Theoretical error analysis and validation in numerical solution of two-dimensional linear stochastic Volterra-Fredholm integral equation by applying the block-pulse functions

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M. Fallahpour1, M. Khodabin1* and K. Maleknejad1

Abstract: In this paper, we introduce an efficient method based on two-dimensional block-pulse functions (2D-BPFs) to approximate the solution of the 2D-linear stochastic Volterra–Fredholm integral equation. Also, we present convergence analysis of the proposed method. Illustrative examples are included to demonstrate the validity and applicability of the proposed method.

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1. Introduction
As we know, in 2D-stochastic integral equations, the values can vary in time and space due to unknown conditions of the surroundings or the medium. Computational complexity of mathematical
operations because of the randomness is the most important obstacle for solving stochastic integral equations in higher dimensions. In this paper, we consider 2D-linear stochastic Volterra–Fredholm integral equation of the second kind

\[
g(x, y) = f(x, y) + \int_0^1 \int_0^1 v_1(x, y, s, t)g(s, t)dsdt \\
+ \int_0^x \int_0^y v_2(x, y, s, t)g(s, t)dsdt \\
+ \int_0^x \int_0^y v_3(x, y, s, t)g(s, t)dB(s)dB(t),
\]

where \((x, y) \in E = [0, 1] \times [0, 1], s \leq x \leq y\). The function \(f(x, y)\) defined on \(E\) and the kernels \(v_i(x, y, s, t); i = 1, 2, 3\) defined on \(E \times E\), in (1) are known functions, whereas \(g(x, y)\) defined on \(E\), is an unknown function and is called the solution of (1). Also, \(B(t)\) is a Brownian motion process and \(\int_0^x \int_0^y v_3(x, y, s, t)g(s, t)dB(s)dB(t)\) is the double Wiener-Itô integral. The condition \(s \leq x \leq y\) is necessary for adaptability to the filtration \(\{F_t, 0 \leq t \leq 1\}\), where \(F_t = \sigma(B(s); 0 \leq s \leq t)\).

Recently, authors Fallahpour, Khodabin, and Maleknejad (2016) have used the BPFs method for solving Equation (1) without investigating the error analysis. Also, authors of Fallahpour, Khodabin, and Maleknejad (2015) have proposed Haar wavelet method to solve 2D-linear stochastic Fredholm integral equation without investigating the error analysis. Here, the BPFs method with the error and convergence analysis is introduced to derive approximate solution of (1). First, for validation the accuracy of the proposed method by some examples in Section 4. Also, in this section, we construct an 95% confidence interval for each solution. Finally, Section 5, gives brief conclusions.

**Lemma 1** (Kuo, 2006) Put \(\phi(t, s) = v(x, y, s, t)g(s, t)\). Let \(\phi\) be a function in \(L^2([0, 1]^2)\). Then, there exists a sequence \(\phi_n\) of off-diagonal step functions such that

\[
\lim_{n \to \infty} \int_a^b \int_a^b |\phi(t, s) - \phi_n(t, s)|^2dsdt = 0.
\]

**Definition 1** (Kuo, 2006) Let \(\phi \in L^2([0, 1]^2)\). Then, the double Wiener-Itô integral of \(\phi\) is defined as follows:

\[
\int_a^b \int_a^b \phi(t, s)dB(t)dB(s) = \lim_{n \to \infty} \int_a^b \int_a^b \phi_n(t, s)dB(t)dB(s) \quad \text{in} \ L^2(\Omega),
\]

where \(\Omega\) is the sample space of random variables.

This paper is organized as follows:

In Section 2, we review the proposed numerical method in Fallahpour et al. (2016) for solving Equation (1) based on BPFs. The error analysis of this method is discussed in Section 3. We show the accuracy of the proposed method by some examples in Section 4. Also, in this section, we construct an 95% confidence interval for each solution. Finally, Section 5, gives brief conclusions.

**2. The BPFs numerical method**

In this section, we review the proposed numerical method to solve 2D-linear stochastic Volterra–Fredholm integral Equation (1) using 2D-BPFs Fallahpour et al. (2016). As we know, for \((x, y) \in ([0, T_1] \times [0, T_2])\), an \((n_1, n_2)\)-set of 2D-BPFs \(\psi_{\alpha_1, \alpha_2}(x, y) (\alpha_1 = 1, 2, \ldots, n_1); (\alpha_2 = 1, 2, \ldots, n_2)\) is defined as follows:
\[
\psi_{a_1,a_2}(x,y) = \begin{cases} 
1, & \text{for } (a_1 - 1)k_1 \leq x < a_1k_1 \\
0, & \text{otherwise}, 
\end{cases}
\]

where \( (n_1, n_2) \) are arbitrary positive integers, \( k_1 = \frac{T_1}{n_1} \) and \( k_2 = \frac{T_2}{n_2} \). The set of 2D-BPFs can be written as a vector \( \Psi(x,y) \), where:

\[
\Psi(x,y) = [\psi_{1,1}(x,y), \ldots, \psi_{1,n_1}(x,y), \ldots, \psi_{2,n_2}(x,y), \ldots, \psi_{n_1,n_2}(x,y)]^T. \tag{2}
\]

From 2D-BPFs elementary properties, it follows that

\[
\Psi(x,y) \Psi^T(x,y)U = U \Psi(x,y), \tag{3}
\]

where \( U \) is an \( n_1n_2 \)-vector and \( \hat{U} = \text{diag}(U) \), for the more details see Jiang and Schaufelberger (1992).

Moreover, for every \( (n_1n_2) \times (n_1n_2) \) matrix \( A \), we have

\[
\Psi^T(x,y)A \Psi(x,y) = \hat{A}^T \Psi(x,y), \tag{4}
\]

where the elements of \( \hat{A} \), \( n_1n_2 \)-vector, are the diagonal entries of matrix \( A \).

For approximating functions \( f(x,y) \), \( v_1(x,y,s,t) \), \( v_2(x,y,s,t) \), \( v_3(x,y,s,t) \) and \( g(x,y) \) with respect to the 2D-BPFs, we have

\[
f(x,y) = F^T \Psi(x,y), \tag{5}
\]

\[
v_1(x,y,s,t) = \Psi^T(x,y) \Gamma_1 \Psi(s,t), \tag{6}
\]

\[
v_2(x,y,s,t) = \Psi^T(x,y) \Gamma_2 \Psi(s,t), \tag{7}
\]

\[
v_3(x,y,s,t) = \Psi^T(x,y) \Gamma_3 \Psi(s,t), \tag{8}
\]

and

\[
g(x,y) = G_3^T \Psi(x,y), \tag{9}
\]

where vectors \( F \) and \( G_3 \) and matrices \( \Gamma_1 \), \( \Gamma_2 \) and \( \Gamma_3 \) are the BPFs coefficients of \( f(x,y) \), \( g(x,y) \), \( v_1(x,y,s,t) \), \( v_2(x,y,s,t) \) and \( v_3(x,y,s,t) \), respectively and \( \Psi(x,y) \) is defined in (2). In (5), \( F \), \( (n_1n_2 \times 1) \)-vector, is a known vector and in (6), (7) and (8), \( \Gamma_1 \), \( \Gamma_2 \) and \( \Gamma_3 \), \( (n_1n_2 \times n_1n_2) \)-matrices are known matrices. Also in (9), \( G_3 \) is unknown vector as follows:

\[
G_3 = [g_{1,1}(x,y), \ldots, g_{1,n_1}(x,y), \ldots, g_{2,n_2}(x,y), \ldots, g_{n_1,n_2}(x,y)]^T.
\]

In Equation (1) to approximate the first 2D-integral, we apply Equations (6) and (9). So, we get
In Fallahpour et al. (2016), it is proved that

\[ D = \begin{pmatrix} n_1 & n_2 \\ n_1 & n_2 \end{pmatrix} \times \begin{pmatrix} n_1 & n_2 \\ n_1 & n_2 \end{pmatrix} \text{ known matrix}, \]

so we conclude that

\[ G_2 = \Gamma_1 \begin{pmatrix} k_1 k_2 & 0 & \cdots & 0 \\ 0 & k_1 k_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_1 k_2 \end{pmatrix} \]

is an \((n_1 n_2)\)-vector, \(\Gamma_1\) is a \((n_1 n_2) \times (n_1 n_2)\) 2D-BPFs coefficients matrix with the elements

\[ v_{a,b,c,d} = \frac{1}{k^4} \int \int \int \int v_1(x,y,s,t) dsdt dsdy dx. \]

Using Equations (7) and (9), we can approximate the second 2D-integral in (1). So, we have

\[
\begin{align*}
\int_0^1 \int_0^1 v_1(x,y,s,t) g(s,t) dsdt & \\
\approx & \int_0^1 \int_0^1 \psi^T(x,y) \Gamma_1 \psi(s,t) \psi^T(s,t) G_1 dsdt \\
= & \psi^T(x,y) \Gamma_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} G_1.
\end{align*}
\]

Now, using Equation (3), we get

\[
\begin{align*}
\int_0^1 \int_0^1 G_1 \psi(s,t) dsdt & \\
= & \psi^T(x,y) \Gamma_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} G_1 P \psi(x,y),
\end{align*}
\]

where \(\Gamma_2 G_1 P\) is an \((n_1 n_2) \times (n_1 n_2)\) matrix, also \(P\) is an \((n_1 n_2) \times (n_1 n_2)\) operational matrix of integration for the 2D-BPFs as Fallahpour et al. (2016).
where $\otimes$ denotes the Kronecker product defined by 

\[
X \otimes Y = (x_{ij}Y),
\]

where $x_{ij}$ is the $ij$-element of $X$ matrix and $O$ is the operational matrix of the 1D-BPFs defined over $[0, T)$ with $k = \frac{T}{n}$ and $I = I_1 = I_2$ as follows:

\[
O = \frac{k}{2} \begin{pmatrix}
1 & 2 & 2 & \ldots & 2 \\
0 & 1 & 2 & \ldots & 2 \\
0 & 0 & 1 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}.
\]

Therefore, using Equation (4) in (11) follows that

\[
\int_0^T \int_0^T v_2(x, y, s, t)g(s, t)dsdt \simeq G_3^T \Psi(x, y),
\]

where $G_3$ is an $(n_1n_2)$-vector with the components equal to the diagonal entries of matrix $\tilde{\Gamma}_2 G_1 P$.

Similarly to approximate the stochastic integral case in (1), we use Equations (8) and (9). So, we have

\[
\int_0^T \int_0^T v_3(x, y, s, t)g(s, t)dB(s)dB(t)
\]

\[
\simeq \int_0^T \int_0^T \Psi^T(x, y)\Gamma_3^2 \Psi(s, t)\tilde{G}_1 dB(s)dB(t)
\]

\[
= \Psi^T(x, y)\Gamma_3^2 \int_0^T \int_0^T \Psi(s, t)\tilde{G}_1 dB(s)dB(t).
\]

Using Equation (3), we get

\[
\int_0^T \int_0^T v_3(x, y, s, t)g(s, t)dB(s)dB(t)
\]

\[
\simeq \Psi^T(x, y)\Gamma_3^2 G_1 P_s \Psi(x, y),
\]

where $\Gamma_3^2 G_1 P_s$ is an $(n_1n_2) \times (n_1n_2)$ matrix, also $P_s$ is an $(n_1n_2) \times (n_1n_2)$ stochastic operational matrix of integration for the 2D-BPFs as Fallahpour et al. (2016).

\[
P_s = O_s(x, n_1) \otimes O_s(n_1x),
\]

where $O_s$ is the $(n \times n)$-stochastic operational matrix of the 1D-BPFs defined over $[0, T)$ with $k = \frac{T}{n}$ and $T = T_1 = T_2$ as follows (Khodabin, Maleknejad, Rostami, & Nouri 2012).
Using Equation (4) in (13) follows that

\[
G_4 \text{ is an } (n_1 \times n_2) \text{-vector with the components equal to the diagonal entries of matrix } \Gamma_3 \tilde{G}_1 P_s.
\]

By substituting Equations (5), (9), (10), (12), and (14) in Equation (1) we get

\[
G_4 \Psi(x, y) \simeq F \Psi(x, y) + G_2 \Psi(x, y) + G_3 \Psi(x, y) + G_4 \Psi(x, y).
\]

Replacing \( \simeq \) with \( = \), Equation (15) gives

\[
G_1 - G_2 - G_3 - G_4 = F.
\]

Clearly, Equation (16) generate a system of \((n_1 \times n_2)\) linear equations with \((n_1 \times n_2)\) unknown variable which can be solved using known methods as direct methods or iterative methods.

### 3. Error analysis

In this section, we present error analysis of the proposed method in Section 2. For convenience, we put \( n_1 = n_2 = n \), and hence \( k_1 = k_2 = \frac{1}{n} \). Also in this section, we consider for simplicity the 2-norms defined in this paper based on following definition.

**Definition 2** Suppose that \( C(A) \) denotes the set of continuous functions on a given set \( A \). If \( f \in C[a, b] \), \( g \in C[a, b]^2 \) and \( v \in C[a, b]^4 \), we can define a norm in \( C[a, b] \), \( C[a, b]^2 \) and \( C[a, b]^4 \) by

\[
||f||_2 = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2},
\]

\[
||g||_2 = \left[ \int_a^b \int_a^b |g(x, y)|^2 \, dx \, dy \right]^{1/2}
\]

and

\[
||v||_2 = \left[ \int_0^b \int_0^b \int_0^b \int_0^b |v(x, y, s, t)|^2 \, dx \, dy \, ds \, dt \right]^{1/2}.
\]
Also, we need the following theorems and definition.

**Definition 3** Suppose that \( f \) maps a convex open set \( E \subset \mathbb{R}^2 \) into \( \mathbb{R} \), \( f \) is differentiable in \( E \), and there is a real number \( M \) such that
\[
\| f' \|_2 \leq M,
\]
for every \( t \in E \). Then
\[
| f(b) - f(a) | \leq M | b - a |,
\]
for all \( a, b \in E \).

**Proof** See Theorem 9.19 in Rudin (1976).

**Theorem 1** Suppose that \( f(s, t) \) is a differentiable function on \( I = [0, 1) \times [0, 1) \) with
\[
\| f' \|_2 \leq M.
\]
Let
\[
\hat{f}_n(s, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i(s) \psi_j(t),
\]
be the 2D-BPFs expansion of \( f(s, t) \) and \( e(s, t) = f(s, t) - \hat{f}_n(s, t) \), then for every \( (s, t) \in I \) we have
\[
\| e \|_2^2 \leq \frac{2}{n^2} \times M^2,
\]
hence
\[
\| e \|_2 = O \left( \frac{1}{n} \right).
\]
**Proof** See Khodabin et al. (2012).

**Theorem 2** Suppose that \( v(x, y, s, t) \) is defined on \( I = [0, 1)^4 \) with
\[
\| v' \|_2 \leq M,
\]
and
\[
\hat{v}_n(x, y, s, t) = \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} v_{abcd} \phi_a(x) \phi_b(y) \xi_c(s) \zeta_d(t),
\]
is 4D-BPFs expansion of \( v(x, y, s, t) \). If
\[
e(x, y, s, t) = v(x, y, s, t) - \hat{v}_n(x, y, s, t),
\]
then for every \( (x, y, s, t) \in I \) we have
\[
\| e \|_2 = O \left( \frac{1}{n} \right).
\]
**Proof** Clearly, we have
\[
e_{a,b,c,d}(x, y, s, t) = v(x, y, s, t) - v_{a,b,c,d}(x, y, s, t)
\]
\[
= \begin{cases} 
  v(x, y, s, t) - v_{a,b,c,d} & \text{for } (x, y, s, t) \in I_{a,b,c,d} \\
  0, & \text{otherwise,}
\end{cases}
\]
where \( I_{a,b,c,d} \) is defined as follows:
\[ I_{a,b,c,d} = \left\{ \frac{a-1}{n} \leq x < \frac{a}{n}, \frac{b-1}{n} \leq y < \frac{b}{n}, \frac{c-1}{n} \leq z < \frac{c}{n}, \frac{d-1}{n} \leq t < \frac{d}{n} \right\}. \]

It can be shown that

\[
\|e_{a,b,c,d}\|_2^2 = \int_{(a-1)n}^{bn} \int_{(b-1)n}^{cn} \int_{(c-1)n}^{dn} e_{a,b,c,d}(x,y,s,t)dt\,dy\,ds\,dx
\]

\[
= \int_{(a-1)n}^{bn} \int_{(b-1)n}^{cn} \int_{(c-1)n}^{dn} (v(x,y,s,t) - v_{a,b,c,d})^2 dt\,dy\,ds\,dx.
\]

Using mean value theorem for 4D-integrals, there exists \((\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in I_{a,b,c,d}\) such that

\[
\|e_{a,b,c,d}\|_2^2 = \frac{1}{n^4} (v(\gamma_1, \gamma_2, \gamma_3, \gamma_4) - v_{a,b,c,d})^2.
\]

We know

\[
v_{a,b,c,d} = \frac{1}{k^6} \int_{(a-1k)}^{ak} \int_{(b-1k)}^{bk} \int_{(c-1k)}^{ck} \int_{(d-1k)}^{dk} v_{a,b,c,d}(x,y,s,t)dt\,dy\,ds\,dx,
\]

therefore, using mean value theorem, there exists \((\theta_1, \theta_2, \theta_3, \theta_4) \in I_{a,b,c,d}\) such that

\[
v_{a,b,c,d} = n^4 \int_{(a-1n)}^{an} \int_{(b-1n)}^{bn} \int_{(c-1n)}^{cn} \int_{(d-1n)}^{dn} v(x,y,s,t)dt\,dy\,ds\,dx,
\]

\[
= n^6 \times \frac{1}{n^6} \times v(\theta_1, \theta_2, \theta_3, \theta_4).
\]

By replacing (18) into (17) and using Definition 3, we get

\[
\|e_{a,b,c,d}\|_2^2 = \frac{1}{n^4} (v(\gamma_1, \gamma_2, \gamma_3, \gamma_4) - v(\theta_1, \theta_2, \theta_3, \theta_4))^2
\]

\[
\leq \frac{1}{n^4} \times 4k^2 \times M^2 = \frac{4M^2}{n^6}
\]

\[
\text{Since}
\]

\[
e(x,y,s,t) = v(x,y,s,t) - \tilde{v}_1(x,y,s,t)
\]

\[
= v(x,y,s,t) - \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c-1}^{n} \sum_{d-1}^{n} \phi_a(x)w_b(y)\psi_c(s)\zeta_d(t)
\]

\[
= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c-1}^{n} \sum_{d-1}^{n} e_{a,b,c,d}(x,y,s,t).
\]
Therefore, we have
\[ \|e\|_2^2 = \iint_0^1 e^2(x, y, s, t) dt ds dy dx \]
\[ = \iint_0^1 \left( \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n e_{abcd}(x, y, s, t) \right)^2 dt ds dy dx \]
\[ = \iint_0^1 \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n e_{abcd}^2(x, y, s, t) dt ds dy dx \]
\[ + 2 \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n \iint_0^1 e_{abcd}(x, y, s, t) \times e_{a'b'c'd'}(x, y, s, t) dt ds dy dx. \]

Since for \( a < a', b < b', c < c' \) and \( d < d' \), we have
\[ I_{a, b, c, d} \cap I_{a', b', c', d'} = \emptyset, \]
then from (19), we obtain
\[ \|e\|_2^2 = \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n \iint_0^1 e_{abcd}^2(x, y, s, t) dt ds dy dx \]
\[ = \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n \|e_{abcd}\|_2^2 \leq n^4 \times \frac{4M^2}{n^6} = \frac{1}{n^2} \times 4M^2, \]
that is mean
\[ \|e\|_2 = O\left(\frac{1}{n}\right). \]

THEOREM 3 For any \( x > 0 \), we have
\[ P(M(t) \geq x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-z^2} \frac{dz}{\sqrt{t}} = 2 \left( 1 - \Phi \left( \frac{x}{\sqrt{t}} \right) \right), \]
where
\[ M(t) = \sup_{0 \leq s \leq t} B(s), \]
and \( \Phi \) is cumulative standard normal distribution function. So for \( x \geq 4 \), we have
\[ \sup_{0 \leq t \leq 1} B(t) < \infty, \]
with probability one.

Proof See Klebaner (2005).
THEOREM 4  Suppose that \( g(s, t) \) is the exact solution of Equation (1) and \( \tilde{g}_n(s, t) \) is the block-pulse series approximate solution of (1) that their elements are obtained by (16) and assume \( \tilde{f}_n(s, t) \) and \( \tilde{v}_i(x, y, s, t) \) are the block-pulse series of \( f(s, t) \) and \( v_i(x, y, s, t) \), \( i = 1, 2, 3 \), respectively. Also assume that

1. \( \|g\|_2 \leq \beta, (s, t) \in [0, 1]^2 \),
2. \( \|v_i\|_2 \leq M_i, i = 1, 2, 3, (x, y, s, t) \in [0, 1]^4 \),
3. \( S(x, y) = \sup_{x \in [0, 1]} |B(x)| \times \sup_{y \in [0, 1]} |B(y)| \),
4. \( M_1 + M_2 + \frac{2M_1 + 2M_2}{n} + \left( M_3 + \frac{2M_3}{n} \right) < 1 \),

then for every \((x, y) \in [0, 1]^2\), we have

\[
\|g - \tilde{g}_n\|_2 = O\left( \frac{1}{n} \right).
\]

Proof  From Equation (1), we get

\[
g(x, y) = f(x, y) - \tilde{f}_n(x, y) = \int_0^1 \left( \int_0^1 \left( v_1(x, y, s, t)g(s, t) - \tilde{v}_{1, n}(x, y, s, t)\tilde{g}_n(s, t) \right)ds \right)dt
\]

\[
+ \int_0^1 \left( \int_0^1 \left( v_2(x, y, s, t)g(s, t) - \tilde{v}_{2, n}(x, y, s, t)\tilde{g}_n(s, t) \right)ds \right)dt
\]

\[
+ \int_0^1 \left( \int_0^1 \left( v_3(x, y, s, t)g(s, t) - \tilde{v}_{3, n}(x, y, s, t)\tilde{g}_n(s, t) \right)ds \right)dt dB(s)dB(t),
\]

then by mean value theorem for 2D-integrals, for every \((x, y) \in [0, 1]^2\) and \((x, y, s, t) \in [0, 1]^4\), we have

\[
\|g - \tilde{g}_n\|_2 \leq \|f - \tilde{f}_n\|_2 + \|v_1g - \tilde{v}_{1, n}\tilde{g}_n\|_2
\]

\[
+ xy\|v_2g - \tilde{v}_{2, n}\tilde{g}_n\|_2
\]

\[
+ B(x)B(y)\|v_3g - \tilde{v}_{3, n}\tilde{g}_n\|_2.
\]

By using Hypotheses 1 and 2 and Theorem 3 we obtain

\[
\|v_1g - \tilde{v}_{1, n}\tilde{g}_n\|_2 \leq \|v_1\|_2 \|g - \tilde{g}_n\|_2 + \|v_1 - \tilde{v}_{1, n}\tilde{g}_n\|_2 (\|g - \tilde{g}_n\|_2 + \|\tilde{g}_n\|_2)
\]

\[
\leq M_1 \|g - \tilde{g}_n\|_2 + \frac{2M_1}{n} (\|g - \tilde{g}_n\|_2 + \|\tilde{g}_n\|_2)
\]

\[
= \left( M_1 + \frac{2M_1}{n} \right) \|g - \tilde{g}_n\|_2 + \frac{2M_1}{n} \beta,
\]

and

\[
\|v_2g - \tilde{v}_{2, n}\tilde{g}_n\|_2 \leq \|v_2\|_2 \|g - \tilde{g}_n\|_2 + \|v_2 - \tilde{v}_{2, n}\tilde{g}_n\|_2 (\|g - \tilde{g}_n\|_2 + \|\tilde{g}_n\|_2)
\]

\[
\leq M_2 \|g - \tilde{g}_n\|_2 + \frac{2M_2}{n} (\|g - \tilde{g}_n\|_2 + \|\tilde{g}_n\|_2)
\]

\[
= \left( M_2 + \frac{2M_2}{n} \right) \|g - \tilde{g}_n\|_2 + \frac{2M_2}{n} \beta.
\]

Similarly for the stochastic case, we get

\[
\|v_3g - \tilde{v}_{3, n}\tilde{g}_n\|_2 \leq \left( M_3 + \frac{2M_3}{n} \right) \|g - \tilde{g}_n\|_2 + \frac{2M_3}{n} \beta.
\]

By substituting (21), (22), and (23) in (20) and using Theorem 2, we can write
By taking sup and the Hypothesis 3, we have
\[ \|g - \hat{g}_n\|_2 \leq \sqrt{\frac{2M}{n}} + \left[ \left( M_1 + \frac{2M_1}{n} \right) \|g - \hat{g}_n\|_2 + \frac{2M_1}{n} \beta \right] + xy \left[ \left( M_2 + \frac{2M_2}{n} \right) \|g - \hat{g}_n\|_2 + \frac{2M_2}{n} \beta \right] + B(x)B(y) \left[ \left( M_3 + \frac{2M_3}{n} \right) \|g - \hat{g}_n\|_2 + \frac{2M_3}{n} \beta \right]. \]

So
\[ \|g - \hat{g}_n\|_2 \leq \frac{\sqrt{2M + 2M_1 \beta + 2M_2 \beta} + \frac{2M_3 \beta}{n} \times S(x, y)}{1 - \left( M_1 + M_2 + \frac{2M_1 + 2M_2}{n} + \left( M_3 + \frac{2M_3}{n} \right) \times S(x, y) \right)} , \]
and by Theorem 4, we get
\[ \|g - \hat{g}_n\|_2 = O \left( \frac{1}{n} \right) . \]

4. Numerical example

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the BPFs method for solving the 2D-linear stochastic Volterra–Fredholm integral equations.

Example 1 Consider the following linear 2D-stochastic Volterra–Fredholm integral equation of the second kind:

\[ \|g - \hat{g}_n\|_2 \leq \frac{\sqrt{2M + 2M_1 \beta + 2M_2 \beta} + \frac{2M_3 \beta}{n} \times S(x, y)}{1 - \left( M_1 + M_2 + \frac{2M_1 + 2M_2}{n} + \left( M_3 + \frac{2M_3}{n} \right) \times S(x, y) \right)} , \]
where

The exact solution of this equation is

For convenience, we put $n_1 = n_2 = n$ so $k_1 = k_2 = \frac{1}{n}$. The solution mean ($\bar{g}(x,y)$), error mean ($\bar{e}(x,y)$) and %95 confidence interval ($L$, $U$) at arbitrary points $(0.1, 0.2)$ and $(0.3, 0.7)$ for some values of $n$ are shown in Tables 1 and 2. You can see 3D-graph of the exact and approximation solutions of Example 1 for various values of arbitrary positive integer of $n$ in Figure 1.

### Example 2

Consider the following linear 2D-stochastic Volterra–Fredholm integral equation of the second kind:

$$g(x, y) = f(x, y) + \int_0^1 \int_0^x xg(s, t)dsdt + \int_0^x xg(s, t)dsdt$$

$$+ \int_0^y \int_0^x (x + y)g(s, t)dB(s)dB(t),$$

where

$$f(x, y) = 1 - x - yx^2 - (x + y)B(x)B(y).$$

The exact solution of this equation is

$$g(x, y) = 1.$$

For convenience, we put $n_1 = n_2 = n$ so $k_1 = k_2 = \frac{1}{n}$. The solution mean ($\bar{g}(x,y)$), error mean ($\bar{e}(x,y)$) and %95 confidence interval ($L$, $U$) at arbitrary points $(0.1, 0.2)$ and $(0.3, 0.7)$ for some values of $n$ are shown in Tables 1 and 2. You can see 3D-graph of the exact and approximation solutions of Example 1 for various values of arbitrary positive integer of $n$ in Figure 1.

### Example 2

Consider the following linear 2D-stochastic Volterra–Fredholm integral equation of the second kind:

$$g(x, y) = f(x, y) + \int_0^1 \int_0^x (x + y + s + t)g(s, t)dsdt$$

$$+ \int_0^y \int_0^x (x + y + s + t)g(s, t)dsdt + \int_0^x xystg(s, t)dB(s)dB(t),$$

where

$$f(x, y) = -\frac{7}{6} - \frac{1}{6} xy(5x^2 + 9xy + 5y^2)$$

$$- 2xy\left(x^2B(x) - \int_0^x B(s)ds\right)\left(yB(y) - \int_0^y B(t)dt\right).$$
The exact solution of this equation is
\[ g(x, y) = x + y. \]

Also for this example, \( \hat{g}(x, y) \), \( \hat{e}(x, y) \) and \( (L, U) \) at arbitrary points \((0.1, 0.2)\) and \((0, 0.6)\) for some values of \( n \) are shown in Tables 3 and 4. 3D-graph of the exact and approximation solutions of this example for some values of \( n \) are shown in Figure 2.

**Example 3** Consider the following linear 2D-stochastic Volterra–Fredholm integral equation of the second kind:

\[
g(x, y) = f(x, y) + \int_0^1 \int_0^1 x \sin(t + y) g(s, t) ds \, dt + \int_0^y \int_0^x x \sin(t + y) g(s, t) ds \, dt + \int_0^y \int_0^x (s + t) \cos(xy) g(s, t) dB(s) dB(t),
\]

For this example, \( \hat{g}(x, y) \), \( \hat{e}(x, y) \) and \( (L, U) \) at arbitrary points \((0.1, 0.2)\) and \((0, 0.6)\) for some values of \( n \) are shown in Tables 3 and 4. 3D-graph of the exact and approximation solutions of this example for some values of \( n \) are shown in Figure 2.
The exact solution of this equation is
\[ f(x, y) = xy + \frac{1}{3} x(\cos(1 + y) + \sin(y) - \sin(1 + y)) + \frac{1}{3} x^4(\cos(2y) + \sin(y) - \sin(2y)) - \cos(xy) \left( x^2 - B(x) - 2 \int_0^x sB(s)ds \right) - \cos(xy) \left( y^2 - B(y) - 2 \int_0^y sB(s)ds \right) \]

The exact solution of this equation is
\[ g(x,y) = xy. \]

The solution, mean \( \bar{g}(x,y) \), error mean \( \bar{e}(x,y) \) and %95 confidence interval \((L, U)\) at arbitrary points \((0.6, 0.8)\) and \((0.4, 0.9)\) for some values of \(n\) are shown in Tables 5 and 6. You can see 3D-graph of the

Table 5. The solutions mean and error mean with %95 confidence interval for Example 3 for some different values of \(n\) in the point \((0.6, 0.8)\)

| \(n\) | \(\bar{g}(x,y)\) | \(\bar{e}(x,y)\) | \((L, U)\)          |
|-------|------------------|------------------|---------------------|
| 2     | 0.178081         | 0.7683170        | (-1.93859, 1.582430) |
| 3     | 0.354656         | 0.125344         | (0.243421, 0.465891) |
| 4     | 0.323693         | 0.156307         | (0.129018, 0.518367) |
| 6     | 0.371893         | 0.140597         | (0.192240, 0.551545) |

Table 6. The solutions mean and error mean with %95 confidence interval for Example 3 for some different values of \(n\) in the point \((0.4, 0.9)\)

| \(n\) | \(\bar{g}(x,y)\) | \(\bar{e}(x,y)\) | \((L, U)\)          |
|-------|------------------|------------------|---------------------|
| 2     | 0.458543         | 0.303459         | (-0.05809, 0.975180) |
| 3     | 0.198560         | 0.320325         | (-0.53132, 0.923039) |
| 4     | 0.192048         | 0.179375         | (0.038456, 0.347021) |
| 6     | 0.192739         | 0.167261         | (0.997053, 0.997090) |

Figure 3. Exact and approximate solution for Example 3 \((n = 6)\).
exact and approximation solutions of Example 3 for various values of arbitrary positive integer of \( n \) in Figure 3.

5. Conclusions
The numerical solution of 2D-stochastic integral equations because of the randomness is very difficult or sometimes impossible as we can say that the solving 2D-stochastic integral equations has been worked very little. In this paper, we have successfully developed the 2D-BPFs numerical method to approximate a solution for 2D-linear stochastic Volterra–Fredholm integral equation in which error analysis and the numerical example show accuracy of this method. The numerical results show that typical convergence rate of the method is \( O\left( \frac{1}{n} \right) \).

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