The Dirichlet Problem for $p$-minimizers on Finely Open Sets in Metric Spaces

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Abstract
We initiate the study of fine $p$-(super)minimizers, associated with $p$-harmonic functions, on finely open sets in metric spaces, where $1 < p < \infty$. After having developed their basic theory, we obtain the $p$-fine continuity of the solution of the Dirichlet problem on a finely open set with continuous Sobolev boundary values, as a by-product of similar pointwise results. These results are new also on unweighted $\mathbb{R}^n$. We build this theory in a complete metric space equipped with a doubling measure supporting a $p$-Poincaré inequality.

Keywords
Dirichlet problem · Doubling measure · Fine continuity · Fine $p$-minimizer · Fine $p$-superminimizer · Fine supersolution · Finely open set · Metric space · Nonlinear fine potential theory · Poincaré inequality · Quasiopen set

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1 Introduction

Superharmonic functions play a fundamental role in the classical potential theory. Unlike harmonic functions (i.e. solutions of the Laplace equation $\Delta u = 0$), they need not be continuous but are finely continuous. In fact, the fine topology is the coarsest topology that makes all superharmonic functions continuous, see Cartan [20]. The fine topology is closely...
related to the Dirichlet boundary value problem for the Laplace equation on open sets. It follows from the famous Wiener criterion [51] that a boundary point \( x_0 \in \partial \Omega \) of a Euclidean domain \( \Omega \) is irregular for \( \Delta u = 0 \) if and only if \( \Omega \cup \{ x_0 \} \) is finely open, i.e. if the complement \( \mathbb{R}^n \setminus \Omega \) is thin at \( x_0 \) in a capacity density sense. In this case, the complement and the boundary are simply too small in the potential theoretical sense to ensure that continuous boundary data enforce continuity of the corresponding solution at \( x_0 \). These facts have lead to the development of fine potential theory and finely (super)harmonic functions associated with \( \Delta u = 0 \) on finely open sets, see the monograph [22] of Fuglede, the papers Fuglede [23–27], Lukeš–Malý [43], Lyons [45, 46], and the book [44] by Lukeš–Malý–Zajíček, which contain additional results and references.

In the nonlinear case, for equations associated with the \( p \)-Laplacian \( \Delta_p \), \( 1 < p \neq 2 \), the first similar study was conducted by Kilpeläinen–Malý [31], who studied \( p \)-fine (super)solutions for such equations on quasiopean subsets of unweighted \( \mathbb{R}^n \). That theory was further extended by Latvala [40, 41], in particular for \( p = n \). Eigenvalue problems for the \( p \)-Laplacian in quasiopean subsets of \( \mathbb{R}^n \) were considered in Fusco–Mukherjee–Zhang [28]. We are not aware of any other papers dealing with \( p \)-fine (super)solutions, and in particular none beyond unweighted \( \mathbb{R}^n \).

The Wiener criterion was extended to the nonlinear theory associated with \( p \)-harmonic functions on open subsets of (unweighted and weighted) \( \mathbb{R}^n \) by Heinonen–Kilpeläinen–Martio [29], Kilpeläinen–Malý [32], Lindqvist–Martio [42], Maz’ya [48] and Mikko–Mikkonen [49], and partially also to metric spaces by Björn [14, 15] and Björn–MacManus–Shamugalingam [16]. It has also been related to fine continuity of \( p \)-superharmonic functions on open sets in much the same way as for the Laplacian. Following this nonlinear development, we define the fine topology on metric spaces using the notion of thinness based on a Wiener type integral, see Definition 3.1.

In this paper we continue our study of fine potential theory on metric spaces, carried through in [7–9], and initiate the study of fine \( p \)-(super)minimizers with \( 1 < p < \infty \). We consider a complete metric space \( X \) equipped with a doubling measure supporting a \( p \)-Poincaré inequality. The function space naturally associated with \( p \)-energy minimizers on such metric spaces is the Sobolev type space \( N^{1,p} \), called the Newtonian space.

The following regularity result for solutions of the Dirichlet problem on finely open sets is our main result, which we obtain as a by-product of more general pointwise results. Even in unweighted \( \mathbb{R}^n \) and for \( \Delta_p u = 0 \), it is more general than the similar Theorem 5.3 in Kilpeläinen–Malý [31].

Here \( \overline{U}^p \) is the fine closure of \( U \).

**Theorem 1.1** Let \( U \subset X \) be finely open and let either \( f \in C(U) \cap N^{1,p}(U) \) or \( f \in C(\overline{U}^p \cap \partial U) \cap N^{1,p}(X) \), where in both cases \( f \) is assumed to be continuous as a function with values in \( \overline{\mathbb{R}} := [-\infty, \infty] \). Then there is a finely continuous solution of the Dirichlet problem in \( U \) with boundary values \( f \).

In the linear axiomatic setting, i.e. for \( p = 2 \), finely (super)harmonic functions and the Dirichlet problem on finely open sets have been rigorously investigated, see the monographs by Fuglede [22] and Lukeš–Malý–Zajíček [44]. As pointed out in [44, p. 389], even in the linear setting the fine boundary can be too small for a fruitful theory of the Dirichlet problem. Thus the use of the metric boundary in Theorem 1.1 is perhaps less unnatural than it may at first seem.

Obviously, some of the linear tools used in [22] and [44] are not available to us, nor in the nonlinear setting of unweighted \( \mathbb{R}^n \) and \( p \neq 2 \). Already the notion of fine
$p$-(super)harmonic functions is not straightforward, and it is an open question whether $p$-(super)minimizers on finely open sets have finely continuous representatives. There are other open problems concerning important properties of such functions, see Section 9 for further discussion.

In metric spaces, there is (in general) no equation to work with (such as the $p$-Laplace equation). Therefore our theory relies on $p$-fine (super)minimizers defined through $p$-energy integrals and upper gradients. This makes our approach essentially independent of the theory in Kilpeläinen–Malý [31] and Latvala [40, 41], even though our main result was inspired by the proof of Theorem 5.3 in [31]. The key arguments in both proofs rely on pasting lemmas and the fine continuity of $p$-superharmonic functions on open sets.

Finely open sets and fine topology are closely related to quasioopen sets and quasitopology, as shown by Fuglede [21]. A similar study on metric spaces is more recent, but the metric space approach seems suitable since it makes it easy to consider the Sobolev type spaces $N^{1,p}$ on nonopen sets, such as finely open and quasioopen sets. These Newtonian spaces were shown in Björn–Björn–Latvala [8] and Björn–Björn–Malý [10] to coincide with the Sobolev spaces developed on quasiopean and finely open sets in $\mathbb{R}^n$ by Kilpeläinen–Malý [31]. Moreover, functions in the spaces $N^{1,p}$ are automatically quasicontinuous, and consequently finely continuous outside a set of zero capacity, both on open and quasiopean sets, see [8, 10], Björn–Björn–Shanmugalingam [13], Björn [15] and Korte [35]. Several of these results play a crucial role in this paper. On unweighted $\mathbb{R}^n$ and for nonlinear fine potential theory, they can be found in the monograph by Malý–Ziemer [47]. See also Heinonen–Kilpeläinen–Martio [29] for many of these results on weighted $\mathbb{R}^n$, as well as [10], Björn–Björn [6] and Lahti [36–39] for further results.

Obstacle problems (and thereby $p$-(super)minimizers, on nonopen sets in metric spaces were studied in Björn–Björn [5] and it was shown therein ([5, Theorem 7.3]) that the theory of obstacle problems is not natural beyond finely open (or quasiopean) sets. In Proposition 5.9, we show that this is true also for the theory of $p$-fine (super)minimizers. Additional fine properties of (super)harmonic functions on open sets were derived in Björn–Björn–Latvala [7, 9].

In addition to fine potential theory, quasiopean sets appear naturally as minimizing sets in shape optimization problems, see e.g. Bucur–Buttazzo–Velichkov [17], Buttazzo–Dal Maso [18], Buttazzo–Shrivastava [19, Examples 4.3 and 4.4] and Fusco–Mukherjee–Zhang [28]. Their importance lies in the fact that they are sub- and superlevel sets of Sobolev functions, see Theorem 3.4.

The outline of the paper is as follows: In Section 2, we recall some definitions from first-order analysis on metric spaces, while the fine topology is introduced in Section 3. Therein, we also give two new characterizations of quasiopean sets, which are probably known to the experts in the field.

In order to be able to study $p$-fine (super)minimizers and the Dirichlet problem on quasiopean sets $U$, we need the appropriate local Newtonian (Sobolev) space $N^{1,p}_{fine-loc}(U)$. We study this space in Section 4, where we also establish a density result that plays a crucial role in the later sections. In Sections 5 and 6, we develop the basic theory of $p$-fine (super)minimizers, obstacle and Dirichlet problems on quasiopean sets.

Finally, in Section 7, we are ready to develop the necessary framework enabling us to obtain Theorem 1.1. We also deduce corresponding pointwise results. In Section 8, we use some of our results to give more information on fine Newtonian spaces. The final Section 9 is devoted to open problems.
2 Notation and Preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We also assume that $1 < p < \infty$.

In this section, we introduce the necessary metric space concepts used in this paper. For brevity, we refer to Björn–Björn–Latvala [7, 9] for more extensive introductions, and references to the literature. See also the monographs Björn–Björn [4] and Heinonen–Koskela–Shanmugalingam–Tyson [30], where the theory is thoroughly developed with proofs.

The measure $\mu$ is doubling if there exists $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in $X$, we have $0 < \mu(2B) \leq C\mu(B) < \infty$, where $\lambda B = B(x_0, \lambda r)$. In this paper, all balls are open.

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length $ds$. A property holds for $p$-almost every curve if the curve family $\Gamma$ for which it fails has zero $p$-modulus, i.e. there is $\rho \in L^p(X)$ such that $\int_{\gamma} \rho ds = \infty$ for every $\gamma \in \Gamma$.

**Definition 2.1** A measurable function $g : X \to [0, \infty]$ is a $p$-weak upper gradient of $f : X \to \mathbb{R} := [-\infty, \infty]$ if for $p$-almost all curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g ds,$$

where the left-hand side is $\infty$ whenever at least one of the terms therein is infinite.

If $f$ has a $p$-weak upper gradient in $L^p_{\text{loc}}(X)$, then it has a minimal $p$-weak upper gradient $g_f \in L^p_{\text{loc}}(X)$ in the sense that $g_f \leq g$ a.e. for every $p$-weak upper gradient $g \in L^p_{\text{loc}}(X)$ of $f$.

**Definition 2.2** Let for measurable $f$,

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p d\mu + \inf_g \int_X g^p d\mu\right)^{1/p},$$

where the infimum is taken over all $p$-weak upper gradients of $f$. The Newtonian space on $X$ is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f-h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere (with values in $\mathbb{R}$), not just up to an equivalence class in the corresponding function space.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $f \in N^{1,p}_{\text{loc}}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$.

The Sobolev capacity of an arbitrary set $E \subset X$ is

$$C_p(E) = C^X_p(E) = \inf_f \|f\|_{N^{1,p}(X)}^p,$$
where the infimum is taken over all \( f \in N^{1,p}(X) \) such that \( f \geq 1 \) on \( E \). A property holds \textit{quasieverywhere} (q.e.) if the set of points for which it fails has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If \( f \in N^{1,p}(X) \), then \( h \sim f \) if and only if \( h = f \) q.e. Moreover, if \( f, h \in N^{1,p}(X) \) and \( f = h \) a.e., then \( f = h \) q.e.

For \( A \subset U \subset X \), where \( U \) is assumed to be measurable, we let
\[
N_0^{1,p}(A, U) = \{ f_A : f \in N^{1,p}(U) \text{ and } f = 0 \text{ on } U \setminus A \}.
\]

If \( U = X \), we write \( N_0^{1,p}(A) = N_0^{1,p}(A, X) \). Functions from \( N_0^{1,p}(A, U) \) can be extended by zero in \( U \setminus A \) and we will regard them in that sense if needed.

\[\text{Definition 2.3 \quad X supports a } p\text{-Poincaré inequality if there exist constants } C > 0 \text{ and } \lambda \geq 1 \text{ such that for all balls } B \subset X, \text{ all integrable functions } f \text{ on } X \text{ and all } p\text{-weak upper gradients } g \text{ of } f,\]

\[
\int_B |f - f_B| \, d\mu \leq C \text{ diam}(B) \left( \int_{\lambda B} g^p \, d\mu \right)^{1/p},
\]

where \( f_B := \frac{1}{\mu(B)} \int_B f \, d\mu \). \hfill (2.2)

In \( \mathbb{R}^n \) equipped with a doubling measure \( d\mu = w \, dx \), where \( dx \) denotes Lebesgue measure, the \( p\)-Poincaré inequality (2.2) is equivalent to the \( p\)-admissibility of the weight \( w \) in the sense of Heinonen–Kilpeläinen–Martio \cite{29}, see Corollary 20.9 in \cite{29} and Proposition A.17 in \cite{4}. Moreover, in this case \( g_u = |\nabla u| \) a.e. if \( u \in N^{1,p}(\mathbb{R}^n) \).

As usual, we will write \( f = f_+ - f_- \), where \( f_\pm = \max\{\pm f, 0\} \).

### 3 Fine Topology and Quasiopen Sets

\textit{Throughout the rest of the paper, we assume that } \( X \) \textit{is complete and supports a } p\text{-Poincaré inequality, that } \mu \textit{is doubling, and that } 1 < p < \infty.\textit{ }

To avoid pathological situations we also assume that \( X \) contains at least two points (and thus must be uncountable due to the Poincaré inequality). In this section we recall the basic facts about the fine topology associated with Newtonian functions.

\textbf{Definition 3.1 \quad A set } E \subset X \textit{ is thin at } x \in X \textit{ if}

\[
\int_0^1 \left( \frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \, dr < \infty.
\]

A set \( V \subset X \) is \textit{finely open} if \( X \setminus V \) is thin at each point \( x \in V \).

In the definition of thinness, we make the convention that the integrand is 1 whenever \( \text{cap}_p(B(x, r), B(x, 2r)) = 0 \). It is easy to see that the finely open sets give rise to a topology, which is called the \textit{fine topology}. Every open set is finely open, but the converse is not true in general. A function \( u : V \to \overline{\mathbb{R}} \), defined on a finely open set \( V \), is \textit{finely continuous} if...
it is continuous when $V$ is equipped with the fine topology and $\mathbb{R}$ with the usual topology. See Björn–Björn [4, Section 11.6] and Björn–Björn–Latvala [7] for further discussion on thinness and the fine topology in metric spaces. The fine interior, fine boundary and fine closure of $E$ are denoted fine-int $E$, $\partial_p E$ and $\overline{E}^p$, respectively.

The following characterization of the fine boundary is from Corollary 7.8 in Björn–Björn [5]. We will mainly use it for finely open sets.

**Lemma 3.2** Let $E \subset X$ be arbitrary. Then the fine boundary of $E$ is

$$\partial_p E = \{x \in E : X \setminus E \text{ is not thin at } x\} \cup \{x \in X \setminus E : E \text{ is not thin at } x\}.$$

The following definition will also be important in this paper.

**Definition 3.3** A set $U \subset X$ is quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $G \cup U$ is open.

A function $u$ defined on a set $E \subset X$ is quasicontinuous if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{E \setminus G}$ is finite and continuous.

The quasiopen sets do not in general form a topology, see Remark 9.1 in [5]. However it follows easily from the countable subadditivity of $C_p$ that countable unions and finite intersections of quasiopen sets are quasiopen. Quasiopen sets have recently been characterized in several ways. Here we summarize the known and some new characterizations. Note in particular the close connection between quasiopen and finely open sets.

**Theorem 3.4** Let $U \subset X$ be arbitrary. Then the following conditions are equivalent:

(i) $U$ is quasiopen;

(ii) $U$ is a union of a finely open set and a set of capacity zero;

(iii) $U$ is $p$-path open, i.e. $\gamma^{-1}(U)$ is relatively open in $[0, l_\gamma]$ for $p$-almost every curve $\gamma : [0, l_\gamma] \to X$;

(iv) $U = \{x : u(x) > 0\}$ for some nonnegative quasicontinuous $u$ on $X$;

(v) $U = \{x : u(x) > 0\}$ for some nonnegative $u \in N^{1,p}(X)$.

**Proof** (i) $\Leftrightarrow$ (ii) This follows from Theorem 1.4 (a) in Björn–Björn–Latvala [9].

(i) $\Rightarrow$ (iii) This follows from Remark 3.5 in Shanmugalingam [50].

(iii) $\Rightarrow$ (i) This follows from Theorem 1.1 in Björn–Björn–Malý [10].

(v) $\Rightarrow$ (iv) By Theorem 1.1 in Björn–Björn–Shanmugalingam [13], $u$ is quasicontinuous and thus (iv) holds.

(iv) $\Rightarrow$ (i) This follows from Proposition 3.4 in [10].

(ii) $\Rightarrow$ (iv) Assume that $V \subset U$ is finely open and $C_p(U \setminus V) = 0$. By Lemma 3.3 in Björn–Björn–Latvala [8], for each $x \in V$ we find a finely continuous $v_x \in N^{1,p}_0(V)$ such that $0 \leq v_x \leq 1$, $v_x(x) = 1$ and $v_x = 0$ outside of $V$. Then $\{y : v_x(y) > 0\}$ is a finely open subset of $V$. Therefore, by the quasi-Lindelöf principle (Theorem 3.4 in [8]), the family $\{v_x : x \in V\}$ contains a countable subfamily $\{v_j\}_{j=1}^\infty$ such that $C_p(Z) = 0$ for the set $Z := \{x \in V : v_j(x) = 0 \text{ for all } j\}$.
Let
\[ v = \sum_{j=1}^{\infty} \frac{2^{-j}v_j}{1 + \|v_j\|_{N^{1,p}(X)}} \in N^{1,p}(X) \quad \text{and} \quad u = \begin{cases} 1 & \text{on } Z \cup (U \setminus V), \\ v & \text{elsewhere}. \end{cases} \]

Since \( u = v \text{ q.e.}, \) also \( u \in N^{1,p}(X). \) Moreover \( U = \{ x : u(x) > 0 \}. \)

Quasiopen, and thus finely open, sets are measurable. If \( U \) is finely open and \( C_p(E) = 0, \) then \( U \setminus E \) is finely open, from which it follows that fine limits do not see sets of capacity zero.

For any measurable set \( E \subset X \) the notion of q.e. in \( E \) can either be taken with respect to the global capacity \( C_p \) on \( X \) or with respect to the capacity \( C_p^E \) determined by \( E \) as the underlying space. However, for a quasiopen set \( U, \) the capacities \( C_p \) and \( C_p^U \) have the same zero sets, and \( C_p \)-quasicontinuity in \( U \) is equivalent to \( C_p^U \)-quasicontinuity, by Propositions 3.4 and 4.2 in [10].

Here we collect some facts on quasicontinuity from [8, Theorem 4.4], [9, Theorem 1.4] and [10, Theorem 1.3]. For further characterizations of quasiopen sets and quasicontinuous functions see [10] and also Theorem 7.2 below.

**Theorem 3.5** Let \( U \) be quasiopen. Then the following are true:

(a) Functions in \( N^{1,p}(U) \) are quasicontinuous.

(b) A function \( u : U \to \mathbb{R} \) is quasicontinuous if and only if it is finite q.e. and finely continuous q.e.

### 4 \( N^{1,p}_{\text{fine-loc}}(U) \) and \( p \)-strict Subsets

From now on we always assume that \( U \) is a nonempty quasiopen set.

In the next section, we will start developing the basic theory of fine superminimizers. For this purpose, we first need to define appropriate local fine Sobolev spaces. Here \( p \)-strict subsets will play a key role, as a substitute for relatively compact subsets of open sets.

Recall that \( V \subset U \) if \( V \) is finely open, then by Lemma 3.3 in Björn–Björn–Latvala [8], \( V \) has a base of fine neighbourhoods consisting only of \( p \)-strict subsets of \( V. \) We recall that functions in \( N^{1,p}_{\text{fine-loc}}(U) \) are finite q.e., finely continuous q.e. and quasicontinuous, by Theorem 4.4 in [8].

Throughout the paper, we consider minimal \( p \)-weak upper gradients in \( U. \) For a function \( u \in N^{1,p}_{\text{fine-loc}}(U) \) we say that \( g_{u,U} \) is a minimal \( p \)-weak upper gradient of \( u \) in \( U \) if

\[ g_{u,U} = g_{u,V} \text{ a.e. in } V \quad \text{for every finely open } p \text{-strict subset } V \subset U, \quad (4.1) \]
where \( g_{u,v} \) is the minimal \( p \)-weak upper gradient of \( u \in N^{1,p}_{\text{loc}}(V) \) with respect to \( V \). (Recall that the local space \( N^{1,p}_{\text{loc}}(U) \) was introduced in Section 2.) If \( u \in N^{1,p}_{\text{loc}}(U) \), then this definition agrees with the definition of \( g_{u,u} \) in Section 2. See [8, Lemma 5.2 and Theorem 5.3] for the existence, a.e.-uniqueness and minimality of \( g_{u,u} \). If \( u \in N^{1,p}_{\text{loc}}(X) \) then the minimal \( p \)-weak upper gradients \( g_{u,u} \) and \( g_{u,v} \) with respect to \( U \) and \( X \), respectively, coincide a.e. in \( U \), see [5, Corollary 3.7] or [8, Lemma 4.3]. For this reason we drop \( U \) from the notation and simply write \( g_u \) from now on.

If follows from the definition of \( N^{1,p}_{\text{fine-loc}}(U) \) and the properties of \( N^{1,p}_{\text{fine-loc}}(U) \) (see Section 2) that functions in \( N^{1,p}_{\text{fine-loc}}(U) \) are defined everywhere in \( U \), and that if \( u \in N^{1,p}_{\text{fine-loc}}(U) \) and \( v = u \) q.e., then also \( v \in N^{1,p}_{\text{fine-loc}}(U) \). Moreover, by the quasi-Lindelöf principle (Theorem 3.4 in [8]) and the existence of a base of fine neighbourhoods in fine-int \( U \), Corollaries 2.20 and 2.21 in [4] extend to functions in \( N^{1,p}_{\text{fine-loc}}(U) \). That is, if \( u, v \in N^{1,p}_{\text{fine-loc}}(U) \) then

\[
\begin{align*}
g_u &= g_u & \text{a.e. on } \{x \in U : u(x) = v(x)\}, \\
g_{\max\{u,v\}} &= g_u\chi_{\{u>v\}} + g_v\chi_{\{v\geq u\}} & \text{a.e.,} \\
g_{\min\{u,v\}} &= g_u\chi_{\{v>u\}} + g_v\chi_{\{u\geq v\}} & \text{a.e.}
\end{align*}
\]

**Remark 4.2** By Proposition 1.48 in [4], \( p \)-weak upper gradients do not see sets of capacity zero. Thus it follows from Theorem 3.4 that \( u \in N^{1,p}_{\text{fine-loc}}(U) \) if and only if \( u \in N^{1,p}(V) \) for every quasiopen \( p \)-strict subset \( V \subseteq U \).

It is not hard to see that \( N^{1,p}_{\text{loc}}(U) \subseteq N^{1,p}_{\text{fine-loc}}(U) \). However, we do not know if the equality \( N^{1,p}_{\text{loc}}(U) = N^{1,p}_{\text{fine-loc}}(U) \) holds for all quasiopen sets \( U \). For the reader’s convenience, we recall in Lemma 4.4 why these spaces coincide for open sets.

**Remark 4.3** One reason for the compact inclusion \( V \subseteq U \) in Definition 4.1 is that if it was replaced by \( V \subset U \), the inclusion \( N^{1,p}_{\text{loc}}(U) \subset N^{1,p}_{\text{fine-loc}}(U) \) might fail. Neither would Lemma 4.4 and Corollary 5.6 below hold.

To see this let \( U = B(0,2) \setminus \{0\} \subset \mathbb{R}^n \), with \( 1 < p < n \), in which case it is easy to see that \( V = B(0, 1) \setminus \{0\} \) is an open \( p \)-strict subset of \( U \), but \( V \notin U \). At the same time \( u(x) = |x|^{(p-n)/(p-1)} \in N^{1,p}_{\text{loc}}(U) \) but \( u \notin N^{1,p}(V) \). Since \( u \) is \( p \)-harmonic in \( U \) also Corollary 5.6 would fail.

**Lemma 4.4** For open \( G \subset X \) we have \( N^{1,p}_{\text{loc}}(G) = N^{1,p}_{\text{fine-loc}}(G) \).

**Proof** First, let \( f \in N^{1,p}_{\text{loc}}(G) \) and let \( V \subseteq G \) be a finely open \( p \)-strict subset. For each \( x \in \partial V \) there is \( r_x > 0 \) such that \( f \in N^{1,p}_{\text{loc}}(B(x, r_x)) \). Since \( \partial V \) is compact there is a finite subcover \( \{B(x_j, r_{x_j})\}_{j=1}^m \) such that \( \partial V \subseteq \bigcup_{j=1}^m B(x_j, r_{x_j}) \). It follows that

\[
\|f\|_{N^{1,p}_{\text{loc}}(\partial V)}^p \leq \sum_{j=1}^m \|f\|_{N^{1,p}(B(x_j, r_{x_j}))}^p < \infty,
\]

and thus \( f \in N^{1,p}_{\text{loc}}(V) \). Hence \( f \in N^{1,p}_{\text{fine-loc}}(G) \).
Conversely, assume that \( f \in N^{1,p}_{\text{fine-loc}}(G) \) and \( x \in G \). Then there is \( r_\times > 0 \) such that \( B(x, r_\times) \subseteq G \). It is straightforward to see that \( B(x, r_\times) \) is a \( p \)-strict subset of \( G \), and thus \( f \in N^{1,p}(B(x, r_\times)) \). Hence \( f \in N^{1,p}_{\text{loc}}(G) \).

The following density result will play a crucial role.

**Proposition 4.5** Let \( E \subset X \) be an arbitrary set and \( 0 \leq u \in N^{1,p}_0(E) \). Then there exist finely open \( p \)-strict subsets \( V_j \subset E \) and bounded functions \( u_j \in N^{1,p}_0(V_j) \) such that

(a) \( V_j \subset V_{j+1} \) and \( 0 \leq u_j \leq u_{j+1} \leq u \) for \( j = 1, 2, \ldots \);

(b) \( \| u - u_j \|_{N^{1,p}(X)} \to 0 \) and \( u_j(x) \to u(x) \) for q.e. \( x \in X \), as \( j \to \infty \).

We may also require that \( u_j \equiv 0 \) outside \( V_j \).

**Proof** Let \( U = \text{fine-int} E \). By Theorem 7.3 in Björn–Björn [5], \( u \in N^{1,p}_0(U) \) and \( u = 0 \) q.e. in \( X \setminus U \). In the rest of the proof we therefore replace \( E \) by \( U \), which is quasiopen by Theorem 3.4. Modifying \( u \) in a set of zero capacity, we can also assume that \( u \equiv 0 \) in \( X \setminus U \).

By truncating and multiplying by a constant and by a cutoff function, we may assume that \( 0 \leq u \leq 1 \) and that \( u \) has bounded support, see the proof of Lemma 5.43 in [4]. As \( U \) is quasiopen and \( u \) is quasi-continuous on \( X \) (by Theorem 3.5), there are open sets \( G_j \) such that \( C_p(G_j) < 2^{-j} \), \( U \cap G_j \) is open and \( u|_{X \setminus G_j} \) is continuous, \( j = 1, 2, \ldots \). We can then also find \( \psi_j \in N^{1,p}(X) \) such that \( 0 \leq \psi_j \leq 1 \), \( \| \psi_j \|_{N^{1,p}(X)} < 2^{-j} \) and \( \psi_j = 1 \) in \( G_j \). Let \( \varphi_k = \min\{1, \sum_{j=k}^{\infty} \psi_j\} \). Then \( \| \varphi_k \|_{N^{1,p}(X)} < 2^{1-k} \) and the sequence \( \{\varphi_k\}_{k=1}^{\infty} \) is decreasing. By dominated convergence, \( \varphi_k(x) \to 0 \) for a.e. \( x \).

Next, let

\[ v_j = (1 - \varphi_j) u, \quad u_j = (v_j - 2^{-j}), \quad \text{and} \quad W_j = \{ x \in X : u_j(x) > 0 \}. \]

As \( \varphi_j \) and \( u \) are bounded it follows from the Leibniz rule [4, Theorem 2.15] that \( v_j \in N^{1,p}(X) \), and thus also \( u_j \in N^{1,p}(X) \). Hence, by Theorem 3.4, \( W_j \) is quasiopen and there is a set \( E_j \) with zero capacity such that \( W_j \setminus E_j \) is finely open. Let \( V_j = W_j \setminus \bigcup_{i=1}^{\infty} E_i \).

Then \( u_j \in N^{1,p}_0(V_j) \) and \( u_j \leq u \).

By the continuity of \( u|_{X \setminus G_j} \) and since \( u_j = 0 \) in the open set \( G_j \), we see that

\[ \overline{V_j} \subseteq \text{supp } u_j = \overline{W_j} \subset \{ x : u(x) \geq 2^{-j} \} \setminus G_j \subset U. \]

Note that \( \text{supp } u_j \) is bounded since \( \text{supp } u \) is bounded. As \( \{\varphi_k\}_{k=1}^{\infty} \) is decreasing, \( \{v_j\}_{j=1}^{\infty} \) is increasing. Since \( v_{j+1} \geq v_j > 2^{-j} \) in \( W_j \), we see that \( u_{j+1} \geq 2^{-j-1} \) in \( W_j \supset V_j \), from which we conclude that \( V_j \) is a \( p \)-strict subset of \( U \) as well as of \( E \).

We next want to show that

\[ u - u_j = (u - v_j) + (v_j - u_j) \to 0 \quad \text{in } N^{1,p}(X). \]

First, \( \| u - v_j \|_{L^p(X)} \to 0 \) and \( \| v_j - u_j \|_{L^p(X)} \leq 2^{-j} \mu(\text{supp } u) \to 0 \). Next, we see that (using the Leibniz rule [4, Theorem 2.15])

\[ g_{v_j - u_j} \leq g_{v_j} + \varphi_j g_u \leq g_{v_j} + \varphi_j g_{v_j}. \tag{4.2} \]

Since \( g_{v_j} \to 0 \) in \( L^p(X) \), \( \varphi_j \to 0 \) a.e., and \( g_u \in L^p(X) \), the right-hand side in Eq. 4.2 tends to 0 in \( L^p(X) \), by dominated convergence. Also

\[ g_{v_j - u_j} \leq (g_u + g_{v_j}) \chi_{\{0 < v_j < 2^{-j}\}} \leq g_u \chi_{\{0 < v_j < 2^{-j}\}} + g_{v_j} \to 0 \quad \text{in } L^p(X), \]

by dominated convergence since \( \chi_{\{0 < v_j < 2^{-j}\}}(x) \to 0 \) for a.e. \( x \). We thus conclude that \( \| u - u_j \|_{N^{1,p}(X)} \to 0 \) as \( j \to \infty \). 
By construction, $V_j \subset V_{j+1}$ and $0 \leq u_j \leq u_{j+1} \leq u$ for $j = 1, 2, \ldots$. It then follows from Corollary 1.72 in [4], that $u_j(x) \to u(x)$ for q.e. $x \in X$, as $j \to \infty$. After replacing $u_j$ by $u_j \chi_{V_j}$ one can also require that $u_j \equiv 0$ on $X \setminus V_j$.

5 Fine (super)minimizers

Definition 5.1 A function $u \in N^{1,p}_{\text{fine-loc}}(U)$ is a fine minimizer (resp. fine superminimizer) in $U$ if

$$\int_V g_u^p \, d\mu \leq \int_V g_{u+\varphi}^p \, d\mu$$

(5.1)

for every finely open $p$-strict subset $V \Subset U$ and for every (resp. every nonnegative) $\varphi \in N^1_0(V)$.

Moreover, $u$ is a fine subminimizer if $-u$ is a fine superminimizer.

By Remark 4.2, we may equivalently consider quasiope $p$-strict subsets $V \Subset U$ in Definition 5.1.

Remark 5.2 It follows from Proposition 5.9 below that if $u \in N^{1,p}_{\text{fine-loc}}(U)$ then $u$ is a fine (super)minimizer in $U$ if and only if it is a fine (super)minimizer in fine-int $U$. On the other hand, this equivalence is not true if we drop the assumption $u \in N^{1,p}_{\text{fine-loc}}(U)$ as seen in Example 8.2 below.

For the reader’s convenience, let us first look at the Euclidean case considered in Kilpeläinen–Malý [31]. By Remark 4.2 and [8, Theorem 1.1] the spaces $N^{1,p}(U)$, $N^{1,p}_{\text{fine-loc}}(U)$ and $N^{1,p}_0(U)$ are equal (up to a.e.-equivalence) to the spaces $W^{1,p}(U)$, $W^{1,p}_{\text{loc}}(U)$ and $W^{1,p}_0(U)$ defined for quasiope subsets of (unweighted) $\mathbb{R}^n$ in [31]. See also Theorem 7.2 below and [31, Theorem 2.10]. This is in particular true for open $U$, in which case $N^{1,p}(U)$ also agrees with the Sobolev space $H^{1,p}(U)$ in Heinonen–Kilpeläinen–Martio [29] (up to refined equivalence classes) also on weighted $\mathbb{R}^n$.

We next show that the fine supersolutions of [31] coincide with our fine superminimizers in $\mathbb{R}^n$. Recall that, for any $v \in N^{1,p}_{\text{fine-loc}}(U)$, with $U \subset \mathbb{R}^n$ quasiope, we have

$$|\nabla v| = g_v \text{ a.e. in } U,$$

(5.2)

where $\nabla v$ is as defined in [31]; see [8, Theorem 5.7]. The proof and the details above apply equally well if $\mathbb{R}^n$ is equipped with a $p$-admissible measure.

Proposition 5.3 Let $U \subset \mathbb{R}^n$ be quasiope and let $u \in N^{1,p}_{\text{fine-loc}}(U)$. Then $u$ is a fine superminimizer in $U$ if and only if $u$ is a fine supersolution of

$$-\text{div}(|\nabla u|^{p-2} \nabla u) = 0$$

(5.3)

in $U$ in the sense of Kilpeläinen–Malý [31, Section 3.1], i.e.

$$\int_V |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \geq 0$$

(5.4)

for all $p$-strict subsets $V \Subset U$ and all bounded nonnegative $\varphi \in N^1_0(V)$.
Proof First, let $u$ be a fine supersolution of Eq. 5.3 in $U$ and let $V \subseteq U$ be a $p$-strict subset of $U$. Let $\varphi \in N_{0}^{1,p}(V)$, $\varphi \geq 0$. Assuming also that $\varphi$ is bounded, we obtain from Eq. 5.4 that
\[
\int_{V} |\nabla u|^{p} \, dx = \int_{V} |\nabla u|^{p-2} \nabla u \cdot \nabla u \, dx \leq \int_{V} |\nabla u|^{p-2} \nabla u \cdot \nabla (u + \varphi) \, dx \\
\leq \left( \int_{V} |\nabla u|^{p} \, dx \right)^{1-1/p} \left( \int_{V} |\nabla (u + \varphi)|^{p} \, dx \right)^{1/p}.
\]
Since $u \in N^{1,p}(V)$, the first integral on the right-hand side is finite, and dividing by it shows that
\[
\int_{V} |\nabla u|^{p} \, dx \leq \int_{V} |\nabla (u + \varphi)|^{p} \, dx.
\]
(5.5)

If $\varphi$ is not bounded, then dominated convergence implies that
\[
\int_{V} |\nabla (u + \varphi)|^{p} \, dx = \lim_{k \to \infty} \int_{V} |\nabla (u + \min\{\varphi, k\})|^{p} \, dx,
\]
and so Eq. 5.5 holds also in this case. Equation 5.2 shows that $u$ is a fine superminimizer in the sense of Definition 5.1.

For the converse implication, assume that $u$ is a fine superminimizer in $U$. Let $V \subseteq U$ be a $p$-strict subset of $U$ and let $\varphi \in N_{0}^{1,p}(V)$ be bounded and nonnegative. Using Eq. 5.2, we have for any $0 < \varepsilon < 1$ that
\[
\int_{V} |\nabla u|^{p} \, dx \leq \int_{V} |\nabla (u + \varepsilon \varphi)|^{p} \, dx,
\]
and therefore
\[
\int_{V} \frac{|\nabla (u + \varepsilon \varphi)|^{p} - |\nabla u|^{p}}{\varepsilon} \, dx \geq 0.
\]
From this the inequality
\[
\int_{U} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \geq 0
\]
follows in the same way as in the proof of Theorem 5.13 in Heinonen–Kilpeläinen–Martio [29].

Lemma 5.4 A function $u$ is a fine minimizer in $U$ if and only if it is both a fine subminimizer and a fine superminimizer in $U$.

Proof Assume that $u$ is both a fine subminimizer and a fine superminimizer in $U$. Let $V \subseteq U$ be a finely open $p$-strict subset and let $\varphi \in N_{0}^{1,p}(V)$. We may assume that $\varphi = 0$ everywhere in $X \setminus V$. Since $\{\varphi \neq 0\}$ are quasiopen $p$-strict subsets of $U$ (by Theorem 3.4), testing Eq. 5.1 with $\varphi_{\pm} \neq 0$ implies that
\[
\int_{\{\varphi \neq 0\}} g_{\mu}^{p} \, d\mu = \int_{\{\varphi_{+} \neq 0\}} g_{\mu}^{p} \, d\mu + \int_{\{\varphi_{-} \neq 0\}} g_{\mu}^{p} \, d\mu \\
\leq \int_{\{\varphi_{+} \neq 0\}} g_{\mu+\varphi_{+}}^{p} \, d\mu + \int_{\{\varphi_{-} \neq 0\}} g_{\mu-\varphi_{-}}^{p} \, d\mu = \int_{\{\varphi \neq 0\}} g_{\mu+\varphi}^{p} \, d\mu,
\]
see Remark 4.2. Adding $\int_{V \cap \{\varphi = 0\}} g_{\mu}^{p} \, d\mu = \int_{V \cap \{\varphi = 0\}} g_{\mu+\varphi}^{p} \, d\mu$ to both sides shows that $u$ is a fine minimizer. The converse implication is trivial.

The following characterization is quite convenient. It also shows that the condition in Definition 5.1 can equivalently be required to hold for arbitrary $V \subset U$. 

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Lemma 5.5  Let \( u \in N_{1-p}^{1, p}(U) \). Then \( u \) is a fine (super)minimizer in \( U \) if and only if

\[
\int_{\{\phi \neq 0\}} g_u^p \, d\mu \leq \int_{\{\phi \neq 0\}} g_{u+\phi}^p \, d\mu
\]

(5.6)

for every (nonnegative) \( \phi \in N_{1-p}^{1, p}(U) \).

Note that for some \( \phi \) the integrals in Eq. (5.6) may be infinite, but then they are always infinite simultaneously. The characterization in Lemma 5.5 is in contrast to the definition of supersolutions in Eq. 5.4, where \( V = U \) is allowed only if \( u \in N_{1-p}^{1, p}(U) \).

Proof  Assume first that \( u \) is a fine superminimizer and that \( \phi \in N_{1-p}^{1, p}(U) \) is nonnegative. By Proposition 4.5, there are finely open \( p \)-strict subsets \( V_j \subset U \) and functions \( \phi_j \in N_{1-p}^{1, p}(V_j) \) such that \( 0 \leq \phi_j \leq \phi \) and

\[
\lim_{j \to \infty} \|\phi_j - \phi\|_{N_{1-p}(X)} = 0.
\]

(5.7)

Since \( u \) is a fine superminimizer, we see that

\[
\int_{V_j} g_u^p \, d\mu \leq \int_{V_j} g_{u+\phi_j}^p \, d\mu = \int_{V_j \cap \{\phi = 0\}} g_{u+\phi_j}^p \, d\mu + \int_{V_j \cap \{\phi \neq 0\}} g_u^p \, d\mu.
\]

As \( u \in N_{1-p}(V_j) \) the last term is finite, and we can thus subtract it from both sides in the inequality obtaining

\[
\int_{\{\phi \neq 0\}} g_u^p \, d\mu \leq \int_{\{\phi \neq 0\}} g_{u+\phi_j}^p \, d\mu + \int_{V_j \cap \{\phi \neq 0\}} g_u^p \, d\mu
\]

\[
\leq \int_{\{\phi \neq 0\}} g_{u+\phi_j}^p \, d\mu + \int_{V_j \cap \{\phi \neq 0\}} g_{u+\phi_j}^p \, d\mu = \int_{\{\phi \neq 0\}} g_{u+\phi_j}^p \, d\mu,
\]

which together with Eq. 5.7 shows that Eq. 5.6 holds.

Conversely, let \( V \subset U \) be a finely open \( p \)-strict subset and \( \phi \in N_{1-p}^{1, p}(V) \) be nonnegative. It then follows from Eq. 5.6 and the fact that \( g_u = g_{u+\phi} \) on \( \{x : \phi(x) = 0\} \), that Eq. 5.1 holds and thus \( u \) is a fine superminimizer. The claim for fine minimizers follows by splitting \( \phi = \phi_+ - \phi_- \) as in the proof of Lemma 5.4.

On open sets, we define (super)minimizers as in [4, Definition 7.7], using Lipschitz test functions with compact support. We shall now see that they coincide with the fine (super)minimizers.

Corollary 5.6  Let \( G \) be an open set. Then \( u \) is a fine (super)minimizer in \( G \) if and only if it is a (super)minimizer in \( G \).

Proof  Since \( G \) is open, \( N_{1-p}^{1, p}(G) = N_{1-p}^{1, p}(G) \) by Lemma 4.4. The equivalence then follows directly from Lemma 5.5 together with a similar characterization for standard (super)minimizers in open sets, see Proposition 3.2 in Björn [1] (or [4, Proposition 7.9]).

Lemma 5.7 (Pasting lemma) Assume that \( U_1 \subset U_2 \) are finely open sets, and that \( u_1 \) and \( u_2 \) are fine superminimizers in \( U_1 \) and \( U_2 \), respectively. Let

\[
u = \begin{cases} u_2 & \text{in } U_2 \setminus U_1, \\ \min\{u_1, u_2\} & \text{in } U_1. \end{cases}
\]
If $u \in N^{1,p}_{\text{fine-loc}}(U_2)$, then $u$ is a fine superminimizer in $U_2$.

**Proof** Assume that $u \in N^{1,p}_{\text{fine-loc}}(U_2)$. Let $V \Subset U_2$ be a finely open $p$-strict subset and $0 \leq \varphi \in N^{1,p}_0(V)$.

Let $\varphi_2 = (u+\varphi-u_2)_+$ and $\varphi_1 = \varphi-\varphi_2$. Lemma 2.37 in [4] implies that $\varphi_2 \in N^{1,p}_0(V)$. It is also easily verified that $\varphi_1 = 0$ outside $U_1$ and hence $\varphi_1 \in N^{1,p}_0(U_1)$. Lemma 5.5 applied to $u_1$, together with the facts that $u = u_1$ when $\varphi_1 > 0$ and $u = u_2$ when $\varphi = \varphi_2 > 0$, therefore yields

\[
\int_{\{\varphi > 0\}} g_p^u \, d\mu = \int_{\{\varphi_1 > 0\}} g_p^{u_1} \, d\mu + \int_{\{\varphi = \varphi_2 > 0\}} g_p^{u_2} \, d\mu
\]

\[
\leq \int_{\{\varphi > 0\}} g_p^{u_1 + \varphi} \, d\mu + \int_{\{\varphi = \varphi_2 > 0\}} g_p^{u_2} \, d\mu
\]

\[
= \int_{\{\varphi = \varphi_1 > 0\}} g_p^{u + \varphi} \, d\mu + \int_{\{\varphi > 0\}} g_p^{u_1 + \varphi} \, d\mu + \int_{\{\varphi = \varphi_2 > 0\}} g_p^{u_2} \, d\mu.
\]

Note that $u + \varphi_1 = u_2$ in $\{x : \varphi(x) > \varphi_1(x) > 0\} = \{x : \varphi(x) > \varphi_2(x) > 0\}$. Summing the last two integrals, we thus obtain

\[
\int_{\{\varphi > 0\}} g_p^u \, d\mu \leq \int_{\{\varphi = \varphi_1 > 0\}} g_p^{u + \varphi} \, d\mu + \int_{\{\varphi > 0\}} g_p^{u_1 + \varphi} \, d\mu
\]

\[
\leq \int_{\{\varphi = \varphi_1 > 0\}} g_p^{u + \varphi} \, d\mu + \int_{\{\varphi > 0\}} g_p^{u_2 + \varphi} \, d\mu,
\]

where in the last step we used Lemma 5.5, applied to $u_2$. Since $u_2 + \varphi_2 = u + \varphi$ in $\{\varphi > 0\}$, we conclude that

\[
\int_V g_p^u \, d\mu = \int_{\{\varphi > 0\}} g_p^u \, d\mu + \int_{V \cap \{\varphi = 0\}} g_p^{u + \varphi} \, d\mu \leq \int_V g_p^{u + \varphi} \, d\mu,
\]

i.e. Eq. 5.1 in the definition of fine superminimizers holds. \hfill \Box

**Corollary 5.8** If $u$ and $v$ are fine superminimizers in $U$, then $\min\{u, v\}$ is also a fine superminimizer in $U$.

Assume that $E$ is an arbitrary measurable set. Then the space $N^{1,p}_{\text{fine-loc}}(E)$ as well as fine minimizers and fine superminimizers in $E$ can be defined in the same way as in Definitions 4.1 and 5.1 (just replacing $U$ by $E$). The following characterization suggests that the notions of fine superminimizers and minimizers might not be very interesting beyond quasiopen sets.

**Proposition 5.9** Let $E$ be measurable and assume that $u \in N^{1,p}_{\text{fine-loc}}(E)$. Then $u$ is a fine (super)minimizer in $E$ if and only if it is a fine (super)minimizer in $V := \text{fine-int } E$.

Note that $u \in N^{1,p}_{\text{fine-loc}}(E)$ can be completely arbitrary on $E \setminus \text{fine-int } E$, even nonmeasurable. Moreover, if in addition $u \in N^{1,p}_{\text{loc}}(X)$ then $g_{u,X}$ and $g_{u,E}$ can differ substantially on $E \setminus \text{fine-int } E$. For this reason we define $g_u$ only on fine-int $E$, using Eq. 4.1 (with $U$ replaced by fine-int $E$). This is enough to make sense of the integrals below.
Proof of Proposition 5.9 Assume that \( u \) is a fine superminimizer in \( V \), and let \( \varphi \in N_0^{1,p}(E) \) be nonnegative. By Theorem 7.3 in Björn–Björn [5] we see that \( \varphi \in N_0^{1,p}(V) \). Hence, by Lemma 5.5,
\[
\int_{\{\varphi \neq 0\}} g_u^p \, d\mu \leq \int_{\{\varphi \neq 0\}} g_{u+\varphi}^p \, d\mu.
\]
Since Proposition 4.5 holds for \( E \), so does Lemma 5.5, from which it follows that \( u \) is a fine superminimizer in \( E \). The converse implication is clear and the proof for fine minimizers is similar. \( \square \)

6 The Obstacle and Dirichlet Problems

The obstacle problem will be a fundamental tool for studying fine minimizers.

Definition 6.1 Assume that \( U \) is bounded and \( C_p(X \setminus U) > 0 \). Let \( f \in N^{1,p}(U) \) and \( \psi : U \to \mathbb{R} \). Then we define
\[
K_{\psi,f}(U) = \{ v \in N^{1,p}(U) : v - f \in N_0^{1,p}(U) \text{ and } v \geq \psi \text{ q.e. in } U \}.
\]
A function \( u \in K_{\psi,f}(U) \) is a solution of the \( K_{\psi,f}(U) \)-obstacle problem if
\[
\int_U g_u^p \, d\mu \leq \int_U g_v^p \, d\mu \quad \text{for all } v \in K_{\psi,f}(U).
\]

The Dirichlet problem is a special case of the obstacle problem, with the trivial obstacle \( \psi \equiv -\infty \). Note that the boundary data \( f \) are only required to belong to \( N^{1,p}(U) \), i.e. \( f \) need not be defined on \( \partial U \) or the fine boundary \( \partial pU \).

Theorem 6.2 Assume that \( U \) is bounded and \( C_p(X \setminus U) > 0 \). Let \( f \in N^{1,p}(U) \) and \( \psi : U \to \mathbb{R} \), and assume that \( K_{\psi,f}(U) \neq \emptyset \). Then there exists a solution \( u \) of the \( K_{\psi,f}(U) \)-obstacle problem, and this solution is unique q.e. Moreover, \( u \) is a fine superminimizer in \( U \).

If \( \psi \equiv -\infty \) in \( U \) or if \( \psi \) is a fine subminimizer in \( U \), then \( u \) is a fine minimizer in \( U \).

If \( \psi \) is a fine subminimizer, then it follows that \( u \) is also a solution of the \( K_{-\infty,f}(U) \)-obstacle problem (as \( u \in N^{1,p}(U) \) is a fine minimizer) and Theorems 1.1, 7.5 and 7.6 are applicable.

Proof The existence and q.e.-uniqueness follow from Theorem 4.2 in Björn–Björn [5].

To show that \( u \) is a fine (super)minimizer in \( U \), let \( V \subseteq U \) be a finely open \( p \)-strict subset and let \( \varphi \in N_0^{1,p}(V) \). If \( \psi \) is not a fine subminimizer and \( \psi \neq -\infty \), then we also require \( \varphi \) to be nonnegative.

It is easily verified that \( v := \max\{u + \varphi, \psi\} \in K_{\psi,f}(U) \). Hence, as \( u \) is a solution of the \( K_{\psi,f}(U) \)-obstacle problem, we get that
\[
\int_U g_u^p \, d\mu \leq \int_U g_v^p \, d\mu = \int_{\{u+\varphi \geq \psi\}} g_{u+\varphi}^p \, d\mu + \int_{\{u+\varphi < \psi\}} g_{\psi}^p \, d\mu
\]
\[
\leq \int_{\{u+\varphi \geq \psi\}} g_{u+\varphi}^p \, d\mu + \int_{\{u+\varphi < \psi\}} g_u^p \, d\mu = \int_V g_{u+\varphi}^p \, d\mu + \int_{U \setminus V} g_u^p \, d\mu. \tag{6.1}
\]
where the second inequality is justified by Lemma 5.5 if $\psi$ is a fine subminimizer, and is trivial otherwise as $u + \varphi \geq \psi$ q.e. in $U$ in that case.

Since $u \in N^{1,p}(U)$, we see that the last integral in Eq. 6.1 is finite and subtracting it from both sides of Eq. 6.1 yields Eq. 5.1 in Definition 5.1 for the above choices of $V$ and $\varphi \in N^{1,p}_0(V)$. As $V$ was arbitrary, it follows that $u$ is a fine superminimizer in $U$. When $\varphi$ is not required to be nonnegative, we conclude that $u$ is a fine minimizer in $U$.

Note that there is a comparison principle for solutions of obstacle problems, see Corollary 4.3 in Björn–Björn [5].

7 Fine Continuity for Solutions of the Dirichlet Problem

In this section we assume that $U$ is a nonempty finely open set. Except for Theorem 7.2, we also assume that $U$ is bounded and that $C_p(X \setminus U) > 0$.

Recall that functions in $N^{1,p}_{\text{fine-loc}}(U)$, and in particular fine superminimizers, are finely continuous q.e. We do not know in general if fine minimizers have finely continuous representatives. However in this section we obtain sufficient conditions for the fine continuity of solutions of the (fine) Dirichlet problem, and deduce Theorem 1.1. The proof of our key Lemma 7.3 below was inspired by the proof of Theorem 5.3 in Kilpeläinen–Mály [31]. As we study fine continuity in this section it is natural to consider only finely open sets $U$.

With continuous boundary data, the solution of the Dirichlet problem in an open set need not be continuous at an irregular boundary point. However, the solution is finely continuous. We demonstrate this by the following example using Corollary 7.7 below.

Example 7.1 Consider, for example, a bounded open set $G \subset X$ with $C_p(X \setminus G) > 0$ and a strongly irregular boundary point $z \in \partial G$, see Björn [2, p. 40] (or [4, Definition 13.1]).

Then $X \setminus G$ is thin at $z$, by the sufficiency part of the Wiener criterion, see Björn–MacManus–Shanmugalingam [16, Theorem 5.1] and Björn [15, p. 370 and Corollary 3.11] (or [4, Theorem 11.24]). Thus $U = G \cup \{z\}$ is finely open. Moreover $C_p(\{z\}) = 0$, by the Kellogg property, see Björn–Björn–Shanmugalingam [11, Theorem 3.9] (or [4, Theorem 10.5]). Hence $C_p(X \setminus U) > 0$.

Since $z$ is strongly irregular, it follows from Theorem 13.13 in [4] that the continuous solution $h$ of the $\tilde{K}_{-\infty,d}(G)$-obstacle problem, with $d(x) = d(x, z)$, does not have a limit at $z$. However, by Corollary 7.7 below, $h$ does have a fine limit.

We will need the following auxiliary result, which may also be of independent interest. In what follows, the notions of fine lim, fine lim sup and fine lim inf are defined using punctured fine neighbourhoods. Note that since

$$\operatorname{cap}_p(B(x, r) \setminus \{x\}, B(x, 2r)) = \operatorname{cap}_p(B(x, r), B(x, 2r)),$$

there are no isolated points in the fine topology, i.e. no singleton sets are finely open.

Theorem 7.2 Let $U \subset V \subset X$ be finely open sets. Assume that $u \in N^{1,p}(U)$ and extend it by 0 to $V \setminus U$. Then the following are equivalent:

(a) $u \in N^{1,p}_0(U, V)$, i.e. $u \in N^{1,p}(V)$;
(b) $u$ is quasicontinuous in $V$;
(c) $u$ is finite q.e. and finely continuous q.e. in $V$;
(d) $u$ is measurable, finite q.e., and $u \circ \gamma$ is continuous for $p$-almost every curve $\gamma : [0, l_\gamma] \to V$;

(e) fine lim$_{U \ni y \to x} u(y) = 0$ for q.e. $x \in V \cap \partial_p U$.

We will only need the equivalence (a) $\iff$ (e) (when proving Lemma 7.3). However, when deducing this equivalence we will rely on several earlier results, which essentially requires us to obtain the full equivalence of (a)–(e).

**Proof** (a) $\Rightarrow$ (b) $\iff$ (c) These implications hold by Theorem 3.5.

(b) $\iff$ (d) This follows from Theorem 1.2 in Björn–Björn–Malý [10].

(d) $\Rightarrow$ (a) Let $g \in L^p(U)$ be a $p$-weak upper gradient of $u$ in $U$, extended by zero to $V \setminus U$. We shall show that $g$ is a $p$-weak upper gradient in $V$. Consider a curve $\gamma$ as in (d) such that none of its subcurves in $U$ is exceptional in Eq. 2.1 for the pair $(u, g)$. Lemma 1.34(c) in [4] implies that $p$-almost every curve has this property. If $\gamma \subset U$ or $\gamma \subset V \setminus U$, there is nothing to prove. Hence by splitting $\gamma$ into two parts, if necessary, and possibly reversing the direction, we may assume that $x = \gamma(0) \in U$ and $y = \gamma(l_\gamma) \notin U$. Let $c = \inf\{t : \gamma(t) \notin U\}$ and $y_0 = \gamma(c)$. By continuity, $u(y_0) = 0$ and we can assume that $c > 0$ as otherwise $u(x) = u(y_0) = u(y)$ and there is nothing to prove. Hence,

$$|u(x) - u(y)| = |u(x) - u(y_0)| = \lim_{\varepsilon \to 0^+} |u(x) - u \circ \gamma(c - \varepsilon)| \leq \int_{\gamma([0, c])} g \, ds \leq \int_{\gamma} g \, ds.$$ 

It follows that $g$ is a $p$-weak upper gradient of $u$ in $V$ and so $u \in N^{1,p}(V)$.

(c) $\Rightarrow$ (e) As $u$ is finely continuous q.e. and $u \equiv 0$ in $V \setminus U$, (e) follows directly.

(e) $\Rightarrow$ (c) Since $u \in N^{1,p}(U)$, it is finely continuous q.e. and finite q.e. in $U$, by Theorem 3.5. Thus $u$ is finite q.e. in $V$ and finely continuous q.e. in $V \setminus \partial_p U$. As $u \equiv 0$ in $V \setminus U$ and (e) holds, $u$ is finely continuous q.e. in $V \cap \partial_p U$. $\square$

We define for any function $u : U \to \mathbb{R}$ the fine lsc-regularization $u_* : \overline{U}^p \to \mathbb{R}$ of $u$ as

$$u_*(x) = \lim_{U \ni y \to x} \inf u(y), \quad \text{if } x \in \overline{U}^p,$$

and the fine usc-regularization $u^* : \overline{U}^p \to \mathbb{R}$ of $u$ as

$$u^*(x) = \lim_{U \ni y \to x} \sup u(y), \quad \text{if } x \in \overline{U}^p.$$

In this paper, we will only regularize Newtonian functions. As these are finely continuous q.e., we have $u = u_* = u^*$ q.e. in $U$. We say that $u$ is finely lsc-regularized if $u = u_*$ in $U$ and finely usc-regularized if $u = u^*$ in $U$. Note that $u_*$ (resp. $u^*$) is finely lsc-regularized (resp. finely usc-regularized) in $U$. Recall also the characterization of $\partial_p U$ in Lemma 3.2.

**Lemma 7.3** Let $z \in U$, $B = B(z, r)$, $f \in N^{1,p}(U)$ and let $u$ be a fine superminimizer in $B \cap U$ such that $u - f \in N^1_0(B \cap U, B)$. Assume that $c \in \mathbb{R}$ is such that

$$f_* \geq c \quad \text{q.e. in } B \cap \partial_p U.$$

If $u_*(z) < c$, then $u_*$ is finely continuous at $z$.

**Proof** Assume that

$$u_*(z) = \lim_{U \ni y \to z} \inf u(x) < c. \quad (7.1)$$
We want to apply the pasting Lemma 5.7 to the fine superminimizers \( c \) and \( u \) in the finely open sets \( B \) and \( B \cap U \), respectively. We therefore show that the function

\[
u_c = \begin{cases} 
  c & \text{in } B \setminus U, \\
  \min\{u, c\} & \text{in } B \cap U 
\end{cases}
\]

belongs to \( N^{1,p}(B) \). This will follow if we can show that \( u_c - c \in N^{1,p}_0(B \cap U, B) \), which we will do using characterization (e) in Theorem 7.2. For this purpose, it suffices to show that

\[
\liminf_{U \ni y \to x} u(y), c = c \quad \text{for q.e. } x \in B \cap \partial \rho U.
\]

(7.2)

Clearly, \( \limsup_{U \ni y \to x} \min\{u(y), c\} \leq c \) everywhere. As \( u - f \in N^{1,p}_0(B \cap U, B) \), Theorem 7.2 shows, shows that for q.e. \( x \in B \cap \partial \rho U \),

\[
\liminf_{U \ni y \to x} u(y) \geq \liminf_{U \ni y \to x} (u - f)(y) + \liminf_{U \ni y \to x} f(y) = f^*(x) \geq c.
\]

Hence, Eq. 7.2 holds and \( u_c \in N^{1,p}(B) \). Therefore, by Lemma 5.7, \( u_c \) is a fine superminimizer in \( B \). As \( B \) is open, \( u_c \) is a superminimizer in \( B \), by Corollary 5.6. It follows from Proposition 7.4 in Kinnunen–Martio [33] (or [4, Proposition 9.4]), that \( u_c \) has a superharmonic representative \( v \) such that \( v = u_c \) q.e. in \( B \). Thus, \( v \) is finely continuous in \( B \), by Björn [15, Theorem 4.4] or Korte [35, Theorem 4.3] (or [4, Theorem 11.38]). As \( v = u_c \) q.e. in \( B \cap U \), we conclude from Eq. 7.1 that

\[
v(z) = \liminf_{x \to z} u_c(x) = \liminf_{x \to z} u(x) = u^*(z) < c.
\]

Since \( v \) is finely continuous at \( z \), there is a fine neighbourhood \( V \) of \( z \) contained in \( B \cap U \) so that \( \sup V v < c \). Hence also

\[
u^*(z) = v(z) = \liminf_{x \to z} u_c(x) = \liminf_{x \to z} u(x) = u^*(z). \quad \square
\]

In what follows, the \( C_p \)-ess lim inf, \( C_p \)-ess lim sup and \( C_p \)-ess lim are taken with respect to the metric topology from \( X \) and up to sets of zero capacity in punctured neighbourhoods. For instance, for a function \( v \) defined in a set \( E \),

\[
C_p \text{- ess lim inf } v(x) := \lim_{r \to 0} C_p \text{- ess inf } v
\]

\[
E \ni x \to E \ni x \in (B(z,r) \setminus \{z\}) : v(x) < k \}
\]

In particular,

\[
C_p \text{- ess lim inf } v(x) = \infty \quad \text{if } C_p(E \cap (B(z,r) \setminus \{z\})) = 0 \text{ for some } r > 0.
\]

Corollary 7.4 Let \( z \in U \), \( f \in N^{1,p}(U) \) and let \( u \) be a fine superminimizer in \( U \) such that \( u - f \in N^{1,p}_0(U) \). If

\[
u^*(z) < C_p \text{- ess lim inf } f(x) \quad \text{or} \quad u^*(z) < C_p \text{- ess lim inf } f^*(x),
\]

(7.3)

then \( u_c \) is finely continuous at \( z \).

Proof It follows directly from the definition of \( f^* \) that

\[
C_p \text{- ess lim inf } f(x) \leq C_p \text{- ess lim inf } f^*(x),
\]

\[
\square
\]
and thus we can without loss of generality assume the latter inequality in Eq. 7.3. We can then find $c > u_*(z)$ and $B = B(z, r)$ such that
\[ f_*(x) \geq c \text{ for all } x \in B \cap \partial p U. \]

Lemma 7.3 concludes the proof.

Note that if $f \in N^{1,p}(X)$ in Corollary 7.4, then $f = f_*$ q.e. on $\overline{U}^p$ and thus
\[ C_p\text{-ess lim inf}_{\partial p U \ni x \to z} f_*(x) = C_p\text{-ess lim inf}_{\partial p U \ni x \to z} f(x) \quad (7.4) \]
in Eq. 7.3.

**Theorem 7.5** Let $f \in N^{1,p}(U)$. Then the finely lsc-regularized solution $h_*$ of the $K_{-\infty,f}(U)$-obstacle problem is finely continuous at each $z \in U$ which satisfies one of the following conditions:

(a) The limit $C_p\text{-ess lim}_{U \ni x \to z} f(x)$ exists.

(b) The equality $C_p\text{-ess lim}_{\partial p U \ni x \to z} f_*(x) = C_p\text{-ess lim}_{\partial p U \ni x \to z} f^*(x)$ holds.

(c) There exists $r > 0$ such that $C_p(B(z, r) \cap \partial p U) = 0$.

**Proof** (a) Assume that $h_*$ (and hence also $h^*$) is not finely continuous at $z$, i.e. that $h_*(z) < h^*(z)$. Then Corollary 7.4, applied to both $h_*$ and $-h^*$, together with the assumption (a) shows that
\[ h_*(z) \geq C_p\text{-ess lim}_{U \ni x \to z} f(x) = -C_p\text{-ess lim}_{U \ni x \to z} (-f)(x) \geq h^*(z), \]
a contradiction. Hence $h_*$ is finely continuous at $z$.

(b) Assume, as in (a), that $h_*$ is not finely continuous at $z$. This time, Corollary 7.4 shows that
\[ h_*(z) \geq C_p\text{-ess lim}_{\partial p U \ni x \to z} f_*(x) = -C_p\text{-ess lim}_{\partial p U \ni x \to z} (-f^*)(x) \geq h^*(z), \]
a contradiction. Hence $h_*$ is finely continuous at $z$.

(c) If $h_*(z) = \infty$, then also $h^*(z) = \infty$ and $h$ is finely continuous at $z$. Otherwise,
\[ h_*(z) < \infty = C_p\text{-ess lim inf}_{\partial p U \ni x \to z} f_*(x), \]
and the conclusion follows from Corollary 7.4.

We can now prove Theorem 1.1.

**Proof of Theorem 1.1** Let $h_*$ be the finely lsc-regularized solution of the Dirichlet problem, i.e. of the $K_{-\infty,f}(U)$-obstacle problem, and let $z \in U$. If $f \in C(U) \cap N^{1,p}(U)$, then condition (a) in Theorem 7.5 holds. Recall that here $f$ is assumed continuous with values in $\mathbb{R}$.

On the other hand, if $f \in C(\overline{U}^p \cap \partial U) \cap N^{1,p}(X)$, then either condition (c) in Theorem 7.5 is fulfilled, or Eq. 7.4 and the continuity of $f$ on $\overline{U}^p \cap \partial U \supset \partial p U \cup (U \cap \partial p U)$ yield
\[ C_p\text{-ess lim}_{\partial p U \ni x \to z} f_*(x) = C_p\text{-ess lim}_{\partial p U \ni x \to z} f(x) = C_p\text{-ess lim}_{\partial p U \ni x \to z} f^*(x), \]
i.e. condition (b) in Theorem 7.5 holds.

Thus, in all cases, $h_*$ is finely continuous at $z$ by Theorem 7.5.
As an application of Theorem 7.5 we obtain the following result. Note that $V$ is finely open since it is the intersection of the finely open set $U$ and the open set $X \setminus \{z\}$. Moreover, $\partial_p V = \partial_p U \cup \{z\}$ as there are no finely isolated points, see Lemma 3.2.

**Theorem 7.6** Let $z \in U$ and $V = U \setminus \{z\}$. Let $f \in N^{1,p}(V)$ and let $h_V$ be a solution of the $K_{-\infty}, f (V)$-obstacle problem. Assume that one of the following holds:

(a) $C_p(\{z\}) > 0$ and \( \lim_{V \ni x \to z} f(x) \) exists, which in particular holds if $f \in N^{1,p}(U)$.

(b) $C_p(\{z\}) = 0$ and $C_p(\lim_{V \ni x \to z} f(x))$ exists.

Then the fine limit

$$\lim_{V \ni x \to z} h_V(x)$$

exists. \hfill (7.5)

**Proof** (a) Note first that if $f \in N^{1,p}(U)$, then $f$ is finely continuous q.e. in $U$ and thus $f(z) = \lim_{V \ni x \to z} f(x)$. Since $h_V - f \in N^{1,p}_0(V)$, Theorem 7.2 implies that

$$\lim_{V \ni x \to z} (h_V - f)(x) = 0$$

and Eq. 7.5 follows.

(b) In this case, Proposition 1.48 in [4] implies that $f, h_V \in N^{1,p}(U)$ (where $h_V(z)$ is defined arbitrarily). Moreover, $h_V - f \in N^{1,p}_0(V) \subset N^{1,p}_0(U)$. Let $h_U$ be a solution of the $K_{-\infty}, f (U)$-obstacle problem. Then $h_U - f \in N^{1,p}_0(U)$ and by the uniqueness part of Theorem 6.2, we conclude that $h_U = h_V$ q.e. in $U$. Theorem 7.5 and the assumption (b) imply that $h_U$ is finely continuous at $z$. As fine limits do not see sets of zero capacity, we get that

$$\lim_{V \ni x \to z} h_V(z) = \lim_{U \ni x \to z} h_U(x) = h_U(z).$$

**Corollary 7.7** Let $G$ be a nonempty bounded open set with $C_p(X \setminus G) > 0$ and let $z \in \partial G$. Let $f \in N^{1,p}(G)$ and assume that the limit

$$f(z) := \lim_{G \ni x \to z} f(x)$$

exists.

Let $h$ be a solution of the $K_{-\infty}, f (G)$-obstacle problem. Then the fine limit

$$\lim_{G \ni x \to z} h(x)$$

exists.

**Proof** If $z$ is a regular point of $\partial G$, then even the metric limit

$$\lim_{G \ni x \to z} h(x) = f(z)$$

exists, by (a) $\Rightarrow$ (g) in Theorem 6.11 in Björn–Björn [3] (or [4, (a) $\Rightarrow$ (i) in Theorem 11.11]). On the other hand, if $z$ is an irregular boundary point of $G$, then $X \setminus G$ is thin at $z$ by the sufficiency part of the Wiener criterion, and $C_p(\{z\}) = 0$ by the Kellogg property (see Example 7.1 for references). Hence $G \cup \{z\}$ is finely open, $f \in N^{1,p}(G \cup \{z\})$, and the claim follows from Theorem 7.6.

In terms of Perron solutions on open sets, Corollary 7.7 yields the following consequence. Here $Pf$ denotes the Perron solution in $G$ with boundary data $f$, see [4, Section 10.3]. Recall that if $f \in C(\partial G)$ then $f$ is resolutive and thus $Pf$ exists, by Theorem 6.1 in Björn–Björn–Shanmugalingam [12] (or [4, Theorem 10.22]).
**Corollary 7.8** Let $G$ be a nonempty bounded open set with $C_p(X \setminus G) > 0$ and let $f \in C(\partial G)$. Then the fine limit

$$\text{fine lim}_{G \ni x \to z} Pf(x)$$

exists for all $z \in \partial G$.

**Proof** Let $\varepsilon > 0$. Then there is $\tilde{f} \in \text{Lip}_c(X)$ such that

$$f \leq \tilde{f} \leq f + \varepsilon \quad \text{on} \quad \partial G.$$

It follows from the definition of Perron solutions, and the resolutivity of continuous functions, that

$$Pf \leq P\tilde{f} \leq Pf + \varepsilon \quad \text{in} \quad G.$$

Moreover, $P\tilde{f}$ is the continuous solution of the $K_{-\infty, \tilde{f}}(G)$-obstacle problem, by Theorem 5.1 in Björn–Björn–Shanmugalingam [12] (or [4, Theorem 10.12]). Using Corollary 7.7 and letting $\varepsilon \to 0$ shows that the fine limit in Eq. 7.6 exists for all $z \in \partial G$. \hfill \Box

### 8 Removability

In this section we assume that $U$ is a quasiopen set.

We conclude the paper by deducing some simple removability results.

**Lemma 8.1** Let $E$ be a set with $C_p(E) = 0$ and $V = U \cup E$. If $u \in N^{1,p}_{\text{fine-loc}}(V)$ is a fine (super)minimizer in $U$, and $u$ is extended arbitrarily to $E$, then $u$ is a fine (super)minimizer in $V$.

**Proof** By Theorem 3.4, $V$ is quasiopen. Let $\varphi \in N^{1,p}_{0}(V)$. Since $\varphi = 0$ q.e. in $X \setminus U$, also $\varphi \in N^{1,p}_{0}(U)$ and the statement follows directly from Lemma 5.5. \hfill \Box

**Example 8.2** (a) It is not enough to assume that $u \in N^{1,p}_{\text{fine-loc}}(U)$ in Lemma 8.1. Consider e.g. $p = 2$, $U = B(0, 1) \setminus \{0\} \subset \mathbb{R}^2$ and

$$u(x) = \log |x|^{(p-2)/(p-1)} \in N^{1,p}_{\text{loc}}(U) = N^{1,2}_{\text{fine-loc}}(U),$$

which is harmonic (and thus a fine minimizer) in $U$. However $u$ has no extension in $N^{1,2}_{\text{loc}}(V) = N^{1,2}_{\text{fine-loc}}(V)$, with $V = B(0, 1)$, and in particular no extension as a fine superminimizer (i.e. as a superminimizer because $V$ is open), even though $C_p(V \setminus U) = 0$.

(b) Even if $U = \text{fine-int} V$, the assumption $u \in N^{1,p}_{\text{fine-loc}}(V)$ cannot be replaced by $u \in N^{1,p}_{\text{fine-loc}}(U)$ in Lemma 8.1. Moreover, fine (super)minimizers on a quasiopen set $V$ can differ from those on its fine interior fine-int $V$.

To see this, let $1 < p < 2$,

$$U = (0, 2) \times (-2, 2) \quad \text{and} \quad V = U \cup \{(0, 0)\}.$$

This time,

$$u(x) = |x|^{(p-2)/(p-1)} \in N^{1,p}_{\text{loc}}(U) = N^{1,p}_{\text{fine-loc}}(U)$$

is a fine minimizer (i.e. a minimizer) in the open set $U$, see Example 7.47 in Heinonen–Kilpeläinen–Martio [29].

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On the other hand, the set
\[ W := \{ (x_1, x_2) : 0 < |x_2| < x_1 < 1 \} \subseteq V \]
is a \( p \)-strict subset of both \( U \) and \( V \). This is easily seen by using
\[ h(x_1, x_2) = \eta(x) \min \left\{ \frac{x_1}{|x_2|}, 1 \right\} \in N_0^{1,p}(U) \]
with a suitable cutoff function \( \eta \), see Example 5.7 in Björn–Björn [3] (or [4, Example 11.10]). However, \( u \notin N_1^{1,p}(W) \) (and even \( u \notin L^p(W) \) if \( 1 < p < \sqrt{2} \)), so \( u \notin N_1^{1,p}(V) \). (Since \( W \not\subset U \), we still have \( u \in N_1^{1,p}(U) \).)

**Corollary 8.3** Let \( G \) be an open set and \( V = G \setminus E \), where \( C_p(E) = 0 \). Assume that \( u \in N_1^{1,p}(V) \) or \( u \in N_1^{1,p}(G) \). Also let
\[ u_*(x) = \liminf_{y \to x} u(y), \quad \text{if} \ x \in V. \]

(a) If \( u \) is a fine superminimizer in \( V \), then \( u_* \) is finely continuous in \( V \).

(b) If \( u \) is a fine minimizer in \( V \), then \( u_* \) is continuous in \( V \), with respect to the metric topology.

**Proof** If \( u \in N_1^{1,p}(V) \) then it follows from Proposition 1.48 in [4] that \( \tilde{u} \in N_1^{1,p}(G) \), where \( \tilde{u} \) is any extension of \( u \) to \( G \). Thus we can assume that \( u \in N_1^{1,p}(G) \). By Lemma 8.1 and Corollary 5.6, \( u \) is a superminimizer in \( G \). It follows from Proposition 7.4 in Kinnunen–Martio [33] (or [4, Proposition 9.4]), that \( u \) has a superharmonic representative \( v \) such that \( v = u \) q.e. in \( G \).

In (a), \( v \) is finely continuous in \( G \), by Björn [15, Theorem 4.4] or Korte [35, Theorem 4.3] (or [4, Theorem 11.38]). In (b), \( v \) is continuous in \( G \), by Kinnunen–Shanmugalingam [34, Proposition 3.3 and Theorem 5.2] (or [4, Theorem 8.14]).

As \( v = u \) q.e., we have \( u_* = v_* = v \) in \( V \), which proves the statement. \( \Box \)

## 9 Open Problems

Fine superminimizers and fine supersolutions can be changed arbitrarily on sets of capacity zero. To fix a precise representative, in potential theory one usually studies pointwise defined finely (super)harmonic functions with additional regularity properties, as used in the proofs of Lemma 7.3 and Corollary 8.3.

In this paper, we do not go further into making a definition of finely (super)harmonic functions in metric spaces. Even in the linear case, there have been several different suggestions for such definitions in the literature, see Lukeš–Malý–Zajíček [44, Section 12.A and Remarks 12.1]. Some definitions have been given in the nonlinear theory on \( \mathbb{R}^n \), but the theory is even less developed and there are many open questions in this context. A few of these are listed below.

**Open problems 9.1** (1) *Is every finely superharmonic function finely continuous?* This is known in the linear case, see [22, Theorem 9.10] and [44, Theorem 12.6]. In the nonlinear case, the best known result is Corollary 7.12 in Latvala [40], which says that the finely superharmonic functions associated with the \( n \)-Laplacian on unweighted \( \mathbb{R}^n \) are approximately continuous.
(2) Does every bounded fine minimizer $u$ have a finely continuous representative $v$ such that $v = u \text{ q.e.}$? Even this special case of (1) is open in the nonlinear case. However, on unweighted $\mathbb{R}^n$, with $p = n \geq 2$, this fact was shown by Latvala [40, Lemma 7.15].

(3) If $u$ is a fine minimizer, is then $u_*$ finite everywhere? This seems to be open even in the linear case on unweighted $\mathbb{R}^n$ (with $p = 2$), since the connection between fine solutions (= fine minimizers) and finely harmonic functions does not seem to have been touched upon in the linear literature.

(4) On unweighted $\mathbb{R}^n$, Latvala [41] showed that $U \setminus E$ is a $p$-fine domain if $U$ is a $p$-fine domain and $C_p(E) = 0$. As an application of this result a strong version of the minimum principle for finely superharmonic functions was obtained. We do not know if the corresponding fine connectedness result holds in our metric setting, or on weighted $\mathbb{R}^n$.

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