Load-Flow Solvability under Security Constraints in DC Distribution Networks

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Abstract—We present sufficient conditions for the load-flow solvability under security constraints in DC distribution networks. In addition, we show that a load-flow solution that fulfills security constraints can be obtained via a convex optimization.

Index Terms—load flow, security constraints, nonsingularity, polynomial optimization, feasibility, DC distribution networks.

I. INTRODUCTION

Recently, DC distribution networks have received increasing attention. In these networks, many control procedures (i) rely on the load-flow solvability, and (ii) require that the load-flow solution (if exists) should satisfy the security constraints. However, due to non-linearity of the load-flow equations, this is hard to study. In this paper, we present sufficient conditions for the load-flow solvability under security constraints in DC distribution networks. Moreover, we show that a load-flow solution that fulfills security constraints can be obtained by solving a convex optimization.

II. PROBLEM FORMULATION

We consider a unipolar DC distribution network that has one slack bus, \(N\) PQ buses, and a generic topology (i.e., radial or meshed). The slack bus is indexed by 0, and the PQ buses are indexed by 1, ..., \(N\). For simplicity of expression, we define:

- \(N = \{0, ..., N\}\) as the set of all buses;
- \(N^{PQ} = N \setminus \{0\}\) as the set of PQ buses;
- \(K(n) \subseteq N\) as the set of neighbouring buses for bus \(n \in N\);
- \(I_{K(n)}(k)\) as the indicator function that has value 1 when \(k \in K(n)\) and 0 otherwise.

Now, let \(v_n, p_n\) be the real-valued nodal voltage and nodal power injection at bus \(n \in N\). In addition, let positive real constant \(g_{nk}\) be the line conductance between buses \(n \in N\) and \(k \in K(n)\). Then, the load-flow equations at PQ buses are formulated as follows:

\[
\sum_{k \in K(n)} g_{nk} v_n - \left(\sum_{k \in K(n)} g_{nk} v_k\right) v_n = p_n, \quad n \in N^{PQ}. \tag{1}
\]

During power system operation, the nodal voltages and branch currents should satisfy security constraints. In the paper, we express the security constraints as follows:

\[
v_{\min} < v_n < v_{\max}, \quad n \in N^{PQ}, \tag{2}
\]

\[
|g_{nk}(v_n - v_k)| < e_{nk}^{\max}, \quad n \in N^{PQ}, \quad k \in K(n), \tag{3}
\]

where \(v_{\min}, v_{\max},\) and \(e_{nk}^{\max}\) are some pre-defined positive real constants. Note that, for the development of our theoretical results, we need the security constraints to be strict inequalities.

With the above grid model and notations, we study the following problem: Given \(v_0\) and \((p_1, ..., p_N)\), is there a unique solution \((v_1, ..., v_N)\) that satisfies the load-flow equations (1) and the security constraints (2)–(3) ?

III. MAIN RESULTS

A. Existence

In Theorem 1 we provide sufficient conditions for the existence of a solution to (1)–(3).

**Theorem 1.** Assume that

1) \(2v_{\min} > v_{\max} > v_0 > v_{\min}\),
2) The following convex optimization P1 is feasible.

**P1** \[
\begin{aligned}
\max_{\alpha_n, \beta_n, v_n, \gamma_n} & \quad \sum_{n=1}^{N} v_n \\
\text{subject to:} & \quad \text{inequalities (3) - (4)}, \\
& \quad \left(\sum_{k \in K(n)} g_{nk}\right)\alpha_n - \sum_{k \in K(n) \setminus \{0\}} g_{nk}\beta_n = p_n + \left(g_{n0}v_0I_{K(n)}(0)\right)v_n, \quad n \in N^{PQ}, \\
& \quad v_n^2 \leq \alpha_n, \quad n \in N^{PQ}, \tag{4}
\end{aligned}
\]

\[
\begin{aligned}
& \quad \beta_n^2 \geq 0, \quad n \in N^{PQ}, \quad k \in K(n) \tag{5} \\
& \quad \beta_{nk}^2 \leq \alpha_n\alpha_k, \quad n \in N^{PQ}, \quad k \in K(n). \tag{6}
\end{aligned}
\]

Then, for any optimizer \(\alpha_n^*, \beta_n^*, v_n^*\) of P1, we have that \((v_1^*, ..., v_N^*)\) is a solution to (1)–(3).

In practice, the condition \(2v_{\min} > v_{\max} > v_0 > v_{\min}\) in Theorem 1 is normally satisfied. Consequently, for any \((p_1, ..., p_N)\) such that P1 is feasible, there exists at least one solution to (1)–(3).

It should be noticed that, Theorem 1 not only gives sufficient conditions on the existence of a solution to (1)–(3), but also shows that a solution can be obtained by solving the convex optimization P1.

**Remark 1.** Consider that the numerical solvers might not be able to handle strict inequalities, we can solve the following convex optimization and verify whether the optimal nodal voltages fulfill the strict inequality (2)–(3).

\[
\begin{aligned}
\max_{\alpha_n, \beta_n, v_n, \gamma_n} & \quad \sum_{n=1}^{N} v_n \\
\text{subject to:} & \quad (4) - (7), \\
& \quad v_n^\min \leq v_n \leq v_n^\max, \quad n \in N^{PQ}, \tag{8} \\
& \quad |g_{nk}(v_n - v_k)| \leq e_{nk}^{\max}, \quad n \in N^{PQ}, \quad k \in K(n). \tag{9}
\end{aligned}
\]
B. Uniqueness

In Theorem 2 we provide sufficient conditions for the uniqueness of the solution to (1)–(3).

**Theorem 2.** If the following polynomial optimization P2 is infeasible, then there exists at most one solution to (1)–(3).

\[
\text{[P2]} \min_{\gamma_n, v_n} \sum_{n=1}^{N} v_n \quad \text{subject to:} \quad (8) - (9),
\]

\[
2 \left( \sum_{k \in K(n)} g_{nk} \right) v_n - \sum_{k \in K(n)} g_{nk} v_k \geq 0, \quad n \in \mathcal{N}^{PQ},
\]

\[
\sum_{n=1}^{N} \gamma_n = 1.
\]

As can be seen, the infeasibility of polynomial optimization P2 is determined by only the security bounds \(v_{\min}, v_{\max}, v_{nk}\) and the slack-bus voltage \(v_0\).

By Theorem 2 under the infeasibility of P2, if power injection \((p_1, \ldots, p_N)\) has a solution to (1)–(3), then this solution must be unique.

**Remark 2.** Due to the non-linear equality constraints, P2 is not convex. This means that the numerical solvers might not be able to directly check the infeasibility of P2. However, we know that P2 is infeasible if any convex relaxation of P2 is infeasible. For efficiency and practicality, we suggest to take the sparsity-exploiting SDP relaxation in (1), which has already been successfully applied to the optimal power flow problem in (2)–(4).

IV. PROOFS

A. Proof of Theorem 2

**Proof.** Since \((v_1^*, \ldots, v_N^*)\) satisfies (2)–(3), we only need to prove that \((v_1^*, \ldots, v_N^*)\) satisfies (1).

First, we show \((v_n^*)^2 = \alpha_n^* \geq v_n\) for some \(n \in \mathcal{N}^{PQ}\). As a result, we have

\[
\left( \sum_{k \in K(n)} g_{nk} \right) (v_n^*)^2 - \left( g_{00} v_0 I_{K(\alpha)}(0) \right) v_n^* \geq \left( \sum_{k \in K(n)} g_{nk} \alpha_k^* - g_{00} v_0 I_{K(\alpha)}(0) \right) v_n^* \\
= \sum_{k \in K(n) \setminus \{0\}} g_{nk} \beta_{nk}^* + p_\alpha.
\]

Observe that the quadratic form \( \left( \sum_{k \in K(n)} g_{nk} \right) (v_n^*)^2 - \left( g_{00} v_0 I_{K(\alpha)}(0) \right) v_n^* \) is convex in \(v_n^*\). Moreover, this quadratic is monotonically increasing with respect to \(v_n^*\), as

\[
\frac{g_{00} v_0 I_{K(\alpha)}(0)}{2 \sum_{k \in K(n)} g_{nk}} \leq \frac{v_n^*}{2} < v_{\min} < v_n^*.
\]

Therefore, \(v_n^*\) can be further increased, which contradicts the optimality of \(v_n^*\). Hence, we must have \((v_n^*)^2 = \alpha_n^*, \forall n \in \mathcal{N}^{PQ}\).

Next, we show \((\beta_{nk}^*)^2 = \alpha_n^* \alpha_k^*, \forall n \in \mathcal{N}^{PQ}, k \in K(n)\). Since \(\beta_{nk} \geq 0, \forall n \in \mathcal{N}^{PQ}, k \in K(n)\), we could show \(\beta_{nk}^* =

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