FACTORIZING DERIVATIVES OF FUNCTIONS
IN THE NEVANLINNA AND SMIRNOV CLASSES

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Abstract. We prove that, given a function $f$ in the Nevanlinna class $\mathcal{N}$ and a positive integer $n$, there exist $g \in \mathcal{N}$ and $h \in \text{BMOA}$ such that $f^{(n)} = gh^{(n)}$. We may choose $g$ to be zero-free, so it follows that the zero sets for the class $\mathcal{N}^{(n)} := \{ f^{(n)} : f \in \mathcal{N} \}$ are the same as those for BMOA$^{(n)}$. Furthermore, while the set of all products $gh^{(n)}$ (with $g$ and $h$ as above) is strictly larger than $\mathcal{N}^{(n)}$, we show that the gap is not too large, at least when $n = 1$. Precisely speaking, the class $\{ gh' : g \in \mathcal{N}, h \in \text{BMOA} \}$ turns out to be the smallest ideal space containing $\{ f' : f \in \mathcal{N} \}$, where “ideal” means invariant under multiplication by $H^\infty$ functions. Similar results are established for the Smirnov class $\mathcal{N}^+$. 

1. Introduction and results

Let $\mathcal{H}(\mathbb{D})$ stand for the set of holomorphic functions on the disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$. Given a class $X \subset \mathcal{H}(\mathbb{D})$ and an integer $n \in \mathbb{N} := \{1, 2, \ldots \}$, we write

$$X^{(n)} := \{ f^{(n)} : f \in X \},$$

where $f^{(n)}$ is the $n$th derivative of $f$. When $n = 1$, we also use the notation $X'$ instead of $X^{(1)}$. Further, we denote by $Z(X)$ the collection of zero sets for $X$; a (discrete) subset $E$ of $\mathbb{D}$ will thus belong to $Z(X)$ if and only if $E = \{ z \in \mathbb{D} : f(z) = 0 \}$ for some non-null function $f \in X$. Now, if $X$ and $Y$ are subclasses of $\mathcal{H}(\mathbb{D})$, we put

$$X \cdot Y := \{ fg : f \in X, g \in Y \}.$$

Finally, a vector space $X$ contained in $\mathcal{H}(\mathbb{D})$ is said to be ideal if

$$H^\infty \cdot X \subset X,$$

where, as usual, $H^\infty$ is the space of bounded holomorphic functions on $\mathbb{D}$.

Our starting point is a result of Cohn and Verbitsky [3] which asserts, or rather implies, that

$$(H^p)^{(n)} = H^p \cdot \text{BMOA}^{(n)}$$

whenever $0 < p < \infty$ and $n \in \mathbb{N}$. Here, we write $H^p$ for the classical (holomorphic) Hardy spaces on the disk, and BMOA for the “analytic subspace” of BMO $= \text{BMO}(\mathbb{T})$, the space of functions with bounded mean oscillation on the boundary $\mathbb{T}$ of $\mathbb{D}$.
circle $\mathbb{T} := \partial \mathbb{D}$. More precisely, BMOA can be defined as $H^1 \cap \text{BMO}$; as to the definitions of (and background information on) $H^p$ and BMO, the reader will find these standard matters in [5, Chapters II and VI].

For $n = 1$, identity (1.1) appeared in Cohn’s earlier paper [2]. On the other hand, [3] extends (1.1) to the case of a fractional derivative and still further; indeed, more general factorization theorems involving tent spaces – and Triebel spaces – are actually established there. It is also shown in [3] that, when factoring $f^{(n)}$ (for $f \in H^p$) in the sense of (1.1), one may choose the $H^p$ factor on the right to be an outer function. As a consequence, one sees that

$$Z\left((H^p)^{(n)}\right) = Z\left(\text{BMOA}^{(n)}\right).$$

In particular, for any fixed $n$, the zero sets for $(H^p)^{(n)}$ are the same for all $p \in (0, \infty)$. This last fact was contrasted in [3] with the Bergman space situation (where, by [7], different $A^p$ spaces have different zero sets). We wish to add, in this connection, that a similar Bergman-type phenomenon (different zero sets for different $p$’s) is also encountered in certain “small” $H^p$-related spaces; namely, it occurs [4] for the star-invariant subspaces $H^p \cap \theta \mathcal{H}_0^\infty$ associated with an inner function $\theta$.

Also related to (1.1) is Aleksandrov and Peller’s work [1] that dealt with the case of $p \in [1, \infty)$ and $n = 1$. There, for a given $f \in H^p$, a weak factorization $f' = \sum_{j=1}^4 g_j h'_j$ was constructed with suitable $g_j \in H^p$ and $h_j \in H^\infty$. Yet another weak factorization theorem from [1], which establishes a connection between BMOA' and $(H^\infty)'$, will be employed in Section 4 below.

The purpose of this paper is to find out whether – and/or to which extent – the (strong) factorization theorem (1.1) carries over to the Nevanlinna class $\mathcal{N}$, or the Smirnov class $\mathcal{N}^+$, in place of $H^p$.

Let us recall that $\mathcal{N}$ is defined as the set of functions $f \in \mathcal{H}(\mathbb{D})$ with

$$\sup_{0 < r < 1} \int_{\mathbb{T}} \log^+ |f(r\zeta)| |d\zeta| < \infty,$$

while $\mathcal{N}^+$ is formed by those $f \in \mathcal{N}$ which satisfy

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \log^+ |f(r\zeta)| |d\zeta| = \int_{\mathbb{T}} \log^+ |f(\zeta)| |d\zeta|.$$

Equivalently, the elements of $\mathcal{N}$ (resp., $\mathcal{N}^+$) are precisely the ratios $u/v$, with $u, v \in H^\infty$ and with $v$ nonvanishing (resp., outer) on $\mathbb{D}$; for this and other characterizations of the two classes, see [5, Chapter II].

As far as factorization theorems of the form (1.1) are concerned, we can hardly expect the behavior of $\mathcal{N}$ or $\mathcal{N}^+$ to mimic that of $H^p$ too closely. In fact, as we shall soon explain, it is the “easy” part of (1.1), i.e., the inclusion

$$\left((H^p)^{(n)}\right) \supset H^p \cdot \text{BMOA}^{(n)}$$

that admits no extension to the Nevanlinna or Smirnov setting. Meanwhile, we remark that (1.3) is indeed easy to deduce, at least for $p = 2$, from the (not so
easy, but readily available) descriptions of \((H^p)^{(n)}\) and \(\text{BMOA}^{(n)}\) as the appropriate Triebel spaces; see [11]. One of these tells us that, for \(\varphi \in \mathcal{H}(\mathbb{D})\),

\[
\varphi \in (H^p)^{(n)} \iff \int_{T} \left( \int_{0}^{1} |\varphi(r\zeta)|^2 (1-r)^{2n-1} dr \right)^{p/2} |d\zeta| < \infty
\]

for all \(n \in \mathbb{N}\) and \(0 < p < \infty\), a fact that has no counterpart for \(\mathcal{N}\) or \(\mathcal{N}^+\). The other, which involves a Carleson measure characterization of \(\text{BMOA}\), will be mentioned in Section 2 below.

Now, to see that the \(\mathcal{N}\) and \(\mathcal{N}^+\) versions of (1.3) actually break down, already for \(n = 1\), we recall Hayman’s and Yanagihara’s results from [6, 12] saying that neither \(\mathcal{N}\) nor \(\mathcal{N}^+\) is invariant with respect to integration. More explicitly, Hayman [6] gave an example of a function \(f \in \mathcal{N}\) whose antiderivative \(F(z) := \int_{0}^{z} f(\zeta) d\zeta\) is not in \(\mathcal{N}\); Yanagihara [12] then strengthened this by showing that \(F\) need not be in \(\mathcal{N}\) even for \(f \in \mathcal{N}^+\). Consequently, \(\mathcal{N}^+\) is not contained in \(\mathcal{N}'\), whence a fortiori

\[
(1.4) \quad \mathcal{N} \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \not\subset (\mathcal{N}^+)'.
\]

Since \(\mathcal{N} \cdot \text{BMOA}'\) (resp., \(\mathcal{N}^+ \cdot \text{BMOA}'\)) contains \(\mathcal{N}\) (resp., \(\mathcal{N}^+\)), we readily deduce from (1.4) that

\[
(1.5) \quad \mathcal{N} \cdot \text{BMOA}' \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \cdot \text{BMOA}' \not\subset (\mathcal{N}^+)'.
\]

A similar conclusion holds for higher order derivatives as well.

We prove, however, that the “difficult” part of (1.1), i.e., the inclusion

\[
(1.6) \quad (H^p)^{(n)} \subset H^p \cdot \text{BMOA}^{(n)}
\]

does remain valid with either \(\mathcal{N}\) or \(\mathcal{N}^+\) in place of \(H^p\).

**Theorem 1.1.** For each \(n \in \mathbb{N}\), we have

\[
\mathcal{N}^{(n)} \subset \mathcal{N} \cdot \text{BMOA}^{(n)} \quad \text{and} \quad (\mathcal{N}^+)^{(n)} \subset \mathcal{N}^+ \cdot \text{BMOA}^{(n)}.
\]

More precisely, given \(f \in \mathcal{N}\) (resp., \(f \in \mathcal{N}^+\), one can find a zero-free function \(g \in \mathcal{N}\) (resp., an outer function \(g \in \mathcal{N}^+\) and an \(h \in \text{BMOA}\) such that \(f^{(n)} = gh^{(n)}\).

It should be mentioned that our method also applies to the meromorphic Nevanlinna class \(\mathcal{N}_{mer}\), defined as the set of quotients \(u/v\), where \(u, v \in H^\infty\) and \(v\) is merely required to be non-null. Moreover, a glance at our proof of Theorem 1.1 will reveal that if the original function \(f\) is of the form \(F/I\), with \(F \in \mathcal{N}^+\) and \(I\) inner, then we may take \(g = G/I^{n+1}\), with \(G\) outer. And again, just as in the \(H^p\) setting, our factorization theorem yields information on the zero sets.

**Corollary 1.2.** We have

\[
\mathcal{Z} (\mathcal{N}^{(n)}) = \mathcal{Z} (\text{BMOA}^{(n)}), \quad n \in \mathbb{N}.
\]

Indeed, Theorem 1.1 shows that every zero set for \(\mathcal{N}^{(n)}\) is a zero set for \(\text{BMOA}^{(n)}\), while the converse is immediate from the fact that \(\text{BMOA} \subset \mathcal{N}\). Furthermore, since
$\mathcal{N}^+$ lies between BMOA and $\mathcal{N}$, as does every $H^p$ with $0 < p < \infty$, Corollary 1.2 obviously implies the identity

$$Z ((\mathcal{N}^+)^{(n)}) = Z (\text{BMOA}^{(n)})$$

and also (1.2).

Finally, restricting ourselves to the case $n = 1$, we wish to take a closer look at the inclusion

$$\mathcal{N}' \subset \mathcal{N} \cdot \text{BMOA}'$$

from Theorem 1.1 along with its $\mathcal{N}^+$ counterpart. We know from (1.5) that the inclusion is proper, and we now stress an important point of distinction between the two sides. Namely, the right-hand side, $\mathcal{N} \cdot \text{BMOA}'$, is ideal (i.e., invariant under multiplication by $H^\infty$ functions), whereas the left-hand side, $\mathcal{N}'$, is not. Moreover, the space $\mathcal{N}'$ is *highly nonideal* in the sense that even the identity function $z$ is not a multiplier thereof! (Otherwise, the formula

$$g = (zg)' - zg', \quad g \in \mathcal{N},$$

would imply that $\mathcal{N}$ is contained in $\mathcal{N}'$, which we know is false.) A similar remark applies to $(\mathcal{N}^+)'$.

Our last result states, then, that $\mathcal{N} \cdot \text{BMOA}'$ is actually the smallest ideal space containing $\mathcal{N}'$, and that the same is true in the $\mathcal{N}^+$ setting.

**Theorem 1.3.** (a) The class $\mathcal{N} \cdot \text{BMOA}'$ is the ideal hull of $\mathcal{N}'$. In other words, $\mathcal{N} \cdot \text{BMOA}'$ is an ideal vector space that contains $\mathcal{N}'$ and is contained in every ideal space $X$ with $\mathcal{N}' \subset X$.

(b) Similarly, $\mathcal{N}^+ \cdot \text{BMOA}'$ is the ideal hull of $(\mathcal{N}^+)'$.

Now let us turn to the proofs.

## 2. Preliminaries

A couple of lemmas will be needed.

**Lemma 2.1.** Let $k \geq 0$ and $l \geq 1$ be integers. If $\varphi \in \text{BMOA}^{(l)}$ and $\psi$ is a function in $\mathcal{H}(\mathbb{D})$ satisfying

$$(2.1) \quad \psi(z) = O((1 - |z|)^{-k}), \quad z \in \mathbb{D},$$

then $\varphi \psi \in \text{BMOA}^{(k+l)}$.

**Proof.** It is known (see, e.g., [8, 10, 11]) that a function $F \in \mathcal{H}(\mathbb{D})$ will be in $\text{BMOA}^{(n)}$, with $n \in \mathbb{N}$, if and only if the measure $|F(z)|^2(1 - |z|)^{2n-1}dx\,dy$ (where $z = x + iy$) is a Carleson measure. The required result follows from this immediately, since (2.1) yields

$$|\varphi(z)\psi(z)|^2(1 - |z|)^{2(k+l)-1} \leq \text{const} \cdot |\varphi(z)|^2(1 - |z|)^{2l-1}$$

for all $z \in \mathbb{D}$. \qed
When $k = 0$, the above lemma reduces to saying that
\begin{equation}
H^\infty \cdot \text{BMOA}^{(n)} \subset \text{BMOA}^{(n)}
\end{equation}
for all $n \in \mathbb{N}$; in other words, $\text{BMOA}^{(n)}$ is an ideal space. This in turn leads to the next observation.

**Lemma 2.2.** For each $n \in \mathbb{N}$, the sets $\mathcal{N} \cdot \text{BMOA}^{(n)}$ and $\mathcal{N}^+ \cdot \text{BMOA}^{(n)}$ are ideal vector spaces.

**Proof.** It is clear that the two sets are invariant under multiplication by $H^\infty$ functions, but maybe not quite obvious that they are vector spaces. It is the linearity property
\[ f_1, f_2 \in \mathcal{N} \cdot \text{BMOA}^{(n)} \Rightarrow f_1 + f_2 \in \mathcal{N} \cdot \text{BMOA}^{(n)} \]
(and a similar fact with $\mathcal{N}^+$ in place of $\mathcal{N}$) that should be verified. To this end, we write
\[ f_j = \frac{u_j}{v_j} \cdot w_j^{(n)} \quad (j = 1, 2), \]
where $u_j, v_j \in H^\infty$ and $w_j \in \text{BMOA}$, and where $v_j$ is zero-free (resp., outer if the $f_j$’s are from $\mathcal{N}^+ \cdot \text{BMOA}^{(n)}$). Note that
\[ f_1 + f_2 = \frac{1}{v_1 v_2} \cdot \left( u_1 v_2 w_1^{(n)} + u_2 v_1 w_2^{(n)} \right). \]
The two terms in brackets, and hence their sum, will be in $\text{BMOA}^{(n)}$ by virtue of (2.2), while the factor $1/(v_1 v_2)$ will be in $\mathcal{N}$ (resp., in $\mathcal{N}^+$). \qed

### 3. Proof of Theorem 1.1

We treat the case of $\mathcal{N}$ first. Take $f \in \mathcal{N}$ and write $f = u/v$, where $u, v \in H^\infty$ and $v$ has no zeros in $\mathbb{D}$. We have then
\begin{equation}
(3.1) \quad f^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}(1/v)^{(k)}. 
\end{equation}
For each $k \in \{0, \ldots, n\}$, Faà di Bruno’s formula (see [9, Chapter 3]) yields
\begin{equation}
(3.2) \quad \left( \frac{1}{v} \right)^{(k)} = \sum C(m_1, \ldots, m_k) v^{-m_1-\cdots-m_k-1} \prod_{j=1}^{k} (v^{(j)})^{m_j},
\end{equation}
where the sum is over the $k$-tuples $(m_1, \ldots, m_k)$ of nonnegative integers satisfying
\begin{equation}
(3.3) \quad m_1 + 2m_2 + \cdots + km_k = k
\end{equation}
and where
\[ C(m_1, \ldots, m_k) = \frac{(-1)^{m_1+\cdots+m_k}}{m_1! \cdots m_k!} \frac{(m_1 + \cdots + m_k)!}{1^{m_1} \cdots k^{m_k}}. \]
For any fixed multiindex $(m_1, \ldots, m_k)$ as above, we clearly have
\begin{equation}
(3.4) \quad v^{-m_1-\cdots-m_k-1} = v^{-n-1} \cdot v^{n-m_1-\cdots-m_k},
\end{equation}
the last factor on the right being bounded. Indeed,
\begin{equation}
(3.5) \quad v^{n-m_1-\cdots-m_k} \in H^\infty,
\end{equation}
since it follows from (3.3) that \( n - m_1 - \cdots - m_k \geq 0 \). We further observe that, for \( j \in \mathbb{N} \),
\begin{equation}
(3.6) \quad v^{(j)}(z) = O((1 - |z|)^{-j}), \quad z \in \mathbb{D}
\end{equation}
(because \( v \in H^\infty \)), and this implies together with (3.3) that
\begin{equation}
(3.7) \quad \prod_{j=1}^{k} [v^{(j)}(z)]^{m_j} = O((1 - |z|)^{-k}), \quad z \in \mathbb{D}.
\end{equation}

Combining (3.2) and (3.4), we see that the \( k \)th summand in (3.1) takes the form
\begin{equation}
(3.8) \quad w_k := \binom{n}{k} \sum C(m_1, \ldots, m_k) u^{(n-k)} v^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} (v^{(j)})^{m_j};
\end{equation}
the sum is understood as in (3.2). We want to show that \( w_k \in \text{BMOA}^{(n)} \), and our plan is to check the corresponding inclusion for each individual term in (3.8). Thus, we claim that the function
\begin{equation}
\Phi_{m_1, \ldots, m_k} := u^{(n-k)} v^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} (v^{(j)})^{m_j}
\end{equation}
satisfies
\begin{equation}
(3.9) \quad \Phi_{m_1, \ldots, m_k} \in \text{BMOA}^{(n)}
\end{equation}
whenever \( 0 \leq k \leq n \) and the \( m_j \)'s are related by (3.3).

First let us verify (3.9) in the case \( k \leq n - 1 \). To this end, we notice that
\( u^{(n-k)} \in (H^\infty)^{(n-k)} \subset \text{BMOA}^{(n-k)} \),
where \( n - k \geq 1 \), while
\begin{equation}
[v(z)]^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} [v^{(j)}(z)]^{m_j} = O((1 - |z|)^{-k}), \quad z \in \mathbb{D},
\end{equation}
by virtue of (3.5) and (3.7). The validity of (3.9) is then guaranteed by Lemma 2.1.

Now if \( k = n \), then the multiindices involved are of the form \((m_1, \ldots, m_n)\) with \( \sum_{j=1}^{n} jm_j = n \). For any such multiindex, at least one of the \( m_j \)'s (say, \( m_l \) with an \( l \in \{1, \ldots, n\} \)) must be nonzero, so that \( m_l \geq 1 \) and
\begin{equation}
(3.10) \quad l(m_l - 1) + \sum_{1 \leq j \leq n, j \neq l} jm_j = n - l.
\end{equation}
Consider the factorization
\begin{equation}
\Phi_{m_1, \ldots, m_n} = v^{(l)} \cdot \left\{ uv^{n-m_1-\cdots-m_n} (v^{(l)})^{m_l-1} \prod_{1 \leq j \leq n, j \neq l} (v^{(j)})^{m_j} \right\}.
\end{equation}
The first factor, $v^{(l)}$, is then in $(H^\infty)^{(l)}$ and hence in BMOA$^{(l)}$, while the second factor (the one in curly brackets) is $O((1 - |z|)^{-n+l})$. The latter estimate is due to (3.6) and (3.10), coupled with the fact that $u$ and $v$ are in $H^\infty$. Applying Lemma 2.1 to the current factorization, we arrive at (3.9), this time with $k = n$.

Now that (3.9) is known to be true, we infer that the functions $w_k$ from (3.8) are all in BMOA$^{(n)}$, whence obviously $\sum_{k=0}^n w_k \in$ BMOA$^{(n)}$. Recalling that $f^{(n)} = v^{-n-1} \sum_{k=0}^n w_k$, we finally conclude that $f^{(n)}$ can be written as $gh^{(n)}$, where $g := v^{-n-1} \in \mathcal{N}$ and $h$ is a BMOA function satisfying $h^{(n)} = \sum_{k=0}^n w_k$.

The case of $\mathcal{N}^+$ is similar. This time, $v$ is taken to be an outer function in $H^\infty$, so $g = v^{-n-1}$ will be an outer function in $\mathcal{N}^+$.

4. Proof of Theorem 1.3

We shall only prove (a), the proof of (b) being similar. We know from Lemma 2.2 that $\mathcal{N} \cdot$ BMOA$'$ is an ideal space. Furthermore, Theorem 1.1 tells us that $\mathcal{N} \cdot$ BMOA$'$ contains $\mathcal{N}'$. It remains to verify that, whenever $X$ is an ideal space with $\mathcal{N}' \subset X$, we necessarily have

$$\mathcal{N} \cdot (H^\infty)' \subset X.$$  

(4.1)

Take any $g \in \mathcal{N}$ and $h \in H^\infty$. Note that

$$gh' = (gh)' - g'h,$$  

(4.2)

where both terms on the right are in $X$. Indeed, $(gh)'$ is obviously in $\mathcal{N}'$ and hence in $X$, while the inclusion $g'h \in X$ is due to the facts that $g' \in \mathcal{N}' \subset X$ and $hX \subset X$ (recall that $X$ is ideal). It now follows from (4.2) that $gh' \in X$, and we have thereby checked that

$$\mathcal{N} \cdot (H^\infty)' \subset X.$$  

(4.3)

Finally, given $\eta \in$ BMOA, we invoke a result of Aleksandrov and Peller [1, Theorem 3.4] to find functions $\varphi_j, \psi_j \in H^\infty$ ($j = 1, 2$) such that $\eta' = \varphi_1 \psi_1' + \varphi_2 \psi_2'$. Letting $g \in \mathcal{N}$ as before, we get

$$g\eta' = g\varphi_1 \psi_1' + g\varphi_2 \psi_2'.$$  

(4.4)

Here, the two terms of the form $g\varphi_j \psi_j'$ are in $\mathcal{N} \cdot (H^\infty)'$, so we infer from (4.3) that they are also in $X$. The right-hand side of (4.4) is therefore in $X$, and so is the left-hand side, $g\eta'$. Thus we conclude that $g\eta' \in X$ for all $g \in \mathcal{N}$ and $\eta \in$ BMOA. This establishes (4.1) and completes the proof.

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