Abstract. Measure Differential Equations (MDE) describe the evolution of probability measures driven by probability velocity fields, i.e. probability measures on the tangent bundle. They are, on one side, a measure-theoretic generalization of ordinary differential equations; on the other side, they allow to describe concentration and diffusion phenomena typical of kinetic equations. In this paper, we analyze some properties of this class of differential equations. We prove a representation result in the spirit of the Superposition Principle by Ambrosio-Gigli-Savaré, and we provide alternative schemes converging to a solution of the MDE, with a particular view to uniqueness/non-uniqueness phenomena.

1. Introduction

The theory of Measure Differential Equations (MDE in brief) has been recently introduced in [9] by F. Camilli, G. Cavagnari, R. De Maio, and B. Piccoli. A Cauchy problem for a MDE is given by

\[
\begin{aligned}
\dot{\mu}_t &= V[\mu_t], \\
\mu_{t=0} &= \mu_0,
\end{aligned}
\]

where \(\mu_0 \in \mathcal{P}(\mathbb{R}^d)\), the space of probability measures on \(\mathbb{R}^d\), and \(V\) is a probability vector field (PVF in brief), i.e. a map assigning to a probability measure \(\mu \in \mathcal{P}(\mathbb{R}^d)\) a probability measure \(V[\mu]\) on the tangent bundle \(T\mathbb{R}^d\). If \(V[\mu] = \mu \otimes \delta_{v(x)}\), for a given Lipschitz continuous vector field \(v\), then (1.1) has a unique solution and it coincides precisely with the unique measure solution of the continuity equation \(\partial_t \mu_t + \text{div}(v \mu_t) = 0\). The study of linear and nonlinear transport equations, in the framework of weak measure solutions, has received a lot of attention in the recent time (see [1–7]). This theory is indeed relatively flexible to describe a large variety of phenomena, as a continuum model for interacting particle systems. The MDE approach can be seen as a further generalization of this technique, when the uncertainty affects not only the position of the particles, but also the law governing their evolution.

Existence of weak measure solutions to (1.1) has been proved in [9] by means of an approximation scheme, called Lattice Approximate Solutions (LAS in brief). The scheme is obtained by discretizing the equation in space, time and velocity and moving convex combinations of Dirac masses through the resulting discrete dynamical system. Uniqueness of solutions to (1.1) is, in general, not expected. However, up to restrict the study to the class of solutions that can be obtained as limits of LASs, in [9, Section 5] the author discusses the uniqueness of a Lipschitz semigroup associated to (1.1) by prescribing the evolution of convex combinations of Dirac measures for a small initial time.

Aim of this paper is to provide a further analysis of (1.1) to better understand certain properties regarding the solutions of the problem. The first result is an extension of the Superposition Principle by Ambrosio-Gigli-Savaré in the context of MDEs. We will provide a representation result for a solution of a MDE, similarly to what occurs for continuity equations with a local vector field (see [1, Theorem 8.2.1]), characterizing a (possibly not
unique) solution of (1.1) with a superposition of integral curves coming from a suitable underlying particle system. In the same spirit, we also provide a consistent probabilistic representation for the LAS scheme in [9].

In the second part of the paper, we consider alternative schemes converging to a solution of the MDE. We first define a semi-discrete in time Lagrangian scheme for (1.1) and we prove that, up to subsequences, it converges to the same limit of the LAS scheme. Moreover, we introduce another semi-discrete in time scheme obtained by taking the barycenter of the PVF at each time step, before moving the mass. We show with an example that this mean velocity scheme may converge to a different solution of (1.1) with respect to the LAS/Lagrangian schemes. This fact highlights the weak framework of the MDE theory, in what concerns uniqueness of solutions. Indeed, unless the analysis is restricted to certain subclasses of measures in the spirit of the Lagrangian flow problem, in general we cannot hope to get uniqueness of the solutions of (1.1), except for the trivial case when the second marginal of the PVF $V$ along the solution $\{\mu_t\}_t$ is atomic, i.e. $V[\mu_t] = \mu_t \otimes \delta_{v(x)}$, and $v$ is Lipschitz continuous.

The paper is organized as follows: in Section 2 we give some preliminaries on optimal transport and measure theory, recalling the MDE setting and the definition of the LAS scheme introduced in [9]; in Section 3 we exploit a Superposition Principle for MDEs and a probabilistic representation construction for the LASs; in Section 4 we provide a Lagrangian approximation scheme; in Section 5, we present another approximating scheme converging to a different solution of (1.1) and finally, in Section 6 we discuss some clarifying examples.

2. Preliminaries and first results

We recall some preliminary definitions and results (we address the reader to [1, 10, 11] as relevant resources regarding optimal transport and measure theory). Given a complete separable metric space $X$, we denote by $\mathcal{P}(X)$ the set of Borel probability measures on $X$, by $\mathcal{P}_p(X)$ the subset of $\mathcal{P}(X)$ whose elements have finite $p$-moment and by $\mathcal{P}_c(X)$ the subset of $\mathcal{P}_p(X)$ whose elements have compact support. We endow the set $\mathcal{P}_p(X)$ with the $p$-Wasserstein distance $W_p^X$, and we consider the metric $W_1^X$ on $\mathcal{P}_c(X)$. In the case $p = 1$, we recall a special duality formula, called the Kantorovich-Rubinstein duality

$$W^X(\mu, \nu) = \sup \left\{ \int_X f \, d(\mu - \nu) : f : X \to \mathbb{R}, \text{Lip}(f) \leq 1 \right\}.$$ 

**Proposition 2.1.** $\mathcal{P}_p(X)$ endowed with the $p$-Wasserstein metric, $W_p^X$, is a complete separable metric space. Moreover, given a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}_p(X)$ and $\mu \in \mathcal{P}_p(X)$, we have that the following are equivalent

1. $\lim_{n \to \infty} W_p^X(\mu_n, \mu) = 0$,
2. $\mu_n \rightharpoonup^* \mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ has uniformly integrable $p$-moments.

In the following, we recall the definition of push-forward (see [1, Section 5.2]) and a disintegration result (see [1, Theorem 5.3.1]).

**Definition 2.2** (Pushforward). Let $X, Y$ be two separable metric spaces, $\mu \in \mathcal{P}(X)$, and $r : X \to Y$ be a Borel map. We define the pushforward measure $r^*_\mu \in \mathcal{P}(Y)$ by $r^*_\mu(B) := \mu(r^{-1}(B))$, for any Borel set $B \subseteq Y$. An equivalent definition is as follows,

$$\langle r^*_\mu, f \rangle := \int_X f(r(x)) \, d\mu(x),$$

for every bounded (or $r^*_\mu$-integrable) Borel function $f : Y \to \mathbb{R}$.

**Theorem 2.3** (Disintegration). Let $X, X$ be complete separable metric spaces, $\mu \in \mathcal{P}(X)$ and $r : X \to Y$ be a Borel map. Then there exists a $r^*_\mu$-a.e. uniquely determined family of

$$X \to \mathcal{P}(Y)$$

such that $\mu = r^*_\mu$.
probability measures \( \{ \mu_x \}_{x \in X} \subset \mathcal{P}(X) \) such that \( \mu_x(X \setminus r^{-1}(x)) = 0 \) for \( r^*_x \mu \)-a.e. \( x \in X \). Furthermore

\[
\int_X f(z) \, d\mu(z) = \int_X \int_{r^{-1}(x)} f(z) \, d\mu_x(z) \, d(r^*_x \mu)(x),
\]

for any Borel map \( f : X \to [0, +\infty] \). We will write \( \mu = (r^*_x \mu) \otimes \mu_x \).

\begin{remark}
As pointed out in [1, Section 5.3], if \( X = X \times Y \) and \( r^{-1}(x) \subseteq \{x\} \times Y \) for all \( x \in X \), then we can identify each measure \( \mu_x \in \mathcal{P}(X \times Y) \) with a measure defined on \( Y \). We will make a strong use of this result throughout the paper.
\end{remark}

We recall now the definition of convolution between measures and product with a coefficient \( a \in \mathbb{R} \). We denote with \( \chi_A \) the characteristic function of \( A \subseteq \mathbb{R}^d \).

\begin{definition}[Convolution]
We define the convolution operator \( \ast : \mathcal{P}^{d} \times \mathcal{P}^{d} \to \mathcal{P}^{d} \) by \( (\mu \ast \nu)(B) := \int_{\mathbb{R}^d} \chi_B(x + y) \, d\nu(y) \, d\mu(x) \), for any Borel set \( B \subseteq \mathbb{R}^d \). Equivalently we may define

\[
\langle \mu \ast \nu, f \rangle := \int_{\mathbb{R}^d} f(x + y) \, d\nu(y) \, d\mu(x),
\]

for any \( \mu \ast \nu \)-integrable Borel function \( f : \mathbb{R}^d \to \mathbb{R} \).

\end{definition}

We define the product operator \( \cdot : \mathbb{R} \times \mathcal{P}^{d} \to \mathcal{P}^{d} \), \( (a, \mu) \mapsto a \cdot \mu \), by

\[
(a \cdot \mu)(B) := \int_{\mathbb{R}^d} \chi_B(ax) \, d\mu(x),
\]

for any Borel set \( B \subseteq \mathbb{R}^d \).

\begin{remark}
Observe that \( \mathcal{P}_e(\mathbb{R}^d) \) is closed w.r.t. convolution and product operators. In particular, as pointed out in [2, Section 6.1] we have that the operation \( \ast \) defines a monoid structure over \( \mathcal{P}_e(\mathbb{R}^d) \).
\end{remark}

\subsection{Recalls on Measure Differential Equations}

In this section we recall some basic definitions introduced in [3] that are at the base of the investigations proposed in this paper.

\begin{definition}
A probability vector field (PVF) is a map \( V : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{T} \mathbb{R}^d) \) consistent with the projection on the first component, i.e. \( \pi_1 V[\mu] = \mu \).

By Theorem 2.3 and Remark 2.4 we can write \( V[\mu] = \mu \otimes \nu_x[\mu] \) for a \( \mu \)-a.e. uniquely determined family of probability measures \( \{ \nu_x[\mu] \}_{x \in \mathbb{R}^d} \subseteq \mathcal{P}(\mathbb{R}^d) \) defined on the fibers \( T_x \mathbb{R}^d = \mathbb{R}^d \).

Given \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \) and a PVF \( V \), we consider the following Cauchy problem

\[
\begin{cases}
\dot{\mu}_t = V[\mu_t], \\
\mu_{t=0} = \mu_0,
\end{cases}
\]

where the nonlocal dynamics is called Measure Differential Equation (MDE). A solution to this problem has to be interpreted as follows.

\begin{definition}
A solution of \( \{ \mu_t \} \) is a map \( \mu : [0, T] \to \mathcal{P}(\mathbb{R}^d), t \mapsto \mu_t \) such that \( \mu_{t=0} = \mu_0 \) and that satisfies for a.e. \( t \in [0, T] \)

\[
\frac{d}{dt} \langle \mu_t, f \rangle = \int_{\mathbb{R}^d} (\nabla f(x) \cdot v) \, dV[\mu_t](x, v),
\]

for any \( f \in C_c^\infty(\mathbb{R}^d) \) such that the right-hand side is defined for a.e. \( t \), the map \( t \mapsto \int_{\mathbb{R}^d}(\nabla f(x) \cdot v) \, dV[\mu_t](x, v) \) belongs to \( L^1([0, T]) \), and the map \( t \mapsto \int_{\mathbb{R}^d} f \, d\mu_t \) is absolutely continuous. Equivalently,

\[
\langle \mu_t - \mu_0, f \rangle = \int_0^t \int_{\mathbb{R}^d} (\nabla f(x) \cdot v) \, dV[\mu_s](x, v) \, ds,
\]

\( \forall f \in C_c^\infty(\mathbb{R}^d) \).
Remark 2.9. Notice that, in the trivial case when \( V[\mu] = \mu \otimes \delta_v(x) \) for some Borel vector field \( v : \mathbb{R}^d \to \mathbb{R}^d \), the MDE in (2.2) reduces to the continuity equation \( \partial_t \mu_t + \text{div}(v \mu_t) = 0 \) (see [9, Section 6]).

We stress that \( V[\mu] \) is a probability measure on \( T^d \) where the components of its elements \((x, v)\) represent, respectively, the position and the infinitesimal displacement. We recall another notion to measure distances between PVFs introduced in [9].

Definition 2.10. Given \( V_i \in \mathcal{P}_c(T^d) \), \( i = 1, 2 \), and denoted by \( \mu_i := \pi_1^\sharp V_i \) the marginal of \( V_i \), we define

\[
W(V_1, V_2) = \inf \left\{ \int_{T^d \times T^d} |v - w| dt(x, v, y, w) : T \in \Pi(V_1, V_2), \pi_{13}^\sharp T \in \Pi_{\text{opt}}(\mu_1, \mu_2) \right\},
\]

where \( \Pi(V_1, V_2) \) is the set of all the transference plans from \( V_1 \) to \( V_2 \) and \( \Pi_{\text{opt}}(\mu_1, \mu_2) \) is the set of the optimal transference plans from \( \mu_1 \) to \( \mu_2 \), and \( \pi_{13} : (T^d)^2 \to (\mathbb{R}^d)^2 \), \((x, v, y, w) \mapsto (x, y)\).

The object \( W \) computes the minimal displacements of the fiber components assuming that marginals \( \mu_i \) are transported in an optimal way. It is important to notice that \( W \) is not a metric since it can vanish for distinct elements in \( \mathcal{P}_c(T^d) \). Moreover, it is easy to verify that

\[
W^{T^d}(V_1, V_2) \leq W(V_1, V_2) + W^{\mathbb{R}^d}(\mu_1, \mu_2).
\]

Considering the problem set in \( \mathcal{P}_c(\mathbb{R}^d) \), we recall here the main assumptions required to have existence and convergence of approximation schemes for solutions of an MDE (see [9]).

(H1) Sublinearity: there exists a constant \( C > 0 \) such that for all \( \mu \in \mathcal{P}_c(\mathbb{R}^d) \),

\[
\sup_{(x, v) \in \text{supp} V[\mu]} |v| \leq C(1 + \sup_{x \in \text{supp} \mu} |x|);
\]

(H2) Continuity of PVF: the map \( V : \mathcal{P}_c(\mathbb{R}^d) \to \mathcal{P}_c(T^d) \) is continuous;

(H3) Local lipschitzianity in \( \mu \)-variable: \( V \) is locally Lipschitz, in particular for every \( R > 0 \) there exists a constant \( L = L(R) > 0 \) such that

\[
W(V[\mu], V[\nu]) \leq L \cdot W^{\mathbb{R}^d}(\mu, \nu),
\]

for every \( \mu, \nu \in \mathcal{P}_c(\mathbb{R}^d) \) such that \( \text{supp}(\mu), \text{supp}(\nu) \subset B(0, R) \), the open ball of radius \( R \) centered in \( 0 \in \mathbb{R}^d \).

In the following, we recall the scheme provided in [9] that has been used in order to prove existence of solutions to (2.2). Let us start introducing some notation. For \( N \in \mathbb{N} \), set

\[
\Delta_N = \frac{T}{N}, \quad \Delta_N^x = \frac{1}{N}, \quad \Delta_N^v = \Delta_N \Delta_N^x = \frac{T}{N^2},
\]

be respectively the time, the velocity and the space-step sizes, noticing that, differently from [9], we set the time step size to \( T/N \) in place of \( 1/N \), for our convenience. Considering the corresponding grid in \([0, T] \times [-N, N]^d \times [-TN, TN]^d \), we denote by \( x_i \) the discretization points in space, and by \( v_i \) the discretization points for the space of velocities. We now build some objects aiming at providing a discrete approximation for \( \mu \in \mathcal{P}_c(\mathbb{R}^d) \) and \( V[\mu] \in \mathcal{P}_c(T^d) \) by concentrating the mass on the points of the grid. Denoting with \( Q = ([0, 1/N]^d \rightleftharpoons \mathbb{R}^d \) and \( Q' = ([0, 1/N]^d \rightleftharpoons \mathbb{R}^d \), we define

\[
\mathcal{A}^x_N(\mu) := \sum_i m^v_i(\mu) \delta_{x_i}, \quad \mathcal{A}^v_N(V[\mu]) := \sum_i \sum_j m^v_{ij}(V[\mu]) \delta_{(x_i, v_j)},
\]
where \(m_i^T(\mu) := \mu(x_i + Q)\) and \(m_{ij}^v(V[\mu]) := V[\mu] \cdot \{(x_i, v) : v \in v_j + Q'\}\). Notice that given \(\mu \in \mathcal{P}_c(\mathbb{R}^d)\), for \(N\) sufficiently large we have
\[
W^d(A_N^x(\mu), \mu) \leq \Delta_N^x, \quad W^d(A_N^v(V[\mu]), V[\mu]) \leq \Delta_N^v.
\]

**Definition 2.11** (Lattice Approximate Solution (LAS)). Let \(V\) be a PVF satisfying \((H1)\). Given \(\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)\), \(T > 0\) and \(N \in \mathbb{N}\), the Lattice Approximate Solution (LAS) \(\mu^N : [0, T] \to \mathcal{P}_c(\mathbb{R}^d)\), \(t \to \mu^N_t\), is defined, by recursion, as follows
\[
\mu^N_0 = A_N^x(\mu_0), \quad \mu^N_{k+1} = \mu^N_k((k+1)\Delta_N) = \sum_{ij} m_{ij}^v(V[\mu^N_k(k\Delta_N)]) \delta_{x_i + k\Delta_N v_j}.
\]

notice that \(\text{supp}(\mu^N_k)\) is contained on the space grid. By time-interpolation we can define \(\mu^N_t\) for all times as
\[
\mu^N_t(k\Delta_N + t) = \sum_{ij} m_{ij}^v(V[\mu^N_t(k\Delta_N)]) \delta_{x_i + t v_j}.
\]

We address the reader to [9] for results granting the convergence of the LAS scheme to a solution of (2.2).

**Remark 2.12.** In general, uniqueness of a solution for (2.2) is not expected (see [9, Example 3]). Indeed, we notice that the set of solutions to the MDE in (2.2), defined by Definition 2.11, coincides with the set of trajectories \(\mu = \{\mu_t\}_{t \in [0, T]} \subseteq \mathcal{P}_c(\mathbb{R}^d)\) satisfying a continuity equation \(\partial_t \mu_t + \nabla (w \cdot \mu_t) = 0\), where \(w : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) is a Borel vector field s.t.
\[
w_i(y) = \int_{\pi_1^{-1}(y)} v \, d\nu_y[\mu_i](v),
\]
for \(\mu_t\text{-a.e. } y\). Where we denoted with \(\nu_y[\mu_i]\) the disintegration of \(V[\mu_i]\) w.r.t. \(\pi_1\). Thus, in order for \(\mu\) to be a solution of (2.2), it is sufficient to follow what \(V[\mu]\) prescribes on the fibers \(T_y \mathbb{R}^d\) in integral average. We will come back to this fact in the next section.

We recall here a definition, used in [9] to derive the uniqueness of solutions to (2.2), when restricting the analysis to a certain class of trajectories. This will be resumed later on in Section 4.

**Definition 2.13.** A Lipschitz semigroup for (2.2) is a map \(S : [0, T] \times \mathcal{P}_c(\mathbb{R}^d) \to \mathcal{P}_c(\mathbb{R}^d)\) such that for every \(\mu, \eta \in \mathcal{P}_c(\mathbb{R}^d)\) and \(t, s \in [0, T]\) the following holds:
(i) \(S_0 \mu = \mu\) and \(S_t S_s \mu = S_{t+s} \mu\);
(ii) the map \(t \to S_t \mu\) is a solution of (2.2);
(iii) for every \(R > 0\) there exists \(C = C(R) > 0\) such that if \(\text{supp}(\mu), \text{supp}(\eta) \subset B(0, R)\) then:
\[
\left\{
\begin{array}{l}
\text{supp}(S_t \mu) \subset B(0, e^{Ct}(R + 1)), \\
W^d(S_t \mu, S_t \eta) \leq e^{Ct}W^d(\mu, \eta), \\
W^d(S_t \mu, S_s \mu) \leq C|t - s|.
\end{array}
\right.
\]

3. A Superposition Principle for MDEs

In this section, we show how to construct a Superposition Principle (see [1, Theorem 8.2.1] for the continuity equation dynamics) adapted to the general framework of MDEs. The procedure is similar to the one used in [6], where the authors provide a representation result for solutions of a continuity equation associated with Carathéodory solutions of a differential inclusion. This result, proved in [6], is exploited in [5], where the authors study optimal control problems in the space of probability measures with microscopic dynamics ruled, precisely, by a differential inclusion.

We split the statement into two parts. In the first part, we see that any measure \(\eta \in
\(\mathcal{P}(\mathbb{R}^d \times \Gamma_{[0,T]})\), concentrated on curves that follow a given PVF \(V\) in integral average, generates a solution of the MDE.

For \(I \subseteq \mathbb{R}\) interval, we denote by \(\Gamma_I\) the set of continuous curves from \(I\) to \(\mathbb{R}^d\) and by \(e_t\) the evaluation operator \(e_t : \mathbb{R}^d \times \Gamma_I \to \mathbb{R}^d, (x, \gamma) \mapsto \gamma(t)\), for \(t \in I\), while \(AC(I; \mathbb{R}^d)\) is the space of absolutely continuous curves from \(I\) to \(\mathbb{R}^d\).

**Theorem 3.1** (Superposition Principle for MDEs - Part I). Let \(T > 0\), \(V : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(T\mathbb{R}^d)\) be a PVF, \(\mu_0 \in \mathcal{P}(\mathbb{R}^d)\). Let \(\eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{[0,T]}\) be concentrated on the set of pairs \((\gamma(0), \gamma) \in \mathbb{R}^d \times \Gamma_{[0,T]}\) such that the following conditions hold

(i) \(\gamma \in AC([0,T]; \mathbb{R}^d)\);

(ii) for a.e. \(t \in [0,T]\) and for \(e_t \sharp \eta\) a.e. \(y \in \mathbb{R}^d\) we have

\[
\int_{e_t^{-1}(y)} \gamma(t) \, d\eta_{t,y}(\gamma) = \int_{\pi_t^{-1}(y)} v \, d\nu_y[e_t \sharp \eta](v),
\]

where \(\eta_{t,y}\) is the disintegration of \(\eta\) w.r.t. \(e_t\), and \(\nu_y[e_t \sharp \eta]\) is the disintegration of \(V[e_t \sharp \eta]\) w.r.t. the projection to the base \(\pi_t\);

(iii) \(\int_0^T \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} |\dot{\gamma}(s)| \, d\eta(x,\gamma) \, dt < +\infty\) and \(\int_0^T \int_{\mathbb{R}^d} |v| \, dV[e_t \sharp \eta](x,v) \, dt < +\infty\).

Then, denoted with \(\mu_t := e_t \sharp \eta\), we have that \(\mu = \{\mu_t\}_{t \in [0,T]} \subseteq \mathcal{P}(\mathbb{R}^d)\) is a solution of the MDE system \((2.2)\).

**Proof.** Let us consider any \(f \in C^\infty_c(\mathbb{R}^d)\). First, we check that \(\int_{\mathbb{R}^d}(\nabla f(x) \cdot v) \, dV[e_s \sharp \eta](x,v)\) is defined for almost every \(s \in [0,T]\). Indeed, immediately by hypothesis \((iii)\),

\[
\int_{\mathbb{R}^d}(\nabla f(x) \cdot v) \, dV[e_s \sharp \eta](x,v) \leq \|\nabla f\|_\infty \int_{\mathbb{R}^d} |v| \, dV[e_s \sharp \eta](x,v) < +\infty
\]

for a.e. \(s \in [0,T]\). Thus, we also have that \(s \mapsto \int_{\mathbb{R}^d}(\nabla f(x) \cdot v) \, dV[e_s \sharp \eta](x,v)\) belongs to \(L^1([0,T])\).

Secondly, the map \(t \mapsto \int_{\mathbb{R}^d} f \, d\mu_s\) is absolutely continuous. Indeed, for \(0 \leq s < t \leq T\) we have

\[
\left| \int_{\mathbb{R}^d} f \, d\mu_s - \int_{\mathbb{R}^d} f \, d\mu_t \right| \leq \int_s^t \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} |(\nabla f(\gamma(\tau)), \dot{\gamma}(\tau))| \, d\eta(x, \gamma) \, d\tau
\]

\[
\leq \int_s^t \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} |\nabla f(\gamma(\tau))| \cdot |\dot{\gamma}(\tau)| \, d\eta \, d\tau
\]

\[
\leq \|\nabla f\|_\infty \int_s^t \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} |\dot{\gamma}(\tau)| \, d\eta \, d\tau < +\infty,
\]

thanks to hypothesis \((iii)\).

Lastly, for a.e. \(t \in [0,T]\),

\[
\frac{d}{dt} \int_{\mathbb{R}^d} f(x) \, d\mu_t(x) = \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) \, d\eta(x, \gamma)
\]

\[
= \int_{\mathbb{R}^d} \nabla f(y) \int_{e_t^{-1}(y)} \dot{\gamma}(t) \, d\eta_{t,y}(\gamma) \, d\mu_t(y)
\]

\[
= \int_{\mathbb{R}^d} \nabla f(y) \int_{\pi_t^{-1}(y)} v \, d\nu_y[\mu_t](v) \, d\mu_t(y)
\]

\[
= \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_t](x,v),
\]

where we used hypothesis \((ii)\).
Remark 3.2. We observe that the second request in item (iii) can be satisfied assuming hypothesis \((H1)\) for the PVF \(V\) together with the hypothesis

\[
\int_0^T \int_{\mathbb{R}^d \times \Gamma([0,T])} \sup |\gamma(t)| \, d\eta(x,\gamma) \, dt < +\infty.
\]

Let us now pass to the other implication. In the second part of the statement that we are going to see, we want to prove the existence of a probabilistic representation \(\eta\) starting from a solution \(\mu\) of the MDE system \((2.2)\) with given PVF \(V\). In this general framework, this can be easily provided thanks to \([1, \text{Theorem 8.3.1}]\) and then applying the Superposition Principle in \([1, \text{Theorem 8.2.1}]\), as described below.

Theorem 3.3 (Superposition Principle for MDEs - Part II). Let \(T > 0\), \(V : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)\) be a PVF, \(\mu_0 \in \mathcal{P}(\mathbb{R}^d)\), \(p > 1\). Let \(\mu = \{\mu_t\}_{t \in [0,T]} \subseteq \mathcal{P}_p(\mathbb{R}^d)\) be an absolutely continuous solution of the MDE system \((2.2)\). Then there exists a probability measure \(\eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma([0,T]))\) such that

(i) \(\eta\) is concentrated on pairs \((x,\gamma)\) such that \(\gamma \in AC([0,T];\mathbb{R}^d)\) is a solution of the ODE \(\dot{\gamma}(t) = w_t(\gamma(t))\) for a.e. \(t \in (0,T)\), \(\gamma(0) = x\), where \(w : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d\), \((t,y) \mapsto w_t(y) = \int_{\pi_1^{-1}(y)} v \, d\nu_y[\mu_t](v)\) for a.e. \(t\) and \(\mu_t\)-a.e. \(y\);

(ii) \(\mu_t = e_t \# \eta\) for any \(t \in [0,T]\).

Straightforwardly, if \(\tilde{\eta} \in \mathcal{P}(\mathbb{R}^d \times \Gamma([0,T]))\) realizes (i) and (ii), then for a.e. \(t \in [0,T]\) and for \(\mu_t\)-a.e. \(y \in \mathbb{R}^d\) we have

\[
\int_{\pi_1^{-1}(y)} \gamma(t) \, d\hat{\eta}_{t,y}(\gamma) = \int_{\pi_1^{-1}(y)} v \, d\nu_y[\mu_t](v),
\]

where \(\hat{\eta}_{t,y}\) is the disintegration of \(\tilde{\eta}\) w.r.t. \(e_t\), and \(\nu_y[\mu_t]\) is the disintegration of \(V[\mu_t]\) w.r.t. the projection on the first component \(\pi_1\).

Proof. Let us take any \(f \in C^\infty_c(\mathbb{R}^d)\). Let \(\mu = \{\mu_t\}_t\) be as in the statement. Then, by Definition \(\ref{def:3.8}\)

\[
(3.1) \quad \frac{d}{dt} \int_{\mathbb{R}^d} f(x) \, d\mu_t(x) = \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_t](x,v)
\]

for a.e. \(t \in [0,T]\).

By \([1, \text{Theorem 8.3.1}]\), there exists \(w : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d\), \((t,y) \mapsto w_t(y)\) such that \(w \in L^1([0,T];L^p_{\mu_t}(\mathbb{R}^d))\) and

\[
(3.2) \quad \frac{d}{dt} \int_{\mathbb{R}^d} f(y) \, d\mu_t(y) = \int_{\mathbb{R}^d} \nabla f(y) \cdot w_t(y) \, d\mu_t(y).
\]

Hence, by \([1, \text{Theorem 8.2.1}]\) there exists a probabilistic representation \(\eta\) concentrated on pairs \((x,\gamma)\) such that \(\gamma\) is an integral solution of the characteristic equation \(\dot{\gamma}(t) = w_t(\gamma(t))\), \(\gamma(0) = x\), and \(\mu_t = e_t \# \eta\). By \((3.1)\) and \((3.2)\) we get

\[
\int_{\mathbb{R}^d} \nabla f(y) \int_{\pi_1^{-1}(y)} v \, d\nu_y[\mu_t](v) \, d\mu_t(y) = \int_{\mathbb{R}^d} \nabla f(y) \cdot w_t(y) \, d\mu_t(y),
\]

concluding the proof of items \((i - ii)\). Let now \(\tilde{\eta}\) be as in the statement. Last property is straightforward since, by \((ii)\) we have

\[
\int_{\mathbb{R}^d} \nabla f(y) \cdot w_t(y) \, d\mu_t(y) = \int_{\mathbb{R}^d} \nabla f(y) \int_{\pi_1^{-1}(y)} \dot{\gamma}(t) \, d\hat{\eta}_{t,y}(\gamma) \, d\mu_t(y),
\]

where \(\hat{\eta}_{t,y}\) the disintegration of \(\tilde{\eta}\) w.r.t. \(e_t\), univocally identified for \(\mu_t\)-a.e. \(y\). \(\square\)
We now complete the analysis concerning the connection between Superposition Principle and MDEs by giving an example of an explicit and consistent construction for a probabilistic representation of the LAS scheme.

Let \( \mu \subseteq \mathcal{P}(\mathbb{R}^d) \) be a solution of the MDE system (2.2) obtained as uniform-in-time limit of LASs \( \{\mu^N\}_{N \in \mathbb{N}} \). We now construct a probabilistic representation for \( \mu \) that is concentrated on uniform limits of the trajectories \( \gamma_{i,j} : [0, T] \to \mathbb{R}^d \), \( \gamma_{i,j}(t) = x_i + tv_j \), where the LASs \( \mu^N \) are concentrated.

Let us start by fixing some notation. Given \( I_1, I_2 \subset \mathbb{R} \) nonempty and compact intervals, with max \( I_1 = \min I_2 \), we define

1. the set of compatible trajectories
   \[
   \mathcal{D}_{I_1, I_2} := \left\{ (x_1, \gamma_1, x_2, \gamma_2) \in \mathbb{R}^d \times I_1 \times \mathbb{R}^d \times I_2 : x_i = \gamma_i(\min I_i), \quad \gamma_i(\max I_i) = \gamma_2(\min I_2) \right\};
   \]
2. the concatenation \( \gamma_1 \star \gamma_2 \in \Gamma_{I_1 \cup I_2} \) of curves \( \gamma_1 \in \Gamma_{I_1}, \gamma_2 \in \Gamma_{I_2} \), with \( \gamma_1(\max I_1) = \gamma_2(\min I_2) \), is a map from \( I_1 \cup I_2 \) to \( \mathbb{R}^d \) defined as follows
   \[
   \gamma_1 \star \gamma_2(t) = \begin{cases} 
   \gamma_1(t), & \text{if } t \in I_1, \\
   \gamma_2(t), & \text{if } t \in I_2;
   \end{cases}
   \]
3. the merge map \( M_{I_1, I_2} : \mathcal{D}_{I_1, I_2} \to \mathbb{R}^d \times I_1 \cup I_2 ; \quad (x_1, \gamma_1, x_2, \gamma_2) \mapsto (x_1, \gamma_1 \star \gamma_2) \). We will omit the subscripts \( I_1, I_2 \) when clear.

**Definition 3.4.** Let \( T > 0 \), and \( \mu^N = \{\mu^N_t\}_{t \in [0, T]} \subseteq \mathcal{P}_c(\mathbb{R}^d) \) be the LAS defined in Definition 2.11. Denote with \( I^N_{a,b} := [a\Delta_N, b\Delta_N] \), for \( a, b \in \mathbb{N}, a \leq b \). We define

1. the following measure in \( \mathcal{P}(\mathbb{R}^d \times \Gamma_{t+1}^\ell) \)
   \[
   \eta^N_{t+1} := \sum_{i,j} m^N_{i,j}(V[\mu^N_{t+1}]) \delta_{(x_i, \gamma^N_{t+1})} = \int_{\mathbb{R}^d} \delta(x, V[\mu^N_{t+1}](x, v)) d\mathcal{L}^N_{t+1} V[\mu^N_{t+1}](x, v),
   \]
   where \( \gamma^N_{t+1} = \gamma^N_{x_i,v_j} \) are the solutions of the LAS characteristic system defined on \( I^N_{t+1} \), thus \( \gamma^N_{i,j}(t) = x_i + v_j(t - \ell \Delta_N) \), for \( t \in I^N_{t+1} \);
2. \( \eta^N_{h+1} := \mu^N_{h\Delta_N} \otimes M_{I^N_{0,h+1}}(\eta^N_{h} \otimes \eta^N_{h+1}) \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{h(h+1)\Delta_N}) \), defined by recursion for \( h = 1, \ldots, N-1 \), where \( \eta^N_{h} \) and \( \eta^N_{h+1} \) are respectively the disintegrations of \( \eta^N_{0} \) and \( \eta^N_{h+1} \) w.r.t. \( e_{h\Delta_N} \);
3. \( \eta^N := \eta^N_{N} \in \mathcal{P}(\mathbb{R}^d \times [0,T]) \).

**Proposition 3.5.** Let \( V \) be a PVF satisfying (H1) and (H2), \( \mu_0 \in \mathcal{P}_c(\mathbb{R}^d) \), and \( \mu \) be a solution of the MDE system (2.2) obtained as uniform-in-time limit of LASs \( \{\mu^N\}_{N} \) for the Wasserstein metric. Let \( \eta^N \) be as in Definition 3.4. Then

1. \( \eta^N \) is a probabilistic representation for the LAS \( \mu^N \), i.e. \( \mu^N_t = e_t \eta^N \) for all \( t \in [0, T] \);
2. \( \eta^N \rightharpoonup^\star \eta \) up to subsequences, and \( \eta \) is a probabilistic representation for \( \mu \).

**Proof.** First we prove that \( \mu^N_t = e_t \eta_{I^N_{t+1}}^N \) for all \( t = 0, \ldots, N-1 \), and \( t \in I^N_{t+1} \). By Definition 2.11
\[
\mu^N_t := \sum_{i,j} m^N_{i,j}(V[\mu^N_{t+1}]) \delta_{x_i + (t - \ell \Delta_N)v_j}. 
\]
Thus, for all \( \varphi \in C_b^0(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} \varphi(x) d\mu^N_t = \sum_{i,j} \varphi(x_i + (t - \ell \Delta_N)v_j) \cdot m^N_{i,j}(V[\mu^N_{t+1}]) = \int_{\mathbb{R}^d \times \Gamma_{t+1}} \varphi(\gamma(t)) d\eta^N_{t+1},
\]
Let us now conclude the proof by showing that \( e_t^N \eta_{t_0}^N = \mu_t^N \) for all \( h = 1, \ldots, N - 1, \) and \( t \in I_{h+1}^N \). Indeed, for all \( \varphi \in C^0_b(\mathbb{R}^d) \),
\[
\int \varphi(\gamma(t)) \, d\eta_{t_0}^N(x, \gamma) = \int \varphi(\gamma(t)) \, d \left[ \mu^N_h \otimes M^N_\nu(\eta_{t_0}^N \otimes \eta_{t-1}^N) \right](x, \gamma)
= \int \varphi(\gamma(t)) \, d \left[ M^N_\nu(\eta_{t_0}^N \otimes \eta_{t-1}^N) \right](x, \gamma) \, d\mu^N_h(y)
= \int \varphi(\gamma_1 \ast \gamma_2(t)) \, d(\eta_{t_0}^N \otimes \eta_{t-1}^N)(x, \gamma_1 \ast \gamma_2) \, d\mu^N_h(y)
= \int \varphi(\gamma_2(t)) \, d\eta_{t_0}^N(x, \gamma_2) \, d\mu^N_h(y)
= \int \varphi(\gamma_2(t)) \, d\eta_{t_0}^N(x, \gamma_2) = \int \varphi(x) \, d\mu^N_t(x),
\]
where in the fourth equality we assumed, without loss of generality, \( t \in I_{h+1}^N \), otherwise we iterate the same procedure for \( \eta_{t_0}^N \). In the last two passages we used what proved before, i.e. \( e_t^N \eta_{t_0}^N = \mu_t^N \).

**Proof of (2).** First, let us prove that the family \( \{ \eta^N \}_N \) is tight, thus there exists \( \eta \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{[0,T]}) \) such that \( \eta^N \rightharpoonup \eta \), up to a non-relabeled subsequence. We proceed in the same way as in [1] Theorem 8.2.1. Indeed, we use [1] Lemma 5.2.2 with
\[
\begin{align*}
\mathbf{r}^1 : (x, \gamma) \mapsto x & \in \mathbb{R}^d, \quad \mathbf{r}^2 : (x, \gamma) \mapsto \gamma - x \in \Gamma_{[0,T]},
\end{align*}
\]
and we notice that \( \mathbf{r}^1 \times \mathbf{r}^2 \) is proper, and by the previous item we have that the family \( \{ \mathbf{r}^1 \mathbf{r}^2 \eta^N \}_N \) is given by the first marginals \( \{ \mu^N_0 \} \) which is tight, furthermore \( \beta^N := \mathbf{r}^2 \mathbf{r}^1 \eta^N \in \mathcal{P}(\Gamma_{[0,T]}) \) satisfy for \( p > 1 \),
\[
\int_{\Gamma_{[0,T]}} \int_0^T |\dot{\gamma}(t)|^p \, dt \, d\beta^N(\gamma) = \int_{\mathbb{R}^d \times \Gamma_{[0,T]}} \int_0^T \frac{d}{ds}(\gamma - x)(T) |\dot{\gamma}(t)|^p \, dt \, d\eta^N(x, \gamma)
\leq T \cdot \max_j |e_j|^p < +\infty.
\]
Hence, the tightness of the family \( \beta^N \) follows by [1] Remark 5.1.5, since for \( p > 1 \) the functional \( \gamma \mapsto \int_0^T |\dot{\gamma}|^p \, dt \) (set to \(+\infty\) if \( \gamma \notin AC^p([0,T];\mathbb{R}^d) \)) or if \( \gamma(0) \neq 0 \) has compact sublevels in \( \Gamma_{[0,T]} \). Thus, the family \( \{ \eta^N \}_N \) is tight.

By weak*-convergence of \( \mu_t^N \) to \( \mu_t \) for all \( t \in [0,T] \) and of \( \eta^N \) to some \( \eta \) up to subsequences, and since from item [1], \( \mu_t^N = e_t^N \eta^N \), then we immediately have that \( \eta \) is a probabilistic representation for \( \mu \), i.e. \( e_t^N \eta = \mu_t \). By construction (see [1] Theorem 5.1.8), we have that \( \eta \) is supported on the pairs \( (x, \gamma) \in \mathbb{R}^d \times \Gamma_{[0,T]} \), where \( \gamma(0) = x \) and \( \gamma \) are the uniform limits of the LASs characteristics where \( \eta^N \) is supported. \( \square \)

4. A semi-discrete Lagrangian scheme for MDE

In this section, we first define a semi-discrete in time Lagrangian scheme for (2.2) and compare it to the LAS scheme in Definition 2.11 showing that they converge to the same limit. Fixed \( T > 0 \), for \( N \in \mathbb{N} \) we set \( \Delta t^N = T/N \) and we define a partition of \([0,T]\) by
\[
\Delta t^N = \{ t_k^N = k \Delta t^N, k = 0, \ldots, N \}.
\]
To simplify the notation, we omit the index \( N \) in \( t_k^N \) and in \( \Delta t^N \) if there is no ambiguity. Given \( \mu_0 \in \mathcal{P}_c(\mathbb{R}^d) \) and a PVF \( V \), we set
\[
\begin{align*}
\mu_0^N := \mu_0; \quad \mu_{t_k}^N := \int_{t_k}^{t_{k+1}} \delta_{x+\nu \Delta t^N} \, d\nu_x[\mu_t^N](v) \, d\mu_t^N(x) = \mu_{t_k}^N \oplus \Delta t \cdot \nu_x[\mu_t^N],
\end{align*}
\]
We now prove that

\[ t \text{ subsequence. Given } \bar{t} \text{ for } x \in \text{supp}(\mu_0), \text{ we denote } \bar{\rho}(\bar{t}) = \int_{\mathbb{R}^d} \delta_x \bar{\mu}_{\bar{t}}(v) d\nu_x(\bar{\mu}_{\bar{t}})(v) d\mu_0(x) \]

(4.3)

where \( x_N^{k+1} := x + \sum_{j=0}^{k} v_j^N \Delta t^N, \text{ for } x \in \text{supp}(\mu_0) \text{ and } v_j^N \in \text{supp}(\nu_x[\bar{\mu}_{\bar{t}}]) \). We extend \( \bar{\mu}^N \) to the interval \([0, T]\) by setting for \( t \in (t_k, t_{k+1}] \)

\[ \bar{\mu}^N_t := \int_{\mathbb{R}^d} \delta_{x + v(t - t_k)} d\nu_x[\bar{\mu}^N_{t_k}](v) d\bar{\mu}^N_{t_k}(x), \]

and we denote \( \bar{\mu}^N = \{\bar{\mu}^N_t\}_{t \in [0, T]} \). Due to the assumptions on \( V \), we have that \( \bar{\mu}^N_t \in \mathcal{P}(\mathbb{R}^d) \) for all \( t \in [0, T] \). Moreover, since for some \( R > 0, \) supp \( (\mu_0) \subset B(0, R) \), then by \( (H1) \) and arguing as in [9, Lemma 3.3] it follows that

\[ \text{supp}(\bar{\mu}^N_t) \subset B(0, e^{CT}(R + 1)), \quad \forall t \in [0, T]. \]  

(4.4)

**Theorem 4.1.** Let \( \mu_0 \in \mathcal{P}_c(\mathbb{R}^d) \). Then, the scheme (4.2) converges, up to a subsequence, to a solution of (2.2).

Moreover, assume that there exists a sequence \( \{N_k\}_{k \in \mathbb{N}} \) such that both the scheme (4.2) and the LAS schemes in Definition 2.11 converge. Then, they converge to the same solution of (2.2).

**Proof.** We first show that sequence (4.2) is equi-Lipschitz continuous in time. For \( f \in \text{Lip}_1(\mathbb{R}^d) \), by \( (H1) \) and (4.4) we have

\[ |\langle \bar{\mu}^N_{t_{k+1}} - \bar{\mu}^N_{t_k}, f \rangle | = \int_{\mathbb{R}^d} |f(x + v\Delta t) - f(x)| d\nu_x[\bar{\mu}^N_{t_k}](v) d\bar{\mu}^N_{t_k}(x) \leq \Delta t \int_{\mathbb{R}^d} |v| d\nu_x[\bar{\mu}^N_{t_k}](v) d\bar{\mu}^N_{t_k}(x) \leq \Delta t C(1 + e^{CT}(R + 1)). \]

It follows that

\[ W^{1,\infty}(\bar{\mu}^N_t, \bar{\mu}^N_s) \leq K|t - s|, \quad \forall t, s \in [0, T], \]

for \( K = K(R, T) > 0 \). By Ascoli-Arzelà Theorem, the sequence \( \{\bar{\mu}^N\}_{N \in \mathbb{N}} \) admits at least a subsequence, still denoted by \( \bar{\mu}^N \), which converges to a measure map \( \bar{\mu} \in \text{Lip}_K([0, T], \mathcal{P}_c(\mathbb{R}^d)) \) such that \( \bar{\mu}_{t=0} = \mu_0 \).

We now prove that \( \bar{\mu} \) is a solution of (2.2). For simplicity we index with \( N \) the converging subsequence. Given \( t \in (t_k^N, t_{k+1}^N) \) and \( f \in C^\infty(\mathbb{R}^d) \cap \text{Lip}_1(\mathbb{R}^d) \) such that \( ||f||_{C^2(\mathbb{R}^d)} \leq 1 \), we have

\[ \langle \bar{\mu}_t - \mu_0, f \rangle - \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot v dV[\bar{\mu}_s](x, v) ds = \langle \bar{\mu}_t - \bar{\mu}_{t_k}, f \rangle + \langle \bar{\mu}_{t_k} - \mu_0, f \rangle - \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot v dV[\bar{\mu}_s](x, v) ds \]

(4.5)

\[ = \sum_{i=0}^{k-1} \left[ \langle \bar{\mu}_{t_{i+1}}^N - \bar{\mu}_{t_i}^N, f \rangle - \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} \nabla f(x) \cdot v dV[\bar{\mu}_s](x, v) ds \right] + \int_{t_k}^{t} \int_{\mathbb{R}^d} \nabla f(x) \cdot v dV[\bar{\mu}_s](x, v) ds \]
Recalling that $\bar{\mu}^N_{t_{i+1}} - \bar{\mu}^N_{t_i}, f) = \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds$,

\[
\langle \bar{\mu}^N_{t_{i+1}} - \bar{\mu}^N_{t_i}, f \rangle - \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds = \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} f(x + v \Delta t) - f(x) \, dV[\bar{\mu}^N_{t_i}](x,v) - \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds
\]

By the Kantorovich-Rubinstein duality, (H3) and the triangular inequality, we get

\[
I_{1,i} \leq \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} |\nabla f(x + (s - t_i)v) - \nabla f(x)| \cdot |v| \, dV[\bar{\mu}_s](x,v) \, ds \leq \left\| f \right\|_{C^2(\mathbb{R}^d)} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} |s - t_i||v|^2 \, dV[\bar{\mu}_s](x,v) \, ds \leq \left\| f \right\|_{C^2(\mathbb{R}^d)} C^2(1 + e^{CT}(R + 1))^2 \Delta t^2.
\]

By the Kantorovich-Rubinstein duality, (H3) and the triangular inequality, we get

\[
I_{2,i} \leq \int_{t_i}^{t_{i+1}} W^{\mathbb{R}^d}(V[\bar{\mu}^N_{t_i}], V[\bar{\mu}_s]) \, ds \leq \int_{t_i}^{t_{i+1}} L \cdot W^{\mathbb{R}^d}(\bar{\mu}^N_{t_i}, \bar{\mu}_s) \, ds \leq \int_{t_i}^{t_{i+1}} L(K(s - t_i) + W^{\mathbb{R}^d}(\bar{\mu}^N_s, \bar{\mu}_s)) \, ds \leq LK \left( \Delta t^2 + \int_{t_i}^{t_{i+1}} W^{\mathbb{R}^d}(\bar{\mu}^N_s, \bar{\mu}_s) \, ds \right).
\]

Replacing the previous estimates in (4.5), we get

\[
\langle \bar{\mu}_t - \mu_0, f \rangle - \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds \leq \sum_{i=0}^{K-1} (I_{1,i} + I_{2,i}) + \langle \bar{\mu}^N_t - \bar{\mu}^N_t, f \rangle - \int_{t_k}^t \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds \leq \langle \bar{\mu}_t - \mu_0, f \rangle + LK \int_0^t W^{\mathbb{R}^d}(\bar{\mu}^N_s, \bar{\mu}_s) \, ds + 2K' T \Delta t,
\]

where $K' = \max\{LK, C^2(1 + e^{CT}(R + 1))^2\}$. Passing to the limit for $N \to \infty$ in the previous inequality and recalling that $W^{\mathbb{R}^d}(\bar{\mu}^N_t, \bar{\mu}_t) \to 0$, we finally get that

\[
\langle \bar{\mu}_t - \mu_0, f \rangle - \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\bar{\mu}_s](x,v) \, ds = 0
\]

for any $f$ and therefore $\mu$ is a solution of (2.2).

Let us now prove the second part of the theorem. Let $\{\mu^N\}$ be a convergent (sub-) sequence generated by the LAS scheme in Definition 2.11 and $\{\bar{\mu}^N\}$ be a convergent one.
generated by the scheme (4.2). Let us denote by \( \mu, \bar{\mu} \) the corresponding limits. Then

\[
W^d(\mu_t, \bar{\mu}_t) \leq W^d(\mu_t, \mu^N_t) + W^d(\bar{\mu}_t, \mu^N_t) + W^d(\mu^N_t, \bar{\mu}^N_t).
\]

Since the first two terms on the right-hand side of the last inequality converge to 0 for \( N \to +\infty \), we have to study only the convergence of the last term. Let \( f \in \text{Lip}_1(\mathbb{R}^d) \), \( t \in (t_k, t_{k+1}] \), then

\[
\langle \mu^N_t - \bar{\mu}^N_t, f \rangle = \int_{\mathbb{R}^d} f(x + (t - t_k)v) d(V[\mu^N_t] - V[\bar{\mu}^N_t])
\]

(4.6)

Notice that this computation holds thanks to the common time-grid shared by the two schemes. For the first term, we have

\[
\int_{\mathbb{R}^d} f(x + (t - t_k)v) d(V[\mu^N_t] - V[\bar{\mu}^N_t]) = \int_{\mathbb{R}^d} f(x + w) d(V^{\Delta t^N}[\mu^N_t] - V^{\Delta t^N}[\bar{\mu}^N_t]),
\]

where we have denoted \( dV^{\Delta t^N}[\eta] = d((t - t_k) \cdot \nu_x[\eta]) d\eta \), referring to the notation in (2.1). Then, we can observe that the map \( \psi : (x, w) \to x + w \) belongs to \( \text{Lip}_1(\mathbb{T}^d, \mathbb{R}^d) \). Since \( f \in \text{Lip}_1(\mathbb{R}^d, \mathbb{R}) \), we have \( f \circ \psi \in \text{Lip}_1(\mathbb{T}^d, \mathbb{R}) \). Then, from the previous inequality and the Kantorovich-Rubinstein duality it follows that

\[
\int_{\mathbb{T}^d} f(x + (t - t_k)v) d(V[\mu^N_t] - V[\bar{\mu}^N_t]) \leq W^{Td}(V^{\Delta t^N}[\mu^N_t], V^{\Delta t^N}[\bar{\mu}^N_t])
\]

\[
\leq \Delta t^N W(V[\mu^N_t], V[\bar{\mu}^N_t]) + W^d(\mu^N_t, \bar{\mu}^N_t) \leq (L \Delta t^N + 1) W^d(\mu^N_t, \bar{\mu}^N_t),
\]

where the last inequality is a consequence of (H3). For the second term in (4.6), by the same argument and (2.3), we found it is bounded by \( \frac{1}{N^2} \). Then

\[
\langle \mu^N_t - \bar{\mu}^N_t, f \rangle \leq \frac{1}{N^2} + (1 + L \Delta t^N) W^d(\mu^N_t, \bar{\mu}^N_t)
\]

\[
\leq (1 + L \Delta t^N)^{k+1} o\left( \frac{1}{N} \right) + \sum_{l=0}^{k} \frac{(1 + L \Delta t^N)^l}{N^2}
\]

\[
\leq e^\frac{LT(k+1)}{N} \cdot o\left( \frac{1}{N} \right) + \frac{e^\frac{LT(k+1)}{N}}{NLT} - 1
\]

and therefore the two schemes converge to the same limit, up to subsequences. \( \square \)

Before giving a further consideration coming as a consequence of the previous theorem, we recall the following result proved in [9, Theorem 5.2].

**Theorem 4.2.** Let \( V \) be a PVF satisfying (H1) and (H3). Assume that, for every \( \mu_0 \) obtained as convex combination of Dirac deltas, the sequence of LASs converges to a unique limit. Then there exists a unique Lipschitz semigroup whose trajectories are limits of LASs.

Then, as a corollary of Theorem 4.1, we get the following.

**Corollary 4.3.** Under the assumptions of Theorem 4.2, there exists a unique Lipschitz semigroup generated by the semi-discrete Lagrangian scheme (4.2) and it coincides with that generated by LASs.
5. A Mean Velocity Scheme for MDE

In this section we provide another approximation scheme for the problem \((2.2)\). As the Lagrangian scheme in Section 4, also this scheme is semi-discrete in time but, due to a different choice of the velocity field, it may converge to a different solution of the MDE (see also Remark 2.12).

We define \(\Delta t^N\) and \(t_k^N\) as in Section 4. Given \(\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)\) and a PVF \(V\) satisfying (H1) – (H3), the new approximation scheme is given iteratively by

\[
\begin{cases}
\hat{\mu}_{t_{0}}^{N}=\mu_{0}; \\
\bar{v}_{t_{j}}(x):=\int_{\mathbb{R}^{d}}vd\nu_{x}[\hat{\mu}_{t_{j}}^{N}](v), \\
\hat{\mu}_{t_{j}+1}^{N}=\hat{\mu}_{t_{j}}^{N}+\Delta t^{N} \cdot \hat{\delta}_{\bar{v}_{t_{j}}(x)},
\end{cases}
\]

for \(j \in \{0, \ldots, N-1\}\). The scheme \((5.1)\) transports the measure distribution \(\hat{\mu}_{t_{j}}^{N}\) by a velocity field obtained as the barycenter of the velocity measure \(\nu_{x}\) at \(\hat{\mu}_{t_{j}}^{N}\).

**Theorem 5.1.** Let \(\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)\). Then, the scheme \((5.1)\) converges, up to a subsequence, to a solution of \((2.2)\).

**Proof.** We first prove that, given \(R > 0\) such that \(\text{supp}(\mu_0) \subset B(0,R)\), there exists \(K = K(R,T) > 0\) such that for \(N\) sufficiently large

\[
\text{sup}(\hat{\mu}_{t_{j}}^{N}) \subset B(0,K), \quad \forall j \in \{0, \ldots, N\}.
\]

Indeed

\[
\text{sup}(\hat{\mu}_{t_{j}}^{N}) \subset B(0,K), \quad \forall j \in \{0, \ldots, N\}.
\]

By construction,

\[
\langle \hat{\mu}_{t_{j}+1}^{N} - \hat{\mu}_{t_{j}}^{N}, f \rangle = \langle \hat{\mu}_{t_{j}+1}^{N} - \hat{\mu}_{t_{j}}^{N}, f \rangle + \sum_{i=1}^{N} \langle \hat{\mu}_{t_{j}+i}^{N} - \hat{\mu}_{t_{j}+i-1}^{N}, f \rangle + \langle \hat{\mu}_{t_{j}}^{N} - \hat{\mu}_{s}^{N}, f \rangle.
\]

By construction,

\[
\text{sup}(\hat{\mu}_{t_{j}+1}^{N}) \subset B(0,K), \quad \forall j \in \{0, \ldots, N\}.
\]

Hence by (H1) and the equi-boundedness of supports, it follows

\[
\langle \hat{\mu}_{t_{j}+1}^{N} - \hat{\mu}_{t_{j}}^{N}, f \rangle \leq \Delta t^{N} \int_{\mathbb{R}^{d}} C(1 + |x|) d\hat{\mu}_{t_{j}}^{N}(x) \leq \Delta t^{N} C(1 + K).
\]

Analogously \(\langle \hat{\mu}_{t_{j}}^{N} - \hat{\mu}_{s}^{N}, f \rangle \leq |t - s| C(1 + K)\) and \(\langle \hat{\mu}_{t_{j}}^{N} - \hat{\mu}_{s}^{N}, f \rangle \leq |t - s| C(1 + K)\).

Hence, taking the supremum for \(f \in \text{Lip}_1(\mathbb{R}^d)\), we have

\[
W^{\mathbb{R}^{d}}(\hat{\mu}_{t_{j}}^{N}, \hat{\mu}_{s}^{N}) \leq |t - s| C(1 + K).
\]
Since the support of $\hat{\mu}^N_{t_j}$ is bounded, uniformly in $N$, it immediately follows that the sequence $\{\hat{\mu}^N_{t_j}\}_{N \in \mathbb{N}}$ have bounded first and second momentum and therefore there exists $\mu \in C([0, T]; \mathcal{P}_c(\mathbb{R}^d))$ such that, up to a subsequence,

$$\sup_{t \in [0, T]} W^{d}_{\text{rd}}(\hat{\mu}^N_{t}, \mu_t) \to 0, \text{ for } N \to +\infty. \quad (5.3)$$

We now prove that $\mu$ is a solution of (2.2). Given $f \in C_c^\infty(\mathbb{R}^d)$ such that $\|f\|_{C^2(\mathbb{R}^d)} \leq 1$, we rewrite

$$\langle \hat{\mu}^N_{t_k} - \hat{\mu}^N_{t_0}, f \rangle = \sum_{j=1}^{k} \langle \hat{\mu}^N_{t_j} - \hat{\mu}^N_{t_{j-1}}, f \rangle,$$

$$= \sum_{j=1}^{k} \int_{\mathbb{R}^d} (f(x + \Delta t \hat{v}_{t_{j-1}}(x)) - f(x)) \, d\hat{\mu}^N_{t_{j-1}} \quad (5.4)$$

$$= \sum_{j=1}^{k} \int_{\mathbb{R}^d} \int_{t_{j-1}}^{t_j} \frac{d}{ds} f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) ds \, d\hat{\mu}^N_{t_{j-1}}(x)$$

$$= \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}^d} \hat{v}_{t_{j-1}}(x) \cdot \nabla f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) d\hat{\mu}^N_{t_{j-1}}(x) ds.$$

We estimate

$$\left| \int_{t_{j-1}}^{t_j} \left( \int_{\mathbb{R}^d} \hat{v}_{t_{j-1}}(x) \cdot \nabla f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) \, d\hat{\mu}^N_{t_{j-1}}(x) - \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x) dV[\mu_s] \right) \, ds \right|$$

$$= \left| \int_{t_{j-1}}^{t_j} \left( \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) \, dV[\hat{\mu}^N_{t_{j-1}}] - \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x) \, dV[\mu_s] \right) \, ds \right|$$

$$\leq \int_{t_{j-1}}^{t_j} \left| \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) \, dV[\hat{\mu}^N_{t_{j-1}}] - \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x) \, dV[\mu_s] \right| \, ds$$

$$+ \int_{t_{j-1}}^{t_j} \left| \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x) \, d\left(V[\hat{\mu}^N_{t_{j-1}}] - V[\mu_s]\right) \right| \, ds$$

$$\leq \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}^d} \|D^2 f\|_{C^2} |\hat{v}_{t_{j-1}}(x)|^2 \, d\hat{\mu}^N_{t_{j-1}}(x) + \int_{t_{j-1}}^{t_j} W^{T^{\mathbb{R}^d}}(V[\hat{\mu}^N_{t_{j-1}}], V[\mu_s]) \, ds$$

$$\leq \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}^d} \|v\|^2 \, dV[\hat{\mu}^N_{t_{j-1}}](x) \, ds + \Delta t \sup_{t \in [t_{j-1}, t_j]} W^{T^{\mathbb{R}^d}}(V[\hat{\mu}^N_{t_{j-1}}], V[\mu_t]).$$

Therefore, by (H1) and [5.2], we have

$$\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left( \int_{\mathbb{R}^d} \hat{v}_{t_{j-1}}(x) \cdot \nabla f(x + (s - t_{j-1}) \hat{v}_{t_{j-1}}(x)) \, d\hat{\mu}^N_{t_{j-1}}(x) \, ds - \int_{0}^{T} \int_{T^{\mathbb{R}^d}} v \cdot \nabla f(x) \, dV[\mu_s](x, v) \, ds \right)$$

$$\leq \sum_{j=1}^{k} \left( \int_{t_{j-1}}^{t_j} \|v\|^2 \, dV[\hat{\mu}^N_{t_{j-1}}](x) \, ds + \Delta t \sup_{t \in [t_{j-1}, t_j]} W^{T^{\mathbb{R}^d}}(V[\hat{\mu}^N_{t_{j-1}}], V[\mu_t]) \right)$$

$$\leq \sum_{j=1}^{k} \frac{\Delta t^2}{2} C' + T \sup_{1 \leq j \leq k} \sup_{t \in [t_{j-1}, t_j]} W^{T^{\mathbb{R}^d}}(V[\hat{\mu}^N_{t_{j-1}}], V[\mu_t])$$

$$\leq \Delta t C' + T \sup_{1 \leq j \leq k} \sup_{t \in [t_{j-1}, t_j]} W^{T^{\mathbb{R}^d}}(V[\hat{\mu}^N_{t_{j-1}}], V[\mu_t]).$$
By (H2), $V$ is uniformly continuous on $B(0, K)$, hence we conclude that the right-hand side vanishes as $N \to +\infty$, thanks to (5.2) and (5.3). If $t_k^N \to t$ for $N \to \infty$, since the term on the left side in (5.4) converges to $(\mu_t - \mu_0, f)$ by construction, the previous estimate implies that $\mu$ is a weak solution to (2.2).

\[ \n \]

\text{Remark 5.2.} For sake of completeness, similarly to Definition 3.4 we can provide an explicit formula to construct a probabilistic representation for the scheme introduced in Section 4 and for the mean velocity one, as follows. Let $P^\alpha := [\alpha \Delta t^N, h \Delta t^N]$, for $a, b \in \mathbb{N}$, $a \leq b$, with $\Delta t^N$ and $t_k$ as in Section 4. Denote respectively with $\tilde{\mu}^N = \{\tilde{\mu}^N_t\}_{t \in [0, T]}$ and $\hat{\mu}^N = \{\hat{\mu}_k^N\}_{k \in \{0, \ldots, N-1\}}$ the schemes defined in (4.2) and (5.1). For $k = 0, \ldots, N - 1$, we define

(A) $\tilde{\eta}_{t_k}^{N+1} := \tilde{\mu}_{t_k}^{N+1} \oplus (-t_k) \cdot \nu_{x}^{\tilde{\mu}_{t_k}^{N+1}} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{t_{k+1}})$, i.e.,

$$\tilde{\eta}_{t_k}^{N+1} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \delta_{(x, x+v(-t_k))} d\nu_{x}^{\tilde{\mu}_{t_k}^{N+1}}(v) d\tilde{\mu}_{t_k}^{N+1}(x) = \int_{\mathbb{R}^d} \delta_{x}(x) dV[\tilde{\mu}_{t_k}^{N+1}](x, v);$$

(B) $\hat{\eta}_{t_k}^{N+1} := \hat{\mu}_{t_k}^{N} \oplus (-t_k) \cdot \hat{\nu}_{x} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{t_{k+1}})$, i.e., $\hat{\eta}_{t_k}^{N+1} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \delta_{(x, x+v(-t_k))} d\hat{\mu}_{t_k}^{N}(x).$

Now, we can build $\tilde{\eta}^N$ and $\hat{\eta}^N$ by applying items (2-3) of Definition 3.4 and replacing item (1) respectively with (A) and (B).

Following the same line as in Proposition 3.5, we can prove that an analogous result holds also for the semi-discrete in time Lagrangian scheme (or the mean velocity one) by replacing the LAS scheme $\mu^N$ with the scheme $\tilde{\mu}^N$ (or $\hat{\mu}^N$) and using the representation $\tilde{\eta}^N$ (or $\hat{\eta}^N$) just provided.

\[ \n \]

\text{6. Examples.}

In this section we present some examples aimed at clarifying the work of the various proposed schemes, in particular we show that the LAS scheme and the mean velocity one in (5.1) may converge to different solutions. For simplicity of computations and without loss of generality, let us set $\Delta_N = \Delta t^N = 1/N$ as a time-step size for all the schemes.

\text{Example 6.1 (Splitting particle).} For every $\mu \in \mathcal{P}_c(\mathbb{R})$ define:

$$B(\mu) = \sup \left\{ x : \mu([- \infty, x]) \leq \frac{1}{2} \right\}.$$

Set $\eta(\mu) = \mu(( - \infty, B(\mu))] - \frac{1}{2}$ so $\mu(\{B(\mu)\}) = \eta(\mu) + \frac{1}{2} - \mu([- \infty, B(\mu)])$. We define $V[\mu] = \mu \otimes \nu_{x}[\mu]$, with

$$\nu_{x}[\mu] = \left\{ \begin{array}{ll} \delta_{-1} & \text{if } x < B(\mu) \\ \frac{1}{\mu(\{B(\mu)\})} (\eta \delta_{1} + \left( \frac{1}{2} - \mu([- \infty, B(\mu)]) \right) \delta_{-1}) & \text{if } x = B(\mu), \mu(\{B(\mu)\}) > 0. \end{array} \right.$$

The solution obtained as limit of LASs, satisfies:

$$\mu_t(A) = \mu_0((- \infty, \mu_0(t) - t]) + t + \mu_0((A \cap B(\mu_0) + t, +\infty] - t)$$

$$+ \frac{1}{\mu_0(\{B(\mu_0)\})} \left( \eta \delta_{B(\mu_0) - t}(A) + \left( \frac{1}{2} - \mu_0([- \infty, B(\mu_0)]) \right) \delta_{B(\mu_0)} - t \right).$$

In particular:

i) The solution with $\mu_0 = \delta_{x_0}$ is given by $\mu_t = \frac{1}{2} \delta_{x_0 + t} + \frac{1}{2} \delta_{x_0 - t}$, as illustrated in Figure 1.

ii) The solution to with $\mu_0 = \chi_{[a,b]}(\lambda$ where $\chi$ is the characteristic function and $\lambda$ is the renormalized Lebesgue measure) is given by $\mu_t = \chi_{[a-t, b+t]}(\lambda + \chi_{[a+b, t, b+t]}(\lambda.$
The same behavior is valid for the scheme (4.2) (see Theorem 4.1). Moreover, it can be verified that the stationary solution \( \{ \delta_{x_0} \} \) is the unique limit of the mean velocity scheme (5.1) when \( \mu_0 = \delta_{x_0} \), while this scheme has the same behavior of the LASs one when \( \mu_0 = \chi_{[a,b]} \). Hence the limit solution depends, in general, on the given approximation scheme.

**Example 6.2.** Let \( d = 1 \), \( V : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{T}) \) a PVF defined by \( V[\mu] := \mu \otimes \omega \), where \( \omega := \frac{1}{2} (\delta_0 + \delta_1) \), and let \( \mu_0 = \delta_0 \). Then, both the LAS and (4.2) schemes give a binomial distribution at every time (see Figure 2) while, as in the previous example, the mean velocity scheme remains stationary. However by the Law of Large Numbers, as \( N \rightarrow +\infty \) all the three schemes univocally converge to the constant solution \( \mu = \{ \delta_0 \}_{t \in [0,T]} \). We refer to [9, Proposition 7.1] for a formal proof.

**Example 6.3.** Let \( d = 1 \), \( V : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{T}) \) a PVF defined by \( V[\mu] := \mu \otimes \omega \), where \( \omega := \frac{1}{2} \chi_{[-1,1]} L = \frac{1}{2} L_{[-1,1]} \), with \( L \) the Lebesgue measure. Let \( \mu_0 = \delta_0 \). Considering the LAS scheme, as illustrated in Figure 3, we notice that for \( N = 1 \), the points \( v_j \) in the discretized space of velocities such that \( m_{ij}^N(V[\mu]) \neq 0 \) are \( v_0 = -1 \) and \( v_1 = 0 \), with equal weight. For \( N = 2 \), we get \( v_0 = -1 \), \( v_1 = -1/2 \), \( v_2 = 0 \) and \( v_3 = 1/2 \), hence we start to give mass also to positive \( x \in \mathbb{R} \), thus obtaining \( \mu_{t=1/2}^2 = \frac{1}{4} \sum_{i=0}^{3} \delta_{-1/2+i/4} \) and
Figure 3. LAS scheme: for $N = 1$ (left) and $N=2$ (right).

$\mu_{t=1}^2 = \sum_{i=-2}^4 \frac{1}{16} (4 - |i - 1|) \delta(-1/2+i/4)$.

Coming to the semi-discrete Lagrangian scheme (4.2), at the first time-step we get the uniform distribution on $[-1,1]$, while afterwards we obtain a normal distribution on $[-t,t]$ (see Figure 4). Reasoning in the same way as in the previous example, by the Law of Large Numbers, the LAS scheme and so also the semidiscrete Lagrangian one converge to the constant solution as $N \to \infty$ (see [9, Proposition 7.1]). Trivially, the mean-velocity scheme shares the same behavior.

Figure 4. Semi-discrete Lagrangian scheme on the left. Mean-velocity scheme on the right.

Example 6.4. Let $d = 1$, $V : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(T\mathbb{R})$ a PVF defined by $V[\mu] := \mu \otimes \delta_{v(x)}$, where $v(x) := 2\sqrt{|x|}$, and $\mu_{t=0} = \delta_{-1}$. Recalling Definition 2.8 by the atomic nature of the PVF $V$ over the fibers $T_x\mathbb{R}$, we deduce that the set of solutions to the MDE coincides with the set of distributional solutions of the continuity equation driven by the vector field $v(\cdot)$. We can thus use the classical Superposition Principle [1, Theorem 8.2.1] to build the trajectories $\mu$ by considering the integral solutions of the underlying ODE $\dot{x}(t) = 2\sqrt{|x(t)|}$, with initial condition $x(0) = -1$. By classical theory we know that this system admits infinite solutions (called Peano’s brush), such as the trivial one $x_{\infty}(t) = -(t-1)^2$ for $t \leq 1$,
\[x_\infty(t) = 0 \text{ for } t \geq 1, \text{ but also the trajectories given by}
\]
\[
x(t) = \begin{cases} 
-(t - 1)^2, & \text{if } t \leq 1, \\
0, & \text{if } 1 \leq t \leq a, \\
(t - a)^2, & \text{if } t \geq a,
\end{cases}
\]
as \(a\) varies in \([1, +\infty[\) (see Figure 5). In particular, among the infinite solutions of the MDE, we have \(\mu^1 = \{\mu^1_t\}_{t \in [0, T]}\), with \(\mu^1_t = \delta_{x_1}(t)\), and \(\mu^2 = \{\mu^2_t\}_{t \in [0, T]}\), with \(\mu^2_t = \delta_{x_\infty}(t)\).

Computing the LAS scheme for \(N = 1\) we get:

1. \(\mu_0 = \delta_{-1}\), hence \(v(-1) = 2\) which belongs to the velocity grid;
2. so we get \(\mu_{|t|=1}^{N=1} = \delta_{-1+2} = \delta_1\), hence \(v(1) = 2\) which belongs to the velocity grid;
3. so we get \(\mu_{|t|=2}^{N=1} = \delta_{1+2} = \delta_3\), hence \(v(3) = 2\sqrt{3}\) which does not belong to the velocity grid. Since \(3 < 2\sqrt{3} < 4\), then the point in the discretized space of velocities for \(N = 1\) such that \(m_{ij}^\nu(V[\mu]) \neq 0\) is \(v_j = 3\);
4. so we get \(\mu_{|t|=3}^{N=1} = \delta_{3+3} = \delta_6\), and so on.

For \(N = 2\) and \(N = 3\), by performing similar computations we obtain the trajectories as represented in Figure 6. We can show that the LAS scheme converges to \(\mu_2\), and thus so does the semidiscrete Lagrangian scheme, up to subsequences. Moreover we notice that, due to the atomic nature of \(V\) over the fibers \(T_x\mathbb{R}\), the mean velocity scheme (5.1) coincides with the semidiscrete Lagrangian one (5.2). Thus, all the three schemes converge, up to subsequences to the same solution \(\mu_2\). Finally we point out that the semidiscrete Lagrangian scheme corresponds to the Euler method for the underlying ODE. We also notice that in our case, for all \(N \in \mathbb{N}\) the grid intersects the critical point \((t, x) = (1, 0)\) where we loose local Lipschitzianity of the vector field. If we perform a perturbation of the grid, shifting it w.r.t. the critical point, then the schemes will converge to \(\mu_1\), up to subsequences.

The lack of uniqueness for the notion of weak solution given in Definition 2.8 and exploited in the examples is not surprising, as already observed in Remark 2.12. Indeed, if the mean velocity field is enough regular, the theory in [1] would grant us the uniqueness of a solution as push-forward of the initial condition. On the other side, if there exist points \(x \in \mathbb{R}^d\) where the PVF is not atomic over \(T_x\mathbb{R}^d\) then it is possible to produce different schemes which converge to different solutions.

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Figure 6. LAS scheme for $N=1, 2, 3$.

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