GELFAND-DICKEY ALGEBRA AND HIGHER SPIN SYMMETRIES ON $T^2 = S^1 \times S^1$

M.B. SEDRA$^1$,
Université Ibn Tofail, Faculté des Sciences, Département de Physique,
Laboratoire de Physique de La Matière et Rayonnement (LPMR), Kénitra, Morocco
Groupement National de Physique de Hautes Energies, GNPHE, Morocco,
Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We focus in this work to renew the interest in higher conformal spins symmetries and their relations to quantum field theories and integrable models. We consider the extension of the conformal Frappat et al. symmetries containing the Virasoro and the Antoniadis et al. algebras as particular cases describing geometrically special diffeomorphisms of the two dimensional torus $T^2$. We show in a consistent way, and explicitly, how one can extract these generalized symmetries from the Gelfand-Dickey algebra. The link with Liouville and Toda conformal field theories is established and various important properties are discussed.

$^1$Associate of ICTP: sedra@ictp.it
1 Introduction

For several years conformal field theories and their underlying Virasoro algebra [1] have played a pioneering role in the study of string theory [2], critical phenomena in certain statistical-mechanical models [3] and integrable systems [4]. The conformal symmetry is generated by the stress energy momentum tensor of conformal spin 2 whose short-distance operator product expansion (OPE) is given by

\[ T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{(z-\omega)} + ... \] (1)

Some years ago, Zamolodchikov opened a new issue of possible extensions of the Virasoro algebra to to higher conformal spins. Among his results, was the discovery of the \( W_3 \) algebra involving, besides the usual spin-2 energy momentum tensor \( T(z) \), a conformal spin-3 conserved current \[ W(z) \] [5, 6]. \( W_3 \)-symmetry is an infinite dimensional algebras extending the conformal invariance (Virasoro algebra) by adding to the energy momentum operator, \( T(z) \equiv W_2 \), a set of conserved currents \( W_s(z) \) of conformal spin \( s > 2 \) with some composite operators necessary for the closure of the algebra.

In the language of 2\( d \) conformal field theory, the above mentioned spin-3 currents of the \( W_3 \)-symmetry are taken in general as primary satisfying the short-distance operator product expansion (OPE) [5, 6]

\[
\begin{align*}
T(z)W(\omega) &= \frac{3W(\omega)}{(z-\omega)^2} + \frac{\partial W(\omega)}{(z-\omega)}, \\
W(z)W(\omega) &= \frac{c/3}{(z-\omega)^3} + \frac{2\Lambda(\omega)}{(z-\omega)} + \frac{\partial T(\omega)}{(z-\omega)^2}} + \frac{1}{(z-\omega)^2}[2\beta\Lambda(\omega) + \frac{3}{10}\beta^2 T(\omega)] \\
&+ \frac{1}{(z-\omega)}[\beta \partial \Lambda(\omega) + \frac{1}{15}\beta^3 T(\omega)]
\end{align*}
\] (2)

where \( \Lambda(\omega) = (TT)(\omega) - \frac{3}{10}\beta^2 T(\omega) \) and \( \beta = \frac{16}{22 + 3c} \). This symmetry which initially was identified as the symmetry of the critical three Potts model, has also been realized as the gauge symmetry of the so-called \( W_3 \) gravity [7]. Since then higher spin extensions of the conformal algebra have been studied intensively by using different methods: field theoretical [8, 9, 10], Lie algebraic [11] or geometrical approach [12]. \( W_3 \)-algebras are also known to arise from the Gelfand-Dickey bracket (second Hamiltonian structure) of the generalized Korteweg-de Vries (KdV) hierarchy [13, 14, 15]. In this context, the correspondence is achieved naturally in terms of pseudo-differential Lax operators, \( \mathcal{L}_n = \sum u_{n-j}\partial^j \), allowing both positive as well as nonlocal powers of the differential \( \partial^j \) where the spin-\( j \) currents \( u_j \) are satisfying integrable non linear differential equations. The integrability of these equations can be traced to the fact that KdV hierarchy’s equations are associated to higher conformal spin symmetries through the Gelfand-Dickey Poisson bracket.
Another kind of higher conformal spin symmetries deals with the Frappat et al. algebra [16] on the bidimensional torus $T^2$. This is generated by the infinite dimensional basis set of mode operators $L_{k,l}$, $k,l \in \mathbb{Z}$ verifying the classical commutation relations

$$[L_{k,l}, L_{r,s}] = [m_0(k - r) - n_0(l - s)]L_{k+r,l+s}$$

(3)

where $n_0$ and $m_0$ are two arbitrary parameters. The above algebra has the property of offering a unified definition of several extensions of the two dimensional conformal algebra. In fact it’s shown that the standard Virasoro algebra as well as the Antoniadis et al. algebras [17] are special cases of eq(3). Later on, a $W_3$-extended conformal symmetry going beyond the Frappat et al. algebra, on the torus $T^2$, is proposed in [18] by adding the spin-3 currents. These several extended conformal symmetries are important as they are associated to higher diffeomorphisms namely $\text{diff}(T^2)$ extending the standard diffeomorphisms, $\text{diff}(S^1)$, involved in the $2d$ conformal invariance.

Note by the way that, contrary to the case of fractional superconformal symmetries, the Frappat et al. symmetries as well as their $W_3$ extensions consist in replacing the manifold $S^1$ by a higher dimensional compact manifold $\mathcal{M}$. This can offer a large framework for studying more general extensions of $2d$-conformal symmetries including fractional superconformal symmetries and their generalizations [19].

The principal goal of this work concerns the study of some nonstandards properties of higher conformal spin symmetries in the bidimensional torus $T^2$. This is motivated, in one hand, by the increasingly important role that play the infinite-dimensional Lie algebras in the development of theoretical physics. Best known examples are given by the Virasoro algebra, which underlies the physics of $2d$ conformal field theories (CFT) and its $W_k$-extensions. On the other hand, it’s today well recognized that $2d$ conformal symmetry and it’s $W_k$ higher spin extensions are intimately related to the algebra of $\text{diff}(S^1)$ and $\text{diff}(T^2)$ respectively. Therefore, we know that $2d$-conformal symmetry and the KdV integrable model are intimately related in the standard $\text{diff}(S^1)$ framework. Such relation is shown to arise from the second hamiltonian structure of the KdV integrable system reproducing then the classical form of the Virasoro algebra.

One wonder whether the area preserving diffeomorphisms on $T^2$ can be related to integrable hierarchy systems in the same spirit. In this context, and after a setup of our conventional notations and basic definitions, we develop a systematic analysis leading to an explicit derivation of the Gelfand-Dickey Poisson bracket. This is based on the theory of pseudo-differentials operators characterized by the following $\text{diff}(T^2)$-non standard Lax differential operators $\{\log H, \}^n + \sum_{i=0}^{n-2} u_{n-i}(z, \omega) \{\log H, \}^i$, where the derivation is given by the hamiltonian vector field $\xi_H \equiv$
{\log H,} playing in \text{diff}(T^2) the same role of the derivation \( \partial_z \) in \text{diff}(S^1). The particular KdV-like Lax differential operator \( \xi_H^2 + u_2 \) associated to the KdV equation is compatible with the conformal-invariant Liouville-like differential equation \( \{ \log \bar{K}, \{ \log H, \phi \} \} = e^\phi \). The compatibility is assured once the current \( u_2(z, \omega) \), shown to satisfy the following KdV-like nonlinear differential equation
\begin{equation}
\partial_t u_2 = \frac{3}{4} \{ \log H, u_2 \} - \frac{3}{4} \{ \log H, u_2 \}^{(3)},
\end{equation}
behaves under a bicomplex change of coordinates like the stress energy momentum tensor on the bidimensional torus \( T^2 \) namely
\[ T(z, \omega) = -\{ \log H, \phi \}^{(2)} - \{ \log H, \phi \}^2. \]
Several important properties are discussed with some concluding remarks at the end.

## 2 Bianalytic fields on the torus \( T^2 \)

In this section we give the general setting of the basic properties of the algebra of bianalytic fields defined on the bidimensional torus \( T^2 \).

1) We start first by noting that the ring \( \mathcal{R} \) of conformal fields defined on the bidimensional torus \( T^2 \), and transforming as primary fields under a special subalgebra of \text{diff}(T^2) \), can be thought of as a generalization of the ring of the Virasoro primary fields of the 2d real space \( \mathbb{R}^2 \cong \mathbb{C} \). The two dimensional torus \( T^2 \) is viewed as a submanifold of the 4d real space \( \mathbb{R}^4 \approx \mathbb{C}^2 \). It is parametrized by two independent complex variables \( z \) and \( \omega \) and their conjugates \( \bar{z} \) and \( \bar{\omega} \) satisfying the constraint equation \( z\bar{z} = \omega \bar{\omega} = 1 \). Solutions of these equations are given by \( z = e^{in\theta}, \omega = e^{im\psi} \) where \( n \) and \( m \) are two integers and where \( \theta \) and \( \psi \) are two real parameters.

2) We identify the ring \( \mathcal{R} \) of bianalytic fields on \( T^2 \) to be \( \mathcal{R} \equiv \hat{\Sigma}^{(0,0)} \) the tensor algebra of bianalytic fields of arbitrary conformal spin. This is an infinite dimensional \( SO(4) \) Lorentz representation that can be written as
\begin{equation}
\hat{\Sigma}^{(0,0)} = \oplus_{k \in \mathbb{Z}} \hat{\Sigma}^{(0,0)}_{(k,k)}
\end{equation}
where the \( \hat{\Sigma}^{(0,0)}_{(k,k)} \)'s are one dimensional \( SO(4) \) irreducible modules corresponding to functions of bianalytic conformal spin \((k,k)\). The upper indices \((0,0)\) carried by the spaces figuring in eq(4) are special values of general indices \((p,q)\) to be introduced later on. The generators of \( \hat{\Sigma}^{(0,0)}_{(k,k)} \) are biperiodic arbitrary functions that we generally indicate by \( f(z,\omega) \). These are bianalytic functions expanded in Fourier series as
\begin{equation}
f(z,\omega) = \sum_{n,m \in \mathbb{Z}} f_{nm} z^n \omega^m, \quad \partial_{\bar{z}} f = \partial_{\bar{\omega}} f = 0
\end{equation}
where the constants \( f_{nm} \) are the Fourier modes of \( f \). This is nothing but a generalization of the usual Laurent expansion of conformal fields on the complex plane \( \mathbb{C} \). Note by the way that the integers \( n \) and \( m \) carried by the Fourier modes \( f_{nm} \) are nothing but the \( U(1) \times U(1) \) Cartan charges of the \( SO(4) \approx SU(2) \times SU(2) \) Lorentz group of the Euclidan space \( \mathbb{R}^4 \). Bianalytic
functions on $\mathbb{C}^2$ carrying $U(1) \times U(1)$ charges $r$ and $s$ as follows

$$f_{(r,s)}(z,\omega) = \sum_{n,m \in \mathbb{Z}} f_{nm} z^{n-r} \omega^{m-s}, \quad r, s \in \mathbb{Z},$$

such that the eq.(5) appears then as a particular example of eq(6). Note also that we can define the constants on $\mathcal{R}$ as been $f_{(0,0)}$ such that

$$\partial_z z f_{(0,0)} = 0 = \partial_\omega f_{(0,0)}.$$

The coefficients $f_{nm}$ are given by

$$f_{nm} = \oint_{c_1} \frac{dz}{2i\pi} z^{-n-l+s} \oint_{c_2} \frac{d\omega}{2i\pi} \omega^{-m-l+s} f_{(r,s)}(z,\omega),$$

where $c_1 \times c_2$ is the contour surrounding the singularity $(z,\omega) = (0,0)$ in the complex space.

3) The special subset $\hat{\Sigma}^{(0,0)}(k,k) \subset \mathcal{R}$ is generated by bianalytic functions $f_{(k,k)}$, $k \geq 2$. They can be thought of as the higher spin currents involved in the construction of the $W$-algebra on $T^2$ [18]. As an example, the following fields

$$W_{(2,2)} = u_{(2,2)}(z,\omega)$$
$$W_{(3,3)} = u_{(3,3)}(z,\omega) - \frac{1}{2} \{\log H, u_{(2,2)}\}$$

are shown to play the same role of the spin-2 and spin-3 conserved currents of the Zamolodchikov $W_3$ algebra [5, 6]. Next we will denote, for simplicity, the fields $u_{(k,k)}(z,\omega)$ of conformal spin $(k,k)$, $k \in \mathbb{Z}$ simply as $u_k(z,\omega)$. The function $\log(H)$ will be also described later.

4) The Poisson bracket on $T^2$ is defined as follows

$$\{f, g\} = \partial_z f \partial_\omega g - \partial_z g \partial_\omega f$$

with $\{z,\omega\} = 1$. We denote $\{f,\cdot\} = \xi_f$ and $\xi_f.g = \{f,\cdot\}.g = \{f, g\} + g\{f,\cdot\}$ equivalently this shows how the Poisson bracket on the torus can play the role of a derivation. For convenience we will adopt the following notation $\xi_H \equiv \xi_{\log H}$ as been the hamiltonian vector field operator associated to the arbitrary function $H$. This hamiltonian operator is shown, in the present context of $T^2$, to replace the standard derivation $\partial_z$ describing the basis of differential operators on the circle $S^1$. We have also

$$\{\log H,\cdot\} \circ f \equiv \xi_H.f = \{\log H, f\} + f \xi_H$$

showing a striking resemblance with action of $S^1$-differential operators ,

$$\partial f = f' + f\partial; \quad f' \equiv (\partial f)$$

Later on the differential operators that will be used are simply integer powers of $\xi_H$ subject to the following rule

$$[\xi_f, \xi_g] = \xi_{[f,g]}$$

Later on the differential operators that will be used are simply integer powers of $\xi_H$ subject to the following rule

$$[\xi_f, \xi_g] = \xi_{[f,g]}$$

Later on the differential operators that will be used are simply integer powers of $\xi_H$ subject to the following rule

$$[\xi_f, \xi_g] = \xi_{[f,g]}$$
5) We introduce new form of the differential operators associated to the symmetry of $T^2$. Before that we have to consider the following conventions notations [15]:

a) $\hat{\Sigma}^{(r,s)}$: This is the space of differential operators of conformal weight $(n,n)$ and degrees $(r,s)$ with $r \leq s$. Typical operators of this space are given by

$$\sum_{k=r}^{s} u_{n-k}(z,\omega)\xi^k_H$$

This is the analogue of $\text{diff}(S^1)$ operators

$$\sum_{k=r}^{s} u_{n-k}\partial^k$$

b) $\hat{\Sigma}^{(0,0)} \subset \mathcal{R}$: This space describes the conserved currents $u_n(z,\omega)$.

c) $\hat{\Sigma}^{(k,k)}$: Is the space of differential operators type,

$$u_{n-k}(z,\omega)\xi^k_H.$$  

(16)

d) The residue operation: $Res$

$$Res(f\xi^{-1}_H) = f.$$  

(17)

6. To summarize, we give here bellow a table containing the essential properties of the mathematical objects involved in $\text{diff}(T^2)$-higher conformal spin symmetries analysis

| Objects $\mathcal{O}$ | The conformal weight $|\mathcal{O}|$ |
|-----------------------|---------------------------------|
| $z, \omega, \partial_z, \partial_\omega$ | $|z| = (-1,0), |\omega| = (0,-1), |\partial_z| = (1,0), |\partial_\omega| = (0,1)$ |
| $L_{k,l}$ | $|L_{k,l}| = (-k,-l)$ |
| $T(z,\omega) \equiv W_2(z,\omega)$ | $|T(z,\omega)| = (2,2)$ |
| $W_s(z,\omega), s = 2, 3, ...$ | $|W_s(z,\omega)| = (s,s)$  

(18)

| $\{f,g\}^{(k)} = \{f,\underbrace{f, ..., f}_{k}, g\}$ | $|\{f,g\}^{(k)}| = (k,k) + k|f| + |g|$ |
| $\{f,g\}^{k} = \underbrace{\{f, g, ..., f, g\}}_{k}$ | $|\{f,g\}^{k}| = (k,k) + k|f| + k|g|$ |

| $\xi_H = \{\log H, \}$ | $|\xi_H| = (1,1)$ |
| $Res$ | $|Res| = (1,1)$ |
3 Higher conformal spin symmetries on $T^2$

3.1 The Frappat et al. conformal symmetry

This is the algebra generated by the infinite dimensional basis set of mode operators $L_{k,l}$, $k, l \in \mathbb{Z}$ satisfying the classical commutation relations (without central extension)

$$[L_{k,l}, L_{r,s}] = [m_0(k - r) - n_0(l - s)]L_{k+r,l+s}$$  \hspace{1cm} (19)

where $n_0$ and $m_0$ are two arbitrary parameters. The above algebra has the property of offering a unified definition of several extensions of the 2d conformal algebra. Here with some particular examples:

1. The Virasoro algebra

Setting for instance $m_0 = 1$ and $n_0 = l = s = 0$, we get the classical Virasoro algebra

$$[L_k, L_r] = L_{k+r} + \frac{c}{12}k(k - 1)\delta_{k+r,0}$$  \hspace{1cm} (20)

for vanishing central charge $c = 0$.

2. The Antoniadis et al algebra

Setting now $m_0 = s$ and $n_0 = r$ one recover the Antoniadis et al algebra given by

$$[L_{k,l}, L_{r,s}] = [ks - lr]L_{k+r,l+s}$$  \hspace{1cm} (21)

3.2 The $W_3$ generalization

Since the Frappat et al. algebra is relevant in describing geometrically special diffeomorphisms of the two dimensional torus $T^2$ and extend successfully the known symmetries namely the Virasoro and the Antoniadis ones, we focussed in previous works [18] to go beyond this extension and improve it much more. In fact the central charge as well as the $W_3$-extension of the Frappat et al. algebra on $T^2$ are built. The first important contribution of [18] consists in computing the missing central charge of the Frappat et al. algebra, we find

$$c(k, l, m_0, n_0) = \frac{c}{12} \prod_{j=0,\pm} [m_0(k - j) - n_0(l - j)]\delta_{k+r,0}\delta_{l+s}$$  \hspace{1cm} (22)

The second important contribution deals with the derivation of the $W_3$-extension of the Frappat et al. algebra. This is generated by the mode operators $L_{k,l}$ and $W_{k,l}$ and reads as

$$[L_{k,l}, W_{r,s}] = [m_0(2r - k) - n_0(2s - l)]W_{k+r,l+s}$$
$$[W_{k,l}, W_{r,s}] = [m_0(k - r) - n_0(l - s)](\alpha L_{k+r,l+s} + \beta \Lambda_{k+r,l+s})$$  \hspace{1cm} (23)

where $\beta = \frac{16}{22+5c}$ and $\alpha = \alpha_1 + \alpha_2$ with

$$\alpha_1 = \frac{1}{12}[m_0(k + r + 3) - n_0(l + s + 3)][m_0(k + r + 2) - n_0(l + s + 2)],$$
$$\alpha_2 = \frac{1}{6}[m_0(k + 2) - n_0(l + 2)][m_0(r + 2) - n_0(s + 2)],$$  \hspace{1cm} (24)
and where $\Lambda_{k+r,l+s}$ is a composite operator given by

$$\Lambda_{k+r,l+s} = \sum_{i,j \in \mathbb{Z}} (L_{i,j}L_{k+r-i,l+s-j} - \frac{3}{10} \alpha_1 L_{k+r,l+s})$$

(25)

The important remark at this level is that this derived \( \text{diff}(T^2) \) extended $W_3$-algebra contains the well known \( \text{diff}(S^1) \) Zamolodchikov $W_3$ algebra [5] as a special subalgebra obtained by setting $n_0 = 0$ and $m_0 = 1$. This new symmetry eqs.(23-25) which extend also the Antoniadis et al. algebra eq.(21) offers moreover a unified definition of several generalizations of the two dimensional $W_3$ algebra exactly as do the Frappat et al. algebra with respect to the Virasoro symmetry.

In the OPE language, we introduce a bianalytic conserved current $T(z,\omega)$ defined on the bi-complex space $\mathbb{C}^2$ parametrized by the complex variables $z$ and $\omega$ and their conjugates $\bar{z}$ and $\bar{\omega}$. The particular field $T(z,\omega)$, generalizing the 2d energy momentum tensor expands in Laurent series as

$$T(z,\omega) = \sum_{k,l \in \mathbb{Z}} z^{-k-2} \omega^{-l-2} L_{k,l}; \quad \partial_z T = \partial_\omega T = 0$$

(26)

or equivalently

$$L_{m,n} = \oint_{c_1} \frac{dz}{2\pi i} z^{m+1} \oint_{c_2} \frac{d\omega}{2\pi i} \omega^{n+1} T(z,\omega),$$

(27)

where $c_1 \times c_2$ is a contour surrounding the singularity $(z,\omega) = (0,0)$ in the complex space. The OPE analogue of the extended Virasoro algebra eqs.(19) and (22) is given by [18]

$$T(1)T(2) = \frac{c}{12} \left\{ \frac{\text{Log}H}{\omega_{12}^2} \right\}^3 + 2 \left\{ \frac{\text{Log}H}{\omega_{12}^2} \right\} T(2) + \{ \text{Log}H, T(2) \} \frac{1}{\omega_{12}^2}$$

(28)

where we have set $T(k) = T(z_k,\omega_k)$ for short and where $H = H(z,\omega)$ is an arbitrary bianalytic function in $\mathbb{C}^2$. A particular choice of this arbitrary function leading to eqs.(19) and (22) is given by $H(z,\omega) = z^{-n_0}\omega^{-m_0}$.

Note that the symbol $\{,\}$ appearing in eq.(28) is the usual Poisson bracket defined, for any pair of bianalytic functions $f(z,\omega)$ and $g(z,\omega)$, as

$$\{ f, g \} = \partial_z f \partial_\omega g - \partial_\omega f \partial_z g$$

(29)

with the following property

$$\{ f, g \}^3 = \{ f, \{ f, f, g \} \},$$

(30)

Note moreover that in establishing eqs.(19) and (22) from the OPE eq.(28) one finds the following expression

$$[L_{k,l}, L_{r,s}] = ((s-l) \frac{\partial \text{Log}H}{\partial \omega} - \frac{\partial \text{Log}H}{\partial \omega} (r-k))L_{k+r,l+s}$$

$$+ \frac{c}{12} \Pi_{j=0}^\infty ((k-j) \frac{\partial \text{Log}H}{\partial \omega} - (l-j) \frac{\partial \text{Log}H}{\partial \omega})$$

(31)
The $W_3$ extended symmetry is generated by a conserved current $W(z, \omega)$ of conformal weight $h_{z,\omega} = (3, 3)$. This is simply seen at the level of the energy-momentum tensor $T(z, \omega)$ who transforms under the bianalytic coordinate change in $\mathbb{C}^2$ as follows

$$z \rightarrow \tau(z, \omega), \quad \omega \rightarrow \sigma(z, \omega)$$

as follows

$$T(z, \omega) = (\partial_z \tau)^2 (\partial_\omega \sigma)^2 \tilde{T}(z, \omega) + \frac{c}{12} S(z, \omega, \tau, \sigma)$$

where $S(z, \omega, \tau, \sigma)$ is the Schwartzian derivative given by

$$S(z, \omega, \tau, \sigma) = \{ \log H, \log (\partial_z \tau \partial_\omega \sigma) \}^{(2)} - \frac{1}{2} \{ \{ \log H, \log (\partial_z \tau \partial_\omega \sigma) \} \}^2$$

One easily observe that the conformal current $T$ exhibits a conformal weight $h_{z,\omega} = (2, 2)$.

In the OPE language, the $W_3$ extension of the FRSTH algebra read in addition to the eq.(28) as

$$T(1)W(2) = 3 \left\{ \log H, \frac{1}{\omega_{12\bar{z}12}} \right\} W(2) + \{ \log H, W(2) \} \frac{1}{\omega_{12\bar{z}12}}$$

$$W(1)W(2) = \frac{c}{360} \left\{ \log H, \frac{1}{\omega_{12\bar{z}12}} \right\}^{(5)} + \frac{1}{3} \left\{ \log H, \frac{1}{\omega_{12\bar{z}12}} \right\}^{(3)} \tilde{T}(2)$$

$$+ \frac{1}{15 \omega_{12\bar{z}12}} \{ \log H, T(2) \}^{(3)} + \frac{1}{3} \{ \log H, \frac{1}{\omega_{12\bar{z}12}} \}^{(2)} \{ \log H, T(2) \}$$

$$+ 2 \beta \left\{ \log H, \frac{1}{\omega_{12\bar{z}12}} \right\} \Lambda(2) + \beta \{ \log H, \Lambda(2) \} \frac{1}{\omega_{12\bar{z}12}},$$

where

$$W(z, \omega) = \sum_{k,l \in \mathbb{Z}} z^{-k} \omega^{-l} W_{k,l};$$

$$\Lambda(z, \omega) = \sum_{k,l \in \mathbb{Z}} z^{-k} \omega^{-l} \Lambda_{k,l};$$

### 3.3 Diff($T^2$) Conformal transformations

Consider an arbitrary finite and bianalytic coordinate change:

$$z \rightarrow \tilde{z} = \sigma(z, \omega)$$

$$\omega \rightarrow \tilde{\omega} = \tau(z, \omega)$$

Under this symmetry, the functions $f_{(r,s)}(z, \omega)$ transform as:

$$f_{(r,s)}(z, \omega) = (\partial_\tau \sigma)^r (\partial_\omega \tau)^s \tilde{f}_{(r,s)}(\sigma, \tau)$$

For an infinitesimal variation

$$\tilde{z} = z - B$$

$$\tilde{\omega} = \omega + A$$

where $A(z, \omega)$ and $B(z, \omega)$ are two arbitrary bianalytic functions of the $U(1) \times U(1)$ charges $(0, -1)$ and $(-1, 0)$ the same as $\omega$ and $z$ respectively, we have

$$\delta f_{(r,s)}(z, \omega) = V(A, B) f_{(r,s)} + (s \partial_\omega A - r \partial_z B) f_{(r,s)}$$

where the vector field $V(A, B) = A \partial_\omega - B \partial_z$, obeys the Lie algebra of diffeomorphisms diff($T^2$) namely

$$[V(A_1, B_1), V(A_2, B_2)] = V(A_1 B_2 - A_2 B_1)$$
with
\[ A = (A_1 \partial_\omega A_2 - B_1 \partial_z B_2) - (1 \leftrightarrow 2) \]
\[ B = (A_1 \partial_\omega B_2 - B_1 \partial_z A_2) - (1 \leftrightarrow 2) \] (42)

In the case of \( \text{diff}(S^1) \), corresponding to set \( A = 0 \) and \( B \neq 0 \) (resp. \( B = 0 \) and \( A \neq 0 \)) the functions \( f_{(r,s)} \) behaves as a two dimensional conformal objects of weight \( h = r \) (resp. \( h = s \)). However the situation, in which both \( A \) and \( B \) are non vanishing independent bianalytic functions, leads to see \( f_{(r,s)} \) as a conformal field of \( \text{diff}(S^1) \times \text{diff}(S^1) \). Note however that \( \text{diff}(T^2) \) is a special case of \( \text{diff}(S^1) \times \text{diff}(S^1) \) corresponding for example to set \( r = s \) at the level of eq(40) and choose the functions \( A \) and \( B \) to transform as
\[ A = \tilde{A}(\partial H) \]
\[ B = \tilde{A}(\partial H) \] (43)

where \( \tilde{A} \) and \( H(z,\omega) \) are respectively arbitrary bianalytic functions of \( U(1) \times U(1) \) charges \((-1,-1)\) and \((n_0,m_0)\) with \( n_0, m_0 \in \mathbb{Z} \). Note by the way that setting \( \tilde{A} = H \), one discovers the area preserving diffeomorphisms algebra of the torus studied in [17, 20]. Further, by imposing the constraint
\[ \partial_\omega H = 0 \quad \text{but} \quad \partial_z H \neq 0, \] (44)
and vice versa \( \partial_\omega H \neq 0 \) but \( \partial_z H = 0 \), the equations reduce to a conformal variation of one of the two \( \text{Diff}(S^1) \) subgroups of \( \text{diff}(T^2) \). We shall consider hereafter the case \( (\partial_z H)(\partial_\omega H) \neq 0 \). With the choice eq(40), eq.(43) takes the following remarkable form :
\[ \delta f_r = \tilde{A}\{\log H, f_r\} + r\{\log H, \tilde{A}\} f_r, \] (45)
where we have set \( f_r = f_{(r,r)} \) for short and where we have used the Poisson bracket defined as:
\[ \{\log H, G\} = (\partial_\omega \log H)\partial_\omega G - (\partial_\omega \log H)\partial_z G. \] (46)

It is interesting to note here that eq.(45) exhibits a striking resemblance with the conformal transformation of a two-dimensional primary field of conformal weight \( h = r \). This is why we refer to the set \( \mathcal{R} \) of bianalytic functions transforming like eq.(45) as been the ring of conformal fields on the torus \( T^2 \). Elements of this ring are then primary fields on \( C^2 \) defining highest weight representations of the conformal algebra on the torus.

4 Higher spin symmetries from the Gelfand-Dickey analysis

We describe here the basic features of the algebra of arbitrary differential operators, refereed hereafter to as \( \hat{\Sigma} \), acting on the ring \( \mathcal{R} \) of analytic functions on \( T^2 \). We show in particular that any such differential operator is completely specified by a conformal weight\(^2\) \((n,n)\), \( n \in \mathbb{Z} \), two integers \( r \) and \( s \) with \( s = r + i, i \geq 0 \) defining the lowest and the highest degrees, respectively,

\(^2\)For a matter of simplicity we denote objects \( X_{(n,n)} \) of conformal weight \( |X_{(n,n)}| = (n,n) \) simply as \( X_n \)
and finally \((1 + r - s) = i + 1\) analytic fields \(u_j(z, \omega)\). Its obtained by summing over all the allowed values of spin (conformal weight) and degrees in the following way:

\[
\hat{\Sigma} = \oplus_{r \leq s} \oplus_{n \in \mathbb{Z}} \hat{\Sigma}^{(r,s)}_n.
\]

(47)

with \(\hat{\Sigma}^{(n,n)}_n \equiv \hat{\Sigma}_n\). Note that the space \(\hat{\Sigma}\) is an infinite dimensional algebra which is closed under the Lie bracket without any condition. A remarkable property of this space is the possibility to introduce six infinite dimensional classes of sub-algebras related to each other by special duality relations. These classes of algebras are given by \(\hat{\Sigma}^\pm_s\), with \(s = 0, +, -\) describing respectively the different values of the conformal spin which can be zero, positive or negative. The \(\pm\) upper indices stand for the sign (positive or negative) of the degrees quantum numbers, for more details see [15].

4.1 Setup of the GD integrable analysis on \(T^2\)

4.1.1 The space \(\hat{\Sigma}^{(r,s)}_n\)

To start let’s precise that the space \(\hat{\Sigma}^{(r,s)}_n\) contains differential operators of fixed conformal spin \((n, n)\) and degrees \((r, s)\), type

\[
L^{(r,s)}_n (u) = \sum_{i=r}^{s} u_{n-i}(z, \omega) \circ \xi^i_H,
\]

(48)

These are \(\xi_H\)’s polynomial differential operators extending the hamiltonian field \(\xi_H = \{logH, \}\). Elements \(L^{(r,s)}_n (u)\) of \(\hat{\Sigma}^{(r,s)}_n\) are a generalization to \(T^2\) of the well known 2\textsuperscript{nd} order Lax differential operator \(\partial^2_z + u_2(z)\) involved in the analysis of the so-called KdV hierarchies and in 2\textsuperscript{d} integrable systems on the circle \(S^1\). The second order example of eq.(48) reads as

\[
L_2 = \xi^2_H + u_2,
\]

(49)

and is suspected to describe, in a natural way, the analogue of the KdV equation on \(T^2\). Moreover, eq.(48) which is well defined for \(s \geq r \geq 0\) may be extended to negative integers by introducing pseudo-differential operators of the type \(\xi^{-k}_H\), \(k > 1\), whose action on the fields \(u_s(z, \omega)\) is given by the Leibnitz rule. It is now important to precise how the operators \(L^{(r,s)}_n (u)\) act on arbitrary functions \(f(z, \omega)\) via the hamiltonian operators \(\xi^k_H\).

Striking resemblance with the standard case [14, 15] leads us to write the following Leibnitz rules for the local and non local differential operators in \(\xi_H\)

\[
\xi^n_H f(z, \omega) = \sum_{s=0}^{n} \epsilon^{s}_{n} \{logH, f\}^{(s)} \xi^{n-s}_H,
\]

(50)

and

\[
\xi^{-n}_H f(z, \omega) = \sum_{s=0}^{\infty} (-)^{s} \epsilon^{s}_{n+s-1} \{logH, f\}^{(s)} \xi^{n-s}_H
\]

(51)
where the $k^{th}$-order derivative $\{\log H, f\}^{(k)} = \{\log H, \{\log H, \ldots \{\log H, f\}\ldots\}\}$ on the torus $T^2$ is the analogue of $f^{(k)} = \frac{\partial^k f}{\partial z^k}$, the $k$th derivative of $f$ in the standard case. Special examples are given by

$$
\begin{align}
\xi^0_H \circ f &= \{\log H, f\} + f \xi_H \\
\xi^2_H \circ f &= \{\log H, f\}^{(2)} + 2\{\log H, f\} \xi_H + f \xi_H^2 \\
\xi^3_H \circ f &= \{\log H, f\}^{(3)} + 3\{\log H, f\}^{(2)} \xi_H + 3\{\log H, f\} \xi_H^2 + f \xi_H^3 \\
\xi^4_H \circ f &= \{\log H, f\}^{(4)} + 4\{\log H, f\}^{(3)} \xi_H + 6\{\log H, f\}^{(2)} \xi_H^2 + 4\{\log H, f\} \xi_H^3 + f \xi_H^4 \\
\xi^{−1}_H \circ f &= f \xi^{−1}_H - \{\log H, f\} \xi^{−2} + \{\log H, f\}^{(2)} \xi^{−3} - \{\log H, f\}^{(3)} \xi^{−4} + ...
\end{align}
$$

As can be checked by using the Leibnitz rule, one have the expected property

$$
\xi^n_H \circ \xi^{−n}_H f(z, \omega) = f(z, \omega) \tag{54}
$$

A natural representation basis of non linear pseudo-differential operators of spin $n$ reads as

$$
\mathcal{P}^{(p,q)}_n[u] = \sum_{i=p}^{q} u_{n-i}(z, \omega) \xi_H^i, \quad p \leq q \leq −1 \tag{55}
$$

This configuration, which is a direct extension of the local Lax operators $\mathcal{L}^{(r,s)}_n(u)$ eq.(48), describes nonlocal differential operators. Later on, we will use another representation of pseudodifferential operators, namely, the Volterra representation. The latter is convenient in the derivation of the Gelfand–Dickey (second hamiltonian structure) of higher conformal spin integrable systems. Note by the way that the non local Leibnitz rule eq.(51) is a special example of the Volterra pseudo-operators as we will show in the forthcoming sections.

### 4.1.2 The algebra of differential operators $\hat{\Sigma}$

This is the algebra of differential operators of arbitrary spins and arbitrary degrees. It’s obtained from $\hat{\Sigma}^{(r,s)}_n$ by summing over all allowed degrees ($r, s$) and conformal weight (spin) $n$ in the following way

$$
\hat{\Sigma} = \bigoplus_{r \leq s} \bigoplus_{n \in \mathbb{Z}} \hat{\Sigma}^{(r,s)}_n \tag{56}
$$

$\hat{\Sigma}$ is an infinite dimensional algebra which is closed under the Lie bracket without any condition. A remarkable property of this space is the possibility to split it into six infinite dimensional
classes of sub-algebras given by \( \hat{\Sigma}^+_s \), with \( s = 0, +, - \). These classes of algebras are describing respectively the different values of the conformal spin which can be zero, positive or negative. They are related to each other by conjugation of the spin and degrees. Indeed, given two integers \( s \geq r \), it’s not difficult to see that the spaces \( \hat{\Sigma}^{(r,s)} \) and \( \hat{\Sigma}^{(-1-r,-1-s)} \) are dual with respect to the pairing product \((,\cdot)\) defined as

\[
(\mathcal{L}^{(r,s)}, \mathcal{P}^{(p,q)}) = \delta_{1+r+q,0} \delta_{1+s+p,0} \text{Res}[\mathcal{L}^{(r,s)} \circ \mathcal{P}^{(p,q)}]
\]

where the residue operation \( \text{Res} \) is given by

\[
\text{Res}[u \xi^j_H] = u_{-1}(z, \omega)
\]

As signaled previously, the residue operation \( \text{Res} \) carries a conformal weight \((-1, -1)\). Note by the way that the \( u_j \)'s currents should satisfy a conformal spin’s duality with respect the following "scalar" product

\[
\langle u_k, u_l \rangle = \delta_{k+l,1} \int dzd\omega u_{1-k}(z, \omega) u_k(z, \omega)
\]

By virtue of this product the one dimensional spaces \( \hat{\Sigma}^{(0,0)}_k \) and \( \hat{\Sigma}^{(0,0)}_{1-k} \) are dual to each other. This leads then to a splitting of the tensor algebra \( \hat{\Sigma}^{(0,0)} \) into two semi-infinite tensor subalgebras \( \hat{\Sigma}^{(0,0)}_+ \) and \( \hat{\Sigma}^{(0,0)}_- \), respectively characterized by positive and negative conformal weights as shown here below

\[
\hat{\Sigma}^{(0,0)}_+ = \bigoplus_{k>0} \hat{\Sigma}^{(0,0)}_k
\]

\[
\hat{\Sigma}^{(0,0)}_- = \bigoplus_{k>0} \hat{\Sigma}^{(0,0)}_{1-k}
\]

We learn in particular that \( \hat{\Sigma}^{(0,0)}_0 \) is the dual of \( \hat{\Sigma}^{(0,0)}_{-1} \). The previous properties shows that the operation \( \langle, \rangle \) carries a conformal weight \( |\langle, \rangle| = (-1, -1) \), consequently this is not a convenient Lorentz scalar product. We need then to introduce a combined Lorentz scalar product \( \langle\langle, \rangle\rangle \) built out of \( \langle, \rangle \) eq.(59) and the pairing product \((,\cdot)\) eq(57) such that \( |\langle\langle, \rangle\rangle| = (0, 0) \) as follows

\[
\langle\langle D^{(r,s)}_m, D^{(p,q)}_n \rangle\rangle = \delta_{n+m,0} \delta_{1+r+q,0} \delta_{1+s+p,0} \int dz \text{Res}[D^{(r,s)}_m \times D^{(-1-s,-1-r)}_-]
\]

With respect to this combined scalar product, one sees easily that the subspaces \( \hat{\Sigma}^{++}, \hat{\Sigma}^{0+} \) and \( \hat{\Sigma}^{+-} \) are dual to \( \hat{\Sigma}^{--}, \hat{\Sigma}^{0-} \) and \( \hat{\Sigma}^{+-} \) respectively. The symbol \( \hat{\Sigma}^{+-} \) correspond for example to Lax operators on \( T^2 \) with positive conformal spin and negative degrees while \( \hat{\Sigma}^{0+} \) corresponds to Lorentz scalar operators of positive degrees. We conclude this section by making the following remarks:

1. \( \hat{\Sigma}^{++} \) is the space of local differential operators with positive definite conformal weight and positive degrees.
2. \( \hat{\Sigma}^{--} \) is the Lie algebra of non local differential operators with negative definite conformal
weight and negative degrees.

3. These two subalgebras $\hat{\Sigma}_{\pm\pm}$ are used to build the Gelfand–Dickey second hamiltonian structure of integrable systems on the Torus $T^2$.

4.2 $W_n$ symmetries from the $\Sigma_{++} \oplus \Sigma_{--}$ algebras on $T^2$

4.2.1 The $\Sigma_{--}$ algebra in the Volterra basis

The Lie algebra $\Sigma_{--}$ discussed previously is in fact isomorphic to the maximal algebra of arbitrary negative definite conformal spin and pure non local pseudo-differential operators $\Sigma^{(-\infty,-1)}$. In the Volterra representation a generic elements of this algebra is given by

$$\Gamma^{(-\infty,-1)}(v) = \sum_{m=0}^{\infty} \alpha(m)\Gamma_{m}^{(-\infty,-1)}(v),$$

(62)

where only a finite number of the coefficients $\alpha(m)$ is non vanishing and where

$$\Gamma_{-m}^{(-\infty,-1)}[v] = \sum_{j=1}^{\infty} \xi_{-j}^{H} \circ v_{j-m}(z,\omega)$$

(63)

More generally, given two integers $r$ and $s$ with $r \geq s \geq 0$, one can built new kinds of coset subalgebras $\Sigma^{(-r,-s)}$ whose generators basis are given in the Volterra representation as

$$\Gamma_{-m}^{(-r,-s)}[v] = \sum_{j=s}^{r} \xi_{-j}^{H} \circ v_{j-m}(z,\omega)$$

(64)

The number of independent fields $v_j(z,\omega)$ involved in the above relation is equal to $(1+r-s)$ and corresponds then to the dimension of $\hat{\Sigma}^{(-r,-s)}$. Note moreover that with respect to the combined scalar product $<<,>>$ eq.(61), the subspace $\Sigma^{(-r,-s)}$ is nothing but the dual of $\Sigma^{(s+1,r+1)}$.

Now, given two differential operators $L_{m}^{(p,q)}[u]$ and $\Gamma_{-n}^{(-1-q,-1-p)}[v]$ belonging to $\hat{\Sigma}^{(p,q)}_{m}$ and $\hat{\Sigma}^{(-1-q,-1-p)}_{-n}$ respectively, their residue pairing reads as

$$\text{Res}[L_{m}^{(p,q)} \circ \Gamma_{-n}^{(-1-q,-1-p)}] = \sum_{i=p}^{q} u_{m-i}(z,\omega)v_{i+1-n}(z,\omega)$$

(65)

The conformal weight of the r.h.s. of the previous equation is $(1+m-n, 1+m-n)$ where the values $m$ and $(-n)$ are the contributions of the operators $L_{m}^{(p,q)}[u]$ and $\Gamma_{-n}^{(-1-q,-1-p)}[v]$.

4.2.2 The Gelfand-Dickey algebra on $T^2$

We show in this section how one can derive the algebra of higher spin currents from the so-called Gelfand-Dickey algebra on the torus $T^2$. This can be done following the same lines of the standard $\mathfrak{sl}_n$-GD algebra on the circle $S^1$ [14, 15]. The first steps concerns the use of the algebra $\Sigma_{++}$ with the particular differential operators

$$L_{n}^{(0,n)}[u] = \sum_{i=0}^{n} u_{n-i}\xi_{H}^{i}$$

(66)
We consider then the \((n + 1)\) fields \(u_j(z, \omega)\) of conformal weight \((j, j)\), \(0 \leq j \leq n\), involved in this equation as the coordinates of a \((n + 1)\) dimensional manifold \(\mathcal{M}^{n+1}\). Let’s denote by \(\mathcal{F}\) the space of differentiable functions on \(\mathcal{M}^{n+1}\) of arbitrary positive conformal spin. We have

\[
\mathcal{F} = \bigoplus_{k \in \mathbb{N}} \mathcal{F}_k
\]

(67)

Note that elements \(F_k\) of \(\mathcal{F}_k\), for \(k\) a positive integer, depend on the \(u_j(z, \omega)\)’s and carry in general a conformal spin index \(k\)

\[
F_k[u] = F_k[u_0(z, \omega), ..., u_n(z, \omega)]
\]

(68)

The functional variation of these objects reads as

\[
\delta F_k[u(z, \omega)] = \int dz' d\omega' \sum_{j=0}^{n} \{ \delta u_j(z', \omega') \frac{\delta F_k[u(z, \omega)]}{\delta u_j(z', \omega')} \}
\]

(69)

A special example consists in setting \(F_k[u(z, \omega)] = u_k(z, \omega)\), with \(0 \leq k \leq n\), we get

\[
\delta u_k(z, \omega) = \int dz' d\omega' \sum_{j=0}^{n} \{ \delta_{jk} \delta(z - z') \delta(\omega - \omega') \delta u_j(z', \omega') \}
\]

(70)

showing among other that

\[
\frac{\delta F_k[u(z, \omega)]}{\delta u_j(z', \omega')} \]

(71)

behaves as a \((k + 1 - j, k + 1 - j)\) conformal weight object. Since we are focusing to derive \(W_n\) symmetries with \((n - 1)\) independent conserved currents on the torus \(T^2\) associated formally to an \(sl_n\)-Lie algebra, one have to impose strong constraints on the spin-(0, 0) and spin-(1, 1) fields \(u_0\) and \(u_1\). This leads then to a reduction of the \((n + 1)\) dimensional manifold \(\mathcal{M}^{(n+1)}\) to an \((n - 1)\)-dimensional submanifold \(\mathcal{M}^{(n-1)}\). Consistency gives

\[
u_0 = 1, \quad u_1 = 0
\]

(72)

The Lax operator eq.(66) reduces then to

\[
\mathcal{L}_{n}^{(0,n)}[u] = \xi_H^n + \sum_{i=0}^{n-2} u_{n-i} \xi_H^i
\]

(73)

Consider next the residue dual of the Lax differential operator eq.(73). This is a pseudo-differential operator which has degrees \((-1, n, 1)\) but unfixed conformal spin, say \((k - n, k)\) an integer. These pseudo-operators reads in terms of the functions \(F_k[u(z, \omega)]\) as

\[
\Gamma_{k-n}^{(-1, n-1)}[F] = \sum_{j=1}^{n+1} \xi_H^{-j} \circ v_{k+j-n}(z, \omega)
\]

(74)

where the \(v_{k+j-n}(z, \omega)\)’s with \(1 \leq j \leq n - 1\) are realized as

\[
v_{k+j-n}(z, \omega) = \frac{\delta F_k[u(z, \omega)]}{\delta u_{2+n-j}(z', \omega')}
\]

(75)
Note that the fields $v_{k+1}$ and $v_k$ define the residue conjugates of $u_0$ and $u_1$ respectively. The explicit determination of the pair of $v$'s fields requires the solving of the following constraint equations

\[ \text{Res}[\mathcal{L}_n^{(0,n)} \circ \Gamma_{k-n}^{(-1-n,-1)}] = \sum_{i=0}^{n-2} u_{n-i}(z,\omega)v_{i+k+1-n}(z,\omega) \]

(76)

\[ \text{Res}[\mathcal{L}_n^{(0,n)} \circ \Gamma_{1-n}^{(-1-n,-1)}] = 0 \]

Algebraic computations lead to

\[ v_{k+1} = 0 \]

(77)

\[ v_k = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=0}^{n-2} \mathcal{H}(j-i-1)(-1)^{j+1}c_j^i[u_{n-j} \times \frac{\delta F_{n,i-j}}{\delta u_{n-i}}]^{(j-i-1)} \]

as a solution of eqs.(76) respectively, where $\mathcal{H}(j)$ is the Heaviside function defined as

\[ \mathcal{H}(j) = \begin{cases} 1, & \text{if } j \geq 0 \\ 0, & \text{elsewhere} \end{cases} \]

(78)

Now given two functionals $F_k[u]$ and $G_l[u]$ depending on the currents $u_0, \ldots, u_n$ with $u_0 = 1$ and $u_1 = 0$ as well as corresponding pseudo-differential operators in the Volterra representation $\Gamma_{k-n}^{(-1-n,-1)}[F]$ and $\Gamma_{1-n}^{(-1-n,-1)}[G]$. With these data, the Gelfand-Dickey bracket reproducing the second hamiltonian structure of integrable systems is given as follows

\[ \{F_k[1], G_l[2]\}_{\mathcal{L}_n^{(0,n)}} = \int dzd\omega \text{Res}[v_{k+n}^{(0,2n-1)}(\mathcal{L},\Gamma) \circ \Gamma_{1-n}^{(-1-n,-1)}] \]

(79)

where we have used the following notation $F_k[u(z_1,\omega_1)] \equiv F_k[1]$ and $G_l[u(z_2,\omega_2)] \equiv G_l[2]$ with the definition

\[ V_{k+n}^{(0,2n-1)}(\mathcal{L},\Gamma) = \mathcal{L}_n^{(0,n)} \circ (\Gamma_{k-n}^{(-1-n,-1)} \circ \mathcal{L}_n^{(0,n)}+ - (\mathcal{L}_n^{(0,n)} \circ \Gamma_{k-n}^{(-1-n,-1)}+ \circ \mathcal{L}_n^{(0,n)} \]

(80)

As usual, the suffix $(\cdot)$ stands for the restriction to the local part of the considered operation (positive powers of $\xi_i^j$). One shows after a straightforward, but lengthy computations, that the previous Gelfand-Dickey bracket reduces to the following important form

\[ \{F_k[1], G_l[2]\}_{GD} = \sum_{i,j=0}^{n-2} \int dzd\omega \left[ \frac{\delta F_k[1]}{\delta u_{n-i}(z',\omega')} \mathcal{D}(n,i,j,z',\omega',u) \frac{\delta G_l[2]}{\delta u_{n-j}(z',\omega')} \right] \]

(81)

where the operator $\mathcal{D}(n,i,j,z',\omega',u)$ is a nonlinear local differential operator of conformal weight $|\mathcal{D}| = (2n - 1 - i - j, 2n - 1 - i - j)$. The knowledge of this operator is a central steps towards computing the Gelfand-Dickey bracket. It’s explicit determination is a tedious work and we will avoid the presentation of all our calculus and restrict our self to the global forms leading to the extended higher conformal spins symmetries on the torus $T^2$.

First of all, we have to underline the important situation for which we can set $F_k = u_k$ and
$G_l = u_l$ with $2 \leq k, l \leq n$. The basic Gelfand-Dickey brackets associated to the higher spin symmetries (conformal and W-extensions) expressed in terms of the canonical fields $u_j, 2 \leq j \leq n$ are given as follows

\[
\{u_k[1], u_l[2]\}_{GD} = \mathcal{D}(n, n - k, n - l, u)\delta(z_1 - z_2)\delta(\omega_1 - \omega_2) \tag{82}
\]

It’s easily seen that $\mathcal{D}(n, n - k, n - l, u)$ is a nonlinear differential operator of conformal weight $|\mathcal{D}(n, n - k, n - l, u)| = (k + l - 1, k + l - 1)$. The particular examples that we are interested in concern the conformal and $W_3$ symmetries associated respectively to the restriction of the order of the Lax operators to $n = 2$ and $3$. For $n = 2$, we have $k = l = 2$, the unique Poisson bracket corresponding the the classical version of the conformal symmetry in $T^2$ is given by

\[
\{u_2[1], u_2[2]\}_{GD} = \mathcal{D}(2, 0, 0, u)\delta(z_1 - z_2)\delta(\omega_1 - \omega_2) \tag{83}
\]

and the differential operator $\mathcal{D}(2, 0, 0, u)$ takes the following form

\[
\mathcal{D}(2, 0, 0, u) = \frac{1}{2} \xi_H^2 + 2u_2(z, \omega)\xi_H + \{\log H, u_2\} \tag{84}
\]

This is a third order differential operator extending the one appearing in the classical version of the Virasoro symmetry. For the nearest values of $z$ and $z'$, the conformal algebra eqs.(83-84) reads as

\[
\{u_2[1], u_2[2]\}_{GD} = \{u_2(z, \omega) + u_2(z', \omega')\}\{\log H, \delta(z - z')\delta(\omega - \omega')\} + \frac{1}{2}\{\log H, \delta(z - z')\delta(\omega - \omega')\}\tag{85}
\]

where \(\{\log H, \delta(z - z')\delta(\omega - \omega')\}\) is the third order derivation of \(\delta(z - z')\delta(\omega - \omega')\). eq.(85) corresponds then to the classical version of the Virasoro algebra on $T^2$. Remark that the central charge corresponding to this algebra is $c = 6$, this is inherited from the simple choice of $u_0 = 1$. If one choices the Lax operator to be

\[
\mathcal{L}_{n}^{(0,n)}[u] = \frac{c}{6} \xi_H^n + \sum_{i=0}^{n-2} u_{n-i} \xi_H^i, \quad \text{with} \quad u_0 = \frac{c}{6} \tag{86}
\]

in this case the bianalytic Virasoro algebra becomes

\[
\{u_2[1], u_2[2]\} = \{u_2 + u_2'\}\{\log H, \delta(z - z')\delta(\omega - \omega')\} + \frac{c}{12}\{\log H, \delta(z - z')\delta(\omega - \omega')\}\tag{87}
\]

Similarly, the $W_3$-extension of the Frappat et al. symmetry on $T^2$ is obtained from the generalized Gelfand-Dickey bracket eq(82) and gives for $n = 3$ three brackets associated to the fields $u_2$ and $u_3$ as follows

\[
\{u_2[1], u_2[2]\}_{GD} = \mathcal{D}(3, 1, 1, u)\delta(z_1 - z_2)\delta(\omega_1 - \omega_2)
\]
\[
\{u_2[1], u_3[2]\}_{GD} = \mathcal{D}(3, 1, 0, u)\delta(z_1 - z_2)\delta(\omega_1 - \omega_2) \tag{88}
\]
\[
\{u_3[1], u_3[2]\}_{GD} = \mathcal{D}(3, 0, 0, u)\delta(z_1 - z_2)\delta(\omega_1 - \omega_2)
\]
where

\[ D(3, 1, 1, u) = 2\xi_H^3 + 2u_2\xi_H + \{\log H, u_2\} \]

\[ D(3, 1, 0, u) = -\xi_H^4 - u_2\xi_H^3 + (3u_3 - 2\{\log H, u_2\})\xi_H + 2\{\log H, u_3\} - \{\log H, u_2\}^{(2)} \]

\[ D(3, 0, 0, u) = 2\frac{2}{3}\xi_H^5 + \frac{4}{3}u_2\xi_H^3 - 3\{\log H, u_2\}\xi_H^2 + (2\{\log H, u_3\} - 2\{\log H, u_2\}^{(2)} - \frac{2}{3}u_2u_2)\xi_H \]

\[ + \{\log H, u_2\}^{(2)} - \frac{2}{3}\{\log H, u_2\}^{(3)} - \frac{2}{3}u_2\{\log H, u_2\}\]

(89)
5 Concluding Remarks

We presented in this paper some important aspects of higher conformal spin symmetries on the bidimensional torus $T^2$. These symmetries, generalizing the Frappat et al. conformal symmetries by adding currents of conformal spin 3 in a non standard way, are also shown to be derived, in their semi-classical form, from the Gelfand-Dickey bracket that we are computing explicitly.

We underline at this level that the obtained $W_3$ extension of the Frappat et al. conformal symmetry on $T^2$ exhibits many remarkable features. The first one concerns the introduction of new kind of derivation inherited from $\text{diff}(T^2)$ and that takes the following form

$$\xi_H \equiv \{ \log H, . \} = \partial_z \log H \partial_\omega - \partial_\omega \log H \partial_z$$

for arbitrary bianalytic function $H(z,\omega)$. These logarithmic derivatives, corresponding to Hamiltonian vector fields, are central in the present study as they are assuring a consistent description of $\text{diff}(T^2)$ symmetries. Besides it’s $\text{diff}(T^2)$ invariance origin, the vector field $\xi_H$ is very useful as it can join in a compact form the bi-complex derivatives $\partial_z$ and $\partial_\omega$. Note that for functions $H$ living on a $n$-dimensional torus $T^n$, the logarithmic derivative can be written as

$$\{ \log H, \} = \sum_{i,j=1}^{n} \Omega_{ij} \partial_i \log H \partial_j$$  \hspace{1cm} (90)

where $H = H(z_1, ..., z_n, \omega_1, ..., \omega_n)$ and $\Omega_{ij} = - \Omega_{ji}$ is the usual $n \times n$ antisymmetric matrix.

The second crucial remark concerning the derived Gelfand-Dickey algebra is that the $n$th order local differential operator used is of the form

$$\mathcal{L}_n^{(0,n)}[u] = \xi_H^n + \sum_{i=0}^{n-2} u_{n-i} \xi_H^i$$  \hspace{1cm} (91)

This is a Lax operator belongings formally to the $A$-series of simple Lie algebra and having $(n-1)$ coordinates functions $\{u_k, k = 2, 3, \ldots n\}$, where we have set $u_0 = 1$ and $u_1 = 0$. This is a natural generalization of the well known differential $sl_2$-Lax operator $\mathcal{L}_{KdV} = \xi_H^2 + u_2$ associated to the $\text{diff}(T^2)$-KdV integrable hierarchy that we will discuss later [22].

We have to underline that the $sl_n$-Lax operators play a central role in the study of integrable models and more particularly in deriving higher conformal spin algebras ($W_k$-algebras) from the extended Gelfand-Dickey second Hamiltonian structure. Since they are also important in recovering 2d conformal field theories via the Miura transformation, we are convinced about the possibility to extend this property, in a natural way, to $\text{diff}(T^2)$ and consider the $T^2$-analogues of the well known 2d conformal models namely: the $sl_2$-Liouville field theory and its $sl_n$-Toda extensions and also the Wess-Zumino-Novikov-Witten conformal model. For instance, consider the KdV Lax operator that we can write by virtue of the Miura transformation as

$$\mathcal{L}_{KdV} = \xi_H^2 + u_2(z,\omega) = (\xi_H + \{ \log H, \phi \}) \times (\xi_H - \{ \log H, \phi \})$$  \hspace{1cm} (92)
where Φ is a Lorentz scalar field. As a result we have

\[ u_2 = -\{\log H, \phi\}^{(2)} - \{\log H, \phi\}^2 \]  

which should describe the classical version of the stress energy momentum tensor of conformal field theory on the torus \( T^2 \). Using \( C \times C \) bicomplex coordinates language, we can write

\[ T(z, \omega) \equiv u_2(z, \omega) = -\{\log H, \phi\}^{(2)} - \{\log H, \phi\}^2 \]  

where \( \{\log H, \phi\}^{(2)} \) is the second order derivative of the Lorentz scalar field \( \phi \) with respect to the \( T^2 \) symmetry. The conservation for this bianalytic conformal current \( T(z, \omega) \), leads to write the following differential equation

\[ \{\log K, \{\log H, \phi\}\} = e^{2\phi} \]  

where \( \tilde{K} = K(\tilde{z}, \tilde{\omega}) \) is an arbitrary bianalytic function of \( \tilde{z} \) and \( \tilde{\omega} \) carrying in general an \((\tilde{n}_0, \tilde{m}_0)\) \( U(1) \times U(1) \) charge. Note also that \( \tilde{K} \) is not necessarily the complex conjugate of the function \( H \) considered earlier. Our experience with conformal field theory and integrable systems leads to conclude that the later "second order" differential equation is nothing but the conformal Liouville like equation of motion on the Torus \( T^2 \). This equation of motion is known to appear in this context as a compatibility relation with the conservation of the stress energy momentum tensor \( T(z, \omega) \) namely \( \{\log K, T(z, \omega)\} = 0 \) or equivalently \( \{\log \tilde{K}, \{\log H, \phi\}\} + \frac{m^2}{\beta} \sum_{i=1}^{n-1} \exp(\beta \alpha_i \phi) = 0 \) (97)

We end this discussion by noting the possibility to interpret the obtained \( \text{diff}(T^2) \)-Gelfand-Dickey symmetries eqs() corresponding to the \( n \)-th order Lax differential operator eq.(91) as been the classical form of the \( W_n \) symmetries behind the \( \{sl_n\} \)-Toda like conformal field theory, generalizing the \( \text{diff}(S^1) \) Toda field theory [23] and whose action is assumed to have the following form

\[ S[\phi] = \int d^2 z d^2 \omega \left( \frac{1}{2} \{\log H, \phi\}, \{\log \tilde{K}, \phi\} - \left( \frac{m^2}{\beta} \right)^2 \sum_{i=1}^{n-1} \exp(\beta \alpha_i \phi) \right) \]  

In this bosonic Toda field theory on \( T^2 \), the Cartan subalgebra-valued scalar field \( \phi \) is given by \( \phi = \sum_{i=1}^{n-1} \alpha_i \phi_i \) where \( \phi_i \) is an \( n \)-component scalar field and \( \{\alpha_i, \ i=1,2,...,n-1\} \) are \((n-1)\)-simple roots for the underlying Lie algebra \( \mathcal{G} \). the parameters \( m \) and \( \beta \) are coupling constants. The derivatives of the scalar field \( \phi \) in the kinetic term are \( \{\log H, \phi\} \) and \( \{\log \tilde{H}, \phi\} \) giving then the extension of \( \text{diff}(S^1) \)-derivatives \( \partial_z \phi \) and \( \tilde{\partial}_z \phi \). The equations of motion that we can derive from the Toda action eq.(96) are summarized as follows

\[ \beta \{\log \tilde{K}, \{\log H, \phi\}\} + m^2 \sum_{i=1}^{n-1} \alpha_i \exp(\beta \alpha_i \phi) = 0 \]  

A particular example is given by the \( \{sl_2\} \) conformal Liouville model eq.(95).
Acknowledgments

I would like to thank the Abdus Salam International Center for Theoretical Physics (ICTP) for hospitality. I present special thanks to the high energy section and to its head Prof. Seif Randjbar-Daemi for considerable scientific help. Best acknowledgements are presented to the office of associates for the invitation and scientific helps. I acknowledge the valuable contributions of OEA-ICTP in the context of NET-62 program

References

[1] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B 241, 333 (1984).
V. S. Dotsenko and V. A. Fateev, Nucl. Phys. B 240, 312 (1984).
P. H. Ginsparg, Les houches Lectures (1988).

[2] M. Green, J. Schwarz, and E. Witten, Superstring Theory, Cambridge, 1986.
S. Randjbar-Daemi and J. A. Strathdee, “Introductory lectures on CFT and Strings,” In Trieste 1989, Proceedings, High energy physics and cosmology 2-79.

[3] C. Itzykson, H. Saluer, J.B. Zuber (eds.): conformal invaraince and applications to statistical machanics. Singapore: World scientific 1988
J. Mussardo, Phys. Rep. C 218, 275-382 (1992).

[4] L.D.Faddeev, L.A.Takhtajan, Hamiltonian methods and the theory of solitons, 1987,
E. Date, M. Kashiwara, M. Jimbo and T. Miwa In ”Nonlinear Integrable Systems”, eds.
M. Jimbo and T. Miwa, World Scientific (1983),
A. Das, Integrable Models, World scientific, 1989.

[5] A. B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205;
V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. B304 (1988) 348;

[6] P. Di Francesco, C. Itzykson and J. B. Zuber, Commun. Math. Phys. 140, 543 (1991).
P. Bouwknegt and K.Schoutens, Phys. Rep. 223 (1993) 183;

[7] K. Schoutens, A. Servin, and P. Van Nieuwenhuizen, Phys. Len. B 243, 245 (1990); E.
Bergshoeff, A. Bilal, and K. S. Stelle, TH 5924/90;

[8] V. A. Fateev and S. Lukyanov, Int. J. Mod. Phys. A 3, SO7 (1987);
E Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, Nucl. Phys.B 304, 348 (1988);
L. Romans, Nucl. Phys.B 352, 829 (1991);
E. Bergshoeff, C.N.Pope, L.J.Romans, E.Sezgin, X.Shen, Phys.Lett.B 245,447 (1990).

[9] V. G. Drinfeld and V. V. Sokolov, Sov. J. Math. 38, 1975 (1985).
[10] E. H. Saidi and M. B. Sedra, Class. Quant. Grav. 10, 1937 (1993);
    Int. J. Mod. Phys. A 9, 891 (1994).

[11] J. Humphreys, (Springer-Verlag, Berlin, 1972).
    V. G. Drinfeld and V. V. Sokolov, J. Sov. Math. 30, 1975 (1984).
    T. Inami and H. Kamro, Commun. Math. Phys 136, 519 (1991);
    Nucl. Phys. B 359, 201 (1991);

[12] A. Bilal, V. V. Fock, and I. I. Kogan, Nucl. Phys. B 359, 635 (1991);
    M. Bershadsky and H. Ooguri, Commun. Math. Phys. 126, 49 (1989);
    G. M. Stotkov, M. Stanishkov, and C. J. Zhu, Nucl. Phys. B 356, 439 (1991).
    E. H. Saidi and M. Zakkari, Phys. Lett. B 281, 507 (1992).

[13] J. L. Gervais, Phys. Lett. B 160, 277 (1985).
    A. Bilal and J. L. Gervais, Phys. Lett. B 206, 412 (1988).
    I. Bakas, Nucl. Phys. B 302, 189 (1988).
    P. Mathieu, Phys. Lett. B 208, 101 (1988).
    K. Yamagishi, Phys. Lett. B 259, 436 (1991).

[14] I. Bakas, Commun. Math. Phys. 123, 627 (1989).

[15] E. H. Saidi and M. B. Sedra, J. Math. Phys. 35, 3190 (1994).

[16] L. Frappat E. Ragoucy, P. Sorba, F. Thuiller, H. Hoegsen, Nucl. Phys. B334 (1990) 250.

[17] I. Antoniadis, P. Ditsas, E. Floratos and J. Iliopoulos, Nucl. Phys. B300 (1988) 549.

[18] E. H. Saidi, M. B. Sedra and A. Serhani, Phys. Lett. B 353, 209 (1995);
    Mod. Phys. Lett. A 10, 2455 (1995).

[19] E. H. Saidi, J. Zerouaoui and M. B. Sedra, Class. Quant. Grav. 12, 1567 (1995);
    Class. Quant. Grav. 12, 2705 (1995).

[20] T. A. Arakelyan and G. K. Savvidy, Phys. Lett. B214 (1988) 350.

[21] K. Toda and S.J. Yu, J. Math. Phys. 41 (2000) 4747; J. Nonlinear Math. Phys. Suppl.
    8(2001)272; Inverse Problems 17 (2001) 1053.

[22] M.B. Sedra, Integrable KdV hierarchies on the Torus $T^2$, work in progress.

[23] P. Mansfield, Nucl. Phys. B 208, 277 (1982).
    P. Mansfield, Nucl. Phys. B 222, 419 (1983).
    J. Evans and T. J. Hollowood, Nucl. Phys. B 352, 723 (1991); [B 382, 662 (1992)].