RESEARCH ARTICLE

Approximate controllability of nonlocal non-autonomous Sobolev type evolution equations

Arshi Meraj* and Dwijendra Narain Pandey

Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, India, PIN - 247667
arshimeraj@gmail.com, dwij.iitk@gmail.com

ARTICLE INFO

Article History:
Received 26 July 2018
Accepted 16 May 2019
Available 30 July 2019

Keywords:
Krasnoselskii fixed point theorem
Evolution system
Approximate controllability
Sobolev type differential equations

AMS Classification 2010:
93B05; 34G20; 34K30

1. Introduction

In this article, we discuss the approximate controllability of nonlocal Sobolev type non-autonomous evolution equations in a separable Hilbert space $X$:

$$\frac{d}{dt}[Ex(t)] + A(t)x(t) = F(t, x(t)) + Bu(t),$$
$$x(0) + G(x) = x_0, \quad x_0 \in D(E), \quad (1)$$

where $A(t)$, $E$ are $X$-valued linear operators with domains are subsets of $X$, and $F$ is $X$-valued function defined over $J \times X$, $G$ is $D(E)$-valued function defined over $C(J, X)$, $J = [0, b]$. The control function $u \in L^2(J, U)$, $U$ is a Hilbert space and $B$ is $X$-valued linear and bounded operator defined over $U$.

The Sobolev type differential equations appears in several fields such as thermodynamics [1], fluid flow via fissured rocks [2], and mechanics of soil [3]. Brill [4] first established the existence of solution for a semilinear Sobolev differential equation in a Banach space. Lightbourne et al. [5] studied a partial differential equation of Sobolev type. Generalization of classical initial condition which is known as nonlocal condition is more effective to obtain better results. Nonlocal Cauchy problem was first considered by Byszewski [6].

Controllability is an important issue in engineering and mathematical control theory. The problem of exact controllability is to show that there exists a control function, that steers the solution of the system from its initial state to the given final state. However in approximate controllability, it is possible to steer the solution of the system from its initial state to arbitrary small neighbourhood of the the final state. Mostly the problem of controllability for various kinds of differential, integro-differential equations and impulsive differential equations are studied for autonomous systems. For more details, we refer to [7] - [13].

The existence of mild solutions for a non-autonomous nonlocal integro-differential equation

*Corresponding Author
is investigated by Yan [14] via Banach contraction principle, Schauder’s fixed point theorem and the theory of evolution families. Haloi et al. [15] generalized the above results for non-autonomous differential equations with deviated arguments by the use of theory of analytic semigroup and Banach fixed point theorem. Alka et al. [16] generalized the results of [15] for instantaneous impulsive non-autonomous differential equations with iterated deviating arguments. Hamdy [17] studied sufficient conditions for controllability of autonomous Sobolev type fractional integro-differential equations with the help of Schauder’s fixed point theorem and the theory of compact semigroup. Mahmudov [18] discussed the approximate controllability of autonomous fractional Sobolev type differential system in Banach space with the help of Schauder’s fixed point theorem. Recently, Haloi [19] established sufficient conditions for approximate controllability of non-autonomous nonlocal delay differential systems with deviating arguments by using theory of compact semigroup and Krasnoselskii fixed point theorem.

To the best of our knowledge, no work yet available on approximate controllability of non-autonomous Sobolev type differential systems, inspired by this, we consider the system (1) to find the sufficient conditions for the approximate controllability. The remaining part of the article is organized as following. Section 2 is concerned with some basic notations and definitions, also we will introduce the expression for mild solutions of the system (1). In section 3, we will study our main results. In section 4, we will present an example to illustrate our results. In last section 5, we will discuss the conclusions.

2. Preliminaries

This section is concerned with some basic assumptions, definitions and theorems required to prove our objectives. For more details, we refer [14, 20] and [21]. Let us denote $C(J, X)$ for the complete norm space of all continuous maps from $J$ to $X$, for a finite constant $r > 0$, let $\Omega_r = \{ x \in C(J, X) : \|x(t)\| \leq r, t \in J \}$. $L^p(J, X) (1 \leq p < \infty)$ is the Banach space of all Bochner integrable functions from $J$ to $X$ with norm $\|x\|_{L^p(J, X)} = \left( \int_0^1 \|x(t)\|^p dt \right)^{1/p}$. Now, we impose the following restrictions (see [4, 20, 21]).

(A1) The operator $A(t)$ is closed, domain of $A(t)$ is dense in $X$ and independent of $t$.

(A2) For $Re(\theta) \leq 0$, $t \in J$, the resolvent operator of $A(t)$ exists and satisfies $\|R(\theta; t)\| \leq \frac{\varsigma}{|\theta|+1}$, for some positive constant $\varsigma$.

(A3) For each fixed $\tau_3 \in J$, there are constants $K \geq 0, \rho \in (0, 1]$ such that $\|A(\tau_1) - A(\tau_2)\|A^{-1}(\tau_3)\| \leq K|\tau_1 - \tau_2|^{\rho}$ for any $\tau_1, \tau_2 \in J$.

(S1) $E$ is closed, bijective operator, and $D(E) \subset D(A)$.

(S2) $E^{-1} : X \rightarrow D(E)$ is compact.

The assumptions (A1), (A2) imply that $-A(t)$ generates an analytic semigroup in $B(X)$, where the symbol $B(X)$ stands for Banach space of all bounded linear operators on $X$. The closed graph theorem with the above assumptions imply that the linear operator $-A(t)E^{-1} : X \rightarrow X$ is bounded, and so for each $t \in J$, $-A(t)E^{-1}$ generates a semigroup of bounded linear operators and hence a unique evolution system $\{S(t_1, t_2) : 0 \leq t_2 \leq t_1 \leq b\}$ on $X$, which satisfies (see [14, 20, 21]):

(i) $S(t_1, t_2) \in B(X)$ and is continuous strongly in $t_1, t_2$ for $0 \leq t_2 \leq t_1 \leq b$.

(ii) $S(t_1, t_2)x \in D(A)$, $x \in X$, $0 \leq t_2 \leq t_1 \leq b$.

(iii) $S(t_1, t_2)S(t_2, t_3) = S(t_1, t_3)$, $0 \leq t_3 \leq t_2 \leq t_1 \leq b$.

(iv) $S(\eta, \eta)$ is identity operator, for $\eta \in J$.

(v) $\|S(t_1, t_2)\| \leq M$, $0 \leq t_2 \leq t_1 \leq b$, for some positive constant $M$.

(vi) For each fixed $t_2$, $\{S(t_1, t_2), t_2 < t_1\}$ is uniformly continuous in uniform operator norm.

(vii) For $0 \leq t_2 < t_1 \leq b$, the derivative $\frac{\partial S(t_1, t_2)}{\partial t_1}$ exists in strong operator topology, is strongly continuous in $t_1$. Moreover,

$$\frac{\partial S(t_1, t_2)}{\partial t_1} + A(t_1)S(t_1, t_2) = 0, 0 \leq t_2 < t_1 \leq b.$$
In this section, we prove the existence of mild solutions and approximate controllability of \( [1] \).

### 3. Main results

In this section, we prove the existence of mild solutions and approximate controllability of \([1]\). For \( x \in C(J, X) \), consider the control function for the system \([1]\) as following:

\[
x(t) = \mathbb{E}^{-1}S(t, 0)\mathbb{E}x_0 + \int_0^t \mathbb{E}^{-1}S(t, s)\mathbb{F}(s)ds.
\]

**Definition 1.** A mild solution of \([1]\) is a function \( x \in C(J, X) \) satisfying the following integral equation

\[
x(\eta) = \mathbb{E}^{-1}S(\eta, 0)\mathbb{E}(x_0 - \mathcal{G}(x)) + \int_0^\eta \mathbb{E}^{-1}S(\eta, \eta')[\mathbb{F}(\eta, x(\eta)) + \mathbb{B}u(\eta)]d\eta, \quad \eta \in J.
\]

For the control \( u \) and initial data \( x_0 \), use \( x_b(x_0, u) \) to denote the state value at time \( b \). The set \( \mathcal{R}(b, x_0) = \{x_b(x_0, u) : u \in L^2(J, U)\} \), is called the reachable set at time \( b \).

**Definition 2.** \([8]\) If \( \mathcal{R}(b, x_0) \) is dense in \( X \), the system \([1]\) is called approximately controllable on \( J \).

Consider the linear control system:

\[
\frac{d}{dt}[\mathbb{E}x(t)] + A(t)x(t) = \mathbb{B}u(t), \quad t \in J, \quad x(0) = x_0.
\]

Corresponding to \([1]\), the controllability operator is given as

\[
\Gamma_b^0 = \int_0^b \mathcal{V}(b, \eta)\mathbb{E}\mathbb{B}^*\mathbb{V}(b, \eta)d\eta,
\]

where \( \mathcal{V}(t, s) := \mathbb{E}^{-1}S(t, s) \), \( * \) denotes the adjoint of the operator. Notice that \( \Gamma_b^0 \) is a bounded linear operator.

**Theorem 2.** \([8]\) The necessary and sufficient conditions for the linear system \([1]\) to be approximately controllable on \( J \) is that, \( \delta R(\delta, \Gamma_b^0) \to 0 \) as \( \delta \to 0^+ \) in the strong operator topology, where \( R(\delta, \Gamma_b^0) := (\delta I + \Gamma_b^0)^{-1}, \delta > 0 \).

Now, we recall the Krasnoselskii fixed point technique.

**Theorem 3.** \([22]\) Let \( S \) is a convex bounded closed subset of a Banach space \( X \). Suppose that \( F_1, F_2 \) be two \( X \)-valued operators defined on \( S \) such that such that \( F_1x + F_2y \in S \) whenever \( x, y \in S \), \( F_1 \) is continuous and compact, and \( F_2 \) is contraction map. Then \( F_1 + F_2 \) has a fixed point in \( S \).

### 3. Main results

**Proof.** By assumption \((H2)\), we get...
\[
\int_0^e \|F(\eta, x(\eta))\| d\eta \leq \int_0^e \left( \|F(\eta, x(\eta))\| - \|F(\eta, 0)\| + \|F(0)\| \right) d\eta \\
\leq \int_0^e (L_1\|x\| + N_1)d\eta \\
\leq (L_1r + N_1)b = K_1.
\]

\[\square\]

This implies, for large enough \(r > 0\), \(F_\lambda(\Omega_r) \subset \Omega_r\) holds.

**Step II:** \(\Phi_\lambda : \Omega_R \to \Omega_R\) is contraction.

For \(x, y \in \Omega_R\) and \(t \in J\), using \((H2)\) and \((H3)\) we obtain

\[
\| (\Phi_\lambda x)(t) - (\Phi_\lambda y)(t) \| \leq \| (V(t, 0) \mathbb{E}G(x)) - \mathbb{G}(y) \| \\
+ \int_0^t \| V(t, s) \| \| F(s, x(s)) - F(s, y(s)) \| ds \\
\leq M_2 M_2 \| x - y \| + M_2 M \\
\int_0^t L_1\|x - y\| ds \\
\leq M_2 M_2 \| x - y \| + M_2 M_1 b \| x - y \| \\
\leq M_2 M (L_2 + L_1 b) \| x - y \| \\
= \Lambda \| x - y \|. \tag{13}
\]

Since \(\Lambda < 1\), therefore \(\Phi_\lambda\) is contraction.

**Step III:** \(\Psi_\lambda\) is continuous in \(\Omega_R\).

Consider \(\{x_n\}\) be a sequence in \(\Omega_R\) with \(\lim_{n \to \infty} x_n = x\) in \(\Omega_R\). From continuity of non-linear term \(\mathcal{F}\) with respect to state variable, we have

\[
\lim_{n \to \infty} \mathcal{F}(\eta, x_n(\eta)) = \mathcal{F}(\eta, x(\eta)), \quad \text{for each } \eta \in J.
\]

So, we can conclude that

\[
\sup_{\eta \in J} \| \mathcal{F}(\eta, x_n(\eta)) - \mathcal{F}(\eta, x(\eta)) \| \to 0 \quad \text{as } n \to \infty. \tag{14}
\]

For \(t \in J\), \((S1)\), \((H3)\), and \((14)\) yield the following

\[
\| (F_\lambda x)(t) \| \leq \| (V(t, 0) \mathbb{E}G(x)) \| \\
+ \int_0^t \| V(t, \eta) \| \| \mathcal{F}(\eta, x(\eta)) \| d\eta \\
+ \int_0^t \| V(t, \eta) \| \mathbb{B}u_\lambda(\eta, x) d\eta \\
\leq \mathcal{M}_2 M (\| \mathbb{E}x_0 \| + \| \mathbb{E}G(0) \|) \\
+ M_2 M K_1 \\
+ M_2 M M_2 K_2 b. \tag{12}
\]

**Theorem 4.** Let the assumptions \((H1)-(H4)\) hold and the functions \(\mathbb{E}G(0)\) is bounded, then a mild solution to the system \((\text{II})\) exists, provided that

\[
\Lambda := \mathcal{M}_2 M (L_2 + L_1 b) < 1. \tag{10}
\]

**Proof.** The proof is divided into the following steps:

**Step I:** For \(\lambda > 0\), we have a constant \(R\) (depends on \(\lambda\)), satisfying \(F_\lambda(\Omega_R) \subset \Omega_R\).

For any positive constant \(r\) and \(x \in \Omega_r\), if \(t \in J\), then by using \((6)\), \((H3)\) and Lemma \((1)\), we have

\[
u_\lambda(t, x) = B^* \nu^*(b, t R(\lambda, \Gamma^b_0)) \left[ x^b - \mathcal{V}(b, 0) \mathbb{E}(x_0 - \mathcal{G}(x)) - \int_0^b \mathcal{V}(b, \eta) \mathcal{F}(\eta, x(\eta)) d\eta \right]
\]

\[
\| u_\lambda(t, x) \| \leq \frac{M_1 M_2 M}{\lambda} \left[ \| x^b \| + M_2 M (\| \mathbb{E}x_0 \| + \| \mathbb{E}G(x) - \mathbb{G}(0) \| + \| \mathbb{E}G(0) \|) \\
+ \| \mathbb{E}G(0) \| \right] \\
+ M_2 M K_1 \\
\leq \frac{M_1 M_2 M}{\lambda} \left[ \| x^b \| + M_2 M (\| \mathbb{E}x_0 \| + \| \mathbb{E}G(0) \|) \\
+ L_2 r + \| \mathbb{E}G(0) \| \right] + M_2 M K_1 \\
:= K_2, \tag{11}
\]

and from \((8)\), \((11)\), we obtain

\[
(F_\lambda x)(t) = \mathcal{V}(t, 0) \mathbb{E}(x_0 - \mathcal{G}(x)) + \int_0^t \mathcal{V}(t, \eta) \mathcal{F}(\eta, x(\eta)) d\eta \\
+ \int_0^t \mathcal{V}(t, \eta) \mathbb{B}u_\lambda(\eta, x) d\eta
\]
\[ \|p(x_n) - p(x)\| \leq M_2 M \| \mathbb{E}G(x_n) - \mathbb{E}G(x)\| \]
\[ + M_2 M \int_0^b \| F(\zeta, x_n(\zeta)) \| d\zeta \]
\[ = M_2 M \| \mathbb{E}G(x_n) - \mathbb{E}G(x)\| \]
\[ + M_2 M b \sup_{\zeta \in J} \| F(\zeta, x_n(\zeta)) \| \]
\[ \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad (15) \]

therefore (15) implies that

\[ \|u_\lambda(\eta, x_n) - u_\lambda(\eta, x)\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad (16) \]

and so

\[ \| (\Psi_\lambda x_n)(t) - (\Psi_\lambda x)(t) \| \leq M_2 M_1 M_2 b \sup_{\eta \in J} \| u_\lambda(\eta, x_n) - u_\lambda(\eta, x)\| \]
\[ \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \]

which means \( \Psi_\lambda \) is continuous in \( \Omega_R \).

**Step IV:** \( \Psi_\lambda : \Omega_R \rightarrow \Omega_R \) is compact. For this we need to show:

(i): The set \( \{(\Psi_\lambda x)(\eta) : x \in \Omega_R\} \) is relatively compact subset of \( X \), for each \( \eta \in J \).

For \( \rho = 0 \), obviously the set \( \{(\Psi_\lambda x)(0) : x \in \Omega_R\} = \{0\} \) is compact subset of \( X \). For fixed \( \rho \in (0, b] \) and \( \xi \in (0, \rho) \), consider an operator \( \Psi_\lambda^\xi \)

on \( \Omega_R \) as following

\[
(\Psi_\lambda^\xi x)(\rho) = \int_0^{\rho-\xi} \mathbb{V}(\rho, \eta)\mathbb{E}u_\lambda(\eta, x)d\eta
+ \int_0^{\rho-\xi} \mathbb{E}^{-1}S(\rho, \rho-\xi)S(\rho-\xi, \eta)\mathbb{E}u_\lambda(\eta, x)d\eta

\leq M_2 M (\rho - \xi)\|u_\lambda\| \sup_{\eta \in (0, \rho-\xi]} \|S(\rho, \eta) - S(\rho, \xi)\| + 2M_2 M_1 \epsilon \|u_\lambda\|.

\]

\[
(J_1) \leq \int_0^{\rho_1 - \epsilon} \mathbb{E}^{-1}[S(\rho_2, \eta) - S(\rho_1, \eta)]
\]
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ + \int_{\rho_1 - \epsilon}^{\rho_1} \mathbb{E}^{-1}[S(\rho_2, \eta) - S(\rho_1, \eta)]
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ \leq M_2 M_1 (\rho_1 - \epsilon)\|u_\lambda\| \sup_{\eta \in (0, \rho_1 - \epsilon]} \|S(\rho_2, \eta) - S(\rho_1, \eta)\| + 2M_2 M_1 \epsilon \|u_\lambda\|.\]

\[
(J_2) \leq \int_0^{\rho_2 - \rho_1} \mathbb{E}^{-1}[S(\rho_2, \eta) - S(\rho_1, \eta)]
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ \leq M_2 M_1 (\rho_2 - \rho_1)\|u_\lambda\| \sup_{\eta \in (0, \rho_2 - \rho_1]} \|S(\rho_2, \eta) - S(\rho_1, \eta)\| + 2M_2 M_1 \epsilon \|u_\lambda\|.

Hence, \( \{(\Psi_\lambda x)(\eta) : x \in \Omega_R\} \) is relatively compact subset of \( X \).

(ii): Now, we show \( \{(\Psi_\lambda x)(\rho) : x \in \Omega_R\} \) is equicontinuous. For any \( x \in \Omega_R \) and \( 0 \leq \rho_1 < \rho_2 \leq b \), we have

\[
\|(\Psi_\lambda x)(\rho_2) - (\Psi_\lambda x)(\rho_1)\| = \int_0^{\rho_2} \mathbb{E}^{-1}S(\rho_2, \eta)
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ - \int_0^{\rho_1} \mathbb{E}^{-1}S(\rho_1, \eta)
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ \leq \int_0^{\rho_1} \mathbb{E}^{-1}[S(\rho_2, \eta) - S(\rho_1, \eta)]
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ + \int_{\rho_1}^{\rho_2} \mathbb{E}^{-1}S(\rho_2, \eta)
\[ \mathbb{E}u_\lambda(\eta, x)d\eta \]
\[ \leq J_1 + J_2.
\]

For \( \rho_1 = 0 \), it is easy to see that \( J_1 = 0 \). When \( \rho_1 > 0 \), let \( \epsilon > 0 \) small enough, we obtain
Hence, \( J_1, J_2 \) as \( \phi_2 \to \phi_1, \varepsilon \to 0 \). As a result, for \( x \in \Omega_R \) as \( \phi_2 \to \phi_1 \), which means that \( \Psi_x : \Omega_R \to \Omega_R \) is equicontinuous. Thus, by Arzelà-Ascoli theorem, \( \Psi_x \) is compact on \( \Omega_R \).

Therefore Krasnoselskii fixed point theorem implies that \( F_\lambda \) has a fixed point, which is a mild solution to the problem \((\ref{eq:1})\).

\[ \int_0^b \| F(\eta, x_\lambda(\eta)) \|^2 d\eta \leq L_\alpha^2 b. \]

Hence \( F(\cdot, x_\lambda(\cdot)) \) is a bounded sequence in \( L^2(J, X) \). So, \( \{ F(\cdot, x_\lambda(\cdot)) : \lambda > 0 \} \) has a subsequence, still denoted by it, converges weakly to some \( F(\cdot) \in L^2(J, X) \). Define

\[ x_\omega = x_b - \mathcal{V}(b, 0)\mathbb{E} x_0 + x_g - \int_0^b \mathcal{V}(b, s)F(s)ds. \]

Now, we get

\[ \| p(x_\lambda) - x_\omega \| \leq \| (b, 0)\mathbb{E}G(x_\lambda) - x_g \| + \mathcal{M} \int_0^b \| F(s, x_\lambda(s)) - F(s) \| ds \to 0 \quad \text{as} \quad \lambda \to 0^+. \label{eq:18} \]

From \((\ref{eq:17}), \(\ref{eq:18}\)), and \((H5)\), we obtain

\[ \| x_\lambda - x_\omega \| \leq \| \lambda R(\lambda, \Gamma_0^\delta) p(x_\lambda) \| \]

\[ \leq \| \lambda R(\lambda, \Gamma_0^\delta) \|\| p(x_\lambda) - x_\omega \| \]

\[ \leq \| \lambda R(\lambda, \Gamma_0^\delta) \|\| p(x_\lambda) - x_\omega \| \to 0 \quad \text{as} \quad \lambda \to 0^+. \]

Hence, \((\ref{eq:11})\) is approximately controllable.

\[ \square \]

### 4. Example

Consider a control system governed by the following partial differential equation:

\[ \frac{\partial}{\partial t}[x(t, z) - x_{zz}(t, z)] + [a(t, z) + \frac{\partial^2}{\partial z^2}]x(t, z) = \mu(t, z) + \sin(x(t, z)), \]

\[ z \in (0, \pi), \quad t \in (0, 1]; \]

\[ x(t, 0) = x(t, \pi) = 0, \quad t \in [0, 1]; \]

\[ x(0, z) + \frac{e^t}{c(1 + e^t)} \cos(x(t, z)) = x_0(z), \]

\[ z \in (0, \pi); \]

where \( X = U = L^2([0, 1] \times [0, \pi], \mathbb{R}), \quad x(0) \in D(\mathcal{A}), \quad a(t, z) \in C^1([0, \pi]) \times [0, 1], \quad J = [0, 1], \), i.e. \( b = 1 \), and \( c \) is positive constant. Define

\[ A(t)x(t, z) = [a(t, z) + \frac{\partial^2}{\partial z^2}]x(t, z), \]

\[ \mathbb{E} x = x - x_{zz}, \]
\[ W(t,s)x = T(t-s)e^{\int_s^t a(\tau)d\tau}x, \quad x \in D(A(t)). \] (21)

Here

\[ T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n, \]

with \( e_n(z) = \sqrt{\frac{2}{n\pi}} \sin(nz), 0 \leq z \leq \pi, n = 1,2,\ldots. \)

The operator \( E \) can be written as following (see [24])

\[ Ex = \sum_{n=1}^{\infty} (1 + n^2) < x, e_n > e_n, \quad x \in D(E). \] (22)

Furthermore for \( x \in X \), we have

\[ E^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} < x, e_n > e_n, \] (23)

which is compact. So, the operator \(-A(t)E^{-1}\) generates a compact evolution system of bounded linear operators that is given as

\[ S(t,s)x = U(t-s)e^{\int_s^t a(\tau)d\tau}x, \] (24)

where

\[ U(t)x = \sum_{n=1}^{\infty} e^{\frac{n^2}{1+n^2} t} < x, e_n > e_n. \]

Hence assumptions (H1), (H4) hold. By putting \( x(t) = x(t,\cdot) \) which means \( x(t)(z) = x(t,z), t \in [0,1], z \in [0,\pi] \) and \( u(t) = \mu(t,\cdot) \) is continuous. Let the bounded linear operator \( \mathbb{B} : U \to X \) is defined as \( \mathbb{B}u(t)(z) = \mu(t,z). \) Further

\[ F(t,x(t))(z) = \sin x(t,z), \]

\[ G(x) = \frac{e^t}{c(1 + e^t)} \cos x. \]

So, the system (19) can be formulated into the abstract form of (11). Note that \( \mathbb{B}G(x) = \frac{b_d}{c(1 + e^t)} \cos x. \) Observe that the functions \( F, G \) satisfies the assumptions (H2), (H3), and also \( F, \mathbb{B}G \) are uniformly bounded. Now it is needed to check the approximately controllability of the associated linear system, for this we show that

\[ \mathbb{B}^{*}\mathcal{V}^{*}(b,s)x = 0, \quad s \in [0,b] \Rightarrow x = 0, \] (25)

where \( \mathcal{V}(t,s) = E^{-1}S(t,s). \) Notice that \( S \) and \( E^{-1} \) are self adjoint. Indeed,

\[ \mathbb{B}^{*}\mathcal{V}^{*}(b,s)x = \mathcal{V}^{*}(b,s)x = S^{*}(b,s)(\mathcal{E}^{-1})^{*}x \]

\[ = e^{\int_{0}^{b} a(\tau)d\tau} \sum_{n=1}^{\infty} \frac{1}{1 + n^2} e^{\frac{-n^2}{1+n^2} (b-s)} \]

\[ < \mathcal{E}^{-1}x, e_n > e_n, \quad n = 1,2,\ldots. \] (26)

This implies that the condition (25) holds, and hence the assumption (H5). Thus by Theorem 5 the system (19) is approximately controllable on \( J. \)

5. Conclusion

In this work, we have obtained that the mild solutions for non-autonomous Sobolev differential equations with nonlocal condition exist mainly by the help of evolution system of bounded linear operators and Krasnosel’skii fixed point technique. Also we have determined the sufficient conditions for approximate controllability by using the controllability of corresponding linear system. The results developed in this article can be extended to the study of existence of mild solutions and approximate controllability for neutral and impulsive differential systems. Moreover the obtained results also can be generalized for fractional Sobolev, neutral and impulsive differential systems.

References

[1] Chen, P.J., & Gurtin, M.E. (1968). On a theory of heat conduction involving two temperatures. Zeitschrift für angewandte Mathematik und Physik, 19(4), 614-627.

[2] Barenblatt, G.I., Zheltov, I.P., & Kochina, I.N. (1960). Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. Journal of Applied Mathematics and Mechanics, 24, 1286-1303.

[3] Taylor, D. (1952). Research on Consolidation of Clays. Massachusetts Institute of Technology Press, Cambridge.

[4] Brill, H. (1977). A semilinear Sobolev evolution equation in a Banach space. Journal of Differential Equations, 24, 412-425.

[5] Lightbourne, J.H., & Rankin, S.M. (1983). A partial differential equation of Sobolev type. Journal of Mathematical Analysis and Applications, 93, 328-337.
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[6] Byszewski, L. (1991). Theorem about the existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications*, 162, 494-505.

[7] Curtain, R.F., & Zwart, H. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York.

[8] Mahmudov, N.I. (2003). Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. *SIAM Journal on Control and Optimization*, 42(5), 1604-1622.

[9] Shaktivel, R., Ren, Y., & Mahmudov, N.I. (2011). On the approximate controllability of semilinear fractional differential systems. *Computers and Mathematics with Applications*, 62, 1451-1459.

[10] Mahmudov, N.I., & Zorlu, S. (2013). Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions. *Boundary Value Problems*, 2013(118), 1-16.

[11] Mahmudov, N.I., & Zorlu, S. (2014). On the approximate controllability of fractional evolution equations with compact analytic semigroup. *Journal of Computational and Applied Mathematics*, 259, 194-204.

[12] Balasubramaniam, P., Vembarasan, V., & Senthilkumar, T. (2014). Approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert Space. *Numerical Functional Analysis and Optimization*, 35(2), 177-197.

[13] Zhang, X., Zhu, C., & Yuan, C. (2015). Approximate controllability of fractional impulsive evolution systems involving nonlocal initial conditions. *Advances in Difference Equations*, 2015(244), 1-14.

[14] Yan, Z. (2009). On solutions of semilinear evolution integro-differential equations with nonlocal conditions. *Tamkang Journal of Mathematics*, 40(3), 257-269.

[15] Haloi, R., Pandey, D.N., & Bahuguna, D. (2012). Existence uniqueness ans asymptotic stability of solutions to non-autonomous semi-linear differential equations with deviated arguments. *Nonlinear Dynamics and Systems Theory*, 12(2), 179-191.

[16] Chadha, A., & Pandey, D.N. (2015). Mild solutions for non-autonomous impulsive semilinear differential equations with iterated deviating arguments. *Electronic Journal of Differential Equations*, 2015(222), 1-14.

[17] Ahmed, H.M. (2012). Controllability for Sobolev type fractional integro-differential systems in a Banach space. *Advances in Differences Equations*, 2012(167), 1-10.

[18] Mahmudov, N.I. (2013). Approximate controllability of fractional Sobolev type evolution equations in Banach space. *Abstract and Applied Analysis*, 2013, doi:10.1155/2013/502839.

[19] Haloi, R. (2017). Approximate controllability of non-autonomous nonlocal delay differential equations with deviating arguments. *Electronic Journal of Differential Equations*, 2017(111), 1-12.

[20] Friedman, A. (1997). Partial Differential Equations. Dover publication.

[21] Pazy, A. (1993). *Semigroup of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, Springer-Verlag, New York.

[22] Granas, A., & Dugundji, J. (2003). *Fixed Point Theory*. Springer-Verlag, New York.

Arshi Meraj obtained her Master’s degree in Mathematics from Indian Institute of Technology Kanpur and pursuing Ph.D. in the department of Mathematics at Indian Institute of Technology Roorkee. Her research interests include fractional differential equations and control theory.

Dwijendra Narain Pandey received his Master’s and Ph.D. degree from Indian Institute of Technology Kanpur. Currently he is Associate Professor in the department of Mathematics at Indian Institute of Technology Roorkee. His area of research interests are functional evolution equations, theory of semigroup of operators, control theory, fractional differential equations and numerical methods for solving differential equations.
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