MULTIPLICITY–FREE SUBVARIETIES OF FLAG VARIETIES

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ABSTRACT. Consider a flag variety $Fl$ over an algebraically closed field, and a sub-
variety $V$ whose cycle class is a multiplicity–free sum of Schubert cycles. We show
that $V$ is arithmetically normal and Cohen–Macaulay, in the projective embedding
given by any ample invertible sheaf on $Fl$.

INTRODUCTION

In this note, we obtain geometric properties of a class of subvarieties of flag va-
rieties, that includes Schubert varieties. By a flag variety, we mean a projective
homogeneous space $Fl = G/P$ where $G$ is a semisimple algebraic group and $P$ is a
parabolic subgroup. Then $Fl$ admits a cellular decomposition into Bruhat cells with
closures the Schubert varieties. Any subvariety $V \subseteq Fl$ is rationally equivalent to a
linear combination of Schubert cycles with uniquely determined integer coefficients;
these are in fact non–negative. We say that $V$ is multiplicity–free if these coefficients
are 0 or 1; in loose words, the numerical invariants of $V$ are as small as possible.

Clearly, Schubert varieties are multiplicity–free. Further examples include: the di-
agonal in $Fl \times Fl$ (and, more generally, those subvarieties of $Fl \times Fl$ that are invariant
under the diagonal $G$–action), the irreducible hyperplane sections of $Fl$ in its small-
est projective embedding, and also the irreducible hyperplane sections of Schubert
varieties in Grassmannians, embedded by their Plücker embedding.

From these examples, we see that multiplicity–free subvarieties may be singular.
But their singularities are well–behaved, as shown by our main result.

**Theorem 1.** Any multiplicity–free subvariety $V$ of a flag variety $Fl$ is normal and
Cohen–Macaulay. Moreover, $V$ admits a flat degeneration in $Fl$ to a reduced Cohen–
Macaulay union of Schubert varieties.

As a consequence, for any globally generated invertible sheaf $\mathcal{L}$ on $Fl$, the restriction
map $H^0(Fl, \mathcal{L}) \rightarrow H^0(V, \mathcal{L})$ is surjective, and $H^n(V, \mathcal{L}) = 0$ for any $n \geq 1$. If, in
addition, $\mathcal{L}$ is ample, then $H^n(V, \mathcal{L}^{-1}) = 0$ for any $n \leq \dim(V) – 1$.

Thus, $V$ is arithmetically normal and Cohen–Macaulay in the projective embedding
given by any ample invertible sheaf on $Fl$.

The latter result is well–known in the case of Schubert varieties, and several proofs
are available, based on methods of Frobenius splitting [8] or standard monomial
theory [6]; the normality of Schubert varieties has also been obtained by Seshadri [10], using an argument of descending induction. This line of argument was taken up in [2] to obtain a version of Theorem 1 concerning orbit closures of spherical subgroups of $G$, in characteristic zero ([loc.cit., Theorem 6 and Corollary 8). The latter assumption plays an essential role there, since the proof relies on the existence of equivariant desingularizations and on the notion of rational singularities.

In the present note, we show how to adapt the arguments of [2] §4 to all multiplicity–free subvarieties, and to arbitrary characteristics. In Section 1, we formulate a criterion for normality and Cohen–Macaulayness of certain subvarieties of arbitrary $G$–varieties (Theorem 2). Section 2 presents some preliminary results on linearized sheaves that are needed for the proof of our criterion, given in Section 3. The final Section 4 presents a proof of Theorem 1, based on that criterion combined with known vanishing theorems for line bundles on unions of Schubert varieties.

This raises the question of characterizing the cycle classes of multiplicity–free subvarieties among all multiplicity–free sums of Schubert cycles. More generally, characterizing the cycle classes of (irreducible) subvarieties of flag varieties is a very natural problem.

For classes of curves, a complete solution follows from the (much more precise) results of [7]; in particular, each positive class of one–cycles arises from an irreducible curve. But this does not extend to higher dimensions: consider, for example, an irreducible surface $V \subset \mathbb{P}^n \times \mathbb{P}^n$. Decompose the cycle class of $V$ as $x[\mathbb{P}^2 \times pt] + y[\mathbb{P}^1 \times \mathbb{P}^1] + z[pt \times \mathbb{P}^2]$ where $x, y, z$ are non–negative integers. Then $y^2 - xz \geq 0$, as follows from the Hodge index theorem. In particular, the multiplicity–free class $[\mathbb{P}^2 \times pt] + [pt \times \mathbb{P}^2]$ does not arise from an irreducible surface (alternatively, one may observe that $(\mathbb{P}^2 \times pt) \cup (pt \times \mathbb{P}^2)$ is not Cohen–Macaulay, and apply Theorem 1.) But any other multiplicity–free class arises from an irreducible surface, since $[\mathbb{P}^2 \times pt] + [\mathbb{P}^1 \times \mathbb{P}^1]$ is the class of the blow–up of a point in $\mathbb{P}^2$, and $[\mathbb{P}^2 \times pt] + [\mathbb{P}^1 \times \mathbb{P}^1] + [pt \times \mathbb{P}^2]$ is the class of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$.

The problem of characterizing the cycle classes of nonsingular subvarieties of flag varieties is also quite natural and open; here very interesting recent results are those of [3], for Grassmannians over the field of complex numbers.

1. A criterion for normality and Cohen–Macaulayness of subvarieties of $G$–varieties

We begin by fixing notation and recalling some results on linear algebraic groups; a general reference is [11].

The ground field $k$ is algebraically closed, of arbitrary characteristic. A variety is a separated integral scheme of finite type over $k$; by a subvariety, we mean a closed subvariety. Given a linear algebraic group $\Gamma$, a $\Gamma$–variety is a variety $Z$ endowed with
a $\Gamma$–action

$$\sigma : \Gamma \times Z \to Z, (g, x) \mapsto g \cdot x.$$  

We will mostly consider $P$–varieties where $P$ is a parabolic subgroup of a semisimple algebraic group $G$, that is, $P$ contains a Borel subgroup $B$ of $G$. Then $P$ is connected, and the homogeneous space $G/P$ is projective; the quotient map $G \to G/P$ is a locally trivial principal $P$–bundle. The number of $B$–orbits in the flag variety $G/P$ is finite, and each of them (a Bruhat cell) is isomorphic to an affine space; the orbit closures are called Schubert varieties.

We fix a Borel subgroup $B \subseteq G$ and consider only parabolic subgroups $P \supseteq B$; these are called standard. The quotient map $P \to P/B$ is a locally trivial principal $B$–bundle, and $P/B$ is a flag variety for a maximal semisimple subgroup of $P$.

A standard parabolic subgroup $P$ is minimal, if $P \neq B$ and $P$ is minimal for this property; equivalently, $P/B$ is isomorphic to the projective line $\mathbb{P}^1$. Then $B$ acts on the cotangent space to $P/B$ at the base point $B/B$ by a character $\alpha$, the simple root associated with $P$; this sets up a bijective correspondence between minimal parabolic subgroups and simple roots. The group $G$ is generated by its minimal parabolic subgroups.

Let $Y$ be a $B$–variety and $P$ a standard parabolic subgroup; we review the construction of the induced variety $P \times^B Y$. The group $B$ acts on the product $P \times Y$ by $b \cdot (p, y) = (p b^{-1}, b y)$. Since $P \to P/B$ is a locally trivial principal $B$–bundle, then so is the quotient

$$q : P \times Y \to (P \times Y)/B = P \times^B Y,$$

and $P \times^B Y$ is a variety. The action of $P$ on itself by left multiplication makes $P \times^B Y$ into a $P$–variety; the projection

$$p_1 : P \times Y \to P$$

yields a $P$–equivariant map

$$f : P \times^B Y \to P/B,$$

a locally trivial morphism. Its fiber at the base point $B/B$ is $B \times^B Y \cong Y$. This defines a closed $B$–equivariant immersion

$$i : Y \to P \times^B Y.$$

Next we assume that $Y$ is a $B$–invariant subvariety of a $G$–variety $X$. Then the map $P \times Y \to X$, $(p, y) \mapsto p \cdot y$ factors through a $P$–equivariant morphism

$$\pi : P \times^B Y \to X$$

with image $P \cdot Y$. One checks that the product map

$$f \times \pi : P \times^B Y \to P/B \times X, (p, y)B \mapsto (pB, py)$$
is a closed immersion. Thus, $\pi$ is proper (so that $P \cdot Y$ is a $P$–subvariety of $X$; in particular, $G \cdot Y$ is a $G$–subvariety), and its fibers identify with closed subschemes of $P/B$.

The normalizer of $Y$ in $G$ is $N_G(Y) = \{ g \in G \mid g \cdot Y = Y \}$, a standard parabolic subgroup of $G$. For any standard parabolic subgroup $P \subseteq N_G(Y)$, the morphism $f \times \pi$ yields an isomorphism

$$P \times^B Y \cong P/B \times Y.$$ 

We say that a minimal parabolic subgroup $P$ raises $Y$ if it is not contained in $N_G(Y)$, that is, $Y$ is strictly contained in $P \cdot Y$. Then we have $\dim(P \cdot Y) = \dim(Y) + 1$, and the morphism $P \times^B Y \to P \cdot Y$ is generically finite.

If $Y$ is not $G$–invariant, then $Y$ is raised by some minimal parabolic subgroup. As a consequence, there exists a sequence $(P_1, \ldots, P_c)$ of such subgroups, so that $G \cdot Y = P_1 \cdots P_c \cdot Y$, where $c$ denotes the codimension of $Y$ in $G \cdot Y$. We say that the $B$–subvariety $Y$ is multiplicity–free, if the morphisms

$$P_1 \times^B P_{i+1} \cdots P_c \cdot Y \to P_1 \cdot P_{i+1} \cdots P_c \cdot Y$$

are birational for all such sequences $(P_1, \ldots, P_c)$. Equivalently, the compositions

$$P_1 \times^B P_2 \times^B \cdots \times^B P_c \times^B Y \to G \cdot Y, \ (p_1, p_2, \ldots, p_c, y)B^c \mapsto p_1p_2 \cdots p_c \cdot y$$

are birational.

If $G$ is simply–laced (i.e., its simple factors have type $A$, $D$ or $E$) and $k$ has characteristic zero, then it suffices to check this condition for one such sequence, by Proposition 5. But this does not extend to arbitrary $G$, as shown by the following geometric variant of Example 4.

**Example.** Let $q$ be a nondegenerate quadratic form on the vector space $k^5$ (where the characteristic of $k$ is not equal to 2). The corresponding special orthogonal group $G = \text{SO}(5)$ acts on the projective space $\mathbb{P}^4 = \mathbb{P}(k^5)$ and leaves invariant the nonsingular quadric $Q = \mathbb{P}(q = 0)$, of dimension 3. Choose a point $p_0 \in Q$ and a line $\ell_0$ in $\mathbb{P}^4$, contained in $Q$ and passing through $p_0$. Let $B$ be the isotropy group in $G$ of the pair $(p_0, \ell_0)$; this is a Borel subgroup of $G$. The nontrivial standard parabolic subgroups are the isotropy groups $P_1$, $P_2$ of $\ell_0$ resp. $p_0$; both are minimal.

Consider the diagonal action of $G$ on $X = Q \times Q$, and the subset $Y = \ell_0 \times (Q \cap p_0^1)$, where $p_0^1$ denotes the hyperplane in $\mathbb{P}^4$, orthogonal to $p_0$ with respect to $q$. Clearly, $Y$ is a $B$–invariant subvariety of codimension 3 in $X$. One checks that $P_1 \cdot P_2 \cdot P_1 \cdot Y = X = P_2 \cdot P_1 \cdot P_2 \cdot Y$ and that the morphism $P_1 \times^B P_2 \times^B P_1 \times^B Y \to X$ is birational, whereas the morphism $P_2 \times^B P_1 \times^B P_2 \times^B Y \to X$ has degree 2.

Returning to the general situation, note that the $B$–subvariety $Y$ is multiplicity–free if and only if
(1) The morphism \( \pi_Y : P \times^B Y \to P \cdot Y \) is birational for any minimal parabolic subgroup \( P \) raising \( Y \), and

(2) the \( B \)-subvariety \( P \cdot Y \) is multiplicity–free.

Note also that any Schubert variety \( S \) in a flag variety \( G/Q \) is multiplicity–free. Indeed,
\[ S \text{ decomposes as a product } P_1 \cdot \cdots \cdot P_d \cdot Q/Q, \text{ where } (P_1, \ldots, P_d) \text{ is a sequence} \]
of minimal parabolic subgroups, and \( d = \dim(S) \). Moreover, \( \pi_S \) is birational for any minimal parabolic subgroup raising \( S \). As a consequence, \( S \) is multiplicity–free, and the map \( P_1 \times^B \cdots \times^B P_d \times^B Q/Q \to S \) is birational, with source a nonsingular projective variety. This morphism is called a standard resolution of the Schubert variety \( S \).

The notion of a multiplicity–free \( B \)-subvariety generalizes that of a multiplicity–free subvariety of a flag variety, as shown by

**Lemma 1.** Let \( Fl = G/Q \) be a flag variety, \( V \subseteq Fl \) a subvariety, and put
\[ Y = \{ g \in G \mid g^{-1}Q/Q \in V \}. \]
Then \( Y \) is a subvariety of \( G \), invariant under the \( Q \)-action by left multiplication. Moreover, \( V \) is normal (resp. Cohen–Macaulay, multiplicity–free) if and only if the \( B \)-subvariety \( Y \) is.

**Proof.** By definition, \( Y \) is the preimage of \( V \) under the map \( G \to G/Q, \ g \mapsto g^{-1}Q \). Since that map is a locally trivial principal bundle for the connected group \( Q \), then \( Y \) is a subvariety, and it is normal (resp. Cohen–Macaulay) if and only if \( V \) is.

To show the equivalence between both notions of multiplicity–freeness, we note that \( V \) is multiplicity–free if and only if its pull–back to \( G/B \) under the natural map \( G/B \to G/Q \) is (since the pull-back of any Schubert variety in \( G/Q \) is a Schubert variety in \( G/B \)). Moreover, \( V \) and its pull–back to \( G/B \) define the same subvariety \( Y \subseteq G \). Thus, we may assume that \( Q = B \).

Let now \( P \) be a minimal parabolic subgroup and consider the natural map
\[ f : G/B \to G/P, \]
a locally trivial fibration with fiber \( \mathbb{P}^1 \). Then the subvariety of \( G \) associated with \( f(V) \subseteq G/P \) (or, equivalently, with \( f^{-1}f(V) \subseteq G/B \)) is \( P \cdot Y \); as a consequence, \( P \) raises \( Y \) if and only if the restriction
\[ f_V : V \to f(V) \]
is generically finite. Moreover, the fibers of \( f_V \) identify to those of \( \pi_Y : P \times^B Y \to P \cdot Y \); in particular, both morphisms have the same degree \( d_V \). Note that \( d_V \neq 0 \) if and only if \( \pi_Y \) (or, equivalently, \( f_V \)) is birational.

Let \( S \subseteq G/B \) be a Schubert variety of positive dimension. Write \( S = P_1 \cdots P_d/B \), where \( P_1, \ldots, P_d \) are minimal parabolic subgroups and \( d = \dim(S) \); let \( P = P_d \) and
Then \( S' = P_1 \cdots P_{d-1} / B \). Then \( S = f^{-1}f(S) \), and the restriction \( S' \rightarrow f(S') = f(S) \) is birational. We thus obtain the equalities of intersection numbers:

\[
\int_{G/B} [V] \cdot [S] = \int_{G/B} [V] \cdot f^*[f(S)] = \int_{G/P} f_*[V] \cdot [f(S)]
= d_V \int_{G/P} [f(V)] \cdot f_*[S'] = d_V \int_{G/B} [f^{-1}f(V)] \cdot [S'],
\]

as follows from the projection formula and from the equalities \( f_*[V] = d(V)[f(V)] \), \( f_*[S'] = [f(S)] \). Moreover, any Schubert variety of positive codimension in \( G/B \) arises as some \( S' \). Thus, \( V \) is multiplicity–free if and only if: \( d_V = 1 \) and \( f^{-1}f(V) \) is multiplicity–free, for any minimal parabolic subgroup \( P \) raising \( Y \). This completes the proof.

We may now state our criterion for normality and Cohen–Macaulayness. It may be regarded as a characteristic–free version of [2] Theorem 5 (see the remarks at the end of Section 3 for more detailed comments).

**Theorem 2.** Let \( X \) be a \( G \)–variety and \( Y \subseteq X \) a multiplicity–free \( B \)–subvariety containing no \( G \)–orbit.

If \( G \cdot Y \) is normal, then \( Y \) is normal as well.

If, in addition, \( G \cdot Y \) is Cohen–Macaulay, then \( Y \) is Cohen–Macaulay as well.

### 2. Induction of linearized sheaves

In this section, we study an analogue of the induction of varieties, concerning linearized sheaves; these may be defined as follows, after [3].

Let \( \Gamma \) be a linear algebraic group, \( Z \) a \( \Gamma \)–variety with action map \( \sigma : \Gamma \times Z \rightarrow Z \), and \( \mathcal{F} \) a quasi–coherent sheaf on \( Z \). A \( \Gamma \)–linearization of \( \mathcal{F} \) is an action of \( \Gamma \) on the symmetric algebra \( S(\mathcal{F}) \) by graded automorphisms, compatibly with its action on \( \mathcal{O}_Z \). Equivalently, \( \mathcal{F} \) is endowed with an isomorphism

\[
\varphi : \sigma^*\mathcal{F} \rightarrow p_2^*\mathcal{F}
\]

of sheaves on \( \Gamma \times Z \), where \( p_2 : \Gamma \times Z \rightarrow Z \) denotes the projection; and, in addition, the following cocycle condition holds:

\[
(\mu \times 1_Z)^*\varphi = p_{23}^*\varphi \circ (1_\Gamma \times \sigma)^*\varphi
\]

as sheaves on \( \Gamma \times \Gamma \times Z \), where \( \mu : \Gamma \times \Gamma \rightarrow \Gamma \) denotes the multiplication, and \( p_{23} : \Gamma \times \Gamma \times Z \rightarrow \Gamma \times Z \) the projection.

The \( \Gamma \)–linearized sheaves on \( Z \) are the objects of a category, with morphisms being those morphisms of sheaves that are compatible with the linearizations. Many operations on sheaves extend to \( \Gamma \)–linearized sheaves; for example, given two such sheaves \( \mathcal{F}, \mathcal{G} \) on \( Z \), the sheaves \( Ext^n_Z(\mathcal{F}, \mathcal{G}) \) are linearized as well. Moreover, for any
Γ–module $V$, the sheaf $\mathcal{F} \otimes_k V$ is Γ–linearized; this yields twists $\mathcal{F}(\chi)$ by characters $\chi : \Gamma \to \mathbb{G}_m$. If, in addition, $Z$ is Cohen–Macaulay, then its dualizing sheaf $\omega_Z$ is Γ–linearized as well.

We will use the following observation: Assume that $\Gamma$ decomposes as a product $\Gamma_1 \times \Gamma_2$ and that the $\Gamma_1$–action on $Z$ admits a $\Gamma_2$–equivariant quotient $q : Z \to Z/\Gamma_1$, which is a locally trivial principal $\Gamma_1$–bundle. Then, for any $\Gamma$–linearized sheaf $\mathcal{F}$ on $Z$, the $\Gamma_1$–invariant direct image $(q_* \mathcal{F})^{\Gamma_1}$ is a $\Gamma_2$–linearized sheaf on $Z/\Gamma_1$, and the canonical map $q^*(q_* \mathcal{F})^{\Gamma_1} \to \mathcal{F}$ is an isomorphism. On the other hand, for any $\Gamma_2$–equivariant sheaf $\mathcal{G}$ on $Z/\Gamma_1$, the sheaf $q^* \mathcal{G}$ is $\Gamma$–linearized, and the canonical map $\mathcal{G} \to q_*^*(q^* \mathcal{G})$ is an isomorphism. Thus, the categories of $\Gamma$-equivariant sheaves on $Z$ and of $\Gamma_1$-equivariant sheaves on $Z/\Gamma_1$ are equivalent.

We now return to a $B$–variety $Y$. Let $P$ be a standard parabolic subgroup. The group $P \times B$ acts on $P \times Y$ via $(p, b) \cdot (q, y) = (pqb^{-1}, by)$; the preceding observation applies to the quotient map $q : P \times Y \to P \times B Y$ by the $B$–action, and also to the projection $p_2 : P \times Y \to Y$, the quotient by the $P$–action. This yields an equivalence between the categories of $B$–linearized sheaves on $Y$, and of $P$–linearized sheaves on $P \times B Y$. Specifically, we obtain the following

**Lemma 2.**  
(1) For any $B$–linearized sheaf $\mathcal{F}$ on $Y$, the sheaf  
$$P \times B \mathcal{F} = q_*^B(p_*^2 \mathcal{F})$$

on $P \times B Y$ is $P$–linearized, and $i^*(P \times B \mathcal{F}) \cong \mathcal{F}$ as $B$–linearized sheaves (where $i : Y \to P \times B Y$ denotes the inclusion).

(2) Conversely, there is a natural isomorphism $\mathcal{G} \cong P \times B i^* \mathcal{G}$ for any $P$–linearized sheaf $\mathcal{G}$ on $P \times B Y$. This yields an equivalence between the categories of $B$–linearized sheaves on $Y$, and of $P$–linearized sheaves on $P \times B Y$.

(3) For any $B$–linearized sheaves $\mathcal{F}, \mathcal{G}$ on $Y$ and for any integer $n$, we have  
$$\text{Ext}^n_{P \times B Y}(P \times B \mathcal{F}, P \times B \mathcal{G}) \cong P \times B \text{Ext}^n_Y(\mathcal{F}, \mathcal{G}).$$

(4) $\mathcal{O}_{P \times B Y} = P \times B \mathcal{O}_Y$ as $P$–linearized sheaves. If, in addition, $Y$ is Cohen–Macaulay, then so is $P \times B Y$, and  
$$\omega_{P \times B Y} = P \times B \omega_Y(\chi)$$
as $P$–linearized sheaves, where $\chi$ is the character of the $B$–action on the fiber of $\omega_{P/B}$ at the base point $B/B$. In particular, if $P$ is a minimal parabolic subgroup with associated simple root $\alpha$, then  
$$\omega_{P \times B Y} = P \times B \omega_Y(\alpha).$$
Proof. (1) Since \( p_2 \) is \( B \)-equivariant and \( P \)-invariant, the sheaf \( p_2^* \mathcal{F} \) is \( P \times B \)-linearized. Thus, \( P \times B \mathcal{F} \) is \( P \)-linearized.

By construction, the \( B \)-linearized sheaf \( i^*(P \times B \mathcal{F}) \) is the \( B \)-invariant direct image of the sheaf \( p_2^* \mathcal{F} \) on \( B \times Y \), under the quotient map \( B \times Y \to B \times B Y \cong Y \). The latter identifies to the action map \( \sigma : B \times Y \to Y \), so that
\[
i^*(P \times B \mathcal{F}) \cong \sigma_B^*(p_2^* \mathcal{F}) \cong \sigma_B^*(\sigma^* \mathcal{F}) \cong \mathcal{F}.
\]

(2) Note that \( q^* \mathcal{G} \) is a \( P \times B \)-linearized sheaf on \( P \times Y \). Since \( p_2 \) is the quotient by the \( P \)-action, we have \( q^* \mathcal{G} = p_2^* \mathcal{F} \) for a unique \( B \)-linearized sheaf \( \mathcal{F} \) on \( Y \). It follows that \( \mathcal{G} \cong q_s^*(q^* \mathcal{G}) = P \times B \mathcal{F} \), so that \( i^* \mathcal{G} \cong \mathcal{F} \) by (1).

(3) Using the local triviality of \( p_2 \) and \( q \), we obtain isomorphisms
\[
q^* \text{Ext}^n_{P \times B Y}(P \times B \mathcal{F}, P \times B \mathcal{G}) \cong \text{Ext}^n_{P \times Y}(q^*(P \times B \mathcal{F}), q^*(P \times B \mathcal{G}))
\]
\[
\cong \text{Ext}^n_{P \times Y}(p_2^* \mathcal{F}, p_2^* \mathcal{G}) \cong p_2^* \text{Ext}^n_{P \times Y}(\mathcal{F}, \mathcal{G}).
\]
Now applying \( q_*^B \) yields our result.

(4) We have \( P \times B \mathcal{O}_Y = q_*^B \mathcal{O}_{P \times Y} \cong \mathcal{O}_{P \times B Y} \).

Recall that the map \( f : P \times B Y \to P/B \) is locally trivial, with fiber \( Y \) at the base point \( B/B \). Thus, if \( Y \) is Cohen–Macaulay, then so is \( P \times B Y \). Then the sheaf \( \omega_{P \times B Y} \) is \( P \)-linearized, and we obtain \( i^* \omega_{P \times B Y} \cong \omega_Y(\chi) \). By (1) and (2), it follows that \( \omega_{P \times B Y} \cong P \times B \omega_Y(\chi) \).

We will also need the following easy result, proved e.g. in [2] p. 294.

Lemma 3. Let \( Y \) be a \( B \)-variety and \( P \) a minimal parabolic subgroup. Then:

1. \( R^n \pi_* \mathcal{F} = 0 \) for any \( n \geq 2 \) and any coherent sheaf \( \mathcal{F} \) on \( P \times B Y \).
2. \( R^1 \pi_* \mathcal{O}_{P \times B Y} = 0 \).

As a final preparation for the proof of Theorem 2, we formulate a slightly stronger version of [2] Lemma 8, which is proved by the same argument.

Lemma 4. Let \( \mathcal{F} \) a coherent \( B \)-linearized sheaf on \( Y \) satisfying the following assumptions.

1. \( \text{Supp} \mathcal{F} \) is \( N_G(Y) \)-invariant.
2. \( \pi_*(P \times B \mathcal{F}) = 0 \) for any minimal parabolic subgroup \( P \) raising \( Y \).

Then \( \text{Supp} \mathcal{F} \) is \( G \)-invariant.

3. Proof of Theorem 2

We may assume that \( G \cdot Y = X \) is normal. We show that \( Y \) is normal by induction on its codimension in \( X \). Thus, we assume that \( Y \neq X \), and that \( P \cdot Y \) is normal for any minimal parabolic subgroup \( P \) raising \( Y \).

Let \( \nu : Z \to Y \) be the normalization, then we have a short exact sequence
\[
(1) \quad 0 \to \mathcal{O}_Y \to \nu_* \mathcal{O}_Z \to \mathcal{F} \to 0
\]
of sheaves on $Y$, where $\text{Supp} \mathcal{F}$ is the non-normal locus. We will show that $\mathcal{F} = 0$.

The normalizer $N_G(Y)$ acts on $Z$, and $\nu$ is equivariant. Therefore, (1) is an exact sequence of $N_G(Y)$–linearized sheaves. Let $P$ be a minimal parabolic subgroup, then we obtain an exact sequence

$$0 \to P \times^B \mathcal{O}_Y \to P \times^B \nu_* \mathcal{O}_Z \to P \times^B \mathcal{F} \to 0$$

of $P$–linearized coherent sheaves on $P \times^B Y$, whence an exact sequence

$$0 \to \pi_*(P \times^B \mathcal{O}_Y) \to \pi_*(P \times^B \nu_* \mathcal{O}_Z) \to \pi_*(P \times^B \mathcal{F}) \to R^1 \pi_*(P \times^B \mathcal{O}_Y)$$

of $P$–linearized sheaves on $P \cdot Y$. Now the map

$$P \times^B \nu : P \times^B Z \to P \times^B Y$$

is the normalization, and $P \times^B (\nu_* \mathcal{O}_Z) = (P \times^B \nu)_* \mathcal{O}_{P \times^B Z}$. Moreover, $P \times^B \mathcal{O}_Y \cong \mathcal{O}_{P \times^B Y}$ by Lemma 3 and $R^1 \pi_* \mathcal{O}_{P \times^B Y} = 0$ by Lemma 3. This yields an exact sequence (2)

$$0 \to \pi_*(P \times^B \nu)_* \mathcal{O}_{P \times^B Z} \to \pi_*(P \times^B \mathcal{F}) \to 0.$$ If $P$ raises $Y$, then $P \cdot Y$ is normal and both morphisms $\pi, \pi \circ (P \times^B \nu)$ are proper and birational. Thus, by Zariski’s main theorem, the maps $\mathcal{O}_{P \cdot Y} \to \pi_* \mathcal{O}_{P \times^B Y}$ and $\mathcal{O}_{P \cdot Y} \to \pi_*(P \times^B \nu)_* \mathcal{O}_{P \times^B Z}$ are isomorphisms. Thus, (2) implies $\pi_*(P \times^B \mathcal{F}) = 0$.

By Lemma 3, we conclude that $\mathcal{F} = 0$.

Next we assume, in addition, that $X$ is Cohen–Macaulay, and we show that $Y$ is Cohen–Macaulay as well. For this, we argue again by induction on $\text{codim}_X(Y) = c$: we assume that $P \cdot Y$ is Cohen–Macaulay for any minimal parabolic subgroup $P$ raising $Y$. We will show that $\text{Ext}_{P \cdot Y}^n(\mathcal{O}_Y, \omega_X) = 0$ for all $n \neq c$.

Let $P$ be a minimal parabolic subgroup raising $Y$. The canonical morphism

$$\varphi : P \times^B X \to X$$

identifies with the projection $p_2 : P/B \times X \to X$, with fibers $\mathbb{P}^1$. By duality for this proper morphism, we obtain an isomorphism

$$R\varphi_* \mathcal{RHom}_{P \times^B X}(\mathcal{O}_{P \times^B Y}, \omega_{P \times^B X}[1]) \cong \mathcal{RHom}_X(R\varphi_* \mathcal{O}_{P \times^B Y}, \omega_X).$$

Now $\varphi_* \mathcal{O}_{P \times^B Y} = \mathcal{O}_{P \cdot Y}$ by normality of $Y$, and $R^n\varphi_* \mathcal{O}_{P \times^B Y} = 0$ for all $n \geq 1$, by Lemma 3. Thus, (3) yields a spectral sequence

$$R^p\varphi_* \text{Ext}_{P \times^B X}^q(\mathcal{O}_{P \times^B Y}, \omega_{P \times^B X}) \Rightarrow \text{Ext}_{P \cdot Y}^{p+q-1}(\mathcal{O}_{P \cdot Y}, \omega_X).$$

But $R^p\varphi_* \mathcal{F} = 0$ for any $p \geq 2$ and any coherent sheaf $\mathcal{F}$, by Lemma 3. Thus, the spectral sequence (3) reduces to short exact sequences

$$0 \to \varphi_* \text{Ext}_{P \times^B X}^n(\mathcal{O}_{P \times^B Y}, \omega_{P \times^B X}) \to \text{Ext}_{P \cdot Y}^{n-1}(\mathcal{O}_{P \cdot Y}, \omega_X) \to R^1 \varphi_* \text{Ext}_{P \times^B X}^{n-1}(\mathcal{O}_{P \times^B Y}, \omega_{P \times^B X}) \to 0.$$
Moreover, $\text{Ext}_X^{n-1}(\mathcal{O}_{P \times Y}, \omega_X) = 0$ for all $n \neq c$, since both $X$ and $P \cdot Y$ are Cohen–Macaulay and $\text{codim}_X(P \cdot Y) = c - 1$. This implies

$$\varphi_* \text{Ext}_P^a(\mathcal{O}_{P \times Y}, \omega_{P \times Y}) = 0$$

for all $n \neq c$. Together with Lemma 2, it follows that

$$\varphi_*(P \times B \text{Ext}_X^a(\mathcal{O}_Y, \omega_X(\alpha))) = 0$$

for all $n \neq c$. But $\text{Ext}_X^a(\mathcal{O}_Y, \omega_X(\alpha)) \cong \text{Ext}_X^a(\mathcal{O}_Y, \omega_X)(\alpha)$ as a $B$–linearized sheaf on $X$, and this sheaf is killed by the ideal sheaf $\mathcal{I}_Y$. This yields for $n \neq c$:

$$\pi_*(P \times B \text{Ext}_X^a(\mathcal{O}_Y, \omega_X)) = 0,$$

Moreover, the sheaf $\text{Ext}_X^a(\mathcal{O}_Y, \omega_X)$ is $N_G(Y)$–linearized, so that

$$\text{Supp} \text{Ext}_X^a(\mathcal{O}_Y, \omega_X)(\alpha) = \text{Supp} \text{Ext}_X^a(\mathcal{O}_Y, \omega_X)$$

is $N_G(Y)$–invariant. By Lemma 3, it follows that $\text{Ext}_X^a(\mathcal{O}_Y, \omega_X) = 0$ for $n \neq c$.

**Remark 1.** Theorem 2 yields a direct proof for normality and Cohen–Macaulayness of Schubert varieties. It also applies to the large Schubert varieties of [1].

Specifically, let $G$ be adjoint, i.e., with trivial center. Then $G$ admits a canonical $G \times G$–equivariant completion $X$, a nonsingular projective variety containing only finitely many $B \times B$–orbits; moreover, the $G \times G$–orbit closures in $X$ are nonsingular. Let $Y \subseteq X$ be a $B \times B$–orbit closure, then $Y$ is multiplicity–free and contains no $G \times G$–orbit, as follows from [12] §2. Since $(G \times G) \cdot Y$ is nonsingular, Theorem 2 yields that $Y$ is normal and Cohen–Macaulay.

This was proved in [1] for the large Schubert varieties, the closures in $X$ of the $B \times B$–orbits in $G$, by a more complicated argument.

**Remark 2.** Recall that a morphism $\pi : \tilde{Z} \to Z$ of varieties is rational if it satisfies the following conditions:

1. $\tilde{Z}$ is normal and Cohen–Macaulay.
2. $\pi$ is proper and birational.
3. $\pi_* \mathcal{O}_{\tilde{Z}} = \mathcal{O}_Z$ and $R^n \pi_* \mathcal{O}_{\tilde{Z}} = R^n \pi_* \omega_{\tilde{Z}} = 0$ for all $n \geq 1$.

Then $Z$ is normal and Cohen–Macaulay, and $\pi_* \omega_{\tilde{Z}} = \omega_Z$, as follows from duality for the morphism $\pi$.

Now, with the notation of Theorem 3, the morphism $\pi_Y : P \times B Y \to P \cdot Y$ is rational, if $G \cdot Y$ is normal and Cohen–Macaulay. Indeed, normality of $Y$ implies $\pi_* \mathcal{O}_{P \times B Y} = \mathcal{O}_{P \cdot Y}$. On the other hand, $R^n \pi_* \mathcal{O}_{P \times B Y} = 0$ for all $n \geq 1$, and $R^n \pi_* \omega_{P \times B Y} = 0$ for all $n \geq 2$, by Lemma 3. In addition, since $Y$ and $X = G \cdot Y$ are Cohen–Macaulay, we have

$$\text{Ext}_P^c(\mathcal{O}_{P \times B Y}, \omega_{P \times B Y}) = \omega_{P \times B Y},$$
and $\text{Ext}^n_{\mathcal{O}_{\mathcal{P} \times \mathcal{B} \mathcal{X}}}(\mathcal{O}_{\mathcal{P} \times \mathcal{B} \mathcal{Y}}, \omega_{\mathcal{P} \times \mathcal{B} \mathcal{X}}) = 0$ for all $n \neq c$. Thus, the exact sequence (4) with $n = c + 1$ yields $R^1\pi_?\omega_{\mathcal{P} \times \mathcal{B} \mathcal{Y}} = 0$ (and this exact sequence with $n = c$ yields an isomorphism $\pi_*\omega_{\mathcal{P} \times \mathcal{B} \mathcal{Y}} \cong \omega_{\mathcal{P} \times \mathcal{B} \mathcal{Y}}$).

Moreover, given an equivariant morphism $\pi: \tilde{Z} \to Z$ of $B$–varieties, the induced morphism $P \times B \pi: P \times B \tilde{Z} \to P \times B Z$ is rational as well, for any standard parabolic subgroup $P$. As a consequence, the morphism $P_1 \times B P_2 \times B \cdots \times B P_d \times B Y \to P_1 P_2 \cdots P_d \cdot Y$ is rational, for any sequence $(P_1, P_2, \ldots, P_d)$ of minimal parabolic subgroups such that each $P_i$ raises $P_{i+1} \cdots P_d \cdot Y$.

When applied to Schubert varieties, this yields an alternative proof for rationality of their standard resolutions.

**Remark 3.** Assume that the characteristic of $k$ is zero. Recall that a variety $Z$ has *rational singularities* if it admits a rational desingularization; then every desingularization of $Z$ is rational. Now let $X$ be a $G$–variety and $Y \subseteq X$ a multiplicity–free $B$–subvariety containing no $G$–orbit; if $G \cdot Y$ has rational singularities, then the same holds for $Y$.

This follows indeed from the proof of [2] Theorem 5. The assumption that $Y$ contains no $G$–orbit was omitted in the statement of that Theorem, but in fact it is used in its proof, as the argument on p. 295, l. 22–25 that would allow to get rid of this assumption, is incorrect (I thank Prof. G. Zwara for pointing out this error to me). As a consequence, Theorem 5 in [loc.cit.] is only proved for regular varieties. This does not affect the subsequent results of [loc.cit.], since they only concern regular varieties.

### 4. Proof of Theorem [4]

By Theorem [2], $V$ is normal and Cohen–Macaulay. We now construct a flat family of subvarieties of $Fl$ with general fiber $V$ and special fiber a reduced Cohen–Macaulay union of Schubert varieties.

Write $Fl = G/P$. Regard $V$ as a point of the Chow variety $\text{Chow}(G/P)$ parametrizing all effective cycles in $G/P$ of appropriate dimension and degree. Recall that $\text{Chow}(G/P)$ is a projective $G$–variety; thus, by Borel’s fixed point theorem, the orbit closure $\overline{B \cdot V}$ contains $B$–fixed points. Choose such a fixed point $V_0$. Then there exist a nonsingular irreducible curve $C$, a closed point $c_0 \in C$, and a morphism $\varphi: C \to \overline{B \cdot V}$ such that: $\varphi(c_0) = V_0$, and $\varphi(C \setminus \{c_0\}) \subseteq B \cdot V$. Since the variety $B$ is rational, we may further assume that $C$ is rational as well ([4] Example 10.1.6). In
particular, \( V \) is rationally equivalent to the cycle \( V_0 \). Since the latter is \( B \)-invariant, it is a multiplicity-free sum of Schubert cycles.

Let \( \mathcal{V} \) be the closure in \( G/P \times C \) of the subset
\[
\{(x, c) \in G/P \times (C \setminus \{c_0\}) \mid x \in \varphi(c)\}.
\]
Let \( p : \mathcal{V} \to C \) be the restriction of the projection \( G/P \times C \to C \). Then \( \mathcal{V} \) is a variety, and the morphism \( p \) is projective and flat. Its fibers at all closed points of \( C \setminus \{c_0\} \) are isomorphic to \( V \), whereas the support of the fiber \( p^{-1}(c_0) \) is contained in the support of \( V_0 \). But by flatness of \( p \), the associated cycle of \( p^{-1}(c_0) \) is rationally equivalent to \( V \), and hence to \( V_0 \). Now the linear independence of Schubert cycles implies the equality of this associated cycle with \( V_0 \).

Next let \( \mathcal{Y} = \{(g, c) \in G \times C \mid (g^{-1}P/P, c) \in \mathcal{V}\} \). This is a subvariety of \( G \times C \); the latter is a \( G \)-variety (for the action by left multiplication on \( G \)), and \( \mathcal{Y} \) is \( B \)-invariant. Clearly, \( G \cdot \mathcal{Y} = G \times C \), and \( \mathcal{Y} \) contains no \( G \)-orbit. Moreover, \( \mathcal{Y} \) is multiplicity-free, as follows from Lemma 1. Thus, \( \mathcal{Y} \) is normal and Cohen–Macaulay by Theorem 2, so that \( \mathcal{V} \) is normal and Cohen–Macaulay as well, by Lemma \([4]\) again. It follows that the fiber \( p^{-1}(c_0) \) is equidimensional and Cohen–Macaulay. On the other hand, this fiber is generically reduced, by the preceding step. Thus, \( p^{-1}(c_0) \) is reduced, and we may identify it to \( V_0 \).

So \( p \) realizes a flat degeneration of \( V \) to \( V_0 \), a reduced Cohen–Macaulay union of Schubert varieties. Now the restriction map \( H^0(G/P, \mathcal{L}) \to H^0(V_0, \mathcal{L}) \) is surjective for any globally generated invertible sheaf \( \mathcal{L} \) on \( G/P \), and \( H^n(V_0, \mathcal{L}) = 0 \) for any \( n \geq 1 \), by \([3]\) (1.9, 1.13, 3.5, 3.7). By semicontinuity, the same holds for \( V \).

If, in addition, \( \mathcal{L} \) is ample, then \( H^n(V_0, \mathcal{L}^{-\nu}) = 0 \) for all \( n \leq \dim(V_0) - 1 = \dim(V) - 1 \) and \( \nu \gg 0 \), since \( V_0 \) is equidimensional and Cohen–Macaulay. But \( V_0 \) is Frobenius split in positive characteristics, so that \( H^n(V_0, \mathcal{L}^{-1}) = 0 \) for all \( n \leq \dim(V) - 1 \) by \([3]\) 1.12, 3.7. By semicontinuity again, the same holds for \( V \).

By \([3]\) Theorem 5, the flag variety \( G/P \) is arithmetically normal in the projective embedding given by any ample invertible sheaf \( \mathcal{L} \). Since \( V \) is normal and the restriction map \( H^0(G/P, \mathcal{L}) \to H^0(V, \mathcal{L}) \) is surjective, it follows that \( V \) is arithmetically normal as well. In addition, \( V \) is Cohen–Macaulay and \( H^n(V, \mathcal{L}^\nu) = 0 \) for \( 1 \leq n \leq \dim(V) - 1 \) and all integers \( \nu \), so that \( V \) is arithmetically Cohen–Macaulay.

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