Supersymmetric indices on $I \times T^2$, elliptic genera, and dualities with boundaries

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Abstract

We study three dimensional $\mathcal{N} = 2$ supersymmetric theories on $I \times M_2$ with 2d $\mathcal{N} = (0, 2)$ boundary conditions at the boundaries $\partial(I \times M_2) = M_2 \sqcup M_2$, where $M_2 = \mathbb{C}$ or $T^2$. We introduce supersymmetric indices of three dimensional $\mathcal{N} = 2$ theories on $I \times T^2$ that couple to elliptic genera of 2d $\mathcal{N} = (0, 2)$ theories at the two boundaries. We evaluate the $I \times T^2$ indices in terms of supersymmetric localization and study dualities on the $I \times M_2$. We consider the dimensional reduction of $I \times T^2$ to $I \times S^1$ and obtain the localization formula of 2d $\mathcal{N} = (2, 2)$ supersymmetric indices on $I \times S^1$. We illustrate computations of open string Witten indices based on gauged linear sigma models. Correlation functions of Wilson loops on $I \times S^1$ agree with Euler pairings in the geometric phase and also agree with cylinder amplitudes for B-type boundary states of Gepner models in the Landau-Ginzburg phase.
1 Introduction

In our previous work [1], we constructed Lagrangians and BPS boundary conditions for 3d $\mathcal{N} = 2$ supersymmetric (SUSY) gauge theories on $S^1 \times D^2$ where $S^1 \times D^2$ is the direct product of a circle and a two dimensional (2d) hemisphere. We defined supersymmetric
indices on $S^1 \times D^2$ and evaluated them in terms of supersymmetric localization. The indices on $S^1 \times D^2$ we have evaluated are $S^1$-extensions of partition functions of 2d $\mathcal{N} = (2, 2)$ theory on $D^2$ \cite{2,3}, those have nice properties; the $K$-theoretic $I$-function \cite{4} appears in the index on $S^1 \times D^2$ which is a q-deformation (trigonometric deformation) of the Givental $I$-function \cite{5} for the moduli space of Higgs branch vacua. For example, see subsequent works for relations between the indices on $S^1 \times D^2$ and the $K$-theoretic $I$-functions \cite{6,7}. For 3d $\mathcal{N} = 4$ gauge theories with an $\mathcal{N} = (2, 2)$ boundary condition, the indices on $S^1 \times D^2$ agree with previous results for the equivariant indices on $S^1 \times \mathbb{C}$, Coulomb gas representations of q-deformation of conformal blocks in qW-algebras \cite{8,9}, and also vertex functions for the $K$-theory of quasimap spaces of Nakajima quiver varieties \cite{10}.

In two dimensions, quantum field theories on spacetimes (worldsheets) with boundaries have attracted much interest in the connections with open strings and D-branes. The hemisphere partition function \cite{2,3} we have mentioned in the above is a typical example of the spacetime with a boundary and it gives Gamma classes, central charges of D-branes and period integrals of mirror Calabi–Yau 3-folds in the string theory. Along with the hemisphere, a basic 2d spacetime with boundaries is the cylinder $I \times S^1$. supersymmetric indices on $I \times S^1$ with states at the boundaries are called open string Witten indices that are related to Euler parings for Calabi-Yau 3-folds in string theory. In this article we study $S^1$-extensions of the open string Witten indices, i.e., three dimensional (3d) $\mathcal{N} = 2$ supersymmetric theories on $I \times T^2$ coupled to 2d $\mathcal{N} = (0, 2)$ boundary theories at the end points of the interval $I$.

This article is organized as follows. In section 2 we study supersymmetric boundary conditions on $I \times M_2$, where $M_2 = \mathbb{C}$ or $T^2$ and construct supersymmetric Lagrangians. This part is essentially same as the analysis in \cite{1}. In section 3 we define a supersymmetric index on $I \times T^2$ by imposing a twisted boundary condition along $T^2$ and impose boundary conditions at the left end and at the right end of $I$. By supersymmetric localization, we will show that the path integral of the index is reduced to multi-contour integrals called the Jeffrey–Kirwan residue. In section 4 we construct 3d theories in which $I \times T^2$ indices are identical to 2d $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (0, 4)$ elliptic genera. In section 5 we study three dimensional dualities; we consider three dimensional IR dualities and put the dual theories on $I \times M_2$. We impose the boundary conditions that cancel the gauge anomaly between the bulk and the boundary and satisfy the ’t Hooft anomaly matching condition. We will show that the $I \times T^2$ indices match in the dual pairs. In section 6 we study chiral algebras associated with simple models on $I \times M_2$; free chiral multiplets with the Dirichlet and Neumann boundary conditions. In section 7 we perform the dimensional reduction of 3d $\mathcal{N} = 2$ theories to 2d $\mathcal{N} = (2, 2)$ theories on $I \times S^1$ and obtain the supersymmetric localization formula for $I \times S^1$ indices. We compare $I \times S^1$ indices with open string Witten indices in the geometric and Landau–Ginzburg (LG) phases. In the section 8 we summarize our results and give comments on future directions.
\section{3d $\mathcal{N} = 2$ theories on $I \times M_2$ with 2d $\mathcal{N} = (0, 2)$ boundary conditions}

In this section we consider 3d $\mathcal{N} = 2$ supersymmetric theory on the direct product of one-dimensional interval and two dimensional flat space $I \times M_2$ with $M_2 = \mathbb{C}$ or $T^2$ and study BPS boundary conditions. We couple the 3d $\mathcal{N} = 2$ theory to boundary 2d $\mathcal{N} = (0, 2)$ theories. The BPS boundary conditions and the boundary interactions we will consider are same as those for $S^1 \times D^2$ introduced in \cite{I}, more precisely those obtained by the flat space limit of $S^1 \times D^2$. We define coordinates $I \times \mathbb{C}$ as follows:

\begin{equation}
I \times \mathbb{C} = \{(x^1, x^2, x^3) \mid x_1 \in [-\pi L, \pi L], x^2, x^3 \in \mathbb{R}\} = \{(x^1, w, \bar{w}) \mid x_1 \in [-\pi L, \pi L], w = x^2 + i x^3, \bar{w} = x^2 - i x^3\}.
\end{equation}

A set coordinates of $T^2$ in $I \times T^2$ is defined by

\begin{equation}
T^2 := \{(x^2, x^3) \mid x^2 + i x^3 \sim x^2 + i x^3 + 2 \pi R \sim x^2 + i x^3 + 2 \pi R \tau\}.
\end{equation}

Here $\tau = \tau_1 + i \tau_2$ is the moduli of the torus $T^2$. The twisted boundary condition for $T^2$ is explained in the next section.

The SUSY transformation $\delta$ of the 3d $\mathcal{N} = 2$ theory is written as $\delta = \epsilon^{\alpha} Q_\alpha + \bar{\epsilon}^{\alpha} \bar{Q}_\alpha$, with the four supercharges $Q_\alpha, \bar{Q}_\alpha$ with $\alpha = 1, 2$. The contractions of spinors are defined below \cite{2.7}. In this paper we choose two component spinors $\epsilon = (\epsilon_1, \epsilon_2)^T$ and $\bar{\epsilon} = (\bar{\epsilon}_1, \bar{\epsilon}_2)^T$ in $\delta$ as

\begin{equation}
\epsilon = \epsilon' \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{\epsilon} = \bar{\epsilon}' \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{equation}

where $\epsilon'$ and $\bar{\epsilon}'$ are Grassmann odd constants. Under appropriate boundary conditions which will be mentioned later, the following two supercharges $Q$ and $\bar{Q}$ are preserved. The SUSY transformation generated by \cite{2.3} is written as

\begin{equation}
\delta = \epsilon' Q + \bar{\epsilon}' \bar{Q} \quad \text{with} \quad Q := Q_2 - Q_1, \quad \bar{Q} := \bar{Q}_2 - \bar{Q}_1.
\end{equation}

We will see \cite{2.3} preserves 2d $\mathcal{N} = (0, 2)$ supersymmetry at the boundaries. Instead of \cite{2.3}, if we choose $\epsilon = \epsilon'(1, -1)^T$ and $\bar{\epsilon} = \bar{\epsilon}'(-1, 1)^T$, 2d $\mathcal{N} = (2, 0)$ supersymmetry is preserved at the boundaries with appropriate boundary conditions.

\subsection{3d $\mathcal{N} = 2$ vector multiplet}

Let $G$ be the gauge group given by a compact Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. We consider a $G$ vector multiplet with BPS boundary conditions at the boundaries $\partial(I \times M_2) =$
$M_{2,L} \sqcup M_{2,R}$, where $M_{2,R} := \{x^1 = \pi L\} \times M_2$ and $M_{2,L} := \{x^1 = -\pi L\} \times M_2$. The integrals at the two boundaries have opposite signs:

$$\int_{M_{2,L} \sqcup M_{2,R}} (\cdots) = -\int_{\{x^1 = -\pi L\} \times M_2} (\cdots) + \int_{\{x^1 = \pi L\} \times M_2} (\cdots). \quad (2.5)$$

The vector multiplet consists of a $g$ valued gauge field $A_\mu$, a real scalar $\sigma$, gaugini $\lambda, \bar{\lambda}$, and an auxiliary field $D$. In our convention, the covariant derivatives and the field strength of the gauge field are defined by

$$D_\mu = \partial_\mu + iA_\mu, \quad [D_\mu, D_\nu] = iF_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \quad (2.6)$$

The SUSY transformation of the 3d $\mathcal{N} = 2$ vector multiplet is given by

$$\delta A_\mu = \frac{i}{2}(\bar{\epsilon}\gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon),$$

$$\delta \sigma = \frac{i}{2}(\bar{\epsilon}\lambda - \bar{\lambda} \epsilon),$$

$$\delta \lambda = -\frac{i}{2} \gamma^{\mu\nu} F_{\mu\nu} \epsilon - D\epsilon + i\gamma_\mu D_\mu \sigma \epsilon,$$

$$\delta \bar{\lambda} = -\frac{i}{2} \gamma^{\mu\nu} F_{\mu\nu} \bar{\epsilon} + D\bar{\epsilon} - i\gamma_\mu D_\mu \bar{\epsilon} \sigma,$$

$$\delta D = -\frac{i}{2} \epsilon\gamma_\mu D_\mu \lambda - \frac{i}{2} D_\mu \bar{\lambda} \gamma_\mu \epsilon + \frac{i}{2}[\epsilon\lambda, \sigma] + \frac{i}{2}[\bar{\lambda} \epsilon, \sigma],$$

where $\mu = 1, 2, 3$. $\gamma^1, \gamma^2, \gamma^3$ are the Pauli matrices and $\gamma^{\mu\nu} := \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Again $\epsilon$ and $\bar{\epsilon}$ are Grassmann odd two components spinors. The contractions of spinors are defined by $\epsilon \lambda = \epsilon^\alpha \lambda_\alpha := \epsilon^T C \lambda$, $\epsilon \gamma_\mu \lambda = \epsilon^\alpha (\gamma^\mu)^\alpha_\beta \lambda_\beta := \epsilon^T C \gamma^\mu \lambda$, where $C := -i\gamma^2$ is a charge conjugation matrix.

A BPS boundary condition at each $x_1 = \pm \pi L$ is given by

$$\sigma = 0, \quad A_1 = 0, \quad \partial_1 A_2 = 0, \quad \partial_1 A_3 = 0, \quad \partial_1 D = 0,$$

$$\lambda_1 - \lambda_2 = 0, \quad \bar{\lambda}_1 - \bar{\lambda}_2 = 0, \quad \partial_1 (\lambda_1 + \lambda_2) = 0, \quad \partial_1 (\bar{\lambda}_1 + \bar{\lambda}_2) = 0. \quad (2.8)$$

Here $\lambda_\alpha, \bar{\lambda}_\alpha$ with $\alpha = 1, 2$ are the components of the gaugini defined by $\lambda = (\lambda_1, \lambda_2)^T$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T$. The SUSY transformations generated by (2.3) are consistent with the boundary condition (2.8).

The restriction of the SUSY transformation (2.7) on a boundary with (2.8) is written as

$$\delta (A_2 + iA_3) = 0,$$

$$\delta (A_2 - iA_3) = 2\bar{\epsilon}' \lambda_1 + 2\epsilon' \bar{\lambda}_1,$$

$$\delta [\hat{D} + iF_{23}] = -2\epsilon' (D_2 + iD_3)\lambda_1,$$

$$\delta [\hat{D} - iF_{23}] = 2\epsilon' (D_2 + iD_3)\bar{\lambda}_1,$$

$$\delta \lambda_1 = -(\hat{D} + iF_{23})\epsilon',$$

$$\delta \bar{\lambda}_1 = (\hat{D} - iF_{23})\epsilon'.$$

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where we defined $\hat{D} := D - iD_1\sigma$. From the boundary condition, $\sigma = 0$ and $A_1 = 0$ at the boundary. (2.9) is same as the SUSY transformation of the 2d $\mathcal{N} = (0, 2)$ $G$ vector multiplet $(A_1, A_2, \lambda_1, \lambda_1, \hat{D})$. The boundary condition (2.8) is preserved by the 2d $\mathcal{N} = (0, 2)$ SUSY transformation.

In 3d $\mathcal{N} = 2$ supersymmetric theories without boundary, the supersymmetric invariant actions of the vector multiplet consist of the Chern–Simons term, the super Yang-Mills term and the Fayet-Iliopoulos (FI) term. We study the super Yang-Mills term and the FI-term in the presence of boundaries. The 3d super Yang-Mills Lagrangian is written as a $Q$-exact form:

$$L_{\text{SYM}} = -Q^2 \text{Tr} \left[ \frac{1}{4} \lambda \lambda \right]$$
$$= - \frac{1}{2} \text{Tr} \left[ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + D^2 \sigma D_\mu \sigma + D^2 + i \lambda \gamma^{\mu} D_\mu \lambda + i \lambda [\lambda, \sigma] \right].$$ (2.10)

$\text{Tr}$ is a trace taken over $g$. The super Yang-Mills action $S_{\text{SYM}}$ is defined by

$$S_{\text{SYM}} := \int_{I \times M_2} L_{\text{SYM}}.$$ (2.11)

With the boundary condition (2.8), the $S_{\text{SYM}}$ is invariant by the SUSY transformation (2.4). Next we consider the FI-term. The action for the FI-term is given by

$$S_{\text{FI}} = \int_{I \times M_2} L_{\text{FI}} = \int_{I \times M_2} i \zeta(D).$$ (2.12)

Here $\zeta$ is the FI-parameter. $S_{\text{FI}}$ is invariant by the SUSY transformation.

### 2.2 3d $\mathcal{N} = 2$ chiral multiplet

Next we consider a chiral multiplet $(\phi, \psi, F)$ in a representation $\mathbf{R}$ of $G$. The anti-chiral multiplet $(\bar{\phi}, \bar{\psi}, \bar{F})$ belongs to the complex conjugate representation $\mathbf{\bar{R}}$ of $G$. The supersymmetric transformation of the chiral multiplet is given by

$$\delta \phi = \bar{\epsilon} \psi,$$
$$\delta \bar{\phi} = \epsilon \bar{\psi},$$
$$\delta \psi = i \gamma^{\mu} \epsilon D_\mu \phi + i \epsilon \sigma \phi + \bar{\epsilon} F,$$
$$\delta \bar{\psi} = i \gamma^{\mu} \bar{\epsilon} D_\mu \bar{\phi} + \bar{\epsilon} \bar{\phi} \sigma \epsilon + \bar{F} \epsilon,$$
$$\delta F = \epsilon (i \gamma^{\mu} D_\mu \psi - i \sigma \psi - i \lambda \phi),$$
$$\delta \bar{F} = \bar{\epsilon} (i \gamma^{\mu} D_\mu \bar{\psi} - i \bar{\psi} \sigma + i \bar{\lambda} \bar{\phi}).$$ (2.13)

If the chiral multiplet $(\phi, \psi, F)$ belongs to a representation of a global symmetry group $G_F$, one can turn on the background gauge fields for the maximal torus of $G_F$. 


The kinetic Lagrangian of the chiral multiplet is written as a Q-exact form:

\[
L_{\chi} = Q^2 (\bar{\phi} F) = -\bar{\phi} D^\mu D_\mu \phi + \bar{\phi} \sigma^2 \phi + i \bar{\phi} D_\phi + \bar{F} F - i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \lambda \bar{\psi}. \tag{2.14}
\]

The kinetic action of the chiral multiplet is defined by

\[
S_{\chi} = \int_{I \times M_2} L_{\chi}. \tag{2.15}
\]

In a generic choice of supersymmetric variation parameters \(\epsilon = (\epsilon_1, \epsilon_2)^T\) and \(\bar{\epsilon} = (\bar{\epsilon}_1, \bar{\epsilon}_2)^T\), the action \(S_{\chi}\) is not invariant under the SUSY transformation in the presence of the boundaries. In the next two subsections, we will study two types of BPS boundary conditions for the chiral multiplet; the Dirichlet (denoted by D) and the Neumann (denoted by N) boundary conditions.

Other parts constructed by chiral multiplets are the superpotential terms. The Lagrangians of the superpotential terms are given by

\[
L_W = \sum_i \frac{\partial W(\phi)}{\partial \phi_i} F_i - \frac{1}{2} \sum_{i,j} \psi_i \psi_j \frac{W(\phi)}{\partial \phi_i \partial \phi_j}, \tag{2.16}
\]

\[
L_{\bar{W}} = \sum_i \frac{\partial \bar{W}(\bar{\phi})}{\partial \bar{\phi}_i} \bar{F}_i - \frac{1}{2} \sum_{i,j} \bar{\psi}_i \bar{\psi}_j \frac{\bar{W}(\bar{\phi})}{\partial \bar{\phi}_i \partial \bar{\phi}_j}. \tag{2.17}
\]

Here \(i\) in the sums\(^3\) labels the chiral multiplets \((\phi_i, \psi_i, F_i)\) in the superpotential \(W(\phi)\). \(\bar{W}(x)\) is the complex conjugate of \(W(x)\). The actions of the superpotential terms are given by

\[
S_W = \int_{I \times M_2} L_W, \quad S_{\bar{W}} = \int_{I \times M_2} L_{\bar{W}}. \tag{2.18}
\]

The SUSY transformations of the superpotential are written as the total derivative:

\[
\delta L_W = -i \partial_\mu \left[ \sum_i (\psi_i \gamma^\mu \epsilon) \frac{\partial W}{\partial \phi_i} \right], \quad \delta L_{\bar{W}} = -i \partial_\mu \left[ \sum_i (\bar{\psi}_i \gamma^\mu \bar{\epsilon}) \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \right]. \tag{2.19}
\]

Then surface terms for the SUSY transformation of the superpotentials are given by

\[
\delta S_W = i \epsilon' \sum_i \int_{\partial(I \times M_2)} (\psi_{1i} - \psi_{2i}) \frac{\partial W}{\partial \phi_i}, \quad \delta S_{\bar{W}} = i \bar{\epsilon}' \sum_i \int_{\partial(I \times M_2)} (\bar{\psi}_{1i} - \bar{\psi}_{2i}) \frac{\partial \bar{W}}{\partial \bar{\phi}_i}. \tag{2.20}
\]

\(^3\)In this article we use two notations for the representations for the flavor symmetry group. For example let \((\phi, \psi, F)\) be a chiral multiplet in the fundamental representation of \(G_F = U(N)\). \((\phi, \psi, F)\) is also expressed as the collection of chiral multiplets \((\phi_i, \psi_i, F_i)\) with \(i = 1, \cdots, N\).
We will see that the surface terms \((2.20)\) vanish under the Dirichlet boundary condition. In this case the supersymmetry is preserved without adding the boundary degrees of freedom. On the other hand, the surface terms \((2.20)\) remain under the Neumann boundary condition. We have to add boundary degrees of freedom to preserve the supersymmetry.

### 2.2.1 Dirichlet boundary condition

First we consider the Dirichlet boundary condition:

\[
\begin{align*}
\phi = 0, & \quad \bar{\phi} = 0, \quad \psi_1 - \psi_2 = 0, \quad \bar{\psi}_1 - \bar{\psi}_2 = 0, \\
\partial_1 F = 0, & \quad \partial_1 \bar{F} = 0, \quad \partial_1 (\psi_1 + \psi_2) = 0, \quad \partial_1 (\bar{\psi}_1 + \bar{\psi}_2) = 0.
\end{align*}
\tag{2.21}
\]

\((2.21)\) is compatible with the SUSY transformation \((2.13)\) with the variation parameters defined by \((2.3)\). The actions \(S_{\chi}\) and \(S_W\) for the chiral multiplet are invariant under the SUSY transformation with the Dirichlet boundary condition.

The restriction of SUSY transformation \((2.13)\) on the boundary with the Dirichlet boundary condition \((2.21)\) is written as

\[
\begin{align*}
\delta \psi_1 &= i \epsilon' \partial_1 \phi + \epsilon' F, \quad \delta F = \epsilon' \left[-2i \partial_1 \psi_1 + 2(D_2 + iD_3)\psi_1\right], \\
\delta \bar{\psi}_1 &= i \epsilon' \partial_1 \bar{\phi} + \epsilon' \bar{F}, \quad \delta \bar{F} = \epsilon' \left[-2i \partial_1 \bar{\psi}_1 + 2(D_2 + iD_3)\bar{\psi}_1\right].
\end{align*}
\tag{2.22}
\]

Note that \((2.22)\) is same as the supersymmetric transformation of the \(\mathcal{N} = (0, 2)\) fermi multiplet given by \((2.30)\), if we take \(E(\phi') := \phi' = \partial_1 \phi\) and \(\psi'_1 := \partial_1 \psi_1\). 

### 2.2.2 Neumann boundary condition

Next we consider the Neumann boundary condition defined by

\[
\begin{align*}
\partial_1 \phi = 0, & \quad \partial_1 \bar{\phi} = 0, \\
\psi_1 + \psi_2 = 0, & \quad \bar{\psi}_1 + \bar{\psi}_2 = 0, \quad \partial_1 (\psi_1 - \psi_2) = 0, \quad \partial_1 (\bar{\psi}_1 - \bar{\psi}_2) = 0.
\end{align*}
\tag{2.23}
\]

\((2.23)\) is compatible with the SUSY transformation with \((2.3)\). At a boundary, the supersymmetric transformation of the chiral multiplet with the Neumann boundary condition gives

\[
\begin{align*}
\delta \phi &= 2 \epsilon' \psi_1, \quad \delta \bar{\phi} = 2 \epsilon' \bar{\psi}_1, \\
\delta \psi_1 &= \epsilon' (D_2 + iD_3) \phi, \quad \delta \bar{\psi}_1 = \epsilon' (D_2 + iD_3) \bar{\phi}.
\end{align*}
\tag{2.24}
\]

The transformation of \((\phi, \psi_1)\) in \((2.24)\) is same as the supersymmetric transformation of the 2d \(\mathcal{N} = (0, 2)\) chiral multiplet.

With the Neumann boundary condition \((2.23)\), the kinetic action of the chiral multiplet \(S_{\chi}\) is invariant by the SUSY transformation. On the other hand, the superpotential term is
not invariant under the SUSY transformation. The SUSY transformations give the following boundary terms:

\[
\delta S_W = 2i \epsilon' \int_{\partial (I \times M_2)} \sum_{\nu'} \psi_{1\nu}' \frac{\partial W}{\partial \phi_{\nu'}} ,
\]

\[
\delta \bar{S}_W = 2i \bar{\epsilon}' \int_{\partial (I \times M_2)} \sum_{\nu'} \bar{\psi}_{1\nu}' \frac{\partial \bar{W}}{\partial \bar{\phi}_{\nu'}} .
\] (2.25)

If the surface terms (2.25) are compensated by the SUSY transformation of appropriate 2d \( \mathcal{N} = (0, 2) \) superpotential terms [11], this cancellation mechanism is analogous to the matrix factorization in the 2d \( \mathcal{N} = (2, 2) \) Landau-Ginzburg models [12] and called the 3d matrix factorization.

In 2d \( \mathcal{N} = (2, 2) \) gauged linear sigma models (GLSMs) with boundaries, the surface term of the superpotential is canceled by brane factors [13]. On the other hand, the dimensional reduction of (2.25) is also canceled by the SUSY transformation of the superpotential in 1d \( \mathcal{N} = 2 \) fermi multiplets. We briefly study these two methods; brane factors and 1d fermi multiplets in section 7.2.2.

### 2.3 2d \( \mathcal{N} = (0, 2) \) theory at boundary and 3d matrix factorization

As we have seen in the previous section, the 3d \( \mathcal{N} = 2 \) theory preserves 2d \( \mathcal{N} = (0, 2) \) supersymmetry at the boundaries except for the superpotential term. 2d \( \mathcal{N} = (0, 2) \) multiplets are necessary to cancel the gauge anomaly and to cancel the boundary terms for the 3d superpotential. We explain 2d \( \mathcal{N} = (0, 2) \) theories at boundaries and couplings between the 2d theories and the 3d theory.

#### \( \mathcal{N} = (0, 2) \) vector multiplet

An \( \mathcal{N} = (0, 2) \) vector multiplet with the gauge group \( G' \) consists of a \( g' \) valued gauge field \( A_i' (i = 2, 3) \), 2d fermions \( \lambda', \bar{\lambda}' \), and an auxiliary \( D' \). The action is given by

\[
S_{2d.vec} = \int_{M_2} \mathcal{L}_{2d.vec} = - \int_{M_2} Q \text{Tr} \left[ \lambda_1' (\hat{D} - i F'_{23}) \right] = \int_{M_2} \text{Tr} \left[ F'^2_{23} + \hat{D}'^2 + 2 \lambda_1' (D_2 + i D_3) \bar{\lambda}_1' \right].
\] (2.26)

Here we defined \( F'^2_{23} : = \partial_2 A_3' - \partial_3 A_2' + i [A_2', A_3'] \). The SUSY transformation is same as (2.9) and \( Q \) is the restriction of (2.4) to (2.9). The FI-term for the 2d \( \mathcal{N} = (0, 2) \) gauge theory is given by

\[
S_{2d.FI} = \int_{M_2} \mathcal{L}_{2d.FI} = \int_{M_2} i \zeta_{2d} (D') .
\] (2.27)
\( \mathcal{N} = (0, 2) \) chiral multiplet

We consider a 2d \( \mathcal{N} = (0, 2) \) chiral multiplet consisting of \((\phi', \psi')\) in a representation of the gauge group \( G' \). The SUSY transformation is same as \([2.24]\), where \((\phi, \psi_1)\) replaced by \((\phi', \psi')\) and the covariant derivative for \( G \) is replaced by that for \( G' \). The Lagrangian of the \( \mathcal{N} = (0, 2) \) chiral multiplet is written as a Q-exact form:

\[
\mathcal{L}_{2d.chi} = -Q \left[ \bar{\phi}'(D_2 - iD_3)\phi' + i\phi'X'\phi' \right] = -\bar{\phi}'(D_2 - iD_3)(D_2 + iD_3)\phi' - 2\bar{\psi}'(D_2 - iD_3)\psi' - 2i\bar{\phi}'\bar{\lambda}_1\psi' + 2i\psi'\lambda_1'\phi' + i\bar{\phi}')(\bar{D} + iF_{23})\phi'.
\] (2.28)

The action of 2d \( \mathcal{N} = (0, 2) \) chiral multiplet is given by

\[
S_{2d.chi} := \int_{M_2} \mathcal{L}_{2d.chi}.
\] (2.29)

\( \mathcal{N} = (0, 2) \) fermi multiplet

The SUSY transformation of the 2d \( \mathcal{N} = (0, 2) \) fermi multiplet \((\psi'_-, F')\) is given by

\[
\delta \psi'_- = i\epsilon' E + \epsilon' F',
\]

\[
\delta F' = \epsilon' \left[ -2i \sum \psi'_+ i \partial_{\phi'_i} E(\phi') + 2(D_2 + iD_3)\psi'_- \right],
\]

\[
\delta \bar{\psi}' = i\epsilon' \bar{E} + \epsilon' \bar{F}',
\]

\[
\delta \bar{F}' = \epsilon' \left[ -2i \sum \bar{\psi}'_+ i \partial_{\bar{\phi}'_i} \bar{E}(\bar{\phi}') + 2(D_2 + iD_3)\bar{\psi}'_- \right].
\] (2.30)

Here \((\phi'_i, \psi'_{+i})\)'s are 2d \( \mathcal{N} = (0, 2) \) chiral multiplets, that can be taken as the boundary values of 3d chiral multiplets with the Neumann boundary condition. \( E \) is a function of \( \phi'_i\)'s. The kinetic term of the fermi multiplet is given by

\[
\mathcal{L}_{fermi} = Q(\bar{\psi}'_- F' - i\bar{E}\psi'_-)
\]

\[
= -2\bar{\psi}'_- (D_2 + iD_3)\psi'_- + F'F' + \bar{E}E + 2i \sum \bar{\psi}'_- i \partial_{\phi'_i} E - 2i \sum \bar{\psi}'_+ i \partial_{\phi'_i} E.
\] (2.31)

The kinetic action of the 2d \( \mathcal{N} = (0, 2) \) fermi multiplet is given by

\[
S_{fermi} := \int_{M_2} \mathcal{L}_{fermi}.
\] (2.32)
The $\mathcal{N} = (0, 2)$ superpotential term is constructed by the fermi multiplets $(\psi'_-, F')$'s and the 2d $\mathcal{N} = (0, 2)$ chiral multiplets $(\phi'_i, \psi'_i)$'s with functions $J^a(\phi')$'s as

$$
\mathcal{L}_J = \sum_a \left( F'^a J^a - 2 \sum_i \psi'_{-a} \psi'_{+i} \frac{\partial J^a}{\partial \phi'_i} \right),
$$

$$
\mathcal{L}_{\bar{J}} = \sum_a \left( \bar{F}'^a \bar{J}^a - 2 \sum_i \bar{\psi}'_{-a} \bar{\psi}'_{+i} \frac{\partial \bar{J}^a}{\partial \bar{\phi}'_i} \right). 
$$

The SUSY transformation of the 2d $\mathcal{N} = (0, 2)$ superpotential is written as

$$
\delta \mathcal{L}_J = 2i\epsilon' \sum_a \left[ - \sum_i \psi'_{+i} \frac{\partial (E^a J^a)}{\partial \phi'_i} + (\partial_2 + i\partial_3)(\psi'_{-a} J^a) \right], 
\tag{2.34}
$$

If $\sum_a E^a J^a = 0$, the superpotential preserves the $\mathcal{N} = (0, 2)$ supersymmetry. There is another possibility to preserve supersymmetry as follows. The surface terms for the 3d $\mathcal{N} = 2$ superpotentials and the SUSY transformation of 2d $\mathcal{N} = (0, 2)$ superpotential are combined as

$$
\delta \int_{I \times M_2} \mathcal{L}_W \bigg|_{x^1 = \pm \pi L} + \delta \int_{\{\pm \pi L\} \times M_2} \mathcal{L}_J 
= \int_{\{\pm \pi L\} \times M_2} 2i\epsilon' \left[ \sum_i \pm \psi'_1 \frac{\partial W}{\partial \phi'_i} - \psi'_{+i} \sum_a \frac{\partial (E^a J^a)}{\partial \phi'_i} \right] 
\tag{2.35}
$$

Then the boundary terms of the SUSY transformation of the 3d superpotential are canceled by the 2d $\mathcal{N} = (0, 2)$ superpotential terms, if the 2d chiral multiplets in $E^a$ and $J^a$ take the boundary values of the 3d chiral multiplets with the Neumann boundary condition; $\phi'_i = \phi'_i|_{x^1 = \pm \pi L}, \psi'_-i = \psi'_-i|_{x^1 = \pm \pi L}$ and satisfy the following relations:

$$
W \bigg|_{x^1 = \pm \pi L} = \pm \sum_a E^a J^a, \quad \bar{W} \bigg|_{x^1 = \pm \pi L} = \pm \sum_a \bar{E}^a \bar{J}^a. 
\tag{2.36}
$$

### 2.4 Anomaly polynomials

The net contributions to anomalies from the fermions in the 3d chiral multiplets and the fermions in the 2d chiral and fermi multiplets are nicely organized as the anomaly polynomials [14]. The 3d and 2d theories have to satisfy the cancellation of the gauge anomalies at both left and right boundaries. Also the 't Hooft anomalies have to match between the IR dual theories. Here let us summarize the contributions to the anomaly polynomials.

- **3d chiral multiplet:**

  A 3d $\mathcal{N} = 2$ chiral multiplet with a charge assignment in Table I contributes to the anomaly polynomial as

  $$
  \pm \frac{1}{2}(Qy + (r - 1)r)^2. 
  \tag{2.37}
  $$
Here + (resp. −) is taken for the Dirichlet (resp. Neumann) boundary condition for the chiral multiplet. y is the field strength for $U(1)_y$ and r is the field strength for the $U(1)_R$ R-symmetry group.

The contribution of the 3d chiral multiplet belongs to a representation R of G:

$$\pm \frac{1}{2} \text{Tr}_R(f)^2. \quad (2.38)$$

Here + (resp. −) is taken for the Dirichlet (Neumann) boundary condition for the chiral multiplet.

• 2d $\mathcal{N} = (0, 2)$ chiral multiplet:

A 2d $\mathcal{N} = (0, 2)$ chiral multiplet with a charge assignment in Table 1 contributes to the anomaly polynomial

$$-(Qy + (r - 1)r)^2. \quad (2.39)$$

• 2d $\mathcal{N} = (0, 2)$ fermi multiplet:

A 2d $\mathcal{N} = (0, 2)$ fermi multiplet with a charge assignment in Table 1 contributes to the anomaly polynomial

$$(Qy + (r - 1)r)^2. \quad (2.40)$$

### 3 SUSY index of 3d $\mathcal{N} = 2$ theory on $I \times T^2$ and localization

#### 3.1 Definition of the index on $I \times T^2$ and SUSY localization formula

In this section we evaluate the SUSY index on $I \times T^2$ in terms of the supersymmetric localization method.
For the later convenience, we introduce the coordinates \((s, t)\) of \(T^2\) defined by
\[
x^2 = s + \tau_1 t, \quad x^3 = t\tau_2,
\]
and take a normalization of an integration measure of \(I \times T^2\) as
\[
\int_{I \times T^2} (\cdots) := \frac{1}{8\pi^3 LR^2} \int_{-\pi L}^{\pi L} dx^1 \int_0^{2\pi R} ds \int_0^{2\pi R} dt \cdots.
\]

We impose the same boundary conditions at the left and the right boundaries \(x^1 = \pm \pi L\) and impose the following twisted boundary conditions for all the fields \(\Psi(x, s, t)\) along the two-dimensional torus \(T^2\):
\[
\Psi(x^1, s, t + 2\pi R) = \prod_i e^{-2\pi iz_i F_i} \Psi(x^1, s, t),
\]
\[
\Psi(x^1, s + 2\pi R, t) = \prod_i e^{-2\pi iz_i F_i} \Psi(x^1, s, t).
\]

The supersymmetric index of the 3d \(N = 2\) supersymmetric theory on \(I \times T^2\) is defined by
\[
Z_{I \times T^2} := \text{Tr}(-1)^F e^{\pi i R P_2} \prod_i e^{2\pi i z_i F_i}
\]
\[
= \text{Tr}(-1)^F e^{\pi i R \tau(P_2 - iP_3)} e^{\pi i \bar{\tau}(P_2 + iP_3)} \prod_i e^{2\pi i z_i F_i}.
\]

Here \(iP_i \simeq \partial_i\) with \(i \in 2, 3, t\) are the generators of translations in the directions \(x^2, x^3\) and \(t\). \(F_i\)'s are the generators of the maximal torus of the flavor symmetry group \(G_F\). \(z_i = z_{ii} - \tau z_{si}\) is the fugacity of the \(U(1)\) flavor charge \(F_i\). The \(I \times T^2\) index (3.4) is a function of the moduli parameter \(\tau\) of the torus \(T^2\), but independent of the complex conjugate \(\bar{\tau}\). This is because a generator of the translation \(P_2 + iP_3\) is written as a \(Q\)-exact form:
\[
\{Q, \bar{Q}\} = 2i(P_2 + iP_3) \simeq 4\partial_{\bar{w}}.
\]

Thus we may write the index as
\[
Z_{I \times T^2} = \text{Tr}(-1)^F e^{\pi i R \tau(P_2 - iP_3)} \prod_i e^{2\pi i z_i F_i}.
\]

From (3.5), it follows that the correlation functions consisting of \(Q\)-closed operators are independent of an anti-holomorphic coordinate \(\bar{w}\) of \(M_2\). In section 6, we compute two point functions of \(Q\)-closed operators in the free chiral multiplet and see this property explicitly.

Although it is possible to introduce 2d vector multiplets at the boundaries and evaluate the \(I \times T^2\) index by localization, we concentrate on the theories without 2d vector multiplets.
to avoid clutter of the localization computation. We summarize the localization formula for the supersymmetric index $Z_{I \times T^2}$ of the 3d $\mathcal{N} = 2$ theories on $I \times T^2$:

$$Z_{I \times T^2}(y; q) = \frac{1}{|W_G|} \sum_{u \in \mathfrak{m}_{\text{diag}}} \text{JK-Res}(Q_u, \eta) Z_{I \times T^2}^{1\text{-loop}}(Q_u, \eta) Z_{T^2_R}^{1\text{-loop}} / u^{\text{rk}(G)} \, du^a. \quad (3.7)$$

Here $T^2_R$ (resp. $T^2_L$) is a boundary torus at $x^1 = \pi L$ (resp. $x^1 = -\pi L$). $G$ is the gauge group for the 3d theory on $I \times T^2$. $|W_G|$ is the cardinality of the Weyl group of $G$. $u$ is a flat connection for the maximal torus of $G$.

$Z_{I \times T^2}^{1\text{-loop}}$, $Z_{T^2_L}^{1\text{-loop}}$ and $Z_{T^2_R}^{1\text{-loop}}$ are the one-loop determinants of the 3d $\mathcal{N} = 2$ theory on $I \times T^2$, the one-loop determinants of the 2d $\mathcal{N} = (0, 2)$ theory on $T^2_L$ and those of the 2d $\mathcal{N} = (0, 2)$ theory on $T^2_R$:

$$Z_{I \times T^2}^{1\text{-loop}} = Z_{3d, \text{vec}, G}(e^{2\pi i u}; q) \prod_{a \in \text{rt}(g)} Z_{\chi, a, \mathbf{R}}(e^{2\pi i u}; y, q), \quad (3.8)$$

$$Z_{T^2_L}^{1\text{-loop}} = \prod_{a \in \chi_{\text{ch}}} Z_{2d, \chi, \mathbf{R}_L}(e^{2\pi i u}, y, q) \prod_{i \in \mathbf{F}} Z_{\text{fermi,} \mathbf{R}_L}(e^{2\pi i u}, y, q), \quad (3.9)$$

$$Z_{T^2_R}^{1\text{-loop}} = \prod_{a \in \chi_{\text{ch}}} Z_{2d, \chi, \mathbf{R}_R}(e^{2\pi i u}, y, q) \prod_{i \in \mathbf{F}} Z_{\text{fermi,} \mathbf{R}_R}(e^{2\pi i u}, y, q), \quad (3.10)$$

where

$$Z_{3d, \text{vec}, G}(x, q) := \left(\frac{2\pi \eta(q)^2}{1}\right)^{\text{rk}(G)} \prod_{\alpha \in \text{rt}(g)} \frac{i^{\theta_1(x^\alpha, q)}}{\eta(q)}, \quad (3.11)$$

$$Z_{\chi, \mathbf{N}, \mathbf{R}}(x, y; q) = Z_{2d, \chi, \mathbf{R}}(x, y; q) := \prod_{Q \in \text{wt}(\mathbf{R})} \prod_{Q' \in \text{wt}(\mathbf{F})} i^{\theta_1(x Q y Q', q)} \eta(q), \quad (3.12)$$

$$Z_{\chi, \mathbf{D}, \mathbf{R}}(x, y; q) = Z_{\text{fermi,} \mathbf{R}}(x, y; q) := \prod_{Q \in \text{wt}(\mathbf{R})} \prod_{Q' \in \text{wt}(\mathbf{F})} i^{\theta_1(x Q y Q', q)} \eta(q). \quad (3.13)$$

The 3d $\mathcal{N} = 2$ one-loop determinant $Z_{I \times T^2}^{1\text{-loop}}$ consists of a 3d $G$ vector multiplet $Z_{3d, \text{vec}, G}$, a 3d chiral multiplets with the Neumann boundary condition $Z_{\chi, \mathbf{N}, \mathbf{R}}$ and a 3d chiral multiplet with the Dirichlet (denoted by D) boundary condition $Z_{\chi, \mathbf{D}, \mathbf{R}}$. $a$ belongs to $a \in \mathbf{D}, \mathbf{N}$. The 2d $\mathcal{N} = (0, 2)$ one-loop determinants $Z_{T^2_L}^{1\text{-loop}}$ and $Z_{T^2_R}^{1\text{-loop}}$ consist of the 2d chiral multiplets $Z_{2d, \chi, \mathbf{R}}(x, y; q)$ and the 2d fermi multiplets $Z_{\text{fermi,} \mathbf{R}}(x, y; q)$. The products are taken over all the multiplets.

$\text{rt}(\mathfrak{g})$ denotes roots of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ and $\text{wt}(\mathbf{R})$ denotes weights of a representation $\mathbf{R}$ of the gauge group. $\mathbf{F}$ denotes a representation of the flavor symmetry group $G_F$. $x^Q := e^{2\pi i} \sum_{a=1}^N Q_a u_a$ and $y^{Q^F} = e^{2\pi i} \sum_i Q_i^{\text{wt}(\mathbf{F})}$. $Q = (Q_1, \ldots, Q_{\text{rk}(G)})$ is a weight of $\mathbf{R}$. $Q^F = (Q^F_1, \ldots, Q^F_{\text{rk}(G_F)})$ is a weight of $\mathbf{F}$. In the path integral formalism, $u$ (resp. $z$) is a flat connection of the maximal torus $G$ (resp. $G_F$) on the torus $T^2$. 

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The theta function $\theta_1(x, q)$ and the eta function $\eta(q)$ are defined by

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

$$\theta_1(x, q) = -i q^{\frac{1}{8}} x^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - xq^{n-1})(1 - x^{-1}q^n),$$

(3.14)

where $q = e^{2\pi i \tau}$.

JK-Res$_{u=u_*} (Q_*, \eta)$ is the Jeffrey-Kirwan (JK) residue defined as follows. When the rk$(G)$ hyperplanes of codimension one, called singular hyperplanes $Q_i(u - u_*) = \sum_{a=1}^{\text{rk}(G)} Q_i^a (u^a - u_*^a) = 0$ ($i = 1, \cdots, \text{rk}(G)$) intersect at a point $u_* = (u_1^*, \cdots, u_{\text{rk}(G)}^*)$ in the $u$-space. The JK residue at the point $u_*$ is defined by

$$\text{JK-Res}_{u=u_*} (Q_*, \eta) \frac{du^1 \wedge \cdots \wedge du^{\text{rk}(G)}}{Q_1(u - u_*) \cdots Q_{\text{rk}(G)}(u - u_*)} = \begin{cases} \\ 1 & \text{if } \eta \in \text{Cone}(Q_1, \ldots, Q_{\text{rk}(G)}), \\ 0 & \text{otherwise}. \end{cases}$$

(3.15)

Here $\text{Cone}(Q_1, \ldots, Q_{\text{rk}(G)}) = \sum_{i=1}^{\text{rk}(G)} \mathbb{R}_{>0} Q_i$ is the cone spanned by gauge charge vectors; $Q_i = (Q_1^i, \ldots, Q_{\text{rk}(G)}^i) \in \mathbb{R}^{\text{rk}(G)}$ ($i = 1, \cdots, \text{rk}(G)$). (3.15) depends on a set of charges $Q_* = (Q_1, \ldots, Q_{\text{rk}(G)})$ and $\eta$. The sum \( \sum_{u_*} \) runs over all the points $u_*$, where $N'$ with $N' \geq \text{rk}(G)$ singular hyperplanes meet at a point and the condition $\eta \in \text{Cone}(Q_1, \ldots, Q_{\text{rk}(G)})$ is satisfied. If $N'$ singular hyperplanes with $N' > \text{rk}(G)$ intersect at a point, we apply the constructive definition of the JK residue in [15]. The condition $N' = \text{rk}(G)$ is satisfied for the models treated in this article.

Note that each one-loop determinant of the 3d multiplet has the same form as the one-loop determinant of the vector, chiral, and fermi multiplet in the 2d $\mathcal{N} = (0, 2)$ elliptic genus in the R-sector [15], respectively and the index is independent of the length of the interval $I$. In other words, the fermionic and the bosonic Kaluza–Klein modes in the $x^1$-direction cancel out, expect for the lowest modes that form the 2d $\mathcal{N} = (0, 2)$ multiplets.

Before we move to technical details of the localization computation, let us briefly recall the principle of supersymmetric localization [16]. The partition function or the index of the supersymmetric theory in the path integral formalism is expressed as

$$Z = \int \mathcal{D}[\Psi] e^{-S[\Psi]}.$$  

(3.16)

We assume the action $S[\Psi]$ is invariant by a fermionic conserved charge (supercharge) $Q$, where $\Psi$ denotes the component fields of the supermultiplets in the theory. Without changing the value of the partition function, one can add one-parameter family of the Q-exact term (action) $\frac{1}{g^2} Q \cdot V[\Psi]$ to the action. If there are more than one Q-exact term, the action is
deformed by a multi-parameter family of Q-exact terms $\sum_i \frac{1}{g_i^2} Q \cdot V_i[\Psi]$. By taking the weak coupling limit $g^2 \to 0$, the path integral is exactly evaluated in the one-loop computation of the fluctuations around the saddle points (zero loci) $\Psi_0$ of the Q-exact term; $Q \cdot V[\Psi_0] = 0$:

$$Z = \lim_{g \to 0} \int \mathcal{D}\Psi e^{-\frac{1}{g^2} Q \cdot V[\Psi]} = \int d\Psi_0 e^{-S[\Psi_0]} Z_{1\text{-loop}}(\Psi_0).$$

(3.17)

Here we expanded fields as $\Psi = \Psi_0 + g \tilde{\Psi}$, where $\tilde{\Psi}$ denotes fluctuations around the saddle point configurations $\Psi_0$. By integrating out $\tilde{\Psi}$, we obtain the one-loop determinant $Z_{1\text{-loop}}$ of $Q \cdot V[\Psi]$ around the saddle point $\Psi_0$. When fermion zero-modes exist, actual localization computation is more involved and one has to treat carefully the zero-mode integral [17, 15].

In our case, the Q-exact terms for the three dimensional part are taken as the super Yang-Mills action (2.10) and the kinetic action of the 3d chiral multiplet (2.15). The Q-exact terms for the two dimensional part are taken as the kinetic action of the 2d chiral multiplet (2.29) and the fermi multiplet (2.32):

$$\sum_i \frac{1}{g_i^2} Q \cdot V_i[\Psi] = \frac{1}{e^2} S_{\text{SYM}} + \frac{1}{g_1^2} S_{\text{chi}} + \sum_{i=L, R} \left( \frac{1}{g^2_{2,i}} S_{2d,\text{chi},i} + \frac{1}{g^2_{3,i}} S_{\text{fermi},i} \right).$$

(3.18)

Here $i = L, R$ express the boundary theories at $T^2_L, T^2_R$, respectively. First we take the limit $g_1^2, g_{2,i}^2, g_{3,i}^2 \to 0$ and then we take the limit $e^2 \to 0$.

The Q-closed actions are the 3d FI-term, and the 3d and 2d superpotential terms. The saddle point configuration of the 3d FI-term is non-zero. On the other hand, the saddle point configuration of superpotential terms is zero. The localization formula does not explicitly depend on the superpotential terms. The superpotentials contribute to the localization formula through the 3d matrix factorization.

### 3.2 Evaluation of the one-loop determinants

First we evaluate the one-loop determinant for the 3d $N = 2$ vector multiplet with the boundary condition (2.8). As we have seen in the previous section, the super Yang-Mills Lagrangian is written as a SUSY transformation. We choose it as a Q-exact term. The saddle point condition for the vector multiplet, i.e., the zero loci of the super Yang–Mills Lagrangian are given by $F_{\mu\nu} = 0$ and $D_\mu \sigma = 0$ and constant values of $\lambda, \bar{\lambda}$.

From the boundary condition (2.8), the saddle point configurations are given by $A_1 = 0$ and $\sigma = 0$ and $F_{23} = 0$. Let $\bar{A}$ be the gauge field which satisfies the saddle point condition $F_{23} = 0$:

$$\bar{A} = \bar{A}_2 dx^2 + \bar{A}_3 dx^3 = \bar{A}_t dt + \bar{A}_s ds,$$

(3.19)

where $\bar{A}_t$ and $\bar{A}_s$ are constants. The covariant derivative with the gauge field (3.19) $D_\mu = \partial_\mu - \bar{A}_\mu \sigma + \bar{A}_\mu^i \gamma^i$:

$$D_\mu = \partial_\mu + \bar{A}_t \sigma + \bar{A}_s \bar{A}_t,$$

(3.20)

where $\sigma = \bar{A}_2 dx^2 + \bar{A}_3 dx^3$ is the boundary condition (2.8). The one-loop determinant is given by:

$$Z_{1\text{-loop}}(\bar{A}) = \int d\bar{A}_0 e^{-S_{\text{SYM}}[\bar{A}_0]} Z_{1\text{-loop}}(\bar{A}_0).$$

(3.21)

Here $\bar{A}_0$ denotes the saddle point configuration and $Z_{1\text{-loop}}(\bar{A}_0)$ is the one-loop determinant of the 3d super Yang-Mills action.

The 3d FI-term contribution is given by:

$$Z_{\text{FI}} = \int d\bar{A}_0 e^{-\frac{1}{2g_1^2} \bar{A}_t^2 + \frac{1}{2g_1^2} \bar{A}_s^2}.$$

(3.22)

The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations.

### 3.3 Calculation of the zero-mode integral

When fermion zero-modes exist, the zero-mode integral is given by:

$$Z_{\text{zero-mode}} = \int d\bar{A}_0 e^{-\frac{1}{2g_1^2} \bar{A}_t^2 + \frac{1}{2g_1^2} \bar{A}_s^2}.$$

(3.24)

The zero-mode integral is evaluated by integrating out the fermion zero-modes and evaluating the path integral at the saddle point configurations.

The localization formula (3.17) is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$

(3.23)

The localization formula is obtained by integrating out the 3d fields and evaluating the path integral at the saddle point configurations. The 3d superpotential contribution is given by:

$$Z_{\text{super}} = \int d\bar{A}_0 e^{-\frac{1}{2g^2_{2,i}} \bar{A}_t^2 + \frac{1}{2g^2_{3,i}} \bar{A}_s^2}.$$
\[ \partial_\mu + i\bar{A}_\mu \text{ is given by} \]
\[ \bar{D}_1 = \partial_1, \]
\[ \bar{D}_2 + i\bar{D}_3 = \frac{i}{\tau_2} \{(\partial_2 - \tau\partial_3) + i(\bar{A}_2 - \tau\bar{A}_3)\} = \frac{i}{\tau_2} \left( \partial_2 - \tau\partial_3 + i\frac{u}{R} \right), \]
\[ \bar{D}_2 - i\bar{D}_3 = -\frac{i}{\tau_2} \{(\partial_2 - \tau\partial_3) + i(\bar{A}_2 - \tau\bar{A}_3)\} = -\frac{i}{\tau_2} \left( \partial_2 - \tau\partial_3 + i\frac{\bar{u}}{R} \right). \] (3.20)

Here \( u \) and \( \bar{u} \) are defined by
\[ 2\pi u := \oint_t \bar{A} - \tau \oint_s \bar{A}, \quad 2\pi \bar{u} := \oint_t \bar{A} - \bar{\tau} \oint_s \bar{A}. \] (3.21)

and \( u, \bar{u} \) take values in a representation of Cartan subalgebra of the gauge group,
\[ u = \sum_{a=1}^{rk(G)} u^a H^a, \quad \bar{u} = \sum_{a=1}^{rk(G)} \bar{u}^a H^a. \] (3.22)

The representation is determined by the matter field on which the covariant derivative acts.
To make the expressions concise, we use same symbol \( \{H_a\}_{a=1}^{rk(G)} \) to express a generator of the Cartan subalgebra of \( g \) and its representation.

We evaluate the one-loop determinants of the fluctuations around the saddle point condition (3.19). We focus on the mode expansions along the interval \( I \) under the boundary condition. Then the fluctuations are expanded as
\[ \tilde{\sigma} = \sum_{\ell=1}^{\infty} \sigma^{(\ell)} \sin \frac{\ell x}{L}, \quad \tilde{A}_1 = \sum_{\ell=1}^{\infty} A^{(\ell)}_1 \sin \frac{\ell x}{L}, \quad \tilde{A}_i = \sum_{\ell=0}^{\infty} A^{(\ell)}_i \cos \frac{\ell x}{L}, \quad (i = 2, 3), \]
\[ \tilde{\lambda}_\alpha = \sum_{\ell=0}^{\infty} \lambda^{(\ell)}_c \cos \frac{\ell x}{L} + (-1)^{\alpha-1} \sum_{\ell=1}^{\infty} \lambda^{(\ell)}_s \sin \frac{\ell x}{L}, \]
\[ \tilde{\lambda}^*_\alpha = \sum_{\ell=0}^{\infty} \bar{\lambda}^{(\ell)}_c \cos \frac{\ell x}{L} + (-1)^{\alpha-1} \sum_{\ell=1}^{\infty} \bar{\lambda}^{(\ell)}_s \sin \frac{\ell x}{L} \quad (\alpha = 1, 2). \] (3.23)

Fields with tilde \( \tilde{\cdot} \) express the fluctuations around the saddle point configuration. Each Kaluza-Klein mode \( \Psi^{(\ell)} \in \{A^{(\ell)}_\mu, \sigma^{(\ell)}, \lambda^{(\ell)}_c, \lambda^{(\ell)}_s, \tilde{\lambda}^{(\ell)}_c, \tilde{\lambda}^{(\ell)}_s\} \) is expanded as
\[ \Psi^{(\ell)} = \sum_{\alpha \in \pi_0(g)} \Psi^{(\ell)}_\alpha E_\alpha. \] (3.24)

Here \( \{H_a, E_\alpha\} \) is the Cartan-Weyl basis for the Lie algebra \( g \) with the normalization \( \text{Tr}(E_\alpha E_\beta) = \delta_{\alpha+\beta,0} \).

In (3.23), the modes labeled by \( \ell \) are functions of the coordinates of the torus \((s, t)\). The twisted boundary condition (3.3) is imposed for the Kaluza-Klein modes \( \Psi^{(\ell)} \)'s:
\[ \Psi^{(\ell)}(s + 2\pi R, t) = \Psi^{(\ell)}(s, t), \quad \Psi^{(\ell)}(s, t + 2\pi R) = \Psi^{(\ell)}(s, t), \]
\[ \rightarrow \Psi^{(\ell)}(s, t) = \sum_{m,n \in \mathbb{Z}} \exp \left( i\frac{m}{R} s + i\frac{n}{R} t \right) \Psi^{(\ell,m,n)}(s, t). \] (3.25)
To perform the path integral for the gauge field, we introduce a gauge fixing term $L_{\text{gf}}$ in the $R_\xi$-gauge with $\xi = 1$ and introduce the Faddeev–Popov ghost $C$ and the anti-ghost $\tilde{C}$:

$$L_{\text{gf}} = \frac{1}{2} \text{Tr}(\bar{D}_\mu \tilde{A}^\mu)^2, \quad L_{\text{gh}} = \text{Tr} \tilde{C}(\bar{D}_\mu \tilde{D}^\mu)C.$$  \hfill (3.26)

We take the Neumann boundary condition for the ghosts at $x^1 = \pm \pi L$ and expand them as

$$C = \sum_{\ell=0}^\infty C^{(\ell)} \cos \frac{\ell x^1}{L}, \quad \tilde{C} = \sum_{\ell=0}^\infty \tilde{C}^{(\ell)} \cos \frac{\ell x^1}{L}. \hfill (3.27)$$

The bosonic fields are combined into a bilinear form

$$\frac{1}{\pi L} \int_{-\pi L}^{\pi L} dx^1 \left( L_{\text{vec}} + L_{\text{gf}} \right)_{\text{bosonic part}}$$

$$= \text{Tr} \left[ \bar{A}^{(0)T} \mathcal{M}_0 A^{(0)} \right] + \frac{1}{2} \sum_{\ell=1}^\infty \text{Tr} \left[ \bar{A}^{(\ell)T} \mathcal{M}_\ell A^{(\ell)} + \sigma^{(\ell)} \left( \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2 \right) \sigma^{(\ell)} \right] + \text{Tr}[D^2], \hfill (3.28)$$

where

$$A^{(0)} = \begin{pmatrix} A_2^{(0)} \\ A_3^{(0)} \end{pmatrix}, \quad A^{(\ell)} = \begin{pmatrix} A_4^{(\ell)} \\ A_2^{(\ell)} \\ A_3^{(\ell)} \end{pmatrix},$$

$$\mathcal{M}_0 = \text{diag}(-\bar{D}_2^2 - \bar{D}_3^2, -\bar{D}_2^2 - \bar{D}_3^2),$$

$$\mathcal{M}_\ell = \text{diag} \left( \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2, \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2, \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2 \right) (\ell = 1, 2, \cdots). \hfill (3.29)$$

Then the bosonic part of the one-loop determinants of the vector multiplet and the gauge fixing term are given by\(^4\)

$$\text{Det}^{-1}(-\bar{D}_2^2 - \bar{D}_3^2) \cdot \prod_{\ell=0}^\infty \text{Det}^{-2} \left( \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2 \right). \hfill (3.30)$$

Here Det denotes the functional determinant with respect to derivatives of the coordinates of the torus and $E_\alpha$'s.

The fermionic part of the vector multiplet and the ghost action is expanded as

$$\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} dx^1 \left( i \tilde{\lambda}^\mu \bar{D}_\mu \lambda + \tilde{C}(D_\mu D^\mu)C \right) = 2\lambda_c^{(0)}(\bar{D}_2 + i\bar{D}_3)\lambda_c^{(0)}$$

$$+ \sum_{\ell=1}^\infty \left( \begin{array}{c} \tilde{\chi}^{(\ell)} \\ \tilde{\chi}^{(\ell)}_s \end{array} \right) \left( \begin{array}{cc} \bar{D}_2 + i\bar{D}_3 & -i\frac{\ell}{L} \\ i\frac{\ell}{L} & \bar{D}_2 - i\bar{D}_3 \end{array} \right) \left( \begin{array}{c} \lambda^{(\ell)}_c \\ \lambda^{(\ell)}_s \end{array} \right)$$

$$+ \sum_{\ell=0}^\infty \tilde{C}^{(\ell)} \left( \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2 \right) C^{(\ell)}. \hfill (3.31)$$

\(^4\)As in the case of localization of the elliptic genera, we absorb the factor $(R\tau_2)^{-1}$ in front of $|n - m\tau + \alpha(u)|^2$ by rescaling $L$ and $D$ in the following one-loop computations.
Then the fermionic part of the one-loop determinant of the vector multiplet and the ghost action is given by

$$\text{Det} \left( \bar{D}_2 + i \bar{D}_3 \right) \text{Det} \left( \bar{D}_2^2 + \bar{D}_3^2 \right) \prod_{\ell=1}^{\infty} \text{Det}^2 \left( \frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2 \right).$$  \hfill (3.32)

From (3.30) and (3.32), we obtain the one-loop determinant of the vector multiplet in (3.11):

$$Z_{3d, vec,G}(x, q) = \text{Det} \left( \bar{D}_2 + i \bar{D}_3 \right) = \prod_{n,m \in \mathbb{Z}} (n + m \tau) \cdot \prod_{n,m \in \mathbb{Z}, \alpha \in \text{rt}(g)} (n + m \tau + \alpha(u))$$

$$= \left( \frac{2\pi \eta(q)^2}{i} \right)^{\text{rk}(G)} \prod_{\alpha \in \text{rt}(g)} \frac{i^{\theta_1(x, q)}}{\eta(q)}.$$  \hfill (3.33)

We used the zeta function regularization in the last line. Next we evaluate the one-loop determinants for the chiral multiplet in a representation $R$ of the gauge group $G$ and in a representation $F$ of a flavor symmetry group $G_F$.

Since we impose the twisted boundary condition with the flavor symmetry in (3.3), the covariant derivative (3.20) is shifted by the background gauge field for the flavor symmetry group:

$$\partial_i + i \bar{A}_i \rightarrow \partial_i + i \bar{A}_i + i A^F_i \text{ for } i = 2, 3. \hfill (3.34)$$

$A^F_i$ with $i = 2, 3$ is the background gauge field on $T^2$ which belongs to a representation of the maximal torus of the flavor symmetry group $G_F$. The fugacity of the flavor symmetry is written in terms of the background gauge field as

$$2\pi z := \oint_t A^F - \tau \oint_s A^F, \quad 2\pi \bar{z} := \oint_t A^F - \bar{\tau} \oint_s A^F. \hfill (3.35)$$

where $z = (z_1, \cdots, z_{\text{rk}(G_F)})$. To make the equations concise, we include $A^F_i$ in the definition of $\bar{D}_i$.

For the Dirichlet boundary condition, the mode expansions of the chiral multiplets along the interval $I$ are expressed as

$$\phi = \sum_{\ell=1}^{\infty} \phi^{(\ell)} \sin \frac{\ell x}{L}, \quad \bar{\phi} = \sum_{\ell=1}^{\infty} \bar{\phi}^{(\ell)} \sin \frac{\ell x}{L},$$

$$\psi_\alpha = \sum_{\ell=0}^{\infty} \psi^{(\ell)}_c \cos \frac{\ell x}{L} + (-1)^{\alpha-1} \sum_{\ell=1}^{\infty} \psi^{(\ell)}_s \sin \frac{\ell x}{L},$$

$$\bar{\psi}_\alpha = \sum_{\ell=0}^{\infty} \bar{\psi}^{(\ell)}_c \cos \frac{\ell x}{L} + (-1)^{\alpha-1} \sum_{\ell=1}^{\infty} \bar{\psi}^{(\ell)}_s \sin \frac{\ell x}{L}, \quad (\alpha = 1, 2). \hfill (3.36)$$
The integration over the interval $I$ gives
\[
\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} dx^1 \bar{\phi} (-\bar{D}^\mu \bar{D}_\mu + iD) \phi = \frac{1}{2} \sum_{\ell=1}^\infty \bar{\phi}^{(\ell)} \left(-\bar{D}_2^2 - \bar{D}_3^2 + \frac{\ell^2}{L^2} + iD\right) \phi^{(\ell)},
\]
\[
\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} dx^1 \bar{\psi} (-i\gamma^\mu \bar{D}_\mu) \psi = -2\bar{\psi}_c^{(0)} (D_2 + iD_3) \psi_c^{(0)}
\]
\[
-\sum_{\ell=1}^\infty \left(\bar{\psi}_c^{(\ell)} \bar{\psi}_s^{(\ell)}\right) \left(\begin{array}{cc}
D_2 + iD_3 & -i\frac{\ell}{L} \\
-i\frac{\ell}{L} & D_2 - iD_3
\end{array}\right) \left(\begin{array}{c}
\psi_c^{(\ell)} \\
\psi_s^{(\ell)}
\end{array}\right).
\] (3.37)

Here we write the zero-mode of the auxiliary field $D$ simply by the same symbol $D$.

For the Neumann boundary condition, the mode expansions along the interval $I$ are given by
\[
\phi = \sum_{\ell=0}^\infty \phi^{(\ell)} \cos \frac{\ell x^1}{L}, \quad \bar{\phi} = \sum_{\ell=0}^\infty \bar{\phi}^{(\ell)} \cos \frac{\ell x^1}{L},
\]
\[
\psi_\alpha = (-1)^{\sigma-1} \sum_{\ell=0}^\infty \psi_c^{(\ell)} \cos \frac{\ell x^1}{L} + \sum_{\ell=1}^\infty \psi_s^{(\ell)} \sin \frac{\ell x^1}{L},
\]
\[
\bar{\psi}_\alpha = (-1)^{\sigma-1} \sum_{\ell=0}^\infty \bar{\psi}_c^{(\ell)} \cos \frac{\ell x^1}{L} + \sum_{\ell=1}^\infty \bar{\psi}_s^{(\ell)} \sin \frac{\ell x^1}{L}, \quad (\alpha = 1, 2).
\] (3.38)

The mode expansions of the kinetic terms are evaluates as
\[
\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} dx^1 \bar{\phi} (-\bar{D}^\mu \bar{D}_\mu + iD) \phi
\]
\[
= \bar{\phi}^{(0)} \left(-\bar{D}_2^2 - \bar{D}_3^2 + iD\right) \phi^{(0)} + \frac{1}{2} \sum_{\ell=1}^\infty \bar{\phi}^{(\ell)} \left(-\bar{D}_2^2 - \bar{D}_3^2 + \frac{\ell^2}{L^2} + iD\right) \phi^{(\ell)},
\]
\[
\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} dx^1 \bar{\psi} (-i\gamma^\mu \bar{D}_\mu) \psi
\]
\[
= -2\bar{\psi}_c^{(0)} (D_2 - iD_3) \psi_c^{(0)} - \sum_{\ell=1}^\infty \left(\bar{\psi}_c^{(\ell)} \bar{\psi}_s^{(\ell)}\right) \left(\begin{array}{cc}
D_2 - iD_3 & -i\frac{\ell}{L} \\
-i\frac{\ell}{L} & D_2 + iD_3
\end{array}\right) \left(\begin{array}{c}
\psi_c^{(\ell)} \\
\psi_s^{(\ell)}
\end{array}\right).
\] (3.39)

From (3.37) and (3.39), we obtain the effect of one-loop determinants with the Dirichlet and the Neumann boundary conditions:
\[
g_{\text{chi,D}}(u, D) = \det (D_2 + iD_3) \prod_{\ell=1}^\infty \det \left(\frac{\ell^2}{L^2} - \bar{D}_2^2 - \bar{D}_3^2\right)
\]
\[
= \prod_{Q \in \text{wt}(R)} \prod_{Q^F \in \text{wt}(F)} \prod_{m,n \in \mathbb{Z}} (n - m\tau + Q(u) + Q^F(z))
\]
Here we used the zeta function regularization.

Again \( Q = (Q_1, \ldots , Q_{\text{rk}(G)}) \) (resp. \( Q^F = (Q^F_1, \ldots , Q^F_{\text{rk}(G^F)}) \)) is a weight of a representation of \( G \) (resp. \( G^F \)). If \( D = 0 \), \( g_{\text{chi},a} \) for \( a \in \{ N, D \} \) becomes the (3.12) and (3.13):

\[
g_{\text{chi},D}(u, D = 0) = Z_{\text{chi},D,R}(x, y; q) = \prod_{Q \in \text{wt}(R)} \prod_{Q^F \in \text{wt}(F)} \frac{i \theta_1(x^{Q} z^{Q^F}, q)}{\eta(q)}, \tag{3.42}
\]

\[
g_{\text{chi},N}(u, D = 0) = Z_{\text{chi},N,R}(x, y; q) = \prod_{Q \in \text{wt}(R)} \prod_{Q^F \in \text{wt}(F)} \frac{i \eta(q)}{\theta_1(x^{Q} y^{Q^F}, q)}. \tag{3.43}
\]

Here we used the zeta function regularization.

In similar way, the one-loop determinants of the 2d \( \mathcal{N} = (0, 2) \) multiplets are computed as

\[
g_{\text{fermi}}(u) = \prod_{Q \in \text{wt}(R)} \prod_{Q^F \in \text{wt}(F)} \prod_{m,n \in \mathbb{Z}} (n - m \tau + Q(u) + Q^F(z)),
\]

\[
g_{2d, \text{chi}}(u, D) = \prod_{Q \in \text{wt}(R)} \prod_{Q^F \in \text{wt}(F)} \prod_{m,n \in \mathbb{Z}} \frac{(n - m \bar{\tau} + Q(\bar{u}) + Q^F(\bar{z}))}{|n - m \tau + Q(u) + Q^F(z)|^2 + iQ(D)}. \tag{3.44}
\]

Here \( R \) and \( F \) are representations of the gauge and flavor symmetry groups of the 2d multiplets on the boundary torus, respectively. Note that \( g_{\text{fermi}}(u) \) and \( g_{2d, \text{chi}}(u, D = 0) \) are given by (3.42) and (3.43) with the zeta function regularization.

Next we perform the integral over the zero-modes of the gaugini \( \lambda, \bar{\lambda} \) and the auxiliary field \( D \), which impose \( D = 0 \) and gives (3.42) and (3.43).

### 3.3 Integration over zero modes

In the previous subsection, we have evaluated the one-loop determinant for the fluctuation around the saddle point locus. When the gaugino zero-modes exist, the path integral over the gaugino zero-modes contributes to the supersymmetric localization procedure. In this
subsection, we will perform the integration over the gaugino zero-modes. The gaugino zero
t modes \( \lambda_{0,0,0}, \tilde{\lambda}_{0,0,0} \) can be expanded as

\[
\lambda_{c,(0,0,0)} = \sum_{a=1}^{\text{rk}(G)} \lambda_{(0)}^a H^a, \quad \tilde{\lambda}_{c,(0,0,0)} = \sum_{a=1}^{\text{rk}(G)} \tilde{\lambda}_{(0)}^a H^a.
\] (3.45)

Then the following combination of the Yukawa couplings saturates the integral over the
gaugino zero-modes:

\[
\begin{aligned}
&\int \prod_{a=1}^{\text{rk}(G)} d\lambda_{(0)}^a d\tilde{\lambda}_{(0)}^a D\tilde{\phi}D\phi D\tilde{\psi}D\psi e^{ \int_{I\times T^2} [\tilde{\phi} (\tilde{D}^\mu \tilde{D}_\mu - iD) \tilde{\psi} + \psi (i\mu D_\mu)] } \\
&\times \frac{1}{(\text{rk}(G))!^2} \left( \int_{I\times T^2} i\tilde{\bar{\psi}} \lambda_{0,0,0} \phi \right)^{\text{rk}(G)} \left( \int_{I\times T^2} i\bar{\psi} \bar{\lambda}_{0,0,0} \phi \right)^{\text{rk}(G)} \\
&= \det h_a(u, D) g_{\text{chi}, a}(u, D).
\end{aligned}
\] (3.46)

Here \( a \in \{D, N\} \) denotes the Dirichlet boundary condition (D) or the Neumann boundary
condition (N). \( h_a^{ab} \) with \( a, b = 1, \ldots, \text{rk}(G) \) is defined by

\[
\begin{aligned}
h_D^{ab} &= \sum_{Q^F \in \text{wt}(g)} \sum_{\ell=1}^{\infty} \sum_{m,n \in \mathbb{Z}} \frac{2Q^a Q^b (n - m\tau + Q(u) + Q^F(z))}{|n - m\tau + Q(u) + Q^F(z)|^2 + \frac{\ell^2}{\tau^2}} \\
&\times \frac{1}{|n - m\tau + Q(u) + Q^F(z)|^2 + \frac{\ell^2}{\tau^2} + iQ(D)},
\end{aligned}
\]

\[
\begin{aligned}
h_N^{ab} &= \sum_{Q^F \in \text{wt}(g)} \sum_{\ell=0}^{\infty} \sum_{m,n \in \mathbb{Z}} \frac{2Q^a Q^b (n - m\tau + Q(u) + Q^F(z))}{|n - m\tau + Q(u) + Q^F(z)|^2 + \frac{\ell^2}{\tau^2}} \\
&\times \frac{1}{|n - m\tau + Q(u) + Q^F(z)|^2 + \frac{\ell^2}{\tau^2} + iQ_1(D)}.
\end{aligned}
\] (3.47)

\( h_a^{ab} \) satisfies the following relations:

\[
\begin{aligned}
\frac{\partial g_a(u, D)}{\partial \bar{u}_a} &= -ih_a^{ab}(u, D) D_b g_a(u, D), \\
\frac{\partial h_a^{ab}(u, D)}{\partial \bar{u}_c} &= \frac{\partial h_a^{ca}(u, D)}{\partial \bar{u}_b} = \frac{\partial h_a^{bc}(u, D)}{\partial \bar{u}_a}.
\end{aligned}
\] (3.48)

The evaluation of integrals over the 3d gaugino zero-modes with the Yukawa couplings
including boundary 2d fields are parallel to (3.46):

\[
\begin{aligned}
h_{2d, \text{chi}} &= \sum_{Q^F \in \text{wt}(g)} \sum_{m,n \in \mathbb{Z}} \frac{2Q^a Q^b}{|n - m\tau + Q(u) + Q^F(z)|^2 + \frac{\ell^2}{\tau^2} + iQ(D)} \left( n - m\tau + Q(\bar{u}) + Q^F(z) \right).
\end{aligned}
\] (3.50)
After performing the path integral for the fluctuations and the gaugino zero-modes, we obtain the following result:

\[
Z_{I \times T^2} = \frac{c}{|W_G|} \lim_{\varepsilon \to 0} \lim_{\tau \to 0} \int_{\mathcal{M} \setminus \Delta_\varepsilon} d^{rk(G)}u \, d^{rk(G)}\bar{u} \int_{\mathbb{R}^{rk(G)}} d^{rk(G)}D \\
\times \det h(u, \bar{u}, D)g(u, \bar{u}, D) \exp \left[ -\frac{1}{2\varepsilon^2} \text{Tr}(D^2) - i\zeta(D) \right],
\]

with

\[
\det h = \prod \det h_a \prod \det h_{2d, \text{chi}}, \quad g = Z_{3d, \text{vec}, G} \prod g_{\text{chi}, a} \prod g_{2d, \text{chi}} \prod g_{\text{fermi}}.
\]

Here the products run over the 3d chiral multiplets, the 2d chiral and the fermi multiplets. \(c\) is an overall constant. For elliptic genera, the overall constant is taken to reproduce the free field computation of the elliptic genera [18, 19, 20]. Since the indices on \(I \times T^2\) do not depend on the length of \(I\), we take the same normalization as the elliptic genera; \(c = (4\pi^2i)^{-rk(G)}\).

\(\mathcal{M}\) is the space of flat connections \(u\) and \(\bar{u}\). \(\Delta_\varepsilon\) is the union of the \(\varepsilon\)-neighborhood around the singular loci of the one-loop determinant defined as follows. First we define \(H_i\) called a singular hyperplane associated with the \(i\)-th 3d chiral multiplet with the Neumann boundary condition or the \(i\)-th 2d \(\mathcal{N} = (0, 2)\) chiral multiplet by

\[
H_i := \{ u = (u_1, \cdots, u_{rk(G)}) | u_i \in \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}, Q_i(u) + Q_i^F(z) = 0 \}.
\]

Then \(\Delta_\varepsilon(H_i)\) is the \(\varepsilon\)-neighborhood of the singular hyperplane \(H_i\):

\[
\Delta_\varepsilon(H_i) := \{ u = (u_1, \cdots, u_{rk(G)}) | |Q_i(u) + Q_i^F(z)| \leq \varepsilon \}.
\]

\(\Delta_\varepsilon\) is the union of \(\Delta_\varepsilon(H_i)\) defined by

\[
\Delta_\varepsilon = \bigcup_i \Delta_\varepsilon(H_i),
\]

where index \(i\) runs over all the singular hyperplanes in the theory. For the higher rank gauge theories, the relations [3.48] and [3.49] satisfy the same properties in order to perform the integration over \(\bar{u}\) and \(D\) in [15]. By repeating the argument in [15], we obtain the expression [3.7] after some tedious computations. For simplicity we shall consider \(U(1)\) gauge theories and evaluate explicitly the integrals of \(\bar{u}\) and \(D\). In this case, [3.51] with \(rk(G) = 1\) is written as

\[
Z_{I \times T^2} = \lim_{\varepsilon \to 0} \int_{\mathcal{M} \setminus \Delta_\varepsilon} \frac{dud\bar{u}}{2\pi} \int_{\mathbb{R}} \frac{dD}{2\pi i} h(u, D)g(u, D) \exp \left[ -\frac{1}{2\varepsilon^2} D^2 - i\zeta D \right].
\]
Figure 1: (a): The integration contour $\Gamma_-$ is specified by the solid black arrow. Any pole arising from $\Delta^+_{\varepsilon}$ does not hit $\Gamma_-$ in the limit $\varepsilon \to 0$. From the disjointness of $\Delta^+_{\varepsilon}$ and $\Delta^-_{\varepsilon}$, any pole approaches to $\Gamma_-$ from the negative imaginary axis in the limit $\varepsilon \to 0$. (b): The decomposition of $\Gamma_-$ to $C_0 + \Gamma_+$. Any pole arising from $\Delta^-_{\varepsilon}$ does not hit $\Gamma_+$ in the limit $\varepsilon \to 0$.

For the rank one gauge theories, we omitted the labels for the Cartan part of the gauge group as $D = D^1$, $u = u^1$, $h = h^{11}$, and so on.

When $u$ locates on a center of the tube $|n - m\tau + Q_i u + Q_i^F(z)| = \varepsilon$, the zero-mode of the auxiliary field $D$ has a pole at $D = iQ_i\varepsilon^2$. The contour of $D$ can be deformed away from the origin of the imaginary axis if the contour does not hit the pole specified as

$$|n - m\tau + Q_i u + Q_i^F(z)|^2 + iQ_i D = 0. \quad (3.57)$$

We define such deformed integration contours $\Gamma_\pm$ by $\Gamma_\pm := \mathbb{R} \pm i\delta$ with $0 < \delta < \varepsilon$. First we take $\Gamma_-$. Eq. (3.48) for the rank one gauge theory is given by

$$\frac{\partial g(u, D)}{\partial \bar{u}} = -iDh(u, D)g(u, D). \quad (3.58)$$

From this relation we can rewrite the integral (3.56) as

$$Z_{I \times T^2} = -\lim_{\varepsilon \to 0} \int_{\Gamma_-} \frac{dD}{2\pi iD} \int_{\mathfrak{g} \setminus \Delta_{\varepsilon}} \frac{dud\bar{u}}{2\pi i} \frac{\partial g(u, D)}{\partial \bar{u}} \exp \left[ -\frac{1}{2e^2}D^2 - i\zeta D \right] \int_{\partial(\mathfrak{g} \setminus \Delta_{\varepsilon}) = \partial\Delta_{\varepsilon}} \frac{du}{2\pi i} g(u, D). \quad (3.59)$$

Here we assume that the $\Delta_{\varepsilon}$ in the rank one gauge theory is decomposed to the disjoint union:

$$\Delta_{\varepsilon} = \Delta^+_{\varepsilon} \sqcup \Delta^-_{\varepsilon}, \quad (3.60)$$

where $\Delta^+_{\varepsilon}$ is the union of the $\varepsilon$-neighborhoods around the singular hyperplanes (=points) $Q_i u + Q_i^F(z) = 0$ with $Q_i > 0$ and $\Delta^-_{\varepsilon}$ is the union of the $\varepsilon$-neighborhoods around the
singular hyperplanes \( Q_i u + Q_i^F(z) = 0 \) with \( Q_i < 0 \). If the condition (3.60) is satisfied, the singular hyperplane arrangements are called projective. For the higher rank gauge theories, the singular hyperplane arrangements mean that weights \( Q = (Q_1, \cdots, Q_{rk(G)}) \) for gauge representations at each singular point \( u \) are contained in a half space of \( \mathbb{R}^{rk(G)} \). In this paper we assume “projective” condition is satisfied.

Since the pole \( D = iQ_i \varepsilon^2 \) with \( Q_i > 0 \) does not hit the integration contour \( \Gamma_\varepsilon \) in the limit \( \varepsilon \to 0 \) as depicted by (a) in Figure 1 and the integrand is bounded, the contribution from a boundary \( \partial \Delta_+^\varepsilon \) in (3.59) vanishes:

\[
\lim_{\varepsilon \to 0} \int_{\Gamma_-} \frac{dD}{2\pi iD} \exp \left[ -\frac{1}{2\varepsilon^2} D^2 - \frac{i\varepsilon}{2} D \right] \oint_{\partial \Delta_+^\varepsilon} \frac{du}{2\pi i} g(u, D) = 0. \tag{3.61}
\]

Next we will see the contribution from \( \Delta_-^\varepsilon \) is written as the contour integral on \( \partial \Delta_-^\varepsilon \). As depicted by Figure 1(b) we decompose the integration contour \( \Gamma_- \) as

\[
\Gamma_- = C_0 + \Gamma_+ \tag{3.62}
\]

Here \( C_0 \) is a small circle around the origin of the \( D \)-plane and the index is expressed as

\[
Z_{I\times T^2} = - \lim_{\varepsilon \to 0} \int_{\Gamma_-} \frac{dD}{2\pi iD} \exp \left[ -\frac{1}{2\varepsilon^2} D^2 - \frac{i\varepsilon}{2} D \right] \oint_{\partial \Delta_-^\varepsilon} \frac{du}{2\pi i} g(u, D) = \lim_{\varepsilon \to 0} \int_{C_0} \frac{dD}{2\pi iD} \exp \left[ -\frac{1}{2\varepsilon^2} D^2 - \frac{i\varepsilon}{2} D \right] \oint_{D_-^\varepsilon} \frac{du}{2\pi i} g(u, D)
\]

\[
- \lim_{\varepsilon \to 0} \int_{\Gamma_+} \frac{dD}{2\pi iD} \exp \left[ -\frac{1}{2\varepsilon^2} D^2 - \frac{i\varepsilon}{2} D \right] \oint_{\partial \Delta_-^\varepsilon} \frac{du}{2\pi i} g(u, D). \tag{3.63}
\]

The last term in (3.63) vanishes due to a similar reason of (3.61). The residue at \( C_0 \) gives an expression:

\[
Z_{I\times T^2} = - \int_{\partial \Delta_-^\varepsilon} \frac{du}{2\pi i} g(u, D = 0). \tag{3.64}
\]

If we choose \( \Gamma_+ \), the same argument gives

\[
Z_{I\times T^2} = \int_{\partial \Delta_+^\varepsilon} \frac{du}{2\pi i} g(u, D = 0). \tag{3.65}
\]

(3.64) and (3.65) are the localization formula (3.7) for the \( G = U(1) \) gauge theories. The sign of \( \delta \) corresponds to \( \eta \) in the JK residues.

### 4 2d \( \mathcal{N} = (2, 2) \) and \( \mathcal{N} = (0, 4) \) theories from 3d \( \mathcal{N} = 4 \) theories

We study the relations between \( I \times T^2 \) indices for 3d \( \mathcal{N} = 4 \) theories and 2d elliptic genera. A similar construction of 2d \( \mathcal{N} = (2, 2) \) and \( \mathcal{N} = (0, 4) \) theories based on 4d \( \mathcal{N} = 2 \) theories on \( T^2 \times S^2 \) was studied in [21].
The 3d $\mathcal{N} = 4$ $G$ vector multiplet is decomposed to 3d $\mathcal{N} = 2$ $G$ vector multiplet and a chiral multiplet $\varphi$ in the adjoint representation. The charge assignments for 3d $\mathcal{N} = 2$ chiral multiplets $q, \tilde{q}$ in the 3d $\mathcal{N} = 4$ multiplet are depicted in Table 2.

| G   | $U(1)_y$ |
|-----|----------|
| $\varphi$ | adj | $-r_1 - r_2$ |
| $q$    | $\mathbb{R}$ | $r_1$ |
| $\tilde{q}$ | $\overline{\mathbb{R}}$ | $r_2$ |

Table 2: The charge assignments of 3d $\mathcal{N} = 2$ multiplets in the 3d $\mathcal{N} = 4$ multiplets. $\varphi$ denotes the chiral multiplet in the $\mathcal{N} = 4$ vector multiplet. $q$ and $\tilde{q}$ denote the chiral multiplets in the 3d $\mathcal{N} = 4$ hypermultiplet. $U(1)_y$ is a flavor symmetry.

We impose the Dirichlet boundary condition for the adjoint chiral multiplets $\varphi$ in the $\mathcal{N} = 4$ vector multiplet and impose the Dirichlet boundary conditions for both $q$ and $\tilde{q}$, or the Neumann boundary conditions for both $q$ and $\tilde{q}$. We take flavor charges as $r_1 = r_2 = \frac{1}{2}$. Then the one-loop determinants of the 3d $\mathcal{N} = 4$ multiplets are given by

\begin{align}
Z_{\text{vec},D}^{\mathcal{N}=4}(x; y; q) &= Z_{\text{vec},D}^{\mathcal{N}=4} Z_{\text{chi},D}^{\mathcal{N}=4} \\
&= \left(2\pi i \eta(q) \theta_1(q, y^{-1})\right)^{rk(G)} \prod_{\alpha \in rt(g)} \frac{\theta_1(x^\alpha, q) \theta_1(x^\alpha y^{-1}, q)}{\eta(q)^2}, \quad (4.1) \\
Z_{\text{hyp},(D,D)}^{\mathcal{N}=4}(x, y, q) &= Z_{\text{chi},D}^{\mathcal{N}=4} Z_{\text{chi},D}^{\mathcal{N}=4} = \prod_{Q \in \text{wt}(\mathbb{R})} \frac{\theta_1(x^Q y^{\frac{1}{2}}, q) \theta_1(x^{-Q} y^{\frac{1}{2}}, q)}{-\eta(q)^2}, \quad (4.2) \\
Z_{\text{hyp},(N,N)}^{\mathcal{N}=4}(x, y, q) &= Z_{\text{chi},N}^{\mathcal{N}=4} Z_{\text{chi},N}^{\mathcal{N}=4} = \prod_{Q \in \text{wt}(\mathbb{R})} \frac{-\eta(q)^2}{\theta_1(x^Q y^{\frac{1}{2}}, q) \theta_1(x^{-Q} y^{\frac{1}{2}}, q)}. \quad (4.3)
\end{align}

Here $D$ and $N$ denote the boundary conditions for a 3d $\mathcal{N} = 2$ chiral multiplet. (4.1), (4.2) and (4.3) agree with the one-loop determinants of the vector, the long fermi, and the hypermultiplet in the 2d $\mathcal{N} = (0, 4)$ elliptic genus, respectively.

Next we choose the Neumann boundary condition for the adjoint chiral multiplet in the 3d $\mathcal{N} = 4$ vector multiplet. We choose the Neumann (resp. Dirichlet) boundary condition for a chiral multiplet in the representation $\mathbb{R}$ (resp. $\overline{\mathbb{R}}$) in the hypermultiplet. Then the 3d $\mathcal{N} = 4$ vector multiplet preserves $\mathcal{N} = (2, 2)$ supersymmetry at the boundaries. From (3.13), the one-loop determinant of the 3d $\mathcal{N} = 4$ vector multiplet $Z_{\text{vec}}^{\mathcal{N}=4}$ and the hypermultiplet $Z_{\text{hyp}}^{\mathcal{N}=4}$ in a representation $\mathbb{R} \oplus \overline{\mathbb{R}}$ are given by

\begin{align}
Z_{\text{vec},N}^{\mathcal{N}=4}(x; q) &= \left(\frac{2\pi \eta(q)^3}{\theta_1(y^{-1}, q)}\right)^{rk(G)} \prod_{\alpha \in rt(g)} \frac{\theta_1(x^\alpha, q)}{\theta_1(x^\alpha y^{-1}, q)},
\end{align}

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\[ Z_{\text{hyp},(N,\mathbb{D})}^{N=4}(x; z, q) = \prod_{Q \in \text{wt}(\mathbb{R})} \frac{\theta_1(x^Q y^{z-1}, q)}{\theta_1(x^Q y^z, q)}. \]  

(4.4)

Here we choose flavor charges as \( r_1 = r \) and \( r_2 = 1 - r \). \[ (4.4) \] agrees with the one-loop determinants of the vector multiplet and the chiral multiplet in the representation \( \mathbf{R} \) for the 2d \( \mathcal{N} = (2, 2) \) elliptic genus.

### 4.1 2d \( \mathcal{N} = (2, 2) \) \( U(N) \) gauge theory from 3d \( \mathcal{N} = 4 \) gauge theory

As an example of 2d \( \mathcal{N} = (2, 2) \) theory, we take a \( G = U(N_c) \) gauge theory with \( \mathbf{R} = \Box^\otimes N_f \) in Table \( \ref{table:2d} \), where \( \Box \) denotes the fundamental representation of \( U(N_c) \). We assume \( N_c \leq N_f \). The 2d index for the 3d \( \mathcal{N} = 4 \) \( U(N_c) \) gauge theory with \( \mathbf{R} = \Box^\otimes N_f \) is given by

\[ Z_{I \times T^2}^{\mathcal{N}=4}(y, z, q; N_c, N_f) = \frac{1}{N_c!} \sum_{u^*} \text{JK-Res}(Q_s, \eta) Z_{\text{vec},(N_c)}^{\mathcal{N}=4} Z_{\text{hyp},(N,\mathbb{D})}^{\mathcal{N}=4} \wedge_{a=1}^{N_f} du^a \]

\[ = \left( \frac{\eta(q)^3}{\theta_1(y^{-1}, q)} \right)^{N_c} \sum_{1 \leq i_1 < \cdots < i_{N_c} \leq N_f} \int_{x_i = z_i} \prod_{a=1}^{N_c} dx_a \]

\[ \times \prod_{1 \leq a \neq b \leq N_c} \frac{\theta_1(x_a y^{-1} y^{-1}, q)}{\theta_1(x_a y^{-1}, q)} \cdot \prod_{a=1}^{N_f} \frac{\theta_1(x_a y^{-1} z_i^{-1}, q)}{\theta_1(x_a z_i^{-1}, q)}. \]  

(4.5)

(4.5) is same as the elliptic genus for the 2d \( \mathcal{N} = (2, 2) \) \( U(N_c) \) gauge theory with \( N_f \) chiral multiplets in the fundamental representation of \( U(N_c) \) in \[ \ref{ref:2d} \]. Here we have taken \( \eta = (1, 1, \cdots, 1) \in \mathbb{R}^{N_c} \). The JK residue is evaluated as

\[ Z_{I \times T^2}^{\mathcal{N}=4}(y, z, q; N_c, N_f) = \sum_{I \subset \{1,2,\ldots,N_f\}} \prod_{a \in \mathcal{I}} \prod_{b \in \{1,\ldots,N_f\} \setminus \mathcal{I}} \frac{\theta_1(z_a^{-1} y \cdot, q)}{\theta_1(z_a^{-1} z_b^{-1}, q)}. \]  

(4.6)

where \( \mathcal{I} := \{i_1, \cdots, i_{N_c}\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_{N_c} \leq N_f \). The sum \( \sum_{\mathcal{I} \subset \{1,2,\ldots,N_f\}} \) runs over all the possible configurations of \( \mathcal{I} \) in \( \{1, \cdots, N_f\} \). Note that \( Z_{I \times T^2}^{\mathcal{N}=4}(N_c, N_f) \) satisfies the following relation:

\[ Z_{I \times T^2}^{\mathcal{N}=4}(y, z, q; N_c, N_f) = Z_{I \times T^2}^{\mathcal{N}=4}(y, z, q; N_f - N_c, N_f). \]  

(4.7)

A pair of \( U(N_c) \) and \( U(N_f - N_c) \) gauge theories is known as a Seiberg-like duality in two dimensions \[ \ref{ref:2d} \], where the Higgs branch is the Grassmann manifold \( \text{Gr}(N_c, N_f) \simeq \text{Gr}(N_f - N_c, N_f) \) in positive FI-parameter regions. The flavor symmetry \( U(1)_y \) is broken to \( \mathbb{Z}_{N_f} \) due to the anomaly.

### 4.2 Mirror of 3d \( \mathcal{N} = 8 \) Super Yang–Mills and M-strings

As an example of 3d \( \mathcal{N} = 4 \) theory on \( I \times T^2 \) leading to a 2d \( \mathcal{N} = (0, 4) \) elliptic genus, we consider the 3d \( \mathcal{N} = 4 \) \( U(N) \) gauge theory with an adjoint hypermultiplet \( (B_1, B_2) \) and a
Table 3: The charge assignments and the boundary conditions. $\varphi$ denotes the adjoint chiral multiplet in the 3d $\mathcal{N} = 4$ vector multiplet. $(B_1, B_2)$ is an adjoint hypermultiplet and $(I, J)$ is a fundamental hypermultiplet. adj is the adjoint representation. $\Box$ (resp. $\square$) denotes the fundamental (resp. anti-fundamental) representation. The b.c. represents the boundary condition. The flavor symmetry group $U(1)_y$ exists for $L = 0$.

|       | $U(N)_{\text{gauge}}$ | $U(1)_{\epsilon_1}$ | $U(1)_{\epsilon_2}$ | $U(1)_y$ | b.c. |
|-------|-----------------------|----------------------|---------------------|-----------|------|
| $\varphi$ | adj                  | 1                    | 1                   | 0         | D    |
| $B_1$   | adj                  | 1                    | 0                   | 0         | N    |
| $B_2$   | adj                  | 0                    | 1                   | 0         | N    |
| $I$     | $\Box$               | $\frac{1}{2}$        | $\frac{1}{2}$      | 0         | N    |
| $J$     | $\Box$               | $\frac{1}{2}$        | $\frac{1}{2}$      | 0         | N    |
| $\psi'_{-L}$ | $\Box$              | 0                    | 0                   | 1         | -    |
| $\psi'_{-R}$ | $\Box$              | 0                    | 0                   | 1         | -    |

fundamental hypermultiplet $(I, J)$. The moduli space of Higgs branch vacua is the ADHM moduli space of the $N$-instantons in the $U(1)$ gauge theory. This theory is known as the mirror dual of the 3d $\mathcal{N} = 8$ super Yang-Mills theory, which flows to the same IR fixed point of the $U(N)_1 \times U(N)_{-1}$ ABJM model [23] describing the world volume theory on $N$-stacks of M2-branes on $\mathbb{C}^4$.

We impose the boundary conditions specified in Table 3. They preserve the supersymmetry of the superpotential term:

$$W = \text{tr}_{\varphi}([B_1, B_2] + IJ).$$

(4.8)

Under the boundary condition in Table 3 we find that the one-loop determinant of each 3d $\mathcal{N} = 4$ multiplet agrees with that of the 2d $\mathcal{N} = (0, 4)$ multiplet. Since 3d multiplets induce the gauge anomalies, we have to introduce fermi multiplets $\psi'_{-L}$ at $x^1 = -\pi L$ and $\psi'_{-R}$ at $x^1 = \pi L$ in Table 3 to cancel the gauge anomaly. In the limit $L \to 0$, $\psi'_{-L}$ and $\psi'_{-R}$ live on the same spacetime and form a long fermi multiplet. An extra $U(1)_y$ flavor symmetry appears in the limit $L = 0$. The charge assignments for the $U(1)_y$ symmetry are depicted in Table 3.

We shall compute the $I \times T^2$ index

$$Z^{ADHM}_{I \times T^2} = \frac{\eta(q)^N}{N!} \int \prod_{a=1}^N du^a \prod_{1 \leq a \neq b \leq N} \theta_1(x_a x_b^{-1}, q) \prod_{a,b=1}^N \theta_1(x_a x_b^{-1} q_1 q_2, q) \prod_{a,b=1}^N \theta_1(x_a x_b^{-1} q_1, q) \theta_1(x_a x_b^{-1} q_2, q)$$

$$\times \prod_{a=1}^N \frac{\theta_1(x_a y, q) \theta_1(x_a^{-1} y, q)}{\theta_1(x_a (q_1 q_2)^{\frac{1}{2}}, q) \theta_1(x_a^{-1} (q_1 q_2)^{\frac{1}{2}}, q)},$$

(4.9)
where \( x_a = e^{2\pi i u_a} \). The fugacities \( q_i = e^{2\pi i \epsilon_i} \) with \( i = 1, 2 \) correspond to the \( \Omega \)-background parameters. We included a formal fugacity \( y \) for \( U(1)_y \) in (4.9) to compare with the M-string partition function. The JK residue computations are same as those of Nekrasov’s \( N \)-instanton partition function in [24]. Then we obtain the result:

\[
Z_{\text{ADHM}}^{I \times T^2} = \sum_{Y: |Y| = N} \prod_{(i,j) \in Y} \theta_1(q_1^{i-\frac{1}{2}j-\frac{1}{2}} y, q) \theta_1(q_1^{-i+\frac{1}{2}j+\frac{1}{2}} y, q) \theta_1(q_1^{\lambda_i^T-j+1} q_2^{\lambda_i-j+1}, q) \theta_1(q_1^{\lambda_i^T-i} q_2^{-\lambda_i-j}, q).
\]

Here the sum is taken over the Young diagrams \( Y = (\lambda_1, \lambda_2, \cdots) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \) and the number of boxes of \( Y \) is \( |Y| = N \). \( Y^T = (\lambda_1^T, \lambda_2^T, \cdots) \) is the transpose of \( Y \). (4.9) and (4.10) reproduce the elliptic genus of M-strings suspended between 2 M5-branes on the single center Taub-NUT space [25], except for the fugacity \( y \). In our case \( y = 1 \) for \( L > 0 \).

5 Three dimensional dualities on \( I \times M_2 \)

5.1 3d \( N = 2 \) SQED and XYZ model

| \( \phi \) | \( U(1)_{\text{gauge}} \) | \( U(1)_y \) | \( U(1)_R \) | b.c. |
|---|---|---|---|---|
| \( \tilde{\phi} \) | 1 | 1 | 0 | N |
| \( \phi \) | -1 | 1 | 0 | N |
| \( \psi^R_-, \psi^L_- \) | 1 | 0 | 0 | - |

Table 4: The charge assignments and boundary conditions for the SQED and the fermi multiplets. \( \phi \) and \( \tilde{\phi} \) denote scalars in the chiral multiplets. \( \psi^R_- \) and \( \psi^L_- \) denote the boundary fermions at \( x^1 = \pi L \) and \( x^1 = -\pi L \), respectively.

| \( \phi_X \) | \( U(1)_y \) | \( U(1)_R \) | b.c. |
|---|---|---|---|
| \( \phi_X \) | -1 | 1 | D |
| \( \phi_Y \) | -1 | 1 | D |
| \( \phi_Z \) | 2 | 0 | N |

Table 5: The charge assignments and boundary conditions for the XYZ model. \( \phi_X, \phi_Y, \phi_Z \) express the scalars in the three chiral multiplets \( X, Y \) and \( Z \).

We consider a simple 3d \( N = 2 \) mirror symmetry; the 3d \( N = 2 \) one-flavor SQED and the XYZ model [26, 27]. The charge assignments of the SQED and the XYZ model are listed in Table 5. We put these theories on the interval and study a duality with the boundaries based on indices and anomaly matching. The Neumann boundary conditions

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for two chiral multiplets \((q, \bar{q})\) in the SQED are specified by \((N, N)\). In the XYZ model, the Dirichlet boundary conditions for chiral multiplets \(X,Y\) and the Neumann boundary condition for a chiral multiplet \(Z\) are specified by \((D, D, N)\). To cancel the gauge anomaly in the SQED, we add one fermi multiplet which couples to 3d \(U(1)\) gauge field on the left and the right boundary, respectively. At these boundaries \(x^1 = \pm \pi L\), the anomaly polynomials are evaluated as

\[
I_{\text{SQED}} = -\frac{1}{2}(f + y - r)^2 - \frac{1}{2}(-f + y - r)^2 + \frac{1}{2}r^2,
\]

\[
I_{\text{XYZ}} = \frac{1}{2}y^2 + \frac{1}{2}y^2 - \frac{1}{2}(2y - r)^2,
\]

\[
I_{\text{fermi}} = f^2,
\]

(5.1)

where \(I_{\text{SQED}}, I_{\text{XYZ}}\) and \(I_{\text{fermi}}\) are the anomaly polynomials for the SQED, the XYZ model and the fermi multiplet, respectively. Then we find that the anomaly polynomials match:

\[
I_{\text{SQED}} + I_{\text{Fermi}} = I_{\text{XYZ}}.
\]

(5.2)

The \(I \times T^2\) index for the SQED with two boundary fermi multiplets is written as

\[
Z_{I \times T^2, (N,N)}^{\text{SQED}} = -\eta(q)^2 \oint_{x=y^{-1}} \frac{dx}{2\pi i x} \frac{\theta_1(x, q)^2}{\theta_1(x, q) \theta_1(x^{-1} y, q)} = \left( \frac{i \theta_1(y^{-1}, q)^2}{\eta(q)} \right)^2 \frac{\eta(q)}{\theta_1(y^2, q)}
\]

(5.3)

Table 6: The charge assignments and boundary conditions for the chiral multiplets in the \(U(N)\) gauge theory and the boundary fermi multiplets. \(\phi\) and \(\tilde{\phi}\) denote scalars in the chiral multiplets. \(SU(N)_y \times SU(N)_{\tilde{y}} \times U(1)_{a}\) is the set of flavor symmetry groups. \(\psi'_{-R}, \psi'_{-L}\) denote the fermions at \(x^1 = \pi L\) and \(x^1 = -\pi L\), respectively. \(\text{det}\) is the determinant representation. 1 is the trivial representation.
Mirror symmetry for one flavor $\mathcal{N} = 2$ SQED is generalized to an Aharony duality \cite{28} for the $U(N)$ gauge theory with $N$ fundamental and anti-fundamental chiral multiplets. The dual theory consists of chiral multiplets $M$, $q$, $\tilde{q}$ with the superpotential $W = \det(M)\tilde{q}q$. The charge assignments of two theories are listed in Table 6 and Table 7. The anomaly polynomials $I_{U(N)+N}$, $I_{\det(M)\tilde{q}q}$, $I_{\text{fermi}}$ of the $U(N)$ gauge theory, the dual theory with $W = \det(M)\tilde{q}q$, and the boundary fermions are given by

$$I_{U(N)+N} = N\Tr(f^2) - (\Tr f)^2 + \frac{N^2}{2}(a - r)^2$$

$$- \frac{N}{2}\Tr(f^2) - \frac{N}{2}\Tr(y)^2 + N(\Tr f)(a - r) - \frac{N^2}{2}(a - r)^2$$

$$- \frac{N}{2}\Tr(\tilde{y})^2 - \frac{N}{2}(2a - r)^2 + \frac{1}{2}(-Na)^2 + \frac{1}{2}(-Na)^2,$$

$$I_{\det(M)\tilde{q}q} = -\frac{N}{2}\Tr(y)^2 - \frac{N}{2}\Tr(\tilde{y})^2 - \frac{N^2}{2}(2a - r)^2$$

$$+ \frac{1}{2}(-Na)^2 + \frac{1}{2}(-Na)^2,$$

$$I_{\text{fermi}} = (\Tr f)^2. \quad (5.4)$$

The anomaly polynomials satisfy a matching condition:

$$I_{U(N)+N} + I_{\text{fermi}} = I_{\det(M)\tilde{q}q}. \quad (5.5)$$

The $I \times T^2$ index for the dual theory is given by

$$Z_{I \times T^2, (N,N,D)}^{W = \det(M)\tilde{q}q} = \left( i \frac{\theta_1(a^{-N})}{\eta(q)} \right)^2 \prod_{i,j=1}^{N} \frac{\eta(q)}{\theta_1(a^2 \tilde{y}_i^{-1} y_j, q)}. \quad (5.6)$$

Here $y_i$'s and $\tilde{y}_i$'s ($i = 1, \cdots, N$) with $\prod_{i=1}^{N} y_i = 1$ and $\prod_{i=1}^{N} \tilde{y}_i = 1$ are fugacties for the $SU(N)_y$ and the $SU(N)_{\tilde{y}}$, respectively. On the other hand, the index for the $U(N)$ gauge
theory is given by

\[
Z_{I \times T^2, (N,N)}^{U(N)+N\text{-flavors}} = \frac{1}{N!} \left( \frac{\eta(q)}{i} \right)^{2N} \sum_{j=1}^{N} \sum_{k=1}^{N} \phi \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} \prod_{1 \leq i \neq j \leq N} \frac{\theta_1(x_i^{-1}, q)}{\eta(q)}
\]

\[
\times \left( \frac{\theta_1(\prod_{i=1}^{N} x_i, q)}{\eta(q)} \right)^2 \prod_{i,j=1}^{N} \frac{(i\eta(q))^2}{\theta_1(x_i a y_j^{-1}, q) \theta_1(x_i^{-1} a y_j, q)}
\]

\[
= \left( \frac{\theta_1(a^{-N})}{\eta(q)} \right)^2 \prod_{i,j=1}^{N} \frac{\eta(q)}{\theta_1(a^2 y_i^{-1} y_j, q)},
\]

(5.7)

where we have chosen \( \eta = (1, \cdots, 1) \) in the JK residue formula. If we choose \( \eta = (-1, \cdots, -1) \), we obtain the same result. Thus we have agreement between the indices of two theories.

5.3 Triality on the interval from 3d Seiberg-like dualities

Figure 2: (a): The decomposition of 3d multiplets \( \{\Phi_i\}_{i=1}^{N_1+N_3} \mapsto (\Phi, \Psi) \) and \( \{\tilde{\Phi}_k\}_{k=1}^{N_2} \mapsto P \) in the theory \( A \) by the boundary conditions. In the right figure of (a), the dashed line denotes the \( N_1 \) chiral multiplets with the Dirichlet boundary condition \( \Psi \). The solid arrows denote the chiral multiplets with the Neumann boundary condition \( \Phi, P \). (b): The decomposition of 3d multiplets \( \{\Phi_i^{\vee}\}_{i=1}^{N_2} \mapsto \Phi^{\vee}, \{\tilde{\Phi}_k^{\vee}\}_{k=1}^{N_1+N_3} \mapsto (P^{\vee}, \Psi^{\vee}) \) and \( M_{ik} \mapsto (M, \Gamma) \) in the theory \( A^{\vee} \).

In this section, we start from two pairs of Seiberg-like dualities and construct three theories on the interval. We will see the three theories satisfy the 't Hooft anomaly matchings and the \( I \times T^2 \) indices agree one another. Our construction is analogous to the relation [29] between the 2d \( \mathcal{N} = (0, 2) \) triality [30] and the twisted compactification on \( S^2 \) with fluxes of 4d \( \mathcal{N} = 1 \) Seiberg dualities [31]. For example, see computations of \( T^2 \times S^2 \) indices in [21].

Let us consider a 3d Seiberg-like dual pair [32]:

\[
G = U(N_c) + \Phi_{i=1,\cdots,N_f} \quad \text{and} \quad \tilde{\Phi}_{k=1,\cdots,N_a},
\]

\[
G = U(N_a - N_c) + \Phi_{k=1,\cdots,N_a}^{\vee}, \tilde{\Phi}_{i=1,\cdots,N_f}^{\vee} \quad \text{and} \quad M_{i=1,\cdots,N_f,k=1,\cdots,N_a}.
\]

(5.8)
by the right quiver in Figure 2(a). To cancel the gauge anomaly of the theory

Table 8: The charge assignments and the boundary conditions in the theory A. The sub-
scripts of the groups correspond to fugacities for these symmetries in the $I \times T^2$ index. Ω
i=1,2’s are fermi multiplets at the left and the right boundaries introduced to cancel the
gauge anomaly.

| $\Phi = \{\Phi_i\}_{i=1}^{N_3}$ | $U(N_3+N_3-N_1)$ gauge | $SU(N_3)_y$ | $SU(N_1)_y'$ | $SU(N_2)_y$ | b.c |
|--------------------------------|--------------------------|--------------|--------------|--------------|------|
| $P := \{\Phi_i\}_{i=1}^{N_3}$ | □                        | □            | □            | □            | N    |
| $\Psi := \{\Phi_i\}_{i=1}^{N_1+N_3}$ | □                        | □            | □            | □            | D    |
| $\Omega_{i=1,2}$               | det                      | 1            | 1            | 1            | -    |

$\Phi^V = \{\Phi_i^V\}_{i=1}^{N_3}$, $P^V := \{\Phi_i^V\}_{i=1}^{N_3}$, $\Psi := \{\Phi_i^V\}_{i=1}^{N_1+N_3}$, $M := \{M_{ik}\}_{i=1}^{N_3}$, $\Gamma = \{M_{ik}\}_{i=1}^{N_1+N_3}$, $\Omega^V_{i=1,2}$’s are fermi multiplets at the left and the right boundaries introduced to cancel the
gauge anomaly.

Table 9: The charge assignments and the boundary conditions for the theory $A^V$ with
$N_c = \frac{N_2+N_3-N_1}{2}$, $N_f = N_1 + N_3$ and $N_a = N_2$. $\Omega^V_{i=1,2}$’s are fermi multiplets at the left and the right boundaries introduced to cancel the
gauge anomaly.

| $\Phi^V = \{\Phi_i^V\}_{i=1}^{N_3}$ | $U(N_1+N_2-N_3)$ gauge | $SU(N_3)_y$ | $SU(N_1)_y'$ | $SU(N_2)_y$ | b.c |
|--------------------------------|--------------------------|--------------|--------------|--------------|------|
| $P^V := \{\Phi_i^V\}_{i=1}^{N_3}$ | □                        | □            | □            | □            | N    |
| $\Psi := \{\Phi_i^V\}_{i=1}^{N_1+N_3}$ | □                        | □            | □            | □            | D    |
| $\Omega^V_{i=1,2}$               | det                      | 1            | 1            | 1            | -    |

Here $G$ denotes the gauge group and we assume $N_f \leq N_a$. In the $U(N_c)$ gauge theory,
$\Phi_i$ with $i = 1, \ldots, N_f$ (resp. $\Phi_k$ with $k = 1, \ldots, N_a$) represent chiral multiplets in the fundamental (resp. anti-fundamental) representation of $U(N_c)$. In the dual $U(N_a - N_c)$
gauge theory, $\Phi_k^V$ with $k = 1, \ldots, N_a$ (resp. $\Phi_i^V$ with $i = 1, \ldots, N_f$) correspond to chiral multiplets in the fundamental (resp. anti-fundamental) representation of $U(N_a - N_c)$. $M_{ik}$ with $k = 1, \ldots, N_a$ and $i = 1, \ldots, N_f$ are mesons and the dual theory has a superpotential
$W = \sum_{i,k} \Phi^V M_{ik} \Phi^V$. 

Now we take $N_c = \frac{N_2+N_3-N_1}{2}$, $N_f = N_1 + N_3$, and $N_a = N_2$ and impose the boundary
conditions depicted in Table 8 and in Table 9. We call these two theories the “theory A”
and the “theory $A^V$”:

Theory $A : G = U(N_2 + N_3 - N_1) + \Phi_{i=1}^{N_3} \text{ and } \tilde{\Phi}_{k=1}^{N_2} ;$ (5.9)

Theory $A^V : G = U(N_1 + N_2 - N_3) + \Phi_{k=1}^{N_2} \text{ and } \tilde{\Phi}_{i=1}^{N_3} \text{ and } M_{i=1}^{N_3} , k=1,\ldots,N_2 .$

Under the boundary conditions, the quiver diagram of the theory A on the interval is depicted
by the right quiver in Figure 2(a). To cancel the gauge anomaly of the theory A, we

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introduce a fermi multiplet \( \Omega_1 \) at \( x^1 = -\pi L \) and another fermi multiplet \( \Omega_2 \) at \( x^1 = \pi L \) in the determinant representation. The boundary conditions in the theory \( A^\mathbf{V} \) on the interval is depicted in Table \([\text{9}]\). The quiver diagram of the theory \( A^\mathbf{V} \) on the interval is depicted by the right quiver in Figure \([2(b)]\). To cancel gauge anomaly of the theory \( A^\mathbf{V} \), we introduce a fermi multiplet \( \Omega_1^\mathbf{V} \) at the left boundary and another fermi multiplet \( \Omega_2^\mathbf{V} \) at the right boundary in the determinant representation. The anomaly polynomials of the theories \( A \), \( A^\mathbf{V} \) and the boundary fermi multiplets are given by

\[
I_{\text{Theory}^A} = \left( \frac{N_2 + N_3 - N_1}{2} \right) \text{Tr}(f^2) - \left( \text{Tr}f \right)^2 + \frac{1}{2} \left( \frac{N_2 + N_3 - N_1}{2} \right)^2 r^2 \quad (\text{vector})
\]

\[
- \frac{1}{2} \left( N_3 \text{Tr}(f^2) + \frac{N_2 + N_3 - N_1}{2} \left[ \text{Tr}(y)^2 + N_2 \left( \frac{N_2 + N_3 - N_1}{N_1 + N_2 + N_3} - 1 \right) r^2 \right] \right) \quad (\Phi)
\]

\[
- \frac{1}{2} \left( N_2 \text{Tr}(f^2) + \frac{N_2 + N_3 - N_1}{2} \left[ \text{Tr}(\bar{y})^2 + N_2 \left( \frac{N_2 + N_3 - N_1}{N_1 + N_2 + N_3} - 1 \right) r^2 \right] \right) \quad (P^\mathbf{V})
\]

\[
+ \frac{1}{2} \left( N_1 \text{Tr}(f^2) + \frac{N_2 + N_3 - N_1}{2} \text{Tr}(y')^2 \right) \quad (\Psi),
\]

\[
I_{\text{Theory}^A^\mathbf{V}} = \left( \frac{N_1 + N_2 - N_3}{2} \right) \text{Tr}(f^2) - \left( \text{Tr}f \right)^2 + \frac{1}{2} \left( \frac{N_1 + N_2 - N_3}{2} \right)^2 r^2 \quad (\text{vector})
\]

\[
- \frac{1}{2} \left( N_2 \text{Tr}(f^2) + \frac{N_1 + N_2 - N_3}{2} \left[ \text{Tr}(y)^2 + N_2 \left( \frac{N_1 + N_2 - N_3}{N_1 + N_2 + N_3} - 1 \right) r^2 \right] \right) \quad (\Phi^\mathbf{V})
\]

\[
- \frac{1}{2} \left( N_1 \text{Tr}(f^2) + \frac{N_1 + N_2 - N_3}{2} \left[ \text{Tr}(\bar{y})^2 + N_1 \left( \frac{N_1 + N_2 - N_3}{N_1 + N_2 + N_3} - 1 \right) r^2 \right] \right) \quad (P^\mathbf{V})
\]

\[
+ \frac{1}{2} \left( N_3 \text{Tr}(f^2) + \frac{N_1 + N_2 - N_3}{2} \text{Tr}(\bar{y})^2 \right) \quad (\Psi^\mathbf{V})
\]

\[
- \frac{1}{2} \left( N_2 \text{Tr}(y)^2 + N_3 \text{Tr}(\bar{y})^2 + N_2 N_3 \left( \frac{N_2 + N_3 - N_1}{N_1 + N_2 + N_3} \right) r^2 \right) \quad (M)
\]

\[
+ \frac{1}{2} \left( N_2 \text{Tr}(y')^2 + N_1 \text{Tr}(\bar{y})^2 + N_1 N_2 \left( \frac{N_1 + N_2 - N_3}{N_1 + N_2 + N_3} \right) r^2 \right) \quad (\Gamma),
\]

\[
I_{\Omega_i} = I_{\Omega_i^\mathbf{V}} = \left( \text{Tr}f \right)^2.
\]

Here each line corresponds to the anomaly contribution from a multiplet specified by the \((\cdot)\). Then the anomaly polynomials satisfy a relation:

\[
I_{\text{Theory}^A} + I_{\Omega_i} = I_{\text{Theory}^A^\mathbf{V}} + I_{\Omega_i^\mathbf{V}}.
\]

Next we compare the \( I \times T^2 \) indices of two theories. From the localization formula, we have the \( I \times T^2 \) indices of two theories \( Z_{\text{Theory}^A}^{T^2} \) and \( Z_{\text{Theory}^A^\mathbf{V}}^{T^2} \):

\[
Z_{\text{Theory}^A}^{I \times T^2}(y, \bar{y}, y') = \frac{(i\eta(q))^{(N_2 + N_3 - N_1)}}{((N_2 + N_3 - N_1)/2)!} \int \frac{dx_a}{2\pi i x_a} \frac{i^2 \theta_1(\prod_{a=1}^{N_2+N_3-N_1} \frac{x_a^2}{\eta(q)^2})}{\eta(q)^2}
\]

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\[
\prod_{a \neq b} \frac{\theta_1(x_a^{-1} x_b)}{\eta(q)} \prod_{a=1}^{N_2+N_3-N_1} \prod_{i=1}^{N_3} \frac{\eta(q)}{\theta_1(x_a y_i^{-1})} \prod_{a=1}^{N_2+N_3-N_1} \prod_{j=1}^{N_2} \frac{\eta(q)}{\theta_1(x_a y_i^{-1})}. \tag{5.13}
\]

If we choose \( \eta \) as \((-1, \ldots, -1)\) in the JK residue operation, the index is given by the residues at \( x_a = \tilde{y}_a \):

\[
Z^{\text{Theory}_a}_{I \times T^2}(y, \tilde{y}, y') = \sum_{\tilde{I} \subset \{1, \ldots, N_2\}} \left( \frac{\theta_1(\prod_{a \in \tilde{I}} \tilde{y}_a)}{\eta(q)} \right)^2 \prod_{a \in \tilde{I}} \prod_{j \in \{1, \ldots, N_3\} \setminus \tilde{I}} \frac{\eta(q)}{\theta_1(y_a y_i^{-1})} \prod_{j=1}^{N_2} \frac{\eta(q)}{\theta_1(y_a^{-1} y_i)}. \tag{5.14}
\]

where we take \( \tilde{I} = \{i_1, \ldots, i_{N_2+N_3-N_1}\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_{N_2+N_3-N_1} \leq N_2 \). The sum \( \sum_{\tilde{I}} \) runs over all the possible \( \tilde{I} \) in \( \{1, \ldots, N_2\} \).

If we choose \( \eta \) as \((1, \ldots, 1)\) in the JK residue formula, the index is expressed by residues at \( x_a = y_a \):

\[
Z^{\text{Theory}_a}_{I \times T^2}(y, \tilde{y}, y') = \sum_{\tilde{I} \subset \{1, \ldots, N_3\}} \left( \frac{\theta_1(\prod_{a \in \tilde{I}} y_a)}{\eta(q)} \right)^2 \prod_{a \in \tilde{I}} \prod_{j \in \{1, \ldots, N_3\} \setminus \tilde{I}} \frac{\eta(q)}{\theta_1(y_a y_i^{-1})} \prod_{j=1}^{N_3} \frac{\eta(q)}{\theta_1(y_a^{-1} y_i)}. \tag{5.15}
\]

where we take \( \tilde{I} = \{i_1, \ldots, i_{N_2+N_3-N_1}\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_{N_2+N_3-N_1} \leq N_3 \). The sum \( \sum_{\tilde{I}} \) runs over all the possible \( \tilde{I} \) in \( \{1, \ldots, N_3\} \). Since the moduli space \( \mathcal{M} \) is compact, \( \tag{5.14} \)

agrees with \( \tag{5.15} \).

The \( I \times T^2 \) index for the dual theory \( A^\vee \) is given by

\[
Z^{\text{Theory}_{a \vee}}_{I \times T^2}(y, \tilde{y}, y') = \frac{(\eta(q))(N_1+N_2-N_3)}{((N_1+N_2-N_3)/2)!} \prod_{i=1}^{N_2} \prod_{j=1}^{N_3} \frac{\eta(q)}{\theta_1(y_i y_j^{-1})} \prod_{j=1}^{N_3} \frac{\eta(q)}{\theta_1(y_j^{-1} y_i)}. \tag{5.16}
\]

\[
\times \sum_{\tilde{y}_a} \frac{d\tilde{y}_a}{2\pi i} \prod_{a=1}^{N_3} \prod_{a \neq b} \frac{\eta(q)}{\theta_1(x_a \tilde{y}_b^{-1})} \prod_{a=1}^{N_3} \prod_{i=1}^{N_2} \frac{\eta(q)}{\theta_1(x_a^{-1} y_i)}, \tag{5.17}
\]

\[
= \prod_{i=1}^{N_2} \prod_{j=1}^{N_3} \frac{\eta(q)}{\theta_1(y_i y_j^{-1})} \prod_{j=1}^{N_3} \frac{\eta(q)}{\theta_1(y_j^{-1} y_i)} \sum_{\tilde{I} \subset \{1, \ldots, N_2\}} \left( \frac{\theta_1(\prod_{a \in \tilde{I}^\vee} \tilde{y}_a)}{\eta(q)} \right)^2 \prod_{a \in \tilde{I}^\vee} \prod_{i=1}^{N_3} \frac{\eta(q)}{\theta_1(y_i).} \tag{5.18}
\]
Here we have chosen $\eta$ as $1, \ldots, 1$ in the JK residue operations. $\mathcal{I}' = \{i_1, \ldots, i_{N_2 + N_3 - N_3} \}$ with $1 \leq i_1 < i_2 < \cdots < i_{N_2 + N_3 - N_3} \leq N_2$. The sum $\sum_{\mathcal{I}'}$ runs over all the possible $\mathcal{I}'$ in $\{1, \ldots, N_2\}$.

For an arbitrary $\mathcal{I}'$ in the sum $\sum_{\mathcal{I}'}$, there exists a unique $\mathcal{I}$ in the sum $\sum_{\mathcal{I}}$ such that $\mathcal{I} = \{1, \ldots, N_2\} \setminus \mathcal{I}'$. Then we have the following identities:

\begin{align}
\prod_{i=1}^{N_2} \prod_{j=1}^{N_3} i \frac{\eta(q)}{\theta_1(y_i y_j)} & \cdot \prod_{a \in \mathcal{I}'} \prod_{i=1}^{N_3} i \frac{\eta(q)}{\theta_1(y_{a_i} y_i)} = \prod_{a \in \mathcal{I}} \prod_{i=1}^{N_3} i \frac{\eta(q)}{\theta_1(y_{a_i} y_i)}, \\
\prod_{a \in \mathcal{I}'} \prod_{i=1}^{N_3} i \frac{\eta(q)}{\theta_1(y_{a_i} y_i)} & = \prod_{a \in \mathcal{I}} \prod_{j \in \{1, \ldots, N_2\} \setminus \mathcal{I}} i \frac{\eta(q)}{\theta_1(y_{a_j} y_i)}, \\
\prod_{i=1}^{N_2} \prod_{j=1}^{N_1} i \frac{\eta(q)}{\theta_1(y_i y_j)} & \cdot \prod_{a \in \mathcal{I}'} \prod_{j=1}^{N_1} i \frac{\eta(q)}{\theta_1(y_{a_j} y'_j)} = \prod_{a \in \mathcal{I}} \prod_{i=1}^{N_1} i \frac{\eta(q)}{\theta_1(y_{a_i} y'_i)}, \\
\theta_1(\prod_{a \in \mathcal{I}} y_a) & = -\theta_1(\prod_{a \in \mathcal{I}} y_a). \tag{5.17}
\end{align}

Applying these identities to (5.14) and (5.16), we obtain the agreement of the $I \times T^2$ indices between the theory $A$ and the theory $A^\vee$:

\begin{equation}
Z_{I \times T^2}^{\text{Theory } A}(y, \bar{y}, y') = Z_{I \times T^2}^{\text{Theory } A^\vee}(y, \bar{y}, y'). \tag{5.18}
\end{equation}

To obtain the third theory in the triality, we take $N_c = \frac{N_2 + N_3 - N_1}{2}, N_f = N_3$ and $N_a = N_1 + N_2$ in (5.8) with $N_3 > N_1 + N_2$. We call the $U(N_2 + N_3 - N_1)$ gauge theory and its Seiberg-like dual as “theory B” and “theory B’’:

\begin{align}
\text{Theory B} : U \left( \frac{N_2 + N_3 - N_1}{2} \right) + \Phi_{i=1, \ldots, N_3} \text{ and } \Phi_{k=1, \ldots, N_1 + N_2}, \tag{5.19} \\
\text{Theory B}^\vee : U \left( \frac{N_1 + N_3 - N_2}{2} \right) + \Phi_{i=1, \ldots, N_3}^\vee \text{ and } M_{i=1, \ldots, N_3, k=1, \ldots, N_1 + N_2}, \\
& \Phi_{i=1, \ldots, N_3}^\vee \text{ and } M_{i=1, \ldots, N_3, k=1, \ldots, N_1 + N_2}.
\end{align}

We put the theory B on the interval and impose the boundary conditions depicted by Table 10. The boundary conditions for the theory B' are depicted by Table 11.

Under the boundary conditions in Table 8 and Table 10, we find that the theory A and the theory B are identical. On the other hand the theory B’ is distinct from the theory A and the theory A’. In a similar way of $I_{\text{Theory } A^\vee}$, we can write down the anomaly polynomial $I_{\text{Theory } B^\vee}$ of the theory B’ which matches with $I_{\text{Theory } A}$ and $I_{\text{Theory } A^\vee}$:

\begin{equation}
I_{\text{Theory } A} + I_{\Omega_i} = I_{\text{Theory } A^\vee} + I_{\Omega_i^\vee} = I_{\text{Theory } B^\vee} + I_{\Omega_i^\vee}. \tag{5.20}
\end{equation}
Table 10: The charge assignments and the boundary conditions for the theory $\mathbf{B}$. $\Omega_{i=1,2}$'s are fermi multiplets at the left and the right boundaries introduced to cancel the gauge anomaly. The theory $\mathbf{B}$ on the interval is identical to the theory $\mathbf{A}$ on the interval.

| $\Phi$ | $U(\frac{N_2+N_3-N_1}{2})_{\text{gauge}}$ | $SU(N_3)_y$ | $SU(N_1)_y'$ | $SU(N_2)_{\bar{y}}$ | b.c |
|---|---|---|---|---|---|
| $\Phi = \{ \Phi_i \}_{i=1}^{N_3}$ | $\square$ | $\square$ | 1 | 1 | N |
| $P := \{ \tilde{\Phi}_i \}_{i=1}^{N_2}$ | $\square$ | $\square$ | 1 | 1 | □ | N |
| $\Psi := \{ \Phi_i \}_{i=N_2+1}^{N_1+N_2}$ | $\square$ | 1 | □ | $\square$ | D |

| $\Omega_{i=1,2}$ | det | 1 | 1 | 1 | - |

Table 11: The charge assignments and the boundary conditions for the theory $\mathbf{B}'$. $\Omega'_{i=1,2}$'s are fermi multiplets at the left and the right boundaries introduced to cancel the gauge anomaly.

| $\Phi'$ := $\{ \Phi'^{\vee}_i \}_{i=1}^{N_1+N_2}$ | $U(\frac{N_1+N_3-N_2}{2})_{\text{gauge}}$ | $SU(N_3)_y$ | $SU(N_1)_y'$ | $SU(N_2)_{\bar{y}}$ | b.c |
|---|---|---|---|---|---|
| $P' := \{ \tilde{\Phi}_i^{\vee} \}_{i=1}^{N_3}$ | $\square$ | $\square$ | 1 | 1 | N |
| $\Psi' := \{ \Phi_i^{\vee} \}_{i=N_1+1}^{N_1+N_2}$ | $\square$ | 1 | 1 | □ | D |
| $M' := \{ M_{ki} \}_{(k,i)=(1,1)}^{(N_3,N_2)}$ | 1 | □ | 1 | □ | N |
| $\Gamma := \{ M_{ki} \}_{(k,i)=(N_2,N_1+1)}^{(N_3,N_1+1)}$ | 1 | □ | □ | 1 | D |

| $\Omega'_{i=1,2}$ | det | 1 | 1 | 1 | - |

Next we evaluate the $I \times T^2$ index of the theory $\mathbf{B}'$:

$$Z^{\text{TheoryB'}}_{I \times T^2}(y, \bar{y}, y') = \frac{(i\eta(q))^{(N_1+N_3-N_2)}}{(N_1 + N_3 - N_2)!} \prod_{i=1}^{N_3} \prod_{k=1}^{N_2} \eta(q) \prod_{i=1}^{N_1} \theta_1(y_i^{-1}y_i') \eta(q)$$

$$\times \sum_{y_a} \frac{\int dxa}{2\pi i xa} \frac{i^{\frac{N_1+N_3-N_2}{2}}}{\prod_{a\neq b}^{N_1+N_3-N_2} \eta(q)} \prod_{i=1}^{N_1} \theta_1(x_a^{-1}x_b) \eta(q) \prod_{j=1}^{N_2} \theta_1(x_a^{-1}y_i) \eta(q)$$

$$= \prod_{i=1}^{N_1} \prod_{k=1}^{N_3} \frac{\eta(q)}{\theta_1(y_i^{-1}y_i')} \prod_{j=1}^{N_1} \theta_1(y_i^{-1}y_i') \eta(q) \prod_{a \in T''} \sum_{y_a} \left( \frac{\theta_1(\prod_{a \in T''} y_a)}{\eta(q)} \right)^2$$

$$\times \prod_{a \in T''} \prod_{i=1}^{N_1} \frac{\eta(q)}{\theta_1(y_a^{-1}y_i')} \prod_{i=1}^{N_2} \theta_1(y_a^{-1}y_i) \eta(q) \prod_{i=1}^{N_3} \frac{\eta(q)}{\theta_1(y_i^{-1}y_i')} \prod_{j=1}^{N_1} \theta_1(y_i^{-1}y_i') \eta(q). \quad (5.21)$$

Here we have chosen $\eta$ as $(-1, \ldots, -1)$ in the JK residue operations. We also define
Figure 3: The decomposition of 3d multiplets \( \{\Phi_i\}_{i=1}^{N_1+N_3} \mapsto (\Phi, \Psi) \) and \( \{\tilde{\Phi}_i\}_{i=1}^{N_2} \mapsto P \) in the theory \( A \) by the boundary conditions. In the right figure in (a), the dashed line denotes the chiral multiplet with the Dirichlet boundary condition \( \Psi \). The solid arrows denote the chiral multiplets with the Neumann boundary condition \( \Phi \), \( P \). (b): The decomposition of 3d multiplets \( \{\Phi^\vee_i\}_{i=1}^{N_2} \mapsto \Phi^\vee \), \( \{\tilde{\Phi}^\vee_i\}_{i=1}^{N_1+N_3} \mapsto (P^\vee, \Psi^\vee) \) and \( M_{ik} \mapsto (M, \Gamma) \) in the theory \( A^\vee \).

\[ I'' = \{i_1, \ldots, i_{N_1+N_3-N_2}\} \] with \( 1 \leq i_1 < i_2 < \cdots < i_{N_1+N_3-N_2} \leq N_3 \). The sum \( \sum_{I''} \) runs over all the possible \( I'' \) in \( \{1, \cdots, N_3\} \). Since there are similar identities of (5.17), the \( I \times T^2 \) indices for theories \( A \) and \( B^\vee \) agree each other. Thus we have shown that the equality of \( I \times T^2 \) indices between theories \( A \), \( A^\vee \) and \( B^\vee \) specified by quiver diagrams in Figure 4:

\[ Z_{I \times T^2}^{\text{Theory } A^\vee}(y, \bar{y}, y') = Z_{I \times T^2}^{\text{Theory } A}(y, \bar{y}, y') = Z_{I \times T^2}^{\text{Theory } B^\vee}(y, \bar{y}, y'). \tag{5.22} \]

Figure 4: The quiver diagrams for the triality of 3d gauge theories with boundaries. (a) and (b) are obtained by a pair of Seiberg-like duality; theory \( A \) and theory \( A^\vee \). (b) and (c) are obtained by a pair of Seiberg-like duality; \( B \) and \( B^\vee \). With the boundary conditions, the theory \( A \) and the theory \( B \) have the same matter content. In the quiver diagrams, we suppressed the boundary fermi multiplets \( \Omega_i \) in (b), \( \Omega^\vee_i \) in (a), \( \Omega^\vee_i \) in (c) with \( i = 1, 2 \).
6 3d theory on the interval and $\beta\gamma$, $bc$ systems

Recently chiral algebras associated with 3d $\mathcal{N} = 2$ theories on a 3d half space $\mathbb{R}_{\leq 0} \times \mathbb{C}$ were studied in [33]. Although the general rules for chiral algebras associated with the 3d theories on $I \times \mathbb{C}$ are not studied yet, the author of [33] considered the simplest model on $I \times \mathbb{C}$, namely, a free chiral multiplet and expected that the chiral algebra for the free chiral multiplet on the interval with the Neumann (resp. Dirichlet ) boundary condition is realized as the $\beta\gamma$-system (resp. the $bc$-system). We study the relation among free chirals on $I \times M_2$, $\beta\gamma$ and $bc$-systems.

The $I \times T^2$ indices for the 3d free chiral multiplet with the Dirichlet (D) and with the Neumann (N) boundary conditions are given by

$$Z_{I \times T^2,N} = \frac{i\eta(q)}{\theta_1(x,q)}, \quad Z_{I \times T^2,D} = \frac{i\theta_1(x,q)}{\eta(q)}. \quad (6.1)$$

Here $x$ is the fugacity of the $U(1)$ flavor symmetry and the fields in the 3d chiral multiplet have the charge +1. We find that $Z_{I \times T^2,N}$ is nothing but the character of the $\beta\gamma$-system with anti-periodic boundary conditions and the weight $(1,0)$ for $(\beta,\gamma)$. On the other hand, $Z_{I \times T^2,D}$ agrees with the character of the $bc$-system with anti-periodic boundary conditions and the weight $(1,0)$ for $(b,c)$ . Thus we have the agreement between the characters of $bc$-system and the supersymmetric indices. Note that (6.1) is same as the 2d $\mathcal{N} = (0,2)$ elliptic genera for the free chiral multiplet and the free fermi multiplet, respectively. This simplest case corresponds to realizations of 2d $\mathcal{N} = (0,2)$ elliptic genera based on the $\beta\gamma$-system and the $bc$-system in [20].

Next we study properties of Q-closed operators. Since we have chosen a supercharge $Q := Q_2 - Q_1$ to define the index and to perform the supersymmetric localization, the index is expected to count the Q-closed operators modulo the Q-exact operators and the boundary conditions. The SUSY transformation of the 3d chiral multiplet by $Q := Q_2 - Q_1$ is written as

$$Q\phi = 0, \quad Q\bar{\phi} = \bar{\psi}_2 - \bar{\psi}_1, \quad Q(\bar{\psi}_2 \pm \bar{\psi}_1) = 0, \quad Q(\psi_1 - \psi_2) = 2(\partial_2 + i\partial_3)\phi, \quad Q(\psi_1 + \psi_2) = 2i\partial_1 \phi. \quad (6.2)$$

Here $F, \bar{F}$ are set to zero by the equations of motion. Then we find that $(\partial_2 - i\partial_3)^n \phi \simeq \partial_w^n \phi$ with $n \geq 0$ are Q-closed operators.

**Neumann boundary condition**

For the Neumann boundary condition (2.23), operators $\bar{\psi}_1 + \bar{\psi}_2$ and $\psi_1 + \psi_2$ are set to zero by the boundary condition. Another Q-closed operator is the first descendant of $\bar{\psi}_1 + \bar{\psi}_2$.
Here we used the equations of motion of $\psi_1, \psi_2$ and the boundary condition $\bar{\psi}_1 + \bar{\psi}_2 = 0$. Then $(\partial_2 - i\partial_3)^n \int_I \bar{\phi} \simeq (\partial_2 - i\partial_3)^n \bar{\phi}^{(0)}$ with $n \geq 1$ are Q-closed operators. The counting of the Q-closed operators $\partial_{\bar{w}}^{n+1} \int_I \bar{\phi}$ and $\partial_w^n \phi$ with $n \geq 0$ is consistent with $Z_{I \times T^2, N}^{\text{free chiral}}$ up to the zero point energy.

Since we expect that the Q-closed operators with the Neumann boundary condition are associated with the $\beta\gamma$-system, the correlation functions of these Q-closed operators should be consistent with the OPE of the $\beta\gamma$-system. On $I \times \mathbb{C}$, the normalized two point functions of $\int_I \partial_w \bar{\phi}$ and $\phi$ are

$$
\langle \int_I \partial_w \bar{\phi}(w) \cdot \phi(x^1, 0) \rangle_{I \times \mathbb{C}} = \langle \partial_w \bar{\phi}^{(0)}(w) \cdot \phi^{(0)}(0) \rangle_{\mathbb{C}} = \frac{1}{2\pi w},
$$

$$
\langle \int_I \partial_w \bar{\phi}(w) \cdot \int_I \partial_w \bar{\phi}(w) \rangle_{I \times \mathbb{C}} = \langle \partial_w \bar{\phi}^{(0)}(w) \cdot \partial_w \phi^{(0)}(0) \rangle_{\mathbb{C}} = 0,
$$

$$
\langle \phi(x^1, w) \phi(0) \rangle_{I \times \mathbb{C}} = \langle \partial_w \bar{\phi}^{(0)}(x^1, w) \cdot \partial_w \phi^{(0)}(0) \rangle_{\mathbb{C}} = 0.
$$

Here $\langle \cdots \rangle_{\mathbb{C}}$ means the correlation functions of a 2d $\mathcal{N} = (0, 2)$ free chiral multiplet with a lowest component scalar $\phi^{(0)}$ on $\mathbb{C}$. We find that these correlation functions of Q-closed operators are independent of coordinates $\bar{w}$ and $x^1$. The correlators in (6.4) are independent of $\bar{w}$, because the translation along $\bar{w}$ is expressed as the anti-commutator of $Q$ and $\bar{Q}$. Two point functions (6.4) match with the OPEs of $\beta\gamma$ system:

$$
\beta(w)\gamma(0) \sim \frac{1}{w}, \quad \beta(w)\beta(0) \sim 0, \quad \gamma(w)\gamma(0) \sim 0.
$$

**The Dirichlet boundary condition**

In this case $\bar{\psi}_1 + \bar{\psi}_2$ is a Q-closed operator. Meanwhile for the Dirichlet boundary condition (2.21), $\phi$ is set to zero. Another Q-closed operator is obtained by the descent equation:

$$
Q \int_I (\psi_1 + \psi_2) = 2i \int_I \partial_1 \phi = 0,
$$

where we used the boundary condition $\phi = 0$ at $x^1 = \pm \pi L$. In a similar way, $\partial_w^n \int_I (\psi_1 + \psi_2)$ is Q-closed.

---

5 The factor $\frac{1}{2\pi}$ in (6.4) can be absorbed to the normalization of the action of the chiral multiplet.
Again we compute two point functions on $I \times \mathbb{C}$ and relate them to the $bc$-system. Two point functions of Q-closed operators are given by

$$\langle \int_I (\psi_1(w) + \psi_2(w)) \cdot (\bar{\psi}_1(0) + \bar{\psi}_2(0)) \rangle_{I \times \mathbb{C}} = 4 \langle \bar{\psi}_c^{(0)}(w) \cdot \psi_c^{(0)}(0) \rangle_{\mathbb{C}} = \frac{1}{\pi w},$$

$$\langle \int_I (\psi_1(w) + \psi_2(w)) \cdot \int_I (\psi_1(0) + \psi_2(0)) \rangle_{I \times \mathbb{C}} = 0,$$

$$\langle (\bar{\psi}_1(x^1, w) + \bar{\psi}_2(x^1, w))(\bar{\psi}_1(0) + \bar{\psi}_2(0)) \rangle_{I \times \mathbb{C}} = 0. \quad (6.7)$$

Here $\langle \cdots \rangle_{\mathbb{C}}$ means the correlation functions in a 2d $\mathcal{N} = (0, 2)$ free fermi multiplet with the fermion $\psi_c^{(0)}$ on $\mathbb{C}$. These correlation functions are consistent with the OPEs of the $bc$ system:

$$b(w)c(0) \sim \frac{1}{w}, \quad b(w)b(0) \sim 0, \quad c(w)c(0) \sim 0. \quad (6.8)$$

7 Dimensional reduction and 2d $\mathcal{N} = (2, 2)$ theories on $I \times S^1$

In section 3, we have studied the supersymmetric localization computation of the indices on $I \times T^2$. In this section we perform the dimensional reduction in the $x^3$-direction and study localization formula of supersymmetric indices for 2d $\mathcal{N} = (2, 2)$ theories on $I \times S^1$. The detailed analysis for the $I \times S^1$ indices is left in our upcoming future work [34].

7.1 SUSY localization formula for 2d $\mathcal{N} = (2, 2)$ theories on $I \times S^1$

First we define the coordinates of $I \times S^1$ as

$$I \times S^1 = \{(x^1, x^2) | -\pi L \leq x^1 \leq \pi L, \ x^2 \sim x^2 + 2\pi R\}. \quad (7.1)$$

In the dimensional reduction, the SUSY transformation, the Lagrangians and boundary conditions at $x^1 = \pm \pi L$ for the 2d theory are naturally originated from the 3d theory in sections 2.1 and section 2.2 with the replacement:

$$A_3(x_1, x_2, x_3) \mapsto \sigma'(x_1, x_2), \quad D_3 \Psi(x^1, x^2, x^3) \mapsto i\sigma' \Psi(x^1, x^2),$$

$$\Psi(x^1, x^2, x^3) \mapsto \Psi(x^1, x^2). \quad (7.2)$$

Here $\sigma'$ is an adjoint scalar in the 2d $\mathcal{N} = (2, 2)$ vector multiplet and $\Psi(x^1, x^2, x^3)$ and $\Psi(x^1, x^2)$ are fields in the 3d $\mathcal{N} = 2$ and the 2d $\mathcal{N} = (2, 2)$ theories. The supersymmetric quantum mechanics at the boundaries $\partial(I \times S^1) = S^1_L \cup S^1_R$ are given by the dimensional reduction of the 2d $\mathcal{N} = (0, 2)$ theories in section 2.3 to the 1d $\mathcal{N} = 2$ theories. In three
dimensions, the surface terms for the bulk 3d superpotential are compensated by SUSY transformation of the boundary 2d $\mathcal{N} = (0, 2)$ superpotentials (2.36). In the same way as (2.36), the surface terms of the bulk 2d superpotential on $I \times S^1$ are canceled by the SUSY transformation of the superpotentials of the 1d Fermi multiplets.

In 2d $\mathcal{N} = (2, 2)$ GLSMs with boundaries, there is another choice of boundary interaction called a Chan–Paton factor or a brane factor that cancels the surface term of the superpotential [13]:

$$W_V = \text{Str}_V P \exp \left( \pm i \oint dx^2 A_2 \right), \quad (7.3)$$

with

$$A_2 = \rho_*(A_2 + i\sigma') + \frac{i}{2} \{Q, Q\} - \frac{1}{2} \sum_i (\psi_{1,i} - \psi_{2,i}) \frac{\partial Q}{\partial \phi_i} + \frac{1}{2} \sum_i (\bar{\psi}_{1,i} - \bar{\psi}_{2,i}) \frac{\partial \bar{Q}}{\partial \bar{\phi}_i}. \quad (7.4)$$

Here $\mathcal{V} = \mathcal{V}_{\text{even}} \oplus \mathcal{V}_{\text{odd}}$ is a $\mathbb{Z}_2$-graded vector space, called a Chan-Paton vector space. $Q \in \text{End}(\mathcal{V})$ is called a matrix factorization or a tachyon profile. $\psi_{1,i}, \psi_{2,i}$ are the first and second components of the fermion $\psi_i = (\psi_{1,i}, \psi_{2,i})^T$ in the $i$-th 2d $\mathcal{N} = (2, 2)$ chiral multiplet. The subscript $i$ in the sum labels the chiral multiplets in the tachyon profile. The surface term of the bulk 2d $\mathcal{N} = (2, 2)$ superpotential $W$, i.e., the dimension reduction of (2.20) is canceled by the SUSY transformation by a Chan–Paton factor $W_V$, if tachyon profiles satisfy the following relations:

$$Q^2 = W \text{id}_\mathcal{V}, \quad \bar{Q}^2 = \bar{W} \text{id}_\mathcal{V}. \quad (7.5)$$

In (7.3), $+$ sign is taken at the right boundary $x^1 = \pi L$ and $-$ is taken at the left boundary $x^1 = -\pi L$.

By introducing Chan-Paton factors, one can change the Neumann boundary condition for chiral multiplets to the Dirichlet boundary condition. For example, see [2] for the localization computation of the hemisphere partition function with the Dirichlet boundary condition. We will see the two methods, i.e., the Neumann boundary condition with a matrix factorization and the Dirichlet boundary condition agree each other in simple examples.

Next we explain the definition of supersymmetric indices on $I \times S^1$. We assume the same boundary condition is imposed at $x^1 = \pm \pi L$. We take the following twisted boundary condition along $S^1$ direction:

$$\Psi(x^1, x^2 + 2\pi R) = \prod_i e^{z_i F_i} \Psi(x^1, x^2). \quad (7.6)$$

Here $F_i$ is the generator of a $U(1)$ flavor symmetry and $z_i$ is the fugacity of $F_i$. Then the
supersymmetric index on $I \times S^1$ is defined by

$$Z_{W_L W_R} : = \text{Tr}(-1)^F e^{2\pi i R F^2} \prod_i e^{-z_i F_i},$$

$$= \text{Tr}(-1)^F e^{2\pi i R \{Q, Q\}} \prod_i e^{-z_i F_i} \quad (7.7)$$

The localization formula of the index is given by

$$Z_{W_L W_R} = \frac{1}{|W_G|} \sum_{u_* \in \mathbb{W}_{\text{ang}}} \text{JK-Res}(Q_*, \eta) \times \text{Str}_{V_L}(e^{-u}) \text{Str}_{V_R}(e^u) \frac{Z_{1\text{-loop}}^{1\text{-loop}} \frac{Z_{1\text{-loop}}^{2\text{-loop}}}{Z_{S_L}^{1\text{-loop}} \wedge \text{rk}(G)} du^a. \quad (7.8)$$

Here $u$ is the saddle point value of $i \not{\partial} (A_2 - \sigma')$. $Z_{1\text{-loop}}^{1\text{-loop}}, Z_{1\text{-loop}}^{2\text{-loop}}$ and $Z_{S_L}^{1\text{-loop}}$ are the one-loop determinants of the 2d $\mathcal{N} = (2, 2)$ theory on $I \times S^1$, the one-loop determinants of the 1d $\mathcal{N} = 2$ theories on $S^1_L$ and $S^1_R$ defined by

$$Z_{1\text{-loop}}^{1\text{-loop}} = \prod_{u \in \mathbb{W}_{\text{ang}}} Z_{i \times S^1}(u), \quad (7.9)$$

$$Z_{S_L}^{1\text{-loop}} = \prod_{u \in \mathbb{W}_{\text{ang}}} Z_{1d.chi,D,R}(u, z), \quad (7.10)$$

$$Z_{S_R}^{1\text{-loop}} = \prod_{u \in \mathbb{W}_{\text{ang}}} Z_{1d.chi,N,R}(u, z). \quad (7.11)$$

The 2d $\mathcal{N} = (2, 2)$ one-loop determinant (7.9) consists of a 2d $\mathcal{N} = (2, 2)$ $G$ vector multiplet $Z_{2d.\text{vec}, G}^{i \times S^1}$, a 2d chiral multiplet $Z_{\text{chi}, N,R}^{i \times S^1}$ with the Neumann boundary condition and a $Z_{\text{chi}, D,R}^{i \times S^1}$ with the Dirichlet boundary condition. The 1d $\mathcal{N} = 2$ one-loop determinants consist of a 1d chiral multiplet $Z_{1d.chi,R}$ and a 1d fermi multiplet $Z_{1d.Fermi, R}$. The products are taken over all the multiplets.

The one-loop determinants of the supermultiplets are given by

$$Z_{2d.\text{vec}, G}^{i \times S^1}(u) = \prod_{\alpha \in \text{t}(g)} 2 \sinh \left( \frac{\alpha(u)}{2} \right), \quad (7.12)$$

$$Z_{\text{chi}, D,R}^{i \times S^1}(u; z) = Z_{1d.Fermi,R}(u, z) = \prod_{Q \in \text{wt}(R)} \prod_{Q \in \text{wt}(F)} 2 \sinh \left( \frac{Q(u) + Q^F(z)}{2} \right), \quad (7.13)$$

$$Z_{\text{chi}, N,R}^{i \times S^1}(u; z) = Z_{1d.chi,R}(u, z) = \prod_{Q \in \text{wt}(R)} \prod_{Q \in \text{wt}(F)} \frac{1}{2 \sinh \left( \frac{Q(u) + Q^F(z)}{2} \right)}. \quad (7.14)$$

We find that the one-loop determinants of the 2d $\mathcal{N} = (2, 2)$ multiplets on $I \times S^1$ are independent of the length of the interval $I$ and agree with the one-loop determinants of 1d $\mathcal{N} = 2$ multiplets on $S^1$ in [33, 24]. A formula without the Dirichlet boundary condition and the boundary 1d multiplets was briefly mentioned in [3].
The derivation of supersymmetric localization formula of the index on $I \times S^1$ is almost parallel to that for the index on $I \times T^2$. But there is a difference coming from non-compactness of the space of $u, \bar{u}$. On $I \times S^1$, $u$ and $\bar{u}$ come from the constant values of $i \oint (A_2 - \sigma')$ that span a non-compact space $\mathcal{M} = (S^1 \times \mathbb{R})^{\text{rk}(G)}$. In addition to the residues around $\partial \Delta_{\veps}$, the residues around $\partial (\mathcal{M} \backslash \Delta_{\veps})$ with $\text{Re}(u_i) = \pm \infty$ possibly contribute to the $I \times S^1$ index. For the non-degenerate case, if the 2d FI-parameter $\zeta$ is contained in the charge cones $\sum_i \mathbb{R}_{>0} Q_i$ at all the singular points $u_\star$, we do not have to take the residues with $\text{Re}(u_i) = \pm \infty$ into account. In such situations, the index is given by the localization formula (7.8) by setting $\eta = \zeta$. On the other hand, $\zeta$ is not contained in charge cones, there is possibly an extra contribution to the index. Here we assume $\zeta$ satisfies the definition of the JK residue and do not consider the extra contribution to the index.

Next we consider the expectation value of Q-closed operators on $I \times S^1$. Each Q-closed operator is a Wilson loop such that a path $C$ is a circle along the $x^2$-direction with $x^3=$constant:

$$W_R = \text{Tr}_R P \exp \left( \oint_C dx^2 (iA_2 - \sigma') \right).$$

(7.15)

Here $R$ is a representation of the 2d gauge group $G$. $W_R$ does not necessarily lie on the boundaries. The saddle point value of the Wilson loop (7.15) is given by

$$W_R|_{\text{saddle point}} = \text{Tr}_R e^u.$$  

(7.16)

Note that a correlation function of Wilson loops is independent of the $x_3$ position. The localization formula for the correlation functions of Wilson loops is given by

$$\left\langle \prod_{i=1}^n W_{R_i} \right\rangle_{W_L W_R} = \frac{1}{|W_G|} \sum_{u_\star \in \text{sing}} \text{JK-Res}(Q_\star, \eta) \left( \prod_{i=1}^n \text{Tr}_R e^u \right) \times \text{Str}_L (e^{-u}) \text{Str}_R (e^u) Z_{I \times S^1}^{1\text{-loop}} Z_{S^1_L}^{1\text{-loop}} Z_{S^1_R}^{1\text{-loop}} \wedge_{a=1}^{\text{rk}(G)} du^a.$$  

(7.17)

One can also insert the Wilson loops $\text{Tr}_F e^z$ for the flavor symmetry group in the correlation functions.

### 7.2 $I \times S^1$ indices, Wilson loops and open string Witten indices

Here we briefly study boundary conditions in $I \times S^1$ indices and compare them with the indices based on the geometric computation and the results in the Gepner models in the CFT computation.

#### 7.2.1 Projective space $\mathbb{P}^{M-1}$

We consider a $U(1)$ GLSM with $M$ chiral multiplets with the gauge charge $+1$ without a superpotential. In the negative FI-parameter region $\zeta < 0$, the supersymmetric vacuum does
not exist. In a generic point in the positive FI parameter region $\zeta > 0$, the moduli space of the Higgs branch vacua is a complex projective space $\mathbb{P}^{M-1}$. We impose the Neumann boundary condition for the $M$ chiral multiplets and introduce a Wilson loop with a charge $a$. From the localization formula (7.17) the vacuum expectation value (vev) of the Wilson loop is given by

$$
\langle e^{ai f(A_{2-i\sigma'})} \rangle = \int_{u=0} \frac{du}{2\pi i} \frac{e^{au}}{(e^{u/2} - e^{-u/2})^M}.
$$

(7.18)

Here we have chosen the FI-parameter $\zeta > 0$ in the JK residue evaluation. Note that (7.18) is invariant under the global gauge transformation $u \mapsto u + 2\pi i$, only if the condition $a + \frac{M}{2} \equiv 0 \pmod{2}$ is satisfied. Especially $M = \text{even}$ is required for $a \in \mathbb{Z}$. When $M = \text{even}$ and $a \in \mathbb{Z}$, the $I \times S^1$ index directly agrees with an index with an $\hat{A}$-class:

$$
\langle e^{ai f(A_{2-i\sigma'})} \rangle = \int_{u=0} \frac{du}{2\pi i u^M} e^{au} \left( \frac{u/2}{\sinh(u/2)} \right)^M
= \int_{\mathbb{P}^{M-1}} \text{ch}(\mathcal{O}(a)) \hat{A}(T\mathbb{P}^{M-1}).
$$

(7.19)

In order for the $I \times S^1$ index to be well-defined at an arbitrary integer $M$, we have to insert a charge $a \pm M/2$ $U(1)$ Wilson loop with $a \in \mathbb{Z}$. If we choose $a + M/2$ with $a \in \mathbb{Z}$, the vev of the operator has a geometric interpretation:

$$
\langle e^{(a + \frac{M}{2})i f(A_{2-i\sigma'})} \rangle = \int_{u=0} \frac{du}{2\pi i u^M} e^{au} \left( \frac{u}{1 - e^{-u}} \right)^M
= \int_{\mathbb{P}^{M-1}} \text{ch}(\mathcal{O}(a)) \text{Td}(T\mathbb{P}^{M-1})
= \left( M - 1 + a \right). \tag{7.20}
$$

Here Td is the Todd class. If we turn on the flavor fugacities for the flavor group $SU(M)$, the vev is modified to the equivariant index of $\mathbb{P}^{M-1}$. The vev of Wilson loops with $e^{-ai f(A_{2-i\sigma'})}, e^{bi f(A_{2-i\sigma'})}$ inserted at the left and the right boundaries is interpreted as the Euler pairing of $\mathcal{O}(b)$ and $\mathcal{O}(a)$:

$$
\langle e^{-ai f(A_{2-i\sigma'})} e^{bi f(A_{2-i\sigma'})} \hat{A}^{M/2} f(A_{2-i\sigma'}) \rangle = \int_{u=0} \frac{du}{2\pi i u^M} e^{(b-a)u} \left( \frac{u}{1 - e^{-u}} \right)^M
= \int_{\mathbb{P}^{M-1}} \text{ch}(\mathcal{O}(a)^\vee) \text{ch}(\mathcal{O}(b)) \text{Td}(T\mathbb{P}^{M-1})
= \chi_{\mathbb{P}^{M-1}}(\mathcal{O}(a), \mathcal{O}(b)). \tag{7.21}
$$

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Next we study a 2d $\mathcal{N} = (2, 2)$ GLSM with a superpotential with two methods. We consider a $U(1)$ gauge theory with chiral multiplets $\phi_i$ with gauge charges $+1$’s for $i = 1, \cdots, M$ and a chiral multiplet $P$ with a gauge charge $-k$. We introduce a superpotential term

$$W = Pf(\phi_1, \cdots, \phi_M), \quad (7.22)$$

where $f(\phi_1, \cdots, \phi_M)$ is a degree $k$ homogeneous polynomial of $\phi_1, \cdots, \phi_M$. At a generic point in the positive FI-parameter region, the moduli space of the Higgs branch vacua is a degree $k$ hypersurface $X$ defined by $f(\phi_1, \cdots, \phi_M) = 0$ in $\mathbb{P}^{M-1}$. The hypersurface $X$ is a Fano $(M-2)$-fold for $k < M$, and a Calabi-Yau $(M-2)$-fold for $k = M$.

Let us study the boundary condition and its consequence in physics. First we consider the geometric phase, i.e., the positive FI-parameter region. We take the Neumann boundary condition for $\phi_i$’s with $i = 1, \cdots, M$ and the Dirichlet boundary condition for $P$. The vev of the Wilson loop is expressed as

$$\langle e^{a_i \oint (A_2 - i \sigma')} \rangle = \oint_{u=0} \frac{du}{2\pi i} e^{au} \frac{e^{-ku/2} - e^{ku/2}}{(e^{u/2} - e^{-u/2})^M} = -\int_X \text{ch} (O(a - (k - M)/2)) \text{Td}(TX), \quad (7.23)$$

where $a - (k - M)/2$ has to be an integer. On the other hand, if we impose the Neumann boundary condition for $P$ and the Dirichlet boundary conditions for $\phi_i$’s, the index is given by

$$\langle e^{a_i \oint (A_2 - i \sigma')} \rangle = \oint_{u=0} \frac{du}{2\pi i} e^{au} \frac{(e^{u/2} - e^{-u/2})^M}{(e^{-u/2} - e^{u/2})^M} = -\sum_{l=0}^{k-1} \oint_{u=2\pi i \frac{l}{k}} \frac{du}{2\pi i} e^{(a + \frac{M-k}{2})u} \frac{(1 - e^{-u})^M}{1 - e^{-ku}}. \quad (7.24)$$

To study a relation with SCFT computation, we consider the case in which the GLSM flows to superconformal field theories, i.e., $k = M$. In the large positive FI parameter region, the GLSM flows to an NLSM with the target space $X$. In this region, $\phi_i$’s parameterize the target space $X$. Then we impose the Neumann boundary conditions for $\phi_i$’s. The correlation function is written as

$$\langle e^{-ai \oint (A_2 - i \sigma')} e^{bi \oint (A_2 - i \sigma')} \rangle = \oint_{u=0} \frac{du}{2\pi i} e^{(b-a)u} \frac{1 - e^{-Mu}}{(1 - e^{-u})^M} = -\int_X \text{ch}(O(-a)) \otimes O(b) \text{Td}(TX). \quad (7.25)$$
Let us reproduce the (7.25) from a matrix factorization; the $\phi_i$’s and $P$ with the Neumann boundary conditions and a matrix factorization (7.5) that makes the superpotential invariant. We take a tachyon profile at the left and the right boundaries as

$$Q = f(\phi)\bar{\eta} + P\eta \text{ with } \{\eta, \bar{\eta}\} = 1, \quad \eta^2 = \bar{\eta}^2 = 0.$$  \hspace{1cm} (7.26)

Here $Q$ acts on the Chan–Paton vector spaces $\mathcal{V}_{L/R} = |0\rangle_{L/R} \oplus |\eta\rangle_{L/R}$ at the left ($L$) and the right ($R$) boundaries. We assign gauge charges $a + \frac{M}{2}$ for $|0\rangle_L$ and $b + \frac{M}{2}$ for $|0\rangle_R$. The saddle point values of the Chan–Paton factors are given by

$$\text{Str}_{\mathcal{V}_L}(e^{-u}) = e^{-au}(e^{\frac{Mu}{2}} - e^{-\frac{Mu}{2}}),$$

$$\text{Str}_{\mathcal{V}_R}(e^u) = e^{bu}(e^{-\frac{Mu}{2}} - e^{\frac{Mu}{2}}).$$ \hspace{1cm} (7.27)

Thus the localization formula for the $I \times S^1$ index with the Chan–Paton factors is given by

$$Z_{\mathcal{W}_L\mathcal{W}_R} = \oint_{u=0} \frac{du}{2\pi i} \frac{\text{Str}_{\mathcal{V}_L}(e^{-u})\text{Str}_{\mathcal{V}_R}(e^u)}{e^{(b-a)u}(1 - e^{-u})^M}.$$  \hspace{1cm} (7.28)

Therefore we obtain the same result as (7.25) up to an overall sign. By comparing (7.25) with (7.28), we find that the two boundary interactions $\mathcal{W}_{\mathcal{V}_L}$ and $\mathcal{W}_{\mathcal{V}_R}$ turn the Neumann boundary condition for $P$ to the Dirichlet one.

Here we make a comment on the derivation of the localization formula. When we impose the Neumann boundary condition for both $P$ and $\phi_i$’s, although the final result (7.28) coincides with the correct result, in the derivation of the localization formula, the singular locus $u = 0$ does not satisfy the condition (3.60), i.e., the singular hyperplane arrangement is not projective. To make the hyperplane arrangements projective and justify the argument around (3.60), we have introduced generic flavor fugacities $z_i$ and have taken $z_i \to 0$ in the final expression.

Next let us consider the Landau-Ginzburg phase, i.e., the negative FI-parameter region. We choose the Neumann boundary condition for $P$ and the Dirichlet boundary conditions for $\phi_i$’s. The index with two Wilson loops is given by

$$\chi_{ab}^{\text{LG}} : = \langle e^{ib} f(A_2 - i\sigma') e^{-ai} f(A_2 - i\sigma') \rangle$$

$$= \sum_{l=0}^{M-1} \oint_{u=\frac{2\pi i l}{M}} \frac{du}{2\pi i} e^{(b-a)u} \frac{(1 - e^{-u})^M}{1 - e^{-Mu}}.$$ \hspace{1cm} (7.29)
For example $M = k = 5$, the $\chi_{ab}^{LG}$ for $a, b = 0, \cdots, 4$ are given by

$$\chi_{ab}^{LG} = \begin{pmatrix} 0 & 5 & -10 & 10 & -5 \\ -5 & 0 & 5 & -10 & 10 \\ 10 & -5 & 0 & 5 & -10 \\ -10 & 10 & -5 & 0 & 5 \\ 5 & -10 & 10 & -5 & 0 \end{pmatrix}. \quad (7.30)$$

We find that $\chi_{ab}^{LG} (a, b = 0, 1, 2, 3, 4)$ correctly reproduce open string Witten indices $I_{\alpha\beta} = \text{Tr}_{\alpha\beta,R}(-1)^F q^{L_{a} - \frac{c}{24}}$ in the Gepner model for the quintic 3-fold in $[30]$. From the open/closed string duality, $I_{\alpha\beta}$’s are calculated by cylinder amplitudes for B-type boundary states in $[37]$.

Let us derive (7.29) from the Neumann boundary conditions with a matrix factorization. We take a tachyon profile $[3]$ as

$$Q = \sum_{i=1}^{M} \left( \phi_i \bar{\eta}_i + \frac{1}{M} P \frac{\partial f(\phi)}{\partial \phi_i} \eta_i \right) \text{ with } \{\eta_i, \bar{\eta}_j\} = \delta_{ij}, \{\eta_i, \eta_j\} = \{\bar{\eta}_i, \bar{\eta}_j\} = 0, \quad (7.31)$$

which acts on the following graded vector space:

$$V = \bigoplus_{i=0}^{M} i \wedge E, \quad (7.32)$$

where $E$ is an $M$-dimensional vector space spanned by $\{\bar{\eta}_i|0\}_{i=1}^{M}$. For simplicity, we suppressed the the subscripts $L/R$ for the left and the right boundaries. Then the saddle point values of the brane factors are given by

$$\text{Str}_{V_L}(e^{-u}) = e^{-(a - \frac{M}{2})u} \sum_{i=0}^{M} (-1)^i \left( \begin{array}{c} M \\ i \end{array} \right) e^{-iu} = e^{-au} (e^\frac{a}{2} - e^-\frac{a}{2})^M, \quad (7.33)$$

$$\text{Str}_{V_R}(e^{u}) = e^{(b - \frac{M}{2})u} \sum_{i=0}^{M} (-1)^i \left( \begin{array}{c} M \\ i \end{array} \right) e^{iu} = e^{bu} (e^\frac{b}{2} - e^-\frac{b}{2})^M. \quad (7.34)$$

Here we assign a gauge charge $a - \frac{M}{2}$ (resp. $b - \frac{M}{2}$) for $|0\rangle_L$ (resp. $|0\rangle_R$). Then the localization formula with two brane factors is given by

$$Z_{W_LW_R} = \sum_{l=0}^{k-1} \oint_{u=2\pi i l} \frac{du}{2\pi i} \frac{\text{Str}_{V_L}(e^{-u})\text{Str}_{V_R}(e^{u})}{(e^{-\frac{Ma}{2}} - e^{\frac{Ma}{2}})(e^\frac{a}{2} - e^-\frac{a}{2})^M} \quad (7.34)$$

Thus we obtain the same result as (7.29) up to an overall sign. By comparing (7.29) with (7.34), we find that the two brane factors $W_{V_L}$ and $W_{V_R}$ turn the Neumann boundary conditions for $\phi_i$’s for $i = 1, \cdots, M$ to the Dirichlet ones.
8 Summary and future directions

We have introduced $I \times T^2$ indices for 3d $\mathcal{N} = 2$ supersymmetric theories coupled to 2d $\mathcal{N} = (0,2)$ boundary theories and have studied properties of the indices. We summarize our results and comments on their implications in future studies.

In section 4 we have studied the 3d $\mathcal{N} = 4$ theory which is mirror dual of the 3d $\mathcal{N} = 8$ super Yang-Mills theory. We find that $I \times T^2$ index for the 3d $\mathcal{N} = 4$ theory agrees with the M-string partition function up to a fugacity $y$. Since the 3d $\mathcal{N} = 8$ super Yang-Mills theory flows to the ABJM model with $\kappa = 1$, an $I \times T^2$ index for the ABJM model with an appropriate boundary condition is expected to reproduce M-string partition functions. In this direction, the authors of [38] studied the level $\kappa = 1$ ABJM model on the interval in the zero length limit and compared it with the M-string partition function. It was found that the partition function of the dimensionally reduced ABJM model partially agrees with the M-string partition function. It is interesting to explore the boundary conditions in the ABJM model and the $I \times T^2$ index relevant to the M-string partition function.

In section 5, we have studied three dimensional dualities with boundaries like the SQED and the XYZ model. In typical cases of 3d dualities between two gauge theories on half spaces $\mathbb{R}_{\leq 0} \times \mathbb{R}^2$ or on $S^1 \times D^2$, it was conjectured in [14] that the Neumann boundary condition for the vector multiplet is mapped to the Dirichlet boundary condition for the vector multiplet in the dual model. The Neumann boundary condition for the vector multiplet is same as the boundary condition (2.8). On the other hand, the Dirichlet boundary condition for the vector multiplet is not treated in this article. In the Dirichlet boundary condition, the global gauge symmetries are preserved at the boundary that are identified with flavor symmetries in the dual model. It would be nice to develop the localization computation on $I \times T^2$ with the Dirichlet boundary condition for the vector multiplet and to study dualities between various combinations of boundary conditions for the supermultiplets.

In section 6, we have treated chiral algebras for simple 3d theories on the interval. For the half spaces $\mathbb{R}_{\leq 0} \times \mathbb{C}$, it is known that chiral algebras are associated to the H-twisted 3d $\mathcal{N} = 4$ gauge theories [39]. In their construction, the gauge theory data correspond to the chiral algebras as follows; the $G$ vector multiplet, the hypermultiplets and the boundary fermi multiplets are associated to the $g_{bc}$-ghost, the symplectic bosons and the fermions. The complex moment map in the 3d gauge theory corresponds to the current for symplectic bosons that enters in the definition of the BRST charge. The gauge anomaly cancellation between 3d and 2d theories corresponds to the nilpotency of the BRST charge. The $S^1 \times D^2$ index corresponds to the vacuum character of the chiral algebra. It is interesting to explore the general rules for the chiral algebras associated with the gauge theories on $I \times M_2$ like the cases for the H-twisted gauge theories on $\mathbb{R}_{\leq 0} \times \mathbb{C}$.

In section 7, we have studied the localization formula for indices on $I \times S^1$. In explicit computations in several examples we have shown that indices on $I \times S^1$ with loop operators
agree with open string Witten indices; Euler pairings in the geometric phase and the cylinder amplitudes of the B-type RR grounds states for the Gepner models in the Landau-Ginzburg phase. To the best of our knowledge, the evaluation of open string Witten indices for the Gepner models based on the GLSMs on $I \times S^1$ is new.

The Euler pairings appear in the physics associated to the geometry of D-branes and they are related to several topics of quantum geometries of target spaces and SUSY cycles. Among them, there is an interesting property between the Kähler potential for a Calabi–Yau $n$-fold, period integrals and Euler pairings:

$$\exp (-i^n K(z, \bar{z})) = \sum_{a,b} \chi^{ab} \int_{A_a} \Omega(z) \int_{A_b} \overline{\Omega}(\bar{z}).$$

(8.1)

Here $K(z, \bar{z})$ is the Kähler potential for the Calabi–Yau $n$-fold. The left and the right hand sides of (8.1) are expressed as partition functions of 2d $\mathcal{N} = (2,2)$ theories as follows. It was conjectured in [40] that $e^{-i^n K(z, \bar{z})}$ is given by an $S^2$ partition function in [41, 42]. Period integrals $\int_{A_a} \Omega(z)$ and their conjugates are given by $D^2$ partition functions in [2, 3]. $\chi^{ab}$ is the inverse matrix of Euler pairings ($I \times S^1$ indices). Since $I \times T^2$ indices and $S^1 \times D^2$ indices in [1] are $S^1$-extensions (q-deformations) of Euler pairings and period integrals, we expect a factorization similar to (8.1) holds between partition functions on closed 3-manifolds, $S^1 \times D^2$ indices and $I \times T^2$ indices. Another future direction is as follows. In three dimensions, quantum differential equations become q-difference equations [4] that annihilate $S^1 \times D^2$ indices, more precisely annihilate $K$-theoretic I-functions in $S^1 \times D^2$ indices. For q-difference equations, counter parts of monodromy matrices are called connection matrices. The relation between monodromy matrices of quantum differential equations and $I \times S^1$ indices imply that connection matrices for q-difference equations for the $K$-theoretic I-functions are described by $I \times T^2$ indices. The relation between the connection matrices and $I \times T^2$ indices will be studied elsewhere.

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