Symmetric polyomino tilings, tribones, ideals, and Gröbner bases

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Abstract

We apply the theory of Gröbner bases to the study of signed, symmetric polyomino tilings of planar domains. Complementing the results of Conway and Lagarias we show that the triangular regions $T_N = T_{3k-1}$ and $T_N = T_{3k}$ in a hexagonal lattice admit a signed tiling by three-in-line polyominoes (tribones) symmetric with respect to the 120° rotation of the triangle if and only if either $N = 27r − 1$ or $N = 27r$ for some integer $r ≥ 0$.

1 Introduction and a summary of main results

Our general objective is to explore signed polyomino tilings which are symmetric with respect to a group of symmetries by the methods of standard (Gröbner) bases of polynomial ideals.

The tiling depicted in Figure 1 illustrating the case $N = 8$ of Theorem 1 shows that a triangular region in a hexagonal lattice may have a signed tiling by congruent copies of the three-in-line tile (tribone). In the same paper Conway and Lagarias showed ([5, Theorem 1.2.]) that neither this nor any other triangular region in the hexagonal lattice can be tiled by tribones (if ‘negative’ tiles are not permitted).

A very nice exposition of these and related results can be found in [13 and 8, Chapter 23].

Theorem 1. (Conway-Lagarias [5, Theorem 11.4]) The triangular region $T_N$ in the hexagonal lattice has a signed tiling by congruent copies by three-in-line tiles (tribones) if and only if $N = 9r$ or $N = 9r + 8$ for some integer $r ≥ 0$.

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Our main results (Theorems 20 and 25) say that the triangular regions $T_N = T_{3k-1}$ and $T_N = T_{3k}$ in a hexagonal lattice admit a tiling by tribones symmetric with respect to the rotation of the triangle through the angle of 120° degrees if and only if either $N = 27r - 1$ or $N = 27r$ for some integer $r \geq 0$. In particular the triangle depicted in Figure 1 does not admit such a tiling.

The method applied for the proof of this theorem is based on the observation that the tile homology group introduced by Conway and Lagarias in [5] (see also Reid [11, Section 2]) is naturally a module over the group ring of the associated group of translations. This group ring is a quotient of a polynomial ring which allows us to reduce the tiling problem to the ‘submodule membership problem’ and apply the theory of Gröbner bases.

Surprisingly enough there are very few applications of the algebraic method based on Gröbner basis to problems of tilings and tessellations and the only reference we are aware of is the paper by Bodini and Nouvel [4]. The fact that the ‘tile homology group’ in the sense of [11] is a module over a polynomial ring offers some obvious technical advantages. One of our objectives is to advertise this approach in the context of signed tilings with symmetries. These problems seem to be particularly well adopted to the algebraic method in light of the fruitful relationship between the theory of Gröbner bases and the theory of invariants of group actions [6], [12].

2 Generalities about lattice tilings

There are three regular lattice tilings of $\mathbb{R}^2$, the triangular lattice $L_\Delta$, square lattice $L_\square$, and the hexagonal lattice $L_{\text{hex}}$, depicted in Figure 2. If $L$ is one of these lattice tilings the associated dual lattice (point set) $L^\circ$ is generated by all barycenters of the elementary cells of $L$.

Let $A(L)$ be the free abelian group generated by all elementary cells of the lattice $L$. A ‘lattice tile’ $P$ (informally a lattice figure in $L$), defined as a finite collection

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*In the remaining case $N = 3k + 1$ there is a hexagon in $T_N$ fixed by the 120°-degrees rotation.
The problem if a given bounded lattice region (lattice figure) $R$ admits a signed tiling with translates of prototiles $\mathcal{R} = \{R_1, \ldots, R_k\}$ is an instance of the subgroup membership problem. Indeed, let $B(\mathcal{R})$ be the subgroup of $A(L)$ generated by all translates of prototiles $R_i$ and let $H(\mathcal{R}) = A(L)/B(\mathcal{R})$ be the associated ‘homology group’. Then (following [4] and [11]) such a tiling exists if and only if $R \in B(\mathcal{R})$ or equivalently if the coset $R + B(\mathcal{R})$ is the zero element in $H(\mathcal{R})$.

Let $G = G(L)$ be the group of all affine transformations that keep the lattice tiling $L$ invariant. Let $\Gamma = \Gamma(L)$ be its subgroup of all translations with this property. By selecting $0 \in L^\circ$ as the zero element, $L^\circ$ is turned into a group and there is a natural identification $\Gamma = L^\circ$.

The group $A(L)$ is a clearly a module over the group ring $\mathbb{Z}[\Gamma]$ (which is isomorphic to the ring $\mathbb{Z}[\mathbb{Z}^2]$ of Laurent polynomials in two variables. This ring can be obtained (in many ways) as a quotient of the semigroup ring $\mathbb{Z}[\mathbb{N}^d] \cong \mathbb{Z}[x_1, \ldots, x_d]$ (for some $d$).

This observation allows us to see the groups $A(L), B(\mathcal{R})$ and $H(\mathcal{R})$ as modules over the polynomial ring $\mathbb{Z}[x_1, \ldots, x_d]$ and to reduce the tiling question to the submodule membership problem [7, Chapter 5]. In turn, in the spirit of [4], one can use the ideas and methods of Gröbner basis theory.

Here we put some emphasis on the use of the ‘submodule membership problem’ as a natural extension of the ‘ideal membership problem’, originally proposed and used by Bodini and Nouvel in [4]. This appears to be a more natural and conceptual approach to the general tiling problems since the module $A(L)$ is no longer required to be monogenic (cyclic over $\mathbb{Z}[P]$) which allows us a greater freedom in choosing the semigroup ring $\mathbb{Z}[P]$. This property will be indispensable in the study of tilings symmetric with respect to a group of symmetries which is the main goal of this paper.

2.1 An example

The reader may find the following example, depicted in Figure 3, as a good illustration of the main problem studied in our paper.

The $(3 \times 3)$ checkerboard $C_{3\times3}$ (Figure 3) is supposed to be paved by translates of two types of prototiles. Each of the cells (elementary squares) is labelled (coordinatized) by a pair $(i, j) \in \mathbb{N}^2$ of integers and each tile (polyomino) is formally a union of a finite
number of elementary cells. In the example depicted in Figure 3 there are two types of prototiles, \( T_1 = \{(0,0), (1,0), (0,1)\} \) and \( T_2 = \{(1,1), (1,0), (0,1)\} \).

Figure 3: A signed tiling of a 3 \( \times \) 3 square symmetric w.r.t. the main diagonal.

The tiling depicted in Figure 3 satisfies the following conditions:

1. The (3 \( \times \) 3) chessboard \( C_{3 \times 3} \) is covered by 3 translated copies of prototile \( T_1 \) and 2 translated copies of prototile \( T_2 \). In general the translated prototiles are not required to be subsets of \( C_{3 \times 3} \).

2. Each (translated) prototile is associated a weight (sign) and the total weight of each of the cells \( (i, j) \in C_{3 \times 3} \) is equal to 1. (The condition that the total weight of cells outside \( C_{3 \times 3} \) is zero is added if prototiles are not necessarily subsets of \( C_{3 \times 3} \).)

3. The tiling is symmetric with respect to the main diagonal of \( C_{3 \times 3} \) in the sense that if a translated prototile \( T + v \) (where \( T \in \{T_1, T_2\} \) and \( v \in \mathbb{N}^2 \)) appears in the decomposition (tiling) with a weight \( w \in \mathbb{Z} \) then the diagonally symmetric prototile \( T' + v' \) is also present with the same weight.

A basic observation is that each polyomino \( P \subset \mathbb{N}^2 \) can be associated a polynomial \( f_P = \sum \{x^iy^j \mid (i, j) \in P\} \), for example \( f_{T_1} = 1 + x + y \) and \( f_{T_2} = x + y + xy \). The decomposition depicted in Figure 3 naturally corresponds to the following decomposition of polynomials in the ring \( \mathbb{Z}[x, y] \) (or in the ring \( \mathbb{Z}[\sigma_1, \sigma_2] \) of symmetric polynomials):

\[
f_{T_1} + [xf_{T_1} + yf_{T_1}] - f_{T_2} + xyf_{T_2} = (1 + x + x^2)(1 + y + y^2) = f_{C_{3 \times 3}}.
\]

Our immediate objective is to use the theory of Gröbner bases to generate such identities. More generally we want to develop and study procedures and algorithms for the systematic analysis (existence and other properties) of decompositions similar to (1).
2.2 Basic facts about polyomino tilings

Informally a polyomino pattern $P$ (or polyomino for short) is a (not necessarily connected) finite region consisting of cells in one of the three regular lattice tilings of the plane (Figure 2). It is sometimes more convenient to describe a polyomino as a collection $P = \{c_1, \ldots, c_k\}$ of elementary cells in the associated lattice $L$. Some authors use the generic name *polyforms* for all three types of polyominos so the $L\square$-polyforms are polyominos in the usual sense [9]; $L\hex$-polyforms are referred to as *polyhexes* etc.

We frequently use a slightly more general (algebraic) definition of a polyomino as a *multiset*, subset of $L$, with multiple and possibly with negative elements. We will tacitly make a distinction between the geometric and algebraic definition by reserving the term ‘weighted polyomino’ for the algebraic version. However most of the time the term ‘polyomino’ is used interchangeably for both kinds of polyomino patterns.

**Definition 2.** A (weighted) polyomino $P$ is a finite weighted subset of $L$ (a multiset) which contains each elementary cell $c \in L$ with some (positive or negative) multiplicity $w_c \in \mathbb{Z}$. In other words $P = \sum w_c c$ is an element of the free abelian group $A(L)$ generated by all cells of the lattice tiling $L$.

2.3 Geometric-algebraic dictionary of polyomino tilings

We have already seen in Section 2.1 an example of the correspondence between a geometric image (Figure 3) and an algebraic expression (equation (1)), based on the correspondence $(i, j) \leftrightarrow x^i y^j$ between the cell labelled by $(i, j) \in \mathbb{N}^2$ and the associated monomial $x^i y^j$.

More generally let $S \cong \mathbb{N}^d$ be a semigroup which acts on the lattice $L$ by translations, which means that there exists a homomorphism $\rho : S \rightarrow \Gamma$ from $S$ to the group $\Gamma = \Gamma(L)$ of all translations that keep the lattice tiling $L$ invariant.

The group $A(L)$ is naturally a module over the semigroup ring $\mathbb{Z}[S] \cong \mathbb{Z}[x_1, \ldots, x_d]$. For example if $S = \Gamma$ then $\mathbb{Z}[S] \cong \mathbb{Z}[\Gamma]$ and $A(L)$ is a $\mathbb{Z}[\Gamma]$-module where $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[x, x^{-1}; y, y^{-1}]$ is the ring of Laurent polynomials.

Let $\mathcal{R} = \{P_1, \ldots, P_k\}$ be a collection of basic tiles (prototiles). Define $B(\mathcal{R})$ as the subgroup of $A(L)$ generated by all translates of the prototiles $P_i$, or equivalently as a $\mathbb{Z}[S]$-submodule of $A(L)$ generated by $\mathcal{R}$.

The following tautological proposition links the idea of the tile homology group of Conway and Lagarias [3] and Reid [11] with the ‘submodule membership problem’ typical for applications of Gröbner bases (as proposed by Bodini and Nouvel [4]).

**Proposition 3.** A polyomino $P$ has a signed tiling by translates of prototiles $\mathcal{R} = \{P_1, \ldots, P_k\}$ if and only if $P \in B(\mathcal{R})$ where $B(\mathcal{R})$ is the $\mathbb{Z}[S]$-submodule of $A(L)$ generated by $\mathcal{R}$. The associated class $[P]$ in the tile homology module

$$H(\mathcal{R}) := A(L)/B(\mathcal{R})$$

is a ‘quantitative measure’ of how far is $P$ from admitting a tiling by $\mathcal{R}$. 

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A modified version of Proposition 3 applies to proper subsets of the lattice tiling $L$. The following proposition serves as an illustration of the simplest case where we restrict our attention to the first quadrant of the $L\square$-lattice tiling. In this case $S = \mathbb{N}^2$ and $\mathbb{Z}[S] = \mathbb{Z}[x, y]$. As in Section 2.1 each polyomino $P \subset \mathbb{N}^2$ is associated a polynomial $f_P = \sum \{x^i y^j \mid (i, j) \in P\}$.

**Proposition 4.** A polyomino pattern $P \subset \mathbb{N}^2$ admits a signed tiling by the first quadrant translates of polyomino patterns $P_1, \ldots, P_k$ if and only if,

$$f_P = h_1 f_{P_1} + \ldots + h_k f_{P_k}$$

for some polynomials $h_1, \ldots, h_k$ with arbitrary integer coefficients or equivalently if,

$$f_P \in \langle f_{P_1}, \ldots, f_{P_k} \rangle.$$

### 2.4 Equivariant polyomino tilings

The group $G = G(L)$ was introduced in Section 2 as the group of all affine transformations that keep the lattice tiling $L$ invariant. The abelian group $A(L)$ is a module over the group ring $\mathbb{Z}[G]$. Since $\Gamma \subset G$ is a normal subgroup, we observe that $G$ acts on $A(L)$ preserving its $\mathbb{Z}[\Gamma]$-module structure as well, provided $G$ acts on the ‘scalars’ from $\mathbb{Z}[\Gamma]$ by conjugation.

Let $Q \subset G$ be a (finite) subgroup of $G$. Assume that the set $\mathcal{R}$ of prototiles is invariant with respect to the group $Q$. Then $Q$ acts on the submodule $B(\mathcal{R})$ and the tile homology module (2). Again, one shouldn’t forget that the action of $Q$ on scalars from $\mathbb{Z}[\Gamma]$ may be non-trivial. Define $B(\mathcal{R})^Q = \text{Hom}_Q(\mathbb{Z}, B(\mathcal{R}))$ as the subgroup (submodule) of $B(\mathcal{R})$ of elements which are invariant under the action of $Q$.

If we restrict our attention to the subring $\mathbb{Z}[\Gamma]^Q \subset \mathbb{Z}[\Gamma]$ of $Q$-invariant elements then the action of $Q$ on scalars from $\mathbb{Z}[\Gamma]^Q$ is trivial and the $\mathbb{Z}[\Gamma]^Q$-module $A(L)$ is a $Q$-module in the usual sense.

An element of the group $A(L)^Q$ is referred to as an *equivariant signed polyomino*. The fundamental problem is to decide when a given polyomino $P \in A(L)^Q$ admits a $Q$-symmetric signed tiling by translates of a $Q$-invariant family of prototiles $\mathcal{R}$. The following criterion is an equivariant analogue of Proposition 3.

**Proposition 5.** Let $\mathcal{R}$ be a $Q$-invariant (finite) set of prototiles. A $Q$-invariant polyomino $P \in A(L)^Q$ has an equivariant, signed tiling by translates of prototiles $\mathcal{R}$ if and only if $P \in B(\mathcal{R})^Q$ where $B(\mathcal{R})$ is the $\mathbb{Z}[S]^Q$-submodule of $A(L)$ generated by $\mathcal{R}$.

The setting of Proposition 5 is exactly the same as before (Proposition 3), however the emphasis is now on the $\mathbb{Z}[\Gamma]^Q$-module structure on $A(L)^Q$ and $B(\mathcal{R})^Q$. In order to apply this criterion one is supposed to determine the ring of invariants $\mathbb{Z}[S]^Q$ and the structure of the module $B(\mathcal{R})^Q$. Both goals can be achieved with the aid of the theory of Gröbner bases, see [6, Section 7] for necessary tools.
3 Hexagonal polyomino with symmetries

3.1 Lattices and semigroup rings

Let $G_{\text{hex}}$ be the group of symmetries of the hexagonal tiling $L_{\text{hex}}$ of the plane depicted in Figure 4. Our objective is to study $L_{\text{hex}}$-tiling problems which are symmetric with respect to some (finite) subgroup of $G_{\text{hex}}$. Our initial focus is on subgroups which act without fixed points (invariant hexagons) so let $S_6$ be the group of all elements in $G_{\text{hex}}$ which keep the vertex $o$ fixed, and let $\mathbb{Z}_3$ be its subgroup generated by the 120°-rotation.

The group $G_{\text{hex}}$ has a free abelian subgroup $D = \Gamma(L_{\text{hex}}) \cong \mathbb{Z}^2$ of rank 2 which is generated by three translations (vectors) $t_x, t_y, t_z$ satisfying the condition $t_x + t_y + t_z = 0$. The associated group ring $P = \mathbb{Z}[D]$ is isomorphic to the ring $\mathbb{Z}[x, y; x^{-1}, y^{-1}]$ of Laurent polynomials in two variables. For our purposes a more convenient representation is $P = \mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle$ (Figure 4) where variables $x, y, z$ correspond to vectors $t_x = \vec{bc}, t_y = \vec{ca}, t_z = \vec{ab}$.

![Figure 4: The hexagonal tiling group $A_{\text{hex}}$ as a module over $P = \mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle$.](image)

Let $A_{\text{hex}}$ be the (infinite dimensional) free abelian group generated by all elementary hexagonal cells of the lattice $L_{\text{hex}}$. The group $A_{\text{hex}}$ is a finitely generated module over the ring $P$, indeed it is generated by the three neighboring cells $a, b, c$ with common vertex $O$, depicted in Figure 4.

For added clarity from here on the lattice $L_{\text{hex}}$ is represented by its dual lattice $L_{\text{hex}}^\circ$ of barycenters of all hexagons (the black dots in Figure 5). Consequently the ‘lattice’ $L_{\text{hex}}^\circ$ is a geometric object (a periodic set of points).

The lattice (discrete subgroup of $\mathbb{Z}^2$) $D$, generated by vectors $t_x, t_y, t_z$ is in this figure represented by white dots. The fact that $A_{\text{hex}}$ is a module over $P$ is simply a reformulation of the fact that the lattice of white dots acts on the set of black dots.

The lattice $D$ is sometimes (Section 3.4) referred to as the $xyz$-lattice. The lattice (discrete group) $E$ generated by black dots is referred to as the $abc$-lattice since it is
generated by vectors $t_a, t_b, t_c$, where $t_a + t_b + t_c = 0$ (Figure 5). The group ring of $E$ is $Q = \mathbb{Z}[a, b, c]/\langle abc - 1 \rangle$.

Note that $E$ has three types of points (Figure 5 on the right) which reflects the fact that the ‘white dot lattice’ $D$ is a sublattice of $E$ of index 3.

### 3.2 The actions of $S_3$ and $\mathbb{Z}_3$ on $\mathbb{Z}[x, y, z]$

Here we collect some basic facts about the symmetric group $S_3$ and the cyclic group $\mathbb{Z}_3$ actions on $\mathbb{Z}[x, y, z]$ induced by the permutations of variables $x, y, z$. As usual for a given $G$-module $M$ the associated submodule of $G$-invariant elements is $M^G$.

Elementary symmetric polynomials are $\sigma_1 = x + y + z, \sigma_2 = xy + yz + zx, \sigma_3 = xyz$.

The $S_3$-invariant polynomials in $\mathbb{Z}[x, y, z]$ which form a $\mathbb{Z}$-basis are $\sigma_3^p = x^p y^p z^p$ (where $p \geq 0$), $\Delta(x^p y^q z^r) = x^p y^q z^r + y^p z^r x^q + z^p x^q y^r$ (for $p \neq q$), and for $p \neq q \neq r \neq p$,

$$H(x^p y^q z^r) = x^p y^q z^r + y^p z^r x^q + z^p x^q y^r + x^p z^q y^r + z^p y^q x^r.$$  \hspace{1cm} (3)

Basic $\mathbb{Z}_3$-invariant polynomials in $\mathbb{Z}[x, y, z]$ are,

$$x^p y^p z^p \quad \text{(where $p \geq 0$)} \quad \text{and} \quad \Delta(x^p y^q z^r) = x^p y^q z^r + y^p z^r x^q + z^p x^q y^r, \hspace{1cm} (4)$$

where $(p, q, r) \neq (p, p, p)$. There is an involution $I$ on the set $\mathbb{Z}[x, y, z]^{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials defined by $I(p(x, y, z)) = p(y, x, z)$. The map $\alpha : \mathbb{Z}[x, y, z]^{S_3} \rightarrow \mathbb{Z}[x, y, z]^{\mathbb{Z}_3}$ is a monomorphism and the image $\text{Im}(\alpha)$ is the fixed point set of the involution $I$.

More explicitly, $\alpha(x^p y^p z^p) = x^p y^p z^p$,

$$\alpha(\Delta(x^p y^q z^r)) = \Delta(x^p y^q z^r) \quad \text{and} \quad \alpha(H(x^p y^q z^r)) = \Delta(x^p y^q z^r) + I(\Delta(x^p y^q z^r)). \hspace{1cm} (5)$$

From here we deduce the following proposition.
Proposition 6. There is a commutative diagram

\[
\begin{array}{c}
0 & \longrightarrow & \langle xyz - 1 \rangle^{S_3} & \longrightarrow & \mathbb{Z}[x, y, z]^{S_3} & \longrightarrow & \mathbb{Z}[\sigma_1, \sigma_2] & \longrightarrow & 0 \\
\downarrow{\alpha'} & & \downarrow{\alpha} & & \downarrow{\alpha''} & & \\
0 & \longrightarrow & \langle xyz - 1 \rangle^{Z_3} & \longrightarrow & \mathbb{Z}[x, y, z]^{Z_3} & \longrightarrow & (\mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle)^{Z_3} & \longrightarrow & 0
\end{array}
\]

where \( \langle xyz - 1 \rangle \subset \mathbb{Z}[x, y, z] \) is the principal ideal generated by \( xyz - 1 \), with the split horizontal exact sequences and injective vertical homomorphisms \( \alpha, \alpha' \) and \( \alpha'' \).

\textbf{Proof:} Since \( \mathbb{Z}[x, y, z]^{S_3} = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3] \) and \( \langle xyz - 1 \rangle^{S_3} = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3](\sigma_3 - 1) \), the exactness of the first row in (6) is an immediate consequence.

More explicitly the description of \( S_3 \)-invariant and \( Z_3 \)-invariant polynomials in \( \mathbb{Z}[x, y, z] \) (3 and 4) allows to describe in a similar fashion invariant polynomials in the ideal (submodule) \( \langle xyz - 1 \rangle = \langle \sigma_3 - 1 \rangle \). For example the basic \( S_3 \)-invariant polynomials in \( \langle \sigma_3 - 1 \rangle^{S_3} \) are,

\[
\sigma_3^p(\sigma_3 - 1), \quad \Delta(x^py^pz^r)(\sigma_3 - 1), \quad H(x^py^pz^r)(\sigma_3 - 1).
\]

Similarly, the basic \( Z_3 \)-invariant polynomials in \( \langle \sigma_3 - 1 \rangle^{Z_3} \) are

\[
\sigma_3^p(\sigma_3 - 1) \quad \text{and} \quad \Delta(x^py^pz^r)(\sigma_3 - 1).
\]

There is an exact sequence of \( \mathbb{Z}[Z_3] \)-modules,

\[
0 \rightarrow \langle xyz - 1 \rangle \rightarrow \mathbb{Z}[x, y, z] \rightarrow \mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle \rightarrow 0
\]

The ideal \( \langle xyz - 1 \rangle \) is as a \( \mathbb{Z} \)-submodule of \( \mathbb{Z}[x, y, z] \) freely generated by binomials

\[
x^py^qz^r(xyz - 1) = x^{p+1}y^{q+1}z^{r+1} - x^py^qz^r. \]

This binomial is \( Z_3 \)-invariant if and only if
$p = q = r$. From here we easily deduce the structure of $\langle xy - 1 \rangle$ as a $\mathbb{Z}[\mathbb{Z}_3]$-module, in particular we observe that there is a decomposition $\langle xy - 1 \rangle \cong T \oplus F$ of $\mathbb{Z}[\mathbb{Z}_3]$-modules where $T$ is a trivial and $F$ a free $\mathbb{Z}[\mathbb{Z}_3]$-module.

It follows that $H^1(\mathbb{Z}_3; \langle xy - 1 \rangle) \cong 0$ and from the long exact sequence of cohomology we obtain the exactness of the second row of (6),

$$0 \rightarrow \langle \sigma_3 - 1 \rangle^{\mathbb{Z}_3} \rightarrow \mathbb{Z}[x, y, z]^{\mathbb{Z}_3} \rightarrow (\mathbb{Z}[x, y, z]/\langle xy - 1 \rangle)^{\mathbb{Z}_3} \rightarrow 0. \quad (10)$$

In particular,

$$(\mathbb{Z}[x, y, z]/\langle xy - 1 \rangle)^{\mathbb{Z}_3} \cong \mathbb{Z}[x, y, z]^{\mathbb{Z}_3}/\langle xy - 1 \rangle^{\mathbb{Z}_3}. \quad (11)$$

The injectivity of $\alpha$ and $\alpha'$ follows from (5). In order to establish the injectivity of $\alpha''$ we observe that (in light of (4) and (8)) $\mathbb{Z}[x, y, z]^{\mathbb{Z}_3}/\langle xy - 1 \rangle^{\mathbb{Z}_3}$ is isomorphic to the submodule of $\mathbb{Z}[x, y, z]^{\mathbb{Z}_3}$ generated by 1 = $x^0y^0z^0$, $\Delta(x^p) = x^py^p + z^p$ and $\Delta(x^py^q) = x^py^q + y^pz^q + z^px^q$ (where $(p, q) \neq (0, 0)$). Similarly, in light of (3) and (7) we observe that $\mathbb{Z}[\sigma_1, \sigma_2] \cong \mathbb{Z}[x, y, z]^{\mathbb{Z}_3}/\langle xy - 1 \rangle^{\mathbb{Z}_3}$ is generated by 1 = $x^0y^0z^0$, $\Delta(x^p)$, $\Delta(x^py^q)$, and $H(x^py^q)$. For example one of these $H$-polynomials (or hexagons) is depicted in Figure 6. From here we observe that $\alpha''$ is injective since it satisfies analogues of formulas (7).

**3.3 The ring $P_{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials**

We begin the analysis of $\mathbb{Z}_3$-invariant hexagonal tilings by describing the structure of the ring $P_{\mathbb{Z}_3} = (\mathbb{Z}[x, y, z]/\langle xy - 1 \rangle)^{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials.

The ring $P$ is as a $\mathbb{Z}$-module freely generated by monomials $x^py^qz^r$ which are not divisible by $xyz$, that is monomials $x^py^qz^r$ which satisfy at least one of conditions $p = 0, q = 0, r = 0$. Each of these monomials is associated a white dot in the $XYZ$-coordinate system (Figure 7).

The ring $P_{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials is as a free $\mathbb{Z}$-module generated by polynomials (‘triangles’) $\Delta(x^py^q) = x^py^q + y^pz^q + z^px^q$, where either $p > 0$ or $q > 0$, and the constant monomial $1 = x^0y^0z^0$. Note that $\Delta(x^py^q)$ is the $\mathbb{Z}_3$-symmetrized version of $x^py^q$; we also write $\Delta(1) = 3 = 1 + 1 + 1$. Moreover, $x^py^q$ is the leading monomial of $\Delta(x^py^q)$ in a monomial order such that $x > y > z$. Consequently, $\Delta(y^k) = \Delta(z^k)$ is almost always recorded as $\Delta(x^k)$.

**Lemma 7.** The following identities hold in the ring $P = \mathbb{Z}[x, y, z]/\langle xy - 1 \rangle$.

$$\Delta(xy)\Delta(x^{p-1}y^{q-1}) = \Delta(x^py^q) + \Delta(x^{p-1}y^{q-2}) + \Delta(x^{p-2}y^{q-1}) \quad (p \geq 2, q \geq 2) \quad (12)$$

$$\Delta(x)\Delta(x^{p-1}y^q) = \Delta(x^py^q) + \Delta(x^{p-1}y^{q+1}) + \Delta(x^{p-2}y^{q-1}) \quad \text{for } p \geq 2 \text{ and } q \geq 1 \quad (13)$$

$$\Delta(x)\Delta(x^py^{q-1}) = \Delta(x^py^q) + \Delta(x^{p+1}y^{q-1}) + \Delta(x^{p-1}y^{q-2}) \quad \text{for } p \geq 1 \text{ and } q \geq 2 \quad (14)$$

$$\Delta(x)\Delta(x^{p-1}) = \Delta(x^p) + \Delta(x^{p-1}y^q) + \Delta(xy^{p-1}) \quad \text{for } p \geq 2 \quad (15)$$

$$\Delta(xy)\Delta(x^{p-1}) = \Delta(x^py^q) + \Delta(x^{p-2}) + \Delta(xy^p) \quad \text{for } p \geq 2. \quad (16)$$
Figure 7: Polynomial $\Delta(x^2y) = x^2y + y^2z + z^2y$ as one of basic $\Delta$-polynomials which generate (over $\mathbb{Z}$) all $\mathbb{Z}_3$-invariant polynomials in $P$.

By the injectivity of the map $\alpha''$ (Proposition 6) the ring $\mathbb{Z}[\sigma_1, \sigma_2]$ can be seen as a subring of the ring $P = \mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle$, so both $P$ and $P^{\mathbb{Z}_3}$ are modules over $\mathbb{Z}[\sigma_1, \sigma_2]$.

**Theorem 8.** The ring $P^{\mathbb{Z}_3} = (\mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle)^{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials is isomorphic (as a module over $P_\sigma = \mathbb{Z}[\sigma_1, \sigma_2]$) to the free module $P_\sigma \cdot 1 \oplus P_\sigma \cdot \theta$ of rank two where $\theta = \Delta(x^2y)$ (Figure 7). Moreover,

$$\Theta = \Theta(\sigma_1, \sigma_2, \theta) := \theta^2 - (\sigma_1 \sigma_2 - 3)\theta + (\sigma_1^3 + \sigma_2^3 - 6\sigma_1 \sigma_2 + 9) = 0$$  \hspace{1cm} (17)

so there is an isomorphism of rings,

$$(\mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle)^{\mathbb{Z}_3} \cong \mathbb{Z}[\sigma_1, \sigma_2, \theta]/\langle \Theta \rangle$$  \hspace{1cm} (18)

where $\langle \Theta \rangle$ is the principal ideal in $\mathbb{Z}[\sigma_1, \sigma_2, \theta]$ generated by $\Theta$.

**Proof:** By definition $\Delta(x) = \sigma_1, \Delta(xy) = \sigma_2, \theta = \Delta(x^2y)$ and $\theta' = \Delta(xy^2)$. Let us show that the ring $P^{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polynomials in $P$ is generated as a $\mathbb{Z}[\sigma_1, \sigma_2]$-module by 1 and $\theta$.

- If $p \geq 2$ and $q \geq 2$ then by the identity (12) (Lemma 7) the polynomial $\Delta(x^p y^q)$ can be expressed in terms of lexicographically smaller $\Delta$-polynomials, multiplied by elements of $\mathbb{Z}[\sigma_1, \sigma_2]$.

- If $p \geq 3$ and $q = 1$ then by Lemma 7 (equation 13) the polynomial $\Delta(x^p y)$ can be also reduced to lexicographically smaller $\Delta$-polynomials.

- If $p \geq 2$ then $\Delta(x^p)$ is by Lemma 7 (equation 15) reducible to lexicographically smaller $\Delta$-polynomials.

It follows that all $\Delta$-polynomials can be expressed in terms of $\theta, \theta', \sigma_1, \sigma_2$ and 1. Since $\theta + \theta' = \sigma_1 \sigma_2 - 3$ (equation 16) we finally conclude that

$$P^{\mathbb{Z}_3} = \mathbb{Z}[\sigma_1, \sigma_2] \cdot 1 + \mathbb{Z}[\sigma_1, \sigma_2] \cdot \theta.$$  \hspace{1cm} (19)

11
The sign “+” in formula (19) can be replaced by “⊕”. Indeed, if \( P + Q \theta = 0 \) for some \( P, Q \in \mathbb{Z}[\sigma_1, \sigma_2] \) then (by interchanging variables \( x \) and \( y \)) we have \( P + Q \theta' = 0 \) which is possible only if \( P = Q = 0 \).

By (19) \( \theta^2 = P + Q \theta \) for some \( P, Q \in \mathbb{Z}[\sigma_1, \sigma_2] \) which by direct calculation leads to equation (17) and the isomorphism (18). □

3.4 The \( abc \)-lattice \( E \) and the \( xyz \)-lattice \( D \)

The ‘white dot’ lattice or the \( xyz \)-lattice \( D \) (Section 3.1) is generated by vectors (translations) \( t_x, t_y, t_z \) (Figures 4 and 5). It is convenient to introduce the \( abc \)-lattice \( E \) as the lattice generated by vectors \( t_a, t_b, t_c \) (Figures 8 and 9).

The lattice \( D \) is a sublattice of \( E \) of index 3. The set of black dots (Figures 5, 8 and 9) is clearly one of the cosets of the quotient lattice \( E/D \cong \mathbb{Z}_3 \).

The fact that \( E/D \cong \mathbb{Z}_3 \) explains why there are three types of dots in these images. In order to avoid clutter we will in subsequent sections continue to draw only black and white dots, however one should keep in mind the whole of the background lattice \( E \) and the presence of ‘invisible’ dots (circled asterisks in Figures 8 and 9).

Figure 8: The lattice \( E \) has three types of dots, black, white, and asterisks in circles.

Let \( \iota : E \to \mathbb{Z}/3\mathbb{Z} \) be the homomorphism which sends the generators \( a, b, c \) to \( 1 \in \mathbb{Z} \).

Then \( D = E_0 = \iota^{-1}(0) \) is the set of white dots and \( E_1 = \iota^{-1}(1) \) is the set of black dots.

3.5 The \( abc \)-ring \( Q \) and the \( xyz \)-ring \( P \)

Let \( Q = \mathbb{Z}[a, b, c]/\langle abc - 1 \rangle \) be the semigroup ring of the lattice \( E \) and let \( Q^{\mathbb{Z}_3} = (\mathbb{Z}[a, b, c]/\langle abc - 1 \rangle)^{\mathbb{Z}_3} \) be the associated ring of \( \mathbb{Z}_3 \)-invariant polynomials. All structure results that apply to the ring \( P = \mathbb{Z}[x, y, z]/\langle xyz - 1 \rangle \) apply to the ring \( Q \) as well. In particular there is an isomorphism (Theorem 8),

\[
(\mathbb{Z}[a, b, c]/\langle abc - 1 \rangle)^{\mathbb{Z}_3} \cong \mathbb{Z}[s_1, s_2, t]/\langle \Theta \rangle \tag{20}
\]

where \( \Theta = \Theta(s_1, s_2, t) \) is the polynomial described in equation (17) and

\[
s_1 = a + b + c \quad s_2 = ab + bc + ca \quad t = \Delta(a^2b) = a^2b + b^2c + c^2a. \tag{21}
\]
The homomorphism \( \iota : E \to \mathbb{Z}/3\mathbb{Z} \) introduced in Section 3.4 allows us to define a \( \mathbb{Z}_3 \)-grading in the rings \( Q \) and \( Q^{\mathbb{Z}_3} \). Indeed, the monomial \( m = a^p b^q c^r \in Q \) is graded by its ‘degree mod 3’ i.e. the mod 3 class of \( \text{deg}(m) = p + q + r \).

The \( xyz \)-ring \( P \), as the group ring of the lattice \( D \) of white dots, corresponds to the elements in the \( abc \)-ring \( Q \) graded by 0. Indeed, this follows from the fact (Figure 9) that \( x = ac^2, y = ba^2, z = cb^2 \).

Moreover, the \( P \)-submodule of \( Q \) of elements graded by 1 (monomials with the degree congruent to 1 mod 3) is precisely the submodule generated by the monomials associated with the black dots.

Recall that the \( P \)-module generated by black dots is precisely the module \( A_{hex} \) from Section 3.1. This observation allows us to reduce the “submodule membership problem” in the \( P \)-module \( A_{hex} \) to the corresponding “ideal membership problem” in the ring \( Q \). We are primarily interested in \( \mathbb{Z}_3 \)-invariant polynomials so in the following section we show how the similar “submodule membership problem” in the \( P^{\mathbb{Z}_3} \)-module \( A^{\mathbb{Z}_3}_{hex} \) can be reduced to the corresponding “ideal membership problem” in the ring \( Q^{\mathbb{Z}_3} \) (Proposition 9).

3.6 The ring \( Q^{\mathbb{Z}_3} \) and \( A^{\mathbb{Z}_3}_{hex} \) as a \( P^{\mathbb{Z}_3} \)-module

The free abelian group \( A_{hex} \), generated by all elementary 2-cells of the hexagonal lattice \( L_{hex} \) (or equivalently all 0-dimensional cells of its dual lattice \( L^\circ_{hex} \)), is a module over the ring \( P = \mathbb{Z}[x, y, z]/(xyz - 1) \).

\( \Delta \)-polynomials already appeared in the description of the ring \( P^{\mathbb{Z}_3} \) of \( \mathbb{Z}_3 \)-invariant polynomials in \( P \) (Figure 7). Following the idea of the ‘Newton polygon construction’, a polynomial \( \Delta(x^py^qz^r) = x^py^q + y^pz^q + z^py^q \) is visualized as a triangle with vertices in the ‘white dot’-lattice, invariant with respect to the action of group \( \mathbb{Z}_3 \).

Similarly the \( \Delta \)-polynomials \( \Delta(a^pb^q) = a^pb^q + b^pc^q + c^pa^q \) are, together with \( 1 = a^0b^0c^0 \), \( \mathbb{Z} \)-generators of the ring \( Q^{\mathbb{Z}_3} \). An immediate consequence is that \( P^{\mathbb{Z}_3} \) is a subring
of $Q^Z_3$.

Finally, the ‘black dot’ $\Delta$-polynomials $\Delta(x^p y^q a) = x^p y^q a + y^p z^q b + z^p x^q c$ (Figure 10) form a $\mathbb{Z}$-basis of the group $A^Z_{hex}$ of $\mathbb{Z}_3$-invariant elements of $A_{hex}$.

The ring $Q^Z_3$ inherits the $\mathbb{Z}_3$-gradation from the ring $Q$. Moreover $P^Z_3$ is precisely the subset of all elements graded by $0 \in \mathbb{Z}_3$ while $A^Z_{hex}$ is generated by ‘black dot triangles’ which are precisely the triangles graded by $1 \in \mathbb{Z}_3$. This characterization is a basis of the following fundamental proposition.

**Proposition 9.** Let $K \subset A^Z_{hex}$ be a $P^Z_3$-submodule of $A^Z_{hex}$. Let $I_K \subset Q^Z_3$ be the ideal in $Q^Z_3$ generated by $K$. Suppose that $p \in A^Z_{hex}$. Then,

$$p \in K \iff p \in I_K.$$  \hspace{1cm} (22)

In other words the ‘submodule membership problem’ is reduced to the ‘ideal membership problem’ in the ring $Q^Z_3$.

**Proof:** The implication $p \in K \Rightarrow p \in I_K$ is clear. The opposite implication is equally easy. Indeed, if $p = \alpha_1 p_1 + \ldots + \alpha_k p_k$ for some elements $p_i \in K$ and homogeneous (in the sense of the $\mathbb{Z}_3$-gradation) elements $\alpha_i \in Q^Z_3$ then we can assume that all $\alpha_i \in P^Z_3$ (the other terms cancel out). $\square$

---

**Figure 10:** $\Delta$-polynomials in the ‘black dot lattice’ generate the group $A^Z_{hex}$.
3.7 Submodule of $A_{\text{hex}}^{\mathbb{Z}_3}$ generated by tribones

A three-in-line hexagonal polyomino or a tribone is a translate of one of the following three types (Figure 10),

\[
\begin{align*}
T_x &= x^{-1} + 1 + x = ab^2 + 1 + ac^2 \\
T_y &= y^{-1} + 1 + y = bc^2 + 1 + ba^2 \\
T_z &= z^{-1} + 1 + z = ca^2 + 1 + cb^2
\end{align*}
\]  

If $A = x^py^qa$ is a ‘black dot’ in the angle $XOY$ then the three basic tribones centered at the point $A$ are

\[
T_x(A) = x^py^qaT_x, \quad T_y(A) = x^py^qaT_y, \quad T_z(A) = x^py^qaT_z. \tag{24}
\]

For example,

\[
T_x(a) = (x^{-1} + 1 + x)a \quad T_y(a) = (y^{-1} + 1 + y)a \quad T_z(a) = (z^{-1} + 1 + z)a \tag{25}
\]

The $\mathbb{Z}_3$-symmetric triplets of tribones, associated to tribones (24), are

\[
\Delta(T_x(A)), \quad \Delta(T_y(A)), \quad \Delta(T_z(A)) \tag{26}
\]

where for example,

\[
\Delta(T_y(A)) = \Delta(x^py^qaT_y) = \Delta(x^py^{q-1}a) + \Delta(x^py^q) + \Delta(x^py^{q+1}a). \tag{27}
\]

**Theorem 10.** The submodule $K_{\text{trib}} \subset A_{\text{hex}}^{\mathbb{Z}_3}$ of $\mathbb{Z}_3$-invariant polyominos (polyhexes) which admit a signed, symmetric tiling by tribones is generated, as a module over $P^{\mathbb{Z}_3}$, by the $\mathbb{Z}_3$-symmetric triplets of tribones,

\[
\Delta(T_x(a)), \Delta(T_y(a)), \Delta(T_z(a)), \Delta(T_x(ax)), \Delta(T_y(ax)), \Delta(T_z(ax)). \tag{28}
\]

**Proof:** Let $A = Ma = x^py^qa \in A_{\text{hex}}$ be a ‘black dot’ in the angle $\angle XOY$ where $M = x^py^q \in P$ is the corresponding ‘white dot’ (Figure 11). The three tribones centered at $A$ are $MaT_x, MaT_y, MaT_z$. We want to show that each of the associated $\mathbb{Z}_3$-symmetric triplets $\Delta(MaT_x), \Delta(MaT_y), \Delta(MaT_z)$ is in the $P^{\mathbb{Z}_3}$-module generated by the six elements listed in the equation (28).

Figure 11 depicts the case where $M = x^4y^3$ and the chosen tribone is $MaT_y = x^4y^3(y^{-1} + 1 + y)a$. Using this particular example we describe a general reduction procedure which allows us to express the triplets $\Delta(MaT_x), \Delta(MaT_y), \Delta(MaT_z)$ by triplets strictly closer to the origin $O$.

Let $\Delta(x) = x + y + z = \sigma_1$ and $\Delta(xy) = xy + yz + zx = \sigma_2$ be the two basic ‘white dot’ triangles ($\Delta$-polynomials) depicted at the center of Figure 11. By translating these triangles we surround the point $A$ by six triangles forming a regular hexagon.

Let us assume that the origin $O$ is not contained in the interior of this hexagon. It follows that one of the six triangles (the shaded triangle $ABC$ in Figure 11) has the
property that the segment $OA$ intersects the side $BC$ opposite to $A$. (Observe that the origin $O$ can be on the segment $BC$ and this happens precisely if $A \in \{ya, xya\}$.)

An immediate consequence is that the lengths of all segments $OB, OC, OX$ are strictly smaller than the length of $OA$. Indeed, by construction $\angle ACO \geq 60^\circ \geq \angle CAO$, and at least one of these inequalities is strict. In our case the triangle $ABC$ is a translate of $\Delta(xy)$ (otherwise we would use $\Delta(x)$). Since $\Delta(xy)X = A + B + C$ we observe that

$$\Delta(xy) \left(y^{-1} + 1 + y \right)X = (y^{-1} + 1 + y)A + (y^{-1} + 1 + y)B + (y^{-1} + 1 + y)C \quad (29)$$

Let $\Delta_Y = \Delta((y^{-1} + 1 + y)Y)$. By symmetrizing the equality (29) and adding (that is by applying the $\Delta$-operator on both sides of (29)) we finally obtain,

$$\Delta(xy) \Delta_X = \Delta_A + \Delta_B + \Delta_C. \quad (30)$$

Summarizing, we see that each $\mathbb{Z}_3$-symmetric triplet of tribones $\Delta_A$ can be expressed in terms of triplets closer to the origin, provided the hexagon associated to $A$ does not contain the origin $O$ in its interior.

The only remaining possibilities are $A' = a$ and $A'' = xa$ which accounts for the six $\mathbb{Z}_3$-symmetric triplets of tribones listed in the theorem. \qed
3.8 The ideal $I_{K_{trib}}$

In this section we express the generating polynomials for the ideal $I_{K_{trib}} \subset Q^{Z^3}$ (listed in (28)) in terms of variables $s_1, s_2, t$ which appear in the description of the ambient ring $Q^{Z^3}$ (equation (20) in Section 3.5).

Proposition 11.

\[
\begin{align*}
\Delta(T_x(a)) &= -3s_1 + 2s_2^2 \\
\Delta(T_y(a)) &= 3s_1 - s_2^2 + s_1 t \\
\Delta(T_z(a)) &= s_1^2 s_2 - s_2^2 - s_1 t \\
\Delta(T_x(ax)) &= -s_1^2 s_2 + 2s_2^2 - s_1 t + s_2^3 t \\
\Delta(T_y(ax)) &= -3s_1 + s_1^2 s_2 - s_2^2 \\
\Delta(T_z(ax)) &= 3s_1 - 2s_1^2 s_2 - s_2^2 + s_1 s_2^3 + s_1 t - s_2^2 t
\end{align*}
\]  

(31)

Proof: The proof is by direct calculations which follow the algorithm described in the proof of Theorem 8. For example

\[
\Delta(T_x(a)) = \Delta(a^2b^2 + a + a^2c^2) = 2\Delta(a^2b^2) + \Delta(a) = 2s_2^2 - 4s_1 + s_1 = 2s_2^2 - 3s_1.
\]

Similarly since

\[
\Delta(a^3b) = \Delta(a)\Delta(a^2b) - \Delta(a^2b) - \Delta(a) = s_1 t - (s_2^2 - 2s_1) - s_1 = s_1 t - s_2^2 + s_1
\]

we deduce that,

\[
\Delta(T_y(a)) = \Delta(c + a + a^3b) = 2s_1 + \Delta(a^3b) = s_1 t - s_2^2 + 3s_1.
\]

4 Calculations

4.1 Auxiliary calculations

Let $I \subset \mathbb{Z}[x, y]$ be the ideal,

\[
I = \langle 1 + x + x^2, 1 + y + y^2, 1 + xy + (xy)^2 \rangle.
\]

(32)

If $p - q \in I$ we say that $p$ and $q$ are congruent mod $I$ and write $p \equiv_I q$. In this section we collect some elementary congruences mod $I$ which are needed for subsequent calculations.

Lemma 12. If $m - n$ is divisible by 3 then,

\[
x^m \equiv_I x^n \quad y^m \equiv_I y^n \quad (xy)^m \equiv_I (xy)^n.
\]

(33)

Lemma 13.

\[
L_k := 1 + x + \ldots x^{k-1} \equiv_I \begin{cases} 
0 & \text{if } k \equiv 0 \mod 3 \\
1 & \text{if } k \equiv 1 \mod 3 \\
1 + x & \text{if } k \equiv 2 \mod 3.
\end{cases}
\]

(34)
Lemma 14.
\[ \Box_k := (1 + x + \ldots x^{k-1})(1 + y + \ldots y^{k-1}) \equiv_I \begin{cases} 0 & \text{if } k \equiv 0 \mod 3 \\ 1 & \text{if } k \equiv 1 \mod 3 \\ x^2y^2 & \text{if } k \equiv 2 \mod 3. \end{cases} \quad (35) \]

The Newton polygon of the polynomial \( \Box_k \) is the square with vertices \( \{(0,0), (0,k-1), (k-1,0), (k-1,k-1)\} \) which has precisely \( k \) integer points on each of its sides. Similarly the polynomial

\[ \Delta_k = L_k + xyL_{k-1} + (xy)^2L_{k-2} + \ldots + (xy)^{k-1}L_1 \quad (36) \]

collects all monomials associated to the integer points in the triangle with the vertices \( \{(0,0), (k-1,0), (k-1,k-1)\} \) (\( \Delta_8 \) is depicted in Figure 12).

Lemma 15.
\[ \begin{align*}
\Delta_{3n-1} & \equiv_I n\Delta_2 & & \equiv_I n(1 + x + xy) \\
\Delta_{3n} & \equiv_I \Delta_{3n-1} & & \equiv_I n\Delta_2 \\
\Delta_{3n+1} & \equiv_I n\Delta_2 + 1
\end{align*} \quad (37) \]

4.2 \( \Delta \)-polynomial of a triangular region

We use the calculations from Section 4.1 to determine the \( \Delta \)-polynomial of a triangular region \( T_N \) depicted in Figure 13 where \( N = 3k - 1 \) is the number of black dots on one of the edges of the triangle. By definition the vertices of the triangular region \( T_N \) are black dots which are associated the monomials,

\[ ax^{k-1}y^{2k-2} \quad by^{k-1}z^{2k-2} \quad cz^{k-1}x^{2k-2}. \]

It is sufficient to determine the \( \equiv_I \) class of the polynomial \( A_k \) described as the sum of all monomials in \( T_N \) which belong to the cone \( xOy \). By inspection we see that \( A_k = B_k + C_k \) where \( B_k \) corresponds to the parallelogram with vertices \( a, ax^{k-1}, ay^{k-1}, a(xy)^{k-1} \) and
Figure 13: Decomposition of $T_N$ into blocks.

$C_k$ is the polynomial associated to the triangle with vertices $axy^k, ax^{k-1}y^k, ax^{k-1}y^{2k-2}$. Since $B_k = a \cdot \Box_k$ and $C_k = axy^k \Delta_{k-1}$ we can use Lemmas [12, 14] and [15] to evaluate $A_k$.

**Proposition 16.**

$$A_k = \begin{cases} 
  dax \Delta_2 & \text{if } k = 3d \\
  a + daxy \Delta_2 & \text{if } k = 3d + 1 \\
  ax^2y^2 + axy^2(d\Delta_2 + 1) & \text{if } k = 3d + 2.
\end{cases} \quad (38)$$

**Proof:**

$$B_k = a \cdot \Box_k \equiv_I \begin{cases} 
  0 & \text{if } k = 3d \\
  a & \text{if } k = 3d + 1 \\
  ax^2y^2 & \text{if } k = 3d + 2.
\end{cases} \quad (39)$$

Since,

$$axy^k \equiv_I \begin{cases} 
  ax & \text{if } k = 3d \\
  axy & \text{if } k = 3d + 1 \\
  axy^2 & \text{if } k = 3d + 2,
\end{cases} \quad (40)$$

$$C_k = axy^k \Delta_{k-1} \equiv_I \begin{cases} 
  dax \Delta_2 & \text{if } k = 3d \\
  daxy \Delta_2 & \text{if } k = 3d + 1 \\
  axy^2(d\Delta_2 + 1) & \text{if } k = 3d + 2.
\end{cases} \quad (41)$$

Since $A_k = B_k + C_k$ the result follows by adding equations (39) and (41). \[\square\]

**Proposition 17.** The $\Delta$-polynomial of $A_k$ is equal to $\Delta(A_k) = P + dQ$ where,

$(k = 3d)$

$$P = 0 \quad Q = 3s_1 - 3s_1^2s_2 + s_1s_2^3$$
\( k = 3d + 1 \) \quad P = s_1 \quad Q = 9s_1 - 6s_1^2s_2 + s_1^3s_2^2 + 4s_1t - 2s_1^2s_2t + s_1t^2 \\
(\textnormal{\textast} k = 3d + 2) \quad P = 11s_1 + s_1^4 - 9s_1^2s_2 + 5s_2^2 + s_1^3s_2^2 - s_1s_2^3 + 4s_1t - 2s_1^2s_2t + s_2^2t + s_1t^2 \\
Q = 24s_1 + s_1^4 - 11s_1^2s_2 + s_1^3s_2^2 - 3s_1^2s_2^3 + 4s_1s_2^3 + 8s_1t - s_1^2t - s_1^2s_2t + 3s_1t^2.

**Proof:** It follows from Proposition \[16\] that,

\[
A_k = \begin{cases}
    d(a^2c^2 + a^3c^4 + a^4c^3) & \text{if } k = 3d \\
    a + d(a^3c + a^4c^3 + a^5c^2) & \text{if } k = 3d + 1 \\
    (a^5c^2 + a^4) + d(a^4 + a^5c^2 + a^6c) & \text{if } k = 3d + 2.
\end{cases}
\tag{42}
\]

The rest of the proof is by direct calculation, by hand or preferably by a computer algebra system. \( \square \)

**4.3 Gröbner basis for the submodule \( K_{trib} \subset A_{hex}^{Z_3} \)**

We want to test if the polynomials \( \Delta(A_k) \) described in Proposition \[17\] belong (for different values of \( k \)) to the submodule \( K_{trib} \) described in Theorem \[10\] (Section 3.7).

In light of Proposition \[9\] this question is reduced to the ‘ideal membership problem’ for the associated ideal \( I_{K_{trib}} \) in the ring \( Q^{Z_3} \cong \mathbb{Z}[s_1, s_2, t]/\langle \Theta \rangle \) and in turn to the ‘ideal membership problem’ for the ideal

\[
J_{K_{trib}} := I_{K_{trib}} \cup \{ \Theta \} \subset \mathbb{Z}[s_1, s_2, t].
\tag{43}
\]

Here \( \Theta \) is the polynomial defined in Theorem \[8\] (equation \[18\]) and again, in the context of the \( abc \)-ring \( Q^{Z_3} \), in Section 3.5.

In this section we determine the Groebner basis for the ideal \( J_{K_{trib}} \).

**Proposition 18.** The Groebner basis \( G = G_{trib} \) of the ideal \( J_{K_{trib}} \subset \mathbb{Z}[s_1, s_2, t] \) with respect to the lexicographic order of variables \( s_1, s_2, t \) is given by the following list of polynomials:

\[
27 + 9t + 3t^2 \quad -27 + t^3 \quad 9s_2 + 3s_2t + s_2t^2 \quad 3s_2^2 \\
\begin{array}{c}
    s_2^2 \\
    s_1s_2^3 \\
    s_1^3s_2 \\
    s_1^3s_2^2 \\
    9 + s_1^3 + s_2^3 + 3t + t^2
\end{array}
\tag{44}
\]

As the first application of Proposition \[18\] we calculate the remainders \( \overrightarrow{P}^G \) and \( \overrightarrow{Q}^G \) of polynomials \( P \) and \( Q \) introduced in Proposition \[17\] on division by the Groebner basis \( G = G_{trib} \).

**Proposition 19.** Let \( P_i \) and \( Q_i \) be the polynomials such that to \( \Delta(A_{3d+i}) = P_i + dQ_i \) (Proposition \[17\]). Then the remainders of these polynomials on division by the Groebner basis \( G = G_{trib} \) are the following:

\[
\begin{align*}
\overrightarrow{P}_0^G &= 0 & \overrightarrow{Q}_0^G &= -s_2^2 \\
\overrightarrow{P}_1^G &= s_1 & \overrightarrow{Q}_1^G &= -s_2^2 \\
\overrightarrow{P}_2^G &= -s_1 & \overrightarrow{Q}_2^G &= -s_2^2
\end{align*}
\tag{45}
\]
5 Main results

Theorem 20. Let $T_N = T_{3k-1}$ be the $\mathbb{Z}_3$-symmetric triangular region in the hexagonal lattice depicted in Figure 14 where $N$ is the number of black dots (hexagons) on the edge of the triangle. Then $T_N$ admits a $\mathbb{Z}_3$-symmetric, signed tiling by three-in-line polyominoes (tribones) if and only if $k = 9r$ for some integer $r$. The first such triangle is $T_{26}$, in particular the triangle $T_8$ shown in Figure 1 does not have a $\mathbb{Z}_3$-symmetric, signed tiling by tribones.

Proof: By Proposition 17 the polynomial $\Delta(A_k)$, equal to the sum of all monomials covered by the triangular region $T_N = T_{3k-1}$, can be expressed (as a polynomial in variables $s_1, s_2, t$) as the sum $\Delta(A_k) = P + dQ$. More explicitly (taking into account the different cases of Proposition 17) we write $\Delta(A_{3d+i}) = P_i + dQ_i$ where $i \in \{0, 1, 2\}$.

By Proposition 19 remainders of these polynomials on division by the Groebner basis $G = G_{\text{trib}}$ are displayed in the table 15.

The leading terms of the Groebner basis (44) are the following,

\[
\begin{align*}
3t^2 & \quad t^3 & \quad s_2 t^2 & \quad 3s_2^2 \\
s_2^2 t & \quad s_4 & \quad 3s_1 & \quad s_1 t \\
s_1 s_2^3 & \quad s_2^2 s_2 & \quad s_2^3 & \quad s_1^3
\end{align*}
\]

(46)

By inspection we see that the polynomial $P_i + dQ_i$ can be reduced to zero if and only $i = 0$ and $d = 3r$ for some integer $r$, or in other words if and only if $k = 3d = 9r$ ($N = 27r - 1$).

5.1 The case $N = 3k$

The triangular region $T_N$ in the case $N = 3k + 1$ has a fixed point (black dot) with respect the $\mathbb{Z}_3$ action so we focus on the remaining case $N = 3k$.

In this case the convex hull of the set of all black dots in the intersection of $T_{3k}$ with the angle $xOy$ is the trapeze with vertices associated the monomials $a, ax^k, ay^{k-1}, ax^{2k-1}y^{k-1}$. We denote the sum of all these monomials by $A_k$. This trapeze is divided into a rhombus, with vertices at $a, ax^{k-1}, ay^{k-1}, a(xy)^{k-1}$ and an equilateral triangle with vertices $ax^k, ax^k y^{k-1}, ax^{2k-1} y^{k-1}$. The sums of monomials in these two regions are respectively $B_k$ and $C_k$, so by definition $A_k = B_k + C_k$.

The mod $I$ class of the polynomial $B_k = a \cdot [k]$ is as in the case $N = 3k - 1$ described by the equality (49). Similarly, $C_k = ax^k \nabla_k$ where $\nabla_k$ is the sum of all monomials in the triangle with vertices at $1, y^{k-1}, (xy)^{k-1}$. This triangle is obtained from $\Delta_k$ by interchanging variables $x$ and $y$ so the following lemma is an immediate consequence of Lemma 15.

Lemma 21.

\[
\begin{align*}
\nabla_{3n-1} & \equiv_I n \nabla_2 \quad \equiv_I n(1 + x + xy) \\
\nabla_{3n} & \equiv_I \nabla_{3n-1} \equiv_I n \nabla_2 \\
\nabla_{3n+1} & \equiv_I n \nabla_2 + 1
\end{align*}
\]

(47)
Similarly,
\[ ax^k \equiv_I \begin{cases} 
  a & \text{if } k = 3d \\
  ax & \text{if } k = 3d + 1 \\
  ax^2 & \text{if } k = 3d - 1,
\end{cases} \tag{48} \]

\[ C_k = ax^k \nabla_k \equiv_I \begin{cases} 
  da \nabla_2 & \text{if } k = 3d \\
  ax(d \nabla_2 + 1) & \text{if } k = 3d + 1 \\
  dax^2 \nabla_2 & \text{if } k = 3d - 1.
\end{cases} \tag{49} \]

and finally,

**Proposition 22.**

\[ A_k = \begin{cases} 
  da \nabla_2 & \text{if } k = 3d \\
  a + ax(d \nabla_2 + 1) & \text{if } k = 3d + 1 \\
  ax^2y^2 + dax^2\Delta_2 & \text{if } k = 3d - 1.
\end{cases} \tag{50} \]

The associated \( \Delta \)-polynomials are given by the following proposition.

**Proposition 23.** The \( \Delta \)-polynomial of \( A_k \) is equal to \( \Delta(A_k) = P + dQ \) where,

\[(k = 3d) \quad \begin{aligned} P &= 0 \\
Q &= s_1^2s_2 - 2s_2^2 \end{aligned} \]

\[(k = 3d + 1) \quad \begin{aligned} P &= -s_1 + s_2^2 \\
Q &= -s_1^2s_2 - 2s_2^2 + s_1s_2^3 - s_2^2t \end{aligned} \]

\[(k = 3d - 1) \quad \begin{aligned} P &= 7s_1 - 5s_1^2s_2 + 3s_2^2 + s_1^2s_2^2 - s_1s_2^3 + 4s_1t - 2s_1^2s_2t + s_2^2t + s_1t^2 \\
Q &= 2s_1^2s_2 + 4s_2^2 - 4s_1s_2^3 + s_2^5. \end{aligned} \]

**Proposition 24.** Let \( P_i \) and \( Q_i \) be the polynomials such that to \( \Delta(A_{3d+i}) = P_i + dQ_i \) (Proposition 23). Then the remainders of these polynomials on division by the Groebner basis \( G = G_{\text{trib}} \) are the following:

\[(k = 3d) \quad \begin{aligned} T_0^G &= 0 \\
Q_0^G &= s_2^2 \end{aligned} \quad \begin{aligned} P_1^G &= -s_1 + s_2^2 \\
Q_1^G &= s_2^2 \\
T_{-1}^G &= s_1 \\
Q_{-1}^G &= s_2^2. \end{aligned} \tag{51} \]

The analysis similar to the proof of Theorem 20 leads to the following result.

**Theorem 25.** Let \( T_N = T_{3k} \) be the \( \mathbb{Z}_3 \)-symmetric triangular region in the hexagonal lattice which has \( 3k \) hexagons on one of its sides. Then \( T_N \) admits a \( \mathbb{Z}_3 \)-symmetric, signed tiling by three-in-line polyominoes (tribones) if and only if \( k = 9r \) for some integer \( r \). The first such triangle is \( T_{27} \).
5.2 Examples illustrating Theorems 20 and 25

Here we describe explicit \( \mathbb{Z}_3 \)-invariant tilings predicted by Theorems 20 and 25.

We begin with the case \( N = 3k - 1 = 27r - 1 \). Figure 13 describes a decomposition of the triangular region \( T_N \) into triangular and rhombic blocks. The side length of the triangle with the vertices \( axy^k, axk - 1y^k, axk - 1y^{2k} \) is \( k - 1 = 9r - 1 \) so by Theorem 1 this triangle admits a signed tiling by tribones. The side length of the rhombus with vertices \( a, axk, ayk - 1, a(xy)^k \) is \( k = 9r \) so it can be paved by tribones. This signed tiling is extended to the whole region \( T_N \) by rotations through the angle of 120° and 240°.

The case \( N = 3k = 27r \) is treated similarly. The intersection of \( T_{3k} \) with the angle \( xOy \) is the trapeze with vertices at \( a, axk, ayk - 1, ax2k - 1y \) (Section 5.1). This trapeze is divided into a rhombus, with vertices at \( a, axk - 1, ayk - 1, a(xy)^k \) and an equilateral triangle with vertices at \( axk, axky - 1, ax2k - 1y^k \). Both figures admit a (signed) tribone tiling. Indeed, the triangle has the side length \( k = 9r \) (and the Theorem 1 applies) while the side length of the rhombus is as before \( k = 9r \).

6 Concluding remarks, examples and questions

Our main objective was to illustrate the method of Gröbner bases in the case of equivariant triboine tilings of the hexagonal lattice. However the method has many other advantages and some of them were mentioned already in the original paper of Bodini and Nouvel [4].

Perhaps the main reason why this method is so well adapted for lattice tiling problem is its close connection with the already developed methods and tools used in lattice geometry.

6.1 Integer-point transform and Brion’s theorem

Following [1] and [2] we define the integer-point transform of a finite subset \( K \subset \mathbb{N}^2 \) as the polynomial \( f_K = \sum \{x^\alpha y^\beta \mid (\alpha, \beta) \in \mathbb{N}^2 \} \).

Brion’s formula, see [1, Chapter 8] and [2, Section 9.3], is a versatile tool for evaluating the integer-point transform of convex polytopes. It allows us to replace a polynomial with a large number of monomials by a very short expression involving only rational functions. It was tacitly used throughout the paper for an independent checking of some of the formulas. More systematic application of these ideas will be presented in [10].

6.2 Integer-point enumeration in polyhedra

Let \( P \) be a convex polytope with vertices in \( \mathbb{N}^d \) and let \( f_P \) be its integer-point transform. The ‘discrete volume’ of \( Q \), defined as the number of integer points inside \( P \), can be...
evaluated as the remainder of $f_P$ on division by the ideal
$$I = \langle x_1 - 1, x_2 - 1, \ldots, x_d - 1 \rangle.$$

Let $J \subset \mathbb{Z}[x_1, \ldots, x_d]$ be an ideal, say the ideal associated to a set $\mathcal{R}$ of prototiles in $\mathbb{N}^d$. Let $G = G_J$ be the Gröbner basis of $J$ with respect to some term order. It may be tempting to ask (at least for some carefully chosen ideals $J$) what is the geometric and combinatorial significance of the remainder $\overline{f_Q^G}$ of the integer-point transform polynomial $f_Q$ on division by the Gröbner basis $G$.

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