How to hunt wild constants

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Abstract

There are now several comprehensive web applications, stand-alone computer programs and computer algebra functions that, given a floating point number such as 6.518670730718491, can return concise nonfloat constants such as $3 \arctan 2 + \ln 9 + 1$ that closely approximate the float. Examples include AskConstants, Inverse Symbolic Calculator, the Maple identify function, MESearch, OEIS, RIES, and WolframAlpha. Usefully often such a result is the exact limit as the float is computed with increasing precision. Therefore these program results are candidates for proving an exact result that you could not derive or conjecture without the program. Moreover, candidates that are not the exact limit can be provable bounds, or convey qualitative insight, or suggest series that they truncate, or provide sufficiently close efficient approximations for subsequent computation. This article describes some of these programs, how they work, and how best to use each of them. Almost everyone who uses or should use mathematical software can benefit from acquaintance with several such programs, because these programs differ in the sets of constants that they can return.

1 Introduction

“...you are in a state of constant learning.”

– Bruce Lee

This article is about numerical mathematical constants that can be computed approximately, rather than about dimensionless or dimensioned physical constants. This article is a more detailed version of a presentation at an Applications of Computer Algebra conference [19].

For real-world problems, we often cannot directly derive exact closed-form results even with the help of computer algebra, but we more often can compute approximate floating-point results – hereinafter called floats. For some such cases there is an exact closed-form expression that the float approximates and that expression is simple enough so that we would like to know it, but we do not know how to derive it or to guess it as a prerequisite to a proof. If we had one or a few plausible concise nonfloat candidates that agree sufficiently closely with the float, then we could concentrate our efforts on attempting to prove that one of these candidates is the exact result.
This article describes text and software tools that provide such candidates. Table 1 lists several such tools in order of presentation, grouped by type. Figure 1 plots the initial publication dates of these tools on a time line. All of the web-based and downloadable tools are free, but some of them require Java, a C compiler, Maple or Mathematica.¹

Table 1: Tables in books, web apps, standalone apps, CAS functions, and CAS apps

| Type         | Name          | Sec. | Prerequisite          | http://www. or https://www. |
|--------------|---------------|------|-----------------------|----------------------------|
| Book table   | Robinson & Potter | 2.1  | Library or buy        | escholarship.org/uc/item/2t95c0bp |
|              | Borwein & Borwein | 2.2  |                       | springer.com/gp/book/9781461585121 |
|              | Steven Finch   | 2.2  |                       | sites.oxy.edu/lengyel/originals/0521818052ws.pdf |
| .pdf table   | Shamo’s Catalog | 3    |                       | [18]                        |
| Web search   | Google, etc.   | 4    | Smartphone or computer | Google.com etc.           |
|              | OEIS           | 5.1  |                       | oeis.org                   |
|              | WolframAlpha   | 5.2  |                       | wolframalpha.com           |
|              | ISC            | 5.3  |                       | wayback.cecm.sfu.ca/projects/ISC/ISCmain.html |
|              | ISC+           | 5.4  |                       | isc.carma.newcastle.edu.au/ |
| Standalone   | MESearch       | 6.1  | Java                  | xuru.org/mesearch/MESearch.asp |
|              | RIES           | 6.2  | C compiler            | maplesoft.com/support/help/Maple/view.aspx |
| In CAS       | identify       | 7.1  | Maple                 | maplesoft.com/support/help/Maple/view.aspx |
|              | identify,findpoly | 7.2  | SymPy                 | docs.sympy.org/0.7.1/modules/mpmath/ |
| CAS apps     | AskConstants   | 8.1  | Mathematica           | AskConstants.org          |
|              | Plouffe’s Inverter | 8.2  | Maple                 | plouffe.fr/Simon%20Plouffe.htm |
| PSLQ         | Custom IR model | 9    | Fortran C++ or CAS    | [1][2]                     |

¹But Mathematica is free and pre-installed on some Raspberry Pi computers that cost only a few dollars.
Float inputs for these tools often arise from computations such as numerical integration, infinite series, iterative equation solving, approximate optimization, etc. The purposes of this article are to explain how best to use these tools, to explain how they work, and to explain how to decide if proposed candidates are promising or else probably impostors.

Section 8 discusses input magnitude limitations and Section 9 discusses some common causes of impostor results, with conclusions in Section 10.

2 Text tables

2.1 Mathematical Constants table by Robinson & Potter

Here is an excerpt from a hand-typed table of about 3000 constants by Robinson and Potter [15]:

| Float | Expression |
|-------|------------|
| 4.2275535337626540809 | $-\psi(1/4) = \gamma + 3\ln 2 + \pi/2$ |
| 0.22755095776884999385 | $4/(\pi^2e^\gamma)$ |
| 0.22756340548747214332 | root of $7xe^x = 2$ |

Notice that:

- Inverse to more common tables of constants, the inputs in the left column are floats and the expressions in the right column are corresponding exact constants.
• The digamma function of 1/4 was negated to make the float positive. Most such tables do this **sign aliasing**, because it is easy to discard the sign of the float, then negate the corresponding nonfloat result. This doubles the potential coverage of the table.

• The inputs are sorted by the **fractional parts** of the absolute values of the input floats rather than by those entire input floats. For a given float $\tilde{x}$, you do a manual search for the fractional part of $|\tilde{x}|$, then decide whether or not the discrepancy with the fractional part of either bracketing entry or of entries slightly further away is small enough to justify further consideration. This **fractional-part aliasing** makes the table applicable to many more examples and is easily inverted mentally to construct the true candidate. For example, we can also easily guess that a numerical result 5.22755 35332 might approach $1 - \psi(1/4)$ as the precision increases.

• The last entry in the above table is an implicit result. Since the popularization of the Lambert $W$ function [5], this result can now be expressed explicitly as $W_0(2/7)$, but the table contains approximate solutions to other equations that cannot yet be expressed explicitly, together with approximate definite integrals, infinite series, and infinite products having no known closed form.

To view a photocopy of the entire table, visit

[https://escholarship.org/uc/item/2t95c0bp](https://escholarship.org/uc/item/2t95c0bp)

**Definition.** *Published constants* are publicly accessible closed-form and/or approximate float constants – either printed or on the web.

**Definition.** *Named constants* are constants having a widely-accepted name, such as Catalan’s constant or the *twin prime* constant. Named constants include those with an OEIS name as described in subsection 5.1.

**Definition.** *Tabulated constants* are formed by systematically applying sets of functions to systematic sets of arguments. Printed function tables for approximate computation usually have equally-spaced arguments that are terminating decimal fractions such as 1.001, 1.002, 1.003, etc. In contrast, to return nonfloat results such as $\arctan(2/3)$, the curated and computer-generated tables described in this article often instead use the set of all reduced fraction arguments whose numerator and denominator magnitudes do not exceed some given integer – perhaps also multiplied by common irrational constants such as $\pi$, $\sqrt{2}$, etc.

**Remark.** The table by Robinson and Potter contains all three of these types.

**Definition.** *Wild constants* are computed float constants for which you do not know an exact closed form for the limit as the precision of the float approaches infinity and you do not know a publication that discusses a float approximation.
2.2 Other printed text tables

- Borwein and Borwein authored a similar printed table of about 100,000 constants [3]. The table contains mostly computer-generated tabulated constants and some dimensionless physical constants.

- The award-winning book by Steven Finch [9] has not only a table of about 10,000 well chosen non-tabulated constants, but it and Volume II [10] contain short essays about those constants, with references to the literature about them.

3 .pdf and .html tables

Michael Shamos posted a .pdf file of a table of about 10,000 constants at http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.366.9997&rep=rep1&type=pdf

Most of these constants are isolated pairs of a float value and a closed form defined by a definite integral, infinite series, or infinite product. Many floats list several corresponding formulas, such as various integrals, series and products that have the same value, which is helpful by suggesting other perhaps more efficient ways to compute more digits.

A web search for “Mathematical constants” can locate other such downloadable tables or .html tables that can be searched directly on the web.

4 Web browsers

General-purpose web browser search engines are helpful for finding candidate nonfloats for particular float inputs.

Here are some input tips for using the Google Search Engine for this purpose:

1. It seems better to use decimal fraction notation than scientific notation.

2. For negative numbers, try both the negative entry and its absolute value.

3. For numbers having magnitude less than 1, try both with and without a leading 0 digit before the decimal point.

4. Some publications have truncated the exact value and some have rounded it, so try both when they differ.

5. Most published constants have 16 or fewer significant digits, because most are computed with 16-digit significands or less, and the author of the published float might have further rounded or truncated the computed result to allow for computational errors or typesetting constraints.

6. Your entry must often have exactly the same digits as a publication – no more and no fewer.

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2 The Microsoft Edge and Apple Safari browsers currently allows a choice of Google, Bing, Yahoo or DuckDuckGo as a search engine.
7. Many relevant publications tend to have exactly 11 digits, including a leading zero before the decimal point if the magnitude is less than 1, because they were taken directly or indirectly from the table in Steven Finch’s book described in subsection 2.2.

8. As you decrease the number of digits down to a few, the number of browser results tends to increase, but so does the number of results that are nearby mathematical constants or are not mathematical constants.

9. Thus one strategy is to use up to the first 16 digits of your decimal float (using a sign if negative and a leading 0 digit for magnitudes < 1), then repeatedly truncate successive trailing digits (and also try rounding where it is different), until you obtain at least one promising result on the first page or two of the results. Then if those results include a name or other helpful clues (such as “Artin’s constant”), branch into a search using that clue. At each digit length you can optionally experiment with omitting a minus sign and or a leading 0 digit.

- If there is more than one plausible result and the associated texts do not indicate equivalence, then repeat the search starting with more digits if you can compute them.

- Somewhat different tips might be appropriate for other search engines, and Google’s search algorithm might evolve to make some of the tips here unnecessary. To learn, experiment with float approximations to a few known mathematical constants that are not too obscure or too well known. For example, try

\[ 0.91596559. \]

5 Free web applications

If you have internet access as you read this, then you might want to try each of the web applications in this section as you read the subsections. With good eyesight, most of them are usable even with most smart phones.

To become acquainted with them, using simple then less simple nonfloat constants from books and articles, try entering their floating-point approximations of varying precisions. Also try random floats having differing numbers of significant digits. As the number of significant digits increases, random floats are decreasingly likely to be the truncated or rounded value of any nonfloat expression simple enough for any of these programs to propose in a reasonable amount of time. An application is generous if it often suggests expressions for such random-number entries, versus parsimonious if it rarely does. It is alright for an application to be generous, but it is important for you to realize it. Generosity can be helpful for determining bounds or efficient sufficiently accurate approximations or for suggesting the dominant terms of a longer exact result.

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3Beware that, for example, rounding 1.546 to 1.55, then subsequently rounding that to 1.6 is different than rounding 1.546 directly to 1.5.
Every float is exactly representable as an easily computed rational number, but floats cannot represent all rational numbers and floats are often approximations to irrational numbers. If your float approaches a limit as the working precision of the calculation approaches infinity and that limit is expressible as a nonfloat expression, then that limit is what you want. Perfect agreement with a mere approximation prevents that desired result.

5.1 The On-line Encyclopedia of Integer Sequences

One of the main purposes of the OEIS site at https://oeis.org/ is to identify integer sequences such as proposing that the truncated sequence 8, 16, 32, 64, . . . might be a subsequence of the sequence whose nth element is $2^n$. Another goal of this database is to provide accurate and comprehensive information about these sequences.

The successive digits of a decimal fraction can be regarded as an integer sequence, and an under-advertised capability of OEIS is to propose nonfloat constant expressions that closely approximate your float. OEIS has sequences of up through about 100 digits or more for about 10,000 constants regarded as important by the many contributors who posted them. To attempt identifying nonfloat candidates for your float, you can enter a comma-separated list of some nonzero leading digits of the absolute value of its decimal fraction value, such as 3, 1, 4, 1, 5, 9, 2, 6. OEIS then attempts to match your entered successive digits with any subsequence of its sequences. It then lists information about each of its integer sequences that contain your sequence as a subsequence. However, it is better to enter your float digits as a positive decimal fraction with a decimal point, such as 3.1415926. (Do not use scientific notation.)

Any matched table entries can contain hyperlinks to references and contain computer-algebra program fragments to compute the constant to any number of digits efficiently. There is also usually a Keyword link labeled “cons” that leads to the precomputed value of the constant up to a median of about 100 significant digits. The references sometimes contain values with thousands of digits.

This exact-match radix search requires you to truncate trailing digits of the absolute value of your float rather than round. However, the hardware or software that computed your float is approximate and almost certainly used rounding rather than truncation. Therefore you should discard enough of your trailing digits to be reasonably confident that your submission matches exactly the leading digits at OEIS.

If your computed float ends with a sequence of digits 9, then also try what you get by adding 1 to the last digit. If your computed float ends with a sequence of digits 0, then also try what you get by subtracting 1 from the last digit, because those small relative changes to your approximate float can dramatically alter the digit sequence.

If you enter too few digits then you will obtain numerous matches, most or all of which are irrelevant. However, with about $10^5$ numbers in the data base, entering 8 or more significant digits is usually sufficient to achieve at most one match, and almost all of the OEIS sequences for floats contain at least that many digits. Therefore you can usually truncate to about 8 significant digits without much chance of false matches or of entering more digits than are in a table entry. However, even if you obtain only one match, you should always subtract all of the digits stored in the table from your float to

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determine how many of yours are correct up to the number of digits stored in the table.

Moreover, if you do obtain more than one match, then it is wise to do this for all of them or to enter more correctly truncated digits if possible to reduce their count to a comfortably inspectable number.

Since OEIS ignores the position of the decimal point, entering 31.41592 also returns \( \pi \) rather than \( 10\pi \). It is your responsibility to notice this and multiply the proposed exact constant by the appropriate power of 10. This base-ten **significand aliasing** increases the number of floats that can be matched.

Since OEIS can match any subsequence of their digit sequences, you can also exploit fractional part aliasing by entering only the fractional part of the absolute value of your float. For example, if your truncated wild float is 9.141592654, then entering it produces no result, whereas entering 0.141592654 produces \( \pi \) and gives its float value, enabling you to recognize that your wild float is probably \( 6 + \pi \). Thus it is worth trying all of your float magnitude, then only its integer part if different.

Beware that fractional part aliasing suffers catastrophic cancellation when the absolute value of the fractional part is small compared to the absolute value of the float, whereas significand aliasing suffers no loss of precision. Therefore it is best if 8 or more significant digits remain after omitting the integer part and truncating dubious trailing digits.

Other than you exploiting these types of aliasing, OEIS makes no effort to identify more general transformations of its constants, such as \( 2/3 \) of a named constant or \( 2/3 \) plus a named constant.

All OEIS integer sequences and float digit sequences are assigned an OEIS name consisting of the letter “A” followed by six digits. Although such names are difficult to remember and do not acknowledge the discoverers, they are unique and serve as a *de facto* standard – particularly when there is no other widely recognized name. In contrast, traditional names can have various spellings, subsets, and orderings of all the discoverers, as well as values differing by a factor of 2, etc, on account of one author using a radius where another uses a diameter, for example.

A complimentary way to use OEIS for named constants is to enter a name, such as *grazing goat*, or an OEIS name such as A133731. If successful, you will find the same information as for entering the truncated float value 1.158728473.

Brief sorted descriptions of all the constants together with links to their pages are at [https://oeis.org/wiki/Index_to_constants#Description](https://oeis.org/wiki/Index_to_constants#Description)

Some of these constants are tabulated, but most are isolated published constants, drawn from many areas by many contributors.

Even if another tool returns a plausible result, OEIS and Steven Finch’s books are good starting points for learning more about an OEIS constant in your result.

### 5.2 WolframAlpha

If you enter a float into WolframAlpha at [https://www.wolframalpha.com/](https://www.wolframalpha.com/) then the results usually include up to three guesses for a nonfloat constant that the float approximates.
A way to test the tools described in this article is to approximate a nonfloat constant to a generous number of significant digits, then see how many of those digits are necessary to produce an equivalent expression, if any. Accordingly, I approximated \(1 + 3/(1 + \sqrt{3} + \pi)\) to 18 digits, then tried entering that and successively fewer digits into Alpha. Table 2 lists the results for floats rounded to 10 and to 11 significant digits, together with entries having the 11 correct digits followed by one and by two intentionally incorrect digits. The underlined digits in the entries differ from the exact expression by at most 0.5 units in the last place.

Table 2: WolframAlpha test for identifying \(1 - \frac{3\sqrt{3}}{1 + \sqrt{3} + \pi}\) from a float approximation

| Entry       | digits | Result (Agrees with input to 0.5 in last bold digit) |
|-------------|--------|-----------------------------------------------------|
| 0.1153442566 | 10     | \[
\frac{11 - 7\pi + 2\pi^2}{\pi(21 + \pi)} \approx 0.11534425649366
\]
|             |        | \[
\frac{1}{\sqrt{92x^4 + 49x^3 - 27x^2 + 37x - 4} \text{ near } x = 0.115344} \approx 0.11534425656239
\]
| 0.11534425658 | 11     | \[
\frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} \approx 0.115344256584835
\] ✓
|             |        | \[
\frac{-2e\ e! + 124 - 35e + 5e^2}{136e} \approx 0.115344256579706
\]
| 0.115344256580 | 12     | \[
\frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} \approx 0.1153442565814835
\] ✓
|             |        | \[
\frac{2(15\sigma_S - 1)}{5(6\sigma_S + 73)} \approx 0.1153442565818196
\]
|             |        | \[
\pi \left( \text{root of } 100x^4 - 152x^3 - x^2 - 27x + 1 \text{ near } x = 0.0367 \right) \approx 0.1153442565828298
\]
| 0.1153442565800 | 13     | \[
\frac{45419\pi}{1237062} \approx 0.1153442565800221712
\]
|             |        | \[
\frac{3e\ e! + 295 - 86e - 9e^2}{94e} \approx 0.15344256579947
\]
|             |        | \[
\frac{1}{45} \left( 4C + 30 - 11\pi - 40\pi\log(2) + 27\pi\log(3) \right) \approx 0.115344256580078601, \text{ where } C \text{ is Catalan’s constant}
\]

Notice that:
1. Alpha did not propose an equivalent for 10 correct digits.

2. Alpha proposed an equivalent expression for 11 correct digits provided they were followed by no more than 1 incorrect digit.

3. Thus Alpha almost always finds expressions that differ from your input by at most a few units in the last place that you entered, making those differences not a good criteria for favoring one alternative over the others.

4. The returned candidates are usually nonequivalent.

So with typically three nonequivalent expressions all having about the same differences from your input, how are we to know which if any are the limit we seek?

By the principle of parsimony, you should prefer the least complex, but the three alternatives for 11 correctly-rounded digits don’t visually differ much in complexity. Here is a way to gain some evidence: If you can compute your wild float to more digits, then do so and round the result enough to be reasonably confident that the last digit or at least the one before it is correctly rounded, then enter that float. Impostors for the lower-precision input are less likely to appear for the higher-precision float. For example, the correctly rounded 18-digit value for the above “domesticated” float is 0.11534425681483524, and entering that returns the three alternatives

\[
\frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} \approx 0.11534425681483524414074, \\
\frac{26534199\pi}{722703038} \approx 0.115344256814835251297, \\
\frac{-678 + 46e + 97e^2}{2(-10 + 167e + 36e^2)} \approx 0.115344256814835236159.
\]

Notice that the first nonfloat is the same as for entering 11 correctly rounded digits in Table 2. The other two nonfloats for 11 digits have been replaced with other expressions and were therefore almost certainly impostors. It is quite possible that the true limit would not appear until higher than 18-digit precision, making the first entry an impostor too that would disappear at higher precision. However, it is typical of true limits that once they appear at a certain precision, they will appear for all higher precisions. Therefore it is advisable to enter increasing numbers of probably-correctly rounded digits until one of the alternatives convincingly persists or you become convinced that Alpha does not model your float. If two or more persist, then perhaps they are equivalent.

Alpha mostly uses the PSLQ integer relation algorithm\(^4\): Given a basis vector \(c = [c_1, c_2, \ldots]\) of nonzero symbolic and/or float constants, the algorithm returns either an indication of failure or a minimal 2-norm vector of integers \([n_1, n_2, \ldots]\) not all 0 such that the inner product \(n_1c_1 + n_2c_2 + \cdots \approx 0.0\) and \(\gcd(n_1, n_2, \ldots) = 1\).

1. One way to exploit this algorithm is to make \(c_1\) be the given float \(\tilde{x}\) and choose commonly occurring nonfloat constants such as 1, \(\pi\), \(\sqrt{2}\), and \(\ln 2\) as the other constants. If the algorithm returns a vector \([n_1, n_2, \ldots]\), then a close-fitting candidate

\(^4\)The LLL algorithm can also be used for this purpose.
is the rational linear combination
\[ \tilde{x} \simeq \frac{n_2}{n_1} c_2 - \frac{n_3}{n_1} c_3 - \cdots. \]

The last entry in Table 2 is of this type.

2. Another way to exploit this algorithm is if
\[ (n_1 c_1 + n_2 c_2 + \cdots) \tilde{x} + (n_k c_k + n_{k+1} c_{k+1} + \cdots) \simeq 0.0, \]
then
\[ \tilde{x} \simeq - \frac{n_1 c_1 + n_2 c_2 + \cdots}{n_k c_k + n_{k+1} c_{k+1} + \cdots}. \]

This linear fractional category of model was used for the correct identifications and some of the incorrect ones in Table 2.

3. Another way to exploit this algorithm is if \( n_1 + n_2 \tilde{x} + \cdots + n_{m+1} \tilde{x}^m \simeq 0.0 \), then \( \tilde{x} \) is approximately one of the zeros of the polynomial \( n_1 + n_2 \tilde{x} + \cdots + n_{m+1} \tilde{x}^m \).
This can be tried for \( m = 1, 2, \ldots \) up to some maximum degree that increases with the number of significant digits. If PSLQ returns a vector of integers, then an approximate polynomial zero algorithm can then be used to determine which zero is closest to the given float. The third alternative for 11 digits in Table 2 reports such an algebraic number candidate.

4. For a power product model of the form \( \tilde{x} \simeq b_1^{r_1} b_2^{r_2} \cdots \) with given positive nonfloat base constants \( b_k \) and unknown rational exponents \( r_k \), if \( n_1 \ln \tilde{x} + n_2 \ln b_1 + n_3 \ln b_2 + \cdots \simeq 0.0 \), then
\[ \tilde{x} \simeq \exp \left( - \frac{n_2 \ln b_1 + n_3 \ln b_2 + \cdots}{n_1} \right) = b_1^{-n_2/n_1} b_2^{-n_3/n_1} \cdots. \]
The irrational factors of an irrational exponent can be considered part of the base to make this category fit a model such as \( \tilde{x} \simeq 2^{r_1} 3^{r_2} \sqrt{5}^{r_3} e^{r_4} \pi = 2^{r_1} \left( 3^{r_2} \pi \right)^{r_3}. \)

5. For a model of the form \( f(\ldots) \), where \( f \) is invertible with respect to at least one of its arguments and that argument model is one of the above types or recursively of this functional type:

- Using predetermined constants for any other arguments, apply an inverse of \( f \) to the given float \( \tilde{x} \) giving \( \tilde{y} \).
- If the argument model is successfully fit to \( \tilde{y} \), giving a nonfloat candidate constant \( y \), then return \( f(y) \).

The computing time grows rapidly with the number of elements \( m \) in the basis vector and the number of words in the float significands. Moreover, the PSLQ algorithm can require the input float to have \( mn \) significant digits of precision to return a correct \( m \) component vector having integer components of up through \( n \) digits even if many of these integer coefficients have much smaller magnitudes, including 0. Therefore, although specialized efforts to identify particular floats have used thousands of digits and hundreds of vector components, the general-purpose programs described here use mostly basis vectors having \( \lesssim 8 \) elements.
More advice for using Alpha to identify floats:

1. If you enter 18 or more significant digits, including any trailing 0 digits, then Alpha uses arbitrary precision rather than 16-digit machine floats. Arbitrary-precision floats are slower but more than 18 digits are necessary for PSLQ to determine many published constants.

2. Alternatively, you can append a suffix of the form ‘n to the significand of your input to indicate that your best guess for the Precision is the value of n, which can be a decimal fraction. For example, if your input for the example in Table 2 was computed by iterative equation solving using 16-digit IEEE binary64 hardware and you guess that most probably about 14 digits are correctly rounded, then you could start by entering 0.1153442565814834’14. This strongly recommended alternative has the advantage that you can enter all of the computed digits, which which are more likely to be correct than the value rounded or truncated to 14 digits, but without having Alpha assume that all but perhaps the last one are correct. In the other direction, if your computed 16-digit machine float is 0.1615000000000000 and you are rather confident that all are correct, then you can enter it more concisely as 0.1615’16. Different from the apostrophe, the accent grave character ‘ is near the upper left corner on many keyboards and might look different on your keyboard.

3. Software does not always display all of the digits in its significands, and internet browsers paste into web applications such as Alpha only the characters that you highlight when copying. For example, Mathematica displays by default only 6 of the approximately 16 significant digits in the widespread IEEE binary64 machine floats, and displays none of the 8 or more stored guard digits in arbitrary-precision floats. Often some of those hardware or guard digits are correct and therefore helpful to try entering. Therefore, learn how to display all of the digits in your software’s floats.\(^5\)

4. Unless you are an exceptionally fast accurate digit typist, highlight then copy the entire float including its hidden digits rather than type it.

5. Then attach to the significand a suffix of the form ‘n, where n is your best guess for the number of significant digits that are correctly rounded. Keep in mind that cancellation and rounding errors often lose one or more digits from the working precision – and that discretization errors such as from quadrature often lose more. Interval arithmetic can provide an upper bound on cancellation and rounding errors, but not necessarily on discretization errors. Significance arithmetic such as Mathematica’s arbitrary-precision float arithmetic provides cancellation and rounding error estimates that are occasionally underestimates but more frequently overestimates.

6. If your first attempt is not satisfying, then try fewer and more for your best estimate for the number of correct digits.

\(^5\)One way to do this in Mathematica is to copy all of the displayed digits then paste into a notebook, then copy all of that fully-displayed float. Another way is to copy the result of InputForm[\text{float}].
7. PSLQ overfits often contain many digits in their rational numbers, such as the first alternative in Table 2 for 13 digits. Several terms containing only simple rational numbers is less suspicious than one term containing a complicated rational number. Information theory suggests that candidates containing nearly as many or more total digits than the float input are particularly dubious.

5.3 Inverse Symbolic Calculator

Inverse Symbolic Calculator (ISC) implemented by Simon Plouffe working with Jonathon and Peter Borwein and others is currently hosted at

http://wayback.cecm.sfu.ca/projects/ISC/ISCmain.html

Plouffe [12] describes its origins.

ISC provides four different techniques to apply to the float that you enter:

5.3.1 Simple Lookup and Browser

This technique searches a precomputed table of sorted truncated 16-digit base-ten significands of the absolute values of the floats, each paired with a character-string description of the constant.

That table has about 30 million pairs that are a mixture of published and tabulated constants. The sorted significands and the corresponding nonfloats are subdivided into about 9000 files whose names include some leading digits of all the significands in that file, with each file containing all of the significands with those leading digits. The search begins with a binary search on the file names in RAM, followed by a search in the appropriate file to perfectly match or bracket the significand of the float that you entered, followed by retrieval of close neighbors of those entries. Here are some input tips:

- ISC Simple Lookup defines a match as all of the entered digits of the significand being identical to that leading portion of a tabulated significand, but ignoring any entered digits beyond 16.

- The last one or more digits of your computed float are often not the correctly truncated ones. To maximize your chance of success, try to compute as many digits as is practical to be reasonably confident that the first 16 digits are correctly truncated, then enter at least those 16 digits. Your float was probably computed with software that rounds rather than truncates, so at least truncate the last computed digit.

- If you cannot compute more than 16 digits, then truncate to as many as your best guess for how many are correctly truncated digits of the limit you seek.

---

6 Mentally truncating digits correctly is easier than mentally rounding them correctly.
• If the least significant truncated digit is a 9, then also try truncating what you get by adding 1 to that 9 digit. If the least significant truncated digit is instead a 0, then also try truncating what you get by subtracting 1 from that 0 digit. Those minimal relative changes to your entered float can dramatically alter the digit sequence.

• You can enter as few as 5 significant digits, but with about 30 million entries having significands 0.1000000000000000 through 0.9999999999999999, the mean distance between table significands is

\[
\frac{0.9999999999999999 - 0.1000000000000000}{30 \times 10^6} \approx 3.3 \times 10^{-8}.
\]

Thus the expected number of matches for 5 entered significant digits is about 1000. It is burdensome to assess more that a few matches even if they contain the limit you seek; and they often do not, because a known closed form does not exist or it is not in the tables or you have exceeded the ISC bound on the number of matches.

• You are offered the opportunity to browse nearby table entries even if there are no matches. Doing so might reveal one or more candidate nonfloats that agree closely and are plausible considering your knowledge of the problem domain. Browse even if there are matches, because your input might not be as correctly truncated as you assumed.

• The table contains many instances of nonfloats that all have the same 16-digit significand value. The corresponding nonfloats are often but not always equivalent to each other within a factor that is a power of 10. Approximating these nonfloats to higher precision can support equivalence or reveal nonequivalence between them.

• If you do not obtain at least one promising result, then also try entering fewer and more truncated digits up through 16, covering what you guess spans most of the possible actual correctly truncated digits of your wild float.

As a test example, the correctly truncated 16-digit value of

\[-10 \sqrt[237]{\frac{2}{3} (8\sqrt{3} - 9)} = -5.123557917376186. \quad (1)\]

Table 3 summarizes the results of entering various further truncated and rounded leading digits to expose the consequences of rounding rather than truncating – and of appending incorrectly truncated digits beyond the sixteenth.
Table 3: ISC Simple Lookup test results for approximated $-10\sqrt{\frac{2}{7}(8\sqrt{3} - 9)}$

| Input       | digits | Treatment       | Result                                      | ?      |
|-------------|--------|-----------------|---------------------------------------------|--------|
| 5.1235      | 5      | truncated       | 38 matches include $12^{1/4}/\sqrt{8 + 9^{3/4}}$ | Too many |
| 5.1236      |        | rounded         |                                             |        |
| 5.123557    | 7      | truncated       | Matches only $12^{1/4}/\sqrt{8 + 9^{3/4}}$   | ✓+     |
| 5.123558    |        | rounded         | 6 matches that do not include nearby $12^{1/4}/\sqrt{8 + 9^{3/4}}$ | All bad |
| 5.1235579173| 11     | truncated       | Matches only $12^{1/4}/\sqrt{8 + 9^{3/4}}$   | ✓+     |
| 5.1235579174|        | rounded         | No match, but browser lists adjacent $12^{1/4}/\sqrt{8 + 9^{3/4}}$, matching 10 digits | ✓      |
| 5.12355791730|       | incorrect last digit | No match, but browser lists adjacent $12^{1/4}/\sqrt{8 + 9^{3/4}}$, matching 10 digits | ✓      |
| 5.12355791737618| 15     | truncated       | Matches only $12^{1/4}/\sqrt{8 + 9^{3/4}}$   | ✓+     |
| 5.12355791737619|       | rounded         | No match, but browser lists adjacent $12^{1/4}/\sqrt{8 + 9^{3/4}}$, matching 14 digits | ✓      |
| 5.1235579173761861|   | incorrect last digit | Matches only $12^{1/4}/\sqrt{8 + 9^{3/4}}$   | ✓+     |
| 5.1235579173761869|   | incorrect last digit | Matches only $12^{1/4}/\sqrt{8 + 9^{3/4}}$   | ✓+     |

For each match, ISC lists the truncated 16-digit base-10 significand following an implicit decimal point, equated to a corresponding nonfloat, such as

$$512355791737618 = 12^{-1/4} / (8 + 9^{3/4})^{1/2}.$$  

To process this match:

1. The ratio of your input 5.123557917376186 to the significand 0.5123557917376186 is 10, so multiply the candidate nonfloat by 10 to de-alias the significand.

2. Then negate that, because our float of interest in equation [1] is negative and we necessarily entered the negative of that value, giving our final result

$$-10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}}.$$  

(2)

3. For verification of our de-aliasing, we should approximate expression [2] to slightly more than 16 significant digits, then truncate to 16 digits, giving $-5.123557917376186$, which agrees with the right side of equation [1] to 16 digits.
4. It is not difficult to transform candidate (2) into our test expression on the left side of (1). But what if our input was a wild constant that happens to agree with an imposter to 16 digits? If we can compute our wild constant to \( n > 16 \) digits, then we should approximate such a candidate nonfloat to more than \( n \) digits for comparison. Thus even though the table is only 16 digits, you are not limited to assessing agreement to only 16 digits and thereby either increasing support for the candidate or exposing impostors.

For candidates resulting from tables, if you can compute your float input to higher precision than used in the table, then it is wise to assess exact and near-exact matches to higher precision than used in the table to help expose impostors.

Beware that the table was assembled over many years from many computed and printed sources that used varying syntactic conventions. For example:

- Gam, Gamma, GAM and GAMMA all denote the Gamma function, whereas gamma denotes the Euler-Mascheroni constant.
- sqrt and sr both denote \( \sqrt{\text{.}} \).
- A polynomial in \( x \) or \( \text{root(a_polynomial_in_x)} \) denotes the zero of that polynomial nearest the given float.
- Non-real expressions often implicitly denote the real parts of those expressions.
- As with our regrettably ambiguous traditional mathematics notation, sometimes \( 16^{\frac{3}{4}} \) means \( 16^{(3/4)} \) rather than \( (16^3)/4 \), \( \sin e + 3 \) means \( \sin(e + 3) \) rather than \( \sin(e) + 3 \), and \( 7/3\pi \) means \( 7/(3\pi) \) rather than \( (7/3)\pi \), etc.

Step 3 above enables you to verify or refute your interpretation of the ambiguous nonfloat syntax, then to try another interpretation if necessary.

Many nonfloats in the table contain either a common name constant such as Bernstein or a red OEIS web link labeled with an OEIS constant having a name of the form “A” with a six-digit suffix. Either way, you can search for their definitions at [https://oeis.org/](https://oeis.org/) as described in subsection 5.1.

### 5.3.2 Smart Lookup

This technique uses the same lookup table about 100 times, but precedes each lookup with a transformation of your float. For example, smart lookup might add \( 3/7 \) or multiply by \( 3/7 \) or apply \( \ln(\ldots) \). For a given number of input digits, the average number of matches is about 100 times as many as for simple lookup, and some transformations lose precision. Therefore smart lookup requires you to enter at least 10 digits, and the transformation of your input is done using 26-digit significands. Thus it is beneficial to enter floats having
up through 26 correctly rounded digits even though the forward table floats have only 16-digit significands.

For example, computed to 26 significant digits,

\[
- \frac{1}{4} - 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}} \simeq -5.3735579173761866715453232.
\] (3)

Applying simple lookup to the absolute value of this float truncated to 16 digits returns no match, and the ISC browser lists no result that agrees to more than 5 digits. However, applying smart lookup to the absolute value of the right side of equation (3), computed with arithmetic that rounds, returns the table

| Function | Result | Precision | Matches |
|----------|--------|-----------|---------|
| K \(-1/4\) | 5.1235579173761866715453232 | 16 | 1 |

The Function entry, K \(-1/4\) is a transformation that was applied to your float, with K representing the nonfloat that you seek and its approximate value 5.3735579173761866715453232. The result entry is the result of applying the transformation to your float as computed with 26-digit significands. The Precision entry is the number of leading digits in the table that matched those in your transformed float. The Matches entry is the number of matches for that transformation.

If you click on the Function entry K \(-1/4\), then ISC shows the result of doing simple lookup with the truncated significand of the transformed Result float, which gives

\[
\frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}},
\] (4)

together with an opportunity to browse around it.

If you want to further investigate one of these nonfloats that have acceptable agreement, then, for example, you should

1. Numerically approximate the nonfloat, preferably to at least 16 digits, giving

\[
K = 12^{1/4} / \sqrt{8 + 9^{3/4}} \simeq 0.512355791737618667.
\]

2. From the ratio of that float to the float in the Result entry, determine a power of 10 by which to multiply the nonfloat:

\[
\frac{5.1235579173761866715453232}{0.512355791737618667} = 10.0000000000000000.
\]

3. Multiply the table nonfloat (4) by that power of 10 giving the magnitude de-aliased nonfloat

\[
y = 10 - \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}}.
\]

4. Solve \(y = K - 1/4\) for K, giving

\[
K = y + 1/4 = 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}} + \frac{1}{4}.
\]
5. Negate this result to de-alias the sign, giving the nonfloat candidate for our float of interest
\[-\left( \frac{1}{4} + 10 \cdot \frac{12^{1/4}}{\sqrt{8} + 9^{3/4}} \right) .\]

More generally there might be several rows in the returned table for several different transformations that resulted in matches. Click on each of the Function entries to see the corresponding nonfloats and optionally browse neighbors.

For this example, entering fewer than 16 digits produced no smart lookup result.

It is worth starting with Simple Lookup first because it is faster and it permits browsing the neighborhood of your input even if there are no matches.

5.3.3 ISC Integer Relation Algorithms

ISC provides an Integer Relation alternative that you should try too even if simple or smart lookup gives a promising result. Integer Relations requires inputs of 16 through 32 significant digits, and it checks for algebraic numbers through fifth degree and for rational linear combinations of the basis vectors

\[ e, \pi, \gamma, \text{Ei}(1), W(1), 1, \sqrt{3} \pi, \ln 3, \ln 2, \gamma, \sqrt{2} \pi, \pi^2, \text{Catalan}, \pi \ln 2, \sqrt{2} \pi^2, (\ln 2)^2, \pi^3, \zeta(3), \pi^2 \ln 2, (\ln 2)^3, \sqrt{3} \pi^3, \sqrt{2} \pi^3 . \]

These basis vectors might seem rather specialized. However, a noticeable portion of wild constants in the printed tables of Section 2 are representable with a subset of at least one of these vectors or with algebraic numbers of degree \( \leq 5 \).

It is wise to compute as many digits as you can up through 32. There is no need to truncate, and it is OK for a few of the last digits to be incorrectly rounded.

5.3.4 Generalized Expansions

You must enter at least 16 significant digits for this option, and it is most appropriate to round your approximate float to where you are reasonably confident that the last digit is correctly rounded.

This option computes a truncated continued fraction from your input float, and computes similar truncated infinite representations such as a truncated infinite product and a truncated Egyptian fraction. This can be helpful because these alternatives might suggest patterns that might be useful for computing the float to arbitrarily many digits more efficiently than the method you used. Moreover, the GFUN package \([17]\) is then applied to these truncated representations to try guessing a generating function for the infinite representation. (It is for you to prove that it is such a generating function.)

Also, you can copy the comma-separated sequence of integers for one of these truncated representations, then paste it at [https://oeis.org/](https://oeis.org/) to attempt learning more about
it, possibly including a closed form. This is most likely to be successful for the continued-fraction representation. A relevant OEIS hint for the sequence 1, 1, 2, 4, 9, 21, 51, 127 is

“Enter about 6 terms, starting with the second term. Leave off the first term or two, because people may disagree about where the sequence begins. Don’t enter too many terms, because you may have more terms than are in the OEIS data base.”

Generalized Expansions and two related tools GCF.TXT by Dougherty-Bliss and Zeilberger and [6] and the Ramanujan Machine by a team of nine authors [14] are intriguing, but adjacent to the topic of this article, so they are not described further here.

5.4 Inverse Symbolic Calculator Plus

Inverse Symbolic Calculator Plus at [https://isc.carma.newcastle.edu.au/](https://isc.carma.newcastle.edu.au/) was a different interface to essentially the same smart lookup and integer-relations models as ISC, ISC+ has had the message “down for maintenance indefinitely” from late 2018 through at least March 2021. However, there is currently a link there titled “The original ISC” that links to the ISC site already discussed in subsection 5.3.

6 Freely downloadable standalone applications

The implemented integer relation models and precomputed table-lookup entries of the software described in this article are based on their implementer’s perception of what patterns of nonfloat constant expressions are most likely to fit the combined needs of the intended users. There is a consequent efficiency to the extent that the implementer’s choices are appropriate for you. However, such an implementation is almost certain to miss some very concise results fitting unimplemented patterns.

The MESearch and RIES programs address this issue by trying all possible expressions composed of user-selected rational numbers, symbolic constants, operators and functions – up to a certain total complexity, time limit or memory limit. Both programs use bidirectional search: The forward table is like the ISC table, and the backward table is like the ISC Smart Lookup transformations. However, rather than using precomputed forward and transformation tables, for each given float both applications build new tables from your selected components in a breadth-first way, interleaving the searching with this table building. The forward table begins with a selected set of simple rational numbers and named constants together with a set of arithmetic operators and functions. More complicated forward expressions are formed by applying all chosen functions and operators to existing entries in an order such that the predicted expression complexity grows approximately monotonically. The backward table starts with the given float, and applies the selected functions in an order such that the estimated resulting complexity also grows approximately monotonically. For functions and operators of more than one operand, all but one of the operands can be taken from the current forward table.
Growth alternates between the forward and backward tables so that at any one time they have roughly the same number of entries.

Even though the initial set of rational numbers is typically quite simple, arithmetic combinations of rational subexpressions can lead to larger numerators and/or denominator magnitudes.

Both programs use only hardware floating point but do a good job of making the best of that limitation. Moreover, as with the 16-digit Inverse Symbolic Calculator table, if you can approximate your wild float and nonfloat candidates to greater than hardware precision, then you can assess the agreement of candidates to greater precision to help distinguish impostors from the limit you seek.

Both programs use a few simplification rules such as employing 0 and 1 identities, doing rational arithmetic, and canceling composition of a function with its inverse. However, neither program uses a computer algebra system, so sometimes the results can be usefully simplified manually or by computer algebra.

Computation ceases at a preset expression complexity limit, memory limit, time limit, or the limit of your patience when you finally interrupt a search. Both programs initially use RAM for the tables, but can resort to a much slower secondary-storage mode when allocated RAM is exhausted.

The fact that the tables are exhaustively built and searched in approximate order of increasing expression complexity has a benefit that simple candidates composed of your selected components are almost always quickly found. However, the open-ended exponential growth of the table sizes and computing time with expression complexity makes it impractical to generate expressions having complexity that is easily achievable by specific integer relation models with sufficiently large-precision inputs. With a medium sized set of selected components, exhaustive search can require overnight to generate an expression containing about 11 or 12 instances of rational numbers, symbolic constants, arithmetic operators, and function invocations.

But the peace of mind knowing that the search was exhaustive for expressions composed of your selected rational numbers, symbolic constants, operators and functions is worth the wait.

In contrast, the other software discussed in this article have a fixed number of table entries and/or integer relation models, and the latter can accommodate expressions having any number of predetermined symbolic constants, with arbitrarily large integers determined by PSLQ. But being non-exhaustive, those tools can and do miss some very simple results that MESearch and/or RIES can determine.

Ignoring the fact that applying some of the transformations to the given float might produce unusable underflow overflow, or non-real results, if the backward table has \( m \) entries and the forward table has \( n \) entries, then the number of potentially recognizable expressions is \( mn \). If both kinds of table entries averaged the same memory space per entry, then for a given amount of memory, \( mn \) is maximized by having \( m = n \). For example to recognize \( 10^{12} \) expressions, it suffices to have only \( 10^6 \) entries in the forward table and \( 10^6 \) entries in the backward table.
6.1 MESeach

MESeach is implemented in Java and freely downloadable from 

http://www.xuru.org/mesearch/MESeach.asp.

The Java Runtime Environment is freely downloadable from https://www.java.com/en/download/ and that is often already installed on most computers.

MESeach uses only invertible functions or operators for the top level function or operator of its backward table, and automatically applies those inverses to the nonfloats in the forward table whose float values closely match a transformed float input.

Figure 2 shows the MESeach input panel for the test float input

$$\frac{1}{\sqrt{5} + 2\sqrt{6} + \sqrt{5}} \approx 0.1857930606004482.$$ (5)

MESeach measures expression complexity by **Length**, defined as the number of occurrences of rational numbers, named constants, functions and operators. Counting the implicit multiply, the Length of the expression on the left side of equation (5) is 12.

MESeach offers a choice of operators, functions, named constants, natural numbers and rational numbers from which to build expressions, and Figure 2 shows my most common choices for these. Choosing only a few components permits you to search to longer expressions before exhausting memory or your patience, but it is important to include components that you expect have a nonnegligible probability of occurring in your problem domain. Functions and operators with more than one argument are particularly costly, and that is why I often initially limit those to “+”, “×”, and “^”. Memory and computing time also grow quickly with the cardinality of the initial set of named constants, natural numbers, and rational numbers. Thus I often initially select only π and e together with very simple natural numbers and fractions.
MESearch only displays candidates whose float value are within ±Tolerance entered under the Target constant near the upper left corner of Figure 2. The default MESearch Tolerance is 1/2 unit in the last entered digit of your Target constant, which is difficult to achieve if your Target constant is all 16 digits of the hardware floats or nearly so. Consequently I often initially choose about 1000 to 10,000 units in the last entered place for Target floats having 16 significant digits, down to about 10 units in the last place for Target floats having 7 or fewer significant digits. Generous tolerances change the result likelihood estimates and are more likely to generate impostors, but unlike overly frugal tolerances, generous tolerances do not tend to lose achievable true limits. I would rather obtain a true limit together with a few impostors, then decide which are most plausible.
and recompute those with smaller tolerances for more accurate likelihood estimates.

MESearch uses your entered absolute tolerance rather than the number of entered
digits, so you should enter all of your computed digits if there are less than 16 or rounded
to 16 digits otherwise, even if any number of the resulting trailing digits are dubious.
Therefore I entered all 16 digits of 0.1857930606004482, for which my entered absolute
tolerance was $10^{-12}$.

Here is a summary of the most important information in the resulting output:

| Length | Expression                                      | Difference | Likely₁   | Likely₂  | Likely₃  |
|--------|-------------------------------------------------|------------|-----------|----------|----------|
| 8      | $((e^{\arccosh(5)})^{(1/2)} + 5^{(1/2)})^{-1}$ | 0.0        | 0.5103    | 0.9135   | 0.9187   |
| 9      | $\text{arctanh} (\cos (\tan (\sin (4/3 / \sinh (\cos (4)))))$ | 6.769e-13  | 3.3e-11   | 0.8515   |
| 9      | $\arctan(\psi(2 - \tan(Ei(logGamma((3/4)^2))))$ | 6.769e-13  | 3.3e-11   | 0.8515   |

Watching the output panel evolve provides some entertainment while you wait and
helps you decide whether and when to terminate the search manually.

The first expression that met the tolerance was at complexity Length 8. The three
Likelihood estimates are based on the fact that the expressions are all possible expres-
sions composed of the selected components, computed in approximately monotonic non-
decreasing order of complexity. The reason for the two vacant Likely₃ entries is that
Likely₃ can be computed only upon completion of searching its Length level, and I
manually halted the search during Length 9.

Clicking a column heading sorts the rows by the values in that column, so I usually
click the middle Likelihood₂, which the implementer Salsamendi states is most often most
correct.

If this input float was a wild float, then having the 0.0 Difference between the entered
float and the value of the Length 8 Expression, together with its two likelihoods of
greater than 90% and one of greater than 50% would lead me to strongly suspect that
the corresponding expression is a true limit. In contrast, the very low values for half the
likelihoods of the two Level 9 candidates and their significantly greater Differences would
lead me to strongly suspect that they are impostors.

The result subexpression $e^{\arccosh(5)}$ having Length 4 is quite different from the
(corresponding test subexpression $\sqrt{5 + 2\sqrt{6}$ having length 7 in the left side of equation
[3], but they are provably equivalent, and MESearch probably would have generated the
latter too at Level $8+(7-4)=11$. Although most people prefer algebraic numbers to be
expressed using radicals or trigonometric functions if possible and practical, representa-
tions using other transcendental functions together with inverses of their relatives often
have lower complexity by most quickly-computed measures.

Salsamendi [16] describes the data structures, algorithm, and time complexity.

### 6.2 RIES (Rilybot Inverse Equation Solver)

RIES is implemented in C, and the source code is freely downloadable from [https://mrob.com/ries](https://mrob.com/ries). Your computer might already have a C or C++ compiler that you

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7The *Mathematica* FunctionExpand function can transform many such cryptic radical expressions to
radical expressions.
are unaware of, and there are several good free ones for Mac OS X, the Unix family, and Windows.

RIES optionally exploits 19-digit IEEE binary80 arithmetic if your CPU and C compiler support it, and the resulting 19 digits is a valuable increment over the typical 16-digit hardware. IEEE binary80 is supported by Intel x86 processors, but not by current or past AMD or ARM processors. The Intel C++ compiler and the free GNU C compiler support this arithmetic, and Microsoft Visual C can be made to do so at the inconvenience of inserting _controlfp_s function calls at appropriate places in the source code.

Here is an example by Bill Gosper (private communication):

Mathematica 12.1 cannot determine a closed form for

$$
\sum_{n=0}^{\infty} \frac{(-1)^n \cos \left( \sqrt{\frac{1}{2} + \left( \frac{1}{2} + n \right)^2} \pi \right)}{\left( \frac{1}{2} + n \right) \left( \frac{1}{2} - \frac{1}{\sqrt{2}} + n \right) \left( \frac{1}{2} + \frac{1}{\sqrt{2}} + n \right)}
$$

Suppose that you compute a 20-digit approximate value 7.0895773641597344051, and you invoke RIES as the command line

```bash
ries -l7 7.0895773641597344051 -s
```

where `-l7` and `-s` are option settings described in the documentation. The following slightly edited lines are displayed as they are computed:

```
Your target value: T = 7.08957736415973441 mrob.com/ries

x = pi + 4 for x = T + 0.0520153  {49}
x = 4 sqrt(pi) for x = T + 0.000238039  {58}
x = 1/(8 + pi) + 7 for x = T + 0.000176411  {81}
x = (sqrt(pi) + pi)/ln(2) for x = T - 0.000106841  {87}
x = e^(e^(sqrt(phi))/5) for x = T - 8.60481e-05  {90}
x = ln(ln(6^8))^2 for x = T + 7.23949e-05  {93}
x = (phi^2)^sqrt(1 + pi) for x = T - 4.72474e-05  {85}
x = e^(e^(pi - 2))^2 - e for x = T + 1.25979e-05  {90}
x = (1/pi + sqrt(pi))^2 + e for x = T - 2.53137e-06  {92}
x = e (sinpi(1/ln(9)) + phi) for x = T + 2.0796e-06  {109}
x = 3^n/(log_3(1 + pi)) + 6 for x = T - 1.32453e-08  {111}
x = 3/pi + 8 - pi^n/x for x = T + 5.08817e-10  {113}
sinpi(sqrt(x)/e) = (1/(8/phi - 1))^2 for x = T - 3.68034e-10  {131}
x = 3((pi^n/4 - 1/7) + phi) - 2 for x = T - 3.51782e-11  {141}
x = 2 e^n/(1 + e) - 1/((1 - 3^n/2) for x = T + 1.65079e-11  {147}
x = ((log_9(1/2 + 8))/4)/e + 7 for x = T - 1.3897e-11  {150}
x = -(pi*cospi(1/sqrt(2))) - pi cospi(1/sqrt(2)) for x = T - 5.28657e-16  {148}
x = pi (1/sinpi(1/sqrt(2) - 1/2) - cospi(1/sqrt(2))) for x = T + 3.05745e-16  {154}
```

24
\[ x = \left(1/\tan(1/\sqrt{8})\right)^4 + 1)(x/2 + \pi) \text{ for } x = T - 1.30104 \times 10^{-18} \{164\} \]

(Stopping now because best match is within 3.07e-18 of target value.)

\[ \log_A(B) = \text{logarithm to base } A \text{ of } B = \frac{\ln(B)}{\ln(A)}, \]
\[ \cos \pi(X) = \cos(\pi \times x) \]
\[ \sin \pi(X) = \sin(\pi \times x) \]
\[ \tan \pi(X) = \tan(\pi \times x) \]
\[ \phi = \text{the golden ratio} = (1+\sqrt{5})/2 \]
\[ A^n/B = \text{Ath root of } B \]

\begin{tabular}{lll}
   --LHS-- & --RHS-- & -Total- \\
   max complexity: & 86 & 81 & 167 \\
   dead-ends: & 363904522 & 872434773 & 123639295 \\
   Time: 204.127 sec \\
   expressions: & 25836915 & 59427019 & 85263934 \\
   Memory: 1351040 KB \\
   distinct: & 7974910 & 9198455 & 17173365 \\
   Total tested: & (7.336e+13) \\
\end{tabular}

The integers in braces at the right ends of lines beginning “x =” are the complexities, which are computed differently from MESearch. Notice that

- RIES rounded the 20 digit input to 1 digit less than the 19 digit arithmetic it is using on this computer.

- RIES displays a candidate only if its absolute difference from the entered float is less than that for the previously displayed candidate. Since candidates are generated in approximate order of non-decreasing complexity, the sequence of displayed candidates is approximately the Pareto set of optima for the conflicting objectives of large agreement and small complexity. Thus usually all but the last few of the displayed candidates are possibly-useful approximations rather than plausible candidates for the limit you seek. (But as with the subsection 6.1 the very last few can be overfit impostors.)

- It is your responsibility to solve the alternative equations

\[
\sin \pi \left( \frac{\sqrt{x}}{e} \right) = \left(1/(8/\phi - 1)\right)^2, \\
\]
\[ x = \left(1/\tan(1/\sqrt{8})\right)^4 + 1)(x/2 + \pi) \]

if you want explicit solutions for them. Doing so for the latter gives

\[
x = -\frac{2\pi \left(1 + \cot \left(\frac{\pi}{2\sqrt{2}}\right)^4\right)}{-1 + \cot \left(\frac{\pi}{2\sqrt{2}}\right)^4} \simeq 7.0895773641597344057, \\
\]

which differs from the 20-digit input only in the last place\[^{8}\]

\[^{8}\] In the other result lines, RIES was able to uniquely invert the backward table entries.
Both of the simpler explicit solutions immediately preceding the last output equation also give the same 20 digit value despite their estimated absolute errors being about 100 times as large. This is probably because the estimated errors rely on linearized error analysis and/or the limitation to hardware floating-point prevents the float values of the nonfloat candidates from being computed more precisely than the float input.

RIES uses a particularly compact data structure that allows it to build particularly large tables before having to resort to slower secondary storage. Compiling with extended precision is slower and reduces the maximum achievable table sizes, but the greatly increased number of recognizable constants makes it worthwhile. To maximize your coverage, compile both a 19-digit and 16-digit version, then try 16 digits if 19 digits exhausts memory or your patience without returning a satisfying result.

7 Functions built into computer algebra systems

7.1 The Maple identify (...) function

Sometimes you want to use a result of a downloadable application in another piece of software. For that purpose, functions are usually better than applications or web applications – because functions have a single output and have no side effects, making them composable, with results that are the same regardless of temporal reorderings due to exploiting commutativity, associativity, etc. Also, functions do not require interactive keystrokes.

The Maple computer algebra system has a function named identify (...) that can take a float as an argument and returns either one nonfloat constant or the input float if the function cannot determine a nonfloat that the function judges sufficiently likely to be the limit you seek. This function is an adaptation by Kevin Hare of a Maple application by Alan Meichsner [11, 4], based on the PSLQ integer relation algorithm.

Default Maple floats are arbitrary-precision with only 10-digit significands. Therefore you must almost always use more precision to have a reasonable chance of success.

The fact that identify (...) results are either the input float or a nonfloat that has very nearly the same value enables identify (...) to map automatically over the parts of non-real constants and over non-constant expressions, trying to replace all or at least some of any floats therein with close nonfloat constants. For example,

\[
\text{identify} \left( \frac{0.27675082 - 1.0369278 I}{8.3118730 x^2 + 11.94535128 y} \right) \rightarrow \frac{0.27675082 - I \zeta(5)}{\sqrt{7} \pi x^2 + 4e \ln(3) y}
\]

where \( I = \sqrt{-1} \) and \( \zeta \) is the Riemann zeta function.

The one float that identify did not convert was a random float. In contrast to applications that allow you to inspect alternatives and reject them all, a function such as identify (...) automatically replaces any number of floats in an expression by nonfloats without your inspection and consent to their complexity and agreement. Therefore it is appropriate for such functions to be parsimonious about the agreement and complexity
of replacements they accept. This makes it a good sign that the random float was not
replaced by a candidate that would almost certainly be an impostor.

To determine your confidence in a Maple identify result, try using successively in-
creasing precisions for the input floats until the result appears to stabilize. Then, assess
the agreements of the replaced floats with their replacements by using the evalf function
on each of those replacements.\[9\]

The identify function has optional arguments that enable you to adjust the integer
relation models, and it is well worth doing so if the defaults do not return a satisfactory
result. For example, although the default maximum degree for seeking algebraic numbers
such as RootOf \( [polynomial, \ldots] \) is 6, you can change that. Many exact constants of
interest are of larger degree, so try larger degrees if an algebraic number is plausible and
you can compute your wild float to high precision.

7.2 The MPMath, SymPy and Sage identify \((\ldots)\), findpoly \((\ldots)\),
and nsimplify \((\ldots)\), functions

Sage is a software system freely downloadable from [http://www.sagemath.org/](http://www.sagemath.org/) that
includes NumPy, SciPy, matplotlib, SymPy, Maxima, GAP, FLINT, R and other pack-
ages. The SymPy computer algebra system in turn includes MPMath, which has an
identify \((\ldots)\) function analogous to Maple’s, except that identification of algebraic num-
bers is done by a separate function named findpoly \((\ldots)\). The SymPy system also has
an nsimplify \((\ldots)\) function somewhat analogous to the NSimplify \([\ldots]\) component of
AskConstants described in subsection \[8.1\].

8 Functions and applications for CAS

8.1 AskConstants and associated functions for Mathematica

AskConstants is an application that I implemented in Mathematica, freely downloadable
from [AskConstants.org](http://www.askconstants.org). AskConstants 4.1 has about 3000 integer relation models, and
bidirectional search with a choice of precomputed tables for which the largest backward
table has about 5 million entries and the largest forward table has about 14 million
entries. The table lookup exploits sign and base two significand aliasing, with automatic
de-aliasing and inversion for close matches.\[10\] Thus ignoring overflow underflow, nonreal
compositions and the aliasing bonus, the largest tables cover about \( (5 \times 10^6) (14 \times 10^6) = 7 \times 10^{13} \) expressions. The backward tables and half of the largest forward table use tabulated
constants. The other half of the largest forward table was generated by exhaustive
table building with the elementary functions and the most commonly-occurring special

\[9\] A replacement subexpression could still be an impostor, but if you are satisfied with the float values
of the replacements, then at least the result is a satisfactory approximation to the original expression,
which was not an exact representation either.

\[10\] The bonus for base 2 aliasing exceeds that of base 10 aliasing because 2 divides a random reduced
numerator or denominator more often than 10 does.
functions to combine advantages of precomputed tables such as ISC and of exhaustive breadth-first tables such as MESearch and RIES.

Figure 4 shows the use of version 4.1 to propose a closed form for the definite integral

\[ \int_0^\infty \frac{dx}{\sqrt{1 + x^2 + (1 + x^2)^{7/2}}} \]

that the \textit{Mathematica} 12.1 \texttt{Integrate} function cannot determine.\footnote{I thank Daniel Lichtblau for this example.}

Figure 3: \texttt{AskConstants} proposes a formula that agrees with 3.26115\ldots to 24 digits

The input field contains the \textit{Mathematica} \texttt{NIntegrate} function to compute the approximate float value displayed below the digit ruler. More commonly, the numerical integration would be done in a notebook, with the float result copied then pasted into the \texttt{AskConstants} input field.

\texttt{AskConstants} displays the candidate result

\[ \frac{\text{ArcCosh}\left[\frac{2}{\sqrt{3}}\right]}{2\sqrt{3}} + \frac{1}{3} \left( -\frac{\pi}{2} + \frac{\text{ArcSinh}\left[1\right]}{\sqrt{2}} \right) \] (6)

on the left side of the Result tab and displays an associated scatter plot of Agreement versus Entropy10 on the right side, where:

- The \textbf{Agreement} of a candidate nonfloat is the number of digits that agree with the input float within 1/2 unit, up to the length of the input significand, including any stored digits that are not ordinarily displayed.
• **Entropy10** is a measure of nonfloat expression complexity that is the sum of the base-ten logarithms of the absolute values of all the nonzero integers occurring in numerators and denominators in an expression, plus about 1.0 per occurrence of a named constant, function or operator – counting duplicates. The implicit units of Entropy10 are also digits.

• The scatter plot shows the Agreement versus Entropy10 for the one accepted candidate as a large dot, with small dots for some computed but rejected candidates for comparison.

• It is desirable to have high agreement, but it is also desirable to have low expression complexity because floats can be approximated arbitrarily well by sufficiently complicated nonfloats that are impostors rather than the limit you seek.

• Therefore

\[
\text{Margin} := \text{Agreement} - \text{Entropy10}
\]

(7)

combines these two competing objectives into one of maximizing Margin, with the largest Margin in the upper left corner of the plot and the smallest Margin in the lower right corner.

• Diagonal Margin intervals are rainbow color-shaded and labeled with adjectives from “Terrible” through “Excellent” as a qualitative guide to the likelihood that a candidate is the limit you seek rather than an impostor\(^\text{12}\)

• Typically, a lower left cluster of rejects is mostly from table lookup, and an upper right cluster is mostly from integer relations.

• As illustrated, hovering the mouse over a plot point lists the nonfloat candidate together with its coordinates and Margin.

• The dashed diagonal line is the lowest acceptable Margin.

• The upper horizontal dashed line is the Precision of the input float estimated by Mathematica’s arbitrary-precision significance arithmetic – or 15.95 for machine floats.

• The horizontal dashed line below it is the smallest acceptable Agreement for the Precision of the given float.

• The large accepted plot point has an Agreement one digit greater than the input Precision because the first normally-undisplayed guard digit in the significand also agreed with the value of the nonfloat computed to larger precision. This “home team advantage” for arbitrary precision floats computed in the AskConstants input field or copied and pasted in the same Mathematica session does not occur for floats copied from outside Mathematica or typed without guard digits.

\(^{12}\)Candidates are not produced in approximate order of increasing complexity, so the helpful likelihood estimates of MESearch are inapplicable. The qualitative adjectives “Terrible” through “Excellent” are based on my experience with test examples.
• Notice that the reject in the extreme upper right corner has even larger Agreement than the accepted result. However, its nonfloat is

\[ \pi \text{ArcTan} \left[ \frac{168443027133}{1616842785493} \right]. \]

Although this could be the exact expression we seek, such large rational number numerators and denominators compared to any in the integrand or integration limits are typical of integer relation overfits. That is why basing acceptance on Agreement alone is especially problematic for integer relation results.

• More generally there might be several accepted candidates. Ones judged equivalent by applying the Mathematica PossibleZeroQ function to their differences are joined by line segments, which can help you assess them more efficiently.

Automatic use of the backward transformation requires an inverse function for the top-level function in the transformation; and as with most mathematics software, Mathematica has very few inverse special functions — probably because they are extremely difficult to implement for nonreal arguments. However, AskConstants directly addresses only real floats, and it is not too difficult to implement most inverse special functions for real arguments and results. Consequently I did that for about 40 special functions. Many of those are multi-branched, such as for BesselJ, Gamma, and Zeta. Consequently I also implemented corresponding functions that return the abscissa and ordinates of the infima and suprema of those functions to accomplish piecewise monotonic partitions of the functions being inverted. The real inverse and infima or suprema functions are separately usable, as are some additional functions analogous to the Mathematica BesselJZero function.

To overcome the limitation to constant expressions, real arguments and results, AskConstants also contains a ProposeBestOrInput function that maps over general expressions and over the real and imaginary parts of non-real constants similar to the Maple identify function.

8.1.1 NSimplify [...] 

Anyone who tests the programs described here with constants having known nonfloat representations might eventually notice that surprisingly often an equivalent proposed candidate is simpler than the published expression. This is because:

• Manual derivations and default computer algebra simplification often produce incompletely simplified expressions.

• Sometimes there is no composition of builtin optional computer algebra transformations that can produce a particularly simple representable equivalent that exists.

Consequently the AskConstants download includes an NSimplify function that supplements the Mathematica FullSimplify function by approximating a nonfloat input expression as a float, then applying the ProposeBestOrInput function to that float. If it

\[^{13}\text{It can also be problematic for table lookup, but less frequently.}\]

\[^{14}\text{Some users might find these functions more frequently useful than AskConstants.}\]
returns a nonfloat having smaller Entropy10 than the given nonfloat and the Mathematica PossibleZeroQ function judges that the difference between the original expression and proposed replacement is 0, then the original is replaced. If that doesn’t succeed, then this process is recursively applied to subexpressions. NSimplify also tries FullSimplify so that the result is at least that successful. This brute-force combination is slow, but it can achieve some dramatic simplifications, and some users might find NSimplify more frequently useful than AskConstants and ProposeBestOrInput.

As an example, applying NSimplify to

\[
\sqrt{\frac{1}{2} - \frac{1}{4}} \rightarrow \sqrt{\frac{4}{2} - \frac{\sqrt{7 - \sqrt{5} + \sqrt{30 - 6\sqrt{5}}}}{4}}
\]

Root \([-97 + 448 #1 - 672 #1^2 + 560 #1^3 - 280 #1^4 + 84 #1^5 - 14 #1^6 + #1^7 &, 1]\]

produced the result

\[
\sin \left( \frac{7\pi}{120} \right)
\]

and the optional trace reported the steps

- Level 2: ProposeBestOrInput \(N\left[\frac{1}{2} - \sqrt{\frac{1}{4}} \cdots ; 26\right]\) \(\mapsto\) \(\sin \left( \frac{7\pi}{120} \right)\), reducing Entropy10 by 14.1; and PossibleZeroQ [difference] \(\mapsto\) True.

- Level 2: ProposeBestOrInput \(N[\text{Root} \{-97 + \cdots + #1^7 &, 1\}, 38]\) \(\mapsto\) \(\frac{1}{2} - 31^{1/7}\), reducing Entropy10 by 18.6; and PossibleZeroQ [difference] \(\mapsto\) True.

Remarks:

1. NSimplify maps over over general expressions, replacing exact constant subexpressions with less complex ones where it can.

2. The Mathematica PossibleZeroQ Function can incorrectly return False, but this would merely cause NSimplify to miss an Entropy10 reduction opportunity.

3. The PossibleZeroQ function can incorrectly return True, causing NSimplify to return a nonequivalent result. However, in this application that also requires the parsimonious AskConstants ProposeBestOrInput function to have generated an impostor nonfloat for a float subexpression, and the product of these two small probabilities is such that I have not yet noticed it happen. I suspect that the overall probability is not much more than that of a nonequivalent result due to a bug in the many built-in Mathematica functions.
8.2 Plouffe’s inverter for Maple

After developing ISC, Simon Plouffe developed Plouffe’s Inverter [13], a similar Maple application with larger tables. As of August 2019, the backward tables contains 405 transformations. The largest version of the forward tables currently contains $69 \times 10^9$ entries on his largest computer, and it continues to grow. Ignoring overflow, underflow, non-real values, sign aliasing and significand aliasing, that version can potentially match $405 \times 69 \times 10^9 \approx 3 \times 10^{13}$ expressions. That version is too large for practical distribution, but if you have already tried other systems and desperately want to exhaust your possibilities, then you can try sending your float constant to him to try.

In contrast, his portable version of the application contains 201 million forward table entries. Thus his portable version can match about $405 \times 201 \times 10^6 \approx 8 \times 10^{10}$ expressions, which is about 24 times as many expressions as Inverse Symbolic Calculator, which can match about $100 \times 30 \times 10^6 = 3 \times 10^9$ expressions. The Maple program without the tables is freely downloadable text that you can copy from [http://plouffe.fr/PlouffeInverter2017.txt](http://plouffe.fr/PlouffeInverter2017.txt) then copy into a plain text editor such as the Windows accessory Notepad, then save as file PlouffeInverter.

There are two choices for the tables. The easiest one, which you might want to try first, is to insert the following fragment into your PlouffeInverter Maple program to use the ISC forward table remotely:

```maple
with(Sockets):
server := 'wayback.cecm.sfu.ca';
s := Sockets:-Open(server, 80);
Sockets:-Write(s, cat('GET /cgi-bin/ipcgi/lookup.pl?number=', valeur,'&lookup_type=simple
'));
text := ReadPage(s);
Sockets:-Close(s);
```

For this alternative the inverter has about four times as many transformations as ISC, so you will be able to match about four times as many expressions.

The other choice is to download about 9,000 GNU-zipped forward table files totaling about 370 gigabytes from [http://plouffe.fr/ip/](http://plouffe.fr/ip/), then unzip them with a compatible unzipper. When unzipped they require a total of slightly more than a terabyte of secondary storage. You will need a fast reliable internet connection and a sufficiently large fast USB-connected external drive, which costs less than $100 USA in 2019.

Either way, there are some file-paths that must be changed, as indicated in comments near the beginning of the program.

Also, his inverter uses the Unix `look` command, for which most implementations do a surprisingly fast search for a leading substring in a sorted file of strings that are not necessarily all the same length. If your computer does not already have that part of the Unix-family operating system, then one easy compact way to acquire the `look` command is to download the free Sage mathematical software system, which is independently advantageous.
9 Custom Integer Relation Models

Wild floats often occur at the high end of a family of problems depending on a parameter \( n \), with known nonfloat values for the low end. For example, a noticeable number of definite integrals have nonfloat representations that can be expressed as a rational linear combination of terms with cofactors that are small positive integer powers of \( \pi \), \( \ln 2 \), \( \ln 3 \), \( \zeta(m) \) with small positive integer \( m \), and low-order polylogarithms with simple arguments such as \( 1/2 \) or \( 1/4 \), perhaps multiplied by \( \sqrt{2} \) or \( \sqrt{3} \). The known closed forms for small \( n \) permits guesses of possible cofactors for the next value of \( n \), and if a guess is successful, then you can use that form to make cofactor guesses for the next value of \( n \).

An increasing number of computer algebra systems have builtin integer-relation solvers, and David Bailey and David Broadhurst [1, 2] describe some particularly efficient implementations in Fortran 90 and C++. Such large models are too time consuming to make them part of the set of more general-purpose integer relation models in tools such as WolframAlpha, AskConstants, or the Maple and MPMath identify functions. However, it is easy to simply invoke a built-in or stand-alone PLSQ function with an input vector containing your float and a set of cofactors.

10 The curse of extreme magnitude

The software described here is most successful at proposing candidate nonfloats for floats whose magnitudes are not extremely different from 1.0. A reason for this is that representations of nonfloat constants having extreme magnitudes often require high-complexity extreme magnitudes of the numerators or denominators of some rational numbers therein, which requires extreme precision input floats for integer relation algorithms or prohibitively extreme magnitude integer subexpressions in lookup tables. Exceptions are functions whose nonzero magnitude can be extremely large or small for arguments that are not extremely large or small, such as \( \Gamma(98/3) \approx 8.26 \times 10^{34} \) or \( \text{erfc}(9) \approx 4.14 \times 10^{-37} \).

Fortunately, most published mathematics constants do not have extreme magnitudes. For example, in Steven Finch’s table of about 10,000 such constants, nonzero magnitudes vary from 0.000111582 through 137.0359 with median about 0.9 and quartiles about 0.4 and 1.9.

11 Some causes of impostors

“1.0000001 is a crowd”

– adapted from James Thurber

Reasons for impostors include:

1. Many functions \( f(x) \) have a stationary value of 1 for some value of \( x \). For example, \( \cos x \), \( \sec x \) and \( \cosh x \), at \( x = 0 \); or \( \tanh x \) and \( \text{erf}(x) \) as \( x \to \infty \). In a neighborhood of the stationary point, such constants that are modeled are likely
to occur as impostors for any that are not modeled, because very low complexity $x$

2. If for some relatively low complexity nonfloat constant $x$ a modeled expression $f(x)$

3. If a sum has some terms with very small magnitudes compared to other terms,
impostors often omit those small-magnitude terms – at least until sufficiently large

12 Conclusions
There are many good tools that can propose nonfloat candidates that your float closely

Most of the programs discussed in this article can propose correct candidates that

Consequently it is wise to try at least:

- the Online Encyclopedia of Integer Sequences\(^{15}\)

- at least one of the general purpose web search engines on your computers or smart

\(^{15}\) This is also a good place to find high precision values and program fragments for efficiently com-

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• one that uses exhaustive breadth-first search (MESearch and RIES).

At the very least you should try all that are built into the computer algebra systems that you already have, together with all of the web-based tools (a browser’s search engine, the Online Encyclopedia of Integer Sequences, Inverse Symbolic Calculator and WolframAlpha) because that is so easy to do.

You might develop favorites that tend to work better for your needs, but it is nice to have all of these alternatives when your favorites don’t succeed and the result is important to you.

The success of your efforts depends strongly on knowing how to groom your float for input, interpret the results, and possibly de-alias and transform the result. These details vary greatly among the tools, and the descriptions of these tools here are intended partly as brief user guides that you can review for tools that you are not already familiar with.

When none of these programs accessible to you can propose a plausible constant, then at least you can have the peace of mind of greatly reducing the chance that your problem has a simple nonfloat result that you did not find.

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