An idealized pulsar magnetosphere: the relativistic force-free approximation

Simon P. Goodwin,1,2 Jonathan Mestel,3 Leon Mestel1* and Geoffrey A. E. Wright1

1Astronomy Centre, Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN1 9QH
2Department of Physics and Astronomy, Cardiff University, 5 The Parade, Cardiff CF24 3YB
3Department of Mathematics, Imperial College of Science, Technology and Medicine, Prince Consort Rd, London SW7 2BZ

Accepted 2003 November 27. Received 2003 November 25; in original form 2003 October 20

ABSTRACT

The non-dissipative relativistic force-free condition should be a good approximation to describe the electromagnetic field in much of the pulsar magnetosphere, but we may plausibly expect it to break down in singular domains. Self-consistent magnetospheric solutions are found with field lines closing both at and within the light-cylinder. In general, the detailed properties of the solutions may be affected critically by the physics determining the appropriate choice of equatorial boundary condition beyond the light-cylinder.

Key words: MHD – relativity – pulsars : general.

1 INTRODUCTION

With canonical values inserted for the neutron star dipole moment and the particle density, and allowing for realistic pair production efficiency near the star in the wind zone, the electromagnetic energy density in the pulsar magnetosphere will be much greater than the kinetic energy density except for very high values of the particle y-values. This suggests that over much of the magnetosphere, the relativistic force-free equation will be a good approximation for determining the magnetic field structure. Equally, experience with the analogous non-relativistic equation suggests that there are likely to be local domains in which non-electromagnetic forces (including possibly inertial forces) are required for force balance. An associated question is whether the dissipation-free conditions can be imposed everywhere, or whether the dynamics of the problem will itself demand local breakdown in the simple plasma condition $E \cdot B = 0$ in singular domains, in addition to the acceleration/pair production domain.

Of the various discussions of the axisymmetric pulsar magnetosphere in the literature, which go back to the early days of pulsar theory (e.g. Scharlemann & Wagoner 1973), we refer particularly to the recent paper by Contopoulos, Kazanas & Fendt (1999, hereafter CKF), and to those by Michel (1973a,b, 1991), Mestel & Wang (1979, hereafter MW), Mestel et al. (1985, hereafter MRWW), Fitzpatrick & Mestel (1988a, hereafter FMa), Fitzpatrick & Mestel (1988b, hereafter FMb), Mestel & Pryce (1992, hereafter MP), and (Mestel & Shibata 1994, hereafter MS). [Much of the relevant material appears in Mestel (1999).] Most models have within the light-cylinder (l-c) a ‘dead zone’, with field lines that close within the l-c, and without any current flow along the (purely poloidal) field, and a ‘wind zone’, with poloidal field lines that cross the l-c, and with poloidal currents maintaining a toroidal field component. The spin-down of the star occurs through the action of the magnetic torques associated with the circulatory flow of current across the l-c.

Although we are here discussing an idealized mathematical model of the pulsar magnetosphere – not even allowing the pulsar to pulse – its structure may nevertheless have great relevance to the class of real pulsars whose magnetic axis is not far from alignment. A striking example of this is PSR0826–34 (Biggs et al. 1985), a pulsar which emits throughout the entire pulse period, suggesting our line of sight is almost along the rotational axis. Other bright pulsars which are suspected on the basis of their profile widths and relatively slow spin-down rates to be near alignment include PSR0031–07, PSR0943+10 and PSR0809+74 (see Lyne & Manchester 1988 and the catalogue in Rankin 1993).

All these pulsars exhibit highly regular subpulse ‘drift’, by which the subpulses march gradually and steadily through the pulse window at a fixed rate. This suggests that their magnetospheres are in a quasi-steady state, and that steady-state models such as those discussed here may have direct physical relevance. Furthermore, the fact that all the above pulsars occasionally – yet apparently spontaneously – switch to one or more alternate ‘drift’ rates, accompanied by a widening (or narrowing) of the pulsar emission profile (Wright & Fowler 1981; Vivekanand & Joshi 1997; van Leeuwen et al. 2002) would indicate that the new steady emission has moved to outer (inner) field lines. Thus, it would appear that more than one steady state is possible in the same pulsar – presumably corresponding to different polar cap shapes, and hence to different ‘dead zone’ boundaries. This would add relevance and motivation to our discussion in Section 6, where we derive magnetosphere solutions for a range of assumptions about the location and properties of the dead zone boundary.

*E-mail: lmestel@sussex.ac.uk
2 THE RELATIVISTIC FORCE-FREE EQUATION

The outer crust of the neutron star of radius $R$ rotates with angular velocity $\alpha$ about the axis defined by the unit vector $k$. We confine attention in this paper to steady, axisymmetric states, and with the magnetic axis parallel rather than antiparallel to the rotation axis $k$. The notation is as in the cited papers (e.g. FMa; FMb). The cylindrical polar coordinate system ($\sigma, \phi, z$) is based on $k$; $t$ is the unit azimuthal, toroidal vector. The poloidal magnetic field $B_\sigma$ is everywhere described by the flux function $P(\sigma, \phi, z)$:

$$B_\sigma = -\nabla P \times \frac{t}{\sigma} = \frac{1}{\sigma} \left( \frac{\partial P}{\partial \sigma}, 0, -\frac{\partial P}{\partial \sigma} \right).$$

By Ampère’s law, $B_\sigma$ is maintained by the toroidal current density

$$j_\phi = (c/4\pi)(n \times B_\sigma) = (c/4\pi)(n \times B).$$

The basic, unperturbed stellar field is assumed to have a dipolar angular dependence and poloidal field strength $B_\sigma$. If, as assumed, $R \ll c, L$ will be found that the magnetospheric currents have slight effect on the form of $P$ near the star, so that it is then well approximated locally by the vacuum form

$$P = -\frac{B_\sigma R^3}{2} \frac{\sigma^2}{(\sigma^2 + z^2)^{3/2}}.$$

In the wind zone of an active magnetosphere there will also be a toroidal component $B_{\phi}$, conveniently written

$$B_{\phi} = -[4\pi S/c](t/\sigma),$$

where $S$ clearly must vanish on the axis. By Ampère’s law, the field (5) is maintained by the poloidal current density

$$j_\rho = (c/4\pi)\nabla \times B_\phi = -\nabla S \times t/\sigma.$$

Thus, $S(\sigma, \phi, z)$ is a Stokes stream function, constant on the poloidal current lines, with $-2\pi S$ measuring the total outflow of charge between the axis and the current line $S$. In a steady state the total outflow of charge must be zero, so there must be a current closing streamline on which $S$ vanishes.

The crust is taken to be a classical perfect conductor, with the electric field

$$E = -(\alpha \rho/c) \phi \times B.$$

In the surrounding magnetosphere, $E$ is in general written as the sum of a ‘corotational’ part of the form (7) and a ‘non-corotational’ part $\nabla \psi$:

$$E = -(\alpha \sigma/c) \phi \times B - \nabla \psi.$$

Here, the function $\psi$ is determined by the magnetospheric physics.

A systematic treatment of the dynamics of a ‘cold’, dissipation-free electron–positron gas has been given by Melatos & Meltrose (1996) (see also Blackman & Field 1993). The significant differences from a normal plasma are due in part to the particles being relativistic, and also through both species having the same rest mass, instead of differing by a factor $A_{in}m_e$ in standard notation. The two-fluid, collisionless and, so, non-dissipative equations, written in terms of ‘lab-frame’ number densities $n^\pm$ and velocities $v^\pm$, and Lorentz factors $\gamma^\pm = (1 - (v^\pm/c)^2)^{-1/2}$, are reduced to one-fluid equations in terms of

$$n = (n^+ + n^-), \quad U = n^+ v^+ + n^- v^-, \quad \Gamma = (1 - U^2/c^2)^{-1/2}, \quad \rho_c = \epsilon(n^+ - n^-).$$

The particle continuity and current continuity equations retain their standard forms

$$\frac{\partial n}{\partial t} + \nabla \cdot (nU) = 0; \quad \frac{\partial \rho_c}{\partial t} + \nabla \cdot j = 0.$$

The equation of bulk motion with velocity $U$ reduces to

$$m \frac{\partial (\rho U)}{\partial t} + mn U \cdot \nabla (\rho U) = \frac{(j - \rho_c U) \times B}{c},$$

provided $j \ll neU$, even if, as expected in a relativistic problem, the ‘convection current’ $\rho_c U$ due to bulk motion of the net charge density is not small compared with the current due to relative motion of the two species. The generalization Ohm’s law reduces to

$$E + \frac{U \times B}{c} = 0.$$

As stated, in this paper we are interested primarily in steady states that are force-free over the bulk of the domain. Just as in a classical plasma, in an $\pm e$-plasma the total particle number density will normally greatly exceed the Goldreich–Julian number density, defined by equation (16), so the ‘perfect conductivity’ and the ‘force-free’ conditions are distinct; a gas satisfying the ‘perfect conductivity’ condition need not be force-free. However, we can easily check that if inertial terms on the left of equation (11) are small compared with either term on the right, then the inertial corrections in equation (12) are a fortiori small. Thus, in the absence of collisions, radiation damping or electron–positron annihilation − all neglected in equations (11) and (12) − the force-free condition implies also the ideal magnetohydrodynamics (MHD; perfect conductivity) condition. To sum up, for our problem, the appropriate approximations to the dynamical equations are the ideal MHD condition

$$E + \frac{U \times B}{c} = 0,$$

implying $E \cdot B = 0$, and the mutual cancellation of the electric and magnetic contributions to the Lorentz force, i.e. the vanishing of the term on the right of equation (11),

$$\frac{(j - \rho_c U) \times B}{c} = \rho_c E + \frac{j \times B}{c} = 0,$$

on use of equation (13).

It should be noted that, in the more general, non-axisymmetric case, with $k$ and $p$ non-aligned, the time-dependent terms will not be negligible everywhere. As discussed by Melatos & Meltrose (1996), at a finite distance far beyond the l-c, the displacement current will dominate over the particle current, and the MHD approximations must break down.

We limit discussion explicitly to the case in which the simple plasma condition $E \cdot B = 0$ is supposed to hold throughout the magnetosphere, all the way from the rigidly rotating, perfectly conducting star out to the l-c (the ‘inner domain’), and beyond into the ‘outer domain’ between the l-c and infinity. Then from equation (8), $B \cdot \nabla \psi = 0$, and the constant value of $\psi$ throughout the stellar crust, implied by equation (7), is propagated into the magnetosphere, yielding

$$E = -(\alpha \rho/c) \times B = (\alpha \rho/c)(-B_z, 0, B_x)$$

$$= (\alpha/c) \nabla P,$$

so that all field lines corotate with the star. This assumes that within the dead zone, there is no vacuum gap separating the negatively

© 2004 RAS, MNRAS 349, 213–224
and positively charged domains (Holloway & Pryce 1981, MRWW; FMa; FMb). In the open field line domain, near the star there has to be a locally non-trivial component of $E$ along $B_p$, able to accelerate the primary electrons to $\gamma$-values high enough for pair production to occur. If, as assumed in this paper, the electron–positron plasma does achieve a steady state (cf. Shibata, Miyazaki & Takahara 1998, 2002), the electric field will again satisfy $E \cdot B = 0$ but will now be given by $E = -\partial \phi / \partial r \times B$, i.e. with the field lines beyond the acceleration domain having individual rotation rates $\alpha \bar{S} t$ that differ from the rotation $\alpha$ of the star (cf. MS, section 4). However, at least for the more rapid rotators this effect will be fairly small, and will for the moment be ignored.

By the Poisson–Maxwell equation, the Goldreich & Julian (1969, hereafter GJ) charge density maintaining the electric field (7) is

$$\rho_e = \frac{\nabla \cdot E}{4\pi} = \frac{-\alpha}{2\pi c} k \cdot \left[ B - \frac{1}{2} r \times (\nabla \times B) \right] = \frac{-\alpha}{2\pi c} \left[ B - \frac{1}{2} \bar{S} \times (\nabla \times B) \right] \phi. \tag{16}$$

With the rotation and magnetic axes aligned, the primary outflowing particles are the negatively charged electrons, yielding a negative current $j_p$ emanating from the polar cap. The stream function $S$ defined by equation (6) begins by increasing from zero on the axis, so that $B_\phi$ is negative; the field lines are twisted backwards with respect to the axis $k$. As the electric force density

$$\rho_e E = -\rho_e (\alpha / c) \phi \times B_p \tag{17}$$

is purely poloidal, in a force-free magnetosphere the toroidal component of the magnetic force density $j_p \times B/c$ vanishes (the ‘torque-free’ condition), so $j_p$ must be parallel to $B_p$, yielding from equation (1) and equation (6) the functional relation

$$S = S(P), \quad j_p = \frac{dS}{dP} B_p; \tag{18}$$

the poloidal current streamlines are identical with the poloidal field lines. The respective contributions of $B_p$ and $B_\phi$ to the poloidal force density are

$$j_\times B_{\phi / c} = (\nabla \times B_\phi) (t \times B_p) / 4\pi \tag{19}$$

and

$$j_\times B_{\phi / c} = \frac{4\pi S}{c^2 \alpha} \frac{dS}{dP} (t \times B_p) = -\frac{4\pi}{c^2 \alpha} S \frac{dS}{dP} \nabla P, \tag{20}$$

on use of equations (18) and equation (1). Using equations (1), (16), (17), (19) and (20), the poloidal component of the force-free equation is

$$\frac{1}{\alpha} (\nabla \times B) \phi \left[ 1 - \left(\frac{\alpha \phi}{c} \right)^2 \right] = \left(\frac{\alpha \phi}{c} \right)^2 \frac{2 B_\phi}{\nabla \times B} = \left(\frac{4\pi}{c^2 \alpha} \right)^2 S \frac{dS}{dP} = 0. \tag{21}$$

Note that the first term in equation (21) combines part of the electric force with the force due to $B_p$; the second term again comes from the electric force, while the third is due to $B_\phi$.

With the sign convention in equation (1), $P$ decreases from zero on the axis, so that initially $dS/dP$ is negative, and the force density (20) due to the toroidal field $B_\phi$ acts toward the axis. Because $S$ must vanish on the current closing streamline, there will in general be an intermediate field line on which $dS/dP$ and so also the volume current density $j_p$ changes sign. Below this field line, the force (20) acts toward the equator. Note that the return of $S$ to zero need not be continuous; closure via a sheet current is not excluded (cf. Section 4). In fact at this stage, we cannot rule out that $S$ increases monotonically from zero to its maximum on a limiting field streamline and then drops to zero, so that all the return current is in the sheet. The actual allowed form for the function $S(P)$ will emerge as part of the global solution.

As in earlier work, we define dimensionless coordinates $(x, z) = (\alpha / c) (\sigma / z, z)$, and normalize $P, S, E, B$ in terms of a standard l-c field strength $B_{\text{n}} = (B / 2) (\alpha / c R)^{1/2}$:

$$P = P B_{\text{n}} (\alpha / c)^2, \quad S = S B_{\text{n}} (c^2 / 4\alpha \pi), \quad (B, E, B) = B_{\text{n}} (E, B). \tag{22}$$

(Once defined, the dimensionless quantities are again immediately written without the bars.) The normalized fields have the form

$$B_p = -\nabla \times t / x, \quad B_\phi = -S / x, \quad E = \nabla P, \tag{23}$$

with $P$ satisfying

$$(x^2 - 1) \frac{\partial^2 P}{\partial x^2} + \left(1 + \frac{x^2}{4(\sigma / z)^2} \right) \frac{\partial P}{\partial x} + (x^2 - 1) \frac{\partial^2 P}{\partial z^2} = S \frac{dS}{dP}. \tag{24}$$

As $(\sigma, z) \rightarrow 0, P$ must reduce to the normalized point dipolar form $-x^2 / (x^2 + z^2)^{3/2}$. The l-c $x = 1$ is a singularity of the differential equation for $P$.

An illustrative solution of equation (24) within the l-c for the non-active case $S = 0$ has been discussed by Michel (1973b, 1991), MW and MP, and will be referred to as the MMWP field. The field lines which reach the equator cross normally, forming a closed domain; the inner domain equatorial boundary condition is

$$B_{\text{n}} (x, 0) = 0, \tag{25}$$

with no equatorial current sheet. The flux function $P$ is most easily constructed as a Fourier cosine integral (cf. the Appendix). If neither the field nor the volume current density are to be non-singular at the l-c, equation (24) with $S = 0$ requires that $B_\phi = -\partial P / \partial x = 0$ when $x = 1$; the field lines cross the l-c normally, and there is a neutral point at the intersection (1, 0) of the l-c and the equator.

The MMWP model is of pedagogic interest through its showing how, in this essentially relativistic problem, the macroscopic electric force acting on the corotating GJ charge density causes a marked deviation of $B$ from the curl-free dipolar form as the l-c is approached. However, the solution is clearly incomplete, as there is no treatment of the domain beyond the l-c. A viable, strictly inactive model will in fact have large vacuum gaps within the l-c (e.g. Smith, Michel & Thacker 2001, and references within). As discussed in Sections 5 and 6, some features of the MMWP field should persist in a realistic active model, such as the large dead zone, and a similar departure from the curl-free structure near and beyond the l-c.

### 3 THE DOMAIN BEYOND THE LIGHT-CYLINDER

Equation (13) and (15) combine into

$$(U - \alpha \sigma \tau) \times B = 0, \tag{26}$$

yielding

$$U = \times B + \alpha \sigma \tau; \tag{27}$$

the sum of corotation with the star plus flow parallel to the total field $B = B_+ + B_\times$, a result familiar from standard stellar wind theory. If $U_p$ were to vanish beyond the l-c, then from equation (9), $n^+(w_t^+ - \alpha \sigma) + n^-(w_t^- - \alpha \sigma) = 0$, requiring that either the electrons or the positrons would have to be superluminal. Thus, along field
lines that cross the l-c, there is not only a net current but also a net particle
flow, as is indeed implied by the term ‘wind zone’. Then, if, as
assumed, the simple ‘perfect conductivity’ condition (26) continues
to hold everywhere beyond the l-c, so that there is strictly no trans-
field motion of the plasma, then in a steady state the field must be
topologically ‘open’, with no field lines crossing the equator.

A CKF-type model has the field lines crossing the equator nor-
mainly within the l-c, as in the MMWP and FMa/FMb fields, but
beyond the l-c, the boundary condition \( B_z = 0 \) is imposed at the equator. In
the northern hemisphere \( B_\phi > 0 \), so by equation (7), near the equator the
electric field \( eB_z \), is in the positive \( z \)-direction. As the
basic field is dipolar, \( B_z \) must change sign at the equator, implying a
locally positive current density \( j_b \) and a magnetic force density that
acts towards the equator. Thus, the poloidal magnetic field pinches,
as in familiar non-relativistic problems, but the combined electric
force (17) and the electromagnetic \( \sigma \times B \) from equation (27), the rotation
velocity is given by

\[
U_\phi = c \left( x + \frac{U_\phi}{c} \frac{B_\phi}{B_\psi} \right).
\]

In a steady state, equation (11) yields the energy integral in a
pressure-free system (MRWW; MS; Contopoulos 1995)

\[
\Gamma[1 - x(U_\phi/c)] = \text{constant},
\]

showing that \( \Gamma \) would become infinite if \( U_\phi = c/x, U_\psi = c/(x^2 - 1) \).
From equation (35) this is equivalent to

\[
|B_\phi/B_\psi| = \sqrt{x^2 - 1} - 1.
\]

Thus, the seemingly exceptional case, with zero equatorial pressure
and so with the force-free equation holding all the way to the equa-
tor, in fact requires that the flow outside the equatorial zone be be
highly relativistic that the neglected inertial terms are not small, so
violating an essential condition for the force-free approximation to
hold. We conclude that, in all cases, the equilibrium conditions for
this model will require at least a local breakdown in force-free condi-
tions. In analogous non-relativistic problems, Lynden-Bell (1996)
has pointed out that a thermal pressure is again required to balance
magnetic pinching forces, exerted locally by an otherwise force-free
field. However, we again emphasize the important difference that,
whereas in a non-relativistic problem the electric stresses are norm-
ally smaller than the magnetic by the factor \((Uc)^2\), in the present
problem beyond the l-c the opposing electric stresses exceed the
pinching poloidal field stresses by the factor \(x^2\), and equilibrium is
possible only through the pressure exerted by the toroidal field.

In the domain near the equator, with \( |B_\phi/B_\psi| \gg 1 \), the relativistic
flow likewise has \( |U_\phi/U_\psi| \gg 1 \), so that \( U_\phi^2 + U_\psi^2 \approx c^2 \), whence
denoting equation (35) (remembering that \( B_\phi \) is negative)

\[
U_\phi/c = \frac{x - b}{x + b} \sqrt{\frac{(1 + b^2 - x^2)}{(1 + b^2)}},
\]

\[
U_\psi/c = \frac{bx + \sqrt{(1 + b^2 - x^2)}}{(1 + b^2)},
\]

where \( b = |B_\phi/B_\psi| \). (The algebraically allowed choice of the opposite
signs before the two radicals would yield \( U_\phi/c = 1 \) at \( x = 1 \), implying
finite \( \Gamma \), and so is rejected.) The outflow of angular momentum
from the star in both hemispheres across a closed surface \( \Sigma \) with
local outward unit normal \( n \) is (e.g. Mestel 1999)

\[
- \int (\sigma B_\phi/4\pi) B_\phi \cdot n \, d\Sigma = -\left( -c/(\alpha) B_\psi^2 \right) \int_0^{\Omega_c} S(P) \, dP,
\]

on use of equations (1), (5) and (22).

4 THE \( S(P) \) RELATION

As noted in Section 2, if there do exist magnetospheric models that
are everywhere non-dissipative, with the field force-free outside

\[
\frac{1}{8\pi^2} [B_\phi^2 + (x^2 - 1)B_\psi^2] + p = p_{eq}.
\]

As the equator is approached, \( p \) steadily increases from the uniform
value (implicit in the force-free assumption), which could in prin-
ciple be zero, to the value \( p_{eq} \). Thus, the model necessarily includes a
thin domain with a non-force-free electromagnetic field.

The limiting case, with the inequality sign in equation (30) re-
placed by equality, appears consistent with zero equatorial pressure
\( p_{eq} \). However, condition (21) assumes that all non-electromagnetic
forces, including inertial forces, are small compared with the dom-
inant terms in the Lorentz force. From equation (27), the rotation
velocity is given by

\[
U_\phi = c \left( x + \frac{U_\phi}{c} \frac{B_\phi}{B_\psi} \right).
\]
singular regions such as the equatorial sheet, then the relevant relation \( S(P) \) should emerge as part of the solution. An early model of the whole magnetosphere by Michel (1973a, 1991) has no dead zone, but a poloidal field that is radial all the way from the star to infinity. In our notation, the Michel field is

\[
P = P_e \left(1 \pm \frac{z}{(x^2 + z^2)^{1/2}}\right), \quad B_z = \pm P_e \frac{x}{(x^2 + z^2)^{1/2}},
\]

\[
B_x = \pm P_e \frac{z}{(x^2 + z^2)^{1/2}}, \tag{40}
\]

with the negative sign applying to the northern hemisphere, and with the critical field line \( P = P_e \) coinciding with the equator. From now on, just the northern hemisphere is considered. Michel’s \( S(P) \) relation is

\[
S = -2P + \frac{P^2}{P_e} = -P_c \frac{x^2}{(x^2 + z^2)^{3/2}}, \quad B_y = -\frac{S}{x}, \tag{41}
\]

so that

\[
\frac{dS}{dP} = -2 \left(1 - \frac{P}{P_e}\right) = -\frac{2z}{(x^2 + z^2)^{1/2}},
\]

\[
S \frac{dS}{dP} = 2P_c \frac{x^2}{(x^2 + z^2)^{3/2}}. \tag{42}
\]

We can easily verify that equation (24) is satisfied. Note that the poloidal field (40) is radial and (away from the equator) independent of the spherical polar angle \( \theta \), and so has no curl; equilibrium is maintained by balance between the electric force given by equations (16) and (17) and the force due to the toroidal field (41). However, with the sign change at the equator, there are again both toroidal and poloidal equatorial sheet currents that respectively maintain the jumps in \( B_x \) and \( B_y \); also \( B_x^2 - (x^2 - 1)B_y^2 = P_c^2c^4 \) when \( z = 0 \), so that by the above discussion, \( P_{c0} > 0 \). Note also that \( dS/dP = 0 \) on the critical line \( P_e \), but is negative for \( 0 < P/P_e < 1 \); in the Michel model, all the return current is in the equatorial sheet.

In the generalized CKF picture, with the poloidal field forced to have a dipolar rather than a radial structure near the star, there is an associated dead zone within the l-c, similar to that found in the MMWP, FMA/FMB and MS fields, and analogous to that in the non-relativistic wind problem (e.g. Mestel & Spruit 1987). The dead zone terminates at the point \((x_c, 0)\); the value of \( x_c (\lesssim 1) \) will be seen to be an extra parameter, fixing the global field structure. Within the dead zone, the field lines close, crossing the equator normally so that the appropriate equatorial boundary condition is \( P_e \propto B_x(x, 0) = 0 \) for \( x < x_c \). The dead zone is bounded by the separatrix field line \( P_s \). In the wind zone outside \( P_s \), the wind flow is along the poloidal field, and so the equatorial boundary condition is \( P_e \propto B_x = 0 \), but now extending through the l-c from \( x_c \) to \( \infty \), with again a charge-current sheet maintaining the discontinuous sign changes in \( E_z, B_z \) and \( B_y \). Thus, the critical field line \( P(x, 0) = P_e \) extending along the equator is the continuation of the separatrix between the wind and dead zones.

Just outside the equatorial charge-current sheet and the separatrix, equation (24) holds all the way in from \( \infty \). At the intersection (1, 0) with the l-c and just outside the sheet, with \( P_s \propto B_x = 0 \) and no singularities in \( P_s \) and \( P_{tc} \), equation (24) requires that the constant value of \( S = 2P_{c0}P \) along \( P_s \) must be zero. However, \( S(P_s) \neq 0 \), because as seen above, beyond the l-c the pinching force exerted by \( B_x \) is necessary for equilibrium; we therefore require, as in the Michel field,

\[
\frac{dS}{dP} = 0 \tag{43}
\]

on the critical field line \( P_e \). This condition is propagated inwards along \( P_e \) on to its continuation, the separatrix \( P_s \) between the wind and dead zones; the equatorial equilibrium conditions beyond the l-c impose a constraint on the global \( S(P) \) relation. From equation (18), condition (43) requires that the poloidal volume current density \( j_x \) falls to zero on \( P_s \), but the finite value for \( S(P_s) \) requires a poloidal current sheet at the equator beyond \( x_c \) and along the separatrix within it.

5 THE DOMAIN WITHIN THE LIGHT-CYLINDER

We have emphasized that, beyond the neutral point \((x_c, 0)\), there is a finite thermal pressure on the equator, necessary for equilibrium. Likewise, it is not obvious that within the l-c, the boundary condition on the field line separatrix between the wind zone (labelled 2) and the dead zone (labelled 1) can be satisfied without a thermal pressure within the dead zone (see Fig. 1).

It is, in fact, easy to generalize equations (21) and (24) to include a pressure gradient; with gravity and inertia still negligible,

\[
\frac{V P}{4\pi} \left\{ \frac{1}{\rho^2} \left(\nabla \times B\right)_0 \left[ 1 - \left(\frac{\alpha \sigma}{c}\right)^2 \right] + \left(\frac{\alpha \sigma}{c}\right)^2 2B_z \right\} + \frac{4\pi}{c^2} \frac{dS}{dP} \right\} - \nabla p = 0. \tag{44}
\]

Thus, \( p = p(P) \); the constant pressure surfaces must coincide with the poloidal field lines. In normalized form, equation (44) becomes

\[
(1 - x^2) \frac{\partial^2 P}{\partial x^2} - \frac{1 + x^2}{\alpha^2} \frac{\partial P}{\partial x} + (1 - x^2) \frac{\partial^2 P}{\partial z^2} = -S \frac{dS}{dP} \tag{45}
\]

Prima facie, there is no obvious objection to the adoption of the simplest case, with \( p \) the same constant on all the field lines within the dead zone and zero in the wind zone, i.e. with a discontinuity in both \( p \) and \( S(P) \) on the separatrix. The equation for \( P \) within the wind zone then remains equation (24) (with signs reversed for convenience):

\[
(1 - x^2) \frac{\partial^2 P}{\partial x^2} - \frac{1 + x^2}{\alpha^2} \frac{\partial P}{\partial x} + (1 - x^2) \frac{\partial^2 P}{\partial z^2} = -\frac{dS}{dP}. \tag{46}
\]
In the dead zone we have the same equation with \( S = 0 \). However, along the separatrix \( P_c \), extending inwards from the equatorial point \((x_c, 0)\), the equilibrium conditions require a discontinuity in \( B_p \) as well as those in \( p \) and \( S(P) \). Writing

\[
n_i = -(t \times (B_p/B_p)),
\]
(47)

the unit normal to the separatrix, we require continuity of

\[
-[(8\pi p + 2B_p^2 + E_i)\delta_{ij} + 2E_iE_j + 2B_iB_j]n_j,
\]
(48)

with \( E \) given by equation (7). The components of equation (48) parallel to \( t \) and to the separatrix are automatically zero. The component normal to \( P_c \) reduces to continuity of

\[
8\pi p + B_p^2[(1 - (\alpha \sigma/c)^2)] + B_p^2,
\]
(49)
i.e. to

\[
8\pi p + B_p^2(1 - x^2) = B_p^2(1 - x^2) + B_p^2.
\]
(50)

It is convenient to normalize \( p \) in units of \( B^2/8\pi \).

For \( x > x_c \), the wind zone 2 extends from the equator \( z = 0 \) to \( z = \infty \); for \( x < x_c \), zone 2 extends from \( z(x) \) defined by the separatrix

\[
P(x, z) = P_c,
\]
(51)
to \( z = \infty \). The separatrix function (51) is not known a priori but must emerge as part of the solution by iteration. We assume provisionally that at the point \((x_c, 0)\) (referred to as \( N \)), the poloidal field \( B_p = 0 \) both just outside and just inside the separatrix. From equation (50), the pressure of the toroidal field is balanced by the thermal pressure \( p \), so that

\[
p = S^2(P_c)/x_c^2.
\]
(52)

For \( x < x_c \), with use of equations (22) and (23), equation (50) then becomes

\[
(\nabla P)P^2 = (\nabla P)^2 + S^2(P_c)(1 - x^2/x_c^2) / (1 - x^2).
\]
(53)

The discontinuity in \( \nabla P \) grows from zero as \( x \) moves in from \( x_c \) but will become a small fraction of \( \nabla P \) for small \( x \).

When \( x < 1 \), the function \( P \) has a simple analytical behaviour near the neutral point \((x, 0)\). The separatrix leaves \((x, 0)\) making an angle \( \theta = 2\pi/3 \) with the outward-pointing equator. To leading order, in both the dead and wind zones, \( P_{x_c} + P_{z_c} = 0 \), which has the local solution

\[
P / P_c = 1 + A_{21}R^{3/2} \sin(3\theta/2),
\]
(54)

where the coefficients \( A_{21} \) apply, respectively, to the wind zone 0 < \( \theta \) < 2\pi/3 and the dead zone 2\pi/3 < \( \theta \) < \( \pi \). It is seen that, along \( \theta = 0 \), \( P = P_c \), and on \( \theta = \pi \), \( P_{x_c} \propto P_{z_c} = 0 \), as required. Across the separatrix \( z = \sqrt{3}(x_c - x) \), \( P \) is continuous, while the jump condition (53) then yields

\[
A^2 = A^2 + 4S^2(P_c)/9P_c^2x_c(1 - x^2).
\]
(55)

At this point, it is instructive to make a comparison with the analogous non-relativistic problem, in which the electric stresses are small by factors \( O(ue)^2 \) and so are negligible, and also inertial forces are still neglected. Suppose that there is again a separatrix passing through the poloidal field neutral point at \((x, 0)\). The balance equation across the separatrix is now continuity of \( 8\pi p + B_p^2 + B_p^2 \); the \( (1 - x^2) \) factors in equation (50) are replaced by unity. If again \( p \) is negligible in the wind zone, then the constant value of \( p \) along the separatrix within the dead zone is again fixed by the condition at the neutral point: \( p = S^2(P_c)/x_c^2 \) in normalized form. The balance condition is then

\[
S^2(P_c)(1 - x^2/x_c^2) + (\nabla P)^2 = (\nabla P)^2.
\]

The presence of the factor \( (1 - x^2/x_c^2) \) in the \( S^2 \) term enables \( B_p \) to be continuous (and zero) at the neutral point \((x, 0)\), but with \( S^2(P_c) \) still non-zero. (Clearly, in the non-relativistic problem, the numerical value of \( x_c \) is of no significance.)

By contrast, in the relativistic problem, near the l-c the electric field strength approaches the poloidal magnetic field strength, so that the factor \( (1 - x^2) \) now appears multiplying both the \( (\nabla P)^2 \) terms in equation (53). If we were now to take \( x_c = 1 \), then the non-vanishing factor \( (1 - x^2) \) would cancel, and equation (53) would reduce to

\[
S^2(P_c) + (\nabla P)^2 = (\nabla P)^2
\]
(56)

along the separatrix. At the point \((x, 0)\), the simultaneous vanishing of \( \nabla P \) and \( \nabla P_c \) would then require that \( S(P_c) = 0 \) (and so also by equation 52 = 0). However, if beyond the l-c the equatorial toroidal component \( B_\phi = -S(P_c)x \) were zero, then the balance condition could not be satisfied (cf. equation 32).

In fact, as discussed in the Appendix, the possible vanishing of \( S \) on the separatrix is appropriate to models with a radically different external equatorial boundary condition. In the present problem, the case \( x_c = 1 \), which yields equation (56), may be incorporated provided that for this limiting case, we allow \( \nabla P \neq 0 \), i.e. the discontinuity in \( B_p \) across the separatrix persists at the equator. If \( x_c = 1 - \epsilon \) with \( \epsilon \ll 1 \), then from equation (53) the equatorial \( B_c \) can be zero on both sides of the separatrix, without requiring that \( S(P_c) = 0 \). However, at a neighbouring separatrix point \( x = x_c - X \) with \( X \ll 1 \), the second term in equation (53) will have climbed from zero at \( N \) to \( S^2(P_c)(X/e + \epsilon) \subset S^2(P_c) \) once \( X \gg \epsilon \). If solutions with \( S(P_c) \neq 0 \) continue to exist as \( x_c \to 1 \), then the separatrix balance condition will imply a steeper and steeper local gradient in \( B_\phi \), indeed tending to a discontinuity when \( x_c = 1 \).

For the limiting case \( x_c = 1 \), an appropriate local model, valid near the critical point \( x = 1, z = 0 \), has \( |\nabla P|^2 \) jumping by a constant value across the separatrix \( \bar{P} = P / P_c = 1 \). New independent variables \( (r, t) \) and the dependent variable \( u \) are introduced, defined by

\[
x = 1 - r \sin t, \quad z = r \cos t, \quad r^2 = (x - 1)^2 + z^2;
\]
(57)

\[
\bar{P} = 1 + u.
\]

When \( r \) and \( u \) are small, equation (46) reduces to

\[
(1 - x^2)(u_{xx} + u_{zz}) - (1 + x^2)u_{x}/x = 0
\]
(58)

in the dead zone. In the contiguous wind zone, the right-hand side is again \( S^2 \), but as this will behave like \( u \), the form (58) is appropriate on both sides of the separatrix. To leading order, this reduces to

\[
u_{rr} + 2u_r / r + 1 / r^2 \sin t [\sin tu_{tt}] = 0.
\]
(59)

The appropriate boundary conditions are:

(i) \( u_{x = 0} \) on \( r = \pi/2 \) normal crossing of the equator;

(ii) \( u \) on the separatrix, leaving the critical point at the, as yet, unknown angle \( t = t_0 \);

(iii) \( (u/r)^2 \) jumps by a constant on \( t = t_0 \).

The solution of equation (59) satisfying these conditions is

\[
\text{Domain a} \ 0 < t < t_0: \quad u = 0,
\]
(60)

\[
\text{Domain b} \ t_0 < t < \pi/2: \quad u = Ar Q_1(\cos t).
\]

where \( Q_1(X) \) is the Legendre function \( Q_1 = -1 + (X/2) \log [(1 + X)/(1 - X)] \), and \( t_0 \), given by \( Q_1(\cos t_0) = 0 \), is 33.5342°. The
constant $A$ then follows from $u/t = S(1)$. In Domain b, $Q_1[\cos t] = 1 - \cos t \log (\tan t/2)$, and $\partial Q_1[\cos t] = \cos t \sin t - \log(\tan t/2)$; whence at points near the l-c (x close to unity),

$$B_x \propto \partial u/\partial z = - A \log(\tan t/2),$$

$$B_c \propto -\partial u/\partial x = A/\sin t,$$

(61)

yielding $B_x = 0$, $B_c \propto A = \text{constant on the equator.}$

An essential step in all the argument is the condition $p = p(P)$, following from equation (44). If instead the thermal pressure $p$ were (illicitly) allowed to vary so as to balance the pressure $B^2 Q_2 = S'(P_c) x^2$ exerted by the external toroidal field, then from equation (50) there would be no discontinuity in $B_c$ at the equator or indeed anywhere along the separatrix.

It is appropriate also to re-emphasize just how crucial to the discussion is the outer domain equatorial boundary condition

$$B_t(x, 0) = 0$$

(62)

for $x > 1$; for it is the consequent local equilibrium requirement $S(P_c) > 0$ that implies a poloidal sheet current within the l-c along the separatrix, leading to the conditions (50) and (53), with a non-vanishing $B_Q$. It is then clear that a non-zero $p(P_c)$ is required for the balance condition to hold at the neutral point.

6 MODEL CONSTRUCTION

The two basic parameters of a pulsar are clearly the angular velocity and the dipole moment of the neutron star. The spin-down rate of an active pulsar will depend on the strength of the circulating poloidal current. In the Michel $S(P)$ relation (41), the form with $S \approx -2P$ valid near the poles derives from taking the current density flowing along the axis as equivalent to the GJ electron charge density moving with the speed $c$, a reasonable upper limit, and indeed one that must be closely approached if the electrons are to generate an electron-positron pair plasma near the star (see, for example, MS, section 3).

We consider only $S(P)$ relations that behave like the Michel form for small $P$.

The only other parameter introduced into the theory is the pressure in the dead zone, which we have taken as uniform. Prima facie, there should be a class of possible functions $S(P)$, each fixing the global field function $P$ and simultaneously the separatrix parameter $P_c$ and the limit $x_c$ of the dead zone, with the value of the required dead zone pressure $p(P_c)$ given by equation (52). Equivalently, a choice of $x_c$ should have an associated class of functions $S(P)$, each fixing the global $P$, the value of $P_c$ and of $p(P_c)$, and the shape of the separatrix $P(x, z) = P_c$ between the star and $x_c$. Intuitively, we expect an increase in $p(P_c)$ to correspond to a decrease in $x_c$.

The functions $P(x, z)$ and $S(P)$ are renormalized in terms of the value $P_c$ pertaining to the separatrix and its extension along the equator $z = 0$, extending from the point $x_c < 1$, through the l-c $x = 1$ to $\infty$:

$$\tilde{P} = P/P_c, \quad \tilde{S} = S/S_c.$$

(63)

Although by convention the unnormalized $P$ is negative, the new normalized form $\tilde{P}$ is positive, and $\tilde{S}$ is negative. (From now on the bar will be dropped, all quantities being assumed renormalized.) On the separatrix and its equatorial continuation, the normalized $P$ is unity. The boundary condition at the origin then becomes $P \approx -(1/(P_c)) \sin^2(\alpha^2 + \beta^2)^{1/2}$.

The equation for $P$ is

$$(x^2 - 1) P_{xx} + (1 + x^2) P_x + (x^2 - 1) P_{zz} = \frac{S}{dS/d\tilde{P}}.$$

(64)

Defining the spherical coordinates $(r, \theta)$ by $z = r \cos \theta$ and $x = r \sin \theta$, it is to be expected that $P \sim r f(\theta)$ as $r \to \infty$, for some power $n$ and function $f$. Such a homogeneous form is consistent with the equatorial condition $P = 1$ on $\theta = \pm \pi$, only if $n = 0$, giving a radial field. The appropriate boundary condition at infinity is thus $\partial P/\partial r = 0$. For large $r$, equation (64) takes the form $\alpha^2 \nabla^2 P = SS'(P)$ which integrates once to require

$$\sin \theta \frac{4P}{d\theta} = S(P) \quad \text{imposing} \quad S(0) = 0.$$

(65)

For the Michel field $P = 1 - \cos \theta$ and $S = 2P - P^2$, but a solution consistent with the dipole at the origin has a different function $S(P)$. It follows that the radial field at infinity must differ from the Michel form. It is, however, found in the next section that the difference is not so great.

6.1 Numerical method

The function $P(x, z)$ satisfies an elliptic, quasi-linear equation with a mixture of Neumann, Dirichlet and Robin boundary conditions. The partial differential equation contains an unknown function $S(P)$ which must be determined by a regularity condition across the singular line $x = 1$. There is a discontinuity in normal derivative across the curve $P = 1$ whose position is unknown a priori. The solution is driven by the dipole of strength $1/P_c$ at the origin. A further numerical difficulty derives from the non-analytical behaviour of the solution near the point $(x_c, 0)$.

The problem is determined by the two parameters $P_c$ and $p$, but for convenience solutions are sought for fixed $P_c$ and $x_c$. This reduces the amount by which the curve $P = 1$ requires adjustment during the solution.

Equation (64) is discretized using a finite difference scheme on a rectangular grid $(m \delta x, n \delta z)$ for $1 \leq m \leq M$ and $0 \leq n \leq N$, and $P_{mn}$ denotes the sought approximation to $P$ at the grid point. The l-c is at $m = m_0$, so that $m_0 \delta x = 1$, and the computational domain is $0 < x < x_m$, $0 < z < z_m$ where $x_m = M \delta x$ and $z_m = N \delta z$.

The position of the dead zone boundary $P = 1$ is marked by its intersections with the grid lines. Away from this boundary and for $m \neq m_0$, second-order centred difference approximations are used for $P_{x}, P_{z}$, and $P_{zz}$, regarding $SS'(P)$ as a known source term. In fact, the dipole at the origin is subtracted out from $P$ before differencing, and the dipole is differentiated analytically.

Equations for $P_{mn}$, and for $P_{mn}$ near the curve $P = 1$ are derived separately as discussed below. The resulting equations are written in a form appropriate for Jacobi iteration. Thus, at each iteration new values of $P_{mn}$ are derived from the old values, for given $S(P)$ and position of the curve $P = 1$. Once the new values of $P_{mn}$ are found, values for $SS'(P)$ are found using

$$SS'(P_{mn}) = (P_{mn} + 1 - P_{mn-1})/\delta x.$$

These values are then interpolated using splines to give the new function $SS'(P)$, which is integrated to give a new value of $S(1)$. For values of $P$ smaller than those reached along the finite length $0 < z < z_m$ the Michel solution was used for $S(P)$. The intersection points of the curve $P = 1$ with the grid lines are then updated by requiring the jump in $|\nabla P|^2$ to be correct. The internal and external values of $\nabla P$ are then used on the next iteration when approximating the derivatives at the grid points close to $P = 1$. The point $(x_c, 0)$ is kept fixed during the iteration, and a local balance there requires the system to converge to suitable values of $P$ and $S(1)$.
Assuming a smooth crossing of the l-c, \( P(x, z) \) is expanded near \( x = 1 \) as a power series in \( (x - 1) \), which is substituted into equation (64). This gives the results
\[
S'(P) = 2P_x \quad \text{and} \quad 4P_{xx} + 2P_{zz} = -(S'),
\]
on \( x = 1 \). This latter equation is discretized and used for \( m = m_0 \) in place of equation (64). Finally, on the edge of the domain, the conditions \( P(0, z) = 0, P_x(x, 0) = 0 \) for \( x < x_c \), while \( P(x, 0) = 1 \) for \( x > x_c \), and \( xP_x + zP_z = 0 \) on \( x = x_m \) and on \( z = z_m \) are imposed.

The above equations are solved iteratively. The numerical behaviour is improved using continuation techniques from a nearby solution, and by the use of under-relaxation. As well as the fundamental parameters \( P_c \) and \( x_c \), the domain size \( x_m \) and \( z_m \), and the grid size \( \delta x \) and \( \delta z \) affect the numerical solution. These latter values can be varied to ensure the computation is robust.

No difficulties are encountered in the vicinity of the l-c nor the dipole. However, when the ‘front’ \( P = 1 \) crosses one of the grid points in the course of the iteration some readjustment occurs and the convergence is slower. Similar effects occur in the numerical solution of Stefan problems with a liquid–solid interface.

Solutions are found for some but not all parameter pairs \( (P_c, x_c) \). For a fixed value of \( x_c \), the scheme converges only if \( P_c \) lies in some interval \( P_1(x_c) < P_c < P_2(x_c) \). As \( x_c \to 1 \), the values of \( P_1 \) and \( P_2 \) decrease, and become proportionally closer. If \( P_c \) is too small the dead zone becomes non-convex, bulging towards the l-c away from the equator. It is believed that solutions cease to exist once \( P \)-lines attempt to cross the l-c three times rather than once. If \( P_c \) is too large, the curve \( P = 1 \) reaches the equator at a value of \( x < x_c \).

Some dead zone \( P \)-lines then cross the equator in the interval \( (x_c, 1) \), and are not linked with the star and such solutions are deemed irrelevant.

The left diagram in Fig. 2 portrays \( P(x, z) \) in the limiting case \( x_c = 1 \) with \( P_c = 1.46 \), for which it is found \( S(1) \approx 0.919 \). An approximately linear variation with distance from \( (1, 0) \) within the dead zone is found, in keeping with equation (60). With step-lengths \( \delta x = \delta z = 0.025 \), the value of \( P(1 - \delta x, 0) \) predicts a value of \( A \) in equation (60) in agreement with the theoretical estimate to two significant figures.

Solutions with \( x_c < 1 \) are similar in appearance, although the local structure around the point \( (x_c, 0) \) differs as discussed above. It was not possible to distinguish the predicted separatrix angles 30° of (54) and 33° of equation (60). The centre panel of Fig. 2 shows the case \( x_c = 0.86 \), \( P_c = 1.65 \) for a larger domain, while the right panel depicts a case with a weaker dipole \( P_c = 2.9 \) and smaller dead zone \( x_c = 0.45 \), with equivalently an increase in the poloidal flux extending to infinity.

The corresponding \( S(P) \) variations are plotted in Fig. 3, along with the quadratic Michel relation. For \( x_c \) some distance from the l-c \( S(P) \) is monotonic and the field on the l-c diverges from the equator. As \( x_c \) approaches unity, a region of negative \( S(P) \) develops near \( P = 1 \), indicating that on the l-c \( P_f \) becomes negative so that the field is converging towards the equator, although it becomes radial further out. The functional form of \( S(P) \) never seems to differ wildly from the Michel solution, but \( P(x, z) \) is affected substantially. This supports an observational point made at the outset of this paper, that magnetosphere currents may need very little alteration to achieve a different field configuration, in particular changes in the polar cap radius. Our solutions here correspond to polar caps which are respectively 20 per cent \( (x_c = 1) \), 30 per cent \( (x_c = 0.86) \) and 70 per cent \( (x_c = 0.45) \) greater than the vacuum dipole case conventionally assumed in interpreting pulsar profile widths.

7 DISCUSSION

7.1 The equatorial boundary condition

After considerable effort we have amended the pioneering study of CKF and constructed a class of models subject in the wind zone to

Figure 3. \( S(P) \), for the cases in Fig. 2. Bottom to top: \( x_c = 0.86, 1, 0.45 \). Michel.

Figure 2. Contours of \( P(x, z) \): (left) \( x_c = 1, P_c = 1.46 \); (middle) \( x_c = 0.86, P_c = 1.65 \); (right) \( x_c = 0.45, P_c = 2.9 \).
the equatorial boundary condition \( B_t = 0 \), i.e. with field lines in the wind zone never crossing the equator, as is indeed required by strict application of the ‘perfect conductivity’, dissipation-free condition. It has been emphasized that, although the constructed magnetospheric field is force-free nearly everywhere, it is non-force-free in the the pinched equatorial zone where the field changes direction, and also along the separatrix between the wind and dead zones. Also, in general, as found earlier in CKF, the current returns to the star partly as a volume current, and partly as a sheet current along the equator and the separatrix.

By definition, the global force-free field condition describes an electromagnetically dominant system. In non-relativistic MHD, we would associate a poloidal field, drawn out so as to have a predominant \( x \)-component, rather with a kinetically-dominant system, with the Reynolds stresses at least locally stronger than the Maxwell stresses. Without such an outward pull, an equatorially pinching, quasi-radial field is likely to be unstable against recurrent reconnection, spontaneously converting into a structure with field lines closing across the equator.

Recall that in the analogous stellar problem (e.g. Mestel & Spruit 1987; Mestel 1999), it is argued that the Alfvenic surface, on which the magnetic and kinetic energy surfaces are comparable, not only defines precisely the extent of the effective corotation, but is also a rough demarcation between the zones with respectively closed and open field lines. Suppose now that in studying the relativistic force-free equation, we consider models with a domain in which some field lines cross the l-c and close beyond it. Gas in this domain cannot corotate with the star, but will flow into the equatorial zone. A hypothetical steady state will clearly require that the accumulating gas be able to flow outwards, crossing the field through a macroresistivity.

The most extreme example will have these field lines approaching the equator normally, so that the equatorial boundary condition \( B_t = 0 \), holding within the l-c, also replaces \( B_t = 0 \), for some way beyond the l-c. However, there is a further significant difference from the non-relativistic problem: a state with the equatorial boundary condition \( B_t = 0 \), \( B_t \neq 0 \) appears to be electromagnetically unstable. If the field is perturbed so that there appear components \( \pm B_t \) on the \( \pm z \) surfaces of the equatorial sheet, then the Maxwell stresses yield \( x \)-components \( (E_x, E_y + B_y B_z)/2\pi \); and provided coupling with the rotating magnetic star is maintained, this becomes \( (1 - x^2)B_t^2/2\pi \). Because \( B_t < 0 \), when \( x > 1 \), the force acts to drive the frozen-in gas in the direction of \( B_t \), so increasing the imposed perturbation. The effect is again due to the electric part of the stress, which beyond the l-c dominates over the poloidal magnetic field contribution. Thus, there may exist steady states, alternative to those discussed in this paper, in which the field at the equator is neither parallel nor perpendicular to the plane, but approaches it obliquely, and with the equatorial gas driven out by a combination of electrical and centrifugal force, and crossing the \( B_t \) field component via a dynamically-driven macrore sistivity.

Any such change in the equatorial boundary condition will react on the global solution of the force-free equation (cf. the Appendix). In MS, two cases were studied, both subject to the special equatorial condition \( B_t = 0 \): (a) the MMWP dead model, with \( S = 0 \), extrapolated beyond the l-c; (b) the special live model, with \( S = -2P + 2P^2/P_d \), where \( P_d \) is the separatrix bordering the dead and wind zones. Note that in this tentative live model, \( S(P_d) = 0 \), all of the current returns as volume current, with no current sheet on the separatrix. In both these cases, it was found impossible to construct a solution that was smoothly continuous on the l-c, and that extended to infinity; the solutions became singular at \( x = 2 \) and \( x = 1.4 \), respectively (cf. Mestel 1999, p. 547 and p. 612).

For the second case, it was proposed that there exists near \( x = 1.4 \) a thin volume domain (taken as cylindrical) in which \( E \cdot B \neq 0 \). A possible physical reason for the effective resistivity is the MHD instability of a local dominantly toroidal field (e.g. Begelman 1998). Prima facie, this allows construction of a solution that is well behaved at both the l-c and infinity. However, it is possible that the failure to find a global solution of the dissipation-free equation might turn out to be just a consequence of the assumed absence of a current sheet on the separatrix. This remains a problem for the future, along with a possible generalization to the cases with an oblique approach of the field to the equator. A fully convincing treatment will require study of the gas and current flow in the equatorial zone.

### 7.2 The inertial terms

The solutions constructed according to the procedure of Section 6 may be regarded as appropriate generalizations of the Michel field, modified by imposition of the dipolar field on the star, with its associated dead zone. The parameters of the problem are assumed to yield the electromagnetic energy density dominating over the kinetic energy density, so that the deviation from force-free conditions is restricted to the thin equatorial domain. As the outflowing gas will be accelerated to high \( \Gamma \)-values, we need to check that the inertial terms remain small far beyond the l-c.

Refer back to the equations (10)–(14). We are interested now in the parameter domain in which the ideal MHD condition (13) remains an adequate approximation to equation (12), but the non-linear inertial term in equation (11) may no longer be negligible. We follow the standard treatment of a perfectly conducting, relativistic wind, flowing in the presence of the asymptotically radial Michel field and the associated toroidal field (Michel 1969; Goldreich & Julian 1970; Li & Melrose 1994; Mestel 1999, section 7.9), using the notation of Melatos and Melrose. In addition to the kinematic relation (27), the continuity condition (10) has the steady-state integral

\[ m_{\eta n k} = \eta \]  

where \( m \) is the electron/positron mass, and the equation of motion (11) yields the modified torque and energy integrals

\[ -\sigma B_\phi/4\pi + \eta \Gamma \sigma U_\phi = -\beta(P)/4\pi \]  

and

\[ \Gamma c^2(1 - \alpha \sigma U_\phi) = H(P). \]

(Recall that in this relativistic problem the electric force density \( E = -\rho_e U \times B/c \) is not negligible; however, it contributes to force balance across the field, but not to the integrals (68) and (69).)

For simplicity, we consider just field lines near to the equator, with

\[ B_{\sigma} = \Phi/\sigma^2, \]

where \( \Phi \) is a measure of the poloidal flux crossing the l-c. The principal results are the prediction of the asymptotic values

\[ U_\phi \simeq c, \quad U_\phi \simeq 0, \quad B_\phi \simeq -(\alpha \sigma/c)B_{\sigma}. \]

and

\[ \Gamma \left( \frac{\alpha^2 \Phi}{4\sigma \rho_\phi c^3} \right)^{1/3} \equiv \sigma^{1/3}. \]
The force-free approximation will remain valid as long as the kinetic energy density is small compared with the electromagnetic, i.e.
\[
\mathcal{R} \equiv \frac{\Gamma n m c^2}{(B^2/4\pi)} \ll 1. \tag{73}
\]
Substitution from equations (67), (70), (71) and (72) yields
\[
\mathcal{R} \simeq \sigma^{-2/3} \simeq \Gamma^{-2}. \tag{74}
\]
The \(\pm\) \(e\)-density \(n\) is written conveniently as the product of a pair-production multiplicity factor \(M\) and an estimate for the local GJ density: \(n = (M(e)(\rho_c)_{\lambda\alpha} \approx M (a B^3/2\pi e c)\). Substitution of numbers yields
\[
\Gamma \simeq [(10^7/M)(B_{11}^3 R_{16}^3/\rho_c^3)]^{1/3} \tag{75}
\]
in terms of a stellar field \(B_1 = 10^{12}\) G, stellar radius \(R_1 = 10^6\) and pulsar rotation period \(P_1 = 1\) s. Thus, even with \(M = 10^3\), the predicted \(\mathcal{R} \ll 1\) for the longest periods, so that the basic approximation leading to (21) and therefore the solutions of section 6—constructed subject to the crucial equatorial boundary condition \(B_1 = 0\) appears to remain valid over the whole domain.

The result is a self-consistency check for the present class of model, but also shows up a basic limitation to its applicability to a real pulsar; for the condition (73) is — not surprisingly — also the condition that the energy flow at infinity be primarily by the Poynting flux rather than the kinetic flux. In fact, the wind flow results are highly model-dependent. Even with the same equatorial boundary condition, the predicted dominantly toroidal field at infinity may be dynamically unstable (e.g. Begelman 1998). Further, experience with other models with non-radial field lines beyond the l-c (MRWW; FMa; FMb; Beskin, Gurevich & Istomin 1993; MS) suggests strongly that in an alternative model with field lines approaching the equator obliquely (cf. Section 7.1), particles streaming into the equator will have acquired much higher \(\Gamma\)-values.

Finally, we emphasize that this paper has been concerned with the conditions for the construction of viable models. In practice, the particle pressure requirements in the equatorial and dead zones may very well put physical limitations on the model parameters: the extent \(x_c\) of the dead zone and the associated separatrix \(P_c\).

ACKNOWLEDGMENTS
The authors thank the referee Professor D. B. Melrose for helpful suggestions on presentation.

REFERENCES
Abramowizt M. C., Stegun I. A., 1965, Handbook of Mathematical Functions Dover, New York
Begelman M. C., 1998, ApJ, 493, 291
Beskin V. S., Gurevich A. V., Iostimov Y. N., 1993, Physics of the Pulsar Magnetosphere. Cambridge Univ. Press Cambridge
Biggs J. D., McCulloch P. M., Hamilton P. A., Manchester R. N., Lyne A. G., 1985, MNRAS, 215, 281
Blackman E. G., Field G. B., 1993, Phys. Rev. Lett., 71, 3481
Contopoulos L. I., 1995, ApJ, 446, 67
Contopoulos L., Kazanas D., Fendt C., 1999, ApJ, 511, 351 (CKF)
Fitzpatrick R., Mastel L., 1988a, MNRAS, 232, 277 (FMA)
Fitzpatrick R., Mastel L., 1988b, MNRAS, 232, 303 (FMB)
Goldreich P., Julian W. H., 1969, ApJ, 157, 869 (GJ)
Goldreich P., Julian W. H., 1970, ApJ, 160, 971
Holloway N. J., Pryce M. H. L., 1981, MNRAS, 194, 95
Jeffreys H., Jeffreys B. S., 1972, Methods of Mathematical Physics, 3rd edn. Cambridge Univ. Press, Cambridge
Li J., Melrose D. B., 1994, MNRAS, 270, 687
Lynden-Bell D., 1996, MNRAS, 279, 389
Lyne A. G., Manchester R. N., 1988, MNRAS, 234, 477
Melatos A., Melrose D. B., 1996, MNRAS, 279, 1168
Mestel L., 1999, Stellar Magnetism. Clarendon Press Oxford
Mestel L., 2001, PASA, 18, 1
Mestel L., Pryce M. H. L., 1992, MNRAS, 254, 355 (MP)
Mestel L., Shibata S., 1994, MNRAS, 271, 621 (MS)
Mestel L., Spruit H. C., 1987, MNRAS, 226, 57
Mestel L., Wang Y.-M., 1979, MNRAS, 188, 799 (MW)
Mestel L., Robertson J. A., Wang Y.-M., Westfold K. C., 1985, MNRAS, 217, 443 (MRRW)
Michel F. C., 1969, ApJ, 158, 727
Michel F. C., 1973a, ApJ, 180, L133
Michel F. C., 1973b, ApJ, 180, 207
Michel F. C., 1991, Theory of Neutron Star Magnetospheres. Univ. Chicago Press, Chicago
Rankin J. M., 1993, ApJS, 85, 145
Scharlemann E. T., Wagoner R. V., 1973, ApJ, 182, 951
Shibata S., Miyazaki J., Takahara F., 1998, MNRAS, 295, L53
Shibata S., Miyazaki J., Takahara F., 2002, MNRAS, 336, 233
Smith I. A., Michel F. C., Thacker P. D., 2001, MNRAS, 322, 209
van Leeuwen A. G. J., Kouwenhoven M. L. A., Ramachandran R., Rankin J. M., Stappers B. W., 2002, A&A, 387, 169
Vivekanand M., Joshi B. C., 1997, ApJ, 477, 431
Wright G. A. E., Fowler L. A., 1981, in Sieber W., Wielebinski R., eds, IAU Symp. 95, Pulsars. Reidel, Dordrecht, p. 221

APPENDIX A: FOURIER METHODS

The present models
Other studies of the pulsar magnetosphere have written the flux function \(P\) as a Fourier integral in \(z\). In the illustrative MMWP model studied in MW and MP, in which the dead zone extends to the l-c, \(P\) is written as a Fourier cosine integral, appropriate to a domain with the equatorial boundary condition \(B_1 \propto \partial P/\partial z = 0\). In the present paper, with the perfectly conducting wind domain extending from \(x_c \leq 1\) to \(\infty\) and so with the equatorial boundary condition \(B_1 \propto \partial P/\partial x = 0\), the appropriate form for the normalized \(P\)-function in the domain \(x_c \leq x \leq \infty\) is the Fourier sine integral
\[
P = \frac{2}{\pi} \int_0^{\infty} f(x, k) \sin k z \, dk, \quad f = \int_0^{\infty} P \sin k z \, dz. \tag{A1}
\]

This is a Fourier representation of the function \(P(x, z)\) in the northern hemisphere and \(-P(x, z)\) in the southern, with \(P(x, 0\pm) = \pm 1\). Using the Fourier theorem (e.g. Jeffreys & Jeffreys 1972), at the discontinuity on the equator, the Fourier representation (A1) should indeed have the value \((+1) + (-1)/2 = 0\). However, this representation will exhibit the Gibbs phenomenon, and may make difficult the accurate construction of behaviour near the equator.

It is convenient to introduce also \(F(x, k)\) defined by
\[
f = \frac{1}{k} + F(x, k). \tag{A2}
\]
Because of the boundary condition \(P = 1\) on \(z = 0\), the Fourier cosine transform of \(\partial P/\partial z\) is

© 2004 RAS, MNRAS 349, 213–224
\[ \int_0^\infty \frac{\partial P}{\partial z} \cos kz \, dz = -1 + k \int_0^\infty P \sin kz \, dz \]
\[ = -1 + k f(x, k) = kF(x, k). \quad (A3) \]

Likewise, application of the Fourier sine transform to the force-free equation (64) yields

\[ (1 - x^2)F'' - \frac{(1 + x^2)}{x} F' - (1 - x^2)k^2 F = -g(x, k), \quad (A4) \]

where as usual \( F = F_z \), and

\[ g(x, k) = \int_0^\infty \frac{dS}{dP} \sin kz \, dz. \quad (A5) \]

In general, the integral (A5) can be converted into one over \( P \):

\[ g(x, k) = \int_1^0 \left( \frac{dS}{dP} \right) \sin kz \, dP. \quad (A6) \]

For a simple illustration, consider the Michel form (41) for \( S(P) \), for which equation (64) has the known solution (40) (again normalized with \( P_c = 1 \), valid everywhere in the absence of a dead zone (i.e. with \( x_c = 0 \)). We quote the known Fourier transformation (Abramowitz & Stegun 1965, equation 9.6.25)

\[ \int_0^\infty \frac{\cos kz}{(x^2 + z^2)^{1/2}} \, dx = \pi^{1/2} \left( \frac{k}{2x} \right)^\nu \frac{K_n(kx)}{\Gamma(\nu + 1/2)}. \quad (A7) \]

Then substitution from equation (42) into equation (A5) and from equation (40) into equations (A1) and (A2) yields after integration by parts

\[ F(x, k) = -xK_\nu(kx), \quad g(x, k) = 2k\pi x^2 \nu \frac{K_{\nu+1}(kx)}{\Gamma(\nu + 1/2)}, \quad (A8) \]

whence equation (A4) is seen to be satisfied from the known properties of Bessel functions.

Now return to the problem with a dipolar field on the star and so with a finite dead zone, terminating at the point \((x_c, 0)\). It emerges from the computations reported in Section 6 that, even when \( x_c \) close to unity, the allowed \( S(P) \) relations do not differ much from the normalized Michel form

\[ S = -2P + P^2. \quad (A9) \]

The appearance of the neutral point \((x_c, 0)\) appears to be associated rather with the marked deviation from the radial structure for \( B_p \), found as the l-c is approached from without. It is of interest to discuss this by application of the Fourier formalism.

Having chosen a constructed model, and adopted the values found for \( P(1, 2) \) and \( P_c(1, 2) \), we then move inwards from \( x = 1 \), keeping \( P \) and \( P_c \) continuous, but using the \( S(P) \) relation (A9). We can write the solution of equation (A4) as

\[ F(x, k) = b_1 F_{\nu} + F_{pi} \quad (A10) \]

where \( b_1 F_{\nu} \) is a complementary function, satisfying equation (A4) with zero right-hand side, and \( F_{pi} \) is a particular integral, conveniently constructed to satisfy \( F_{pi}(1, k) = 0 \). As in the MMWP problem, \( F_{\nu} \) is started off from \( x = 1 \) by the non-singular series

\[ F_{\nu} = 1 + \frac{k^2}{4} (1 - x^2) + \frac{k^4}{36} (1 - x)^3 + \cdots, \quad (A11) \]

and is then continued inwards. (Note that the function \( F_{\nu} \) for given \( k \) will remain unchanged during the whole operation.)

With \( g(x, k) \) prescribed and \( F_{\nu} \) known, the particular integral is given by

\[ F_{pi} = F_{\nu} \int_x^1 dx \left[ \frac{x}{(1 + x)F_{\nu}^2} \int_x^1 \left[ \frac{g(v)F_{\nu}(v)/v}{1 - v} \right] dv \right]. \quad (A12) \]

Near the l-c, \( F_{pi} \) is best computed from the series

\[ F_{pi} = \frac{g(1, k)}{2} (1 - x) + \frac{g'(1, k)}{8} (1 - x)^5 + \cdots. \quad (A13) \]

found simply from equation (A4) and its first derivative.

With the particular choice for the particular integral, \( F(1, k) = b_1 F_{\nu}(1, k) + F_{pi}(1, k) = b_1 \), so \( b_1 \) is fixed for given \( S(P) \) from the previously constructed solution for \( x > 1 \). For the present problem, from the given values of \( P \) on the l-c, continuity of \( F_z \) and so of \( P \), is ensured by the construction of \( F_{pi} \). Numerical continuation of \( F_{pi} \) from equation (A12) will depend on substitution into \( g \) of the values for \( P \) emerging from the inward integration.

The model with \( x_c = 0.86 \) was chosen for study. The inward integration is to be halted when a value \( x = x_c \) is reached for which, using equation (A3) together with equation (A10),

\[ \frac{\partial P}{\partial z} |_{z=0} = 0 = \frac{\partial P}{\partial z} |_{z=\pi/2} \]
\[ = kF(x_c, k) dk \]
\[ = \int_0^\infty [k b_1 F_{\nu}(x_c, k) + kF_{pi}(x_c, k)] dk \]
\[ = \int_0^\infty k \left[ F(1, k)F_{\nu}(x_c, k) + F_{pi}(x_c, k) \right] dk. \quad (A14) \]

However, the numerical work yielded only partially satisfactory results. The preliminary integrations confirmed that the complementary function dominates i.e. the solution is indeed determined primarily by the distribution of \( P \) on the l-c – but the predicted value for \( x_c \) was \( \approx 0.82 \). Attempts to bring \( x_c \) closer to the value 0.86 found in the global integration of Section 6 by including higher \( k \)-values led to oscillating or unstable back transforms. We conclude that in a problem with the boundary condition \( P(x, 0) = 0 \), requiring the Fourier sine integral (A1), but with \( P(x, 0) = 1 \), the method is in general inappropriate for study of the field structure near the equator.

The equatorial boundary condition \( B_z = 0 \)

For the models of this paper, recall that in the inner domain the dead zone terminates within the l-c at the point \( N \) with coordinates \((x_c, 0)\) with \( x_c \leq 1 \). Within the dead zone, the field lines cross the equator normally, so that \( \partial P/\partial z = 0 \) for \( x < x_c, z = 0 \); whereas the condition that the wind zone be perfectly conducting enforces a fully open field crossing the l-c, so that \( \partial P/\partial z = 0 \) for \( x > x_c, z = 0 \). This in turn yields a non-zero \( S \) both at the outer domain equator and on its continuation as the separatrix \( P_c \), between the wind and dead zones, so that much of the current returns to the star as a sheet. Simultaneously, the conditions of equilibrium require a gas pressure both at the equator beyond \( x_c \) and within the dead zone.

In the somewhat more complicated model of MS, there is again a force-free domain extending beyond the l-c, but now the boundary condition \( \partial P/\partial z \propto B_z(x, 0) = 0 \) is supposed to hold in the outer domain \( x > 1 \) also. This is the appropriate approximation if, for example, a strong dynamically-driven macroresistivity allows the radial wind flow along the equator to cross the field lines. As in MW and MP, the solution for \( P \) can now be written as a Fourier cosine integral. The constraint (18) still holds away from the equator, but a local departure from torque-free conditions, associated with the flow of gas within the equatorial zone, allows \( S(P) \) to vary along the equator. Further, with \( B_z(x, 0) = 0 \), the arguments in Section 3 requiring a non-zero \( S \) at the equator and so along the separatrix.
between the wind and dead zones no longer hold. It is tentatively postulated that the current closure condition can now be satisfied by the simple choice \( S(P_c) = 0 \), allowing \( S(P) \) to go to zero continuously, without there needing to be a sheet current along \( P_c \). There is no pinched, high-pressure equatorial sheet, nor is there a mandatory introduction of pressure into the dead zone.

However, as noted in Section 7, these simplifications come at a price; for the case studied, with no current sheet on the separatrix, and with the equatorial boundary condition \( \partial P / \partial z \propto B_z = 0 \) holding everywhere, a solution that is well behaved and continuous at the l-c blows up before it can reach infinity. In MP, it is shown that imposition of smooth continuity at the l-c yields a singularity in the solution at less than two l-c radii. Equivalently, we can apply standard asymptotic theory to the Fourier integral, demanding that the Fourier transform behave properly at infinity; there then appears an incompatibility of sign between solutions within and without the l-c (Mestel 2001). Resolution of the dilemma may be as suggested in MP, with local breakdown in the simple plasma condition \( E \cdot B = 0 \), not only on the equator, but also in a thin, dissipative volume domain, idealized as a cylindrical shell symmetric about the rotation axis. Further work is needed to test whether, instead, relaxation of the condition \( S(P_c) = 0 \) will allow construction of a global solution of equation (64) subject to the condition \( B_z(x, 0) = 0 \).

This paper has been typeset from a \TeX/\LaTeX\ file prepared by the author.