Quantum one-way versus classical two-way communication in XOR games

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Abstract
In this work, we give an example of exponential separation between quantum and classical resources in the setting of XOR games assisted with communication. Specifically, we show an example of a XOR game for which $O(n)$ bits of two-way classical communication are needed in order to achieve the same value as can be attained with $\log n$ qubits of one-way communication. We also find a characterization for the value of a XOR game assisted with a limited amount of two-way communication in terms of tensor norms of normed spaces.

Keywords XOR games · Communication complexity · Tensor norms

Mathematics Subject Classification 46B28 · 46B28

1 Introduction and main results

From the foundational point of view, one of the main goals in Quantum Information is to quantify the difference in performance between quantum and classical resources for a given task. In particular, this quantification has been thoroughly studied in the context of Bell inequalities (see, e.g., review [1]). In the XOR games bipartite scenario, two separate parties, Alice and Bob, are given inputs $x$ and $y$ and they answer their outputs $a, b = \pm 1$ with certain probability, therefore generating a correlation. The set of correlations they can generate when sharing a quantum state and performing local measurements on it is different from the set they can generate when using only classical
resources, and this difference can be witnessed with the so-called Bell inequalities [2]. There is an abundance of results studying Bell inequalities. In the XOR game case, Tsirelson [3], using Grothendieck’s inequality, showed that the separation between both types of resources is bounded by a constant in the bipartite case, whereas in [4] (see also [5]) the authors show that in the tripartite case this separation can be unbounded. A survey for the situation in the case of general (non-XOR) games can be seen in [6].

This comparison between quantum and classical resources was soon extended to different settings. In this note, we are interested in a scenario where communication is allowed.

In the typical communication complexity scenario [7], one studies the minimum number of bits that Alice and Bob have to exchange in order to correctly (up to a bounded probability of error) compute a Boolean function for any pair of inputs \( x \) and \( y \). In [8], quantum communication complexity was introduced. The author analyzed the limitations and advantages of allowing Alice and Bob to communicate using quantum messages in order to accomplish a communication task. In [9], another model was introduced, in which the players are allowed to share an unlimited amount of entanglement while they communicate using a classical channel. Both scenarios can be related since the combination of 2 classical bits and an ebit can be used to teleport a qubit [10]. Another variant would be to combine both entanglement and quantum communication, which again can be related to the last model using teleportation.

In this paper, we consider the use of quantum messages without entanglement and its comparison with the use of classical messages. In this scenario, partial Boolean functions have been found for which quantum and classical communication complexities are exponentially separated [11,12].

In the recent paper [13], the authors related the communication complexity and Bell inequality settings, and showed a general method to obtain Bell inequality violations starting from large enough quantum over classical communication complexity advantage.

In [14], the authors continued this line of research and introduced a new setting, communication assisted games, which, from a conceptual point of view, can be seen as a mixture of the two previously defined scenarios (quantum nonlocality and communication complexity problems).

In this new setting, Alice and Bob try to win a XOR game using shared classical randomness together with the communication of a limited number of bits, either classical or quantum. This setting is very much related to the study of distributional complexity, with a change in emphasis: In the definition of distributional complexity, the probability of winning is fixed, and one studies the amount of communication needed. Opposite to this, in the definition of our new setting one fixes in advance the amount of communication, and calculates the probability of winning for that amount of communication.

In [14], the authors studied the one-way communication case in the communication assisted game setting. In this paper, we focus on the two-way communication case. Specifically, in [14] the authors achieved exponential separation with an example of a game for which \( O(n) \) bits of one-way (classical) communication are needed in order to achieve the same value as the one that can be attained with \( \log n \) qubits of one-way
(quantum) communication. They left open the question of the existence of a game for which one can obtain the same order of exponential separation between the one-way quantum communication and the general (two way) classical communication.

In our work, we answer this question positively. We show that actually the same game appearing in [14] achieves this exponential separation, which, as in the one-way case, is the maximum possible separation, up to a logarithmic factor.

We state next some notation needed for the statement of our main result. Given two finite sets $X$, $Y$, a bipartite XOR game $T$ with $X$ and $Y$ as input sets for Alice and Bob, respectively, is a linear functional described by a matrix $(T_{x,y})_{(x,y)\in X\times Y}$, where $\sum_{x,y} |T_{x,y}| = 1$. It describes the situation where Alice and Bob are asked the pair of questions $(x, y)$ with probability $|T_{x,y}|$ and, in order to win the game, they must output answers $a, b \in \{\pm 1\}$ verifying $ab = \text{sgn}(T_{x,y})$.

We call $L_{tw,c}$ (respectively, $Q_{ow,c}$) to the convex set of the correlations Alice and Bob can generate when they are allowed the use of shared randomness and $c$-bits of two-way classical communication (respectively, $c$ qubits of one-way communication). Then, given a XOR game $T$ we can consider the following two quantities:

$$\omega_{tw,c}(T) = \sup_{P \in L_{tw,c}} |\langle T, P \rangle|$$

and

$$\omega^*_w(T) = \sup_{P \in Q_{tw,c}} |\langle M, P \rangle|.$$

With this notation, our main result can be stated.

**Theorem 1** For every $n \in \mathbb{N}$, there exist an XOR game $T$ with $2^{2n}$ inputs for Alice and $2^{n^2}$ inputs for Bob such that, for every $k \in \mathbb{N}$,

$$\frac{\omega^*_w(T)}{\omega_{tw,c}(T) \log k} \geq \frac{C}{\log k},$$

where $C$ is a constant independent of $n, k$.

This result implies the above mentioned exponential separation: Alice and Bob need to communicate $k = O(n)$ classical bits to obtain the same value as the one obtained with $\log n$ qubits.

The lower bound for the quantum communication value in our result is the same lower bound as in [14]. For the sake of completeness, we sketch the proof in our paper. The main technical part of our proof is to upper bound the two-way classical communication value. To prove this upper bound we rely on techniques from the local theory of Banach spaces, in particular on a careful use of the Khintchine and double Khintchine inequalities. Also, careful reasoning is needed when handling the dependencies appearing between a message and the previous and following messages.

Our second result is Theorem 3, a characterization of $\omega_{tw,c}(T)$, the value of the game when $c$ bits of two-way classical communication are used, in terms of tensor norms. Although not strictly needed for Theorem 1, this second result was the starting point of this research and lies behind our ideas. Similar techniques have been used before to characterize the classical value of a XOR game [3,6] or the value of a XOR game with one-way classical communication [14].
Since the statement of the result requires several previous technical definitions, we postpone it until Sect. 4.

This paper is organized as follows: in Sect. 2, we will present the form of a general two-way protocol explicitly, with the properties and the dependences of the corresponding messages that are being sent. In Sect. 3, we will present the proof of Theorem 1. This proof does not require tensor norms, although, as we said before, it is the tensor norm idea that lies behind our reasonings. Finally, in Sect. 4, we state and prove Theorem 3, our characterization of the two-way communication value via tensor norms. In order to do this, we previously state the needed notions from Banach space theory and tensor norm theory.

2 Two-way classical communication

For the sake of completeness, and in order to fix our notation, in this section we describe randomized classical communication protocols, and the model associated with them, in the particular case of XOR games.

We consider a protocol with \( t \) rounds of two-way classical communication between Alice and Bob. In round \( i \), first Alice will send \( c_i \) bits to Bob and, after receiving them, Bob will send \( d_i \) bits to Alice. After that, the round \( i + 1 \) can begin.

We consider general randomized protocols and, therefore, the messages each agent sends are random variables depending on the previous inputs of the corresponding agent and on a possible shared randomness.

That is, we can view the first message \( m_1 \) of Alice as an application

\[
M_1: X \rightarrow \mathbb{R}^{2^{c_1}},
\]

such that, for every \( x \in X \), \( M_1(x) := \left( M_1^{m_1}(x) \right)_{m_1=1}^{2^{c_1}} \) is a probability distribution on the possible messages \( m_1 \) sent by Alice when she receives input \( x \).

Bob’s first message is a mapping

\[
N_1: Y \times \left[ 2^{c_1} \right] \rightarrow \mathbb{R}^{2^{d_1}},
\]

such that, for every \( y \in Y \) and \( m_1 \in \left[ 2^{c_1} \right] \), \( N_1(y, m_1) := \left( N_1^{n_1}(y, m_1) \right)_{n_1=1}^{2^{d_1}} \) is a probability distribution on the possible messages \( n_1 \) sent by Bob when he receives input \( y \) and message \( m_1 \) from Alice.

Similarly, Alice’s and Bob’s last messages are mappings

\[
M_t: X \times \left[ 2^{d_1} \right] \times \cdots \times \left[ 2^{d_{t-1}} \right] \rightarrow \mathbb{R}^{2^{c_t}},
\]

and

\[
N_t: Y \times \left[ 2^{c_1} \right] \times \cdots \times \left[ 2^{c_t} \right] \rightarrow \mathbb{R}^{2^{d_t}}.
\]

After they interchange messages, Alice and Bob produce \( \pm 1 \)-valued outputs \( a(x, n_1, \ldots, n_t) \), \( b(y, m_1, \ldots, m_t) \).
We will use the notation $\bar{m}, \bar{n}$ for the multi-indices $(m_1, \ldots, m_t), (n_1, \ldots, n_t)$.

Therefore, Alice’s strategy is a function

$$a: X \times [2^{d_1}] \times \cdots \times [2^{d_t}] \longrightarrow \{ \pm 1 \} \times \mathbb{R}^{2^{c_1}} \times \cdots \times \mathbb{R}^{2^{c_t}}$$

$$a(x, \bar{n}) = (a(x, \bar{n}), M_1^{m_1}(x), M_2^{m_2}(x, n_1), \ldots, M_t^{m_t}(x, n_1, \ldots, n_{t-1})),$$

which can be seen as a tensor

$$\bar{a} = (\bar{a}(x, \bar{m}, \bar{n}))_{x, \bar{m}, \bar{n}} = \sum_{x, \bar{m}, \bar{n}} a(x, \bar{n}) M_1^{m_1}(x) M_2^{m_2}(x, n_1)$$

$$\cdots M_t^{m_t}(x, n_1, \ldots, n_{t-1}) e_x \otimes e_{m_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_1} \otimes \cdots \otimes e_{n_t}. \quad (1)$$

Similarly, Bob’s strategy is given by a function

$$b: Y \times [2^{c_1}] \times \cdots \times [2^{c_t}] \longrightarrow \{ \pm 1 \} \times \mathbb{R}^{2^{d_1}} \times \cdots \times \mathbb{R}^{2^{d_t}}$$

$$b(y, \bar{m}) = (b(y, \bar{m}), N_1^{n_1}(y), N_2^{n_2}(y, m_1), \ldots, N_t^{n_t}(y, m_1, \ldots, m_t)),$$

which can be seen as a tensor

$$\bar{b} = (\bar{b}(y, \bar{m}, \bar{n}))_{y, \bar{m}, \bar{n}} = \sum_{y, \bar{m}, \bar{n}} b(y, \bar{m}) N_1^{n_1}(y) N_2^{n_2}(y, m_1)$$

$$\cdots N_t^{n_t}(y, m_1, \ldots, m_t) e_y \otimes e_{m_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_1} \otimes \cdots \otimes e_{n_t}. \quad (2)$$

In future reasonings, we will need the following result, which follows easily from the definitions.

**Lemma 1** The tensors $\bar{a}, \bar{b}$ given in Eqs. (1) and (2) verify

$$\sup_{x} \sum_{m_1} \sup_{n_1} \cdots \sup_{n_{t-1}} \sum_{m_t} \sup_{n_t} |a(x, \bar{n}) M_1^{m_1}(x) \cdots M_t^{m_t}(x, n_1, \ldots, n_{t-1})| \leq 1,$$

and

$$\sup_{y,m_1} \sum_{n_1} \sum_{m_2} \cdots \sum_{m_t} \sum_{n_t} |b(y, \bar{m}) N_1^{n_1}(y) \cdots N_t^{n_t}(y, m_1, \ldots, m_t)| \leq 1.$$

**Proof** For the first case, bound $a(x, \bar{n})$ by 1 and recall that fixing $x, n_1, \ldots, n_i$ makes

$$\sum_{m_i} M_t^{m_t}(x, n_1, \ldots, n_{i-1}) \leq 1$$

for all $i$. Proceed similarly for the second case. \(\square\)
3 Proof of Theorem 1

The game appearing in Theorem 1 is the same that was already used in [14] to prove a similar bound for the one-way communication value. We recall the precise definition of the game here:

**Definition 1** We consider the XOR game $T$ where $X = \{\pm 1\}^n \times \{\pm 1\}^n$ and $Y = \{\pm 1\}^{n^2}$. Alice’s inputs will be named $\tilde{x} = (x, z) \in \{\pm 1\}^n \times \{\pm 1\}^n$ and Bob’s inputs $y \in \{\pm 1\}^{n^2}$. Then, the coefficients $T_{\tilde{x}, y} = T_{(x, z), y}$ take the following form:

$$T_{(x, z), y} = \frac{1}{L} \sum_{i, j=1}^{n} x_i z_j y_{ij},$$

where $L$ is a normalization factor in order to fulfill $\sum_{xyz} |T_{(x, z), y}| = 1$, which means $L = \sum_{xyz} \sum_{ij} x_i z_j y_{ij} |.$

That is, the probability of question $(\tilde{x}, y)$ is $\frac{1}{L} |\sum_{ij} x_i z_j y_{ij}|$ and the condition that the players have to fulfill with their answers in that case is $ab = \text{sign} \sum_{i, j} x_i z_j y_{ij}$.

**Remark 1** The following estimate for the value of $L$ is given in [14, Lemma 5.3]:

$$\frac{1}{\sqrt{2}} n 2^{n^2 + 2n} \leq L \leq n 2^{n^2 + 2n}.$$

In order to prove Theorem 1 we need to show a lower bound for the value with quantum communication and an upper bound for the value with classical communication. The quantum value was already proven in [14].

**Proposition 1** [14, Proposition 5.6] Let $T$ be the XOR game defined in Definition 1. Then,

$$\omega^*_{ow, \log n}(T) \geq \frac{C}{\sqrt{n}},$$

where $C$ is a constant independent of $n$.

A detailed proof of this result can be seen in [14, Proposition 5.6]. For the sake of clarity, we sketch next the main idea:

For every $\tilde{x} = (x, z) \in X$, we define the $n$-dimensional states:

$$|\varphi_x\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i |i\rangle$$

and

$$|\varphi_z\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} z_j |j\rangle.$$

Then, we consider the operator $\rho_{\tilde{x}} = |\varphi_x\rangle \langle \varphi_z|$ which is, in general, not self-adjoint.

Also, for every $y \in Y$ we consider the (non-self-adjoint) operator $A_y$ in $M_n$, whose matrix in the canonical basis is $(y_{i, j})_{i, j=1}^{n} = (r_{i, j}(y))_{i, j=1}^{n}.$
Then, Alice will send one of the positive components of $\rho_\tilde{x} = |\varphi_x\rangle\langle \varphi_z|$, properly normalized, and Bob will measure with one of the self-adjoint components of $A_y$, again properly normalized.

Our main contribution is the upper bound for the value with two-way classical communication. To make the proof easier to follow, we state first some lemmas. Some of them were already used in [14], but we recall them here for completeness and the convenience of the reader.

First, we state Khintchine and Double Khintchine inequalities in the precise form we will use. A proof of the double Khintchine inequality can be found in [15, p. 455].

**Theorem 2 (Khintchine inequalities)** For $1 \leq p < \infty$ there exist constants $a_p, b_p \geq 1$ such that

$$a_p^{-1} \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{y \in \{\pm 1\}^n} \frac{1}{2^n} \left| \sum_{i=1}^{n} \alpha_i y_i \right|^p \right)^{\frac{1}{p}} \leq b_p \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}}$$

(3)

for every $n \in \mathbb{N}$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

Moreover,

$$a_p^{-2} \left( \sum_{i,j=1}^{n} |\alpha_{i,j}|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{x,z \in \{\pm 1\}^n} \frac{1}{2^{2n}} \left| \sum_{i,j=1}^{n} \alpha_{i,j} x_i z_j \right|^p \right)^{\frac{1}{p}} \leq b_p^2 \left( \sum_{i,j=1}^{n} |\alpha_{i,j}|^2 \right)^{\frac{1}{2}}$$

(4)

for every $n \in \mathbb{N}$ and all $\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{n,n} \in \mathbb{C}$.

In our reasonings, we actually need the transposed version of both Khintchine inequalities. We state the precise result.

**Lemma 2** Let $1 < p < \infty$ and let $p'$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for every $n \in \mathbb{N}$ and for every sequence of numbers $(\alpha(y))_{y \in \{-1,1\}^n}$,

$$\left( \sum_{i=1}^{n} \left( \sum_{y \in \{-1,1\}^n} y_i \alpha(y) \right)^2 \right)^{\frac{1}{2}} \leq b_{p'}^2 \left( 2^n \right)^{\frac{1}{p'}} \left( \sum_{y \in \{-1,1\}^n} |\alpha(y)|^p \right)^{\frac{1}{p}},$$

where $b_{p'}$ is the constant appearing in Lemma 2 for $p'$.

Moreover, for every $n \in \mathbb{N}$ and for every finite sequence of numbers $(\alpha(x,z))_{(x,z) \in \{-1,1\}^n \times \{\pm 1\}^n}$,

$$\left( \sum_{i,j=1}^{n} \left( \sum_{(x,z)} x_i z_j \alpha(x,z) \right)^2 \right)^{\frac{1}{2}} \leq b_{p'}^2 \left( 2^{2n} \right)^{\frac{1}{p'}} \left( \sum_{(x,z)} |\alpha(x,z)|^p \right)^{\frac{1}{p}},$$
where the sums in \((x, z)\) are over \([±1]^n \times [±1]^n\) and \(b_{p′}\) is again the constant appearing in Lemma 2 for \(p′\).

\textbf{Proof} The second statement follows from (4). The proof can be seen in [14, Lemma 5.4]. The proof of the first statement is done similarly, using (3) rather than (4). \qed

We will also need the following simple consequence of Holder’s inequality.

\textbf{Lemma 3} For every \(1 < p < \infty\) and for every finite sequence of real numbers \((\alpha_i)_{i=1}^d\),

\[
\sum_{i=1}^d |\alpha_i| \leq d^{1/p'} \left( \sum_{i=1}^d |\alpha_i|^p \right)^{1/p},
\]

where \(\frac{1}{p} + \frac{1}{p'} = 1\).

We state and prove one more technical simple result.

\textbf{Lemma 4} Let \(\mathbf{a}, \mathbf{b}\) be as in Eqs. (1), (2). Then, for every \((x, y) \in X \times Y\)

\[
\sum_{\mathbf{m}, \mathbf{n}} |\mathbf{a}(x, \mathbf{m}, \mathbf{n})\mathbf{b}(y, \mathbf{m}, \mathbf{n})| \leq 1.
\]

\textbf{Proof} Recalling the definitions of \(\mathbf{a}, \mathbf{b}\), we have

\[
\sum_{\mathbf{m}, \mathbf{n}} |\mathbf{a}(x, \mathbf{m}, \mathbf{n})\mathbf{b}(y, \mathbf{m}, \mathbf{n})|
\]

\[
= \sum_{\mathbf{m}, \mathbf{n}} |a(x, \mathbf{m})M_1^{n_1}(x, n_1)M_2^{n_2}(x, n_1) \cdots M_t^{n_t}(x, n_1, \ldots, n_{t-1})b(y, \mathbf{m})N_1^{n_1}(y)N_2^{n_2}(y, m_1)
\]

\[
\cdots N_1^{n_t}(y, m_1, \ldots, m_t)|
\]

\[
\leq \sum_{\mathbf{m}, \mathbf{n}_1, \ldots, \mathbf{n}_{t-1}} |b(y, \mathbf{m})|M_1^{n_1}(x, n_1)M_2^{n_2}(x, n_1) \cdots M_t^{n_t}(x, n_1, \ldots, n_{t-1})
\]

\[
N_1^{n_1}(y)N_2^{n_2}(y, m_1) \cdots N_{t-1}^{n_{t-1}}(y, m_1, \ldots, m_{t-1}) \sum_{n_t}|a(x, \mathbf{n})|N_t^{n_t}(y, m_1, \ldots, m_t)
\]

\[
\leq \sum_{\mathbf{m}, \mathbf{n}_1, \ldots, \mathbf{n}_{t-1}} |b(y, \mathbf{m})|M_1^{n_1}(x, n_1)M_2^{n_2}(x, n_1) \cdots M_t^{n_t}(x, n_1, \ldots, n_{t-1})
\]

\[
N_1^{n_1}(y)N_2^{n_2}(y, m_1) \cdots N_{t-1}^{n_{t-1}}(y, m_1, \ldots, m_{t-1})
\]

\[
= \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_{t-1}} M_1^{n_1}(x)M_2^{n_2}(x, n_1) \cdots M_t^{n_t}(x, n_1, \ldots, n_{t-1})
\]

\[
N_1^{n_1}(y)N_2^{n_2}(y, m_1) \cdots N_{t-1}^{n_{t-1}}(y, m_1, \ldots, m_{t-1}) \sum_{\mathbf{m}_t}|b(y, \mathbf{m})|M_t^{n_t}(x, n_1, \ldots, n_{t-1})
\]

\[
\leq \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_{t-1}} M_1^{n_1}(x)M_2^{n_2}(x, n_1) \cdots M_t^{n_t}(x, n_1, \ldots, n_{t-1})
\]

\[
N_1^{n_1}(y)N_2^{n_2}(y, m_1) \cdots N_{t-1}^{n_{t-1}}(y, m_1, \ldots, m_{t-1}) \leq 1.
\]
To see the last inequality, it is enough to keep on summing in the same order, that is, in \( n_{t-1} \), then in \( m_{t-1} \), then in \( n_{t-2} \), etc. \( \square \)

Now we can upper bound the value of \( T \) with two-way classical communication. We have

**Proposition 2** Let \( T \) be the XOR game from Definition 1. Then,

\[
\omega_{tw, \log k}(T) \leq \frac{4\sqrt{2}e^{5/2} (\log k)^{3/2}}{n}.
\]

**Proof** We assume there are \( t \) rounds of communication with a total amount of bits exchanged of \( \log k \). Therefore, \( \log k = \sum_{i=1}^{t} c_i + d_i \), where \( c_i, d_i \) are as in Sect. 2. We also assume that Alice starts the communication, the other case being similar.

As explained in Sect. 2, it is enough to bound the quantity

\[
\sum_{\tilde{x}, y} T_{\tilde{x}, y} \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \bar{b}(y, \bar{m}, \bar{n}),
\]

when \( \bar{a}(\tilde{x}, \bar{m}, \bar{n}), \bar{b}(y, \bar{m}, \bar{n}) \) are as in Eqs. (1) and (2).

We have

\[
\sum_{\tilde{x}, y} T_{\tilde{x}, y} \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \bar{b}(y, \bar{m}, \bar{n}) \leq \sum_{\bar{m}, \bar{n}} \left| \sum_{\tilde{x}, y} T_{\tilde{x}, y} \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \bar{b}(y, \bar{m}, \bar{n}) \right|^{1/p}
\]

\[
\leq k^{1/p} \left( \sum_{\bar{m}, \bar{n}} \left| \sum_{\tilde{x}, y} T_{\tilde{x}, y} \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \bar{b}(y, \bar{m}, \bar{n}) \right|^{p} \right)^{1/p}
\]

\[
= \frac{k^{1/p}}{L} \left( \sum_{\bar{m}, \bar{n}} \left( \sum_{(x, z), y} \sum_{i, j} x_i z_j y_{ij} \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \bar{b}(y, \bar{m}, \bar{n}) \right)^p \right)^{1/p}
\]

\[
= \frac{k^{1/p}}{L} \left( \sum_{\bar{m}, \bar{n}} \left( \sum_{i, j} \left( \sum_{(x, z)} x_i z_j \bar{a}(\tilde{x}, \bar{m}, \bar{n}) \right) \left( \sum_{y} y_{ij} \bar{b}(y, \bar{m}, \bar{n}) \right) \right)^p \right)^{1/p},
\]

where the second inequality follows from Lemma 3.
We note now that, for every choice of \( m, n \),

\[
\left| \sum_{i,j} \left( \sum_{(x,z)} x_i z_j \bar{a}(x, z, m, n) \right) \left( \sum_{y} y_{ij} \bar{b}(y, m, n) \right) \right|
\]

\[
\leq \left( \sum_{i,j} \left( \sum_{(x,z)} x_i z_j \bar{a}(x, z, m, n) \right)^2 \right)^{1/2} \left( \sum_{y} \left( \sum_{ij} y_{ij} \bar{b}(y, m, n) \right)^2 \right)^{1/2}
\]

\[
\leq b_{p'}^3 \left( 2^{2n+2n^2} \right)^{1/p'} \left( \sum_{x,z} |\bar{a}(x, z, m, n)|^{p} \right)^{1/p} \left( \sum_{y} |\bar{b}(y, m, n)|^{p} \right)^{1/p},
\]

where the first inequality follows from Cauchy–Schwartz inequality and the second one follows from Lemma 2.

Using this, we have that

\[
\sum_{\bar{x}, y} T_{\bar{x}, y} \bar{a}(\bar{x}, m, n) \bar{b}(y, m, n)
\]

\[
\leq \frac{k_{p'}}{L} b_{p'}^3 \left( 2^{2n+2n^2} \right)^{1/p'} \left( \sum_{m,n} \sum_{x,z} |\bar{a}(x, z, m, n)|^{p} \left( \sum_{y} |\bar{b}(y, m, n)|^{p} \right) \right)^{1/p}
\]

\[
= \frac{k_{p'}}{L} b_{p'}^3 \left( 2^{2n+2n^2} \right)^{1/p'} \left( \sum_{x,z} \sum_{y} \bar{a}(x, z, m, n) \bar{b}(y, m, n) \right)^{1/p}
\]

\[
\leq \frac{k_{p'}}{L} b_{p'}^3 \left( 2^{2n+2n^2} \right)^{1/p'} \left( 2^{2n+2n^2} \right)^{1/p},
\]

where in the last inequality we have used Lemma 4 and the simple fact that, for every \( 1 < p < \infty \), if

\[
\sum_{m,n} |\bar{a}(x, z, m, n)\bar{b}(y, m, n)| \leq 1,
\]

then also

\[
\sum_{m,n} |\bar{a}(x, z, m, n)\bar{b}(y, m, n)|^p \leq 1.
\]

To finish, we use that \( L \geq \frac{1}{\sqrt{2}} 2n^2 + 2n \) by Remark 1. We also use that \( b_{p'} \leq \sqrt{2ep'} \) (see [15, Section 8.5]) and we make the choice \( p' = \log k \). Then, we have:
\[ \sum_{\vec{x}, \vec{y}} T_{\vec{y}, \vec{m}, \vec{n}} \tilde{a}(\vec{x}, \vec{m}, \vec{n}) \tilde{b}(\vec{y}, \vec{m}, \vec{n}) \leq \frac{4e^{5/2}(\log k)^{3/2}}{n}. \]

Now, Propositions 2 and 1 together prove Theorem 1.

4 The value of a game with two-way classical communication as a tensor norm

The purpose of this section is to show that the value of any XOR game assisted with a general two-way classical communication protocol can be described by a norm in the tensor of certain Banach spaces. In order to make this work self-contained, the required notions and definitions from Banach space theory and tensor norm theory will be presented here.

Given a normed space \( X \), denote by \( \| \cdot \|_X \) its norm, and by \( B_X = \{ x \in X \text{ such that } \|x\|_X \leq 1 \} \) its unit ball. The dual space consists of the linear and continuous maps from \( X \) to the scalar field (\( \mathbb{R} \) in our case) and it is denoted by \( X^* \). The norm of the dual space has the natural expression \( \|x^*\|_{X^*} = \sup_{x \in B_X} \| \langle x^*, x \rangle \| \).

All Banach spaces considered in this article are finite dimensional. In particular, we are interested in the spaces \( \ell^R_1 \) and \( \ell^R_\infty \), and their combination which we describe below.

Given \( R \in \mathbb{N} \) and a Banach space \( X \), we will define the spaces \( \ell^R_1(X) \) and \( \ell^R_\infty(X) \): As vector spaces, they are just the spaces whose elements are sequences of \( R \) elements in \( X \). Given one such element \( u = \{x_i\}_{i=1}^R \) with \( x_i \in X \), their norms are defined as follows:

\[
\|u\|_{\ell^R_1(X)} = \sum_{i=1}^R \|x_i\|_X,
\]

\[
\|u\|_{\ell^R_\infty(X)} = \max_{1 \leq i \leq R} \|x_i\|_X.
\]

With this definition at hand, we will consider the spaces \( \ell^R_1(\ell^S_\infty) \), \( \ell^R_\infty(\ell^S_1) \) and further concatenation of these spaces. For example, the element

\[
z = \{z(x_1, a_1, x_2, a_2, \ldots, x_t, a_t)\}_{x_1,a_1,\ldots,x_t,a_t} \in \mathbb{R}^{R_1S_1 \cdots R_tS_t}
\]

(5)

can be seen as an element in the space \( \ell^R_\infty(\ell^S_1(\ldots \ell^R_\infty(\ell^S_1) \ldots)) \). Considered in that space, the norm of \( z \) is

\[
\|z\|_{\ell^R_\infty(\ell^S_1(\ldots \ell^R_\infty(\ell^S_1) \ldots))} = \max_{x_1} \sum_{a_1} \ldots \max_{x_t} \sum_{a_t} |z(x_1, a_1, \ldots, x_t, a_t)|.
\]

Note the similarity of this expression with the one appearing in Lemma 1.
Recall that a sequence of \( R \) elements in \( X \), \( u = \{x_i\}_{i=1}^R \) can be naturally seen as an element in the tensor product \( R \otimes X \), the identification being \( u = \sum_{i=1}^R e_i \otimes x_i \), where \( e_i \) are the vectors of the canonical basis of \( R \). Hence, the element \( z \) mentioned in (5) can be algebraically identified with an element in \( R_1 \otimes R_{S_1} \otimes \cdots \otimes R_t \otimes R_{S_t} \).

Given two finite-dimensional Banach spaces \( X \) and \( Y \), the tensor product \( X \otimes Y \) can be endowed with different norms compatible with the norm structure of \( X \) and \( Y \), giving rise to different Banach spaces. This is the core idea of tensor norm theory. In this work, we will need the so-called \( \epsilon \)-norm. The following definition of the \( \epsilon \)-norm, together with basic properties thereof, can be seen, for instance, in [15,16].

Given two normed spaces \( X \) and \( Y \) and an element \( u = \sum_{i=1}^L x_i \otimes y_i \) in \( X \otimes Y \), the \( \epsilon \)-norm of \( u \) is defined by:

\[
\| u \|_{X \otimes Y} = \sup \left\{ \left( \sum_{i=1}^L |x^*(x_i)||y^*(y_i)| \right)^{1/2} : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}.
\] (6)

We will use the notation \( X \otimes_{\epsilon} Y \) to refer to the space \( X \otimes Y \) endowed with the \( \epsilon \)-norm.

Some basic notions about convexity will also be needed. Recall that a set \( A \) is convex if given \( x \) and \( y \) in \( A \), then \( \lambda x + (1 - \lambda) y \) is in \( A \) for all \( \lambda \in [0, 1] \). Given a set with \( n \) elements \( B = \{x_1, \ldots, x_n\} \), we define the convex hull of \( B \) as:

\[
\text{co}(B) = \left\{ \sum_{i=1}^n \alpha_i x_i \text{ such that } x_i \in A, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1 \right\}.
\]

An extreme point of a set \( A \) is a point which does not lie in any open line segment joining two points in the set. That is, if \( y \) is an extreme point of \( A \) and we can write \( y = \lambda x_1 + (1 - \lambda) x_2 \) with \( x_1 \) and \( x_2 \) in \( A \), and with \( x_1 \neq x_2 \), then \( \lambda \) is either 0 or 1. It is well known and easy to see that every convex set coincides with the convex hull of its extreme points.

The proof of the following lemma follows immediately from the definitions involved.

**Lemma 5** Denoting by \( \{e_i\}_{i=1}^R \) to the elements of the canonical basis of \( R \), we have:

1. The extreme points of \( B_{\ell_\infty^R} \) are exactly the elements of the form \( \sum_{i=1}^R a_i e_i \), where \( a_i = \pm 1 \) for every \( i \).
2. The extreme points of \( B_{\ell_1^R} \) are exactly the elements of the form \( a_i e_i \), where \( a_i = \pm 1 \).
3. Given a Banach space \( X \), the extreme points of \( B_{\ell_\infty^R}(X) \) are exactly the elements of the form \( \sum_{i=1}^R e_i \otimes x_i \), where \( x_i \) is an extreme point of \( B_X \) for every \( i \), and we use the tensor notation to identify \( \ell_\infty^R(X) \) and \( \ell_\infty^R \otimes \ell_\infty X \).
4. Given a Banach space \( X \), the extreme points of \( B_{\ell_1^R}(X) \) are exactly the elements of the form \( e_i \otimes x_i \), where \( x_i \) is an extreme point of \( B_X \) and we use the tensor notation as above.

In the reasonings below, it will be useful to write \( e_i \otimes x_i \) as \( \sum_{j=1}^R \delta_{i,j} e_j \otimes x_i \), where we define \( \delta_{i,j} \) by
The following result characterizes the extreme points of the unit ball of the space $\ell^R_1(\ell^S_1(\ldots \ell^R_1(\ell^S_1(\ldots))).$ In order to simplify its application later, we state it and prove it for the particular values of $R_1, S_1, \ldots, R_t, S_t$ for which we will apply it.

**Lemma 6** Consider two finite sets $X, Y$, with cardinals $R, S$, respectively, and positive integers $c_1, \ldots, c_t, d_1, \ldots, d_t$. Using again the notation $\overline{m}, \overline{n}$ for the multi-indices $(m_1, \ldots, m_t), (n_1, \ldots, n_t)$, the extreme points of the unit ball of $\ell^R_1(\ell^{2c_1}_2(\ldots (\ell^{2c_t}_2) \ldots))$ are exactly the elements of the form:

$$\sum_{x, \overline{m}, \overline{n}} a_x, \overline{m} \delta_{m_1, M_1(x)} \delta_{m_2, M_2(x, n_1)} \ldots \delta_{m_t, M_t(x, n_1, \ldots, n_{t-1})} e_x \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t},$$

where $x \in X, \overline{m} \in [2^{c_1}] \times \cdots \times [2^{c_t}], \overline{n} \in [2^{d_1}] \times \cdots \times [2^{d_t}], a_x, n_1, \ldots, n_t \in \{\pm 1\}$ for all $x, n_1, \ldots, n_t$ and $M_1: X \rightarrow [2^{c_1}], M_2: X \times [2^{d_1}] \rightarrow [2^{c_2}]$ and so on, are functions.

Similarly, the extreme points of the unit ball of $\ell^S_2(\ell^{2d_1}_2(\ell^{2d_2}_2(\ldots (\ell^{2c_t}_2) \ldots)))$ are exactly the elements of the form:

$$\sum_{y, \overline{m}, \overline{n}} b_y, \overline{m} \delta_{n_1, N_1(y, m_1)} \delta_{n_2, N_2(y, m_1, m_2)} \ldots \delta_{n_t, N_t(y, m_1, m_2, \ldots, m_t)} e_y \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t},$$

where, similarly as above, $y \in Y, \overline{m} \in [2^{c_1}] \times \cdots \times [2^{c_t}], \overline{n} \in [2^{d_1}] \times \cdots \times [2^{d_t}], b_y, m_1, \ldots, m_t \in \{\pm 1\}$ for all $x, m_1, \ldots, m_t$ and $N_1: Y \times [2^{c_1}] \rightarrow [2^{d_1}], N_2: Y \times [2^{c_1}] \times [2^{c_2}] \rightarrow [2^{d_2}]$ and so on, are functions.

**Proof** The proof follows easily from Lemma 5 and induction. For the sake of clarity, we write out the proof for the case of $\ell^S_2(\ell^{2d_1}_2(\ell^{2d_2}_2))$, which corresponds to $t = 2$ in the second statement of the lemma.

First note that according to Lemma 5 and the notation following it, the extreme elements of the unit ball of $\ell^{2d_2}_2(\ell^{2d_2}_2)$ are of the form

$$\sum_{m_2, n_2=1}^{2^{c_2}, 2^{d_2}} b_{m_2, n_2} \delta_{n_2, N_2(m_2)} e_{m_2} \otimes e_{n_2},$$

where $N_2: [2^{c_2}] \rightarrow [2^{d_2}]$ runs over all possible functions and $b_{m_2} \in \{\pm 1\}$ for all $m_2$. 

\[ \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
Then, with the aid of the $\delta$ notation, the extreme points of the of the unit ball of $\ell_1^{2d_1}$ ($\ell_2^{2d_2}$ ($\ell_1^{2d_2}$)) can be written as

$$2^{d_1} \sum_{n_1=1}^{2^{d_1}} \sum_{n_2,m_2} z_{m_2} \delta_{n_2,N_2(m_2)} \delta_{n_1,n_0} e_{n_1} \otimes e_{m_2} \otimes e_{n_2},$$

where $n_0 \in [2^{d_1}]$.

Finally, to describe the extreme points of the unit ball of $\ell_2^{S_1} (\ell_2^{2d_1} (\ell_2^{2d_2} (\ell_1^{2d_2}))))$, first note that $\mathbb{R}^{S_1} = \mathbb{R}^S \otimes \mathbb{R}^2$.

Then, applying again Lemma 5, for every $y$ and $m_1$, we obtain that the extreme points of the unit ball of $\ell_2^{S_1} (\ell_2^{2d_1} (\ell_2^{2d_2} (\ell_1^{2d_2}))))$ are exactly those of the form

$$2^{d_1} \sum_{y,m_1=1}^{2^{d_1}} e_y \otimes e_{m_1} \otimes \left( \sum_{n_1=1}^{2^{d_1}} \sum_{n_2,m_2} b_{m_2} \delta_{n_2,N_2(m_2)} \delta_{n_1,n_0} e_{n_1} \otimes e_{m_2} \otimes e_{n_2} \right).$$

In that expression, the functions $b_{m_2}$, $\delta_{n_2,N_2(m_2)}$ and $\delta_{n_1,n_0}$ depend also on $y$ and $m_1$, and therefore we can rewrite the formula above as:

$$\sum_{y,m_1,m_2,n_1,m_2,n_2} b_{y,m_1,m_2} \delta_{n_1,N_1(m_1,y)} \delta_{n_2,N_2(y,m_1,m_2)} e_y \otimes e_{m_1} \otimes e_{n_1} \otimes e_{m_2} \otimes e_{n_2},$$

where $N_2: Y \times [2^{c_1}] \times [2^{c_2}] \to [2^{d_2}]$ and $N_1: Y \to [2^{c_1}]$ are functions and $b_{y,m_1,m_2} \in \{\pm 1\}$ for all $y, m_1, m_2$.

We still need some more notation before we can state our last result. A XOR game $T$ with coefficients $(T_{x,y})_{x \in X, y \in Y}$ can be considered as an element

$$T \in \ell_1^R \otimes \ell_1^S,$$

where $R, S$ are the cardinals of $X, Y$, respectively. As it is well known (see, for instance, [6]) the value of $T$ without communication coincides with the norm $\|T\|_{\ell_1^R \otimes \ell_1^S}$.

In order to consider the value of $T$ in the presence of classical communication, we need a more involved construction.

Given $T$ and positive integers $c_1, \ldots, c_t, d_1, \ldots, d_t$ as above, using the multi-index notation as before, we can consider the tensor $T \otimes id \otimes \cdots \otimes id \in \ell_1^R (\ell_2^{c_1} (\ell_2^{d_1} (\ell_2^{d_t} (\ell_1^{d_t})))) \otimes \ell_2^{S_1} (\ell_2^{c_2} (\ell_2^{d_2} (\ell_2^{d_t} (\ell_1^{d_t}))))$ defined by

$$T \otimes id \otimes \cdots \otimes id := \sum_{x,y,m,r} T_{x,y}(e_x \otimes e_{m_1} \otimes e_{n_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_t})$$

$$\otimes (e_y \otimes e_{m_1} \otimes e_{n_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_t})$$

$$= \sum_{x,y,m,r,m',n'} \delta_{m_1,m'} \cdots \delta_{m_t,m'} \delta_{n_1,n'} \cdots \delta_{n_t,n'} T_{x,y}$$

$$(e_x \otimes e_{m_1} \otimes e_{n_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_t}) \otimes (e_y \otimes e_{m_1} \otimes e_{n_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_t})$$
In the case of one-way communication, it was proven in [14] that the value of $T$ when using $c_1$ bits of one-way communication from Alice to Bob is $\|T \otimes id\|_{\ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})}$, that is, the $\epsilon$ norm of $T \otimes id$ when considered as an element in

$$\ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})$$

The analog of this statement for the case of several rounds of two-way communication is our last result. We consider, as before, two finite sets $X$, $Y$ with cardinals $R$, $S$, respectively, and a XOR game $T = (T_{x,y})_{x,y \in X,Y}$. We consider a general two-way protocol as defined in Sect. 2 in which there is a total amount of $c$-bits of two-way communication exchanged, in $t$ different rounds. The messages sent by Alice to Bob use $c_1$ to $c_t$ bits, respectively, and the ones sent by Bob to Alice, $d_1$ to $d_t$, respectively. Hence $\sum_{i=1}^t c_i + \sum_{i=1}^t d_i = c$.

**Theorem 3** Consider $= (T_{x,y})_{x,y \in X,Y}$ and $\omega_{tw,c}(T)$ as above. Then, the following holds:

$$\omega_{tw,c}(T) = \|T \otimes id \otimes id \otimes \cdots \otimes id\|_{\ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})} \epsilon \ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})$$

**Proof of Theorem 3** Considering the supremum below in the possible strategies of Alice and Bob, we have

$$\omega_{tw,c}(T) = \sup_{x,m_1,\ldots} \sum_{x,m_1,\ldots} T_{x,y} M_{1}^{m_1}(x) N_{1}^{m_1}(y, m_1)$$

$$\cdots M_{t}^{m_t}(x, m_1, \ldots, n_{t-1}) N_{t}^{m_t}(y, m_1, \ldots, m_t) a(x, m) b(y, m)$$

$$= \sup_{x,m_1,\ldots} \sum_{x,m_1,\ldots} \delta_{m_1,m_1'} \cdots \delta_{m_t,m_t'} \delta_{n_1,n_1'} \cdots \delta_{n_t,n_t'} T_{x,y} M_{1}^{m_1}(x) N_{1}^{m_1}(x, m_1)$$

$$\cdots M_{t}^{m_t}(x, m_1, \ldots, n_{t-1}) N_{t}^{m_t}(y, m_1, \ldots, m_t) a(x, m) b(y, m_1, \ldots, m_t)$$

$$= \sup_{x,m_1,\ldots} (T \otimes id \otimes \cdots \otimes id) \sum_{x,m_1,\ldots} a(x, m) M_{1}^{m_1}(x)$$

$$\cdots M_{t}^{m_t}(x, m_1, \ldots, n_{t-1}) e_{x} \otimes e_{m_1} \otimes \cdots \otimes e_{m_t} \otimes e_{n_1} \otimes \cdots \otimes e_{n_t}$$

$$\otimes \sum_{y,m_1',\ldots,m_t',n_1',\ldots,n_t'} b(y, m_1', \ldots, m_t') N_{1}^{m_1'}(x, m_1') \cdots N_{t}^{m_t'}(y, m_1', \ldots, m_t')$$

$$e_{y} \otimes e_{m_1'} \otimes \cdots \otimes e_{m_t'} \otimes e_{n_1'} \otimes \cdots \otimes e_{n_t'}).$$

We recommend the reader to write the formula above in the case $t = 2$.

It follows now immediately from the definitions and Lemma 1 that

$$\omega_{tw,c}(T) \leq \|T \otimes id \otimes \cdots \otimes id\|_{\ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})} \epsilon \ell^1_{\infty}(\ell^2_{\infty} \otimes \ell^2_{\infty})$$

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In order to prove the reverse inequality, note first that it follows from the definitions that
\[ \|T\|_{\ell^R,\ell^S_{1,\infty}}(\ell^2_{c_1}(\ell^2_{d_1}(\cdots(\ell^2_{c_t}(\ell^2_{d_t}))))^{\otimes \epsilon \ell^6}) \]
coincides with
\[ \sup \{ \langle T \otimes id \otimes \cdots \otimes id | a \otimes b \rangle \} \]
when \( a \in B_{\ell^R, \ell^S_{1,\infty}}(\ell^2_{c_1}(\ell^2_{d_1}(\cdots(\ell^2_{c_t}(\ell^2_{d_t}))))^{\otimes \epsilon \ell^6}) \)
and \( b \in B_{\ell^R_{1,\infty}}(\ell^2_{c_1}(\ell^2_{d_1}(\cdots(\ell^2_{c_t}(\ell^2_{d_t}))))^{\otimes \epsilon \ell^6}) \).
It follows now from compactness and convexity that the supremum above is actually a maximum which will be attained on extreme points \( a, b \) of the respective unit balls.

Now, Remark 6 tells us that the extreme points of \( B_{\ell^R_{1,\infty}}(\ell^2_{c_1}(\ell^2_{d_1}(\cdots(\ell^2_{c_t}(\ell^2_{d_t}))))^{\otimes \epsilon \ell^6}) \)
have the form
\[ \sum_{x,n_1,m_1,\ldots,n_t} a_{x,n_1,\ldots,n_t} \delta_{m_1,m_1(x)} \delta_{m_2,m_2(x,n_1)} \cdots \delta_{m_t,m_t(x,n_1,\ldots,n_{t-1})} e_x \otimes e_{y_1} \otimes e_{\delta_1} \otimes \cdots \otimes e_{y_t} \otimes e_{\delta_t} \]
which can be seen according to (1) as a deterministic strategy for Alice in which the final answer is \( a_{x,n_1,\ldots,n_t} \) and the messages that she has sent are \( m_1(x), m_2(x,n_1), \ldots \) and \( m_t(x,n_1,\ldots,n_{t-1}) \).

We proceed similarly for the extreme points of \( B_{\ell^R_{1,\infty}}(\ell^2_{c_1}(\ell^2_{d_1}(\cdots(\ell^2_{c_t}(\ell^2_{d_t}))))^{\otimes \epsilon \ell^6}) \).

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