A Round and Bipartize Approximation Algorithm for Vertex Cover

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Abstract

The vertex cover problem is a fundamental and widely studied combinatorial optimization problem. It is known that its standard linear programming relaxation is integral for bipartite graphs and half-integral for general graphs. As a consequence, the natural rounding algorithm based on this relaxation computes an optimal solution for bipartite graphs and a 2-approximation for general graphs. This raises the question of whether one can obtain improved bounds on the approximation ratio, depending on how close the graph is to being bipartite.

In this paper, we consider a round-and-bipartize algorithm that exploits the knowledge of an induced bipartite subgraph to attain improved approximation ratios. Equivalently, we suppose that we have access to a subset of vertices $S$ whose removal bipartizes the graph.

If $S$ is an independent set, we prove a tight approximation ratio of $1 + 1/\rho$, where $2\rho - 1$ denotes the odd girth of the contracted graph $\tilde{G} := G/S$ and thus satisfies $\rho \geq 2$. We show that this is tight for any graph and independent set by providing a family of weight functions for which this bound is attained. In addition, we give tight upper bounds for the fractional chromatic number and the integrality gap of such graphs, both of which also depend on the odd girth.

If $S$ is an arbitrary set, we prove a tight approximation ratio of $(1 + 1/\rho)(1 - \alpha) + 2\alpha$, where $\alpha \in [0, 1]$ denotes the total normalized dual sum of the edges lying inside of the set $S$. As an algorithmic application, we show that for any efficiently $k$-colorable graph with $k \geq 4$ we can find a bipartizing set satisfying $\alpha \leq 1 - 4/k$. This provides an approximation algorithm recovering the bound of $2 - 2/k$ in the worst case (i.e., when $\rho = 2$), which is best possible for this setting when using the standard relaxation.

1 Introduction

1.1 Vertex Cover

We focus in this paper on the vertex cover problem, an instance of which is given by a weighted graph $G = (V, E, w)$, where $w : V \mapsto \mathbb{R}^+$ is a non-negative weight function on the vertices. The optimization problem is to find the minimal weight subset of vertices $S \subset V$ that covers every edge of the graph, i.e.

$$\min \left\{ w(S) \middle| S \subset V, |S \cap \{i,j\}| \geq 1 \quad \forall (i,j) \in E \right\}.$$ 

We denote by $OPT$ an optimal subset of vertices for this problem, and by $w(OPT)$ the total weight of that solution. A natural linear programming relaxation, as well as its dual, is given by:

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For a given graph $G$, we denote the primal linear program by $P(G)$ and the dual by $D(G)$. An important property of $P(G)$ is the fact that any extreme point solution $x^*$ is half-integral, i.e. $x^*_v \in \{0, \frac{1}{2}, 1\}$ for any $v \in V$, see [4, 37, 50]. This gives rises to a straightforward 2-approximation algorithm for the vertex cover problem by solving $P(G)$ and outputting the vertices whose LP variable is set to at least a half, hence returning $S := \{v \in V \mid x^*_v \geq 1/2\}$. This solution is a 2-approximation because $w(S) \leq 2w(OPT)$, see [23]. Moreover, it turns out that $P(G)$ is in fact integral for any bipartite graph, hence solving it directly outputs $OPT$ [32].

The vertex cover problem is known to be NP-complete [29] and APX-complete [41]. Moreover, it was shown to be NP-hard to approximate within a factor of 7/6 in [25], a factor later improved to 1.36 in [17]. It is in fact NP-hard to approximate within a factor of $2 - \varepsilon$ for any fixed $\varepsilon > 0$ if the unique games conjecture is true [39]. Several different algorithms achieving approximation ratios of $2 - o(1)$ have been found for the weighted and unweighted versions of the problem: [28, 24, 10, 36, 9, 27]. Another large body of work is interested in finding exact fixed parameter tractable algorithms for the decision version: [13, 6, 14, 15, 18, 39, 40, 49, 19].

The integrality gap of the standard linear relaxation $P(G)$ for a graph on $n$ vertices equals to $2 - 2/n$. In fact, a more fine-grained analysis shows that it is equal to $2 - 2/\chi^f(G)$, where $\chi^f(G)$ is the fractional chromatic number of the graph [48]. An integrality gap of $2 - \varepsilon$ is proved for a large class of linear programs in [4]. In [11], it is shown that any linear program which approximates vertex cover within a factor of $2 - \varepsilon$ requires super-polynomially many inequalities.

1.2 Problem definition and motivation

We consider the following set-up. We are given a weighted non-bipartite graph $G = (V,E)$ and an optimal solution $x^* \in \{0, \frac{1}{2}, 1\}$ to $P(G)$. We denote by $V_\alpha := \{v \in V \mid x^*_v = \alpha\}$ the vertices taking value $\alpha$ and by $G_\alpha = G[V_\alpha]$ the subgraph of $G$ induced by the vertices $V_\alpha$ for any $\alpha \in \{0, \frac{1}{2}, 1\}$. Moreover, we suppose that we have knowledge of a vertex-induced bipartite subgraph of $G_{1/2}$. Equivalently, we suppose that we have access to a set of vertices $S \subset V_{1/2}$ whose removal bipartizes $G_{1/2}$. The question of finding such a good set algorithmically is also tackled in a later section.

We consider a simple rounding algorithm, detailed in Algorithm 1.1. It first solves $P(G)$, takes the vertices assigned value one by the linear program to the solution and removes all the integral nodes from the graph to arrive at $G_{1/2}$. The algorithm then takes all the vertices in the set $S$ to the solution, removes them from the graph and solves another (now integral) linear program on the bipartite remainder. The vertices with LP value one are then also added to the solution. The question studied is the following: what is the worst-case approximation ratio and which weight functions are attaining it?

\textbf{Algorithm 1.1 Round and Bipartize for Vertex Cover}

\begin{itemize}
  \item Solve the linear program $P(G)$ to get $V_0$, $V_{1/2}$ and $V_1$
  \item Solve the integral linear program $P(G_{1/2} \setminus S)$ to get $W \subset V_{1/2}$
  \item return $V_1 \cup S \cup W$
\end{itemize}

The problem of algorithmically finding such a set $S$ for the unweighted version is related to...
the odd cycle transversal number of a graph, denoted by \( \text{oct}(G) \), see [43, 31]. This is defined as the minimal number of vertices to be removed in order to make the graph bipartite. Another relevant concept is the odd cycle packing number \( \text{ocp}(G) \), defined as the maximum number of vertex-disjoint odd cycles of \( G \) and satisfying \( \text{ocp}(G) \leq \text{oct}(G) \). An interesting observation of [22] shows that the maximum subdeterminant of the vertex-edge incidence matrix of \( P(G) \) is equal to \( 2^{\text{ocp}(G)} \). The case where \( \text{ocp}(G) = 0 \) corresponds to the bipartite case, and the constraint matrix is then totally unimodular, i.e., its maximum subdeterminant is equal to one, in which case the vertices of the linear relaxation are known to be integral. The related maximum stable set problem has been studied in terms of the \( \text{ocp}(G) \) in [12, 6, 10].

Note that such a bipartizing set always exists: we may simply take \( S = V_{1/2} \), which recovers the standard 2-approximation algorithm for vertex cover. Our analysis thus falls into the framework of beyond the worst-case analysis [44]. It also has some interesting connections to learning-augmented algorithms, which have access to some prediction in their input, obtained for instance through machine learning. This prediction is assumed to come without any worst-case guarantees, and the goal is then to take advantage of it by making the algorithm perform better when this prediction is good, while still keeping robust worst-case guarantees. This framework has been particularly fruitful for online algorithms: [8, 35, 32, 33, 1, 2]. In our case, assuming a prediction on the set \( S \), robustness is guaranteed since we can never do worse than a 2-approximation. In fact, even if the predicted set is not bipartizing, one may simply greedily add vertices to it until it becomes so while still guaranteeing a 2-approximation.

Structurally, for the vertex cover problem (as for the stable set problem), there is a significant difference between a bipartite and a non-bipartite graph. The former is polynomial time solvable due to the integrality of the linear relaxation, whereas the latter only has a 2-approximation. However, the polyhedron is still well understood for this setting, since any extreme point solution is guaranteed to be half-integral. Moreover, non-bipartite graphs have an exact characterization stating that they all contain at least one odd cycle. A very natural parameter to look at in this setting is then the length of the shortest odd cycle, also named the odd girth of the graph. In [23], it is shown that graphs with a large odd girth also admit a small cardinality bipartizing set. We show in this paper that, in the setting we consider, the odd girth is in fact the right parameter to look at, since it determines tight bounds on the approximation ratio.

A key property implying the integrality of the polyhedron of a bipartite graph is the total unimodularity of the constraint matrix, see for instance [46, 47]. We hope that the techniques introduced in this paper might allow to tackle other problems in approximation algorithms by exploiting the fact that a substructure of the problem instance could have a totally unimodular constraint matrix. One technique which might benefit from this is iterative rounding, which requires solving a linear program at each iteration [34]. Having a better analysis for the case where the linear program becomes integral could potentially allow to get better guarantees and running times for certain approximation algorithms, since iterative rounding can terminate by simultaneously rounding on all the non-zero integral variables at this step without losing solution quality.

We now state a high-level view of the main ideas of our analysis. We first show that one can assume without loss of generality that we work on graphs where the solution \( x_v = 1/2 \) for every \( v \in V \) is optimal, and characterize the weight functions satisfying this assumption. We lower bound \( w(OPT(G)) \) by the weight of the optimal LP solution and split the total cost of the algorithm into two parts: \( w(S) \) and \( w(OPT(G \setminus S)) \). In order to get a hold of the second term, we construct feasible solutions depending on the structure of the graph, as well as on a dual understanding of the weight space. The minimum weight of these feasible solutions is then a valid upper bound. Tightness is shown by explicitly constructing classes of weight functions matching the obtained upper bounds.
1.3 Contributions and organization of the paper

The rest of the paper is organized as follows. We introduce the setting and needed notations in Section 2, devoted to preliminaries. In Section 3, we introduce the weight space polytope we work with and compute its set of vertices.

In Section 4.1, we focus on the case where \( S \) is a single vertex, i.e. \( S = \{v_p\} \), and prove that the approximation ratio is bounded by \( 1 + 1/\rho \) where \( 2\rho - 1 \) is the odd girth of the graph \( G \). We moreover prove this bound is tight and exhibit a convex subset of the weight space attaining it for any such graph. Note that \( \rho \geq 2 \), hence the absolute worst-case ratio is \( 3/2 \), which is smaller than 2. In addition, the ratio tends to one as the odd girth increases. A simple example is the odd cycle \( C_{2n-1} \), for which this framework yields a tight \( 1 + 1/n \) approximation.

In Section 4.2, we focus on the case where \( S = I \) is now an arbitrary independent set. We generalize the previous result and prove that the approximation ratio is bounded by \( 1 + 1/\rho \), where \( 2\rho - 1 \) now denotes the odd girth of the graph \( \tilde{G} := G/S \), where all the vertices in \( S \) are contracted into a single node. We also show this bound is tight and provide a convex subset of the weight space attaining it for any such graph and independent set. As in the previous setting, the absolute worst-case for the ratio is \( 3/2 \) and goes to one as the odd girth of the contracted graph increases. These are in fact the graphs admitting a 3-coloring, where each color class can be used as the bipartizing set.

In Section 4.3, we consider the most general case of \( S \) being an arbitrary set. As previously, \( 2\rho - 1 \) denotes the odd girth of the contracted graph \( \tilde{G} \), if the latter does in fact contain an odd cycle. We prove an upper bound on the approximation ratio of \( (1 + 1/\rho)(1 - \alpha) + 2\alpha \), where \( \alpha \in [0, 1] \) is the total normalized dual sum of the edges contained inside of the set \( S \). Note that this is simply a linear interpolation between \( 1 + 1/\rho \) and 2 with respect to this \( \alpha \) parameter. In the case where \( \tilde{G} \) is bipartite, we similarly prove an approximation ratio of \( 1 + \alpha \), which is a linear interpolation between 1 and 2 with respect to \( \alpha \). In addition, for any \( \alpha \in [0, 1] \) and any \( \rho \geq 2 \), we show that these bounds are tight and can be attained.

In Section 5, we prove that any graph which can be \( k \)-colored in polynomial time admits an efficiently findable bipartizing set \( S \) satisfying \( \alpha \leq 1 - 4/k \). This gives rises to a \( 2 - 2/k \) approximation algorithm for this setting in the worst case, a previously known bound which is best possible when working with the standard linear relaxation, since it coincides with the integrality gap. A nice example here is the class of planar graphs, known to be efficiently 4-colorable \( [3] \), and for which a bipartizing set with \( \alpha = 0 \) can thus be found.

In Section 6, we analyze the integrality gap and the fractional chromatic number for graphs which, as in Sections 4.1 and 4.2, contain an independent set whose removal yields a bipartite graph. Equivalently, these are simply the graphs with chromatic number three. We provide exact formulas in the case where \( S \) is a single vertex and tight upper bounds for the general case.

2 Preliminaries

We introduce in this section the setting, known results and notation that we use throughout the paper. We define \( \mathbb{R}_+ \) to be the non-negative real numbers and \( [k] := \{1, \ldots, k\} \) to be the natural numbers up to \( k \in \mathbb{N} \). For a vector \( x \in \mathbb{R}^n \), we denote the sum of the coordinates on a subset by \( x(A) := \sum_{i \in A} x_i \) for any \( A \subset [n] \).

The primal standard linear programming relaxation on a graph \( G = (V, E) \) is denoted by \( P(G) \). A key property of \( P(G) \) was introduced by Nemhauser and Trotter in \( [38] \). It essentially allows to reduce an optimal solution \( x^* \in \{0, \frac{1}{2}, 1\}^V \) to a fully half-integral solution by looking at the subgraph induced by the half-integral vertices. As before, we denote by \( V_\alpha := \{v \in V \mid x^*_v = \alpha\} \)
the vertices taking value \( \alpha \) and by \( \mathcal{G}_\alpha = \mathcal{G}[V_\alpha] \) the subgraph of \( \mathcal{G} \) induced by the vertices \( V_\alpha \) for any \( \alpha \in \{0, \frac{1}{2}, 1\} \).

**Theorem 2.1** (Nemhauser, Trotter [38]). Let \( x^* \in \{0, \frac{1}{2}, 1\}^V \) be an optimal extreme point solution to the linear programming relaxation \( P(\mathcal{G}) \). Then,

\[
w(\text{OPT}(\mathcal{G}_{1/2})) = w(\text{OPT}(\mathcal{G})) - w(V_1).
\]

A proof of this theorem can also be found in [20], Lemma 9.17. It directly implies that it is sufficient for us to consider without loss of generality graphs where the fully half-integral solution is optimal. This is formalized in the next lemma.

**Lemma 2.1.** Let \( x^* \in \{0, \frac{1}{2}, 1\}^V \) be an optimal solution to \( P(\mathcal{G}) \). If \( U \subset V_{1/2} \) is a feasible vertex cover on \( \mathcal{G}_{1/2} \) with approximation ratio at most \( \phi \), i.e., \( w(U) \leq \phi w(\text{OPT}(\mathcal{G}_{1/2})) \), then

\[
w(U) + w(V_1) \leq \phi w(\text{OPT}(\mathcal{G})).
\]

**Proof.** The proof is an easy consequence of Theorem 2.1 and the fact that \( \phi \geq 1 \):

\[
w(U) + w(V_1) \leq \phi w(\text{OPT}(\mathcal{G}_{1/2})) + w(V_1) \leq \phi (w(\text{OPT}(\mathcal{G}_{1/2})) + w(V_1)) = \phi w(\text{OPT}(\mathcal{G})).
\]

\( \square \)

We will thus in the rest of the paper assume that we work on weighted graphs satisfying the fact that the solution \((\frac{1}{2}, \ldots, \frac{1}{2})\) is optimal and introduce the weight space formally in Section 3.

For a given set \( S \subset V \), we define \( \mathcal{G}' := \mathcal{G} \setminus S = (V', E') \) to be the graph obtained by removing the set \( S \) and all the incident edges to it. Hence,

\[
E = E' \cup \delta(S) \cup E[S]
\]

where \( \delta(S) := \{(u, v) \in E \mid u \in S, v \notin S\} \) and \( E[S] := \{(u, v) \in E \mid u \in S, v \in S\} \). We also denote by \( \mathcal{G}' := \mathcal{G}/S = (\bar{V}, \bar{E}) \) the graph obtained by contracting all the vertices in \( S \) into a single new node \( v^S \in \bar{V} \). We allow for multiple edges, but no self-loops. The only edges present in \( E \) but not in \( \bar{E} \) are thus the ones with both endpoints in \( S \), i.e. \( E[S] \). We also denote the neighbourhood of a vertex \( v \in V \) by \( N(v) \).

If \( S \subset V \) is such that \( \mathcal{G} \setminus S \) is a bipartite graph, we consider the following simple algorithm.

First, take the vertices in \( S \subset V \) to the cover and remove them from the graph. Then solve the integral linear program \( P(\mathcal{G} \setminus S) \) and take the vertices having LP value one to the cover. The **approximation ratio**, given a weight function \( w : V \to \mathbb{R}_+ \) is thus defined as

\[
R(w) := \frac{w(S) + w(\text{OPT}(\mathcal{G} \setminus S))}{w(\text{OPT}(\mathcal{G}))}. \tag{2.1}
\]

For simplicity of notation, we omit the dependence on \( w \) of \( \text{OPT}(\mathcal{G}) \) and \( \text{OPT}(\mathcal{G} \setminus v_p) \).

## 3 Weight Space

Given a graph \( \mathcal{G} = (V, E) \), for which weight functions \( w : V \to \mathbb{R}_+ \) is the fully half-integral solution \( x = (\frac{1}{2}, \ldots, \frac{1}{2}) \) an optimal solution to the linear program \( P(\mathcal{G}) \)? We answer this question with the help of complementary slackness.

**Lemma 3.1.** Let \( \mathcal{G} = (V, E) \) be a graph and let \( w : V \to \mathbb{R}_+ \) be a weight function. The feasible solution \( x_v = \frac{1}{2} \) for every \( v \in V \) to the linear program \( P(\mathcal{G}) \) is optimal if and only if there exists \( y \in \mathbb{R}_+^E \) satisfying \( y(\delta(v)) = w_v \) for every \( v \in V \).
Proof. By complementary slackness, a feasible solution \( x \in \mathbb{R}_+^V \) to the primal \( P(\mathcal{G}) \) and a feasible solution \( y \in \mathbb{R}_+^E \) to the dual \( D(\mathcal{G}) \) are optimal if and only if:

\[
x_v > 0 \implies y(\delta(v)) = w_v \quad \forall v \in V
\]

and

\[
y_v > 0 \implies x_u + x_v = 1 \quad \forall e = (u,v) \in E.
\]

If \( x = (\frac{1}{2}, \ldots, \frac{1}{2}) \) is an optimal solution, then, by strong duality, there exists an optimal dual solution \( y \) and this solution needs to satisfy \( y(\delta(v)) = w_v \) for every \( v \in V \). Conversely, if there exists a dual solution \( y \) satisfying \( y(\delta(v)) = w_v \) for every \( v \in V \), then the pair \( x = (\frac{1}{2}, \ldots, \frac{1}{2}) \) and \( y \) satisfy both the conditions of complementary slackness, implying that \( x \) is optimal for \( P(\mathcal{G}) \).

We now define the polyhedron \( Q^{Y,W} \subset \mathbb{R}_+^E \times \mathbb{R}_+^V \):

\[
Q^{Y,W} := \left\{ (y,w) \in \mathbb{R}_+^E \times \mathbb{R}_+^V \mid y(\delta(v)) = w_v \quad \forall v \in V \right\}.
\]

The desired weights are the projection of this polyhedron to the subspace of weights by Lemma 3.1

\[
\text{proj}_W(Q^{Y,W}) := \{ w \in \mathbb{R}_+^V \mid \exists y \in \mathbb{R}_+^E \text{ s.t. } y(\delta(v)) = w_v \quad \forall v \in V \}.
\]

This polyhedron can be computed using standard techniques such as the Fourier-Motzkin elimination. Moreover, notice that this is in fact a polyhedral cone, since for every \( (y,w) \in Q^{Y,W} \) and any \( \lambda \in \mathbb{R}_+ \), we have that \( \lambda(y,w) \in Q^{Y,W} \). Without loss of generality, in order to work with a bounded space, we can now intersect the projection with a hyperplane to get a polytope. We make the normalization ensuring that the optimal linear programming solution has objective value one, i.e. \( w(V)/2 = 1 \):

\[
Q^W := \text{proj}_W(Q^{Y,W}) \cap \left\{ w \in \mathbb{R}_+^V \mid w(V) = 2 \right\}.
\]

Notice also that any dual solution \( y \in \mathbb{R}_+ \) satisfying \( y(\delta(v)) = w_v \) for every \( v \in V \) now has a total sum of \( y(E) = 1 \), since

\[
2 = \sum_{v \in V} w_v = \sum_{v \in V} y(\delta(v)) = \sum_{v \in V} \sum_{e \in E} y_e 1_{\{e \in \delta(v)\}} = \sum_{e \in E} y_e \sum_{v \in V} 1_{\{e \in \delta(v)\}} = 2 y(E).
\]

It turns out that the vertices of the polytope \( Q^W \) have a one-to-one correspondence with the edges of the graph \( \mathcal{G} = (V,E) \). We denote by \( \mathbb{I}_v \in \mathbb{R}^V \) the indicator vector of a vertex \( v \in V \).

**Theorem 3.1.** The polytope \( Q^W \) of a graph \( \mathcal{G} = (V,E) \) can be expressed as:

\[
Q^W = \text{conv}\left( \left\{ \mathbb{I}_u + \mathbb{I}_v \mid (u,v) \in E \right\} \right).
\]

Moreover, \( \mathbb{I}_u + \mathbb{I}_v \) is an extreme point of \( Q^W \) for every edge \( (u,v) \in E \).

**Proof.** Let \( w \in Q^W \) and \( y \in \mathbb{R}_+^E \) such that \( y(\delta(v)) = w_v \) for every \( v \in V \), as well as \( y(E) = 1 \). Observe that:

\[
w = \sum_{(u,v) \in E} y_{(u,v)} (\mathbb{I}_u + \mathbb{I}_v).
\]
By looking at this equality coordinate by coordinate, for a fixed vertex \( v \in V \), the contribution from the left hand side is \( w_v \), whereas the contribution from the right hand side is \( y(\delta(v)) \). We have thus managed to decompose an arbitrary \( w \in Q^W \) into a convex combination of the vectors \( \{ \mathbb{1}_u + \mathbb{1}_v \mid (u, v) \in E \} \).

Let \( (u, v) \in E \) and let us now show that \( \bar{w} := \mathbb{1}_u + \mathbb{1}_v \) is an extreme point of \( Q^W \). Firstly, it is clear that \( \bar{w} \in Q^W \), the dual solution satisfying complementary slackness being \( y_{(u,v)} = 1 \) and \( y_e = 0 \) for every \( e \in E \setminus (u, v) \), and the sum of all the weights being indeed equal to two. Suppose for contradiction that there exist distinct \( w^1, w^2 \in Q^W \) such that \( \bar{w} = \frac{1}{2}(w^1 + w^2) \).

Notice that, since the weight vectors are required to be non-negative, \( \bar{w} = w^1 = 1 - \epsilon \) and \( w^1 = w^2 = 1 - \epsilon \) for some \( \epsilon > 0 \). However, the fully half-integral solution is now not an optimal LP solution on the weights \( w^1 \) and \( w^2 \). Indeed, on the weight function \( w^1 \), the feasible solution \( V \setminus \{ u \} \) has objective value \( 1 - \epsilon \), whereas the fully half-integral solution has weight 1. Similarly, on the weight function \( w^2 \), the feasible solution \( V \setminus v \) has objective value \( 1 - \epsilon \). By Lemma 3.1, \( w^1, w^2 \notin Q^W \), leading to a contradiction. The weight function \( \bar{w} \) thus cannot be written as a non-trivial convex combination of two other points in the polytope \( Q^W \) and is therefore an extreme point (or a vertex) of \( Q^W \).

We will in the rest of the paper assume that every weight function satisfies \( w \in Q^W \), and thus also that \( w(V) = 2 \) and \( y(E) = 1 \) for every corresponding dual solution \( y \in \mathbb{R}_+^E \). We now end this section by showing that this normalization of the weight space allows us to get a convenient lower bound on \( w(OPT(G)) \).

**Lemma 3.2.** Let \( G = (V, E) \) be a graph. For any \( w \in Q^W \),

\[
w(OPT(G)) \geq 1.
\]

**Proof.** Since \( w \in Q^W \), we know that the fully half-integral solution is an optimal linear programming solution, showing \( w(V)/2 \leq w(OPT(G)) \), by feasibility of \( OPT(G) \). The result follows again by the definition of \( Q^W \), where we normalized the sum of the weights to be equal to two: \( w(V) = 2 \). Another way this bound can be obtained is by considering the corresponding dual solution \( y \in \mathbb{R}_+^E \) satisfying \( y(\delta(v)) = w_v \) for every \( v \in V \). Any feasible vertex cover needs to count the dual value on every edge at least once, showing \( w(OPT(G)) \geq y(E) = 1 \). \( \Box \)

## 4 Round and Bipartize

### 4.1 One Vertex to Bipartite

We focus in this section on graphs where the removal of one vertex leads to a bipartite graph. The framework is as follows.

- We are given a non-bipartite graph \( G = (V, E) \) and a vertex \( v_p \in V \) such that \( G \setminus v_p = (A \cup B, E') \) is a bipartite graph.

- Every weight function \( w : V \to \mathbb{R}_+ \) considered satisfies \( w \in Q^W \).

- We assume that we round on the vertex \( v_p \) in the first step. Since \( G \setminus v_p \) is a bipartite graph, solving the linear program \( P(G \setminus v_p) \) directly outputs \( OPT(G \setminus v_p) \).
The approximation ratio, given weights \( w \in Q^W \) and the fact that we round on \( v_p \) in the first step, is the following quantity:

\[
R(w) := \frac{w(v_p) + w(OPT(G \setminus v_p))}{w(OPT(G))}.
\]

Our goal is to pick a weight function \( w \in Q^W \) maximizing the approximation ratio, and we in fact show in this section that

\[
R(w) \leq 1 + \frac{1}{\rho} \quad \forall w \in Q^W
\]

where \( 2\rho - 1 \) is the odd girth of \( G \) and thus satisfies \( \rho \geq 2 \). We also prove that this bound is tight and is attained for a certain class of weight functions \( W \subset Q^W \), which are related to all the shortest odd cycles of length \( 2\rho - 1 \):

\[
R(w) = 1 + \frac{1}{\rho} \quad \forall w \in W.
\]

We now dive deeper into the structure of the graph \( G \setminus v_p \). By assumption, this graph admits a bipartition \( A \cup B \) of the vertices. Let us assume that it has \( k \) connected components \( A_1 \cup B_1, \ldots, A_k \cup B_k \), all of which are bipartite as well, where \( A = \bigcup_i A_i \) and \( B = \bigcup_i B_i \). We now fix an arbitrary such component \( A_i \cup B_j \).

- If \( v_p \) has an incident edge to both \( A_j \) and \( B_j \), then this component contains (if including \( v_p \)) an odd cycle of \( G \). This holds since any path between a node in \( A_j \) and a node in \( B_j \) has odd length.

- If \( v_p \) has incident edges with only one side, we assume without loss of generality that this side is \( A_j \). One could simply switch both sides in the other case while still keeping a valid bipartition of the graph \( G \setminus v_p \).

- If \( v_p \) does not have incident edges with either of the two sides, then \( A_j \cup B_j \) is a connected component of the original graph \( G \). We call such components dummy components and denote by \( A_q \cup B_d \) the bipartite graph formed by taking the union of all the dummy components.

We denote \( N_A(v_p) := N(v_p) \cap A \) and \( N_B(v_p) := N(v_p) \cap B \). We now split the graph into layers, where each layer corresponds to the nodes at the same shortest path distance from \( N_A(v_p) \). More precisely, we define

\[
\mathcal{L}_i := \left\{ v \in A \cup B \mid d(N_A(v_p), v) = i \right\} \quad \text{for } i \in \{0, \ldots, q\}
\]

(4.1)

where \( d(N_A(v_p), v) \) represents the unweighted shortest path distance between \( v \) and a vertex in \( N_A(v_p) \). The parameter \( q \) is defined to be the maximal finite distance from \( N_A(v_p) \) in the graph \( G \). An important observation is the fact that these layers are alternatingly included in one side of the bipartition, see Figure[1] for an illustration of the construction.

If the graph \( G \) is not connected, note that \( d(N_A(v_p), v) = \infty \) for the vertices \( v \) lying in dummy components. In order to add the dummy components to the layers and keep alternation between the two sides of the bipartition, we define the last two layers to either be \( \{\mathcal{L}_{q+1} := A_d, \mathcal{L}_{q+2} := B_d\} \) or \( \{\mathcal{L}_{q+1} := B_d, \mathcal{L}_{q+2} := A_d\} \), depending on which side of the bipartition the last connected layer \( \mathcal{L}_q \) lies. We now have that \( \mathcal{L}_i \subset A \) if \( i \) is even, and \( \mathcal{L}_i \subset B \) if \( i \) is odd. In fact,

\[
A = \bigcup_{i=0}^{\lfloor t/2 \rfloor} \mathcal{L}_{2i} \quad \text{and} \quad B = \bigcup_{i=1}^{\lfloor t/2 \rfloor} \mathcal{L}_{2i-1},
\]

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Figure 1: The layers of a bipartite graph $\mathcal{G} \setminus v_p = (A \cup B, E')$ with $\rho = 4$. The blue square vertices correspond to $N(v_p)$, where the two left ones are $L_0 = N_A(v_p)$ and the two right ones are $N_B(v_p)$.

where the parameter $l \in \mathbb{N}$ represents the index of the last layer: if $\mathcal{G}$ is connected, then $l = q$, otherwise $l = q + 2$. Notice also that $L_0 = N_A(v_p)$. However, $N_B(v_p)$ may now have several different vertices in different layers, see Figure 1.

Let $C \subseteq V$ be an arbitrary odd cycle of $\mathcal{G}$. Notice that this cycle contains $v_p$, a vertex from $N_A(v_p)$ and a vertex from $N_B(v_p)$, since $\mathcal{G} \setminus v_p$ is bipartite and therefore does not contain an odd cycle. Any odd cycle $C$ in $\mathcal{G}$ thus corresponds to an odd path between a vertex in $N_A(v_p) = L_0$ and a vertex in $N_B(v_p)$. By the assumption that the shortest odd cycle length of $\mathcal{G}$ is $2\rho - 1$, the first layer having a non-empty intersection with $N_B(v_p)$ is $L_{2\rho - 3}$: $L_i \cap N_B(v_p) = \emptyset \quad \forall i < 2\rho - 3$.

A shortest odd cycle of length $2\rho - 1$ therefore corresponds to an odd path of length $2\rho - 3$ between $L_0$ and a vertex in $L_{2\rho - 3} \cap N_B(v_p)$, see Figure 1 for an illustration. We now define edges connecting two consecutive layers $L_i$ and $L_{i+1}$ as follows:

$$E[L_i, L_{i+1}] := \{(u, v) \in E' \mid u \in L_i, v \in L_{i+1}\} \quad \forall i \in \{0, \ldots, l - 1\}.$$ We also denote by

$$\delta_A(v_p) = \{(v_p, u) \in E \mid u \in A\}, \quad \delta_B(v_p) = \{(v_p, u) \in E \mid u \in B\}$$

the incident edges to $v_p$ respectively connecting to $A$ and $B$.

**Lemma 4.1.** Let $\mathcal{G} = (V, E)$ be a non-bipartite graph, let $v_p \in V$ such that $\mathcal{G} \setminus v_p = \left(\bigcup_{i=0}^{l-1} L_i, E'\right) = (A \cup B, E')$ is bipartite, and let $(y, w) \in Q^{Y,W}$. Then,

$$w\left(OPT(\mathcal{G} \setminus v_p)\right) \leq y(E') + \min \left\{y(\delta_A(v_p)), y(\delta_B(v_p)), \min_{j \in [\rho-2]} y(E[L_{2j-1}, L_{2j}])\right\}.$$ 

**Remark.** When $\rho = 2$, there are only two terms in the outer minimum, which are $y(\delta_A(v_p))$ and $y(\delta_B(v_p))$.

**Proof.** We prove this lemma by constructing $\rho$ explicit vertex covers of $\mathcal{G} \setminus v_p$ and illustrate those in Figure 2. Firstly, notice that taking one side of the bipartition is a feasible vertex cover. The cover $A$ has weight $y(E') + y(\delta_A(v_p))$, whereas the cover $B$ has weight $y(E') + y(\delta_B(v_p))$. This holds because every edge $(u, v) \in E'$ only has one endpoint in both these covers, meaning that the dual on every such edge is counted exactly once, leading to a contribution of $y(E')$. The
Figure 2: An illustration of the ρ feasible covers constructed in the proof of Lemma 4.1. The square vertices correspond to N(v_p), where the two left ones are N_A(v_p) and the two right ones are N_B(v_p).

vertices in N_A(v_p) give an additional contribution of y(δ_A(v_p)) for the cover A. Similarly, the vertices in N_B(v_p) incur an additional cost of y(δ_B(v_p)) for the cover B. Therefore,

\[ w(OPT(G \setminus v_p)) \leq y(E') + \min \left\{ y(\delta_A(v_p)), y(\delta_B(v_p)) \right\}. \]  (4.2)

We now construct ρ−2 additional covers with the help of the layers. If ρ ≠ 2, fix a j ∈ [ρ−2], and start the cover S_j by taking the two consecutive layers L_{2j−1} and L_{2j}. Complete this cover by taking remaining layers alternatively (hence always skipping one) until covering every edge of the graph. Notice that this cover has an empty intersection with N(v_p). By counting its weight in terms of the dual values on the edges, S_j counts every edge in E' once, except for the edges in E[L_{2j−1}, L_{2j}], which it counts twice. The weight of the cover is thus

\[ w(S_j) = y(E') + y(E[L_{2j−1}, L_{2j}]). \]

Since this is true for any j ∈ [ρ−2], we get that

\[ w(OPT(G \setminus v_p)) \leq y(E') + \min_{j \in [ρ−2]} y(E[L_{2j−1}, L_{2j}]). \]

Combining this with the bound (4.2) finishes the proof of the lemma.

Theorem 4.1. Let G = (V, E) be a graph and v_p ∈ V such that G \setminus v_p = (A ∪ B, E') is bipartite. For any w ∈ Q^W, the approximation ratio satisfies:

\[ R(w) \leq 1 + \frac{1}{ρ} \]

where 2ρ − 1 is the odd girth of G.

Proof. Let w ∈ Q^W and let y ∈ R^E be the corresponding dual solution. By Lemma 3.2 and
Lemma 4.1 the approximation ratio satisfies:

\[ R(w) = \frac{w(v_p) + w(OPT(G \setminus v_p))}{w(OPT(G))} \leq y(\delta(v_p)) + y(E') + \min \left\{ y(\delta_A(v_p)), y(\delta_B(v_p)), \min_{j \in [\rho-2]} y(E[L_{2j-1}, L_{2j}]) \right\} \]

\[ = 1 + \min \left\{ y(\delta_A(v_p)), y(\delta_B(v_p)), \min_{j \in [\rho-2]} y(E[L_{2j-1}, L_{2j}]) \right\} \leq 1 + \frac{1}{\rho}. \]

The last equality follows from the fact that \( E = E' \cup \delta(v_p) \) and \( y(E) = 1 \). The last inequality follows from the following argument. Notice that the outer minimum term can be seen as a minimum over \( \rho \) values \( y(E_i) \) for \( E_i \subseteq E \). Moreover, these sets of edges \( \{E_i\}_{i \in \rho} \) are pairwise disjoint and sum to at most one, since the total sum of the edges of the graph is \( y(E) = 1 \). This minimum can thus be upper bounded by \( 1/\rho \).

We now prove that this bound is tight and is attained for a certain class of weight functions. Let \( C = \{C_1, \ldots, C_k\} \) be all the shortest odd cycles (of length \( 2\rho - 1 \)) of the graph \( G \). Notice that every odd cycle (and thus in particular every \( C \in C \)) of the graph \( G \) contains the vertex \( v_p \), since \( G \setminus v_p \) is bipartite. For every \( C \in C \), we define the following weight function \( w^C : V \to \mathbb{R}_+ \), that we call the basic weight function corresponding to the odd cycle \( C \):

\[ w^C(v) = \begin{cases} 
2/\rho & \text{if } v = v_p \\
1/\rho & \text{if } v \in C \setminus v_p \\
0 & \text{if } v \notin C 
\end{cases} \]

It is not hard to see that \( w^C \in Q^W \). Indeed, construct the dual solution certifying that by setting both dual edges incident to \( v_p \) to 1/\( \rho \). After that, alternatingly set the dual edges to 0 and 1/\( \rho \) along the odd cycle. For any edge outside of \( C \), set its dual value to 0. We now define:

\[ \mathcal{W} := \left\{ w : V \to \mathbb{R}_+ \mid w = \sum_{C \in C} \lambda^C w^C; \sum_{C \in C} \lambda^C = 1; \lambda^C \geq 0 \forall C \in C \right\} \]

One may see this as the convex hull of the basic weight functions corresponding to each shortest odd cycle. Since each \( w^C \in Q^W \), and \( Q^W \) is a convex polytope, we have that \( \mathcal{W} \subseteq Q^W \).

**Theorem 4.2.** Let \( G = (V, E) \) be a graph and \( v_p \in V \) such that \( G \setminus v_p = (\bigcup_{i=0}^l L_i, E') \) is bipartite. For any weight function \( w \in \mathcal{W} \), the approximation ratio satisfies

\[ R(w) = 1 + \frac{1}{\rho} \]

where \( 2\rho - 1 \) is the odd girth of \( G \).

**Proof.** Let \( C \) be the set of all the shortest odd cycles (of length \( 2\rho - 1 \)) of the graph \( G \) and let \( w = \sum_{C \in C} \lambda^C w^C \in \mathcal{W} \). Notice that, for any subset of vertices \( S \subseteq A \cup B \), we can count its weight as

\[ w(S) = \sum_{v \in S} w_v = \sum_{v \in S} \sum_{C \in C} \lambda^C 1_{\{v \in C\}} = \frac{1}{\rho} \sum_{C \in C} \lambda^C \sum_{v \in S} 1_{\{v \in C\}} = \frac{1}{\rho} \sum_{C \in C} \lambda^C |S \cap C|. \quad (4.3) \]
Notice also that every odd cycle $C \in \mathcal{C}$ intersects each layer $L_i$ for $i \in \{0, \ldots, 2\rho - 3\}$ exactly once. Therefore, by (4.3),

$$w(L_i) = \frac{1}{\rho} \quad \forall i \in \{0, \ldots, 2\rho - 3\}.$$ 

We now claim that

$$w(OPT(G)) = 1.$$ 

The fact that $w(OPT(G)) \geq 1$ follows from Lemma 3.2. For the reverse inequality, notice that it is possible to take a feasible cover by taking exactly $\rho$ layers in addition to the zero weight vertices, for instance $L_0 \cup L_2 \cup L_3 \cup L_5 \cdots \cup L_{2\rho-3}$, showing $w(OPT(G)) \leq 1$. Similarly, we claim

$$w(OPT(G \setminus v_p)) = \frac{\rho - 1}{\rho}.$$ 

After removal of $v_p$, every cycle $C \in \mathcal{C}$ becomes a path of length $2\rho - 3$ (and thus consisting of $2\rho - 2$ vertices), with one vertex in each layer $L_i$ for $i \in \{0, \ldots, 2\rho - 3\}$. By feasibility, $OPT(G \setminus v_p)$ has to contain at least $\rho - 1$ vertices for every such path. Using (4.3), we infer $w(OPT(G \setminus v_p)) \geq (\rho - 1)/\rho$. For the reverse inequality, taking $\rho - 1$ layers alternatively, such as $L_0 \cup L_2 \cup L_4 \cdots \cup L_{2\rho-4}$, as well as the zero weight vertices, builds a feasible cover of weight exactly $(\rho - 1)/\rho$. Finally, notice that

$$w(v_p) = \frac{2}{\rho}$$

because every $C \in \mathcal{C}$ contains $v_p$. By combining the three equalities, we get

$$R(w) = \frac{w(v_p) + w(OPT(G \setminus v_p))}{w(OPT(G))} = 1 + \frac{1}{\rho}.\]

\[4.2\] Independent Set to Bipartite

We focus in this section on graphs $G = (V, E)$ where there exists an independent set $I \subseteq V$ such that $G' := G \setminus I = (V \setminus I, E \setminus \delta(I))$ is a bipartite graph. The algorithm we consider is a two-step rounding process: we first round on the nodes in $I$, remove them from the graph, and
then solve the (integral) linear program on the bipartite graph $G \setminus I$. Given a weight function $w \in Q^W$, the approximation ratio is defined as:

$$R(w) := \frac{w(I) + w(OPT(G \setminus I))}{w(OPT(G))}.$$  

For a feasible vertex cover $U \subset V \setminus I$ of the bipartite graph $G' = G \setminus I$, we define

$$E_U := \left\{ (u, v) \in E' \mid u \in U, v \in U \right\} \cup \left\{ (u, v) \in E \mid u \in U, v \in I \right\}.$$  

Note that the second term is a subset of edges of the original graph $G$. In particular, $E_U$ might have a non-empty intersection with $I$.

**Definition 4.1.** Let $G = (V, E)$ and $I \subset V$. Two feasible vertex covers $U_1, U_2 \subset V \setminus I$ of $G' = G \setminus I$ are edge-separate with respect to $G$ if

$$E_{U_1} \cap E_{U_2} = \emptyset.$$  

See Figure 5 for an illustration of three pairwise edge-separate covers. If the graph $G$ is clear from the context, we will sometimes simply say that the covers are edge-separate.

**Lemma 4.2.** Let $G = (V, E)$ be a graph and $I \subset V$ be an independent set such that $G \setminus I$ is bipartite. If there exist $k$ pairwise edge-separate feasible vertex covers of the graph $G \setminus I$, then

$$R(w) \leq 1 + \frac{1}{k}$$  

for any weight function $w \in Q^W$.

**Proof.** Let $(y, w) \in Q^{Y,W}$ be such that $w(V) = 2$ and $y(E) = 1$. We use the fact that $I$ is an independent set: its weight is equal to the sum of the dual values of the edges incident to it:

$$w(I) = y(\delta(I)).$$  

(4.4)

Let $U \subset V \setminus I$ be a feasible cover of $G \setminus I$. We compute the weight of this cover with the help of the dual variables:

$$w(U) = y(E \setminus \delta(I)) + y(E_U).$$  

(4.5)

We now explain how this formula is obtained. By feasibility of $U$, it counts the dual value of each edge in the graph $G \setminus I$ at least once, leading to a contribution of $y(E') = y(E \setminus \delta(I))$. The edges $(u, v) \in E'$ which are counted twice are in fact the ones where both endpoints lie in $U$, i.e. $u \in U$ and $v \in U$. The final quantity we need to account for are the edges in $E \setminus E'$ which also contribute to the weight of the cover. These are in fact the edges $(u, v) \in E$ incident to $I$ but with one endpoint in $U$, i.e. $u \in U$, $v \in I$, and are counted once. Therefore, any feasible cover $U$ incurs a necessary cost of $y(E') = y(E \setminus \delta(I))$, with a surplus of $y(E_U)$.

We now let $U_1, \ldots, U_k \subset V \setminus I$ be the $k$ pairwise edge-separate feasible covers of $G \setminus I$. Since these are all feasible, we get an upper bound on the weight of $OPT(G \setminus I)$:

$$w(OPT(G \setminus I)) \leq \min_{i \in [k]} w(U_i) = y(E \setminus \delta(I)) + \min_{i \in [k]} y(E_{U_i}) \leq y(E \setminus \delta(I)) + \frac{1}{k}. $$  

(4.6)

The last inequality follows from the assumption that the subsets of edges $E_{U_i}$ are pairwise disjoint: suppose that $y(E_{U_i}) > 1/k$ for every $i \in [k]$. Then, $y(E) \geq \sum_{i \in [k]} y(E_{U_i}) > 1$, which is a contradiction. Combining Lemma 3.2, (4.4) and (4.6), we get the desired bound:

$$R(w) = \frac{w(I) + w(OPT(G \setminus I))}{w(OPT(G))} \leq y(\delta(I)) + y(E \setminus \delta(I)) + \min_{i \in [k]} y(E_{U_i}) \leq 1 + \frac{1}{k}.$$  

$\square$
Definition 4.2. Let $G = (V, E)$ be a graph and $I \subset V$ be a subset of vertices. We denote by $\tilde{G} := G/I = (\tilde{V}, \tilde{E})$ the graph obtained by contracting the vertices in $I$ into a single new node $v_I \in \tilde{V}$, where multiple edges are allowed, but self-loops are not.

Remark. Observe that if $I$ is an independent set, the edge set $\tilde{E}$ of the graph $\tilde{G}$ has a one-to-one correspondence with the edge set $E$ of the original graph $G$.

Theorem 4.3. Let $G = (V, E)$ be a graph and $I \subset V$ be an independent set such that $G \setminus I = (A \cup B, E')$ is bipartite. For any $w \in Q^W$, the approximation ratio satisfies

$$R(w) \leq 1 + \frac{1}{\rho}$$

where $2\rho - 1$ is the odd girth of the vertex-contracted graph $\tilde{G} = G/I$.

Remark. The odd girth of $\tilde{G}$ is at most the odd girth of $G$, but can be strictly smaller, see Figure 4 for an example.

Proof. By Lemma 4.2 it is enough to construct $\rho$ edge-separate covers of $G \setminus I$ in order to prove the theorem. Observe that it is enough to work in the contracted graph $\tilde{G} := G/I$. Indeed, constructing $k$ edge-separate feasible covers of $G \setminus I$ with respect to $\tilde{G}$ directly implies that one can construct $k$ edge-separate feasible covers of $G \setminus I$ with respect to $G$, since the edge sets of $G$ and $\tilde{G}$ are the same.

First, notice that $\tilde{G}$ does indeed contain an odd cycle. Fix an arbitrary odd cycle $C$ of $G$ and observe that $C \cap I \neq \emptyset$, since $I$ breaks all the odd cycles by assumption. Since $I$ is an independent set, there exists an odd path between two nodes of $C \cap I$. This odd path becomes an odd cycle when the vertices in $I$ get contracted in the graph $\tilde{G}$. The odd girth is thus well-defined.

Moreover, $\tilde{G}$ is now a graph which is one vertex away from being bipartite: if we denote by $v^I$ the contracted vertex, then $\tilde{G} \setminus v^I = G \setminus I = (A \cup B, E')$. We can thus decompose the vertex set of the graph $G \setminus I$ into the layers

$$L_i := \left\{ v \in A \cup B \mid d\left(N_A(v^I), v\right) = i \right\} \text{ for } i \in \{0, \ldots, l\}$$

as explained in (4.1). We now construct the $\rho$ edge-separate covers in the same fashion as in the proof of Lemma 4.1, and these are illustrated in Figures 2 and 5.

The first two covers are both sides of the bipartition $A$ and $B$, where $E_A = \{ (u, v^I) \in \tilde{E} \mid u \in A \}$ and $E_B = \{ (u, v^I) \in \tilde{E} \mid u \in B \}$. If $\rho > 2$, the remaining $\rho - 2$ covers are constructed as follows: fix a $j \in [\rho - 2]$, and start the cover $U_j$ by taking the two consecutive layers $L_{2j-1}$ and $L_{2j+1}$.
Figure 5: The $\rho = 3$ edge-separate covers of $G \setminus v^I$ constructed in the proof of Theorem 4.3. Observe that only the three bottom edges are required to be covered. For each cover $U \subset V \setminus v^I$, the blue edges correspond to $E_U$. Notice that they are pairwise disjoint.

and $L_{2j}$. Complete it by taking remaining layers alternatingly (hence always skipping one) until covering every edge of the graph. In this case,

$$E_{U_j} = E[L_{2j-1}, L_{2j}] = \{(u, v) \in \tilde{E} \mid u \in L_{2j-1}, v \in L_{2j}\}$$

Clearly, $E_A, E_B, E_{U_1}, \ldots, E_{U_{\rho-2}}$ are all pairwise disjoint, implying that the covers constructed are pairwise edge-separate. By Lemma 4.2 this finishes the proof of the theorem.

We now show that this bound is tight and is attained for a class of weight functions $w \in Q^W$. Let $\tilde{C}$ be all the shortest odd cycles (of length $2\rho - 1$) of the graph $\tilde{G}$, each of which is containing $v^I$. For every such cycle $C \in \tilde{C}$, we define the following dual function on the edges $y^C : \tilde{E} \to \mathbb{R}_+$: set both dual edges incident to $v^I$ to $1/\rho$ and then alternatively set the dual edges to $0$ and $1/\rho$ along the odd cycle, see Figure 6. For any edge outside of $C$, set its dual value to $0$. We now take the convex hull of all these functions:

$$\mathcal{Y} := \left\{ y : \tilde{E} \to \mathbb{R}_+ \mid y = \sum_{C \in \tilde{C}} \lambda^C y^C; \quad \sum_{C \in \tilde{C}} \lambda^C = 1; \quad \lambda^C \geq 0 \quad \forall C \in \tilde{C} \right\}.$$ 

Because of the one-to-one correspondence between the edge sets $\tilde{E}$ and $E$, we can naturally define a weight function on the original vertex set once we fix a $y \in \mathcal{Y}$:

$$w(v) := y(\delta(v)) \quad \forall v \in V.$$ 

We define the space of all such weight functions as:

$$\mathcal{W} := \{w : V \to \mathbb{R}_+ \mid w(v) = y(\delta(v)) \quad \forall v \in V; \quad y \in \mathcal{Y}\}.$$ 

**Theorem 4.4.** Let $G = (V, E)$ be a graph and $I \subset V$ be an independent set such that $G \setminus I = (A \cup B, E')$ is bipartite. For any $w \in \mathcal{W}$, the approximation ratio satisfies

$$R(w) = 1 + \frac{1}{\rho}$$

where $2\rho - 1$ is the odd girth of the vertex-contracted graph $G/I$.

**Proof.** The proof is essentially the same as the one of Theorem 4.2. Using the same arguments, we get that $w(I) = 2/\rho, w(OPT(G \setminus I)) = w(OPT(\tilde{G} \setminus v^I)) = (\rho - 1)/\rho$ and $w(OPT(\tilde{G})) = 1$, showing

$$R(w) = \frac{w(I) + w(OPT(G \setminus I))}{w(OPT(\tilde{G}))} = 1 + \frac{1}{\rho}.$$ 

\[\square\]
4.3 Arbitrary Set to Bipartite

We now consider the setting where we are given an arbitrary graph $G = (V, E)$ and a subset of vertices $S \subset V$ such that $G' = G \setminus S = (A \cup B, E')$ is a bipartite graph. The algorithm considered is again a two-step rounding process: first round on the vertices in $S$ and take them to the cover, remove $S$ and solve the linear program on $G \setminus S$ to get an integral solution.

Let $(y, w) \in Q_{Y,W}$ such that $w(V) = 2$ and $y(E) = 1$. The approximation ratio is

$$R(w) := \frac{w(S) + w(OPT(G \setminus S))}{w(OPT(G))}.$$  

Denote the set of edges with both endpoints in the set $S$ by

$$E[S] := \{(u, v) \in E \mid u \in S, v \in S\}.$$  

Our guarantee on the approximation ratio will depend on the total sum of the dual variables on these edges. We denote this sum by

$$\alpha := y(E[S]).$$

In addition, the guarantee that we get on the approximation ratio depends on whether the contracted graph $G := G/S$ is bipartite or not. If $G$ contains an odd cycle, we denote its odd girth by $2\rho - 1$ and show that the approximation ratio is determined by a linear interpolation between $1 + 1/\rho$ and $2$ with respect to the $\alpha$ parameter. Hence, for $\alpha = 0$, we recover the bound of the previous section, whereas for $\alpha = 1$, we recover the $2$-approximation. Similarly, if $G$ is bipartite, we show that the approximation ratio is determined by a linear interpolation between $1$ and $2$ with respect to $\alpha$.

**Theorem 4.5.** Let $G = (V, E)$ be a graph and $S \subset V$ such that $G \setminus S = (A \cup B, E')$ is bipartite. For any $w \in Q_{W}$, the approximation ratio satisfies

$$R(w) \leq \left(1 + \frac{1}{\rho}\right)(1 - \alpha) + 2\alpha \quad \text{with } \alpha \in [0, 1] \text{ and } \rho \geq 2$$

if the contracted graph $G/S$ contains an odd cycle and $2\rho - 1$ denotes its odd girth.

**Proof.** Recall that $w(V) = 2$ and $y(E) = 1$, meaning that $w(OPT(G)) \geq y(E) = 1$ by Lemma 3.2. We decompose the weight of $S$ with respect to the dual variables. The edges in $E[S]$ are counted twice, whereas the edges in $\delta(S)$ are counted once, giving:

$$w(S) = 2\alpha + y(\delta(S)).$$  

(4.7)
Consider the contracted graph $\tilde{G} = G/S = (\tilde{V}, \tilde{E})$ and denote by $v^S$ the contracted node. The edge set of this graph is now
\[ \tilde{E} = \delta(S) \cup E' \]
since the edges in $E[S]$ have been collapsed. This graph contains by assumption an odd cycle, but is now one vertex away from being bipartite, if we were to remove $v^S$. By Lemma 4.1, we can construct $\rho$ edge-separate covers $U_1, \ldots, U_\rho \subset \tilde{V} \setminus v^S$ on the bipartite graph $G'$ with respect to $\tilde{G}$. These covers are still edge-separate with respect to the original graph $G$. This allows us to get an upper bound on the weight of $OPT(G \setminus S)$:
\[ w(OPT(G \setminus S)) \leq \min_{i \in [\rho]} w(U_i) = y(E') + \min_{i \in [\rho]} y(E_U_i) \leq y(E') + \frac{1 - \alpha}{\rho}. \tag{4.8} \]
The first inequality holds since every $U_i$ is a feasible cover of $G \setminus S$. The second equality holds by counting the weight of a cover $U_i$ in terms of the dual edges as explained in [4.5]. The last inequality holds because the edge sets $\{E_{U_i}\}_{i \in [\rho]}$ are pairwise disjoint, and their total dual sum is at most $1 - \alpha$.

Combining Lemma 3.2, [4.7] and [4.8], and using $y(E) = \alpha + y(\delta(S)) + y(E') = 1$, we get
\[
R(w) = \frac{w(S) + w(OPT(G \setminus S))}{w(OPT(G))} \leq 2\alpha + y(\delta(S)) + y(E') + \frac{1 - \alpha}{\rho} \\
= 1 + \alpha + \frac{1 - \alpha}{\rho} = \left(1 + \frac{1}{\rho} \right) (1 - \alpha) + 2\alpha.
\]

What happens if the contracted graph $G/S$ does not contain an odd cycle and is thus bipartite? In that case, we can get an exact formula on $w(OPT(G \setminus S))$ without having the additive term $(1 - \alpha)/\rho$ in (4.8). The approximation ratio is then determined by a linear interpolation between 1 and 2 with respect to the $\alpha$ parameter.

**Theorem 4.6.** Let $G = (V, E)$ be a graph and $S \subset V$ such that $G \setminus S = (A \cup B, E')$ is bipartite. For any $w \in Q^W$, the approximation ratio satisfies
\[ R(w) \leq 1 + \alpha \quad \text{with } \alpha \in [0, 1] \]
if the contracted graph $G/S$ is bipartite.
Proof. The only change with respect to the previous proof is the bound on $w(OPT(\mathcal{G} \setminus S))$ in [4.8]. We denote the contracted graph by $\tilde{\mathcal{G}} = \mathcal{G} / S$ and by $v^S$ the contracted vertex. Suppose $\tilde{\mathcal{G}}$ admits the bipartition $(\tilde{A} \cup \tilde{B}, \tilde{E})$ and assume without loss of generality that $v^S \in \tilde{A}$. Note that $\tilde{E} = E' \cup \delta(v^S)$.

Any feasible cover of $\mathcal{G} \setminus S$ needs to count the dual value of every edge in $E'$ at least once. Taking the cover $\tilde{A} \setminus v^S$ counts every edge in $E'$ exactly once, showing that $w(OPT(\mathcal{G} \setminus S)) = y(E')$. Hence, using $y(E) = \alpha + y(\delta(S)) + y(E') = 1$, we get

$$R(w) \leq 2\alpha + y(\delta(S)) + y(E') = 1 + \alpha.$$ $\square$

We now show that these two bounds are tight for any values of $\alpha \in [0, 1]$ and $\rho \geq 2$.

**Theorem 4.7.** Let $\alpha \in [0, 1]$ and let $\rho \geq 2$. There exists a non-bipartite graph $\mathcal{G} = (V, E)$, with weights $(y, w) \in Q^{V,W}$, and a set $S \subseteq V$ with $y(E[S]) = \alpha$, where $\mathcal{G} / S$ has odd girth $2\rho - 1$ and which satisfies

$$R(w) = \left(1 + \frac{1}{\rho}\right)(1 - \alpha) + 2\alpha.$$ 

Proof. An example of such a graph can be constructed as follows. We first construct $\mathcal{G} / S$: take an odd cycle of length $2\rho - 1$ with a distinguished node $v^S$ and assign dual value $(1 - \alpha)/\rho$ to both edges incident to it. Alternatively assign dual values $0$ and $(1 - \alpha)/\rho$ along the odd cycle for the remaining edges. In order to construct $\mathcal{G}$, replace $v^S$ by a triangle $S$ with dual edges set to $\alpha$, 0 and 0, where the two previous incident edges to $v^S$ are adjacent to the endpoints of the edge with value $\alpha$. Note that we replace it with a triangle instead of a single edge in order to avoid $\mathcal{G}$ becoming bipartite. Similarly to the proof of Theorem 1.2 one can check that

$$w(S) = 2\alpha + \frac{2(1 - \alpha)}{\rho}; \quad w(OPT(\mathcal{G} \setminus S)) = \frac{(1 - \alpha)(\rho - 1)}{\rho}; \quad w(OPT(\mathcal{G})) = 1.$$ 

Therefore,

$$R(w) = 2\alpha + \frac{2(1 - \alpha)}{\rho} + \frac{(1 - \alpha)(\rho - 1)}{\rho} = \left(1 + \frac{1}{\rho}\right)(1 - \alpha) + 2\alpha.$$ $\square$

**Theorem 4.8.** Let $\alpha \in [0, 1]$. There exists a non-bipartite graph $\mathcal{G} = (V, E)$, with weights $(y, w) \in Q^{V,W}$, and a set $S \subseteq V$ with $y(E[S]) = \alpha$, where $\mathcal{G} / S$ is bipartite and which satisfies

$$R(w) = 1 + \alpha.$$ 

Proof. Let $\mathcal{G}$ be an arbitrary odd cycle. Consider an arbitrary edge $(u, v) \in E$ and assign it dual value $\alpha$. The set $S$ is defined to be $S = \{u, v\}$. Assign dual value zero to the edge $(u, w) \in E$, where $w$ is the second neighbour of $u$ in the cycle. For the remaining edges, arbitrarily assign dual values, while ensuring that they sum up to $1 - \alpha$. The fact that one edge is equal to zero is necessary in order to get the exact formula $w(OPT(\mathcal{G})) = 1$, a feasible cover showing $w(OPT(\mathcal{G})) \leq 1$ being the following: take both endpoints of the edge $(u, v)$ and take remaining vertices alternatively (hence always skipping one) along the odd cycle. All the edges are counted once, except for $(u, w)$, which is counted twice but has value zero. Moreover, $w(S) = 2\alpha + y(\delta(S))$ and $w(OPT(\mathcal{G} \setminus S)) = y(E')$, where $E'$ is the edge set of the bipartite graph $\mathcal{G} \setminus S$. Therefore,

$$R(w) = 2\alpha + y(\delta(S)) + y(E') = 1 + \alpha.$$ $\square$
5 Algorithmic application

A natural question to ask is what is, for a general graph, the best possible set $S$ bipartizing the graph one can find? By best possible, we mean here a set $S$ with a low value for the $\alpha$ parameter. In fact, once such a set $S$ is found, there can also be freedom in the choice of the dual solution in order to optimize the $\alpha$ parameter. This motivates the following definition.

**Definition 5.1.** Let $(G, S, y, w)$ be a graph with a bipartizing set $S \subset V$, weights $w \in Q^W$ and a dual solution $y \in \mathbb{R}_+^E$. A tuple $(G', S', y', w')$ is approximation preserving if

$$w(S) + w(OPT(G \setminus S)) \leq w'(S') + w'(OPT(G' \setminus S')).$$

Moreover, we say that $\alpha \in [0, 1]$ valid for $S \subset V$ if there exists an approximation preserving $(G', S', y', w')$ such that $\alpha = y'(E[S'])$.

Finding a valid $\alpha \in [0, 1]$ would directly allow us to use it in the bound of Theorem 4.5 where the $\rho$ parameter would correspond to the one of the approximation preserving graph. We present here an application if a coloring of a graph can be found efficiently.

**Theorem 5.1.** Let $G = (V, E)$ be a graph with weights $w \in Q^W$ that can be $k$-colored in polynomial time for $k \geq 4$. There exists an efficiently findable set $S \subset V$ bipartizing the graph and a valid $\alpha$ such that

$$\alpha \leq 1 - 4/k.$$

**Proof.** Let us denote by $V_1, \ldots, V_k$ the $k$ independent sets defining the color classes of the graph $G$. We assume without loss of generality that they are ordered by weight $w(V_1) \leq w(V_2) \cdots \leq w(V_k)$. Since $w(V) = 2$, the two color classes with the largest weights satisfy $w(V_{k-1}) + w(V_k) \geq 4/k$. We define the bipartizing set to be the remaining color classes: $S := V_1 \cup \cdots \cup V_{k-2}$. We denote by $y \in \mathbb{R}_+^E$ the dual solution satisfying complementary slackness and $y(E) = 1$.

We now define an approximation preserving tuple $(G', S', y', w')$ in the following way. Let $G' = K_k$ be the complete graph on $k$ vertices, denoted by $\{v_1, \ldots, v_k\}$. The weights are defined to be

$$w'(v_i) := w(V_i) \quad \text{and} \quad y'(v_i, v_j) := y(E[V_i, V_j])$$

for every $i, j \in [k]$. These clearly satisfy the complementary slackness condition $y'(\delta(v_i)) = w'(v_i)$ for every $i \in [k]$. The bipartizing set is defined to be $S' := \{v_1, \ldots, v_{k-2}\}$. This tuple is approximation preserving since $w(S) = w'(S')$ and $w(OPT(G \setminus S)) \leq w'(OPT(G' \setminus S'))$. In order to prove the theorem, we still need to tweak the dual solution $y'$ to ensure $\alpha := y'(E[S']) \leq 1 - 4/k$.

Observe that $w'(v_{k-1}) + w'(v_k) \geq 4/k$.

- If $y'(v_{k-1}, v_k) = 0$, then the result follows since in that case $y'(\delta(S')) \geq 4/k$ and thus $y'(E[S']) \leq 1 - 4/k$.

- If $y'(E[S']) = 0$, then the result trivially follows as well.

Suppose thus that $y'(E[S']) > 0$ and $y'(v_{k-1}, v_k) > 0$. Pick an arbitrary edge $(v_i, v_j) \in E[S']$ satisfying $y'(v_i, v_j) > 0$ and consider the 4-cycle $(v_i, v_j, v_{k-1}, v_k)$. Notice that alternatively increasing and decreasing the dual values on the edges of this cycle by a small amount $\epsilon > 0$ gives another feasible dual solution satisfying the complementary slackness condition. More formally, we set $\epsilon := \min\{y'(v_i, v_j), y'(v_{k-1}, v_k)\}$. These clearly satisfy the complementary slackness condition.

Formally, we set $\epsilon := \min\{y'(v_i, v_j), y'(v_{k-1}, v_k)\}$, decrease $y'(v_i, v_j)$ and $y'(v_{k-1}, v_k)$ by $\epsilon$, while increasing $y'(v_i, v_{k-1})$ and $y'(v_k, v_j)$ by the same amount. Observe that this leads to either $(v_i, v_j)$ or $(v_{k-1}, v_k)$ dropping to dual value zero. We can repeat this procedure until either $y'(E[S']) = 0$ or $y'(v_{k-1}, v_k) = 0$, finishing the proof of the theorem. \qed
Setting $\alpha \leq 1 - 4/k$ and $\rho \geq 2$ in the bound of Theorem 4.5 directly gives us an approximation algorithm of ratio $R(w) \leq 2 - 2/k$ for this setting. This recovers a result given by Hochbaum in [27] and is in fact best possible when working with the standard linear relaxation. This holds since we are lower bounding $w(OPT)$ by comparing it to the value of the optimal LP solution in Lemma 3.2 and the fact that the integrality gap of the linear relaxation on the complete graph with $k$ vertices is $2 - 2/k$.

6 Integrrality Gap and Fractional Chromatic Number

In this section, we consider as in Sections 4.1 and 4.2 graphs $G = (V, E)$ having the property that there exists an independent set $I \subset V$ whose removal yields a bipartite graph. Equivalently, this is simply the class of graphs with chromatic number three. For the case where $I = \{v_p\}$ is a single vertex, we provide explicit formulas for the integrality gap and the fractional chromatic number, and show that they are again determined by the odd girth of the graph, generalizing a result known for the cycle graph. For the case of an arbitrary 3-colorable graph, we provide tight upper bounds. A key result that we use in this section is given by Singh in [48], which relates the integrality gap with the fractional chromatic number of a graph.

The latter is denoted as $\chi_f(G)$ and is defined as the optimal solution of the following primal-dual linear programming pair. We denote by $I \subset 2^V$ the set of all independent sets of the graph $G$. Solving these linear programs is however NP-hard because of the possible exponential number of independent sets.

$$\min \sum_{I \in I} y_I$$

subject to

$$\sum_{I \in I, v \in I} y_I \geq 1 \quad \forall v \in V$$

$$y_I \geq 0 \quad \forall I \in I$$

$$\max \sum_{v \in V} z_v$$

subject to

$$\sum_{v \in I} z_v \leq 1 \quad \forall I \in I$$

$$z_v \geq 0 \quad \forall v \in V$$

Note that $\chi_f(G) = 2$ if and only if $G$ is bipartite.

**Theorem 6.1** (Singh, [48]). Let $G = (V, E)$ be a graph. The integrality gap of the vertex cover linear programming relaxation $P(G)$ is:

$$IG(G) = 2 - \frac{2}{\chi_f(G)},$$

where $\chi_f(G)$ is the fractional chromatic number of the graph $G$.

We first focus on graphs with the existence of a single vertex whose removal produces a bipartite graph. The following theorem generalizes the result given for the cycle graph in [4] and turns out to be the same formula as for series-parallel graphs [21].

**Theorem 6.2.** Let $G = (V, E)$ be a non-bipartite graph and $v_p \in V$ such that $G \setminus v_p = (A \cup B, E')$ is bipartite. Then,

$$\chi_f(G) = 2 + \frac{1}{\rho - 1},$$

where $2\rho - 1$ is the odd girth of $G$.

**Proof of Theorem 6.2.** We prove this theorem by constructing feasible primal and dual solutions of objective value $2 + 1/(\rho - 1)$. By strong duality, these two solutions are then optimal for their respective linear programs, hence proving the theorem.
Figure 8: An optimal dual solution constructed in the proof of Theorem 6.2. Each node on a shortest odd cycle is assigned a fractional value of $1/(\rho - 1)$.

We first construct the dual solution. Let $C$ be the set of all the shortest odd cycles of $G$. For any such cycle $C \in C$, define the dual solution $z^C \in \mathbb{R}^V$ by

$$z^C_v = \begin{cases} 1/\rho - 1 & \text{if } v \in C \\ 0 & \text{if } v \in V \setminus C \end{cases}$$

This solution is feasible since any independent set in an odd cycle of length $2\rho - 1$ has size at most $\rho - 1$. Indeed, fix an independent set $I \in \mathcal{I}$, then:

$$\sum_{v \in I} z^C_v = \sum_{v \in I \cap C} \frac{1}{\rho - 1} = \frac{|I \cap C|}{\rho - 1} \leq 1.$$ 

Moreover, the objective value of this solution is:

$$\sum_{v \in V} z^C_v = \sum_{C \in C} \frac{1}{\rho - 1} = \frac{2\rho - 1}{\rho - 1} = 2 + \frac{1}{\rho - 1}.$$ 

Let us now construct the primal solution. We will do so by constructing $2\rho - 1$ independent sets $I_k \in \mathcal{I}$ and assigning to each of them a fractional value of $y(I_k) = 1/(\rho - 1)$. All the other independent sets are assigned value zero. We split the bipartite graph $G \setminus v_p$ into the layers $L_i := \{v \in A \cup B \mid d(N_A(v_p), v) = i\}$ for $i \in \{0, \ldots, l\}$ as explained in (4.1). As a reminder, any shortest odd cycle corresponds to a path between $L_0 = N_A(v_p)$ and $L_{2\rho - 3} \cap N_B(v_p)$. The original vertex set $V$ is thus decomposed into $\{v_p\} \cup L_0 \cup \cdots \cup L_l$, where each layer is an independent set and only has edges going out to $v_p$ or its two neighbouring layers.

Let us first focus on the subgraph consisting of the vertices in $\{v_p\} \cup \bigcup_{i=0}^{2\rho - 3} L_i$, where any shortest odd cycle has exactly one vertex per layer (per abuse of notation, we say that $\{v_p\}$ is also a layer in this situation). For convenience of indexing, we rename these layers as $\hat{L}_1, \ldots, \hat{L}_{2\rho - 1}$ where $\hat{L}_1 = v_p$ and $\hat{L}_i = L_{i-2}$ for $i > 1$. We now create $2\rho - 1$ independent sets on this subgraph in the following way. The first independent set is defined as $U_1 = \hat{L}_1 \cup \hat{L}_4 \cup \hat{L}_6 \cdots \cup \hat{L}_{2\rho - 2}$, where we take the first layer $\hat{L}_1$, skip two before taking the next one and then continue by taking the remaining layers alternatingly (hence always skipping one), see Figure 9. Note that the layer following $\hat{L}_{2\rho - 1}$ is assumed to be $\hat{L}_1$. This procedure generates in fact a distinct independent set by starting at $\hat{L}_k$ for any $k \in [2\rho - 1]$ and we denote the corresponding independent set by
Notice that each layer is contained in exactly $\rho - 1$ of the constructed independent sets $\{U_k \mid k \in [2\rho - 1]\}$.

We now focus on the subgraph consisting of the vertices in $\bigcup_{i > 2\rho - 3} \mathcal{L}_i$. We can construct two different independent sets there by taking either the odd or even indexed layers, i.e.

$$R_1 := \bigcup_{i \text{ odd, } i > 2\rho - 3} \mathcal{L}_i \quad \text{and} \quad R_2 := \bigcup_{i \text{ even, } i > 2\rho - 3} \mathcal{L}_i.$$ 

We now define our final $2\rho - 1$ independent sets on the full graph as:

$$I_k := \begin{cases} U_k \cup R_1 & \text{if } v_p \notin U_k \\ U_k \cup R_2 & \text{if } v_p \in U_k \end{cases} \quad \forall k \in [2\rho - 1].$$

These are in fact independent sets: in the first case, the first layer in $R_1$ is $\mathcal{L}_{2\rho - 1}$ whereas the last layer in $U_k$ has index at most $2\rho - 3$, meaning that there are no two neighbouring layers. In the second case, since $v_p \in U_k$, we have that $\mathcal{L}_{2\rho - 3} \notin U_k$, by construction of $U_k$. The last layer in $U_k$ thus has index at most $2\rho - 4$, whereas the first layer in $R_2$ is $\mathcal{L}_{2\rho - 2}$, meaning again that there are no two neighbouring layers. In addition, there is no edge between $v_p$ and $R_2$, because the only even indexed layer having edges sent to $v_p$ is $\mathcal{L}_0 = N_A(v_p)$.

We now define our primal solution as

$$y(I_k) = \frac{1}{\rho - 1} \quad \forall k \in [2\rho - 1],$$

and $y(I) = 0$ for every other independent set $I \in \mathcal{I}$. We now show this is a feasible solution, i.e. that every vertex $v \in V$ belongs to at least $\rho - 1$ independent sets in $\{I_k \mid k \in [2\rho - 1]\}$.
For \( v \in \{v_p\} \cup \bigcup_{i=0}^{2p-3} L_i \), such a vertex lies by construction in exactly \( \rho - 1 \) independent sets \( \{U_k \mid k \in [2p-1]\} \), and thus also of \( \{I_k \mid k \in [2p-1]\} \). For \( v \in \bigcup_{i=2p-3}^{\rho-1} L_i \), if \( v \) belongs to an even indexed layer, then it is contained in \( \rho - 1 \) of the desired independent sets. If it belongs to an odd indexed layer, then it is contained in \( \rho \) of them. Therefore, \[
\sum_{I \in \mathcal{I}, v \in I} y_I = \sum_{k=1}^{2p-1} y(I_k) 1_{\{v \in I_k\}} = \frac{1}{\rho - 1} \sum_{k=1}^{2p-1} 1_{\{v \in I_k\}} \geq 1.
\]
The objective value of this primal solution is clearly \( 2 + 1/(\rho - 1) \). We have constructed feasible primal and dual solutions with the same objective value. By strong duality, this finishes the proof of the theorem.

We now consider the case where \( G = (V, E) \) is a graph with chromatic number \( \chi(G) = 3 \).

**Theorem 6.3.** Let \( G = (V, E) \) be a 3-colorable graph with color classes \( V = V_1 \cup V_2 \cup V_3 \). Then, \[
\chi^f(G) \leq 2 + \min_{i \in \{1,2,3\}} \frac{1}{\rho_i - 1}
\]where \( 2\rho_i - 1 \) is the odd girth of the contracted graph \( G/V_i \) for each \( i \in \{1,2,3\} \). Moreover, equality holds if one color class only contains one vertex.

**Proof.** We prove this theorem by constructing three feasible solutions of value \( 2 + 1/(\rho_i - 1) \) for each \( i \in \{1,2,3\} \) to the primal linear program of the fractional chromatic number on the graph \( G \).

Fix an \( i \in \{1,2,3\} \) and consider the graph \( \tilde{G} := G/V_i = (\tilde{V}, \tilde{E}) \) with odd girth \( 2\rho_i - 1 \). We denote the contracted node by \( \tilde{v} \in \tilde{V} \). Since this graph is bipartite if we were to remove \( \tilde{v} \), we know that its fractional chromatic number is equal to \( 2 + 1/(\rho_i - 1) \) by Theorem 6.2. Let \( \{I_k, k \in [2\rho_i - 1]\} \) be the independent sets in the support of the optimal primal solution of the graph \( \tilde{G} \) constructed in the proof of this theorem. For each of these independent sets, we extend them to the original graph in the following way:

\[
I_k = \begin{cases} 
\tilde{I}_k & \text{if } \tilde{v} \notin \tilde{I}_k \\
(\tilde{I}_k \setminus \tilde{v}) \cup V_i & \text{if } \tilde{v} \in \tilde{I}_k.
\end{cases}
\]

In words, if \( \tilde{v} \) happens to belong to \( \tilde{I}_k \), we replace it by \( V_i \) to get a valid independent set in the original graph. Assigning fractional value \( y(I_k) = 1/(\rho_i - 1) \) for every \( k \in [2\rho_i - 1] \) yields a feasible primal solution with objective value \( 2 + 1/(\rho_i - 1) \). Since we can do this for every \( i \in \{1,2,3\} \), and the optimal minimum value of the primal linear program is at most the objective value of any of these feasible solutions, the proof is finished.

Moreover, this upper bound is in fact tight, since it holds with equality when one of the color classes only contains one vertex by Theorem 6.2.

It is now straightforward to extend this result for the integrality gap by Theorem 6.1.

**Corollary 6.1.** Let \( G = (V, E) \) be a 3-colorable graph with color classes \( V = V_1 \cup V_2 \cup V_3 \). The integrality gap \( IG(G) \) of the standard linear programming relaxation \( P(G) \) satisfies:

\[
IG(G) \leq 1 + \min_{i \in \{1,2,3\}} \frac{1}{2\rho_i - 1}
\]

where \( 2\rho_i - 1 \) is the odd girth of the contracted graph \( G/V_i \) for each \( i \in \{1,2,3\} \). Moreover, equality holds if one color class only contains one vertex.

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