Close to the Giant Magnons

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Abstract

We consider the most general string configurations on the $R_t \times S^3$ subspace of $AdS_5 \times S^5$, described by the Neumann-Rosochatius integrable system. Under some restrictions on the parameters of the solution and in an appropriate limit, they correspond to small deviation from the known finite-size giant magnon solutions with one and two angular momenta. Analyzing the finite-size effect on the dispersion relation, we find that the leading correction is modified in a way similar to the $\gamma$-deformed case $R_t \times S^3_\gamma$. The subleading correction for a string with one angular momentum is also found. It depends on the same parameter, which describes the generalization of the leading correction.
1 Introduction

The AdS/CFT duality [1] has become a major subject of investigations in contemporary high energy physics. The most developed part of this correspondence is between type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM) in four dimensions. Recent developments in this direction are mainly based on the integrable structures discovered in both theories.

Integrability on the SYM side appears in the calculations of conformal dimensions, which are related to the string energies according to the AdS/CFT correspondence. The remarkable observation by Minahan and Zarembo [2] is that the conformal dimension of an operator composed of the scalar fields in $\mathcal{N} = 4$ SYM, in the planar limit, can be computed by diagonalizing the Hamiltonian of one-dimensional integrable spin chain. This result has been further extended to the full $PSU(2,2|4)$ sector [3] and the Bethe ansatz equations which are supposed to hold for all loops have been conjectured [4, 5].

The type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ is described by a nonlinear sigma model with $PSU(2,2|4)$ symmetry [6]. This sigma model has been shown to have an infinite number of local and nonlocal conserved currents [7], and some of the conserved charges such as energy and angular momenta have been computed explicitly from the classical integrability. These results based on the classical integrability provide valuable information on the AdS/CFT duality in the domain of large t’Hooft coupling constant.

Various classical string solutions played an important role in testing and understanding the AdS/CFT duality in the case under consideration. The classical giant magnon [8] discovered in $R_t \times \text{S}^2$ gave a strong support for the conjectured all-loop $SU(2)$ spin chain and made it possible to get a deep insight in the AdS/CFT duality. In addition, this solution is related to the integrable sine-Gordon model. It was extended to the dyonic giant magnon, which corresponds to a string moving on $R_t \times \text{S}^3$ and related to the complex sine-Gordon model [9, 10]. Further extensions to $\text{AdS}_5 \times \text{S}^5$ have been also worked out [11, 12, 13, 14].

The reduction of the classical string dynamics to the Neumann-Rosochatius (NR) integrable system [15, 16, 11], has proved to be an useful tool for describing a large class of string solutions on $\text{AdS}_5 \times \text{S}^5$ and other backgrounds, including giant magnons and spiky strings. It can be used also for studying the finite-size effects, related to the wrapping interactions in the dual field theory [17].

The finite-size effect for the giant magnon has been computed from the $S$-matrix in [17] and has been shown to be consistent with the classical string result to the leading order. The finite-size effect for the giant magnon has been first found by solving the string sigma model in a uniform and conformal gauges [18] and, subsequently, many related results, such as gauge independence [19], multi giant magnon states [20] and quantization of finite-size giant magnon [21], have been derived. This result has been also related to explicit solutions of the sine-Gordon equation in a finite-size space [22]. The dispersion relation for the finite-size
dyonic giant magnon was obtained in [23].

In this article, we will consider string configurations on the $R_t \times S^3$ background, which depend on one more parameter, compared to infinite or finite-size giant magnons with one or two angular momenta. Our aim is to find the corresponding generalizations of the finite-size corrections to the dispersion relations. To this end, in section 2 we introduce the classical string action on $R_t \times S^3$ and the corresponding NR system. In section 3 we compute the conserved quantities and angular differences of interest. In section 4 we obtain the dispersion relation for the finite-size dyonic string up to the leading order. Section 5 is devoted to the derivation of the subleading correction to the energy-charge relation for the finite-size string with one nonzero angular momentum. We conclude the paper with some remarks in section 6. Appendix A contains information about the elliptic integrals appearing in the calculations, the $\epsilon$-expansions used and the solutions for the parameters.

2 Strings on $R_t \times S^3$ and the NR Integrable System

Here we will briefly remaid how the string dynamics on $R_t \times S^3$ can be reduced to the one of the NR integrable system [15, 16, 11]. We begin with the Polyakov string action

$$S^P = -\frac{T}{2} \int d^2 \xi \sqrt{-\gamma^{ab} G_{ab}}, \quad G_{ab} = g_{MN} \partial_a X^M \partial_b X^N,$$

(2.1)

$$\partial_a = \partial/\partial \xi^a, \quad a, b = (0, 1), \quad (\xi^0, \xi^1) = (\tau, \sigma), \quad M, N = (0, 1, \ldots, 9),$$

and choose conformal gauge $\gamma^{ab} = \eta^{ab} = diag(-1, 1)$, in which the Lagrangian and the Virasoro constraints take the form

$$L_s = \frac{1}{2} (G_{00} - G_{11})$$

(2.2)

$$G_{00} + G_{11} = 0, \quad G_{01} = 0.$$  

(2.3)

We embed the string in $R_t \times S^3$ subspace of $AdS_5 \times S^5$ as follows

$$Z_0 = \text{Re}^{it(\tau, \sigma)}, \quad W_j = R r_j(\tau, \sigma) e^{i\phi_j(\tau, \sigma)}, \quad \sum_{j=1}^{2} W_j \bar{W}_j = R^2,$$

where $R$ is the common radius of $AdS_5$ and $S^5$, and $t$ is the AdS time. For this embedding, the metric induced on the string worldsheet is given by

$$G_{ab} = -\partial_a Z_0 \partial_b Z_0 + \sum_{j=1}^{2} \partial_a W_j \partial_b \bar{W}_j = R^2 \left[ -\partial_a \partial_b t + \sum_{j=1}^{2} (\partial_a r_j \partial_b r_j + r_j^2 \partial_a \phi_j \partial_b \phi_j) \right].$$

The corresponding string Lagrangian becomes

$$\mathcal{L} = L_s + \Lambda_s \left( \sum_{j=1}^{2} r_j^2 - 1 \right),$$

where $\Lambda_s$ is the level of the corresponding Virasoro algebra.
where \( \Lambda_s \) is a Lagrange multiplier. In the case at hand, the background metric does not depend on \( t \) and \( \phi_j \). Therefore, the conserved quantities are the string energy \( E_s \) and two angular momenta \( J_j \), given by

\[
E_s = - \int d\sigma \frac{\partial L_s}{\partial (\partial_0 t)}, \quad J_j = \int d\sigma \frac{\partial L_s}{\partial (\partial_0 \phi_j)}.
\]

(2.4)

In order to reduce the string dynamics to the NR integrable system, we use the ansatz

\[
t(\tau, \sigma) = \kappa \tau, \quad r_j(\tau, \sigma) = r_j(\xi), \quad \phi_j(\tau, \sigma) = \omega_j \tau + f_j(\xi),
\]

\[
\xi = \alpha \sigma + \beta \tau, \quad \kappa, \omega_j, \alpha, \beta = \text{constants}.
\]

(2.5)

It can be shown that after integrating once the equations of motion for \( f_j \), which gives (prime is used for \( d/d\xi \))

\[
f_j' = \frac{1}{\alpha^2 - \beta^2} \left( \frac{C_j}{r_j^2} + \beta \omega_j \right), \quad C_j = \text{constants},
\]

(2.6)

one ends up with the following effective Lagrangian for the coordinates \( r_j \)

\[
L_{NR} = (\alpha^2 - \beta^2)^2 \sum_{j=1}^{2} \left[ r_j'^2 - \frac{1}{(\alpha^2 - \beta^2)^2} \left( \frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] + \Lambda_s \left( \sum_{j=1}^{2} r_j^2 - 1 \right).
\]

(2.7)

This is the Lagrangian for the NR integrable system.

The Virasoro constraints (2.3) give the conserved Hamiltonian \( H_{NR} \) and a relation between the embedding parameters and the integration constants \( C_j \):

\[
H_{NR} = (\alpha^2 - \beta^2)^2 \sum_{j=1}^{2} \left[ r_j'^2 + \frac{1}{(\alpha^2 - \beta^2)^2} \left( \frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2,
\]

\[
\sum_{j=1}^{2} \omega_j C_j + \beta \kappa^2 = 0.
\]

(2.8)

(2.9)

On the ansatz (2.5), \( E_s \) and \( J_j \) defined in (2.4) take the form

\[
E_s = \frac{\sqrt{\lambda} \kappa}{2\pi \alpha} \int d\xi, \quad J_j = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{\alpha^2 - \beta^2} \int d\xi \left( \frac{\beta}{\alpha} C_j + \alpha \omega_j r_j^2 \right),
\]

(2.10)

where we have used that the string tension and the ’t Hooft coupling constant \( \lambda \) in the dual \( \mathcal{N} = 4 \) SYM are related by

\[
TR^2 = \frac{\sqrt{\lambda}}{2\pi}.
\]
3 Conserved Quantities and Angular Differences

If we introduce the variable
\[ \chi = 1 - r_1^2 = r_2^2, \]
and use (2.9), the first integral (2.8) can be rewritten as
\[ \chi' = 4 \omega_1^2 \alpha \chi - \frac{2 - (1 + v^2)W - u^2}{1 - u^2} \chi^2 \]
\[ - \frac{1 - (1 + v^2)W + [(vW - uK)^2 - K^2]}{1 - u^2} \chi - \frac{K^2}{1 - u^2} \]
\[ = \frac{4 \omega_1^2 (1 - u^2)}{\alpha^2 (1 - v^2)^2} (\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n), \]
(3.1)

where
\[ v = \frac{\beta}{\alpha}, \quad u = \frac{\omega_2}{\omega_1}, \quad W = \left( \frac{\kappa}{\omega_1} \right)^2, \quad K = \frac{C_2}{\alpha \omega_1}. \]

We are interested in the case when
\[ 0 < \chi_m < \chi < \chi_p < 1, \quad \chi_n < 0. \]

The three equations following from (3.1) are
\[ \chi_p + \chi_m + \chi_n = \frac{2 - (1 + v^2)W - u^2}{1 - u^2}, \]
\[ \chi_p \chi_m + \chi_p \chi_n + \chi_m \chi_n = \frac{1 - (1 + v^2)W + (vW - uK)^2 - K^2}{1 - u^2}, \]
\[ \chi_p \chi_m \chi_n = - \frac{K^2}{1 - u^2}. \]
(3.2)

We will use them essentially to find the dispersion relations for the string configurations under consideration.

Correspondingly, the conserved quantities (2.10) transform to
\[ E = \kappa \int_{-r}^r d\xi \frac{(1 - v^2)\sqrt{W}}{\sqrt{1 - u^2}} \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \]
\[ J_1 = \frac{J_1}{\delta \alpha} = \frac{1}{\sqrt{1 - u^2}} \int_{\chi_m}^{\chi_p} \frac{[1 - v(vW - uK) - \chi]}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} d\chi, \]
\[ J_2 = \frac{J_2}{\delta \alpha} = \frac{1}{\sqrt{1 - u^2}} \int_{\chi_m}^{\chi_p} \frac{(u\chi - vK) d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \]
(3.3)
Computing the angular differences
\[ p \equiv \Delta \phi_1 = \phi_1(r) - \phi_1(-r), \quad \tilde{p} \equiv \Delta \phi_2 = \phi_2(r) - \phi_2(-r), \] (3.4)
one finds
\[ p = \int_{-r}^{r} d\xi f'_1 = \frac{1}{\sqrt{1 - u^2}} \int_{\chi_m}^{\chi_p} \left( \frac{vW - uK}{1 - \chi} \right) \frac{d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \] (3.5)
\[ \tilde{p} = \int_{-r}^{r} d\xi f'_2 = \frac{1}{\sqrt{1 - u^2}} \int_{\chi_m}^{\chi_p} \left( \frac{K}{\chi} - uW \right) \frac{d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}. \] (3.6)

By using the formulas for the elliptic integrals given in Appendix A, we can rewrite (3.3), (3.5) and (3.6) in their final form
\[ E = 2(1 - v^2)\sqrt{W} \sqrt{\frac{1}{1 - u^2} K(1 - \epsilon)}, \]
\[ J_1 = \frac{2}{\sqrt{1 - u^2}} \left( \frac{1 - v(vW - uK)}{\sqrt{\chi_p - \chi_n}} \right) K(1 - \epsilon) - \sqrt{\chi_p - \chi_n} E(1 - \epsilon) \],
\[ J_2 = \frac{2}{\sqrt{1 - u^2}} \left( u \sqrt{\chi_p - \chi_n} E(1 - \epsilon) - \frac{vK - u\chi_n}{\sqrt{\chi_p - \chi_n}} K(1 - \epsilon) \right), \] (3.7)
\[ p = \frac{2}{\sqrt{1 - u^2} \sqrt{\chi_p - \chi_n}} \left( \frac{vW - uK}{1 - \chi_p} \Pi \left( \frac{\chi_p - \chi_m}{1 - \chi_p} | 1 - \epsilon \right) - vK(1 - \epsilon) \right), \]
\[ \tilde{p} = \frac{2}{\sqrt{1 - u^2} \sqrt{\chi_p - \chi_n}} \left( \frac{K}{\chi_p} \Pi \left( \frac{\chi_m - 1}{\chi_p} | 1 - \epsilon \right) - uvK(1 - \epsilon) \right). \]

4 Finite-Size Dyonic String

In order to find the leading finite-size correction to the energy-charge relation for the string with two angular momenta (dyonic string), we have to consider the limit \( \epsilon \to 0 \) in (3.2) and (3.7). The behavior of the complete elliptic integrals in this limit is given in Appendix A. For the parameters in the solution, we will use the ansatz
\[ \chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2} \log(\epsilon)) \epsilon, \]
\[ \chi_m = \chi_{m1} \epsilon, \]
\[ \chi_n = \chi_{n1} \epsilon, \]
\[ v = v_0 + (v_1 + v_2 \log(\epsilon)) \epsilon, \]
\[ u = u_0 + (u_1 + u_2 \log(\epsilon)) \epsilon, \]
\[ W = 1 + W_1 \epsilon, \]
\[ K = K_1 \epsilon. \] (4.1)
Replacing (4.1) into (3.2), one finds four equations for the coefficients in the expansions of $\chi_p$ and $\chi_m$. They are solved by

\[
\begin{align*}
\chi_{p0} &= 1 - \frac{v_0^2}{1 - u_0^2}, \\
\chi_{p1} &= -\frac{1}{(1 - u_0^2)(1 - v_0^2 - u_0^2)} \left[ v_0 (-2K_1 u_0 (1 - u_0^2)^2 \\
&\quad + 2(1 - v_0^2 - u_0^2) \left( v_0 u_0 u_1 + (1 - u_0^2) v_1 \right) + v_0 (1 - u_0^2)(1 - v_0^2 - 2u_0^2) W_1 \right] ,
\end{align*}
\] (4.2)

\[
\begin{align*}
\chi_{p2} &= -\frac{2v_0 (v_0 u_0 u_2 + (1 - u_0^2)v_2)}{(1 - u_0^2)^2} \\
\chi_{m1} &= -\frac{2v_0 u_0 K_1 + (1 - v_0^2) W_1}{1 - v_0^2 - u_0^2} - \chi_{n1}.
\end{align*}
\]

As a next step, we impose the condition $p$ to be finite. It leads to the relation

\[
p = \arcsin \left( \frac{2v_0 \sqrt{1 - v_0^2 - u_0^2}}{1 - u_0^2} \right),
\] (4.3)

as well as to four equations for the parameters involved.

From the requirement $J_2$ to be finite, one derives the equality

\[
J_2 = \frac{2u_0 \sqrt{1 - v_0^2 - u_0^2}}{1 - u_0^2},
\] (4.4)

and two more equations.

The equations (4.3), (4.4) are solved by

\[
\begin{align*}
v_0 &= \frac{\sin(p)}{\sqrt{J_2^2 + 4 \sin^2(p/2)}}, & u_0 &= \frac{J_2}{\sqrt{J_2^2 + 4 \sin^2(p/2)}}.
\end{align*}
\] (4.5)

After the replacement of (4.2) into the remaining six equations, one finds that one of them is satisfied identically, while the others can be solved with respect to $v_1$, $v_2$, $u_1$, $u_2$, $W_1$, leading to the following form of the energy-charge relation in the considered approximation

\[
\mathcal{E} - J_1 = \frac{2 \sqrt{1 - v_0^2 - u_0^2}}{1 - u_0^2} - \left( \frac{(1 - v_0^2 - u_0^2)^{3/2}}{2(1 - u_0^2)} + \sqrt{1 - v_0^2 - u_0^2} \chi_{n1} \right) \epsilon.
\] (4.6)

To the leading order, the expansion for $J_1$ gives

\[
\epsilon = 16 \exp \left( - \frac{2 - \frac{2v_0^2}{1 - u_0^2} + J_1 \sqrt{1 - v_0^2 - u_0^2}}{1 - v_0^2} \right).
\] (4.7)
By using (4.5) and (4.7), (4.6) can be rewritten as
\[
E - J_1 = \sqrt{J_2^2 + 4\sin^2(p/2)} - \frac{16\sin^2(p/2) (\sin^2(p/2) + 2\chi_{n1})}{\sqrt{J_2^2 + 4\sin^2(p/2)}} \quad (4.8)
\]
\[
\exp \left[ -2 \left( J_1 + \sqrt{J_2^2 + 4\sin^2(p/2)} \right) \sqrt{J_2^2 + 4\sin^2(p/2)} \sin^2(p/2) \right] .
\]
Thus, the new parameter in the dispersion relation is \( \chi_{n1} \) (\( \chi_n = \chi_{n1} \varepsilon \)), reflecting the fact that we are considering more general string solution. For \( J_2 = 0 \), (4.8) simplifies to
\[
E - J_1 = 2 \sin \frac{p}{2} \left[ 1 - 4 \left( \sin^2 \frac{p}{2} + 2\chi_{n1} \right) e^{-2 - J_1 \csc \frac{p}{2}} \right] . \quad (4.9)
\]
If we set \( \chi_{n1} = 0 \), (4.9) reduces to the finite-size giant magnon dispersion relation first found in [18], while (4.8) reproduces the result of [23].

Now, we are going to show how the equalities (4.9) and (4.8) can be related to the ones found for giant magnons in \( T\!sT\)-transformed \( AdS_5 \times S^5 \) in [24] and [25] respectively. To this end, let us consider the leading term in the expansion for the angular difference \( \tilde{p} \)
\[
\tilde{p} = -\frac{2K_1}{\chi_{n1}} \sqrt{\frac{1}{1 - v_0^2 - u_0^2 + (1 - u_0^2)\chi_{n1}} - \frac{1}{1 - v_0^2 - u_0^2}} \quad (4.10)
\]
\[
\times \arctan \sqrt{-\frac{(1 - u_0^2)\chi_{n1}}{1 - v_0^2 - u_0^2 + (1 - u_0^2)\chi_{n1}} + \frac{u_0v_0}{\sqrt{1 - v_0^2 - u_0^2}} \log \left( \frac{\varepsilon}{16} \right) }.
\]
If instead of \( \chi_{n1} \) we introduce the angle \( \Phi \) as
\[
\frac{\Phi}{2} = \arctan \sqrt{-\frac{(1 - u_0^2)\chi_{n1}}{1 - v_0^2 - u_0^2 + (1 - u_0^2)\chi_{n1}}}, \quad (4.11)
\]
this gives
\[
\chi_{n1} = -\frac{1 - v_0^2 - u_0^2}{1 - u_0^2} \sin^2(\Phi/2) = -\sin^2(p/2) \sin^2(\Phi/2), \quad (4.12)
\]
and the first term in (4.10) takes the form
\[
\frac{2K_1 (1 - u_0^2)}{(1 - v_0^2 - u_0^2)^{3/2}} \Phi \csc \Phi. \quad (4.13)
\]
If we impose the natural condition (4.13) to be an angle proportional to the angle \( \Phi \), this gives
\[
K_1 = \tilde{\Lambda} \frac{(1 - v_0^2 - u_0^2)^{3/2}}{2(1 - u_0^2)} \sin \Phi = \tilde{\Lambda} \frac{\sin^4(p/2)}{\sqrt{J_2^2 + 4\sin^2(p/2)}} \sin \Phi, \quad (4.14)
\]
where $\tilde{\Lambda}$ does not depend on $\Phi$. In the approximation under consideration, we do not see any criterion to fix the parameter $\tilde{\Lambda}$. However in the next section, we will show that $\tilde{\Lambda} = \pm 1$ must be fulfilled. As a result, the relation between the angles $\tilde{p}$ and $\Phi$ becomes

$$\tilde{p} + \mathcal{J}_1 + \frac{\sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)}}{\mathcal{J}_2^2 + 4 \sin^4(p/2)} \sin(p) = \pm \Phi. \quad (4.15)$$

In accordance with (4.12), the dispersion relation (4.8) takes the form

$$\mathcal{E} - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)} - \frac{16 \sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)}} \cos \Phi \quad (4.16)$$

$$\exp \left[ -2 \left( \mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)} \right) \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2) \sin^2(p/2)} \right],$$

which is the same as in [25]. For $\mathcal{J}_2 = 0$, (4.16) reduces to the form found in [24].

## 5 Subleading Correction

Here our aim is to obtain the subleading correction to the dispersion relation of a string with one angular momentum $\mathcal{J}_1$, i.e. to (4.9). Now, we will use the following expansions for the parameters

$$\chi_p = \chi_{p0} + \left( \chi_{p1} + \chi_{p2} \log(\epsilon) \right) \epsilon + \left( \chi_{p20} + \chi_{p21} \log(\epsilon) + \chi_{p22} \log^2(\epsilon) \right) \epsilon^2,$$

$$\chi_m = \chi_{m1} \epsilon + \left( \chi_{m20} + \chi_{m21} \log(\epsilon) \right) \epsilon^2,$$

$$\chi_n = \chi_{n1} \epsilon + \left( \chi_{n20} + \chi_{n21} \log(\epsilon) \right) \epsilon^2,$$

$$v = v_0 + (v_1 + v_2 \log(\epsilon)) \epsilon + \left( v_{20} + v_{21} \log(\epsilon) + v_{22} \log^2(\epsilon) \right) \epsilon^2,$$

$$u = \left( u_1 + u_2 \log(\epsilon) \right) \epsilon + \left( u_{20} + u_{21} \log(\epsilon) + u_{22} \log^2(\epsilon) \right) \epsilon^2,$$

$$W = 1 + W_1 \epsilon + \left( W_{20} + W_{21} \log(\epsilon) \right) \epsilon^2,$$

$$K = K_1 \epsilon + \left( K_{20} + K_{21} \log(\epsilon) \right) \epsilon^2,$$

where $u_0$ is set to zero in accordance with (4.5).

Replacing (5.1) into (3.2), one finds that one of the equations leads to

$$K_1 = \pm \sqrt{-\chi_{p0} \chi_{m1} \chi_{n1}}. \quad (4.5)$$

Further on, we will use the plus sign. We point out that the dispersion relation does not depend on this sign.
The solutions of the remaining nine equations are

\[
\begin{align*}
\chi_{p0} &= 1 - v_0^2, \\
\chi_{p1} &= -v_0 (2v_1 + v_0 W_1), \\
\chi_{p2} &= -2v_0 v_2, \\
\chi_{m1} &= -W_1 - \chi_{n1}, \\
\chi_{p20} &= -\frac{1}{1 - v_0^2} \left[ v_1^2 + v_0 \left( 2 \left( 1 - v_0^2 \right) (v_1 W_1 + v_20) - v_0^3 \left( u_1^2 + W_{20} \right) - 2u_1 \sqrt{\left( 1 - v_0^2 \right) \chi_{n1} (W_1 + \chi_{n1})} + v_0 \left( u_1^2 - v_1^2 + W_{20} + \chi_{n1} (W_1 + \chi_{n1}) \right) \right) \right], \\
\chi_{p21} &= \frac{1}{1 - v_0^2} \left[ - \left( 1 - v_0^2 \right) (2v_0 v_2 W_1 + 2v_1 v_2 + v_0 (2v_0 u_1 u_2 + 2v_21 + v_0 W_{21})) \right] \\
&+ 2v_0 u_2 \sqrt{\left( 1 - v_0^2 \right) \chi_{n1} (W_1 + \chi_{n1})}, \\
\chi_{p22} &= - \left( v_0^2 u_2^2 + v_0^2 + 2v_0 v_{22} \right), \\
\chi_{m20} &= -\frac{1}{1 - v_0^2} \left[ (1 - v_0^2) \left( W_{20} + \chi_{n20} \right) - v_0^2 \chi_{n1} (W_1 + \chi_{n1}) \right] \\
&+ 2v_0 u_1 \sqrt{\left( 1 - v_0^2 \right) \chi_{n1} (W_1 + \chi_{n1})}, \\
\chi_{m21} &= -\frac{1}{1 - v_0^2} \left[ (1 - v_0^2) (W_{21} + \chi_{n21}) + 2v_0 u_2 \sqrt{\left( 1 - v_0^2 \right) \chi_{n1} (W_1 + \chi_{n1})} \right],
\end{align*}
\]

where

\[
K_1 = \sqrt{\left( 1 - v_0^2 \right) \chi_{n1} (W_1 + \chi_{n1})}.
\]

From the condition \( p \) to be finite, we obtain

\[
p = \arcsin \left( 2v_0 \sqrt{1 - v_0^2} \right),
\]

solved by

\[
v_0 = \cos \frac{p}{2},
\]

and nine more equations as well. Three of them can be solved independently, leading to

\[
\begin{align*}
v_1 &= \frac{1}{4} v_0 \left( 1 - v_0^2 + 2\chi_{n1} \right) \left( 1 - \log(16) \right), \\
v_2 &= \frac{1}{4} v_0 \left( 1 - v_0^2 + 2\chi_{n1} \right), \\
W_1 &= - \left( 1 - v_0^2 + 2\chi_{n1} \right).
\end{align*}
\]

Our next condition \( J_2 = 0 \), gives

\[
\begin{align*}
u_1 &= \frac{v_0 K_1}{1 - v_0^2} \log(4), \\
u_2 &= -\frac{v_0 K_1}{2(1 - v_0^2)},
\end{align*}
\]
where
\[ K_1 = \sqrt{-(1-v_0^2)(1-v_0^2+\chi_{n1})\chi_{n1}}, \]
and another three equations. Thus, a system of nine equations remains to be solved, using the results obtained so far. It turns out that one of them is satisfied identically, while the others can be solved with respect to \( v_{02}, v_{21}, v_{22}, u_{02}, u_{21}, u_{22}, W_{02}, W_{21} \).

The parameters \( \chi_{n20}, \chi_{n21}, K_{20}, K_{21} \), are still undetermined. To fix them, let us consider the expansion for the angular difference \( \tilde{\phi} \). To the leading order, we have
\[ \tilde{\phi} = 2 \arctan \sqrt{-\frac{-\chi_{n1}}{1-v_0^2+\chi_{n1}}}, \]
solved by
\[ \chi_{n1} = -(1-v_0^2)\sin^2\tilde{\phi} = -\sin^2\frac{p}{2}\sin^2\frac{\tilde{\phi}}{2}, \]
and leading to
\[ K_1 = \frac{1}{2} \sin^3\frac{p}{2} \sin \tilde{\phi}. \]  
(5.2)
Comparing (5.2) with (4.14) taken at \( J_2 = 0 \), one finds \( \tilde{\Lambda} = 1, \tilde{\phi} = \Phi \) (see also (4.15)).

Now, we impose the condition the equality \( \tilde{\phi} = \Phi \) to be valid in the subleading order also. This gives four equations, which can be solved with respect to \( \chi_{n20}, \chi_{n21}, K_{20} \) and \( K_{21} \). The solutions for all parameters in their final form are given in appendix A.

Replacing all solutions for the parameters appearing in the expansion for \( E - J_1 \), we find the following dispersion relation up to the subleading order correction
\[ E - J_1 = 2 \sin \frac{p}{2} \left( 1 - 4 \cos \tilde{\phi} \sin^2\frac{p}{2} e^{-2-J_1 \csc \frac{\tilde{\phi}}{2}} \right) \]
\[ - \left\{ 4 \sin^3\frac{p}{2} [13 + 9 \cos p + 16 \cos \tilde{\phi} + (1 + 3 \cos p) \cos 2\tilde{\phi} \right] 
- 64 (1 - \log 2) \log 2 (1 + \cos p) \sin^2 \tilde{\phi} \right\] 
\[ + 4 J_1 (3 - 2 \cos p - \log 16 + (\log 16 - 1) (\cos 2\tilde{\phi} + 2 \cos 2p \sin^2 \tilde{\phi})) \]
\[ - 16 J_1^2 \sin \frac{p}{2} \cos \frac{p}{2} \cos 2\tilde{\phi} \} e^{-4-2J_1 \csc \frac{\tilde{\phi}}{2}}, \]
where \( \tilde{\phi} \neq 0 \), describes the deviation of the energy-charge relation corrections from the ordinary giant magnon case. This is our final result for a finite-size string with one (large) angular momentum.
6 Concluding Remarks

In this paper we considered the most general string configurations on the $R_t \times S^3$ subspace of $AdS_5 \times S^5$, described by the NR integrable system. Imposing appropriate conditions on the parameters involved, we restrict ourselves to string solutions, which contain as particular cases the finite-size giant magnons with one and two angular momenta. Taking the limit in which the modulus of the elliptic integrals is close to one, we found the corrections to the dispersion relations. For the dyonic string, this is done to the leading order. For the string with one nonzero angular momentum, the subleading correction is also obtained. In both cases, the corrections depend on one more parameter $\Phi$ or $\tilde{p}$. When $\Phi = 0$, the finite-size giant magnons energy-charge relations should be reproduced. It is true to the leading order. However, this is not the case, when we consider the subleading correction. Indeed, according to [15], in conformal gauge, it is given by

$$ -8 \sin \frac{p}{2} \left[ (7 + 6 \cos \phi) \sin^2 \frac{p}{2} + 2 J_1 (2 + 3 \cos \phi) \sin \frac{p}{2} + 2 J_1^2 \cos^2 \frac{p}{2} \right] e^{-4-2J_1 \csc \frac{\theta}{2}}, \quad (6.1) $$

while from (5.3) we obtain

$$ -8 \sin \frac{p}{2} \left[ 3 (5 + 2 \cos \phi) \sin^2 \frac{p}{2} + 2 J_1 \sin \frac{p}{2} - 2 J_1^2 \cos^2 \frac{p}{2} \right] e^{-4-2J_1 \csc \frac{\theta}{2}}. \quad (6.2) $$

Thus, only the terms containing $J_1^2$ coincide up to a sign. Similar discrepancy was observed in [22] too, where by using different approach, the authors found agreement with (6.1) for the leading term $\propto J_1^2$, but disagreement at the subleading in $J_1$ orders. In (6.2), we have additional sign difference. At the moment, we do not know the reason, and can not explain why these three results are different. Obviously, this issue should be studied in more detail.

The approach we used here, can be applied to find the finite-size effects on the dispersion relations for different string configurations in $AdS_5 \times S^5$ (generalized spiky strings for example), as well as the finite-size effects in other backgrounds. In particular, this problem for dyonic giant magnons in the $\gamma$-deformed $AdS_4 \times CP_3$ is under investigation [26].

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A Elliptic Integrals, $\epsilon$-Expansions and Solutions for the Parameters

The elliptic integrals appearing in the main text are given by

\[
\int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} = \frac{2}{\sqrt{\chi_p - \chi_n}} K(1 - \epsilon),
\]

\[
\int_{\chi_m}^{\chi_p} \frac{\chi d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} = \frac{2\chi_n}{\sqrt{\chi_p - \chi_n}} K(1 - \epsilon) + 2\sqrt{\chi_p - \chi_n} E(1 - \epsilon),
\]

\[
\int_{\chi_m}^{\chi_p} \frac{d\chi}{\chi \sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} = \frac{2}{\chi_p \sqrt{\chi_p - \chi_n}} \Pi\left(1 - \frac{\chi_m}{\chi_p} | 1 - \epsilon\right),
\]

\[
\int_{\chi_m}^{\chi_p} \frac{(1 - \chi) d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} = \frac{2}{(1 - \chi_p) \sqrt{\chi_p - \chi_n}} \Pi\left(-\frac{\chi_p - \chi_m}{1 - \chi_p} | 1 - \epsilon\right),
\]

where

\[
\epsilon = \frac{\chi_m - \chi_n}{\chi_p - \chi_n}.
\]

We use the following expansions for the complete elliptic integrals [27]

\[
K(1 - \epsilon) = -\frac{1}{2} \log\left(\frac{\epsilon}{16}\right) - \frac{1}{4} \left(1 + \frac{1}{2} \log\left(\frac{\epsilon}{16}\right)\right) \epsilon - \frac{3}{128} \left(7 + 3 \log\left(\frac{\epsilon}{16}\right)\right) \epsilon^2 + \ldots,
\]

\[
E(1 - \epsilon) = 1 - \frac{1}{4} \left(1 + \log\left(\frac{\epsilon}{16}\right)\right) \epsilon - \frac{1}{64} \left(13 + 6 \log\left(\frac{\epsilon}{16}\right)\right) \epsilon^2 + \ldots,
\]

\[
\Pi(-n|1 - \epsilon) = \frac{2\sqrt{n} \arctan(\sqrt{n}) - \log\left(\frac{\epsilon}{16}\right)}{2(1 + n)} - \frac{2 - 4\sqrt{n} \arctan(\sqrt{n}) + (1 - n) \log\left(\frac{\epsilon}{16}\right)}{8(1 + n)^2} \epsilon
\]

\[
- \frac{21 - 12n - 5n^2 - 48\sqrt{n} \arctan(\sqrt{n}) + 3(3 - (6 + n)n) \log\left(\frac{\epsilon}{16}\right)}{128(1 + n)^3} \epsilon^2 + \ldots, \quad n > 0.
\]

We use also the equality [28]

\[
\Pi(\nu|m) = \frac{q_1}{q} \Pi(\nu_1|m) - \frac{m}{q \sqrt{-\nu \nu_1}} K(m),
\]

where

\[
q = \sqrt{(1 - \nu) \left(1 - \frac{m}{\nu}\right)}, \quad q_1 = \sqrt{(1 - \nu_1) \left(1 - \frac{m}{\nu_1}\right)},
\]

\[
\nu = \frac{\nu_1 - m}{\nu_1 - 1}, \quad \nu_1 < 0, \quad m < \nu < 1.
\]
For the parameters, we use the following expansions

\[ \chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2} \log(\epsilon)) \epsilon + (\chi_{p20} + \chi_{p21} \log(\epsilon) + \chi_{p22} \log^2(\epsilon)) \epsilon^2, \]

\[ \chi_m = \chi_{m1} \epsilon + (\chi_{m20} + \chi_{m21} \log(\epsilon)) \epsilon^2, \]

\[ \chi_n = \chi_{n1} \epsilon + (\chi_{n20} + \chi_{n21} \log(\epsilon)) \epsilon^2, \]

\[ v = v_0 + (v_1 + v_2 \log(\epsilon)) \epsilon + (v_{20} + v_{21} \log(\epsilon) + v_{22} \log^2(\epsilon)) \epsilon^2, \]

\[ u = u_0 + (u_1 + u_2 \log(\epsilon)) \epsilon + (u_{20} + u_{21} \log(\epsilon) + u_{22} \log^2(\epsilon)) \epsilon^2, \]

\[ W = 1 + W_1 \epsilon + (W_{20} + W_{21} \log(\epsilon)) \epsilon^2, \]

\[ K = K_1 \epsilon + (K_{20} + K_{21} \log(\epsilon)) \epsilon^2. \]

Considering the case of a string with two angular momenta, we neglect the terms proportional to \( \epsilon^2 \), thus obtaining the leading correction to the dispersion relation only. To find the subleading correction for a string with one angular momentum, we set \( u_0 = 0 \).

The solutions for the parameters for the first case, \( J_2 \neq 0 \), are given by \( (\tilde{\Lambda} = 1) \)

\[ \chi_{p0} = \sin^2(p/2), \]

\[ \chi_{p1} = -\frac{1}{16 (J_2^2 + 4 \sin^2(p/2))} \left[ \frac{1}{J_2^2 + 4 \sin^2(p/2)} \right] \left[ \cos \Phi \left( \cos p \left( 15 + 8 J_2^2 + 60 \log 2 \right) \right) \right. \]

\[ - \left. (1 + \log 16) \left( 10 + (6 + J_2^2) \cos 2p - \cos 3p \right) \right. \]

\[ + J_2^2 \left( \log 16 - 7 + J_2^2 \left( 256 - 2 \right) \right) \sin^2 p \]

\[ - \frac{16 J_2 \sin \Phi \sin^3(p/2) \cos(p/2) (5 + 2 J_2^2 - 4 \cos p - \cos 2p) \log 2}{\sqrt{J_2^2 + 4 \sin^2(p/2)}} \right], \]

\[ \chi_{p2} = \frac{1}{8 \left( J_2^2 + 4 \sin^2(p/2) \right) \left( J_2^2 + 4 \sin^2(p/2) \right)} \left[ \sin(p/2) \sin p \left( \cos(p/2) \left( J_2^2 + 10 + 15 \cos p - (6 + J_2^2) \cos 2p + \cos 3p \right) \cos \Phi \right. \right. \]

\[ - \left. J_2 \sqrt{J_2^2 + 4 \sin^2(p/2)} \left( 5 + 2 J_2^2 - 4 \cos p - \cos 2p \right) \sin(p/2) \sin \Phi \right) \right], \]

\[ \chi_{m1} = \sin^2(p/2) \cos^2(\Phi/2), \]

\[ \chi_{n1} = -\sin^2(p/2) \sin^2(\Phi/2), \]
In the equalities above, the angle $\Phi$ is related to the angle $\tilde{\rho}$, defined in (3.4), according to

$$\Phi = \tilde{\rho} + \mathcal{J}_2 \frac{\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)}}{\mathcal{J}_2^2 + 4 \sin^4(p/2)} \sin p.$$
The solutions for the parameters for the second case, $J_2 = 0$, are as follows

$$
\chi_{p0} = \sin^2 \frac{p}{2}, \quad \chi_{p1} = \frac{1}{8} \cos \tilde{p} \sin^2 p (1 + \log 16), \quad \chi_{p2} = -\frac{1}{8} \cos \tilde{p} \sin^2 p,
$$

$$
\chi_{p20} = \frac{1}{512} [ \cos 3p (-1 + 8 \log 2 (-1 + \log 4 (-3 + \log 16))) + 32 \log 2 (-1 - 4 \log^2 2 + \cos 2p (-1 + \log 4)^2 + \log 16 
+2 \left(2 \cos \tilde{p} + \cos 2\tilde{p} (-1 + \log 4)^2 \right) \sin^2 p + \cos p (1 + 16 (3 - \log 16) \log^2 2 + \log 256 
+8 \ \cos 2\tilde{p} (1 + 8 \log^2 2 (3 + \log 16) + \log 256) \sin^2 p)] ,
$$

$$
\chi_{p21} = \frac{1}{128} \left[ 2 \cos^2 \frac{p}{2} (5 - 8 \cos \tilde{p} + \cos 2p (1 - 12 \log 2 (-1 + \log 4)) 
+ 2 \cos p (-3 + \log 16 + 4 \cos \tilde{p}) + \log 16 (-5 + \log 64)) 
- 4 \cos 2\tilde{p} (1 + \log 16 (-2 + \log 8) + \cos p (1 + 12 \log^2 2 + \log 64)) \sin^2 p \right],
$$

$$
\chi_{p22} = \frac{1}{128} \sin^2 p \left[ \cos p (4 - 8 \log 2 + \cos 2\tilde{p} (2 + \log 256)) - 2 (-5 + \log 256) \sin^2 \tilde{p} \right],
$$

$$
\chi_{m1} = \cos^2 \frac{\tilde{p}}{2} \sin^2 \frac{p}{2},
$$

$$
\chi_{m20} = \frac{1}{64} \left[ 3 + \cos 2p (-1 + \log 16 (-1 + \log 4)) + \log 4096 
+ 8 \log^2 2 \left(-1 + 4 \cos 2\tilde{p} \sin^2 \frac{p}{2} \right) + 4 \cos \tilde{p} (1 + \log 16) \sin^2 p 
+ 2 \cos p \left(-1 - \log 16 + 2 \cos 2\tilde{p} (1 + 8 \log^2 2 + \log 16) \sin^2 \frac{p}{2} \right) \right],
$$

$$
\chi_{m21} = -\frac{1}{4} \cos^2 \frac{\tilde{p}}{2} \sin^2 \frac{p}{2} \left[ 1 + \cos p \cos \tilde{p} - (1 + \cos p) (1 - \cos \tilde{p}) \log 16 \right],
$$

$$
\chi_{n1} = -\sin^2 \frac{\tilde{p}}{2} \sin^2 \frac{p}{2},
$$

$$
\chi_{n20} = -\frac{1}{2} \sin \frac{p}{2} [1 + \log 16 (\log 4 + \cos p (1 + \log 4)) 
- \cos^4 \frac{p}{2} (1 + 16 \log^2 2 + \log 16) - (8 \log^2 2 + \cos p (1 + 8 \log^2 2 + \log 16)) \sin^2 \frac{p}{2} \sin^2 \frac{\tilde{p}}{2}],
$$

$$
\chi_{n21} = -\frac{1}{4} \left[ 1 - \cos p \cos \tilde{p} - (1 + \cos p) (1 + \cos \tilde{p}) \log 16 \right] \sin^2 \frac{p}{2} \sin^2 \frac{\tilde{p}}{2},
$$
\[v_0 = \cos \frac{p}{2}, \quad v_1 = \frac{1}{4} (1 - \log 16) \cos \tilde{p} \cos \frac{p}{2} \sin^2 \frac{p}{2}, \quad v_2 = \frac{1}{4} \cos \tilde{p} \cos \frac{p}{2} \sin^2 \frac{p}{2},\]

\[v_{20} = \frac{1}{512} \cos \frac{p}{2} \left[-37 + \cos 2p (5 - 8 \log 2 (7 + \log 16 (-5 + \log 16))) + 8 \log 2 (23 - 36 \log 2 + \log^2 16) + 4 (\cos 2\tilde{p} (19 - 8 \log 2 (13 + 8 \log 2 (-3 + \log 4))) - 32 \cos \tilde{p} (-1 + \log 4)) \sin^2 \frac{p}{2}\right],\]

\[v_{21} = \frac{1}{256} \cos \frac{p}{2} \left[-23 - 24 \log 2 (-3 + \log 4) + \cos 2p (7 + 8 \log 2 (-5 + \log 64)) + 4 (8 \cos \tilde{p} + \cos 2\tilde{p} (13 + 48 (-1 + \log 2) \log 2)) \sin^2 \frac{p}{2}\right],\]

\[v_{22} = -\frac{1}{128} \cos \frac{p}{2} \left[12 + \cos (p + 2\tilde{p}) + 8 (-2 + \cos (p + 2\tilde{p})) \log 2 - 8 \cos p (-1 + \log 4) + 2 \cos 2\tilde{p} (-5 + \log 256) + \cos (p - 2\tilde{p}) (1 + \log 256) \sin^2 \frac{p}{2}\right],\]

\[u_1 = \frac{1}{2} \log 2 \sin p \sin \tilde{p}, \quad u_2 = -\frac{1}{8} \sin p \sin \tilde{p},\]

\[u_{20} = \frac{1}{16} \sin p \left[-1 + \log 16 \right] \left[\sin \tilde{p}\right],\]

\[u_{21} = \frac{1}{32} \sin p \left[-2 \sin \tilde{p} - (1 + \log 16 (-2 + \log 8) + 4 \cos p \log 2 (1 + \log 8)) \sin 2\tilde{p}\right],\]

\[u_{22} = \frac{1}{64} \left[-2 + \cos p + 4 (1 + \cos p) \log 2 \sin p \sin 2\tilde{p}\right],\]

\[W_1 = -\cos \tilde{p} \sin^2 \frac{p}{2},\]

\[W_{20} = \frac{1}{64} \left[11 - 16 \cos \tilde{p} + \log 16 (-7 + \log 16) + \cos 2p (1 - 16 \log^2 2 + \log 4096) - 4 \cos 2\tilde{p} (1 + \log 16 (-1 + \log 16)) \sin^2 \frac{p}{2}\right],\]

\[W_{21} = \frac{1}{16} \left[5 + 3 \cos p - \cos 2\tilde{p} + \cos p \cos 2\tilde{p} - 16 (1 + \cos p) \log 2 \sin^2 \tilde{p}\right],\]
\[ K_1 = \frac{1}{2} \sin^3 \frac{p}{2} \sin \bar{\rho}, \]

\[ K_{20} = \frac{1}{8} \sin \frac{p}{2} \sin \bar{\rho} \left[ 2 + 8 \log^2 2 + \log 16 + \cos p \left( 1 + 8 \log^2 2 + \log 256 \right) \right. \]

\[ - \cos^4 \frac{p}{2} \left( 3 + \log^2 16 + \log 4096 \right) \]

\[ - (1 + \log 16 + \log^2 16 + \cos p \left( 3 + \log^2 16 + \log 4096 \right)) \sin^2 \frac{p}{2} \sin^2 \frac{\bar{\rho}}{2}, \]

\[ K_{21} = -\frac{1}{16} \sin^3 \frac{p}{2} \sin \bar{\rho} \cos \bar{\rho} \left[ 1 + \log 256 + \cos p \left( 3 + \log 256 \right) \right]. \]

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