Data-Driven Inverse of Linear Systems and Application to Disturbance Observers

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Abstract—This work develops a data-based construction of inverse dynamics for LTI systems. Specifically, the problem addressed here is to find an input sequence from the corresponding output sequence based on pre-collected input and output data. The problem can be considered as a reverse of the recent use of the behavioral approach, in which the output sequence is obtained for a given input sequence by solving an equation formed by pre-collected data. The condition under which the problem gives a solution is investigated and turns out to be $L$-delay invertibility of the plant and a certain degree of persistent excitation of the data input. The result is applied to form a data-driven disturbance observer. The plant dynamics augmented by the data-driven disturbance observer exhibits disturbance rejection without the model knowledge of the plant.

I. INTRODUCTION

Seminal work of the behavioral approach [1] for dynamic systems has been receiving growing attention in recent years [2]–[4]. Instead of using state space matrices for linear dynamics, which are often obtained from first principles on the target dynamics, the behavioral approach uses a set of equations to determine if a pair of input and output belongs to the target dynamics. The equations involve Hankel matrices for linear dynamics, which are obtained by input/output trajectories that satisfy a condition of persistency of excitation. This approach is suitable when data are available but applying first principles for modeling is not straightforward. Recent advances in this area include [3], [5]–[7].

Particularly, as given in [8], the fundamental result of this approach is that we can determine a future output for a given input using Hankel matrices built by using previously collected input and output data. No identification of system matrices (typically $A$, $B$, $C$, $D$) is necessary. A question that this paper is concerned is the inverse of what is just described: can we determine the input that generated a given output using previously collected input and output data? In other words, the question pertains to finding conditions and methods how we can build an inverse dynamics of the given system using collected data. If this is successfully carried out, an immediate application is to build a disturbance observer [9], [10] with data, that identifies the disturbance that affected the output. Another application is to analyze if a desired output is feasible when input is constrained.

Similar questions have been asked and answered in [8], [11]. However, the result is restricted to the case where $D$ is invertible. This work does not assume the invertibility of $D$. Instead the notion of $L$-delay invertibility is invoked from the literature [12].

This paper is organized as follows: Section II provides a brief review of the invertibility of discrete-time LTI systems. Section III gives main results. An application of disturbance observer is given in Section IV. Section V concludes the paper.

II. REVIEW OF LTI SYSTEM INVERTIBILITY

Consider a discrete-time linear-time-invariant system

\begin{align}
  x(k + 1) &= Ax(k) + Bu(k) \\
  y(k) &= Cx(k) + Du(k)
\end{align}

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and $k \in \mathbb{Z}$ is the discrete time index. Then the transfer function matrix of given system (1) is given by $G(z) = C(zI_n - A)^{-1}B + D$.

Definition 1 (see [12]). A proper transfer function matrix $\hat{G}(z)$ is an $L$-delay inverse system for given system (1), where $L$ is a nonnegative integer, if

\begin{equation}
  \hat{G}(z)G(z) = \frac{1}{z^L}I_m.
\end{equation}

Note that $G(z)$ is an $p \times m$ transfer function matrix and $\hat{G}(z)$ is a $m \times p$ transfer function matrix. Thus, a necessary condition for (2) is that $m \leq p$.

Definition 2 (see [12]). The system (1) is invertible if it has an $L$-delay inverse for some finite $L$. The least integer $L$ for which an $L$-delay inverse exists will be called the inherent delay of the invertible system and is denoted by $L_0$. 

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For single input and single output systems of (1), the inherent delay $L_0$ is the relative degree. Define, for an integer $t > 0$,

$$T_t := \begin{bmatrix} D & 0_{p \times m} & 0_{p \times m} & \cdots & 0_{p \times m} \\ CB & D & 0_{p \times m} & \cdots & 0_{p \times m} \\ CAB & CB & D & \cdots & 0_{p \times m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{t-1}B & CA^{t-2}B & CA^{t-3}B & \cdots & D \end{bmatrix}$$

(3)

In addition, define $T_0 = D$. Then, the following theorem gives the necessary and sufficient condition for the existence of an $L$-delay inverse for the system of (1).

**Theorem 1** (see [12]). The following are equivalent:

1. The system (1) has an $L$-delay inverse.
2. There exists $L$ such that $\text{rank}(T_t) - \text{rank}(T_{t-1}) = m$, where $\text{rank}(T_{t-1}) = 0$.
3. The rank of the matrix $G(z)$ is $m$.

As a result of Theorem 1, if the system has $L$-delay inverse, then the first $m$ columns of $T_L$ are independent.

### III. DATA-BASED CONSTRUCTION OF INVERSE SYSTEMS

We introduce notations commonly used in the literature for data-driven system and control. For a given signal $f(k) \in \mathbb{R}^q$, $k \in \mathbb{Z}$, and $T \in \mathbb{N}$, let $f_{[k,k+T]}$ be defined by

$$f_{[k,k+T]} := \begin{bmatrix} f(k) \\ f(k+1) \\ \vdots \\ f(k+T) \end{bmatrix} \in \mathbb{R}^{q(T+1)}.$$

Using the vectors $u_{[k,k+L]}$ and $y_{[k,k+L]}$, the system of (1) can be written as

$$y_{[k,k+L]} = O_L x(k) + T_L u_{[k,k+L]}$$

(5)

where the matrix $O_L$ is defined by

$$O_L = \left[ C^T (CA)^T \cdots (CA^L)^T \right]^T.$$  

(6)

When the $L$-delay inverse exists for the system of (1), there exists an $m \times p(L+1)$ matrix $\mathcal{K}$ satisfying

$$\mathcal{K} T_L = \begin{bmatrix} I_m & 0_{m \times mL} \end{bmatrix}.$$  

Such a matrix $\mathcal{K}$ exists due to Rouché-Capelli theorem [13] with the fact that

$$\text{rank}(T_L) = \text{rank} \left( \begin{bmatrix} T_L \\ I_m & 0_{m \times mL} \end{bmatrix} \right).$$

which follows from the fact that the first $m$ columns of $T_L$ are independent. Pre-multiplying $\mathcal{K}$ to both sides of (5) yields

$$u(k) = -\mathcal{K} O_L x(k) + \mathcal{K} y_{[k,k+L]},$$

(7)

which represents the input $u(k)$ by the state and sequence of outputs. State update equation with respect to $y_{[k,k+L]}$ is obtained by substituting (7) in (1a). Then, the following dynamics is obtained:

$$x(k+1) = \tilde{A} x(k) + \tilde{B} y_{[k,k+L]}$$

(8a)

$$u(k) = \tilde{C} x(k) + \tilde{D} y_{[k,k+L]}$$

(8b)

where $\tilde{A} = A - BK \mathcal{O}_L$, $\tilde{B} = BK$, $\tilde{C} = -\mathcal{K} \mathcal{O}_L$, and $\tilde{D} = \mathcal{K}$.

In order to provide more specific arguments, we first remind readers of Hankel matrix and the notion of persistent excitation (PE) for a signal $f(k) \in \mathbb{R}^q$. Denote by $\mathcal{H}_t(f_{[i,j]})$ the $q \times (j - i - t + 2)$ dimensional Hankel matrix constructed using the values of $f(i), \ldots, f(j)$:

$$\mathcal{H}_t(f_{[i,j]}) = \begin{bmatrix} f(i) & f(i + 1) & \cdots & f(j - t + 1) \\ f(i + 1) & f(i + 2) & \cdots & f(j - t + 2) \\ \vdots & \vdots & \ddots & \vdots \\ f(i + t - 1) & f(i + t) & \cdots & f(j) \end{bmatrix}.$$  

The notion of persistency of excitation of a segment from a signal is defined as follows.

**Definition 3** (see [1]). The vector $f_{[0,T-1]} = \mathbb{R}^{qT}$ from signal $f(k) \in \mathbb{R}^q$ is persistently exciting of order $L$ if the matrix $\mathcal{H}_t(f_{[0,T-1]}) \in \mathbb{R}^{qt \times (T-t+1)}$ has full row rank.

As hinted by (8), data-based construction of inverse dynamics may involve input and output sequences of different length.

**Definition 4.** A pair of vectors $u_{[0,T-1]} \in \mathbb{R}^{mT}$ and $y_{[0,T-1+L]} \in \mathbb{R}^{p(T+L)}$ is a trajectory of the system (1) if and only if there exists $u_{[T,T-1+L]}$ such that the pair $u_{[0,T-1+L]}$ and $y_{[0,T-1+L]}$ is a length $T+L$ trajectory of the system (1).

Now, for some $T > 0$ and $L \geq 0$, denote the input and output data collected from an experiment for the system (1) by $u_{[0,T+L-1]}$ and $y_{[0,T+L-1]}$, respectively.

**Lemma 1.** Let the system (1) be controllable. Assume that $u_{[0,T-1]}$ is persistently exciting of order $n + t + L$. Then, for any $t_0$, a pair of length $t$ input $u_{[t_0,t_0+t-1]}$ and length $t + L$ output $y_{[t_0,t_0+t+L-1]}$ is a trajectory of (1) if and only if there is $g \in \mathbb{R}^T$ such that

$$\begin{bmatrix} u_{[t_0,t_0+t-1]} \\ y_{[t_0,t_0+t+L-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_t(u_{[0,T-1]}) \\ \mathcal{H}_{t+L}(y_{[0,T+L-1]}) \end{bmatrix} g.$$  

(9)
Proof. Let \( u_{[t_0,t_0+t-1]} \) and \( y_{[t_0,t_0+t+L-1]} \) be a trajectory of (1). Then, by Definition 4, there exists \( u_{[t_0+t_0+t+L-1]} \) such that the pair \( u_{[t_0,t_0+t+L-1]} \) and \( y_{[t_0,t_0+t+L-1]} \) is a trajectory of (1). Then, by [5, Theorem 1] there exists \( g \) such that

\[
\begin{bmatrix}
  u_{[t_0,t_0+t+L-1]} \\
  y_{[t_0,t_0+t+L-1]}
\end{bmatrix} = \begin{bmatrix}
  \mathcal{H}_{t+L}(u^d_{[0,T+L-1]}) \\
  \mathcal{H}_{t+L}(y^d_{[0,T+L-1]})
\end{bmatrix} g.
\]

(10)

Hence (9) is satisfied. Conversely, let \( u_{[t_0,t_0+t-1]} \) and \( y_{[t_0,t_0+t+L-1]} \) satisfy (9) for some \( g \). Set the subsequent sequence of \( u \) as

\[
u_{[t_0+t_0+t+L-1]} = \mathcal{H}_L(u^d_{[t,T+L-1]})^T g.
\]

(11)

Then, \( u_{[t_0,t_0+t+L-1]} \) and \( y_{[t_0,t_0+t+L-1]} \) satisfy (10). Again due to [5, Theorem 1] the pair is the trajectory of (1), and due to Definition 4, the pair \( u_{[t_0,t_0+t-1]} \) and \( y_{[t_0,t_0+t+L-1]} \) is a trajectory of (1).

Denote by \( U_p, U_f, Y_p \) and \( Y_{f_L} \) the following four Hankel matrices formed by the data \( u^d_{[0,T+L-1]}, \)
\( y^d_{[0,T+L-1]} \) for some \( T_p > 0 \) and \( T_f > 0 \) satisfying \( T > T_p + T_f - 1 \)

\[
U_p = \mathcal{H}_{T_p}(u^d_{[0,T_p-T_f-1]}), \quad U_f = \mathcal{H}_{T_f}(u^d_{[0,T_p-T_f-1]}), \quad Y_p = \mathcal{H}_{T_p}(y^d_{[0,T_p-T_f-1]}), \quad Y_{f_L} = \mathcal{H}_{T_f}(y^d_{[0,T_p-T_f-1]}).
\]

Lemma 2. Let the system of (1) be observable and have an L-delay inverse with \( L \geq 0 \). Assume that \( T_p \) be greater than or equal to the observability index of (1). Then,

\[
\mathcal{N}
\begin{bmatrix}
  U_p \\
  Y_p \\
  U_f \\
  Y_{f_L}
\end{bmatrix}
= \mathcal{N}
\begin{bmatrix}
  U_p \\
  Y_p \\
  Y_{f_L}
\end{bmatrix}
\]

(12)

where \( \mathcal{N}(A) \) is the null space of matrix \( A \).

Proof. Denote \( [x(T_p), x(T_p+1), \ldots, x(T_f+L)] \) by \( X \), and define \( v(k) = y^f_{[k,k+L]} \) for the simplicity of derivation. First, we show that \( U_f \) is determined by \( X \) and \( \mathcal{H}_{T_f}(v|_{T_p,T_f-1}) \) by (7) and (8) as

\[
U_f = \begin{bmatrix}
  \hat{C}A \\
  \vdots \\
  \hat{C}A^{T_f-1}
\end{bmatrix} X + \begin{bmatrix}
  \hat{D} & 0_{m \times p(L+1)} & \ldots & 0_{m \times p(L+1)} \\
  \hat{C}B & \hat{D} & \ldots & 0_{m \times p(L+1)} \\
  \vdots & \ddots & \ddots & \vdots \\
  \hat{C}A^{T_f-2}B & \hat{C}A^{T_f-3}B & \ldots & \hat{D}
\end{bmatrix} \times \mathcal{H}_{T_f}(v|_{T_p,T_f-1}).
\]

Again using (3), (6), and (8), \( X \) is determined by \( U_p \) and \( Y_p \) as

\[
X = A^{T_p}T_{T_p-1}(Y_p - T_{T_p-1}U_p) + C_{T_p-1}U_p
\]

(13)

where

\[
C_{T_p-1} = \begin{bmatrix}
  A^{T_p-1}B & A^{T_p-2}B & \ldots & AB & B
\end{bmatrix}
\]

and \( \hat{\cdot} \) implies the left-inverse of a matrix. Finally, let \( M_f = \mathbb{R}^{p(L+1) \times p(T_f + L)} \) be a matrix that includes the identity matrix of size \( p(L+1) \) from its \( p(L+1) \) 1-th column and all other elements are zero. Then, it is easily seen that

\[
\mathcal{H}_{T_f}(v|_{T_p,T_f-1}) = MY_{f_L}
\]

(14)

where \( M^T = [M_1^T, M_2^T, \ldots, M_k^T] \). Therefore, \( \mathcal{H}_{T_f}(v|_{T_p,T_f-1}) \) is determined by \( Y_{f_L} \). Combining (13) and (14), we arrive that \( U_f \) linearly depends on \( U_p, Y_p \) and \( Y_{f_L} \). Therefore, (12) follows.

Now we state the main result of this paper.

Theorem 2. Let the system of (1) be controllable, observable and have an L-delay inverse with \( L \geq 0 \). Assume that \( T_p \) be greater than or equal to the observability index of (1) and \( u^d_{[0,T-1]} \) is PE of order \( n + T_p + T_f + L \). Then, for any trajectory \( u_{[t_0,t_0+T_p-1]} \) and \( y_{[t_0,t_0+T_p+T_f+L-1]} \) of (1) where \( t_0 \) is arbitrary, the input sequence \( u_{[t_0+T_p,t_0+T_p+T_f+T_f-1]} \) is uniquely determined by

\[
u_{[t_0+T_p,t_0+T_p+T_f+T_f-1]} = U_f g,
\]

(15)

where \( g \) is a solution of

\[
\begin{bmatrix}
  U_p \\
  Y_p \\
  Y_{f_L}
\end{bmatrix} g =
\begin{bmatrix}
u_{[t_0,t_0+T_p-1]} \\
  y_{[t_0,t_0+T_p+T_f+L-1]}
\end{bmatrix}
\]

(16)

Proof. We first claim that (16) always has a solution. Since \( u_{[t_0,t_0+T_p-1]} \) and \( y_{[t_0,t_0+T_p+T_f+T_f-1]} \) are (different length) trajectory of (1), there exists a \( T_f \) long sequence \( u^* \) that satisfies

\[
\begin{bmatrix}
  U_p \\
  Y_p \\
  Y_{f_L}
\end{bmatrix} g =
\begin{bmatrix}
u_{[t_0,t_0+T_p-1]} \\
  y_{[t_0,t_0+T_p+T_f+T_f-1]}
\end{bmatrix}
\]

(17)

by Definition 4 and Lemma 1, which proves the first claim.

We note that, for any solution \( g \) to (16), the vector \( U_f g \) is unique. Indeed, if \( g_1 \) and \( g_2 \) are two solutions of (16), then \( g_1 - g_2 \in \mathcal{N}(U_p^T Y_p^T Y_{f_L}^T) \). This in turn implies that \( g_1 - g_2 \in \mathcal{N}(U_f) \) by (12), so that \( U_f(g_1 - g_2) = 0_{T_f \times 1} \).

Therefore, set \( u_{[t_0+t_0+T_p+T_p+T_f-1]} = U_f g \) for any
solution $g$ of (16). Then, it holds that
\[
\begin{bmatrix}
    U_p \\
    Y_p \\
    U_f \\
    Y_{fl}
\end{bmatrix}
\begin{bmatrix}
    u_{[t_0,t_0+T_p-1]} \\
    y_{[t_0,t_0+T_p-1]} \\
    u_{[t_0+T_p,t_0+T_p+T_f-1]} \\
    y_{[t_0+T_p,t_0+T_p+T_f+L-1]}
\end{bmatrix},
\]
(18)
which implies that the pair $u_{[t_0,t_0+T_p+T_f-1]}$ and $y_{[t_0,t_0+T_p+T_f+L-1]}$ are trajectories of (1) by Lemma 1.

Theorem 2 provides a foundation to answer the question posed in Introduction, i.e., determining the input that generated a given output using previously collected data, since it states that $T_f$ long input $u_{[t_0+T_p,t_0+T_p+T_f-1]}$ is uniquely determined for any pair $u_{[t_0,t_0+T_p-1]}$ and $y_{[t_0,t_0+T_p+T_f+L-1]}$. Figure 1 illustrates this with color and shape coded signals when $T_p = 4$, $T_f = 3$ and $L = 2$. Signals in black circle are required to determine the signal in red triangle.

![Fig. 1: Graphical illustration of sequences in Theorem 2](image)

The necessity of $T_p$ long pair of previous $u$ and $y$ is for the system state $x(t_0 + T_p)$ identification. From this perspective, an interpretation of the theorem is given as follows. Any vector $g$ belonging to
\[
\mathcal{G} := \left\{ g : \begin{bmatrix} U_p \\ Y_p \end{bmatrix} g = \begin{bmatrix} u_{[t_0,t_0+T_p-1]} \\ y_{[t_0,t_0+T_p-1]} \end{bmatrix} \right\}
\]
contains the information of the state $x(t_0 + T_p)$, so that the set
\[
\left\{ \begin{bmatrix} U_f \\ Y_{fl} \end{bmatrix} g : g \in \mathcal{G} \right\}
\]
is a collection of all possible $T_f$-long input and $(T_f+L)$-long output sequences starting from $x(t_0 + T_p)$. Thus, any observed $y_{[t_0+T_p,t_0+T_p+T_f+L-1]}$ must belong to this set, i.e., there exists $g \in \mathcal{G}$ and
\[
y_{[t_0+T_p,t_0+T_p+T_f+L-1]} = Y_{fl} g.
\]
This $g$ is identified by solving (16). The corresponding $u_{[t_0+T_p,t_0+T_p+T_f+T_f-1]}$ is obviously given as in (15) and unique.

The result of Theorem 2 is written in the form of algorithm. See Algorithm 1.

**Algorithm 1: Input Estimation**

1. **Setup:** Set $T_p, T_f, L \in \mathbb{N}_0$ with $T_p \geq 4$ (observability index of the system) and $L \geq L_0$.
   Make $U_p, U_f, Y_p, Y_{fl}$ from persistently exciting input of order $n + T_p + T_f + L$. Get $u_{[-T_p-1]}$ and $y_{[-T_p-1]}$. Set $k = 0$.
2. **Input:** $y_{[k,k+T_f+L-1]}$
3. Solve $g$ for
   \[
   \begin{bmatrix}
    Y_p \\
    U_p \\
    Y_{fl}
\end{bmatrix} g = \begin{bmatrix}
    y_{[k-k-T_p,k-1]} \\
    u_{[k-T_p,k-1]} \\
    y_{[k,k+T_f+L-1]} 
\end{bmatrix}.
   \]
4. Set $u_{[k,k+T_f+T_p-1]} = U_p g$.
5. **Output:** $\hat{u} = u_{[k,k+T_f+T_p-1]}$
6. Update $k \leftarrow k + T_f$
7. Repeat from 2

To initiate Algorithm 1, $u_{[-T_p-1]}$ and $y_{[-T_p-1]}$ must be obtained. If the system is initially at rest, both can be set to zeros. An example is provided.

**Example 1.** As an example, we consider the system of (1) with
\[
A = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix},
\]
(19a)
\[
C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
(19b)
The system has $L$-inverse with $L = 1$. Applying $u_{[0,99]}$ as shown in Figure 2(a) to the system with $x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ yields $y_{[0,99]}$ as shown in Figure 2(b).

![Fig. 2: Inputs and outputs](image)
\[ L = 1, \] and make Hankel matrices with data generated by a 30 long random input sequence satisfying PE of order \( n + T_p + T_f \) ... 2.\]

The signals \( u_0, y, d, u \) are, respectively, command input, output, disturbance, and the actual input to the plant.

\[ \hat{u} \]

\[ \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix} \]

Fig. 3: Estimation of inputs

\[ L = 1 \text{ step delay. Note that } \hat{u} \text{ is computed in batches of length } T_f = 3. \text{ Iteration } k \text{ gives } \hat{u}[3k,3k+2]. \]

Algorithm 1 receives \( y[k,k+T_f+L-1] \) as an input and computes \( \hat{u} = u[k,k+T_f-1] \) at a time. This may be cumbersome to implement for real-time applications. We provide Algorithm 2 tailored for real-time application. Here, the input is modified to use \( y \) available up to the current instance and compute \( \hat{u} \) of length 1.

**Algorithm 2: Input Estimation (with } T_f = 1 )**

1. **Setup:** \( T_p, L \in \mathbb{N}_0 \) with \( T_p \geq (\text{observability index of the system}) \) and \( L \geq L_0 \). Make Hankel matrices \( U_p, U_f, Y_p, Y_f_L \) from persistently exciting input of order \( n + T_p + 1 + L \). Get \( u[-T_p-L,-L-1] \) and \( y[-T_p-L,-1] \). Set \( k = 0 \).

2. **Input:** \( y(k) \)

3. Solve \( g \) for

\[
\begin{bmatrix} Y_p \\ U_p \\ Y_f_L \end{bmatrix} g = \begin{bmatrix} y[k-T_p-L,k-L-1] \\ u[k-T_p-L,k-L-1] \\ y[k-L,k] \end{bmatrix}.
\]

4. Set \( u(k-L) = U_f g \).

5. **Output:** \( \hat{u}(k) = u(k-L) \)

6. Update \( k \leftarrow k + 1 \)

7. Repeat from 2

**Example 2.** Consider the system of (1) with

**Example 3.** The proposed inversion of a linear system can also be used for feasibility analysis of a desired output when input is constrained. Consider a desired output \( y^*_{[0,T_f+L-1]} \) and an input constraint set \( \mathcal{U} \subset \mathbb{R}^m \). By inverting the system for \( y^*_{[0,T_f+L-1]} \), the input sequence \( u_{[0,T_f-1]} \) is obtained. If \( U_f g \in \mathcal{U} \) then the desired output is feasible. Otherwise, it is not.

**IV. Application to Disturbance Observer**

Consider the block diagram of Figure 6, where \( G(z) \) is the plant and the block \'DBINV\' is the data-based inversion block of \( G(z) \) implemented by Algorithm 2. The signals \( u_0, y, d, u \) are, respectively, command input, output, disturbance, and the actual input to the plant.
Note that the transfer function from $u_0$ to $y$ is
\[ Y(z) = G(z)[U_0(z) + (1 - z^{-L})D(z)] \tag{21} \]
where $Y$, $U_0$, and $D$ are $z$-transform of the corresponding signals, which can also be seen from the Figure 6 by
\[
\hat{u}(k) = d(k - L) + \Delta(k - L), \quad \hat{d}(k) = d(k - L), \\
\Delta(k) = u_0(k) - \hat{d}(k) = u_0(k) - d(k - L).
\]
From (21) it is seen that, when $d(k)$ is slowly varying, the effect of $d$ is approximately removed from the input.

This is exactly the idea of the model-based disturbance observer [10]. In this sense, we refer the proposed structure as data-driven disturbance observer.

Example 4. Consider the block diagram of Figure 6 where a realization of $G(z)$ is given by the system of (19). We assume that the signal $u_0$ is the same as the signal shown in Figure 2(a) and the signal $d$ is given in Figure 7(a). The corresponding output is shown in Figure 7(b). Clearly, the effect of disturbance is removed after some transients.

The simulation results agree with the derivation in (21) and we emphasize that no information of $G(z)$ is directly used, and thus, the term of data-driven disturbance observer is justified.

V. Conclusions

A data-based construction of inverse dynamics has been developed for discrete-time LTI systems. The notion of $L$-delay inverse is invoked from the literature, and it is combined with the system description method from behavioral approach. The outcome is data-based representation of inverse dynamics, which is similar to that of LTI systems, but differs in that input is estimated with some delay. The result is applied to build disturbance observers from collected input and output data that satisfy a level of persistency of excitation condition. Applications and extension of this result seem to be immense.

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