A single far-field uniquely determines the shape of a scattering screen

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Abstract

We consider the problem of fixed frequency acoustic scattering from a sound-soft flat screen. More precisely the obstacle is restricted to a two-dimensional plane and it scatters acoustic waves to three-dimensional space. The model is particularly relevant in the study and design of reflecting sonars and antennas. Our main result is that given the plane where the screen is located, a single far-field pattern determines the exact shape of the screen. It is true even for screens whose shape is an arbitrary simply connected smooth domain. This is in stark contrast with earlier single measurement inverse scattering results where only polygonal scatterers are determined, or other very restrictive a priori conditions are imposed.

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1 Introduction

1.1 Antennas

The motivation for the study of wave scattering from thin and large objects lies in the antenna theory. The starting point for this was when the Prussian Academy announced an open competition about who could be the first to show the existence or non-existence of electromagnetic (EM) waves in 1879. The existence of these waves were predicted fifteen years earlier by the mathematical theory of James Clerk Maxwell [31]. The competition was won in 1882 by young Heinrich Hertz, in favour of Maxwell’s theory. He did this by constructing a dipole antenna radiating
EM waves which he could measure. It is needless to mention the importance which this experiment together with Maxwell’s theory has had for modern society. Hertz’s antenna consisted of two identical perfectly conducting planar bodies, in his case squares, which create radiating EM waves. Since, by reciprocity, radiating antennas are identical to receiving antennas, the theory of antennas is closely connected to EM scattering and inverse scattering theory.

A key question in antenna design for scientific radio arrays is how to choose the antenna topology so that its impedance and radiation pattern are frequency independent (FI) over a wide range of frequencies and, simultaneously, the radiation pattern supports beamforming. Well-known examples of FI antennas include log-periodic, log-spiral, and UHF fractal antennas on high-frequencies. While proven good for extremely wide band work, these are heavy and complicated structures and thus not cost-efficient for extremely large arrays.

Instead of relying on traditional antenna forms, we aim to derive general principles for designing antennas with frequency independent characteristics. A major step in such a design strategy is to solve the inverse scattering problem: given a far-field, which antenna shape produces it? In this paper we study the technically easier acoustic scattering problem.

1.2 Mathematical background

The problem of inverse scattering with reduced measurement data has gained a lot of interest lately. Traditionally determining a scatterer from far-field measurements requires sending all possible incident waves and recording the corresponding far-field patterns. The method of using complex geometrical optics solutions and infinitely many far-field measurements in the fixed frequency setting was pioneered by Sylvester and Uhlmann in [37], and was the first method for uniquely determining an arbitrary smooth enough scattering potential by far-field measurements. The field has grown extremely fast since then, almost to the point of saturation, and we will only point the reader towards the surveys in [39] for references up to 2003, which gives a good picture of the situation except for scattering in two dimensions, which was solved by Bukhgeim [8] in 2007 and improved by several authors, e.g. [3, 7, 14, 18, 22].

In many applications the scatterer is impenetrable, or we are only interested in its shape or location. The shape determination problem is known as Schiffer’s problem in the literature [12]. M. Schiffer showed that a sound-soft obstacle (with non-empty interior) can be uniquely determined by infinitely many far-field patterns. The proof appeared as a private communication in the monograph by Lax and Phillips [27]. Linear sampling [11] and factorization [24] methods were developed and they are very well suited for shape determination, also from the numerical point of view. These were applied in the context of screens in electromagnetic scattering to determine the shape and location of the screen [9], also numerically. However these methods require the full use of infinitely many far-field patterns.

There was still much to improve: counting dimensions shows that a single far-field (a mapping $\mathbb{R}^{n-1} \rightarrow \mathbb{C}$) should be enough to determine
the shape (a manifold of dimension \( n - 1 \)). Colton and Sleeman reduced the requirements to finitely many far-field patterns [13]. It is widely conjectured that the uniqueness for Schiffer’s problem follows from a single far-field pattern [12, 23], and the situation for a general shape is wide open. This brings in the current results. Various authors proved at roughly the same time in the recent past that polyhedral sound-soft obstacles are uniquely determined by a single far-field pattern in various settings [1, 10, 16, 28–30, 35]. Part of these results apply also for screens as long as the screen is polygonal. So far there is no proof for the unique determination of an obstacle’s shape by one far-field pattern without restrictive a priori assumptions. The results in [19] come very close: the obstacle can be any Lipschitz domain as long as its boundary is not an analytic manifold. It does not allow screens, which is our focus.

An alternative approach which has gained interest recently, is to consider what can be determined with less data, e.g. one measurement, in the setting of penetrable scatterers which were usually treated with various methods based on the Sylvester–Uhlmann [37] or Bukhgeim [8] papers. Much of the recent work taking this point of view uses unique continuation results and precise analysis on the behaviour of Fourier transforms of the characteristic functions of various shapes [2, 4–6, 20, 21, 34]. A very interesting point of view is determining the so-called convex scattering support [25, 26] by one far-field measurement. Again, none of the above are applicable to screens per se.

Our work in this paper shows that a single far-field uniquely determines the shape of a smooth flat screen. Our methods are based on new ideas, which are partly motivated by the study of certain integral operators in [32, 33]. Our method shows that the far-field is actually the restriction of the two-dimensional Fourier transform of a function supported on the screen to a ball of radius \( k \), the wavenumber. We then show that the shape of the screen is exactly the support of that function. This latter part involves a delicate analysis of the Taylor coefficients of the scattered wave at the screen, but it leads to our main theorem: that Schiffer’s problem is uniquely solvable for flat screens on a plane in three dimensions.

### 1.3 Definitions and Theorems

Let us go forward to the mathematics. We start by defining what we mean by a screen and the scattering problem from screens. Then we state our three main theorems. They give representation formulas for the scattered wave, the far-field pattern, and the unique solvability of Schiffer’s problem for determining the shape of a scattering screen using a single incident wave. In Section 2 we prove the representation formulas, and then in Section 3 we solve the inverse problem.

We consider the scattering of a two dimensional sound-soft and flat obstacle \( \Omega \) in three dimensional space. We will assume that \( \Omega \) is an open subset of \( \mathbb{R}^2 \times \{0\} \).

**Definition 1.1.** We call a set \( \Omega \subset \mathbb{R}^3 \) a screen, if \( \Omega = \Omega_0 \times \{0\} \) for some simply connected bounded smooth domain \( \Omega_0 \subset \mathbb{R}^2 \) which we call its shape.
The scattering of acoustic waves by $\Omega$ leads to the study of the Helmholtz equation $(\Delta + k^2)u = 0$ where the wave number $k$ is given by the positive constant $k = \omega/c$ where $c$ is the constant speed of sound in the background fluid (air, water, etc) and $\omega$ is the angular frequency of the wave.

The pressure of the total wave vanishes on the boundary of a sound-soft obstacle, and the total wave is a sum of the incident and scattered waves. This leads to the following set of partial differential equations.

**Definition 1.2.** We define the direct scattering problem for a screen $\Omega$ as follows. Given an incident wave $u_i$ satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ and a screen $\Omega$, the direct scattering problem has a solution if there is $u_s \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ that satisfies the following conditions

\begin{align*}
(\Delta + k^2)u_s &= 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, & (1.1) \\
u_i(x) + u_s(x) &= 0, & x \in \Omega, & (1.2) \\
r \left( \frac{\partial}{\partial r} - ik \right) u_s &= 0, & r \to \infty, & (1.3)
\end{align*}

where $r = |x|$ and the limit is uniform over all directions $\hat{x} = x/r \in \mathbb{S}^2$ as $r \to \infty$.

There are a few things above that we should clarify. By $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ we mean the set of distributions $\psi$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ for which $\psi|_U \in H^1(U)$ for any bounded convex open set $U \subset \mathbb{R}^3 \setminus \overline{\Omega}$. Secondly, since strictly speaking $u_s$ is not defined on $\Omega$, by (1.2) we mean that the Sobolev trace of $u_s$ both from above ($x_3 > 0$) and below ($x_3 < 0$) coincides, and is equal to $-u_i$ on $\Omega$.

We shall start by showing a representation formula (1.4) for solutions $u_s$ of the direct scattering problem for the screen. This is mainly done so that the reader would get a better intuition about this type of problems. This formula is well known, and it gives a unique solution to the direct problem [36]. After that we will show that the far-field, defined below, corresponding to a single given non-trivial incident wave uniquely determines the screen $\Omega$. We remark that the far-field pattern exists and is unique for each $u_s$ satisfying the following assumptions. See [12] for reference.

**Definition 1.3.** Let $u_s$ satisfy the Sommerfeld radiation condition of (1.3) and the Helmholtz equation $(\Delta + k^2)u_s = 0$ outside a ball $B \subset \mathbb{R}^3$. We say that $u_s^{\infty}$ is the far-field of $u_s$ if

$$ u_s(x) = \frac{e^{ik|x|}}{|x|} \left( u_s^{\infty}(\hat{x}) + O\left( \frac{1}{|x|} \right) \right) $$

uniformly over $\hat{x}$ as $x \to \infty$.

We define some notation which will be useful throughout the whole text.

- $x, y, \ldots$ represent variables in $\mathbb{R}^3$, and we associate to them various projections described below.
- $x', y', \ldots$ mean variables in $\mathbb{R}^2$ or projections to $\mathbb{R}^2$. For example if $x = (1, 2, 3) \in \mathbb{R}^3$ then in that context $x' = (1, 2) \in \mathbb{R}^2$, but we could have $dy'$ in an integral over a subset of $\mathbb{R}^2$ without having to define the variable $y$ separately.
• \( x^0, y^0, \ldots \) denote lifts to \( \mathbb{R}^3 \), meaning \( x^0 = (x',0) \). For example if \( x' = (-1,-2) \) then \( x^0 = (-1,-2,0) \). This notation can also be used as a projection \( \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\} \). So if \( x = (1,2,3) \) then \( x^0 = (1,2,0) \). Essentially \( x^0 = (x')^0 = x^0 \) and \( x^{0'} = (x')' = x' \) but we do not use this combined notation explicitly.

• \( \Phi \) is reserved for the fundamental solution to \( (\Delta + k^2) \), defined in Lemma 2.2.

• \( u^+, u^- \) mean the function \( u \) restricted to \( \mathbb{R}^2 \times \mathbb{R}_+ \) and \( \mathbb{R}^2 \times \mathbb{R}_- \), respectively. If their variable is in \( \mathbb{R}^2 \times \{0\} \) then they are the two-sided limits (traces) as \( x_3 \to 0 \). We often use \( \partial_3 u^+ \) and \( \partial_3 u^- \). These are simply the derivatives in the \( x_3 \)-direction of \( u^+ \) and \( u^- \), respectively. Often this is evaluated on \( \mathbb{R}^2 \times \{0\} \) where it then denotes the one-sided derivative, i.e. the trace of \( \partial_3 u^\pm \).

• \( \tilde{H}^{-1/2}(\Omega_0) \): this is the set of \( H^{-1/2}(\mathbb{R}^2) \) distributions whose support is contained in \( \overline{\Omega_0} \), where we recall that \( \Omega_0 \) signifies the shape of a screen \( \Omega \).

Let us discuss the direct scattering problem \( (1.1)–(1.3) \) first. In Section 2, Proposition 2.4, we will show the well-known representation formula
\[
    u_s(x) = \int_{\mathbb{R}^2} \Phi(x, y^0) \rho(y') dy'
\]
for all \( x \in \mathbb{R}^3 \setminus \overline{\Omega} \), where
\[
    \rho(y') = \partial_3 u^+_s(y^0) - \partial_3 u^-_s(y^0)
\]
is an element of \( \tilde{H}^{-1/2}(\Omega_0) \) and the integral in \( (1.4) \) is interpreted as a distribution pairing between \( \rho \) and the smooth test function \( \Phi \) restricted to the screen. Taking the trace \( x \to \Omega \) in \( (1.4) \) and recalling that \( u_s = -u_i \) on \( \Omega \) in the sense of traces, \( (1.2) \), we get
\[
    u_i(x) = -\int_{\mathbb{R}^2} \Phi(x, y^0) \rho(y') dy'.
\]
Now, for any candidate solution \( u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega}) \), it solves the direct problem \( (1.1)–(1.3) \) if and only if \( \rho \), as defined above, is in \( \tilde{H}^{-1/2}(\Omega_0) \) and is the solution to \( (1.6) \). More precisely, given \( \rho \) solving the integral equation, we can define \( u_s \) by \( (1.4) \), and it would solve the direct scattering problem. This was shown in Theorem 2.5 in [36]. Theorem 2.7 in the same source proves that \( (1.6) \) has a unique solution \( \rho \in \tilde{H}^{-1/2}(\Omega_0) \) given any \( u_i \in H^{1/2}(\Omega_0) \).

Our main contributions are the following.

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^3 \) be a screen and \( u_s \) satisfy the direct scattering problem for some incident field \( u_i \) and screen \( \Omega \). Then its far-field has the representation
\[
    u^\infty_s(\hat{x}) = \frac{1}{4\pi} \left\langle (\partial_3 u^+_s - \partial_3 u^-_s)(y^0), e^{-ik\hat{x} \cdot y^0} \right\rangle_{y'},
\]
(1.7)
for \( \hat{x} \in S^2 \). If \( \partial_3 u^+ - \partial_3 u^- \) is integrable on \( \Omega \), this formula is equivalent to

\[
\begin{align*}
\hat{u}^\infty(\hat{x}) &= \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-ik\hat{x}^0 \cdot y} (\partial_3 u^+ - \partial_3 u^-)(y^0) dy^0
\end{align*}
\]

**Theorem 1.5.** Let \( \Omega, \tilde{\Omega} \subset \mathbb{R}^3 \) be screens and \( k \in \mathbb{R}_+ \). Let \( u_i \) be an incident wave and \( u_s, \tilde{u}_s \) be scattered waves that satisfy the direct scattering problem for screens \( \Omega, \tilde{\Omega} \), respectively.

If \( u_i \) is not antisymmetric with respect to \( \mathbb{R}^2 \times \{0\} \) and \( u^\infty_s = \tilde{u}^\infty_s \), then \( \Omega = \tilde{\Omega} \). If it is antisymmetric then \( u^\infty_s = \tilde{u}^\infty_s = 0 \) for any screens \( \Omega, \tilde{\Omega} \).

**2 Representation theorems**

In this section we will prove that solutions to the direct scattering problem satisfy (1.4). In essence we present the well-known but very condensed argument of [36] in more detail for the convenience of the readers. We will start with representation formulas for smooth functions and then approximate the \( H^1 \)-smooth \( u_s \). At the end of the section we will prove Theorem 1.4.

**Lemma 2.1.** Let \( D \subset \mathbb{R}^3 \) be a bounded domain whose boundary is piecewise of class \( C^1 \) and let \( \nu \) denote the unit normal vector to the boundary \( \partial D \) directed to the exterior of \( D \). Then, for \( u, v \in C^2(D) \) we have Green’s second formula

\[
\int_D (v \Delta u - u \Delta v) dx = \int_{\partial D} (\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu}) ds
\]

where \( ds \) is the surface measure of \( \partial D \).

**Proof.** Theorem 3 in Appendix C.2 of [17]. \( \square \)

**Lemma 2.2.** Let \( D \subset \mathbb{R}^3 \) be a bounded domain whose boundary is piecewise of class \( C^1 \) and \( k \in \mathbb{R}_+ \). Let

\[
\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}
\]

for \( x, y \in \mathbb{R}^3, x \neq y \). Then for any \( \varphi \in C^2(\overline{D}) \) and \( x \in \mathbb{R}^3 \setminus \partial D \) we have

\[
\int_D \Phi(x, y)(\Delta + k^2)\varphi(y)dy = \int_{\partial D} (\Phi(x, y)\partial_y \varphi(y) - \varphi(y)\partial_y \Phi(x, y))ds(y) + \begin{cases} 
0, & x \in \mathbb{R}^3 \setminus \overline{D}, \\
-\varphi(x), & x \in D.
\end{cases}
\]

**Proof.** We have \((\Delta + k^2)\varphi\) bounded and \( y \mapsto \Phi(x, y) \) integrable for any \( x \), so

\[
\int_D \Phi(x, y)(\Delta + k^2)\varphi(y)dy = \lim_{r \to 0} \int_{D \setminus B(x, r)} \Phi(x, y)(\Delta + k^2)\varphi(y)dy.
\]
Green’s second formula (2.1) applied to the integral on the right gives

\[ \ldots = \int_{D \setminus B(x,r)} (\Delta + k^2) \Phi(x,y) \varphi(y) dy \]
\[ + \int_{S(x,r) \cap \partial D} (\Phi(x,y) \partial_r \varphi(y) - \varphi(y) \partial_r \Phi(x,y)) ds(y) \]
\[ + \int_{\partial D \setminus B(x,r)} (\Phi(x,y) \partial_r \varphi(y) - \varphi(y) \partial_r \Phi(x,y)) ds(y). \]

The first integral here vanishes because \((\Delta_y + k^2) \Phi(x,y) = 0\) when \(y \neq x\).

The integral over \(\partial D \setminus B(x,r)\) gives the second term in the claim when \(r \to 0\) because \(\Phi, \partial \Phi\) are integrable since \(x \notin \partial D\). Let us estimate the first term in the first boundary integral. We have

\[ \int_{S(x,r) \cap \partial D} \Phi(x,y) \partial_r \varphi(y) ds(y) = \int_{S(x,r) \cap \partial D} e^{ikr/4\pi r^2} \partial_r \nu(y) ds(y) \]

and by the ML-inequality we have

\[ \left| \int_{S(x,r) \cap \partial D} \Phi(x,y) \partial_r \varphi(y) ds(y) \right| \leq \frac{1}{4\pi r} \sup_{y \in S(x,r) \cap \partial D} |\nabla \varphi(y)| 4\pi r^2 \to 0 \]
as \(r \to 0\) because \(|\nabla \varphi|\) has a uniform bound in \(\partial D\). In the last integral we have \(\partial_r u \Phi(x,y) = -\partial_r (e^{ikr/4\pi r}) = -i k e^{ikr/4\pi r} + e^{ikr/4\pi r^2}\). The integral involving \(i k e^{ikr/4\pi r}\) can be estimated as above to conclude that it vanishes when \(r \to 0\). The remaining integral is

\[ -\frac{ikr}{4\pi r^2} \int_{S(x,r) \cap \partial D} \varphi(y) ds(y) \]
\[ = -\frac{ikr}{4\pi r^2} \int_{S(x,r) \cap \partial D} (\varphi(y) - \varphi(x)) ds(y) - \frac{ikr}{4\pi r^2} \varphi(x) s(S(x,r) \cap \partial D). \]

We have \(|\varphi(y) - \varphi(x)| \leq \sup_{x \in \partial D} |\nabla \varphi(x)||x - y|\) so the absolute value of the first integral above can be estimated as

\[ \ldots \leq \sup_{x \in \partial D} |\nabla \varphi| \frac{1}{4\pi r^2} \int_{S(x,r) \cap \partial D} |x - y| dy = \sup_{x \in \partial D} |\nabla \varphi| \frac{1}{4\pi r^2} r s(S(x,r) \cap \partial D) \to 0 \]
as \(r \to 0\). The remaining term proves the claim in each of the cases \(x \in D, x \in \mathbb{R}^3 \setminus \overline{D}\).

\[ \square \]

**Lemma 2.3.** Let \(D \subset \mathbb{R}^3\) be a bounded domain with smooth boundary and \(k \in \mathbb{R}_+\). Let \(u_s \in H^1(D)\) with \((\Delta + k^2) u_s \in L^2(D)\). Then

\[ u_s(x) = -\int_D \Phi(x,y)(\Delta + k^2) u_s(y) dy \]
\[ + \int_{\partial D} (\Phi(x,y) \partial_r u_s(y) - u_s(y) \partial_r \Phi(x,y)) ds(y) \quad (2.3) \]
for \( x \in D \) in the distribution sense. For \( x \in \mathbb{R}^3 \setminus \overline{D} \) we have

\[
0 = -\int_D \Phi(x, y)(\Delta + k^2)u_s(y)dy + \int_{\partial D} (\Phi(x, y)\partial_n u_s(y) - u_s(y)\partial_n \Phi(x, y))ds(y) \tag{2.4}
\]

in the distribution sense. Here the boundary integrals involving \( \partial_n u_s \) are to be interpreted as distribution pairings between a \( H^{-1/2}(\partial D) \) function and a test function.

**Proof.** We will prove only the first case, namely \( x \in D \). The second one follows similarly. Let \( (\varphi_j)_{j=0}^\infty \) be a sequence of smooth functions defined on \( D \) such that

\[
\|u_s - \varphi_j\|_{H^1(D)} + \|((\Delta + k^2)(u_s - \varphi_j))\|_{L^2(D)} \to 0
\]
as \( j \to \infty \). Such a sequence exists, for example by convolving \( u_s \) with a mollifier \( \psi \), as in \( \varphi_j = (u_s * \psi_{1/j})|_D \).

We have \( \Phi(x, y) = \Psi(x - y) \) for \( \Psi(z) = \exp(ik|z|)/(4\pi|z|) \) which is locally integrable in \( \mathbb{R}^3 \). Hence the first term in the right-hand side of (2.3), equal to \( \Psi * (\Delta + k^2)u_s \), can be approximated by \( \Psi * (\Delta + k^2)\varphi_j \) in the \( L^2(D) \)-sense.

For any \( x \in D \) the second integral in (2.3) is well defined because \( y \mapsto \Phi(x, y) \) and \( y \mapsto \partial_n \Phi(x, y) \) are smooth on the smooth manifold \( \partial D \).

Moreover the \( x \)-dependence is smooth, so the mapping

\[
u_s \mapsto \int_{\partial D} u_s(y)\partial_n \Phi(x, y)ds(y)
\]
is bounded \( H^1(D) \to H^{1/2}(\partial D) \to C^0(D) \) and similarly

\[
u_s \mapsto \int_{\partial D} \Phi(x, y)\partial_n u_s(y)ds(y)
\]
is bounded \( H^1(D) \to H^{-1/2}(\partial D) \to C^0(D) \) when the integral is interpreted as a distribution pairing between a \( H^{-1/2}(\partial D) \)-function and a test function. The continuity does not necessarily hold up to the boundary. Because \( \varphi_j \to u_s \) in \( H^1(D) \) and the trace operators map \( \text{Tr} : H^1(D) \to H^{1/2}(\partial D) \), \( \partial_n : H^1(D) \to H^{-1/2}(\partial D) \), so the boundary integrals with \( u_s \) replaced by \( \varphi_j \) converge to the corresponding ones in \( C^0(D) \), namely uniformly over compact subsets of \( D \).
In conclusion, for a test function \( \psi \in C_0^\infty(D) \) we have

\[
(u_s, \psi) = \lim_{j \to \infty} \langle \varphi_j, \psi \rangle
\]

\[
= \lim_{j \to \infty} \left\langle - \int_D \Phi(x, y) (\Delta + k^2) \varphi_j(y) dy + \int_{\partial D} (\Phi(x, y) \partial_n \varphi_j(y) - \varphi_j(y) \partial_n \Phi(x, y)) ds(y), \psi(x) \right\rangle
\]

\[
= \left\langle - \int_D \Phi(x, y) (\Delta + k^2) u_s(y) dy + \int_{\partial D} (\Phi(x, y) \partial_n u_s(y) - u_s(y) \partial_n \Phi(x, y)) ds(y), \psi(x) \right\rangle
\]

so the equality holds in \( \mathcal{D}'(D) \).

**Proposition 2.4.** Let \( \Omega \subset \mathbb{R}^3 \) be a screen, \( k \in \mathbb{R}_+ \) and \( \Phi \) the fundamental solution from Lemma 2.2. Let \( u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \Omega) \). If \( (\Delta + k^2)u_s = 0 \) in \( \mathbb{R}^3 \setminus \Omega \) and it satisfies the Sommerfeld radiation condition, then

\[
u_s(x) = \int_{\mathbb{R}^2} \Phi(x, y')(\partial_3 u_s^+ - \partial_3 u_s^-)(y') dy',
\]

(2.5)

for \( x \in \mathbb{R}^3 \setminus \Omega \). Also \( y' \mapsto (\partial_3 u_s^+ - \partial_3 u_s^-)(y') \) is in \( \widetilde{H}^{-1/2}(\Omega_0) \), and more precisely the integral above represents the distribution pairing of a \( \widetilde{H}^{-1/2}(\Omega_0) \)-function with the smooth test function \( \Phi \) restricted to \( \mathbb{R}^2 \times \{0\} \) on the \( y \)-variable.

**Proof.** Fix \( x \in \mathbb{R}^3 \setminus \Omega \). Let \( D \subset \mathbb{R}^3 \) be a bounded smooth domain for which \( x \in D \) and \( \Omega \subset \partial D \) and furthermore we want this set to be on top of \( \Omega \), namely that its boundary normal pointing to the interior at \( \Omega \) is \( e_3 \) and not \(-e_3\). Let \( R > \sup_{z \in D} |x - z| \). We will use the formulas of Lemma 2.3 on \( D \), which has \( \Omega \) on its boundary, and \( B(x, R) \setminus \overline{D} \).

Firstly note that since \( (\Delta + k^2)u_s = 0 \) only the boundary integrals on the right-hand sides of (2.3) and (2.4) remain. We will se the first integral as is, namely

\[
u_s(x) = \int_{\partial D} (\Phi(x, y) \partial_n^D u_s(y) - u_s(y) \partial_n^D \Phi(x, y)) ds(y),
\]

(2.6)

where we denote by \( \partial_n^D \) the internal boundary normal derivative of \( D \), applied to functions on \( D \). We will have the integrals in (2.4) to be over the set \( B(x, R) \setminus \overline{D} \). The boundary of this set is \( S(x, r) \cup \partial D \), and the boundary normal pointing to its interior is \(-e_3\) on \( \Omega \subset \partial(B(x, R) \setminus \overline{D}) \). We will split the boundary integral accordingly, and in the integral over \( \partial D \) we denote by \( \partial_n^{ex} \) the external boundary normal derivative applied to
function on $B(x, R) \setminus \overline{D}$. In conclusion (2.4) becomes

$$0 = \int_{S(x, R)} (\Phi(x, y)\partial_{\nu}u_s(y) - u_s(y)\partial_{\nu}\Phi(x, y))ds(y)
+ \int_{\partial D} (\Phi(x, y)(-\partial_{\nu}^{D_s})u_s(y) - u_s(y)(-\partial_{\nu}^{D_s})\Phi(x, y))ds(y). \tag{2.7}$$

Finally, by interior elliptic regularity we see that $u_s$ is continuous (in fact real analytic) in some neighbourhood of $x$. Also, because $x$ is outside of $\partial D$ and $S(x, R)$, the individual boundary integrals above are continuous. Hence the equality in the sense of distributions is in fact a pointwise equality for continuous functions. In other words, both of (2.6) and (2.7) hold as continuous functions. We still remind that the integrals involving $\partial_{\nu}u_s$ represent distribution pairings for an element of $H^{-1/2}(\partial D)$ with that of a smooth $\Phi$.

Let us add (2.6) and (2.7). By smoothness, $\partial_{\nu}^{D}\Phi = \partial_{\nu}^{D_s}\Phi$. Note that two-sided Sobolev traces of $H^1$-functions yield identical results, so the integrals of $u_s\partial_{\nu}^{D}\Phi$ and $u_s\partial_{\nu}^{D_s}\Phi$ in (2.6) and (2.7) cancel out. The sum then gives

$$u_s(x) = \int_{S(x, R)} (\Phi(x, y)\partial_{\nu}u_s(y) - u_s(y)\partial_{\nu}\Phi(x, y))ds(y)
+ \int_{\partial D} \Phi(x, y)(\partial_{\nu}^{D_s} u_s - \partial_{\nu}^{D_s} u_s)(y)ds(y). \tag{2.8}$$

Note that as $R \to \infty$ the first integral in (2.8) vanishes because $u_s$ satisfies the Sommerfeld radiation condition. Also, $u_s$ is $C^1$ outside of $\overline{\Omega}$ by elliptic interior regularity, so the second integral’s integrand is zero when $y \notin \overline{\Omega}$. Thus, letting $R \to \infty$ gives

$$u_s(x) = \int_{\Omega} \Phi(x, y)(\partial_{\nu}^{D_s} u_s - \partial_{\nu}^{D_s} u_s)(y)dy$$

which implies the claim as $\partial_{\nu}^{D_s} u_s = \partial_{\nu} u_s^+$ and $\partial_{\nu}^{D_s} u_s = \partial_{\nu} u_s^-$ on $\Omega \subset \mathbb{R}^2 \times \{0\}$. Furthermore, as above, since $u_s$ is $C^1$ outside of $\overline{\Omega}$, we see that $\partial_{\nu} u_s^+ - \partial_{\nu} u_s^- = 0$ outside of $\overline{\Omega}$, so the integrand in the statement is in $H^{-1/2}(\Omega_0)$, as claimed. \qed

With the proposition above, we are almost ready to prove the formula for the far-field of a wave scattered by a screen, Theorem 1.4. But first let us prove a lemma.

**Lemma 2.5.** Let $k \in \mathbb{R}_+$ and $K \subset \mathbb{R}^3$ be a nonempty compact set. Then

$$\lim_{r \to \infty} \sup_{|x| = r} \sup_{y \in K} |x| \left| \partial_{\nu}^\alpha \left( \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x} \cdot y} \right) \right| = 0$$

for any multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 1$. Recall that $\hat{x} = x/|x|$.

**Proof.** The case of $|\alpha| = 0$ is well known, see for example the proof of Theorem 2.5 in [12]. For $|\alpha| = 1$ we will instead show the equivalent statement with $\partial_{\nu}^\alpha$ replaced by $\nabla_y$. Recall the following differentiation rules
\[ \nabla_y [x-y]^s = -s \frac{|x-y|^{s-1}}{|x-y|} \text{ for all } s \in \mathbb{R}, \]
\[ \nabla_y e^{ik|x-y|} = -ik \frac{x-y}{|x-y|} e^{ik|x-y|}, \text{ and} \]
\[ \nabla_y e^{-ik\hat{x} y} = -ik \hat{x} e^{-ik\hat{x} y}. \]

These imply that
\[ \nabla_y \left( \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x} y} \right) \]
\[ = -ik \frac{x-y}{|x-y|} e^{ik|x-y|} + \frac{x-y}{|x-y|^2} e^{ik|x-y|} + ik \hat{x} e^{ik|x|} e^{-ik\hat{x} y} \]
\[ = -ik \left( \frac{x-y}{|x-y|} - \hat{x} \right) e^{ik|x-y|} - ik \hat{x} \left( \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x} y} \right) \]
\[ + \frac{x-y}{|x-y|^2} e^{ik|x-y|}. \]

Let us consider the three types of terms above. To prove the estimate, let us take the absolute value and multiply by \(|x|\). The last one gives
\[ |x| \left| \frac{x-y}{|x-y|} e^{ik|x-y|} \right| = \frac{|x|}{|x-y|^2} \to 0 \]
uniformly as \( y \in K, |x| = r \) and \( r \to \infty \). The first term gives
\[ |x| \left| \frac{x-y}{|x-y|} \right| e^{ik|x-y|} = k |x| \frac{x-y}{|x-y|} - \frac{x}{|x|} \]
where can still estimate
\[ \frac{|x-y|}{|x-y|} \frac{x}{|x|} = \frac{x-y}{|x|} - \frac{|x|}{|x|} \frac{y}{|y|} \leq \frac{|x-y|}{|x|} + \frac{|y|}{|x|} \leq 2 \frac{|y|}{|x|} \]
because \(|x| - |x-y| \leq |y|\) by the triangle inequality. Thus the first term also tends to zero uniformly as \( r \to \infty \). Lastly, the second one is estimated as
\[ |x| \left| -ik \hat{x} \left( \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x} y} \right) \right| = k |x| \left| \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x} y} \right| \]
which tends to zero uniformly because this is the case \(|\alpha| = 0\) proven at the beginning of this proof.

**Proof of Theorem 1.4.** By the definition of the far-field there is a finite constant \( C > 0 \) independent of \( x \) such that
\[ \left| u_\infty(\hat{x}) - |x| e^{-ik|x|} u_+(x) \right| \leq \frac{C}{|x|} \]
when \(|x| \to \infty\). Let us denote \( \rho(y') = (\partial_3 u_+^+(y') - \partial_3 u_-^-(y'))(y') \). Then (2.5) gives
\[ u_\infty^+(\hat{x}) = \lim_{|x| \to \infty} |x| e^{-ik|x|} (\rho(y'), \Phi(x, y'))_{y'} \]

11
If \( \rho \) is integrable then similar to the formula (3.1) in the statement. We can rewrite

\[
|x|e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle = \left( \rho(y'), \frac{|x|e^{-ik|x|}\Phi(x, y^0) - e^{-ik\hat{y}^0}/(4\pi)}{4\pi} \right)_{y'} + \frac{1}{4\pi} \langle \rho(y'), e^{-ik\hat{y}^0} \rangle_{y'}.
\]

We can write the \( C^1 \)-test function on the second line as

\[
|x|e^{-ik|x|}\Phi(x, y^0) - e^{-ik\hat{y}^0}/(4\pi) = \frac{e^{-ik|x|}|x|}{4\pi} \left( \frac{e^{ik|y^0|}}{|x - y^0|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{y}^0} \right)
\]

which converges to zero in the \( C^1 \) topology over \( y' \), and a fortiori \( y^0 \), restricted to any compact set by Lemma 2.5. Note that the \( C^1 \)-seminorms are taken with respect to the \( y' \)-variable, and the absolute value makes the \( e^{-ik|x|} \) that doesn’t appear in the lemma disappear. Hence the application of the lemma is allowed. Elements of \( H^{-1/2}(\Omega_0) \) act well on \( C^1 \)-functions, so the distribution pairing with \( \rho \) and the test function tend to zero. Thus

\[
\lim_{|x| \to \infty} |x|e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle_{y'} = \frac{1}{4\pi} \langle \rho(y'), e^{-ik\hat{y}^0} \rangle_{y'}
\]
as claimed. \( \square \)

## 3 Solving the inverse problem

We are ready to tackle the inverse problem in this section.

**Lemma 3.1.** Let \( k \in \mathbb{R} \) and \( \rho \in \mathcal{E}'(\mathbb{R}^2) \) be a distribution of compact support. Let\(^1\)

\[
u^\infty(x) = \frac{1}{4\pi} \langle \rho, e^{-ik\hat{y}^0} \rangle
\]

for \( \hat{x} \in S^2 \) and where the distribution pairing is over the variable \( y' = (y_1, y_2) \in \mathbb{R}^2 \). Then \( \rho \) is uniquely determined by \( \nu^\infty \).

**Proof.** The operator mapping \( \rho \mapsto \nu^\infty \) is bounded and linear \( \mathcal{E}'(\mathbb{R}^2) \to C^0(S^2) \). This is because \( \hat{x} \mapsto (y' \mapsto \exp(-ik\hat{x} \cdot y')) \) is continuous \( S^2 \to \mathcal{E}(\mathbb{R}^2) \). So it is enough to show that \( \rho = 0 \) if \( \nu^\infty = 0 \). Let us assume the latter. For \( \xi' \in \mathbb{R}^2 \) we have

\[
\hat{\rho}(\xi') = \frac{1}{2\pi} \langle \rho, e^{-i\xi' \cdot y'} \rangle
\]

where the distribution pairing is over the variable \( y' \in \mathbb{R}^2 \). This looks similar to the formula (3.1) in the statement. We can rewrite

\[-ik\hat{x} \cdot y^0 = -ik(\hat{x}_1, \hat{x}_2, \hat{x}_3) \cdot (y_1, y_2, 0) = -i(k\hat{x}_1, k\hat{x}_2) \cdot (y_1, y_2).\]

\(^1\)If \( \rho \) is integrable then \( \nu^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-ik\hat{x} \cdot y^0} \rho(y')dy' \).
Thus

\[ u_s^\infty(\hat{x}) = \frac{1}{2} \hat{\rho}(k\hat{x}_1, k\hat{x}_2). \]  

(3.2)

The left-hand side is zero for all \( \hat{x} \in S^2 \). When \( \hat{x} \) goes through the whole of \( S^2 \), the sum including only two of the squares, \( \hat{x}_1^2 + \hat{x}_2^2 \), goes through the whole interval \((0, 1)\). Alternatively

\[ \hat{\rho}(\xi') = 2u_s^\infty \left( \xi_1/k, \xi_2/k, \sqrt{k^2 - \xi_1^2 - \xi_2^2/k} \right) = 0 \]

for all \( |\xi'| \leq k \). Since \( \rho \) has compact support, \( \hat{\rho} \) can be extended to an entire function on \( \mathbb{C}^2 \). Since it vanishes on an open subset of \( \mathbb{R}^2 \) it must be the zero function. Hence \( u_s^\infty = 0 \) implies \( \rho = 0 \).

\[ \text{Lemma 3.2. Let } (\Delta + k^2)u_i = 0 \text{ in } \mathbb{R}^3. \text{ Let } \Omega \subset \mathbb{R}^3 \text{ be a screen and } u_s \text{ satisfy the direct scattering problem 1.2. Denote} \]

\[ \rho(x') = \partial_j u_j^i(x^0) - \partial_3 u_j^i(x^0) \]

for \( x' \in \mathbb{R}^2 \) and its properties are given in Proposition 2.4. If \( u_i(x', x_3) \neq -u_i(x', -x_3) \) for some \( x \in \mathbb{R}^3 \) then

\[ \Omega_0 = \text{supp } \rho \]

(3.3)

for the shape \( \Omega_0 \) of the screen \( \Omega \).

**Proof.** The function \( \rho \) is a well-defined \( H^{-1/2}(\Omega_0) \)-function by Proposition 2.4 so in particular \( \text{supp } \rho \subset \Omega_0 \). It remains to prove that \( \Omega_0 \subset \text{supp } \rho \).

Assume the contrary, that \( \Omega_0 \) is not contained in the support of \( \rho \). Then neither is \( \Omega_0 \) because if \( \Omega_0 \subset \text{supp } \rho \) then \( \Omega_0 \subset \text{suppp } \rho = \text{supp } \rho \).

Because \( \Omega_0 \) is an open set and \( \text{supp } \rho \) is closed there is \( x_0' \in \Omega_0 \) and \( r > 0 \) such that \( B(x_0', r) \subset \Omega_0 \setminus \text{supp } \rho \).

Let us study the behaviour of \( u_s \) in the tube \( B(x_0', r) \times \mathbb{R} \). We have \( \rho = 0 \) on \( B(x_0', r) \). Recall formula (1.4), which combined with the vanishing of \( \rho \) implies that \((\Delta + k^2)u_s = 0\) in the whole tube, and interior elliptic regularity implies that \( u_s \) is smooth there. In addition the formula implies that \( u_s(x_1, x_2, x_3) = u_s(x_1, x_2, -x_3) \) for all \( x \) in the tube. The vanishing of \( \rho \) gives \( \partial_3 u_j^i = \partial_3 u_j^i \) on the base of the tube. These two imply that actually \( \partial_3 u_j(\cdot, 0) = 0 \) for \( x' \in B(x_0', r) \).

We have the following

\[ u_s = -u_i, \]

(3.4)

\[ \partial_3 u_s = 0 \]  

(3.5)

on \( B(x_0', r) \times \{0\} \). Let us calculate the higher order derivatives. Note that \( \partial_3^2 \) and \((\Delta + k^2)\) commute, and \((\Delta + k^2)u_s = 0 \) in the tube. Thus

\[ 0 = \partial_3^2(\Delta + k^2)u_s = (\Delta + k^2)\partial_3^2 u_s = (\Delta' + k^2)\partial_3^2 u_s + \partial_3^{i+2} u_s \]

in the tube, and we denote \( \Delta' = \partial_3^2 \). This gives \( \partial_3^{i+2} u_s = -(\Delta' + k^2)\partial_3^i u_s \). Let us restrict ourselves to \( B(x_0', r) \times \{0\} \) next. By induction and (3.4)–(3.5) we see that

\[ \partial_3^j u_s = \begin{cases} 
    (-1)^{j+1}(\Delta' + k^2)^j u_s, & j \in 2\mathbb{N}, \\
    0, & j \in 2\mathbb{N} + 1 
\end{cases} \]
on $B(x'_0, r) \times \{0\}$. This can still be simplified! Recall that $u_i$ is an incident wave, so $(\Delta + k^2)u_i = 0$ everywhere. This means that $(\Delta' + k^2)u_i = -\partial^2_i u_i$, and a fortiori $(\Delta' + k^2)'u_j = (-\partial^2_i)' u_i$ everywhere by the commutating of $\partial^2_i$ and $(\Delta' + k^2)$. This implies

$$\partial^2_i u_i = \begin{cases} \partial^2_i u_i, & j \in 2\mathbb{N}, \\ 0, & j \in 2\mathbb{N} + 1. \end{cases} \tag{3.6}$$

The other derivatives, $\partial_1$ and $\partial_2$ commute with each other and $\partial_3$, so finally we have

$$\partial^\alpha u_s = \begin{cases} -\partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N} + 1 \end{cases} \tag{3.7}$$

on $B(x'_0, r) \times \{0\}$ for all multi-indices $\alpha \in \mathbb{N}^3$. Let us define

$$\bar{u}_i(x) = \frac{1}{2} \left( u_i(x_1, x_2, x_3) + u_i(x_1, x_2, -x_3) \right)$$

for all $x \in \mathbb{R}^3$. This satisfies the Helmholtz equation everywhere, and is an incident wave because $u_i$ is one. We see that

$$\partial^\alpha \bar{u}_i(x) = \frac{1}{2} \left( \partial^\alpha u_i(x_1, x_2, x_3) + (-1)^{\alpha_3} \partial^\alpha u_i(x_1, x_2, -x_3) \right)$$

so

$$\partial^\alpha \bar{u}_i = \begin{cases} \partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N} + 1 \end{cases} \tag{3.8}$$

on $B(x'_0, r) \times \{0\}$. By (3.7) we see immediately that $\partial^\alpha u_s = -\partial^\alpha \bar{u}_i$ on the base of the tube for all $\alpha \in \mathbb{N}^3$. Both functions $u_s$ and $\bar{u}_i$ satisfy the Helmholtz equation not only in the tube but also in $\mathbb{R}^3 \setminus \overline{B}(0, R)$, where $R > 0$ is large enough that $\overline{\Omega} \subset B(0, R)$. Solutions of the Helmholtz equation are real-analytic. Because their Taylor-expansions at $(x'_0, 0)$ are equal, the functions are equal in the component of $(B(x'_0, r) \times \mathbb{R}) \cup (\mathbb{R}^3 \setminus \overline{B}(0, R))$ that contains $(x'_0, 0)$, so in particular $u_s = -\bar{u}_i$ in all of $\mathbb{R}^3 \setminus \overline{B}(0, R)$.

The function $u_s$ satisfies the Sommerfeld radiation condition, so so does $\bar{u}_i$. On the other hand $(\Delta + k^2)\bar{u}_i = 0$ in all of $\mathbb{R}^3$, so $\bar{u}_i$ is the zero function$^2$, which means that $u_i$ is antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$, a contradiction. Hence $\overline{\Omega} \subset \text{supp } \rho$. \hfill \square

The solution to the inverse problem of determining a screen $\Omega$ from the knowledge of a single incident wave $u_i$ and the corresponding far-field $u_\infty$ scattered from the screen comes from a combination of determining $\rho$ from the far-field, and then $\Omega$ from $\rho$. There is a slight surprise, namely that the problem is only solvable for incident waves that are not too (anti)symmetric. However, one sees that antisymmetry is not the deciding factor: what matters is whether $u_i$ is identically zero on the screen. By

$^2$Use e.g. (2.2) for a large ball whose radius grows to infinity. The boundary integral decreases to zero as was seen for the first integral in (2.8).
a similar argument as that at the end of the proof of Lemma 3.2, we see that if \( u_i = 0 \) on a non-empty open subset of \( \mathbb{R}^2 \times \{0\} \) then \( u_i(x', x_3) = -u_i(x', -x_3) \) for all \( x \in \mathbb{R}^3 \). It is interesting to see that partial invisibility is achieved inside thickened screens as long as the incident plane wave comes from a direction almost parallel to the screen’s normal [15]. The direction of incident waves seems very important in scattering from objects that are thin in one direction.

**Proof of Theorem 1.5.** Theorem 1.4 and Lemma 3.1 imply that \( \rho = \tilde{\rho} \) when \( u_\infty^s = \tilde{u}_\infty^s \). If \( u_i \) is not antisymmetric with respect to \( \mathbb{R}^2 \times \{0\} \) then \( \Omega_0 = \text{supp } \rho = \text{supp } \tilde{\rho} = \tilde{\Omega}_0 \) by Lemma 3.2. Because \( \Omega_0 \) is a smooth domain, we have \( \Omega_0 = \text{int } \overline{\Omega_0} \), and similarly for \( \tilde{\Omega}_0 \). Thus the equation above implies \( \Omega_0 = \tilde{\Omega}_0 \) and by lifting, \( \Omega = \tilde{\Omega} \).

If \( u_i \) is antisymmetric then \( u_i = 0 \) everywhere on \( \mathbb{R}^2 \times \{0\} \) and \( u_i = 0 \) satisfies all conditions of the direct scattering problem. Since solutions to the direct scattering problem (1.2) are unique by [36, Thms 2.5–2.7], this is the only solution. Thus \( u_s = \tilde{u}_s = 0 \) and the same holds for their far-fields. This is irrespective of the shape of \( \Omega, \tilde{\Omega} \subset \mathbb{R}^2 \).

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**References**

[1] G. Alessandrini and L. Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc., 35 (2005), 1685–1691.

[2] E. Blästen, *Nonradiating sources and transmission eigenfunctions vanish at corners and edges*, SIAM J. Math. Anal. 50 (2018), no. 6, 6255–6270.

[3] E. Blästen, O. Yu. Imanuvilov and M. Yamamoto, *Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials*, Inverse Probl. Imaging 9 (2015), no. 3, 709–723.

[4] E. Blästen and H. Liu, *On corners scattering stably, nearly non-scattering interrogating waves, and stable shape determination by a single far-field pattern*, Indiana Univ. Math. J., in press, 2019.

[5] E. Blästen and H. Liu, *Recovering piecewise constant refractive indices by a single far-field pattern*, Inverse Problems, accepted 2020, http://iopscience.iop.org/10.1088/1361-6420/ab958f.

[6] E. Blästen, L. Päivärinta and J. Sylvester, *Corners always scatter*, Comm. Math. Phys., 331 (2014), 725–753.
[7] E. Blåsten, L. Tzou and J. Wang, Uniqueness for the inverse boundary value problem with singular potentials in 2D, Math. Z. (2019), https://doi.org/10.1007/s00209-019-02436-0.
[8] A. L. Bukhgeim, Recovering a potential from Cauchy data in the two-dimensional case, J. Inverse Ill-Posed Probl. 16 (2008), no. 1, 19–33.
[9] F. Cakoni, D Colton and E. Darrigrand, The inverse electromagnetic scattering problem for screens, Inverse Problems 19 (2003), no. 3, 627–642.
[10] J. Cheng and M. Yamamoto, Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves, Inverse Problems, 19 (2003), 1361–1384.
[11] D. Colton and A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, Inverse Problems 12 (1996), no. 4.
[12] D. Colton and R. Kress, Inverse acoustic and electromagnetic scattering theory, Applied Mathematical Sciences, 93, Springer-Verlag, Berlin, 1992.
[13] D. Colton and B. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, IMA J. Appl. Math., 31 (1983), 253–259.
[14] D. Dos Santos Ferreira, C. E Kenig and M. Salo, Determining an unbounded potential from Cauchy data in admissible geometries, Comm. Partial Differential Equations 38 (2013), no. 1, 50–68.
[15] Y. Deng, H. Liu and G. Uhlmann, On regularized full- and partial-cloaks in acoustic scattering, Comm. Partial Differential Equations, 42 (2017), no. 6, 821–851.
[16] J. Elschner and M. Yamamoto, Uniqueness in determining polyhedral sound-hard obstacles with a single incoming wave, Inverse Problems, 24 (2008), 035004 (7pp).
[17] L. C. Evans, Partial differential equations, second edition, Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 2010.
[18] C. Guillarmou and L. Tzou, Calderón inverse problem with partial data on Riemann surfaces, Duke Math. J. 158 (2011), no. 1, 83–120.
[19] N. Honda, G. Nakamura and M. Sini, Analytic extension and reconstruction of obstacles from few measurements for elliptic second order operators, Math. Ann., 355 (2013), no. 2, 401–427.
[20] G. Hu, M. Salo and E. Vesalainen, Shape identification in inverse medium scattering problems with a single far-field pattern, SIAM J. Math. Anal., 48 (2016),152–165.
[21] M. Ikehata, Reconstruction of a source domain from the Cauchy data, Inverse Problems, 15 (1999), 637–645.
[22] O. Yu. Imanuvilov, G. Uhlmann and M. Yamamoto, The Calderón problem with partial data in two dimensions, J. Amer. Math. Soc. 23 (2010), no. 3, 655–691.
[23] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2nd edition, Springer-Verlag, New York, 2006.

[24] A. Kirsch and N. Grinberg, *The factorization method for inverse problems*, Oxford Lecture Series in Mathematics and its Applications, 36, Oxford University Press, Oxford, 2008.

[25] S. Kusiak and J. Sylvester, *The scattering support*, Comm. Pure Appl. Math., 56 (2003), 1525–1548.

[26] S. Kusiak and J. Sylvester, *The convex scattering support in a background medium*, SIAM J. Math. Anal., 36 (2005), 1142–1158.

[27] P. Lax and R. Phillips, *Scattering Theory*, Academic Press, New York and London, 1967.

[28] H. Liu, M. Petri, L. Rondi and J. Xiao, *Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements*, J. Differential Equations, 262 (2017), no. 3, 1631–1670.

[29] H. Liu, L. Rondi and J. Xiao, *Mosco convergence for $H(\text{curl})$ spaces, higher integrability for Maxwell’s equations, and stability in direct and inverse EM scattering problems*, J. Eur. Math. Soc. (JEMS), 21 (2019), no. 10, 2945–2993.

[30] H. Liu and J. Zou, *Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers*, Inverse Problems, 22 (2006), 515–524.

[31] J. C. Maxwell, *On physical lines of force*, Philosophical Magazine, 90 (1861), 11–23.

[32] L. Päivärinta and S. Rempel, *A deconvolution problem with the kernel $1/|x|$ on the plane*, Appl. Anal. 26 (1987), no. 2, 105–128.

[33] L. Päivärinta and S. Rempel, *Corner singularities of solutions to $\Delta^{+1/2}u = f$ in two dimensions*, Asymptotic Anal. 5 (1992), no. 5, 429–460.

[34] L. Päivärinta, M. Salo and E. V. Vesalainen, *Strictly convex corners scatter*, Rev. Mat. Iberoam. 33 (2017), no. 4, 1369–1396.

[35] L. Rondi, *Stable determination of sound-soft polyhedral scatterers by a single measurement*, Indiana Univ. Math. J., 57 (2008), 1377–1408.

[36] E. P. Stephan, *Boundary integral equations for screen problems in $\mathbb{R}^3$*, Integral Equations Operator Theory 10 (1987), no. 2, 236–257.

[37] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) 125 (1987), no. 1, 153–169.

[38] H. Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[39] G. Uhlmann, *Inside Out: Inverse Problems and Applications*, Cambridge University Press, Cambridge, 2003.