ABP maximum principles for fully nonlinear integro-differential equations with unbounded inhomogeneous terms

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Abstract
Aleksandrov–Bakelman–Pucci maximum principles are studied for a class of fully nonlinear integro-differential equations of order $\sigma \in [2 - \varepsilon_0, 2)$, where $\varepsilon_0$ is a small constant depending only on given parameters. The goal of this paper is to improve an estimate of Guillen and Schwab (Arch Ration Mech Anal 206(1):111–157, 2012) in order to avoid the dependence on $L^1$ norm of the inhomogeneous term.

Mathematics Subject Classification 35R09 · 47G20

1 Introduction
In this paper, we study the fully nonlinear nonlocal equation of the form:

$$
\begin{cases}
\mathcal{M}^- u(x) \leq f(x) & \text{in } B_1, \\
u(x) \geq 0 & \text{in } \mathbb{R}^n \setminus B_1,
\end{cases}
$$

where by setting

$$
\delta(u, x, y) := u(x + y) + u(x - y) - 2u(x),
$$

we define the minimal fractional Pucci operator:

$$
\mathcal{M}^- u(x) := \inf\left\{ (2 - \sigma) \int_{\mathbb{R}^n} \frac{\delta(u, x, y) y^T A y}{|y|^{n+\sigma+2}} dy \right\},
$$

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Throughout this paper, we suppose that
\[ n \geq 2, \quad 0 < \lambda \leq n \Lambda, \]
and define \( B_r := \{ y \in \mathbb{R}^n : |y| < r \} \). Furthermore, \( Q_r \subset \mathbb{R}^n \) denotes the open cube with its center at 0 and its side-length \( r \). We also set \( B_r(x) := x + B_r \) and \( Q_r(x) := x + Q_r \).

We will also use the maximal Pucci operator defined by
\[
\mathcal{M}^+ u(x) := \sup_{\lambda \leq \text{Tr}(A) \text{ and } O \leq \Lambda \leq \text{Id}} \left\{ (2 - \sigma) \int_{\mathbb{R}^n} \delta(u, x, y) \frac{y^T A y}{|y|^{n+\sigma+2}} \, dy \right\}.
\]

The main purpose of this paper is to show the Aleksandrov–Bakelman–Pucci (ABP for short) maximum principle depending only on \( L^n \) norm of \( f^+ := \max\{f, 0\} \).

**Theorem 1.1** There exist constants \( \hat{C} > 0 \) and \( \epsilon_0 \in (0, 1) \) depending only on \( n, \lambda \) and \( \Lambda \) such that if \( u \in L^\infty(\mathbb{R}^n) \cap \text{LSC}(\mathbb{R}^n) \) is a viscosity supersolution of (1) with \( \sigma \in [2 - \epsilon_0, 2) \) and \( f \in C(\overline{B}_1) \), then it follows that
\[
- \inf_{B_1} u \leq \hat{C} \| f^+ \|_{L^n(\{u \leq 0\})}.
\]

The ABP maximum principle plays a fundamental role in the regularity theory of fully nonlinear equations (see [1, 2] for instance). The ABP maximum principle for the second order equation presents a bound for the infimum of supersolutions by \( L^n \) norm of inhomogeneous terms. In the case of nonlocal equations, the ABP maximum principle is investigated in [3, 5, 7] and references therein. More precisely, for viscosity solutions, Caffarelli and Silvestre prove that the maximum is estimated from above by Riemann sums of \( f \) instead of their \( L^n \) norms in [3]. A more quantitative version of the ABP maximum principle is established by Guillen and Schwab in [5]. For an application of results in [5], \( W^{\sigma, \lambda} \)-estimate is obtained by Yu [8], which was inspired by earlier works by [4, 6]. The ABP maximum principle for strong solutions of second order equations with integral terms is obtained by Mou and Świech although the nonlocal part is regarded as a lower order term in [7].

We shall recall the estimate shown by Guillen and Schwab from [5]:

**Theorem 1.2** There exists a constant \( C_0 = C_0(n, \lambda) \) such that if \( u \in L^\infty(\mathbb{R}^n) \cap \text{LSC}(\mathbb{R}^n) \) is a viscosity supersolution of (1) with \( \sigma \in (0, 2) \) and \( f \in C(\overline{B}_1) \), then it follows that
\[
- \inf_{B_1} u \leq C_0 \| f^+ \|_{L^\infty(\{u \leq 0\})}^{(2-\sigma)/2} \| f^+ \|_{L^n(\{u \leq 0\})}^{\sigma/2},
\]
where \( \Gamma_\sigma \) is defined in Sect. 2.

Here, we shall briefly explain our proof of Theorem 1.1. In [5], Guillen and Schwab introduce the fractional order envelope \( \Gamma_\sigma \) and its Riesz potential \( P \), which interprets the fractional order equation as a second order partial differential equation. Applying the ABP maximum principle for second order equations, they show that \( P \) is bounded from below by \( L^n \) norm of \( f \) as follow:
\[
- \inf P \leq \overline{C} \| f \|_{L^n(\{u \leq \Gamma_\sigma\})}.
\]
A more precise statement of the above is presented in Lemma 2.3. In order to estimate
\( \inf \Gamma_\sigma \) by \( \inf P \), they obtain the lower bound of
\[
|\{ \Gamma_\sigma \leq \inf \Gamma_\sigma/2 \} \cap (B_r(x_0) \setminus B_{r/2}(x_0))|,
\]
which is derived from a simple property of the equation. Here, \( x_0 \) is a point such that
\( \inf \Gamma_\sigma = \Gamma_\sigma(x_0) < 0 \), and \( r \in (0, -\Gamma(x_0)/(2f(x_0))) \). Hence it follows from the estimate that
\[
-\inf \Gamma_\sigma \leq C\|f^+\|_\infty^{(2-\sigma)/2}(\inf P)^{\sigma/2}.
\]

Theorem 1.2 follows from above two inequalities.

In our proof of Theorem 1.1, we introduce a new iteration procedure, which is based on
an argument to show the weak Harnack inequality in the regularity theory for fully non-
linear PDE originated by Caffarelli in [1], in order to obtain the bound of
\[
|\{ \Gamma_\sigma \leq \inf \Gamma_\sigma/2 \} \cap (Q_r(x_0) \setminus Q_{r/4}\sigma(x_0))|
\]
from the ABP maximum principle. Using this bound, one can show a new ABP maximum
principle from the original one. We inductively obtain a sequence of constants in ABP
maximum principles, which yields (2) by taking the limit of this sequence.

We notice that \{ \( u = \Gamma_\sigma \) \} is replaced by \{ \( u \leq 0 \) \} in (2) while \{ \( u = \Gamma_\sigma \) \} \subset \{ \( u \leq 0 \) \}. However (2) has enough information to prove Hölder continuity estimates.

**Remark 1.3** Although we will prove Theorem 1.1 under the assumption that \( f \in C(\overline{B_1}) \),
it remains to hold if we assume \( f \in C(B_1) \cap L^\infty(B_1) \). Indeed, this is observed as follows:
Since we may suppose that \( \inf_{B_1} u < 0 \), we can find \( x_0 \in B_1 \) such that \( \inf_{B_1} u = u(x_0) \). We
choose small \( \eta > 0 \) such that \( x_0 \in B_{1-\eta} \). Notice that \( f \in C(B_1) \cap L^\infty(B_1) \subset C(\overline{B_{1-\eta}}) \).
Applying Theorem 1.1 to \( u(x) - \inf_{\mathbb{R}^n \setminus B_{1-\eta}} u \), we have
\[
u(x_0) \leq -\inf_{\mathbb{R}^n \setminus B_{1-\eta}} u + \tilde{C}(1-\eta)^{\sigma-1}\|f\|_{L^\infty([u \geq 0] \cap B_{1-\eta})}.
\]

Letting \( \eta \to 0 \), we obtain (2).

## 2 Preliminaries

Throughout this paper, we let \( |\cdot| \) be the Euclidean norm.

For a measurable subset \( A \) of \( \mathbb{R}^n \), \( |A| \) is its Lebesgue measure, and \( \chi_A \) is its indicator
function. We denote by \( S^{n-1} \) the \( n-1 \) dimensional unit sphere. We write \( u^- := \max\{-u, 0\} \). We say \( Q \subset Q_1 \) is a dyadic cube if there exists \( m \in \mathbb{N} \) such that \( Q \) is
obtained by dividing \( Q_1 \) to \( 2^mn \) cubes, and \( \tilde{Q} \) is the predecessor of \( Q \) if \( Q \) is one of \( 2^n \) cubes
constructed from dividing \( \tilde{Q} \). We recall a lemma named the Calderón–Zygmund cube
decomposition.

**Lemma 2.1** (Lemma 4.2 in [2]) Let \( A \subset B \subset Q_1 \) be measurable sets and \( 0 < \delta < 1 \) such that

(a) \( |A| \leq \delta \),

(b) if \( Q \) is a dyadic cube such that \( |A \cap Q| > \delta|Q| \), then \( \tilde{Q} \subset B \), where
\( \tilde{Q} \) is the predecessor of \( Q \).
Then, $|A| \leq \delta |B|$.

We recall the definition of viscosity solutions. We say that $\varphi$ touches $u$ from below at $x$ in a neighborhood $U$ whenever

$$u(x) = \varphi(x) \quad \text{and} \quad u(y) \geq \varphi(y) \quad \text{for } y \in U.$$  

**Definition 2.2**

$u \in LSC(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a viscosity supersolution of $\mathcal{M}^-u = f$ in $B_1$ if whenever $\varphi$ touches $u$ from below at $x \in B_1$ in some neighborhood $U$ for $\varphi \in C^2(U)$,

$$v := \begin{cases} 
\varphi & \text{in } U \\
u & \text{in } \mathbb{R}^n \setminus U
\end{cases}$$

satisfies that $\mathcal{M}^-v(x) \leq f(x)$.

Hereafter we often use the following convention: “$u$ satisfies $\mathcal{M}^-u \leq f$” means “$u$ is a viscosity supersolution of $\mathcal{M}^-u = f$”.

We introduce some notations from [5]: for $\alpha \in (0, 2)$,

$$\mathcal{A}(\alpha) := \pi^{\alpha/2} \frac{\Gamma((n - \alpha)/2)}{\Gamma(\alpha/2)},$$

we note that $\mathcal{A}(\alpha)/\alpha$ converges to a positive constant as $\alpha \to 0$. Hence, we can find a constant $c_0 = c_0(n) > 0$ such that

$$\left(\frac{\sqrt{n}}{2}\right)^{n+2-\alpha} \cdot \frac{\mathcal{A}(2 - \sigma)}{8(1 - 4^{(2-\sigma)/n})} \geq c_0 \quad (\forall \sigma \in (0, 2)). \quad (4)$$

Let $v$ be a bounded function satisfying

$$\int_{\mathbb{R}^n} \frac{|\delta(v, x, y)|}{|y|^{n+\sigma}} \, dy < \infty \quad (x \in \mathbb{R}^n).$$

We define its fractional order hessian by

$$D^\sigma v(x) := \frac{(n + \sigma - 2)(n + \sigma)}{2} \mathcal{A}(2 - \sigma) \int_{\mathbb{R}^n} \frac{y \otimes y}{|y|^{n+\sigma+2}} \delta(v, x, y) \, dy,$$

which is a real symmetric $n \times n$ matrices valued function in $\mathbb{R}^n$. The first eigenvalue of $D^\sigma v$ is defined by

$$E_\sigma v(x) := \inf_{\tau \in S^{n-1}} \{(D^\sigma v(x)\tau) \cdot \tau\}.$$

We consider the fractional order envelope $\Gamma_\sigma$ of $u$, which is an analogue of the classical convex envelope, defined by

$$\Gamma_\sigma(x) := \sup\{v(x) : E_\sigma(v) \geq 0 \text{ in } B_3, \text{ and } v \leq -u^- \text{ in } \mathbb{R}^n\}.$$

We can see that $\Gamma_\sigma$ is the unique viscosity solution of the obstacle problem:
\[
\begin{aligned}
\begin{cases}
\min\{E_{\sigma}(\Gamma_{\sigma}), -u - \Gamma_{\sigma}\} = 0 & \text{in } B_3, \\
\Gamma_{\sigma} = 0 & \text{in } \mathbb{R}^n \setminus B_3.
\end{cases}
\end{aligned}
\]

We also consider the Riesz potential \( P \) of \( \Gamma_{\sigma} \),
\[
P(x) := A(2 - \sigma) \int_{\mathbb{R}^n} \frac{\Gamma_{\sigma}(y)}{|x - y|^{n-(2-\sigma)}} \, dy, \quad (x \in \mathbb{R}^n).
\]

The following estimate is proved in (6.1) in [5].

**Lemma 2.3** There exist \( C, R > 1 \) depending only on \( n \) and \( \lambda \) such that if \( u \) satisfies (1) with \( \sigma \in (0, 2) \) and \( f \in C(B_1^\lambda) \), then it follows that
\[
-\inf_{B_R} P \leq C\|f\|_{L^\infty(u=\Gamma_{\sigma})},
\]

where
\[
\begin{aligned}
\supp \eta & \subset B_{2/\sqrt{n}}, \\
\eta & \leq -2 \text{ in } Q_3 \text{ and } \|\eta\|_{\infty} \leq M_1, \\
\text{for every } \sigma \in (1, 2), \text{ we have } \mathcal{M}^+ \eta(x) & \leq M_2 \xi(x) \text{ in } \mathbb{R}^n,
\end{aligned}
\]

where \( \xi \) is a continuous function with support inside \( B_{1/4} \) and such that \( 0 \leq \xi \leq 1 \).

### 3 Proof of Theorem 1.1

Set
\[
\mu := \frac{1}{(64M_2^2\sqrt{n})^\pi}, \tag{6}
\]

where \( M_2 \) is the constant in Proposition 2.4. We then freeze small \( \varepsilon_0 \in (0, 1) \) such that for \( \sigma \in (1, 2) \),
\[
C_0 M_2^{2-\sigma/2} \leq \varepsilon_0^{-\sigma/(2n)}, \tag{7}
\]
\[
(256 \sqrt{n} \varepsilon_0^{-1/n})^\mu (\varepsilon_0^{-1/2})^{-\sigma-1} M_1^{1+\mu^{-1}\log 2} \leq \varepsilon_0^{-1/n}, \tag{8}
\]
\[
\varepsilon_0^{-1} \leq \varepsilon_0^{-1/n}, \tag{9}
\]
\[
2^\mu (\varepsilon_0^{-1/2} M_1^{1+\mu^{-1}\log 2} \varepsilon_0^{1/2n}) \leq 2^{-1}. \tag{10}
\]
where \( C_0, c_0 \) and \( \overline{C} \) are constants from Theorem 1.2, (4) and Lemma 2.3, respectively, and \( M_1 \) and \( M_2 \) are those from Proposition 2.4.

We will see that Theorem 1.1 holds true when
\[
\hat{C} := \frac{c_0^{1/n}}{C_0}
\]
in view of Lemma 3.1 below, by sending \( i \to \infty \) in (11).

Throughout this paper, we remind that this \( e_0 \in (0,1) \) satisfies (7)–(10) in the above.

Lemma 3.1 For any \( i = 0, 1, \ldots \) and \( u \) satisfying (1) with \( \sigma \in [2 - e_0, 2) \) and \( f \in C(\bar{B}_1) \), it follows that
\[
-\inf_{B_1} u \leq C_i \|f^+\|^{(2-\sigma)/2}_{L^{\infty}([u \leq 0])} \|f^+\|^{\sigma/2}_{L^\infty([u \leq 0])} + \epsilon_0^{1/n} \|f^+\|_{L^\infty([u \leq 0])},
\]
where \( C_0 \) is from Theorem 1.2, and \( C_{i+1} := 2^{-1} C_i = 2^{-(i+1)} C_0 \).

In the case of \( i = 0 \), (11) follows directly from (3).

Assuming (11) for all viscosity supersolutions of (1) when \( i = \tilde{i} - 1 \), we shall show (11) for those when \( i = \tilde{i} \).

First, we prove the following lemma:

Lemma 3.2 For \( h \in C(\bar{B}_{2\sqrt{n}}) \), we assume
\[
\|h\|_{L^{\infty}(B_{2\sqrt{n}})} \leq (C_{i-1} e_0^{\sigma/2n})^{-2/(2-\sigma)} =: L_{i-1},
\]
and
\[
\|h\|_{L^s(B_{2\sqrt{n}})} \leq \frac{\epsilon_0^{1/n}}{64\sqrt{n}} =: M_3.
\]

Let \( v \in L^\infty(\mathbb{R}^n) \cap LSC(\mathbb{R}^n) \) be any nonnegative viscosity supersolution of
\[
\mathcal{M}^- v(x) \leq h(x) \quad \text{in } B_{2\sqrt{n}}.
\]
If \( v \) satisfies
\[
\inf_{Q_1} v \leq 1,
\]
then it follows that
\[
|\{v \leq M_1\} \cap Q_1| > \mu_0,
\]
where \( M_1 \) is the constant in Proposition 2.4, and \( \mu \) is defined by (6).

**Proof** Let \( \eta \) be the function in Proposition 2.4. We observe that \( w := v + \eta \) satisfies
\[
\begin{align*}
\mathcal{M}^- w(x) &\leq h(x) + M_2 \xi(x) =: g(x) \quad \text{in } B_{2\sqrt{n}}, \\
w(x) &\geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\sqrt{n}}.
\end{align*}
\]
Since \( \eta \leq -2 \) in \( Q_3 \), and (14), we obtain \( -\inf_{B_{2\sqrt{n}}} w \geq 1 \).

From the definition of \( \tilde{i} \), we can apply (11) with \( i = \tilde{i} - 1 \) to \( w(2\sqrt{n}x) \), to derive
Let \( v \) be as in Lemma 3.2. Although the next lemma is rather standard, we give a proof for the reader’s convenience.

By recalling that

\[
\text{supp } \xi \subset B_{1/4},
\]

we obtain

\[
\inf_{B_{2n}} w \leq C_{i-1} (2\sqrt{n})^{\sigma/2} \|g\|_{L^\infty(v \leq 0)}^{(2-\sigma)/2} \|g\|_{L^\infty(v \leq 0)}^{\sigma/2} + \frac{(2\sqrt{n})^{\sigma-1}}{\epsilon_0^{1/n}} \|g\|_{L^\infty(v \leq 0)}.
\]

Since \( C_{i-1} = 2^{-i+1} C_0 \leq C_0 \), by hypotheses (12) and (7), we have

\[
C_{i-1} (2\sqrt{n})^{\sigma/2} \|g\|_{L^\infty(v \leq 0)}^{(2-\sigma)/2} \leq C_{i-1} (2\sqrt{n})^{\sigma/2} (\|f\|_{L^\infty(v \leq 0)} + M_2^{2-\sigma/2})
\]

\[
\leq (2\sqrt{n})^{\sigma/2} (\epsilon_0^{-\sigma/2n} + C_0 M_2^{2-\sigma/2})
\]

\[
\leq 2(2\sqrt{n})^{\sigma/2} \epsilon_0^{-\sigma/2n}
\]

Hence, we have

\[
1 \leq 2 \left( (2\sqrt{n})^{\sigma/2} \epsilon_0^{-1/n} \|g\|_{L^\infty(v \leq 0)} \right)^{\sigma/2} + (2\sqrt{n})^{\sigma-1} \epsilon_0^{-1/n} \|g\|_{L^\infty(v \leq 0)}. 
\]  

(16)

Now, we claim

\[
\|g\|_{L^\infty(v \leq 0)} > \frac{\epsilon_0^{1/n}}{32\sqrt{n}}.
\]

Indeed, if not, we have a contradiction because it follows from (16) that

\[
1 \leq 2 \left( \frac{2\sqrt{n}}{32\sqrt{n}} \right)^{\sigma/2} + \frac{(2\sqrt{n})^{\sigma-1}}{32\sqrt{n}} \leq \frac{9}{16}.
\]

Hence, since \( 0 \leq \xi \leq 1 \), \( \text{supp } \xi \subset B_{1/4} \), by assumption (13), we have

\[
\frac{\epsilon_0^{1/n}}{64\sqrt{n}} < \|g\|_{L^\infty(v \leq 0)} \leq \|f\|_{L^\infty(B_1)} + M_2 \|\xi\|_{L^\infty(v \leq 0)}
\]

\[
\leq \frac{\epsilon_0^{1/n}}{64\sqrt{n}} + M_2 |\{v \leq 0\} \cap B_{1/4}|^{1/n}.
\]

Therefore, we obtain

\[
\frac{\epsilon_0}{(64M_2\sqrt{n})^{n}} < |\{v \leq 0\} \cap B_{1/4}| \leq |\{u \leq M_1\} \cap Q_1|.
\]

By recalling that \( \mu = (64M_2\sqrt{n})^{-n} \) is given in (6), (15) is now proved.

Although the next lemma is rather standard, we give a proof for the reader’s convenience.

**Lemma 3.3** Let \( v \) be as in Lemma 3.2. Then

\[
|\{v > M^k\} \cap Q_1| \leq (1 - \mu_0)^k
\]

for \( k = 1, 2, \ldots \), where \( M_1 \) and \( \mu \) are constants from Proposition 2.4 and (6), respectively.

**Proof** For \( k = 1 \), (17) is trivial due to (15). Suppose that (17) holds for \( k - 1 \). Let

\[
A := \{v > M^k\} \cap Q_1, \quad B := \{v > M^{k-1}\} \cap Q_1.
\]

(17) will be proved if we show that
Proof of Lemma 3.1.
We will prove (11) with $\mu_0$. Hence
\[
|A| \leq (1 - \mu_0)|B| \tag{18}
\]
and (18) will be deduced by applying Lemma 2.1. We need to check conditions there. Clearly $A \subset B \subset Q_1$ and $|A| \leq |\{v > M\} \cap Q_1| \leq (1 - \mu_0)$. We now prove that if $Q$ is a dyadic cube such that
\[
|A \cap Q| \geq (1 - \mu_0)|Q|, \tag{19}
\]
then $\tilde{Q} \subset B$, where $\tilde{Q}$ is the predecessor of $Q$. If not, there exists some $Q = Q_{1/2^j}(x^*)$ satisfying (19), and $\tilde{Q} \not\subset B$. We then find $\tilde{x} \in \tilde{Q}$ such that $v(\tilde{x}) \leq M_1^{k-1}$.

Setting functions
\[
\tilde{v}(x) := \frac{v(x^* + 2^{-j}x)}{M_1^{k-1}} \quad \text{and} \quad \tilde{h}(x) := \frac{h(x^* + 2^{-j}x)}{2^j M_1^{k-1}},
\]
we observe that $\tilde{v}$ satisfies $\mathcal{M}^{-}\tilde{v} \leq \tilde{h}$ in $B_{2\sqrt{n}}$. Notice $\tilde{v} \geq 0$ by definition. Since $\tilde{x} \in Q_{3/2^j}(x^*)$, we have $\inf_{\tilde{Q}} \tilde{v} \leq 1$. Moreover, we have
\[
\|\tilde{h}\|_{L^{\infty}(B_{2\sqrt{n}})} \leq \frac{\|h\|_{L^{\infty}(B_{2\sqrt{n}})}}{2^j M_1^{k-1}} \leq \|h\|_{L^{\infty}(B_{2\sqrt{n}})} \leq L_1^{-1}
\]
and
\[
\|\tilde{h}\|_{L^q(B_{2\sqrt{n}})} \leq \frac{2\|h\|_{L^q(B_{2\sqrt{n}})}}{2^j M_1^{k-1}} \leq \|h\|_{L^q(B_{2\sqrt{n}})} \leq M_3.
\]
Since $\tilde{v}$ satisfies the hypotheses of Lemma 3.2, it follows that
\[
\mu_0 \leq |\{\tilde{v} \leq M_1\} \cap Q_1| = 2^n |\{v \leq M_1^k\} \cap Q_{1/2^j}(x^*)|
\]
Hence $|Q \setminus A| > \mu_0 |Q|$, which contradicts (19). \qed

Proof of Lemma 3.1
We will prove (11) with $\tilde{i}$ for any viscosity supersolution $u$ of (1). Let $\tilde{g} \in C_0(\mathbb{R}^n)$ be an arbitrary function satisfying $\tilde{g} \geq f^+ \chi_{\{u \leq 0\}} \cap B_1$ in $\mathbb{R}^n$. Without loss of generality, we can assume that $u$ satisfies
\[
\mathcal{M}^{-} u \leq f^+ \chi_{\{u \leq 0\}} \cap B_1 \leq \tilde{g} \quad \text{in} \; \mathbb{R}^n,
\]
because $-u^-$ satisfies the above, instead of $u$. We shall show
\[
-\inf_{B_1} u \leq C_i \|\tilde{g}\|_{L^\infty(\mathbb{R}^n)} \|\tilde{g}\|_{L^q(\mathbb{R}^n)} \|\tilde{g}\|_{L^p(\mathbb{R}^n)} \|\tilde{g}\|_{L^q(\mathbb{R}^n)} + \varepsilon_0 \|\tilde{g}\|_{L^q(\mathbb{R}^n)},
\]
where $\|\tilde{g}\|_{L^\infty(\mathbb{R}^n)} := \|\tilde{g}\|_{L^\infty(\mathbb{R}^n)}$ and $\|\tilde{g}\|_{L^p(\mathbb{R}^n)} := \|\tilde{g}\|_{L^p(\mathbb{R}^n)}$. By taking the infimum of $\tilde{g}$ so that $f^+ \chi_{\{u \leq 0\}} \leq \tilde{g}$, this inequality implies (11) when $i = \tilde{i}$ for $u$.

Since we may suppose $\inf_{B_1} u < 0$, we can find $x_0 \in B_1$ such that
\[
\inf_{B_1} u = u(x_0) < 0 \quad (x_0 \in B_1).
\]
For $r \in (0, 1)$, setting
\[ N_0 := \frac{r^\alpha}{L_{i-1}} \| \tilde{g} \|_\infty + \frac{r^{\alpha-1}}{M_3} \| \tilde{g} \|_n, \]

where \( L_{i-1} \) and \( M_3 \) are constants from Lemma 3.2, we apply Lemma 3.3 to

\[
u_r(x) := \frac{u(x_0 + rx) - u(x_0)}{N_0} \quad \text{and} \quad h_r(x) := \frac{r^\alpha \tilde{g}(x_0 + rx)}{N_0}
\]

to obtain that for \( k = 1, 2, \ldots, \)

\[(1 - \mu \xi_0)^k r^\nu \geq r^n \{ u_r > M_1^k \} \cap Q_1
\]

\[= \{ u > u(x_0) + M_1^k N_0 \} \cap Q_r(x_0). \quad (20)\]

We choose a large \( k_0 \in \mathbb{N} \) such that

\[(1 - \mu \xi_0)^{k_0} \leq \frac{1}{2} < (1 - \mu \xi_0)^{k_0-1}. \quad (21)\]

Next, setting \( r_0 := \min\{s_1, s_2, 1\} \), where

\[s_1 := \left( \frac{-u(x_0)M_3}{4M_1^{k_0} \| g \|_n} \right)^{1/(\sigma-1)} \quad \text{and} \quad s_2 := \left( \frac{-u(x_0)L_{i-1}}{4M_1^{k_0} \| \tilde{g} \|_\infty} \right)^{1/\sigma},
\]

we easily observe that for any \( 0 < r \leq r_0, \)

\[M_1^{k_0} N_0 \leq M_1^{k_0} \left( \frac{s_2^2 \| g \|_\infty}{L_{i-1}} + \frac{s_1^{\sigma-1} \| \tilde{g} \|_n}{M_3} \right) = - \frac{u(x_0)}{2}.
\]

Thus, (20) together with (21) yields

\[\frac{r^n}{2} \geq \left\{ \left\{ u > \frac{u(x_0)}{2} \right\} \cap Q_r(x_0) \right\} \quad (0 < r \leq r_0). \quad (22)\]

Therefore, we have

\[\left\{ \left\{ u \leq \frac{u(x_0)}{2} \right\} \cap Q_r(x_0) \right\} \geq \frac{r^n}{2}.
\]

Let \( r_l := 4^{-1/n} r_0 \). For each \( l = 0, 1, \ldots, \), we have

\[\left\{ \left\{ u \leq \frac{u(x_0)}{2} \right\} \cap Q_r(x_0) \right\} \geq \frac{r^n}{2} - |Q_{l+1}| = \frac{r^n}{4}.
\]

Note that \( \Gamma_\sigma \leq u \) in \( \mathbb{R}^n \) and \( \Gamma_\sigma(x_0) = u(x_0) \). Hence, setting

\[A_l := \left\{ \Gamma_\sigma \leq \frac{\Gamma_\sigma(x_0)}{2} \right\} \cap \left( Q_r(x_0) \setminus Q_{l+1}(x_0) \right),
\]

we see that \( |A_l| \geq r^n/4 \) for \( l = 0, 1, \ldots \). Noting \( |y| \leq 2^{-1/\sqrt{n r_l}} \) for \( y \in Q_n \), we thus calculate as follows:
\[-P(x_0) = \mathcal{A}(2 - \sigma) \int_{\mathbb{R}^n} -\Gamma_\sigma(x_0 + y) |y|^{-n+2-\sigma} dy\]
\[\geq \mathcal{A}(2 - \sigma) \sum_{l=0}^{\infty} \int_{\{x_0+y\in A_l\}} \frac{1}{2} \Gamma_\sigma(x_0) |y|^{-n+2-\sigma} dy\]
\[\geq \mathcal{A}(2 - \sigma) \left( \frac{\sqrt{n}}{2} \right)^{-n+2-\sigma} \sum_{l=0}^{\infty} |A_l| r_l^{-n+2-\sigma} / 8\]
\[\geq \left( \frac{\sqrt{n}}{2} \right)^{-n+2-\sigma} \frac{-\Gamma_\sigma(x_0) A(2 - \sigma) r_0^{-2-\sigma}}{8 (1 - 4^{-2-\sigma}) / n}\]
\[\geq - c_0 \Gamma_\sigma(x_0) r_0^{-2-\sigma},\]

where \(c_0\) is from (4). Using (5) and (23), we obtain
\[- \inf_{B_1} u = -\Gamma_\sigma(x_0) \leq - c_0^{-1} r_0^{-2-\sigma} P(x_0) \leq c_0^{-1} \mathcal{C} r_0^{-2-\sigma} \|f^+\|_{L^\infty(u=\Gamma_\sigma)} \]
\[\leq c_0^{-1} \mathcal{C} r_0^{-2-\sigma} \|\hat{g}\|_n.\]

We shall consider three cases of \(r_0 > 0\). If we suppose \(r_0 = 1\), then (2) holds true for \(\hat{C} = c_0^{-1} \mathcal{C}\). Thus, due to (9), we conclude the proof.

We next assume \(r_0 = s_1\). In this case, it follows that
\[- \inf_{B_1} u \leq c_0^{-1} \mathcal{C} \left( \frac{4M_1^{k_0} \|\hat{g}\|_n}{-M_3 \inf_{B_1} u} \right)^{(2-\sigma)(\sigma-1)-1} \|\hat{g}\|_n,\]
which implies
\[- \inf_{B_1} u \leq (c_0^{-1} \mathcal{C})^{\sigma-1} (4M_1^{k_0} M_3^{-1})^{2-\sigma} \|\hat{g}\|_n.\]

Because of \(\varepsilon_0 \geq 2 - \sigma\) and (21), we have
\[k_0(2 - \sigma) \leq k_0 \varepsilon_0 \leq \varepsilon_0 - \frac{\varepsilon_0 \log 2}{\log(1 - \mu \varepsilon_0)} \leq \varepsilon_0 + \frac{\log 2}{\mu},\]
where the last inequality is valid because \(z \leq - \log(1 - z)\) for \(0 \leq z < 1\).

Recalling that \(M_3 = (64 \sqrt{n})^{-1} \varepsilon_0^{1/n}\), by (8), we conclude that
\[- \inf_{B_1} u \leq (256 \sqrt{n})^{-1} \varepsilon_0^{1/n} (c_0^{-1} \mathcal{C})^{\sigma-1} M_1^{1 + \mu^{-1} \log \varepsilon_0} \|\hat{g}\|_n \leq \varepsilon_0^{-1/n} \|\hat{g}\|_n.\]  
(24)

In the case of \(r_0 = s_2\), we have
\[- \inf_{B_1} u \leq c_0^{-1} \mathcal{C} \left( \frac{4M_1^{k_0} \|\hat{g}\|_\infty}{-L_{-1} \inf_{B_1} u} \right)^{(2-\sigma)/\sigma} \|\hat{g}\|_n.\]

Hence
\[- \inf_{B_1} u \leq (c_0^{-1} \mathcal{C})^{\sigma/2} (4M_1^{k_0} L_{-1}^{-1})^{(2-\sigma)/2} \|\hat{g}\|_\infty \|\hat{g}\|_n.\]
Recall that $L_{i-1} = (C_{i-1}e_0^{2/2n})^{-2/(2-\sigma)}$. It follows from (10) that

$$-\inf_{B_1} u \leq 2^{i_0} (e_0^{-1} C)^{\sigma/2} M_{i+1}^{\sigma/2} \log^2 2e_0^{\sigma/2n} C_{i-1} \|\tilde{g}\|_{L^{\infty}}^{(2-\sigma)/2} \|\tilde{g}\|_{L^{n}}^{\sigma/2}$$

$$\leq 2^{i_0} C_{i-1} \|\tilde{g}\|_{L^{\infty}}^{(2-\sigma)/2} \|\tilde{g}\|_{L^{n}}^{\sigma/2}$$

$$= C_i \|\tilde{g}\|_{L^{\infty}}^{(2-\sigma)/2} \|\tilde{g}\|_{L^{n}}^{\sigma/2}. \quad (25)$$

In any case, from (24) and (25), we have

$$-\inf_{B_1} u \leq C_i \|\tilde{g}\|_{L^{\infty}}^{(2-\sigma)/2} \|\tilde{g}\|_{L^{n}}^{\sigma/2} + e_0^{-1/n} \|\tilde{g}\|_{L^{n}}.$$

Taking the infimum of $\tilde{g}$ satisfying $\tilde{g} \geq f^+ \chi_{\{u \leq 0\} \cap B_1}$ in $\mathbb{R}^n$, (11) is proved with $i$. \hfill \Box

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