Stacky Formulations of Einstein Gravity

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Abstract

This is an investigation of "stacky" structures for Einstein gravity together with an alternative reformulation in the language of formal moduli problems. Our constructions are essentially based on [1, 3, 4]. In the first part of the paper, we first revisit the aspects of (vacuum) Einstein gravity on a Lorentzian 3-manifold $M$ with cosmological constant $\Lambda = 0$. Next, we shall provide a realization of the moduli space of Einstein’s field equations as a certain stack. We indeed construct the stack of (vacuum) Einstein gravity in $n$-dimensional set-up with vanishing cosmological constant by using the homotopy theoretical formulation of stacks. The treatment in [3], in fact, leads to a certain "stacky" formulation of Yang–Mills fields on Lorentzian manifolds. Therefore, we intend to adopt such a language to provide similar "stacky" formulations in the case of Einstein gravity. With this new formulation, we also upgrade the equivalence of certain 2+1 quantum gravities with gauge theory to the isomorphism between the corresponding moduli stacks. The second part of the paper, on the other hand, is designed as a detailed survey on formal moduli problems. It is in particular devoted to formalize Einstein gravity in the language of formal moduli problems and to study the algebraic structure of observables in terms of factorization algebras.

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Part I
Stack of Einstein Gravity

1 Introduction

It is an undoubted fact that moduli theory plays a significant role in analyzing field theories. Throughout the discussion we shall focus on the classical side of the story, indeed. The reason why one may prefer to adopt a moduli theoretic approach is relatively simple: A classical field theory can be described by a piece of data $(M, F_M, S, G)$ where $F_M$ denotes the space of fields on some base manifold $M$, $S$ is a smooth action functional on $F_M$, and $G$ is a certain group encoding the symmetries of the system. Then the standard folklore suggests that the key information about the system is encoded in the critical locus $\text{crit}(S)$ of $S$, modulo symmetries. Therefore, the problem of interest boils down to the analysis of the properties of this moduli space. Of course, moduli theory has some natural questions related to "bad quotients" or "bad intersections". There are some classical techniques to deal with these sorts of problems, but nowadays some people prefer to use a relatively new technology, namely derived algebraic geometry \[49\]. DAG combines higher categorical objects and homotopy theory with many tools from homological algebra. Hence, it can be considered as a higher categorical/homotopy theoretical refinement of classical algebraic geometry. Consequently, DAG suggests new and alternative perspectives in physics as well. In that respect, the formulation of certain gauge theories in the language of derived algebraic geometry, developed by Costello and Gwilliam \[1,10\], provides new and fruitful insights to encode the formal geometry of the associated moduli space of the theory. Adopting such an approach, we would like to present a similar type of analysis in the case of Einstein gravity. Before explaining the derived formulation of a classical field theory in a general set-up \[1\], we shall first briefly recall how to define a classical field theory in Lagrangian formalism \[20\]:

**Definition 1.0.1.** A classical field theory on a manifold $M$ consists of the following data:

(i) The space $F_M$ of fields of the theory, which is defined to be the space $\Gamma(M, F)$ of sections of a particular sheaf $F$ on $M$,

(ii) The action functional $S : F_M \to k$ ($\mathbb{R}$ or $\mathbb{C}$) that captures the behavior of the system under consideration.

Furthermore, if we want to describe a quantum system, as a third component, we need to employ the path integral formalism. This part, however, is beyond the scope of the current discussion. Instead, we refer to \[20,37,41\].

**Remark 1.0.1.** As briefly mentioned above, to encode the dynamics of the system in a well-established manner, we need to study the critical locus $\text{crit}(S)$ of $S$. One can determine $\text{crit}(S)$ by employing variational techniques for the functional $S$. This leads to define $\text{crit}(S)$ to be the space of solutions to the Euler-Lagrange equations modulo symmetries. Therefore, a classical field theory can be thought of as a study of the moduli space $\mathcal{E}L$ of solutions to E-L equations.

In the language of derived algebraic geometry, on the other hand, we would like to define the notion of a classical field theory in the following way \[1\]:

**Definition 1.0.2.** Let $F_M$ be the space of fields for some base manifold $M$, and $S : F_M \to k$ a smooth action functional as above. A (perturbative) classical field theory is a sheaf of derived stacks on $M$ equipped with a symplectic form of cohomological degree $-1$.

For a complete discussion of all concepts mentioned in the Definition 1.0.2, see Appendix of \[5\] or Chapter 3 of \[1\]. For a quick introduction to the notion of a derived stack, see \[13\]. The shifted symplectic geometry (known also as PTVV’s shifted symplectic geometry) is originally developed by T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi in \[16\]. An accessible overview to the basics of this structure can be found in \[17\] or \[18\]. As stressed in \[1\], one can unpackage the formidable Definition 1.0.2 as follows:

i. Describing a classical field theory sheaf theoretically \[20\] boils down to the study of the moduli space $\mathcal{E}L$ of solutions to the Euler-Lagrange equations (and hence the critical locus of the action functional) which is in fact encoded by a certain moduli functor.
ii. As stressed in [21], a moduli functor, however, would not be representable in a generic situation due to certain problems, such as the existence of degenerate critical points or non-freeness of the action of the symmetry group on the space of fields [19]. In order to avoid problems of this kind (and to capture the perturbative behavior at the same time), one may adopt the language of derived algebraic geometry. Hence, one may need to replace the naïve notion of a moduli problem by a so-called formal moduli problem in the sense of Lurie [7].

Motivation, Einstein gravity and the outline. Having adopted the suitable language, our intentions are (i) to show that, in the case of a certain Einstein gravity, one can view the moduli space $\mathcal{EL}$ of solutions to E-L equations as a suitable stack, and (ii) to upgrade the equivalence of a certain 2+1 quantum gravity with gauge theory to a stack isomorphism. On the other hand, (iii) the construction of the formal moduli problem of Einstein gravity, and the algebraic structure of observables of the theory in the context of derived algebraic geometry are investigated. The detailed outline is as follows:

1. Starting with the usual metric formalism (aka 2nd order formalism) in 2+1 dimensions, the vacuum Einstein field equations with cosmological constant $\Lambda = 0$ read as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (1.1)$$

Then, notice that after contracting with $g^{\mu\nu}$, one has $R = 0$. Therefore, it follows directly from substituting back into equation 1.1 that the moduli space $\mathcal{E}L$ of solutions to those field equations turns out to be the moduli space $\mathcal{E}(M)$ of Ricci-flat ($R_{\mu\nu} = 0$) Lorentzian metrics on $M$. In other words, it is just the moduli of flat geometric structures on $M$. With this interpretation in hand, it follows that Lorentzian spacetime is locally modeled on $(ISO(2, 1), \mathbb{R}^{2+1})$ where $\mathbb{R}^{2+1}$ denotes the usual Minkowski spacetime [8, 25]. In the 2+1 dimensional case, on the other hand, Weyl tensor is identically zero. Then the Riemann tensor can locally be expressed in terms of $R$ and $R_{\mu\nu}$, and so we locally have $R_{\mu\nu\sigma\rho} = 0$ as well. That is, any solution of the vacuum Einstein field equations in 3-dimensions with vanishing cosmological constant is locally flat. Thus, the metric is locally equivalent to the standard Minkowski metric $\eta_{\mu\nu}$. In a more physical point of view, on the other hand, the vanishing of $R_{\mu\nu\sigma\rho}$ means that 3D spacetime does not have any local degrees of freedom: there are no gravitational waves in the classical theory, and no gravitons in the quantum theory. For details, see [8, 9].

2. For the more general case of $\Lambda \neq 0$, $\mathcal{E}L$ turns out to be the moduli of Lorentzian metrics of constant curvature where the sign of this constant curvature depends on that of $\Lambda$. In fact, the field equations, in this case, read as

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R. \quad (1.2)$$

Hence, the corresponding spacetime is locally modeled as either $(SO(2, 2), dS_3)$ for $\Lambda > 0$ or $(SO(3, 1), AdS_3)$ for $\Lambda < 0$. Analyzing the cases when $\Lambda \neq 0$, however, is beyond the scope of the current discussion.

3. Employing the Cartan’s formalism [8, 9, 2], one can reinterpret 2+1 gravity in the language of gauge theory. Basics of this interpretation will be briefly discussed in Section 2.1. Roughly speaking, the study of 2+1 gravity in fact boils down to that of $ISO(2, 1)$ Chern Simons theory on $M$ with the action functional $CS : \mathcal{A} \to S^3$ of the form

$$CS[A] = \int_M \langle A, dA + \frac{2}{3}A \wedge A \rangle \quad (1.3)$$

where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on the Lie algebra $\mathfrak{g}$ of $ISO(2, 1)$, and the gauge group $G$ is locally of the form $Map(U, ISO(2, 1))$. This group acts on the space $\mathcal{A}$ of $ISO(2, 1)$-connections on $M$ in a natural way: For all $g \in G$ and $A \in \mathcal{A}$, we set

$$A \cdot \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot d\rho. \quad (1.4)$$
The corresponding E-L equation, in this case, turns out to be
\[ F_A = 0, \] (1.5)
where \( F_A = dA + A \wedge A \) is the curvature two-form on \( M \). Furthermore, under gauge transformations, the curvature 2-form \( F_A \) behaves as follows:
\[ F_A \mapsto F_A \cdot \rho := \rho^{-1} \cdot F_A \cdot \rho \quad \text{for all } \rho \in G. \] (1.6)
Notice that this case involves the non-compact gauge group ISO(2, 1), and hence the required treatment is slightly different from the case of compact gauge groups. Recall that in the case of \( G := SU(2) \), in particular, one has a unique bilinear form on \( g \), which is in fact the Killing form (up to a scaling constant). As outlined in [23] (see ch. 25), the compact case \( G = SU(2) \) can be analyzed by means of highly non-trivial theorems of Atiyah and Bott [38], but this is a rather different and deep story per se. Thus, the analysis of the compact case is beyond scope of the current discussion.

4. Using gauge theoretic interpretation, on the other hand, the physical phase space of \( 2+1 \) dimensional gravity on \( M = \Sigma \times (0, \infty) \) (with \( \Lambda = 0 \)) can be realized as the moduli space \( \mathcal{M}_{\text{flat}} \) of flat ISO(2, 1)-connections on \( \Sigma \) [2] in a way that we have a map
\[ \phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{\text{flat}} \] (1.7)
sending a flat pseudo-Riemannian metric \( g \) to the corresponding (flat) gauge field \( A^g \), where \( A^g \) is a Lie algebra iso(2, 1)-valued 1-form. Indeed, this 1-form is constructed by combining the vierbein \( e^a \) and the spin connection \( \omega^a \) as follows:
\[ A_i := P_a e_i^a + J_a \omega_i^a \] (1.8)
where \( A^a = A_a(x^i)dx^i \) in a local coordinate chart \( x = (x^i) \) such that \( P_a \) and \( J_a \) correspond to translations and Lorentz generators for the Lie algebra of the Poincaré group ISO(2, 1) for \( a = 1, 2, 3 \). For an introduction to the gauge theoretic interpretation, see [2, 8, 9] or section 6 of [6]. As indicated above, the associated Chern-Simons theory has a non-compact gauge group ISO(2, 1), and hence the analysis of Atiyah and Bott is no longer available. But, instead, one will have the following identification [50]:
\[ \mathcal{M}_{\text{flat}} \cong T^* (\text{Teich}(\Sigma)) \] (1.9)
where \( \text{Teich}(\Sigma) \) denotes the Teichmüller space associated to the closed surface \( \Sigma \) of genus \( g > 1 \). In that case, \( \mathcal{M}_{\text{flat}} \) becomes a \( 12g - 12 \) dimensional symplectic manifold with the standard symplectic structure on the cotangent bundle. The identification allows to employ the canonical/geometric quantization [53, 54] of the cotangent bundle in order to manifest the quantization of the phase space (even if this manifestation is by no means unique) [2].

**Remark 1.0.2.** In a genuine quantum gravity, one seeks for the construction of a quantum Hilbert space by quantizing the honest moduli space \( \mathcal{E}(M) \) of solutions to the vacuum Einstein field equations on \( M \). In the gauge theoretic formulation, on the other hand, one can actually quantize the phase space \( \mathcal{M}_{\text{flat}} \) of the Chern-Simons theory associated to \( 2+1 \) dimensional gravity in the sense of the naïve discussion above ([6], sec. 6). That is, to construct a quantum theory of gravity, a possible strategy we may have is as follows: First, we translate everything into gauge theoretical framework, and view everything as a gauge theory. Then, one may try to "quantize" the corresponding gauge theory. When \( \Lambda = 0 \), as discussed above, the \( 2+1 \) gravity corresponds to the Chern-Simons theory with gauge group \( G = ISO(2, 1) \). When \( \Lambda \neq 0 \), on the other hand, one has a Chern-Simons theory with either \( G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) for \( \Lambda < 0 \) or \( G = SL(2, \mathbb{C}) \) for \( \Lambda > 0 \). As indicated before, analyzing the cases when \( \Lambda \neq 0 \) is beyond the scope of the current discussion. In any case, however, we end up with the following question: Are the resulting theories equivalent (in some sense)?

**Definition 1.0.3.** We say that the quantum gravity is **equivalent** to gauge theory in the sense of the canonical formalism if the map \( \phi \) above is an isomorphism:
\[ \phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{\text{flat}}. \] (1.10)
As indicated in section 6 of [6], one has the equivalence of the quantum gravity with a gauge theory in the case of a vacuum Einstein gravity on \( M \) with vanishing cosmological constant where \( M = \Sigma \times (0, \infty) \) and \( \Sigma \) is a closed Riemann surface of genus \( g > 1 \). Indeed, this result (and much more) were established via the works of Mess [25], Goldman [26] et al. The main idea behind the construction is, roughly speaking, as follows: there is a one-to-one correspondence between the moduli space \( \mathcal{M}_{\text{flat}} \) of flat \( G \)-connections on \( \Sigma \) and the moduli space

\[
\text{Hom}(\pi_1(\Sigma), G)/G.
\]

Here, \( \text{Hom}(\pi_1(\Sigma), G)/G \) denotes the moduli space of representations of the surface group \( \pi_1(\Sigma) \) in \( G \) where \( G \) acts on \( \text{Hom}(\pi_1(\Sigma), G) \) by conjugation [28]. The construction of such an isomorphism in 1.0.3 is essentially based on the Teichmüller theoretic treatment [27] of representations of the surface group \( \pi_1(\Sigma) \), and the global topology of the space \( \text{Hom}(\pi_1(\Sigma), G)/G \) in the cases where \( G = \text{ISO}(2, 1) \) or \( G = \text{PSL}(2, \mathbb{R}) \cong \text{SO}_0(2, 1) \) [26].

Key achievements of the current work. [3] describes a certain "stacky" formulation of Yang–Mills fields on Lorentzian manifolds. The construction uses some techniques naturally appearing in the homotopy theory of stacks [4], [1, 10], on the other hand, provide a formulation of classical field theories in the language of formal moduli problems, and analyze the algebraic structure of observables in some classical fields theories. In that respect, our main intention is in fact to provide similar constructions in the case of a certain Einstein gravity. Consequently, we upgrade the equivalence mentioned above to the "stacky" level. The main results of this paper can be outlined as follows.

1. In the first part of the paper, we employ the techniques in [3, 4] to show that
   
   (a) There is a suitable moduli stack of Einstein gravity (Theorem 5.0.1).
   
   (b) The isomorphism \( \phi \), in fact, induces an isomorphism of the corresponding stacks (Theorem 6.0.1).

2. The second part of the paper is essentially designed as a detailed survey on formal moduli problems, the structure of observables and factorization algebras. The appearances of these concepts in various classical field theories are also discussed. In that respect, it also includes the realizations of these concepts in the case of a particular 2+1 Einstein gravity. This part mainly relies on the derived geometric constructions especially in the case of Chern-Simons theory. Therefore, we first revisit basic constructions in [1, 10], and then show that 3D Einstein gravity arises as a natural example of these constructions. To this end,
   
   (a) we present an obvious formal moduli problem in the case of 3D Cartan theory of the vacuum Einstein gravity with/without cosmological constant.
   
   (b) Furthermore, in this 3D scenario, we revisit the algebraic structure of observables in the language of factorization algebras as well.

It should be noted that, in this part, the constructions related to 3D Einstein gravity are just the particular cases of the constructions given for Chern-Simons theory [1, 10, 11, 30].
2 A review: The equivalence of 2+1 quantum gravity with gauge theory

Before investigating the stacky behaviour, we shall try to elaborate the contribution of Mess and Goldman in constructing the equivalence indicated above. This section is also designed to provide a brief guideline to the existing literature.

2.1 An introduction to Cartan’s formalism and gauge theoretic interpretation of 2+1 gravity

In this section we shall revisit certain aspects of Cartan’s formalism in a rather succinct way, and we refer to [51, 52] for details. Cartan’s formalism, roughly speaking, consists of the following data:

1. A section \( e_i^a \) of the orthonormal frame bundle \( LM \) over \( M \) for each \( i \). That is,
   \[
   e_i^a \in \Gamma(M, LM)
   \] (2.1)
   where \( i \) labels the space indices with respect to the local chart \( (U_i, x) \) around the point \( p \in M \) and a’s are called Lorentz indices labeling vectors in the orthonormal basis \( \{e_1^1, e_1^2, ..., e_{\dim M}^i\} \) over \( U_i \). Here, each fibre
   \[
   LM_p = \{ (e_1^1(p), ..., e_{\dim M}^i(p)) : e_1^1(p), ..., e_{\dim M}^i(p) \text{ forms a basis for } TM_p \}
   \] (2.2)
   of \( LM \) is isomorphic to \( GL(n, \mathbb{R}) \). Such \( e_i^a \) are called vierbein.

2. A \( SO(2,1) \)-connection (or the spin connection) one-form \( \omega^a_{\ b} \) on \( M \). That is,
   \[
   \omega^a_{\ b} \in \Omega^1(M) \otimes so(2, 1)
   \] (2.3)
   where \( \omega_i \) is a Lie algebra-valued connection 1-form on \( LM \) such that \( \omega^a_i := (e_i^a)^* \omega_i \).

3. Compatibility conditions on metric:
   \[
   g_{ij} = e_i^a e_j^b \eta_{ab} \quad \text{and} \quad g^{ij} e_i^a e_j^b = \eta^{ab}
   \] (2.4)
   where \( \eta \) denotes the usual Minkowski metric.

The punchline is the following observation [2]: In 2+1 dimensional gravity, vierbein and spin connection can be considered as a pair \( (e_i^a, \omega^a_{\ b}) \) such that they could be combined into a certain gauge field \( A \) with the gauge group \( ISO(2,1) \) where \( \omega^a_{\ b} \) in fact plays the role of the so-called \( SO(2,1) \)-part of the connection \( A \) (or say the Lorentz-part), while \( e_i^a \) corresponds to translation generators of the Lie algebra \( iso(2, 1) \) of \( ISO(2,1) \). For some technical reasons, vierbein is supposed to be invertible in order to avoid the non-degeneracy on the metric.

Now, the usual Einstein-Hilbert action

\[
\mathcal{I}_{EH}[g] := \frac{1}{16\pi G} \int_M \! dx^3 \sqrt{-g}R
\] (2.5)

can be re-expressed by employing Cartan’s formalism as follows [2]:

\[
\mathcal{I}_{EH}[e, \omega] = \int_M \! e^a \wedge \left( d\omega^a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right)
\] (2.6)

where \( e^a = e_i^a dx^i \) and \( \omega^a = \frac{1}{2} \epsilon_{abc} \omega_{ij} dx^i \) together with an invariant non-degenerate, bilinear form \( \langle \cdot, \cdot \rangle \) on the Lie algebra \( iso(2,1) \) (with its generators \( J^a \) and \( P^a \) corresponding to Lorentz and translation generators resp.) defined as

\[
\langle J_a, P_b \rangle = \delta_{ab} \quad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0
\] (2.7)

with the structure relations for the Lie algebra given as

\[
[J_a, J_b] = \epsilon_{abc} J^c, \quad [J^a, P^b] = \epsilon_{abc} P^c, \quad [P^a, P^b] = 0.
\] (2.8)
Setting a gauge field as

$$A_i = P_\alpha e^\alpha_i + J_\alpha \omega^\alpha_i$$

(2.9)

where \( A^\alpha = A_i(x)dx^i \) in a local coordinate chart \( x = (x^i) \) such that \( P_\alpha \) and \( J_\alpha \) correspond to translations and Lorentz generators resp. as above, one can define a Chern-Simons theory with gauge group \( G = ISO(2, 1) \) in accordance with the above bilinear form \( \langle \cdot, \cdot \rangle \) so that the usual Chern-Simons action

$$CS[A] = \int_M \langle A, dA + \frac{2}{3} A \wedge A \rangle = \int_M \langle A, \frac{1}{3} dA + \frac{2}{3} F \rangle = T^\rho_{EH}[e, \omega]$$

(2.10)

becomes exactly the same expression as \( T^\rho_{EH} \). For computational details, see [2, 8, 9]. Note that obtaining the same action functional is just one part of the whole story, and one also requires to verify that the diffeomorphism invariance of 2+1 gravity must also be encoded in some way in \((e, \omega)\)-formalism. As stressed explicitly in [2, 8, 9], the notions of invariance in these two formalisms, i.e. the 2\(^\text{nd}\)-order (metric) formalism and 1\(^\text{st}\)-order \((e, \omega)\)-formalism, are related to each other in the sense that the invariance under spacetimes diffeomorphisms in metric formalism corresponds to the invariance under the corresponding gauge transformations, so-called local Lorentz transformations and local translations in \((e, \omega)\)-formalism. Due to the rather expository nature of this section, we cross our fingers and avoid the derivation of those correspondences to save some space and time! For a systematic treatment, we again refer to [2, 8, 9].

Having adopted Cartan’s formalism with the above observations, one can manifestly associate to 2+1 vacuum Einstein gravity with \( \Lambda = 0 \) on \( M = \Sigma \times (0, \infty) \) a particular gauge theory, namely Chern-Simons theory with the gauge group \( G = ISO(2, 1) \). Herein \( \Sigma \) is a closed Riemann surface of genus \( g > 1 \). Furthermore, using the natural gauge theoretic approach, we have the corresponding field equations [15] in \((e, \omega)\)-language for \( T^\rho_{EH} \):

$$de^a + \epsilon^{abc} \omega_b \wedge e_c = 0$$

(2.11)

$$d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c = 0$$

(2.12)

where the equation 2.12 corresponds to the fact that the torsion \( T^a = 0 \) and \( \omega_b \) serves as a so-called soldering form [2] through which one can make sense of the notion of torsion. The equation 2.11, on the other hand, corresponds to the fact that \( \omega \) is indeed a flat \( SO(2, 1) \)-connection (or equivalently that the curvature of the metric \( g_{ij} = e^a_i e^a_j \) vanishes). Furthermore, once we impose the field equations, we have the following observations:

1. \( e \) can be realized as a cotangent vector to the point \( \omega \) in the space \( X \) of flat \( SO(2, 1) \)-connections on \( \Sigma \). As explained in [15], this follows naïvely from the following observation: Given a smooth curve \( \omega(s) \) in \( X \), then by imposing the EOMs in equation 2.11, we get

$$d\omega^a(s) + \frac{1}{2} \epsilon^{abc} \omega_b(s) \wedge \omega_c(s) = 0,$$

(2.13)

and taking derivatives would give

$$d\left(\frac{d\omega^a}{ds}\right) + \epsilon^{abc} \omega_b(s) \wedge \frac{d\omega^c}{ds} = 0.$$

(2.14)

Then, from equation 2.12 we have \( \frac{d\omega^a}{ds} = e^a \). Therefore, if we consider the canonical/geometric quantization (in the sense of [53, 54]) of the cotangent bundle with coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) where \( p_i = \dot{q}_i \) for each \( i \), then \( \omega_i \)'s in fact play the role of coordinates \( q_i \)'s as \( e_i \)'s are viewed as momenta \( p_i \).

2. The observation above implies that a solution \((e, \omega)\) determines a point in the cotangent bundle \( T^*X \), and hence one can realize the Poincaré group \( ISO(2, 1) \) as the cotangent bundle over the Lorentz group \( SO(2, 1) \). That is,

$$ISO(2, 1) = T^*(SO(2, 1)).$$

(2.15)
3. As we will discuss in more detail soon (cf. Section 2.2), there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat}$ of flat $G$-connections on $\Sigma$ and the moduli space $\text{Hom}(\pi_1(\Sigma), G)/G$ of representations of the surface group $\pi_1(\Sigma)$ in $G$ [28] where $G$ acts on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation.

4. Furthermore, we will have the following isomorphisms from Mess [25] and Goldman [26]. These are based on Teichmüller theoretic treatment [27] of representations of the surface group $\pi_1(\Sigma)$ in the cases where $G = \text{ISO}(2, 1)$ or $G = \text{PSL}(2, \mathbb{R}) \cong \text{SO}_0(2, 1)$.

$$\mathcal{M}_{flat} \cong \text{Hom}(\pi_1(\Sigma), \text{ISO}(2, 1))/\sim$$

$$\cong T^*\left(\text{Hom}(\pi_1(\Sigma), \text{PSL}(2, \mathbb{R}))/\sim\right)$$

$$\cong T^*(\text{Teich}(\Sigma))$$

(2.16)

where $\text{Teich}(\Sigma)$ denotes the Teichmüller space associated to the closed surface $\Sigma$ of genus $g > 1$. Note that this observation will be the crucial if one requires the invertibility of the map $\phi$ (cf. Theorem 2.4.1).

### 2.2 The holonomy representation of flat $G$-connections

As explicitly studied in [44], there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat}$ of flat $G$-connections on $\Sigma$ and the moduli space $\text{Hom}(\pi_1(\Sigma), G)/G$ of (holonomy) representations of the surface group $\pi_1(\Sigma)$ in $G$ [28]. That is, we have

$$\mathcal{M}_{flat} \cong \text{Hom}(\pi_1(\Sigma), G)/G,$$  

(2.17)

where $G$ acts on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation. The correspondence is well-known and based on the techniques emerging in the theory of principal $G$-bundles [35, 36]. Sketch of the idea is as follows:

1. Let $A$ be a $G$-connection on a principal $G$-bundle $P \xrightarrow{\pi} \Sigma$. Given a smooth path $\gamma$ in $\Sigma$, for any $p \in \pi^{-1}(\gamma(0))$ there exists a unique horizontal path $\tilde{\gamma}_p$ starting with $\tilde{\gamma}_p(0) = p$, and hence we have the standard parallel transport map

$$T_\gamma : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(1)), \quad p \mapsto \tilde{\gamma}_p(1).$$

(2.18)

Note that if $\gamma$ is a loop, then the corresponding lift $\tilde{\gamma}_p$ lands in the same fiber, i.e. $\tilde{\gamma}_p(1) \in \pi^{-1}(\gamma(0))$ (but not necessarily hits the same point, i.e. $\tilde{\gamma}_p(1) \neq p$), and hence by definition of a principal $G$-bundle the parallel transport map becomes

$$T_\gamma : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(0)), \quad p \mapsto \tilde{\gamma}_p(1) = p \cdot g \text{ for some } g \in G$$

(2.19)

which yields the definition of the holonomy group

$$\text{Hol}_p(\gamma, A) = \{g, \in G : T_\gamma(p) = p \cdot g\}.$$  

(2.20)

As $p$ varies along the fiber via the action of $G$, for any $h \in G$, one has

$$\text{Hol}_{ph}(\gamma, A) = h^{-1} \cdot \text{Hol}_p(\gamma, A) \cdot h.$$  

(2.21)

2. Note that we haven’t used the flatness of $A$ in the above construction. The flatness of $A$ will come into play in accordance with the following propositions.

**Proposition a.** The connection $A$ is flat if and only if the holonomy group $\text{Hol}_p(\gamma, A)$ depends only on the homotopy class of $\gamma$ in $\pi_1(\Sigma)$.

**Proposition b.** The holonomy of a flat $A$-connection for a contractible loop $\gamma_0$ is trivial:

$$\text{Hol}_p(\gamma_0, A) = \{e\}.  \quad (2.22)$$

Now, one can define a well-defined map, compatible with the actions on both sides, as follows [44]:

$$\mathcal{M}_{flat} \rightarrow \text{Hom}(\pi_1(\Sigma), G)/G, \quad [A] \mapsto \left(\rho[A] : [\gamma] \mapsto g_\gamma\right).$$  

(2.23)
3. Converse of the map requires a constructive argument (for more details, see [28], sec. 2.3) in the following sense: To a given representation \( \rho : \pi_1(\Sigma) \to G \), one assigns a flat principal \( G \)-bundle \( P_\rho \to \Sigma \) as follows. First, consider the universal cover \( \tilde{\Sigma} \to \Sigma \). Notice that \( \pi_1(\Sigma) \) acts on \( \tilde{\Sigma} \) via deck transformations because of the fact that for the universal cover \( \tilde{\Sigma} \to \Sigma \) one has

\[
\text{Deck}(\tilde{\Sigma}) \cong \pi_1(\Sigma) \quad \text{and} \quad \tilde{\Sigma} \to \tilde{\Sigma}/\text{Deck}(\tilde{\Sigma}) \cong \Sigma,
\]

and hence \( \tilde{\Sigma} \to \Sigma \) in fact admits a principal \( \pi_1(\Sigma) \)-bundle structure. Now, given a representation \( \rho : \pi_1(\Sigma) \to G \), consider the space \( \tilde{\Sigma} \times G \) and a right \( \pi_1(\Sigma) \)-action on \( \tilde{\Sigma} \times G \) as follows: for all \( \gamma \in \pi_1(\Sigma) \) and \( (x, g) \in \tilde{\Sigma} \times G \), we define

\[
(x, g) \cdot \gamma := (x \cdot \gamma, \rho(\gamma^{-1}) \cdot g)
\]

where \( \gamma \) acts on \( x \in \tilde{\Sigma} \) via deck transformation as indicated above. Then, we can introduce an equivalence relation, and hence a quotient space as

\[
P_\rho := \tilde{\Sigma} \times G/\sim
\]

where \( (x, g) \sim (y, h) \iff y = x \cdot \gamma \) and \( h = \rho(\gamma^{-1}) \cdot g \) for some \( \gamma \in \pi_1(\Sigma) \). Finally, from [28],

\[
P_\rho \xrightarrow{q} \Sigma, \quad [(x, g)] \mapsto q(x),
\]

indeed defines a flat principal \( G \)-bundle with a natural right \( G \)-action on \( P_\rho \) given by

\[
[(x, g)] \cdot h := [(x, gh)] \quad \text{for any} \ h \in G.
\]

The existence of the inverse of the map

\[
\phi : \mathcal{E}(M) \to \mathcal{M}_\text{flat},
\]

which leads to the equivalence of quantum gravity with a gauge theory in the sense discussed before, is essentially related to the analysis of topological components of the space \( \text{Hom}(\pi_1(\Sigma), G)/G \) in the case of \( G = \text{PSL}(2; \mathbb{R}) \), which was studied in [26]. Now, our next task is, in a rather expository manner, to elaborate the role of a particular component of \( \text{Hom}(\pi_1(\Sigma), \text{PSL}(2; \mathbb{R}))/\text{PSL}(2; \mathbb{R}) \), namely the Fuchsian representations, in proving the existence of such \( \phi^{-1} \).

### 2.3 Fuchsian representations of the surface group in \( \text{PSL}(2; \mathbb{R}) \)

In [26], Goldman originally investigates the global topology of the space \( \text{Hom}(\pi_1(\Sigma), G)/G \) in the case of \( G = \text{PSL}(2; \mathbb{R}) \) where \( \Sigma \) is a closed orientable surface of genus \( g > 1 \) (no a priori complex structure is assumed in the first place). The results in [26], in fact, depend on the study of certain characteristic classes. For details, you may visit [28], sec. 4, as well. According to the previous observations, to a given representation \( \rho \in \text{Hom}(\pi_1(\Sigma), G)/G \), we assign a flat principal \( G \)-bundle \( P_\rho \to \Sigma \) in a well-established manner. If \( G = \text{PSL}(2, \mathbb{R}) \), since \( \text{PSL}(2, \mathbb{R}) \) acts on \( \mathbb{R}P^1 \cong S^1 \) by orientation-preserving projective transformations, one can also define the associated \( \mathbb{R}P^1 \)-bundle on \( \Sigma \) with the structure group \( \text{PSL}(2, \mathbb{R}) \) as

\[
P_\rho \times \mathbb{R}P^1/\text{PSL}(2, \mathbb{R}) \xrightarrow{\pi} \Sigma,
\]

and hence we have the Euler number \( e \) associated to this \( \mathbb{R}P^1 \)-bundle which induces the map

\[
e : \text{Hom}(\pi_1(\Sigma), \text{PSL}(2; \mathbb{R}))/\sim \to H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.
\]

Goldman’s results related to the connected components are as follows.

**Theorem 2.3.1.** The connected components of \( \text{Hom}(\pi_1(\Sigma), \text{PSL}(2; \mathbb{R}))/\sim \) are the preimages \( e^{-1}(n) \) of the map \( e \) where \( n \in \mathbb{Z} \) such that

\[
|n| \leq |\chi(\Sigma)| = 2g - 2.
\]

Furthermore, it has precisely \( 4g - 3 \) components and the maximal component \( e^{-1}(2g - 2) \) consists of discrete and faithful representations (which can be identified with \( \text{Teich}(\Sigma) \)).
This motivates the following definition [28].

**Definition 2.3.1.** A representation \( \rho \in Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/\sim \) is called **Fuchsian** if it is *discrete and faithful*, i.e. \( \rho \) is injective, its image \( \rho(\pi_1(\Sigma)) \) is a discrete subgroup of \( PSL(2;\mathbb{R}) \) such that the quotient space \( PSL(2,\mathbb{R})/\rho(\pi_1(\Sigma)) \) is compact.

With this definition in hand, from Goldman’s theorem we have the following observation:

**Corollary 2.3.1.** \( \rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R}) \) is Fuchsian \( \iff \) \( e(\rho) = 2g - 2 \) \( \iff \) \( e(T\Sigma) \).

Here, \( e(T\Sigma) \) denotes the Euler number associated to the tangent bundle on \( \Sigma \), which is in fact equal to \(-\chi(\Sigma)\). Note also that for the representations with \( e(\rho) = 2g - 2 \), the corresponding \( \mathbb{R}P^1 \)-bundle is isomorphic to the (unit) tangent bundle \( T\Sigma \) over \( \Sigma \) for which one has \( 2g - 2 = e(T\Sigma) \). Furthermore, we have the following corollary which can be taken as the definition of a Fuchsian representation as well. For more details, we refer to [25, 2, 28].

**Corollary 2.3.2.** \( \rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R}) \) is Fuchsian \( \iff \) It arises from the holonomy of a hyperbolic structure on \( \Sigma \).

**Remark 2.3.1.** As indicated at the beginning of the current section, we do not assume any a priori Riemannian structure on \( \Sigma \). Now, if \( \Sigma \) is a closed Riemann surface of genus \( g > 1 \), then by the Uniformization Theorem, \( \Sigma \) admits a unique hyperbolic structure inherited from the one on the upper half plane \( \mathbb{H} \). Therefore, \( \Sigma \) is locally modeled on \( (Isom(\mathbb{H}),\mathbb{H}) \). That is, \( \Sigma \) is locally isometric to \( \mathbb{H}/\Gamma \) for some discrete subgroup \( \Gamma \subset Isom(\mathbb{H}) \cong PGL(2,\mathbb{R}) \). Therefore, the choice of a hyperbolic structure (which is indeed parametrized by \( Teich(\Sigma) \)) defines automatically a Fuchsian representation \( \rho \) of surface group \( \pi_1(\Sigma) \) in \( PSL(2,\mathbb{R}) \).

### 2.4 Outline of the construction for the desired equivalence

Now, we are in the place of discussing the results stated in Mess’ paper [25] related to both \( (i) \) the existence of \( \phi^{-1} \) (which leads to the desired equivalence of quantum gravity with gauge theory) and \( (ii) \) the construction of the quantum Hilbert space \( \mathcal{H}_E(M) \) associated to the classical phase space \( \mathcal{E}(M) \) of \( 2+1 \) gravity with \( \Lambda = 0 \). The outline is as follows.

1. Mess establishes, by using Thurston theory (and Teichmüller theory), the following relation: For a closed Riemann surface \( \Sigma \) of genus \( g > 1 \),
   \[
   Hom(\pi_1(\Sigma), ISO(2,1))/ISO(2,1) \cong T^*(Hom(\pi_1(\Sigma), PSL(2;\mathbb{R}))/PSL(2;\mathbb{R})).
   \] (2.33)
   As stressed in Remark 2.3.1, there is a one-to-one correspondence between hyperbolic structures on \( \Sigma \) and Fuchsian representation of \( \pi_1\Sigma \), and hence we have
   \[
   Hom(\pi_1(\Sigma), ISO(2,1))/\sim \cong T^*(Teich(\Sigma)).
   \] (2.34)

2. One has the following theorem through which the equivalence of quantum gravity with gauge theory can be established in the sense of Definition 1.0.3.

   **Theorem 2.4.1.** ([25], Prop. 2) \( \rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R}) \) with \( \Sigma \) a closed Riemann surface of genus \( g > 1 \), there exits a flat Lorentzian manifold \( M \) of the form \( \Sigma \times (0,\infty) \) and holonomy \( \psi : \pi_1(\Sigma) \to ISO(2,1) \) such that \( \psi = \rho \).

3. As briefly stressed above, one can alternatively reformulate such an Einstein gravity, especially in dimension \( 2+1 \), as a particular gauge theory, *Chern-Simons* theory with the gauge group being the Poincaré group \( ISO(2,1) \). Hence, in the gauge theoretical interpretation, we can realize the classical physical phase space of Einstein gravity as that of Chern-Simons theory, namely *the moduli space* \( \mathcal{M}_{flat} \) of *flat* \( ISO(2,1) \)-connections on \( M \). Furthermore, it follows directly from the Cartan’s geometric formulation of Einstein-Hilbert action and the analysis of the corresponding EOMs that any flat metric in fact defines a corresponding flat gauge connection. Thus, one has a canonical moduli space of flat connections (not invertible in the first place)
   \[
   \phi : \mathcal{E}(M) \longrightarrow \mathcal{M}_{flat}, \quad g \longmapsto A^\flat.
   \] (2.35)
In that respect, we say that the quantum gravity is equivalent to gauge theory in the sense of the canonical formalism if this canonical map is, in fact, an isomorphism. If \( \mathcal{E}(M) \) denotes the moduli space of \( 2+1 \) dimensional (vacuum) Einstein gravity with the vanishing cosmological constant on a Lorentzian 3-manifold \( M = \Sigma \times (0, \infty) \) where \( \Sigma \) is a closed Riemann surface of genus \( g > 1 \), then from Theorem 2.4.1, the map

\[
\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{\text{flat}}
\]  

is an isomorphism where

\[
\mathcal{M}_{\text{flat}} \cong \text{Hom}(\pi_1(\Sigma), \text{ISO}(2, 1)) / \sim \cong T^\ast(\text{Teich}(\Sigma)).
\]  

Note that as indicated in Remark 2.3.1, the choice of a hyperbolic structure on \( \Sigma \) gives rise to a certain Fuchsian representation \( \rho \) of surface group \( \pi_1(\Sigma) \) in \( \text{PSL}(2, \mathbb{R}) \). Hence, Theorem 2.4.1 applies once \( \Sigma \) is endowed with a Riemann surface structure.

4. These observations together with the equivalence of quantum gravity with gauge theory implies that quantization of \( 2+1 \) gravity in the case of \( \Lambda = 0 \) and \( M = \Sigma \times (0, \infty) \) as above boils down to the canonical quantization of the cotangent bundle \( T^\ast(\text{Teich}(\Sigma)) \) for which the associated quantum Hilbert space \( \mathcal{H}_{\mathcal{E}(M)} \) is defined as

\[
\mathcal{H}_{\mathcal{E}(M)} = \mathcal{L}^2(\text{Teich}(\Sigma)).
\]  

More details and further discussions can be found in section 3.1 of [2].

3 Pre-stacky formulation for Einstein Gravity

Before discussing the stacky formulation of \( \mathcal{E}(M) \), we first recall, in a very brief and expository fashion, the notion of a moduli problem, and try to explain why one requires to employ the stacky refinement of the theory. A moduli problem is a problem of constructing a classifying space (or a moduli space \( \mathcal{M} \)) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) and the families of objects up to their intrinsic symmetries. In other words, a moduli space serves as a solution space of a given moduli problem of interest. In general, the set of isomorphism classes of objects that we would like to classify is not able to provide sufficient information to encode the geometric properties of the moduli space itself. Therefore, we expect a moduli space to behave well enough to capture the underlying geometry. Thus, this expectation leads the following wish-list for a "fine" moduli space \( \mathcal{M} \):

1. \( \mathcal{M} \) is supposed to serve as a parameter space in a sense that there must be a one-to-one correspondence between the points of \( \mathcal{M} \) and the set of isomorphism classes of objects to be classified:

\[
\{\text{points of } \mathcal{M}\} \leftrightarrow \{\text{isomorphism classes of objects in } \mathcal{C}\}
\]  

2. The existence of universal classifying object, say \( T \), through which all other objects parametrized by \( \mathcal{M} \) can also be reconstructed. This, in fact, makes the moduli space \( \mathcal{M} \) even more sensitive to the behavior of "families" of objects on any base object \( B \) manifested by a certain representative morphism \( B \rightarrow \mathcal{M} \). That is, for any family

\[
\pi : X := \{X_b \in \text{Ob}(\mathcal{C}) : \pi^{-1}(b) = X_b, \ b \in B\} \rightarrow B
\]  

parametrized by some base scheme \( B \), there exits a unique morphism \( f : B \rightarrow \mathcal{M} \) such that one has the following fibered product diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & T \\
\downarrow \pi & & \downarrow \\
B & \xrightarrow{f} & \mathcal{M}
\end{array}
\]  

where \( X = B \times_\mathcal{M} T \). That is, the family \( X \) can be uniquely obtained by pulling back the universal object \( \tau \) along the morphism \( f \). For an accessible overview, see [14], and rather complete treatments can be found in [56] and [21].
In the language of category theory, on the other hand, a moduli problem can be formalized as a certain functor

\[ \mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Sets} \quad (3.4) \]

which is called the moduli functor where \( \mathcal{C}^{\text{op}} \) is the opposite category of the category \( \mathcal{C} \) and \( \text{Sets} \) is the category of sets. In order to make the argument more transparent, we take \( \mathcal{C} \) to be the category \( \text{Sch} \) of \( k \)-schemes. Note that for each scheme \( U \in \text{Sch} \), \( \mathcal{F}(U) \) is the set of isomorphism classes parameterized by \( U \), and for each morphism \( f : U \to V \) of schemes, we have a morphism \( \mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U) \) of sets. Together with the above formalism, the existence of a fine moduli space corresponds to the representability of the moduli functor \( \mathcal{F} \) in the sense that

\[ \mathcal{F} = \text{Hom}_{\text{Sch}}(\cdot, \mathcal{M}) \quad \text{for some } \mathcal{M} \in \text{Sch}. \quad (3.5) \]

If this is the case, then we say that \( \mathcal{F} \) is represented by \( \mathcal{M} \).

In many cases, however, the moduli functor is not representable in the category \( \text{Sch} \) of schemes. This is essentially where the notion of stack comes into play. The notion of stack can be thought of as a first instance such that the ordinary notion of category no longer suffices to define such an object. To make sense of this new object in a well-established manner and enjoy the richness of this new structure, we need to introduce a higher categorical notion, namely a 2-category [21, 22]. The theory of stacks, therefore, employs higher categorical techniques and notions in a way that provides a mathematical treatment for the representability problem by re-defining the moduli functor as a stack, a particular groupoid-valued pseudo-functor with local-to-global properties,

\[ \mathcal{X} : \mathcal{C}^{\text{op}} \to \text{Grpds} \quad (3.6) \]

where \( \text{Grpds} \) denotes the 2-category of groupoids with objects being categories \( \mathcal{C} \) in which all morphisms are isomorphisms (these sorts of categories are called groupoids), 1-morphisms being functors \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) between groupoids, and 2-morphisms being a natural transformations \( \psi : \mathcal{F} \Rightarrow \mathcal{F}' \) between two functors.

**Remark 3.0.1.** In order to make sense of local-to-global (or "glueing") type arguments, one requires to introduce an appropriate notion of topology on a category \( \mathcal{C} \). Such a structure is manifestly given in [21] and called a Grothendieck topology \( \tau \). Furthermore, a category \( \mathcal{C} \) equipped with a Grothendieck topology \( \tau \) is called a site. Note that if we have a site \( \mathcal{C} \), then we can define a sheaf on \( \mathcal{C} \) in a well-established manner as well.

**Definition 3.0.1.** Let \( \mathcal{C} \) be a category in which all products exist. A Grothendieck topology \( \tau \) on \( \mathcal{C} \) consists of the following data.

1. For each object \( U \) in \( \mathcal{C} \), a collection of families \( \{U_i \xrightarrow{f_i} U\} \) of morphisms in \( \mathcal{C} \), denoted by \( \tau(U) \).
2. If \( V \xrightarrow{f} U \) is an isomorphism, then \( \{V \xrightarrow{f_i} U\} \in \tau(U) \).
3. If the family \( \{U_i \xrightarrow{f_i} U\} \in \tau(U) \) and for each \( i \in I \) one has a family \( \{U_{ij} \xrightarrow{f_{ij}} U_i\} \in \tau(U_i) \), then \( \{U_{ij} \xrightarrow{f_{ij} \circ f_{ij}} U\} \in \tau(U) \). \( (3.7) \)
4. Given a family \( \{U_i \xrightarrow{f_i} U\} \in \tau(U) \) and a morphism \( V \to U \) with the base change diagram

\[
\begin{array}{ccc}
V \times_U U_i & \longrightarrow & U_i \\
\downarrow & & \downarrow \\
V & \underset{f}{\longrightarrow} & U
\end{array}
\]

then \( \{V \times_U U_i \to V\} \) is in \( \tau(V) \).

Here, the families \( \{U_i \to U\} \in \tau(U) \) are called covering families for \( U \) in the Grothendieck topology \( \tau(U) \).
A motivating example. In the case of \( \mathcal{C} = \text{Top} \), for any topological space \( X \), the Grothendieck topology \( \tau(X) \) corresponds to the usual notion of open coverings \( \{ U_i \subseteq X \}_i \) of \( X \) with the maps \( \varphi_i \) being the usual inclusions (or open embeddings) such that

\[
X \subseteq \bigcup_i U_i. \tag{3.9}
\]

In that case, moreover, the fibered product \( U_{ij} := U_i \times_X U_j \) in fact corresponds to the intersection of open subsets \( U_i \) and \( U_j \) in \( X \).

**Definition 3.0.2.** [39] Given two covariant functor between categories \( F, G : \mathcal{A} \rightarrow \mathcal{B} \), a natural transformation is the data of morphisms

\[
m_A : F(A) \rightarrow G(A) \quad \text{for all objects } A \in \mathcal{A} \tag{3.10}
\]

such that for each morphism \( A \xrightarrow{f} A' \) in \( \mathcal{A} \), the following diagram commutes:

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
m_A \downarrow & & \downarrow m_{A'} \\
G(A) & \xrightarrow{G(f)} & G(A')
\end{array}
\] \tag{3.11}

Note that for a contravariant functor, one has the same definition with arrows \( F(f) \) and \( G(f) \) being reversed.

**Definition 3.0.3.** [21] Let \( \mathcal{C} \) be a category. A prestack \( \mathcal{X} : \mathcal{C}^{\text{op}} \rightarrow \text{Grpds} \) consists of the following data.

1. For any object \( U \) in \( \mathcal{C} \), an object \( \mathcal{X}(U) \) in \( \text{Grpds} \). That is,

\[
U \mapsto \mathcal{X}(U) \tag{3.12}
\]

where \( \mathcal{X}(U) \) is a groupoid, i.e. a category in which all morphisms are isomorphisms.

2. For each morphism \( U \xrightarrow{f} V \) in \( \mathcal{C} \), a contravariant functor of groupoids

\[
\mathcal{X}(V) \xrightarrow{\mathcal{X}(f)} \mathcal{X}(U). \tag{3.13}
\]

Note that \( \mathcal{X}(f) \) is indeed a functor of categories, and hence one requires to provide an action of \( \mathcal{X}(f) \) on objects and morphisms of \( \mathcal{X}(V) \) in a compatible fashion in the sense of [39], Chapter 1.

3. Given a composition of morphisms \( U \xrightarrow{f} V \xrightarrow{g} W \) in \( \mathcal{C} \), there is an invertible natural transformation between two functors

\[
\phi_{g,f} : \mathcal{X}(g \circ f) \Rightarrow \mathcal{X}(f) \circ \mathcal{X}(g) \tag{3.14}
\]

such that the following diagram commutes (encoding the associativity).

\[
\begin{array}{ccc}
\mathcal{X}(h \circ g \circ f) & \xrightarrow{\phi_{h,g,f}} & \mathcal{X}(g \circ f) \circ \mathcal{X}(h) \\
\phi_{h\circ g,f} \downarrow & & \downarrow \phi_{g,f} \circ \mathcal{X}(h) \\
\mathcal{X}(f) \circ \mathcal{X}(h \circ g) & \xrightarrow{id_{\mathcal{X}}(f) \circ \phi_{h,g}} & \mathcal{X}(f) \circ \mathcal{X}(g) \circ \mathcal{X}(h)
\end{array}
\] \tag{3.15}

Due to the Condition 3., the prestack \( \mathcal{X} \) is indeed an object in \( PFunc(\mathcal{C}, \text{Grpds}) \). It is just a pseudo-functor (See [21], ch. 2 for the general definition). Or, equivalently it is also called a presheaf of groupoids.
Proposition 3.0.1. Given a Lorentzian n-manifold $M$, let $\mathcal{C}$ be the category of open subsets of $M$ with morphisms being canonical inclusions between open subsets whenever $U \subset V$. Then, the following assignment
\[ \mathcal{E} : \text{C}^{\text{op}} \rightarrow \text{Grpds} \] (3.16)
defines a presheaf where

1. For each object $U$ of $\mathcal{C}$, $\mathcal{E}(U)$ is a groupoid of Ricci-flat pseudo-Riemannian metrics on $U$ where objects form the set
\[ \text{FMet}(U) := \{ g \in \Gamma(U, \text{Met}_M) : R_{\mu\nu} = 0 \}. \] (3.17)
Here $\text{Met}_M$ denotes metric "bundle" on $M$, and the set of morphisms is defined by
\[ \text{Hom}_{\mathcal{E}(U)}(g, g') := \{ \varphi \in \text{Isom}(U) : g' = \varphi^* g \} \] (3.18)
where $\text{Isom}(U) \subset \text{Diff}(U)$ denotes the group of isometries of $U$ and it acts on $\text{FMet}(U)$ by pullback. Furthermore, we denote this action by $g \cdot \varphi := \varphi^* g$ and a morphism $g \sim \to g' = \varphi^* g$ in $\text{Hom}_{\mathcal{E}(U)}(g, g')$ by $(g, \varphi)$. The composition of two morphisms is given as
\[ (g \cdot \varphi) \cdot \psi = g \cdot (\varphi \circ \psi) \] (3.19)
where $(g \cdot \varphi) \cdot \psi = \psi^* \varphi^* g = (\varphi \circ \psi)^* g = g \cdot (\varphi \circ \psi)$ for any $\varphi, \psi \in \text{Isom}(U)$.

2. To each morphism $U \xrightarrow{f} V$ in $\mathcal{C}$, i.e., $f : U \leftrightarrow V$ with $U \subset V$, one assigns
\[ \mathcal{E}(V) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(U). \] (3.20)
Here $\mathcal{E}(f) \in \text{Fun}(\mathcal{E}(V), \mathcal{E}(U))$ is a functor of categories whose action on objects and morphisms of $\mathcal{E}(V)$ is given as follows:

(a) For any object $g \in \text{Ob}(\mathcal{E}(V)) = \text{FMet}(V)$,
\[ g \xrightarrow{\mathcal{E}(f)} f^* g \] (3.21)
where
\[ \mathcal{E}(f)(g) := f^* g \in \text{FMet}(U). \] (3.22)
Notice that the pullback of a Ricci-flat metric, in general, may no longer be Ricci-flat. But, in the case of particular canonical inclusions $f : U \leftrightarrow V$ with $U, V$ open subsets, if a metric $g$ is Ricci-flat on $V$, so is $f^* g$ on $U$. This is because $f^* g$ is just the restriction $g|_U$ of $g$ to the open subset $U$.

(b) For any morphism $(g, \varphi) \in \text{Hom}_{\mathcal{E}(V)}(g, g')$ with $\varphi \in \text{Isom}(V)$ such that $g' = \varphi^* g$, when restricted to $U$, both $g$ and $g'$ does still lie in the orbit space of $g$ with $\varphi \circ f := f^* \varphi$ being just the restriction of $g$ to the smaller open subset $U$ in $V$, and hence we set
\[ \left( g \xrightarrow{\sim} (g, \varphi) \xrightarrow{\mathcal{E}(f)} f^* g \right) = \left( g|_U \xrightarrow{(f^* g, \varphi)} \varphi^* g|_U = f^* \varphi^* g = (\varphi \circ f)^* g \right) \] (3.23)
where
\[ \mathcal{E}(f)(g, \varphi) := (f^* g, \varphi \circ f) = (g|_U, \varphi|_U) \] (3.24)
is a morphism in $\mathcal{E}(U)$ as $\varphi|_U$ gives an isometry of $U$. By using $f^* \varphi := \varphi \circ f = \varphi|_U$, we indeed have
\[ (g, \varphi) \xrightarrow{\mathcal{E}(f)} (f^* g, f^* \varphi) \in \text{Hom}_{\mathcal{E}(U)}(f^* g, f^* g'). \] (3.25)

3. Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$, there exists an invertible natural transformation (arising naturally from properties of pulling-back)
\[ \phi_{h \circ f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h) \] (3.26)
together with the compatibility condition 3.14.
\textit{Proof}. Besides the construction instructed in (1) and (2), we need to show that

\begin{itemize}
  \item[(i)] Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$, that is
    \begin{equation}
    \begin{tikzcd}
    \text{U} \arrow{r}{f} & \text{V} \arrow{r}{h} & \text{W},
    \end{tikzcd}
    \tag{3.27}
    \end{equation}
there is an invertible natural transformation $\psi_{h,f}: E(h \circ f) \Rightarrow E(f) \circ E(h)$ given schematically as
    \begin{equation}
    \begin{tikzcd}
    E(W) \arrow[r, bend right=15, swap, \psi_{h,f}] & E(f) \circ E(h)
    \end{tikzcd}
    \tag{3.28}
    \end{equation}

  \item[(ii)] Given a composition diagram of morphisms $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ in $\mathcal{C}$, the associativity condition holds in the sense that the following diagram commutes:
    \begin{equation}
    \begin{tikzcd}
    E(p \circ h \circ f) \arrow[rr, bend right=15, swap, \psi_{p,h,f}] & & E(h \circ f) \circ E(p) \arrow[ll, bend left=15, \psi_{h,f} \circ id_{E(p)}] \\
    E(f) \circ E(p \circ h) \arrow[rr, bend left=15, \psi_{p,h} \circ \psi_{f,h} \circ id_{E(f)}] & & E(f) \circ E(h) \circ E(p)
    \end{tikzcd}
    \tag{3.29}
    \end{equation}
\end{itemize}

\textit{Proof of (i)}: First, we need to analyze objectwise: for any object $g \in \text{Ob}(E(W)) = \text{FMet}(W)$, we have the following \textit{strong condition} by which the rest of the proof will become rather straightforward:
    \begin{equation}
    E(h \circ f)(g) = (h \circ f)^* g = f^* h^* g = (E(f) \circ E(h))(g) \in \text{FMet}(U).
    \tag{3.30}
    \end{equation}
As we have identical metrics $E(h \circ f)(g) = E(f) \circ E(h)(g)$ for any $g \in \text{FMet}(W)$, there is, by construction, a unique identity map
    \begin{equation}
    (E(h \circ f)(g), id_U) \in \text{Hom}_{\text{FMet}(U)}(E(h \circ f)(g), E(f) \circ E(h)(g))
    \tag{3.31}
    \end{equation}
such that
    \begin{equation}
    E(h \circ f)(g) \xrightarrow{\sim} E(f) \circ E(h)(g) = id_U^* (E(h \circ f)(g)) = E(h \circ f)(g).
    \tag{3.32}
    \end{equation}
Thus, one has the natural choice of collection of morphisms $\{m_g : E(h \circ f)(g) \longrightarrow E(f) \circ E(h)(g)\}$ defined as
    \begin{equation}
    m_g = (E(h \circ f)(g), id_U) \quad \text{for all } g \in \text{FMet}(W),
    \tag{3.33}
    \end{equation}
for which the following diagram commutes: Just for the sake of notational simplicity, we let
    \begin{equation}
    \mathcal{F} := E(h \circ f) \quad \text{and} \quad \mathcal{G} := E(f) \circ E(h),
    \tag{3.34}
    \end{equation}
then for each morphism $g \xrightarrow{(g, \phi)} g'$ in $E(W)$, we have
    \begin{equation}
    \begin{tikzcd}
    \mathcal{F}(g) \arrow{r}{\mathcal{F}((g, \phi))} & \mathcal{F}(g') \\
    \mathcal{G}(g) \arrow{r}{\mathcal{G}((g, \phi))} & \mathcal{G}(g')
    \end{tikzcd}
    \tag{3.35}
    \end{equation}
In fact, the commutativity follows from the following observation (thanks to the strong condition 3.30 we obtained above): By using the definition of action of the functor $\mathcal{E}(f)$ on $\mathcal{E}(W)$, we get

$$\mathcal{F}((g, \phi)) = \mathcal{E}(h \circ f)((g, \phi))$$

$$= ((h \circ f)^* g, (h \circ f)^* \phi)$$

$$= (f^* \circ h^* g, f^* \circ h^* \phi)$$

$$= (\mathcal{E}(f) \circ \mathcal{E}(h)(g), f^* \circ h^* \phi)$$

$$= \mathcal{E}(f) \circ \mathcal{E}(h)((g, \phi))$$

$$= \mathcal{G}((g, \phi)).$$

(3.36)

which implies the commutativity of the diagram. Furthermore, it is clear from the strong condition 3.30, and hence the construction that $\psi_{h,f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h)$ is in fact invertible. In other words, we have $\mathcal{E}(h \circ f) \cong \mathcal{E}(f) \circ \mathcal{E}(h)$ up to an invertible natural transformation. In the language of 2-categories, on the other hand, these $\psi_{h,f}$’s are called 2-isomorphims.

Proof of (ii): Now, associativity follows directly from the following observations: If $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$ is a commuting diagram, then what we have shown so far are as follows.

1. $\mathcal{F}(g) = \mathcal{G}(g)$ for any $g \in \text{Ob}(\mathcal{E}(W)) = \text{FMet}(W)$

(3.37)

2. $\mathcal{F}((g, \phi)) = \mathcal{G}((g, \phi))$ for any $g \xrightarrow{(g, \phi)} g'$ in $\mathcal{E}(W)$.

(3.38)

where $\mathcal{F} := \mathcal{E}(h \circ f)$ and $\mathcal{G} := \mathcal{E}(f) \circ \mathcal{E}(h)$. Now, let $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ be a commutative diagram in $\mathcal{C}$, then it suffices to show that associativity condition in the sense introduced above holds both objectwise and morphismwise in a compatible manner:

- Let $g \in \text{Ob}(\mathcal{E}(Z)) = \text{FMet}(Z)$, then we have

$$\mathcal{E}(p \circ (h \circ f))(g) = \mathcal{E}(h \circ f) \circ \mathcal{E}(p)(g) \quad \text{from } 3.37 \text{ with } \psi_{p,hof}$$

(3.39)

$$= \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p)(g) \quad \text{from } 3.37 \text{ with } \psi_{h,f} \circ \text{id}_{\mathcal{E}(p)}$$

(3.40)

$$= \mathcal{E}(f) \circ \mathcal{E}(p \circ h)(g) \quad \text{from } 3.37 \text{ with } \text{id}_{\mathcal{E}(f)} \circ \psi_{p,h}$$

(3.41)

$$= \mathcal{E}((p \circ h) \circ f)(g), \quad \text{from } 3.37 \text{ with } \psi_{poh,f}$$

(3.42)

which gives the commutativity of the diagram objectwise.

- Let $g \xrightarrow{(g, \phi)} g'$ in $\mathcal{E}(Z)$, then we have

$$\mathcal{E}(p \circ (h \circ f))((g, \phi)) = \mathcal{E}(h \circ f) \circ \mathcal{E}(p)((g, \phi)) \quad \text{from } 3.38 \text{ with } \psi_{p,hof}$$

(3.43)

$$= \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p)((g, \phi)) \quad \text{from } 3.38 \text{ with } \psi_{h,f} \circ \text{id}_{\mathcal{E}(p)}$$

(3.44)

$$= \mathcal{E}(f) \circ \mathcal{E}(p \circ h)((g, \phi)) \quad \text{from } 3.38 \text{ with } \text{id}_{\mathcal{E}(f)} \circ \psi_{p,h}$$

(3.45)

$$= \mathcal{E}((p \circ h) \circ f)((g, \phi)), \quad \text{from } 3.38 \text{ with } \psi_{poh,f}$$

(3.46)

which completes the proof.

\[\square\]
4 Towards the stacky formulation

We will study the stacky nature of the prestack $E$ in Proposition 3.0.1 in the language of homotopy theory as discussed in [3, 4]. This homotopy theoretical approach is essentially based on the model structure [40] on the (2-) category $Grpds$ of groupoids and the category $Psh(C, Grpds)$ of presheaves in groupoids. Furthermore, one also requires to adopt certain simplicial techniques and some practical results from [3, 4] in order to establish the notion of a stack in the language of homotopy theory. For an introduction to simplicial techniques, see [10], Appendix A.

4.1 A digression on (co-)simplicial objects

The following discussion is based on [10, 22]. Let $\Delta$ denote the category of finite ordered sets with objects being finite ordered sets

$$[n] := \{0 < 1 < 2 < \cdots < n\}$$

(4.1)

together with the morphisms $f : [n] \to [m]$ being non-decreasing functions. Note that the set $[n]$ corresponds to $\Delta^n$, the usual $n$-simplex in $\mathbb{R}^{n+1}$, given as a set

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \text{ and } 0 \leq x_k \leq 1 \text{ for all } k\},$$

(4.2)

and hence the map $f : [n] \to [m]$ induces a linear map

$$f_* : \Delta^n \to \Delta^m, \quad e_k \mapsto e_{f(k)}.$$ (4.3)

As addressed in [10], each map $f : [n] \to [m]$ can be factored into a surjection followed by an injection such that

1. Any injection can also be factored into a sequence $\{d^n_i\}$ of coface maps where $d^n_i : [n-1] \to [n]$ is a map of simplices given as

$$d^n_i(j) = \begin{cases} j & \text{if } j \leq i \\ j+1 & \text{else} \end{cases}$$ (4.4)

2. Each surjection can also be factored into a sequence $\{s^n_i\}$ of codegeneracy maps where $s^n_i : [n+1] \to [n]$ is a map of simplices given as

$$s^n_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{else} \end{cases}$$ (4.5)

Remark 4.1.1. Geometrically speaking, the coface map $d^n_i$ in fact "injects" the $i^{th}$ $(n-1)$-simplex into an $n$-simplex depicted, for instance, as follows:

\[\begin{array}{c}
\Delta^1 : [v_0, v_1] \\
\downarrow d^2_1 \\
\Delta^2 : [w_0, w_1, w_2]
\end{array}\]

(4.6)

where $w_0 = (d^2_0)_*(v_0)$ and $w_2 = (d^2_0)_*(v_1)$. Codegeneracy map $s^n_i$, on the other hand, does collapse an edge $[i, i+1]$ of an $(n+1)$-simplex, and hence it projects an $(n+1)$-simplex onto an $n$-simplex:

\[\begin{array}{c}
w_2 \\
\downarrow s^1_1 \\
w_0 \quad \text{collapsed} \quad w_1
\end{array}\]

(4.7)

where $v_0 = (s^1_1)_*(w_0)$ and $v_1 = (s^1_1)_*(w_1) = (s^1_1)_*(w_2)$. 18
**Definition 4.1.1.** Let $\mathcal{C}$ be a category. A (co-)simplicial object in $\mathcal{C}$ is a (covariant) contravariant functor

$$X_{\bullet} : \Delta^{op} \to \mathcal{C}. \tag{4.8}$$

If $\mathcal{C} = \text{Sets}$, then $X_{\bullet} \in \text{Fun}(\Delta, \text{Sets})$ is called a simplicial set, and the image $X_{\bullet}([n])$ of $[n]$ is called the set of $n$-simplices and is denoted by $X_n$. That is, we have

$$[n] \xrightarrow{X_{\bullet}} X_{\bullet}([n]) =: X_n, \tag{4.9}$$

and for each morphism $f : [n] \to [m]$, one has

$$X_m \xrightarrow{X_{\bullet}(f)} X_n. \tag{4.10}$$

**Lemma 4.1.1.** \cite{22} Any morphism $f \in \text{Hom}_{\Delta}([n], [m])$ can be written as a composition of the coface $d^n_i$ and codegeneracy $s^n_i$ maps such that the following relations hold:

1. $d_{j}^{n+1} \circ d_{i}^{n} = d_{i}^{n+1} \circ d_{j}^{n-1} \quad \text{if} \quad 0 \leq i < j \leq n + 1. \tag{4.11}$
2. $s_{i}^{n-1} \circ d_{i}^{n} = d_{i}^{n-1} \circ s_{i}^{n-2} \quad \text{if} \quad 0 \leq i < j \leq n - 1. \tag{4.12}$
3. $s_{j}^{n-1} \circ d_{j}^{n} = s_{j}^{n-1} \circ d_{j+1}^{n} = id_{[n-1]} \quad \text{if} \quad 0 \leq j \leq n - 1. \tag{4.13}$
4. $s_{i}^{n-1} \circ d_{i}^{n} = d_{i}^{n-1} \circ s_{i}^{n-2} \quad \text{if} \quad n \leq i > j + 1 > 0. \tag{4.14}$
5. $s_{i}^{n-1} \circ s_{j}^{n} = s_{i}^{n-1} \circ s_{j+1}^{n} \quad \text{if} \quad 0 \leq i < j \leq n - 1. \tag{4.15}$

Furthermore, given a cosimplicial object $X_{\bullet}$ in $\mathcal{C}$, a covariant functor from $\Delta$ to $\mathcal{C}$, for any object $[n]$ in $\Delta$, one has a sequence of objects in $\mathcal{C}$

$$X_n := X_{\bullet}([n]) \tag{4.16}$$

together with the morphisms

$$X_{\bullet}(d^n_i) : X_{n-1} \to X_n \quad \text{and} \quad X_{\bullet}(s^n_i) : X_{n+1} \to X_n$$

such that by abusing the notation and using $d^n_i$ and $s^n_i$ in places of the maps $X_{\bullet}(d^n_i)$ and $X_{\bullet}(s^n_i)$ respectively, Lemma 4.1.1 relating $d^n_i$ and $s^n_i$’s is also viable, and hence we can introduce the following diagram, namely the cosimplicial diagram in $\mathcal{C}$, which encodes the structure of cosimplicial object $X_{\bullet}$ in terms of its simplices $X_n$ along with the corresponding coface and codegeneracy maps:

$$X_{\bullet} = \left( X_0 \xrightarrow{d_0^1} X_1 \xrightarrow{s_0^1} X_0 \xrightarrow{d_1^2} X_2 \cdots \right). \tag{4.17}$$

To simplify the notation, in general, we omit the codegeneracy maps and write the cosimplicial diagram in $\mathcal{C}$ in a rather compact way:

$$X_{\bullet} = \left( X_0 \xrightarrow{d_0^1} X_1 \xrightarrow{d_1^2} X_2 \cdots \right). \tag{4.18}$$

Note that the cosimplicial diagram for $X_{\bullet}$ can also have a geometric interpretation in terms of the usual simplices as follows: In order to make the geometric realization more transparent we assume $\mathcal{C} := \text{Sets}$, and hence consider $X_{\bullet}$ as a cosimplicial object in $\text{Sets}$. Let $x$ be an object in $X_0$, $h : d_1^0(x) \to d_0^0(x)$ a morphism in $X_1$. Then $x$ and $h$ can be represented as 0- and 1-simplices in $X_{\bullet}$ respectively such that, by using the properties of $d^n_i$ and $s^n_i$, we pictorially have

$$X_{\bullet} \xrightarrow{d_1^1} \xrightarrow{d_0^0} \quad \text{1-simplex } \Delta^1$$

and

$$X_{\bullet} \xrightarrow{d_2^1} \xrightarrow{d_1^1} \xrightarrow{d_0^0} \quad \text{2-simplex } \Delta^2 \tag{4.19}$$
Remark 4.1.2. Note that the existence of such a geometric interpretation above requires the following algebraic conditions.

\[(a) \quad s_0^0(h) = id_x, \quad \text{and} \quad (b) \quad d_0^2 \circ d_1^2(h) = d_1^2(h).\]  

Indeed, condition (a) follows from the property 4.13 in the Lemma 4.1.1, namely

\[s_0^0 \circ d_1^1(x) = s_0^0 \circ d_0^1(x) = id_0(x) = x.\]  

and hence \(s_0^0(h) : s_0^0 \circ d_1^1(x) \rightarrow s_0^0 \circ d_0^1(x)\) is just the morphism \(id_x : x \rightarrow x\). Furthermore, condition (b) corresponds to the commutativity of the following diagram together with the property 4.11 in the Lemma 4.1.1:

\[
\begin{array}{ccc}
  d_2^2 \circ d_1^1(x) & \xrightarrow{d_2^2(h)} & d_2^3 \circ d_1^3(x) \\
  \downarrow \text{" = " by 4.11} & & \downarrow \text{" = " by 4.11} \\
  d_1^1 \circ d_1^1(x) & \xrightarrow{d_1^1(h)} & d_1^3 \circ d_1^3(x)
\end{array}
\]

4.2 Main ingredients of model categories

The notion of a model structure, which was originally defined by Quillen [46], serves as a particular mathematical treatment for abstracting homotopy theory in a way that one can localize the given category \(\mathcal{C}\) by formally inverting a special class of morphisms, namely the weak equivalences, such that this extra structure formally encodes the localization procedure. In that respect, naïvely speaking, a model structure consists of three distinguished classes of morphisms, namely weak equivalences \(W\), fibrations \(F\) and cofibrations \(CF\) along with certain axioms and compatibility conditions. This structure eventually leads to localization \(W^{-1}\mathcal{C}\) of the given category \(\mathcal{C}\), also called the homotopy category \(Ho(\mathcal{C})\) of \(\mathcal{C}\). We manifestly follow the treatment of the subject as discussed in [40].

Given a category \(\mathcal{C}\), denote by \(\text{Map}(\mathcal{C})\) the category of morphisms of \(\mathcal{C}\) with objects being morphisms \(f\) in \(\mathcal{C}\), and morphisms between \(f\) and \(g\) being a pair

\[(\phi_f, \phi_g) \in \text{Ob}(\text{Map}(\mathcal{C})) \times \text{Ob}(\text{Map}(\mathcal{C}))\]  

such that the following diagram commutes: for any two morphism \(f : A \rightarrow B\) and \(g : C \rightarrow D\) in \(\mathcal{C}\), we have

\[
\begin{array}{ccc}
  A & \xrightarrow{\phi_f} & C \\
  f \downarrow & & \downarrow g \\
  B & \xrightarrow{\phi_g} & D
\end{array}
\]

Definition 4.2.1. Let \(\mathcal{C}\) be a category.

1. A morphism \(f : A \rightarrow B\) in \(\mathcal{C}\) (or an object \(f \in \text{Ob}(\text{Map}(\mathcal{C}))\)) is called a retract of a morphism \(g : C \rightarrow D\) in \(\mathcal{C}\) if there exists a retraction of objects in the sense that one has retractions \(r : C \rightarrow A\) and \(r' : D \rightarrow B\) such that the following diagram commutes:

\[
\begin{array}{ccc}
  A & \xrightarrow{\ i \ } & C & \xrightarrow{\ r \ } & A \\
  f \downarrow & & \downarrow g & & \downarrow f \\
  B & \xrightarrow{\ i' \ } & D & \xrightarrow{\ r' \ } & B
\end{array}
\]

where \(r \circ i = id_A\) and \(r' \circ i' = id_B\) are retractions of objects.
2. **A functorial factorization** is an ordered pair \((F, G)\) of functors

\[
F, G : \text{Map}(C) \to \text{Map}(C)
\]  

such that for any morphism \(f\) in \(C\), we have

\[
f = G(f) \circ F(f).
\]

3. Let \(i \in \text{Hom}_C(A, B)\) and \(p \in \text{Hom}_C(C, D)\). We say that \(i\) has the **left lifting property** with respect to \(p\), and \(p\) has the **right lifting property** with respect to \(i\) if for any commutative diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{g} & D
\end{array}
\]

(4.28)

there is a lift \(h : B \to C\) commuting the following diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{g} & D
\end{array}
\]

(4.29)

Note that in the language of standard homotopy theory, it is equivalently said that the map \(p : C \to D\) has the homotopy lifting property. It is then called a *fibration*. Similarly, we say that the map \(i : A \to B\) has the homotopy extension property. Such map \(i\), on the other hand, is called a *cofibration*. Inspired by these notions naturally emerging in standard homotopy theory (on the category \(\text{Top}\) of topological spaces \([24]\)), we have the following abstraction which allows us to make sense of homotopy theory on an arbitrary category in a rather axiomatic way \([40]\).

**Definition 4.2.2.** A model structure on a category \(C\) (in which both initial and final objects exist) consists of three subcategories of \(\text{Map}(C)\), so-called weak equivalences \(W\), fibrations \(F\) and cofibrations \(C\), and two functorial factorizations \((\alpha, \beta)\) and \((\gamma, \delta)\) along with certain axioms given as

(i) (2-out-3 axiom of weak equivalences) If \(f\) and \(g\) are morphisms in \(C\) such that \(g \circ f\) is defined and any two morphisms in the set \(\{f, g, g \circ f\}\) are weak equivalences, then so is the third.

(ii) If \(f\) and \(g\) are morphisms in \(C\) such that \(f\) is a retract of \(g\) and \(g\) is a weak equivalence (cofibration or fibration respectively), so is \(f\).

(iii) \(f \in \text{Map}(C)\) is called a **trivial cofibration** (or fibration respectively) if \(f \in C \cap W\) (or \(f \in F \cap W\) resp.) Then we have

- \(i \in C \cap W\) has the left lifting property with respect to \(p \in F\). That is, each trivial cofibration has the left lifting property w.r.t fibrations.
- \(i \in C \cap W\) has the left lifting property with respect to \(p \in F \cap W\). That is, each cofibration has the left lifting property w.r.t trivial fibrations.

(iv) (The existence of weak factorization system) For all \(f \in \text{Map}(C)\), we have

\[
(\alpha, \beta) \in C \times (F \cap W) \quad \text{and} \quad (\gamma, \delta) \in (C \cap W) \times C.
\]

(4.30)

Then, a category \(C\) (in which both initial and final objects exist) is called a **model category** if it admits a model structure.
As we stressed before, we intend to introduce a (localization) functor

\[ C \to W^{-1}C \]  \hspace{1cm} (4.31)

such that all elements of \( W \) become invertible in \( W^{-1}C \), called the homotopy category of \( C \) and denoted by \( \text{Ho}(C) \). The complete treatment will not be given, but instead we shall introduce a prototype example which in fact captures the essence of the item. For a concrete construction, we refer to [22, 40].

**Example 4.2.1.** [24] Let \( \text{Top} \) be the category of topological spaces with morphisms being continuous functions \( f : X \to Y \) between topological spaces. Then, it admits a model structure where we set

\[ W := \{ f : X \to Y : \pi_i X \xrightarrow{f} \pi_i Y \text{ is an isomorphism for all } i \} \]  \hspace{1cm} (4.32)

\[ \mathcal{F} := \{ f : X \to Y : f \text{ has homotopy lifting property w.r.t } \mathbb{D}^n \} = \{ \text{Serre's fibrations} \} \]  \hspace{1cm} (4.33)

Hence, in the case of compactly generated topological spaces, we have

\[ \text{Ho}(\text{Top}) \simeq CW \]  \hspace{1cm} (4.34)

where \( CW \) denotes the category of CW-complexes. For more examples, see [24].

Similarly, the (2-)category \( \text{Grpds} \) of groupoids will have a model structure for which the weak equivalences are set to be the equivalences of groupoids, namely fully faithfully essentially surjective functors between groupoids. Therefore, the next task will be to elaborate the model structures on \( \text{Grpds} \) and \( \text{Psh}(C, \text{Grpds}) \), and then provide an alternative definition of a stack with the aid of such model structures [3, 4].

**Theorem 4.2.1.** The category \( \text{Gpds} \) admits a model structure where

1. A morphism \( F : C \to D \) between two groupoids (i.e. a functor between two particular categories) is said to be a weak equivalence if it is fully faithful and essentially surjective. Recall that [Va]

   (a) A functor \( F : C \to D \) is called fully faithful if for any objects \( A, B \in C \), the map

   \[ \text{Hom}_C(A, B) \to \text{Hom}_D(F(A), F(B)) \]  \hspace{1cm} (4.35)

   is a bijection of sets.

   (b) A functor \( F : C \to D \) is called essentially surjective if for any objects \( D \in D \), there exists an object \( A \) in \( C \) such that one has an isomorphism of objects

   \[ F(A) \xrightarrow{\sim} D. \]  \hspace{1cm} (4.36)

2. A morphism \( F : C \to D \) between two groupoids is called fibration if for each object \( A \) in \( C \) and each morphism \( \phi : F(A) \xrightarrow{\sim} D \) in \( D \), there exist an object \( B \) and a morphism \( f : A \xrightarrow{\sim} B \) in \( C \) such that \( F(f) = \phi \). That is,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & F(A) \\
\downarrow \phi & & \downarrow \phi = F(f) \\
B & \xrightarrow{\sim} & D = F(B)
\end{array}
\]  \hspace{1cm} (4.37)

3. A morphism \( F : C \to D \) between two groupoids is called cofibration if it is injective on objects.
A model structure on $PSh(C, Grpds)$, on the other hand, can also be defined in a similar fashion. In fact, it admits a so-called global model structure given as follows [3, 4]:

**Lemma 4.2.1.** The category $PSh(C, Grpds)$ admits a model structure where

1. A morphism $\phi : X \to Y$ in $PSh(C, Grpds)$, which is indeed a natural transformation between two functors given schematically as

   ![Diagram](image)

   is called a weak equivalence if for each object $A$ in $C$, the morphism

   $\phi(A) : X(A) \to Y(A)$

   is a weak equivalence in $Grpds$.

2. A morphism $\phi : X \to Y$ in $PSh(C, Grpds)$ is called a fibration if for each object $A$ in $C$, the morphism

   $\phi(A) : X(A) \to Y(A)$

   is a fibration in $Grpds$.

3. A morphism $\phi : X \to Y$ in $PSh(C, Grpds)$ is called a cofibration if it has the left lifting property w.r.t. all trivial fibrations ($p \in F \cap W$). That is, as we indicated before, there is a lift $h : Y \to X'$ commuting the following diagram.

   ![Diagram](image)

Furthermore, as addressed in [3, 4], by using a suitable localization of $PSh(C, Grpds)$, one can also define another model structure on $PSh(C, Grpds)$, namely a local model structure. This structure indeed allows us to encode the local-to-global process. In other words, studying the local model structure instead of the global one allows us to make sense of gluing properties of presheaves $X \in PSh(C, Grpds)$. This essentially will lead to the description of stacks in the language of model categories and homotopy theory.

**Theorem 4.2.2.** [4]. Let $C$ be a site. There exists a model structure on $PSh(C, Grpds)$ which is obtained by localizing the global model structure with respect to the morphisms of the form

$S := \{ \text{hocolim}_{PSh(C, Grpds)}(U_\bullet) \to U : \{U_i \to U\} \text{ is a covering family of } U\}$

where $U : \text{Hom}_C(\cdot, U)$ is the standard Yoneda functor and $U_\bullet$ denotes the simplicial diagram in $PSh(C, Grpds)$. That is,

$U_\bullet := \left( \coprod_i U_i \rightrightarrows \coprod_{ij} U_{ij} \rightrightarrows \coprod_{ijk} U_{ijk} \cdots \right)$.  

(4.43)
4.3 Revisiting the homotopy-theoretical definition of a stack

Unfortunately, we are not able to provide either a complete mathematical treatment or a proof of Theorem 4.2.2 or Theorem 4.2.1 at this stage. But, instead we refer to [4]. As stressed in [3], moreover, we have the following definition/theorem which allows us to formulate the classical notion of a Deligne-Mumford stack [29] in the language of homotopy theory.

**Definition 4.3.1.** Let \( \mathcal{C} \) be a site. A **stack** is a prestack \( \mathcal{X} : \mathcal{C}^{op} \rightarrow \text{Grpds} \) such that for each covering family \( \{ U_i \rightarrow U \} \) of \( U \) the canonical morphism

\[
\mathcal{X}(U) \longrightarrow \text{holim}_{\text{Grpds}}(\mathcal{X}(U_\bullet))
\]  

is a weak equivalence (and hence equivalence of categories) in \( \text{Grpds} \) where

\[
\mathcal{X}(U_\bullet) := \left( \prod_i \mathcal{X}(U_i) \Rightarrow \prod_{ij} \mathcal{X}(U_{ij}) \Rightarrow \prod_{ijk} \mathcal{X}(U_{ijk}) \Rightarrow \cdots \right)
\]

is the cosimplicial diagram in \( \text{Grpds} \) (cf. Diagram 4.18) and \( U_{i_1 \cdots i_m} \) denotes the fibered product of \( U_{i_1} \)'s in \( U \), that is

\[
U_{i_1 \cdots i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m}.
\]

**Remark 4.3.1.**

1. The weak equivalences in Definition 4.3.1 are the ones introduced in Theorem 4.2.1, namely those morphisms in \( \text{Grpds} \) which are fully faithful and essentially surjective.

2. We haven’t discussed the notion of “\( \text{holim}_{\text{Grpds}}(\cdot) \)” in detail. For a complete construction of this item we refer to section 2 of [4]. The following lemma ([4], corollary 2.11), on the other hand, does provide an explicit characterization of “\( \text{holim}_{\text{Grpds}}(\cdot) \)” as a particular groupoid.

**Lemma 4.3.1.** Given a cosimplicial diagram \( X_\bullet \) in Grpds of the form

\[
X_\bullet = \left( X_0 \Rightarrow X_1 \Rightarrow X_2 \Rightarrow \cdots \right)
\]

where each \( X_i \) is a groupoid, then \( \text{holim}_{\text{Grpds}}(X_\bullet) \) is a groupoid for which

(i) **objects** are the pairs \( (x, h) \) where \( x \) is an object in \( X_0 \), \( h : d_1^1(x) \rightarrow d_0^1(x) \) a morphism in \( X_1 \) such that

\[
(a) \quad s_0^0(h) = id_x,
\]

\[
(b) \quad d_0^2 \circ d_2^1(x) = d_1^1(x).
\]

Note that as we discussed in Lemma 4.1.1 and Remark 4.1.2, \( x \) and \( h \) can be realized as 0- and 1-simplices in \( X_\bullet \) respectively such that, by using the properties of \( d_1^1 \) and \( s_0^0 \), those conditions correspond to the commutativity of the diagram

\[
d_2^1 \circ d_1^1(x) \xrightarrow{d_2^1(h)} d_2^1 \circ d_0^1(x) = d_0^1 \circ d_1^1(x) \xrightarrow{d_0^1(h)} d_0^1 \circ d_0^1(x)
\]

\[
\text{"} = \text{" } \text{ by Lemma 4.1.1}
\]

\[
d_1^1 \circ d_1^1(x) \xrightarrow{d_1^1(h)} d_2^1 \circ d_0^1(x)
\]

\[
\text{"} = \text{" } \text{ by Lemma 4.1.1}
\]

and hence we pictorially have

\[
\begin{align*}
x & \quad \xrightarrow{h} \quad \bullet \quad \xrightarrow{s_0^0} \quad d_0^1(x) \quad \xrightarrow{d_1^1} \quad d_1^1(x) \quad \xrightarrow{d_2^1(h)} \quad d_2^1 \circ d_0^1(x) \\
& \quad \xrightarrow{d_1^1(h)} \quad \bullet \quad \xrightarrow{d_0^1(h)} \quad d_0^1(x) \quad \xrightarrow{d_2^1(h)} \quad d_2^1 \circ d_0^1(x)
\end{align*}
\]
(ii) morphisms are the arrows of pairs \((x, h) \to (x', h')\) that consist of a morphism \(f : x \to x'\) in \(X_0\) such that the following diagram commutes.

\[
\begin{array}{ccc}
d_1^1(x) & \xrightarrow{d_1^1(f)} & d_1^1(x') \\
\downarrow h & & \downarrow h' \\
d_0^1(x) & \xrightarrow{d_0^1(f)} & d_0^1(x')
\end{array}
\] (4.52)

Here, \(d^n\)'s are in fact covariant functors between groupoids.

Remark 4.3.2. Given a cosimplicial diagram \(X\) in \(Grpd\), Lemma 4.3.1 indeed serves as an equivalent definition of \(\text{holim}_{Grpd}(X)\). Therefore, throughout the discussion, for those who are not comfortable with the construction of \(\text{holim}_{Grpd}(X)\) presented in [4] -involving homotopy theory, model structures etc...- we simply define the homotopy limit \(\text{holim}_{Grpd}(X)\) of a cosimplicial diagram \(X\) in \(Grpd\) as a particular groupoid with the properties outlined in Lemma 4.3.1.

5 Stacky formulation for Einstein Gravity

Assume that \(C\) is the category described in Proposition 3.0.1. Let

\[
\mathcal{E} : C^{\text{op}} \longrightarrow Grpd
\] (5.1)

be a prestack defined in Proposition 3.0.1 encoding the moduli space of solutions to the vacuum Einstein field equations with \(\Lambda = 0\) on a Lorentzian manifold \(M\). Now, inspired by the approach presented in [3, 4], we shall provide the stacky structure on \(\mathcal{E}\) in accordance with Definition 4.3.1 and Lemma 4.3.1.

Theorem 5.0.1. \(\mathcal{E} : C^{\text{op}} \longrightarrow Grpd\) is a stack.

Proof. As in the case of [3], we first endow \(C\) with an appropriate Grothendieck topology \(\tau\) (cf. Definition 3.0.1) by defining the covering families \(\{U_i \to U\}_i\) of \(U\) in \(C\) to be "good" open covers \(\{U_i \subseteq U\}\) meaning that the fibered products

\[
U_{i_1, i_2, \ldots, i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m}
\] (5.2)

corresponding to the intersection of those open subsets \(U_i\)'s in \(U\) are either empty or open subsets diffeomorphic to \(\mathbb{R}^{(n-1)+1}\) (and hence lie in \(C\)) where the morphisms

\[
U_i \hookrightarrow U
\] (5.3)

are the canonical inclusions (and hence a morphism in \(C\)) for each \(i\). Therefore, we clearly have the same (even simpler) site structure on \(C\) discussed in [3]. Let \(U\) be an object in \(C\). Given \(\{U_i \subseteq U\}\) a covering family for \(U\), one has the following cosimplicial diagram in \(Grpd\)

\[
\mathcal{E}(U) := \left( \prod_i \mathcal{E}(U_i) \xrightarrow{\prod_i \mathcal{E}(U_{ij})} \prod_{i,j,k} \mathcal{E}(U_{ijk}) \right) \cdots
\] (5.4)

where \(U_{i_1, i_2, \ldots, i_m}\) denotes the fibered product of \(U_{i_n}\)'s in \(U\), that is

\[
U_{i_1, i_2, \ldots, i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m},
\] (5.5)

which, in that case, corresponds to the usual intersection of \(U_i\)'s in \(U\). Note that for a family

\[
\{g_i\} \in \prod_i \mathcal{E}(U_i),
\] (5.6)
where $\mathcal{E}(U_i) = FMet(U_i)$, the coface maps $d_0^i$ and $d_1^i$ correspond to the suitable restrictions of each component, namely

$$g_j|_{U_{ij}} \text{ and } g_j|_{U_{ij}}.$$  \hfill (5.7)

Now, it follows from the Lemma 4.3.1 that $\text{holim}_{\text{Grpds}}(\mathcal{E}(U_\bullet))$ is indeed a particular groupoid and can be defined as follows:

1. **Objects** are the pairs $(x, h)$ where

$$x := \{g_i\} \in \prod_i \mathcal{E}(U_i), \quad \hfill (5.8)$$

i.e. a family of Ricci-flat pseudo-Riemannian metrics on $U_i$’s, and hence we pictorially have the following observation:

$$\begin{tikzpicture}
  \node (g1) at (0,0) {$\{g_i|_{U_{ij}}\}$};
  \node (d2) at (2,0) {$\{g_k|_{U_{ijk}}\}$};
  \node (gj) at (2,-1) {$\{g_j|_{U_{ij}}\}$};
  \node (d1) at (0,-1) {$\{g_k|_{U_{ijk}}\}$};
  \node (ijk) at (1,-2) {$\phi_{ij}$};
  \node (gj) at (0,-2) {$\{g_j|_{U_{ijk}}\}$};
  \node (g1) at (2,-2) {$\{g_i|_{U_{ij}}\}$};

  \draw[->] (g1) to node [above] {$d_1^i$} (gj);
  \draw[->] (d1) to node [above] {$d_2^i$} (d2);
  \draw[->] (d1) to node [above] {$\sim$} (gj);

  \draw[->] (ijk) to node [above] {$\phi_{ij}$} (gj);
  \draw[->] (ijk) to node [above] {$\phi_{ij}$} (g1);
  \draw[->] (gj) to node [above] {$\phi_{ij}$} (g1);
\end{tikzpicture}$$  \hfill (5.9)

where $g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}}$ for some $\varphi_{ij} \in Isom(U_{ij})$. Notice that for all $i, j, k$ we have

$$g_k|_{U_{ijk}} = \varphi_{jk}^* g_j|_{U_{ijk}} = \varphi_{jk}^* \varphi_{ij}^* g_i|_{U_{ijk}} = (\varphi_{ij} \circ \varphi_{jk})^* g_i|_{U_{ijk}}.$$  \hfill (5.10)

which means that there exists a morphism $\varphi_{ik} : g_i|_{U_{ijk}} \overset{\sim}{\rightarrow} g_k|_{U_{ijk}}$. Therefore, we define the morphism $h$ in $\prod \mathcal{E}(U_{ij})$ as a family

$$h := \{g_j|_{U_{ij}} \sim (g_i|_{U_{ij}} \varphi_{ij}) \rightarrow g_j|_{U_{ij}} : g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}} \text{ with } \varphi_{ij} \in Isom(U_{ij})\}$$  \hfill (5.11)

where $g_k|_{U_{ijk}} = (\varphi_{ij} \circ \varphi_{jk})^* g_i|_{U_{ijk}}$ and $s_0^i(h) : \{g_i\} \rightarrow \{g_i\}$, which is just the identity morphism. As a remark, the conditions in the definition of the family $\{h\}$ correspond to those in Lemma 4.3.1 (4.48 and 4.49). Therefore, the objects of $\text{holim}_{\text{Grpds}}(\mathcal{E}(U_\bullet))$ must be of the form

$$(x, h) = \left(\{g_i \in FMet(U_i)\}, \{\varphi_{ij} \in Isom(U_{ij})\}\right)$$  \hfill (5.12)

where $\{g_i\}$ is an object in $\prod \mathcal{E}(U_i)$, and for each $i, j$, $\varphi_{ij} := (g_i|_{U_{ij}}, \varphi_{ij})$ is a morphism in $\prod \mathcal{E}(U_{ij})$ satisfying

(i) $g_j|_{U_{ij}} = \varphi_{ij}^* g_i|_{U_{ij}} \text{ with } \varphi_{ij} \in Isom(U_{ij}),$  \hfill (5.13)
(ii) $\text{On } U_{ijk}, \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ (cocycle condition),  \hfill (5.14)
(iii) $s_0^i(h) : \{g_i\} \rightarrow \{g_i\}, \text{ the identity morphism.}$  \hfill (5.15)

2. A morphism in $\text{holim}_{\text{Grpds}}(\mathcal{E}(U_\bullet))$

$$(x, h) \rightarrow (x', h')$$  \hfill (5.16)

consists of the following data:

(a) a morphism $x \overset{\sim}{\rightarrow} x'$ in $\prod \mathcal{E}(U_i)$, that is,

$$\{g_i\} \overset{\sim}{\rightarrow} \{g'_i\}$$  \hfill (5.17)

where $g_i, g'_i \in FMet(U_i)$ such that $g'_i = \varphi_i^* g_i$ for some $\varphi_i \in Isom(U_i)$,
(b) together with the commutative diagram

\[
\begin{array}{ccc}
g_i|_{U_{ij}} & \xrightarrow{\varphi_i|_{U_{ij}}} & g'_i|_{U_{ij}} \\
h = \varphi_j & & h' = \varphi'_j \\
g_j|_{U_{ij}} & \xrightarrow{\varphi_j|_{U_{ij}}} & g'_j|_{U_{ij}}
\end{array}
\]

(5.18)

In fact, it follows from the fact that \(g_j|_{U_{ij}} = \varphi'_j g_i|_{U_{ij}}\) and \(g'_j|_{U_{ij}} = \varphi''_j g'_i|_{U_{ij}}\) we have

\[
(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}) \cdot \varphi'_j = \varphi'_j\varphi_i g_i|_{U_{ij}} \\
= \varphi''_j g'_i|_{U_{ij}} \\
= g'_j|_{U_{ij}}.
\]

(5.19)

and, on the other hand, one also has

\[
(g_i|_{U_{ij}} \cdot \varphi_j|_{U_{ij}}) \cdot \varphi'_j = \varphi'_j|_{U_{ij}} g_i|_{U_{ij}} \\
= \varphi''_j g'_i|_{U_{ij}} \\
= g'_j|_{U_{ij}}.
\]

(5.20)

which imply the commutativity of the diagram, and hence one can also deduce the following relation:

\[
(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}) \cdot \varphi'_j = (g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}) \cdot \varphi'_j
\]

(5.21)

\[
\iff
\]

\[
g_i|_{U_{ij}} \cdot (\varphi_j \circ \varphi_i|_{U_{ij}}) = g_i|_{U_{ij}} \cdot (\varphi_i|_{U_{ij}} \circ \varphi'_j)
\]

(5.22)

\[
\iff
\]

\[
\varphi'_j = \varphi^{-1}_i|_{U_{ij}} \circ \varphi_j \circ \varphi_i|_{U_{ij}}
\]

(5.23)

Thus, a morphism in \(\text{holim}_{\text{Grpds}}(E(U_\bullet))\) from \(\{g_i\}, \{\varphi_{ij}\}\) to \(\{g'_i\}, \{\varphi'_{ij}\}\) is a family

\[
\left\{ \varphi_i \in \text{Isom}(U_i) : g'_i = \varphi'_i g_i \text{ and } \varphi'_{ij} = \varphi^{-1}_i|_{U_{ij}} \circ \varphi_j \circ \varphi_i|_{U_{ij}} \text{ for all } i,j. \right\}
\]

(5.24)

Now, for a covering family \(\{U_i \subseteq U\}\) of \(U\), the canonical morphism

\[
\Psi : E(U) \longrightarrow \text{holim}_{\text{Grpds}}(E(U_\bullet))
\]

(5.25)

is defined as a functor of groupoids where

- for each object \(g\) in \(FMet(U)\),
  \[
  g \xmapsto{\Psi} \left( \{g|_{U_i}\}, \{\varphi_{ij} = id\} \right),
  \]
  together with the trivial cocyle condition.

- for each morphism \(g \xrightarrow{\sim}_{(g,\varphi)} \varphi^* g\) with \(\varphi \in \text{Isom}(U)\),
  \[
  (g \xrightarrow{\sim}_{(g,\varphi)} \varphi^* g) \xmapsto{\Psi} \left( \{\varphi_i : = \varphi|_{U_i}\} \right)
  \]
  (5.27)

where \(\varphi|_{U_i}\) trivially satisfies the desired relation in 5.24 for being a morphism in \(\text{holim}_{\text{Grpds}}(E(U_\bullet))\).
Claim: $\Psi$ is a fully faithful and essentially surjective functor between groupoids (cf. Theorem 4.2.1).

Proof of claim:

(i) Given a family of objects $\{g_i\}$ with the family of transition functions $\{\varphi_{ij} \in Isom(U_{ij})\}$ such that

$$g_j|_{U_{ij}} = \varphi_{ij}^*g_i|_{U_{ij}}$$

along with the cocycle condition

$$\text{On } U_{ijk}, \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik},$$

we need to show that these are patched together to form a metric $g \in \text{FMet}(U)$. In fact, this follows from the analysis of geometric structures [8] on objects in $\mathcal{C}$ together with the approach employed in [3]. The punchline is the following lemma:

Lemma 5.0.1. All cocycles are trivializable on manifolds diffeomorphic to $\mathbb{R}^n$.

Proof. It directly follows from the Quillen-Suslin theorem (known also as Serre’s conjecture) which algebraically states that every finitely generated projective module over a polynomial ring $k[x_1, \ldots, x_n]$ with $k$ a principle ideal domain is free [47]. Geometrically, on the other hand, finitely generated (free) projective modules over a polynomial ring in fact correspond to (trivial) vector bundles over an affine space $\mathbb{A}^n_k$. As an affine space $\mathbb{A}^n_k$ is topologically contractible, it admits no non-trivial topological (holomorphic or smooth resp.) vector bundles (for an introductory discussion see [39], ch. 14). The Quillen-Suslin theorem proves that this is also the case for algebraic vector bundles. 

Therefore, by Lemma 5.0.1, $\{\varphi_{ij} = \text{id}\}$ for all $i, j$, and hence it follows from the property

$$\varphi'_{ij} = \varphi^{-1}_i|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}} \quad \text{for all } i, j,$$

that we have the following observation:

$$\text{id} = \varphi_{ij} = \varphi^{-1}_i|_{U_{ij}} \circ \varphi_{ij} |_{U_{ij}} \iff \text{there exists } \varphi \text{ such that } \varphi|_{U_i} = \varphi_i$$

As we have a trivial cocycle condition with $\varphi_{ij} = \text{id}$, $g_i$’s are glued together by transition functions $\varphi_{ij}$ along with the trivial cocycle condition to form $g \in \text{FMet}(U)$ so that $g|_{U_i} = g_i$ and $\varphi|_{U_i} = \varphi_i$ for all $i$. Therefore, $\Psi$ is essentially surjective.

(ii) We need to show that the map

$$\text{Hom}_{E(U)}(g,g') \rightarrow \text{Hom}_{\text{holim}_{\mathcal{Grpds}}(\mathbb{E}(U\cdot))}(\Psi(g), \Psi(g'))$$

is a bijection of sets. Let $g \xrightarrow{(g,\varphi)} \varphi^*g$ be a morphism in $\mathbb{E}(U)$. Then $\Psi$ sends $(g,\varphi)$ to a family of morphisms

$$\{\varphi_i := \varphi|_{U_i}\}$$

where $g'_i = \varphi^*_i g_i$ with the condition $\varphi'_{ij} = \varphi_i^{-1}|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_j|_{U_{ij}}$ for all $i, j$. Notice that as $\varphi_i := \varphi|_{U_i} = \text{id}$ for all $i$ implies that $\varphi$ must be the identity mapping in the first place, and hence we conclude that $\Psi$ is a injective on morphisms. Surjectivity, on the other hand, follows from the fact that the functor

$$C^\infty(\cdot, U) : C^{op} \rightarrow \text{Sets}, \quad V \mapsto C^\infty(V, U)$$

is a sheaf (for the fpqc-topology [3, 21, 39]). Indeed, $\iota : U_i \hookrightarrow U$ is an open embedding and $\varphi_i$ is a diffeomorphism (isometry) of $U_i$, one has a morphism

$$\iota \circ \varphi_i : U_i \rightarrow U \quad \text{in } C^\infty(U_i, U)$$

together with the suitable compatibility condition as above, and since $C^\infty(\cdot, U)$ is a sheaf on $\mathcal{C}$ in the sense of [21], by local-to-global properties of $C^\infty(\cdot, U)$ there is a diffeomorphism (isometry) $\varphi$ such that $\varphi|_{U_i} = \varphi_i$. Therefore, $\Psi$ is a surjective on morphisms as well. This completes the proof of claim.
With the claim in hand together with the Definition 4.3.1, the canonical morphism
\[ \Psi : \mathcal{E}(U) \rightarrow \text{holim}_{\text{Grpds}}(\mathcal{E}(U_\bullet)) \quad (5.35) \]
is a weak equivalence (cf. Theorem 4.2.1) in $\text{Grpds}$, and this completes the proof of Theorem 5.0.1

□

**Definition 5.0.1.** The stack in Theorem 5.0.1
\[ \mathcal{E} : C^{op} \rightarrow \text{Grpds} \quad (5.36) \]
is called the *moduli stack of flat Lorentzian structures* (or the moduli stack of solutions to the vacuum Einstein field equations with $\Lambda = 0$, i.e. that of Ricci-flat pseudo-Riemannian metrics) on $M$.

We sometimes call it directly the *moduli stack of Einstein gravity*.

### 6 Stacky equivalence of 3D quantum gravity with a gauge theory

As we have already discussed in the introduction (cf. Definition 1.0.3) that one can introduce the notion of *equivalence* between quantum gravity and a gauge theory if the corresponding moduli spaces are isomorphic. That is, one requires
\[ \phi : \mathcal{E}(M) \rightarrow \mathcal{M}_{\text{flat}}. \quad (6.1) \]

In fact, with the help of Theorem 2.4.1, one has such an equivalence of the quantum gravity with a gauge theory in the case of 2+1 dimensional vacuum Einstein gravity (with vanishing cosmological constant) on a Lorentzian 3-manifold $M$ of the form $\Sigma \times (0, \infty)$ where $\Sigma$ is a closed Riemann surface of genus $g > 1$. Now, we would like to show that the isomorphism $\phi$ naturally induces an *isomorphism of associated stacks*.

We shall first revisit [3] and introduce a particular stack similar to $BG_{\text{con}}$ (Example 2.11 in [3]). This helps us to view the space $\mathcal{M}_{\text{flat}}$ as a certain stack. Of course we first need to introduce the "flat" counterpart of this classifying stack $BG_{\text{con}}$. Just for simplicity we use $\mathcal{M}$ for the flat case whose construction is the same as that of $BG_{\text{con}}$. Note that the main ingredients of this structure are encoded by the theory of principal $G$-bundles in the following sense: Let $P \rightarrow M$ be a principal $G$-bundle on a 3-manifold $M$ as above, and $\sigma \in \Gamma(U, P)$ a local trivializing section. We then schematically have
\[ P \xrightarrow{\delta \ G} P \xrightarrow{\sigma} \pi \xrightarrow{\pi} M \quad (6.2) \]

where $\delta \ G$ denotes the right $G$-action on the smooth manifold $P$ via diffeomorphisms of $P$. Let $\omega$ be a Lie algebra-valued connection one-form on $P$ and $A := \sigma^* \omega$ its local representative, i.e., the Lie algebra-valued connection 1-form on $M$. $A$ is also called a *local Yang-Mills field*. Then the space of fields is defined to be the infinite-dimensional space $\mathcal{A}$ of all $G$-connections on a principal $G$-bundle over $M$, i.e. $\mathcal{A} = \Omega^1(M) \otimes \mathfrak{g}$. Furthermore, the Chern-Simons action functional $CS : \mathcal{A} \rightarrow S^1$ is given by
\[ CS[A] = \int_M \langle A, dA + \frac{2}{3} A \wedge A \rangle \quad (6.3) \]
where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on its Lie algebra $\mathfrak{g}$. The gauge group is locally of the form $G = \text{Map}(U, G)$ that acts on the space $\mathcal{A}$ as follows: For all $\rho \in G$ and $A \in \mathcal{A}$, we set
\[ A \cdot \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot d\rho \cdot \rho. \quad (6.4) \]
The corresponding E-L equation in this case turns out to be
\[ F_A = 0, \] (6.5)
where \( F_A = dA + A \wedge A \) is the curvature two-form on \( M \) associated to \( A \). Furthermore, under the gauge transformation, the curvature 2-form \( F_A \) behaves as follows:
\[ F_A \mapsto F_A \bullet \rho := \rho^{-1} \cdot F_A \cdot \rho \quad \text{for all } \rho \in G. \] (6.6)

Note that the moduli space \( \mathcal{M}_{\text{flat}} \) of flat connections on \( M \), i.e. \( A \in \mathcal{A} \) with \( F_A = 0 \), modulo gauge transformations emerges in many other areas of mathematics, such as topological quantum field theory, low-dimensional quantum invariants [55] or (infinite dimensional) Morse theory [31, 33, 48]. Note that for the gravitational interpretation (in the case of vanishing cosmological constant), one requires to consider the case of \( G = ISO(2, 1) \) [6].

**Lemma 6.0.1.** Let \( \mathcal{C} \) be the category in Proposition 3.0.1 such that \( M \) is a Lorentzian 3-manifold topologically of the form \( \Sigma \times \mathbb{R} \) where \( \Sigma \) a closed Riemann surface of genus \( g > 1 \). The following assignment
\[ \mathcal{M} : C^{\text{op}} \to \text{Grpds} \] (6.7)
defines a stack corresponding to the space \( \mathcal{M}_{\text{flat}} \) where

1. For each object \( U \) of \( \mathcal{C} \), \( \mathcal{M}(U) \) is a groupoid of flat \( G \)-connections on \( U \) with objects being the elements of the set \( \Omega^1(U, \mathfrak{g})_{\text{flat}} \) of Lie algebra-valued 1-forms on \( U \) such that \( F_A = 0 \), and morphisms form the set
\[ \text{Hom}_{\mathcal{M}(U)}(A, A') = \{ \rho \in G : A' = A \bullet \rho \} \] (6.8)
where the action of the gauge group \( G \), which is locally of the form \( C^{\infty}(U, G) \), on \( \Omega^1(U, \mathfrak{g})_{\text{flat}} \) is defined as above: For all \( \rho \in G \) and \( A \in \mathcal{A} \), we set
\[ A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot d\rho. \] (6.9)
Furthermore, we denote a morphism
\[ A \xrightarrow{\sim} A' = A \bullet \rho \] (6.10)
in \( \text{Hom}_{\mathcal{M}(U)}(A, A') \) by \( (A, \rho) \).

2. To each morphism \( U \xrightarrow{f} V \) in \( \mathcal{C} \), i.e. \( f : U \hookrightarrow V \) with \( U \subset V \), one assigns
\[ \mathcal{M}(V) \xrightarrow{\mathcal{M}(f)} \mathcal{M}(U). \] (6.11)
Here \( \mathcal{M}(f) \in \text{Fun}(\mathcal{M}(V), \mathcal{M}(U)) \) is a functor of categories whose action on objects and morphisms of \( \mathcal{M}(V) \) is given as follows.

(a) For any object \( A \in \text{Ob}(\mathcal{M}(V)) = \Omega^1(V, \mathfrak{g})_{\text{flat}} \),
\[ A \xrightarrow{\mathcal{M}(f)} f^* A \, (= A|_U) \] (6.12)
where \( \mathcal{M}(f)(A) := f^* A \in \Omega^1(U, \mathfrak{g})_{\text{flat}} \). Here we use the fact that the pullback (indeed the restriction to an open subset \( U \) in our case) of a flat connection in the sense that \( F_A = 0 \) is also flat.

(b) For any morphism \( (A, \rho) \in \text{Hom}_{\mathcal{M}(V)}(A, A') \) with \( \rho \in G \) such that \( A' = A \bullet \rho \), it follows from the fact that
\[ f^*(A \bullet \rho) = f^* A \bullet f^* \rho \] (6.13)
where \( f^* \rho = \rho \circ f \in C^{\infty}(U, G) \), we conclude that \( f^*(A \bullet \rho) \) lies in the orbit space of \( f^* A \), and hence we get
\[ \left( A \xrightarrow{(A, \rho)} A' = A \bullet \rho \right) \xrightarrow{\mathcal{M}(f)} \left( f^* A \xrightarrow{(f^* A \bullet f^* \rho)} f^* (A \bullet \rho) = f^* A \bullet f^* \rho \right) \] (6.14)
where \( \mathcal{M}(f)(A, \rho) := (f^* A, f^* \rho) \) is a morphism in \( \mathcal{M}(U) \). Note that the identity 6.13 we mentioned above can indeed be proven by just local computations of the pullback of a connection \( A \) together with the action \( A \bullet \rho \).

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Proof. This is similar to the proofs of Proposition 3.0.1 and Theorem 5.0.1 in the special case where \( n = 3 \) and \( M \) as above. For a complete treatment to the generic case (i.e., without flatness requirement), see Examples 2.10 and 2.11 in [3]. For the flat case, on the other hand, one has exactly the same proof with \( \Omega^1(U, g)_{flat} \) instead of \( \Omega^1(U, g) \) thanks to the fact that the pullback of a flat connection by a canonical inclusion \( U \hookrightarrow V \) between open subsets is also flat.

To sum up, we have the following observations so far:

1. Before the "stacky" constructions, by Theorem 2.4.1, we already have an isomorphism of moduli spaces

\[
\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat}
\]

(6.15)

in the case of vacuum Einstein general relativity with the cosmological constant \( \Lambda = 0 \) on a Lorentzian 3-manifold of the form \( M = \Sigma \times (0, \infty) \) where \( \Sigma \) is a closed Riemann surface of genus \( g > 1 \).

2. Let \( \mathcal{C} \) be the category in Proposition 3.0.1. In Theorem 5.0.1, we have constructed the stack of flat Lorentzian structures on a generic \( n \)-manifold

\[
\mathcal{E} : \mathcal{C}^{op} \rightarrow \text{Grpds}.
\]

(6.16)

3. In Lemma 6.0.1, we have introduced the classifying stack of principal \( G \)-bundles with flat connections on \( \Sigma \)

\[
\mathcal{M} : \mathcal{C}^{op} \rightarrow \text{Grpds}
\]

(6.17)

where \( \mathcal{C} \), in that case, involves particular choices of dimension \( (n = 3) \) and the form of a manifold \( M := \Sigma \times (0, \infty) \).

Given a closed Riemann surface \( \Sigma \) of genus \( g > 1 \), we now intend to show that if \( \mathcal{C} \) is the category in Proposition 3.0.1 with the special case \( (n = 3) \) where \( M \) is a Lorentzian 3-manifold of the form \( \Sigma \times (0, \infty) \), then there exists an invertible natural transformation

\[
\mathcal{E} \xrightarrow{\Phi} \mathcal{M}
\]

(6.18)

between these two stacks \( \mathcal{E} \) and \( \mathcal{M} \). This eventually provides a stacky extension of the isomorphism between the underlying moduli spaces

\[
\phi : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{M}_{flat}.
\]

(6.19)

**Theorem 6.0.1.** Let \( \mathcal{E} \) and \( \mathcal{M} \) be as above. Then there exists an invertible natural transformation

\[
\Phi : \mathcal{E} \Rightarrow \mathcal{M}.
\]

(6.20)

Proof. Given a closed Riemann surface \( \Sigma \) of genus \( g > 1 \), let \( \mathcal{C} \) be the category of open subsets of \( \Sigma \times (0, \infty) \) with morphisms being the canonical inclusions whenever \( U \subset V \). Recall that, as we explained in section 1, any solution of the vacuum Einstein field equations with vanishing cosmological constant is locally flat. This means that the metric is locally equivalent to the standard Minkowski metric \( \eta_{\mu \nu} \). Then, we first notice that it follows from gauge theoretic realization of 2+1 gravity (with \( \Lambda = 0 \)) in Cartan’s formalism that any solution to the vacuum Einstein field equation with \( \Lambda = 0 \) on any open subset of \( M \) defines a flat \( ISO(2,1) \)-connection, and thus for any object \( U \) in \( \mathcal{C} \), we have a natural map

\[
\Phi_U : \mathcal{E}(U) \rightarrow \mathcal{M}(U)
\]

(6.21)

which is indeed a functor of groupoids defined as follows:

1. To each \( g \in FMet(U) \), a solution to the vacuum Einstein field equations (with \( \Lambda = 0 \)) over \( U \), one assigns the corresponding flat \( ISO(2,1) \)-connection \( A^g \) in \( \Omega^1(U, iso(2,1))_{flat} \) described by the Cartan’s formalism. That is,

\[
g \underset{\Phi_U}{\rightarrow} A^g.
\]

(6.22)
2. Note that, as we addressed in Section 2.1, Cartan’s formalism not only provides the equivalence between the Einstein-Hilbert action functional for such a 2+1 gravity and the one for Chern-Simons theory with the gauge group \( ISO(2,1) \), but also encodes the symmetries of each theory in the sense that the diffeomorphism invariance of 2+1 gravity theory does correspond to the gauge invariance behaviour of the associated Chern-Simons theory (and vice versa). It means that the equivalence classes \([g]\) of flat pseudo-Riemannian metrics correspond to the gauge equivalence classes of the associated connections \([A^g]\). Therefore, for any \( g' \in [g] \), i.e. \( g' = \varphi^* g \) for some isometry \( \varphi \), the corresponding connections

\[
A^g \quad \text{and} \quad A^{\varphi^* g}
\]

are also gauge equivalent, and hence lie in the same equivalence class. That is, there exist \( \rho_{\varphi} \in G \), a gauge transformation associated to \( \varphi \), such that

\[
A^{\varphi^* g} = A^g \cdot \rho_{\varphi}.
\]

In other words, such a correspondence can also be expressed as the following commutative diagram:

\[
\begin{array}{ccc}
A^g & \xrightarrow{\rho_{\varphi}} & A^{\varphi^* g} = A^g \cdot \rho_{\varphi} \\
\downarrow \varphi^* & & \\
g & \xrightarrow{\varphi} & \varphi^* g
\end{array}
\]

together with a group isomorphism

\[
Diff(U) \longrightarrow C^\infty(U, G), \quad \varphi \mapsto \rho_{\varphi},
\]

where \( Diff(U) \) is endowed with the usual composition, and the group operation on \( C^\infty(U, G) \) is given by the pointwise multiplication. Here, we shall consider \( \varphi \in Isom(U) \subset Diff(U) \).

3. To each morphism \((g, \varphi) : g \longrightarrow \varphi^* g \) in \( \text{Hom}_E(U(g, g')) \), \( \Phi_U \) associates a morphism

\[
\begin{array}{ccc}
A^g & \xrightarrow{\sim} & A^g \cdot \rho_{\varphi} (= A^{\varphi^* g}) \\
\downarrow \rho_{\varphi} & & \\
A^g & \xrightarrow{\rho_{\varphi}} & A^{\varphi^* g} = A^g \cdot \rho_{\varphi}
\end{array}
\]

where \( \rho_{\varphi} \in C^\infty(U, ISO(2,1)) \) is a gauge transformation corresponding to \( \varphi \) in accordance with the diagram in 6.25. Therefore, for any morphism \( f : U \hookrightarrow V \) in \( C \), using 6.26, one also has the following commutative diagram:

\[
\begin{array}{ccc}
Diff(V) & \xrightarrow{f^* ( = |_U)} & Diff(U) \\
\downarrow \text{by 6.26} & & \downarrow \text{by 6.26} \\
C^\infty(V, ISO(2,1)) & \xrightarrow{f^* ( = |_U)} & C^\infty(U, ISO(2,1))
\end{array}
\]

4. Functoriality. Given a composition of morphisms \( g \xrightarrow{(g, \varphi)} \varphi^* g \xrightarrow{(\varphi \circ \psi, \psi)} \psi^* \varphi^* g \) in \( E(U) \), that is

\[
(g \cdot \varphi) \cdot \psi = g \cdot (\varphi \circ \psi)
\]

we have the following commutative diagram

\[
\begin{array}{ccc}
g & \xrightarrow{\varphi} & \varphi^* g \\
\downarrow \varphi & & \downarrow \psi \\
g & \xrightarrow{(g, \varphi)} & \varphi^* g \\
\downarrow \rho_{\varphi} & & \downarrow \rho_{\psi} \\
A^g & \xrightarrow{\rho_{\varphi} \rho_{\psi}} & A^{\varphi^* g} = A^{(\varphi \circ \psi)^* g}
\end{array}
\]
where, using the commutativity,
\[ A^g \bullet \rho_{\varphi \circ \psi} = A^{(\varphi \circ \psi)^*} g = A^{\varphi^*} g \bullet \rho_{\psi} = (A^g \bullet \rho_{\varphi}) \bullet \rho_{\psi}, \]
and hence, with the abuse of notation by using just \( \varphi \) in place of \((g, \varphi)\) (similarly for \( \psi \) and \( \varphi \circ \psi \)), one has
\[ \rho_{\varphi} \cdot \rho_{\psi} = \rho_{\varphi \circ \psi} \]
which gives the desired functoriality in the sense that
\[ \Phi_U(g, \varphi \circ \psi) = \rho_{\varphi} \circ \rho_{\psi} = (\Phi_U(g, \varphi) \cdot \Phi_U(g, \psi)). \]

Now, we need to show that for each morphism \( f : U \to V \) in \( \mathcal{C} \), i.e. \( f : U \hookrightarrow V \) with \( U \subset V \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{E}(V) & \xrightarrow{\Phi_V} & \mathcal{M}(V) \\
\downarrow{\mathcal{E}(f)} & & \downarrow{\mathcal{M}(f)} \\
\mathcal{E}(U) & \xrightarrow{\Phi_U} & \mathcal{M}(V)
\end{array}
\]

In fact, the commutativity follows from the definition 6.21 of the morphism \( \Phi_U \): Let \( g \in \text{FMet}(V) \), then we get, from the construction and the restriction functor \( \cdot|_U \), the natural diagram

\[
g \xrightarrow{\rho_{\varphi}} A^g \\
g|_U \xrightarrow{\varphi|_U} A^g|_U = A^f|_U
\]

Hence, a direct computation yields
\[ (\mathcal{M}(f) \circ \Phi_V)(g) = f^* A^g \\
= A^g|_U \\
= A^{f^*} g \quad \text{from 6.35} \\
= \Phi_U(f^* g) \\
= (\Phi_U \circ \mathcal{E}(f))(g), \]
which gives an "objectwise" commutativity of the diagram. Similarly, for any morphism
\[ (g, \varphi) : g \to \varphi^* g \in \text{Hom}_{\mathcal{E}(V)}(g, g') \]
one has another natural diagram again from the definition and the restriction functor as above

\[
\varphi \xrightarrow{\rho_{\varphi}} \\
\varphi|_U \xrightarrow{\rho_{\varphi|_U}} \rho_{\varphi|_U} = \rho_{\varphi|_U}
\]

Therefore, we obtain
\[ (\mathcal{M}(f) \circ \Phi_V)(g, \varphi) = (f^* A^g, f^* \rho_{\varphi}) \\
= (A^{f^*} g, \rho_{\varphi}|_U) \quad \text{from 6.35} \\
= (A^{f^*} g, \rho_{\varphi|_U}) \quad \text{from 6.38} \\
= (A^{f^*} g, \rho_{f^* \varphi}) \\
= \Phi_U(f^* g, f^* \varphi) \\
= (\Phi_U \circ \mathcal{E}(f))(g, \varphi), \]

(6.39)
which implies the desired "morphismwise" commutativity.

Therefore, \( \Phi \) defines a natural transformation between \( \mathcal{E} \) and \( \mathcal{M} \) along with the collection

\[
\{ \Phi_U : \mathcal{E}(U) \to \mathcal{M}(U) \}_{U \in \text{Ob}(\mathcal{C})}
\]

of natural maps defined by means of Cartan geometric formulation of Einstein gravity together with the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}(V) & \xrightarrow{\Phi_V} & \mathcal{M}(V) \\
\mathcal{E}(f) \downarrow & & \downarrow \mathcal{M}(f) \\
\mathcal{E}(U) & \xrightarrow{\Phi_U} & \mathcal{M}(V)
\end{array}
\]

for each morphism \( f : U \to V \) in \( \mathcal{C} \), i.e. \( f : U \to V \) with \( U \subset V \).

The inverse construction, on the other hand, essentially follows from Mess' result (cf. Theorem 2.4.1) that for each object \( U \) in \( \mathcal{C} \), the map \( \Phi_U \) is indeed invertible and the inverse map

\[
\Phi_U^{-1} : \mathcal{M}(U) \to \mathcal{E}(U)
\]

is defined as follows: Once we choose a hyperbolic structure on a closed orientable surface \( \Sigma \) of genus \( g > 1 \) and view it as a Riemannian surface, then a flat connection \( A \) defines the holonomy representation of such a hyperbolic structure, and hence a Fuchsian representation (cf. section 2.3 and corollary 2.3.2). Thus, by Theorem 2.4.1, there exist a suitable flat pseudo-Riemannian manifold \( M \) whose flat structure given by a flat pseudo-Riemannian metric denoted by \( g_A \) such that \( M = \Sigma \times (0, \infty) \) whose surface group representation agrees with the former one. Therefore, we have a well-defined assignment on objects

\[
\Phi_U^{-1} : \mathcal{M}(U) \to \mathcal{E}(U), \quad A \mapsto g_A
\]

such that due to the fact that surface group representation agrees with the former one (cf. Theorem 2.4.1), the flat connection \( A^{\phi_A} \) associated to \( g_A \) is exactly the connection we started with, i.e.

\[
\Phi_U \circ \Phi_U^{-1} : A \mapsto g_A \mapsto A^{\phi_A} = A.
\]

Then, by using the similar analysis as above, it is rather straightforward to check that we have a well-defined assignment \( \Phi_U^{-1} \) on both objects and morphisms together with appropriate commutative diagram analogous to the one in 6.34, and hence \( \Phi_U^{-1} \) is functor of groupoids as well. Thus, by construction, \( \Phi^{-1} \) is indeed a natural transformation that serves as an inverse of the natural transformation \( \Phi \) between two stacks. Therefore, one has an invertible natural transformation \( \Phi \)

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\Phi} & \mathcal{M} \\
\mathcal{C}^{\text{op}} & \xrightarrow{\Phi} & \text{Grpds.}
\end{array}
\]

This completes the proof of Theorem 6.0.1.
Part II
Formal Moduli Problems and 3D Einstein Gravity

7 Towards the derived geometry of Einstein gravity

As outlined in Section 3, one can define a certain moduli functor $\mathcal{E}\mathcal{L}$ corresponding to a given classical field theory as follows. Let $\mathcal{C}$ be the category, we set

$$\mathcal{E}\mathcal{L} : \mathcal{C}^{\text{op}} \to \text{Sets}, \ U \mapsto \mathcal{E}\mathcal{L}(U),$$

(7.1)

where $\mathcal{E}\mathcal{L}(U)$ is the set of isomorphism classes of solutions to the E-L equations over $U$. More precisely, $\mathcal{E}\mathcal{L}(U)$ is the moduli space $\mathcal{E}\mathcal{L}(U)/\mathcal{G}$ of solutions to the E-L equations modulo gauge transformations. But, as we succinctly discussed in Section 3, the quotient space might be "bad" in the sense that it may fail to live in the same category. In other words, the moduli functor $\mathcal{E}\mathcal{L}$, in general, is not representable in $\mathcal{C}$. In order to circumvent the problem, we introduce the "stacky" version of $\mathcal{E}\mathcal{L}$ as the quotient moduli stack $[\mathcal{E}\mathcal{L}/\mathcal{G}] : \mathcal{C}^{\text{op}} \to \text{Grpds}$, $U \mapsto [\mathcal{E}\mathcal{L}/\mathcal{G}](U)$,

(7.2)

where $[\mathcal{E}\mathcal{L}/\mathcal{G}](U)$ is the groupoid of solutions to the E-L equations over $U$. Even if this explains the emergence of stacky language in Definition 1.0.2 in a rather intuitive way, the discussion above is just the tip of the iceberg and is still too naïve to capture the notion of a derived stack. We need further concepts in order to enjoy the richness of Definition 1.0.2, such as the formal neighborhood of a point in a derived stack, a formal moduli problem (in the sense of [7]), $\mathcal{L}_\infty$ algebras, the Maurer-Cartan equation for a $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ and the associated Maurer-Cartan formal moduli problem $B\mathfrak{g}$ etc... For an expository introduction to derived stacks, see [13]. The following material is mainly based on [1].

7.1 Formal moduli problems

In our setup, formal moduli problems are constructed to capture the formal geometries of moduli spaces of solutions to certain defining differential equations. The main motivation of the current digression on introducing the notion of a formal moduli problem can be outlined as follows. Consider a classical data $(M, F_M, S, G)$ where $F_M$ denotes the space of fields on some base manifold $M$, $S$ is a smooth action functional on $F_M$, and $G$ is a certain group encoding the symmetries. We define a perturbative classical field theory on $M$ to be the sheaf $\mathcal{E}\mathcal{L}$ of derived stacks on $M$: To each open subset $U$ of $M$, one assigns

$$U \mapsto \mathcal{E}\mathcal{L}(U) \in \text{dStk}$$

(7.3)

where $\text{dStk}$ denotes the $\infty$-category of derived stacks [13, 22] and $\mathcal{E}\mathcal{L}(U)$ is given in the functor of points formalism as

$$\mathcal{E}\mathcal{L}(U) : \text{cdga}^{\leq 0}_k \to \text{sSets} \ (\text{or} \ \text{\infty-Grpds})$$

(7.4)

where $\text{cdga}^{\leq 0}_k$ and $\text{sSets} (\text{or} \ \text{\infty-Grpds})$ denote the category of commutative differential graded $k$-algebras and the $\infty$-category of simplicial sets ($\infty$-groupoids) respectively. Here $\mathcal{E}\mathcal{L}(U)(R)$ is the simplical set of solutions to the defining relations (i.e. EL-equations) with values in $R$. In other words, the points of $\mathcal{E}\mathcal{L}(U)$ form an $\infty$-groupoid. For more details on $\infty$-categories or related concepts, see [22].

As discussed above, in order to circumvent certain problems we work with the derived moduli space of solutions instead of the naïve one. Furthermore, we also intend to capture the perturbative behavior of the theory. Hence, this derived moduli space is defined as a formal moduli problem

$$\mathcal{E}\mathcal{L}(U) : \text{dgArt}_k \to \text{Ssets}$$

(7.5)

where $\text{dgArt}_k$ is the ($\infty$-)category of dg artinian algebras, where morphisms are simply maps of dg commutative algebras (cf. Appendix A of [1] or [22]).
Remark 7.1.1. In order to remember the perturbative behavior around the solution $p \in \mathcal{L}(U)$, we employ the notion of a formal neighborhood of a point (cf. [1], Appendix A). This concept essentially helps us to make the scheme structure sensitive enough to encode small thickenings of a point obtained by adding infinitesimal directions. To keep track such infinitesimal directions assigned to a point $p$, it is in fact more suited to use dg artinian algebras as a local model for the scheme structure instead of the usual commutative $k$-algebras. That is, the scheme structure, informally speaking, is locally modeled on a kind of nilpotent commutative dg-algebra such that the structure consists of points with infinitesimal directions attached to them. Furthermore, every formal moduli functor can be manifested by using the language of $L_\infty$-algebras in the sense of [7], which will be stressed below. Now, we intend to elaborate the content of Lurie’s theorem.

Definition 7.1.1. A differential graded Artinian algebra $(A, m)$ is a commutative differential graded algebra

$$A = \bigoplus_{n \in \mathbb{Z}_{\leq 0}} A^n$$

over a field $k$ concentrated in degrees $\leq 0$ such that

1. Each graded component $A^j$ is finite dimensional and $A^j = 0$ for $j << 0$,

2. $A$ has an unique maximal ideal $m$ such that $A/m = k$ and $m^N = 0$ for large $N$.

Definition 7.1.2. A formal moduli problem (or a particular derived stack) is an $\infty$-functor (of $\infty$-categories)

$$\mathcal{F} : dgArt_k \longrightarrow Ssets$$

such that

1. $\mathcal{F}(k)$ is contractible.

2. $\mathcal{F}$ maps surjective morphisms of dg Artinian algebras to fibrations of simplicial sets.

3. Let $A, B, C$ be dg Artinian algebras, and $B \to A$ and $C \to A$ surjective morphisms, then there exists a fibered product $B \times_A C$ such that

$$\mathcal{F}(B \times_A C) \to \mathcal{F}(B) \times_{\mathcal{F}(A)} \mathcal{F}(C)$$

is a weak equivalence.

Now, we shall be interested in constructing a particular kind of formal moduli problem which can be defined as the simplicial set of solutions to the so-called Maurer-Cartan equations. This concept, in fact, frequently emerges in the theory of $L_\infty$-algebras. Therefore, we shall first provide a brief introduction to the theory of $L_\infty$-algebras.

A digression on the theory of $L_\infty$-algebras. Informally speaking, an $L_\infty$ algebra $\mathfrak{g}$ can be considered as a certain dg Lie algebra endowed with a sequence $\{\ell_n\}$ of multilinear maps of (cohomological) degree $2 - n$ as

$$\ell_n : g^ {\otimes n} \longrightarrow g,$$

which are called $n$-brackets with $n = 1, 2, ...$ such that each bracket satisfies a certain graded anti-symmetry condition and an $n$-Jacobi rule (for a complete treatment, see App. A of [1] or [43]). In order to motivate the notion of a $L_\infty$ algebra, we shall first investigate differential graded Lie algebras. Note that all kinds of $L_\infty$ algebras we shall be interested in are indeed differential graded Lie algebras.

Definition 7.1.3. A differential graded Lie algebra $\mathfrak{g}$ over a ring $R$ is a dg $R$-module $(\mathfrak{g}, d)$ where $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^n$ together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that for all $X, Y, Z \in \mathfrak{g}$, one has

1. (Graded anti-symmetry) $[X, Y] = -(-1)^{deg(X)deg(Y)}[Y, X]$.

2. (Graded Leibniz rule) $d[X, Y] = [dX, Y] + (-1)^{deg(X)}[X, dY]$.

3. (Graded Jacobi rule) $X, [Y, Z]] = [[X, Y], Z] + (-1)^{degXdegY}[Y, [X, Z]]$.  

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Example 7.1.1. Let $M$ be a smooth manifold and $\mathfrak{h}$ a Lie algebra. Then there exists a natural dgla structure (which will be central and will appear in the context of gauge theories) given as follows:

$$g := \Omega^*(M) \otimes \mathfrak{h},$$

where the differential is the usual de Rham differential $d_{dR}$ and the bracket $[\cdot, \cdot]$ is given by

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_\mathfrak{h}.$$  

(7.10) 

(7.11)

Definition 7.1.4. An $L_\infty$ algebra over $R$ is a $\mathbb{Z}$-graded, projective $R$-module

$$g = \bigoplus_{n \in \mathbb{Z}} g^n$$

(7.12) 

equipped with a sequence

$$\{ \ell_n : g^\otimes n \to g \}$$

(7.13) 

of multilinear maps of (cohomological) degree $2 - n$, which are called $n$-brackets with $n = 1, 2, \ldots$, such that each bracket satisfies the following conditions:

1. Graded anti-symmetry: For all $n$ and for $i = 1, \ldots, n - 1$ one has

$$\ell_n(a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n) = -(-1)^{\text{deg}(a_i)\text{deg}(a_{i+1})} \ell_n(a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_n).$$

(7.14) 

2. $n$-Jacobi rule: For all $n,$

$$0 = \sum_{k=1}^n (-1)^k \left( \sum_{i_1 < i_2 < \cdots < i_k < j_{k+1} < \cdots < j_n} (-1)^{\text{sign}(\sigma)} \ell_{n-k+1}(\ell_k(a_{i_1}, \ldots, a_{i_k}, a_{j_{k+1}}, \ldots, a_{j_n})) \right)$$

(7.15) 

where $\{i_1, i_2, \ldots, i_k, j_k+1, \ldots, j_n\} = \{1, 2, \ldots, n\}$ and $(-1)^{\text{sign}(\sigma)}$ denotes the sign of the permutation for assigning the element of the set $\{i_1, i_2, \ldots, i_k, j_k+1, \ldots, j_n\}$ to the element of $\{1, 2, \ldots, n\}.$

Remark 7.1.2. From the definition of an $L_\infty$ algebra, one can conclude that $\ell_1^2 = 0$, and $\ell_2$ satisfies the conditions in the Definition 7.1.3. Therefore, we also write $\ell_1 := d$ and $\ell_2 := [\cdot, \cdot].$

A first natural example of $L_\infty$ algebras. One can revisit Example 7.1.1 and interpret $g$ as a $L_\infty$ algebra in the following way:

$$g := \Omega^*(M) \otimes \mathfrak{h},$$

(7.16) 

where the only non-zero multilinear maps are $\ell_1 := d_{dR}$ and $\ell_2 := [\cdot, \cdot]$ such that

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_\mathfrak{h}$$

for all $\alpha \otimes X, \beta \otimes Y \in \Omega^*(M) \otimes \mathfrak{h}.$

Definition 7.1.5. For an $L_\infty$ algebra $g$, the Maurer-Cartan (MC) equation is given as

$$d \alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^\otimes n) = 0$$

(7.17)

where $\alpha$ is an element of degree 1.

Note that when we reconsider the case $g := \Omega^*(M) \otimes \mathfrak{h}$, the MC equation reduces to

$$d_{dR} A + \frac{1}{2} [A, A] = 0 \quad \text{where} \quad A \in \Omega^1(M) \otimes \mathfrak{h}.$$ 

(7.18)

The end of digression.
We now like to present a construction for a particular kind of formal moduli problem, which can be defined as the simplicial set of solutions to the Maurer-Cartan equations.

**Definition 7.1.6.** Let \( g \) be an \( L_\infty \) algebra, \((A, m)\) a dg Artinian algebra. We define the simplicial set \( MC(g \otimes m) \) of solutions to the Maurer-Cartan equation in \( g \otimes m \) as follows:

\[
MC(g \otimes m) \in Fun(\Delta, Sets)
\]

(7.19)

where an \( n \)-simplex in the set \( MC(g \otimes m)_n \) of \( n \)-simplices is an element

\[
\alpha \in g \otimes m \otimes \Omega^*(\Delta^n)
\]

(7.20)

of cohomological degree 1 that satisfies the Maurer-Cartan equation 7.1.5, i.e.

\[
da \alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0.
\]

(7.21)

**Remark 7.1.3.** In Definition 7.1.6, \( \alpha \) is in fact an element of the tensor product complex \( g \otimes m \otimes \Omega^*(\Delta^n) \) of dg algebras which is defined as

\[
g \otimes m \otimes \Omega^*(\Delta^n) = \bigoplus_k (g \otimes m \otimes \Omega^*(\Delta^n))^k
\]

(7.22)

where \( g = \bigoplus_k g^k \) with the differential \( d_g \), \( m = \bigoplus_k m^k \) with the differential \( d_A \) and \( \Omega^*(\Delta^n) \) is the usual de Rham complex on the \( n \)-simplex \( \Delta^n \) with the de Rham differential \( d_{dR} \). Here \( \Delta^n \) denotes an \( n \)-simplex in \( \mathbb{R}^{n+1} \) given as a set

\[
\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \text{ and } 0 \leq x_k \leq 1 \text{ for all } k\}.
\]

(7.23)

Therefore, the degree \( k \) component of \( g \otimes m \otimes \Omega^*(\Delta^n) \) is given by

\[
(g \otimes m \otimes \Omega^*(\Delta^n))^k = \bigoplus_{p+q+r = k} g^p \otimes m^q \otimes \Omega^r(\Delta^n),
\]

(7.24)

and hence we obtain the **total complex associated to the triple complex**

\[
g \otimes m \otimes \Omega^*(\Delta^n) = \bigoplus_k \bigoplus_{p+q+r = k} g^p \otimes m^q \otimes \Omega^r(\Delta^n)
\]

(7.25)

with the **total differential** \( d^k_{tot} : (g \otimes m \otimes \Omega^*(\Delta^n))^k \to (g \otimes m \otimes \Omega^*(\Delta^n))^{k+1} \) defined by

\[
d^k_{tot} = \sum_{p+q+r = k} d^p_{1,q,r} + (-1)^p d^p_{2,q,r} + (-1)^{p+q} d^p_{3,q,r}
\]

(7.26)

where

\[
d^p_{1,q,r} = d^p_g \otimes id^q_A \otimes id^r_{dR}, \quad d^p_g : g^p \to g^{p+1}
\]

(7.27)

\[
d^p_{2,q,r} = id^p_g \otimes d^q_A \otimes id^r_{dR}, \quad d^q_A : m^q \to m^{q+1}
\]

(7.28)

\[
d^p_{3,q,r} = id^p_g \otimes id^q_A \otimes d^r_{dR}, \quad d^r_{dR} : \Omega^r \to \Omega^{r+1}.
\]

(7.29)

For a more concrete treatment to the notions like double/triple complexes and their total complexes, see [22], Chapter 12. In order to illustrate the situation related to the triple complexes and motivate the structure of such "higher dimensional" cochain complexes, one can consider a rather simple setting in which \( A \) is assumed to be an ordinary \( k \)-algebra. Note that \( A \) can be viewed as a complex that is concentrated at degree 0, and all other components are trivial with differential being zero. Hence, in this situation, we can consider \( g \otimes m \otimes \Omega^*(\Delta^n) \) as a double complex and write \( g \otimes \Omega^*(\Delta^n) \) instead. Furthermore, we diagrammatically have
\[ \cdots \to g^p \otimes \Omega^{r+1}(\Delta^n) \xrightarrow{d^p,r+1} g^{p+1} \otimes \Omega^{r+1}(\Delta^n) \xrightarrow{d^p,r} \cdots \]
\[ \cdots \to \Omega^r(\Delta^n) \xrightarrow{d^p,r} \Omega^{r+1}(\Delta^n) \xrightarrow{d^p,r} \cdots \]
(7.30)

where \( d^p,r = d^p_g \otimes \text{id}_{\Omega^r} \) and \( d^p,r = \text{id}_{d^p_g} \otimes d^r_d \) for all \( p, r \). Note that each square in the diagram is commutative, and hence different parts of the differential are compatible. For the precise structural relations, we again refer to [22], Ch.12.

**Definition 7.1.7.** Given an \( \mathcal{L}_\infty \) algebra \( g \), we can define a functor \( B_g \in \text{Fun}(dgArt_k, sSets) \) associated to \( g \) as follows.

\[ B_g : dgArt_k \to sSets, \quad (A, m) \mapsto B_g[(A, m)] := MC(g \otimes m) \]
(7.31)

where the set of \( n \)-simplicies is defined as above (cf. Definition 7.1.6):

\[ MC(g \otimes m)_n = \left\{ \alpha \in \bigoplus_{p+q+r=1} g^p \otimes m^q \otimes \Omega^r(\Delta^n) : \right. \]
\[ \left. \quad d\alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0. \right\} \]
(7.32)

**Lemma 7.1.1.** The functor \( B_g \) is a formal moduli problem.

**Theorem 7.1.1.** [7] Every formal moduli problem is represented by a Maurer-Cartan functor \( B_g \) for some differential graded Lie algebra (or an \( \mathcal{L}_\infty \) algebra) \( g \) up to a weak equivalence. More precisely, there exists an equivalence of \( \infty \)-categories

\[ dgla_k \sim \text{Moduli}_k \subset \text{Fun}(dgArt_k, sSets) \]
(7.33)

where \( dgla_k \) and \( \text{Moduli}_k \) denote \( \infty \)-categories of differential graded Lie algebras over \( k \) and that of formal moduli problems over \( k \) respectively with \( k \) being a field of characteristic zero.

**Remark 7.1.4.** Here, \( dgla_k \) is in fact an \( \infty \)-category arising as the homotopy category with weak equivalences being (chains of) quasi-isomorphisms of the underlying \( dg \) \( k \)-modules. In that respect, two \( dgla 's \) \( g \) and \( g' \) induce equivalent formal moduli problems provided that they are related to each other by a chain of quasi-isomorphisms. That is,

\[ B_g \sim B_{g'} \iff \exists \phi = \{ \phi_i \} : g \to g'. \]
(7.34)

where each \( \phi_i \) is a degree-wise quasi-isomorphism.

### 7.2 Sheaf of formal moduli problems

Having introduced the notion of a formal moduli problem compatible with the language of \( \mathcal{L}_\infty \) algebras, it turns out that a formal moduli problem is an unexpectedly tractable notion - thanks to the Lurie’s theorem 7.1.1- in the sense that all kinds of formal moduli problems \( F \), up to weak equivalences, can be represented in a relatively simple form:

\[ F = B_g \]
(7.35)

for some \( dgla \) \( g \). Note that we are interested in particular formal moduli problems that define derived moduli spaces of solutions to the Euler-Lagrange equations on an open subset \( U \) of \( M \). Therefore, we shall next seek for a well-defined notion of a “local” formal moduli problem with suitable local-to-global properties. The structure one requires is called a local \( \mathcal{L}_\infty \) algebra [1, 57]. This will serve as a sheaf of \( \mathcal{L}_\infty \) algebras associated to “local” formal moduli problems.
Definition 7.2.1. Let $M$ be a manifold. A local $\mathcal{L}_\infty$ algebra on $M$ consists of the following data:

1. A graded vector bundle $L \xrightarrow{\pi} M$ over $M$ where $L = \bigoplus L^n$ with the space of smooth sections being denoted by
   $$\mathcal{L} := \Gamma(M, L).$$
   Furthermore, we denote the space of local sections over an open subset $U$ of $M$ by
   $$\mathcal{L}(U) := \Gamma(U, L).$$

2. A differential operator $d : \mathcal{L} \to \mathcal{L}$ of cohomological degree 1 such that $d^2 = 0$.

3. A sequence $\{\ell_n : \mathcal{L} \otimes^n \to \mathcal{L}\}$ of multilinear maps of (cohomological) degree $2 - n$ with $n \geq 2$ such that $d$ along with the sequence $\{\ell_n\}$ endow $\mathcal{L}$ with the structure of an $\mathcal{L}_\infty$ algebra.

We have the following immediate and prototype example. Let $\mathfrak{h}$ be a Lie algebra, $L$ the exterior algebra bundle over $M$

$$L := \bigwedge^* T^* M \otimes \mathfrak{h} \longrightarrow M$$

such that the corresponding sections are $\mathfrak{h}$-valued 1-forms where for all open subset $U$ of $M$,

$$\mathcal{L}(U) = \Omega^*(U) \otimes \mathfrak{h}.$$

Note that one can revisit Example 7.1.1 and interpret $\mathcal{L}(U)$ as an $\mathcal{L}_\infty$ algebra with the structure maps $\{\ell_n\}$ where the only non-zero multilinear maps on $\mathcal{L}$ are $d := d_{dR}$ and $\ell_2 := [,]$, which is given by

$$[\alpha \otimes X, \beta \otimes Y] := \alpha \wedge \beta \otimes [X, Y]_\mathfrak{h}.$$

Now we are in the place of introducing manifestly the following sheaf $\mathcal{B}L$ of formal moduli problems associated to a given local $\mathcal{L}_\infty$ algebra $L$. For the proof of being indeed a sheaf, we refer to [1]: Let $M$ be a manifold and $L$ a local $\mathcal{L}_\infty$ algebra on $M$. Then, we set

$$\mathcal{B}L : \text{Opens}_{op} M \longrightarrow \text{Moduli}_{k}, \quad U \mapsto \mathcal{B}L(U)$$

where $\text{Opens}_{op} M$ is the category of open subsets of $M$ with morphisms being canonical inclusions, and $\text{Moduli}_k$ is as in Theorem 7.1.1. Here, for all open subset $U$ of $M$, $\mathcal{B}L(U)$ is the formal moduli problem

$$\mathcal{B}L(U) : dg\text{Art}_k \longrightarrow s\text{Sets}, \quad (A, m_A) \mapsto \mathcal{B}L(U)[(A, m_A)] := MC(\mathcal{L}(U) \otimes m_A)$$

such that the set $MC(\mathcal{L}(U) \otimes m_A)[n]$ of $n$-simplicies is defined as in 7.32 with the replacement of $g$ by $\mathcal{L}(U)$.

8 Examples of formal moduli problems

We first intend to summarize what we have done so far and provide a kind of a recipe to motivate the derived geometric interpretation of a classical field theory. We then concentrate on a particular 2+1 dimensional Einstein gravity.

i. Employing the above approaches, describing a classical field theory boils down to the study of the moduli space $\mathcal{E}L$ of solutions to the Euler-Lagrange equations (the critical locus of action functional). This is in fact encoded by a certain moduli functor.

ii. As stressed in Section 3, a moduli functor, however, would not be representable in general due to the existence of degenerate critical points or non-freeness of the action of the symmetry group on the space of fields. In order to avoid these sorts of problems (and to capture the perturbative behavior at the same time), one may replace the naïve notion of a moduli problem by a formal moduli problem as addressed in Section 7.2.
iii. A formal moduli problem $\mathcal{F}$, on the other hand, turns out to be unexpectedly tractable notion in the sense that understanding $\mathcal{F}$, at the end of the day, boils down to finding a suitable $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ (a dgla, in fact) such that $\mathcal{F}$ can be represented by the Maurer-Cartan functor $\mathcal{B}\mathfrak{g}$ associated to $\mathfrak{g}$.

iv. Having obtained an appropriate $\mathcal{L}_\infty$ algebra $\mathfrak{g}$, we can analyze the structure of $\mathfrak{g}$ so as to encode the aspects of the theory.

We shall first briefly give two relatively tractable examples. For details, see \cite{1}.

**Example 8.0.1.** Consider a free scalar massless field theory on a Riemannian manifold $M$ with space of field being $C^\infty(M)$ and the action functional being of the form

$$S(\phi) := \int_M \phi \Delta \phi. \tag{8.1}$$

The corresponding E-L equation in this case turns out to be

$$\Delta \phi = 0, \tag{8.2}$$

and hence the moduli space $EL$ of solutions to the E-L equations is the moduli space of harmonic functions

$$\{ \phi \in C^\infty(M) : \Delta \phi = 0 \}. \tag{8.3}$$

Now, having employed the derived enrichment $\mathcal{E}L$ of $EL$ as described above, we need to find a suitable $\mathcal{L}_\infty$ algebra $\mathcal{E}$ whose Maurer-Cartan factor $\mathcal{B}\mathcal{E}$ represents the formal moduli problem $\mathcal{E}L$. The answer is as follows: We define $\mathcal{E}$ to be the two-term complex concentrated in degree 0 and 1

$$\mathcal{E} : C^\infty(M) \xrightarrow{\Delta} C^\infty(M)[-1], \tag{8.4}$$

equipped with a sequence $\{ \ell_n \}$ of multilinear maps where $\ell_1 := \Delta$ and $\ell_i = 0$ for all $i > 1$. The Maurer-Cartan equation, on the other hand, turns out to be

$$\Delta \phi = 0. \tag{8.5}$$

Hence, the set of 0-simplices of the simplicial set $\mathcal{B}\mathcal{E}(A)$ for $A$ ordinary Artinian algebra is given as

$$\{ \phi : \Delta \phi = 0 \}. \tag{8.6}$$

For further details and interpretation of other simplices, see chapter 2 of \cite{10} or chapter 4 of \cite{1}.

**Example 8.0.2.** We shall revisit Chern-Simons gauge theory on a closed, orientable 3-manifold $X$ with the gauge group $H$. As usual, Let $P \to X$ be a principal $H$-bundle on $X$, $\mathfrak{h}$ denote the Lie algebra of $H$. Suppose $A \in A := \Omega^1(X) \otimes \mathfrak{h}$ is the Lie algebra-valued connection 1-form on $X$ such that the Chern-Simons action functional $CS : A \to S^1$ is given by

$$CS[A] = \int_X \langle A, d_A A + \frac{2}{3} A \wedge A \rangle \tag{8.7}$$

where $\langle \cdot, \cdot \rangle$ is a certain bilinear form on $\mathfrak{h}$. Here, the gauge group $\mathcal{G}$ is locally of the form $Map(U, H)$ with the usual action on the space $A$. The corresponding E-L equation, in this case, turns out to be

$$F_A := d_A A + A \wedge A = 0, \tag{8.8}$$

where $F_A$ is the curvature two-form on $X$ associated to $A$. Hence, the critical locus of $CS$ modulo gauge transformations is the set

$$\{ [A] \in \Omega^1(X) \otimes \mathfrak{h} : d_A A + A \wedge A = 0 \}. \tag{8.9}$$

As before, we define a suitable $\mathcal{L}_\infty$ algebra $\mathfrak{g}$ encoding the formal moduli problem as follows:

$$\mathfrak{g} := \Omega^*(X) \otimes \mathfrak{h}[1], \tag{8.10}$$
where the only non-zero multilinear maps are \( \ell_1 := d_{dR} \) and \( \ell_2 := [\cdot, \cdot] \) given as in Example 7.1.1. Notice that the Maurer-Cartan equation, in this case, becomes

\[
d_{dR}A + \frac{1}{2}[A, A] = 0, \tag{8.11}
\]

and hence the corresponding the Maurer-Cartan functor \( Bg \) yields the desired result. We shall elaborate the construction below. A relatively complete treatment can be found in chapter 4 of [1], chapter 5 of [11], or [30]. Furthermore, as stressed in [11], the space of all fields associated to the theory, which is encoded by a particular \( \mathcal{L}_\infty \) algebra \( g \), can also be interpreted in the Batalin-Vilkovisky formalism as follows:

- The space of degree \(-1\) fields, so-called ghosts, corresponds to the space
  \[
  \Omega^0(X) \otimes \mathfrak{h} = Map(X, H). \tag{8.12}
  \]

- The space of degree \(0\) fields, so-called fields, corresponds to the space
  \[
  \Omega^1(X) \otimes \mathfrak{h}. \tag{8.13}
  \]

- The space of degree \(1\) fields, so-called anti-fields, corresponds to the space
  \[
  \Omega^2(X) \otimes \mathfrak{h}. \tag{8.14}
  \]

- The space of degree \(2\) fields, so-called anti-ghosts, corresponds to the space
  \[
  \Omega^3(X) \otimes \mathfrak{h}. \tag{8.15}
  \]

### 8.1 Formal moduli problem of Chern-Simons theory

A formal moduli problem encoding \textit{deformation theory} of the flat \( H \)-bundles \( P \to M \) on a closed orientable 3-manifold \( M \) can be defined as follows [1]: Let \( \nabla \) be a flat connection on \( P \). Define an \( \mathcal{L}_\infty \) algebra \( g \) to be

\[
g := \Omega^*(M) \otimes \mathfrak{h}, \tag{8.16}
\]

where the only non-zero multilinear maps are \( \ell_1 := d_{d\nabla} \) and \( \ell_2 := [\cdot, \cdot] \) given as in Example 7.1.1. Here, \( d_{d\nabla} \) denotes the covariant derivative defined as a coupling of the de Rham differential \( d_{dR} \) with the connection \( \nabla \):

\[
d_{d\nabla} := d_{dR} + [\nabla, \cdot]. \tag{8.17}
\]

**Remark 8.1.1.** As \( \nabla \) is flat, the differential \( d_{d\nabla} \) squares zero, and hence one can form the following complex

\[
\cdots \to \Omega^i(M, \mathfrak{h}) \xrightarrow{d_{d\nabla}} \Omega^{i+1}(M, \mathfrak{h}) \to \cdots \tag{8.18}
\]

where \( \Omega^i(M, \mathfrak{h}) \) is just the short-hand standard notation for \( \Omega^i(M) \otimes \mathfrak{h} \) for all \( i \), and will be used repeatedly throughout the discussion below.

Starting with the 0-simplicies, let \( (R, m_R) \) be a dg Artinian algebra with the maximal ideal \( m_R \). As the difference \( \nabla' - \nabla \) is again \( \mathfrak{h} \)-valued 1-form on \( P \) for any \( \mathfrak{h} \)-valued 1-form \( \nabla' \), the space \( \Omega^i(M, \mathfrak{h}) \) is in fact affine. A deformation of \( \nabla \) is then given by an element

\[
A \in \Omega^1(M, \mathfrak{h}) \otimes m_R^0. \tag{8.19}
\]

Therefore, the curvature \( F_{\nabla'} \) (or just \( F(A) \)) of the deformed connection \( \nabla' := \nabla + A \) is given by

\[
F(A) = d_{dR}(\nabla + A) + \frac{1}{2}[\nabla + A, \nabla + A]
\]

\[
= d_{dR}\nabla + \frac{1}{2}[\nabla, \nabla] + d_{dR}A + \frac{1}{2}[A, A] + [\nabla, A]
\]

\[
= d_{d\nabla}A + \frac{1}{2}[A, A] \quad (\text{from } F_{d\nabla} = 0) \tag{8.20}
\]
such that from the usual Bianchi identity, we have

\[ 0 = d_{\nabla'} F_{\nabla'} = d_{\text{deformed}} F_{\nabla'} + [\nabla', F_{\nabla'}] = d_{\text{deformed}} F_{\nabla'} + [\nabla', F_{\nabla'}] + [A, F_{\nabla'}] = d_{\nabla} F_{\nabla} + [A, F(A)] = 0. \] (8.21)

Now we denote \( F_{\nabla} \) by \( F(A) \) to emphasize the connection \( A \) deforming \( \nabla \), then the computation above gives

\[ d_{\nabla} F(A) + [A, F(A)] = 0. \] (8.22)

Note that even if it captures the information about the deforming connection \( A \), the notation \( F(A) \) could be misleading in the sense that while \( F(A) \) stands for the curvature of the deformed connection \( \nabla' := \nabla + A \) (deformed by \( A \)), \( F_A \) denotes the curvature 2-form for the connection \( A \). Now, one can define the following formal moduli problem \( Bg \in \text{Moduli}_k \) for the Chern-Simons theory.

**Lemma 8.1.1.** Let \((R, m_R)\) be a dg Artinian algebra with the maximal ideal \( m_R \) and \( n \in \mathbb{Z}_{\geq 0} \). Then the set of \( n \)-simplicies of \( Bg(R) \) is given by

\[ Bg(R)_n = \left\{ A \in \bigoplus_{p+q+r=1} \Omega^p(M, h) \otimes m_R^q \otimes \Omega^r(\Delta^n) : d_{\nabla} A + d_R A + d_{\text{deformed}} + \frac{1}{2}[A, A] = 0 \right\} \]

where \( d_{\nabla}, d_R \) and \( d_{\text{deformed}} \) denote the differentials on dg algebras \( \Omega^p(M, h) \), \( R \) and \( \Omega^r(\Delta^n) \) respectively. Furthermore, the choice of each \( \mp \) sign can be determined as instructed in Remark 7.1.3.

**Proof.** For the construction and details we refer to Costello and Gwilliam’s book [1], ch. 4.3, pg.30. \( \Box \)

**Remark 8.1.2.** One can recover the usual moduli space of flat \( H \)-connection from Lemma 8.1.1 together with some extra structures being manifested by higher simplicies. This essentially relates the gauge equivalent solutions in the following way:

1. Let \((R, m_R)\) be an ordinary Artinian algebra with the maximal ideal \( m_R \), and \( n = 0 \). Note that \( \Omega^*(\Delta^0) \cong k \) if \( * \neq 0 \), else it is 0. As \( R \) can be viewed a dg algebra concentrated in degree 0 (i.e. \( q = 0 \)), and \( n = 0 \) with \( d_R = 0 = d_{\text{deformed}} \), the only possible scenario in which one can form a cohomological degree 1 element is when \( p = 1 \). The set of 0-simplicies, therefore, is given as

\[ Bg(R)_0 = \left\{ A \in \Omega^1(M, h) \otimes m_R : d_{\nabla} A + \frac{1}{2}[A, A] = 0 \right\} \] (8.23)

which is the usual moduli space of flat \( H \)-connections where \( F(A) := d_{\nabla} A + \frac{1}{2}[A, A] \) is the curvature of deformed connection as above.

2. Assume \((R, m_R)\) is again an ordinary Artinian algebra (i.e. \( q = 0 \)) with the maximal ideal \( m_R \), and \( n = 1 \). Now, since \( n = 1 \), only \( \Omega^0(\Delta^1) \) and \( \Omega^1(\Delta^1) \) will survive (i.e. \( r \in \{0, 1\} \)). Thus, one has two possible configurations to form cohomologicaly degree 1 element:

\[ (p, q, r) \in \{(1, 0, 0), (0, 0, 1)\} \] (8.24)

Therefore, a generic degree 1 element \( A \) in \( Bg(R)_1 \) has a decomposition

\[ \left( \Omega^1(M, h) \otimes m \otimes \Omega^0(\Delta^1) \right) \oplus \left( \Omega^0(M, h) \otimes m \otimes \Omega^1(\Delta^1) \right) \] (8.25)

where \( \Omega^0(\Delta^1) \cong C^\infty([0, 1]) \) and \( \Omega^1(\Delta^1) = \text{span}_k \{dt\} \) such that \( A \) can be expressed in a local chart \((U, x)\) as

\[ A = A_0(t) + A_1(t) \cdot dt. \] (8.26)

Herein \( A_0(t) = a_0(x, t)dx^i \) with \( a_0(x, t) \) is \( h \otimes m \)-valued smooth function on \([0, 1]\), and \( A_1(t) \) is a smooth \( h \otimes m \)-valued function on \( M \) parametrized by \( t \). It follows from the properties of
triple complexes outlined in Remark 7.1.3 and the definition of the Maurer-Cartan equation 7.1.5 that one can obtain the following equations

\[ \begin{align*}
  d_{\nabla} A_0(t) + \frac{1}{2} [A_0(t), A_0(t)] &= 0 \\
  \frac{dA_0(t)}{dt} + [A_1(t), A_0(t)] &= 0,
\end{align*} \]

(8.27)

(8.28)

together with the commutative diagram

Here, Equation 8.27 implies that \( \{A_0(t)\} \) defines a flat family of connections while Equation 8.28 implies that the gauge equivalence classes of the family \( \{A_0(t)\} \) are independent of \( t \) up to homotopy defined by \( A_1(t) \).

3. Higher simplicies provide an enriched and refined structure through which one can capture further relations between equivalences, and relations between such relations etc... In other words, higher simplicial structures in derived stacks allow us to encodes "symmetries between symmetries" and "symmetries between symmetries between symmetries" type argument. Different layers of the simplicial structure encode different levels of symmetries/equivalences. Each set of simplicies of a derived stack records further relations, and hence it is able to encode the "higher symmetries" argument above. For more details on the interpretation of higher simplicial structures in the case of gauge or free field theories, see [1], ch.4.

8.2 Formal moduli problem of 3D Einstein gravity

As an immediate application to the formulation of gauge theories in the language of formal moduli problems [1], we have a natural formal moduli problem for the Cartan geometric formulation of a 2+1 dimensional Einstein gravity.

**Corollary 8.2.1.** The construction in Lemma 8.1.1 defines a natural formal moduli problem for the 2+1 Cartan's geometric formulation of vacuum Einstein gravity theory with vanishing cosmological constant.

**Proof.** As manifestly analyzed in Proposition 3.0.1 and Theorem 5.0.1, we have the following groupoid-valued functor

\[ \mathcal{E} : \mathcal{C}^{op} \longrightarrow \text{Grpds} \]

(8.30)

which defines a *stack* where for each object \( U \) of \( \mathcal{C} \), \( \mathcal{E}(U) \) is a groupoid of (Ricci) flat pseudo-Riemannian metrics on \( U \) with objects being elements of the set \( \text{FMet}(U) \)

\[ \text{FMet}(U) := \{ g \in \Gamma(U, Met_M) : R_{\mu\nu} = 0 \} \]

(8.31)

of (Ricci) flat pseudo-Riemannian metrics on \( U \) where \( Met_M \) denotes metric "bundle" on \( M \) and \( R_{\mu\nu} \) is the Ricci-tensor. Now, thanks to the Cartan formalism, which is briefly explained in Section 2.1, one can reformalize such a gravity theory in which the Einstein-Hilbert action is presented as

\[ I'_{EH} = \int_M \epsilon^a \wedge \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) \]

(8.32)
where $\omega \in \Omega^1(LM, \mathfrak{so}(2,1))$ and $e \in \Omega^1(LM, \mathbb{R}^{2+1})$ are $\mathfrak{so}(2,1)$-valued Ehresmann connection 1-form on the frame bundle $LM$ on $M$, and $e \in \Omega^1(LM, \mathbb{R}^{2+1})$ is the coframe field. The variation of this action, on the other hand, with respect to $\omega$ and $e$ independently yields

$$\delta I_{EH}[e, \omega] = \int_M tr(\delta \omega \wedge \Omega[\omega] + \delta e \wedge d_e e), \quad (8.33)$$

and thus the corresponding field equations are of the form

$$0 = \Omega[\omega] = d\omega + \frac{1}{2}[\omega, \omega] \quad (8.34)$$
$$0 = d_\omega e = de^a + [\omega, e]. \quad (8.35)$$

Note that [15], solving 8.35 for $\omega$ as a function of $e$ and rewriting 8.34 as an equation of $\omega[e]$ give rise to the usual vacuum Einstein field equations with vanishing cosmological constant. Now, set the corresponding Cartan connection

$$A \in \Omega^1(LM, \mathfrak{so}(2,1)) \quad (8.36)$$

which can be expressed uniquely as a decomposition

$$A = \omega + e \in \Omega^1(LM, \mathfrak{so}(2,1)) \oplus \Omega^1(LM, \mathbb{R}^{2+1}). \quad (8.37)$$

Let $\mathfrak{g}$ be an $\mathcal{L}_\infty$ algebra

$$\mathfrak{g} := \Omega^*(M) \otimes \mathfrak{so}(2,1), \quad (8.38)$$

where the only non-zero multilinear maps are $\ell_1 := d_\nabla$ and $\ell_2 := [\cdot, \cdot]$ given as in Example 7.1.1. Here, $d_\nabla$ denotes the covariant derivative defined as a coupling of the de Rham differential $d_{\mathbb{R}}$ with the fixed flat connection $\nabla$ on $LM$:

$$d_\nabla := d_{\mathbb{R}} + [\nabla, \cdot]. \quad (8.39)$$

Then, from Lemma 8.1.1, we have a formal moduli problem $\mathcal{B}_{\mathfrak{g}} \in \text{Moduli}_k$

$$\mathcal{B}_{\mathfrak{g}} : dg_\text{Arts} \longrightarrow s\text{Sets}, \quad (R, m_R) \longmapsto \mathcal{B}_{\mathfrak{g}}(R) \quad (8.40)$$

such that for each $n$ the set of $n$-simplicies of $\mathcal{B}_{\mathfrak{g}}(R)$ is given by

$$\mathcal{B}_{\mathfrak{g}}(R)_n = \left\{ A \in \bigoplus_{p+q+r=1} \Omega^p(M, \mathfrak{so}(2,1)) \otimes m_R^p \otimes \Omega^r(\Delta^n) : d_\nabla A = d_{\mathbb{R}} A + d_{\mathbb{R}} A + \frac{1}{2}[A, A] = 0 \right\}. \quad (8.41)$$

Note that the standard moduli data can be recovered by considering the set of 0-simplicies of $\mathcal{B}_{\mathfrak{g}}(R)$ in the case of $(R, m_R)$ being an ordinary Artinian algebra with the maximal ideal $m_R$. Indeed, we have

$$\mathcal{B}_{\mathfrak{g}}(R)_0 = \left\{ A \in \Omega^1(M, \mathfrak{so}(2,1)) \otimes m_R : d_\nabla A + \frac{1}{2}[A, A] = 0 \right\}. \quad (8.41)$$

When $A$ has the unique decomposition, one has a reductive splitting [51, 52]:

$$F_A = 0 \iff \Omega[\omega] = 0 \text{ and } d_\omega e = 0. \quad (8.42)$$

These are the desired defining relations. □

**Remark 8.2.1.** We should point out that all formal moduli constructions for Einstein gravity work with non-zero cosmological constant as well. As we stressed before, one has a Chern-Simons theory with either $G = SL(2, \mathbb{R}) \times SL(2; \mathbb{R})$ for $\Lambda < 0$ or $G = SL(2, \mathbb{C})$ for $\Lambda > 0$. Therefore, we end up with exactly the same constructions with different gauge groups.
9 The structure of observables on formal moduli problems

9.1 A naïve discussion on factorization algebras

[10, 1] study factorization algebras to provide a generalization of the Kontsevich’s deformation quantization approach to quantum mechanics. In other words, while deformation quantization essentially encodes the nature of observables in one-dimensional quantum field theories, factorization algebra formalism provides an n-dimensional generalization of this approach. To be more precise, we first recall how to describe observables in classical mechanics and those in corresponding quantum mechanical system. Let \( (M, \omega) \) be a symplectic manifold (a phase space), then we define the space \( \mathcal{A}^\text{cl} \) of classical observables on \( M \) to be the space \( C^\infty(M) \) of smooth functions on \( M \). Hence, \( \mathcal{A}^\text{cl} \) forms a Poisson algebra with respect to the Poisson bracket \( \{ \cdot, \cdot \} \) on \( C^\infty(M) \) given by

\[
\{ f, g \} := -\omega(X_f, X_g) = X_f(g) \quad \text{for all } f, g \in C^\infty(M),
\]

where \( X_f \) is the Hamiltonian vector field associated to \( f \in C^\infty(M) \). Here, \( X_f \) is defined implicitly by the equation

\[
\iota_{X_f} \omega = df,
\]

where \( \iota_{X_f} \) denotes the usual contraction operator. Employing geometric quantization formalism [41, 53, 54], a quantization concept boils down to the study of representation theory of (a certain subalgebra \( A \) of) classical observables in the sense that one can construct a quantum Hilbert space \( \mathcal{H} \) and a Lie algebra homomorphism \(^1\)

\[
Q : \mathcal{A} \subset (C^\infty(M), \{ \cdot, \cdot \}) \longrightarrow (\text{End}(\mathcal{H}), [\cdot, \cdot])
\]

together with Dirac’s quantum condition: For all \( f, g \in \mathcal{A} \) we have

\[
[Q(f), Q(g)] = -i\hbar Q(\{f, g\})
\]

where \( [\cdot, \cdot] \) denotes the usual commutator on \( \text{End}(\mathcal{H}) \).

In accordance with the above set-up, while classical observables form a Poisson algebra, the space \( \mathcal{A}^q \) of quantum observables forms an associative algebra which is related to classical one by the quantum condition 9.4. Deformation quantization, in fact, serves as a mathematical treatment that captures this correspondence. In other words, it essentially encodes the procedure of deforming commutative structures to non-commutative ones for a general Poisson manifolds [42].

Factorization algebras, on the other hand, are algebro-geometric objects which are manifestly described sheaf theoretically as follows:

**Definition 9.1.1.** A prefactorization algebra \( \mathcal{F} \) on a manifold \( M \) consists of the following data:

- For each open subset \( U \subseteq M \), a cochain complex \( \mathcal{F}(U) \).
- For each open subsets \( U \subseteq V \) of \( M \), a cochain map \( \iota_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \).
- For any finite collection \( U_1, \ldots, U_n \) of pairwise disjoint open subsets of \( V \subseteq M \), \( V \) open in \( M \), there is a morphism

\[
\iota_{U_1,\ldots,U_n;V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)
\]

**i.** Invariance under the action of symmetric group \( S_n \) permuting the ordering of the collection \( U_1, \ldots, U_n \) in the sense that

\[
\iota_{U_1,\ldots,U_n;V} = \iota_{U_{\sigma(1)},\ldots,U_{\sigma(n)};V} \quad \text{for any } \sigma \in S_n.
\]

That is, the morphism \( \iota_{U_1,\ldots,U_n;V} \) is independent of the ordering of open subsets \( U_1, \ldots, U_n \), but it depends only on the family \( \{U_i\} \).

\(^1\)A Lie algebra homomorphism \( \beta : g \rightarrow \mathfrak{h} \) is a linear map of vector spaces such that \( \beta([X,Y]_g) = [\beta(X), \beta(Y)]_\mathfrak{h} \). Keep in mind that, one can easily suppress the constant "-i\hbar" in 9.4 into the definition of \( Q \) such that the quantum condition 9.4 becomes the usual compatibility condition that a Lie algebra homomorphism satisfies.
ii. Associativity condition: if \( U_{i_1} \circ \cdots \circ U_{i_m} \subset V_i \) and \( V_1 \circ \cdots \circ V_k \subset W \) where \( U_{i_j} \) (resp. \( V_i \)) are pairwise disjoint open subsets of \( V_i \) (resp. \( W \)) with \( W \) open in \( M \), then the following diagram commutes.

\[
\begin{array}{ccc}
\bigotimes_{i=1}^k \otimes_{j=1}^n \mathcal{F}(U_{i,j}) & \longrightarrow & \bigotimes_{i=1}^k \mathcal{F}(V_i) \\
\downarrow & & \downarrow \\
\mathcal{F}(W) & & 
\end{array}
\]  

(9.7)

With this definition in hand, a prefactorization algebra behaves like a co-presheaf except the fact that we use tensor product instead of a direct sum of cochain complexes. Furthermore, we can define a factorization algebra once we impose certain local-to-global conditions on a prefactorization algebra analogous to the ones imposed on presheaves [34]. For a complete discussion, we refer to Ch. 3 of [10] or [34].

Factorization algebras, in fact, serve as \( n \)-dimensional counterparts to those objects which are realized in deformation quantization formalism. In particular, one recovers observables in classical/quantum mechanics when restricts to the case of \( n = 1 \) [10]. For instance, in a particular gauge theory, holonomy observables, namely Wilson line operators, can be formalized in terms of such objects. These are, in fact, the ones that are related to Witten’s Knot invariants. They actually arise from the analysis of certain partition functions in three-dimensional Chern-Simons theory [55]. In this approach to perturbative quantum field theories, quantum observables in these types of theories form a factorization algebra. It turns out that a factorization algebra of quantum observables is related to a (commutative) factorization algebra of associated classical observables in the following sense:

**Theorem 9.1.1.** (Weak quantization Theorem [10]): For a classical field theory and a choice of BV quantization,

1. The space \( \text{Obs}^q \) of quantum observables forms a factorization algebra over the ring \( \mathbb{R}[\hbar] \).
2. \( \text{Obs}^{cl} \cong \text{Obs}^q \mod \hbar \) as a homotopy equivalences where \( \text{Obs}^{cl} \) denotes the associated factorization algebra of classical observables.

Note that the theorem above is just a part of the story, and it is indeed weak in the sense that it is not able to capture the data related to Poisson structures. To provide a correct \( n \)-dimensional analogue of deformation quantization approach, we need to refine the notion of a classical field theory in such a way that the richness of this new set-up become visible. This is where derived algebraic geometry comes into play.

As we discussed above, the space of classical observables forms a (commutative) factorization algebra. This allows us to employ certain cohomological methods encoding the structure of observables in the following sense [10]: Factorization algebra \( \text{Obs}^{cl} \) of observables can be realized as a particular assignment analogous to a co-sheaf of cochain complexes as mentioned above. That is, for each open subset \( U \subset M \) of \( M \), \( \text{Obs}^{cl}(U) \) has a \( \mathbb{Z} \)-graded structure

\[
\text{Obs}^{cl}(U) = \bigoplus_{i \in \mathbb{Z}} \text{Obs}^{cl}_i(U) : \cdots \longrightarrow \text{Obs}^{cl}_{-1}(U) \longrightarrow \text{Obs}^{cl}_{0}(U) \xrightarrow{d_0} \text{Obs}^{cl}_1(U) \longrightarrow \cdots
\]

together with suitable connecting homomorphisms \( d_i : \text{Obs}^{cl}_i(U) \rightarrow \text{Obs}^{cl}_{i+1}(U) \) for each \( i \). Each cohomology group \( H^i(\text{Obs}^{cl}(U)) \) encodes the structure of observables as follows:

- "Physically meaningful" observables are the closed ones with cohomological degree 0, i.e., \( \mathcal{O} \in \text{Obs}^{cl}_0(U) \) with \( d_0 \mathcal{O} = 0 \). (and hence \([\mathcal{O}] \in H^0(\text{Obs}^{cl}(U))\).)
- \( H^1(\text{Obs}^{cl}(U)) \) contains anomalies, i.e., obstructions for classical observables to be lifted to the quantum level. In gauge theory, for instance, there exist certain classical observables respecting gauge symmetries such that they do not admit any lift to quantum observables respecting gauge symmetries. This behaviour is indeed encoded by a non-zero element in \( H^1(\text{Obs}^{cl}(U)) \)
\begin{itemize}
  \item $H^n(\text{Obs}^S(U))$ with $n < 0$ can be interpreted as symmetries, higher symmetries of observables etc. via higher categorical arguments.
  \item $H^n(\text{Obs}^S(U))$ with $n > 1$ has no clear physical interpretation.
\end{itemize}

### 9.2 Construction of Chevalley-Eilenberg complexes

In this section, we shall only present a treatment for the construction of Chevalley-Eilenberg complexes and the corresponding homology/cohomology modules in the case of ordinary Lie algebras $\mathfrak{g}$. The reason of this restriction is just to make the argument more tractable and avoid complicated expressions, which possibly arise from the internal gradings and higher structural relations as in 7.14 or 7.15. Generalizations to dglas or $L_\infty$ algebras are relatively straightforward procedures. Hence, we refer to [1] (App. A), [32] (ch. 21-23) or [7] (ch. 2.2). Now, the current discussion is based on constructions presented in [45].

**Definition 9.2.1.** Let $\mathfrak{g}$ be a Lie algebra over a field $k$ of characteristic zero. We define the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ as

$$U\mathfrak{g} := \text{Tens}(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y])$$

where $\text{Tens}(\mathfrak{g}) := \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n} = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \oplus \mathfrak{g}^{\otimes i} \oplus \cdots$.

Now, by using the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ one can introduce an $U\mathfrak{g}$-module

$$V_i(\mathfrak{g}) := U\mathfrak{g} \otimes_k \bigwedge^i \mathfrak{g} \quad \text{for all } i,$$

along with the natural maps

$$V_0(\mathfrak{g}) \rightarrow U\mathfrak{g}/(\mathfrak{g} \cong k)$$

$$V_1(\mathfrak{g}) \rightarrow V_0(\mathfrak{g}), \quad u \otimes x \mapsto ux.$$

where $V_0(\mathfrak{g}) = U\mathfrak{g}$ and $V_1(\mathfrak{g}) = U\mathfrak{g} \otimes_k \mathfrak{g}$. Hence, we have an exact sequence

$$V_1(\mathfrak{g}) \rightarrow V_0(\mathfrak{g}) \rightarrow k \rightarrow 0.$$

**Definition 9.2.2.** For $k > 1$, we define a morphism $d : V_k(\mathfrak{g}) \rightarrow V_{k-1}(\mathfrak{g})$ as

$$d(u \otimes x_1 \wedge \cdots \wedge x_k) := \sum_{i=1}^k (-1)^{i+1}ux_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k$$

$$+ \sum_{i<j} (-1)^{i+j}u \otimes [x_i,x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k.$$ 

**Lemma 9.2.1.** $d^2 = 0$ and $(V_*(\mathfrak{g}), d)$ is indeed a chain complex.

**Proof.** Let $k = 2$ and $u \otimes x_1 \wedge x_2$ given. Then we have

$$d(d(u \otimes x_1 \wedge x_2)) = d(ux_1 \otimes x_2 - ux_2 \otimes x_1 - u \otimes [x_1,x_2])$$

$$= ux_1x_2 - ux_2x_1 - u(x_1x_2 - x_2x_1)$$

$$= 0.$$

The rest follows from the immediate induction on $k$. See [45] for the explicit expressions. \hfill \Box

The chain complex $V_*(\mathfrak{g}) := U\mathfrak{g} \otimes_k \Lambda^* \mathfrak{g}$ is sometimes called the Chevalley-Eilenberg (C-E) complex or the standard complex. It essentially serves as a suitable projective resolution $\mathcal{P}$ \footnote{A projective resolution $\mathcal{P}$ of a module $N$ is a free resolution of $N$ such that the functor $\text{Hom}_{\text{Mod}}(A, \cdot)$ is exact.} for the base field $k$ in order to define $\text{Tor}^k_{U\mathfrak{g}}(k, M)$ and $\text{Ext}^k_{U\mathfrak{g}}(k, M)$ for any $\mathfrak{g}$-module $M$. In that respect, we have the following observations from [45].
\textbf{Theorem 9.2.1.} $V_*(g) \rightarrow k$ is a projective resolution of the (trivial) $g$-module $k$.

\textbf{Corollary 9.2.1.} Let $V_*(g)$ be as above.

1. If $M$ is a right $g$-module, then the Lie algebra homology modules
\begin{equation}
H^i_{\text{Lie}}(g, M) = \text{Tor}^U_{i+1}(k, M)
\end{equation}
are the homology of the chain complex
\begin{equation}
M \otimes_{Ug} V_*(g) = M \otimes_{Ug} Ug \otimes_k \bigwedge^* g \cong M \otimes_k \bigwedge^* g
\end{equation}
We denote this tensor product complex by
\begin{equation}
k \otimes_{Ug}^L M : \cdots \rightarrow M \otimes_k \bigwedge^2 g \rightarrow M \otimes_k g \rightarrow M.
\end{equation}

2. If $M$ is a left $g$-module, then the Lie algebra cohomology modules
\begin{equation}
H^i_{\text{Lie}}(g, M) = \text{Ext}^i_U(k, M)
\end{equation}
are the cohomology of the chain complex
\begin{equation}
\text{Hom}_g(V_*(g), M) = \text{Hom}_g(Ug \otimes_k \bigwedge^* g, M) \cong \text{Hom}_k(\bigwedge^* g, M)
\end{equation}
together with an isomorphism
\begin{equation}
\bigwedge^i g^* \otimes_k M \cong \text{Hom}_k(\bigwedge^i g, M).
\end{equation}
where $g^*$ is the dual space of $g$. We denote this tensor product complex by
\begin{equation}
\mathbb{R}\text{Hom}_{Ug}(k, M) : \cdots \rightarrow g^* \otimes_k M \rightarrow \bigwedge^2 g^* \otimes_k M \rightarrow \cdots
\end{equation}
In this complex, an $n$-cochain is just $k$-multilinear map
\begin{equation}
f : \bigwedge^n g \rightarrow M
\end{equation}
together with the coboundary maps $\delta : \text{Hom}_k(\bigwedge^n g, M) \rightarrow \text{Hom}_k(\bigwedge^{n+1} g, M)$ defined as follows: For any $x_1 \wedge \cdots \wedge x_{n+1}$ and $f \in \text{Hom}_k(\bigwedge^n g, M)$,
\begin{equation}
\delta f (x_1 \wedge \cdots \wedge x_{n+1}) := \sum (-1)^{i+1} x_if(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{n+1}) + \sum_{i < j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{n+1}).
\end{equation}

\textbf{Remark 9.2.1.} In the context of derived algebraic geometry, we have the following correspondences:

1. The tensor product complex $k \otimes_{Ug}^L M$ introduced in 9.16 defines the derived tensor product of $g$-modules $k$ and $M$ over $Ug$. Note that this in fact boils down to the construction of left derived functor associated to the right exact functor $\cdot \otimes_R A$ for any $R$-module $A$. For an accessible introduction to left/right derived functors, see for instance [39], ch. 23.

2. The other complex $\mathbb{R}\text{Hom}_{Ug}(k, M)$ defined in 9.20, on the other hand, is in fact right derived functor associated to the left exact functor $\text{Hom}_R(A, \cdot)$ for any $R$-module $A$.

\textbf{Remark 9.2.2.} Let $M := k$ be the trivial $g$-module. Then we denote the resulting complexes in 9.16 and 9.20 respectively by
\begin{equation}
C_*(g) \cong \bigwedge^* g \quad \text{and} \quad C^*(g) = \text{Hom}_k(\bigwedge^* g, k) \cong \text{Hom}_k(C_*(g), k).
\end{equation}
They are sometimes referred as $C-E$ complexes as well. As every commutative algebra should be interpreted as an algebra of functions on a certain space [1], we have the following naïve observations capturing the geometric realizations and the roles of $C-E$ complexes in the context of derived algebraic geometry:
1. $C^*(\mathfrak{g})$ can be viewed as an algebra $\mathcal{O}(B\mathfrak{g})$ of functions on the classifying space $B\mathfrak{g}$.

2. $C_* (\mathfrak{g})$ can be considered as the space of distributions on $B\mathfrak{g}$.

As outlined in [32], one can also make sense of these definitions and geometric interpretations of $C^*(\mathfrak{g})$ and $C_* (\mathfrak{g})$ in the case of $\mathfrak{g}$ being a differential graded Lie algebra (or even being an $\mathcal{L}_\infty$ algebra [1]). Therefore, one can provide almost the same constructions with some modifications according to the graded structure of $\mathfrak{g}$. Note that, all kinds of $\mathcal{L}_\infty$ algebras we shall be interested in are, in fact, differential graded Lie algebras.

### 9.3 Factorization algebra of observables

As addressed in [1], in the case of classical field theories, one can make a reasonable measurement only on those fields which are the solutions to the Euler-Lagrange equations. Observables, therefore, are defined as functions

$$\mathcal{O} : EL \longrightarrow k$$

(9.23)
on the moduli space $EL$ of solutions to the Euler-Lagrange equations. Now, we intend to extend this idea to the derived setting and provide an appropriate treatment with the notion of observables on a derived moduli stack of solutions to the E-L equations.

Given a classical field theory, let $\mathcal{L}$ be the corresponding local $\mathcal{L}_\infty$ algebra on a manifold $M$. As outlined in Section 7.2 one can define a sheaf $\mathcal{B}\mathcal{L}$ of formal moduli problem

$$\mathcal{B}\mathcal{L} : \text{Opens}^{op}_{M} \longrightarrow \text{Moduli}_k, \quad U \longmapsto \mathcal{B}\mathcal{L}(U)$$

(9.24)
where $\mathcal{B}\mathcal{L}(U)$ can be considered as a derived space of solutions to field equations of the theory. As noted in Remark 9.2.2, the C-E complex $C^*(\mathcal{L}(U))$ associated to $\mathcal{L}(U)$ can be interpreted as an *algebra* of functions on a derived space of solutions to the field equations over $U$. Then, we can define the space of observables over $U$ in the following natural way [1]:

**Definition 9.3.1.** The space $\text{Obs}^{cl}(U)$ of observables with support on an open subset $U$ of $M$ is defined to be a commutative differential graded $k$-algebra

$$\text{Obs}^{cl}(U) := C^*(\mathcal{L}(U)).$$

(9.25)

Note that it follows directly from the properties of PDEs and the construction of $\mathcal{B}\mathcal{L}$ that if $U_1, \ldots, U_n$ are pairwise disjoint open subsets of $U$, then restrictions of solutions over $U$ to each $U_i$ induce a natural map

$$\mathcal{B}\mathcal{L}(U) \longrightarrow \mathcal{B}\mathcal{L}(U_1) \times \cdots \times \mathcal{B}\mathcal{L}(U_n),$$

(9.26)
such that each function $f$ over $\mathcal{B}\mathcal{L}(U_i)$ can be pulled-back via the natural map above, and hence one obtains a morphism

$$\text{Obs}^{cl}(U_1) \otimes \cdots \otimes \text{Obs}^{cl}(U_n) \longrightarrow \text{Obs}^{cl}(U).$$

(9.27)
Therefore, the assignment $\text{Obs}^{cl}$ admits the structure of pre-factorization algebra. Furthermore, as $\mathcal{B}\mathcal{L}$ is a sheaf, it induces a local-to-global property on $\text{Obs}^{cl}$ in a natural way. Thus, this observation essentially gives a sketch of the proof of the following proposition.

**Proposition 9.3.1.** The assignment

$$\text{Obs}^{cl} : U \longmapsto \text{Obs}^{cl}(U)$$

(9.28)
is a factorization algebra of observables.

### 9.4 Factorization algebra of observables for 3D Einstein gravity

Let $\mathfrak{g}$ be an $\mathcal{L}_\infty$ algebra

$$\mathfrak{g} := \Omega^*(M) \otimes \text{iso}(2, 1),$$

(9.29)
where the only non-zero multilinear maps are $\ell_1 := d\nabla$ and $\ell_2 := [\cdot, \cdot]$ given as in Example 7.1.1. Here, $d\nabla$ denotes the covariant derivative defined as a coupling of the de Rham differential $d_{dR}$ with the fixed flat connection $\nabla$ on $LM$:

$$d\nabla := d_{dR} + [\nabla, \cdot] .$$

(9.30)
Then, as we discussed before, from Lemma 8.1.1, we have a formal moduli problem $B_g \in Moduli_k$

$$B_g : dgArt_k \longrightarrow sSets, \ (R, m_R) \longmapsto B_g(R)$$ \tag{9.31}

for the vacuum Einstein gravity with vanishing cosmological constant in 3D Cartan formalism. Then, the space of functions over an open subset $U$ of $M$ can be defined to be a C-E complex associated to $dgla g(U) = \Omega^*(U) \otimes \mathfrak{iso}(2, 1)$. That is, a factorization algebra of observables for this 3D Einstein gravity is given by

$$\text{Obs}_{GR}^c : U \longmapsto \text{Obs}_{GR}^c(U)$$ \tag{9.32}

where $\text{Obs}_{GR}^c(U) = C^*(\Omega^*(U) \otimes \mathfrak{iso}(2, 1))$.

**Remark 9.4.1.** As we pointed out before, non-zero cosmological constants would yield different gauge groups, and hence different Lie algebras. Therefore, a factorization algebra of observables for 3D Einstein gravities with non-vanishing cosmological constant would involve, instead of $\mathfrak{iso}(2, 1)$, either $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2; \mathbb{R})$ for $\Lambda < 0$ or $\mathfrak{sl}(2, \mathbb{C})$ for $\Lambda > 0$.

### 10 Conclusion

This paper provides an investigation of "stacky" formulations and derived geometric interpretations for Einstein gravities in various scenarios. Inspired by the constructions in [1, 3, 4], we have the following results, observations and comments.

(i) Theorem 5.0.1 provides a construction of the moduli stack of a certain Einstein gravity in $n$-dimensional set-up.

(ii) In Theorem 6.0.1, we concentrate on 3D theories in a particular scenario. We upgrade an equivalence of certain $2+1$ quantum gravities with gauge theory to an isomorphism of corresponding stacks in the case where $M$ is topologically of the form $\Sigma \times (0, \infty)$ and $\Sigma$ is a closed Riemann surface of genus $g > 1$.

(iii) Inspired by what has been already done for Chern-Simons theory [1], we introduce derived geometric constructions for 3D Einstein gravity as a natural example of the formulations in [1, 10]. In that respect, Corollary 8.2.1 is an immediate observation which directly follows from [1]. It essentially provides an obvious formal moduli problem in the case of 3D Cartan theory of the vacuum Einstein gravity with vanishing cosmological constant. It is indeed straightforward to observe that this is just a particular case of the construction given for Chern-Simons theory [1, 10, 11, 30]. Therefore, the second part of the paper can also be viewed as a detailed survey on the construction of formal moduli problems in the case of various classical field theories, including a certain $2+1$ Einstein gravity as a particular example.

(iv) Once we adopt derived geometric interpretation of a classical field theory, the algebraic structure of observables becomes transparent in some way. Indeed, it can be described naturally in terms of a certain factorization algebra on the formal moduli problem of interest. In that respect, Section 9 is devoted to provide a survey on factorization algebras of observables. As a natural example of the constructions in [1, 10], we also present the factorization algebra of observables in 3D Cartan theory of (vacuum) Einstein gravity with/without cosmological constant.

A possible future direction, on the other hand, would be to elaborate further geometric structures on the (derived) stack of Einstein gravity in the context of symplectic (or Poisson) derived spaces [12, 16, 18]. Or one can also use the similar analysis to provide "stacky" formulations for other classical field theories.
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