Benedicks effect in a relativistic simple fluid

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Abstract

According to standard thermophysical theories, cross effects are mostly present in multicomponent systems. In this paper we show that for relativistic fluids an electric field generates a heat flux even in the single component case. In the non-relativistic limit the effect vanishes and Fourier’s law is recovered. This result is novel and may have applications in the transport properties of very hot plasmas.
I. INTRODUCTION

It is well known that when one considers an ionized plasma as a binary mixture in the presence of an electromagnetic field, direct and cross effects arise. Of special interest, are the thermoelectric effects which have been discussed in Ref. [1]. The Benedicks effect, also called electrothermic effect, corresponds to a heat flux caused by an electric field. It’s history can be traced back to the experimental works developed by Carl Axel Fredrik Benedicks in the second decade of the twentieth Century in homogeneous systems [2]. These experiments showed the existence of the reciprocal cross effect of the Thomson (thermoelectric) effect (1856) which consists of the generation of a flow of charge due to a temperature gradient. The Benedicks effect should not be confused with the Peltier effect, which is strictly related to inhomogeneous systems.

Here, we present what we consider an interesting result which may lead to applications. If one takes a single component charged relativistic gas subject to the presence of an electric field, in addition to the electric current a heat flux appears driven by such field. This effect is absent in the non-relativistic domain and suggests the possibility of having additional cross-like effects in relativistic fluids. This effect may also impact the dynamics of hot plasmas possibly affecting stability and other properties of such systems.

Several authors have addressed the problem of heat conduction in charged fluids. In Ref. [3] a calculation of the thermoelectric effect, together with the electric current associated with a temperature gradient is performed using Anderson and Witting’s collisional model for a binary mixture. Also, in the non-relativistic scenario a reciprocal effect giving the electrical current due to a temperature gradient is usually obtained assuming stationary state in Boltzmann’s equation [4], a procedure that will be carefully discussed in the last section of the present work.

In this work we establish the relativistic heat flux for a simple charged fluid in the presence of an electric field starting from a relativistic Boltzmann equation using Marle’s relaxation time approach to describe the collisional effects. To accomplish this task, a correction to the relativistic local equilibrium distribution function is obtained in Sect. II in terms of the gradients in the system to first order in the Knudsen parameter. Section III is devoted to the explicit calculation of the heat flux and the corresponding transport coefficients. The interpretation and discussion of these results are given in Section IV together with final
II. THE RELATIVISTIC BOLTZMANN EQUATION

We start the calculation by setting up the kinetic equation for the system in consideration which is a dilute gas of non-degenerate charged particles at a temperature high enough so that \( z = \frac{kT}{m^2c^2} \) is at least close to unity and relativistic effects in the individual dynamics are relevant. The system as a whole has a hydrodynamic four velocity which we denote by \( U^\mu \) and its constituents have charge \( q \) and mass \( m \). The relativistic Boltzmann equation in the relaxation time approximation reads \( 3 \)

\[
\nu^\alpha \frac{\partial f}{\partial x^\alpha} + \dot{\nu}^\alpha \frac{\partial f}{\partial v^\alpha} = - \frac{f - f^{(0)}}{\tau}
\]

The structure of the BGK-like collision kernel was first suggested by Marle and is considered here as an adequate approximation since it will lead to the general structure of the distribution function as well as the dissipative fluxes without introducing all the complicated expressions and calculations that are involved when considering the complete collision kernel. The complete calculation would perhaps yield relevant corrections to the particular values of the transport coefficients but both the structure of the fluxes as well as the general behavior of the coefficients will remain unaltered. Here \( f \) is the distribution function, \( f^{(0)} \) the local equilibrium solution, and \( \tau \) a characteristic relaxation time. The molecular dynamic variables are the individual particle four-velocity \( v^\mu \) and the corresponding acceleration \( \dot{v}^\mu \), both referred to the laboratory frame.

Since the fluid is charged, in presence of an electrostatic field the acceleration in the second term on the left side of Eq. \( 1 \) may be written as

\[
\dot{v}^\mu = \frac{q}{m} v_\alpha F^{\alpha \mu}
\]

where

\[
F^{\alpha \mu} = \begin{pmatrix}
0 & 0 & 0 & -\frac{\phi^1}{c} \\
0 & 0 & 0 & -\frac{\phi^2}{c} \\
0 & 0 & 0 & -\frac{\phi^3}{c} \\
\frac{\phi^1}{c} & \frac{\phi^2}{c} & \frac{\phi^3}{c} & 0
\end{pmatrix}
\]
is the usual electromagnetic field tensor and thus

$$m\dot{v}^\mu = \begin{cases} -qv^4 \left( \frac{\phi^\mu}{c} \right) & \mu = 1, 2, 3 \\ -qv^4 \frac{\phi^\mu}{c} & \mu = 4 \end{cases} \quad (4)$$

Here a coma indicates a partial derivative whereas a semicolon will denote a covariant one. For the special relativistic fluid, we consider a flat Minkowski spacetime with a $++--$ signature such that the covariant derivative will coincide with the ordinary one in cartesian coordinates. The dot represents a proper time derivative.

Substitution of Eq. (2) in Eq. (1) leads to

$$v^\alpha f_{,\alpha} + \frac{q}{m} v_\alpha F^{\alpha\mu} \frac{\partial f}{\partial v^\mu} = -\frac{f - f^{(0)}}{\tau} \quad (5)$$

which will be solved, following the standard Chapman-Enskog procedure, by introducing the assumption that the distribution function can be written as

$$f = f^{(0)} + f^{(1)} \quad (6)$$

where $f^{(1)}$ is of first order in the gradients. By substituting Eq. (6) in Eq. (5) and keeping only terms up to first order in the gradients, one can write the deviation from the equilibrium solution as follows

$$f^{(1)} = -\tau \left\{ v^\alpha f^{(0)}_{,\alpha} + \left( \frac{q}{m} v_\alpha F^{\alpha\mu} \right) \frac{\partial f^{(0)}}{\partial v^\mu} \right\} \quad (7)$$

The presence of the local equilibrium distribution function on the right side of Eq. (7) follows from the fact that its derivatives are already first order in the gradients since the local equilibrium assumption asserts that its space and time dependence is given only though the gradients and time derivatives of the state variables $n$, $T$ and $U^\rho$. It must also be taken into account that the time derivatives of such variables are in turn written in terms of gradients by introducing Euler’s relativistic equations, a step necessary in order to assure existence of the first order solution [6, 7].

In order to establish the explicit dependence of $f^{(1)}$ with the gradients, we start by using the following identity

$$v^\alpha = v^\beta h^\alpha_\beta + \gamma_{(k)} U^\alpha \quad (8)$$

where $h^\alpha_\beta$ projects in the hyperplane orthogonal to the hydrodynamic velocity, that is $h^\alpha_\beta U^\beta = 0$. Also, for the last equality use has been made of the identity $v^\beta U_\beta = -\gamma_{(k)} c^2$ which is
valid since, being the contraction a scalar, it can be calculated in a comoving frame where $\mathcal{U}^\nu = [\vec{0}, c]$ and $v^\alpha \equiv K^\alpha$, with

$$K^\alpha = \gamma(k) \left[ k^\ell, c \right] \quad \ell = 1, 2, 3$$  \hspace{1cm} (9)

denoting the chaotic or peculiar velocity $[5, 7]$ and $\gamma(k) = (1 - k^2/c^2)^{-1/2}$ being the usual Lotenz factor with $k = \sqrt{k^\ell k^\ell}$. By noticing that $v^\alpha f_{,\alpha}^{(0)}$ is also an invariant quantity, the calculation is carried out in the comoving frame where the projector has the simple form

$$h^\alpha_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (10)

so that one can write in general

$$v^\alpha f_{,\alpha}^{(0)} = v^\alpha f_{,\alpha}^{(0)} \big|_{CF} = \gamma(k) k^{\delta\ell}_i f_{,\ell}^{(0)} \big|_{CF} + \gamma(k) c f_{4,\alpha}^{(0)} \big|_{CF}$$  \hspace{1cm} (11)

where $i, \ell = 1, 2, 3$ and the derivatives are calculated in the comoving frame, denoted by the subscript $CF$, where local equilibrium is satisfied. Since locally the equilibrium distribution function is a Juttner function given by $[5, 8]$

$$f^{(0)} = \frac{n}{4\pi c^3 z \mathcal{K}_{\frac{1}{2}} (z)} \exp \left( \frac{\mathcal{U}^\beta v_\beta}{zc^2} \right)$$  \hspace{1cm} (12)

one has

$$\frac{\partial f^{(0)}}{\partial n} = \frac{f^{(0)}}{n} \quad \frac{\partial f^{(0)}}{\partial \mathcal{U}^\alpha} = v_\alpha \frac{f^{(0)}}{zc^2} \quad \frac{\partial f^{(0)}}{\partial v^\alpha} = \frac{U_\alpha}{zc^2} f^{(0)}$$  \hspace{1cm} (13)

and

$$\frac{\partial f^{(0)}}{\partial T} = f^{(0)} \left( 1 - \frac{\gamma(k)}{z} \frac{G \left( \frac{1}{z} \right)}{z} \right)$$  \hspace{1cm} (14)

where $G \left( \frac{1}{z} \right) = \frac{K_n \left( \frac{1}{z} \right)}{K_2 \left( \frac{1}{z} \right)}$ and $\mathcal{K}_n \left( \frac{1}{z} \right)$ is the modified $n$-th order Bessel function of the second kind depending on the relativistic parameter $z = \frac{kt}{mc^2}$. Thus,

$$f_{,\alpha}^{(0)} \big|_{CF} = f^{(0)} \left[ \frac{n_{,\alpha}}{n} + \frac{T_{,\alpha}}{T} \left( 1 - \frac{\gamma(k)}{z} \frac{G \left( \frac{1}{z} \right)}{z} \right) + \frac{v_\beta}{zc^2} \mathcal{U}_\beta \right]$$  \hspace{1cm} (15)

and for $f_{4,\alpha}^{(0)} = \frac{1}{z} \frac{\partial f^{(0)}}{\partial t}$ we have

$$f_{4,\alpha}^{(0)} \big|_{CF} = f^{(0)} \left[ \frac{1}{n} \frac{\partial n}{\partial t} \big|_{CF} + \frac{T}{T} \left( 1 - \frac{\gamma(k)}{z} \frac{G \left( \frac{1}{z} \right)}{z} \right) \frac{\partial T}{\partial t} \big|_{CF} + \frac{K_{,\beta}}{zc^2} \frac{\partial \mathcal{U}^\beta}{\partial t} \big|_{CF} \right]$$  \hspace{1cm} (16)
Following Hilbert’s procedure \[7\], the time derivatives are to be written in terms of the gradients of the state variables via Euler’s (lower order) equations:

\[
\frac{\partial n}{\partial t} + \mathbf{U}^\ell n_{,\ell} = -nU^\mu_{,\mu} \tag{17}
\]

\[
\frac{\partial U^\mu}{\partial t} + \mathbf{U}^\ell U^\mu_{,\ell} = -h^{\mu\nu}mc^2z\left(\frac{T_\nu}{T} + \frac{n_\nu}{n}\right) + n\frac{q}{\tilde{\rho}}U_\nu F^{\nu\mu} \tag{18}
\]

\[
\frac{\partial T}{\partial t} + \mathbf{U}^\ell T_{,\ell} = -\frac{kT}{C_n(z)}U^\alpha_{,\alpha} \tag{19}
\]

where \(C_n\) is the specific heat for constant \(n\) and for and ideal gas \(\tilde{\rho} = \frac{n\rho}{2} + \frac{p^2}{c^2} = nmG\left(\frac{1}{z}\right)\).

The set given by Eqs. \((17)-(19)\) corresponds to the local equilibrium, zero order in the gradients, relativistic hydrodynamic equations which can be obtained in the kinetic theory framework by multiplying Boltzmann’s equation by collisional invariants and integrating in velocity space. Such a procedure can be found for the relativistic case in Ref. \[5\] for example. In order to obtain such relations in the comoving frame, as required by Eq. \((16)\), we consider \(\mathbf{U}^\nu = [\vec{0}, c]\) and Eq. \((10)\) which leads to

\[
\left.\frac{\partial n}{\partial t}\right|_{CF} = 0 \tag{20}
\]

\[
\left.\frac{\partial U^\ell}{\partial t}\right|_{CF} = -\frac{1}{G\left(\frac{1}{z}\right)} \left[ h^{\ell\nu}c^2z\left(\frac{T_\nu}{T} + \frac{n_\nu}{n}\right) - \frac{q}{m}cF^{4\ell}\right] \tag{21}
\]

\[
\left.\frac{\partial U^4}{\partial t}\right|_{CF} = 0 \tag{22}
\]

\[
\left.\frac{\partial T}{\partial t}\right|_{CF} = 0 \tag{23}
\]

Introducing Eqs. \((20)\) to \((23)\) in Eq. \((16)\) one obtains

\[
v^\alpha f^{(0)}_{,\alpha} = f^{(0)}(0)k^\ell h^\ell_i \left\{ \left(1 - \frac{\gamma(k)}{G\left(\frac{1}{z}\right)}\right)\frac{n_{,\ell}}{n} + \frac{\gamma(k)}{G\left(\frac{1}{z}\right)}\left[\frac{q}{zmc^2}cF^{4\ell}\right] + \frac{T_\ell}{T}\left(1 - \frac{\gamma(k)}{z} - \frac{\gamma(k)}{G\left(\frac{1}{z}\right)} - \frac{G\left(\frac{1}{z}\right)}{z}\right) \right\} \tag{24}
\]

where a term proportional to the velocity gradient is ignored from now on since it will not couple with the heat flux. This fact is sustained by Curie’s principle by means of which only
fluxes and forces of the same tensorial rank are coupled to each other in the constitutive equations.

On the other hand, the second term in Eq. (7) is calculated using the fact that the field tensor only has electric potential components ($4 - \ell$ with $\ell$ running up to 3) such that

$$\left( \frac{q}{m} v_\alpha F^{\alpha \mu} \right) \frac{u_\mu}{zc^2} = \frac{q}{m} v_\alpha F^{\alpha 4} \frac{u_4}{zc^2} = \frac{q}{m} v_\alpha F^{\alpha \mu} \frac{u_\mu}{zc^2}$$

(25)

The quantity is scalar and therefore can be calculated in the comoving frame, so that

$$\left( \frac{q}{m} v_\alpha F^{\alpha \mu} \right) \frac{u_\mu}{zc^2} = \gamma(k) \frac{q}{m} k_\ell F^{4 \ell} \frac{c}{zc^2}$$

(26)

and thus, since $F^{4 \ell} = \frac{\phi^\ell}{c} = -F^{\ell 4}$

$$f^{(1)} = -\tau f^{(0)} \gamma(k) h_\ell^k k^k \left\{ \frac{1}{z} \left( \frac{\gamma(k)}{G \left( \frac{1}{z} \right)} - 1 \right) \frac{q}{mc^2} \phi_\ell + \left( 1 - \frac{\gamma(k)}{G \left( \frac{1}{z} \right)} \right) \frac{n_\ell}{T} \right\}$$

(27)

The function in Eq. (27) is the correction to the equilibrium distribution function to first order in the gradients in the presence of an electric field. This part of the distribution yields the dissipative terms in the hydrodynamic equations to the Navier-Stokes level. In the next section, $f^{(1)}$ will be introduced in the expression for the heat flux in the relativistic case in order to establish the general structure of the corresponding constitutive equation as well as the particular value of the transport coefficients for the relaxation approximation.

III. CONSTITUTIVE EQUATION FOR THE HEAT FLUX

An analysis of both the concept and the formal expression for the heat flux in relativistic kinetic theory in relation to the flux of chaotic energy, as initially conceived by R. Clausius [9], can be found in Ref. [1]. In this context it can be shown that the heat flux in the comoving frame is given by

$$J_{[\nu]} = mc^2 h_\nu \int k^\mu f^{(1)} \gamma_{(k)}^2 d^4 K$$

(28)

and can be expressed in an arbitrary frame using a Lorentz transformation. From (28) and considering the structure of Eq. (27) it can be seen that three independent driving forces
will be present in the flux namely, the temperature, particle density and electric potential gradients. Of particular interest in this work is the term that corresponds to the electric field which in this relaxation time approximation is given by

\[ J_{\ell}^{Q,E} = -\tau q h_{\ell}^i \phi^i \int k^2 f^{(0)} \left( \frac{\gamma_k}{G \left( \gamma_k \right)} - 1 \right) \gamma_k^3 d^* K, \]  

(29)

Notice that this flux vanishes in the non-relativistic limit where both \( \gamma_k \) and \( G \left( \frac{1}{z} \right) \) are unity. Thus, the heat flux in the single component fluid due to the thermoelectric effect is only present for non-negligible values of the parameter \( z \) and can be written as

\[ J_{\ell}^{Q,E} = L_{TE} h_{\ell}^i \phi^i \]  

(30)

where the corresponding transport coefficient is given by

\[ L_{TE} = n \tau c^2 q \left\{ 5z + \frac{1}{G \left( \frac{1}{z} \right)} - G \left( \frac{1}{z} \right) \right\} \]  

(31)

For small \( z \), in the non-relativistic limit one has

\[ L_{TE} = n \tau c^2 q \frac{5}{2} (z^2 - z^3 + ...) \]  

(32)

or equivalently

\[ L_{TE} \sim \frac{5}{2} n \tau \left( \frac{kT}{m} \right)^2 \frac{q}{c^2} \]  

(33)

The general form for the heat flux including all the gradients is given by

\[ J_{\mu}^{Q} = -h^{\mu\nu} \left( L_{TT} \frac{T}{T} + L_{Tn} \frac{n}{n} + L_{TE} \phi \phi \right) \]  

(34)

where the transport coefficients corresponding to the first two terms were previously obtained [12]. In the non-relativistic limit, the ordering of the effects with respect to \( c \) can be extracted from the following expression

\[ J_{\mu}^{Q} \sim -\frac{5}{2} n \tau \frac{k^2}{m} h^{\mu\nu} \left[ \frac{T}{T} + \frac{kT}{mc^2} \frac{n}{n} + \frac{q}{mc^2} \phi \phi \right] \]  

(35)

from where it can be clearly seen that the only effect that survives in the non-relativistic limit is the one corresponding to Fourier’s law and the other two terms are relevant only when the thermal energy and/or the electric energy are comparable in magnitude to the rest mass of the molecules. It is worthwhile to remind the reader at this point that the results obtained in this section are valid for single component gases, no binary mixture needs to be considered.
IV. SUMMARY AND FINAL REMARKS

In this paper, we have studied the heat flux in a single component relativistic charged gas in the presence of an electric field. The calculation was performed using Marle’s kernel for the Boltzmann equation. The result is a heat flux coupled to three forces, namely the gradients of the temperature, density and electric potential. Only the temperature gradient contribution does not vanish in the non-relativistic limit while the other two effects are strictly relativistic.

Some authors have argued in favor of a cross-like relationship between temperature gradients and electric field in the non-relativistic single component fluid [4]. In order for such an effect to be established, one would have to set the partial time derivative in Boltzmann’s equation to zero. This is usually justified by assuming a steady state in the gas, however the Chapman-Enskog procedure arises from an ordering scheme in Boltzmann’s equation in which the time derivative of the equilibrium distribution function results as a first order in the gradients term and thus must be considered when calculating the first order correction. Also, if the Benedicks effect would exist in the non-relativistic single component fluid in steady state, one should expect it to be recoverable as a limit of the complete solution.

By inspection of the structure of the heat flux given by Eq. (34), one can draw interesting conclusions regarding the physics of dissipative simple relativistic charged gases. If for the sake of simplicity one ignores the particle density for a homogeneous system, the constitutive equation has a structure similar to the one leading to Tolman’s law in the case of a gravitational field [13]. Reasoning in a similar way as in Ref. [13], if one assumes a vanishing heat flux, then to lowest order in $z$:

$$\frac{T_{\mu}}{T} = -\frac{q}{mc^2} \phi_{,\mu}$$

in such a way that a temperature gradient can be sustained in the absence of a heat flux by means of an electric field for the simple charged gas. It is interesting to notice that for the case of negative and positive charged particles, the required electric potential gradient reverses direction for the same temperature gradient.

Another interesting question arises here regarding the possibility of a relativistic version of a Wiedemann-Franz like relation for a single component fluid. This could lead to experimental tests for the strictly relativistic effects here presented, as well as for the pertinent generalizations of this formalism including magnetic fields.
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