Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order

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Abstract: In this work we derive for the first time the complete gravitational cubic-in-spin effective action at the next-to-leading order for the interaction of generic compact binaries via the effective field theory for gravitating spinning objects and its extension to this sector. This sector, which enters at the fourth and a half post-Newtonian (4.5PN) order for rapidly rotating compact objects, completes finite size effects up to this order, and is the first sector completed beyond the current state of the art for generic compact binary dynamics at the 4PN order. At this order in spins with gravitational nonlinearities we have to take into account additional terms, which arise from a new type of worldline couplings, due to the fact that at this order the Tulczyjew gauge for the rotational degrees of freedom, which involves the linear momentum, can no longer be approximated only in terms of the four-velocity. One of the main motivations for us to tackle this sector is also to see what happens when we go to a sector, which corresponds to the gravitational Compton scattering with quantum spins of three halves, and maybe possibly also get an insight on the inability to uniquely fix its amplitude from factorization when spins of five halves and higher are involved. A general observation that we can clearly make already is that even-parity sectors in the order of the spin are easier to handle than odd ones. In the quantum context this corresponds to the greater ease of dealing with bosons compared to fermions.
1 Introduction

Since the first detection of gravitational waves (GWs) from a binary black hole coalescence was announced in 2016 it has become increasingly pressing to provide high precision theoretical predictions for the modeling of GW templates. The latter significantly rely on implementing analytical results obtained within the post-Newtonian (PN) approximation of classical Gravity \[1\] via the Effective-One-Body approach \[2\]. In particular in recent years we have made a remarkable progress in pushing the precision frontier for the orbital dynamics of compact binaries, i.e. whose components are generic compact objects. The complete state of the art to date for the orbital dynamics of a generic compact binary is shown in table \[1\].

As a measure for the loop computational scale we show in table \[1\] the number of \(n\)-loop graphs that enter at the \(N^n\)LO in \(l\) powers of the spin, i.e. up to the \(l\)th spin-induced multipole moment, in the sectors completed to date. The count is based on computations carried out with the effective field theory (EFT) of PN Gravity \[3\], which use the Kaluza-Klein decomposition of the field from \[4\], that has considerably facilitated high precision computations within the EFT approach \[4–16\]. As can be seen the current complete state of the art is at the 4PN order, whereas the next-to-leading order (NLO) cubic-in-spin sector that enters at the 4.5PN order is evaluated in this paper. All of the sectors at the current state of the art (but the top right entry at the 4PN order for the non-rotating case) are available in the public “\textsc{EFTofPNG}” code at https://github.com/miche-levi/pncbc-eftofpng \[17\].

Let us stress that in order to attain a certain level of PN accuracy, the various sectors should be tackled across the diagonals of table \[1\], rather than along the axes, namely

\[
\begin{array}{ll}
1 & \text{Introduction} \\
2 & \text{The EFT of gravitating spinning objects} \\
3 & \text{The essential computation} \\
\quad 3.1 & \text{One-graviton exchange} \\
\quad 3.2 & \text{Two-graviton exchange} \\
\quad 3.3 & \text{Cubic self-interaction} \\
4 & \text{New features from gauge of rotational DOFs} \\
5 & \text{The cubic-in-spin action at the next-to-leading order} \\
6 & \text{Conclusions} \\
\end{array}
\]

\[1\]
Table 1. The complete state-of-the-art of PN Gravity theory for the orbital dynamics of generic compact binaries. Each PN correction enters at the order $n + l + \text{Parity}(l)/2$, where the parity is 0 or 1 for even or odd $l$, respectively. We elaborate on the meaning of the numerical entries and the gray area in the text below.

| $l$ | $n$ | $(N^0)\text{LO}$ | $(N^1)\text{LO}$ | $(N^2)\text{LO}$ | $(N^3)\text{LO}$ | $(N^4)\text{LO}$ |
|-----|-----|------------------|------------------|------------------|------------------|------------------|
| $S^0$ | 1   | 0                | 3                | 0                | 25               |
| $S^1$ | 2   | 7                | 32               |                  |                  |
| $S^2$ | 2   | 2                | 18               |                  |                  |
| $S^3$ | 4   | 24               |                  |                  |                  |
| $S^4$ | 3   |                  |                  |                  |                  |

progress must be made by going in parallel to higher loops and to higher orders of the spin. In general, the former involves more challenges of computational loop technology and tackling associated divergences, whereas the latter necessitates an improvement of the fundamental understanding of spin in gravity, and tackling finite size effects with spin [18]. These enter first at the 2PN order [19] from the LO spin-induced quadrupole. Within the EFT approach whose extension to the spinning case was first approached in [20], finite size effects include as additional parameters the Wilson coefficients, that correspond to the multipole deformations of the object due to its spin, as in [21] for the quadrupole.

With a considerable time gap from the LO result, the NLO spin-squared interaction was treated in a series of works [11, 22–25], where the result in [11] was derived within the formulation of the EFT for gravitating spinning objects introduced there. The LO cubic- and quartic-in-spin interactions were first tackled in [24, 26] for black holes. In [10], based on the formulation presented in [11], these were derived for generic compact objects, where also the quartic-in-spin interaction was completed. Only specific pieces of the latter results were recovered in [27] via S-matrix combined with EFT techniques, whereas [28] which treated only cubic-in-spin effects, also provided the LO effects in the energy flux. The work in [29] then also derived for the case of black holes the LO sectors to all orders in spin. Finally, the NNLO spin-squared interaction was derived in [13]. Notably the latter results together with the complete quartic-in-spin results for generic compact objects in [10], both at the 4PN order, were derived so far exclusively within the EFT formulation of spinning gravitating objects [11].

Recently, there has also been a surge of interest in harnessing modern advances in scattering amplitudes to the problem of a coalescence of a compact binary. Notably, a new implementation for the non-rotating case to the derivation of classical potentials was carried out in [30, 31]. Further, based on a new quantum formalism introduced in [32] for massive particles of any spin, new approaches to the computation of spin effects of black holes in the classical potential were put forward in [33, 34] and then in [35, 36]. In these approaches classical effects with spin to the $l$th order correspond to amplitudes involving a quantum spin of $s = l/2$. In particular as of the one-loop level the gravitational Compton
The gravitational Compton scattering relevant as of the one-loop level. The gravitational Compton amplitude involves two massive spinning particles and two massless gravitons, where factorization constraints do not uniquely determine the amplitude for $s > 2$ [32]. The gray area in table 1 then corresponds in the quantum context to where the gravitational Compton scattering with a spin $s > 1$ is required.

Notably, the gray area in table 1 also corresponds to, as was already pointed out in [11], where we can no longer take the linear momentum $p_\mu$, with which the generic formulation in [11] was derived, to be its leading approximation given by $m \frac{u_\mu}{\sqrt{u^2}}$, as was done in all past spin sectors tackled, but we have to take into account corrections to the linear momentum from the non-minimal coupling part of the spinning particle action. Can we then get a well-defined result? Can we get an insight from examining this new feature at the classical level on the non-uniqueness of fixing the graviton Compton amplitude with $s > 2$?

This work builds on the formalism of the EFT for gravitating spinning objects introduced in [11] and the implementation on [10] to compute the cubic-in-spin interaction at the NLO, that enters at the 4.5PN order for maximally-rotating compact objects, beyond the current state of the art of PN theory in general and with spins in particular [37], and is the leading sector in the intriguing gray area of table 1. We compute the complete sector, taking into account all interactions that include all possible multipoles up to the octupole. Thus beyond pushing the state of the art in PN theory, there are two conceptual objectives that we get to address in this work: 1. To learn how the difference from the leading linear momentum to its correction affects the results; 2. To see whether this difference is related with the non-uniqueness of the gravitational Compton amplitude of higher spin states, or to get any possible insight on this non-uniqueness.

The paper is organized as follows. In section 2 we go over the formulation from [11], and the necessary ingredients to evaluate this sector. In section 3 we present the essential computation, where the linear momentum assumes its leading approximation in terms of the four-velocity, as done in all past evaluations of spin sectors. In section 4 we find the new contributions arising from the correction to the leading linear momentum, which matters as of this sector, and gives rise to a new type of worldline-graviton coupling. In section 5 we compute the final action of this sector, and finally we conclude in section 6 with some observations and questions.
2 The EFT of gravitating spinning objects

Let us consider the ingredients that are required in order to carry out the evaluation of this sector, that contains spins up to cubic order along with first gravitational nonlinearities. This evaluation will build on the EFT of gravitating spinning objects formulated in [11], and its implementation from LO up to the state of the art at the 4PN order in [10–13, 37]. We will also use here the Kaluza-Klein decomposition of the metric [4, 38] to scalar, vector and symmetric tensor components, which was adopted in all high order PN computations both with and without spins for its facilitating virtues [18], and follow conventions consistent with the abovementioned works. Further, we follow similar gauge choices, notational and pictorial conventions as presented in [11].

The effective action we start from is that of a two-particle system [18], with each of the particles described by the one-particle effective action of a spinning particle, that was provided in [11]. This effective action contains a pure gravitational piece, from which the propagators and self-interacting vertices are derived. The Feynman rules for the propagator and the time insertions on the propagators are given e.g. in eqs. (5)-(10) of [9], and for the cubic gravitational vertices in eqs. (2.10)-(2.13), and (2.15) of [12]. Further, for each of the two particles the worldline action of a spinning particle is considered from [11], where its spin-induced non-minimal coupling part was constructed, and then gauge freedom of the rotational DOFs is incorporated into the action. We recall that this action has the following form:

$$S_{pp}(\sigma) = \int d\sigma \left[ -m\sqrt{u^2} - \frac{1}{2}\hat{S}_{\mu\nu} \hat{\Omega}^{\mu\nu} - \frac{\hat{S}_{\mu\nu} p_{\nu}}{p^2} D_{\mu} + L_{SI} \right],$$

(2.1)

given in terms of four velocity $u_{\mu}$, the linear momentum $p_{\mu}$ and the generic rotational DOFs, denoted with a hat e.g. $\hat{S}_{\mu\nu}$, and where the label “SI” stands for the spin-induced part of the action, which for the sector evaluated here will consist of its two leading terms given by

$$L_{SI} = -\frac{C_{ES} S^2}{2m} E_{\mu\nu} S^\mu S^\nu - \frac{C_{BS} S^3}{6m^2} D_\lambda B_{\mu\nu} S^\mu S^\nu S^\lambda,$$

(2.2)

where here it is the spin vector $S^\mu$ that is used, as described in detail in [11, 18]. We recall that in eq. (2.1) there is an extra term, which appears in the action from the restoration of gauge freedom of the rotational DOFs. This term, which is essentially the Thomas precession as discussed in detail in [11] (and recovered recently as “Hilbert space matching” in [36, 39]), contributes to all orders in the spin as of the LO spin-orbit sector, and in particular also to finite size spin effects, though it does not encode any UV physics, but rather in the context of an effective action just accounts for the fact that a relativistic gravitating object has an extended measure.

Since we compute here the complete NLO cubic-in-spin sector our graphs will contain all multipoles in the presence of spin up to the spin-induced octupole, i.e. also including the mass, spin and spin-induced quadrupole. For this reason we need to use Feynman rules of worldline-graviton coupling to NLO for all of these multipoles, where in this work we
need to derive further new rules for the octupole couplings. The Feynman rules required for the mass couplings are given in eqs. (64), (67), (68), (79), (81), (93), (95) of [7]. Next, we approach the Feynman rules linear in spin, noting we that we have first kinematic contributions as noted in eq. (5.28) of [11], that are linear in the spin but have no field coupling, which we will take into account in section 5.

The Feynman rules required for the linear-in-spin couplings are given in eqs. (5.29)-(5.31) of [11], and eqs. (2.31)-(2.34) of [12]. For the spin quadrupole couplings the rules are given in eqs. (2.18)-(2.24) of [13], and for the LO spin octupole couplings they are found in eqs. (2.19),(2.20) of [10]. As we noted in addition to the abovementioned Feynman rules, further rules are required here for the spin-induced octupole worldline-graviton coupling. The two Feynman rules of the scalar and vector components of the KK decomposition, which appeared already at LO in [10] should be extended to higher PN order, and further we will have new rules that enter for the one-graviton coupling of the tensor component of the KK fields, and a couple of two-graviton couplings, involving again the scalar and the vector components of the KK fields, which appeared at LO.

The extended rules for the one-graviton couplings are then given as follows:

\[ \int dt \left[ \frac{C_{BS}^3}{12m^2} S^i S^j \epsilon_{klm} S^m \left( \partial_i \partial_j \partial_l A_k (1 + \frac{v^2}{2}) + v^l (\partial_t A_{i,jk} + \partial_t A_{i,ij}) + \right. \right. \]
\[ + \left. \left. v^l v^m (A_{i,jkn} - A_{n,ijk}) \right) - \frac{v^l}{2} A_{k,jln} (S^m v^n + S^n v^m) \right] \], \hspace{1cm} (2.3)\]

\[ \int dt \left[ \frac{C_{BS}^3}{3m^2} S^i S^j \epsilon_{klm} S^m v^l \left( \partial_i \partial_j \partial_k \phi (1 + \frac{v^2}{2}) - \frac{1}{2} v^i v^n \partial_j \partial_k \partial_n \phi \right) \right] , \hspace{1cm} (2.4)\]

where the rectangular boxes represent the spin-induced octupole.

The new Feynman rules required here are given as follows:

\[ \int dt \left[ \frac{C_{BS}^3}{12m^2} S^i S^j \epsilon_{klm} S^m \partial_i \partial_l \left( \partial_j \sigma_{kn} v^n - \partial_n \sigma_{jk} v^n - \partial_l \sigma_{jk} \right) \right] \], \hspace{1cm} (2.5)\]

for the one-graviton coupling, where for the two-graviton couplings we get:

\[ \int dt \left[ \frac{C_{BS}^3}{12m^2} S^i S^j \epsilon_{klm} S^m \left( 2 \phi_{i,ijl} A_k - \phi A_{k,ijl} + 6 \phi_{i,l} A_{k,j} + 6 \phi_{j} A_{k,i,l} + 4 \phi_{i,l} A_{k,l} + 4 \phi_{i} A_{k,l} + 2 \phi_{k} A_{j,i,l} + 2 \phi_{k} A_{j,l} - \delta_{ij} \phi_{n} A_{k,n} \right) \right] , \hspace{1cm} (2.6)\]

\[ \int dt \left[ \frac{C_{BS}^3}{3m^2} S^i S^j \epsilon_{klm} S^m v^l \left( 3 \phi_{i,j} \phi_{ik} + 3 \phi_{k} \phi_{ij} - 3 \phi \phi_{ij,k} - \delta_{ij} \phi_{kn} \phi_{n} \right) \right] , \hspace{1cm} (2.7)\]

We note that in these rules the spin is already fixed to the canonical gauge and all indices are Euclidean. Notice the complexity of these couplings with respect to the other worldline
couplings at the NLO level, and notice also the dominant role that the gravitomagnetic vector plays in the coupling to the odd-parity octupole, similar to the situation in the coupling to the spin dipole. Note that this is the first sector which necessitates to take the curved Levi-Civita tensor into account.

For this sector there is no need to extend the non-minimal coupling part of the spinning particle action and add higher dimensional operators beyond what was provided in [11], but we need to pay special attention to the new feature that differentiates this specific sector from all the spin sectors which were tackled in the past. In this sector it is no longer sufficient to use the leading approximation for the linear momentum $p_\mu$ in terms of the four-velocity $u_\nu$ all throughout, rather one has to take into account the subleading term in the linear momentum, which is linear in Riemann and quadratic in the spin and becomes relevant exactly once we get to the level that is non-linear in gravity and cubic in the spins, i.e. at this sector, as was already explicitly noted in [11]. We will address in detail the particular contributions coming from this new feature in section 4 below after we have done the essential computation, which requires only the leading approximation to the linear momentum, similar to what was considered in all past PN computations with spin, in the following section.

3 The essential computation

In this section we carry out the perturbative expansion of the effective action in terms of Feynman graphs, and provide the value of each diagram, under the leading approximation of the linear momentum. At the NLO level, i.e. up to the $G^2$ order, with spins all of the three relevant topologies are realized even when the beneficial KK decomposition of the field is used, as discussed in [6, 7, 11, 18]. As shown in figures 2-4 below (drawn using Jaxodraw [40, 41] based on [42]) there is a total of $50 = 10 + 16 + 24$ graphs making up the sector, distributed among the relevant topologies of one- and two-graviton exchanges and cubic self-interaction, respectively. As shown in table 1 about half of the total graphs require a one-loop evaluation (the highest loop in this sector). We note that as we go into the nonlinear regime of the sector, the options for the make up of the interaction become more intricate.

At the one-graviton exchange level we only have two kinds of interaction contributing, similar to the LO in [10], namely either an octupole-monopole or a quadrupole-dipole interaction. As noted in [10] there are nice analogies among these interactions according to the parity of the multipole moments involved. Following these analogies the relevant graphs of one-graviton exchange are easily constructed. Yet, once we proceed to the level nonlinear in the gravitons further types of interactions emerge. In particular, there are also interactions involving the various multipoles on two different points of the same worldline, which add up to interactions that are cubic in the spin, such as a spin and a spin-induced quadrupole, or two spin dipoles on the same worldline, as can already be seen as of the NLO spin-squared sector [11, 13].

We note that all the graphs in this sector should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. For more specific details on the
Figure 2. The one-graviton exchange Feynman graphs, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels 1 $\leftrightarrow$ 2 exchanged. At the one-graviton exchange level we only have two kinds of interactions contributing, similar to the LO in [10], namely either a quadrupole-dipole or an octupole-monopole one. As noted in [10] there are nice analogies among these interactions according to the parity of the multipole moments involved. Following these analogies the relevant graphs here are easily constructed. Notice that we have here the four graphs that appeared at the LO with the quadratic time insertions on the propagators at graphs (a7)-(a10), and a new octupole coupling involving the tensor component of the KK fields at graph (a3).

3.1 One-graviton exchange

As can be seen in figure 2 we have 10 graphs of one-graviton exchange in this sector, the majority of which already involve time derivatives to be applied. Consistent with former works by one of the authors we keep all of the higher order time derivative terms that emerge in the evaluations of the graphs, and they will be treated properly via redefinitions of the position and the rotational variables as shown in [43]). Notice that we have here the 4 graphs that appeared at the LO with the quadratic time insertions on the propagators at graphs 1(a7)-(a10), and a new octupole coupling involving the tensor component of the KK fields at graph 1(a3).

The graphs in figure 2 are evaluated as follows:

$$
\text{Fig. 1(a1)} = -\frac{1}{2} C_{1(BS)} G \frac{m_2}{m_1^3} \left[ \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 (2S_2^2 \vec{v}_1 \cdot \vec{n} + 3 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 10(\vec{S}_1 \cdot \vec{n})^2 \vec{v}_1 \cdot \vec{n}) \\
+ 2 \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_2^2 - 5(\vec{S}_1 \cdot \vec{n})^2) \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left( S_2^2 (v_1^2 + v_2^2) \\
- \vec{S}_1 \cdot \vec{v}_1 (\vec{S}_1 \cdot \vec{v}_1 - 5 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}) - 5(\vec{S}_1 \cdot \vec{n})^2 (v_1^2 + v_2^2) \right) \right]
$$
\[ - \vec{v}_1 \cdot \vec{v}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_1 \left( S_1^2 - 5(S_1 \cdot \vec{n})^2 \right) \]
\[ + \frac{1}{3} C_{(BS)} \frac{G m_2}{r^4 m_1} \left[ 2 \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left( \vec{S}_1 \cdot \vec{v}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \right) \right. \]
\[ + \vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_2 \left( S_1^2 - 3(S_1 \cdot \vec{n})^2 \right) + \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left( S_1^2 - 3(S_1 \cdot \vec{n})^2 \right) \]
\[ - 3 \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left( \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) - 3 \vec{S}_1 \cdot \vec{a}_1 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \]
\[ - 3 \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n}. \]  
(3.1)

\[ \text{Fig. 1(a2)} = \frac{1}{2} C_{(BS)} \frac{G m_2}{r^4 m_1} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2(v_1^2 + 3v_2^2) - \vec{S}_1 \cdot \vec{v}_1 \left( \vec{S}_1 \cdot \vec{v}_1 - 5 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \right) \right. \]
\[ - 5(S_1 \cdot \vec{n})^2(v_1^2 + 3v_2^2) \], \quad (3.2)

\[ \text{Fig. 1(a3)} = -C_{(BS)} \frac{G m_2}{r^4 m_1} \left[ \left( \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} v_2 - \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right) \left( S_1^2 - 5(S_1 \cdot \vec{n})^2 \right) \right] \]
\[ + C_{(BS)} \frac{G m_2}{r^4 m_1} \left[ \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left( \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \right. \]
\[ + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \]. \quad (3.3)

\[ \text{Fig. 1(a4)} = \frac{3}{2} C_{(ES)} \frac{G m_2}{r^4 m_1} \left[ 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \left( \vec{S}_1 \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{n} \left( 3v_1^2 + v_2^2 \right) \right) \right. \]
\[ + 2 \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \left( 2S_1^2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \right) \]
\[ - \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \left( S_1^2 \left( 5v_1^2 + v_2^2 - 10(\vec{v}_1 \cdot \vec{n})^2 \right) - 2 \vec{S}_1 \cdot \vec{v}_1 \left( \vec{S}_1 \cdot \vec{v}_1 - 5 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \right) \right. \]
\[ - 5(S_1 \cdot \vec{n})^2 \left( 3v_1^2 + v_2^2 \right) \] \]
\[ \left. + C_{(ES)} \frac{G m_2}{r^4 m_1} \left[ 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \left( \vec{S}_1 \cdot \vec{a}_1 + \vec{S}_1 \cdot \vec{v}_1 \right) + 2 \vec{S}_2 \cdot \vec{v}_2 \times \vec{S}_1 \cdot \vec{v}_1 + 4 \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \vec{S}_1 \cdot \vec{S}_1 + 2 \vec{S}_2 \cdot \vec{v}_2 \times \vec{a}_2 \left( S_1^2 - 3(S_1 \cdot \vec{n})^2 \right) \right. \]
\[ - 6 \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \vec{S}_2 \vec{a}_1 \vec{n} - 2 \vec{S}_1 \cdot \vec{S}_1 \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \]
\[ + \vec{S}_1 \cdot \vec{a}_1 \vec{S}_1 \cdot \vec{n} \left. \right) \right] \]
\[ - 4C_{(ES)} \frac{G m_2}{r^4 m_1} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \vec{S}_2 \cdot \vec{v}_2 \left. \right] \right] , \quad (3.4)

\[ \text{Fig. 1(a5)} = -\frac{3}{2} C_{(ES)} \frac{G m_2}{r^4 m_1} \left[ 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \left( \vec{S}_1 \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{n} v_1^2 \right) \right. \]
\[ - 6 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \left( S_1^2 \vec{v}_2 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \]
\[ - 5(S_1 \cdot \vec{n})^2 \vec{v}_2 \cdot \vec{n} \right. \]
\[ - 5(S_1 \cdot \vec{n})^2 \vec{v}_2 \cdot \vec{n} \right. \]
\[ + 10 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} - 5(S_1 \cdot \vec{n})^2 \vec{v}_1^2 \]
\[ - 3 \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \vec{S}_2 \cdot \vec{v}_2 \left( S_1^2 - 5(S_1 \cdot \vec{n})^2 \right) \vec{v}_1 \cdot \vec{v}_2 \right] \]
\[ - 3C_{(ES)} \frac{G m_2}{r^4 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \right] \]
\[ -\frac{1}{2}C_{1(ES^2)} \frac{G}{r^2 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 (4 \dot{\vec{S}}_1 \cdot \vec{v}_2 - 3 \dot{\vec{S}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \right. \\
+ \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_2 (4 \dot{\vec{S}}_1 \cdot \vec{v}_2 - 3 \dot{\vec{S}}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - 3 \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n}) \\
+ \dot{\vec{S}}_1 \cdot \vec{v}_1 \vec{n} + \dot{\vec{S}}_1 \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - 2 \dot{\vec{S}}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_1 - \dot{\vec{S}}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
- 3 \dot{\vec{S}}_1 \cdot \vec{S}_2 \cdot \vec{n} (\vec{S}_1 \cdot \vec{v}_1 \vec{n} - \vec{S}_1 \cdot \vec{n} \vec{v}_1) \\
+ 4 \left( \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{a}_2 + \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2 \right) \left( \vec{n}^2 - 3 (\vec{S}_1 \cdot \vec{n})^2 \right) \\
- 3 \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{n} \left( 2 \vec{S}_1^2 \vec{a}_1 \cdot \vec{n} + 4 \dot{\vec{S}}_1 \cdot \vec{v}_1 \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - \dot{\vec{S}}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \right) \\
- \vec{S}_1 \cdot \vec{a}_1 \vec{S}_1 \cdot \vec{n} - 3 \vec{S}_2 \cdot \vec{v}_2 \vec{n} - 4 \dot{\vec{S}}_1 \cdot \vec{v}_1 \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \\
+ 3 \vec{S}_2 \cdot \vec{v}_2 \cdot \vec{n} \left( 8 \dot{\vec{S}}_1 \cdot \vec{v}_2 \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \\
+ 2C_{1(ES^2)} \frac{G}{r^2 m_1} \left[ \left( \vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_2 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \right) \vec{S}_1 \cdot \vec{n} \\
+ \left( \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_2 \right) \vec{S}_1 \cdot \vec{n} - 2 \left( \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} + \vec{S}_2 \cdot \vec{a}_2 \times \vec{n} \right) \vec{S}_1 \cdot \vec{n} \right], \]

(3.5)

Fig. 1(a6) = \(-3C_{1(ES^2)} \frac{G}{r^2 m_1} \left[ 2\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} \vec{v}_1 + 2\vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_2^2 \right. \\
+ \left( \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \vec{v}_1 \cdot \vec{n} - \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \vec{v}_2 \cdot \vec{n} \right) \left( \vec{S}_1^2 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right) \right] \\
+ C_{1(ES^2)} \frac{G}{m_1 r^2} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} \vec{v}_1 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \\
- 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \left( \vec{S}_1 \cdot \vec{v}_1 + \vec{S}_1 \cdot \vec{a}_1 \right) - 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{v}_1 \\
+ 3 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \left( \vec{S}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right) + 3 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \\
- 6 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{v}_2 \vec{S}_1 \cdot \vec{n} + 3 \vec{S}_2 \cdot \vec{v}_2 \times \vec{a}_1 \vec{S}_1 \vec{v}_2 \vec{S}_1 \cdot \vec{n} \\
- 3 \vec{S}_2 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} - 3 \vec{S}_2 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} - 3 \vec{S}_2 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \\
- 2 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{a}_1 \vec{S}_1 \cdot \vec{n} \\
+ 3 \vec{S}_2 \cdot \vec{a}_1 \times \vec{n} \left( \vec{S}_1^2 \vec{v}_2 \times \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \right] \\
- C_{1(ES^2)} \frac{G}{r^2 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \vec{v}_2 + 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \\
- 2 \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \left( \vec{S}_1^2 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \right], \]

(3.6)

Fig. 1(a7) = \(\frac{1}{2}C_{1(ES^3)} \frac{G m_2}{r^2 m_1} \left[ \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left( \vec{S}_1^2 \vec{v}_2 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} - 5 (\vec{S}_1 \cdot \vec{n})^2 \vec{v}_2 \cdot \vec{n} \right) \right. \\
- \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left( \vec{S}_1^2 \left( \vec{v}_1 \cdot \vec{v}_2 - 5 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right) + 2 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{v}_2 \\
- 10 \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 10 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
- 5 \left( \vec{S}_1 \cdot \vec{n} \right)^2 \left( \vec{v}_1 \cdot \vec{v}_2 - 7 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right) \right] \]
\[\begin{align}
+ \frac{1}{6}C_{1(BS)} \frac{G m_2}{r^4 m_1^3} & \left( \hat{S}_1 \cdot \hat{v}_1 \times \hat{a}_2 (\hat{S}_1 \cdot \hat{n})^2 \right) \\
+ 6 \hat{S}_1 \cdot \hat{v}_2 \times \hat{n} (\hat{S}_1 \cdot \hat{S}_2 \hat{v}_1 \cdot \hat{n} + \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} + \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} - 5 \hat{S}_1 \cdot \hat{n} \hat{S}_1 \cdot \hat{n} \hat{v}_2 \cdot \hat{n}) \\
+ 3 \hat{S}_1 \cdot \hat{a}_2 \times \hat{n} (\hat{S}_1^2 \hat{v}_1 \cdot \hat{n} + 2 \hat{S}_1 \cdot \hat{v}_1 \hat{S}_1 \cdot \hat{n} - 5 (\hat{S}_1 \cdot \hat{n})^2 \hat{v}_1 \cdot \hat{n}) \\
+ 3 \hat{S}_1 \cdot \hat{v}_2 \times \hat{n} (\hat{S}_1^2 \hat{v}_2 \cdot \hat{n} + 2 \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} - 5 (\hat{S}_1 \cdot \hat{n})^2 \hat{v}_2 \cdot \hat{n}) \\
- \frac{1}{6}C_{1(BS)} \frac{G m_2}{r^4 m_1^3} & \left[ 2 \hat{S}_1 \cdot \hat{a}_2 \times \hat{n} (\hat{S}_1 \cdot \hat{S}_1 - 3 \hat{S}_1 \cdot \hat{n} \hat{S}_1 \cdot \hat{n}) \\
+ \hat{S}_1 \cdot \hat{a}_2 \times \hat{n} (\hat{S}_1^2 - 3 (\hat{S}_1 \cdot \hat{n})^2) \right], \\
& \text{(3.7)}
\end{align}\]

\[\begin{align}
\text{Fig. 1(a8)} &= \frac{1}{2}C_{1(BS)} \frac{G m_2}{r^4 m_1^3} \left[ 2 \hat{S}_1 \cdot \hat{v}_1 \times \hat{v}_2 \left( \hat{S}_1^2 \hat{v}_1 \cdot \hat{n} + 2 \hat{S}_1 \cdot \hat{v}_1 \hat{S}_1 \cdot \hat{n} - 5 (\hat{S}_1 \cdot \hat{n})^2 \hat{v}_1 \cdot \hat{n} \right) \\
&+ \hat{S}_1 \cdot \hat{v}_1 \times \hat{n} \left( \hat{S}_1^2 \left( \hat{v}_1 \cdot \hat{v}_2 - 5 \hat{v}_1 \cdot \hat{n} \hat{v}_2 \cdot \hat{n} \right) + 2 \hat{S}_1 \cdot \hat{v}_1 \hat{S}_1 \cdot \hat{v}_2 \\
&- 10 \hat{S}_1 \cdot \hat{v}_1 \hat{S}_1 \cdot \hat{n} \hat{v}_2 \cdot \hat{n} - 10 \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} \hat{v}_1 \cdot \hat{n} \\
&- 5 (\hat{S}_1 \cdot \hat{n})^2 (\hat{v}_1 \cdot \hat{v}_2 - 7 \hat{v}_1 \cdot \hat{n} \hat{v}_2 \cdot \hat{n}) \right) \right] \\
- \frac{1}{6}C_{1(BS)} \frac{G m_2}{r^4 m_1^3} & \left[ 2 \hat{S}_1 \cdot \hat{v}_1 \times \hat{v}_2 \left( \hat{S}_1 \cdot \hat{S}_1 - 3 \hat{S}_1 \cdot \hat{n} \hat{S}_1 \cdot \hat{n} \right) \\
&- (\hat{S}_1 \cdot \hat{v}_2 \times \hat{a}_1 - \hat{S}_1 \cdot \hat{v}_1 \times \hat{v}_2) (\hat{S}_1^2 - 3 (\hat{S}_1 \cdot \hat{n})^2) - 6 \hat{S}_1 \cdot \hat{v}_1 \times \hat{n} (\hat{S}_1 \cdot \hat{S}_1 \cdot \hat{v}_2 \cdot \hat{n} \\
&+ \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} + \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} - 5 \hat{S}_1 \cdot \hat{n} \hat{S}_1 \cdot \hat{n} \hat{v}_2 \cdot \hat{n} \\
&- 3 (\hat{S}_1 \cdot \hat{a}_1 \times \hat{n} + \hat{S}_1 \cdot \hat{v}_1 \times \hat{n}) (\hat{S}_1^2 \hat{v}_2 \cdot \hat{n} + 2 \hat{S}_1 \cdot \hat{v}_2 \hat{S}_1 \cdot \hat{n} - 5 (\hat{S}_1 \cdot \hat{n})^2 \hat{v}_2 \cdot \hat{n}) \right], \\
& \text{(3.8)}
\end{align}\]
Note that almost all these graphs contain higher order time derivatives terms, notably
Figure 3. The two-graviton exchange Feynman graphs, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e., with the worldline labels 1 ↔ 2 exchanged. These graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, in particular here at the nonlinear level there are also interactions involving the various multipoles on two different points of the same worldline, which add up to interactions that are cubic in the spin, such as a spin dipole and a spin-induced quadrupole or two spin dipoles on the same worldline as can already be seen as of the NLO spin-squared sector [11, 13]. Consequently notice that there are nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as shown in graphs (b14)-(b16). We also have here two new two-graviton–octupole couplings in graphs (b1), (b2).

second order time derivatives, where graph 1(a10) even contains third order ones.

Further notice that the value of graph 1(a5) is unique in that it also contains time derivatives of the spin, which appeared already in graph 2(a) of the LO in [10], but eventually did not contribute at the LO. At this order, as we will see here in section 5 these terms actually contribute.

3.2 Two-graviton exchange

As can be seen in figure 3 we have 16 graphs of two-graviton exchange in this sector. Here the majority of the graphs do not involve time derivatives. We have here two new two-graviton–octupole couplings in graphs 1(b1), 1(b2), and on the other hand we have here nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as in graphs 1(b14)-1(b16).

The graphs in figure 3 are evaluated as follows:

Fig. 2(b1) = \frac{1}{3} C_{1(BS^3)} \frac{G^2 m_2^2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ 13 S_1^2 - 75 (\vec{S}_1 \cdot \vec{n})^2 \right],

(3.11)

Fig. 2(b2) = \frac{1}{3} C_{1(BS^3)} \frac{G^2 m_2^2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ 4 S_1^2 - 27 (\vec{S}_1 \cdot \vec{n})^2 \right],

(3.12)

Fig. 2(b3) = - C_{1(BS^3)} \frac{G^2 m_2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right],

(3.13)

Fig. 2(b4) = - C_{1(BS^3)} \frac{G^2 m_2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right],

(3.14)
Fig. 2(b5) = 8C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ 3\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_2 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_2 \cdot \mathbf{v}_2 \times \mathbf{n} \left[ 2S_1^2 - 9(S_1 \cdot \mathbf{n})^2 \right] \right], \\
(3.15)

Fig. 2(b6) = C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ -14 \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \left( \mathbf{S}_1 \cdot \mathbf{v}_1 - 3S_1 \cdot \mathbf{n} \mathbf{v}_1 \cdot \mathbf{n} \right) \\
+ 3\mathbf{S}_2 \cdot \mathbf{v}_1 \times \mathbf{n} \left( 3S_1^2 + 10(\mathbf{S}_1 \cdot \mathbf{n})^2 \right) \\
+ 18C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \mathbf{S}_1 \cdot \mathbf{n} \right], \\
(3.16)

Fig. 2(b7) = 2C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ 2\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_2 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \left( \mathbf{S}_1 \cdot \mathbf{v}_2 - 3S_1 \cdot \mathbf{n} \mathbf{v}_2 \cdot \mathbf{n} \right) \\
+ \mathbf{S}_2 \cdot \mathbf{v}_2 \times \mathbf{n} \left( 2S_1^2 - 3(S_1 \cdot \mathbf{n})^2 \right), \\
(3.17)

Fig. 2(b8) = -C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ 2\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_2 \mathbf{S}_1 \cdot \mathbf{n} + 3\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \left( \mathbf{S}_1 \cdot \mathbf{v}_2 - 2S_1 \cdot \mathbf{n} \mathbf{v}_2 \cdot \mathbf{n} \right) \\
+ \mathbf{S}_2 \cdot \mathbf{v}_2 \times \mathbf{n} \left( 5S_1^2 - 12(S_1 \cdot \mathbf{n})^2 \right), \\
(3.18)

Fig. 2(b9) = -C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ \mathbf{S}_1 \cdot \mathbf{v}_1 \times \mathbf{n} \left[ S_1^2 - 3(S_1 \cdot \mathbf{n})^2 \right] \right], \\
(3.19)

Fig. 2(b10) = C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} \left[ 3\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_2 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_2 \cdot \mathbf{v}_2 \times \mathbf{n} \left( 2S_1^2 - 9(S_1 \cdot \mathbf{n})^2 \right) \right], \\
(3.20)

Fig. 2(b11) = -4C_{1(ES^2)} \frac{G^2}{r^5} \left[ 3\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_2 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_2 \cdot \mathbf{v}_2 \times \mathbf{n} \left( 2S_1^2 - 9(S_1 \cdot \mathbf{n})^2 \right) \right], \\
(3.21)

Fig. 2(b12) = -4C_{1(ES^2)} \frac{G^2}{r^5} \left[ \mathbf{S}_2 \cdot \mathbf{v}_1 \times \mathbf{n} \left[ S_1^2 - 3(S_1 \cdot \mathbf{n})^2 \right] \right], \\
(3.22)

Fig. 2(b13) = -12C_{1(ES^2)} \frac{G^2}{r^5} \left[ 2\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_2 \cdot \mathbf{v}_1 \times \mathbf{n} \left( S_1^2 - 5(S_1 \cdot \mathbf{n})^2 \right) \right] \\
+ 12C_{1(ES^2)} \frac{G^2}{r^5} \left[ \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \mathbf{S}_1 \cdot \mathbf{n} + \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \mathbf{S}_1 \cdot \mathbf{n} \right], \\
(3.23)

Fig. 2(b14) = 2\frac{G^2}{r^5} \left[ \mathbf{S}_1 \cdot \mathbf{v}_2 \times \mathbf{n} \left[ 2\mathbf{S}_1 \cdot \mathbf{S}_2 - 3\mathbf{S}_1 \cdot \mathbf{n} \mathbf{S}_2 \cdot \mathbf{n} \right] \right], \\
(3.24)

Fig. 2(b15) = -\frac{G^2}{r^5} \left[ \mathbf{S}_1 \cdot \mathbf{v}_1 \times \mathbf{n} \left[ \mathbf{S}_1 \cdot \mathbf{S}_2 - 3\mathbf{S}_1 \cdot \mathbf{n} \mathbf{S}_2 \cdot \mathbf{n} \right] \right], \\
(3.25)

Fig. 2(b16) = -\frac{G^2}{r^5} \left[ 2\mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{n} \mathbf{S}_1 \cdot \mathbf{v}_1 - \mathbf{S}_1 \cdot \mathbf{v}_1 \times \mathbf{n} \left( 5\mathbf{S}_1 \cdot \mathbf{S}_2 - 9\mathbf{S}_1 \cdot \mathbf{n} \mathbf{S}_2 \cdot \mathbf{n} \right) \\
+ 3\mathbf{S}_2 \cdot \mathbf{v}_1 \times \mathbf{n} \left( S_1^2 - (S_1 \cdot \mathbf{n})^2 \right) \right]. \\
(3.26)

3.3 Cubic self-interaction

As can be seen in figure 4 we have 24 graphs of cubic self-interaction in this sector, 6 of which contain time-dependent self-interaction, similar to what we have in the odd parity spin-orbit sector [7, 11, 12]. Similar to the nonlinear graphs of two-graviton exchange, these graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, and we have here nonlinearities originating from gravitons sourced
Figure 4. The Feynman graphs at one-loop level, i.e. with cubic self-gravitational interaction, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels 1 ↔ 2 exchanged. Similar to the nonlinear graphs of two-graviton exchange, these graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, and we have here nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as shown in graphs (c4)-(c8). We also have here cubic vertices containing time derivatives, similar to what we have in the NLO odd parity spin-orbit sector [7, 11, 12].

strictly from minimal coupling to the worldline as shown in graphs (c4)-(c8). This sector required using tensor one-loop integrals of up to order 5.

The graphs in figure 4 are evaluated as follows:

Fig. 3(a1) = \(-\frac{16}{3}C_{1(BS^3)}\frac{G^2 m_2^2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 - 6(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.27)

Fig. 3(a2) = \(-\frac{3}{2}C_{1(BS^3)}\frac{G^2 m_2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.28)

Fig. 3(a3) = \(\frac{3}{2}C_{1(BS^3)}\frac{G^2 m_2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.29)

Fig. 3(a4) = \(-\frac{1}{3}C_{1(BS^3)}\frac{G^2 m_2^2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 6(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.30)

Fig. 3(a5) = \(-\frac{1}{8}C_{1(BS^3)}\frac{G^2 m_2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.31)

Fig. 3(a6) = \(-\frac{1}{3}C_{1(BS^3)}\frac{G^2 m_2^2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 - 6(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.32)

Fig. 3(a7) = \(-\frac{1}{8}C_{1(BS^3)}\frac{G^2 m_2}{r^5 m_1^3} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.33)

Fig. 3(b1) = \(\frac{1}{2}C_{1(ES^3)}\frac{G^2 m_2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[ S_1^2 + 3(\vec{S}_1 \cdot \vec{n})^2 \right] \), (3.34)
Fig. 3(b2) = \(-8C_{1(ES^2)}\frac{G^2 m_2}{r^3 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - 3(S_1 \cdot \vec{n})^2) \right], \quad (3.35)

Fig. 3(b3) = 4C_{1(ES^2)}\frac{G^2 m_2}{r^3 m_1} \left[ 4 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 9(S_1 \cdot \vec{n})^2) \right], \quad (3.36)

Fig. 3(b4) = \frac{1}{2} C_{1(ES^2)}\frac{G^2 m_2}{r^4 m_1} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 3(S_1 \cdot \vec{n})^2 \right] - 2C_{1(ES^2)}\frac{G^2 m_2}{r^4 m_1} \vec{S}_1 \cdot \vec{S}_1 \times \vec{n}, \quad (3.37)

Fig. 3(b5) = 8C_{1(ES^2)}\frac{G^2 m_2}{r^4 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 3(S_1 \cdot \vec{n})^2) \right] - 4C_{1(ES^2)}\frac{G^2 m_2}{r^4 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \right], \quad (3.38)

Fig. 3(b6) = 4C_{1(ES^2)}\frac{G^2}{r^4} \left[ 2\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} - \vec{S}_2 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_1 - 6\vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}) \right] + \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} (2S_1^2 - 9(S_1 \cdot \vec{n})^2) \right] - 12C_{1(ES^2)}\frac{G^2}{r^4} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \right], \quad (3.39)

Fig. 3(b7) = -\frac{3}{8} C_{1(ES^2)}\frac{G^2 m_2}{r^5 m_1} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[ S_1^2 - 5(S_1 \cdot \vec{n})^2 \right], \quad (3.40)

Fig. 3(b8) = 2C_{1(ES^2)}\frac{G^2 m_2}{r^5 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - 3(S_1 \cdot \vec{n})^2) \right], \quad (3.41)

Fig. 3(b9) = \frac{1}{4} C_{1(ES^2)}\frac{G^2}{r^5} \left[ 4\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \right] + 3\vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - 5(S_1 \cdot \vec{n})^2), \quad (3.42)

Fig. 3(c1) = \frac{3}{8} C_{1(ES^2)}\frac{G^2 m_2}{r^5 m_1} \left[ \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 5(S_1 \cdot \vec{n})^2) \right] - C_{1(ES^2)}\frac{G^2 m_2}{r^5 m_1} \left[ \vec{S}_1 \cdot \vec{S}_1 \times \vec{n} \vec{S}_1 \cdot \vec{n} \right], \quad (3.43)

Fig. 3(c2) = -2C_{1(ES^2)}\frac{G^2 m_2}{r^5 m_1} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - 3(S_1 \cdot \vec{n})^2) \right], \quad (3.44)

Fig. 3(c3) = -\frac{1}{4} C_{1(ES^2)}\frac{G^2}{r^5} \left[ 4\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_1 - 3\vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}) \right] + 3 \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 5(S_1 \cdot \vec{n})^2) \right] + C_{1(ES^2)}\frac{G^2}{r^5} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \right], \quad (3.45)

Fig. 3(c4) = 4\frac{G^2}{r^5} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} - 3\vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{n})^2 \right], \quad (3.46)

Fig. 3(c5) = 4\frac{G^2}{r^5} \left[ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{S}_2 \right], \quad (3.47)
far, only the first correction, that is we now consider also beyond the leading term, which was the only one required for lower order spin sectors thus to cubic order in the spin in this sector, we should take into account in the linear momentum in this work.

\[ u \] rather than the four-velocity where we have denoted the leading approximation to the linear momentum as \( \bar{u} = \nu \),

\[ \text{present the spin supplementary condition (SSC) given by} \]

\[ S \]

\[ \text{tion initially taken in the covariant gauge as formulated by Tulczyjew in} \]

\[ \text{tetrad. This gauge is distinguished among possible covariant gauges as the only gauge of} \]

\[ p \]

\[ \text{Therefore the difference between using} \]

\[ \| \text{as of the NLO of the sector cubic in the spins, namely the sector that we are studying} \]

\[ \text{in this work.} \]

\[ \text{Let us then find how these new feature transpires in this sector. Since we are working to cubic order in the spin in this sector, we should take into account in the linear momentum beyond the leading term, which was the only one required for lower order spin sectors thus far, only the first correction, that is we now consider also} \]

\[ \Delta p_\kappa \equiv p_\kappa - \bar{p}_\kappa \approx \frac{C_{ES}^2}{2m} S^\mu S^\nu \left( \frac{2}{u} R_{\mu \nu \kappa \lambda} u^\alpha - \frac{1}{u^3} R_{\mu \nu \beta} u^\alpha u^\beta u_\kappa \right), \]

where we have denoted the leading approximation to the linear momentum as \( \bar{p}_\kappa \equiv \frac{\bar{u}}{u} u_\kappa \). Let us also note that due to eq. (4.8) of [11] at this order the spin vectors and the spin vectors can be used interchangeably.
Hence, the part that is linear in the spin in the action of the spinning particle actually gives rise to a new type of worldline-graviton couplings that are cubic in the spin, due to its dependence in the linear momentum. We recall that the relevant part of the Lagrangian is given as follows \[ L_S = \frac{1}{2} \hat{\Sigma}_{ab} \hat{\Sigma}_{ab}^{\mu} + \frac{1}{2} \hat{\Sigma}_{ab} \omega_{\mu}^{ab} u^{\mu} + \frac{\hat{\Sigma}_{ab} b^b D p^a}{p^2} \frac{D \lambda}{D \lambda}, \] (4.3)

where the hatted DOFs represent the generic rotational DOFs. Therefore the new contributions arise from substituting in the gauge, which we choose here as the canonical gauge formulated in [11] as
\[ \hat{\Lambda}^a_{[0]} = \delta^a_0, \quad \hat{\Sigma}_{ab} (p_b + p_0) = 0, \] (4.4)

into the linear-in-spin couplings, as well as from the extra term that enters from minimal coupling, appearing last in eq. (4.3), which was found in [11] to be related with the gauge of the rotational DOFs, and stands for the Thomas precession as was noted in section 2.

Let us stress again that the subtlety here is not about switching from the covariant gauge, but rather about advancing from using in the gauge \( u^\nu \) to \( p^\nu \) as the basic covariant gauge, which is necessary as of this nonlinear order in gravity and cubic order in spins.

Working out explicitly this part of the action in terms of the local spin variable in the canonical gauge similarly to the derivations in [11], and keeping only terms that lead to new cubic-in-spin couplings, we obtain here the following contribution:
\[ L_{S \to S^3} = \omega_{\mu}^{ij} u^\mu \frac{\hat{\Sigma}_{ij} \dot{p}_i}{p} - \omega_{ij}^{\mu} u^{\mu} \frac{\hat{\Sigma}_{ij}^k p^k}{p (p + p^0)} - \hat{\Sigma}_{ij}^k \dot{p}_i^j \] (4.5)

where in principle all the indices here are in the locally flat frame. In order to obtain the new cubic-in-spin couplings we only need to substitute in the correction to the linear momentum from eq. (4.2) to linear order, keeping in mind that all the contributions at the zeroth order are taken into account in section 3 above, and section 5 below.

At this point it becomes clear that the first two terms in eq. (4.5) give rise to new two-graviton couplings, and that the last term gives rise to new one-graviton couplings containing higher order time derivatives. The resulting new Feynman rules for the one-graviton couplings are then given as follows:

\[ \int dt \left[ \frac{C_{ES}^2}{4m^2} S^i S^j \epsilon_{klm} \left[ 2 S^m a^k \left( A_{i,jl} - A_{j,il} + \delta_{ij} (A_{n,ln} - A_{l,n}) \right) \right] 
+ \hat{S}_m v^k \left( A_{i,jl} - A_{j,il} + \delta_{ij} (A_{n,ln} - A_{l,n}) \right) \right] \right], \] (4.6)

\[ \int dt \left[ \frac{C_{ES}^2}{2m^2} S^i S^j \epsilon_{klm} \left[ 2 S^m a^k \left( \phi_{i,l j} v^j - \phi_{j,l i} v^j + \delta_{ij} \left( \partial_{l} \phi_{,l} + \phi_{,m n} v^l \right) \right) \right] 
+ \hat{S}_m v^k \left( \phi_{i,l j} v^j + \delta_{ij} \partial_{l} \phi_{,l} + \delta_{jl} \left( \partial_{l} \phi_{,i} + \phi_{,i,n} v^l \right) \right) \right], \] (4.7)
Figure 5. The extra one- and two-graviton exchange Feynman graphs, which appear at the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. These graphs contain a new type of worldline-graviton couplings, which we dub as the “composite” octupole ones, and obviously yield similar graphs to the corresponding graphs with the “elementary” spin-induced octupole in figure 2 (a1), (a2) and in figure 3 (b1), (b2).

where a black square mounted on an oval blob represents this new type of “composite” cubic-in-spin worldline couplings. Notice that all these rules contain accelerations and even time derivatives of spins similar to the acceleration terms that appear first in the rules for the spin-orbit sector [11].

For the new two-graviton couplings we get the following rules:

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig5a1} \\
\includegraphics[width=0.2\textwidth]{fig5a2}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{Fig. 4(a1)} &= -C_{1(ES^2)} \frac{G m_2}{r^3 m_1^2} \left[ 2 \vec{S}_1 \cdot \vec{v}_2 \times \vec{a}_1 \left( 2 \vec{S}_1^2 - 3 \left( \vec{S}_1 \cdot \vec{n} \right)^2 \right) \\
&\quad + 6 \vec{S}_1 \cdot \vec{a}_1 \times \vec{n} \left( \vec{S}_1^2 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) - \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{v}_2 \\
&\quad - \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left( 2 \vec{S}_1^2 - 3 \left( \vec{S}_1 \cdot \vec{n} \right)^2 \right) + 3 \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left( \vec{S}_1^2 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \right], \\
\text{Fig. 4(a2)} &= \frac{1}{2} C_{1(ES^2)} \frac{G m_2}{r^3 m_1^2} \left[ 2 \vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 \left( \vec{S}_1^2 - 3 \left( \vec{S}_1 \cdot \vec{n} \right)^2 \right) + 2 \vec{S}_1 \cdot \vec{v}_2 \times \vec{a}_1 \vec{S}_1^2 \right].
\end{align*}
\]
and 43 to get the final effective action where we have:

$$L_{NLO} = L_{NLO}^{S_i^2 S_2} + L_{NLO}^{S_1^2} + (1 \leftrightarrow 2),$$

(5.1)

where we have:

$$L_{NLO}^{S_i^2 S_2} = \frac{1}{4} \frac{G^2}{r^5} L_{(1)} + \frac{3}{2} \frac{C_1(ES^2)}{r^4 m_1} L_{(2)} + \frac{1}{4} \frac{C_1(ES^2)}{r^5} L_{(3)} + \frac{G^2 m_2}{r^5 m_1} L_{(4)}$$

$$+ \frac{G^2}{r^4} L_{(5)} + \frac{C_1(ES^2)}{r^4 m_1} L_{(6)} + \frac{G^2}{r^4} L_{(7)} + 14 C_1(ES^2) \frac{G^2 m_2}{r^4 m_1} L_{(8)}$$

$$+ \frac{1}{2} C_1(ES^2) \frac{G}{r^2 m_1} L_{(9)} - \frac{1}{2} C_1(ES^2) \frac{G}{r m_1} L_{(10)},$$

(5.2)

with the following pieces:

$$L_{(1)} = - \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \left( 17 \vec{S}_1 \cdot \vec{v}_1 - 55 \vec{S}_1 \cdot \vec{v}_2 - 9 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} - 15 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right)$$

$$- \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left( 19 \vec{S}_1 \cdot \vec{S}_2 + 63 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right)$$

$$+ 81 \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left( \vec{S}_1 \cdot \vec{S}_2 - 7 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right) + 20 \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} S_1^2 - 64 \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} S_2^2,$$

(5.3)

$$L_{(2)} = - 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \left( \vec{S}_1 \cdot \vec{v}_1 \left( \vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \right) - \vec{S}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right)$$

$$- \vec{S}_1 \cdot \vec{n} \left( \vec{v}_1^2 - \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 5 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right)$$

(5.4)
\[
\begin{align*}
+ 2\vec{S}_1 \cdot \vec{v}_2 \left(\vec{S}_1 \cdot \vec{v}_1 \left(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\right) - \vec{S}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n}\right) \\
- \vec{S}_1 \cdot \vec{n} \left(\vec{v}_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2^2 - 5 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) - 2 \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 S_1^2 \vec{v}_2 \cdot \vec{n} \\
+ \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \left(3\vec{v}_1^2 - 10(\vec{v}_1 \cdot \vec{n})^2 + 5 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
- 2\vec{S}_1 \cdot \vec{v}_1 \left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - 5 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 5 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) - 10\vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
- 5(\vec{S}_1 \cdot \vec{n})^2 (\vec{v}_1^2 + 7 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
+ \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \left(S_1^2 (4\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2^2 + 10(\vec{v}_1 \cdot \vec{n})^2 - 5 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
+ 2\vec{S}_1 \cdot \vec{v}_1 \left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - 5\vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} + 5\vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) + 10\vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
+ 5(\vec{S}_1 \cdot \vec{n})^2 (\vec{v}_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 - 7\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
+ \vec{v}_1 \cdot \vec{v}_2 \times \vec{n} \vec{S}_2 \cdot \vec{v}_1 \left(3 S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right),
\end{align*}
\]

\[L_{(3)} = -4 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 44 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} \\
+ 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} + 3 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \\
- 3 \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2) - \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (29 S_1^2 - 129(\vec{S}_1 \cdot \vec{n})^2),
\]

\[L_{(4)} = + 18 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} (\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - 9 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}) \\
+ 6\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} + 17 \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} S_1^2 + \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} (7 S_1^2 - 42(\vec{S}_1 \cdot \vec{n})^2),
\]

\[L_{(5)} = + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n},
\]

\[L_{(6)} = + \frac{1}{2} \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \left(\vec{S}_1 \cdot \vec{v}_2 + 3 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
+ \frac{1}{2} \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \left(\vec{S}_1 \cdot \vec{v}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{v}_2 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
+ 3 \vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_1 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + \frac{1}{2} \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \left(\vec{S}_1 \cdot \vec{v}_2 + 3 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
+ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \left(\vec{S}_1 \cdot \vec{v}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}\right) \\
+ \frac{1}{2} \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \left(\vec{S}_1 \cdot \vec{v}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{v}_2 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}\right) \\
- \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 (\vec{S}_1 \cdot \vec{S}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n}) \\
- \frac{3}{2} \vec{S}_2 \cdot \vec{a}_1 \times \vec{v}_2 (3 S_1^2 - (\vec{S}_1 \cdot \vec{n})^2) - \vec{S}_2 \cdot \vec{a}_2 \times \vec{v}_2 S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2) \\
- \frac{1}{2} (\vec{S}_2 \cdot \vec{v}_1 \times \vec{a}_2 + \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2)(5 S_1^2 - 9(\vec{S}_1 \cdot \vec{n})^2) \\
- (\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_2)(\vec{S}_1 \cdot \vec{v}_1 - 3 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}) \\
+ \frac{3}{2} \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 + 2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{n}
\]

\(–20–\)
\[
\begin{align*}
&+ S_1 \cdot a_1 \cdot \vec{v}_1 \cdot \vec{n} - 2 S_1 \cdot \vec{n} \cdot \vec{v}_1 - \vec{a}_1 - \dot{S}_1 \cdot \vec{n} \cdot v_1^2 - \dot{S}_1 \cdot \vec{v}_1 \cdot \vec{v}_2 \cdot \vec{n} - \dot{\vec{S}}_1 \cdot \vec{v}_2 \cdot \vec{n} \\
&- \vec{S}_1 \cdot \vec{n} \cdot \vec{v}_1 \cdot \vec{v}_2 + 5 \dot{S}_1 \cdot \vec{n} \cdot \vec{v}_1 \cdot \vec{n} \cdot \vec{v}_2 \cdot \vec{n} - 2 \dot{\vec{S}}_1 \cdot \vec{n} \] \\
&+ \frac{3}{2} \dot{S}_1 \cdot \vec{S}_2 \times \vec{n} \left( \vec{S}_1 \cdot \vec{n} \cdot \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{n} \cdot v_1^2 - \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \right) \\
&+ \frac{3}{2} \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \left( 2 S_1^2 \cdot a_1 \cdot \vec{n} + 4 \vec{S}_1 \cdot \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{S}_1 \cdot \vec{n} \right) \\
&- \vec{S}_1 \cdot \vec{a}_1 \cdot \vec{S}_1 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{S}_1 \cdot \vec{n} - 10 \vec{S}_1 \cdot \vec{S}_1 \cdot \vec{n} - \vec{S}_2 \cdot \vec{n} \\
&- \frac{3}{2} \vec{S}_2 \cdot \vec{S}_2 \times \vec{n} \left( 8 S_1^2 \cdot a_1 \cdot \vec{n} + 2 \vec{v}_2 \cdot \vec{n} \left( 3 \dot{S}_1 \cdot \vec{S}_1 - 5 \vec{S}_1 \cdot \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) - \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right) \\
&- \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \\
&- \frac{3}{2} \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \left( 3 S_1^2 \cdot v_1 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{S}_1 \cdot \vec{n} - 5 \vec{S}_1 \cdot \vec{S}_1 \cdot \vec{n} \right) \\
&- \frac{3}{2} \vec{S}_2 \cdot \vec{a}_2 \times \vec{n} \left( 2 S_1^2 \cdot a_1 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{v}_1 \cdot \vec{S}_1 \cdot \vec{n} + 5 \vec{S}_1 \cdot \vec{n} \right), \quad (5.8)
\end{align*}
\]

\[
L_{(7)} = + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \\
L_{(8)} = + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n}, \quad (5.9)
\]

\[
L_{(9)} = + 4 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + 4 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \\
+ 2 \vec{S}_2 \cdot \vec{n} \cdot \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \cdot \vec{v}_1 \\
+ 3 \vec{S}_2 \cdot \vec{n} \left( \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \right) \\
+ \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} - \vec{S}_2 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \\
- \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} - \vec{S}_2 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \\
- 3 \vec{v}_2 \cdot \vec{n} \left( \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} \right) \\
- 2 \vec{S}_2 \cdot \vec{n} \cdot \vec{S}_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{v}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} - 6 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \\
+ 2 \vec{S}_1 \cdot \vec{n} \left( \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} + \vec{S}_1 \cdot \vec{n} \right) + 2 \vec{S}_1 \cdot \vec{n} \left( \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \right) \\
- 6 \left( \vec{S}_2 \cdot \vec{S}_2 \times \vec{n} + \vec{S}_2 \cdot \vec{n} \right) \left( \vec{S}_1 \cdot \vec{S}_1 + \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \right), \quad (5.11)
\]

\[
L_{(10)} = + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \cdot \vec{S}_1 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \vec{S}_1 \cdot \vec{n}, \quad (5.12)
\]

and also:

\[
F_{\text{NLO}}^{S_1} = - \frac{1}{2} C_1(ES^2) \frac{G^2 m_2}{r^4 m_2} L_{[1]} + C_1(ES^2) \frac{G^2 m_2}{r^4 m_2} L_{[2]} + \frac{1}{2} C_1(BS^2) \frac{G m_2}{r^4 m_2} L_{[3]}
\]

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\begin{align}
&+ \frac{1}{4} C_{1(BS^3)} \frac{G^2 m_2}{r^3 m_1} L_{[4]} + \frac{1}{3} C_{1(BS^3)} \frac{G^2 m^2_2}{r^3 m_1^2} L_{[5]} + \frac{1}{2} C_{1(ES^2)} \frac{G m_2}{r^3 m_1^2} L_{[6]} \\
&- 3C_{1(ES^2)} \frac{G^2 m_2}{r^4 m_1} L_{[7]} + \frac{1}{6} C_{1(BS^3)} \frac{G m_2}{r^3 m_1^2} L_{[8]} - \frac{1}{6} C_{1(ES^3)} \frac{G m_2}{r^2 m_1^2} L_{[9]},
\end{align}

with the pieces:

\begin{align}
L_{[1]} &= + \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 3(S_1 \cdot \vec{n})^2) - 3 \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - (\vec{S}_1 \cdot \vec{n})^2), \\
L_{[2]} &= + \bar{S}_1 \cdot \vec{v}_2 \times \vec{n} \left( 5S_1^2 - 6(S_1 \cdot \vec{n})^2 \right),
\end{align}

\begin{align}
L_{[3]} &= - \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 \vec{v}_1 \cdot \vec{n} S_1 \cdot \vec{n} - 2 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n}) \\
&+ \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 (v_1^2 - 2 \vec{v}_1 \cdot \vec{v}_2 + 2 v_2^2 - 5 \vec{v}_1 \cdot \vec{n} v_2 - \vec{n}) \\
&- \bar{S}_1 \cdot \vec{v}_1 (S_1 \cdot \vec{v}_1 - 2\vec{S}_1 \cdot \vec{v}_2 - 5 \vec{S}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 - \vec{n}) ) - 10 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
&- 5(S_1 \cdot \vec{n})^2 (v_1^2 - 2 \vec{v}_1 \cdot \vec{v}_2 + 2 v_2^2 - 7 \vec{v}_1 \cdot \vec{n} \vec{v}_2 - \vec{n}) \\
&- \bar{S}_1 \cdot \vec{v}_2 \times \vec{n} (S_1^2 (v_2^2 - 5 \vec{v}_2 \cdot \vec{v}_1 - \vec{n}) \\
&- \bar{S}_1 \cdot \vec{v}_2 (S_1 \cdot \vec{v}_1 - 2\vec{S}_1 \cdot \vec{v}_2 - 5 \vec{S}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 - \vec{n}) ) - 10 \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n} \\
&- 5(S_1 \cdot \vec{n})^2 (v_2^2 - 7 \vec{v}_1 \cdot \vec{n} \vec{v}_2 - \vec{n}) ) + \vec{v}_1 \cdot \vec{v}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 (S_1^2 - 5(S_1 \cdot \vec{n})^2),
\end{align}

\begin{align}
L_{[4]} &= + \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 5(S_1 \cdot \vec{n})^2) - 10 \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} (S_1^2 - 5(S_1 \cdot \vec{n})^2),
\end{align}

\begin{align}
L_{[5]} &= + 3 \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} (S_1^2 - 7(S_1 \cdot \vec{n})^2) - \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} (4 S_1^2 - 27(S_1 \cdot \vec{n})^2),
\end{align}

\begin{align}
L_{[6]} &= + 2 \bar{S}_1 \cdot \vec{v}_1 \times a_1 \left( S_1^2 - 3 \left( S_1 \cdot \vec{n} \right)^2 \right) - 6 \vec{S}_1 \cdot \vec{v}_2 \times a_1 \left( S_1^2 - 2 \left( S_1 \cdot \vec{n} \right)^2 \right) \\
&- 6 \bar{S}_1 \cdot a_1 \times \vec{n} \left( S_1^2 \vec{v}_1 \vec{n} + \bar{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} \right) \\
&- \bar{S}_1 \cdot a_1 \times \vec{n} \left( 2\vec{S}_1 \cdot \vec{v}_1 - 3\vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n} (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) \right) \\
&+ 3 \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} \left( S_1^2 - 2 \left( S_1 \cdot \vec{n} \right)^2 \right) \\
&- 3 \bar{S}_1 \cdot \vec{v}_1 \times \vec{n} \left( S_1^2 \vec{v}_1 \vec{n} - \vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \right),
\end{align}

\begin{align}
L_{[7]} &= + \hat{\vec{S}}_1 \cdot \vec{S}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n},
\end{align}

\begin{align}
L_{[8]} &= + 2 \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 (\hat{\vec{S}}_1 \cdot \vec{S}_1 - 3 \hat{\vec{S}}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n}) + \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 (S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2) \\
&+ \vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_2 (S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2) + \vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_2 (S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2)
\end{align}
where $S$ redefinitions that we will have. From position shifts in lower order sectors we will have: consider the redefinition of the spins to linear order. contributing only at the next-to-NNLO (NNNLO) level. Therefore, here it is sufficient to leading spin redefinition is of higher PN order, terms quadratic in the leading redefinition are linear in the spin but have no field coupling. Those are required here to NLO as follows: example we recall that we have kinematic contributions as noted in eq. (5.28) of \[9\], that are linear in the spin but have no field coupling. Those are required here to NLO as follows:

$$L_{\text{kin}} = -\vec{S} \cdot \vec{\Omega} - \frac{1}{2} \left( 1 + \frac{3}{4} v^2 \right) \epsilon_{ijk} S_k v^j a^i,$$

(5.23)

where $S_{ij} = \epsilon_{ijk} S_k$, and $\Omega_{ij} = \epsilon_{ijk} \Omega_k$. At LO, e.g., we define the following shift of the positions, $\Delta \vec{y}_I$, according to

$$\vec{y}_1 \rightarrow \vec{y}_1 + \frac{1}{2m_1} \vec{S}_1 \times \vec{v}_1,$$

(5.24)

and similarly for particle 2 with $1 \leftrightarrow 2$ to remove the leading accelerations.

Note that as of the NLO linear-in-spin level higher order time derivatives of spin also appear, where it was shown how to generically treat these in section 5 of \[43\]. Yet, since the leading spin redefinition is of higher PN order, terms quadratic in the leading redefinition contribute only at the next-to-NNLO (NNNLO) level. Therefore, here it is sufficient to consider the redefinition of the spins to linear order.

To recap, let us list the additional contributions coming from lower order variable redefinitions that we will have. From position shifts in lower order sectors we will have:

1. The LO (1.5PN) position shift in eq. (5.24) implemented to linear order on the NLO quadratic-in-spin (spin1-spin2 + spin-squared) sectors.

2. The above LO position shift implemented to quadratic order on the Newtonian and LO spin-orbit sectors.
3. The above LO position shift to cubic order implemented on the Newtonian sector.

4. The NLO position shift at 2.5PN order in eq. (6.20) of [11] implemented to linear order on the LO quadratic-in-spin sectors.

5. The NLO position shifts at 3PN order in eqs. (6.30), (6.43) of [11] implemented to linear order on the \textit{shifted} LO spin-orbit sector.

The leading redefinition of spin (of 2PN order) in eq. (6.21) of [11] will not contribute to our sector. From spin redefinitions, i.e. rotations of the spin, we will have then:

1. The spin redefinitions at 2.5PN order in eqs. (6.31), (6.44) of [11] implemented to linear order on the LO quadratic-in-spin sectors.

2. The spin redefinitions at 3PN order, which were required at the LO cubic-in-spin sector [10], implemented to linear order on the LO spin-orbit sector.

In a forthcoming publication we will present the full details of these redefinitions and the contributions which add up to get the reduced effective action.

6 Conclusions

In this work we derived for the first time the complete NLO cubic-in-spin PN effective action for the interaction of generic compact binaries via the EFT formulation for gravitating spinning objects in [11] and its extension to the leading sector, where gravitational nonlinearities are considered at an order in the spins that is beyond quadratic. This sector, which enters at the 4.5PN order for rapidly rotating compact objects, completes finite size effects up to this order, and is the first sector completed beyond the current state of the art for generic compact binary dynamics at the 4PN order. Once again we see that the EFT of gravitating spinning objects has enabled pushing the state of the art in PN Gravity. Yet the analysis in this work indicates that going beyond this sector into the intriguing gray area of table 1 may become impossibly intricate.

We have seen that at this order in spins with nonlinearities in gravity we have to take into account additional terms, which arise from a new type of worldline couplings, due to the fact that at this order the Tulczyjew gauge, which involves the linear momentum, can no longer be approximated only in terms of the four-velocity, as the latter approximation differs from the linear momentum by an order $O(RS^2)$. The correction gives rise to new “composite” couplings from the gauge of rotational DOFs. It is interesting to consider whether these new couplings have an insightful physical interpretation.

As we noted in section 1 one of the main motivations for us to tackle this sector was also to see what happens when we go to a sector at order higher than quadratic in the spins and nonlinear in gravity, which corresponds to a gravitational Compton scattering with quantum spins of $s \geq 3/2$, and to possibly also get an insight on the non-uniqueness of fixing its amplitude from factorization when spins of $s \geq 5/2$ are involved [32]. From [11] and the analysis in section 4 we can see that going to an order quintic in spins, or
in the quantum case to $s = 5/2$, exactly corresponds to where the correction to $p_\mu$ in eq. (4.2) has to be taken into account at quadratic order. We will discuss this interesting connection between the classical and the quantum levels at a forthcoming publication. A general observation that we can clearly make already is that even-parity sectors in $l$, see table 1, are easier to handle than odd ones. In the quantum context this corresponds to the greater ease of dealing with bosons compared to fermions.

Unless all the additional terms from section 4 conspire to cancel out eventually, we obtain an effective action that differs from that with the gauge used in lower spin sectors, involving only the four-velocity. Yet, it could be that when computing the consequent observable quantities, such as the binding energy, or the EOMs, one finds that this difference does not show up, and the two gauges are physically equivalent. In a forthcoming publication we will present the resulting Hamiltonian, EOMs, and finally gauge invariant quantities, such as the binding energy, and get an answer for this question.

At the moment it is not clear whether computations carried out within an amplitudes framework can capture all the classical effects derived in this paper. The generic results in this work can serve to streamline such a framework, as that which was initiated in [35, 36], or provide a crosscheck for the conjectured result for the scattering angle at one-loop level in the restricted case of black holes with aligned spins in [39].

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