinitesimal broadening to the energy levels, and which must be sent to zero after the limit $N \to \infty$. There is a class of important observables—including the statistics of eigenfunctions and energy levels—which simply cannot be defined on infinite lattices: The mere formulation of statistics of normalized extended wave-functions in a closed system requires the understanding of the thermodynamic limit of finite-size instances. In consequence, in order to address these questions one has to either study large but finite system or to work on the simultaneous limit $N \to \infty$, $\eta \to 0$, $N^\phi \eta = \text{cst}$. This motivated the authors of Refs. [18–20] to put forward the “inflationary population dynamics” approximation scheme mentioned above to deal with this situation.

In this paper we propose a novel approach to study the Anderson model on Bethe lattices (both RRGs and loop-less Cayley trees). This strategy consists in finding the solution of the self-consistent recursion relations for the Green’s functions directly on random instance of large but finite sizes. This approach is well-known both in statistical physics and computer science, and more precisely, in the context of spin-glasses and combinatorial optimization problems, and goes under the name of “Belief Propagation” (BP) or “Message Passing” [30], and is generically believed to provide an accurate and robust
II. MODEL AND STATE OF THE ART

The model we focus on consists of non-interacting spinless electrons in a disordered potential:

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) - \sum_{i=1}^{N} \epsilon_i c_i^\dagger c_i,$$

where the first sum runs over all the nearest neighbors sites of the lattice, the second sum runs over all N sites; $c_i^\dagger$, $c_i$ are fermionic creation and annihilation operators, and $t$ is the hopping kinetic energy scale, which we take equal to 1. The on-site energies $\epsilon_i$ are i.i.d. random variables uniformly distributed in the interval $[-W/2, W/2]$:

$$p(\epsilon) = \frac{1}{W} \theta \left( \frac{W}{2} - |\epsilon| \right).$$

As anticipated in the introduction, we will focus on two types of Bethe lattices with a tree-like structure. The first is defined as a $(k+1)$-RRG, i.e., a lattice chosen uniformly at random among all graphs of N sites where each of the sites has connectivity $k+1$. The properties of such random graphs have been extensively studied (see Ref. [50] for a review). A RRG can be essentially viewed as a finite portion of a tree wrapped onto itself. It is known in particular that for large number of sites any finite portion of such a graph is a tree with a probability going to one as $N \to \infty$, and that the RRG has large loops of typical length of order $\ln N$ [50]. Hence the RRG ensemble can be thought as describing a tight-binding model on a lattice that has locally a tree-like structure but does not possess a boundary. The model [1] is then a sum of two random matrices, $\mathcal{H} = \mathcal{C} + \mathcal{D}$. $\mathcal{C}$ is the connectivity matrix of the RRG, $c_{ij} = -t$ if sites $i$ and $j$ are connected and zero otherwise. $\mathcal{D}$ is the diagonal matrix corresponding to the on-site random energies, $D_{ii} = \epsilon_i \delta_{ij}$. It is known from previous studies that the former ensemble of sparse random matrices belongs to the GOE universality class (with fully delocalized eigenvectors) [51, 52], while the latter is described by definition by Poisson statistics (with fully localized eigenvectors).

The second type of lattice that we will consider is a (non-random) finite portion of $n_g$ generations of an infinite loop-less tree of connectivity $k+1$ (also known as Cayley tree). A finite fraction of the sites of a Cayley tree belong to the boundary and only have connectivity equal to 1 (more precisely, for a Cayley tree of $n_g$ generations, the number of boundary sites is $(k+1)k^{n_g-1}$, while the total number of sites is $1 + (k+1)(k^{n_g}-1)/(k-1)$) [50]. Note that while the RRG is statistically translationally invariant, the Cayley tree is not translationally invariant even in absence of disorder, since the properties of a given site depend on its distance from the boundary (or, equivalently, from the root) of the tree.

Localization on the RRG was first studied by Abou-Chacra, Anderson and Thouless [6] and then later by many others, see 12, 20, 27, 28, 30, 35, 42, 54, 55 and Refs. therein. Many similarities, but also few important differences, with the 3d behavior have been found. As mentioned above, the differences mainly concern the critical properties. Contrary to the finite dimensional case, the critical behavior is not power-law-like but instead exponential, i.e., one finds essential singularities approaching the localization transition from the delocalized regime [27, 51, 52]. Moreover, the IPR, is found to have a discontinuous jump at the transition from a
... at the transition. Arguments based on supersymmetric field theory indicate that the level statistics should display a transition from GOE to Poisson statistics concomitant with the localization transition. However, the first numerical studies didn’t fully support this claim. Moreover, the arguments of Refs. [18–20] indicates that the two transitions might actually not coincide. As discussed above, evidences of an intermediate phase, which is delocalized but not ergodic were first found in. These findings triggered a lot of activity. In Refs. [16, 17], on the numerical study of the spectrum of fractal dimensions of finite size systems, it was conjectured that the eigenstates are multifractal in the whole delocalized phase. More recently, the authors of Refs. [18–20] combined exact diagonalization and semi-analytical calculations to claim the existence of a fractal in the whole delocalized phase. More recently, these results have been confirmed by the numerical investigations of Refs. [28–30] of the level and eigenfunction statistics on the delocalized side of the Anderson transition on the RRG and similar sparse random lattices unveiled the existence of very strong finite size effects with a characteristic crossover scale $N_c(W)$ associated to a pronounced non-monotonous behavior of the observables as a function of $N$. Such correlation volume is found to diverge exponentially fast at the Anderson transition, thus possibly explaining the discrepancy between theoretical results and numerics. The origin of the non-monotonocity has been traced back to the localized nature of the Anderson critical point in the limit of infinite dimensions. For $N < N_c$, the system flows towards the Anderson transition fixed point, whose properties on the RRG are analogous to the localized phase, whereas for $N > N_c$ the system approaches the $N \to \infty$ ergodic behavior. The conclusion of the investigations of Refs. [23–30] is thus that the system is ergodic in the whole delocalized phase, but is characterized by dramatic and non-trivial finite-size effects even very far from the critical point, which give rise to an apparent non-ergodic behavior in a crossover region where the correlation volume is larger than the accessible system sizes. Nonetheless, some aspects of the problem cannot be explained by this scenario and must be analyzed more carefully. As we stressed in the introduction, important questions on the nature of this cross-over region remain unanswered.

On the other hand, the properties of the Anderson model on the Cayley tree have been much less studied. Monthus and Garel studied numerically the statistics of transmission amplitudes on a Cayley tree, finding that it has a multifractal form in the delocalized phase. More recently, these results have been confirmed by the analysis of where it was shown that the delocalized phase have subtle properties and is, in fact, non-ergodic, with wave-functions presenting a multifractal behavior.

In the following, without loss of generality, we focus on the $k = 2$ case (i.e., total connectivity $k + 1 = 3$) and (mostly) on the middle of the spectrum, $E = 0$. Previous studies of the transmission properties and dissipation propagation determined that the localization transition takes place at $W_L \approx 18.2$, while previous analysis of the spectral properties have suggested the presence of the non-ergodic delocalised phase in the range $10 \approx W_E < W < W_L$.

III. EXACT DIAGONALIZATION ON THE RRG

In order to analyze the statistics of energy levels and wave-functions amplitudes, and clarify its relationship with the localization transition, we have diagonalized the Hamiltonian on the RRG for several system sizes $N = 2^n$, from $n = 6$ to $n = 15$, and for several values of the disorder strength $W$ on the delocalized side of the Anderson transition $W < W_L \approx 18.2$. For each $N$ and $W$, we have averaged over both the on-site quenched disorder and on RRG realizations, taking (at least) $2^{22−n}$ different samples. Since we are interested in $E = 0$, we only focused on 1/8 of the eigenstates centered around the middle of the band (we have checked that taking 1/16 or 1/32 of the states does not alter the results, but yields a poorer statistics).

A. Level statistics

We have studied the statistics of level spacings of neighboring eigenvalues: $s_m = E_{m+1} - E_m \geq 0$, where $E_m$ is the energy of the $m$-th eigenstate in the sample. In the delocalized regime, level crossings are forbidden. Hence the eigenvalues are strongly correlated and the level statistics is expected to be described by Random Matrix Theory (more precisely, several results support a general relationship between delocalization and the Wigner’s surmise of the GOE). Conversely, in the localized phase wave-functions close in energy are exponentially localized on very distant sites and do not overlap. Thus there is no level-repulsion and eigenvalues should be distributed similarly to random points thrown on a line (Poisson statistics). In order to avoid difficulties related to the unfolding of the spectrum, we follow and measure the ratio of adjacent gaps,

$$r_m = \frac{\min\{s_m, s_{m+1}\}}{\max\{s_m, s_{m+1}\}},$$

and obtain the probability distribution $\Pi(r)$, which displays a universal form depending on the level statistics. In particular $\Pi(r)$ is expected to converge to its GOE and Poisson counterpart in the extended and localized regime, allowing to discriminate between the two phases as $\langle r \rangle$ changes from $\langle r \rangle_{\text{GOE}} \sim 0.53$ to $\langle r \rangle_{\text{P}} \sim 0.39$ respectively.

The GOE-Poisson transition can also be captured by correlations between nearby eigenstates such as the mu-
ual overlap between two subsequent eigenvectors, defined as:

$$q_m = \sum_{i=1}^{N} |\langle i|m\rangle||\langle i|m+1\rangle|.$$ 

In the GOE regime the wave-functions amplitudes are i.i.d. Gaussian random variables of zero mean and variance $1/N$ \[55], hence $\langle q\rangle$ converges to $\langle q\rangle_{\text{GOE}} = 2/\pi$. Conversely in the localized phase two successive eigenvector are typically peaked around very distant sites and do not overlap, and therefore $\langle q\rangle \propto N \rightarrow \infty$. At first sight this quantity seems to be related to the statistics of wave-functions’ coefficients rather than to energy gaps. Nonetheless, in all the random matrix models that have been considered in the literature so far, one empirically finds that $\langle q\rangle$ is directly associated to the statistics of gaps between neighboring energy levels. Perhaps the best example of that is provided by the generalization of the Rosenzweig-Porter random matrix model of [21,22], where there is a whole region of the parameter space where wave-functions are delocalized but multifractal and strongly correlated, while the statistics of neighboring gaps is still described by the GOE ensemble. In this case one numerically finds that $\langle q\rangle$ converges to its GOE universal value $2/\pi$ irrespective of the fact that wave-functions amplitudes are not uncorrelated Gaussian random variables of variance $1/N$.

In Fig. 1 we show the behavior of the average value of the ratio of adjacent gaps, $\langle r\rangle$, and of (the logarithm of) the typical value of the mutual overlap between subsequent eigenvectors, $q_{\text{typ}} = e^{\langle \ln q \rangle}$, as a function of the disorder $W$, for several system sizes $N = 2^n$, with $n$ from 6 to 15. As expected, for small (resp. large) enough disorder we recover the universal values $\langle r\rangle_{\text{GOE}} \approx 0.53$ and $q_{\text{typ}}^{\text{GOE}} = 2/\pi$ (resp. $\langle r\rangle_p \approx 0.39$ and $q_{\text{typ}}^{\text{P}} \rightarrow 0$) corresponding to GOE (resp. Poisson) statistics. However, as pointed out in [15] the different curves corresponding to different values of $N$ cross much before the localization transition, occurring at $W_L \approx 18.2$, as indicated by the vertical dashed line in the plot. This behavior was interpreted in terms of an intermediate delocalized but non-ergodic phase [15]. Nevertheless, analyzing carefully the data, we realized that the crossing point is in fact slowly but systematically drifting towards larger values of $W$ as $N$ is increased, as also observed [28,29].

This is clearly unveiled by Fig. 2 where we plot the behavior of $q_{\text{typ}}$ and $\langle r\rangle$ as a function of $n = \log_2 N$, for several values of the disorder belonging to the range where the curves of $\langle r\rangle$ and $q_{\text{typ}}$ for different $n$ cross, i.e., $10 \lesssim W \lesssim 16$. One indeed observes that in this region $q_{\text{typ}}$ and $\langle r\rangle$ become non-monotonic functions of $n$. The position of the minimum of $q_{\text{typ}}$ (highlighted by dashed vertical lines in the left panel of Fig. 2) naturally defines a characteristic system size, $N_c(W) = 2^{n_c(W)}$, governing the crossover from Poisson to GOE statistics (on the scale of the mean level spacing): For $N < N_c(W)$ one has indeed that $q_{\text{typ}}$ decreases as the system size is increased, as expected for localized wave-functions, whereas for $N > N_c(W)$ it is an increasing function of $n$ and eventually converges to the GOE universal value. The same non-monotonic behavior as a function of the system size is found for $\langle r\rangle$ (right panel of Fig. 2), and has been previously observed in Refs. [15,28,29].
The IPR of the eigenfunction $|m\rangle$ is defined as $\Upsilon_2^{(m)} = \sum_{i=1}^{N} |\langle i|m\rangle|^2$. In the full extended regime wave-functions are uniformly spread over all the sites of the RRG, thus $\langle i|m\rangle$ are random variables of order $1/\sqrt{N}$, due to normalization, and $\langle \Upsilon_2 \rangle$ vanishes as $1/N$ for $N \to \infty$. Conversely in the localized phase wave-functions are localized on $O(1)$ sites and $\langle \Upsilon_2 \rangle$ approaches a constant value in the thermodynamic limit (in particular, in the infinite disorder limit one has that $\langle \Upsilon_2 \rangle \to 1$).

A related—and less fluctuating—observable is provided by the support set, recently introduced in \cite{16,17} as a measure of wave-functions ergodicity. For an eigenvector $|m\rangle$ with sites ordered according to $|\langle i|m\rangle| > |\langle i+1|m\rangle|$, it is defined as the sets of sites $i < S_{\epsilon}^{(m)}$ such that:

$$\sum_{i=1}^{S_{\epsilon}^{(m)}} |\langle i|m\rangle|^2 \leq 1 - \epsilon < \sum_{i=1}^{S_{\epsilon}^{(m+1)}} |\langle i|m\rangle|^2.$$  

The scaling of $S_{\epsilon}$ for $N \to \infty$ and $\epsilon$ arbitrary small but finite allows to discriminate between the extended and the localized regimes, as $S_{\epsilon}$ is $N$-independent for localized wave-functions while it diverges as $N$ for $N \to \infty$ for fully delocalized states.

In the intermediate extended non-ergodic phase the eigenstates are supposed to be be delocalized on a subset of $N^D$ sites. One therefore expects that the disorder-dependent fractal exponent $D$ describing the scaling of the support set with the system size as $\langle S_{\epsilon} \rangle \sim N^D$ should be strictly smaller than one in the intermediate delocalized non-ergodic phase \cite{10,20}. In fact one can show \cite{16,17,19,21} that the exponent $D$ coincides with the fractal dimension $D_1$. Similarly the IPR should behave as $\langle \Upsilon_2 \rangle \sim N^{-D_2}$, with $D_2 \in (0,1)$. (See below for a precise definition of the fractal exponents $D_q$.)

We have measured the wave-functions’ amplitudes from ED of the Hamiltonian (1) on the RRG for several values of the disorder strength $W$ and for several system sizes $N = 2^n$, and computed the typical value of the IPR, $\Upsilon_2^{typ} = e^{\langle \ln \Upsilon_2 \rangle}$, and the average value of the support set, $\langle S_{\epsilon} \rangle$.\footnote{One should focus in the regime where $\epsilon$ is arbitrary small but finite. In practice we have taken $\epsilon \in (10^{-4},10^{-3})$.} As explained in the previous section, averages are taken over (at least) $2^{2^n-\epsilon}$ different realizations of the disorder and over 1/8 of the eigenstates centered around the middle of the band.

The flowing fractal exponents $D_2$ and $D_1$ describing the scaling of the typical value of the IPR and of the average value of the support set with $N$ can then be approximately evaluated as:

$$D_2(W,n) \simeq -\frac{\ln \Upsilon_2^{typ}(W,n) - \ln \Upsilon_2^{typ}(W,n-1)}{\ln 2},$$

$$D_1(W,n) \simeq \frac{\ln \langle S_{\epsilon}(W,n) \rangle - \ln \langle S_{\epsilon}(W,n-1) \rangle}{\ln 2}.$$  

In Fig. 3 the numerical values of $D_2$ and $D_1$ are plotted as a function of the disorder $W$ for several system sizes. $D_2$ and $D_1$ show a remarkably similar—although slightly less clean—behavior compared to the one of $(r)$ and $q^{typ}$ of Figs. 1 and 2. At fixed $N$, $D_2$ and $D_1$ decreases as $W$ is increased. At fixed and small enough disorder, they both grows with $N$ and seem to approach the standard value $1$ for $N \to \infty$, corresponding to fully delocalized wave-functions. Conversely, at fixed and large enough disorder, $D_2$ and $D_1$ decrease to zero as the system size is increased, implying that $\langle \Upsilon_2 \rangle, \langle S_{\epsilon}(n) \rangle \to \text{cst}$ for $N \to \infty$, as expected for localized eigenstates. Although different curves corresponding to different values of $N$ cross much before the localization transition, a careful analysis of the data shows that the crossing point is in fact slowly but systematically drifting towards larger values of $W$ as $N$ is increased. As for $(r)$ and $q^{typ}$, the $n$-dependence of $D_2$ and $D_1$ at fixed $W$ is in fact non-monotonic. The characteristic crossover scales over which the non-monotonicity of $D_2$ and $D_1$ is observed is within our numerical accuracy the same as the one found above for the level statistics on the scale of the mean level spacing. This suggests that convergence to the conventional ergodic behavior in the delocalized phase of RRG, with Wigner-Dyson statistics for the energy levels and $1/N$ scaling of the IPR, is governed by a unique characteristic correlation volume\footnote{28, 29}.
An eigenstate $|m\rangle$ and its coefficients $w_m(i) = |\langle i|m\rangle|^2$ can be characterized by the moments (i.e., generalized IPR) $(\Upsilon_q) = \sum_{i=1}^{N}|w_m(i)|^q \propto N^{-\tau(q)} = N^{-D_q(q-1)}$. $(\Upsilon_1 = 1$ for the normalization and $(\Upsilon_2$ is the standard IPR defined above). For ergodic systems, in the limit $N \to \infty$, all the wave-function amplitudes are of $O(1/N)$, corresponding to $\tau(q) = q - 1$. Conversely, finding that the ratio $D_q = \tau(q)/(q-1)$ depends on $q$ (and is different from one) is a signatures of non-ergodic states. In this case, the eigenfunctions are called multifractal. It is customary to characterize the amplitudes $w_m(i)$ by the spectrum of fractal dimensions $f(\alpha)$, defined in the following way: The number $N(\alpha)$ of sites $i$ that have amplitudes scaling as $N^{-\alpha}$ behaves as $N(\alpha) \sim N(f(\alpha))$. As a result, one has that:

$$\Upsilon_q = \sum_{i=1}^{N} w_i^q \sim \int d\alpha \exp[(f(\alpha) - q\alpha) \ln N].$$

Then, in the thermodynamic limit, the saddle point computation of $\Upsilon_q$ leads to the following Legendre transform formula:

$$\alpha = \frac{d\tau}{dq}, \quad f'(\alpha) = q, \quad \tau(q) = q\alpha - f'(\alpha).$$

$f(\alpha)$ is a convex function of $\alpha$. The value $q = 0$ is associated with the most probable value $\alpha_m$ of the wave-function coefficients, where the singularity spectrum reaches its maximum, $f(\alpha_m) = 1$. The value $q = 1$ is associated with the point $\alpha_1$ such that $f(\alpha_1) = \alpha_1$, and $f'(\alpha_1) = 1$. In the $N \to \infty$ limit, a finite support $0 < \alpha_- < \alpha < \alpha_+$ where $f(\alpha) > 0$ is a signature of multifractality, while for ergodic states, $f(\alpha) = -\infty$ unless for $\alpha = 1$, where $f(1) = 1$, and $(\alpha_- < \alpha_1 < 1 < \alpha_{\text{max}} < \alpha_+) \to 1$.

The behaviour at low and strong disorder is as expected: At low enough disorder (see App. A and Fig. 26) the support of the singularity spectrum clearly shrinks as $N$ is increased, and $f_N(\alpha)$ eventually converge to a $\delta$-function for large $N$, $\lim_{N \to \infty} f_N(\alpha) = \delta(\alpha - 1)$ (see also Fig. 27), corresponding to full ergodicity; whereas in the localized regime (see App. A and Fig. 28) for $W = 19$, $f_N(\alpha)$ gets broader as the system size is increased and shows a shape which is reminiscent of the triangular form typically observed in the insulating phase.

We now focus on the putative intermediate phase. In the top panel of Fig. 4 we plot the singularity spectrum for $W = 13$—deep in the crossover non-ergodic regime—and for several system sizes $N = 2^n$, with $n$ from 8 to 13. (More information and details are given in App. A) In the following we will focus in particular on the $N$-dependence of two specific points of the singularity spectrum: The point $\alpha_1$ (associated to $q = 1)$ where $f_N(\alpha_1) = \alpha_1$, and $f'_N(\alpha_1) = 1$; And the lower edge of the support of $f_N(\alpha)$, $\alpha_-$. The bottom left panel provides a zoom of the same curves in the region (a), close to the lower edge of the support of $f_N(\alpha)$, while the bottom right panel shows the plots of $\alpha - f_N(\alpha)$ in the region (b), allowing to identify the position of $\alpha_1$. These plots clearly demonstrate that the evolution of $f_N(\alpha)$ is non-monotonic: For small enough sizes (i.e., $n \lesssim n_c(W)$) the support of $f_N(\alpha)$ gets broader, and $\alpha_-$ and $\alpha_1$ decrease and as $n$ is increased, as for non-ergodic states. Conversely, for larger sizes (i.e., $n \gtrsim n_c(W)$) the support of $f_N(\alpha)$ shrinks back, and $\alpha_-$ and $\alpha_1$ increase with $n$. A similar behavior is observed in the whole crossover region, $W \gtrsim 10$. The crossover scale governing the non-monotonic behavior of the singularity spectrum coincides, within our numerical accuracy, with the feature found above from the non-monotonic behavior of the level statistics and of the IPR. See Fig. 5 for a summary of the numerical observations discussed above.

C. The characteristic crossover scale

The numerical results presented in this section suggest the emergence of a unique characteristic scale which controls the transition from a phase characterized by Poisson statistics—localization—lack-of-ergodicity to one displaying GOE statistics—delocalization—ergodicity for the Anderson model on RRGs of finite size. Such crossover scale is already very large well below the Anderson localization, resulting in a broad crossover region where finite size effects are extremely important. As mentioned above, in such crossover region all observables and probes introduced in the previous sections share the
same non-monotonic behavior as a function of the system size [25–27].

This is highlighted in Fig. 5 where we plot the $n_c(W)$ extracted from different observables related, to the statistics of the gaps (i.e., $\langle \rho \rangle$ and $q^{typ}$), and to wave-functions ergodicity (i.e., $D_2$, $D_1$, $\alpha_1$, and $\alpha_-$) for $W = 13$. All the different curves show a very similar non-monotonic shape. The position of the minimum, $n_c(W)$, seems to depend very weakly on the choice of the observable.

This is confirmed by the main panel of Fig. 6 where we plot the characteristic crossover scales, $n_c(W)$, extracted from different probes, showing that, within our numerical accuracy, they all yield the same dependence on the disorder strength $W$.

The non-monotonic behavior has been interpreted in [25] in terms of the nature of the Anderson critical point on the RRG, which has properties similar to that of the localized phase [5, 27, 31, 32], with critical level statistics of Poisson form and strongly localized critical wave-functions. The observables of systems of size $N \ll N_c(W)$ would then first flow upon increasing $N$ towards the critical values, which tend, for $d \to \infty$, to the ones of the localized phase (i.e., $\langle \rho \rangle_c = (\rho)_p \approx 0.39$, $q^{typ} = 0$, $D_{2c} = 0$, $D_{1c} = 0$, $\alpha_- = 0$, $\alpha_1 = 0$). Then, when $N$ becomes larger than the correlation volume $N_c$, the observables flow towards their standard values in the delocalized, fully ergodic, phase.

The black curve of Fig. 6 shows a fit of the data of the form $n_c \simeq A/(W_L - W)^\nu$, with $A \approx 20$ and $\nu \approx 0.6$, implying an exponential divergence of the correlation volume at the transition point. Although these data are not sufficient to allow for an accurate estimation of $\nu$, the value of the exponent is not too far from the one predicted by the supersymmetric analysis, $\nu = 1/2$ [26, 27]. Note, however, that recently a different expression has been proposed for $n_c \simeq A/(W_L - W)$, with $\nu = 1$ [20]. Our numerical data are clearly too far from $W_L$ to address this controversy.

The gray thick curve of Fig. 6 corresponds to the estimation of the crossover scale given by Eq. (10) obtained from the convergence of the probability distribution of the Local Density of States (LDoS) within the BP approximation explained below (see Sec. IV B and Fig. 13 for more details).

Finally, in the inset of Fig. 6 we show the evolution with $n$ of the crossing point of the curves of $q^{typ}(W)$ of Fig. 1 for two subsequent system sizes. The crossing point moves very slowly—although in a systematic way—towards larger values of the disorder as $N$ is increased, and seems to approach $W_L$ in the infinite size limit.

The numerical results presented here are compatible with the idea that the Anderson model on the RRG is fully ergodic in the whole delocalized phase in the limit of infinite size, and that standard metallic behavior is eventually restored for system sizes larger than the correlation volume, as suggested in [28, 29] and in agreement with the analytical predictions of Refs. [26, 27]. However this conclusion is based on the extrapolation of the numerical results obtained for finite systems, and relies on the assumption that no singularity occurs for $N \gg N_c$. In fact, this conjecture has been questioned in Refs. [18–20, 29], where it has been put forward that there exists a first-order transition in the thermodynamic limit between ergodic and non-ergodic states (with a finite jump of, e.g., $D_1$ and $D_2$) at $W_E \approx 10$. In the following we propose a new approach to deal with this controversy and to answer
the open questions raised in the introduction.

IV. BP SOLUTION OF THE ITERATION EQUATIONS FOR THE GREEN’S FUNCTIONS ON THE RRG AND ON THE CAYLEY TREE

As discussed in the introduction, the Anderson model on tree-like structures allows, in principle, for an exact solution in the limit of infinite lattices [6], which yield the probability distribution function of the diagonal elements of the resolvent matrix, defined as \( \mathcal{G}(z) = (\mathcal{H} - zI)^{-1} \).

In order to obtain the recursive equations, the key objects are the so-called cavity Green’s functions, \( G_{i \rightarrow j}(z) = [(\mathcal{H}_{i \rightarrow j} - zI)^{-1}]_{ii} \), i.e., the diagonal elements on site \( i \) of the resolvent matrix of the modified Hamiltonian \( \mathcal{H}_{i \rightarrow j} \) where the edge between the site \( i \) and one of its neighbors \( j \) has been removed.

Take a given site \( i \) and its neighbors \( \{j_1, \ldots, j_{k+1}\} \) living on an infinite tree. If one removes the site \( i \) from the graph, then the sets \( \{j_1, \ldots, j_{k+1}\} \) are uncorrelated, since the lattice would break in \( k + 1 \) semi-infinite disconnected branches. One then obtains (e.g., by Gaussian integration) the following iteration relations for the cavity Green’s functions [6,33]:

\[
G_{i \rightarrow j_p}(z) = -\epsilon_i - z - t \sum_{j_p \in \partial i/j_p} G_{j_p \rightarrow i}(z),
\]

where \( z = E + i\eta \), \( \eta \) is an infinitesimal imaginary regulator that smoothens out the pole-like singularities in the right hand sides, \( \epsilon_i \) is the on-site random energy taken from the distribution \( \mathbb{P}\), and \( \partial i/j \) denotes the set of all \( k + 1 \) neighbors of \( i \) except \( j \). (Note that for each site with \( k + 1 \) neighbors one can define \( k + 1 \) cavity Green’s functions and \( k + 1 \) recursion relations of this kind.) After that the solution of Eqs. (4) has been found, one can finally obtain the diagonal elements of the resolvent matrix of the original problem on a given site \( i \) as a function of the cavity Green’s functions on the neighboring sites [33]:

\[
G_{i}^{-1}(z) = -\epsilon_i - z - t \sum_{j_q \in \partial i} G_{j_q \rightarrow i}(z).
\]

In the following we will (mostly) focus on the middle of the spectrum \( (E = 0) \) and set \( t = 1 \).

The statistics of the diagonal elements of the resolvent gives—in the \( \eta \rightarrow 0^+ \) limit—the spectral properties of \( \mathcal{H} \). In particular, the probability distribution of the LDoS at energy \( E \) is given by:

\[
\rho(E) = \frac{1}{\pi} \Im G_i(z) \propto \frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \frac{\eta}{\sigma^2} \delta(E - E_m),
\]

from which the average Density of States (DoS) is simply given by \( \rho = (1/N) \sum_i \rho_i = 1/(N\pi)\text{Tr}\Im G \). Similarly, the IPR can be expressed as:

\[
\Upsilon_2 = \lim_{\eta \rightarrow 0^+} \frac{\eta}{\pi\rho(E)N} \sum_{i=1}^N |G_i(z)|^2.
\]

Note that Eqs. (4) and (5) are exact on Cayley trees, even for finite lattices of \( n_L \) generations, due to the absence of loops. This is not true instead, on the RRG. Indeed, in this case, when site \( i \) is removed from the graph, the neighbors \( \{j_1, \ldots, j_{k+1}\} \) are not truly decoupled, since they are still connected by some (typically large) loop present somewhere in the system. Since the average size of the loops scales as \( \ln N \), it is reasonable to expect that Eqs. (4) and (5) become asymptotically exact in the thermodynamic limit as the cavity Green’s functions on sites \( \{j_1, \ldots, j_{k+1}\} \) become uncorrelated in absence of site \( i \) if the typical length of the loops which connect them is larger than the correlation length. This has been in fact proven rigorously in Ref. [39] using the local convergence of RRGs to Cayley trees. One can then argue that the recursion equations provide an approximate solution for the diagonal elements of the resolvent matrix for the Anderson model on RRGs of \( N \) sites, and that the quality of the approximation should improve as \( N \) is increased.

Since the Green’s functions \( G_{i \rightarrow j} \) and \( G_i \) are random variables, Eqs. (4) and (5) naturally lead to functional equations on their probability distribution \( Q(G) \) and \( P(G) \). Let us first focus on the RRG, where the sites of the lattice are statistically translationally invariant due to the absence of boundaries. From Eq. (4) one first gets the self-consistent functional equation for the probability distributions of the cavity Green’s functions in the \( N \rightarrow \infty \) limit (averaged over the on-site disorder and on different realizations of the random lattice):

\[
Q(G) = \int d\rho(\epsilon) \prod_{i=1}^k dQ(G_i) \delta(G^{-1} + \epsilon + z + \sum_{i=1}^k G_i),
\]

where \( \rho(\epsilon) \) is the probability distribution of the on-site random energy, Eq. (2). Once the fixed point of Eq. (8) is obtained, using Eq. (9) one can compute the probability distribution of the diagonal elements of the resolvent:

\[
P(G) = \int d\rho(\epsilon) \prod_{i=1}^{k+1} dQ(G_i) \delta(G^{-1} + \epsilon + z + \sum_{i=1}^{k+1} G_i).
\]

This set of functional equations can be solved numerically with an arbitrary degree of precision using a population dynamics algorithm [6,15,20,33,60].

For Cayley trees, Eqs. (8) and (9) are valid only in the bulk, in the proximity of the root, and at finite \( \eta \). Indeed, due to the presence of the boundary, the lattice is not statistically invariant by translation. In order to write the functional iteration equations for the probability distributions of the Green’s functions one needs then to take into account the position of the sites inside the tree, as explained in detail in App. [3]

In agreement with previous results [6,20,27,31,33], we find that in the localized phase, \( W > W_L \approx 18.2 \),
the iteration equations (1) and (8) are unstable with respect to the imaginary regulator $\eta$: $P(G)$ is singular and the average DoS vanishes in the $\eta \to 0^+$ limit. Conversely, in the metallic phase the probability distribution converges to a stable non-singular distribution function, provided that $\eta < \eta_c(W)$, where $\eta_c(W)$ is an energy scale which is finite in the whole delocalized phase and vanishes exponentially as $\eta_c(W) \approx \exp[-\text{cst}/(W_L - W)^\nu]$ for $W \to W_L$. For $\eta < \eta_c(W)$ the typical value of $\langle \ln G \rangle$, defined as $\langle \ln G \rangle_{\text{typ}} = e^{\langle \ln \ln G \rangle}$, also converges to a $\eta$-dependent finite value $\langle \ln G \rangle_{\text{typ}}$ which vanishes exponentially for $W \to W_L$ with the same exponent $\nu$ and behaves as $\langle \ln G \rangle_{\text{typ}}(W) \propto \eta_c(W)^b$, with $b \approx 1.12$ in our numerics, see Fig. 13. Similarly, $\langle |G|^2 \rangle$ converges to a finite value (which diverges exponentially for $W \to W_L$) in the whole delocalized phase, and the IPR goes to zero.

However, this analysis is carried out when the limit $\eta \to 0^+$ is taken after the thermodynamic limit $N \to \infty$. Recently in Refs. [18][22] it has been put forward that taking the limit $N \to \infty$ first does not allow to detect the existence of delocalized but non-ergodic states (if they exist). Indeed, for multifractal states the wave-functions typically occupy a fraction of $N^D$ sites (with $0 < D < 1$), implying the existence of an energy scale $\eta_c(N)$ which decreases as $N^{D-1}$ but stays much larger than the mean level spacing $\delta = 1/(\rho N)$. This should be the hallmark of the non-ergodic extended phase [18][22]. In order to deal with this situation, one should instead take the simultaneous limits $N \to \infty$, $\eta \to 0$, with $N^\phi \eta = \text{cst}$ (with $\phi \leq D$). This motivated the authors of Refs. [18][22] to propose an analytical approximate method, based on RSB and called “inflationary population dynamics”, which consists in modifying the iteration relations (1) and (8) in a way that allows to distinguish the multifractal states.

The other issue of taking the $N \to \infty$ limit from the start consists in the fact that several important observables related to the statistics of wave-functions and energy levels, are simply not defined on infinite lattices. In order to ascertain their properties one should instead understand their scaling behavior with $N$ in the limit of very large sizes.

In this paper we propose a novel and alternative strategy to overcome these issues. The idea is to solve directly Eqs. (4) and (5) on random instances of large but finite sizes. In practice, we first generate the lattice [a random realization of the RRG or the (non-random) Cayley tree] and draw the random on-site energies from Eq. (2). Then we find the fixed point of Eqs. (4), which becomes a system of $(k + 1)N$ coupled equation for the cavity Green’s functions [61]. This can be done iteratively with arbitrary precision in a time which scales linearly with $N$. Finally, using Eqs. (5) one obtains the diagonal elements of the resolvent matrix on each site. We repeat this procedure several times to average over different realizations of the disorder.

This strategy is well known in statistical physics and information theory and goes under the name of “Belief Propagation” (BP) or “Message Passing” algorithm [36], and has been—and still currently is—widely used in particular in the context of random optimization and inference problems, and spin glass models on sparse random graphs. As already said above, the BP approach is exact on the Cayley tree, since in this case Eqs. (4) and (5) are exact due to the absence of loops. Conversely, on the RRG the iteration equations become asymptotically exact in the $N \to \infty$ limit only. Although there is a rigorous proof of the convergence of the BP solution for the Anderson model on the RRG in the large $N$ limit [59], there is no rigorous estimate of the error at large but finite $N$. However, in most cases studied in the literature the BP approach has proven as a powerful, accurate and controlled approximation and in general provides good estimations of local and average quantities, which improve as the system size is increased [36].

The BP approach has several advantages:

(a) The fixed point of Eqs. (4) can be found in a time which scales linearly with the size of the system. This allows to investigate lattices of huge size (e.g., up to $N = 2^{29}$) i.e., several orders of magnitude larger than what can be achieved by the most efficient available algorithms of ED [30]. This allows to overcome finite size effects even deep inside the intermediate supposedly non-ergodic region;

(b) Although the starting point is provided formally by the same set of local equations both for the RRG and the Cayley tree, the BP algorithm gives in general substantially different fixed point solutions for the two cases, since this it is sensitive to the presence of loops, boundaries, and to the structure of the lattice, thereby allowing to disclose the difference between the two kinds of tree-like graphs;

(c) Within the BP approach it is natural and straightforward to define observables related to the eigenfunctions and energy level statistics which can be expressed in terms of the Green’s functions defined on the sites of a random instance of finite size $N$. Moreover, one can easily investigate the properties of those observables on an energy scale which scales in a non-trivial way with the system size (provided that it stays larger than the mean level spacing $\delta$). As a consequence, this method allows to unveil the existence of an energy scale which stays larger than $\delta$ but decreases with $N$, which is the hallmark of the non-ergodic extended phase [18][22].

The rest of the paper is devoted to the BP analysis. In the next section we compare the results found within the BP approach to EDs (up to the accessible system sizes, $N = 2^{15}$) for several values of the disorder, and establish its accuracy and the domain of validity. In particular, we show that, provided that the imaginary regulator $\eta$ stays larger than the mean level spacing $\delta$, the BP approach yields an excellent estimation of local and average observables, and also accounts for sample-to-sample
fluctuations due to different realizations of the disorder, and spatial fluctuations due to the local environment. In particular, we study the convergence of the probability distribution of the LDoS in the limit of large sizes for the Anderson model on the RRG and on the Cayley tree. Finally, in Secs. [IVD] and [IVE] we focus on two observables, i.e., the level compressibility [37] and the overlap correlation function, related respectively to the statistics of energy gaps and of wave-functions’ amplitudes [21, 38–45], and analyze their behavior on the RRG and on the Cayley tree.

A. Test of accuracy and domain of applicability of the BP approach

Differently from more standard applications of BP in statistical physics and information theory, in the present case the iteration equations [4] and [5] are ill-defined in the limit \( \eta \to 0^+ \), due to the presence of pole-like singularities in the right hand sides. One then needs needs to consider the simultaneous limit \( N \to \infty \) and \( \eta \to 0^+ \). This unusual situation deserves a more careful analysis of the convergence properties of the BP approximation and of its domain of applicability.

In Fig. 7 we plot the behavior of the typical value of the imaginary part of the Green’s functions, \( \text{Im} G_{\text{typ}} = \text{exp}[\langle \ln \text{Im} G \rangle] \), as a function of the imaginary regulator measured in units of the mean level spacings \( \eta/\delta \), with \( \delta = 1/(N\rho) \) for three different system sizes, \( N = 2^n \) with \( n = 11, 13, 15 \), averaged over few realizations of the disorder and of the RRG at \( W = 6 \). The continuous lines and filled symbols correspond to the results obtained from ED, whereas the dashed lines and empty symbols correspond to the result obtained using the BP approximation on the same sets of random instances. The vertical dashed lines show the positions of \( \eta_c/\delta \) for the different system sizes. The full orange dashed lines represent the behavior \( \text{Im} G_{\text{typ}} \propto \eta \) for \( \eta < \delta \) and \( \text{Im} G_{\text{typ}} \propto (\eta/\eta_c)^{\beta} \) for \( \eta > \delta \). The horizontal full orange line shows the \( \eta \)-independent asymptotic value \( \text{Im} G_0^{\text{typ}} \) obtained from Eqs. [8] and [9] for \( \eta \ll \eta_c \).

FIG. 7: Logarithm of the typical values of the imaginary part of the Green’s functions, \( \text{Im} G_{\text{typ}} = \text{exp}[\langle \ln \text{Im} G \rangle] \), as a function of the imaginary regulator measured in units of the mean level spacings \( \eta/\delta \), with \( \delta = 1/(N\rho) \) for three different system sizes, \( N = 2^n \) with \( n = 11, 13, 15 \), averaged over few realizations of the disorder and of the RRG at \( W = 6 \). The continuous lines and filled symbols correspond to the results obtained from ED, whereas the dashed lines and empty symbols correspond to the result obtained using the BP approximation on the same sets of random instances. The vertical dashed lines show the positions of \( \eta_c/\delta \) for the different system sizes. The full orange dashed lines represent the behavior \( \text{Im} G_{\text{typ}} \propto \eta \) for \( \eta < \delta \) and \( \text{Im} G_{\text{typ}} \propto (\eta/\eta_c)^{\beta} \) for \( \eta > \delta \). The horizontal full orange line shows the \( \eta \)-independent asymptotic value \( \text{Im} G_0^{\text{typ}} \) obtained from Eqs. [8] and [9] for \( \eta \ll \eta_c \).

1) For \( \eta < \delta \) the typical LDoS is proportional to the imaginary regulator and vanishes as a constant times \( \eta \): If the broadening of the energy levels is smaller than the typical distance between the \( \delta \)-peaks the system looks as if it was localized. In this regime \( \text{Im} G_{\text{typ}} \) is essentially size-independent, although huge sample-to-sample fluctuations are observed.

2) For \( \delta < \eta < \eta_c \) the typical value of \( \text{Im} G \) reaches a \( \eta \)-independent (and size-independent provided that \( N \) is large enough) plateau value. The threshold \( \eta_c(W) \) corresponds to the value of \( \eta \) below which the solution of the functional self-consistent equations [8] and [9] for the Green’s functions obtained in the thermodynamic limit yields a stable (non-singular) \( \eta \)-independent function, and the plateau coincides with the value of \( \text{Im} G_{\text{typ}} \) obtained from this stable probability distribution (orange horizontal line, \( \text{Im} G_0^{\text{typ}} \approx -1.525 \) for \( W = 6 \)). The position of \( \eta_c/\delta = N\rho \eta_c \) is highlighted by the vertical dashed lines for the different system sizes (\( \rho \approx 0.123 \) and \( \eta_c \approx 8 \cdot 10^{-3} \) for \( W = 6 \)). The plateau regime shrinks as the system size is decreased since \( \delta \) is proportional to \( 1/N \). For too small systems (e.g., \( N = 2^{11} \)) the typical level spacing becomes larger than \( \eta_c \) and the plateau regime disappears.

3) For \( \eta > \eta_c \) the typical value of the LDoS grows as \( \text{Im} G_{\text{typ}} \approx \text{Im} G_0^{\text{typ}} + (\eta/\eta_c)^{\beta} \). The exponent \( \beta \) is the same found from Eqs. [8] and [9], and describes the approach of \( \text{Im} G_{\text{typ}} \) to its limiting value as \( \eta \) is decreased below \( \eta_c \) (\( \beta \approx 0.095 \) for \( W = 6 \)).

Furthermore, we notice that the BP approach (dashed lines and empty symbols) provides a very good approximation of the exact result provided that \( \eta \) is larger than few mean level spacings. Conversely, as expected, BP fails completely for \( \eta < \delta \).

Upon increasing the disorder strength, the average DoS decreases (e.g., \( \rho \approx 0.104 \) for \( W = 8 \) and \( \rho \approx 0.0824 \) for \( W = 11 \)) and \( \eta_c \) grows extremely fast (e.g., \( \eta_c \approx 10^{-3} \) for \( W = 8 \), \( \eta_c \approx 8 \cdot 10^{-5} \) for \( W = 11 \)). Hence one needs larger and larger system sizes to be able to
observe the plateau. For example, at $W = 11$ the plateau regime is not visible even for the largest available system, $N = 2^{15}$, while it already starts to appear at $W = 8$ for $N = 2^{15}$ (see Fig. 8). In both cases, we still notice an agreement between the exact results and the BP approximation as far as $\eta/\delta > 1$.

On the basis of these observations, from now on we will set the imaginary regulator to be few level spacings, $\eta = c\delta$, with $c \gtrsim 1$. (Recent results [62] suggest that in fact the Anderson model on the RRG might display uncommon features in the regime $\eta \ll \delta$. Here we do not consider such regime and focus on the more standard situation $\eta > \delta$ only.)

In Fig. 9, we show the values of the typical (left panels) and average (right panels) DoS for several random realizations of the Hamiltonian $H(1)$, for two values of the disorder, $W = 8$ (top panels) and $W = 11$ (bottom panels), for different system sizes, $N = 2^n$ with $n = 11, \ldots, 15$, and $c = 8$. The continuous lines and filled symbols correspond to the values obtained from ED, while the dashed lines and empty symbols represent the results found with the BP approximation. These data show that BP correctly reproduces not only average quantities but also accounts for sample-to-sample fluctuations in an extremely satisfactory way. Moreover, one can check that the relative error of the BP results on average quantities decreases with $N$ (roughly as $1/\sqrt{N}$). We also find that the relative error of the BP approximation decreases with the disorder strength (see also Fig. 10). Although this might seem surprising at first, one can rationalize this observation by recalling that the errors done by the BP approximation are due to the presence of loops of finite size (i.e., smaller than the correlation length $\ln N_c$) where a resonance between two sites belonging to the same loop occurs [62]. The number of such loops in the large $N$ limit is given asymptotically by some known distribution function and stays of $O(1)$ [50]. When $W$ is increased, the probability that two sites belonging to a short loop are in resonance decreases, and the accuracy of the BP results improves.

In Fig. 10, we plot $\text{Im} G_i$ for the first 100 sites $i$ of a spe-
FIG. 11: Average (red, orange, light green and dark green) and typical (violet, indigo, blue, turquoise) DoS, \( \rho(E) = \langle \text{Im} G(\epsilon) \rangle / \pi \) and \( \rho^{\text{typ}}(E) = e^{(\text{Im} \text{Im} G(\epsilon))}/\langle \text{Im} G(\epsilon) \rangle \), as a function of \( E/\delta \) at \( W = 11 \) for two specific realizations of the RRG and of the random energies of \( \mathcal{N} \).\( W \langle \rho(\epsilon), \rho^{\text{typ}}(\epsilon) \rangle \) are likely to belong to short loops and to be in resonance due to the local environment. Only very small discrepancies on some specific sites are observed. Those sites and filled symbols correspond to the values obtained from ED, whereas dashed lines and empty symbols represents the results obtained using the BP approximation.

FIG. 12: Probability distribution functions of \( \ln \text{Im} G \) at \( W = 12 \) for the Anderson model on RRGs of several sizes \( N = 2^n \), with \( n = 11 \) (turquoise), \( n = 12 \) (magenta), \( n = 13 \) (red), \( n = 14 \) (light green), \( n = 15 \) (blue), \( n = 16 \) (brown), \( n = 18 \) (violet), \( n = 20 \) (orange), \( n = 22 \) (maroon), \( n = 24 \) (black), \( n = 26 \) (dark green), \( n = 28 \) (black), and \( n = 29 \) (yellow), averaged over several realizations of the RRG and of the on-site disorder (and for \( c = 8 \)). Full curves correspond to the results of the BP approach, whereas symbols represent the PDFs obtained from ED (but averaged over \( 2^{17-n} \) realizations only). The dashed light blue curve shows the solution of the functional self-consistent equations valid in the thermodynamic limit, Eqs. (5) and (6), found via population dynamics for \( \eta < \eta_c \).

In this section we focus on the convergence of the probability distribution of the LDoS obtained from the BP approach for large but finite systems. As mentioned above, one of the advantage of BP is that the system of coupled equations (4) and (5) can be easily solved by iteration in a linear time in \( N \), thereby allowing to access system size several order of magnitude larger than the ones currently accessible via ED. In Fig. 12 we show the probability distributions of the imaginary part of the Green’s functions of the Anderson model on the RRG, for \( N = 2^n \) with \( n = 11, \ldots, 29 \) at \( W = 12 \) (deep into the putative delocalized non-ergodic phase), averaged over many independent realization of the disorder (as in the previous section we set \( \eta = c\delta \) with \( c = 8 \)). We observe that:

order 1. It also reproduces correctly the fluctuations between different random instances due to different random realization of the graph and of the quenched diagonal elements of the Hamiltonian, and works nicely over the whole energy range from energies of order \( 1/N \) up to energies of order 1. The relative error of the BP approximation decreases as the system size \( N \) and the disorder strength \( W \) are increased.

B. Convergence of the distribution of the LDoS

All in all, these findings shows that the BP approach yields a powerful, efficient and accurate approximation for the Green’s functions of the Anderson model on the RRG, not only at the level of average quantities, but also at the local scale, provided that the imaginary regulator is scales as the mean level spacing times a constant of
$P(\ln \text{Im}\mathcal{G})$ converges to a stable, non-singular, size independent distribution for large enough sizes (say, for $N \gtrsim 2^{28}$);

Convergence occurs when $\eta$ becomes smaller than an energy scale $\eta_c(W)$ which remains finite, and which coincides with the scale below which the solution of Eqs. [8] and [9] becomes stationary;

The stationary probability distribution at large $N$ turns out to be the same (within our numerical accuracy) as the one found from Eqs. [8] and [9] for $\eta < \eta_c$;

For the system sizes accessible via ED ($N = 2^{11}, \ldots, 2^{15}$) we find an excellent agreement between the BP results and the exact distributions;

Since $(1/N) \sum_{i=1}^{N} |\mathcal{G}_i|^2$ converges to a size-independent finite value, from Eq. [7] one has that the IPR goes to zero as $\eta \propto 1/N$.

We find the very same scenario for all values of the disordered strength $W \lesssim 13.5$. For larger values of the disorder the correlation volume $N_c(W)$ would be required to observe the convergence to a stationary distribution,

$$N_c(W) = \frac{c}{\rho(W)\eta_c(W)},$$

becomes exceedingly large, due to the fact that $\eta_c$ becomes exponentially small as one move closer to $W_L$. Interestingly, such estimation of the crossover size obtained from the convergence of the probability distribution of the LDoS within the BP approach, Eq. [10], is plotted in Fig. 6 as a gray thick line, showing that it accounts quite well for the scale on which the ED data exhibit the non monotonicity. In Fig. 13 we plot the inverse of the characteristic crossover length $n_c^{-1} = 1/\log_2 N_c(W)$ (red curve) obtained from the convergence of the distribution of the LDoS, Eq. [10]. The dashed black line corresponds to a fit of the data as $n_c^{-1} \approx a(W_L - W)^\nu$, with $a \approx 0.022$, $\nu \approx 0.6$, and $W_L \approx 18.2$. The plot also shows the inverse of the logarithm of the asymptotic value of $\text{Im}\mathcal{G}^{\text{typ}}$, $1/\log_2(\text{Im}\mathcal{G}_0^{\text{typ}}/\rho)$ (maroon), of the inverse Thouless energy, $1/\log_2 E_{Th}^{-1}$ (magenta), and of the value of the plateau if the overlap correlation function $K_2(E)$ at small energy, $1/\log_2 q_2$ (violet), see Sec. IV E for a precise definition of the last two quantities.

C. Spectral fractal exponents

The drastically different behavior observed on the RRG and on the Cayley tree is clearly illustrated by Figs. 15 and 16. In Figs. 15 and 16, we show the evolution with the system size of the typical DoS, $\rho^{\text{typ}} = \epsilon(\ln \text{Im}\mathcal{G})/|\text{Im}\mathcal{G}|$ and of the IPR [Eq. [8]], averaged over several realizations of the disorder and of the RRG for several values of $W$, which give access directly to the fractal exponents $D_1$ and $D_2$.

The plots show that for small enough system sizes the Anderson model on the RRG behaves as if it was in a non-ergodic extended phase: $\rho^{\text{typ}}$ and $\langle \Upsilon_2 \rangle$ show apparent power-law behaviors, $\rho^{\text{typ}} \propto N^{1-D_1}$ and $\langle \Upsilon_2 \rangle \propto N^{-D_2}$. However, for large enough sizes [i.e., larger than the crossover scale $N_c(W) = c/(\rho(W)\eta_c(W))$, Eq. [10]]
FIG. 14: Probability distribution functions of \(\ln \text{Im} G\) at \(W = 12\) for the Anderson model on the loop-less Cayley tree of \(n_g\) generations, with \(n_g = 15\) (brown), \(n_g = 17\) (violet), \(n_g = 19\) (orange), \(n_g = 21\) (maroon), \(n_g = 23\) (dark green), \(n_g = 25\) (black), and \(n_g = 27\) (gray), averaged over several realizations of the on-site disorder (and for \(c = 8\)). The typical value of \(\text{Im} G\) decreases as \(N^{1-D_1}\) (with \(D_1 \approx 0.98\) for \(W = 12\)). The tails of the distributions exhibit a power-law \(P(\text{Im} G) \propto \text{cst}/(\text{Im} G)^{1+\mu}\) with an exponent \(\mu \approx 0.52\) up to the cut-off at \(1/n\).

FIG. 15: Typical value of the DoS, averaged over many independent realizations of the on-site disorder and of the RRG, as a function of the system size \(n = \ln N/\ln 2\) for several values of the disorder strength. The continuous curves give the results of the BP approximation. The symbols correspond to the values obtained from ED up to the largest accessible system sizes \((N = 2^{15})\). The dotted-dashed black line shows the fits \(\rho^{\text{typ}} \propto N^{1-D_1}\) over the range of \(N\) where one observes an apparent power-law dependence and a multifractal behavior. The red dashed straight line and empty circles represents the behavior of \(\rho^{\text{typ}}\) as a function of \(n = \ln N/\ln 2\) for the Anderson model on the Cayley tree at \(W = 12\) and measured at the root of the tree.

FIG. 16: \(N\) times the IPR, averaged over many independent realizations of the on-site disorder and of the RRG, as a function of the system size \(n = \ln N/\ln 2\) for several values of the disorder strength. The continuous curves give the results of the BP approximation. The symbols correspond to the values obtained from ED up to the largest accessible system sizes \((N = 2^{15})\). The dotted-dashed black line shows the fits \(\langle \Upsilon_2 \rangle \propto N^{-D_2}\) over the range of \(N\) where one observes an apparent power-law dependence and a multifractal behavior. The red dashed straight line represents the behavior of \(\langle \Upsilon_2 \rangle\) as a function of \(n = \ln N/\ln 2\) for the Anderson model on the Cayley tree at \(W = 12\) and measured at the root of the tree.

The \(N\)-dependence of \(\rho^{\text{typ}}\) and \(\langle \Upsilon_2 \rangle\) saturates to a \(N\)-independent value—which coincides with the ones found from the solution of Eqs. \((8)\) and \((9)\)—and ergodicity is restored. Again we observe an excellent agreement between the BP approximation (continuous curves) and the results obtained from ED (filled symbols) up to the accessible system sizes. Yet, due to the fact that the crossover volume \(N_c(W)\) grows exponentially fast as \(W\) is increased and is already very large far below \(W_L\), the recovery of ergodicity is only visible via ED for moderately weak disorder, \(W \lesssim 8\).

It is important to stress that the properties of the crossover region are highly unusual, as the apparently non-ergodic behavior can be characterized by a set of multifractal exponents, e.g., \(D_1\) and \(D_2\), which are well-defined over a broad range of \(N\) and depend on the disorder in a non-trivial way. In order to interpret these results, we also plot the evolution with the system size of the typical DoS and of the IPR at \(W = 12\) at the root of Cayley trees of \(n_g\) generations (see below for a precise definition of these quantities), showing that the spectral fractal dimensions found at the root of the Cayley tree turn out to be surprisingly close to the apparent multifractal exponents observed on the RRG for \(N < N_c\). The same behavior is found at all disorder strengths.

In fact, as discussed above, the Cayley tree is not translationally invariant and sites at different distances from the root are not equivalent, it is instructive to study the behavior of the typical DoS and of the IPR at a given
where \( N_{\ell} = (k + 1) k^{\ell-1} \) is the total number of sites \( i_{\ell} \) belonging to the \( \ell \)-th generation of the tree. As already noticed in [24] the appropriate scaling variable characterizing the position of the sites on a Cayley tree of \( n_g \) generations is the dimensionless distance from the root, \( x = \ell/n_g \), with \( 0 \leq x \leq 1 \). It was shown in [24] that for a given disorder strength \( x \) controls the spectrum of wave-functions’ multifractal exponents. In Fig. 17 we plot the evolution with the system size of \( \rho_{\ell}^{\text{typ}} \) and \( N(\Upsilon_2^{(x)}) \) at the root of the tree, \( x = 0 \) (orange), for \( x = 1/4 \) (red), \( x = 1/2 \) (magenta), \( x = 3/4 \) (violet), and for the whole tree (black) at \( W = 4 \) and \( W = 12 \) showing that the Anderson model on the Cayley tree displays a non-ergodic multifractal behavior at all scales in the whole delocalized phase (except at small enough disorder and sufficiently close to the root [24] [49]), \( \rho_{\ell}^{\text{typ}} \propto N_{\ell}^{-D_{\ell}^{(x)}} \) and \( (\Upsilon_2^{(x)}) \propto N_{\ell}^{-D_{2}^{(x)}} \), with spectral fractal dimensions \( D_{\ell}^{(x)}(W) \) and \( D_{2}^{(x)}(W) \) which decrease as \( x \) is increased (i.e., when one moves closer to the boundary of the tree, consistently with localization of wave-functions at the boundary [24] [53]) and as \( W \) is increased (the spectral fractal dimensions \( D_{1,2}^{(x)} \) all vanish at the Anderson transition at \( W_L \)).

In Figs. 18 and 19 we plot the behavior of \( D_1 \) and \( D_{1,2}^{(x)}(W) \) as a function of the disorder strength for the Anderson model on the Cayley tree for four different positions inside the lattice, \( x = 0, x = 1/4, x = 1/2, \) and \( x = 3/4 \). The spectral fractal dimensions \( D_1 \) and \( D_2 \) of the whole tree are controlled by the one of the leaves \( D_{1,2}^{(x=1)} \), since the boundary contains roughly half of the total sites. We also show on the same plot the apparent spectral fractal dimensions \( D_1 \) and \( D_2 \) measured on the RRG in the non-ergodic crossover region, for \( N < N_c(W) \), which, as anticipated above, turn out to be close to the spectral fractal dimension found at the root of the Cayley tree, \( D_{1,2}(W) \approx D_{1,2}^{(x=0)}(W) \), at the same disorder strength. (Note that the root of the Cayley tree displays a transition at \( W = W_T \approx 6 \), below which we find that \( D_{1,2}^{(x=0)} = 1 \), see, e.g., the top panels of Fig. 17). This transition is tightly related to the ones recently discussed in [19] [20] [24] and will be analyzed in full details in a forthcoming paper [49].

In conclusion, the analysis of the convergence of the LDoS indicate that the Anderson model on the RRG is fully ergodic in the whole delocalized phase, ergodicity being eventually restored on a finite energy scale \( \eta_c(W) \) (resp., a finite system size \( N_c(W) \)) which becomes exponentially small (resp., exponentially large) as \( W_L \) is approached, while the Anderson model on the loop-less Cayley tree displays a genuine multifractal (non-ergodic) behavior in the whole delocalized phase, as already discovered in [24] [24]. However, the non-ergodic crossover region observed on the RRG is highly non-trivial: The

\[
\rho_{\ell}^{\text{typ}} = e^{(\ln \Im G_i)} \langle \Im G \rangle,
\]

\[
\Upsilon_2^{(\ell)} = \lim_{\eta \to 0^+} \frac{\eta}{\pi \rho N_{\ell}} \sum_{i=1}^{N_{\ell}} |G_{i\ell}|^2,
\]
apparent multifractal behavior observed on the RRG for $N < N_c(W)$ seems to be controlled by the the multifractal behavior found at the root of the Cayley, giving rise to non-trivial disorder-dependent fractal exponents.

D. The level compressibility

In order to obtain more information on the level and eigenfunctions’ statistics of the Anderson model on the RRG and on the Cayley tree, and to clarify the differences between the two types of lattices, in the remaining part of this section we study two specific observables related to the statistics of energy levels and wave-functions’ coefficients, which can be easily expressed in terms of the elements of the resolvent matrix, and computed within the BP approach.

Here we start by focusing on the level compressibility, $\chi_N(E)$ [37] for the number of energy levels inside the interval $[-E/2, E/2]$, which, as explained below, is a suitable probe to distinguish between ergodic, localized, and multifractal states [38–42]. To this aim, we first introduce the number of energy levels inside an energy interval of width $E$ (and centered around zero):

$$\mathcal{N}_N(E) = \int_{-E/2}^{E/2} \sum_{m=1}^{N} \delta(E' - E_m) \, dE',$$

where $E_m$ are the eigenvalues of the Hamiltonian. The level compressibility is defined as the ratio between the variance of $\mathcal{N}_N(E)$, characterizing the fluctuations of energy level within $[-E/2, E/2]$, and its average [37]:

$$\chi_N(E) = \frac{(\mathcal{N}_N(E))^2 - \mathcal{N}_N(E)^2}{\mathcal{N}_N(E)},$$

where $\overline{\cdot}$ denotes the average over the disorder.

Let us focus on the behavior of $\chi_N(E)$ when the energy interval is measured in units of the mean level spacings: $E = s\delta$. In the standard ergodic metallic phase, described by the Wigner-Dyson statistics, energy levels strongly repel each other, and the variance scales as $(\mathcal{N}_N(E))^2 - \mathcal{N}_N(E)^2 \propto \ln N \mathcal{N}_N(E)$ [37]. Hence the level compressibility vanishes as $\chi_N(E) \propto \ln N / N$ for large $N$. Conversely, in the localized phase energy levels are thrown as random points on a line and are described by a Poisson distribution. Hence $(\mathcal{N}_N(E))^2 - \mathcal{N}_N(E)^2 = \mathcal{N}_N(E)$ and $\chi_N(E) \to 1$ for $N \to \infty$. Finally, for non-ergodic multifractal states the variance of the number of energy levels inside an interval should scale linearly with the average [38–41], and $\chi_N(E)$ is expected to converge to a (system-dependent) constant between 0 and 1 in the large $N$ limit (at least in simplest scenarios).

The level compressibility in the Anderson model on the RRG has been recently studied in the thermodynamic limit in [12]. However, in this case the limit $N \to \infty$ is taken from the start, while the $s \to 0^+$ and $\eta \to 0^+$ limits are taken after the thermodynamic limit. As already explained above, this strategy does not allow to detect the existence of the putative delocalized non-extended states. One should instead study the behavior of $\chi_N(E)$ at finite $N$, letting $s$ scale as $N^\sigma$, with $\sigma \leq 0 \leq 1$, thereby enabling to scan the statistics of energy levels on all scales, from that of the mean level spacing ($\sigma = 0$) up to energies of order one ($\sigma = 1$). This can be easily achieved in the framework of the BP approximation, since $\mathcal{N}_N(E)$ can be expressed in a simple way in terms of the Green’s functions defined on the nodes and on the edges of the lattice. The calculation on the RRG, which is carried out in full details in App. C, yield:

$$\mathcal{N}_N(E) = \lim_{\eta \to 0^+} \left\{ \frac{1 - k}{2\pi} \sum_{i=1}^{N} \left[ \Psi_i(z_+) - \Psi_i(z_-) \right] \right. + \left. \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{j \in \partial i} \left[ \psi_{i \to j}(z_+) - \psi_{i \to j}(z_-) \right] \right\},$$

where $z_\pm = \pm E/2 + \im \eta$, and the angles $\Psi_i\rightarrow_j(z)$ and $\psi_i\rightarrow_j(z)$ are defined as the phases of $G_{i\rightarrow j}(z)$ and $G_{i\rightarrow j}(z)$ respectively, $G_{i\rightarrow j}(z) = |G_{i\rightarrow j}(z)| e^{i\Psi_{i\rightarrow j}(z)}$, and $G_{i\rightarrow j}(z) = |G_{i\rightarrow j}(z)| e^{i\psi_{i\rightarrow j}(z)}$ (we have chosen here to put the branch-cut in the complex plane along the negative real axis). A very similar expression can be obtained for the Cayley tree, Eq. (C6). In fact, while in the latter case Eq. (C6) is an exact formula for $\mathcal{N}_N(E)$, one should keep in mind that due to the presence of loops Eq. (11) only provides an approximate expression for the number of
energy levels on RRGs of finite size (which is expected to become asymptotically exact in the $N \to \infty$ limit).

In order to analyze the scaling properties of the level compressibility $\chi_N(E)$ we need then to compute the average of $N\chi_N(E)$ and its fluctuations over many independent random instances of large but finite size, using Eqs. (4), (5), and (11), and investigate their asymptotic behavior in the limit of large $N$. Hence, three simultaneous limits are involved: $N \to \infty$, $\eta = c\delta \to 0^+$ (with $c = 8$ as above), and $E = s\delta \to 0^+$, where $\delta = 1/(N\rho(W))$ is the mean level spacings around the middle of the band. (Note that it does not make much sense to take $s$ smaller than $c$, since the broadening of the $\delta$-peaks of the DoS smoothens-out the information on individual levels on energy intervals smaller than $\eta$.)

As far as the existence of the putative non-delocalized phase is concerned, the scaling behavior of the level compressibility on the scale of the mean level spacing only [i.e., for $s = O(1)$] might be uninformative: Consider, for instance, the model of Ref. [21] of the Rosenzweig-Porter type, where an intermediate mixed phase can be explicitly realized in some region of the parameter space. It can be shown that in such phase the level statistics on the scale of the mean level spacing is still described by the GOE ensemble, whereas a crossover to Poisson statistics takes place on a scale $N^{Dz-1}$ which goes to zero with $N$ but stays much larger than $\delta$. In order to be able to describe this situation, we let $s$ be equal to $s = cN^\gamma$, and consider several values of $\sigma \in (0, 1)$. This allows to probe the statistics of energy levels at all scales $E \propto N^{\sigma-1}$ spanning the whole energy range from the scale of the mean level spacing ($\sigma = 0$) up to energies of $O(1)$ ($\sigma \to 1$).

The results for $\chi_N(E)$ for the Anderson model on the RRG are plotted in Figs. 20 and 21 for $W = 11$ and $W = 12$ respectively, as a function of the energy measured in units of the mean level spacing $\eta$, for several system sizes, $N = 2^n$, with $n = 11, \ldots, 26$. The level compressibility has been averaged over many independent realizations of the on-site disorder and of the RRG. From the top panels we notice that at large enough energy (and/or small enough $N$), $\chi_N(E)$ seems to approach a constant value between zero and one ($\chi \approx 0.49$ for $W = 11$ and $\chi \approx 0.57$ for $W = 12$), which is a typical signature of non-ergodic multifractal states. However, when the energy is decreased below a certain value, $\chi_N(E)$ departs from the plateau value and decreases to zero. The energy at which $\chi_N(E)$ reaches the plateau grows proportionally to $N$ as the system size is increased. Hence, if the system size is too small (i.e., $N < N_\chi(W)$) one is not able to observe the departure from the plateau and the system behaves as if it was in a genuine non-ergodic phase, with a well defined value of $\chi \in (0, 1)$. We also show the data obtained from ED (filled symbols) up to the largest available system size, $N = 2^{15}$, (averaged however over much fewer samples). They are in reasonably good agreement within the numerical accuracy with the BP results.

In the bottom panel we plot the rescaled level com-

---

**FIG. 20:** Top panel: Level compressibility, $\chi_N$, (averaged over many independent realizations of the on-site disorder and of the RRG) plotted as a function of $E/\delta$ for the Anderson model on the RRG at $W = 11$ and for several system sizes $N = 2^n$ with $n = 11, \ldots, 26$ (and for $c = 8$). Continuous lines show the results found with the BP approach while full symbols represent the data obtained from BP (averaged over $2^{15-n}$ samples only). Bottom panel: Same data as above with a rescaling of the $y$-axis as $N\chi_N/\ln N$. The vertical dashed lines spot the values of the energy at which the curves corresponding to different sizes deviate from the scaling function.

**FIG. 21:** Top panel: Level compressibility, $\chi_N$, (averaged over many independent realizations of the on-site disorder and of the RRG) plotted as a function of $E/\delta$ for the Anderson model on the RRG at $W = 12$ and for several system sizes $N = 2^n$ with $n = 11, \ldots, 26$ (and for $c = 8$). Continuous lines show the results found with the BP approach while full symbols represent the data obtained from BP (averaged over $2^{15-n}$ samples only). Bottom panel: Same data as above with a rescaling of the $y$-axis as $N\chi_N/\ln N$. The vertical dashed lines spot the values of the energy at which the curves corresponding to different sizes depart from the scaling function.
pressibility, $N \chi_N(E)/\ln N$, which should collapse onto a $N$-independent scaling function in the limit of large sizes if the Wigner-Dyson statistics is recovered. This is precisely what we observe in the bottom panels, which exhibit a nice collapse for small enough energies and large enough sizes. The values of the energy at which the curves corresponding to different $N$ deviate from the scaling function are spotted as vertical dashed line, and are found to scale proportionally to $N$ for large enough sizes. This behavior indicates that, provided that $N$ is sufficiently large, ergodicity and GOE statistics are eventually recovered in the delocalized phase of the Anderson model on the RRG on an energy scale which remains finite (and which vanished exponentially at $W_L$).

The situation is drastically different on the Cayley tree, as shown in Fig. 22. We indeed observe that, when the number of generations $n_g$ of the tree is increased, the level compressibility approaches asymptotically a function which is roughly constant and which stretches to larger and larger values of the energy as the system size is increased. This is a clear signature of multifractal non-ergodic states characterized by sub-Poissonian statistics on all energy scales [38, 40, 41]. The plateau value of $\chi$ (green dashed lines) increases as $W$ is increased, and is already large at small disorder (e.g., $\chi \approx 0.77$ for $W = 2$), and is very close to unity at moderate disorder strength ($\chi \approx 1$ at $W = 12$), compatible with the localization of wave-functions close to the boundary of the tree.

E. The overlap correlation function

Another very useful probe of the statistics of the eigenfunctions which allows to distinguish between ergodic, localized, and multifractal states is provided by the overlap correlation function between eigenstates at different energy levels [21, 40, 43–45], defined as:

$$K_2(E) = \frac{N \sum_i \sum_{m,m'} |\langle i|m\rangle|^2 |\langle i|m'\rangle|^2 \delta[E - (E_m - E_{m'})]}{\sum_{m,m'} \delta[E - (E_m - E_{m'})]}$$

(12)

where $\langle i|m\rangle$ is the amplitude of the eigenvector $|m\rangle$ on site $i$.

For eigenfunctions of GOE matrices $K_2(E) = 1$ identically, independently on $E$ on the entire spectral bandwidth. In the standard (ergodic) metallic phase $K_2(E)$ has a plateau at small energies, $K_2(E) \approx q_2$ for $E < E_{Th}$, followed by a fast-decay which is described by a power-law, $K_2(E) \sim E^{-\gamma}$, with a system-dependent exponent [43]. The height of the plateau is larger than one, which implies an enhancement of correlations compared to the case of independently fluctuating Gaussian wave-functions. The Thouless energy, $E_{Th}$, which separates the plateau from the power-law decay stays finite in the thermodynamic limit and extends to larger energies as one goes deeply into the metallic phase, and corresponds to the energy range over which GOE-like correlations establish [43].

The behavior of the overlap correlation function for multifractal wave-functions is instead drastically different, as shown in [21]: The plateau is present only in a narrow energy interval $E < E_{Th} \sim \delta N^{D_2}$ which shrinks to zero in the thermodynamic limit as $N^{C_2-1}$, while its height grows $N^{1-D_2}$. This can be interpreted recalling that multifractal wave-functions typically occupy a fraction $N^D$ of the total sites, which implies the existence of an energy scale, $E_{Th}$, which decreases with $N$ but stays much larger than the mean level spacing, beyond which eigenfunctions poorly overlap with each other and the statistics is no longer GOE.

For any given random instance of the Hamiltonian, the overlap correlation function [12] can be easily expressed in terms of the Green’s functions computed at energies $\pm E/2$ as:

$$K_2(E) = \lim_{n \to 0^+} \frac{N \sum_i \text{Im} G_i(-E/2) \text{Im} G_i(E/2)}{\sum_i \text{Im} G_i(-E/2) \sum_i \text{Im} G_i(E/2)} .$$

In order to determine the scaling properties of the overlap correlation function, we have computed the average of $K_2(E)$ over many independent realizations of the disorder for the Anderson model on the RRG and on the Cayley tree, using the expression above where the Green’s functions are evaluated at the fixed point solution of the BP equations, and for energy differences varying from the scale of the mean level spacing up to energy differences of $O(1)$.
The results for the RRG are plotted in Fig. 23 for $W = 11$ and $W = 12$, showing that the $N$-dependence of $K_2(E)$ saturates for large enough $N$ and that the curves converge to a $N$-independent limiting function characterized by a plateau at small energy followed by a fast decrease [$K_2(E) \sim (E_{Th}/E)$] at large energy corresponding to the onset of level repulsion (with $\gamma \approx 1$ independently of $W$) [20]. The crossover from the plateau to the power-law decay takes place on the energy scale $E_{Th}$ (vertical red dashed lines), which stays finite in the thermodynamic limit and represents the width of the energy band within which GOE-like correlations are established [43]. This behavior is very similar to the one found in the metallic phase of the 3d Anderson model close to the critical point. In particular, the fact that the plateau survives in the $N \to \infty$ limit and extends to larger energies as one goes deeply into the conducting phase is a clear signature of ergodic states [13][45]. However, the fact that its value is much larger than one is an apparent manifestation of the enhancement of correlations and of the fact that wave-functions show significant deviations from uncorrelated Gaussian random variables. We again observe an excellent agreement between the results obtained using the BP approximation and EDs (note, however, that the BP approximation does not allow to access energies smaller than the broadening of the energy levels, $\delta\eta$, for the reasons explained above). Nevertheless, at $W = 11$ and $W = 12$, deep into the non-ergodic-like crossover regime, the largest system sizes via ED are too small to allow to observe the convergence of $K_2(E)$.

The Thouless energy is found to be proportional to the energy scale $\eta_L(W)$ below which the probability distribution of the local DoS converges to a stable non-singular distribution (see Fig. 13), and thus vanishes exponentially at $W_L$. Moreover, $E_{Th}$ turns out to coincide (within our numerical accuracy) with the energy scale below which the Wigned-Dyson asymptotic scaling of the level compressibility is recovered (vertical dashed lines of Figs. 20 and 21), indicating that the energy band within which the statistics of energy levels is described by the Wigner-Dyson statistics coincides with the one over which wave-functions correlations are GOE-like and $K_2(E)$ has a plateau.

The situation on the Cayley tree is completely different. In this case, as shown in Fig. 24, $K_2(E)$ presents all the distinctive features typically observed for multifractal states: the Thouless energy decreases with the system size as $\delta N^{D_2} \propto N^{D_1-1}$ whereas the height of the plateau grows as $N^{1-D_2}$. The curves of $K_2(E)/N^{1-D_2}$ for different $N$ collapse (for large enough $N$ and small enough energies) onto the same curve once the energies are rescaled by $E_{Th}$. In fact, as discussed above, the value of $D_2$ is actually very close to zero at moderate disorder strength ($D_2 \approx 0.002$ at $W = 12$) and is already very small at weak disorder ($D_2 \approx 0.04$ for $W = 2$). Note that the power-law decay from the plateau, $K_2(E) \sim (E_{Th}/E)^\gamma$, observed on the Cayley tree is quite different with respect to the RRG: We find that the exponent $\gamma$ is greater than one and slowly increases with $W$ ($\gamma \approx 1.35$ at $W = 2$).
and $\gamma \approx 1.95$ at $W = 12$). Interestingly, in the region where the fractal exponents $D_1$ and $D_2$ are close to zero ($W \gtrsim 10$, see Figs. 18 and 19), the value of the exponent $\gamma$ is very close to $\gamma \approx 2$, which is the same found in the whole delocalized non-ergodic phase of the random matrix model of the Rosenzweig-Porter type of Ref. 21-22.

V. RECAP OF THE MAIN RESULTS, CONCLUSIONS, AND PERSPECTIVES

In this paper we have studied the Anderson model on two different kinds of Bethe lattices, the RRG and the loop-less Cayley tree, focusing in particular on the ergodic properties of the delocalized phase on these two lattices. Our analysis is based on a novel approach which consists in solving the iteration relations for the Green’s functions directly on random instances of large but finite sizes. We start this section by giving below a sketchy summary of the main results.

1) Exact diagonalization on the RRG: Characteristic crossover scale.
In Sec. [11] we have presented an accurate numerical analysis of several observables and probes associated to level and eigenfunction statistics that display different universal behaviors in the ergodic and non-ergodic regimes (such as the ratio of adjacent gaps, the overlap between eigenvectors corresponding to subsequent eigenvalues, the IPR, the wave-functions’ support set, and their spectrum of fractal dimensions). We performed EDs on the delocalized side of the Anderson transition on RRGs of size $N$ from $2^6$ to $2^{15}$. Our results clearly show the existence of a characteristic system size governing finite size effects, $N_c(W)$, as already observed in 13, 25, 29, which diverges much faster than a power-law approaching the localization transition (as predicted by the supersymmetric analysis 27) and is already very large far from it. The most important observation is that the behavior of all the considered observables, both those associated to the statistics of energy levels on the scale of the mean level spacings, and those related to the statistics of wave-functions, is governed by the correlation volume $N_c(W)$ (see Fig. [21]), suggesting that the crossover from Poisson statistics and multifractal wave-functions to GOE statistics and ergodic wave-functions occurs concomitantly.

2) BP solution: Convergence of the local density of states and fractal exponents.
In Sec. [IV] we discussed the results found computing the BP solution of the self-consistent iteration equations for the Green’s functions of the Anderson model on the RRG and on the Cayley tree on very large but finite instances of size $N$ from $2^{11}$ to $2^{29}$ sites. In Sec. [IVA] we have shown that the results obtained using the BP approximation on the RRG are in excellent agreement with the exact solution obtained from ED (up to the largest system sizes accessible via ED, $N = 2^{15}$), provided that the imaginary regulator $\eta$ is of the order of the mean level spacing, i.e., $\eta = \delta$, with $\delta = 1/(N \rho(W))$ [where $\rho(W)$ is the average DoS at the center of the band]. We show in particular that the BP solution provides a tight and controlled approximation not only for average and/or global quantities, but also for local observables, and accounts accurately for sample to sample and spatial fluctuations. (Note that BP is exact on the Cayley tree due to the absence of loops.)

In Sec. [IV.B] we focused on the probability distribution of the LDoS obtained within the BP approach on the RRG, and showed that the dependence on the system size of $P(\text{Im}G)$ saturates for large enough sizes (i.e., $N \gg N_c(W)$ or, equivalently, for $\eta$ smaller than a disorder-dependent energy scale $\eta_c(W)$ which stays finite in the delocalized phase and vanishes exponentially at $W_L$), and convergence to a stationary, size independent, stable, non-singular, probability distribution is observed (at least up to the largest accessible disorder strength $W \lesssim 13.5$). Interestingly, the crossover scale $N_c(W)$ obtained from the convergence of the LDoS within the BP approach, Eq. (10), accounts very well for the scale above which ergodic behavior emerges (see Fig. 6).

Conversely, we observed that the Anderson model on the Cayley tree displays a genuine multifractal, non-ergodic behavior at all scales in the whole delocalized phase, in agreement with [24, 25]. We computed the fractal exponents $D_1$ and $D_2$ associated to the spectral statistics, which exhibit a non-trivial dependence on the position inside the tree [24, 49], and we showed that the apparent non-ergodic features observed on the RRG for $N < N_c$ seems to be controlled by the multifractal properties of the region close to the root of the Cayley tree at the same disorder strength.

3) Level compressibility and overlap correlation function.
In Secs. [IV.D] and [IV.E] we focused on two spectral probes, such as the level compressibility $\chi_N(E) = \partial \rho(N)/(\partial N)$ and the overlap correlation function $K_2(E)$, associated respectively with the statistics of level spacings and eigenfunctions that display very different scaling behavior in the delocalized, localized and intermediate mixed phase [21, 38-45]. These observables can be easily expressed in terms of the Greens’ functions obtained from the BP solution of the Anderson model on the RRG and on the Cayley tree. Their analysis on the RRG reveal the existence of an energy scale, $E_{Th}(W)$, which remains finite in the whole delocalized phase, corresponding to the window in energy within which the Wigner-Dyson level statistics is recovered and eigenfunctions exhibit GOE-like correlations, corresponding to a size-independent plateau of $K_2(E)$ at small energy separation 43. Such energy scale vanishes exponentially fast approaching $W_L$, and is in fact proportional to $N_c^{-1}$. Hence, for $N < N_c(W)$ the mean level spacing is larger than $E_{Th}(W)$ and the system looks like as if it were in an intermediate non-ergodic delocalized phase.

Conversely, on the Cayley tree the behavior of $\chi_N(E)$ and $K_2(E)$ is fully consistent with the existence of gen-
unely multifractal states in the whole delocalized phase (with localization of the wave functions close to the boundary of the tree). In particular, energy levels on the Cayley tree exhibit a sub-Poissonian statistics (in fact, very close to Poissonian already very far from \( W_L \)), while the analysis of eigenfunctions’ correlations show the existence of an energy scale which decreases with \( N \) (as \( N^{D_2-1} \)) but stays larger than the mean level spacing, which is the hallmark of non-ergodic extended states.

All in all, the results presented in this paper support in a coherent way the idea that the Anderson model on the RRG becomes fully ergodic in the whole delocalized phase: ergodicity and GOE statistics are eventually recovered in the thermodynamic limit in the whole extended phase, implying that the GOE-ergodic/Poisson-non-ergodic transition of the energy levels and eigenvalues is concomitant with Anderson localization, in agreement with the recent results of \([28,30]\) and with the predictions of \([26,27]\) based on supersymmetric field theory. Nonetheless, ergodicity establishes on a system size (resp., energy scale) which becomes exponentially large (resp., small) as the localization transition is approached, and exceeds the system sizes accessible via ED well before the localization transition, resulting in a very wide crossover region in which the system looks as if it were in a mixed (delocalized but non-ergodic) phase for all practical purposes, i.e. on finite but large length and time scales (volumes smaller than \( N_c \) and times smaller than \( h/E_{Th}(W) \)).

Furthermore, the apparent non-ergodic-like crossover region observed on the RRG for \( N < N_c \) has highly non-trivial properties, and is characterized by a set of effective disorder-dependent fractal exponents which are independent on \( N \) in a broad range of system sizes. Such apparent multifractal behavior seems to be controlled by the one of the root of the Cayley tree at the same disorder strength. Indeed, a genuine non-ergodic extended phase is found in the Anderson model on the loop-less Cayley tree in the whole delocalized side, as predicted in \([24,25]\). The properties of such phase will be discussed in more details in a forthcoming paper \([19]\).

On the basis of the analogy between Anderson localization on Bethe lattices and Many-Body Localization \([7,11]\), these phenomena might play a very important role and lead to highly non-trivial behaviors in the delocalized phase of many-body interacting disordered systems exhibiting MBL \([48,49]\).

Given the difficulty of the questions we are addressing, it is natural to dwell about possible limitations of our analysis. For instance, there is the possibility that for some reason the BP approach starts to fail in some region of the parameters space, and in particular within the putative delocalized non-ergodic phase, \( W ≥ W_E \) and \( N \) very large. However, besides the fact that an excellent agreement between the BP approximation and ED results is found for all observables and probes considered and that BP passed successfully all the numerical tests of Sec. \([V,A]\) there are no examples in the literature of other models where something similar might happen. On the contrary, the BP approximation is expected on general grounds to improve as \( N \) is increased \([36]\). Yet, although there is a rigorous proof of the convergence of the BP solution for the Anderson model on the RRG in the large \( N \) limit \([59]\), there is no rigorous estimate of the error at large but finite \( N \). It would be very interesting in this respect to characterize in a quantitative way the convergence of both local and average observables obtained from the BP approximation. In standard statistical mechanics models one generally finds that the finite-size corrections of BP for global quantities, such as, e.g., the free-energy, are of order \( O(1/N) \) (in the replica-symmetric phase) \([63]\). Here instead our numerical results suggest that, up to the moderately large size accessible via ED, global observables approach their exact values as \( 1/\sqrt{N} \). Further work is necessary to obtain more definite conclusions.

Another point worth mentioning is that all results discussed above are valid for \( η > δ \), where the simultaneous limits \( N → ∞ \) and \( η ∝ 1/N → 0^+ \) are taken. Recent studies of the LDoS on the delocalized side of the Anderson model on the RRG seem to suggest that its statistical properties might be unusual in the regime \( η ≪ δ \) \([62]\). As discussed above, the BP approach is not applicable to this situation and here we only focused on the more standard case \( η > δ \).

Another related interesting perspective would be to benchmark the BP framework onto the random matrix models of the Rosenzwieg-Porter type of Ref. \([21,22]\), which is characterized by a whole region of the parameter space where wave-functions are delocalized but truly multifractal. Preliminary results (which will be discussed in a forthcoming work \([64]\)) indicate that in this case BP is able to detect correctly the presence of the delocalized non-ergodic states.

Appendix A: Multifractality

In this appendix we give more information and details on the computation of the spectrum of fractal dimensions of wave-functions coefficients. In order to obtain \( f_N(α) \), we have computed the average of the moments \( ⟨ Y_q(n)⟩ \), for different system sizes \( N = 2^n \), with \( n \) from 6 to 15, and for 400 different values of \( q \) in the interval \((-3, 5)\). Data are averaged over (at least) \( 2^{22-n} \) samples, and over \( 1/8 \) of the eigenstates around the middle of the band. For each value of the disorder strength \( W \), \( τ_N(q) \) is obtained as (minusc) the derivative of the logarithm of the moments with respect to the logarithm of the system size, which
for different values of $W = 13$) and for other values of the disorder in the delocalized phase ($W = 2$). In order to check the accuracy of our numerical procedure, in Fig. 25 we verify that the non-trivial symmetry relation of Eq. (A3), for $W = 5$ and $W = 13$ and for $n = 10$ (coinciding red continuous curves and blue dashed curves). Similar plots are found for other values of $W$ in the delocalized phase and for other system sizes.

FIG. 25: Verification of the symmetry relation of Eq. (A3), for $W = 5$ and $W = 13$ and for $n = 10$ (coinciding red continuous curves and blue dashed curves). Similar plots are found for other values of $W$ in the delocalized phase and for other system sizes.

can be approximately evaluated as:

$$\tau_N(q) = -\frac{\ln\langle Y_q(n)\rangle - \ln\langle Y_q(n - \delta n)\rangle}{\delta n \ln 2}.$$ (A1)

We then have computed $\alpha_N(q)$ as the derivatives of $\tau_N(q)$ with respect to $q$:

$$\alpha_N(q) = \frac{\tau_N(q + \delta q) - \tau_N(q)}{\delta q}.$$ (A2)

For simplicity, in most of the cases we have chosen $\delta n = 1$, and we have used $\delta q = 5 \cdot 10^{-5}$. Finally, we evaluate numerically the Legendre transform $f_N(\alpha_N) = \alpha_N(q) - \tau_N(q)$, where $\tau_N(q)$ and $\alpha_N$ are given by Eqs. (A1) and (A2).

As demonstrated in [16, 17], in the region of extended states the spectrum of fractal dimensions should obey the following symmetry relation:

$$f(1 + \alpha) = f(1 - \alpha) + \alpha.$$ (A3)

In order to check the accuracy of our numerical procedure, in Fig. 25 we verify that the non-trivial symmetry relation (A3) is indeed nicely fulfilled for $f_N(\alpha)$ for two values of the disorder in the delocalized phase ($W = 5$ and $W = 13$) and for $N = 2^1$. Similar outcomes are found for different values of $W$ in the extended regime and for other values of $N$.

2 Note that we have performed an annealed computation (logarithm of the average) instead of the quenched one (average of the logarithm). One can show that the spectrum of fractal dimensions obtained using the two definitions coincide as far as $f(\alpha) > 0$, i.e., in the whole support $\alpha \in (\alpha_-, \alpha_+)$. For $W = 5$, $W = 10$, and $W = 13$ where we have considered smaller values of $\delta n$ in order to obtain more precise results.

FIG. 26: Spectrum of fractal dimensions $f_N(\alpha)$ for $W = 5$ and for different system sizes $N = 2^n$ with $n$ from 9 to 13. The inset shows a zoom of the same curves in the region close to $\alpha = 1$. The straight line $f(\alpha) = \alpha$ (black dashed line) is tangent to $f_N(\alpha)$ in $\alpha_1$.

FIG. 27: Main panel: $D_2$, $\alpha_m$, $\alpha_1$, and $\alpha_{cross}$ as a function of $n = \log_2 N$. Inset: $D_2$, $\alpha_m$, $\alpha_1$, and $\alpha_{cross}$ approach 1 exponentially in $n$ on the same characteristic scale.

In the following we will focus in particular on the $N$-dependence of four specific points of the singularity spectrum: the most probable value $\alpha_m$ where $f_N(\alpha)$ reaches its maximum, $f_N(\alpha_m) = 1$; the point $\alpha_1$ (associated to $q = 1$) where $f_N(\alpha_1) = \alpha_1$, and $f_N(\alpha_1) = 1$; the lower edge of the support of $f_N(\alpha)$, $\alpha_-$; the point $\alpha_{cross}$ where the spectra of fractal dimensions for two subsequent system sizes cross.

In Fig. 26 the singularity spectrum is plotted for $W = 5$ and for several system sizes $N = 2^n$ with $n$ from 9 to 13 (the inset shows a zoom of the same curves in the region close to $\alpha = 1$). One clearly observes that the support of $f_N(\alpha)$ shrinks as $N$ is increased.

From Fig. 25 we determine the value of $\alpha_1$ (where...
by taking into account their distance from the root. This can be done by introducing at each generation \( \ell \) the probability distributions of two types of cavity Green’s functions, \( U_\ell(\overline{G}) \) and \( V_\ell(\overline{G}) \) defined, respectively, in absence of the edge with a site of the previous or the next generation. These functions must satisfy the following functional equations:

\[
U_\ell(\overline{G}) = \int dp(\epsilon) \prod_{i=1}^{k} dU_{\ell+1}(\overline{G}_i) \delta \left( \overline{G}^{-1} + \epsilon + z + \sum_{i=1}^{k} \overline{G}_i \right),
\]

\[
V_\ell(\overline{G}) = \int dp(\epsilon) \prod_{i=1}^{k-1} dU_{\ell+1}(\overline{G}_i) \, dV_{\ell-1}(\overline{G}_0)
\]

\[
\times \delta \left( \overline{G}^{-1} + \epsilon + z + \sum_{i=1}^{k-1} \overline{G}_i + \overline{G}_0 \right),
\]

with \( \ell = 0, \ldots, n_g \) and \( \ell' = 1, \ldots, n_g - 1 \), with the initial condition at the boundary:

\[
U_{n_g}(\overline{G}) = \int dp(\epsilon) \delta \left( \overline{G}^{-1} + \epsilon + z \right),
\]

and with the prescription that \( V_0(\overline{G}) \equiv U_0(\overline{G}) \). From the equations above, one can finally obtain the probability distributions of the Green’s functions at any generation of the tree:

\[
R_\ell(\overline{G}) = \int dp(\epsilon) \prod_{i=1}^{k} dU_{\ell+1}(\overline{G}_i) \, dV_{\ell-1}(\overline{G}_0)
\]

\[
\times \delta \left( \overline{G}^{-1} + \epsilon + z + \sum_{i=1}^{k} \overline{G}_i + \overline{G}_0 \right),
\]

\[
R_0(\overline{G}) = \int dp(\epsilon) \prod_{i=1}^{k+1} dU_{1}(\overline{G}_i) \delta \left( \overline{G}^{-1} + \epsilon + z + \sum_{i=1}^{k+1} \overline{G}_i \right),
\]

with \( \ell = 1, \ldots, n_g \). Note that deep in the bulk of the tree, in the limit \( n_g \to \infty \) at finite \( \eta \), the probability distributions becomes \( \ell \)-independent, we recover the functional equations \([8]\) and \([9]\) found for infinite RRGs. However, if one consider the simultaneous limits \( N \to \infty \) and \( \eta \propto 1/N \to 0^+ \), the fixed point of Eqs. \([8]\) and \([9]\) is never reached and is immaterial as far as the spectral statistics is concerned.

### Appendix C: Calculation of the number of energy levels \( \mathcal{N}_N(E) \)

In this appendix we show how to express the number of energy levels inside the interval \([-E/2,E/2], \mathcal{N}_N(E)\), in terms of the Green’s functions and the cavity Green’s functions defined within the BP approach. In order to do this, one can proceed in two equivalent ways, either using the representation of the Heaviside step function \( \theta(x) \) (for \( x \in \mathbb{R} \)) in terms of the discontinuity of the complex logarithm along the negative real axis, \( \theta(x) = \)

![Image](image.png)
\( \lim_{\eta \to 0^+} [\ln(x + i\eta) - \ln(x - i\eta)] \), as done in [32], or starting directly from the definition of the density of state \( N\rho_N(E) = \lim_{\eta \to 0^+} \text{Tr Im} \mathcal{G} \). Here we follow the second path, and write:

\[
N\rho_N(E) = \frac{1}{\pi} \lim_{\eta \to 0^+} \text{Im} \sum_{i=1}^N \int \mathcal{D} \phi \phi_i^2 e^{-\frac{i}{2} \sum_{j,k} \phi_j (\mathcal{H} - z)_{jk} \phi_k} Z(z),
\]

where the “partition function” \( Z(z) \) is defined as:

\[
Z(z) = \int \mathcal{D} \phi e^{-\frac{i}{2} \sum_{j,k} \phi_j (\mathcal{H} - z)_{jk} \phi_k} = \frac{\pi N/2}{\text{det}(\mathcal{H} - z)},
\]

and \( z = E + i\eta \). From the expressions above, it is straightforward to rewrite the DoS as:

\[
N\rho_N(E) = \frac{1}{\pi i \eta \to 0^+} \left[ \frac{\partial \ln Z(z)}{\partial E} - \frac{\partial \ln Z(z^*)}{\partial E} \right].
\]

Inserting this equation into the definition of the number of energy levels within the interval \([-E/2, E/2]\), \( N_N(E) = N \int_{-E/2}^{E/2} \rho_N(E') dE' \), one finally ends up with:

\[
N_N(E) = \frac{1}{\pi i \eta \to 0^+} \left[ \ln Z(E/2 + i\eta) - \ln Z(-E/2 + i\eta) - \ln Z(E/2 - i\eta) + \ln Z(-E/2 - i\eta) \right].
\]

The “generalized free-energy” \( F(z) = \ln Z(z) \) can be easily computed within the BP approach as a sum of local contributions involving the Green’s functions defined on the nodes of the RRG and the cavity Green’s functions defined on the links of the RRG. More precisely it can be shown that \( \ln Z(z) \) can be written as a sum of a site and a link contributions [30, 60]:

\[
F(z) = \sum_{i=1}^N \Delta F_s^{(i)}(z) - \sum_{(i,j)} \Delta F_l^{(i\rightarrow j)}(z),
\]

where \( \Delta F_s^{(i)}(z) \) is the “free-energy shift” corresponding to the addition of site \( i \) to the lattice:

\[
e^{\Delta F_s^{(i)}} = \int \mathcal{D} \phi \prod_j \mathcal{D} \phi_j e^{(\epsilon_i + z) \frac{\phi_i^2}{2} - \sum_j \left[ \frac{\phi_j^2}{2\epsilon_j^2} - i\epsilon_j \phi_j \right]} \prod_j \mathcal{D} \phi_j e^{-\sum_j \frac{\phi_j^2}{2\epsilon_j^2}} = \sqrt{2\pi G_i},
\]

where the index \( j \) runs over the \( k + 1 \) neighbors of \( i \), and \( \Delta F_l^{(i\rightarrow j)}(z) \) is the “free-energy shift” corresponding to the addition of the link between sites \( i \) and \( j \):

\[
e^{\Delta F_l^{(i\rightarrow j)}} = \int \mathcal{D} \phi_i \mathcal{D} \phi_j e^{-\frac{\phi_i^2}{2\epsilon_i^2} - \frac{\phi_j^2}{2\epsilon_j^2} + i\epsilon_{ij} \phi_i \phi_j} \int \mathcal{D} \phi_i \mathcal{D} \phi_j e^{-\frac{\phi_i^2}{2\epsilon_i^2} + \frac{\phi_j^2}{2\epsilon_j^2}} = (1 - t^2 G_{i\rightarrow j} G_{j\rightarrow i})^{-1/2}.
\]

In fact, the addition of a site \( i \) can be equivalently viewed as a two-step process: first the cavity iteration involving the site \( i \) and only \( k \) of its \( k + 1 \) neighbors (say, sites \( \{j_1, \ldots, j_k\} \) and then the addition of the link between the cavity site \( i \) and the missing neighbors \( j_{k+1} \). Then one has that [30, 60]:

\[
\Delta F_s^{(i)}(z) = \Delta F_{\text{iter}}^{(i\rightarrow j)}(z) + \Delta F_l^{(i\rightarrow j)}(z),
\]

which implies that the “free-energy” (C2) can be equivalently rewritten as:

\[
F(z) = \frac{1}{2} \sum_{i=1}^N \Delta F_s^{(i)}(z) + \sum_{(i,j)} \Delta F_{\text{iter}}^{(i\rightarrow j)}(z),
\]

where the “iteration free-energy shift” reads:

\[
e^{\Delta F_{\text{iter}}^{(i\rightarrow j)}} = \sqrt{2\pi G_{i\rightarrow j}}.
\]

Plugging the “free-energy shifts” into Eqs. (C2) and (C3) one finds two equivalent expressions for the generalized free-energy:

\[
F(z) = \frac{1}{2} \sum_{i=1}^N \ln[2\pi G_i(z)] + \frac{1}{2} \sum_{(i,j)} \ln[1 - t^2 G_{i\rightarrow j}(z) G_{j\rightarrow i}(z)] + \frac{1}{4} \sum_{i=1}^N \ln[G_i(z)] + \sum_{(i,j)} \ln[G_{i\rightarrow j}(z)] - \frac{N}{2} \ln(2\pi),
\]

Using the iteration equations (4) and (5), by noticing that \( G_{i\rightarrow j}^{-1} = G_{i\rightarrow j}^{-1} - t^2 G_{i\rightarrow j} \), one can explicitly show that these two expressions are in fact the same. Furthermore, since \( G_i(z^*) = G_i^*(z) \) one has that:

\[
\ln \frac{G_i(z)}{G_i^*(z)} = 2i\Psi_i(z),
\]

where \( G_i(z) = |G_i(z)| e^{i\Psi_i(z)} \). From now on we choose to define the angles in the interval \([-\pi, \pi]\), i.e., we place the branch-cut of the logarithm along the negative real axis. In fact, since the imaginary part of the Green’s functions are all positive for \( \eta > 0 \) by definition, all the \( \Psi_i \) and \( \psi_{i\rightarrow j} \) involved in the equations will fall in the interval \([-\pi, \pi]\).

Hence, plugging the second line of Eq. (C4) into Eq. (C1) one finds Eq. (11) given in the main text. Equivalently, from the first line of Eq. (C4) one gets:

\[
N_N(E) = \frac{1}{\pi i \eta \to 0^+} \left\{ \sum_{i=1}^N \left[ \Psi_i(z_+) - \Psi_i(z_-) \right] + \sum_{(i,j)} \left[ \varphi_{i\rightarrow j}(z_+) - \varphi_{i\rightarrow j}(z_-) \right] \right\},
\]

where \( z_{\pm} = \pm E/2 + i\eta \), and the angle \( \varphi_{i\rightarrow j}(z) \) is defined as the phase of \( 1 - t^2 G_{i\rightarrow j}(z) G_{j\rightarrow i}(z) \).

For a random diagonal Hamiltonian, \( \mathcal{H} = -\epsilon_i \delta_{ij} \) (i.e., \( t = 0 \)), for which one has that \( G_i = G_{i\rightarrow j} = (-\epsilon_i - z)^{-1} \), one can explicitly check using the representation of the Heaviside step function in terms of the discontinuity
of the complex logarithm along the negative real axis that both Eqs. [11] and [C6] both give back \( \mathcal{N}(E) = \int_{E/2}^{E/2} \delta(E' + \epsilon_i) dE' \).

The computation of \( F(z) \) on the Cayley tree is even easier, since one can obtain its expression directly by integrating out progressively the sites starting from the boundary. This yields:

\[
Z(z) = \prod_{\ell=1}^{n_\eta} \prod_{i=1}^{N_\ell} \sqrt{2\pi G_{i\ell\rightarrow i\ell-1}(z)} \sqrt{2\pi G_0(z)},
\]

where \( N_\ell = (k + 1)^k \ell - 1 \) is the total number of sites \( i_\ell \) belonging to the \( \ell \)-th generation of the tree. Plugging this expression into Eq. [C1] one finally obtains:

\[
\mathcal{N}(E) = \frac{1}{\pi} \lim_{\eta \to -i} \left\{ \psi_\eta(z_+) - \psi_\eta(z_-) \right\}
\sum_{\ell=1}^{n_\eta} \sum_{i=1}^{N_\ell} \left[ \psi_{i\ell\rightarrow i\ell-1}(z_+) - \psi_{i\ell\rightarrow i\ell-1}(z_-) \right],
\]

(C6)

Acknowledgments

We thank I. Aleiner, B. L. Altshuler, E. Bogomolny, J.-P. Bouchaud, C. Castellani, Y. Fyodorov, T. Garel, L. Ioffe, V. Kravtsov, P. Le Doussal, G. Lemarié, A. D. Mirlin, C. Monthus, M. Muller, V. Oganessian, G. Parisi, V. Ros, A. Scardicchio, G. Sernierian, K. S. Tikhonov, S. Warzel for useful inputs, remarks and discussions. This research was partially supported by a grant from the Simons Foundation (# 454935 Giulio Biroli). Marco Tarzia is a member of the Institut Universitaire de France.

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