Energy Conservation for the Weak Solutions of the Compressible Navier–Stokes Equations

CHENG YU

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Abstract

In this paper, we prove the energy conservation for the weak solutions of the compressible Navier–Stokes equations for any time \( t > 0 \), under certain conditions. The results hold for the renormalized solutions of the equations with constant viscosities, as well as the weak solutions of the equations with degenerate viscosity. Our conditions do not depend on the dimensions. The energy may be conserved on the vacuum for the compressible Navier–Stokes equations with constant viscosities. Our results are the first ones on energy conservation for the weak solutions of the compressible Navier–Stokes equations.

1. Introduction

This paper deals with the energy conservation for the weak solutions of the compressible Navier–Stokes equations, namely

\[
\rho_t + \text{div}(\rho \mathbf{u}) = 0 \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P - 2\mu \Delta \mathbf{u} - \lambda \nabla \text{div} \mathbf{u} = 0,
\]

as well as the following equations with degenerate viscosity:

\[
\rho_t + \text{div}(\rho \mathbf{u}) = 0 \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P - 2\nu \text{div}(\rho \mathbf{D} \mathbf{u}) = 0,
\]

respectively with initial data

\[
\rho|_{t=0} = \rho_0(x), \quad \rho \mathbf{u}|_{t=0} = \mathbf{m}_0(x) = \rho_0 \mathbf{u}_0,
\]

where \( P = \rho^\gamma \), \( \gamma > 1 \) denotes the pressure, \( \rho \) is the density of fluid, \( \mathbf{u} \) stands for the velocity of fluid and \( \mathbf{D} \mathbf{u} = \frac{1}{2} [\nabla \mathbf{u} + \nabla^T \mathbf{u}] \) is the strain tensor. The viscosity coefficients \( \mu, \lambda \) satisfy \( \mu > 0 \) and \( 2\mu + N\lambda \geq 0 \). For the sake of simplicity we
will consider the case of a bounded domain with periodic boundary conditions in $\mathbb{R}^N$, namely $\Omega = \mathbb{T}^N$, $N = 2, 3$. Here we define $u_0 = 0$ on the set $\{x|\rho_0(x) = 0\}$. Without loss of generality, we will fix $2\nu = 1$ in (1.2) from now on.

As we all know, the global existence of weak solution to (1.1) and (1.3) was established in [11,12,19] for any $\gamma > \frac{N}{2}$. The weak solution of (1.2)–(1.3) was studied in [1–4,18,21,24]. The purpose of this paper is to provide the sufficient conditions for the energy conservation of the weak solutions of (1.2) first, then extend our result to the weak solutions of (1.1).

A weak solution $(\rho, u)$ to (1.2)–(1.3) constructed in [24] satisfies the following energy inequality:

$$\int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \int_0^T \int_\Omega \rho |D u|^2 \, dx \, dt \leq \int_\Omega \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx,$$

(1.4)
even if $\rho \geq \rho_0 > 0$. A natural question to ask is when a weak solution $(\rho, u)$ of the compressible Navier–Stokes equations (1.2)–(1.3) satisfies, not only (1.4), but the stronger version

$$\int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \int_0^T \int_\Omega \rho |D u|^2 \, dx \, dt = \int_\Omega \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx.$$

(1.5)

It is well known that if a solution is smooth enough, then it conserves the energy. Thus, our question is connected to the regularity of weak solutions. However, the regularity of global weak solutions of (1.2) remains mostly open. Naturally, of particular interest is the question: how badly behaved can $(\rho, u)$ be while still keeping its energy conservation? In mathematics, how much regularity is needed for a weak solution to ensure energy equality (1.5)? We can ask the same question for the weak solutions of equation (1.1). The main contribution of our paper is to provide the first result on this topic of the compressible Navier–Stokes equations for any $t > 0$.

As we mentioned before, the question to ask concerns how much regularity is necessary to conserve energy. In the context of the incompressible Navier–Stokes equations, the pioneering study was done by Serrin [22]. He proved that a weak solution $u$ conserves its energy globally, provided $u \in L^p(0, T; L^q(\Omega))$, where

$$\frac{2}{p} + \frac{N}{q} \leq 1,$$

where $N$ is the dimension. Later, Shinbrot [23] proved the same conclusion if

$$\frac{2}{p} + \frac{2}{q} \leq 1$$

and $q \geq 4$. Meanwhile, in the context of incompressible Euler equations, this question is linked to a famous conjecture of Onsager [15]: energy should be conserved if the solution is Hölder continuous with the exponent greater than 1/3, while solutions with less regularity possibly dissipate energy. The first part was solved
by \([7,8,10]\), while significant progress has recently been made on the second part \([5,6]\). Very recently, Isett \([14]\) solved the second part of the Onsager conjecture. Feireisl et al. \([13]\) gave sufficient conditions on the regularity of solutions to the inhomogeneous incompressible Euler and the compressible isentropic Euler systems in order for the energy to be conserved in the distribution sense. Leslie and Shvydkoy \([16]\) showed that the energy balance relation for density dependent incompressible Navier–Stokes equations holds for weak solutions if the velocity, density and pressure belong to a range Besov spaces of smoothness \(\frac{1}{3}\). To the best of our knowledge, there is no available result for the weak solutions of the compressible Navier–Stokes equations.

The sufficient conditions for energy conservation are addressed in this paper. Our approach relies on the idea of Vasseur and Yu \([24]\) and Yu \([25]\). Compared to the incompressible Navier–Stokes equations, we are not able to deduce that \(u(t, x)\) is continuous at time \(t = 0\). Our alternative way for the compressible Navier–Stokes equations is to gain the continuity of \(\rho(t)\) and \((\sqrt{\rho}u)(t)\) in the strong topology at \(t = 0\). To this end, we need \(\nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))\). Lucky, BD entropy (see \([1–3]\)) gives us such an estimate on density for the degenerate compressible Navier–Stokes equations. However, we need an additional condition on the density to study the weak solutions of \((1.1)\). Meanwhile, it is crucial to rely on Lemma 2.3 to handle the nonlinear compositions \((\rho u)\), and \(\rho u \otimes u\). Here we have to mention that our approach allows us to handle vacuum states for the weak solutions of \((1.1)\).

The question addressed here is motivated by the fact that energy conservation is fundamental both in the physical theory as well as in the mathematical study of the fluid dynamics. It is natural, therefore, to seek a rigorous theory which accommodates this question. The results of this paper effectively achieve this goal by providing a certain condition for the weak solution, for any \(t > 0\). To address our main result, we now give a precise definition and discussion of our weak solutions \((\rho, u)\) to the initial value problem \((1.2)–(1.3)\) in the following sense:

**Definition 1.1.** The \((\rho, u)\) is called a global weak solution to \((1.2)–(1.3)\), if \((\rho, u)\) satisfies the following, for any \(t \in [0, T]\):

- \((1.2)\) holds in \(\mathcal{D}'((0, T) \times \Omega))\) satisfying
  
  \[
  \rho \geq 0, \quad \rho \in L^\infty([0, T]; L^\gamma(\Omega)),
  \]
  
  \[
  \nabla \rho^\frac{\gamma}{2} \in L^2(0, T; L^2(\Omega)), \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)),
  \]
  
  \[
  \sqrt{\rho}u \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\rho} \nabla u \in L^2(0, T; L^2(\Omega));
  \]

- \((1.3)\) holds in \(\mathcal{D}'(\Omega))\).

The global existence of a weak solution to \((1.2)\) under the above definition was proved in \([24]\). The energy inequality \((1.4)\) holds for almost every \(t \in [0, T]\) if the density is bounded away from zero. Here we state the following result on the energy conservation for the weak solution:

**Theorem 1.1.** Let \((\rho, u)\) be a weak solution of \((1.2)–(1.3)\) in the sense of Definition 1.1. Moreover, if

\[
0 < \underline{\rho} \leq \rho(t, x) \leq \bar{\rho} < \infty, \quad (1.6)
\]
\[ \mathbf{u} \in L^p(0, T; L^q(\Omega)) \quad \text{for any } \frac{1}{p} + \frac{1}{q} \leq \frac{5}{12} \text{ with } q \geq 6, \quad (1.7) \]

and

\[ \sqrt{\rho_0} \mathbf{u}_0 \in L^4(\Omega), \quad (1.8) \]

then such a weak solution \((\rho, \mathbf{u})\) satisfies (1.5) for any \(t \in [0, T]\).

**Remark 1.1.** The condition (1.8) could be replaced by (1.11). We will give details about this issue at the end of this paper.

**Remark 1.2.** Regarding the incompressible Navier–Stokes equations, Shinbrot [23] has shown that a weak solution can retain its energy if

\[ \mathbf{u} \in L^p(0, T; L^q(\Omega)) \quad \text{for any } \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \text{ with } q \geq 4. \]

A new proof of Shinbrot’s result is given in [25]. Our condition (1.10) for the compressible version is slightly stronger than the incompressible ones, because we have to rely on the commutator lemma (Lemma 2.3) to handle the terms \((\rho \mathbf{u})_t\) and \(\text{div}(\rho \mathbf{u} \otimes \mathbf{u})\), which needs additional regularity on \(\mathbf{u}\).

Next we extend Theorem 1.1 to the weak solution of (1.1) in what follows. The global existence of renormalized weak solutions to (1.1) was established in [11, 12, 19] for any \(\gamma > \frac{N}{2}\). In particular, we need the additional restriction (1.9) for density, which allows us to have the continuity of \(\rho(t)\) and \((\sqrt{\rho} \mathbf{u})(t)\) in the strong topology at \(t = 0\). Meanwhile, \(\mathbf{u}\) is uniformly bounded in \(L^2(0, T; H^1(\Omega))\) for the viscosity constants case. Thus, our approach allows us to handle vacuum states. With the restriction (1.9) in mind, we can modify the proof of Theorem 1.1 to show the result. One slight difference is to show \(W = 0\) in (2.26) for getting the continuity of \(\rho(t)\) and \((\sqrt{\rho} \mathbf{u})(t)\) in the strong topology at \(t = 0\). We will give details regarding this point at the end of this paper.

**Theorem 1.2.** Let \((\rho, \mathbf{u})\) be a weak solution of (1.1) and (1.3) in the sense of [11, 20]. Moreover, if

\[ 0 \leq \rho(t, x) \leq \bar{\rho} < \infty, \quad \text{and} \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)), \quad (1.9) \]

\[ \mathbf{u} \in L^p(0, T; L^q(\Omega)) \quad \text{for any } \frac{1}{p} + \frac{1}{q} \leq \frac{5}{12}, \text{ and } q \geq 6, \quad (1.10) \]

and

\[ \mathbf{u}_0 \in L^k(\Omega), \quad \frac{1}{k} + \frac{1}{q} \leq \frac{1}{2}, \quad (1.11) \]

then such a weak solution \((\rho, \mathbf{u})\) satisfies

\[ \int_\Omega \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + 2\mu \int_0^T \int_\Omega |\nabla \mathbf{u}|^2 \, dx \, dt \]

\[ + \lambda \int_0^T \int_\Omega |\text{div} \mathbf{u}|^2 \, dx \, dt = \int_\Omega \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx \]

for any \(t \in [0, T]\).

**Remark 1.3.** An interesting point is that the density may vanish in (1.9). Thus, Theorem 1.2 indicates that the energy may be conserved even on the vacuum.
2. Proof of Main Results

The main object of this section is to prove our main results, including Theorem 1.1 and Theorem 1.2. We devote Section 2.1 to a proof of Theorem 1.1, and Section 2.2 to a proof of Theorem 1.2.

2.1. Proof of Theorem 1.1

Note that, for any weak solution \((\rho, u)\), with condition (1.6), Theorem 1.1 satisfies

\[
\|u\|_{L^\infty(0, T; L^2(\Omega))} \leq C < \infty, \quad \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \leq C < \infty.
\]  

(2.1)

Now, let us give an estimate on \(\rho_t\) in the following lemma:

**Lemma 2.1.** For any weak solution \((\rho, u)\) in the sense of Definition 1.1, with additional conditions (1.6)–(1.10), \(\rho_t\) is bounded in \(L^p(0, T; L^{\frac{2q}{q+2}}(\Omega)) + L^2(0, T; L^2(\Omega))\). In particular, \(\rho_t\) is bounded in \(L^2(0, T; L^{\frac{2q}{q+2}}(\Omega))\) if \(p \geq 2\).

**Proof.** Note that

\[
\rho_t = -2\sqrt{\rho} u \cdot \nabla \sqrt{\rho} - \sqrt{\rho} \sqrt{\rho} \text{div} u,
\]

consequently \(\rho_t\) is bounded in \(L^p(0, T; L^{\frac{2q}{q+2}}(\Omega)) + L^2(0, T; L^2(\Omega))\), thanks to the estimates in definition 1.1, plus (1.6) and (1.10).

**Remark 2.1.** For any weak solution \((\rho, u)\) satisfied by B–D entropy [1–3], \(\sqrt{\rho}\) is bounded in \(L^\infty(0, T; L^2(\Omega))\). Thus, Lemma 2.1 gives us that \(\rho_t\) is locally integrable. It is crucial to have the estimate of \(\sqrt{\rho}\) for making commutator estimates.

Here we make use of \(L^p - L^q\) inequality to deduce the following lemma directly:

**Lemma 2.2.** If \(u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; L^q(\Omega))\), then there exists some \(\alpha \in (0, 1)\) such that \(u \in L^{r}(0, T; L^{s}(\Omega))\), for any

\[
\frac{1}{r} = \frac{1}{p} - \alpha,
\]

and

\[
\frac{1}{s} = \frac{\alpha}{2} + \frac{1 - \alpha}{q}.
\]

Now, let us to define

\[
\overline{f}(t, x) = \eta_{\varepsilon} \ast f(t, x), \quad t > \varepsilon,
\]

where \(\eta_{\varepsilon} = \frac{1}{\varepsilon^{N+1}} \eta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)\), and \(\eta(t, x) \geq 0\) is a smooth even function compactly supported in the space–time ball of radius 1, and with an integral equal to 1.

The following lemma (proved by Lions [19]) is crucial to the current pages, and is adopted following a statement in [17]:
Lemma 2.3. Let $\partial$ be a partial derivative in space or time. Let $f, g \in L^p(\mathbb{R}^+ \times \Omega)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have
\[
\|\partial(fg) - \partial(f g)\|_{L^r(\mathbb{R}^+ \times \Omega)} \leq C \|\partial f\|_{L^p(\mathbb{R}^+ \times \Omega)} \|g\|_{L^q(\mathbb{R}^+ \times \Omega)}
\]
for some constant $C > 0$ independent of $\varepsilon, f$ and $g$, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,
\[
\partial(fg) - \partial(f g) \to 0 \quad \text{in } L^r(\mathbb{R}^+ \times \Omega)
\]
as $\varepsilon \to 0$, if $r < \infty$.

With Lemma 2.1 and Lemma 2.3 in hand, we are ready to show Theorem 1.1. Here we introduce a new function $\Phi_1 = \psi(t)u$. Note that $\psi(t) \in \mathcal{D}(0, +\infty)$ is a test function, where $\mathcal{D}(0, +\infty)$ is a class of all smooth compactly supported functions in $(0, +\infty)$. In particular, this function vanishes close $t = 0$, however it is needed to extend the result for $\psi(t) \in \mathcal{D}(-1, +\infty)$. Note that, $\psi(t)$ is compactly supported in $(0, \infty)$. $\Phi_1$ is well defined on $(0, \infty)$ for $\varepsilon$ small enough. Using this to test the second equation in (1.2), one obtains
\[
\int_0^T \int_\Omega \Phi \left( (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho \gamma - \text{div}(\rho \mathbb{D}u) \right) \, dx \, dt = 0,
\]
which in turn yields
\[
\int_0^T \int_\Omega \psi(t) \Phi \left( (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho \gamma - \text{div}(\rho \mathbb{D}u) \right) \, dx \, dt = 0, \quad (2.2)
\]
where we used the fact $\eta(-t, -x) = \eta(t, x)$.

The first term in (2.2) shows that
\[
\int_0^T \int_\Omega \psi(t) \Phi \left( (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \rho \gamma - \text{div}(\rho \mathbb{D}u) \right) \, dx \, dt = A + \int_0^T \int_\Omega \psi(t) \rho \partial_t |\mathbb{u}|^2 dx \, dt + \int_0^T \int_\Omega \psi(t) \rho_t |\mathbb{u}|^2 dx \, dt. \quad (2.3)
\]

Similarly, the second term in (2.2) gives us
\[
\int_0^T \int_\Omega \psi(t) \Phi \text{div}(\rho u \otimes u) \, dx \, dt
\]
\[
= \int_0^T \int_\Omega \psi(t) \left( \text{div}(\rho u \otimes u) - \text{div}(\rho \mathbb{u}) \otimes \mathbb{u} \right) \mathbb{u} \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \psi(t) \text{div}(\rho u \otimes \mathbb{u}) \mathbb{u} \, dx \, dt
\]

\[
= B + \int_0^T \int_\Omega \psi(t) \rho \mathbf{u} \cdot \nabla \frac{\mathbf{u}^2}{2} \, dx + \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{u})|\mathbf{u}|^2 \, dx \, dt \\
= B + \int_0^T \int_\Omega \psi(t) \rho \frac{u^2}{2} \, dx + \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{u})|\mathbf{u}|^2 \, dx \, dt. \tag{2.4}
\]

Thanks to Lemma 2.1, the last term of the right-hand side in (2.3) and the second term of the right-hand side in (2.4) are well defined. Combining (2.3) and (2.4), the first two terms in (2.2) are given by

\[
\int_0^T \int_\Omega \psi(t) \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) \, dx \, dt + A + B. \tag{2.5}
\]

Next, the last term in (2.2) gives us

\[
\int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{D} \mathbf{u}) \mathbf{u} \, dx \, dt = \int_0^T \int_\Omega \psi(t) \left( \text{div}(\rho \mathbf{D} \mathbf{u}) - \text{div}(\rho \mathbf{D} \mathbf{u}) \right) \mathbf{u} \, dx \, dt \\
+ \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{D} \mathbf{u}) \mathbf{u} \, dx \, dt \\
= D - \int_0^T \int_\Omega \psi(t) \rho |\mathbf{D} \mathbf{u}|^2 \, dx \, dt. \tag{2.6}
\]

We estimate the third term in (2.2) as follows:

\[
\int_0^T \int_\Omega \psi(t) \nabla \rho \gamma^{-1} \mathbf{u} \, dx \, dt \\
= \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \nabla \rho \gamma^{-1} \mathbf{u} \, dx \, dt \\
= \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \left( \rho \nabla \rho \gamma^{-1} - \rho \nabla \rho \gamma^{-1} \right) \mathbf{u} \, dx \, dt \\
+ \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \rho \nabla \rho \gamma^{-1} \mathbf{u} \, dx \, dt \\
= E - \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{u}) \rho \gamma^{-1} \, dx \, dt. \tag{2.7}
\]

The second term in the last equality of (2.7) shows that

\[
\frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{u}) \rho \gamma^{-1} \, dx \, dt \\
= \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \left( \text{div}(\rho \mathbf{u}) - \text{div}(\rho \mathbf{u}) \right) \rho \gamma^{-1} \, dx \, dt \\
+ \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \text{div}(\rho \mathbf{u}) \rho \gamma^{-1} \, dx \, dt \\
= E_{21} - \frac{\gamma}{\gamma - 1} \int_0^T \int_\Omega \psi(t) \rho_t \rho \gamma^{-1} \, dx \, dt. \tag{2.8}
\]
Meanwhile, the last term in (2.8) is as follows:

\[
\frac{\gamma}{\gamma - 1} \int_0^T \int_{\Omega} \psi(t) \rho_t \rho^{\gamma - 1} \, dx \, dt \\
= \frac{\gamma}{\gamma - 1} \int_0^T \int_{\Omega} \psi(t) (\rho_t - \rho_t) \rho^{\gamma - 1} \, dx \, dt \\
+ \frac{\gamma}{\gamma - 1} \int_0^T \int_{\Omega} \psi(t) \left( \rho_t \rho^{\gamma - 1} - \rho_t \rho^{\gamma - 1} \right) \, dx \, dt \\
+ \frac{\gamma}{\gamma - 1} \int_0^T \int_{\Omega} \psi(t) \rho_t \rho^{\gamma - 1} \, dx \, dt \\
= E_{31} + E_{32} + \frac{1}{\gamma - 1} \int_0^T \int_{\Omega} \psi(t) (\rho^{\gamma})_t \, dx \, dt \\
\]  
(2.9)

Thanks to (2.5)–(2.9), (2.2) gives us

\[
\int_0^T \int_{\Omega} \psi(t) \left( \frac{1}{2} \rho |\vec{u}|^2 + \frac{\rho^{\gamma}}{\gamma - 1} \right)_t \, dx \, dt \\
+ \int_0^T \int_{\Omega} \psi(t) \rho |\vec{D} u|^2 \, dx \, dt + A + B - D + E - E_{21} + E_{31} + E_{32} = 0.
\]

This yields

\[
- \int_0^T \int_{\Omega} \psi_t \left( \frac{1}{2} \rho |\vec{u}|^2 + \frac{\rho^{\gamma}}{\gamma - 1} \right) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \psi(t) \rho |\vec{D} u|^2 \, dx \, dt + R_\varepsilon(t, x) = 0,
\]  
(2.10)

where \( R_\varepsilon(t, x) = A + B - D + E - E_{21} + E_{31} + E_{32} \).

Note that given (1.6) and (1.10), one obtains

\[
\int_0^T \int_{\Omega} \frac{1}{2} \rho |\vec{u}|^2 \psi_t \, dx \, dt \to \int_0^T \int_{\Omega} \frac{1}{2} \rho |\vec{u}|^2 \psi_t \, dx \, dt \quad \text{as} \quad \varepsilon \to 0. \]  
(2.11)

Thanks to (1.6) and (2.1), we find that, for any \( \varepsilon \) that tends to zero,

\[
\int_0^T \int_{\Omega} \psi(t) \rho |\vec{D} u|^2 \, dx \, dt \to \int_0^T \int_{\Omega} \psi(t) \rho |\vec{D} u|^2 \, dx \, dt.
\]  
(2.12)

The next step is to make use of Lemma 2.3 to prove

\[
R_\varepsilon(t, x) \to 0
\]  
(2.13)

as \( \varepsilon \) goes to zero. First, we assume that \( u \) is bounded in \( L^r(0, T; L^2(\Omega)) \). We will improve this restriction later. Under this restriction, by Lemma 2.1, \( \rho_t \) is uniformly bounded in \( L^r(0, T; L^{\frac{2r}{r + 2}}(\Omega)) \) + \( L^2(0, T; L^2(\Omega)) \). Thus, Lemma 2.3 gives us

\[
|A| \leq \| \psi(t) \|_{L^\infty(0, T)} \int_0^T \int_{\Omega} \left| \vec{u} (\rho \vec{u})_t - (\rho \vec{u})_t \right| \, dx \, dt \\
\leq C \| \psi(t) \|_{L^\infty(0, T)} \| \rho_t \|_{L^2(0, T; L^{\frac{2r}{r + 2}}(\Omega))} \| \vec{u} \|_{L^2(0, T; L^r(\Omega))}^2
\]
for any \( r \geq 4 \) and \( s \geq 6 \).

Meanwhile, for any \( u \in L^p(0, T; L^q(\Omega)) \), by Lemma 2.2, one obtains that for \( 0 < \alpha < 1 \),

\[
\frac{1 - \alpha}{p} = \frac{1}{r}, \quad \text{and} \quad \frac{\alpha}{2} + \frac{1 - \alpha}{q} = \frac{1}{s},
\]

with \( q \geq 6 \). This gives us, for any \( 0 < \alpha < 1 \),

\[
\left( \frac{1}{p} + \frac{1}{q} \right) (1 - \alpha) \leq \left( \frac{1}{p} + \frac{1}{q} \right) (1 - \alpha) + \frac{\alpha}{12} \leq \frac{5}{12} (1 - \alpha),
\]

and thus

\[
\frac{1}{p} + \frac{1}{q} \leq \frac{5}{12},
\]

with \( q \geq 6 \).

Thanks to Lemma 2.3 again, as \( \varepsilon \) tends to zero, we have

\[
A \to 0.
\]

An argument similar to that given above in the control for \( A \) shows that the terms \( B, D, E_{21} \) in \( R_\varepsilon(t, x) \) here converge to zero, as \( \varepsilon \to 0 \), in particular,

\[
B - D - E_{21} \to 0.
\]

Note that \( \rho_t \) is bounded in \( L^p(0, T; L^{\frac{2q}{q+2}}(\Omega)) + L^2(0, T; L^2(\Omega)) \) and \( \rho \) is bounded in \( L^\infty(0, T; \Omega) \), so we conclude that \( E_{31}, E_{32} \) goes to zero as \( \varepsilon \) tends to zero. Similar argument can show that \( E \to 0 \) as \( \varepsilon \) goes to zero. Thus,

\[
R_\varepsilon \to 0
\]
as \( \varepsilon \to 0 \).

We are ready to pass to the limits in (2.10). Letting \( \varepsilon \) go and to zero, and using (2.11)–(2.13), what we have proved is that in the limit

\[
- \int_0^T \int_\Omega \psi_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho \gamma}{\gamma - 1} \right) \, dx \, dt + \int_0^T \int_\Omega \psi(t) \rho |D u|^2 \, dx \, dt = 0,
\]

for any test function, \( \psi \in \mathcal{D}(0, \infty) \).

The final step is to extend our result (2.17) for the test function \( \psi(t) \in \mathcal{D}(-1, \infty) \). To this end, it is necessary for us to have the continuity of \( \rho(t) \) and \( (\sqrt{\rho} u)(t) \) in the strong topology at \( t = 0 \). Adopting a similar argument to that of Vasseur and Yu [24], what we expected can be done.

Here we remark that \( \sqrt{\rho} u \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \). Using (1.6) and

\[
\rho_t = -\text{div}(\rho u),
\]

we have

\[
\rho_t \in L^2(0, T; H^{-1}(\Omega)).
\]
Note that, $\nabla \sqrt{\rho}$ is bounded in $L^\infty(0, T; L^2(\Omega))$ and $\rho$ is bounded away from infinity. Thus, the identity

$$\nabla \rho = 2 \sqrt{\rho} \nabla \sqrt{\rho}$$

gives us

$$\nabla \rho \in L^2(0, T; L^2(\Omega)). \tag{2.19}$$

From (2.18) and (2.19), one obtains

$$\rho \in C([0, T]; L^2(\Omega)), \tag{2.20}$$

thanks to Theorem 3 on page 287 of book [9].

Using

$$\sqrt{\rho}_t = -\text{div}(\sqrt{\rho} \mathbf{u}) + \frac{1}{2} \sqrt{\rho} \text{div} \mathbf{u},$$

we deduce

$$\sqrt{\rho}_t \in L^2(0, T; H^{-1}(\Omega)).$$

Because $\nabla \sqrt{\rho}$ is bounded in $L^\infty(0, T; L^2(\Omega))$, we have that

$$\sqrt{\rho} \in C([0, T]; L^2(\Omega)). \tag{2.21}$$

Meanwhile, we see

$$\text{ess lim sup}_{t \to 0} \int_\Omega |\sqrt{\rho} \mathbf{u} - \sqrt{\rho_0} \mathbf{u}_0|^2 \, dx$$

$$\leq \text{ess lim sup}_{t \to 0} \left( \int_\Omega \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx - \int_\Omega \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx \right)$$

$$+ \text{ess lim sup}_{t \to 0} \left( 2 \int_\Omega \sqrt{\rho_0} \mathbf{u}_0 (\sqrt{\rho_0} \mathbf{u}_0 - \sqrt{\rho} \mathbf{u}) \, dx + \int_\Omega \left( \frac{\rho_0^\gamma}{\gamma - 1} - \frac{\rho^\gamma}{\gamma - 1} \right) \right). \tag{2.22}$$

Using (1.4) and the convexity of $\rho \mapsto \rho^\gamma$, we have

$$\text{ess lim sup}_{t \to 0} \int_\Omega |\sqrt{\rho} \mathbf{u} - \sqrt{\rho_0} \mathbf{u}_0|^2 \, dx$$

$$\leq 2 \text{ess lim sup}_{t \to 0} \int_\Omega \sqrt{\rho_0} \mathbf{u}_0 (\sqrt{\rho_0} \mathbf{u}_0 - \sqrt{\rho} \mathbf{u}) \, dx$$

$$= W. \tag{2.23}$$

To show the continuity of $\sqrt{\rho} \mathbf{u}(t)$ in the strong topology at $t = 0$, we need $W = 0$. To this end, we introduce the time evolution of the integral averages

$$t \in (0, T) \mapsto \int_\Omega (\rho \mathbf{u})(t, x) \cdot \psi(x) \, dx,$$
which is defined by

\[
\frac{d}{dt} \int_{\Omega} (\rho u)(t, x) \cdot \psi(x) \, dx = \int_{\Omega} \rho u \otimes u : \nabla \psi \, dx + \int_{\Omega} \rho' \text{div} \psi \, dx + \int_{\Omega} \rho \mathbb{D}u \nabla \psi \, dx,
\]

(2.24)

where \( \psi(x) \in C_0^\infty(\Omega) \) is a test function. All estimates from (1.4), (1.6) and (1.10) imply that (2.24) is continuous function with respect to \( t \in [0, T] \). On the other hand, we have

\[
\rho u \in L^\infty(0, T; L^2(\Omega)),
\]

and hence

\[
\rho u \in C([0, T]; L^2_{\text{weak}}(\Omega)).
\]

(2.25)

We consider \( W \) as follows:

\[
W = 2 \text{ess lim sup} \int_{\Omega} \sqrt{\rho_0 u_0} \sqrt{\rho} (\sqrt{\rho_0 u_0} - \rho u) \, dx
\]

\[
\leq 2 \text{ess lim sup} \int_{\Omega} \sqrt{\rho_0 u_0} \sqrt{\rho} (\sqrt{\rho_0 u_0} - \rho_0 u_0) \, dx
\]

\[
+ 2 \text{ess lim sup} \int_{\Omega} \sqrt{\rho_0 u_0} \sqrt{\rho} (\rho_0 u_0 - \rho u) \, dx,
\]

(2.26)

where we used (1.6).

Using (2.21) and (2.25) in (2.26), one deduces \( W = 0 \) provided that \( \sqrt{\rho_0 u_0} \in L^4(\Omega) \). Thus, we have

\[
\text{ess lim sup} \int_{\Omega} |\sqrt{\rho} u - \sqrt{\rho_0} u_0|^2 \, dx = 0,
\]

which gives us

\[
\sqrt{\rho} u \in C([0, T]; L^2(\Omega)).
\]

(2.27)

By (2.20) and (2.27), we get

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho'}{\gamma - 1} \right) \, dx \, dt = \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{\rho_0'}{\gamma - 1} \right) \, dx.
\]

(2.28)

Choosing the following test function for (2.17):

\[
\psi_\tau(t) = \psi(t) \quad \text{for} \ t \geq \tau + \frac{1}{K}, \quad \psi_\tau(t)
\]

\[
= \frac{t}{\tau} \quad \text{for} \ t \leq \tau, \quad \psi_\tau \text{ is a } C^1 \text{ smooth function,}
\]
we then get

\[- \int_{\tau + \frac{1}{K}}^{T} \int_{\Omega} \psi_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx \, dt + \int_{0}^{T} \int_{\Omega} \psi_t (t) \rho |Du|^2 \, dx \, dt \]

\[- \int_{\tau}^{\tau + \frac{1}{K}} \int_{\Omega} (\psi_t)_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx \, dt \]

\[= \frac{1}{\tau} \int_{0}^{\tau} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx \, dt. \quad (2.29)\]

Note that,

\[\left| \int_{\tau}^{\tau + \frac{1}{K}} \int_{\Omega} (\psi_t)_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx \, dt \right| \leq \frac{C}{K} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx \to 0\]

as \( K \) goes to large. Letting \( K \to \infty \), from (2.29) we derive

\[- \int_{\tau + \frac{1}{K}}^{T} \int_{\Omega} \psi_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx \, dt + \int_{\tau}^{T} \int_{\Omega} \psi_t (t) \rho |Du|^2 \, dx \, dt \]

\[= \frac{1}{\tau} \int_{0}^{\tau} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx \, dt. \quad (2.30)\]

By (2.28), passing to the limit as \( \tau \to 0 \) in (2.30), one obtains

\[- \int_{0}^{T} \int_{\Omega} \psi_t \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx \, dt + \int_{0}^{T} \int_{\Omega} \psi (t) \rho |Du|^2 \, dx \, dt \]

\[= \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx. \quad (2.31)\]

Taking

\[\psi (t) = \begin{cases} 
0 & \text{if } t \leq \tilde{t} - \frac{\epsilon}{2} \\
\frac{1}{2} + \frac{t - \tilde{t}}{\epsilon} & \text{if } \tilde{t} - \frac{\epsilon}{2} \leq t \leq \tilde{t} + \frac{\epsilon}{2} \\
1 & \text{if } t \geq \tilde{t} + \frac{\epsilon}{2},
\end{cases} \quad (2.32)\]

(2.31) then gives, for every \( \tilde{t} \geq \frac{\epsilon}{2}, \)

\[\frac{1}{\epsilon} \int_{\tilde{t} - \frac{\epsilon}{2}}^{\tilde{t} + \frac{\epsilon}{2}} \left( \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx \right) \, dt \]

\[+ \int_{\tilde{t} - \frac{\epsilon}{2}}^{\tilde{t} + \frac{\epsilon}{2}} \int_{\Omega} \left( \frac{1}{2} + \frac{t - \tilde{t}}{\epsilon} \right) \rho |Du|^2 \, dx \, dt \]

\[+ \int_{\tilde{t} + \frac{\epsilon}{2}}^{T} \int_{\Omega} \rho |Du|^2 \, dx \, dt = \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx. \quad (2.33)\]
The second term on left hand side in (2.33) is controlled as follows:

$$\int_{t-\frac{\epsilon}{2}}^{t+\frac{\epsilon}{2}} \int_\Omega \left( \frac{1}{2} + \frac{t - \tilde{t}}{\epsilon} \right) \rho |D\mathbf{u}|^2 \, dx \, dt \leq \int_{t-\frac{\epsilon}{2}}^{t+\frac{\epsilon}{2}} \int_\Omega \rho |D\mathbf{u}|^2 \, dx \, dt \to 0,$$

as $\epsilon \to 0$. Thanks to the Lebesgue point Theorem, (2.33) gives us

$$\int_\Omega \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) \, dx + \int_0^T \int_\Omega \rho |D\mathbf{u}|^2 \, dx \, dt = \int_\Omega \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx.$$

This ends our proof of Theorem 1.1.

### 2.2. Proof of Theorem 1.2

Now, let us address the proof of Theorem 1.2. We can modify the above proof slightly to show Theorem 1.2. The main advantage of condition $0 < \rho \leq \rho(t, x)$ is to have $\nabla \mathbf{u} \in L^2(0, T; L^2(\Omega))$ in Theorem 1.1. Thus, we can drop this restriction for the constant viscosities case.

The one difference is to show $W = 0$ in (2.26). In fact, we have

$$W = 2 \text{ess lim sup } \int_\Omega \mathbf{u}_0 (\rho_0 \mathbf{u}_0 - \sqrt{\rho_0} \rho \mathbf{u}) \, dx$$

$$\leq 2 \text{ess lim sup } \int_\Omega \mathbf{u}_0 (\rho_0 \mathbf{u}_0 - \rho \mathbf{u}) \, dx$$

$$+ 2 \text{ess lim sup } \int_\Omega \mathbf{u}_0 (\sqrt{\rho} - \sqrt{\rho_0}) \sqrt{\rho} \mathbf{u} \, dx.$$

Applying a similar argument, we can show

$$\rho \mathbf{u} \in C([0, T]; L^2_{\text{weak}}(\Omega)), \quad \sqrt{\rho} \in C([0, T]; L^2(\Omega)).$$

To show $W = 0$, we need $\mathbf{u}_0 \in L^k$ where $\frac{1}{k} + \frac{1}{q} \leq \frac{1}{2}$. We have also to mention here that this is true for the degenerate viscosity.

After modifying the above argument, we are able to follow the same one to show Theorem 1.2.

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