Gauge-invariant perturbations around symmetry-reduced sectors of general relativity: applications to cosmology

Bianca Dittrich\(^1\) and Johannes Tambornino\(^{1,2}\)

\(^1\) Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada
\(^2\) Institut für Physik, RWTH Aachen, D-52056 Aachen, Germany

E-mail: bdittrich@perimeterinstitute.ca and jtambornino@perimeterinstitute.ca

Received 21 March 2007, in final form 26 July 2007
Published 30 August 2007
Online at stacks.iop.org/CQG/24/4543

Abstract

We develop a gauge-invariant canonical perturbation scheme for perturbations around symmetry-reduced sectors in generally covariant theories, such as general relativity. The central objects of investigation are gauge-invariant observables which encode the dynamics of the system.

We apply this scheme to perturbations around a homogeneous and isotropic sector (cosmology) of general relativity. The background variables of this homogeneous and isotropic sector are treated fully dynamically which allows us to approximate the observables to an arbitrary high order in a self-consistent and fully gauge-invariant manner. Methods to compute these observables are given.

The question of back-reaction effects of inhomogeneities onto a homogeneous and isotropic background can be addressed in this framework. We illustrate the latter by considering homogeneous but anisotropic Bianchi-I cosmologies as perturbations around a homogeneous and isotropic sector.

PACS numbers: 04.25.−g, 04.60.−m, 04.60.Ds, 98.80.Jk

1. Introduction

General relativity has two very challenging features: first, the dynamics of the theory is highly nonlinear; second, general relativity is a diffeomorphism-invariant and background-independent theory. These two features make it very difficult to construct gauge-invariant observables, that is, to extract physical predictions. Diffeomorphism invariance of the theory includes invariance under time reparametrizations, therefore observables have to be constants of motion. Hence finding gauge-invariant observables is intimately related to solving the
dynamics of the theory. But because of the highly nonlinear structure of the theory, it is quite hopeless to solve general relativity exactly. Indeed, so far there are almost\(^3\) no gauge-invariant observables known [1].

Therefore, we think that it is important to develop perturbation schemes in order to attack the dynamics of the theory. The difficulty for a perturbation scheme in general relativity is again the control over the gauge dependence of the results, which becomes already quite challenging at second order (see, for instance [2]).

In this work, we will develop a manifestly gauge-invariant perturbation theory in the canonical framework. This perturbation theory allows the calculation of observables to an arbitrary high order. In an earlier work [9], we introduced a calculation scheme for gauge-invariant observables for perturbations around a fixed background spacetime, which in the canonical framework corresponds to perturbations around a fixed phase-space point. In this work, we will rather consider an expansion around a whole symmetry-reduced sector of the theory. This sector will be treated non-perturbatively and could for instance describe homogeneous and isotropic fields (i.e. cosmology), spherically symmetric fields (i.e. black holes) or even midi-superspaces such as cylindrically symmetric gravitational waves. The latter example has infinitely many degrees of freedom, but is solvable [3, 4]. Note that perturbations around a fixed phase-space point arise as a special case of the scheme developed here.

We think that the methods developed here are useful for a quantum theory of gravity as well as for approaching classical dynamics, for instance for cosmological perturbation theory. In a quantum theory of gravity, gauge-invariant (i.e. physical) observables are the central object. The construction and interpretation of the observables of the theory is therefore one of the key open issues, which is also related to the problem of time, see, for instance [5–10]. Approximation methods for gauge-invariant observables provide an explicit way to construct observables and may also help to test (quantum) interpretation of these as well as to discuss phenomenological implications [8, 9]. The approximation scheme developed in this work shows explicitly how to relate local field observables, that is, the observables used in (quantum) field theory on curved spacetime, to gauge-invariant observables in full general relativity. It could therefore be helpful in obtaining a better understanding of quantum field theory on curved spacetime as a limit to the full theory of quantum gravity.

Using an approximation scheme for observables around a whole (symmetry-reduced) sector of the theory allows one to explore properties of gauge-invariant observables better than in a perturbation scheme around a fixed phase-space point. First of all, one can now incorporate results from symmetry-reduced (exactly solvable) models. The degrees of freedom describing these sectors are treated non-perturbatively. Second, as we will see, this approach provides a gauge-invariant (perturbative) description of dynamical processes also in closed universes, where for instance the notion of scattering amplitudes is problematic.

Moreover, since the early days of quantum gravity mini- and midi-superspace models gained by symmetry reduction of the full theory played an important role, especially for describing quantum cosmology ([11, 12] and references therein). However, because of the nonlinear dynamics of the theory, which leads to a coupling of non-symmetric and symmetric modes, the question arises whether the results gained from these models are reliable [13]. For different attempts to derive symmetry-reduced quantized models from the quantized full theory, see, for instance [14–16]. Since the coupling between non-symmetric and symmetric modes arises through the dynamics, we expect that the issues raised here become important if one wants to match solutions to the Hamiltonian constraint in the symmetry-reduced model.

---

\(^3\) For pure gravity, only ten ADM charges related to the Poincaré symmetries are known.
with solutions to the Hamiltonian constraint in the full theory. We provide such a matching for classical gauge-invariant observables, however the observables in the full theory have higher order corrections due to the non-symmetric modes. These corrections can be given explicitly, thus allowing us for an estimation of errors, which arise if one ignores the non-symmetric modes.

Recently, there is also a growing interest to incorporate (linear) perturbations [17] into the framework of loop quantum cosmology [18]. Here we hope that a clear gauge-invariant formulation of perturbations of symmetry-reduced models might help to understand for instance issues related to the gauge dependence of these approaches. Since the scheme developed in this work is consistent to any order, it proves that linear perturbations can be understood as lowest order of fully gauge-invariant perturbations governed by nonlinear dynamics. Similar gauge issues are present in recent developments in background-independent approaches to the graviton propagator [19].

A gauge-invariant canonical perturbation theory could be also fruitful in classical applications, such as the second (and higher) order perturbation theory around cosmological solutions [20] or black holes. The main difficulty here is to control the gauge dependence of the results. This gauge dependence can be understood from the fact that one has to identify spacetime points in the ‘physical’ (non-symmetric) universe with spacetime points in the ‘background’ universe, around which the perturbation is taken. This identification can be related to a choice of coordinates for the ‘physical’ universe. One might wonder why we attempt to develop a perturbation theory in the canonical formalism, where one would expect the problem to be even worse due to the foliation for the ‘physical’ and ‘background’ universe one has to choose in the canonical framework.

The resolution is that we use observables as central objects, i.e. we attempt to approximate directly a gauge-invariant observable of the full theory and do not consider (the difference of) fields on two different manifolds representing the perturbed and unperturbed spacetime. Observables in the canonical formalism correspond to phase-space functions, gauge-invariant observables are invariant under the action of the constraints (the gauge generators).

The phase-space of general relativity is just a representation of the space of all spacetimes (i.e. solutions of the Einstein equations). Gauge-invariant phase-space functions give the same value on spacetimes which are related by a diffeomorphism. Hence by considering gauge-invariant phase-space functions, we do not need to worry about the identification process between points in the perturbed and unperturbed spacetime.

In order to approximate gauge-invariant phase-space functions, we have to declare which variables are to be considered small. This choice is done in such a way that the approximate observables coincide with the exact observables if evaluated on the symmetry-reduced sector of the phase-space. Indeed, the zeroth-order variables can be defined by an averaging procedure. First-order phase-space functions vanish on symmetric spacetimes, higher order phase-space functions are products of first-order phase-space functions. Note that the splitting of phase-space variables into zeroth and first order is done on the gauge-variant level. Generically, a gauge-invariant phase-space function is a sum of terms of different order.

This approach has similarities to the \((1+3)\) covariant perturbation theory [21] which introduces a preferred timelike observer congruence. In this work, we rather have to declare a part of the degrees of freedom, as for instance scalar fields or the longitudinal modes of the gravitational field, to be used as clocks, which might be a more general procedure. However, we think that it should be possible to recover the \((1+3)\) covariant perturbation theory if one manages to match the choice of clocks to the preferred observer congruence. On the general relation between observables in the canonical theory and ‘covariant’ observables, see [7].
A key feature of this work is that we keep the zeroth-order variables as fully dynamical phase-space variables and not just as parameters describing the background universe as one would do in a perturbation around a fixed phase-space point. Indeed, we have to keep the zeroth-order variables as canonical variables to allow for a consistent gauge-invariant framework to higher than linear order. Moreover, this provides a very natural description for back-reaction effects: these arise as higher order corrections to observables arising through averaging of (time-evolved) phase-space variables. Since this approach is gauge-invariant, it could shed some light on the discussion of whether these back-reactions are measurable effects or caused by a specific choice of gauge, see, for instance [22–24]. As already mentioned we have to choose clocks, which also define the hypersurfaces (by physical criteria, e.g. by demanding that a scalar field is constant on these hypersurfaces) over which the averaging is performed. Therefore, the observables describing the back-reaction effects depend on the choice of clocks. However, as we will see, one can find relations between the gauge-invariant observables corresponding to one choice of clocks and the gauge-invariant observables corresponding to another choice of clocks.

Let us shortly describe the main ideas of the approach developed here: we will approximate a special class of gauge-invariant observables, known as complete observables [6, 7, 25]. The complete observables have the advantage that they describe the dynamics of the theory by giving the evolution of certain non-gauge-invariant observables (the partial observables) with respect to other non-gauge-invariant observables (the clock variables). In [25], methods to compute these complete observables were developed, in particular a power series was derived.

This power series serves as a starting point for our approximation scheme. We will divide the phase-space variables into two sets, namely variables of zeroth and first order. This division can be implemented by using a projection operator on the space of phase-space functions which projects onto the symmetry-reduced sector. First-order variables are projected to zero. For cosmological applications, this projection operator is defined by an averaging procedure. Now one can define approximate complete observables of order $k$ by omitting in the power series all terms of order higher than $k$.

Moreover, one can define evolution equations for these complete observables and even gauge-invariant functions that generate this evolution (i.e. Hamiltonians). As we will see, the calculation of the complete observables up to a certain order can be cast into a form where ‘free propagation’ (i.e. the linear propagation of the first-order variables on the symmetry-reduced background and the evolution of the zeroth-order variables in the symmetry-reduced sector) is perturbed by interaction processes between first-order fields and between first-order fields and zeroth-order variables.

In section 2, we will shortly summarize the necessary details concerning complete observables for a general gauge system (in a canonical description). Furthermore, we will define gauge-invariant Hamiltonians, which generate the physical time evolution with respect to the clocks. Section 3 introduces the approximation scheme for a general system around a symmetry-reduced sector. There we also define approximate gauge-invariant observables of a certain order and consider first properties of approximate complete observables.

In section 4, we consider this scheme for perturbations around a sector describing isotropic and homogeneous cosmologies. We derive and interpret the power series expressions for the complete observables. Then we consider the transformation between complete observables defined with respect to different choices of clocks (which can be understood as representing different families of observers). Section 6 defines lapse and shift functions, which allows us to compare this canonical approach to covariant approaches. Lapse and shift functions determine foliations of spacetimes. These foliations are defined by the choice of clock variables, i.e. by physical conditions.
Section 7 gives explicit examples for clock variables. We consider clocks built from the gravitational degrees of freedom and clocks defined by the matter field degrees of freedom. For the former case, we can give a set of clock variables which is related to the longitudinal gauge (see, for instance [17]), the dynamics of the corresponding complete observables is considered in appendix B.

In section 8, we consider the equations of motion for the first-order complete observables (associated with the scalar modes) and a method to find these complete observables. Then we consider the lowest order back-reaction effects, that is, second-order complete observables associated with zeroth-order functions. We evaluate the back-reaction effect for a very simple toy model, namely a Bianchi-I-universe treated as a perturbation of a homogeneous and isotropic universe, in section 10.

The appendix contains a definition of the tensor mode decomposition (appendix A), the equations of motion for the first-order complete observables associated with the scalar modes with longitudinal gauge clocks (appendix B) and tensor modes (appendix C), as well as a discussion of issues related to the so-called linearization instabilities (appendix D).

2. Complete observables

In this section, we will give a short introduction to complete observables for finite dimensional systems. The generalization to infinite dimensional systems is straightforward. For further details, we refer the reader to [7, 9, 25]. A very short summary of the necessary details on constraint systems can be found in [25], for a longer introduction, see, for instance [26]. In order to explain some of the properties of complete observables, we will express them using a power series, which we are now going to derive.

To this end, we consider a first-class (non-degenerate) constraint system with constraints \( \{C_j\}_{j=1}^m \). Since the system is first class, all the constraints generate gauge transformations [26]. Assume that we can find a set of phase-space functions \( \{T_K\}_{K=1}^m \) (called clock variables) such that the determinant of the matrix \( A^K_j := \{T^K, C_j\} \) is always non-vanishing (as a function of phase-space). In this case, we can define an equivalent set of first-class constraints \( \hat{C}_K \) by multiplying the original constraints with the inverse of the matrix \( A^K_j \):

\[
\hat{C}_K := C_j (A^{-1})^j_K,
\]

where here and in the following we sum over repeated indices. These new constraints are (weakly, i.e. modulo terms proportional to the constraints) conjugated to the clock variables, that is, they satisfy

\[
\{T^K, \hat{C}_L\} \simeq \delta^K_L,
\]

where by ‘\( \simeq \)’ we denote that this equation holds modulo terms proportional to the constraints.

From property (2.3) one can prove\(^4\) that the constraints \( \hat{C}_K \) are weakly Abelian, i.e. their Poisson bracket is proportional to terms at least quadratic in the constraints

\[
\{\hat{C}_K, \hat{C}_L\} = O(C^2).
\]

\(^4\) Compute \( \{T^K, \hat{C}_L\} \) directly and using the Jacobi identity. Comparing the two results, one can conclude that the structure functions \( f^L_{KM} \) defined by \( \{\hat{C}_K, \hat{C}_L\} = f^L_{KM} \hat{C}_M \) have to vanish on the constraint hypersurface.
Hence the gauge transformations generated by these constraints commute on the constraint hypersurface. This allows us to define gauge-invariant observables \( F_{[f; T^K]}(\tau) \), called complete observables, by a power series (assuming that this power series converges):

\[
F_{[f; T^K]}(\tau) = \sum_{r=0}^{\infty} \frac{1}{r!} \left\{ f, \hat{C}_{K_1}, \ldots, \hat{C}_{K_r} \right\} (\tau^K - T^{K_1}) \ldots (\tau^{K_r} - T^{K_r}).
\] (2.5)

Here, \( f \) is a phase-space function (called the partial observable) and \( \{ \tau^K \}_{K=1}^{m} \) are a set of constants. Due to the properties (2.3) and (2.4), the Poisson bracket of the series (2.5) with the constraints vanishes at least weakly. Hence the complete observable \( F_{[f; T^K]}(\tau) \) is (weakly) gauge invariant, that is, a (weak) Dirac observable.

Furthermore, we have that the complete observable coincides with the partial observable \( f \) if restricted to the (gauge fixing) hypersurface defined by \( \{ T^K = \tau^K \}_{K=1}^{m} \). Therefore, we can understand the complete observables as gauge-invariant extensions of gauge-restricted functions, where the gauge is given by \( \{ T^K = \tau^K \}_{K=1}^{m} \). (In this way, one can find a definition of complete observables alternative to the power series (2.5), see [25].) From this observation, it follows that associating complete observables to partial observables is an algebraic morphism, we have

\[
F_{[f + \hbar \hat{H}; T^K]}(\tau) \simeq F_{[f; T^K]}(\tau) \cdot F_{[\mathbf{0}; T^K]} + F_{[\hbar \mathbf{H} T^K]}(\tau)
\] (2.6)
as can be also seen in a more elaborate way by using the power series (2.5).

The value of the complete observables is constant on the gauge orbits. In generally covariant systems, such as general relativity, the orbits describing time evolution (i.e. foliations of spacetimes) are part of the gauge orbits, or in other words coordinate time evolution is actually a gauge transformation. Hence gauge-invariant observables are constant in coordinate time.

Nevertheless, one can vary the parameters \( \tau^K \) in the complete observable \( F_{[f; T^K]}(\tau) \). Then the value of the phase-space function \( F_{[f; T^K]}(\tau) \) does change. We can interpret this change as a time evolution with respect to the chosen clock variables \( T^K \). Indeed, the complete observable \( F_{[f; T^K]}(\tau) \) gives the value of the phase-space function \( f \) at that ‘moment’ at which the clocks \( T^K \) show the values \( \tau^K \). Using the representation of the complete observable as the power series (2.5), one can derive the ‘equations of motion’

\[
\frac{\partial}{\partial \tau^M} F_{[f; T^K]}(\tau) \simeq F_{[\{f, \hat{C}_{M}; T^K\}]}(\tau)
\] (2.7)

which replace the usual time evolution equations. The usual initial conditions are now replaced by

\[
F_{[f; T^K]}(\tau)\{T^M = \tau^M\}_{M=1}^{m} = f.
\] (2.8)

This closes our short introduction to complete observables. In the remainder of this section, we define generalizations of energy-like observables.

Here we understand energy as a phase-space function which generates the physical time evolution in (one of the parameters) \( \tau^M \). Hence we ask whether there exists gauge-invariant phase-space functions \( H_M \) which generate the evolution (2.7), that is, these functions should satisfy

\[
\frac{\partial}{\partial \tau^M} F_{[f; T^K]}(\tau) \simeq \{ F_{[f; T^K]}(\tau), H_M(\tau) \},
\] (2.9)

where we allowed for a \( \tau \)-dependence of these \( \tau \)-generators. If the variation in say \( \tau^1 \) corresponds to time evolution (with respect to the physical clock \( T^1 \)), one could interpret the corresponding generator \( H_1(\tau) \) as a possibly \( \tau \)-dependent ‘physical Hamiltonian’ [27]
Indeed, in the case that the clock variables $T^K$ Poisson commute, it is possible to find functions $H_K(\tau)$ that satisfy \eqref{eq:clock-condition}. However, this equation is not satisfied for all phase-space functions $f$, we have to exclude functions which depend on the clocks $T^K$ and functions which do not commute with the clocks $T^K$. We will look for the gauge-invariant functions $H_L(\tau)$ by assuming that they can be written as complete observables $F_{[h_L, T^K]}(\tau)$. The partial observables $h_L$ have to satisfy
\begin{equation}
\frac{\partial}{\partial \tau^L} F_{[f; T^K]}(\tau) \simeq F_{[\{f; \hat{C}_L; T^K\}])(\tau) \simeq \{F_{[f; T^K]}, F_{[h_L, T^K]}(\tau)\} \simeq F_{[[f, h_L]; T^K]}(\tau),
\end{equation}
where $\{f, g\}$ denotes the Dirac bracket with respect to the gauge $\{T^K = \tau^K\}$:
\begin{equation}
\{f, h_L\} \simeq \{f, \hat{C}_K\}[T^K, h_L] + \{f, T^K\}[\hat{C}_K, h_L] - \{f, \hat{C}_K\}[T^K, T^M][\hat{C}_M, h_L].
\end{equation}

For the last equation in \eqref{eq:clock-condition}, we use the general property
\begin{equation}
\{F_{[f; T^K]}, F_{[g; T^K]}\} \simeq F_{[\{f, g\}; T^K]}(\tau)
\end{equation}
which can be proved by using the power series \eqref{eq:power-series}. Here, one needs to prove equation \eqref{eq:partial-observables} only up to terms at least linear in the clocks $T^K$, or in other words on the gauge-restricted surface $\{T^K = \tau^K\}$. Then one can use that the right-hand side has to be gauge-invariant (because the left-hand side is), hence it can be written as the gauge-invariant extension of a $\{T^K = \tau^K\}$-gauge-restricted function.

Comparing equation \eqref{eq:clock-condition} with \eqref{eq:partial-observables}, we see that the functions $h_L$ have to satisfy
\begin{equation}
\{f, \hat{C}_L\} \simeq \{f, h_L\} - \{f, \hat{C}_K\}[T^K, h_L] + \{f, T^K\}[\hat{C}_K, h_L] - \{f, \hat{C}_K\}[T^K, T^M][\hat{C}_M, h_L] + O((T - \tau)),
\end{equation}
where $O(T - \tau)$ denotes terms vanishing on the gauge-fixing hypersurface $\{T^K = \tau^K\}$.

This equation can be satisfied if we choose $h_L = -P_L$, where $P_L$ is a phase-space function such that $\{T^K, P_L\} = \delta^K_L + O(C)$, i.e. $P_L$ has to be a momentum (weakly) conjugated to $T^K$ and to commute with the other clocks. Furthermore, we have the condition that $f$ has to commute with $P_L$ as well as with the clocks $\{T^K \}_{k=1}^m$ and that the clocks have to be Abelian. (However, the ‘momenta’ $\{P_L \}_{L=1}^m$ do not have to Poisson-commute with each other.) Note that we can add to $h_L = -P_L$ arbitrary functions vanishing on the constraint hypersurface, for instance $\hat{C}_L$. Such additions do not change the associated complete observable (on the constraint hypersurface). Also, if one is interested only in the generator associated with one specific $T^K$ parameter, say $\tau^1$, it is sufficient to specify $P_1$.

Therefore, we will assume that the clock variables $\{T^K \}_{k=1}^m$ are Abelian. In this case, one can find (locally) symplectic coordinate charts of the form $\{[T^K \}_{k=1}^m, \{P_K \}_{k=1}^m, \{q^a, p_a\}\}$, where $P_K$ are conjugated to $T^K$ and the set $\{q^a, p_a\}$ denotes the remaining symplectic coordinates.

Then we can define complete observables $F_{[\{P_L; T^K\}])(\tau)$ that generate the evolution in the parameters $\tau^K$ according to
\begin{equation}
\frac{\partial}{\partial \tau^K} F_{[f; T^K]}(\tau) \simeq \{F_{[f; T^K]}(\tau), F_{[\{P_L; T^K\}])(\tau)\} \simeq F_{[[f, \{P_L\}; T^K]](\tau)}
\end{equation}
for $f$ a function of $\{q^a, p_a\}$ only. (It is actually sufficient that $f$ commutes with the clocks $\{T^K \}_{k=1}^m$ and with the momentum $\Pi_L$.) Note however that the complete observables associated with the set $\{q^a, p_a\}$ give a complete set of gauge-invariant observables. Indeed, the restriction to these coordinates as partial observables is natural: the complete observable associated with
a clock $T^K$ is a constant $\tau^K$, hence we cannot generate evolution in this constant via a Poisson bracket with a generating function. The canonical momenta $\Pi_K$ can be solved for by using the constraints.

In general it may happen that the Dirac bracket $\{f, -P_L\}^*$ may depend on the clocks also if $f$ does not depend on the clocks. This will occur if the physical Hamiltonian $F([-P_L; T_K]) (\tau)$ is $\tau$-dependent. This is analogous to the situation of an explicit time-dependent Hamiltonian in usual classical mechanics. As in the latter case, we have to add to the time evolution equations a piece that takes care of this $\tau$-dependence (we will also add a piece, which takes care of the possibly dependence of $f$ on the momenta $P_K$):

$$\frac{\partial}{\partial \tau} F_{[f, T_K]} (\tau) \simeq \{F([-P_L; T_K]) (\tau), F([-P_L; T_K]) (\tau)\} + F_{[[f, T_K]] C_\nu, P_L; T_K]} (\tau) \quad (2.15)$$

holds for arbitrary phase-space functions $f$.

A $\tau$-generator $F_{[-P_L; T_K]} (\tau)$ does not depend on $\tau$ if the momentum $P_L$ Poisson-commutes with the constraints, i.e. if $P_L$ is gauge invariant. This corresponds to a time-independent Hamiltonian in classical mechanics, i.e. systems with conserved energy. One can always find a set of Abelian clocks $T^K_m$ and a set of conjugated momenta $P_K_m$ such that $l = \min(m, n - m)$ of the momenta are gauge-invariant and do not vanish identically on the constraint hypersurface\(^5\). Here, $2n$ is the dimension of the phase-space.

We want to emphasize that given a set of clocks the $\tau$-generators $F_{[-P_L; T_K]} (\tau)$ are not uniquely determined. To have uniquely determined $\tau$-generators (modulo constants and constraints), we have to specify the momenta conjugated to the clocks or alternatively the partial observables we want to evolve according to (2.14). One choice of momenta would be $P_L = C_L$. In this case, the $\tau$-generators would vanish on the constraint hypersurface. This is consistent with equation (2.14) because functions $f$ allowed in this equation have to be gauge invariant, which means that the associated complete observables do not depend on the parameters $\tau^K$.

In summary, we see that complete observables can be used to express dynamics in a gauge-invariant way and lead furthermore to a generalization of the notions of time (which is specified by using clocks) and energy (which is specified by the clocks and a subset of partial observables for which we want to describe dynamics).

3. Approximate complete observables

We will now develop a perturbation scheme for the complete observables. For such a scheme, we need to specify which kind of quantities we perturb in, i.e. the quantities assumed to be small. In this approach, these quantities will be phase-space-dependent, therefore we will get a good approximation in certain regions of phase-space. One example of such quantities are deviations from a fixed phase-space point (serving as a background, e.g. a phase-space point describing flat space). We considered a perturbation scheme for this case in [9].

In this work, we are interested in deviations from a whole (symmetry-reduced) sector of the phase-space. In general relativity such a sector could for instance correspond to homogeneous and isotropic spacetimes. In general we will describe the sector, we want to perturb around, by a linear projection operator $\mathcal{P}$, which acts on the space of phase-space functions. The projection property means that $\mathcal{P} : \mathcal{P} = \mathcal{P}$.

\(^5\) This statement has to be understood locally, i.e. the phase-space function in question may only be defined locally. The proof [28] basically uses that one can locally always find a polarization of the phase-space in which $(n - m)$ of the symplectic coordinate pairs are gauge invariants [29].
In particular, we can apply the projection operator to some set of symplectic coordinates (which can be understood as phase-space functions) \((\chi^a, \pi_a)\) with \(a = 1, \ldots, n\), where \(2n\) is the dimension of the phase-space. Then we can write
\[
\chi^a = \mathcal{P} \cdot \chi^a + (\text{Id} - \mathcal{P}) \cdot \chi^a, \quad \pi^a = \mathcal{P} \cdot \pi^a + (\text{Id} - \mathcal{P}) \cdot \pi^a.
\]
(3.1)
The sector we perturb around is given by the vanishing of the ‘fluctuations’ \((\text{Id} - \mathcal{P}) \cdot \chi^a\) and \((\text{Id} - \mathcal{P}) \cdot \pi^a\).

For our purposes, we will assume that the projection operator \(\mathcal{P}\) is of the following form\(^6\): there exist symplectic coordinate charts \((\chi^a, \pi^a)\)\(^6\) such that the projection on the symplectic coordinates is given by
\[
\mathcal{P} \cdot \chi^a = 0, \quad \mathcal{P} \cdot \pi^a = 0
\]
for \(a\) in some subset \(I\) of the index set \([1, \ldots, n]\) and
\[
\mathcal{P} \cdot \chi^b = \chi^b, \quad \mathcal{P} \cdot \pi^b = \pi^b
\]
for the remaining indices \(b \notin I\).

The projection operator acts on a general phase-space function \(f\) by setting all fluctuation variables \((\chi^a, \pi^a)_{a \in I}\) in \(f\) to zero, that is,
\[
(\mathcal{P} \cdot f)(\chi^a, \pi^a; \chi^b, \pi^b) = f(\mathcal{P} \cdot \chi^a, \mathcal{P} \cdot \pi^a; \mathcal{P} \cdot \chi^b, \mathcal{P} \cdot \pi^b) = f(0, 0; \chi^b, \pi^b),
\]
where the index \(a\) takes values in the set \(I\) and the index \(b\) in \([1, \ldots, n]/I\).

The fluctuation variables \((\chi^a, \pi^a)_{a \in I}\) will be considered to be small, that is, functions linear in these variables are defined to be of first order, functions quadratic in these variables of second order and so on. The variables \((\chi^b, \pi^b)_{b \notin I}\) are defined to be of zeroth order.

It is sometimes more convenient not to work with the symplectic coordinates \((\chi^a, \pi^a)\)\(^6\) but with some other set \((\chi^\prime, \pi^\prime)\). The projection operator acts of course also on this set and we can define the (degenerate) coordinates
\[
X^\prime = (\text{Id} - \mathcal{P}) \cdot \chi^\prime, \quad P^\prime = (\text{Id} - \mathcal{P}) \cdot \pi^\prime, \quad X^\prime = \mathcal{P} \cdot \chi^\prime, \quad P^\prime = \mathcal{P} \cdot \pi^\prime.
\]
(3.5)
Since \(\chi^\prime = X^\prime + x^\prime\) and \(\pi^\prime = P^\prime + p^\prime\) we can expand (suitable) phase-space functions in the fluctuation variables \((x^\prime, p^\prime)\) and introduce a classification of phase-space functions by defining the variables \((x^\prime, p^\prime)\) to be of first order. This classification will coincide with the previous one if the coordinates \((\chi^a, \pi^a)\)\(^n\) and \((\chi^\prime, \pi^\prime)\) are related by a linear symplectic transformation, which we will assume to be the case.

The coordinates (3.5) do not need to be symplectic anymore, the Poisson brackets have to be determined by
\[
[X^\prime, P^\prime] = \{\mathcal{P} \cdot \chi^\prime, \mathcal{P} \cdot \pi^\prime\}, \quad [x^\prime, p^\prime] = \{(\text{Id} - \mathcal{P}) \cdot \chi^\prime, (\text{Id} - \mathcal{P}) \cdot \pi^\prime\}.
\]
(3.6)
The Poisson brackets between fluctuation variables \((x^\prime, p^\prime)\) and the ‘sector’ variables \((X^\prime, P^\prime)\) vanish. Moreover, we have the condition on the fluctuation variables that \(\mathcal{P} \cdot x^\prime = P^\prime = 0\).

Also the coordinates \((X^\prime, P^\prime)\) will be in general highly degenerate.

The Poisson bracket of a phase-space function of order \(l\) with a phase-space function of order \(k\) can in general consist of a term of order \((l + k)\) and of a term of order \((l + k - 2)\) (or \(l > k \geq 1\)). The higher order term can arise through the Poisson bracket between the zeroth-order variables \((X^\prime, P^\prime)\). The Poisson bracket of a phase-space function of order \(l\) with a zeroth-order term is of order \(l\) (or vanishes).

Note that perturbations around a fixed phase-space point \(m_0\) arise as a special case: the projection operator is given by \(\mathcal{P} \cdot f = f(m_0) \cdot 1\), that is, \(\mathcal{P}\) maps all functions to constant

---

\(^6\) This assumption can be cast into the language of Poisson embeddings, see [16]. It ensures that the kinematics of the symmetry-reduced system and the symmetry-reduced sector embedded into the full phase-space coincide.
functions, where the constant is given by the evaluation of the phase-space function at the phase-space point \( m_0 \). Symplectic coordinates \((\chi', \pi')\) with properties (3.2) and (3.3) can be found starting from any symplectic coordinate chart \((\chi', \pi')\) and defining \( \chi'' = \chi' - \chi'(m_0) \) and \( \pi'' = \pi' - \pi'(m_0) \). The index set \( I \) in (3.2) and (3.3) coincides with the set \( \{1, \ldots, n\} \).

For the following, we will introduce some notation in order to specify terms of a certain order in a phase-space function \( f \): with \([k]f\) we will denote all terms which are of order \( k \) in \( f \), with \([k]f \) we will denote all terms in \( f \) which are of order less than or equal to \( k \).

We define gauge-invariant observables of order \( k \) as phase-space functions which commute with the constraints modulo terms of order \( k \) (and modulo constraints). Gauge-invariant phase-space functions of order \( k \) can be obtained from phase-space functions \( F \) which are exactly gauge-invariant by omitting all terms of order higher than \( k \), i.e. by truncating to \([k]F\):

\[
([k] F, C_I) = [F, C_I] + [O(k + 1)],
\]

where by \( O(l) \) we denote terms of order \( l \) or higher. Here the lowest order term on the right-hand side will in general appear through the Poisson bracket of the \( O(k + 1) \) term with the first-order term \([1] C_I \) of the constraint. All other terms are of higher order.

In particular, we can find approximate complete observables of order \( k \) by considering their truncation to order \( k \). In the following, we will assume that the weakly Abelianized constraints, as defined in equation (2.2), \( \hat{C}_K \) and the clocks \( T^K \) can be divided into two subsets \( \{[\hat{C}_H]_{\text{H}}, [\hat{C}_I]_{\text{I}}\} \) and \( \{[T^H]_{\text{H}}, [T^I]_{\text{I}}\} \), such that the clocks \( T^H \) are of zeroth order and the clocks \( T^I \) are of first order. For the constraints \( \hat{C}_H \), we assume that for an arbitrary first-order function \([1] f\)

\[
([1] f, \hat{C}_H) = O(1),
\]

which is satisfied if the constraints \( \hat{C}_H \) do not have a first-order term \([1] \hat{C}_H = 0 \), however they may have a zeroth-order term. For the constraints \( \hat{C}_I \), we will assume that the zeroth-order terms vanish and that the first-order terms do not vanish. Consider the power series for the complete observables (2.5)

\[
F_{[1], [T^I]}(\tau) \simeq \sum_{r=0}^{\infty} \frac{1}{r!} \left\{ [f, \hat{C}_{K_i}], \ldots, [f, \hat{C}_{K_r}] \right\} (\tau^{K_i} - T^{K_i}) \ldots (\tau^{K_r} - T^{K_r})
\]

\[
\simeq \sum_{s=0}^{\infty} \sum_{r=0}^{s} \frac{1}{(r - s)!} \left\{ [f, \hat{C}_{H_i}], \ldots, [f, \hat{C}_{H_{s-r+i}}], [\hat{C}_{H_{s-r+i}}], \ldots, [\hat{C}_{H_r}] \right\} \times (\tau^{H_i} - T^{H_i}) \ldots (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \times (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \ldots (\tau^{H_r} - T^{H_r})
\]

\[
\simeq \sum_{s=0}^{\infty} \sum_{r=0}^{s} \frac{1}{p^r q^s} \left\{ [f, \hat{C}_{H_i}], \ldots, [f, \hat{C}_{H_{s-r+i}}], [\hat{C}_{H_{s-r+i}}], \ldots, [\hat{C}_{H_r}] \right\} (\tau^{H_i} - T^{H_i}) \ldots (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \times (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \ldots (\tau^{H_r} - T^{H_r}),
\]

where we used that we can rearrange the constraints in any order and that the clocks \( T^H \) commute (at least weakly) with the constraints \( \hat{C}_I \). Now set the parameters \( \tau^I \) to zero. Then for the zeroth-order complete observable associated with a zeroth-order function \([0] f\), we have

\[
([0] f, [T^H], \tau^I = 0) \simeq \sum_{q=0}^{\infty} \frac{1}{q!} ([0] f, \hat{C}_{H_i}, \ldots, [0] f, \hat{C}_{H_r}] (\tau^{H_i} - T^{H_i}) \ldots (\tau^{H_{r+q}} - T^{H_{r+q}})
\]

\[
\simeq \sum_{s=0}^{\infty} \sum_{r=0}^{s} \frac{1}{p^r q^s} \left\{ [0] f, \hat{C}_{H_i}, \ldots, [0] f, \hat{C}_{H_{s-r+i}}], [\hat{C}_{H_{s-r+i}}], \ldots, [\hat{C}_{H_r}] \right\} (\tau^{H_i} - T^{H_i}) \ldots (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \times (\tau^{H_{s-r+i}} - T^{H_{s-r+i}}) \ldots (\tau^{H_r} - T^{H_r}),
\]

(3.10)
where the second equation holds due to our assumption on the constraints \( \tilde{C}_H \) to have vanishing first-order parts. There only appear zeroth-order variables in the second line in (3.10), hence we can say that the zeroth-order complete observables associated with a zeroth-order function\(^7\) are the complete observables of the symmetry-reduced sector. The next higher order correction to this complete observable is a second-order term and can be considered as the correction (back-reaction) term to the dynamics of the symmetry-reduced sector due to deviations from symmetry (in the initial values).

One can also consider for instance the first-order complete observable associated with a first-order function. As we will see these observables describe the propagation of linear perturbations (which are linearly gauge-invariant) on the symmetry-reduced sector.

One can also consider for instance the first-order complete observable associated with a first-order function. As we will see these observables describe the propagation of linear perturbations (which are linearly gauge-invariant) on the symmetry-reduced sector.

Note that this approach allows us to find gauge-invariant observables to any order \( k \) by omitting in the series for the complete observables all terms of order higher than \( k \). For this the assumptions we made on the clocks \( \{ [T^H]_{H \in \mathcal{H}}, [T^I]_{I \in \mathcal{I}} \} \) and the constraints \( \{ [\tilde{C}_H]_{H \in \mathcal{H}}, [\tilde{C}_I]_{I \in \mathcal{I}} \} \) are not strictly necessary. However, we will see that with these conditions the computation of the complete observables is similar to the usual perturbative calculations involving the ‘free’ propagation of perturbations and their interaction as well as the interaction of the zeroth-order variables with the perturbations.

If the power series (3.9) for the complete observable converges, it defines an exact gauge-invariant observable which coincides with the approximate Dirac observable \([k]F[f; T^K](\tau)\) modulo terms of order \((k + 1)\). If the power series does not converge in some phase-space region, this might be due to the fact that the clock variables \( T^K \) do not provide a good parametrization of the gauge orbits in this phase-space region [25]. In this case one can try to find a set of new clock variables \( T'^K \), with a better behaviour in this respect and such that (3.8) is satisfied also for these new clocks. Assume that the complete observable \( F_{[f'; T';S]}(\tau^H, \tau^I = 0) \) associated with these new clock variables and the partial observable \( f' := [k]F_{[f; T]}(\tau^H, \tau^I = 0) \) can be defined. This complete observable will also coincide with \([k]F_{[f'; T']}(\tau)\) modulo terms of order \( k \), as can be seen by examining the power series (3.9) for a complete observable and using that \( f' \) Poisson-commutes modulo terms of order \( k \) with the constraints.

### 4. Application to cosmology

We will now apply the formalism to general relativity. The sector we perturb around will be the sector describing homogeneous and isotropic (FLRW) cosmologies with a minimally coupled scalar field. We will work with (complex) connection variables [30, 31], however the formalism is independent of the choice of variables and can be also applied to real connection variables or the metric (ADM) variables [32]. We will follow the conventions in [31].

The canonical variables are fields on a spatial manifold \( \Sigma \), the coordinates of which we will denote by \( \{ \sigma^{a} \}_{a=1}^{3} \). We will assume that \( \Sigma \) is a compact manifold and diffeomorphic to the 3-Torus \( T^3 = S^1 \times S^1 \times S^1 \) (in other words, the fields are assumed to be periodic). For convenience we will assume that the coordinate length of each spatial direction is equal to \( L = 1 \).

The configuration variables are given by a (complex) connection \( \{ A^{j}_{a} \}_{j,a=1}^{3} \), where latin letters from the beginning of the alphabet denote spatial indices and from the middle of the

\(^7\) Note that the complete observables associated with a zeroth-order function have vanishing first order, hence the zeroth-order complete observable coincides with the first-order complete observable in this case. The zeroth-order complete observable associated with a zeroth-order function is therefore a gauge-invariant observable of first order.
In particular, $D_a$ can be implemented by a projection operator

$$\gamma \frac{1}{2} 2K_{ab} e^b_j e^a_i. + \text{space metric can be calculated from the triads by } q_{ab} = e^b_i e^a_j. \text{ The conjugated momenta } E^a_j \text{ are constructed out of the triads}

$$\gamma \frac{1}{2} E^a_j = \frac{1}{2} \delta^a_j + \epsilon_{abc} e^b_i e^c_j. \text{ where } \epsilon_{abc} \text{ and } \epsilon_{jhi} \text{ are totally antisymmetric tensors with } \epsilon_{123} = \epsilon_{132} = 1. \text{ This gives the following relation between the momenta } E^a_j \text{ and the 3-metric}

$$\gamma \frac{1}{2} 3 \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} 3 \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \epsilon_{abc} e^b_i e^c_j. \text{ The Poisson brackets between the phase-space variables are}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}

$$\gamma \frac{1}{2} \delta^a_j \delta^a_j \text{ where }\}
The Poisson brackets between the homogeneous variables and between the fluctuation variables can be found by using the projection (4.11):

\[ \{ A, E \} = \frac{\kappa}{3}, \quad \{ a^b_{\sigma}, e^d_{\sigma'} \} = \kappa \delta^b_{a} \delta^d_{d} (\sigma, \sigma') - \frac{\kappa}{3} \delta^b_{a} \delta^d_{d}, \]

\[ \{ \Phi, \Pi \} = \gamma, \quad \{ \phi, \rho(\sigma') \} = \gamma \delta(\sigma, \sigma') - \gamma. \quad (4.12) \]

The Poisson bracket between a homogeneous variable and a fluctuation variable vanishes.

In the following, we will raise and lower the indices with the Kronecker symbols \( \delta_{ab} \) or \( \delta_{ab} \), respectively (and not with the background metric \( Q_{ab} := \mathcal{E}_{ab} \)).

It will be convenient to work with the Fourier-transformed variables, using these the Poisson bracket relations simplify. For any field \( f(\sigma) \), we define

\[ f(k) = \int_{\Sigma} \exp(-ik \cdot \sigma) f(\sigma) \, d\sigma, \]

where \( k \cdot \sigma := k_{\alpha} \sigma^{\alpha} \) and the wave vector \( k \) takes values in \( 2\pi \mathbb{Z}^3 \). The inverse transform is

\[ f(\sigma) = \sum_{k \in [2\pi \mathbb{Z}]} \exp(ik \cdot \sigma) f(k). \]

Now the homogeneous variables are given by the \((\frac{1}{2} \times \text{trace of the}) k = 0 \) modes of the fields. The Poisson brackets for the Fourier modes of the fluctuation variables are

\[ \{ a_{ab}(k), e^{cd}(k') \} = \kappa \delta^{c}_{a} \delta^{d}_{b} \delta_{k \cdot k'} - \frac{\kappa}{3} \delta^{c}_{a} \delta^{d}_{b} \delta_{k,0} \delta_{k',0}, \]

\[ \{ \phi(k), \rho(k') \} = \gamma \delta_{k \cdot k'} - \gamma \delta_{k,0} \delta_{k',0}, \quad (4.15) \]

where the additional terms on the right-hand side implement that \( a_{\alpha}^{\alpha}(0) = e_{\alpha}^{\alpha}(0) = \phi(0) = \rho(0) = 0 \).

We want to remark that the Fourier-transformed variables can be used to define the symplectic coordinates used in section 3 in which the projection operator \( \mathcal{P} \) maps part of the symplectic coordinates to zero and leaves the other coordinates invariant. The homogeneous part of the coordinates would be given by \( \sqrt{3}A, \sqrt{3}E; \Phi, \Pi \). The symplectic pairs that are mapped to zero are given by \( (a_{ab}(k), e^{ab}(-k)) \) and \( (\phi(k), \pi(-k)) \) for \( k \neq 0 \) and

\[ \left( \sqrt{\frac{1}{2}} \left( \frac{1}{2} a_{11}(0) - a_{22}(0) + \frac{1}{2} a_{33}(0) \right), \sqrt{\frac{1}{2}} \left( \frac{1}{2} e^{11}(0) - e^{22}(0) + \frac{1}{2} e^{33}(0) \right) \right), \]

\[ \left( \frac{1}{2} a_{11}(0) - a_{33}(0), \frac{1}{2} e^{11}(0) - e^{33}(0) \right). \]

We expand the constraints \( C_j \) (where \( C_0 = S, C_{Da} = D_a, C_{Ga} = G_a \)) in the homogeneous and fluctuation variables in order to find the \( m \)-th order parts \( \text{to}(m)C_j \), taking the fluctuation variables as first-order and the homogeneous variables as zero-order quantities. The zeroth order of the constraints vanishes except for the scalar constraint

\[ (0)S = \kappa^{-1} 6\beta^2 \Lambda^2 E^2 + \gamma^{-1} \left( \frac{1}{2} \Pi^2 + E^3 V(\Phi) \right). \]

The first-order parts of the constraints are given by

\[ (1)G_b = \kappa^{-1} \left( \partial_\sigma e^a_b + \beta A e^{ba}_{ac} e^{ac} + \beta \mathcal{E}_{ac} a_{bc} a^{ac} \right), \]

\[ (1)D_a = \kappa^{-1} \left( E (\partial_\sigma a^b_b - \partial_\sigma a^b_b) - A \partial_\sigma e^b_a \right) + \gamma^{-1} \Pi \partial_\sigma \phi, \]

\[ (1)S = \kappa^{-1} \left( 2E \partial_\sigma \partial_\rho e^b_a + 2\beta A e^b_a \partial_\rho e^b_a + 4\beta^2 A E^2 a^b_a + 4\beta^2 A^2 E e^a_a \right) + \gamma^{-1} \left( \Pi \partial_\rho + E^3 V(\Phi) \phi + E^3 V(\Phi) e^a_a \right), \]

where in the Fourier-transformed quantities the partial derivative stands for \( (\partial_\sigma f)(k) = ik_\sigma f(k) \). Note that the first order of the zero modes of the diffeomorphism constraints \( D_a \) vanish, this is related to the linearization instabilities of backgrounds with compact spatial
slices and Killing vectors [34]. In the following, we will ignore the integrated diffeomorphism constraints $D_c(0)$ and show in appendix D that one can indeed deal with these constraints after one has computed the complete observable with respect to all the other constraints. Another way to circumvent the problem of linearization instabilities is to couple the system to massless scalar fields and perturb around non-homogeneous field configurations of these scalar fields. This will be explained in section 7.1.

The first-order part of the zero mode scalar constraint vanishes too, however the overall scalar constraints have a zero-order component. For this reason, we do not have a linearization instability corresponding to the scalar constraint: rather than viewing the second-order part of the integrated scalar constraint as a restriction on the first-order variables, we see it as a correction to the zeroth-order part, signifying a back-reaction effect of the perturbation variables onto the homogeneous variables. This viewpoint is possible because the zeroth-order variables are part of the phase-space, which differs from a perturbative approach around a fixed background.

Furthermore, we need the integrated second order of the scalar constraint:

\[
(2)S(0) = \sum_k \kappa^{-1}(2\beta E e^{\alpha\beta}(\partial_0 a_{bc}(k) - \partial_\alpha a_{bc}(k)))e^{\beta\gamma}d\phi(k) + \beta^2 E^2(a_{\alpha}(k)\partial_k a_{\alpha}(k))
\]

\[
- a_{\alpha}(k)\phi(k) + 2\beta^2 A^2(e^{\alpha\beta}(k)\phi(k) - e^{\alpha}(k)\phi(k))
\]

\[
+ 2\beta^2 A\beta E^2(a_{\alpha}(k)\partial_k a_{\alpha}(k) + a_{\alpha}(k)\partial_k a_{\alpha}(k))
\]

\[
+ \gamma^{-1}\left(\frac{1}{2}\beta_2 + \frac{1}{2}E^2\beta^{\alpha\beta}(\partial_0 \phi)(k)(\partial_\alpha \phi)(k) + E^2 V.(\Phi)e^{\alpha}(k)\phi(k)
\]

\[
+ \frac{1}{2}E^3 V.(\Phi)\phi(k)\phi(-k) + \frac{1}{2}E V.(\Phi)(e^{\alpha}(k)\phi(k) - e^{\alpha}(k)\phi(k))\right).
\]  \hspace{1cm} (4.19)

To construct complete observables we need to choose a set of clock variables $T^K(k)$ where $K \in \{0, Da, Ga; a = 1, 2, 3\}$ and $k \in 2\pi T^J$. This has to be done in a way such that at least the zeroth order of the matrix

\[
A^j_1(k, k') := \{T^K(k), C_j(k')\}
\]  \hspace{1cm} (4.20)
is invertible. This will ensure that at least to the lowest order the clock variables provide a good parametrization of the gauge orbits (and that is the reason we have to exclude the integrated diffeomorphism constraints, since these start at second order).

We will choose the clock $T^0(0)$ to have a non-vanishing zeroth-order part and a vanishing first-order part. All the other clocks should have vanishing zero-order parts. This will ensure that the new constraints $C_k(k)$ and the clocks $T^K$ have a similar structure as explained in equation (3.8). (We are a bit more general here.) The structure of the matrix $A$ is then as follows. The zeroth order of the matrix is of diagonal block form, that is,

\[
(0)A^j_1(0, k') = 0 \text{ if } j \neq 0 \text{ or } k' \neq 0,
\]

\[
(0)A^K_1(k, 0) = 0 \text{ if } K \neq 0 \text{ or } k \neq 0.
\]  \hspace{1cm} (4.21)

Since we will quite often need to exclude the index combinations $(j = 0 \text{ and } k = 0)$ as well as $(K = 0 \text{ and } k = 0)$ from the set of indices to sum over, we will introduce indices $(\hat{j}, \hat{K}, \hat{k})$ to signify that these do not assume the values $(j = 0 \text{ and } k = 0)$ or $(K = 0 \text{ and } k = 0)$. (Also these indices do not include $j, K = Da$ and $k = 0$, since we excluded the integrated diffeomorphism constraints.)

Moreover, the first order of the matrix element $A^j_1(0, 0) = \{T^0(0), C_0(0)\}$ vanishes. This structure of the matrix $A$ ensures that the constraint
\[
\tilde{C}_0(0) := \sum_{k'} C_j(k')(A^{-1})^j_0(k', 0) \\
= C_0(0)(A^{-1})^0_0(0, 0) + \sum_{k'} C_j(k')(A^{-1})^j_0(k', 0)
\]

(4.22)

has vanishing first order. To see this, we have to convince ourselves that the first order of \((A^{-1})^0_0(0, 0)\) and the zeroth order of \((A^{-1})^j_0(k', 0)\) are vanishing. The latter follows from the fact that \(A\) is of diagonal block form (4.21), hence the inverse has the same kind of block form. Furthermore, the first order of the inverse of \(A\) can be expanded as

\[
(1)(A^{-1})^0_0(k', 0) = -\sum_{k''} (A^{-1})^j_0(k', k'') A^k_{m} k'' \tilde{C}^m_0(0, 0) \quad \text{and}
\]

(4.23)

\[
(1)(A^{-1})^j_0(k', k) = -\sum_{k''} (A^{-1})^j_0(k', k'') T^k_{m}(k'') \tilde{C}_0(0)(A^{-1})^0_0(0, 0, 0).
\]

For \(j = 0\) and \(K = 0\), the sum in the last line collapses to just one term

\[
(1)(A^{-1})^0_0(0, 0) = -\sum_{k''} (A^{-1})^0_0(0, 0) (1)(T^0_0(0), C_0(0)) (0)(A^{-1})^0_0(0, 0, 0) = 0,
\]

(4.24)

where we used that the first order of \(C_0(0)\) and \(T^0_0(0)\) vanishes.

Hence the first order of the constraint \(\tilde{C}_0(0)\) vanishes. Also this constraint is the sole one among the \(\tilde{C}_K(k)\) with a non-vanishing zeroth-order part:

\[
\tilde{C}_K(k) = \sum_{k'} (0) C_j(k') (A^{-1})^j_1(k', k) = (0) C_0(0)(A^{-1})^1_0(0, k)
\]

vanishes for \(K \neq 0\) or \(k \neq 0\) because of the block diagonal form of the zeroth order of \(A\).

Let us consider the series for the complete observable \(F_{[f; Y]}(\tau)\) associated with a function \(f\) and with parameter values \(\tau^0(0) = \tau\) and \(\tau^k(\tilde{k}) = 0\):

\[
F_{[f; Y]}(\tau) \simeq \sum_{r=0}^\infty \sum_{k_1, \ldots, k_r} \frac{1}{r!} \{ \cdots \{ [f, \tilde{C}_{K_1}(k_1)], \ldots [\tilde{C}_{K_r}(k_r)] \} \} (\tau^{K_1}(k_1) \\
- T^{K_1}(k_1)) \cdots (\tau^{K_r}(k_r) - T^{K_r}(k_r)) \\
\simeq \sum_{r=0}^\infty \sum_{k_1, \ldots, k_r} \frac{1}{(r-s)!} [f, \tilde{C}_0(0)]_{(r-s)} (\tilde{C}_{K_1}(k_1), \ldots, \tilde{C}_{K_r}(k_r)) \\
\times (\tau - T^0(0))^{r-s} \times (-T^{K_1}(\tilde{k}_1)) \cdots (-T^{K_r}(\tilde{k}_r)) \\
\simeq \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{k_1, \ldots, k_p} \frac{1}{q!} q^\alpha q^\alpha \tilde{C}_r(0)(\tau - T^0(0))^q (\tilde{C}_{K_1}(k_1), \ldots, \tilde{C}_{K_r}(k_r)) \\
\times (-T^{K_1}(\tilde{k}_1)) \cdots (-T^{K_r}(\tilde{k}_p)),
\]

(4.26)

where we denote by \([\cdots]_q\) iterated Poisson brackets \([f, g]_q = \{ [f, g]_{q-1}, g \} \) and \([f, g]_0 = f\). In the first step we used the fact that we can arrange the constraints \(\tilde{C}_K(k)\) in any order since they commute up to terms quadratic in the constraints. In the second step, we exploited that \(T^0(0)\) commutes with the constraints \(\tilde{C}_K(k)\) up to terms proportional to the constraints. The result can be interpreted in the following way: the complete observable \(F_{[f; Y]}(\tau)\) can be calculated by first finding the complete observable corresponding to the single constraint \(\tilde{C}_0(0)\) with parameter value \(\tau\) and then computing the complete observable associated with this result with respect to the remaining constraints. One can also choose to perform the calculation the other way around, i.e. first compute the complete observable with respect to the
constraints \( \hat{C}_K(\hat{k}) \) and then to deal with the constraint \( \hat{C}_0(0) \). Here one uses the fact that the clocks \( T^K(\hat{k}) \) commute with the constraint \( \hat{C}_0(0) \) up to terms proportional to the constraints

\[
\mathcal{F}_{f: T^K}(\tau) \approx \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{q!p!} \left\{ \cdots \left[ f, \hat{C}_{k_1}(\hat{k}_1), \cdots, \hat{C}_{k_p}(\hat{k}_p) \right] \right\} \\
\times \left( -T^{k_1}(\hat{k}_1) \cdots -T^{k_p}(\hat{k}_p), \hat{C}_0(0) \right)_q (\tau - T^0(0))^g.
\] (4.27)

Assume that \( f \) is a zeroth- or first-order quantity. Expanding the complete observables (4.26) and (4.27) up to a certain order \( m \), we see that we need the constraints \( \hat{C}_0(0) \) up to an order \( m \) for \( f \) zeroth order and up to an order \( m + 1 \) for \( f \) first order. We need the remaining constraints up to an order \( m \) for \( f \) first order and up to an order \( m - 1 \) for \( f \) zeroth order. The reason for this is that the lowest order in \( \{ g, \hat{C}_K(\hat{k}) \} \) is \((n - 1)\) if \( g \) has lowest order \( n \). However, for each Poisson bracket with such a constraint, the expression gets multiplied with the clock variable \( T^K(\hat{k}) \) which is at least of first order.

On the other hand, \( \{ g, \hat{C}_0(0) \} \) is at least of order \( n \) if \( g \) has lowest order \( n \), so one does not lose any order in the Poisson brackets with the constraint \( \hat{C}_0(0) \). Hence it is crucial that the constraint \( \hat{C}_0(0) \) has a vanishing first-order part, otherwise a perturbational calculation in the usual sense is not possible.

We can interpret expression (4.26) in the following way: first, we have to evolve the partial observable \( f \) with respect to the constraint \( \hat{C}_0(0) \) which generates time evolution with respect to the clock \( T^0(0) \). If we want to calculate the complete observable to a certain order \( m \), we have to calculate this evolution up to terms of order \( m \). This evolution can be broken up into ‘free evolution’ described by the zeroth-order part (for the zeroth-order variables) and second-order part (for the first-order variables) of \( \hat{C}_0(0) \) and ‘interaction processes’ described by the higher order parts (and the second-order part for the interaction of the homogeneous variables with the inhomogeneities).

In a second step, we have to calculate the gauge-invariant extension of the (clock) time-evolved function. This requires only a finite number of terms in the second (outer) sum since we can omit all terms with more than \( m \) factors of the (inhomogeneous) clocks \( T^K(\hat{k}) \).

In the case that \( f \) is of zeroth order, i.e. a ‘background variable’, the zeroth-order complete observable corresponds to the complete observable in the isotropic and homogeneous model. Higher (than first) order complete observables take into account the back-reaction of the fluctuations onto the background.

For \( f \) first order, the first-order complete observable describes (linearized) propagation of first-order gauge-invariant perturbations. The time evolution of these perturbations is expressed with respect to a physical clock (defined by a homogeneous variable). This is different from the usual theory of perturbations on a fixed background [17].

5. Transformation between different sets of clock variables

In this section, we will explore the dependence of the complete variables on the choice of clock variables. Since one can understand the complete observables also as gauge-invariant extensions of gauge-restricted functions, this will also enable us to connect different gauges.

To derive a relation between the complete observables with respect to two different sets of clock variables \( \{ T^K(\hat{k}) \} \) and \( \{ T^{rK}(\hat{k}) \} \), we will come back to the interpretation of the complete observables: the complete observables \( \mathcal{F}_{f: T^K}(\tau^K) \) assigns to a phase-space point \( x \) the value of the function \( f \) at the point \( y \) on the gauge orbit through \( x \) at which the clocks \( T^K(\hat{k}) \) coincide with the parameter values \( \tau^K(\hat{k}) \), that is, \( T^K(\hat{k})|_y = \tau^K(\hat{k}) \).
If one replaces in $F_{[f,T_\tau]}(\tau^K)$ the $\tau^K(k)$ parameters by the complete observables $F_{[T^x(k),T^\tau]}(\tau^L)$, one will get the value of $f$ at that point $z$ on the gauge orbit through $x$ at which the clocks $T^K(k)$ coincide with the complete observables $F_{[T^x(k),T^\tau]}(\tau^L)$. Note that whereas $T^K(k)$ changes along the gauge orbit the complete observable $F_{[T^x(k),T^\tau]}(\tau^L)$ is constant along the gauge orbit, so with the assumption that the clocks provide a good parametrization of the gauge orbit the point $z$ is uniquely determined. Hence $z$ is a point on the gauge orbit through $x$ which has to satisfy

$$T^K(k) = F_{[T^x(k),T^\tau]}(\tau^L).$$

(5.1)

The complete observable $F_{[T^x(k),T^\tau]}(\tau^L)$ on the right-hand side gives the value of $T^K(k)$ on that point $y'$ on the gauge orbit through $x$ on which the clocks $T^L(k')$ coincide with the parameter values $\tau^L(k')$. Hence the point $z$ in (5.1) has to coincide with the point $y'$, characterized by the condition $T^L(k)(y') = \tau^L(k)$. Therefore, we can conclude that

$$F_{[f,T^\tau]}(F_{[T^x(k),T^\tau]}(\tau^L)) = F_{[f,T^\tau]}(\tau^L).$$

(5.2)

This gives us a relation between the complete observables with respect to two different sets of clock variables. If one wants to use this formula in order to obtain $F_{[f,T^\tau]}(\tau^L)$ from $F_{[f,T^\tau]}(\tau^K)$ one needs the functional dependence of the complete observable $F_{[f,T^\tau]}$ on the parameter values $\tau^K(k)$. However, with the exception of the parameter $\tau^0(0)$ we set these clock parameters to zero. One can nevertheless use formula (5.2) if one Taylor-expands the left-hand side around some fixed values for the parameters $\tau^K(k)$. This would give a formula connecting complete observables with respect to clock variables $T^K(k)$ with complete observables with respect to $T^L(k)$.

A simpler way to obtain a formula for $F_{[f,T^\tau]}(\tau^L)$ as a function of complete observables with respect to the clocks $T^K(k)$ is to start with the power series for $F_{[f,T^\tau]}(\tau^L)$:

$$F_{[f,T^\tau]}(\tau^L) \simeq \sum_{r=0}^{\infty} \sum_{k_1,\ldots,k_r} \frac{1}{r!} \left\{ f, \hat{C}', \cdots \right\},$$

(5.3)

where $\hat{C}'(k) = \sum_k C_j(k')(A^{-1})_j^k(k',k)$ and $A^K_j(k,k') = \{ T^K(k), C_j(k') \}$. Now we take both sides of equation (5.3) as ‘input function’ $f$ for the complete observable $F_{[f,T^\tau]}(\tau^K)$. The left-hand side does not change, since the complete observable associated with a Dirac observable is given by the Dirac observable itself. Therefore, we get

$$F_{[f,T^\tau]}(\tau^L) \simeq \sum_{r=0}^{\infty} \sum_{k_1,\ldots,k_r} \frac{1}{r!} F_{[\cdots(f,\hat{C}'x_1(k_1),\ldots),\hat{C}',\cdots]}(\tau^K)$$

$$\times (\tau^K(k_1) - T^K(k_1)) \cdots (\tau^K(k_r) - T^K(k_r))$$

(5.4)

as a formula for a complete observable with respect to the clocks $T^L$ as a function of complete observables with respect to the clocks $T^K$.

In a first-order approximation (for $f$ first order) and for the case that the clocks $T^0(0) = T^0(0)$ and the corresponding parameter values $\tau^0(0) = \tau^0(0) = \tau$ coincide, formula (5.4) reduces to

$$(1) F_{[f,T^\tau]}(\tau) \simeq (1) F_{[f,T^\tau]}(\tau) = \sum_k F_{[\cdots(f,\hat{C}'x(k),\cdots)]}(\tau) (1) F_{[T^\tau x(k),T^\tau]}(\tau),$$

(5.5)

Here we set all the other parameter values $\tau^K(k) = T^K(k) = 0$ to zero and we used that $\tau = F_{[\tau^0,0],T^\tau}(\tau)$. 

Gauge-invariant perturbations around symmetry-reduced sectors of general relativity

4559
If we have found the first-order complete observables associated with some basis of phase-space functions with respect to one set of clocks $T^K(k)$, we can calculate the first-order complete observables with respect to another set of clocks with the help of formula (5.5). Formulae for higher order complete observables can be derived by expanding (5.4) to the appropriate order.

6. Lapse and shift functions

As explained earlier, the complete observables $F_{I,T^j}(\tau^K)$ can also be understood as gauge-invariant extensions of the phase-space function $f$ using the gauge $T^K(k) = \tau^K(k) \forall K, k$. Considering the complete observables just for one fixed set of parameters $\tau^K$ would correspond to a ‘frozen time’ picture. The time evolution is generated by a constraint, that is, time evolution is a gauge transformation. Fixing all gauge degrees of freedom would also mean to consider a fixed time. However, we can choose a one-parameter family of gauge fixings, as for instance $T^0(0) = \tau, T^K(\hat{k}) = 0$ for $\tau$ in (some subset of) $\mathbb{R}$, that would represent a varying time.

A phase-space point on the constraint hypersurface gives rise to a solution of the equation of motion, that is, a spacetime manifold. The one-parameter family of gauge fixings defines a foliation of this spacetime manifold, as well as spatial coordinates on each of the leaves of the foliation. Hence we can find (phase-space-dependent) lapse functions and shift vectors using this foliation and characterize our choice for the clock variables and the one-parameter family of $\tau$-parameters.

Lapse and shift can be used to construct the four-dimensional metric using the four-dimensional coordinates given by the foliation. This allows one to compare the results of this approach to (covariant) methods utilizing gauge fixing.

From (2.7), we see that the (gauge) generator for a translation in the $\tau^0(0)$ parameter is given by $\tilde{C}_0(0)$. Hence the lapse function $^8 N^0$ and shift vector $N^D_a$ can be read off as the coefficients in front of the scalar and diffeomorphism constraint. We will also define $N^{Ga}$ as the coefficient in front of the Gauss constraint.

This motivates the definition

$$ N^j(k) := (A^{-1})^j_0(-k, 0), $$

so that we can write

$$ \tilde{C}_0(0) = \sum_k N^j(k) C_j(-k) $$

for our ‘time evolution’ generator $\tilde{C}_0(0)$. Note that if one uses a gauge fixing one would restrict the consideration to the gauge-fixing hypersurface $\{T^K(k) = \tau^K(k)\}$, so if one compares (6.1) to gauge fixing one should omit all terms vanishing on this hypersurface.

Assume that one has given a covariant gauge on the spacetime metric and that this gauge can be formulated as a set of restrictions on the 3-metric $\{R_\alpha(q_{cd}) = 0, \alpha \in I\}$ (or the triad variables), where $I$ is an appropriate index set and of restrictions on the form of the lapse and shift functions. The longitudinal gauge is often employed for cosmological perturbations [17]. To find clock variables reproducing this gauge, one can proceed in the following way. Because the gauge fixed hypersurfaces are described by $T^K(k) = \tau^K(k) = 0$, the restrictions on the 3-metric $R_\alpha(q_{cd})$ can be directly taken as a subset of the clock variables $\tilde{T}^k$ involving the 3-metric (or the triad variables). For the rest of the clock variables one makes the ansatz

$^8$ Note that we are using the scalar constraint $S$ here, which has density weight 2. Usually one defines the lapse function $^\perp N^a$ as the coefficient in front of the Hamiltonian constraint $C_\perp = (\sqrt{q})^{-1} C$, which has density weight 1. Hence we have $N^\perp = \sqrt{q} N^0$.
that these are functions of the conjugated momenta. The explicit form of these functions has to be determined such that the resulting lapse and shift functions, calculated according to equation (6.1), are of the desired form at least on the constraint and gauge fixed hypersurface. In this way, clocks reproducing the longitudinal gauge can be found, see appendix B.

7. Clock variables and Hamiltonians

The clock variables should be chosen such that the zeroth order of the matrix \( A^K_j (k, k') \) is invertible. Once one has found such clock variables one can in principle define new clock variables by multiplying the old clock variables with the zeroth order of the inverse matrix \((A^{-1})^K_j (k, k')\). These new clock variables will lead to a new matrix, the zeroth order of which will be given by the identity matrix (with the exception of the entry \( A^0_0 (0, 0) \) which might differ from 1).

However, these new clock variables might not be very convenient to deal with, since also the first-order clocks \( T^i (k) \) will depend on the homogeneous variables, leading to additional terms for the higher orders of the matrix \( A^K_j (k, k') \) coming from the Poisson bracket between the homogeneous variables in the clock variables and the constraints.

Here we will specify the ‘inhomogeneous’ clock variables \( T^i (k) \). Hence we allow zero-momentum \( k \) for the Gauss clock \( T_G \), but not for the scalar clock \( T^0 \) and not for the diffeomorphism clock \( T_D \) (because of the linearization instabilities). One choice for the clock variables is

\[
T_G = \epsilon^{abc} e_{bc} = \epsilon^{abc} (AT e_{bc} + LT e_{bc} + TL e_{bc}), \\
T_D = -W^{-2} (\frac{1}{2} W^{-2} \partial^a \delta^{bc} \partial_a e_{bc} + W^{-2} \partial^a \delta^{bc} \partial_a e_{bc}) e_{de} \\
= -W^{-2} (-\partial^a LL e_{de} d + \frac{1}{2} \partial^a T e_{de} d - \partial_a TL e_{de} d - \partial_d LT e_{de} d), \\
\]

(7.1)

where \( W := \sqrt{-\delta^{bc} e_{bc}} \). Here we introduced a tensor mode decomposition for the gravitational variables (with respect to the flat metric \( \delta_{ab} \)), the notation is explained in appendix A.

This set of clock variables is obviously Abelian and leads to the following zeroth order for the Poisson brackets between the clocks and the constraints:

\[
\{ T_G (k), (1) G_b (k') \} = 2\beta E E^{\delta_i \delta_{k, k'}} e_{bc} \\
\{ T_G (k), (1) S (k') \} = 0 \\
\{ T_D (k), (1) G_b (k') \} = 0 \\
\{ T_D (k), (1) D_b (k') \} = 2E^{\delta_i \delta_{k, k'}} \\
\{ T^0 (k), (1) G_b (k') \} = 0 \\
\{ T^0 (k), (1) D_b (k') \} = 0 \\
\{ T^0 (k), (1) S (k') \} = -4\beta^2 AE^2 \delta_{k, k'}.
\]

(7.2)

By adding to the Gauss clock a term proportional to the diffeomorphism clock, we can define new clock variables which lead to a diagonal (except for the \( \delta_{k, k'} \) factor) matrix \((A^0)_{ij} (k, k')\). The clock variables are obviously Abelian and lead to the following zeroth order for the Poisson brackets between the clocks and the constraints:

\[
T_G = \epsilon^{abc} e_{bc} - \epsilon^{abc} \partial_b T^D c = \epsilon^{abc} (e_{bc} - W^{-2} \partial_b \partial_a e_{bc} - W^{-2} \partial_b \partial_a e_{bc}) \\
= \epsilon^{abc} (AT + LT + TL) e_{bc} - \epsilon^{abc} (W^{-2} \partial_b \partial_a e_{bc} + W^{-2} \partial_b \partial_a e_{bc}) e_{de}.
\]

(7.3)

where we assume \( k \neq 0 \). The \( T_G (k = 0) \) clock is still given by \( \epsilon^{abc} e_{bc} \).
With this new Gauss clock, we have
\[
\begin{align*}
[T^{Ga}(k), (1)G_b(k')] &= 2\beta E\delta^a_0\delta_{k,-k'} &\{T^{Ga}(k), (1)D_b(k')\} &= 0 \\
[T^{Ga}(k), (1)S(k')] &= 0 \\
[T^{Da}(k), (1)G_b(k')] &= 0 &\{T^{Da}(k), (1)D_b(k')\} &= E\delta^a_0\delta_{k,-k'} \\
[T^{Da}(k), (1)S(k')] &= 0 \\
[T^0(k), (1)G_b(k')] &= 0 &\{T^0(k), (1)D_b(k')\} &= 0 \\
[T^0(k), (1)S(k')] &= -4\beta^2 A E^2 \delta_{k,-k'}
\end{align*}
\] (7.4)
for the zeroth order of the Poisson brackets between the clocks and the constraints. As one can see from formula (5.4), which gives the relation between complete observables using different sets of clock variables, the complete observables using either the clocks (7.1) or (7.3) coincide (for parameter values \(\tau^k(\hat{k}) = 0\)). The reason for this is that both sets of clocks define the same gauge-fixing surface.

Assume that one has chosen a clock \(T^0(0)\), as for instance \(T^0(0) = \Phi\). Then we can define gauge-invariant functions that generate the evolution for the complete observables in the \(\tau = \tau^0(0)\) parameter. According to section 2, we have to find a momentum \(P_0\) conjugated to \(T^0(0)\), which has to commute (weakly) with the other clocks \(T^k(\hat{k})\). For the above example we could choose \(P_0 = \Pi\). In general, we will assume that \(P_0\) has a non-vanishing zeroth-order part and a vanishing first-order part.

(Since we are only interested in the generator for the evolution in \(\tau^0(0)\), we do not have to specify momenta conjugated to the clocks \(T^k(\hat{k})\). However, a natural choice would be to choose the first order of the constraints \((1)\hat{C}_k(\hat{k})\). For issues arising because of the linearization instabilities, see appendix D.)

Then we can define the physical Hamiltonian as \(H_0(\tau) := \hat{F}_{[b_0, T^{T^1}_T]}(\tau)\), where
\[
h_0 = -P_0 \simeq -P_0 + \hat{C}_0(0).
\] (7.5)
With this physical Hamiltonian, we can write
\[
\frac{d}{d\tau} \hat{F}_{[f, T^{T^1}_T]}(\tau) = \{\hat{F}_{[f, T^{T^1}_T]}(\tau), H_0(\tau)\}
\] (7.6)
for functions \(f\) that Poisson-commute with \(P_0\) and the clocks \(T^K\). If \(f\) does not commute with \(P_0\), we have to add a term \(\hat{F}_{[f, P_0, T^{T^1}_T]}(\tau)\) to the right-hand side of equation (7.6). This term has the same purpose as the additional time derivative \(\partial_\tau f\) that appears in the explicit time-dependent Hamiltonian systems in classical mechanics, where the time evolution equations are given by
\[
\frac{d}{dt} F = [F, H] + \frac{\partial}{\partial t} f.
\] (7.7)
In general, the Hamiltonian \(H_0(\tau)\) will have a non-vanishing zeroth-order part (as long as one does not choose \((0)P_0 = (0)\hat{C}_0(0)\)) and a vanishing first-order part. If one interprets \(H_0(\tau)\) as an energy, this shows that also the zeroth-order variables contribute to this energy, \(H_0(\tau)\) will be \(\tau\)-independent (i.e. energy is conserved) if \(P_0\) is a gauge-invariant function. This would be the case for \(P_0 = \Pi\) and constant potential for the scalar field.

Another choice for the clock variables, which is, as we will see in appendix B, related to the longitudinal gauge [17] is given by
\[
T^{Ga} = e^{abc} e_{bc} \\
= e^{abc} (AT e_{bc} + LT e_{bc} + TL e_{bc}),
\]
\[ T^{Da} = -W^{-2} \left( -\frac{1}{2} W^{-2} \delta^a_d \partial_e \partial_e - \frac{1}{2} \delta^a_d \delta^b_e \partial_e \partial_e + \frac{1}{2} \delta^b_e \delta^a_d \partial_e \partial_e \right), \]
\[ T^0 = W^{-2} \left( \frac{1}{2} W^{-2} \delta^e_d + \frac{1}{2} \delta^e_d \partial_e \partial_e \right), \]
\[ \{ T^{Da} (k), (1) G_b (k') \} = 2 \beta E \delta^a_b \delta_{k,-k'}, \]
\[ \{ T^{Da} (k), (1) S (k') \} = 0, \]
\[ \{ T_0 (k), (1) G_b (k') \} = 0, \]
\[ \{ T_0 (k), (1) D_b (k') \} = E \delta^a_b \delta_{k,-k'}, \]
\[ \{ T_0 (k), (1) S (k') \} = 2 E \delta_{k,-k'}. \] (7.8)

This set of clocks differs from (7.1) only in the scalar clocks. Note that the scalar clocks and the diffeomorphism clocks do not commute. The zeroth-order parts of the Poisson brackets between constraints and clocks are given by

\[ \{ T^{Ga} (k), (1) G_b (k') \} = 2 \beta E \delta^a_b \delta_{k,-k'}, \]
\[ \{ T^{Ga} (k), (1) S (k') \} = 0, \]
\[ \{ T^{Da} (k), (1) D_b (k') \} = E \delta^a_b \delta_{k,-k'}, \]
\[ \{ T^{Da} (k), (1) S (k') \} = 0, \]
\[ \{ T_0 (k), (1) G_b (k') \} = 0, \]
\[ \{ T_0 (k), (1) D_b (k') \} = -W^{-2} A \delta_b \delta_{k,-k'}, \]
\[ \{ T_0 (k), (1) S (k') \} = 2 E \delta_{k,-k'}. \] (7.9)

For these clocks, which are related to the longitudinal gauge, we cannot give a physical Hamiltonian along the lines of section 2. The reason for this is that the clocks do not Poisson-commute with each other. However, according to equation (2.11), which gives the Dirac bracket, the term that arises because of the non-Abelianess of the clocks is

\[ \sum_{k,k'} \{ f, \tilde{C}_k (k) \} \{ T^K (k), T^M (k') \} \{ \tilde{C}_M (k'), h_0 \}. \] (7.10)

This term is at least of second order if \( f \) and \( h_0 \) are gauge-invariants to first order. Under these conditions, a physical Hamiltonian valid for the linearized theory can be defined in the same way as for the Abelian clocks (7.1).

7.1. Scalar fields as clocks

The clock variables we introduced so far are quite non-local, i.e. they require for their definition inverse derivatives (in the form of \( W^{-2} = |k|^{-2} \)) or the inverse of the matrix \( A^K \) requires inverse derivatives. One way to avoid this is to use scalar fields as clocks. Scalar fields are used quite often as clocks, see, for example [7, 8, 22, 35–37]. As explained in [7] using scalar fields as clocks can lead to huge simplifications for the calculation of complete observables. This is related to the fact that scalar fields provide a local characterization of spacetime points (as opposed to for instance the longitudinal modes of the metric fields, which rather characterize a foliation of spacetime).

On the one hand we are interested in a homogeneous background, on the other hand we want to use the values of four scalar fields to define a coordinate system, i.e. at least the scalar fields \( \varphi^A, A = 1, 2, 3 \) defining the spatial coordinates have to be non-homogeneous (at zeroth order). However, we want to preserve the homogeneity of the Hamiltonian constraints at zeroth order, therefore we will choose the scalar fields used as ‘spatial coordinate system’ to be massless, that is, having vanishing potential. Since we assumed the spatial manifold \( \Sigma \) to have topology \( S^1 \times S^1 \times S^1 \), we will assume that the scalar fields \( \varphi^A, A = 1, 2, 3 \) take values in \( S^1 \) (parametrized by the values of the interval \([0, 1]\)).

We will use another scalar field \( \varphi^0 \), taking values in \( \mathbb{R} \), as the time coordinate, that is, the clock for the scalar constraint. The division into background variables and perturbations is
now
\[
\begin{align*}
A^i_a(\sigma) &= λβδ^i_a + a^i_a(σ) β δ^i_a, \\
E^i_a(σ) &= E β^{-1} δ^i_a + ε^i_a(σ) β^{-1} δ^i_a,
\end{align*}
\]
\[
\begin{align*}
ψ^0(σ) &= Φ^0 + ϕ^0(σ), & π_0(σ) &= Π_0 + ρ_0(σ), \\
ψ^A(σ) &= δ^A_a σ^a + φ^A(σ), & π_A(σ) &= 0 + ρ_A(σ),
\end{align*}
\]
(7.11)
where $π_0, π_A$ are the momenta conjugate to the scalar fields $ψ^0, ψ^A$, respectively. The integrals of $ψ^0, ρ^0$ over the spatial manifold are constrained to vanish, this is not the case for the variables $ϕ_A, ρ_A$. Note that for the fields $ψ^A, π_A$ we perturb around a fixed value and not around their averaged value. Also this fixed (background) value is an inhomogeneous function on the spatial manifold for the scalar field whereas for the momentum it is just given by the constant function 0. Thus, we treat these fields differently from the scalar field $ψ^0$ which is used as a clock for the scalar constraint. This changes the action of the projection operator $P$ as defined in equation (4.11). The action of $P$ on the scalar field $ψ^0$ and its conjugated momentum is still given by the second line of (4.11) whereas for the scalar fields $ψ^A$ and their conjugated momenta we define $(P · ψ^A(σ)) := δ^A_a σ^a$ and $(P · π_A(σ)) := 0$. On these fields $ψ^A, π_A$ the projection operator acts therefore in the same way as in a perturbation around a fixed background. The quantities $a^i_a, ε^i_a, ϕ^0, ρ_0, φ^A, ρ_A$ as defined by equation (7.11) are of first order.

As one can check there are solutions to the equations of motion where all the first-order perturbations are exactly zero, in particular the values $δ^A_a σ^a$ of the scalar fields $ψ^A$ are constant in time (but inhomogeneous in space). The other fields evolve in the same way as in a completely homogeneous FRW universe. That is, to zeroth order the scalar fields $ϕ^0$ just define a spatial physical coordinate system but do not participate in the dynamics.

The Poisson brackets between the gravitational variables are as before (4.12), for the matter fields we have
\[
\{ϕ^0, Π_0\} = γ_0, \quad \{ϕ^0(σ), ρ_0(σ')\} = γ_0 δ(σ, σ') - γ_0, \quad \{ϕ^A(σ), ρ_A(σ')\} = γ_A δ(σ, σ'),
\]
(7.12)
where $γ_0, γ_A$ are the coupling constants for the scalar fields $ψ^0, ψ^A$, respectively. The matter parts of the diffeomorphism and scalar constraints are given by
\[
\begin{align*}
\text{max} D_a &= γ_0^{-1} π_0 δ_a ϕ^0 + \sum_{A=1,2,3} γ_A^{-1} π_A δ_a φ^A \\
&= γ_0^{-1} π_0 δ_a ϕ^0 + \sum_{A=1,2,3} γ_A^{-1} ρ_A δ_a φ^A + O(2), \\
\text{max} S &= \frac{1}{2} γ_0^{-1} (π_0^2 + q q^{ab} δ_a ϕ^0 δ_b ϕ^0 + 2 q V_0(ϕ^0)) + \frac{1}{2} \sum_A γ_A^{-1} (π_A^2 + q q^{ab} δ_a φ^A δ_b φ^A) \\
&= γ_0^{-1} \left( \frac{1}{2} Π_0^2 + E^3 V_0(Φ^0) \right) + \frac{1}{2} \sum_A γ_A^{-1} E^2 + γ_0^{-1} (π_0 ρ_0 + E^3 V_0(Φ^0) ϕ^0) + E^2 V_0(Φ^0)e^{a} + \sum_A γ_A^{-1} (ε^{ab} δ_a δ_b φ^A + E^2 δ^a δ_b φ^A) + O(2).
\end{align*}
\]
(7.13)
The zeroth-order part of the diffeomorphism constraints vanishes as before, the zeroth-order part of the matter part of the scalar part is given by the first line of the last equation in (7.13). (The gravitational part of the zeroth-order constraints does not change.)

9 This is necessary in order to get a well-defined ‘physical coordinate system’ at zeroth order.
We could add more scalar fields, these will have the same kind of contribution as the scalar field $\phi_0$. Note that the first order of the integrated diffeomorphism constraint does not vanish anymore

$$ (1)^{D_a}(k = 0) = \int_\Sigma (1)^{D_a}(\sigma) \, d\sigma = \int_\Sigma \sum_{A=1,2,3} \gamma^{-1}_A \rho_A \delta^A_a \, d\sigma, $$

so the problem of the linearization instabilities does not occur for this choice of background (because the background values of the scalar fields break the translational symmetry).

The first order of the integrated scalar constraint is still vanishing, showing that there actually exists an exact solution of the equations of motion where all the perturbation variables vanish identically for all times.

Now we can choose as clock variables

$$ T^{Ga}(k) = e^{abc}e_{bc}(k), \quad T^{Da}(k) = \sum_A \phi^A(k) \delta^a_A, \quad T^0(k) = \phi^0(k) $$

for all wave vectors $k = 0$ and $k \neq 0$. As parameter values one has to choose $\tau^{Ga}(k) = 0$, $\tau^{Da}(k) = \int_\Sigma \exp(-ik\sigma) \sigma a \, d\sigma$ and $\tau^0(k) = 0$ for $k \neq 0$ as well as $\tau^0(0) = \tau$.

The zeroth order of the Poisson bracket between the Gauss clock and the constraints is as before (7.2), for the Poisson brackets between the diffeomorphism and scalar clock and the constraints we have

$$ (0)^{T^{Da}(k), Db(k')} = \delta^a_b \delta_{k,-k'}, \quad (0)^{T^0(k), S(k')} = \Pi_0 \delta_{k,-k'} $$

with all the other (zeroth order) Poisson brackets vanishing. Hence the zeroth order of the matrix $A^k_j(k, k')$ is invertible on phase-space points where $\Pi_0 \neq 0$ and $E \neq 0$ (from the commutator of the Gauss clock with the Gauss constraint).

These considerations show that one can apply the perturbative formalism also if one uses scalar fields as clock variables. Moreover, the problem of linearization instabilities does not occur.

8. First-order perturbations: scalar modes

Let us consider the first order of the complete observables in more detail. We will assume that we are dealing with one scalar field and use the gravitational fields to define the non-homogeneous clock variables. Starting from the partial differential equation (2.7) for the complete observables, we will derive the equations of motion for the scalar mode perturbations (to first order). The equations of motion for the tensor modes are derived in appendix C.

Consider the ‘time evolution’ equation

$$ \frac{d}{d\tau} [f; T^0](\tau) = F_{[\{f, \tilde{C}_0(0)\}; T^0]}(\tau), $$

where $\tau = \tau^0(0)$ is the parameter associated with the clock $S^T(k = 0)$. To simplify the formulae, we will introduce the notation $[[f]](\tau) := F_{[f; T^0]}(\tau)$ and suppress the dependence from the choice of clock variables.

Hence we have

$$ \frac{d}{d\tau} [[f]](\tau) \simeq [[[f, \tilde{C}_0(0)]]](\tau) \simeq \sum_{k'} [[[f, C_j(k')]]](\tau) [[(A^{-1})^0_0(k', 0)]](\tau) $$

as the differential equation satisfied by the complete observable associated with $f$. With the introduction of lapse and shift functions (see section 6)

$$ \Lambda^j(k) := (A^{-1})^0_0(-k, 0), $$

$$ \Lambda^j(k)$$
we obtain the following system of differential equations:

\[
\frac{d}{d\tau}([\phi(k)](\tau)) \simeq (0)\left[\lambda^0(0)(\tau)\right]^{(1)}\left[\rho(k)\right](\tau) + (1)\left[\lambda^0(k)(\tau)\right]^{(0)}\left[\Pi\right](\tau),
\]

\[
\frac{d}{d\tau}([\rho(k)](\tau)) \simeq (0)\left[\lambda^0(0)(\tau)\right]^{(1)}\left[2\partial^a\partial_a\phi(k) - 2V'(\Phi)e^a_a(k)\right] - E^3V''(\Phi)\phi(k)](\tau) - (1)\left[\lambda^0(k)(\tau)\right]^{(0)}\left[3E^3V'(\Phi)](\tau)\right)
\]

\[
- \Xi \left[\partial_aN^a(k)\right](\tau) \simeq (0)\left[\Pi\right](\tau)
\]

for the first-order complete observables associated with the fluctuations \(\phi(k), \rho(k)\) in the scalar field and its conjugate momentum. Here we assume \(k \neq 0\).

Now on the right-hand side of equations (8.4), there also appear functions of the gravitational degrees of freedom, so in principle one would have to add differential equations for these gravitational degrees of freedom. However, if we are dealing with only one scalar field in our system, there should be only one unconstrained scalar mode degree of freedom. Indeed, we can use

\[
[[C_j(\tau)](\tau) \simeq 0, \quad [[T^\Phi(\tau)](\tau) \simeq 0
\]

(8.6)

to express the (first-order complete observables associated with the) lapse and shift functions as well as \(e^a_a\) as functions of the (first-order complete observables associated with the) scalar field \(\phi\) and its conjugated momentum \(\rho\) and the homogeneous variables.

For instance, for the choice (7.1) of clock variables we have

\[
\partial_\tau T^a = \epsilon_{abc} \partial^a \partial^b \partial^c,
\]

\[
\partial_\tau T^a_{Da} = -W^{-2}\left(-\frac{3}{2} \partial_\tau \partial_b \partial^b + \frac{1}{2} \partial_\tau \partial_\tau \partial^b (\partial^a + T^a_b)ight)
\]

\[
T^0 = \partial_\tau T^{0}_{Ab}.
\]

(8.7)

(8.8)

(8.9)

Hence we can use

\[
[[AT e^{a_b}(k)] \simeq 0, \quad [[T^{e_{ab}}(k)] \simeq 0, \quad [[L_e^{a_b}(k)] \simeq 0.
\]

(8.10)

Furthermore, using the first order of the constraints (4.18) and the relation (8.10) gives us

\[
[[AT e^{a_b}(k)] \simeq 0 + O(2), \quad [[T^{e_{ab}}(k)] \simeq -\frac{K}{\gamma} E^{-1} \Pi \phi(k) + O(2),
\]

\[
[[L_e^{a_b}(k)] \simeq -\frac{K}{\gamma} (4\beta^2 A E^2)^{-1} (\Pi \rho(k) + E^3 V'(\Phi)\phi(k)) + \frac{K}{\gamma} E^{-1} \Pi \phi(k) + O(2).
\]

(8.11)

This shows that we can express the gravitational scalar modes as a combination of the matter scalar modes.

The lapse and shift functions (for \(k \neq 0\)) for the clock variables (7.1) can be found to be (here we abbreviate \(N := (0)\lambda^0(0)\))

\[
(1)N^Da(k) = -N \sum_k (A^{-1}T^a_{Da})(-k, k')(1)A^0_b(k', 0)
\]

\[= -N \sum_k E^{-1} \delta_{-k', -k}(T^a_{Da}(k'), S(0))
\]

\[= -NE2\beta^2 W^{-2} (\beta^a L a^b - \frac{1}{2} \beta^a T a^b) + O(T),
\]

\[= -NE2\beta^2 W^{-2} (\beta^a L a^b - \frac{1}{2} \beta^a T a^b) + O(T).
\]
\begin{align}
\mathcal{N}^0(k) &= -N \sum_k (A^{-1})^0_k (-k,k') (A^0_k (k',0) \\
&= -N \sum_k (-4\beta^2 AE^2)^{-1} \delta_{-k,-k} \{T^0(k), T^0(0)\} \\
&= -N \frac{1}{2} A^{-1T} a^b + O(T),
\end{align}

where \( O(T) \) denotes terms which vanish with \( T^\hat{k} (\hat{k}) \). Using equation (8.11), the differential equations for the scalar matter field becomes

\[
\frac{d}{d\tau} [\phi(k)](\tau) \simeq (1) \left[ N(\rho(k) + \frac{1}{2\gamma} (AE)^{-1}\Pi^2 \phi(k)) \right](\tau),
\]

\[
\frac{d}{d\tau} [\rho(k)](\tau) \simeq (1) \left[ N(E^2 \partial^a \partial_a \phi(k) - 3 K^2 \Phi^2 \phi(k) - E^2 V''(\Phi) \phi(k) \right.
\]

\[
+ \frac{1}{2\gamma} (AE)^{-1}\Pi^2 \rho(k)) \right](\tau).
\]

These equations have to be supplemented with the differential equation for the homogeneous variables \( H = A, E, \Pi \) or \( \Phi \), here it is sufficient to consider the zeroth order of this equation:

\[
\frac{d}{d\tau} [H](\tau) \simeq \phi(k).
\]

Assume that one can find the general solution for these differential equations in dependence on initial data for some parameter value \( \tau = \tau_0 \). Since the differential equation (8.13) is linear in the fluctuation fields, such a solution for instance for the scalar field can be written as

\[
[\phi(k)]((\tau - \tau_0)) \simeq G_1((\tau - \tau_0); k; [H]((\tau_0))) [\phi(k)]((\tau_0))
\]

\[
+ G_2((\tau - \tau_0); k; [H]((\tau_0))) [\rho(k)]((\tau_0)),
\]

where \( G_1, G_2 \) can be understood as generalized (free) propagator functions. Now the complete observables restricted to the gauge-fixing surface have to satisfy

\[
[\phi(k)]((\tau - T^0(0)) \simeq \Phi(k),
\]

\[
[H](\tau)_{\tau = 0} \simeq H.
\]

For the complete observable associated with the scalar field, we therefore have

\[
[\phi(k)]((\tau - T^0(0)) \simeq G_1((\tau - T^0(0)); k; H) \phi(k) + G_2((\tau - T^0(0)); k; H) \rho(k). \]

Now we only have to determine \([\phi(k)]((\tau - T^0(0)))\) away from the hypersurface \( T^\hat{k} (\hat{k}) = 0 \). This is done by replacing \( \phi(k) \) and \( \rho(k) \) in (8.17) by their first-order gauge-invariant extensions in the \( T^\hat{k} (\hat{k}) \)-direction

\[
\phi(k) \rightarrow \phi(k) - \sum_k [\phi(k), \tilde{C}_k (\hat{k})] T^\hat{k} (\hat{k}),
\]

\[
\rho(k) \rightarrow \rho(k) - \sum_k [\rho(k), \tilde{C}_k (\hat{k})] T^\hat{k} (\hat{k}).
\]
Hence the first-order complete observable associated with the matter scalar mode $\phi(k)$ is given by

\[
^{(1)}[[\phi(k)](\tau)] \simeq G_1((\tau - T^0(0)); k; H) \left( \phi(k) - \sum_{\hat{k}}^{(0)} \{\phi(k), \hat{\mathcal{C}}_{\hat{k}}(\hat{k})\} T^\hat{k}(\hat{k}) \right)
\]

\[
+ G_2((\tau - T^0(0)); k; H) \left( \rho(k) - \sum_{\hat{k}}^{(0)} \{\rho(k), \hat{\mathcal{C}}_{\hat{k}}(\hat{k})\} T^\hat{k}(\hat{k}) \right). \tag{8.19}
\]

The first-order complete observables can be understood to describe the ‘free’ propagation of the perturbations on the cosmological background. Higher order complete observables will take care of interaction processes\(^{10}\) given by the higher (than second) order parts of the constraint $\hat{\mathcal{C}}_0(0)$ as well as of back-reaction terms arising from the coupling of homogeneous and inhomogeneous variables in the constraint. In the following section, we will consider second-order complete observables associated with a zeroth-order phase-space function, which capture the lowest order back-reaction effects.

9. Back-reaction terms

Here we want to consider the back-reaction effects of the inhomogeneities onto the homogeneous variables. To this end, we have to find the complete observables associated with a homogeneous variable up to second order. (The first order of such a complete observable vanishes.)

Omitting in (4.26) with $f$ a function of zeroth order all terms of third order and higher, we find

\[
^{[2]} F_{[f; T^x]}(\tau) \simeq ^{(0)} F_{[f; T^x]}(\tau) + G + I + J, \tag{9.1}
\]

where

\[
^{(0)} F_{[f; T^x]}(\tau) = \sum_{q=0}^{\infty} \frac{1}{q!} \left\{ f^{(q)}(0) \hat{C}_0(0) \right\} (\tau - T^0(0))^q =: \alpha^{(q)}_{\text{free}}(f),
\]

\[
G = \sum_{\hat{k}_1} \left\{ \alpha^{(T^0(0))}_{\text{free}}(f), \hat{\mathcal{C}}_{\hat{k}_1}(\hat{k}_1) \right\} (-T^\hat{k}_1(\hat{k}_1))
\]

\[
+ \sum_{\hat{k}_1, \hat{k}_2} \frac{1}{2!} \left\{ \alpha^{(T^0(0))}_{\text{free}}(f), \hat{\mathcal{C}}_{\hat{k}_1}(\hat{k}_1), \hat{\mathcal{C}}_{\hat{k}_2}(\hat{k}_2) \right\} (-T^\hat{k}_1(\hat{k}_1)) (-T^\hat{k}_2(\hat{k}_2)). \tag{9.2}
\]

Hence $G$ is the gauge-invariant extension to second order with respect to the $\hat{\mathcal{C}}_{\hat{k}}(\hat{k})$ constraints of the zeroth-order term. The last two terms in (9.1) are given by

\[
I = \sum_{q=0}^{\infty} \sum_{q_1, q_2, q_3=0} \left\{ \alpha^{(T^0(0))}_{\text{free}}(f), \hat{\mathcal{C}}_{\hat{k}_1}(\hat{k}_1), \hat{\mathcal{C}}_{\hat{k}_2}(\hat{k}_2) \right\} (\tau - T^0(0))^q,
\]

\[
J = \sum_{\hat{k}_1} \left\{ I, \hat{\mathcal{C}}_{\hat{k}_1}(\hat{k}_1) \right\} (-T^\hat{k}_1(\hat{k}_1))
\]

\[
+ \sum_{\hat{k}_1, \hat{k}_2} \left\{ I, \hat{\mathcal{C}}_{\hat{k}_1}(\hat{k}_1), \hat{\mathcal{C}}_{\hat{k}_2}(\hat{k}_2) \right\} (-T^\hat{k}_1(\hat{k}_1)) (-T^\hat{k}_2(\hat{k}_2)). \tag{9.3}
\]

See [9] for an explicit example of a complete observable taking into account interaction processes in a second-order approximation around flat space.
Using the identity
\[ \frac{t^{q_1+q_2+1}}{(q_1+q_2+1)!} = \frac{1}{q_1! q_2!} \int_0^t \frac{1}{s-t} (s-t)^{q_1+q_2}, \]  
we can rewrite the term \( I \) as
\[ I = \int_0^{\tau} ds \, \alpha_{\text{free}}^{(m)}(g) \left( \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \} \right), \]  
where we define the free evolution of a higher order term \((m)\) as
\[ \alpha_{\text{free}}^{(m)}(g) = \sum_{q=0}^{\infty} \frac{1}{q!} \{ \alpha_{\text{free}}^{(m)}(g) \}^q \]  
(9.6)

We have also for the free evolution the factorization property \( \alpha_{\text{free}}^{(m)}(f \cdot g) = \alpha_{\text{free}}^{(m)}(f) \cdot \alpha_{\text{free}}^{(m)}(g) \), hence the free evolution of a higher order term is determined by the free evolution of its zero- and first-order constituents. Note that for the free evolution of a first-order term one can drop all second and higher order terms which might arise, i.e.
\[ \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \} = \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \}. \]  
(9.7)

In particular, the free propagation for the linear terms is linear, i.e. we can express the free evolution of a first-order term via the propagator functions used in section 8.

By using that
\[ \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \hat{T}^{\hat{k}}(\hat{k}) \} = O(\mathcal{C}) + O(2) \quad \text{and} \quad \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \hat{\mathcal{C}}_{\hat{k}}(\hat{k}) \} = O(\mathcal{C}^2) + O(2), \]  
we obtain for the term \( I + J \)
\[ I + J \simeq \int_0^{\tau} ds \, \alpha_{\text{free}}^{(m)}(g) \left( \left\{ \alpha_{\text{free}}^{(m)}(f) \right\}, \{ \tilde{\mathcal{C}}_0(0) \} \right) \]
\[ + \frac{1}{2} \left\{ \left\{ \alpha_{\text{free}}^{(m)}(f) \right\}, \{ \tilde{\mathcal{C}}_0(0) \} \right\} \left[ \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_1) \}, \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_2) \} \right] \]
\[ + \frac{1}{2} \left\{ \left\{ \alpha_{\text{free}}^{(m)}(f) \right\}, \{ \tilde{\mathcal{C}}_0(0) \} \right\} \left[ \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_1) \}, \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_2) \} \right] \left( \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_1) \}, \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}_2) \} \right). \]  
(9.9)

The second and third term project out of \( \left\{ \alpha_{\text{free}}^{(m)}(f) \right\}, \{ \tilde{\mathcal{C}}_0(0) \} \) all terms proportional to the clock variables \( \hat{T}^{\hat{k}}(\hat{k}) \). Furthermore, because of (9.8) one does not need to evolve terms proportional to the linearized constraints \( \{ \tilde{\mathcal{C}}_{\hat{k}}(\hat{k}) \} \), hence one is left with the evolution of the scalar mode and the two tensor modes.

In summary, we learn that the gauge-invariant second-order back-reaction effect \( F_{\hat{f},\hat{T}^{\hat{k}}}(\tau) \) consists of two pieces: one is the gauge-invariant extension of the homogeneous term \( \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \} \) to the appropriate order, the other piece comes about through the ‘interaction’ of the homogeneous variables with the second-order part of the time generating constraint \( \{ \tilde{\mathcal{C}}_0(0) \} \). Here one needs to consider only the first-order gauge-invariant terms (i.e. the first-order physical modes) that arise in this interaction.

Note that for the second-order complete observable \( \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \} \) associated with a zeroth-order function \( f \) we do not need to consider the integrated vector constraints (related to the linearization instabilities). Since these start at second order and \( \{ \alpha_{\text{free}}^{(m)}(f) \}, \{ \tilde{\mathcal{C}}_0(0) \} \) does not contain a first-order term, these second-order complete observables are already invariant modulo second-order terms with respect to the integrated vector constraints.

Higher order complete observables can be calculated by expanding the power series for complete observables (4.26) systematically and by using the identity (9.4) repeatedly. This
will result in a Feynman-graph-like expansion, that is, a sum of terms describing different interaction processes generated by the higher order terms of the constraint $\tilde{C}_0(0)$.

In the following section, we will calculate for a very simple model the lowest order back-reaction effect onto the isotropic and homogeneous geometry explicitly.

10. Bianchi-I as a model for perturbations

Here, similar to [38] we will consider anisotropic but homogeneous cosmologies of Bianchi-I-type as a perturbation of isotropic and homogeneous cosmologies. We will calculate the lowest order effect of the anisotropies onto the isotropic and homogeneous variables. The Bianchi-I model (with a massless scalar field) is solvable, therefore one can compare the results to the exact model.

10.1. The model

For the Bianchi-I model, the connection $\tilde{A}_i^j$ is given by $\tilde{A}_i^j = \beta \text{diag}(A_1, A_2, A_3)$ and the canonical momentum $\tilde{E}_i^j$ is given by $\tilde{E}_i^j = \beta^{-1} \text{diag}(E_1, E_2, E_3)$ (see [38]). Further on we assume that there is a massless and homogeneous scalar field $\Phi$ and its canonical momentum $\Pi$.

The Gauss constraint and the diffeomorphism constraint are trivially fulfilled, the Hamiltonian constraints reduce to the following (weight 2 version):

$$C = \frac{2\beta^2}{\kappa} (E_1 A_1 E_2 A_2 + E_1 A_1 E_3 A_3 + E_2 A_2 E_3 A_3) + \frac{1}{2\gamma} \Pi^2,$$

(10.1)

where $\beta = i/2$ is the Immirzi parameter and $\kappa$ and $\gamma$ are coupling constants. The symplectic structure is given by

$$\{ A_i, E_j \} = \kappa \delta_{ij}, \quad \{ \Phi, \Pi \} = \gamma,$$

(10.2)

where $i, j = 1, 2, 3$.

10.2. Expansion around the isotropic sector

In order to expand the model around the isotropic sector, we define

$$A := \frac{1}{3} \sum_i A_i, \quad E := \frac{1}{3} \sum_i E_i,$$

$$a_i := A_i - \frac{1}{3} \sum_k A_k, \quad e_i := E_i - \frac{1}{3} \sum_k E_k.$$

(10.3)

(10.4)

This can be implemented through a projector $\mathcal{P}$ which projects a phase-space function $f$ onto its averaged value: $\mathcal{P}(f)(A_1, A_2, A_3; E_1, E_2, E_3) = f(A, A, A; E, E, E)$. We will call the isotropic variables $A$ and $E$ zeroth-order variables and the ‘fluctuations’ $a_i$ and $e_i$ first-order variables.

The symplectic structure in these variables reduces to

$$\{ A, E \} = \frac{1}{3} \kappa, \quad \{ a_i, e_j \} = (\delta_{ij} - \frac{1}{3}) \kappa, \quad \{ \Phi, \Pi \} = \gamma.$$

(10.5)

The fluctuation variables are not completely independent of each other and fulfil the following relations (by construction):

$$\sum_i a_i = 0, \quad \sum_i e_i = 0.$$

(10.6)
In these variables, the Hamiltonian constraint can be split into four parts:

\[ \begin{align*}
\mathcal{C} &= (0)\mathcal{C} + (2)\mathcal{C} + (3)\mathcal{C} + (4)\mathcal{C} \\
(0)\mathcal{C} &= \frac{6\beta^2}{\kappa} A^2 E^2 + \frac{1}{2\gamma} \Pi^2 \\
(2)\mathcal{C} &= \frac{2\beta^2}{\kappa} A\sum_i a_i e_i - \frac{\beta^2}{\kappa} A^2 \sum_i e_i e_i - \frac{\beta^2}{\kappa} E^2 \sum_i a_i a_i \\
(3)\mathcal{C} &= - \frac{2\beta^2}{\kappa} \left( A \sum_i a_i e_i e_i + E \sum_i a_i a_i e_i \right) \\
(4)\mathcal{C} &= \frac{\beta^2}{\kappa} \sum_{i \neq j} a_i e_i a_j e_j.
\end{align*} \] (10.7)

10.3. Exact solution

Fortunately, this model can be solved exactly and we can compare the exact solution with the one that we will derive perturbatively later. We choose lapse \( N = 1 \) and can obtain the exact solution by solving the following first-order system of differential equations:

\[ \begin{align*}
\dot{A}_i &:= \{ A, C \} = 2\beta^2 A_i (A_j E_j + A_k E_k), \\
\dot{E}_i &:= \{ E, C \} = -2\beta^2 E_i (A_j E_j + A_k E_k), \\
\dot{\Phi} &:= \{ \Phi, C \} = \Pi, \\
\dot{\Pi} &:= \{ \Pi, C \} = 0,
\end{align*} \] (10.12)

where the indices \( i, j, k \) on the right-hand side of these equations are mutually different. (The Einstein sum convention does not apply here and throughout this section.) The dot refers to the derivative in coordinate time \( t \) (with the choice \( N = 1 \)). One can see that \( A_i E_i \) is a Dirac observable, i.e. Poisson-commutes with \( C \) for \( i = 1, 2, 3 \).

We can easily solve this system of differential equations and obtain the following solutions:

\[ \begin{align*}
A_i(t) &= A_i \exp\left[ -2\beta^2 A_i E_i t \right] \exp\left[ \frac{2\beta^2}{\kappa} \sum_j A_j E_j t \right], \\
E_i(t) &= E_i \exp\left[ 2\beta^2 A_i E_i t \right] \exp\left[ -2\beta^2 \sum_j A_j E_j t \right], \\
\Phi(t) &= \Pi t + \Phi, \\
\Pi(t) &= \Pi.
\end{align*} \] (10.16)

To calculate complete observables, we have to specify a clock variable in order to get rid of the coordinate time \( t \). For our purposes it is convenient to choose \( T = \frac{\Phi}{\Pi} \) because it evolves linearly in coordinate time: \( T(t) = t + \frac{\Phi}{\Pi} \). Inverting this relation and inserting it into (10.16)–(10.19) leads to the following complete observables:

\[ F_{[A_i, T = \frac{\Phi}{\Pi}]}(\tau) = A_i \exp\left[ -2\beta^2 A_i E_i \left( \tau - \frac{\Phi}{\Pi} \right) \right] \exp\left[ \frac{2\beta^2}{\kappa} \sum_j A_j E_j \left( \tau - \frac{\Phi}{\Pi} \right) \right], \] (10.20)
\[ F_{E,T}(\tau) = E_i \exp \left[ 2 \beta^2 A_i E_i \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right] \exp \left[ -2 \beta^2 \sum_j A_i E_j \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right], \quad (10.21) \]

\[ F_{\Phi,T}(\tau) = \Pi \tau, \quad (10.22) \]

\[ F_{\Pi,T}(\tau) = \Pi. \quad (10.23) \]

For the isotropic variables, the observables read as follows:

\[ F_{A,T}(\tau) = \frac{1}{3} \sum_i A_i \exp \left[ -2 \beta^2 A_i E_i \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right] \exp \left[ 2 \beta^2 \sum_j A_j E_j \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right], \quad (10.24) \]

\[ F_{E,T}(\tau) = \frac{1}{3} \sum_i E_i \exp \left[ 2 \beta^2 A_i E_i \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right] \exp \left[ -2 \beta^2 \sum_j A_j E_j \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right]. \quad (10.25) \]

To be able to compare these exact results with the perturbative calculation later we can Taylor-expand (10.25) around \( E_i = E, A_i = A \). For the complete observable associated with \( E \), we obtain

\[ F_{E,T}(\tau) = \exp \left[ -\omega \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \right] \times \left[ E + \frac{2}{3} \beta^2 A \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \sum_i e_i e_i - \frac{2}{3} \beta^2 E \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \sum_i a_i a_i + \frac{1}{6} \beta^2 \omega A \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \sum_i e_i e_i + \frac{1}{6} \beta^2 \omega E \left( \tau - \Phi \frac{\Phi}{\Pi} \right) \sum_i a_i a_i \right] + O(3), \quad (10.26) \]

where we introduced the abbreviation \( \omega = 4 \beta^2 A E \) and \( O(3) \) denotes terms of order 3 and higher in the anisotropic fluctuations. The first-order term vanishes due to the condition \( \sum_i e_i = \sum_i a_i = 0. \)

### 10.4. Using the perturbative approach

Now that we know the exact solution, we can try to reproduce these results order per order using the perturbative approach to complete observables. Here we will only consider the lowest non-trivial order of the complete observable associated with the homogeneous variable \( E \). In the first step we have to specify a clock function. If we choose \( T = \Phi \frac{\Phi}{\Pi} \) (in order to fulfil \( \{T,C\} = 1 \)) as a clock, the analysis gets as simple as possible, because \( C := (\{T,C\})^{-1} C = C \).

In this case, the complete observable associated with an arbitrary phase-space function \( f \) is (formally) given by

\[ F_{f,T}(\tau) = \sum_0^{\infty} \frac{(\tau - \Phi \frac{\Phi}{\Pi})^n}{n!} \{ f, C \}_n = \sum_0^{\infty} \frac{(\tau - \Phi \frac{\Phi}{\Pi})^n}{n!} \{ f, 0^2 C + \ldots C + \ldots \}_n. \quad (10.27) \]
We can evaluate this sum order per order in the fluctuation variables. For all zeroth-order quantities \( f \), we obtain

\[
^{(0)}F[^{f,T}_{\Phi_1^2}](\tau) = \sum_{n=0}^{\infty} \frac{\left( \tau - \frac{\Phi^2}{\Pi} \right)^n}{n!} \{ f, (^{(0)}C) \}_n =: \alpha'^{(0)}_{\text{lin}}(f),
\]

(10.28)

(1) \( F[^{f,T}_{\Phi_1^2}](\tau) = 0 \),

(10.29)

(2) \( F[^{f,T}_{\Phi_1^2}](\tau) = \sum_{n=0}^{\infty} \sum_{n_1+n_2+n_3=0} \frac{\left( \tau - \frac{\Phi^2}{\Pi} \right)^n}{n!} \{ \{ f, (^{(0)}C) \}_{n_1}, (^{(2)}C), (^{(0+2)}C) \}_{n_2} \}.
\]

(10.30)

For first-order quantities \( f \), we get

\[
^{(1)}F[^{f,T}_{\Phi_1^2}](\tau) = \sum_{n=0}^{\infty} \frac{\left( \tau - \frac{\Phi^2}{\Pi} \right)^n}{n!} \{ f, (^{(0+2)}C) \}_n =: \alpha'^{(0+2)}_{\text{free}}(f).
\]

(10.31)

Expression (10.30) can be interpreted as follows: a zeroth-order quantity \( f \) propagates with respect to the ‘free’ Hamiltonian constraint \( (^{(0)}C) \), then there is an ‘interaction’ with the second-order Hamiltonian constraint \( (^{(2)}C) \) which yields terms quadratic in the fluctuations. After the interaction, zeroth-order quantities evolve according to the ‘free’ Hamiltonian constraint \( (^{(0)}C) \) and first-order quantities according to the sum of the zeroth- and second-order Hamiltonian constraint \( (^{(0)}C) + (^{(2)}C) \) (where one can ignore all higher than first-order quantities that appear in evolving the first-order quantity).

The interpretation of (10.31), which can be seen as the ‘free’ propagation of a first-order quantity \( f \), is similar.

We will compute the second order of \( F[^{E,T}_{\Phi_1^2}](\tau) \). Using identity (9.4), the expression (10.30) can be reformulated as follows:

\[
^{(2)}F[^{f,T}_{\Phi_1^2}](\tau) = \int_0^{\tau} \frac{\tau - \frac{\Phi^2}{\Pi}}{\alpha'^{(2)}_{\text{free}}(E)} \{ \{ \alpha'^{(2)}_{\text{lin}}(E), (^{(2)}C) \} \}.
\]

(10.32)

The first step is to calculate

\[
\alpha'^{(2)}_{\text{lin}}(A), \alpha'^{(2)}_{\text{lin}}(E),
\]

(10.33)

\[
\alpha'^{\text{free}}(a), \alpha'^{\text{free}}(e).
\]

(10.34)

The first two quantities (10.33) are the solutions to the differential equations

\[
\dot{A} := \{ A, (^{(0)}C) \} = 4\beta^2 A^2 E,
\]

(10.35)

\[
\dot{E} := \{ E, (^{(0)}C) \} = -4\beta^2 A E^2.
\]

(10.36)

These can easily be found to be

\[
\alpha'^{(2)}_{\text{lin}}(A) = A \exp(\omega t),
\]

(10.37)

\[
\alpha'^{(2)}_{\text{lin}}(E) = E \exp(-\omega t),
\]

(10.38)

\[
\omega := 4\beta^2 A E.
\]

(10.39)

The second set of quantities can be found by solving

\[
\dot{a} = \{ a, (^{(2)}C) \} = 2\beta^2 A E a - 2\beta^2 A^2 e,
\]

(10.40)

\[
\dot{e} = \{ e, (^{(2)}C) \} = -2\beta^2 A E e + 2\beta^2 E^2 a.
\]

(10.41)
where the time-dependence of the homogeneous variables is given by $A(t) = \alpha_{mC}(A)$ and $E(t) = \alpha_{mC}(E)$. The solution to this set of differential equations is given by

$$\alpha'_{\text{free}}(a_k) = a_k \exp[\omega t] - \frac{\omega}{2} \left( a_k + \frac{A}{E} e_k \right) t \exp[\omega t].$$  \hspace{1cm} (10.42)

$$\alpha'_{\text{free}}(e_k) = e_k \exp[-\omega t] + \frac{\omega}{2} \left( e_k + \frac{E}{A} a_k \right) t \exp[-\omega t].$$  \hspace{1cm} (10.43)

Now we can calculate (10.32) step by step:

$$\{\alpha_{mC}(E), \{C\} \} = \frac{1}{3} \exp[-\omega t] \left[ 2\beta^2 A (1 - \omega t) \sum_i e_i e_i 
- 2\beta^2 E \sum_i e_i a_i + 2\beta^2 \omega t \frac{E^2}{A} \sum_i a_i a_i \right].$$

$$\alpha'_{\text{free}}^{-\frac{\pi}{3}} \left[ \{\alpha'_{mC}(E), \{C\} \} \right] = \frac{2}{3} \beta^2 A \exp \left[ -\omega \left( \tau - \frac{\Phi}{\Pi} \right) \right] \left[ \sum_i e_i e_i - \frac{E}{A} \sum_i a_i a_i \right]$$

$$+ \omega \left( \tau - \frac{\Phi}{\Pi} \right) \left[ \frac{3}{2} \sum_i e_i e_i + \frac{E}{A} \sum_i e_i a_i - \frac{1}{2} \frac{E^2}{A^2} \sum_i a_i a_i \right]$$

$$+ \frac{1}{2} \omega^2 \left( \tau - \frac{\Phi}{\Pi} \right)^2 \left[ \sum_i e_i e_i + \frac{E^2}{A^2} \sum_i a_i a_i + 2 \frac{E}{A} \sum_i e_i a_i \right]$$

$$+ \omega t \left[ - \frac{5}{2} \sum_i e_i e_i - \frac{E}{A} \sum_i e_i a_i + \frac{3}{2} \frac{E^2}{A^2} \sum_i a_i a_i \right]$$

$$- 2\omega \left( \tau - \frac{\Phi}{\Pi} \right) \left[ \sum_i e_i e_i + \frac{E^2}{A^2} \sum_i a_i a_i + 2 \frac{E}{A} \sum_i e_i a_i \right]$$

$$+ \frac{3}{2} \omega^2 \tau^2 \left[ \sum_i e_i e_i + \frac{E^2}{A^2} \sum_i a_i a_i + 2 \frac{E}{A} \sum_i e_i a_i \right].$$  \hspace{1cm} (10.44)

This gives the second-order contribution

$$(2) F_{\{E, \tau = \frac{\Phi}{\Pi}\}}(\tau) = \int_0^{\tau - \frac{\Phi}{\Pi}} d\tau \alpha'_{\text{free}}^{-\frac{\pi}{3}} \left[ \{\alpha'_{mC}(E), \{C\} \} \right]$$

$$= \exp \left[ -\omega \left( \tau - \frac{\Phi}{\Pi} \right) \right] \left[ \frac{2}{3} \beta^2 A \left( \tau - \frac{\Phi}{\Pi} \right) \sum_i e_i e_i - \frac{2}{3} \beta^2 E \left( \tau - \frac{\Phi}{\Pi} \right) \sum_i a_i a_i \right]$$

$$+ \frac{\beta^2}{6} \omega A \left( \tau - \frac{\Phi}{\Pi} \right)^2 \sum_i e_i e_i + \frac{\beta^2}{6} \omega^2 E \frac{E^2}{A} \left( \tau - \frac{\Phi}{\Pi} \right)^2 \sum_i a_i a_i$$

$$+ \frac{\beta^2}{3} \omega E \left( \tau - \frac{\Phi}{\Pi} \right)^2 \sum_i a_i e_i. \hspace{1cm} (10.45)$$

As expected, (10.45) coincides with (10.26), the result we obtained by just Taylor-expanding the exact expression. This simple model shows that it is, at least in principle, possible to calculate back-reaction effects in cosmological applications of general relativity using the methods developed in this work.
Gauge-invariant perturbations around symmetry-reduced sectors of general relativity

As an illustration we compare in figure 1 the exact complete observable with the zeroth-and second-order complete observable associated with \( E \). As one can see there the (relative) difference of the second-order approximation to the zeroth-order approximation grows with ‘time’ \( \tau \), as does the difference of the second-order approximation to the exact solution.

\textbf{11. Discussion and conclusions}

We presented a gauge-invariant canonical scheme for perturbations around symmetry-reduced sectors of gauge systems. This scheme is applied to general relativity, in particular to perturbations around the cosmological sector. It can be used to calculate the dynamics up to an arbitrary high order in the fluctuation variables.

The central objects in this perturbative scheme are complete observables. These complete observables are gauge-invariant (i.e. Dirac) observables. In the canonical formalism, this means that the complete observables have to be invariant under (coordinate) time reparametrizations, i.e. constants of motion. Nevertheless, one can express the dynamics of the theory using the complete observables. To this end, one has to choose a set of physical clocks. Evolution of dynamical entities can then be understood as an evolution in relation to the clocks. One can even define generalized Hamiltonians, that is, gauge-invariant phase-space functions that generate this evolution for the complete observables.

Different choices of clocks can be interpreted as different setups for a physical measurement. We gave explicit formulae relating complete observables associated with different choices of clocks in section 5. In particular, the clocks define the hypersurfaces over which the averaging (4.11) is defined. The complete observables evaluated on a phase-space point (i.e. evaluated on a spacetime satisfying the Einstein equations) give the values of the partial observables, such as the averaged densitized triad \( E \) and its fluctuations \( e^a_i \), on the hypersurface determined by \( \{ T^K(k) = \tau^K(k) \} \). Gauge invariance of the complete
observables ensures that the value of the complete observable does not change if we evaluate the complete observable on spacetimes related by diffeomorphisms. From the complete observables associated with all the phase-space variables (which will give an overcomplete basis of observables), one can calculate the values of all kinds of physical observables on the hypersurface \( \{ T^k(k) = \tau^k(k) \} \). For instance one might wish to consider instead of the averaged densitized triad some averaged function of the (spacetime\(^{11}\)) metric components: in this case, one has first to express the partial observable in question as a function \( f \) of the zeroth- and first-order phase-space variables and then to consider the complete observable associated with this function (expanded to the appropriate order). Note that this complete observable can be expressed as the same function \( f \) of the complete observables associated with the zeroth- and first-order complete observables.

We considered in more detail first- and second-order complete observables. Higher complete observables can be calculated by similar methods. The zeroth-order complete observables coincide with the Dirac observables of the symmetry-reduced model. The dynamics of the first-order complete observables is related to the theory of cosmological linear perturbations (see [17] and references therein), the difference is that in our case dynamics is expressed with respect to a (global) clock variable, which can however be related to choosing a (zeroth order) lapse function.

Second-order complete observables associated with zeroth-order partial observables describe back-reaction effects. We calculated these back-reaction effects in section 10 for a simple model. To allow for an explicit expression for these effects in interesting (inflationary) scenarios further approximations, as for instance slow roll or long wavelength approximations, need to be implemented. This can be done by introducing small parameters, which characterize these approximations [39]. The power series for the complete observables has then to be expanded to a certain order in the fluctuation variables and to a certain order in these new parameters.

A perturbative scheme around a whole phase-space sector has several advantages compared to an expansion around a fixed phase-space point. First, some of the degrees of freedom, namely those corresponding to the symmetry-reduced sector are treated non-perturbatively. These degrees of freedom are used to define the clock in relation to which we express the dynamics of the theory. Compared to defining a field theory on a fixed background, where the background is used to define time, the clock in this approach is a fully dynamical entity, as one would expect in the full theory. Indeed, one would expect problems with gauge consistency to higher order if one uses a background time, since such a time cannot take into account back-reaction effects onto the physical clock.

Also a quantization of the theory should not only quantize the perturbations on a fixed background but rather consider the quantized perturbations on a quantized geometry, described by the (quantized) symmetry-reduced sector. The variables describing the symmetry-reduced sector contribute for instance to the Hamiltonian generating the physical time evolution. From this perspective, it might be fruitful to reconsider certain concepts, like energy and vacua, from quantum field theory on curved spacetime.

As in the case for perturbations around a fixed background, where the background is given in certain coordinates, the symmetry-reduced sector is described by using a certain type of coordinates. That is, there exist configurations which are not included in the symmetry-reduced sector, but which are nevertheless physically symmetric. These configurations can be obtained by a ‘non-symmetric’ coordinate transformation from a symmetric configuration. Degrees of freedom gained in this manner correspond to gauge degrees of freedom.

\(^{11}\) The 4-metric can be computed by using lapse and shift functions defined in section 6, see also [7].
However, by expanding around a sector describing a family of solutions in different coordinate systems one can investigate the non-perturbative effects of such gauge degrees of freedom compared to an expansion around a smaller sector (which could be just one phase-space point), which would treat some of these gauge degrees of freedom perturbatively [40]. That is, we can embed symmetry-reduced models into each other and explore in this way the reliability of symmetry reduction.

Acknowledgments

We are grateful to Thomas Thiemann for discussions and for suggesting we consider the Bianchi-I model. Furthermore, we would like to thank Laurent Freidel and Stefan Hofmann for discussions. JT thanks the German National Merit Foundation for financial support and the Perimeter Institute for Theoretical Physics for hospitality. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

Appendix A. Tensor mode decomposition

Similar to the longitudinal and transversal modes for a vector field on \( \mathbb{R}^3 \), one can introduce tensor modes for a tensor field. For a proof of the completeness of these modes, see [33]. To begin with we define the projector onto the transversal modes of a vector field by

\[
(p \cdot v)_a := p^b \cdot v_b := (\delta^b_a + W^{-2} \cdot \delta_a b) \cdot v_b.
\]  

(A.1)

This allows us to introduce the following basis of tensor modes:

\[
\begin{align*}
(LTP \cdot T)_{ab} &= (\delta^c_a - p^c_a) \cdot p^d_b \cdot T_{cd} & \text{2 left long. right transv. modes} \\
(LTP \cdot T)_{ab} &= (\delta^c_a - p^c_a) \cdot (\delta^d_b - p^d_b) \cdot T_{cd} & \text{1 left and right long. mode} \\
(TLP \cdot T)_{ab} &= p^c_a \cdot (\delta^d_b - p^d_b) \cdot T_{cd} & \text{2 left transv. right long. modes} \\
(TP \cdot T)_{ab} &= \frac{1}{2} p^a_b \cdot p^c_d \cdot T_{cd} & \text{1 symm. transv. trace part mode} \\
(ATP \cdot T)_{ab} &= \frac{1}{2} (p^a_b \cdot p^c_d - p^a_d \cdot p^c_b) \cdot T_{cd} & \text{1 antisymm. transv. mode} \\
(STTP \cdot T)_{ab} &= \frac{1}{2} (p^a_b \cdot p^c_d + p^a_d \cdot p^c_b - p_{ab} p^{cd}) \cdot T_{cd} & \text{2 symm. transv. tracefree modes}.
\end{align*}
\]  

(A.2)

Using the projector property \( p \cdot p = p \), it is easy to see that the projectors \( X P \) are orthogonal to each other and satisfy \( X P \cdot T P = \delta^{XY} X P \). Furthermore, the set of projectors is complete, that is,

\[
\sum_X X P_{cd} = \delta^c_a \delta_d^b.
\]  

(A.3)

We will denote the tensor modes by \( X T_{ab} := (X P \cdot T)_{ab} \).

Appendix B. First-order perturbations: scalar modes in longitudinal gauge

Here we will derive the equations of motion for the first-order scalar perturbations with the choice (7.8)

\[
T^G a = e^{ab} e_{bc} = e^{ab} (AT e_{bc} + LT e_{bc} + TL e_{bc}),
\]
\[ T^{\text{D}a} = -W^{-2}\left(-\frac{1}{2}W^{-2}\tilde{g}^{a}\partial_{a}\partial_{e} + \frac{1}{2}\tilde{g}^{a}\delta_{de} - \partial_{a}\delta^{a}_{e} - \partial_{a}\delta^{a}_{e}\right)e^{de} \]
\[ = -W^{-2}\left(-\tilde{a}^{aLl}e^{l}_{d} + \frac{1}{2}\tilde{g}^{a}T_{a}^{e}e^{de} - \partial_{a}T^{L}e^{de} - \partial_{a}L^{T}e^{de}\right), \]
\[ T^{0} = W^{-2}\left(\frac{1}{2}\tilde{a}^{a} - \tilde{a}^{L}L^{a}_{d}\right), \quad (B.1) \]

for the clock variables. As we will see this choice is related to the longitudinal gauge: in the notation of section 8, we have
\[ [[T^{\text{D}a}]] = \left[ -W^{-2}\left(-\frac{1}{2}W^{-2}\tilde{g}^{a}\partial_{a}\partial_{e} + \frac{1}{2}\tilde{g}^{a}\delta_{de} - \partial_{a}\delta^{a}_{e} - \partial_{a}\delta^{a}_{e}\right)e^{de} \right] \simeq 0. \quad (B.2) \]

(Note that the brackets \([[]]\) now refer to complete observables with respect to the clock variables \((B.1)\).) We want to translate this condition on the triad perturbation to a condition on the spatial metric \(q_{ab}\). With the relation
\[ \det(q)q^{ab} = \beta^{2}E_{j}^{a}E_{j}^{b}, \quad (B.3) \]

between the densitized triad variables \(E_{j}^{a}\) and the inverse metric \(g^{ab}\) (where \(\det(q)\) is the determinant of the metric), we get
\[ q_{ab} = E\delta_{ab} + e^{a}\delta_{ab} - e_{ab} - e_{ba} + O(2) \equiv E\delta_{ab} + h_{ab} + O(2). \quad (B.4) \]

Hence, \(e_{ab} + e_{ba} = -h_{ab} + h_{cd}\delta_{ad} + O(2)\) and together with \([[e_{ab}]] = [[e_{ba}]]\) from the Gauss clock (in \((B.1)\), we obtain
\[ 2^{(1)}[[T^{\text{D}a}]] = \left[ -W^{-2}\left(-\frac{1}{2}W^{-2}\tilde{g}^{a}\partial_{a}\partial_{e} + \frac{1}{2}\tilde{g}^{a}\delta_{de} - \partial_{a}\delta^{a}_{e} - \partial_{a}\delta^{a}_{e}\right)(h^{de} - \delta^{de}h^{e}) \right] \]
\[ \simeq 0. \quad (B.5) \]

This is the longitudinal gauge condition for the 3-metric (see \([17]\)). Furthermore, we can calculate the lapse and shift functions introduced in section 6 according to
\[ (1)\mathcal{N}^{j}(k) = -\mathcal{N}\sum_{k}^{(0)}(A^{-1})_{k}^{j}(-k', k')\{T^{K}(k'), (2)C_{0}(0)\}, \quad (B.6) \]

where we used the definition \(\mathcal{N} = (0)\mathcal{N}^{q}(0) = g^{ab}(0)(T^{0}(0), C_{0}(0)))^{-1}\). This gives for the longitudinal clocks
\[ (1)[[\mathcal{N}^{0}(k)]] \simeq (1)[[-NE^{-1}\frac{1}{2}e^{b}_{b}(k)]] \]
\[ (1)[[\mathcal{N}^{D_{a}}(k)]] \simeq 0 \]
\[ (1)[[\mathcal{N}^{C_{a}}(k)]] \simeq (1)[[\mathcal{N}^{a}e^{c}_{e}(k)]] \quad (B.7) \]

In particular, we see that the shift function \((1)[[\mathcal{N}^{D_{a}}(k)]]\) vanishes, which is the other gauge condition in the longitudinal gauge (without coupling to spin one (vector) matter fields). Hence we can see our first-order observables as gauge-invariant extensions of longitudinal gauge-restricted functions.

Since we only consider a coupling to a matter scalar field, i.e. matter without anisotropic stress, the first-order lapse function can be related to the scalar metric perturbation. We do not get any contribution from the matter part of the scalar constraint to the first-order lapse function for the following reason: according to \((B.6)\) the first-order lapse function involves the Poisson bracket of the second-order scalar constraint with the clocks. For scalar field matter, the second-order matter part of the scalar constraint involves only terms of the following form: either products of the trace modes \(T^{\text{a}}_{\text{a}b} := \sum_{k}^{(0)}(T^{0}(0), C_{0}(0)))^{-1}\) vanishes, which is the other gauge condition in the longitudinal gauge (without coupling to spin one (vector) matter fields). Hence we can see our first-order observables as gauge-invariant extensions of longitudinal gauge-restricted functions.
trace mode \( T^T e_{ab} \) and the scalar trace-free triad mode \( T^F e_{ab} := \frac{1}{2} T e_{ab} - \Lambda L e_{ab} \). Hence, \( T^G a \) and \( T^{D a} \) commute with the second-order matter part of the scalar constraint and the Poisson bracket with the scalar clock \( T^0 = W^{-2} T^F a^b \) gives a term proportional to \( T^F e^b \) which we can ignore since it is proportional to the divergence of the diffeomorphism clock \( \partial_a T^{D a} \). Therefore, we do not get any contribution to the first-order lapse function from the matter part of the constraint.

This would change if the second-order matter part would include a coupling between the triad perturbation and a matter perturbation of the form \( e_{ab} f^a_{\delta} \), where \( f^a_{\delta} \) represents some first-order matter term (with non-diagonal elements). However, such a term would lead to anisotropic stress, i.e. non-diagonal terms in the spatial part of the energy–momentum tensor. (This can be seen by recalling [41] that for non-derivative couplings the spatial part of the energy–momentum tensor is given by the variation of the matter part of the Hamiltonian constraint with respect to the spatial metric: \( T^{ab} = -2 \frac{\partial}{\partial e_{\perp ab}} e^\text{matter} \).

The lapse and shift functions (B.7) can be used to derive, in the same way as in section 8, the equations of motion for the scalar matter field \( (1)[\phi] [(\tau) \) :

\[
\begin{align*}
\frac{d}{d\tau} (1)[\phi] (\tau) &\simeq (1)[N^0 (\rho - \frac{1}{2} E^{-1} \Pi e^b_0)] (\tau), \\
\frac{d}{d\tau} (0)[\rho] (\tau) &\simeq (1)[N (E^2 \phi^a \partial_a \phi - \frac{1}{3} E^2 V' (\Phi) e^a_a - E^3 V'' (\Phi) \phi + \frac{3}{2} E^2 V' (\Phi) e^b_0)] (\tau).
\end{align*}
\]

(B.8)

These equations lead to the following wave equation for the scalar mode:

\[
\begin{align*}
\frac{d^2}{d\tau^2} (1)[\phi] (\tau) &\simeq (0)[N^0] (\tau) \frac{d}{d\tau} (0)[N^0] (\tau) \frac{d}{d\tau} (1)[\phi] (\tau) \\
&\quad + (1)[N^0 (E^2 \phi^a \partial_a \phi - \frac{1}{3} E^2 V' (\Phi) e^a_a - E^3 V'' (\Phi) \phi)] \\
&\quad - (0)[\frac{2}{3} N \Pi] (\tau) \frac{d}{d\tau} (1)[E^{-1} e^b_0] (\tau) \\
&\quad - (1)[\frac{2}{3} N E^{-1} e^b_0] (\tau) \frac{d^2}{d\tau^2} (0)[\Pi] (\tau).
\end{align*}
\]

(B.9)

Using \( (1)[\bar{T}^0 (\kappa)] = 0 + O(2) \) and \( (1)[\bar{C} (\kappa)] \simeq 0 + O(2) \), one can replace the metric mode \( (1)[e^a_0] \) in (B.9) by some combination of the scalar field \( (1)[\phi] \) and its first \( \tau \)-derivative, however this does not lead to a simple equation.

Another possibility is to derive a wave equation for the metric mode \( (1)[e^a_0] \) in the same way as for the scalar field mode:

\[
\begin{align*}
\frac{d}{d\tau} (1)[e^a_0] (\tau) &\simeq (1)[-N^0 4 \beta^2 E^2 a^a_a] (\tau), \\
\frac{d}{d\tau} (0)[a^a_a] (\tau) &\simeq (1)[N^0 (\delta^a \delta^0 e^a_\epsilon - 4 \beta^2 A^2 e^a_\epsilon + 8 \beta^2 A e^a_\epsilon + 3 \frac{K^2}{2} E^2 V' (\Phi) \phi)] (\tau).
\end{align*}
\]

(B.10)

12 For conformal time, that is if one chooses the clock \( T^0 (0) \) in such a way that \( (0) \) \( N = [T^0 (0), \bar{C}_0 (0)] \) \(-1 = E^{-1} \), the wave equation (B.9) coincides with the wave equation for the (first-order gauge-invariant) scalar field in [17]. The metric scalar mode \( \psi (k) \) used in [17] can be computed to be \( \psi = - \frac{1}{6} E^{-1} e^a_\epsilon \). Also the (first-order) gauge-invariant Bardeen potential [42] is given by \( \psi = (1)[-\frac{1}{6} E^{-1} e^a_\epsilon] \).
Hence we obtain the wave equation
\[
\frac{d^2}{d\tau^2}[[e^a_μ](τ)] \simeq (0)[N](τ) \frac{d}{d\tau}[[N]](τ) \frac{d}{d\tau}[[e^a_μ]](τ) + (0)[N][N](τ) \frac{d}{d\tau}[[e^a_μ]](τ) \]
\[
+ \left( \left[ -N^2 4\beta^2 E^2 \left( \partial^a \partial^e e_c^e - 4\beta^2 A^2 e_c^e + 3\frac{K}{\gamma} E^2 V(\Phi) \right) \right] \right) (τ),
\]
where we can replace the matter scalar field by
\[
(1)[\phi]](τ) \simeq \left( \left[ \frac{\gamma}{\kappa} \Pi^{-\frac{1}{3}} A e^c_μ \right] (τ) + \left[ \frac{2}{3} (N^2 4\beta^2 E^2)^{-1} \right] (τ) \frac{d}{d\tau}[[e^a_μ]](τ) \right) (B.11)
\]
Using the time evolution of the homogeneous variables \(A, E, \Phi, \Pi\) and specializing to conformal time \((N = E^{-1})\), we therefore have as the wave equation\(^{13}\) for \([E^{-1} e^a_μ]\)
\[
\frac{d^2}{d\tau^2}[[E^{-1} e^a_μ]](τ) \simeq \left( \left[ \frac{4\beta^2 \partial^a \partial_e e_c^e - 2 E^2 \left( N^2 4\beta^2 E^2 \right)^{-1} \right] (τ) \frac{d}{d\tau}[[E]](τ) \frac{d}{d\tau}[[E^{-1} e^a_μ]](τ) \right)
\]
\[
+ (0)[N][E][N][E](τ) \frac{d}{d\tau}[[E^{-1} e^a_μ]](τ) \]
\[
- (0)[2\Pi][E][\Pi][E](τ) \frac{d}{d\tau}[[E^{-1} e^a_μ]](τ) \]
\[
- (0)[E][E][E][E](τ) \frac{d}{d\tau}[[E^{-1} e^a_μ]](τ) (B.12)
\]

### Appendix C. First-order perturbations: tensor modes

Here we will derive the differential equation for the tensor modes of the gravitational field, that is, the \(ST\) modes. The \(ST\) modes \(ST_{a}^{ab}\) and \(ST_{ab}^{S}\) are already gauge-invariants of first order, since they commute with the first-order part of the constraints. Hence according to equation \((5.5)\), which connects the first-order complete observables with respect to different choices of clock variables, the first-order complete observable associated with the \(ST\) modes are independent of the choice of clock variables \(T^k(k)\).

Indeed, in the differential equation \((8.2)\) for the first-order complete observables associated with the \(ST\) modes (where \(N^{-1} = (0)[T^0(0), S(0)]\))
\[
\frac{d}{d\tau}[[ST_{a}^{ab}]](τ) \simeq (1)[[ST_{a}^{ab}], \hat{C}_0(0)]]](τ) \simeq (1)[[N][ST_{a}^{ab}], S(0)]]](τ),
\]
\[
\frac{d}{d\tau}[[ST_{a}^{ab}]](τ) \simeq (1)[[ST_{a}^{ab}], \hat{C}_0(0)]]](τ) \simeq (1)[[N][ST_{a}^{ab}], S(0)]]](τ),
\]
all terms which may depend on the choice of \(T^k(k)\) variables drop out. Here we used the definition \((4.22)\) of \(\hat{C}_0(0)\) and the fact that the \(ST\) modes commute with the first-order part of the constraints.

With the second-order scalar constraint \((4.19)\), we get
\[
\frac{d}{d\tau}[[ST_{a}^{ab}]](τ) \simeq \left[ N \left( 2\beta E D_{ab}^{fe} ST_{a}^{fe} - 2 \left( \beta A^2 + \frac{K}{\gamma} \frac{\gamma}{\kappa} V(\Phi) \right) ST_{a}^{ab} + 2\beta^2 AE ST_{a}^{ab} \right) \right],
\]
\[
\frac{d}{d\tau}[[ST_{a}^{ab}]](τ) \simeq \left[ N \left( -2\beta E D_{ab}^{fe} ST_{a}^{fe} + 2\beta^2 E^2 ST_{a}^{ab} - 2\beta^2 AE ST_{a}^{ab} \right) \right],
\]
\(^{13}\) This wave equation coincides with the wave equation for the Bardeen potential \(\Psi = (1)[E^{-1} e^a_μ]\) in \([17]\).
where $D^f_{ab} = \frac{1}{2} \varepsilon^{efc} \delta_b^c (h_{ab} \delta_d^f + h_{da} \delta_b^f)$. If $D$ acts on $STT$ modes it simplifies to $D^f_{ab} = \varepsilon^f_{ae} \delta_b^a$, moreover on $STT$ modes we have $(D \cdot D)^f_{ab} = -\delta^f_b \delta^f_a$.

We want to derive a wave equation for the $STT_{ab}$ modes. To this end, we have to calculate the second ($\tau$-) time derivative of $STT_{ab}$. In the process, we also have to take into account the $\tau$-dependence of the homogeneous variables and to replace the $STT_{ab}$ modes by the $\tau$-derivative of the $STT_{ab}$ modes with the help of the second of the equation (C.2). The resulting equation of motion for the $STT_{ab}$ modes is

$$
\frac{d^2}{d\tau^2} [STT_{ab}] \simeq (0)[N^{-1}] \frac{d}{d\tau} (0)[N] - (0)[N] 8 \beta^2 A E \left(4 \beta^2 E^2 \delta^c \partial_c + 4 \beta^2 E^3 \kappa \gamma V(\Phi) + 16 \beta^4 A^2 \gamma^2 \right) [STT_{ab}].
$$

(C.3)

The wave equation simplifies if we consider instead of $STT_{ab}$ the rescaled variable $E^{-1} STT_{ab}$. (The correspondence to the metric variables is given by $E^{-1} TT_{ab} = -2E^{-1} STT_{ab}$, where $h_{ab}$ is the deviation of the spatial metric $q_{ab}$ from the isotropic and homogeneous background $g_{ab} = E \delta_{ab} + h_{ab} =: Q_{ab} + h_{ab}$. Hence the rescaled variable corresponds to $(2) Q_{icc} = TT_{abc}$.)

$$
\frac{d^2}{d\tau^2} [E^{-1} STT_{ab}] \simeq (0)[N^{-1}] \frac{d}{d\tau} (0)[N] - (0)[N] 4 \beta^2 E^2 \delta^c \partial_c E^{-1} STT_{ab}].
$$

(C.4)

Specializing to conformal time ($N = E^{-1}$), we find

$$
\frac{d^2}{d\tau^2} [E^{-1} STT_{ab}] \simeq (0)[-1] \frac{d}{d\tau} (0)[-1] - 4 \beta^2 \delta^c \partial_c [-1] E^{-1} STT_{ab}].
$$

(C.5)

Appendix D. Linearization instabilities

So far we have left out the discussion of the integrated diffeomorphism constraints, which start with second-order terms (if one does not use three massless scalar fields as clocks for the diffeomorphism constraints).

Here we will show for a general first-class constraint system that one can apply the complete observable method with respect to a subset of the constraints, also if these do not form a subalgebra. In our case, this subset is given by all the constraints with the exception of the integrated diffeomorphism constraints. After constructing these (partially invariant) complete observables, one can use these to get fully invariant observables.

We will explain the procedure for a finite dimensional system, the generalization to field systems is straightforward. To start with, consider a first-class constraint system with constraints $\{C_i\}_{i=1}^N$ and subdivide this set into two subsets of constraints $\{C_{a\alpha}\}^{M}_{\alpha=1}, \{C_{b\alpha}\}^{N-M}_{\alpha=1}$. We want to apply the complete observable method to the first subset $\{C_{a\alpha}\}^{M}_{\alpha=1}$ of the constraint set. Hence we choose a set of clock variables $\{T^A\}^{M}_{A=1}$ and define, in the same way as we would deal with the full set of constraints, the new constraints

$$
\hat{C}_A := (A^{-1})^a_\alpha C_{a\alpha}, \quad \text{where} \quad A^A_{\alpha} = \{T^A, C_{a\alpha}\}.
$$

(D.1)

These new constraints satisfy

$$
\{T^B, \hat{C}_A\} = g^B_{\alpha} + \lambda^{BC}_A C_{C\alpha}.
$$

(D.2)

for some phase-space functions $\lambda^{BC}_A$. 


Furthermore, we want to add to the constraints \{\hat{C}_A\} another set of constraints such that we get a complete system of constraints and such that these added constraints commute at least on the constraint hypersurface with the clock variables \{T^A\}. Therefore, we define the constraints
\[
\hat{C}_\alpha := C_\alpha - \{T^A, C_\alpha\} \hat{C}_A
\]
which satisfy
\[
\{T^B, \hat{C}_\alpha\} = O(C) \quad \text{(D.4)}
\]
We want to consider the Poisson algebra of the new constraints \(\hat{C}_A\) and \(\hat{C}_\alpha\). To this end, we define the structure functions \(\tilde{f}\) by
\[
\{\hat{C}_A, \hat{C}_B\} = \tilde{f}_{AB} \hat{C}_C + \tilde{f}_{A\beta} \hat{C}_\beta,
\]
\[
\{\hat{C}_A, \hat{C}_\beta\} = \tilde{f}_{A\beta} \hat{C}_C + \tilde{f}_{AB} \hat{C}_\alpha,
\]
\[
\{\hat{C}_\gamma, \hat{C}_B\} = \tilde{f}_{\gamma B} \hat{C}_C + \tilde{f}_{\gamma \beta} \hat{C}_\beta.
\]
Then using the Jacobi identity, we can calculate
\[
O(C) = \{\hat{C}_A, \{\hat{C}_B, \hat{C}_C\}\} = \{\hat{C}_B, \{\hat{C}_A, \hat{C}_C\}\} + \{\hat{C}_C, -\delta^C_A + O(C)\}
\]
\[
= \hat{f}_{AB} + O(C). \quad \text{(D.6)}
\]
Hence we have \(\hat{f}_{AB} = O(C)\) and therefore
\[
\{\hat{C}_A, \hat{C}_B\} = O(C^2) + \hat{f}_{AB} \hat{C}_\alpha. \quad \text{(D.7)}
\]
Similarly
\[
\{T^A, \{\hat{C}_B, \hat{C}_\alpha\}\} = \{T^A, \hat{f}_{B\alpha} \hat{C}_C + \hat{f}_{B\beta} \hat{C}_\beta\}
\]
\[
= \hat{f}_{A\beta} + O(C). \quad \text{(D.8)}
\]
On the other hand,
\[
\{T^A, \{\hat{C}_B, \hat{C}_\alpha\}\} = \{\hat{C}_B, \{T^A, \hat{C}_\alpha\}\} + \{\hat{C}_\alpha, \{\hat{C}_B, T^A\}\} = O(C), \quad \text{(D.9)}
\]
hence \(\hat{f}_{A\beta} = O(C)\) and we have that
\[
\{\hat{C}_B, \hat{C}_\alpha\} = O(C^2) + \hat{f}_{B\alpha} \hat{C}_\gamma. \quad \text{(D.10)}
\]
In the same way one can prove that
\[
\{\hat{C}_\alpha, \hat{C}_\beta\} = O(C^2) + \hat{f}_{\alpha \beta} \hat{C}_\gamma. \quad \text{(D.11)}
\]
In summary, we learn that the constraints \(\{\hat{C}_A\}\) are weakly Abelian modulo terms proportional to the constraints \(\hat{C}_\alpha\) and that the constraint set \(\{\hat{C}_\alpha\}\) forms an ideal (modulo terms quadratic in the constraints) of the full constraint algebra.

Assume that we have a phase-space function \(f\) that is (weakly) invariant under the constraints \(\{\hat{C}_A\}\). Then we can use the power series for complete observables (2.5) just with the constraints \(\{\hat{C}_A\}\) and compute the complete observable associated with \(f\):
\[
F_{\{f; T^A\}}(\tau^A) \simeq \sum_{k=0}^{\infty} \frac{1}{k!} \{\cdots \{ f, \hat{C}_{A_k}, \cdots \}, \hat{C}_{A_k}\} (\tau^{A_1} - T^{A_1}) \cdots (\tau^{A_k} - T^{A_k}). \quad \text{(D.12)}
\]
Because of the properties (D.7) and (D.10), the resulting function is (weakly) invariant under both sets of constraints \(\{\hat{C}_A\}\) and \(\{\hat{C}_\alpha\}\). (Just consider the Poisson bracket of (D.12) with constraints \(\hat{C}_A\) and \(\hat{C}_\alpha\)\).


That is to find a fully gauge-invariant observable we have to start with a partial observable \( f \) that is invariant under the constraints \( \{C_\alpha \} \). However, it is also possible to calculate complete observables (D.12) associated with non-invariant functions \( f \) and then to find fully gauge-invariant observables using that

\[
F_{f_1 f_2, \tau_1} \simeq F_{f_1, \tau_1} (\hat{\tau}) \cdot F_{f_2, \tau_1} (\hat{\tau}) + F_{f_1 f_2, \tau_1} (\hat{\tau}).
\]

Hence we can obtain a fully gauge-invariant observable by combining complete observables associated with non-invariant partial observables \( f_i \) algebraically such that the same algebraic combination of \( f_i \) is invariant under the constraints \( \hat{C}_\alpha \).

Applied to our situation this means that we can ignore the integrated diffeomorphism constraints in computing complete observables. However, in the end we have to find combinations of partial observables that are invariant under the (altered) integrated diffeomorphism constraints \( \hat{C}_{Da} (0) \) defined by

\[
\hat{C}_{Da} (0) = C_{Da} (0) - \sum_{K, k} (T^K (k), C_{Da} (0)) \hat{C}_K (k).
\]

(The sum does not include the values \( K, k = (Db, 0) \).) Note that also the altered integrated diffeomorphism constraints start at second order.

Since the integrated diffeomorphism constraints generated coordinate translations, functions of the form \( f(k) \) are invariant under the constraints \( C_{Da} (0) \). Here we denote by \( f(k) \) the Fourier transformation of a field variable \( f(\sigma) \). Therefore, such functions are a good point to start to look for functions which are invariant under the constraints (D.14).

Concerning the discussion of the gauge-invariant \( \tau \)-generators \( H_0 (\tau) = F_{(−P_0, \tau)} \hat{\tau} \) in section 7, the results do not change if we demand that \( f \) in equation (7.6) and \( P_0 \) are invariant (to the order in question) under the altered integrated diffeomorphism constraints \( \hat{C}_{Da} (0) \).

References

[1] Torre C G 1993 Gravitational observables and local symmetries Phys. Rev. D 48 2373 (Preprint gr-qc/9306030)
[2] Bruni M, Matarrese S, Mollerach S and Sonego S 1997 Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond Class. Quantum Grav. 14 2585–606 (Preprint gr-qc/9609040)
[3] Kuchar K V 1971 Canonical quantization of cylindrical gravitational waves Phys. Rev. D 4 955–86
[4] Torre C 1991 A complete set of observables for cylindrically symmetric gravitational fields Class. Quantum Grav. 8 1895–911
[5] Kuchar K V 1991 The problem of time in canonical quantization Conceptual Problems of Quantum Gravity ed A Ashtekar and J Stachel (Boston, MA: Birkhäuser) pp 141–71
[6] Rovelli C 1990 Quantum mechanics without time: a model Phys. Rev. D 42 2638
Rovelli C 1991 Time in quantum gravity: an hypothesis Phys. Rev. D 43 442
Rovelli C 1991 Is there incompatibility between the ways time is treated in general relativity and in standard quantum mechanics Conceptual Problems in Quantum Gravity ed A Ashtekar and J Stachel (Boston, MA: Birkhäuser) p 126
Rovelli C 1991 What is observable in classical and quantum gravity? Class. Quantum Grav. 8 1895
Rovelli C 2002 Partial observables Phys. Rev. D 65 124013
Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[7] Dittrich B 2006 Partial and complete observables for canonical general relativity Class. Quantum Grav. 23 6155 (Preprint gr-qc/0507106)
[8] Giddings S B, Marolf D and Hartle J B 2006 Observables in effective gravity Phys. Rev. D 74 064018 (Preprint hep-th/0512200)
[9] Dittrich B and Tambornino J 2007 A perturbative approach to Dirac observables and their spacetime algebra Class. Quantum Grav. 24 757 (Preprint gr-qc/0610040)
[10] Hellmann F, Mondragon M, Perez A and Rovelli C 2006 Multiple-event probability in general-relativistic quantum mechanics Preprint gr-qc/0610140
[34] Fischer A and Marsden J 1973 Linearization stability of the Einstein equations Bull. Am. Math. Soc. 79 997–1003
Moncrief V 1975 Spacetime symmetries and linearization stability of the Einstein equations: I J. Math. Phys. 16 493–8
Moncrief V 1976 Spacetime symmetries and linearization stability of the Einstein equations: II J. Math. Phys. 17 1893–902

[35] Kuchař K V and Torre C G 1991 Harmonic gauge in canonical gravity Phys. Rev. D 44 3116–23

[36] Smolin L 1993 Time, measurement and information loss in quantum cosmology College Park 1993, Directions in General Relativity vol 2, ed B L Hu (Cambridge: Cambridge University Press) (Preprint gr-qc/9301016)

[37] Thiemann T et al 2006 Solving the problem of time in general relativity and cosmology with phantoms and k-essence Preprint astro-ph/0607380

[38] Bojowald M, Hernández H I and Morales-Tecotl H A 2006 Perturbative degrees of freedom in loop quantum gravity: anisotropies Class. Quantum Grav. 23 3491–516 (Preprint gr-qc/0511058)

[39] Tambornino J 2007 A perturbative approach to Dirac observables in diffeomorphism invariant theories Diploma Thesis RWTH Aachen

[40] Dittrich B in preparation

[41] Kuchař K 1976 Dynamics of tensor fields in hyperspace: III J. Math. Phys. 17 801–20

[42] Bardeen J M 1980 Gauge invariant cosmological perturbations Phys. Rev. D 22 1882