Representations for weighted Moore-Penrose inverses of partitioned adjointable operators

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Abstract

For two positive definite adjointable operators $M$ and $N$, and an adjointable operator $A$ acting on a Hilbert $C^*$-module, some properties of the weighted Moore-Penrose inverse $A_{MN}^\dagger$ are established. If $A = (A_{ij})$ is $1 \times 2$ or $2 \times 2$ partitioned, then general representations for $A_{MN}^\dagger$ in terms of the individual blocks of $A_{ij}$ are studied. In the case when $A$ is $1 \times 2$ partitioned, a unified representation for $A_{MN}^\dagger$ is presented. In the $2 \times 2$ partitioned case, an approach to the construction of the Moore-Penrose inverse from the non-weighted case to the weighted case is provided. Some results known for matrices are extended to the general setting of operators on Hilbert $C^*$-modules.

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Introduction

The weighted Moore-Penrose inverse of an arbitrary (singular and rectangular) matrix has many applications in the weighted linear least-squares problems, statistics, neural network, numerical analysis and so on. For a partitioned matrix $A = (A_{ij})$, it has been of interest to derive general expressions for the weighted Moore-Penrose inverse of $A$ in terms of the individual blocks of $A_{ij}$. If $A = (A_{11}, A_{12})$ is a $1 \times 2$ partitioned matrix, then some formulas for the (non-weighted) Moore-Penrose inverse $A^\dagger$, such as Cline [2] and Mihalyffy [8] are well-known. In the weighted case, a formula for $A_{MN}^\dagger$ of a $1 \times 2$ partitioned matrix $A$ was given by Miao [6]. Later, this formula was reproved by Chen [1], Wang and Zheng [10] by using different methods. Recently, another formula for $A_{MN}^\dagger$ has been obtained by the first author [11]. In this paper, in the general context of Hilbert $C^*$-module operators, we will provide a unified representation for $A_{MN}^\dagger$ (see Theorem 3.4 below). As a result, the equivalence of the formulas for $A_{MN}^\dagger$ given respectively in [6] and [11] is derived.

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If \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) is a \( 2 \times 2 \) partitioned matrix, then things may become much more complicated. Most works in literature concerning representations for \( A^\dagger \) were carried out under certain restrictions on the blocks of \( A_{ij} \). In 1991, a general expression for \( A^\dagger \) without any restriction imposed on the blocks of \( A_{ij} \), was given by Miao in [7]. Since then, more than twenty years has passed. However, due to the complexity revealed in [3, 5, 7], there has not been much progress concerning the generalization of Miao’s result [7] from the non-weighted case to the weighted case. In this paper, we make such an effort in the general setting of Hilbert \( C^* \)-module operators.

The paper is organized as follows. In Section 1 in the general setting of Hilbert \( C^* \)-module operators, we will establish some properties on weighted Moore-Penrose inverses. Following the line initiated in [12], in Section 2 we will study the relationship between weighted Moore-Penrose inverses \( A_{MN}^\dagger \), where \( A \) is fixed, while \( M \) and \( N \) are variable. In Section 3 we will study unified representations for weighted Moore-Penrose inverses of \( 1 \times 2 \) partitioned adjointable operators. In Section 4, an approach, initiated in [11] for \( 1 \times 2 \) partitioned adjointable operators, is applied to study the general expressions for weighted Moore-Penrose inverses of \( 2 \times 2 \) partitioned adjointable operators. Our key point is the construction of a commutative diagram in page 16 through which the main results of [7] are generalized from the non-weighted case to the weighted case.

1 Weighted Moore-Penrose inverses of adjointable operators

In this section, in a general setting of adjointable operators on Hilbert \( C^* \)-modules, we establish some properties on weighted Moore-Penrose inverses, most of which are known for matrices. Throughout this paper, \( \mathfrak{A} \) is a \( C^* \)-algebra, \( \mathbb{C} \) is the complex field, and \( \mathbb{C}^{m \times n} \) is the set of \( m \times n \) complex matrices. By a projection, we mean an idempotent and a self-adjoint element in a certain \( C^* \)-algebra. For any Hilbert \( \mathfrak{A} \)-modules \( H \) and \( K \), let \( \mathcal{L}(H, K) \) be the set of adjointable operators from \( H \) to \( K \). If \( H = K \), then \( \mathcal{L}(H, H) \), which we abbreviate to \( \mathcal{L}(H) \), is a unital \( C^* \)-algebra, whose unit is denoted by \( I_H \). For any \( A \in \mathcal{L}(H, K) \), the range and the null space of \( A \) are denoted by \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \), respectively.

Throughout, the notations of “\( \oplus \)” and “\( \dagger \)” are used with different meanings. For any Hilbert \( \mathfrak{A} \)-modules \( H_1 \) and \( H_2 \), let

\[
H_1 \oplus H_2 = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mid h_i \in H_i, i = 1, 2 \right\},
\]

which is also a Hilbert \( \mathfrak{A} \)-module whose \( \mathfrak{A} \)-valued inner product is given by

\[
\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \text{ for any } x_i \in H_1 \text{ and } y_i \in H_2, i = 1, 2.
\]

If both \( H_1 \) and \( H_2 \) are submodules of a Hilbert \( \mathfrak{A} \)-module \( H \) such that \( H_1 \cap H_2 = \{0\} \), then we define

\[
H_1 + H_2 = \{ h_1 + h_2 \mid h_i \in H_i, i = 1, 2 \} \subseteq H.
\]
If furthermore $H = H_1 + H_2$, then we call $P_{H_1,H_2}$ the oblique projector along $H_2$ onto $H_1$, where $P_{H_1,H_2}$ is defined by

$$P_{H_1,H_2}(h) = h_1, \text{ for any } h = h_1 + h_2 \in H \text{ with } h_i \in H_i, i = 1, 2.$$  

**Lemma 1.1.** (cf. [4, Theorem 3.2] and [13, Remark 1.1]) Let $H, K$ be two Hilbert $\mathfrak{A}$-modules and $A \in \mathcal{L}(H, K)$. Then the closeness of any one of the following sets implies the closeness of the remaining three sets:

$$\mathcal{R}(A), \mathcal{R}(A^*), \mathcal{R}(AA^*), \mathcal{R}(A^*A).$$

Furthermore, if $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(A^*A)$ and with respect to the $\mathfrak{A}$-valued inner product, the following orthogonal decompositions hold:

$$H = \mathcal{N}(A) + \mathcal{R}(A^*), \; K = \mathcal{R}(A) + \mathcal{N}(A^*). \quad (1.1)$$

Throughout the rest of this section, $H, K$ and $L$ are three Hilbert $\mathfrak{A}$-modules.

**Definition 1.1.** An element $M$ of $\mathcal{L}(K)$ is said to be positive definite, if $M$ is positive and invertible in $\mathcal{L}(K)$.

**Proposition 1.2.** Let $M \in \mathcal{L}(K)$ be positive definite. Then with the inner-product given by

$$\langle x, y \rangle_M = \langle x, My \rangle, \text{ for any } x, y \in K,$$  

$K$ also becomes a Hilbert $\mathfrak{A}$-module.

**Proof.** With respect to $\langle \cdot, \cdot \rangle_M$, $K$ is clearly an inner-product $\mathfrak{A}$-module [4, P. 2]. We prove that $K$ is complete with respect to the norm induced by

$$\|x\|_M \overset{def}{=} \|\langle x, x \rangle_M\|^\frac{1}{2} = \|M^\frac{1}{2}x\|, \text{ for any } x \in K. \quad (1.3)$$

In fact, if we let $C_1 = \|M^{-\frac{1}{2}}\|^{-1} > 0$ and $C_2 = \|M^\frac{1}{2}\| > 0$, then by (1.3) we can get

$$C_1 \|x\| \leq \|x\|_M \leq C_2 \|x\|, \text{ for any } x \in K,$$

which means that $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent norms on $K$. Since $K$ is assumed to be complete with respect to the original norm $\|\cdot\|$, the completeness of $K$ with respect to the induced norm $\|\cdot\|_M$ follows.

**Remark 1.1.** We use the notation $K_M$ to denote the Hilbert $\mathfrak{A}$-module with the inner-product given by (1.2), and call $K_M$ the weighted space (with respect to $M$). Following the notation, for any positive definite element $N$ of $\mathcal{L}(H)$, $T \in \mathcal{L}(H, K)$, $x \in H$ and $y \in K$, we have

$$\langle Tx, y \rangle_M = \langle Tx, My \rangle = \langle x, T^*My \rangle = \langle x, N^{-1}T^*My \rangle_N.$$  

So if we regard $T$ as an element of $\mathcal{L}(H_N, K_M)$, then

$$T^\# = N^{-1}T^*M, \quad (1.4)$$

where $T^\# \in \mathcal{L}(K_M, H_N)$ is the adjoint operator of $T \in \mathcal{L}(H_N, K_M)$.

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1 The reader should be aware that as sets, $\mathcal{L}(H, K)$ and $\mathcal{L}(H_N, K_M)$ are the same.
**Definition 1.2.** Let $A \in \mathcal{L}(H, K)$ be arbitrary, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be two positive definite operators. The weighted Moore-Penrose inverse $A_{MN}^\dagger$ (if it exists) is the element $X$ of $\mathcal{L}(K, H)$, which satisfies

$$AXA = A, \ XAX = X, (MAX)^* = MAX \text{ and } (NXA)^* = NXA.$$  \hfill (1.5)

If $M = I_K$ and $N = I_H$, then $A_{MN}^\dagger$ is denoted simply by $A^\dagger$, which is called the Moore-Penrose inverse of $A$.

**Theorem 1.3.** Let $A \in \mathcal{L}(H, K)$ be arbitrary, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be two positive definite operators. The self-adjoint, the last two equalities in (1.5) hold. Conversely, suppose that $\mathcal{R}(A)$ is closed in $K$, then $\mathcal{R}(A)$ is also closed in $K_M$, so by [13, Theorem 2.2 and Proposition 2.4] there exists uniquely an element $X \in \mathcal{L}(K_M, H_N)$ satisfying

$$AXA = A, \ XAX = X, (AX)^\# = AX \text{ and } (XA)^\# = XA.$$  \hfill (1.6)

By (1.4) we get $(AX)^\# = M^{-1}(AX)^*M$ and $(XA)^\# = N^{-1}(XA)^*N$. As $M$ and $N$ are self-adjoint, the last two equalities in (1.5) hold. □

**Remark 1.2.** Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K), N \in \mathcal{L}(H)$ be positive definite. As in the finite-dimensional case [9, Theorem 1.4.4], by [13, Theorem 2.2] we have

$$\mathcal{R}(A_{MN}^\dagger) = \mathcal{R}(A^\#) = \mathcal{R}(N^{-1}A^*M) = N^{-1}\mathcal{R}(A^*),$$

$$\mathcal{N}(A_{MN}^\dagger) = \mathcal{N}(A^\#) = \mathcal{N}(N^{-1}A^*M) = N^{-1}\mathcal{N}(A^*).$$

**Proposition 1.4.** (cf. [11, Lemma 0.1]) Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be positive definite. Then $A_{MN}^\dagger$ is the unique element $X$ of $\mathcal{L}(K, H)$ which satisfies

$$A^*MAX = A^*M, \ \mathcal{R}(NX) \subseteq \mathcal{R}(A^*).$$  \hfill (1.6)

**Proof.** By [13, Proposition 2.4] we know that $A_{MN}^\dagger$ is the unique element $X$ of $\mathcal{L}(K_M, H_N)$ which satisfies

$$AX = A_{MN}^\dagger \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A^\#).$$  \hfill (1.7)

In view of (1.4), we know that (1.6) can be rewritten as

$$A^\#AX = A^\#; \ \mathcal{R}(X) \subseteq \mathcal{R}(A^\#).$$  \hfill (1.8)

Since $A^\#AA_{MN}^\dagger = A^\#$ and $(A_{MN}^\dagger)^\#A^\#AX = (AA_{MN}^\dagger)^\#AX = A_{MN}^\dagger AX = AX$, the equivalence of (1.7) and (1.8) follows. □

**Lemma 1.5.** (cf. [11, Lemma 0.3]) Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K)$ be positive definite. Then for any $X \in \mathcal{L}(K, H)$, the following two statements are equivalent:

(i) $AXA = A, (MAX)^* = MAX$;
(ii) $A^*MAX = A^*M$.

If condition (i) is satisfied, then for any positive definite element $N \in \mathcal{L}(H)$, $X$ has the form

$$X = A^\dagger_{MN} + (I_H - A^\dagger_{MN}A)Y, \text{ for some } Y \in \mathcal{L}(K,H). \tag{1.9}$$

**Proof.** (1) Let $N$ be any positive definite element of $\mathcal{L}(H)$. By (1.4) we know that conditions (i) and (ii) can be rephrased respectively as

$$AXA = A, \ (AX)^\# = AX, \tag{1.10}$$

$$A^\#AX = A^\#. \tag{1.11}$$

Suppose that (1.10) is satisfied. Then

$$A^\#AX = A^\#(AX)^\# = (AXA)^\# = A^\#. $$

Conversely, if (1.11) is satisfied, then it is easy to show that $(AXA - A)^\#(AXA - A) = 0$, so $AXA = A$. Furthermore, 

$$(AX)^\# = X^\#A^\# = X^\#A^\#AX = (A^\#AX)^\#X = (A^\#)^\#X = AX.$$ 

(2) Suppose that $X \in \mathcal{L}(K,H)$ is given such that (1.11) is satisfied. Then

$$A^\#A(X - A^\dagger_{MN}) = A^\# - A^\# = 0 \implies (A(X - A^\dagger_{MN}))^\#A(X - A^\dagger_{MN}) = 0,$$

so $A(X - A^\dagger_{MN}) = 0$; or equivalently, $A^\dagger_{MN}A(X - A^\dagger_{MN}) = 0$, hence there exists $Y \in \mathcal{L}(K,H)$ such that $X - A^\dagger_{MN} = (I_H - A^\dagger_{MN}A)Y$. \qed

**Definition 1.3.** An element $X$ of $\mathcal{L}(K,H)$ is said to be a $(1,3)$-inverse of $A \in \mathcal{L}(H,K)$, written $X \in A\{1,3\}$, if $AXA = A$ and $(AX)^* = AX$.

**Proposition 1.6.** Let $A \in \mathcal{L}(H,K)$ have a closed range. Then for any $X \in (AA^*)\{1,3\}$, we have $A^\dagger = A^*X$.

**Proof.** Put $Y = A^*X$. By (1.6) it is sufficient to verify that

$$A^*AY = A^*, \ \mathcal{R}(Y) \subseteq \mathcal{R}(A^*).$$

The second condition is obviously satisfied. Replacing $A, M$ with $AA^*$ and $I_K$ respectively, by “(i)$\implies$ (ii)” in Lemma 1.5 we obtain $AA^*AA^*X = AA^*$, therefore

$$A^*AY = A^*AA^*X = A^1(AA^*AA^*X) = A^1AA^* = A^*.$$ \qed

## 2 Relationship between weighted Moore-Penrose inverses

Throughout this section, $H$ and $K$ are two Hilbert $\mathfrak{A}$-modules, $M \in \mathcal{L}(K)$ and $N_1, N_2 \in \mathcal{L}(H)$ are three positive definite operators. The purpose of this section is to generalize [12, Lemma 2.4] from the finite-dimensional case to the Hilbert $C^*$-module case. For any $A \in \mathcal{L}(H,K)$, if $\mathcal{R}(A)$ is closed, then as in [12] we define

$$R_{M;N_1,N_2} = I_H + (I_H - A^\dagger_{MN_1}A)N_1^{-1}(N_2 - N_1) = A^\dagger_{MN_1}A + (I_H - A^\dagger_{MN_1}A)N_1^{-1}N_2. \tag{2.1}$$
Lemma 2.1. Let \( A \in \mathcal{L}(H, K) \) have a closed range. The operator \( R_{M;N_1,N_2} \) defined by (2.1) is invertible.

Proof. Let \( P = A_{M,N_1}^1 A, S = (I_H - P)N_1^{-1}N_2(I_H - P), H_1 = (I_H - P)H \) and \( S|_{H_1} : H_1 \to H_1 \) be the restriction of \( S \) to \( H_1 \).

First, we prove that \( S|_{H_1} \in \mathcal{L}(H_1) \) is invertible. By the last condition in (1.5) we get

\[
N_1 S = (I_H - P)^* N_2 (I_H - P) = (N_2^1 (I_H - P))^* (N_2^1 (I_H - P)).
\] (2.2)

As \( P \) is idempotent, we have \( \mathcal{N}(S) = \mathcal{N}(N_1 S) = \mathcal{N}(I_H - P) = \mathcal{R}(P) \), which means that \( \mathcal{N}(S|_{H_1}) = \mathcal{R}(P) \cap H_1 = \{0\} \). Furthermore, since \( \mathcal{R}((N_2^2 (I_H - P)) \) is closed, we may apply Lemma 1.1 to (2.2) to conclude that

\[
\mathcal{R}(S|_{H_1}) = \mathcal{R}(S) = N_1^{-1} \mathcal{R}(N_1 S) = N_1^{-1} \mathcal{R}((I_H - P)^* N_2^1)
\]

\[
= \mathcal{R}(N_1^{-1}(I_H - P)^*) = \mathcal{R}((I_H - P)N_1^{-1}) = \mathcal{R}(I_H - P) = H_1.
\]

This completes the proof of the invertibility of \( S|_{H_1} \).

Next, let

\[
Y = P + (S|_{H_1})^{-1}(I_H - P) - (S|_{H_1})^{-1}(I_H - P)N_1^{-1}N_2 P.
\]

Then since \( R_{M;N_1,N_2} = P + (I_H - P)N_1^{-1}N_2 P + S \), it is easy to verify that \( R_{M;N_1,N_2} Y = Y R_{M;N_1,N_2} = I_H \).

Lemma 2.2. (cf. [12, Lemma 2.4]) Suppose that \( A \in \mathcal{L}(H, K) \) has a closed range. Then \( A_{M,N_2}^1 = R_{M;N_1,N_2}^{-1} A_{M,N_1}^1 \), where \( R_{M;N_1,N_2} \) is defined by (2.1).

Proof. Let \( A^* = N_1^{-1} A^* M \in \mathcal{L}(K_M, H_{N_1}) \) be the conjugate operator of \( A \in \mathcal{L}(H_{N_1}, K_M) \). To simplify the notation, we define

\[
X = R_{M;N_1,N_2}^{-1} \cdot (I_H - A_{M,N_1}^1 A)N_1^{-1}N_2.
\] (2.3)

Then

\[
(I_H - A_{M,N_1}^1 A)N_1^{-1}A^* = (I_H - A_{M,N_1}^1 A)A^* M^{-1} = 0,
\]

so by (2.3) we have

\[
X N_2^{-1} \mathcal{R}(A^*) = 0.
\] (2.4)

Since \( A_{M,N_1}^1 A(I_H - A_{M,N_1}^1 A) = 0 \), by (2.1) and (2.3) we have

\[
X(I_H - A_{M,N_1}^1 A) = (R_{M;N_1,N_2}^{-1} \cdot A_{M,N_1}^1 A + X)(I_H - A_{M,N_1}^1 A)
\]

\[
= R_{M;N_1,N_2}^{-1} \cdot (A_{M,N_1}^1 A + (I_H - A_{M,N_1}^1 A)N_1^{-1}N_2)(I_H - A_{M,N_1}^1 A)
\]

\[
= R_{M;N_1,N_2}^{-1} \cdot R_{M;N_1,N_2} \cdot (I_H - A_{M,N_1}^1 A) = I_H - A_{M,N_2}^1 A.
\] (2.5)

As \( I_H - A_{M,N_1}^1 A \) is the oblique projector of \( H \) along \( N_2^{-1} \mathcal{R}(A^*) \) onto \( \mathcal{N}(A) = \mathcal{R}(I_H - A_{M,N_1}^1 A) \), in view of (2.4) and (2.5) we conclude that \( I_H - A_{M,N_2}^1 A = X \). Furthermore, by (2.3) and (2.1) we have

\[
I_H - A_{M,N_2}^1 A = X = R_{M;N_1,N_2}^{-1} \cdot (I_H - A_{M,N_1}^1 A)N_1^{-1}N_2
\]

\[
= R_{M;N_1,N_2}^{-1} \cdot R_{M;N_1,N_2} - A_{M,N_1}^1 A = I_H - R_{M;N_1,N_2}^{-1} \cdot A_{M,N_1}^1 A.
\] (2.6)
Lemma 3.1. Let 

\[ A_{MN_2}^\dagger A = R_{M_1, N_1, N_2}^{-1} \cdot A_{MN_1}^\dagger. \] (2.7) \n
Note that \( AA_{MN_1}^\dagger = AA_{MN_2}^\dagger \) is the oblique projector of \( K \) along \( M^{-1}N(A^*) \) onto \( \mathcal{R}(A) \), so if we multiply \( A_{MN_1}^\dagger \) from the right on both sides of (2.7), then we may obtain

\[ A_{MN_2}^\dagger = A_{MN_2}^\dagger AA_{MN_2}^\dagger = A_{MN_2}^\dagger AA_{MN_1}^\dagger = R_{M_1, N_1, N_2}^{-1} \cdot A_{MN_1}^\dagger. \]

Remark 2.1. With the notation of Lemma 2.2 by (2.6) we obtain

\[ (I_H - A_{MN_2}^\dagger A)N_2^{-1} = R_{M_1, N_1, N_2}^{-1} \cdot (I_H - A_{MN_1}^\dagger A)N_1^{-1}. \] (2.8)

3 Unified representations for weighted Moore-Penrose inverses of \( 1 \times 2 \) partitioned operators

Throughout this section, \( H_1, H_2 \) and \( H_3 \) are three Hilbert \( \mathfrak{A} \)-modules, \( A \in \mathcal{L}(H_1, H_3) \) and \( B \in \mathcal{L}(H_2, H_3) \) are arbitrary, \( M_1 \in \mathcal{L}(H_3) \) and

\[ N = \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2) \] (3.1)

are two positive definite operators, where \( N_1 \in \mathcal{L}(H_1), L \in \mathcal{L}(H_2, H_1) \) and \( N_2 \in \mathcal{L}(H_2) \). By [11] Section 5 we know that both \( N_1 \) and \( S(N) \) are positive definite, where \( S(N) \) is the Schur complement of \( N \) defined by

\[ S(N) = N_2 - L^* N_1^{-1} L. \]

When \( A \) has a closed range, we put

\[ C = (I_{H_3} - AA_{MN_1}^\dagger) B \in \mathcal{L}(H_2, H_3). \] (3.2)

Lemma 3.1. Let \( A \in \mathcal{L}(H_1, H_3) \) have a closed range. Then

(i) \( \mathcal{R}(A^*) = \mathcal{R}(A^*) \oplus \mathcal{R}(C^*) \);

(ii) \( \mathcal{R}(A^*) = \mathcal{N}(\{(I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}\}). \)

Proof. (i) For any \( \xi, \eta \in H_3 \), let \( \zeta = (AA_{MN_1}^\dagger)^* \xi + (I_{H_3} - AA_{MN_1}^\dagger)^* \eta \). Then \( A^* \zeta = A^* \xi \) and \( C^* \zeta = C^* \eta \), so \( (A^*) \zeta \in \mathcal{R}(A_{MN_1}^\dagger) \).

(ii) As \( AA_{MN_1}^\dagger A = A \), we have

\[ \mathcal{R}(A^*) = \mathcal{R}((A_{MN_1}^\dagger A)^*) = \mathcal{N}(I_{H_1} - (A_{MN_1}^\dagger A)^*) \]

\[ = \mathcal{N}(N_1(I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}) = \mathcal{N}((I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}). \]

Although the technique lemma in [1] ([1] Lemma 0.2) is no longer true in the infinite-dimensional case, we can still provide a formula for \( (A, C)_{MN}^\dagger \) by following the line in [1] together with some modifications.
Theorem 3.2. (cf. [1 Theorem 1.1]) Let $C$ be defined by (3.2), and suppose that $R(A, R(C)$ and $R(A, C)$ are all closed. Then

$$(A, C)_{MN}^1 = \left( A_{MN_1}^\dagger - \frac{(I_{H_1} - A_{MN_1}^\dagger)N_1^{-1}LU}{U} \right),$$

where

$$S = N_2 - L^*(I_{H_1} - A_{MN_1}^\dagger)N_1^{-1}L = S(N) + L^*A_{MN_1}^\dagger AN_1^{-1}L \in \mathcal{L}(H_2),$$

$$U = C_{MS}^\dagger - (I_{H_2} - C_{MS}^\dagger)S^{-1}L^*A_{MN_1}^\dagger \in \mathcal{L}(H_3, H_2).$$

Proof. Note that $A_{MN_1}^\dagger A$ is a projection on the weighted space $(H_1)_{N_1}$, so for any $\xi \in H_2$, we have

$$\langle L^*A_{MN_1}^\dagger AN_1^{-1}L\xi, \xi \rangle = \langle (A_{MN_1}^\dagger A)(N_1^{-1}L\xi), N_1^{-1}L\xi \rangle_{N_1} \geq 0,$$

hence $L^*A_{MN_1}^\dagger AN_1^{-1}L$ is positive definite. Note also that

$$C^*MA = B^*(I_{H_3} - AA_{MN_1}^\dagger)^*MA = B^*M(I_{H_3} - AA_{MN_1}^\dagger)A = 0.$$  \hfill (3.6)

Now let $N_3$ be any positive definite element of $\mathcal{L}(H_2)$. For any $X_1 \in \mathcal{L}(H_3, H_1)$ and $X_2 \in \mathcal{L}(H_3, H_2)$, by Proposition 1.4 we know that $(\begin{array}{c} X_1 \\ X_2 \end{array}) = (A, C)_{MN}^1$ if and only if

$$\begin{pmatrix} A^* \\ C^* \end{pmatrix} M(A, C) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A^* \\ C^* \end{pmatrix} M, \quad R \left( N \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right) \subseteq R \left( \begin{pmatrix} A^* \\ C^* \end{pmatrix} \right).$$

Combining the above two conditions with (3.6), we may apply Lemma 3.1 to conclude that $(\begin{array}{c} X_1 \\ X_2 \end{array}) = (A, C)_{MN}^1$ if and only if the following four equations hold:

$$A^*M X_1 = A^*M;$$

$$C^*MC X_2 = C^*M;$$

$$(I_{H_1} - A_{MN_1}^\dagger)N_1^{-1}(N_1X_1 + LX_2) = 0; \hfill (3.10)$$

$$(I_{H_2} - C_{MN_3}^\dagger)N_3^{-1}(L^*X_1 + N_2X_2) = 0. \hfill (3.11)$$

By (3.9) we have

$$X_1 = A_{MN_1}^\dagger + (I_{H_1} - A_{MN_1}^\dagger)Y_1, \hfill (3.12)$$

$$X_2 = C_{MN_3}^\dagger + (I_{H_2} - C_{MN_3}^\dagger)Y_2, \hfill (3.13)$$

for some $Y_1 \in \mathcal{L}(H_3, H_1)$ and $Y_2 \in \mathcal{L}(H_3, H_2)$. It follows from (3.10) and (3.12) that

$$(I_{H_1} - A_{MN_1}^\dagger)Y_1 + (I_{H_1} - A_{MN_1}^\dagger)N_1^{-1}LX_2 = 0.$$ 

Combining the above equality with (3.12) we get

$$X_1 = A_{MN_1}^\dagger - (I_{H_1} - A_{MN_1}^\dagger)N_1^{-1}LX_2. \hfill (3.14)$$
It follows from (3.11), (3.14) and (3.18) that
\[(I_{H_2} - C_{M N_3}^\dagger) N_3^{-1} L^* A_{M N_1}^\dagger + (I_{H_2} - C_{M N_3}^\dagger) N_3^{-1} S X_2 = 0. \tag{3.15}\]

By (3.13) we have
\[(I_{H_2} - C_{M N_3}^\dagger) N_3^{-1} S X_2 = (I_{H_2} - C_{M N_3}^\dagger) N_3^{-1} S C_{M N_3}^\dagger + (I_{H_2} - C_{M N_3}^\dagger) N_3^{-1} S (I_{H_2} - C_{M N_3}^\dagger) Y_2. \tag{3.16}\]

So, if we let \(N_3 = S\), then by the above equality we get
\[(I_{H_2} - C_{M S}^\dagger) X_2 = (I_{H_2} - C_{M S}^\dagger) Y_2. \tag{3.17}\]

The expression for \(U\) given by (3.5) follows from (3.13), (3.17) and (3.15) by letting \(N_3 = S\). The conclusion then follows from (3.14).

Theorem 3.3 below was proved in [1, 6, 10] for matrices by using different methods. In the context of Hilbert \(C^*\)-module operators, we can give a general proof as follows:

**Theorem 3.3.** Under the conditions of Theorem 3.2 we have
\[(A, B)_{M N}^\dagger = \left( A_{M N_1}^\dagger - \left( D + (I_{H_1} - A_{M N_1}^\dagger A) N_1^{-1} L \right) \tilde{U} \right), \tag{3.18}\]

where
\[
D = A_{M N_1}^\dagger B \in \mathcal{L}(H_2, H_1), \tag{3.19}
\]
\[
\tilde{S} = N_2 - L^*(I_{H_1} - A_{M N_1}^\dagger A) N_1^{-1} L + D^* N_1 D - D^* L - L^* D \in \mathcal{L}(H_2), \tag{3.20}
\]
\[
\tilde{U} = C_{M S}^\dagger + (I_{H_2} - C_{M S}^\dagger C)(\tilde{S})^{-1}(D^* N_1 - L^*) A_{M N_1}^\dagger \in \mathcal{L}(H_3, H_2). \tag{3.21}
\]

**Proof.** Let \(T = \begin{pmatrix} I_{H_1} & -D \\ 0 & I_{H_2} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2)\). Then \(T\) is invertible with \(T^{-1} = \begin{pmatrix} I_{H_1} & D \\ 0 & I_{H_2} \end{pmatrix}\).

In view of (3.12) and (3.19), we have
\[(A, B) T = (A, C), \tag{3.22}\]

which means that \(\mathcal{R}(A, B) = \mathcal{R}(A, C)\) is closed, so \((A, B)_{M N}^\dagger\) exists. Furthermore, by (1.6) and (3.22) we know that \((A, B)_{M N}^\dagger\) is the unique solution \(\tilde{X} = \left( \frac{\tilde{X}_1}{\tilde{X}_2} \right) \in \mathcal{L}(H_3, H_1 \oplus H_2)\) to the equation
\[(A, C)^* M (A, C) T^{-1} \tilde{X} = (A, C)^* M, \tag{3.23}\]
\[R(T^* N T \cdot T^{-1} \tilde{X}) \subseteq R((A, C)^*). \tag{3.24}\]

It follows from (1.6) that \(T^{-1} \tilde{X} = (A, C)_{M N}^\dagger\), where
\[
\tilde{N} = T^* N T = \begin{pmatrix} N_1 & L - N_1 D \\ L^* - D^* N_1 & N_2 - D^* L - L^* D + D^* N_1 D \end{pmatrix}. \tag{3.25}\]
By the definition of $D$ we get $(I_{H_1} - A_{MN_1}^\dagger A)D = 0$ and
\[
D^*N_1(I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1} = B^*(A_{MN_1}^\dagger)^*N_1(I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1} = B^*(A_{MN_1}^\dagger)^*(I_{H_1} - A_{MN_1}^\dagger A)^* = 0.
\]
In view of (3.4), if we replace $N_2$, $L$ with $N_2 - D^*L - L^*D + D^*N_1D$ and $L - N_1D$ respectively, and define
\[
\tilde{S} = (N_2 - D^*L - L^*D + D^*N_1D) - (L - N_1D)^*(I - A_{MN_1}^\dagger A)N_1^{-1}(L - N_1D)
\]
then by Theorem 3.2 we conclude that
\[
T^{-1}\tilde{X} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}
\]
with
\[
V_2 = C_{MS}^\dagger - (I_{H_2} - C_{MS}^\dagger C)(\tilde{S})^{-1}(L - N_1D)^*(I - A_{MN_1}^\dagger A)N_1^{-1}A_{MN_1}^\dagger, \quad (3.26)
\]
\[
V_1 = A_{MN_1}^\dagger - (I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}(L - N_1D)V_2
\]
\[
= A_{MN_1}^\dagger - (I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}LV_2. \quad (3.27)
\]
As $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = T(\begin{pmatrix} V_1 \\ V_2 \end{pmatrix})$, (3.21) and (3.18) then follow from (3.26) and (3.27).

Now we are ready to give a unified representation for $(A, B)_{MN}^\dagger$ in terms of $C_{MN_3}^\dagger$, where $N_3 \in \mathcal{L}(H_2)$ can be an arbitrary positive definite operator.

**Theorem 3.4.** Under the conditions of Theorem 3.2 we have
\[
(A, B)_{MN}^\dagger = \begin{pmatrix} A_{MN_1}^\dagger - (D + (I_{H_1} - A_{MN_1}^\dagger A)N_1^{-1}L)V \\ V \end{pmatrix}, \quad (3.28)
\]
where $N_3 \in \mathcal{L}(H_2)$ is arbitrary positive definite, $D$ and $\tilde{S}$ are defined by (3.19) and (3.20) respectively, and
\[
R_{M;N_3,\tilde{S}} = I_{H_2} + (I_{H_2} - C_{MN_3}^\dagger C)N_3^{-1}(\tilde{S} - N_3),
\]
\[
V = R_{M;N_3,\tilde{S}}^{-1}\left(C_{MN_3}^\dagger + (I_{H_2} - C_{MN_3}^\dagger C)N_3^{-1}(D^*N_1 - L^*)A_{MN_1}^\dagger\right). \quad (3.29)
\]

**Proof.** By Lemma 2.2 we have $C_{MS}^\dagger = R_{M;N_3,\tilde{S}}^{-1}C_{MN_3}^\dagger$. Furthermore, by (2.5) we can get
\[
(I_{H_2} - C_{MS}^\dagger C)(\tilde{S})^{-1} = R_{M;N_3,\tilde{S}}^{-1}(I_{H_2} - C_{MN_3}^\dagger C)N_3^{-1}.
\]
The conclusion then follows from (3.18) and (3.21).

In the special case of the preceding theorem where $N_3 = S(N)$, we regain the main technique result of [11] as follows:

**Theorem 3.5.** (cf. [11] Theorem 5.1) Under the conditions of Theorem 3.2 we have
\[
(A, B)_{MN}^\dagger = \begin{pmatrix} A_{MN_1}^\dagger - (\Sigma + N_1^{-1}L)\Omega \\ \Omega \end{pmatrix}, \quad (3.30)
\]

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where $C$ and $D$ are defined by (3.2) and (3.19) respectively, and

$$
\Sigma = A^\dagger_{MN}(B - AN_1^{-1}L) = D - A^\dagger_{MN}AN_1^{-1}L, \quad (3.31)
$$

$$
Y = (I - C^\dagger_{MS(N)}C)S(N)^{-1}, \quad (3.32)
$$

$$
\Omega = (I + Y\Sigma^*N_1\Sigma)^{-1}(Y\Sigma^*N_1 \cdot A^\dagger_{MN} + C^\dagger_{MS(N)}). \quad (3.33)
$$

Proof. Let $\tilde{S}$ be given by (3.20) and define

$$
\Delta = \tilde{S} - S(N) = L^*A^\dagger_{MN}AN_1^{-1}L + D^*N_1D - D^*L - L^*D. \quad (3.34)
$$

By definition we have

$$
\Sigma^* = D^* - L^*A^\dagger_{MN}AN_1^{-1}, \quad \text{so} \quad \Sigma^*N_1 = D^*N_1 - L^*A^\dagger_{MN}A. \quad (3.35)
$$

It follows that $\Sigma^*N_1A^\dagger_{MN} = D^*N_1A^\dagger_{MN} - L^*A^\dagger_{MN}$. Therefore,

$$
(I - C^\dagger_{MS(N)}C)S(N)^{-1}(D^*N_1 - L^*)A^\dagger_{MN} = Y\Sigma^*N_1A^\dagger_{MN}. \quad (3.36)
$$

By the definition of $D$, we have $A^\dagger_{MN}AD = D$, so by (3.35) and (3.31) we have

$$
\Sigma^*N_1 \Sigma = (D^*N_1 - L^*A^\dagger_{MN}A)(D - A^\dagger_{MN}AN_1^{-1}L)
\begin{align*}
&= D^*N_1D - D^*(A^\dagger_{MN}A)^*L - L^*A^\dagger_{MN}AD + L^*A^\dagger_{MN}AN_1^{-1}L \\
&= D^*N_1D - D^*L - L^*D + L^*A^\dagger_{MN}AN_1^{-1}L = \Delta. \quad (3.37)
\end{align*}
$$

It follows that

$$
R_{M,S(N),\tilde{S}} = I + Y\Delta = I + Y\Sigma^*N_1\Sigma. \quad (3.38)
$$

Finally, by the definitions of $D$ and $\Sigma$ we get

$$
D + (I - A^\dagger_{MN}A)N_1^{-1}L = \Sigma + N_1^{-1}L. \quad (3.39)
$$

Expression (3.33) for $\Omega$ follows from (3.29), (3.38) and (3.36). Formula (3.30) for $(A, B)^\dagger_{MN}$ then follows from (3.28) and (3.31).

4 Representations for weighted Moore-Penrose inverses of $2 \times 2$ partitioned operators

4.1 Non weighted case

Following the line initiated in [7], in this section we study the representations for the (non-weighted) Moore-Penrose inverse $A^\dagger$ of a general $2 \times 2$ partitioned operator matrix

$$
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2, K_1 \oplus K_2), \quad (4.1)
$$

where $H_1, H_2, K_1$ and $K_2$ are four Hilbert $\mathfrak{A}$-modules, $A_{11} \in \mathcal{L}(H_1, K_1)$, $A_{12} \in \mathcal{L}(H_2, K_1)$, $A_{21} \in \mathcal{L}(H_1, K_2)$ and $A_{22} \in \mathcal{L}(H_2, K_2)$. In the case when $A_{11}$ has a closed range, let $S(A)$ be the Schur complement of $A$ defined by

$$
S(A) = A_{22} - A_{21}A_{11}^\dagger A_{12} \in \mathcal{L}(H_2, K_2). \quad (4.2)
$$
4.1.1 Special case

**Lemma 4.1.** Suppose that $A_{11}$ has a closed range. Then both $F_1(A)^\dagger$ and $F_2(A)^\dagger$ exist, where

\[
F_1(A) = \begin{pmatrix} -A_{11}^\dagger A_{12} & 0 \\ A_{11}^\dagger A_{11} & 0 \end{pmatrix} \in \mathcal{L}(H_2, H_1 \oplus H_2),
\]

\[
F_2(A) = -A_{21}A_{11}^\dagger I_{K_2} \in \mathcal{L}(K_1 \oplus K_2, K_2).
\]

Furthermore, the following equalities hold:

(i) $F_1(A)^\dagger \cdot \begin{pmatrix} A_{11}^\dagger A_{11} & A_{11}^\dagger A_{12} \\ 0 & 0 \end{pmatrix} = F_1(A)^\dagger - (0, I_{H_2})$;

(ii) $F_2(A)^\dagger = F_2(A)^\dagger - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix}$.

**Proof.** By definition we have $F_2(A)F_2(A)^* = I_{K_2} + (A_{21}A_{11}^\dagger)(A_{21}A_{11}^\dagger)^*$, which is invertible, hence by Proposition 1.6 we have

\[
F_2(A)^\dagger = F_2(A)^* \cdot (F_2(A)F_2(A)^*)^{-1}.
\]

It follows from (4.1) and (4.5) that

\[
\begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} F_2(A)^\dagger = \begin{pmatrix} -(A_{21}A_{11}^\dagger)^* \\ -(A_{21}A_{11}^\dagger)(A_{21}A_{11}^\dagger)^* \end{pmatrix}(F_2(A)F_2(A)^*)^{-1}
= \left[ F_2(A)^* - \begin{pmatrix} 0 \\ F_2(A)F_2(A)^* \end{pmatrix} \right](F_2(A)F_2(A)^*)^{-1} = F_2(A)^\dagger - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix}.
\]

The proof of (i) is similar. □

**Theorem 4.2.** (cf. [7, Theorem 2]) Suppose that both $A_{11}$ and $S(A)$ have closed ranges, and

\[
(I_{K_1} - A_{11}A_{11}^\dagger)A_{12} = 0, \ A_{21}(I_{H_1} - A_{11}A_{11}^\dagger) = 0.
\]

Then

\[
A^\dagger = X_L(A) \text{diag}(A_{11}^\dagger, 0) X_R(A) + F_1(A) S(A)^g F_2(A),
\]

where $F_1(A)$ and $F_2(A)$ are defined by (4.3) and (4.4) respectively, and

\[
S(A)^g = S(A)^\dagger_{[F_2(A)(F_2(A)^*)^{-1}F_1(A)^*]^{-1}}, \ F_1(A) \in \mathcal{L}(K_2, H_2),
\]

\[
X_L(A) = I_{H_1\oplus H_2} - F_1(A) [I_{H_2} - S(A)^gS(A)] F_1(A)^\dagger \in \mathcal{L}(H_1 \oplus H_2),
\]

\[
X_R(A) = I_{K_1\oplus K_2} - F_2(A)^\dagger [I_{K_2} - S(A)S(A)^g] F_2(A) \in \mathcal{L}(K_1 \oplus K_2).
\]

**Proof.** It follows from (4.1), (4.3), (4.4) and (4.6) that

\[
AF_1(A) = \begin{pmatrix} 0 \\ S(A) \end{pmatrix} \quad \text{and} \quad F_2(A)A = (0, S(A)),
\]

(4.11)
which implies that
\[ AX_L(A) = A \text{ and } X_R(A)A = A. \] (4.12)

To simplify the notation, let
\[ \lambda_1(A) = I_{H_2} - S(A)^g S(A) \text{ and } \lambda_2(A) = I_{K_2} - S(A)S(A)^g. \] (4.13)

Then by (ii) of Lemma 4.11 we have
\[
A \text{ diag}(A_{11}^†, 0) X_R(A) = \begin{pmatrix} A_{11} A_{11}^† & 0 \\ A_{21} A_{11}^† & 0 \end{pmatrix} X_R(A)
\]
\[
= \begin{pmatrix} A_{11} A_{11}^† & 0 \\ A_{21} A_{11}^† & 0 \end{pmatrix} - \begin{pmatrix} A_{11} A_{11}^† & 0 \\ A_{21} A_{11}^† & 0 \end{pmatrix} F_2(A)^\dagger \lambda_2(A) F_2(A)
\]
\[
= \begin{pmatrix} A_{11} A_{11}^† & 0 \\ A_{21} A_{11}^† & 0 \end{pmatrix} - F_2(A)^\dagger \lambda_2(A) F_2(A) + \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A). \] (4.14)

Furthermore, by the first equality in (4.11) we get
\[
AF_1(A) S(A)^g F_2(A) = \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} S(A) S(A)^g F_2(A)
\]
\[
= \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} (I_{K_2} - \lambda_2(A)) F_2(A) = \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} F_2(A) - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A) \] (4.15)
\[
= \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} - \begin{pmatrix} A_{21} A_{11}^† & 0 \\ -A_{21} A_{11}^† & I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A).
\]

Now let Z be the right side of (4.7). Then by the first equality in (4.12), (4.14) and (4.15), we get
\[
AZ = \text{ diag}(A_{11} A_{11}^†, I_{K_2}) - F_2(A)^* (F_2(A) F_2(A)^*)^{-1} \lambda_2(A) F_2(A), \] (4.16)

which means that \((AZ)^* = AZ\), since by the definitions of \(S(A)^g\) and \(\lambda_2(A)\) we have
\[
\lambda_2(A)^* = (F_2(A) F_2(A)^*)^{-1} \lambda_2(A) (F_2(A) F_2(A)^*).
\]

As \(\lambda_2(A)(0, S(A)) = 0\), we may combine (4.16) with the second equality in (4.11) to get
\[
AZA = \text{ diag}(A_{11} A_{11}^†, I_{K_2}) A = A.
\]

Similarly, as \(F_1(A)^\dagger = (F_1(A)^* F_1(A))^{-1} F_1(A)^*\) and
\[
\text{ diag}(A_{11}^† A_{11}, I_{H_2}) X_L(A) = X_L(A) - \text{ diag}(I_{H_2} - A_{11}^† A_{11}, 0),
\]
we can prove that
\[
ZA = \text{ diag}(A_{11}^† A_{11}, I_{H_2}) - F_1(A) \lambda_1(A) (F_1(A)^* F_1(A))^{-1} F_1(A)^*
\]
with \((ZA)^* = ZA\) and \(ZA Z = Z\), therefore \(Z = A^\dagger\).
Corollary 4.3. Let \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{L}(K_1 \oplus K_2) \) be positive, where \( A_{ij} \in \mathcal{L}(K_j, K_i) \) for \( i, j = 1, 2 \). If both \( \mathcal{R}(A_{11}) \) and \( \mathcal{R}(S(A)) \) are closed, then

\[
A^\dagger = X_L(A) \text{diag}(A_{11}^\dagger, 0) X_R(A) + F_1(A) S(A)g F_1(A)^*, \tag{4.17}
\]

where \( F_1(A) \) is defined by (4.3), \( S(A)g, X_L(A) \) and \( X_R(A) \) are given respectively as (4.8), (4.9) and (4.10) by letting \( F_2(A) \) be replaced with \( F_1(A)^* \). In addition, a \( \{1,3\} \)-inverse of \( A \) can be given by

\[
A^{(1,3)} = \text{diag}(A_{11}^\dagger, 0) X_R(A) + F_1(A) S(A)g F_1(A)^*. \tag{4.18}
\]

Proof. Since \( A \) is positive, by [13, Corollary 3.5] we have

\[
A_{11} \geq 0, \quad A_{12} = A_{11} A_{11}^\dagger A_{12} \quad \text{and} \quad S(A) \geq 0. \tag{4.19}
\]

As \( (A_{11}^\dagger)^* = A_{11}^{\dagger*} \), conditions in (4.6) are satisfied. Note that in this case \( F_3(A) = (F_1(A))^* \), (4.17) follows from (4.7). Let \( A^{(1,3)} \) be the operator given by (4.18). As \( AX_L(A) = A \) we have \( AA^{(1,3)} = AA^\dagger \), so \( A^{(1,3)} \) is a \( \{1,3\} \)-inverse of \( A \). \( \square \)

4.1.2 General case

Let

\[
E = AA^* \overset{def}{=} \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{22} \end{pmatrix} \in \mathcal{L}(K_1 \oplus K_2). \tag{4.20}
\]

If \( (A_{11}, A_{12}) \) has a closed range, then as \( E_{11} = (A_{11}, A_{12})(A_{11}, A_{12})^* \), by Lemma 1.1 and Proposition 4.6 we know that \( E_{11}^\dagger \) exists such that \( (A_{11}, A_{12})^\dagger = (A_{11}, A_{12})^* E_{11}^\dagger \). Let \( S(E) = E_{22} - E_{12}^* E_{11} E_{12} \) be the Schur complement of \( E \). Assuming further that both \( A \) and \( S(E) \) have closed ranges, then for any \( \{1,3\} \)-inverse \( E^{(1,3)} \) of \( E \), we have \( A^\dagger = A^* E^{(1,3)} \). In particular, by (4.18) we have

\[
A^\dagger = A^* \cdot \left[ \text{diag}(E_{11}^\dagger, 0) X_R(E) + F_1(E) S(E)g F_1(E)^* \right], \tag{4.21}
\]

where

\[
F_1(E) = \left( -E_{11}^\dagger E_{12} \right) \in \mathcal{L}(K_2, K_1 \oplus K_2), \tag{4.22}
\]

\[
S(E)^g = S(E)_{[F_1(E)^* F_1(E)]^{-1}} F_1(E)^* F_1(E) \in \mathcal{L}(K_2), \tag{4.23}
\]

\[
X_R(E) = I_{K_1} K_2 - (F_1(E)^*)^\dagger (I_{K_2} - S(E) S(E)^g) F_1(E)^* \in \mathcal{L}(K_1 \oplus K_2). \tag{4.24}
\]

4.2 The weighted case

Following the line initiated in [11] for \( 1 \times 2 \) partitioned operators, in this subsection we provide an approach to the construction of Moore-Penrose inverses of \( 2 \times 2 \) partitioned operators from the non-weighted case to the weighted case. A detailed description of our idea can be illustrated as follows.
For any Hilbert \( \mathfrak{A} \)-module \( X \), and any projection \( P \) of \( \mathcal{L}(X) \), let \( X_1 = PX \) and \( X_2 = (I_X - P)X \), and define \( \lambda_X : X \rightarrow X_1 \oplus X_2 \) by

\[
\lambda_X(x) = \begin{pmatrix} Px \\ x - Px \end{pmatrix}, \text{ for any } x \in X. \tag{4.25}
\]

Then \( \lambda_X \) is a unitary operator with \( \lambda_X^* = \lambda_X^{-1} \), where \( \lambda_X^{-1} : X_1 \oplus X_2 \rightarrow X \) is given by

\[
\lambda_X^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2, \text{ for any } x_1 \in X_i, i = 1, 2.
\]

Now let \( H_1 \) and \( H_2 \) be two Hilbert \( \mathfrak{A} \)-modules,

\[
N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2) \tag{4.26}
\]

be a positive definite operator, where \( N_{11} \in \mathcal{L}(H_1), N_{12} \in \mathcal{L}(H_2, H_1) \) and \( N_{22} \in \mathcal{L}(H_2) \). Let \( S(N) = N_{22} - N_{12}^* N_{11}^{-1} N_{12} \) be the Schur complement of \( N \). Define

\[
a = N_{11}^{-1} N_{12}, \quad P = \begin{pmatrix} I_{H_1} & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = (H_1 \oplus H_2)_N. \tag{4.27}
\]

Then \( P^2 = P \) and \( NP = P^* N \), so \( P^* = N^{-1} P^* N = P \), which means that \( P \in \mathcal{L}(X) \) is a projection of \( \mathcal{L}(X) \), where \( X \) is the weighted space defined by \( (4.27) \) whose inner-product is given by

\[
\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle_N = \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, N \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle = \langle x_1, N_{11} x_2 + N_{12} y_2 \rangle + \langle y_1, N_{12}^* x_2 + N_{22} y_2 \rangle
\]

for any \( x_i \in H_1 \) and \( y_i \in H_2, i = 1, 2 \). By \( (4.27) \) we have

\[
X_1 = PX = \left\{ \begin{pmatrix} h_1 + a h_2 \\ 0 \end{pmatrix} \bigg| h_i \in H_i \right\} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \bigg| u \in H_1 \right\}, \tag{4.28}
\]

\[
X_2 = (I_X - P)X = \left\{ \begin{pmatrix} -a h_2 \\ h_2 \end{pmatrix} \bigg| h_2 \in H_2 \right\}. \tag{4.29}
\]

With the inner products inherited from \( X \), both \( X_1 \) and \( X_2 \) are Hilbert \( \mathfrak{A} \)-modules. Let \( j_{H_1} : (H_1)_{N_{11}} \rightarrow X_1 \) and \( j_{H_2} : (H_2)_{S(N)} \rightarrow X_2 \) be defined by

\[
j_{H_1}(h_1) = \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \quad \text{and} \quad j_{H_2}(h_2) = \begin{pmatrix} -a h_2 \\ h_2 \end{pmatrix}, \text{ for any } h_i \in H_i, i = 1, 2.
\]

It is easy to verify that both \( j_{H_1} \) and \( j_{H_2} \) are unitary operators with

\[
j_{H_1}^{-1} \begin{pmatrix} h_1 \\ 0 \end{pmatrix} = h_1 \quad \text{and} \quad j_{H_2}^{-1} \begin{pmatrix} -a h_2 \\ h_2 \end{pmatrix} = h_2, \text{ for any } h_i \in H_i, i = 1, 2.
\]

Let \( j_{H_1} \oplus j_{H_2} : (H_1)_{N_{11}} \oplus (H_2)_{S(N)} \rightarrow X_1 \oplus X_2 \) be the associated unitary operator defined by

\[
(j_{H_1} \oplus j_{H_2}) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} j_{H_1}(h_1) \\ j_{H_2}(h_2) \end{pmatrix} = \begin{pmatrix} h_1 \\ 0 \\ -a h_2 \\ h_2 \end{pmatrix}, \text{ for any } h_i \in H_i, i = 1, 2.
\]
Then clearly, 
\[(j_{H_1} \oplus j_{H_2})^\# = (j_{H_1} \oplus j_{H_2})^{-1} = j_{H_1}^{-1} \oplus j_{H_2}^{-1} = j_{H_1}^\# \oplus j_{H_2}^\#\]  

Now suppose that \(K_1\) and \(K_2\) are two additional Hilbert \(\mathfrak{A}\)-modules, and \(M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix} \in \mathcal{L}(K_1 \oplus K_2)\) is a positive definite operator, where \(M_{11} \in \mathcal{L}(K_1), M_{12} \in \mathcal{L}(K_2, K_1)\) and \(M_{22} \in \mathcal{L}(K_2)\). Let \(S(M) = M_{22} - M_{12}^* M_{11}^{-1} M_{12}\) be the Schur complement of \(M\), and define

\[b = M_{11}^{-1} M_{12}, \quad Q = \begin{pmatrix} I_{K_1} & b \\ 0 & 0 \end{pmatrix} \text{ and } Y = (K_1 \oplus K_2)^M.\]  

Similarly, define \(Y_1 = QY, Y_2 = (I_Y - Q)Y, \lambda_Y : Y \to Y_1 \oplus Y_2, j_{K_1} : (K_1)_{M_{11}} \to Y_1\) and \(j_{K_2} : (K_2)_{S(M)} \to Y_2\).

With the notation as above and suppose further that \(A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2, K_1 \oplus K_2)\), where \(A_{11} \in \mathcal{L}(H_1, K_1), A_{12} \in \mathcal{L}(H_2, K_1), A_{21} \in \mathcal{L}(H_1, K_2)\) and \(A_{22} \in \mathcal{L}(H_2, K_2)\). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
(H_1)_{N_{11}} \oplus (H_2)_{S(N)} & \xrightarrow{j_{H_1} \oplus j_{H_2}} & X_1 \oplus X_2 \\
B & \xrightarrow{j_{K_1} \oplus j_{K_2}} & Y_1 \oplus Y_2 \xrightarrow{\lambda_Y^{-1}} A \\
(K_1)_{M_{11}} \oplus (K_2)_{S(M)} & & (K_1 \oplus K_2)^M
\end{array}
\]

where

\[B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = (j_{K_1}^{-1} \oplus j_{K_2}^{-1}) \circ \lambda_Y \circ A \circ \lambda_X^{-1} \circ (j_{H_1} \oplus j_{H_2}),\]  

with

\[
\begin{align*}
B_{11} &= A_{11} + M_{11}^{-1} M_{12} A_{21}, \\
B_{12} &= A_{12} + M_{11}^{-1} M_{12} A_{22} - A_{11} N_{11}^{-1} N_{12} - M_{11}^{-1} M_{12} A_{21} N_{11}^{-1} N_{12}, \\
B_{21} &= A_{21}, \\
B_{22} &= A_{22} - A_{21} N_{11}^{-1} N_{12}.
\end{align*}
\]  

Since \(\lambda_X, \lambda_Y, j_{H_1} \oplus j_{H_2}\) and \(j_{K_1} \oplus j_{K_2}\) are all unitary operators, by [4.32] we get

\[A_{MN}^\dagger = \lambda_X^{-1} \circ (j_{H_1} \oplus j_{H_2}) \circ B_{\text{diag}(M_{11}, S(M)), \text{diag}(N_{11}, S(N))}^\dagger \circ (j_{K_1}^{-1} \oplus j_{K_2}^{-1}) \circ \lambda_Y.\]  

So if we let

\[B_{\text{diag}(M_{11}, S(M)), \text{diag}(N_{11}, S(N))}^\dagger = \begin{pmatrix} (B_{11})_{11} & (B_{11})_{12} \\ (B_{12})_{21} & (B_{12})_{22} \end{pmatrix},\]

where

\[
\begin{align*}
(B_{11})_{11} &\in \mathcal{L}((K_1)_{M_{11}}, (H_1)_{N_{11}}), \\
(B_{11})_{12} &\in \mathcal{L}((K_2)_{S(M)}, (H_1)_{N_{11}}), \\
(B_{12})_{21} &\in \mathcal{L}((K_1)_{M_{11}}, (H_2)_{S(N)}), \\
(B_{12})_{22} &\in \mathcal{L}((K_2)_{S(M)}, (H_2)_{S(N)}),
\end{align*}
\]
then by (4.37) we conclude that $A_{MN}^*(4.38)–(4.36)$ we have

\[
A_{MN}^* = \begin{pmatrix}
(A_{MN}^*)_{11} & (A_{MN}^*)_{12} \\
(A_{MN}^*)_{21} & (A_{MN}^*)_{22}
\end{pmatrix}
\]

with $(A_{MN}^*)_{11} \in \mathcal{L}(K_1, H_1)$, $(A_{MN}^*)_{12} \in \mathcal{L}(K_2, H_1)$, $(A_{MN}^*)_{21} \in \mathcal{L}(K_1, H_2)$, and $(A_{MN}^*)_{22} \in \mathcal{L}(K_2, H_2)$, such that

\[
\begin{align*}
(A_{MN}^*)_{11} &= (B^*)_{11} - N_{11}^{-1} N_{12} (B^*)_{21}, \\
(A_{MN}^*)_{12} &= (B^*)_{11} M_{11}^{-1} M_{12} + (B^*)_{12} - N_{11}^{-1} N_{12} (B^*)_{21} M_{11}^{-1} M_{12} - N_{11}^{-1} N_{12} (B^*)_{22}, \\
(A_{MN}^*)_{21} &= (B^*)_{21}, \\
(A_{MN}^*)_{22} &= (B^*)_{21} M_{11}^{-1} M_{12} + (B^*)_{22}. 
\end{align*}
\]

(4.38) (4.39) (4.40) (4.41)

Note that $(H_1)_{N_{11}}, (H_2)_{S(N)}, (K_1)_{M_{11}}$ and $(K_2)_{S(M)}$ are all Hilbert $\mathfrak{A}$-modules, the Moore-Penrose inverse of $B_{11} \in \mathcal{L}((H_1)_{N_{11}}, (K_1)_{M_{11}})$ equals $(B_{11})_{M_{11},N_{11}}^\dagger$, and the adjoint operator $B_{11}^\dagger$ of $B_{11} \in \mathcal{L}((H_1)_{N_{11}}, (K_1)_{M_{11}})$ equals $N_{11}^{-1} B_{11}^\dagger M_{11} \in \mathcal{L}(K_1, H_1)$. Since formula (4.21) is valid for any Hilbert $\mathfrak{A}$-module operators, we may use this formula to get a concrete expression for $B_{diag(M_{11},S(M)),diag(N_{11},S(N))}^\dagger$, and then obtain an expression for $A_{MN}^*$ by (4.38)–(4.41).

5 A numerical example

Example 5.1. Let $M = \begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$, $N = \begin{pmatrix}
2 & 1 & 1 & 0 \\
1 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$ and $A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}$

with

\[
A_{11} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, A_{12} = \begin{pmatrix}
1 & -1 \\
1 & 3
\end{pmatrix}, A_{21} = \begin{pmatrix}
0 & -2 \\
0 & 0
\end{pmatrix} \text{ and } A_{22} = \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}.
\]

Then $M_{11} = \begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}$, $N_{11} = \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}$, $S(M) = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 1
\end{pmatrix}$ and $S(N) = \begin{pmatrix}
\frac{1}{3} & 0 \\
0 & 1
\end{pmatrix}$. By (4.33)–(4.36) we have

\[
B_{11} = \begin{pmatrix}
1 & -1 \\
0 & 0
\end{pmatrix}, B_{12} = \begin{pmatrix}
0 & 0 \\
1 & 3
\end{pmatrix}, B_{21} = \begin{pmatrix}
0 & -2 \\
0 & 0
\end{pmatrix}, B_{22} = \begin{pmatrix}
-\frac{2}{3} & -2 \\
0 & 0
\end{pmatrix}.
\]

Note that the matrix $B = (B_{ij})_{1 \leq i,j \leq 2}$, regarded as an element of

\[
\mathcal{L}((H_1)_{N_{11}} \oplus (H_2)_{S(N)}, (K_1)_{M_{11}} \oplus (K_2)_{S(M)})
\]

\[
= \mathcal{L}((H_1 \oplus H_2)_{diag(N_{11},S(N)), (K_1 \oplus K_2)_{diag(M_{11},S(M))}}),
\]

whose conjugate $B^\#$ is given by

\[
B^\# = \text{diag}(N_{11},S(N))^{-1} \cdot B^* \cdot \text{diag}(M_{11},S(M)) = \begin{pmatrix}
2 & 0 & \frac{1}{3} & 0 \\
-2 & 0 & -\frac{2}{3} & 0 \\
0 & 3 & -1 & 0 \\
0 & 3 & 1 & 0
\end{pmatrix}.
\]
Let $E = BB^\# = \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} \in \mathcal{L}((K_1)_{M_{11}} \oplus (K_2)_{S(M)})$, where

$$E_{11} = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}, E_{12} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \text{ and } E_{22} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

By direct computation we have

$$(E_{11})_{M_{11},M_{11}}^\dagger = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}, F_1(E) = \begin{pmatrix} -(E_{11})_{M_{11},M_{11}}^\dagger E_{12} \\ I_{K_2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{6} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_1(E)^\# = S(M)^{-1} \cdot F_1(E)^* \cdot \text{diag}(M_{11}, S(M)) = \begin{pmatrix} -1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F_1(E)^\# \cdot F_1(E) = \begin{pmatrix} \frac{47}{36} & 0 \\ 0 & 1 \end{pmatrix}, S(E) = E_{22} - E_{21} \cdot (E_{11})_{M_{11},M_{11}}^\dagger \cdot E_{12} = \begin{pmatrix} \frac{7}{3} & 0 \\ 0 & 0 \end{pmatrix},$$

$$Z_1 \overset{\text{def}}{=} S(M)F_1(E)^\# F_1(E) = \begin{pmatrix} \frac{47}{72} & 0 \\ 0 & 1 \end{pmatrix}, Z_2 \overset{\text{def}}{=} S(M)(F_1(E)^\# F_1(E))^{-1} = \begin{pmatrix} \frac{47}{36} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Note that $((K_2)_{S(M)})F_1(E)^\# F_1(E) = (K_2)Z_1$ and $((K_2)_{S(M)}) (F_1(E)^\# F_1(E))^{-1} = (K_2)Z_2$, so by the Lemma we have

$$S(E)^g = S(E)_{Z_2,Z_1}^\dagger = \begin{pmatrix} \frac{3}{7} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Let $T = \begin{pmatrix} -1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix} \in \mathcal{L}((K_1)_{M_{11}}, (K_2)_{S(M)})$ and $Z_3 = \text{diag}(M_{11}, S(M))$. As

$$F_1(E)^\# = (T, I_{K_2}) \in \mathcal{L}((K_1)_{M_{11}} \oplus (K_2)_{S(M)}, (K_2)_{S(M)}) = \mathcal{L}((K_1 \oplus K_2)Z_3, (K_2)_{S(M)}),$$

if we replace $H_1, H_2, H_3, A, B, N_1, L$ and $N_2$ with $K_1, K_2, K_2, T, I_{K_2}, S(M), M_{11}, 0$ and $S(M)$ respectively, then we may apply Theorem 3.3 to get

$$(F_1(E)^\#)_{S(M),Z_3}^\dagger = \begin{pmatrix} T_{S(M),M_{11}}^\dagger \tilde{U} - \tilde{D} \tilde{U} \end{pmatrix},$$

where

$$D = T_{S(M),M_{11}}^\dagger = \begin{pmatrix} -\frac{9}{11} & 0 \\ -\frac{6}{11} & 0 \end{pmatrix}, \quad \tilde{S} = S(M) + D^* M_{11} D = \begin{pmatrix} \frac{47}{36} & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = I_{K_2} - TT_{S(M),M_{11}}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{S(M),\tilde{S}}^\dagger = C,$$

$$\tilde{U} = C_{S(M),\tilde{S}}^\dagger + (I_{K_2} - C_{S(M),\tilde{S}}^\dagger \tilde{S})^{-1} D^* M_{11} T_{S(M),M_{11}}^\dagger = \begin{pmatrix} \frac{36}{7} & 0 \\ 0 & 1 \end{pmatrix}.$$
Therefore,

\[
(F_1(E)^\#)^\dagger_{S(M),Z_3} = \begin{pmatrix}
-\frac{9}{17} & 0 \\
-\frac{6}{17} & 0 \\
\frac{36}{17} & 0 \\
0 & 1
\end{pmatrix}.
\]

It follows from (4.24) that

\[
X_R(E) = I_{K_1} \oplus K_2 - (F_1(E)^\#)^\dagger_{S(M),Z_3} \left( I_{K_2} - S(E)S(E)^g \right) F_1(E)^\# = \text{diag}(1, 1, 1, 0),
\]

hence by (4.21) we get

\[
B^\dagger \cdot \text{diag}(M_{11},S(M)) = B^\# \cdot \text{diag}(E_1^\dagger_{M_{11},M_{11}},0) X_R(E) + F_1(E)S(E)^g F_1(E)^\# = \begin{pmatrix}
(B^\dagger)_{11} & (B^\dagger)_{12} \\
(B^\dagger)_{21} & (B^\dagger)_{22}
\end{pmatrix},
\]

where

\[
(B^\dagger)_{11} = \begin{pmatrix}
\frac{4}{7} & \frac{1}{17} \\
-\frac{3}{7} & \frac{1}{12}
\end{pmatrix}, \quad (B^\dagger)_{12} = \begin{pmatrix}
-\frac{1}{17} & 0 \\
-\frac{1}{14} & 0
\end{pmatrix},
\]

\[
(B^\dagger)_{21} = \begin{pmatrix}
\frac{9}{17} & \frac{13}{28} \\
-\frac{3}{11} & \frac{5}{28}
\end{pmatrix}, \quad (B^\dagger)_{22} = \begin{pmatrix}
-\frac{9}{17} & 0 \\
\frac{3}{11} & 0
\end{pmatrix}.
\]

It follows from (4.38)–(4.41) that

\[
(A^\dagger_{MN})_{11} = \begin{pmatrix}
\frac{1}{7} & -\frac{2}{7} \\
\frac{3}{14} & \frac{5}{28}
\end{pmatrix}, \quad (A^\dagger_{MN})_{12} = \begin{pmatrix}
\frac{3}{7} & 0 \\
-\frac{11}{28} & 0
\end{pmatrix},
\]

\[
(A^\dagger_{MN})_{21} = \begin{pmatrix}
\frac{9}{14} & \frac{13}{28} \\
\frac{3}{14} & \frac{5}{28}
\end{pmatrix}, \quad (A^\dagger_{MN})_{22} = \begin{pmatrix}
-\frac{9}{28} & 0 \\
\frac{3}{28} & 0
\end{pmatrix},
\]

therefore,

\[
A^\dagger_{MN} = \begin{pmatrix}
\frac{1}{7} & -\frac{2}{7} & \frac{3}{7} & 0 \\
-\frac{3}{14} & \frac{5}{28} & -\frac{11}{28} & 0 \\
\frac{6}{14} & \frac{13}{28} & -\frac{9}{28} & 0 \\
-\frac{3}{14} & \frac{5}{28} & \frac{3}{28} & 0
\end{pmatrix}.
\]

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