Convex Trace Functions on Quantum Channels and the Additivity Conjecture

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(Dated: May 25, 2009)

We study a natural generalization of the additivity problem in quantum information theory: given a pair of quantum channels, then what is the set of convex trace functions that attain their maximum on unentangled inputs, if they are applied to the corresponding output state?

We prove several results on the structure of the set of those convex functions that are “additive” in this more general sense. In particular, we show that all operator convex functions are additive for the Werner-Holevo channel in $3 \times 3$ dimensions, which contains the well-known additivity results for this channel as special cases.

I. INTRODUCTION AND MAIN DEFINITION

For quite some time, the additivity conjecture has been one of the most notorious open problems in quantum information theory; it has been settled only recently in a breakthrough paper by Hastings [1]. The original conjecture can be stated in several equivalent ways [2]; one possible formulation is via the minimum output entropy of a quantum channel $\Phi$, defined as

$$S_{\min}(\Phi) := \min_\rho S(\Phi(\rho)) = \min_\rho \left( -\Phi(\rho) \log \Phi(\rho) \right),$$

where the minimization is over all input states $\rho$, and $S$ is von Neumann entropy. The intuition is that $S_{\min}(\Phi)$ is a measure of noisiness of the channel $\Phi$.

The original additivity conjecture stated that

$$S_{\min}(\Phi \otimes \Omega) = S_{\min}(\Phi) + S_{\min}(\Omega) \quad (1)$$

for all channels $\Phi$ and $\Omega$; that is, the minimum output entropy of a pair of channels should be the sum of the individual minimum output entropies. During the years that the problem has been studied, it turned out to be convenient to generalize the additivity problem to $p$-Rényi entropies: For $p > 0$, $p \neq 1$, and density matrices $\rho$, define [3]

$$S_p^{\min}(\Phi) := \min_\rho \frac{1}{1-p} \log \Tr(\Phi(\rho)^p),$$

and then the question is whether

$$S_p^{\min}(\Phi \otimes \Omega) = S_p^{\min}(\Phi) + S_p^{\min}(\Omega) \quad (2)$$

holds true in general, for all $p > 0$. Due to the limit $S_1^{\min}(\Phi) := \lim_{p \to 1} S_p^{\min}(\Phi) = S_{\min}(\Phi)$, (2) is a natural generalization of (1).

Quite surprisingly, it turned out that the conjectured equalities [2] and (1) are both false in general. They were subsequently disproved by constructing counterexample channels, first for $p > 1.79$ [4], then for $p > 2$ [5], then for $p > 1$ and $p \approx 0.333$ [6, 7], and finally for $p = 1$ [1], killing the original conjecture [1]. For detailed expositions of the problem and its history, see for example [8] or [9].

Despite those no-go results, it has been shown that additivity holds for many interesting classes of channels and several values of $p$, for example for the cases that one of the channels is the identity channel [9, 10] or a unital qubit channel [11]. Even if additivity fails in general, its validity in special cases is still interesting in its own and potentially useful for channel coding problems, cf. [12].

The main goal of this paper is to show that some of those results for special channels have a natural interpretation within a more general framework.

To motivate our more general definition, notice first that the additivity conjecture can equivalently be stated as the assertion that entanglement does not help to produce pure outputs. In fact, Equation (1) holds if and only if the map

$$\rho \mapsto S(\Phi \otimes \Omega(\rho))$$

attains its global minimum at an unentangled input state $\rho$: since $S(\sigma \otimes \rho) = S(\sigma) + S(\rho)$ for density operators $\sigma$ and $\rho$, we get

$$S_{\min}(\Phi \otimes \Omega) \leq S(\Phi \otimes \Omega(\rho_\Phi \otimes \rho_{\Omega})) = S_{\min}(\Phi) + S_{\min}(\Omega)$$

if $\rho_\Phi$ and $\rho_{\Omega}$ are the minimizers for the two channels, i.e. $S_{\min}(\Phi) = S(\Phi(\rho_\Phi))$ and similarly for $\rho_{\Omega}$. This means that the inequality “$\leq$” in (1) is always true.
On the other hand, as von Neumann entropy is concave, the global minimum will be attained at some pure input state; also, \( \rho_a \) and \( \rho_2 \) can be chosen pure. Thus, the fact that \( \rho_a \otimes \rho_1 \) is indeed the global minimizer, i.e. “=” holds in (1), is equivalent to the fact that no other entangled input state can produce even smaller output entropy.

Equation (2) can be reformulated in a similar way: additivity for \( p \)-Rényi entropy with \( p > 1 \) holds true if and only if the function

\[
\rho \mapsto \text{Tr} (\Phi \otimes \Omega (\rho)^p)
\]

attains its global maximum at an unentangled input state \( \rho \). Thus, we have two variations of the same general problem: *Compute the trace of a convex function of the output, and decide whether this expression attains its global maximum at an unentangled input state.* For von Neumann entropy \( S \), this function is \( x \log x \), while for the \( p \)-Rényi entropy \( S_p \) with \( p > 1 \), this function is \( x^p \).

It is natural to ask what happens if the problem is generalized. What if one takes another convex function, different from \( x \log x \) or \( x^p \)? We use Definition 1 to study the generalized problem. In this definition and all of the following, applying a (convex) function \( f : [0, 1] \rightarrow \mathbb{R} \) to a density matrix \( \sigma := \Phi \otimes \Omega (\rho) \) is meant in the sense of spectral calculus: diagonalizing \( \sigma = U \text{diag}(\lambda_1, \ldots, \lambda_n)U^\dagger \), we define

\[
f(\sigma) := U \begin{pmatrix} f(\lambda_1) & & \\
& \ddots & \\
& & f(\lambda_n) \end{pmatrix} U^\dagger
\]

such that in particular \( \text{Tr} f(\sigma) = \sum_{i=1}^n f(\lambda_i) \), where the sum is over all eigenvalues \( \lambda_i \) of \( \sigma \).

**Definition 1 (Additive Functions on Q-Channels)**

Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a convex function, and let \( \Phi \) and \( \Omega \) be quantum channels. We say that \( f \) is additive for \((\Phi, \Omega)\) if there exists some unentangled input state \( \rho_u \) such that

\[
\text{Tr} f(\Phi \otimes \Omega (\rho_u)) \geq \text{Tr} f(\Phi \otimes \Omega (\sigma))
\]

for all input states \( \sigma \).

There are some simple consequences of this definition. First note that if \( f \) is convex as a real function, then it is automatically a “convex trace function” on the density operators in the sense that

\[
\text{Tr} f(\lambda \rho + (1 - \lambda) \sigma) \leq \lambda \text{Tr} f(\rho) + (1 - \lambda) \text{Tr} f(\sigma),
\]

see [13]. Thus, \( \text{Tr} f(\cdot) \) attains its maximum on pure input states, i.e. \([4]\) is equivalent to the existence of pure states \( \psi_1 \) and \( \psi_2 \) such that

\[
\text{Tr} f(\Phi (\psi_1) \otimes \Omega (\psi_2)) \geq \text{Tr} f(\Phi \otimes \Omega (\varphi)) \quad \forall \text{pure states } \varphi.
\]

Yet, in contrast to von Neumann or \( p \)-Rényi entropy, there is in general no way to further simplify the expression on the left-hand side by splitting the function of the tensor product into two addends or factors.

Clearly, this definition captures the additivity problems as special cases:

**Lemma 2** Let \((\Phi, \Omega)\) be a pair of quantum channels. Then additivity for \( p \)-Rényi entropy

\[
S_p^{\min}(\Phi \otimes \Omega) = S_p^{\min}(\Phi) + S_p^{\min}(\Omega)
\]

holds if and only if the function \( f_p \) is additive for \((\Phi, \Omega)\), where

\[
f_p(x) := \begin{cases} x^p & \text{if } p > 1, \\ x \log x & \text{if } p = 1, \\ -x^p & \text{if } 0 < p < 1. \end{cases}
\]

Hence proving the additivity conjecture for a pair of channels is equivalent to showing that \( x \log x \) is additive. Is there any reason why this more general approach could help? In fact, there is a popular example in matrix analysis where a similar strategy turned out to be successful, which is Löwner’s theory of operator convex functions (\([14, 15]\)).

A real function \( f \) is called *operator convex* if

\[
f(\lambda \rho + (1 - \lambda) \sigma) \leq \lambda f(\rho) + (1 - \lambda) f(\sigma)
\]

for all self-adjoint operators \( \rho \) and \( \sigma \) and \( 0 < \lambda < 1 \). This is an operator inequality, meaning that the difference of the right- and left-hand side is positive semidefinite. Clearly, operator convex functions are convex, but the converse turns out to be false. For example, \( x^3 \) is convex, but not operator convex.

Given some convex function \( f \), it can be difficult to decide directly from the definition \([14]\) whether \( f \) is operator convex. By contrast, it turns out that there is a simple characterization of the set of all operator convex functions, which can be stated elegantly in terms of integral representations or complex analysis. This is an unexpected result, since the definition \([14]\) itself involves only a linear-algebraic inequality.

Thus, it seems reasonable to hope that something similar might happen in the case of the additivity problem, at least for special classes of (highly symmetric) channels: possibly the class of additive functions for a channel pair, defined by the linear-algebraic inequality \([3]\), is also simple to characterize. As we will show in this paper, this speculation turns out to be true for the Werner-Holevo channel in \( 3 \times 3 \) dimensions at least.

We start by giving some simple examples.

**II. SOME EXAMPLES**

**Example 3** Let \( a > 0 \) and \( b, c \in \mathbb{R} \). A convex function \( f : [0, 1] \rightarrow \mathbb{R} \) is additive for a pair of channels if and
only if the function

\[ af(x) + bx + c \]

is additive for that pair of channels. In particular, linear functions \( f(x) = bx + c \) are additive (for every pair of quantum channels).

**Proof.** It is clear that scaling a function with \( a > 0 \) does not change the location of its global maximum. If \( \Phi : \mathcal{S}(\mathcal{H}_1^d) \rightarrow \mathcal{S}(\mathcal{H}_2^d) \) and \( \Omega : \mathcal{S}(\mathcal{H}_1^d) \rightarrow \mathcal{S}(\mathcal{H}_2^d) \) are arbitrary quantum channels, and if \( \rho \) is an arbitrary state on \( \mathcal{H}_1^d \otimes \mathcal{H}_1^d \), then it holds for \( f(x) := bx + c \)

\[
\text{Tr}\left( \Phi \otimes \Omega(\rho) \right) = \text{Tr}(b \cdot \Phi \otimes \Omega(\rho) + c \cdot 1) = b + c \cdot \dim \mathcal{H}_2^d \cdot \dim \mathcal{H}_2^d.
\]

Thus, \( \text{Tr}(\Phi \otimes \Omega(\rho)) \) is constant, and can be added to any function without changing its additivity properties. In particular, \( f \) itself is additive, as every unentangled input \( \rho_u \) satisfies [3]. \( \square \)

**Example 4** For channels of the form \( \Phi \otimes 1 \), every convex function is additive.

That is, if \( \Phi \) is an arbitrary quantum channel, and \( 1 \) is the identity channel on some Hilbert space, then every convex function \( f : [0,1] \rightarrow \mathbb{R} \) is additive for \( (\Phi,1) \).

In particular, as is well-known, von Neumann entropy and the p-Rényi entropies are additive for such channels for all \( p > 0 \).

**Proof.** The proof closely follows the lines of [4]. Suppose \( \rho'_{12} = (\Phi \otimes 1)(\rho_{12}) \). We may choose \( \rho_{12} \) to be pure. Let \( U_{13} \) be a unitary dilation of \( \Phi \), such that

\[
\rho'_{12} = \text{Tr}_3(U_{13} \otimes 1_2)(\rho_{12} \otimes |\varphi_3(\varphi_3)) U_{13}^\dagger \otimes 1_2).
\]

As seen in Example 3 we may assume that \( f(0) = 0 \). Since the expression right of \( \text{Tr}_3 \) is a pure state, the spectrum will not change if we replace \( \text{Tr}_3 \) by \( \text{Tr}_{12} \) up to possible multiplicity of the eigenvalue zero. Thus,

\[
\text{Tr}(\rho'_{12}) = \text{Tr}(\text{Tr}_{12}(U_{13} \otimes 1_2)(\rho_{12} \otimes |\varphi_3(\varphi_3)) U_{13}^\dagger \otimes 1_2)) = \text{Tr}(\text{Tr}_{13}(\rho_1 \otimes |\varphi_3(\varphi_3) U_{13}^\dagger)),
\]

where \( \rho_1 := \text{Tr}_2 \rho_{12} \). By convexity, this expression is maximized if \( \rho_1 \) is pure, i.e. \( \rho_{12} = \rho_1 \otimes \rho_2 \). \( \square \)

It is well-known [7] that the minimum output p-Rényi entropy of a pair of quantum channels is at least as large as that of one of its constituents, i.e. \( S_{\rho}^{\text{min}}(\Phi \otimes \Omega) \geq S_{\rho}^{\text{min}}(\Phi) \). The following lemma generalizes this statement, and yields an analogous property for all convex functions.

**Lemma 5 (Single Channel Bound)** If \( f : [0,1] \rightarrow \mathbb{R} \) is a convex function with \( f(0) = 0 \), then

\[
\max_{\rho} \text{Tr}(\Phi \otimes \Omega(\rho)) \leq \max_{\rho} \text{Tr}(\Phi(\rho))
\]

and similarly for \( \Omega \).

**Remark.** If \( f(0) \neq 0 \), then the bound is \( \max_{\rho} \text{Tr}(\Phi(\rho)) + d_{\Phi}(d_\Omega - 1) f(0) \), where \( d_\Phi \) and \( d_\Omega \) denote the dimensions of the output Hilbert spaces of \( \Phi \) and \( \Omega \) respectively.

**Proof.** If \( |0\rangle \) denotes an arbitrary pure state on the output Hilbert space of \( \Omega \), and \( \{\lambda_i\}_{i=1}^{d_\Omega} \) is the spectrum of \( \Phi(\rho) \), then the spectrum of \( \Phi(\rho) \otimes |0\rangle\langle 0| \) is \( \{\lambda_1, \ldots, \lambda_{d\Phi}, 0, 0, \ldots, 0\} \), with \( d_{\Phi}d_\Omega - d_{\Phi} \) zeroes. Thus,

\[
\max_{\rho} \text{Tr}(\Phi \otimes \Omega(\rho)) = \max_{\rho} \text{Tr}(\Phi \otimes 1(1 \otimes \Omega(\rho))) = \max_{\rho' = 1 \otimes \Omega(\rho)} \text{Tr}(\Phi(\rho')) \leq \max_{\rho'} \text{Tr}(\Phi \otimes 1(\rho')) \leq \text{Tr}(\Phi(\rho)) \otimes |0\rangle\langle 0| = \text{Tr}(\Phi(\rho)) + d_{\Phi}(d_\Omega - 1) f(0).
\]

The equality in (\*) follows from Example 3. \( \square \)

The first counterexample channel to the additivity conjecture for the p-Rényi entropy (for \( p > 4.79 \)) has been given by Werner and Holevo [3]. In dimension \( d \), the Werner-Holevo channel \( \Phi_d \) is defined as

\[
\Phi_d(\rho) := \frac{1}{d - 1} (1 - \rho^T).
\]

It has the useful covariance property

\[
\Phi_d(U \rho U^\dagger) = U \Phi_d(\rho) U^\dagger
\]

for any unitary \( U \). As a simple example, we derive a necessary condition for additivity for this channel in dimension \( d = 3 \):

**Example 6** If \( f : [0,1] \rightarrow \mathbb{R} \) is a convex function with

\[
f\left(\frac{1}{3}\right) + 8 f\left(\frac{1}{12}\right) > 5 f(0) + 4 f\left(\frac{1}{4}\right),
\]

then \( f \) is not additive for the Werner-Holevo channel pair \( (\Phi_3, \Phi_3) \).

**Proof.** If \( |\psi\rangle, |\varphi\rangle \in \mathbb{C}^d \) are arbitrary pure states, the output \( \Phi_d \otimes \Phi_d(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|) \) has a \((2d - 1)\)-fold degenerate eigenvalue \( 0 \), and a \((d - 1)^2\)-fold degenerate eigenvalue \( 1/(d - 1)^2 \). Due to the covariance property [8], this is true for all pure states and does not depend on \( |\psi\rangle \) or \( |\varphi\rangle \). Thus,

\[
\text{Tr}(\Phi_d \otimes \Phi_d(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)) = (2d - 1)f(0) + (d - 1)^2 f\left(\frac{1}{(d - 1)^2}\right).
\]
On the other hand, if we input a maximally entangled state \( \rho_m \), it is shown in \cite{4} that the output \( \Phi_3 \otimes \Phi_4(\rho_m) \) has a single eigenvalue \( (2 - 2/d)/(d-1)^2 \) and a \( (d^2 - 1) \)-fold degenerate eigenvalue \((1 - 2/d)/(d-1)^2\), such that

\[
\text{Tr}f(\Phi_3^2(\rho)) = \left(\frac{2 - \frac{2}{d}}{(d-1)^2}\right) + (d^2 - 1)f\left(\frac{1 - \frac{2}{d}}{(d-1)^2}\right).
\]

Comparing both expressions for \( d = 3 \), we see that \( f \) is not additive for \( (\Phi_3, \Phi_4) \) if the stated inequality holds.\(\square\)

It is clear that we get more similar inequalities for \( d \geq 4 \), but in most cases, these inequalities seem to be weaker.

The following lemma shows that the multiplicativity problem of the minimum output rank also fits into Definition \cite{14}. We need this result later in the proof of Example \cite{14}.

**Lemma 7 (Minimum Output Rank)**

The convex function

\[
\delta_0(x) := \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \in (0, 1] 
\end{cases}
\]

is not for all channels additive.

**Proof.** The function \( \delta_0 \) is related to the minimum output rank of quantum channels \( \Phi \):

\[
\max_\rho \text{Tr} \delta_0(\Phi(\rho)) = d - \min_\rho \text{rank}(\Phi(\rho)),
\]

where \( d \) is the dimension of the output Hilbert space of \( \Phi \). It has been shown in \cite{3} that the minimum output rank is not multiplicative; there exist channels \( \Phi \) and \( \Omega \) such that

\[
\min_\rho \text{rank}(\Phi \otimes \Omega(\rho)) < \min_\rho \text{rank}(\Phi(\rho)) \cdot \min_\rho \text{rank}(\Omega(\rho)),
\]

which means that \( \text{rank}(\Phi \otimes \Omega(\rho)) \) does not achieve its global minimum at tensor product input states \( \rho \). Consequently, \( \text{Tr} \delta_0(\Phi \otimes \Omega(\rho)) \) achieves its global maximum at entangled input states \( \rho \).\(\square\)

## III. ON THE STRUCTURE OF ADDITIVE FUNCTIONS

Since every convex function on \([0, 1]\) is bounded, the sup norm distance \( \|f - g\|_\infty := \sup_{x \in [0,1]} |f(x) - g(x)| \) can be used as a distance measure on the set of convex functions \( \mathcal{F} \) on the unit interval. This way, we get a notion of “open” and “closed” sets in \( \mathcal{F} \). Formally, we get the relative topology of \( \mathcal{F} \) within the larger Banach space of bounded functions on \([0, 1]\).

The next lemma shows that the set \( M \subset \mathcal{F} \) of additive functions for a fixed pair of channels is a closed cone, where “cone” refers to the simple property that \( \alpha f \in M \) holds for every \( \alpha \geq 0 \).

**Lemma 8** With respect to the \( \| \cdot \|_\infty \)-norm topology, the set of additive functions on a pair of channels \( (\Phi, \Omega) \) is a closed cone.

**Proof.** The cone property is trivial: a function \( f \) is additive for \( (\Phi, \Omega) \) if and only if \( \alpha f \) is additive for \( (\Phi, \Omega) \).

On the other hand, a function \( f \) is not additive for \( (\Phi, \Omega) \) if and only if there exists an entangled state \( \rho \), such that

\[
\text{Tr}f(\Phi(\Omega(\rho))) > \text{Tr}f(\Phi(\Omega(\rho_1)) \otimes \Omega(\rho_2)) \quad \forall \rho_1, \rho_2.
\]

It is clear that there is some \( \varepsilon > 0 \) such that for any convex function \( g : [0, 1] \rightarrow \mathbb{R} \) with \( \|f - g\|_\infty < \varepsilon \), equation \( 7 \) still holds if \( f \) is replaced by \( g \). This shows that the set of non-additive functions for \( (\Phi, \Omega) \) is open.\(\square\)

In the following, we will often show that a sequence of additive convex functions \( \{f_n\} \) on \([0, 1]\) converges pointwise to a limit function \( f \), and then refer to Lemma \( 8 \) to conclude that \( f \) must be additive, too. In fact, it is shown in \cite{16, Corollary 1.3.8} that in this case, pointwise convergence implies uniform convergence, and the limit function must be convex. Hence this kind of reasoning is justified.

It is a natural question whether the set of additive or non-additive functions has interesting properties. One useful property is convexity. It is not clear in general if the set of additive functions for a given arbitrary pair of channels is convex. However, convexity holds for the special class of unitarily covariant channels. In accordance with \cite{17}, we call a channel \( \Phi \) unitarily covariant if for every unitary \( U \), there exists a unitary \( V \) such that

\[
\Phi(U \rho U^\dagger) = V \Phi(\rho)V^\dagger \quad \forall \rho.
\]

Sometimes a different class of channels is studied: a channel \( \Phi \) is called irreducibly covariant (cf. \cite{18, 19}) if there are irreducible unitary representations \( U_g, V_g \) of a group \( G \) such that

\[
\Phi(U_g \rho U_g^\dagger) = V_g \Phi(\rho)V_g^\dagger
\]

for all \( g \in G \) and all \( \rho \). Unitarily covariant channels need not be irreducibly covariant, and vice versa; for example, if \( \Phi \) and \( \Omega \) are \( d \)-dimensional unitarily covariant channels with \( V = U \), then the tensor product channel \( \Phi \otimes \Omega \) is irreducibly covariant with respect to \( U(d) \times U(d) \), but it is in general not unitarily covariant. For a counterexample in the opposite direction, define a channel \( \Omega \) on \( \mathbb{C}^2 \) via \( \Omega(\rho) := (\text{Tr} \rho)|0\rangle \langle 0| \), where \( |0\rangle \in \mathbb{C}^2 \) is some normalized vector. Then \( \Omega \) is unitarily covariant in the sense of Equation \( 8 \) (with \( V = I \) for every \( U \)), but it is not irreducibly covariant, since any group representation \( V_g \) satisfying Equation \( 9 \) must leave the subspace spanned by \( |0\rangle \) invariant.

The Werner-Holevo channel \( \Omega \) is an example of a unitarily covariant channel due to \cite{10}.
Lemma 9 (Unitarily Covariant Channels)
If $\Phi$ and $\Omega$ are unitarily covariant channels, then the set of additive functions on $(\Phi, \Omega)$ is convex (and due to Lemma 3 a closed convex cone).

Proof. Let $f$ and $g$ be additive convex functions for $(\Phi, \Omega)$. We have to prove that $f + g$ is also additive for $(\Phi, \Omega)$.

Due to the unitary covariance of $\Phi$ and $\Omega$, the eigenvalues of $\Phi \otimes \Omega(\rho)$ and $\Phi \otimes \Omega(U \otimes V \rho U^\dagger \otimes V^\dagger)$ are the same for every unitary $U$ and $V$. Thus, $\text{Tr} f(\Phi \otimes \Omega(\rho))$ depends only on the Schmidt coefficients of the pure state $\rho$ (and similarly for $g$). Since $f$ is additive, the expression $\text{Tr} f(\Phi \otimes \Omega(\rho))$ attains its global maximum at every pure unentangled input state $\rho$ at once. The same is true for $g$; thus, $f$ and $g$ have a global maximizer in common. It follows that $f + g$ must have the same global maximizer, namely, an unentangled state. $\square$

The minimum output entropy additivity conjecture is known to hold true for the Werner-Holevo channel, defined in (5), in arbitrary dimensions. According to Datta [20] and Alicki and Fannes [21], the same is true for the additivity of the $p$-Rényi entropy for $1 < p \leq 2$, but additivity does not hold if $p > 4.79$ (cf. [4]). Moreover, additivity also holds in the domain $0 < p < 1$, as remarked in [3].

Due to Lemma 2, those additivity results are related to the functions $x \log x$ and $x^p$ for $1 < p \leq 2$ as well as $-x^p$ for $0 < p < 1$. An interesting observation is that all these functions are operator convex as defined in (7).

Thus, the following theorem contains many known results on the Werner-Holevo channel as special cases:

Theorem 10 (Werner-Holevo Channel)
Every operator convex function $f : [0, \infty) \to \mathbb{R}$ is additive for the Werner-Holevo channel ( tensored with itself) in dimension 3.

We conjecture that this is also true for the Werner-Holevo channel in larger dimensions $d \geq 4$ and for more than two factors; yet, it seems that the original calculations in [20] cannot be so easily adapted to that general case. Also, numerically it seems that it is sufficient that $f$ is operator convex on $[0, 1]$ (instead of $[0, \infty)$), but the proof is more difficult.

Proof. It is well-known [14] that every operator convex function $g$ on $(-1, 1)$ has an integral representation of the form

\[ g(t) = g(0) + g'(0) t + \frac{g''(0)}{2} \int_{-1}^{1} \frac{t^2 - 1}{1 - \lambda t} d\mu(\lambda), \]

where $\mu$ is some probability measure on $[-1, 1]$. Therefore, if $f$ is operator convex on $[0, \infty)$, then it is in particular operator convex on $(0, 1)$, and we can shift the above expression by substituting $x := \frac{t + 1}{2}$ to obtain

\[ f(x) = \alpha + \beta x + \gamma \int_{-1}^{1} \frac{(2x - 1)^2}{1 - \lambda(2x - 1)} d\mu(\lambda), \]

where $\alpha + \beta x = f \left( \frac{1}{2} \right) + \frac{1}{2} f' \left( \frac{1}{2} \right) (2x - 1)$, and $\gamma = \frac{1}{2} f'' \left( \frac{1}{2} \right) \geq 0$. Moreover, the measure $\mu$ must vanish on $(0, 1]$, because $f_\lambda(x) := \frac{(2x - 1)^2}{a_3(x)}$ has a pole in the positive reals for every $\lambda \in (0, 1]$, but $f$ is by assumption defined on all of $[0, \infty)$. For the same reason, $\mu$ must vanish at $\lambda = -1$.

According to Lemma 2, it is thus sufficient to show that the functions $\alpha + \beta x$ and $f_\lambda$ are additive for every $\lambda \in (-1, 0]$; then, it follows that $f$ must be additive, too.

But the function $\alpha + \beta x$ is trivially additive (as shown in Lemma 3). Let $\Phi_3$ be the Werner-Holevo channel in dimension $d = 3$ as defined in (5). From [22], we know the eigenvalues of the output $\Phi_3 \otimes \Phi_3(\rho)$ if the input has Schmidt coefficients $(\lambda_1, \lambda_2, \lambda_3)$: There are 6 eigenvalues of the form

\[ e_{\alpha\beta} := \frac{1 - \lambda_\alpha - \lambda_\beta}{4}, \quad (\alpha \neq \beta, \quad \alpha, \beta = 1, 2, 3) \]

and 3 eigenvalues of the form

\[ G_\alpha := \frac{1}{3} \cos^2 \left( \frac{\theta}{6} - \frac{2\pi(\alpha - 1)}{6} \right) \quad (\alpha = 1, 2, 3), \]

where $\tan \theta = \frac{\sqrt{4(2 - t)}}{t - \pi}$, and $t = \lambda_1 \lambda_2 \lambda_3$. Hence,

\[ \text{Tr} f_\lambda(\Phi_3 \otimes \Phi_3(\rho)) = \sum_{\alpha \neq \beta} f_\lambda(e_{\alpha\beta}) + \sum_{\alpha = 1}^{3} f_\lambda(G_\alpha). \]

Since the set

\[ \left\{ \left. \frac{1 - \lambda_\alpha - \lambda_\beta}{4} \right| \sum_{i=1}^{3} \lambda_i = 1 \right\} \subset \mathbb{R}^6 \]

is convex, the convex function $\sum_{\alpha \neq \beta} f_\lambda(e_{\alpha\beta})$ attains its maximum on the extremal points, i.e. those points where, up to permutation, $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$.

Due to the simple form of the functions $f_\lambda$, it is easy to show with some analysis that the function $\sum_{\alpha = 1}^{3} f_\lambda(G_\alpha)$ attains its global maximum at $\theta \in [0, \pi] \setminus \{\theta = \pi\}$, corresponding to $t = \lambda_1 \lambda_2 \lambda_3 = 0$. In fact, this expression is constant in $\theta$ for $\lambda = 0$, and it is increasing in $\theta$ if $-1 < \lambda < 0$.

In summary, $\text{Tr} f_\lambda(\Phi_3 \otimes \Phi_3(\rho))$ attains its global maximum on the states with Schmidt coefficients $(1, 0, 0)$, i.e. on the unentangled states. Thus, $f_\lambda$ is additive for every $\lambda \in (-1, 0]$ for two copies of the Werner-Holevo channel in dimension 3. The claim follows. $\square$

Consider the set $\mathcal{U}$ of functions that are additive for all pairs of unitarily covariant channels $(\Phi, \Omega)$. According to Lemma 3, the set $\mathcal{U}$ is a closed convex cone. It is an interesting problem to determine the set $\mathcal{U}$ explicitly. In the light of Theorem 10 and due to the fact that the most natural closed convex subset of the convex functions is the set of operator convex functions, the following conjecture seems natural:
Conjecture 11 (Additivity & Operator Convexity)
The set of functions \( \mathcal{U} \) that are additive for all unitarily covariant channels agrees with the set of operator convex functions on some interval \( I \subset \mathbb{R} \).

It seems that for a fixed pair of channels, the set of additive functions does not have a simple description in general, and several natural conjectures on the structure of the set of additive functions fail. For example, it is easy to construct convex functions \( f \) and \( g \) such that \( f \) and \( f + g \) are additive for the Werner-Holevo channel pair \((\Phi_3, \Phi_3)\), but such that \( g \) is not additive for \((\Phi_3, \Phi_3)\). Also, there are additive functions \( f \) and \( g \) such that \( \max\{f, g\} \) is not additive (cf. Theorem 19).

In the following, we will prove some more results on the set of functions that are additive for certain sets of channels. We will assume that the channel sets have the following property:

Definition 12 (Channel Classes) In the remainder of the paper, a channel class \( \mathcal{C} \) is a set of channels which is closed with respect to tensor products, and which contains all maximally depolarizing channels. That is,

- \( \Phi, \Omega \in \mathcal{C} \Rightarrow \Phi \otimes \Omega \in \mathcal{C} \),
- \( \Sigma_\sigma \in \mathcal{C} \) for all \( \sigma = \frac{1}{2} \mathbf{1} \), where \( \Sigma_\sigma(A) := \text{Tr}(A) \).

Examples of channel classes are

- the set of all channels, and
- the set of irreducibly covariant channels: tensor products of irreducibly covariant channels are again irreducibly covariant [14], and \( \Sigma_\sigma \) is irreducibly covariant if \( \sigma \) is proportional to the identity.

The set of unitarily covariant channels is not a channel class. However, the set of all channels which can be written as tensor products of unitarily covariant channels is a channel class.

We are interested in the set of functions that are additive for all channels in a given channel class \( \mathcal{C} \) (we call them the “functions that are additive for \( \mathcal{C} \)). According to Lemma 8, the set of those functions is a closed cone (cf. Theorem 19).

Proof. Let \( \vec{\mu} \) be an arbitrary probability vector, and let \( f \) be additive for \( \mathcal{C} \). Let \( \Phi, \Omega \in \mathcal{C} \) be arbitrary channels. We have to show that the function \( \tilde{f}(x) := \sum_{i=1}^{n} f(\mu_i x) \) is additive for \((\Phi, \Omega)\).

Let \( \sigma \) be a \( n \times n \) density operator with eigenvalues \( \mu_1, \ldots, \mu_n \). Consider the channel \( \Phi_3 \otimes \Omega \otimes \Sigma_\sigma \), and let \( |\psi\rangle \) be an input vector for this tripartite channel. It has a Schmidt decomposition

\[
|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\Phi_3\rangle \otimes |i\Sigma_\sigma\rangle,
\]

where \( \{|i\Phi_3\rangle\} \) and \( \{|i\Sigma_\sigma\rangle\} \) are orthonormal bases on the input Hilbert spaces for the channels \( \Phi_3 \otimes \Omega \) and \( \Sigma_\sigma \) respectively. The corresponding output is

\[
\Phi_3 \otimes \Omega \otimes \Sigma_\sigma(|\psi\rangle \langle \psi|) = \sum_{ij} \lambda_i \lambda_j \Phi_3 \otimes \Omega(|i\Phi_3\rangle \langle j\Phi_3|) \otimes \Sigma(|i\Sigma_\sigma\rangle \langle j\Sigma_\sigma|) = \sum_i \lambda_i \Phi_3 \otimes \Omega(|i\Phi_3\rangle \langle i\Phi_3|) \otimes \sigma.
\]

The trace of a convex function on that output attains its maximum, due to convexity, in the extremal case where, up to permutation, \( \lambda_1 = 1 \) and \( \lambda_2 = \lambda_3 = \ldots = 0 \). This means that we may choose the input to be unentangled between \( \Phi \otimes \Omega \) and \( \Sigma_\sigma \). In this case, if the output \( \Phi_3 \otimes \Omega(|\psi\rangle \langle \psi|) \) has spectrum \( \{\alpha_1, \ldots, \alpha_N\} \), then the output \( \Phi_3 \otimes \Omega \otimes \Sigma_\sigma(|\psi\rangle \langle \psi| \otimes |\varphi\rangle \langle \varphi|) \) has spectrum \( \{\alpha_i \mu_j\}_{i,j} \).

Since \( f \) is additive for \( \mathcal{C} \), it is in particular additive for the channel pair \((\Phi \otimes \Sigma_\sigma, \Omega)\) as long as \( \mathcal{C} \) is closed with respect to tensor products with \( \Sigma_\sigma \). If this is the case, the expression \( \text{Tr}(f(\Phi \otimes \Sigma_\sigma(|\psi\rangle \langle \psi| \otimes |\varphi\rangle \langle \varphi|)) \) attains its global maximum at an unentangled input state \( |\psi\rangle \).

But

\[
\text{Tr}(f(\Phi \otimes \Sigma_\sigma(|\psi\rangle \langle \psi|))) = \sum_i \tilde{f}(\alpha_i) = \sum_i \sum_j f(\mu_j \alpha_i) = \text{Tr}(f(\Phi \otimes \Sigma_\sigma(|\psi\rangle \otimes |\varphi\rangle \langle \varphi|))
\]

and so the expression \( \text{Tr}(f(\Phi \otimes \Omega(|\psi\rangle \langle \psi|)) \) attains its global maximum at unentangled input states \( |\psi\rangle \). It follows that \( f \) is additive for \((\Phi, \Omega)\). In particular, we get that \( f \) is additive for \( \mathcal{C} \) for every \( n \in \mathbb{N} \) if we insert \( \vec{\mu} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \).

As a simple example application, we find that functions which are additive for all channels must be continuous at zero:

Example 14 If a convex function \( f : [0, 1] \to \mathbb{R} \) is additive for all channels, then it is continuous at zero.

Proof. Let \( \mathcal{C} \) be the class of all channels. Suppose that \( f \) is additive for \( \mathcal{C} \), but not continuous at zero. Since \( f \) is convex, the limit \( y := \lim_{x \to 0} f(x) \) exists and is less than \( f(0) \). If \( f \) is additive for \( \mathcal{C} \), then the function

\[
g(x) := \frac{f(x) - y}{f(0) - y}
\]

is additive for \( \mathcal{C} \) as well for every probability vector \( (\mu_1, \ldots, \mu_n) \).
is additive for $C$ as well due to Example 8. Since $f$ is continuous on $(0, 1)$ and
\[
g \left( \frac{x}{n} \right) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{f \left( \frac{x}{n} \right) - y}{f(0) - y} & \text{if } x \in (0, 1), \end{cases}
\]
the sequence of functions $\{g \left( \frac{x}{n} \right)\}_{n \in \mathbb{N}}$ converges to the function $\delta_0$ introduced in Lemma 14. We know from Theorem 13 that the functions $g \left( \frac{x}{n} \right)$ are additive for $C$ for every $n \in \mathbb{N}$. Moreover, according to Lemma 8, the set of additive functions for $C$ is closed. Thus, $\delta_0$ must be additive, which contradicts Lemma 7. □

We will later see that this result is not valid for $x = 1$: in Lemma 17 we show that there exist additive functions that are discontinuous at $x = 1$.

Here is another interesting example which in some sense “interpolates” between the von Neumann and $p$-Rényi entropies:

**Example 15 (Distorted Entropy and $p$-Purity)**
Let $\frac{1}{2} \leq p \leq 1$ and $C$ a channel class. If the function
\[
x^p \log x
\]
is additive for $C$, then the function $-x^p$ is additive for $C$ as well; consequently, the minimum output $p$-Rényi entropy is additive for all channel pairs in $C$.

**Proof.** Notice that $x^p \log x$ is convex on $[0, 1]$ if and only if $\frac{1}{2} \leq p \leq 1$, which explains the choice of the interval for $p$. Suppose that $x^p \log x$ is additive for $C$. Then, $\left( \frac{1}{n} \right)^p \log \frac{1}{n}$ is additive for $C$ as well for every $n \in \mathbb{N}$ according to Theorem 13. As multiplication with a constant does not affect additivity, it follows that $x^p \log x - x^p \log n$ is additive for $C$ as well, and so is
\[
-x^p + \frac{x^p \log x}{\log n} \quad \text{for every } n \in \mathbb{N}.
\]
Taking the limit $n \to \infty$, the claim follows from Lemma 8. □

Here are some more consequences of Theorem 13. The proofs are very similar to the proof of Example 14 and thus omitted.

**Lemma 16 (von Neumann Entropy, Analyticity)**
Let $f : [0, 1] \to \mathbb{R}$ be a convex function.

- If $f(x) = ax \log x + O(x)$ with $a \neq 0$ and $f$ is additive for a class of channels $C$, then $x \log x$ is additive for $C$ as well, i.e. the minimum output von Neumann entropy is additive for $C$.
- If $f$ has a non-linear analytic extension to a complex neighbourhood of zero, then there exist channels $(\Phi, \Omega)$ such that $f$ is not additive for $(\Phi, \Omega)$.

This shows that von Neumann entropy plays some kind of special role: if any function that behaves like $x \log x$ for small $x$ is additive, then von Neumann entropy is automatically additive as well. The second part of the lemma concerns possible functions $f$ that are additive for all channels: this possibility is ruled out for many functions, for example, say, for $f(x) = \frac{1}{ax}$ for $a > -1$. The main idea to prove the second part is the fact that, after subtracting a linear function, analytic functions can be approximated by a monomial $a \cdot x^m$, but the functions $x^m$ violate additivity for some channels as shown, for example, in [3].

A convex function on $[0, 1]$ is automatically continuous on $(0, 1)$, but it may be discontinuous at the endpoints. We have shown in Example 14 that functions that are additive for all channels are continuous at $x = 0$. In contrast, the following simple arguments show that additive functions may be discontinuous at $x = 1$.

**Lemma 17** If $\Phi$ and $\Omega$ are quantum channels such that $\Phi \otimes \Omega$ outputs a pure state, then every convex function is additive for $(\Phi, \Omega)$.

Consequently, if $f, g : [0, 1] \to \mathbb{R}$ are convex functions that differ only at $x = 1$, then $f$ is additive for any pair of channels if and only if $g$ is additive for that pair of channels.

**Proof.** Let $f : [0, 1] \to \mathbb{R}$ be a convex function, and suppose there exists some input state $\rho_0$ and a pure state $|\varphi\rangle$ such that $\Phi \otimes \Omega(\rho_0) = |\varphi\rangle \langle \varphi|$. Denoting the minimum output entropy of a channel $\Phi$ by $S_{\min}(\Phi)$ as in the introduction, it is well-known [7] and in fact proven in the present paper in Lemma 5 that
\[
0 = S_{\min}(\Phi \otimes \Omega) \geq S_{\min}(\Phi),
\]
and so $\Phi$ (and by the same argument, $\Omega$) outputs a pure state, too. Taking the tensor product of the corresponding inputs, we get an unentangled (pure tensor product) input state $\rho_0$ for $\Phi \otimes \Omega$ such that $\Phi \otimes \Omega(\rho_0)$ is pure as well. Due to the Schur convexity [14] of the map $(\lambda_1, \ldots, \lambda_n) \mapsto \frac{1}{\sum_{i=1}^n f(\lambda_i)}$, the state $\tilde{\rho}_0$ is a maximizer of the map $\rho \mapsto \text{Tr} f(\Phi \otimes \Omega(\rho))$, and so $f$ is additive for $(\Phi, \Omega)$.

Let now $\Phi$ and $\Omega$ be arbitrary quantum channels. If $\Phi \otimes \Omega$ outputs a pure state, then both $f$ and $g$ are additive for $(\Phi, \Omega)$. On the other hand, if $\Phi \otimes \Omega$ does not output a pure state, then the eigenvalues of every output are strictly less than 1, and $\text{Tr} f(\sigma) = \text{Tr} g(\sigma)$ for every $\sigma$. □

Thus, modifying a convex function at $x = 1$ does not affect its additivity property. For example, the following function $\delta_1$ is additive for all channels:
\[
\delta_1(x) := \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases}
\]
IV. PIECEWISE LINEAR FUNCTIONS

The simplest functions that have not yet been studied before in the context of additivity are the piecewise linear functions. More in detail, while linear functions \( f(x) = ax + b \) are additive for all channels according to Lemma 3, the simplest examples of functions with unknown additivity properties are those functions \( f \) which are the maximum of two linear functions,

\[
f(x) := \begin{cases} 
  ax + b & \text{if } x \leq x_0 \\
  cx + d & \text{if } x > x_0
\end{cases}
\]

with \( ax_0 + b = cx_0 + d \), and \( a < c \) to ensure continuity and convexity. We call \( x_0 \) the kink of \( f \).

Fig. 1 shows what such functions look like. Are those functions additive? In this section, we give a partial answer to this question. We first note a simple consequence of Theorem 13: there we have shown that if \( f \) is additive, then \( f \left( \frac{a}{b} \right) \) must be additive as well. It is natural to conjecture that more generally, \( f(\lambda x) \) must then always be additive for every \( \lambda \in [0,1] \). While it is not clear if this holds true in general, we can prove it for the case that \( f \) is differentiable at zero:

**Lemma 18** Let \( f : [0,1] \rightarrow \mathbb{R} \) be a convex function which is differentiable at zero. If \( C \) is a class of channels which is closed with respect to tensor products with \( \Sigma_\sigma \) for all \( \sigma \) (cf. Definition 12), and if \( f \) is additive for \( C \), then the function

\[
x \in [0,1] \mapsto \sum_{i=1}^{n} f(\mu_i \cdot x)
\]

is additive for \( C \) as well for every sub-probability vector \( (\mu_1,\ldots,\mu_n) \), i.e. if \( \mu_i \geq 0 \) for every \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} \mu_i \leq 1 \).

In particular, \( f(\lambda x) \) is additive for \( C \) for all \( \lambda \in [0,1] \).

**Proof.** Without loss of generality, we may assume that \( f(0) = 0 \), otherwise we can add some constant to \( f \) without changing its additivity properties.

Let \( m := 1 - \sum_{i=1}^{n} \mu_i \), then \( \left( \frac{m}{N}, \ldots, \frac{m}{N}, \mu_1, \ldots, \mu_n \right) \) is a probability vector for every \( N \in \mathbb{N} \). According to Theorem 13 the function

\[
N \cdot f\left( \frac{m}{N} \right) + \sum_{i=1}^{n} f(\mu_i x)
\]

must then be additive for \( C \) for every \( N \in \mathbb{N} \). Since \( f \) is by assumption differentiable at zero, the limit \( \lim_{h \to 0} \frac{f(b)-f(a)}{h} \) exists and equals \( f'(0) \). Hence

\[
\lim_{N \to \infty} N \cdot f\left( \frac{m}{N} \right) = mx \lim_{N \to \infty} N \cdot \frac{m}{N} f\left( \frac{m}{N} \right) = mf'(0).
\]

Due to the closedness property of the additive functions as shown in Lemma 3, it follows that the function

\[
\sum_{i=1}^{n} f(\mu_i x) + mf'(0) x
\]

is additive for \( C \). But \( mf'(0)x \) is a linear function that we may subtract without affecting additivity due to Example 3.

We now use this lemma to prove our result on piecewise linear functions: if those functions are additive or not depends only on the location of the kink.

**Theorem 19 (Piecewise Linear Functions)**

There is a global constant \( \frac{1}{4} \leq \gamma \leq 1 \) such that the following holds true: if \( f \) is the maximum of two linear functions as plotted in Fig. 1 with kink at \( x_0 \), then

\[
f \text{ is additive for all channels} \iff x_0 \geq \gamma.
\]

Similarly, for every channel class \( C \) which is closed with respect to tensor products with \( \Sigma_\sigma \) for all \( \sigma \), there is a constant \( 0 \leq \gamma \leq 1 \) with the same property.

It is natural to conjecture that \( \gamma = 1 \) holds; in this case, no function of this type would be additive for all channels.

**Proof.** For simplicity, we assume that \( C \) is the class of all channels; the more general case is completely analogous. It is sufficient to consider the piecewise linear functions

\[
g_{x_0}(x) := \begin{cases} 
  0 & \text{if } x \leq x_0 \\
  x - x_0 & \text{if } x > x_0
\end{cases}
\]

since every function which is the maximum of two linear functions can be transformed into this form without affecting its additivity properties, if it has kink at \( x_0 \). Explicitly, if \( f \) is defined as in (10), then the function \( g \) defined by

\[
g(x) := \frac{f(x) - (ax + b)}{c - a}
\]

has this form, and shares the additivity property with \( f \) due to Example 3.

Thus, additivity of \( f \) (resp. \( g \)) depends only on the location of the kink. Now suppose \( g_i \) is additive for some
Let $t \in [0, 1]$. As $g_t$ is differentiable at zero, it follows from Lemma 15 that $g_t(\lambda x)$ is additive as well for every $\lambda \in (0, 1)$. It is elementary to see that

$$
\frac{1}{\lambda} g_t(\lambda x) = g_{t'}(x),
$$

and so $g_{t'}$ is additive, or equivalently $g_t$ for every $t' \geq t$. This shows that there is some constant $\gamma \in [0, 1]$ such that $g_{\gamma t}$ is additive if and only if the kink $x_0$ is larger than or equal to $\gamma$.

Finally, if $\frac{1}{2} < x_0 < \frac{1}{2}$, then $g_{x_0}$ is not additive according to Example 6. This shows that $\gamma \geq \frac{1}{2}$.

Here is a recipe how to improve the lower bound on $\gamma$ (or in the best case to prove that $\gamma = 1$): find an example of a pair of channels such that the maximum output eigenvalue $\Lambda$ is attained at an entangled input state. Then $\gamma \geq \Lambda$. In fact, the proof above (or rather its reference to Example 6) exploits this fact for a pair of Werner-Holevo channels in dimension $3 \times 3$.

V. CONCLUSIONS

In this paper, we have studied the problem whether a given convex trace function, if it is applied to the output of a bipartite quantum channel, attains its maximum at an unentangled input state. This problem generalizes the minimum output entropy additivity problem in a natural way: for example, there is a single channel bound on the output capacity (Lemma 3), additivity always holds if one of the channels is the identity channel (Example 1), and the study of the minimum output rank (Lemma 7) and the largest output eigenvalue (Theorem 19) have natural interpretations in our more general framework.

In Theorem 11 we have shown that all operator convex functions on $[0, \infty)$ are additive for the Werner-Holevo channel in $3 \times 3$ dimensions, which contains the well-known additivity results for this channel as special cases. Since the set of functions that are additive for all unitarily covariant channels is convex (Lemma 5), it is natural to conjecture that this set of functions can be classified further, possibly in a way as stated in Conjecture 11.

We have also shown some additional structural properties of the set of additive functions (e.g. Lemma 5), or Theorem 13, drawing new connections between functions like $x^p \log x$ and the $p$-Rényi entropies, and also yielding partial reasons why von Neumann entropy seems to play a special role for additivity (cf. Lemma 10).

Even though the original additivity conjecture has recently been disproved [1], it is still interesting to study additivity for special classes of channels. Moreover, the transition from additivity to non-additivity (say, the dimensionality of the channels) is still not well understood, and the history of the additivity problem shows that introducing new entropy notions (like $p$-Rényi entropy) can be useful. This is why we are confident that our framework of additive convex functions might be helpful in some instances of this problem.

Acknowledgments. The author would like to thank N. Ay, J. Eisert, D. Gross, T. Krüger, R. Seiler, A. Szkoła, R. Werner, and C. Witte for helpful discussions – and especially Ra. Siegmund-Schultze for his never-ending enthusiasm for the additivity conjecture.

Special thanks go to A. Winter for his kind hospitality during a visit to Bristol and for many discussions.

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