Characterizing closed curves on Riemann surfaces
via homology groups of coverings

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Abstract

Let $S$ be a hyperbolic oriented Riemann surface of finite type. The main purpose of this paper is to show that non-trivial geometric intersection between closed curves on $S$ is detected by some symplectic submodules they naturally determine in the homology groups of the compactifications of unramified $p$-coverings of $S$, for $p \geq 2$ a fixed prime. In particular, this gives a characterization of simple closed curves on $S$ in terms of homology groups of $p$-coverings.

In Section 4, we define a $p$-adic Reidemeister pairing on the fundamental group of $S$ and show that the free homotopy classes of two loops have trivial geometric intersection if and only if they are orthogonal with respect to this pairing.

As an application, we give a geometric argument to prove that oriented surface groups are conjugacy $p$-separable (a combinatorial proof of this fact was recently given by Paris [15]).

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1 Introduction

In the papers [12] and [21], Stallings and Jaco established the equivalence between the Poincaré conjecture, now a celebrated theorem by Perelman, and the following group-theoretical statement:

$\star$ Let $S_g$ be a closed oriented surface of genus $g \geq 2$. Let $F_g$ be a free group of rank $g$. Let $\eta: \pi_1(S_g, s_0) \to F_g \times F_g$ be an epimorphism. Then, there is a non-trivial element in the kernel of $\eta$ which may be represented by a simple closed curve in $S_g$.

Of course, it is still an interesting problem to provide a group-theoretic proof of the above statement. The first step in this direction is to give an algebraic characterization of simple closed curves on the closed Riemann surface $S_g$.

A program in this sense was formulated by Turaev [22]. Progress in this direction have been recently accomplished by Chas, Gadgil and Krongold (see [4], [5], [6]), who characterize simple closed curves on a Riemann surface in terms of the Goldman Lie algebra.
In Section 2, we give an elementary criterion to characterize simple closed curves on any hyperbolic Riemann surface in terms of the intersection pairing on the closures of normal unramified p-coverings of the given Riemann surface, for a fixed prime \( p \). The proof is based on the hyperbolic geometry of the \( p \)-adic solenoid, developed by means of some elementary pro-\( p \) group theory.

In Section 4, we refine further this characterization by means of a \( p \)-adic Reidemeister pairing. This is a quite natural pro-\( p \) version of the classical one (cf. Section 3 [10]). We show that, an element \( \gamma \in \pi_1(S_g, s_0) \) contains a simple closed curve in its free homotopy class if and only if it is singular for the \( p \)-adic Reidemeister pairing. In particular, whether the free homotopy class of \( \gamma \) contains or not an embedded representative can be determined purely in terms of the group structure of the fundamental group \( \pi_1(S_g, s_0) \).

In Section 5, we apply the above characterization of simple closed curves to show that it is possible to distinguish homotopy classes of closed curves on a hyperbolic Riemann surface in terms of the first homology groups of its normal unramified \( p \)-coverings. An easy consequence is conjugacy \( p \)-separability of surface groups.

2 Characterization of simple closed curves

Let \( S_g \) be a compact oriented Riemann surface without boundary of genus \( g \) and let \( S_{g,n} := S_g - \{ P_1, \ldots, P_n \} \) be the same surface from which \( n \) distinct points have been removed. We assume that \( \chi(S_{g,n}) = 2 - 2g - n < 0 \). Even if most of the stated results hold also in the non-orientable case, for simplicity, we only consider the orientable case.

A closed curve on \( S_{g,n} \) is a continuous map from the circle \( S^1 \) to the surface \( S_{g,n} \). For a fixed base point \( a \), let us denote by \( \Pi_{g,n} \) the fundamental group of \( S_{g,n} \). Then, it is well known that the set of homotopy classes of closed curves on \( S_{g,n} \) identifies with the quotient of the sets \( \{ \{ \alpha, \alpha^{-1} \} | \alpha \in \Pi_{g,n} \} \) and \( \{ \langle \alpha \rangle | \alpha \in \Pi_{g,n} \} \) by the action induced by inner automorphisms of \( \Pi_{g,n} \).

Given a covering map \( p: S' \to S_{g,n} \) and a closed curve \( \gamma: S^1 \to S_{g,n} \), the connected components of the fiber product \( S^1 \times_{S_{g,n}} S' \) are homeomorphic to \( S^1 \). Let us then call the restriction \( \tilde{\gamma}: S^1 \to S' \) to a connected component of the pull-back map \( \gamma': S^1 \times_{S_{g,n}} S' \to S' \) an elevation of \( \gamma \) to \( S' \) (this terminology is borrowed from [8]).

**Definition 2.1.** For \( K \) a finite index subgroup of \( \Pi_{g,n} \), let \( p_K: S_K \to S_{g,n} \) be the associated covering, \( \overline{S_K} \) the compact Riemann surface obtained filling in the punctures of \( S_K \) and \( H_1(\overline{S_K}) \) the first homology group of \( \overline{S_K} \) with \( \mathbb{Z} \)-coefficients. Let us then define \( V_\gamma^K \) to be the submodule of \( H_1(\overline{S_K}) \) generated by the cycles supported on the elevations of \( \gamma \) to \( S_K \).

For a normal finite index subgroup \( K \) of \( \Pi_{g,n} \), let \( G_K \) be the deck transformation group of \( p_K: S_K \to S_{g,n} \). Given a closed curve \( \gamma \) in \( S_{g,n} \), the group \( G_K \) then acts naturally and transitively on the set of elevations of \( \gamma \) to \( S_K \) and \( V_\gamma^K \) is a \( G_K \)-invariant submodule of \( H_1(\overline{S_K}) \).

For a closed curve \( \gamma \) on \( S_{g,n} \), let \( \tilde{\gamma} \) be an element of \( \Pi_{g,n} \) whose free homotopy class contains the closed curve \( \gamma \) and let \( k > 0 \) be the smallest integer such that \( \tilde{\gamma}^k \in K \).
Then, the submodule \( V^K_\gamma \) can also be characterized as the image of the normal subgroup \( \langle \tilde{\gamma}^k \rangle^\Pi_{g,n} \cap K \) in the homology group \( H_1(\Sigma_k) \), where we denote by \( \langle \tilde{\gamma}^k \rangle^\Pi_{g,n} \) the smallest normal subgroup of \( \Pi_{g,n} \) which contains \( \tilde{\gamma}^k \). Let us also define \( V^K_{\gamma'} := V^K_\gamma \).

For an integer \( s > 0 \), an \( s \)-power of a closed curve \( \gamma : S^1 \to S_{g,n} \) is a closed curve \( \gamma^s : S^1 \to S_{g,n} \) which factors through a continuous map \( S^1 \to S^1 \) of degree \( s \) and \( \gamma \). A closed curve \( \gamma \) on the Riemann surface \( S_{g,n} \) is called non-power or primitive if it is not homotopic to an \( s \)-power of a closed curve, for \( s > 1 \). Let us observe that an elevation of a non-power closed curve is also non-power.

A simple closed curve (briefly s.c.c.) on \( S_{g,n} \) is an embedded circle \( S^1 \hookrightarrow S_{g,n} \). An interesting problem is that of establishing when a non-power closed curve \( \gamma \) on \( S_{g,n} \) has in its homotopy class an embedded representative. In this section, we will give a characterization of this property in terms of the homology of finite unramified coverings of \( S_{g,n} \).

More generally, we will be able to determine in this way when the geometric intersection number \( |\gamma \cap \gamma'|_G \) between two closed curves \( \gamma \) and \( \gamma' \) is zero or one.

In order to formulate the main result, it is convenient to introduce some natural topologies on the fundamental group \( \Pi_{g,n} \).

A profinite topology on a group \( G \) is a topology (making it a topological group) for which a neighborhood basis of the identity consists of finite index subgroups. The usual way to define such a topology on the group \( G \) is to fix a class of finite groups \( C \) in the sense of the definition below:

**Definition 2.2.** A class of finite groups (cf. Definition 3.1 in [1]) is a full subcategory \( C \) of the category of finite groups which is closed under taking subgroups, homomorphic images and extensions (meaning that a short exact sequence of finite groups is in \( C \) whenever its exterior terms are). We always assume that \( C \) contains a nontrivial group.

Since \( C \) contains a nontrivial group, it will contain \( \mathbb{Z}/p \) for some prime \( p \geq 2 \) and then it easily follows that \( C \) contains all the finite groups whose order is a power of \( p \).

The finite groups whose order is a power of \( p \) are nilpotent and form a class of finite groups by themselves (denoted sometimes \( (p) \)). So these provide the minimal examples.

Given a group \( G \), then we say that a subgroup \( H \leq G \) is \( C \)-open if it contains a normal subgroup \( N \) of \( G \) such that the quotient group \( G/N \) belongs to \( C \). In this case, we write \( H \leq_C G \) and \( N \triangleleft_C G \), respectively. A fundamental system of neighborhoods of the identity for the pro-\( C \) topology on the group \( G \) is given by any cofinal system of \( C \)-open normal subgroups. We say that a subgroup \( H \) of \( G \) is \( C \)-closed if \( H \) is a closed subset in the pro-\( C \) topology of \( G \) and that a map of groups \( G \to G' \) is \( C \)-continuous if it is continuous in the pro-\( C \) topologies of \( G \) and \( G' \). Note that a monomorphism is always \( C \)-continuous.

The pro-\( C \) completion of a group \( G \) is defined to be the inverse limit:

\[
\hat{G}^C := \lim_{\leftarrow} G/N.
\]

Let us endow the finite groups \( G/N \) with the discrete topology and the group \( \hat{G}^C \) with the subspace topology induced by the natural monomorphism \( \hat{G}^C \hookrightarrow \prod_{N \triangleleft_C G} G/N \). Then,
the profinite group $\hat{G}^e$ is a compact, Hausdorff, totally disconnected, topological group and a fundamental system of neighborhoods of the identity is provided by the kernels of the natural epimorphisms $\hat{G}^e \to G/N$, for $N \triangleleft G$. There is a natural homomorphism of groups $G \to \hat{G}^e$ with dense image and the pro-$\mathcal{C}$ topology on $G$ is, by definition, the weakest topology for which this map is continuous.

With the above notations and definitions, let us now state the main result of the paper:

**Theorem 2.3.** Let $\mathcal{C}$ be a class of finite groups. Then, a pair of (not necessarily distinct) closed curves $\gamma$ and $\gamma'$ on a Riemann surface $S_{g,n}$ have trivial geometric intersection if and only if, for a cofinal system of $\mathcal{C}$-open subgroups $\{K\}$ of $\Pi_{g,n}$, there holds $(x,y)_K = 0$, for all $x \in V^K_\gamma$ and all $y \in V^K_{\gamma'}$, where $(\cdot, \cdot)_K$ is the intersection pairing on the first integral homology group of the closed Riemann surface $\overline{S}_K$.

Taking $\gamma' = \gamma$ in Theorem 2.3, we get the aforementioned characterization of s.c.c.'s:

**Corollary 2.4.** The homotopy class of a non-power closed curve $\gamma$ on $S_{g,n}$ contains a simple closed curve if and only if, for a cofinal system of $\mathcal{C}$-open subgroups $\{K\}$ of $\Pi_{g,n}$, the associated submodules $V^K_\gamma$ of $H_1(\overline{S}_K)$ are totally isotropic.

Thanks to Corollary 4.4 in [10], we can also give an algebraic criterion to decide whether two non-homotopic s.c.c.'s $\alpha$ and $\beta$ on $S_{g,n}$ have geometric intersection one.

**Corollary 2.5.** Let $\alpha$ and $\beta$ be non-homotopic s.c.c.'s on $S_{g,n}$. Then $\alpha$ and $\beta$ are homotopic to s.c.c.'s which meet transversally in a single point if and only, for some $\tilde{\alpha}, \tilde{\beta} \in \Pi_{g,n}$, whose free homotopy classes contain, respectively, $\alpha$ and $\beta$, and for a cofinal system of $\mathcal{C}$-open subgroups $\{K\}$ of $\Pi_{g,n}$, the associated submodules $V^K_{[\tilde{\alpha}, \tilde{\beta}]}$ of $H_1(\overline{S}_K)$ are totally isotropic.

The case in which Theorem 2.3 and its corollaries are more interesting is undoubtedly when $\mathcal{C}$ is the class of finite $p$-groups. In particular, the applications given in Section 5 make use only of this case.

### 3 The pro-$\mathcal{C}$ hyperbolic solenoid

Let us fix a complete, non-singular metric on $S_{g,n}$ of constant curvature $-1$. A closed curve $\gamma$ on $S_{g,n}$ is peripheral if it is homotopic to a power of a s.c.c. bounding a 1-punctured disc on $S_{g,n}$. Peripheral closed curves are characterized by the property that they have trivial geometric intersection with any closed curve on the surface. With the chosen metric, all non-peripheral closed curves on $S_{g,n}$ have a unique geodesic representative in their homotopy class.

Let $\mathbb{D} \to S_{g,n}$ be the universal covering space. The choice of a point $\tilde{a} \in \mathbb{D}$, lying above the base point $a \in S_{g,n}$, identifies the fundamental group $\Pi_{g,n}$ with the covering transformation group of $\mathbb{D} \to S_{g,n}$. The space $\mathbb{D}$, with the induced metric, can then be identified with the Poincaré disc and the fundamental group $\Pi_{g,n}$ with a discrete subgroup...
of $\text{Aut}(\mathbb{D}) \cong \text{PSL}_2(\mathbb{R})$. Note that all elements of $\Pi_{g,n}$ are identified with hyperbolic elements of $\text{PSL}_2(\mathbb{R})$, except those containing a peripheral curve in their homotopy class, which are instead identified with parabolic elements.

For $\mathcal{C}$ a given class of finite groups, let $\hat{\Pi}^\mathcal{C}_{g,n}$ be the pro-$\mathcal{C}$ completion of the fundamental group $\Pi_{g,n}$. Since $\mathcal{C}$ contains the class of finite $p$-groups, for some prime $p > 1$, by Theorem A.1, there is a natural monomorphism $\Pi_{g,n} \hookrightarrow \hat{\Pi}^\mathcal{C}_{g,n}$. By means of this monomorphism, let us identify $\Pi_{g,n}$ with its image in the pro-$\mathcal{C}$ group $\hat{\Pi}^\mathcal{C}_{g,n}$.

The pro-$\mathcal{C}$ hyperbolic solenoid $S^\mathcal{C}_{g,n}$ is defined to be the inverse limit space:

$$S^\mathcal{C}_{g,n} := \lim_{\leftarrow} S_{K} \cong \lim_{\leftarrow} [\mathbb{D} \times (\Pi_{g,n}/K)]/\Pi_{g,n} = \mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}/\Pi_{g,n},$$

where an element $x \in \Pi_{g,n}$ acts on the product $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}$ by the formula $x \cdot (d, \beta) = (x \cdot d, x \cdot \beta)$. This is a generalization of the hyperbolic $n$-punctured, genus $g$ solenoid. For $\mathcal{C} = (p)$, the pro-$p$ solenoid $S^\mathcal{C}_{g,n}$ is called the $p$-adic solenoid. For $\mathcal{C}$ the class of all finite groups, the pro-$\mathcal{C}$ solenoid is denoted simply by $S_{g,n}$ and called the solenoid.

From the above realization of the pro-$\mathcal{C}$ solenoid and the fact that the natural homomorphism $\Pi_{g,n} \to \hat{\Pi}^\mathcal{C}_{g,n}$ is injective, it follows that the inverse limit of the natural covering maps $\mathbb{D} \to \mathbb{D}/K$ is a $\Pi_{g,n}$-equivariant embedding $\mathbb{D} \to S^\mathcal{C}_{g,n}$ with dense image. Moreover, under the isomorphism described above, the image of the Poincaré disc in the solenoid identifies with the image of $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}$ in the quotient $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}/\Pi_{g,n}$. Let us identify $\mathbb{D}$ with its image in the $p$-adic solenoid. In this way, we get the preferred analytic leaf $\mathbb{D}$ of the solenoid. The other leaves are obtained translating this one by the action of $\hat{\Pi}^\mathcal{C}_{g,n}$.

From the above construction, it is clear that the preferred leaf is determined by the canonical monomorphism $\Pi_{g,n} \hookrightarrow \hat{\Pi}^\mathcal{C}_{g,n}$ (a discretification of $\hat{\Pi}^\mathcal{C}_{g,n}$). Twisting this canonical monomorphism by the inner automorphism $\text{inn}_\gamma : x \mapsto \gamma x \gamma^{-1}$, for $\gamma \in \hat{\Pi}^\mathcal{C}_{g,n}$, we get another realization of the pro-$\mathcal{C}$ solenoid whose preferred leaf is instead the translated leaf $\gamma \cdot \mathbb{D}$:

$$S^\mathcal{C}_{g,n} \cong \gamma \cdot \mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}/\Pi_{g,n},$$

where now an element $x \in \Pi_{g,n}$ acts on the product $\gamma \cdot \mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}$ by the formula $x \cdot (d, \beta) = (\text{inn}_\gamma(x) \cdot d, \text{inn}_\gamma(x) \cdot \beta)$.

**Proposition 3.1.** The pro-$\mathcal{C}$ solenoid $S^\mathcal{C}_{g,n}$ is a connected Hausdorff topological space endowed with a natural continuous action of the pro-$\mathcal{C}$ group $\hat{\Pi}^\mathcal{C}_{g,n}$, with quotient the Riemann surface $S_{g,n}$.

**Proof.** The pro-$\mathcal{C}$ solenoid contains the disc $\mathbb{D}$ as a dense connected subspace and a topological space containing a connected dense subset is connected. \hfill $\Box$

If $\Lambda^\mathcal{C}$ is the set of primes which occur as orders of groups in $\mathcal{C}$, then there holds $\hat{\mathbb{Z}}^\mathcal{C} \cong \prod_{p \in \Lambda^\mathcal{C}} \mathbb{Z}_p$. For every prime $p \in \Lambda^\mathcal{C}$, there is a natural epimorphism $\hat{\Pi}^\mathcal{C}_{g,n} \to \hat{\Pi}^{(p)}_{g,n}$. It follows that the pro-cyclic subgroup topologically generated by an element $\gamma \in \hat{\Pi}^\mathcal{C}_{g,n}$ inside the pro-$\mathcal{C}$ completion $\hat{\Pi}^\mathcal{C}_{g,n}$ is isomorphic to $\hat{\mathbb{Z}}^\mathcal{C}$. Let us denote this group simply by $\gamma \hat{\mathbb{Z}}^\mathcal{C}$. 


Let us describe the closure inside the pro-$\mathcal{C}$ solenoid $\mathbb{S}^\mathcal{C}_{g,n}$ of a hyperbolic line $\ell$ of $\mathbb{D}$ (also called a geodesic), obtained as the inverse image of a closed geodesic curve in $S_{g,n}$:

**Proposition 3.2.** Let $\ell$ be a geodesic on the Poincaré disc $\mathbb{D}$ whose image in $S_{g,n}$ is a closed curve. Let us then identify $\ell$ with its canonical image in the pro-$\mathcal{C}$ solenoid $\mathbb{S}^\mathcal{C}_{g,n}$. The closure $\overline{\ell}$ of this geodesic inside $\mathbb{S}^\mathcal{C}_{g,n}$ is naturally isomorphic to the one-dimensional pro-$\mathcal{C}$ solenoid $\mathbb{R} \times \hat{\mathbb{Z}}^\mathcal{C} / \mathbb{Z}$, where $z \in \mathbb{Z}$ acts on $\mathbb{R} \times \hat{\mathbb{Z}}^\mathcal{C}$ by the formula $z \cdot (r, s) = (z + r, z + s)$. In particular, the closure $\overline{\ell}$ has no self-intersection points.

**Proof.** There is a non-power hyperbolic element $\gamma \in \Pi_{g,n}$ whose action on $\mathbb{D}$ has for axis the given geodesic $\ell$. Let then $\gamma^\mathbb{Z} \cong \mathbb{Z}$ be the cyclic subgroup generated by $\gamma$ in $\Pi_{g,n}$.

The geodesic $\ell$ identifies with the subspace $\ell \times \gamma^\mathbb{Z} / \gamma^\mathbb{Z}$ of the solenoid $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n} / \Pi_{g,n}$. Let us then show that its closure there identifies with the image of the closed subspace $\ell \times \gamma^\mathbb{Z}$ of $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}$ in the pro-$\mathcal{C}$ solenoid and that this image is isomorphic to the quotient of the closed subspace $\ell \times \gamma^\mathbb{Z}$ by its stabilizer $\gamma^\mathbb{Z} \cap \Pi_{g,n}$, for the action of the group $\Pi_{g,n}$ on the space $\mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n}$ described above. There is at least a natural continuous map:

$$\varphi_\gamma: \ell \times \gamma^\mathbb{Z} \to \mathbb{D} \times \hat{\Pi}^\mathcal{C}_{g,n} / \Pi_{g,n} \equiv \mathbb{S}^\mathcal{C}_{g,n}.$$

Let us observe that, since the element $\gamma \in \Pi_{g,n}$ is non-power, the cyclic subgroup $\gamma^\mathbb{Z}$ is self-centralizing in $\Pi_{g,n}$. Therefore, there holds the identity $\gamma^\mathbb{Z} \cap \Pi_{g,n} = \gamma^\mathbb{Z}$, otherwise stated, the subgroup $\gamma^\mathbb{Z}$ is closed in the pro-$\mathcal{C}$ topology of $\Pi_{g,n}$. It follows that the map $\varphi_\gamma$ is injective since two points $(d, \beta)$ and $(d', \beta')$ of $\ell \times \gamma^\mathbb{Z}$ are mapped to the same point of the $p$-adic solenoid if and only if there is an $x \in \Pi_{g,n}$ such that $(x \cdot d, x\beta) = (d', \beta')$ and this happens only if $x \in \gamma^\mathbb{Z} \cap \Pi_{g,n} = \gamma^\mathbb{Z}$.

Now, the quotient space $\ell \times \gamma^\mathbb{Z} / \gamma^\mathbb{Z}$ is homeomorphic to the quotient space $\mathbb{R} \times \hat{\mathbb{Z}}^\mathcal{C} / \mathbb{Z}$, described in the statement of the proposition, and this is a compact Hausdorff topological space. Therefore, the map $\varphi_\gamma$ is a homeomorphism onto its image $\text{Im} \varphi_\gamma$, which then is a closed subset of the pro-$\mathcal{C}$ solenoid, because this is a Hausdorff space. Since the image of $\varphi_\gamma$ contains the geodesic $\ell$ as a dense subset, it follows that $\text{Im} \varphi_\gamma$ is actually the closure of $\ell$ inside the pro-$\mathcal{C}$ solenoid $\mathbb{S}^\mathcal{C}_{g,n}$.

We can now improve the cyclic case of a classical result by Scott [17]:

**Theorem 3.3.** Let $\gamma$ be a non-power, non-peripheral closed curve on the Riemann surface $S_{g,n}$. Then, for any fixed prime $p$, there is a normal, unramified $p$-covering $p_1: S_L \to S_{g,n}$ such that every elevation of $\gamma$ to $S_L$ is homotopic to a non-separating simple closed curve.

**Proof.** Let us assume that the closed curve $\gamma$ is immersed in $S_{g,n}$ and is the geodesic representative in its homotopy class. Let $\ell$ be a lift of $\gamma$ to $\mathbb{D}$ which is then a hyperbolic line. By Proposition 3.2, the closure $\overline{\ell}$ of this geodesic inside the $p$-adic solenoid $\mathbb{S}^{(p)}_{g,n}$ is naturally isomorphic to $\mathbb{R} \times \mathbb{Z}_p / \mathbb{Z}$ and has no self-intersection points.
For a given finite index normal subgroup $K$ of $\Pi_{g,n}$, let us denote by $\gamma_K$ the image of $\ell$ in $S_K$. The inverse limit $\lim_{\leftarrow \, K \trianglelefteq_p \Pi_{g,n}} \gamma_K$ is then naturally identified with the closure $\overline{\ell}$ of the geodesic $\ell$ in the $p$-adic solenoid $S_{g,n}^p$ and hence it has no self-intersection points.

For a geodesic $\delta$ on a Riemann surface or in the $p$-adic solenoid $S_{g,n}^p$, let $\delta \cap_s \delta$ be the set of its transversal self-intersection points. There holds then

$$\lim_{\leftarrow \, K \trianglelefteq_p \Pi_{g,n}} (\gamma_K \cap_s \gamma_K) = \lim_{\leftarrow \, K \trianglelefteq_p \Pi_{g,n}} \gamma_K \cap_s \lim_{\leftarrow \, K \trianglelefteq_p \Pi_{g,n}} \gamma_K = \overline{\ell} \cap_s \overline{\ell} = \emptyset,$$

where we use the fact that inverse limits commute with inverse limits and, in particular, with intersections. Since, for all finite index normal subgroups $K$ of $\Pi_{g,n}$, the set $\gamma_K \cap_s \gamma_K$ is finite and the inverse limit of a system of non-empty finite sets is non-empty, we conclude that there is a $p$-open normal subgroup $H$ of $\Pi_{g,n}$ such that $\gamma_H \cap_s \gamma_H = \emptyset$.

By Lemma 3.10 [3], there is a characteristic unramified $p$-covering $p_{LH}: S_L \to S_H$ such that the connected components of $p_{LH}^{-1}(\alpha)$ are non-separating simple closed curves for any given non-peripheral s.c.c. $\alpha$ on $S_H$. Thus, the induced normal unramified $p$-covering $p_L: S_L \to S_{g,n}$ has the desired properties.

**Remarks 3.4.**

(i) With the notations of Theorem 3.3, let $\tilde{\gamma} \in \Pi_{g,n}$ be an element whose free homotopy class contains $\gamma$. In case $\Pi_{g,n}$ is a free group, i.e. for $n \geq 1$, Theorem 3.3 follows from the fact that, by Theorem 5.7 [16], there is a $p$-open subgroup $H$ of $\Pi_{g,n}$ which contains the cyclic group $\tilde{\gamma}\mathbb{Z}$ as a free factor. In the closed surface case instead, the result is new and can be expressed, group-theoretically, by saying that there is a $p$-open subgroup $H$ of $\Pi_g$, containing $\tilde{\gamma}$, such that $\tilde{\gamma}$ appears as the stable letter in a HNN-extension presentation of $H$ (just take $H = \langle L, \tilde{\gamma} \rangle$).

(ii) The existence of a finite unramified covering with the properties stated in Theorem 3.3 follows as well from the fact that finitely generated subgroups of a surface group $\Pi_{g,n}$ are geometric, i.e. they can be realized as fundamental groups of subsurfaces of finite coverings of $S_{g,n}$ (see [17] and [18]). However, in the result of Scott, no condition was imposed on the covering transformation group.

In Proposition 3.2, we described the closure in the pro-$C$ solenoid of a geodesic $\ell \subset \mathbb{D}$ which projects to a closed curve in $S_{g,n}$. A much subtler question is to describe the intersection of the closures of two such geodesics. This is done in the following theorem whose proof will require a deeper result in the theory of surface groups (Theorem A.2).

**Theorem 3.5.** Let $\ell$ and $\ell'$ be distinct geodesics of $\mathbb{D}$ which project to closed curves of $S_{g,n}$ and let, respectively, $\overline{\ell}$ and $\overline{\ell'}$ be their closures in the pro-$C$ solenoid $\mathbb{S}_{g,n}^C$. Then, there holds $\overline{\ell} \cap \overline{\ell'} = \ell \cap \ell'$, i.e. their intersection is empty when $\ell$ and $\ell'$ are disjoint geodesics of $\mathbb{D}$ and otherwise they meet in the single point $\ell \cap \ell'$.

**Proof.** For $p \in \Lambda_C$, there is a normal unramified covering $\pi_p: \mathbb{S}_{g,n}^C \to S_{g,n}^p$ with covering transformation group the kernel of the natural epimorphism $\hat{\Pi}_{g,n} \to \hat{\Pi}_{g,n}^p$ and the images...
\( \pi_p(\ell) \) and \( \pi_p(\ell') \) are the closures of the geodesics \( \ell \) and \( \ell' \) in the \( p \)-adic solenoid. Therefore, it is enough to prove the theorem for \( \mathcal{C} \) the class of finite \( p \)-groups, for \( p > 1 \) a prime.

We can assume that the geodesics \( \ell \) and \( \ell' \) in \( \mathbb{D} \) are the axes, respectively, of two non-power hyperbolic elements \( \gamma \) and \( \gamma' \) of \( \Pi_{g,n} \), whose free homotopy classes contain the images of \( \ell \) and \( \ell' \) in the Riemann surface \( S_{g,n} \).

By Proposition 3.2, we then know that the curves \( \ell \) and \( \ell' \) in \( S_{g,n}^{(p)} \) identify with the images of \( \ell \times \tilde{\gamma}^p \) and \( \ell' \times \tilde{\gamma}'^p \) in the quotient \( \mathbb{D} \times \hat{\Pi}_{g,n}^{(p)}/\Pi_{g,n} = S_{g,n}^{(p)} \).

For \( s, t \in \mathbb{Z}_p \), the images of two leaves \( \ell \times \tilde{\gamma}^s \) and \( \ell' \times \tilde{\gamma}'^t \) in the quotient \( \mathbb{D} \times \hat{\Pi}_{g,n}^{(p)}/\Pi_{g,n} \) can possibly intersect only if there exists an \( f \in \Pi_{g,n} \) such that \( f \cdot \tilde{\gamma}'^t = \tilde{\gamma}^s \), i.e. only if:

\[
\tilde{f} = \tilde{\gamma}^s \tilde{\gamma}'^{-t} \in \Pi_{g,n} \cap \tilde{\gamma}^p \cdot \tilde{\gamma}'^p.
\]

If \( \tilde{\gamma}' = \tilde{\gamma}^\pm 1 \), then \( \ell = \ell' \), against the hypothesis, hence \( \tilde{\gamma}' \neq \tilde{\gamma}^\pm 1 \).

For a subgroup \( H \) of a group \( G \), let us denote by \( Z_G(H) \) the centralizer of \( H \) in \( G \). Since \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) are non-power elements of \( \Pi_{g,n} \), for all \( m \in \mathbb{Z} \setminus \{0\} \), there hold \( Z_{\Pi_{g,n}}(\tilde{\gamma}^m) = \tilde{\gamma}^Z \) and \( Z_{\Pi_{g,n}}(\tilde{\gamma}'^m) = \tilde{\gamma}'^Z \). Therefore, we have the identities:

\[
\tilde{\gamma}^Z \cap \tilde{\gamma}'^Z = \{1\}, \quad \tilde{\gamma}^Z \cap \Pi_{g,n} = \tilde{\gamma}^Z \quad \text{and} \quad \tilde{\gamma}'^Z \cap \Pi_{g,n} = \tilde{\gamma}'^Z. \quad (\star)
\]

Thus, we can apply Theorem [A.2] and conclude that \( f = \tilde{\gamma}^u \tilde{\gamma}'^{-v} \), for some \( u, v \in \mathbb{Z} \). But now, the identity \( \tilde{\gamma}^Z \cap \tilde{\gamma}'^Z = \{1\} \) implies that \( f \) admits a unique expression \( f = \tilde{\gamma}^s \tilde{\gamma}'^{-t} \), with \( s, t \in \mathbb{Z}_p \). It follows that \( s = u, t = v \in \mathbb{Z} \). Thus, the leaves \( \ell \times \tilde{\gamma}^s \) and \( \ell' \times \tilde{\gamma}'^t \) are in the \( \Pi_{g,n} \)-orbit of the the leaves \( \ell \times 1 \) and \( \ell' \times 1 \), respectively, and their image in \( S_{g,n}^{(p)} \) is contained in the preferred leaf \( \mathbb{D} \times 1 \), where their intersection is identified with \( \ell \cap \ell' \). \( \square \)

**Proof of Theorem 2.3.** Before we start with the proof, let us make the following obvious remark. For a given closed curve \( \gamma \) on \( S_{g,n} \), let \( \tilde{\gamma} \in \Pi_{g,n} \) be an element of the fundamental group whose free homotopy class contains \( \gamma \). Let \( K \) be a characteristic finite index subgroup of \( \Pi_{g,n} \) and let \( k > 0 \), be the smallest integer such that \( \tilde{\gamma}^k \in K \). Then, for a power \( \gamma^s \) of \( \gamma \) such that \( s \) divides \( k \), the pull-backs of the closed curves \( \gamma \) and \( \gamma^s \) to \( S_K \) have the same set of components. More generally, if \( \gamma^s \) is a power of \( \gamma \) and \( m \) is the g.c.d. of \( s \) and \( k \), then each elevation of \( \gamma^s \) to \( S_K \) is an \( s/m \)-power of one of \( \gamma \).

Two closed curves on a Riemann surface have non-trivial geometric or algebraic intersection if and only if any given powers of them have the same property. By the above remarks, it is then enough to prove Theorem 2.3 when both \( \gamma \) and \( \gamma' \) are non-power closed curves. If one of the two curves is peripheral, then either \( \gamma \) is homotopic to \( \gamma' \) or the two curves have disjoint representatives in their homotopy classes. In both cases, the conclusion of the theorem holds trivially. Let us then assume that neither of the two curves is peripheral and that both closed curves \( \gamma \) and \( \gamma' \) are geodesics for the given metric on \( S_{g,n} \). In this way, the geometric intersection number \( |(\gamma \cap \gamma')|_G = 0 \) if and only if either \( \gamma \) and \( \gamma' \) are disjoint or \( \gamma = \gamma' \) is a s.c.c.. In all these cases, the conclusion of the theorem holds for any covering \( \hat{S} \to S_{g,n} \) and there is nothing to prove.

Let us consider the case when \( \gamma = \gamma' \) and \( |(\gamma \cap \gamma)|_G \neq 0 \). By replacing \( S_{g,n} \) with a suitable \( \mathcal{C} \)-covering \( \hat{S} \to S_{g,n} \) and \( \gamma = \gamma' \) with two distinct elevations \( \tilde{\gamma} \) and \( \tilde{\gamma}' \), such that
[γ ∩ γ′]_G \neq 0$, we are then reduced to consider only the case when the two given curves $γ$ and $γ'$ are a pair of distinct non-power, non-peripheral, closed curves with non-trivial geometric intersection. Let us assume, moreover, as above that the given curves are geodesic for the fixed complete metric on $S_{g,n}$.

There are lifts $ℓ$ and $ℓ'$ of $γ$ and $γ'$, respectively, to $ℤ$ such that $ℓ \neq ℓ'$ and $ℓ \cap ℓ' \neq \emptyset$. These are geodesics in $ℤ$ and are the axes of two elements $γ$ and $γ'$ of $Π_{g,n}$ whose free homotopy classes contain, respectively, $γ$ and $γ'$. Then, the axes $ℓ$ and $ℓ'$ intersect in a single point $P$. Let us denote by $P$ the image of $P$ in the surface $S_{g,n}$.

For a given finite index normal subgroup $K$ of $Π_{g,n}$, let us denote by $γ_K$, $γ'_{K}$ and $P_K$, respectively, the images of $ℓ$, $ℓ'$ and $P$ in the surface $S_{g,n}$.

Let us identify, as in the proof of Theorem 3.3, the inverse limits $\lim_{←K⊂G} γ_K$ and $\lim_{←K⊂G} γ'_{K}$ with the closures $\overline{ℓ}$ and $\overline{ℓ}'$ of $ℓ$ and $ℓ'$, respectively, in the pro-$ℤ$ hyperbolic solenoid. By Theorem 3.5 there holds:

$$\overline{ℓ} \cap \overline{ℓ}' = \lim_{K⊂G} γ_K \bigcap \lim_{K⊂G} γ'_{K} = \lim_{K⊂G} (γ_K \cap γ'_{K}) = P.$$ 

As above, here, we are using the fact that inverse limits commute with inverse limits and, in particular, with intersections.

By Theorem 3.3 and the above arguments, there is then a $G$-open subgroup $L$ of $Π_{g,n}$ such that the curves $γ_L$ and $γ'_{L}$ are simple and all their intersection points lie below the point $P$ and, in particular, above the point $P$. This implies that the intersection indices at these points are all equal.

Therefore, if $π_L$ and $γ'_{L}$ are cycles of the homology group $H_1(S_L, ℤ)$ supported on $γ_L$ and $γ'_{L}$, respectively, there holds $(π_L, γ'_{L})_L = k$, for some integer $k \neq 0$, and the same holds for every $G$-open subgroup $K$ of $Π_{g,n}$ contained in $L$.

\[ \square \]

4 An algebraic characterization of simple closed curves via a pro-$G$ Reidemeister pairing

Given a standard presentation $Π_g = \langle α_1, \ldots, α_g, β_1, \ldots, β_g | \prod_{i=1}^g [α_i, β_i] \rangle$ of the fundamental group of a closed oriented surface $S_g$, there are various algorithms which permit to determine whether there is a s.c.c. in the free homotopy class of an element of $Π_g$ (for a survey on this subject, see for instance §3 of [7]).

In this section, we show that there is a pairing on the fundamental group $Π_g$ with values in the $p$-adic group ring of the pro-$p$ completion (more generally, in the pro-$G$ completion) of $Π_g$ whose singular elements are precisely the loops which contain a s.c.c. in their free homotopy class. This pairing is entirely determined by the group structure of $Π_g$.

Let $K$ be a finite index subgroup of $Π_g$. The cup product on the first homology group $H_1(K)$ can be recovered from the group structure of $K$. Indeed, by the description of the lower central series of one-relation groups given in [13], there is a short exact sequence:

$$0 \rightarrow ℤ \xrightarrow{Δ} K/[K, K] \wedge K/[K, K] \xrightarrow{ψ} [K, K]/[[K, K], K] \rightarrow 1,$$
where the epimorphism $\psi$ is induced by the assignment $\alpha \wedge \beta \mapsto [\alpha, \beta]$, for all $\alpha, \beta \in K$, and the monomorphism $\phi$ is defined by $\phi(1) = \sum_{i=1}^{g_K} a_i \wedge b_i$, where $g_K$ is the genus of $S_K$ and $\{a_i, b_i\}_{i=1}^{g_K}$ is any set of generators for the homology group $H_1(S_K, \mathbb{Z}) = K/[K, K]$ for which the identity $\langle a_i, b_i \rangle = \delta_{ij}$, with $\delta_{ij}$ Kronecker’s delta, holds.

Since the integral homology of a closed surface group is torsion free, there is a natural isomorphism $H^1(K) := H^1(K, \mathbb{Z}) \cong \text{Hom}(H_1(K), \mathbb{Z})$. An easy computation then shows that the cap product on the integral cohomology group is the integral dual of $\phi$:

$$\phi^*: H^1(K) \wedge H^1(K) \rightarrow \mathbb{Z}.$$ 

There is also a canonical isomorphism $H_1(K) \cong \text{Hom}(H^1(K), \mathbb{Z})$. After identifying $H_1(K)$ with $H^1(K)^*: = \text{Hom}(H^1(K), \mathbb{Z})$ by means of the latter isomorphism, the non-degenerate skew-symmetric form $\phi^*$ on $H^1(K)$ establishes a canonical isomorphism between $H^1(K)$ and $H_1(K)$, defined by $x \mapsto x^* := \phi^*(x \wedge \cdot) \in H^1(K)^* \equiv H_1(K)$, for $x \in H^1(K)$. The cup product on $H_1(K)$ is then described by the formula:

$$\langle x^*, y^* \rangle_K = \phi^*(x \wedge y), \quad \text{for all } x^*, y^* \in H_1(K).$$

In this way, we get a description of the cup product on $H_1(K)$ only in terms of the group structure of $K$.

For $K$ a finite index normal subgroup of $\Pi_g$, let us usual $G_K := \Pi_g/K$ and let $\mathbb{Z}[G_K]$ be the integral group ring of $G_K$. The Reidemeister pairing (cf. Section 3 in [10])

$$\Phi_K: H_1(K) \times H_1(K) \rightarrow \mathbb{Z}[G_K]$$

is defined by $\Phi_K(a, b) = \sum_{h \in G_K} \langle a, h \cdot b \rangle_K h$, for $a, b \in H_1(K)$.

There is a natural involution $\tau_K: \mathbb{Z}[G_K] \rightarrow \mathbb{Z}[G_K]$, defined extending linearly the map $h \mapsto h^{-1}$ on $G_K$. With respect to the involution $\tau$, the $\mathbb{Z}[G_K]$-valued form $\Phi_K(\cdot, \cdot)$ is sesquilinear, skew-hermitian and non-degenerate.

By Lemma 3.1 in [10], if $K'$ is a finite index normal subgroup of $\Pi_g$ contained in $K$, there is a commutative diagram of Reidemeister pairings:

$$
\begin{array}{ccc}
H_1(K') \times H_1(K') & \xrightarrow{\Phi_{K'}} & \mathbb{Z}[G_{K'}] \\
\downarrow q_* \times q_* & & \downarrow q_* \\
H_1(K) \times H_1(K) & \xrightarrow{\Phi_K} & \mathbb{Z}[G_K],
\end{array}
$$

where $q_*: H_1(K') \rightarrow H_1(K)$ and $q_*: \mathbb{Z}[G_{K'}] \rightarrow \mathbb{Z}[G_K]$ are the natural maps.

Let us switch to $\mathbb{Q}_p$ coefficients, so that the natural map $q_*: H_1(K', \mathbb{Q}_p) \rightarrow H_1(K, \mathbb{Q}_p)$ is surjective. For a class of finite groups $\mathcal{C}$, let us then define:

$$
H^1_{\mathcal{C}}(\mathbb{Q}_p) := \varprojlim_{K \subset \mathcal{C} \Pi_g} H_1(K, \mathbb{Q}_p) \quad \text{and} \quad \mathbb{Q}_p[[\hat{\Pi}_g^\mathcal{C}]] := \varprojlim_{K \subset \mathcal{C} \Pi_g} \mathbb{Q}_p[G_K].
$$

The set of involutions $\{\tau_K\}_{K \subset \mathcal{C} \Pi_g}$ also forms an inverse system and its inverse limit is an involution $\hat{\tau}^\mathcal{C}: \mathbb{Q}_p[[\hat{\Pi}_g^\mathcal{C}]] \rightarrow \mathbb{Q}_p[[\hat{\Pi}_g^\mathcal{C}]]$, which, for $h \in \hat{\Pi}_g^\mathcal{C}$, is just given by $h \mapsto h^{-1}$.
Taking the inverse limit $\Phi_\varepsilon := \lim_{\rightarrow} K \to \Phi_K \otimes \mathbb{Q}_p$, then, with respect to $\hat{\cdot}$, we get a $\mathbb{Q}_p[[\hat{\Pi}_g^\varepsilon]]$-valued non-degenerate, sesquilinear, skew-hermitian form on $H_1^\varepsilon(\mathbb{Q}_p)$:

$$\Phi_\varepsilon : H_1^\varepsilon(\mathbb{Q}_p) \times H_1^\varepsilon(\mathbb{Q}_p) \to \mathbb{Q}_p[[\hat{\Pi}_g^\varepsilon]].$$

Let us now show how $\Phi_\varepsilon$ induces a pairing on the pro-$\mathcal{C}$ surface group $\hat{\Pi}_g^\varepsilon$. Let us denote by $\hat{K}$ an open normal subgroup of $\hat{\Pi}_g^\varepsilon$ and let $K := i^{-1}(\hat{K})$, where $i : \Pi_g \to \hat{\Pi}_g^\varepsilon$ is the natural embedding. For an element $\alpha \in \hat{\Pi}_g^\varepsilon$, let then $\nu_K(\alpha)$ be the minimal natural number such that there holds $\alpha^{\nu_K(\alpha)} \in \hat{K}$. Let us define a continuous map

$$h_\varepsilon : \hat{\Pi}_g^\varepsilon \to H_1^\varepsilon(\mathbb{Q}_p)$$

as follows. Let $\{\hat{K}_i\}_{i \in \mathbb{N}}$ be a countable nest of open normal subgroups of $\hat{\Pi}_g^\varepsilon$ forming a base of neighborhoods of the identity and let $h_i : \hat{K}_i \to H_1(K_i, \mathbb{Q}_p)$ be the natural homomorphism, for $i \in \mathbb{N}$. Then, let us define, for $\alpha \in \hat{\Pi}_g^\varepsilon$:

$$h_\varepsilon(\alpha) := \left(\ldots, \frac{1}{\nu_{K_i}(\alpha)}h_i(\alpha^{\nu_{K_i}(\alpha)}), \ldots, \frac{1}{\nu_{K_1}(\alpha)}h_1(\alpha^{\nu_{K_1}(\alpha)})\right) \in H_1^\varepsilon(\mathbb{Q}_p).$$

In Section 5 we will show that the restriction of the map $h_\varepsilon$ to $\Pi_g$ is injective (cf. Remark 5.4). Composing the map $h_\varepsilon$ with the pairing $\Phi_\varepsilon$, we get a continuous pairing

$$\mathcal{R}_\varepsilon : \hat{\Pi}_g^\varepsilon \times \hat{\Pi}_g^\varepsilon \to \mathbb{Q}_p[[\hat{\Pi}_g^\varepsilon]],$$

which we call the pro-$\mathcal{C}$ Reidemeister pairing on $\hat{\Pi}_g^\varepsilon$. By restriction, we get as well a pairing on the discrete surface group $\Pi_g$. By the definition, it is clear that the value of $\mathcal{R}_\varepsilon(\alpha, \beta)$ only depends on the conjugacy classes of $\alpha$ and $\beta$ in $\hat{\Pi}_g^\varepsilon$. Therefore, if we denote by $\mathcal{L}_{\mathcal{C}^{or}}$ the set of homotopy classes of oriented closed curves on $S_g$, we get also a well defined pro-$\mathcal{C}$ Reidemeister pairing:

$$\mathcal{R}_\varepsilon : \mathcal{L}_{\mathcal{C}^{or}} \times \mathcal{L}_{\mathcal{C}^{or}} \to \mathbb{Q}_p[[\hat{\Pi}_g^\varepsilon]].$$

**Proposition 4.1.** There is a character $\chi_p : \text{Aut}(\hat{\Pi}_g^\varepsilon) \to \mathbb{Z}_p^*$ such that, for all $f \in \text{Aut}(\hat{\Pi}_g^\varepsilon)$ and $\alpha, \beta \in \hat{\Pi}_g^\varepsilon$, there holds $\mathcal{R}_\varepsilon(f(\alpha), f(\beta)) = \chi_p(f) \cdot \mathcal{R}_\varepsilon(\alpha, \beta)$. For $f \in \text{inn}(\hat{\Pi}_g^\varepsilon)$, there holds $\chi_p(f) = 1$. Thus, the character $\chi_p$ descends to a character $\bar{\chi}_p : \text{Out}(\hat{\Pi}_g^\varepsilon) \to \mathbb{Z}_p^*$.

**Proof.** For $\hat{K}$ an open characteristic subgroup of $\hat{\Pi}_g^\varepsilon$, let us consider the effect of the automorphism $f$ on the short exact sequence

$$0 \to \mathbb{Z}_p \overset{\phi}{\to} (\hat{\Pi}_g^\varepsilon/\hat{\Pi}_g^\varepsilon) \otimes \mathbb{Z}_p \overset{\psi}{\to} (\hat{\Pi}_g^\varepsilon/[[\hat{\Pi}_g^\varepsilon, \hat{\Pi}_g^\varepsilon]]) \otimes \mathbb{Z}_p \to 1.$$

There holds $f_* \circ \phi(1) = \sum_{i=1}^{2\kappa} f_*(a_i) \wedge f_*(b_i) = \chi_p(f) \sum_{i=1}^{2\kappa} a_i \wedge b_i$, for some $\chi_p(f) \in \mathbb{Z}_p^*$, and thus $f_* \circ \phi = \chi_p(f) \cdot \phi$. By double duality, there then holds $f_*(\Phi_K \otimes \mathbb{Q}_p) \circ f_* = \chi_p(f) \cdot (\Phi_K \otimes \mathbb{Q}_p)$. 


Since open characteristic subgroups form a base of neighborhoods of the identity of $\hat{\Pi}_g^\varepsilon$, passage to the inverse limit yields the claim of the proposition.

As observed above, the value of $\mathcal{R}_\varepsilon(\alpha, \beta)$ only depends on the conjugacy classes of $\alpha$ and $\beta$ in $\hat{\Pi}_g^\varepsilon$. In particular, the pairing is invariant under inner automorphisms of $\hat{\Pi}_g^\varepsilon$ and so the second part of the proposition follows as well.

**Remark 4.2.** If we identify the profinite completion $\hat{\Pi}_g$ of $\Pi_g$ with the étale fundamental group of $C \times \mathbb{Q} \text{Spec}(\mathbb{Q})$, where $C$ is a projective smooth curve of genus $g$ defined over $\mathbb{Q}$, there is a natural faithful representation $\rho_C: G_\mathbb{Q} \hookrightarrow \text{Out}(\hat{\Pi}_g)$, where $G_\mathbb{Q}$ is the absolute Galois group (cf. (ii) Theorem C in [11] and Theorem 7.7 in [4]). It is then not difficult to see that the character $\bar{\chi}_p$ restricts on $G_\mathbb{Q}$ to the $p$-adic cyclotomic character. In particular, this implies that both characters $\chi_p$ and $\bar{\chi}_p$ are surjective.

Let $h: K \to H_1(K)$ be the natural map. From the definition of the standard Reidemeister pairing $\Phi_K$ given above, it is then clear that, for $\alpha, \beta \in \Pi_g$, the submodules $V^K_\alpha$ and $V^K_\beta$ are reciprocally orthogonal if and only if $\Phi_K(h(\alpha^K_{\nu K}(\alpha)), h(\beta^K_{\nu K}(\beta))) = 0$. Similarly, $V^K_\gamma$ is totally isotropic if and only if $\Phi_K(h(\gamma^K_{\nu K}(\gamma)), h(\gamma^K_{\nu K}(\gamma))) = 0$. Combining this remark with Theorem 4.3 and its corollaries, we get the following characterization of simple closed curves on $S_g$. For a closed curve $\gamma$ on $S_g$, let us denote by $\tilde{\gamma} \in \hat{\Pi}_g$ an element whose free homotopy class contains $\gamma$ and by $\vec{\gamma} \in \mathcal{L}^{or}$ the homotopy class of $\gamma$ with a fixed orientation:

**Theorem 4.3.** Let $\mathcal{C}$ be a class of finite groups and $g \geq 2$.

(i) A pair of closed curves $\alpha$ and $\beta$ on the closed Riemann surface $S_g$ have trivial geometric intersection if and only if there holds $\mathcal{R}_\varepsilon(\vec{\alpha}, \vec{\beta}) = 0$.

(ii) The homotopy class of a non-power closed curve $\gamma$ on $S_g$ contains a simple closed curve if and only if there holds $\mathcal{R}_\varepsilon(\vec{\gamma}, \vec{\gamma}) = 0$.

(iii) Let $\alpha$ and $\beta$ be non-homotopic s.c.c.’s on $S_g$. Then $\alpha$ and $\beta$ are homotopic to s.c.c.’s which meet transversally in a single point if and only, for some $\tilde{\alpha}, \tilde{\beta} \in \Pi_g$, there holds $\mathcal{R}_\varepsilon([\tilde{\alpha}, \tilde{\beta}], [\tilde{\alpha}, \tilde{\beta}]) = 0$.

A consequence of Theorem 4.3 is that the topology of the closed Riemann surface $S_g$ is entirely determined by the natural embedding $\Pi_g \hookrightarrow \hat{\Pi}_g^\varepsilon$ and by the algebraic structure of $\hat{\Pi}_g^\varepsilon$, for any class $\mathcal{C}$ of finite groups.

5 Distinguishing closed curves

As a first consequence of Theorem 2.3, we have the following result:

**Theorem 5.1.** Let $\gamma$ and $\gamma'$ be closed curves not homotopic to powers of peripheral s.c.c.’s on the Riemann surface $S_{g,n}$ and let $p \in \mathbb{N}$ be a fixed prime. Then, the two curves are non-homotopic, if and only if, there is a normal, unramified $p$-covering $p_K: S_K \to S_{g,n}$ such that the $G_K$-invariant submodules $V^K_\gamma$ and $V^K_{\gamma'}$ of $H_1(S_K)$ are distinct.
Proof. One implication is obvious. So let us suppose that, for all normal, finite, unramified $p$-coverings $p_K: S_K \rightarrow S_{g,n}$, there holds $V_\gamma^K = V_\gamma'^K$. In particular, there holds $\langle x, y \rangle_K = 0$, for all $x \in V_\gamma^K$, $y \in V_\gamma'^K$ and all $p$-open normal subgroups $K$ of $\Pi_{g,n}$.

From Theorem 2.3 it follows that $\gamma$ and $\gamma'$ have trivial geometric intersection. But then, for powers of non-peripheral s.c.c.’s, with trivial geometric intersection, the above condition implies that $\gamma$ and $\gamma'$ are homotopic to some powers of the same s.c.c. $\tilde{\gamma}$. Let us then assume that the closed curve $\gamma$ is homotopic to a power $\tilde{\gamma}^s$ of $\tilde{\gamma}$ and the closed curve $\gamma'$ is homotopic to a power $\tilde{\gamma}'^t$ of $\tilde{\gamma}$.

If $\tilde{\gamma}$ is homologically non-trivial on the closed surface $S_g = \overline{S_{g,n}}$, the homology classes of $\gamma$ and $\gamma'$ generate the same submodule of $H_1(S_g)$ only if $s = t$ and the curves $\gamma$ and $\gamma'$ are then homotopic.

If instead $\tilde{\gamma}$ is homologically trivial on the closure $S_g$ of $S_{g,n}$, the s.c.c. $\tilde{\gamma}$ is separating and bounds on each side a subsurface of negative Euler characteristic (because $\tilde{\gamma}$ is non-peripheral). It is then not difficult to see (cf. the proof of Lemma 3.10) that there is an abelian, unramified, $p$-covering $p_K: S_K \rightarrow S_{g,n}$ such that all connected components of $p_K^{-1}(\tilde{\gamma})$ are non-separating and are mapped bijectively onto $\tilde{\gamma}$ by $p_K$. In particular, they have non-trivial integral homology classes in $H_1(\overline{S_K})$. As above, $V_\gamma^K = V_{\gamma'}^K$ then implies that $s = t$ and that $\gamma$ and $\gamma'$ are homotopic. \(\square\)

Remark 5.2. In terms of the homology of $p$-coverings, peripheral closed curves on $S_{g,n}$ are characterized by the property that, for some fixed $p$ and every $p$-open normal subgroup $K$, there holds $V_\gamma^K = \{0\}$. The necessity of the condition is obvious. That it is also sufficient immediately follows from Theorem 2.3 and the fact that having trivial geometric intersection with any closed curve on $S_{g,n}$ characterizes peripheral closed curves. In particular, in terms of the homology of normal $p$-coverings it is possible to distinguish a peripheral curve from a non-peripheral curve but not a peripheral curve from another.

For arbitrary closed curves, we then derive the following weaker separation property:

Theorem 5.3. Let $\alpha$ and $\beta$ be non-peripheral closed curves on $S_{g,n}$. Then, the two curves are non-homotopic if and only if, for any fixed prime $p$, there is a normal, unramified $p$-covering $p_K: S_K \rightarrow S_{g,n}$ such that every cycle of $H_1(\overline{S_K})$ supported on an elevation of $\alpha$ to $S_K$ is distinct from every cycle supported on an elevation of $\beta$ to $S_K$.

Proof. The case when a power of one curve is homotopic to a power of the other can be treated as in the proof of Theorem 5.1. Let us then assume that no power of one curve is homotopic to a power of the other.

By Theorem 5.3, there is a $p$-open normal subgroup $L$ of $\Pi_{g,n}$ such that each component of the pull-backs of $\alpha$ and $\beta$ to $S_L$ is homotopic to a power of a simple closed curve. Since no power of $\alpha$ is homotopic to a power of $\beta$, all components of the pull-backs of $\alpha$ and $\beta$ to $S_L$ are pairwise non-homotopic.

At this point, we can apply Theorem 5.1 and conclude that there is a characteristic, unramified $p$-covering $p_{KL}: S_K \rightarrow S_L$ such that, if $\gamma_\alpha$ and $\gamma_\beta$ are, respectively, elevations of $\alpha$ and $\beta$ to $S_L$, the submodules $V_{K,L,\gamma_\alpha}$ and $V_{K,L,\gamma_\beta}$ of $H_1(\overline{S_K})$, generated by the cycles supported, respectively, on the elevations of $\gamma_\alpha$ and $\gamma_\beta$ from $S_L$ to $S_K$, are distinct.
Let $G_{K,L}$ be the Galois group of the covering $p_{KL}: S_K \to S_L$. If some cycle $\tilde{\alpha}$ of $H_1(S_K)$, supported on an elevation of $\alpha$ to $S_K$, were in the same homology class of a cycle $\tilde{\beta}$, supported on an elevation of $\beta$ to $S_K$, this would imply that the $G_{K,L}$-orbit of $\tilde{\alpha}$ generates the same submodule of $H_1(S_K)$ as the $G_{K,L}$-orbit of $\tilde{\beta}$, in contrast with the above conclusion.

**Remark 5.4.** By Lemma 3.10 \[3\], if the subgroup $K$ of Theorem 5.3 is contained in $[\Pi_y, \Pi_y] \Pi_y^\ell$, for some integer $\ell \geq 2$, then, for $\gamma$ a s.c.c. on $S_y$, the inverse image $p_{KL}^{-1}(\gamma)$ does not contain separating curves or cut pairs. So that two cycles of $H_1(S_K)$ supported on distinct connected component of $p_{KL}^{-1}(\gamma)$ are also distinct. Together with Theorem 5.3 this implies that the map $h_\phi: \Pi_y \to H_1^\ell(\mathbb{Q}_p)$, defined in Section 4, is injective.

Let $p$ be a prime. A group $G$ is conjugacy $p$-separable if, whenever $x$ and $y$ are non-conjugate elements of $G$, there exists some finite $p$-quotient of $G$ in which the images of $x$ and $y$ are non-conjugate.

An almost immediate consequence of Theorem 5.3 is conjugacy $p$-separability of fundamental groups of oriented Riemann surface. This was well known for open surfaces but, in the closed surface case, it was proved only recently by Paris \[15\].

**Theorem 5.5.** The fundamental group $\Pi_{g,n}$ of an oriented surface is conjugacy $p$-separable.

**Proof.** If $\Pi_{g,n}$ is abelian, the result is trivial. So, let us assume $2g - 2 + n > 0$ and let be given $\alpha, \beta \in \Pi_{g,n}$ belonging to distinct conjugacy classes.

Let us consider first the case when both $\alpha$ and $\beta$ contain peripheral curves in their homotopy classes bounding, respectively, the punctures $P_i$ and $P_j$, with $1 \leq i_1, i_2 \leq n$. Since their conjugacy classes are distinct, either $i_1 \neq i_2$ and $n \geq 2$, or $i_1 = i_2$ and $\alpha$ is conjugated to $\beta^s$, for some $s \neq t \in \mathbb{N}$.

Let $N$ be the kernel of the natural epimorphism $\Pi_{g,n} \to \Pi_y$ induced by filling in the punctures of $S_{g,n}$ and let $L := N[\Pi_{g,n}, \Pi_{g,n}]\Pi_{g,n}$, then there holds $\alpha, \beta \in L$. If $n \geq 1$, the surface $S_L$ associated to $L$ has at least two punctures, otherwise, if $n \geq 2$, the surface $S_L$ has at least three punctures. In any case, the loops $\alpha$ and $\beta$ lift to loops on $S_L$ which define distinct non-trivial homology classes $\tilde{\alpha}$ and $\tilde{\beta}$ in $H_1(S_L, \mathbb{Z}/p^n) = H_1(L, \mathbb{Z}/p^n)$, for some $s \in \mathbb{N}^+$. An element of $\Pi_{g,n}$ in the conjugacy class of $\alpha$ lifts to $S_L$ to a peripheral loop bounding a puncture lying above $P_i$ and determines a cycle of $H_1(S_L, \mathbb{Z}/p^n)$ in the $G_L$-orbit of $\tilde{\alpha}$. Moreover, all elements in the $G_L$-orbit of $\tilde{\alpha}$ are of this type.

It is then clear that, whether or not $i_1 \neq i_2$, the images of $\alpha$ and $\beta$ in the homology group $H_1(L, \mathbb{Z}/p^n)$ are in distinct $G_L$-orbits. Therefore, also their images in the finite $p$-quotient $\Pi_{g,n}/[L, L]Lp^n$, which contains $L/[L, L]Lp^n \cong H_1(L, \mathbb{Z}/p^n)$ as a subgroup, are non-conjugate.

Let us then consider the case in which one of the two elements, say $\alpha$, contains a peripheral curve in its homotopy class and $\beta \in [\Pi_{g,n}, \Pi_{g,n}]$. Let $L$ be defined as above. Then, there holds as well $\alpha, \beta \in L$ and their images $\tilde{\alpha}$ and $\tilde{\beta}$ in $H_1(L, \mathbb{Z}/p)$ are in distinct $G_L$-orbits. As above, it follows that the conjugacy classes of $\alpha$ and $\beta$ are separated by the finite $p$-quotient $\Pi_{g,n}/[L, L]Lp^n$. 


The case in which one of the two elements, say \( \alpha \), contains a peripheral curve in its homotopy class and \( \beta \) instead a non-separating curve is straightforward.

Let us then assume that neither of the two elements contain a peripheral curve in its homotopy class. By Theorem 5.3, there is an oriented hyperbolic Riemann surface \( S \) of finite type and by \( \pi \) that surface groups are residually finite, which is more in the spirit of the present paper:

\[ \Pi \rightarrow \tilde{\Pi} \]

A first basic property of the pro-\( p \) completion of a surface group is given in the following theorem. Although this is well known, we prefer to adapt here to the pro-\( p \) case the simple geometric proof by Hempel (cf. [9]) that surface groups are residually finite, which is more in the spirit of the present paper:

**Theorem A.1.** The canonical homomorphism \( \Pi \rightarrow \tilde{\Pi} \) is injective. Otherwise stated, oriented surface groups are residually \( p \)-groups for all primes \( p \geq 2 \).

**Proof.** It is enough to show that, for a given \( 1 \neq \alpha \in \Pi \) and a map \( f : (S^1, \ast) \rightarrow (S, \ast) \) representing \( \alpha \), there is a characteristic unramified \( p \)-covering \( \pi : S' \rightarrow S \) such that \( f \) does not lift to a map \( \tilde{f} : (S^1, \ast) \rightarrow (S', \ast) \), where we denote also by \( \ast \) a base point on \( S' \) lying over the base point on \( S \) for \( \Pi \). It is not restrictive to assume that \( f \) is an immersion whose image has transversal self-intersection points.

Let us proceed by induction on the cardinality of the singular set \( s(f) \) of the map \( f \).

If \( f \) is an embedding, there is a characteristic unramified \( p \)-covering \( \pi : S' \rightarrow S \) such that the connected components of \( \pi^{-1}(f(S^1)) \) are non-separating simple closed curves (cf. Lemma 3.10 [3]). Let \( S'' \rightarrow S' \) be the unramified \( p \)-covering with covering transformation group isomorphic to the homology \( H_1(S', \mathbb{Z}/p) \). Then, the composition \( \pi' : S'' \rightarrow S \) is a characteristic unramified \( p \)-covering with the desired property.

If \( s(f) \neq \emptyset \), let \( U \) be a regular neighborhood of \( f(S^1) \) in \( S \). There is a simple loop \( g : (S^1, \ast) \rightarrow (U, \ast) \) which represents a non-trivial element of \( \Pi \).

By the preceding case, there is a characteristic unramified \( p \)-covering \( \pi : S' \rightarrow S \) such that \( g \) does not lift to a map \( \tilde{g} : (S^1, \ast) \rightarrow (S', \ast) \). If \( f \) does not lift too to \( S' \), we are done. If \( f \) does lift to a map \( \tilde{f} : (S^1, \ast) \rightarrow (S', \ast) \), then \( s(\tilde{f}) \subseteq s(f) \). If \( s(\tilde{f}) = s(f) \), then \( \pi|_{\tilde{f}(S^1)} \)
would be an embedding, and \( \pi \) would map a neighborhood of \( \tilde{f}(S^1) \) homeomorphically onto \( U \). But then \( g \) would lift to \( S' \). Therefore, there holds \( s(\tilde{f}) \subseteq s(\tilde{f}) \) and the proof follows by induction.

By Theorem A.1, we can and do identify the group \( \Pi \) with its image in the pro-\( p \) completion \( \hat{\Pi}^{(p)} \). The second result we need is the following:

**Theorem A.2.** Let \( \delta_1, \delta_2 \in \Pi \) be elements such that the cyclic subgroups \( \delta_1^p \) and \( \delta_2^p \) are \( p \)-closed in \( \Pi \) and let \( \delta_1^{\tilde{p}} \) and \( \delta_2^{\tilde{p}} \) be the pro-cyclic subgroups topologically generated by \( \delta_1 \) and \( \delta_2 \) in the pro-\( p \) completion \( \hat{\Pi}^{(p)} \). Then, there holds:

\[
\Pi \cap \delta_1^{\tilde{p}} \cdot \delta_2^{\tilde{p}} = \delta_1^p \cdot \delta_2^p.
\]

**Proof.** If \( \delta_1 = \delta_2^{\pm 1} \), the statement is just a different formulation of the fact that \( \delta_1^p \) and \( \delta_2^p \) are \( p \)-closed in \( \Pi \). Let us then assume \( \delta_1 \neq \delta_2^{\pm 1} \) which then implies that \( \delta_1^{\tilde{p}} \cap \delta_2^{\tilde{p}} = \{1\} \).

If \( \Pi \) is a free group, the equality in the statement of the theorem is a particular case of Theorem 6.1 [16]. However, a more direct and self-contained proof immediately follows from the lemma:

**Lemma A.3.** Let \( H_1 \) and \( H_2 \) be finitely generated subgroups of a free group \( F \) which are \( p \)-closed in \( F \) and such that \( H_1 \cap H_2 = \{1\} \). Then the set \( H_1 \cdot H_2 \) is \( p \)-closed in \( F \).

**Proof.** The proof is an adaptation to the pro-\( p \) topology of Niblo’s proof of Theorem 3.2 [14]. By Theorem 5.7 [16], there is a \( p \)-open subgroup \( U \) of \( F \) such that \( H_1 \) is a free factor of \( U \). Since \((H_1 \cdot H_2) \cap U \) is \( p \)-closed if and only if \( H_1 \cdot H_2 \) is \( p \)-closed, we may assume that \( F = K \ast H_1 \) and then form the double \( F \ast_{H_1} F = K \ast H_1 \ast K \) along the subgroup \( H_1 \). Let \( \tau_i: F \hookrightarrow F \ast_{H_1} F \), for \( i = 1, 2 \), be the natural \( p \)-continuous monomorphism which identifies \( F \), respectively, with the first and the second factor. The assignment \( x \mapsto \tau_1^{-1}(x) \tau_2(x) \) then defines a \( p \)-continuous map \( \eta: F \to F \ast_{H_1} F \) and, by Lemma 3.1 [14], there holds:

\[
\eta^{-1}(\langle \tau_1(H_2), \tau_2(H_2) \rangle) = H_1 \cdot H_2.
\]

It remains to show that \( \langle \tau_1(H_2), \tau_2(H_2) \rangle \) is closed in the pro-\( p \) topology of \( F \ast_{H_1} F \). Let us consider the natural \( p \)-continuous epimorphism \( \mu: F \ast_{H_1} F \to F \) which identifies the two factors of the double. By hypothesis, there holds \( H_1 \cap H_2 = \{1\} \) and then \( \mu^{-1}(H_2) \cap H_1 = \{1\} \). From the Kurosh Subgroup Theorem (cf. Theorem 14 in [19]), it follows:

\[
\mu^{-1}(H_2) = (\mu^{-1}(H_2) \cap F) * (\mu^{-1}(H_2) \cap F) * D = \tau_1(H_2) * \tau_2(H_2) * D,
\]

where \( D \) is some free factor. Since \( \mu \) is \( p \)-continuous, \( \mu^{-1}(H_2) \) is \( p \)-closed in \( F \ast_{H_1} F \) and then so is its free factor \( \tau_1(H_2) * \tau_2(H_2) = \langle \tau_1(H_2), \tau_2(H_2) \rangle \) (cf. Corollary 5.6 [16]).

Let us then consider the case when \( \Pi \) is a hyperbolic closed surface group. The idea is to reduce the proof to Lemma A.3 as done above for \( \Pi \) free. In the case under consideration, at least the group \( F = \langle \delta_1, \delta_2 \rangle \) is free, because it has infinite index in \( \Pi \). This follows from
the fact that a subgroup of infinite index of a hyperbolic oriented surface group identifies
with the fundamental group of a non-compact oriented Riemann surface, which is free.

We cannot directly replace $\Pi$ by $F$ in the proof of the theorem, because we do not
know whether $F$ is closed in the pro-$p$ topology of $\Pi$ and so whether its closure $\overline{F}$ in $\hat{\Pi}^{(p)}$
coincides with the pro-$p$ completion of $F$.

Let us then observe that the closure $\overline{F}$ of $F$ in the pro-$p$ completion $\hat{\Pi}^{(p)}$ has infinite
index. Indeed, since $\Pi$ is a closed surface group, its pro-$p$ completion $\hat{\Pi}^{(p)}$ is a Demuškin
pro-$p$ group and so are its open subgroups. Now, a 2-generated Demuškin pro-$p$ group
is soluble, therefore $\overline{F}$ is not open in $\hat{\Pi}^{(p)}$ and so is of infinite index. From part (b) of
Exercise 5, Ch. I, §4.5, [20], it then follows that $\overline{F}$ is a free pro-$p$ group.

Now, the intersection $F' = \overline{F} \cap \Pi$ has also infinite index in $\Pi$ and hence is a free group.
Moreover, the subgroup $F'$ is closed in the pro-$p$ topology of $\Pi$ and so its closure $\overline{F}$ in $\hat{\Pi}^{(p)}$
coincides with its pro-$p$ completion. Thus, by replacing $\Pi$ with $F'$, in the required equality
$\Pi \cap \delta_1^{\hat{p}p} \cdot \delta_2^{\hat{p}p} = \delta_1^{\hat{p}} \cdot \delta_2^{\hat{p}}$, we are finally reduced to the case when $\Pi$ is a free group which we
already treated in Lemma A.3. \hfill $\square$

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