MULTIPLICITY THEOREMS IN FRÉCHET SETTING

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Abstract. We prove multiplicity theorems for Keller $C^1_c$-functionals on Fréchet spaces and Finsler manifolds which are invariant under the action of a discrete subgroup. For such functionals we evaluate the minimal number of critical points by applying the Lusternik-Schnirelmann category.

1. Introduction

Functions which are invariant under a group action on manifolds or linear spaces may have more critical points overall. This is due to the fact that an invariant function may be viewed as a function on the quotient space which commonly has the richer topology than the ambient space. The study of critical points of invariant functions in the Banach setting has a long history and involves theories such as Ljusternik-Schnirelman category theory, genus and cogenus theory and various topological index theories, etc, see for example [4, 3, 5, 12].

In aforementioned theories, an effectual technique in obtaining critical points of a differentiable functional is based on deformation arguments along gradient or pseudogradient vector fields. Deformation flows generated by such vector fields are broadly obtained by solving a Cauchy problem. The other method of locating critical points is based on variational principles among which the Ekeland variational principle is the prominent one. It states the existence of a certain minimizing sequence on a complete metric space along which we reach the infimum value of the minimization problem.

An attempt to extend multiplicity results to more general context of Fréchet manifolds was made in [2], where the Ljusternik-Schnirelman category theorem was proved for Fréchet Finsler manifolds. As mentioned in [2], there are several significant obstructions to extend the deformation technique to the Fréchet setting: lack of general solvability theory of ODEs. In a Fréchet space a linear ODE may have no solution for any initial value condition. Besides, if solutions to initial value problem exist they may not be unique. In addition, because of the deficient topological structures of the duals of Fréchet spaces, cotangent bundles do not admit smooth manifold structures and consequently the notion of pseudogradient vector fields and Finsler structures on cotangent bundles make no sense.

In this regard, in [2] it was proved a deformation result which is not implemented by considering pseudogradient flows for the case of Fréchet Finsler manifolds. Also, it was introduced the Palais-Smale condition on manifolds by using an auxiliary function to detour

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the need of a smooth structure on cotangent bundles. In this paper by applying the deformation result we extend the Ljusternik-Schnirelmann theorem for the case differentiable functionals on Fréchet Finsler manifolds which are invariant under a compact Lie group action (Theorem 4.2). For the case Fréchet spaces an analogous deformation result and a Ljusternik-Schnirelmann theorem are not available yet. Therefore, we consider another multiplicity problem, namely evaluating the minimal number of the critical orbits of a functional which is invariant under the action of a discrete group of a Fréchet space (see Theorem 3.3). To prove this theorem we employ the Ekeland variational principle.

2. Preliminaries

In this section we briefly recall the basic concepts of the theory of Fréchet spaces and establish our notations.

By \( U \subseteq T \) we mean that \( U \) is an open subset of a topological space \( T \). If \( S \) is another topological space, then we denote by \( C^p(T, S) \) the set of continuous mappings from \( T \) into \( S \).

We denote by \( \mathcal{B}_F \) a Fréchet space whose topology is defined by a sequence of seminorms \((\|\cdot\|_{F, n})_{n \in \mathbb{N}}\), which we can always assume to be increasing (by considering \( \max_{k \leq n} \|\cdot\|_{F, n} \), if necessary). Moreover, the complete translation-invariant metric

\[
d_F(x, y) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|x - y\|_{F, n}}{1 + \|x - y\|_{F, n}}
\]

induces the same topology on \( F \). We denote by \( B_d(x, r) \) an open ball with center \( x \) and radius \( r > 0 \). We do not write the metric \( \rho \) when it does not cause confusion. In what follows we consider only Fréchet spaces over the field \( \mathbb{R} \) of real numbers. Throughout the paper we assume that \( F, E \) are Fréchet spaces and \( L(E, F) \) is the set of all continuous linear mappings from \( E \) to \( F \).

A bornology \( \mathcal{B}_F \) on \( F \) is a covering of \( F \) satisfying the following axioms:

1. \( \mathcal{B}_F \) is stable under finite unions;
2. if \( A \in \mathcal{B}_F \) and \( B \subseteq A \), then \( B \in \mathcal{B}_F \).

The compact bornology on \( F \) is the family \( \mathcal{B}_F^c \) of relatively compact subsets of \( F \) having the set of all compact subsets of \( F \) as a base, in the sense that every \( B \in \mathcal{B}_F^c \) is contained in some compact set.

Let \( \mathcal{B}_E^c \) be the compact bornology on \( E \). We endow the vector space \( L(E, F) \) with the \( \mathcal{B}_E^c \)-topology which is the topology of uniform convergence on all compact subsets of \( E \). This is a Hausdorff locally convex topology which can be defined by the family of all seminorms obtained as follows:

\[
\|L\|_{B,n} := \sup\{\|L(e)\|_{F,n} : e \in B\},
\]

where \( B \in \mathcal{B}_E^c \) and \( n \in \mathbb{N} \).

Let \( \varphi : U \subseteq E \rightarrow F \) be a mapping. If the directional derivatives

\[
\mathcal{D}_\varphi(x)h = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t}
\]

exist for all \( x \in U \) and all \( h \in E \), and the induced map \( \mathcal{D}_\varphi(x) : U \rightarrow L(E, F) \) is continuous for all \( x \in U \), then we say that \( \varphi \) is a Keller \( C^1_c \) mapping (or differentiable of class Keller \( C^1_c \)) see [7].

A \( C^1_c \)-Fréchet manifold \( M \) is a manifold modeled on a Fréchet space \( F \) with an atlas of coordinate charts such that the coordinate transition functions are all Keller \( C^1_c \)-mappings.
Definition 2.1. [9] Let $\mathbb{F}$ be a Fréchet space, $T$ a topological space and $V = T \times \mathbb{F}$ the trivial bundle with fiber $\mathbb{F}$ over $T$. A Finsler structure for $V$ is a collection of continuous functions $\| \|_{\mathbb{F},n} : V \to \mathbb{R}^+$, $n \in \mathbb{N}$, such that

1. (F1): For $b \in T$ fixed, $\| x \|_{\mathbb{F},n}^b := \| (b, x) \|_{\mathbb{F},n}$ is a collection of seminorms on $\mathbb{F}$ which gives the topology of $\mathbb{F}$.

2. (F2): Given $k > 1$ and $x_0 \in T$, there exists a neighborhood $W$ of $x_0$ such that

$$\frac{1}{k} \| x \|_{\mathbb{F},n}^{x_0} \leq \| x \|_{\mathbb{F},n}^w \leq k \| x \|_{\mathbb{F},n}^{x_0} \quad \text{for all} \quad w \in W, n \in \mathbb{N}, x \in \mathbb{F}.$$

Suppose $M$ is a $C^1$-Fréchet manifold modeled on $\mathbb{F}$. Let $\pi_M : TM \to M$ be the tangent bundle and let $\|\|_{M,n} : TM \to \mathbb{R}^+$ be a collection of continuous functions, $n \in \mathbb{N}$. We say that $\{\|\|_{M,n}\}_{n \in \mathbb{N}}$ is a Finsler structure for $TM$ if for a given $x \in M$ there exists a bundle chart $\psi : U \times \mathbb{F} \simeq TM |_U$ with $x \in U$ such that

$$\{\|\|_{M,n} \circ \psi^{-1}\}_{n \in \mathbb{N}}$$

is a Finsler structure for $V = U \times \mathbb{F}$.

A Fréchet Finsler manifold is a Fréchet manifold together with a Finsler structure on its tangent bundle. Regular (in particular paracompact) manifolds admit Finsler structures.

If $\{\|\|_{M,n}\}_{n \in \mathbb{N}}$ is a Finsler structure for $M$ then we can obtain a graded Finsler structure, denoted by $\{\|\|_{M,n}\}_{n \in \mathbb{N}}$, that is $\|\|_{M,i} \leq \|\|_{M,i+1}$ for all $i \in \mathbb{N}$.

We define the length of a $C^1$-curve $\gamma : [a, b] \to M$ with respect to the $n$-th component by

$$L_n(\gamma) = \int_a^b \| \gamma'(t) \|_{M,n} \, dt.$$  

The length of a piecewise path with respect to the $n$-th component is the sum over the curves constituting to the path. On each connected component of $M$, the distance is defined by

$$\rho_n(x, y) = \inf_{\gamma} L_n(\gamma),$$

where infimum is taken over all piecewise $C^1$-curve connecting $x$ to $y$. Thus, we obtain an increasing sequence of metrics $\rho_n(x, y)$ and define the distance $\rho$ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x, y)}{1 + \rho_n(x, y)}. \quad (2.1)$$

The distance $\rho$ defined by (2.1) is a metric for $M$ which is bounded by $1$. Furthermore, the topology induced by this metric coincides with the original topology of $M$ (see [9]).

3. Multiplicity theorems for Fréchet spaces

Let $L$ be a topological group with the identity element $e$. A continuous action of $L$ on a Fréchet space (or manifold) $\mathbb{F}$ is a mapping $A : L \times \mathbb{F} \to \mathbb{F}$, $A(l, m)$ written as $l \cdot m$, such that $e \cdot m = m$ and $(l_1 \ast l_2) \cdot m = l_1 \cdot (l_2 \cdot m)$ for all $l_1, l_2 \in L$ and all $m \in \mathbb{M}$ (here $\ast$ denotes the operation of $L$).

A set $A \subset \mathbb{F}$ is called $L$-invariant, if $l \cdot m \in A$ for all $m \in A$ and all $l \in L$. A functional $\varphi : \mathbb{F} \to \mathbb{R}$ is called $L$-invariant if $\varphi(l \cdot m) = \varphi(m)$ for all $l \in L$ and $m \in \mathbb{M}$. A map $\Phi : \mathbb{F} \to \mathbb{F}$ is called $L$-equivalent if $\Phi(l \cdot m) = l \cdot \Phi(m)$ for all $m \in \mathbb{M}$ and all $l \in L$. The set of fixed points of $\mathbb{F}$ under the action is defined as

$$\text{Fix}(L) := \{ x \in \mathbb{F} \mid l \cdot x = x, \forall l \in L \}.$$
Let $G$ be a discrete subgroup of a Fréchet space $F$ and let $q : F \to F/G$ be the canonical surjection. A subset $A \subset F$ is called $q$-saturated if $A = q^{-1} \circ q(A)$. Suppose the space $F_1$ generated by $G$ has the dimension $n$. Let $F_2$ be a topological complement of $F_1$, so that $F$ is isomorphic to $F_1 \times F_2, F \cong F_1 \times F_2$. Let $T^n$ be the $n$-torus, then $G \cong \mathbb{Z}^n$ and $q(F) \cong T^n \times F_2$.

Let $\varphi : F \to \mathbb{R}$ be a $G$-invariant functional of class Keller $C^1_c$, and $c$ a critical point of $\varphi$. The set $q^{-1}(q(c))$ consists of the critical points of $\varphi$ (since $\varphi'$ is $G$-invariant) and is called a critical orbit of $\varphi$ through $c$.

We will extend the Palais-Smale compactness condition ([1, Definition 3.2]) to $G$-invariant functionals.

**Definition 3.1.** Let $\varphi : F \to \mathbb{R}$ be a $G$-invariant functional of class Keller $C^1_c$. We say that $\varphi$ satisfies (PS)$_G$-condition if for every sequence $(x_n) \subset F$ such that $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$, the sequence $(q(x_n))$ contains a convergent subsequence.

The Lusternik-Schnirelmann category $\text{Cat}_{\tau}A$ of a subset $A$ of a topological space $T$ is the minimal number of closed sets that cover $A$ and each of which is contractible to a point in $T$. If $\text{Cat}_{\tau}A$ is not finite, we write $\text{Cat}_{\tau}A = \infty$.

The following result concerning the Lusternik-Schnirelmann category of subsets of ANR spaces will be applied in the proof of Lemma 3.1.

**Theorem 3.1** ([10], Theorem 6.3). Let $T$ be an ANR space and $A \subseteq T$. Then there exists a neighborhood $\mathcal{U}$ of $A$ such that $\text{Cat}_T\mathcal{U} = \text{Cat}_T A$.

Let $\text{Co}(F)$ be the set of compact subsets of $F$. Define the sets

$$A_i = \{ A \subset F \mid A \subset C(T), \text{Cat}_{q(F)}q(A) \geq i \},$$

(3.1)

for $i \in \mathbb{N}$. In view of the property (3) of Lemma 3.3 each $A_i$ is a deformation invariant class of subsets of $T$. The $i$-th Lusternik-Schnirelmann minimax value of $\varphi$ is defined by

$$\mu_i = \inf_{A \in A_i} \sup_{x \in A} \varphi(x).$$

(3.2)

It is easy to see that the sequence of numbers $\mu_i$ is increasing.

The proofs of the following two lemmas are based on the standard arguments, see for example [13, Lemma 3.2, Lemma 3.3].

Let $\text{CB}(F)$ be the family of all nonempty closed and bounded subsets of $F$. We define the Hausdorff metric $d_{\text{H}}$ on $\text{CB}(F)$ by

$$d_{\text{H}}(A, B) = \max \left\{ \sup_{a \in A} d_F(a, B), \sup_{b \in B} d_F(b, A) \right\}.$$

**Lemma 3.1.** The space $(\mathcal{A}_i, d_{\text{H}})$ is a complete metric space.

**Proof.** The space $(\text{CB}(F), d_{\text{H}})$ is complete since $F$ is complete, cf. [8]. Thus, we only need to prove that $\mathcal{A}_i$ is closed in $\text{CB}(F)$. Let $(A_k) \subset \mathcal{A}_i$. Suppose $A \in \text{CB}(F)$ and $d_{\text{H}}(A_k, A) \to 0$. By [14, Corollary 1.2.13] and [14, Proposition 1.2.14] the space $q(F) \cong T^n \times F_2 \cong (S^1)^n \times F_2$ is an ANR. Therefore, by Theorem 3.1 there exists a closed neighborhood $\mathcal{U}$ of $A$ such that

$$\text{Cat}_{q(F)}q(A) = \text{Cat}_{q(F)}(\mathcal{U}).$$

As $q^{-1}(\mathcal{U})$ is a closed neighborhood of the compact set $A$, there exists $k$ such that $A_k \subset \mathcal{U}$. Thereby,

$$\text{Cat}_{q(F)}q(A) \geq \text{Cat}_{q(F)}(A_k) \geq i.$$
Therefore, $A \in \mathcal{A}_i$. \hfill \Box

**Lemma 3.2.** Let $\varphi : \mathbb{F} \to \mathbb{R}$ be a $\mathbf{G}$-invariant functional of class Keller $C^1_c$. Then, the function

$$
\psi : \mathcal{A}_i \to \mathbb{R}
$$

$$
\psi(A) = \max_{x \in A} \varphi(x)
$$

is lower semicontinuous.

**Proof.** Let $(A_k) \subset \mathcal{A}_i$. Suppose $A \in \mathcal{A}_i$ and $d_H(A_k, A) \to 0$. For each $x \in A$, there exists a sequence $(x_k) \subset \mathbb{F}$ such that $x_k \in A_k$ and $x_k \to x$. Thus,

$$
\varphi(x) = \lim_{k \to \infty} \varphi(x_k) \leq \lim_{k \to \infty} \psi(A_k),
$$

and as $x \in A$ is arbitrary we have

$$
\psi(A) \leq \lim_{k \to \infty} \psi(A_k).
$$

\hfill \Box

We shall need the following strong version of Ekeland’s variational principle.

**Theorem 3.2 (\cite{6}, Theorem 4.7).** Let $(\mathbf{M}, \mathbf{m})$ be a complete metric space. Let a functional $\phi : \mathbf{M} \to (-\infty, \infty]$ be lower semi-continuous, bounded from below and not identical to $\infty$. Let $\varepsilon > 0$ be an arbitrary real number, $m \in \mathbf{M}$ a point such that

$$
\phi(m) \leq \inf_{x \in \mathbf{M}} \phi(x) + \varepsilon.
$$

Then for an arbitrary $r > 0$, there exists a point $m_r \in \mathbf{M}$ such that

1. (EK1) $\phi(m_r) \leq \phi(m)$;
2. (EK2) $\mathbf{m}(m_r, m) < r$;
3. (EK3) $\phi(m_r) < \phi(x) + \frac{\varepsilon}{r} \mathbf{m}(m_r, x) \forall x \in \mathbf{M}\setminus \{m_r\}$.

Also, we will apply the following properties of Lusternik-Schnirelmann category.

**Lemma 3.3.** \cite[Proposition 2.2]{13} Let $\mathbf{T}$ be a topological space, $A, B \subset \mathbf{T}$. Then

1. If $A \subset B$, then $\text{Cat}_T A \leq \text{Cat}_T B$.
2. $\text{Cat}_T (A \cup B) \leq \text{Cat}_T A + \text{Cat}_T B$.
3. If $A$ is closed and $\mathcal{H} : [0, t_0] \times \mathbf{T} \to \mathbf{T}$ is a deformation, then $\text{Cat}_T A \leq \text{Cat}_T (\mathcal{H}(t_0, A))$.
4. If $\mathbf{T}$ is a Finsler manifold, then there exists a neighborhood $U$ of $A$ such that $\text{Cat}_T U = \text{Cat}_T A$.
5. If $\mathbf{T}$ is a connected Finsler manifold and $A$ is closed then $\text{Cat}_T A \leq \dim A + 1$, where $\dim$ is the covering dimension.

**Theorem 3.3.** Let $\mathbf{G}$ be a discrete subgroup of a Fréchet space $\mathbb{F}$. Assume that the dimension of the space generated by $\mathbf{G}$ is a finite number $n$. Let $\varphi : \mathbb{F} \to \mathbb{R}$ be a $\mathbf{G}$-invariant functional of class Keller $C^1_c$. If $\varphi$ is bounded below and satisfies the (PS)$_\mathbf{G}$-condition, then $\varphi$ has $n + 1$ critical orbits.

**Proof.** Consider the increasing sequence of the Lusternik-Schnirelmann minimax values $\mu_i = \inf_{A \in \mathcal{A}_i} \sup_{x \in A} \varphi(x)$, $1 \leq i \leq n + 1$. Define the sets

$$
S_{\mu_i} := \{ x \in \mathbb{F} \mid \varphi'(x) = 0 \& \varphi(x) = \mu_i \}.
$$
We claim that if $\mu_i = \mu_k = \mu$ for some $k, i \leq k \leq n + 1$, then $S_{\mu_i}$ contains $k - i + 1$ critical orbits. This concludes the proof of the theorem. We prove the claim by contradiction. Suppose that $S_{\mu_i}$ contains $m$ distinct critical orbits $q(x_1), \cdots, q(x_m)$ and $m \leq k - i$. Pick the positive number $r$ so that on $B(x_j, 2r), 1 \leq j \leq m$, the canonical surjection $q$ is injective. Define the set

$$B_r := \bigcup_{j=1}^{m} B(x_j + g, r).$$

We show that there exists $\varepsilon, 0 < \varepsilon^2 < r^2$ such that

$$\|\varphi'(x)\|_B > \varepsilon \quad \forall B \in B^c_F,$$  

(3.3)

if $x \in \varphi^{-1}([\mu - \varepsilon^2, \mu + \varepsilon^2]) \setminus B_r$. Because, if (3.3) is not valid, then there exists a sequence $(x_j) \subset F \setminus B_r$ such that

$$|\varphi(x_j)| \leq \mu + \frac{1}{j} \text{ and } |\varphi'(x)| \|_{B,n} \leq \frac{1}{j} \quad \forall n \in \mathbb{N} \text{ and } \forall B \in B^c_F.$$ 

Since $\varphi$ satisfies the (PS)$_F$-condition we may assume that $q(x_j) \to q(\tilde{x})$ for some $\tilde{x} \in F$. As $\varphi$ and $\varphi'$ are $G$-invariant, we may suppose that $x_j \in [0,1]^n \times \mathbb{F}_2$. Whence, $x_j \to \tilde{x}$ yields $\tilde{x} \in F \setminus B_r$ and $\varphi(\tilde{x}) = \mu$ and $\varphi'(\tilde{x}) = 0$ which is impossible because $B_r$ is a neighborhood of $S_{\mu_i}$. There exists $A \in A_k$ such that

$$\psi(A) = \max_A \varphi \leq \mu + \varepsilon^2.$$ 

This is achievable by the definition of $\mu_k$. Let $A = A \setminus B_{2r}$. In virtue of Lemma 3.3 we obtain

$$k \leq \text{Cat}_{q(F)} q(A)$$

$$\leq \text{Cat}_{q(F)} (q(A) \cup q(B_{2r}))$$

$$\leq \text{Cat}_{q(F)} q(A) + \text{Cat}_{q(F)} q(B_{2r})$$

$$\leq \text{Cat}_{q(F)} q(A) + m \quad \text{by the definition of the category}$$

$$\leq \text{Cat}_{q(F)} q(A) + k - i.$$ 

Thus, $A \in A_i$.

By Lemma 3.1, the space $(A_i, d_H)$ is complete. Also, by Lemma 3.2 the function $\psi : A_i \to \mathbb{R}$ is lower semicontinuous. So, we can employ the Ekeland variational theorem 3.2. By the latter theorem there exists $A \in A_i$ such that

(P1) $\psi(A) \leq \varphi(A) \leq \mu + \varepsilon^2$,

(P2) $d_H(A, A) \leq \varepsilon$,

(P3) $\psi(S) > \psi(A) - \varepsilon d_H(A, S) \quad S \in A_i, S \neq A.$

As $A \cap B_{2r} = \emptyset$ and $d_H(A, A) \leq \varepsilon \leq r$, then $A \cap B_{2r} = \emptyset$. Also, the set

$$C := \{s \in A \mid \mu - \varepsilon^2 \leq \varphi(s)\}$$

is a subset of $\varphi^{-1}([\mu - \varepsilon^2, \mu + \varepsilon^2]) \setminus B_r$. The set $C$ is closed and as $\varphi$ is continuous, then it is compact. By (3.3) for each $y \in C$ there exists $h_{B,y} \in B$ such that

$$\langle \varphi'(y), h_{B,y}\rangle < -\varepsilon.$$  

(3.4)

Since $\varphi'$ is continuous, it follows from (3.4) that there exists $r_y > 0$ such that for all $g \in G$ and all $h \in F$ with $\|h\|_{F,n} < r_y$ we obtain

$$\langle \varphi'(y + g + h), h_{B,y}\rangle < -\varepsilon.$$
Since $C$ is compact, we can find a subcovering $C_1, \ldots, C_n$ defined by

$$C_i = B(y_i, r_{y_i}) \quad 1 \leq i \leq n.$$ 

Define the functions $\phi_i : F \to [0, 1]$ by

$$\phi_i(x) = \begin{cases} 
\sum_{g \in G} d_F(x + g, CC_i) \\ \sum_{k=1}^n \sum_{g \in G} d_F(x + g, CC_k) 
\end{cases} x \in \bigcup_{i=1}^n C_j, \\
0 \text{ otherwise.}$$

Fix a $G$-invariant continuous function $\phi : F \to [0, 1]$ such that

$$\phi(x) = \begin{cases} 
1 & \mu \leq \varphi(x), \\
0 & \varphi(x) \leq \mu - \varepsilon. 
\end{cases}$$

Let $r_{\min} = \min_{1 \leq i \leq n} r_{y_i}$. Define the curve $I \in C([0, 1] \times F, F)$ by

$$I(t, x) = x + tr_{\min} \phi(x) \sum_{i=1}^n \psi_i(x)(h_{B,y_i}).$$

For all $x \in F$, all $g \in G$ and all $t \in [0, 1]$ we have

$$I(t, x + g) = I(t, x) + g.$$ 

Lemma 3.3 implies that

$$\text{Cat}_{q(F)}(q(I(1, A))) \geq \text{Cat}_{q(F)}(q(A)) \geq i,$$

whence, as $I(1, A)$ is compact, $I(1, A) \in A_i$. By the mean value theorem (see [7]) and (P3) for each $y \in C$, there is $t \in (0, 1)$ such that

$$\varphi(I(1, A)) - \varphi(x) = \langle \varphi'(I(t, A)), r_{\min} \phi(x) \sum_{i=1}^n \psi_i(x)(h_{B,y_i}) \rangle$$

$$= r_{\min} \phi(x) \sum_{i=1}^n \psi_i(x) \langle \varphi'(x + tr_{\min} \phi(x) \sum_{i=1}^n \psi_i(x)(h_{B,y_i})), h_{B,y_i} \rangle$$

$$\leq -\varepsilon r_{\min} \phi(x). \quad (3.5)$$

If $x \in C$, then $\phi(x) = 0$ and $\varphi(I(1, X)) = \varphi(x)$. Let $y \in A$ so that $\varphi(I(1, Y)) = \psi(C)$. Then,

$$\mu \leq \varphi(I(1, Y)) \leq \varphi(y).$$

Thus, $y \in C$ and $\phi(y) = 1$. It follows from (3.5) that

$$\varphi(I(1, y)) - \varphi(y) \leq -\varepsilon r_{\min}.$$ 

Therefore,

$$\psi(S) + \varepsilon r_{\min} \leq \varphi(y) \leq \psi(A).$$

However, $d_H(A, S) \leq r_{\min}$ by the definition of $S$. Hence,

$$\psi(S) + \varepsilon d_H(A, S) \leq \psi(A)$$

which contradicts (P3) and concludes the proof. \qed
4. Multiplicity theorems for Fréchet manifolds

Henceforth we assume that $\mathcal{M}$ is a connected $C^1$-Fréchet manifold modeled on $\mathbb{F}$ endowed with a complete Finsler metric $\rho$ (2.1), and that $\varphi : \mathcal{M} \to \mathbb{R}$ is a non-constant Keller $C^1_c$-functional. Let $x \in \mathcal{M}$, we shall say that $x$ is a critical point of $\varphi$ if $(\varphi \psi^{-1})'(\psi(x)) = 0$ for a chart $(x \in U, \psi)$ and hence for every chart whose domain contains $x$.

Let $(\|\cdot\|_{M,n})_{n \in \mathbb{N}}$ be a graded Finsler structure on $T\mathcal{M}$. Define a closed unit semi-ball centered at the zero vector $0_x$ of $T_x\mathcal{M}$ by

$$\mathcal{B}^n(0_x, 1) = \{ h \in T_x\mathcal{M} : \| h \|_{M,n}^n \leq 1 \}$$

for each $x \in \mathcal{M}$ and each $\| h \|_{M,n}^n$. Let

$$\mathcal{B}_\infty(0_x) = \bigcap_{n=1}^\infty \mathcal{B}^n(0_x, 1).$$

The set $\mathcal{B}_\infty(0_x)$ is not empty and infinite because it can be identified with a convex neighborhood of the zero of the Fréchet space $U \times \mathbb{F}$, where $U$ is an open neighborhood of $x$.

Let $\varphi : \mathcal{M} \to \mathbb{R}$ be a $C^1$-functional and $x \in \mathcal{M}$. Define

$$\Phi_\varphi(x) = \inf \{ \varphi'(x, h) : h \in \mathcal{B}_\infty(0_x) \}. \quad (4.1)$$

**Definition 4.1** (The $(PS)_c$-condition for Fréchet manifolds, [2]). We say that a $C^1$-functional $\varphi : \mathcal{M} \to \mathbb{R}$ satisfies the Palais-Smale condition at a level $c \in \mathbb{R}$, $(PS)_c$ in short, in a set $A \subset \mathcal{M}$ if any sequence $(m_i)_{i \in \mathbb{N}} \subset A$ such that

$$\varphi(m_i) \to c \quad \text{and} \quad \Phi_\varphi(m_i) \to 0,$$

has a convergent subsequence.

We denote by $\text{Cr}(\varphi)$ the set of critical points of $\varphi$, and for $c \in \mathbb{R}$

$$\text{Cr}(\varphi, c) = \{ x \in \text{Cr}(\varphi), \varphi(x) = c \},$$

$$\varphi^c = \{ x \in \mathcal{M} : \varphi(x) \leq c \}.$$

A mapping $\mathcal{H} \in C([0,1] \times \mathcal{M} \to \mathcal{M})$ is called a deformation if $\mathcal{H}(0, x) = x$ for all $x \in \mathcal{M}$. Let $C$ be a subset of $\mathcal{M}$, we say that $\mathcal{H}$ is a $C$-invariant for an interval $I \subset [0,1]$ if $\mathcal{H}(t, x) = x$ for all $x \in C$ and all $t \in I$.

A family $\mathcal{F}$ of subset of $\mathcal{M}$ is said to be deformation invariant if for each $A \in \mathcal{F}$ and each deformation $\mathcal{H}$ for $\mathcal{M}$, $\mathcal{H}_I(x) := \mathcal{H}(1, x)$, it follows that

$$\mathcal{H}_I(A) \subset \mathcal{F}.$$
Lemma 4.1. Let \( \varphi : \mathbb{M} \rightarrow \mathbb{R} \) be an \( \mathbb{L} \)-invariant functional of class Keller \( C^1_c \). Let \((\mathcal{U}, \psi)\) be an arbitrary chart and \( \varphi_\psi \) the local representative of \( \varphi \) in the chart. Then
\[
\langle \varphi_\psi'(\psi(l \cdot x)), \psi(y) \rangle = \langle \varphi'(\psi(x), \psi(l^{-1} \cdot y)) \rangle \quad \forall x, y \in \mathcal{U}; \forall l \in \mathbb{L}.
\]

Proof. Let \( \varphi \) be \( \mathbb{L} \)-invariant and satisfies the Palais-Smale condition at all levels. Let \( \Phi \) be a Keller \( \mathbb{L} \)-invariant functionals, we will extend the deformation results [2, Lemma 3.1, Corollary 3.5] to make the deformation \( \mathbb{L} \)-equivalent.

Theorem 4.1. Assume \( \varphi : \mathbb{M} \rightarrow \mathbb{R} \) is \( \mathbb{L} \)-invariant. Let \( B \) and \( A \) be closed disjoint, \( \mathbb{L} \)-invariant subsets of \( \mathbb{M} \) and let \( A \) be compact. Suppose \( k > 1 \) and \( \epsilon > 0 \) are such that
\[
\Phi_\varphi(x) < -2\epsilon(1 + k^2) \quad \forall x \in A.
\]
Then there exist \( t_0 > 0 \) and \( \mathbb{L} \)-equivalent deformation \( \mathcal{H} \) for \([0, t_0]\) such that
\[
\begin{align*}
(1) \quad & \mathcal{H}(t, x) = x, \quad \forall t \in B, \forall t \in [0, t_0), \\
(2) \quad & \rho(\mathcal{H}(t, x), x) \leq kt, \quad \forall x \in M, \\
(3) \quad & \varphi(\mathcal{H}(t, x)) - \varphi(x) \leq -2\epsilon(1 + k^2)t, \quad \forall x \in M.
\end{align*}
\]

Corollary 4.1. Let \( \varphi : \mathbb{M} \rightarrow \mathbb{R} \) be a Keller \( C^1_c \) closed non-constant function. Suppose \( \varphi \) is \( \mathbb{L} \)-invariant and satisfies the Palais-Smale condition at all levels.
\[
\begin{align*}
(1) \quad & \text{If for } c \in \mathbb{R} \text{ and } \delta > 0 \text{ we have } \varphi^{-1}[c - \delta, c + \delta] \cap \text{Cr}(\varphi) = \emptyset, \\
& \text{then there exists } t_1 < t_0 \text{ and } 0 < \epsilon < \delta \text{ such that } \mathcal{H}(t_1, \varphi^{c+\epsilon}) \subset \varphi^{c-\epsilon}. \quad (4.2)
\end{align*}
\]
\[
\begin{align*}
(2) \quad & \text{If } \varphi \text{ has finitely many critical points, and for } c \in \mathbb{R} \text{ if } U \text{ is an } \mathbb{L} \text{-invariant, open neighborhood of Cr}(\varphi, c) \text{ (} U = \emptyset \text{ if Cr}(\varphi, c) = \emptyset), \text{then there exist } t_1 < t_0 \text{ and } \epsilon > 0 \text{ such that } \\
& \mathcal{H}(t_1, \varphi^{c+\epsilon} \setminus U) \subset \varphi^{c-\epsilon}. \quad (4.3)
\end{align*}
\]

The proofs of Theorem 4.1 and Corollary 4.1 are just modifications of the proofs of Lemma 3.1 and Corollary 3.5 in [2]. We just need to show that if \( \mathcal{H} \) is a deformation obtained in [2, Lemma 3.1], then \( \mathcal{H}_t(t, m) = \int_{\mathbb{L}} l^{-1} \cdot \mathcal{H}(t, l \cdot m) d\mu \) is \( \mathbb{L} \)-equivalent. To see the latter, let \( g \in \mathbb{L} \). Then,
\[
\begin{align*}
\mathcal{H}_t(t, g \cdot m) &= \int_{\mathbb{L}} l^{-1} \cdot \mathcal{H}(t, g \cdot (l \cdot m)) d\mu \\
&= g \cdot \int_{\mathbb{L}} (l \ast g)^{-1} \cdot \mathcal{H}(t, ((l \ast g) \cdot m)) d\mu \\
&= g \cdot \mathcal{H}_t(t, m).
\end{align*}
\]

Now we extend the Ljusternik–Schnirelmann theorem [2, Theorem 3.10].
Given $i \in \mathbb{N}$, we define the sets
\[ K_i := \{ A \in M \mid A \text{ is compact and } L\text{-invariant with } \text{Ind}(A) \geq i \}, \]
and the values $c_i$ by
\[ c_i := \inf_{A \in K_i} \max_{x \in A} \varphi. \]

**Theorem 4.2.** Let $L$ be a compact Lie group that acts on a $C^1$-Fréchet Finsler manifold $M$. Let $\varphi : M \rightarrow \mathbb{R}$ be a non-constant closed, $L$-invariant functional of class Keller $C^1$. Given any $n \in \mathbb{N}$ such that $n \leq m$ for some $m \in \mathbb{N}$ such that $c := c_n = c_k > -\infty$. If $\varphi$ satisfies the (PS)$_c$-condition, then $\text{Ind}(\text{Cr}(\varphi, c)) \geq m - n + 1$.

**Proof.** From Definition 4.1 it follows directly that the set $\text{Cr}(\varphi, c)$ is compact and it is $L$-invariant by Lemma 4.1. In virtue of (I.4) we may find an $L$-invariant, open neighborhood $U$ of $\text{Cr}(\varphi, c)$ such that $\text{Ind}(\overline{U}) = \text{Ind}(\text{Cr}(\varphi, c))$. By the definitions of the values $c_i$, we may find $A \in K_i$ such that
\[ \max_{A} \varphi \leq c + \epsilon. \]

Let $A = A \setminus U$. Then by (I.2) and (I.3) we will have
\begin{align*}
i \leq & \text{Ind}(A) \\
\leq & \text{Ind}(A) + \text{Ind}(A \cap \overline{U}) \\
\leq & \text{Ind}(A) + \text{Ind}(U) \\
= & \text{Ind}(A) + \text{Ind}(\text{Cr}(\varphi, c)) \tag{4.4}
\end{align*}

Observe that $A \subseteq \varphi^{c+\epsilon} \setminus U$. Thus, by Theorem there is an $L$-equivalent deformation $H$ such that by applying Corollary 4.1 for some $0 < t_0 < 1$ we will get
\[ A := H(t_0, A) \subseteq \varphi^{c-\epsilon}. \]

Since $A$ is compact and $L$-invariant, and $H(t_0, \cdot)$ is $L$-equivalent, it follows that $A$ is compact and $L$-invariant and
\[ \max_{A} \varphi \leq c - \epsilon. \]

Moreover, $\text{Ind}(A) \leq n - 1$ by the definition of $c_n = c$. Thus, by (I.3)
\[ \text{Ind}(A) \leq \text{Ind}(A) \leq n - 1. \tag{4.5} \]

It follows from (4.5) and (4.4) that $\text{Ind}(\text{Cr}(\varphi, c)) \geq m - n + 1$. Furthermore, $\text{Cr}(\varphi, c) \neq \emptyset$ by (I.1). \qed

**References**

1. K. Eftekharinasab, *A Generalized Palais-Smale Condition in the Fréchet space setting*, Proceedings of the International Geometry Center, Vol. 11, No. 1 (2018) 1-11.
2. K. Eftekharinasab and I. Lastivka, *A Lusternik-Schnirelmann type theorem for $C^1$-Fréchet manifolds*, J. Indian Math. Soc., Vol. 88., Nos(3-4)(2021), 309-322.
3. K-C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhauser, 1993
4. K-C. Chang, *Methods in Nonlinear Analysis*, Springer-Verlag, 2005.
5. L. Gasinski and N. Papageorgiou, *Nonlinear Analysis*, Chapman and Hall/CRC, 2005.
6. D. Hyers, G. Isac, and T. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific, 1997.
7. H. Keller, *Differential calculus in locally convex spaces*, Berlin: Springer-Verlag, 1974.
8. J. Munkres, *Topology*, Pearson College Div; 2nd edition, 2000.
9. T. N. Subramaniam, *Slices for the actions of smooth tame Lie groups*, PhD thesis, Brandeis University, 1984.
10. R. Palais, *Lusternik-Schnirelman theory on Banach manifolds*, Topology, Vol. 5, 115-132.
11. R. Palais, *The principle of symmetric criticality*, Commun. Math. Phys. 69 (1979), 19-30.
12. N. Papageorgiou and S. Th. Kyritsi-Yiallourou, *Handbook of Applied Analysis*, Springer-Verlag, 2009.
13. A. Szulkin, Ljusternik-Schnirelmann theory on $C^1$-manifolds, *Annales de l’I. H. P., section C*, 5 (1988), 119–139.
14. J. Van Mill, *Infinite-Dimensional Topology. Prerequisites and Introduction*, North-Holland Mathematical Library Volume 43, 1988.

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