Global Existence and Asymptotic Behavior for a Reaction–Diffusion System with Unbounded Coefficients

Mohamed Majdoub and Nasser-Eddine Tatar

Abstract. We consider a reaction–diffusion system which may serve as a model for a ferment catalytic reaction in chemistry. The model consists of a system of reaction–diffusion equations with unbounded time-dependent coefficients and different polynomial reaction terms. An exponential decay of the globally bounded solutions is proved. The key tool of the proofs are properties of analytic semigroups and some inequalities.

Mathematics Subject Classification. 35K57, 35A01, 35B40, 35Q92.

Keywords. Analytic semi-group, exponential decay, fractional operator, reaction–diffusion system, sectorial operator.

1. Introduction

In the past 3 decades, a strong interest in the questions of global existence and asymptotic behavior of solutions to various classes of reaction–diffusion systems has been witnessed. A particular attention was paid to autonomous systems and different sufficient conditions were assumed to ensure global existence of classical solutions. A main difficulty in proving global existence is the lack of maximum principle estimates or invariant regions. Recently, the global existence of classical solutions was shown in [2] for quasi-positive nonlinearities having a (slightly super-)quadratic growth and obeying a mass control assumption. This, in particular, includes the cases of mass conservation and mass dissipation. See also [12] for initial data of low regularity.
Our goal in this paper is to study the following non-autonomous system:

\[
\begin{aligned}
&u_t - (d_1 \Delta - b_1)u = a_1(t)w^m - a_2(t)u^n v^k, \\
v_t - d_2 \Delta v = (a_1(t) + a_3(t)) w^m - a_2(t)u^n v^k, \\
w_t - (d_3 \Delta - b_2)w = - (a_1(t) + a_3(t)) w^m - a_4(t)w + a_2(t)u^n v^k, \\
(\partial u/\partial \nu) = (\partial v/\partial \nu) = (\partial w/\partial \nu) = 0,
\end{aligned}
\]  

(1.1)

where \( t > 0, \ x \in \Omega \subset \mathbb{R}^N, \ N \geq 1 \), \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \partial/\partial \nu \) denotes the outward normal derivative on \( \partial \Omega \). The diffusion coefficients \( d_i, \ i = 1, 2, 3 \) are assumed positive constants and \( u_0, v_0 \) and \( w_0 \) are given nonnegative bounded initial functions. The coefficients \( a_i(t) \) are of the form \( t^{\sigma_i} h_i(t) \), where \( \sigma_i \geq 0, \ i = 1, 2, 3 \) and \( \sigma_4 = 0 \). The functions \( h_i(t) \) are nonnegative continuous functions. The powers \( m, n \) and \( k \) are assumed greater than one.

The problem of global existence and asymptotic behavior of solutions to reaction–diffusion systems has attracted considerable attention in the mathematical community and the literature is very extensive.

In [1], the author proves the global existence for strongly coupled semi-linear parabolic systems of arbitrary even order under linear time-dependent boundary conditions. Feng [3] uses the method of upper and lower solutions to establish the global existence of nonnegative solutions to certain classes of reaction–diffusion systems. In [11], the authors study some reaction–diffusion systems with natural structure conditions and nonlinear diffusion operators of porous media type. They obtained global existence of weak solutions provided that the reactive terms are bounded in \( L^1 \). In [14] the authors investigate the asymptotic behavior of a reaction–diffusion system through its corresponding ordinary differential equation. The fundamental question was: under what conditions, does global existence for the ODE imply global existence for the reaction–diffusion system. They gave negative answers and affirmative ones for several typical examples. In [16], the global existence for a reaction–diffusion system was obtained by assuming that the nonlinearity is locally Lipschitz and satisfies a Lyapunov-type condition.

Note that the question of blow-up of the solutions to some reaction–diffusion systems that preserve nonnegativity and for which the total mass of the components is uniformly bounded was discussed in [17].

Let us now discuss some recent papers proposing similar and complementary approaches, and considering additional features. In [8], the authors consider an iterative system of singular multipoint boundary value problems on time scales. Sufficient conditions are derived for the existence of infinitely many positive solutions by applying Krasnoselskii’s cone fixed point theorem in a Banach space. In [13], the author studies the regularity of the true solution to the homogeneous BVP of double-sided fractional diffusion advection reaction equation with variable coefficients on a bounded interval. It is shown that, under suitable conditions, the true solution exists and can be represented in the form of fractional integral. Moreover, it is proved that usually, this integral representation cannot be further improved even with smooth coefficients and right-hand side function in the equation. He also obtained the
precise bound for this integral representation. This is in sharp contrast to the case of integer-order elliptic PDEs. In [21], a Keller–Segel chemotaxis model is described by a system of nonlinear partial differential equations. The authors adapt a convection diffusion equation for the cell density coupled with a reaction–diffusion equation for chemoattractant concentration. They obtained global existence, uniqueness and boundedness of the weak solution for the problem, with the help of the Galerkin method.

For a survey on global existence, blow-up and asymptotic behavior of reaction–diffusion systems, we refer to [4,14,19].

Systems with time dependent nonlinearities were considered in [7] by Kahane. Specifically, a system of the form

\[
\begin{align*}
-u_t + Lu &= f(x,t,u,v) \quad \text{in } \Omega \times (0,\infty) \\
-v_t + Mv &= g(x,t,u,v) \quad \text{in } \Omega \times (0,\infty),
\end{align*}
\]

where \(L\) and \(M\) are uniformly elliptic operators, with boundary conditions of Robin type is studied. He proved that the solution converges to the stationary state, that is the solution of the limiting elliptic problem. To this end he assumed that

\[
\begin{align*}
\bar{f}(x,u,v) &= \lim_{x \to \Omega} f(x,t,u,v) \\
\bar{g}(x,u,v) &= \lim_{x \to \Omega} g(x,t,u,v)
\end{align*}
\]

uniformly in \(\Omega\) and \((u,v)\) in any bounded subset of the first quadrant in \(\mathbb{R}^2\). The matrix formed by the partial derivatives \(\bar{f}_u, \bar{f}_v, \bar{g}_u\) and \(\bar{g}_v\) satisfies a column diagonal dominance type condition. In the present work we do not make such restrictions.

Let us return now to our system (1.1). In the case \(m = n = k = 1, b_1 = b_2 = 0\) and \(a_i(t) \equiv a_i\) are constants, this problem has been studied by Wang [22]. If, moreover, \(a_3 = a_4 = 0\), then this problem may also serve as a model for sugar transporting into red blood cells (see Ruan [20], Rothe [19], Feng [3], Morgan [16] and references therein). In particular, Wang proved a convergence result and an exponential decay result in \(C^\mu(\bar{\Omega}), \mu \in [0,2)\). Unfortunately, his method seems to be not valid for the present problem (1.1) because of the unboundedness of the coefficients.

To our knowledge, a system with time-dependent coefficients like (1.1) is still not well studied. Nevertheless, using similar ideas as in [2], we are able to prove global existence of classical solutions to (1.1) provided that \(0 \leq m \leq 2 + \varepsilon, n, k \geq 0\) and \(n + k \leq 2 + \varepsilon\) for sufficiently small \(\varepsilon > 0\). More precisely, we assume that

\[
\begin{align*}
0 \leq m &\leq 2 + \varepsilon \quad \text{and} \quad n + k \leq 2 + \varepsilon, \\
u_0, v_0, w_0 &\in L^1(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad u_0, v_0, w_0 \geq 0
\end{align*}
\]

\(\textbf{Theorem 1.1.} \) Suppose that assumption (1.3) is fulfilled. Then, there exists \(\varepsilon > 0\) such that if (1.2) is satisfied, (1.1) possesses a unique classical global solution.

Our main task here is to study the asymptotic behavior and the decay rate of global classical solutions obtained in Theorem 1.1. To this end we shall
adapt the methods used in Hoshino [5] for the former task and the methods used in Kirane and Tatar [10] for the latter goal. To state our results in a clear way, we shall fix some notations. For \( \alpha \in (0, 1) \), we denote by
\[
q^* = q^*(\alpha) = \begin{cases} \frac{2}{\alpha} - 1, & \text{if } \frac{1}{2} \leq \alpha < 1 \\ 2, & \text{if } 0 < \alpha < \frac{1}{2} \end{cases}
\]
and \( q \) the Lebesgue conjugate exponent of \( q^* \). Let \( r = \frac{1-\alpha}{\alpha} \). First, we state a convergence result of the solutions in the space \( C^\mu(\Omega), \mu \in [0, 2) \). This result will be needed in the sequel. Without loss of generality, we shall assume that \( b_1 = b_2 = b \).

**Theorem 1.2.** Suppose that \( 1 - q\alpha > 0, 1 + q(\sigma_i - \alpha)l > 0, i = 1, 2, 3 \) for some \( l \) such that \( 1 < l < \min\{m, n, k\}, h_i \in L^{q^*}(0, \infty), i = 1, 2, 3, 4 \) and \( a_1(t) \leq Ca_3(t) \) for some positive constant \( C \), \( \forall t > 0 \). Then, for every \( \mu \in [0, 2) \)
\[
u(t) \to 0, v(t) \to v_{\infty} \text{ and } w(t) \to 0 \text{ in } C^\mu(\Omega) \text{ as } t \to \infty
\]
where \( v_{\infty} = |\Omega|^{-1} \int_\Omega h_4(s) + b w(s) dx ds \).

**Remark 1.3.** The proof of Theorem 1.2 is similar to the proof of Theorem 2.2 in [5] with minor modifications that will be clear from the proof of our next and main result. It is therefore omitted.

To state our next result, we suppose that \( N \) and \( p \) satisfy
\[
2\alpha \leq \frac{N}{p} \text{ and } \max \left\{ 1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha} \right\} < \min\{m, n\}. \tag{1.4}
\]

**Theorem 1.4.** Assume that the hypotheses of Theorem 1.2 and (1.4) are fulfilled. If
\[
(l-1) \int_0^\infty h(s) ds < \log \left( \frac{1+C_{\alpha}^{-1}}{C_0} \right)
\]
for some constant \( l \) such that \( \max \left\{ 1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha} \right\} < l < \min\{m, n\} \) and a constant \( C_0 = C_0(\|u_0\|_p, \|w_0\|_p) \) and a function \( h \) to be determined in the proof, then for \( \mu \in [0, 2) \) and \( N, p \) such that \( 0 \leq \mu < 2\alpha - \frac{N}{p} \) we have
\[(a) \|u\|_{C^\mu(\Omega)}, \|w\|_{C^\mu(\Omega)} \leq Ce^{-(b-\varepsilon)t} \left( \|u_0\|_p + \|w_0\|_p \right), \text{ where } 0 < \varepsilon < b, \text{ as } t \to \infty.
\]
\[(b) (i) \text{ If } d_2 \lambda < lb, \|
u - v_{\infty}\|_{C^\mu(\Omega)} \leq Ce^{-\min\{(b-\varepsilon), d_2 \lambda\}t} \text{ as } t \to \infty.
\]
\[(ii) \text{ If } d_2 \lambda \geq lb \text{ and for } i = 1, 2, 3
\]
\[
\int_0^t e^{q^*\rho s} h_i^{q^*}(s) ds = O(e^{q^*\tilde{\rho}t}) \text{ as } t \to \infty
\]
for some \( \rho > d_2 \lambda - l(b - \varepsilon) \) and \( \tilde{\rho} < d_2 \lambda \), then
\[
\|v - v_{\infty}\|_{C^\mu(\Omega)} \leq Ce^{\min\{(b-\varepsilon), d_2 \lambda - \tilde{\rho}\}t} \text{ as } t \to \infty.
\]
2. Preliminaries

In this section we present the notation that will be used in this paper and prepare some material which will be useful in our proofs.

By \(W^{l,p}(\Omega)\) we denote the usual Sobolev space of order \(l \geq 0\) for \(1 \leq p \leq \infty\). The space \(C^\mu(\Omega), \mu \geq 0\) is the Banach space of \([\mu]-\)times continuously differentiable functions in \(\Omega\) whose \([\mu]-\)th order derivatives are Hölder continuous with exponent \(\mu - [\mu]\).

**Definition 2.1.** For \(p \in (1, \infty)\), we define \(D(A_p) = D(B_p) = D(G_p) = \{y \in W^{2,p}(\Omega) : \partial y/\partial \nu \mid_{\partial \Omega} = 0\}\), \(A_py = -(d_1 \Delta - b_1)y, B_py = -d_2 \Delta y, G_py = -(d_3 \Delta - b_2)y\).

The operators \(-A_p, -B_p\) and \(-G_p\) defined in this way are sectorial operators (see Henry [4]) and generate analytic semigroups \(\{e^{-tA_p}\}_{t \geq 0}\), \(\{e^{-tB_p}\}_{t \geq 0}\) and \(\{e^{-tG_p}\}_{t \geq 0}\), respectively.

**Definition 2.2.** By \(Q_0\) and \(Q_+\) we designate the following projection operators, for \(y \in L^p(\Omega), p \in (1, \infty)\)

\[
Q_0y = |\Omega|^{-1} \int_\Omega y(x)dx \quad \text{and} \quad Q_+y = y - Q_0y
\]

where \(|\Omega|\) is the volume of \(\Omega\).

**Definition 2.3.** The restriction of \(B_p\) onto \(Q_+L^p(\Omega)\) will be denoted by \(B_{p+}\)

i.e. \(B_{p+} = B_p \mid_{Q_+L^p(\Omega)}\).

For \(p \in (1, \infty)\), the operator \(B_{p+}\) generates an analytic semigroup denoted by \(\{e^{-tB_{p+}}\}_{t \geq 0}\) in \(Q_+L^p(\Omega)\).

We define the fractional powers of the above operators in the usual way (see Henry [4]).

**Lemma 2.4.** Let \(A\) be a sectorial operator in \(X = L^p(\Omega), 1 \leq p < \infty\) with \(D(A) = X^1 \subset W^{m,p}(\Omega)\) for some \(m \geq 1\). Then, for \(0 \leq \alpha \leq 1\), we have \(X^\alpha \subset C^\mu\) when \(0 \leq \mu < \max - \frac{N}{p}\).

**Lemma 2.5.** Let \(\lambda\) denote the least positive eigenvalue of the Laplacian with homogeneous Neumann boundary condition. Let \(p \in (1, \infty)\). For every \(\alpha \in [0, 1)\) there exist positive constants \(C_i\), \(i = 1, 2, 3\) such that for \(t > 0\) and \(y \in L^p(\Omega)\)

\[
\|A_\alpha^p e^{-tA_p}y\|_p \leq C_1 t^{-\alpha} e^{-b_1 t} \|y\|_p, \quad \|B_\alpha^p e^{-tB_p+Q+y}\|_p \leq C_2 t^{-\alpha} e^{-d_2 t} \|Q+y\|_p, \quad \|G_\alpha^p e^{-tG_p y}\|_p \leq C_3 t^{-\alpha} e^{-b_3 t} \|y\|_p.
\]

**Lemma 2.6.** Let \(p \in (1, \infty), l \geq 1\) and \(\alpha \in [0, 1)\). Then, there exists a positive constant \(C\) such that

\[
\|y\|_{pl} \leq C \|A_\alpha^p y\|_p^\theta \|y\|_p^{1-\theta},
\]

where \(\theta\) satisfies

\[
\frac{N(l-1)}{2pl\alpha} < \theta < 1.
\]
See Henry [4] for the proofs of Lemmas 2.4, 2.5, 2.6.

Lemma 2.7. Let \( \alpha \in [0, 1) \) and \( \beta \in \mathbb{R} \). There exists a positive constant \( C = C(\alpha, \beta) \) such that
\[
\int_0^t s^{-\alpha} e^{\beta s} ds \leq \begin{cases} 
Ce^{\beta t} & \text{if } \beta > 0 \\
C(t + 1) & \text{if } \beta = 0 \\
Ci & \text{if } \beta < 0.
\end{cases}
\]

See Hoshino and Yamada [6], for instance, for the proof of Lemma 2.7.

Lemma 2.8. Let \( \mu, \nu, \tau \) and \( z > 0 \), then
\[
z^{1-\nu} \int_0^z (z - \xi)^{\nu-1} \xi^{\mu-1} e^{-\tau \xi} d\xi \leq C\tau^{-\mu}
\]
where \( C \) is a constant independent of \( z \).

See Michalski [15] or Kirane and Tatar [9] for the proof of Lemma 2.8. Let \( y : [a, b] \to [0, \infty) \) be a continuous function satisfying
\[
y(t) \leq M + \int_a^t \lambda(s) g(y(s)) ds, \quad a \leq t \leq b
\]
where \( M > 0, \lambda : [a, b] \to [0, \infty) \) is continuous and \( g : [0, \infty) \to [0, \infty) \) is continuous nondecreasing with
\[g(y) > 0, \text{ for all } y > 0.\]

We define
\[
G_M(y) = \int_y^\infty \frac{ds}{g(s)}, \quad y > 0.
\]
The following Bihari-type inequality is classical and can be found in many references. See for instance [18].

Lemma 2.9. Under the above assumptions on \( y, \lambda, g \), we have
\[
y(t) \leq G_M^{-1} \left( \int_a^t \lambda(s) ds \right),
\]
for \( a \leq t \leq T \leq b \) with \( T > a \) satisfies
\[
\int_a^T \lambda(s) ds < \int_M^\infty \frac{ds}{g(s)} ds.
\]

Remark 2.10. The classical Gronwall’s inequality corresponds to \( g(y) = y \).
3. Proof of Theorem 1.1

Define \( z = (u, v, w) \), we rewrite (1.1) as
\[
\begin{aligned}
&u_t - d_1 \Delta u = f_1(t, z), \\
v_t - d_2 \Delta v = f_2(t, z), \\
w_t - d_3 \Delta w = f_3(t, z), \\
y_t - \Delta y = f_4(t, z), \quad y(0) = 0,
\end{aligned}
\] (3.1)

where \( f_4 := -f_1 - f_2 - f_3 \) and the last equation in \( y \) is added to fulfill technical requirements as it will be clear below. From (3.1) we deduce that
\[
(u + v + w + y)_t - \Delta (d_1 u + d_2 v + d_3 w + y) = 0.
\] (3.2)

It follows that
\[
u(x, t) + v(x, t) + w(x, t) + y(x, t) - \Delta m(x, t) = g(x),
\] (3.3)

where
\[
m(x, t) = \int_0^t [d_1 u(x, s) + d_2 v(x, s) + d_3 w(x, s) + y(x, s)] ds,
\] (3.4)

and
\[
g(x) = u_0(x) + v_0(x) + w_0(x).
\] (3.5)

Therefore,
\[
\|z\|_{L^\infty_{x, t}} \leq \|\Delta m\|_{L^\infty_{x, t}} + \|g\|_{L^\infty_{x, t}} \quad \text{and} \quad \|m\|_{L^\infty_{x, t}} \leq C_T,
\] (3.6)

where \( C_T \) depends continuously in \( T > 0 \). Using (3.3), one can write
\[
B(x, t) \partial_t m - \Delta m = g(x),
\] (3.7)

where
\[
0 < B \leq B(x, t) = \frac{u(x, t) + v(x, t) + w(x, t) + y(x, t)}{d_1 u(x, t) + d_2 v(x, t) + d_3 w(x, t) + y(x, t)} \leq \overline{B} < \infty.
\]

Lemma 3.5 in [2] implies
\[
|m(x, t) - m(x', t)| \leq C_T |x - x'|^\delta, \quad \forall 0 < t < T,
\] (3.8)

for some \( \delta \in (0, 1) \). Equation (3.3) also implies
\[
\partial_t m - \Delta m = (d_1 - 1) u + (d_2 - 1) v + (d_3 - 1) w + g(x).
\] (3.9)

Using the Hölder continuity of \( m \) and [2, Lemma 1.1, p. 283], we infer
\[
|\nabla m| \leq C_T |z|^{1-\delta}.
\] (3.10)

Applying [2, Lemma 1.1, p. 283] to each equation of \( q \in \{u, v, w, y\} \) leads to
\[
|\nabla q| \leq C_T (1 + |z|^{3/2}).
\] (3.11)

Using [2, Lemma 3.9, p. 295] together with (3.10) and (3.11) gives
\[
|\Delta m| \leq C_T |\nabla m|^{1/2} \left( |\nabla u| + |\nabla v| + |\nabla w| + |\nabla g| \right)^{1/2} \\
\leq C_T |z|^{1-\delta}(1 + |z|^{3/2})^{1/2} \\
\leq C_T \left( 1 + |z|^{(2 - \delta)/2} + \frac{3}{2} \right).
\]
From the above estimate and (3.6), we obtain that
\[ \|z\|_{L^\infty_{x,t}} \leq C_T \left( 1 + \|z\|_{L^\infty_{x,t}}^{1-\delta} + \frac{3}{4} \right). \]
Since \( \frac{1-\delta}{2(2-\delta)} + \frac{3}{4} < 1 \) then \( \|z\|_{L^\infty_{x,t}} \leq C_T \). Consequently \( T_{max} = \infty \).

4. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. We proceed in several steps.

A. The decay rate of \( \|u\|_{C^\beta(\bar{\Omega})} \) and \( \|w\|_{C^\beta(\bar{\Omega})} \):

Clearly we have the integral equations associated with the first and third equation of (1.1)

\[ u(t) = e^{-tA_p}u(0) + \int_0^t e^{-(t-s)A_p} \left\{ a_1(s)w^m - a_2(s)u^nv^k \right\} ds \] (4.1)

\[ w(t) = e^{-tG_p}w(0) + \int_0^t e^{-(t-s)G_p} \left\{ -(a_1(s) + a_3(s))w^m - a_4(s)w 
+ a_2(s)u^nv^k \right\} ds. \] (4.2)

Applying \( A_\alpha p \) and \( G_\alpha p \), \( 0 < \alpha < 1 \) to both sides of (4.1) and (4.2), respectively, we infer from Lemma 2.8 that

\[ \|A_\alpha^\alpha u\|_p \leq C_1 t^{-\alpha} e^{-bt} \|u_0\|_p + C_1 \int_0^t (t-s)^{-\alpha} e^{-b(t-s)} s_\sigma^1 h_1(s) \|w^m\|_p ds \]

\[ + C_1 \int_0^t (t-s)^{-\alpha} e^{-b(t-s)} s_\sigma^2 h_2(s) \|u^nv^k\|_p ds \]

and

\[ \|G_\alpha^\alpha w\|_p \leq C_3 t^{-\alpha} e^{-bt} \|w_0\|_p \]

\[ + C_3 \int_0^t (t-s)^{-\alpha} e^{-b(t-s)} (s_\sigma^1 h_1(s) + s_\sigma^3 h_3(s)) \|w^m\|_p ds \]

\[ + C_3 \int_0^t (t-s)^{-\alpha} e^{-b(t-s)} h_4(s) \|w\|_p ds \]

\[ + C_3 \int_0^t (t-s)^{-\alpha} e^{-b(t-s)} a_2(s) \|u^nv^k\|_p ds. \]

As \( 0 \leq v \leq M \), for some \( M > 0 \), we have for all \( t \geq \delta > 0 \)

\[ e^{bt} \|A_\alpha^\alpha u\|_p \leq C_1 \delta^{-\alpha} \|u_0\|_p + C_1 \int_0^t (t-s)^{-\alpha} e^{bs} s_\sigma^1 h_1(s) \|w^m\|_p ds \]
Lemma 2.9 and the uniform boundedness of $w$ allow us to write
$$
\|w^m\|_p = \|w\|_{mp}^m \leq C \|G^\alpha_p w\|_p^{m\theta} \cdot \|w\|_p^{m(1-\theta)} \leq C_4 \|G^\alpha_p w\|_p^{m\theta}.
$$
with $\frac{N(m-1)}{2p\alpha} < \theta < 1$.

In the same way, we have
$$
\|u^n\|_p \leq C_5 \|A^\alpha_p u\|_p^{n\theta}, \quad \text{with} \quad \frac{N(n-1)}{2p\alpha} < \theta < 1.
$$

Let us choose $l$ such that
$$
\max \left\{ 1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha} \right\} < l < \min\{m, n\}
$$
and $\theta = \frac{l}{m}$ in (4.5) and $\theta = \frac{l}{n}$ in (4.6). Then
$$
\|w^m\|_p \leq C_4 \|G^\alpha_p w\|_p^l \quad \text{and} \quad \|u^n\|_p \leq C_5 \|A^\alpha_p u\|_p^l.
$$

Taking (4.7) into account in (4.3) and (4.4), we obtain
$$
e^{bt} \|A^\alpha_p u\|_p \leq C_1 \delta^{-\alpha} \|u_0\|_p + C_6 \int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{r_1} h_1(s) \left(e^{bs} \|G^\alpha_p w\|_p\right)^l ds
$$
$$
+ C_7 \int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{r_2} h_2(s) \left(e^{bs} \|A^\alpha_p u\|_p\right)^l ds.
$$

and
$$
e^{bt} \|G^\alpha_p w\|_p \leq C_3 \delta^{-\alpha} \|w_0\|_p + C_3 \int_0^t (t-s)^{-\alpha} h_4(s) e^{bs} \|w\|_p ds
$$
$$
+ C_8 \int_0^t (t-s)^{-\alpha} e^{b(1-l)s} (s^{r_1} h_1 + s^{r_3} h_3)(s) \left(e^{bs} \|G^\alpha_p w\|_p\right)^l ds
$$
$$
+ C_9 \int_0^t (t-s)^{-\alpha} e^{b(1-l)s} s^{r_2} h_2(s) \left(e^{bs} \|A^\alpha_p u\|_p\right)^l ds.
$$
The second term in the right-hand side of (4.9) may be estimated in the following manner, for $0 < \varepsilon < b$

$$\int_0^t (t-s)^{-\alpha} h_4(s) e^{b s} \|w\|_p \, ds = \int_0^t (t-s)^{-\alpha} e^{\varepsilon s} e^{(b-\varepsilon)s} h_4(s) \|w\|_p \, ds$$

$$\leq \left( \int_0^t (t-s)^{-\alpha \varepsilon} e^{\varepsilon q s} \, ds \right)^{\frac{1}{q}} \left( \int_0^t h_4^q(s) \left( e^{(b-\varepsilon)s} \|w\|_p \right)^{q^*} \, ds \right)^{\frac{1}{q^*}}$$

$$\leq C_{10} e^{\varepsilon t} \left( \int_0^t h_4^q(s) \left( e^{(b-\varepsilon)s} \|w\|_p \right)^{q^*} \, ds \right)^{\frac{1}{q^*}} \quad (4.10)$$

We have used the Hölder inequality, Lemma 7 and the embedding $D(G_1^\alpha) \subset L^p$ to derive the last inequalities in (4.10). Multiplying (4.8) and (4.9) by $e^{-\varepsilon t}$, setting

$$U(t) = e^{(b-\varepsilon)t} \|A^\alpha_P u\|_p, \quad W(t) = e^{(b-\varepsilon)t} \|G^\alpha_P w\|_p$$

and taking into account (4.10) we find for $t \geq \delta > 0$

$$U(t) \leq C_1 \delta^{-\alpha} \|u_0\|_p + C_6 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^\sigma_1 h_1(s) W(s)^l \, ds$$

$$+ C_7 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^\sigma_2 h_2(s) U(s)^l \, ds,$$

and

$$W(t) \leq C_3 \delta^{-\alpha} \|w_0\|_p + C_{10} \left( \int_0^t h_4^q(s) W(s)^{q^*} \, ds \right)^{\frac{1}{q^*}}$$

$$+ C_8 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} (s^\sigma_1 h_1 + s^\sigma_2 h_3)(s) W(s)^l \, ds$$

$$+ C_9 \int_0^t (t-s)^{-\alpha} e^{-(b-\varepsilon)(l-1)s} s^\sigma_2 h_2(s) U(s)^l \, ds.$$
and

\[ W(t) \leq C_3 \delta^{-\alpha} \|w_0\|_p + C_{10} \left( \int_0^t h_4^q(s) W(s) ds \right)^{\frac{1}{q'}} + C_{13} \delta^{-\alpha} \left( \int_0^t h_1^q(s) W(s) ds \right)^{\frac{1}{q'}} + C_{14} \delta^{-\alpha} \left( \int_0^t h_3^q(s) W(s) ds \right)^{\frac{1}{q'}} + C_{15} \delta^{-\alpha} \left( \int_0^t h_2^q(s) U(s) ds \right)^{\frac{1}{q'}}. \]

Now we use the inequality

\[(x_1 + x_2 + \ldots + x_n)^r \leq n^{r-1}(x_1^r + x_2^r + \ldots + x_n^r)\]

to get

\[ U(t)^{q^*} \leq 3^{q^*-1} \left( C_1 \delta^{-\alpha} \|u_0\|_p \right)^{q^*} + 3^{q^*-1} \left( C_{11} \delta^{-\alpha} \right)^{q^*} \int_0^t h_1^q(s) W(s) ds + 3^{q^*-1} \left( C_{12} \delta^{-\alpha} \right)^{q^*} \int_0^t h_2^q(s) U(s) ds, \quad (4.11) \]

and

\[ W(t)^{q^*} \leq 5^{q^*-1} \left( C_3 \delta^{-\alpha} \|w_0\|_p \right)^{q^*} + 5^{q^*-1} C_{10}^{q^*} \int_0^t h_4^q(s) W(s) ds + 5^{q^*-1} \left( C_{13} \delta^{-\alpha} \right)^{q^*} \int_0^t h_1^q(s) W(s) ds + 5^{q^*-1} \left( C_{14} \delta^{-\alpha} \right)^{q^*} \int_0^t h_3^q(s) W(s) ds + 5^{q^*-1} \left( C_{15} \delta^{-\alpha} \right)^{q^*} \int_0^t h_2^q(s) U(s) ds. \quad (4.12) \]

Putting \( F(t) = U(t)^{q^*} + W(t)^{q^*} \), we infer from (4.11) and (4.12) that

\[ F(t) \leq C_0 \left( \|u_0\|_p, \|w_0\|_p \right) + \int_0^t h(s)(F(s) + F(s)^t)ds, \quad t \geq \delta > 0 \]

where \( C_0 \left( \|u_0\|_p, \|w_0\|_p \right) = 3^{q^*-1} \left( C_1 \delta^{-\alpha} \|u_0\|_p \right)^{q^*} + 5^{q^*-1} \left( C_3 \delta^{-\alpha} \|w_0\|_p \right)^{q^*} \) and

\[ h(s) = \max \left\{ (C_{17} + C_{19})h_1^q(s), C_{20}^q h_3^q(s), (C_{18} + C_{21})h_2^q(s), C_{22} h_4^q(s) \right\} \]
where $C_i, i = 17, 18, \ldots, 22$ are the coefficients of the integral terms in (4.11) and (4.12) in the order.

Let $G(z) = \int_{z_0}^{z} \frac{dy}{y + y^l}$. Then, by Lemma 2.9 we may conclude that

\[
F(t) \leq G^{-1} \left[ G \left( C_0 \left( \|u_0\|_p, \|w_0\|_p \right) \right) + \int_0^t h(s) ds \right] \\
\leq C_0 \left( 1 + C_0^{l-1} \right) \frac{1}{\theta} \int_0^t \frac{h(s) ds}{e^\theta} \left[ 1 - \frac{C_0}{1 + C_0^{l-1}} e^{(l-1) \int_0^t h(s) ds} \right]^{\frac{1}{l-1}}.
\]

From our assumptions on $\|u_0\|_p, \|w_0\|_p$ and $h(t)$ we deduce that

\[
\|A_p^\alpha u\|_p \leq Ce^{-(b-\epsilon)t} \left( \|u_0\|_p + \|w_0\|_p \right) \tag{4.13}
\]

and

\[
\|G_p^\alpha w\|_p \leq Ce^{-(b-\epsilon)t} \left( \|u_0\|_p + \|w_0\|_p \right). \tag{4.14}
\]

The decay rates in $C^\mu(\overline{\Omega}), \mu \in [0, 2)$ follow from Lemma 2.7.

B. The decay rate of $\|v - v_\infty\|_{C^\alpha(\overline{\Omega})}$:

As in Hoshino [5], let us write

\[
v - v_\infty = (Q_0 v(t) - v_\infty) + Q_+ v(t),
\]

and estimate the terms in the right-hand side separately.

a. The estimation of $Q_0 v(t) - v_\infty$.

Integrating the second equation in (1.1) over $(0, t) \times \Omega$, we have

\[
\int_\Omega v(x, t) dx + \int_0^t \int_\Omega \{ (a_1(s) + a_3(s))w^m - a_2(s)u^n v^k \} dx ds = \int_\Omega v_0(x) dx.
\]

It appears then that

\[
|Q_0 v(t) - v_\infty| = |\Omega|^{-1} \int_t^\infty \int_\Omega \{ (a_1(s) + a_3(s))w^m - a_2(s)u^n v^k \} dx ds.
\]

In the rest of the proof, $C$ will denote a generic positive constant which may be different at different occurrences.

Using (4.7), (4.13), (4.14) and the Hölder inequality we see that

\[
|Q_0 v(t) - v_\infty| \\
\leq C \int_t^\infty \left\{ (s^{\sigma_1} h_1 + s^{\sigma_3} h_3)(s)e^{-l(b-\epsilon)s} + s^{\sigma_2} h_2(s) M^k e^{-l(b-\epsilon)s} \right\} ds \\
\leq C \int_t^\infty \left\{ \sum_{i=1}^{3} s^{\sigma_i} h_i(s) \right\} e^{-l(b-\epsilon)s} ds.
\]
We also used Lemma 7 in the last inequality.

b. The estimation of $Q_+v(t)$:

To estimate $Q_+v(t)$ let us apply $B^\alpha_{p^+}Q_+$ to the integral equation associated with the second equation in (1.1). We find for all $t \geq \delta > 0$

$$B^\alpha_{p^+}Q_+v(t) = B^\alpha_{p^+}e^{-(t-\delta)B^\alpha_{p^+}Q_+}v(\delta) + \int_\delta^t B^\alpha_{p^+}e^{-(t-s)B^\alpha_{p^+}Q_+} \left[ (a_1(s) + a_3(s))w^m - a_2(s)u^nw^k \right] ds.$$

Taking the $L^p$-norm and using the second inequality in Lemma 5, we obtain

$$\|B^\alpha_{p^+}Q_+v(t)\|_p \leq C_2(t-s)^{-\alpha}e^{-d_2(t-s)^{\lambda}} \|Q_+v(\delta)\|_p + C_2 \|Q_+\|_{L^p(\Omega)\to L^p(\Omega)} \int_\delta^t (t-s)^{-\alpha}e^{-d_2\lambda(t-s)} \left\{ (s^{\sigma_1}h_1 + s^{\sigma_3}h_3) \|w^m\| + s^{\sigma_2}h_2M^k \|u^n\| \right\} ds.$$

Next, using (4.7), (4.13) and (4.14) we see that for all $t \geq \delta + T$

$$\|B^\alpha_{p^+}Q_+v(t)\|_p \leq Ce^{-d_2\lambda t} \|Q_+v(\delta)\|_p + Ce^{-d_2\lambda t} \int_\delta^t (t-s)^{-\alpha}e^{d_2\lambda s}e^{-(l(\delta-s)s)} \left( \sum_{i=1}^{3} s^{\sigma_i}h_i(s) \right) ds$$

$$\leq Ce^{-d_2\lambda t} \left\{ \|Q_+v(\delta)\|_p + \int_0^{t-\delta} (t-\delta-s)^{-\alpha}e^{\left[ (l\delta - l\cdot(\delta-s))s \right]} \left( \sum_{i=1}^{3} (s+\delta)^{\sigma_i}h_i(s+\delta) \right) ds \right\}.$$

(4.16)

(i) If $d_2\lambda < lb$, then choose $\varepsilon$ such that $0 < l\varepsilon < lb - d_2\lambda$. Hence, we may apply Lemma 8, together with the Hölder inequality to get

$$\|B^\alpha_{p^+}Q_+v(t)\|_p \leq Ce^{-d_2\lambda t} \text{ for all } t \geq \delta + T.$$

(4.17)

(ii) If $d_2\lambda \geq lb$, then $l(b-\varepsilon) - d_2\lambda < 0$. Multiplying by $e^{\rho(s+\delta)}e^{-\rho(s+\delta)}$, with $\rho > d_2\lambda - l(b-\varepsilon)$, the integrand in (4.16), using the Hölder inequality and Lemma 8, we see that

$$\|B^\alpha_{p^+}Q_+v(t)\|_p \leq Ce^{-d_2\lambda t} \left\{ \|Q_+v(\delta)\|_p + e^{-\tilde{\rho}t} \right\}$$

$$\leq Ce^{-(d_2\lambda-\tilde{\rho}t)}.$$

The conclusion follows from (4.15), (4.17) and (4.18).
5. Concluding Remarks

Some remarks are in order:

- The condition \(a_1(t) \leq Ca_3(t)\) needed in Theorem 1.2 has not been used in Theorem 1.4. So, the decay rates hold provided one may prove a convergence result without this condition.
- The assumption \(\max\left\{1, \frac{N(m-1)}{2p\alpha}, \frac{N(n-1)}{2p\alpha}\right\} < \min\{m, n\}\) may be relaxed somewhat using different \(l_1\) and \(l_2\) and applying Lemma 9 with \(p = 2\).
- We also have exponential decay in (b) (ii) without the growth condition on \(\sigma_i\), \(i = 1, 2, 3\) in case \(\sigma_i = 0\), \(i = 1, 2, 3\).
- It is possible to obtain sharper estimates using comparison results by replacing the bounds \(M^k\) with \(v_{\infty} - Ce^{-(b-\varepsilon)t}\) in (4.3) and \(v_{\infty}^k + Ce^{-(b-\varepsilon)t}\) in (4.4), for large values of \(t\).

Acknowledgements

The authors would like to express their gratitude to the Editors for handling the paper and to the anonymous referee for the attentive reading and insightful suggestions which improved the manuscript. We also express our deep thanks to Bao Quoc Tang for providing us with the proof of Theorem 1.1 and for interesting discussions. The second author is very grateful to the IRC for Intelligent Manufacturing and Robotics in King Fahd University of Petroleum and Minerals for its continuous support through Project No: SB201006.

Author contributions All authors reviewed the manuscript.

Funding Funding information is not applicable/no funding was received.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. No data sets were generated or analyzed during the current study.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.
References

[1] Amann, H.: Global existence for semilinear parabolic systems. J. Reine Angew. Math. 360, 47–83 (1985)

[2] Fellner, K., Morgan, J., Tang, Bao, Tang, Bao Q.: Global classical solutions to quadratic systems with mass control in arbitrary dimensions. Ann. Inst. Henri Poincaré Anal. Non Linéaire 37, 281–307 (2020)

[3] Feng, W.: Coupled system of reaction–diffusion equations and applications in Carrier facilitated diffusion. Nonlinear Anal. 17(3), 285–311 (1991)

[4] Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, vol. 840. Springer-Verlag, Berlin-New York (1981)

[5] Hoshino, H.: On the convergence properties of global solutions for some reaction–diffusion systems under Neumann boundary conditions. Diff. Integll. Eqs. 9(4), 761–778 (1996)

[6] Hoshino, H., Yamada, Y.: Solvability and smoothing effect for semilinear parabolic equations. Funkc. Ekvac. 34, 475–494 (1994)

[7] Kahane, C.S.: On the asymptotic behavior of solutions of nonlinear parabolic systems under Robin type boundary conditions. Funkc. Ekvac. 26, 51–78 (1983)

[8] Khuddush, M., Prasad, K. Rajendra., Vidyasagar, K.V.: Infinitely many positive solutions for an iterative system of singular multipoint boundary value problems on time scales. Rend. Circ. Mat. Palermo 2(71), 677–696 (2022)

[9] Kirane, M., Tatar, N.-E.: Global existence and stability of some semilinear problems. Arch. Math. Brno Tomus 36, 1–12 (2000)

[10] Kirane, M., Tatar, N.-E.: Convergence rates for a reaction–diffusion system. Zeit. Anal. Anw. (J. Anal. Math.) 202, 347–357 (2000)

[11] Laamri, E.H., Pierre, M.: Global existence for reaction–diffusion systems with nonlinear diffusion and control of mass. Ann. Inst. Henri Poincaré Anal. Non Linéaire 34, 571–591 (2017)

[12] Laamri, E.H., Perthame, B.: reaction–diffusion systems with initial data of low regularity. J. Diff. Equa. 269, 9310–9335 (2020)

[13] Li, Y.: Integral representation bound of the true solution to the BVP of double-sided fractional diffusion advection reaction equation. Rend. Circ. Mat. Palermo 2(71), 407–428 (2022)

[14] Martin, R. H., Pierre , M.: Nonlinear reaction–diffusion systems, in Nonlinear Equations in the Applied Sciences, Math. Sci. Engrg., 185 W. F. Ames and C. Rogers, Editors, Academic Press, Boston, MA , pp. 363-398 (1992)

[15] Michalski, M. W.: Derivatives of non integer order and their applications, Dissertationes Mathematicae, Polska Akademia Nauk, Instytut Matematyczny, Warszawa (1993)

[16] Morgan, J.: Global existence for semilinear parabolic systems. SIAM J. Math. Anal. 20(5), 1128–1144 (1989)

[17] Pierre, M., Schmitt, D.: Blowup in reaction–diffusion systems with dissipation of mass. SIAM J. Math. Anal. 28, 259–269 (1997)

[18] Pinto, M.: Integral inequalities of Bihari-type and applications. Funkc. Ekvac. 33(3), 387–404 (1990)

[19] Rothe, F.: Global Solutions of reaction–diffusion Systems. Lecture Notes in Mathematics, vol. 1072. Springer-Verlag, Berlin (1984)
[20] Ruan, W.: Bounded solutions for reaction–diffusion systems with nonlinear boundary conditions. Nonlinear Anal. 14(12), 1051–1077 (1990)

[21] Slimani, A., Bouzettouta, L., Guesmia, A.: Existence and uniqueness of the weak solution for Keller-Segel model coupled with Boussinesq equations. Demonstr. Math. 54, 558–575 (2021)

[22] Wang, M.X.: Asymptotic behavior of solutions to some reaction–diffusion systems. Chin. J. Contemp. Math. 18(3), 249–260 (1997)

Mohamed Majdoub
Department of Mathematics, College of Science
Imam Abdulrahman Bin Faisal University
P.O. Box 1982
Dammam
Saudi Arabia
e-mail: mmajdoub@iau.edu.sa;
med.majdoub@gmail.com

and

Basic and Applied Scientific Research Center
Imam Abdulrahman Bin Faisal University
P.O. Box 1982
31441 Dammam
Saudi Arabia

Nasser-Eddine Tatar
Department of Mathematics, Interdisciplinary Research Center for Intelligent Manufacturing and Robotics
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia
e-mail: tatarn@kfupm.edu.sa

Received: June 20, 2022.
Revised: September 29, 2022.
Accepted: March 21, 2023.