Effective pair interaction between impurity particles induced by a dense Fermi gas

David Mitrouskas and Peter Pickl

Abstract

We study the dynamics of a small number of impurity particles coupled to the ideal Fermi gas in a $d$-dimensional box. The impurities interact with the fermions via a two-body potential $\lambda v(x)$ where $\lambda$ is a coupling constant and $v(x)$ for instance a screened Coulomb potential. After taking the large-volume limit at positive Fermi momentum $k_F$ we consider the regime of high density of the fermions, that is, $k_F$ large compared to one. For coupling constants that scale like $\lambda^2 \sim k_F^{(2-d)}$ we show that the impurity particles effectively decouple from the fermions but evolve with an attractive pair interaction among each other which is induced by fluctuations in the Fermi gas.

1 Introduction and Main Result

The presence of a surrounding medium can change the behaviour of quantum particles in a drastic way. Besides altering individual properties like mass and charge, the presence of an environment might also lead to medium-induced interactions between the particles. A prominent example of this effect is the phonon-mediated interaction between two repulsive polarons. The induced interaction is attractive and has the potential to overcompensate the repulsion between the polarons which may even cause the formation of a new quasi particle, the so-called bipolaron [1, 3, 6]. Other examples arise in the theory of ultra cold atoms, for instance, Casimir-type forces between heavy fermions placed into a Fermi sea [15] or effective interactions between Fermi polarons [5, 13] or angulons [12]. The effect of fermion-mediated interactions is known also for dilute mixtures of Bose-Fermi gases [7, 10, 16] for which experimental observations have been reported in [2, 4].

In this article we are interested in medium-induced interactions for a system of impurity particles immersed into a dense ideal Fermi gas. To this end we analyze the many-body Schrödinger time-evolution of $n \geq 2$ impurity particles coupled to a large number of fermions via a suitable two-body potential, for instance a screened Coulomb potential. Our main result shows that the impurity particles effectively decouple from the fermions if the number of fermions (per unit volume) becomes large. The presence of the Fermi gas, however, leaves
its trace as an attractive interaction between the impurities. Physically, one can think of the induced interaction as being mediated by the creation and annihilation of electron-hole pairs in the Fermi sea.

This work generalizes our previous findings in \cite{8, 9}. While \cite{8} studies a single impurity that effectively decouples from the Fermi gas, in \cite{9} we considered the case $n = 2$ and derived the emergence of an effective interaction. Apart from generalizing to $n \geq 3$, our present analysis adds several other important improvements: (i) We extend the result to three spatial dimensions (in \cite{8, 9} we focused on the two-dimensional case). (ii) We treat pair potentials with a Coulomb singularity whereas our earlier results were restricted to bounded potentials. (iii) We obtain improved error estimates and a more transparent proof. (iv) Most importantly, in our opinion, we discuss the effective interaction among the impurities in more detail. This allows us to show that it adds a non-trivial effect at leading order to the effective dynamics (see Proposition 1.3).

The article is organized as follows. In the next two sections we introduce the microscopic and effective models, respectively. In Section 1.3 we state our main results in which we compare the time-evolved states of the microscopic model and the effective model. All proofs are postponed to Section 2.

1.1 The model

We consider a system of $n \geq 2$ impurity particles and $N$ fermions in a $d$-dimensional cube $\Lambda = [0, L]^d$ with periodic boundary conditions. To this model we assign the Hilbert space $\mathcal{H}_n \otimes \mathcal{H}_N^-$ where $\mathcal{H}_n = L^2(\Lambda)^{\otimes n}$ describes the states of the impurities with coordinates $y_1, \ldots, y_n$ and $\mathcal{H}_N^- = \bigwedge^N L^2(\Lambda)$ (the subspace of all anti-symmetric wave functions in $\mathcal{H}_N$) is the state space for the fermions with coordinates $x_1, \ldots, x_N$. The Hamiltonian is given by

\[
H = \sum_{i=1}^n (-\Delta_{y_i}) + \sum_{i<j}^n w(y_i - y_j) + \sum_{i=1}^N (-\Delta_{x_i}) + \lambda \sum_{i=1}^n \sum_{j=1}^N v_L(y_i - x_j)
\]

where $h_n^0$ and $T_N$ acting only on the tensor components $\mathcal{H}_n$ and $\mathcal{H}_N^-$, respectively, that is, they have to be understood as $h_n^0 \otimes 1$ and $1 \otimes T_N$. For the pair potentials $w$ and $v_L$ we suppose the following properties.

(Aw) $w$ is a real-valued even function on $\Lambda$ that satisfies $w^2 \leq c(1 - \Delta)$ as an operator inequality on $L^2(\Lambda)$ for some constant $c \in [0, 1)$. 

There is a rotational invariant function $\hat{v}_\infty(k) : \mathbb{R}^d \to \mathbb{R}$ (not depending on $L$) with

$$|\hat{v}_\infty(k)| \leq (k^2 + R)^{-1}$$

for some $R > 0$ and $\hat{v}_L(k) = \hat{v}_\infty(k)$ for all $k \in (2\pi/L)\mathbb{Z}^d$.\(^1\)

It is well known that, under these conditions, $H$ defines a self-adjoint operator.

In Proposition 1.3 we also require that $|\hat{v}_\infty(k)| \geq (1 + R)^{-1}$ for all $k^2 \leq 1$. Hence it is suggestive to think of $\hat{v}_\infty(k) = (k^2 + R)^{-1}$, which, up to some constants, is the Fourier transform of a Yukawa potential.

In our main results we choose the coupling constant $\lambda$ proportional to $\rho^{(2-d)/2d}$ with $\rho = NL^{-d}$, and then analyze the regime $\rho \gg 1$. To be more precise, we will first take the large-volume limit $L \to \infty$ with $\rho > 0$ constant and then consider $\rho \gg 1$. As will be explained in Section 1.2, the scaling of $\lambda$ is chosen such that we have a non-trivial effective dynamics.

Our goal is to analyze the solution of the time-dependent Schrödinger equation

$$\begin{cases}
    i \frac{d}{dt} \Psi(t) = H \Psi(t) \\
    \Psi(0) = \Psi_0
\end{cases}$$

for initial states $\Psi_0 \in \mathcal{H}_n \otimes \mathcal{H}_N^{-}$ of the form

$$\Psi_0(y_1, \ldots, y_n, x_1, \ldots, x_N) = \xi_0(y_1, \ldots, y_n) \otimes \Omega_0(x_1, \ldots, x_N).$$

We assume that the fermions are initially in the ground state of the non-interacting Fermi gas (the Fermi sea), that is, the ground state of the kinetic energy operator $T_N$. The $n$-body wave function $\xi_0 \in \mathcal{H}_n$ can be chosen more generally. Our only requirement is that its kinetic energy is of order one with respect to $\rho \gg 1$, by which we ensure an important separation of scales between the impurity particles and the fast fermions. No statistics are imposed on the impurities.

Instead of having $N$ as a free model parameter, it is more convenient to choose a Fermi momentum $k_F > 0$, and then fix $N$ in terms of $k_F$ and $L$ by

$$N = N(k_F, L) = |B_F|$$

with

$$B_F = \{ k \in (2\pi/L)\mathbb{Z}^d : |k| \leq k_F \}.$$
This implies the non-degeneracy of the free fermionic ground state, given by the anti-symmetric product of all plane waves with momenta inside the Fermi ball \(B_F\),

\[
\Omega_0 = \bigwedge_{k \in B_F} \varphi_k \in \mathcal{H}_{N(k_F,L)}^-,
\quad \varphi_k(x) = \frac{\exp(ikx)}{L^{d/2}} \in L^2(\Lambda).
\] (7)

Clearly, \((T_N - E^0(k_F, L))\Omega_0 = 0\) with eigenvalue \(E^0(k_F, L) = \sum_{k \in B_F} k^2\).

Replacing the sum by its Riemann integral, one obtains the useful relation between the Fermi momentum and the average density,

\[
\frac{N(k_F, L)}{L^d} = V_d k_F^d + o(1)
\] (8)

where \(o(1)\) vanishes as \(L \to \infty\), and the constants equal \(V_1 = 1/\pi\), \(V_2 = 1/(4\pi)\) and \(V_3 = 1/(6\pi^2)\). From (8) one infers that \(\varrho \gg 1\) is equivalent to \(k_F \gg 1\).

### 1.2 Effective \(n\)-body model

For non-vanishing interaction between the impurities and the fermions (i.e., for \(v_L \neq 0\)), one can not expect that the time-evolved wave function \(\Psi(t) = e^{-iHt}\Psi_0\) exhibits the same product form as the initial state \(\Psi_0 = \xi_0 \otimes \Omega_0\). Nevertheless, by including an effective interaction among the impurities, we shall show that the product structure is approximately preserved in the limit of large \(k_F\). To this end, we compare \(\Psi(t)\) with the product wave function

\[
\Psi_{\text{eff}}(t) = e^{-i\hat{h}_n t} \xi_0 \otimes e^{-iE(k_F, L)t} \Omega_0
\] (9)

where \(E(k_F, L)\) is the energy shift

\[
E(k_F, L) = E^0(k_F, L) + n\lambda \hat{\nu}_L(0) \frac{N(k_F, L)}{L^d},
\] (10)

and \(h_n\) the an operator on \(\mathcal{H}_n\) defined by

\[
h_n = h_n^0 - \sum_{i<j} \lambda^2 W_{k_F}(|y_i - y_j|) - n\lambda^2 W_{k_F}(0).
\] (11)

Here, we introduced the effective interaction potential

\[
W_{k_F}(r) = V_d^2 \int_{|k| \leq k_F} d^d k \int_{|l| > k_F} d^d l \frac{[\hat{v}_\infty(l - k)]^2}{l^2 - k^2 + (l - k)^2 + 1} \cos((l - k) \cdot \hat{r} \hat{a})
\] (12)

4
for $r \geq 0$ and $\hat{a} \in \mathbb{R}^d$ an arbitrary vector of unit length (since $\hat{v}_\infty$ is rotational invariant the direction of $\hat{a}$ is irrelevant). In Lemma 1.1 below we show that $W_{k_F}(r)$ is a bounded function and thus $h_n$ is self-adjoint and generates the unitary time evolution $e^{-i h_n t}$.

Before we discuss $W_{k_F}(r)$ in more detail, let us comment on the physical interpretation of the effective dynamics defined by (9):

(a) There is no interaction between the impurities and the fermions.
(b) The time-evolution of the fermions is stationary.
(c) The impurities evolve with an additional pair interaction described by the potential $-\lambda^2 W_{k_F}(r)$. (The last term in (11) only adds a constant phase shift.)

The heuristic picture behind the effective interaction is that it is caused by particle-hole excitations in the Fermi sea. One of the impurities produces a particle-hole excitation in the Fermi sea and then a different impurity annihilates the particle-hole excitation again. After such a second-order process the impurity particles are obviously correlated with each other but not with the Fermi sea. We remark that this mechanism is in principle similar to the effect of vacuum polarization in QED if one interprets the Fermi sea as the vacuum.

Since we are interested in the limit of large density, it is important to understand the properties of the effective potential for large values of $k_F$. In our main results we choose the coupling constant $\lambda$ such that

$$\lambda^2 = k_F^{(2-d)}.$$  

This is motivated by the next lemma stating that for such $\lambda$, the effective potential $\lambda^2 W_{k_F}(r)$ is of order one with respect to $k_F \gg 1$. In Proposition 1.3 we shall use this property to show that the effective interaction can not be omitted in the effective dynamics.

**Lemma 1.1.** Let $W_{k_F}(r)$ be defined by (12) with $|\hat{v}_\infty(k)| \leq (k^2 + R)^{-1}$ for all $k \in \mathbb{R}^d$ and some $R > 0$, and $|\hat{v}_\infty(k)| \geq (1 + R)^{-1}$ for all $k^2 \leq 1$. It follows that there are constants $C > 1$ and $c > 0$ such that

$$\sup_{k_F \geq 1} \sup_{r \in [0, \infty)} |k_F^{(2-d)} W_{k_F}(r)| \leq C \text{ and } \inf_{k_F \geq 1} \inf_{r \in [0, c]} k_F^{(2-d)} W_{k_F}(r) \geq C^{-1}. (14)$$

Below we visualize these qualitative properties by showing a numerical computation for $k_F^{(2-d)} W_{k_F}(r)$ as a function of $r \geq 0$. We set $d = 2$ and $\hat{v}_\infty(k) = 1/2$ for $k^2 \leq 1$ and zero

\footnote{Similar processes can of course involve more than two impurities which would lead to more complicated effective interactions. In our setting, however, these would be of subleading order and thus need not be taken into account in the effective dynamics.}
otherwise. The graph is plotted in $k_F$-independent units of length (x-axis) and energy (y-axis). Since the single points converge rapidly for growing values of $k_F$ we only depict it for one value. The picture is qualitatively the same in other dimensions.

![Figure 1: Effective pair potential $k_F^{(2-d)}W_{k_F}(r)$ in $k_F$-independent units](image)

### 1.3 Main results

We are now ready to state our main theorem which provides an estimate for the large-volume limit of the norm of $\Psi(t) - \Psi_{\text{eff}}(t)$. For reasons explained in the remarks, we only consider $d \in \{2, 3\}$. Note that we indicate the norm resp. the scalar product on the Hilbert space $\mathcal{H}_n \otimes \mathcal{H}_N^-$ by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ whereas for the spaces $\mathcal{H}_n$ and $\mathcal{H}_N^-$, we add additional subscripts.

**Theorem 1.2**. Let $d \in \{2, 3\}$, $n \geq 2$ and assume $(Aw)$ and $(Av_L)$. For $L, k_F > 0$ choose $N = N(k_F, L)$ as in (6) and let $\Omega_0$ be given by (7). Let further $\xi_0 \in \mathcal{H}_n$ obey the conditions

$$\|\xi_0\|_{\mathcal{H}_n} = 1 \quad \text{and} \quad q_{\xi_0} := \sup_{k_F \geq 1} \sup_{L > 0} \sum_{i=1}^n \langle \xi_0, (-\Delta_{y_i})\xi_0 \rangle_{\mathcal{H}_n} < \infty,$$

and let $E(k_F, L)$ and $h_n$ be defined by (10) and (11), respectively. For $|\lambda| = k_F^{(2-d)/2}$ the wave functions $\Psi(t) = e^{-iHt}\xi_0 \otimes \Omega_0$ and $\xi(t) = e^{-ih_nt}\xi_0$ satisfy the following property. There exists
a constant $C(n, R, q_{\xi_0}) > 0$ such that

$$\limsup_{L \to \infty} \| \Psi(t) - \xi(t) \otimes e^{-iE(k_F,L)t} \Omega_0 \| \leq C(n, R, q_{\xi_0}) \frac{(1 + |t|)(\ln k_F)^3}{\sqrt{k_F}}$$  \hspace{1cm} (16)

for all $k_F \geq 2$ and $t \in \mathbb{R}$.

Since all wave functions on the left side of (16) are normalized to one, the bound is meaningful if the right side is small compared to one. This is the case for $k_F \gg 1$ as long as $|t| \ll \sqrt{k_F(\ln k_F)^{-3}}$.

**Remarks.** 1.1. As a simple corollary one gets convergence of the reduced densities (in trace norm distance), $\gamma^{(n)}_{\Psi(t)} = \text{Tr}_{\mathcal{H}_N} |\Psi(t)\rangle \langle \Psi(t)|$ and $\mu^{(N)}_{\Psi(t)} = \text{Tr}_{\mathcal{H}_N} |\Psi(t)\rangle \langle \Psi(t)|$ towards $|\xi(t)\rangle \langle \xi(t)|$ and $|\Omega_0\rangle \langle \Omega_0|$, respectively:

$$\limsup_{L \to \infty} \left( \text{Tr}_{\mathcal{H}_N} \left| \gamma^{(n)}_{\Psi(t)} - |\xi(t)\rangle \langle \xi(t)| \right| + \text{Tr}_{\mathcal{H}_N} \left| \mu^{(N)}_{\Psi(t)} - |\Omega_0\rangle \langle \Omega_0| \right| \right) \leq C(n, R, q_{\xi_0}) \frac{(1 + |t|)(\ln k_F)^3}{\sqrt{k_F}}.$$  \hspace{1cm} (17)

1.2. Our proof provides a more general statement than Theorem 1.2. We shall show that for $d \in \{1, 2, 3\}$, $\lambda \in \mathbb{R}$ and all $k_F \geq 2$, it holds that

$$\limsup_{L \to \infty} \| \Psi(t) - \xi(t) \otimes e^{-iE(k_F,L)t} \Omega_0 \| \leq C(n, R, q_{\xi_0}) \Gamma(d, k_F, \lambda, t)$$  \hspace{1cm} (18)

with

$$\Gamma(d, k_F, \lambda, t) = |\lambda|(1 + |t|)(1 + \lambda^2 k_F^{(d-2)}) k_F^{(d-3)/2} (\ln k_F)^{1/2}$$

$$+ \lambda^2 (k_F^{(d-3)} \ln k_F + |t|(1 + \lambda^2 k_F^{(d-2)}) k_F^{(d-3)} \ln k_F + |t| \sqrt{\gamma(d, k_F)})$$

$$+ |\lambda|^3 |t| (k_F^{(3d-7)/2} \ln k_F + \sqrt{\gamma(d, k_F)} k_F^{(d-3)/2} (\ln k_F)^{1/2} + \gamma(d, k_F))$$  \hspace{1cm} (19)

and

$$\gamma(d, k_F) = \begin{cases} k_F^{(2d-5)} (\ln k_F)^3 & \text{for } d = 2, 3 \\ k_F^{-2} (\ln k_F)^3 & \text{for } d = 1 \end{cases}.$$  \hspace{1cm} (20)

Setting $|\lambda| = k_F^{(2-d)/2}$ leads to (16).

1.3. For $d = 1$ the natural choice of the coupling constant (such that the effective interaction is of order one) is $\lambda^2 = k_F$. In this case, however, the error term $\lambda^2 \sqrt{\gamma(d, k_F)}$ is not small,
and thus the right side in (18) does not provide a useful bound. Nonetheless, we expect that a modification of the proof would allow the derivation of Theorem 1.2 also for $d = 1$ and $\lambda^2 = k_F$. Since we prefer to keep the presentation at a considerable length, we omit this case and restrict our analysis to $d \in \{2, 3\}$.

1.4. Theorem 1.2 is valid also for $n = 1$ if one uses $h_{n=1} = -\Delta_y - \lambda^2 W_{k_F}(0)$. This improves our findings in [9].

Our second result shows that the effective interaction in (11) adds a non-negligible effect to the dynamics. In other words, a result like Theorem 1.2 can not be true if one replaces the $n$-body Hamiltonian $h_n$ by

$$\tilde{h}_n = \tilde{h}_n^0 - nk_F^{(2-d)} W_{k_F}(0).$$

(21)

**Proposition 1.3.** Assume the same conditions as in Theorem 1.2, and in addition, let $|\hat{v}_\infty(k)| \geq 1/(1 + R)$ for all $k^2 \leq 1$ and assume

$$\inf_{k_F \geq 1} \inf_{L > 0} \sum_{i < j}^{n} k_F^{(2-d)} \langle \xi_0, W_{k_F}(|y_i - y_j|) \xi_0 \rangle_{H_n} \geq c_0 \quad \text{and} \quad \sup_{k_F \geq 1} \sup_{L > 0} \||h_0^0 \xi_0\|_{H_n} \leq C_0$$

(22)

for some constants $c_0, C_0 > 0$. Then there exist times $t_1 > t_0 > 0$ (depending on $c_0, C_0, n, R$ and $q_{\xi_0}$) such that

$$\liminf_{L \to \infty} \||\Psi(t) - \exp(-i\tilde{h}_n t) \xi_0 \otimes e^{-iE(k_F, L)t} \Omega_0\| \geq \frac{c_0}{2} t$$

(23)

for all $t \in (t_0, t_1)$ and all $k_F$ large.

We emphasize that assumptions (15) and (22) are not very restrictive and, in particular, the three conditions are consistent with each other. This is thanks to Lemma 1.1 that allows us to locate some mass of $\xi_0$ inside the non-vanishing positive core of the effective potential while keeping the values of $||h_0^0 \xi_0||$ and $q_{\xi_0}$ of order one as $k_F$ tends to infinity.

We conclude this section with a short sketch of the strategy behind the proof of Theorem 1.2. The starting point is to use the fundamental theorem of calculus to write

$$\left(1 - e^{i(H - E(k_F, L))t} e^{-i\tilde{h}_n t}\right) \xi_0 \otimes \Omega_0$$

$$= -i \int_0^t ds e^{i(H - E(k_F, L)s)} \left(V_{N+n} - n \lambda \hat{v}_L(0) \frac{N(k_F, L)}{L^d} + h_0^0 - h_n\right) \xi(s) \otimes \Omega_0. $$

(24)

For $\lambda = o(k_F (\ln k_F)^{-3/2})$, the upper bound is still useful, but this case is less interesting since the effective potential would be of subleading order and, in particular, of the same order as the error.
To estimate the norm of the part with $V^{(\text{exc})}_{N+n}$, we need to use the unitary $e^{i(H-E(k_F,L))s}$. The heuristic idea is that the operator $T^{(\text{exc})}_N := T_N - E^0(k_F,L)$ in

$$H - E(k_F,L) = T^{(\text{exc})}_N + h^0 + V^{(\text{exc})}_{N+n}$$

produces a large energy shift when applied to states orthogonal to $\Omega_0$. This, in turn, leads to phase cancellations (or destructive interference) in (24) and thus suppresses the value of the norm. The obvious way to exploit such phase cancellations is to employ the identity

$$i e^{i(H-E(k_F,L))s} = e^{i(H-E(k_F,L))s} e^{-iT^{(\text{exc})}_N s} \left( \frac{d}{ds} e^{iT^{(\text{exc})}_N s} \right) \left( T^{(\text{exc})}_N \right)^{-1}$$

and then use integration by parts (note that $V^{(\text{exc})}_{N+n} \Omega_0$ is orthogonal to $\Omega_0$ and thus (26) can be applied). This leads to a perturbation type expansion for the first part in (24) that involves terms with expressions like $V^{(\text{exc})}_{N+n} (T^{(\text{exc})}_N)^{-1} V^{(\text{exc})}_{N+n} \xi(s) \otimes \Omega_0$. However, since not all of the terms in this expansion are sufficiently small, we need to use the unitary again and proceed by a second integration by parts. To do that, we now have to sort the terms into a component along $\Omega_0$ and all other components orthogonal to $\Omega_0$. In the first component there are no more phase cancellations since $T^{(\text{exc})}_N \Omega_0 = 0$ (the points of stationary phase so to say). This part is canceled by the second term in (24), which follows from

$$h_n = h^0_n - \langle \Omega_0, V^{(\text{exc})}_{N+n} (T^{(\text{exc})}_N)^{-1} V^{(\text{exc})}_{N+n} \Omega_0 \rangle_{\mathcal{H}_N},$$

and it is this cancellation that determines the choice of $h_n$. For all other terms, the ones orthogonal to $\Omega_0$, we can proceed by a second expansion via integration by parts. All terms that are obtained by this expansion are then estimated separately.

As a final remark let us mention that a similar strategy was recently used in [11, 14] to study the quantum fluctuations of the dynamics of a strongly coupled polaron. In this model, the term without the oscillating phase leads to an effective quadratic interaction between the phonons inside the polaron cloud.

## 2 Proofs

We first introduce the formalism of second quantization and then state some preliminary estimates for sums of different transition amplitudes. The bounds for the transition amplitudes are required throughout the proof of Theorem 1.2. (To shorten the presentation of this section, we provide some more bounds and all proofs in Appendix A.)
2.1 Second quantization

We think of the Hilbert space $H^N = H_n \otimes H_{m-N}$ as the $N$-particle sector of $H_n \otimes F$ with $F$ the fermionic Fock space

\[ F = \bigoplus_{m=0}^{\infty} H_m, \quad H_m = \bigwedge^m L^2(\Lambda). \]  

This way we can profit from the use of the formalism of second quantization, which strictly speaking is not necessary, but simplifies many computations.

For plane waves $\varphi_k(x) = L^{-d/2} \exp(ikx)$, $k \in (2\pi/L)\mathbb{Z}^d$, we define the creation and annihilation operators $a_k, a_k^* : H_n \otimes F \to H_n \otimes F$ through

\[
(a_k \Psi)^{(m)}(y_1, \ldots, y_n, x_1, \ldots, x_m) = \sqrt{m+1} \int_\Lambda \varphi_k(x) \Psi^{(m+1)}(y_1, \ldots, y_n, x_1, \ldots, x_m, x), \\
(a_k^* \Psi)^{(m)}(y_1, \ldots, y_n, x_1, \ldots, x_m) = \sum_{j=1}^m (-1)^j \varphi_k(x_j) \Psi^{(m-1)}(y_1, \ldots, y_j, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m),
\]

where $\Psi = (\Psi^{(m)})_{m \geq 0}$ with $\Psi^{(m)} \in H_n \otimes H_m$. They satisfy the usual canonical anticommutation relations

\[
a_k a_l + a_l a_k^* = \delta_{kl}, \quad a_k a_l + a_l a_k = 0 \quad \forall k, l \in (2\pi/L)\mathbb{Z}^d. \]  

In terms of creation and annihilation operators we can write the Hamiltonian $H$ as an operator on the Hilbert space $H_n \otimes F$. For that purpose, define

\[
\mathbb{H} = h_n^0 + T + V - E^0(k_F, L)
\]

with

\[
T = \sum_k k^2 a_k^* a_k, \quad V = \sum_{i=1}^n V^{(i)}, \quad V^{(i)} = \lambda L^{-d} \sum_{k \neq l} \hat{v}_L(l - k) e^{i(k-l)y} a_l^* a_k.
\]

Restricting $\mathbb{H}$ to the $N$-particle sector yields

\[
\mathbb{H} | H_n \otimes H_{N(k_F, L)} = H - E(k_F, L)
\]

with $E(k_F, L)$ defined in (10). Note that for ease of notation we subtracted the energy.
\[ E^0(k_F, L) \] and also the momentum-conserving part of the interaction

\[ \lambda \sum_{i=1}^{n} L^{-d} \sum_{k,l} \hat{\nu}_L(l - k)e^{i(k-l)y} \delta_{kl} a_i^* a_k = \frac{n \lambda \hat{\nu}_L(0)}{L^d} \sum_{k} a_k^* a_k. \] 

(33)

In terms of the Fock space vacuum \(|0\rangle = (1, 0, 0, \ldots)\), the Fermi sea (7) is given by \( \Omega_0 = (\prod_{k \in B_F} a_k^*) |0\rangle \). Because of the anti-commutation relations, it satisfies

\[
\begin{aligned}
    a_k \Omega_0 &= 0 \quad \text{for all } k \in B_{F}^c, \\
    a_k^* \Omega_0 &= 0 \quad \text{for all } k \in B_F.
\end{aligned}
\]

(34)

Moreover, for momenta \( l'_1, \ldots, l'_n, l_1, \ldots, l_n \in B_{F}^c \) and \( k'_1, \ldots, k'_m, k_1, \ldots, k_m \in B_F \) (with integers \( n + m \geq 1 \)) the anti-commutation relations together with (34) imply the Wick formula

\[
\begin{aligned}
    \langle a_{l'_1}^* a_{l'_2}^* \cdots a_{l'_n}^* a_{k'_1} \cdots a_{k'_m} \Omega_0, a_{l_1} \cdots a_{l_n} a_{k_1} \cdots a_{k_m} \Omega_0 \rangle_{F_N} &= \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} \delta_{l'_i l_{\sigma(i)}} \right) \left( \sum_{\tau \in S_m} \text{sgn}(\tau) \prod_{j=1}^{m} \delta_{k'_j k_{\tau(j)}} \right),
\end{aligned}
\]

(35)

where \( \text{sgn}(\sigma) \in \{-1, 1\} \) is the sign of the permutation \( \sigma \in S_n \) (the symmetric group).

### 2.2 Preliminary bounds

Throughout this section we use the notation

\[
\sum_{(l,k) \in A} f(l,k) = \sum_{k} \sum_{l} \chi_{A}(l,k) f(l,k),
\]

(36)

where \( \chi_{A} \) denotes the characteristic function \( \chi_{A}(l,k) = 1 \) for \( (l,k) \in A \) and \( \chi_{A}(l,k) = 0 \) otherwise. To state the next lemma, let us introduce the set of momentum pairs

\[ T_F = \{(l,k) \in B_F^c \times B_F \} \subset (2\pi/L)\mathbb{Z}^d \times (2\pi/L)\mathbb{Z}^d. \]

(37)

**Lemma 2.1.** Let \( d \in \{1, 2, 3\} \) and assume \((A v_L)\). There is a constant \( C > 0 \) (depending on \( R \)) such that for all \( k_F \geq 2 \) the following estimates hold.

\[
\lim_{L \to \infty} \left( L^{-2d} \sum_{(l,k) \in T_F} |\hat{\nu}_L(l - k)|^2 \right) \leq C k_F^{(d-1)},
\]

(38a)
\[
\lim_{L \to \infty} \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{v}_L(l-k)|^2}{l^2 - k^2 + 1} \right) \leq C k_F^{(d-2)}, \tag{38b}
\]
\[
\lim_{L \to \infty} \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{v}_L(l-k)|^2}{(l^2 - k^2 + 1)^2} \right) \leq C k_F^{(d-3)} \ln k_F, \tag{38c}
\]
\[
\lim_{L \to \infty} \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{v}_L(l-k)|^2}{(l^2 - k^2 + (l-k)^2 + 1)^2} \right) \leq C k_F^{(d-3)} (\ln k_F)^2. \tag{38d}
\]

These bounds will be frequently used in the next section. Their derivation is postponed to Appendix A. Some more bounds, of similar type, are provided in Lemma 2.2.

### 2.3 Proof of Theorem 1.2

In this section we derive the bound stated in Remark 1.2, that is, we keep \(\lambda \in \mathbb{R}\) arbitrary and consider \(d \in \{1, 2, 3\}\). Theorem 1.2 is a consequence of this bound for \(|\lambda| = k_F^{(2-d)/2}\).

Our goal is to estimate the norm difference
\[
\| e^{-i(H-E)t} \xi_0 \otimes \Omega_0 - \xi(t) \otimes \Omega_0 \| \tag{39}
\]
with \(\xi(t) = e^{-ih_n t} \xi_0\). For shorter notation, we omit from now on the arguments in
\[
E = E(k_F, L), \quad E^0 = E^0(k_F, L) \quad \text{and} \quad N = N(k_F, L). \tag{40}
\]

To start we employ (32) and use the fundamental theorem of calculus to get
\[
(1 - e^{iHt} e^{-ih_n t}) \xi_0 \otimes \Omega_0 = - \int_0^t ds \frac{d}{ds} (e^{iHs} e^{-ih_n s}) \xi_0 \otimes \Omega_0 =: \Phi(t) + \phi(t) \tag{41}
\]
with
\[
\Phi(t) = -i \int_0^t ds \ e^{iHs} \nabla \xi(s) \otimes \Omega_0 \quad \phi(t) = i \int_0^t ds \ e^{iHs} (h_n - h_n^0) \xi(s) \otimes \Omega_0. \tag{42}
\]

In the first part of the proof we use two integration by parts in order to expand the state \(\Phi(t)\) into several contributions. In the second part we estimate these contributions separately. In particular we single out one contribution which is canceled by \(\phi(t)\).

**Decomposition of \(\Phi(t)\)**
We define the resolvent type operator

\[ R = (T - E^0 + P_f^2 + 1)^{-1} \upharpoonright \mathcal{H}_n \otimes \mathcal{H}_{N}^- \]  

with \( P_f = \sum_k k a_k^* a_k \) the momentum operator of the fermions. Since \( T \upharpoonright \mathcal{H}_n \otimes \mathcal{H}_{N}^\uparrow \geq E^0 \), the operator \( R \) is bounded. Here we deviate slightly from the strategy explained at the end of Section 1.3, where we used \( (T - E^0)^{-1} \) instead of \( R \). While the plus one is added in order to avoid a potential singularity from contributions whose energy excitation vanishes in the limit \( L \to \infty \), the momentum part \( P_f^2 \) needs to be included for important cancellations.  

With the aid of \( R \) we can rewrite

\[ \Phi(t) = -\int_0^t ds \ e^{iE s} e^{-i(T-E^0+1+P_f^2)s} \left( \frac{d}{ds} e^{i(T-E^0+1+P_f^2)s} \right) R V \xi(s) \otimes \Omega_0 \]  

and integrate by parts. This leads to

\[ \Phi(t) = \sum_{i=0}^3 \Phi_i(t) \]  

\[ \Phi_0(t) = -e^{iE s} R V \xi(s) \otimes \Omega_0 \]  

\[ \Phi_1(t) = i \int_0^t ds \ e^{iE s} R V (-1 + h^0_n - h_n) \xi(s) \otimes \Omega_0, \]  

\[ \Phi_2(t) = i \int_0^t ds \ e^{iE s} R \{ h^0_n, V \} - P_f^2 V \} \xi(s) \otimes \Omega_0, \]  

\[ \Phi_3(t) = i \int_0^t ds \ e^{iE s} V R V \xi(s) \otimes \Omega_0, \]  

where we used \( \frac{d}{ds} \xi(s) = -ih_n \xi(s) \) and

\[ \frac{d}{ds} e^{iE s} e^{-i(T-E^0+1+P_f^2)s} = ie^{iE s} (h^0_n + V - 1 - P_f^2) e^{-i(T-E^0+1+P_f^2)s}. \]  

In line (45d) we proceed by decomposing the state \( V R V \xi(s) \otimes \Omega_0 \) according to the number of holes in the Fermi sea \( \Omega_0 \). (A hole is a an unoccupied momentum mode with \( k \in B_F^\uparrow \); since we only consider states in \( \mathcal{H}_{N}^- \), a state with \( m \) holes is automatically a state with \( m \) particle-hole pairs where particle refers to an occupied mode with \( k \in B_F^\downarrow \).) We use that the operator \( V R V \) changes the number of holes at most by two, and thus the state \( V R V \xi(s) \otimes \Omega_0 \) has at most two holes. Introducing the orthogonal projector \( P_{(m)} \) in \( \mathcal{H}_n \otimes \mathcal{H}_{N}^- \) (acting trivially in

---

4As we allow for singular interaction potentials \( v(x) \), the gain in kinetic energy of the impurity after interacting with a fermion can be very large. Adding the operator \( P_f^2 \) in \( R \) will lead to a cancellation of this gain in kinetic energy, cf. (45c).
that projects onto the closed subspace
\[ \text{ran} P^{(m)} = \left\{ \Psi \in \mathcal{H}_n \otimes \mathcal{H}_{N_m}^- : \sum_{k \in B_F} \| a_k^* \Psi \|^2 = m \| \Psi \|^2 \right\} \] (47)
(the subspace of all states with exactly \( m \) holes), we obtain
\[ \Phi_3(t) = \sum_{m=0}^{2} i \int_{0}^{t} ds \ e^{iHs} P^{(m)} \mathcal{V}R\mathcal{V} \xi(s) \otimes \Omega_0 =: \sum_{m=0}^{2} \Phi_{3m}(t). \] (48)

In the contribution \( \Phi_{30} \) we need to expand a second time via integration by parts. Proceeding similarly as in (44), one verifies
\[ \Phi_{32}(t) = -i \sum_{i=0}^{3} \Phi_{32;i}(t) \] (49)
with
\[ \Phi_{32;0}(t) = \left[ e^{iHs} RP^{(j)} \mathcal{V}R\mathcal{V} \xi(s) \otimes \Omega_0 \right]_0^t, \] (50a)
\[ \Phi_{32;1}(t) = -i \int_{0}^{t} ds \ e^{iHs} RP^{(j)} \mathcal{V}R\mathcal{V} (-1 + h_n^0 - h_n) \xi(s) \otimes \Omega_0, \] (50b)
\[ \Phi_{32;2}(t) = -i \int_{0}^{t} ds \ e^{iHs} RP^{(j)} \{ [h_n^0, \mathcal{V}R\mathcal{V}] - \mathcal{P}_1^{2} \mathcal{V}R\mathcal{V} \} \xi(s) \otimes \Omega_0, \] (50c)
\[ \Phi_{32;3}(t) = -i \int_{0}^{t} ds \ e^{iHs} \mathcal{V}R\mathcal{V} RP^{(j)} \mathcal{V}R\mathcal{V} \xi(s) \otimes \Omega_0. \] (50d)

In the last line, we can decompose again in terms of the number of holes, i.e.
\[ \Phi_{32;3}(t) = \sum_{m=1}^{3} \Phi_{32;3m}(t), \quad \Phi_{32;3m}(t) = -i \int_{0}^{t} ds \ e^{iHs} P^{(m)} \mathcal{V}R\mathcal{V} P^{(2)} \mathcal{V}R\mathcal{V} \xi(s) \otimes \Omega_0, \] (51)
since the state \( \mathcal{V}R\mathcal{V} P^{(2)} \mathcal{V}R\mathcal{V} \xi(s) \otimes \Omega_0 \) contains \( m \in \{1, 2, 3\} \) holes.

Collecting everything we obtain the decomposition
\[ \Phi(t) = \sum_{i=0}^{2} (\Phi_i(t) + \Phi_{32;i}(t)) + \Phi_{30}(t) + \Phi_{31}(t) + \sum_{m=1}^{3} \Phi_{32;3m}(t). \] (52)

Before we proceed, let us lay out the motivation behind this expansion. When estimating
the separate contributions, the idea is that every $R$ should give a factor $k_F^{-1}$ whereas every $V$ leads to a factor $\lambda k_F^{(d-1)/2}$ (modulo some log factors). This explains for instance why the norm of $\Phi_0(t)$ can be bounded by a constant times $\lambda k_F^{(d-2)/2}$ while the norm of $\Phi_{32;33}(t)$ can be bounded by a constant times $|t| |\lambda|^3 k_F^{(3d-7)/2}$. Even though this simple rule is the correct intuition, it also oversimplifies the situation somewhat as it is not applicable to each term in the expansion. (It does not apply whenever a $V$ does not change the number of holes in the state it acts on, which happens for instance in $\Phi_{31}(t).$)

**Notation.** For the second part of the proof it is helpful to introduce further notation.

- For $i, j, u \in \{1, ..., n\}$ we set
  \[ K^{(i)}_{lk} = e^{i(k-l)y}, \quad K^{(i,j)}_{nm,lk} = K^{(i)}_{nm} K^{(j)}_{lk}, \quad K^{(i,j,u)}_{sr,nm,lk} = K^{(i)}_{sr} K^{(j)}_{nm} K^{(u)}_{lk}. \]  

- To simply the notation, we write from now on $\hat{v}_L(k) = \hat{v}(k).$ Later on we shall also use the abbreviations
  \[ \hat{v}_{lk} = \hat{v}(l-k), \quad \varepsilon_{lk} = l^2 - k^2 + (l-k)^2. \] 

- Moreover we write
  \[ g(t, k_F, L) \lesssim f(t, k_F, L) \]  

to indicate that there is a constant $C > 0$ independent of the parameters $t, k_F$ and $L$ such that $g(t, k_F, L) \leq C f(t, k_F, L)$ for all $t \in \mathbb{R}, k_F \geq 2$ and $L > 0.$ The constant $C$ is allowed to depend on the fixed model parameters $d, n, w, R$ and on the initial state $\xi_0.$

**Estimates for the different contributions in $\Phi(t) + \phi(t)$**

**Term $\Phi_0(t).** We use this first term to warm up with some simple computations. Since $a_t^* a_k \Omega_0$ is a simultaneous eigenstate of $T$ and $P^2_t$ with eigenvalues $E_0 + l^2 - k^2$ and $(l-k)^2$, respectively, we have

\[ (T - E^0)a_t^* a_k \Omega_0 = (l^2 - k^2)a_t^* a_k \Omega_0, \quad P^2_t a_t^* a_k \Omega_0 = (l-k)^2 a_t^* a_k \Omega_0, \]  

and thus also

\[ Ra_t^* a_k \Omega_0 = (l^2 - k^2 + (l-k)^2 + 1)^{-1} a_t^* a_k \Omega_0. \]

Applying the Wick rule (35) as well as $\|K^{(i)}_{lk}(\xi(s))\|_{\mathcal{H}_n} = \|\xi(s)\|_{\mathcal{H}_n} = 1$, one easily verifies

\[ \|R^{(i)}(\xi(s) \otimes \Omega_0)\|^2 \]
To bound the remaining norm, use $h^2$. This leads to

$$
\Phi
$$

which provides the first error term in (19).

**Term $\Phi_1(t)$**. Following similar steps as in the above computation, one shows that

$$
\|RV^{(i)}(-1 + h^0_n - h_n)\xi(s) \otimes \Omega_0\|^2 \\
\lesssim L^{-2d} \sum_{(l,k) \in T_F} \frac{\lambda^2 |\widehat{v}(l-k)|^2}{(l^2 - k^2 + (l-k)^2 + 1)^2} (1 + \|h^0_n - h_n\|_{\mathcal{H}}^2).
$$

To bound the remaining norm, use $h^0_n - h_n = \sum_{i<j} \lambda^2 W_{k_F}(y_i - y_j) + n\lambda^2 W_{k_F}(0)$ and $|W_{k_F}(r)| \leq W_{k_F}(0)$. Hence we can apply (38b) and (38d) to find the bound

$$
\limsup_{L \to \infty} \|\Phi_1(t)\| \lesssim |t|(1 + \lambda^2 k_F^{(d-2)})|\lambda|k_F^{(d-3)/2}/(\ln k_F)^{1/2}.
$$

**Term $\Phi_2(t)$**. Here we need to evaluate

$$
P^{2}_{t}V^{(i)}\xi(s) \otimes \Omega_0 = L^{-d} \sum_{(l,k) \in T_F} \widehat{v}(l-k)(l-k)^2 K_{l+k}^{(i)}\xi(s) \otimes a^*_t a_k \Omega_0
$$

and

$$
[h^0_n, V^{(i)}]\xi(s) \otimes \Omega_0 = L^{-d} \sum_{(l,k) \in T_F} \widehat{v}(l-k)(l-k)^2 K_{l+k}^{(i)}\xi(s)a^*_t a_k \Omega_0 \\
+ L^{-d} \sum_{(l,k) \in T_F} \widehat{v}(l-k)K_{l+k}^{(i)}(k-l) \cdot (-2i\nabla y_i)\xi(s) \otimes a^*_t a_k \Omega_0.
$$

Taking the difference, the terms proportional to $(l-k)^2$ cancel out, which is the reason for including $P^2_t$ in $R$ (see the remark in Footnote 4). Thus we obtain

$$
R\{[h^0_n, V^{(i)}] - P^2_t V^{(i)}\}\xi(s) \otimes \Omega_0
$$
\[ = L^{-d} \sum_{(l, k) \in T_F} \tilde{v}(l-k) K_{lk}^{(i)} (k-l) \cdot (\mathbf{-} 2i \nabla_{y_i}) \xi(s) \otimes Ra_l a_k \Omega_0, \] (65)

the norm of which we can estimate by

\[ \| R \{ [h_n^0, \mathbb{V}^{(i)}] - P_F^2 \mathbb{V}^{(i)} \} \xi(s) \otimes \Omega_0 \|^2 \]

\[ \lesssim \left( L^{-2d} \sum_{(l, k) \in T_F} \frac{\lambda^2 |\tilde{v}(l-k)|^2 (l-k)^2}{(l^2 - k^2 + (l-k)^2 + 1)^2} \right) \| \nabla_{y_i} \xi(s) \|^2 \mathcal{H}_n. \] (66)

To bound the norm involving the gradient we use the assumption that \( w^2 \leq c(1 - \Delta) \) for some \( c \in [0, 1) \). This implies

\[ \sum_{i=1}^{n} \langle \xi(s), (-\Delta_{y_i}) \xi(s) \rangle_{\mathcal{H}_n} \lesssim 1 + \langle \xi(s), h_n^0 \xi(s) \rangle_{\mathcal{H}_n}, \] (67)

and since \( |W_{k_F}(r)| \leq W_{k_F}(0) \lesssim k_F^{(d-2)} \), we can proceed by

\[ \langle \xi(s), h_n^0 \xi(s) \rangle \lesssim \lambda^2 k_F^{(d-2)} + \langle \xi(0), h_n^0 \xi(0) \rangle_{\mathcal{H}_n} \]

\[ \lesssim \lambda^2 k_F^{(d-2)} + \sum_{i=1}^{n} \langle \xi(0), (-\Delta_{y_i}) \xi(0) \rangle_{\mathcal{H}_n} \lesssim \lambda^2 k_F^{(d-2)} + 1, \] (68)

where the last step follows from Assumption (15).

In combination with (38d) we can now take the large-volume limit in (66) to find

\[ \limsup_{L \to \infty} \| \Phi_2(t) \| \lesssim |t|(1 + \lambda^2 k_F^{(d-2)}) |\lambda| k_F^{(d-3)/2} (\ln k_F)^{1/2}. \] (69)

**Term** \( \phi(t) + \Phi_{30}(t) \). The contribution \( \Phi_{30}(t) \) is the one that determines the effective Hamiltonian \( h_n \). To see this, use

\[ \langle \Omega_0, \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \Omega_0 \rangle_{\mathcal{H}_n^+} = L^{-2d} \sum_{(l, k) \in T_F} \frac{|\tilde{v}(l-k)|^2}{l^2 - k^2 + (l-k)^2 + 1} e^{i(l-k)(y_j-y_i)}, \] (70)

and \( P^{(0)} = 1 \otimes |\Omega_0 \rangle \langle \Omega_0 | \), in order to compute

\[ \Phi_{30}(t) = \sum_{i=1}^{n} \int_{0}^{t} ds \, e^{iHs} P^{(0)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \]

\[ = i \sum_{i=1}^{n} \int_{0}^{t} ds \, e^{iHs} \left( L^{-2d} \sum_{(l, k) \in T_F} \frac{\lambda^2 |\tilde{v}(l-k)|^2}{l^2 - k^2 + (l-k)^2 + 1} \right) \xi(s) \otimes \Omega_0 \] (71a)
\[
+ i \sum_{i<j}^{n} \int_0^t ds \ e^{i\overline{\mathcal{H}}s} \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{\lambda^2 |\hat{v}(l-k)|^2}{l^2 - k^2 + (l-k)^2 + 1} \cos((k-l) \cdot (y_i - y_j)) \right) \xi(s) \otimes \Omega_0.
\]

(71b)

In the large-volume limit, the expressions in parenthesis converge to \( \lambda^2 W_{kF}(0) \) and \( \lambda^2 W_{kF}(|y_i - y_j|) \), respectively (since the Riemann sums converge to the corresponding integrals). Because \( \hat{v}_{\infty}(k) \) is rotational invariant, we can then replace the argument in the cosine by \((l-k) \cdot \hat{a}|y_i - y_j|\) for any unit vector \( \hat{a} \in \mathbb{R}^d \). Since

\[
\phi(t) = i \int_0^t ds \ e^{i\overline{\mathcal{H}}s} \left( - \sum_{i<j} \lambda^2 W_{kF}(|y_i - y_j|) - n \lambda^2 W_{kF}(0) \right) \xi(s) \otimes \Omega_0,
\]

we get a complete cancellation between \( \Phi_{30}(t) \) and \( \phi(t) \), that is

\[
\lim_{L \to \infty} \| \phi(t) + \Phi_{30}(t) \| = 0.
\]

(73)

We emphasize that this is crucial since the norm of \( \Phi_{30}(t) \) is of order \( \lambda^2 k_F^{(d-2)} \) which is of order one if we choose \( \lambda^2 = k_F^{(2-d)} \).

**Term \( \Phi_{31}(t) \).** Abbreviating \( \varepsilon_{lk} = l^2 - k^2 + (l-k)^2 \) and \( \hat{v}_{lk} = \hat{v}(l-k) \), we compute

\[
P^{(1)} \Psi^{(i)} \mathcal{R} \Psi^{(j)} \xi(s) \otimes \Omega_0 = L^{-2d} \sum_{m \in B_F} \sum_{(l,k) \in T_F} \frac{\lambda^2 \hat{v}_{km} \hat{v}_{lk} K^{(i,j)}_{km,lk}(s) \otimes a_m a^*_l \Omega_0}{\varepsilon_{lk} + 1}
\]

\[
+ L^{-2d} \sum_{n \in (B_F)^c} \sum_{(l,k) \in T_F} \frac{\lambda^2 \hat{v}_{nm} \hat{v}_{lk} K^{(i,j)}_{nl,lk}(s) \otimes a^*_n a_k \Omega_0}{\varepsilon_{lk} + 1},
\]

(74a)

(74b)

where we utilized the identity

\[
P^{(1)} a^*_m a_m a^*_k a_k \Omega_0 = \delta_{kn} \chi_{B_F}(m) a_m a^*_l \Omega_0 + \delta_{ml} \chi_{B_F}(n) a^*_m a_k \Omega_0 \quad \forall \ (l, k) \in B_F^c \times B_F.
\]

(75)

Using \( \| K^{(i,j)}_{km,lk}(s) \|_{\mathcal{N}} = 1 \) we proceed in the first line with

\[
\| (74a) \|^2 \leq L^{-4d} \sum_{m,m' \in B_F} \sum_{(l,k) \in T_F} \sum_{(l',k') \in T_F} \frac{\lambda^4 |\hat{v}_{km}| |\hat{v}_{m'k'}| |\hat{v}_{lk}| |\hat{v}_{l'k'}|}{(\varepsilon_{lk} + 1)(\varepsilon_{l'k'} + 1)} \left| \langle a_m a^*_l \Omega_0, a_m a^*_l \Omega_0 \rangle_{\mathcal{N}} \right|
\]

\[
= \lambda^4 \left( L^{-2d} \sum_{(l,m) \in T_F} \left( L^{-d} \sum_{k \in B_F} \frac{|\hat{v}(l-k)| |\hat{v}(k-m)|}{l^2 - k^2 + (l-k)^2 + 1} \right)^2 \right),
\]

(76)
and similarly in the second line with

\[ \| (74b) \|^2 \leq \lambda^4 \left( L^{-2d} \sum_{(n,k) \in T_F} \left( L^{-d} \sum_{l \in B'_F} \frac{\lambda^2 |\hat{v}(l - k)| |\hat{v}(n - l)|}{l^2 - k^2 + (l - k)^2 + 1} \right)^2 \right). \] (77)

The two remaining expressions are estimated in Lemma 2.2, (122a) and (122b). This implies

\[ \limsup_{L \to \infty} \| \Phi_{21}(t) \| \lesssim |t| \lambda^2 \sqrt{\gamma(d, k_F)} \] (78)

with \( \gamma(d, k_F) \) defined in (20).

**Term \( \Phi_{32;0}(t) \).** Here we have

\[ P^{(2)} R^{(i)}_n R^{(j)}_m \xi(s) \otimes \Omega_0 = L^{-2d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^2 |\hat{v}_{nm}\hat{v}_{lk}|}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K^{(i,j)}_{nm,lk}(s) \otimes a_{n}^* a_{m} a_{l}^* a_{k} \Omega_0, \] (79)

which follows from

\[ P^{(2)} a_{n}^* a_{m} a_{l}^* a_{k} \Omega_0 = \chi_{T_F}(n, m) a_{n}^* a_{m} a_{l}^* a_{k} \Omega_0 \quad \forall (l, k) \in B'_F \times B_F. \] (80)

Using the basic inequality

\[ \left| \frac{\hat{v}_{nm}\hat{v}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \right| \leq \frac{|\hat{v}_{nm}\hat{v}_{lk}|^2}{2(\varepsilon_{nm} + 1)^2(\varepsilon_{lk} + 1)^2} + \frac{|\hat{v}_{nm}\hat{v}_{lk}|^2}{2(\varepsilon_{nm} + 1)^2(\varepsilon_{lk} + 1)^2}, \] (81)

we can estimate the norm by

\[ \| P^{(2)} R^{(i)}_n R^{(j)}_m \xi(s) \otimes \Omega_0 \|^2 \leq L^{-4d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^4 |\hat{v}_{nm}\hat{v}_{lk}|}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)^2(\varepsilon_{lk} + 1)^2} \times \left| \langle a_{n}^* a_{m} a_{l} a_{k} \Omega_0, a_{n}^* a_{m} a_{l} a_{k} \Omega_0 \rangle \right| \]

\[ \leq 4 \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{\lambda^2 |\hat{v}_{lk}|^2}{(l^2 - k^2 + 1)^2} \right)^2. \] (82)

In the last step we used that the scalar product provides four different possibilities to cancel the primed summation.
With the aid of Lemma 2.1 we get
\[
\limsup_{L \to \infty} \| \Phi_{32,0}(t) \| \lesssim \lambda^2 k_F^{(d-3)} \ln k_F. \tag{83}
\]

**Term** $\Phi_{32,1}(t)$. Proceeding similarly as in the previous computation, we obtain
\[
\| P(2) \nabla^{(i)} R\nabla^{(j)} (-1 + h_0 - h) \xi(s) \otimes \Omega_0 \|_2^2
\lesssim L^{-4d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \left| \frac{\lambda^2 \hat{v}_{nm} \hat{v}_{lk}}{(\bar{\varepsilon}_{nm} + \bar{\varepsilon}_{lk} + 1)(\bar{\varepsilon}_{lk} + 1)} \right|^2 (1 + \|(h_n - h_0^0)\xi_0\|_{\mathcal{H}_n}^2). \tag{84}
\]

From here we can follow analogous steps as for $\Phi_1(t)$, in order to find
\[
\limsup_{L \to \infty} \| \Phi_{32,1}(t) \| \lesssim |t|(1 + \lambda^2 k_F^{(d-2)}) \lambda^2 k_F^{(d-3)} \ln k_F. \tag{85}
\]

**Term** $\Phi_{32,2}(t)$. Similarly as in $\Phi_2(t)$, here it is important that certain contributions cancel each other. To see this, we compute
\[
P^{(2)}[h^0, \nabla R \nabla] \Omega_0 = \sum_{i,j=1}^n \nabla^{(i)} R[h^0, \nabla^{(j)}] \Omega_0 + R[h^0, \nabla^{(i)}] \nabla^{(j)} \Omega_0
\]
\[
= \sum_{i,j=1}^2 L^{-2d} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \hat{v}_{nm} \hat{v}_{lk} K_{nm,lk}^{(i,j)} (l - k + n - m)^2 a_n^* a_m R a_l^* a_k \Omega_0
\]
\[
+ \sum_{i,j=1}^n L^{-2d} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \hat{v}_{nm} \hat{v}_{lk} G_{nm,lk}^{(i,j)} a_n^* a_m R a_l^* a_k \Omega_0 \tag{86}
\]
where
\[
G_{nm,lk}^{(i,j)} = K_{nm,lk}^{(i,j)} \left(2(n - m) \cdot (-i \nabla_{y_i}) + 2(l - k) \cdot (-i \nabla_{y_j})\right). \tag{87}
\]

If we subtract
\[
P_t^2 P^{(2)}[h^0, \nabla R \nabla] \Omega_0 = \sum_{i,j=1}^n L^{-2d} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \hat{v}_{nm} \hat{v}_{lk} K_{nm,lk}^{(i,j)} (l - k + n - m)^2 a_n^* a_m R a_l^* a_k \Omega_0,
\]
\[
\tag{88}
\]
the terms proportional to $(l - k + n - m)^2$ cancel each other. The difference can thus be bounded by
\[
\| R\{ P^{(2)}[h^0, \nabla R \nabla] - P_t^2 P^{(2)}[h^0, \nabla R \nabla] \} \xi(s) \otimes \Omega_0 \|^2
\]
\[20\]
\begin{align}
&\lesssim L^{-4d} \sum_{(n,m)\in T_F} \sum_{(l,k)\in T_F} \frac{\lambda^4|\hat{v}_{nm}\hat{v}_{lk}|^2}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)^2(\varepsilon_{lk} + 1)^2}((n-m)^2 + (l-k)^2) \sum_{i=1}^n \|\nabla_{y_i}\xi(s)\|^2_{\mathcal{H}_n} \\
&\lesssim \lambda^4 \left( L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}_{lk}|^2(l-k)^2}{(l^2 - k^2 + (l-k)^2 + 1)^2} \right) \left( L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}_{lk}|^2}{(l^2 - k^2 + 1)^2} \right) \left( 1 + \lambda^2 k_F^{(2-d)} \right) 
\end{align}

where we employed (68) in the last step. Using Lemma 2.1 we get

\begin{align}
\limsup_{L\to\infty} \|\Phi_{32;1}(t)\| \lesssim |t|(1 + \lambda^2 k_F^{(2-d)}) \lambda^2 k_F^{(d-3)}(\ln k_F)^{3/2}.
\end{align}

**Term** $\Phi_{32;3}(t)$. We have

\begin{align}
P^{(3)}R^{(i)}P^{(2)}R^{(j)}R^{(k)}\xi(s) \otimes \Omega_0
= L^{-3d} \sum_{(s,r)\in T_F} \sum_{(n,m)\in T_F} \sum_{(l,k)\in T_F} \frac{\lambda^3 \hat{v}_{sr}\hat{v}_{nm}\hat{v}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K^{(i,j,k)}_{sr,nm,lk} \xi(s) \otimes a^*_r a^*_n a_m a_l a_k \Omega_0 \tag{91}
\end{align}

which is a direct consequence of

\begin{align}
P^{(3)}a^*_r a^*_n a_m a_l a_k \Omega_0 = \chi_{T_F}(s,r) \chi_{T_F}(n,m) a^*_r a_m a_k \Omega_0 \quad \forall (l,k) \in B_F^c \times B_F. \tag{92}
\end{align}

Using

\begin{align}
\left| \frac{\hat{v}_{s'r'}\hat{v}_{n'm'}\hat{v}_{l'k'}}{(\varepsilon_{n'm'} + \varepsilon_{l'k'} + 1)(\varepsilon_{l'k'} + 1)(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \right|
\leq \frac{|\hat{v}_{s'r'}\hat{v}_{n'm'}\hat{v}_{l'k'}|^2}{2(\varepsilon_{n'm'} + 1)^2(\varepsilon_{l'k'} + 1)^2} + \frac{|\hat{v}_{sr}\hat{v}_{nm}\hat{v}_{lk}|^2}{2(\varepsilon_{nm} + 1)^2(\varepsilon_{lk} + 1)^2},
\end{align}

we proceed by

\begin{align}
\|P^{(3)}R^{(i)}P^{(2)}R^{(j)}R^{(u)}\xi(s) \otimes \Omega_0\|^2 
\leq L^{-6d} \sum_{(s,r)\in T_F} \sum_{(n,m)\in T_F} \sum_{(l,k)\in T_F} \sum_{(s',r')\in T_F} \sum_{(n',m')\in T_F} \sum_{(l',k')\in T_F} \frac{\lambda^6|\hat{v}_{sr}\hat{v}_{nm}\hat{v}_{lk}|^2}{(\varepsilon_{nm} + 1)^2(\varepsilon_{lk} + 1)^2} \|K^{(i,j,u)}_{sr,nm,lk}\xi(s)\|^2_{\mathcal{H}_n} \\
\times \left| \left\langle a^*_r a^*_n a_m a_l a_k \Omega_0, a^*_s a^*_r a^*_n a_m a_l a_k \Omega_0 \right\rangle_{\mathcal{H}_N} \right|. \tag{94}
\end{align}

In the last expression we employ $\|K^{(i,j,u)}_{sr,nm,lk}\xi(s)\|_{\mathcal{H}_n} = 1$ and use the fact that the scalar
product provides 36 different combinations to cancel the primed summations. This gives
\[
(94) \lesssim \lambda^6 \left( L^{-2d} \sum_{(s,r) \in T_F} |\hat{\nu}_{sr}|^2 \right)^2 \left( L^{-2d} \sum_{(n,m) \in T_F} |\hat{\nu}_{nm}|^2 \right)^2 \left( L^{-2d} \sum_{(l,k) \in T_F} |\hat{\nu}_{lk}|^2 \right). \tag{95}
\]
By Lemma 2.1 we obtain
\[
\lim_{L \to \infty} \sup_{\nu_{22;23}} \|\Phi_{22;23}(t)\| \lesssim |t| |\lambda|^3 k_F^{3d-7/2} \ln k_F. \tag{96}
\]
**Term \( \Phi_{32;31}(t) \).** Next we consider the contributions with one hole. Here one verifies
\[
P(1)\nu^{(i)} P(2)\nu^{(j)} P(3)\nu^{(u)} \xi(s) \otimes \Omega_0
\]
\[
= 2L^{-3d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \lambda^3 |\hat{\nu}_{nm}|^2 |\hat{\nu}_{lk}| \frac{K_{mn,nn,lm}^{(i,j,u)}(s) \otimes a_i^* a_k \Omega_0}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \tag{97a}
\]
\[
+ 2L^{-3d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \lambda^3 |\hat{\nu}_{nm}|^2 |\hat{\nu}_{lk}| \frac{K_{kl,nn,lm}^{(i,j,u)}(s) \otimes a_i^* a_m \Omega_0}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \tag{97b}
\]
\[
+ 2L^{-3d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \lambda^3 |\hat{\nu}_{n}\hat{\nu}_{nm}|^2 |\hat{\nu}_{lk}| \frac{K_{mn,nm,lm}^{(i,j,u)}(s) \otimes a_i^* a_m \Omega_0}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \tag{97c}
\]
\[
- 2L^{-3d} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \lambda^3 |\hat{\nu}_{n}\hat{\nu}_{nm}|^2 |\hat{\nu}_{lk}| \frac{K_{mn,nm,lm}^{(i,j,u)}(s) \otimes a_i^* a_k \Omega_0}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} \tag{97d}
\]
since
\[
P(1)\sigma_{a_l^* a_r^* a_m a_i^* a_k} \Omega_0 = \delta_{sm} \delta_{rn} a_i^* a_k \Omega_0 + \delta_{sl} \delta_{rt} a_i^* a_m \Omega_0 + \delta_{sk} \delta_{rn} a_i^* a_m \Omega_0 \quad \forall l, n \in B_F^c, \quad k, m \in B_F. \tag{98}
\]
The first line is estimated by
\[
\| (97a) \|^2 \leq 4\lambda^6 \left( L^{-2d} \sum_{(n,m) \in T_F} \frac{|\hat{\nu}(n - m)|^2}{m^2 - n^2 + 1} \right)^2 \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{\nu}(l - k)|^2}{(l^2 - k^2 + 1)} \right)^2, \tag{99}
\]
and the same bound holds for \( \| (97b) \|^2 \). Similarly one finds
\[
\| (97c) \|^2 \leq 4\lambda^6 \left( L^{-2d} \sum_{(n',k') \in T_F} \frac{|\hat{\nu}(n' - k')|^2}{(n'^2 - k'^2 + 1)} \right) \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{\nu}(l - k)|^2}{(l^2 - k^2 + 1)} \right)^2, \tag{100}
\]
22
which holds as well for $\| (97d) \|^2$. In total we get

$$
\limsup_{L \to \infty} \| \Phi_{32;31}(t) \| \lesssim |t| |\lambda|^3 k_F^{(3d-7)/2} (\ln k_F)^{1/2}.
$$

(101)

**Term** $\Phi_{32;32}(t)$. Lastly we come to the contributions with two holes. A straightforward computation leads to

$$
P^{(2)V^{(i)}} R^{(j)} R^{(u)} \xi(s) \otimes \Omega_0
$$

$$
= L^{-3d} \sum_{s \in B_F} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^3 \hat{\varphi}_{nl} \hat{\varphi}_{nm} \hat{\varphi}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K_{sn,nm,lk}^{(i,j,u)} \xi(s) \otimes a_s^* a_m a_l^* a_k \Omega_0
$$

(102a)

$$
+ L^{-3d} \sum_{s \in B_F} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^3 \hat{\varphi}_{nl} \hat{\varphi}_{nm} \hat{\varphi}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K_{r,l,nm,lk}^{(i,j,u)} \xi(s) \otimes a_r^* a_s^* a_m a_k \Omega_0
$$

(102b)

$$
+ L^{-3d} \sum_{r \in B_F} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^3 \hat{\varphi}_{ml} \hat{\varphi}_{nm} \hat{\varphi}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K_{m,r,nm,lk}^{(i,j,u)} \xi(s) \otimes a_r a_s^* a_m a_l^* \Omega_0
$$

(102c)

$$
+ L^{-3d} \sum_{r \in B_F} \sum_{(n,m) \in T_F} \sum_{(l,k) \in T_F} \frac{\lambda^3 \hat{\varphi}_{kl} \hat{\varphi}_{nm} \hat{\varphi}_{lk}}{(\varepsilon_{nm} + \varepsilon_{lk} + 1)(\varepsilon_{lk} + 1)} K_{r,k,nm,lk}^{(i,j,u)} \xi(s) \otimes a_r^* a_s^* a_m a_l^* \Omega_0
$$

(102d)

where we used

$$
P^{(2)} a_s^* a_r^* a_m a_l^* a_k \Omega_0 = \chi_{B_F}(s) (\delta_{rn} a_s^* a_m a_l^* a_k \Omega_0 + \delta_{rl} a_s^* a_m a_l^* a_k \Omega_0)
$$

$$
+ \chi_{B_F}(r) (\delta_{sm} a_r a_s^* a_m a_l^* a_k \Omega_0 + \delta_{sk} a_r a_s^* a_m a_l^* a_k \Omega_0) \quad \forall l, n \in B_F, k, m \in B_F.
$$

(103)

We estimate

$$
\| (102a) \|^2 \leq \left( L^{-2d} \sum_{(l,k) \in T_F} \frac{\lambda^2 |\hat{\varphi}_{lk}|^2}{(\varepsilon_{lk} + 1)^2} \right) \left( L^{-2d} \sum_{(s,m) \in T_F} \frac{\lambda^2 |\hat{\varphi}_{nm}|^2 |\hat{\varphi}_{ls}|^2}{(\varepsilon_{nm} + 1)^2} \right)
$$

(104a)

$$
+ L^{-2d} \sum_{s,l \in B_F} \left( L^{-2d} \sum_{(n,m) \in T_F} \frac{\lambda^3 |\hat{\varphi}_{nl}| |\hat{\varphi}_{nm}| |\hat{\varphi}_{ls}|}{(\varepsilon_{nm} + 1)^2} \right)^2
$$

(104b)

$$
+ L^{-2d} \sum_{k,m \in B_F} \left( L^{-2d} \sum_{l,n \in B_F} \frac{\lambda^3 |\hat{\varphi}_{lk}| |\hat{\varphi}_{nm}| |\hat{\varphi}_{ln}|}{(\varepsilon_{nm} + 1)^2} \right)^2
$$

(104c)

$$
+ \left( L^{-3d} \sum_{(l,k) \in T_F} \sum_{n \in B_F} \frac{\lambda^3 |\hat{\varphi}_{nk}| |\hat{\varphi}_{ln}|}{(\varepsilon_{nk} + 1)^2} \right)^2,
$$

(104d)

and in close analogy, one derives the same bound also for (102b)-(102d).

With the aid of Lemma 2.1 (38c) and Lemma 2.2, we can estimate the above expressions

---

23
and obtain
\[
\limsup_{L \to \infty} \| \Phi_{22,22}(t) \| \lesssim |t| \| \lambda \|^3 \left( \sqrt{\gamma(d,k_F)} k_F^{(d-3)/2} (\ln k_F)^{1/2} + \gamma(d,k_F) \right). \tag{105}
\]

This completes the derivation of inequality (18) and thus the proof of Theorem 1.2.

### 2.4 Proof of Proposition 1.3

Set \( \lambda^2 = k_F^{(2-d)} \). We first derive a lower bound for the norm difference \( \| (e^{-ih_nt} - e^{-i\tilde{h}_n t}) \xi_0 \|_{\mathcal{H}_n} \).

To this end we compute
\[
(1 - e^{i\tilde{h}_n t} e^{-ih_nt}) \xi_0 = i \int_0^t ds e^{i\tilde{h}_n s} (h_n - \tilde{h}_n) e^{-ih_ns} \xi_0 \\
= it(h_n - \tilde{h}_n) \xi_0 - \int_0^t ds \int_0^s dr e^{i\tilde{h}_n r} (h_n(h_n - \tilde{h}_n) - (h_n - \tilde{h}_n) h_n) e^{-ihr} \xi_0. \tag{106}
\]

With the Cauchy–Schwarz inequality we can use (106) to estimate
\[
\| (e^{-ih_nt} - e^{-i\tilde{h}_n t}) \xi_0 \|_{\mathcal{H}_n} \geq |\langle \xi_0, (1 - e^{i\tilde{h}_n t} e^{-ih_nt}) \xi_0 \rangle |
\[
\geq t \left| \sum_{i<j} \langle \xi_0, \lambda^2 W_{k_F}(|y_i - y_j|) \xi_0 \rangle_{\mathcal{H}_n} \right|
\[
- \sum_{i<j} \int_0^t ds \int_0^s dr \langle e^{-i\tilde{h}_n r} \xi_0, (\tilde{h}_n \lambda^2 W_{k_F}(|y_i - y_j|) - \lambda^2 W_{k_F}(|y_i - y_j|) h_n) e^{-ihr} \xi_0 \rangle_{\mathcal{H}_n}. \tag{107}
\]

By condition (22) we know that the first summand is bounded from below by \( c_0 t \). The absolute value in (107), in turn, is estimated from above by
\[
\left| \int_0^t ds \int_0^s dr \ldots \right| \leq \frac{t^2}{2} \lambda^2 W_{k_F}(0) (\| \tilde{h}_n \xi_0 \|_{\mathcal{H}_n} + \| h_n \xi_0 \|_{\mathcal{H}_n}) \tag{108}
\]

where we used \( |W_{k_F}(r)| \leq W_{k_F}(0) \) and \( \| \tilde{h}_n e^{-i\tilde{h}_n r} \xi_0 \| = \| \tilde{h}_n \xi_0 \| \) and the same for \( \tilde{h}_n \) replaced by \( h_n \). With \( \lambda^2 W_{k_F}(0) \leq C \) and the second assumption in (22), this implies
\[
\left| \int_0^t ds \int_0^s dr \ldots \right| \leq Ct^2 \tag{109}
\]

24
for some constant $C > 0$ independent of $k_F$ and $L$. Hence there is a time $t_1 > 0$ such that for all $t \in (0, t_1)$,

$$\|(e^{-iht} - e^{-i\tilde{h}_n t})\xi_0\|_{\mathcal{F}_n} \geq c_0 t - \frac{n^2}{2} Ct^2 \geq \frac{3c_0}{4} t.$$  

(110)

In combination with Theorem 1.2 this leads to

$$\liminf_{L \to \infty} \|\Psi(t) - \exp(-i\tilde{h}_n t)\xi_0 \otimes e^{-iE\Omega_0}\| \geq \frac{3c_0}{4} t - C \frac{(1 + t)(\ln k_F)^{3/2}}{\sqrt{k_F}}$$  

(111)

for all $t \in (0, t_1)$. Thus we can find a time $t_0$ such that for $t \in (t_0, t_1)$ and $k_F$ large enough the right side is bounded from below by $c_0 t/2$. This concludes the proof of the proposition.

### 2.5 Proof of Lemma 1.1

For $\mu \geq 1$ we set

$$W_{k_F}^{\leq \mu}(r) = V_d^2 \int_{|k| \leq k_F} d^d k \int_{|l| \geq k_F} d^d l \frac{\tilde{v}_\infty(l - k)^2}{l^2 - k^2 + (l - k)^2 + 1} \cos((l - k) \cdot r \hat{a}) \chi(0, \mu)(|l - k|)$$

(112)

where $\chi(0, \mu)(|l - k|) = 1$ for $|l - k| \leq \mu$ and zero otherwise. Below we show that there is a constant $c_1 > 0$ such that

$$W_{k_F}^{\leq \mu}(0) \geq c_1 k_F^{(d-2)}.$$  

(113)

Moreover, by Lemma 2.1, it follows that there is a constant $C_2 > 0$ such that

$$\left| \frac{d}{dr} W_{k_F}^{\leq \mu}(r) \right| \leq \mu V_d^2 \int_{|k| \leq k_F} d^d k \int_{|l| \geq k_F} d^d l \frac{\tilde{v}_\infty(l - k)^2}{l^2 - k^2 + 1} \chi(0, \mu)(|l - k|) \leq C_2 \mu k_F^{(d-2)}.$$  

(114)

Combining the two estimates, one concludes that $k_F^{(2-d)} W_{k_F}^{\leq \mu}(r)$ is uniformly bounded from below for some small ball around $r = 0$. More precisely,

$$\inf_{k_F \geq 1} \inf \left\{ k_F^{(2-d)} W_{k_F}^{\leq \mu}(r) : 0 \leq r \leq c_1/(2C_2\mu) \right\} \geq \frac{c_1}{2}.$$  

(115)

To get a lower bound for $W_{k_F}(r)$, we write

$$W_{k_F}(r) = W_{k_F}^{\leq \mu}(r) + W_{k_F}^{\geq \mu}(r)$$

(116)

and use that $W_{k_F}^{\leq \mu}(r)$ exceeds the absolute value of $W_{k_F}^{\geq \mu}(r)$ on a small ball if we choose $\mu$
large enough (since the size of this ball may shrink with $\mu$ we keep $\mu$ fixed with respect to $k_F$). To this end we shall show that there is a constant $C_3 > 0$ such that

$$\sup_{k_F \geq 1} \sup_{r \geq 0} |k_F^{(2-d)} W_{k_F}^\leq \mu(r)| \leq C_3 \mu^{-1}$$

(117)

for all $\mu \geq 1$. Together with (115) this proves the lower bound in (14) since (setting $c = c_1/(2C_2\mu)$)

$$\inf_{k_F \geq 1} \inf_{r \in [0,c]} k_F^{(2-d)} W_{k_F}(r) \geq \inf_{k_F \geq 1} \inf_{r \in [0,c]} k_F^{(2-d)} \left( W_{k_F}^\leq \mu(r) - |W_{k_F}^\geq \mu(r)| \right) \geq \frac{c_1}{2} - C_3 \mu^{-1},$$

(118)

which is strictly positive for $\mu \geq 4C_3/c_1$.

It remains to show (113) and (117).

Proof of (117). Since $|W_{k_F}^{\geq \mu}(r)| \leq W_{k_F}^{\geq \mu}(0)$ and $l^2 - k^2 \geq k_F(|l| - |k|)$, it is sufficient to show

$$J = \int_{|k| \leq k_F} |\hat{v}_\infty(l - k)|^2 \frac{1}{|l| - |k| + k_F^{-1}} \chi_{(0,\infty)}(|l - k|) \lesssim \mu^{-1} k_F^{(d-1)}. \quad (119)$$

This is done in Appendix A: Comparing with (127) we see that $J = \sum_{m=1}^{M+1} J_{2m}$ which is shown to be bounded by $J \lesssim \mu^{(d-4)} k_F^{(d-1)} + k_F^{(d-2)}$. For $k_F$ large enough this implies (117).

Proof of (113). Here we estimate

$$W_{k_F}^{\leq \mu}(0) \geq \frac{V_d}{2k_F(1 + R)^2} \int_{|k| \leq k_F} |\hat{v}_\infty(l - k)|^2 \frac{1}{|l| - |k| + k_F^{-1}} \chi_{(0,1)}(|l - k|), \quad (120)$$

where we used $\chi_{(0,\mu)}(|k - l|) \geq \chi_{(0,1)}(|l - k|)$, $|\hat{v}(k)|^2 \geq 1/(1 + R)^2$ for $k^2 \leq 1$, $|l| + |k| \leq 2k_F$ and $(l - k)^2 \leq 1$. Next we use that the right side is bounded from below by a positive constant times

$$k_F^{(d-2)} \int_{k_F^{-1/2}}^{k_F} \int_{k_F^{-1/2}}^{k_F} ds \frac{1}{r^{-s + k_F^{-1}}}. \quad (121)$$

Evaluating the remaining expression we get the desired bound, $W_{k_F}^{\leq \mu}(0) \geq c_1 k_F^{(d-2)}$ for some constant $c_1 > 0$ and all $\mu \geq 1$. 

26
Appendix A

Lemma 2.2. Let \( d \in \{1, 2, 3\} \), assume \((A_{\nu L})\) and let \( \gamma(d, k_F) \) be defined by (20). There exists a constant \( C > 0 \) (depending on \( R \)) such that for all \( k_F \geq 2 \),

\[
\lim_{L \to \infty} L^{-2d} \sum_{(n,k) \in T_F} \left( L^{-d} \sum_{l \in B_F^c} \frac{|\hat{v}_L(l-k)| |\hat{v}_L(n-l)|}{l^2 - k^2 + 1} \right)^2 \leq C \gamma(d, k_F), \quad (122a)
\]

\[
\lim_{L \to \infty} L^{-2d} \sum_{(l,m) \in T_F} \left( L^{-d} \sum_{k \in B_F} \frac{|\hat{v}_L(l-k)| |\hat{v}_L(m-k)|}{l^2 - k^2 + 1} \right)^2 \leq C \gamma(d, k_F), \quad (122b)
\]

\[
\lim_{L \to \infty} L^{-3d} \sum_{(l,k) \in T_F} \frac{|\hat{v}_L(l-k)| |\hat{v}_L(n-k)| |\hat{v}_L(l-n)|}{(l^2 - k^2 + 1)(n^2 - k^2 + 1)} \leq C \gamma(d, k_F), \quad (122c)
\]

\[
\lim_{L \to \infty} L^{-2d} \sum_{k,m \in B_F} \left( L^{-2d} \sum_{l,n \in B_F} \frac{|\hat{v}_L(l-k)| |\hat{v}_L(n-m)| |\hat{v}_L(l-n)|}{(l^2 - k^2 + 1)(n^2 - m^2 + 1)} \right)^2 \leq C \gamma(d, k_F)^2, \quad (122d)
\]

\[
\lim_{L \to \infty} L^{-2d} \sum_{r,l \in B_F^c} \left( L^{-2d} \sum_{(n,m) \in T_F} \frac{|\hat{v}_L(l-m)| |\hat{v}_L(n-m)| |\hat{v}_L(r-n)|}{(l^2 - m^2 + 1)(n^2 - m^2 + 1)} \right)^2 \leq C \gamma(d, k_F)^2. \quad (122e)
\]

In the remainder of this appendix we provide the proof of Lemmas 2.1 and 2.2. To derive the stated bounds, we replace the expression to be estimated by its Riemann integral and proceed by estimating the latter.

Proof of (38a). The integral

\[
J_1 = \int_{|k| \leq k_F} \int_{|l| \geq k_F} \frac{d^d k \ d^d l}{(1 + |l-k|^2)^2} \quad (123)
\]

is bounded from above by

\[
\int_{|k| \leq k_F} d^d k \int_{|l| \geq k_F} d^d l \frac{1}{(1 + |l-k|^2)^2} \lesssim \int_{k_F-s}^k ds \int_0^\infty dr (1+r)^{-2} \lesssim k_F^{(d-1)}. \quad (124)
\]

Proof of (38b). Let us first note estimate (for \( m = 1, 2 \))

\[
\int_{|k| \leq k_F} d^d k \int_{|l| \geq k_F} d^d k \frac{\chi(0,a)(|l-k|)}{(|l| - |k| + k_F^{-1})^m} \lesssim (k_F a)^{(d-1)} \int_{-a}^a ds \int_0^a dr \frac{1}{(r-s + k_F^{-1})^m}. \quad (125)
\]

(One of the angle integrations gives a factor \((a/k_F)^{d-1}\) while the radial component of each
variable is bounded by a constant times $k_F$.)

In (38b) we use $l^2 - k^2 \geq k_F(|l| - |k|)$ and consider the integral

$$J_2 = \int_{|l| \geq k_F} \int_{|k| \leq k_F} d^d k \frac{\left| \hat{\varphi}_\infty(k - l) \right|^2}{|l| - |k| + k_F^{-1}}. \tag{126}$$

To estimate such expressions, we discretize them, that is, for $M$ the smallest integer larger than $\ln k_F$, we write

$$J_2 = \sum_{m=0}^{M+1} J_{2,m}, \quad J_{2,m} = \int_{|k| \leq k_F} \int_{|l| \geq k_F} d^d l \frac{\left| \hat{\varphi}_\infty(l - k) \right|^2}{|l| - |k| + k_F^{-1}} \chi_{[a_m, a_{m+1})}(|l - k|), \tag{127}$$

with $a_0 = 0$, $a_m = \mu k_F^{(m-1)/M}$ for $m = 1, \ldots, M + 1$ and some large enough $\mu$ (for instance $\mu \geq 10$) and $a_{M+2} = \infty$. Note that in the following estimates the value of the constant $C$ does not depend on $\mu$ or $M$.

For $m = 0$ we use $|\hat{\varphi}_\infty(l - k)| \leq R^{-1}$ and then apply (125) to get

$$J_{2,0} \lesssim (k_F \mu)^{(d-1)} \int_{-\mu}^{\mu} ds \int_{0}^{1} dr \frac{1}{r - s + k_F^{-1}}. \tag{128}$$

Since for $a > 1 > 2\varepsilon > 0$,

$$\int_{-a}^{0} ds \int_{0}^{a} dr \frac{1}{r - s + \varepsilon} \leq 5a \ln(3a) + \varepsilon \ln(\varepsilon^{-1}), \tag{129}$$

we obtain

$$J_{2,0} \lesssim \mu^{d}(1 + \ln \mu) k_F^{(d-1)}. \tag{130}$$

For $m = 1, \ldots, M$ we use $|\hat{\varphi}_\infty(l - k)|^2 \leq \mu^{-4} k_F^{-4(m-1)/M}$ and then proceed with (125) and (129). This gives

$$J_{2,m} \lesssim \mu^{-4} k_F^{-4(m-1)/M} \left( k_F \mu_{k_F^{m/M}}^{m/M} \right)^{(d-1)} \int_{-\mu k_F^{m/M}}^{0} ds \int_{0}^{\mu k_F^{m/M}} dr \frac{1}{r - s + k_F^{-1}}$$

$$\lesssim \mu^{(d-4)} k_F^{(d-1)} k_F^{-(d-4)m/M} \ln(\mu k_F^{m/M}). \tag{131}$$
Using \( \ln(\mu k_F^{m/M}) \lesssim k_F^{m/(2M)} \) we can easily bound the sum of the \( m \) dependent factors by

\[
\sum_{m=1}^{M} \left( k_F^{(d-7)/M} \right)^m = \frac{k_F^{(d-7)/M} - k_F^{(d-7)/(M+1)/M}}{1 - k_F^{(d-7)/M}} \lesssim 1 \tag{132}
\]

which leads to \( \sum_{m=1}^{M} J_{2,m} \lesssim \mu^{(d-4)} k_F^{(d-1)} \).

In the last term we use that for \( \mu \) large enough (for instance \( \mu \geq 10 \)), we have \(|l - k| \geq |l| - |k| \geq |l|/2 \) and \( |l| \geq 5k_F \). By this one verifies that

\[
J_{2,M+1} \lesssim \int_0^{k_F} ds \int_0^{s^{d-1}} dr \frac{1}{r} \lesssim k_F^{(2d-5)} \lesssim k_F^{(d-2)}. \tag{133}
\]

Combining all estimates proves (38b).

**Proof of (38c).** This bound is derived in close analogy to the previous one. The only difference is that one needs to use

\[
\int_0^{k_F} ds \int_{-a}^a dr \frac{1}{(r - s + \varepsilon)^2} \leq 2 \ln(2a) + \ln(\varepsilon^{-1}) \tag{134}
\]

instead of (129). We omit the details.

**Proof of (38d).** Since the contribution with \(|l - k| \leq 1\) is smaller than the expression on the left side of (38c), it is sufficient to consider

\[
J_3 = \int_{|k| \leq k_F} d^dk \int_{|l| > k_F} d^dl \frac{\hat{v}_\infty(l - k)^2(l - k)^2}{(l^2 - k^2 + (l - k)^2 + 1)^2} \chi_{[1, \infty]}(|l - k|). \tag{135}
\]

(Note that we do not estimate the denominator as before.) We discretize again by writing

\[
J_3 = \sum_{m=1}^{M+1} J_{3,m}, \quad J_{3,m} = \int_{|k| \leq k_F} d^dk \int_{|l| > k_F} d^dl \frac{\hat{v}_\infty(l - k)^2(l - k)^2}{(l^2 - k^2 + (l - k)^2 + 1)^2} \chi_{[a_m, a_{m+1}]}(|l - k|), \tag{136}
\]

with \( a_m = \mu k_F^{(m-1)/M} \) for \( m = 1, ..., M + 1 \) and some large enough \( \mu \), \( a_{M+2} = \infty \) and \( M \) the smallest integer larger than \( \ln k_F \).

For \( m = 1, ..., M \) we use \( l^2 - k^2 + (l - k)^2 + 1 \geq k_F (|l| - |k| + k_F^{-1}) \) and \( |\hat{v}_\infty(l - k)|^2 |l - k|^2 \lesssim \ldots \)
(\mu k_F^{(m-1)/M})^{-2}. By means of (125) and (134) we then get

\begin{align*}
J_{3,m} & \lesssim k_F^{(d-3)}(\mu k_F^{(m-1)/M})^{-2}(\mu k_F^{m/M})^{(d-1)} \int_{-\mu k_F^{m/M}}^{0} \int_{0}^{\mu k_F^{m/M}} ds \int_{0}^{1} \frac{1}{(r - s + k_F^{-1})^2} \\
& \lesssim \mu^{(d-3)} k_F^{(d-3)} k_F^{(d-3)m/M} \ln(\mu k_F). \tag{137}
\end{align*}

The sum over the \(m\) dependent factors is bounded by

\begin{align*}
\sum_{m=1}^{M} k_F^{m(d-3)/M} & \leq M \leq 1 + \ln k_F \tag{138}
\end{align*}

and thus we have \(\sum_{m=1}^{M} J_{3,m} \lesssim \mu^{(d-3)} k_F^{(d-3)}(1 + \ln k_F)^2\).

In the last term we bound the factor \((l - k)^2\) by the denominator. This gives

\begin{align*}
J_{3,M+1} & \lesssim \int_{|l| \leq k_F} d^d k \int_{|l| > k_F} d^d l \frac{[\hat{v}_\infty(l - k)]^2}{l^2 - k^2 + (l - k)^2 + 1} \chi_{[\mu k_F, \infty]}(|l - k|). \tag{139}
\end{align*}

From here, we use \(l^2 - k^2 \geq k_F^{-1}(|l| - |k|)\) and \(|l - k| \geq |l| - |k| \geq |l|/2\) and \(|l| \geq 5k_F\) (which is true for instance for \(\mu \geq 10\)). Hence we get

\begin{align*}
J_{3,M+1} & \lesssim k_F^{-1} \int_{|l| \leq k_F} d^d k \int_{|l| > k_F} d^d l \frac{[\hat{v}_\infty(l - k)]^2}{|l| - |k| + k_F^{-1}} \chi_{[\mu k_F, \infty]}(|l - k|) \\
& \lesssim k_F^{-1} \int_{s=0}^{k_F} ds s^{d-1} \int_{5k_F}^{\infty} dr r^{d-4} \lesssim k_F^{(2d-4)}. \tag{140}
\end{align*}

**Proof of (122a).** To estimate the integral

\begin{align*}
J_4 = \int_{|l| \leq k_F} d^d k \int_{|n| \geq k_F} d^d n \left( \int_{|l| \geq k_F} d^d l \frac{[\hat{v}_\infty(l - n)] [\hat{v}_\infty(l - k)]}{l^2 - k^2 + 1} \right)^2, \tag{141}
\end{align*}

we employ

\begin{align*}
\int_{\mathbb{R}^d} d^d n |\hat{v}_\infty(l - n)| |\hat{v}_\infty(l' - n)| \lesssim 1. \tag{142}
\end{align*}
Thus we have

$$J_4 \lesssim \int_{|l| \geq k_F} d^d l \left( \int_{|k| \leq k_F} d^d k \left( \int_{|l| \geq k_F} d^d l \frac{\hat{\nu}_\infty(l-k)}{|l| - |k| + k_F^{-1}} \right)^2 \right) \lesssim k_F^{-2} \int_{|l| \geq k_F} d^d k \left( \int_{|k| \geq k_F} d^d l \frac{\hat{\nu}_\infty(l-k)}{|l| - |k| + k_F^{-1}} \right)^2.$$  \hspace{1cm} (143)

In the last expression we estimate the integrand by

$$\left( \int_{|l| \geq k_F} d^d l \frac{\hat{\nu}_\infty(l-k)}{|l| - |k| + k_F^{-1}} \right)^2 \lesssim A_0(k)^2 + \left( \sum_{m=1}^M A_m(k) \right)^2 + A_{M+1}(k)^2 \hspace{1cm} (144)$$

with

$$A_m(k) = \int_{|l| \geq k_F} d^d l \frac{\hat{\nu}_\infty(l-k)}{|l| - |k| + k_F^{-1}} \chi_{[a_m, a_{m+1})}(|l-k|)$$ \hspace{1cm} (145)

where $M$ is the smallest integer larger than $\ln k_F$, and

$$a_0 = 0, \quad a_m = 10 k_F^{(m-1)/M} \quad (m = 1, \ldots, M+1), \quad a_{M+2} = \infty. \hspace{1cm} (146)$$

The $A_m(k)$ are estimated by

$$A_0(k) \lesssim \int_{k_F}^{k_F+10} dr \frac{1}{r - |k| + k_F^{-1}} \chi_{[-10,0)}(|k| - k_F)$$

$$\lesssim \ln(k_F + 10 - |k| + k_F^{-1}) - \ln(k_F - |k| + k_F^{-1}) \chi_{[-10,0)}(|k| - k_F)$$

$$\lesssim \ln(20 + k_F^{-1}) - \ln(k_F^{-1}) \chi_{[-10,0)}(|k| - k_F) \lesssim \ln k_F \chi_{[-10,0)}(|k| - k_F),$$ \hspace{1cm} (147)

which leads to

$$k_F^{-2} \int_{|k| \leq k_F} d^d k A_0(k)^2 \lesssim k_F^{(d-3)} (\ln k_F)^2.$$ \hspace{1cm} (148)

For $m = 1, \ldots, M$ we use $|\hat{\nu}_\infty(l-k)| \lesssim k_F^{-2m/M}$ and get

$$A_m(k) \lesssim k_F^{(d-1)m/M} k_F^{-2m/M} \int_{k_F}^{k_F+10 k_F^{m/M}} \frac{1}{r - |k| + k_F^{-1}} \chi_{[-10 k_F^{m/M}, 0)}(|k| - k_F)$$

$$\lesssim k_F^{(d-3)m/M} \left( \ln(20 k_F^{m/M} + k_F^{-1}) - \ln(k_F^{-1}) \right) \chi_{[-10 k_F^{m/M}, 0)}(|k| - k_F)$$
\[ \lesssim k_F^{(d-3)m/M} \ln k_F \chi_{[-10k_F^{m/M}, 0)}(|k| - k_F). \]  

(149)

We thus find

\[ k_F^{-2} \int_{|k| \leq k_F} d^d k \left( \sum_{m=1}^M A_m(k) \right)^2 \lesssim k_F^{(d-3)} (\ln k_F)^2 \sum_{n,m=1}^M k_F^{(d-3)m/M} k_F^{(d-2)n/M} \]

\[ \lesssim \begin{cases} 
  k_F^{(d-3)} (\ln k_F)^3 & (d = 1, 2) \\
  k_F (\ln k_F)^3 & (d = 3) 
\end{cases}. \]  

(150)

In the last summand we use \(|l| \geq 5k_F\) to get

\[ \int_{|k| \geq k_F} d^d k A_{M+1}(k)^2 \leq k_F^{-2} \int_{|k| \geq k_F} d^d k \left( \int_{5k_F}^{\infty} r^{d-4} dr \right)^2 \lesssim k_F^{3d-8}. \]  

(151)

**Proof of (122b).** The derivation is almost the same as the previous one. Consider

\[ J_5 = \int_{|m| \leq k_F} d^d m \int_{|l| \geq k_F} d^d l \left( \int_{|k| \leq k_F} d^d k \frac{\tilde{\nu}_\infty(k - m)}{|l|^2 - k^2 + 1} \right)^2 \]

\[ \lesssim k_F^{-2} \int_{|l| \geq k_F} d^d l \left( \int_{|k| \leq k_F} d^d k \frac{\tilde{\nu}_\infty(l - k)}{|l| - |k| + k_F^{-1}} \right)^2 \]

\[ \lesssim k_F^{-2} \int_{|l| \geq k_F} d^d l \left( B_0(l)^2 + \left( \sum_{m=1}^M B_m(l) \right)^2 + B_{M+1}(l)^2 \right), \]  

(152)

with

\[ B_m(l) = \int_{|l| \geq k_F} d^d k \frac{\tilde{\nu}_\infty(l - k)}{|l| - |k| + k_F^{-1}} \chi_{a_m, a_{m+1}}(|l| - k) \]  

(153)

and \((a_m)_{m=0}^{M+2}\) defined by (146).

The \(B_m(l)\) are estimated by

\[ B_0(l) \lesssim \int_{k_F^{-10}}^{k_F} ds \frac{1}{|l| - s + k_F^{-1}} \chi_{[0, 10)}(|l| - k_F) \]

\[ = (\ln(|l| - k_F + 10 + k_F^{-1}) - \ln(|l| - k_F + k_F^{-1})) \chi_{[0, 10)}(|l| - k_F) \]

\[ \lesssim (\ln(20 + k_F^{-1}) - \ln(k_F^{-1})) \chi_{[-a_1, 0)}(|l| - k_F) \lesssim \chi_{[-10, 0)}(|l| - k_F) \ln k_F, \]  

(154)
which leads to
\[
 k_F^{-2} \int_{|l| \geq k_F} d^d l \, B_0(l)^2 \lesssim k_F^{(d-3)} (\ln k_F)^2. \tag{155}
\]

For \( m = 1, \ldots, M \), we have
\[
 B_m(l) \lesssim k_F^{(d-3)m/M} \int_{k_F - 10 k_F^{m/M}}^{k_F} ds \, \frac{1}{|l| - s + k_F^{-1} \chi_{[0,10k_F^{m/M}]}(|l| - k_F)}
\]
\[
 \lesssim k_F^{(d-3)m/M} \chi_{[0,10k_F^{m/M}]}(|l| - k_F) \ln k_F \tag{156}
\]
and thus get
\[
 k_F^{-2} \int_{|l| \geq k_F} d^d l \left( \sum_{m=1}^{M} B_m(l) \right)^2 \lesssim k_F^{(d-3)} (\ln k_F)^2 \sum_{n,m=1}^{M} k_F^{(d-3)m/M} k_F^{(d-3)m/M} \lesssim \begin{cases} 
 k_F^{(d-2)} (\ln k_F)^3 & (d = 1) \\
 k_F^{2d-5} (\ln k_F)^2 & (d = 2, 3) 
\end{cases}. \tag{157}
\]

In \( B_{M+1}(l) \) we use \(|l| \geq 5k_F\) and proceed by
\[
 k_F^{-2} \int_{|l| \geq k_F} d^d l B_{M+1}(l) \lesssim k_F^{-2} \int_{|l| \geq k_F} d^d l \left( k_F^{d/3} \right)^2 \lesssim k_F^{(3d-8)} \lesssim \begin{cases} 
 k_F^{(d-4)} & (d = 1, 2) \\
 k_F^{(d-3)} & (d = 3) 
\end{cases}. \tag{158}
\]

**Proof of (122c)-(122e).** Here we can use \(|\hat{v}_\infty(l - n)| \leq R^{-1}\) and
\[
 J_6 = \int_{|k| \leq k_F} d^d k \int_{|l| \geq k_F} d^d l \int_{|n| \geq k_F} d^d n \frac{\hat{v}_\infty(l - k) |\hat{v}_\infty(n - k)| \hat{v}_\infty(l - n)}{(l^2 - k^2 + 1)(n^2 - k^2 + 1)}
\]
\[
 \lesssim \int_{|k| \leq k_F} d^d k \left( \int_{|l| \geq k_F} d^d l \frac{\hat{v}_\infty(l - k)}{l^2 - k^2 + 1} \right)^2 \tag{159}
\]
which has been estimated in (143).

Similarly, also for
\[
 J_7 = \int_{|k| \leq k_F} d^d k \int_{|m| \leq k_F} d^d m \left( \int_{|l| \geq k_F} d^d l \int_{|n| \geq k_F} d^d n \frac{\hat{v}_\infty(l - k) |\hat{v}_\infty(n - m)| \hat{v}_\infty(l - n)}{(l^2 - k^2 + 1)(n^2 - m^2 + 1)} \right)^2
\]
\[
\leq \left( \int_{|k| \leq k_F} d^d k \left( \int_{|l| \geq k_F} d^d l \frac{|\hat{v}_\infty (l-k)|}{l^2 - k^2 + 1} \right)^2 \right)^2
\]

(160)

and

\[
J_8 = \int_{|r| \geq k_F} d^d r \int_{|l| \geq k_F} d^d l \left( \int_{|n| \geq k_F} d^d n \int_{|m| \leq k_F} d^d m \frac{|\hat{v}_\infty (l-m)| |\hat{v}_\infty (n-m)| |\hat{v}_\infty (r-n)|}{(l^2 - m^2 + 1)(n^2 - m^2 + 1)} \right)^2 
\leq \left( \int_{|m| \leq k_F} d^d m \left( \int_{|l| \geq k_F} d^d l \frac{|\hat{v}_\infty (l-m)|}{l^2 - m^2 + 1} \right)^2 \right)^2.
\]

(161)

Acknowledgments

We are grateful to Maximilian Jeblick for his contributions at an earlier stage of this project.

References

[1] A. Camacho-Guardian, L.A. Peña Ardila, T. Pohl and G.M. Bruun. Bipolarons in a Bose–Einstein Condensate. Phys. Rev. Lett. 121, 013401. (2018)

[2] B.J. DeSalvo, K. Patel, G. Cai and C. Chin. Observation of fermion-mediated interactions between bosonic atoms. Nature 568, 61–64. (2019)

[3] J.T. Devreese and A.S. Alexandrov. Fröhlich polaron and bipolaron: recent developments. Reports on Progress in Physics72, 066501. (2009)

[4] H. Edri, B. Raz, N. Matzliah, N. Davidson and R. Ozeri. Observation of spin-spin fermion-mediated interactions between ultracold bosons. Phys. Rev. Lett. 124, 163401 (2020)

[5] T. Enss, B. Tran, M. Rautenberg, M. Gerken, E. Lippi, M. Drescher, B. Zhu, M. Weidemüller and M. Salmhofer. Scattering of two heavy Fermi polarons: Resonances and quasibound states. Phys. Rev. A 102, 063321. (2020)

[6] R.L. Frank, E.H. Lieb, R. Seiringer and L.E. Thomas. Ground state properties of multi-polaron systems. In XVIIth International Congress on Mathematical Physics, Proceedings of the ICMP 2012, A. Jensen (ed.), 477–485, World Scientific, Singapore. (2013)

[7] B. Huang. Bose-Einstein condensate immersed in a Fermi sea: Theory of static and dynamic behavior across phase separation. Phys. Rev. A 101, 063618. (2020)

[8] M. Jeblick, D. Mitrouskas, S. Petrat and P. Pickl. Free time evolution of a tracer particle coupled to a Fermi gas in the high-density limit. Commun. Math. Phys. 356, 143–187. (2017)
[9] M. Jeblick, D. Mitrouskas and P. Pickl. Effective dynamics of two tracer particles coupled to a Fermi gas in the high-density limit. Chapter in *Macroscopic limits of quantum systems, Springer Proceedings in Mathematics & Statistics*. (2018)

[10] J.J. Kinnunen and G.M. Bruun. Induced interactions in a superfluid Bose-Fermi mixture. *Phys. Rev. A* 91, 041605(R). (2015)

[11] N. Leopold, D. Mitrouskas, S. Rademacher, B. Schlein and R. Seiringer. Landau-Pekar equations and quantum fluctuations for the dynamics of a strongly coupled polaron. Preprint: arXiv:2005.02098

[12] X. Li, E. Yakaboylu, G. Bighin, R. Schmidt, M. Lemeshko and A. Deuchert. Intermolecular forces and correlations mediated by a phonon bath. *J. Chem. Phys.* 152, 164302. (2020)

[13] S.I. Mistakidis, G.C. Katsimiga, G.M. Koutentakis and P. Schmelcher. Repulsive Fermi polarons and their induced interactions in binary mixtures of ultracold atoms. *New J. Phys.* 21 043032. (2018)

[14] D. Mitrouskas. A note on the Fröhlich dynamics in the strong coupling limit. *Lett. Math. Phys.* 111, 45. (2021)

[15] Y. Nishida. Casimir interaction among heavy fermions in the BCS-BEC crossover. *Phys. Rev. A* 79, 013629. (2009)

[16] D.H. Santamore and E. Timmermans. Fermion-mediated interactions in a dilute Bose-Einstein condensate. *Phys. Rev. A* 78, 013619. (2008)

(David Mitrouskas)
INSTITUTE OF SCIENCE AND TECHNOLOGY (IST) AUSTRIA
AM CAMPUS 1, 3400 KLOSTERNEUBURG, AUSTRIA
E-mail address: david.mitrouskas@ist.ac.at

(Peter Pickl)
FACHBEREICH MATHEMATIK, UNIVERSITÄT TÜBINGEN
AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY
E-mail address: p.pickl@uni-tuebingen.de