Abstract

We show that the splitting feature of the Einstein tensor, as the first term of the Lovelock tensor, into two parts, namely the Ricci tensor and the term proportional to the curvature scalar, with the trace relation between them is a common feature of any other homogeneous terms in the Lovelock tensor. Motivated by the principle of general invariance, we find that this property can be generalized, with the aid of a generalized trace operator which we define, for any inhomogeneous Euler–Lagrange expression that can be spanned linearly in terms of homogeneous tensors. Then, through an application of this generalized trace operator, we demonstrate that the Lovelock tensor analogizes the mathematical form of the Einstein tensor, hence, it represents a generalized Einstein tensor. Finally, we apply this technique to the scalar Gauss–Bonnet gravity as an another version of string–inspired gravity.

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1 Introduction

Among scalar Lagrangians, field equations based on a Lagrangian quadratic in the curvature tensor have had a long history in the theory of gravitation [1]. Perhaps a legitimate mathematical motivation to examine gravitational theories built on non–linear Lagrangians has been the phenomenological character of the Einstein theory, that is the dependence of the Einstein tensor and Lagrangian on the derivatives of the metric, which leaves room for such amendments and the dimension [2]. Actually, the Einstein Lagrangian is not the most general second order Lagrangian allowed by the principle of general invariance, and indeed, through this principle the latter generalization can be performed up to any order, and a general scalar Lagrangian is a higher derivative Lagrangian.

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Despite Einstein’s gravitational theory successes, its main difficulties become manifest when the curvature of space–time is not negligible, e.g., in the very early universe with distances of the order of Planck’s length, where an Euclidean topological structure is quite unlikely. At such distances, even the fluctuations of quantum gravitation will be extremely violent and probably produce an ever changing, dynamic topology [3]. This perhaps allows non–linear gravitational Lagrangians to be considered as alternative theories.

Nowadays, it is also well known that Einstein’s gravity when treated as a fundamental quantum gravity leads to a non–renormalizable theory. In order to permit renormalization of the divergences, quantum gravity has indicated that the Einstein–Hilbert action should be enlarged by the inclusion of higher order curvature terms [5]. In fact, it has been shown [4] that the Lagrangian  \[ L = \frac{1}{\kappa^2} (R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}) \]

which, by the Gauss–Bonnet (GB) theorem, is the most general quadratic Lagrangian in, and up to, four dimensions and has the required Newtonian limit, solves the renormalization problem and is multiplicatively renormalizable [6] and asymptotically free [7]; however it is not unitary‡ within usual perturbation theory [8]. Actually, its particle spectrum contains a further massive scalar spin–two ghost, which has either negative energy or a negative norm, and the existence of negative energy excitations in a model leads to causality violation [8]. However, the lack of unitarity is the main reason for not considering higher order gravities as the best candidate for the quantum gravity description.

The theory of superstrings, in its low energy limits, also suggests the above inclusions, and in order to be ghost–free it is shown [9] that it must be in the form of dimensionally continued GB densities, that is the Lovelock Lagrangian [10,11],

\[
\mathcal{L} = \frac{1}{\kappa^2} \sum_{0<n<\frac{D}{2}} \frac{1}{2^n} c_n \delta^{\alpha_1...\alpha_{2n}}_{\beta_1...\beta_{2n}} R_{\alpha_1\alpha_2}^{\beta_1...\beta_{2n}} R_{\alpha_2\alpha_{n-1}}^{\beta_2...\beta_{2n-1}} \beta_{2n} \equiv \sum_{0<n<\frac{D}{2}} c_n L^{(n)} \]

where we set \( c_1 \equiv 1 \) and the other \( c_n \) constants to be of the order of Planck’s length to the power \( 2(n-1) \), for the dimension of \( \mathcal{L} \) to be the same as \( L^{(1)} \). Symbol \( \delta^{\alpha_1...\alpha_p}_{\beta_1...\beta_p} \) is the generalized Kronecker delta symbol, see, e.g., Ref. [12], which is identically

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\( \circ \) See, for example, Refs. [4] and references therein.

\( \ast \) \( \alpha, \beta \) and \( \kappa^2 \equiv \frac{16\pi G}{c^4} \) are constants, and field equations are shown as \( G^{(\text{gravitation})}_{\alpha\beta} = \frac{1}{2} \kappa^2 T_{\alpha\beta} \), where the definition \( \delta (L_m \sqrt{-g}) \equiv -\frac{1}{2} \sqrt{-g} T_{\alpha\beta} \delta g^{\alpha\beta} \) is used.

\( \dagger \) A characteristic property of unitarity is the scalar product, or norm, invariance.

\( \dagger \) Our conventions are a metric of signature +2, \( R^{\mu}_{\nu\alpha\beta} = -\Gamma^{\mu}_{\nu\alpha\beta} + \cdots \), and \( R_{\mu\nu} \equiv R^\rho_{\rho\mu\nu} \).
zero if \( p > D \). The maximum value of \( n \) is related to space–time dimension, \( D \), by

\[
n_{\text{max}} = \begin{cases} 
\frac{D}{2} - 1 & \text{even } D \\
\frac{D-1}{2} & \text{odd } D 
\end{cases}.
\]

(1.2)

An important aspect of this suggestion is [13] that it does not arise in attempts to quantize gravity. The above ghost–free property, and the fact that the Lovelock Lagrangian is the most general second order Lagrangian which, the same as the Einstein–Hilbert Lagrangian, yields the field equations as second order equations, have stimulated interests in Lovelock gravity and its applications in the literature, see, e.g., Refs. [14,15] and references therein. The Lovelock Lagrangian obviously reduces to the Einstein–Hilbert Lagrangian in four dimensions and its second term is the GB invariant, \( L^{(2)} = \frac{1}{\kappa^2} (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) \).

We have noticed that each term of the Lovelock tensor, that is \( G^{(n)}_{\alpha\beta} \), where the Lovelock tensor is [10,11]

\[
G_{\alpha\beta} = - \sum_{0<n<\frac{D}{2}} \frac{1}{2n+1} c_n g_{\alpha\mu} \delta^{\mu\alpha_1...\alpha_{2n}}_{\beta_1...\beta_{2n}} R_{\alpha_1\alpha_2...\alpha_{2n}} R_{\alpha_{2n-1}\alpha_2...\alpha_2} \equiv \sum_{0<n<\frac{D}{2}} c_n G^{(n)}_{\alpha\beta},
\]

(1.3)

has also the following remarkable properties. We mean, each term of the \( G^{(n)}_{\alpha\beta} \) can be rewritten in a form that analogizes the Einstein tensor with respect to the Ricci and the curvature scalar tensors, namely \( G^{(n)}_{\alpha\beta} = R^{(n)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(n)} \) where

\[
R^{(n)}_{\alpha\beta} \equiv \frac{n}{2n} \delta^{\alpha_1\alpha_2...\alpha_{2n}}_{\beta_1...\beta_{2n}} R_{\alpha_1\alpha_2...\alpha_{2n}} R_{\beta_{2n-1}\alpha_2...\alpha_2},
\]

(1.4)

\[
R^{(n)} = \frac{1}{2n} \delta^{\alpha_1...\alpha_{2n}}_{\beta_1...\beta_{2n}} R_{\alpha_1\alpha_2...\alpha_{2n}} R_{\beta_{2n-1}\alpha_2...\alpha_2},
\]

(1.5)

\( R^{(1)}_{\alpha\beta} \equiv R_{\alpha\beta} \) and \( R^{(1)} = R \).

The proof of these can easily be done using the definition of the generalized Kronecker delta symbol and the properties of the Riemann–Christoffel tensor. An

‡ We have neglected the cosmological term, and \( G^{(1)}_{\alpha\beta} = G_{\alpha\beta} \) that is, the Einstein tensor.
alternative and more basic approach is to notice that it is what one gets in the process of varying the action, \( \delta \int L^{(n)} \sqrt{-g} \, d^{D}x \), where its Euler–Lagrange expression will be

\[
\frac{\delta L^{(n)}}{\delta g_{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} L^{(n)} = \frac{1}{\kappa^2} G^{(n)}_{\alpha\beta} .
\]

Hence, from relations (1.1), (1.4) and (1.5), one can easily show that

\[
\frac{\delta L^{(n)}}{\delta g_{\alpha\beta}} = \frac{1}{\kappa^2} R^{(n)}_{\alpha\beta} \quad \text{and} \quad L^{(n)} = \frac{1}{\kappa^2} R^{(n)} . \tag{1.7}
\]

Almost the same procedure has, perhaps, been carried by Lovelock [10], but he then proceeded from this to derive equation (1.3).

Before we continue, we should indicate that one does not need necessarily to restrict oneself to relations (1.4), (1.5) and (1.7) in any kind of generic field equations and may take any term in the final result of \( \frac{\delta L}{\delta g_{\alpha\beta}} \) which appears to be a scalar multiple of \( g_{\alpha\beta} \) out of it, see Ref. [16].

Although the above derivation is straightforward, what is not so obvious at the first sight is that there also exists a relation between \( R^{(n)}_{\alpha\beta} \) and \( R^{(n)} \) analogous to that which exists between the Ricci tensor and the curvature scalar, namely

\[
\frac{1}{n} \text{trace } R^{(n)}_{\alpha\beta} = R^{(n)} , \tag{1.8}
\]

where the trace means the standard contraction of any two indices that is, for example, \( \text{trace } A_{\mu\nu} \equiv g^{\alpha\beta} A_{\alpha\beta} \).

Hence, the splitting feature of the Einstein tensor, as the first term of the Lovelock tensor, into two parts with the trace relation between them is a common feature of any other terms in the Lovelock tensor, in which each term alone is a homogeneous Lagrangian. Indeed, this property has been resulted through the variation procedure. Thus, motivated by the principle of general invariance, one also needs to consider what might happen if the Lagrangian under consideration, and hence its relevant Euler–Lagrange expression is an inhomogeneous tensor, as for example, the (whole) Lovelock Lagrangian, \( \mathcal{L} \), which is constructed of terms with a mixture of different orders.

In this case, the relevant Euler–Lagrange expression can easily be written by analogy with the form of \( G^{(n)}_{\alpha\beta} \), for example, \( \mathcal{G}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} \) where

\[
\mathcal{R}_{\alpha\beta} \equiv \sum_{0 < n < \frac{D}{2}} c_n R^{(n)}_{\alpha\beta} \quad \text{and} \quad \mathcal{R} \equiv \sum_{0 < n < \frac{D}{2}} c_n R^{(n)}.
\]

(1.9)

But a similar relation to equation (1.8) does not apparently hold between \( \mathcal{R}_{\alpha\beta} \) and
due to the factor $\frac{1}{n}$.

To overcome this issue, in the next section we will introduce a generalized trace as an extra mathematical tool for Riemannian manifolds, which will slightly alter the original form of the trace relation and modify it sufficiently to enable us to deal with the above difficulty. Then, in Section 3, we will consider the case of the inhomogeneous Lovelock tensor, where also some discussions will be presented, and in the last section, we will apply our technique to a more interested case of the scalar GB gravity.

## 2 Generalized Trace

In this section, we will define a generalized trace, which we will denote by $\text{Trace}^*$, for tensors whose components are homogeneous functions of the metric and its derivatives.

But first, as either of $g_{\mu\nu}$ or $g^{\mu\nu}$ can be chosen as a base for counting the homogeneity degree numbers, we will choose, without loss of generality and as a convention, the homogeneity degree number (HDN) of $g^{\mu\nu}$ as $[+1]$; hence, the HDN of $g_{\mu\nu}$ will be $[-1]$ since $g^{\mu\nu} g_{\mu\alpha} = \delta^\nu_\alpha$. So, as contravariant and covariant tensors are mapped into each other in a one-to-one manner by the metric, their HDNs are different by $\pm 2$. Similarly, we will choose the HDN of $g^{\mu\nu,\alpha}$ as $[+1]$. Therefore, from $g_{\alpha\beta,\rho} = -g_{\alpha\mu} g_{\beta\nu} g^{\mu\nu,\rho}$, the HDN of $g_{\mu\nu,\alpha}$ will be $[-1]$ as well. To specify the HDNs of higher derivatives of the metric, one may consider $\partial_\alpha$ as if with the HDN of zero, and hence for $\partial^\alpha \left( = g^{\alpha\beta} \partial_\beta \right)$ as if with $[+1]$. By the very elementary property of homogeneous functions, the HDN of a term consists of cross functions is obviously found by adding the HDN of each of the cross functions. For convenience, the HDNs, $h$, of some homogeneous functions of the metric and its derivatives are given in Table 1, and whenever necessary, we will show the HDN of a function in brackets attached to the upper left hand side, e.g., $[+1]g^{\mu\nu}$ and $[-1]g_{\mu\nu}$.

Now, for a general \( \binom{N}{M} \) tensor, e.g., $A^{\alpha_1...\alpha_N}_{\beta_1...\beta_M}$, which is a homogeneous function of degree $h$ with respect to the metric and its derivatives, we define

\[
\text{Trace}^{[h]} A^{\alpha_1...\alpha_N}_{\beta_1...\beta_M} := \begin{cases} 
\frac{1}{h - \frac{N}{2} + \frac{M}{2}} \text{trace}^{[h]} A^{\alpha_1...\alpha_N}_{\beta_1...\beta_M} & \text{when } h - \frac{N}{2} + \frac{M}{2} \neq 0 \\
\text{trace}^{[h]} A^{\alpha_1...\alpha_N}_{\beta_1...\beta_M} & \text{when } h - \frac{N}{2} + \frac{M}{2} = 0.
\end{cases}
\]  

\[ (2.1) \]

* These definitions match our HDN conventions of $[+1]g^{\mu\nu}$ and $[+1]g^{\mu\nu,\alpha}$ to comply with our needs.
Table 1: The HDNs of useful homogeneous functions.

| Function                                      | The HDN |
|-----------------------------------------------|---------|
| \( g^{\mu\nu} \) (our convention)            | +1      |
| \( g_{\mu\nu} \)                             | -1      |
| \( g^{\mu\nu},_\alpha \) (our convention)    | +1      |
| \( g^{\mu\nu},_\alpha \)                     | +2      |
| Operator \( \partial_\alpha \)               | as if 0 |
| Operator \( \partial^\alpha \)               | as if +1|
| \( g^{\mu\nu},_\alpha \) \( \equiv \) \( \Delta \) \( (g_{\mu\nu}) \) | -D      |
| \( \Gamma_\alpha^{\mu\nu} \)                | 0       |
| \( R_\alpha^{\beta\mu\nu} \)                | 0       |
| \( R_{\alpha\beta\mu\nu} \) \& \( C_{\alpha\beta\mu\nu} \) | -1      |
| \( R_{\mu\nu} \)                             | 0       |
| \( R^{\mu\nu} \)                             | +2      |
| \( R \)                                      | +1      |
| \( L^{(n)} \) \& \( R^{(n)} \)               | +n      |
| \( R^{(n)}_{\mu\nu} \) \& \( G^{(n)}_{\mu\nu} \) | \( n-1 \) |

Contravariant and covariant components of a tensor obviously have different HDNs, however, by the above definition, the equality of their Traces is still retained. For example, \( \text{Trace} \, A_{\mu\nu} = \text{Trace} \, A^{\mu\nu} \equiv A \) regardless of what the HDN, just as for the trace operator that is, \( \text{trace} \, A_{\mu\nu} = \text{trace} \, A^{\mu\nu} \equiv A_\alpha^\alpha \). Note that using our definition it follows that

\[
\begin{cases}
  A = \frac{1}{h+1} A_\alpha^\alpha & \text{for } h \neq -1 \\
  A = A_\alpha^\alpha & \text{for } h = -1 \text{,} \\
\end{cases}
\]  \hspace{1cm} (2.2)

where the factor of \( \frac{1}{h+1} \) has entered because we have taken \( [h] \, A_{\mu\nu} \), and therefore used the fact that the HDNs of both \( A \) and \( A_\alpha^\alpha \) are \( [h+1] \).

In general, the generalized trace has, by its definition, all of the properties of the usual trace, especially its invariance under a similarity transformation (for similar tensors) if the transformation does not change the HDN of the tensor, and its basis independence for linear operators on a finite dimensional Hilbert space. However, as we will show in the following, it cannot act as a linear operator† when

† That is, for example, \( \text{trace} \left( a_1 \, A_{\mu\nu} + a_2 \, B_{\mu\nu} \right) = a_1 \, \text{trace} \, A_{\mu\nu} + a_2 \, \text{trace} \, B_{\mu\nu} \).
the coefficients of linearity themselves are any scalar homogeneous functions of degree \( h' \neq 0 \).

By the definition of the Trace, for cases when \( h' \neq 0 \), we have, for example,

\[
\text{Trace} \left( {^\prime}h C^\prime [h] A_{\mu \nu} \right) = \frac{1}{h' + h + 1} \text{trace} \left( {^\prime}h C^\prime [h] A_{\mu \nu} \right) \quad \text{for } h' + h \neq -1
\]

\[
= \frac{[h']C}{h' + h + 1} \text{trace} [h] A_{\mu \nu} ,
\]

and using the definition once again, we get

\[
\text{Trace} \left( {^\prime}h C^\prime [h] A_{\mu \nu} \right) = \begin{cases} 
\frac{h+1}{h'h+1} [h']C \text{Trace} [h] A_{\mu \nu} \quad \text{for } h \neq -1 \\ 
\frac{1}{h'} [h']C \text{Trace} [h] A_{\mu \nu} \quad \text{for } h = -1
\end{cases} \tag{2.3}
\]

Alternatively, we obtain

\[
\text{Trace} \left( {^\prime}h C^\prime [h] A_{\mu \nu} \right) = \frac{1}{[h']C} \text{Trace} [h] A_{\mu \nu} \quad \text{for } h = -1 .
\]

(2.4)

Obviously, these extra factors can be made equal to one, only when \( h = -1 \) and \( h' = +1 \), or when \( h = 0 \) and \( h' = -1 \), as in second equation of (2.3) or in equation (2.4), respectively.

Therefore, due to these extra factors the Trace is not a linear operator as mentioned above.

It is necessary to emphasize that for our immediate purposes, which led us initially to define a generalized trace, the Trace indeed has the distributive law of the usual trace in the cases when there are either no coefficients of linearity, or when coefficients are included with their associated tensors, and or when coefficients are meant to be scalars with \( h' = 0 \).

To justify the way that we have defined a generalized trace, other than that it satisfies our need for dealing with inhomogeneous Lagrangians, one can show that this definition also has a link with Euler’s theorem for homogeneous functions.

Suppose \( A(g^{\mu \nu}) \) is a homogeneous scalar function of degree \([h]\), i.e. \( A(\lambda g^{\mu \nu}) = \lambda^h A(g^{\mu \nu}) \). Euler’s theorem states that \( g^{\mu \nu} \frac{\partial A}{\partial g^{\mu \nu}} = h A \). As a rough and ready argument, define \( \frac{\partial A}{\partial g^{\mu \nu}} \equiv A_{\mu \nu} \), where \( A_{\mu \nu} \) is of degree \( h - 1 \), then \( g^{\mu \nu} A_{\mu \nu} \) denotes
its usual trace. Also, define $A \equiv \text{Trace } A_{\mu \nu}$. Then, from Euler’s theorem, one can derive $\text{trace}^{[h-1]} A_{\mu \nu} = h \text{ Trace } A_{\mu \nu}$, or

$$\text{Trace}^{[h-1]} A_{\mu \nu} = \frac{1}{h} \text{trace } A_{\mu \nu} \quad \text{ when } h \neq 0 ,$$

which is the same as our definition (2.1). When $h = 0$, which means that $A$ does not depend on the metric and its derivatives, Euler’s theorem is trivial, that is $\frac{\partial A}{\partial g_{\mu \nu}} = 0$. Therefore, the best and consistent choice is to make no distinction between $\text{Trace}$ and the trace for $[0] A$.

On the other hand, using Table 1, it is straightforward to relate [14] the orders $n$ in any Lagrangian, as in $L^{(n)}$, that represents its HDN.* So, one may refer to Lagrangians with their HDNs rather than their orders. As an immediate efficiency, consider the following example. In order to amend the Lagrangian of sixth order gravity [17], Berkin et al [18] discussed that the Lagrangian term of $R \Box R$ is a third order Lagrangian based on the dimensionality scale, for two derivatives are dimensionally equivalent to one Riemann–Christoffel tensor or any one of its contractions. However, it can be better justified on account of the above regard, since it has the HDN three.

Using the above aspect, the special case of $h = 0$, similar to relation (1.1), corresponds to $c_0 L^{(0)} \equiv 2\Lambda/\kappa^2$, a constant, which produces the cosmological term, $-\Lambda \Box g_{\mu \nu}$ or equivalently $\Lambda \Box g^{\mu \nu}$, in the field equations. Hence, the exception value in our definition of generalized trace, equation (2.1), for an scalar maybe related to the cosmological term difficulty, see Ref. [14] for more details. Nevertheless, with our choice of definition for the generalized trace, one has

$$\text{Trace}^{[+1]} g^{\mu \nu} = \text{trace } g^{\mu \nu} = D \quad \text{ and } \quad \text{Trace}^{[-1]} g_{\mu \nu} = \text{trace } g_{\mu \nu} = D .$$

Finally, as an example, if one applies the definition of generalized trace on equation (1.4) a relation similar to equation (1.8) will be obtained, but in even more analogous form for each order, namely

$$\text{Trace } R^{(1)}_{\alpha \beta} = R^{(1)} = \kappa^2 L^{(1)}$$
$$\text{Trace } R^{(2)}_{\alpha \beta} = R^{(2)} = \kappa^2 L^{(2)}$$
$$\text{Trace } R^{(3)}_{\alpha \beta} = R^{(3)} = \kappa^2 L^{(3)}$$

$$\vdots$$
$$\text{Trace } R^{(n)}_{\alpha \beta} = R^{(n)} = \kappa^2 L^{(n)} .$$

* This choice is as to be consistent with our HDN conventions of $[+1] g^{\mu \nu}$ and $[+1] g^{\mu \nu} , \alpha$, and with our definition of $\text{Trace}$.  

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where, by equation (2.2), we generally have

\[ R^{(n)} = \frac{1}{n} R^{(n)} \rho \rho, \]  

(2.7)

and, for example, \( R^{(1)} = R^{(1)} \rho \rho \equiv R \), as expected.

3 Inhomogeneous Lovelock Tensor and Discussions

As a simple evident of efficiency of the Trace operator, we will apply it to the Lovelock tensor.

In the case of inhomogeneous Lovelock Lagrangian, we have shown that the Lovelock tensor can be written as \( G_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R \). Furthermore, by substituting for \( R^{(n)} \) from equation (1.5) and using equation (1.1), we have \( \mathcal{L} = \frac{1}{\kappa^2} \mathbb{R} \) and also by substituting for it from equation (2.6), then using the distributivity of the Trace, we get \( \mathbb{R} = \text{Trace} \, R_{\alpha \beta} \), which is exactly in the same form as the equations of (2.6).

Hence, in higher dimensional space–times, the Lovelock tensor, which reduces to the Einstein tensor in four dimensions and its other useful and interest properties have been summarized in the introduction, analogizes the mathematical form of the Einstein tensor as well. Implicitly, in higher order gravities, where the geometry is represented by the Lovelock tensor, the field equations can be written as

\[ G_{\alpha \beta} = \frac{1}{2} \kappa^2 T_{\alpha \beta}. \]  

(3.1)

We therefore classify the Lovelock tensor, as a generalized Einstein tensor, and call \( \mathcal{L} \), \( R_{\alpha \beta} \) and \( \mathbb{R} \) the generalized Einstein’s gravitational Lagrangian, generalized Ricci tensor and generalized curvature scalar, respectively.

Recently, it has been claimed [20] that a distinct, but equivalent, derivation of the gravitational dynamics can be obtained for a Lovelock–type action from the trace of a Bianchi–type identity satisfied by a fourth rank tensor which is a polynomial in curvature. Besides, the trace of such a tensor is equal to the corresponding Lagrangian. This has been demonstrated [20] separately for the Einstein gravity, and also for the GB term alone. However, we know that the unity of physics during its development must be maintained by the correspondence principle. That is, in every new physical theory the previous one is contained as a limiting case. Indeed, gravitational theories based on a Lagrangian which is only purely quadratic in the curvature tensor have been strongly criticized [19] as nonviable theories. The two main objections against these Lagrangians are as the metric based on them does
not satisfy the flat space limit at asymptotically large distances; and disagreement with observations follows when the matter is incorporated. Therefore, one should demand that Einstein’s gravity must be maintained as a limiting case of non-linear gravitational Lagrangians. On this ground, if one wants to perform the approach of Ref. [20] for the Einstein plus the GB term gravity, or in general for the (whole) Lovelock polynomial terms together, one must employ the Trace operator instead of the usual trace one, for the same reasons explained in the introduction.

More applications and evident of usefulness of this Trace operator have recently been shown in Refs. [14,16,21]. This is perhaps the main reason why we have been encouraged to present the details of its definition, for a wider audience by its publication.

In this work, our main effort has been to define a generalized trace as an extra mathematical tool, by which, and as its application, we have shown that the analogy of the Einstein tensor can be generalized to any inhomogeneous Euler–Lagrange expression if it can be spanned linearly in terms of homogeneous tensors, e.g., the Lovelock tensor. The significant use of this new operator is that one can apply it to achieve the Lovelock gravity as a generalization of the Einstein gravity. However, we should emphasis that by making such a new definition for generalized trace, we do not really mean to change the essential or inherent properties of the original Lovelock tensor. The underlying hope motivating our work is to grasp better insight and understanding of the properties and abilities of the Lovelock gravity.

Besides the very well known classical successes of Einstein’s theory, the above analogy may be used as a part of a programme to impose a total analogy of Einstein’s gravity on Lovelock’s gravity [2], wherein the latter can then be considered as a generalized Einstein’s gravity. A tentative suggestion that may relate higher order gravities under some kind of transformation, e.g., conformal and or Legendre–like transformations, see, for example, Refs. [22]. Hence, a major task will be to construct, and hence to achieve, a generalized counterpart for each essential term used in Einstein’s gravity, especially the metric, for which the task is under investigation [23]. This, we believe, would give a better view on higher order gravities, and would also let straightaway to apply the results of Einstein’s theory to Lovelock’s theory. At least, such a generalization is of potential importance as it gives an alternative framework in order to derive consequences analogous to those obtained in general relativity for the generalized theories of gravities.

Almost in the same spirit, a further analogy in the mathematical form of the alternative form of Einstein’s field equations and the relevant alternative form of Lovelock’s field equations have shown that the price for this analogy is to accept the existence of the trace anomaly of the energy–momentum tensor even in
classical treatments [16]. Investigation has indicated [2,16] that there is an interesting similarity between the trace anomaly relation suggested by Duff [24] and the constraint relation that coefficients of any generic second order Lagrangian must satisfy in order to hold the desired analogy. That is, one may speculate that the origin of Duff’s suggested relation is classically due to the covariance of the form of Einstein’s equations [16]. And, in Ref. [14], a dimensional dependent version of Duff’s trace anomaly relation has been derived based on this analogy, in where its important achievements have also been itemized.

Implicitly, one must note that, wherever a term such as $g_{\alpha\beta} R$ is involved in an equation, its analogous counterpart, $g_{\alpha\beta} \Re$, may not lead to the correct corresponding equation in its generalized form. For example, in a $D$–dimensional space–time, a traceless Ricci tensor is usually defined as

$$Q_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{D} g_{\alpha\beta} R . \quad (3.2)$$

Whereas, the corresponding generalized Traceless Ricci tensor can only be defined as

$$Q_{\alpha\beta} \equiv \Re_{\alpha\beta} - \frac{1}{D} g_{\alpha\beta} \sum_{0<n<\frac{D}{2}} n c_n R^{(n)} , \quad (3.3)$$

and obviously, it is not the covariant form of the former relation. However, our main concern is the analogy in the fundamental equations of any theory of gravity.

In the next section we will apply the Trace operator to the scalar Gauss–Bonnet (SGB) gravity.

4 Scalar Gauss–Bonnet Gravity

The Lovelock Lagrangian in its total form has hardly been used in physical backgrounds. As, we have applied [14,16] the Trace operator for the first plus second terms of the Lovelock Lagrangian, that is the GB gravity. Also, we have employed this technique for third order term of the Lovelock Lagrangian [21]. But, despite the successes of the Lovelock gravity, especially the GB gravity, one may performs to consider also account of scalars, e.g., in simplest case, dilaton. Indeed, the SGB gravity as another version of string–inspired gravity, has been suggested, Refs. [25] and references therein, to be used recently as a possibility for gravitational dark energy, in order to explain the observed acceleration of the universe. Also, inclusion of higher order terms has been considered in Refs. [26]. This scenario exhibits several features of cosmological interest for late universe [15,25,26].
and applications in the early universe, see, e.g., Ref. [27] and references therein. Hence, it may be of more interest and or useful for physicists to apply the Trace operator to the SGB gravity, where this section is devoted to.

Typically the low–energy limit of the string theory characterizes scalar fields and their couplings to various curvature terms. We consider the effective action given by

\[ S = \int \left[ L^{(\text{SGB})} + L_m \right] \sqrt{-g} \, d^D x , \tag{4.1} \]

where \( L^{(\text{SGB})} \equiv L^{(1)} + L^{(\phi)} + L^{(0)} + f(\phi)L^{(2)} \), and where \( L^{(1)} = \frac{1}{\kappa^2} R \) the usual Einstein–Hilbert Lagrangian, \( L^{(\phi)} \equiv -\frac{1}{\kappa^2} \gamma \partial_\mu \phi \partial^\mu \phi \) with \( \gamma = \pm 1 \) for the canonical scalar and or phantom, \( L^{(0)} \equiv -\frac{1}{\kappa^2} V(\phi) \) and \( L^{(2)} = \frac{1}{\kappa^4} (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) \) is the GB invariant that gives a total derivative in four dimensions.

We are interested in the variation over the metric \( g^{\alpha\beta} \), which after some manipulation it gives*

\[ G^{(\text{SGB})}_{\alpha\beta} = \frac{1}{2} \kappa^2 T_{\alpha\beta} , \tag{4.2} \]

where we have arranged the result as \( G^{(\text{SGB})}_{\alpha\beta} = G^{(1)}_{\alpha\beta} + G^{(\phi)}_{\alpha\beta} + G^{(0)}_{\alpha\beta} + f(\phi)G^{(2)}_{\alpha\beta} \) and \( G^{(f(\phi))}_{\alpha\beta} \equiv R_{\alpha\beta}^{(\text{SGB})} - \frac{1}{2} g_{\alpha\beta} R^{(\text{SGB})} \), and where also details of the corresponding Euler–Lagrange expressions are \( G^{(1)}_{\alpha\beta} \equiv G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \),

\[ G^{(\phi)}_{\alpha\beta} \equiv R^{(\phi)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(\phi)} \equiv \left( -\frac{\gamma}{2} \partial_\alpha \phi \partial^\beta \phi \right) - \frac{1}{2} g_{\alpha\beta} \left( -\frac{\gamma}{2} \partial_\mu \phi \partial^\mu \phi \right) , \]

\[ G^{(0)}_{\alpha\beta} \equiv R^{(0)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(0)} \equiv \left[ -\frac{V(\phi)}{D-2} g_{\alpha\beta} \right] - \frac{1}{2} g_{\alpha\beta} \left[ -\frac{V(\phi)}{D-2} \right] = \frac{1}{2} V(\phi) g_{\alpha\beta} , \]

see Ref. [14] for details,

\[ G^{(2)}_{\alpha\beta} \equiv R^{(2)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(2)} \]

\[ \equiv \left[ 2R R_{\alpha\beta} - 4 \left( R_{\alpha\mu} R_{\beta}^{\mu \nu} + R_{\alpha\mu\beta\nu} R^{\mu\nu} \right) + 2R_{\alpha\mu\nu} R_{\beta}^{\mu\nu} \right] - \frac{1}{2} g_{\alpha\beta} \kappa^2 L^{(2)} \]

* See Ref. [2] for a few useful equations.
\[ G^{(f(\phi))}_{\alpha\beta} \equiv R^{(f(\phi))}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(f(\phi))} \]
\[ \equiv \left[ -2R f(\phi)_{;\alpha\beta} + 4R_{\alpha\mu} f(\phi)_{;\beta}^{\mu} + 4R_{\beta\mu} f(\phi)_{;\alpha}^{\mu} - 4R_{\alpha\beta} f(\phi)_{;\mu}^{\mu} \right. \]
\[ + 4R_{\alpha\beta\mu\nu} f(\phi)^{;\mu\nu} - \frac{1}{2} g_{\alpha\beta} a R f(\phi)_{;\mu}^{\mu} - \frac{1}{2} g_{\alpha\beta} b R^{\mu\nu} f(\phi)_{;\mu\nu} \]
\[ - \frac{1}{2} g_{\alpha\beta} \left[ (-4 - a) R f(\phi)_{;\mu}^{\mu} + (8 - b) R^{\mu\nu} f(\phi)_{;\mu\nu} \right], \]

where \( a \) and \( b \) are arbitrary constants, \( (; \mu\nu) \) means \( (; \mu; \nu) \) and so on.

All the Ricci–like tensors and the Ricci–like scalars are defined in a way to satisfy the necessarily corresponding trace relations, namely \( \text{Trace} R^{(\text{SGB})}_{\alpha\beta} = R^{(\text{SGB})} \), \( \text{Trace} R_{\alpha\beta} = R \), \( \text{Trace} R^{(\phi)}_{\alpha\beta} = R^{(\phi)} \), \( \text{Trace} R^{(0)}_{\alpha\beta} = R^{(0)} \), \( \text{Trace} R^{(2)}_{\alpha\beta} = R^{(2)} \) and finally \( \text{Trace} R^{(f(\phi))}_{\alpha\beta} = R^{(f(\phi))} \) when the constants \( a \) and \( b \) are:

\[ a = - \frac{2}{D - 2} \quad \text{and} \quad b = \frac{8}{D - 2} \quad (4.3) \]

for \( D \neq 2 \). That is, the latter trace relation depends on the dimension of space–time, as one may have expected due to the GB term, with which a similar deduction has been arisen in Ref. [14].

Therefore, we have been able to develop generalized Einstein tensor technique for the SGB gravity via the Trace operator.

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