GENERALIZED SEMIMAGIC SQUARES  
FOR DIGITAL HALFTONING  

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ABSTRACT. Completing Aronov et al.’s study on zero-discrepancy matrices for digital halftoning, we determine all \((m,n,k,l)\) for which it is possible to put \(mn\) consecutive integers on an \(m \times n\) board (with wrap-around) so that each \(k \times l\) region has the same sum. For one of the cases where this is impossible, we give a heuristic method to find a matrix with small discrepancy.

A semimagic square is a square matrix whose entries are consecutive integers and which has equal row and column sums. One way to generalize this millennia-old concept is to specify the sums on regions other than rows and columns. Ingenious constructions of squares satisfying various sum constraints have been described by many professional and amateur mathematicians. While most of them are interested in adding more and more constraints to make their squares impressive, one can generally consider sum conditions on any set of regions.

Aronov et al. \[1\] took up this problem for square regions: is there an \(n \times n\) matrix with entries \(0, \ldots, n^2 - 1\) such that every \(k \times k\) region has the same sum? It is amusing to note that this variant of the classical problem is motivated by an engineering question of finding good dither matrices for digital halftoning, a method to approximate a continuous-tone image by a binary image for printing (see their paper for details). They showed \[1, \text{Theorem 1}\], using what they call constant-gap matrices, that the answer is yes if \(k\) and \(n\) are even or if \(n\) is an integer power of \(k\), and no if \(k\) and \(n\) are relatively prime or if \(k\) is odd and \(n\) is even. We will solve this problem completely by determining all \((n,k)\) for which such matrices exist (Section 1). Our construction of the matrices is much simpler even for the cases that have already been settled positively. We also give counterexamples to Asano et al.’s conjecture on the smallest possible discrepancy when \(n\) is odd and \(k = 2\) (Section 2).

Definitions. For a positive integer \(N\), we write \([N] = \{0, 1, \ldots, N-1\}\). The remainder when an integer \(x\) is divided by \(N\) belongs to \([N]\) and is denoted by \(x \mod N\).

We consider the slightly generalized setting where the matrices and regions are rectangles instead of squares. Let \(m\) and \(n\) be positive integers. For an \(m \times n\) matrix \(D\) and index \((i,j) \in [m] \times [n]\), we denote the \((i,j)\)th entry of \(D\) by \(D(i,j)\). Any set \(R \subseteq [m] \times [n]\) of indices is called a region. The sum of the numbers on \(R\) is denoted by \(D(R) = \sum_{(i,j) \in R} D(i,j)\). The discrepancy of \(D\) with respect to a set \(\mathcal{R}\) of regions is the difference between the maximum and minimum \(D(R)\) as \(R\) varies in \(\mathcal{R}\). When it is zero, \(D\) is said to be \(\mathcal{R}\)-uniform.

The translate of \(R\) by \((a,b) \in \mathbb{Z}^2\) is denoted by

\[(1) \quad R + (a,b) = \{ ((i+a) \mod m, (j+b) \mod n) : (i,j) \in R \} \subseteq [m] \times [n].\]

The set of all translates of \(R\) is denoted by \(\overline{R} = \{ R + (a,b) : (a,b) \in \mathbb{Z}^2 \}\).
By an $m \times n$ table we mean an $m \times n$ matrix in which each element of $[mn]$ appears exactly once. We are interested in tables with small (or zero) discrepancy with respect to $[k] \times [l]$, the set of all $k$-by-$l$ rectangles (Figure 1).

1. **Zero discrepancy**

The greatest common divisor of positive integers $x$ and $y$ is denoted by $\gcd(x, y)$. The goal of this section is to show the following:

**Theorem 1.** Let $m$, $n$, $k$, $l$ be positive integers with $k < m$ and $l < n$. Let $k' = \gcd(k, m)$ and $l' = \gcd(l, n)$. Then there exists a $[k'] \times [l']$-uniform $m \times n$ table if and only if $k'$ and $l'$ are greater than 1 and $k'l'(mn - 1)$ is even.

This is an immediate consequence of the following Lemmas 2 and 3.

**Lemma 2.** A $[k] \times [l]$-uniform $m \times n$ matrix is $[\gcd(k, m)] \times [\gcd(l, n)]$-uniform.

**Proof.** Let $D$ be a $[k] \times [l]$-uniform $m \times n$ matrix. We will show that $D$ is $[k'] \times [l']$-uniform, where $k' = \gcd(k, m)$. We get the conclusion of the lemma by repeating the same argument with rows and columns switched.

For each $(i, j) \in [m] \times [n]$, the regions $[k'] \times [l] + (i, j)$ and $[k'] \times [l] + (i + k, j)$ have the same sum on $D$, because each of them combined with $[k - k'] \times [l] + (i + k', j)$ makes a $k \times l$ rectangle. Thus for each $(i, j) \in [m] \times [n]$, the rectangles

$[k'] \times [l] + (i + qk, j), \quad q \in [m/k']$,

all have the same sum on $D$. Since $k' = \gcd(k, m)$, these $m/k'$ rectangles cover the strip $[m] \times [l] + (0, j)$ without overlap. Hence,

$$
\frac{m}{k'} \cdot D([k'] \times [l] + (i, j)) = \sum_{q \in [m/k']} D([k'] \times [l] + (i + qk, j)) = D([m] \times [l] + (0, j)) = \frac{1}{k} \sum_{r \in [m]} D([k] \times [l] + (r, j)).
$$

Since the rightmost side is a constant independent of $(i, j)$ by $[k] \times [l]$-uniformity, so is the leftmost side. Thus $D$ is $[k'] \times [l']$-uniform. \qed

**Lemma 3.** Let $m$ and $n$ be positive integers, and let $k < m$ and $l < n$ be their positive divisors, respectively. Then there exists a $[k] \times [l]$-uniform $m \times n$ table if and only if $k$ and $l$ are greater than 1 and $kl(mn - 1)$ is even.
If \( P \) the desired matrix.

**Lemma 4.** Let \( D \) be a \([k] \times [l]\)-uniform \( m \times n \) matrix. It is easy to see that \( D(R) = kl(mn - 1)/2 \) for each \( R \in [k] \times [l] \). Since \( D(R) \) must be an integer, the second claim follows. For the first claim, assume \( k = 1 \) for contradiction (the case \( l = 1 \) is similar). Then \( D([1] \times [l]) = D([1] \times [l]) + (0, 1) \) and hence \( D(0, 0) = D(0, l) \), contradicting the assumption that \( D \) is a table.

For the converse, we use the building blocks provided by the following lemma:

**Lemma 4.** Let \( k > 1 \) and \( l > 0 \) be integers and let \( n \) be a positive multiple of \( l \). If \( kl(n-1) \) is even, then there exists a \([k] \times [l]\)-uniform \( k \times n \) matrix in which each row is a permutation of \([n]\).

**Proof.** A \([k] \times [l]\)-uniform \( k \times n \) matrix and a \([k'] \times [l']\)-uniform \( k' \times n \) matrix stacked vertically make a \([k+k'] \times [l+l']\)-uniform \((k+k') \times n\) matrix. Also, a \([k] \times [l]\)-uniform matrix is \([k] \times [l']\)-uniform for any multiple \( l' \) of \( l \). Therefore, it suffices to construct the desired matrix \( P \) for the cases \((k,l) = (2,1), (3,1) \) and \((3,2)\) (Figure 2). If \((k,l) = (2,1)\), let

\[
\begin{align*}
P(0,j) &= j, & P(1,j) &= n - 1 - j.
\end{align*}
\]

If \((k,l) = (3,1)\), then \( n \) is odd by the assumption; let

\[
\begin{align*}
P(0,j) &= j, & P(1,j) &= \left(j + \frac{n-1}{2}\right) \mod n, & P(2,j) &= (-2j-1) \mod n.
\end{align*}
\]

If \((k,l) = (3,2)\), let

\[
\begin{align*}
P(0,j) &= P(1,j) = \left[\frac{j}{2}\right] + \frac{n}{2}(j \mod 2), & P(2,j) &= n - 1 - j.
\end{align*}
\]

It is easy to verify that \( P \) is \([k] \times [l]\)-uniform in each case. \(\square\)

**Proof of the “if” part of Lemma 3.** We may assume without loss of generality that \( l(mn-1) \) is even. In this case, both \( kl(n-1) \) and \( l(m/k - 1) \) are even, so by Lemma 4, there are a \([k] \times [l]\)-uniform \( k \times n \) matrix \( P \) whose rows are permutations of \([n]\), and a \([l] \times [1]\)-uniform \( l \times (m/k) \) matrix \( Q \) whose rows are permutations of \([m/k]\). Define an \( m \times l \) matrix \( T \) by

\[
T(a,j) = Q(j, \lfloor a/k \rfloor)k + (a \mod k).
\]

Then \( T \) is \([k] \times [l]\)-uniform and its columns are permutations of \([m]\). Define an \( m \times n \) matrix \( D \) by

\[
D(a,b) = P(a \mod k, b)m + T(a, b \mod l)
\]
\[ D = \begin{bmatrix} P & P \\ P & P \end{bmatrix} \times 9 + \begin{bmatrix} T & T & T & T \end{bmatrix} \]

(Figure 3). Since \( P \) and \( T \) are \([k] \times [l]\)-uniform, so is \( D \). To see that \( D \) is a table, suppose that \( D(a, b) = D(a', b') \). By (7) and (8) we see that

\[
\begin{align*}
P(a \mod k, b) &= P(a' \mod k, b'), \\
Q(b \mod l, \lfloor a/k \rfloor) &= Q(b' \mod l, \lfloor a'/k \rfloor), \\
a \mod k &= a' \mod k.
\end{align*}
\]

Since \( P \)'s rows are permutations, the first and the third equation imply that \( b = b' \). Since \( Q \)'s rows are permutations, this and the second equation imply that \( a = a' \). □

In the above, we constructed the uniform table as a linear combination of two uniform matrices with smaller entries. This idea is due to Euler [3] who gave a construction of a semimagic square (that is, a \((1 \times [n] \cup [n] \times 1)\)-uniform \( n \times n \) table) from a pair of special \((1 \times [n] \cup [n] \times 1)\)-uniform matrices called Latin squares.

2. Finding low-discrepancy tables by ranking

In this section, we confine ourselves, as Asano et al. [2] did, to the case where \( k = l = 2 \) and \( m = n \). Theorem 1 states that in this case a uniform table exists if and only if \( n \) is even. For odd \( n \)'s, they construct a table with discrepancy \( 2n \), and conjecture that it is the smallest possible. This is refuted by our Figures 1 and 4. Figure 1 was discovered by an exhaustive search. We describe briefly how Figure 4 was obtained.

Define \( f : [0, 1]^2 \to \mathbb{R} \) by \( f(x, y) = g(x) + g(y) \), where

\[
g(x) = \begin{cases} 1 - (4x - 1)^2 & \text{if } x \leq 1/2, \\
-1 + (4x - 3)^2 & \text{if } x \geq 1/2
\end{cases}
\]

(Figure 5). Let \( \alpha, \beta \in [0, 1] \) and define \( s : [n]^2 \to [0, 1]^2 \) by

\[
s(i, j) = \left( \frac{i + \alpha}{n}, \frac{j + \beta}{n} \right).
\]
Let $H$ be the $n \times n$ table whose $(i,j)$th entry is the rank of $f(s(i,j))$ (with some tie-breaking rule):

$$H(i,j) = \left| \{ (i',j') \in [n]^2 : f(s(i',j')) < f(s(i,j)) \} \right| \quad \text{or} \quad \left( f(s(i',j')) = f(s(i,j)) \right) \text{ and } ni' + j' < ni + j \right|.$$ 

Finally, define the desired matrix $D$ by

$$D((i+j) \mod n, (i-j) \mod n) = H(i,j).$$

Figure 4. A $31 \times 31$ table whose discrepancy with respect to $[2] \times [2]$ is 27.

To see intuitively why $D$ has small discrepancy, note that a $2 \times 2$ region in $D$ corresponds to the region in $H$ (or its translate) shown in Figure 6. These four cells are mapped by $s$ to two nearby points $(x, y)$ and another two points $(x+1/2, y+1/2)$ (the coordinates are modulo 1). Since $f(x, y) = -f(x+1/2, y+1/2)$, the sum of the values of $f$ at these four points is almost zero. Thus, assuming that taking the ranks does not distort the distribution of values too much, we can expect that $D$ has low discrepancy. We add the displacement $(\alpha, \beta)$ in order to reduce the chance of ties in the ranking which seem to work adversely.

As Aronov et al. [1] point out, our problem is analogous to a common situation in discrete geometry where we try to arrange discrete objects so that they look close to some “balanced” continuous distribution. The constraint peculiar to our problem is
that we have to use each number in \([mn]\) exactly once. The ranking technique used here may be applicable to other problems with this constraint. However, analyzing its performance seems to be hard: although our computer experiment for several \(n\)'s suggests that the above method achieves sublinear \(2 \times 2\) discrepancy, we have no proof yet.

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