THE $t$-STABILITY NUMBER OF A RANDOM GRAPH

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Abstract. Given a graph $G = (V, E)$, a vertex subset $S \subseteq V$ is called $t$-stable (or $t$-dependent) if the subgraph $G[S]$ induced on $S$ has maximum degree at most $t$. The $t$-stability number $\alpha_t(G)$ of $G$ is the maximum order of a $t$-stable set in $G$. We investigate the typical values that this parameter takes on a random graph on $n$ vertices and edge probability equal to $p$. For any fixed $0 < p < 1$ and fixed non-negative integer $t$, we show that, with probability tending to 1 as $n \to \infty$, the $t$-stability number takes on at most two values which we identify as functions of $t$, $p$ and $n$. The main tool we use is an asymptotic expression for the expected number of $t$-stable sets of order $k$. We derive this expression by performing a precise count of the number of graphs on $k$ vertices that have maximum degree at most $k$.

1. Introduction

Given a graph $G = (V, E)$, a vertex subset $S \subseteq V$ is called $t$-stable (or $t$-dependent) if the subgraph $G[S]$ induced on $S$ has maximum degree at most $t$. The $t$-stability number $\alpha_t(G)$ of $G$ is the maximum order of a $t$-stable set in $G$. The main topic of this paper is to give a precise formula for the $t$-stability number of a dense random graph.

The notion of a $t$-stable set is a generalisation of the notion of a stable set. Recall that a set of vertices $S$ of a graph $G$ is stable if no two of its vertices are adjacent. In other words, the maximum degree of $G[S]$ is 0, and therefore a stable set is a 0-stable set.

The study of the order of the largest $t$-stable set is motivated by the study of the $t$-improper chromatic number of a graph. A $t$-improper colouring of a graph $G$ is a vertex colouring with the property that every colour class is a $t$-stable set, and the $t$-improper chromatic number $\chi_t(G)$ of $G$ is the least number of colours necessary for a $t$-improper colouring of $G$. Obviously, a 0-improper colouring is a proper colouring of a graph, and the 0-improper chromatic number is the chromatic number of a graph.

The $t$-improper chromatic number is a parameter that was introduced and studied independently by Andrews and Jacobson [4], Harary and Fraughnau (née Jones) [10, 11], and by Cowen et al. [7]. The importance of the $t$-stability number in relation to the $t$-improper chromatic number comes from the following obvious inequality: if $G$ is a graph that has $n$ vertices, then

$$\chi_t(G) \geq \frac{n}{\alpha_t(G)}.$$}

The $t$-improper chromatic number also arises in a specific type of radio-frequency assignment problem. Let us assume that the vertices of a given graph represent transmitters and an edge between two vertices indicates that the corresponding transmitters interfere. Each interference creates some amount of noise which we denote by $N$. Overall, a transmitter can tolerate up to a specific amount of noise which we denote by $T$. The problem now is to
assign frequencies to the transmitters and, more specifically, to assign as few frequencies as possible, so that we minimise the use of the electromagnetic spectrum. Therefore, any given transmitter cannot be assigned the same frequency as more than $T/N$ nearby transmitters — that is, neighbours in the transmitter graph — as otherwise the excessive interference would distort the transmission of the signal. In other words, the vertices/transmitters that are assigned a certain frequency must form a $T/N$-stable set, and the minimum number of frequencies we can assign is the $T/N$-improper chromatic number.

Given a graph $G = (V, E)$, we let $S_t = S_t(G)$ be the collection of all subsets of $V$ that are $t$-stable. We shall determine the order of the largest member of $S_t$ in a random graph $G_{n,p}$. Recall that $G_{n,p}$ is a random graph on a set of $n$ vertices, which we assume to be $V_n := \{1, \ldots, n\}$, and each pair of distinct vertices is present as an edge with probability $p$ independently of every other pair of vertices. Our interest is in dense random graphs, which means that we take $0 < p < 1$ to be a fixed constant.

We say that an event occurs asymptotically almost surely (a.a.s.) if it occurs with probability that tends to $1$ as $n \to \infty$.

1.1. Related background. The $t$-stability number of $G_{n,p}$ for the case $t = 0$ has been studied thoroughly for both fixed $p$ and $p(n) = o(1)$. Matula [18, 19, 20] and, independently, Grimmett and McDiarmid [9] were the first to notice and then prove asymptotic concentration of the stability number using the first and second moment methods. For $0 < p < 1$, define $b := 1/(1 - p)$ and

$$\alpha_{0,p}(n) := 2 \log_b n - 2 \log_b \log_b n + 2 \log_b(e/2) + 1.$$ 

For fixed $0 < p < 1$, it was shown that for any $\varepsilon > 0$ a.a.s.

$$\lfloor \alpha_{0,p}(n) - \varepsilon \rfloor \leq \alpha_0(G_{n,p}) \leq \lfloor \alpha_{0,p}(n) + \varepsilon \rfloor,$$

showing in particular that $\chi(G_{n,p}) \geq (1 - \varepsilon)n/\alpha_{0,p}(n)$. Bollobás and Erdős [4] extended (1) to hold with $p(n) > n^{-\delta}$ for any $\delta > 0$ (and $p(n)$ bounded away from unity). Much later, with the use of martingale techniques, Frieze [3] showed that for any $\varepsilon > 0$ there exists some constant $C_\varepsilon$ such that if $p(n) \geq C_\varepsilon/n$ then (1) holds a.a.s.

Efforts to determine the chromatic number of $G_{n,p}$ took place in parallel with the study of the stability number. For fixed $p$, Grimmett and McDiarmid showed that, for any $\varepsilon > 0$, the expected number $C_{n,j}$ of $j$-colourings of $G_{n,p}$ satisfies the following a.a.s.:

$$C_{n,j} \to \begin{cases} 0 & \text{if } j \leq (1 - \varepsilon)n/\alpha_{0,p}(n) \\ \infty & \text{if } j \geq (1 + \varepsilon)n/\alpha_{0,p}(n) \end{cases},$$

furthermore, they conjectured that $\chi(G_{n,p}) \sim n/\alpha_{0,p}(n)$ a.a.s. This conjecture remained one of the most tantalising open problems in random graph theory for over a decade, until Bollobás [2] and Matula and Kučera [17] used martingales to establish the conjecture. It was crucial for this work to obtain strong upper bounds on the probability of nonexistence in $G_{n,p}$ of a stable set with just slightly fewer than $\alpha_{0,p}(n)$ vertices. Luczak [15] fully extended the result to hold for sparse random graphs; that is, for the case $p(n) = o(1)$ and $p(n) \geq C/n$ for some large enough constant $C$.

Consult Bollobás [3] or Janson, Łuczak and Ruciński [14] for a detailed survey of these as well as related results.

For the case $t \geq 1$, the first results on the $t$-stability number were developed indirectly as a consequence of broader work on hereditary properties of random graphs. A graph property — that is, an infinite class of graphs closed under isomorphism — is said to be hereditary if every induced subgraph of every member of the class is also in the class. For any given $t$, the
class of graphs that are \( t \)-stable is an hereditary property. As a result of study in this more general context, it was shown by Scheinerman \cite{23} that, for fixed \( p \), there exist constants \( c_{p,1} \) and \( c_{p,2} \) such that \( c_{p,1} \ln n \leq \alpha_t(G_{n,p}) \leq c_{p,2} \ln n \) a.a.s. This was further improved by Bollobás and Thomason \cite{5} who characterised, for any fixed \( p \), an explicit constant \( c_p \) such that \((1 - \varepsilon)c_p \ln n \leq \alpha_t(G_{n,p}) \leq (1 + \varepsilon)c_p \ln n \) a.a.s. For any fixed hereditary property, not just \( t \)-stability, the constant \( c_p \) depends upon the property but essentially the same result holds. Recently, Kang and McDiarmid \cite{15} considered \( t \)-stability separately, but also treated the situation in which \( t = t(n) \) varies (i.e. grows) in the order of the random graph. They showed that, if \( t = o(\ln n) \), then a.a.s.

\[
(1 - \varepsilon)2 \log_b n \leq \alpha_t(G_{n,p}) \leq (1 + \varepsilon)2 \log_b n
\]

(where \( b = 1/(1 - p) \), as above). In particular, observe that the estimation \cite{2} for \( \alpha_t(G_{n,p}) \) and the estimation \cite{1} for \( \alpha_0(G_{n,p}) \) agree in their first-order terms. This implies that as long as \( t = o(\ln n) \) the \( t \)-improper and the ordinary chromatic numbers of \( G_{n,p} \) have roughly the same asymptotic value a.a.s.

1.2. The results of the present work. In this paper, we restrict our attention to the case in which the edge probability \( p \) and the non-negative integer parameter \( t \) are fixed constants. Restricted to this setting, our main theorem is an extension of \cite{1} and a strengthening of \cite{2}.

**Theorem 1.** Fix \( 0 < p < 1 \) and \( t \geq 0 \). Set \( b := 1/(1 - p) \) and

\[
\alpha_t(n) := 2 \log_b n + (t - 2) \log_b \log_b n + \log_b(t^2 + t) + t \log_b(2bp/e) + 2 \log_b(e/2) + 1.
\]

Then for every \( \varepsilon > 0 \) a.a.s.

\[
[\alpha_t(n) - \varepsilon] \leq \alpha_t(G_{n,p}) \leq [\alpha_t(n) + \varepsilon].
\]

We shall see that this theorem in fact holds if \( \varepsilon = \varepsilon(n) \) as long as \( \varepsilon \gg \ln \ln n/\sqrt{\ln n} \).

We derive the upper bound with a first moment argument, which is presented in Section 3. To apply the first moment method, we estimate the expected number of \( t \)-stable sets that have order \( k \). In particular, we show the following.

**Theorem 2.** Fix \( 0 < p < 1 \) and \( t \geq 0 \). Let \( \alpha_t^{(k)}(G) \) denote the number of \( t \)-stable sets of order \( k \) that are contained in a graph \( G \). If \( k = O(\ln n) \) and \( k \to \infty \) as \( n \to \infty \), then

\[
\mathbb{E}(\alpha_t^{(k)}(G_{n,p})) = \left( e^{2n^2b^k} - k^{t - 2} \frac{2(tbp/e)}{1} \right)^{k/2} (1 + o(1))^k.
\]

Using this formula, we will see in Section 3 that the expected number of \( t \)-stable sets with \([\alpha_t(n) + \varepsilon] + 1 \) vertices tends to zero as \( n \to \infty \).

The key to the calculation of this expected value is a precise formula for the number of degree sequences on \( k \) vertices with a given number of edges and maximum degree at most \( t \). We obtain this formula by the inversion formula of generating functions — applied in our case to the generating function of degree sequences on \( k \) vertices and maximum degree at most \( t \). This formula is an integral of a complex function that is approximated with the use of an analytic technique called saddle-point approximation. Our proof is inspired by the application of this method by Chvátal \cite{6} to a similar generating function. This is presented in Section 2.

The lower bound in Theorem 1 is derived with a second moment argument in Section 4.

We remark that Theorems 1 and 2 are both stated to hold for the case \( t = 0 \) (if we assume that \( 0^0 = 1 \)) in order to stress that these results generalise the previous results of
Matula [18, 19, 20] and Grimmett and McDiarmid [9]. Our methods indeed apply for this special case, however in our proofs our main concern will be to establish the results for \( t \geq 1 \).

Finally, in Section 5 we give a fairly precise formula for the \( t \)-improper chromatic number of \( G_{n,p} \). For \( t = 0 \), that is, for the chromatic number, McDiarmid [21] gave a fairly tight estimate on \( \chi(G_{n,p}) = \chi_0(G_{n,p}) \) proving that for any fixed \( 0 < p < 1 \) a.a.s.

\[
\frac{n}{\alpha_{0,p}(n) - 1 - o(1)} \leq \chi_0(G_{n,p}) \leq \frac{n}{\alpha_{0,p}(n) - 1 - \frac{1}{2} - \frac{1}{1-(1-p)^{1/2}} + o(1)}.
\]

Panagiotou and Steger [22] recently improved the lower bound showing that a.a.s.

\[
\chi_0(G_{n,p}) \geq \frac{n}{\alpha_{0,p}(n) - 2 - \frac{2}{\ln b} + o(1)}.
\]

In Section 5, we improve the upper bound and we generalise (for \( t \geq 1 \)) both this improved upper bound and the lower bound of Panagiotou and Steger.

**Theorem 3.** Fix \( 0 < p < 1 \) and \( t \geq 0 \). Then a.a.s.

\[
\frac{n}{\alpha_{t,p}(n) - 1 - \frac{2}{\ln b} + o(1)} \leq \chi_t(G_{n,p}) \leq \frac{n}{\alpha_{t,p}(n) - 2 - \frac{2}{\ln b} - o(1)}.
\]

### 2. Counting degree sequences of maximum degree \( t \)

Given non-negative integers \( k, t \) with \( t < k \), we let

\[
C_{2m}(t, k) := \sum_{\{d_1, \ldots, d_k\} : \sum_i d_i = 2m, d_i \leq t} \frac{1}{\prod_i d_i!}.
\]

(Here, the \( d_i \) are non-negative integers.) Given a fixed degree sequence \( (d_1, \ldots, d_k) \) with \( \sum_i d_i = 2m \), the number of graphs on \( k \) vertices \( (v_1, \ldots, v_k) \) where \( v_i \) has degree \( d_i \) is at most

\[
\frac{1}{\prod_i d_i! m!2^m}.
\]

See for example [3] in the proof of Theorem 2.16 or Section 9.1 in [14] for the definition of the configuration model, from which the above claim follows easily. Therefore, \( C_{2m}(t, k)(2m)!/(m!2^m) \) is an upper bound on the number of graphs with \( k \) vertices and \( m \) edges such that each vertex has degree at most \( t \).

In the proof of Theorem 2 we will need fairly tight estimates for \( C_{2m}(t, k) \), when \( 2m \) is close to \( tk \). In particular, as we will see in the next section (proof of Lemma 7) we will need a tight estimate for \( C_{2m}(t, k) \) when \( t - \ln k/\sqrt{k} < 2m/k < t - 1/(\sqrt{k} \ln k) \), since it turns out that in this range the expected number of \( t \)-stable sets having \( m \) edges is maximised. However, in Theorem 5 below we will give an asymptotic estimate of \( C_{2m}(t, k) \) for a much wider range of the ratio \( 2m/k \).

For \( t \geq 1 \), note that \( C_{2m}(t, k) \) is the coefficient of \( z^{2m} \) in the following generating function:

\[
G(z) = R_t(z)^k = \left( \sum_{i=0}^t \frac{z^i}{i!} \right)^k.
\]

Cauchy’s integral formula gives

\[
C_{2m}(t, k) = \frac{1}{2\pi i} \int_C \frac{R_t(z)^k}{z^{2m+1}} dz,
\]

where the integration is taken over a closed contour containing the origin.

Before we state the main theorem of this section, we need the following lemma.
Lemma 4. Fix \( t \geq 1 \). The function \( r R_t'(r)/R_t(r) \) is strictly increasing in \( r \) for \( r > 0 \). For each \( y \in (0, t) \), there exists a unique positive solution \( r_0 = r_0(y) \) to the equation \( r R_t'(r)/R_t(r) = y \) and furthermore the function \( r_0(y) \) is a continuous bijection between \((0, t)\) and \((0, \infty)\). Thus, if we set

\[
s(y) = r_0(y) \frac{d}{dx} \frac{xR_t'(x)}{R_t(x)} \bigg|_{x=r_0(y)},
\]

then \( s(y) > 0 \).

Theorem 5. Assume that \( t \geq 1 \) is fixed and \( k \to \infty \). If \( m \) and \( k \) are such that \( \varepsilon \leq 2m/k < t - \ln^7 k/k \) for any \( \varepsilon > 0 \), and \( r_0 \) and \( s \) are defined as in Lemma 4, then uniformly

\[
C_{2m}(t, k) = \frac{1}{\sqrt{2\pi k s(2m/k)}} \frac{R_t(r_0(2m/k))^k}{r_0(2m/k)^{2m}} (1 + o(1)).
\]

In the proof of the theorem (as well as in later sections), we make frequent use of the following lemma. Proofs of both Lemmas 4 and 6 are postponed until the end of the section.

Lemma 6. If \( y = y(k) \to t \) as \( k \to \infty \) (and \( y < t \)) and \( r_0 \) and \( s \) are defined as in Lemma 4, then

\[
\begin{align*}
(3) & \quad r_0 = \frac{t}{t - y} + O(1), \\
(4) & \quad \frac{dr_0}{dy} = \frac{r_0^2}{t} \left( 1 + O \left( \frac{1}{r_0} \right) \right), \quad \text{and} \\
(5) & \quad s = \frac{t}{r_0} \left( 1 + O \left( \frac{1}{r_0} \right) \right).
\end{align*}
\]

Proof of Theorem 5. The proof is inspired by [6]. Throughout, we for convenience drop the subscript and write \( R(z) \) in the place of \( R_t(z) \). Let \( C \) be the circle of radius \( r_0 \) centred at the origin. Using polar coordinates, we obtain

\[
C_{2m}(t, k) = \frac{1}{2\pi i} \int_C \frac{R(r_0 e^{i\varphi})^k}{r_0^{2m+1} e^{2m\varphi} e^{i\varphi}} d(r_0 e^{i\varphi}) = \frac{1}{2\pi r_0^{2m}} \int_{-\pi}^{\pi} R(r_0 e^{i\varphi})^k e^{i2m\varphi} d\varphi.
\]

We let \( \delta = \delta(k) := \ln k \sqrt{r_0/k} \) and write

\[
C_{2m}(t, k) = \frac{1}{2\pi r_0^{2m}} \left( \int_{-\pi}^{2\pi-\delta} \frac{R(r_0 e^{i\varphi})^k}{e^{i2m\varphi}} d\varphi + \int_{-\delta}^{\delta} \frac{R(r_0 e^{i\varphi})^k}{e^{i2m\varphi}} d\varphi \right).
\]

Note that, since \( 2m/k < t - \ln^7 k/k \), it follows from (3) that \( \delta \to 0 \) as \( k \to \infty \). We shall analyse the two integrals of (6) separately; however, we must also consider two cases for the behaviour of \( 2m/k \):

(A) There is some \( \varepsilon' > 0 \) such that \( 2m/k < t - \varepsilon' \) for any \( k \).

(B) \( 2m/k \to t \) as \( k \to \infty \) (and \( 2m/k < t - \ln^7 k/k \)).

Observe that in Case (A) there are positive constants \( a, b \) such that \( a \leq r_0 \leq b \) for \( k \) sufficiently large, whereas \( r_0 \to \infty \) as \( k \to \infty \) in Case (B) by (3).
To begin, we consider the first integral of (6) and we wish to show that it makes a negligible contribution to the value of $C_{2n}(t, k)$. Note that

$$|R(t_0e^{iφ})|^2 = \left( \sum_{j=0}^{t} \frac{r_0^j}{j!} \cos(jφ) \right)^2 + \left( \sum_{j=0}^{t} \frac{r_0^j}{j!} \sin(jφ) \right)^2$$

$$= \sum_{0 \leq j_1, j_2 \leq t} \frac{r_0^{j_1+j_2}}{j_1!j_2!} (\cos(j_1φ) \cos(j_2φ) + \sin(j_1φ) \sin(j_2φ))$$

$$= \sum_{0 \leq j_1, j_2 \leq t} \frac{r_0^{j_1+j_2}}{j_1!j_2!} \cos((j_1 - j_2)φ)$$

(7)

$$= R(t_0)^2 - \sum_{0 \leq j_1 < j_2 \leq t} \frac{2r_0^{j_1+j_2}}{j_1!j_2!} (1 - \cos((j_1 - j_2)φ)).$$

If we are in Case [A], then (since $|\cos| \leq 1$)

$$|R(t_0e^{iφ})|^2 \leq R(t_0)^2 - 2r_0 (1 - \cos φ) = R(t_0)^2 \left( 1 - \frac{2r_0}{R(t_0)^2} (1 - \cos φ) \right)$$

so that

$$|R(t_0e^{iφ})|^k \leq R(t_0)^k \left( 1 - \frac{2r_0}{R(t_0)^2} (1 - \cos φ) \right)^{k/2} \leq R(t_0)^k \exp \left( -\frac{r_0k}{R(t_0)^2} (1 - \cos φ) \right).$$

It follows that

$$\left| \int_{\delta}^{2π-δ} \frac{R(t_0)e^{iφ}}{e^{j2mφ}} dφ \right| \leq \int_{\delta}^{2π-δ} \left| R(t_0)e^{iφ} \right| dφ \leq 2πR(t_0)^k \exp \left( -\frac{r_0k}{R(t_0)^2} (1 - \cos δ) \right)$$

(8)

$$= 2πR(t_0)^k \exp \left( -\frac{r_0}{R(t_0)^2} \cdot \frac{kδ^2}{\ln k} \cdot \frac{1 - \cos δ}{δ^2} \cdot \ln k \right).$$

If instead we are in Case [B] then $t_0 \to ∞$ as $k \to ∞$ and hence, from (7),

$$|R(t_0e^{iφ})|^2 \leq R(t_0)^2 \left( 1 - \frac{2r_0^{2t-1}}{R(t_0)^{2t}} \left( 1 - \cos φ \right) \right) = R(t_0)^2 \left( 1 - \left( 1 + o(1) \right) \frac{2t}{r_0^2} (1 - \cos φ) \right).$$

It follows that for $k$ large enough

$$\left| \int_{\delta}^{2π-δ} \frac{R(t_0)e^{iφ}}{e^{j2mφ}} dφ \right| \leq 2πR(t_0)^k \left( 1 - \left( 1 + o(1) \right) \frac{2t}{r_0^2} (1 - \cos δ) \right)^{k/2} \leq 2πR(t_0)^k \exp \left( -\frac{tk}{2r_0} (1 - \cos δ) \right)$$

(9)

$$= 2πR(t_0)^k \exp \left( -\frac{t}{2} \cdot \frac{kδ^2}{r_0 \ln k} \cdot \frac{1 - \cos δ}{δ^2} \cdot \ln k \right).$$

Since $δ \to 0$, we have that $(1 - \cos δ)/δ^2 \to 1/2$. By the choice of $δ$, we also have that $kδ^2/(r_0 \ln k) \to ∞$ as $k \to ∞$, and it follows, for both Cases [A] and [B] from Inequalities (8) and (9) that

$$\left| \int_{\delta}^{2π-δ} \frac{R(t_0)e^{iφ}}{e^{j2mφ}} dφ \right| < R(t_0)^k/k,$$

for large enough $k$.
In order to precisely estimate the second integral of \((6)\), we consider the following function:

\[
 f(\varphi) := R(r_0e^{i\varphi}) \exp\left(-\frac{2m}{k}\varphi\right) = \exp\left(-\frac{2m}{k}\varphi\right) \left(\sum_{j=0}^{t} \frac{r_0^j}{j!} (\cos(j\varphi) + i\sin(j\varphi))\right).
\]

The importance of the function \(f\) is that

\[
 \int_{-\delta}^{\delta} R(r_0e^{i\varphi})k e^{i2m\varphi} d\varphi = \int_{-\delta}^{\delta} f(\varphi) k d\varphi.
\]

We will show that the real part of \(f(\varphi)^k\) is well approximated by \(R(r_0)^k \exp(-sk\varphi^2/2)\) when \(|\varphi|\) is small. The imaginary part can be ignored as the integral approximates a real quantity.

In order to apply Taylor’s Theorem, we shall need the first, second and third derivatives of \(f\) with respect to \(\varphi\). First,

\[
 f'(\varphi) = \exp\left(-\frac{2m}{k}\varphi\right) \left(\sum_{j=0}^{t} \frac{r_0^j}{j!} \left(\frac{2m}{k} - j\right) (\sin(j\varphi) - i\cos(j\varphi))\right).
\]

Note that

\[
 f'(0) = -i \left(\frac{2m}{k} \sum_{j=0}^{t} \frac{r_0^j}{j!} - \sum_{j=0}^{t} \frac{r_0^j}{j!} j\right) = -i \left(\frac{2m}{k} R(r_0) - r_0R'(r_0)\right) = 0
\]

by the choice of \(r_0\). Next,

\[
 f''(\varphi) = -i \frac{2m}{k} f'(\varphi) + \exp\left(-\frac{2m}{k}\varphi\right) \left(\sum_{j=0}^{t} \frac{r_0^j}{j!} \left(\frac{2m}{k} - j\right) j(\cos(j\varphi) + i\sin(j\varphi))\right).
\]

Therefore,

\[
 f''(0) = -i \frac{2m}{k} f'(0) + \sum_{j=0}^{t} \frac{r_0^j}{j!} \left(\frac{2m}{k} - j\right) j
\]

\[
 = \frac{2m}{k} \sum_{j=1}^{t} \frac{r_0^j}{j!} j - \sum_{j=1}^{t} \frac{r_0^j}{j!} j(j - 1) - \sum_{j=1}^{t} \frac{r_0^j}{j!} j
\]

\[
 = \left(\frac{r_0R'(r_0)}{R(r_0)}\right) r_0R'(r_0) - r_0^2 R''(r_0) - r_0R'(r_0)
\]

\[
 = -r_0 \left(\frac{-r_0R'(r_0)^2}{R(r_0)} + r_0R''(r_0) + R'(r_0)\right)
\]

\[
 = -R(r_0)r_0 \left(\frac{(r_0R''(r_0) + R'(r_0))R(r_0) - r_0R'(r_0)^2}{R(r_0)^2}\right)
\]

\[
 = -R(r_0)r_0 d \left(\frac{xR'(x)}{R(x)}\right) \bigg|_{x=r_0} = -R(r_0)s(2m/k).
\]
Thus, \( f''(0) < 0 \) by Lemma 4. Last, we have

\[
f'''(\phi) = - i \frac{2m}{k} f''(\phi) - i \frac{2m}{k} \exp\left(-i \frac{2m}{k} \phi \right) \left( \sum_{j=0}^{t} \frac{r_0^j}{j!} \left( \frac{2m}{k} - j \right) j(\sin(j\phi) + i\sin(j\phi)) \right) + \exp\left(-i \frac{2m}{k} \phi \right) \left( \sum_{j=0}^{t} \frac{r_0^j}{j!} \left( \frac{2m}{k} - j \right) j^2(-\sin(j\phi) + i\sin(j\phi)) \right).
\]

Since \( \epsilon < 2m/k \), there is a positive constant \( a \) such that \( a \leq r_0 \), for \( k \) sufficiently large. Clearly, \( f(0) = R(r_0) > a^t/t! > 0 \). The continuity of \( f \) on the compact set \(-\pi \leq \phi \leq \pi\) implies that there is a positive constant \( \delta_0 \) such that whenever \( |\phi| \leq \delta_0 \) we have \( Re(f(\phi)) > 0 \). Since the first two derivatives of \( \text{Im}(f(\phi)) \) with respect to \( \phi \) vanish when \( \phi = 0 \), and also \( \text{Im}(f(0)) = 0 \), Taylor’s Theorem implies that

\[
|\text{Im}(f(\phi))| \leq \sup_{|\phi| \leq \delta_0} |\text{Im}(f'''(\phi))| \frac{\phi^3}{6}
\]

if \( |\phi| \leq \delta_0 \). Now, note that \( Re(f(\phi)) \) and \( \text{Im}(f'''(\phi)) \) can be considered as polynomials of degree \( t \) with respect to \( r_0 \). The leading term of \( Re(f(\phi)) \) is

\[
Re\left( \exp\left(-i \frac{2m}{k} \phi \right) \left( \cos(t\phi) + i\sin(t\phi) \right) \right) \frac{r_0^t}{t!};
\]

thus, if \( r_0 \to \infty \) as \( k \to \infty \), then \( \text{Re}(f(\phi)) = \Omega(r_0^t) \). On the other hand, using the derivative computations above and simplifying, it follows that the leading term of \( \text{Im}(f'''(\phi)) \) is

\[
\text{Im}\left( \exp\left(-i \frac{2m}{k} \phi \right) \left( \sin(t\phi) + i\cos(t\phi) \right) \right) \left( \frac{t - 2m}{k} \right)^3 \frac{r_0^t}{t!};
\]

If \( r_0 \to \infty \) as \( k \to \infty \), then, by [3], \( t - 2m/k = (1+o(1))t/r_0 \) and thus \( \text{Im}(f'''(\phi)) = O(r_0^{t-1}) \). So, there exists \( c_1 > 0 \) such that for every \( \phi \) with \( |\phi| \leq \delta_0 \)

\[
\sup_{|\phi| \leq \delta_0} \left| \frac{\text{Im}(f'''(\phi))}{\text{Re}(f(\phi))} \right| < \frac{c_1}{r_0},
\]

and therefore

\[
\left| \frac{\text{Im}(f(\phi))}{\text{Re}(f(\phi))} \right| \leq \frac{c_1 \phi^3}{6r_0},
\]

for any \( \phi \) with \( |\phi| \leq \delta_0 \). On the other hand, we have (see pages 15–16 of [6] for the details)

\[
\left| \frac{\text{Re}(z^k)}{\text{Re}(z)^k} - 1 \right| \leq \epsilon \left( k, \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right| \right),
\]

with

\[
\epsilon(k, x) = (1 + x)^k - 1 - xk \leq e^{xk} - 1
\]

(for \( x \geq 0 \)). Since \( \epsilon(k, x) \) increases in \( x \) for \( x \geq 0 \), we have

\[
1 - \epsilon \left( k, \frac{c_1 \delta^3}{6r_0} \right) \leq \frac{\text{Re}(f(\phi)^k)}{\text{Re}(f(\phi))^k} \leq 1 + \epsilon \left( k, \frac{c_1 \delta^3}{6r_0} \right),
\]

whenever \( |\phi| \leq \delta \leq \delta_0 \).

Next, we approximate the function \( \ln \text{Re}(f(\phi)) \). First,

\[
\frac{d}{d\phi} \left( \ln \text{Re}(f(\phi)) \right) \bigg|_{\phi=0} = \frac{\text{Re}(f'(\phi))}{\text{Re}(f(\phi))} \bigg|_{\phi=0} = 0.
\]
Second, we have
\[ \frac{d^2}{d\varphi^2} (\ln \Re(f(\varphi))) = \frac{d}{d\varphi} \left( \frac{\Re(f'(\varphi))}{\Re(f(\varphi))} \right) = \frac{\Re(f''(\varphi))\Re(f(\varphi)) - \Re(f'(\varphi))^2}{\Re(f(\varphi))^2}; \]
therefore, by Equation (11),
\[ \frac{d^2}{d\varphi^2} (\ln \Re(f(\varphi))) \bigg|_{\varphi=0} = \frac{\Re(f''(0))\Re(f(0)) - \Re(f'(0))^2}{\Re(f(0))^2} = -\frac{R(r_0)s}{R(r_0)} = -s. \]
Third, note that routine calculations give that (for $|\varphi| \leq \delta_0$)
\[ \frac{d^3}{d\varphi^3} (\ln \Re(f(\varphi))) = \frac{\Re(f'''(\varphi))\Re(f(\varphi))^2 - 3\Re(f''(\varphi))\Re(f'(\varphi))\Re(f(\varphi)) + 2\Re(f'(\varphi))^3}{\Re(f(\varphi))^3}. \]
If, as we did earlier for $\Re(f(\varphi))$ and $\Im(f''(\varphi))$, we consider $\Re(f'(\varphi))$, $\Re(f''(\varphi))$ and $\Re(f'''(\varphi))$ as polynomials with respect to $r_0$, we can show that $\Re(f'(\varphi)) = O(r_0^{-1})$, $\Re(f''(\varphi)) = O(r_0^{-1})$ and $\Re(f'''(\varphi)) = O(r_0^{-1})$. It then follows that there exists $c_2 > 0$ such that for every $\varphi$ with $|\varphi| \leq \delta_0$
\[ \left| \frac{d^3}{d\varphi^3} (\ln \Re(f(\varphi))) \right| \leq \frac{c_2}{r_0}. \]
Therefore, Taylor’s Theorem implies that for every $\varphi$ with $|\varphi| \leq \delta_0$ we have
\[ \left| \ln \Re(f(\varphi)) - \left( \ln R(r_0) - \frac{s\varphi^2}{2} \right) \right| \leq \frac{c_2\varphi^3}{6r_0}. \]
It follows that
\[ \exp \left( -\frac{c_2k\delta^3}{6r_0} \right) \leq \frac{\Re(f(\varphi))^k}{R(r_0)^k\exp(-sk\varphi^2/2)} \leq \exp \left( \frac{c_2k\delta^3}{6r_0} \right). \]
If $r_0 \to \infty$ as $k \to \infty$, then the condition that $2m/k < t - \ln^7 k/k$ and \[(13)\]
together imply that $r_0 < tk/\ln^7 k + O(1)$. Therefore, $k\delta^3/r_0 = \sqrt{r_0/\ln^3 k} \to 0$ as $k \to \infty$, and we have
\[ \exp \left( \frac{c_2k\delta^3}{6r_0} \right) = 1 + o(1) \quad \text{and} \quad \epsilon \left( k, \frac{c_1\delta^3}{6r_0} \right) \leq \exp \left( \frac{c_1k\delta^3}{6r_0} \right) - 1 = o(1), \]
proving that
\[ \Re(f(\varphi)^k) = R(r_0)^k \exp(-sk\varphi^2/2) + O(1) \]
uniformly for $|\varphi| \leq \delta$. Thus, Inequality \[(10)\] along with the above imply that
\[ \int_C \frac{R(r_0e^{i\varphi})}{e^{2m\varphi}} d\varphi = \int_{-\delta}^{\delta} \frac{R(r_0e^{i\varphi})}{e^{2m\varphi}} d\varphi + \int_{-\delta}^{\delta} f(\varphi)^k d\varphi \]
\[ = R(r_0)^k \left( \int_{-\delta}^{\delta} \exp(-sk\varphi^2/2) d\varphi + o(1) \right). \]
Setting $y = \sqrt{k}\varphi$, observe that
\[ \int_{-\delta}^{\delta} \exp \left( -\frac{sk\varphi^2}{2} \right) d\varphi = \frac{1}{\sqrt{k}} \int_{-\delta\sqrt{k}}^{\delta\sqrt{k}} \exp \left( -\frac{sy^2}{2} \right) dy. \]
Since $\delta\sqrt{k} \to \infty$ as $k \to \infty$, we obtain
\[ \int_{-\delta\sqrt{k}}^{\delta\sqrt{k}} \exp \left( -\frac{sy^2}{2} \right) dy = \sqrt{\frac{2\pi}{s}} + o(1). \]
Thus, Equation (13) becomes
\[ \int_C \frac{R(r_0 e^{i\varphi})^k}{e^{2m\varphi}} d\varphi = \sqrt{\frac{2\pi}{ks}} R(r_0)^k (1 + o(1)). \]
and the result follows.

2.1. Proofs of the technical lemmas.

Proof of Lemma 4. Note that
\[ \lim_{r \to 0} \frac{r R'_t(r)}{R_t(r)} = 0, \quad \lim_{r \to \infty} \frac{r R'_t(r)}{R_t(r)} = t. \]
As \( 0 < 2m/k < t \), the lemma follows if we show that the function \( r R'_t(r)/R_t(r) \) is strictly increasing in \( r \) for \( r \geq 0 \).

First, if \( t = 1 \), then \( r R'_t(r)/R_t(r) = r/(r + 1) \) and it is easy to see that it is strictly increasing in \( r \) for \( r \geq 0 \). Indeed, its derivative with respect to \( r \) is \( 1/(r + 1)^2 \) and this is positive for \( r \geq 0 \).

For each pair \( x, y \) of reals such that \( 0 < x < y \), we will prove by induction on \( t \) that
\[ \frac{x R'_t(x)}{R_t(x)} < \frac{y R'_t(y)}{R_t(y)}. \]
Our inductive hypothesis is to assume that the above holds for some \( t \). Note that \( R_{t+1}(x) = R_t(x) + x^{t+1}/(t+1)! \) and \( R'_{t+1}(x) = R'_t(x) + x^t/t! \). Therefore, our aim is to prove that
\[ \frac{x R'_t(x) + x^{t+1}/t!}{R_t(x) + x^{t+1}/(t+1)!} < \frac{y R'_t(y) + y^{t+1}/t!}{R_t(y) + y^{t+1}/(t+1)!}. \]
To simplify notation, we set
\[ a(r) := r R'_t(r) = r + r^2 + \cdots + \frac{r^t}{(t-1)!}, \]
\[ b(r) := R_t(r) = 1 + r + \frac{r^2}{2!} + \cdots + \frac{r^t}{t!}, \]
so that Inequality (14) rewrites as follows:
\[ a(x)b(y) + a(x)\frac{y^{t+1}}{(t+1)!} + \frac{x^{t+1}}{t!} b(y) < a(y)b(x) + b(x)\frac{y^{t+1}}{t!} + a(y)\frac{x^{t+1}}{(t+1)!}. \]
Note that our inductive hypothesis is equivalent to \( a(x)b(y) < a(y)b(x) \). If we set
\[ g_1(x, y) := a(x)\frac{y^{t+1}}{(t+1)!} + \frac{x^{t+1}}{t!} b(y) \quad \text{and} \quad g_2(x, y) := a(y)\frac{x^{t+1}}{(t+1)!} + b(x)\frac{y^{t+1}}{t!}, \]
then it suffices to prove that \( g_2(x, y) - g_1(x, y) > 0 \). Now, note that
\[ (t+1)b(x) - a(x) = (t+1) \sum_{i=0}^{t} \frac{x^i}{i!} - \sum_{i=0}^{t} \frac{x^i}{(i-1)!} = \sum_{i=0}^{t} \frac{x^i}{i!}(t + 1 - i). \]
It follows that
\[ (g_2(x, y) - g_1(x, y))\frac{(t+1)!}{x^{t+1} y^{t+1}} = \frac{(t+1)b(x) - a(x)}{x^{t+1}} - \frac{(t+1)b(y) - a(y)}{y^{t+1}} \]
\[ = \sum_{i=0}^{t} \frac{x^{i-t-1}}{i!}(t + 1 - i) - \sum_{i=0}^{t} \frac{y^{i-t-1}}{i!}(t + 1 - i). \]
This last expression is positive because \( x < y \) and hence \( x^{i-t-1} > y^{i-t-1} \) for every \( i \leq t \). Since \( x > 0 \), it follows that \( g_2(x, y) - g_1(x, y) \) is positive as well. \( \square \)

**Proof of Equation \((3)\).** First, note that \( r_0 = r_0(y) \to \infty \) as \( k \to \infty \) by Lemma \(4\). So

\[
r_0 R'(r_0) = \frac{r_0 t}{(t-1)!} \left( 1 + \frac{t-1}{r_0} + O\left(\frac{1}{r_0^2}\right) \right),
\]

\[
R(r_0) = \frac{r_0^t}{t!} \left( 1 + \frac{t}{r_0} + O\left(\frac{1}{r_0^2}\right) \right).
\]

Thus,

\[
\frac{r_0 R'(r_0)}{R(r_0)} = t \frac{1 + \frac{t-1}{r_0} + O\left(\frac{1}{r_0^2}\right)}{1 + \frac{t}{r_0} + O\left(\frac{1}{r_0^2}\right)} = t \left( 1 + \frac{t-1}{r_0} + O\left(\frac{1}{r_0^2}\right) \right) \left( 1 - \frac{t}{r_0} + O\left(\frac{1}{r_0^2}\right) \right)
\]

\[
= t \left( 1 - \frac{t-y}{t} + O\left(\frac{1}{r_0^2}\right) \right) = t \left( 1 - \frac{1}{r_0} + O\left(\frac{1}{r_0^2}\right) \right).
\]

Since \( r_0 R'(r_0)/R(r_0) = y = t(1-(t-y)/t) \), we obtain

\[(15) \quad 1 - \frac{t-y}{t} = 1 - \frac{1}{r_0} + O\left(\frac{1}{r_0^2}\right),\]

which can be rewritten as

\[
r_0 = \frac{t}{t-y} \left( 1 + O\left(\frac{1}{r_0}\right) \right),
\]

and this implies the desired expression. \( \square \)

**Proof of Equation \((4)\).** A more careful treatment of the computations for the proof of \((2)\) shows that the \( O(1/r_0^2) \) error term in \((15)\) may instead be written \( \eta(1/r_0)/r_0^2 \) where \( \eta \) is a power series with positive radius of convergence. Then, by differentiating both sides of \((15)\) with respect to \( y \) and applying the chain rule to its right-hand side, we obtain

\[
\frac{1}{t} = \frac{d}{dr_0} \left( 1 - \frac{1}{r_0} + \eta \left(\frac{1}{r_0}\right) \frac{1}{r_0^2} \right) \frac{dr_0}{dy}.
\]

We have that

\[
\frac{d}{dr_0} \left( 1 - \frac{1}{r_0} + \eta \left(\frac{1}{r_0}\right) \frac{1}{r_0^2} \right) = \frac{1}{r_0^2} - \eta \left(\frac{1}{r_0}\right) \frac{2}{r_0^3} - \eta' \left(\frac{1}{r_0}\right) \frac{1}{r_0^4} = \frac{1}{r_0^2} + O\left(\frac{1}{r_0^3}\right)
\]

and the lemma immediately follows. \( \square \)

**Proof of Equation \((5)\).** By the definition of \( r_0 \), it follows from the chain rule that

\[
1 = \frac{d}{dy} \frac{r_0 R'(r_0)}{R(r_0)} = \frac{d}{dr_0} \frac{r_0 R'(r_0)}{R(r_0)} \frac{dr_0}{dy}.
\]

Thus,

\[
\frac{d}{dx} \frac{x R'(x)}{R(x)} \bigg|_{x=r_0(y)} = \left( \frac{dr_0(y')}{dy'} \bigg|_{y'=y} \right)^{-1},
\]

implying that

\[
s(y) = r_0(y) \left( \frac{dr_0(y')}{dy'} \bigg|_{y'=y} \right)^{-1} = \frac{t}{r_0(y)} \left( 1 + O\left(\frac{1}{r_0(y)}\right) \right)
\]

as required. \( \square \)
3. The expected number of \( t \)-stable sets of order \( k \) - proof of Theorem 2

In this section, we give an asymptotic expression for the expected number of \( t \)-stable subsets of \( V_n \) of order \( k \) in \( G_{n,p} \), proving Theorem 2. We will assume that \( k = k(n) = O(\ln n) \) and also \( k \to \infty \) as \( n \to \infty \). Towards the end of the section, we will specify \( k \) and derive the upper bound of Theorem 1 by a first moment argument.

Let \( A \) be a subset of \( V_n \) that has order \( k \). If \( \alpha_t^{(k)}(G_{n,p}) \) denotes the number of subsets of \( V_n \) of order \( k \) that are \( t \)-stable, then

\[
\mathbb{E}(\alpha_t^{(k)}(G_{n,p})) = \binom{n}{k} \mathbb{P}(A \in S_t).
\]

Partitioning according to the number of edges that \( A \) induces, we have

\[
\mathbb{P}(A \in S_t) = \sum_{m=0}^{\lfloor tk/2 \rfloor} \mathbb{P}(A \in S_t, e(A) = m).
\]

By the definition of \( C_{2m}(t,k) \) (given at the beginning of Section 2), it follows that

\[
\mathbb{P}(A \in S_t, e(A) = m) \leq p^m(1-p)^{t/2-m}C_{2m}(t,k)\frac{(2m)!}{m!2^m} =: f(m).
\]

First, we find the value of \( m \) for which the expression \( f(m) \) on the right-hand side of (17) is maximised. If \( m^* \) is such that \( f(m^*) = \max\{f(m) : 0 \leq 2m \leq tk \} \), it turns out that the following holds.

**Lemma 7.** \( 2m^* = tk - \sqrt{tk/\ln k} + o(\sqrt{k}) \).

**Proof.** Let \( \lambda_m = \lambda_m(t,k) = f(m+1)/f(m) \). Thus,

\[
\lambda_m = \frac{p}{1-p} \frac{C_{2m+2}(t,k) 1 \cdot (2m+2)(2m+1)}{2} \leq \frac{p}{1-p} \frac{C_{2m+2}(t,k)}{C_{2m}(t,k)} (2m+1).
\]

We will estimate \( \lambda_m \) for all \( m \) with \( 0 \leq 2m \leq tk \) and treat three separate cases:

(A) \( 2m < tk - \sqrt{tk/\ln k} \);

(B) \( 2m > tk - \sqrt{tk/\ln k} \); and

(C) \( tk - \sqrt{tk/\ln k} \leq 2m \leq tk - \sqrt{tk/\ln k} \).

We will use Theorem 5 in Case (C) as we will determine those values \( m \) for which \( \lambda_m \approx 1 \) within that range. In the other cases we will use a cruder argument, which is nonetheless sufficient for our purposes.

**Case (A).** We will show that \( \lambda_m > 1 \) for any such \( m \). We set \( S_{2m}(t,k) = (2m)!C_{2m}(t,k) \). Note that this is equal to the number of ways of allocating \( 2m \) labelled balls into \( k \) bins so that each bin does not receive more than \( t \) balls — we also denote the set of such allocations by \( S_{2m}(t,k) \). We have

\[
\frac{C_{2m+1}(t,k)}{C_{2m}(t,k)} = \frac{S_{2m+1}(t,k)}{S_{2m}(t,k)} \frac{1}{(2m+2)(2m+1)}.
\]

We will obtain a lower bound on the left-hand side, by first obtaining a lower bound on the ratio \( S_{2m+1}(t,k)/S_{2m}(t,k) \). Let us consider \( 2m+2 \) distinct balls which we label 1, \ldots, 2m+1, 2m+2. We construct an auxiliary bipartite graph whose parts are \( S_{2m}(t,k) \) and \( S_{2m+2}(t,k) \). If \( c \in S_{2m}(t,k) \) and \( c' \in S_{2m+2}(t,k) \), then \( c, c' \) forms an edge in the auxiliary graph if \( c' \) restricted to balls 1, \ldots, 2m is \( c \). So any \( c' \in S_{2m+2}(t,k) \) is adjacent to exactly one configuration \( c \in S_{2m}(t,k) \), that is, its degree in the auxiliary graph is equal to 1. Also, if \( e(c) \) is the number of non-full bins in a configuration \( c \in S_{2m}(t,k) \), then \( e(c) \) has at least...
$e(c)(e(c) - 1)$ neighbours in $S_{2m+2}(t,k)$. This is the case since there are at least $e(c)(e(c) - 1)$ ways of allocating balls $2m+1$ and $2m+2$ into the non-full bins of $c$, therefore giving a lower bound on the number of configurations in $S_{2m+2}(t,k)$ whose restriction on the first $2m$ balls is $c$. But $2m < tk - \sqrt{k} \ln k$ and therefore $e(c) \geq \sqrt{k} (\ln k)/t$. These observations imply that for $k$ large enough

$$S_{2m+2}(t,k) \geq \frac{k \ln^2 k}{2t^2} S_{2m}(t,k),$$

and therefore

$$\frac{C_{2(m+1)}(t,k)}{C_{2m}(t,k)} = \frac{S_{2(m+1)}(t,k)}{S_{2m}(t,k)} \frac{1}{(2m+2)(2m+1)} \geq \frac{k \ln^2 k}{2(2m+2)(2m+1)} = \Omega\left(\frac{\ln^2 k}{m}\right).$$

So $\lambda_m = \Omega(\ln^2 k) > 1$ in Case (A).

**Case (B)** We treat this case similarly. We consider an auxiliary bipartite graph as above. Let $c \in S_{2m}(t,k)$ be a configuration of balls $1, \ldots, 2m$. Since there are at most $\sqrt{k}/\ln k$ places available in the non-full bins, there are at most $k/\ln^2 k$ ways of allocating balls $2m+1$ and $2m+2$ into the non-full bins of $c$. In other words, the degree of any vertex in $S_{2m}(t,k)$ is at most $k/\ln^2 k$. Also, as above, the degree of any vertex/configuration $c' \in S_{2m+2}(t,k)$ is equal to one. Therefore,

$$\frac{S_{2m+2}(t,k)}{S_{2m}(t,k)} \leq \frac{k}{\ln^2 k}.$$ 

Substituting this into (18), we obtain

$$\frac{C_{2(m+1)}(t,k)}{C_{2m}(t,k)} \leq \frac{k}{\ln^2 k} \frac{1}{(2m+2)(2m+1)}.$$ 

Therefore, in Case (B) we have

$$\lambda_m = O\left(\frac{k}{m \ln^2 k}\right) = O\left(\frac{1}{\ln^2 k}\right) = o(1).$$

**Case (C)** In this range, we need more accurate estimates, as we will identify those $m$ for which $\lambda_m$ is approximately equal to 1. We appeal to Theorem 5 for asymptotic estimates of $C_{2m}(t,k)$ and $C_{2m+2}(t,k)$ and write $\lambda_m = (1 + o(1))\tilde{\lambda}_m$ where

$$\tilde{\lambda}_m = \frac{p}{1-p} \left(\frac{s(2m/k)}{s(2(m+1)/k)}\right)^{1/2} \left(\frac{R(r_0(2(m+1)/k))}{R(r_0(2m/k))}\right)^k \frac{r_0(2m/k)^{2m}}{r_0(2(m+1)/k)^{2m+2}(2m+1)}.$$ (19)

Writing $2m = tk - xk$, we have $x = o(1)$. So, by (3) and (4), uniformly for every $z \in [t-x, t-x+2/k]$, we have

$$\frac{d r_0}{d y} \bigg|_{y=z} = \frac{t}{x^2} (1 + o(1));$$

thus, the Mean Value Theorem yields

$$r_0(2(m+1)/k) = r_0(2m/k) + \frac{2t}{x^2k} (1 + o(1)) = r_0(2m/k) \left(1 + \frac{2}{xk}(1 + o(1))\right).$$

So, since $xk \rightarrow \infty$ as $k \rightarrow \infty$, Equation (20) and (5) yield

$$\left(\frac{s(2m/k)}{s(2(m+1)/k)}\right)^{1/2} = 1 + o(1).$$ (21)
To estimate the third ratio of \((19)\), we write \(r_0(2(m+1)/k) = r_0(2m/k)(1+\eta)\) where \(\eta = (2/xk)(1+o(1))\) by \((20)\). We also write

\[
R(r_0(2(m+1)/k)) = \frac{r_0^t(2(m+1)/k)}{t!} \sum_{t=0}^{\infty} \frac{t!}{(t-\ell)!} \frac{1}{r_0^t(2(m+1)/k)}.
\]

Note that

\[
\sum_{t=0}^{\infty} \frac{t!}{(t-\ell)!} \frac{1}{r_0^t(2(m+1)/k)} = \sum_{t=0}^{\infty} \frac{t!}{(t-\ell)!} \frac{(1+\eta)^-\ell}{r_0^t(2m/k)} = \sum_{t=0}^{\infty} \frac{t!}{(t-\ell)!} \frac{1-\ell\eta(1+O(\eta))}{r_0^t(2m/k)}.
\]

Since this last big-O term is \(o(1/k)\), it follows that

\[
\frac{R(r_0(2(m+1)/k))}{r_0(2(m+1)/k)^t} = \frac{1}{t!} \left( 1 + \frac{t}{r_0(2m/k)}(1-\eta) + \frac{t(t-1)}{r_0^2(2m/k)} + o(1/k) \right)
\]

and similar calculations show that

\[
\frac{R(r_0(2m/k))}{r_0(2m/k)^t} = \frac{1}{t!} \left( 1 + \frac{t}{r_0(2m/k)} + \frac{t(t-1)}{r_0^2(2m/k)} + o(1/k) \right).
\]

So the third ratio in \((19)\) becomes

\[
\left( \frac{R(r_0(2(m+1)/k))}{R(r_0(2m/k))} \right)^k = \left( \frac{r_0(2(m+1)/k)}{r_0(2m/k)} \right)^{tk} \left( 1 - \frac{t\eta}{r_0(2m/k)} + o(1/k) \right)^k
\]

\[
= \left( \frac{r_0(2(m+1)/k)}{r_0(2m/k)} \right)^{tk} e^{-2(1+o(1))}
\]

where the last equality holds by the fact that

\[
\frac{t\eta t}{r_0(2m/k) x} = \frac{t(2/xk)k}{x} (1+o(1)) = 2(1+o(1)).
\]

Since \(xk \to \infty\), we have by \((20)\) and \((3)\) that

\[
r_0(2(m+1)/k) = r_0(2m/k)(1+o(1)) = (1+o(1))t/x.
\]

So using \((20)\) and \((22)\) we can write the product of the third and the fourth terms in \((19)\) as follows:

\[
\left( \frac{R(r_0(2(m+1)/k))}{R(r_0(2m/k))} \right)^k \times \frac{r_0^{2m}(2m/k)}{r_0^{2m+2}(2m+1/k)}
\]

\[
e^{-2} \left( \frac{r_0(2(m+1)/k)}{r_0(2m/k)} \right)^{tk} \left( \frac{1+o(1)}{r_0^2(2m+1/k)} \right)
\]

\[
e^{-2} \left( 1 + \frac{2}{xk} (1+o(1)) \right)^{xk} \frac{x^2}{t^2} (1+o(1)) = \frac{x^2}{t^2} (1+o(1)).
\]

If \(x \geq \omega(k)/\sqrt{k}\), where \(\omega(k) \to \infty\), then substituting this last equation and \((21)\) into \((19)\) and recalling that \(\lambda_m = (1+o(1))\hat{\lambda}_m\), we obtain

\[
\lambda_m = \Omega(1) \frac{x^2}{t^2} (2m+1) = \Omega \left( \frac{(\omega(k)^2 m}{k} \right) = \Omega(\omega(k)^2) \to \infty.
\]

If \(x \leq 1/(\omega(k)/\sqrt{k})\), then these substitutions yield

\[
\lambda_m = O(1) \frac{x^2}{t^2} (2m+1) = O \left( \frac{m}{\omega(k)^2 k} \right) = O \left( \frac{1}{\omega(k)^2} \right) = o(1).
\]
Assume now that \( x = \alpha / \sqrt{k} \), for some \( \alpha = \Theta(1) \). In this case,
\[
\lambda_m = \frac{p}{1 - p} \frac{\alpha^2}{t^2} (tk - xk + 1)(1 + o(1)) = \frac{p}{1 - p} \frac{\alpha^2}{t} (1 + o(1)) \frac{b^{1/(1-p)}}{t} (1 + o(1)).
\]
Thus for any fixed \( \alpha' < \sqrt{tk/bp} < \alpha'' \) and for \( k \) large enough we have \( tk - \alpha'' \sqrt{k} \leq 2m^* \leq tk - \alpha' \sqrt{k} \). Putting all these different cases together, we deduce that, if \( m^* \) is such that \( f(m^*) \) is maximised over the set \( 0 \leq 2m \leq tk \), then \( 2m^* = tk - \sqrt{tk/bp} + o(\sqrt{k}) \). This concludes the proof of Lemma 7.

Before we proceed to the proof of Theorem 2, let us use Lemma 7 to compute a precise asymptotic expression for \( f(m^*) \). First, observe that
\[
p^{m^*(1 - p)(\frac{t}{2} - m^*)} = b^{-\left(t\frac{k}{2}\right)}(bp)^{tk/2 - \sqrt{tk/bp} + o(\sqrt{k})} = b^{-\left(t\frac{k}{2}\right)}(bp)^{tk/2} \left(1 + O\left(\frac{1}{\sqrt{k}}\right)\right)^k.
\]
For the second part of the expression for \( f(m^*) \), note that, by Theorem 5,
\[
C_{2m^*}(t, k) = \frac{1}{\sqrt{2\pi s(2m^*/k)}} R(r_0(2m^*/k))^k R_0(2m^*/k)2^m^* (1 + o(1)).
\]
By (3), we have
\[
r_0(2m^*/k) = \sqrt{tk/bp} + o(\sqrt{k}).
\]
Thus, by (5), \( s(2m^*/k) = \Theta(1/\sqrt{k}) \). Now, it follows that
\[
R(r_0(2m^*/k)) = \frac{r_0^t(2m^*/k)}{t!} \sum_{t=0}^{t} t! \frac{1}{(t - \ell)!} \frac{1}{r_0^t(2m^*/k)} = \frac{r_0^t(2m^*/k)}{t!} \left(1 + \frac{t}{r_0(2m^*/k)} + O\left(\frac{1}{r_0(2m^*/k)}\right)\right)
\]
\[
= \frac{r_0^t(2m^*/k)}{t!} \left(1 + \sqrt{\frac{t}{bpk} + o\left(\frac{1}{\sqrt{k}}\right)}\right);
\]
therefore,
\[
R(r_0(2m^*/k))^k = \frac{(r_0(2m/k))^t}{t!^k} e^{\sqrt{tk/bp} + o(\sqrt{k})}.
\]
Substituting this into (24), we obtain
\[
C_{2m^*}(t, k) = \Theta(k^{1/4}) \frac{(r_0(2m^*/k))^t}{t!^k} e^{\sqrt{tk/bp} + o(\sqrt{k})} = \Theta(k^{1/4}) \left(\sqrt{tk/bp} + o(\sqrt{k})\right)^{\sqrt{tk/bp} + o(\sqrt{k})} e^{\sqrt{tk/bp} + o(\sqrt{k})} \frac{1}{t!^k}
\]
\[
= \frac{1}{t!^k} \left(1 + O\left(\frac{\ln k}{\sqrt{k}}\right)\right)^k.
\]
For the last part of the expression for \( f(m^*) \), we apply Stirling’s formula to obtain
\[
\frac{(2m^*)!}{m^*!2^{m^*}} = \frac{(2m^*/e)^{2m^*}}{(m^*/e)^{m^*}} \sqrt{2\pi(2m^*)} e^{o(1)} \frac{1}{2m^*} = \Theta(1) \left( \frac{2m^*}{e} \right)^{m^*} \frac{(tk - \sqrt{tk/bp} + o(\sqrt{k})}{e}^{tk/2-\sqrt{tk/bp}/2+o(\sqrt{k})}.
\]
(26)
\[
= \Theta(1) \left( \frac{tk}{e} \right)^{tk/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k.
\]

Now, substituting (23), (25) and (26) into the expression for \( f \) (given in (17)), we obtain the following:
\[
f(m^*) = b^{-k}(2)(bp)^{tk/2} \frac{1}{t!^k} \left( \frac{tk}{e} \right)^{tk/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k
\]
(27)
\[
= \left( b^{-k+1} \left( \frac{tbpk}{e} \right) \frac{1}{t!^2} \right)^{k/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k.
\]

**Upper bound on** \( \mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \). By (16) and (27), we deduce that
\[
\Pr(A \in S_t) \leq \left( \frac{tk}{2} + 1 \right)f(m^*) = \left( b^{-k+1} \left( \frac{tbpk}{e} \right) \frac{1}{t!^2} \right)^{k/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k.
\]
(28)
Thus, we obtain,
\[
\mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \leq \left( \frac{n}{k} \right) \left( b^{-k+1} \left( \frac{tbpk}{e} \right) \frac{1}{t!^2} \right)^{k/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k
\]
(29)
\[
= \left( e^{2n^2b^{-k+1}k} \left( \frac{tbpk}{e} \right) \frac{1}{t!^2} \right)^{k/2} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k.
\]

Now, if we set \( k = \lfloor \alpha_t(p,n) + \varepsilon(n) \rfloor \) for some function \( \varepsilon(n) \gg \ln \ln n / \sqrt{\ln n} \), then, substituting this into (29), we obtain
\[
\mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \leq \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right) b^{-\varepsilon} \left( 1 + O \left( \ln \frac{k}{\sqrt{k}} \right) \right)^k = o(1),
\]
thus proving the right-hand side inequality in Theorem [1].

**Lower bound on** \( \mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \). To derive the lower bound on \( \mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \), we observe
\[
\mathbb{E} \left( \alpha_t^{(k)}(G_{n,p}) \right) \geq \left( \frac{n}{k} \right) \Pr(A \in S_t, e(A) = m^*).\]

Let \((d_1, \ldots, d_k)\) be a degree sequence such that, for every \( 1 \leq i \leq k \), \( d_i \leq t \) and \( \sum_i d_i = 2m^* \). By Theorem 2.16 in [3], with \( \lambda := \frac{1}{m^*} \sum_i \binom{d_i}{2} \), the number of graphs with this degree sequence is
\[
(1 + o(1))e^{-\lambda/2-\lambda^2/4} \frac{(2m^*)!}{m^*!2^{m^*}}.
\]
But, since \( d_i \leq t \) for every \( i \), then using the estimate from Lemma \([7]\) we obtain \( \lambda \leq t^2k/2m^* \leq 2t \) for \( k \) large enough. So the total number of graphs on \( k \) vertices, \( m^* \) edges and with maximum degree at most \( t \) is at least

\[
\frac{e^{-t^2}}{2} C_{2m^*}(t,k) \frac{(2m^*)!}{m^!*2m^*}.
\]

Since \( k = O(\ln n) \), we have \( \binom{n}{k} = \Omega(\sqrt{1/k}(n\epsilon/k)^k) \). Hence, using \([27]\), we obtain

\[
\mathbb{E}(\alpha_t^k(G_{n,p})) \geq \binom{n}{k} P(A \in S_t, e(A) = m^*) \geq \binom{n}{k} \frac{e^{-t^2}}{2} f(m^*)
\]

\[\text{(30)}\]

\[
\quad = \left( e^2 n^2 b^{-k+1} k t^{-2} \left( \frac{tbp}{e} \right) \frac{1}{t!^2} \right)^{k/2} \left( 1 + O \left( \frac{\ln k}{\sqrt{k}} \right) \right)^k.
\]

If \( k = \lfloor \alpha_{t,p}(n) - \varepsilon(n) \rfloor \) for some function \( \varepsilon(n) \) satisfying \( \ln \ln n/\sqrt{\ln n} \ll \varepsilon(n) \ll \ln n \), then by \( \text{(30)} \) we obtain

\[\text{(31)}\]

\[
\mathbb{E}(\alpha_t^k(G_{n,p})) \geq \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right)^{k/2} \left( 1 + O \left( \frac{\ln k}{\sqrt{k}} \right) \right)^k = n^{\varepsilon(n)(1+o(1))} \to \infty.
\]

In the next section, we use a sharp concentration inequality to show moreover that the following holds.

**Lemma 8.** If \( \varepsilon(n) \gg \ln \ln n/\sqrt{\ln n} \) is a function that satisfies \( \limsup_{n \to \infty} \varepsilon(n) < 2 \), then

\[
\mathbb{P}(\alpha_t(G_{n,p}) < \lfloor \alpha_{t,p}(n) - \varepsilon(n) \rfloor) = \exp \left( -n^{\varepsilon(n)(1+o(1))} \right).
\]

This implies the left-hand side inequality of Theorem \([1]\)

4. A SECOND MOMENT CALCULATION - PROOF OF LEMMA \(8\)

Let \( (x_n) \) be a bounded sequence of real numbers such that for

\[
k = 2 \log_b n + (t - 2) \log_b \log_b n + x_n \in \mathbb{N}
\]

we have \( \mathbb{E}(\alpha_t^k(G_{n,p})) \to \infty \) as \( n \to \infty \). In this section, we prove that a.a.s. there is a \( k \)-subset of \( V_n \) which is \( t \)-stable, using a second moment argument. For this, we use Janson’s Inequality \([12, 13]\) or Theorems 2.14, 2.18 in \([14]\):

\[\text{(32)}\]

\[
\mathbb{P}(\alpha_t^k(G_{n,p}) = 0) \leq \exp \left( - \frac{\mathbb{E}^2(\alpha_t^k(G_{n,p}))}{\mathbb{E}(\alpha_t^k(G_{n,p})) + \Delta} \right),
\]

where

\[
\Delta = \sum_{A,B \subseteq V_n, k-1 \geq |A \cap B| \geq 2} \mathbb{P}(A,B \in S_t).
\]

Let \( p(k, \ell) \) be the probability that two \( k \)-subsets of \( V_n \) that overlap on exactly \( \ell \) vertices are both in \( S_t \). We write

\[
\Delta = \sum_{\ell=2}^{k-1} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} p(k, \ell) + \sum_{\ell=k-1}^{k-[(t+3) \log_b \log_b n]+1} \binom{n}{k} \binom{k}{\ell} \binom{n-k}{k-\ell} p(k, \ell) =: \Delta_1 + \Delta_2.
\]
We conclude the proof by showing that

\[ \Delta_1 = O\left(\frac{\ln^5 n}{n^2}\right) \mathbb{E}^2(\alpha^{(k)}_t(G_{n,p})) \text{ and } \Delta_2 = o(\mathbb{E}(\alpha^{(k)}_t(G_{n,p}))). \]

Thus, if \( \mathbb{E}(\alpha^{(k)}_t(G_{n,p})) = n^{\varepsilon(n)(1+o(1))} \), where \( \varepsilon(n) \) satisfies \( \limsup_{n \to \infty} \varepsilon(n) < 2 \), then it follows that \( \Delta_1 = o(\mathbb{E}(\alpha^{(k)}_t(G_{n,p}))) \) and therefore

\[ \frac{\mathbb{E}(\alpha^{(k)}_t(G_{n,p}))}{\mathbb{E}(\alpha^{(k)}_t(G_{n,p})) + \Delta} = 1 + o(1). \]

So Lemma 8 follows from (32) by substituting the expression for \( \mathbb{E}(\alpha^{(k)}_t(G_{n,p})) \) from (31).

**Bounding \( \Delta_1 \).** Let us begin by bounding \( \Delta_1 \), first estimating \( p(k, \ell) \). Let \( A \) and \( B \) be two \( k \)-subsets of \( V_n \) that overlap on exactly \( \ell \) vertices, i.e. \( |A \cap B| = \ell \). Then \( p(k, \ell) = \mathbb{P}(A, B \in S_t) = \mathbb{P}(A \in S_t \mid B \in S_t) \mathbb{P}(B \in S_t) \). In what follows, we estimate the first factor in terms of \( \mathbb{P}(A \in S_t) \). If \( C = A \cap B \), we have

\[ \mathbb{P}(A \in S_t \mid B \in S_t) = \sum_{m=0}^{[\ell/2]} \mathbb{P}(A \in S_t \mid e(C) = m, B \in S_t) \mathbb{P}(e(C) = m \mid B \in S_t). \]  

Let \( \hat{C} \) be a specific realisation of \( C \) and suppose that \( e(\hat{C}) = e \). Also let \( \hat{C}_0 \) be the realisation of \( C \) with \( e(\hat{C}_0) = 0 \). Observe now that

\[ \mathbb{P}(A \in S_t \mid \hat{C}, B \in S_t) \leq \mathbb{P}(A \in S_t \mid \hat{C}_0, B \in S_t), \]

as all the instances in the first (conditional) space are included in the second one, with the same weight. Moreover, note that \( \mathbb{P}(A \in S_t \mid \hat{C}_0, B \in S_t) = \mathbb{P}(A \in S_t \mid e(C) = 0) \), because conditional on \( e(C) = 0 \) (or equivalently on \( \hat{C}_0 \)) the events \( A \in S_t \) and \( B \in S_t \) are independent. Hence,

\[ \mathbb{P}(A \in S_t \mid e(C) = m, B \in S_t) = \sum_{\hat{C}: e(\hat{C}) = m} \mathbb{P}(A \in S_t \mid \hat{C}, B \in S_t) \mathbb{P}(\hat{C} \mid B \in S_t) \leq \mathbb{P}(A \in S_t \mid \hat{C}_0, B \in S_t) \sum_{\hat{C}: e(\hat{C}) = m} \mathbb{P}(\hat{C} \mid B \in S_t) \]

\[ \leq \mathbb{P}(A \in S_t \mid e(C) = 0, B \in S_t) = \mathbb{P}(A \in S_t \mid e(C) = 0). \]

Also,

\[ \mathbb{P}(A \in S_t) \geq \mathbb{P}(A \in S_t \mid e(C) = 0) \mathbb{P}(e(C) = 0) \]

yielding

\[ \mathbb{P}(A \in S_t \mid e(C) = 0) \leq (1 - p)^{-\lfloor \ell/2 \rfloor} \mathbb{P}(A \in S_t). \]

Combining (33), (34) and (35), we obtain

\[ \mathbb{P}(A \in S_t \mid B \in S_t) \leq b^{\ell/2}(\mathbb{P}(A \in S_t)). \]

Therefore,

\[ p(k, \ell) = \mathbb{P}(A \in S_t \mid B \in S_t) \mathbb{P}(B \in S_t) \leq b^{\ell/2} (\mathbb{P}(A \in S_t))^2. \]

On the other hand, for every \( \ell \leq k \),

\[ \binom{k}{\ell}\binom{n-k}{k-\ell} \leq k^\ell \frac{k^\ell}{(n-k)^\ell} \binom{n}{k}. \]
Using the estimate of \((36)\) along with the above inequality, we have

\[
\Delta_1 \leq \left( \binom{n}{k} \mathbb{P}(A \in S_t) \right)^2 \sum_{\ell=2}^{k-\left[(\ell+3) \log_b \log_b n\right]} \left( \frac{k^2}{n-k} \right)^{\ell} b^{\ell/2}.
\]

(37)

If we set \(s_\ell = (k^2/(n-k))^{\ell/2} b^{\ell/2}\), then \(s_{\ell+1}/s_\ell = b^{\ell}/(n-k)\). So the sequence \(\{s_\ell\}\) is strictly decreasing for \(\ell < \log_b (n-k) - 2 \log_b k\) and is strictly increasing for \(\ell > \log_b (n-k) - 2 \log_b k\). So

\[
\max \{ s_\ell : 2 \leq \ell \leq k - \left[(\ell+3) \log_b \log_b n\right] \} \leq \max \{ s_2, s_2, s_{2 \log_b n-4.5 \log_b \log_b n} \}.
\]

We have that \(s_2 = bk^4/(n-k)^2\), but

\[
s_{2 \log_b n-4.5 \log_b \log_b n} \leq \left( \frac{k^2}{n-k} b^{\log_b n - 2.25 \log_b \log_b n} \right)^{2 \log_b n - 4.5 \log_b \log_b n} \leq \left( \frac{4 \log_b^2 n}{\log_b 2.25} \right)^{2 \log_b n - 4.5 \log_b \log_b n} \leq \left( \frac{4 \log_b^2 n}{\log_b 2.25} \right)^{\log_b n} = o(s_2).
\]

Thus, Inequality \((37)\) now becomes for \(n\) large enough

\[
\Delta_1 \leq \frac{bk^5}{(n-k)^2} \mathbb{E}^2(\alpha_t^{(k)}(G_{n,p})) = O\left( \frac{\ln^5 n}{n^2} \right) \mathbb{E}^2(\alpha_t^{(k)}(G_{n,p})).
\]

Bounding \(\Delta_2\). Now, we will show that \(\Delta_2 = o(\mathbb{E}(\alpha_t^{(k)}(G_{n,p})))\). First, we have

\[
\left( \frac{k}{\ell} \right) \frac{n-k}{k-\ell} \leq (kn)^{k-\ell}
\]

We now give a rough estimate on \(p(k, \ell)\). If \(A, B\) are two \(k\)-sets of vertices that overlap on \(\ell\) vertices (and if \(\text{deg}_S(v)\) denotes the number of neighbours of \(v\) in \(S\)), then

\[
\mathbb{P}(B \in S_t \mid A \in S_t) \leq \mathbb{P}(\forall v \in B \setminus A, \text{deg}_{A \cap B}(v) \leq \ell) \leq \left( \frac{\ell}{\ell - t} \right)^{k-\ell} (1 - p)^{\ell-t} \leq \left( \frac{k^{\ell-\ell}}{\ell - t} \right)^{k-\ell} \leq b^{(\ell \log_b k + t - k + [(\ell+3) \log_b \log_b n])} (\ell - t)^{k-\ell} \leq b^{(\ell - 2 \log_b n + (\ell + 3) \log_b \log_b n + \Theta(1))} \leq \left( \frac{\log_b^{\ell+6} n}{n^2} \right)^{k-\ell}.
\]

Substituting these estimates into the expression for \(\Delta_2\), we obtain

\[
\Delta_2 \leq \frac{n}{k} \mathbb{P}(A \in S_t) \sum_{\ell=2}^{k-[(\ell+3) \log_b \log_b n]} \left( \frac{k^2 \log_b^{\ell+6} n}{n^2} \right)^{k-\ell} \leq \mathbb{E}(\alpha_t^{(k)}(G_{n,p})) \left( \frac{k \log_b^{\ell+6} n}{n} \right) = o(\mathbb{E}(\alpha_t^{(k)}(G_{n,p}))).
\]
5. The $t$-improper chromatic number

5.1. The upper bound. For a fixed $0 < \varepsilon < 1$, we set $\hat{\alpha}_{t,p}(n) = [\alpha_{t,p}(n) - 1 - \varepsilon]$. First, we will show the following.

**Lemma 9.** A.a.s. for all $V' \subseteq V_n$ with $|V'| \geq n/\ln^3 n$, we have $\alpha_t(G_{n,p}[V']) \geq \hat{\alpha}_{t,p}(|V'|)$.

**Proof.** Note that (31) implies that for any $V' \subseteq V_n$ with $|V'| \geq n/\ln^3 n$, we have

$$\mathbb{P} \left( \alpha_t(G_{n,p}[V']) < \hat{\alpha}_{t,p}(|V'|) \right) = \exp \left( -|V'|^{1+\varepsilon+o(1)} \right) \leq \exp \left( -\left( \frac{n}{\ln^3 n} \right)^{1+\varepsilon+o(1)} \right).$$

Since there are at most $2^n$ choices for $V'$, the probability that there exists a set $V' \subseteq V_n$ with $|V'| \geq n/\ln^3 n$ and $\alpha_t(G_{n,p}[V']) < \hat{\alpha}_{t,p}(|V'|)$ is at most $2^n \exp \left( -(n/\ln^3 n)^{1+\varepsilon+o(1)} \right) = o(1)$.

We consider the following algorithm for $t$-improperly colouring $G_{n,p}$. Let $V' = V_n$. While $|V'| \geq n/\ln^3 n$, we choose and remove a $t$-stable set from $G_{n,p}[V']$ of size $\hat{\alpha}_t(|V'|)$. At the end, we obtain a collection of $t$-stable sets and each of them will form a colour class. The above lemma implies that a.a.s. we will be able to perform this algorithm, and end up with a set of at most $n/\ln^3 n$ vertices. We give a different colour to each of these vertices. Thus, if the above algorithm “runs” for $f(n)$ steps, then $\chi_t(G_{n,p}) \leq f(n) + n/\ln^3 n$.

Since $\alpha_{t,p}(s) - 1 - \varepsilon$ is strictly increasing for all $s$ that are sufficiently large, for these $s$ the function $\alpha_{t,p}(s)$ is non-decreasing. It is easy to see that

$$\hat{\alpha}_{t,p} \left( \left\lfloor \frac{n}{\ln^3 n} \right\rfloor \right) = 2 \log_b n \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right) = \hat{\alpha}_{t,p}(n) \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right).$$

Since $\hat{\alpha}_{t,p}(\lfloor n/\ln^3 n \rfloor) \leq \hat{\alpha}_{t,p}(s) \leq \hat{\alpha}_{t,p}(n)$ for all integers $n/\ln^3 n \leq s \leq n$,

$$\hat{\alpha}_{t,p}(s) = \hat{\alpha}_{t,p}(n) \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right),$$

and therefore

$$f(n) = \frac{n}{\hat{\alpha}_{t,p}(n)} \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right).$$

Assume that there are $n_i$ vertices available when we have removed $i$ $t$-stable sets from $V_n$. Thus, the $t$-stable set that will be picked during the $(i+1)$th iteration will have size $\hat{\alpha}_{t,p}(n_i)$. Since the colouring algorithm stops as soon as there are less than $n/\ln^3 n$ vertices available, the following inequality holds:

$$\sum_{i=0}^{f(n)-2} \hat{\alpha}_{t,p}(n_i) \leq n \left( 1 - \frac{1}{\ln^3 n} \right) \leq n.$$

Note that for all $i \geq 0$, $n_i = n - \sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j)$. Therefore,

$$\log_b n_i = \log_b \left( n - \sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j) \right) = \log_b n + \log_b \left( 1 - \frac{\sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j)}{n} \right).$$
We have
\[
\sum_{i=0}^{f(n)-2} \log_b \left( 1 - \frac{\sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j)}{n} \right) = \frac{1}{\ln b} \sum_{i=0}^{f(n)-2} \frac{n}{\hat{\alpha}_{t,p}(n_i)} \ln \left( 1 - \frac{\sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j)}{n} \right) \frac{\hat{\alpha}_{t,p}(n_i)}{n}
\]
\[
\leq n(1 + o(1)) \sum_{i=0}^{f(n)-2} \ln \left( 1 - \frac{\sum_{j=0}^{i-1} \hat{\alpha}_{t,p}(n_j)}{n} \right) \frac{\hat{\alpha}_{t,p}(n_i)}{n}
\]
\[
= \frac{n(1 + o(1))}{\hat{\alpha}_{t,p}(n) \ln b} \int_0^1 \ln(1 - x)dx = \frac{n(1 + o(1))}{\hat{\alpha}_{t,p}(n) \ln b}.
\]
So
\[
f(n) - 2 \sum_{i=0}^{f(n)-2} 2 \log_b n_i = (f(n) - 1)2 \log_b n - \frac{2n(1 + o(1))}{\hat{\alpha}_{t,p}(n) \ln b}
\]
Also,
\[
\log_b \log_b n_i \geq \log_b \log_b \left( \frac{n}{\ln^3 n} \right) = \log_b \log_b n + \log_b \left( 1 - \frac{3 \ln n}{\log_b n} \right)
\]
\[
= \log_b \log_b n - O \left( \frac{\ln \ln n}{\ln n} \right).
\]
Moreover, \( \log_b \log_b n_i \leq \log_b \log_b n \) so, for every \( t \geq 0 \),
\[
(t - 2) \sum_{i=0}^{f(n)-2} \log_b \log_b n_i \geq (f(n) - 1)(t - 2) \log_b \log_b n - O \left( \frac{f(n) \ln \ln n}{\ln n} \right).
\]
Now, Equality (11) and Inequality (12) imply that for every \( t \geq 0 \) we have
\[
\sum_{i=0}^{f(n)-2} \hat{\alpha}_{t,p}(n_i) \geq (f(n) - 1) \left( \alpha_{t,p}(n) - \varepsilon - 2 \right) - \frac{2n(1 + o(1))}{\hat{\alpha}_{t,p}(n) \ln b} - O \left( \frac{f(n) \ln \ln n}{\ln n} \right)
\]
\[
\geq (f(n) - 1) \left( \alpha_{t,p}(n) - \varepsilon - 2 \right) - \frac{2n(1 + o(1))}{f(n) \hat{\alpha}_{t,p}(n) \ln b} - O \left( \frac{\ln \ln n}{\ln n} \right)
\]
\[
\geq (f(n) - 1) \left( \alpha_{t,p}(n) - \varepsilon - 2 - \frac{2}{\ln b} - o(1) \right).
\]
So by (40) we obtain
\[
f(n) - 1 \leq \frac{n}{\alpha_{t,p}(n) - \varepsilon - 2 - 2/\ln b - o(1)}.
\]

5.2. The lower bound. This proof is the generalisation of a proof of the lower bound on the chromatic number of a dense random graph given recently by Panagiotou and Steger [22]. We let \( \alpha_C(n) = 2 \log_b n + (t-2) \log_b \log_b n - C \), where \( C = C_n > 2 \log_b n + (t-2) \log_b \log_b n - \alpha_{t,p}(n) \) is some function which is \( \Theta(1) \), such that \( \alpha_C(n) \) is integral. We specify \( C \) at a later stage. Let \( r = r_C := \lfloor n/\alpha_C(n) \rfloor \). By Theorem 11 a.a.s. there are no \( t \)-stable sets in \( G_{n,p} \) of size more than \( \alpha_{t,p}(n) + 1 \). We will estimate the expected number of \( t \)-improper colourings of \( G_{n,p} \) with \( r \) colours such that each colour set has size at most \( \alpha_{t,p}(n) + 1 \). In particular, we show that, if \( C < 2 \log_b n + (t-2) \log_b \log_b n - \alpha_{t,p}(n) + 1 + 2/\ln b - \varepsilon \), then this expectation converges to zero, proving that \( \chi_t(G_{n,p}) > r_C \) a.a.s.

\[\text{Note that } \int \ln(1-x)dx = -(1-x)\ln(1-x) + 1 - x.\]
Let $D$ denote the set of $r$-tuples of positive integers $(k_1, \ldots, k_r)$ such that $\sum_{i=1}^r k_i = n$ and $k_i \leq \alpha_{t,p}(n) + 1$ for all $i$. For some $(k_1, \ldots, k_r) \in D$, let $P = (P_1, \ldots, P_r)$ denote a partition of $V_n$ into $r$ non-empty parts $P_1, \ldots, P_r$ such that $|P_i| = k_i$. From (28), we obtain

$$
\Pr(P_i \in S_t) \leq \left( b^{-k_i+1} \left( \frac{tbpk_i}{e} \right)^t \frac{1}{t!^2} \right)^{k_i/2} \left( 1 + O \left( \frac{\ln k_i}{\sqrt{k_i}} \right) \right)^{k_i}.
$$

$$
\Pr(P_i \in S_t, \forall i) = \prod_{i=1}^r \Pr(P_i \in S_t) \leq \prod_{i=1}^r \left( b^{-k_i+1} \left( \frac{tbpk_i}{e} \right)^t \frac{1}{t!^2} \right)^{k_i/2} \left( 1 + O \left( \frac{\ln k_i}{\sqrt{k_i}} \right) \right)^{k_i}
= b^{-(\sum_{i=1}^r k_i^2/2)+n/2} \left( \frac{tbp}{e} \right)^{tn/2} \left( \prod_{i=1}^r k_i^{tk_i/2} \right) \frac{1}{t!^n} (1 + o(1))^n
= \left( \frac{tb^{1+1/t}}{et!^2/t} \right)^{tn/2} \left( \prod_{i=1}^r k_i^{tk_i/2} \right) (1 + o(1))^n,
$$
uniformly over all $(k_1, \ldots, k_r) \in D$. So, if $X_{t,r} = X_{t,r}(G_{n,p})$ denotes the number of $t$-improper colourings with $r$ colours and with each colour class of size at most $\alpha_{t,p}(n) + 1$, then

$$
\mathbb{E}(X_{t,r}) = \frac{1}{r!} \left( \frac{tb^{1+1/t}}{et!^2/t} \right)^{tn/2} \sum_{(k_1, \ldots, k_r) \in D} \left( \frac{n}{k_1 \cdots k_r} \right)^{-\sum_{i=1}^r \left( \frac{k_i^2}{2} - \frac{t}{2} k_i \log_b k_i \right)} (1 + o(1))^n.
$$

We call a partition where all parts differ by at most one pairwise balanced. In the next subsection, we give a routine proof of the following property of balanced partitions.

**Lemma 10.** For large enough $n$, the function

$$
h(P) := -\sum_{i=1}^r \left( \frac{k_i^2}{2} - \frac{t}{2} k_i \log_b k_i \right),
$$

where $P = \{P_1, \ldots, P_r\}$ is a partition of $V_n$ with $|P_i| = k_i$, is maximised over $D$ when $P$ is a balanced partition.

Let $B$ be a balanced partition. Then all parts have sizes either equal to $\alpha_C(n)$ or to $\alpha_C(n) + 1$ and there are less than $\alpha_C(n)$ parts that take the latter quantity. Then

$$
h(B) = -\frac{n}{\alpha_C(n)} \left( \frac{\alpha_C^2(n)}{2} - \frac{t}{2} \alpha_C(n) \log_b \alpha_C(n) \right) + o(n)
= -\frac{1}{2} n \alpha_C(n) + \frac{t}{2} n \log_b \alpha_C(n) + o(n)
= -n \log_b n - \frac{t}{2} n \log_b \log_b n + \frac{Cn}{2} + \frac{t}{2} n \log_b 2 + \frac{t}{2} n \log_b \log_b n + o(n)
= -n \log_b n + n \log_b \log_b n + \frac{Cn}{2} + \frac{t}{2} n \log_b 2 + o(n).
$$

Also, for any $(k_1, \ldots, k_r) \in D$, we have (for $n$ large enough)

$$
\binom{n}{k_1 \cdots k_r} \leq \frac{n!}{(\alpha_C(n))^{n/2}} \frac{n^n}{\sqrt{2\pi n \alpha_C(n)}} \leq \frac{n^n}{(\alpha_C(n))^{n+r/2}}
= \beta^n \log_b n - n \log_b \alpha_C(n) - \frac{t}{2} \log_b \alpha_C(n)
= \beta^n \log_b n - n \log_b 2 - n \log_b \log_b n + o(n).
$$
since \( r \log_b \alpha_C(n) \leq (n/\alpha_C(n)) \log_b \alpha_C(n) = o(n) \). Finally, \( r! \geq r^r e^{-r} \) and therefore

\[
(46) \quad \frac{1}{r!} \leq b^{-r \log_b r + r \log_b e} = b^{-\frac{n}{\alpha_C(n)}} \log_b \left( \frac{n}{\alpha_C(n)} \right) + o(n) = b^{-\frac{\log_b n}{\alpha_C(n)}} + o(n) = b^{-\frac{1}{2} + o(n)}.
\]

As there are at most \( \binom{n}{r} \leq \left( \frac{en}{r} \right)^r \leq \left( 2e\alpha_C(n) \right)^r \leq b^{r \log_b \alpha_C(n) + O(r)} = b^{o(n)} \) summands in (43), we obtain from (44), (45) and (46) that

\[
E(X_{t,r}) \leq \left( \frac{tb^{1+1/t}b}{et^{2/t}} \right)^{tn/2} b^{C \log_b n - n \log_b 2 - \frac{n}{2}(1 + o(1))^n} = b^{\frac{1}{2(t \log_b (2t/p) + t - 2 \log_b t + C - 2 \log_b 2)}(1 + o(1))^n}.
\]

Therefore, if \( C = C_n = -\log_b (t/t^2) - t \log_b (2bp/e) - \log_b (1/4) - \epsilon \), i.e. if \( \alpha_C(n) > \alpha_{t,p}(n) - 1/2 \ln b + \epsilon \) for an arbitrary \( \epsilon > 0 \), then \( E(X_{t,r}) = o(1) \). Thus, a.a.s.

\[
\chi_t(G_{n,p}) \geq \frac{n}{\alpha_{t,p}(n) - 1} - \frac{2}{\ln n} + \epsilon.
\]

5.3. **Proof of Lemma 10**. Suppose \( h(P), P, k_i \) are defined as in Lemma 10 and furthermore assume that the parts of \( P \) are ordered by increasing size, i.e., \( k_1 \leq \cdots \leq k_r \). Let \( \tilde{P} = \{\tilde{P}_1, \ldots, \tilde{P}_r\} \) be a partition of \( V_n \) where for some \( v \in P_r \) we have \( \tilde{P}_1 = P_1 \cup \{v\} \) and \( \tilde{P}_r = P_r \setminus \{v\} \), whereas \( \tilde{P}_i = P_i \) for all \( 1 < i < r \). In other words, we obtain \( \tilde{P} \) by moving a vertex from \( P_r \) to \( P_i \). Lemma 10 easily follows from the repeated application of the following.

**Lemma 11.** For large enough \( n \), it holds that, if \( k_1 < k_r - 1 \), then \( h(\tilde{P}) > h(P) \).

**Proof.** First, \( k_1 \leq \alpha_C(n) \) and \( k_r \geq \alpha_C(n) + 1 \), since the number of parts is \( r = [n/\alpha_C(n)] \).

\[
2(h(\tilde{P}) - h(P)) = -((k_1 + 1)^2 + t(k_1 + 1) \log_b (k_1 + 1) - (k_r - 1)^2 + t(k_r - 1) \log_b (k_r - 1) - k_1^2 - tk_1 \log_b k_1 + k_r^2 - tk_r \log_b k_r - 2(k_r - k_1 - 1) + t((k_1 + 1) \log_b (k_1 + 1) - k_1 \log_b k_1) + t((k_r - 1) \log_b (k_r - 1) - k_r \log_b k_r)\)
\]

(47)

Note that

\[
(k_1 + 1) \log_b (k_1 + 1) = (k_1 + 1) \log_b k_1 + (k_1 + 1) \log_b (1 + 1/k_1) \geq (k_1 + 1) \log_b k_1 + (k_1 + 1) (1/k_1 - 1/(2k_1^2)) = k_1 \log_b k_1 + \log_b k_1 + 1 + O\left(1/k_1\right),
\]

and similarly, since \( k_r \geq \alpha_C(n) + 1 \to \infty \) as \( n \to \infty \),

\[
(k_r - 1) \log_b (k_r - 1) = k_r \log_b k_r - \log_b k_r + 1 - o(1).
\]

Substituting these estimates into (47), we obtain

\[
2(h(\tilde{P}) - h(P)) \geq 2(k_r - k_1 - 1) - t(\log_b k_r - \log_b k_1) + O\left(1/k_1\right).
\]

Assume first that \( k_r - k_1 \leq \ln \ln n \). Then \( \log_b (k_r/k_1) \leq \log_b (k_r/(k_r - \ln \ln n)) = \log_b (1 + \ln \ln n/(k_r - \ln \ln n)) = o(1) \). But \( k_r - k_1 - 1 \geq 1 \) and \( k_1 \geq \alpha_C(n) + 1 \ln \ln n \) and therefore the right-hand side of (48) is positive for \( n \) large enough. If, on the other hand \( k_r - k_1 > \ln \ln n \), we write \( \log_b k_r - \log_b k_1 = \log_b (k_r - k_1) + \log_b (1 + k_1/(k_r - k_1) - \log_b k_1 \]

\[
= \log_b (k_r - k_1) + \log_b (1/k_1 + 1/(k_r - k_1))
\]

\[
= \log_b (k_r - k_1) + \log_b (1/k_1 + o(1)) \leq \log_b (k_r - k_1) + 1.
\]
So
\[2(k_r - k_1 - 1) - t(\log_b k_r - \log_b k_1) + O(1/k_1) \geq 2(k_r - k_1 - 1) - t(\log_b (k_r - k_1) - 1) \to \infty\]
as \(n \to \infty\) and, by (48), \(h(\tilde{P}) - h(P) > 0\) for \(n\) large enough.

\[\square\]

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THE $t$-STABILITY NUMBER OF A RANDOM GRAPH

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