Uniqueness of static photon surfaces: Perturbative approach

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Abstract

A photon surface $S$ is defined as a three-dimensional timelike hypersurface such that any null geodesic initially tangent to $S$ continues to be included in $S$, like $r = 3M$ of the Schwarzschild spacetime. Using analytic solutions to static perturbations of a Schwarzschild spacetime, we examine whether a nonspherical spacetime can possess a distorted static photon surface. It is shown that if the region outside of $r = 3M$ is vacuum, no distorted photon surface can be present. Therefore, we establish the perturbative uniqueness for an asymptotically flat vacuum spacetime with a static photon surface. It is also pointed out that if matter is present in the outside region, there is a possibility that a distorted photon surface could form.

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I. INTRODUCTION

A remarkable property of a black hole is the existence of an event horizon. But in predicting observational image around a black hole, the location of closed circular orbits of null geodesics (say, photon rings) is even more important. In a Schwarzschild spacetime, a photon ring is located at $r = 3M$, and all photon rings form a photon sphere due to spherical symmetry. More generally, a photon sphere is defined as a static timelike surface $S$ with spherically symmetric geometry such that any null geodesic initially tangent to $S$ will remain tangent to $S$ \[1, 2\]. The photon sphere has important effects on particles/waves traveling around a static black hole, and as a consequence, on the gravitational lensing effects \[1, 3–5\] and black hole shadows \[6\] (see also Ref. \[7\]).

A related notion is the photon surface, which was proposed as generalization of the photon sphere \[2\]. The photon surface is defined as a timelike hypersurface $S$ such that any null geodesic initially tangent to $S$ continues to be included in $S$. The photon surface is a broader notion compared to the photon sphere: it may be dynamical or may not be spherically symmetric. But spacetimes possessing photon surfaces are fairly restricted. For example, a Kerr spacetime does not possess a photon surface. Although there are null geodesics each of which remains on a $r = \text{const.}$ surface in the Boyer-Lindquist coordinates, the $r$ value depends on the angular momentum of null geodesics \[8\] (see also Sec. 5.8 of Ref. \[9\]). As a result, there is a photon region in which null geodesics staying on $r = \text{const.}$ surfaces distribute like layers. On the inner/outer boundary of this region, prograde/retrograde null geodesics rotate on the equatorial plane. In the nonrotation limit, this photon region becomes infinitely thin and reduces to a photon surface. In this manner, a photon surface is formed as a special limit of general cases.

In this paper, we consider a static photon surface in a static vacuum spacetime. A static photon surface naturally arises when the spacetime is spherically symmetric with strong gravitational source. Our question is whether a nonspherical spacetime can possess a static photon surface or not. Specifically, we consider a vacuum spacetime with a (possibly distorted) static photon surface as an inner boundary. Here, we do not take care of the inside region of the photon surface: There may be a nonspherical star composed of unusual matter or an event horizon surrounded by a naked singularity, and so on, which causes distortion of a photon surface/spacetime from spherical symmetry.
A partial answer was given as the uniqueness theorem for a (generalized) photon sphere. The author of Ref. [10] redefined the notion of a photon sphere as a static photon surface on which the time lapse function $N = \sqrt{-g_{tt}}$ is constant. Then, analyzing the Einstein equations in the outside region with the boundary conditions on the photon sphere, the spacetime is shown to be spherically symmetric. Namely, an asymptotically flat vacuum spacetime that allows the presence of a photon sphere is only the Schwarzschild spacetime [10, 11]. This uniqueness theorem was generalized to electrovacuum spacetimes [12, 13] and to other spacetimes [14–16].

The constancy of the time lapse function $N$ on a photon sphere would be imposed by technical reason. The question here is what happens if we relax this condition, i.e., if we consider a surface on which the time lapse function $N$ is nonconstant. As a first step toward this direction, we adopt the perturbative approach. Namely, we consider a static perturbation of a Schwarzschild spacetime and study whether a static photon surface exists or not in a distorted Schwarzschild spacetime.

Analyses of static perturbations are useful, because if absence of a regular solution is proved, no regular branch of static solutions starting from the background solution exists, and therefore, a restriction on the solution space is obtained. For example, the perturbative uniqueness was shown for a large class of higher-dimensional static black hole solutions [17]. Conversely, if there is a regular solution to a static perturbation, we notice the possibility that an unknown solution branch exists. One example is the static perturbation at the threshold of the Gregory-Laflamme instability of black string spacetimes [18], which motivated the numerical constructions of nonuniform black string solutions in the nonlinear regime [19, 20].

This paper is organized as follows. In the next section, we summarize the geometrical properties of the photon surface, which will be used to judge its existence in a perturbed Schwarzschild spacetime. In Sec. III, we discuss what the absence/presence of the first-order solution can tell us on the nonexistence/existence of a fully nonlinear solution. In Sec. IV, analytic solutions to static perturbations of a Schwarzschild spacetime are derived. In Sec. V, we examine whether a photon surface exists or not in a distorted Schwarzschild spacetime. It is shown that a photon surface vanishes once the spacetime is distorted, if the region outside of $r = 3M$ is vacuum. We also point out that it may be possible to realize a distorted photon surface if matter is present outside of $r = 3M$. Section VI is devoted to a conclusion. In the Appendix, the detailed calculations on the photon surface condition.
in a perturbed Schwarzschild spacetime are explained. To simplify the notation, we use the geometrical units $c = G = 1$.

II. PHOTON SURFACE

In this section, we review the geometrical properties of a photon surface. Let $S$ be a timelike three-dimensional hypersurface in a spacetime $(M, g_{ab})$ and $n^a$ be the unit normal to $S$. The induced metric $h_{ab}$ and the extrinsic curvature $\chi_{ab}$ of $S$ are introduced by

$$h_{ab} = g_{ab} - n_a n_b, \quad (1)$$

$$\chi_{ab} = h_a^{\ c} \nabla_c n_b = \frac{1}{2} \mathcal{L}_n h_{ab}, \quad (2)$$

where $\mathcal{L}_n$ denotes the Lie derivative with respect to $n^a$. In Ref. [2], three expressions of the necessary and sufficient condition for $S$ to be a photon surface are presented. The first expression is that affine-parametrized null geodesics of the submanifold $S$ are simultaneously affine-parametrized null geodesics of the spacetime $M$. The second expression is that for arbitrary null vectors $k^a$ tangent to $S$,

$$\chi_{ab} k^a k^b = 0 \quad (3)$$

holds. The third expression is that the hypersurface $S$ is umbilical, i.e.,

$$\chi_{ab} \propto h_{ab}. \quad (4)$$

Each of these three expressions is equivalent to the condition for $S$ to be a photon surface, and the proofs are given in Ref. [2].

Here, we discuss intuitive interpretation for these conditions. It is useful to recall the variational principle for affine-parametrized null geodesics. The action is given by

$$S = \int L d\lambda, \quad L = \frac{1}{2} g_{ab} k^a k^b. \quad (5)$$

Consider a null geodesic of $M$ whose path is included in $S$. If we slightly deform its path, the action is unchanged to first order, and this holds also when the deformation is restricted on $S$. Therefore, the null geodesic under consideration is also a null geodesic of the submanifold $S$. Conversely, for a hypersurface $S$ to be a photon surface, each null geodesic of the submanifold $S$ has to be a null geodesic of $M$ (i.e., the first expression). This means that the action $S$
must be stationary when its path is deformed toward outside of \( S \), i.e., in the direction of \( n^a \). Since change in the Lagrangian \( \mathcal{L} \) in this deformation is proportional to \( \chi_{ab} k^a k^b \), the condition (3) must hold (i.e., the second expression). If the condition (4) is satisfied, the condition (3) is obviously satisfied as well. Because of the arbitrariness of \( k^a \), no other form of \( \chi_{ab} \) is allowed (i.e., the third expression).

Among the three expressions, the third one, Eq. (4), is used in this paper in order to identify the (non)existence of a photon surface in a distorted Schwarzschild spacetime. In what follows, the condition (4) is referred as the “photon surface condition”.

III. WHAT A PERTURBATION CAN TELL US

In this paper, we study the first-order metric perturbation of a Schwarzschild spacetime. Before starting the analysis, we discuss what absence/presence of a solution of a first-order perturbation can tell us on the nonexistence/existence of a fully nonlinear solution in this section.

In the perturbation theory, the metric is expanded as

\[
g_{ab} = g^{(0)}_{ab} + \epsilon g^{(1)}_{ab} + \epsilon^2 g^{(2)}_{ab} + \cdots,
\]

with a small expansion parameter \( \epsilon \) and \( g^{(0)}_{ab} \) is a background spacetime metric. In our case, \( g^{(0)}_{ab} \) is the Schwarzschild solution of the vacuum Einstein equation,

\[
R_{ab} = 0,
\]

and the perturbed metric is also required to satisfy this equation. Substituting the formula (3) into Eq. (7) and collecting the same-order terms with respect to \( \epsilon \), the zeroth-order equation is trivially satisfied and the first-order and higher-order equations have the following form:

\[
\mathcal{L} \left( g^{(1)}_{ab} \right) = 0; \quad (8a)
\]

\[
\mathcal{L} \left( g^{(n)}_{ab} \right) = S_n \left( g^{(1)}_{ab}, \ldots, g^{(n-1)}_{ab} \right), \quad (n = 2, 3, \ldots), \quad (8b)
\]

where \( \mathcal{L} \) is the linear operator and \( S_n \left( g^{(1)}_{ab}, \ldots, g^{(n-1)}_{ab} \right) \) is sum of the products of \( g^{(1)}_{ab}, \ldots, g^{(n-1)}_{ab} \) with \( n \)th order (see [21] for a similar expression in the second-order case).
In our problem, we prepare a regular solution of $g^{(1)}_{ab}$ and look for a photon surface that satisfies the photon surface condition, $h_{ab} \propto \chi_{ab}$. The photon surface is assumed to deviate from $r = 3M$ as $\epsilon$ is increased. Suppose it is shown that the photon surface condition is not compatible with any nontrivial first-order perturbation. Namely, we consider the case that the only first-order solution consistent with the presence of a photon surface is $g^{(1)}_{ab} = 0$ and the photon surface does not distort up to $O(\epsilon)$. In this situation, we can show that the $n$th-order perturbation also vanishes for arbitrary $n$ using the mathematical induction. Assuming $g^{(i)}_{ab} = 0$ for all $i = 1, \ldots, n - 1$, the source term $S_n$ of Eq. (8b) becomes zero and the solution space of $g^{(n)}_{ab}$ is same as that of $g^{(1)}_{ab}$. Setting $\epsilon' = \epsilon^n$, the perturbation is effectively same as the first-order perturbation with respect to $\epsilon'$. Then, by the same reason as the first-order case, we have $g^{(n)}_{ab} = 0$ and the photon surface does not distort up to $O(\epsilon^n)$. Therefore, the absence of a nontrivial first-order solution implies the absence of a nontrivial solution at any order. There is no regular sequence of fully nonlinear solutions branching from the background spacetime.

Note that although any regular solution sequence branching from the Schwarzschild solution can be ruled out in the above manner, there remains a possibility that a sequence of regular spacetimes approaching the Schwarzschild limit in a singular way might exist at least logically (see Sec. 3.3 of [17] for such an example in the context of the ordinary static black hole uniqueness). Therefore, showing the nonexistence of perturbative solutions does not give a complete proof of the uniqueness near the Schwarzschild limit in the solution space. Such a possibility is left open in this paper.

On the other hand, suppose a first-order solution with a distorted photon surface can be constructed. In this case, the existence of the first-order solution does not guarantee the existence of a fully nonlinear solution, because there is a possibility that the second-order (or higher-order) perturbation is incompatible with the photon surface condition. In other words, the existence of a first-order solution is a necessary condition for the existence of a fully nonlinear solution, but not a sufficient condition. Therefore, we have to interpret the existence of the first-order solution as an indication for the possible existence of a fully nonlinear solution.
IV. STATIC PERTURBATION OF SCHWARZSCHILD SPACETIME

Now, we study static first-order perturbations of a Schwarzschild spacetime. The background metric is given in the coordinates \((t, r, \theta, \phi)\) as follows:

\[
\hat{s}^2 = -e^{2\nu(0)} dt^2 + e^{2\mu(0)} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),
\]

(9)

\[
e^{2\nu(0)} = e^{-2\mu(0)} = 1 - \frac{2M}{r}.
\]

(10)

Static distortion of a Schwarzschild spacetime is represented by time-independent even-parity perturbations [22]. After a suitable gauge transformation (i.e., the Regge-Wheeler gauge [22]), the metric of a distorted Schwarzschild spacetime is written in diagonal form:

\[
\hat{s}^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + e^{2\psi} r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),
\]

(11)

\[
\nu = \nu^{(0)} + \nu^{(1)} + \cdots,
\]

(12a)

\[
\mu = \mu^{(0)} + \nu^{(1)} + \cdots,
\]

(12b)

\[
\psi = \psi^{(1)} + \cdots,
\]

(12c)

with a small expansion parameter \(\epsilon\). The first-order functions are expanded with the angular eigenmodes,

\[
\nu^{(1)} = -\mu^{(1)} = -\sum_{\ell, m} H^{(1)}_{\ell m}(r) Y_{\ell m}(\theta, \phi),
\]

(13a)

\[
\psi^{(1)} = \sum_{\ell, m} K^{(1)}_{\ell m}(r) Y_{\ell m}(\theta, \phi),
\]

(13b)

with the spherical harmonics \(Y_{\ell m}(\theta, \phi)\), where \(\ell\) and \(m\) are integers satisfying \(\ell \geq 0\) and \(-\ell \leq m \leq \ell\). Here, the first-order equations with different \((\ell, m)\) values decouple, and each mode can be treated separately. In what follows, we consider a single mode and write the radial functions as \(H^{(1)}\) and \(K^{(1)}\) for brevity.

As discussed in [23], the radial equations for a fixed \(\ell\) have the same form for arbitrary \(m\) due to the spherical symmetry of the background spacetime. In the case of \(m = 0\), the

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1 Although the perturbed metric in the Regge-Wheeler gauge has nonzero (tr) component in general, this vanishes in the static case due to the \(T\) symmetry of the Einstein’s equation (i.e., the symmetry under the time-reversal transformation, \(t \rightarrow -t\)).
first equality in Eq. (13a) is found from the difference between the $\theta\theta$ and $\phi\phi$ components of the Einstein equations (7). We consider the case $\ell \geq 2$ because the $\ell = 0$ and $\ell = 1$ modes correspond to coordinate transformations and shift of the mass value. The radial equations for the first-order quantities are

\[
r^2 e^{2\nu(0)} H^{(1)}_{,rr} + 2r \left( r e^{2\nu(0)} \right)_{,r} H^{(1)}_{,r} - r^2 \left( e^{2\nu(0)} \right)_{,rr} K^{(1)}_{,r} - \ell(\ell + 1)H^{(1)} = 0,
\]

(14a)

\[
r^2 e^{2\nu(0)} \left( H^{(1)}_{,rr} - 2K^{(1)}_{,rr} \right) + 2r \left( r e^{2\nu(0)} \right)_{,r} H^{(1)}_{,r} \]
\[- r \left[ r \left( e^{2\nu(0)} \right)_{,r} + 4e^{2\nu(0)} \right] K^{(1)}_{,r} + \ell(\ell + 1)H^{(1)} = 0,
\]

(14b)

\[
e^{2\nu(0)} \left( H^{(1)}_{,r} - K^{(1)}_{,r} \right) + \left( e^{2\nu(0)} \right)_{,r} H^{(1)} = 0,
\]

(14c)

\[
r^2 e^{2\nu(0)} K^{(1)}_{,rr} - 2r e^{2\nu(0)} H^{(1)}_{,r} + r \left[ r \left( e^{2\nu(0)} \right)_{,r} + 4e^{2\nu(0)} \right] K^{(1)}_{,r} \]
\[- 2 \left( r e^{2\nu(0)} \right)_{,r} H^{(1)} - (\ell^2 + \ell - 2)K^{(1)} = 0,
\]

(14d)

where ,$r$ denotes the derivative with respect to $r$. For $m = 0$, these equations are derived from $tt, rr, r\theta$ components and the sum of the $\theta\theta, \phi\phi$ components of the Einstein equations, respectively.

It is convenient to introduce a normalized radial coordinate as

\[x = \frac{r}{2M}.
\]

(15)

From Eqs. (14a) and (14b), the equation for $H^{(1)}$ is derived as

\[x(x - 1)H^{(1)}_{,xx} + (2x - 1)H^{(1)}_{,x} - \left\{ \ell(\ell + 1) + \frac{1}{x(x - 1)} \right\} H^{(1)} = 0.
\]

(16)

The solution to this equation is

\[H^{(1)}(x) = \alpha_\ell P^2_\ell(2x - 1) + \beta_\ell Q^2_\ell(2x - 1),
\]

(17)

This solution was first presented in Eq. (35) of Ref. [22], but a typo is included: this formula is not for $M$ but for $H$.  

8
where $P_\ell^\mu(z)$ and $Q_\ell^\mu(z)$ denote the associated Legendre functions of the first and second kinds,

$$P_\ell^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+z}{1-z}\right)^{\mu/2} {}_2F_1\left(-\ell, \ell + 1; 1 - \mu; \frac{1-z}{2}\right),$$  \hspace{1cm} (18a)

$$Q_\ell^\mu(z) = \frac{\sqrt{\pi} \Gamma(\ell + \mu + 1)}{2^{\ell+1} \Gamma(\ell + 3/2)} \frac{(1-z^2)^{\mu/2}}{z^{\ell+\mu+1}} {}_2F_1\left(\frac{\ell + \mu + 1}{2}, \frac{\ell + \mu + 2}{2}; \ell + \frac{3}{2}; \frac{1}{z^2}\right).$$  \hspace{1cm} (18b)

Here, ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Once $H^{(1)}$ is obtained, $K^{(1)}$ is calculated by the formula

$$K^{(1)}(x) = \frac{1}{(\ell^2 + \ell - 2)} \left[ H^{(1)}_{xx} + \left(\frac{1}{x-1} + \frac{1}{x} + \ell^2 + \ell - 2\right) H^{(1)}\right],$$  \hspace{1cm} (19)

which is derived by eliminating $H^{(1)}_{rr}$, $K^{(1)}_{rr}$ and $K^{(1)}_r$ from Eqs. (14a)–(14d).

The first term $\alpha_\ell P_\ell^2(2x - 1)$ of the solution (17) is regular on the horizon $x = 1$ but is divergent at infinity, $x \to \infty$, while the second term $\beta_\ell Q_\ell^2(2x - 1)$ decays at infinity but is divergent on the horizon. In this paper, we consider distortion of a Schwarzschild spacetime in the region outside of the photon surface, $r \gtrsim 3M$. When the spacetime is vacuum in this region, we have to set $\alpha_\ell = 0$ and adopt only the second term $\beta_\ell Q_\ell^2(2x - 1)$ to make the perturbation regular. Such a perturbation would be generated by, e.g., nonspherical distribution of matter or naked singularities in the region $r < 3M$. On the other hand, if we consider the distortion of a Schwarzschild spacetime due to matter outside of $r = 3M$, the perturbation in the neighborhood of $r = 3M$ is represented by both terms of Eq. (17) with nonvanishing $\alpha_\ell$ and $\beta_\ell$.

V. DISTORTION OF PHOTON SURFACE

In this section, we examine whether a static photon surface exists in a distorted Schwarzschild spacetime.

A. Photon surface condition

We rewrite the photon surface condition $\chi_{ab} \propto h_{ab}$ with perturbative quantities. Let a static photon surface in a distorted Schwarzschild spacetime be given as

$$r = f(\theta, \phi),$$  \hspace{1cm} (20)
where
\[ f = f^{(0)} + \epsilon f^{(1)} + \cdots , \quad \text{with} \quad f^{(0)} = 3M. \] (21)
Here, \( f^{(1)} \) is assumed to be dependent on the angular coordinates, since the perturbation of the Schwarzschild spacetime has the possibility to cause the distortion of the coordinate shape of the photon surface.

For this surface, we calculate the photon surface condition, \( \chi_{ab} \propto h_{ab} \). We present the details of calculations in the Appendix and just show the results here:
\[
\begin{align*}
  f^{(1)}_{,\phi\phi} &= \cot \theta f^{(1)}_{,\phi}, \\
  f^{(1)}_{,\phi\theta} &= \sin^2 \theta f^{(1)}_{,\theta\theta} - \sin \theta \cos \theta f^{(1)}_{,\phi}, \\
  (\nu^{(1)}_{,r} - \psi^{(1)}_{,r}) \bigg|_{r=3M} &= \frac{1}{3M^2} \left( f^{(1)} - f^{(1)}_{,\theta\theta} \right). \tag{22c}
\end{align*}
\]
The first two equations (22a) and (22b) are solved as
\[ f^{(1)} = \sin \theta \left( \alpha e^{i\phi} + \beta e^{-i\phi} \right) + \gamma \cos \theta + \delta, \] (23)
where \( \alpha, \beta, \gamma \) and \( \delta \) are integral constants.\(^3\) This is a linear combination of the four spherical harmonics \( Y_{\ell m}(\theta, \phi) \) with \( (\ell, m) = (1, \pm 1), (1, 0), \) and \( (0, 0) \). Then, the right-hand side of Eq. (22c) has the \( \ell = 0 \) and 1 modes, while the left-hand side has the \( \ell \geq 2 \) modes from the assumption. Therefore, each side of Eq. (22c) becomes zero, and the right-hand side gives
\[ f^{(1)} = 0. \] (24)
Namely, in the Regge-Wheeler gauge, the coordinate position of a distorted photon surface, if it exists, must remain at \( r = 3M \) to first order. The left-hand side implies that the metric perturbation must satisfy
\[ (\nu^{(1)}_{,r} - \psi^{(1)}_{,r}) \bigg|_{r=3M} = 0 \] (25)
in order for \( r = 3M \) to be a photon surface.

B. Vacuum case

Equation (25) indicates that a photon surface can exist only when the metric perturbation satisfies a special property. Here, we examine whether this property is satisfied if the spacetime is vacuum outside of \( r = 3M \).

\(^3\) The solution (23) means that if the umbilical condition \( \chi_{ab} \propto h_{ab} \) is restricted to the spatial part, allowed distortion is shift of the position toward three directions and uniform expansion.
TABLE I: The values of Eq. (27a), \( \frac{d}{d\tilde{x}} Q^2_\ell^{(1)}(\tilde{x})/Q^2_\ell^{(1)}(\tilde{x}) \big|_{\tilde{x}=2} \), in the vacuum case and the values of \( \alpha_\ell/\beta_\ell \) to satisfy the condition (26) in the presence of matter for \( \ell = 2, ..., 5 \).

| \( \ell \) | \( \frac{d}{d\tilde{x}} Q^2_\ell^{(1)}(\tilde{x})/Q^2_\ell^{(1)}(\tilde{x}) \big|_{\tilde{x}=2} \) (Numerical Value) | \( \alpha_\ell/\beta_\ell \) (Numerical Value) |
|---|---|---|
| 2 | \( \frac{108-\text{arctanh}(1/2)-64}{81-\text{arctanh}(1/2)} \) | \( \frac{26}{35} \cdot \text{arctanh} \left( \frac{1}{2} \right) \) (2.847 \times 10^{-2}) |
| 3 | \( \frac{1485-\text{arctanh}(1/2)-818}{810-\text{arctanh}(1/2)} \) | \( \frac{322}{885} \cdot \text{arctanh} \left( \frac{1}{2} \right) \) (1.121 \times 10^{-3}) |
| 4 | \( \frac{25920-\text{arctanh}(1/2)-14240}{109835-\text{arctanh}(1/2)} \) | \( \frac{5444}{9885} \cdot \text{arctanh} \left( \frac{1}{2} \right) \) (5.966 \times 10^{-5}) |
| 5 | \( \frac{182385-\text{arctanh}(1/2)-100186}{62370-\text{arctanh}(1/2)} \) | \( \frac{37202}{67725} \cdot \text{arctanh} \left( \frac{1}{2} \right) \) (3.564 \times 10^{-6}) |

Rewriting with \( H^{(1)} \) and \( K^{(1)} \) and using Eq. (14c), Eq. (25) is rewritten as

\[
\frac{H^{(1)}}{H^{(1)}} \bigg|_{\tilde{x}=2} = -\frac{1}{3},
\]  

where we introduced \( \tilde{x} = 2x - 1 \) with \( x = r/2M \). The assumption that the region outside of \( r = 3M \) is vacuum corresponds to \( \alpha_\ell = 0 \) in the solution of \( H^{(1)} \) in Eq. (17). In this situation, the left-hand side of Eq. (26) is calculated as

\[
\frac{d}{d\tilde{x}} Q^2_\ell^{(1)}(\tilde{x}) \bigg|_{\tilde{x}=2} = -\frac{3\ell + 1}{6} - \frac{1}{4} \left[ \frac{d}{dz} \right. F_1(a, b; c; z) \bigg]_{z=1/4} \]  

with

\[
a = \frac{\ell + 3}{2}, \quad b = \frac{\ell}{2} + 2, \quad c = \ell + \frac{3}{2}.
\]  

The explicit values of the formula (27a) are shown in Table I for \( \ell = 2, ..., 5 \). Because both \( 2F_1(a, b; c; z) \) and \( \frac{d}{dz} F_1(a, b; c; z) \) are positive at \( z = 1/4 \), the values of the formula (27a) are smaller than \(-1/3\) for arbitrary \( \ell \geq 2 \). Therefore, the condition (26) cannot be satisfied if the outside region is vacuum.

The result here indicates that once a Schwarzschild spacetime is distorted due to matter/naked singularities inside of \( r = 3M \), the photon surface vanishes. Therefore, the perturbative uniqueness holds for a spacetime with a static photon surface in the case that the outside region is vacuum.

C. Nonvacuum case

Here, we discuss the case that the outside spacetime is not vacuum and the distortion of a photon surface is caused by matter distribution outside of \( r = 3M \), as well as that inside
of $r = 3M$. In such a case, the perturbation in the vicinity of $r = 3M$ is given by Eq. (17) with nonvanishing $\alpha_\ell$ and $\beta_\ell$. The introduction of the extra parameter $\alpha_\ell$ enables one to satisfy the condition (26). Namely, the condition (26) becomes an equation for $\alpha_\ell/\beta_\ell$, and the explicit values of the solutions are listed in Table I for $\ell = 2, \ldots, 5$.

Perturbative distortion of a photon surface is possible, in principle, if we discard the vacuum assumption outside of $r = 3M$. However, the requirement for fine-tuning of $\alpha_\ell/\beta_\ell$ indicates that one has to distribute matter in a very special manner in order to realize a distorted photon surface.

VI. CONCLUSION

In this paper, we have examined whether a distorted Schwarzschild spacetime can possess a static photon surface. The photon surface condition $\chi_{ab} \propto h_{ab}$ requires the metric perturbation to have a special property described by Eq. (25). If the region outside of $r = 3M$ is vacuum, the regular solution of the metric perturbation does not satisfy this condition for any $\ell \geq 2$. Therefore, we have proved the perturbative uniqueness theorem: There is no solution branch of a spacetime possessing a distorted photon surface that regularly connects to the Schwarzschild solution, if the outside spacetime is vacuum and asymptotically flat.

From this perturbative uniqueness, two possibilities are indicated on the existence of a distorted photon surface. One is that a spacetime solution with a distorted photon surface exists but it does not regularly connect to the Schwarzschild solution in the solution space. The other is that there is no solution of a spacetime with a distorted photon surface at all. Among these two, the first possibility is difficult to imagine and would be unlikely. For this reason, we propose the following conjecture: If an asymptotically flat, vacuum spacetime possesses a static photon surface, the spacetime is the Schwarzschild spacetime. Up to now, the uniqueness theorem has been proved for a spacetime with a (generalized) photon sphere, i.e., a photon surface given by a contour surface of the time lapse function, $N = \sqrt{-g_{tt}}$. Our result indicates that it would be possible to eliminate the assumption of the constancy of the lapse function on a photon surface in proving the uniqueness theorem. This direction is worth challenging.

On the other hand, if we consider a perturbation generated by matter outside of $r = 3M$, the perturbative distortion of a photon surface has turned out to be possible although fine-
tuning between the amplitudes of two independent perturbative solutions is required. The fine-tuning indicates that matter has to be distributed in a special way in order to realize a distorted photon surface. Here, one has to keep in mind that the existence of a first-order perturbative solution does not necessarily guarantee the presence of a fully nonlinear solution because the nonlinearity may invalidate it. Therefore, whether a photon surface can exist in the nonlinear regime is an interesting next issue, and it is worth trying to construct explicit solutions. Also, clarifying whether such a distorted photon surface exists in a realistic astrophysical context would be an interesting remaining problem.

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Appendix A: Calculation of photon surface condition

In this appendix, we present calculations to derive the perturbative photon surface condition, Eqs. (22a)–(22c). Suppose a photon surface is given by \( r = f(\theta, \phi) \) in a distorted Schwarzschild spacetime with a metric (11). We would like to calculate the photon surface condition \( \chi_{ab} \propto h_{ab} \) for this surface. It is convenient to introduce coordinates \((\tilde{r}, \tilde{\theta}, \tilde{\phi})\) such that the photon surface is given by \( \tilde{r} = 0 \) and the radial coordinate basis \( \partial_{\tilde{r}} \) is orthogonal to \( \tilde{r} = \text{const.} \) slices. If such coordinates are introduced, the induced metric \( h_{ab} \) on the \( \tilde{r} = \text{const.} \) slice coincides with \( g_{ab} \) except for the \( \tilde{r}\tilde{r} \) component, and the extrinsic curvature is calculated by

\[
\chi_{ab} = \frac{1}{2\sqrt{g_{\tilde{r}\tilde{r}}}} \partial_{\tilde{r}} h_{ab}. \tag{A1}
\]

Then, the photon surface condition is equivalent to \( \partial_{\tilde{r}} h_{ab} \propto h_{ab} \).

For this reason, we consider the following coordinate transformation,

\[
\begin{align*}
\tilde{r} &= r + f(\tilde{\theta}, \tilde{\phi}), \\
\tilde{\theta} &= \theta - p(\tilde{r}, \tilde{\theta}, \tilde{\phi}), \\
\tilde{\phi} &= \phi - q(\tilde{r}, \tilde{\theta}, \tilde{\phi}),
\end{align*}
\tag{A2}
\]
and require \( p = q = 0 \) on the photon surface so that \( \theta = \tilde{\theta} \) and \( \phi = \tilde{\phi} \) on \( \tilde{r} = 0 \). In order to make the metric components \( g_{\tilde{r}\tilde{\phi}} \) and \( g_{\tilde{\theta}\tilde{\theta}} \) vanish, we further require

\[
\begin{align*}
    f_{\tilde{\theta}\tilde{\phi}} e^{2\mu} &= r^2 e^{2\psi} \left( (1 - p_{\tilde{\theta}}) p_{\tilde{\phi}} - \sin^2 \theta q_{\tilde{\phi}} q_{\tilde{\theta}} \right), \\
    f_{\tilde{\phi}\tilde{\phi}} e^{2\mu} &= r^2 e^{2\psi} \left[ \sin^2 \theta (1 - q_{\tilde{\phi}}) q_{\tilde{\phi}} - p_{\tilde{\phi}} p_{\tilde{\phi}} \right].
\end{align*}
\]

In particular, this relation is reduced to

\[
\begin{align*}
    p_{\tilde{\phi}} &= \frac{f_{\tilde{\phi}}}{f^2} e^{2(\mu - \psi)}, \\
    q_{\tilde{\phi}} &= \frac{f_{\tilde{\phi}}}{f^2} e^{2(\mu - \psi)} \sin^{-2} \theta,
\end{align*}
\]

on the photon surface \( \tilde{r} = 0 \). Under these conditions on \( p \) and \( q \), the nonvanishing metric components become

\[
\begin{align*}
    g_{tt} &= -e^{2\nu}, \\
    g_{\tilde{r}\tilde{r}} &= e^{2\mu} + r^2 e^{2\psi} (p_{\tilde{r}}^2 + \sin^2 \theta p_{\tilde{\theta}}^2), \\
    g_{\tilde{\theta}\tilde{\theta}} &= e^{2\mu} f_{\tilde{\theta}}^2 + r^2 e^{2\psi} \left[ (1 - p_{\tilde{\theta}})^2 + \sin^2 \theta q_{\tilde{\theta}}^2 \right], \\
    g_{\tilde{\phi}\tilde{\phi}} &= e^{2\mu} f_{\tilde{\phi}}^2 + r^2 e^{2\psi} \left[ p_{\tilde{\phi}}^2 + \sin^2 \theta (1 - q_{\tilde{\phi}})^2 \right], \\
    g_{\tilde{r}\tilde{\phi}} &= e^{2\mu} f_{\tilde{\theta}} f_{\tilde{\phi}} - r^2 e^{2\psi} \left[ (1 - p_{\tilde{\theta}}) p_{\tilde{\phi}} + \sin^2 \theta (1 - q_{\tilde{\phi}}) q_{\tilde{\theta}} \right].
\end{align*}
\]

In what follows, we adopt the perturbative approximation and keep the quantities up to the first order in \( \epsilon \). For example, since the derivatives of the metric components and the function \( f \) with respect to the angular coordinates, \( \tilde{\theta} \) and \( \tilde{\phi} \), are first order, their products (like \( f_{\tilde{\theta}}^2 \) or \( f_{\tilde{\phi}} f_{\tilde{\theta}} \)) are ignored. The nonvanishing induced metric components on the photon surface \( \tilde{r} = 0 \) are

\[
\begin{align*}
    h_{tt} &= -e^{2\nu}, \\
    h_{\tilde{\theta}\tilde{\theta}} &= f^2 e^{2\psi}, \\
    h_{\tilde{\phi}\tilde{\phi}} &= f^2 e^{2\psi} \sin^2 \theta.
\end{align*}
\]

On the other hand, the nonvanishing components of \( \partial_{\tilde{r}} h_{ab} \) on the \( \tilde{r} = 0 \) surface are

\[
\begin{align*}
    \partial_{\tilde{r}} h_{tt} &= -2 e^{2\nu} v_{\tilde{r}}, \\
    \partial_{\tilde{r}} h_{\tilde{\theta}\tilde{\theta}} &= 2 f^2 \left[ e^{2\nu} \left( \frac{1}{f} + \psi_{\tilde{r}} \right) - e^{2\mu} \frac{f_{\tilde{\theta}}}{f^2} \right], \\
    \partial_{\tilde{r}} h_{\tilde{\phi}\tilde{\phi}} &= 2 f^2 \left[ e^{2\nu} \left( \frac{1}{f} + \psi_{\tilde{r}} \right) \sin^2 \theta - e^{2\mu} \frac{f_{\tilde{\phi}}}{f^2} \sin \theta \cos \theta f_{\tilde{\phi}} \right], \\
    \partial_{\tilde{r}} h_{\tilde{r}\tilde{\phi}} &= -2 e^{2\mu} \left( f_{\tilde{\theta}} - \cot \theta f_{\tilde{\phi}} \right).
\end{align*}
\]
Since $h_{\tilde{\theta}\tilde{\phi}} = 0$, the condition $\partial_{\tilde{\theta}} h_{\tilde{\theta}\tilde{\phi}} \propto h_{\tilde{\theta}\tilde{\phi}}$ implies $f_{\tilde{\theta}\tilde{\phi}} = \cot \theta f_{\tilde{\phi}}$, which is identical to Eq. (22a). The other conditions $\partial_{\tilde{\tau}} h_{tt}/h_{tt} = \partial_{\tilde{\theta}} h_{\tilde{\theta}\tilde{\phi}}/h_{\tilde{\theta}\tilde{\phi}} = \partial_{\tilde{\theta}} h_{\tilde{\phi}\tilde{\phi}}/h_{\tilde{\phi}\tilde{\phi}}$ are written as

$$- f^2 e^{2(\psi - \mu)} \left( \nu_{\tilde{\tau}} - \psi_{\tilde{\tau}} - \frac{1}{f} \right) = f_{\tilde{\theta}\tilde{\phi}} = \frac{f_{\tilde{\theta}\tilde{\phi}}}{\sin^2 \theta} + \cot \theta f_{\tilde{\phi}}. \quad (A8)$$

The second equality is equivalent to Eq. (22b). Separating the first equality to the zeroth-order and first-order equations taking account of Eq. (21), we find that the zeroth-order equation is satisfied by $f^{(0)} = 3M$, and the first-order equation is reduced to Eq. (22c).

Although calculations become more tedious, the perturbative photon surface condition can be derived also in the original $(t, r, \theta, \phi)$ coordinates. We have checked that the same equations are obtained in this way.

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