Spectra of Heterotic Strings on Orbifolds

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Abstract

We obtain the spectrum of heterotic strings compactified on orbifolds, focusing on its algebraic structure. Affine Lie algebra provides its current algebra and representations. In particular, the twisted spectrum and the Abelian charge are understood. A twisted version of algebra is used in the homomorphism from the orbifold action to the group action. The relation between the conformal weight and the mass gives a useful rule.

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1. INTRODUCTION

Weakly coupled heterotic strings compactified on orbifolds \[1, 2\] have long garnered a great deal of attention due to their several desirable features of low energy theory. They contribute to the understanding of such things as gauge groups, matter spectra, number of families, and small supersymmetries, to name a few. They are very predictive, compared to, for example, field theoretic orbifold models, because string theory places restrictions on possible spectra in the bulk and at the fixed points.

In the process of building a model, the most severe obstacle is the fact that there are too many possible string vacua. It is expected that a realistic orbifold compactification includes more than one Wilson line in order to have a sufficiently small group and number of families \[3, 4\]. In distinguishing the gauge groups and matter spectra, it is currently known that there are more than \(\sim 10^7\) possibilities in the simplest \(T^6/\mathbb{Z}_3\) orbifolds with two Wilson lines \[5, 6\]. However, such a large estimate is due to a lack of understanding of symmetries. We recently classified all the gauge groups and untwisted matter spectra with the aid of some group theoretical methods and observed that the number is dramatically lower \[6\].

However, a difficulty still lies in understanding the twisted sector spectrum. We know that the modular invariance condition requires the twisted strings, characterized by the periodic boundary condition up to automorphisms. The twisted strings form another independent Hilbert space, whose structure is poorly understood. We obtain the twisted string spectrum from the mass shell condition of strings supplemented by GSO projection, but it remains unclear how to determine and analyze its algebraic structure. Again, the task of finding abelian generators was a trial-and-error job.

In this paper, we seek such a structure with the aid of algebraic tools. In this endeavor, the rich algebraic structure provided by the affine Lie algebra proves to be useful. The vertex operator construction \[7\] teaches us that the heterotic string spectrum is the representation of affine Lie algebra.

In particular, the twisted sector states form the representations of a twisted version of affine Lie algebra, defined in the same manner as twisting the physical states. When the twisting is inner automorphism, with a suitable change of basis as we will see, the algebra is isomorphic to the original one, making it easy to obtain twisted representations \[8, 9\].

This also provides us a systematic way to obtain and classify groups and spectra. We
will see that the affine Lie algebra plays a crucial role in understanding the Abelian group embedded in the non-Abelian group as well. The related anomalous $U(1)$ is also an important issue \[10, 11, 12\].

This paper focuses on modular invariant theory and level one algebra. However, the discussion still holds for those without modular invariance or those with higher level algebras \[13\]. Most of the mathematical facts used here can be found in Ref. \[8, 9, 14\]. We will follow convention on naming roots and weights of $\mathfrak{g}$, which is also presented in the appendix.

**II. AFFINE LIE ALGEBRA**

Consider an algebra $\mathfrak{g}$, whose generators satisfy the commutation relation,

$$[T^a_m, T^b_n] = i f^{abc} T^c_{m+n} + m \delta_{m+n,0} \delta^{ab} K. \quad (1)$$

The group indices $a, b, c$ run over the dimensions of the algebra $d = \dim \mathfrak{g}$ and the ‘mode’ indices $m, n$ are integers. This is an extension of the simple Lie algebra $\mathfrak{g}$ for which $m = n = 0$. Without $K$ term, $T^a_m$ are understood as infinitesimal generators of mapping $S^1 \rightarrow \mathfrak{g}$, from those of simple Lie algebra $T^a_0$. This procedure is known as an affinization, so this algebra is called the *affine Lie algebra*, or the *Kac–Moody algebra*. We use the overlined letter for the objects of the simple Lie algebra.

We introduce two additional generators. One is the *central element* $K$, commuting with all the generators

$$[K, T^a_m] = 0, \quad (2)$$

which is made of a linear combination of the Cartan generators $H$ and turns out to be unique. The other is the *grade operator* $D$ which has the following commutation relations

$$[D, T^a_m] = m T^a_m, \quad [D, K] = 0. \quad (3)$$

We collectively denote the weights of $\mathfrak{g}$ as the eigenvalues of $(H_0, K, D)$. Inspecting the Killing form, the natural inner product is

$$(\lambda, k, n) \cdot (\lambda', k', n') = \lambda \cdot \lambda' + kn' + k'n. \quad (4)$$

We have simple roots of $\mathfrak{g}$ by extending those $\bar{\alpha}^i$ of the simple Lie algebra $\bar{\mathfrak{g}}$, with the highest
where $r$ is the rank of $\mathfrak{g}$. The $\alpha^0$ will raise and lower the eigenvalue $D$ because of the $(0, 0, 1)$ component.

It is useful to define the dual vector $\alpha^\vee = 2\alpha/\alpha^2$. Constructing the Cartan matrix $A^{ij} = 2\alpha^i \cdot \alpha^j$, we obtain the Dynkin diagram of $\mathfrak{g}$, which is the same as the extended Dynkin diagram of $\bar{\mathfrak{g}}$.

With the positive numbers $a_i$ known as the dual Coxeter labels, we have the unique expansion $\bar{\theta}^\vee = \sum a_i \bar{\alpha}^i$ and $a_0 = 1$. The sum $g = a_0 + \sum a_i$ is called the dual Coxeter number. An example for $E_8$ and $SO(8)$ will be given in Figs. 1 and 2. For a list of Dynkin diagrams and Coxeter numbers, see Ref. [9]. We also define fundamental weights $\Lambda_i$ as $\alpha^\vee \cdot \Lambda_j = \delta^i_j$, therefore

$$\Lambda_i = (\bar{\Lambda}, \frac{1}{2} a_i \bar{\theta}^2, 0)$$

and the Weyl vector $\rho = \sum_i^r \Lambda_i + g \Lambda_0$. The fundamental weight has a relation with the inverse of the Cartan matrix $\bar{\Lambda}_i \cdot \bar{\Lambda}_j = A_{ij}^{-1}$, sometimes referred to as ‘quadratic form’ in the literature. From the definition of the Weyl vector, it follows that

$$g\bar{\theta}^2 \delta^{ab} = f^{acd} f^{bde},$$

the quadratic Casimir for the adjoint representation. We will follow convention when referring to the simple roots and the fundamental weights of reference [9] and their explicit form can also be found in the appendix.

By triangular decomposition, we separate generators into the Cartan subalgebra and the ladder operators

$$[H^i_m, H^j_n] = m \delta^{ij} \delta_{m+n,0} K,$$

$$[H^i_m, E^\alpha_n] = \bar{\alpha}^i E^\alpha_{m+n},$$

$$[E^\alpha_m, E^{\bar{\beta}}_n] = \pm E^{\alpha + \bar{\beta}}_{m+n} \text{ for } \alpha + \bar{\beta} \in \bar{\mathfrak{g}} \text{ root},$$

$$[E^\alpha_n, E^{-\alpha}_{-n}] = \bar{\alpha} \cdot H_0 + nK.$$
Again we see that these are extended relations of the simple Lie algebra. Note that the
grade can also be raised and lowered by generator $T_n^a$ with $n \neq 0$. For example, the Cartan
subalgebra does not mutually commute anymore, and by

$$[H_n^i, H_{-n}^j] = \delta^{ij} n K,$$

each of which raise and lower the eigenvalue of $D$ by $nk'$, in accord with the product \[12\].

We have the highest weight module, whose highest weight vector is annihilated by all the
positive roots including the positive grade operator,

$$E_0^\bar{\alpha} |\Lambda\rangle = 0, \text{ for all } \bar{\alpha} > 0, \quad (13)$$

$$T_n^a |\Lambda\rangle = 0, \text{ for all } n > 0. \quad (14)$$

The complete vectors in the module are obtained by a series of lowering operations. Owing to
the creation operator for $n$, such a weight system is, generally speaking, infinite dimensional.

Here we are interested in a finite dimensional submodule. For any weight $\lambda = (\bar{\lambda}, k', n)$
of a highest weight module,

$$\frac{2}{\alpha^2} [E_n^\bar{\alpha}, E_{-n}^{-\bar{\alpha}}]|\lambda\rangle = \frac{2}{\alpha^2} (\bar{\alpha} \cdot \bar{\lambda} + nk')|\lambda\rangle \quad (15)$$

from the relation \[12\]. From the $SU(2)$ representation theory, for a finite dimensional
module, all of the eigenvalues are integers. So it follows from simple Lie algebra $\mathfrak{g}$ that each
eigenvalue in the sum is a separate integer. Applying to the highest weight state, in the
$\alpha^0 = (-\bar{\theta}, 0, 1)$ direction,

$$\left(-\bar{\Lambda} \cdot \bar{\theta} + \frac{2}{\bar{\theta}^2} k'\right)|\Lambda\rangle. \quad (16)$$

Now the eigenvalues are nonnegative integers. Therefore, the quantity $k \equiv 2k'/\bar{\theta}^2$, called
the level, is a nonnegative integer.

We are interested in the level one algebra, therefore hereafter we set $k = 1$, although
we explicitly bare the letter $k$ for extension. For the highest weight $\bar{\Lambda} = \sum t_i \bar{\Lambda}_i$, $t_i$ is a
nonnegative integer, by the definition of the fundamental weight. Then the above relation
leads to the so-called integrability condition

$$0 \leq \sum_{i=1}^{r} a_i t_i \leq k. \quad (17)$$
The above relation naturally extended to affine Lie algebra in a tidy form. From \( \Lambda = \sum t_i \Lambda_i = (\bar{\Lambda}, \frac{1}{2} \theta^2 (t_0 + \sum_{i=1}^r a_i t_i), 0) \), the middle element being the level, we have an equivalent relation

\[
k = t_0 + \sum_{i=1}^r a_i t_i.
\]

It is noted that for the level one algebra only a few can satisfy this condition. For the \( SU(n) \) algebra, the dual Coxeter label corresponding to a fundamental weight is always 1; thus every representation \( \Lambda_i \), having dimension \( \binom{n}{1} \), is possible. However, the other groups have \( a_i = 1 \) only for the outer most nodes of the Dynkin diagram. For example, for \( E_6 \), \( \Lambda_1(27) \) and \( \Lambda_6(27) \) have \( a_i = 1 \) thus can satisfy this condition.

A corollary of this observation is that, in the \( k = 1 \) case the adjoint representation, which can become an adjoint Higgs field used to break typical \( SU(5) \) and \( SO(10) \) unified theory, cannot satisfy this condition. The weight vector for the adjoint representation of \( SU(n) \) is \( \Lambda_1 + \Lambda_{n-1} \), hence it has the sum of the dual Coxeter labels greater than 1: \( a_1 + a_{n-1} = 2 > 1 = k \). Of course for the higher level algebra, we can have such adjoint representation [13].

### III. HETEROTIC STRING

#### A. Current algebra

Heterotic string theory has closed strings with one worldsheet supersymmetry on the right movers. For the left movers, in which we are interested in this section, on top of the ten spacetime bosonic degrees of freedom, 16 extra bosons (or 32 fermions) are needed to cancel the conformal anomaly. The modular invariant theory allows the gauge group \( SO(32) \) or \( E_8 \times E_8 \) [15].

One way to denote the group degrees of freedom is by state vectors \( |p \rangle \). In the bosonic description, they represent charges in given directions, thus spanning the root space. In the low energy limit, massive modes of the order of the string scale are decoupled. Resorting to the mass shell condition (in unit \( \alpha' = 1 \)),

\[
\frac{M_L^2}{4} = \frac{p^2}{2} + \tilde{N} - 1
\]

and the modular invariance, one notes that the zero mode vector \( p \) lives in the even and self-dual lattice, which turns out to be the \( SO(32) \) or \( E_8 \times E_8 \) lattice. The Cartan generators
are provided by oscillators $\tilde{\alpha}_{i-1}|0\rangle$. Combined with massless right movers, they constitute the fields for supergravity coupled with this gauge group.

Despite their clear spectra, the role of massive modes is not transparent. To have a better understanding, we look at another equivalent description. In conformal field theory, every state has one to one correspondence with a local ‘vertex operator’. They look like $T^i(z)\psi(\bar{z})e^{ik\cdot X}$ with $T^i(z)$ being

$$H^i(z) = \partial_z X^i, \quad (20)$$
$$E^\alpha(z) = \alpha^i E^i = c^\alpha : e^{i\alpha \cdot X} :, \quad (21)$$

where $c^\alpha$ is the two-cocycle determining the sign of the commutator as in (11) and colons denoting the conventional normal ordering. The operator product expansion (whose expectation value is a two-point correlation function) between two currents is, as $z \to 0$,

$$T^a(z)T^b(0) \sim \frac{k^\prime \delta^{ab}}{z^2} + \frac{ic^{abc}}{z}T^c(0). \quad (22)$$

With the mode expansion $T^a(z) = \sum T^a_n z^{-n-1}$, and identifying $c^{abc} \equiv f^{abc}$ they satisfy the commutation relation of the affine Lie algebra $\{19, 12\}$. This normalization also fixes the level $k = 2k^\prime/\bar{\theta}^2 = 1$. In the heterotic string theory embedding, we have only the level one vertex operators. However, we can make higher level algebra by embedding it into the product of level one algebras, for example $\{13\}$.

The two dimensional conformal symmetry, which the string theory possesses, is realized by the analytic transformation $z^\prime = f(z)$ (i.e. independent of $\bar{z}$) $\{17\}$. (Since the all the currents are from the left mover, we will focus on analytic currents in $z$ for the present purpose.) Under it, the vertex operator $O(z)$ transforms as

$$O(z^\prime) = \left( \frac{\partial z^\prime}{\partial z} \right)^{-h} O(z), \quad (23)$$

where we define $h$ as conformal weight. If an operator transforms definitely under (23), we call it primary operator. We can check that the currents $\{20, 21\}$ are primary operators of conformal weight one. The mapping from a state to a vertex operator corresponds to shrinking (conformal transformation) the ‘in’ and ‘out’ states of a cylindrical Feynman diagram into points. All the quantum number is kept and we will see that especially the mass is converted into the conformal weight.
By Sugawara construction \[16, 17\], we construct a worldsheet energy–momentum tensor $T(z)$ from the generators of the algebra $g$,

$$T(z) = \frac{1}{\beta} \sum_{a=1}^{d} : T^a(z) T^a(z) : .$$

(24)

This is the generator of the conformal symmetry \[23\]. The normalization $\beta$ will be fixed by requiring for $T^a(z)$’s to transform as the primary fields of conformal weight one \[17\],

$$T(z) T^a(0) \sim \frac{T^a(0)}{z^2} + \frac{\partial T^a(0)}{z} .$$

(25)

which implies

$$[L_m, T^a_{-n}] = n T^a_{m-n} .$$

(26)

Doing Fourier expansion $T(z) = \sum L_n z^{-n-2}$, we have its components

$$L_n = \frac{1}{\beta} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{d} : T^a_{m+n} T^a_{-m} : .$$

(27)

Now we have another way to express conformal vacuum \[14\],

$$L_n |\Lambda\rangle = 0, \quad n > 0 .$$

(28)

Acting $L_{-1}$ we have

$$L_{-1} |\Lambda\rangle = \frac{2}{\beta} T^a_{-1} T^a_{0} |\Lambda\rangle .$$

Using \[11\] and \[26\], we get

$$T^a_{0} |\Lambda\rangle = \frac{2}{\beta} (i f^{bac} T^c_0 + k' \delta^{ab}) T^a_{0} |\Lambda\rangle$$

$$= \frac{1}{\beta} (i f^{bac} i f^{dea} T^d_0 + 2 k' T^b_{0}) |\Lambda\rangle$$

$$= \frac{1}{\beta} (\bar{\theta}^2 g + \bar{\theta}^2 k) T^b_{0} |\Lambda\rangle ,$$

(29)

where in the last line we used the property of the quadratic Casimir and $g$ \[8\], and definition of the level $k$. Therefore we have the normalization $\beta = \bar{\theta}^2 (k + g)$. Finally we have the Virasoro algebra,

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{\epsilon^g}{2} (m^3 - m) \delta_{m+n,0} ,$$

(30)

with the conformal anomaly,

$$\epsilon^g = \frac{kd}{k + g \theta^2 / 2} .$$

(31)

We usually normalize $\bar{\theta}^2 = 2$. In addition, for $k = 1$, with a simple relation $g + 1 = d/r$ \[9\], it coincides with the rank $c^g = r$. 
B. Finding the state

We want to find the physical state \(|\Lambda\rangle\),

\[ L_n |\Lambda\rangle = 0, \quad n > 0. \]

Then, the worldsheet Hamiltonian is \(L_0\) which gives rise to the mass squared operator of string states and has a lower bound. Note the commutator (26)

\[ [L_m, T^a_{-n}] = n T^a_{m-n}. \]

Putting \(m = 0\), we see that \(|\Lambda\rangle\) is an eigenstate of \(L_0\) and \(D\) up to an additive constant. We use additive normalization \(L_0|1\rangle = D|1\rangle = 0\), where \(|1\rangle\) is an uncharged physical vacuum (the ground state of a unitary conformal field theory). As a corollary, the gauge boson generators should be zero modes since they should commute with the worldsheet Hamiltonian \(L_0\), and not change the mass. By (28), the eigenvalue \(h_\Lambda\) is positive definite and minimal. We have

\[ \frac{M^2}{4} = h_\Lambda - \frac{c}{24} = n. \quad (32) \]

instead of (19). The oscillator part \(\tilde{N}\) is going to be contained in \(h_\Lambda\) later. Note that, in the additive normalization, all the fields of given conformal field theory contribute to the conformal anomaly \(c\). We have not specified the whole, i.e. spacetime degrees of freedom, as we will see in (65), so that \(c = c^g + c^s\).

The eigenvalue of \(L_0\), or the conformal weight \(h_\Lambda\) is obtained if we apply (27)

\[ L_0 |\Lambda\rangle = \frac{1}{\theta^2(k+g)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^d : T^a_{m+n} T^a_{m-n} : |\Lambda\rangle = \frac{1}{\theta^2(k+g)} \sum_{a=1}^d T^a_0 T^a_0 |\Lambda\rangle = \frac{C_r}{\theta^2(k+g)} |\Lambda\rangle. \quad (33) \]

Here \(C_r\) is the quadratic Casimir in the representation of \(\Lambda\) and explicitly expressed in terms of fundamental weights, using the property of the Weyl vector \([9]\). Therefore we obtain so-called the Freudenthal–de Vries strange formula \([18]\),

\[ h_\Lambda = \frac{\Lambda \cdot (\Lambda + 2\rho)}{\theta^2(k+g)}. \quad (34) \]

Explicitly with \(\Lambda = \sum_1^r t_i \Lambda_i\) and \(\Lambda_i \cdot \Lambda_0 = 0\), we have

\[ h_\Lambda = \frac{1}{\theta^2(k+g)} \sum_{i,j=1}^r (t_i + 2)t_j A^{-1}_{ij}. \quad (35) \]
In case the state $|\Lambda\rangle$ consists of a single fundamental weight $\Lambda_i$, (we will consider more general case soon) it reduces to

$$h_{\Lambda_i} = \frac{1}{2}\Lambda_i^2 = A^{-1}_{ii} \quad \text{(no summation of } i)\text{,} \tag{36}$$

using the property of the Coxeter labels [9].

The problem of finding states satisfying the mass shell condition (32) is converted to the finding $h_{\Lambda}$ satisfying it. Now the task is to seek the explicit form of the highest weight vector $\Lambda$ by reading off the inverse Cartan matrix $A^{-1}$. This is convenient because $A^{-1}$ is uniquely defined, independent of the basis vector one uses. The corresponding highest weight vector is obtained from the following relation;

$$p = \bar{\Lambda}_i = \sum_j A^{-1}_{ij} \bar{\alpha}^j{\check{\nu}} \tag{37}$$

where we carefully use $\bar{\alpha}^j{\check{\nu}}$ as a simple (dual) root of original $E_8 \times E_8$, in which the algebra is embedded. The complete weight vectors are obtained by successive lowering with the aid of the Cartan matrix.

In general the state is charged under more than one simple (and Abelian) algebra. Typically, it is the fixed point algebra resulting from the breaking of $E_8 \times E_8$ or $SO(32)$. Denote the whole algebra as $\bigoplus h$ and the conformal weight of each simple algebra as $h_{\Lambda^h}$. Since the conformal weight is additive, we may replace the $h_{\Lambda}$ in (36) as the sum over the whole algebra,

$$h_{\Lambda} = \sum h_{\Lambda^h}. \tag{38}$$

Also because such algebras are disconnected, we can easily write the weight vector $p$ in (37) as also the sum of the weight vectors of each algebra

$$p = \sum \bar{\Lambda}^h_i. \tag{39}$$

We will see shortly that this still holds for the Abelian groups.

Now we can interpret massive modes in terms of algebraic operators. The grade raising generators $H^i_{-n}$ and $E^\alpha_{-n}(n > 0)$ correspond to $\alpha^i_{-n-1}|0\rangle$ and $|p\rangle$ with $p^2/2 = n + 1$, respectively, by (32). For explicit spectra with $n = 1$, see [19]. With more compact dimensions, we may generalize this idea to the case of Narain compactification [20].
IV. TWISTING ALGEBRA

A. Twisted algebra

Let us define shift vector $v$ by which the action on states is

$$\omega = \exp(2\pi iv \cdot \text{ad}_H)$$  \hspace{1cm} (40)

where $\text{ad}$ is the commutation ‘adjoint’ operation $\text{ad}_A B = [A, B]$. Noting that $[H^i, \bar{E}^\alpha] = \bar{\alpha}^i E^\alpha$, we have

$$\omega |p\rangle = \exp(2\pi iv \cdot p)|p\rangle.$$  

The order of automorphism $l$ is the minimal integer such that $\omega^l = 1$, so that $lv$ belongs to the (dual) weight lattice. It is known that every inner automorphism of finite order can be represented by such a shift [9]. For the case of $E_8$, the Dynkin diagram possesses no symmetry, thus the only automorphism is inner automorphism [25]. Such an automorphism naturally comes from the twisted string current

$$T^a(e^{2\pi i z}) = e^{2\pi i\eta^a}T^a(z)$$  \hspace{1cm} (41)

with $l\eta \in \mathbb{Z}$. The same terminology is used for the algebra. They satisfy the twisted algebra

$$[T^a_{m+\eta^a}, T^b_{n+\eta^b}] = i f^{abc}T^c_{m+n+\eta^c+\eta^b} + (m + \eta^a)\delta_{m+n+\eta^a+\eta^b,0}\delta^{ab}K.$$  \hspace{1cm} (42)

Under the triangular decomposition [9,12], the twisting [10] leaves the Cartan generators invariant. Thus, we still have the integer moded generators. This should be true also for raising generators by some redefinition. To absorb the twist in the $\bar{\alpha}$ direction, we define

$$\bar{E}_{n}^\alpha = E_{n+\bar{\alpha} \cdot v}^\alpha.$$  \hspace{1cm} (43)

To compensate for this, we require

$$\bar{H}_n^i = H_n^i + v^i \delta_{n,0}K,$$

$$\bar{K} = K,$$

$$\bar{D} = D - v \cdot H_0.$$  \hspace{1cm} (44-46)

These newly defined generators satisfy the commutation relation of untwisted ones [9,12]; the twisted algebra is isomorphic to the untwisted algebra [11]. In the Kaluza–Klein theory,
where the bosonic description is based, the charge and the mass are not distinguished. Thus, the ladder operator \( \hat{D} \) carries mass \( n + v \cdot \vec{\alpha} \). Ultimately, we will be interested in the zero modes, or the vanishing \( \hat{D} \) eigenvalues.

The weight of a twisted state is the eigenvalue of \( \tilde{\hat{H}}_0 \) (Recall that \( H_n \) with \( n \neq 0 \) is the ladder operator in the \( D \) direction.) This has an effect

\[
|p\rangle \rightarrow |\tilde{p}\rangle = |p + v\rangle.
\]

(47)

Thus

\[
L_0 |\tilde{\Lambda}\rangle = h_{\tilde{\Lambda}} |\tilde{\Lambda}\rangle = h_{\tilde{\Lambda}} |\tilde{p}\rangle.
\]

(48)

So, we interpret that this is the representation of the twisted algebra. In the \( E_8 \times E_8 \) theory, we cannot treat two \( E_8 \) groups separately in the twisted sector because of the relation \( (44) \).

The mass shell condition for the highest representation is

\[
\frac{M_L^2}{4} = \frac{(p + v)^2}{2} + \tilde{N} - \frac{c}{24} = h_{\tilde{\Lambda}} - \frac{c}{24}.
\]

(49)

Satisfied by the highest weight \( \tilde{\Lambda} \), we have the same explicit vector as \( (37) \)

\[
p + v = \sum_j (A^h)^{-1}_{ij} \tilde{\alpha}^j.\]

(50)

Again we will deal with the unbroken subgroup \( h \) arising from breaking the original group \( g \), which is typically \( E_8 \times E_8 \) or \( SO(32) \) and in which \( h \) is embedded. Then here, we use the inverse Cartan matrix \( (A^h)^{-1} \) of the subgroup \( h \) and original \( g \) dual root \( \tilde{\alpha}^j \). Now the resulting \( p \) does not necessarily belong to the untwisted zero mode roots satisfying \( p^2 = 2 \), which is the case when the eigenvalue of \( \tilde{D} \), or \( M_L^2 \) vanishes but not \( D \). Also for the state charged under semisimple and Abelian groups, the conformal weight and weight vector are additive as in \( (38,39) \) with \( p \) replaced by \( p + v \).

The twisted algebra depends on the shift vector \( v \) only. The argument is further extended to the \( k \)th twisted sector, in which the only change is the effective shift vector \( kv \). We also have the symmetry \( v \rightarrow -v \), meaning that we always have the antiparticle\((-v)\) which has the complex conjugate representation from that of the particle\((v)\). The chirality comes from right movers which have only one helicity by Gliozzi–Scherk–Olive (GSO) projection. A complete chiral state consists of both.
FIG. 1: The extended $E_8$ Dynkin diagram and the ordering. The italic numbers are the dual Coxeter labels.

**B. Fixed point algebra**

Consider the invariant subalgebra under shifting (40), whose elements satisfy

$$p \cdot v \in \mathbb{Z}. \quad (51)$$

This is known as fixed point algebra and will be unbroken subalgebra under orbifolding.

Given $v$ of finite order, there are only a few such fixed point algebras possible [6]. We can easily observe this from the Dynkin diagram. Expressing the shift vector in terms of the fundamental weight

$$v = \frac{1}{l} \sum_{i=1}^{r} s_i \bar{\Lambda}_i. \quad (52)$$

We can show that the following is always satisfied [9]

$$l = s_0 + \sum_{i=1}^{r} a_i s_i, \quad s_i \in \mathbb{Z}_{\geq 0}, \quad (53)$$

with the dual Coxeter label $a_i$ previously defined. By the other way around, from a given shift vector we can always find an equivalent one of this form. This ‘dominant’ form is most convenient because we can track the group theoretical origin of the action. We see that if

$$\frac{s_i}{l} = \tilde{\alpha}^{i'} \cdot v \quad (54)$$

is nonzero, then the corresponding root is a broken root as can be seen from the relation (51). The nonzero integral value also passes the condition, but this is not the case because the above restriction implies $s_i < l$. Note the Cartan generators are untouched and will provide the $U(1)$ generators.

Only a few set of integers $\{s_0, s_i\}$ satisfy the relation (53) by the given order $l$, since $a_i$’s are positive. To find the unbroken (fixed point) algebra, we only need to find a set of nonnegative integers whose sum is order $l$ and then delete the circles corresponding to the nonzero element. We can extend this argument for more than one shift vector [6]. These extra shift vector(s) are provided by Wilson lines.
V. ABELIAN CHARGE

Recall that under the shift (40) the Cartan generators remain invariant, even when their roots are prevented by the condition (51). This means that in the fixed point algebra, they play the role of $U(1)$ generators and the rank is preserved. The corresponding charge generator $q_i$ that projects the state vector to give $U(1)$ charge $Q_i$

$$Q_i = q_i \cdot p,$$  \hspace{1cm} (55)

or $\tilde{p}$ instead of $p$ in the twisted sector. We use the same index $i$ since we have one to one correspondence between $\tilde{\Lambda}_i$ and $U(1)$ subgroups. The Abelian generators are proportional to the fundamental weights used (corresponding to $s_i \neq 0$) in the shift vector (52),

$$q_i \propto \tilde{\Lambda}_i$$ \hspace{1cm} (56)

if the extended root of the original algebra is projected out $s_0 \neq 0$. This is true because $q_i$ should be orthogonal to the rest of the (simple) roots, otherwise this vector would be the root vector of the corresponding nonabelian group. If the extended root survives $s_0 = 0$, we can always find the following Abelian generators $q$ (as many as the number of Abelian groups in the fixed point group). By making linear combinations between the fundamental weights used in the shift vector (52), allowing the negative coefficient of $s'_i$ we have

$$q \propto \sum s'_i \tilde{\Lambda}_i, \quad s'_i \in \mathbb{Z}$$ \hspace{1cm} (57)

satisfying

$$q \cdot \tilde{\theta} = 0,$$

for it should be orthogonal to extended root $-\tilde{\theta}$ of the original algebra.

The normalization of $q_i$’s, related to the level $k$ and determined by normalization of the current $T^a(z)$ [11]. The corresponding vertex operator in this direction is $q_i \cdot \partial_z X$, and has a different coefficient from (20). From (22), by fixing normalization of $f^{abc}$, as (8), the relative normalization of the $z^{-2}$ term should be $k = q_i^2$ in this direction. For Abelian groups, the structure constants vanish, and the normalization has to be fixed in another way. However, at the compactification scale of an orbifold, this $U(1)$ generator is embedded in $E_8 \times E_8$ groups and thus has definite normalization

$$q_i^2 = k$$ \hspace{1cm} (58)
to 1, as discussed before. The conformal weight for a state is

$$h_{Q_i} = \frac{1}{2} Q_i^2 = \frac{1}{2} (q_i \cdot p)^2.$$  \hspace{1cm} (59)

Comparing to the similar relation (36), we can determine a $U(1)$ charged piece of vector $p$. Interestingly, it is also proportional to $q_i$: The other parts of $p$ are fundamental weights of the unbroken nonabelian group, which should not be charged under this $U(1)$,

$$q_i \cdot p = q_i \cdot r, \quad r \propto \tilde{A}_i \propto q_i.$$  \hspace{1cm} (60)

This means that we can decompose the shift vector into completely disconnected parts. The resulting state vector is

$$p = \sum A^{-1}_{ij} \alpha^{j\nu} + r.$$  \hspace{1cm} (61)

The normalization of $q$ is fixed by (59). In general, states may be charged under more than one $U(1)$’s: then the vector is simply the addition of each $U(1)$ part.

There are potential anomalous $U(1)$’s. Since all the $U(1)$ generators belong to the original $SO(32)$ or $E_8 \times E_8$, by redefinition we can always absorb anomalies into one $U(1)$. This is cancelled by the Green–Schwarz (GS) mechanism, and the charges of the whole spectrum satisfy a specific ‘universality’ condition. It also fixes normalization [10, 11, 12], and our normalization gives the correct answer. The statement of Ref. [22] is for all theory if we have at least one anomalous $U(1)$, the GS mechanism fixes the normalization in four dimensional theory, regardless of the origin of group breaking, which in this case is orbifolding.

We have stressed that the highest weight vector is the sum of the highest weight vectors of disconnected semisimple parts. This also holds true for the Abelian group, where now we have $c = 1/k = 1$, the ‘rank’ of the Abelian group, and this is natural from the relation (31) with $d = 1$ and $g = 0$. It follows that the conformal weight of a given state is the sum of conformal weights of each simple or Abelian group.

VI. ORBIFOLDING SPACETIME

A. Compactification on orbifold

To obtain a realistic theory, we compactify the string on an orbifold. Practically, we define the orbifold as a torus modded by a finite order automorphism $T^n/\mathbb{P}$. Let $n$ be even
and pair the coordinates to complexify $Z^i = 2^{-1/2}(X^{2i-1} + iX^{2i})$. We define the twist $\theta$ of $\bar{P}$ by a rotation on the diagonal entries of the spacetime group $SO(8)$ (the massless little group of Lorentz $SO(1,9)$)

$$\theta Z^i = \exp(2\pi i\phi_i)Z^i$$

(62)

up to lattice translations. Because the action of $\bar{P}$ is a finite order and defined on the torus, it is known that there are only thirteen kinds of lattice and twisting for at least $N = 1$ supersymmetry [1].

In this orbifold theory we have the twisted sector, since we have closed strings modulo $\theta$. There is another condition

$$v^2 - \phi^2 = 0, \mod 2/l.$$ (63)

They are required by the modular invariance of the string loop amplitude [1, 23].

The breaking of the gauge group occurs when we associate the orbifold twist $\theta$ and the shift vector $v$. From [13] we see that the grade (equivalently the mass squared, or eigenvalue of worldsheet Hamiltonian) of the raising operator is changed by $\bar{\alpha} \cdot v$ because of shifting [40]. Physically, the zero mode of this operator, corresponding to roots of gauge symmetry, should commute with the Hamiltonian. Equivalently, the state $|\phi\rangle$ should be invariant under [40]. So the resulting unbroken algebra is a fixed point algebra, obeying the condition (51).

The matter spectrum is totally determined by the mass shell condition (19), supplemented by the generalized GSO projection below. All of the matter spectrum forms the highest weight module of the fixed point algebra. In the untwisted sector, the matter representation is decomposed according to the transformation property of $p \cdot v$. In each twisted sector, for each highest weight representation

$$\sum h_{\Lambda_i} - \frac{c}{24} = 0.$$ (64)

where $c = c^g + c^s$ is the total conformal anomaly of the gauge group [41]. For the spacetime degrees of freedom in the $k$th twisted sector, it is given by zeta function regularization,

$$-\frac{1}{24}c^g = \sum_i \sum_{n=1}^{\infty} (n + k\phi_i) = \sum_i \left(-\frac{1}{24} + \frac{1}{4}k\phi_i(1 - k\phi_i)\right),$$ (65)

where $i$ runs over the real spacetime bosonic degrees of freedom. The sign is opposite for fermions. We adjusted $k\phi$ modulo integer to lie in $0 < k\phi_i < 1$. By definition, the total zero point energy is in $0 < c/24 < 1$. Inspecting the metric tensor of weights, one can
see that there are only a few states with $h < \frac{1}{2}$, so that only a few can be simultaneous representations under more than one group.

In general, the orbifold action is not free, that is, we have fixed points. The number of such localized spectra is that of the fixed points $\chi$, from the Lefschetz fixed point theorem,

$$\chi = \text{det}(1 - \theta) = \prod 4 \sin^2(\pi \phi_i),$$

over the compact dimension. Here, $\theta$ is regarded as the rotation matrix of the lattice and it has integral elements. The twisted string center of mass cannot have momentum and is localized around fixed points.

However, when the order of the orbifold is non-prime, this naive $\chi$ does not work. By modular transformations some sectors can mix. By successive shifting, a fixed point does not remain at that fixed point any longer. For example, in the $T^6/Z_4$ orbifold, all the fixed points under the shift $\phi = \frac{1}{4}(2 1 1 0)$ are not fixed points under the shift $2\phi = \frac{1}{2}(2 1 1 0)$. Here, we have a nontrivial fixed representation. The remedy is to count the number of effective fixed points (or tori). This can be read off by integrating out the partition function. The projection

$$(\Delta_m)^n = \exp \left[2\pi i n(\tilde{N} - N + (p + mv) \cdot v - (s + m\phi) \cdot \phi - \frac{1}{2}(mv^2 - m\phi^2))\right],$$

projects out the invariant states under the orbifold action in the $(\theta^m, \theta^n)$ twisted states \[2, 11\]. (Also the vector $p$ should belong to the $E_8 \times E_8$ or $SO(32)$ lattice.) Thus, the number of effective fixed points in the $\theta^m$th twisted sector is

$$P_m = \frac{1}{l} \sum_{n=0}^{l-1} \tilde{\chi}_{mn}(\Delta_m)^n.$$

The new $\tilde{\chi}_{mn}$ is slightly different from the number of fixed point $\chi$ because of the spin structure \[2\], but can be fixed by modular invariance. This is a heterotic string version of the GSO projection condition.

### B. Spacetime, or Lorentz symmetry

Now consider states carrying the spacetime index. Here we focus on the massless states, which are provided by worldsheet fermions that transform under the Lorentz symmetry $SO(8)$. For the full description of a supersymmetric right mover, we need supersymmetric
FIG. 2: The extended $SO(8)$ Dynkin diagram. The italic numbers are the dual Coxeter labels.

conformal algebra. In the Neveu–Schwarz (NS) sector, the lowest lying state transforms as vector $\mathbf{8}$. The Ramond (R) sector has the lowest lying state, transforming as spinorial $\mathbf{8}$. When we bosonize them with four bosons, we have a unified description. Similar to the weight vector of the gauge group $\mathfrak{p}$, the state vector is denoted by $s$. The NS vector has the component $(\pm 1 \ 0 \ 0 \ 0)$ up to permutations and the R spinor has $(\pm \frac{1}{2} \ \pm \frac{1}{2} \ \pm \frac{1}{2} \ \pm \frac{1}{2})$ with an even number of minus signs. These are all massless. Not only these massless states, but all the excited states form the representation of the $SO(8)$ affine Lie algebra.

We can easily examine the symmetry breaking of $SO(8)$ using the fixed point algebra as discussed before. For example, the $T^4/Z_3$ twist is given by $\phi = \frac{1}{3}(2 \ 1 \ 1 \ 0) = \frac{1}{3}(\Lambda_1 + \Lambda_3 + \Lambda_4)$. (We are using the fundamental weights of $SO(8)$.) Deleting the corresponding nodes from the extended Dynkin diagram, one notes that the surviving group is $SU(3) \times U(1)^2$.

Accordingly, this leads to branching $\mathbf{8} = \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}$, which can be checked by the transformation property of $s \cdot \phi$. The $SU(3)$ is the holonomy group and the remaining $U(1)$’s are the $R$ symmetry and the noncompact $SO(2)$ Lorentz symmetry in the light cone gauge. When we compactify on a four dimensional orbifold $T^4/Z_l$, with shift $\frac{1}{l}(1 \ 1 \ 0 \ 0) = \frac{1}{l}\Lambda_2$ the unbroken Lorentz symmetry $SO(4) = SU(2) \times SU(2)$ is not explicitly manifest.

The same argument applies to the twisted sector. The orbifold action (62) is basically the same as shifting (10). Thus under the twisting, they are shifted as $s \rightarrow s + \phi$, satisfying the relation (14). The bosonic degrees of freedom have the same Lorentz symmetry. Since we know every inner automorphism can be converted into a shift vector, without knowing the concrete representation, we have a corresponding conformal weight $h_\Lambda$. Alternatively, we may understand the spacetime contribution in terms of an oscillator. Through orbifolding, now we have a shifted oscillator mode number by $\phi_i$ affecting a mass shell condition such as $h_\Lambda$; however then the algebraic property becomes less transparent.

Note that, to the bosonic degrees of freedom, the spacetime twisting (62) does touch the
Cartan generator itself. Care must be taken in dealing with them. Using the fact that every
finite inner automorphism can be representation, the spacetime twisting can be converted
to twisting with shift vector, which have a suitable basis.

We may use this fact for orbifolding and obtaining the massless spectrum in other super-
string theories.

VII. EXAMPLES

A. Standard embedding

Take an example of $E_8 \times E_8$ theory compactified on $T^6/\mathbb{Z}_3$ orbifold with twist $\phi = \frac{1}{3}(2\ 1\ 1\ 0) = \frac{1}{3}(\bar{\Lambda}_1 + \bar{\Lambda}_3 + \bar{\Lambda}_4)$ in the $SO(8)$ basis. Deleting the corresponding nodes in the Dynkin diagram, the resulting spacetime degrees of freedom are $SU(3) \times U(1) \times SO(2)$ as before. For the standard embedding of the group degrees of freedom, $v = \frac{1}{3}(2\ 1\ 1\ 0^5; 0^8) = \frac{1}{3}(\bar{\Lambda}_2; 0)$ in the $E_8$ basis, we obtain an unbroken group as

$$SU(3) \times E_6 \times E_8.$$ 

To find the matter spectrum, we need only check a few. From the integrability condition the candidates are,

$$h_3 = \frac{1}{3}, \quad h_{27} = \frac{2}{3}, \quad h_{(3, 27)} = h_3 + h_{27} = 1$$

The antiparticles have the same conformal weights. The GSO projection condition determines which one survives. Under the spacetime and the gauge group, $(SU(3); SU(3) \times E_6; E_8)$, the untwisted sector has

$$3(1; 3, 27; 1)$$

The multiplicity 3 came from the right movers. Numbers are in boldface, except those for spacetime representation.

In the twisted sector, we have the zero point energy $-\frac{c}{24} = -\frac{1}{24}(c^g + c^s) = -\frac{2}{3}$ from eq.(65). The $\bar{\Lambda}_1 = 27$ of $E_6$ has conformal weight $\frac{2}{3}$. The $\bar{\Lambda}_2 = 3$ of $SU(3)$ has $\frac{1}{3}$, regardless of whether they come from the Lorentz or gauge group. So the combination of $(3; 3, 1; 1)$ under the Lorentz group $SU(3)$ and gauge group $SU(3)$ also has a total conformal weight.
of $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. The corresponding highest weight vectors $\tilde{p}$ are

$$(1, 27; 1) : \sum_j (A^{E_6})^{-1}_{1j} \tilde{\alpha}^j = (\frac{2}{3} - \frac{2}{3} - \frac{2}{3} 0^6; 0^8)$$

$$(3, 1; 1) : \sum_j (A^{SU(3)})^{-1}_{2j} \tilde{\alpha}^j = (-\frac{1}{3} \frac{1}{3} - \frac{2}{3} 0^6; 0^8).$$

The $\tilde{\alpha}^j = \bar{\alpha}^{j'}$ is a simple root in the $E_8 \times E_8$ basis and each inverse Cartan matrix $(A^h)^{-1}$ is of the simple subgroups $E_6$ and $SU(3)$. Here we suppressed the eigenvalue of $(D, K)$. Note that the vector $p = \tilde{p} - v$ belongs to $E_8 \times E_8$ lattice, not necessarily being a root with $p^2 = 2$.

With multiplicity from the number of fixed points $\chi = 27$ from eq. (66), we have

$$27(3; 3, 1; 1) + 27(1; 1, 27; 1),$$

which survives under the projection (68). The antiparticles come from the second twisted sector with twist $2\phi$, or equivalently $-\phi$.

**B. Models having an Abelian group**

Consider again the $T^6/Z_3$ example with the shift vector $v = \frac{1}{3}(2 \ 0^7; 1 \ 1 \ 0^6) = \frac{1}{3}(\bar{\Lambda}_7; \bar{\Lambda}_1)$. We can check that the modular invariance condition is satisfied and the resulting gauge group is $SO(14) \times U(1) \times E_7 \times U(1)$. The two $U(1)$ generators are $q_7 = \frac{1}{2}(\bar{\Lambda}_7; 0)$ and $q'_1 = \frac{1}{2\sqrt{2}}(0; \bar{\Lambda}_1)$ by the normalization (58). Note that this gives correct normalization [12] for the GS mechanism.

In view of the branching rule, in the untwisted sector we obtain

$$3(1; 14, 1) + 3(1; 64, 1) + 3(1; 56, 1) + 3(1; 1, 1).$$

In the twisted sector, the zero point energy is still $-\frac{c}{24} = -\frac{2}{3}$. The $SO(14)$ vector with $h_{14} = \frac{1}{2}$ alone cannot be massless, but should have other components to fulfill the mass shell condition. The missing mass is provided by other vectors $r_7$ and $r'_1$ charged under $U(1)$'s.

The corresponding highest weight vector has the form

$$\tilde{p} = \sum_j (A^{SO(14)})^{-1}_{1j} \tilde{\alpha}^j + r_7 + r'_1.$$

The first term is $\bar{\Lambda}_1$ of $SO(14)$. The $r_7$ and $r'_1$ are also proportional to $(\bar{\Lambda}_7; 0)$ and $(0; \bar{\Lambda}_1)$, respectively. They are completely fixed by the condition

$$h_{Q} = \frac{1}{2}(q_7 \cdot r_7)^2 + \frac{1}{2}(q'_1 \cdot r'_1)^2 = \frac{1}{6},$$

20
and the generalized GSO projection condition. The resulting vector is
\[
\tilde{p} = (0 \ 1 \ 0^6; 0^8) + (-\frac{1}{3})(1 \ 0^7; 0^8) + \frac{1}{3}(0^8; 1 \ 1 \ 0^6),
\]
and charged as \((1; 14; 1)\). The Lorentz 3 of \(SU(3)\) can contribute \(h = \frac{1}{3}\) and it provides another charged state, \((3; 1; 1)\). In addition, there is a state which is a singlet under the whole nonabelian group \((1; 1; 1)\). They all have multiplicity \(\chi = 27\).

\[C. \hspace{1em} \text{Non-prime orbifold}\]

We take the \(T^6/Z_4\) orbifold example with \(\phi = \frac{1}{4}(2 \ 1 \ 1 \ 0)\) and the standard embedding in the group space. The resulting spacetime symmetry is \(SU(2) \times U(1)^2\) and the group degree of freedom is \(E_6 \times SU(2) \times U(1) \times E_8\). The \(U(1)\) generator is \(q_2 = \frac{1}{\sqrt{6}}(\bar{\Lambda}_2; 0)\). In the untwisted sector, by the branching rule we have
\[
2(1; 27, 2; 1) + 2(1; \bar{27}, 1; 1) + 2(1; 27, 1; 1) + 2(1; 1, 2; 1)
\]
with the multiplicity 2 from the spacetime \(SU(2)\) doublet right movers.

In the first twisted sector the zero point energy is \(-\frac{11}{16}\). We have \(h_{27} = \frac{2}{3}, \ h_2 = \frac{1}{2}\) which determines the nonabelian part of the state only. The missing mass is from the \(U(1)\) charge. By the same argument, the weight vector have the \(U(1)\) parts \(q\) as \(\frac{1}{12}(\bar{\Lambda}_2; 0), \ \frac{1}{2}(\bar{\Lambda}_2; 0)\) respectively.

Let us see the second twisted sector whose effective shift is \(2\phi = \frac{1}{2}(2 \ 1 \ 1)\). Its zero point energy is \(-\frac{4}{3}\). We expect a vectorlike spectrum since the \(k\)th twisted sector spectrum is the same as that of the \((l - k)\)th. The Abelian charge for \(27\) is \(\frac{1}{6}(\bar{\Lambda}_2; 0)\).

Resorting to the projection condition \((68)\), we have ten \(27\)'s and six \(\bar{27}\)'s. Also, we observe that the modular invariance condition completely determines the \(U(1)\) part of a given vector.

\[VIII. \hspace{1em} \text{DISCUSSION}\]

We have seen that the spectra of heterotic strings on orbifolds can be obtained from a simple relation between conformal weight and mass. This relation is natural in view of conformal field theory. The connection between conformal field theory and affine Lie algebra is provided by Sugawara construction.
This proves to be useful when we are interested in the twisted sector. By orbifolding with a shift vector, we associated the spacetime point group action with the group degree of freedom. This twisting mixes the massless and massive states, and they are represented by twisted affine Lie algebra. In fact, they are elements of another, independent Hilbert space of twisted states. The conformal weights, hence the explicit states in the twisted state, can be easily obtained because the twisted algebra is isomorphic to untwisted one.

Also we stress that this is the only systematic way to obtain Abelian generators. The corresponding conformal weights and representation vectors are treated on an equal footing as those of nonabelian cases. It is hoped that this will reveal the role of anomalous $U(1)$.

We observe that the resulting spectrum is strongly related to branching rules \cite{18}. For the purely group degree of freedom, there are some mathematical discussions on branching rules of affine Lie algebra. From the partition function (called a character in algebra) we can relate representations between a given algebra and an embedded subalgebra. However, in string theory some characters are different. The low energy fields acquire chirality due to right movers and the multiplicity comes from the invariance property of states under the point group. However this nature is reflected in the same way and also leads to the same rule.

If such a unified branching rule is accessible, we are able to have a better understanding of, for example, the anomaly freedom of orbifold theory \cite{23}: we know that a representation of subalgebras embedded from an anomaly-free representation is also anomaly-free. The modular invariance is crucial for an anomaly-free theory. Using the branching function \cite{18} and anomaly polynomial \cite{24} it will be observed that the absence of anomaly originates from the fact that there is no modular form of weight two.

We can apply this idea to other theories described on the lattice: the Narain compactification and the free fermionic formulation just by interpreting shift and twisting fields. Also it holds for orbifold compactification of other theories, for example Type II strings. The method we present in this paper will, we hope, provide a guideline for a top-down approach of model building.

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APPENDIX A: ALGEBRAIC CONVENTIONS

In this appendix, we define some algebraic elements used in the examples, following [9] where the complete list is available. The orthogonal representation for simple roots of $E_8$ are

\[ \bar{\alpha}_1 = ( 0 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\alpha}_2 = ( 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\alpha}_3 = ( 0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 0 ) \]
\[ \bar{\alpha}_4 = ( 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 ) \]
\[ \bar{\alpha}_5 = ( 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 0 ) \]
\[ \bar{\alpha}_6 = ( 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 ) \]
\[ \bar{\alpha}_7 = ( \frac{1}{2} \ -\frac{1}{2} \ -\frac{1}{2} \ -\frac{1}{2} \ -\frac{1}{2} \ -\frac{1}{2} \ -\frac{1}{2} \ \frac{1}{2} ) \]
\[ \bar{\alpha}_8 = ( 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 ). \]

The extended root $\bar{\alpha}_0 = (-1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) = -\bar{\theta}$ is defined as the negative of highest root. All the root vectors are self-dual $\bar{\alpha}^i = \bar{\alpha}^{i\vee}$ because $\bar{\alpha}^2 = 2$. Accordingly, we have fundamental weights,

\[ \bar{\Lambda}_1 = ( 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_2 = ( 2 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_3 = ( 3 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_4 = ( 4 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_5 = ( 5 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_6 = ( \frac{5}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ -\frac{1}{2} ) \]
\[ \bar{\Lambda}_7 = ( 2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 ) \]
\[ \bar{\Lambda}_8 = ( \frac{3}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} ). \]

They are used in the standard form of shift vector in [52]. We have also used this method to break the spacetime Lorentz group $SO(8)$, whose simple roots and fundamental weights
are
\[\bar{\alpha}_1 = ( 1 -1 0 0 ) \quad \bar{\Lambda}_1 = ( 1 0 0 0 )\]
\[\bar{\alpha}_2 = ( 0 1 -1 0 ) \quad \bar{\Lambda}_2 = ( 1 1 0 0 )\]
\[\bar{\alpha}_3 = ( 0 0 1 -1 ) \quad \bar{\Lambda}_3 = ( \frac{1}{2} \frac{1}{2} \frac{1}{2} -\frac{1}{2} )\]
\[\bar{\alpha}_4 = ( 0 0 1 1 ), \quad \bar{\Lambda}_4 = ( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} ).\]

Now we define the quadratic form matrices, defined as the inverses of the Cartan matrices \(A_{ij} = 2a^i \cdot a^j\). They are used for finding the conformal weight in (30) and obtaining the corresponding highest weight vector in (37,50). We take examples of \(SU(3), E_6\) and \(E_8\) used in the standard embedding of \(T^6/Z_3\) orbifold,

\[
(A^{SU(3)})^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad (A^{E_6})^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}
\]

\[
(A^{E_8})^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}.\]

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