NON–FLAT CONFORMAL BLOW–UP PROFILES FOR THE 1D CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. For the critical one-dimensional nonlinear Schrödinger equation, we construct blow-up solutions that concentrate a soliton at the origin at the conformal blow-up rate, with a non-flat blow-up profile. More precisely, we obtain a blow-up profile that equals $|x| + i\kappa x^2$ near the origin, where $\kappa$ is a universal real constant. Such profile differs from the flat profiles obtained in the same context by Bourgain and Wang [1].

1. INTRODUCTION

We consider the mass critical Schrödinger equation in one dimension

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (1.1)$$

Recall that the mass $M(u)$, the momentum $J(u)$ and the energy $E(u)$ of a solution $u$ of (1.1) are formally conserved

$$M(u) = \int_{\mathbb{R}} |u|^2 \, dx,$$

$$J(u) = 3 \int_{\mathbb{R}} (\partial_x u)\overline{u} \, dx,$$

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}} |u|^6 \, dx.$$

The Cauchy problem for (1.1) is locally well-posed in $L^2(\mathbb{R})$: for any $u_0 \in L^2(\mathbb{R})$, there exists a unique maximal solution $u$ of (1.1) in $C([0, T), L^2(\mathbb{R}))$ satisfying $u(0, \cdot) = u_0$ (see e.g. [2] for details). If in addition $u_0 \in H^1(\mathbb{R})$, then the solution belongs to $C([0, T), H^1(\mathbb{R}))$; moreover, if the maximal time of existence $T > 0$ is finite then

$$\liminf_{t \uparrow T} (T - t)^{\frac{1}{2}} \|\partial_x u(t)\|_{L^2} > 0.$$

Let $Q$ be the positive, even, $H^1$ solution of $-Q'' + Q = Q^5$ on $\mathbb{R}$ given by

$$Q(x) = 3^4 (\cosh(2x))^{-\frac{7}{2}}. \quad (1.2)$$

It is well-known [23] that if the $H^1$ solution $u$ satisfies $\|u(t)\|_{L^2} < \|Q\|_{L^2}$, then it is global and bounded in $H^1$.

The pseudo-conformal symmetry of the equation implies that if $u(t, x)$ is a solution of (1.1) then

$$u(t, x) = \frac{1}{|t|^\frac{7}{2}} u \left( -\frac{1}{t} \frac{x}{|t|} \right) e^{i\frac{|x|^2}{8t}}.$$
is also a solution of (1.1). By this symmetry, the solitary wave solution \( u(t, x) = e^{it}Q(x) \) yields the explicit blow-up solution

\[
S(t, x) = \frac{1}{|t|^{\frac{1}{2}}} Q \left( \frac{x}{t} \right) e^{-\frac{|x|^2}{4t} + it} \tag{1.3}
\]

defined on \((−∞, 0) \times \mathbb{R}\). This solution is called the minimal mass solution since

\[
\| S(t) \|_{L^2} = \| Q \|_{L^2}, \quad \lim_{t \uparrow 0} \| \partial_x S(t) \|_{L^2} = +\infty.
\]

Moreover, for \( t < 0 \) close to 0,

\[
\| \partial_x S(t) \|_{L^2} \sim \frac{C}{|t|^{\frac{1}{2}}}
\]

and \(|t|^{-\frac{1}{2}}\) is thus called the conformal blow-up rate. In addition, it was proved in [15] that \( S \) is the unique (up to the invariances of (1.1)) blow-up solution with the minimal mass.

The general blow-up problem for (1.1) is still largely open, but a lot of information is known for blow-up solutions close to the ground state \( Q \), more precisely under the following assumption on the mass of the initial data

\[
\int Q^2 < \int u_0^2 < \int Q^2 + \delta, \tag{1.4}
\]

where \( \delta > 0 \) is small. We recapitulate the main known results, in the context of the one-dimensional space variable with even symmetry on the initial data. Suppose that \( u_0 \in H^1(\mathbb{R}) \) is even, satisfies (1.4) and that the corresponding solution \( u \) blows up in finite time \( 0 < T < +\infty \), then the following properties hold.

**Orbital stability** [23]: There exist \( \gamma(t) \) and \( \lambda(t) \) such that

\[
\sup_{[0, T]} \| e^{-i\gamma(t)} \lambda^{\frac{1}{2}}(t)u(t, \lambda(t)x) - Q \|_{H^1} \text{ is small as } \delta \text{ is small.}
\]

**Asymptotic stability** [16] [17]: There exist sequences \( (\gamma_n) \), \( (\lambda_n) \) and \( (t_n) \) with \( t_n \uparrow T \), such that

\[
\lim_{n \to +\infty} e^{-i\gamma_n} \lambda_n^{\frac{1}{2}} u(t_n, \lambda_n x) \to Q \text{ in } \dot{H}^1 \text{ as } n \to +\infty.
\]

**Dichotomy and gap for the blow-up rate** [20]: The solution \( u \)

- either blows up with the loglog rate

\[
\lim_{t \to T} \frac{T - t}{\log|\log(T - t)|} \| \partial_x u(t) \|_{L^2} = \frac{\| Q' \|_{L^2}}{\sqrt{2\pi}}, \tag{1.5}
\]

- or blows up faster than the conformal rate

\[
\liminf_{t \to T} (T - t) \| \partial_x u(t) \|_{L^2} > 0. \tag{1.6}
\]

**Blow-up profile** [16]: There exist \( \gamma(t) \), \( \lambda(t) \) and \( r_* \in L^2(\mathbb{R}) \) such that

\[
u(t) - e^{i\gamma(t)} \lambda_*^{\frac{1}{2}}(t)Q \left( \frac{x}{\lambda(t)} \right) \to r_* \text{ in } L^2(\mathbb{R}) \text{ as } t \uparrow T.
\]

Moreover,

\[
(r_* \in H^1(\mathbb{R}) \text{ and } r_*(0) = 0) \iff (1.6) \text{ holds} \tag{1.7}
\]
Generic log-log blow-up rate \([17]\): The set of initial data in \(H^1\) satisfying (1.3) and such that the solution blows up with the loglog blow-up rate (1.5) is open in \(H^1\). It contains all the initial data with negative energy such that (1.4) holds.

Existence of non-generic blow-up \([1, 9, 18]\): For \(A > 0\) large, if \(r_* \in H^A(\mathbb{R})\) satisfies
\[
(1 + |x|^2)^A r_* \in L^2(\mathbb{R}) \quad \text{and} \quad r_*^{(k)}(0) = 0 \quad \text{for any} \quad 0 \leq k \leq A, \quad (1.8)
\]
then, there exist \(\tau > 0\) and a solution \(u\) of (1.1) on \([-\tau, 0)\) such that
\[
\lim_{t \uparrow 0} u(t) - S(t) = r_* \quad \text{in} \quad H^1(\mathbb{R}).
\]

The minimal mass solution \([18]\) and the non-generic blow-up solutions constructed in \([1]\) have the conformal blow-up rate \(|\|\partial_x u(t)\|_{L^2} \sim C(T - t)^{-1}\) for \(t \sim T\). The existence of solutions blowing up strictly faster than the conformal rate is an open problem, which is equivalent, by the pseudo–conformal symmetry, to the existence of global solutions blowing up in infinite time. The only known example \([18]\) contains at least two bubbles, which means that (1.4) is not satisfied.

The log-log rate is physically more relevant than the conformal rate since it was proved to be stable under the assumption (1.4) (the set of blow-up solutions exhibited in \([17]\) is open in \(H^1\)). However, it is interesting mathematically to further investigate the conformal blow-up case. For example, the conformal blow-up rate is seen as a frontier separating solutions with the loglog rate and scattering, as discussed in \([1, 9, 18]\).

In this paper, we focus on the blow-up profile \(r_*\) as defined in \([10]\), in the conformal blow-up case.

**Theorem 1.** There exist \(\kappa \in \mathbb{R}\), \(\delta_0 > 0\) and \(C > 0\) with the following property. For any \(\delta \in (0, \delta_0)\), there exist \(\tau > 0\) and a solution \(u \in C([-\tau, 0), H^1(\mathbb{R}))\) of (1.1) which blows up at time 0 with the asymptotics
\[
\lim_{t \uparrow 0} |t| \|\partial_x u(t)\|_{L^2} = \|Q\|_{L^2}, \quad (1.9)
\]
\[
\lim_{t \uparrow 0} \{u(t) - S(t)\} = \lim_{t \uparrow 0} \left\{u(t) - \frac{e^t}{|t|^\frac{1}{2}} Q \left(\frac{x}{t}\right)\right\} = r_* \quad \text{in} \quad L^2(\mathbb{R}), \quad (1.10)
\]
where the function \(r_* \in H^1(\mathbb{R})\) satisfies \(\|r_*\|_{L^2} \leq C\delta^\frac{1}{2}\) and
\[
r_* (x) = |x| + i\kappa x^2 \quad \text{for all} \quad x \in (-\delta, \delta). \quad (1.11)
\]

**Remark 1.1.** This example illustrates the fact that a blow-up profile in the conformal case need not be flat at a blow-up point as one could have expected from previous constructions of conformal blow-up solutions in \([1, 9, 18]\); see \([18]\).

**Remark 1.2.** The constant \(\kappa\) is defined in \([2, 8]\); see also Remark 2.1.

**Remark 1.3.** We expect that the proof of Theorem 1 can be extended to more general blow-up profiles of the form
\[
r_* (x) = z_1 (|x| + i\kappa x^2) + z_2 |x|^3 + w_*(x) \quad (1.12)
\]
in the neighborhood of 0 where \(z_1, z_2 \in \mathbb{C}\) and \(w_*\) is a term of order \(x^4\) at \(x = 0\), with suitable regularity properties. We do not pursue this issue here for the sake of simplicity.
It is an open problem to determine all possible blow-up profiles, in terms of regularity and behavior at the blow-up point. On the one hand, in the one-dimensional case, from [16], the condition $r^*_0(0) = 0$ is necessary for $r^*_s$ to be a blow-up profile (see (1.7)). On the other hand, we do not know whether the implicit relation $r''_s(0) = 2i\kappa r'_s(0)$ in (1.12) is necessary or is only a technical assumption due to our approach.

Remark 1.4. The question of the blow-up rate and the blow-up profile has been studied for other nonlinear dispersive or wave PDEs; see for example the following articles: [3, 4, 5, 6, 7, 8, 10, 11, 12, 21]. Analogies between the present work and [12], devoted to the critical generalized Korteweg-de Vries equation, are discussed in the sketch of the proof.

Sketch of the proof. To analyse the blow-up phenomenon, we introduce the rescaled variable

$$u(t, x) = \lambda^{-\frac{1}{2}}(s) e^{i\gamma_0(s)} U(s, y), \quad y = \frac{x}{\lambda}, \quad ds = \frac{dt}{\lambda^2},$$  

(1.13)

where $y \in \mathbb{R}$ and $s \gg 1$ (which means that $t$ is close to the blow up time) and where the scaling and phase parameters $\lambda > 0$ and $\gamma_0$ are functions to be determined. By direct computation,

$$i\partial_t u + \partial_x^2 u + |u|^4 u = \lambda^{-\frac{3}{2}} e^{i\gamma} \mathcal{E}(U),$$  

(1.14)

where

$$\mathcal{E}(U) := i\partial_s U + \partial_y^2 U - U + |U|^4 U - i\frac{\lambda}{\lambda_0} \Lambda U - (\gamma - 1) U.$$  

(1.15)

It is easily checked using $Q'' + Q^5 = Q$ that the function $U(s, y) = e^{-ib(s)} \frac{\lambda^2}{\lambda} Q(y)$ satisfies $\mathcal{E}(U) = 0$ for the special choice of parameters

$$\gamma(s) = s, \quad \lambda(s) = \frac{1}{s}, \quad b(s) = \frac{1}{s},$$  

(1.16)

which corresponds to the minimal mass solution $S(t)$ defined in (1.3), and thus to $r^*_s = 0$ in (1.10). Any residual profile $r^*_s$ satisfying a strong flatness condition such as (1.8) will decouple from the soliton and in such case it is enough to consider an approximate solution containing only a rescaled soliton, as in [1, 18]. The main new ingredient of the proof of Theorem 1 consists in constructing a suitable approximate solution $V$ of $\mathcal{E}(V) = 0$. In contrast with the flat case, to generate the specific asymptotic behavior (1.10)–(1.11), where the residual profile is expected to interact strongly with the bubble, the approximate solution $V$ has to include additional terms. We set

$$V(s, y) = e^{-ib(s)} \frac{\lambda^2}{\lambda} Q(y) + \theta(s, y) \lambda^{-\frac{3}{2}} (\varphi_1(y) \cos \gamma + i\psi_1(y) \sin \gamma),$$  

(1.17)

with a suitable cut-off function $\theta$ ($\theta \equiv 1$ close to the soliton) and where

$$\gamma(s) \sim s, \quad \lambda(s) \sim \frac{1}{s}, \quad b(s) \sim \frac{1}{s}.$$  

We now discuss the choice of the pair of functions $(\varphi_1, \psi_1)$; the form of $V$ in (1.17) and the choice of the cut-off function $\theta$ are justified later. By linearization, we obtain formally

$$\mathcal{E}(V) \sim -\lambda^{-\frac{3}{2}} [(\psi_1 + L_+ \varphi_1) \cos \gamma + i(\varphi_1 + L_- \psi_1) \sin \gamma],$$

where $L_+ = \partial_y^2 + 1, L_- = -\partial_y^2 + 1$. The choice of the functions $(\varphi_1, \psi_1)$ is different from the flat case, where the cut-off was simply $\theta(s, y) = 1$.
where the operators $L_\pm$ are defined in (2.1) and where we have discarded higher order terms. It is thus natural to consider the system

$$\begin{pmatrix} L_+ & 1 \\ 1 & L_- \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0. \tag{1.18}$$

From [19, Proposition 2.1.4], the system (1.18) does not have any non zero bounded solution $(\varphi, \psi)$. However, it follows from ODE arguments (see Appendix A) that this system has an even solution $(\varphi_1, \psi_1)$ with the following asymptotic behavior as $|y| \to +\infty$,

$$\varphi_1(y) \sim |y|, \quad \psi_1(y) \sim -|y|. \tag{1.19}$$

(See Lemma 2.2 for a precise statement.) We now justify that the asymptotic behavior (1.19) of $(\varphi_1, \psi_1)$ as $|y| \to +\infty$, the multiplicative factor $\lambda^{\frac{3}{2}}$ in (1.17) and the choice of a suitable cut-off function $\theta$ generate the first term of the profile $r_\ast$ in the original variables $(t, x)$. Indeed, from (1.17), the asymptotic behavior of $V(s, y)$ for $|y|$ large corresponds to the function $R_\ast(s, y) = \lambda^{\frac{3}{2}} e^{-i\gamma |y|}$ and after the change of variable (1.13), we recognize the first term $|x|$ in the profile $r_\ast$ defined by (1.11). For a different multiplicative factor $\lambda^\alpha$, $\alpha \neq \frac{3}{2}$, the ansatz would be either too large (the approximate solution is not close to $Q$ in $L^2$) or too small (the solution is simply $S(t, x)$). The second term $i\kappa x^2$ in (1.11) is due to an additional correction term in the definition of $V$ (see Section 3.2) which is needed in our method to obtain manageable error terms.

Note that the idea of using a pair $(\varphi_1, \psi_1)$ solution of (1.18) and the choice of a cut-off function $\theta$ related to the original variable (see Section 3.2) was inspired by the construction technique introduced in [12] (see the definition of $A_1$ and formula (1.16) in [12]), devoted to the existence of a new blow-up rate for the critical generalized KdV equation. We also refer to [10, 11] and references therein for more blow-up results concerning that equation.

Once the approximate solution $V$ is constructed in Section 3 (see Lemma 3.1 and Proposition 3.2 for its main properties), the proof scheme follows the ones in [14, 21, 13]. For a sequence of solutions of (1.1) constructed backwards in time close to the approximate solution, we establish uniform estimates for the parameters (by modulation and ODE techniques, see Sections 4 and 6) and for the error term (by energy arguments inspired by [21], see Section 5). In Section 7, passing to the strong limit in $L^2$, we obtain a solution of (1.1) as stated in Theorem 1.

Compared to previous works on blowup for the nonlinear Schrödinger equation, a key difficulty comes from oscillations in time through the phase parameter $\gamma$, due to that fact that the solitonic bubble has a phase $\gamma \to +\infty$ at the blow-up time, while the profile $r_\ast(x)$ is time independent. Several time dependent oscillatory quantities (like (3.9) and (4.27)) are introduced to use cancellations in mean value in time while estimating the parameters.

**Notation.** Set

$$f(u) = |u|^4 u, \quad F(u) = \frac{1}{6} |u|^6.$$

We denote

$$\langle g, h \rangle = \Re \int g \overline{h}.$$

Set

$$\Sigma = \{ g \in H^1(\mathbb{R}); xg \in L^2(\mathbb{R}) \}.$$
Let $\Lambda$ be the generator of $L^2$-scaling in 1D:
$$\Lambda g = \frac{1}{2} g + x \partial_x g.$$  
In particular,
$$\langle g_1, \Lambda g_2 \rangle + \langle g_2, \Lambda g_1 \rangle = 0,$$
for all $g_1, g_2 \in \Sigma$, hence
$$\langle g, \Lambda g \rangle = 0,$$
for all $g \in \Sigma$. In addition,
$$\langle \partial_x^2 g, \Lambda g \rangle = -\|\partial_x g\|_{L^2}^2,$$
for all $g \in H^2(\mathbb{R})$ with $\partial_x g \in \Sigma$. More generally, for $k \in \mathbb{Z}$, we let
$$\Lambda_k g = \frac{1 - k}{2} g + x \partial_x g,$$
so that $\Lambda = \Lambda_0$.

We define the vector space
$$\mathcal{Y} = \left\{ g \in C^\infty(\mathbb{R}, \mathbb{R}); g \text{ is even and} \right. $$
$$\left. \exists q \in \mathbb{N}, \forall p \in \mathbb{N}, \exists C > 0, \forall x \in \mathbb{R}, |g^{(p)}(x)| \leq C|x|^q e^{-|x|} \right\}$$
where $\langle x \rangle = (1 + x^2)^{1/4}$.

Moreover, we fix an even function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi \geq 0$, $\chi(x) = 0$ for $|x| \leq 1/2$ and $\chi(x) = 1$ for $|x| \geq 1$, and we define the functions $\mu_k$, $k \geq 0$, by
$$\mu_k(x) = |x|^k \chi(x).$$
(1.24)

We let
$$\mathcal{Z}_k = \mathcal{Y} + \text{span}\left\{\mu_0, \ldots, \mu_k\right\}.$$  
(1.25)

Using the operators $\Lambda_j$ defined by (1.23), it is not difficult to see that for any $j, k \in \mathbb{N}$,
$$\Lambda_j \mathcal{Z}_k \subset \mathcal{Z}_k$$
$$\Lambda_{2k+3} \mathcal{Z}_{k+1} \subset \mathcal{Z}_k$$
$$\Lambda_1 \mathcal{Z}_0 \subset \mathcal{Y}.$$  
(1.27)  (1.28)

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## 2. Linearization around the soliton

### 2.1. Linearized operator.**

The linearization of (1.1) around the special solution $u(t, x) = e^{it} Q(x)$ involves the following Schrödinger operators:
$$L_+ := -\partial_x^2 + 1 - 5Q^4, \quad L_- := -\partial_x^2 + 1 - Q^4.$$  
(2.1)

Denote by $\rho \in \mathcal{Y}$ the unique radial $H^1$ solution of
$$L_+ \rho = \frac{x^2}{4} Q.$$
See [24, Appendix B]. We recall the generalized null space relations [24, Appendix B]

\[ L - Q = 0, \quad L_+(\Lambda Q) = -2Q, \quad L_-(x^2Q) = -4\Lambda Q, \quad L_+\rho = \frac{x^2}{4}Q. \quad (2.2) \]

We check that

\[ \langle \rho, Q \rangle = -\frac{1}{2} \langle L_+\rho, \Lambda Q \rangle = -\frac{1}{8} \langle x^2Q, \Lambda Q \rangle = \frac{1}{8} \int x^2Q^2 > 0. \quad (2.3) \]

For any \( a \in \mathbb{R} \), we set

\[ Q_a = Q + a\rho \]

so that

\[ Q''_a - Q_a + Q_4^5 + a\frac{x^2}{4}Q = Q_5^5 - 5aQ^4\rho. \quad (2.4) \]

We will use the following well-known coercivity properties.

**Lemma 2.1.** There exists a constant \( \zeta_0 > 0 \) such that

\[ \langle L_+g, g \rangle \geq \zeta_0 \|g\|^2_{H^1} - \frac{1}{\zeta_0} \left( \langle g, Q \rangle^2 + \langle g, y^2Q \rangle^2 \right) \]

and

\[ \langle L_-g, g \rangle \geq \zeta_0 \|g\|^2_{H^1} - \frac{1}{\zeta_0} \left( \langle g, \rho \rangle^2 + \langle g, \Lambda Q \rangle^2 \right) \]

for any real-valued function \( g \in H^1(\mathbb{R}) \).

**Proof.** The estimates are immediate consequences of Propositions 2.9 and 2.10 in [24]. \( \square \)

2.2. **Linearized system.** For the construction of the approximate blow-up solution, we will use systems involving the operators \( L_+ \) and \( L_- \), as introduced below.

**Lemma 2.2.** The following properties hold.

(i) There exists a unique pair of even functions \((\varphi_1, \psi_1) \in Z^2_1\) such that

\[ \begin{cases} 
\psi_1 + L_+\varphi_1 = 0 \\
\varphi_1 + L_-\psi_1 = 0 
\end{cases} \quad (2.5) \]

and satisfying

\[ \varphi_1(x) = \mu_1(x) + c_1 + v_1(x), \quad \psi_1(x) = -\mu_1(x) - c_1 + w_1(x) \]

where \( c_1 \in \mathbb{R} \) and \( v_1, w_1 \in \mathcal{Y} \).

(ii) Given a pair of even functions \((g, h) \in \mathcal{Y}^2\), there exists a unique pair of even functions \((\varphi, \psi) \in Z^2_0\) solution of the system

\[ \begin{cases} 
\psi + L_+\varphi = g \\
\varphi + L_-\psi = h 
\end{cases} \quad (2.6) \]

and satisfying

\[ \varphi(x) = c + v(x), \quad \psi(x) = -c + w(x) \]

where \( c \in \mathbb{R} \) and \( v, w \in \mathcal{Y} \).
The proof of Lemma 2.2 is given in Appendix A. The key ingredient is a result from [19] concerning the non existence of a resonance pair for (2.5), i.e. the non existence of a pair of non zero bounded functions satisfying (2.6).

We define \( \alpha_1 \in \mathbb{R} \) by
\[
\langle \Lambda_3 \varphi_1 - x^2 Q^4 \varphi_1 + \alpha_1 \rho, Q \rangle = 0 \tag{2.7}
\]
and \( \kappa \in \mathbb{R} \) by
\[
\kappa = -\frac{c_1}{2}. \tag{2.8}
\]

Remark 2.1. The constant \( \kappa \) appears in the statement of Theorem 1. From the definition of \( c_1 \) in part (i) of Lemma 2.2, it does not seem clear how to determine \( \kappa \) explicitly.

Now, we define another pair of functions that will be useful in the construction of the approximate rescaled blow-up solution.

Lemma 2.3. There exists a unique pair of even functions \((\varphi_2, \psi_2) \in Z_2^2\) solution of the system
\[
\begin{align*}
\psi_2 + L_+ \varphi_2 &= -\Lambda_3 \varphi_1 - x^2 Q^4 \varphi_1 \\
\varphi_2 + L_- \psi_2 &= -\Lambda_3 \varphi_1 + x^2 Q^4 \varphi_1 - \alpha_1 \rho
\end{align*} \tag{2.9}
\]
and satisfying
\[
\varphi_2(x) = \kappa x^2 + c_2 + v_2(x), \quad \psi_2(x) = -\kappa x^2 - c_2 + w_2(x)
\]
where \( c_2 \in \mathbb{R} \) and \( v_2, w_2 \in \mathcal{Y} \).

Proof. Note from (1.27) with \( k = 0 \) that \( \Lambda_3 \varphi_1, \Lambda_3 \psi_1 \in Z_0 \). More precisely, from Lemma 2.2 we have
\[
\begin{align*}
-\Lambda_3 \psi_1 - x^2 Q^4 \varphi_1 &= c_1 + g_2, \\
-\Lambda_3 \varphi_1 + x^2 Q^4 \varphi_1 - \alpha_1 \rho &= -c_1 + h_2
\end{align*}
\]
where \( g_2, h_2 \in \mathcal{Y} \). In particular, \((\varphi_2, \psi_2)\) solves (2.9) if and only if setting \( \tilde{\varphi}_2 = \varphi_2 + \frac{1}{2}c_1 x^2, \tilde{\psi}_2 = \psi_2 - \frac{1}{2}c_1 x^2 \), the pair \((\tilde{\varphi}_2, \tilde{\psi}_2)\) satisfies
\[
\begin{align*}
\dot{\tilde{\varphi}}_2 + L_+ \tilde{\varphi}_2 &= \dot{g}_2 \\
\dot{\tilde{\psi}}_2 + L_- \tilde{\psi}_2 &= \dot{h}_2
\end{align*}
\]
where
\[
\dot{g}_2 = g_2 - \frac{5}{2} c_1 x^2 Q^4, \quad \dot{h}_2 = h_2 + \frac{1}{2} c_1 x^2 Q^4.
\]
As \( \dot{g}_2, \dot{h}_2 \in \mathcal{Y} \), by (11) of Lemma 2.2 there exists a unique pair of functions \((\tilde{\varphi}_2, \tilde{\psi}_2)\) solutions in \( Z_0^2 \) of this system, satisfying for a constant \( c_2 \in \mathbb{R} \) and \( v_2, w_2 \in \mathcal{Y} \),
\[
\tilde{\varphi}_2 = c_2 + v_2, \quad \tilde{\psi}_2 = -c_2 + w_2.
\]
Thus, setting \( \varphi_2 = \tilde{\varphi}_2 - \frac{1}{2}c_1 x^2, \psi_2 = \tilde{\psi}_2 + \frac{1}{2}c_1 x^2 \), the pair \((\varphi_2, \psi_2)\) solves (2.9). The uniqueness of the solution \((\varphi_2, \psi_2)\) with the specified asymptotic behavior is deduced from part (i) of Lemma 2.2 \( \square \)

Let \( \alpha_2 \in \mathbb{R} \) be defined by (see (2.3))
\[
\langle \Lambda_5 \varphi_2 - x^2 Q^4 \varphi_2 + \alpha_2 \rho, Q \rangle = 0. \tag{2.10}
\]
3. Approximate blow-up solution in rescaled variables

In the rest of this paper, \( \delta \in (0, 1) \) is a constant. For the sake of clarity, we also introduce the bootstrap constant

\[
K = \delta^{-\frac{1}{2}}.
\]

(3.1)

This constant will be taken large to close some bootstrap estimates by a continuity argument in the proof of Proposition 6.1.

Let \( s_0 > 1 \) be a constant to be chosen sufficiently large, possibly depending on \( K \). Let \( n \geq s_0 \) be an integer. Throughout this paper, except in Section 7, the notation \( a \lesssim b \) means that \( a \leq Cb \), where the constant \( C > 0 \) is independent of \( K \) and \( n \).

3.1. Rescaled variables. Define

\[
T_n = -\frac{1}{n}, \quad S_n = n.
\]

(3.2)

We introduce the change of variables \((s, y) \mapsto (t, x)\) for \( s \in [s_0, S_n] \),

\[
x = \lambda(s)y, \quad dt = \lambda^2(s)ds, \quad T_n - t = \int_s^{S_n} \lambda^2(\tau)d\tau,
\]

(3.3)

where \((\gamma, \lambda, b, a)\) are \( C^1 \) functions of \( s \in [s_0, S_n] \) satisfying the following bootstrap estimates, for any \( s \in [s_0, S_n] \)

\[
\begin{align*}
|\gamma(s) - s| & \leq 2s^{-\frac{7}{4}}, \\
|b(s) - s^{-1}| & \leq 2s^{-\frac{7}{4}}, \\
\left| \frac{b(s)}{\lambda(s)} - 1 \right| & \leq 2Ks^{-1}, \\
|a(s)| & \leq (1 + |\alpha_1|)s^{-\frac{5}{2}},
\end{align*}
\]

(3.4)

where \( \alpha_1 \) is given by (2.7). It follows from (3.3) that

\[
|\lambda(s) - s^{-1}| \leq 3s^{-\frac{7}{4}},
\]

(3.5)

for \( s_0 \) large. We also introduce the multiplier

\[
M_b(s, y) = e^{ib(s)y^2}.
\]

For future use, we also introduce the notation, for \( k = 1, 2, 3 \),

\[
j_k(s) = \int_s^{S_n} \lambda^{k+1}(\tau)d\tau.
\]

(3.6)

Using (3.6), we estimate, for \( s \) large,

\[
j_k(s) = \int_s^{S_n} \lambda^{k+1}(\tau)d\tau
\]

\[
= \int_s^{S_n} \tau^{-k-1}d\tau + \int_s^{S_n} O(\tau^{-\frac{4k+7}{4}})d\tau
\]

\[
= \frac{1}{k} (s^{-k} - n^{-k}) + (s^{-1} - n^{-1})O(s^{-\frac{4k+1}{4}}).
\]

Thus,

\[
\frac{1}{2k}(s^{-k} - n^{-k}) \leq j_k(s) \leq \frac{2}{k}(s^{-k} - n^{-k}).
\]

(3.7)
Note that $T_n - t(s) = j_1(s)$ and thus

$$T_n - t(s) = (s^{-1} - n^{-1}) + (s^{-1} - n^{-1})O(s^{-\frac{4}{3}}),$$

so that using $T_n = -n^{-1},$

$$|t(s) + s^{-1}| \lesssim s^{-\frac{4}{3}}, \quad s^{-1} \lesssim |t(s)| \lesssim s^{-1}. \quad (3.8)$$

Now, we introduce bootstrap estimates on the time derivative of the parameters. In order to determine $a$ at a sufficient order of precision, we define the function $\Omega = \Omega(s)$ by

$$\Omega = \alpha_1 b \lambda^\frac{2}{3} \cos \gamma + \alpha_2 b \lambda^\frac{2}{3} \sin \gamma + \alpha_3 \lambda^3 \sin 2\gamma + \alpha_4 b^2 \lambda^\frac{2}{3} \sin \gamma \quad (3.9)$$

where $\alpha_1$ and $\alpha_2$ are defined by (2.7), (2.10) and

$$\alpha_3 \int \rho Q = -2 \int Q^2 \varphi_1 \psi_1, \quad \alpha_4 \int \rho Q = -\frac{1}{4} \int y^4 Q^5 \psi_1. \quad (3.10)$$

Let

$$m_\gamma = \gamma_s - 1, \quad m_\lambda = \frac{\lambda_s}{\lambda} + b, \quad m_b = b_s + b^2 - a, \quad m_a = a_s - \Omega, \quad (3.11)$$

and

$$\vec{m} = \begin{pmatrix} m_\gamma \\ m_\lambda \\ m_b \\ m_a \end{pmatrix}. \quad (3.12)$$

We impose the following bootstrap estimates

$$|\vec{m}(s)| \leq K s^{-3}. \quad (3.13)$$

Remark 3.1. The relations $m_\gamma = 0, m_\lambda = 0$ and $m_b = 0$ give the standard approximate equations for the parameters $\gamma, \lambda$ and $b$ for the conformal blow-up. For example, for the minimal mass blow-up solution (1.3), one has

$$\gamma(s) = s, \quad \lambda(s) = \frac{1}{s}, \quad b(s) = \frac{1}{s}, \quad a(s) = 0.$$ 

The function $\Omega$ allows us to keep track of some oscillatory terms in the behavior of $a_s$ that cannot be estimated in absolute value, but that can be easily integrated in time. This leads to a refined estimate of $a$ (see the last line of (3.4)), which we use in the equation of $b$ (see the definition of $m_b$).

3.2. Definition of an approximate rescaled blow-up solution. First, let $\Theta_0 : \mathbb{R} \to [0, 1]$ be an even smooth function with compact support such that

$$\Theta_0(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases} \quad (3.14)$$

and $C_0 = \|\Theta_0\|_{L^2}^2 > 0$. Define

$$\Theta(x) = \Theta_0(x/\delta)$$

so that

$$\Theta(x) = \begin{cases} 1 & \text{if } |x| < \delta \\ 0 & \text{if } |x| > 2\delta \end{cases}$$

and

$$\|\Theta\|_{L^2}^2 = C_0 \delta.$$
We define
\[ \theta[\Gamma](y) = \Theta(\lambda y). \]  
where
\[ \Gamma = (\gamma, \lambda, b, a, j_1, j_2, j_3) \in \mathbb{R}^7 \cap \{ \lambda > 0 \}. \]
When \( \Gamma \) is a function of \( s \), we will write by abuse of notation
\[ \theta(s, y) = \theta[\Gamma(s)](y) = \Theta(\lambda(s) y). \]

Now, we introduce
\[ W[\Gamma](y) = M_b Q_a(y) + \theta[\Gamma](y) (A[\Gamma](y) + iB[\Gamma](y)) \]
where
\[ A[\Gamma](y) = A_1[\Gamma](y) + A_2[\Gamma](y) = \lambda \phi_1(y) \cos \gamma + \lambda \phi_2(y) \sin \gamma, \]
\[ B[\Gamma](y) = B_1[\Gamma](y) + B_2[\Gamma](y) = \lambda \psi_1(y) \sin \gamma - \lambda \psi_2(y) \cos \gamma. \]
Recall that the functions \( \phi_1, \psi_1 \in \mathcal{Z}_1 \) are defined in Lemma 2.2 and the functions \( \phi_2, \psi_2 \in \mathcal{Z}_2 \) are defined in Lemma 2.3.

When \( \Gamma \) is a function of \( s \), we write by abuse of notation
\[ W(s, y) = W[\Gamma(s)](y) = M_{-b(s)} Q_a(s)(y) + \theta(s, y) (A(s, y) + iB(s, y)) \]
where
\[ A(s, y) = A_1(s, y) + A_2(s, y) = \lambda \phi_1(s) \cos \gamma(s) + \lambda \phi_2(s) \sin \gamma(s), \]
\[ B(s, y) = B_1(s, y) + B_2(s, y) = \lambda \psi_1(s) \sin \gamma(s) - \lambda \psi_2(s) \cos \gamma(s). \]

**Remark 3.2.** The function \( W \) is the main part of the approximate solution. It contains the modified bubble \( Q_a \), multiplied by the quadratic phase \( M_{-b} \). It also contains the term \( \theta(A + iB) \) which will generate the expected blow-up profile at \( x = 0 \). That term is not multiplied by \( M_{-b} \) which is consistent with the result of convergence of the residual profile in [10].

We also introduce
\[ Z[\Gamma](y) = \lambda \phi e^{-i\gamma} \left[ -ij_1(\nu_1 + \nu_2)(\lambda y) - \frac{1}{2} j_1^2(\nu_1'' + \nu_2')(\lambda y) \right. \]
\[ \left. - ij_2 c_1 \Theta'(\lambda y) + j_3 c_2 \Theta''(\lambda y) \right] \]
where the functions \( \nu_1 \) and \( \nu_2 \) are explicitly defined as
\[ \nu_1(x) = \Theta'(x)(|x| + ikx^2) + 2\Theta'(x)(\text{sign}(x) + 2ikx), \]
\[ \nu_2(x) = f(\Theta(x)(|x| + ikx^2)). \]
Again, when \( \Gamma \) is a function of \( s \), we write by abuse of notation
\[ Z(s, y) = Z[\Gamma(s)](y). \]

**Remark 3.3.** The function \( Z \) is a correction term in the approximate solution, to take into account error terms generated by \( W \), as the nonlinear term \( \nu_2 \) for example.

Note that the expression of this correction term is simpler in the variable \( (t, x) \). Setting
\[ z(t, x) = \lambda^{-\frac{1}{2}}(s)e^{iy(s)} Z(s, y), \]
one has
\begin{equation}
z(t, x) = -i(T_n - t)(\nu_1(x) + \nu_2(x)) - \frac{1}{2}(T_n - t)^2(\nu_1''(x) + \nu_2''(x))
- i\left(\int_{s(t)}^{S_n} \lambda^3 c_1 \Theta''(x) + \left(\int_{s(t)}^{S_n} \lambda^4 \right)c_2 \Theta''(x),
\right)
\end{equation}
where following (3.3), \( s \) is the function of \( t \) defined by
\[ T_n - t = \int_{s(t)}^{S_n} \lambda^2 = j_1(s(t)) \]
(so that \( s'(t)\lambda^2(s(t)) = 1 \)).

Let \( V \) be the approximate rescaled blow-up solution defined by
\[ V[\Gamma](y) = W[\Gamma](y) + Z[\Gamma](y). \]
It is immediate by the definitions of \( W \) and \( Z \) that \( V \) is a locally Lipschitz function of \( \Gamma \) with values in \( L^2_y \). The exact expressions of \( W \) and \( Z \) will be justified by the computations in the proof of Proposition 3.2. As above, the notation
\[ V(s, y) = W(s, y) + Z(s, y) \]
will be used when \( \Gamma \) is a function of \( s \).

We denote
\[ I(s) = [-2\delta\lambda^{-1}, 2\delta\lambda^{-1}], \quad J(s) = [-2\delta\lambda^{-1}, -\delta\lambda^{-1}] \cup [\delta\lambda^{-1}, 2\delta\lambda^{-1}]. \]
We now establish some pointwise bounds on \( V, W \) and \( Z \).

**Lemma 3.1.** Under the bootstrap estimates (3.3),
\begin{align*}
|A| + |B| & \leq s^{-\frac{3}{2}}(1 + |y|) + s^{-\frac{3}{2}}(1 + y^2), \\
|\theta A| + |\theta B| & \leq s^{-\frac{3}{2}}(1 + |y|)1_{I(s)} \lesssim \delta s^{-\frac{3}{2}}1_{I(s)}, \\
|\partial_y(\theta A)| + |\partial_y(\theta B)| & \leq s^{-\frac{3}{2}}1_{I(s)}, \\
|W| + (1 + |y|)|\partial_y W| & \lesssim e^{-\frac{\lambda y^2}{2}} + \delta s^{-\frac{3}{2}}1_{I(s)}, \\
|Z| + (1 + |y|)|\partial_y Z| & \lesssim \delta^{-1}s^{-\frac{3}{2}}1_{J(s)} + s^{-\frac{3}{2}}(|y|^3 + 1)1_{I(s)},
\end{align*}
\begin{align*}
|V - M_{-\delta}Q| + |V - Q| + ||V - Q| & \lesssim \delta s^{-\frac{3}{2}}, \\
|V| + (1 + |y|)|\partial_y V| & \lesssim e^{-\frac{\lambda y^2}{2}} + \delta s^{-\frac{3}{2}}1_{I(s)},
\end{align*}
Assuming in addition (3.13), it holds
\begin{equation}
|\partial_y V| \lesssim s^{-\frac{3}{2}}(1 + |y|1_{I(s)}) + s^{-\frac{3}{2}}1_{J(s)}. \tag{3.27}
\end{equation}

**Proof.** Since \( \Theta(x) = \Theta_0(x/\delta) \), we see that
\[ |\Theta(x)| \lesssim 1_{\{|x| < 2\delta\}}, \quad x \in \mathbb{R}, \tag{3.28}
\]
and, for \( j \geq 1 \),
\[ |\Theta^{(j)}(x)| \lesssim \delta^{-j}1_{\{|x| < 2\delta\}}, \quad x \in \mathbb{R}. \tag{3.29}
\]
Thus, for \( j \geq 1 \), by (3.4),
\[ |\theta| \lesssim 1_{I(s)}, \quad |\partial_y \theta| \lesssim \delta^{-j} s^{-j}1_{J(s)}. \]
Moreover, we observe for future reference that
\begin{equation}
Q|M_b\theta - 1| \lesssim Q|1 - \theta| + Q|M_b - 1| \lesssim e^{-\frac{\lambda y^2}{2}} + |b| \lesssim s^{-1}. \tag{3.30}
\end{equation}
Estimate (3.21) follows directly from the bootstrap assumptions (3.3) and the properties of the functions $(\varphi_1, \psi_1)$, $(\varphi_2, \psi_2)$ given in Lemmas 2.2, 2.3. Thus,
\[
|\theta A| + |\theta B| \lesssim \left[ s^{-\frac{7}{2}}(1 + |y|) + s^{-\frac{7}{2}}(1 + y^2) \right] I_1(s)
\lesssim s^{-\frac{7}{2}}(1 + |y|) I_1(s) \lesssim \delta s^{-\frac{7}{4}} I_1(s),
\]
which is (3.21). We also check that
\[
|\partial_y(\theta A)| + |\partial_y(\theta B)| \lesssim \left[ s^{-\frac{7}{2}} + \delta^{-1} s^{-\frac{7}{2}}(1 + |y|) + \delta^{-1}s^{-\frac{7}{2}}(1 + y^2) \right] I_1(s) \lesssim s^{-\frac{7}{4}} I_1(s),
\]
which is (3.22). Now, the pointwise estimate (3.23) for $W$ follows directly from the expression of $W$, the definition of $Q$ in (3.24) and (3.21). Besides, (3.22) implies the pointwise estimate (3.23) for $(1 + |y|)|\partial_y W|$. Using (3.28)-(3.29), it is not difficult to estimate from (3.17) and (3.4),
\[
|z(t, x)| \lesssim (T_n - t) \left[ \delta^{-1} I_1(\delta < |x| < \delta) + |x|^5 I_1(|x| < \delta) \right]
+ (T_n - t)^2 \left[ \delta^{-3} I_1(\delta < |x| < \delta) + |x|^3 I_1(|x| < \delta) \right],
\]
and
\[
|\partial_x z(t, x)| \lesssim (T_n - t) \left[ \delta^{-2} I_1(\delta < |x| < \delta) + |x|^4 I_1(|x| < \delta) \right]
+ (T_n - t)^2 \left[ \delta^{-4} I_1(\delta < |x| < \delta) + |x|^2 I_1(|x| < \delta) \right].
\]
In particular,
\[
|z(t, x)| \lesssim (T_n - t) \left[ \delta^{-1} I_1(\delta < |x| < \delta) + \delta^5 I_1(|x| < \delta) \right]
+ (T_n - t)^2 \left[ \delta^{-3} I_1(\delta < |x| < \delta) + \delta^4 I_1(|x| < \delta) \right],
\]
and
\[
|\partial_x z(t, x)| \lesssim (T_n - t) \left[ \delta^{-2} I_1(\delta < |x| < \delta) + \delta^4 I_1(|x| < \delta) \right]
+ (T_n - t)^2 \left[ \delta^{-4} I_1(\delta < |x| < \delta) + \delta^2 I_1(|x| < \delta) \right].
\]
Using (3.31)-(3.32), we obtain
\[
|Z(s, y)| \lesssim s^{-\frac{7}{2}} \left[ \delta^{-1} I_1(s) + s^{-3} |y|^5 I_1(s) \right] + s^{-\frac{7}{2}} \left[ \delta^{-1} I_1(s) + s^{-3} |y|^3 I_1(s) \right]
\lesssim s^{-\frac{7}{2}} \delta^{-1} I_1(s) + s^{-\frac{7}{2}} |y|^3 I_1(s),
\]
and
\[
|\partial_y Z(s, y)| \lesssim s^{-\frac{7}{2}} \left[ \delta^{-2} I_1(s) + s^{-4} |y|^4 I_1(s) \right] + s^{-\frac{7}{2}} \left[ \delta^{-4} I_1(s) + s^{-2} |y|^2 I_1(s) \right]
\lesssim s^{-\frac{7}{2}} \delta^{-2} I_1(s) + s^{-\frac{7}{2}} |y|^2 I_1(s).
\]
This proves (3.24) for $s$ large. We obtain from (3.21) and (3.24) that
\[
|V - M_{\lambda b} Q_a| \lesssim \delta s^{-\frac{7}{4}},
\]
and (3.25) follows from
\[
|Q_a - Q| + |M_{\lambda b} Q - Q| \lesssim |a| + |b| \lesssim s^{-1}.
\]
By (3.23) and (3.24), we obtain (3.26).

Finally, we prove (3.27). By a direct computation,
\[
\partial_s W = Q_a \partial_s M_{\lambda b} + a_s M_{\lambda b} + (A + iB) \partial_s \theta + \theta (\partial_s A + i \partial_s B).
\]
Using (3.11), it follows from elementary calculations and the estimates $|a_s| \lesssim s^{-\frac{7}{2}}$, $|b_s| \lesssim s^{-\frac{7}{2}}$, $|\frac{a_s}{A}| \lesssim s^{-1}$ and $|\partial_s \theta| \lesssim s^{-1} I_1(s)$ (by assumptions (3.3) and (3.13)) that
\[
|Q_a \partial_s M_{\lambda b} + a_s M_{\lambda b} + (A + iB) \partial_s \theta| \lesssim s^{-\frac{7}{2}}.
\]
The terms in the expansion of $\theta(\partial_s A + i\partial_s B)$ can all be estimated by similar arguments, so we give the details for one term only, $\theta \partial_s A_1$. We have

$$\theta \partial_s A_1 = \theta \left( \frac{3}{2} \lambda s A_1 - \gamma_s \lambda^\frac{3}{2} \varphi_1 \sin \gamma \right).$$

Since $|\theta A_1| \lesssim s^{-\frac{1}{2}}$, we see that

$$\frac{|\lambda s}{\lambda} A_1 \lesssim s^{-\frac{3}{2}}.$$

Moreover, $|\gamma_s| \lesssim 1$ and $|\theta \varphi_1| \lesssim (1 + |y|) 1_{I(s)}$, so that

$$|\gamma_s \lambda^\frac{3}{2} \theta \varphi_1 \sin \gamma| \lesssim s^{-\frac{3}{2}} (1 + |y|) 1_{I(s)}.$$

and so

$$|\partial_s W| \lesssim s^{-\frac{3}{2}} (1 + |y| 1_{I(s)}).$$

We now estimate $\partial_s Z$. Writing $\partial_s Z = -i \gamma_s Z + \tilde{Z}$, where

$$\tilde{Z} = \frac{\lambda}{\lambda} \Lambda Z + \lambda^\frac{3}{2} e^{-i \gamma} \left[ i \lambda^2 (\nu_1 + \nu_2)(\lambda y) + \left( \int_s^{\infty} \lambda^2 (\nu_1'' + \nu_2'')(\lambda y) + i \lambda^3 c_1 \Theta''(\lambda y) + \lambda^4 c_2 \Theta''(\lambda y) \right) \right].$$

By (3.24),

$$|\gamma_s Z| \lesssim |Z| \lesssim \delta^{-1} s^{-\frac{3}{2}} 1_{I(s)} + s^{-\frac{3}{2}}.$$

Moreover, similarly as in the proof of (3.24), we see that $|\tilde{Z}| \lesssim s^{-\frac{3}{2}} 1_{I(s)} \lesssim s^{-\frac{3}{2}}$. Thus,

$$|\partial_s Z| \lesssim \delta^{-1} s^{-\frac{3}{2}} 1_{I(s)} + s^{-\frac{3}{2}}.$$

By summing the estimates of $\partial_s W$ and $\partial_s Z$, we obtain (3.27).

3.3. Equation of the approximate rescaled blow-up solution. The rest of this section is devoted to the proof of the following result. Recall that the notation $E$ is defined by (1.15).

**Proposition 3.2.** Under the bootstrap estimates (3.4),

$$E(V) = S_0 + R$$

where

$$S_0 = \bar{m} \cdot \bar{\rho}_0, \quad \bar{\rho}_0 = M_{-b} \begin{pmatrix} -Q_a \\ -i \lambda Q_a - \frac{b}{2} y^2 Q_a \\ \frac{1}{2} y^2 Q_a \\ i \rho \end{pmatrix}$$

and

$$R = \bar{m} \cdot \bar{\rho}_m + R_\Gamma,$$

where $\bar{\rho}_m = \bar{\rho}_m[\Gamma] \in (L^2)^4$ and $R_\Gamma = R_\Gamma[\Gamma] \in L^2$ are locally Lipschitz functions of $\Gamma$ with values in $L^2_y$ and

$$\|\bar{\rho}_m[\Gamma]\|_{L^2} + \|R_\Gamma[\Gamma]\|_{L^2} \to 0 \quad \text{as} \ (\lambda, b, a, j_1, j_2, j_3) \to 0. \quad (3.36)$$
Moreover, assuming further \((3.13)\), the error term \(\mathcal{R}[\Gamma(s)](y) = \mathcal{R}(s, y)\) satisfies
\[
\begin{align*}
\|\mathcal{R}(s)\|_{H^1} &\lesssim s^{-3}, \\
\|y\mathcal{R}(s)\|_{L^2} &\lesssim s^{-2}, \\
|(M_b(s)\mathcal{R}(s), iQ)| &\lesssim s^{-4}.
\end{align*}
\] (3.37)

(3.38)

(3.39)

3.4. Contribution of the main part of the approximate solution. Using the equation \((2.4)\) of \(Q_a\) and the definition of \(\Omega\) in \((3.9)\), we decompose \(\mathcal{E}(W)\) as
\[
\mathcal{E}(W) = S_0 + S_1 + S_2 + R_3 + R_4 + R_5.
\] (3.40)

where
\[
\begin{align*}
S_0 &= M_{-b}(-m_\gamma Q_a + (m_b - 2bm_\lambda)\frac{1}{2} y^2 Q_a + im_\alpha \rho - im_\lambda \Lambda Q_a), \\
S_1 &= \theta_{yy} A + 2\theta_y A_y + i(\theta_{yy} B + 2\theta_y B_y), \\
S_2 &= f(W) - f(M_{-b}Q_a) - 5Q^4\theta A + by^2 Q^4 B - iQ^4 \theta B + iby^2 Q^4 A \\
&\quad + i\theta(\alpha_3 \lambda^3 \sin 2\gamma + \alpha_4 b^2 \lambda^4 \sin \gamma)\rho, \\
R_3 &= \theta(G + iH), \\
R_4 &= M_{-b}(Q_a^5 - Q_a^5 - 5aQ^4 \rho + a^2y^2\frac{4}{4} \rho), \\
R_5 &= i(M_{-b} - \theta)\Omega\rho,
\end{align*}
\]

and
\[
\begin{align*}
G &= -\partial_s B - L_+ A - b(\Lambda B + y^2 Q^4 B) - m_\gamma A + m_\lambda \Lambda B, \\
H &= \partial_s A - L_- B + b(\Lambda A - y^2 Q^4 A) - m_\gamma B - m_\lambda \Lambda A \\
&\quad + (\alpha_1 b\lambda^3 \cos \gamma + \alpha_2 b\lambda^4 \sin \gamma)\rho.
\end{align*}
\]

We refer to Appendix \([3]\) for the proof of \((3.40)\).

Remark 3.4. In the expression \((3.40)\), errors terms designated by \(R_k\) will be estimated directly in \((3.7)\) while errors terms designated by \(S_k\) will be corrected by some terms from the equation of \(Z\), before being estimated.

Note also that the contribution of \(i\Omega\rho\) from the definition of \(m_a\) has been split between \(S_2\) and \(R_3\) (with an error term \(R_5\) due to the quadratic phase and the localization \(\theta\)).

3.5. Contribution of the correction part of the approximate solution. To write the equation of \(Z\), it is simpler to derive first the equation of \(z\) in the original variables \((t, x)\). Indeed, by \((3.17)\), we have
\[
\begin{align*}
i\partial_t z + \partial_x^2 z &= -\nu_1 - \nu_2 - \frac{1}{2}(T_n - t)^2(\nu_1^{(4)} + \nu_2^{(4)}) \\
&\quad - \lambda(s(t))c_1\Theta'' - i\left(\int_{s(t)}^{S_n} \lambda^3 c_1 \Theta^{(4)}\right) \\
&\quad - i\lambda^2(s(t))c_2\Theta'' + \left(\int_{s(t)}^{S_n} \lambda^4 c_2 \Theta^{(4)}\right).
\end{align*}
\]

Thus, by \((1.14)\), one has
\[
\mathcal{E}(Z) = T_1 + T_2 + R_6 + R_7.
\] (3.41)
where
\[ T_1 = -\lambda^2 e^{-i\gamma} \left[ \nu_1(\lambda y) + \lambda c_1 \Theta''(\lambda y) + i\lambda^2 c_2 \Theta''(\lambda y) \right], \]
\[ T_2 = -\lambda^2 e^{-i\gamma} \nu_2(\lambda y), \]
\[ R_6 = \lambda^2 e^{-i\gamma} \left[ -\frac{1}{2} \left( \int_s^{S_n} \Theta'(\lambda y) \right)^2 \nu_1^{(4)}(\lambda y) + \nu_2^{(4)}(\lambda y) \right] \]
\[ + \left( \int_s^{S_n} -ic_1 \lambda^3 + c_2 \lambda^4 \Theta^{(4)}(\lambda y) \right], \]
\[ R_7 = f(Z). \]

3.6. Equation of the approximate rescaled blow-up solution. We set
\[ R_8 = f(W + Z) - f(W) - f(Z), \]
so that, since \( V = W + Z \),
\[ \mathcal{E}(V) = \mathcal{E}(W) + \mathcal{E}(Z) + R_8. \]
Setting
\[ R_1 = S_1 + T_1, \quad R_2 = S_2 + T_2 \]
and
\[ R = \sum_{k=1}^{8} R_k, \]
from (3.40) and (3.41), we have obtained the expected form (3.35) for \( \mathcal{E}(V) \). To complete the proof of Proposition 3.2, it remains to prove the estimates (3.37), (3.38) and (3.39) on each of the error term \( R_k \), \( k = 1, \ldots, 8 \). We also need to verify the structure and the estimates in (3.36). We state and prove the following technical estimates to be used in the next subsection.

Lemma 3.3. With \( \theta \) defined by (3.16), \( Z_k \) defined by (1.25) and under the assumption (3.4), the following properties hold.

(i) For \( R \in \mathcal{Y} \) and \( s \) large,
\[
\| \theta R \|_{L^2} \lesssim 1,
\]
\[
\| (\partial_y^j \theta) R \|_{L^2} \lesssim e^{-\sqrt{s}} \text{ if } j \geq 1.
\]

(ii) Let \( j, k \in \mathbb{N} \). For any \( R \in \mathcal{Z}_k \), for \( s \) large,
\[
\| (\partial_y^j \theta) R \|_{L^2} \lesssim \delta^{\frac{j}{s} + k - j},
\]
\[
\| \partial_y (\partial_y^j \theta) R \|_{L^2} \lesssim \delta^{\frac{j}{s} + k - j}, \text{ if } j \geq 1,
\]
\[
\| \partial_y (\theta R) \|_{L^2} \lesssim 1 \text{ if } k = 0.
\]

In particular,
\[
\| (\partial_y^j \theta) R \|_{H^1} \lesssim \delta^{\frac{j}{s} + k - j},
\]
for all \( j, k \geq 0 \) and \( R \in \mathcal{Z}_k \).

Proof. Proof of (i). Let \( R \in \mathcal{Y} \), in particular \( R^2(y) \lesssim e^{-\frac{1}{2} |y|} \). The first inequality is immediate since \( 0 \leq \theta \leq 1 \). If \( j \geq 1 \), then with the notation (3.19)
\[
\int |\partial_y^j \theta|^2 R^2(y) dy \lesssim \lambda^2 j^{-2j} \int_{|y|} e^{-\lambda^2 |y|} dy \lesssim e^{-\frac{1}{2} \frac{1}{|y|} \lambda^2} \lesssim e^{-2\sqrt{s}},
\]
where we used $\lambda \leq \delta$ for $s$ large, and $\lambda(s) \leq \frac{2\delta}{\delta^2}$ for $s$ large (by (3.3)).

Proof of (i). Let $j, k \geq 0$ and $R \in \mathbb{Z}_k$, in particular $R^2(y) \lesssim (y)^{2k}$. We calculate

$$\int |\partial_j^\alpha \partial_k^\beta R^2(y) dy \lesssim \lambda^2 |\delta - 2j| \int_{l(s)} \langle y \rangle^{2k} dy \lesssim \lambda^{2} |\delta - 2j| \left(\frac{\delta}{\lambda}\right)^{1+2k} \int_{-2}^{2} \langle x \rangle^{2k} dx$$

where we have used the change of variable $x = \frac{\lambda}{\delta}y$ and $\frac{\delta}{\lambda} \leq 1$ for $s$ large. The first estimate then follows from (3.4). The remaining two estimates follow easily from the first one and from part (i).

\[ \Box \]

3.7. Estimates of the error terms. Estimates of $\mathcal{R}_1$. We develop using the expression of $\nu_1$

$$\mathcal{T}_1 = -\lambda \hat{\nu} e^{-i\gamma [\nu_1(\lambda y) + \lambda c_1 \Theta''(\lambda y) + i\lambda^2 c_2 \Theta''(\lambda y)]}$$

$$= -\theta_{yy} \left[ \lambda \hat{\nu} |y| \cos \gamma + \kappa \lambda \hat{\nu} y^2 \sin \gamma - i\lambda^2 \hat{\nu} |y| \sin \gamma + i\kappa \lambda \hat{\nu} y^2 \cos \gamma \right.$$

$$+ \lambda^2 c_1 \cos \gamma + \lambda^2 c_2 \sin \gamma - i\lambda^2 c_1 \sin \gamma + i\lambda^2 c_2 \cos \gamma \bigg]$$

$$- 2\theta_y \left[ \lambda \hat{\nu} (\text{sign} y) \cos \gamma + 2\kappa \lambda \hat{\nu} y \sin \gamma - i\lambda^2 (\text{sign} y) \sin \gamma + 2i\kappa \lambda \hat{\nu} y \cos \gamma \right].$$

Thus,

$$\mathcal{R}_1 = \mathcal{S}_1 + \mathcal{T}_1 = \theta_{yy} \left[ \lambda \hat{\nu} (\varphi_1 - |y| - c_1) \cos \gamma + \lambda \hat{\nu} (\varphi_2 - \kappa y^2 - c_2) \sin \gamma \right.$$

$$+ i\theta_{yy} \left[ \lambda \hat{\nu} (\psi_1 + |y| + c_1) \sin \gamma - \lambda \hat{\nu} (\psi_2 + \kappa y^2 + c_2) \cos \gamma \bigg]$$

$$+ 2\theta_y \left[ \lambda \hat{\nu} (\varphi'_1 - \text{sign} y) \cos \gamma + \lambda \hat{\nu} (\varphi'_2 - 2\kappa y) \sin \gamma \right.$$

$$+ 2i\theta_y \left[ \lambda \hat{\nu} (\psi'_1 + \text{sign} y) \sin \gamma - \lambda \hat{\nu} (\psi'_2 + 2\kappa y) \cos \gamma \right].$$

Observe that $\theta_y$ and $\theta_{yy}$ vanish outside $J(s)$, hence on $[-1, 1]$ for $s$ large. In particular, $\mathcal{R}_1 = \chi(y) \mathcal{R}_1$. From the asymptotics in Lemmas 2.2 and 2.3 it follows that the functions

$$\chi(\varphi_1 - |y| - c_1), \chi(\psi_1 + |y| + c_1), \chi(\varphi_2 - \kappa y^2 - c_2), \chi(\psi_2 + \kappa y^2 + c_2),$$

$$\chi(\varphi'_1 - \text{sign} y), \chi(\psi'_1 + \text{sign} y), \chi(\varphi'_2 - 2\kappa y), \chi(\psi'_2 + 2\kappa y)$$

all belong to $\mathcal{Y}$. Therefore, it follows from Lemma 3.3 that

$$||\mathcal{R}_1||_{H^1} + ||y \mathcal{R}_1||_{L^2} + ||\langle M_b \mathcal{R}_1, iQ \rangle||_{H^1} \lesssim e^{-\sqrt{\gamma}},$$

for $s$ large. Besides, it is immediate by its expression above that $\mathcal{R}_1|_{\Gamma} \in L^2$ is a locally Lipschitz function of $\Gamma$ with values in $L^2$ and $||\mathcal{R}_1|_{\Gamma}||_{L^2} \to 0$ as $\lambda \to 0$.

Estimates of $\mathcal{R}_2$. We decompose $\mathcal{R}_2 = \mathcal{S}_2 + \mathcal{T}_2 = \mathcal{R}_{2,1} + \mathcal{R}_{2,2}$, where

$$\mathcal{R}_{2,1} = \mathcal{S}_2 - f(\theta(A + iB))$$

$$\mathcal{R}_{2,2} = f(\theta(A + iB)) + \mathcal{T}_2.$$

Given $X, Y \in \mathbb{R}$, we expand $f(Q_a + X + iY) - f(Q_a)$. Since

$$|Q_a + X + iY|^4 = \left[ (Q_a + X)^2 + Y^2 \right]^2$$

$$= (Q_a + X)^4 + 2(Q_a + X)^2Y^2 + Y^4$$

$$= 4Q_a^3X + 6Q_a^2X^2 + 4Q_aX^3 + 2Q_a^2Y^2 + 4Q_aXY^2$$

$$+ Q_a^4 + |X + iY|^4,$$
we have the general identity
\[
f(Q_a + X + iY) = 5Q^4_aX + iQ^4_aY + 10Q^4_aX^2 + 2Q^3_aY^2 + 4iQ^3_aXY + 10Q^3_aX^3 + 2iQ^2_aY^3 + 6iQ^2_aX^2Y + 6Q^2_aXY^2 + 5Q_aX^4 + 4iQ_aX^3Y + 6Q_aX^2Y^2 + 4iQ_aXY^3 + Q_aY^4 + f(Q_a) + f(X + iY).
\]
We apply the above identity to the real-valued functions \(X, Y\) such that
\[
M_b\theta(A + iB) = X + iY,
\]
i.e.
\[
X = \Re(M_b\theta(A + iB)), \quad Y = \Im(M_b\theta(A + iB)),
\]
and so,
\[
M_b W = Q_a + X + iY.
\]
Thus,
\[
f(W) = M_{-b} [f(Q_a + X + iY) - f(Q_a)] + f(M_{-b}Q_a)
\]
\[
= M_{-b} (5Q^4_aX + iQ^4_aY)
\]
\[
+ M_{-b} (10Q^3_aX^2 + 2Q^3_aY^2 + 4iQ^3_aXY)
\]
\[
+ M_{-b} (10Q^2_aX^3 + 2iQ^2_aY^3 + 6iQ^2_aX^2Y + 6Q^2_aXY^2)
\]
\[
+ M_{-b} (5Q_aX^4 + 4iQ_aX^3Y + 6Q_aX^2Y^2 + 4iQ_aXY^3 + Q_aY^4) + f(M_{-b}Q_a) + f(\theta(A + iB)).
\]
We compute
\[
R_{2,1} = f(W) - f(M_{-b}Q_a) - f(\theta(A + iB))
\]
\[
- 5Q^4\theta A + by^2Q^4\theta B - iQ^4\theta B + iby^2Q^4\theta A
\]
\[
+ i\theta(\alpha_3\lambda^3 \sin 2\gamma + \alpha_4b^2\lambda^2 \sin \gamma)\rho
\]
\[
= M_{-b} (5Q^4_aX + iQ^4_aY) - 5Q^4\theta A + by^2Q^4\theta B - iQ^4\theta B + iby^2Q^4\theta A
\]
\[
+ M_{-b} (10Q^3_aX^2 + 2Q^3_aY^2 + 4iQ^3_aXY)
\]
\[
+ M_{-b} (10Q^2_aX^3 + 2iQ^2_aY^3 + 6iQ^2_aX^2Y + 6Q^2_aXY^2)
\]
\[
+ M_{-b} (5Q_aX^4 + 4iQ_aX^3Y + 6Q_aX^2Y^2 + 4iQ_aXY^3 + Q_aY^4)
\]
\[
+ i\theta(\alpha_3\lambda^3 \sin 2\gamma + \alpha_4b^2\lambda^2 \sin \gamma)\rho.
\]
We observe that
\[
M_{-b}(5Q^4X + iQ^4Y) = Q^4 M_{-b} \left[3M_b\theta(A + iB) + 2M_b\theta(A + iB)\right]
\]
\[
= 3Q^4\theta(A + iB) + 2M_{-2b}Q^4\theta(A - iB),
\]
thus
\[
M_{-b}(5Q^4_aX + iQ^4_aY) - 5Q^4\theta A + by^2Q^4\theta B - iQ^4\theta B + iby^2Q^4\theta A
\]
\[
= 2 \left(M_{-2b} - 1 + \frac{1}{2}by^2\right) Q^4\theta(A - iB) + M_{-b}(Q^4 - Q^4)(5X + iY);
and so,

\[ R_{2,1} = 2 (M_{-b} - 1 + \frac{3}{4}by^2) Q^4 \theta(A - iB) + M_{-b} (Q_a^4 - Q^4) (5X + iY) \]

\[ + M_{-b} (10Q_a^3 X^2 + 2Q_a^3 Y^2 + 4iQ_a^3 XY) \]

\[ + M_{-b} (10Q_a^3 X^3 + 2iQ_a^3 Y^3 + 6iQ_a^3 X^2 Y + 6Q_a^3 XY^2) \]

\[ + M_{-b} (5Q_a X^4 + 4iQ_a X^3 Y + 6Q_a X^2 Y^2 + 4iQ_a XY^3 + Q_a Y^4) \]

\[ + i\theta(\alpha_3 \lambda^3 \sin 2\gamma + \alpha_4 b^2 \lambda^2 \sin \gamma) \rho. \]

Therefore, by Lemma 3.10, it is easy to estimate \( R_{2,1} \) in \( H^1 \) and \( yR_{2,1} \) in \( L^2 \)

\[ \| R_{2,1} \|_{H^1} + \| yR_{2,1} \|_{L^2} \lesssim s^{-3}. \]

Now, we write

\[ M_b R_{2,1} = 2M_b (M_{-b} - 1 + \frac{3}{4}by^2 + \frac{5}{2}y^4) Q^4 \theta(A - iB) \]

\[ + (Q_a^4 - Q^4) (5X + iY) + 4i(Q_a^3 - Q^3) XY \]

\[ + 10Q_a^3 X^3 + 2iQ_a^3 Y^3 + 6iQ_a^3 X^2 Y^2 \]

\[ + 5Q_a X^4 + 4iQ_a X^3 Y + 6Q_a X^2 Y^2 + 4iQ_a XY^3 + Q_a Y^4 \]

\[ + i[M_b \theta - 1](\alpha_3 \lambda^3 \sin 2\gamma + \alpha_4 b^2 \lambda^2 \sin \gamma) \rho \]

\[ - (M_b \theta - 1) \frac{5}{4} y^4 Q^4 (A - iB) + i\frac{5}{4} y^4 Q^4 B_2 \]

\[ + 4iQ^3 (XY - A_1 B_1) \]

\[ + 10Q_a^3 X^2 + 2Q_a^3 Y^2 - \frac{5}{4} y^4 Q^4 A \]

\[ + i(\alpha_3 \lambda^3 \sin 2\gamma + \alpha_4 b^2 \lambda^2 \sin \gamma) \rho + 4iQ^3 A_1 B_1 + i\frac{5}{4} y^4 Q^4 B_1. \]

We note that, by the bootstrap assumption, the first six lines are easily seen to be \( O(s^{-4}) \) in \( L^2 \). For the seventh line, we first observe that

\[ X = \{ \Re [(M_b \theta - 1)(A + iB)] + A_2 \} + A_1, \]

\[ Y = \{ \Im [(M_b \theta - 1)(A + iB)] + B_2 \} + B_1. \]

Thus,

\[ XY - A_1 B_1 = \{ \Re [(M_b \theta - 1)(A + iB)] + A_2 \} \{ \Im [(M_b \theta - 1)(A + iB)] + B_2 \} \]

\[ + A_1 \{ \Im [(M_b \theta - 1)(A + iB)] + B_2 \} + B_1 \{ \Re [(M_b \theta - 1)(A + iB)] + A_2 \}, \]

and from (3.30), one obtains

\[ |Q^3 (XY - A_1 B_1)| \lesssim s^{-4}. \]

The next-to-last line only contains real-valued terms, so its projection on \( iQ \) vanishes. Last, the projection of the last line on \( iQ \) vanishes because of the choice (3.10) of \( \alpha_3 \) and \( \alpha_4 \). Therefore,

\[ \langle M_b R_{2,1}, iQ \rangle = O(s^{-4}) \]

Now, we estimate \( R_{2,2} \). We introduce a notation

\[ V_*(s, y) = e^{-i\gamma} \left( \lambda^2 |y|^2 + ik\lambda^2 y^2 \right) = A_* + iB_* , \]
where
\[
A_* = \lambda^2 |y| \cos \gamma + \kappa \lambda^2 y^2 \sin \gamma,
\]
\[
B_* = -\lambda^2 |y| \sin \gamma + \kappa \lambda^2 y^2 \cos \gamma,
\]
so that
\[
T_2 = -f(V_*) = -f(\theta(A_* + iB_*)).
\]
Using this notation, we estimate
\[
|R_{2,2}| = |f(\theta(A + iB) - f(\theta(A_* + iB_*))|
\leq \theta^5 \left( |A - A_*| + |B - B_*| \right) \left( A^4 + B^4 + A_*^4 + B_*^4 \right).
\]
By the asymptotics in Lemmas 2.2 and 2.3 and $\theta(s,y) = 0$ for $\lambda|y| \geq 2\delta$, we have
\[
|A - A_*| + |B - B_*| \lesssim \lambda^2, \quad \theta(A^4 + B^4 + A_*^4 + B_*^4) \lesssim \lambda^6(y^4 + 1).
\]
This implies the following pointwise estimate
\[
|R_{2,2}| \lesssim \theta \lambda^{\frac{10}{3}}(y^4 + 1).
\]
Similarly, we have
\[
|\partial_y R_{2,2}| \lesssim \theta \lambda^{\frac{10}{3}}(y^4 + 1).
\]
Thus, we obtain
\[
\|R_{2,2}\|_{H^1} \lesssim \delta^{\frac{2}{3}}s^{-3},
\]
\[
\|yR_{2,2}\|_{H^1} \lesssim \delta^{\frac{2}{3}}s^{-2},
\]
and
\[
|(M_\gamma R_{2,2}, iQ)| \lesssim s^{-\frac{10}{3}}.
\]
Besides, it is immediate that all the terms in the expressions of $S_2[\Gamma]$ and $T_2[\Gamma]$ are locally Lipschitz functions of $\Gamma$ with values in $L^2_y$. We also observe that $f(W) - f(M_{-1}Q_a)$, as well as all the other terms in $S_2[\Gamma]$ and $T_2[\Gamma]$ converge to 0 in $L^2$ as $(\lambda, b, a) \to 0$.

Estimates of $R_3$. We compute $G$ and $H$. It follows from direct computations that
\[
- \partial_s B_1 - L_+ A_1 - b(\Lambda B_1 + y^2 Q^4 B_1) - m_{\gamma} A_1 + m_{\lambda} \Lambda B_1
= -\lambda^2 b(A_3 \psi_1 + y^2 Q^4 \psi_1) \sin \gamma - m_{\gamma} \lambda^2 (\varphi_1 + \psi_1) \cos \gamma + m_{\lambda} \lambda \lambda^2 (A_3 \psi_1) \sin \gamma
\]
and
\[
\partial_s A_1 - L_- B_1 + b(\Lambda A_1 - y^2 Q^4 A_1) - m_{\gamma} B_1 - m_{\lambda} \Lambda A_1
= \lambda^2 b(A_3 \varphi_1 - y^2 Q^4 \varphi_1) \cos \gamma - m_{\gamma} \lambda^2 (\varphi_1 + \psi_1) \sin \gamma - m_{\lambda} \lambda \lambda^2 (A_3 \varphi_1) \cos \gamma.
\]
Similarly, using the definition of $(\varphi_2, \psi_2)$, it follows that
\[
- \partial_s B_2 - L_+ A_2 - b(\Lambda B_2 + y^2 Q^4 B_2) - m_{\gamma} A_2 + m_{\lambda} \Lambda B_2
= \lambda^2 (A_3 \psi_1 + y^2 Q^4 \psi_1) \sin \gamma + \lambda^2 b(A_3 \psi_2 + y^2 Q^4 \psi_2) \cos \gamma
- m_{\gamma} \lambda^2 (\varphi_2 + \psi_2) \sin \gamma - m_{\lambda} \lambda \lambda^2 (A_3 \psi_2) \cos \gamma
\]
and
\[
\partial_s A_2 - L \cdot B_2 + b(\Lambda A_2 - y^2 Q^4 A_2) - m_{\gamma} B_2 - m_{\lambda} A_2 \\
= -\lambda \hat{\sigma}(\Lambda_3 \varphi_1 - y^2 Q^4 \varphi_1 + \alpha_1 \rho) \cos \gamma + \lambda \hat{\sigma} b(\Lambda_5 \varphi_2 - y^2 Q^4 \varphi_2) \sin \gamma \\
+ m_{\gamma} \lambda \hat{\sigma}(\varphi_2 + \psi_2) \cos \gamma - m_{\lambda} \lambda \hat{\sigma}(\Lambda_5 \varphi_2) \sin \gamma.
\]
Summing up the previous identities, we obtain
\[
G = -\lambda \hat{\sigma} \left( \frac{b}{\lambda} - 1 \right) (\Lambda_3 \varphi_1 + y^2 Q^4 \varphi_1) \sin \gamma \\
+ \lambda \hat{\sigma} b(\Lambda_5 \varphi_2 + y^2 Q^4 \varphi_2) \cos \gamma \\
- m_{\gamma} \lambda \hat{\sigma} [(\varphi_1 + \psi_1) \cos \gamma + \lambda (\varphi_2 + \psi_2) \sin \gamma] \\
+ m_{\lambda} \lambda \hat{\sigma} [(\Lambda_3 \varphi_1) \sin \gamma - \lambda (\Lambda_5 \varphi_2) \cos \gamma]
\]
and
\[
H = \lambda \hat{\sigma} \left( \frac{b}{\lambda} - 1 \right) (\Lambda_3 \varphi_1 - y^2 Q^4 \varphi_1 + \alpha_1 \rho) \cos \gamma \\
+ \lambda \hat{\sigma} b(\Lambda_5 \varphi_2 - y^2 Q^4 \varphi_2 + \alpha_2 \rho) \sin \gamma \\
- m_{\gamma} \lambda \hat{\sigma} [(\varphi_1 + \psi_1) \sin \gamma - \lambda (\varphi_2 + \psi_2) \cos \gamma] \\
- m_{\lambda} \lambda \hat{\sigma} [(\Lambda_3 \varphi_1) \cos \gamma + \lambda (\Lambda_5 \varphi_2) \sin \gamma].
\]
We estimate \(\theta H\) in \(H^1\). First, by (3.4), we have \(|\frac{b}{\lambda} - 1| \leq Ks^{-1}\). Moreover, by (1.24) with \(k = 0\), \(\Lambda_3 \varphi_1 - y^2 Q^4 \varphi_1 + \alpha_1 \rho \in \mathcal{Z}_0\). Thus, using Lemma 3.3 we have
\[
\lambda \hat{\sigma} \left| \frac{b}{\lambda} - 1 \right| \|\theta(\Lambda_3 \varphi_1 - y^2 Q^4 \varphi_1 + \alpha_1 \rho)\|_{H^1} \lesssim \delta \frac{s}{s}.
\]
For the second term in \(H\), we observe that by the asymptotics of \(\varphi_2\) in Lemma 2.2 with \(k = 0\), \(\Lambda_5 \varphi_2 - y^2 Q^4 \varphi_2 + \alpha_2 \rho \in \mathcal{Z}_0\) and so using Lemma 3.3
\[
\lambda \hat{\sigma} |b| \|\theta(\Lambda_5 \varphi_2 - y^2 Q^4 \varphi_2 + \alpha_2 \rho)\|_{H^1} \lesssim \delta \frac{s}{s}.
\]
By the asymptotics in Lemmas 2.2 and 2.3 and by (1.27), we have
\[
\varphi_1 + \psi_1 \in \mathcal{Y}, \quad \varphi_2 + \psi_2 \in \mathcal{Y}, \quad \Lambda_3 \varphi_1 \in \mathcal{Z}_0, \quad \Lambda_5 \varphi_2 \in \mathcal{Z}_0.
\]
Thus, by Lemma 3.3, using also the estimates \(|m_{\gamma}| + |m_{\lambda}| \leq Ks^{-3}\) from (3.13), we have
\[
|m_{\gamma} \lambda \hat{\sigma} (||\theta(\varphi_1 + \psi_1)||_{H^1} + \lambda ||\theta(\varphi_2 + \psi_2)||_{H^1}) \lesssim Ks^{-\frac{5}{2}}
\]
and
\[
|m_{\lambda} \lambda \hat{\sigma} (||\theta(\Lambda_3 \varphi_1)||_{H^1} + \lambda ||\theta(\Lambda_5 \varphi_2)||_{H^1}) \lesssim \delta \frac{s}{s} Ks^{-4}.
\]
Therefore, we have proved that \(||\theta H||_{H^1} \equiv \delta \frac{s}{s} Ks^{-3}\) for \(s\) large, and the analogous estimate \(||\theta G||_{H^1} \equiv \delta \frac{s}{s} Ks^{-3}\) for \(G\) is proved similarly. Since \(R_3 = \theta(G + iH)\), we obtain
\[
||R_3||_{H^1} \equiv \delta \frac{s}{s} Ks^{-3}.
\]
Now, we observe that by the definitions of \(\alpha_1\) and \(\alpha_2\) (see (2.47), (2.41)) the terms in the first two lines in (3.46) are orthogonal to \(Q\). Thus, by (3.4) and (3.13) we have
\[
|\langle H, Q \rangle| \lesssim Ks^{-\frac{5}{2}},
\]
for \(s\) large, so that using also the intermediate calculations in (3.30)
\[
|\langle R_3, iQ \rangle| = |\langle \theta H, Q \rangle| = |\langle H, Q \rangle| + |\langle H, (1-\theta)Q \rangle| \lesssim Ks^{-\frac{5}{2}}.
\]
Thus, using also (3.28), (3.29) we see that
\[ |\langle M_b \mathcal{R}_3, iQ \rangle| \leq |\langle \mathcal{R}_3, iQ \rangle| + |(M_b - 1)\mathcal{R}_3, iQ \rangle| \lesssim \delta^{\frac{5}{2}} Ks^{-4}. \]
Moreover,
\[ \|y\mathcal{R}_3\|_{L^2} = \|y\mathcal{R}_3\|_{L^2(\Omega)} \leq 2\delta\lambda^{-1}\|\mathcal{R}_3\|_{L^2} \lesssim \delta^{\frac{3}{2}} Ks^{-2}. \]
Besides, we see from \( \mathcal{R}_3 = \theta(G + iH) \) and the expressions of \( G \) and \( H \) in (3.30)–(3.31) that \( \mathcal{R}_3 = \tilde{m} \cdot \tilde{\rho}_3 + \mathcal{R}_{r,3} \), where the functions \( \tilde{\rho}_3[\Gamma] \) and \( \mathcal{R}_{r,3}[\Gamma] \) are locally Lipschitz functions of \( \Gamma \) with values in \( L^p_0 \) and converge to 0 in \( L^2 \) as \( (\lambda, b) \to 0 \).

Estimates of \( \mathcal{R}_4 \). Since \( Q, \rho \in \mathcal{V} \), using the definition of \( \mathcal{R}_4 \), and (5.4), it follows that
\[ |\mathcal{R}_4| + |\partial_y \mathcal{R}_4| \lesssim a^2 e^{-\frac{1}{2}|y|} \lesssim s^{-5} e^{-\frac{1}{2}|y|}. \]
Thus,
\[ \|\mathcal{R}_4\|_{H^1} + \|y\mathcal{R}_4\|_{L^2} \lesssim s^{-5} \]
and
\[ |\langle M_b \mathcal{R}_4, iQ \rangle| \lesssim s^{-5}. \]
Besides, it is immediate by its definition that \( \mathcal{R}_4[\Gamma] \) is a locally Lipschitz function of \( \Gamma \) with values in \( L^p_0 \) and \( \|\mathcal{R}_4[\Gamma]\|_{L^2} \to 0 \) as \( a \to 0 \).

Estimates of \( \mathcal{R}_5 \). Since \( \rho \in \mathcal{V} \), using \( |\Omega| \lesssim s^{-\delta} \) and (3.4) we have
\[ |\mathcal{R}_5| + |\partial_y \mathcal{R}_5| \lesssim s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|} ( |1 - \theta| + |\theta_y| + b) \]
\[ \lesssim s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|} e^{-\frac{1}{4}\delta} + s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|}. \]
Thus,
\[ \|\mathcal{R}_5(s)\|_{H^1} + \|y\mathcal{R}_5(s)\|_{L^2} \lesssim s^{-\frac{5}{2}}. \]
Next, we see that
\[ |\Im(\mathcal{R}_5(s))| \lesssim s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|} ( |1 - \theta| + b^2) \]
\[ \lesssim s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|} e^{-\frac{1}{4}\delta} + s^{-\frac{5}{2}} e^{-\frac{1}{4}|y|}. \]
Thus,
\[ |\langle M_b \mathcal{R}_5(s), iQ \rangle| \lesssim s^{-\frac{5}{2}}. \]
Also, it is immediate by its definition that \( \mathcal{R}_5[\Gamma] \) is a locally Lipschitz function of \( \Gamma \) with values in \( L^p_0 \) and \( \|\mathcal{R}_5[\Gamma]\|_{L^2} \to 0 \) as \( (\lambda, b) \to 0 \).

Estimates of \( \mathcal{R}_6 \). It follows by easy calculations that
\[ |\nu^{(4)}_1(x)| \lesssim \delta^{-5} 1_{\{\delta < |x| < 2\delta\}}, \]
\[ |\nu^{(5)}_1(x)| \lesssim \delta^{-6} 1_{\{\delta < |x| < 2\delta\}}, \]
\[ |\nu^{(4)}_2(x)| \lesssim \delta 1_{\{|x| < 2\delta\}}, \]
\[ |\nu^{(5)}_2(x)| \lesssim 1_{\{|x| < 2\delta\}}. \]
Next, by (5.4)
\[ s^2 \left( \int_s^{S_n} \lambda^2 \right)^2 + s^2 \int_s^{S_n} \lambda^3 + s^3 \int_s^{S_n} \lambda^4 \lesssim 1. \]
Thus, using also (3.28), (3.29) we see that
\[ |\mathcal{R}_6(s, y)| \lesssim s^{-\frac{5}{2}} \delta^{-5} 1_{\{s\}} + s^{-\frac{5}{2}} \delta 1_{\{s\}}, \]
and
\[ |\partial_y R_6(s, y)| \lesssim s^{-\frac{14}{5}} \delta^{-6} 1_{J(s)} + s^{-\frac{14}{5}} 1_{I(s)}. \]

Therefore the error term \( R_6 \) satisfies
\[ \| R_6(s) \|_{H^1} \lesssim \delta^{-\frac{14}{5}} s^{-4}, \]
and
\[ \| (M_b R_6(s), iQ) \| \lesssim \delta s^{-\frac{2}{5}}, \]
for \( s \) large. Moreover,
\[ \| y R_6 \|_{L^2} = \| y R_6 \|_{L^2(I(s))} \lesssim 2\delta \lambda^{-1} \| R_6 \|_{L^2} \lesssim \delta^{-\frac{2}{5}} s^{-3}. \]

Also, we rewrite \( R_6 \) as
\[ R_6[\Gamma] = \lambda^2 e^{-i\gamma} \left[ -\frac{1}{2} j_1(4\lambda^4 y + \nu_2(4\lambda y)) + (-ic_1 j_2 + c_2 j_3) \Theta(4\lambda y) \right]. \]

Thus, it is immediate that \( R_6[\Gamma] \) is a locally Lipschitz function of \( \Gamma \) with values in \( L^2_y \) and \( \| R_6[\Gamma] \|_{L^2} \to 0 \) as \( (\lambda, j_1, j_2, j_3) \to 0. \)

Estimates of \( R_7 \). By change of variable, \( x = \lambda y \), the error term \( R_7 \) satisfies
\[ \| R_7 \|_{L^2} \lesssim \lambda^2 \| z^5 \|_{L^2}, \quad \| \partial_y R_7 \|_{L^2} = \lambda^2 \| \partial_x (z^5) \|_{L^2}. \]

From (3.33) and (3.34),
\[ \| z^5 \|_{L^2} \lesssim (T_n - t)^5 \delta^{-\frac{4}{5}} + (T_n - t)^{10} \delta^{-\frac{8}{11}}, \]
and
\[ \| \partial_x (z^5) \|_{L^2} \lesssim (T_n - t)^5 \delta^{-\frac{4}{5}} + (T_n - t)^{10} \delta^{-\frac{8}{11}}, \]
Since \( (T_n - t) \lesssim (s(t))^{-1} \), it follows that
\[ \| R_7 \|_{H^1} \lesssim \delta^{-\frac{2}{5}} s^{-7} \lesssim s^{-6}, \]
for \( s \) large, and so
\[ \| (M_b R_7(s), iQ) \| \lesssim s^{-6} \]
for \( s \) large. Moreover, since \( R_7 = f(Z) \) and \( Z \) is supported in \( I(s) \),
\[ \| y R_7 \|_{L^2} = \| y R_7 \|_{L^2(I(s))} \lesssim 2\delta \lambda^{-1} \| R_7 \|_{L^2} \lesssim s^{-5}. \]

Estimates of \( R_8 \). Using (3.23), (3.24) and \( e^{-\frac{1}{5} t i} 1_{I(s)} \lesssim e^{-\delta t} e^{-\frac{1}{5} t} \), it follows that
\[ (|W| + |\partial_y W|)(|Z| + |\partial_y Z|) \lesssim s^{-\frac{2}{5}} e^{-\frac{1}{5} t} + s^{-2} 1_{I(s)} + s^{-5} (|y|^3 + 1) 1_{I(s)}. \]
Moreover,
\[ (|W| + |\partial_y W| + |Z| + |\partial_y Z|)^3 \lesssim \left[ e^{-\frac{1}{5} t} + s^{-\frac{2}{5}} 1_{I(s)} \right]^3 \]
\[ \lesssim e^{-\frac{2}{5} t} + s^{-\frac{2}{5}}. \]
Thus, by the definition of \( R_8 \), it holds
\[ |R_8| + |\partial_y R_8| \lesssim (|W| + |\partial_y W|)(|Z| + |\partial_y Z|)|W| + |\partial_y W| + |Z| + |\partial_y Z|)^3 \]
\[ \lesssim s^{-\frac{2}{5}} e^{-\frac{1}{5} t} + s^{-\frac{2}{5}} 1_{I(s)} + s^{-\frac{2}{5}} (|y|^3 + 1) 1_{I(s)}. \]
Therefore,
\[ \| R_8(s) \|_{H^1} \lesssim \delta^\frac{2}{5} s^{-3}, \]
\[ \| y R_8(s) \|_{L^2} \lesssim \delta^\frac{2}{5} s^{-2}, \]
and
\[ |\langle M_b \mathcal{R}_8(s), iQ \rangle| \lesssim s^{-\frac{9}{2}}, \]
for \( s \) large.

Finally, we rewrite \( \mathcal{R}_7 + \mathcal{R}_8 \) as
\[ (\mathcal{R}_7 + \mathcal{R}_8)[\Gamma] = f(W + Z) - f(W) \]
and it follows that \( (\mathcal{R}_7 + \mathcal{R}_8)[\Gamma] \) is a locally Lipschitz function of \( \Gamma \) with values in \( L^2_y \) and converges to 0 as \( \lambda \to 0 \).

Therefore, Proposition 3.2 is proved.

4. Modulation around the approximate blow-up solution

4.1. Decomposition by modulation of the parameters. We recall that given \( T \in \mathbb{R} \) and an initial data \( u_T \in H^3 \cap \Sigma \) such that \( \partial_x u_T \in \Sigma \), there exists a solution \( u \) of (1.1) defined on some interval \( I \ni T \) with \( u(T) = u_T \), in the same regularity class, i.e. \( u \in C(I, H^3 \cap \Sigma) \), \( \partial_x u \in C(I, \Sigma) \); \( u \in C^1(I, H^1) \). This follows from standard arguments, see e.g. [2, Proof of Lemma 5.6.2].

We look for a decomposition of \( u \) on \( I \) of the form
\[ u(t, x) = \lambda^{-\frac{5}{2}}(s) e^{i\gamma(s)}(V[\Gamma(s)](y) + \varepsilon(s, y)), \]
where \( V \) is defined in (3.18), \( S > 0 \) and \( (\gamma, \lambda, b, a) \) are \( C^1 \) parameters such that
\[ \begin{cases} 
\langle \varepsilon, M_{-b}Q \rangle = 0 \\
\langle \varepsilon, M_{-b}^2 Q \rangle = 0 \\
\langle \varepsilon, i M_{-b} \Lambda Q \rangle = 0 \\
\langle \varepsilon, i M_{-b} \rho \rangle = 0
\end{cases} \]
(4.2)

Remark 4.1. The choice of orthogonality relations (4.2) is standard and corresponds to observations in [24] and [21].

Using (1.14) and (3.35) with \( \tilde{m} \) defined in (3.12), we see that \( \varepsilon \) and \( (\gamma, \lambda, b, a) \) satisfy
\[ 0 = \lambda^{\frac{5}{2}} e^{-i\gamma} \left[ i \partial_t u + \partial_x^2 u + f(u) \right] = \mathcal{E}(V + \varepsilon) \]
\[ = \mathcal{E}(\varepsilon) + f(V + \varepsilon) - f(V) - f(\varepsilon) + \mathcal{E}(V) \]
\[ = i \partial_t \varepsilon + \partial_x^2 \varepsilon - \varepsilon + f(V + \varepsilon) - f(V) + i b \Lambda \varepsilon - i m \Lambda \varepsilon - m \gamma \varepsilon + S_0 + \mathcal{R}. \]
(4.3)

From its definition, it is easy to check that \( V \in C(H^3 \cap \Sigma) \), \( \partial_x V \in C(\Sigma) \), \( V \in C^1(H^1) \). Thus, by formula (4.3), \( \varepsilon \in C(H^3 \cap \Sigma) \), \( \partial_x \varepsilon \in C(\Sigma) \), \( \varepsilon \in C^1(H^1) \), and the above equation makes sense in \( L^2 \) and in \( H^1_{loc} \).

4.2. Bootstrap assumption. The estimates of Section 3 on the approximate solution \( V \), given in Lemma 3.1 and Proposition 3.2 were established assuming the bootstrap estimates (3.4) and (3.13). Bootstrap estimates on the function \( \varepsilon \) will also be necessary. Below, we recapitulate all the bootstrap estimates to be used later.
First, we observe that since $u$ is locally Lipschitz. Second, since $V$ and $\partial_y V$ are bounded (see (3.26)), for any $\Phi \in H^1$, the map $\lambda \mapsto \int \Phi(x) V_1 |\lambda| \left( \frac{x}{\lambda} \right) k \left( \frac{x}{\lambda} \right) dx$ is locally Lipschitz and similarly for $V_2$. It follows that $\mathcal{M}$ is locally Lipschitz. Since $p_1', p_2', q_1', q_2'$ are arbitrary, the result follows by using (4.5).
Let $g, h \in \mathcal{Y}$ be time-independent functions and let

$$
\mathcal{I}[g, h](s) = \langle \varepsilon(s), M_{-b}(g + ih) \rangle = \langle \varepsilon[\Gamma(s)], M_{-b}(g + ih) \rangle.
$$

We give a general estimate on $\frac{d\mathcal{I}}{ds}$ that will be used to establish the modulation equations.

**Lemma 4.2.** On its interval of definition, the function $\mathcal{I}$ is $C^1$ and satisfies

$$
\frac{d\mathcal{I}}{ds} = \langle \varepsilon, iM_{-b}(L_-g + iL_+h) \rangle
$$

$$
- m_1 \langle Q, h \rangle + \frac{1}{2} (m_6 - 2m_\lambda) \langle y^2Q, h \rangle - m_2 \langle \rho, g \rangle + m_\lambda \langle \Lambda Q, g \rangle
$$

$$
+ \tilde{m} \cdot \tilde{\rho}_\lambda + \mathcal{R}_\mathcal{I},
$$

where $\tilde{\rho}_\lambda(s) = \rho_\lambda[\Gamma(s)], \mathcal{R}_\mathcal{I}(s) = \mathcal{R}_\mathcal{I}[\Gamma(s)]$ are Lipschitz functions of $\Gamma$ converging to 0 as $(\lambda, b, a, j_1, j_2, j_3) \to 0$ and $d(u[\Gamma]; \Gamma) \to 0$.

Moreover, assuming \[\mathbb{I}\],

$$
|\tilde{\rho}_\lambda| \lesssim s^{-1}, \quad |\mathcal{R}_\mathcal{I}(s)| \lesssim s^{-3},
$$

and

$$
\frac{d\mathcal{I}}{ds} = \langle \varepsilon, iM_{-b}(L_-g + iL_+h) \rangle
$$

$$
+ 4 \lambda^2 \left[ \langle \varepsilon_1, 5Q^3\phi_1h \rangle \cos \gamma + \langle \varepsilon_2, Q^3\phi_1h \rangle \sin \gamma \right]
$$

$$
- 4 \lambda^2 \left[ \langle \varepsilon_1, Q^3\phi_1g \rangle \sin \gamma + \langle \varepsilon_2, Q^3\phi_1g \rangle \cos \gamma \right]
$$

$$
- m_2 \langle Q, h \rangle + \frac{1}{2} (m_6 - 2m_\lambda) \langle y^2Q, h \rangle - m_2 \langle \rho, g \rangle + m_\lambda \langle \Lambda Q, g \rangle
$$

$$
+ \langle M_\varepsilon\mathcal{R}, h - ig \rangle + O(K^{\frac{3}{4}}s^{-4}).
$$

**Proof.** Denote $\mathcal{I}(s) = \mathcal{I}[g, h](s)$. We differentiate $\mathcal{I}$ using \[\mathbb{I}\]. First,

$$
\frac{d\mathcal{I}}{ds} = \langle \partial_\varepsilon \varepsilon, M_{-b}(g + ih) \rangle - \frac{1}{4} b_\varepsilon \langle \varepsilon, iy^2M_{-b}(g + ih) \rangle
$$

$$
= \langle -\partial_\varepsilon^2 \varepsilon + \varepsilon - f(V + \varepsilon) + f(V), iM_{-b}(g + ih) \rangle
$$

$$
- b \langle \Lambda \varepsilon, M_{-b}(g + ih) \rangle - \frac{1}{4} b_\varepsilon \langle \varepsilon, iy^2M_{-b}(g + ih) \rangle
$$

$$
+ \langle m_\lambda \Lambda \varepsilon - im_\gamma \varepsilon, M_{-b}(g + ih) \rangle
$$

$$
- \langle S_0, iM_{-b}(g + ih) \rangle - \langle \mathcal{R}, iM_{-b}(g + ih) \rangle.
$$

Integrating by parts and using \[\mathbb{I}\], we have

$$
\langle -\partial_\varepsilon^2 \varepsilon, iM_{-b}(g + ih) \rangle = -\langle \varepsilon, iM_{-b}(g'' + ih'') \rangle - b \langle \varepsilon, yM_{-b}(g' + ih') \rangle
$$

$$
- \frac{1}{2} b_\varepsilon \langle \varepsilon, M_{-b}(g + ih) \rangle + \frac{1}{4} b^2 \langle \varepsilon, iy^2M_{-b}(g + ih) \rangle
$$

and

$$
-b \langle \Lambda \varepsilon, M_{-b}(g + ih) \rangle = \frac{1}{2} b \langle \varepsilon, M_{-b}(g + ih) \rangle + b \langle \varepsilon, yM_{-b}(g' + ih') \rangle
$$

$$
- \frac{1}{2} b^2 \langle \varepsilon, iy^2M_{-b}(g + ih) \rangle.
$$
Thus, for $s$ large,

$$
(-\partial_t^2 \tilde{\varepsilon}, iM_{-b}(g + ih)) - b(\Delta \varepsilon, M_{-b}(g + ih)) - \frac{1}{4}b_s \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle
$$

$$
= -\langle \varepsilon, iM_{-b}(g'' + ih'') \rangle - \frac{1}{4}(b_s + b^2) \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle
$$

$$
= -\langle \varepsilon, iM_{-b}(g'' + ih'') \rangle - \frac{1}{4}a \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle - \frac{1}{4}m_b \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle
$$

$$
= -\langle \varepsilon, iM_{-b}(g'' + ih'') \rangle + O(K^{\frac{4}{s}}s^{-\frac{2}{s}})
$$

(4.10)

where we have estimated, using (4.4),

$$
|a \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle| \lesssim |a| \| \varepsilon \|_{L^2} \lesssim K^{\frac{4}{s}}s^{-\frac{2}{s}},
$$

$$
|m_b \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle| \lesssim |m_b| \| \varepsilon \|_{L^2} \lesssim K^{\frac{2}{s}}s^{-\frac{5}{s}}.
$$

Now, we decompose the term

$$
-\langle f(V + \varepsilon) - f(V), iM_{-b}(g + ih) \rangle = \langle f(M_b V + M_b \varepsilon) - f(M_b V), h - ig \rangle.
$$

We extend the notation (3.43) by setting

$$
\tilde{X} = X + \mathcal{R}(M_b Z) = \mathcal{R}(M_b V - Q_a)
$$

$$
\tilde{Y} = Y + 3(M_b Z) = \mathfrak{Z}(M_b V - Q_a)
$$

$$
E_1 = \mathcal{R}(M_b \varepsilon), \quad E_2 = \mathfrak{Z}(M_b \varepsilon),
$$

so that

$$
M_b V = Q_a + \tilde{X} + i\tilde{Y},
$$

$$
M_b V + M_b \varepsilon = Q_a + \tilde{X} + E_1 + i(\tilde{Y} + E_2).
$$

(4.12)

Using the identity (3.42) twice (once with $(\tilde{X} + E_1, \tilde{Y} + E_2)$ and once with $(\tilde{X}, \tilde{Y})$), we obtain

$$
f(M_b V + M_b \varepsilon) - f(M_b V)
$$

$$
= f(Q_a + \tilde{X} + E_1 + i\tilde{Y} + iE_2) - f(Q_a + \tilde{X} + i\tilde{Y})
$$

$$
= 5Q_a^2 E_1 + iQ_a^4 E_2
$$

$$
+ 10Q_a^2((\tilde{X} + E_1)^2 - \tilde{X}^2) + 2Q_a^3((\tilde{Y} + E_2)^2 - \tilde{Y}^2)
$$

$$
+ 4iQ_a^3((\tilde{X} + E_1)(\tilde{Y} + E_2) - \tilde{X}\tilde{Y}) + \tilde{R}_1,
$$

(4.13)

where

$$
\tilde{R}_1 = 10Q_a^2((\tilde{X} + E_1)^3 - \tilde{X}^3) + 2iQ_a^3((\tilde{Y} + E_2)^3 - \tilde{Y}^3)
$$

$$
+ 6iQ_a^2((\tilde{X} + E_1)^2(\tilde{Y} + E_2) - \tilde{X}^2\tilde{Y}) + 6Q_a^2((\tilde{X} + E_1)(\tilde{Y} + E_2)^2 - \tilde{X}\tilde{Y}^2)
$$

$$
+ 5Q_a((\tilde{X} + E_1)^4 - \tilde{X}^4) + 4iQ_a((\tilde{X} + E_1)^3(\tilde{Y} + E_2) - \tilde{X}\tilde{Y}^3)
$$

$$
+ 6Q_a((\tilde{X} + E_1)^2(\tilde{Y} + E_2)^2 - \tilde{X}\tilde{Y}^2)
$$

$$
+ 4iQ_a((\tilde{X} + E_1)(\tilde{Y} + E_2)^3 - \tilde{X}\tilde{Y}^3) + Q_a((\tilde{Y} + E_2)^4 - \tilde{Y}^4)
$$

$$
+ f(\tilde{X} + E_1 + i\tilde{Y} + iE_2) - f(\tilde{X} + i\tilde{Y}).$$
Recall that by (4.14), $|E_1| + |E_2| \lesssim \|\varepsilon\|_{L^\infty} \lesssim K^{\frac{1}{2}} s^{-2}$ and thus by (3.21), (3.24), we have

$$|\tilde{X}| + |\tilde{Y}| + |E_1| + |E_2| \lesssim s^{-\frac{2}{2}} + s^{-\frac{3}{2}}(|y| + s^{-1}|y|^2)1_{f(s)} + \delta^{-1} s^{-\frac{2}{2}} 1_{J(s)}$$

(4.14)

Therefore,

$$|\tilde{R}_1| \lesssim (|\tilde{X}| + |\tilde{Y}| + |E_1| + |E_2|)^3 + (|\tilde{X}| + |\tilde{Y}| + |E_1| + |E_2|)^5$$

$$\lesssim (s^{-\frac{2}{2}}(1 + |y|^2))^3 + (s^{-\frac{3}{2}}(1 + |y|^2))^5 + (\delta^{-3} s^{-\frac{3}{2}} + \delta^{-5} s^{-\frac{5}{2}})1_{J(s)}$$

$$\lesssim s^{-\frac{2}{2}}(1 + |y|^{10}) + \delta^{-3} s^{-\frac{2}{2}} 1_{J(s)}.$$

Since $g, h \in \mathcal{Y}$, it follows that for $s$ large

$$|(\tilde{R}_1, h - ig)| \lesssim s^{-\frac{2}{2}}.$$

For the first line in the last identity in (4.13) (recalling $|Q_{\alpha} - Q| \lesssim |\alpha\rho| \lesssim s^{-\frac{2}{2}} |\rho|$), we have

$$\langle 5Q_a^4 E_1 + iQ_a^2 E_2, h - ig \rangle - \langle 5Q_a^4 E_1 + iQ_a^2 E_2, h - ig \rangle \lesssim |\alpha||\varepsilon||_{L^2} \lesssim K^{\frac{1}{2}} s^{-\frac{2}{2}};$$

and (using the definition of $E_1, E_2$)

$$\langle 5Q_a^4 E_1 + iQ_a^2 E_2, h - ig \rangle = \langle Q_a^4 E_1, 5h \rangle - \langle Q_a^4 E_2, g \rangle = \langle Q_a^4 M_\theta(5h), 5h \rangle = -\langle \varepsilon, iM_{-\theta}(Q_a^4 g + 5Q_a^4 h) \rangle.$$

We now estimate the first quadratic term in the last identity in (4.13). We first estimate

$$|(|Q_a^3 - Q_1^3)|((\tilde{X} + E_1)^2 - \tilde{X}^2)| \lesssim |\alpha\rho|((\tilde{X}^2 + E_1^2) \lesssim s^{-\frac{1}{2}}$$

by (4.14). Next, we write

$$10Q^3((\tilde{X} + E_1)^2 - \tilde{X}^2) = \tilde{R}_2 + 20Q^3 A_1 \varepsilon_1,$$

where

$$\tilde{R}_2 = 20Q^3(\tilde{X} E_1 - A_1 \varepsilon_1) + 10Q^3 E_1^2.$$

On the one hand, using (3.20) and (3.21),

$$Q^2|\theta(A + iB) - (A + iB)| \lesssim s^{-1} Q|A + iB| \lesssim s^{-\frac{1}{2}},$$

and in particular

$$Q^2|X - A| \lesssim s^{-\frac{1}{2}}.$$

Hence, using (3.24),

$$Q^2|\tilde{X} - A_1| \lesssim s^{-\frac{2}{2}} + Q^2|A_2| + Q^2|Z| \lesssim s^{-\frac{5}{2}}.$$

Moreover, by (4.14) and (3.30),

$$|E_1| \lesssim K^{\frac{1}{2}} s^{-2}, \quad Q|E_1 - \varepsilon_1| = Q|R((M_\theta - 1)\varepsilon)| \lesssim s^{-1} \|\varepsilon\|_{L^\infty} \lesssim K^{\frac{1}{2}} s^{-3}.$$

Therefore, using also (3.20), we obtain

$$Q^3|\tilde{X} E_1 - A_1 \varepsilon_1| \lesssim Q^3|((\tilde{X} - A_1)E_1| + Q^3|A_1(E_1 - \varepsilon_1)| \lesssim K^{\frac{1}{2}} s^{-\frac{2}{2}}.$$

Since also

$$Q^3 E_1^2 \lesssim K^{\frac{1}{2}} s^{-4},$$

we estimate

$$|(\tilde{R}_2, h - ig)| \lesssim K^{\frac{1}{2}} s^{-\frac{2}{2}} + K^{\frac{1}{2}} s^{-4} \lesssim K^{\frac{1}{2}} s^{-4}.$$
On the other hand,

$$\langle 20Q^3A_1 \varepsilon_1, h - ig \rangle = \langle 20Q^3A_1 \varepsilon_1, h \rangle = 20\lambda^2 \langle \varepsilon_1, Q^3 \varphi_1 h \rangle \cos \gamma.$$ 

Combining the above two estimates, we obtain

$$\langle 10Q^3((\tilde{X} + E_1)^2 - \tilde{X}^2), h - ig \rangle = 20\lambda^2 \langle \varepsilon_1, Q^3 \varphi_1 h \rangle \cos \gamma + O(K^{\frac{1}{2}}s^{-4}).$$

Similarly, for the other two quadratic terms in the last identity in (4.13), we obtain

$$\langle 2Q^3((\bar{Y} + E_2)^2 - \bar{Y}^2), h - ig \rangle = 4\lambda^2 \langle \varepsilon_2, Q^3 \psi_1 h \rangle \sin \gamma + O(K^{\frac{1}{2}}s^{-4}),$$

$$\langle 4iQ^3(Q + E_1)(\bar{Y} + E_2) - \bar{X}Y \rangle, h - ig \rangle = -4\lambda^2 \langle \varepsilon_1, Q^3 \varphi_1 g \rangle \sin \gamma$$

$$- 4\lambda^2 \langle \varepsilon_2, Q^3 \varphi_1 g \rangle \cos \gamma + O(K^{\frac{1}{2}}s^{-4}).$$

Summing up the above calculations, we see that

$$\langle -f(V + \varepsilon) + f(V), iM_{-b}(g + ih) \rangle = -\langle \varepsilon, iM_{-b}(Q^3 g + 5iQ^4 h) \rangle$$

$$+ 4\lambda^2 \langle \langle \varepsilon_1, 5Q^3 \varphi_1 h \rangle \cos \gamma + \langle \varepsilon_2, Q^3 \psi_1 h \rangle \sin \gamma \rangle$$

$$- 4\lambda^2 \langle \langle \varepsilon_1, Q^3 \psi_1 g \rangle \sin \gamma + \langle \varepsilon_2, Q^3 \varphi_1 g \rangle \cos \gamma \rangle + O(K^{\frac{1}{2}}s^{-4}).$$

Therefore, combining this estimate with (4.10), we obtain for the first two lines of the right-hand side of (4.9)

$$\langle -\partial_y^2 \varepsilon + \varepsilon - f(V + \varepsilon) + f(V), iM_{-b}(g + ih) \rangle$$

$$= \langle \varepsilon, iM_{-b}(Q^3 g + 5iQ^4 h) \rangle$$

$$- \langle b\langle \varepsilon, M_{-b}(g + ih) \rangle - \frac{1}{4}b_\gamma \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle \rangle$$

$$+ 4\lambda^2 \langle \langle \varepsilon_1, 5Q^3 \varphi_1 h \rangle \cos \gamma + \langle \varepsilon_2, Q^3 \psi_1 h \rangle \sin \gamma \rangle$$

$$- 4\lambda^2 \langle \langle \varepsilon_1, Q^3 \psi_1 g \rangle \sin \gamma + \langle \varepsilon_2, Q^3 \varphi_1 g \rangle \cos \gamma \rangle + O(K^{\frac{1}{2}}s^{-4}).$$

For the next term in the right-hand side of (4.9), we use (4.4),

$$\langle m_A \Lambda \varepsilon - im_{\gamma} \varepsilon, M_{-b}(g + ih) \rangle \lesssim \langle \langle m_A \rangle \rangle \langle |m_{\gamma}| \rangle \| \varepsilon \|_{H^1} \lesssim K^{\frac{1}{2}}s^{-5}.$$ 

Last, using the expression of $S_0$, we estimate using (4.4) and $|a| \lesssim s^{-\frac{3}{2}}$,

$$-\langle S_0, iM_{-b}(g + ih) \rangle = -m_{\gamma} \langle Q_a, h \rangle + \frac{1}{2}(m_b - 2bm_{\lambda}) \langle y^2 Q_a, h \rangle$$

$$- m_a \langle \rho, g \rangle + m_{\lambda} \langle \Lambda Q_a, g \rangle$$

$$= -m_{\gamma} \langle Q_a, h \rangle + \frac{1}{2}(m_b - 2bm_{\lambda}) \langle y^2 Q_a, h \rangle$$

$$- m_a \langle \rho, g \rangle + m_{\lambda} \langle \Lambda Q_a, g \rangle + O(Ks^{-\frac{3}{2}}).$$

As a first consequence of these calculations, we see that $\frac{d\tilde{\mu}}{dt}$ has the form (4.6) where $\tilde{\mu}_I$ and $R_I$ converge to 0 as $(\lambda, b, a, j_1, j_2, j_3) \to 0$ and $\|\varepsilon\|_{H^1} = d(\mu_1; \Gamma) \to 0$. Moreover, we obtain (4.8) for $s$ large, and using $\lambda^\frac{1}{2}\|\varepsilon\|_{L^2} \lesssim K^{\frac{1}{2}}s^{-\frac{3}{2}}$ and (3.37), we check that $\tilde{\mu}_I$ and $R_I$ defined in (4.6) satisfy the estimates (4.7).

Finally, the Lipschitz property of $\tilde{\mu}_I$ and $R_I$ follows from Lemma 4.1 through a tedious but elementary examination of all the error terms. As a first example, we observe that

$$m_b \langle \varepsilon, iy^2 M_{-b}(g + ih) \rangle = \tilde{m} \cdot \tilde{\rho}.$$
where
\[
\tilde{\rho} = \begin{pmatrix}
0 \\
0 \\
\langle \varepsilon, iy^2 M \cdot \ell (g + ih) \rangle \\
0
\end{pmatrix}
\]
and \( \Gamma \mapsto \langle \varepsilon, iy^2 M \cdot \ell (g + ih) \rangle \) is clearly locally Lipschitz by Lemma 4.1. To conclude, we treat a typical nonlinear term from (4.11) to replace \( \tilde{\rho} \) and then apply again Lemma 4.1. For the second term, we use (4.11) to replace \( \tilde{G} \) and then apply again Lemma 4.1.

We are now in a position to actually prove the existence of a decomposition (4.1) on some time interval for initial data close \( V[G] \).

**Lemma 4.3.** There exists \( \omega_0 > 0 \) such that for any \( S > 0, \omega \in (0, \omega_0) \) and \(( u^{in}, \Gamma^{in} ) \in H^1 \times (\mathbb{R}^6 \cap \{ \lambda > 0 \}) \), if
\[
|\langle \ell \rangle | + |d(\ell) \Gamma^{in}| < \omega,
\]
and \( u \) is the solution of (4.1) with \( u(j_1^{in}) = u^{in} \), then there exist \( \bar{s} \subset (-\infty, S) \) and \( \Gamma \in C^1([\bar{s}, S]) \) such that \( \Gamma(S) = \Gamma^{in} \),
\[
|\langle \ell \rangle | + |d(\ell) \Gamma^{in}| < \omega
\]
and which satisfies the following system of ODEs
\[
\begin{cases}
\partial_t \langle \varepsilon, M \cdot \ell \rho \rangle = 0 \\
\partial_t \langle \varepsilon, M \cdot \ell y^2 Q \rangle = 0 \\
\partial_t \langle \varepsilon, M \cdot \ell \Delta Q \rangle = 0 \\
\partial_t \langle \varepsilon, M \cdot \ell B \rangle = 0 \\
\partial_s j_1 = -\lambda^2 \\
\partial_s j_2 = -\lambda^3 \\
\partial_s j_3 = -\lambda^4
\end{cases}
\tag{4.17}
\]
where \( \varepsilon = \varepsilon[G] \) is defined in (4.10).

**Proof.** It follows from Lemma 4.2 applied with the following choice of \((g, h)\)
\[
(0, \rho), \quad (y^2 Q, 0), \quad (0, \Delta Q), \quad (Q, 0),
\]
that the system (4.17) is equivalent to
\[
D[\Gamma] \partial_t \Gamma = F[\Gamma]
\]
where
\[
D = D_0 + D_{\Gamma},
\]
\[
D_0 = \begin{pmatrix}
0 & -\frac{1}{2} \langle y^2 Q, \rho \rangle & 0 & 0 & 0 & 0 \\
0 & -\langle \Delta Q, y^2 Q \rangle & 0 & 0 & 0 & 0 \\
4 \langle Q, \Delta Q \rangle & 0 & -\langle y^2 Q, \Delta Q \rangle & 0 & 0 & 0 \\
0 & -\langle \Delta Q, Q \rangle & 0 & -\langle \rho, Q \rangle & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
and
\[
D_{\Gamma} = \begin{pmatrix}
\langle Q, \rho \rangle \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
and

\[ \mathbf{D}_\Gamma \to 0 \quad \text{and} \quad \mathbf{F}[\Gamma] \to 0 \]

as \( \| (\lambda, b, a, j_1, j_2, j_3) \| + \| \varepsilon \|_{H^1} \to 0 \). By (1.21) and (2.3), the matrix \( \mathbf{D}_0 \) is invertible. Moreover, using also Lemma 4.1, \( \mathbf{D}_\Gamma \) and \( \mathbf{F} \) are locally Lipschitz. For small \( \| (\lambda, b, a, j_1, j_2, j_3) \| \) and \( \| \varepsilon \|_{H^1} \), the matrix \( \mathbf{D} \) is therefore invertible and the result follows from the Cauchy-Lipschitz theorem. \( \square \)

The estimates in Lemma 4.2 allow us to improve the estimates in (4.4) related to the equations satisfied by the parameters. Recall that \( m_\gamma, m_\lambda, m_b \) and \( m_a \) are defined in (3.11). We also prove a refined estimate on \( m_a = a_s - \Omega \) (see Remarks 3.1 and 4.2).

**Lemma 4.4.** Under the assumption (1.3), for \( s \) large

\[ |\bar{m}| \lesssim s^{-3}. \tag{4.18} \]

In particular,

\[ \| S_0 \|_{H^1} + \| y S_0 \|_{L^2} \lesssim s^{-3}. \tag{4.19} \]

Moreover,

\[ m_a = -\alpha_5 \lambda^\frac{7}{2} \left[ \langle \varepsilon, Q^4 \psi \rangle \sin \gamma + \langle \varepsilon_2, Q^4 \varphi \rangle \cos \gamma \right] + O(K^\frac{5}{2} s^{-4}) \tag{4.20} \]

for \( s \) large, where \( \alpha_5 = 32 \left( \int y^2 Q^2 \right)^{-1} \).

**Proof.** All the estimates in the proof hold for \( s \) large (possibly depending on \( K \)). By the orthogonality relations (4.2), we have

\[ \mathcal{I}[Q, 0] = \mathcal{I}[y^2 Q, 0] = \mathcal{I}[0, \lambda Q] = \mathcal{I}[0, \rho] = 0. \tag{4.21} \]

We use the simplified estimate (4.6) to derive the estimates (4.18) on \( m_a, m_\lambda, m_b \) and \( m_\gamma \). Using first (4.6) for \( \mathcal{I}[Q, 0] = 0 \), i.e. \( g = Q \) and \( h = 0 \), together with the identities \( L_- Q = 0, \langle Q, \lambda Q \rangle = 0 \) and \( \langle \rho, Q \rangle = \frac{1}{8} \int y^2 Q^2 \neq 0 \) (by (2.2), (1.21) and (2.3)), we obtain

\[ |m_a| \lesssim s^{-3}. \tag{4.22} \]

Next, using (4.6) for \( \mathcal{I}[y^2 Q, 0] = 0 \), together with the identities\( L_-(y^2 Q) = -4 \lambda Q, \langle \varepsilon, i M_{-b} \lambda Q \rangle = 0, \langle \lambda Q, y^2 Q \rangle = -\int y^2 Q^2 \neq 0 \) (by (2.2), (1.21), and (2.3)), we obtain

\[ |m_\lambda| \lesssim s^{-3}. \tag{4.23} \]

Similarly, using (4.6) for \( \mathcal{I}[0, \lambda Q] = 0 \), together with the identities \( L_+(\lambda Q) = -2Q, \langle \varepsilon, i M_{-b} Q \rangle = 0, \langle \lambda Q, y^2 Q \rangle = -\int y^2 Q^2 \neq 0 \) (by (2.2), (1.21), and (2.3)), we obtain

\[ |m_b| \lesssim s^{-3}. \tag{4.24} \]

Last, using (4.6) for \( \mathcal{I}[0, \rho] = 0 \), the identities \( L_+ \rho = \frac{y^2}{2} Q, \langle \varepsilon, i M_{-b} y^2 Q \rangle = 0 \) and \( \langle \rho, Q \rangle = \frac{1}{8} \int y^2 Q^2 \neq 0 \) (by (2.2), (1.21), and (2.3)), and with the estimate (3.39), we obtain

\[ |m_\gamma| \lesssim s^{-3}. \tag{4.25} \]

Thus we have proved (1.15).

Finally, using (4.8) for \( \mathcal{I}[Q, 0] = 0 \), together with the identities \( L_- Q = 0, \langle Q, \lambda Q \rangle = 0 \) and \( \langle \rho, Q \rangle = \frac{1}{8} \int y^2 Q^2 \neq 0 \) (by (2.2), (1.21) and (2.3)) and with the estimate (3.39), we obtain (4.20). \( \square \)
Lemma 2.2 (ii) that there exists a unique pair of even functions \((\varphi_0, \psi_0)\) such that
\[
\begin{align*}
\varphi_0 + L_+ \varphi_0 &= -Q^4 \psi_1 \\
\varphi_0 + L_- \psi_0 &= Q^4 \varphi_1
\end{align*}
\] (4.26)
and satisfying
\[
\varphi_0(x) = c + v(x), \quad \psi_0(x) = -c + w(x)
\]
where \(c \in \mathbb{R}\) and \(v, w \in \mathcal{Y}\). We define
\[
\mathcal{J} = \cos \gamma \int \varepsilon_1 \psi_0 \Theta_0(y \sqrt{\lambda}) + \sin \gamma \int \varepsilon_2 \varphi_0 \Theta_0(y \sqrt{\lambda}).
\] (4.27)
Since \(\varphi_0\) and \(\psi_0\) need not be in \(L^2\), we introduce the cut-off function \(\Theta_0(y \sqrt{\lambda})\) in the definition of \(\mathcal{J}\). (Recall that the function \(\Theta_0\) is defined in (5.14).)

Lemma 4.5. Under the assumption (1.4), for \(s\) large
\[
|\mathcal{J}| \lesssim K^4 s^{-\frac{7}{4}}
\] (4.28)
and
\[
\mathcal{J}_s = \langle \varepsilon_1, Q^4 \psi_1 \rangle \sin \gamma + \langle \varepsilon_2, Q^4 \varphi_1 \rangle \cos \gamma + O(K^4 s^{-\frac{7}{4}}).
\] (4.29)

Proof. By (5.13),
\[
|\mathcal{J}| \lesssim \|\Theta_0(y \sqrt{\lambda})\|_{L^2} \|\varepsilon\|_{L^2} \lesssim \lambda^{-\frac{1}{4}} \|\varepsilon\|_{L^2} \lesssim K^4 s^{-\frac{7}{4}},
\]
which proves the first statement.

Next, setting \(\Theta_1(s, y) = \Theta_0(y \sqrt{\lambda})\) and \(\Theta_2(s, y) = y \sqrt{\lambda} \Theta_0'(y \sqrt{\lambda})\) (so that it holds \(y \partial_y \Theta_1(y) = \Theta_2(y)\)), we obtain by direct differentiation
\[
\mathcal{J}_s = \cos \gamma \int \left( \partial_s \varepsilon_1 \psi_0 \Theta_1 + \frac{1}{2} \varepsilon_1 \lambda_s \psi_0 \Theta_2 + \gamma_s \varepsilon_2 \varphi_0 \Theta_1 \right) + \sin \gamma \int \left( \partial_s \varepsilon_2 \varphi_0 \Theta_1 + \frac{1}{2} \varepsilon_2 \lambda_s \varphi_0 \Theta_2 - \gamma_s \varepsilon_1 \psi_0 \Theta_1 \right)
= \cos \gamma \mathcal{J}^1_s + \sin \gamma \mathcal{J}^2_s.
\]
Using (5.10), (5.20), the identity \(\Lambda(\psi_0 \Theta_1) = (\Lambda \psi_0) \Theta_1 + \psi_0 \Theta_2\) and integration by parts, we deduce that
\[
\mathcal{J}^1_s = -\int \varepsilon_2 \partial^2_y \psi_0 \Theta_1 - \psi_0 \Theta_1 + 2 \partial_y \psi_0 \partial_y \Theta_1 + \psi_0 \partial^2_y \Theta_1
\]
\[
- \int 3(f(V + \varepsilon) - f(V)) \psi_0 \Theta_1
+ \int (b - m \lambda) \varepsilon_1 [(\Lambda \psi_0) \Theta_1 + \psi_0 \Theta_2] + \int m \gamma \varepsilon_2 \psi_0 \Theta_1
- \int 3 S_0 \psi_0 \Theta_1 - \int 3 R \psi_0 \Theta_1 + \frac{1}{2} \int \varepsilon_1 \lambda_s \psi_0 \Theta_2 + \int \gamma_s \varepsilon_2 \varphi_0 \Theta_1.
\]
Using (4.26), $b - m_\lambda + \frac{\lambda}{\lambda} = 0$, and $\gamma_s - 1 = m_\gamma$, we obtain
\[
J_s^1 = \int \varepsilon_2 Q^4 \varphi_1 \Theta_1 - \int (3(f(V + \varepsilon) - f(V)) - Q^4 \varepsilon_2) \psi_0 \Theta_1
- \int \varepsilon_2(2\partial_y \psi_0 \partial_y \Theta_1 + \psi_0 \partial_y^2 \Theta_1)
+ \int (b - m_\lambda)\varepsilon_1 (\Lambda \psi_0) \Theta_1 + \frac{1}{2} \psi_0 \tilde{\Theta}_1)
+ \int m_\gamma \varepsilon_2 \psi_0 \Theta_1
- \int \Re S_0 \psi_0 \tilde{\Theta}_1 + \int m_\gamma \varepsilon_2 \varphi_0 \tilde{\Theta}_1.
\]
We estimate, using (4.4) and $\varphi_0, \psi_0 \in Z_0$,
\[
\left| \int \varepsilon_2 \partial_y \psi_0 \partial_y \Theta_1 \right| \lesssim \|\varepsilon\|_{L^2} \|\partial_y \psi_0\|_{L^2} \|\partial_y \tilde{\Theta}_1\|_{L^\infty} \lesssim \sqrt{\lambda} \|\varepsilon\|_{L^2} \lesssim K^{\frac{1}{2}} s^{-\frac{3}{2}},
\]
and
\[
\left| \int \varepsilon_2 \psi_0 \partial_y^2 \Theta_1 \right| \lesssim \|\varepsilon\|_{L^2} \|\psi_0\|_{L^\infty} \|\partial_y^2 \tilde{\Theta}_1\|_{L^2} \lesssim \lambda^2 \|\varepsilon\|_{L^2} \lesssim K^{\frac{1}{4}} s^{-\frac{7}{2}}.
\]
By similar estimates,
\[
\left| \int (b - m_\lambda)\varepsilon_1 (\Lambda \psi_0) \Theta_1 + \frac{1}{2} \psi_0 \tilde{\Theta}_1 \right| + \int m_\gamma \varepsilon_2 \psi_0 \tilde{\Theta}_1 \lesssim \|S_0\|_{L^2} + \|\Re\|_{L^2} \|\psi_0\|_{L^\infty} \|\tilde{\Theta}_1\|_{L^2} \lesssim s^{-\frac{7}{2}}.
\]
Thus,
\[
J_s^1 = \int \varepsilon_2 Q^4 \varphi_1 \Theta_1 - \int (3(f(V + \varepsilon) - f(V)) - Q^4 \varepsilon_2) \psi_0 \tilde{\Theta}_1 + O(K^{\frac{1}{4}} s^{-\frac{5}{2}}).
\]
Similarly,
\[
J_s^2 = \int \varepsilon_1 Q^4 \psi_1 \tilde{\Theta}_1 + \int (\Re (f(V + \varepsilon) - f(V)) - 5Q^4 \varepsilon_1) \varphi_0 \tilde{\Theta}_1 + O(K^{\frac{1}{4}} s^{-\frac{5}{2}}).
\]
We claim that
\[
\int |f(V + \varepsilon) - f(V) - [5Q^4 \varepsilon_1 + iQ^4 \varepsilon_2]\| \tilde{\Theta}_1 \lesssim K^{\frac{1}{4}} s^{-3},
\]
and the conclusion follows. The claim is an immediate consequence of the definition of $\tilde{\Theta}_1$ and the pointwise estimate
\[
|f(V + \varepsilon) - f(V) - [5Q^4 \varepsilon_1 + iQ^4 \varepsilon_2]| \lesssim K^{\frac{1}{4}} s^{-\frac{3}{2}} e^{-\frac{4\lambda}{s}}
+ s^{-\frac{5}{2}} (s^{-5} + |y|^5 + s^{-5}|y|^{10}).
\] (4.30)
Since $f(V + \varepsilon) - f(V) = M_{\lambda b}[f(M_b V + M_b \varepsilon) - f(M_b V)]$, it follows from formula (4.13) that
\[
f(V + \varepsilon) - f(V) - [5Q^4 \varepsilon_1 + iQ^4 \varepsilon_2] = 5Q^4[M_{\lambda b} E_1 - \varepsilon_1] + iQ^4[M_{\lambda b} E_2 - \varepsilon_2]
+ M_{\lambda b}(Q^4 - Q^4)(5 E_1 + i E_2) + \Re_3,
\]
Moreover, using (3.30) and (4.15), we see that the bootstrap estimates (4.4) hold for apply to the solution \( u \). In this section, we derive energy estimates in the original variables \((t,x)\) rather than the rescaled variables \((s,y)\). Finally, where, from (4.13),

\[
\lambda \geq s^{-5} e^{-\frac{1}{2} \delta} (\delta^{-5} + |y|^5 + s^{-5}|y|^5).
\]

Using the first estimate in (4.14), we deduce that

\[
\lambda \geq s^{-3} e^{-\frac{1}{4} \delta} + s^{-\frac{15}{2}} (\delta^{-5} + |y|^5 + s^{-5}|y|^5).
\]

Moreover, using (3.30) and (4.15), we see that

\[
Q^4 [M-\varepsilon E_1 - \varepsilon |V + |E_2| \leq s^{-1} \| \varepsilon \| L^{\infty} e^{-|y|} \leq K \frac{4}{s} s^{-3} e^{-|y|}.
\]

Finally, \( |Q_4^4 - Q_4^1| \leq |a| e^{-|y|} \leq s^{-2} e^{-|y|} \), so that

\[
|Q_4^4 - Q_4^1| (5E_1 + iE_2) \leq s^{-2} e^{-|y|} \| \varepsilon \| L^{\infty} \leq K \frac{4}{s} s^{-3} e^{-|y|}.
\]

This proves the estimate (4.30) and completes the proof. □

**Lemma 4.6.** Under the assumption (4.4), for \( s \) large

\[
\left| \frac{d}{ds} [a + \alpha_3 \lambda \frac{s^2}{\lambda} J] - \Omega \right| \leq K \frac{s}{s} s^{-4}.
\]

**Proof.** We have

\[
\frac{d}{ds} [a + \alpha_3 \lambda \frac{s^2}{\lambda} J] = \Omega + m_a + \frac{3}{2} \alpha_3 \lambda s \frac{s^2}{\lambda} J + \alpha_3 \lambda \frac{s^2}{\lambda} J.
\]

Since \( |\alpha_3| \leq |b| + |m_a| \leq s^{-1} \), the result follows from Lemmas 4.4 and 4.5. □

5. Energy estimates

Consider a solution \( u \) of (1.1) as in Section 4.1 with \( T = T_n = -\frac{1}{\lambda} \) as defined in (3.2) and \( S = S_n = n \). Assume that \( u \) can be decomposed as in Section 4 and that the bootstrap estimates (4.4) hold for \( s \leq S_n \) close to \( S_n \). In particular (3.3) and (3.13) also hold; hence, the estimates deduced in Lemma 5.1 and Proposition 5.2 apply to the solution \( u \). In this section, we derive energy estimates in the original variables \((t,x)\) rather than the rescaled variables \((s,y)\). The main reason is the simplicity of equation (5.4) below, compared to (4.3) which involves the operator \( \Lambda \). However, we will sometimes reintroduce the notation \( \varepsilon \) to take advantage of the spectral properties of the operators \( L_{\pm} \) and of the related orthogonality conditions (4.2): see Lemma 2.1.

We let

\[
v(t,x) = \lambda^{-\frac{4}{3}}(s) V(s,y)
\]

\[
\eta(t,x) = \lambda^{-\frac{4}{3}}(s) \varepsilon(s,y).
\]

Note that

\[
u(t,x) = e^{i\gamma(s(t))}(v(t,x) + \eta(t,x)).
\]

The equation for \( \eta \) is now

\[
0 = i\partial_t \eta + \partial_{xx} \eta - \frac{1}{\lambda^2(s(t))} \eta + f(v + \eta) - f(v) - \frac{m_\gamma(s(t))}{\lambda^2(s(t))} \eta + Q_0 + P;
\]

where, from (4.13),

\[
M_\delta \tilde{R}_3 = 10 Q_4^4((\tilde{X} + E_1)^2 - \tilde{X})^2 + 2 Q_4^4((\tilde{Y} + E_2)^2 - \tilde{Y})^2
\]

\[
+ 4i Q_4^4((\tilde{X} + E_1)(\tilde{Y} + E_2) - \tilde{X} \tilde{Y}) + \tilde{R}_1
\]

and thus, using the expression of \( \tilde{R}_1 \),

\[
|\tilde{R}_3| \leq e^{-|y|} (|\tilde{X}| + |\tilde{Y}| + |E_1| + |E_2|)^2 + (|\tilde{X}| + |\tilde{Y}| + |E_1| + |E_2|)^5.
\]
where
\[ P(t, x) = \lambda^{-\frac{3}{2}} R(s, y), \quad Q_0(t, x) = \lambda^{-\frac{3}{2}} S_0(s, y). \]
(5.5)

Since \( \eta \in C(H^3) \cap C^1(H^1) \), the above equation makes sense in \( H^1 \).

It follows from the change of variable (5.2), the estimate on \( t(s) \) written in (3.8) and the bootstrap assumption (4.4) on \( \varepsilon \) that
\[ \|\eta\|_{L^2} \lesssim K^\frac{1}{4} |t|^2, \]
\[ \|\partial_x \eta\|_{L^2} \lesssim K^\frac{1}{4} |t|, \]
\[ \|x\eta\|_{L^2} \lesssim K^\frac{1}{4} |t|^2. \]
(5.6)

In addition, it follows from (5.1) and (3.26) that
\[ |v| + |x| |\partial_x v| \lesssim \lambda^{-\frac{3}{2}} e^{-\frac{|x|}{\lambda}} + \delta 1_{\{|x| \leq 2\delta\}}, \]
\[ |\partial_x v| \lesssim \lambda^{-\frac{3}{2}} e^{-\frac{|x|}{\lambda}} + \delta \lambda^{-1} 1_{\{|x| \leq 2\delta\}}. \]
(5.7)

In particular,
\[ \lambda^{\frac{3}{2}} \|v\|_{L^\infty} + \lambda^{\frac{3}{2}} \|x\partial_x v\|_{L^\infty} + \| |x|^2 v\|_{L^\infty} + \|v\|_{L^2} + \|x\partial_x v\|_{L^2} \lesssim 1. \]
Moreover, (5.5) and (3.37)-(3.38) yield
\[ \|P\|_{L^2} + \|xP\|_{L^2} \lesssim |t|, \]
\[ \|\partial_x P\|_{L^2} \lesssim 1, \]
(5.8)

while (5.5) and (4.19) yield
\[ \|Q_0\|_{L^2} \lesssim |t|, \]
\[ \|xQ_0\|_{L^2} \lesssim |t|^2, \]
\[ \|\partial_x Q_0\|_{L^2} \lesssim 1. \]
(5.9)

We define
\[ N = \int x^2 |\eta|^2 \]
\[ H = \int \{ |\partial_x \eta|^2 + \lambda^{-2} |\eta|^2 - 2 [F(v + \eta) - F(v) - \Re(f(v)\bar{\eta})] \} \]
\[ K = \Im \int x(\partial_x \eta)\bar{\eta} \]
and
\[ G = \lambda^2 H + 2K + \frac{1}{4 \lambda^2} N. \]

First, we provide direct upper bounds and coercivity estimates on the quantities \( N, H \) and \( K \). Second, we give estimates on the time derivatives of these quantities using (4.4).

**Lemma 5.1.** Assuming (4.4),
\[ N(t) \lesssim K^\frac{1}{2} |t|^4, \]
\[ |H(t)| \lesssim K^\frac{1}{2} |t|^2, \]
\[ |K(t)| \lesssim K^\frac{1}{2} |t|^3. \]
(5.10)
for $t < T_n$ close to $T_n$. Moreover, there exist a constant $\zeta > 0$ such that
\begin{align}
\zeta \left( \| \partial_x \eta(t) \|^2_{L^2} + \lambda^{-2}(t) \| \eta(t) \|^2_{L^2} \right) &\leq \mathcal{H}(t), \\
\zeta \| \eta(t) \|^2_{L^2} &\leq \mathcal{G}(t),
\end{align}
for $t < T_n$ close to $T_n$.

**Proof.** The estimates (5.10) for $\mathcal{N}$ and $\mathcal{K}$ follow directly from (5.6). Next, we note that
\[ \mathcal{H} = \lambda^{-2} \left\{ |\partial_y \xi|^2 + |\xi|^2 - 2 |F(V + \epsilon) - F(V) - \Re(f(V)^\tau)| \right\}. \]
We claim that
\[ |V + \epsilon|^6 - |V|^6 - 6 \Re(|V|^4 V \tau) = 15 \Re(|V|^2 \tau)^2 + 3 |\langle V \mathcal{V} \rangle|^2 + O(|\epsilon|^3 |V|^3) + O(|\epsilon|^6). \]
This is immediate if $|V| \leq 2|\epsilon|$, and follows by developing $|1 + \frac{1}{V}|^6$ if $|V| > 2\epsilon$. Next, by (3.25),
\[ |V| \mathcal{V} = \mathcal{V}^2 + |V|(V - Q)\tau + Q(|V| - Q)\tau = \mathcal{V}^2 + O(s^{-\frac{1}{2}}|\epsilon|). \]
It follows that
\[ \lambda^2 \mathcal{H} = \langle L_+ \epsilon_1, \epsilon_1 \rangle + \langle L_- \epsilon_2, \epsilon_2 \rangle + O(s^{-\frac{1}{2}} \| \epsilon \|^3_{H^1}) + O(|\epsilon|^3_{H^1}). \]
This proves in particular the estimate (5.10) for $\mathcal{H}$. Next, by (4.2),
\[ |\langle \epsilon, Q \rangle| + |\langle \epsilon, y^2 Q \rangle| + |\langle \epsilon, i \rho \rangle| + |\langle \epsilon, i \Lambda Q \rangle| \leq s^{-1} \| \epsilon \|^2_{L^2}. \]
Similarly,
\[ |\langle \epsilon, Q \rangle| + |\langle \epsilon, y^2 Q \rangle| + |\langle \epsilon, i \rho \rangle| + |\langle \epsilon, i \Lambda Q \rangle| \leq s^{-1} \| \epsilon \|^2_{L^2}. \]
By Lemma 2.1 and (4.3),
\[ \lambda^2 \mathcal{H} \geq \zeta_0 \| \epsilon \|^3_{H^1} + O(s^{-\frac{1}{2}} \| \epsilon \|^3_{H^1}), \]
and estimate (5.11) follows for $s$ large, letting $\zeta = \frac{1}{2} \zeta_0$.

We finally prove (5.12). We observe first that
\[ \lambda^2 \int |\partial_x \eta|^2 + b \mathcal{N} \int x(\partial_x \eta) \eta + \frac{1}{4} \frac{b^2}{\lambda^2} \int x^2 |\eta|^2 = \lambda^2 \int |\partial_x \left( \eta e^{i \frac{\lambda}{4} x^2} \right)|^2. \]
Thus,
\[ \mathcal{G} = \lambda^2 \mathcal{H} + b \mathcal{N} + \frac{1}{4} \frac{b^2}{\lambda^2} \mathcal{N} \]
\[ = \lambda^2 \left\{ |\partial_x \left( \eta e^{i \frac{\lambda}{4} x^2} \right)|^2 + \lambda^{-2} |\eta|^2 - 2 |F(v + \eta) - F(v) - \Re(f(v)^\tau)| \right\}. \]
We let $\epsilon_b = M_b \epsilon$ and $V_b = M_b V$, so that
\[ v(t, x) = \lambda^{-\frac{1}{2}}(s)V(s, y) = \lambda^{-\frac{1}{2}}(s)M_b(s, y)V_b(s, y), \]
\[ \eta(t, x) = \lambda^{-\frac{1}{2}}(s)\eta(s, y) = \lambda^{-\frac{1}{2}}(s)M_b(s, y)\epsilon_b(s, y), \]
and
\[ \mathcal{G} = \int \{ |\partial_y \epsilon_b|^2 + |\epsilon_b|^2 - 2 |F(V_b + \epsilon_b) - F(V_b) - \Re(f(V_b)^\tau)| \} \].
Since
\[ |\langle \epsilon_b, Q \rangle| + |\langle \epsilon_b, y^2 Q \rangle| + |\langle \epsilon_b, i \rho \rangle| + |\langle \epsilon_b, i \Lambda Q \rangle| = 0 \]
by \( (4.2) \), we deduce, following the proof of \((5.11)\), that
\[
G \geq \zeta \| \varepsilon b \|_{H^1}^2 \geq \zeta \| \varepsilon b \|_{L^2}^2 = \zeta \| \eta \|_{L^2}^2.
\]
This proves \((5.12)\). (Note that \(G\) does not control \(\| \partial_x \eta \|_{L^2}^2\).) \(\square\)

**Proposition 5.2.** Assuming \((4.4)\),
\[
\frac{d}{dt} \mathcal{N}(t) \lesssim \| \partial_x \eta(t) \|_{L^2} \| x \eta(t) \|_{L^2} + K^{\frac{1}{4}} |t|^3,
\]
\[
\frac{d}{dt} \mathcal{H}(t) \lesssim |t|^{-3} \| \eta(t) \|_{L^2}^2 + K^{\frac{3}{4}} |t|,
\]
\[
\frac{d}{dt} \mathcal{G}(t) \lesssim K^{\frac{3}{4}} |t|^3,
\]
for \(t < T_n\) close to \(T_n\).

To prove Proposition \(5.2\) we will use the next lemmas. In the following lemmas, we assume that \(\varepsilon\) is sufficiently smooth, namely \(\varepsilon \in C([S_1, S_2], H^2(\mathbb{R})) \cap C^1([S_1, S_2], L^2(\mathbb{R}))\) and \(\varepsilon \in C([S_1, S_2], L^2(\mathbb{R}, |y|^2 dy))\), so that the calculations in the proof are valid. Of course, we will apply the lemma to appropriately smooth solutions.

**Lemma 5.3.** Assuming \((4.3)\),
\[
\frac{d}{dt} \mathcal{N} = 4\mathcal{K} + O(K^{\frac{3}{4}} |t|^3),
\]
for \(t < T_n\) close to \(T_n\).

**Lemma 5.4.** Assuming \((4.3)\),
\[
\frac{d}{dt} \mathcal{H}(t) = 2 \frac{b}{\lambda^2} \| \varepsilon \|_{L^2}^2 - 2 \frac{b}{\lambda^2} \langle f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 v^2 \bar{\eta}, \Lambda v \rangle + O(K^{\frac{3}{4}} |t|),
\]
for \(t < T_n\) close to \(T_n\). In particular,
\[
\frac{d}{dt} \mathcal{H}(t) \lesssim |t|^{-3} \| \eta \|_{L^2}^2 + K^{\frac{3}{4}} |t|.
\]

**Lemma 5.5.** Assuming \((4.3)\),
\[
\frac{d}{dt} \mathcal{K} = 2 \int [\partial_x \eta]^2 - 4R \int [F(v + \eta) - F(v) - f(v) \bar{\eta}] + 2 \langle f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 v^2 \bar{\eta}, \Lambda v \rangle + O(K^{\frac{3}{4}} |t|^2),
\]
for \(t < T_n\) close to \(T_n\).

**Proof of Lemma 5.3.** We give the formal argument, the complete proof requires truncation of \(x^2\), see for instance \([2, \text{Proof of Proposition 6.5.1}]\). Multiplying the equation \((5.4)\) by \(x^2 \bar{\eta}\) and taking the imaginary part, we obtain
\[
\frac{1}{2} \frac{d}{dt} \mathcal{N} = 2\mathcal{K} - 3 \int [f(v + \eta) - f(v)] x^2 \bar{\eta} - 3 \int [Q_0 + \mathcal{P}] x^2 \bar{\eta};
\]
and so, since \(|f(v + \eta) - f(v)| \lesssim (|v|^4 + |\eta|^4)|\eta|\),
\[
\left| \frac{d}{dt} \mathcal{N} - 4\mathcal{K} \right| \lesssim |x|^{\frac{1}{2}} \| x \|_{L^\infty} \| \eta \|_{L^2}^2 + \| \eta \|_{L^\infty} \mathcal{N} + (\| xQ_0 \|_{L^2} + \| x\mathcal{P} \|_{L^2}) \sqrt{\mathcal{N}}.
\]
Using \((5.6) - (5.9)\), we deduce

\[
\left| \frac{d}{dt}N - 4K \right| \lesssim K^{\frac{1}{2}}|t|^4 + K^{\frac{5}{2}}|t|^6 + K^{\frac{3}{2}}|t|^3,
\]

which completes the proof. \(\square\)

**Proof of Lemma 5.4.** Using the identities

\[
\begin{align*}
\partial_t F(v) &= \Re([v^4\varphi]\varphi\bar{\varphi}) \\
\partial_t F(v + \eta) &= |v + \eta|^4\Re((v + \eta)\partial_t(\varphi + \bar{\varphi})) \\
\partial_t(\Re(f(\varphi\bar{\varphi}))) &= \Re((3|v|^4\eta + 2|v|^2v^2\bar{\varphi})\partial_t\varphi + |v|^4v^2\partial_t\bar{\varphi}),
\end{align*}
\]

it follows from elementary calculations that (recall that \(dt = \lambda^2ds\))

\[
\frac{d}{dt} \mathcal{H} = -2\langle f(v + \eta) - f(v) - 3|v|^4\eta - 2|v|^2v^2\bar{\varphi}, \partial_t v \rangle - 2\partial_t v - \lambda^{-2}\partial_t V(s, y).
\]

We first estimate \(\mathcal{H}_1\). Note that

\[
\partial_t v(t, x) = \lambda^{-\frac{2}{5}}\partial_t V(s, y) - \frac{\lambda s}{\lambda} \lambda^{-\frac{2}{5}}\Lambda V(s, y).
\]

Since \(\Lambda V(s, y) = \lambda^{\frac{2}{5}}\Lambda v(t, x)\), we deduce that

\[
\partial_t v(t, x) = \lambda^{-\frac{2}{5}}\partial_t V(s, y) - \frac{\lambda s}{\lambda} \lambda^{-2}\Lambda v(t, x);
\]

and so (recall that \(dx = \lambda dy\))

\[
\mathcal{H}_1 = \mathcal{H}_4 + 2\frac{\lambda s}{\lambda} \lambda^{-2}\langle f(v + \eta) - f(v) - 3|v|^4\eta - 2|v|^2v^2\bar{\varphi}, \partial_t v \rangle.
\]

We have

\[
|f(V + \varepsilon) - f(V)| - 3|V|^4\varepsilon - 2|V|^2V^2\bar{\varphi}| \lesssim (\varepsilon^3 + |V|^3)|\varepsilon|^2,
\]

for all \(V, \varepsilon \in \mathbb{C}\). (see the proof of \((5.13)\)). Thus by \((5.20)\) and \(\|\varepsilon\|^2 \lesssim K^{\frac{3}{2}}s^{-6} \lesssim s^{-\frac{3}{5}}\),

\[
\mathcal{H}_4 \lesssim s^4s^{-\frac{3}{5}} \int [(1 + |y|)(1 + 1) + \delta^{-1}1_{J(s)}(e^{-\frac{2\varepsilon}{\lambda}} + s^{-\frac{2}{5}})|\varepsilon|^2.
\]

Since \(|(1 + |y|)(1 + 1)\delta^{-1}1_{J(s)}(e^{-\frac{2\varepsilon}{\lambda}} + s^{-\frac{2}{5}})| \lesssim 1\), we deduce that

\[
\mathcal{H}_4 \lesssim s^{\frac{2}{5}}\|\varepsilon\|^2 \lesssim K^{\frac{3}{2}}s^{-\frac{2}{5}} \lesssim s^{-\frac{3}{5}} \lesssim |t|,
\]

where we used \((5.21)\).

Next, using \((5.21)\) and

\[
(i\partial_{xx} \eta - \lambda^{-2}\eta + f(v + \eta) - f(v), \partial_{xx} \eta - \lambda^{-2}\eta + f(v + \eta) - f(v)) = 0,
\]

which completes the proof. \(\square\)
we obtain
\[
\frac{1}{2} \mathcal{H}_2 = \lambda^{-2} m_\gamma \langle \partial_{xx} \eta - \lambda^{-2} \eta + f(v + \eta) - f(v), i \eta \rangle \\
- \langle \partial_{xx} \eta - \lambda^{-2} \eta + f(v + \eta) - f(v), i P + i Q_0 \rangle
\]
\[
= \mathcal{H}_5 + \mathcal{H}_6.
\]
We have
\[
\mathcal{H}_5 = \lambda^{-2} m_\gamma \langle f(v + \eta) - f(v), i \eta \rangle.
\]
Since
\[
|f(v + \eta) - f(v)| \lesssim (|v|^4 + |\eta|^4)|\eta| \lesssim \lambda^{-2}|\eta|
\]
by (5.6), (5.7) and (4.4), we see that
\[
\mathcal{H}_5 \lesssim |\eta|^2 \lambda^{-2} \eta^2.
\]
Using again (5.24), we arrive at (5.18). The estimate (5.19) also follows.

Proof of Lemma 5.20. We first observe that
\[
\frac{d}{dt} \mathcal{K} = -2 \Im \int \Lambda \bar{\eta} \partial_\gamma \eta.
\]
It now follows from (5.4) that
\[
\frac{d}{dt} \mathcal{K} = -2 \Re \int \Lambda \bar{\eta}^2 \eta + 2 \lambda^{-2} \Re \int \Lambda \bar{\eta} - 2 \Re \int \Lambda \bar{\eta} [f(v + \eta) - f(v)]
\]
\[
+ 2 \lambda^{-2} m_\gamma \Re \int \Lambda \bar{\eta} - 2 \Re \int \Lambda \bar{\eta} Q_0 - 2 \Re \int \Lambda \bar{\eta} P.
\]
Using (1.21) and (1.22), we deduce that
\[
\frac{d}{dt} \mathcal{K} = 2 \int |\partial_x \eta|^2 - 2 \Re \int \Lambda \Pi [f(v + \eta) - f(v)] - 2 \Re \int \Lambda \Pi \mathcal{Q}_0 - 2 \Re \int \Lambda \Pi \mathcal{P}.
\]
Using (1.20) we deduce that
\[
\frac{d}{dt} \mathcal{K} = 2 \int |\partial_x \eta|^2 - 2 \Re \int \Lambda \Pi [f(v + \eta) - f(v)] + 2 \Re \int \Pi \Lambda \mathcal{Q}_0 + 2 \Re \int \Pi \Lambda \mathcal{P}.
\]
We write
\[
-2 \Re \int \Lambda \Pi [f(v + \eta) - f(v)] = -\Re \int \Pi [f(v + \eta) - f(v)]
\]
and integrating by parts
\[
-2 \Re \int x \partial_x \Pi [f(v + \eta) - f(v)] = 2 \Re \int [F(v + \eta) - F(v) - f(v) \Pi]
\]
\[
+ 2 \Re \int x \partial_x \Pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
and so, multiplying by \(x\) and integrating by parts
\[
-2 \Re \int x \partial_x \Pi [f(v + \eta) - f(v)] = 2 \Re \int [F(v + \eta) - F(v) - f(v) \Pi]
\]
\[
+ 2 \Re \int x \partial_x \Pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
We rewrite this last equation in the form
\[
-2 \Re \int x \partial_x \Pi [f(v + \eta) - f(v)] = 2 \Re \int [F(v + \eta) - F(v) - f(v) \Pi]
\]
\[
+ 3 \Re \int [2 \Lambda \Pi - \Pi] [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
Thus we see that
\[
-2 \Re \int \Lambda \Pi [f(v + \eta) - f(v)] = -\Re \int \Pi [f(v + \eta) - f(v)]
\]
\[
+ 2 \Re \int [F(v + \eta) - F(v) - f(v) \Pi]
\]
\[
- \Re \int \Pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
\[
+ 2 \Re \int \Lambda \Pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
which we rewrite in the form
\[
-2 \Re \int \Lambda \Pi [f(v + \eta) - f(v)] = -4 \Re \int [F(v + \eta) - F(v) - f(v) \Pi]
\]
\[
+ 2 \Re \int \Lambda \Pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 \eta^2]
\]
Thus we arrive at the expression
\[
\frac{d}{dt} K = 2 \int |\partial_x \eta|^2 - 4 \Re \int [F(v + \eta) - F(v) - f(v)v] + 2 \Re \int \Lambda \pi [f(v + \eta) - f(v) - 3|v|^4 \eta - 2|v|^2 v^2 \eta] + 2 \Re \int \eta \Lambda Q_0 + 2 \Re \int \eta \Lambda P.
\]

On the other hand, using (5.10), (5.17) and (5.18), we obtain by the Cauchy-Schwarz inequality
\[
2 \Re \int \eta \Lambda Q_0 + 2 \Re \int \eta \Lambda P = O(K^{\frac{4}{3}} |t|^3),
\]
which completes the proof. \(\square\)

End of the proof of Proposition 5.2. The estimate (5.14) is a consequence of (5.17) and the estimate (5.15) is already proved in (5.19).

We finally prove (5.16). By the expression of \(G = \lambda^2 H + bK + \frac{1}{4} \frac{b^2}{\lambda^2} N\), \(dt = \lambda^2 ds\), and the definitions of \(m_\lambda\) and \(m_b\), we have
\[
\frac{dG}{dt} = \lambda^2 \frac{dH}{dt} + b \frac{dK}{dt} + \frac{1}{4} \frac{b^2}{\lambda^2} \frac{dN}{dt} - 2b \frac{b^2}{\lambda^2} K + 2m_\lambda H + m_b + a \frac{K}{\lambda^2} N + \frac{1}{2} \frac{1}{\lambda^2} \left( \frac{a}{\lambda} + \frac{m_b}{\lambda} - \frac{m_\lambda b}{\lambda} \right) b N,
\]
where the last term comes from the identity
\[
\frac{d}{ds} \left( \frac{b}{\lambda} \right) = \frac{b_x - \frac{2 \lambda}{\lambda} b}{\lambda} = \frac{a}{\lambda} + \frac{m_b}{\lambda} - \frac{m_\lambda b}{\lambda}, \tag{5.25}
\]
see (3.11). Combining (5.14), (5.18), (5.20) with the expressions of \(H\) and \(K\), the resulting cancellations imply that
\[
\lambda^2 \frac{dH}{dt} + b \frac{dK}{dt} + \frac{1}{4} \frac{b^2}{\lambda^2} \frac{dN}{dt} - 2b \frac{b^2}{\lambda^2} K = 0 + O(K^{\frac{4}{3}} |t|^3).
\]
By (5.10) and (5.18),
\[
|m_\lambda| |H| + \left| \frac{m_b}{\lambda^2} \right| |K| \lesssim K^{\frac{4}{3}} |t|^4,
\]
and
\[
\left| \frac{a}{\lambda^2} K \right| + \left| \frac{1}{\lambda^2} \left( \frac{a}{\lambda} + \frac{m_b}{\lambda} - \frac{m_\lambda b}{\lambda} \right) b N \right| \lesssim K^{\frac{4}{3}} |t|^\frac{7}{2}.
\]
Thus, we obtain (5.16). \(\square\)

6. Proof of the uniform estimates

We consider \(n \geq s_0\), where \(s_0\) will be chosen sufficiently large later. Recall from (3.2) that \(T_n = -\frac{1}{n}\) and \(S_n = n\). For \(\beta \in (-1, 1)\), we set
\[
\begin{align*}
\epsilon_n & = 0, \quad \gamma_n = n, \quad a_n = \alpha_n \frac{\pi}{\lambda} \sin n, \\
\lambda_n & = b_n = n^{-1} - \alpha_n \frac{\pi}{\lambda} \cos n + \beta n^{-\frac{3}{2}}, \\
\Gamma_n & = (\gamma_n, \lambda_n, b_n, a_n, 0, 0, 0), \\
u_n(x) & = \lambda_n \frac{\pi}{\lambda} e^{i\gamma_n} V[\Gamma_n](y), \quad y = \frac{x}{\lambda_n},
\end{align*}
\tag{6.1}
\]
and
\[
\begin{align*}
\frac{d}{dt} \lambda_n \frac{\pi}{\lambda} e^{i\gamma_n} V[\Gamma_n](y) & = \lambda_n \frac{\pi}{\lambda} \frac{\dot{\lambda}}{\lambda} e^{i\gamma_n} V[\Gamma_n](y), \\
\frac{d}{dt} \frac{1}{\lambda^2} |\lambda_n|^2 & = \lambda_n \frac{\pi}{\lambda} \frac{\dot{\lambda}}{\lambda} + \frac{1}{\lambda^2} \left( \frac{\dot{\lambda}}{\lambda} - \frac{\pi}{\lambda} \frac{\dot{\lambda}}{\lambda} \right) |\lambda_n|^2.
\end{align*}
\tag{6.2}
\]
where \( V[\Gamma_{n}^{\infty}] \) is the blow-up approximate solution defined in (3.18) with the parameter \( \Gamma_{n}^{\infty} \) defined in (6.1) (recall that \( V \) depends on the constant \( \delta = K^{-2} \)). We consider the solution \( u_{n} \) of (1.1) with initial data \( u_{n}(T_{n}) = u_{n}^{\infty} \). Using Lemma 4.3 with \( u(t, x) = u_{n}(T_{n} + t, x) \) and \( S = S_{n} \), there exist a time \( \bar{s}_{n} < S_{n} \) and a \( C^{1} \) function \( \Gamma \) such that \( u_{n} \) decomposes as in (4.11) with (4.17). Since \( \varepsilon_{n}(S_{n}) = \varepsilon_{n}^{\infty} = 0 \), integrating the first four equations in (4.17), we obtain the orthogonality relations (4.2). Note that the bootstrap estimate (4.4) is satisfied with strict inequalities at \( s = S_{n} \) (for \( n \) large) and thus by continuity, it is also satisfied on \([\bar{s}_{n}, S_{n}]\) after possibly increasing \( \bar{s}_{n} < S_{n} \). Note that there exists \( \tau_{K} \) large depending only on \( K \) such that if (4.4) is satisfied on a time interval \([\tau, S_{n}]\), for \( \tau_{K} < \tau < S_{n} \), then the assumption

\[
|\lambda(s) b(s), a(s), j_{1}(s), j_{2}(s), j_{3}(s)| + |d(u_{n}[\Gamma(s)]; \Gamma(s))| < \omega,
\]

of Lemma 4.3 is satisfied for all \( s \in [\tau, S_{n}] \) so that the decomposition can be extended to \([\tau', S_{n}]\) for some \( \tau_{K} < \tau' < \tau \). Thus, assuming \( s_{0} \geq \tau_{K} \), we may define

\[
s_{*}(s, \beta, n) = \inf\{s \in [s_{0}, S_{n}] : (4.4) \text{ holds for } u_{n} \text{ on } [s, S_{n}]\}.
\]

In particular, possibly taking \( s_{0} \) larger, depending on \( K \), we may apply the estimates proved in Sections 3, 4, and 5 to the solution \( u_{n} \) on the rescaled time interval \([s_{*}, S_{n}]\).

The next proposition, which is the main result of this section, shows that for \( K \) large, there exists at least a value of \( \beta = \beta_{n} \) such that (4.4) holds on \([s_{0}, S_{n}]\) where \( s_{0} \) is independent of \( n \).

**Proposition 6.1.** For all sufficiently large \( K \), there exists \( s_{0} \geq \tau_{K} \) such that for all \( n > s_{0} \), it holds \( s_{*}(\beta_{n}, n) = s_{0} \) for some \( \beta_{n} \in (-1, 1) \).

The rest of this section is devoted to the proof of Proposition 6.1. Recall that the estimate of \( m_{n} \) in (4.4) was strictly improved for \( K \) large by Lemma 4.4. In Section 6.2, we improve the bootstrap estimate of \( \varepsilon \) in (4.4). In Section 6.3, we improve the bootstrap estimates of \( \gamma \), \( a \) and \( \frac{b}{a} \) in (4.4). Last, in Section 6.4, we prove by a contradiction argument that there exists at least one value of \( \beta \in (-1, 1) \) such that \( s_{*}(\beta_{n}, n) = s_{0} \).

### 6.1. Closing the bootstrap estimates on the error term.

In this subsection, we prove the following result.

**Lemma 6.2.** For all \( s \in [s_{*}, S_{n}] \),

\[
\|\varepsilon(s)\|_{H^{1}} \lesssim K^{\frac{7}{4}}s^{-2},
\]

\[
\|\varepsilon(s)\|_{L^{2}} \lesssim K^{\frac{3}{4}}s^{-1}.
\]

**Proof.** Define \( t_{*} < 0 \) such that \( T_{n} - t_{*} = \int_{s_{*}}^{S_{n}} \lambda^{2} \). Let \( t \in [t_{*}, T_{n}] \). Integrating (5.10) on \([t, T_{n}]\), using \( G(T_{n}) = 0 \), we obtain

\[
|G(t)| \lesssim K^{|t|}.
\]

Thus, by (5.12),

\[
\|\eta(t)\|_{L^{2}}^{2} \lesssim G(t) \lesssim K^{|t|}.
\]

Inserting this estimate in (5.15), we obtain

\[
\left| \frac{d}{dt} H(t) \right| \lesssim K^{\frac{7}{4}}|t|.
\]
Integrating the latter estimate on \([t, T_n]\), using \(\mathcal{H}(T_n) = 0\), we obtain

\[|\mathcal{H}(t)| \lessapprox K^{\frac{1}{4}}|t|^2.\]

By (6.11), this gives

\[\|\partial_x \eta(t)\|_{L^2}^2 \lessapprox K^{\frac{1}{4}}|t|^2.\]

Thus, by change of variable \(\|\varepsilon(s)\|_{H^1} \lessapprox K^{\frac{1}{4}}s^{-2}\) on \([s_*, S_n]\). We continue using (5.14), the above estimate and (5.6) for \(\|x \eta\|_{L^2}\),

\[
\frac{d}{dt} N(t) \lessapprox \|\partial_x \eta\|_{L^2} \|x \eta\|_{L^2} + K^{\frac{1}{4}}|t|^3 \lessapprox K^{\frac{1}{4}}|t|^3.
\]

Integrating this estimate on \([t, T_n]\), using \(N(T_n) = 0\), we obtain

\[N(t) = \|x \eta(t)\|_{L^2}^2 \lessapprox K^{\frac{1}{4}}|t|^4.\]

The proof of the lemma is complete. \(\square\)

6.2. Closing the parameter estimates.

**Lemma 6.3.** For all \(s \in [s_*, S_n]\),

\[
\begin{align*}
|\gamma(s) - s| & \lessapprox s^{-2}, \\
|a(s) - \alpha_1 s^{-\frac{3}{2}} \sin \gamma(s)| & \lessapprox K^{\frac{1}{4}}s^{-3}, \\
|b(s) - \lambda(s) - 1| & \lessapprox K^{\frac{1}{4}}s^{-1}.
\end{align*}
\]

(6.4)

**Proof.** Integrating on \([s, S_n]\) the estimate \(|\gamma_s - 1| \lessapprox s^{-3}\) in (4.18) and using \(\gamma_{in} = n = S_n\), we obtain

\[|\gamma(s) - s| \lessapprox s^{-2},\]

which is the estimate for \(\gamma\) in (6.4).

Now, by (4.31), we have

\[
\frac{d}{ds}[a + \alpha_2 \lambda^{\frac{3}{2}} J] = \Omega + O(K^{\frac{1}{4}}s^{-4}).
\]

Integrating, using \(J(S_n) = 0\), then (4.28),

\[
\begin{align*}
|a(S_n) - a(s) - \int_s^{S_n} \Omega| & \lessapprox \lambda^{\frac{3}{2}}|J(s)| + K^{\frac{1}{4}}s^{-3} \lessapprox K^{\frac{1}{4}}s^{-3}.
\end{align*}
\]

(6.5)

We now calculate \(\int \Omega\) using the definition of \(\Omega\) in (4.9). First, by \(1 = \gamma_s - m_\gamma\)

\[
\int_s^{S_n} b \lambda^{\frac{3}{2}} \cos \gamma = - \int_s^{S_n} m_\gamma b \lambda^{\frac{3}{2}} \cos \gamma + \int_s^{S_n} b \lambda^{\frac{3}{2}} \frac{d}{ds} \sin \gamma
\]

\[= O(s^{-\frac{3}{2}}) + \int_s^{S_n} b \lambda^{\frac{3}{2}} \frac{d}{ds} \sin \gamma.
\]

Next,

\[
\int_s^{S_n} b \lambda^{\frac{3}{2}} \frac{d}{ds} \sin \gamma = [b \lambda^{\frac{3}{2}} \sin \gamma](S_n) - [b \lambda^{\frac{3}{2}} \sin \gamma](s) - \int_s^{S_n} \frac{d}{ds} (b \lambda^{\frac{3}{2}}) \sin \gamma;
\]
then, using (4.13) and (4.4) (in particular, $|a| \lesssim s^{-\frac{5}{2}}$)
\[
\int_s^{S_n} \frac{d}{ds} (b\lambda^2 s) \sin \gamma = -\frac{5}{2} \int_s^{S_n} b^2 \lambda^2 s \sin \gamma + \int_s^{S_n} \lambda^2 \left(a + m_b + \frac{3}{2} b m_\lambda\right) \sin \gamma \\
= -\frac{5}{2} \int_s^{S_n} b^2 \lambda^2 s \sin \gamma + O(s^{-3}).
\]
We integrate by parts again
\[
\int_s^{S_n} b^2 \lambda^2 s \sin \gamma = -\int_s^{S_n} m_b b^2 \lambda^2 s \sin \gamma - \int_s^{S_n} b^2 \lambda^2 s \frac{d}{ds} \cos \gamma \\
= O(s^{-\frac{1}{2}}) - \int_s^{S_n} b^2 \lambda^2 s \frac{d}{ds} \cos \gamma.
\]
Next,
\[
\int_s^{S_n} b^2 \lambda^2 s \frac{d}{ds} \cos \gamma = [b^2 \lambda^2 s \cos \gamma](S_n) - [b^2 \lambda^2 s \cos \gamma](s) - \int_s^{S_n} \frac{d}{ds} (b^2 \lambda^2 s) \cos \gamma \\
= O(s^{-\frac{1}{2}}) - \int_s^{S_n} \frac{d}{ds} (b^2 \lambda^2 s) \cos \gamma;
\]
then, using (4.4) and (4.13)
\[
\int_s^{S_n} \frac{d}{ds} (b^2 \lambda^2 s) \cos \gamma = -\frac{7}{2} \int_s^{S_n} b^3 \lambda^2 s \cos \gamma + \int_s^{S_n} b \lambda^2 \left(2a + 2m_b + \frac{3}{2} b m_\lambda\right) \cos \gamma \\
= O(s^{-\frac{1}{2}}).
\]
Collecting the above identities, we obtain
\[
\int_s^{S_n} b\lambda^2 s \cos \gamma = [b\lambda^2 s \sin \gamma](S_n) - [b\lambda^2 s \sin \gamma](s) + O(s^{-3}).
\]
The other three terms in (3.9) can also be calculated by integration by parts (once only) and they are all $O(s^{-3})$. Hence, in the end we obtain
\[
\int_s^{S_n} \Omega = \alpha_1 [b\lambda^2 s \sin \gamma](S_n) - \alpha_1 [b\lambda^2 s \sin \gamma](s) + O(s^{-3}) \\
= \alpha_1 S_n^{-\frac{3}{2}} \sin \gamma(S_n) - \alpha_1 s^{-\frac{3}{2}} \sin \gamma(s) + O(s^{-3}),
\]
where we used (4.4) in the last line. Using now
\[
a(S_n) = \alpha_1 S_n^{-\frac{3}{2}} \sin \gamma(S_n),
\]
as specified in (5.1), we finally obtain from (5.5),
\[
|a(s) - \alpha_1 s^{-\frac{3}{2}} \sin \gamma(s)| \lesssim K^{-\frac{3}{2}} s^{-3},
\]
which is the desired estimate related to $a$ in (6.4).

Last, we observe that by the calculation (5.25) and (4.18)
\[
\frac{d}{ds} \left(\frac{b}{\lambda}\right) = \frac{a}{\lambda} + O(s^{-2}),
\]
so that using (6.6)
\[
\left| \frac{d}{ds} \left(\frac{b}{\lambda}\right) - \alpha_1 s^{-\frac{3}{2}} \sin \gamma \right| \lesssim K^{-\frac{3}{2}} s^{-2}.
\]
Integrating on \([s, S_n]\) and using \(\lambda(S_n) = b(S_n)\) from (6.1), we obtain
\[
\left| \frac{b(s)}{\lambda(s)} - 1 \right| \lesssim K^{\frac{1}{2}} s^{-1} + \left| \int_s^{S_n} (s')^{-\frac{3}{2}} \sin \gamma(s') \right|.
\]
We observe that, using \(1 = \gamma_s - m_\gamma\) and then integrating by parts
\[
\int_s^{S_n} (s')^{-\frac{3}{2}} \sin \gamma(s') ds' = -\int_s^{S_n} m_\gamma(s')^{-\frac{3}{2}} \sin \gamma(s') ds' \
+ \int_s^{S_n} (s')^{-\frac{3}{2}} \gamma(s') \sin \gamma(s') ds' \
= -\int_s^{S_n} m_\gamma(s')^{-\frac{3}{2}} \sin \gamma(s') ds' - \frac{3}{2} \int_s^{S_n} (s')^{-\frac{3}{2}} \cos \gamma(s') ds' \
- S_n^{-\frac{3}{2}} \cos \gamma(S_n) + s^{-\frac{3}{2}} \cos \gamma(s);
\]
and so, applying (4.18)
\[
\left| \int_s^{S_n} (s')^{-\frac{3}{2}} \sin \gamma(s') ds' \right| \lesssim s^{-\frac{3}{2}}.
\]
Thus, we have proved
\[
\left| \frac{b(s)}{\lambda(s)} - 1 \right| \lesssim K^{\frac{1}{2}} s^{-1},
\]
which is the estimate related to \(\frac{b}{\lambda}\) in (6.4).

Using Lemmas 4.4, 6.2 and 6.3, we see that if \(K\) is sufficiently large (independent of \(n\)), then on \([s_*, S_n]\),
\[
\begin{align*}
|\gamma(s) - s| & \leq \frac{1}{2} s^{-\frac{7}{4}}, \\
\left| \frac{b(s)}{\lambda(s)} - 1 \right| & \leq \frac{1}{2} K s^{-1}, \\
|\alpha(s) - \alpha_1 s^{-\frac{7}{4}} \sin \gamma(s)| & \leq \frac{1}{2} K^{\frac{1}{2}} s^{-3}, \\
|\tilde{m}(s)| & \leq \frac{1}{2} K s^{-3}, \\
||\epsilon(s)||_{H^1} & \leq \frac{1}{2} K^\frac{1}{2} s^{-2}, \\
||\gamma\epsilon(s)||_{L^2} & \leq \frac{1}{2} K^\frac{1}{2} s^{-1},
\end{align*}
\]
which means that all the estimates in [4.4] are strictly improved except the estimate of \(b\). We now consider a value of \(K\) sufficiently large so that (6.8) holds on \([s_*, S_n]\).

6.3. **Closing the bootstrap by a contradiction argument.** Closing the estimate of \(b\) in (4.4) requires a specific contradiction argument related to the choice of the free parameter \(\beta \in (-1, 1)\).

We set
\[
g(s) = g_\beta(s) = s^{\frac{7}{4}} \left( b(s) - s^{-1} + \alpha_1 s^{-\frac{7}{4}} \cos \gamma(s) \right)
\]
so that
\[
|g(s)| \leq 1 \iff |b(s) - s^{-1} + \alpha_1 s^{-\frac{7}{4}} \cos \gamma(s)| \leq s^{-\frac{7}{4}}.
\]
Hence, by \((6.1)\) and \((4.4)\),
\[ g(S_n) = \beta \quad \text{and} \quad |g(s)| \leq 1. \]

Using \((4.18)\) and the estimate on \(a\) in \((4.4)\), we have
\[ b_s + b^2 - \alpha_1 s^{-2} \sin \gamma = m_0 + a - \alpha_1 s^{-2} \sin \gamma = O(Ks^{-3}). \]

Thus, by direct computation, and \((4.4)\)
\[ g_s = \frac{7}{4} s^{-1} g + s^\frac{7}{4} \left( -b^2 + s^{-2} - \alpha_1 m_0 s^{-2} \sin \gamma - \frac{5}{2} \alpha_1 s^{-2} \cos \gamma + O(Ks^{-3}) \right) \]
\[ = -\frac{g}{4s} + O(Ks^{-\frac{3}{2}}), \]
where we have used, by \((4.4)\),
\[ s^\frac{7}{4}(-b^2 + s^{-2}) = -s^\frac{7}{4}(b + s^{-1})(b - s^{-1}) = -\frac{2}{s}g + O(s^{-\frac{7}{4}}). \]

This estimate implies the following properties, for \(s\) large,
\[ g(s) = 1 \implies g_s(s) < 0 \quad \text{and} \quad g(s) = -1 \implies g_s(s) > 0. \quad (6.9) \]

Here, \(s_0\) large is fixed so that all the previous estimates hold, and we work for any \(n > s_0\). For the sake of contradiction, assume that for any \(\beta \in (-1, 1)\), \(s_*(\beta, n) = s_*(\beta) > s_0\). We claim now that
\[ |g(s_*(\beta))| = 1. \]

Indeed, if \(|g(s_*(\beta))| < 1\), then by \((6.8)\), the bootstrap estimate \((4.4)\) is satisfied with strict inequalities at \(s = s_*\), which is a contradiction with the definition of \(s_*\) by continuity. By \((6.9)\), the map \(s_*\) is continuous on \((-1, 1)\). We deduce that the map
\[ g \circ s_* : \beta \in (-1, 1) \mapsto g(s_*(\beta)) \in \{-1, 1\} \]
is continuous. Moreover, from \((6.9)\), for \(\beta\) close to 1, \(g(s_*(\beta)) = 1\) and for \(\beta\) close to \(-1\), \(g(s_*(\beta)) = -1\). Thus,
\[ g \circ s_*((-1, 1)) = \{-1, 1\}, \]
which is a contradiction since \((-1, 1)\) is connected. Therefore, for any \(n\) large, there exists \(\beta_n \in (-1, 1)\) such that for any \(s \in [s_0, S_n]\), \(|g(s)| < 1\), which completes the proof of Proposition \(6.1\).

---

### 7. Proof of the Main Result

Let
\[ r_*(x) = \Theta(x) \left( |x| + i\kappa |x|^2 \right). \quad (7.1) \]

It follows that \(r_*(x) = |x| + i\kappa |x|^2\) on \((-\delta, \delta)\) and \(|r_*|_{L^2} \leq C\delta^\frac{7}{2}\), for some constant independent of \(\delta\). In this section, we consider \(K\) sufficiently large to satisfy the assumptions of Proposition \(6.1\). This corresponds to choosing \(\delta > 0\) sufficiently small.

In the rest of this section, the implicit constants in inequalities may depend on \(\delta\), but are independent of \(n\).
7.1. Construction of a sequence of solutions. In the next proposition, we construct a sequence of solutions \((u_n)\) of (1.1), defined on \([t_1, T]\), where \(t_1 < 0\) is independent of \(n\), satisfying uniform estimates related to the blow-up behavior expected in Theorem [1].

**Proposition 7.1.** There exists \(t_1 < 0\) such that, for all \(n > 1\) large, there exists \(\beta_n \in (-1, 1)\) with the following property. If \(u_n\) be the solution of (1.1) such that \(u_n(T_n) = u_n^0\), where \(u_n^0\) is defined in (6.1)–(6.2), then \(u_n\) exists on \([t_1, T]\) and satisfies the following uniform estimates

\[
\left\| u_n(t) - \frac{e^i}{|t|^{\frac{3}{2}}} Q\left(\frac{x}{t}\right) - r_n \right\|_{L^2} \lesssim |t|^{\frac{3}{4}},
\]

(7.2)

\[
|u_n(t)|_{H^1} \lesssim |t|^{-1}.
\]

(7.3)

\[
|zu_n(t)|_{L^2} \lesssim 1.
\]

(7.4)

**Remark 7.1.** We observe by change of variable that

\[
\lim_{t \to 0} \left\{ S(t) - \frac{e^i}{|t|^{\frac{3}{2}}} Q\left(\frac{x}{t}\right) \right\} = 0 \quad \text{in} \quad L^2(\mathbb{R}),
\]

where \(S\) is defined in (1.3), which justifies the equality of the limits in (1.10).

**Proof.** We use Proposition [6.1] passing from the variable \((s, y)\) to the variable \((t, x)\), and using (1.4). Let \(t_0 = t_0(n)\) be defined by \(T_n - t_0 = J_1(s_n)\). By (5.8), we have the estimate \(|t_0| \gtrsim s_n^{-1}\), and so there exists \(t_1\) independent of \(n\) such that \(t_0(n) \leq t_1 < 0\).

We now consider any \(t \in [t_1, T]\). For the sake of readability, we do not mention the \(n\) dependence in the following formulas. Let

\[
w(t, x) = e^{i\gamma} \lambda^{-\frac{3}{2}} W(s, y),
\]

\[
z(t, x) = e^{i\gamma} \lambda^{-\frac{1}{2}} Z(s, y).
\]

We recall from (5.3) and \(V = W + Z\) that

\[u = w + z + e^{i\gamma} \eta.\]

First, from (5.6), we know that

\[
|\eta(t)|_{L^2} \lesssim |t|^2, \quad |\partial_x \eta(t)|_{L^2} \lesssim |t|, \quad |xz(t)|_{L^2} \lesssim |t|^2.
\]

Second, we estimate \(z\). From (6.24), recall that

\[
|Z| \lesssim s^{-\frac{3}{2}} 1_{J(s)} + s^{-\frac{3}{2}} (|y|^3 + 1) 1_{J(s)} \lesssim s^{-\frac{3}{2}} 1_{J(s)}
\]

and

\[
|\partial_y Z| \lesssim (1 + |y|)^{-1} \left[ s^{-\frac{3}{2}} 1_{J(s)} + s^{-\frac{3}{2}} (|y|^3 + 1) 1_{J(s)} \right] \lesssim s^{-\frac{5}{2}} 1_{J(s)}.
\]

Thus,

\[
|Z(t)|_{L^2} \lesssim s^{-1}, \quad |\partial_y Z(t)|_{L^2} \lesssim s^{-2}, \quad |yZ(t)|_{L^2} \lesssim 1.
\]

By change of variable and (6.8), we obtain

\[
|z(t)|_{H^1} \lesssim |t|, \quad |xz(t)|_{L^2} \lesssim |t|.
\]

Last, we decompose \(w\) as

\[
w = q + r, \quad q(t, x) = e^{i\gamma} \lambda^{-\frac{3}{2}} M_b Q_a(y), \quad r(t, x) = e^{i\gamma} \lambda^{-\frac{1}{2}} \theta(s, y)(A + iB)(y).
\]
By the change of variable $z = \frac{\lambda}{t}$, we have
\[
\left\| q(t) - \frac{e^{\frac{i}{t}}}{|t|^\frac{3}{2}} Q \left( \frac{x}{t} \right) \right\|_{L^2} = \left\| e^{i\bar{\lambda} - \frac{i}{t} \bar{\lambda} - \frac{3}{2} \bar{\lambda}^2} Q_\alpha \left( \frac{z}{\lambda} \right) - Q(z) \right\|_{L^2}
\]
where
\[
\bar{\gamma} = \gamma - \frac{1}{t}, \quad \bar{\lambda} = \frac{\lambda}{|t|}.
\]
By (4.4) and (3.8), we have
\[
\left| \bar{\gamma} \right| + \left| \bar{\lambda} - 1 \right| + |b| + |a| \lesssim |t|^\frac{2}{3}.
\]
Writing
\[
\left\| e^{i\bar{\lambda} - \frac{i}{t} \bar{\lambda} - \frac{3}{2} \bar{\lambda}^2} Q_\alpha \left( \frac{z}{\lambda} \right) - Q(z) \right\|_{L^2} \lesssim \left\| e^{i\bar{\lambda} - \frac{i}{t} \bar{\lambda} - \frac{3}{2} \bar{\lambda}^2} Q_\alpha \left( \frac{z}{\lambda} \right) - Q \left( \frac{z}{\lambda} \right) \right\|_{L^2} + \left\| Q \left( \frac{z}{\lambda} \right) - Q(z) \right\|_{L^2},
\]
it follows from usual computations, using the decay properties of $Q$ for the second term on the right-hand side, that
\[
\left\| q(t) - \frac{e^{\frac{i}{t}}}{|t|^\frac{3}{2}} Q \left( \frac{x}{t} \right) \right\|_{L^2} \lesssim |t|^\frac{2}{3}.
\]
Let
\[
A_\alpha(s, y) = \lambda^\frac{3}{2} |y| \cos \gamma + \kappa \lambda^\frac{5}{2} |y|^2 \sin \gamma, \\
B_\alpha(s, y) = -\lambda^\frac{3}{2} |y| \sin \gamma + \kappa \lambda^\frac{5}{2} |y|^2 \cos \gamma,
\]
so that
\[
\tau_\alpha(t, x) = e^{i\gamma} \lambda^{-\frac{1}{2}} \theta(s, y)(A_\alpha + iB_\alpha)(y).
\]
Now, note that by change of variable $x = \lambda y$,
\[
\| r - r_\alpha \|_{L^2} = \| \theta((A - A_\alpha) + i(B - B_\alpha)) \|_{L^2}.
\]
By the asymptotics of the functions $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ in Lemmas 2.2 and 2.3 we have
\[
\| \theta((A - A_\alpha) + i(B - B_\alpha)) \|_{L^2} \leq \lambda^\frac{3}{2} \| \theta(\varphi_1 - |y|) \|_{L^2} + \lambda^\frac{5}{2} \| \theta(\psi_1 + |y|) \|_{L^2} \]
\[
+ \lambda^\frac{5}{2} \| \theta(\varphi_2 + \kappa |y|^2) \|_{L^2} + \lambda^\frac{7}{2} \| \theta(\psi_2 + \kappa |y|^2) \|_{L^2} \lesssim \lambda^\frac{3}{2} \| \theta \|_{L^2} \lesssim \delta^\frac{3}{4} |t|.
\]
Thus
\[
\| r(t) - r_\alpha \|_{L^2} \lesssim \delta^\frac{3}{4} |t|.
\]
This estimate completes the proof of (7.3).

Finally, by (4.4), for $s$ large,
\[
\| W(s) \|_{H^1} \lesssim 1, \quad \| y W(s) \|_{L^2} \lesssim \delta^\frac{3}{2} s,
\]
and thus, by change of variable,
\[
\| u(t) \|_{H^1} \leq |t|^{-1}, \quad \| x w(t) \|_{L^2} \lesssim \delta^\frac{4}{5},
\]
which completes the proof of (7.3)–(7.4). \qed
7.2. Compactness argument. Finally, passing to the limit as \( n \to +\infty \) in the sequence of solutions constructed in Proposition 7.1, we construct a solution \( u \) of (1.1) satisfying the conclusions of Theorem 1.

By the bound (7.3)–(7.4), there exists a subsequence \( u_{nk}(t_1) \) of \( u_n(t_1) \) and \( u_0 \in \Sigma \) such that

\[
 u_{nk}(t_1) \to u_0 \quad \text{weakly in } H^1 \text{ as } k \to +\infty,
\]

\[
 u_{nk}(t_1) \to u_0 \quad \text{strongly in } L^2 \text{ as } k \to +\infty.
\]

Let \( u \) be the \( H^1 \) solution of (1.1) such that \( u(t_1) = u_0 \). By the local wellposedness of the Cauchy problem in \( L^2 \) for (1.1) (see e.g. [2, Theorem 4.7.1]), the solution \( u(t) \) exists on \( [t_1, 0) \) and for any \( t \in [t_1, 0) \),

\[
 u_{nk}(t) \to u(t) \quad \text{strongly in } L^2 \text{ as } k \to +\infty.
\]

Passing to the limit in (7.2), we obtain on \( [t_1, 0) \),

\[
 \| u(t) - \frac{e^t}{|t|^\frac{1}{2}} Q \left( \frac{x}{t} \right) - r_* \|_{L^2} \lesssim |t|^\frac{1}{2},
\]

\[
 \| u(t) \|_{H^1} \lesssim |t|^{-1}.
\]

It follows from the first estimate above that \( \{ u(t); t \in [t_1, 0] \} \) is not compact in \( L^2 \). Therefore, the maximal time of existence of the \( L^2 \) solution \( u(t) \) is 0. By persistence of regularity (see e.g. [2, Theorem 4.7.1]), \( u(t) \) blows up in \( H^1 \) at time \( t = 0 \). Next, by the Gagliardo-Nirenberg inequality \( \| g \|_{L^6}^2 \lesssim \| \partial_x g \|_{L^2} \| g \|_{L^2}^2 \), we obtain

\[
 \| u(t) - \frac{e^t}{|t|^\frac{1}{2}} Q \left( \frac{x}{t} \right) - r_* \|_{L^6} \lesssim |t|^{\frac{1}{2}}.
\]

By \( E(Q) = 0 \), we have for any \( t \neq 0 \),

\[
 \left\| \frac{e^t}{|t|^\frac{1}{2}} Q \left( \frac{x}{t} \right) \right\|_{L^6}^6 = \frac{\| Q' \|_{L^2}^2}{t^2}
\]

and thus

\[
 \lim_{t \to 0} t^2 \| u(t) \|_{L^6}^6 = \| Q' \|_{L^2}^2.
\]

Therefore, by conservation of the energy \( E(u(t)) \), we obtain (1.9). This completes the proof of Theorem 1.

APPENDIX A.

In this appendix, we prove Lemma 2.2. First, we prove the following ODE result.

**Lemma A.1.** Let \( m > 0 \). Let \( P_j \) for \( 1 \leq j \leq 4 \) \( \in C(\mathbb{R}, \mathbb{R}) \) and suppose there exists \( \delta > 0 \) such that

\[
 |P_j(x)| \leq Ce^{-(2m+\delta)x}, \quad x \geq 0 \quad (A.1)
\]

If \( u, v \in C^2(\mathbb{R}, \mathbb{R}) \) satisfy

\[
 \begin{cases}
 -u'' + m^2 u + P_1 u + P_2 v = 0 \\
 -v'' + P_3 u + P_4 v = 0
\end{cases}
\]

then there exist constants \( c, d_0, d_1 \in \mathbb{R} \) such that

(i) for \( x \geq 0 \), \( |u(x) - ce^{mx}| \lesssim e^{-mx} \),

(ii) for \( x \geq 0 \), \( |v(x) - d_0 - d_1 x| \lesssim e^{-mx} \).
Proof. Setting \( u' = mw \) and \( v' = mz \), we have

\[
\frac{d}{dx}(u^2 + w^2 + v^2 + z^2) = 4muw + 2mvz + \frac{P_1}{m}uw + \frac{P_2}{m}vw + \frac{P_3}{m}uz + \frac{P_4}{m}vz.
\]

It follows that for every \( \nu > 0 \),

\[
\frac{d}{dx}(u^2 + w^2 + v^2 + z^2) \leq 2(m + \nu)(u^2 + w^2 + v^2 + z^2)
\]

for all sufficiently large \( x \); and so

\[
u^2 + w^2 + v^2 + z^2 \lesssim e^{2(m+\nu)x}, \quad x \geq 0.
\]

It follows that

\[
|P_j(|u| + |v|)| \lesssim e^{-(m+\delta-\nu)x}, \quad x \geq 0, \quad (A.3)
\]

for \( j = 1, 2, 3, 4 \). Next, from the variation of the parameter formula, we see that there exist constants \( u_1 \) and \( u_2 \) such that

\[
\begin{aligned}
&u(x) = u_le^{mx} + u_2e^{-mx} + \frac{1}{2m} \int_0^x (e^{m(x-s)} - e^{-m(x-s)})(P_1u + P_2v)(s) \, ds. \quad (A.4)
&
\end{aligned}
\]

Applying \( (A.3) \) with \( \nu < \delta \) and setting

\[
c = u_1 + \frac{1}{2m} \int_0^\infty e^{-ms}(P_1u + P_2v)(s) \, ds, \quad (A.5)
\]

we deduce from \( (A.4) \) that

\[
\begin{aligned}
&u(x) - ce^{mx} = u_2e^{-mx} - \frac{1}{2m} \int_0^x e^{-m(x-s)}(P_1u + P_2v)(s) \, ds
= -\frac{1}{2m} \int_x^\infty e^{m(x-s)}(P_1u + P_2v)(s) \, ds. \quad (A.6)
&
\end{aligned}
\]

It follows easily from \( (A.3) \) (with \( \nu < \delta \)) and \( (A.6) \) that

\[
|u(x) - ce^{mx}| \lesssim e^{-mx}, \quad x \geq 0. \quad (A.7)
\]

This proves the first part of the statement.

Set now

\[
d_1 = v'(0) + \int_0^\infty (P_3u + P_4v)(\tau) \, d\tau,
\]

which is well defined by \( (A.3) \) (with \( \nu < \delta \)). It follows from the second equation in \( (A.2) \) that

\[
v'(s) - d_1 = -\int_s^\infty (P_3u + P_4v)(\tau) \, d\tau.
\]

Using again \( (A.3) \), we see that

\[
\left| \int_s^\infty (P_3u + P_4v)(\tau) \, d\tau \right| \lesssim e^{-ms},
\]

and we define

\[
d_0 = v(0) - \int_0^\infty \int_s^\infty (P_3u + P_4v)(\tau) \, d\tau ds.
\]

It follows that

\[
v(x) - d_0 - d_1x = \int_x^\infty \int_s^\infty (P_3u + P_4v)(\tau) \, d\tau ds,
\]

so that \( |v(x) - d_0 - d_1x| \lesssim e^{-mx} \). This completes the proof. \( \Box \)
We turn to the proof of Lemma A.2. Set
\[ \tilde{\varphi} = \varphi + \psi, \quad \tilde{\psi} = \varphi - \psi, \quad \tilde{g} = g + h, \quad \tilde{h} = g - h. \]
We see by the expressions of \( L_+ \) and \( L_- \) and direct computations that the system (2.6) is equivalent to
\[
\begin{cases}
-\tilde{\varphi}'' + 2\tilde{\varphi} - 3Q^4\tilde{\varphi} - 2Q^4\tilde{\psi} = \tilde{g} \\
-\tilde{\psi}'' - 3Q^4\tilde{\psi} - 2Q^4\tilde{\varphi} = \tilde{h}.
\end{cases}
\] (A.8)
Recall that we consider only even solutions.

**Lemma A.2.** The following properties hold.

(i) There exists a unique pair of even functions \((\tilde{\varphi}_1, \tilde{\psi}_1) \in \mathcal{Y} \times Z_1\) such that
\[
\begin{cases}
-\tilde{\varphi}_1'' + 2\tilde{\varphi}_1 - 3Q^4\tilde{\varphi}_1 - 2Q^4\tilde{\psi}_1 = 0 \\
-\tilde{\psi}_1'' - 3Q^4\tilde{\psi}_1 - 2Q^4\tilde{\varphi}_1 = 0
\end{cases}
\] (A.9)
and satisfying
\[ \tilde{\psi}_1(x) = \mu_1(x) + c_1 + \tilde{w}_1(x) \] (A.10)
where \( c_1 \in \mathbb{R} \) and \( \tilde{w}_1 \in \mathcal{Y} \).

(ii) Given \( (\tilde{g}, \tilde{h}) \in \mathcal{Y} \times \mathcal{Y} \), there exists a unique even solution \((\tilde{\varphi}, \tilde{\psi}) \in \mathcal{Y} \times Z_0\) of (A.8).

**Proof.** Proof of part (i). Define the solutions \((f_1, g_1)\) and \((f_2, g_2)\) of system (A.9) corresponding to the initial data \( f_1(0) = 1, f_2(0) = 0, g_1(0) = 0, g_2(0) = 0 \) and \( f_2(0) = 0, f_2'(0) = 0, g_2(0) = 1, g_2'(0) = 0 \). From (i) of Lemma A.1 and standard ODE arguments for the regularity and the decay of the derivatives, it follows that there exist \( a_1, a_2 \in \mathbb{R} \) and \( w_1, w_2 \in \mathcal{Y} \) such that
\[
f_1(x) = a_1 \mu_0(x)e^{\sqrt{\mathcal{L}}x} + w_1(x) \quad \text{and} \quad f_2(x) = a_2 \mu_0(x)e^{\sqrt{\mathcal{L}}x} + w_2(x).
\]
Therefore, there exists a linear combination of the pairs \((f_1, g_1)\) and \((f_2, g_2)\), denoted by \((f, g)\), such that \( f \in \mathcal{Y} \). Since the solutions \((f_1, g_1)\) and \((f_2, g_2)\) are independent, the pair \((f, g)\) is non zero and satisfies the system (A.9). By (i) of Lemma A.1 there exist \( b_0, b_1 \in \mathbb{R} \) and \( z \in \mathcal{Y} \) such that
\[
g(x) = b_1 \mu(x) + b_0 + z(x).
\]
Recall from [19, Proposition 2.1.4] that there exist no nonzero bounded solution of (A.9), which means exactly that \( b_1 \neq 0 \). Thus, \((\tilde{\varphi}_1, \tilde{\psi}_1) = \frac{1}{b_1}(f, g)\) satisfies (A.9) and (A.10).

The uniqueness part of (i) also follows from the result of non existence of a non zero bounded solution of (A.9) in [19, Proposition 2.1.4].

Proof of part (ii). Let \((\tilde{g}, \tilde{h}) \in \mathcal{Y} \times \mathcal{Y}\) be a pair of even functions. We first construct a solution of (A.8) in \( \mathcal{Y} \times Z_1\), using a reformulation of the problem.

Indeed, integrating twice the second equation in (A.8), and using that \( \tilde{\psi} \) is even, we obtain
\[
\tilde{\psi} + 3 \int_0^x \int_0^y Q^4 \tilde{\psi} = - \int_0^x \int_0^y (2Q^4 \tilde{\varphi} + \tilde{h}) + C
\]
for some constant \( C \). Now, we will solve the system
\[
\begin{cases}
-\tilde{\varphi}'' + 2\tilde{\varphi} - 3Q^4\tilde{\varphi} - 2Q^4\tilde{\psi} = \tilde{g} \\
\tilde{\psi} + 3 \int_0^x \int_0^y Q^4 \tilde{\psi} = - \int_0^x \int_0^y (2Q^4 \tilde{\varphi} + \tilde{h})
\end{cases}
\] (A.11)
where for simplicity the constant $C$ was chosen equal to 0. We rewrite the second equation in (A.11) in the equivalent form

$$Q^2\psi - T(Q^2\psi) = -Q^2 \int_0^x \int_0^y (2Q^4\bar{\varphi} + \bar{h}),$$  \hspace{1cm} (A.12)$$

where we have defined, for any $u \in L^2(\mathbb{R})$

$$Tu = -3Q^2 \int_0^x \int_0^y Q^2u.$$

It is clear that $T \in \mathcal{L}(L^2(\mathbb{R}))$ and that $T$ is a compact operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. We also observe that if the function $u$ is even, then $Tu$ is also even. Moreover, the adjoint of $T$ is given by (use Fubini’s theorem)

$$T^*u = -3Q^2 \int_x^\infty \int_y^\infty Q^2u. \hspace{1cm} (A.13)$$

We claim that $I - T^*$ has a trivial kernel. Indeed, suppose $u \in L^2(\mathbb{R})$ and $T^*u = u$. By the definition of $T_*$, given $R \geq 0$ and $x > R$, we have by the Cauchy-Schwarz inequality and (1.2)

$$|T^*u(x)| \leq 3Q^2 \int_x^\infty \|u\|_{L^2(\{x>y\})} \left(\int_y^\infty Q^4\right)^{\frac{1}{2}} \leq e^{-2x\|u\|_{L^2(\{x>R\})}} \int_x^\infty e^{-2y}dy \lesssim e^{-4x\|u\|_{L^2(\{x>R\})}}.$$

By the above estimate and $T^*u = u$, we deduce that $u(x) = 0$ for $x$ large. The same vanishing property holds for $v = Q^{-2}u$, which satisfies the ODE $v'' = -3Q^4v$. By uniqueness, $v \equiv 0$ and so $u \equiv 0$ on $\mathbb{R}$. By Fredholm’s alternative, $I - T$ is invertible on $L^2(\mathbb{R})$. Therefore, we may write equation (A.12) in the form

$$\tilde{\psi} = -Q^{-2}(I - T)^{-1} \left(Q^2 \int_x^\infty \int_y^\infty (2Q^4\bar{\varphi} + \bar{h})\right), \hspace{1cm} (A.14)$$

We let

$$v = -\bar{\varphi}'' + 2\bar{\varphi},$$

i.e.

$$\bar{\varphi} = (-\partial_{xx} + 2)^{-1}v, \hspace{1cm} (A.15)$$

where the inverse is in the sense of $L^2(\mathbb{R})$. The first equation in (A.11) becomes

$$v - 3Q^4(-\partial_{xx} + 2)^{-1}v = \bar{g} + 2Q^4\tilde{\psi}.$$  

Using (A.14), we rewrite this last equation in the form

$$v - 3Q^4(-\partial_{xx} + 2)^{-1}v = \bar{g} - 2Q^2(I - T)^{-1} \left(Q^2 \int_x^\infty \int_y^\infty \bar{h}\right) + \frac{4}{3}Q^2(I - T)^{-1}T(Q^2(-\partial_{xx} + 2)^{-1}v).$$

Set

$$T_1v = 3Q^4(-\partial_{xx} + 2)^{-1}v + \frac{4}{3}Q^2(I - T)^{-1}T(Q^2(-\partial_{xx} + 2)^{-1}v)$$

$$= \frac{5}{3}Q^4(-\partial_{xx} + 2)^{-1}v + \frac{4}{3}Q^2(I - T)^{-1}(Q^2(-\partial_{xx} + 2)^{-1}v)$$

$$= \frac{1}{3}Q^2[5 + 4(I - T)^{-1}](Q^2(-\partial_{xx} + 2)^{-1}v).$$
We claim that

\[ I - T_1 \]

with

\[ \tilde{w} \]

On \( L^2(\mathbb{R}) \), the operator \( v \mapsto Q^2(-\partial_{xx} + 2)^{-1}v \) is compact and the operator \( v \mapsto Q^2[5 + 4(I - T)^{-1}]v \) is continuous. Thus, \( T_1 \) is a compact operator. Moreover,

\[ T_1^* u = \frac{1}{3}(-\partial_{xx} + 2)^{-1}[Q^2(5 + 4(I - T^*)^{-1})Q^2 u]. \]

We claim that \( I - T_1^* \) has a trivial kernel. Indeed, suppose \( u \in L^2(\mathbb{R}) \) and \( T_1^* u = u \). In particular, \( u \in H^2(\mathbb{R}) \) and

\[ -u'' + 2u = \frac{1}{3}[Q^2(5 + 4(I - T^*)^{-1})Q^2 u]. \]

Setting

\[ u_1 = (I - T^*)^{-1}Q^2 u, \]  

we obtain

\[ -u'' + 2u = \frac{4}{3}Q^2 u_1 + \frac{5}{3}Q^4 u. \]  

We rewrite equation (A.17) in the form

\[ -\frac{T^* u_1}{Q^2} = u - \frac{u_1}{Q^2}. \]

Applying (A.13) and differentiating twice, we obtain

\[ 3Q^2 u_1 = \left( u - \frac{u_1}{Q^2} \right)'' \]  

Last, we set

\[ w = -\frac{2}{3}\left( u - \frac{u_1}{Q^2} \right), \]  

so that

\[ w'' = -2Q^2 u_1. \]  

Moreover, (A.19) implies

\[ Q^2 u_1 = Q^4 u + \frac{3}{2}Q^4 w. \]  

Substituting (A.21) in (A.18) and (A.20), we obtain the system

\[
\begin{aligned}
- w'' + 2u - 3Q^4 u - 2Q^4 w &= 0, \\
- w'' - 2Q^4 u - 3Q^4 w &= 0.
\end{aligned}
\]

This system does not have any non trivial \( L^2 \) solution by [19 Corollary 2.1.3], so we conclude that \( u \equiv 0 \). This proves that \( I - T_1^* \) has a trivial kernel as claimed.

By Fredholm’s alternative, \( I - T_1 \) is invertible and we define the even function

\[ v = (I - T_1)^{-1}\left[ \tilde{g} - 2Q^2(I - T)^{-1}\left( Q^2 \int_0^x \int_0^y \tilde{h} \right) \right], \]

so that equation (A.16) is satisfied. Let \( \tilde{\varphi} \) be given by (A.14) and let \( \tilde{\psi} \) be given by (A.15). Observe that \( \tilde{\varphi} \in H^2(\mathbb{R}) \) and \( Q^2 \tilde{\psi} \in L^2(\mathbb{R}) \). Moreover, the pair \((\tilde{\varphi}, \tilde{\psi})\) satisfies (A.11). By the second equation in (A.11), we see that \( \tilde{\psi} \in H^2_{\text{loc}}(\mathbb{R}) \) and \( \tilde{\psi}'' \in L^2(\mathbb{R}) \). From the above, it follows that \((\tilde{\varphi}, \tilde{\psi})\) satisfy (A.8). Integrating twice the second equation in (A.8), we deduce that \( \tilde{\psi}(x) = d_1\mu_1(x) + d_0 + w(x) \) with \( d_1, d_0 \in \mathbb{R} \) and \( w \in \mathcal{Y} \), and it follows easily that \( \tilde{\varphi} \in \mathcal{Y} \). Therefore, the pair
$(\tilde{\varphi} - d_1\tilde{\varphi}_1, \tilde{\psi} - d_1\tilde{\psi}_1)$ is a solution of (A.8) in $\mathcal{Y} \times \mathcal{Z}_0$. Finally, uniqueness follows again from [19, Proposition 2.1.4]. □

Let $(\tilde{\varphi}_1, \tilde{\psi}_1)$ given by part (i) of Lemma A.2. It follows that that the pair

$$(\varphi_1, \psi_1) = (\tilde{\varphi}_1 + \tilde{\psi}_1, \tilde{\varphi}_1 - \tilde{\psi}_1)$$

is an even solution of the system

$$\begin{cases}
\psi_1 + L_+ \varphi_1 = 0 \\
\varphi_1 + L_- \psi_1 = 0
\end{cases}$$

(A.23)

such that

$$\varphi_1 - \mu_1 - c_1 \in \mathcal{Y}, \quad \psi_1 + \mu_1 + c_1 \in \mathcal{Y},$$

(A.24)

which proves part (i) of Lemma 2.2.

Part (ii) of Lemma 2.2 follows similarly from part (ii) of Lemma A.2.

Appendix B.

In this appendix, we justify the identity (3.40). Recall that

$$W(s, y) = M_{-b}Q_a + \theta(A + iB)$$

and that

$$m_\gamma = \gamma_s - 1, \quad m_\lambda = \frac{\lambda_s}{\lambda} + b, \quad m_b = b_s + b^2 - a, \quad m_a = a_s - \Omega.$$ 

Some useful calculations

$$\begin{cases}
\partial_s M_{-b} = -iM_{-b}b_s \frac{\partial^2}{\partial y^2} \\
\partial_y M_{-b} = -iM_{-b}b \frac{\partial^2}{\partial y^2} \\
\partial_y^2 M_{-b} = M_{-b}(-i^2 b^2 - b^2 \frac{\partial^2}{\partial y^2}) \\
\partial_s Q_a = a_s \rho \\
\partial_y^2 Q_a = Q_a - a^2 Q - 5aQ^4 \\
\partial_s \theta = \lambda_s y \theta'(\lambda y) = y \frac{\partial_s}{\partial y} \theta
\end{cases}$$

(B.1)

We first rewrite $\mathcal{E}(W)$. We replace $-W - (\gamma_s - 1)W$ by $-\gamma_s W$, so

$$\mathcal{E}(W) - f(W) = i\partial_s W + \partial_y^2 W - i\frac{\lambda_s}{\lambda} \Lambda W - \gamma_s W$$

Next, using $W = M_{-b}Q_a + \theta(A + iB)$, we obtain

$$\mathcal{E}(W) - f(W) = i\partial_s (M_{-b}Q_a) + \partial_y^2 (M_{-b}Q_a) - i\frac{\lambda_s}{\lambda} \Lambda(M_{-b}Q_a) - \gamma_s M_{-b}Q_a + i\partial_s (\theta(A + iB)) + \partial_y^2 (\theta(A + iB)) - i\frac{\lambda_s}{\lambda} \Lambda(\theta(A + iB)) - \gamma_s (\theta(A + iB))$$
We next use $\partial_s(uv) = u\partial_s v + \partial_s uv$, $\partial_{yy}(uv) = u\partial_{yy}v + 2\partial_y u \partial_y v + \partial_{yy} uv$ and $\Lambda(uv) = u\Lambda v + yv \partial_y u$, and we obtain

$$E(W) - f(W) = i(\partial_s M_{-b}) Q_a + iM_{-b} \partial_s Q_a$$
$$+ \partial_{yy} M_{-b} Q_a + 2\partial_y M_{-b} \partial_y Q_a + M_{-b} \partial_{yy} Q_a$$
$$- i\frac{\lambda_s}{\lambda} M_{-b} \Lambda Q_a - i\frac{\lambda_s}{\lambda} y \partial_y M_{-b} Q_a - \gamma_s M_{-b} Q_a$$
$$+ i\partial_y \theta(A + iB) + i\theta(\partial_y A + i\partial_y B)$$
$$+ \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})$$
$$- i\frac{\lambda_s}{\lambda} \theta(\Lambda A + i\Lambda B) - i\frac{\lambda_s}{\lambda} y \theta_y(A + iB) - \gamma_s \theta(A + iB)$$

Using now the calculations (3.1):

$$E(W) - f(W) = M_{-b} b \frac{y^2}{4} Q_a + iM_{-b} \partial_s \rho$$
$$+ M_{-b} \left( - \frac{b}{2} - b^2 \frac{y^2}{4} Q_a - 2iM_{-b} b \frac{y}{2} \partial_y Q_a \right.$$
$$+ \left. M_{-b} \left( Q_a - a \frac{y^2}{4} Q - Q^5 - 5aQ^4 \rho \right) \right.$$
$$- \frac{\lambda_s}{\lambda} M_{-b} \Lambda Q_a - \frac{\lambda_s}{\lambda} b \frac{y^2}{2} Q_a - \gamma_s M_{-b} Q_a$$
$$+ \frac{\lambda_s}{\lambda} y \theta_y(A + iB) + i\theta(\partial_y A + i\partial_y B)$$
$$+ \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})$$
$$- \left. i\frac{\lambda_s}{\lambda} \theta(\Lambda A + i\Lambda B) - i\frac{\lambda_s}{\lambda} y \theta_y(A + iB) - \gamma_s \theta(A + iB) \right)$$

We cancel the terms $i\frac{\lambda_s}{\lambda} y \theta_y(A + iB) - i\frac{\lambda_s}{\lambda} y \theta_y(A + iB)$ and factorize $M_{-b}$:

$$E(W) - f(W) = M_{-b} \left( b \frac{y^2}{4} Q_a + i\partial_s \rho \right.$$
$$- \frac{b}{2} Q_a - b^2 \frac{y^2}{4} Q_a - iby \partial_y Q_a + Q_a - a \frac{y^2}{4} Q - Q^5 - 5aQ^4 \rho$$
$$- \frac{\lambda_s}{\lambda} \Lambda Q_a - \frac{\lambda_s}{\lambda} b \frac{y^2}{2} Q_a - \gamma_s Q_a$$
$$+ i\theta(\partial_y A + i\partial_y B) + \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})$$
$$- \left. i\frac{\lambda_s}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB) \right)$$

We use $Q_a - \gamma_s Q_a = -m_\gamma Q_a$; $-ib\Lambda Q_a$; $b + \frac{\lambda_s}{\lambda} = m_\lambda$ and $-a \frac{y^2}{4} Q = -a \frac{y^2}{4} Q_a + a^2 \rho$:

$$E(W) - f(W) = M_{-b} \left( b \frac{y^2}{4} Q_a + i\partial_s \rho \right.$$
$$- m_\lambda \Lambda Q_a - b^2 \frac{y^2}{4} Q_a - a \frac{y^2}{4} Q - Q^5 - 5aQ^4 \rho + a^2 \rho \frac{y^2}{4} Q_a - m_\gamma Q_a$$
$$+ i\theta(\partial_y A + i\partial_y B) + \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})$$
$$- \left. i\frac{\lambda_s}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB) \right)$$
We reorder the terms in $y^2 Q_a$ and use $b_s - b^2 - a - 2b \frac{\Delta}{\lambda} = m_b - 2bm$:

\[
\mathcal{E}(W) = M_{-b} \left( (m_b - 2bm) \frac{y^2}{4} Q_a + ia_s \rho \right.
\]

\[
- im \lambda \Lambda Q_a - Q^5 - 5aQ^4 \rho + a^2 \frac{y^2}{4} \rho - m \gamma Q_a \right.
\]

\[
+ i \theta (\partial_s A + i \partial_s B) + \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})
\]

\[
- i \frac{\lambda}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
\]

Using now $a_s = m_a + \Omega$ and reordering

\[
\mathcal{E}(W) = M_{-b} \left( -m \gamma Q_a + (m_b - 2bm) \frac{y^2}{4} Q_a + im_a \rho - im \lambda \Lambda Q_a \right)
\]

\[
- Q^5 - 5aQ^4 \rho + a^2 \frac{y^2}{4} \rho + i\Omega \rho \right)
\]

\[
+ i \theta (\partial_s A + i \partial_s B) + \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})
\]

\[
- i \frac{\lambda}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
\]

Using $M_{-b}(-Q^5 - 5aQ^4 \rho + a^2 \frac{y^2}{4} \rho) = R_4 - f(M_{-b}Q_a)$, we find:

\[
\mathcal{E}(W) = M_{-b} \left( -m \gamma Q_a + (m_b - 2bm) \frac{y^2}{4} Q_a + im_a \rho - im \lambda \Lambda Q_a \right)
\]

\[
+ iM_{-b} \Omega \rho + R_4 + f(W) - f(M_{-b}Q_a)
\]

\[
+ i \theta (\partial_s A + i \partial_s B) + \theta_{yy}(A + iB) + 2\theta_y(A_y + iB_y) + \theta(A_{yy} + iB_{yy})
\]

\[
- i \frac{\lambda}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
\]

Taking into account the definitions of $S_0$, $S_1$ and $R_3$, this is equivalent to

\[
\mathcal{E}(W) = S_0 + S_1 + R_4 + R_5 + i \theta \Omega \rho + f(W) - f(M_{-b}Q_a)
\]

\[
+ i \theta (\partial_s A + i \partial_s B) + \theta(A_{yy} + iB_{yy}) - i \frac{\lambda}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
\]

(B.2)

Next, using the definitions of $L_+$ and $L_-$, $m_\gamma = \gamma_s - 1$ and $m_\lambda - b = \frac{\lambda}{\lambda}$, we rewrite

\[
G = -\partial_s B + A_{yy} + 5Q^4 A - by^2 Q^4 B - \gamma_s A + \frac{\lambda}{\lambda} \Lambda B,
\]

\[
H = \partial_s A + B_{yy} + Q^4 B - by^2 Q^4 A - \gamma_s B - \frac{\lambda}{\lambda} \Lambda A
\]

\[
+ (\alpha_1 b \lambda \frac{\gamma}{\gamma} \cos \gamma + \alpha_2 b \lambda \frac{\gamma}{\gamma} \sin \gamma) \rho.
\]

It follows that

\[
R_3 = i \theta (\partial_s A + i \partial_s B) + \theta(A_{yy} + iB_{yy}) - i \frac{\lambda}{\lambda} \theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
\]

\[
+ \theta(5Q^4 A - by^2 Q^4 B) + i \theta(Q^4 B - by^2 Q^4 A)
\]

\[
+ i \theta (\alpha_1 b \lambda \frac{\gamma}{\gamma} \cos \gamma + \alpha_2 b \lambda \frac{\gamma}{\gamma} \sin \gamma) \rho
\]
Using the definition of $S_2$ and $\Omega$, we deduce that

\[
S_2 + R_3 = i\theta(\partial_s A + i\partial_s B) + \theta(A_{yy} + iB_{yy})
- i\frac{\lambda_s}{\lambda}\theta(\Lambda A + i\Lambda B) - \gamma_s \theta(A + iB)
+ f(W) - f(M - iQa) + i\theta\Omega\rho
\]

Identities (B.2) and (B.3) immediately yield (3.40).

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