M-Accretive Realisations of Skew-Symmetric Operators

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Abstract. We consider skew-symmetric operators $A_0$ on a Hilbert space $H$ and characterise all (nonlinear) m-accretive restrictions of $A := -A_0^*$ in terms of the ‘deficiency spaces’ $\ker(1 \pm A)$. The results are illustrated by several examples and applied to a partial differential equation with an impedance type boundary condition.

1 Introduction

In mathematical physics, the realisation of certain – frequently differential – operators play an essential role. For instance, in quantum mechanics, selfadjoin realisations of symmetric operators are of key interest while for well-posedness of partial differential equations, different realisations like selfadjoint, skew-selfadjoint and accretive ones are a common subject of study. For all these cases, different frameworks and techniques have been developed.

The probably most prominent technique to characterise all selfadjoint extensions of symmetric operators $S$ goes back to von Neumann, [24], and involves the so-called deficiency spaces $\ker(i \pm S^*)$ and the Cayley transform associated with $S$ (see e.g. [25, Chapter 4], [19, Chapter 13.2]). Other techniques involving forms (see e.g. [20]) or boundary triplets ([6, Chapter 3], [19, Chapter 14]) have been used to study selfadjoint, skew-selfadjoint and accretive realisations. Note that the framework of boundary triplets and the generalised concept of boundary relations has been successfully used in spectral theory for linear operators and more generally linear relations (see e.g. [4, 2] and the references therein).

We emphasise that in all the references above, the realisations are assumed to be linear. So far, the topic of nonlinear realisations seems not to have been addressed extensively in the literature. We are only aware of a paper by posilicano [18], where m-accretive nonlinear extensions of accretive symmetric operators are considered and by one of the authors in [22], where nonlinear m-accretive realisations of skew-symmetric block operator matrices are considered. We also mention [1], where m-accretive realisations were used to study port-Hamiltonian systems with nonlinear boundary feedback.

In the present paper, we provide a characterisation of all possible nonlinear m-accretive restrictions of an operator $A := -A_0^*$, where $A_0$ is a skew-symmetric operator on some Hilbert space. The character-
isation follows the spirit of the von Neumann approach for symmetric operators mentioned above, by considering the spaces \( \ker(1 \pm A) \) and contractions between them.

The motivation to study such realisations lies in the study of evolution equations, where m-accretivity (or equivalently m-dissipativity) is a key assumption for differential operators to ensure well-posedness. Indeed, in the language of \( C_0 \)-semigroups, m-accretive operators correspond to generators of contractive semigroups by the famous theorem of Lumer-Phillips in the linear case (see e.g. [5, Theorem 3.15]) and by the work of Komura in the nonlinear case. [8] (also note the extension to Banach spaces by Crandall and Pazy. [3]). Moreover, m-accretive operators play a crucial role in the theory of evolutionary equations or inclusions, see e.g. [11], [12, Chapter 7], [23] and [21].

The article is structured as follows. In Section 2 we collect some preliminary results on skew-symmetric operators and m-accretive relations. These results are used to prove the main result in Section 3 (Theorem 3.1) providing a characterisation of all m-accretive realisations of \( A := -A^* \supseteq A_0 \) for some skew-symmetric operator \( A_0 \). Section 4 is devoted to two examples. Firstly, we discuss the derivative on an interval and use our abstract characterisation result to describe all m-accretive realisations of the derivative in terms of boundary conditions. Secondly, we apply the characterisation result to block operator matrices, which naturally occur in many equations of mathematical physics. The characterisation is done in terms of abstract boundary data spaces, which turn out to be equivalent to classical trace spaces in case of gradient, divergence or rotation on smooth domains. We hereby provide another and simpler proof of the main result of [22]. The article is concluded by a concrete example illustrating the applicability of the abstract result to differential equations with impedance type boundary conditions.

Throughout, all Hilbert spaces are without loss of generality assumed to be real and their inner product is denoted by \( \langle \cdot, \cdot \rangle \) with suitable indices.

## 2 Preliminaries

Throughout, let \( H \) be a Hilbert space and \( A_0 : \text{dom}(A_0) \subseteq H \to H \) be a skew-symmetric closed linear operator and set \( A := -A_0^* \supseteq A_0 \). For a closed operator \( B : \text{dom}(B) \subseteq X \to Y \) between to Hilbert spaces \( X \) and \( Y \) we equip \( \text{dom}(B) \) with the graph inner product. In that way, \( \text{dom}(B) \) becomes a Hilbert space itself. Since \( A_0 \subseteq A \) we clearly have \( \text{dom}(A_0) \subseteq \text{dom}(A) \) and \( \text{dom}(A_0) \) is a closed subspace of \( \text{dom}(A) \). We first compute the orthogonal complement of this subspace.

**Lemma 2.1.** We have

\[
\text{dom}(A) = \text{dom}(A_0) \oplus_A \ker(1 - A^2) = \text{dom}(A_0) \oplus_A \ker(1 - A) \oplus_A \ker(1 + A),
\]

where the orthogonal sum is taken with respect to the graph inner product in \( \text{dom}(A) \).

**Proof.** We start to compute the orthogonal complement of \( \text{dom}(A_0) \) in \( \text{dom}(A) \). We have

\[
u \in \text{dom}(A_0)^\perp_{\text{dom}(A)} \iff \forall u \in \text{dom}(A_0) : \langle u, v \rangle_{\text{dom}(A)} = 0,
\]

\[\text{dom}(A) = \text{dom}(A_0) \oplus_A \ker(1 - A^2) = \text{dom}(A_0) \oplus_A \ker(1 - A) \oplus_A \ker(1 + A),\]

Note that a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle_H \) can also be viewed as a real Hilbert space by restricting scalar multipliers to \( \mathbb{R} \) and using \( \langle \cdot, \cdot \rangle := \text{Re}\langle \cdot, \cdot \rangle \) as inner product without affecting the topology of \( H \).
and thus, the assertion follows.

The latter shows \( \ker(1 - \pi) \subseteq \ker(1 - A^2) \) and hence, the first orthogonal decomposition follows. For the second equality, we note that \( \ker(1 \pm A) \subseteq \ker(1 - A^2) \): Indeed, for \( u \in \ker(1 \pm A) \) we have \( u \in \dom(A) \) and \( u \pm Au = 0 \). The latter implies \( Au = \mp u \in \dom(A) \) and \( A^2 u = \mp Au = u \). Moreover, the subspaces \( \ker(1 + A) \) and \( \ker(1 - A) \) are orthogonal in \( \dom(A) \). Indeed, for \( u \in \ker(1 + A) \), \( v \in \ker(1 - A) \) we have

\[
\langle u, v \rangle_{\dom(A)} = \langle u, v \rangle_H + \langle Au, Av \rangle_H = \langle u, v \rangle_H + \langle u, -v \rangle_H = 0.
\]

It remains to prove that each element \( u \in \ker(1 - A^2) \) can be decomposed in \( u = v + w \) for some \( v \in \ker(1 - A) \), \( w \in \ker(1 + A) \). We set \( v := \frac{1}{2}(1 + A)u \in \dom(A) \) and \( w := \frac{1}{2}(1 - A)u \in \dom(A) \). Then clearly \((1 - A)v = \frac{1}{2}(1 - A^2)u = 0 \) and \((1 + A)w = \frac{1}{2}(1 - A^2)u = 0 \) and \( v + w = u \), which shows the desired decomposition.

**Definition.** We define the orthogonal projections according to the decomposition in Lemma 2.1 by

\[
\pi_0 : \dom(A) \rightarrow \dom(A_0),
\]

\[
\pi_1 : \dom(A) \rightarrow \ker(1 - A),
\]

\[
\pi_{-1} : \dom(A) \rightarrow \ker(1 + A),
\]

respectively. Similarly, we denote the canonical embeddings by \( \iota_0, \iota_1 \) and \( \iota_{-1} \), respectively.

**Remark 2.2.** Note that \( \pi_0 = \iota_{0}^{*} \), \( \pi_1 = \iota_{1}^{*} \) and \( \pi_{-1} = \iota_{-1}^{*} \), where the adjoints are computed with respect to the graph inner product of \( A \), i.e. in the Hilbert space \( \dom(A) \). Moreover, for \( u \in \dom(A) \) we have

\[
Au = A\pi_0 u + A\pi_1 u + A\pi_{-1} u = A_0 \pi_0 u + \pi_1 u - \pi_{-1} u.
\]

**Lemma 2.3.** For \( u \in \dom(A) \) we have

\[
\langle Au, u \rangle_H = \|\pi_1 u\|_H^2 - \|\pi_{-1} u\|_H^2.
\]

**Proof.** We compute

\[
\langle Au, u \rangle_H = \langle A_0 \pi_0 u + \pi_1 u - \pi_{-1} u, \pi_0 u + \pi_1 u + \pi_{-1} u \rangle_H
\]

\[
= \|\pi_1 u\|_H^2 - \|\pi_{-1} u\|_H^2 + \langle A_0 \pi_0 u, \pi_1 u + \pi_{-1} u \rangle_H + \langle \pi_1 u, \pi_0 u \rangle_H - \langle \pi_{-1} u, \pi_0 u \rangle_H,
\]

where we have used \( \langle A_0 \pi_0 u, \pi_0 u \rangle_H = 0 \) due to the skew-symmetry of \( A_0 \). Moreover, we compute

\[
\langle A_0 \pi_0 u, \pi_1 u + \pi_{-1} u \rangle_H = -\langle \pi_0 u, A \pi_1 u + A \pi_{-1} u \rangle_H
\]

\[
= -\langle \pi_0 u, \pi_1 u \rangle_H + \langle \pi_0 u, \pi_{-1} u \rangle_H
\]

and thus, the assertion follows. \( \square \)
The kernels $\ker(1 - A)$ and $\ker(1 + A)$ are also closed subspaces of the Hilbert space $H$ and thus, there exist orthogonal projections

\[ P_{\ker(1 - A)} : H \rightarrow \ker(1 - A), \]
\[ P_{\ker(1 + A)} : H \rightarrow \ker(1 + A). \]

The next lemma relates these projections with the projections $\pi_1$ and $\pi_{-1}$.

**Lemma 2.4.** Let $u \in \text{dom}(A)$. Then

\[ \pi_1 u = \frac{1}{2} P_{\ker(1 - A)} (1 + A) u, \]
\[ \pi_{-1} u = \frac{1}{2} P_{\ker(1 + A)} (1 - A) u. \]

**Proof.** The reasoning for the second equality being analogous, we only show the first. It suffices to check that $\pi_1 u - \frac{1}{2} (1 + A) u$ is orthogonal to $\ker(1 - A)$ with respect to $H$. For this, let $v \in \ker(1 - A)$ and compute

\[
\langle \pi_1 u - \frac{1}{2} (1 + A) u, v \rangle_H = \frac{1}{2} \langle \pi_1 u - u, v \rangle_H + \frac{1}{2} \langle \pi_1 u - Au, v \rangle_H
\]
\[
= \frac{1}{2} \langle \pi_1 u - u, v \rangle_H + \frac{1}{2} \langle A\pi_1 u - Au, Av \rangle_H
\]
\[
= \frac{1}{2} \langle \pi_1 u - u, v \rangle_{\text{dom}(A)} = 0,
\]
where we have used that $Av = v$ and $A\pi_1 u = \pi_1 u$. \qed

We conclude this section by stating some useful facts on $m$-accretive operators. For later use we recall the definition in the framework of binary relations:

**Definition.** Let $B \subseteq H \times H$. Then $B$ is called *accretive*, if

\[
\forall (u, v), (x, y) \in B : \langle y - x, u - v \rangle_H \geq 0.
\]

Moreover, $B$ is *$m$-accretive*, if $B$ is accretive and $1 + B$ is onto; i.e., for each $f \in H$ there exists $(u, v) \in B$ such that

\[ u + v = f. \]

**Remark 2.5.** For an accretive operator $B$, we have that $1 + B$ is onto if and only if $\lambda + B$ is onto for some $\lambda > 0$. Moreover, the celebrated Theorem of Minty, see [9] or [21, Theorem 17.1.7], states that an accretive operator $B$ is $m$-accretive if and only if it is maximal in the set of accretive relations. In particular, if $B$ is $m$-accretive and $C$ is accretive with $B \subseteq C$, then $B = C$.

For later purposes, we state the following simple observation.

**Proposition 2.6.** Let $B \subseteq H \times H$ be accretive. Then the following statements are equivalent:
(i) $B$ is $m$-accretive,
(ii) $B$ is closed and $\text{ran}(1 + B)$ is dense in $H$.

Proof. (i) $\Rightarrow$ (ii): If $B$ is $m$-accretive, then $\text{ran}(1 + B) = H$. For showing the closedness of $B$, let $(x_n, y_n)_n$ be a sequence in $B$ with $x_n \to x$ and $y_n \to y$ in $H$ for some $x, y \in H$. By assumption, we find $(u, v) \in B$ such that $u + v = x + y$.

Then we estimate for each $n \in \mathbb{N}$
\[
\|x_n - u\|^2_H = \langle x_n - u, x_n + y_n - (u + v) \rangle - \langle x_n - u, y_n - v \rangle_H
\leq \langle x_n - u, x_n + y_n - (x + y) \rangle_H,
\]
where we have used the accretivity of $B$. Letting $n \to \infty$ in the latter inequality, we infer
\[
\|x - u\|^2_H \leq 0
\]
and hence, $x = u$. The latter implies $y = v$ and thus, $(x, y) = (u, v) \in B$, which shows that $B$ is closed.

(ii) $\Rightarrow$ (i): Let $f \in H$. By assumption we find a sequence $(f_n)_n$ in $\text{ran}(1 + B)$ such that $f_n \to f$. Moreover, we find $(u_n, v_n) \in B$ with $u_n + v_n = f_n$ for each $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$ we estimate
\[
\|u_n - u_m\|^2_H = \langle u_n - u_m, f_n - f_m \rangle_H - \langle u_n - u_m, v_n - v_m \rangle_H
\leq \langle u_n - u_m, f_n - f_m \rangle_H
\leq \|u_n - u_m\|_H \|f_n - f_m\|_H
\]
and hence, $\|u_n - u_m\|_H \leq \|f_n - f_m\|_H \to 0$ as $n, m \to \infty$. Thus, we find $u \in H$ with $u_n \to u$ as $n \to \infty$. The latter implies
\[
v_n = f_n - u_n \to f - u
\]
and thus, as $B$ is closed, we infer $(u, f - u) \in B$, which yields, $f \in \text{ran}(1 + B)$ and thus, $B$ is $m$-accretive.

\[
\square
\]

3 A Characterisation of $m$-Accretive Restrictions

As in the previous section, let $A_0 : \text{dom}(A_0) \subseteq H \to H$ be closed and skew-symmetric on a Hilbert space $H$ and set $A := -A_0^*$. We recall the decomposition from Lemma 2.1
\[
\text{dom}(A) = \text{dom}(A_0) \oplus_A \ker(1 - A) \oplus_A \ker(1 + A).
\]

We aim to prove the following theorem, which provides a characterisation of all $m$-accretive restrictions of $A$.

Theorem 3.1. Let $B \subseteq A$. Then $B$ is $m$-accretive if and only if there exists a mapping
\[
h : \ker(1 - A) \to \ker(1 + A)
\]
with
\[ \forall x, y \in \ker(1 - A) : \|h(x) - h(y)\|_H \leq \|x - y\|_H \]
such that
\[ \text{dom}(B) = \{u \in \text{dom}(A) ; h(\pi_1 u) = \pi_{-1} u\} . \]

We start to prove the following partial result of this characterisation.

**Proposition 3.2.** Let
\[ h : \ker(1 - A) \to \ker(1 + A) \]
with
\[ \forall x, y \in \ker(1 - A) : \|h(x) - h(y)\|_H \leq \|x - y\|_H \]
and define
\[ B : \{u \in \text{dom}(A) ; h(\pi_1 u) = \pi_{-1} u\} \subseteq H \to H, \]
\[ u \mapsto Au. \]

Then \( B \) is \( m \)-accretive.

**Proof.** We first show that \( B \) is accretive. For doing so, let \( u, v \in \text{dom}(B) \) and we compute, using Lemma 2.3

\[
(B(u) - B(v), u - v)_H = (A(u - v), u - v)_H
\]
\[
= \|\pi_1(u - v)\|_H^2 - \|\pi_{-1}(u - v)\|_H^2
\]
\[
= \|\pi_1 u - \pi_1 v\|_H^2 - \|h(\pi_1 u) - h(\pi_1 v)\|_H^2
\]
\[
\geq 0.
\]

Next, we prove that \( B \) is closed. For doing so, let \( (u_n)_n \) in \( \text{dom}(B) \) such that \( u_n \to u \) and \( B(u_n) \to v \) for some \( u, v \in H \). Since \( B \subseteq A \), we infer that \( u \in \text{dom}(A) \), \( v = Au \) and \( u_n \to u \) in \( \text{dom}(A) \). Consequently, \( \pi_1(u_n) \to \pi_1(u) \) and \( \pi_{-1}(u_n) \to \pi_{-1}(u) \) in \( \text{dom}(A) \) and hence, also in \( H \). Thus,
\[
h(\pi_1 u) = \lim_{n \to \infty} h(\pi_1(u_n)) = \lim_{n \to \infty} \pi_{-1}(u_n) = \pi_{-1} u,
\]
and hence, \( u \in \text{dom}(B) \) with \( v = Au = B(u) \), which shows the closedness of \( B \). In order to prove the \( m \)-accretivity of \( B \), it suffices to show that \( \text{ran}(1 + B) \) is dense in \( H \) by Proposition 2.6. For doing so, let \( f \in H \) and define \( v := \Pi_{\ker(1 - A)} f \in \ker(1 - A) \). Then \( f - v \in \ker(1 - A)^\perp_H = \overline{\text{ran}}(1 + A_0) \) and thus, we find a sequence \( (w_n)_n \) in \( \text{dom}(A_0) \) with
\[
w_n + A_0 w_n \to f - v \quad (n \to \infty).
\]

Now, define
\[ u_n := w_n + \frac{1}{2} v + h(\frac{1}{2} v). \]

Then \( u_n \in \text{dom}(A) \) with \( \pi_1 u_n = \frac{1}{2} v, \pi_{-1} u_n = h(\frac{1}{2} v) \) and thus, \( u_n \in \text{dom}(B) \) for each \( n \in \mathbb{N} \). Moreover,
\[
u_n + B(u_n) = w_n + \frac{1}{2} v + h(\frac{1}{2} v) + A_0 w_n + \frac{1}{2} v - h(\frac{1}{2} v)
\]
\[
= w_n + A_0 w_n + v \to f \quad (n \to \infty)
\]
and hence, \( f \in \overline{\text{ran}}(1 + B) \), proving the density of \( \text{ran}(1 + B) \). \( \square \)
With this proposition at hand, we are able to prove Theorem 3.1.

**Proof of Theorem 3.1.** Employing Proposition 3.2, it suffices to prove that for an \( m \)-accretive operator \( B \subseteq A \) we find a function \( h : \ker(1 - A) \to \ker(1 + A) \) with the desired properties. So, let \( B \subseteq A \) be \( m \)-accretive. Using Lemma 2.4, we obtain for each \( u, v \in \text{dom}(B) \)

\[
0 \leq \langle B (u) - B (v), u - v \rangle_H = \langle A(u - v), u - v \rangle_H = \|\pi_1 u - \pi_1 v\|_H^2 - \|\pi_{-1} u - \pi_{-1} v\|_H^2
\]

and thus,

\[
\forall u, v \in \text{dom}(B) : \|\pi_{-1} u - \pi_{-1} v\|_H \leq \|\pi_1 u - \pi_1 v\|_H.
\]

The latter yields, that we can define a mapping

\[
h : \{\pi_1 u ; u \in \text{dom}(B)\} \subseteq \ker(1 - A) \to \ker(1 + A),
\]

\[
\pi_1 u \mapsto \pi_{-1} u
\]

which is well-defined and contractive. Next, we show that \( \{\pi_1 u ; u \in \text{dom}(B)\} = \ker(1 - A) \). Indeed, by Lemma 2.4 we have that

\[
\{\pi_1 u ; u \in \text{dom}(B)\} = \{\frac{1}{2} P_{\ker(1 - A)} (1 + A) u ; u \in \text{dom}(B)\}
\]

\[
= \{\frac{1}{2} P_{\ker(1 - A)} (1 + B) (u) ; u \in \text{dom}(B)\}
\]

\[
= \text{ran} P_{\ker(1 - A)} = \ker(1 - A),
\]

where we have used that \( 1 + B \) is onto. Thus, \( h : \ker(1 - A) \to \ker(1 + A) \) is a contractive mapping and by definition

\[
\text{dom}(B) \subseteq \{u \in \text{dom}(A) ; h(\pi_1 u) = \pi_{-1} u\}.
\]

Denote now by \( C \) the restriction of \( A \) to the set \( \{u \in \text{dom}(A) ; h(\pi_1 u) = \pi_{-1} u\} \). Then \( C \) is \( m \)-accretive by Proposition 3.2 and clearly, \( B \subseteq C \). By the maximality of \( B \), we infer \( B = C \) and thus, indeed

\[
\text{dom}(B) = \{u \in \text{dom}(A) ; h(\pi_1 u) = \pi_{-1} u\}.
\]

The mapping \( h \) in Theorem 3.1 can be computed in terms of the operator \( B \).

**Proposition 3.3.** Let \( B \subseteq A \) be \( m \)-accretive and \( h : \ker(1 - A) \to \ker(1 + A) \) as in Theorem 3.1. Then

\[
h(v) = \pi_{-1}((1 + B)^{-1}(2v)) = \frac{1}{2} P_{\ker(1 + A)} ((1 - A) (1 + B)^{-1}(2v)) \quad (v \in \ker(1 - A)).
\]

In particular, \( h \) is uniquely determined. Moreover, \( -A_0 \subseteq B \) if and only if \( h(0) = 0 \) and \( B \) is linear if and only if \( h \) is linear.

**Proof.** For \( v \in \ker(1 - A) \) we have that \( u := (1 + B)^{-1}(2v) \in \text{dom}(B) \). Then \( v = \frac{1}{2} (1 + B) (u) \in \ker(1 - A) \) and hence,

\[
v = P_{\ker(1 - A)} \left( \frac{1}{2} (1 + B) (u) \right) = \frac{1}{2} P_{\ker(1 - A)} ((1 + A) u) = \pi_1(u),
\]
where we have used Lemma 2.4. In consequence,
\[ h(v) = h(\pi_1(u)) = \pi_{-1}(u) = \pi_{-1}((1 + B)^{-1}(2v)). \]
The second asserted equality is a consequence of Lemma 2.4. Moreover, \(-A_0 \subseteq B\) if and only if \(\text{dom}(A_0) \subseteq \text{dom}(B)\). Since the elements in \(\text{dom}(A_0)\) are precisely those \(u \in \text{dom}(A)\) with \(\pi_1 u = \pi_{-1} u = 0\), we derive that \(-A_0 \subseteq B\) if and only if \(h(0) = 0\). Finally, the formula for \(h\) shows, that \(h\) is linear if \(B\) is linear. If on the other hand \(h\) is linear, then \(\text{dom}(B) \subseteq \text{dom}(A)\) is a subspace and hence, \(B\) is linear. \(\square\)

4 Two Examples

We illustrate our findings of the previous section by two examples. First, we consider the derivative on a bounded interval and discuss \(m\)-accretive realisation of it and second, we study block-operator matrices, which naturally arise in the study of evolutionary equations and most (if not all) equations of mathematical physics.

4.1 The Derivative

Throughout, let \(a, b \in \mathbb{R}\) with \(a < b\) and consider the operator
\[ \partial_0 : H^1_0([a, b]; \mathbb{R}) \subseteq L^2([a, b]; \mathbb{R}) \to L^2([a, b]; \mathbb{R}), \quad f \mapsto f', \]
where \(H^1_0([a, b]; \mathbb{R})\) denotes the Sobolev-space \(H^1([a, b]; \mathbb{R})\) with vanishing boundary conditions; that is,
\[ H^1_0([a, b]; \mathbb{R}) = \{ f \in L^2([a, b]; \mathbb{R}) : f' \in L^2([a, b]; \mathbb{R}), f(a) = f(b) = 0 \}. \]
Here, \(f'\) denotes the derivative of \(f\) in the distributional sense and the point-evaluations \(f(a)\) and \(f(b)\) are well-defined due to Sobolev’s embedding theorem. We note, that
\[ H^1_0([a, b]; \mathbb{R}) = \overline{C^\infty_c([a, b]; \mathbb{R})}^{H^1([a, b]; \mathbb{R})}, \]
where \(C^\infty_c([a, b])\) denotes the space of arbitrarily differential functions having compact support in the open interval \([a, b]\) and the closure is taken in the Sobolev space \(H^1\), which is given by
\[ H^1([a, b]; \mathbb{R}) := \{ f \in L^2([a, b]; \mathbb{R}) : f' \in L^2([a, b]; \mathbb{R}) \} \]
equipped with the graph inner product of the weak derivative. It is well-known that \(\partial_0\) is skew-symmetric and that its adjoint is given by
\[ \partial_0^*= -\partial, \]
where
\[ \partial : H^1([a, b]; \mathbb{R}) \subseteq L^2([a, b]; \mathbb{R}) \to L^2([a, b]; \mathbb{R}), \quad f \mapsto f'. \]
We begin to compute the spaces \(\ker(1 \pm \partial)\) and the orthogonal projections \(\pi_{\pm1}\).
Lemma 4.1. We have
\[
\ker(1 \pm \partial) = \{ t \mapsto ce^{\mp t} ; c \in \mathbb{R} \}.
\]
Moreover,
\[
\pi_{-1} u = (t \mapsto \frac{u(a)e^{-a} - u(b)e^{-b}}{e^{-2a} - e^{-2b}} e^{-t}),
\]
\[
\pi_{1} u = (t \mapsto \frac{u(b)e^{b} - u(a)e^{a}}{e^{2b} - e^{2a}} e^{t}).
\]

Proof. The asserted equality for the kernels is clear. Let now \( u \in H^1([a,b]; \mathbb{R}) \). Then \( (\pi_{-1} u)(t) = ce^{-t} \) for some \( c \in \mathbb{R} \) and satisfies
\[
\langle t \mapsto e^{-t}, u - \pi_{-1} u \rangle_{H^1([a,b]; \mathbb{R})} = 0.
\]
The latter gives
\[
\int_{a}^{b} e^{-t}(u(t) - u'(t)) \, dt = \int_{a}^{b} e^{-t}ce^{-t} \, dt + \int_{a}^{b} e^{-t}ce^{-t} \, dt = 2c \int_{a}^{b} e^{-2t} \, dt = c(e^{-2a} - e^{-2b}).
\]
The integral on the left hand side gives
\[
\int_{a}^{b} e^{-t}(u(t) - u'(t)) \, dt = \int_{a}^{b} e^{-t}u(t) \, dt - \left( u(b)e^{-b} - u(a)e^{-a} + \int_{a}^{b} e^{-t}u(t) \, dt \right) = u(a)e^{-a} - u(b)e^{-b}
\]
and hence,
\[
c = \frac{u(a)e^{-a} - u(b)e^{-b}}{e^{-2a} - e^{-2b}}.
\]
The formula for \( \pi_{1} u \) follows by analogous arguments.

Since \( \ker(1 - \partial) \) and \( \ker(1 + \partial) \) are both one-dimensional, mappings \( h : \ker(1 - \partial) \to \ker(1 + \partial) \) are induces by mappings \( g : \mathbb{R} \to \mathbb{R} \) via
\[
h(t \mapsto ce^{t}) = (t \mapsto g(c)e^{-t}).
\]

Lemma 4.2. A function \( h : \ker(1 - \partial) \to \ker(1 + \partial) \) satisfies
\[
\forall x, y \in \ker(1 - \partial) : \| h(x) - h(y) \|_{L^2([a,b])} \leq \| x - y \|_{L^2([a,b])},
\]
if and only if
\[
\forall c, d \in \mathbb{R} : \| g(c) - g(d) \| \leq e^{a+b}|c - d|,
\]
where \( h \) and \( g \) are linked via (1).
Proof. The proof is straightforward. For \(x, y \in \ker(1 - \partial)\) we find \(c, d \in \mathbb{R}\) such that \(x = (t \mapsto ce^{t}), y = (t \mapsto de^{t})\). We then compute

\[
\|h(x) - h(y)\|_{L^2([a,b])}^2 = \int_a^b |g(c) - g(d)|^2 e^{-2x} \, dx
\]

and on the other hand

\[
\|x - y\|_{L^2([a,b])}^2 = \int_a^b |c - d|^2 e^{2x} \, dx = |c - d|^2 \frac{1}{2}(e^{2b} - e^{2a})
\]

and thus, the assertion follows with the help of the equality

\[
\frac{e^{2b} - e^{2a}}{e^{-2a} - e^{-2b}} = e^{2(a+b)}.
\]

**Theorem 4.3.** Let \(B \subseteq \partial\). Then \(B\) is \(m\)-accretive if and only if there exists a function \(g : \mathbb{R} \to \mathbb{R}\) with

\[
\forall c, d \in \mathbb{R} : |g(c) - g(d)| \leq e^{a+b}|c - d|
\]

such that

\[
\text{dom}(B) = \{u \in H^1([a,b]; \mathbb{R}) : \frac{u(a)e^{-a} - u(b)e^{-b}}{e^{-2a} - e^{-2b}} = g \left( \frac{u(b)e^{b} - u(a)e^{a}}{e^{2b} - e^{2a}} \right) \}.
\]

**Proof.** The statement is a direct application of Theorem 3.1 in combination with Lemma 4.1 and Lemma 4.2.

**Remark 4.4.** (a) We note that in the symmetric case when \(a = -b\) with \(b > 0\), we have that

\[
e^{a+b} = 1
\]

and thus, \(g\) has to be a contraction.

(b) Note that in the linear case, we can write the boundary condition as

\[
\left( \frac{e^{-b}}{e^{-2a} - e^{-2b}} + g \frac{e^b}{e^{2b} - e^{2a}} \right) u(b) = \left( \frac{e^{-a}}{e^{-2a} - e^{-2b}} + g \frac{e^a}{e^{2b} - e^{2a}} \right) u(a)
\]

for a real number \(g \in \mathbb{R}\) with \(|g| \leq e^{a+b}\). Note that

\[
\frac{e^{-a}}{e^{-2a} - e^{-2b}} + g \frac{e^a}{e^{2b} - e^{2a}} = \frac{e^a}{e^{2b} - e^{2a}} \left( e^{2b} + g \right) \neq 0
\]

for \(|g| \leq e^{a+b} < e^{2b}\) and thus, the boundary condition can be written as

\[
\left( \frac{e^{-b}}{e^{-2a} - e^{-2b}} + g \frac{e^b}{e^{2b} - e^{2a}} \right) \left( \frac{e^{-a}}{e^{-2a} - e^{-2b}} + g \frac{e^a}{e^{2b} - e^{2a}} \right)^{-1} u(b) = u(a).
\]
Note that the term on the left hand side can be rewritten as
\[
\left( e^{b-a} \frac{e^{2a} + g}{e^{2b} + g} \right) u(b) = u(a).
\]
Using that the mapping
\[
[-e^{a+b}, e^{a+b}] \ni g \mapsto e^{b-a} \frac{e^{2a} + g}{e^{2b} + g} \in [-1, 1]
\]
is bijective, we obtain that (2) is equivalent to
\[
cu(b) = u(a)
\]
for some $|c| \leq 1$, which recovers the well-known characterisation of linear boundary conditions for $m$-accretive realisations of the derivative, see e.g. [7, Theorem 7.2.4] or [15, Theorem 2.8]. It is noteworthy that in the nonlinear situation, we cannot expect to formulate all boundary conditions as $u(a) = f(u(b))$ for a suitable $f$.

4.2 Block-Operator Matrices

In this section we inspect the following setting. We assume that $H = H_0 \oplus H_1$ for two Hilbert spaces $H_0, H_1$. Moreover, let $G_0 : \text{dom}(G_0) \subseteq H_0 \to H_1$ and $D_0 : \text{dom}(D_0) \subseteq H_1 \to H_0$ be two closed, densely defined linear operators such that
\[
G := -D_0^* \supseteq G_0 \quad \text{and} \quad D := -G_0^* \supseteq D_0.
\]
We consider the following skew-symmetric operator on $H$
\[
A_0 := \begin{pmatrix}
0 & D_0 \\
G_0 & 0
\end{pmatrix}
\]
with its negative adjoint
\[
A = -A_0^* = \begin{pmatrix}
0 & D \\
G & 0
\end{pmatrix}.
\] (3)

Remark 4.5. The prototype for those operators $G_0$ and $D_0$ are the gradient and the divergence with vanishing boundary values. More precisely, let $\Omega \subseteq \mathbb{R}^n$ open and set
\[
\begin{align*}
\text{grad}_0 : H_0^1(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^n, \quad f \mapsto (\partial_j f)_{j \in \{1, \ldots, n\}}, \\
\text{div}_0 : \text{dom}(\text{div}_0) \subseteq L_2(\Omega)^n &\to L_2(\Omega), \quad \Phi \mapsto \sum_{j=1}^n \partial_j \Phi_j,
\end{align*}
\]
where
\[
\text{dom}(\text{div}_0) = C^\infty_c(\Omega) \cap \text{dom}(\text{div}).
\]
and \( \text{div} := -\text{grad}_0^*, \text{grad} := -\text{div}_0^* \) are the usual distributional divergence and gradient on \( L_2(\Omega) \). In this situation we have

\[
H_0 = L_2(\Omega), \quad H_1 = L_2(\Omega)^n, \quad G_0 = \text{grad}_0, \quad D_0 = \text{div}_0, \quad G = \text{grad}, \quad D = \text{div}
\]

and the resulting block operator matrix is given by

\[
A = \begin{pmatrix}
0 & \text{div} \\
\text{grad} & 0
\end{pmatrix}.
\]

Those operators naturally occur in first-order formulations of classical partial differential equations in mathematical physics, in particular for the wave and heat equation. We emphasise that the same construction works for the rotation, yielding the block operator matrix

\[
A = \begin{pmatrix}
0 & -\text{curl} \\
\text{curl} & 0
\end{pmatrix}
\]

and similar for the gradient and divergence on higher-order tensor fields, allowing to treat Maxwell’s equations, the equation of elasticity and coupled problems thereof. For more details we refer to \cite{11, 13, 16, 14, 21}.

Following \cite{17}, we introduce the following spaces.

**Definition.** For operators \( G \) and \( D \) as above we set

\[
\mathcal{BD}(G) := \{ u \in \text{dom}(DG) ; DGu = u \},
\]

\[
\mathcal{BD}(D) := \{ v \in \text{dom}(GD) ; GDv = v \}.
\]

Then \( \mathcal{BD}(G) = \text{dom}(G_0)^\perp \text{dom}(G) \) and \( \mathcal{BD}(D) = \text{dom}(D_0)^\perp \text{dom}(D) \) and hence, both spaces are closed subspaces of \( \text{dom}(G) \) and \( \text{dom}(D) \), respectively. Moreover, \( G_{BD} : \mathcal{BD}(G) \to \mathcal{BD}(D) \) and \( D_{BD} : \mathcal{BD}(D) \to \mathcal{BD}(G) \) with \( G_{BD}u = Gu \) and \( D_{BD}v = Dv \) are unitary with \( D_{BD}^* = G_{BD} \). For \( u \in \text{dom}(G) \) and \( v \in \text{dom}(D) \) we denote the orthogonal projections on \( \mathcal{BD}(G) \) and \( \mathcal{BD}(D) \) by \( u_{BD} \) and \( v_{BD} \), respectively.

We begin to compute the projections onto \( \ker(1 \pm A) \).

**Lemma 4.6.** Let \( A \) be as in \cite{9}. For \((u, v) \in \text{dom}(A)\) we have

\[
\pi_1 \begin{pmatrix}
u \\
v
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
u + Dv_{BD} \\
G\nu_{BD} + v_{BD}
\end{pmatrix}, \quad \pi_{-1} \begin{pmatrix}
u \\
v
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
u - Dv_{BD} \\
-G\nu_{BD} + v_{BD}
\end{pmatrix}.
\]

**Proof.** Let \((u, v) \in \text{dom}(A)\); that is, \( u \in \text{dom}(G) \) and \( v \in \text{dom}(D) \). Then we can decompose \( u = u_0 + u_{BD} \) and \( v = v_0 + v_{BD} \) where \( u_0 \in \text{dom}(G_0) \) and \( v_0 \in \text{dom}(D_0) \) and \( u_0 \perp u_{BD} \) and \( v_0 \perp v_{BD} \) in \( \text{dom}(G) \) and \( \text{dom}(D) \) respectively. We recall from Lemma \cite{24}

\[
\pi_1 \begin{pmatrix}
u \\
v
\end{pmatrix} = \frac{1}{2} P_{\ker(1-A)} \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \right) \begin{pmatrix} u \\
v
\end{pmatrix}.
\]


We compute for \((x, y) \in \ker(1 - A)\)

\[
\langle u + Dv, x \rangle_{H_0} + \langle v + Gu, y \rangle_{H_1} = \langle u_0, x \rangle_{H_0} + \langle u_{BD}, x \rangle_{H_0} + \langle D_0 v_0, x \rangle_{H_0} + \langle Dv_{BD}, x \rangle_{H_0} + \langle v_0, y \rangle_{H_1} + \langle v_{BD}, y \rangle_{H_1} + \langle G_0 u_0, y \rangle_{H_1} + \langle Gu_{BD}, y \rangle_{H_1} = \langle u_0, x \rangle_{H_0} + \langle u_{BD}, x \rangle_{H_0} - \langle v_0, Gx \rangle_{H_1} + \langle Dv_{BD}, x \rangle_{H_0} + \langle v_0, y \rangle_{H_1} + \langle v_{BD}, y \rangle_{H_1} - \langle u_0, Dy \rangle_{H_0} + \langle Gu_{BD}, y \rangle_{H_1}.
\]

Using \(\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Dy \\ Gx \end{pmatrix}\), we derive \(x = Dy\) and \(y = Gx\) and in particular \(x \in BD(G)\) and \(y \in BD(D)\). The latter gives

\[
\langle u + Dv, x \rangle_{H_0} + \langle v + Gu, y \rangle_{H_1} = \langle u_{BD}, x \rangle_{H_0} + \langle Dv_{BD}, x \rangle_{H_0} + \langle v_{BD}, y \rangle_{H_1} + \langle Gu_{BD}, y \rangle_{H_1} = \langle u_{BD}, x \rangle_{H_0} + \langle Dv_{BD}, Dv \rangle_{H_0} + \langle v_{BD}, y \rangle_{H_1} + \langle Gu_{BD}, Gx \rangle_{H_1} = \langle u_{BD}, x \rangle_{BD(G)} + \langle v_{BD}, y \rangle_{BD(D)}.
\]

Hence,

\[
\langle \begin{pmatrix} 1 + \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_{BD} + Dv_{BD} \\ G_{BD} u_{BD} + v_{BD} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rangle_{H_0 \times H_1} = \langle u_{BD}, x \rangle_{BD(G)} + \langle v_{BD}, y \rangle_{BD(D)} - \langle u_{BD} + Dv_{BD}, x \rangle_{H_0} - \langle Gu_{BD} + v_{BD}, y \rangle_{H_1} = \langle Gu_{BD}, Gx \rangle_{H_1} + \langle Dv_{BD}, Dy \rangle_{H_0} - \langle Dv_{BD}, x \rangle_{H_0} - \langle Gu_{BD}, y \rangle_{H_1} = 0
\]

and using that \(\begin{pmatrix} u_{BD} + Dv_{BD} \\ Gu_{BD} + v_{BD} \end{pmatrix} \in \ker(1 - A)\), we infer

\[
\pi_1 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \pi_{\ker(1 - A)} \left( \begin{pmatrix} 1 + \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} u_{BD} + Dv_{BD} \\ Gu_{BD} + v_{BD} \end{pmatrix}.
\]

The second formula follows by replacing \(G\) and \(D\) by \(-G\) and \(-D\), respectively. \(\square\)

**Corollary 4.7.** Let \(A\) be as in \(\footnote{3}\). Then each mapping \(h : \ker(1 - A) \to \ker(1 + A)\) is uniquely determined by a mapping \(f : BD(G) \to BD(D)\) via

\[
h(u, v) = (f(u), -Gf(u)) \quad ((u, v) \in \ker(1 - A)).
\]

Moreover,

\[
|h|_{\text{Lip}} = |f|_{\text{Lip}},
\]

where the Lipschitz-seminorm of \(h\) is computed in \(H = H_0 \times H_1\) and the Lipschitz-seminorm of \(f\) is computed in \(BD(G)\).
Proof. If \( h(u, v) = (h_1(u, v), h_2(u, v)) \) is given, then define \( f(u) := h_1(u, Gu) \) for each \( u \in BD(G) \). Note that \( f \) is well-defined, since \((u, Gu) \in \ker(1 - A)\) for each \( u \in BD(G) \). Moreover,

\[
h(u, v) = h(u, Gu) = (f(u), -Gf(u)) \quad ((u, v) \in \ker(1 - A)),
\]

since \( h \) attains values in \( \ker(1 + A) \) and thus, the second coordinate of \( h(u, v) \) is given by \(-Gh_1(u, v)\) according to Lemma 4.6. Similarly, if \( f \) is given, we set \( h : \ker(1 - A) \to \ker(1 + A) \) by \( h(u, v) := (f(u), -Gf(u)) \), which is well-defined by Lemma 4.6. Finally, we observe that for \((u, v) \in \ker(1 \pm A)\) we have

\[
\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{H_0 \times H_1}^2 = \|u\|_{H_0}^2 + \|v\|_{H_1}^2 = \|u\|_{H_0}^2 + \|Gu\|_{H_1}^2 = \|u\|_{BD(G)}^2,
\]

from which we derive the last assertion. \( \Box \)

Next, we provide another characterisation of \( m \)-accretive relations on a Hilbert space.

**Lemma 4.8.** Let \( M \subseteq H \times H \) for some Hilbert space \( H \). Then \( M \) is \( m \)-accretive, if and only if there exists \( f : H \to H \) Lipschitz-continuous with \( |f|_{\text{Lip}} \leq 1 \) such that

\[
M = 2(1 + f)^{-1} - 1.
\]

In particular

\[
v = f(u) \Leftrightarrow (u + v, u - v) \in M \quad (u, v \in H).
\]

**Proof.** Assume first that \( M \) is \( m \)-accretive and set

\[
f := \left( \frac{1}{2}(M + 1) \right)^{-1} - 1.
\]

Observe that for \( u, v \in H \) we have that

\[
(u, v) \in f \Leftrightarrow (u + v, u - v) \in M.
\]

Thus, for \((u, v), (x, y) \in f\) we estimate

\[
0 \leq \langle (u + v) - (x + y), (u - v) - (x - y) \rangle \\
= \langle (u - x) + (v - y), (u - x) - (v - y) \rangle \\
= \|u - x\|^2 - \|v - y\|^2
\]

and hence,

\[
\|v - y\| \leq \|u - x\|.
\]

This proves that \( f \) is a Lipschitz-continuous mapping with \( |f|_{\text{Lip}} \leq 1 \). To prove that \( \text{dom}(f) = H \), we observe that \( u \in \text{dom}(f) \) if and only if \((u + f(u), u - f(u)) \in M\). The later is equivalent to

\[
(u + f(u), 2u) \in 1 + M
\]
and since $1 + M$ is onto, the assertion follows.

If conversely $f$ is given, we set

$$M := 2(1 + f)^{-1} - 1.$$ 

Then for $u, v \in H$ we have

$$(u, v) \in M \iff f \left( \frac{1}{2}(u + v) \right) = \frac{1}{2}(u - v).$$

Hence, for $(u, v), (x, y) \in M$ we estimate

$$
(u - x, v - y) \\
= \left( \frac{1}{2}(u + v) + \frac{1}{2}(u - v) - \frac{1}{2}(x + y) \right) - \frac{1}{2}(x - y) \\
= \frac{1}{4} \text{Re} \langle ((u + v) - (x + y)) + ((u - v) - (x - y)) \rangle \\
= \frac{1}{4} \| (u + v) - (x + y) \|^2 - \frac{1}{4} \| (u - v) - (x - y) \|^2 \\
\leq 0,
$$

and hence, $M$ is accretive. For showing $m$-accretivity, let $v \in H$. We set $u := f(\frac{1}{2}v) + \frac{1}{2}v$. Then $(u, v) \in 1 + M$ since,

$$f \left( \frac{1}{2}(u + (v - u)) \right) = f \left( \frac{1}{2}v \right) = u - \frac{1}{2}v = \frac{1}{2}(u - (v - u)),
$$

which shows $(u, v - u) \in M$.

\begin{proof}
Theorem 4.9. Let $C \subseteq A$ with $A$ as in (3) Then the following statements are equivalent:

(i) $C$ is $m$-accretive,

(ii) there exists $h : \ker(1 - A) \to \ker(1 + A)$ with $|h|_{\text{Lip}} \leq 1$ such that

$$\text{dom}(C) = \{ x \in \text{dom}(A) ; h(\pi_1 x) = \pi_{-1} x \}.$$

(iii) there exists $f : BD(G) \to BD(G)$ with $|f|_{\text{Lip}} \leq 1$ such that

$$\text{dom}(C) = \{ (u, v) \in \text{dom}(A) ; f(\frac{1}{2}(u_{BD} + Dv_{BD})) = \frac{1}{2}(u_{BD} - Dv_{BD}) \}.$$

(iv) there exists an $m$-accretive relation $M \subseteq BD(G) \times BD(G)$ such that

$$\text{dom}(C) = \{ (u, v) \in \text{dom}(A) ; (u_{BD}, Dv_{BD}) \in M \}.$$

In either case we have

$$f(u) = h_1((u, Gu)) \quad (u \in BD(G)), \quad M = 2(1 + f)^{-1} - 1.$$ 
\end{proof}
Proof. The equivalence of (i) and (ii) follows from Theorem 3.1. The equivalence of (ii) and (iii) follows from Corollary 4.7. For the equivalence of (iii) and (iv), we observe that
\[(u_{BD}, Dv_{BD}) \in M \iff f(1/2(u_{BD} + Dv_{BD})) = 1/2(u_{BD} - Dv_{BD})\]
and thus, the statement follows from Lemma 4.8.

Remark 4.10. Note that the equivalence of (i) and (iv) in the latter theorem is the main result of [22].

We conclude this section with the study of linear $m$-accretive relations, which by Theorem 4.9 correspond to linear $m$-accretive restrictions of the operator $A$ given in (3). We start with the following simple observation.

Lemma 4.11. Let $X, Y$ be Hilbert spaces and $M \subseteq X \times Y$ be a closed subspace. Then there exists $S \in \mathcal{L}(X; X \times Y)$ and $T \in \mathcal{L}(Y; X \times Y)$ such that
\[(u, v) \in M \iff Su = T v \quad (u \in X, v \in Y).\]

Proof. Since $M$ is a closed subspace of the Hilbert space $X \times Y$, we have $(u, v) \in M$ if and only if $P_M(u, v) = 0$ for each $u \in X, v \in Y$. Hence, we may set $S := P_M(\iota_X)$ and $T := -P_M(\iota_Y)$, where $\iota_X$ and $\iota_Y$ denote the canonical embeddings of $X$ and $Y$ in $X \times Y$, respectively. Then
\[Su = T v \iff P_M(\iota_X u + \iota_Y v) = 0 \iff (u, v) \in M.\]

Since linear $m$-accretive relations $M$ on a Hilbert space $X$ are closed (see e.g. Proposition 2.6), $m$-accretive relations can be described by operator equalities as in the lemma above. The natural question which arises is: when is a relation, which is determined by two operators as above, $m$-accretive?

Proposition 4.12. Let $X$ be a Hilbert space and $Y$ a normed space. Let $S, T \in \mathcal{L}(X, Y)$ and consider the relation
\[M := \{(u, v) \in X \times X ; Su = T v\}.\]

Then $M$ is $m$-accretive, if and only if the following three properties are satisfied
\begin{itemize}
  \item $\text{ran}(T - S) \subseteq \text{ran}(T + S)$,
  \item $T + S$ is one-to-one,
  \item $\| (S + T)^{-1}(T - S) \| \leq 1$.
\end{itemize}

Proof. By Lemma 4.8 we know that $M$ is maximal monotone if and only if $f := (1/2)(M + 1))^{-1} - 1$ defines a Lipschitz-continuous mapping on $X$ with $|f|_{\text{Lip}} \leq 1$. Since $M$ is linear, we need to show that $f \in L(X)$ with $\|f\| \leq 1$. We recall from Lemma 4.8 that
\[(u, v) \in f \iff (u + v, u - v) \in M \iff S(u + v) = T(u - v) \iff (S + T)v = (T - S)u.\]

Then $\text{dom}(f) = X$ is equivalent to $\text{ran}(T - S) \subseteq \text{ran}(S + T)$. Moreover, $f$ is a mapping if and only if $(0, v) \in f$ implies $v = 0$. The latter is equivalent to the injectivity of $S + T$. Finally, the condition $\|f\| \leq 1$ is equivalent to $\|(S + T)^{-1}(T - S)\| \leq 1$, which shows the assertion.
If we apply the latter proposition to linear restrictions of the block operator $A$ in (3) we obtain the following characterisation.

**Corollary 4.13.** Let $C \subseteq A$ be a linear restriction of the block operator $A$ given in (3). Then the following statements are equivalent:

(i) $C$ is $m$-accretive,

(ii) There exists a normed space $Y$ and operators $S, T \in L(\mathcal{BD}(G), Y)$ with

- $\text{ran}(T - S) \subseteq \text{ran}(T + S)$,
- $T + S$ is one-to-one,
- $\|(S + T)^{-1}(T - S)\| \leq 1$,

such that

$\text{dom}(C) = \{(u, v) \in \text{dom}(A) ; Su_{BD} = T Du_{BD}\}$.

5 Application to the Wave Equation with Impedance Boundary Conditions

We consider the wave equation on a open set $\Omega \subseteq \mathbb{R}^n$ with impedance type conditions. Employing the framework of evolutionary equations, we may write the equation as a system of the form

$$(\partial_t M(\partial_t) + A) U = F,$$

where $\partial_t$ stands for the temporal derivative and $M$ is a material law operator, incorporating all physical coefficients. The operator $A$ is a suitable restriction of the block operator matrix

$$\begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix},$$

where grad and div are defined as in the previous section, that is,

- $\text{grad} : H^1(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)^n, \ f \mapsto (\partial_j f)_{j \in \{1, \ldots, n\}},$
- $\text{div} : H(\text{div}, \Omega) \subseteq L^2(\Omega)^n \to L^2(\Omega), \ \Phi \mapsto \sum_{i=1}^n \partial_i \Phi_i,$

where

$$H(\text{div}, \Omega) := \{\Phi \in L^2(\Omega)^n ; \sum_{i=1}^n \partial_i \Phi_i \in L^2(\Omega)\}$$

and the derivatives are computed in the distributional sense. We recall that in case of bounded set $\Omega$ with Lipschitz-boundary $\Gamma$, we may define the trace operator

$$\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma), \ f \mapsto f|_{\Gamma},$$
which is a surjective bounded operator, see e.g. [10, Chapter 1, Theorem 1.2]. Moreover, \( \ker(\gamma_0) = H^1_0(\Omega) \) and so, we may restrict \( \gamma_0 \) to the space \( \mathcal{BD}(\text{grad}) := H^1_0(\Omega)^⊥ \) and obtain an isomorphism between the spaces \( \mathcal{BD}(\text{grad}) \) and \( H^{1/2}(\Gamma) \). Hence, we can equip \( H^{1/2}(\Gamma) \) with the equivalent norm

\[
\| \gamma_0 f \|_{H^{1/2}(\Gamma)} := \| f \|_{H^1(\Omega)} \quad (f \in \mathcal{BD}(\text{grad}))
\]

such that \( \gamma_0 : \mathcal{BD}(\text{grad}) \to H^{1/2}(\Gamma) \) becomes unitary. In a similar way, we may define a trace operator on \( H(\text{div}, \Omega) \) by setting

\[
\gamma_n : H(\text{div}, \Omega) \to H^{-1/2}(\Gamma),
\]

where \( H^{-1/2}(\Gamma) \) denotes the dual space of \( H^{1/2}(\Gamma) \) and

\[
\langle \gamma_n \Phi, \gamma_0 f \rangle := \langle \Phi, \text{grad} f \rangle_{L^2(\Omega)^n} + \langle \text{div} \Phi, f \rangle_{L^2(\Omega)} \quad (\Phi \in H(\text{div}, \Omega), f \in H^1(\Omega)).
\]

(4)

For \( f \in C^1(\Omega) \cap H^1(\Omega) \) and \( \Phi \in C^1(\Omega)^n \cap H(\text{div}, \Omega) \) one obtains \( \gamma_0 f = f|_\Gamma \) and \( \gamma_n \Phi = \Phi|_G \cdot n \), where \( n \) denotes the unit outward directed normal on \( \Gamma \). The definition of \( \gamma_n \) gives

\[
\ker(\gamma_n) = \{ \Phi \in H(\text{div}, \Omega) : \forall f \in H^1(\Omega) : \langle \Phi, \text{grad} f \rangle = -\langle \text{div} \Phi, f \rangle_{L^2(\Omega)} \} = \text{dom}(\text{div}0),
\]

where \( \text{div}0 \) is the closure of of \( \text{div}|_{C^\infty(\Omega)^n} \). Hence, we may restrict \( \gamma_n \) to \( \text{dom}(\text{div}0)^⊥ \cap \mathcal{BD}(\text{div}) = \mathcal{BD}(\text{div}) \) and obtain an isomorphism from \( \mathcal{BD}(\text{div}) \) to \( H^{-1/2}(\Gamma) \). Then (4) reads as

\[
\langle \gamma_n \Phi, \gamma_0 f \rangle = \langle \Phi, \text{grad} f \rangle_{L^2(\Omega)^n} + \langle \text{div} \Phi, f \rangle_{L^2(\Omega)} = \langle \Phi, \text{grad} f \rangle_{L^2(\Omega)^n} + \langle \Phi, \text{div} \text{grad} f \rangle_{L^2(\Omega)} = \langle \Phi, \text{grad} \mathcal{BD} f \rangle_{\mathcal{BD}(\text{div})} = \langle \text{div} \mathcal{BD} \Phi, f \rangle_{\mathcal{BD}(\text{grad})} \quad (f \in \mathcal{BD}(\text{grad}), \Phi \in \mathcal{BD}(\text{div})).
\]

In particular, keeping in mind that we have renormed \( H^{1/2}(\Gamma) \), we have

\[
\| \gamma_n \Phi \| = \| \text{div} \mathcal{BD} \Phi \|_{\mathcal{BD}(\text{grad})} = \| \Phi \|_{\mathcal{BD}(\text{div})} \quad (\Phi \in \mathcal{BD}(\text{div}))
\]

and hence, \( \gamma_n \) is unitary from \( \mathcal{BD}(\text{div}) \) to \( H^{-1/2}(\Gamma) \). In order to formulate boundary conditions of impedance type for the operator \( \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \) one needs to compare the traces \( \gamma_0 f \) and \( \gamma_n \Phi \) for \( f \in \mathcal{BD}(\text{grad}) \) and \( \Phi \in \mathcal{BD}(\text{div}) \). We consider the following type of boundary conditions

\[
Sg = Th, \quad (g \in H^{1/2}(\Gamma), h \in H^{-1/2}(\Gamma))
\]

where \( S \in L(H^{1/2}(\Gamma), Y) \) and \( T \in L(H^{-1/2}(\Gamma), Y) \) for some normed space \( Y \). Then a direct consequence of Corollary 4.13 is the following result.

**Proposition 5.1.** Let \( \Omega \) be a bounded Lipschitz domain. Let \( S \in L(H^{1/2}(\Gamma), X) \) and \( T \in L(H^{-1/2}(\Gamma), X) \) for some normed space \( X \). Consider \( C \subseteq \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \) with

\[
\text{dom}(C) := \{ (f, \Phi) \in H^1(\Omega) \times H(\text{div}, \Omega) : S\gamma_0 f = T\gamma_n \Phi \}.
\]

Then \( C \) is \( m \)-accretive if and only if
• $S\gamma_0 + T\gamma_n \text{grad}_{BD}$ is one-to-one on $\text{bd}(\text{grad})$,
• $\text{ran}(T\gamma_n \text{grad}_{BD} - S\gamma_0) \subseteq \text{ran}(T\gamma_n \text{grad}_{BD} + S\gamma_0)$,
• $\|(T\gamma_n \text{grad}_{BD} + S\gamma_0)^{-1}(T\gamma_n \text{grad}_{BD} - S\gamma_0)\| \leq 1$.

**Proof.** Note that since $\text{ker}(\gamma_0) = H^1_0(\Omega)$ and $\text{ker}(\gamma_n) = \text{dom}(\text{div}_0)$ we have

$$\text{dom}(C) := \{ (f, \Phi) \in H^1(\Omega) \times H(\text{div}, \Omega) : S\gamma_0 f_{BD} = T\gamma_n \Phi_{BD} \} = \{ (f, \Phi) \in H^1(\Omega) \times H(\text{div}, \Omega) : S\gamma_0 f_{BD} = T\gamma_n \text{div}_{BD} \text{grad}_{BD} \Phi_{BD} \}$$

and now the assertion follows from Corollary 4.13 applied to $G = \text{grad}$ and $D = \text{div}$.

Typically, the trace operators are compared via the pivot space $L^2(\Gamma)$. Indeed, one can show that the embedding

$$\iota : H^{1/2}(\Gamma) \to L^2(\Gamma), \quad g \mapsto g$$

is continuous, injective and has dense range, see [10, Chapter 2, Theorem 4.9]. Consequently, the dual operator

$$\iota' : L^2(\Gamma) \to H^{-1/2}(\Gamma)$$

is also continuous injective and has dense range (here we identified $L^2(\Gamma)'$ with $L^2(\Gamma)$ by the usual Riesz isomorphism). Impedance type boundary conditions then typically take the form

$$K\iota\gamma_0 f = \gamma_n \Phi,$$

for an operator $K \in L(L^2(\Gamma))$.

**Lemma 5.2.** Let $\Omega$ be a bounded Lipschitz domain. Let $f \in H^1(\Omega)$, and $\Phi \in H(\text{div}, \Omega)$ and $K \in L(L^2(\Gamma))$. Then $K\iota\gamma_0 f = \gamma_n \Phi$ if and only if

$$\text{div}_{BD} \Phi_{BD} = \kappa^* K\kappa f_{BD},$$

where $\kappa := \iota\gamma_0 : \text{bd}(\text{grad}) \to L^2(\Gamma)$.

**Proof.** Assume first that $K\gamma_0 f = \gamma_n \Phi$; that is, for each $g \in H^{1/2}(\Gamma)$ we have

$$\langle \gamma_n \Phi, g \rangle = \langle K\gamma_0 f, \iota g \rangle_{L^2(\Gamma)}.$$  

The we compute for each $v \in \text{bd}(\text{grad})$

$$\langle \text{div}_{BD} \Phi_{BD}, v \rangle_{\text{bd}(\text{grad})} = \langle \gamma_n \Phi, \gamma_0 v \rangle = \langle K\gamma_0 f, \iota \gamma_0 v \rangle_{L^2(\Gamma)} = \langle \kappa^* K\kappa f_{BD}, v \rangle_{\text{bd}(\text{grad})},$$

where we have used $\gamma_0 f = f_{BD}$ and $\gamma_n \Phi = \Phi_{BD}$. The latter gives $\text{div}_{BD} \Phi_{BD} = \kappa^* K\kappa f_{BD}$. Assume now conversely, that $\text{div}_{BD} \Phi_{BD} = \kappa^* K\kappa f_{BD}$ and let $g \in H^{1/2}(\Gamma)$. Then we find $v \in \text{bd}(\text{grad})$ such that $g = \gamma_0 v$. Hence,

$$\langle \gamma_n \Phi, g \rangle = \langle \gamma_n \Phi, \gamma_0 v \rangle = \langle \gamma_n \Phi, \gamma_0 f \rangle = \langle K\gamma_0 f, \iota\gamma_0 v \rangle_{L^2(\Gamma)}.$$
\[
\begin{align*}
&= (\text{div}_{BD} \Phi_{BD}, v)_{BD(\text{grad)}} \\
&= (\kappa^* K\kappa f_{BD}, v)_{BD(\text{grad})} \\
&= (K\kappa f_{BD}, \kappa v)_{L^2(\Gamma)} \\
&= (K\gamma_0 f, \gamma_0)_{L^2(\Gamma)}
\end{align*}
\]
and hence, \( \gamma_0 \Phi = K\gamma_0 f. \)

**Proposition 5.3.** Let \( \Omega \) be a bounded Lipschitz domain. Let \( K \in L(L^2(\Gamma)) \). Consider \( C \subseteq \left( \begin{array}{cc} 0 & \text{div} \\ \text{grad} & 0 \end{array} \right) \) with

\[
\text{dom}(C) := \{ (f, \Phi) \in H^1(\Omega) \times H(\text{div}, \Omega) ; K\gamma_0 f = \gamma_0 \Phi \}.
\]

Then \( C \) is \( m \)-accretive if and only if \( K \) is accretive.

**Proof.** By Lemma 5.2 we have

\[
(f, \Phi) \in \text{dom}(C) \iff \text{div}_{BD} \Phi_{BD} = \kappa^* K\kappa f_{BD}.
\]

Moreover, by Theorem 4.9(iv) \( C \) is \( m \)-accretive if and only if \( \kappa^* K\kappa \) is \( m \)-accretive, which is equivalent to the accretivity of \( \kappa^* K\kappa \), since it is a bounded operator. Finally, \( \kappa^* K\kappa \) is accretive, if and only if \( K \) is accretive, since \( \kappa \) has a dense range.

**Remark 5.4.** As the previous example shows, classical trace spaces can be incorporated within the framework of the corresponding \( BD \)-spaces. Note however, that \( BD \)-spaces can also be used, when no traces are available, for instance if the boundary of the underlying domain is not smooth enough. Thus, \( BD \)-spaces provide a unified framework to formulate abstract boundary conditions without any restrictions on the underlying domain.

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