Berestycki-Lions conditions on ground state solutions for Kirchhoff-type problems with variable potentials *

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Abstract

By introducing some new tricks, we prove that the nonlinear problem of Kirchhoff-type
\[ \begin{cases} - (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^3; \\ u \in H^1(\mathbb{R}^3), \end{cases} \]
admits two class of ground state solutions under the general “Berestycki-Lions assumptions” on the nonlinearity $f$ which are almost necessary conditions, as well as some weak assumptions on the potential $V$. Moreover, we also give a simple minimax characterization of the ground state energy. Our results improve and complement previous ones in the literature.

Keywords: Kirchhoff-type problem; Ground state solution; Pohožaev manifold

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1 Introduction

In this paper, we consider the following nonlinear problem of Kirchhoff-type:
\[ \begin{cases} - (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^3; \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1} \]
where $a, b > 0$ are two constants, $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy
(V1) $V \in C(\mathbb{R}^3, [0, \infty));$
(V2) $V_\infty := \liminf_{|y| \to \infty} V(y) \geq V(x)$ for all $x \in \mathbb{R}^3;$$

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(F1) \( f \in C(\mathbb{R}, \mathbb{R}) \) and there exists a constant \( C_0 > 0 \) such that
\[
|f(t)| \leq C_0 \left(1 + |t|^5\right), \quad \forall t \in \mathbb{R};
\]

(F2) \( f(t) = o(t) \) as \( t \to 0 \) and \( |f(t)| = o\left(|t|^5\right) \) as \( |t| \to +\infty \).

Clearly, under assumptions (V1), (V2), (F1) and (F2), weak solutions to \( (1.1) \) correspond to critical points of the energy functional defined in \( H^1(\mathbb{R}^3) \) by
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[a|\nabla u|^2 + V(x)u^2\right] \, dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)^2 - \int_{\mathbb{R}^3} F(u) \, dx,
\]
where and in the sequel, \( F(t) := \int_0^t f(s) \, ds \). We say a nontrivial weak solution \( \bar{u} \) to \( (1.1) \) is a ground state solution if \( I(\bar{u}) \leq I(v) \) for any nontrivial solution \( v \) to \( (1.1) \).

There have been many works about the existence of nontrivial solutions to \( (1.1) \) by using variational methods, see for example, \[1, 5, 6, 7, 8, 9, 11, 12, 13, 18, 20, 21, 23, 24, 25, 27, 28, 29, 32, 34, 35\] and the references therein. A typical way to deal with \( (1.1) \) is to use the mountain-pass theorem. For this purpose, one usually assumes that \( f(t) \) is subcritical and superlinear at \( t = 0 \) and 4-superlinear at \( t = \infty \) in the sense that
\[
(SF) \lim_{|t| \to \infty} \frac{F(t)}{t^4} = \infty,
\]
and satisfies the Ambrosetti-Rabinowitz type condition
\[
(AR) \ f(t)t \geq 4F(t) \geq 0, \; \forall \; t \in \mathbb{R};
\]
or the following variant convex condition
\[
(S1) \ f(t)/|t|^3 \text{ is strictly increasing for } t \in \mathbb{R} \setminus \{0\}.
\]
In fact, under (SF) and (AR) (or (S1)), it is easy to verify the Mountain Pass geometry and the boundedness of (PS) sequences for \( I \).

When \( f(t) \) is not 4-superlinear at \( t = \infty \), following the procedure of Ruiz \[26\] in which the nonlinear Schrödinger-Poisson system was dealt with, Li and Ye \[19\] first proved that the following special form of \( (1.1) \)
\[
\begin{cases}
- (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + u = |u|^{p-2}u, & x \in \mathbb{R}^3; \\
u \in H^1(\mathbb{R}^3),
\end{cases}
\]
has a ground state positive solution if \( 3 < p < 6 \), by using a minimizing argument on a Nehari-Pohožaev manifold obtained by combining the Nehari manifold and the corresponding Pohožaev identity. Subsequently, by introducing a new Nehari-Pohožaev manifold differing from \[19\] and using Jeanjean’s monotonicity trick \[14\] and a suitable approximating method, Guo \[10\] generalized Li and Ye’s result to \( (1.1) \), where \( V \) and \( f \) satisfy (V1), (V2), (F1), (F2) and
(V3') $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta \in (0, 1)$ such that
\[ |\nabla V(x) \cdot x| \leq \frac{\theta a}{2|x|^2}, \quad \forall \, x \in \mathbb{R}^3 \setminus \{0\}; \]

(S2) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and \( \left( \frac{f(t)}{t} \right)' > 0 \).

Applying Guo’s result to (1.3), the condition $3 < p < 6$ in [19] can be relaxed to $2 < p < 6$. More recently, Tang and Chen [31] introduced some new skills to weaken (V3') and (S2) to the following conditions

(V3) $V \in C^1(\mathbb{R}^3, \mathbb{R})$, and
\[ |\nabla V(x) \cdot x| \leq \frac{a}{2|x|^2}, \quad \forall \, x \in \mathbb{R}^3 \setminus \{0\}; \]

(S3) $f \in C(\mathbb{R}, \mathbb{R})$ and $\frac{f(t)+6F(t)}{t^2}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

We remark that (SF), (AR), (S1)-(S3) are all global growth conditions. Inspired by the fundamental paper [4], Azzollini [2] proved that the “limit problem” associated with (1.1)
\[
\begin{align*}
-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V_\infty u &= f(u), \quad x \in \mathbb{R}^3; \\
 u &\in H^1(\mathbb{R}^3),
\end{align*}
\]
has a ground state positive solution if $f$ satisfies the Berestycki-Lions type assumptions: (F1), (F2) and the following local assumption

(F3) there exists $s_0 > 0$ such that $F(s_0) > \frac{1}{4} V_\infty s_0^2$.

Obviously, (F1)-(F3) are satisfied by a very wide class of nonlinearities. In particular, only local conditions on $f(t)$ are required. Moreover, in view of [2], (F1)-(F3) are “almost” necessary for the existence of a nontrivial solution of problem (1.4). This kind of conditions were first introduced by Berestycki and Lions [4] for the study of the nonlinear scalar field equation
\[
-\Delta u + V_\infty u = f(u), \quad u \in H^1(\mathbb{R}^N).
\]

To prove the above result, Azzollini considered the following constrained minimization problem
\[
m^\infty := \inf_{u \in M^\infty} I^\infty(u),
\]
where
\[
I^\infty(u) := \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V_\infty u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^3} F(u) \, dx
\]
is the energy functional associated with (1.4), and
\[
M^\infty := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{P}^\infty(u) = 0 \}
\]
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is the Pohožaev manifold, and $\mathcal{P}^\infty$ is the Pohožaev functional defined by
\[
\mathcal{P}^\infty(u) = \frac{a}{2} \| \nabla u \|^2_2 + \frac{3}{2} V_\infty \| u \|^2_2 + \frac{b}{2} \| \nabla u \|^4_2 - 3 \int_{\mathbb{R}^3} F(u) dx. \tag{1.7}
\]

Azzollini first proved that $I^\infty$ possesses a minimizer $u^\infty$ on $\mathcal{M}^\infty \cap H^1_r(\mathbb{R}^3)$, it is also a minimizer on $\mathcal{M}^\infty$ by Schwarz symmetrization, then verified that $u^\infty$ is a critical point of $I^\infty$ by means of the Lagrange multipliers Theorem.

In another paper [3], Azzollini, by means of a rescaling argument, established a general relationship between solutions of (1.4) and (1.5). That is
\[
u \in C_2(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3)
\]
is a solution to (1.4) if and only if there exist $v \in C_2(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3)$ satisfying (1.5) and $t > 0$ such that $t^2 a + t b \| \nabla v \|^2_2 = 1$ and $u(x) = v(tx)$. With this relationship and the results obtained in [4, 15] in hand, Azzollini [3] also concluded the same results as [2]. Following [3], Lu [22] proved that (1.4) has infinitely many distinct radial solutions if $f$ is odd and satisfies (F1)-(F3).

The approach used in [2, 3] is valid only for autonomous equations, it does not work any more for nonautonomous equation (1.1) with $V \neq$ constant. In the present paper, based on [2, 4, 16, 30], we shall develop a new approach to look for a ground state solution for (1.1) by using (F3) instead of (S3). Our results improve and generalize the Azzollini’s results in [2, 3] on autonomous equation (1.4). More precisely, we have the following theorem.

**Theorem 1.1.** Assume that $V$ and $f$ satisfy (V1)-(V3) and (F1)-(F3). Then problem (1.1) has a ground state solution.

To prove Theorem 1.1 we will use an idea from Jeanjean and Tanaka [16], that is an approximation procedure to obtain a bounded (PS)-sequence for $I$, instead of starting directly from an arbitrary (PS)-sequence. More precisely, firstly for $\lambda \in [1/2, 1]$ we consider a family of functionals $I_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by
\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ a |\nabla u|^2 + V(x) u^2 \right] dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \lambda \int_{\mathbb{R}^3} F(u) dx. \tag{1.8}
\]
These functionals have a Mountain Pass geometry, and denoting the corresponding Mountain Pass levels by $c_\lambda$. Let
\[
A(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ a |\nabla u|^2 + V(x) u^2 \right] dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2, \quad B(u) = \int_{\mathbb{R}^3} F(u) dx.
\]
Then $I_\lambda(u) = A(u) - \lambda B(u)$. Unfortunately, $B(u)$ is not sign definite under (F1)-(F3), it prevents us from employing Jeanjean’s monotonicity trick [14] used in [16]. Thanks to the work of Jeanjean and Toland [17], $I_\lambda$ still has a bounded (PS)-sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ at level $c_\lambda$ for almost every $\lambda \in [1/2, 1]$. However, there is no more a monotone dependence of $c_\lambda$ upon $\lambda \in [1/2, 1]$ in this case, while it plays a crucial role in Jeanjean’s monotonicity
trick. To show that the bounded sequence \( \{u_n(\lambda)\} \) converges weakly to a nontrivial critical point of \( I_\lambda \), one usually has to establish the following strict inequality

\[
c_\lambda < \inf_{K^\infty_\lambda} I^\infty_\lambda,
\]

where

\[
I^\infty_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a |\nabla u|^2 + V_\infty u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \lambda \int_{\mathbb{R}^3} F(u) \, dx
\]

and

\[
K^\infty_\lambda := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : (I^\infty_\lambda)'(u) = 0 \}.
\]

In view of the results in \([2, 3]\), for every \( \lambda \in [1/2, 1] \), there exists \( w^\infty_\lambda \in K^\infty_\lambda \) such that

\[
I^\infty_\lambda(w^\infty_\lambda) = \inf_{K^\infty_\lambda} I^\infty_\lambda.
\]

Since \( V(x) \leq V_\infty \) but \( V(x) \not\equiv V_\infty \), it is standard to show (1.9) if \( w^\infty_\lambda > 0 \). However, there is no more information on the sign of \( w^\infty_\lambda \) from the results in \([2, 3]\). Therefore, it becomes nontrivial to show (1.9). To overcome this difficulty we use a strategy introduced in \([30]\). Let

\[
P^\infty_\lambda(u) = \frac{a}{2} \|\nabla u\|^2 + \frac{3V_\infty}{2} \int_{\mathbb{R}^3} u^2 \, dx + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - 3\lambda \int_{\mathbb{R}^3} F(u) \, dx
\]

and

\[
M^\infty_\lambda := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : P^\infty_\lambda(u) = 0 \}.
\]

We first prove that problem (1.4) has a solution \( \bar{u}^\infty \in H^1(\mathbb{R}^3) \) such that \( I^\infty(\bar{u}^\infty) = \inf_{M^\infty_\lambda} I^\infty \). By means of the translation invariance for \( \bar{u}^\infty \) and a crucial inequality related to \( I(u), I(u_t) \) and \( P(u) \) (the IIP inequality in short, see Lemma 2.2 where \( u_t(x) = u(x/t) \), it plays an important role in many places of this paper), we can find \( \bar{\lambda} \in [1/2, 1] \) and prove directly the following crucial inequality

\[
c_\lambda < m^\infty_\lambda := \inf_{M^\infty_\lambda} I^\infty_\lambda, \quad \lambda \in (\bar{\lambda}, 1].
\]

In particular, it is not required any information on sign of \( \bar{u}^\infty \) in our arguments. Then applying (1.14) and a precise decomposition of bounded (PS)-sequences, we can get a nontrivial critical point \( u_\lambda \) of \( I_\lambda \) which possesses energy \( c_\lambda \) for almost every \( \lambda \in (\bar{\lambda}, 1] \).

In the proof of Theorem 1.1, a crucial step is to show that problem (1.3) has a solution \( \bar{u}^\infty \in H^1(\mathbb{R}^3) \) such that \( I^\infty(\bar{u}^\infty) = \inf_{M^\infty} I^\infty \). With the help of the Lions’ concentration compactness, the IIP inequality established in Lemma 2.2 the “least energy squeeze approach” and some subtle analysis, we can prove a more general conclusion. In fact, we shall conclude that (1.11) has a solution \( \bar{u} \in M \) such that \( I(\bar{u}) = \inf_M I \) if \( f \) satisfies (F1)-(F3) and \( V \) satisfies (V1), (V2) and the following decay assumption on \( V \):
(V4) \( V \in C^1(\mathbb{R}^3, \mathbb{R}) \) and \( t \mapsto 3V(tx) + \nabla V(tx) \cdot (tx) + \frac{a}{4|tx|^2} \) is nonincreasing on \((0, \infty)\) for every \( x \in \mathbb{R}^3 \setminus \{0\} \);

where

\[
\mathcal{M} := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{P}(u) = 0 \}
\]

(1.15)

and

\[
\mathcal{P}(u) := \frac{a}{2} \| \nabla u \|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x]u^2 \, dx \\
+ \frac{b}{2} \| \nabla u \|_4^4 - 3 \int_{\mathbb{R}^3} F(u) \, dx.
\]

(1.16)

Actually the equality \( \mathcal{P}(u) = 0 \) is nothing but the Pohožaev identity related with equation (1.1). More precisely, we have the following theorem.

**Theorem 1.2.** Assume that \( V \) and \( f \) satisfy (V1), (V2), (V4) and (F1)-(F3). Then problem (1.1) has a solution \( \bar{u} \in H^1(\mathbb{R}^3) \) such that \( \mathcal{I}(\bar{u}) = \inf_{\mathcal{M}} \mathcal{I} = \inf_{u \in \Lambda} \max_{t > 0} \mathcal{I}(u_t) > 0 \), where \( u_t(x) := u(x/t) \) and \( \Lambda = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left[ \frac{1}{2} V_\infty u^2 - F(u) \right] \, dx < 0 \right\} \).

As a consequence of Theorem 1.2 we have the following corollary.

**Corollary 1.3.** Assume that \( f \) satisfies (F1)-(F3). Then problem (1.4) has a solution \( \bar{u} \in H^1(\mathbb{R}^3) \) such that \( \mathcal{I}_\infty(\bar{u}) = \inf_{\mathcal{M}_\infty} \mathcal{I}_\infty = \inf_{u \in \Lambda} \max_{t > 0} \mathcal{I}_\infty(u_t) > 0 \).

Remark 1.4. As a consequence of Theorem 1.2 the ground state value \( m := \inf_{\mathcal{M}} \mathcal{I} \) has a minimax characterization \( m = \inf_{u \in \Lambda} \max_{t > 0} \mathcal{I}(u_t) \) which is much simpler than the usual characterizations related to the Mountain Pass level.

Our approach to show Theorem 1.2 is different from the ones used in [2, 3]. Moreover, Theorem 1.2 generalizes the Azzollini’s results in [2, 3] on autonomous equation (1.4) to (1.1) with \( V \neq \text{constant} \). In particular, such an approach could be useful for the study of other problems where radial symmetry of bounded sequence either fails or is not readily available.

Remark 1.5. There are indeed many functions which satisfy (V1)-(V3). For example

i). \( V(x) = \alpha - \frac{\beta}{|x|^\sigma + 1} \) with \( \alpha > \beta > 0, \sigma \geq 2 \) and \( a \geq 2\sigma\beta \);

ii). \( V(x) = \alpha - \frac{\beta \sin^2 |x|}{|x|^\sigma + 1} \) with \( \alpha > \beta > 0 \) and \( a \geq 4\beta \);

iii). \( V(x) = \alpha - \beta e^{-|x|^\sigma} \) with \( \alpha > \beta > 0, \sigma > 0 \) and \( a e^{(\sigma+2)/\sigma} \geq 2\beta(\sigma + 2)^{(\sigma+2)/\sigma} \sigma^{-2/\sigma} \).

In particular, if \( \alpha > \beta > 0, \sigma \geq 2 \) and \( a \geq 2\sigma\beta(3 + \sigma) \), then \( V(x) = \alpha - \frac{\beta}{|x|^\sigma + 1} \) also satisfies (V4).
Applying Theorem 1.1 to the following perturbed problem:

\[
\begin{cases}
- (a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + [V_\infty - \varepsilon h(x)] u = f(u), & x \in \mathbb{R}^3; \\
u \in H^1(\mathbb{R}^3),
\end{cases}
\tag{1.17}
\]

where \( V_\infty \) is a positive constant and the function \( h \in C^1(\mathbb{R}^3, \mathbb{R}) \) verifies:

(H1) \( h(x) \geq 0 \) for all \( x \in \mathbb{R}^3 \) and \( \lim_{|x| \to \infty} h(x) = 0 \);

(H2) \( \sup_{x \in \mathbb{R}^3} [-|x|^2 \nabla h(x) \cdot x] < \infty \).

Then we have the following corollary.

**Corollary 1.6.** Assume that \( h \) and \( f \) satisfy (H1), (H2) and (F1)-(F3). Then there exists a constant \( \varepsilon_0 > 0 \) such that problem (1.17) has a ground state solution \( \bar{u}_\varepsilon \in H^1(\mathbb{R}^3) \setminus \{0\} \) for all \( 0 < \varepsilon \leq \varepsilon_0 \).

Throughout the paper we make use of the following notations:

- \( H^1(\mathbb{R}^3) \) denotes the usual Sobolev space equipped with the inner product and norm
  \[
  (u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx, \quad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^3);
  \]

- \( H^1_0(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : |x| = |y| \Rightarrow u(x) = u(y)\} \);

- \( L^s(\mathbb{R}^3)(1 \leq s < \infty) \) denotes the Lebesgue space with the norm \( \|u\|_s = (\int_{\mathbb{R}^3} |u|^s dx)^{1/s} \);

- For any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), \( u_t(x) := u(t^{-1}x) \) for \( t > 0 \);

- For any \( x \in \mathbb{R}^3 \) and \( r > 0 \), \( B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\} \);

- \( C_1, C_2, \ldots \) denote positive constants possibly different in different places.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries, and give the proof of Theorem 1.2. In Section 3, we complete the proof of Theorem 1.1.

## 2 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. To this end, we give some useful lemmas. Since \( V(x) \equiv V_\infty \) satisfies (V1), (V2) and (V4), thus all conclusions on \( \mathcal{I} \) are also true for \( \mathcal{I}_\infty \). For (1.4), we always assume that \( V_\infty > 0 \). First, by a simple calculation, we can verify Lemma 2.1.

**Lemma 2.1.** Assume that (V4) holds. Then one has

\[
3t^3[V(x) - V(tx)] - (1 - t^3)\nabla V(x) \cdot x \geq -\frac{a(1 - t)^2(2 + t)}{4|x|^2}, \quad \forall t \geq 0, \quad x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.1}
\]
Lemma 2.2. Assume that (V1), (V2), (V4), (F1) and (F2) hold. Then
\[ I(u) \geq I(u_t) + \frac{1 - t^3}{3} P(u) + \frac{b(1 - t)^2(1 + 2t)}{12} \| \nabla u \|^4_2, \quad \forall \ u \in H^1(\mathbb{R}^3), \ t > 0. \tag{2.2} \]

Proof. According to Hardy inequality, we have
\[ \| \nabla u \|^2_2 \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} \, dx, \quad \forall \ u \in H^1(\mathbb{R}^3). \tag{2.3} \]

Note that
\[ I(u_t) = \frac{a t}{2} \| \nabla u \|^2_2 + \frac{t^3}{2} \int_{\mathbb{R}^3} V(tx) u^2 \, dx + \frac{b t^2}{4} \| \nabla u \|^4_2 - t^3 \int_{\mathbb{R}^3} F(u) \, dx. \tag{2.4} \]

Thus, by (1.2), (1.16), (2.1), (2.3) and (2.4), one has

\[
I(u) - I(u_t) = \frac{a(1 - t)}{2} \| \nabla u \|^2_2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - t^3 V(tx)] u^2 \, dx + \frac{b(1 - t^2)}{4} \| \nabla u \|^4_2 \\
+ (t^3 - 1) \int_{\mathbb{R}^3} F(u) \, dx \\
= \frac{1 - t^3}{3} \left\{ \frac{a}{2} \| \nabla u \|^2_2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 \, dx + \frac{b}{2} \| \nabla u \|^4_2 - 3 \int_{\mathbb{R}^3} F(u) \, dx \right\} \\
+ \frac{a(1 - t)^2(1 + 2t)}{6} \| \nabla u \|^2_2 + \frac{1}{6} \int_{\mathbb{R}^3} \{ 3t^3 [V(x) - V(tx)] - (1 - t^3) \nabla V(x) \cdot x \} u^2 \, dx \\
+ \frac{b(1 - t)^2(1 + 2t)}{12} \| \nabla u \|^4_2 \\
\geq \frac{1 - t^3}{3} P(u) + \frac{b(1 - t)^2(1 + 2t)}{12} \| \nabla u \|^4_2.
\]

This shows that (2.2) holds.

From Lemma 2.2 we have the following two corollaries.

Corollary 2.3. Assume that (F1) and (F2) hold. Then
\[ I^\infty(u) \geq I^\infty(u_t) + \frac{1 - t^3}{3} P^\infty(u) + \frac{b(1 - t)^2(1 + 2t)}{12} \| \nabla u \|^4_2, \quad \forall \ u \in H^1(\mathbb{R}^3), \ t > 0. \tag{2.5} \]

Corollary 2.4. Assume that (V1), (V2), (V4), (F1) and (F2) hold. Then for \( u \in \mathcal{M} \)
\[ I(u) = \max_{t > 0} I(u_t). \tag{2.6} \]

Lemma 2.5. Assume that (V1), (V2) and (V4) hold. Then there exist two constants \( \gamma_1, \gamma_2 > 0 \) such that
\[ \gamma_1 \| u \|^2_2 \leq a \| \nabla u \|^2_2 + \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 \, dx \leq \gamma_2 \| u \|^2_2, \quad \forall \ u \in H^1(\mathbb{R}^3). \tag{2.7} \]
Proof. Let $t \to \infty$ in (2.1), and using (V2), one has
\[
3V(x) + \nabla V(x) \cdot x \leq 3V_\infty + \frac{a}{2|x|^2}, \quad \forall \, x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.8}
\]
Then it follows from (2.3) and (2.8) that there exists $\gamma_2 > 0$ such that the second inequality in (2.7) holds.

Next, we prove that the first inequality holds. By (2.1), one has
\[
3V(x) + \nabla V(x) \cdot x \geq 3V(tx) - \frac{a}{4|x|^2} \left(1 - \frac{1}{t}\right)^2 \left(1 + \frac{2}{t}\right), \quad \forall \, t > 0, \ x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.9}
\]
It is easy to see that there exist $\varepsilon_0 > 0$ and $t_0 > 0$ such that
\[
\left(1 - \frac{1}{t_0}\right)^2 \left(1 + \frac{2}{t_0}\right) \leq 1 - \varepsilon_0,
\]
which, together with (2.9), implies
\[
3V(x) + \nabla V(x) \cdot x \geq 3V(t_0x) - \frac{a}{4|x|^2}(1 - \varepsilon_0), \quad \forall \, x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.10}
\]
By (V2), there exists $R_0 > 0$ such that $V(x) \geq V_\infty/2$ for all $x \geq |t_0R_0|$. Choose $\alpha_0 \in (0, V_\infty/2)$ such that
\[
\varepsilon_0a - 3\alpha_0 \left(\frac{4\pi R_0}{3}\right)^{2/3} S^{-1} \geq \frac{\varepsilon_0a}{2}. \tag{2.11}
\]
Then it follows from (V1), (2.3), (2.10), (2.11) and Sobolev inequality that
\[
\begin{align*}
& a \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 \, dx \\
& \geq a \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} \left[3V(t_0x) - \frac{a}{4|x|^2}(1 - \varepsilon_0)\right] u^2 \, dx \\
& \geq a \|\nabla u\|_2^2 + 3\alpha_0 \int_{|x| \geq R_0} u^2 \, dx - (1 - \varepsilon_0)a\|\nabla u\|_2^2 \\
& = 3\alpha_0 \|u\|_2^2 + \varepsilon_0a\|\nabla u\|_2^2 - 3\alpha_0 \int_{|x| \leq R_0} u^2 \, dx \\
& \geq 3\alpha_0 \|u\|_2^2 + \varepsilon_0a\|\nabla u\|_2^2 - 3\alpha_0 \left(\frac{4\pi R_0}{3}\right)^{2/3} \left(\int_{|x| \leq R_0} u^6 \, dx\right)^{1/3} \\
& \geq 3\alpha_0 \|u\|_2^2 + \varepsilon_0a\|\nabla u\|_2^2 - 3\alpha_0 \left(\frac{4\pi R_0}{3}\right)^{2/3} S^{-1}\|\nabla u\|_2^2 \\
& \geq \min \left\{3\alpha_0, \frac{\varepsilon_0a}{2} \right\} \|u\|_2^2 := \gamma_1\|u\|_2^2, \quad \forall \, u \in H^1(\mathbb{R}^3). \tag{2.12}
\end{align*}
\]

To show $\mathcal{M} \neq \emptyset$, we define a set $\Lambda$ as follows:
\[
\Lambda = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left[\frac{1}{2}V_\infty u^2 - F(u)\right] \, dx < 0 \right\}. \tag{2.13}
\]


Lemma 2.6. Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then \( \Lambda \neq \emptyset \) and

\[
\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \mathcal{P}^\infty(u) \leq 0 \text{ or } \mathcal{P}(u) \leq 0 \} \subset \Lambda.
\] (2.14)

Proof. In view of the proof of [11 Theorem 2], (F3) implies \( \Lambda \neq \emptyset \). Next, we have two cases to distinguish:

1. \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) and \( \mathcal{P}^\infty(u) \leq 0 \), then (1.17) implies \( u \in \Lambda \).

2. Let \( t = 0 \) and \( t \to \infty \) in (2.14), respectively, and using (V2), one has

\[
-\frac{a}{4|x|^2} + 3V_\infty \leq 3V(x) + \nabla V(x) \cdot x \leq 3V_\infty + \frac{a}{2|x|^2}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.
\] (2.15)

For \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) and \( \mathcal{P}(u) \leq 0 \), then it follows from (1.16), (2.3) and (2.15) that

\[
3 \int_{\mathbb{R}^3} \left[ \frac{1}{2}V_\infty u^2 - F(u) \right] dx
= \mathcal{P}(u) - \frac{a}{2}\|
abla u\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} [3(V(x) - V_\infty) + \nabla V(x) \cdot x] u^2 dx - \frac{b}{2}\|
abla u\|_2^4
\leq -\frac{a}{2}\|
abla u\|_2^2 + \frac{a}{8} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx - \frac{b}{2}\|
abla u\|_2^4
\leq -\frac{b}{2}\|
abla u\|_2^4 < 0,
\]

which implies \( u \in \Lambda \). 

\[\Box\]

Lemma 2.7. Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then for any \( u \in \Lambda \), there exists a unique \( t_u > 0 \) such that \( u_{t_u} \in \mathcal{M} \).

Proof. Let \( u \in \Lambda \) be fixed and define a function \( \zeta(t) := \mathcal{I}(u_t) \) on \((0, \infty)\). Clearly, by (1.16) and (2.4), we have

\[
\zeta'(t) = 0 \iff \frac{a}{2}\|
abla u\|_2^2 + \frac{t^2}{2} \int_{\mathbb{R}^3} [3V(tx) + \nabla V(tx) \cdot tx] u^2 dx
+ \frac{bt}{2}\|
abla u\|_2^4 - 3t^2 \int_{\mathbb{R}^3} F(u) dx = 0
\iff \mathcal{P}(u_t) = 0 \iff u_t \in \mathcal{M}.
\]

It is easy to verify, using (V1), (V2) and the definition of \( \Lambda \), that \( \lim_{t \to 0} \zeta(t) = 0 \), \( \zeta(t) > 0 \) for \( t > 0 \) small and \( \zeta(t) < 0 \) for \( t \) large. Therefore \( \max_{t \in (0, \infty)} \zeta(t) \) is achieved at \( t_u > 0 \) so that \( \zeta'(t_u) = 0 \) and \( u_{t_u} \in \mathcal{M} \).

Now we pass to prove that \( t_u \) is unique for any \( u \in \Lambda \). In fact, for any given \( u \in \Lambda \), let \( t_1, t_2 > 0 \) such that \( u_{t_1}, u_{t_2} \in \mathcal{M} \). Then \( \mathcal{P}(u_{t_1}) = \mathcal{P}(u_{t_2}) = 0 \). Jointly with (2.2), we have

\[
\mathcal{I}(u_{t_1}) \geq \mathcal{I}(u_{t_2}) + \frac{t_3^3 - t_2^3}{3t_1^3} \mathcal{P}(u_{t_1}) + \frac{b(t_2^3 - t_1^3)^2(t_1 + 2t_2)}{12t_1} \|
abla u\|_2^4
= \mathcal{I}(u_{t_2}) + \frac{b(t_2^3 - t_1^3)^2(t_1 + 2t_2)}{12t_1} \|
abla u\|_2^4
\] (2.16)
\[ I(u_{t_2}) \geq I(u_t) + \frac{t_3^2 - t_1^2}{3t_2} \mathcal{P}(u_{t_2}) + \frac{b(t_2^2 - t_1^2)^2(t_2 + 2t_1)}{12t_2} \| \nabla u \|_2^4 \]

\[ = I(u_t) + \frac{b(t_2^2 - t_1^2)^2(t_2 + 2t_1)}{12t_2} \| \nabla u \|_2^4. \]  

(2.17)

(2.16) and (2.17) imply \( t_1 = t_2 \). Therefore, \( t_u > 0 \) is unique for any \( u \in \Lambda \).

**Corollary 2.8.** Assume that (F1)-(F3) hold. Then for any \( u \in \Lambda \), there exists a unique \( t_u > 0 \) such that \( u_{t_u} \in \mathcal{M}^\infty \).

Combining Corollary 2.4 with Lemma 2.7, we have the following lemma.

**Lemma 2.9.** Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then

\[ \inf_{u \in \mathcal{M}} I(u) := m = \inf_{u \in \Lambda} \max_{t > 0} I(u_t). \]

Similar to [29, Lemma 2.10], we have the following lemma.

**Lemma 2.10.** Assume that (V1), (V2), (F1) and (F2) hold. If \( u_n \rightharpoonup \bar{u} \) in \( H_1(\mathbb{R}^3) \), then

\[ I(u_n) = I(\bar{u}) + I(u_n - \bar{u}) + \frac{b}{2} \| \nabla \bar{u} \|_2^2 \| \nabla (u_n - \bar{u}) \|_2^2 + o(1) \]  

(2.18)

and

\[ \mathcal{P}(u_n) = \mathcal{P}(\bar{u}) + \mathcal{P}(u_n - \bar{u}) + b \| \nabla \bar{u} \|_2^2 \| \nabla (u_n - \bar{u}) \|_2^2 + o(1). \]  

(2.19)

**Lemma 2.11.** Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then

(i) there exists \( \rho_0 > 0 \) such that \( \| \nabla u \|_2 \geq \rho_0 \), \( \forall u \in \mathcal{M} \);

(ii) \( m = \inf_{u \in \mathcal{M}} I(u) > 0 \).

**Proof.** (i). Since \( \mathcal{P}(u) = 0 \), \( \forall u \in \mathcal{M} \), by (F1), (F2), [16], (2.7) and Sobolev embedding inequality \( S\|u\|_6^6 \leq \| \nabla u \|_2^2 \), one has

\[ \gamma_1 \|u\|_2^2 + b \| \nabla u \|_2^4 \leq a \| \nabla u \|_2^2 + \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx + b \| \nabla u \|_2^4 \]

\[ = 6 \int_{\mathbb{R}^3} F(u) dx \]

\[ \leq \frac{\gamma_1}{2} \|u\|_2^2 + C_1 \|u\|_6^6 \leq \frac{\gamma_1}{2} \|u\|_2^2 + C_1 S^{-3} \| \nabla u \|_2^6, \]  

(2.20)

which implies

\[ \| \nabla u \|_2 \geq \rho_0 \leq \left( \frac{bS^3}{C_1} \right)^{1/2}, \quad \forall u \in \mathcal{M}. \]  

(2.21)

(ii). By (2.22) with \( t \to 0 \), we have

\[ I(u) = I(u) - \frac{1}{3} \mathcal{P}(u) \geq \frac{b}{12} \| \nabla u \|_2^4, \quad \forall u \in \mathcal{M}. \]  

(2.22)

This, together with (2.21) shows that \( m = \inf_{u \in \mathcal{M}} I(u) > 0 \).
Lemma 2.12. Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then \( m \leq m^\infty \).

Proof. In view of Lemma 2.6 and Corollary 2.8 we have \( \mathcal{M}^\infty \neq \emptyset \). Arguing indirectly, we assume that \( m > m^\infty \). Let \( \varepsilon := m - m^\infty \). Then there exists \( u^\infty_\varepsilon \) such that

\[
u^\infty_\varepsilon \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > I^\infty(u^\infty_\varepsilon).
\]  

(2.23)

In view of Lemmas 2.6 and 2.7, there exists \( t_\varepsilon > 0 \) such that \( (u^\infty_\varepsilon)_{t_\varepsilon} \in \mathcal{M} \). Thus, it follows from (V2), (1.2), (1.6), (2.5) and (2.23) that

\[m^\infty + \frac{\varepsilon}{2} > I^\infty((u^\infty_\varepsilon)_{t_\varepsilon}) \geq I((u^\infty_\varepsilon)_{t_\varepsilon}) \geq m.\]

This contradiction shows the conclusion of Lemma 2.12 is true. \( \square \)

Lemma 2.13. Assume that (V1), (V2), (V4) and (F1)-(F3) hold. Then \( m \) is achieved.

Proof. In view of Lemmas 2.6, 2.7 and 2.11 we have \( \mathcal{M} \neq \emptyset \) and \( m > 0 \). Let \( \{u_n\} \subset \mathcal{M} \) be such that \( I(u_n) \to m \). Since \( \mathcal{P}(u_n) = 0 \), then it follows from (2.2) with \( t \to 0 \), we have

\[m + o(1) = I(u_n) \geq \frac{b}{12} \|\nabla u_n\|^4_{L^2}. \]

(2.24)

This shows that \( \{\|\nabla u_n\|_2\} \) is bounded. Next, we prove that \( \{\|u_n\|\} \) is also bounded. From (F1), (F2), (1.10), (2.7) and Sobolev embedding inequality, one has

\[
\gamma_1 \|u_n\|^2 \leq a \|\nabla u_n\|^2_2 + \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u_n^2 \, dx + b \|\nabla u_n\|^4_{L^2}
\]

\[= 6 \int_{\mathbb{R}^3} F(u) \, dx \]

\[\leq \frac{\gamma_1}{2} \|u\|^2_2 + C_3 \|u\|^6_6 \leq \frac{\gamma_1}{2} \|u\|^2_2 + C_3 S^{-3} \|\nabla u\|^6_{L^2}. \]

(2.25)

Hence, \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). Passing to a subsequence, we have \( u_n \rightharpoonup \bar{u} \) in \( H^1(\mathbb{R}^3) \). Then \( u_n \to \bar{u} \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) for \( 2 \leq s < 2^* \) and \( u_n \to \bar{u} \) a.e. in \( \mathbb{R}^3 \). There are two possible cases: 

i). \( \bar{u} = 0 \) and ii). \( \bar{u} \neq 0 \).

Case i). \( \bar{u} = 0 \), i.e. \( u_n \to 0 \) in \( H^1(\mathbb{R}^3) \). Then \( u_n \to 0 \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) for \( 2 \leq s < 2^* \) and \( u_n \to 0 \) a.e. in \( \mathbb{R}^3 \). By (V2) and (2.15), it is easy to show that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} [V_\infty - V(x)] u_n^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_n^2 \, dx = 0.
\]

(2.26)

From (1.2), (1.6), (1.7), (1.16) and (2.26), one can get

\[I^\infty(u_n) \to m, \quad \mathcal{P}^\infty(u_n) \to 0. \]

(2.27)

From Lemma 2.11 (i), (1.7) and (2.27), one has

\[a \rho_0^2 \leq a \|\nabla u_n\|^2_2 + 3V_\infty \|u_n\|^2_2 + \frac{b}{2} \|\nabla u_n\|^4_{L^2} = 6 \int_{\mathbb{R}^3} F(u_n) \, dx + o(1). \]

(2.28)
Using (F1), (F2), (2.28) and Lions’ concentration compactness principle [33, Lemma 1.21], we can prove that there exist \( \delta > 0 \) and a sequence \( \{y_n\} \subset \mathbb{R}^3 \) such that \( \int_{B_1(y_n)} |u_n|^2 \, dx > \delta \).

Let \( \tilde{u}_n(x) = u_n(x + y_n) \). Then we have \( \|\tilde{u}_n\| = \|u_n\| \) and

\[
P^\infty(\tilde{u}_n) = o(1), \quad I^\infty(\tilde{u}_n) \to m, \quad \int_{B_1(0)} |\tilde{u}_n|^2 \, dx > \delta. \tag{2.29}
\]

Therefore, there exists \( \hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\} \) such that, passing to a subsequence,

\[
\begin{cases}
\hat{u}_n \to \hat{u}, & \text{in } H^1(\mathbb{R}^3); \\
\hat{u}_n \to \hat{u}, & \text{in } L^s_{\text{loc}}(\mathbb{R}^3), \quad \forall s \in [1, 6); \\
\hat{u}_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^3.
\end{cases} \tag{2.30}
\]

Let \( w_n = \hat{u}_n - \hat{u} \). Then (2.30) and Lemma 2.10 yield

\[
I^\infty(\hat{u}_n) = I^\infty(\hat{u}) + I^\infty(w_n) + \frac{b}{2} \|\nabla \hat{u}\|_2^2 \|\nabla w_n\|_2^2 + o(1) \tag{2.31}
\]

and

\[
P^\infty(\hat{u}_n) = P^\infty(\hat{u}) + P^\infty(w_n) + b\|\nabla \hat{u}\|_2^2 \|\nabla w_n\|_2^2 + o(1). \tag{2.32}
\]

Set

\[
\Psi_0(u) = \frac{4a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4}{12}. \tag{2.33}
\]

Then one has,

\[
\Psi_0(w_n) \leq m - \Psi_0(\hat{u}) + o(1), \quad P^\infty(w_n) \leq -P^\infty(\hat{u}) + o(1). \tag{2.34}
\]

If there exists a subsequence \( \{w_{n_i}\} \) of \( \{w_n\} \) such that \( w_{n_i} = 0 \), then going to this subsequence, we have

\[
I^\infty(\hat{u}) = m, \quad P^\infty(\hat{u}) = 0. \tag{2.35}
\]

Next, we assume that \( w_n \neq 0 \). We claim that \( P^\infty(\hat{u}) \leq 0 \). Otherwise, if \( P^\infty(\hat{u}) > 0 \), then (2.34) implies \( P^\infty(w_n) < 0 \) for large \( n \). In view of Lemma 2.6 and Corollary 2.8, there exists \( t_n > 0 \) such that \( (w_n)_{t_n} \in \mathcal{M}^\infty \). From (1.6), (1.7), (2.5), (2.33) and (2.34), we obtain

\[
m - \Psi_0(\hat{u}) + o(1) \geq \Psi_0(w_n) = I^\infty(w_n) - \frac{1}{3} P^\infty(w_n)
\]

\[
\geq I^\infty((w_n)_{t_n}) - \frac{t_n^3}{3} P^\infty(w_n)
\]

\[
\geq m^\infty - \frac{t_n^3}{3} P^\infty(w_n) \geq m^\infty,
\]

which implies \( P^\infty(\hat{u}) \leq 0 \) due to \( m \leq m^\infty \) and \( \Psi_0(\hat{u}) > 0 \). Since \( \hat{u} \neq 0 \) and \( P^\infty(\hat{u}) \leq 0 \), in view of Lemma 2.6 and Corollary 2.8, there exists \( t_\infty > 0 \) such that \( \hat{u}_{t_\infty} \in \mathcal{M}^\infty \). From (1.6), (1.7), (2.5), (2.29), (2.33) and the weak semicontinuity of norm, one has

\[
m = \lim_{n \to \infty} \left[ I^\infty(\hat{u}_n) - \frac{1}{3} P^\infty(\hat{u}_n) \right]
\]

13
\[
\begin{align*}
\lim_{n \to \infty} \Psi_0(\hat{u}_n) & \geq \Psi_0(\hat{u}) \\
= \mathcal{I}^\infty(\hat{u}) - \frac{1}{3} \mathcal{P}^\infty(\hat{u}) & \geq \mathcal{I}^\infty(\hat{u}_{t_\infty}) - \frac{t_\infty^3}{3} \mathcal{P}^\infty(\hat{u}) \\
\geq m^\infty - \frac{t_\infty^3}{3} \mathcal{P}^\infty(\hat{u}) & \geq m - \frac{t_\infty^3}{3} \mathcal{P}^\infty(\hat{u}) \geq m,
\end{align*}
\]

which implies (2.35) holds also. In view of Lemmas 2.6 and 2.7 there exists \( \hat{t} > 0 \) such that \( \hat{u}_{\hat{t}} \in \mathcal{M} \), moreover, it follows from (V2), (1.21), (1.6), (2.35) and Corollary 2.3 that

\[
m \leq \mathcal{I}(\hat{u}_{\hat{t}}) \leq \mathcal{I}^\infty(\hat{u}_{\hat{t}}) \leq \mathcal{I}(\hat{u}) = m.
\]

This shows that \( m \) is achieved at \( \hat{u}_{\hat{t}} \in \mathcal{M} \).

Case ii). \( \bar{u} \neq 0 \). Let \( v_n = u_n - \bar{u} \). Then Lemma (2.10) yields

\[
\mathcal{I}(u_n) = \mathcal{I}(\bar{u}) + \mathcal{I}(v_n) + \frac{b}{2} \| \nabla \bar{u} \|^2_2 \| \nabla v_n \|^2_2 + o(1) \tag{2.36}
\]

and

\[
\mathcal{P}(u_n) = \mathcal{P}(\bar{u}) + \mathcal{P}(v_n) + b \| \nabla \bar{u} \|^2_2 \| \nabla v_n \|^2_2 + o(1). \tag{2.37}
\]

Set

\[
\Psi(u) = \frac{4a \| \nabla u \|^2_2 + b \| \nabla u \|^4_2}{12} - \frac{1}{6} \int_{\mathbb{R}^3} \nabla V(x) \cdot xu^2 \, dx. \tag{2.38}
\]

Then it follows from (2.3) and (2.1) with \( t = 0 \) that

\[
\Psi(u) = \frac{4a \| \nabla u \|^2_2 + b \| \nabla u \|^4_2}{12} - \frac{1}{6} \int_{\mathbb{R}^3} \nabla V(x) \cdot xu^2 \, dx \\
\geq \frac{4a \| \nabla u \|^2_2 + b \| \nabla u \|^4_2}{12} - \frac{a}{12} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} \, dx \\
\geq \frac{b}{12} \| \nabla u \|^2_2, \quad \forall \, u \in H^1(\mathbb{R}^3). \tag{2.39}
\]

Since \( \mathcal{I}(u_n) \to m \) and \( \mathcal{P}(u_n) = 0 \), then it follows from (1.21), (1.16), (2.36), (2.37) and (2.38) that

\[
\Psi(v_n) \leq m - \Psi(\bar{u}) + o(1), \quad \mathcal{P}(v_n) \leq -\mathcal{P}(\bar{u}) + o(1). \tag{2.40}
\]

If there exists a subsequence \( \{v_{n_i}\} \) of \( \{v_n\} \) such that \( v_{n_i} = 0 \), then going to this subsequence, we have

\[
\mathcal{I}(\bar{u}) = m, \quad \mathcal{P}(\bar{u}) = 0, \tag{2.41}
\]

which implies the conclusion of Lemma 2.13 holds. Next, we assume that \( v_n \neq 0 \). We claim that \( \mathcal{P}(\bar{u}) \leq 0 \). Otherwise \( \mathcal{P}(\bar{u}) > 0 \), then (2.40) implies \( \mathcal{P}(v_n) < 0 \) for large \( n \). In view of
Lemmas 2.6 and 2.7, there exists $t_n > 0$ such that $(v_n)_{t_n} \in \mathcal{M}$. From (1.2), (1.16), (2.2), (2.38) and (2.40), we obtain

$$m - \Psi(\bar{u}) + o(1) \geq \Psi(v_n) = \mathcal{I}(v_n) - \frac{1}{3} \mathcal{P}(v_n)$$

$$\geq \mathcal{I}((v_n)_{t_n}) - \frac{t_n^3}{3} \mathcal{P}(v_n)$$

$$\geq m - \frac{t_n^3}{3} \mathcal{P}(v_n) \geq m,$$

which implies $\mathcal{P}(\bar{u}) \leq 0$ due to $\Psi(\bar{u}) > 0$. Since $\bar{u} \neq 0$ and $\mathcal{P}(\bar{u}) \leq 0$, in view of Lemmas 2.6 and 2.7, there exists $\bar{t} > 0$ such that $\bar{u} \bar{t} \in \mathcal{M}$. From (1.2), (1.16), (2.2), (2.38), (2.39) and the weak semicontinuity of norm, one has

$$m = \lim_{n \to \infty} \left[ \mathcal{I}(u_n) - \frac{1}{3} \mathcal{P}(u_n) \right]$$

$$= \lim_{n \to \infty} \Psi(u_n) \geq \Psi(\bar{u})$$

$$= \mathcal{I}(\bar{u}) - \frac{1}{3} \mathcal{P}(\bar{u}) \geq \mathcal{I}(\bar{u}_t) - \frac{\bar{t}^3}{3} \mathcal{P}(\bar{u})$$

$$\geq m - \frac{\bar{t}^3}{3} \mathcal{P}(\bar{u}) \geq m,$$

which implies (2.41) also holds. 

**Lemma 2.14.** Assume that (V1), (V2), (V4) and (F1)-(F3) hold. If $\bar{u} \in \mathcal{M}$ and $\mathcal{I}(\bar{u}) = m$, then $\bar{u}$ is a critical point of $\mathcal{I}$.

**Proof.** Similar to the proof of [30, Lemma 2.13], we can prove this lemma only by using

$$\mathcal{I}(\bar{u}_t) \leq \mathcal{I}(\bar{u}) - \frac{b(1-t)^2(1+2t)}{12} \|\nabla \bar{u}\|_2^4$$

$$= m - \frac{b(1-t)^2(1+2t)}{12} \|\nabla \bar{u}\|_2^4, \quad \forall \ t > 0. \quad (2.42)$$

and

$$\varepsilon := \min \left\{ \frac{b(1-T_1)^2(1+2T_1)}{36} \|\nabla \bar{u}\|_2^4, \frac{b(1-T_2)^2(1+2T_2)}{36} \|\nabla \bar{u}\|_2^4, 1, \frac{\rho_0}{8} \right\}$$

instead of [30] (2.40) and $\varepsilon$, respectively. 

**Proof of Theorem 1.2.** In view of Lemmas 2.9, 2.13 and 2.14, there exists $\bar{u} \in \mathcal{M}$ such that

$$\mathcal{I}(\bar{u}) = m = \inf_{u \in \Lambda} \max_{t > 0} \mathcal{I}(u_t), \quad \mathcal{I}'(\bar{u}) = 0.$$

This shows that $\bar{u}$ is a ground state solution of (1.1).
3 Proof of Theorem 1.1

In this section, we assume that \( V(x) \not\equiv V_\infty \) and give the proof of Theorem 1.1.

**Proposition 3.1.** \([17]\) Let \( X \) be a Banach space and let \( J \subset \mathbb{R}^+ \) be an interval, and

\[
\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,
\]

be a family of \( C^1 \)-functional on \( X \) such that

(i) either \( A(u) \to +\infty \) or \( B(u) \to +\infty \), as \( \|u\| \to \infty \);

(ii) \( B \) maps every bounded set of \( X \) into a set of \( \mathbb{R} \) bounded below;

(iii) there are two points \( v_1, v_2 \) in \( X \) such that

\[
\tilde{c}_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \tag{3.1}
\]

where

\[
\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2 \}.
\]

Then, for almost every \( \lambda \in J \), there exists a sequence such that

(i) \( \{u_n(\lambda)\} \) is bounded in \( X \);

(ii) \( \Phi_\lambda(u_n(\lambda)) \to c_\lambda \);

(iii) \( \Phi'_\lambda(u_n(\lambda)) \to 0 \) in \( X^* \), where \( X^* \) is the dual of \( X \).

**Lemma 3.2.** \([10]\) Assume that (V1)-(V3), (F1) and (F2) hold. Let \( u \) be a critical point of \( I_\lambda \) in \( H^1(\mathbb{R}^3) \), then we have the following Pohožaev type identity

\[
P_\lambda(u) := \frac{a}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx
\]

\[
+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - 3\lambda \int_{\mathbb{R}^3} F(u) dx = 0. \tag{3.2}
\]

Correspondingly, we also let

\[
P_\infty^\lambda(u) = \frac{a}{2} \|\nabla u\|_2^2 + \frac{3V_\infty}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - 3\lambda \int_{\mathbb{R}^3} F(u) dx, \tag{3.3}
\]

for \( \lambda \in [1/2, 1] \). Set

\[
\mathcal{M}_\lambda^\infty := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : P_\lambda^\infty(u) = 0 \}, \quad m^\infty_\lambda := \inf_{u \in \mathcal{M}_\lambda^\infty} \mathcal{I}_\lambda^\infty(u). \tag{3.4}
\]

By Corollary 2.3 we have the following lemma.
Lemma 3.3. Assume that (F1) and (F2) hold. Then
\[
I_\lambda^\infty (u) \geq I_\lambda^\infty (u_t) + \frac{1-t^3}{3} P_\lambda^\infty (u) + \frac{b(1-t)^2(1+2t)}{12} \|\nabla u\|_2^4,
\]
\[
\forall u \in H^1(\mathbb{R}^3), \quad t > 0.
\]
(3.5)

Lemma 3.4. Assume that (V1)-(V3) and (F1)-(F3) hold. Then

(i) there exists $T > 0$ independent of $\lambda$ such that $I_\lambda ((u_1^\infty)_T) < 0$ for all $\lambda \in [0.5, 1]$;

(ii) there exists a positive constant $\kappa_0$ independent of $\lambda$ such that for all $\lambda \in [0.5, 1],
\[
c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) \geq \kappa_0 > \max \{I_\lambda(0), I_\lambda ((u_1^\infty)_T)\},
\]
where $\Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = (u_1^\infty)_T \}$;

(iii) $c_\lambda$ is bounded for $\lambda \in [0.5, 1]$;

(iv) $m_\lambda^\infty$ is non-increasing on $\lambda \in [0.5, 1]$;

(v) $\limsup_{\lambda \to \lambda_0} c_\lambda \leq c_{\lambda_0}$ for $\lambda_0 \in (0.5, 1]$.

Since $m_\lambda^\infty = I_\lambda^\infty (u_1^\infty)$ and $\int_{\mathbb{R}^3} F(u_1^\infty)dx > 0$, then the proof of (i)-(iv) in Lemma 3.4 is standard, (v) can be proved similar to [13] Lemma 2.3, so we omit it.

In view of Corollary 1.3, $I_1^\infty = I^\infty$ has a minimizer $u_1^\infty \neq 0$ on $\mathcal{M}_1^\infty = \mathcal{M}^\infty$, i.e.
\[
u_1^\infty \in \mathcal{M}_1^\infty, \quad (I_1^\infty)'(u_1^\infty) = 0 \quad \text{and} \quad m_1^\infty = I_1^\infty(u_1^\infty),
\]
(3.6)
where $m_1^\infty$ is defined by (3.4). Since (1.4) is autonomous, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, then there exist $\bar{x} \in \mathbb{R}^3$ and $\bar{r} > 0$ such that
\[
V_\infty - V(x) > 0, \quad |u_1^\infty(x)| > 0 \quad \text{a.e.} \quad |x - \bar{x}| \leq \bar{r}.
\]
(3.7)

Lemma 3.5. Assume that (V1)-(V3) and (F1)-(F3) hold. Then there exists $\hat{\lambda} \in [1/2, 1)$ such that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\hat{\lambda}, 1]$.

Proof. It is easy to see that $I_\lambda ((u_1^\infty)_t)$ is continuous on $t \in (0, \infty)$. Hence for any $\lambda \in [1/2, 1]$, we can choose $t_\lambda \in (0, T)$ such that $I_\lambda ((u_1^\infty)_{t_\lambda}) = \max_{t \in (0, T]} I_\lambda ((u_1^\infty)_t)$. Setting
\[
\gamma_0(t) = \begin{cases} (u_1^\infty)(tT), & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}
\]
Then $\gamma_0 \in \Gamma$ defined by Lemma 3.4 (ii). Moreover
\[
I_\lambda ((u_1^\infty)_{t_\lambda}) = \max_{t \in [0,1]} I_\lambda (\gamma_0(t)) \geq c_\lambda.
\]
(3.8)
Let

\[ \zeta_0 := \min \{3\bar{r}/8(1 + |\bar{x}|), 1/4\}. \]  

(3.9)

Then it follows from (3.9) that

\[ |x - \bar{x}| \leq \frac{\bar{r}}{2} \quad \text{and} \quad s \in [1 - \zeta_0, 1 + \zeta_0] \Rightarrow |sx - \bar{x}| \leq \bar{r}. \]  

(3.10)

Since \( P^\infty(u_1^\infty) = 0 \), then \( \int_{\mathbb{R}^N} F(u_1^\infty)dx > 0 \). Let

\[ \tilde{\lambda} := \max \left\{ \frac{1}{2}, 1 - \frac{\min_{s \in [1 - \zeta_0, 1 + \zeta_0]} \int_{\mathbb{R}^3} [V_\infty - V(sx)]|u_1^\infty|^2dx}{2(1 - \zeta_0)^{-3}T^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx}, 1 - \frac{b\zeta_0^2 \|
abla u_1^\infty\|^2}{12T^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx} \right\} \]

Then it follows from (3.7) and (3.10) that \( 1/2 \leq \tilde{\lambda} < 1 \). We have two cases to distinguish:

Case i). \( t_\lambda \in [1 - \zeta_0, 1 + \zeta_0] \). From (1.8), (1.10), (3.5)-(3.8), (3.9) and Lemma 3.4 (iv), we have

\[ m_\lambda^\infty \geq m_1^\infty = I_1^\infty(u_1^\infty) \geq I_1^\infty((u_1^\infty)_{t_\lambda}) \]

\[ = I_\lambda((u_1^\infty)_{t_\lambda}) - (1 - \lambda) t_\lambda^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx + \frac{t_\lambda^3}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)]|u_1^\infty|^2dx \]

\[ \geq c_\lambda - (1 - \lambda)T^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx \]

\[ + \frac{(1 - \zeta_0)^3}{2} \min_{s \in [1 - \zeta_0, 1 + \zeta_0]} \int_{\mathbb{R}^3} [V_\infty - V(sx)]|u_1^\infty|^2dx \]

\[ > c_\lambda, \quad \forall \lambda \in (\tilde{\lambda}, 1]. \]

Case ii). \( t_\lambda \in (0, 1 - \zeta_0) \cup (1 + \zeta_0, T] \). From (1.8), (1.10), (3.5), (3.6), (3.8), (3.9) and Lemma 3.4 (iv), we have

\[ m_\lambda^\infty \geq m_1^\infty = I_1^\infty(u_1^\infty) \geq I_1^\infty((u_1^\infty)_{t_\lambda}) + \frac{b(1 - t_\lambda)^2(1 + 2t_\lambda)}{12} \|
abla u_1^\infty\|^2 \]

\[ = I_\lambda((u_1^\infty)_{t_\lambda}) - (1 - \lambda)t_\lambda^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx \]

\[ + \frac{t_\lambda^3}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda x)]|u_1^\infty|^2dx + \frac{b(1 - t_\lambda)^2(1 + 2t_\lambda)}{12} \|
abla u_1^\infty\|^4 \]

\[ \geq c_\lambda - (1 - \lambda)T^3 \int_{\mathbb{R}^3} F(u_1^\infty)dx + \frac{b\zeta_0^2}{12} \|
abla u_1^\infty\|^4 \]

\[ > c_\lambda, \quad \forall \lambda \in (\tilde{\lambda}, 1]. \]

In both cases, we obtain that \( c_\lambda < m_\lambda^\infty \) for \( \lambda \in (\tilde{\lambda}, 1] \). \hfill \Box

Lemma 3.6. Assume that (V1)-(V3) and (F1)-(F3) hold. Let \( \{u_n\} \) be a bounded \((PS)_{c_\lambda}\) sequence for \( I_\lambda \) with \( \lambda \in [1/2, 1] \). Then there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), an integer \( l \in \mathbb{N} \cup \{0\} \), and \( u_\lambda \in H^1(\mathbb{R}^3) \) such that

(i) \( A_\lambda^2 := \lim_{n \to \infty} \|
abla u_n\|_2^2 \) exists, \( u_n \rightharpoonup u_\lambda \) in \( H^1(\mathbb{R}^3) \) and \( \mathcal{E}'_\lambda(u_\lambda) = 0 \);
(ii) \( w^k \neq 0 \) and \( (\mathcal{E}_\lambda^\infty)'(w^k) = 0 \) for \( 1 \leq k \leq l \);

(iii) \[
c + \frac{bA_\lambda^4}{4} = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{E}_\lambda^\infty(w^k);
\]

\[
A_\lambda^2 = \|\nabla u_\lambda\|_2^2 + \sum_{k=1}^{l} \|\nabla w^k\|_2^2,
\]

where

\[
\mathcal{E}_\lambda(u) = \frac{a + bA_\lambda^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 \, dx - \lambda \int_{\mathbb{R}^3} F(u) \, dx
\]

and

\[
\mathcal{E}_\lambda^\infty(u) = \frac{a + bA_\lambda^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{V_\infty}{2} \int_{\mathbb{R}^3} u^2 \, dx - \lambda \int_{\mathbb{R}^3} F(u) \, dx.
\]

where we agree that in the case \( l = 0 \) the above holds without \( w^k \).

Analogous to the proof of [19, Lemma 3.4], we can prove Lemma 3.6, so we omit it here.

**Lemma 3.7.** Assume that (V1)-(V3) and (F1)-(F3) hold. Then for almost every \( \lambda \in [\bar{\lambda}, 1] \), there exists \( u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\} \) such that

\[
\mathcal{T}_\lambda'(u_\lambda) = 0, \quad \mathcal{T}_\lambda(u_\lambda) = c_\lambda.
\]

**Proof.** Under (V1)-(V3) and (F1)-(F3), Lemma 3.4 implies that \( \mathcal{T}_\lambda(u) \) satisfies the assumptions of Proposition 3.4 with \( X = H^1(\mathbb{R}^3) \) and \( \Phi_\lambda = \mathcal{T}_\lambda \). So for almost every \( \lambda \in [0.5, 1] \), there exists a bounded sequence \( \{u_n(\lambda)\} \subset H^1(\mathbb{R}^3) \) (for simplicity, we denote the sequence by \( \{u_n\} \) instead of \( \{u_n(\lambda)\} \)) such that

\[
\mathcal{T}_\lambda(u_n) \rightarrow c_\lambda > 0, \quad \|\mathcal{T}_\lambda'(u_n)\| \rightarrow 0.
\]

By Lemma 3.6 there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), and \( u_\lambda \in H^1(\mathbb{R}^3) \) such that \( A_\lambda^2 := \lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2 \) exists, \( u_n \rightharpoonup u_\lambda \) in \( H^1(\mathbb{R}^3) \) and \( \mathcal{E}_\lambda'(u_\lambda) = 0 \), and there exist \( l \in \mathbb{N} \) and \( w^1, \ldots, w^l \in H^1(\mathbb{R}^3) \setminus \{0\} \) such that \( (\mathcal{E}_\lambda^\infty)'(w^k) = 0 \) for \( 1 \leq k \leq l \),

\[
c_\lambda + bA_\lambda^4 = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{E}_\lambda^\infty(w^k)
\]

and

\[
A_\lambda^2 = \|\nabla u_\lambda\|_2^2 + \sum_{k=1}^{l} \|\nabla w^k\|_2^2.
\]

Since \( \mathcal{E}_\lambda'(u_\lambda) = 0 \), then we have the Pohožaev identity referred to the functional \( \mathcal{E}_\lambda 

\[
\mathcal{P}_\lambda(u_\lambda) := \frac{a + bA_\lambda^2}{2} \|\nabla u_\lambda\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u_\lambda^2 \, dx - 3\lambda \int_{\mathbb{R}^3} F(u_\lambda) \, dx = 0.
\]
From (V3) and Hardy inequality
\[ a\|\nabla u_\lambda\|_2^2 \geq \frac{a}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2}dx \geq \int_{\mathbb{R}^3} \nabla V(x) \cdot xu^2dx. \quad (3.20) \]

It follows from (3.18), (3.19) and (3.20) that
\[ \mathcal{E}_\lambda(u_\lambda) = \mathcal{E}_\lambda(u_\lambda) - \frac{1}{3} \tilde{\mathcal{P}}_\lambda(u_\lambda) = \frac{a + bA^2}{3} \|\nabla u_\lambda\|_2^2 - \frac{1}{6} \int_{\mathbb{R}^3} \nabla V(x) \cdot xu^2dx \geq \frac{bA^2}{3} \|\nabla u_\lambda\|_2^2. \quad (3.21) \]

Since \((\mathcal{E}_\lambda^\infty)'(w^k) = 0\), then we have the Pohožaev identity referred to the functional \(\mathcal{E}_\lambda^\infty\)
\[ \tilde{\mathcal{P}}_\lambda^\infty(w^k) := \frac{a + bA^2}{2} \|\nabla w^k\|_2^2 + \frac{3V^\infty}{2} \int_{\mathbb{R}^3} (w^k)^2dx - 3\lambda \int_{\mathbb{R}^3} F(w^k)dx = 0. \quad (3.22) \]

Thus, from (3.18) and (3.22), we have
\[ 0 = \tilde{\mathcal{P}}_\lambda^\infty(w^k) \geq \mathcal{P}_\lambda^\infty(w^k). \quad (3.23) \]

Since \(w^k \neq 0\) and \(w^k \in \Lambda\), in view of Lemmas 2.6 and 2.7, there exists \(t_k > 0\) such that \((w^k)_{t_k} \in \mathcal{M}_\lambda^\infty\). From (1.10), (1.12), (3.3), (3.14), (3.18), (3.22) and (3.23), one has
\[ \mathcal{E}_\lambda^\infty(w^k) = \mathcal{E}_\lambda^\infty(w^k) - \frac{1}{3} \tilde{\mathcal{P}}_\lambda^\infty(w^k) = \frac{a + bA^2}{3} \|\nabla w^k\|_2^2 - \frac{1}{12} \|\nabla w^k\|_2^2 \geq \frac{bA^2}{4} \|\nabla w^k\|_2^2 + \mathcal{I}_\lambda^\infty((w^k)_{t_k}) - \frac{bA^2}{12} \mathcal{P}_\lambda^\infty(w^k) \geq \frac{bA^2}{4} \|\nabla w^k\|_2^2 + m^\infty. \quad (3.24) \]

It follows from (3.17), (3.18), (3.21) and (3.24) that
\[ c_\lambda + \frac{bA^4}{4} = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{E}_\lambda^\infty(w^k) \geq lm^\infty + \frac{bA^2}{4} \left[ \|\nabla u_\lambda\|_2^2 + \sum_{k=1}^{l} \|\nabla w^k\|_2^2 \right] \]
\[ = lm^\infty + \frac{bA^4}{4}, \quad \forall \lambda \in (\bar{\lambda}, 1], \]

which, together with Lemma 3.5 implies that \(l = 0\) and \(\mathcal{E}_\lambda(u_\lambda) = c_\lambda + \frac{bA^4}{4}\). Hence, it follows from (3.18) that \(A_\lambda = \|\nabla u_\lambda\|_2\), and so \(\mathcal{I}_\lambda(u_\lambda) = 0\) and \(\mathcal{I}_\lambda(u_\lambda) = c_\lambda\).

**Proof of Theorem 1.2**. In view of Lemma 3.7, there exist two sequences \(\{\lambda_n\} \subset (\bar{\lambda}, 1]\) and \(\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)\), denoted by \(\{u_n\}\), such that
\[ \lambda_n \to 1, \quad \mathcal{I}_\lambda^{\infty}(u) = 0, \quad \mathcal{I}_\lambda^{\infty}(u) = c_\lambda. \quad (3.25) \]
From (V3), (1.8), (2.3), (3.25) and Lemma 3.4 (v), one has
\[
c_1 \geq c_{\lambda n} = I_{\lambda n}(u_n) - \frac{1}{3} P_{\lambda n}(u_n) \\
= \frac{a}{3} \|\nabla u_n\|_2^2 - \frac{1}{6} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_n^2 dx + \frac{b}{6} \|\nabla u_n\|^4 \\
\geq \frac{b}{6} \|\nabla u_n\|^4.
\] (3.26)

This shows that \(\{\|\nabla u_n\|_2\}\) is bounded. Next, we demonstrate that \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^3)\). By (V1), (V2), (F1), (F2), (1.8), (3.25), (3.26) and the Sobolev embedding inequality, we have
\[
\gamma_1^2 \|u_n\|^2 \leq \int_{\mathbb{R}^3} [a |\nabla u_n|^2 + V(x) u_n^2] \, dx \\
\leq 2c_{\lambda n} + 2\lambda_n \int_{\mathbb{R}^3} F(u_n) dx \\
\leq 2c_1 + \frac{\gamma_1^2}{4} \|u_n\|^2 + C_5 \|u_n\|^6_6 \\
\leq 2c_1 + \frac{\gamma_1^2}{4} \|u_n\|^2 + C_5 S^{-3} \|\nabla u_n\|^6_2,
\]
where \(\gamma_1^2\) is a positive constant. Hence, \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^3)\). In view of Lemma 3.4 (v), we have \(\lim_{n \to \infty} c_{\lambda n} = c_* \leq c_1\). Hence, it follows from (1.2), (1.8) and (3.25) that
\[
I(u_n) \to c_*, \quad I'(u_n) \to 0.
\] (3.27)

This shows that \(\{u_n\}\) satisfy (3.16) with \(I_{\lambda} = I\) and \(c_\lambda = c_*\). In view of the proof of Lemma 3.7, we can show that there exists \(\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}\) such that
\[
I'(\bar{u}) = 0, \quad 0 < I(\bar{u}) \leq c_1.
\] (3.28)

Let
\[
\mathcal{K} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I'(u) = 0\}, \quad \hat{m} := \inf_{u \in \mathcal{K}} I(u).
\]

Then (3.28) shows that \(\mathcal{K} \neq \emptyset\) and \(\hat{m} \leq c_1\). For any \(u \in \mathcal{K}\), Lemma 3.2 implies \(P(u) = P_1(u) = 0\). Hence it follows from (3.21) that \(I(u) = I_1(u) > 0\), and so \(\hat{m} \geq 0\). Let \(\{u_n\} \subset \mathcal{K}\) such that
\[
I'(u_n) = 0, \quad I(u_n) \to \hat{m}.
\] (3.29)

In view of Lemma 3.5 \(\hat{m} \leq c_1 < m_{1}^\infty\). By a similar argument as in the proof of Lemma 3.7, we can prove that there exists \(\tilde{u} \in H^1(\mathbb{R}^N) \setminus \{0\}\) such that
\[
I'(\tilde{u}) = 0, \quad I(\tilde{u}) = \hat{m}.
\] (3.30)

This shows that \(\tilde{u}\) is a nontrivial least energy solution of (1.1). \(\square\)
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