THE EVENTUAL PARACANONICAL MAP OF A VARIETY OF MAXIMAL ALBANESE DIMENSION

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Abstract. Let $X$ be a smooth complex projective variety such that the Albanese map of $X$ is generically finite onto its image. Here we study the so-called eventual $m$-paracanonical map of $X$, whose existence is implied by the results of [4] (when $m = 1$ we also assume $\chi(K_X) > 0$).

We show that for $m = 1$ this map behaves in a similar way to the canonical map of a surface of general type, as described in [6], while it is birational for $m > 1$. We also describe it explicitly in several examples.

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1. Introduction

Let $X$ be a smooth complex projective variety and let $a: X \to A$ a map into an abelian variety that is generically finite onto its image and such that $a^*: \text{Pic}^0(A) \to \text{Pic}^0(X)$ is injective; we identify $\text{Pic}^0(A)$ with a subgroup of $\text{Pic}^0(X)$ via $a^*$.

Let $L \in \text{Pic}(X)$ be a line bundle such that $h^0(L \otimes \alpha) > 0$ for every $\alpha \in \text{Pic}^0(A)$. We may consider the pull back $L^{(d)}$ to the étale cover $X^{(d)} \to X$ induced by the $d$-th multiplication map of $A$. In [4] it is shown that for $d$ large and divisible enough and $\alpha \in \text{Pic}^0(A)$ general the map given by $|L^{(d)} \otimes \alpha|$ does not depend on $\alpha$ and is obtained by base change from the so-called eventual map associated to $L$, a generically finite map $\phi: X \to Z$, uniquely determined up to birational isomorphism, such that $a$ factors through it.

Here we consider the case when $a: X \to A$ is the Albanese map and $L = mK_X$, $m \geq 1$, and study the eventual $m$-paracanonical map, which is intrinsically attached to $A$. (For $m = 1$ one needs to assume also that $\chi(K_X) > 0$, since by the generic vanishing theorem $\chi(K_X)$ is the generic value of $h^0(K_X \otimes \alpha)$).

After recalling (§2) the main properties of the eventual map, in §3 we concentrate on the case $m = 1$ and prove that the relations between the numerical invariants of $X$ and of the paracanonical image $Z$ are...
completely analogous to those between a surface of general type and its canonical image (see [6]). In §4 we prove that the eventual \( m \)-canonical map is always birational for \( m > 1 \), and therefore \( m = 1 \) is the only interesting case.

The last section contains several examples, that show that the eventual paracanonical map is really a new object and needs not be birational or coincide with the Albanese map, and can have arbitrarily large degree. We finish by posing a couple of questions.

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Conventions: We work over the complex numbers.

“map” means rational map.

Given maps \( f: X \to Y \) and \( g: X \to Z \), we say that \( g \) is composed with \( f \) if there exists a map \( h: Y \to Z \) such that \( g = h \circ f \). Given a map \( f: X \to Y \) and an involution \( \sigma \) of \( X \), we say that \( f \) is composed with \( \sigma \) if \( f \circ \sigma = f \).

We say that two dominant maps \( f: X \to Z \), \( f': X \to Z' \) are birationally equivalent if there exists a birational isomorphism \( h: Z \to Z' \) such that \( f' = h \circ f \).

A generically finite map means a rational map that is generically finite onto its image. A smooth projective variety \( X \) is of maximal Albanese dimension if its Albanese map is generically finite.

The symbol \( \equiv \) denotes numerical equivalence of line bundles/Q-divisors.

2. Preliminaries

In this section we fix the set-up and recall some facts and definitions. More details can be found in [1], [3], [4].

Let \( X \) be a smooth projective map, let \( a: X \to A \) be a generically finite map to an abelian variety of dimension \( q \). Given a line bundle \( L \) on \( X \), the continuous rank \( h^0_a(L) \) of \( L \) (with respect to \( a \)) is defined as the minimum value of \( h^0(L \otimes \alpha) \) as \( \alpha \) varies in \( \text{Pic}^0(A) \).

Let \( d \) be an integer and denote by \( \mu_d: A \to A \) the multiplication by \( d \); the following cartesian diagram defines the variety \( X^{(d)} \) and the
maps $a_d$ and $\tilde{\mu}_d$.

\[
\begin{array}{ccc}
X^{(d)} & \xrightarrow{\tilde{\mu}_d} & X \\
\downarrow a & & \downarrow a \\
A & \xrightarrow{\mu_d} & A
\end{array}
\]  

(2.1)

We set $L^{(d)} := \tilde{\mu}_d^* L$; one has $h^0_a(L^{(d)}) = d^2 h^0_a(L)$.

We say that $a$ is strongly generating if the induced map $\text{Pic}^0(A) \to \text{Pic}^0(X)$ is an inclusion; notice that if $a$ is strongly generating, then the variety $X^{(d)}$ is connected for every $d$.

**Definition 2.1.** We say that a certain property holds generically for $L$ (with respect to $a$) iff it holds for $L \otimes \alpha$ for general $\alpha \in \text{Pic}^0(A)$; similarly, we say that a property holds eventually (with respect to $a$) for $L$ iff it holds for $L^{(d)}$ for $d$ sufficiently large and divisible. For instance, we say that $L$ is eventually generically birational if for $d$ sufficiently large and divisible $L^{(d)} \otimes \alpha$ is birational for $\alpha \in \text{Pic}^0(A)$ general.

In [4] we have studied the eventual generic behaviour of a line bundle $L$ with $h^0_a(L) > 0$:

**Theorem 2.2** (Thm. A of [4]). Let $X$ be a smooth projective variety, let be a strongly generating and generically finite map $a: X \to A$ to an abelian variety $A$, and let $L \in \text{Pic}(X)$ be such that $h^0_a(L) > 0$.

Then there exists a generically finite map $\varphi: X \to Z$, unique up to birational equivalence, such that:

(a) the map $a: X \to A$ is composed with $\varphi$.

(b) for $d \geq 1$ denote by $\varphi^{(d)}: X^{(d)} \to Z^{(d)}$ the map obtained from $\varphi: X \to Z$ by taking base change with $\mu_d$; then the map given by $|L^{(d)} \otimes \alpha|$ is composed with $\varphi^{(d)}$ for $\alpha \in \text{Pic}^0(A)$ general.

(c) for $d$ sufficiently large and divisible, the map $\varphi^{(d)}$ is birationally equivalent to the map given by $|L^{(d)} \otimes \alpha|$ for $\alpha \in \text{Pic}^0(A)$ general.

The map $\varphi: X \to Z$ is called the eventual map given by $L$ and the degree $m_L$ of $\varphi$ is called the eventual degree of $L$.

### 3. Eventual behaviour of the paracanonical system

With the same notation as in §2 we assume that $X$ is a smooth projective variety of maximal Albanese dimension with $\chi(K_X) > 0$ and we consider $L = K_X$. Since $h^0_a(K_X) = \chi(K_X)$ by generic vanishing, Theorem 2.2 can be applied to $L = K_X$ taking as $a: X \to A$ the Albanese map of $X$.
We call the eventual map \( \varphi : X \to Z \) given by \( K_X \) the eventual paracanonical map, its image \( Z \) the eventual paracanonical image and its degree \( m_X := m_{K_X} \) the eventual paracanonical degree.

The following result is analogous to Thm. 3.1 of [6].

**Theorem 3.1.** In the above set-up, denote by \( \widetilde{Z} \) any smooth model of \( Z \). Then one of the following occurs:

(a) \( \chi(K_{\widetilde{Z}}) = \chi(K_X) \)
(b) \( \chi(K_{\widetilde{Z}}) = 0 \).

Furthermore, in case (b) the Albanese image of \( X \) is ruled by tori.

**Proof.** We adapt the proof of Thm. 3.1 of [6].

Assume that \( \chi(K_{\widetilde{Z}}) > 0 \). Up to replacing \( X \) by a suitable smooth birational model, we may assume that the induced map \( \varphi : X \to \widetilde{Z} \) is a morphism. Recall that by Theorem 2.2, the Albanese map of \( X \) is composed with \( \varphi \). Fix \( \alpha \in \text{Pic}^0(A) \) general, so that in particular the map given by \( |K_X \otimes \alpha| \) is composed with \( \varphi \).

Pick \( 0 \neq \omega \in H^0(K_{\widetilde{Z}} \otimes \alpha) \), denote by \( \varphi^*\omega \in H^0(K_X \otimes \alpha) \) the pullback of \( \omega \) regarded as a differential form with values in \( \alpha \), and set \( D := \text{div}(\omega) \). The Hurwitz formula gives \( \text{div}(\varphi^*\omega) = \varphi^*D + R \), where \( R \) is the ramification divisor of \( \varphi \). On the other hand, by assumption we have that \( \text{div}(\varphi^*\omega) = \varphi^*H_0 + F \), where \( H_0 \) is an effective divisor of \( \widetilde{Z} \) and \( F \) is the fixed part of \( |K_X \otimes \alpha| \). Hence we have:

\[
\varphi^*D + R = \varphi^*H_0 + F.
\]

We want to use the above relation to show that \( H_0 \leq D \). Let \( \Gamma \) be a prime divisor of \( \widetilde{Z} \), let \( \Gamma' \) be a component of \( \varphi^*\Gamma \) and let \( e \) be the ramification index of \( \varphi \) along \( \Gamma' \), that is, \( e \) is the multiplicity of \( \Gamma' \) in \( \varphi^*\Gamma \). Note that \( \Gamma' \) appears in \( R \) with multiplicity \( e - 1 \). If \( a \) is the multiplicity of \( \Gamma \) in \( D \) and \( b \) is the multiplicity of \( \Gamma \) in \( H_0 \), then comparing the multiplicity of \( \Gamma' \) in both sides of (3.1) we get \( ea + e - 1 \geq eb \), namely \( a \geq b \). This shows that \( H_0 \leq D \) and we have the following chain of inequalities:

\[
h^0(H_0) \leq h^0(K_{\widetilde{Z}} \otimes \alpha) \leq h^0(K_X \otimes \alpha) = h^0(H_0).
\]

So all the inequalities in (3.2) are actually equalities and, by the generality of \( \alpha \), we have \( \chi(K_{\widetilde{Z}}) = h^0(K_{\widetilde{Z}} \otimes \alpha) = h^0(K_X \otimes \alpha) = \chi(K_X) \)

Assume \( \chi(K_{\widetilde{Z}}) = 0 \), instead. Then the Albanese image of \( \widetilde{Z} \) is ruled by tori by [9, Thm. 3]. Let \( \beta : \widetilde{Z} \to A \) be the map such that \( a = \beta \circ \varphi \). By the universal property of the Albanese map, we have that \( \beta \) is the Albanese map of \( \widetilde{Z} \), hence \( a(X) = \beta(\widetilde{Z}) \) is also ruled by tori. \[\Box\]
Corollary 3.2. In the above set-up, denote by \( n \) the dimension of \( X \). Then:

(i) \( \text{vol}(K_X) \geq m_X n! \chi(K_X) \)

(ii) if \( \chi(K_Z) > 0 \), then \( \text{vol}(K_X) \geq 2m_X n! \chi(K_X) \)

In particular, the stronger inequality (ii) holds whenever the Albanese image of \( X \) is not ruled by tori.

Proof. (i) Follows directly by Corollary 3.12 of [4].

For (ii) observe that if \( \chi(K_Z) > 0 \), then \( \chi(K_X) = \chi(K_Z) \) by Theorem 3.1; the Main Theorem of [1] gives \( \text{vol}(K_Z) \geq 2n! \chi(K_Z) = 2n! \chi(K_X) \) and, arguing as in Corollary 3.12, we have inequality (ii).

Finally, the last sentence in the statement is a consequence of Theorem 3.1. □

One can be more precise in the surface case:

Proposition 3.3. In the above set-up, assume that \( \dim X = 2 \) and \( X \) is minimal of general type. Then one of the following cases occurs:

(a) \( p_g(X) = p_g(Z) \), \( q(X) = q(Z) \) and \( m_X \leq 2 \)

(b) \( \chi(Z) = 0 \), \( 2 \leq m_X \leq 4 \) and the Albanese image of \( X \) is ruled by tori.

Proof. Assume \( \chi(Z) > 0 \); then by Theorem 3.1 we have \( \chi(Z) = \chi(X) \) and Corollary 3.2 (ii) gives \( K_X^2 \geq 4m_X \chi(X) \). Hence \( m_X \leq 2 \) follows immediately by the Bogomolov-Miyaoka-Yau inequality \( K_X^2 \leq 9 \chi(K_X) \).

We have \( q(Z) = q(X) \), since the Albanese map of \( X \) is composed with \( \varphi: X \to Z \), so \( p_g(X) = \chi(X) + q(X) - 1 = \chi(Z) + q(Z) - 1 = p_g(Z) \).

Assume now that \( \chi(Z) = 0 \); in this case the Albanese image of \( X \) is ruled by tori by Theorem 3.1 and Corollary 3.2 (i) gives \( K_X^2 \geq 2m_X \chi(X) \). Using the Bogomolov-Miyaoka-Yau inequality as above we get \( m_X \leq 4 \). On the other hand \( m_X > 1 \), since \( \varphi \) cannot be birational because \( \chi(X) > \chi(Z) \).

Remark 3.4. For the case of a minimal smooth threefold of general type \( X \), it is proven in [7] that the inequality \( K_X^3 \leq 72 \chi(X) \) holds. Hence the arguments used to prove Proposition 3.3 above show that in this case we have \( m_X \leq 6 \) if \( \chi(Z) > 0 \), and \( m_X \leq 12 \) if \( \chi(Z) = 0 \).

4. Eventual behaviour of the \( m \)-paracanonical system for \( m > 1 \)

In this section we consider the eventual \( m \)-paracanonical map for varieties of general type and maximal Albanese dimension, for \( m \geq 2 \).
Our main result implies that for \( m \geq 2 \) the eventual \( m \)-paracanonical map does not give additional information on the geometry of \( X \):

**Theorem 4.1.** Let \( X \) be a smooth projective \( n \)-dimensional variety of general type and maximal Albanese dimension; denote by \( a : X \to A \) the Albanese map.

Then there exists a positive integer \( d \) such that the system \( |mK_X(d) \otimes \alpha| \) is birational for every \( m \geq 2 \) and for every \( \alpha \in \text{Pic}^0(A) \).

In particular \( mK_X \) is eventually generically birational for \( m \geq 2 \).

**Remark 4.2.** The question whether the \( m \)-canonical system of a variety of maximal Albanese dimension is birational has been considered by several authors.

The answer is positive for \( m \geq 3 \) ([10], [8], [16]).

The case \( m = 2 \) has been studied in [2] for a variety \( X \) of maximal Albanese dimension with \( q(X) > \dim X \), proving that if \( |2K_X| \) is not birational then either \( X \) is birational to a theta divisor in a p.p.a.v. or there exists a fibration \( f : X \to Y \) onto an irregular variety with \( \dim Y < \dim X \). If \( X \) is a smooth theta divisor in a p.p.a.v., then it is easy to check directly that \( |2K_X \otimes \alpha| \) is birational for \( 0 \neq \alpha \in \text{Pic}^0(X) \) and that for every \( d \geq 2 \) the system \( |K_X(d) \otimes \alpha| \) is birational for every \( \alpha \in \text{Pic}^0(X) \).

**Warning:** in this section all the equalities of divisors that we write down are equalities of \( \mathbb{Q} \)-divisors up to numerical equivalence. Moreover, in accordance with the standard use in birational geometry, we switch to the additive notation and write \( L + \alpha \) instead of \( L \otimes \alpha \).

We fix a very ample divisor \( H \) on \( A \) and we set \( M := a^*H \); we use the notation of diagram 2.1 and set \( M_d := a_d^*H \) for \( d > 1 \). By [13] Prop 2.3.5, we have that \( \tilde{\mu}_d^*M \equiv d^2M_d \).

Moreover, since the question is birational, we may assume that the map \( f : X \to Y \) to the canonical model (which exists by [3]) is a log resolution.

To prove Theorem 4.1 we use the following preliminary result, that generalizes [3] Lem. 2.4:

**Lemma 4.3.** In the above set-up, there exists \( d \gg 0 \) such that

\[
K_{X(d)} = L + (n - 1)M_d + E,
\]

where \( L \) and \( E \) are \( \mathbb{Q} \)-divisors such that \( L \) is nef and big and \( E \) is effective with normal crossings support.

**Proof.** By [12] Cor.1.5] the fibers of \( f \) are rationally chain connected, hence they are contracted to points by \( a \); it follows that the Albanese map of \( X \) descends to a morphism \( \overline{\alpha} : Y \to A \).
Setting \( M := \varpi^* H \), there exists \( \epsilon > 0 \) such that for \( |t| < \epsilon \) the class \( K_Y - tM \) is ample and therefore \( f^* K_Y - tM \) is nef and big on \( X \). By the definition of canonical model we have \( K_X = f^* K_Y + E \), with \( E \) an effective \( \mathbb{Q} \)-divisor. In addition, the support of \( E \) is a normal crossings divisor since \( f \) is a log resolution. Choosing \( d \) such that \( \frac{n-1}{td^2} < \epsilon \) and pulling back to \( X \), we have

\[
K_{X(d)} = \tilde{\mu}_d^* K_X = \tilde{\mu}_d^* (f^* K_Y - \frac{n-1}{d^2} M) + \frac{n-1}{d^2} \tilde{\mu}_d^* M + \tilde{\mu}_d^* E.
\]

The statement now follows by observing that:

- \( \frac{n-1}{td^2} \tilde{\mu}_d^* M = (n-1)M_d \),
- \( L := \tilde{\mu}_d^* (f^* K_Y - \frac{n-1}{d^2} M) = \tilde{\mu}_d^* f^* K_Y - (n-1)M_d \) is the class of a nef and big line bundle,
- \( \tilde{\mu}_d^* E \) is an effective \( \mathbb{Q} \)-divisor with normal crossings support.

\[\square\]

**Proof of Thm. 4.1.** Since \( X \) is of maximal Albanese dimension, \( K_X \) is effective and \( |2K_X| \subset |mK_X| \) for \( m \geq 2 \), hence it is enough to to prove the statement for \( m = 2 \). We may also assume \( n \geq 2 \), since for \( n = 1 \) the statement follows easily by Riemann-Roch.

By Theorem 2.2 and Lemma 4.3 we can fix \( d \gg 0 \) such that:

- \( K_{X(d)} = L + (n-1)M_d + E \), with \( L \) nef and big and \( E \) is effective with normal crossings support.
- for \( \alpha \in \text{Pic}^0(A) = \text{Pic}^0(X) \) general the map given by \( |2K_{X(d)} + \alpha| \) coincides with \( \varphi^{(d)} \) (notation as in Theorem 2.2)

Write \( E = [E] + \Delta \), so that \( \Delta \geq 0 \) is a \( \mathbb{Q} \)-divisor with \( |\Delta| = 0 \) and normal crossings support. Set \( D := K_{X(d)} - [E] = L + (n-1)M_d + \Delta \). We are going to show that if \( C \subset X \) is the intersection of \( (n-1) \) general elements of \( |M_d| \), then the map induced by \( |K_{X(d)} + D + \alpha| \) restricts to a generically injective map on \( C \) for \( \alpha \in \text{Pic}^0(A) \) general. Since \( |K_{X(d)} + D + \alpha| \subseteq |2K_{X(d)} + \alpha| \) and the map \( \varphi^{(d)} \) is composed with the map \( a_d : X^{(d)} \to A \) by Theorem 2.2 this will prove the statement.

First of all we prove that \( |K_{X(d)} + D + \alpha| \) restricts to a complete system on \( C \) by showing that \( h^1(\mathcal{I}_C(K_{X(d)} + D + \alpha)) = 0 \) for every \( \alpha \in \text{Pic}^0(A) \). To this end we look at the resolution of \( \mathcal{I}_C \) given by the Koszul complex:

\[
0 \to \mathcal{O}_{X^{(d)}}(-(n-1)M_d) \to \cdots \to \bigwedge^2 \mathcal{O}_{X^{(d)}}(-M_d)^{\oplus n-1} \to \mathcal{O}_{X^{(d)}}(-M_d)^{\oplus n-1} \to \mathcal{I}_C \to 0
\]

Twisting (4.1) by \( K_{X^{(d)}} + D + \alpha \) we obtain a resolution of \( \mathcal{I}_C(K_{X^{(d)}} + D + \alpha) \) such that every term is a sum of line bundles numerically equivalent.
to a line bundle of the form
\[ K_{X(d)} + D - iM_d = K_{X(d)} + L + (n - 1 - i)M_d + \Delta \]
for some \( 1 \leq i \leq n - 1 \). Since the hypotheses of Kawamata-Viehweg’s vanishing theorem apply to each of these line bundles, it follows that \( \mathcal{I}_C(K_{X(d)} + D + \alpha) \) has a resolution of length \( n - 2 \) such that all the sheaves appearing in the resolution have zero higher cohomology. Hence the hypercohomology spectral sequence gives \( h^1(\mathcal{I}_C(K_{X(d)} + D + \alpha)) = 0 \) as claimed.

Finally observe that by construction \( (K_{X(d)} + D + \alpha)|_C \) has degree strictly greater than \( 2g(C) + 1 \) and therefore it is very ample. \( \Box \)

5. Examples, remarks and open questions

In this section we give some examples of the eventual behaviour of the paracanonical system; we keep the notation of \( \S 2 \) and \( \S 3 \). All our examples are constructed as abelian covers; we use the notation and the general theory of [15].

Let \( \Gamma \) be a finite abelian group, let \( \Gamma^* := \text{Hom}(\Gamma, \mathbb{C}^*) \) be its group of characters, and let \( \pi: X \to Y \) be a flat \( \Gamma \)-cover with \( Y \) smooth and \( X \) normal. One has a decomposition \( \pi_*\mathcal{O}_X = \oplus_{\chi \in \Gamma^*} L^{-1}_\chi \), where \( L_\chi \) is a line bundle and \( \Gamma \) acts on \( L^{-1}_\chi \) via the character \( \chi \). In particular, we have \( L_1 = \mathcal{O}_Y \).

Our main observation is the following:

Lemma 5.1. In the above setup, assume that \( X \) and \( Y \) are smooth of maximal Albanese dimension; denote by \( \overline{\pi}: Y \to A \) the Albanese map. If the following conditions hold:

(a) the map \( \pi \) is totally ramified, i.e., it does not factor through an étale cover \( Y' \to Y \) of degree \( > 1 \);
(b) for every \( 1 \neq \chi \in \Gamma^* \), \( h^1(L^{-1}_\chi) = 0 \);
(c) there exists precisely one element \( \overline{\chi} \in \Gamma^* \) such that \( h^0_\alpha(K_Y + L_{\overline{\chi}}) > 0 \).

then:

(i) the map \( a := \overline{\pi} \circ \pi \) is the Albanese map of \( X \);
(ii) the eventual paracanonical map \( \varphi: X \to Z \) is composed with \( \pi: X \to Y \);
(iii) if in addition \( K_Y \otimes L_{\overline{\chi}} \) is eventually generically birational (e.g., if \( a \) is generically injective), then \( \pi \) is birationally equivalent to \( \varphi \).

Proof. We use freely the notation of \( \S 2 \).
(i) First of all, by condition (b) we have
\[ q(X) = h^1(\mathcal{O}_X) = \sum \chi h^1(L_X^{-1}) = q(Y), \]
hence the induced map \( \text{Alb}(X) \to A \) is an isogeny. On the other hand, the fact that \( \pi \) is totally ramified implies that the map \( \text{Pic}^0(Y) \to \text{Pic}^0(X) \) is injective. It follows that \( \text{Alb}(X) \to A \) is an isomorphism and the map \( a = \overline{a} \circ \pi \) is the Albanese map of \( X \).

(ii) Denote by \( \pi^{(d)} : X^{(d)} \to Y^{(d)} \) the \( \Gamma \)-cover induced by \( \pi : X \to Y \) taking base change with \( Y^{(d)} \to Y \). There is a cartesian diagram:

\[
\begin{array}{ccc}
X^{(d)} & \longrightarrow & X \\
\pi^{(d)} \downarrow & & \downarrow \pi \\
Y^{(d)} & \longrightarrow & Y \\
\pi_d \downarrow & & \downarrow \pi \\
A & \longrightarrow & A \\
\end{array}
\] (5.1)

By Theorem 2.2, it is enough to show that for every \( d \) the system \( |K_{X^{(d)}} \otimes \alpha| \) is composed with \( \pi^{(d)} : X^{(d)} \to Y^{(d)} \) for \( \alpha \in \text{Pic}^0(A) \) general or, equivalently, that \( |K_{X^{(d)}} \otimes \alpha| \) is \( \Gamma \)-invariant for \( \alpha \in \text{Pic}^0(A) \) general. One has \( \pi^{(d)}_* \mathcal{O}_{X^{(d)}} = \bigoplus_{\chi \in \Gamma} (L_X^{(d)})^{-1} \). Hence by the formulae for abelian covers we have \( h^0_{\alpha}(K_{X^{(d)}}) = \sum_{\chi \in \Gamma} h^0_{\alpha}(K_{Y^{(d)}} \otimes L_X^{(d)}) = h^0_{\alpha}(K_{Y^{(d)}} \otimes L_X^{(d)}) \), because of condition (c). So, for \( \alpha \in \text{Pic}^0(A) \) general, we have \( H^0(K_{X^{(d)}} \otimes \alpha) = H^0(K_{Y^{(d)}} \otimes L_X^{(d)} \otimes \alpha) \); it follows that \( \Gamma \) acts trivially on \( \mathbb{P}(H^0(K_{X^{(d)}} \otimes \alpha)) \) and therefore the map given by \( |K_{X^{(d)}} \otimes \alpha| \) factors through \( \pi^{(d)} \).

(iii) The proof of (ii) shows that eventually \( \varphi^{(d)} \) is birationally equivalent to the composition of \( \pi^{(d)} \) with the map given by \( |K_{Y^{(d)}} \otimes L_X^{(d)} \otimes \alpha| \) for \( \alpha \in \text{Pic}^0(A) \) general. So if \( |K_Y \otimes L_X| \) is eventually generically birational, then \( \varphi^{(d)} \) and \( \pi^{(d)} \) are birationally equivalent for \( d \) sufficiently large and divisible. \hfill \Box

**Example 5.2** (Examples with \( m_X = 2 \)). Every variety of maximal Albanese dimension \( Z \) with \( \chi(Z) = 0 \) and \( \dim Z = n \geq 2 \) occurs as the eventual paracanonical image for some \( X \) of general type with \( m_X = 2 \).

Let \( L \) be a very ample line bundle on \( Z \), pick a smooth divisor \( B \in |2L| \) and let \( \pi : X \to Z \) be the double cover given by the equivalence relation \( 2L \equiv B \). Since \( Z \) is of maximal Albanese dimension, \( K_Z \) is effective and therefore \( |K_Z \otimes L| \) is birational. By Kodaira vanishing we
have \( h^1(Z, L^{-1}) = 0 \) and
\[
\chi(K_Z \otimes L) = h^0(Z, K_Z \otimes L) = h^0(Z, K_Z \otimes L).
\]
So by continuity the system \(|K_Z \otimes L|\) is birational for general \( \alpha \in \text{Pic}^0(Z) \). By Lemma 3.5 of [4], \( K_{Z(\alpha)} \otimes L^{(d)} \) is generically birational for every \( d \) and therefore \( \pi: X \to Z \) is the eventual paracanonical map by Lemma 5.1.

**Remark 5.3.** If in Example 5.2 we take a variety \( Z \) whose Albanese map is not generically injective, then the eventual paracanonical map of the variety \( X \) is neither an isomorphism nor coincides with the Albanese map of \( X \).

Varieties \( Z \) with \( \chi(Z) = 0 \) whose Albanese map is generically finite of degree \( > 1 \) do exist for \( \dim Z \geq 2 \). Observe that it is enough to construct 2-dimensional examples, since in higher dimension one can consider the product of \( Z \) with any variety of maximal Albanese dimension. To construct an example with \( \dim Z = 2 \) one can proceed as follows:

- take a curve \( B \) with a \( \mathbb{Z}_2 \)-action such that the quotient map \( \pi: B \to B' := B/\mathbb{Z}_2 \) is ramified and \( g(B') > 0 \);
- take \( E \) an elliptic curve, choose a \( \mathbb{Z}_2 \)-action by translation and denote by \( E' \) the quotient curve \( E/\mathbb{Z}_2 \);
- set \( Z := (B \times E)/\mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) acts diagonally on the product.

The map \( f: Z \to B' \times E' \) is a ramified double cover and it is easy to see that the Albanese map of \( Z \) is the composition of \( f \) with the inclusion \( E' \times B' \to E' \times J(B') \). Indeed, by the universal property of the Albanese map there is a factorization \( Z \to \text{Alb}(Z) \to E' \times J(B') \). The map \( \text{Alb}(Z) \to E' \times J(B') \) is an isogeny, since \( q(Z) = g(B') + 1 \); in addition, the dual map \( E' \times J(B') \to \text{Pic}^0(Z) \) is injective, since \( f \) is ramified. We conclude that \( \text{Alb}(Z) \to E' \times J(B') \) is an isomorphism and therefore the Albanese map of \( Z \) has degree 2.

**Example 5.4** (Examples with \( m_X = 4 \) and \( \chi(Z) = 0 \)). For \( i = 1, 2 \) let \( A_i \) be abelian varieties of dimension \( q_i \geq 1 \) and set \( A = A_1 \times A_2 \).

For \( i = 1, 2 \) let \( M_i \) be a very ample line bundle on \( A_i \) and choose \( B_i \in |2M_i| \) general; we let \( \pi: X \to A \) be the \( \mathbb{Z}_2 \)-cover with branch divisors \( D_1 = \text{pr}_1^* B_1 \), \( D_2 = \text{pr}_2^* B_2 \) and \( D_3 = 0 \), given by relations \( 2L_1 \equiv D_2 \) and \( 2L_2 \equiv D_1 \), where \( L_1 = \eta_1 \boxtimes M_2 \), \( L_2 = M_1 \boxtimes \eta_2 \), with \( 0 \neq \eta_i \in \text{Pic}^0(A_i)[2] \). One has \( L_3 = L_1 \otimes L_2 \), so that \( L_3 \) is very ample. It is easy to check that \( h^1(L_j^{-1}) = 0 \) for \( j = 1, 2, 3 \) and \( h^0(L_j) > 0 \) if and only if \( j = 3 \). So the assumptions of Lemma 5.1 are satisfied and arguing as in Example 5.2 we see that \( \pi: X \to A \) is both the Albanese map and the eventual paracanonical map of \( X \).
An interesting feature of this example is that for \(d\) even the cover \(\pi^{(d)}: X^{(d)} \to A\) is the product of two double covers \(f^{(d)}_i: X_i \to A_i, i = 1, 2\), branched on the pull back of \(B_i\). If \(q_1, q_2 \geq 2\), then \(q(X^{(d)}) = q(X)\), but if, say, \(q_1 = 1\) then \(q(X^{(d)})\) grows with \(d\).

In particular, if \(q_1 = q_2 = 1\) then for \(d\) even the surface \(X^{(d)}\) is a product of bielliptic curves.

**Example 5.5** (Examples with large \(m_X\)). A variation of Example 5.4 gives examples with higher values of \(m_X\). For simplicity, we describe an example of a threefold \(X\) with \(m_X = 8\), but it is easy to see how to modify the construction to obtain example with \(m_X = 2^k\) for every \(k\) (the dimension of the examples will also increase, of course).

For \(i = 1, \ldots, 3\), let \(A_i\) be elliptic curves, let \(M_i\) be a very ample divisor on \(A_i\) and let \(B_i \in |2M_i|\) be a general divisor. Set \(A = A_1 \times A_2 \times A_3\) and choose \(\eta_1, \eta_2 \in \text{Pic}^0(A)[2]\) distinct such that, setting \(\eta_3 = \eta_1 + \eta_2\), then \(\eta_j|_{A_i} \neq 0\) for every choice of \(i, j \in \{1, 2, 3\}\). Set \(\Gamma := \mathbb{Z}_2^3\) and denote by \(\gamma_1, \gamma_2, \gamma_3\) the standard generators of \(\Gamma\). We let \(\pi: X \to A\) be the \(\Gamma\)-cover given by the following building data:

\[
D_{\gamma_i} = \text{pr}_i^* B_i, \quad i = 1, \ldots, 3, \quad D_\gamma = 0 \quad \text{if} \quad \gamma \neq \gamma_i,
\]

and

\[
L_i = \text{pr}_i^* M_i \otimes \eta_i, \quad i = 1, \ldots, 3.
\]

It is immediate to see that the reduced fundamental relations (cf. [15, Prop. 2.1]):

\[
2L_i \equiv D_{\gamma_i}, \quad i = 1, \ldots, 3
\]

are satisfied. The corresponding cover \(\pi: X \to A\) satisfies the assumptions of Lemma 5.1 and arguing as in the previous cases we see that \(\pi: X \to A\) is both the Albanese map and the eventual paracanonical map of \(X\).

**Question 5.6.** The statement of Theorem 3.1 shows an analogy between the eventual behaviour of the paracanonical system of varieties of maximal Albanese dimension and the canonical map (cf. [6, Thm. 3.1]). As in the case of the canonical map, the previous examples show that it is fairly easy to produce examples of case (b) of Theorem 3.1 (cf. Example 5.2). On the other hand, we do not know any examples of case (a) with \(m_X > 1\).

For instance, in the 2-dimensional case by Proposition 3.3 \(X\) would be a surface of general type and maximal Albanese dimension with an involution \(\sigma\) such that \(q(X/\sigma) = q(X)\), \(p_g(X/\sigma) = p_g(X)\) and such that the eventual paracanonical map of \(X/\sigma\) is birational. If such an \(X\) exists, then it has Albanese map of degree \(2k\), with \(k \geq 2\). Indeed, the
Albanese map of $X$ is composed with $\sigma$ and $K_{X/\sigma}^2 \leq \frac{1}{2} K_X^2 \leq \frac{9}{2} \chi(X) = \frac{9}{2} \chi(X/\sigma) < 5 \chi(X/\sigma)$, where the last inequality but one is given by the Bogomolov-Miyaoka-Yau inequality. Theorem 6.3 (ii) of [4] implies that the Albanese map of $X/\sigma$ has degree $k > 1$.

**Question 5.7.** In the surface case the eventual paracanonical degree is at most 4 by Proposition 3.3. In order to give a bound for higher dimensional varieties in the same way, one would need to bound the volume of $K_X$ in terms of $\chi(K_X)$; however Example 8.5 of [4] shows that this is not possible for $n \geq 3$. So, it is an open question whether it is possible to give a bound on $m_X$, for $X$ a variety of fixed dimension $n > 2$. Note that Example 5.5 shows that this bound, if it exists, has to increase with $n$.

**References**

[1] M.A. Barja, *Generalized Clifford Severi inequality and the volume of irregular varieties*. Duke Math. J. 164 (2015), no. 3, 541–568.

[2] M.A. Barja, M. Lahoz, J.C. Naranjo, G. Pareschi, *On the bicanonical map of irregular varieties*. J. Algebraic Geom. 21 (2012), no. 3, 445–471.

[3] M.A. Barja, R. Pardini, L. Stoppino, *Surfaces on the Severi line*, Journal de Mathématiques Pures et Appliquées, (2016), no. 5, 734–743.

[4] M.A. Barja, R. Pardini, L. Stoppino, *Linear systems on irregular varieties*, preprint.

[5] C. Birkar, P. Cascini, C.D. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*. J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[6] A. Beauville, *L’application canonique pour les surfaces de type général*. Inv. Math. 55 (1979), 121–140.

[7] J.A. Chen, M. Chen, D.Q. Zhang, *The 5-canonical system on 3-folds of general type*. J. Reine Angew. Math. 603 (2007), 165–181.

[8] J.A. Chen, C. Hacon, *Linear series on irregular varieties*, In “Algebraic geometry in East Asia (Kyoto, 2001)”, 143–153, World Sci. Publ., River Edge, NJ (2002).

[9] L. Ein, R. Lazarsfeld, *Singularities of Theta divisors, and birational geometry of irregular varieties*, J. Amer. Math. Soc. 10, 1 (1997), 243–258.

[10] Z. Jiang, M. Lahoz, S. Tirabassi, *On the Iitaka fibration of varieties of maximal Albanese dimension*. Int. Math. Res. Not. (2013) no. 13, 2984–3005.

[11] M. Green, R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math., 90, 1987, 416–440.

[12] C. Hacon, J. McKernan, *On Shokurov’s rational connectedness conjecture*, Duke Math. J. 138 (2007), no. 1, 119-136.

[13] H. Lange, C. Birkenhake Complex Abelian Varieties. Second edition. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004.
[14] R. Lazarsfeld, Positivity in algebraic geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 48. Springer-Verlag, Berlin, 2004.
[15] R. Pardini, Abelian covers of algebraic varieties, J. reine angew. Math. 417 (1991), 191–213.
[16] G. Pareschi, M. Popa, Regularity on abelian varieties III: relationship with Generic Vanishing and applications in Grassmannians, Moduli Spaces and Vector Bundles (D.A. Ellwood and E. Previato, eds.), 141–168, Clay Mathematics Proceedings 14, Amer. Math. Soc., Providence, RI, 2011, 141–168.