Parameter estimation for the Lomax distribution using the E-Bayesian method

A Fitrilia¹,², I Fithriani¹,³ and S Nurrohmah¹,⁴

¹ Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Depok 16424, Indonesia.

Email: ṡaldilafitrilia@gmail.com, Ṣida.fithriani@gmail.com, Ṣnurrohmah@sci.ui.ac.id

Abstract. In this paper, we will find the parameter estimator from Lomax distribution on one of censored data, that is right censored data type II. Lomax distribution is also called Pareto type II distribution. The parameter will be estimated is the shape parameter with the assumption of scale parameters (β) has been known. The parameter estimation method used in this study is E-Bayesian estimation method. E-Bayesian estimation is an expectation of Bayes estimation, in order to obtain Bayes estimation expectations is by calculating the mean of Bayes estimators. E-Bayesian estimation method is used to estimate failure rate. The estimation will use prior Gamma as conjugate prior distribution from Lomax distribution and Loss function will be used is balanced squared error loss function (BSELF). Thus, the main purpose of this study is to find the parameter estimator of the Lomax distribution on the right censored data type II using the E-Bayesian method. The final result of this study, we get the likelihood function from Lomax distribution on the right censored data type II and the parameter estimator from Lomax distribution on the right censored data type II using the E-Bayesian method.

1. Introduction

The Lomax distribution was first introduced by K. S. Lomax in 1954 [1]. Lomax distribution is a probability distribution used in business, economics, and actuarial science. The Lomax distribution has two parameters and denoted by Lomax(α, β) where α is a form parameter and β is the scale parameter. Lomax distribution has probability density function (pdf) that defined as

\[ f(x; \alpha, \beta) = \frac{\alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{-(\alpha + 1)}, \quad x > 0, \quad (\alpha, \beta > 0) \tag{1} \]

and cumulative distribution function (cdf) that defined as

\[ F(x; \alpha, \beta) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad x > 0, \quad (\alpha, \beta > 0) \tag{2} \]

The survival function or reliability function of the Lomax distribution defined as the probability of an object can survive until time t is
\[ S(t) = 1 - F(t) = 1 - \left( 1 - \left( 1 + \frac{t}{\beta} \right)^{-\alpha} \right) = \left( 1 + \frac{t}{\beta} \right)^{-\alpha}, \quad t > 0 \]  

(3)

The hazard function of the Lomax distribution represents the probability of a component failing at time \( t \) is

\[ h(t) = \frac{f(t)}{S(t)} = \frac{\alpha \left( 1 + \frac{t}{\beta} \right)^{-(\alpha+1)}}{\beta} = \frac{\alpha \left( 1 + \frac{t}{\beta} \right)^{-1}}{\beta}, \quad t > 0 \]

(4)

Mean and variance of Lomax distribution is

\[ \mu = \frac{\beta}{\alpha - 1}, \quad \alpha > 1 \]  

(5)

and

\[ \sigma^2 = \frac{\alpha \beta^2}{(\alpha - 1)^2(\alpha - 2)}, \quad \alpha > 2 \]  

(6)

Since the parameters of \( \alpha \) and \( \beta \) are unknown, parameter estimation is necessary, one of the parameter estimation that can be used is the E-Bayesian estimation method. E-Bayesian estimation method are used to estimate failure rates, furthermore they are suitable for censored data with small sample sizes and high reliability [2]. Estimated parameters using the Bayesian approach, which to obtain E-Bayesian estimate of \( \alpha \) (i.e. expectation of the Bayes estimates of \( \alpha \)) is obtained by calculating the mean of the Bayes estimator.

2. Balanced square error loss function (BSELF)

The estimator of a function, using BSEL is actually a combination of the maximum likelihood estimator of a function and the Bayes estimator using the SEL. Ahmadi et al. [3] introduced a balanced loss function (BLF) derived from Zellner [4] in the form

\[ L_{\mu, \omega, \delta_0}(\Lambda(\alpha), \delta) = \omega q(\delta) \rho(\delta_0, \delta) + (1 - \omega) q(\alpha) \rho(\Lambda(\alpha), \delta) \]  

(7)

where \( q(\cdot) \) is a suitable positive weight function and \( \rho(\Lambda(\alpha), \delta) \) is any Loss function when estimating \( \Lambda(\alpha) \) with \( \delta \). The parameter \( \delta_0 \) is the selected prior estimator of \( \Lambda(\alpha) \), obtained for example from the maximum likelihood criterion, and \( 0 \leq \omega < 1 \). If \( \rho(\Lambda(\alpha), \delta) = (\delta - \Lambda(\alpha))^2 \) and \( q(\alpha) = 1 \) are substituted in equation (5), obtained BSELF, given as

\[ L_{\omega, \delta_0}(\Lambda(\alpha), \delta) = \omega (\delta - \delta_0)^2 + (1 - \omega)(\delta - \Lambda(\alpha))^2 \]  

(8)

The corresponding Bayes estimator of the function \( \Lambda(\alpha) \) is given as

\[ \delta_{\omega, \Lambda, \delta_0}(x) = \omega \delta_0 + (1 - \omega) E(\Lambda(\alpha) \mid x) \]  

(9)
3. Likelihood function from Lomax distribution on the right censored data type II

In the right censored data type II, there are \( r \) observations of \( n \) samples observed, and the study is stopped after the \( r \)th failure that occurred. In this censored, \( r \) is determined before the data is collected. The joint probability density function of right censored data type II of \( X_1, X_2, \ldots, X_r \) for \( r < n \) is

\[
f(x_1, x_2, \ldots, x_r) = \frac{n!}{(n-r)!} \left( \prod_{i=1}^{r} f(x_i) \right) [S(x_r)]^{n-r} \tag{10}\]

then the joint probability density function of Lomax distribution on the right censored data type II is

\[
f(x_1, x_2, \ldots, x_r) = \frac{n!}{(n-r)!} \frac{1}{\beta^r \prod_{i=1}^{r} \left( 1 + \frac{x_i}{\beta} \right)} \frac{1}{\prod_{i=1}^{r} \left( 1 + \frac{x_i}{\beta} \right)^{\alpha}} \left( \frac{x_r}{\beta} \right)^{\alpha(n-r)} \tag{11}\]

Thus the likelihood function of the Lomax distribution on the right censored data type II can be written as

\[
L(x|\alpha, \beta) = \frac{n!}{(n-r)!} \alpha^r v(x; \beta) e^{-M\alpha} \tag{12}\]

where \( x = (x_1, x_2, \ldots, x_r) \), \( v(x; \beta) = \frac{1}{\beta^r \prod_{i=1}^{r} \left( 1 + \frac{x_i}{\beta} \right)} \)

and \( M = M(x; \beta) = \sum_{i=1}^{r} \ln \left( 1 + \frac{x_i}{\beta} \right) + (n-r) \ln \left( 1 + \frac{x_r}{\beta} \right) \)

4. Bayesian estimation of Lomax distribution on the right censored data type II

Bayes estimation is estimation method that combines prior information with information obtained from sample data. Prior information is obtained from the distribution of parameters, while the information from the data is summarized in the likelihood function. The combination of prior information and information from the data will produce posterior information (posterior distribution). The results are expressed in terms of this posterior distribution which will be the basis in the Bayes method for estimating the parameters.

From (12), assuming \( \beta \) is known, the maximum likelihood estimator of the parameter \( \alpha \) is given as

\[
\ln L(x|\alpha) = \ln \frac{n!}{(n-r)!} + r \ln \alpha + \ln v(x; \beta) + (-M\alpha) \\
\frac{d}{d\alpha} \ln L(x|\alpha) = 0
\]

so that the maximum likelihood estimator of the parameter \( \alpha \) is obtained

\[
\hat{\alpha}_{ML} = \frac{r}{M} \tag{13}\]
To search for Bayes estimator, we need to find the prior distribution first. According to Box and Tiao [5], the conjugate prior distribution, refers to the reference model analysis, especially in the creation of likelihood function. So in the defining of conjugate prior always thought about the defining of the prior distribution pattern that has the conjugate form with the probability density function of the likelihood construction. According to Castillo [6], if \( F \) is the class of the sampling distribution \( p(y|\theta) \) and \( P \) is the class of the prior distribution for \( \theta \), then class \( P \) is said to conjugate for \( F \) if the posterior probability \( p(\theta|y) \) has the same distribution form with the prior probability function \( p(\theta) \) for all \( p(y|\theta) \in F \). In this study, Gamma conjugate prior is used to find Bayes estimators. Likelihood function of the Lomax distribution on the right censored data type II is

\[
f(x|\alpha, \beta) = \frac{n!}{(n-r)!} \alpha^r \beta^r (1 - \beta)^{n-r} e^{-\beta x}
\]

Assuming \( \beta \) is known, we can use the Gamma distribution as an conjugate prior distribution of \( \alpha \) with shape and scale parameter \( a \) and \( b \) respectively and its probability density function given by

\[
g(\alpha|a, b) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-ba}, \quad \alpha > 0
\]

The likelihood function and the probability density function of Gamma distribution has the same functional form where \( r = a - 1 \) and \( b = M \). Therefore, the Gamma Distribution is used as the conjugate prior distribution of the Lomax distribution on the right censored data type II. Furthermore, we will find the posterior distribution density function

\[
q(\alpha|x) = \frac{\int_0^\infty f(x|\alpha, \beta) g(\alpha|a, b) d\alpha}{\int_0^\infty f(x|\alpha, \beta) g(\alpha|a, b) d\alpha} = \frac{\alpha^{r+a-1} e^{-(b+M)\alpha} (b + M)^{r+a}}{\Gamma(r + \alpha)}
\]

(16)

with \( \kappa = \frac{(b+T)^{r+a}}{\Gamma(r+a)} \). Posterior probability density function in (16) is the probability density function of Gamma distribution with parameters \( r + a \) and \( b + T \). Since the resulting posterior distribution is Gamma distributed, just like the prior distribution which Gamma distributed, it is proven that the Gamma distribution is the conjugate prior of the Lomax distribution on the right censored data type II. The Bayes estimator of \( \alpha \) with balanced square error loss function (BSELF) can be given as

\[
\hat{\alpha}_B(a, b) = \omega \hat{\delta}_0 + (1 - \omega) E(\Lambda(\alpha)|x)
\]

with \( \hat{\delta}_0 \) is maximum likelihood estimator and \( E(\Lambda(\alpha)|x) \) is Bayes estimator with square error loss function.

\[
E(\Lambda(\alpha)|x) = \int_0^\infty \Lambda(\alpha) q(\Lambda(\alpha)|x) d\alpha = \frac{(r + a)}{(b + M)}
\]

(17)

Then the Bayes estimator for \( \alpha \) is obtained as

\[
\hat{\alpha}_B(a, b) = \omega \left( \frac{r}{M} \right) + (1 - \omega) \left( \frac{r + a}{b + M} \right)
\]

(18)
5. E-Bayesian estimation of Lomax distribution on the right censored data type II

E-Bayesian estimation methods are used to estimate failure rates. E-Bayesian estimation is an expectation of Bayes estimation, in order to obtain Bayes estimation expectations is by calculating the mean of Bayes estimators.

Suppose that life an object is Lomax distributed on the right censored data type II with a joint probability density function is (11). Let the prior distribution of \( \alpha \) be the conjugate \(-\Gamma(a,b)\) distribution with the probability density function is (15) where \( \Gamma(a) = \int_{0}^{\infty} x^{a-1}e^{-x}dx \) is a function of Gamma, \( a > 0, b > 0 \), and \( a \) and \( b \) are hyperparameters. According to Han [7], the hyperparameters \( a \) and \( b \) must be choose to guarantee that \( g(a|a, b) \) is a decreasing function of \( a \). The derivative of \( g(a|a, b) \) with respect to \( a \) is

\[
\frac{dg(a|a, b)}{da} = \frac{b^a}{\Gamma(a)}[(a - 1)a^{a-2}e^{-ba} + a^{a-1}(-b)e^{-ba}]
\]

Thus, for \( 0 < a < 1, b > 0 \), the prior \( g(a|a, b) \) is a decreasing function of \( a \). Assuming that the hyperparameters \( a \) and \( b \) in \( g(a|a, b) \) are independent and \( \pi(a, b) = \pi_1(a)\pi_2(b) \). The E-Bayesian estimator of \( \alpha \) (the expectation of the Bayes estimator of \( \alpha \)) can be written as

\[
\hat{\alpha}_{EB} = E(\alpha|X) = \int_{0}^{\infty} \int \hat{\alpha}_B(a,b)\pi(a,b)dadb
\]

where \( \tilde{g} \) is the set of all possible values of \( a \) and \( b \) where the prior density function is a derivative of \( a \). \( \hat{\alpha}_B(a,b) \) is the Bayes estimator of \( \alpha \) given by (18). For more detail see Han [2].

The E-Bayesian estimates of \( \alpha \) are derived depending on three different distributions of the hyperparameters \( a \) and \( b \). These distribution are used to investigate the impact of the different prior distributions on E-Bayesian estimation of \( \alpha \). The prior distribution used must be conjugate to make it easier to find the posterior distribution. If the prior distribution is not conjugate, we will have difficulty to find the posterior distribution. The following distributions of \( a \) and \( b \) may be used

\[
\pi_1(a, b) = \frac{2(c - b)}{c}, \quad 0 < a < 1, 0 < b < c
\]
\[
\pi_2(a, b) = \frac{1}{c}, \quad 0 < a < 1, 0 < b < c
\]
\[
\pi_3(a, b) = \frac{2b}{c^2}, \quad 0 < a < 1, 0 < b < c
\]

We can obtained the E-Bayesian estimate of \( \alpha \) based on \( \pi_1(a, b), \pi_2(a, b), \) and \( \pi_3(a, b) \) by using (18) and (21) in (20), (18) and (22) in (20), and (18) and (23) in (20) respectively with \( \hat{\alpha}_{EB} \) is the E-Bayesian estimator of \( \alpha \), \( \hat{\alpha}_B(a,b) \) is the Bayes estimator of \( \alpha \) given by (18), and \( a, b \) is hyperparameters. The corresponding E-Bayesian estimate of \( \alpha \) are respectively.

For \( \pi_1(a, b) \),

\[
\hat{\alpha}_{EB1} = \int_{0}^{1} \int \hat{\alpha}_B(a,b)\pi_1(a,b)dadb = \int_{0}^{1} \int \hat{\alpha}_B(a,b)\pi_1(a,b)dadb
\]

\[
= \int_{0}^{1} \int \omega \left( \frac{r}{M} \right) + (1 - \omega) \left( \frac{r + a}{b + M} \right) \times \frac{2(c - b)}{c^2} dadb
\]
\[ \hat{\alpha}_{EB1} = \omega \left( \frac{r}{M} \right) + \frac{2}{c} (1 - w) \left( r + \frac{1}{2} \right) \left( \frac{c + M}{s} \ln \left( \frac{c + M}{s} \right) - 1 \right) \]  

(24)

For \( \pi_2(a, b) \),

\[ \hat{\alpha}_{EB2} = \int_0^c \int_0^c \hat{\alpha}_B(a,b) \pi_2(a,b) db da = \int_0^c \int_0^c \hat{\alpha}_B(a,b) \pi_2(a,b) db da \]
\[ = \int_0^c \int_0^c \left[ \omega \left( \frac{r}{M} \right) + (1 - w) \left( \frac{r + a}{b + M} \right) \right] \times \frac{1}{c} db da \]
\[ \hat{\alpha}_{EB2} = \omega \left( \frac{r}{M} \right) + \frac{(1 - w)}{c} \left( r + \frac{1}{2} \right) \ln \left( \frac{c + M}{M} \right) \]  

(25)

For \( \pi_3(a, b) \),

\[ \hat{\alpha}_{EB3} = \int_0^c \int_0^c \hat{\alpha}_B(a,b) \pi_3(a,b) db da = \int_0^c \int_0^c \hat{\alpha}_B(a,b) \pi_3(a,b) db da \]
\[ = \int_0^c \int_0^c \left[ \omega \left( \frac{r}{M} \right) + (1 - w) \left( \frac{r + a}{b + M} \right) \right] \times \frac{2b}{c^2} db da \]
\[ \hat{\alpha}_{EB3} = \omega \left( \frac{r}{M} \right) + \frac{2}{c} (1 - w) \left( r + \frac{1}{2} \right) \left( 1 - \frac{M}{c} \ln \left( \frac{c + M}{c} \right) \right) \]  

(26)

6. Conclusion
In this paper, the E-Bayesian method is used to estimate the \( \alpha \) parameter of the Lomax distribution in the right censored data type II. Since E-Bayesian estimation is an expectation of Bayes estimation, in order to obtain Bayes estimation expectations is by calculating the mean of Bayes estimators, it is necessary to find the Bayes estimation first. Loss function used in this study is balanced square error loss function (BSELF) which is a combination of the maximum likelihood estimator of a function and the Bayes estimator using the square error loss function (SELF).

7. References
[1] K. S. Lomax 1954 J. Am. Stat. Association 49 847
[2] Ming Han 2008 App. Math. Model. 33 1915
[3] Jafar Ahmadi, M. J. Jozani, Eric Marchand & Ahmad Parsian 2008 J. Stat. Plan. Inference 1180
[4] Arnold Zellner 1994 Statistical Desicion Theory and Methods V 337-390
[5] G. E. P. Box & G. C. Tiao 1973 Bayesian inference in statistical analysis (Philippines: Addison-Wesley Publishing Company, Inc)
[6] Enrique D. Castillo 2007 Process optimization a statistical approach (New York: Springer Science + Business Media, LCC)
[7] Ming Han 1997 The structure of hierarchical prior distribution and its applications 31-40
[8] Essam K. Al-Hussaini & Mohamed Hussein 2011 Open J. of Stat 28-38
[9] Hassan M. Okasha 2013 J. Egyptian Math. Soc. 489
[10] Hesham M. Reyad & Soha O. Ahmed 2014 Int. J. Adv. Math. Sci. 3 108-120

Acknowledgments
We would like to thank the DRPM Universitas Indonesia for supporting in our research with Hibah PITTA UI 2018.