STABLE MAPS INTO THE CLASSIFYING SPACE OF THE GENERAL LINEAR GROUP

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1. Introduction

In this note we give a definition of stable maps into the classifying stack $BGL_r$ of the general linear group. To support our belief that the definition is the correct one, we show that there are natural boundary morphisms between the moduli groupoids parameterizing stable maps to $BGL_r$. We warn the reader beforehand that we do not as yet have applications of our constructions. Our purpose in this note is mainly to explain the combinatorics of stable maps into $BGL_r$.

Before giving an idea of what stable maps to $BGL_r$ are, we first recall the case of stable maps into a complex projective variety $V$. Let $g \in \mathbb{N}_0$, and let $S$ be a finite set. A stable $S$-pointed map of genus $g$ consists of a prestable curve $C$ together with nonsingular points $(x_s)_{s \in S}$ on it and a morphism $f : C \to V$ of algebraic varieties such that the collection of these data has at most finitely many automorphisms.

Stable maps into $V$ are a fundamental notion for defining Gromov-Witten invariants for $V$. Namely, there is a Deligne-Mumford stack $\overline{M}_{g,S}(V)$ parameterizing all $S$-pointed stable maps of genus $g$, and Gromov-Witten invariants of $V$ are defined by means of intersection theory on $\overline{M}_{g,S}(V)$.

The moduli stack $\overline{M}_{g,S}(V)$ breaks up into connected components $\overline{M}_{g,S}(V, \beta)$ where $\beta$ runs through some monoid. Stable maps which belong to that component are called of class $\beta$. Given two stable maps, say $S_1 \cup \{*,\} \cup S_2$-pointed, of genus $g_i$ and class $\beta_i$ for $i = 1, 2$ such that the points $*$ are mapped to the same point in $V$, one obtains a new stable map which is $S_1 \cup S_2$-pointed, of genus $g_1 + g_2$ and class $\beta_1 + \beta_2$, by clutching together the two curves at the points $*$. This yields a morphism

$$\overline{M}_{g_1,S_1 \cup \{*,\}}(V, \beta_1) \times_V \overline{M}_{g_2,S_2 \cup \{*,\}}(V, \beta_2) \to \overline{M}_{g_1+g_2,S_1 \cup S_2}(V, \beta_1 + \beta_2)$$

which is an instance of a boundary morphism. A general point in its image belongs to a stable map whose underlying curve has two smooth irreducible components joined in one double point such that the restriction of the map to the $i$-th component is a $S_i$-pointed stable map of genus $g_i$ and class $\beta_i$. The combinatorial type of such a stable map is a decorated graph containing two vertices labeled by $(g_1, \beta_1)$ and $(g_2, \beta_2)$ and half-edges (or “tails”) labeled by elements of $S_1$ and $S_2$ respectively:

\begin{center}
\begin{tikzpicture}
  \node (S1) at (0,0) {$S_1$};
  \node (S2) at (2,0) {$S_2$};
  \node (b1) at (1,0) {$\beta_1$};
  \node (b2) at (1.5,0) {$\beta_2$};
  \node (g1) at (0.5,-1) {$g_1$};
  \node (g2) at (1.5,-1) {$g_2$};
  \draw (S1) -- (b1) -- (b2) -- (S2);
  \draw (b1) -- (g1) -- (b2) -- (g2);
\end{tikzpicture}
\end{center}
More generally, to each $S$-pointed stable map of genus $g$ and class $\beta$ one can associate some decorated graph $\tau$, its combinatorial type. For each combinatorial type there is a stack $\overline{M}(V, \tau)$, defined as fiber product of some stacks $\overline{M}_{g_i, S_i}(V, \beta_i)$, and a boundary morphism

$$\overline{M}(V, \tau) \to \overline{M}_{g, S}(V, \beta)$$

generalizing the clutching described above. The collection of boundary morphisms is responsible for the rich algebraic structure of quantum cohomology of $V$ (cf. [M]).

By the very definition of the stack $\text{BGL}_r$ a morphism from a curve $C$ to $\text{BGL}_r$ is nothing else but a vector bundle of rank $r$ over $C$. Thus it is clear that the generic stable map to $\text{BGL}_r$ should be a pair consisting of a smooth curve together with a vector bundle of rank $r$ on it. Now it is well known that given a family of stable curves which is generically smooth and has a singular special fiber, a vector bundle defined on the complement of that fiber does not necessarily extend to a vector bundle over the whole family, not even if one allows for an étale base change of the family. This is just like in the situation of morphisms into a variety $V$, where it may happen that a morphism to $V$, defined on the complement of the special fiber, does not extend to a morphism from the whole family to $V$, even after base change.

However there is always a family of semistable curves which outside the special fiber coincides with the given one, such that the morphism extends to that new family, at least after an étale base change. The same solution applies to vector bundles, i.e. to morphisms into $\text{BGL}_r$. Namely there exists a semistable family such that the given vector bundle extends after étale base change. Furthermore the new family and the extension can be chosen in such a way that the restriction of the vector bundle on those chains of projective lines in the semistable special fiber which are contracted under stabilization satisfies a certain condition which we call “admissibility” (cf. Definition 2.2). Accordingly, we define $S$-pointed stable maps of genus $g$ and degree $d$ to $\text{BGL}_r$ as semistable $S$-pointed curves of genus $g$ together with vector bundles of rank $r$ and degree $d$ satisfying that condition.

It is straightforward to define families of stable $S$-pointed maps of genus $g$ and degree $d$ to $\text{BGL}_r$, and pull back and isomorphisms of these. Thus we obtain a groupoid over $\mathbb{C}$ which we denote by $\overline{M}_{g, S}(\text{BGL}_r, d)$.

To give an idea of what the boundary morphisms are, let us first consider the analogue of the example above. Thus we are given two curves, together with vector bundles of rank $r$ on them, and on each of the curves one special point. Now if we join the curves by identifying the two special points, we need an additional datum to obtain a vector bundle on the new curve. This datum consists in an isomorphism between the respective fibers of two given bundles at the special points. However isomorphisms between two vector spaces may degenerate in families. In other words, after choosing a bases of the two vector spaces, an isomorphism between them is given by an element in $\text{GL}_r$ which is noncompact and therefore in general a family of isomorphisms cannot be extended to special points. Our solution of this problem is a certain compactification $K\text{GL}_r$ of $\text{GL}_r$ which we have constructed in an earlier paper [K1].
To be more precise, let us give a definition of the groupoid \( \overline{M}(BGL_r, \tau) \), where \( \tau \) is the decorated graph

\[
\begin{array}{c}
S_1 \quad d_1 \quad d_2 \\
g_1 \quad \emptyset \quad g_2 \\
\end{array}
\]

which corresponds to the example considered above. First we observe that one can interpret the diagonal morphism \( \Delta : BGL_r \to BGL_r \times BGL_r \) as follows. There is a commutative diagram

\[
\begin{array}{ccc}
BGL_r & \xrightarrow{\cong} & \text{Isom}(pr_1^*E, pr_2^*E) \\
& \searrow & \downarrow \Delta \\
& & BGL_r \times BGL_r
\end{array}
\]

where the horizontal arrow is an isomorphism, \( E \) is the universal vector bundle over \( BGL_r \), and \( pr_i : BGL_r \times BGL_r \to BGL_r \) is the \( i \)-th projection. We define

\[
BGL_r^r := KGL(pr_1^*E, pr_2^*E)
\]

as the fiberwise compactification of the locally trivial GL\(_r\)-bundle \( \text{Isom}(pr_1^*E, pr_2^*E) \to BGL_r \times BGL_r \) and let

\[
\overline{\Delta} : BGL_r^r \to BGL_r \times BGL_r
\]

be the natural projection. With this notation the groupoid \( \overline{M}(BGL_r, \tau) \) is defined by the Cartesian diagram

\[
\begin{array}{c}
\overline{M}(BGL_r, \tau) \\
\longrightarrow \overline{M}_{g_1, S_1 \cup \{ * \}}(BGL_r, d_1) \times \overline{M}_{g_2, S_2 \cup \{ * \}}(BGL_r, d_2) \\
\downarrow \overline{\Delta} \\
BGL_r \\
\end{array}
\]

The existence of a natural boundary morphism

\[
\overline{M}(BGL_r, \tau) \to \overline{M}_{g,S}(BGL_r, d)
\]

where \( g = g_1 + g_2, S = S_1 \cup S_2 \) and \( d = d_1 + d_2 \) then follows from results in \([K2]\) and \([K3]\).

Under this boundary morphism, a point of \( \overline{M}(BGL_r, \tau) \) which lies over the complement of \( BGL_r \) in \( BGL_r^r \) is mapped to a stable map to \( BGL_r \) whose underlying curve not stable but only semistable, and the stabilizing map contracts a chain of projective lines. The restriction of the underlying bundle of the stable map to this chain of projective lines yield a tuple of degrees and we in \([K]\) below we give a careful description of which strata of \( BGL_r^r \) (or of \( KGL_r \)) correspond to which tuple of degrees.

All the groupoids considered here are in fact Artin stacks. Unfortunately they are neither of finite type over \( \mathbb{C} \) nor separated. This is to be expected since already the moduli stack of vector bundles over a smooth curve are non-separated Artin stacks which are not of finite type. Still they might be usefull e.g. for defining Gromov-Witten invariants of \( BGL_r \) if we succeed to find a kind of Shatz stratification for them. We hope to address this question in a future paper.
It should be mentioned that A. Schmitt \[\text{Sch}\] and J. Li \[\text{Li}\] have constructed varieties parameterizing a certain subset of all stable maps to \(\text{BGL}_r\). However they do not define boundary morphisms and we doubt whether that would be possible at all in their context.

We would also like to mention work of D. Abramovich, A. Corti and A. Vistoli \[\text{AV} \\text{ACV}\] where stable maps to \(\text{BG}\) for finite groups \(G\) are defined. In \[\text{K4}\] we have established some connection between their construction and ours, but the exact relationship remains mysterious.

2. Stable maps

First we recall some definitions from \[\text{K2}\]

**Definition 2.1.** A chain of projective lines is a connected prestable curve over a field, whose components are projective lines and whose associated graph is linear. An irreducible component of \(R\) is called extremal, if it intersects only one other component.

**Definition 2.2.** A vector bundle \(E\) of rank \(r\) on a chain \(R\) of projective lines is called admissible, if the following two properties are satisfied:

1. For each irreducible component \(R_i\) of \(R\) there exists a number \(d_i \geq 1\) such that the restriction of \(E\) to \(R_i \cong \mathbb{P}^1\) is isomorphic to the bundle \(d_iO \oplus (r - d_i)O(1)\).
2. If a global section of \(E\) vanishes in two smooth points lying in different extremal components of \(R\), then it vanishes everywhere.

It is not difficult to see that if there exists an admissible vector bundle \(E\) of rank \(r\) on a chain \(R\) of projective lines, then the length of that chain is bounded by \(r\) (cf. section 3 in \[\text{K2}\]).

**Definition 2.3.** Let \(C'\) be a prestable curve over a field \(k\). A Gieseker vector bundle of rank \(r\) on \(C'\) is a pair \((C \to C', E)\), where \(C\) is a prestable curve over \(k\), \(C \to C'\) is a morphism and \(E\) is a rank \(r\) vector bundle on \(C\) which satisfy the following properties:

1. The morphism \(C \to C'\) is an isomorphism over the complement of the singular points of \(C'\).
2. The fiber of \(C \to C'\) over a singular point of \(C'\) is either a point or a chain of projective lines.
3. The restriction of \(E\) to each of the nontrivial fibers of \(C \to C'\) is admissible.

Now we come to the main definition of this note:

**Definition 2.4.** Let \(S\) be a finite set. A stable \(S\)-pointed map to \(\text{BGL}_r\) is a tupel \((C, x_i \mid i \in S, E)\), where \((C, x_i \mid i \in S)\) is a prestable \(S\)-pointed curve over a field and \(E\) is a vector bundle on \(C\) such that if \((C, (x_i)) \to (C^{\text{st}}, (y_i))\) denotes the stabilization morphism (cf. \[\text{M}\] V.1.6) then \((C \to C^{\text{st}}, E)\) is a Gieseker vector bundle of rank \(r\) on \(C^{\text{st}}\).

3. Combinatorial types

Before defining the combinatorial type of a stable \(S\)-pointed map to \(\text{BGL}_r\), let me first recall some definitions from \[\text{M}\].

**Definition 3.1.** A (finite) graph \(\tau\) consists of the data \((F_\tau, V_\tau, \partial_\tau, j_\tau)\), where \(F_\tau\) is a (finite) set (of flags), \(V_\tau\) is a finite set (of vertices), \(\partial_\tau : F_\tau \to V_\tau\) is the boundary map, and \(j_\tau : F_\tau \to F_\tau\) is an involution.
An isomorphism $\tau \xrightarrow{\sim} \sigma$ between two graphs consists of two bijections $F_\tau \xrightarrow{\sim} F_\sigma$, $V_\tau \xrightarrow{\sim} V_\sigma$, compatible with $\partial$ and $j$.

The fixed points of $j_\tau$ form the set $S_\tau$ of tails, the pairs $(f, f')$ with $f \in F_\tau$ and $f' = j_\tau(f) \neq f$ form the set $E_\tau^{or}$ of oriented edges of $\tau$. If $e = (f, f') \in E_\tau^{or}$ is an oriented edge, then $j_\tau(e) := (j_\tau(f), j_\tau(f'))$ is the edge with the opposite orientation. The two-element orbits of $j_\tau$ (viewed as an inversion of $F_\tau$ or of $E_\tau^{or}$) form the set $E$ of (un-oriented) edges of $\tau$. For $v \in V_\tau$ let $F_\tau(v) := \partial^{-1}(v)$ denote the set of flags starting from $v$, and let $|v| := |F_\tau(v)|$ denote the valence of $v$. The topological realization of a graph $\tau$ is denoted by $||\tau||$.

Definition 3.2. A modular graph is a graph $\tau$ together with a map $g : V_\tau \rightarrow \mathbb{N}_0$, $v \mapsto g_v$. A modular graph $\tau$ is called stable, if $|v| \geq 3$ for all $v$ with $g_v = 0$ and $|v| \geq 1$ for all $v$ with $g_v = 1$.

There is a standard way to associate a modular graph $(\tau, g)$ to a prestable $S$-pointed curve $(C, x_i \mid i \in X)$ (cf. [M], III. 2). The vertices $v$ of $\tau$ correspond to the irreducible components $C_v$ of $C$, and $g_v$ is the genus of $C_v$. The un-oriented edges $e$ of $\tau$ correspond to the singular points $x_e$ of $C$. A prestable $S$-pointed curve is stable if and only if $(\tau, g)$ is stable.

Definition 3.3. A chain-type is a tupel $d = (d_1, \ldots, d_q)$ of integers $d_i \geq 1$, where $q \geq 0$. The number $q$ is called the length of the chain-type $d$ and the integer $|d| := \sum_{i=1}^{q} d_i$ is called its degree. The (empty) chain-type $d = ()$ of length zero will also be denoted by the symbol $\emptyset$.

For a chain-type $d = (d_1, \ldots, d_q)$ we denote by $j(d)$ the reversed chain-type $(d_q, \ldots, d_1)$. The reversed empty chain-type is again empty.

Definition 3.4. A chain-graph (for rank $r$) is a tupel $(\tau, g, d, \textbf{d})$ where $(\tau, g)$ is a stable modular graph and $d$, $\textbf{d}$ are maps

$$d : V_\tau \rightarrow \mathbb{Z}$$

$$\textbf{d} : E_\tau^{or} \rightarrow \{\text{chain-types of degree} \leq r\}$$

where $\textbf{d}$ has the property that $\textbf{d}(j_\tau(e)) = j(d(e))$ for every oriented edge $e$ of $\tau$.

Definition 3.5. The combinatorial type of an $S$-labeled stable map $(C, x_i \mid i \in S, E)$ to $\text{BGL}_r$ is the chain-graph $(\tau, g, d, \textbf{d})$, where $(\tau, g)$ is the stable modular graph associated to the stabilization $C^{st}$ of $C$ (in particular, $S_\tau$ is identified with $S$) and the maps $d$, $\textbf{d}$ are characterized by the following properties.

1. Let $v \in V_\tau$ and let $C_v$ be the associated component of the stabilized curve $C^{st}$. There exists a unique component of $C$ which is mapped isomorphically onto $C_v$. By a slight abuse of notation we denote that component again by the symbol $C_v$. We require that the restriction of $E$ to $C_v$ is of degree $d(v)$.

2. Let $e = (f, f')$ be an oriented edge of $\tau$ and let $x_e \in C^{st}$ be the corresponding singular point. If the fiber of $C \rightarrow C^{st}$ over $x_e$ consists of just one point, then $d(e) = \emptyset$. Else the fiber $R$ is a chain of length $q \geq 1$ of projective lines. To the flags $f$ and $f'$ there are associated components $C_f$, $C_{f'}$ of $C^{st}$ together with points $x_f \in C_f$ and $x_{f'} \in C_{f'}$. By abuse of notation we denote by $(C_f, x_f)$ and $(C_{f'}, x_{f'})$ also the pointed components of $C$ which are mapped isomorphically to the corresponding pointed components of $C^{st}$. Let $R_1, \ldots, R_q$ be the irreducible components of the chain $R$ numbered in such
a way that \( R_1 \) meets the point \( x_f \) and \( R_q \) meets the point \( x_{f'} \) and such that \( R_i \cap R_j \)
is empty for \( |i - j| > 1 \). Then \( \mathbf{d}(e) = (\deg(E|_{R_1}), \ldots, \deg(E|_{R_q})) \).

It is clear that for any chain-graph \((\tau, g, d, \mathbf{d})\) for rank \( r \) there exists an \( S_r \)-labeled stable map to \( \text{BGL}_r \) whose combinatorial type is \((\tau, g, d, \mathbf{d})\).

4. The groupoid of stable maps to \( \text{BGL}_r \)

Let \( S \) be a finite set and let \( r \geq 1 \). An \( S \)-labeled stable map over a scheme \( T \) to \( \text{BGL}_r \)
is a tuple \((C/T, x_i \mid i \in S, E)\), where \((C/T, x_i \mid i \in S)\) is a prestable \( S \)-pointed curve and \( E \) is a vector bundle of rank \( r \) on \( C \) such that for each geometric point \( t \) of \( T \) the fiber \((C_t, x_i(t) \mid i \in S, E_t)\) is a stable map to \( \text{BGL}_r \).

A morphism
\[
(C/T, x_i \mid i \in S, E) \to (C'/T', x_i' \mid i \in S, E')
\]
between two \( S \)-labeled stable maps to \( \text{BGL}_r \) consists of the following data:

1. A morphism \( T \to T' \),
2. An isomorphism \( f : C \sim C' \times_{T'} T \) which for each \( i \in S \) maps the section \( x_i \) to the pull back of the section \( x_i' \),
3. An isomorphism \( E \sim f^*(\mathcal{O}_T \otimes_{\mathcal{O}_{T'}} E') \).

For integers \( g \geq 0 \) and \( d \) let \( \mathcal{M}_{g,S}(\text{BGL}_r, d) \) be the groupoid (over the category of schemes) whose objects are \( S \)-labeled stable maps \((C/T, x_i \mid i \in S, E)\) to \( \text{BGL}_r \) where \( C/T \) is a \( T \)-curve of genus \( g \) and the restriction of the vector bundle \( E \) to each fiber of \( C \to T \) is of degree \( d \).

5. Generalized isomorphisms and Gieseker vector bundle data

This section and the following are preparatory to section \( \text{IV} \) below, where we will define an analogue of the groupoid \( \mathcal{M}(V, \tau) \) and the boundary morphism \( \mathcal{M}(V, \tau) \to \mathcal{M}_{g,S}(V, \beta) \) (cf. \textit{ref} V. 4.7) for the case when the variety \( V \) is replaced by the stack \( \text{BGL}_r \).

Let \( F_1 \) and \( F_2 \) be two \( r \)-dimensional \( k \)-vector spaces. In \textit{ref} we have defined a smooth compactification \( \text{KGL}(F_1, F_2) \) of the variety \( \text{Isom}(F_1, F_2) \cong \text{GL}_n \) of linear isomorphisms from \( F_1 \) to \( F_2 \) which has properties analogous to the so called “wonderful compactification” of adjoint linear groups introduced by De Concini and Procesi. We call the \( k \)-valued points of \( \text{KGL}(F_1, F_2) \) \textit{generalized isomorphisms} from \( F_1 \) to \( F_2 \). The complement of \( \text{Isom}(F_1, F_2) \) inside \( \text{KGL}(F_1, F_2) \) is a normal crossing divisor with smooth irreducible components
\[
Y_i = Y_i(F_1, F_2) \quad \text{and} \quad Z_i = Z_i(F_1, F_2) \quad (i \in [0, r - 1])
\]
For subsets \( I, J \) of the set \([0, r - 1] \) we write \( \overline{O}_{I,J}(F_1, F_2) \) for the intersection of all \( Z_i \) and \( Y_j \), where \( i \) and \( j \) run through the set \( I \) and \( J \) respectively. The variety \( \overline{O}_{I,J}(F_1, F_2) \) is non-empty if and only if
\[
\min(I) + \min(J) \geq r
\]
(here we use the convention that the minimum of the empty subset of \([0, r - 1] \) is \( r \)). A pair \((I, J)\) of subsets of \([0, r - 1] \) which satisfies this property is called a \( GI \)-type (for the rank \( r \)).

A \( k \)-rational point of \( \overline{O}_{I,J}(F_1, F_2) \) is called a \textit{generalized isomorphism of type} \((I, J)\). There is an isomorphism
\[
\text{inv} : \begin{cases} 
\text{KGL}(F_1, F_2) & \to \text{KGL}(F_2, F_1) \\
\Phi & \mapsto \Phi^{-1}
\end{cases}
\]
with \( \text{inv}^2 = \text{id} \), whose restriction to \( \text{Isom}(F_1, F_2) \) is the inversion. We have \( \text{inv}(Y_i(F_1, F_2)) = Z_i(F_2, F_1) \) for all \( i \in [0, r - 1] \). Thus if \( \Phi \) is a generalized isomorphism of type \((I, J)\) then \( \Phi^{-1} \) is a generalized isomorphism of type \((J, I)\).

**Definition 5.1.** Let \((D, x_1, x_2)\) be a two-pointed prestable curve over a field \(k\). A *Gieseker vector bundle datum* on \((D, x_1, x_2)\) is a pair \((F, \Phi)\) where \(F\) is a vector bundle on \(D\) with fibers \(F_1\) and \(F_2\) in \(x_1\) and \(x_2\) respectively, and \(\Phi\) is a generalized isomorphism from \(F_1\) to \(F_2\).

Let \((D, x_1, x_2)\) be a two-pointed prestable curve and let \(C'\) be the prestable curve which arises by identifying the two points \(x_1\) and \(x_2\). Thus we have a morphism \(D \to C'\) which is the partial normalization of the curve \(C'\) at a singular point \(x\).

In our paper [K2] we have shown that a Gieseker vector bundle datum \((F, \Phi)\) on \(D\) induces a Gieseker vector bundle \((C \to C', E)\) on \(C'\). Moreover the construction of \((C \to C', E)\) yields a distinguished singular point \(y \in C\) which is mapped to the point \(x\), and one can recover the data \((D, x_1, x_2, F, \Phi)\) from the data \((C \to C', E, y)\).

Let \((C \to C', E)\) be the Gieseker vector bundle associated to the Gieseker vector bundle datum \((F, \Phi)\). Let \(R\) be the fiber over \(x\) of the morphism \(C \to C'\). Then \(R\) is either a single point or a chain of projective lines. In the second case we may consider \(D\) as a subscheme of \(C\) and may number the components \(R_1, \ldots, R_m\) of \(R\) such that \(R_1\) meets \(D\) in \(x_1\) and \(R_m\) meets \(D\) in \(x_2\) and \(R_i \cap R_j = \emptyset\) if \(|i - j| > 1\). Let \(d = \emptyset\) if \(R\) is reduced to a point and \(d = (\deg(E|_{R_1}), \ldots, \deg(E|_{R_m}))\) else.

Now what can be said about the chain-type \(d\), if the generalized isomorphism \(\Phi\) is chosen in \(\mathcal{O}_{I,J}(F_1, F_2)\) for some GI-type \((I, J)\)? To answer this question we define a mapping

\[
\{\text{GI-types}\} \to \{\text{chain-types}\}
\]

as follows: Let \((I, J)\) be a GI-type, with \(I = \{i_1, \ldots, i_p\}\), \(J = \{j_1, \ldots, j_q\}\) where \(p, q \geq 0\) and

\[
0 \leq i_1 \leq \cdots \leq i_p < i_{p+1} := r
\]

\[
0 \leq j_1 \leq \cdots \leq j_q < j_{q+1} := r
\]

The GI-type \((I, J)\) is mapped to the chain-type \((d_1, \ldots, d_{p+q})\), where

\[
d_m = \begin{cases} 
  i_{m+1} - i_m & \text{if } m \in [1, p] \\
  j_{p+q-m+2} - j_{p+q-m+1} & \text{if } m \in [p+1, p+q]
\end{cases}
\]

The mapping can be visualized by the following picture

\[
\begin{array}{cccccccc}
0 & \cdots & i_1 & \cdots & i_p & rr & \cdots & j_q & \cdots & j_2 & \cdots & j_1 & \cdots & 0 \\
\hline
& & d_1 & \cdots & d_p & \cdots & d_{p+1} & \cdots & d_{p+q} & \cdots & \cdots & \cdots & \cdots & \\
\end{array}
\]

The answer to the above question can now be formulated as follows: If \(\Phi\) is generic in \(\mathcal{O}_{I,J}(F_1, F_2)\) (more precisely, if \(\Phi\) is contained in \(\mathcal{O}_{I,J}(F_1, F_2)\)) but not in \(\mathcal{O}_{I',J'}(F_1, F_2)\) for \(I'\) strictly larger than \(I\) or \(J'\) strictly larger than \(J\) then the chain-type \(d\) is the image of the GI-type \((I, J)\) by the mapping

\[
\{\text{GI-types}\} \to \{\text{chain-types}\}
\]

defined above.
6. The groupoid of Gieseker vector bundle data

Let $T$ be a scheme and let $F_1$ and $F_2$ be two vector bundles of rank $r$ on $T$. In [K1] we have defined a smooth $T$-scheme $\text{KGL}(F_1, F_2)$, whose fiber over at a point $t \in T$ is a compactification of the variety $\text{Isom}(F_{1,t}, F_{2,t})$, which is isomorphic to $\text{KGL}(F_{1,t}, F_{2,t})$. A section of $\text{KGL}(F_1, F_2) \rightarrow T$ is called a generalized isomorphism from $F_1$ to $F_2$. For each GI-type $(I, J)$ there exists a natural closed subscheme $\overline{\text{O}}_{I,J}(F_1, F_2)$ of $\text{KGL}(F_1, F_2)$ whose fiber over $t \in T$ is the variety $\overline{\text{O}}_{I,J}(F_{1,t}, F_{2,t})$. A section of $\overline{\text{O}}_{I,J}(F_1, F_2) \rightarrow T$ is called a generalized isomorphism of type $(I, J)$ from $F_1$ to $F_2$. Also the isomorphism $\text{inv}: \text{KGL}(F_1, F_2) \rightarrow \text{KGL}(F_2, F_1)$, $\Phi \mapsto \Phi^{-1}$ generalizes to the relative situation and has the properties $\text{inv}^2 = \text{id}$ and $\text{inv}(\overline{\text{O}}_{I,J}(F_1, F_2)) = \text{inv}(\overline{\text{O}}_{J,I}(F_2, F_1))$. All these objects commute (in an obvious sense) with base change $T' \rightarrow T$.

**Definition 6.1.** Let $(D/T, x_1, x_2)$ be a two-pointed pre-stable curve over a scheme $T$ and let $(I, J)$ be a GI-type for a rank $r \in \mathbb{N}$. A Gieseker vector bundle datum of rank $r$ and type $(I, J)$ on $(D/T, x_1, x_2)$ is a pair $(F, \Phi)$, where $F$ is a vector bundle of rank $r$ on $D$ and $\Phi$ is a section of $\overline{\text{O}}_{I,J}(x_1^*F, x_2^*F) \rightarrow T$.

An isomorphism

$$(F, \Phi) \sim (F', \Phi')$$

between two Gieseker vector bundle data of type $(I, J)$ on $(D/T, x_1, x_2)$ is an isomorphism $F \sim F'$ such that the induced isomorphism $\overline{\text{O}}_{I,J}(x_1^*F, x_2^*F) \sim \overline{\text{O}}_{I,J}(x_1^*F', x_2^*F')$ carries $\Phi$ to $\Phi'$.

We denote by $\text{GVBD}_{I,J}$ the groupoid whose objects are Gieseker vector bundle data of rank $r$ and type $(I, J)$ on $(D/T, x_1, x_2)$ and whose arrows are isomorphisms between Gieseker vector bundle data.

Let $(I, J)$ and $(I', J')$ be two GI-types which are mapped to the same chain-type. Then by similar arguments as used in the proof of [K3] 4.2 it can be shown that there is a canonical isomorphism of groupoids

$$\beta_{I', J'}^{I,J}: \text{GVBD}_{I,J} \sim \text{GVBD}_{I', J'}.$$ 

Moreover, we have $\beta_{I', J'}^{I,J} = \text{id}$ and if $(I'' J'')$ is a third GI-type with the same associated chain-type as $(I, J)$ and $(I', J')$, then $\beta_{I'', J''}^{I,J'} \circ \beta_{I', J'}^{I,J} = \beta_{I'', J''}^{I,J} \circ \beta_{I', J'}^{I,J}$.

Let $d$ be a chain-type. The family of isomorphisms $(\beta_{I', J'}^{I,J})_{(I,J), (I', J') \rightarrow d}$ enables us to define a new groupoid $\text{GVBD}_d$ as follows. The objects of $\text{GVBD}_d$ are the union of all objects of the groupoids $\text{GVBD}_{I,J}$ where $(I, J)$ runs through all GI-types which are mapped to the chain-type $d$. A morphism from an object $(F, \Phi)$ of $\text{GVBD}_{I,J}$ to an object $(F', \Phi')$ of $\text{GVBD}_{I', J'}$ is a family $(f_{A,B})_{(A,B) \rightarrow d}$ where

$$f_{A,B}: \beta_{A,B}^{I,J}(F, \Phi) \rightarrow \beta_{A,B}^{I', J'}(F', \Phi')$$

is a morphism in $\text{GVBD}_{A,B}$ such that $\beta_{A', B'}^{A,B}(f_{A,B}) = f_{A', B'}$ for all GI-types $(A, B), (A', B')$ mapped to $d$.

**Remark 6.2.** Let $(I, J)$ be a GI-type and let $d$ be its associated chain-type. Then the inclusion $\text{GVBD}_{I,J} \hookrightarrow \text{GVBD}_d$ is an equivalence of categories. The advantage of $\text{GVBD}_d$
over $\text{GVBD}_{I,J}$ is that its definition depends only on the chain-type $d$ and does not involve the choice of of a GI-type over $d$.

**Definition 6.3.** An object $(F, \Phi)$ of the groupoid $\text{GVBD}_d$ is called a *Gieseker vector bundle datum of type $d$* on the two-pointed curve $(D/T, x_1, x_2)$.

Let $C'$ be the prestable $T$-curve constructed from $D$ by identifying the two sections $x_1$ and $x_2$ and let $x$ be the section of $C'$ given by the composition $T \xrightarrow{x} D \to C'$. Let $\text{GVB}$ be the following groupoid: The objects of $\text{GVB}$ are Gieseker vector bundles $(C \to C', E)$ of rank $r$ on $C'$ such that $C \to C'$ is an isomorphism over the complement of $x(T)$. An arrow $(C_1 \to C', E_1) \to (C_2 \to C', E_2)$ between two such objects consists of a $C'$-isomorphism $f : C_1 \cong C_2$ and an isomorphism $E_1 \cong f^*E_2$.

In section 5 we have already mentioned that a Gieseker vector bundle datum on a two-pointed prestable curve over a field induces a Gieseker vector bundle on the prestable curve obtained by identifying the two marked points. This construction also works in families over an arbitrary base scheme $T$ (cf. [K2]) and gives for each GI-type $(I, J)$ a natural morphism of groupoids

$$\nu_{I,J} : \text{GVBD}_{I,J} \to \text{GVB}.$$

The morphisms $\nu_{I,J}$ are compatible with the isomorphisms $\beta_{I,J}^{I',J'}$ in the sense that all triangles

$$
\begin{array}{ccc}
\text{GVBD}_{I,J} & \xrightarrow{\beta_{I,J}^{I',J'}} & \text{GVBD}_{I',J'} \\
\downarrow{\nu_{I,J}} & & \downarrow{\nu_{I',J'}} \\
\text{GVB} & & \text{GVB}
\end{array}
$$

commute.

It follows that for each chain-type $d$ the family $(\nu_{I,J})_{(I, J) \to d}$ induces a morphism of groupoids

$$\nu_d : \text{GVBD}_d \to \text{GVB}.$$

### 7. Boundary morphisms

In this section we will define for each chain-graph (cf. section 3)

$$\tau = (\tau, g_v \mid v \in V_\tau, \delta_v \mid v \in V_\tau, d_e \mid e \in E_\tau)$$

a groupoid (over the category of schemes) $\overline{M}(\text{BGL}_r, \tau)$ together with a boundary morphism

$$\nu_\tau : \overline{M}(\text{BGL}_r, \tau) \to \overline{M}_{g,S}(\text{BGL}_r, d),$$

where $g = \sum g_v + \dim H_1(\|\tau\|)$ is the genus of the combinatorial type, $d = \sum d_v + \sum |d_e|$ is its total degree and $S = S_\tau$ is the set of its tails.

We start with an auxiliary definition.

**Definition 7.1.** A *GI-graph* (for rank $r$) is a tuple

$$\gamma = (\tau, g_v \mid v \in V_\tau, I_f \mid f \in F_\tau \setminus S_\tau)$$

where $(\tau, g_v \mid v \in V_\tau)$ is a stable modular graph, where the $\delta_v$ are integers and where for each oriented edge $(f, f')$ the pair $(I_f, I_{f'})$ is a GI-type for the rank $r$. 

Before defining the groupoid $\overline{M}(BGL_r, \tau)$ we will first define the groupoid $\overline{M}(BGL_r, \gamma)$ for a GI-graph $\gamma$. We will see below that to $\gamma$ there is associated a chain-graph $\tau$, and the relationship between the groupoids $\overline{M}(BGL_r, \gamma)$ and $\overline{M}(BGL_r, \tau)$ is analogous to the relationship between GVBD$_{r,j}$ and GVBD$_d$ described in section [3].

Let

$$\gamma = (\tau, g_v \mid v \in V_\tau, \delta_v \mid v \in V_\tau, I_f \mid f \in F_\tau \setminus S_\tau)$$

be a GI-graph and let $T$ be a scheme. An object of $\overline{M}(BGL_r, \gamma)$ over $T$ is a family

$$((C_v/T, (x_i)_{i \in F_\tau(v)}, F_v) \mid v \in V_\tau ; \Phi_e \mid e \in E^\text{or}_\tau)$$

where

1. $(C_v/T, (x_i)_{i \in F_\tau(v)}, F_v)$ is an $F_\tau(v)$-labeled stable map over $T$ to $BGL_r$ such that $C_v/T$ is a curve of genus $g$ over $T$.
2. For every oriented edge $e = (f, f')$ of $\tau$ with $f \in F_\tau(v)$ and $f' \in F_\tau(v')$, $\Phi_e$ is a generalized isomorphism of type $(I_f, I_{f'})$ from $x_f^*F_v$ to $x_{f'}^*F_{v'}$.
3. For every oriented edge $e$ the equality $\Phi_{j(e)} = \Phi_e^{-1}$ holds.
4. For each vertex $v \in V_\tau$ we have $\deg F_v = \delta_v$.

An arrow

$$((C_v/T, (x_i)_{i \in F_\tau(v)}, F_v) \mid v \in V_\tau ; \Phi_e \mid e \in E^\text{or}_\tau)$$

$$\rightarrow ((C'_v/T, (x'_i)_{i \in F_\tau(v)}, F'_v) \mid v \in V_\tau ; \Phi'_e \mid e \in E^\text{or}_\tau)$$

between two objects of $\overline{M}(BGL_r, \gamma)$ over $T$ is a family of isomorphisms

$$(C_v/T, (x_i)_{i \in F_\tau(v)}, F_v) \cong (C'_v/T, (x'_i)_{i \in F_\tau(v)}, F'_v)$$

of stable maps over $T$ to $BGL_r$ (cf. section [4]), where $v$ runs through the set $V_\tau$ of vertices of $\tau$, such that for each oriented edge $e = (f_1, f_2)$ with $f_1 \in F_\tau(v_1), f_2 \in F_\tau(v_2)$ the induced isomorphism

$$\text{KGL}(x^*_f F_{v_1}, x^*_f F_{v_2}) \cong \text{KGL}(x'^*_f F'_{v_1}, x'^*_f F'_{v_2})$$

maps the generalized isomorphism $\Phi_e$ to $\Phi'_e$.

If $T' \rightarrow T$ is a morphism of schemes then an object of $\overline{M}(BGL_r, \gamma)$ over $T$ pulls back to one over $T'$ and an arrow between objects over $T$ induces one between their respective pull back over $T'$. This defines $\overline{M}(BGL_r, \gamma)$ as a groupoid over the category of schemes.

To a GI-graph $\gamma$ we associate the chain-graph

$$\tau = (\tau, g_v \mid v \in V_\tau, d_v \mid v \in V_\tau, d_e \mid e \in E^\text{or}_\tau)$$

where $d_v = \delta_v - \sum_{f \in F_\tau(v)}(r \min(I_f))$, and for each oriented edge $e = (f, f')$ the chain-type $d_e$ is the image of the GI-type $(I_f, I_{f'})$. This yields a mapping

$$\{ \text{GI-graphs} \} \rightarrow \{ \text{chain-graphs} \}$$

which obviously is surjective and finite to one.

Let $\tau$ be a chain-graph and let $\gamma$ and $\gamma'$ be two GI-graphs mapped to $\tau$. Using arguments similar as in the proof of [3, 4.2], one shows that there is a canonical isomorphism

$$\beta^\gamma_{\gamma'} : \overline{M}(BGL_r, \gamma) \rightarrow \overline{M}(BGL_r, \gamma')$$

We have $\beta^\gamma_1 = \text{id}$ and $\beta^\gamma_{\gamma''} \circ \beta^\gamma_{\gamma'} = \beta^\gamma_{\gamma''}$ for any further GI-graph $\gamma''$ mapped to $\tau$. 
We use the family $(\gamma^\prime)_{\gamma, \gamma^\prime \to \tau}$ to define the groupoid $\mathcal{M}(\text{BGL}_r, \tau)$ over the category of schemes. The construction is completely analogous to the construction of GVBD in section 6. The objects of $\mathcal{M}(\text{BGL}_r, \tau)$ are the union of objects of $\mathcal{M}(\text{BGL}_r, \gamma)$ where $\gamma$ runs through all GI-graphs mapped to $\tau$. Let $\mathcal{E}$ and $\mathcal{E}'$ be two objects in $\mathcal{M}(\text{BGL}_r, \tau)$ and assume that $\mathcal{E}$ belongs to $\mathcal{M}(\text{BGL}_r, \gamma)$ and $\mathcal{E}'$ to $\mathcal{M}(\text{BGL}_r, \gamma')$. An arrow of $\mathcal{M}(\text{BGL}_r, \tau)$ from $\mathcal{E}$ to $\mathcal{E}'$ is then a family of arrows

$$f_\pi : \beta^\tau_\pi(\mathcal{E}) \to \beta^\tau_{\pi'}(\mathcal{E}')$$

in $\mathcal{M}(\text{BGL}_r, \pi)$ where $\pi$ runs through all GI-graphs over $\tau$ such that for all pairs $\pi, \pi'$ of GI-graphs over $\tau$ we have $\beta^\tau_\pi(f_\pi) = f_{\pi'}$.

Remark 7.2. Of course for each GI-graph $\gamma$ mapped to a chain-graph $\tau$ the inclusion $\mathcal{M}(\text{BGL}_r, \gamma) \hookrightarrow \mathcal{M}(\text{BGL}_r, \tau)$ is an equivalence of categories and the reason why we defined $\mathcal{M}(\text{BGL}_r, \tau)$ in the way we did is that we want to make apparent that it does not depend on any choice of a GI-graph $\gamma$ over $\tau$.

The boundary morphism

$$\nu_\tau : \mathcal{M}(\text{BGL}_r, \tau) \to \mathcal{M}_{g,S}(\text{BGL}_r, d)$$

is now defined as follows: With the help of the constructions in [K2] we get a canonical morphism $\nu_\gamma : \mathcal{M}(\text{BGL}_r, \gamma) \to \mathcal{M}_{g,S}(\text{BGL}_r, d)$ for every GI-graph $\gamma$ over $\tau$ such that for any two GI-graphs $\gamma$ and $\gamma'$ the diagram

$$\begin{array}{ccc}
\mathcal{M}(\text{BGL}_r, \gamma) & \xrightarrow{\beta^\gamma_{\gamma'}} & \mathcal{M}(\text{BGL}_r, \gamma') \\
\downarrow{\nu_\gamma} & & \downarrow{\nu_{\gamma'}} \\
\mathcal{M}_{g,S}(\text{BGL}_r, d) & & 
\end{array}$$

Therefore the family $(\nu_\gamma)_{\gamma \to \tau}$ induces a morphism $\nu_\tau : \mathcal{M}(\text{BGL}_r, \tau) \to \mathcal{M}_{g,S}(\text{BGL}_r, d)$.

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