Galilean Lee Model of the Delta Function Potential

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The scattering cross section associated with a two dimensional delta function has recently been the object of considerable study. It is shown here that this problem can be put into a field theoretical framework by the construction of an appropriate Galilean covariant theory. The Lee model with a standard Yukawa interaction is shown to provide such a realization. The usual results for delta function scattering are then obtained in the case that a stable particle exists in the scattering channel provided that a certain limit is taken in the relevant parameter space. In the more general case in which no such limit is taken finite corrections to the cross section are obtained which (unlike the pure delta function case) depend on the coupling constant of the model.

I. Introduction

The problem of scattering by a delta function in two dimensions is of considerable interest for a number of reasons, not the least of which is the fact that it lacks a dimensional parameter. This leads directly to the appearance of divergences in the calculation of bound state energies and scattering amplitudes, a fact which seriously complicates the task of physical interpretation. Although a delta function potential occurs in the relevant wave equation for the case of spin one-half Aharonov-Bohm scattering [1], it appears there in conjunction with $1/r^2$ terms with coefficients such that a cancellation of all divergences occurs. Since this requires a somewhat delicate limiting process (namely, the limit of vanishing flux tube radius must be taken at the end of the calculation), it is important to note that no such limiting process suffices to yield a finite result for the pure delta function potential. Such a goal can only be achieved by a) limiting consideration to the attractive delta function and b) requiring that there be a bound state associated with the scattering channel. The latter step is frequently justified by pointing out that the delta function is so singularly attractive that a bound state is a natural expectation. The crucial point is that a type of renormalization is carried out by which divergences are combined into a physical parameter (i.e., the bound state energy) in such a way that the relevant scattering amplitude can be written as a finite function of the scattering energy and the bound state energy. Just as the physical mass is not amenable to calculation in covariant field theory, so also in this case the bound state energy for the delta function potential cannot be calculated from first principles.

This renormalization program has been carried out for the two dimensional delta function and finite results obtained [2,3]. Bender and Mead [4] have gone one step further by considering the attractive delta function in $D$-dimensional space with the $D = 2$ result obtained as a limit. They assert that refs.[2] and [3] obtain the wrong cross section and infer from this that it is essential that the two dimensional result be obtained as a limit of the arbitrary $D$ case. However, Cavalcanti [5] has recently pointed out that the results of refs.[2-4] are identical provided only that a calculational error in ref.[3] is corrected. The two dimensional delta function thus seems to be reasonably well understood within the framework of conventional Schrödinger analysis.

Since Aharonov-Bohm scattering is well known to be the two particle sector of a Galilean invariant pure Chern-Simons gauge theory, it is natural to ask what the corresponding Galilean field theory [6] of the delta function should be. This paper examines that question and shows that a Yukawa coupling in such a theory provides a realization of the delta function potential in the limit in which a direct (or contact) interaction is obtained. In the following section the properties of Galilean field theories are briefly reviewed and the Galilean invariant trilinear interaction term constructed. The theory obtained from this process is essentially the Galilean version of the Lee model and has been discussed previously by Lévy-Leblond [7]. In section III the two particle scattering sector is considered and the corresponding Hilbert space constructed. This allows one to calculate the two particle scattering matrix and thereby obtain a formal expression for the cross section. In IV the various limits of the latter are considered and the renormalization carried out. The Conclusion summarizes some of the principal results obtained.

II A Galilean Model

One begins the construction of an appropriate Galilean covariant model by the determination of the relevant free particle Lagrangian. Using the fact that the invariant quantity in Galilean relativity is $E - p^2/(2M)$ where $M$ is the particle mass, it is straightforward to infer the free particle Lagrangian

$$\mathcal{L}_0 = \psi^\dagger i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M} - U_0 |\psi$$
where $U_0$ is generally referred to as the internal energy parameter. The fields $\psi$ and $\psi^\dagger$ satisfy the equal time commutation relation
\[
[\psi(x, t), \psi^\dagger(x', t)] = \delta(x - x')
\]
while $\psi$ has the additional property that it annihilates the vacuum state $|0\rangle$, i.e.,
\[
\psi(x, t)|0\rangle = 0.
\]
It may be noted that no reference is made here to the number of spatial dimensions although all calculations of scattering amplitudes will consider only the case of two spatial dimensions.

The construction of the interaction Lagrangian term requires that specification be made of the particles (or fields) participating in the interaction. Since a Yukawa (or trilinear) type of interaction is to be used here, it is convenient to employ the notation of the Lee model. The latter considers three particles $V$, $N$, and $\theta$ with the allowed interactions
\[
V \leftrightarrow N + \theta. \tag{1}
\]
It is to be noted that the Bargmann superselection rule on the mass requires that the mass parameters of the particles satisfy the relation
\[
M = M + m
\]
where $M$, $M$, and $m$ denote the masses of the $V$, $N$, and $\theta$ particles respectively. One additional remark has to do with the fact that although the original Lee model considered the $V$ and $N$ fields to be fermionic and the $\theta$ to be bosonic, such distinctions are not included here since they have no impact on the calculations to be presented.

To determine the most general interaction Lagrangian consistent with Eq.(1) and the requirement of Galilean invariance one makes use of the general transformation law
\[
U(g)^{-1}\psi(x, t)U(g) = \exp[iM\gamma(g; x, t)]\psi(x', t')
\]
where $U(g)$ is the unitary operator associated with the Galilean transformation
\[
\begin{align*}
x' &= R x + vt + a \\
t' &= t + b.
\end{align*}
\]
The parameters $R$, $a$, $b$, and $v$ refer respectively to rotations, spatial translations, time translations, and Galilean boosts with $\gamma(g; x, t)$ given by
\[
\gamma(g; x, t) = \frac{1}{2}v^2 t + v \cdot Rx.
\]
This leads to the desired interaction term which is found to be of the form [7]
\[
L_I(x, t) = g_0 \int d|y| f(|y|) V^\dagger(x, t)N(x + \frac{m}{M}y, t)\theta(x - \frac{M}{M}y, t) + h.c. \tag{2}
\]
where $g_0$ is a coupling constant and $f(|y|)$ is a form factor which can be arbitrarily prescribed without breaking Galilean invariance. Although the local limit in which $f(y)$ becomes a delta function will eventually be taken in order to make contact with the delta function potential, it is essential to retain for now the regularized version (2).

In order to facilitate the calculation of the scattering matrix it is desirable to transform all operators to momentum space. With the definition
\[
\psi(x, t) = \int \frac{dp}{2\pi} e^{ip \cdot x}\psi(p, t)
\]
the nonvanishing commutation relations among the $V$, $N$, and $\theta$ fields are of the form
\[
[\psi(p, t), \psi^\dagger(p', t)] = \delta(p - p').
\]
This leads to the result that the Hamiltonian is
\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \] (3)

where \[ \mathcal{H}_0 = \int dp \{ V(p) \left( \frac{p^2}{2M} + U_0 \right) V(p) + N^\dagger(p) \frac{p^2}{2M} N(p) + \theta^\dagger(p) \frac{p^2}{2m} \theta(p) \} \] (4)

and

\[ \mathcal{H}_I = -\frac{g_0}{2\pi} \int dp dq \{ f(\omega) V(p) N\left( \frac{M}{\lambda} p + q \right) \theta \left( \frac{m}{\lambda} p - q \right) + \text{h.c.} \} \] (5)

where \( \omega = |q| \). With the derivation of this result the scattering amplitude in the \( N\theta \) sector can be readily calculated.

III The Scattering Matrix

As a preliminary to the calculation of the \( N\theta \) scattering amplitude it should be observed that the Hamiltonian (3-5) implies that the vacuum as well as the single \( N \) and single \( \theta \) states are unmodified by the interaction. Specifically

\[ \mathcal{H}|0\rangle = 0 \]

while \( N^\dagger(p)|0\rangle \) and \( \theta^\dagger(p)|0\rangle \) are both eigenvectors of \( \mathcal{H} \) corresponding to eigenvalues \( p^2/2M \) and \( p^2/2m \) respectively.

The sector which contains an \( N\theta \) pair is nontrivial since it is linked by the interaction to the single \( V \) state. Since one is generally interested in the case in which the initial configuration of the particles is one of given momentum, it is appropriate to isolate one term in the expression for the \( N\theta \) state as consisting of \( N \) and \( \theta \) particles with momenta \( \frac{M}{\lambda} p + k \) and \( \frac{m}{\lambda} p - k \) respectively. Thus one writes

\[ |P, k^{(+)}\rangle = N^\dagger\left( \frac{M}{\lambda} p + k \right) \theta^\dagger\left( \frac{m}{\lambda} p - k \right)|0\rangle + \zeta V^\dagger(P)|0\rangle + \int dq g(q) N^\dagger\left( \frac{M}{\lambda} p + q \right) \theta^\dagger\left( \frac{m}{\lambda} p - q \right)|0\rangle \] (6)

with \( \zeta \) and \( g(q) \) to be determined from the eigenvalue equation

\[ \mathcal{H}|P, k^{(+)}\rangle = E|P, k^{(+)}\rangle. \] (7)

The notation employed here is intended to indicate that the left hand side of (6) is an outgoing state which in the remote past consisted of an \( N\theta \) pair at essentially infinite spatial separation.

Upon comparison of the coefficients of the various terms in (7) it is found that the energy \( E \) is given by

\[ E = \frac{p^2}{2M} + \frac{k^2}{2\mu} \]

where \( \mu \) is the reduced mass

\[ \mu = \frac{Mm}{\lambda^2}. \]

It also follows that \( \zeta \) and \( g(q) \) are related by the equations [9]

\[ (U_0 - \frac{k^2}{2\mu})\zeta = \frac{g_0}{2\pi} f(\omega_k) + \frac{g_0}{2\pi} \int dq g(q) f(\omega_q) \]

and

\[ g(q)(q^2 - k^2)**(1/2) = \frac{g_0}{2\pi} \zeta f^*(\omega_q). \]

These lead to the result

\[ g(q) = \frac{(\frac{g_0}{2\pi})^2 f^*(\omega_q) f(\omega_k)}{(\frac{q^2 - k^2}{2\mu})[U_0 - \frac{k^2}{2\mu} - \frac{g_0}{2\pi} \int \frac{dq}{2\pi} \frac{2\mu [f(\omega_q)]^2}{q^2 - k^2}].} \] (8)

It is to be noted that in writing (8) \( k^2 \) is always to be taken to mean \( k^2 + i\epsilon \) in accordance with the outgoing wave boundary condition.
The scattering amplitude (8) can be used to construct the scattering matrix upon noting that incoming states \( |P, k(-)\rangle \) are readily obtained by changing the sign of the \( ie \) term. Upon adopting a consistent normalization for two particle states this yields the result for the scattering matrix \( S_{fi} \) in that sector

\[
S_{fi} = \langle P', k(-) | P, k(+) \rangle
\]

\[
= (2\pi)^4 \delta(P - P') \left[ \delta(k - k') + 4\pi i \mu \delta(k^2 - k'^2) \right] \frac{f^*(\omega_{k'}) f(\omega_k)}{U_0 - \frac{k^2 + ie}{2\mu} - \frac{g_0^2}{2\pi^2} \int \frac{d\omega}{(2\pi)^2} \frac{2\mu |f(\omega)|^2}{\omega^2 - k^2 - ie} } .
\]

The result (9) leads directly to an expression for the cross section in the form

\[
\frac{d\sigma}{d\phi} = (2\pi)^3 (\frac{\mu^2}{k}) \left[ \frac{f^*(\omega_{k'}) f(\omega_k)}{U_0 - \frac{k^2 + ie}{2\mu} - \frac{g_0^2}{2\pi^2} \int \frac{d\omega}{(2\pi)^2} \frac{2\mu |f(\omega)|^2}{\omega^2 - k^2 - ie} } \right]^2 .
\]

Although one clearly is most interested in the local limit of (10) (i.e., the case in which \( f(\omega) \) is constant), the resulting divergence in the integral over \( q \) in that limit means that that limiting procedure must be handled with some caution. This is the renormalization process to which attention is now directed.

**IV Renormalization and the Delta Function Limit**

In order to deal with the issue of how to take the local limit \( f(\omega) = 1 \) it should be noted that the relevant quantity is

\[
G_V^{-1}(U) = U_0 - U - \frac{g_0^2}{2\pi^2} \int \frac{d\omega}{(2\pi)^2} \frac{2\mu |f(\omega)|^2}{\omega^2 - 2\mu U}
\]

where

\[
U \equiv E - \frac{P^2}{2\mathcal{M}}
\]

which also happens to be the inverse propagator for the \( V \) particle. The application of the usual renormalization procedure requires either that the inverse propagator vanish on the physical sheet in the case of a stable (i.e., bound state) particle or that its real part vanish for the case of a resonance. Since this can happen in the local limit only in the case \( U_0 > 0 \), it is henceforth assumed that this condition is satisfied [10]. One also observes from the form of \( G_V^{-1}(U) \) that the behavior of the integrand which appears in that quantity guarantees that the bound state condition will be satisfied for some \( U < 0 \). Taking this value of \( U \) to be \(-E_0\) there follows that

\[
U_0 = -E_0 + \frac{g_0^2}{2\pi^2} \int \frac{d\omega}{(2\pi)^2} \frac{2\mu |f(\omega)|^2}{\omega^2 - 2\mu E_0} .
\]

Using this result to eliminate \( U_0 \) from \( G_V^{-1}(U) \) one obtains the renormalized form

\[
G_V^{-1} = -(U + E_0) \left[ 1 + g_0^2 \int \frac{d\omega}{(2\pi)^2} \frac{2\mu}{(\omega^2 - 2\mu(U))} \right]
\]

where the local limit has been taken in the convergent integral over \( q \).

This result upon insertion in (10) yields the differential cross section as

\[
\frac{d\sigma}{d\phi} = \frac{2\pi}{k} \left[ -i\pi + \log \frac{k^2}{2\mu E_0} - \frac{k^2}{2\mu} + E_0 \right]^{-2}
\]

and the total cross section result

\[
\sigma = \frac{4\pi^2}{k(\frac{k^2}{\mu E_0} - \frac{k^2}{g_0^2/2\pi})} .
\]
This can also be expressed in terms of a renormalized coupling constant \( g \) with \((g/2\pi)^2\) defined as the residue of the scattering amplitude at the pole. Thus

\[
g^2 = \frac{g_0^2}{1 + \frac{\mu g_0^2}{2\pi E_0}}
\]

or, equivalently,

\[
g_0^2 = \frac{g^2}{1 - \frac{\mu g^2}{2\pi E_0}}.
\]

The result (11) coincides in the limit \( g_0 \to \infty \) with the results obtained previously for the delta function [2-5]. This is, of course, expected since the equation of motion for \( V(x,t) \) implied by the Hamiltonian (3-5) is of the form

\[
\left( i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M} - U_0 \right) V(x,t) = -g_0 \int dy f(|y|)N(x + \frac{m}{M}y,t)\theta(x - \frac{M}{M}y,t)
\]

and has the property that for \( U_0, g_0 \to \infty \) with \( \frac{g_0^2}{U_0} \) fixed, it becomes an equation of constraint. Thus Eq.(2) is in that limit

\[
\mathcal{L}_I = \lambda \left| \int dy f(|y|)N(x + \frac{m}{M}y,t)\theta(x - \frac{m}{M}y,t) \right|^2
\]

where \( \lambda > 0 \) is defined by

\[
\lambda = \frac{g_0^2}{U_0}.
\]

Using the approach of III it follows that Eq.(12) yields the usual momentum space equations for the delta function potential, thereby establishing that the above limit is indeed the field theory of the delta function potential.

V Conclusion

In this work a Galilean field theory of the two-dimensional delta function has been presented. A noteworthy extension of the delta function problem has also been achieved by using as the framework for this discussion a trilinear or Yukawa coupling. The explicit and exact expression for the S matrix has been derived in the two particle sector of the model and subsequently used to derive the cross section. It was found that while the cross section for this extended model has correction terms to the delta function potential, there is agreement in the limit in which a contact (or quadrilinear) coupling is obtained. The renormalization process has been seen to play an essential role in achieving the goal of taking the local (i.e., delta function) limit.

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[6] Of course one could also construct an appropriate theory in Minkowski space. Since, however, this would require at some point that a nonrelativistic limit be taken, it is clearly desirable to build in that limit at the outset (i.e., to construct a Galilean theory).
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[8] It should be noted that the bare (i.e., unrenormalized) internal energies of the $N$ and $\theta$ fields have been taken to be zero. That this can be done without loss of generality is a consequence of the fact that $\psi$ can always be replaced by $\psi e^{-iU_0 t}$, thereby eliminating the internal energy term for each of these fields. That this cannot also be done for the $V$ field is a consequence of the fact that the interaction term constrains the set of allowable gauge transformations.

[9] It should be noted that there is a sign error in the corresponding calculation of ref.[7].

[10] Alternatively stated, the local field theory with $U_0 < 0$ is a null theory.