Meta-uncertainty for particle image velocimetry

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Abstract
Uncertainty quantification for particle image velocimetry (PIV) is critical for comparing experimentally measured flow fields with computational fluid dynamics results, and model design and validation. However, PIV features a complex measurement chain with coupled, non-linear error sources, and quantifying the uncertainty is challenging. Multiple assessments show that none of the current methods can reliably measure the actual uncertainty across a wide range of experiments, and estimates can vary. Because the current methods differ in assumptions regarding the measurement process and calculation procedures, it is not clear which method is best to use for an experiment where the error distribution is unknown. To address this issue, we propose a method to estimate an uncertainty method’s sensitivity and reliability, termed the Meta-Uncertainty. The novel approach is automated, local, and instantaneous, and based on perturbation of the recorded particle images. We developed an image perturbation scheme based on adding random unmatched particles to the interrogation window pair considering the signal-to-noise of the correlation plane. Each uncertainty scheme’s response to several trials of random particle addition is used to estimate a reliability metric, defined as the rate of change of the inter-quartile range of the uncertainties with increasing levels of particle addition. We also propose applying the meta-uncertainty as a weighting metric to combine uncertainty estimates from individual schemes, based on ideas from the consensus forecasting literature. We use planar and stereo PIV measurements across a range of canonical flows to assess the performance of the uncertainty schemes. Further, a novel method is introduced to assess an uncertainty scheme’s performance based on a quantile comparison of the error and uncertainty distributions, generalizing the current method of comparing the RMS of the two distributions. The results show that the combined uncertainty method outperforms the individual methods, and this work establishes the meta-uncertainty as a useful reliability assessment tool for PIV uncertainty quantification.

Keywords: particle image velocimetry, uncertainty, meta-uncertainty

(Some figures may appear in colour only in the online journal)
1. Introduction

Over the past decade, there has been increasing effort in particle image velocimetry (PIV) to develop a-posteriori uncertainty quantification methodologies for local and instantaneous displacement measurements [1]. These efforts aim to provide uncertainties to PIV measurements that can be used for comparison to CFD results and for model design and validation. Parallel efforts included methods for propagating these uncertainties to derived quantities such as turbulence statistics [2], velocity derivatives [3] and pressure [4, 5], as well as to stereo PIV [6] and volumetric PTV [7] measurements. There has also been recent progress in uncertainty quantification in background oriented schlieren (BOS), an image-based density measurement technique, where the apparent displacement of a dot pattern is used to measure the density field. Since the displacement is the first measurement performed in a typical BOS measurement chain, there has also been related advances in displacement uncertainty quantification for BOS using methods from PIV, along with propagation of displacement uncertainties through the density integration chain [8], and using the uncertainties to improve the density integration process by weighted least squares minimization [9]. Since developments in PIV uncertainty quantification also find applicability in other measurement modalities, this highlights a need for robust and accurate methods for estimating the displacement uncertainty.

The measurement uncertainty of a quantity (e.g. velocity in PIV) represents the interval expected to contain the true value. This uncertainty depends on all the factors in the overall measurement chain. Since PIV involves a complex measurement chain from image recording through processing and post-processing, the final measurement can suffer from a multitude of error sources such as particle size, seeding density, shear, noise, out-of-plane motion, and processing algorithms, to name a few [10]. These error sources can combine in a coupled and non-linear manner to affect the final measurement uncertainty, and also depend on the final quantity of interest, whether the displacement, shear, pressure, or density. Even just for displacement uncertainty, while there have been many methods proposed in the literature, none perform well under all situations.

PIV displacement uncertainty methods are commonly classified into direct and indirect methods. Indirect methods predict the displacement uncertainty by calibrating the variation of uncertainty to various image parameters (such as particle size, density, shear, noise) and signal-to-noise ratio (SNR) metrics of the cross-correlation plane (such as the peak to peak ratio (PPR), mutual information (MI) and others [11–13]). Monte-Carlo simulations with synthetic images are used to obtain the calibration [11–14]. The performance of all indirect methods relies on the calibration process, which must be accurate and reflect all possible experimental scenarios in a typical measurement.

Direct methods estimate the uncertainty directly based on image or correlation plane properties without calibration. Presently, three direct methods are available to estimate the displacement uncertainty—image matching (IM) [15], correlation statistics (CS) [16], and moment of correlation (MC) [17]. In brief, IM or particle disparity proposed by Sciachitano et al [15] estimates the uncertainty in the displacement using a statistical analysis of the disparity between the measured positions of particles in the two frames after a converged iterative deformation interrogation procedure. The performance of this method is sensitive to the accuracy of the particle position estimation and deteriorates with increasing seeding density, noise, and out-of-plane motion. CS, proposed by Wiencke [16], estimates the uncertainty again using the image disparity but at a pixel level. The correlation peak’s asymmetry at the end of a converged window deformation procedure is used to measure the correlation error, and propagating the standard deviation of this error through the sub-pixel estimator provides the uncertainty. The CS method relies on the correlation plane statistics and performs better at higher seeding densities and larger interrogation windows. MC, proposed by Bhattacharya et al [17], predicts the uncertainty by estimating the second-order moment of the PDF of displacements contributing to the cross-correlation plane. The PDF is estimated as the generalized cross-correlation [18, 19]. MC first finds the PDF from the inverse Fourier transform of the phase of the complex cross-correlation plane [20, 21], followed by Gaussian filtering, gradient correction, and scaling by the effective number of particles contributing to the cross-correlation. This method also works better with high seeding densities and large interrogation windows, and small interrogation windows can lead to an over-prediction of the uncertainty.

However, multiple previous works show that none of the PIV uncertainty quantification methods perform well under all situations [22, 23]. While the direct methods are sensitive to elemental error sources [22], they can under-predict the random error [23]. In addition, direct methods can predict different uncertainties for the same flow field [8, 17], and as a result, no PIV uncertainty method is universally consistent and robust. Further, it is often impossible to choose the correct estimate in an experiment because the actual random error is unknown, and these potentially incorrect estimates in displacement uncertainty can propagate to derived quantities with detrimental implications for further analysis. Therefore, it is not clear which method to use for an experiment where the error is unknown.

A similar problem also exists in the consensus forecasting literature when assessing the risk/reliability associated with competing models that predict a future quantity based on incomplete information in the present [24–26]. In these applications, the variance of the fluctuations of each model prediction provides the ‘risk’/volatility’. The model fusion idea has also been developed in the context of multi-fidelity
uncertainty quantification for complex engineering systems [27–29].

In this work, we adapt these ideas to the problem of PIV uncertainty quantification and introduce a concept termed the ‘Meta-Uncertainty’, which characterizes the properties of the uncertainty quantification method and the uncertainty estimate. We also propose a method to estimate this meta-uncertainty in a local, instantaneous, and automated manner. The method is based on SNR properties of the cross-correlation plane, and is detailed below.

Past works on the theory of PIV cross-correlation show the correlation plane comprises the correlation between corresponding particles (the displacements between images of the same particle recorded in the two consecutive frames), and the non-corresponding particles (including image noise) [30–33]. The corresponding correlation leads to the primary correlation peak that determines the displacement on which we are interested to quantify the uncertainty. It has also been shown that the SNR of this correlation plane can be determined from the correlating particles using metrics such as the N_F1F2 and the MI [13]. Since the correlation measurement depends primarily on the particle properties (such as shape, intensities, and displacements), it is expected that perturbing the particle distribution in the interrogation window can sufficiently perturb the measurement (and the error) to provide an accurate error distribution.

Further, since the uncertainty estimates are in-turn based on the cross-correlation plane, the particle perturbation will also lead to a corresponding perturbation in the uncertainty estimates. Therefore, we estimate the meta-uncertainty by perturbing the particle images in an interrogation window pair and assessing the variation of the uncertainty estimates to this perturbation. Each trial of the image perturbation will lead to a corresponding perturbation in the uncertainty estimate for each uncertainty scheme. By repeating the image perturbation over several trials and over different levels, we quantify the response/distribution of each uncertainty scheme to the image perturbation, with the width of the distribution related to the sensitivity of the corresponding uncertainty quantification scheme. The probability distribution of the uncertainty estimates to this particle perturbation represents the Meta-Uncertainty.

However, the perturbation should be large enough to provide a variation of the displacement uncertainty without significantly affecting the cross-correlation plane and displacement estimate on which we are trying to calculate the uncertainty bounds. To examine this, three perturbation methods are explored in this work: (a) random particle removal, (b) paired particle removal, and (c) random particle addition, and we assess their effect using SNR metrics of the cross-correlation, such as the peak ratio and MI. Testing these methods on the images shows that the third method of random particle addition can provide an uncertainty perturbation without affecting the SNR of the underlying cross-correlation plane, which constitutes the measurement on which we are reporting the uncertainty.

We then use descriptive statistics of the distribution, such as the inter-quartile range (IQR) and the rate of change of the IQR with increasing particle addition, to estimate a reliability metric (the meta-uncertainty) for each uncertainty scheme. A broader distribution of uncertainty estimates and a higher meta-uncertainty will characterize schemes that are more sensitive to the perturbations.

Finally, we apply the meta-uncertainty to develop a new uncertainty quantification scheme for PIV that combines the estimates from the individual schemes weighted by the inverse of their meta-uncertainty. Similar to consensus forecasting, where the aim is to combine estimates from different models based on a meta-analysis of the individual models [25, 26, 34], this approach aims to fuse the prediction from multiple uncertainty models into a new, more robust, and reliable estimate. The hypothesis is that different models utilize different aspects of the information associated with the measurement, and therefore their combination provides a better estimate than each individual model. In the present context, the forecast quantity is the displacement uncertainty, the individual models are the uncertainty quantification methods, and the meta-uncertainty provides the weights for each model. While the proposed framework is general and can apply to many individual uncertainty schemes, here, we will consider only the three direct displacement uncertainty schemes—IM [15], CS [16], and MC [17]. We assess the performance of the meta-uncertainty estimation method and the combined uncertainty scheme with synthetic and experimental planar and stereo PIV images.

2. Methodology

Figure 1 shows the overall meta-uncertainty based combination method which consists of three major steps: (a) the estimation of the meta-uncertainty for each uncertainty method, (b) calculation of weights based on the response function, and (c) calculating the combined uncertainty. The following sections detail the procedure for each step.

The meta-uncertainty is based on the uncertainty scheme’s PDF and describes its response to a perturbation in the input intensity distribution. Since the PIV uncertainty methods rarely have a closed-form expression, we estimate the PDF using a Monte-Carlo simulation procedure. Further, to estimate the true/parent PDF requires the knowledge of all inputs—a set of all possible PIV images—which is not possible. Therefore, we perturb the intensity distributions of the interrogation window pair for several trials to generate a local population of particle image pairs and estimate the corresponding uncertainty.

For the image perturbation procedure to be valid, it must be able to provide a variation of the uncertainty estimates without an appreciable change in the underlying SNR metrics of the cross-correlation estimator (such as the PPR, MI and others [11–13]) whose uncertainty we are trying to estimate. There are several potential methods to perturb the particle images. We adopt a method of adding random and unpaired particles to the interrogation window pair. Analysis with synthetic and experimental images showed that this method best accomplished perturbing the images with a negligible change.
in the SNR metrics. Results for the perturbed images and the corresponding uncertainty ratios obtained using the other perturbation methods: removing paired particles and removing random particles, are provided in appendix A, figures A1–A3.

2.1. Particle perturbation
Images are perturbed by adding random/unpaired particles whose positions are drawn from a uniform distribution, and the shapes are approximated by a Gaussian intensity profile, as is standard in PIV image generation. The parameters of the Gaussian profile such as the peak intensity and the particle diameter are based on their average values across the identified particles for each frame. Particle identification and centroid-estimation are performed using methods commonly used in particle tracking velocimetry [35, 36]. Following this, we add a set of unpaired particles to the interrogation window pair as shown in figure 1(a), with the number of unpaired particles specified as a fraction of the seeding density. Figure 2 shows a sample particle image pair with the perturbation.

2.2. Meta-uncertainty calculation
The perturbed window pair is then cross-correlated, and the uncertainty is estimated using all three individual methods as shown in figure 1(b). We repeat this procedure for several trials to build a PDF of the ratio of the perturbed to original uncertainties for each estimator \( \sigma_{\Delta x, n}^{pert} / \sigma_{\Delta x, n}^{orig} \). Figure 3 shows sample distributions of the uncertainty schemes and statistics such as the median and quartiles. Each level of particle addition results in a distribution of uncertainties due to the perturbation. These distributions become wider with increasing level of particle addition at a different rate of increase for each method. The width of the distribution represents the sensitivity of each

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**Figure 1.** Illustration of the meta-uncertainty based combination methodology.

**Figure 2.** Example of a perturbed image pair, with the red circles indicating the location of the added particles.
Figure 3. Effect of particle addition on the PDF of the ratio of resampled to original uncertainties. Q1 and Q3 represent the lower quartile (25th percentile) and upper quartile (75th percentile) respectively.

Figure 4. Variation of the IQR of uncertainty ratios with particle addition percentage for a grid point, with the corresponding weights obtained from the straight line fit.

scheme to particle perturbation, and therefore a scheme with a wider PDF is less reliable compared to a scheme with a narrower PDF. However, the response and relative sensitivity of each scheme will vary with the local image and flow conditions.

2.3. Weight calculation

We calculate the IQR for each particle perturbation level from the distributions, and the rate of change of this IQR with particle perturbation (from a linear regression as shown in figure 1(c)) provides the weight for each scheme. The IQR is used in place of the RMS because it is less sensitive to outliers. Equation (1) provides the weight of the x component, with a similar equation for the y component. The slope (and the weight) calculation depends on the number of points used for the regression and the individual scheme’s response. Figure 4 shows a sample result for the IQR variation with five particle addition levels and the corresponding weights. These results are consistent with figure 3 with MC showing the highest rate of increase and therefore assigned the lowest weight, with CS showing the lowest rate of increase and therefore assigned the highest weight. This weight is essentially a local and instantaneous reliability assessment metric for each uncertainty scheme, and the proposed method allows for an automated way to estimate this metric for arbitrary particle images. As mentioned earlier, past works on the theory of PIV cross-correlation show the correlation plane SNR depends primarily on the particle properties (such as shape, intensities, and displacements), and therefore perturbing the particle distribution in the interrogation window can sufficiently perturb the measurement (and the error) to provide an accurate error distribution. While image perturbations are used to assess the sensitivity in this work, alternate methods
Table 1. Summary of processing parameters for all datasets.

|                | Turbulent boundary layer (TBL) | Laminar separation bubble (LSB) | Stagnation flow (SF) | Vortex ring (VR) | Jet flow (JF) |
|----------------|--------------------------------|---------------------------------|---------------------|-----------------|--------------|
| WS 1 (% overlap, No. of passes) | 64 × 64 (75%, 2) | 64 × 64 (75%, 4) | 64 × 64 (75%, 4) | 64 × 64 (75%, 1) | 32 × 32 (87.5%, 4) |
| WS 2 (% overlap, No. of passes) | 64 × 64 (87.5%, 1) | 64 × 64 (75%, 1) | 32 × 32 (75%, 1) | 32 × 32 (75%, 1) | 16 × 16 (75%, 3) |

Figure 5. Datasets used for assessment on planar PIV images. (a) Turbulent boundary layer [37], (b) laminar separation bubble [38], (c) laminar stagnation flow [39], (d) vortex ring [40], and (e) jet flow [22].

may be explored. The relative weights for each scheme can vary across grid points within the same flow field, and across flow-fields:

\[ w_x = \left( \frac{\Delta IQR_x}{\Delta \text{particle }\%} \right)^{-1}. \]  

(1)

2.4. Combined uncertainty calculation

Finally, we calculate the combined uncertainty as the weighted average of the individual uncertainty schemes as shown in figure 1(d),

\[ \sigma_{x,\text{comb}} = \sum_{n=1}^{N} w_{x,n} \sigma_{x,n} \]  

(2)

for the x-component where \( \sigma_{x,n} \) represents the individual uncertainty, \( w_{x,n} \) represents the corresponding weights, \( n \) represents the subscript for each method, and \( N \) represents the total number of methods (here, \( N = 3 \)). When two random variables are added, their expectations add linearly, and the standard deviations add in quadrature. Here the uncertainty is itself the random variable, therefore it is added linearly. In the next section, we will assess the method’s performance with synthetic and experimental images from planar and stereo PIV experiments.

3. Results

3.1. Planar PIV

Planar PIV measurements from several canonical flows are used to test the uncertainty quantification methods over a wide variation of image and flow conditions. The datasets used are: a TBL (PIV Challenge 2003B) [37], a LSB (PIV Challenge 2005B) [38], laminar SF [39], a VR (fourth PIV Challenge)
Figure 6. PDFs of the error and the individual and combined uncertainty estimates for consolidated results from all datasets.

[40], and the unsteady inviscid core of a jet [22]. For each dataset [17], we processed the images with two processing routines (WS1 and WS2 as listed in Table 1) to provide a further variation in the testing, using the open-source PIV code PRANA [41]. Figure 5 shows the displacement contours from all flow fields, and the error analysis used a true solution, based on details from the respective publications.

We test the combination framework shown on the datasets using a Monte-Carlo simulation by randomly choosing a dataset and a processing setting, a snapshot within the dataset, and a grid point within the snapshot. For the chosen grid point, particle addition is used to calculate the meta-uncertainty, and the individual uncertainties are combined using a weighted average using the procedure shown in Figure 1. Particle addition was performed for five levels of the seeding density from 5% to 25% and over 100 trials for each perturbation level (as shown in Figures 3 and 4). This procedure is repeated for 1000 grid points for each dataset and processing setting, and the errors and uncertainties across all datasets (10 000 grid points in total) are merged to calculate the corresponding statistics. This section discusses the merged statistics, and the appendix contains the individual dataset results.

Figure 6 shows the PDFs of the error and uncertainty distributions in the form of violin plots, with the individual schemes in blue, the error in black, and the combined uncertainty scheme in orange. It is to be noted that these differ from the violin plots of the uncertainty ratio shown in Figure 3, which is an intermediate quantity used to estimate the weights for the combination method. Also shown are the statistics such as the median (circles), quartiles (triangles), and the root mean square (RMS) (straight line) of each distribution. In PIV uncertainty, the error distribution of a single measurement is assumed to be Gaussian with the standard uncertainty being the standard deviation of the displacement measurement. Sciacchitano et al [22] showed that when the error distribution for each grid point is modeled as a zero-mean Gaussian random variable with the standard deviation representing the local uncertainty, the RMS of a composite error distribution based on assimilating the individual error distributions should match the RMS of the corresponding composite uncertainty distributions. Therefore, the RMS analysis only requires that the individual error distributions for each grid point be Gaussian, and does not place a restriction on the composite distributions. However, the assumption that an individual error measurement be Gaussian is still a limitation in the analysis. For these results, the RMS of the error distribution is 0.08 pix., with the combined uncertainty scheme predicting an RMS of 0.07 pix., while the RMS of the individual uncertainty schemes being 0.05 pix. for IM, 0.06 pix. for MC, and 0.1 pix. for CS. Therefore, the combined scheme provides the best estimate of the RMS error and can compensate for the under-prediction by IM and MC and the over-prediction by CS. However, there is still a 0.01 pix. discrepancy between the RMS estimates of the error and the combined scheme, indicating that there is room for further improvement of the method.

As a final note, the error/uncertainty distributions are skewed towards zero, due to a combination of the flow fields used in this study and the processing algorithms. In PIV with multi-pass processing, the error linearly increases with displacement till 0.5 pixels [42], and the multi-grid deformation algorithms used in this processing bound the displacement range to within ±0.1 pixels on the last pass [43, 44]. Further, from Figure 9, it can be seen that most of the grid points in the datasets lie in regions with low displacement and low shear because of the spatial distribution of the flow fields, and these are two of the important elemental sources of error and uncertainty [10]. For these reasons, the distributions are skewed towards lower values. The distributions will look quite different if, for example, a single-pass processing was used on a dataset where the bulk of the flow features velocity gradients and high displacements. The error/uncertainty distributions for each of the individual datasets are shown in Appendix B, figure B1.

3 PRANA: PIV research and analysis (available at: https://github.com/aether-lab/prana/)
The PDF of weights assigned to each individual scheme are shown in figure 7 with the individual schemes in blue, and the black dashed line represents the case when all schemes are equally weighted. CS is assigned the highest weights for most cases, followed by IM, and then by MC, consistent with the sample result shown in figure 4. However, even though CS is assigned a higher weight and over-predicts the RMS of the error, the combined effect of IM and MC, which under-predict the error, brings down the RMS of the combined scheme to be close to the RMS of the error. This highlights the advantage of the meta-model as even if one uncertainty scheme (CS) is more robust to perturbations in the correlation plane SNR and hence has a higher weight, the method can compensate with the weighting of other schemes. Finally, the weight distributions are broad, denoting that there are several grid points for which CS could be assigned a lower weight than either IM and MC. Therefore, the meta-uncertainty calculation is also able to capture the variation in the image and flow conditions across the datasets.

We also introduce a new method to compare the error and uncertainty distributions, based on a quantile–quantile comparison. This is a generalization of the method proposed by Sciacchitano et al. [22] for comparing the RMS of the error and uncertainty distributions, to address the sensitivity of the RMS calculations to outliers. Consider a set of error measurements \( \epsilon_i \), where \( i \) represents the grid point under consideration, with each error drawn from a corresponding distribution \( f_i \). Also, let the distribution of all the error measurements be \( f_e \). For each error measurement, we have an estimate of the uncertainty \( U_{n,i} \) where \( n \) represents an uncertainty method. This uncertainty measurement represents the standard deviation of the error distribution \( f_{\epsilon,i} \), with \( \epsilon_{n,i} \) representing the estimated error. Finally, we can also define a combined distribution of each uncertainty scheme’s estimated errors as \( f_{\hat{e}} \). Our aim then is to compare the distributions of the true error distribution, \( f_e \), with that of the estimated error distribution, \( f_{\hat{e}} \), for each uncertainty scheme. To enable this comparison, we need a model for the estimated error distributions \( f_{\hat{e,i}} \) at each grid point. Following the analysis of Sciacchitano et al [22], if we assume this distribution to be a zero-mean Gaussian random variable, \( f_{\hat{e,i}} = \mathcal{N}(0, U_{n,i}) \), then the overall distribution of the estimated error becomes the convolution of these individual distributions.

If the uncertainty estimate \( U_{n,i} \) is correct, then the RMS of the above distribution must equal that of the error distribution, consistent with the previous result of Sciacchitano et al [22]. It is to be noted that the Gaussian assumption applies to the error distribution from individual grid points \( (f_{\epsilon,i}) \), while those in figure 6 are composite distributions obtained from sampling several Gaussian distributions. The composite distribution need not be Gaussian for the analysis to be valid. Further details regarding the assumptions can be found in appendix A of Sciacchitano et al [22]. Further, since the grid points are drawn randomly from the datasets over space and time, the correlation between uncertainty estimates between successive grid points is expected to be negligible.

Therefore, in this work, we compare the distributions instead of the RMS values for a more rigorous comparison and to reduce the effect of outliers.

The following procedure is used to estimate \( f_{\hat{e}} \). For each grid point and uncertainty method \( U_{n,i} \), we draw several (here 1000) random values of \( \epsilon_{n,i} \) from the corresponding normal distribution. These estimated error values from all grid points are assimilated to provide a PDF of the estimated error distribution \( f_{\hat{e}} \). Then the true and estimated error distribution are compared using a quantile–quantile plot. Quantiles divide the probability distribution of a random variable into intervals with equal probabilities. In the plots shown in figure 8, the distributions of the true and estimated error have been divided into 100 quantiles (or percentiles), and the values of the error distributions at each quantile level is compared. The x-axis represents the quantiles of \( f_e \) and the y-axis represents the quantiles of \( f_{\hat{e}} \), with each curve corresponding to an uncertainty scheme, and the black line representing the 1:1 variation. The orange curve corresponding to the combined
scheme is overall closest to the black line, showing that the combined uncertainty scheme best approximates the true error distribution. The quantile-quantile plots for each of the individual datasets are shown in Appendix B, figure B2.

To complement the statistical analysis, we investigate the variation of the RMS error and uncertainty as functions of the element error sources such as fractional displacement and shear [10, 14]. To perform the comparison, we bin the errors and uncertainties based on their corresponding values of the displacement and shear, as estimated from the true solution. Then we calculate the bin-wise RMS of the error and uncertainties to calculate the variation of these statistics with the elemental error sources. Figure 9 shows these results, along with the number of measurements corresponding to each bin. The results show that (a) the errors/uncertainties increase with fractional displacement and (to a lesser extent) with shear, which is consistent with PIV theory [32, 33, 42], and (b) the combined scheme provides an RMS uncertainty that is, on average, the closest to the RMS error. However, MC performs better for low values of the velocity gradients, since the large uncertainties predicted by CS for these measurements shift the combined estimates upward.

In summary, the analysis on planar PIV datasets showed that the combined uncertainty scheme based on the meta-uncertainty better represented the error distribution in terms of the RMS, quantiles, and the effect of error sources such as fractional displacement and shear. In the next section, we demonstrate the performance of the method for stereo PIV images of a VR.

3.2. Stereo PIV

The performance of the meta-uncertainty-based framework is also tested with stereo PIV images by utilizing the uncertainty quantification methodology introduced by Bhattacharya et al [6]. The method propagates the planar PIV uncertainties for each camera through the stereo-reconstruction process, accounting for uncertainties in the mapping function coefficients from the self-calibration procedure [45]. The analysis here uses the VR dataset from Case E of the 4th PIV Challenge [40] (center and left cameras), similar to Bhattacharya et al [6]. Figure 10 shows displacement contours for the three displacement components, with 50 snapshots used for the analysis.

We assess the uncertainty schemes using the Monte-Carlo procedure detailed before, with the additional step of propagating the perturbed planar uncertainties through the stereo-reconstruction procedure to calculate the corresponding stereo uncertainties, the IQR and the weights. Therefore, the full measurement chain was used to calculate the meta and combined uncertainties.

Figure 11 shows the distribution of weights, errors, and uncertainties, and figure 11(a) is consistent with the planar results, with CS being assigned the highest weight, followed by IM and MC. Further, the relative distribution of the weights is nearly identical for the three displacement components. From the error and uncertainty distributions shown in figure 11(b), we see that the RMS of the combined uncertainty method is again the closest to the RMS error for the in-plane component $U$ and $V$, similar to the planar data, and slightly
Figure 10. Spatial variation of displacement components for the stereo PIV dataset.

Figure 11. Results of applying the meta-uncertainty model to stereo PIV images. (a) PDF of weights, (b) error and uncertainty distributions, and (c) quantile–quantile comparison of the true and estimated error.
Finally, the quantile-quantile plots in figure 11(c) show that the combined uncertainty best approximates the true error distribution for the in-plane displacement components, while IM and MC perform better for the out-of-plane components. The deviation of the combined uncertainty closely follows that of the CS curve because of the high weights assigned to CS. Therefore, in situations without an obvious choice for the best individual scheme, the combined scheme offers minor performance improvement. However, the performance of the individual schemes varies for the vast majority of the experiments, and when the error distribution is not available, it is impossible to guess the best method. Therefore, the combined method offers the most robust estimate of the uncertainty for a general experiment without a true solution. Overall, these results establish that the meta-uncertainty based combination framework also performs well for stereo PIV measurements.

4. Summary and conclusions

This work introduced a reliability metric of PIV uncertainty quantification methods termed the meta-uncertainty and an automated, local, and instantaneous method for its estimation. The meta-uncertainty describes the sensitivity of an uncertainty quantification method to perturbation in the input images, with a more sensitive scheme possessing a higher meta-uncertainty and lower reliability. Random/unpaired particles are added to perturb the images and estimates the uncertainty using each method over several trials and different particle addition levels. The PDF of the uncertainty provides a statistical measure of the response function (the meta-uncertainty), and the rate of change of the IQR of the individual uncertainty schemes with particle addition provides the reliability metric.

In addition, this work also introduced a framework for combining individual uncertainty estimates based on the meta-uncertainty, similar to consensus forecasting. Since the uncertainty estimation methods differ in their use of information regarding the displacement estimation process, we hypothesized that combining the individual estimates should provide a better estimate of the uncertainty. The individual estimates were combined using a weighted average, with the weights based on the inverse of the rate of change of IQR, with a more sensitive/less reliable scheme assigned a lower weight.

Both the meta-uncertainty estimation and the combination framework were tested with the direct uncertainty methods—IM, MC, and CS—with planar and stereo PIV images of several canonical flows, which offer a range of error and uncertainty sources. The planar PIV dataset included a TBL, LSB, laminar SF, VR, and IF, two processing settings for each flow field, and the stereo PIV dataset used was a VR. We calculated the individual and combined uncertainties for grid points randomly sampled from these datasets, and results showed that the combined uncertainty best represented the true error distribution in terms of the overall RMS and the variation of RMS with error sources such as displacement and shear. Further, a new method was introduced to compare the error and uncertainty estimates by generalizing the RMS comparison method of Sciacchitano et al. to quantiles of the error and uncertainty distribution for a more rigorous comparison that is less sensitive to outliers.

The results showed that the error distribution based on the combined method predicts the true error distribution better than the individual uncertainty methods. For the stereo PIV dataset, the meta-uncertainty based combined method showed the best performance for the in-plane components, with a slight over-prediction in the RMS error for the out-of-plane component. However, since an individual uncertainty scheme performance varies significantly for different experiments, the combined method likely provides the best potential estimate of the uncertainty for a general experiment.

It was seen that the CS method was assigned a higher weight most of the time, because it was the least sensitive to the perturbations. It is to be noted that the sensitivity of the scheme is a function of the image and displacement conditions, and therefore a different behavior may be expected in other situations. The weight calculation method proposed in this work is automated and local, and can therefore handle a variety of conditions that may be encountered in a PIV experiment. Further, the major contribution of this work is to introduce the concept of the sensitivity of an uncertainty quantification scheme, a method for its estimation, and a demonstration of its use for uncertainty combination. However, there are certainly limitations to this approach and therefore many avenues of improvement to the method from the sensitivity calculation to the weighting.

A major limitation of the method is the computational cost involved in estimating the meta-uncertainty with approximately 1000 perturbation trials performed for each grid point and uncertainty scheme. Therefore, future work could reduce this computational cost by developing approximate theoretical models of the individual schemes’ response functions to accelerate the computations. Machine learning based neural-network models can also improve the combination framework. Another aspect for improvement is in considering the covariance between the individual uncertainty schemes during the weighting and combination steps. Finally, the use of meta-uncertainty to uncertainty combination is just one application of the concept, and the meta-uncertainty concept can also improve the individual schemes themselves by analyzing their response to particle perturbations, and can be a stand-alone tool in reliability assessments of uncertainty estimates for other calculations such as uncertainty propagation, de-noising and other post-processing routines. In conclusion, this paper establishes the meta-uncertainty as a useful reliability assessment tool for PIV uncertainty quantification and the combination framework as a successful estimator of the uncertainty for cross-correlation PIV processing.
Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors. The codes implementing the methodology are available for download from: https://github.itap.purdue.edu/lrajendr/meta-uncertainty.

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Appendix A. Methods for perturbing the particle images

Figure A1. Illustration of methods used to perturb the particle images. (a) Removing paired particles, (b) removing unpaired particles, (c) adding unpaired particles.

Figure A2. Effect of image perturbation on the uncertainty ratios for the three methods.
Figure A3. Effect of particle perturbation on correlation plane SNR metrics for the three methods.

Appendix B. Planar uncertainty distributions for individual datasets

Figure B1. Error and uncertainty distributions for each of the planar PIV datasets. For each violin plot, the left (darker) and right (lighter) halves correspond to the results for WS1 and WS2 processing, respectively.
Figure B2. Quantile-Quantile comparisons of the true and estimated error distributions based on the individual and combined uncertainty estimates for each planar PIV dataset.

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