Synchronization of Rössler Oscillators on Scale-free Topologies

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We study the synchronization of Rössler oscillators as prototype of chaotic systems, when they are coupled on scale-free complex networks. We find that the underlying topology crucially affects the global synchronization properties. Especially, we show that the existence of loops facilitates the synchronizability of the system, whereas Rössler oscillators do not synchronize on tree-like topologies beyond a certain size. By considering Cayley trees, modified by various shortcuts, we find that also the distribution of shortest path lengths between two oscillators plays an important role for the global synchronization.

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I. INTRODUCTION

Synchronization is an ubiquitous phenomenon in nature, ranging from flashing fireflies in the Australian forest [1], crickets chirping in unison [2] in natural systems, tremor in Parkinson’s disease or epilepsy in medical applications [3], laser arrays [4], or Josephson junctions in physics [5], electrochemical oscillators in chemistry [6] and designed synchronization in robotics. In particular, synchronization properties of limit-cycle oscillators were studied in a number of papers (for a review see [7]), but even systems which are individually chaotic like Rössler oscillators, can synchronize under certain conditions. Rössler oscillators are sometimes treated as prototype of chaotic systems. According to a conjecture of Calenbuhr and Mikhailov [8], the behavior of Rössler oscillators shows some universal features. For a certain class of interactions and under the influence of noise, clusters of synchronizing oscillators form above a certain threshold of the coupling strength between the oscillators, and, for larger couplings, after an intermittent phase, the whole set of oscillators synchronizes.

Rössler oscillators were studied for different interaction schemes and on different geometries [10]. In this paper we study the conditions for Rössler oscillators to synchronize on scale-free network topologies. Scale-free networks seem to be realized in a number of natural and artificial systems like genetic or proteomic networks, the world-wide-web and the internet. Synchronization is certainly one of the important dynamical processes, running on these networks, as it is supposed to be a necessary ingredient for the efficient organization and functioning of coupled individual units, that, after all, lead to well coordinated behavior in time. Therefore we are interested in the compatibility of scale-free topologies with synchronization, in particular for the case that the individual dynamics is chaotic. While usually the synchronization transition is studied as a function of the coupling strength or the system size, we describe here, in addition to the usual approach, a transition to the synchronized phase as a function of the topology. As we shall see, when a tree becomes too large in size to allow for synchronization, synchronization becomes possible again beyond a critical threshold in the coupling, when a critical number of loops and shortcuts is introduced into the tree, while it is impossible for arbitrary couplings on a tree beyond a certain size.

In section I we introduce the model and define the order parameters that are used to distinguish the phases with and without the condensates of synchronized oscillators. In the second section we describe details of the simulations and summarize the results in section III.

II. THE MODEL

We consider a system of \( N \) Rössler oscillators, distributed on the nodes of a scale-free network, generated with the growth algorithm of Bárabasi and Albert [12] (see below). Each individual oscillator is described by the following set of dynamical equations that was originally proposed by Rössler [13] as a ”model of a model” for describing the trajectory of flow, satisfying the Lorenz equation [14],

\[
\begin{align*}
\dot{x}_i & = -\omega y_i - z_i \\
\dot{y}_i & = \omega x_i + ay_i \\
\dot{z}_i & = b - cz_i + x_i z_i .
\end{align*}
\]

For \( \omega = 1, a = 0.15, b = 0.2, c = 8.5 \) the system is in the chaotic state. Among various possibilities of coupling these oscillators, we choose

\[
\begin{align*}
\dot{x}_i & = -y_i - z_i \\
\dot{y}_i & = x_i + ay_i + \epsilon (\bar{y}_i - y_i) \\
\dot{z}_i & = b + (x_i - c)z_i + \epsilon (\bar{z}_i - x_i z_i) ,
\end{align*}
\]

where \( \bar{x}_i, \bar{y}_i, \bar{z}_i \) are averages defined as

\[
\bar{x}_i = \frac{1}{k_i} \sum_{j=0}^{N} A_{ij} x_j
\]
and accordingly for $\bar{y}_i$ and $\bar{z}_i$, $k_i$ denotes the degree of node $i$, $A_{ij}$ is the adjacency matrix i.e.$A_{ij} = 1$ if $i$ and $j$ are connected and 0 otherwise, that is the only place where the topology of the network enters. For $k_i = N - 1$ and $A_{ij} = 1$ for all $i,j \in \{1,...,N\}$ the system corresponds to a globally coupled population of Rössler oscillators as it was considered in [3]. In our description the population is partially coupled rather than globally. It is coupled along the links of the scale-free network, therefore the driving force towards the common synchronized state is produced by nearest-neighbors, whose number varies according to the scale-free degree-distribution. Since our averages are still node-dependent, the stability analysis of [3], derived for $x = \frac{1}{N} \sum_{j=0}^{N} x_j$, ($\bar{y}, \bar{z}$ alike,) does not immediately apply. For this case of global coupling with driving force that tries to reduce the difference from the common synchronized state ($\bar{x}, \bar{y}, \bar{z}$), one expects a globally synchronized stable state for $\epsilon = 1, a < 1$, so that all deviations from global averages exponentially decrease with time [3]. In our scheme the force drives to node-dependent average values over nearest neighbors whose number is neither regular nor $N - 1$, i.e. all-to-all. Nevertheless we find a result quite similar to the all-to-all case: a global attractor to a synchronized state as long as $\epsilon < 1.25$. The stability is evident on the level of numerical simulations.

In order to check how the results depend on the nonlinear terms of our coupling scheme, we made also some tests for the linear vector coupling defined according to

$$
\begin{align*}
\dot{x}_i &= -y_i - z_i + \epsilon (\bar{x}_i - x_i) \\
\dot{y}_i &= x_i + ay_i + \epsilon (\bar{y}_i - y_i) \\
\dot{z}_i &= b + (x_i - c)z_i + \epsilon (\bar{z}_i - z_i),
\end{align*}
$$

(4)
i = 1,...,N, as it was used in [11].

### A. Choice of order parameters

As a first indicator for a partially or fully synchronized state, we measure the histogram of instantaneous pair distances $d_{ij}(t)$ between all pairs of nodes as a function of (simulation) time, defined by [11]

$$
d_{ij} = \left[ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right]^{1/2},
$$

(5)
i,j = 1,...,N. A fully synchronized state shows up as a sharp peak in the distribution of $d_{ij}$, since the pair distances between any two nodes approach zero. No synchronization or desynchronization in the opposite case, lead to a broad distribution. As order parameters in the usual sense (varying between 0 and 1), 0 for the desynchronized phase and 1 for the fully synchronized phase, we choose two order parameters $r$ and $s$, as proposed in [11], defined in the following way.

$$
r(t) = \frac{1}{N(N - 1)} \sum_{j \neq i,i,j = 1}^{N} \Theta(\delta - d_{ij}(t)),
$$

(6)
and

$$
s(t) = 1 - \frac{1}{N} \prod_{i=1}^{N} \prod_{j=1,j \neq i}^{N} \Theta(d_{ij}(t) - \delta),
$$

(7)
where $\Theta(x)$ is the Heavyside function, i.e. $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise. The parameter $\delta$ is a small number to account for the finite numerical accuracy, e.g. $\delta = 0.0001$, so that two states in phase space lying inside a sphere of radius $\delta$ are considered as mutually being synchronized. The parameter $r(t)$ gives the fraction of pairs of elements $(i,j)$ which are synchronized at time $t$ (i.e., $d_{ij} \leq \delta$). This fraction is one if all possible pairs are synchronized and zero if no pair is synchronized, intermediate values $0 < r < 1$ reflect partial synchronization. The second order parameter $s(t)$ is more sensitive to partial synchronization. The second term on the r.h.s. of Eq.4 only contributes to the fraction if node $i$ has no other node within a distance of $\delta$. Therefore $s$ is already 1 when the total number of states is partitionized in synchronized pairs without synchronization between the pairs. In general we have $r < s < 1$ (as it is confirmed in the figures below) if some elements form clusters while others are still isolated. From a simultaneous measurement of $r$ and $s$ it is possible to obtain some information about the partial synchronization that is usually a precursor to the fully synchronized state. We measured in general all three functions $d_{ij}$, $r$, and $s$ as a function of the number of iterations.

### III. Measurements and results

#### A. Generating the topology

For the scale-free topology we used the growth algorithm of Bárabasi and Albert [12], later referred to as the BA model. In each step, one node with $m$ edges is added to the network. It is connected to $m$ of the formerly generated nodes according to preferential attachment. In our simulations we chose $m$ between 1 and 10. For testing the role of the loops we used the regular topology of a Cayley tree with $z$ edges at each node, $z = 3,...,6$. The tree structure was then modified in various ways as we shall see below. We also made some runs on a small-world topology, starting from a regular ring topology with $k = 2$ neighbors and randomly adding shortcuts to each node with probability $p = 0.01$ according to the algorithm proposed by Newman and Watts [15].

#### B. Choice of parameters

For the parameters of the individual Rössler oscillators we chose $a = 0.15$, $b = 0.20$, $c = 8.5$ throughout all simulations to make sure that the individual systems are in the chaotic regime. The total number $N$ of oscillators
was varied between 10, 50, 200 up to 500 on the scale-free topology, and \( N = 190 \) on the Cayley tree. The parameter \( m \) of the growth algorithm varied between 1 and 10, the coupling strength \( \epsilon \) was out of the interval \([0.1, 1.2]\). In the numerical simulations of Eq.\( \ref{eq:2} \) we used the fourth order Runge-Kutta methods with a typical time-step size of \( dt = 0.001 \) (when \( N = 200 \)). Variation of \( dt \) between \( 10^{-12} \leq dt \leq 10^{-1} \) led to qualitatively the same results.

C. Results for \( m = 1 \)

Fig.1 a) displays the results for the histogram of distances for \( m = 1 \), \( N = 200 \) and various couplings \( \epsilon \) up to 1.25, above which the numerical integration becomes unstable. The distributions are broad and do not indicate any synchronized state. This result is further supported by measurements of \( r \) and \( s \) as shown in Fig. 2 a) and b), respectively, \( r \) stays zero for \( m = 1 \), while \( s \) increases from \( \epsilon = 0.1 \) on, indicating some partial synchronization. The value of \( N = 200 \) seems to represent the large-\( N \) limit, for the considered range of \( \epsilon \) and \( m \), since we obtained the same result for \( r \) and \( s \) for \( N = 300, 400, 500 \). On the other hand, for smaller systems, \( N < 20 \), we do see a fully synchronized state when the coupling \( \epsilon \) exceeds a critical threshold. As value for \( \delta \) in Eq.\( \ref{eq:6} \) that accounts for the finite numerical accuracy, we choose \( \delta = 0.0001 \). For too small values of \( \delta \) we observe large variations in the long-time behavior of \( r \) and \( s \), for too large \( \delta \), the values of \( r \) and \( s \) are stable over long times, but their values do depend on \( \delta \). In between, i.e. for \( 0.0001 < \delta < 0.1 \) we find a plateau for the values of \( r \) and \( s \), that is, the behavior becomes independent of the size of \( \delta \).

D. Results for \( m \) larger than one but still integer

If we keep the number \( N \) of oscillators fixed to 200, we observe for \( m > 1 \) a fully synchronized state above a critical threshold in the coupling \( \epsilon \); this threshold is the larger the smaller \( m \), again \( r < s \) in general, as seen from Fig.2 a) and b).

E. Results for intermediate noninteger \( m \)

One of the main differences between the Bárabasi-Albert networks with parameter \( m = 1 \) and \( m > 1 \) is the tree-like structure for \( m = 1 \) and the existence of loops for \( m > 1 \). In order to check whether it is only the loops that facilitate synchronization and how many loops are needed, we generalized the growth algorithm to non-integer values of \( m \) in the following way. We introduce an additional probability \( p_m \) for a new node to have...
we have characterized via the parameter synchronized state as a function of "topology", parameterized via the parameter $m$ for two values of the coupling strength $r$.

![FIG. 3: Histogram of pair distances on a BA-model with $1 < \langle m \rangle < 2$ with $N = 200$ and $\epsilon = 0.9$](image)

![FIG. 4: Order parameters $r$ (a) and $s$ (b) as function of the parameter $m$ that is used to distinguish different topologies, for two values of the coupling strength](image)

$m = 1$ edges and $1 - p_m$ for having $m = 2$ edges attached to the nodes of the network when it is introduced during the growth process. The distribution of pair distances of oscillators for $1 < \langle m \rangle < 2$ is displayed in Fig. 4. For given $N$ and $\epsilon$ we therefore observe a transition to a fully synchronized state as a function of "topology", parameterized via the parameter $m$. For $N = 200$ and $\epsilon = 0.9$, we have $1.35 \leq m_c \leq 1.4$, cf. Fig 4. Fig 4(a) shows that the position of the transition, now in $m$ rather than in $\epsilon$, depends on the coupling strength for fixed $N$. The smaller $\epsilon$, the larger $m_c$. An interesting feature is seen in Fig. 4(b), where $s$ is plotted as a function of $m$. For $\epsilon = 0.3$ and $m = 1$, the finite value of $s$ indicates some partial synchronization, $s$ then drops to zero at $m = 2$ and increases to 1 for $m = 3$. As we have argued above, $s = 1$ does not necessarily imply full synchronization, but some partial one, at least. The behavior of $s$ is non-monotonic as a function of $m$. A similar non-monotonic behavior of $s$ as function of time was observed in [9] for Rössler oscillators, for which a partial synchronization was followed by desynchronization, before the full synchronization set in.

**F. Rössler oscillators on a Cayley tree**

From the former results we conclude that a certain number of loops facilitates synchronization on scale-free networks, the larger $m$, the more loops [16], the smaller the coupling strength needed for synchronization. In order to check whether it is the mere number of loops that facilitate synchronization or also the type of loops, we studied $N$ Rössler oscillators on a Cayley tree whose regular structure was modified in a controlled way by adding a) edges to construct a given number of triangles at random locations, b) shortcuts between the outermost nodes, c) shortcuts between the outermost nodes and the central node with probability $p_1$, see Fig 5. For $N \geq 187$ and arbitrary values of $\epsilon$ (more precisely, for $\epsilon$ as large as $\epsilon = 0.9$), Rössler oscillators do not synchronize on a Cayley tree, neither for shortcuts according to a) or b), but for method c) and $p_1 > 0.9$ they do so, as it is seen in Fig 5. As it turns out also from measurements on a small-world topology with $p = 0.01$, synchronization in all of our measurements goes along with a distribution of shortest path lengths that looks Poissonian like, cf. Fig 5(a). On the other hand, it is not the average shortest path length that is conclusive alone. As Fig. 5 shows, the average is almost the same in a) and b) while the distribution is different and those of Fig 5(a) correspond to desynchronization on a Cayley tree without or with shortcuts between the outermost nodes (apart from $p_1 = 0.99$), while those of Fig 5(b) lead to synchronization apart from the BA model with $m = 1$, since in this case the loops are totally absent. This shows that neither the existence of loops alone nor the existence of a certain dis-
FIG. 6: Distance distributions for Rössler oscillators on a Cayley tree with additional edges according to c) with different interconnection probabilities $p_1$, as further explained in the text

FIG. 7: Histograms of shortest paths of length $\ell$ for various networks with Rössler oscillators. The envelope of the distributions looks similar for a) with $p = 0.99$ and b) for $m = 1, m = 2$ and on the small world network

IV. SUMMARY AND CONCLUSIONS

We studied an ensemble of Rössler oscillators on sale-free networks constructed by the Bárábasi-Albert growth algorithm. In contrast to the usual investigations we studied the transition from the desynchronized or partially synchronized state to the fully synchronized state as a function of the network topology, parameterized by the $m$, the number of newly attached edges in the growth algorithm. For the tree topology ($m = 1$) and given coupling strength $\epsilon$, there is a fully synchronized state below some critical size $N$ that disappears for larger $N$. This result is similar to synchronization of Kuramoto oscillators on Cayley trees which is possible for small enough size $N$ and coordination number $z$ [17]. Above a certain number of nodes, the tree of Rössler oscillators is no longer synchronizable however large the coupling strength is, but it is then the parameter $m$ that introduces loops and shortcuts into the tree, and along with this allows for full synchronization again when $m$ exceeds a certain value that depends on $N$ and $\epsilon$. The threshold in $\epsilon$ depends on $N$ and $m$, and vice versa, the threshold in $m$ is sensitive to $N$ and $\epsilon$. Small $N$, large $\epsilon$ (chosen from the stability regime) and large $m$ favor synchronization. These qualitative results are not specific for our choice of nonlinear couplings between the Rössler oscillators, but also hold for the vector coupling scheme of Eq[4]. Moreover, numerical simulations of Rössler oscillators on Cayley trees with different artificially introduced shortcuts suggest that it is not only the mere number of loops that favors synchronization, but loops that provide real shortcuts such as those between outermost and central nodes.

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