BRST symmetry and the convolutional double copy

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ABSTRACT: Motivated by the results of Anastasiou et al., we consider the convolutional double copy for BRST and anti-BRST covariant formulations of gravitational and gauge theories in more detail. We give a general BRST and anti-BRST invariant formulation of linearised \( \mathcal{N} = 0 \) supergravity using superspace methods and show how this may be obtained from the square of linearised Yang-Mills theories. We demonstrate this relation for the Schwarzschild black hole and the ten-dimensional black string solution as two concrete examples.

KEYWORDS: Black Holes, BRST Quantization, Scattering Amplitudes, Supergravity Models

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1 Introduction

The emergence in recent years of a powerful “double copy” relation between gravitational and gauge theory scattering amplitudes [1, 2], mirroring earlier such relations found in string theory [3], has led to a wide variety of applications, including the ability to calculate higher-loop supergravity amplitudes and in gravitational wave physics (see [4] and references cited therein). Given the by now well-established double copy relation for scattering amplitudes, a clear question worth pursuing is whether this relation is reflected at the level of classical solutions (see [5] and references cited therein). Most of the explicit such relationships that have been found, such that the Kerr-Schild double copy [6] or the Weyl double copy [7], work for a special class of gravitational solutions. However, the convolutional double copy [8–10] posits a square relationship between gauge and gravitational
theories at the level of the fields, albeit at the linearised level. We shall be focusing on this approach in this paper.

The idea of a convolutional double copy goes back to [11], where the global U-duality groups are analysed. However, more significantly it is shown in [8] that the convolutional product of gauge theory symmetries reproduces the symmetries of the gravitational theory at the linear level. That there may be a convolutional square relation between gauge theory and gravitational fields in position space is not so surprising if we recall the double copy relation for scattering amplitudes, which are derived in momentum space. In [10], taking the low-energy limit of the bosonic string ($\mathcal{N} = 0$ supergravity) as a concrete gravitational theory, it is found that in order to have a matching of the degrees of freedom in this theory, i.e. $(D^2 - 2D + 2)$, and the square of the Yang-Mills theory, i.e. $(D - 1)^2$, as well as to be able to disentangle the dilaton and the trace of the metric, one ought to consider a BRST covariant formulation for each of the theories. Thus, a set of ghost fields are introduced for the gravitational theory, as well as the gauge theories. This sets up a dynamical equivalence between gauge theories squared and gravitational theories.\footnote{The idea that BRST ghosts are needed in order to establish the double copy relations was presaged in the work of Siegel in [12].} However, an important assumption in this construction is that all fields are sourced by non-trivial currents, which makes it possible formally to invert the d’Alembertian operator appearing in the equations of motion.

In this paper, we revisit [10] and consider the procedure outlined there more generally and in more detail. In particular, we present a detailed study of a BRST and anti-BRST covariant formulation of the $\mathcal{N} = 0$ theory. We show that the 2-form gauge invariance may be neatly treated using the superspace method developed in [13, 14]. The BRST and anti-BRST Lagrangian is then constructed by adding to the original gauge-invariant Lagrangian a gauge-fixing contribution that is by construction BRST and anti-BRST invariant. One of the facts that we emphasise in this work is that because the inverse of the d’Alembertian operator may be used freely, this means that the source term is in principle ambiguous in the sense that there is some freedom to choose which terms are taken to be contributing to the $\mathcal{N} = 0$ supergravity sources and which terms are not. This is particularly important when dealing with explicit solutions.

In section 2, we derive a general BRST and anti-BRST invariant formulation of the $\mathcal{N} = 0$ theory and show how the action used in [10] may be obtained as a special case of this general formulation. In section 3, we reconsider in greater detail the convolutional relation studied in [10]. In contrast to [10], we do not constrain the gravitational field ansätze using the equations of motion, because as explained above terms may be absorbed into the source term, making such conditions arbitrary. We rely solely on the BRST and anti-BRST transformations to constrain the ansätze. We derive the double copy relation for the sources in section 3.3. Our results are broadly in agreement with those of [10]. We highlight the fact that there is freedom in choosing the coefficients in the ansätze. In particular, the choice made by [10] leads to a breakdown of the double copy relation when the Yang-Mills gauge-fixing parameter $\xi = 1$. This behaviour is an artefact of the
choice made in [10] and there is another choice for which this behaviour is not observed and moreover the equation for the gravitational source term is simpler. It should be noted that, similar consideration, such as not using equations of motion to impose constraints on the double copy dictionary has already appeared in earlier works, for example in [15–18].

In section 4, we demonstrate the convolutional double copy relation for two classical solutions: the Schwarzschild black hole and the ten-dimensional black string solution. Black holes have been considered in this context in [16, 19–21]. However, here, we derive the gravitational fields from the ansätze derived in section 3. Interestingly, in order to obtain a convolutional double copy description of the black string, we need to use Yang-Mills fields that lie in an $SU(2)$ subgroup of the gauge group. This contrasts with the double copy description of the black hole, where it suffices to consider Yang-Mills fields in a $U(1)$ subgroup of the gauge group. We also discuss the BPS limit of the black string, for which it turns out that only an abelian $U(1)$ subgroup of the Yang-Mills gauge group is required.

After conclusions in section 5, we include also an appendix, where we summarise some basic results about the convolution product that is employed in the double-copy relations throughout the paper. We also discuss the notion of the convolution inverse, which plays a role in the double-copy relations, and in particular we highlight the fact that care must be taken ensure that the convolution inverse actually exists for the functions one is using.

2 BRST symmetry and $\mathcal{N} = 0$ supergravity

2.1 The linearised theory

The theory that arises as the low-energy limit of the bosonic string is sometimes referred to as “$\mathcal{N} = 0$ supergravity”. Its field content comprises the metric $g_{\mu\nu}$, the dilaton $\varphi$ and the antisymmetric tensor potential $B_{\mu\nu}$, with the equations of motion following from the Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{12} e^{-a\varphi} H^2 \right],$$

where $H = dB$ and

$$a = \sqrt{\frac{8}{D - 2}},$$

with $D$ being the dimension of the spacetime.

Our discussion will be restricted to the linearisation of the theory around a Minkowski spacetime background, with the metric written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$  (2.3)

At this order, the Lagrangian is simply given by

$$\mathcal{L}(h_{\mu\nu}, \varphi, B_{\mu\nu}) = -\frac{1}{2} h^{\mu\nu} G^\text{lin}_{\mu\nu} - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho},$$

where

$$G^\text{lin}_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu} + \partial_{\sigma} \partial_{(\mu} h_{\nu)} \sigma - \frac{1}{2} \partial_{\mu} \partial_{\nu} h - \frac{1}{2} \partial_{\rho} \partial_{\sigma} h^{\rho\sigma} \eta_{\mu\nu} + \frac{1}{2} \Box h \eta_{\mu\nu}.$$  (2.5)
is the linearised Einstein tensor, \( h = \eta^{\mu\nu} h_{\mu\nu} \), and indices are raised and lowered using the Minkowski metric. This Lagrangian is invariant under linearised diffeomorphisms and 2-form gauge transformations, namely under
\[
\delta h_{\mu\nu} = 2 \partial(\mu \xi_\nu), \quad \delta B_{\mu\nu} = 2 \partial(\mu \lambda_\nu), \quad \delta \varphi = 0.
\] (2.6)

### 2.2 General formulation invariant under BRST and anti-BRST

In a BRST formulation, the diffeomorphism and 2-form gauge invariances are handled by the introduction of diffeomorphism ghosts \( c_\mu \) and \( \bar{c}_\mu \), and 2-form gauge invariance ghosts \( d_\mu \) and \( \bar{d}_\mu \). The implementation of the BRST formulation for the 2-form gauge symmetry is a little more complicated than that for the diffeomorphism invariance, because of the “ghosts for ghosts” phenomenon, stemming from the fact that gauge parameters \( \lambda_\mu \) in eq. (2.6) that are themselves of the form \( \lambda_\mu = \partial_\mu \lambda \) do not contribute. The upshot from this is that in addition to the ghost \( d_\mu \) and antighost \( \bar{d}_\mu \) fields one must also introduce fields \( d \), \( \bar{d} \) and \( \eta \). We shall discuss this in detail below, but first we consider the ghost \( c_\mu \) and antighost \( \bar{c}_\mu \) in the gravity sector.

In the gravity sector, the BRST transformations in the linearised theory are as follows:
\[
Q h_{\mu\nu} = 2 \partial(\mu c_\nu), \quad Q c_\mu = 0, \quad Q \bar{c}_\mu = W_\mu, \quad Q W_\mu = 0.
\] (2.7)

There are also anti-BRST transformations
\[
\bar{Q} h_{\mu\nu} = 2 \partial(\mu \bar{c}_\nu), \quad \bar{Q} c_\mu = 0, \quad \bar{Q} \bar{c}_\mu = -W_\mu, \quad \bar{Q} W_\mu = 0.
\] (2.8)

Later, \( W_\mu \) will be seen to be the gauge-fixing functional for the diffeomorphism symmetry. The BRST and anti-BRST transformations can be seen to anticommute when acting on any of the fields, i.e. \( \{Q, \bar{Q}\} = 0 \). Note that this is an off-shell statement: no equations of motion are involved.

The BRST and anti-BRST description of the 2-form gauge symmetry requires a lengthier discussion, which follows in the next subsection.

### 2.3 BRST and anti-BRST for the 2-form gauge invariance

Our discussion here is based upon the approach of [13, 14]. We define a superspace exterior derivative operator\(^2\)
\[
\hat{d} = d + s + \bar{s},
\] (2.9)

where
\[
d = dx^\mu \frac{\partial}{\partial x^\mu}, \quad s = d\theta \frac{\partial}{\partial \theta}, \quad \bar{s} = d\bar{\theta} \frac{\partial}{\partial \bar{\theta}}.
\] (2.10)

We also define a super 2-form \( B \) as
\[
B = B - d_\mu dx^\mu \wedge d\theta - \bar{d}_\mu dx^\mu \wedge d\bar{\theta} - d d\theta \wedge d\theta - \bar{d} d\bar{\theta} \wedge d\bar{\theta} - \eta d\theta \wedge d\bar{\theta},
\] (2.11)
where $B$ is just the usual bosonic 2-form,

$$B = \frac{1}{2}B_{\mu\nu}\,dx^\mu \wedge dx^\nu, \quad (2.12)$$

and $d_\mu, \bar{d}_\mu, d, \bar{d}$ and $\eta$ are ghost fields with ghost numbers $+1, -1, +2, -2$ and 0 respectively.

Fields are taken to be functions of $x^\mu, \theta$ and $\bar{\theta}$, with theta expansions of the form

$$B_{\mu\nu}(x, \theta, \bar{\theta}) = B_{\mu\nu}(x) + QB_{\mu\nu}(x) \theta + \tilde{Q}B_{\mu\nu}(x) \bar{\theta} + \cdots, \quad (2.13)$$

e etc. Here, $Q$ denotes the BRST operator and $\tilde{Q}$ the anti-BRST operator. In what follows, it will be assumed that after exterior derivatives are taken, $\theta$ and $\bar{\theta}$ are set to zero.

We now find that $\mathcal{H} \equiv \mathcal{d}S$ is given by

$$\mathcal{H} = H + \left(\frac{1}{2}QB_{\mu\nu} - \partial_{[\mu}d_{\nu]}\right) dx^\mu \wedge dx^\nu \wedge d\theta + \left(\frac{1}{2}\bar{Q}B_{\mu\nu} - \partial_{[\mu}\bar{d}_{\nu]}\right) dx^\mu \wedge dx^\nu \wedge d\bar{\theta} - (\partial_\mu d - \bar{Q}d_\mu) dx^\mu \wedge d\theta \wedge d\bar{\theta} + (\partial_\mu \bar{d} - Q\bar{d}_\mu) dx^\mu \wedge d\theta \wedge d\bar{\theta} + (Q\eta + Qd) d\theta \wedge d\bar{\theta} - (Q\eta + \bar{Q}d) d\theta \wedge d\bar{\theta} - Qd \partial_\mu d \theta \wedge d\bar{\theta} + \bar{Q}d \partial_\mu \bar{d} \theta \wedge d\bar{\theta} + (\partial_\mu \eta - \bar{Q}d_\mu) dx^\mu \wedge d\theta \wedge d\bar{\theta} - (\partial_\mu \bar{\eta} - Q\bar{d}_\mu) dx^\mu \wedge d\theta \wedge d\bar{\theta} + (Q\bar{d} \partial_\mu \bar{d} \theta \wedge d\bar{\theta} - (Q\bar{d} + \bar{Q}d) \partial_\mu \bar{d} \theta \wedge d\bar{\theta} + \bar{Q}d \partial_\mu d \bar{\theta} \wedge d\bar{\theta}, \quad (2.14)$$

where $H = \mathcal{d}B = \frac{1}{6}H_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$ is the purely bosonic 3-form field strength.

Requiring that all the components of $\mathcal{H}$ with projections in the $\theta$ and $\bar{\theta}$ directions must vanish gives the equations

$$QB_{\mu\nu} = 2\partial_{[\mu}d_{\nu]}, \quad Qd_\mu = \partial_\mu d, \quad Qd = 0, \quad QB_{\mu\nu} = 2\partial_{[\mu}\bar{d}_{\nu]}\bar{d}, \quad \bar{Q}d_\mu = \partial_\mu \bar{d}, \quad \bar{Q}d = 0, \quad (2.16)$$

together with

$$Q\bar{d} + \bar{Q}\eta = 0, \quad \bar{Q}d + Q\eta = 0, \quad \partial_\mu \eta - Q\bar{d}_\mu - \bar{Q}d_\mu = 0. \quad (2.17)$$

These last three equations can be rewritten as separated BRST and anti-BRST transformations by introducing auxiliary ghost fields $\tau$ and $\bar{\tau}$ and an auxiliary bosonic field $K_\mu$, such that eqs. (2.17) become\(^3\)

$$Q\bar{d} = -\bar{\tau}, \quad \bar{Q}\eta = \bar{\tau}, \quad \bar{Q}d = -\tau, \quad Q\eta = \tau, \quad (2.18)$$

$$Q\bar{d}_\mu = K_\mu + \frac{1}{2}\partial_\mu \eta, \quad \bar{Q}d_\mu = -K_\mu + \frac{1}{2}\partial_\mu \eta, \quad (2.19)$$

\(^3\)Note that in \cite{22}, the last equation in (2.17) is written in terms of two distinct auxiliary fields $B_\mu$ and $\bar{B}_\mu$ as

$$Q\bar{d}_\mu = B_\mu, \quad \bar{Q}d_\mu = -\bar{B}_\mu$$

together with the constraint $B_\mu - \bar{B}_\mu - \partial_\mu \eta = 0$ that would then have to be imposed by hand. Instead, here, we are using just the single auxiliary field $K_\mu$, in terms of which the fields of \cite{22} are given by $B_\mu = K_\mu + \frac{1}{2}\partial_\mu \eta$ and $\bar{B}_\mu = K_\mu - \frac{1}{2}\partial_\mu \eta$. The advantage of our current approach using just $K_\mu$ is that there is no need to impose any constraint equation.
where \( K_\mu \) is for now arbitrary. Eventually, it will acquire the interpretation of being an auxiliary field whose algebraic equation follows after the introduction of a gauge-fixing and ghost action. It follows also that

\[
Q \tau = 0, \quad \bar{Q} \tau = 0, \quad Q \bar{\tau} = 0, \quad \bar{Q} \bar{\tau} = 0,
\]

\[
QK_\mu = -\frac{1}{2} \partial_\mu \tau, \quad \bar{Q}K_\mu = \frac{1}{2} \partial_\mu \bar{\tau}.
\] (2.20)

Again, \( \tau \) and \( \bar{\tau} \) are arbitrary at this stage, and will later be seen to be auxiliary fields that are determined by the form of the gauge-fixing and ghost action.

2.4 General class of BRST and anti-BRST invariant theories

We can now construct a general class of theories by considering a Lagrangian

\[
\mathcal{L} = \mathcal{L}(h_{\mu\nu}, \phi, B_{\mu\nu}) + \hat{\mathcal{L}},
\] (2.21)

where \( \mathcal{L}(h_{\mu\nu}, \phi, B_{\mu\nu}) \) is the set of (gauge-invariant) kinetic terms for \( B_{\mu\nu}, h_{\mu\nu} \) and \( \phi \), and

\[
\hat{\mathcal{L}} = \bar{Q}QX = -\bar{Q}QX,
\] (2.22)

where \( X \) is the most general quadratic function built from the fields \( h_{\mu\nu}, \phi, B_{\mu\nu}, c_\mu, \bar{c}_\mu, d_\mu, \bar{d}_\mu, d, \bar{d}, \eta \), consistent with having vanishing ghost number and being Lorentz invariant. We also impose the restriction that the Lagrangian should not contain any terms with more than two derivatives. The most general such function can be taken to be

\[
X = \beta_1 \bar{d} d + \beta_2 \bar{d}^\mu d_\mu + \beta_3 B^{\mu\nu} B_{\mu\nu} + \gamma_1 \bar{c}^\mu c_\mu + \gamma_2 h^{\mu\nu} h_{\mu\nu} + \gamma_3 h^2
+ \beta_4 \bar{d}^\mu c_\mu + \bar{\beta}_4 \bar{d}^\mu d_\mu + \beta_5 h \eta + \beta_6 h \phi.
\] (2.23)

\( \hat{\mathcal{L}} \) constructed by this means will comprise the gauge-fixing and ghost terms in the full Lagrangian \( \mathcal{L} \) given in eq. (2.21).

Using eqs. (2.16), (2.18), (2.19) and (2.20), we then find from (2.22) that

\[
\hat{\mathcal{L}} = \beta_1 \bar{\tau} \tau + \beta_2 \left[ \bar{\tau} \partial^\mu d_\mu - \partial^\mu \bar{\tau} d_\mu - \bar{\partial} \bar{d} + K^\mu K_\mu - \frac{1}{4} (\partial \eta)^2 \right]
+ 4\beta_3 (\bar{d}^\mu \partial_\mu c_\nu + 4h^{\mu\nu} \partial_\mu h_\nu) + \gamma_4 h \eta
+ \beta_4 \left( \bar{\tau} \partial^\mu c_\mu + W^\mu K_\mu + \frac{1}{2} W^\mu \partial_\mu \eta \right)
+ \bar{\beta}_4 \left( -\partial^\mu \bar{c}_\mu + W^\mu K_\mu - \frac{1}{2} W^\mu \partial_\mu \eta \right)
+ 2\beta_5 W^\mu \partial_\mu \eta
- 2\beta_6 W^\mu \partial_\mu \phi.
\] (2.24)

Recall that in this formulation we have \( \hat{\mathcal{L}} = Q\bar{Q}X = -\bar{Q}QX \) off-shell and with no need for imposing any constraint, and so \( \hat{\mathcal{L}} \) is manifestly invariant under both \( Q \) and \( \bar{Q} \) transformations.
Solving for the auxiliary fields $W_\mu$ and $K_\mu$, we find

\begin{align}
W_\mu &= \frac{1}{\Delta} \left\{ -8\beta_2 \left( \gamma_2 \partial^\nu h_{\nu\mu} + \gamma_3 \partial_\mu h + \frac{1}{2} \beta_6 \partial_\mu \varphi \right) + 4\beta_3 \left( \beta_4 + \bar{\beta}_4 \right) \partial^\nu B_{\nu\mu} \\
&\quad + \beta_2 \left( \beta_4 - \bar{\beta}_4 - 4\beta_5 \right) \partial_\mu \eta \right\}, \\
K_\mu &= \frac{1}{\Delta} \left\{ 4\left( \beta_4 + \bar{\beta}_4 \right) \left( \gamma_2 \partial^\nu h_{\nu\mu} + \gamma_3 \partial_\mu h + \frac{1}{2} \beta_6 \partial_\mu \varphi \right) - 8\beta_3 \gamma_1 \partial^\nu B_{\nu\mu} + \\
&\quad -\frac{1}{2} \left( \beta_4 + \bar{\beta}_4 \right) \left( \beta_4 - \bar{\beta}_4 - 4\beta_5 \right) \partial_\mu \eta \right\},
\end{align}

where

\[ \Delta = (\beta_4 + \bar{\beta}_4)^2 - 4\beta_2 \gamma_1. \]  

We can also solve the algebraic equations of motion for the auxiliary fields $\tau$ and $\bar{\tau}$ that follow from the Lagrangian (2.24), finding

\begin{align}
\tau &= -\frac{1}{\beta_1} \left[ \beta_2 \partial^\mu d_\mu + (\beta_4 - 2\beta_5) \partial^\mu c_\mu \right], \\
\bar{\tau} &= \frac{1}{\beta_1} \left[ \beta_2 \partial^\mu \bar{d}_\mu + (\beta_4 + 2\beta_5) \partial^\mu \bar{c}_\mu \right].
\end{align}

### 2.5 Specialisation to ref. [10]

It is instructive to compare the general expressions we have obtained in sections 2.3 and 2.4 with the form of the BRST transformations and Lagrangian discussed in [10]. Scaling up the Lagrangian in [10] by a factor of 2 in order to agree with our normalisation, it becomes

\[ \mathcal{L}' = \mathcal{L}(h_{\mu\nu}, \varphi, B_{\mu\nu}) + \hat{\mathcal{L}}', \]

where \( \mathcal{L}(h_{\mu\nu}, \varphi, B_{\mu\nu}) \) is given by eq. (2.4) and

\[ \hat{\mathcal{L}}' = -2\bar{c}_\mu \square c_\mu - 2\bar{d}\square d_\mu + \left( 2 - \frac{2m_d}{\xi_d} \right) \bar{d}_\mu \partial^\mu \partial^\nu d_\nu + 2m_d \bar{d}\square d + \frac{1}{\xi_h} \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h \right)^2 \\
+ \frac{1}{\xi_B} \left( \partial^\nu B_{\nu\mu} + \partial_\mu \eta' \right)^2. \]

Note that in matching the BRST transformations in [10] with our transformations in section 2.3 we can see that the normalisations of \( c_\mu, d_\mu \) and \( d \) (and their conjugates) are the same, but we must introduce a scaling factor that relates our \( \eta \) ghost and the corresponding \( \eta' \) ghost in [10]. From a matching of the BRST transformations we deduce that our parameters in the \( \hat{\mathcal{L}}' \) Lagrangian (2.30), and the relation between \( \eta \) and \( \eta' \), must be chosen so that

\begin{align}
\beta_4 &= -\bar{\beta}_4 = 2\beta_5, \\
\beta_6 &= 0, \\
\gamma_3 &= -\frac{1}{2} \gamma_2, \\
\gamma_1 &= 2\gamma_2 \xi_h, \\
\beta_2 &= 2\beta_3 \xi_B, \\
\beta_1 &= -\beta_2 \xi_d, \\
\eta' &= \frac{1}{2} \xi_B \eta.
\end{align}

Consistency also requires that \( m_d = \frac{1}{2} \xi_B \).
After substituting the equations of motion of auxiliary fields in $\hat{L}$, and comparing the Lagrangians (i.e. $\hat{L}$ and $\hat{L}'$), we find that we must take $\beta_3 = \gamma_2 = -\frac{1}{2}$. Thus, to summarise, we find that the BRST formulation in [10] corresponds to the specialisation of our general formulation in which
\begin{equation}
\beta_1 = \xi_B \xi_d, \quad \beta_2 = -\xi_B, \quad \beta_3 = -\frac{1}{2}, \quad \beta_4 = -\bar{\beta}_4 = 2\beta_5, \quad \beta_6 = 0,
\end{equation}
\begin{equation}
\gamma_1 = -\xi_h, \quad \gamma_2 = -\frac{1}{2}, \quad \gamma_3 = \frac{1}{4}.
\end{equation}

Furthermore, in order for the formulation in [10] to exhibit both BRST and anti-BRST invariance, it is necessary that the parameters $m_d$ and $\xi_B$ obey the relation
\begin{equation}
m_d = \frac{1}{2} \xi_B.
\end{equation}
Note that with the choices in eqs. (2.32) for the coefficients, we have
\begin{equation}
W_\mu = \frac{1}{\xi_h} \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h \right), \quad K_\mu = \frac{1}{\xi_B} \partial^\nu B_{\nu\mu}.
\end{equation}
Thus $Qc_\mu = W_\mu$ and $Q\bar{d}_\mu = K_\mu + \frac{1}{2} \partial_\mu \eta' + \xi_B (\partial^\nu B_{\nu\mu} + \partial \eta')$ can be seen to have the standard forms for De Donder and 2-form Lorenz gauge-fixing functionals.

We finish this section by summarising the BRST and anti-BRST transformations of the fields in this specialisation. These will be needed in section 3.2. The BRST transformations are given by
\begin{equation}
\begin{aligned}
Q h_{\mu\nu} &= 2 \partial_\nu \epsilon_\mu, & Q c_\mu &= 0, & Q \bar{c}_\mu &= \frac{1}{\xi_h} \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h \right), & Q \varphi &= 0, \\
Q B_{\mu\nu} &= 2 \partial_\nu d_\mu, & Q d_\mu &= \partial_\mu d, & Q \bar{d}_\mu &= 0, \\
Q \bar{d}_\mu &= \frac{1}{\xi_B} (\partial^\nu B_{\nu\mu} + \partial_\mu \eta'), & Q \bar{d} &= \frac{1}{\xi_d} \partial^\mu \bar{d}_\mu, & Q \eta' &= \frac{m_d}{\xi_d} \partial^\mu d_\mu.
\end{aligned}
\end{equation}
The anti-BRST transformations are:
\begin{equation}
\begin{aligned}
\bar{Q} h_{\mu\nu} &= 2 \partial_\nu \bar{c}_\mu, & \bar{Q} \bar{c}_\mu &= 0, & \bar{Q} c_\mu &= -\frac{1}{\xi_h} \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h \right), & \bar{Q} \varphi &= 0, \\
\bar{Q} B_{\mu\nu} &= 2 \partial_\nu \bar{d}_\mu, & \bar{Q} \bar{d}_\mu &= \partial_\mu \bar{d}, & \bar{Q} \bar{d} &= 0, \\
\bar{Q} \bar{d}_\mu &= -\frac{1}{\xi_B} (\partial^\nu B_{\nu\mu} + \partial_\mu \eta'), & \bar{Q} \bar{d} &= -\frac{1}{\xi_d} \partial^\mu \bar{d}_\mu, & \bar{Q} \eta' &= -\frac{m_d}{\xi_d} \partial^\mu \bar{d}_\mu.
\end{aligned}
\end{equation}

3 $\mathcal{N} = 0$ Supergravity from the Square of Yang-Mills

As outlined in the introduction, there are various realisations of the double copy idea at the level of solutions and theories, each with its own features and challenges. Here, we will be focusing on the convolutional double copy of [8]. The basic idea here is that in momentum space, the double copy relation satisfied by the scattering amplitudes is one of a product, i.e. a gravitational amplitude is roughly obtained by taking the product of two
Yang-Mills amplitudes. Therefore, at the level of fields, expressed in position space, it is natural to expect that the double copy relation will be one of convolution. More precisely, at the linearised level, one expects a relation of the form
\[ H_{\mu\nu} = A_{\mu}^a \ast \phi_{ab}^{-1} \ast \tilde{A}_{\nu}^b \equiv A_{\mu}^a \circ \tilde{A}_{\nu}^b, \]  
(3.1)
where \( H_{\mu\nu} \) is a composite field containing the metric perturbation \( h_{\mu\nu} \), two-form field \( B_{\mu\nu} \) and the dilaton \( \varphi \), and \( \phi_{ab} \) is the spectator bi-adjoint scalar field \([23, 24]\), which is used to soak up the potentially different gauge indices. \( \ast \) denotes a convolution, while \( \circ \) denotes a convolutive product with the pseudo-inverse of the bi-adjoint scalar suppressed, for brevity. (See appendix A for a more detailed discussion of the convolution product.) Thus, in this way one can view the gravitational theory as the square of a gauge theory. Moreover, the linearised gravitational symmetries may be derived from those of the gauge theory.

Off-shell, the gravitational theory has \((D^2 - 2D + 2)\) degrees of freedom: \( D(D - 1)/2 \) from the metric, \((D - 1)(D - 2)/2 \) from the 2-form and 1 from the scalar field. In contrast, each one of the gauge theories has \((D - 1)\) degrees of freedom, which come from the gauge field. Taking the convolutive product of gauge theory fields then gives \((D - 1)^2\) degrees of freedom, which is one short of the gravitational theory. This problem is resolved in \([10]\) by including ghost fields to take care of the gauge symmetry, leading to BRST covariant theories. We discussed the BRST covariant formulation of the gravitational theory in section 2, and we shall review the BRST formulation of Yang-Mills theory in 3.1. The gravitational theory, then, has fields \((h_{\mu\nu}, B_{\mu\nu}, \varphi, c_\mu, \bar{c}_\mu, d_\mu, \bar{d}_\mu, d, \bar{d}, \eta)\) and the two Yang-Mills theories have fields \((A_\mu, c, \bar{c})\) and \((\tilde{A}_\mu, \tilde{c}, \bar{\tilde{c}})\) (see section 3.1). The convolutive product of the ghost fields of the Yang-Mills theories gives 4 degrees of freedom, which accounts for the degrees of freedom in \((d, \bar{d}, \eta)\) and a combination of \( h \) and \( \varphi \), which we can simply think of as accounting for \( \varphi \). The convolutive product of the ghost sector and the gauge fields then contributes \(4(D - 1)\) degrees of freedom, which account for the gravitational ghost fields \((c_\mu, \bar{c}_\mu, d_\mu, \bar{d}_\mu)\). Finally, now, the \((D - 1)^2\) degrees of freedom coming from the convolutive product of the gauge fields accounts for the degrees of freedom in \( h_{\mu\nu} \) and \( B_{\mu\nu} \).

In addition, the introduction of the ghost fields allows one to disentangle the degrees of freedom in \( h \) and \( \varphi \). Crucially, the formalism developed in \([10]\) requires the presence of sources so that one can work with the non-local operator \( \Box^{-1} \),
\[ \Box^{-1} \Box = \Box \Box^{-1} = 1. \]  
(3.2)
Thus, this construction cannot deal with source-free vacuum solutions, such as gravitational waves.

### 3.1 BRST formulation of linearised Yang-Mills

Following the proposal in \([10]\) for deriving the linearised \( \mathcal{N} = 0 \) supergravity theory as the convolution product of two copies of linearised Yang-Mills, we begin by reviewing the BRST formulation for the linearised Yang-Mills theories. Thus we begin with the Lagrangian
\[ \mathcal{L}_A = \text{tr} \left( -\frac{1}{4} F^{\mu\nu}_A F_{\mu\nu} + \frac{1}{\xi} (\partial^\mu A_\mu)^2 - \bar{c} \Box c \right), \]  
(3.3)
where at the linearised level we can just take $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. The equations of motion for the Yang-Mills and ghost fields are, after introducing sources, given by

$$\Box A_\mu - \frac{\xi + 1}{\xi} \partial_\mu \partial_\nu A_\nu = j_\mu(A), \quad \Box c^\alpha = j^\alpha(c),$$

(3.4)

where, following [10], the ghost $c$ and antighost $\tilde{c}$ are grouped into an $\text{OSp}(2)$ doublet $c^\alpha$, with

$$c^1 = c, \quad c^2 = \tilde{c}, \quad c_1 = \bar{c}, \quad c_2 = -c.$$  

(3.5)

The constant $\xi$ parameterises a family of Lorenz gauge fixings. Making momentum space expansions

$$A_\mu = \int d^4 p A_\mu(p) e^{ip \cdot x}, \quad j_\mu = \int d^4 p J_\mu(p) e^{ip \cdot x},$$

(3.6)

we have

$$-p^2 A_\mu + \frac{\xi + 1}{\xi} p_\mu p_\nu A_\nu = J_\mu.$$  

(3.7)

The propagator $G_{\mu\nu}$ has the property that

$$A_\mu = G_{\mu\nu} J^\nu.$$  

(3.8)

Making the ansatz that

$$G_{\mu\nu} = \alpha \eta_{\mu\nu} + \beta p_\mu p_\nu,$$  

(3.9)

we can plug $J_\mu$ from eq. (3.7) into eq. (3.8), and hence solve for $\alpha$ and $\beta$. The result is that the propagator is given by

$$G_{\mu\nu} = \frac{\eta_{\mu\nu} - \frac{\xi + 1}{\xi} p_\mu p_\nu}{p^2}.$$  

(3.10)

Taking $\xi = -1$ corresponds to the Feynman-'t Hooft propagator. Note that if we make no gauge fixing (i.e. take $\xi = \infty$), the propagator becomes singular, as expected.

The theory (3.3) is invariant under the BRST transformations

$$QA_\mu = \partial_\mu c, \quad Qc = 0, \quad Q\tilde{c} = \frac{1}{\xi} \partial^\mu A_\mu.$$  

(3.11)

It is also invariant under the anti-BRST transformations

$$\bar{Q}A_\mu = \partial_\mu \bar{c}, \quad \bar{Q}\bar{c} = 0, \quad \bar{Q}c = -\frac{1}{\xi} \partial^\mu A_\mu.$$  

(3.12)

### 3.2 BRST, anti-BRST and the convolutional double copy

Following [10], we begin by writing ansätze for the fields of linearised $\mathcal{N} = 0$ supergravity as the convolution product of two copies of linearised Yang-Mills (one untilded and one tilded), as follows:

$$\varphi = A^\rho \circ \tilde{A}_\rho + \alpha_1 c^\alpha \circ \tilde{c}_\alpha + \alpha_2 \Box^{-1} (\partial A \circ \partial \tilde{A}),$$  

(3.13)

$$h_{\mu\nu} = A_{(\mu} \circ \tilde{A}_{\nu)} + a_1 \Box^{-1} \partial_{(\mu} \partial_{\nu} (A^\rho \circ \tilde{A}_\rho) + a_2 \Box^{-1} \partial_{(\mu} \partial_{\nu} (c^\alpha \circ \tilde{c}_\alpha) + a_3 \Box^{-1} (\partial A \circ \partial_{(\mu} \tilde{A}_{\nu)} + \partial_{(\mu} A_{\nu)} \circ \partial \tilde{A}) + a_4 \Box^{-2} \partial_{(\mu} \partial_{\nu} (\partial A \circ \partial \tilde{A}) + c_{(1} e_{2} c^{\alpha} \circ \tilde{c}_{\alpha} + b_3 \Box^{-1} (\partial A \circ \partial \tilde{A})),$$  

(3.14)

$$B_{\mu\nu} = A_{[\mu} \circ \tilde{A}_{\nu]} + \gamma \Box^{-1} (\partial A \circ \partial_{[\mu} \tilde{A}_{\nu]} - \partial_{[\mu} A_{\nu]} \circ \partial \tilde{A}).$$  

(3.15)
Thus eq. (3.21) implies
\[ Q(c^\alpha \circ \tilde{c}_\alpha) = -\frac{1}{\xi}(c \circ \partial \tilde{A} + \partial A \circ \tilde{c}) , \quad \bar{Q}(\bar{c}^\alpha \circ \bar{c}_\alpha) = -\frac{1}{\bar{\xi}}(\bar{c} \circ \partial \bar{\tilde{A}} + \partial A \circ \bar{\tilde{c}}). \] (3.16)

Note that the Yang-Mills gauge and ghost fields in the ansätze (3.13), (3.14) and (3.15) should be thought of as carrying gauge-group indices that are suppressed and traced over in the convolution products.

Constraints on the constant coefficients in the ansätze follow by requiring that the fields \( h_{\mu \nu}, \varphi \) and \( B_{\mu \nu} \) should satisfy their BRST and anti-BRST transformation rules, as discussed in section 2, as a consequence of the BRST and anti-BRST transformation rules (3.11) and (3.12). For definiteness, we shall assume in this discussion that the parameters in the gauge-fixing and ghost Lagrangian \( \hat{L} \) in eq. (2.24) have been specialised as in eqs. (2.32) and (2.33), so that the gauge-fixing functionals \( W_\mu \) and \( K_\mu \) take the forms given in eqs. (2.34). The BRST and anti-BRST transformations in this case are summarised in (2.35) and (2.36), respectively.

Starting with the dilaton field \( \varphi \), the requirement that \( Q \varphi = 0 \) implies
\[ 1 - \frac{\alpha_1}{\xi} + \alpha_2 = 0. \] (3.17)
We obtain the identical condition from considering instead \( \bar{Q} \varphi = 0 \).

From \( Q h_{\mu \nu} = 2 \partial (\mu \circ c_\nu) \), we deduce that
\[ b_1 - \frac{b_2}{\xi} + b_3 = 0 \] (3.18)
and
\[ c_\mu = \frac{1}{2} (1 + a_3) (c \circ \tilde{A}_\mu + \tilde{A}_\mu \circ \bar{c}) + \frac{1}{2} \left(a_1 - \frac{a_2}{\xi} + a_3 + a_4\right) \Box^{-1} \partial (c \circ \partial \tilde{A} + \partial A \circ \bar{c}). \] (3.19)

Using instead \( \bar{Q} h_{\mu \nu} = 2 \partial (\mu \bar{c}_\nu) \) gives the same condition (3.18) and also
\[ \bar{c}_\mu = \frac{1}{2} (1 + a_3) (\bar{c} \circ \tilde{A}_\mu + \tilde{A}_\mu \circ \bar{c}) + \frac{1}{2} \left(\bar{a}_1 - \frac{a_2}{\bar{\xi}} + a_3 + a_4\right) \Box^{-1} \partial (\bar{c} \circ \partial \tilde{A} + \partial A \circ \bar{c}). \] (3.20)

So in fact \( \bar{c}_\mu \) is precisely what one would get by conjugating the expression for \( c_\mu \).

From \( \bar{Q} \bar{c}_\mu = \xi_{\hat{h}}^{-1} \left( \partial^\mu h_{\nu \mu} - \frac{1}{2} \partial_\mu h \right) \) we then find
\[ \frac{1 + a_3}{2\xi} = \frac{1 + a_3}{2\xi_{\hat{h}}}, \] (3.21)
\[ 1 + a_1 - \frac{a_2}{\xi} + 2a_3 + a_4 = \frac{a_2 + (2 - D)b_2}{\xi_{\hat{h}}}, \] (3.22)
\[ \frac{1}{\xi} \left(a_1 - \frac{a_2}{\xi} + a_3 + a_4\right) = \frac{1}{2\xi_{\hat{h}}} \left[a_4 + (2 - D)b_3\right], \] (3.23)
\[ 0 = a_1 - 1 + (2 - D)b_1. \] (3.24)
Thus eq. (3.21) implies \( \xi_{\hat{h}} = \xi \). Note that eqs. (3.22), (3.23) and (3.24) also imply eq. (3.18).
The upshot is that in the $\varphi$ and $h_{\mu\nu}$ sectors, we may think of $\alpha_1$, $a_1$, $a_2$, $a_3$ and $a_4$ as being freely specifiable, with $\alpha_2$, $b_1$, $b_2$ and $b_3$ then being determined by eqs. (3.17), (3.22), (3.23) and (3.24).

From the ansatz for the $B_{\mu\nu}$ field in eq. (3.15), we can read off from the BRST transformation $QB_{\mu\nu} = 2\partial_{[\mu}d_{\nu]}$ that
\[
d_{\mu} = \frac{1}{2}(\gamma + 1)(c \circ \tilde{A}_{\mu} - A_\mu \circ \tilde{c}) + \beta \Box^{-1} \partial_{(\mu}(c \circ \partial \tilde{A} - \partial A \circ \tilde{c}),
\]
where $\beta$ is an as-yet undetermined constant. Considering instead $\bar{Q}B_{\mu\nu} = 2\partial_{[\mu}\bar{d}_{\nu]}$, we deduce
\[
d_{\mu} = \frac{1}{2}(\gamma + 1)(\bar{c} \circ \tilde{A}_{\mu} - A_\mu \circ \tilde{\bar{c}}) + \bar{\beta} \Box^{-1} \partial_{(\mu}(\bar{c} \circ \partial \tilde{A} - \partial A \circ \tilde{\bar{c}}),
\]
where $\bar{\beta}$ is an (a priori different) as-yet undetermined constant.

Plugging $d_{\mu}$ given in eq. (3.25) into $Qd_{\mu} = \partial_{\mu}d$ implies that
\[
d = -(\gamma + 1 + 2\beta)c \circ \tilde{c}.
\]
Similarly, from $\bar{Q}d_{\mu} = \partial_{\mu}\bar{d}$ we obtain
\[
\bar{d} = -(\gamma + 1 + 2\bar{\beta})\bar{c} \circ \tilde{\bar{c}}.
\]

Plugging $\tilde{d}_{\mu}$ into $Q\tilde{d}_{\mu} = \xi_{B}^{-1}(\partial^{\nu}B_{\nu\mu} + \partial_{\mu}\eta')$ implies that we must have
\[
\xi_{B} = \xi,
\]
and also gives\footnote{Note that it is the rescaled field $\eta' = \frac{1}{2}\xi_{B}\eta$ that appears in eq. (2.30), and not the field $\eta$ that is defined in section 2.3.}
\[
\eta' = -\frac{1}{2}(\gamma + 1 + 2\bar{\beta})\xi_{B}(c \circ \tilde{c} + \bar{c} \circ \tilde{\bar{c}}).
\]
From $\bar{Q}d_{\mu} = -\xi_{B}^{-1}(\partial^{\nu}B_{\nu\mu} - \partial_{\mu}\eta')$ we again learn that $\xi_{B} = \xi$, and also we find
\[
\eta' = -\frac{1}{2}(\gamma + 1 + 2\beta)\xi_{B}(c \circ \tilde{c} + \bar{c} \circ \tilde{\bar{c}}).
\]
Comparing with eq. (3.30) therefore implies that the constants $\beta$ and $\bar{\beta}$ are equal,
\[
\bar{\beta} = \beta.
\]

Plugging $\tilde{d}$ from eq. (3.28) into $Q\tilde{d} = \xi_{d}^{-1}\partial^{\mu}\tilde{d}_{\mu}$ gives
\[
\frac{\gamma + 1 + 2\beta}{\xi} = \frac{\gamma + 1 + 2\beta}{2\xi_{d}},
\]
and hence (assuming $(\gamma + 1 + 2\beta) \neq 0$) that
\[
\xi_{d} = \frac{1}{2}\xi.
\]
The same conclusion results from $Qd = -\xi_{d}^{-1}\partial^{\mu}d_{\mu}$.
Plugging $\eta'$ given in eq. (3.30) into $Q\eta' = (m_d/\xi_d) \partial^\mu d^\mu$ gives

$$\left(\gamma + 1 + 2\beta\right) \left(\frac{\xi_B}{\xi} - \frac{m_d}{\xi_d}\right) = 0, \quad (3.35)$$

and so (assuming $(\gamma + 1 + 2\beta) \neq 0$),

$$\frac{\xi_B}{\xi} = \frac{m_d}{\xi_d}. \quad (3.36)$$

Putting all the above together, we have

$$\xi_B = \xi, \quad m_d = \xi_d = \frac{1}{2}\xi. \quad (3.37)$$

Note that these results imply, in particular, that the requirement in eq. (2.33) which we previously found to be necessary in order that $Q$ and $\bar{Q}$ anticommute is indeed obeyed.

Moreover, in the Kalb-Ramond sector, the coefficients $\beta$ and $\gamma$ are freely specifiable at this point.

### 3.3 Field equations and the convolutional double copy

Having extracted the various constraints on the coefficients in the convolutional double copy ansätze (3.13), (3.14) and (3.15) that follow from requiring a consistency with the BRST and anti-BRST transformations, we can now turn to a consideration of the field equations. For the fields $\varphi$, $h_{\mu\nu}$ and $B_{\mu\nu}$, the equations of motion will be those following from the Lagrangian $L'$ given by eqs. (2.29), (2.4) and (2.30), subject also to the condition (2.33). After including source currents, the equations of motion are

$$\square \varphi = j(\varphi), \quad (3.38)$$

$$\square h_{\mu\nu} - \frac{\xi_B + 2}{\xi_B} \left(2\partial^\rho \partial_{(\mu}h_{\nu)\rho} - \partial_{(\mu}h_{\nu)}\right) = j_{\mu\nu}(h), \quad (3.39)$$

$$\square B_{\mu\nu} + \frac{\xi_B + 2}{\xi_B} 2\partial^\rho \partial_{(\mu}B_{\nu)\rho} = j_{\mu\nu}(B). \quad (3.40)$$

We shall consider current sources for the ghost fields too, which thus obey the equations of motion

$$\square e_{\mu} = j_{\mu}(e_{\nu}), \quad \square \bar{e}_{\mu} = \bar{j}_{\mu}(\bar{e}_{\nu}), \quad (3.41)$$

After considering terms involving $\xi_B$ and $m_d$, the equations of motion for $\eta'$ can be obtained:

$$\square \eta' = j(\eta'). \quad (3.41)$$

The idea now is to substitute the convolutional double copy ansätze into the equations of motion. This first of all provides a verification that the equations for the fields of the $\mathcal{N} = 0$ supergravity (together with its ghost fields) are indeed satisfied by virtue of the equations of motion for the Yang-Mills fields (and their ghosts). Additionally, one can read
off the expressions for the source currents of the $\mathcal{N} = 0$ supergravity in terms of the source currents for the Yang-Mills theories.

It should be noted that by taking the divergence of the Yang-Mills equation

$$\Box A_{\mu} - \frac{\xi + 1}{\xi} \partial_{\mu} \partial A = j_{\mu}$$

we find $\Box \partial A = -\xi \partial j$, where $\partial j \equiv \partial_{\mu} j^{\mu}$, and hence $\partial A = -\xi \Box^{-1} \partial j$. Plugging this back into (3.42) allows us to solve for $A_{\mu}$ in terms of $j_{\mu}$, with

$$A_{\mu} = \Box^{-1} j_{\mu} - (\xi + 1) \Box^{-2} \partial_{\mu} \partial j.$$  

From this follow the results that

$$A^\rho \circ \tilde{A}_\rho = \Box^{-2} j^\rho \circ \tilde{j}_\rho + (\xi^2 - 1) \Box^{-3} \partial j \circ \partial \tilde{j}, \quad \partial A \circ A^\rho = \xi^2 \Box^{-2} \partial j \circ \partial \tilde{j}.$$  

Consider first the equation of motion for the dilaton. Plugging the ansatz (3.13) into the equation of motion (3.38) and using the above results gives

$$\Box \varphi = \Box^{-1} j^\rho \circ \tilde{j}_\rho + \alpha_1 \Box^{-1} j^{\alpha}(c) \circ \tilde{j}_{\alpha}(\tilde{c}) + [(1 + \alpha_2) \xi^2 - 1] \Box^{-2} \partial j \circ \partial \tilde{j},$$

and hence\(^5\)

$$j(\varphi) = \Box^{-1} j^\rho \circ \tilde{j}_\rho + \alpha_1 \Box^{-1} j^{\alpha}(c) \circ \tilde{j}_{\alpha}(\tilde{c}) + [(1 + \alpha_2) \xi^2 - 1] \Box^{-2} \partial j \circ \partial \tilde{j}.$$  

Now consider the equation of motion for the graviton. Plugging the ansatz (3.14) into the equation of motion (3.39), we can obtain an expression for the graviton current (energy-momentum tensor):

$$j_{\mu\nu}(h) = \Box^{-1} j_{(\mu} \circ \tilde{j}_{\nu)} + b_2 \eta_{\mu\nu} \Box^{-1} j^{\alpha}(c) \circ \tilde{j}_{\alpha}(\tilde{c}) + e_1 \left( \partial A \circ \partial_{(\mu} \tilde{A}_{\nu)} + \partial_{(\mu} A_{\nu)} \circ \partial \tilde{A} \right)$$

$$+ e_2 \partial_{(\mu} \eta_{\nu)} A^\rho \circ \tilde{A}_\rho + e_3 \Box^{-1} \partial_{\mu} \partial_{\nu} (\partial A \circ \partial \tilde{A}) + e_4 \eta_{\mu\nu} \partial A \circ \partial \tilde{A}$$

$$+ e_5 \partial_{\mu} \partial_{\nu} (c^{\alpha} \circ \tilde{c}_{\alpha}) + b_1 \eta_{\mu\nu} \Box^{-1} j^\rho \circ \tilde{j}_\rho,$$  

where the $e_i$ coefficients are given by\(^6\)

$$e_1 = -\frac{1 + 2a_3}{\xi}, \quad e_2 = \frac{1 + (D - 2)b_1](\xi + 2) - 2a_1}{\xi},$$

$$e_3 = \frac{(D - 2)(2 + \xi)}{\xi} b_3 - \left(1 + \frac{\xi^2}{\xi^2} + 2a_4 \xi \right), \quad e_4 = \left(1 - \frac{1}{\xi^2}\right) b_1 + b_3,$$

$$e_5 = \frac{(D - 2)(\xi + 2)}{\xi} b_2 + 2a_2.$$  

\(^5\)In [10], the choice $\alpha_1 = 1/\xi$ was made, which implies, given the condition (3.17) found previously, that $\alpha_2 = -1 + 1/\xi^2$ and hence the last term in (3.46) vanishes.

\(^6\)We have already used the equation $\xi_k = \xi$ that was derived previously from the BRST and anti-BRST invariance conditions.
In view of the relation (3.43), and the consequent equations (3.44), the graviton current \( \hat{j}_{\mu\nu}(h) \) in eq. (3.47) can be expressed entirely in terms of the Yang-Mills and Yang-Mills ghost source currents, as

\[
\begin{align*}
\hat{j}_{\mu\nu}(h) &= \Box^{-1} j_{[\mu} \circ \hat{j}_{\nu]} + b_2 \eta_{\mu\nu} \Box^{-1} j^a(c) \circ \hat{j}_a(c) - e_1 \xi \Box^{-2} \left( \partial j \circ \partial_{(\mu} j_{\nu)} + \partial_{(\mu} j_{\nu)} \circ \partial j \right) \\
&+ [2e_1 \xi (\xi + 1) + e_2 (\xi^2 - 1) + e_3 \xi^2] \Box^{-3} \partial_{\mu} \partial_{\nu} (\partial j \circ \partial j) + e_2 \Box^{-2} \partial_{\mu} \partial_{\nu} (j^\rho \circ \hat{j}_\rho) \\
&+ e_4 \eta_{\mu\nu} \Box^{-2} (\partial j \circ \partial \hat{j}) + e_5 \Box^{-2} \partial_{\mu} \partial_{\nu} (j^a(c) \circ \hat{j}_a(c)) + b_1 \eta_{\mu\nu} \Box^{-1} j^\rho \circ \hat{j}_\rho.
\end{align*}
\]

(3.49)

For the Kalb-Ramond field \( B_{\mu\nu} \) we find, plugging the ansatz (3.15) into the equation of motion (3.40) that the source current \( j_{\mu\nu}(B) \) is given by\(^7\)

\[
\begin{align*}
\hat{j}_{\mu\nu}(B) &= \Box^{-1} j_{[\mu} \circ \hat{j}_{\nu]} - \frac{2\gamma - 1}{\xi} \left( \partial A \circ \partial_{[\mu} \hat{A}_{\nu]} - \partial_{[\mu} \hat{A}_{\nu]} \circ \partial \hat{A} \right).
\end{align*}
\]

(3.50)

Written purely in terms of Yang-Mills source currents we therefore have

\[
\begin{align*}
\hat{j}_{\mu\nu}(B) &= \Box^{-1} j_{[\mu} \circ \hat{j}_{\nu]} + (2\gamma + 1) \Box^{-2} (\partial j \circ \partial_{[\mu} \hat{j}_{\nu]} \circ \partial \hat{j}) - \frac{1}{\xi} \left( \partial \hat{A} \circ \partial_{[\mu} \hat{A}_{\nu]} - \partial_{[\mu} \hat{A}_{\nu]} \circ \partial \hat{A} \right).
\end{align*}
\]

(3.51)

In what has been done so far, we have simply given the general expressions for the source currents \( j(\varphi) \), \( j_{\mu\nu}(h) \) and \( j_{\mu\nu}(B) \) for \( \varphi \), \( h_{\mu\nu} \) and \( B_{\mu\nu} \) in terms of the source currents of the Yang-Mills sectors. This has not resulted in any further restrictions on the coefficients in the original convolutional double copy ansätze (3.13), (3.14) and (3.15): the only restrictions are those we already derived in section 3.2 by considering the BRST and anti-BRST transformations. Since the number of such restrictions is less than the number of coefficients in the double-copy ansätze, one is free to impose some further restrictions by requiring the expressions for the source currents \( j(\varphi) \), \( j_{\mu\nu}(h) \) and \( j_{\mu\nu}(B) \) to take simplified forms. For example, one natural such choice could be to require that the terms appearing on the right-hand sides of eqs. (3.46), (3.49) and (3.51) should involve only the operator \( \Box^{-1} \) itself, and not the higher inverse powers \( \Box^{-2} \) or \( \Box^{-3} \).

This can indeed be done. Ostensibly, achieving this would require imposing a total of seven conditions on the remaining six free parameters in the double copy ansätze. However, after imposing the conditions from the BRST and anti-BRST transformations, as we did in section 3.2, it turns out that requiring that the five constants \( e_1, e_2, e_3, e_4 \) and \( e_5 \) in eqs. (3.48) should vanish imposes only four conditions on the remaining free parameters. Thus we obtain a unique solution, which is given by

\[
\begin{align*}
a_1 &= 0, & a_2 &= -\frac{\xi + 2}{2\xi}, & a_3 &= -\frac{1}{2}, & a_4 &= -\frac{1 + \xi}{\xi}, \\
b_1 &= \frac{1}{2 - \hat{D}}, & b_2 &= \frac{1}{(2 - \hat{D})\xi}, & b_3 &= \frac{1}{(2 - \hat{D})\xi^2}, \\
a_1 &= \frac{1}{\xi}, & a_2 &= -1 + \frac{1}{\xi^2}, & \gamma &= -\frac{1}{2}.
\end{align*}
\]

(3.52)

\(^7\)We have used here the already-derived result that \( \xi_B = \xi \).
With these choices, the source currents in (3.46), (3.49) and (3.51) become simply

\[ j(\varphi) = \Box^{-1} j^\rho \circ \tilde{j}_\rho + \alpha_1 \Box^{-1} j^\alpha(c) \circ \tilde{j}_\alpha(c), \]
\[ j_{\mu\nu}(h) = \Box^{-1} j_{\mu\nu} \circ \tilde{j}_{\nu}, \]
\[ j_{\mu\nu}(B) = \Box^{-1} j_{[\mu} \circ \tilde{j}_{\nu]} . \]

In [10], different choices were made for the parameters in the convolution product ansatz (3.14) for \( h_{\mu\nu} \). Specifically, in [10] the structures involving \( \partial j \) or \( \partial \tilde{j} \) in \( j_{\mu\nu}(h) \) given in eq. (3.49) were required to be absent, together with requiring \( a_4 = 0 \). After imposing also the conditions for BRST and anti-BRST invariance, this leads to the conditions given in eqs. (18a), (18b) and (18c) in [10], namely

\[ a_1 = \frac{1}{1 - \xi}, \quad a_2 = \frac{1 + \xi}{2(1 - \xi)}, \quad a_3 = - \frac{1}{2}, \quad a_4 = 0, \]
\[ b_1 = - \frac{\xi}{(2 - D)(1 - \xi)}, \quad b_2 = - \frac{1}{(2 - D)(1 - \xi)}, \quad b_3 = - \frac{(1 + \xi)}{(2 - D)\xi}. \]

With this choice for the parameters the ansatz for \( h_{\mu\nu} \) becomes singular if the gauge parameter \( \xi = 1 \). There is no such singularity for other choices, such as in eq. (3.52).

We shall consider yet another convenient choice for the parameters in section 4, when we construct the Schwarzschild black hole and black string as convolutional double copies.

4 Black hole and black string from Yang-Mills

In this section we show how the Schwarzschild black hole and the black string solution can be obtained explicitly, at the linearised level, from convolution products of solutions of two Yang-Mills theories. In order to do this, it is useful first to construct the explicit expression for the energy-momentum tensor \( T_{\mu\nu} \) in terms of the source current \( j_{\mu\nu}(h) \) that we found in eq. (3.49).

The equation of motion (3.39) for \( h_{\mu\nu} \) was obtained by varying the Lagrangian \( L' \) given by eqs. (2.29), (2.4) and (2.30), subject also to the condition (2.33), and then subtracting out the trace. Prior to subtracting the trace, the equation of motion takes the form

\[ \Box h_{\mu\nu} - \frac{\xi}{\xi_h} + \frac{2}{\xi_h} (2\partial^\rho \partial(\mu h)_{\nu}) - \partial_h(\partial_h h) + \frac{\xi}{\xi_h} + \frac{1}{\xi_h} \partial^\rho \partial^\sigma h_{\rho\sigma} = - 16\pi T_{\mu\nu}, \]

where \( T_{\mu\nu} \) is the energy-momentum tensor. Taking out the trace, and comparing with eq. (3.39), we see that \( T_{\mu\nu} \) is given in terms of the source current \( j_{\mu\nu}(h) \) by

\[ T_{\mu\nu} = - \frac{1}{16\pi} (j_{\mu\nu}(h) - \frac{1}{2} \eta_{\rho\sigma} j_{\rho\sigma}(h) \eta_{\mu\nu}). \]

*For convenience, we set Newton’s constant \( G = 1 \).
From eq. (3.49) we therefore have that
\[
T_{\mu\nu} = -\frac{1}{16\pi} \left\{ \Box^{-1} j_{(\mu} \circ \tilde{j}_{\nu)} + h_1 \Box^{-2} \partial_{(\rho} \partial_{\nu)} (j^\rho \circ \tilde{j}_{\nu}) + h_2 \Box^{-2} \left( \partial j \circ \partial (\mu \tilde{j}_{\nu}) + \partial (\mu \tilde{j}_{\nu}) \circ \partial j \right) \\
+ h_3 \Box^{-3} \partial_{\rho} \partial_{(\nu} (\partial j \circ \tilde{j}_{\nu}) + h_4 \Box^{-2} \partial_{\rho} \partial_{\nu} (j^\alpha (c) \circ \tilde{j}_\alpha (c)) + h_5 \eta_{\mu\nu} \Box^{-1} j^\rho \circ \tilde{j}_\rho \\
+ h_6 \eta_{\mu\nu} \Box^{-2} (\partial j \circ \partial j) + h_7 \eta_{\mu\nu} \Box^{-1} j^\alpha (c) \circ \tilde{j}_\alpha (c) \right\}, \tag{4.3}
\]
where
\[
\begin{align*}
   h_1 &= e_2, & h_2 &= -e_1 \xi, & h_3 &= 2e_1 \xi (\xi + 1) + e_2 (\xi^2 - 1) + e_3 \xi^2, \\
   h_4 &= e_5, & h_5 &= \left( 1 - \frac{1}{2} D \right) b_1 - \frac{1}{2} e_2 - \frac{1}{2}, \\
   h_6 &= -e_1 \xi^2 - \frac{1}{2} e_2 (\xi^2 - 1) - \frac{1}{2} e_3 \xi^2 + \left( 1 - \frac{1}{2} D \right) e_4 \xi^2, & h_7 &= \left( 1 - \frac{1}{2} D \right) b_2 - \frac{1}{2} e_5. \tag{4.4}
\end{align*}
\]

We shall find it convenient, for what follows, to make a choice for the \( a_i \) and \( b_i \) parameters in the ansatz (3.14) for \( h_{\mu\nu} \), which are related to \( e_i \) via (3.48), such that \( T_{\mu\nu} \) in eq. (4.3) has a form that is well adapted to the construction of the linearised Schwarzschild black hole and black string. Specifically, we shall choose the free parameters in the convolutional double-copy ansätze so that
\[
h_1 = h_2 = h_4 = h_5 = h_7 = 0 \tag{4.5}
\]
in the expression (4.3) for the energy-momentum tensor \( T_{\mu\nu} \). We must also satisfy the equations (3.22), (3.23) and (3.24) that came from requiring BRST and anti-BRST invariance. We find that a unique solution exists, with
\[
a_1 = a_2 = a_4 = 0, & a_3 = -\frac{1}{2}, & b_1 = -b_3 = \frac{1}{2 - D}, & b_2 = 0. \tag{4.6}
\]
The remaining non-vanishing \( h_i \) coefficients are then given by
\[
h_3 = -1, & h_6 = 0. \tag{4.7}
\]
To summarise, with the \( a_i \) and \( b_i \) coefficients in the double-copy ansatz (3.14) for \( h_{\mu\nu} \) being chosen as in eq. (4.6), the energy-momentum tensor \( T_{\mu\nu} \) is given in terms of the Yang-Mills source currents by
\[
T_{\mu\nu} = -\frac{1}{16\pi} \left[ \Box^{-1} j_{(\mu} \circ \tilde{j}_{\nu)} - \Box^{-3} \partial_{\rho} \partial_{\nu} (\partial j \circ \tilde{j}) \right], \tag{4.8}
\]
For the Kalb-Ramond field \( B_{\mu\nu} \), we see from the expression for \( j_{\mu\nu} (B) \) in eq. (3.50) that if we choose the free parameter \( \gamma \) to be equal to \( -\frac{1}{2} \), then
\[
j_{\mu\nu} (B) = \Box^{-1} j_{[\mu} \circ \tilde{j}_{\nu]} . \tag{4.9}
\]
For the dilaton, with \( \alpha_2 \) determined by the condition of BRST invariance in eq. (3.17), we can view \( \alpha_1 \) as a free parameter. We shall then choose \( \alpha_1 = 1/\xi \) for convenience, as we did in eqs. (3.52), and so
\[
j (\varphi) = \Box^{-1} j^\rho \circ \tilde{j}_\rho + \frac{1}{\xi} \Box^{-1} j^\alpha (c) \circ \tilde{j}_\alpha (c) . \tag{4.10}
\]
4.1 Schwarzschild black hole

For definiteness, we shall consider the four-dimensional Schwarzschild black hole, but the
discussion generalises immediately to the more general case of the Tangherlini black hole
in arbitrary dimension. Starting from the standard Schwarzschild form of the metric,
\[ ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \]
(4.11)
it can be recast in the isotropic form by defining a new radial variable \( \rho \) such that
\[ r = \rho \left(1 + \frac{M}{2\rho}\right)^2. \]
(4.12)
The Schwarzschild metric then becomes
\[ ds^2 = - \left(1 - \frac{M}{2\rho}\right)^2 dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 dx^i dx^i, \]
(4.13)
where \( x^i = \rho n^i \) and \( n^i \) is a unit vector in the Cartesian 3-space, with \( dn^i dx^i = d\Omega_2^2 \). To
linear order in \( M \) the metric (4.13) takes the form \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) with \( h_{\mu\nu} \) having the
non-vanishing components
\[ h_{00} = \frac{2M}{\rho}, \quad h_{ij} = \frac{2M}{\rho} \delta_{ij}. \]
(4.14)
The four Minkowskian coordinates are \( x^\mu = (t, x^i) = (t, \mathbf{x}) \). It follows from eqs. (4.14) that \( \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h_{00} \eta_{\mu\nu} \) is given by
\[ \bar{h}_{00} = \frac{4M}{\rho}, \quad \bar{h}_{ij} = 0, \quad \bar{h}_{0i} = 0. \]
(4.15)
In isotropic coordinates the linearised Schwarzschild metric therefore obeys the De Donder
gauge condition \( \partial^\mu \bar{h}_{\mu\nu} = 0 \). From the linearised Einstein equation \( \Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \), we can read off, bearing in mind that \( \Box \frac{1}{\rho} = \partial_\rho \partial_\rho \frac{1}{\rho} = -4\pi \delta^{(3)}(\mathbf{x}) \) where \( \rho = |\mathbf{x}| \), that the
energy-momentum tensor that acts as the source for the Schwarzschild solution has only a
\( T_{00} \) component, given by
\[ T_{00} = M \delta^{(3)}(\mathbf{x}). \]
(4.16)
It is now straightforward to see that if we choose Yang-Mills source currents to lie in
a U(1) subgroup of the gauge group, and to take the form
\[ j_\mu = (\Box\delta^{(4)}(x), 0, 0, 0), \quad \tilde{j}_\mu = -16\pi M (\delta^{(3)}(\mathbf{x}), 0, 0, 0), \]
(4.17)
then since \( \partial^\mu \tilde{j}_\mu = 0 \), only the first term in the expression (4.8) for the energy-momentum
tensor will survive, and we shall indeed have
\[ T_{\mu\nu} = M \delta^0_\mu \delta^0_\nu \delta^{(3)}(\mathbf{x}), \]
(4.18)
which is of precisely the correct form for the energy-momentum tensor (4.16) for a point-mass located at the origin. Note that the tilded Yang-Mills current is of the form that corresponds to a point charge of strength $16\pi M$. The untilded Yang-Mills current is just the D’Alembertian acting on the four-dimensional delta function $\delta^{(4)}(x)$. As noted in appendix A, $\delta^{(4)}(x)$ is the identity for convolution products, and the D’Alembertian just serves to cancel the $-\frac{1}{c^2}$ occurring in the expression for $T_{\mu\nu}$ in terms of the Yang-Mills currents.

The Kalb-Ramond source current given in eq. (4.9) will indeed vanish for the choice of Yang-Mills source currents in eqs. (4.17).

For the dilaton, if we take

$$j^0(c) = (t, 0) \Box \delta^4(x), \quad \tilde{j}_\alpha(\tilde{c}) = -16\pi \xi M (\tilde{\iota}, 0) \delta^{(3)}(\mathbf{x}),$$

(4.19)

where $\iota$ and $\tilde{\iota}$ are Grassmann-valued constants with $\iota \tilde{\iota}$ giving 1, then the contribution from the $c$ and $\tilde{c}$ Yang-Mills ghost currents will cancel the contribution from the Yang-Mills gauge currents in eq. (4.10), and so we can arrange for $j(\varphi)$ to vanish.

In summary, we have seen that we can choose the Yang-Mills sources so that the energy-momentum tensor for $h_{\mu\nu}$ has the appropriate form for a point-mass source for the linearised Schwarzschild solution, while there will be no sources for $B_{\mu\nu}$ or $\varphi$, and so these fields will vanish.

### 4.2 Ten-dimensional black string solution

The ten-dimensional solution describing a black string centred on the origin of the eight-dimensional transverse space was constructed in [25]. In the notation we shall use, it is given by [26]

$$ds^2 = W^{-\frac{2}{3}} (-f dt^2 + dx^2) + W^{\frac{1}{3}} (f^{-1} dr^2 + r^2 d\Omega_7^2),$$

$$W = 1 + \frac{k \sinh^2 \delta}{r^6}, \quad f = 1 - \frac{k}{r^6}, \quad \varphi = \frac{1}{2} \log W,$$

$$H = \lambda e^\varphi * \omega_7, \quad \lambda = 3k \sinh 2\delta,$$

(4.20)

where $*$ denotes the ten-dimensional Hodge dual, and $\omega_7$ denotes the volume form on the unit 7-sphere. The constants $k$ and $\delta$ characterise the mass per unit length and the 3-form charge of the string.

Introducing asymptotically-Minkowskian coordinates $X^M$ with

$$X^0 = t, \quad X^1 = x, \quad X^I = y^I, \quad 2 \leq I \leq 9,$$

(4.21)

where $y^I = r n^I$ with $n^I n^I = 1$ and $dn^I dn^I = d\Omega_7^2$, we can write the black string metric up to linear order in $k$ as $g_{MN} = \eta_{MN} + h_{MN}$, where the non-zero components of $h_{MN}$ are given by

$$h_{00} = \frac{k}{r^6} \left(1 + \frac{3s^2}{4}\right), \quad h_{11} = -\frac{3s^2}{4r^6}, \quad h_{IJ} = \frac{ks^2}{4r^6} \delta_{IJ} + \frac{k}{r^3} y^I y^J,$$

(4.22)
and we are employing the abbreviations $s$ and $c$, where\textsuperscript{9}
\begin{equation}
  s = \sinh \delta, \quad c = \cosh \delta.
\end{equation}
From eqs. (4.22) it follows that $\tilde{h}_{MN} \equiv h_{MN} - \frac{1}{2} \eta_{MN}$ is given by
\begin{equation}
  \tilde{h}_{00} = \frac{k c^2}{r^6}, \quad \tilde{h}_{11} = -\frac{k s^2}{r^6}, \quad \tilde{h}_{IJ} = \frac{k}{r^8} y^I y^J.
\end{equation}

Note that $\tilde{h}_{MN}$ given by eqs. (4.24) does not satisfy the De Donder gauge condition $\partial^M \tilde{h}_{MN} = 0$. We can easily perform a coordinate transformation that puts it in De Donder gauge, at the linear order to which we are working, by means of a diffeomorphism $\xi^M$ such that $\delta h_{MN} = \partial_M \xi_N + \partial_N \xi_M$. Specifically, if we choose the diffeomorphism parameter to be given by
\begin{equation}
  \xi^0 = 0, \quad \xi^1 = 0, \quad \xi^I = \frac{k y^I}{12 r^6},
\end{equation}
then the gauge-transformed $\tilde{h}_{MN}$ will have components given by
\begin{equation}
  \tilde{h}_{00} = \frac{k}{r^6} \left( c^2 + \frac{1}{6} \right), \quad \tilde{h}_{11} = -\frac{k}{r^6} \left( s^2 + \frac{1}{6} \right), \quad \tilde{h}_{IJ} = 0,
\end{equation}
and all off-diagonal components zero too. It is easily seen that this $\tilde{h}_{MN}$ tensor is in De Donder gauge. Since we have
\begin{equation}
  \Box \frac{1}{r^6} = \partial_I \partial_I \frac{1}{r^6} = -\partial_I \left( \frac{6 y^I}{r^8} \right),
\end{equation}
then by integrating over the interior of a sphere centred on the origin we see that
\begin{equation}
  \Box \frac{1}{r^6} = -6 \Omega_7 \delta^{(8)}(y),
\end{equation}
where $\Omega_7 = \frac{1}{3} \pi^4$ is the volume of the unit 7-sphere. It follows that at the linearised order the energy-momentum tensor, which is related to $h_{MN}$ by $\Box h_{MN} = -16 \pi T_{MN}$ in De Donder gauge, is given by
\begin{equation}
  T_{00} = \frac{3 k \Omega_7}{8 \pi} \left( c^2 + \frac{1}{6} \right) \delta^{(8)}(y), \quad T_{11} = -\frac{3 k \Omega_7 s^2}{8 \pi} \left( s^2 + \frac{1}{6} \right) \delta^{(8)}(y),
\end{equation}
with all other components vanishing.

For the construction of this black string solution in terms of the convolution double copy we shall employ Yang-Mills fields in an SU(2) subgroup of the gauge group, rather than the U(1) subgroup that sufficed for the previous black hole example.\textsuperscript{10} We therefore now denote the Yang-Mills currents by $J^{(a)}_M$ and $\tilde{J}^{(a)}_M$, where the index $a$ will range over 1,
2 and 3. (We enclose the index in parentheses to avoid the risk of ambiguities when we assign explicit numerical values to $a$ in what will follow.)

It can now be seen that if we choose the Yang-Mills currents $j^{(a)}_M$ and $\tilde{j}^{(a)}_M$ in the convolutional double copy to be given by

$$j^{(1)}_M = (1, 1, 0, \ldots, 0) \Box \delta^{(10)}(X), \quad \tilde{j}^{(1)}_M = -\frac{k \Omega_7}{2} (1, -1, 0, \ldots, 0) \delta^{(8)}(y),$$

$$j^{(2)}_M = (1, -1, 0, \ldots, 0) \Box \delta^{(10)}(X), \quad \tilde{j}^{(2)}_M = -\frac{k \Omega_7}{2} (1, 1, 0, \ldots, 0) \delta^{(8)}(y),$$

$$j^{(3)}_M = (c, s, 0, \ldots, 0) \Box \delta^{(10)}(X), \quad \tilde{j}^{(3)}_M = -6k \Omega_7 (c, -s, 0, \ldots, 0) \delta^{(8)}(y),$$

then for the parameter choices in which the energy-momentum tensor is given by eq. (4.8), we see that $T_{\mu \nu}$ in eq. (4.8) is precisely equal to the energy-momentum tensor we found in eq. (4.29) for the black string. The reason for needing both the $a = 1$ and $a = 2$ currents, which contribute equally to $T_{MN}$, will be seen below when we consider the source current $J_{MN}(B)$ for the Kalb-Ramond field.

It can be seen from the expression for the 3-form field strength $H$ in eqs. (4.20) that one can write the 2-form potential $B$ in a gauge where it is given just by

$$B_{01} = -\frac{ksc}{W} r^6 = -\frac{ksc}{r^6} + \mathcal{O}(k^2).$$

Clearly this is written in a Lorenz-like gauge where $\partial^M B_{MN} = 0$. At the linear order in $k$ it obeys $\Box B_{01} = 6k \Omega_7 sc \delta^{(8)}(y)$. Comparing with eq. (3.40) we see that the source current $j_{MN}(B)$ is given by

$$j_{01}(B) = 6k \Omega_7 sc \delta^{(8)}(y).$$

It can be seen that this is precisely consistent with what we obtain by substituting the Yang-Mills source currents of eqs. (4.30) into eq. (3.51). Note that the $a = 1$ Yang-Mills currents would contribute an additional, unwanted, term to $j_{MN}(B)$, which is cancelled by an equal but opposite contribution from the $a = 2$ currents.

The dilaton in the solution in eqs. (4.20) is given by

$$\varphi = -\frac{ksc^2}{2y^6},$$

at the linear order in $k$, and so from (3.38) we find that the dilaton current is given by

$$j(\varphi) = 3k \Omega_7 s^2 \delta^{(8)}(y).$$

We see from eqs. (4.30) and (4.10) that we can obtain this current in the double copy construction by choosing the Yang-Mills ghost currents

$$j^a(c) = (c, 0) \Box \delta^{(10)}(X), \quad \tilde{j}_a(\bar{c}) = \mu (\bar{c}, 0) \delta^{(8)}(y),$$

with $\bar{c}$ giving 1 and the constant $\mu$ chosen to be

$$\mu = \xi k \Omega_7 (1 - 9c^2).$$
4.3 BPS string solution

It is interesting also to consider the BPS limit of the black string, which is obtained by sending the charge parameter $\delta$ to infinity while simultaneously sending $k$ to zero, holding the product $q \equiv k \sinh^2 \delta$ fixed. Thus in the BPS limit one has

$$ks^2 \rightarrow q, \quad kc^2 \rightarrow q.$$  \hspace{1cm} (4.37)

It can be seen from eq. (4.29) that the energy-momentum tensor for the BPS string is given by

$$T_{00} = \frac{3q\Omega \tau}{8\pi} \delta^{(8)}(y), \quad T_{11} = -\frac{3q\Omega \tau}{8\pi} \delta^{(8)}(y),$$  \hspace{1cm} (4.38)

with all other components vanishing. Since one can always make a see-saw rescaling of the expressions for the untilded and tilded Yang-Mills currents in (4.30), of the form $j_M^{(a)} \rightarrow k^{1/2} j_M^{(a)}$ and $\tilde{j}_M^{(a)} \rightarrow k^{-1/2} \tilde{j}_M^{(a)}$, it is easily seen that the BPS string solution can be obtained as the convolutional double copy of just U(1) Yang-Mills sources, with

$$j_M^{(3)} = (1, 1, 0, \ldots, 0) \Box \delta^{(10)}(X), \quad \tilde{j}_M^{(3)} = -6q\Omega\tau (1, -1, 0, \ldots, 0) \delta^{(8)}(y).$$  \hspace{1cm} (4.39)

The ghost currents can again be chosen as in eq. (4.35), with the constant $\mu$, following from eq. (4.36), given by

$$\mu = -9\xi q\Omega\tau.$$  \hspace{1cm} (4.40)

5 Discussion and conclusions

In this paper, we have explored in greater detail some of the ideas presented in [10] for obtaining “$\mathcal{N} = 0$ supergravity” (i.e. the low-energy limit of the bosonic string) as a convolutional double copy of Yang-Mills squared. A crucial aspect of the construction in [10] is the inclusion of the ghosts and antighosts of a BRST description, in order to capture fully the degrees of freedom of the theory. Accordingly, our discussion has been centred around the BRST description of the theory and its gauge invariances, with an emphasis on the anti-BRST as well as the BRST symmetries of the system.

We have illustrated the application of the convolutional double copy in two examples. The first, which has been discussed previously, is the construction of the four-dimensional static black hole solution of $\mathcal{N} = 0$ supergravity (i.e. the Schwarzschild black hole) as a double copy of two Yang-Mills solutions, one of which describes a U(1) static point charge and the other a U(1) configuration corresponding to a $\Box \delta^{(4)}(x)$ source. Our second example, which is more involved, is the ten-dimensional black string solution of $\mathcal{N} = 0$ supergravity. We showed how this can be constructed as a double copy involving non-abelian SU(2) Yang-Mills solutions in the two copies. As far as we are aware, this is the first example that has been obtained in the literature in which multi-component Yang-Mills configurations are necessary in order to describe a $\mathcal{N} = 0$ supergravity solution. We showed also how the BPS string solution arises as a limit in which the SU(2) Yang-Mills reduces to U(1).

All of the discussion in our paper has been, as in [10], at the linearised level. There have been some discussions that push beyond the linear level, such as the construction up to
cubic order in the Lagrangian, in [17]. It would be of interest to pursue these investigations further.

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A Convolution product and convolution inverse

A.1 Basic properties of the convolution

The convolution is defined, for functions \( f(x) \) and \( g(x) \) in a \( D \)-dimensional spacetime, by

\[
(f \ast g)(x) = \int d^D y f(x - y) g(y).
\]  

(A.1)

The convolution product is associative, so \( f \ast (g \ast h) = (f \ast g) \ast h \). For functions obeying appropriate boundary conditions, one has

\[
\partial_\mu (f \ast g) = (\partial_\mu f) \ast g = f \ast (\partial_\mu g).
\]  

(A.2)

If \( \mathcal{F}_k \) denotes the Fourier transform with respect to the \( D \)-momentum \( k \), so that

\[
\mathcal{F}_k(f(x)) \equiv \int d^D x f(x) e^{-i k \cdot x},
\]  

(A.3)

then

\[
\mathcal{F}_k(f \ast g) = \mathcal{F}_k(f) \mathcal{F}_k(g).
\]  

(A.4)

The convolution inverse \( \tilde{f}(x) \) of a function \( f(x) \) is defined by

\[
(f \ast \tilde{f})(x) = \delta^D(x).
\]  

(A.5)

Note that the function \( f(x) = \delta^D(x) \) is equal to its own convolution inverse. In other words, \( \delta^D(x) \) is the identity in the convolution product. At the level of Fourier transforms, one can see from (A.4) that the convolution inverse \( \tilde{f} \) and the original function \( f \) are such that

\[
\mathcal{F}_k(\tilde{f}) = \frac{1}{\mathcal{F}_k(f)}.
\]  

(A.6)

A.2 Non-existence of convolution inverse of \( \frac{1}{r} \)

One might think that a natural expectation for the Yang-Mills configurations that would give rise to the Schwarzschild black hole as the convolutional double copy would be to take point-particle solutions for the left and right Yang-Mills potentials, as, for example, in [16]. Taking four dimensions for simplicity, the Yang-Mills potentials for point charge
particles would be of the form $A^a_0 \sim \phi$, and $\tilde{A}^\alpha_0 \sim \phi$. In order to get a linearised metric perturbation of the form $h_{00} \sim \frac{\alpha}{r}$, this would then require that the spectator field $\phi^{-1}_{ab}$ in eq. (3.1) should have the general form of the convolution inverse of $\frac{1}{r}$. However, it can easily be seen that the convolution inverse of $\frac{1}{r}$ does not exist. This can be seen in terms of the Fourier transform, as follows. The Fourier transform of $\frac{1}{r}$ is given by

$$F_k \left( \frac{1}{r} \right) = \frac{\delta(k^0)}{k^2},$$

and so it follows from eq. (A.6) that the Fourier transform of the convolution inverse of $\frac{1}{r}$ would be

$$\frac{k^2}{\delta(k^0)}.$$  

This is a function that is “infinite almost everywhere”, and thus does not really make sense. Thus while it is true that one could write a formal expression for the Fourier transform of the convolution products of two $\frac{1}{r}$ functions with a “convolution inverse” of $\frac{1}{r}$ as

$$\frac{\delta(k^0)}{k^2} \times \frac{k^2}{\delta(k^0)} \times \frac{\delta(k^0)}{k^2} = \frac{\delta(k^0)}{k^2},$$

conveying the notion that one would thereby obtain a result of $\frac{1}{r}$ for the linearised metric contribution, this seems to be more akin to a mnemonic that lacks mathematical justification.

The non-existence of the convolution inverse of $\frac{1}{r}$ can also be seen directly in position space. Adopting the notation that the position 4-vector $x^\mu$ has components $(x^0, \mathbf{x})$, we see from eq. (A.5) that the convolution inverse of the function $\frac{1}{r}$ would be a function $\tilde{f}(x)$ such that

$$\int d^4 y \frac{\tilde{f}(y)}{|x-y|} = \delta^4(x)$$

$$= \delta(x^0) \delta^3(\mathbf{x}).$$

The integral on the left-hand side is independent of $x^0$. However, the delta function on the right-hand side includes a factor $\delta(x^0)$. Thus, there cannot exist any function $f(x)$ such that it is the convolution inverse of $f(x) = \frac{1}{|x|}$.

A.3 Non-existence of convolution pseudo-inverse of $\frac{1}{r}$

The convolution pseudo-inverse $F$ of a function $f$ can be defined by the equation

$$f * F * f = f.$$  

Writing this out explicitly, this means

$$\int d^D y \int d^4 z f(x - y - z) F(z) f(y) = f(x).$$
If we consider the previous four-dimensional example, and now try to calculate the convolution pseudo-inverse \( F(x) \) of the function \( f(x) = \frac{1}{r} \), where \( r = |x| \), we would then require that

\[
\int d^4y \int d^4z \frac{1}{|x - y - z|} \frac{1}{|y|} F(z) = f(x) .
\]

(A.13)

The \( y \) integration does not involve dependence on the unknown function \( F(z) \) that we are trying to solve for, and so for this we may first consider just the \( y \) integration, defining

\[
G(x - z) \equiv \int d^4y \frac{1}{|x - y - z|} \frac{1}{|y|} .
\]

(A.14)

There are two convergence problems here. First of all, the integrand does not depend on \( y^0 \), and so the \( y^0 \) integration will give infinity. Over and above that, the remaining spatial integrations will be of the form

\[
\int d^4y \frac{1}{|w - y|} \frac{1}{|y|} ,
\]

(A.15)

where \( w = x - z \). This integration will diverge at large \( |y| \).

In short, the “convolutive pseudo-inverse” of \( 1/r \) would appear not to exist. In general, it would appear that convolution inverses of functions that do not depend on all \( D \) spacetime coordinates do not exist.

A.4 Convolutional double copy ansatz using well-defined convolution inverse

For the reasons seen in the previous subsections, it does not seem to be possible to construct a convolutional double copy for an object such as a black hole by taking the product of tilded and untilded Yang-Mills point charges. For this reason, the kinds of ansätze we have used in this paper have been along the lines of those employed in [19, 20]. Such an ansatz is asymmetric, in the sense that one copy of the Yang-Mills potential is taken to be that of a point charge, while the other is taken to be of the form of a full \( D \)-dimensional delta function \( \delta^D(x) \). The spectator field is taken to be the convolution inverse of this, and so it exists and is in fact also \( \delta^D(x) \).

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References

[1] Z. Bern, J.J.M. Carrasco and H. Johansson, New Relations for Gauge-Theory Amplitudes, *Phys. Rev. D* **78** (2008) 085011 [arXiv:0805.3993] [insPIRE].

[2] Z. Bern, J.J.M. Carrasco and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [insPIRE].

[3] H. Kawai, D.C. Lewellen and S.H.H. Tye, A Relation Between Tree Amplitudes of Closed and Open Strings, *Nucl. Phys. B* **269** (1986) 1 [insPIRE].
[4] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, The SAGEX Review on Scattering Amplitudes. Chapter 2: An Invitation to Color-Kinematics Duality and the Double Copy, arXiv:2203.13013 [inSPIRE].

[5] D.A. Kosower, R. Monteiro and D. O’Connell, The SAGEX Review on Scattering Amplitudes. Chapter 14: Classical Gravity from Scattering Amplitudes, arXiv:2203.13025 [inSPIRE].

[6] R. Monteiro, D. O’Connell and C.D. White, Black holes and the double copy, JHEP 12 (2014) 056 [arXiv:1410.0239] [inSPIRE].

[7] A. Luna, R. Monteiro, I. Nicholson and D. O’Connell, Type D Spacetimes and the Weyl Double Copy, Class. Quant. Grav. 36 (2019) 065003 [arXiv:1810.08183] [inSPIRE].

[8] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes and S. Nagy, Yang-Mills origin of gravitational symmetries, Phys. Rev. Lett. 113 (2014) 231606 [arXiv:1408.4434] [inSPIRE].

[9] A. Anastasiou, L. Borsten, M.J. Duff, A. Marrani, S. Nagy and M. Zoccali, Are all supergravity theories Yang-Mills squared?, Nucl. Phys. B 934 (2018) 606 [arXiv:1707.03234] [inSPIRE].

[10] A. Anastasiou, L. Borsten, M.J. Duff, S. Nagy and M. Zoccali, Gravity as Gauge Theory Squared: A Ghost Story, Phys. Rev. Lett. 121 (2018) 211601 [arXiv:1807.02486] [inSPIRE].

[11] L. Baulieu, Perturbative Gauge Theories, Phys. Rept. 129 (1985) 1 [inSPIRE].

[12] M.J. Perry and E. Teo, BRS and anti-BRS symmetry in topological Yang-Mills theory, Nucl. Phys. B 392 (1993) 369 [hep-th/9209072] [inSPIRE].

[13] L. Borsten, I. Jubb, V. Makwana and S. Nagy, Gauge $\times$ gauge on spheres, JHEP 06 (2020) 096 [arXiv:1911.12324] [inSPIRE].

[14] G.L. Cardoso, S. Nagy and S. Nampuri, A double copy for $\mathcal{N}=2$ supergravity: a linearised tale told on-shell, JHEP 10 (2016) 127 [arXiv:1609.05022] [inSPIRE].

[15] G. Cardoso, S. Nagy and S. Nampuri, Multi-centered $\mathcal{N}=2$ BPS black holes: a double copy description, JHEP 04 (2017) 037 [arXiv:1611.04409] [inSPIRE].

[16] G. Cardoso, S. Nagy and S. Nampuri, Comments on the double copy construction for gravitational theories, PoS CORFU2017 (2018) 177 [arXiv:1803.07670] [inSPIRE].

[17] L. Bonora and R.P. Malik, BSRT, anti-BSRT and gerbes, Phys. Lett. B 655 (2007) 75 [arXiv:0707.3922] [inSPIRE].
[23] A. Hodges, New expressions for gravitational scattering amplitudes, *JHEP* 07 (2013) 075 [arXiv:1108.2227] [inSPIRE].

[24] F. Cachazo, S. He and E.Y. Yuan, Scattering of Massless Particles: Scalars, Gluons and Gravitons, *JHEP* 07 (2014) 033 [arXiv:1309.0885] [inSPIRE].

[25] G.T. Horowitz and A. Strominger, Black strings and P-branes, *Nucl. Phys. B* 360 (1991) 197 [inSPIRE].

[26] M.J. Duff, H. Lü and C.N. Pope, The Black branes of M-theory, *Phys. Lett. B* 382 (1996) 73 [hep-th/9604052] [inSPIRE].