Minimal $P$-symmetric period problem of first-order autonomous Hamiltonian Systems

Chungen Liu and Ben-Xing Zhou
School of Mathematics and LPMC, Nankai University,
Tianjin 300071, P. R. China

Abstract: Let $P \in Sp(2n)$ satisfying $P^k = I_{2n}$, we consider the minimal $P$-symmetric period problem of the autonomous nonlinear Hamiltonian system

$$\dot{x}(t) = JH'(x(t)).$$

For some symplectic matrices $P$, we show that for any $\tau > 0$ the above Hamiltonian system possesses a $k\tau$ periodic solution $x$ with $k\tau$ being its minimal $P$-symmetric period provided $H$ satisfies the Rabinowitz’s conditions on the minimal period conjecture, together with that $H$ is convex and $H(Px) = H(x)$.

Key Words: Maslov $P$-index, Relative Morse index, Minimal $P$-symmetric period, Hamiltonian system

1 Introduction and main result

In this paper, we study the following first-order autonomous Hamiltonian system with $P$-boundary condition:

$$\begin{cases}
\dot{x} = JH'(x), x \in \mathbb{R}^{2n} \\
x(\tau) = Px(0).
\end{cases}$$

(1.1)

where $\tau > 0$, $P \in Sp(2n)$, and $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ is the Hamiltonian function satisfying $H(Px) = H(x)$, $\forall x \in \mathbb{R}^{2n}$. $H'(x)$ denote its gradient, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix, $I_n$ is the identity matrix on $\mathbb{R}^n$. Without confusion, we shall omit the subindex of the identity matrix.

A solution $(\tau, x)$ of the problem (1.1) is called a $P$-solution of the Hamiltonian systems. The problem (1.1) has relation with the closed geodesics on Riemannian manifold (cf. [17]) and symmetric periodic solution or the quasi-periodic solution problem (cf. [18]). In addition, the
There are constants \( \mu > 0 \) such that \( \frac{H(x)}{x} \rightarrow +\infty \) as \( x \rightarrow 0 \) and \( \frac{H(x)}{|x|^2} \rightarrow 0 \) as \( x \rightarrow +\infty \). F. Clarke and I. Ekeland proved a result on the corresponding minimal period problem for some given \( T \) in [5]; I. Ekeland and H. Hofer gave a criterion for the conjecture in [10] which is unfortunately not easy to check.

For \( P \)-boundary problem, S. Tang [28] and the first author of this paper proved that for any \( 0 < \tau < \frac{\pi}{\max_{t \in [0, \tau]} \| J_S P(t) x(t) \| \} \), there exists a nonconstant \( P \)-solution with its minimal \( P \)-symmetric period \( k\tau \) or \( \frac{k\tau}{k+1} \) via the iteration theory of Maslov \( P \)-index. In [22], the first author C. Liu transformed some periodic boundary problem for nonlinear delay differential systems and some nonlinear delay Hamiltonian systems to \( P \)-boundary problems of Hamiltonian systems as above, we also refer [8, 14, 16, 19] and references therein for the background of \( P \)-boundary problems in \( N \)-body problems.

Let \( P \in Sp(2n) \) and \( k \in \mathbb{N} = \{0, 1, 2, \cdots \} \), we say \( P \) satisfies \((P)_k\) condition, if \( P^k = I_{2n} \) and for each integer \( m \) with \( 1 \leq m \leq k-1 \), \( P^m \neq I \). If \( P \) satisfies \((P)_k\) condition, a \( P \)-solution \((\tau, x)\) can be extended as a \( k\tau \)-periodic solution \((k\tau, x^k)\). We say that a \( T \)-periodic solution \((T, x)\) of the Hamiltonian system in \((1.1)\) is \( P \)-symmetric if \( x(\frac{T}{2}) = Px(0) \). \( T \) is the \( P \)-symmetric period of \( x \). \( T \) is called the minimal \( P \)-symmetric period of \( x \) if \( T = \min \{ \lambda > 0 \mid x(t + \frac{\lambda}{k}) = Px(t), \forall t \in \mathbb{R} \} \).

We assume the following conditions on \( H \) in our arguments:

(H0) \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \forall x \in \mathbb{R}^{2n} \);

(H1) \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \) with \( H(Px) = H(x) \), \( \forall x \in \mathbb{R}^{2n} \);

(H2) \( H(x) \geq 0 \), \( \forall x \in \mathbb{R}^{2n} \);

(H3) \( H(x) = o(|x|^2) \) as \( |x| \to 0 \);

(H4) There are constants \( \mu > 2 \) and \( R_0 > 0 \) such that

\[ 0 < \mu H(x) \leq (H'(x), x), \ \forall |x| \geq R_0; \]

(H5) \( H''(x) > 0 \), \( \forall x \in \mathbb{R}^{2n} \);

In [35], Rabinowitz proved that the Hamiltonian system in \((1.1)\) possesses a non-constant prescribed period solution provided \( H \) satisfying (H0) and (H2)-(H4). Because a \( \tau/k \)-periodic function is also a \( \tau \)-periodic function, moreover, in [35] Rabinowitz proposed a conjecture: under the conditions (H0) and (H2)-(H4), for any \( \tau > 0 \), the Hamiltonian system in \((1.1)\) possesses a \( \tau \)-periodic with \( \tau \) being its minimal period. Since then, there were many papers on this minimal period problem (cf. [8], [9], [1], [10], [31], [32], [33], [9], etc.). In 1997, D. Dong and Y. Long [9] developed a new method on this prescribed minimal period solution problem and discovered the intrinsic relationship between the minimal period and the indices of a solution. Based upon the work of [8], G. Fei, Q. Qiu, T. Wang and others applied this method to various problems of Rabinowitz’s conjecture (cf. [12], [13], [24], etc.). In fact, under conditions \( H(0) = 0 \) and \( H(x) > 0 \), \( \forall x \in \mathbb{R}^{2n} \setminus \{0\} \); \( \frac{H(x)}{|x|^2} \rightarrow +\infty \) as \( x \rightarrow 0 \) and \( \frac{H(x)}{|x|^2} \rightarrow 0 \) as \( x \rightarrow +\infty \), F. Clarke and I. Ekeland proved a result on the corresponding minimal period problem for some given \( T \) in [5]; I. Ekeland and H. Hofer gave a criterion for the conjecture in [10] which is unfortunately not easy to check.
of this paper improved the result that for every \( \tau > 0 \), there exists a nonconstant \( P \)-solution with its minimal \( P \)-symmetric period \( k\tau + \frac{k\tau}{k+1} \).

For \( n \in \mathbb{N}, k > 0 \), denote by

\[
Sp(2n) \equiv Sp(2n, \mathbb{R}) = \{ M \in L(\mathbb{R}^{2n}) \mid M^TJM = J \},
\]

\[
\mathcal{P}_\tau(2n) \equiv \{ \gamma \in C([0, \tau], Sp(2n)) \mid \gamma(0) = I \},
\]

\[
Sp(2n)_k \equiv \{ P \in Sp(2n) \mid P \text{ satisfies } (P)_k \text{ condition} \},
\]

\[
\Omega(M) \equiv \{ N \in Sp(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \text{ and } \nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap U \}.
\]

Denote by \( \Omega^0(M) \) the path connected component of \( \Omega(M) \) which contains \( M \).

**Lemma 1.1.** If \( P \in Sp(2n)_k \), then there exists a matrix

\[
I_{2p} \circ R \left( \frac{2\pi}{k}\right)^{\circ j_1} \circ \cdots \circ R \left( \frac{2\pi}{k}\right)^{\circ j_r} \in \Omega^0(P^{-1}),
\]

with \( p + \sum_{m=1}^r j_m = n \).

**Proof.** For \( P \in Sp(2n)_k \), we have \( \sigma(P^{-1}) = \sigma(P) \subseteq \{ 1, e^{\frac{2\pi i}{k}}, e^{\frac{4\pi i}{k}}, \cdots, e^{\frac{2(k-1)\pi i}{k}} \} \subseteq U \).

By the Theorem 1.8.10 in [29], there exists \( M_1(\omega_1) \circ M_2(\omega_2) \cdots \circ M_s(\omega_s) \in \Omega^0(P^{-1}) \) where \( M_i(\omega_i) \) is a basic normal form of some eigenvalue of \( P^{-1}, 1 \leq i \leq s \). And the following are the basic normal forms for eigenvalues in \( U \).

Case 1. \( N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \lambda = \pm 1, b = \pm 1,0. \)

Since \( P \in Sp(2n)_k \), we have \( b = 0 \) and \( \lambda \in \{ -1,1 \} \cap \sigma(P^{-1}) \).

Case 2. \( R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi) \).

Since \( P \in Sp(2n)_k \), we have \( \theta \in \{ \frac{2\pi}{k}, \frac{4\pi}{k}, \cdots, \frac{2(k-1)\pi}{k} \} \).

Case 3. \( N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, b_1 \in \mathbb{R}, b_2 \neq b_3. \)

From direct computation, it is easy to check that the matrix

\[
T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\]

satisfies that \( TR(\theta)T^{-1} = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} \). Then

\[
\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} N_2(\omega, b) \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \]

\[
= \begin{pmatrix} TR(\theta)T^{-1} & TbT^{-1} \\ 0 & TR(\theta)T^{-1} \end{pmatrix}, \text{ where}
\]

\[
TbT^{-1} = \begin{pmatrix} \frac{1}{2}(b_1 + b_4) - \frac{\sqrt{-1}}{2}(b_2 - b_3) & \frac{1}{2}(b_1 + b_3) - \frac{\sqrt{-1}}{2}(b_1 - b_4) \\ \frac{1}{2}(b_2 + b_3) + \frac{\sqrt{-1}}{2}(b_1 - b_4) & \frac{1}{2}(b_1 + b_4) + \frac{\sqrt{-1}}{2}(b_2 - b_3) \end{pmatrix}.
\]

Denoted by

\[
\begin{pmatrix} TR(i\theta)T^{-1} & X(i) \\ 0 & TR(k\theta)T^{-1} \end{pmatrix} = \begin{pmatrix} TR(\theta)T^{-1} & TbT^{-1} \\ 0 & TR(\theta)T^{-1} \end{pmatrix}^i, \ i \in \mathbb{N}
\]
where \( X(i) = \begin{pmatrix} x_1(i) & x_2(i) \\ x_3(i) & x_4(i) \end{pmatrix} \) and \( X(1) = T b T^{-1} = \begin{pmatrix} x_1(1) & x_2(1) \\ x_3(1) & x_4(1) \end{pmatrix} \). By direct computation, we have

\[
X(k) = k e^{\sqrt{i-1}(k-1)\theta} x_1(1),
\]

\[
x_4(k) = k e^{-\sqrt{i-1}(k-1)\theta} x_4(1).
\]

Thus, from \( P^k = I \) we have \( X(k) = 0 \), so \( x_1(1) = x_4(1) = 0 \), i.e.

\[
\frac{1}{2} (b_1 + b_4) - \frac{\sqrt{i-1}}{2} (b_2 - b_3) = 0,
\]

\[
\frac{1}{2} (b_1 + b_4) + \frac{\sqrt{i-1}}{2} (b_2 - b_3) = 0.
\]

Then we have \( b_2 = b_3 \), which is contradict to the definition of the basic normal form \( N_2(\omega, b) \).

Therefore, from (Case1)-(Case3), we get \( M_i(\omega_i) = R(\theta_i) \) where \( \theta_i \in \{0, \frac{2\pi}{k}, \frac{4\pi}{k}, \ldots, \frac{2(k-1)\pi}{k}\} \), \( 1 \leq i \leq s \). And the lemma is proved.

For the notations in Lemma 1.1 we define

\[
Sp(2n)_k(r, p; j_1, j_2, \ldots, j_r) \equiv \left\{ P \in Sp(2n)_k \mid k - 2 \sum_{m=1}^{r} m \cdot j_m > 1, r < \frac{k}{2} \right\}.
\]

Now we state the main result of this paper.

**Theorem 1.1.** Suppose \( P \in Sp(2n)_k(r, p; j_1, j_2, \ldots, j_r) \), and the Hamiltonian function \( H \) satisfies (H1)-(H5), then for every \( \tau > 0 \), the system (1.1) possesses a non-constant \( P \)-solution \((\tau, x)\) such that the minimal \( P \)-symmetric period of the extended \( k\tau \)-periodic solution \((k\tau, x^k)\) is \( k\tau \).

In order to prove the above result, we need to obtain the relationship between the Maslov \( P \)-index and Morse index. Thus we organize this paper as follows, in Section 2, we recall the definition and properties of the Maslov \( P \)-index theory, and we also list out the relationship between the Morse index and the Maslov \( P \)-index (see [27], [28], [23], [11] and [12]). In Section 3, we first study the iteration formula of Maslov index of paths \( \xi \in \mathcal{P}_r(2n) \) such that \( \xi(\tau) = P^{-1} \) in detail, then we will give the complete proof of the main result.

## 2 Preliminaries

In this section, we give a brief introduction to the Maslov \( P \)-index and its iteration properties, and then give the relationship between Maslov \( P \)-index and the relative Morse index which is studied by the first author of this paper in [23].

Maslov \( P \)-index was first studied in [7] and [21] independently for any symplectic matrix \( P \) with different treatment. The first author and S. Tang in [27], [28] defined the Maslov \((P, \omega)\)-index \((i_P^\omega(\gamma), \nu_P^\omega(\gamma))\) for any symplectic path \( \gamma \in \mathcal{P}_r(2n) \). And then the first author of this paper used relative index theory to develop Maslov \( P \)-index in [23] which is consistent with the definition
When the symplectic matrix \( P = \text{diag}\{-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa\} \), \( 0 \leq \kappa \in \mathbb{N} \leq n \), the \((P, \omega)\)-index theory and its iteration theory were studied in \([8]\) and then be successfully used to study the multiplicity of closed characteristics on partially symmetric convex compact hypersurfaces in \( \mathbb{R}^{2n} \). Here we use the notions and results in \([21, 27, 28]\).

For \( \omega \in \mathbb{U} \), then the Maslov \((P, \omega)\)-index of a symplectic path \( \gamma \in \mathcal{P}_\tau(2n) \) is defined as a pair of integers (cf. \([27]\))

\[
(i_\omega^P(\gamma), \nu_\omega^P(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\},
\]

where the index part

\[
i_\omega^P(\gamma) = i_\omega(P^{-1}\gamma * \xi) - i_\omega(\xi),
\]

\( \xi \in \mathcal{P}_\tau(2n) \) such that \( \xi(\tau) = P^{-1} \) and the nullity

\[
\nu_\omega^P(\gamma) = \dim \ker(\gamma(\tau) - \omega P).
\]

Suppose \( B(t) \in C(\mathbb{R}, \mathcal{L}_\kappa(\mathbb{R}^{2n})) \), if \( \gamma \in \mathcal{P}_\tau(2n) \) is the fundamental solution of the linear Hamiltonian systems

\[
\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n},
\]

we also call \((i^P_{\omega}(\gamma), \nu^P_{\omega}(\gamma))\) the Maslov \((P, \omega)\)-index of \( B(t) \), denoting \((i^P_{\omega}(B), \nu^P_{\omega}(B)) = (i^P_{\omega}(\gamma), \nu^P_{\omega}(\gamma))\), just as in \([21, 27, 28]\). If \( x \) is a \( P \)-solution of \((1.1)\), then the Maslov \((P, \omega)\)-index of the solution \( x \) is defined to be the Maslov \((P, \omega)\)-index of \( B(t) = H''(x(t)) \) and denoted by \((i^P_{\omega}(x), \nu^P_{\omega}(x))\).

When \( \omega = 1 \), we omit the subindex, denoted by \((i^P(\gamma), \nu^P(\gamma))\) or \((i^P(B), \nu^P(B))\) for simplicity.

For \( m \in \mathbb{N} \), we extend the definition of \( x(t) \) which is the solution of \((1.1)\) to \([0, +\infty)\) by

\[
x(t) = P^j x(t - j\tau), \quad \forall j \tau \leq t \leq (j + 1)\tau, \quad j \in \mathbb{N}
\]

and define the \( m \)-th iteration \( x^m \) of \( x \) by

\[
x^m = x|_{[0,m\tau]}.
\]

If \( P \) satisfies \((P)_k\) condition, then \( x^k \) becomes an \( k\tau\)-periodic solution of the Hamiltonian system in \((1.1)\). We know that the fundamental solution \( \gamma_x \in \mathcal{P}_\tau(2n) \) carries significant information about \( x \). For any \( \gamma \in \mathcal{P}_\tau(2n) \), S. Tang and the first author of this paper have defined the corresponding \( m \)-th iteration path \( \gamma^m : [0, m\tau] \rightarrow Sp(2n) \) of \( \gamma \) in \([27]\) by

\[
\gamma^m(t) = \begin{cases} 
\gamma(t), & t \in [0, \tau], \\
P\gamma(t - \tau)P^{-1}\gamma(\tau), & t \in [\tau, 2\tau], \\
P^2\gamma(t - 2\tau)(P^{-1}\gamma(\tau))^2, & t \in [2\tau, 3\tau], \\
P^3\gamma(t - 3\tau)(P^{-1}\gamma(\tau))^3, & t \in [3\tau, 4\tau], \\
\cdots, \\
P^{m-1}\gamma(t - (m-1)\tau)(P^{-1}\gamma(\tau))^{m-1}, & t \in [(m-1)\tau, m\tau].
\end{cases}
\]

If the matrix function \( B(t) \) in the linear Hamiltonian system \([23]\) satisfies \( P^TB(t + \tau)P = B(t) \), the iteration of its fundamental solution \( \gamma \) is defined in the same way.
Corresponding we set
\[ i^\omega_{\omega_0}^p(\gamma, m) = i^\omega_{\omega_0}(\gamma^m), \quad \nu^\omega_{\omega_0}^p(\gamma, m) = \nu^\omega_{\omega_0}(\gamma^m). \]

If the subindex \( \omega = 1 \), we simply write \((i^p(\gamma, m), \nu^p(\gamma, m))\), and omit the subindex 1 when there is no confusion. In the sequel, we use the notions \((i(\gamma), \nu(\gamma))\) and \((i(\gamma, m), \nu(\gamma, m))\) to denote the Maslov-type index and the iterated index of symplectic path \( \gamma \) with the periodic boundary condition which were introduced by Y. Long and his collaborators (cf. [25], [26], [29], [34], etc.).

In [27, 28], S. Tang and the first author of this paper obtained the important Bott-type formula and iteration inequalities for Malsov \((P, \omega)\)-index as follows.

**Lemma 2.1.** ([22], Bott-type iteration formula) For any \( \tau > 0, \gamma \in \mathcal{P}_\tau(2n) \) and \( m \in \mathbb{N} \), there hold
\[ i^\omega_{\omega_0}^p(\gamma, m) = \sum_{\omega^n = \omega_0} i^\omega_{\omega_0}(\gamma), \quad \nu^\omega_{\omega_0}^p(\gamma, m) = \sum_{\omega^n = \omega_0} \nu^\omega_{\omega_0}(\gamma), \] (2.5)
where \( P \in Sp(2n) \) and \( \omega_0 \in U \).

**Lemma 2.2.** ([28]) For any path \( \gamma \in \mathcal{P}_\tau(2n), P \in Sp(2n) \) and \( \omega \in U \setminus \{1\} \), it always holds that
\[ i^P(\gamma, 1) + \nu^P(\gamma, 1) - n + i_1(\xi) - i_\omega(\xi) \leq i_\omega^P(\gamma) \leq i^P(\gamma, 1) + n - \nu^P(\gamma) + i_1(\xi) - i_\omega(\xi). \] (2.6)

**Lemma 2.3.** ([28], iteration inequality) For any path \( \gamma \in \mathcal{P}_\tau(2n), P \in Sp(2n) \) and \( m \in \mathbb{N} \),
\[ m(i^P(\gamma, 1) + \nu^P(\gamma, 1) - n) + n - \nu^P(\gamma, 1) + mi_1(\xi) - i(\xi, m) \leq i^p_m(\gamma, m) \]
\[ \leq m(i^P(\gamma, 1) + n) - n - (\nu^m_m, \gamma, m) - \nu^P(\gamma, 1) + mi_1(\xi) - i(\xi, m). \] (2.7)

Let \( e(M) \) be the elliptic height of symplectic matrix \( M \) just as the same in [29], the following lemma is important for the proof of Theorem [1.1].

**Lemma 2.4** ([28]). For any path \( \gamma \in \mathcal{P}_\tau(2n), P \in Sp(2n) \), set \( M = \gamma(\tau) \) and extend \( \gamma \) to \([0, \infty)\) by \([24]\). Then for any \( m \in \mathbb{N} \) we have
\[ \nu^p(\gamma, 1) - \nu^{p(m+1)}(\gamma, m+1) - \nu(\xi, m+1) - \frac{e(P^{-1}M)}{2} - \frac{e(P^{-1})}{2} \leq i^{p(m+1)}(\gamma, m+1) - i^P(\gamma, 1) \]
\[ \leq \nu^P(\gamma, 1) - \nu^{p(m+1)}(\gamma, m+1) - \nu(\xi, m) + \frac{e(P^{-1}M)}{2} + \frac{e(P^{-1})}{2}. \] (2.8)

Let \( S_{kT} = \mathbb{R}/(kT\mathbb{Z}), \ W_P = \{ z \in W^{1/2, 2}(S_{kT}, \mathbb{R}^{2n}) \mid z(t + \tau) = Pz(t) \} \) be a closed subspace of \( W^{1/2, 2}(S_{kT}, \mathbb{R}^{2n}) \). It is also a Hilbert space with norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \) as in \( W^{1/2, 2}(S_{kT}, \mathbb{R}^{2n}) \). We denote by \( \| \cdot \|_s \) the \( L^s \)-norm for \( s \geq 1 \). By the well-known Sobolev embedding theorem, we have the following embedding property: for any \( s \in [1, +\infty) \), there is a constant \( \alpha_s > 0 \) such that
\[ \| z \|_s \leq \alpha_s \| z \|, \quad \forall z \in W_P. \] (2.9)
Let $\mathcal{L}_s(W_P)$ and $\mathcal{L}_c(W_P)$ denote the space of the bounded self-adjoint linear operator and compact linear operator on $W_P$. We define two operators $A, B \in \mathcal{L}_s(W_P)$ by the following bilinear forms:

$$
\langle Ax, y \rangle = \int_0^\tau (-J\dot{x}(t), y(t))dt, \quad \langle Bx, y \rangle = \int_0^\tau (B(t)x(t), y(t))dt.
$$

(2.10)

Suppose that $\cdots \leq \lambda_{-j} \leq \cdot \cdots \leq \lambda_{-1} < 0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ are all nonzero eigenvalues of $A$ (count with multiplicity), correspondingly, $e_j$ is the eigenvector of $\lambda_j$ satisfying $\langle e_j, e_i \rangle = \delta_{ji}$. We denote the kernel of $A$ by $W^0_P$ which is exactly the space $\ker_p (P - I)$. For $m \in \mathbb{N}$, define the finite dimensional subspace of $W_P$ by

$$
W^m_P = W^-_m \oplus W^0_P \oplus W^+_m
$$

with $W^-_m = \{ z \in W_P | z(t) = \sum_{j=1}^m a_j e_j(t), a_j \in \mathbb{R} \}$ and $W^+_m = \{ z \in W_P | z(t) = \sum_{j=1}^m a_j e_j(t), a_j \in \mathbb{R} \}$. Suppose $P_m$ is the orthogonal projections $P_m : W_P \to W^m_P$ for $m \in \mathbb{N} \cup \{0\}$. Then $\{ P_m | m = 0, 1, 2, \cdots \}$ is the Galerkin approximation sequence respect to $A$.

For a self-adjoint operator $T$, we denote by $M^*(T)$ the eigenspaces of $T$ with eigenvalues belonging to $(0, +\infty), \{0\}$ and $(-\infty, 0)$ with $* = +, 0$ and $* = -, \text{ respectively}$. And the dimension of eigenspaces $M^*(T)$ is denoted by $m^*(T) = \dim M^*(T)$. Similarly, we denote by $M^*_d(T)$ the eigenspaces of $T$ with eigenvalues belonging to $(d, +\infty), (-d, d)$ and $(-\infty, -d)$ with $* = +, 0$ and $* = -, \text{ respectively}$. We denote $m^*_d(T) = \dim M^*_d(T)$. For any adjoint operator $L$, we denote $L^\delta = (L|_{\operatorname{Im}L})^{-1}$.

The following theorem gives the relationship between the Maslov $P$-index and Morse index for any $P \in Sp(2n)$.

**Theorem 2.1.** ([23], Lemma 3.2 and Theorem 4.6) For $P \in Sp(2n)$, suppose that $B(t) \in \mathcal{C}(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$ and $P^TB(t + \tau)P = B(t)$ with the Maslov $P$-index $(i^P(B), \nu^P(B))$. For any constant $0 < d \leq \frac{1}{4}\| (A - B) \|^2 \|^{-1}$, there exists an $m_0 > 0$ such that for $m \geq m_0$, there holds

$$
m^-_d(P_m(A - B)P_m) = m + i^P(B),
$$

$$
m^0_d(P_m(A - B)P_m) = \nu^P(B),
$$

$$
m^+_d(P_m(A - B)P_m) = m + \dim \ker_p (P - I) - i^P(B) - \nu^P(B),
$$

where $B$ is the operator defined by $B(t)$.

For the operators $A$ and $B$ defined in (2.10), there is another description of the Maslov $P$-index as follows.

**Lemma 2.5** ([23]). For any two operators $B_1, B_2 \in \mathcal{C}(\mathbb{R}, \mathcal{L}_s(2n))$ with $B_i(t + \tau) = (P^{-1})^TB_i(t)P^{-1}, i = 1, 2$ and $B_1 < B_2$, there holds

$$
i^P(B_2) - i^P(B_1) = \sum_{s \in (0, 1)} \nu^P((1 - s)B_1 + sB_2).
$$

(2.12)

**Remark 2.1.** Suppose that $B > 0$, we have

$$
i^P(B) = \sum_{s \in [0, 1]} \nu^P(sB).
$$

(2.13)
3 The proof of Theorem 1.1

In [23], the following result was proved.

**Theorem 3.1.** ([23]) Suppose \( P \in \text{Sp}(2n)_k \), and the Hamiltonian function \( H \) satisfies \((H1)-(H4)\), then for every \( \tau > 0 \), the system (1.1) possesses a nonconstant \( P \)-solution \((\tau, x)\) satisfying

\[
\dim \ker R(P - I) + 2 - \nu^P(x) \leq i^P(x) \leq \dim \ker R(P - I) + 1.
\]

Before the proof of Theorem 1.1, we need to get the information about the iteration Maslov index for paths connecting \( I \) and \( P^{-1} \). Firstly, from Lemma 1.1 we recall that for any \( P \in \text{Sp}(2n)_k \), there exists \( (p, j_1, j_2, \ldots, j_r) \in \mathbb{N}^{r+1} \) such that \( p + \sum_{m=1}^{r} j_m = n \) and \( I_2 \omega \circ R\left(\frac{2\pi}{k}\right)^{\circ j_1} \circ \cdots \circ R\left(\frac{2\pi}{k}\right)^{\circ j_r} \in \Omega^0(P^{-1}) \). We remind that here \( \mathbb{N} = \{0, 1, \cdots, \} \).

**Lemma 3.1.** For \( P \in \text{Sp}(2n)_k(r, p; j_1, j_2, \cdots, j_r) \) and \( \xi \in \mathcal{P}_r(2n) \) with \( \xi(\tau) = P^{-1} \), there holds

\[
(k + 1)i(\xi) - i(\xi, k + 1) = \sum_{m=1}^{r} (k - 2m)j_m - kp.
\]

**Proof.** By Theorem 9.3.1 in [29] (also [34]), we have

\[
i(\xi, k + 1) = (k + 1)i(\xi) + S^+_{(P^{-1})}(1) - C(P^{-1}) + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{k + 1}{2\pi}\right)^{-}S^{-}_{(P^{-1})}(e^{\sqrt{-1}\theta})
\]

\[
= (k + 1)i(\xi) + kS^+(1) - (k + 2)C(P^{-1}) + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{k + 1}{2\pi}\right)^-{S}^{-}_{(P^{-1})}(e^{\sqrt{-1}\theta}),
\]

where \( S^+(1) = p \); \( C(P^{-1}) = \sum_{m=1}^{r} j_m \); \( \sigma(P^{-1}) \), \( \sigma \) is a splitting number of \( M \in \text{Sp}(2n) \) at \( \omega \in U \); \( C(M) = \sum_{\omega \in (0, 2\pi)} S^{-}_{(P^{-1})}(e^{\sqrt{-1}\theta}) \) and \( E(a) = \min\{m \in \mathbb{Z} | m \geq a\} \). One can see these notions in Chapter 9 of [29].

For \( P \in \text{Sp}(2n)_k(r, p; j_1, j_2, \cdots, j_r) \), \( I_2 \omega \circ R\left(\frac{2\pi}{k}\right)^{\circ j_1} \circ \cdots \circ R\left(\frac{2\pi}{k}\right)^{\circ j_r} \in \Omega^0(P^{-1}) \) with \( 2r < k \). By direct computation, for \( \theta \in (0, \pi) \cup (\pi, 2\pi) \), we have

\[
S^+(1) = p ;
\]

\[
C(P^{-1}) = \sum_{m=1}^{r} j_m ;
\]

\[
S^-(e^{\sqrt{-1}\theta}) = \begin{cases} j_m, & \text{if } \theta = \frac{2m\pi}{k}, 1 \leq m \leq r, e^{\frac{2m\pi}{k}} \in \sigma(P^{-1}) ; \\ 0, & \text{otherwise} . \end{cases}
\]

So

\[
\sum_{\theta \in (0, 2\pi)} E\left(\frac{k + 1}{2\pi}\right)^{-}S^{-}_{(P^{-1})}(e^{\sqrt{-1}\theta}) = E\left(\frac{k + 1}{k}\right)j_1 + E\left(\frac{2(k + 1)}{k}\right)j_2 + \cdots + E\left(\frac{r(k + 1)}{k}\right)j_r
\]

\[
= 2j_1 + 3j_2 + \cdots + (r + 1)j_r.
\]
Then
\[ i(\xi, k + 1) = (k + 1)i(\xi) + kp - (k + 2)(j_1 + j_2 + \cdots + j_r) + 2(2j_1 + 3j_2 + \cdots + (r + 1)j_r) \]
\[ = (k + 1)i(\xi) + kp - (k - 2)j_1 - (k - 4)j_2 - \cdots - (k - 2r)j_r, \] (3.7)
thus \((k + 1)i(\xi) - i(\xi, k + 1) = \sum_{m=1}^{r}(k - 2m)j_m - kp. \]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.**

Suppose that \((k\tau, x^k)\) is the \(k\tau\)-periodic solution extended by \(P\)-solution \((\tau, x)\) in Theorem 3.1. If \(k\tau\) is not the minimal \(P\)-symmetric period of \((k\tau, x^k)\), i.e., \(\tau > \min\{\lambda > 0 \mid x(t + \lambda) = Px(t), \forall t \in \mathbb{R}\}\), then there exists some \(l \in \mathbb{N}\) such that
\[ T = \frac{\tau}{l} = \min\{\lambda > 0 \mid x(t + \lambda) = Px(t), \forall t \in \mathbb{R}\}. \]
Thus \(x(\tau - T) = x(0)\), both \((l - 1)T\) and \(kT\) are the period of \(x\). Since \(kT\) is the minimal \(P\)-symmetric period, we obtain \(kT \leq (l - 1)T\) and then \(k \leq l - 1\).

Note that \(x_{[0,kT]}\) is the \(k\)-th iteration of \(x_{[0,T]}\). Suppose \(\gamma \in \mathcal{P}_T(2n)\) is the fundamental solution of the following linear Hamiltonian system
\[ \dot{z}(t) = JB(t)z(t) \] (3.8)
with \(B(t) = H''(x_{[0,T]}(t))\). Suppose \(\xi\) be any symplectic path in \(\mathcal{P}_T(2n)\) such that \(\xi(T) = P^{-1}\), since \(P^k = I\), then
\[ \nu(\xi, 1) = \nu(\xi, k + 1) = \nu(\xi, l). \] (3.9)
All eigenvalues of \(P\) and \(P^{-1}\) are on the unit circle, then the elliptic height
\[ e(P^{-1}) = e(P) = 2n. \] (3.10)
Since the system (1.1) is autonomous, we have
\[ \nu_1(x_{[0,kT]}) \geq 1 \text{ and } \nu^{P^{-1}}(\gamma, l - 1) = \nu_1(x_{[0,(l-1)T]}) \geq 1. \] (3.11)
By Lemma 2.4, \(P^{l-1} = I\) and (3.9)-(3.11), we have
\[ i^l(\gamma, l - 1) = i^{P^{l-1}}(\gamma, l - 1) \]
\[ \leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + \nu(\xi, 1) - \nu(\xi, l) + \frac{e(P^{-1}\gamma(T))}{2} + \frac{e(P^{-1})}{2} - \nu^{P^{-1}}(\gamma, l - 1) \]
\[ \leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + \frac{e(P^{-1}\gamma(T))}{2} + n - 1 \]
\[ \leq i^{P^l}(\gamma, l) - i^P(\gamma, 1) + 2n - 1. \] (3.12)
Note that \(i^{P^l}(\gamma, l) = i^P(x_{[0,\tau]}) \leq \dim \ker_{\mathbb{R}}(P - I_{2n}) + 1\), here we write \(i^P(x_{[0,\tau]})\) for \(i^P(x)\) to remind the solution \(x\) is defined in the interval \([0, \tau]\). By the definition of Maslov \(P\)-index,
\[ i^l(\gamma, l - 1) = i_1(\gamma, l - 1) + n. \]
So we get
\[ i_1(\gamma, l - 1) \leq \dim \ker_{\mathbb{R}}(P - I) - i_P(\gamma, 1) + n. \]  
(3.13)

By the condition (H5) and Remark 2.1, we have
\[ i_P(\gamma, 1) = i_P(B) = \sum_{s \in [0,1)} \nu^P(sB) = \sum_{s \in [0,1)} \dim \ker_{\mathbb{R}}(\gamma_B(sT) - P). \]  
(3.14)

Here we remind that \( B(t) \) and \( \gamma_B \) are defined in (3.8). Since \( \gamma_B(0) = I \), so \( \dim \ker_{\mathbb{R}}(\gamma_B(sT) - P) = \dim \ker_{\mathbb{R}}(P - I) \) when \( s = 0 \). Thus we have
\[ i_P(\gamma, 1) \geq \dim \ker_{\mathbb{R}}(P - I). \]  
(3.15)

From (3.12), it implies
\[ i_1(\gamma, l - 1) \leq n. \]  
(3.16)

By the convex condition (H5), we also have
\[ i_1(x|_{[0,kT]}) \geq n \quad \text{and} \quad i_1(x|_{[0,(l-1)T]}) \geq n. \]  
(3.17)

We set \( m = \frac{t-1}{k} \). Note that \( x|_{[0,(l-1)T]} \) is the \( m \)-th iteration of \( x|_{[0,kT]} \). By (3.11), (3.16), (3.17) and Lemma 4.1 in [26], we obtain \( m = 1 \) and then \( k = l - 1 \). From the above process (3.12)-(3.13) and (3.15)-(3.17), we obtain \( k = l - 1 \) provided \( e(P^{-1}\gamma(T)) = 2n \), and
\[ i_P(\gamma, 1) = \dim \ker_{\mathbb{R}}(P - I); \]
\[ i_P(\gamma, l) = i_P(\gamma, k + 1) = \dim \ker_{\mathbb{R}}(P - I) + 1, \]  
(3.18)
\[ \nu^{P^k} (\gamma, k) = \nu^P (\gamma, 1) = 1. \]

Here we remind that the left inequality in (2.7) of Lemma 2.3 holds independent of the choice of \( \xi \in P_r(2n) \), then for any \( \xi \in P_r(2n) \) we have
\[ i_P(\gamma, k + 1) \geq (k + 1)(i_P(\gamma, 1) + \nu^P(\gamma, 1) - n) + n - 1 + (k + 1)i_1(\xi) - i(\xi, k + 1). \]  
(3.19)

By the condition \( P \in Sp(2n)_k(r; p; j_1, j_2, \ldots, j_r) \), we get
\[ \dim \ker_{\mathbb{R}}(P - I) = 2p, \]  
(3.20)
\[ k - 2 \sum_{m=1}^{r} m \cdot j_m \geq 1. \]  
(3.21)

Applying (3.18), (3.20) and Lemma 3.1 to (3.19), we get
\[ k - 2 \sum_{m=1}^{r} m \cdot j_m \leq 1. \]  
(3.22)

It is contradict to the inequality (3.21). So the minimal \( P \)-symmetric period of \((k\tau, x^k)\) is \( k\tau \).

**Remark 3.1.** Note that \( e(P^{-1}\gamma(T)) = e((P^{-1}\gamma(T))^l) = e(P^{-1}\gamma(T)) = 2n \) is required in the above proof. If \( e(P^{-1}\gamma(T)) \leq 2n - 2 \), we get \( i_1(\gamma, l - 1) < n \) by taking the same process as (3.12)-(3.13). It contradicts to the second inequality of (3.17). At this moment, the minimal \( P \)-symmetric period of \((k\tau, x^k)\) is \( k\tau \).

The condition (H5) can be replaced by a weaker condition: \( H''(x(t)) \geq 0 \) and \( \int_0^T H''(x(t))dt > 0 \) for the \( P \)-solution \((\tau, x)\) in Theorem 3.1.
References

[1] A. Ambrosetti, V. Coti Zelati, *Solutions with minimal period for Hamiltonian systems in a potential well*, Ann. Inst. H. Poincaré Anal. Non Linéaire 4 (1987), 242–275.

[2] A. Ambrosetti, G. Mancini, *Solutions of minimal period for a class of convex Hamiltonian systems*, Math. Ann. 255 (1981), 405–421.

[3] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems*, in Progress in Nonlinear Differential Equations and Their Application, Vol. 6, (1993).

[4] K. C. Chang, J. Liu and M. Liu, *Nontrivial periodic solutions for strong resonance Hamiltonian systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 14(1) (1997), 103–117.

[5] F. Clarke, I. Ekeland, *Hamiltonian trajectories having prescribed minimal period*, Comm. Pure. Appl. Math. 33 (1980), no. 3, 103–116.

[6] A. Chenciner and R. Montgomery, *A remarkable periodic solution of the three body problem in the case of equal masses*, Ann. of Math. 152: 3 (2000), 881–901.

[7] Y. Dong, *P-index theory for linear Hamiltonian systems and multiple solutions for nonlinear Hamiltonian systems*, Nonlinearity 19: 6 (2006), 1275–1294.

[8] Y. Dong and Y. Long, *Closed characteristics on partially symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$*, J. Diff. Equa. 196 (2004), 226-248.

[9] D. Dong and Y. Long, *The iteration formula of the Maslov-type index theory with applications to nonlinear Hamiltonian systems*, Trans. Amer. Math. Soc. 349 (1997), 2619–2661.

[10] I. Ekeland, H. Hofer, *Periodic solutions with prescribed minimal period for convex autonomous Hamiltonian systems*, Invent. Math. 81 (1985), 155–188.

[11] G. Fei, *Relative Morse index and its application to Hamiltonian systems in the presence of symmetries*, J. Diff. Equa. 122 (1995), 302–315.

[12] G. Fei and Q. Qiu, *Minimal period solutions of nonlinear Hamiltonian systems*, Nonlinear Analysis, Theory, Method Applications 27: 7 (1996), 821–839.

[13] G. Fei, S. K. Kim, T. Wang *Minimal period estimates of period solutions for superquadratic Hamiltonian systems*, J. Math. Anal. Appl. 238 (1999), 216–233.

[14] D. Ferrario and S. Terracini, *On the existence of collisionless equivariant minimizers for the classical n-body problem*, Invent. Math. 155: 2 (2004), 305–362.

[15] N. Ghoussoub, *Location, multiplicity and Morse indices of min-max critical points*, J. Reine Angew Math. 417 (1991), 27–76.
[16] X. Hu and P. Wang, Conditional Fredholm determinant for the $S$-periodic orbits in Hamiltonian systems, Journal of Functional Analysis 261 (2011), 3247–3278.

[17] X. Hu and S. Sun, Index and stability of symmetric periodic orbits in Hamiltonian systems with application to figure-eight orbit, Comm. Math. Phys. 290 (2009), 737–777.

[18] X. Hu and S. Sun, Morse index and the stability of closed geodesics, Sci. China Math. 53: 5 (2010), 1207–1212.

[19] X. Hu and S. Sun, Stability of relative equilibria and Morse index of central configurations, C. R. Acad. Sci. Paris 347 (2009), 1309–1312.

[20] A. Lazer and S. Solomini, Nontrivial solution of operator equations and Morse indices of critical points of min-max type, Nonlinear Anal. 12 (1988), 761–775.

[21] C. Liu, Maslov $P$-index theory for a symplectic path with applications, Chin. Ann. Math. 4 (2006), 441–458.

[22] C. Liu, Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems, J. Diff. Equa. 252 (2012), 5712–5734.

[23] C. Liu, Relative index theories and applications, Top. Meth. Nonl. Anal, to appear.

[24] C. Liu, Minimal period estimates for brake orbits of nonlinear symmetric Hamiltonian systems, Discrete and continuous dynamical systems 27(1) (2010), 337–355.

[25] C. Liu and Y. Long, An optimal increasing estimate of the iterated Maslov-type indices, Chinese Sci. Bull. 42 (1997), 2275–2277.

[26] C. Liu and Y. Long, Iteration inequalities of the Maslov-type index theory with applications, J. Diff. Equa. 165 (2000), 355–376.

[27] C. Liu and S. Tang, Maslov $(P, \omega)$-index theory for symplectic paths, Advanced Nonlinear Studies 15 (2015), 963–990.

[28] C. Liu and S. Tang, Iteration inequalities of the Maslov $P$-index theory with applications, Nonlinear Analysis 127 (2015), 215–234.

[29] Y. Long, Index theory for symplectic paths with application, Progress in Mathematics, Vol. 207, Birkhäuser Verlag, 2002.

[30] Y. Long, Bott formula of the Maslov-type index theory, Pacific J. Math. 187 (1999), 113–149.

[31] Y. Long, The minimal period problem for classical Hamiltonian systems with even potentials, Ann. Inst. H. Poincaré Anal. non. linéaire 10(1993), 605–626.

[32] Y. Long, The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems, J. Diff. Equa. 111(1994), 147–174.
[33] Y. Long, *On the minimal period for periodic solutions of nonlinear Hamiltonian systems*, Chinese Ann. of Math. 18B(1997), 481–484.

[34] Y. Long and C. Zhu, *Closed characteristics on compact convex hypersurfaces in $R^{2n}$*, Annals of Math. 155 (2002), 317–368.

[35] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. 31 (1978), 157–184.

[36] P. H. Rabinowitz, *Minimax method in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., No.65, A. M. S., Providence(1986).

[37] S. Solimini, *Morse index estimates in min-max theorems*, Manuscriptum Math. 63 (1989), 421–453.