THE DIFFERENTIAL ENERGY DISTRIBUTION OF THE UNIVERSAL DENSITY PROFILE OF DARK HALOS

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ABSTRACT

We study the differential energy distribution of dark matter halos by carrying out a cosmological N-body simulation. We give an analytical formula for the differential energy distribution of dark matter in the halos obtained by the numerical simulation. From the analytical formula we reconstruct the density profile described by the Navarro, Frenk, & White (NFW) profile. The NFW profile is consistent with the analytical formula of our fractional mass distribution. We find a parameter in our analytical formula for the differential energy distribution that is related to the slope of the inner cusp of the dark halo. We obtain the distribution function for the NFW profile, which has a sharp cutoff at the high binding energy. We discuss physical reasons for the form of the analytical formula.

1. INTRODUCTION

It is interesting to understand how the density profiles of galaxies and clusters of galaxies have formed. Navarro, Frenk, & White (1995, 1996, 1997; NFW) have shown in their N-body simulations of cold dark matter (CDM) in the standard biased CDM and four power-law spectra with indices $n = 0, -0.5, \text{ and } -1$, open CDM ($\Omega_0 = 0.1$) with power-law spectra ($n = 0 \text{ and } -1$), and $\Lambda$CDM cosmology, that density profiles of dark halos have a universal profile described as

$$\rho(r) \propto \frac{1}{(r/r_s)(1 + r/r_s)^2}. \quad (1)$$

Several investigators have shown that the formula provides a good fit to their numerical results (Cole & Lacey 1996; Tormen, Bouchet, & White 1997; Huss, Jain, & Steinmetz 1999; Thomas et al. 1998). However, some other simulations (Fukushige & Makino 1997; Moore et al. 1998; Okamoto & Habe 1999, 2000) indicate that their density profiles have steeper inner cusps than the NFW profile. Jing (2000) gives his numerical results that steeper cusps of density profile are found in recent mergers and in dark halos with substructures by a large set of high-resolution cosmological simulations.

Subramanian, Cen, & Ostriker (2000) discussed the general theoretical grounds of the density profiles. They discussed that there is a possible connection between the slope of the inner region of a dark halo and the formation epoch, and proposed that there is a possible relation to cosmological parameters.

However, physical reasons for the NFW profile have not been explored, although they are very important to understanding the formation of galaxies and clusters of galaxies.

We study the differential energy distribution of the NFW profile, $dM/de$, where $e$ is the binding energy. The differential energy distribution was studied to understand the virialization process of the $N$-body system (van Albada 1982). Binney (1982) shows that $dM/de$ of the elliptical galaxies is approximated by the Boltzmann factor if the surface bright-

ness of elliptical galaxies obeys the de Vaucouleurs' $r^{1/4}$ law and their mass-to-light ratio is constant. It is interesting to study the phase-space structure of virialized objects with the NFW profile, since the differential energy distribution may give insight into the relaxation process (Binney & Tremain 1987).

In this paper we calculate the formation of clusters of galaxies by an N-body simulation, and we study the differential energy distribution of the clusters in our numerical results. We find an analytical formula for the fractional mass distribution that is defined by the differential distribution divided by the mass of the object, fitted by an analytical formula. We show that the NFW profile is consistent with our analytical formula for the fractional mass distribution.

Using the iteration method, we construct the density profile from the analytical formula to show how the slope of the cusp of the density changes with parameters in our analytical formula.

In the next section, we illustrate our numerical method. In § 3, we present our numerical results. We show the iteration method and give results from this method in § 4, and we summarize and discuss our results in § 5.

2. NUMERICAL SIMULATION

We use the Hofman & Ribak (1991) procedure to set initial conditions in order to have a massive dark halo near the center of a simulation box. The cosmology is the standard cold dark matter (SCDM) model (e.g., Davis et al. 1985; $\Omega = 1$, $\sigma_8 = 0.67$, $H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and $h = 0.5$).

Numerical simulations are carried out using theGRAPE-PESPH code. GRAPE is a special-purpose hardware for calculating gravitation between N-body particles (Sugimoto et al. 1990). We combined smoothed particle hydromatams (SPH; Monaghan 1992) with GRAPE. We select massive halos for which mass is as large as that of clusters of galaxies, and then calculate their density profile and the fractional mass distribution.

The masses of a CDM and a SPH particle are $5.89 \times 10^{11} M_\odot$ and $3.10 \times 10^{10} M_\odot$, respectively. The gravitational softening length is 100 kpc. The number of
both CDM and SPH particles is 29,855. The size of the simulation box is 80 Mpc.

3. NUMERICAL RESULTS

3.1. Density Distribution

Before we show our numerical results, we summarize the NFW profile. NFW proposed that the profile of the dark halo of cosmological object in their numerical results as

$$\rho(r) = \frac{\delta_c}{(r/r_s)(1 + r/r_c)^2},$$  \hspace{1cm} (2)

where $r_c = r_{200}/c$, $c$ is a dimensionless parameter, and $\rho_{cr} = (3H_0^2)/(8\pi G)$ is the critical density of the universe. Since the mass of the halo is $M(r_{200}) = 200 \times 4\pi/3\rho_{cr}r_{200}^3$, there is a relation between $\delta_c$ and $c$, as

$$\delta_c = \frac{200}{3} \left[ \ln(1 + c) - c/[1 + c] \right],$$  \hspace{1cm} (3)

where $M(r_{200})$ is the mass for which the averaged density inside $r_{200}$ is 200 times $\rho_{cr}$.

This density profile has a cusp in the inner region, $\rho(r) \propto r^{-1}$; $\rho(r) \propto r^{-3}$ in the outer region. NFW97 showed that the parameter $c$ decreases with halo mass.

Table 1 shows the physical values of our simulated clusters. The units of mass and X-ray temperature are $10^{15} M_\odot$ and $10^8 K$, respectively.

Figure 1 shows a density profile of our simulated typical rich cluster, CLc, and the NFW profile with $c = 4.4$. Solid line shows our numerical result and dashed line is given by eq. (5), with $q = 0.667$ and $\epsilon_0 = 1.47$. Figure 2 shows the fractional mass distribution of CLc.

In Table 2 we give $N_0$, $q$, and $\epsilon_0$ of our numerical results for rich clusters.

We should confirm the consistency between $N(\epsilon)$ given by equation (5) and the NFW profile. We give a fractional mass distribution from the NFW profile as follows for the comparison.

We assume that the phase-space distribution function $f(x, v)$ depends on $\epsilon$. At a radius $r$, the velocity of a dark

![Fig. 1](image1.png)

**Fig. 1**—Density profile of one of simulated clusters, CLc, and the NFW profile with $c = 4.4$. Solid line shows our numerical result and dashed line the NFW profile. Radial distance is normalized by $r_{200}$. Our simulated clusters agree well with the NFW formula between the gravitational softening length and $r_{200}$.

![Fig. 2](image2.png)

**Fig. 2**—Fractional mass distribution of our simulated rich cluster, CLc. Solid line shows our numerical result, and dashed line is given by eq. (5) with $q = 0.667$ and $\epsilon_0 = 1.47$.

| Mass  | $T$ | $c$ |
|-------|-----|-----|
| $10^{15} M_\odot$ | $10^8 K$ |
| CLa   | 3.05 | 0.953 | 2.50 |
| CLb   | 2.80 | 0.890 | 2.72 |
| CLc   | 3.36 | 1.29  | 4.41 |
| CLc   | 1.60 | 0.796 | 2.31 |
| CLd   | 0.831| 0.540 | 2.67 |
| CLe   | 3.71 | 1.57  | 6.34 |
| CLf   | 2.31 | 0.654 | 2.75 |

**TABLE 1**

**THE PHYSICAL VALUES OF OUR SIMULATED CLUSTERS**

Note:—Each value is calculated at redshift $z = 0$, except for CLc.
matter particle of binding energy $\varepsilon$ is $v = \sqrt{2(\Psi - \varepsilon)}^{1/2}$. The density profile can be given as (Binney & Tremaine 1987)

$$\rho(r) = 4\pi \int_{\Psi(r_\varepsilon)}^{\Psi(r)} f(\varepsilon)\sqrt{2(\Psi - \varepsilon)}^{1/2} d\varepsilon,$$

where $r_\varepsilon$ is the edge of the dark halo. From this equation, we can give $f(\varepsilon)$ as

$$f(\varepsilon) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\varepsilon} \int_{\varepsilon_{\min}}^{\varepsilon} \frac{d\rho/d\Psi}{(\varepsilon - \Psi)^{1/2}} d\Psi,$$

where $\varepsilon_{\min} = \Psi(r_\varepsilon)$.

Table 2

| $N_0$ | $q$ | $\varepsilon_0$ |
|-------|-----|-----------------|
| CLa   | 0.309 | 0.612 | 1.27 |
| CLb   | 0.253 | 0.680 | 1.21 |
| CLc ($z = 0$) | 0.190 | 0.667 | 1.47 |
| CLc ($z = 0.25$) | 0.230 | 0.606 | 1.05 |
| CLc ($z = 0.5$) | 0.305 | 0.600 | 0.672 |
| CLd   | 0.276 | 0.699 | 1.26 |
| CLe   | 0.117 | 0.503 | 2.03 |
| CLf   | 0.258 | 0.580 | 1.41 |

Note—Each value is calculated at redshift $z = 0$, except for CLc.

Equation (6) gives mass $M$ as

$$M(r) = 16\pi^2 \int_{0}^{r} r^2 dr \int_{\Psi(r_\varepsilon)}^{\Psi(r)} f(\varepsilon)\sqrt{2(\Psi - \varepsilon)}^{1/2} d\varepsilon.$$  

From equation (8), the differential energy distribution is

$$\frac{dM(\varepsilon)}{d\varepsilon} = f(\varepsilon)g(\varepsilon),$$

where

$$g(\varepsilon) = 16\pi^2 \int_{0}^{r_{\varepsilon}(\varepsilon)} [2(\Psi - \varepsilon)]^{1/2}r^2 dr,$$

and $r_{\varepsilon}(\varepsilon)$ is the maximum radius that can reached by a particle of binding energy $\varepsilon$.

If we assume that the density profile is the NFW profile, we get $dM/d\varepsilon$ from equations (7), (9), and (10) for the NFW profile.

Figure 3 shows the fractional mass distribution of NFW obtained in this way and $N(\varepsilon)$ given by equation (5). We show that $N(\varepsilon)$ given by equation (5) is consistent with the NFW profile. We note that the cutoff at high binding energy seen in Figure 2 is not an artifact due to limitations of our numerical resolutions.

Figure 4 shows the distribution function of the NFW profile obtained by the above method. Since lower binding energy particles evaporate from the dark halo, their fraction become small. This form of function is similar to the distribution function of the King model, $f_k(\varepsilon) \propto (e^{\varepsilon_{\min} - 1})$ (Binney 1982). On the other hand, there is a peak and sharp cutoff at high binding energy. This part corresponds to the central cusp. Although the distribution function of the NFW profile has a cutoff at high energy, the distribution function of the King model does not have such a cutoff. This is an important difference between them.

4. The Fractional Mass Distribution, Density Profile, and Distribution Function

We find that the NFW profile satisfies the fractional mass distribution given by equation (5) with $q = 0.6-0.7$. We study how the density profile changes when we change the parameters $q$ and $\varepsilon_0$ in equation (5). In this study, we use an iteration method as shown in the next subsections.

4.1. The Iteration Method

Binney (1982; see also Binney & Tremaine 1987) studied the phase-space structure of galaxies for which the surface brightness is the de Vaucouleurs’ $r^{1/4}$ law. We apply his method to our study of the phase-space structure of a dark halo with the NFW profile. We obtain the density profile and the phase-space distribution that are consistent with equation (5) using the iteration method. The procedure is as follows.

We assume that the dark halo is spherically symmetric. From equation (5), we assume $dM(\varepsilon)/d\varepsilon$ given by the equation

$$\frac{dM(\varepsilon)}{d\varepsilon} = M_0 \left[ 1 - (1 - q) \frac{\varepsilon}{\varepsilon_0} \right]^{q/(1-q)},$$

where $M_0 = 1.0$. We use a trial function of a density profile to calculate a gravitational potential in $g_0(\varepsilon)$,

$$\rho_0(x) = \frac{\rho_c}{[1 + (x/x_0)^2]^{3/2}},$$
where \( x_i = r_i/r_{200} = 0.1 \). This trial function is similar to the \( \beta \)-model of the X-ray surface brightness of clusters of galaxies. We obtain a relative potential from this trial density function. The relative potential is written as

\[
\Psi(x) = 4\pi \int_0^x x \rho_0(x') x'^2 dx' + \int_x^{x_0} x \rho_0(x') x' dx' + \Phi_{x_0},
\]

(13)

where \( x = r/r_{200} \), and \( x_0 = 10 \) is assumed; \( \Phi_{x_0} \) is chosen to be \( f > 0 \) for \( \epsilon > 0 \) and \( f = 0 \) for \( \epsilon \leq 0 \). We obtain \( g_0(\epsilon) \) by equation (10) for \( \rho_0 \). From equation (9),

\[
f_0(\epsilon) = \frac{dM(\epsilon)/d\epsilon}{g_0(\epsilon)}.
\]

(14)

Next, we calculate the new density as

\[
\rho_{i+1}(x) = (1 - x)\rho_i(x) + 2\pi \int_{\Psi_i(\epsilon)}^{\Psi_{i+1}(\epsilon)} f(\epsilon)[2(\Psi_i - \epsilon)]^{1/2} d\epsilon,
\]

(15)

where \( i \) is an iterative index and \( x \) is an arbitrary constant of convergence. Here we assume \( x = 0.5 \). Next, we again follow the step of equations (13)–(15) using \( \rho_1 \) instead of \( \rho_0 \), and we obtain \( \rho_2 \). We continue until \( \rho_i \) converges. In this way, we obtain \( \rho \) and \( f(\epsilon) \) that are consistent with equation (11).

### 4.2. The Density Profile and the Distribution Function

Using the Binney’s iteration method, we reconstruct the mass density profile from equation (11).

We show our result for \( q = 0.67 \) and \( \epsilon = 1.4 \) in Figure 5. A solid line shows the iteration method and a dashed line the NFW profile. We confirm that \( dM(\epsilon)/d\epsilon \) well characterizes the NFW profile.

In Figure 6, we show density profiles for different \( q \) but \( \epsilon = 1.4 \). Smaller \( q \) (e.g., \( q = 0.5 \)) result in a shallower core in the inner region. On the other hand, larger \( q \) (\( q > 0.67 \)) make a cusp steeper than in the NFW profile; the density profile approaches \( \rho \propto r^{-2} \) in the inner part for \( q \rightarrow 1 \).

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**Figure 5.** Density profile constructed by the iteration method, assuming eq. (11). Solid line shows our result for \( q = 0.67 \) and \( \epsilon_0 = 1.4 \), and dashed line shows the NFW profile of \( \epsilon = 6.59 \). Radial distance is normalized by \( r_{200} \).

**Figure 6.** Density profiles for various \( q \). Dotted-dashed, dotted, short-dashed, dashed, and solid lines are for \( q = 0.25,0.5,0.67,0.75, \) and 1.0, respectively. Density profile with smaller \( q \) has a flat core. On the other hand, one approaches \( \rho \propto r^{-2} \) in \( r < 0.1 \) for \( q \rightarrow 1 \).

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*For various values of \( \epsilon_0 \), the density profiles are similar to the NFW for \( q = 0.6 \), as shown in Figure 7. The absolute value of the density depends on \( \epsilon_0 \). Therefore, the slope of the cusp depends only on \( q \), not \( \epsilon_0 \).*

**Figure 7.** Density profiles for various \( \epsilon \). Dotted, short-dashed, dashed, and solid lines are for \( \epsilon_0 = 0.5,1.0,1.5, \) and 2.0, respectively. Density profiles are self-similar. However, their normalizations are different.

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**Figure 8.** We show \( f(\epsilon) \) for various \( q \). Dotted-dashed, dotted, short-dashed, dashed, and solid lines are for \( q = 0.25,0.5,0.67,0.75, \) and 1.0, respectively.
We analyze the universal density profile of dark halo proposed by NFW and its differential energy distribution. Our main results are summarized as follows:

1. We study the fractional mass function, \( N(\epsilon) \), for a dark halo obtained by our numerical simulation and find its analytical formula, which is given by equation (5).
2. We show that the NFW profile is given by equation (11).
3. We show that the slope of the cusp in the density profile changes with the value of the parameter \( q \) in the analytical formula.

We can see that \( N(\epsilon) \) shows the statistical property of the NFW profile. If the NFW profile is universal, \( q = 0.6 \sim 0.7 \) in equation (5). Different \( q \) will make the slope of a cusp different. Since \( q \) plays an important role, we should make clear what physical process determines \( q \). Recent high-resolution numerical simulations (Okamoto & Habe 1999, 2000) show a steeper cusp, \( \rho \propto r^{-1.5} \), than the NFW profile. This profile corresponds to \( q = 0.75 \sim 0.8 \). The isothermal profile, \( \rho \propto r^{-2} \), corresponds to \( q = 1 \).

We study \( f(\epsilon) \) for the NFW profile. The formula for this is not isothermal nor the King formula, \( f_K \propto e^{-\epsilon^2} - 1 \). The \( f(\epsilon) \) for the NFW profile have an energy cutoff at the high end of \( \epsilon \). We should study the reason why \( f(\epsilon) \) has such a form. Lynden-Bell (1967) studied the distribution function \( f(\epsilon) \) of a virialized system. Maximizing the Boltzmann entropy of the system, the resulting distribution is an isothermal profile, \( \rho \propto r^{-2} \). In this case the system has infinite extend and infinite mass. This is not realistic for astronomical objects. Cosmological simulations have shown that galaxies and clusters of galaxies formed in these simulations have a more rapid radial decline than isothermal in the outer part.

We note that the form of the distribution function relates to the cusp profile. For \( q < 1 \), there is a maximum binding energy \( \epsilon_{\text{max}} \) and a sharp peak in \( f(\epsilon) \) at \( \epsilon_{\text{max}} \). Since there is \( \epsilon_{\text{max}} \), the phase-space volume occupied by dark matter is more limited than in the isothermal profile. This may be the reason why the cusp profile differs from \( \rho \propto r^{-2} \). For \( q < 1 \), the distribution function \( f(\epsilon) \) changes with \( \epsilon \) values as shown in Figure 8.

We note that the form of equations (5) and (11) are similar to the Tsallis escort distribution,

\[
\frac{P(E, T)}{P(0, T)} = \left[ 1 - (1 - q) \frac{E}{T} \right]^{q/(1 - q)},
\]

where \( T \) is temperature parameter and \( q \) is the entropic index (Tsallis, Mendes, & Plastino 1998). Tsallis’ nonextensive generalized statistics (Tsallis 1988) are given attention in the area of statistics of multifractal systems. In a nonextensive system (long-range microscopy memory, long-range forces, fractal spacetime) the following generalized entropy has been proposed:

\[
S_q = k \frac{1 - \sum p_i^q}{q - 1} \left( \sum_i p_i = 1, q \in \mathbb{R} \right),
\]

where \( k \) is a positive constant. Optimization of \( S_q \) yields for the canonical ensemble

\[
\begin{align*}
 p_i &= Z_q^{-1} \left[ 1 - (1 - q) \epsilon_i / T^q \right]^{1/(1 - q)}, \\
 Z_q &= \sum_i \left[ 1 - (1 - q) \epsilon_i / T^q \right]^{1/(1 - q)}.
\end{align*}
\]

and when \( q \to 1 \) the Boltzmann-Gibbs result is recovered. In these statistics, an expected value of any physical variable is given by the Tsallis escort distribution,

\[
\langle A \rangle_q = \frac{\sum p_i^q A_i}{\sum p_i^q},
\]

where \( \{ A_i \} \) are the eigenvalues of an arbitrary observable \( A \).

The escort distribution, \( P_l = P_l / \sum P_l \), is similar to the form of equations (5) and (11).

It is expected that long-range interaction will make a system nonextensive. Our case may be one of these. Our results suggests that the differential energy distribution of collisionless particles is described by the escort distribution of the Tsallis statistics. This may indicate that the energetic process of collisionless particles must be a stochastic process.

In the Tsallis escort distribution function, there is a maximum value of \( \epsilon \) for \( p_l > 0 \) for \( 0 < q < 1 \). This case is called superextensive. The shallower cusp profile in the NFW profile shows the superextensive property of the dark matter distribution.

We show that the NFW profile corresponds to \( q = 0.6 \sim 0.7 \). Lavagno et al. (1998) have recently shown that a fraction of the peculiar velocity of cluster of galaxies (Bahcall & Oh 1996) is well explained by the Tsallis escort integral,

\[
P(>v) = \int_0^{\epsilon_{\text{max}}} \left[ 1 - (1 - q)v/v_{\text{esc}} \right]^{q/(1 - q)} dv
\]

They obtained \( q = 0.23 \) to fit the fraction of the peculiar velocity of clusters of galaxies, which is smaller than in our
case. There is a conjecture that the $q$ of a system approaches unity when the system proceeds to relaxation (Tsallis 1999). Our results in which $q = 0.6–0.7$ are consistent with this conjecture, since the dark matter distribution in a cluster of galaxies is a more relaxed system than the large-scale motion of clusters of galaxies. The value of $q$ must be related to the degree of relaxation of collisionless particles. Here $q = 1$ corresponds to an isothermal distribution in the Tsallis statistics. Gravitational systems such as clusters of galaxies with the universal profile may be nonextensive because they formed recently.

We should study the reason why the dark halo has a value of $q = 0.6–0.7$ in hierarchical clustering scenarios. It would be interesting to study the differential energy distribution of a self-gravitational system formed in circumstances without hierarchical clustering in order to make clear what mechanism determines the $q$ of the gravitational system.

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