Potential theory and forcing

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February 22, 2022

Abstract

We isolate a property of capacities which leads to construction of proper forcings, and prove that among others, the Newtonian capacity enjoys this property.

1 Introduction

We will be concerned with outer regular subadditive capacities on Polish spaces. These are functions $c : \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ satisfying the following demands:

1. $c(0) = 0$, $A \subset B \to c(A) \leq c(B)$, $c(A \cup B) \leq c(A) + c(B)$
2. $c(A) = \inf \{c(O) : O \subset X \text{ is open and } A \subset O\}$ for every set $A \subset X$
3. $c(\bigcup A) = \sup \{c(A_n) : n \in \omega\}$ whenever $A_n : n \in \omega$ is an inclusion-increasing sequence of subsets of $X$
4. $c(K) < \infty$ for all compact sets $K \subset X$

where $X$ is a Polish space. The typical representatives of this class are the Lebesgue measure or the Newtonian capacity, as well as other capacities used in the potential theory [1].

Let $I_c = \{ A \subset X : c(A) = 0 \}$. This is a $\sigma$-ideal. In the spirit of [11], [2] we will be interested in the forcing features of the factor forcing $P_{I_c}$ of Borel $I$-positive subsets of $X$ ordered by inclusion, in particular in the status of properness of the forcing [11], [9]. The following property of the capacity $c$ will be instrumental:

Definition 1.1. The capacity $c$ is stable if for every open set $A \subset X$ there is a Borel set $\tilde{A} \supset A$ of equal capacity such that for every $c$-positive set $B \subset X \setminus \tilde{A}$, it is the case that $c(A \cup B) > c(A)$.

*2000 AMS subject classification 03E40, 31C15.
†Partially supported by GA CR grant 201-03-0933 and NSF grant DMS 0300201.
It turns out that very many capacities share this property. The set \( \tilde{A} \) is always one that has been long known and studied—for the Newtonian capacity \(^{11}\), it is \( \tilde{A} = A \cup \{ x \in \mathbb{R}^3 : \text{the potential of } A \text{ is } \geq 1 \text{ at } x \} \); for the Steprâns capacities \(^{10}\), it is the set \( \tilde{A} = A \cup \{ x \in X : \text{the set } A \text{ has upper density } 1 \text{ at } x \} \). We include a list of examples in Section 5.

**Theorem 1.2.** Suppose that \( c \) is an outer regular subadditive stable capacity on some Polish space \( X \). Then

1. the forcing \( P_Ic \) is proper
2. \((ZF+AD+)\) the capacity \( c \) is continuous in increasing wellordered unions, and every set has a Borel subset of the same capacity.

The finer forcing properties of the posets \( P_Ic \) are shrouded in mystery except for a couple of general observations:

- the forcings \( P_Ic \) are capacitable and therefore bounding \(^2\) 7.13
- the ideal \( I_c \) is generated by \( G_\delta \) sets and therefore the forcing \( P_Ic \) makes the ground model reals meager \(^2\) 2.17
- The poset \( P_Ic \), where \( c \) is the Newtonian capacity, is nowhere c.c.c. \(^4\) Theorem 4.6
- If \( c \) is strongly subadditive then the forcing \( P_Ic \) preserves Lebesgue outer measure \(^12\).

An interesting feature of the proofs is that they can be combined in the following sense. If \( \{ c_m : m \in n \} \) is a finite collection of countably subadditive submeasures on some Polish space \( X \) let \( b \) be their join, the submeasure defined by \( b(A) = \inf \{ \sum m c_m(B_m) : A = \bigcup m B_m \} \). This is the largest submeasure smaller than all of the submeasures \( \{ c_m : m \in n \} \). We do not know if a join of a collection of capacities must be a capacity. However, we do know that the \( \sigma \)-ideal \( I_b \) is generated by the union of the ideals \( I_{c_m} : m \in n \) and

**Theorem 1.3.** Suppose that \( \{ c_m : m \in n \} \) is a finite collection of outer regular strongly subadditive stable capacities, and \( b \) is their join. Then

1. the forcing \( P_Ib \) is proper
2. \((ZF+AD+)\) the submeasure \( b \) is continuous in increasing wellordered unions of uncountable cofinality, in particular the ideal \( I_b \) is closed under wellordered unions. Every set has a Borel subset of the same submeasure.

Note the additional assumption of strong subadditivity on the capacities concerned. Not all stable capacities are strongly subadditive—the Steprâns capacities are not, while the Newtonian capacity is. Another possible combination is the following.
Theorem 1.4. Suppose that \( \{c_m : m \in n\} \) is a finite collection of outer regular strongly subadditive stable capacities on a compact Polish space \( X \) with a metric \( d \), and let \( s > 0 \) be a real number. Let \( I \) be the \( \sigma \)-ideal generated by the ideals \( I_m : m \in n \) and the sets of finite \( s \)-dimensional Hausdorff measure. Then

1. the forcing \( P_I \) is proper
2. (ZF+AD+) the ideal \( I \) is closed under wellordered unions, and every \( I \)-positive set has a Borel \( I \)-positive subset.

The corresponding result for just the ideal of \( \sigma \)-finite Hausdorff measure sets was proved in [11]. Further variations are possible, but not effortless—for example we can further adjoin the ideal of sets of \( \sigma \)-finite \( s \)-dimensional packing measure, but we do not know how to adjoin ideals of sets of \( \sigma \)-finite measure for two different Hausdorff measures and preserve properness and closure under wellordered unions. We have not studied the properties of the resulting forcings. We do not know if they are bounding. We do not know if there are ideals \( I, J \) such that the forcings \( P_I \) and \( P_K \) are proper while the forcing \( P_K \) is not, where \( K \) is the ideal generated by \( I \cup J \).

Perhaps the most important open problem regards the relationship between stability and other properties of capacities.

Question 1.5. Is every outer regular subadditive capacity stable? Is every outer regular strongly subadditive capacity stable? Is the factor forcing \( P_I \) proper for every outer regular subadditive capacity? Is every outer regular capacity continuous in increasing wellordered unions under AD+?

Our notation follows the set theoretic standard of [6]. AD denotes the use of the Axiom of Determinacy, AD+ is a technical strengthening of AD due to W. Hugh Woodin.

Special thanks go to Murali Rao of University of Florida, who showed that the capacities encountered in potential theory are stable. Without this result I would have never considered writing the present paper.

2 Stable capacities

Once and for all fix a Polish space \( X \) with a countable basis \( B \) for its topology, closed under finite unions. Let \( c \) be an outer regular subadditive stable capacity with the \( A \mapsto \tilde{A} \) operation as indicated in Definition 1.1. Let \( P \) be a forcing adding a point \( \dot{x} \in \dot{X} \).

Consider an infinite game \( G \) between Players I and II. In the beginning, Player I indicates an initial condition \( p_{ini} \in P \) and then produces a sequence \( D_k : k \in \omega \) of open dense subsets of the forcing \( P \) as well as a \( c \)-null set \( A \). Player II produces a sequence \( p_{ini} \geq p_0 \geq p_1 \geq \ldots \) such that \( p_k \in D_k \) and \( p_k \) decides the membership of the point \( x \) in the \( k \)-th basic open subset of the space \( X \) in some fixed enumeration. Player II wins if, writing \( g \subset P \) for the filter his conditions generate, the point \( \dot{x}/g \) falls out of the set \( A \).
In order to complete the description of the game, we have to describe the exact schedule for both players. At round $k \in \omega$, Player I indicates the open dense set $D_k \subset P_k$, and sets $A(k, l) \in B$ for $l \in k$ so that $c(A(k, l)) \leq 2^{-l}$ and $l \in k_0 \in k_1$ implies $A(k_0, l) \subset A(k_1, l)$ and $c(A(k_1, l)) - c(A(k_0, l)) \leq 2^{-k_0}$. In the end, let $A(l) = \bigcup_k A(k, l)$ and recover the set $A \subset X$ as $A = \bigcap_l A(l)$. The continuity of the capacity in increasing unions shows that $c(A(l)) \leq 2^{-l}$ and so $c(A) = 0$. Note that apart from the open dense sets, Player I has only countably many moves at his disposal. Still, he can produce a superset of any given $c$-null set as his final set $A \subset X$. Player II is allowed to tread water, that is, to wait for an arbitrary finite number of rounds (place trivial moves) before placing the next condition $p_k$ on his sequence.

**Lemma 2.1.** Player II has a winning strategy in the game $G$ if and only if $P \models \dot{x}$ falls out of all ground model coded $c$-null sets.

**Proof.** The key point is that the payoff set of the game $G$ is Borel in the (large) tree of all legal plays, and therefore the game is determined by $\mathbb{C}$. A careful computation will show that the winning condition for Player I is in fact a union of an $F_\sigma$ and a $G_\delta$ set.

For the left-to-right direction, if there is some condition $p \in P$ and a $c$-null Borel set $B \subset X$ such that $p \models \dot{x} \in B$, then Player I can win by indicating $p_{ini} = p$, producing some null set $A \supset B$, and on the side producing an increasing sequence $M_i : i \in \omega$ of countable elementary submodels of some large structure and playing in such a way that the sets $D_k : k \in \omega$ enumerate all open dense subsets of the poset $P$ in the model $M = \bigcup_i M_i$, and $\{p_k : k \in \omega\} \subset M$. In the end, this must bring success: this way, Player II’s filter $g \subset P$ is $M$-generic containing the condition $p$, by the forcing theorem applied in the model $M$, $M[g] \models \dot{x}/g \in A$, and by Borel absoluteness $\dot{x}/g \in A$ as desired.

The right-to-left direction is harder. Suppose that $P \models \dot{x}$ falls out of all ground model coded $c$-null sets, and $\sigma$ is a strategy for Player I. By the determinacy of the game $G$, it will be enough to find a counterplay against the strategy $\sigma$ winning for Player II. The following claim will be used repeatedly.

**Claim 2.2.** Suppose that $p \in P$ is a condition. There is a real number $\epsilon(p) > 0$ such that for every set Borel $B \subset X$ there is a condition $q \leq p$ forcing the point $\dot{x}$ out of $B$.

**Proof.** If this failed for some condition $p \in P$, then for each number $i \in \omega$ there would be a Borel set of capacity $\leq 2^{-i}$ such that $p \models \dot{x} \in B_i$. But then, the set $B = \bigcap_i B_i$ is a Borel set of zero capacity and $p \models \dot{x} \in B$. This is a contradiction. $\square$

Let $p_{ini} \in P$ be the initial condition indicated by the strategy $\sigma$, and let $l \in \omega$ be a number such that $2^{-l} < \epsilon(p_{ini})$. We will construct a counterplay such that in the end, the point $\dot{x}/g$ falls out of the set $A(l)$. Consider the tree $T$ of all partial plays $\tau$ of the game $G$ respecting the strategy $\sigma$ such that they end at some round $k$ with Player II placing a condition $p \in P$ as his last move such that
Claim 2.3. \( T \) \( \in \) the required play in the tree \( X \) subset of \( r \) such that \( \bigcap \) \( i > \) the continuity of the capacity in increasing unions there would have to be a \( r \) satisfied after Player II places the move by Player I, deciding whether the point \( \dot{x} \) \( \in \) the appropriate basic open subset of \( X \) or not. This will be the next nontrivial move of Player II past \( \tau \) in the required play in the tree \( T \) extending \( \tau \), we just have to decide at which round to place that move in order to make the condition (*) hold.

Assume for contradiction that for no round \( k > \bar{k} \) the condition (*) will be satisfied after Player II places the move \( r \) at the round \( k \). Then for every number \( k > \bar{k} \) there is a Borel set \( B(k) \supset A(k, l) \) such that \( c(B(k)) \leq c(A(k, l)) + 2^{-k} \) such that \( r \models \dot{x} \in \bar{B}(k) \).

Claim 2.3. \( c(\bigcap_k B(k) \cup B) = c(B) \).

Proof. Note \( B = \bigcup_k A(k, l) \) is an increasing union. If the claim failed, by the continuity of the capacity in increasing unions there would have to be a number \( i > k \) such that \( c(\bigcap_k B(k) \cup A(i, l)) > c(B) + 2^{-i} \). However, \( B_i \supset \bigcap_k B(k) \cup A(i, l) \) and \( c(B(i)) \leq c(A(i, l)) + 2^{-i} \leq c(B) + 2^{-i} \), contradiction. \( \square \)

By the properties of the tilde operation, it must be the case that \( c(\bigcap_k B(k) \setminus \bar{B}) = 0 \). At the same time, \( r \models \dot{x} \in \bigcap_k B_k \setminus \bar{B} \). This contradicts the assumption that \( P \models \dot{x} \) falls out of all ground model coded \( c \)-null sets! \( \square \)

Corollary 2.4. The forcing \( P_{\text{gen}} \) is proper.

Proof. Note that the forcing \( P_{\text{gen}} \) forces the generic point \( \dot{x}_{\text{gen}} \in \hat{X} \) to fall out of all ground model coded \( c \)-null sets. Let \( \sigma \) be the corresponding winning strategy for Player II in the game \( G \). Let \( M \) be a countable elementary submodel of a large structure containing the strategy \( \sigma \), and let \( B \in P_{\text{gen}} \cap M \) be an arbitrary condition. We must prove \( \square \) that the set \( \{ x \in B : x \text{ is } M\text{-generic} \} \) is \( L_c \)-positive. Suppose that \( A \in L_c \) is a \( c \)-null set, and simulate a play of the game \( G \) in which Player I indicates \( B = p_{\text{ini}} \), enumerates all open dense subsets of \( P_{\text{gen}} \) in the model \( M \) and produces the set \( A \) or some of its \( c \)-null supersets, and Player II follows his strategy \( \sigma \). By elementarity, all the moves in this play are in the model \( M \), therefore the filter \( g \subset M \cap P_{\text{gen}} \) Player II created is \( M \)-generic. \( \square \)
Lemma 2.6. \( x \) can tread water for an arbitrary number of steps before playing a precise schedule of the two players is similar to the game \( G \). Thus the set of all generic points in the set \( B \) is \( I_c \)-positive as desired.

In fact, a second look will show that the collection of generic points of the set \( B \) has the same capacity as the set \( B \) itself. \( \square \)

**Corollary 2.5.** In the choiceless Solovay model, the ideal \( I_c \) is closed under wellordered unions.

**Proof.** Let \( \kappa \) be an inaccessible cardinal and let \( G \subset \text{Coll}(\omega, < \kappa) \) be a generic filter. The Solovay model \( N \) is then defined as \( V[\mathbb{R}^{|G|}] \). See [6] for basic properties of this model.

Suppose that \( \langle A_\alpha : \alpha \in \lambda \rangle \) is a wellordered collection of \( c \)-null sets in the model \( N \) and \( B \) is its union. We must prove that \( c(B) = 0 \). By a standard homogeneity argument we may assume that the collection is definable from ground model parameters in the model \( N \). Suppose for contradiction that \( c(B) > 0 \). Then in \( N \) there must be a point \( x \in B \) which falls out of all ground model coded \( c \)-null Borel sets, and in \( V \) there must be a forcing \( P \) of size \( < \kappa \) and a \( P \)-name \( \dot{x} \) such that \( V \models P \models \dot{x} \) falls out of all ground model coded \( c \)-null sets and \( \text{Coll}(\omega, < \kappa) \models \dot{x} \in \dot{B} \). There must be a condition \( p \in P \) and an ordinal \( \alpha \in \lambda \) such that \( p \models \text{Coll}(\omega, < \kappa) \models \dot{x} \in A_\alpha \). In the model \( N \), look at the set \( C = \{ x \in X : \exists g \in P \ P \models g \wedge x = \dot{x}/g \wedge g \ is \ V \text{-generic} \} \). By the forcing theorem and a standard homogeneity argument it must be the case that \( C \subset A_\alpha \). The proof will be complete once we show \( c(C) > 0 \).

This is more or less the same as the previous proof. Use Lemma 2.1 in the ground model to find a winning strategy \( \sigma \) for Player II in the game \( G \) associated with the name \( \dot{x} \). Apply a wellfoundedness argument to see that this strategy is still winning in the model \( N \). Now given a \( c \)-null set \( D \subset X \) in the model \( N \), find a play of the game \( G \) in which Player I indicates the initial condition \( p = p_{ini} \), enumerates all the dense subsets of the forcing \( P \) in the ground model, and produces some \( c \)-null superset of the set \( D \). The resulting point \( x = \dot{x}/g \) falls into the set \( C \setminus D \), showing that the set \( C \) cannot be \( c \)-null. \( \square \)

A little bit of extra work will show that actually the capacity \( c \) is continuous in increasing wellordered unions in the choiceless Solovay model.

There is a related integer game. Suppose that \( c \) is an outer regular stable capacity, \( B \subset X \) is a set and \( \varepsilon > 0 \) is a real number. The infinite game \( H(B, \varepsilon) \) is played between Players I and II. Player I creates an open set \( A \) such that \( c(A) \leq \varepsilon \) and Player II creates a point \( x \in X \). Player II wins if \( x \in B \setminus A \). The precise schedule of the two players is similar to the game \( G \). At round \( k \in \omega \), Player I plays a basic open set \( A(k) \subset X \) such that \( c(A(k)) \leq \varepsilon \) and \( k_0 \in k_1 \) implies \( A(k_0) \subset A(k_1) \) and \( c(A(k_1)) - c(A(k_0)) \leq 2^{-k_0} \). In the end, the set \( A \) is recovered as \( \bigcup_k A(k) \). For Player II, fix some Borel bijection \( \pi : 2^\omega \to X \). Player II can tread water for an arbitrary number of steps before playing a nontrivial move, a bit (0 or 1). Let \( y \in 2^\omega \) be the sequence of bits he got in the end; the point \( x \in X \) is recovered as \( x = \pi(y) \).

**Lemma 2.6.** \( c(B) < \varepsilon \) implies that Player I has a winning strategy in the game \( H(B, \varepsilon) \) which in turn implies that \( c(B) \leq \varepsilon \).
Proof. The first implication is trivial: if \( c(B) < \epsilon \) then Player I can win by producing any open subset of capacity \( < \epsilon \) covering the set \( B \), disregarding Player II’s moves completely.

The second implication is harder. Fix a winning strategy \( \sigma \) for Player I in the game \( H(B, \epsilon) \). We must produce a set of capacity \( \leq \epsilon \) covering the set \( B \).

For every partial play \( \tau \) of the game respecting the strategy \( \sigma \) let \( A(\tau) \) be the resulting set \( A \) in the infinite extension of the play \( \tau \) in which Player I follows the strategy \( \sigma \) and Player II makes no nontrivial moves past \( \tau \). Also, for a number \( j \in \omega \) and a bit \( b \in 2 \) let \( \tau jb \) be the finite play extending \( \tau \), respecting the strategy \( \sigma \), in which Player II placed only one nontrivial move past \( \tau \), and it was the bit \( b \) at round \( j \), and it was also the last move of the play \( \tau jb \). Now fix a finite play \( \tau \) respecting the strategy \( \sigma \) and a bit \( b \in 2 \). An argument identical to that of Claim 2.3 will show that

\[
c(\bigcap_j \bar{A}(\tau jb) \cup A(\tau)) = c(A(\tau))
\]

and the definitory property of the set \( \bar{A}(\tau) \) implies that

\[
c(\bigcap_j \bar{A}(\tau jb) \setminus \bar{A}(\tau)) = 0.
\]

Let \( C_{\tau b} = \bigcap_j \bar{A}(\tau jb) \setminus \bar{A}(\tau) \). We claim that the set \( B \) is covered by the set \( \bar{A}(0) \cup \bigcup_{\tau, b} C_{\tau b} \), which has capacity \( \leq \epsilon \).

And indeed, suppose for contradiction that \( x \in B \) is some point such that \( x \notin \bar{A}(0) \cup \bigcup_{\tau, b} C_{\tau b} \), and choose some binary sequence \( y \in 2^\omega \) so that \( x = \pi(y) \).

Consider the tree \( T \) of all partial plays \( \tau \) following the strategy \( \sigma \) such that \( x \notin \bar{A}(\bar{A}(\tau)) \) and in the course of the play \( \tau \) Player II generated an initial segment of the sequence \( y \in 2^\omega \). An argument just like in Lemma 2.1 reveals that \( 0 \in \tau \) and every play in the tree \( T \) can be extended to a longer play still in the tree \( T \) in which Player II made one more nontrivial move. Any infinite branch through the tree \( T \) constitutes a counterplay against the strategy \( \sigma \) in which Player II won, a contradiction.

Corollary 2.7. (ZF+AD) Every set has a Borel subset of the same capacity.

Proof. It will be enough to produce an analytic set of the same capacity, since then an obvious application of Choquet’s capacitability theorem gives an \( F_\sigma \) subset of the same capacity. And it will be really enough to produce an analytic subset of arbitrarily close smaller capacity.

So let \( B \subseteq X \) be a set and let \( 0 < \epsilon < c(B) \) be a real number. By the previous lemma and the determinacy assumption, Player II has a winning strategy \( \sigma \) in the game \( H(B, \epsilon) \). Let \( A \) be the set of all possible points \( x \in X \) which result from a play of the game in which Player II follows the strategy \( \sigma \). Note that

- \( A \subseteq B \) since the strategy \( \sigma \) was winning for Player II.
• $A$ is analytic by its definition

• $c(A) \geq \epsilon$ since the strategy $\sigma$ obviously remains winning for Player II in the game $G(A, \epsilon)$.

The Corollary follows.

**Corollary 2.8.** (ZF+$AD$+) The capacity $c$ is continuous in increasing wellordered unions.

**Proof.** Work with AD+. The corollary is proved by induction on length of the wellordered union in question. The stages of countable cofinality are handled by the definitory properties of a capacity, the successor stages, the stages corresponding to singular ordinals and ordinals $\geq \Theta$ are all nearly trivial. We are left with the case of a regular uncountable cardinal $\kappa \in \theta$. By a theorem of Steel [3], there is a set $Y \subset 2^\omega$ and a prewellordering $\prec$ on it such that every analytic subset of $Y$ meets only $< \kappa$ many classes.

Suppose $(A_\alpha : \alpha \in \kappa)$ is an increasing sequence of subsets of the space $X$ with union $A \subset X$. We must produce an ordinal $\alpha \in \kappa$ such that $c(A) = c(A_\alpha)$. Consider the capacity $c^*$ on the space $X \times 2^\omega$ given by $c^*(B) = c(\text{projection of the set } B \text{ into the } X \text{ coordinate})$. It is easy to verify that this is an outer regular subadditive stable capacity. Consider the set $B \subset X \times 2^\omega$ given by $\langle x, y \rangle \in B$ iff $y \in Y$ and $x \in A_{|y|}$. It is clear that $c^*(B) = c(A)$.

Now the previous corollary applied to the capacity $c^*$ gives a Borel set $C \subset B$ of the same $c^*$ capacity. The projection of the set $C$ into the $2^\omega$ coordinate is an analytic subset of the set $Y$, and so it meets only $< \kappa$ many classes of the prewellorder $\prec$, bounded by some ordinal $\alpha \in \kappa$. The projection of the set $C$ into the $X$ coordinate is then a subset of the set $A_\alpha$ and by the definition of the capacity $c^*$ it has capacity equal to that of the set $A$. Thus $c(A_\alpha) = c(A)$ as desired. \qed

## 3 Joins of capacities

Let $\{c_m : m \in n\}$ be a finite collection of submeasures on some Polish space $X$, and let $b$ be their join, $b(A) = \inf\{\Sigma_{m \in n} c_m(A_m) : A \subset \bigcup_{m \in n} A_m\}$, with the associate collection $I_b = \{A \subset X : b(A) = 0\}$.

**Claim 3.1.** 1. $b$ is a submeasure

2. $I_b$ is the $\sigma$-ideal generated by the collection $\bigcup_{m \in n} I_{c_m}$.

**Proof.** For (1), let $A = \bigcup_{k \in \omega} A_k$ be a countable union of sets; we must show that $b(A) \leq \Sigma_k b(A_k)$. Let $\epsilon > 0$ be a real number and argue that $\Sigma_k b(A_k) + \epsilon \geq b(A)$. For every number $k$ find sets $B^m_k : m \in n$ such that $A_k \subset \bigcup_{m \in n} B^m_k$ and $\Sigma_{m \in n} c_m(B^m_k) < b(A_k) + \epsilon \cdot 2^{-k-1}$. Consider the sets $C^m = \bigcup_{k \in \omega} A^m_k$ for $m \in n$. It is clear that $A = \bigcup_{m \in n} C^m$ and by the countable subadditivity of the submeasures $c_m$ it is the case that $\Sigma_m c_m(C^m) \leq \Sigma_m c_m(A^m_k) \leq \Sigma_k b(A_k) + \epsilon$ and therefore $b(A) \leq \Sigma_k b(A_k) + \epsilon$ as desired.
For (2), it is clear that $\bigcup_m I_{c_m} \subset I_d$. On the other hand, if $A \subset X$ is a set such that $b(A) = 0$, for every number $k \in \omega$ and $m \in n$ choose sets $A^m_k$ so that $\bigcup_m A^m_k = A$ and $\sum_m c_m(A^m_k) \leq 2^{-k}$. By a counting argument, for every point $x \in X$ there must be a number $m \in n$ such that the point $x$ belongs to infinitely many of the sets $A^m_k : k \in \omega$. In other words, $A = \bigcup_m B^m$ where $B^m = \{x \in X : \exists k \in \omega x \in A^m_k\} = \bigcap_{k \in \omega} \bigcup_{k \geq 1} A^m_k$. It is clear from the last expression and the subadditivity of the submeasure $c_m$ that $c_m(B^m) = 0$. Thus we expressed the set $A$ as a union of sets of respective zero submeasures as desired.

We will now prove Theorem 1.3. Note the extra assumption of strong subadditivity for the capacities. We do not know if it is necessary, however our economical proofs do use it in one small, absolutely critical point. For the record let us state

**Definition 3.2.** A capacity $c$ is strongly subadditive if $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$ for all sets $A, B$.

**Claim 3.3.** Suppose that $c$ is a strongly subadditive capacity and $B, A_n, B_n : n \in \omega$ are sets such that $A_n \subset B_n \cap B$ and $c(B_n) - c(A_n) \leq \epsilon_n$ for all $n$ and some real numbers $\epsilon_n$. Then $c \left( \bigcup_n B_n \cap B \right) - c(B) \leq \Sigma_n \epsilon_n$.

**Proof.** First note that for every number $n \in \omega$, $c(B_n \cap B) - c(B) \leq \epsilon_n$. Namely, by the strong subadditivity $c(B_n \cap B) \leq c(B_n) + c(B)$ and therefore $c(B_n \cap B) \leq c(B_n) + c(B) - c(B_n \cap B) \leq c(B_n) + c(B) - c(A_n) \leq c(B) + \epsilon_n$.

Now argue that $c(B_0 \cup B_1 \cup B) - c(B) \leq \epsilon_0 + \epsilon_1$, the rest follows by the continuity of the capacity under increasing wellordered unions. But this is just like the situation in the previous paragraph: $c((B_0 \cup B) \cup (B_1 \cup B)) + c((B_0 \cap B_1) \cup B) \leq c(B_0 \cup B) + c(B_1 \cup B)$ and $c(B_0 \cup B_1 \cup B) \leq c(B_0 \cup B) + c(B_1 \cup B) - c((B_0 \cap B_1) \cup B) \leq c(B) + \epsilon_0 + c(B) + \epsilon_1 - c(B) = c(B) + \epsilon_0 + \epsilon_1$. □

Suppose that $\{c_m : m \in n\}$ are outer regular strongly subadditive stable capacities on a Polish space $X$ and let $b$ be their join, with the associated tilde operation. We will abuse the notation to use the same tilde to denote this operation for any of the capacities. Which capacity is concerned will be always clear from the index of the set: $\tilde{B}(m)$ denotes the $c_m$ tilde of the set $B(m)$.

Suppose that $P$ is a forcing adding a point $\hat{x} \in \hat{X}$. Consider the infinite game $G$ between Players I and II. In the beginning, Player I indicates an initial condition $p_{1i} \in P$ and then produces a sequence $D_k : k \in \omega$ of open dense subsets of the forcing $P$ as well as a $b$-null set $A$. Player II produces a sequence $p_{imi} \geq p_0 \geq p_1 \geq \ldots$ such that $p_k \in D_k$ and the condition $p_k$ decides the membership of the point $\hat{x}$ in the $k$-th basic open subset of the space $X$ in some fixed enumeration, generating some filter $g \subset P$. Player II wins if the realization $\hat{x}/g$ falls out of the set $A$.

In order to complete the description of the game, we have to describe the exact schedule for both players. At round $k \in \omega$, Player I indicates the open dense set $D_k \subset P_{l_k}$ and sets $A(k, l, m) \in B$ for $l \in k$ and $m \in n$ so that
Every node of the tree can be extended to a longer one.

Suppose that

Claim 3.5.

\[ G \]

strategy \( \sigma \) such that for every collection Player II has a winning strategy in the game Lemma 3.4.

Proof. As in the proof of Lemma 2.1 the game

If this failed for some condition \( p \) is a condition \( q \) is a condition \( p \) is a condition. There is a real number \( \epsilon(p) > 0 \) such that for every collection \( \{B_m : m \in n\} \) of Borel sets with \( c_m(B_m) \leq \epsilon \) there is a condition \( q \leq p \) such that \( q \models \not\exists x \in \bigcup_m B_m \).

Proof. If this failed for some condition \( p \in P \), then for every number \( k \in \omega \) there would be a collection \( \{B_m^k : m \in n\} \) of Borel sets such that \( c_m(B_m^k) \leq 2^{-k} \) and \( x \models \not\exists x \in \bigcup_{m \in n} B_m^k \). For every number \( m \in n \), by the subadditivity of the capacity \( c_m \) it is the case that the set \( C_m = \bigcap_k \bigcup_{j > k} B_m^j \) has \( c_m \)-capacity zero, and by the choice of the sets \( B_m^k \) it is the case that \( p \models \not\exists x \in \bigcup_{m \in n} C_m \). However, the latter set has \( b \)-null measure zero, contradiction.

Let \( p_m \in P \) be the initial condition indicated by the strategy \( \sigma \), and let \( l \in \omega \) be a number such that \( 2^{-l} < \epsilon(p_m) \). We will construct a counterplay such that in the end, the point \( x/g \) falls out of all the sets \( A(l, m) : m \in n \).

Consider the tree \( T \) of all partial plays \( \tau \) of the game \( G \) respecting the strategy \( \sigma \) such that they end at some round \( k \) with Player II placing a condition \( p \in P \) as his last move such that

\[(**)\]

for every collection \( \{B_m : m \in n\} \) of Borel sets such that for every number \( m \in n \), \( B_m \supseteq A(k, l, m) \), \( c_m(B_m) - c(A(k, l, m)) \leq 2^{-k} \) and \( c_m(B_m) \leq 2^{-l} \), there is a condition \( q \leq p \) such that \( q \models \not\exists x \in \bigcup_m B_m \).

Note that every infinite play whose initial segments form an infinite branch through the tree \( T \) Player II won in the end, because for no number \( k \in \omega \) and \( m \in n \) and no condition \( p \in g \) it could be the case that \( p \models \exists x \in A(k, l, m) \) by the condition \((**)\) and therefore \( x/g \notin A(l, m) \). Now the play \( \{p_m\} \) is in the tree \( T \) by the choice of the number \( l \in \omega \), and so it will be enough to show that every node of the tree can be extended to a longer one.
Suppose \( \tau \in T \) is a finite play of length \( \bar{k} \), ending with a nontrivial move \( p \in P \) of Player II and some basic open sets \( A(k, l, m) : m \in n \) indicated by the strategy \( \sigma \), satisfying the property \((**)\). Consider the infinite play extending \( \tau \) in which Player I follows the strategy \( \sigma \) and Player II places only trivial moves past \( \tau \). Let \( B(m) \subset X \) be the open set produced as \( A(l, m) \) in that play, and consider the set \( \hat{B}(m) \). Clearly, \( c_m(\hat{B}(m)) = c_m(B(m)) \leq c_m(A(\bar{k}, l, m)) + 2^{-k} \), and by the property \((**)\) there is a condition \( q \leq p \) forcing \( \dot{x} \notin \bigcup_m B(m) \). Let \( r \leq q \) be a condition in the appropriate open dense set indicated by Player I, deciding whether the point \( \dot{x} \) belongs to the appropriate basic open subset of \( X \) or not. This will be the next nontrivial move of Player II past \( \tau \) in the required play in the tree \( T \) extending \( \tau \), we just have to decide at which round to place that move in order to make the condition \((**)\) hold.

Assume for contradiction that for no round \( k > \bar{k} \) the condition \((**)\) will be satisfied after Player II places the move \( r \) at the round \( j \). Then for every number \( k > \bar{k} \) and \( m \in n \) there are Borel sets \( B(k, m) \supset A(k, l, m) \) such that \( c_m(B(m)) \leq c_m(A(k, l, m)) + 2^{-k} \leq 2^{-\bar{k}} \), and such that \( r \models \dot{x} \in \bigcup_{m \in n} B(k, m) \).

**Claim 3.6.** \( c_m(\bigcap_{k \geq \bar{k}} B(i, m) \cup B(m)) = c_m(B(m)) \).

**Proof.** This is the only point in the proof where the strong subadditivity is used. For every number \( k > \bar{k} \), it is the case that \( c_m(\bigcup_{i > k} B(i, m) \cup B(m)) \leq c_m(B(m)) + \Sigma_{i > k} 2^{-i} = c_m(B(m)) + 2^{-k} \) by Claim 3.5, so the intersection of these sets must have capacity equal to \( c_m(B(m)) \). \( \square \)

By the properties of the tilde operation, for every number \( m \in n \) it must be the case that \( c_m(\bigcap_{k \geq \bar{k}} B(i, m) \setminus B(m)) = 0 \). Since \( r \models \dot{x} \notin \bigcup_{m \in n} B(m) \), it must be that \( r \models \dot{x} \notin \bigcup_{m \in n} \bigcap_{i > k} B(i, m) \), and we can find a condition \( s \leq r \) and numbers \( k_m : m \in n \) such that \( s \models \dot{x} \notin \bigcup_{m \in n} \bigcup_{i > k_m} B(i, m) \). Choose a natural number \( k \) larger than all the numbers \( k_m : m \in n \). Then \( s \models \dot{x} \notin \bigcup_{m \in n} B(k, m) \), contradicting the choice of the sets \( B(k, m) \)!

**Corollary 3.7.** The forcing \( P_I \) is proper.

**Proof.** Same as in Corollary 2.4. \( \square \)

**Corollary 3.8.** Every analytic set has a Borel subset of the same \( b \)-submeasure.

Note that in the previous section this followed immediately from Choquet’s theorem, but here some work is necessary.

**Proof.** We will use the following general fact.

**Claim 3.9.** If \( I \) is a \( \sigma \)-ideal generated by Borel sets and \( R_I \) is the partial order of analytic \( I \)-positive sets ordered by inclusion then in the \( R_I \)-extension there is a generic point \( \dot{x}_{\text{gen}} \) such that for every analytic set \( A \) in the ground model, \( P \models A \in \dot{G} \iff \dot{x}_{\text{gen}} \in A \), where \( \dot{G} \) is a name for the generic filter.
This should be compared to the basic Lemma 2.1.1 of [11]. Of course the partial order $R_{l_1}$ in the end turns out to have $P_{l_1}$ as a dense subset, but we will know that only after we prove the current Corollary, and in the proof it is necessary to consider the poset $R_{l_1}$ itself.

To prove the claim, first define the name $\hat{x}_{gen} \in \dot{X}$ as the unique point belonging to all basic open sets $B \subset X$ such that $B \in \dot{G}$. It is not difficult to show that this is well defined. Suppose that $A \in R_{l_1}$ is an analytic set and argue that $A \Vdash \hat{x}_{gen} \in \dot{A}$. Just let $f : \omega^\omega \to A$ be a continuous surjection and in the generic extension let $T \subset \omega^{<\omega}$ be defined by $t \in T \iff f'' \{ y \in \omega^\omega : t \subset y \} \in G$. The $\sigma$-additivity of the ideal $I$ will show that this tree has no terminal nodes, and if $y \in \omega^\omega$ is any infinite branch then $f(y) \in A$ must be the generic point $\hat{x}_{gen}$. On the other hand, if some condition $B \in R_{l_1}$ forces $\hat{x}_{gen} \in \dot{A}$ then it must be the case that $A \cap B \notin I$ and it is a common lower bound of the conditions $B$ and $A$. This happens because if $A \cap B$ were an $I$-small set then it would have an $I$-small Borel superset $C$, and the condition $B \setminus C$ would force $\hat{x}_{gen} \in \dot{B} \setminus \dot{C}$ and $\hat{x}_{gen} \notin \dot{A}$, contradiction.

To prove the Corollary, suppose that $A \subset X$ is a $b$-positive analytic set, let $M$ be a countable elementary submodel of a large enough structure, and let $B = \{ x \in A : x$ is $M$-generic point for the forcing $R_{l_1} \}$. An argument similar to Lemma 2.4 and Corollary 2.4 will show that $b(B) = b(A)$. Moreover, the set $B \subset A$ is Borel since it is in one-to-one Borel correspondence with the Borel collection of $M$-generic filters on the poset $M \cap R_{l_1}$. The Corollary follows.

Corollary 3.10. In the choiceless Solovay model, the ideal $I_c$ is closed under wellordered unions.

The proof is the same as in the case of Corollary 2.6.

There is again an associated integer game. Suppose that $B \subset X$ is a set and $\epsilon > 0$ is a real number. The infinite game $H(B, \epsilon)$ is played between Players I and II. Player I creates an open set $A$ such that $b(A) \leq \epsilon$ and Player II creates a point $x \in X$. Player II wins if $x \in B \setminus A$. The precise schedule of the two players is again similar to the game $G$, with a small change. In the beginning, Player I indicates positive rational numbers $q_m : m \in n$. Later, at round $k \in \omega$, Player I plays open basic sets $A(k, m) \subset X$ for $m \in n$ such that $c_m(A(k, m)) \leq q_m$ and $k_0 \in k_1$ implies $A(k_0, m) \subset A(k_1, m)$ and $c_m(A(k_1, m)) - c(A(k_0, m)) \leq 2^{-k_0}$. In the end, the sets $A(m) \subset X$ are recovered as $A(m) = \bigcup_k A(k, m)$ and the set $A$ as $A = \bigcup_{m \in \omega} A(m)$. For Player II, fix some Borel bijection $\pi : 2^\omega \to X$. Player II can tread water for an arbitrary number of steps before playing a nontrivial move, a bit (0 or 1). Let $y \in 2^\omega$ be the sequence of bits he got in the end; the point $x \in X$ is recovered as $x = \pi(y)$.

Lemma 3.11. $b(B) < \epsilon$ if and only if Player I has a winning strategy in the game $H(B, \epsilon)$.

Proof. This is very similar to Lemma 2.0. The left-to-right implication is trivial—if $c(B) < \epsilon$ then Player I has a winning strategy which ignores Player II’s moves entirely. For the converse suppose $\sigma$ is a winning strategy for Player I. We will
produce sets $B_m : m \in \mathbb{N}$ such that $B \subset \bigcup_m B_m$ and $\Sigma_m c_m(B_m) < \varepsilon$. In fact, if $q_m : m \in \mathbb{N}$ are the rational numbers indicated by the strategy $\sigma$ at its first move, the sets $B_m : m \in \mathbb{N}$ will satisfy $c_m(B_m) \leq q_m$.

Use the notation parallel to that in Lemma 2.6. For a finite play $\tau$ observing the strategy $\sigma$ and a number $m \in \omega$ let $A_m(\tau) \subset X$ be the set resulting as $A_m$ in the infinite extension of the play $\tau$ in which Player I follows his strategy $\sigma$ and Player II makes no nontrivial moves past the play $\tau$. Let also $\tau_{ja}$ be the finite extension of the play $\tau$ in which Player I follows the strategy $\sigma$ and Player II makes only trivial moves except for the last $j$-th round when he places the bit $a$. As in Claim 3.5, for every number $m$, every bit $a$, and every finite play $\tau$ we have

$$c_m(\bigcap_k \bigcup_{j > k} \tilde{A}(m)(\tau_{ja}) \cup A(m)(\tau)) = c_m(A(m)(\tau))$$

and by the definitory property of the tilde operation

$$c_m(\bigcap_k \bigcup_{j > k} \tilde{A}(m)(\tau_{ja}) \setminus \tilde{A}(m)(\tau)) = 0$$

Let $C_{\tau am} = \bigcap_k \bigcup_{j > k} (\tilde{A}(m)(\tau_{ja}) \setminus \tilde{A}(m)(\tau))$: this is a set of $c_m$ capacity zero and therefore the set $B_m = A(m)(0) \cup \bigcup_{\tau, a} C_{\tau am}$ has $c_m$-capacity $\leq q_m$. We claim that $B \subset \bigcup_m B_m$, which will complete the proof of the Lemma.

Suppose for contradiction this fails and there is a point $x = \pi(y) \in B \setminus \bigcup_m B_m$. We will produce a counterplay against the strategy $\sigma$ in which Player II produces this point $x$ and wins. Consider the tree $T$ of all partial plays $\tau$ of the game in which Player I follows his strategy $\sigma$, Player II produced an initial segment of the binary sequence $y \in 2^\omega$, and $x \notin \bigcup_m A(m)(\tau)$. As in the proof of Lemma 3.5, $0 \in T$ and every node in the tree $T$ can be extended into a node with one more nontrivial move by Player II. Any infinite branch of the tree $T$ containing infinitely many nontrivial moves by Player II forms the desired counterplay winning for Player II.

**Corollary 3.12.** (ZF+DC+AD) Every set has a Borel subset of the same $b$ submeasure.

**Proof.** Let $B \subset X$ be a set, $b(B) = \varepsilon$. First, produce an analytic subset $A \subset B$ of the same submeasure. By the previous lemma and a determinacy argument, Player II has a winning strategy $\sigma$ in the game $H(B, \varepsilon)$. Let $A$ be the set of all points $x \in X$ which can result from some counterplay against the strategy $\sigma$. The set $A$ is analytic by its definition, it is a subset of the set $B$ since the strategy $\sigma$ is winning for Player II, and it has $b$-submeasure $\geq \varepsilon$ since the strategy $\sigma$ remains winning for Player II in the game $H(A, \varepsilon)$.

The second step is to produce a Borel subset of the set $A$ of the same submeasure. There are two ways to argue here. Either note that Corollary 3.8 is proved in ZF+DC. Or use the following corollary and the fact that every analytic set is a union of $\aleph_1$ many Borel sets. □
Corollary 3.13. (ZF+AD+) The submeasure $b$ is continuous in increasing wellordered unions of uncountable cofinality.

Proof. This is parallel to the proof of the uncountable regular length case of Corollary 2.8. We omit the proof. □

4 Adjoining a Hausdorff measure

Suppose that $X$ is a compact metric space with metric $d$ and \{${c_m : m \in \mathbb{N}}$\} is a finite collection of outer regular strongly subadditive stable capacities on it, and let $s > 0$ be a real number. Consider the $\sigma$-ideal $I$ generated by the sets of zero capacity for one of the capacities in the collection, and the sets of finite $s$-dimensional Hausdorff measure. There are several notational issues. As in the previous section, each capacity $c_m$ comes with a tilde operation associated to it as in Definition 3.1. We are going to use the same tilde character for all of them, and exactly which one is to be applied will be immediately clear from the context. Regarding the Hausdorff measure, it is important to note that every set of a given diameter can be covered by a basic open set of an arbitrarily close larger diameter. If $E$ is a collection of basic open sets, its weight, $w(E)$, is the number $\Sigma_{O \in E} \text{diam}^s(O)$.

Suppose that $P$ is a forcing adding a point $\hat{x} \in \hat{X}$. The infinite game $G$ between Players I and II is defined in the following fashion. In the beginning, Player I indicates an initial condition $p_{\text{ini}} \in P$ and then produces a sequence $D_k : k \in \omega$ of open dense subsets of the forcing $P$ as well as a set $A$ in the ideal $I$. Player II produces a sequence $p_{\text{ini}} \geq p_0 \geq p_1 \geq \ldots$ such that $p_k \in D_k$ and $p_k$ decides the membership of the point $\hat{x} \in X$ in the $k$-th basic open set in some fixed enumeration. Player II wins if, writing $g \subset P$ for the filter generated by his conditions, the realization $\hat{x}/g$ falls out of the set $A$.

In order to complete the description of the game, we have to describe the exact schedule for both players. At round $k \in \omega$, Player I indicates the open dense set $D_k \subset P_{\omega_k}$, sets $A(k, l, m) \in \mathcal{B}$ for $l \in k$ and $m \in n$ and finite sets $E(j, k, l)$ for $j, l \in k$ so that:

- $c_m(A(k, l, m)) \leq 2^{-l}$ and $l \in k_0 \in k_1$ implies $A(k_0, l, m) \subset A(k_1, l, m)$ and $c_m(A(k_1, l, m)) - c(A(k_0, l, m)) \leq 2^{-k_0}.

- $E(j, k, l)$ is a finite collection of basic open sets of diameter $\leq 2^{-j}$ and weight $\leq l + 1$ and if $j, l \in k_0 \in k_3$ are numbers then the collection $E(j, k_1, l) \setminus E(j, k_0, l)$ consists only of sets of diameter $\leq 2^{-k_0}$.

In the end, let $A(l, m) = \bigcup_k A(k, l, m)$ and $A(m) = \bigcap_l A(l, m)$. Just as in the proof of Claim 3.1, $c_m(A(m)) = 0$. Let also $E(j, l) = \bigcup_k E(j, k, l)$ and $E(l) = \bigcup_j E(j, l)$. Clearly, the set $E(l)$ has $s$-dimensional measure \leq l, this for every number $l \in \omega$. The set $A \subset X$ is recovered as $A = \bigcup_{m \in n} A(m) \cup \bigcup_{l \in \omega} E(l)$.

Note that apart from the open dense sets, Player I has only countably many moves at his disposal. Still, he can produce a superset of any given $I$-small set.
as his final set $A \subset X$. Player II is allowed to tread water, that is, to wait for an arbitrary finite number of rounds (place trivial moves) before placing the next condition $p_k$ on his sequence.

**Lemma 4.1.** Player II has a winning strategy in the game $G$ if and only if $P \models \dot{x}$ falls out of all ground model coded $I$-small sets.

**Proof.** The left-to-right direction is proved just as in the previous cases. For the right-to-left direction, suppose that $P \models \dot{x}$ falls out of all Borel ground model coded sets in the ideal $I$. By the Borel determinacy, it will be enough to produce a counterplay against a given Player I’s strategy $\sigma$, winning for Player II.

Let $p_{ini}$ be the initial condition indicated by the strategy $\sigma$ and let $\epsilon(p_{ini}) > 0$ be the real number from Claim 3.5. Let $l \in \omega$ be a number such that $2^{-l} < \epsilon(p_{ini})$. We will produce a counterplay such that the resulting point $x \in X$ falls out of the set $\bigcup_{m \in n} A(l, m)$ and out of all sets $\bigcup E(k_i, i)$ for all numbers $i \in \omega$ where $k_i : i \in \omega$ indexes the rounds at which Player II placed the $i$-th nontrivial move. Such a counterplay will certainly result in Player II’s victory.

Consider the tree $T$ of all partial plays $\tau$ of the game $G$ in which Player I follows his strategy $\sigma$, with the following properties. The play $\tau$ ends at round $k$ with a nontrivial move $p \in P$ of Player II, and indexing by $\langle k_i : i \in \bar{i} \rangle$ the rounds at which Player II placed nontrivial moves,

\[(***)\] for every system $\{ B_m : m \in n, F_i : i \in \bar{i} \}$ such that

1. each $B_m \supset A(\bar{k}, l, m)$ is a Borel subset of the space $X$ such that $c_m(B_m) - c_m(A(\bar{k}, l, m)) \leq 2^{-k}$ and $c_m(B_m) \leq 2^{-l}$
2. each $F_i$ is a collection of basic open sets of the space $X$ such that $E(k_i, \bar{k}, i) \subset F_i$ and $w(F_i) \leq i + 1$ and the open sets in the set $F_i \setminus E(k_i, \bar{k}, i)$ have diameters $\leq 2^{-k}$

there is a condition $q \leq p$ such that $q \models \dot{x} \notin \bigcup_{m \in n} B_m \cup \bigcup_{i \in \bar{i}} \bigcup F_i$.

It is not difficult to see that if $\tau$ is a counterplay against the strategy $\sigma$ in which Player II made infinitely many nontrivial moves and the initial segments of the play $\tau$ form an infinite branch through the tree $T$, then the play $\tau$ is winning for Player II. Thus it is enough to show that every node of the tree $T$ has a proper extension still in the tree $T$.

Let $\tau \in T$ be a finite play ending at some round $\bar{k}$, with similar notational use as in the condition $(***)$. To find its nontrivial extension in the tree $T$, consider its infinite extension in which Player II places only trivial moves past $\tau$. It will result in some sets $A(l, m) : m \in \omega$ and $E(k_i, i) : i \in \bar{i}$. By the property $(***)$ there will be a condition $q \leq p$ forcing the point $\dot{x}$ out of the sets $A(l, m) : m \in n$ as well as out of the sets $E(k_i, i) : i \in \bar{i}$. Let $r \leq q$ be a condition in the appropriate open dense set indicated by Player I, deciding whether the point $\dot{x}$ belongs to the appropriate basic open set or not. This will be the next nontrivial move of Player II past $\tau$ in the required extension of the play $\tau$ in the tree $T$, we just have to show that there is a round at which it can be placed so that the condition $(***)$ is preserved.
Suppose for contradiction that for every round $k > \bar{k}$ the condition (***) fails if Player II places the move $r$ at that round. Thus there must be sets $B(k, m) \supset A(k, l, m) : m \in n$ and $F(k, i) \supset E(k, k, i) : i \in i$ and a set $F(k, i)$ as in (***) such that $r \Vdash \bar{x} \in \bigcup_m B(k, m) \cup \bigcup_{i \in i} F(k, i) \cup \bigcup F(k, i)$. By the choice of the condition $q$ it must be the case that actually $r \Vdash \bar{x} \in \bigcup_{m \in n} (B(k, m) \backslash \tilde{A}(l, m)) \cup \bigcup_{i \in i} \bigcup (F_i \backslash E(k, k, i)) \cup \bigcup F(k, i)$. As in the previous section, $c_m(\bigcap_k \bigcup_{j > k}(B(j, m) \backslash \tilde{A}(l, m))) = 0$ for every number $m \in n$ and therefore there is a condition $s \leq r$ and a number $k > \bar{k}$ such that $s \Vdash \bar{x} \notin \bigcup_{m \in n} \bigcup_{j > k} B(j, m)$. The set $\bigcap_{j > k} (\bigcup_{i \in i} (F(j, i) \backslash E(k, j, i)) \cup F(j, i))$ has Hausdorff measure $\leq \Sigma_{n+1} i + 1$ by the definition of the Hausdorff measure, and therefore there is a number $j' > j$ and a condition $t \leq s$ forcing $\bar{x} \notin \bigcup_{i \in i} F(j', i)$ and therefore $\bar{x} \notin \bigcup_{m \in n} B(j', m) \cup \bigcup_{i \in i} F(j', i)$. This contradicts the choice of the sets $B(j', m)$ and $F(j', i)$.

The Lemma follows! \qed

The proofs of the following corollaries are essentially identical to the previous sections. We leave out the proofs.

**Corollary 4.2.** The forcing $P_l$ is proper.

**Corollary 4.3.** Every analytic $I$-positive set has an $I$-positive Borel subset.

**Corollary 4.4.** In the choiceless Solovay model, the ideal $I$ is closed under wellordered unions.

Again, there is a related integer game and we get

**Corollary 4.5.** $(ZF+AD^+)$ Every $I$-positive set has a Borel $I$-positive subset.

**Corollary 4.6.** $(ZF+AD^+)$ the ideal $I$ is closed under wellordered unions.

## 5 Examples

### 5.1 Potential theory

The main result of this section is due to Murali Rao. It turns out that most if not all capacities arising in potential theory are stable. We will use a general approach to potential spaces exposed in [1], Section 2.3.

**Definition 5.1.** Let $M$ be a space with a positive measure $\nu$, and let $n \in \omega$. A *kernel* on $\mathbb{R}^n \times M$ is a function $g : \mathbb{R}^n \times M \to \mathbb{R}$ such that $g(\cdot, y)$ is lower semicontinuous for every point $y \in M$ and $g(x, \cdot)$ is $\nu$-measurable for each point $x \in \mathbb{R}^n$.

**Definition 5.2.** For every $\nu$-measurable function $f : M \to \mathbb{R}$ let $Gf : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $Gf(x) = \int_M g(x, y)f(y)d\nu(y)$.

Now let $p \geq 1$ be a real number. Associated with it is the uniformly convex Banach space $L^p(\nu)$ and its subset $L^p_0(\nu)$ consisting of non-negative functions. We are ready to define the capacity $c = c_{p, \mathbb{R}^n}$ on $\mathbb{R}^n$:
**Definition 5.3.** For every set $E \subset \mathbb{R}^n$ let $\Omega_E = \{ f \in L^p_+(\nu) : \forall x \in E \ G_f(x) \geq 1 \}$ and let $c_{g,p}(E) = \inf \{ \int_M f^p d\nu : f \in \Omega_E \}$.

It turns out that the function $c_{g,p}$ is an outer regular subadditive capacity, see [1], Propositions 2.3.4-6 and 2.3.12. It is not immediately clear if it has to be strongly subadditive, even though in many cases including the Newtonian capacity it is. Most capacities in potential theory are obtained in this way; we just mention the most notorious examples.

**Example 5.4.** The Newtonian capacity results from a Newton kernel and $p = 2$. The Newton kernel is a special case of Riesz kernels with $\alpha = 2$, see below. This is perhaps not the simplest way of viewing this classical capacity. A simpler definition can be found in [7] 30.B.

**Example 5.5.** The Riesz capacities result from Riesz kernels. If $0 < \alpha < n$ is a real number, the Riesz kernel $I_\alpha : \mathbb{R}^n \to \mathbb{R}$ is given by

$$I_\alpha(x) = a_\alpha \int_0^\infty t^{\frac{n-\alpha}{2}} e^{-\frac{\pi|x|^2}{t}} \frac{\gamma_\alpha}{|x|^{n-\alpha}} dt$$

for certain constants $a_\alpha, \gamma_\alpha$; the above setup will yield the $\alpha$-th Riesz capacity by letting $M = \mathbb{R}^n$, $\nu =$the Lebesgue measure, and $g(x, y) = I_\alpha(x - y)$.

**Example 5.6.** The Bessel capacities result from Bessel kernels. If $\alpha > 0$ then the Bessel kernel $G_\alpha : \mathbb{R}^n \to \mathbb{R}$ is given by

$$G_\alpha(x) = a_\alpha \int_0^\infty t^{\frac{\alpha}{2}} e^{-\frac{\pi|x|^2}{t}} \frac{\gamma_\alpha}{|x|^{n+\alpha}} dt.$$ 

Then proceed similarly as in the case of Riesz capacities.

We will now show that the capacities obtained in this way are stable. The key tool is the following description of the closure of the set $\Omega_E$ in the space $L^p(\nu)$:

**Fact 5.7.** [1], Proposition 2.3.9. Let $E \subset \mathbb{R}^n$ be a set. Then $\overline{\Omega}_E = \{ f \in L^p_+(\nu) : \forall x \in E \ G_f(x) \geq 1 \}$.

Since the set $\overline{\Omega}_E \subset L^p(\nu)$ is closed and convex, the uniform convexity of the Banach space $L^p(\nu)$ implies that there is a unique function $f \in \overline{\Omega}_E$ with the smallest norm. The function $f$ is called the potential function of the set $E$, and clearly $\int_M f^p d\nu = c(E)$. We will write $f = f_E$.

For every set $A \subset \mathbb{R}^n$ let $\tilde{A} = A \cup \{ x \in \mathbb{R}^n : G_{f_E}(x) \geq 1 \}$. The following claim immediately implies that this set works as demanded by Definition 1.1.

**Claim 5.8.** Let $A \subset B \subset \mathbb{R}^n$ be arbitrary sets. Then $c(A) < c(B)$ if and only if $c(B \setminus \tilde{A}) > 0$.

**Proof.** On one hand, if $c(B \setminus \tilde{A}) = 0$ then $f_A \in \overline{\Omega}_B$ and therefore $c(B) \leq \int_M f_A^p d\nu = c(A)$. On the other hand, suppose $c(A) = c(B)$. Since $A \subset B$, it is the case that $f_B \in \overline{\Omega}_A$. Since $c(A) = c(B)$, it is the case that the norms of
the functions \( f_B \) and \( f_A \) coincide. By the uniqueness of the function of minimal norm in the set \( \Omega_A \) it must be the case that \( f_A = f_B \). Thus \( B \setminus \tilde{A} \subseteq \{ x \in B : Gf_A(x) < 1 \} = \{ x \in B : Gf_B(x) < 1 \} \) and the capacity of the latter set is zero by the definition of the set \( \Omega_B \). The claim follows.

Slight variations of the above definitions are in use in potential theory. A typical small change is the replacement of the Banach space \( L^p(\nu) \) with \( l^q \) in the case of Besov capacities and Lizorkin-Triebel capacities [1] Chapter 4. The results and proofs mentioned above apply again in these cases. The above definitions can be further generalized to yield capacities on spaces other than \( \mathbb{R}^n \).

5.2 Stepräns capacities

In [10] Stepräns implicitly used the following method to construct an interesting family of capacities. They are all subadditive, outer regular and stable. They are generally not strongly subadditive. Fremlin in [3] derived a weaker property for many of the Stepräns capacities which is nevertheless strong enough to make Theorems 1.3 and 1.4 go through for them.

**Definition 5.9.** Let \( X \) be a set. Let \( f \leq g \) for functions \( f, g \in \mathbb{R}^X \) denote the coordinatewise ordering. A **good norm on** \( X \) **is a norm** \( n : \mathbb{R}^X \to \mathbb{R}^+ \) **with the following properties:**

- it respects the absolute value: for all functions \( f, g : X \to \mathbb{R}, |f| \leq |g| \) implies \( n(f) \leq n(g) \)
- \( n(1) = 1 \)
- if \( X \) is a finite set we demand \( |f| < |g| \) implies \( c(f) < c(g) \).

Note that a good norm \( n \) on a set \( X \) generates a probability submeasure \( c_n \) on \( X \) given by \( c_n(A) = n(\chi_A) \).

**Definition 5.10.** If \( X, Y \) are sets and \( n, m \) are good norms on each respectively, their **iteration** \( n \ast m \) is the good norm on \( X \times Y \) described by \( (n \ast m)(f) = n(x \mapsto m(y \mapsto f(x, y))) \).

Note that the iteration is an associative but not necessarily a commutative operation.

**Definition 5.11.** Suppose that \( X_i : i \in \omega \) is a sequence of finite sets and \( n_i : i \in \omega \) is a sequence of good norms on the respective sets. Write \( m_i = n_0 \ast n_1 \ast \cdots \ast n_i \), so \( m_i \) is a good norm on \( X_0 \times X_1 \times \cdots \times X_i \). By the **limit** of the sequence \( m_i \) we mean the good norm \( k = \lim_i m_i \) on \( X = \prod_i X_i \) given by the following:

- Suppose first that \( f \in \mathbb{R}^X \) is a nonnegative step function, i.e. there is \( j \in \omega \) such that the value \( f(x) \) depends only on \( x \upharpoonright j + 1 \) and so we can write \( f^\ast(x \upharpoonright j) = f(x) \). Then let \( k(f) = m_j(f^\ast) \). Note that by the multiplicativity property this does not depend on the choice of the number \( j \).
If \( g \in \mathbb{R}^X \) is a nonnegative lower semicontinuous function, then \( g = \sup_n f_n \) for some sequence of nonnegative step functions \( f_0 \leq f_1 \leq \ldots \).

Let \( k(g) = \sup_n f_n \). A compactness argument will show that this does not depend on the choice of the sequence \( f_n : n \in \omega \). Note that a limit \( h \) of an increasing sequence \( \{g_n : n \in \omega \} \) of lower semicontinuous functions is again lower semicontinuous, and by a compactness argument again \( k(h) = \sup_n k(g_n) \).

Finally, if \( h \in \mathbb{R}^X \) is an arbitrary function then let \( k(h) = \inf \{k(g) : |h| \leq g \text{ and } g \text{ is lower semicontinuous} \} \).

A part of the above construction can be performed even for functions \( n_i \) which respect the absolute value and are subadditive, i.e. they miss multiplicativity from the properties of a norm. However, the multiplicativity of the norms is critical in Stepans's construction in that it makes it possible to prove the following sweeping theorem:

**Theorem 5.12.** Suppose \( k \) is a limit of a sequence of norms as described in Definition 5.11. Then the derived submeasure is an outer regular stable capacity.

**Proof.** We will begin with a number of definitions. Suppose \( k \) is obtained from good norms \( n_i : i \in \omega \) on finite sets \( X_i : i \in \omega \). Set \( X = \prod_i X_i \). Let \( \langle X \rangle^\omega \) denote the set of all finite sequences \( t \) such that \( t(i) \in X_i \) whenever \( i \in \text{dom}(t) \).

For every sequence \( t \in \langle X \rangle^\omega \) let \( O_t \subset X \) be the basic open set determined by the sequence. Let \( k_t \) be the good norm on \( O_t \) which is the limit of the sequence \( n_i * n_{i+1} * \ldots \), where \( i = \text{dom}(t) \); we will often apply it to functions \( f \in \mathbb{R}^X \) with the convention \( k_t(f) = k_t(f \upharpoonright O_t) \). It is not necessary but instructive to observe

\[ k_t(f) = k(f)/k(O_t) \text{ for every function } f \in \mathbb{R}^X \text{ with support } O_t. \]

**Claim 5.13.**

\[ k_t(f) = k(f)/k(O_t) \text{ for every function } f \in \mathbb{R}^X \text{ with support } O_t. \]

**Claim 5.14.** Suppose \( t \in \langle X \rangle^\omega \) and \( \{s_m : m \in n \} \) is a finite set of its mutually incomparable extensions. Suppose \( f : X \to \mathbb{R} \) is a function with support \( \bigcup_m O_{s_m} \). Then the \( k_t \)-norm of the function \( f \) depends only on its \( k_{s_m} \)-norms for \( m \in n \).

**Proof.** We will deal with the case of \( \{s_m : m \in n \} \) being the collection of all immediate successors of \( t \). The general case then follows by induction on the size of the tree \( \{u : t \subset u \subset s_m \text{ for some } m \in n \} \).

Let \( i = \text{dom}(t) \). Suppose first that \( f \in \mathbb{R}^X \) is a step function, so the value \( f(x) \) depends only on \( x \upharpoonright j \) for some number \( j > i \). Let \( g : X \to \mathbb{R}^+ \) be the function defined by \( g(y) = k_{l \rightarrow y}(f) \). The definitions immediately imply that \( k_t(f) = n_i(g) \). The case of lower semicontinuous functions and arbitrary functions are similar.

**Claim 5.15.** Suppose \( \{t_n : n \in \omega \} \) is a collection of mutually incomparable sequences in \( \langle X \rangle^\omega \), and let \( f \) be a function with support \( \bigcup_n O(t_n) \). Then \( k(f) = \sup_m k(f \upharpoonright \bigcup_{n \in m} O(t_n)) \).
Proof. Suppose for contradiction that \( k(f) > (1 + \epsilon) \sup_m(f \upharpoonright \bigcup_{n \in \omega} O(t_n)) \). For every number \( n \in \omega \) find a lower semicontinuous function \( f_n \) with domain \( O(t_n) \) such that \( f_n \geq f \upharpoonright O(t_n) \) and \( k_n(f_n) < (1 + \epsilon)k_n(f \upharpoonright O(t_n)) \). Let \( g_m = \sup_{n \in \omega} f_n \). By the previous Claim and multiplicativity it follows that \( k(g_m) < (1 + \epsilon)k(f \upharpoonright \bigcup_{n \in \omega} O(t_n)) \). Now the functions \( g_m : m \in \omega \) form an increasing sequence of l.s.c. functions; write \( g \) for their pointwise supremum. We have \( g > f \) and \( k(g) = \sup_m k(g_m) \leq (1 + \epsilon) \sup_m k(f \upharpoonright \bigcup_{n \in \omega} O(t_n)) < k(f) \), a contradiction.

Now we are finally ready to deal with the derived capacity \( c \). It is immediate from the definitions that \( c \) is an outer regular submeasure. If \( O \subset X \) is an open set then \( c(O) = \sup \{ c(P) : P \subset O \text{ is clopen} \} \), and so \( c \) is continuous in increasing unions of open sets and countably subadditive. The key in the rest of the proof will be the following density property:

Claim 5.16. Suppose that \( B \subset X \) is a \( c \)-positive set, and \( \epsilon > 0 \). Then there is a sequence \( t \in \langle X \rangle^{<\omega} \) such that \( c(B) \cap O(t) > (1 - \epsilon)c(O_t) \).

Proof. Suppose this fails for some set \( B \subset X, c(B) > 0 \), and a number \( \epsilon > 0 \). By induction on \( n \in \omega \) build sets \( K_n : n \in \omega \) and \( O_n : n \in \omega \) so that:

- \( K_n \subset \langle X \rangle^{<\omega} \) is a set of mutually incomparable sequences, and \( O_n = \bigcup\{ O_t : t \in K_n \} \). \( K_0 = \{0\} \) and \( O_0 = X \).
- every node in \( K_n \) has some extension in \( K_{n+1} \), and every node in \( K_{n+1} \) has some initial segment in \( K_n \), so \( O_{n+1} \subset O_n \).
- for every node \( t \in K_n \) the set \( O_{n+1} \cap O_t \) covers the set \( B \cap O_t \), and \( c(O_{n+1} \cap O_t) < (1 - \epsilon)c(O_t) \). Such a set certainly exist by our assumption.

The multiplicativity of the norms and Claim 5.15 can then be used to show that \( c(O_n) < (1 - \epsilon)^n \). Thus \( c(\bigcap_n O_n) = 0 \). On the other hand, the last item implies inductively that \( B \subset \bigcap_n O_n \), a contradiction.

Now we are ready to show that \( c \) is continuous in increasing unions. Suppose \( B = \bigcup_n B_n \) is an increasing union of sets, and for contradiction assume that \( (1 - \epsilon)c(B) > \sup_n c(B_n) \). Consider the set \( K = \{ t \in \langle X \rangle^{<\omega} : \exists n c(B_n \cap O_t) > (1 - \epsilon)c(O_t) \} \) and let \( L \subset K \) be a subset of it consisting of mutually incomparable sequences such that every element of \( K \) is an extension of an element of \( L \). Let \( C = B \setminus \bigcup\{ O(t) : t \in L \} \) and observe that \( c(C) = 0 \). If \( c(C) > 0 \) then by the countable subadditivity of the submeasure \( c \) there would have to be a number \( n \) such that \( c(C \cap B_n) > 0 \), and the previous claim applied to the set \( C \cap B_n \) would yield a sequence \( t \in \langle X \rangle^{<\omega} \) contradicting the choice of the sets \( K, L \). Now for each sequence \( t \in K \) choose a number \( n_t \) such that \( c(B_{n_t} \cap O_t) > (1 - \epsilon)c(O_t) \) and consider the set \( D = \bigcup\{ B_{n_t} \cap O_t : t \in L \} \). Clearly, for each \( t \in L \) it is the case that \( c(D \cap O_t) \geq (1 - \epsilon)c(O_t) \geq (1 - \epsilon)c(B \cap O_t) \) and therefore by the above Claims and multiplicativity it is the case that \( c(D) > (1 - \epsilon)c(B \setminus C) = (1 - \epsilon)c(B) \).

By Claim 5.15 writing \( D_M = \bigcup_{t \in M} D \cap O_t \) for every set \( M \subset K \) there must
be a finite set \( M \subset K \) such that \( c(D_M) > (1 - \epsilon)c(B) \). However, since the sets \( B_n : n \in \omega \) form an increasing sequence, there must be a number \( m \in \omega \) such that \( D_M \subset B_m \). Thus \( c(B_m) > (1 - \epsilon)c(B) > \sup_n c(B_n) \), a contradiction.

To see that the capacity \( c \) is stable, let \( A \subset X \) be a set. For every number \( \epsilon > 0 \) let

\[
A_\epsilon = \{ x \in X : \exists n c(A \cap O_{x,n}) > (1 - \epsilon)c(O_{x,n}) \}
\]

and let \( \tilde{A} = A \cup \bigcap \epsilon A_\epsilon \). We claim that this set works as demanded by Definition 1.1. First note that \( c(\tilde{A}) = c(A) \): every set \( A_\epsilon \) is open and by the multiplicativity it has capacity \( \leq (1 - \epsilon)c(A) \), and therefore \( c(\bigcap \epsilon A_\epsilon) \leq c(A) \). Moreover by Claim 5.10, \( c(A \setminus \bigcap \epsilon A_\epsilon) = 0 \) and therefore \( c(A \cup \bigcap \epsilon A_\epsilon) = c(\bigcap \epsilon A_\epsilon) \leq c(A) \) as required. Now suppose that \( B \subset X \setminus \tilde{A} \) is a positive capacity set. By the countable additivity of the capacity \( c \), there is a number \( \epsilon \) such that the set \( B \setminus \tilde{A} \) has positive capacity. Use Claim 5.10 to find a sequence \( t \in (X)^{<\omega} \) such that \( c(B \setminus \tilde{A} \cap O_t) > (1 - \epsilon)c(O_t) \). It is now clear that \( c(B \cap O_t) > (1 - \epsilon)c(O_t) \geq c(A \cap O_t) \) and it follows that \( c(A \cup B) > c(A) \) as required.

References

[1] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*. Springer Verlag, New York, 1996.

[2] Ilijas Farah and Jindřich Zapletal. Four and more. *Annals of Pure and Applied Logic*. accepted.

[3] David Fremlin. On a construction of Steprāns. preprint.

[4] Wolfhard Hansen. Semi-polar sets and quasi-balayage. *Mathematische Annalen*, 257:495–517, 1981.

[5] Stephen Jackson. The weak square property. *J. Symbolic Logic*, 66:640–657, 2001.

[6] Thomas Jech. *Set Theory*. Academic Press, San Diego, 1978.

[7] Alexander S. Kechris. *Classical Descriptive Set Theory*. Springer Verlag, New York, 1994.

[8] D. Anthony Martin. *A purely inductive proof of Borel determinacy*, pages 303–308. Number 42 in Proceedings of Symposia in Pure Mathematics. American Mathematical Society, Providence, 1985.

[9] Saharon Shelah. *Proper and Improper Forcing*. Springer Verlag, New York, second edition, 1998.

[10] Juris Steprāns. Many quotient algebras of the integers modulo co-analytic ideals. preprint.

[11] Jindřich Zapletal. *Descriptive Set Theory and Definable Forcing*. Memoirs of American Mathematical Society. AMS, Providence, 2004.

[12] Jindřich Zapletal. Two preservation theorems. 2005. preprint.