Off-shell supergravity-matter couplings in three dimensions

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Abstract

We develop the superspace geometry of $\mathcal{N}$-extended conformal supergravity in three space-time dimensions. General off-shell supergravity-matter couplings are constructed in the cases $\mathcal{N} \leq 4$. 

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# Contents

1 Introduction 2

2 Geometry of \( \mathcal{N} \)-extended conformal supergravity 4

2.1 The algebra of covariant derivatives 5
2.2 Super-Weyl transformations 8
2.3 Coupling to a vector multiplet 8

3 Matter couplings in \( \mathcal{N} = 1 \) supergravity 10

4 Matter couplings in \( \mathcal{N} = 2 \) supergravity 13

4.1 Complex basis for spinor covariant derivatives 13
4.2 Scalar and vector multiplets 16
4.3 Matter couplings 17
4.4 Conformal compensators 20

5 Matter couplings in \( \mathcal{N} = 3 \) supergravity 22

5.1 Elaborating on the \( \mathcal{N} = 3 \) superspace geometry 23
5.2 Covariant projective multiplets 25
5.3 Analytic projection operator 30
5.4 Vector multiplet prepotential 32
5.5 Supersymmetric action principle 34
5.6 Conformal compensators 35
5.7 Locally supersymmetric sigma-models 37

6 Matter couplings in \( \mathcal{N} = 4 \) supergravity 38

6.1 Elaborating on the \( \mathcal{N} = 4 \) superspace geometry 38
6.2 Covariant projective multiplets 41
6.3 Hybrid projective multiplets 45
1 Introduction

Supergravity in three space-time dimensions was introduced as early as 1977 \cite{1,2}. Its simplest version with two supercharges ($\mathcal{N} = 1$) and the corresponding matter couplings became a textbook subject by 1983 \cite{3}. In the mid-1980s, topologically massive $\mathcal{N}$-extended supergravity theories were constructed \cite{4,5,6,7,8} in which supersymmetric Lorentz Chern-Simons terms were interpreted as extended conformal supergravity. More recently, (gauged) nonlinear sigma-models with $\mathcal{N}$ local supersymmetries were constructed in the on-shell component approach \cite{9,10}. The on-shell approach was also used in \cite{11,12} to construct $\mathcal{N} = 6$ and $\mathcal{N} = 8$ conformal supergravities and their couplings to ABJM type theories and BLG M2-branes respectively. Surprisingly, to the best of our knowledge, no results have appeared on general off-shell supergravity-matter couplings in the interesting cases $\mathcal{N} = 3$ and $\mathcal{N} = 4$. It is clear that such results should be based on appropriate superspace techniques, and the latter have not yet been developed. The present paper is aimed at filling the existing gap.

Recently, there have appeared exciting results on massive 3D supergravity \cite{13,14,15} which is a supersymmetric extension of the so-called new massive 3D gravity \cite{16}. A unique feature of this approach is that the (super)gravity action is a parity-preserving higher-derivative variant of 3D (super)gravity which respects unitarity. So far only the $\mathcal{N} = 1$ massive supergravity version has been fully elaborated \cite{14}, and linearized results
are available, e.g., in the $\mathcal{N} = 4$ case [15]. To go beyond the linearized approximation, one option is to develop, as mentioned in [15], an $\mathcal{N}$-extended superconformal tensor calculus. We believe, however, that developing superspace techniques may lead to a more adequate setting, at least in the cases $\mathcal{N} \leq 4$.

As regards the cases $\mathcal{N} = 3$ and $\mathcal{N} = 4$, our approach is a natural generalization of the off-shell formulations for general supergravity-matter theories with eight supercharges in five [17, 18] and four [19, 20, 21] dimensions. These formulations build in part on the techniques from projective superspace which were originally developed for extended Poincaré supersymmetry in [23, 24, 25] (see also [26] for a review). The matter couplings in [17, 18, 19, 20, 21] are described in terms of the so-called covariant projective multiplets which are curved-superspace extensions of the superconformal projective multiplets introduced for the first time in [28, 29]. This is in accord with the general principle that matter couplings in Poincaré supergravity can equivalently be described as conformal supergravity coupled to superconformal matter [30, 31, 32]. In three dimensions, therefore, a first step toward developing superspace settings for $\mathcal{N} = 3, 4$ supergravity theories should consist in a construction of superconformal projective multiplets and their self-couplings. This has recently been achieved as part of more general results on off-shell 3D $\mathcal{N} \leq 4$ rigid superconformal sigma-models [33].

Projective superspace [23, 24, 25] is less well known than harmonic superspace [34, 35]. For any number of space-time dimensions in which they exist, $D \leq 6$, the projective and the harmonic superspace approaches use the same supermanifold. For instance, in the case of 4D $\mathcal{N} = 2$ supersymmetry, they make use of the isotwistor superspace $\mathbb{R}^{4|8} \times \mathbb{C}P^1$ introduced originally by Rosly [36]. The relationship between the rigid harmonic and projective superspace formulations is spelled out in [37] (see also [38] for a recent review). Essentially, they differ in (i) the structure of off-shell supermultiplets used; and (ii) the supersymmetric action principle chosen. This makes the two approaches rather complementary. As emphasized in [17, 18, 19, 21], the difference deepens in the context of supergravity. Projective superspace is suitable for developing covariant geometric formulations for supergravity-matter systems [17, 18, 19, 21], similar to the famous Wess-Zumino approach for 4D $\mathcal{N} = 1$ supergravity [39, 40]. Harmonic superspace offers prepotential formulations [41, 35], similar to the Ogievetsky-Sokatchev approach to 4D $\mathcal{N} = 1$ supergravity [42].

1 Similar ideas have been developed in two dimensions [22].

2 The term “projective superspace” was coined in 1990 [25]. The modern projective-superspace terminology was introduced in 1998 [27].

3 General superconformal couplings of projective multiplets has also been given in [28, 29].
In the case of 3D rigid supersymmetry, \( \mathcal{N} = 3 \) and \( \mathcal{N} = 4 \) harmonic superspaces were introduced by Zupnik [43, 44, 45, 46]. This approach was used, in particular, to describe ABJM models in \( \mathcal{N} = 3 \) harmonic superspace [47]. No harmonic superspace formulation for supergravity in three dimensions has yet been constructed. Three-dimensional \( \mathcal{N} = 3 \) and \( \mathcal{N} = 4 \) projective superspace approaches have recently been developed [33] to describe general superconformal field theories. It should be mentioned that the 3D \( \mathcal{N} = 4 \) projective superspace \( \mathbb{R}^{3|8} \times \mathbb{C}P^1 \) was introduced by Lindström and Roček in 1988 [24] as a direct generalization of their four-dimensional construction [23]. It follows from the analysis in [33] that two mirror copies of \( \mathbb{C}P^1 \) are required to provide a natural superspace setting for general off-shell \( \mathcal{N} = 4 \) supermultiplets.

In this paper, the 3D \( \mathcal{N} \leq 4 \) supergravity-matter couplings are formulated in terms of superspace and superfields. The issue of component reduction will be addressed in a separate publication.

This paper is organized as follows. In section 2 we develop the superspace geometry of \( \mathcal{N} \)-extended conformal supergravity in three space-time dimensions. Matter couplings in supergravity theories with \( \mathcal{N} = 1, 2, 3 \) and 4 are studied, on a case-by-case basis, in sections 3 to 6. Our final comments and conclusions are given in section 7. The main body of the paper is accompanied by two appendices. Our 3D notation and conventions are collected in Appendix A. Appendix B is devoted to the derivation of the left projection operator.

## 2 Geometry of \( \mathcal{N} \)-extended conformal supergravity

In this section we develop a formalism of differential geometry in a curved three-dimensional \( \mathcal{N} \)-extended superspace, which is locally parametrized by real bosonic \( (x^m) \) and real fermionic \( (\theta^I_\mu) \) coordinates

\[
z^M = (x^m, \theta^I_\mu), \quad m = 0, 1, 2, \quad \mu = 1, 2, \quad I = 1, \ldots, \mathcal{N},
\]

(2.1)

that is suitable to describe \( \mathcal{N} \)-extended conformal supergravity. A natural condition upon such a geometry is that it should reduce to that of \( \mathcal{N} \)-extended Minkowski superspace \( \mathbb{R}^{3|2\mathcal{N}} \) in a flat limit. We recall that the spinor covariant derivatives \( D^I_\alpha \) associated with Minkowski superspace satisfy the anti-commutation relations

\[
\{D^I_\alpha, D^J_\beta\} = 2i \delta^{IJ} (\gamma^c)_{\alpha\beta} \partial_c.
\]

(2.2)
An explicit realization of $D^I_\alpha$ is

$$D^I_\alpha = \frac{\partial}{\partial \theta^I_\alpha} + i (\sigma^b)_{\alpha\beta} \theta^\beta_I \partial_b . \quad (2.3)$$

As usual, there is no need to distinguish between upper and lower $SO(N)$ vector indices.

As compared to the 4D supersymmetry, the three-dimensional case has an important specific feature which is due to the fact that 3D spinors are real. This feature is the conjugations rule: given a superfield $F$ of Grassmann parity $\epsilon(F)$, it holds that

$$(D^I_\alpha F)^* = -(-)^{\epsilon(F)} D^I_\alpha \bar{F},$$
with $\bar{F} := (F)^*$ the complex conjugate of $F$.

### 2.1 The algebra of covariant derivatives

We choose the structure group to be $SL(2, \mathbb{R}) \times SO(N)$, and denote by $M_{ab} = -M_{ba}$ and $N_{IJ} = -N_{JI}$ the corresponding generators. The covariant derivatives have the form:

$$D_A \equiv (D_a, D^I_\alpha) = E_A + \Omega_A + \Phi_A . \quad (2.5)$$

Here $E_A = E_A^M(z) \partial_M$ is the supervielbein, with $\partial_M = \partial/\partial z^M$,

$$\Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = -\Omega_A^b M_b = \frac{1}{2} \Omega_A^{\beta\gamma} M_{\beta\gamma} , \quad M_{ab} = -M_{ba} , \quad M_{\alpha\beta} = M_{\beta\alpha} \quad (2.6)$$

is the Lorentz connection, and

$$\Phi_A = \frac{1}{2} \Phi_A^{KL} N_{KL} , \quad N_{KL} = -N_{LK} \quad (2.7)$$

is the $SO(N)$-connection. The Lorentz generators with two vector indices ($M_{ab}$), with one vector index ($M_a$) and with two spinor indices ($M_{\alpha\beta}$) are related to each other by the rules: $M_a = \frac{1}{2} \varepsilon_{abc} M^{bc}$ and $M_{\alpha\beta} = (\gamma^a)_{\alpha\beta} M_a$ (for more details see Appendix A). The generators of $SL(2, \mathbb{R}) \times SO(N)$ act on the covariant derivatives as follows:\footnote{The operation of (anti)symmetrization of $n$ indices is defined to involve a factor of $(n!)^{-1}$.}

$$[M_{ab}, D^I_\alpha] = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_\alpha^\beta D^I_\beta , \quad [M_a, D^I_\alpha] = -\frac{1}{2} (\gamma_a)_\alpha^\beta D^I_\beta ,$$

$$[M_{\alpha\beta}, D^I_\gamma] = \varepsilon_{\gamma(\alpha} D^I_{\beta)} , \quad [M_{ab}, D_c] = 2 \delta_{[b}^c \delta_{a]} D_{d]} , \quad [M_a, D_b] = \varepsilon_{abc} D^c ,$$

$$[N_{KL}, D^I_\alpha] = 2 \delta^K_{[I} D_{\alpha]} , \quad [N_{KL}, D_a] = 0 . \quad (2.8a)$$

$$[M_{ab}, D^I_\alpha] = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_\alpha^\beta D^I_\beta , \quad [M_a, D^I_\alpha] = -\frac{1}{2} (\gamma_a)_\alpha^\beta D^I_\beta ,$$

$$[M_{\alpha\beta}, D^I_\gamma] = \varepsilon_{\gamma(\alpha} D^I_{\beta)} , \quad [M_{ab}, D_c] = 2 \delta_{[b}^c \delta_{a]} D_{d]} , \quad [M_a, D_b] = \varepsilon_{abc} D^c ,$$

$$[N_{KL}, D^I_\alpha] = 2 \delta^K_{[I} D_{\alpha]} , \quad [N_{KL}, D_a] = 0 . \quad (2.8b)$$

$$[M_{ab}, D^I_\alpha] = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_\alpha^\beta D^I_\beta , \quad [M_a, D^I_\alpha] = -\frac{1}{2} (\gamma_a)_\alpha^\beta D^I_\beta ,$$

$$[M_{\alpha\beta}, D^I_\gamma] = \varepsilon_{\gamma(\alpha} D^I_{\beta)} , \quad [M_{ab}, D_c] = 2 \delta_{[b}^c \delta_{a]} D_{d]} , \quad [M_a, D_b] = \varepsilon_{abc} D^c ,$$

$$[N_{KL}, D^I_\alpha] = 2 \delta^K_{[I} D_{\alpha]} , \quad [N_{KL}, D_a] = 0 . \quad (2.8c)$$
The supergravity gauge group is generated by local transformations of the form
\[ \delta K \mathcal{D}_A = [K, \mathcal{D}_A], \quad K = K^C(z) \mathcal{D}_C + \frac{1}{2} K^{cd}(z) \mathcal{M}_{cd} + \frac{1}{2} K^{PQ}(z) \mathcal{N}_{PQ}, \] (2.9)
with all the gauge parameters obeying natural reality conditions but otherwise arbitrary. Given a tensor superfield \( T(z) \), it transforms as follows:
\[ \delta_K T = K T. \] (2.10)

The covariant derivatives satisfy the (anti)commutation relations
\[ [\mathcal{D}_A, \mathcal{D}_B] = T^{C}_{AB} \mathcal{D}_C + \frac{1}{2} R_{AB}^{ KL} \mathcal{N}_{KL} + \frac{1}{2} R_{AB}^{ cd} \mathcal{M}_{cd}, \] (2.11)
with \( T^{C}_{AB} \) the torsion, \( R_{AB}^{ cd} \) the Lorentz curvature and \( R_{AB}^{ KL} \) the SO(\( N \)) curvature. The torsion and the curvature are related to each other by the Bianchi identities:
\[ \sum_{[ABC]} [\mathcal{D}_A, [\mathcal{D}_B, \mathcal{D}_C]] = 0. \] (2.12)

To describe conformal supergravity, we impose conventional constraints on the torsion. They are:
\[ T^I_{\alpha}{}^J{}^c = 2i \delta^I_J (\gamma^c)_{\alpha\beta}, \quad \text{(dimension 0)} \] (2.13a)
\[ T^I_{\alpha}{}^J{}^K = 0, \quad T^I_{\alpha}{}^b{}^c = 0, \quad \text{(dimension 1/2)} \] (2.13b)
\[ T_{ab}{}^c = 0, \quad \varepsilon^\gamma T_{a\beta}{}^{[JK]} = 0. \quad \text{(dimension 1)} \] (2.13c)

We emphasize that for any \( N \) the torsion has no dimension-1/2 components (this differs from Howe’s formulation for 4D \( N \)-extended conformal supergravity [19]). The above constraints have been introduced in [18]. However, no explicit solution to the Bianchi identities has been given in [18]. Solutions to some of the Bianchi identities are implicit in the results of [18].

Under the conventional constraints (2.13a)–(2.13c), the solution to the Bianchi identities is given by the following algebra of covariant derivatives:
\[ \{ \mathcal{D}_\alpha^I, \mathcal{D}_\beta^J \} = 2i \delta^I_J (\gamma^c)_{\alpha\beta} \mathcal{D}_c - 2i \varepsilon_{\alpha\beta} C^{\delta I J} \mathcal{M}_{\gamma\delta} - 4i S_{IJ} M_{\alpha\beta} \\
+ \left( i \varepsilon_{\alpha\beta} X_{IJKL} + 4i \varepsilon_{\alpha\beta} S^K_{[I} \delta^J]_{\gamma\delta} + i C_{\alpha\beta}^{ KL} \delta^{IJ} - 4i C_{\alpha\beta}^{ K(I} \delta^{J)L} \right) N_{KL}, \] (2.14a)
\[ [\mathcal{D}_\alpha, \mathcal{D}_\gamma^K] = - \left( \varepsilon_{(\alpha} C_{\beta)\gamma}^{ KL} + \varepsilon_{\delta(\alpha} C_{\beta)\gamma}^{ KL} + 2 \varepsilon_{\gamma(\alpha} \varepsilon_{\beta)\delta} S^{KL} \right) \mathcal{D}_\delta^{KL} \\
+ \frac{1}{2} R_{\alpha\beta\gamma}^{ K \delta\rho} \mathcal{M}^{\delta\rho} + \frac{1}{2} R_{\alpha\beta\gamma}^{ K P Q} \mathcal{N}_{PQ}, \] (2.14b)

\(^5\)The results are presented to dimension-3/2 in the torsion and curvature. We plan to give a complete solution to the Bianchi identities elsewhere.
Here all the dimension-1 components are real and satisfy the symmetry properties
\[ X^{IJKL} = X^{[IJKL]} , \quad S^{I} = S^{(I)} , \quad C_{a}^{IJ} = C_{a}^{[IJ]} . \] (2.15)

The torsion superfield of $S^{I}$ can be decomposed into its trace and traceless parts as
\[ S^{I} = S^{\delta}_{\delta}^{I} + S^{I} , \quad S = \frac{1}{N} \delta_{I} S^{I} , \quad \delta_{I} S^{I} = 0 . \] (2.16)

The dimension-3/2 components of the torsion and the curvature are
\[ T^{\alpha \beta \gamma}_{\delta} = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_{\alpha \beta} \varepsilon_{\delta \gamma} T^{\gamma}_{\alpha \beta \delta} , \]
\[ R^{\alpha \beta \gamma}_{\delta \rho} = i C^{\alpha \beta \gamma}_{\delta \rho} - \frac{4i}{3} \varepsilon_{c(\alpha} (D^{K})_{\gamma)} - \frac{4(N - 1)i}{3N} \varepsilon_{\gamma(\alpha} S_{\beta)}^{K} , \]
\[ R^{\alpha \beta \gamma}_{\delta \rho} = 4 \varepsilon_{\gamma(\alpha} C^{\alpha \beta \gamma}_{\delta \rho} - \frac{16}{3} \varepsilon_{\gamma(\delta \varepsilon_{\rho})_{\beta}} - \frac{4}{3} \varepsilon_{\rho(\delta \varepsilon_{\beta})_{\gamma}} (D^{K})_{\gamma} S^{K} + \frac{16(N - 1)}{3N} \varepsilon_{\gamma(\delta \varepsilon_{\rho})_{\beta} S_{\gamma}^{K} } , \]
\[ R^{\alpha \beta \gamma}_{\delta \rho} = 2 \varepsilon_{\gamma(\alpha} \left( - 2 C^{\alpha \beta \gamma}_{\delta \rho} + 3 T^{\gamma}_{\beta} P^{Q} K + 8 (D_{\gamma})^{P} S^{Q} K + \frac{(5N - 8)}{N} S_{\beta}^{P} \delta^{Q} K \right) + C^{\alpha \beta \gamma}_{\delta \rho} K + 2 C^{\alpha \beta \gamma}_{\delta \rho} P^{Q} K . \] (2.17)

The superfields $C_{\alpha \beta \gamma}^{K P Q}, T^{\alpha \beta \gamma}_{\delta \rho} K, T^{\alpha \beta \gamma}_{\delta \rho} P Q, S_{\alpha}^{K}$ are defined through the differential constraints satisfied by the dimension-1 torsion and curvature superfields. At dimension-3/2 the Bianchi identities imply
\[ D_{\alpha}^{I} S^{JK} = 2 T^{I}_{\alpha}^{(JK)} + S^{(J} K^{)I} - \frac{1}{N} S_{\alpha}^{I} \delta^{JK} , \]
\[ D_{\alpha}^{I} C^{JK}_{\beta} = 3 C^{JK}_{\alpha} + 3 T^{I}_{\alpha}^{JK} + 4 (D^{I}_{\beta})^{K} \delta^{J} K^{I} + \frac{(N - 4)}{N} S^{I}_{\beta} \delta^{JK} \]
\[ + C^{JK}_{\alpha} - 2 S^{J} K^{I} , \]
\[ D_{\alpha}^{I} X^{JKLP} = X_{\alpha}^{IJKLP} - 4 C^{JKLP} S^{I} . \] (2.18)

The symmetry properties of the superfields $T^{I}_{\alpha}^{JK}, C^{JK}_{\alpha} I, T^{I}_{\alpha}^{JK}, X_{\alpha}^{IJKLP}$ are
\[ T^{I}_{\alpha}^{JK} = T^{I}_{\alpha}^{[JK]} , \quad \delta^{JK} T^{I}_{\alpha}^{JK} = T_{\alpha}^{[IJK]} = 0 , \]
\[ C^{JK}_{\alpha} I = C^{JK}_{\alpha} I , \quad C^{IJK}_{\alpha} I = C^{IJK}_{\alpha} I , \]
\[ C^{IJK}_{\alpha} = C_{\alpha}^{IJK} , \quad X_{\alpha}^{IJKLP} = X_{\alpha}^{[IJKLP]} . \] (2.19)

A remarkable result in superfield supergravity is Dragon’s theorem [50] which states that the curvature is completely determined by the torsion. More precisely, this result concerns the Lorentz curvature and it does not necessarily apply to the curvature associated with

\[ 7 \]
the $R$-symmetry subgroup of the structure group. This is exactly what happens in three dimensions for $N \geq 4$. The antisymmetric tensor $X^{IJKL}$ appears only in the $SO(N)$ curvature but not as a component of the torsion.

In this paper we will often use the well-known rule for integration by parts in super-space: given a vector superfield $V = V^A \mathcal{D}_A$, it holds that

$$\int d^3x \, d^{2N}\theta \, E \, (-1)^{\varepsilon_A} \left\{ \mathcal{D}_A V^A - (-1)^{\varepsilon_B} T_{AB} \, V^A \right\} = 0 \, , \quad E^{-1} = \text{Ber}(E_A) \, . \quad (2.20)$$

In particular, the fact that the torsion has no dimension-1/2 components implies the following useful result:

$$\int d^3x \, d^{2N}\theta \, E \, \mathcal{D}_\alpha^I V^I_\alpha = 0 \, . \quad (2.21)$$

### 2.2 Super-Weyl transformations

The constraints (2.13a)–(2.13c) can be shown to be invariant under arbitrary super-Weyl transformations of the form

$$\delta \sigma \mathcal{D}_a = \sigma \mathcal{D}_a + \frac{i}{8}(\gamma^a)^{\gamma\delta}(\mathcal{D}_\gamma^I \sigma) \mathcal{D}_\delta^I \sigma + \varepsilon_{abc}(\mathcal{D}_a^b \sigma) \mathcal{M}^c + \frac{i}{16}(\gamma^a)^{\gamma\delta}([\mathcal{D}_\gamma^I, \mathcal{D}_\delta^J] \sigma) \mathcal{N}^{KL} \, , \quad (2.22a)$$

where $\sigma$ is a real unconstrained superfield. This leads to

$$\delta \sigma S^{IJ} = \sigma S^{IJ} - \frac{i}{8}([\mathcal{D}_\gamma^I, \mathcal{D}_\delta^J] \sigma) \, , \quad (2.22b)$$

This invariance is essential for the geometry under consideration to describe conformal supergravity.

### 2.3 Coupling to a vector multiplet

We now couple the multiplet of conformal supergravity to an Abelian $\mathcal{N}$-extended vector multiplet $V = d z^M \mathcal{V}_M = E^A \mathcal{V}_A$, with $\mathcal{V}_A := E_A^M \mathcal{V}_M$. For this we modify the covariant derivatives as

$$\mathcal{D}_A \, \rightarrow \, \mathcal{D}_A := \mathcal{D}_A + V_A \mathcal{Z} \, , \quad [\mathcal{Z}, \mathcal{D}_A] = 0 \, , \quad (2.23)$$
with $V_A(z)$ the gauge connection associated with a generator $Z^6$. The gauge transformation of $V_A$ is

$$\delta V_A = -D_A \tau, \quad (2.24)$$

with $\tau(z)$ an arbitrary scalar superfield.

The algebra of covariant derivatives is

$$[D_A, D_B] = T_{AB}^C D_C + \frac{1}{2} R_{AB}^{cd} M_{cd} + \frac{1}{2} R_{AB}^{KL} N_{KL} + F_{AB} Z. \quad (2.25)$$

Here $F_{AB}$ is the gauge-invariant field strength, and the torsion and curvatures are the same as above. The field strength satisfies the Bianchi identities

$$\sum_{[ABC]} (D_A F_{BC} - T_{AB}^D F_{DC}) = 0. \quad (2.26)$$

To describe an irreducible vector multiplet, we have to impose covariant constraints on $F_{AB}$. Their structure is different for $\mathcal{N} = 1$ and for $\mathcal{N} > 1$.

In the $\mathcal{N} = 1$ case, one imposes the covariant constraint \[51\]

$$F_{\alpha\beta} = 0. \quad (2.27)$$

Then, from the Bianchi identities one gets

$$F_{a\beta} = -\frac{1}{2}(\gamma_a)_{\beta}^{\gamma} W_{\gamma}, \quad (2.28a)$$

$$F_{ab} = \frac{i}{4} \varepsilon_{abc}(\gamma^c)^{\gamma\delta} D_{\gamma} W_{\delta}, \quad (2.28b)$$

together with the dimension-2 differential constraint on the spinor field strength

$$D^{\alpha} W_{\alpha} = 0. \quad (2.29)$$

For $\mathcal{N} > 1$ one imposes the following dimension-1 covariant constraint \[52, 53, 43\]

$$F^{I}_{\alpha\beta} = 2i\varepsilon_{\alpha\beta} W^{IJ}, \quad W^{IJ} = -W^{JI} \quad (2.30)$$

which is a natural generalization of the 4D $\mathcal{N} > 1$ constraints \[54, 55\]. The Bianchi identities are solved by the following expressions for the field strengths

$$F_{aA} = -\frac{1}{(\mathcal{N}-1)} (\gamma_a)^{\beta} D_{\beta} W^{IJ}, \quad (2.31a)$$

$$F_{ab} = \frac{i}{4\mathcal{N}(\mathcal{N}-1)} \varepsilon_{abc}(\gamma^c)^{\rho\tau} [D^{K}_{\rho} \cdot D^{L}_{\tau}] W_{KL} + \frac{2}{\mathcal{N}} \varepsilon_{abc} C^{cKL} W_{KL}. \quad (2.31b)$$

\[6\]For $\mathcal{N} > 1$, one can interpret $Z$ as a central charge.
The case \( N = 2 \) is special in the sense that the field strength \( W^{IJ} \) and the torsion \( C^{cKL} \) are proportional to the antisymmetric tensor \( \varepsilon^{IJ} \) (normalized as \( \varepsilon^{12} = 1 \)),

\[
W^{IJ} = \varepsilon^{IJ} G , \quad C^{cKL} = \varepsilon^{KL} C^c .
\]  

(2.32)

As a result, the components of \( F_{AB} \) become

\[
F^{IJ}_{a\beta} = 2i\varepsilon_{a\beta} \varepsilon^{IJ} G ,
\]

(2.33a)

\[
F^{IJ}_{a\beta} = -\varepsilon^{JK}(\gamma_a)_{\beta} \gamma D_{\gamma K} G ,
\]

(2.33b)

\[
F_{ab} = \varepsilon_{abc} \left( \frac{1}{4} \varepsilon^{KL} D_{\gamma K} D_{\delta L} + 2 \varepsilon^{c} \right) G .
\]

(2.33c)

Further analysis of the Bianchi identities shows that \( G \) obeys the dimension-2 constraint

\[
\left( \varepsilon^{(I} D_{\gamma J)} D_{\gamma K} - 4i\varepsilon^{(I} S_{K J)} \right) G = 0 .
\]

(2.34)

Unlike eq. (2.34), in the case \( N > 2 \) the field strength \( W^{IJ} \) is constrained by the dimension-3/2 Bianchi identity

\[
D_I W^{JK} = D_{I} W^{JK} - \frac{1}{(N - 1)} \left( \delta^{IJ} D_{\gamma L} W_{KL} - \delta^{IK} D_{\gamma L} W_{JL} \right) .
\]

(2.35)

This constraint can be shown to define an on-shell multiplet for \( N > 4 \).

The concept of super-Weyl transformations introduced in subsection 2.2 can be extended to the gauge-covariant derivatives (2.23). The key observation is that the one-form \( V = dz^M V_M \) is invariant under the super-Weyl transformation, and this determines the super-Weyl transformation law of \( V_A \) defined by \( V = E^A V_A \). After that one can read of the transformation law of the field strength. In the case \( N = 1 \) one finds

\[
\delta_\sigma W_\alpha = \frac{3}{2} \sigma W_\alpha ,
\]

(2.36)

while for \( N > 1 \) the field strength \( W^{IJ} \) transforms as

\[
\delta_\sigma W^{IJ} = \sigma W^{IJ} .
\]

(2.37)

### 3 Matter couplings in \( N = 1 \) supergravity

The geometry of 3D \( N = 1 \) supergravity and its matter couplings have been studied in the literature [1, 2, 3, 6, 56, 53, 57]. Here the structure group coincides with the 3D
Lorentz group. The algebra of covariant derivatives becomes

$$\{D_\alpha, D_\beta\} = 2iD_{\alpha\beta} - 4iSM_{\alpha\beta}, \quad (3.1a)$$

$$[D_{\alpha\beta}, D_\gamma] = -2\varepsilon_{\gamma(\alpha}D_{\beta)} + 2\varepsilon_{\gamma(\alpha}C_{\beta)\rho\sigma}M^{\rho\sigma} + \frac{2}{3}((D_\gamma S)M_{\alpha\beta} - 4(D_{(\alpha}S)M_{\beta)\gamma)), \quad (3.1b)$$

$$[D_\alpha, D_\beta] = \frac{1}{2}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\left\{-iC_{\alpha\beta\gamma}D_\gamma - \frac{4i}{3}(D_\alpha S)D_\beta + iD_{(\alpha}C_{\beta)\gamma\delta}M^{\gamma\delta}
- \left(\frac{2i}{3}(D^2S) + 4S^2\right)M_{\alpha\beta}\right\}. \quad (3.1c)$$

The torsion and the curvature are expressed in terms of a dimension-1 scalar $S$ and a dimension-3/2 totally symmetric spinor $C_{\alpha\beta\gamma} = C^{(\alpha\beta\gamma)}$, in complete agreement with, e.g., [3]. They obey the constraint

$$D_\alpha C_{\beta\gamma\delta} = D_{(\alpha}C_{\beta)\gamma\delta} - i\varepsilon_{\alpha(\beta}D_{\gamma\delta)S}. \quad (3.2)$$

As is seen from (3.1a), the vector derivative can be re-defined by $D_{\alpha\beta} \rightarrow D_{\alpha\beta} - 2SM_{\alpha\beta}$ such that the relation (3.1a) takes the same form as in flat superspace [3].

The super-Weyl transformation of the covariant derivatives, given e.g. in [53, 58], is

$$\delta_\sigma D_\alpha = \frac{1}{2}\sigma D_\alpha + (\gamma_\alpha D^\alpha_S)M_{\alpha\beta}, \quad (3.3a)$$

$$\delta_\sigma D_\alpha = \sigma D_\alpha + \frac{i}{2}(\gamma_\alpha)\gamma_\delta(D_\gamma\sigma)D_\delta + \varepsilon_{abc}(D^b\sigma)M^c. \quad (3.3b)$$

The induced transformation of the torsion is:

$$\delta_\sigma S = \sigma S - \frac{i}{4}D_\gamma D_\gamma\sigma, \quad (3.4a)$$

$$\delta_\sigma C_{\alpha\beta\gamma} = \frac{3}{2}\sigma C_{\alpha\beta\gamma} + \frac{1}{2}D_{(\alpha\beta}D_{\gamma)}\sigma. \quad (3.4b)$$

With the technical tools presented, it is easy to derive a locally supersymmetric and super-Weyl invariant action principle. It is constructed in terms of a purely imaginary Lagrangian $L$ whose super-Weyl transformation is

$$\delta_\sigma L = 2\sigma L, \quad (3.5)$$

modulo total derivatives. The action is

$$S = \int d^3x d^2\theta E L, \quad E^{-1} = \text{Ber}(E_A^M), \quad (L)^* = -L. \quad (3.6)$$

It is super-Weyl invariant since $\delta_\sigma E = -2\sigma E$. 

11
Let us construct a nonlinear sigma-model coupled to $\mathcal{N} = 1$ conformal supergravity. Its dynamics will be described by real scalar superfields $\varphi^\mu$ taking values in a Riemannian manifold $\mathcal{M}$. Consider the kinetic term

$$L_0 = -\frac{1}{2} g_{\mu\nu}(\varphi)(D^\alpha \varphi^\mu)(D_\alpha \varphi^\nu), \quad (3.7)$$

where $g_{\mu\nu}(\varphi)$ is the metric on the target space. We are looking for a Lagrangian of the form $L = L_0 + \ldots$ that transforms homogeneously as $(3.5)$ modulo total derivatives.

Postulating the super-Weyl transformation of $\varphi^\mu$

$$\delta_\sigma \varphi^\mu = \frac{1}{2} \sigma \chi^\mu(\varphi), \quad (3.8)$$

for some vector field $\chi = \chi^\mu(\varphi)\partial_\mu$ on $\mathcal{M}$, we find

$$\delta_\sigma L_0 = -\frac{1}{2} g_{\mu\nu}(\varphi)(D^\alpha \varphi^\mu)(D_\alpha \varphi^\nu) \left( \nabla_\chi \chi^\nu(\varphi) + \delta_\chi^\nu \right) \sigma - \frac{1}{2} g_{\mu\nu}(\varphi)(D^\alpha \varphi^\mu)(D_\alpha \sigma). \quad (3.9)$$

In the case that $\sigma = \text{const}$, the action $S_0 = \int d^3x d\theta \theta E L_0$ is invariant only if

$$\nabla_\mu \chi^\nu = \delta_\mu^\nu \implies \chi^\mu(\varphi) = \partial_\mu f(\varphi), \quad f(\varphi) := \frac{1}{2} g_{\mu\nu}(\varphi)\chi^\mu(\varphi)\chi^\nu(\varphi). \quad (3.10)$$

We see that $\chi = \chi^\mu(\varphi)\partial_\mu$ is a homothetic conformal Killing vector field such that $\chi^\mu$ is the gradient of a function over the target space. Therefore, the sigma-model target space $\mathcal{M}$ is a Riemannian cone [59], as in the rigid superconformal case [33, 60]. Now, the variation $(3.9)$ becomes

$$\delta_\sigma L_0 = 2\sigma L_0 - \frac{1}{2}(D^\alpha f)(D_\alpha \sigma) = 2\sigma L_0 + \frac{1}{2} f(D^\alpha D_\alpha \sigma) - \frac{1}{2} D^\alpha(f D_\alpha \sigma). \quad (3.11)$$

It remains to recall the transformation law $(3.4a)$ as well as to notice that eq. $(3.10)$ implies the homogeneity condition

$$\chi^\mu \partial_\mu f = 2f. \quad (3.12)$$

We then observe that

$$\delta_\sigma \left( iS f(\varphi) \right) = 2\sigma \left( iS f(\varphi) \right) + \frac{1}{4}(D^\alpha D_\alpha \sigma) f(\varphi), \quad (3.13)$$

and therefore

$$L := -\frac{1}{2} g_{\mu\nu}(\varphi)(D^\alpha \varphi^\mu)(D_\alpha \varphi^\nu) - 2iS f(\varphi) \quad (3.14)$$

is the required Lagrangian.
The above Lagrangian can be modified by adding a potential term
\[ \mathcal{L} = -\frac{1}{2} \left( g_{\mu\nu}(\varphi)(\mathcal{D}^\alpha \varphi^\mu)(\mathcal{D}_\alpha \varphi^\nu) + 4iSf(\varphi) \right) + iV(\varphi) \, . \] (3.15)
For the action to be super-Weyl invariant, the potential should satisfy the homogeneity condition
\[ \chi^\mu(\varphi)V_\mu(\varphi) = 4V(\varphi) \, . \] (3.16)

In the rigid supersymmetric case, the Lagrangian (3.15) reduces to that corresponding to the general \( \mathcal{N} = 1 \) superconformal sigma-model [33].

In the case of a single scalar superfield \( \varphi \), the general form for (3.15) is
\[ \mathcal{L} = -\frac{1}{2} \left( (\mathcal{D}^\alpha \varphi)(\mathcal{D}_\alpha \varphi) + 4iS\varphi^2 \right) + \lambda i\varphi^4 \, , \quad \lambda = \text{const} \, . \] (3.17)
We can choose \( \varphi \) to be a superconformal compensator, if we think of Poincaré supergravity as conformal supergravity coupled to the compensator. Then we should use \( -\mathcal{L} \) as a supergravity Lagrangian, and interpret \( \lambda \) as a cosmological constant.

### 4 Matter couplings in \( \mathcal{N} = 2 \) supergravity

Three-dimensional \( \mathcal{N} = 2 \) supergravity and its matter couplings have not been studied as thoroughly as in the \( \mathcal{N} = 1 \) case. This is partly due to the fact that 3D \( \mathcal{N} = 2 \) supergravity can be obtained by dimensional reduction from that with \( \mathcal{N} = 1 \) supersymmetry in four dimensions, and therefore much is known about the component structure of 3D \( \mathcal{N} = 2 \) supergravity-matter systems. However, there are several reasons to achieve a better understanding of the superspace geometry of \( \mathcal{N} = 2 \) supergravity, for instance, in the context of massive 3D supergravity [13, 14, 15].

#### 4.1 Complex basis for spinor covariant derivatives

The \( R \)-symmetry subgroup of the \( \mathcal{N} = 2 \) superspace structure group is \( \text{SO}(2) \cong \text{U}(1) \). Instead of dealing with the anti-Hermitian generator \( \mathcal{N}_{KL} = -\mathcal{N}_{LK} \) of \( \text{SO}(2) \), as defined in subsection 2.1, it is convenient to introduce a scalar Hermitian generator \( \mathcal{J} \) defined by
\[ \mathcal{N}_{KL} = i\varepsilon_{KL}\mathcal{J} \, , \quad \mathcal{J} = -\frac{i}{2}\varepsilon^{PQ}\mathcal{N}_{PQ} \, , \] (4.1)
\footnote{The antisymmetric tensors \( \varepsilon^{IJ} = \varepsilon_{IJ} \) are normalized as \( \varepsilon^{12} = \varepsilon_{12} = 1 \). The normalization of \( \varepsilon_{IJ} \) is nonstandard as compared with the definitions given in Appendix A.}
which acts on the covariant derivatives as
\[
[J, D_\alpha^I] = -i \varepsilon^{IJ} D_{\alpha J} .
\] (4.2)

It is also convenient to switch to a complex basis for the spinor covariant derivatives, \(D_\alpha^I \rightarrow (D_\alpha, \bar{D}_\alpha)\), in which \(D_\alpha\) and \(\bar{D}_\alpha\) have definite U(1) charges. We define
\[
D_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^1 - i D_\alpha^2) , \quad \bar{D}_\alpha = -\frac{1}{\sqrt{2}}(D_\alpha^1 + i D_\alpha^2) ,
\] (4.3)
such that
\[
[J, D_\alpha] = D_\alpha , \quad [J, \bar{D}_\alpha] = -\bar{D}_\alpha .
\] (4.4)

The SO(2) connection and the corresponding curvature take the form
\[
\frac{1}{2} \Phi_A^{KL} N_{KL} = i \Phi_A J , \quad \frac{1}{2} R_{AB}^{KL} N_{KL} = i R_{AB} J .
\] (4.5)

Given a complex superfield \(F\) and its complex conjugate \(\bar{F} := (F)^*\), the following rule for complex conjugation holds
\[
(D_\alpha F)^* = (-1)^{\varepsilon(F)} \bar{D}_\alpha \bar{F} ,
\] (4.6)
which can be compared with \((2.4)\).

In the \((D, \bar{D})\) basis introduced, the supergravity algebra \((2.14a)\) and \((2.14b)\) takes the form
\[
\{D_\alpha, D_\beta\} = -4 R M_{\alpha \beta} , \quad \{D_\alpha, \bar{D}_\beta\} = 4 R M_{\alpha \beta} ,
\] (4.7a)
\[
\{D_\alpha, \bar{D}_\beta\} = -2i D_{\alpha \beta} - 2 C_{\alpha \beta} J - 4 i \varepsilon_{\alpha \beta} S J + 4 i S M_{\alpha \beta} - 2 \varepsilon_{\alpha \beta} C^{\gamma \delta} M_{\gamma \delta} ,
\] (4.7b)
\[
[D_{\alpha \beta}, D_\gamma] = -i \varepsilon_{\gamma (\alpha} C_{\beta) \delta} D^\delta - i C_{\gamma (\alpha} D_{\beta) J} - 2 \varepsilon_{\gamma (\alpha} S D_{\beta)} - 2i \varepsilon_{\gamma (\alpha} \bar{R} D_{\beta)} + 2 \varepsilon_{\gamma (\alpha} \bar{R} D_{\beta)}
+ 2 \varepsilon_{\gamma (\alpha} C_{\beta) \delta \rho} M^\delta \rho - \frac{4}{3} \left( 2 D_{\alpha \beta} S + i \bar{D}_{\alpha \beta} R \right) M_{\gamma \delta} + \frac{1}{3} \left( 2 D_{\gamma \delta} S + i \bar{D}_{\gamma \delta} R \right) M_{\alpha \beta}
+ \left( C_{\alpha \beta \gamma} + \frac{1}{3} \varepsilon_{\gamma (\alpha} \left( 8 (D_{\beta)} S + i \bar{D}_{\beta)} R \right) \right) J .
\] (4.7c)

Here we have accounted for the fact that in the \(N = 2\) case
\[
X^{IJKL} = 0 , \quad C_a^{IJ} = \varepsilon^{IJ} C_a ,
\] (4.8)
as well as we have defined the scalar torsion superfields
\[
R := -\frac{i}{2} (S^{11} - S^{22} + 2i S^{12}) , \quad \bar{R} := \frac{i}{2} (S^{11} - S^{22} - 2i S^{12}) ,
\] (4.9)
\[
S := \frac{1}{2} \delta_{IJ} S^{IJ} = \frac{1}{2} (S^{11} + S^{22}) .
\] (4.10)
The \( U(1) \) charges of \( R \) and its conjugate are
\[
\mathcal{J} \bar{R} = 2 \bar{R} , \quad \mathcal{J} R = -2 R ,
\] (4.11)
while the real fields \( S \) and \( C_a \) are obviously neutral. The dimension-3/2 differential constraints on the dimension-1 torsion superfields are
\[
\mathcal{D}_\alpha \bar{R} = 0 , \quad \bar{\mathcal{D}}_\alpha R = 0 ,
\] (4.12)
\[
\mathcal{D}_\alpha C_{\beta\gamma} = i C_{\alpha\beta\gamma} - \frac{1}{3} \varepsilon_{\alpha(\beta} \left( \bar{\mathcal{D}}_{\gamma)} \bar{R} + 4i \mathcal{D}_{\gamma} S \right) \] (4.13)
where we have defined the completely symmetric complex spinors
\[
C_{\alpha\beta\gamma} := \frac{1}{\sqrt{2}} (C_{\alpha\beta\gamma}^1 - i C_{\alpha\beta\gamma}^2) , \quad \bar{C}_{\alpha\beta\gamma} := -\frac{1}{\sqrt{2}} (C_{\alpha\beta\gamma}^1 + i C_{\alpha\beta\gamma}^2) ,
\] (4.14)
which are charged under the \( U(1) \)-group
\[
\mathcal{J} C_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} , \quad \mathcal{J} \bar{C}_{\alpha\beta\gamma} = -\bar{C}_{\alpha\beta\gamma} .
\] (4.15)
It follows from (4.13) that \( S \) is a real linear superfield,
\[
(\bar{\mathcal{D}}^2 - 4R)S = (\mathcal{D}^2 - 4\bar{R})S = 0 .
\] (4.16)

The super-Weyl transformation of the covariant derivatives becomes
\[
\delta_\sigma \mathcal{D}_\alpha = \frac{1}{2} \mathcal{D}_\alpha + (\mathcal{D}^\beta \sigma) \mathcal{M}_{\alpha\beta} - (\mathcal{D}_\alpha \sigma) \mathcal{J} ,
\] (4.17a)
\[
\delta_\sigma \bar{\mathcal{D}}_\alpha = \frac{1}{2} \bar{\mathcal{D}}_\alpha + (\bar{\mathcal{D}}^\beta \sigma) \bar{\mathcal{M}}_{\alpha\beta} - (\bar{\mathcal{D}}_{\alpha} \sigma) \mathcal{J} ,
\] (4.17b)
\[
\delta_\sigma \mathcal{D}_a = \mathcal{D}_a - \frac{i}{2} (\gamma_a)^{\gamma\delta} (\mathcal{D}_{\gamma} \sigma) \mathcal{D}_\delta - \frac{i}{2} (\gamma_a)^{\gamma\delta} (\bar{\mathcal{D}}_{\gamma} \sigma) \mathcal{D}_\delta + \varepsilon_{abc} (\mathcal{D}^b \sigma) \mathcal{M}^c
- \frac{i}{8} (\gamma_a)^{\gamma\delta} ([\mathcal{D}_{\gamma}, \mathcal{D}_\delta] \sigma) \mathcal{J} .
\] (4.17c)
From here we can read off the transformation of the torsion
\[
\delta_\sigma S = \sigma S + \frac{i}{8} [\mathcal{D}_\gamma, \mathcal{D}_\gamma] \sigma ,
\] (4.17d)
\[
\delta_\sigma C_a = \sigma C_a + \frac{i}{8} (\gamma_a)^{\gamma\delta} [\mathcal{D}_{\gamma}, \mathcal{D}_\delta] \sigma ,
\] (4.17e)
\[
\delta_\sigma R = \sigma R + \frac{1}{4} \mathcal{D}^2 \sigma , \quad \delta_\sigma \bar{R} = \sigma \bar{R} + \frac{1}{4} \bar{\mathcal{D}}^2 \sigma ,
\] (4.17f)
where we have defined
\[
\mathcal{D}^2 := \mathcal{D}^\alpha \mathcal{D}_\alpha , \quad \mathcal{D}^2 := \bar{\mathcal{D}}^\alpha \mathcal{D}_\alpha .
\] (4.18)
4.2 Scalar and vector multiplets

We now wish to study in some detail the properties of scalar and vector multiplets. Consider a covariantly chiral scalar Φ, \( \bar{D}_\alpha \Phi = 0 \), which is a primary field under the super-Weyl group, \( \delta_\sigma \Phi = w_\sigma \Phi \). Then its super-Weyl weight \( w \) and its U(1) charge have the same value and opposite signs,

\[
\bar{D}_\alpha \Phi = 0, \quad \mathcal{J}_\Phi = -w \Phi, \quad \delta_\sigma \Phi = w_\sigma \Phi. \quad (4.19)
\]

Consider now a complex scalar Ψ with the properties

\[
\mathcal{J}_\Psi = (2 - w)\Psi, \quad \delta_\sigma \Psi = (w - 1)\sigma \Psi, \quad (4.20)
\]

for some constant super-Weyl weight \( w \). Then, the superfield

\[
\Phi = \bar{\Delta} \Psi, \quad \bar{\Delta} := -\frac{1}{4}(\bar{D}^2 - 4R) \quad (4.21)
\]

is characterized by the properties (4.19). The operator \( \bar{\Delta} \) is the \( \mathcal{N} = 2 \) chiral projection operator. The fact that the explicit form of \( \bar{\Delta} \) is identical to that for the chiral projection operator in 4D \( \mathcal{N} = 1 \) supergravity [39, 40], is not surprising since the anticommutator \( \{D_\alpha, D_\beta\} \) in (4.7a) is algebraically identical to that in the 4D \( \mathcal{N} = 1 \) case.

We next turn to a complex linear superfield Σ. It is defined to obey the constraint

\[
(\bar{D}^2 - 4R)\Sigma = 0 \quad (4.22)
\]

and no reality condition. If Σ is chosen to transform homogeneously under the super-Weyl transformations, then its U(1) charge is determined by the super-Weyl weight,

\[
\delta_\sigma \Sigma = w_\sigma \Sigma \implies \mathcal{J}_\Sigma = (1 - w)\Sigma. \quad (4.23)
\]

This follows from the identity

\[
\delta_\sigma(\bar{D}^2 - 4R) = \sigma(\bar{D}^2 - 4R) + 2(\bar{D}^\alpha \sigma)\bar{D}_\alpha - 2(\bar{D}^\alpha \sigma)\bar{D}_\alpha \mathcal{J} + 2(\bar{D}^\alpha \sigma)\bar{D}_\beta \mathcal{M}_{\alpha\beta} - (\bar{D}^2 \sigma) + (\bar{D}^2 \sigma)\mathcal{J}, \quad (4.24)
\]

Indeed, using the relations (4.23) allows us to prove that

\[
\delta_\sigma \left( (\bar{D}^2 - 4R)\Sigma \right) = (1 + w)\sigma(\bar{D}^2 - 4R)\Sigma = 0. \quad (4.25)
\]

Unlike the chiral and the complex linear superfields, the superconformal transformation law of the vector multiplet is uniquely fixed. Consider an Abelian vector multiplet
described by its gauge-invariant field strength $G$. The latter is real, $(G)^* = G$, and obeys the constraint:  

$$ (\bar{D}^2 - 4R)G = 0 \quad \overset{G\rightarrow G^*}{\Rightarrow} \quad (\bar{D}^2 - 4\bar{R})G = 0 \ , $$  

which is equivalent to eq. (2.34). Since $G$ is neutral under the group U(1), $\mathcal{J}G = 0$, eq. (4.23) tells us that the super-Weyl transformation of $G$ is

$$ \delta_\sigma G = \sigma G \ . $$  

The constraint (4.26) can be solved in terms of a real unconstrained prepotential $V$,

$$ G = i\bar{D}^\alpha \mathcal{D}_\alpha V \ , $$  

which is defined modulo arbitrary gauge transformations of the form:

$$ \delta V = \lambda + \bar{\lambda} \ , \quad \mathcal{J}\lambda = 0 \ , \quad \bar{\mathcal{D}}_\alpha \lambda = 0 \ . $$  

It is consistent to consider the gauge prepotential $V$ to be inert under the super-Weyl transformations,

$$ \delta_\sigma V = 0 \ . $$  

### 4.3 Matter couplings

We are now prepared to introduce interesting matter couplings in $\mathcal{N} = 2$ supergravity. Let us first elaborate on locally supersymmetric and super-Weyl invariant actions. Given a real Lagrangian $\mathcal{L}$ with the super-Weyl transformation law

$$ \delta_\sigma \mathcal{L} = \sigma \mathcal{L} \ , $$  

the action

$$ S = \int d^3x d^2\theta d^2\bar{\theta} E \mathcal{L} \ , \quad E^{-1} = \text{Ber}(E_A^M) \ , $$  

is invariant under the supergravity gauge group. It is also super-Weyl invariant since the corresponding transformation law of $E$ is $\delta_\sigma E = -\sigma E$. 

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8Eq. (4.26) is a 3D version of the constraint defining the 4D $\mathcal{N} = 1$ tensor multiplet [64].
The existence of covariantly chiral superfields in $\mathcal{N} = 2$ conformal supergravity implies that the action (4.32) can also be rewritten as an integral over a chiral subspace, for instance, using the approach developed in [61].

$$S = \int d^3 x d^2 \theta d^2 \bar{\theta} E \mathcal{L} = \int d^3 x d^2 \theta \mathcal{E} \bar{\Delta} \mathcal{L}.$$  

(4.33)

Here $\mathcal{E}$ denotes the chiral density, $\bar{D}_\alpha \mathcal{E} = 0$, and $\bar{\Delta}$ the chiral projection operator (4.21). As follows from (4.20) and (4.31), the chiral superfield $\bar{\Delta} \mathcal{L}$ has super-Weyl weight $+2$ and $U(1)$ charge $-2$. Thus the chiral density has the properties

$$\mathcal{J} \mathcal{E} = 2 \mathcal{E}, \quad \delta_\sigma \mathcal{E} = -2 \sigma \mathcal{E}.$$  

(4.34)

The construction (4.33) allows us to introduce a different action principle. Given a chiral Lagrangian $\mathcal{L}_c$ of super-Weyl weight two

$$\bar{D}_\alpha \mathcal{L}_c = 0, \quad \delta_\sigma \mathcal{L}_c = 2 \sigma \mathcal{L}_c,$$

(4.35)

the following chiral action

$$S_c = \int d^3 x d^2 \theta d^2 \bar{\theta} E \frac{\mathcal{L}_c}{R} = \int d^3 x d^2 \theta \mathcal{E} \mathcal{L}_c$$

(4.36)

is locally supersymmetric and super-Weyl invariant. The first representation in (4.36) is analogous to that derived by Zumino [40] in 4D $\mathcal{N} = 1$ supergravity.

We now wish to uncover conditions on the target space geometry under which a 3D $\mathcal{N} = 2$ rigid supersymmetric sigma-model [63] can be coupled to conformal supergravity. Consider the $\mathcal{N} = 2$ locally supersymmetric sigma-model action

$$S = \int d^3 x d^2 \theta d^2 \bar{\theta} E K(\Phi^I, \bar{\Phi}^J), \quad \bar{D}_\alpha \Phi^I = 0,$$

(4.37)

where the dynamical variables $\Phi^I$ are covariantly chiral scalar superfields, $K(\Phi, \bar{\Phi})$ is the Kähler potential of a Kähler manifold $\mathcal{M}$. As usual, we denote by $g_{IJ}(\Phi, \bar{\Phi})$ the Kähler metric on $\mathcal{M}$. We postulate the super-Weyl transformation of the chiral superfields

$$\delta_\sigma \Phi^I = \frac{1}{2} \sigma \chi^I(\Phi),$$

(4.38)

There should exist a prepotential formulation for 3D $\mathcal{N} = 2$ supergravity that is similar to that developed many years ago for 4D $\mathcal{N} = 1$ supergravity [62]. In such a formulation eq. (4.33) could be derived using a covariant chiral representation.
where $\chi^\mu := (\chi^I, \bar{\chi}^\bar{J})$ is a holomorphic vector field on the target space. The action (4.37) can be seen to be invariant provided the Kähler potential satisfies the condition

$$\chi^I(\Phi) K_I(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (4.39)$$

This condition turns out to imply that $\chi^I$ is a homothetic holomorphic Killing vector on the target space. It has the properties

$$\nabla_I \chi^J = \delta^J_I , \quad \nabla_I \bar{\chi}^J = \bar{\partial}_I \chi^J = 0 , \quad (4.40a)$$

$$\chi_I := g_{IJ} \bar{\chi}^J = \partial_I K , \quad g_{IJ} = \bar{\partial}_I \bar{\partial}_J K , \quad (4.40b)$$

where $K$ can be chosen to be

$$K = g_{IJ} \chi^I \bar{\chi}^J . \quad (4.41)$$

These properties mean that the target space $\mathcal{M}$ is a Kählerian cone [59].

There is an important consistency condition. For the chirality condition $\bar{D}_\alpha \Phi^I = 0$ to be super-Weyl invariant, $\delta_\sigma (\bar{D}_\alpha \Phi^I) = 0$, the U(1) charge of $\Phi^I$ is uniquely fixed as

$$\mathcal{J} \Phi^I = -\frac{1}{2} \chi^I(\Phi) . \quad (4.42)$$

The sigma-model action (4.37) is invariant under local U(1) transformations, as a consequence of (4.39).

The sigma-model (4.37) can be generalized to include a superpotential.

$$S = \int d^3x d^2 \theta d^2 \bar{\theta} E K(\Phi^I, \bar{\Phi}^\bar{J}) + \int d^3x d^2 \theta d^2 \bar{\theta} E \left\{ \frac{W(\Phi^I)}{R} + \text{c.c.} \right\} , \quad (4.43)$$

with $W(\Phi)$ a holomorphic scalar field on the target space. It should obey the homogeneity condition

$$\chi^I(\Phi) W_I(\Phi) = 4 W(\Phi) \quad (4.44)$$

for the second term in (4.43) to be locally supersymmetric and super-Weyl invariant. The theory (4.43) is a locally supersymmetric extension of the general 3D $\mathcal{N} = 2$ superconformal sigma-model presented in [33].

Local complex coordinates, $\Phi^I$, on $\mathcal{M}$ can be chosen in such a way that $\chi^I = \Phi^I$. Then $K(\Phi^I, \bar{\Phi}^\bar{J})$ obeys the following homogeneity condition:

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (4.45)$$
Locally supersymmetric nonlinear sigma-models can also be generated from self-couplings of vector multiplets. Consider a system of Abelian vector multiplets described by real field strengths $G^i$, with $i = 1, \ldots, n$, constrained by

$$(\bar{D}^2 - 4R)G^i = (D^2 - 4\bar{R})G^i = 0, \quad i = 1, \ldots, n.$$  \hspace{1cm} (4.46)

Their dynamics can be described by an action of the form

$$S = \int d^3 x d^2 \theta d^2 \bar{\theta} E L(G^i).$$ \hspace{1cm} (4.47)

We know that the constraints (4.46) require the super-Weyl transformation law $\delta_\sigma G^i = \sigma G^i$. The action is therefore super-Weyl invariant provided the Lagrangian is a homogeneous function of $G^i$ of first degree,

$$G^i L_i(G) = L(G).$$ \hspace{1cm} (4.48)

This theory is a local supersymmetric extension of the $\mathcal{N} = 2$ superconformal model presented in [33]. In the case of a single vector multiplet, there is a super-Weyl invariant action generated by the Lagrangian $L(G) \propto (-G \ln G + 4V_S)$. Such an action describes an improved vector multiplet [52] which is a 3D version of the 4D $\mathcal{N} = 1$ improved tensor multiplet [65].

The vector multiplet model (4.47) can be generalized to include gauge-invariant Chern-Simons couplings

$$S_{CS} = \int d^3 x d^2 \theta d^2 \bar{\theta} E \left\{ L(G^i) + \frac{1}{2} m_{ij} V^i G^j \right\}, \quad m_{ij} = m_{ji} = (m_{ij})^* = \text{const}.$$ \hspace{1cm} (4.49)

Here $V^i$ is the gauge prepotential for $G^i$ defined as in (4.28).

### 4.4 Conformal compensators

As is well-known, Poincaré supergravity can be realised as conformal supergravity coupled to a compensating supermultiplet (or compensator) [30]. Different choices of compensator lead, in general, to different off-shell formulations for Poincaré supergravity, as has been shown in detail in the case of 4D $\mathcal{N} = 1$ supergravity [49, 3]. In complete analogy with 4D $\mathcal{N} = 1$ supergravity, there are three different types of compensator for $\mathcal{N} = 2$ supergravity in three dimensions: (i) a chiral scalar $\Phi$ and its conjugate $\bar{\Phi}$; (ii) a real linear scalar $G$; (iii) a complex linear scalar $\Sigma$ and its conjugate $\bar{\Sigma}$. Here we briefly discuss these choices.
In the case (i), the compensator $\Phi$ can be chosen to have super-Weyl weight $1/2$,
$$\delta_\sigma \Phi = \frac{1}{2} \sigma \Phi . \tag{4.50}$$
The freedom to perform the super-Weyl and local $U(1)$ transformations can be used to impose the gauge
$$\Phi = 1 . \tag{4.51}$$
Such a gauge fixing is accompanied by the consistency conditions
$$0 = \bar{D}_\alpha \Phi = -\frac{i}{2} \Phi_\alpha , \quad 0 = \{ D_\alpha , \bar{D}_\beta \} \Phi = -\Phi_{\alpha\beta} + C_{\alpha\beta} - 2i\epsilon_{\alpha\beta} S , \tag{4.52}$$
and therefore
$$\Phi_\alpha = S = 0 , \quad \Phi_{\alpha\beta} = C_{\alpha\beta} . \tag{4.53}$$
The formulation is the analogue of old-minimal 4D $\mathcal{N} = 1$ supergravity (see [3, 81] for reviews).

Another choice for compensator is the field strength of a vector multiplet, $G = \bar{G}$, which is subject to the linear constraint (4.26) and has the super-Weyl transformation (4.27). The super-Weyl gauge freedom can be used to impose the condition
$$G = 1 . \tag{4.54}$$
The local $U(1)$ group remains unbroken. The resulting geometry is characterized by the properties
$$R = \bar{R} = 0 \iff \{ D_\alpha , D_\beta \} = \{ \bar{D}_\alpha , \bar{D}_\beta \} = 0 . \tag{4.55}$$
This is clearly the 3D analogue of 4D $\mathcal{N} = 1$ new minimal supergravity (see [3] for a review).

A conformal compensator can be chosen to be a complex linear superfield $\Sigma$ which obeys the constraint (4.22) and is characterized by the local $U(1)$ and super-Weyl transformation properties (4.23). These local symmetries can be used to impose the gauge condition
$$\Sigma = 1 \tag{4.56}$$
which implies some restrictions on the geometry. To describe such restrictions, it is useful to split the covariant derivatives as
$$D_\alpha = \nabla_\alpha + iT_\alpha J , \quad \bar{D}_\alpha = \bar{\nabla}_\alpha + iT_\alpha \bar{J} \tag{4.57}$$
where we have renamed the original U(1) connection $\Phi_\alpha$ as $T_\alpha$. The operators $\nabla_\alpha$ and $\bar{\nabla}_\alpha$ have no U(1) connection. In the gauge (4.56), the constraint $(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0$ turns into

$$R = -\frac{i(w-1)}{4} \left( \bar{\nabla}_\alpha \bar{T}^\alpha - iw_T \bar{T}^\alpha \right).$$  \hspace{1cm} (4.58)

We see that $R$ becomes a descendant of $T_\alpha$ and its complex conjugate. Eq. (4.58) is not the only constraint which is induced by the gauge fixing (4.56). By evaluating $\{D_\alpha, D_\beta\} \Sigma$ and $\{D_\alpha, \bar{D}_\beta\} \Sigma$ and then setting $\Sigma = 1$ gives

$$\nabla_{(a} T_{b)} = 0, \quad S = \frac{1}{8} \left( \nabla^\alpha T_\alpha - \nabla^\alpha \bar{T}_\alpha + 2i T^\alpha \bar{T}_\alpha \right),$$  \hspace{1cm} (4.59a)

$$\Phi_{\alpha\beta} = C_{\alpha\beta} + \frac{i}{2} \nabla_{(a} \bar{T}_{\beta)} + \frac{i}{2} \bar{\nabla}_{(a} T_{\beta)} + T_{(a} \bar{T}_{\beta)},$$  \hspace{1cm} (4.59b)

If we define a new vector covariant derivative $\nabla_a$ by $D_a = \nabla_a + i\Phi_a$, then the algebra of the covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}_\alpha)$ proves to be

$$\{\nabla_\alpha, \nabla_\beta\} = -2i T_{(\alpha} \nabla_{\beta)} - i(w-1) \left( \nabla^\gamma T_\gamma + iw T^\gamma T_\gamma \right) M_{\alpha\beta},$$  \hspace{1cm} (4.60a)

$$\{\nabla_\alpha, \bar{\nabla}_\beta\} = -2i \nabla_{\alpha\beta} - i\bar{T}_\beta \nabla_\alpha + i T_\alpha \bar{\nabla}_\beta - 2\varepsilon_{\alpha\beta} C_{\gamma\delta} M_{\gamma\delta} + \frac{i}{2} \left( \nabla^\gamma T_\gamma - \nabla^\gamma \bar{T}_\gamma + 2i T^\gamma \bar{T}_\gamma \right) M_{\alpha\beta}. $$  \hspace{1cm} (4.60b)

The emerging formulation for 3D $\mathcal{N} = 2$ supergravity is analogous to 4D $\mathcal{N} = 1$ non-minimal supergravity (see [3] for a review).

The procedure of de-gauging described in this subsection is completely similar to that presented in the book [3] which in turn closely followed Howe’s approach [49].

5 Matter couplings in $\mathcal{N} = 3$ supergravity

To the best of our knowledge, three-dimensional $\mathcal{N} = 3$ supergravity in superspace is terra incognita. Here we set out to explore this continent.

In this and the following sections, we build on the projective-superspace formulations for general 5D $\mathcal{N} = 1$ and 4D $\mathcal{N} = 2$ supergravity-matter theories which were developed in [17] [18] [19] [20] [21], as well as on the recent results obtained in [33] concerning the off-shell $\mathcal{N} = 3$ and $\mathcal{N} = 4$ rigid superconformal sigma-models in three dimensions.
5.1 Elaborating on the $\mathcal{N} = 3$ superspace geometry

In accordance with the geometric formulation developed in section 2, the structure group of $\mathcal{N} = 3$ conformal supergravity is $\text{SL}(2, \mathbb{R}) \times \text{SO}(3)$, with the spinor derivatives $\mathcal{D}_\alpha^I$ transforming in the defining (vector) representation of $\text{SO}(3)$. In order to define a large class of matter multiplets coupled to supergravity, however, it is convenient to switch to an isospinor notation using the isomorphism $\text{SO}(3) \cong \text{SU}(2)/\mathbb{Z}_2$. As usual, this is achieved by replacing any $\text{SO}(3)$ vector index by a symmetric pair of $\text{SU}(2)$ spinor indices, $\mathcal{D}_\alpha^I \rightarrow \mathcal{D}_\alpha^{ij} = \mathcal{D}_{ji}^I$. In this subsection, our isospinor notation is defined and the supergravity algebra is rewritten using this notation.

Isospinor indices are raised and lowered with the aid of the $\text{SU}(2)$ invariant antisymmetric tensors $\varepsilon^{ij}$ and $\varepsilon_{ij}$ ($\varepsilon^{12} = \varepsilon_{21} = 1$) according to the rule

$$\psi^i = \varepsilon^{ij} \psi_j, \quad \psi_i = \varepsilon_{ij} \psi^j.$$  \hfill (5.1)

Given a real isovector $V_I$, we associate with it the symmetric isospinor $V_{ij}$ defined by

$$V_I \rightarrow V_{ij} := (\tau^I)_{ij} V_I = V_{ji}, \quad V_I = (\tau_I)^{ij} V_{ij}, \quad (V_{ij})^* = V^{ij}.$$  \hfill (5.2)

see Appendix A for the definition of the $\tau$-matrices. The normalization of the $\tau$-matrices is such that

$$A_I^I B_I = A_{ij} B_{ij},$$  \hfill (5.3)

for any isovectors $A_I$ and $B_I$ and the associated symmetric isospinors $A_{ij}$ and $B_{ij}$. Consider now an antisymmetric second-rank isotensor, $A_{IJ} = -A_{JI}$. Its counterpart with isospinor indices, $A_{ijkl} = -A_{klij} = A_{IJ}(\tau^I)_{ij}(\tau^J)_{kl}$ can be decomposed as

$$A_{ijkl} = \frac{1}{2} \varepsilon_{jk} A_{ik} + \frac{1}{2} \varepsilon_{ik} A_{jk}, \quad A_{ijkl} = -\frac{1}{2} \varepsilon_{jl} A^{ik} - \frac{1}{2} \varepsilon_{ik} A^{jl}, \quad A_{ij} = A_{ji}.$$  \hfill (5.4)

In particular, if $A_{IJ} = -A_{JI}$ and $B_{IJ} = -B_{JI}$ are two antisymmetric isotensors, and $A_{ij}$ and $B_{ij}$ are their isospinor counterparts, then it holds that

$$\frac{1}{2} A_{IJ} B_{IJ} = \frac{1}{2} A^{kl} B_{kl}.$$  \hfill (5.5)

Finally, let us derive the isospinor analogue of the completely antisymmetric third-rank tensor $\varepsilon_{IJK}$ ($\varepsilon^{123} = 1$). The definition $\varepsilon_{ijklpq} = \varepsilon_{IJK}(\tau^I)_{ij}(\tau^J)_{kl}(\tau^K)_{pq}$ leads to

$$\varepsilon_{ijklpq} = -\frac{1}{\sqrt{2}} \left( \varepsilon_p(k \varepsilon_l(i \varepsilon_j)q) + \varepsilon_q(k \varepsilon_l(i \varepsilon_j)p) \right).$$  \hfill (5.6)
We are now ready to rewrite the results obtained in section 2 for the case \( \mathcal{N} = 3 \) in the isospinor notation introduced. The covariant derivatives are

\[
\mathcal{D}_A \equiv (\mathcal{D}_a, \mathcal{D}_\alpha^b) = E_A + \Omega_A + \Phi_A ,
\]

where the SO(3) connection \( \Phi_A \) takes the form

\[
\Phi_A = \frac{1}{2} \Phi_A^{KL} \mathcal{N}_{KL} = \frac{1}{2} \Phi_A^{kl} \mathcal{J}_{kl} .
\]

Here we have introduced the SU(2) generator \( \mathcal{J}_{kl} \) which is obtained from \( \mathcal{N}_{KL} \) as

\[
\mathcal{N}_{KL} \rightarrow \mathcal{N}_{ijkl} = \frac{1}{2} \varepsilon_{ji} \mathcal{J}_{ik} + \frac{1}{2} \varepsilon_{ik} \mathcal{J}_{jl} , \quad \mathcal{N}_{ijkl} = -\frac{1}{2} \varepsilon_{jl} \mathcal{J}^{ik} - \frac{1}{2} \varepsilon_{ik} \mathcal{J}^{jl} .
\]

It acts on the spinor covariant derivatives \( \mathcal{D}_\alpha^b := \mathcal{D}_a^l (\tau_i)^{lj} \) as follows

\[
[\mathcal{J}^{kl}, \mathcal{D}_\alpha^b] = \varepsilon^{i(k} \mathcal{D}_a^{lj} + \varepsilon^{j(k} \mathcal{D}_a^{li} .
\]

In the \( \mathcal{N} = 3 \) case under consideration, the dimension-1 components of the torsion and the curvature can be rewritten as

\[
C_a^{ij} \rightarrow C_a^{ijkl} = -\frac{1}{2} \varepsilon^{ik} C_a^{jl} - \frac{1}{2} \varepsilon^{jl} C_a^{ik} , \quad C_a^{ij} = C_a^{ji} ,
\]

\[
S_a^{ij} \rightarrow S_a^{ijkl} = S^{ijkl} - \varepsilon^{i(k} \varepsilon^{j(l} S^{)ij} , \quad S_a^{ijkl} = S^{(ijkl)} .
\]

The algebra of spinor covariant derivatives becomes

\[
\{ \mathcal{D}_a^i, \mathcal{D}_b^j \} = -2i \varepsilon^{i(k} \varepsilon^{j(l} (\gamma^c)_{\alpha \beta} D_c - i \varepsilon_{\alpha \beta} (\varepsilon^{j(l} S^{ikpq} + \varepsilon^{ik} S^{jpq}) J_{pq} + 2i \varepsilon_{\alpha \beta} S \left( \varepsilon^{ik} \mathcal{J}^{jl} + \varepsilon^{ik} \mathcal{J}^{jl} \right)
\]

\[
- i \varepsilon^{i(k} \varepsilon^{j(l} C_{\alpha \beta}^{pq} J_{pq} + i C_{\alpha \beta}^{j(l} \mathcal{J}^{i)} + i C_{\alpha \beta}^{j(l} \mathcal{J}^{i)} + i C_{\alpha \beta}^{j(l} \mathcal{J}^{i)} + i C_{\alpha \beta}^{j(l} \mathcal{J}^{i)}
\]

\[
- i C_{\alpha \beta}^{ij} \mathcal{J}^{kl} - i C_{\alpha \beta}^{kl} \mathcal{J}^{ij} + i \varepsilon_{\alpha \beta} (\varepsilon^{ik} C^{\gamma \delta jl} + \varepsilon^{il} C^{\gamma \delta ik}) M_{\gamma \delta}
\]

\[
- 4i (S^{ijkl} - \varepsilon^{i(k} \varepsilon^{j(l} S) M_{\alpha \beta} ,
\]

where we have taken into account the fact that \( X^{IJKL} = 0 \) for \( \mathcal{N} = 3 \).

The dimension-3/2 Bianchi identities become

\[
\mathcal{D}_a^{ij} S_a^{klpq} = -\frac{1}{2} \varepsilon^{j(l} T_a^{ikpq} - \frac{1}{2} \varepsilon^{ik} T_a^{jlpq} - \frac{1}{2} \varepsilon^{jq} T_a^{iklp} - \frac{1}{2} \varepsilon^{ip} T_a^{jklq}
\]

\[
- \frac{1}{2} S_a^{kl} \varepsilon^{p(i} \varepsilon^{j)q} - \frac{1}{2} S_a^{pq} \varepsilon^{k(i} \varepsilon^{j)l} + \frac{1}{3} S_a^{ij} \varepsilon^{k(p} \varepsilon^{q)l} ,
\]

\[
\mathcal{D}_a^{ij} C_a^{kl} = \sqrt{2} \varepsilon^{i(k} \varepsilon^{j(l} C_{\alpha \beta}^{kl} + C_{\alpha \gamma}^{k(i} \varepsilon^{j)l} + C_{\beta \gamma}^{l(i} \varepsilon^{j)k} + \frac{2 \sqrt{2}}{3} \varepsilon^{i(k} \varepsilon^{j(l} \varepsilon_{\alpha \beta} C_{\gamma}^{kl} + 2 \varepsilon_{\alpha \beta} T_a^{ijkl}
\]

\[
- \frac{4}{3} \varepsilon_{\alpha \beta} \left( (\mathcal{D}_a^{k(i} S_a^{j)l} + (\mathcal{D}_a^{l(i} S_a^{j)k} + \frac{1}{9} \varepsilon_{\alpha \beta} \left( S_a^{k(i} \varepsilon^{j)l} + S_a^{l(i} \varepsilon^{j)k} \right) .
\]

\[
(5.13)
\]

\[
(5.14a)
\]

\[
(5.14b)
\]
Here the dimension-3/2 component superfields possess the symmetry properties
\[
\mathcal{T}_\alpha^{ijkl} = \mathcal{T}_\alpha^{ijkl}, \quad \mathcal{S}_{\alpha}^{ij} = \mathcal{S}_{\alpha}^{ji}, \\
C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}, \quad C_{\alpha\beta\gamma}^{ij} = C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}^{ij}. \quad (5.15a)
\]
\[
C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}^{ij}, \quad C_{\alpha\beta\gamma}^{ij} = C_{\alpha\beta\gamma}^{ji} = C_{(\alpha\beta\gamma)}^{ij}. \quad (5.15b)
\]

We conclude by giving the super-Weyl transformation in isospinor notation. It holds
\[
\delta_\sigma D_{\alpha}^{ij} = \frac{1}{2}\sigma D_{\alpha}^{ij} + (D_{\beta}^{ij}) \mathcal{M}_{\alpha\beta} - (D_{\alpha}^{k(\sigma})\mathcal{J}^{j)k}), \quad (5.16a)
\]
\[
\delta_\sigma D_{a} = \sigma D_{a} + \frac{i}{2}(\gamma_a)^{\gamma\delta}(D_{\gamma}^{kl}) \sigma \mathcal{D}_{\delta kl} + \varepsilon_{abc}(D^{b\sigma})\mathcal{M}^c + \frac{i}{16}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{\gamma}^{(k})\mathcal{D}_{\delta p}]\sigma)\mathcal{J}_{kl}. \quad (5.16b)
\]
The super-Weyl transformation laws of the torsion superfields are
\[
\delta_\sigma S^{ijkl} = \sigma S^{ijkl} - \frac{i}{8}[D_{(ij)}^{\gamma}D_{\gamma}^{kl}]\sigma, \quad \delta_\sigma S = \sigma S - \frac{i}{24}[D^{ijkl}, D_{\gamma kl}]\sigma, \quad (5.16c)
\]
\[
\delta_\sigma C_{a}^{ij} = \sigma C_{a}^{ij} - \frac{i}{8}(\gamma_a)^{\gamma\delta}[D_{\gamma}^{k(i]}D_{\delta k)]\sigma. \quad (5.16d)
\]

### 5.2 Covariant projective multiplets

In this section we introduce a large family of \( \mathcal{N} = 3 \) (matter) supermultiplets coupled to conformal supergravity – covariant projective multiplets. One of the simplest projective multiplets, the so-called \( \mathcal{O}(2) \) multiplet, is naturally associated with the field strength of a \( \mathcal{N} = 3 \) vector multiplet. Although being the simplest in the family, it displays many properties of the general projective multiplets. We therefore start by considering this particular multiplet, and then turn to the general case.

The antisymmetric field strength of the vector multiplet, \( W^{IJ} \), is equivalently described by the symmetric isospinor \( W^{ij} \) which originates as
\[
W^{IJ} \rightarrow W^{ijkl} = -\frac{1}{2}\varepsilon^{jl}W^{ik} - \frac{1}{2}\varepsilon^{ik}W^{jl}. \quad (5.17)
\]
In terms of \( W^{ij} \) the Bianchi identity \( \Box \) turns into the analyticity constraint
\[
\mathcal{D}_{\alpha}^{(ij}W^{kl)} = 0. \quad (5.18)
\]

Let us introduce a complex commuting isospinor, \( v^i \in \mathbb{C}^2 \setminus \{0\} \), and use it to define the derivative
\[
\mathcal{D}_{\alpha}^{(2)} := v_i v_j \mathcal{D}_{\alpha}^{ij}, \quad (5.19)
\]
\[\text{Our conventions for isospinor bosonic variables and projective multiplets differ slightly from [19, 21], but agree with those adopted in [38].}\]
as well as the superfield
\[ W^{(2)} := v_i v_j W^{ij} . \] (5.20)

Then, the constraint (5.18) is equivalent to
\[ \mathcal{D}^{(2)}_\alpha W^{(2)} = 0 . \] (5.21)

The superscripts, which are attached to \( W^{(2)} \) and \( \mathcal{D}^{(2)}_\alpha \), indicate the degree of homogeneity in \( v^i \). Similarly to the local superspace coordinates \( z^M \), the isospinor \( v^i \) is defined to be inert under the local structure-group transformations. Its sole role is to package the field strength \( W^{ij} \) into an index-free object. This interpretation of \( v^i \) as a book-keeping device is discussed in detail in [19].

In accordance with (5.13), the spinor covariant derivatives \( \mathcal{D}^{(2)}_\alpha \) satisfy the algebra
\[ \{ \mathcal{D}^{(2)}_\alpha , \mathcal{D}^{(2)}_\beta \} = -4i S^{(4)}_{\alpha \beta} + 2i C^{(2)}_{\alpha \beta} J^{(2)} , \] (5.22a)
where we have defined
\[ C^{(2)}_{\alpha \beta} := v_i v_j C_{\alpha \beta ij} , \quad S^{(4)} := v_i v_j v_k S^{ijkl} , \quad J^{(2)} := v_i v_j J^{ij} . \] (5.22b)

It follows from (5.22a) that the constraint (5.21) is consistent. Indeed, the SU(2) transformation
\[ J_{ij} W_{kl} = -\varepsilon_{k(i} W_{j)l} - \varepsilon_{l(i} W_{j)k} \] (5.23)
implies \( J^{(2)} W^{(2)} = 0 \).

Under the infinitesimal supergravity gauge transformation,
\[ \delta_K \mathcal{D}_A = [K, \mathcal{D}_A] , \quad K = K^C(z) \mathcal{D}_C + \frac{1}{2} K^{cd}(z) \mathcal{M}_{cd} + \frac{1}{2} K^{kl}(z) J_{kl} , \] (5.24)
the field strength \( W^{ij} \) changes as
\[ \delta_K W^{ij} = K^C \mathcal{D}_C W^{ij} + W^{li} (K^{ij})_l . \] (5.25)

In terms of \( W^{(2)} \), this transformation law can be rewritten in the form:
\[ \delta_K W^{(2)} = \left( K^C \mathcal{D}_C + \frac{1}{2} K^{ij} J_{ij} \right) W^{(2)} , \] (5.26a)
\[ K^{ij} J_{ij} W^{(2)} = -\left( K^{(2)} \theta^{(-2)} - 2 K^{(0)} \right) W^{(2)} . \] (5.26b)
Here we have denoted
\[ K^{(2)} := K^{ij} \ v_i v_j , \quad K^{(0)} := \frac{v_i u_j}{(v, u)} K^{ij} , \quad (v, u) := v^i u_i \] (5.27)
and also introduced the differential operator
\[ \partial^{(-2)} := \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i} . \] (5.28)
The expressions in (5.27) and (5.28) involve a new isospinor \( u_i \) which is subject to the condition \((v, u) \neq 0\), but otherwise completely arbitrary. By definition, \( W^{(2)} \) is independent of \( u_i \). The variation \( \delta_K W^{(2)} \) can be seen to be independent of \( u_i \) as well, in spite of the fact that each of the two terms on the right-hand side of (5.26b) involves \( u_i \).

In accordance with (2.37), the super-Weyl transformation of \( W^{(2)} \) is
\[ \delta_\sigma W^{(2)} = \sigma W^{(2)} . \] (5.29)
It may be seen that the analyticity constraint (5.21) and the functional form of \( W^{(2)} \) uniquely determine the super-Weyl transformation law of \( W^{(2)} \). This is similar to the properties of the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) vector multiplets.

The condition \((v, u) \neq 0\) means that \( v^i \) and \( u^i \) form a basis for \( \mathbb{C}^2 \). Therefore the isospinors \( v^i \) and \( u^i \) can be used to define a new basis for the isospinor indices, with the aid of the completeness relation
\[ \delta^i_j = \frac{1}{(v, u)} \left( v^i u_j - v^j u^i \right) . \] (5.30)
Specifically, associated with a symmetric valence-\( n \) isospinor \( T^{i_1 \ldots i_n} = T^{(i_1 \ldots i_n)} \) is a set of \( (n+1) \) index-free objects
\[ T^{(n-2m)} := T^{i_1 \ldots i_{n-m} i_{n-m+1} \ldots i_n} v_{i_1} \cdots v_{i_{n-m}} \frac{u_{i_{n-m+1}}}{(v, u)} \cdots \frac{u_{i_n}}{(v, u)} , \quad m = 0, 1, \ldots, n \] (5.31)
which are homogeneous in \( v \) and \( u \) of degrees \( n - 2m \) and 0, respectively.\footnote{In some situations, in order to avoid possible misunderstanding, it would be more precise to use the notation \( T^{(n-m,m)} \) instead of \( T^{(n-2m)} \). Such a notation is not used in this paper.} For example, starting from the spinor covariant derivatives \( D_{\alpha}^{ij} \), we generate
\[ D_{\alpha}^{(2)} := v_i v_j D_{\alpha}^{ij} , \quad D_{\alpha}^{(0)} := \frac{v_i u_j}{(v, u)} D_{\alpha}^{ij} , \quad D_{\alpha}^{(-2)} := \frac{u_i u_j}{(v, u)^2} D_{\alpha}^{ij} . \] (5.32)
Applying this rule to the SU(2) generators \( J^{ij} \) gives
\[ J^{(2)} := v_i v_j J^{ij} , \quad J^{(0)} := \frac{v_i u_j}{(v, u)} J^{ij} , \quad J^{(-2)} := \frac{u_i u_j}{(v, u)^2} J^{ij} . \] (5.33)
We are now prepared to define general projective multiplets. A covariant projective supermultiplet of weight \( n \), \( Q^{(n)}(z,v) \), is defined to be a Lorentz-scalar superfield that lives on the curved \( \mathcal{N} = 3 \) superspace \( \mathcal{M}^{3|6} \), is holomorphic with respect to the isospinor variables \( v^i \) on an open domain of \( \mathbb{C}^2 \setminus \{0\} \), and is characterized by the following conditions: (i) it obeys the covariant analyticity constraint

\[
\mathcal{D}^{(2)}_{\alpha} Q^{(n)} = 0 ;
\]

(ii) it is a homogeneous function of \( v \) of degree \( n \), that is,

\[
Q^{(n)}(z,c v) = c^n Q^{(n)}(z,v) , \quad c \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\} ;
\]

(iii) supergravity gauge transformations act on \( Q^{(n)} \) as follows:

\[
\delta_K Q^{(n)} = \left( K^C D_C + \frac{1}{2} K^{ij} \mathcal{J}_{ij} \right) Q^{(n)} ,
\]

\[
K^{ij} \mathcal{J}_{ij} Q^{(n)} = - \left( K^{(2)} \partial^{(-2)} - n K^{(0)} \right) Q^{(n)} .
\]

Note that by construction, \( Q^{(n)} \) is independent of \( u \), i.e. \( \partial Q^{(n)}/\partial u^i = 0 \). One can see that \( \delta_K Q^{(n)} \) is also independent of the isotwistor \( u \), \( \partial(\delta_K Q^{(n)})/\partial u^i = 0 \), due to (5.35). It is also important to note that eq. (5.36) implies that

\[
\mathcal{J}^{(2)} Q^{(n)} = 0 ,
\]

and hence the covariant analyticity constraint (5.34) is indeed consistent.

The analyticity constraint (5.34) and the homogeneity condition (5.35) are consistent with the interpretation that the isospinor \( v^i \in \mathbb{C}^2 \setminus \{0\} \) is defined modulo the equivalence relation \( v^i \sim c v^i \), with \( c \in \mathbb{C}^* \), hence it parametrizes \( \mathbb{C}P^1 \). Therefore, the projective multiplets live in \( \mathcal{M}^{3|6} \times \mathbb{C}P^1 \).

There exists a real structure on the space of projective multiplets. Given a weight-\( n \) projective multiplet \( Q^{(n)}(v^i) \), its smile conjugate, \( \tilde{Q}^{(n)}(v^i) \), is defined by

\[
Q^{(n)}(v^i) \longrightarrow \tilde{Q}^{(n)}(\bar{v}_i) \longrightarrow \tilde{Q}^{(n)}(\bar{v}_i \rightarrow -v_i) \equiv: \tilde{Q}^{(n)}(v^i) ,
\]

with \( \tilde{Q}^{(n)}(\bar{v}_i) := \overline{Q^{(n)}(v^i)} \) the complex conjugate of \( Q^{(n)}(v^i) \), and \( \bar{v}_i \) the complex conjugate of \( v^i \). One can show that \( \tilde{Q}^{(n)}(v) \) is a weight-\( n \) projective multiplet. In particular, \( \tilde{Q}^{(n)}(v) \) obeys the analyticity constraint \( \mathcal{D}^{(2)} \tilde{Q}^{(n)} = 0 \), unlike the complex conjugate of \( Q^{(n)}(v) \). One can also check that

\[
\tilde{Q}^{(n)}(v) = (-1)^n Q^{(n)}(v) .
\]
Therefore, if \( n \) is even, one can define real projective multiplets, \( \tilde{Q}^{(2n)} = Q^{(2n)} \). Note that geometrically, the smile-conjugation is complex conjugation composed with the antipodal map on the projective space \( \mathbb{C}P^1 \).

Let \( Q^{(n)}(z, v) \) be a projective supermultiplet of weight \( n \). Assuming that it transforms homogeneously under the super-Weyl transformations, the analyticity constraints uniquely fix its transformation law:

\[
\delta_\sigma Q^{(n)} = \frac{n}{2} \sigma Q^{(n)} .
\] (5.40)

Our definition of the 3D \( \mathcal{N} = 3 \) projective multiplets given above is reminiscent of the covariant projective multiplets in 4D \( \mathcal{N} = 2 \) conformal supergravity \([21]\) or 5D \( \mathcal{N} = 1 \) conformal supergravity \([18]\). However, the three-dimensional case has two specific features as compared to four and five dimensions. First of all, the analyticity constraint \( (5.34) \) is formulated in terms of two spinor operators, \( D^{(2)}_\alpha \), while the 4D projective multiplets are annihilated by four derivatives \( D^{(1)}_\alpha := D_i^\alpha v_i \) and \( \bar{D}^{(1)}_\dot{\alpha} := \bar{D}_i^{\dot{\alpha}} v_i \). Secondly, the operators \( D^{(2)}_\alpha \) are quadratic in the isotwistor variables \( v_i \), while their four-dimensional analogues, \( D^{(1)}_\alpha \) and \( \bar{D}^{(1)}_\dot{\alpha} \), are linear in \( v_i \).

We now list several projective multiplets that can be used to describe superfield dynamical variables. A natural generalization of the field strength \( W^{(2)}(v) \) is a real \( \mathcal{O}(2n) \) multiplet, with \( n = 1, 2, \ldots \). It is described by a real weight-2n projective superfield \( H^{(2n)}(v) \) of the form:

\[
H^{(2n)}(v) = H^{i_1 \ldots i_{2n}} v_{i_1} \ldots v_{i_{2n}} = \hat{H}^{(2n)}(v) .
\] (5.41)

The analyticity constraint \( (5.34) \) is equivalent to

\[
D^{(ij)}_\alpha H^{k_1 \ldots k_{2n}} = 0 .
\] (5.42)

The reality condition \( \hat{H}^{(2n)} = H^{(2n)} \) is equivalent to

\[
\overline{H^{i_1 \ldots i_{2n}}} = H_{i_1 \ldots i_{2n}} = \varepsilon_{i_1 j_1} \cdots \varepsilon_{i_{2n} j_{2n}} H^{j_1 \ldots j_{2n}} .
\] (5.43)

The field strength of the vector multiplet is a real \( \mathcal{O}(2) \) multiplet. For \( n > 1 \), the real \( \mathcal{O}(2n) \) multiplet can be used to describe an off-shell (neutral) hypermultiplet.

An off-shell (charged) hypermultiplet can be described in term of the so-called \textit{arctic} weight-\( n \) multiplet \( \Upsilon^{(n)}(v) \) which is defined to be holomorphic in the north chart \( \mathbb{C} \), of the projective space \( \mathbb{C}P^1 = \mathbb{C} \cup \{ \infty \} \):

\[
\Upsilon^{(n)}(v) = (v^1)^n \Upsilon^{[n]}(\zeta) , \quad \Upsilon^{[n]}(\zeta) = \sum_{k=0}^{\infty} \Upsilon_k \zeta^k ,
\] (5.44)
and its smile-conjugate antarctic multiplet \( \tilde{\Upsilon}^{(n)}(v) \),
\[
\tilde{\Upsilon}^{(n)}(v) = (v^2)^n \tilde{\Upsilon}^{[n]}(\zeta) = (v^1 \zeta)^n \tilde{\Upsilon}^{[n]}(\zeta), \quad \tilde{\Upsilon}^{[n]}(\zeta) = \sum_{k=0}^{\infty} \tilde{\Upsilon}_k \frac{(-1)^k}{\zeta^k}.
\] (5.45)

Here we have introduced the inhomogeneous complex coordinate \( \zeta = v^2/v^1 \) on the north chart of \( \mathbb{C}P^1 \). The pair consisting of \( \Upsilon^{[n]}(\zeta) \) and \( \tilde{\Upsilon}^{[n]}(\zeta) \) constitutes the so-called polar weight-\( n \) multiplet. The spinor covariant derivative \( D^{(2)}_\alpha \) can be represented as
\[
D^{(2)}_\alpha = (v^1)^2 D^{[2]}_\alpha, \quad D^{[2]}_\alpha(\zeta) = D^{22} - 2\zeta D^{12} + \zeta^2 D^{11}.
\] (5.46)

It follows from this representation that the analyticity condition (5.34) relates, in a non-trivial way, the superfield coefficients \( \Upsilon_k \) in the series (5.44).

Our last example is the real tropical multiplet \( U^{(2n)}(v) \) of weight \( 2n \) defined by
\[
U^{(2n)}(v) = (iv^1v^2)^n U^{[2n]}(\zeta) = (v^1)^2 (i \zeta)^n U^{[2n]}(\zeta), \quad U^{[2n]}(\zeta) = \sum_{k=-\infty}^{\infty} U_k \zeta^k, \quad U_k = (-1)^k U_{-k}.
\] (5.47)

As will be shown below, the case \( n = 0 \) can be used to describe a gauge prepotential of the vector multiplet.

### 5.3 Analytic projection operator

In this subsection we show how to engineer covariant projective multiplets.

The torsion superfield \( S^{(4)} \) was defined in subsection 5.2, eq. (5.22b). It proves to be a real \( O(4) \) multiplet. Indeed, the equation (5.14a) implies the relation \( D^{(4)}_\alpha S^{(4)} = 0 \) which is equivalent to the analyticity constraint \( D^{(2)}_\alpha S^{(4)} = 0 \). It is easy to see that \( S^{(4)} \) does not enjoy the super-Weyl transformation law (5.40). As follows from eq. (5.16c), its super-Weyl transformation is inhomogeneous,
\[
\delta_\sigma S^{(4)} = 2\sigma S^{(4)} - \frac{i}{4} \left( D^{(4)} - 4iS^{(4)} \right) \sigma,
\] (5.48)

where \( D^{(4)} \) is defined by
\[
D^{(4)} := D^{(2)}_\gamma D^{(2)}_\gamma.
\] (5.49)

The appearance in (5.48) of the following differential operator
\[
\Delta^{(4)} := \frac{i}{4} \left( D^{(4)} - 4iS^{(4)} \right)
\] (5.50)
is not accidental. This operator turns out to be a \(\mathcal{N} = 3\) analytic projection operator. In particular, \(\Delta^{(4)}\) is such that the constraint \(\mathcal{D}_\alpha^{(2)}\mathcal{S}^{(4)} = 0\) is preserved under the super-Weyl transformations, \(\delta_\sigma(\mathcal{D}_\alpha^{(2)}\mathcal{S}^{(4)}) = 0\). This follows from the transformation rule

\[
\delta_\sigma\mathcal{D}_\alpha^{(2)} = \frac{1}{2}\sigma\mathcal{D}_\alpha^{(2)} + (\mathcal{D}^{(2)}\beta\sigma)\mathcal{M}_{\alpha\beta} + (\mathcal{D}_\alpha^{(2)}\sigma)\mathcal{J}^{(0)} - (\mathcal{D}_\alpha^{(0)}\sigma)\mathcal{J}^{(2)},
\]

the identity

\[
\mathcal{D}_\alpha^{(2)}\Delta^{(4)} = \frac{1}{2}C_{\alpha\beta}^{(2)}\mathcal{D}^{(2)}\beta\mathcal{J}^{(2)} + \frac{1}{6}(\mathcal{D}^{(2)}\beta C_{\alpha\beta}^{(2)})\mathcal{J}^{(2)} - \mathcal{S}^{(4)}\mathcal{D}_\beta^{(2)}\mathcal{M}_{\alpha\beta},
\]

and the obvious relation \(\mathcal{J}^{(0)}\mathcal{S}^{(4)} = -2\mathcal{S}^{(4)}\). Note that the above super-Weyl transformations of \(\mathcal{D}_\alpha^{(2)}\) follows from (5.16a).

Let us formulate more precisely what we mean by ‘analytic projection operator.’ First of all, we have to introduce the concept of isotwistor superfields, following [19]. By definition, a weight-\(n\) isotwistor superfield \(U^{(n)}\) is a tensor superfield (with suppressed Lorentz indices) that lives on \(\mathcal{M}^{8|6}\), is holomorphic with respect to the isospinor variables \(v^i\) on an open domain of \(\mathbb{C}^2 \setminus \{0\}\), is a homogeneous function of \(v^i\) of degree \(n\),

\[
U^{(n)}(cv) = c^n U^{(n)}(v), \quad c \in \mathbb{C}^*,
\]

and is characterized by the supergravity gauge transformation

\[
\delta_K U^{(n)} = \left(K^C\mathcal{D}_C + \frac{1}{2}K^{ab}\mathcal{M}_{ab} + \frac{1}{2}K^{ij}\mathcal{J}_{ij}\right)U^{(n)},
\]

\[
\mathcal{J}_{ij} U^{(n)} = -\left(v_i v_j \theta^{(-2)} - \frac{n}{(v, u)}v_i u_j\right)U^{(n)} \implies \mathcal{J}^{(2)}U^{(n)} = 0. \quad (5.53b)
\]

It is clear that any weight-\(n\) projective multiplet is an isotwistor superfield, but not vice versa. If \(U^{(n-4)}\) is a Lorentz scalar, it follows from (5.52) that the weight-\(n\) isotwistor superfield \(Q^{(n)} := \Delta^{(4)}U^{(n-4)}\) obeys the analyticity constraint \(\mathcal{D}_\alpha^{(2)}Q^{(n)} = 0\), and therefore it is a projective multiplet. One can also prove that if under the super-Weyl transformations \(U^{(n-4)}\) varies as a primary field of special weight,

\[
\delta_\sigma U^{(n-4)} = \frac{(n - 2)}{2}\sigma U^{(n-4)},
\]

then \(\Delta^{(4)}U^{(n-4)}\) also transforms homogeneously according to eq. (5.40). The derivation of this property requires some straightforward algebra making use of eq. (5.51) and the relations

\[
[\mathcal{J}^{(2)}, \mathcal{D}_\alpha^{(2)}] = 0, \quad [\mathcal{J}^{(0)}, \mathcal{D}_\alpha^{(2)}] = -\mathcal{D}_\alpha^{(2)}, \quad \mathcal{J}^{(0)}U^{(n-4)} = -\frac{(n - 4)}{2}U^{(n-4)}. \quad (5.55)
\]
As a simple application of the construction described, we note that one can build a weight-4 projective superfield, $\Delta^{(4)}P$, from a $v$-independent scalar superfield $P$. The only condition that $P$ has to satisfy is to have weight one under super-Weyl transformations $\delta_\sigma P = \sigma P$.

The careful reader should have noticed that the explicit form of the analytic projector operator is formally equivalent to that of the antichiral projection operator

$$\Delta = -\frac{1}{4}(\mathcal{D}^2 - 4\bar{R}) \tag{5.56}$$

in $\mathcal{N} = 2$ conformal supergravity, see subsection 4.2. This is not surprising if one notes that the anti-commutation relation (5.22a) reduces to

$$\{\mathcal{D}^{(2)}_\alpha, \mathcal{D}^{(2)}_\beta\} U^{(n)} = -4iS^{(4)}\mathcal{M}_{\alpha\beta}U^{(n)} \tag{5.57}$$

when acting on an arbitrary isotwistor superfield $U^{(n)}$. This result is analogous to the first anti-commutation relation in (4.7a),

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} U = -4\bar{R}\mathcal{M}_{\alpha\beta}U, \tag{5.58}$$

for any $\mathcal{N} = 2$ tensor superfield. The relations (5.57) and (5.58) show an analogy between $\mathcal{N} = 3$ projective multiplets, $\mathcal{D}^{(2)}_\alpha Q^{(n)} = 0$, and $\mathcal{N} = 2$ antichiral superfields, $\mathcal{D}_\alpha \bar{\Psi} = 0$. In particular, both $Q^{(n)}$ and $\bar{\Psi}$ must be scalar with respect to the Lorentz group.

### 5.4 Vector multiplet prepotential

In this subsection we show that the constraints obeyed by the $\mathcal{N} = 3$ vector-multiplet field strength $W^{ij}$ can be solved in terms of a real weight-zero tropical prepotential $V(v^i)$ defined modulo arbitrary gauge transformations of the form

$$\delta V = \lambda + \bar{\lambda}, \tag{5.59}$$

where $\lambda(v^i)$ is an arctic weight-zero multiplet. Conceptually, this is similar to the situation in 4D $\mathcal{N} = 2$ conformal supergravity in which the covariantly chiral field strength of a vector multiplet is also given in terms of a weight-zero real tropical prepotential [66, 20], as an extension of the rigid-supersymmetric constructions given in [25, 28]. Technically, in 4D $\mathcal{N} = 2$ rigid supersymmetry, the idea to describe the massless vector multiplet in terms of a tropical multiplet appeared for the first time in [25]. The transformation law (5.59) is a locally supersymmetric version of that given in [25].

32
the 3D solution which we are going to present turns out to differ significantly from its four-dimensional counterpart.

We start from the real weight-zero tropical multiplet $V(v^i)$ and associate with it a weight-two isotwistor superfield $W^{(2)}(w^i)$ defined by

$$W^{(2)}(w) := \frac{1}{8\pi i} \oint_{\gamma} (v, dv) \left\{ (w, v)^2 D^{(-2)} D_{\alpha}^{(-2)} \right. - 4 \frac{(w, v)(w, u)}{(v, u)} D^{(-2)} D_{\alpha}^{(0)} \\
+ 4 \frac{(w, u)^2}{(v, u)^2} D^{(0)} D_{\alpha}^{(0)} - 4i (w, v)^2 S^{(-4)} + 8i \frac{(w, v)(w, u)}{(v, u)} S^{(-2)} \\
- 16i \frac{(w, u)^2}{(v, u)^2} S^{(0)} + 8i \frac{(w, u)^2}{(v, u)^2} S^{(2)} \right\} V(v),$$

for some closed integration contour $\gamma$. Here the integrand involves the superfields

$$S^{(-4)} := \frac{u_i u_j u_k u_l}{(v, u)^4} S^{ijkl}, \quad S^{(-2)} := \frac{v_i u_j u_k u_l}{(v, u)^3} S^{ijkl}, \quad S^{(0)} := \frac{v_i v_j u_k u_l}{(v, u)^2} S^{ijkl}$$

which are defined in accordance with our general conventions introduced earlier. It follows from (5.60) that $W^{(2)}(w)$ has the following functional form: $W^{(2)}(w) = W^{ij} w_i w_j$, for some real SU(2) triplet $W^{ij}$. The field strength (5.60) is indeed invariant under the gauge transformations (5.59).

A crucial property of (5.60) is that it does not depend on the auxiliary isospinor $u^i$. This property can be proved considering an infinitesimal transformation $\delta u^i = \alpha v^i$ and then making use of the analyticity condition $D^{(-2)} V = 0$ in conjunction with the anticommutation relations for the spinor covariant derivatives.

The fact that (5.60) is independent of $u^i$ can be used to derive two important implications. First of all, it allows us to prove the invariance of $W^{(2)}$ under the gauge transformation (5.59). Secondly, it makes it possible to prove that $W^{(2)}(w)$ is a projective multiplet. Indeed, let us choose $u^i = w^i$ in (5.60) and also re-label $w^i \rightarrow v^i$ and $v^i \rightarrow \hat{v}^i$. Then (5.60) turns into

$$W^{(2)}(v) = \Delta^{(4)} \oint_{\gamma} \frac{(\hat{v}, d\hat{v})}{2\pi(\hat{v}, \hat{v})^2} V(\hat{v}).$$

This representation makes it manifest that $W^{(2)}(v)$ is a projective multiplet.

We postulate the prepotential $V$ to be inert under the super-Weyl transformations,

$$\delta_\sigma V = 0$$

This leads to the correct transformation law for $W^{(2)}$.\footnote{To prove this the reader should use eq. (5.52) and the fact that $w_i w_j J^{ij} \oint (v, dw) V(z, v) = 0$.}
5.5 Supersymmetric action principle

With the results obtained in the previous subsections, we are now prepared to formulate a locally supersymmetric and super-Weyl invariant action principle.

Similarly to the off-shell supergravity-matter systems with eight supercharges in four and five dimensions \([18, 21]\), our Lagrangian \(\mathcal{L}^{(2)}\) is chosen to be a real weight-2 projective multiplet, with the following super-Weyl transformation law

\[
\delta_{\sigma} \mathcal{L}^{(2)} = \sigma \mathcal{L}^{(2)} .
\]

(5.64)

Associated with \(\mathcal{L}^{(2)}\) is the action

\[
S(\mathcal{L}^{(2)}) = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \int d^{3}x d^{6}\theta EC^{(-4)} \mathcal{L}^{(2)} , \quad E^{-1} = \text{Ber}(E_{A}^{M}) .
\]

(5.65)

Here the superfield \(C^{(-4)}\) is required to be a Lorentz-scalar isotwistor superfield of weight \(-4\) such that the following two conditions hold:

\[
\delta_{\sigma} C^{(-4)} = -\sigma C^{(-4)} ,
\]

(5.66a)

\[
\Delta^{(4)} C^{(-4)} = 1 .
\]

(5.66b)

These conditions prove to guarantee that the action (5.65) is invariant under the supergravity gauge and the super-Weyl transformations. The invariance of \(S\) under the supergravity gauge transformations can be proven in complete analogy to the 5D and 4D cases \([17, 18, 19, 21]\). To show that the action (5.65) is super-Weyl invariant, it is necessary to make use of the super-Weyl transformation laws (5.64) and (5.66a), as well as to use the observation that

\[
\delta_{\sigma} E = 0 ,
\]

(5.67)

which is similar to the 4D \(\mathcal{N} = 2\) case.

All information about a concrete dynamical system is encoded in its Lagrangian \(\mathcal{L}^{(2)}\). It may look somewhat odd that the action (5.65) also involves the ‘compensating’ superfield \(C^{(-4)}\), in principle one and the same for all dynamical systems. The important point, however, is that the action (5.65) does not depend on \(C^{(-4)}\) if the Lagrangian \(\mathcal{L}^{(2)}\) is independent of \(C^{(-4)}\). To prove this statement, let us represent the Lagrangian as \(\mathcal{L}^{(2)} = \Delta^{(4)} U^{(-2)}\), for some isotwistor superfield \(U^{(-2)}\) of weight \(-2\). Then, making use of eq. (5.66b) allows us to rewrite the action in the form

\[
S = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \int d^{3}x d^{6}\theta E U^{(-2)} .
\]

(5.68)
This representation makes manifest the fact that the action does not depend on $C^{(-4)}$.

A natural choice for $C^{(-4)}$ is available if the theory under consideration possesses an Abelian vector multiplet such that its field strength $W^{ij}$ is nowhere vanishing, that is $W := \sqrt{W^{ij}W_{ij}} \neq 0$. Such a vector multiplet may be a conformal compensator. Since the super-Weyl transformation of $W$ is

$$\delta_\sigma W = \sigma W ,$$  \hspace{1cm} (5.69)

we immediately observe that $C^{(-4)}$ can be chosen as

$$C^{(-4)} := \frac{W}{\Sigma^{(4)}}, \quad \Sigma^{(4)} := \Delta^{(4)}W .$$  \hspace{1cm} (5.70)

Indeed, the condition (5.66a) holds since the super-Weyl transformation of $\Sigma^{(4)}$ is

$$\delta_\sigma \Sigma^{(4)} = 2\sigma \Sigma^{(4)} .$$  \hspace{1cm} (5.71)

The condition (5.66b) holds, since $\Sigma^{(4)}$ is an $O(4)$ multiplet.

More generally, given a real weight-$n$ isotwistor superfield $U^{(n)}$, with the super-Weyl transformation law (5.54), it is possible to define $C^{(-4)}$ as

$$C^{(-4)} = \frac{U^{(n)}}{\Delta^{(4)}U^{(n)}},$$  \hspace{1cm} (5.72)

provided $(\Delta^{(4)}U^{(n)})^{-1}$ is well defined.

The action (5.65) has the following important property:

$$S\left(W^{(2)}(\lambda + \bar{\lambda})\right) = 0 ,$$  \hspace{1cm} (5.73)

with $W^{(2)}$ a real $O(2)$ multiplet and $\lambda$ an arctic weight-zero multiplet.

### 5.6 Conformal compensators

As is well known, conformal supergravity is a useful starting point to construct Poincaré supergravity theories [30]. This is achieved by coupling the (Weyl multiplet) (i.e. the multiplet of conformal supergravity) to a compensating matter multiplet (compensator). The latter allows one to gauge away part of the local symmetries by imposing appropriate gauge conditions. In 4D $\mathcal{N} = 2$ supergravity, two compensators are required of which one is a vector multiplet (see [32] and references therein). In the case of $\mathcal{N} = 3$
supergravity in three dimensions, the vector multiplet can be chosen as a compensator. Its field strength $W^{ij}$ must be nowhere vanishing, that is $W := \sqrt{W^{ij}W_{ij}} \neq 0$. Then, the super-Weyl gauge freedom can be used to impose the gauge condition $W = 1$. After that, the local SU(2) symmetry allows one to set $W^{ij} = w^{ij}$, for some constant SU(2) triplet $w^{ij}$ of unit length.

The supergravity Lagrangian is

$$L^{(2)}_{\text{SUGRA}} = \frac{1}{\kappa^2} W^{(2)} \ln \frac{W^{(2)}}{i\bar{\Upsilon}^{(1)}\bar{\Upsilon}^{(1)}} + \frac{\xi}{\kappa^2} V W^{(2)} ,$$

(5.74)

with $\kappa^2$ and $\xi$ the gravitational and cosmological constants, respectively. The cosmological term is described by a U(1) Chern-Simons term. The action is invariant under the gauge transformations (5.59). The first term in (5.74) is (minus) the Lagrangian for a massless improved vector multiplet coupled to conformal supergravity. Its 4D $\mathcal{N} = 2$ counterpart was given in [20] as a locally supersymmetric extension of the projective-superspace formulation [23] for the 4D $\mathcal{N} = 2$ improved tensor multiplet [32, 74].

The supergravity action can equivalently be described by the following Lagrangian

$$\tilde{L}^{(2)}_{\text{SUGRA}} = \frac{1}{\kappa^2} V \left\{ \mathbb{W}^{(2)} + \xi W^{(2)} \right\} ,$$

(5.75)

where

$$\mathbb{W}^{(2)} := W^{ij} v_i v_j = \Delta^{(4)} \oint \frac{(\hat{v}, d\hat{v})}{2\pi(v, \hat{v})^2} \ln \frac{W^{(2)}(\hat{v})}{i\bar{\Upsilon}^{(1)}(\hat{v})\bar{\Upsilon}^{(1)}(\hat{v})} , \quad D^{(2)}_a \mathbb{W}^{(2)} = 0$$

(5.76)

is a composite real $O(2)$ multiplet. The contour integral in (5.76) can be evaluated using the technique developed in [68].

In the case $\xi = 0$, we can construct a dual supergravity formulation by considering the first-order model

$$L^{(2)}_{\text{first-order}} = \frac{1}{\kappa^2} U^{(2)} \left( \ln \frac{U^{(2)}}{i\bar{\Upsilon}^{(1)}\bar{\Upsilon}^{(1)}} - 1 \right) ,$$

(5.77)

where $U^{(2)}$ is a real weight-two tropical multiplet. Varying the first-order action with respect to $\Upsilon^{(1)}$ and its conjugate gives $U^{(2)} = W^{(2)}$, and then we return to the original formulation. On the other hand, varying the first-order action with respect to $U^{(2)}$ gives $U^{(2)} = i\bar{\Upsilon}^{(1)}\bar{\Upsilon}^{(1)}$, and we arrive at the dual formulation

$$L^{(2)}_{\text{SUGRA, dual}} = -\frac{i}{\kappa^2} \Upsilon^{(1)}\bar{\Upsilon}^{(1)}$$

(5.78)

in which the compensator is an off-shell hypermultiplet.

If the cosmological constant is non-zero, $\xi \neq 0$, then the theory (5.74) proves to be self-dual under a different type of duality transformation that is similar to the one considered in [20].
5.7 Locally supersymmetric sigma-models

The Lagrangian in (5.65) is required to be a real weight-two covariant projective multiplet with the super-Weyl transformation law (5.64). Otherwise $L^{(2)}$ may be completely arbitrary. This freedom in the choice of $L^{(2)}$ means that practically any off-shell $\mathcal{N} = 3$ rigid superconformal theory [33] can be coupled to $\mathcal{N} = 3$ conformal supergravity.

We consider a system of interacting weight-one arctic multiplets, $\Upsilon^{(1)I}(v)$, and their smile-conjugates, $\tilde{\Upsilon}^{(1)I}(v)$, described by a Lagrangian of the form [29]:

$$L^{(2)} = i K(\Upsilon^{(1)I}, \tilde{\Upsilon}^{(1)J}) .$$  \hspace{1cm} (5.79)

Here $K(\Phi^{I}, \bar{\Phi}^{J})$ is a real function of $n$ complex variables $\Phi^{I}$, with $I = 1, \ldots, n$, satisfying the homogeneity condition

$$\Phi^{I} \frac{\partial}{\partial \Phi^{I}} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) .$$  \hspace{1cm} (5.80)

The function $K(\Phi^{I}, \bar{\Phi}^{J})$ can be interpreted as the Kähler potential of a Kählerian cone $\mathcal{M}$ written in special complex coordinates in which the homothetic conformal Killing vector field $\chi^{I}(\Phi)$ has the form $\chi^{I}(\Phi) = \Phi^{I}$.

There exists a more geometric formulation of the theory (5.79) described in detail in [67]. It is realized in terms of a single weight-one arctic multiplet $\Upsilon^{(1)}$ and $n - 1$ weight-zero arctic multiplets $\Xi^{i}$. The corresponding Lagrangian is

$$K(\Upsilon^{(1)I}, \tilde{\Upsilon}^{(1)J}) = \Upsilon^{(1)} \tilde{\Upsilon}^{(1)} \exp \left\{ K(\Xi^{i}, \bar{\Xi}^{j}) \right\} ,$$  \hspace{1cm} (5.81)

where the original variables $\Upsilon^{(1)I}$ are related to the new ones by a holomorphic reparametrization. The arctic variables $\Upsilon^{(1)}$ and $\Xi^{i}$ parametrize a holomorphic line bundle over a Kähler-Hodge manifold with Kähler potential $K(\varphi^{i}, \bar{\varphi}^{j})$, see [67] for more details.

Consider a system of $n$ Abelian vector multiplets, and let $W^{(2)}_{I}$ be their field strengths, $I = 1, \ldots, n$. Its dynamics can be described by a Lagrangian of the form

$$L^{(2)} = L(W^{(2)}_{I}) ,$$  \hspace{1cm} (5.82)

where $L$ is a real homogeneous function of degree +1,

$$W^{(2)}_{I} \frac{\partial}{\partial W^{(2)}_{I}} L = L .$$  \hspace{1cm} (5.83)

\footnote{The action generated by the Lagrangian (5.79) is real due to (5.39).}
The vector multiplet model (5.82) can be generalized to include a Chern-Simons term
\[
\mathcal{L}^{(2)}_{CS} = \mathcal{L}(W_I^{(2)}) + \frac{1}{2} m^{IJ} W_I W_J^{(2)}, \quad m^{IJ} = m^{JI} = (m^{IJ})^* = \text{const}. 
\] (5.84)
Here \(V_I\) is the weight-zero tropical prepotential for the field strengths \(W_I\), eq. (5.62). The action associated with \(\mathcal{L}^{(2)}_{CS}\) is invariant under gauge transformations \(\delta V_I = \lambda_I + \tilde{\lambda}_I\), with \(\lambda_I\) arctic weight-zero multiplets.

6 Matter couplings in \(\mathcal{N} = 4\) supergravity

The structure of multiplets in 3D \(\mathcal{N} = 4\) supersymmetry is largely determined by the fact that the Lie algebra of the \(R\)-symmetry group is reducible, \(so(4) \cong su(2) \oplus su(2)\).

6.1 Elaborating on the \(\mathcal{N} = 4\) superspace geometry

Within the geometric formulation developed in section 2, the structure group of \(\mathcal{N} = 4\) conformal supergravity is \(\text{SL}(2, \mathbb{R}) \times \text{SO}(4)\), with the spinor derivatives \(\mathcal{D}_\alpha^I\) transforming in the defining (vector) representation of \(\text{SO}(4)\). In order to define a large class of matter multiplets coupled to supergravity, it is advantageous to make use of the isomorphism \(\text{SO}(4) \cong (\text{SU}(2)_L \times \text{SU}(2)_R)/\mathbb{Z}_2\) and switch to an isospinor notation, \(\mathcal{D}_\alpha^I \rightarrow \mathcal{D}_\alpha^{\bar{i}i}\), by replacing each \(\text{SO}(4)\) vector index by a pair of isospinor ones. We use the notation \(\psi_i\) and \(\chi^\bar{i}\) to denote the isospinors which transform under the defining representations of \(\text{SU}(2)_L\) and \(\text{SU}(2)_R\), respectively. The rules for raising and lowering isospinor indices are spelled out in Appendix A. The algebraic structure underlying the correspondence \(\mathcal{D}_\alpha^I \rightarrow \mathcal{D}_\alpha^{\bar{i}i}\) is also explained in Appendix A. For completeness, here we only repeat the definition. Associated with a real \(\text{SO}(4)\) vector \(V_I\) is a second-rank isospinor \(V^{\bar{i}i}\) defined as
\[
V_I \rightarrow V^{\bar{i}i} := (\tau^I)_\bar{i}^i V_I, \quad V_I = (\tau_I)^{\bar{i}i} V^{\bar{i}i}, \quad (V^{\bar{i}i})^* = V^{\bar{i}i}, \quad (6.1)
\]
see Appendix A for the definition of the \(\tau\)-matrices. If \(V_I\) and \(U_I\) are two \(\text{SO}(4)\) vectors, and \(V^{\bar{i}i}\) and \(U^{\bar{i}i}\) the associated second-rank isospinors, then
\[
V^I U_I = V^{\bar{i}i} U^{\bar{i}i}. \quad (6.2)
\]
Along with the relations (6.1) and (6.2), we need a few more general results. Given an antisymmetric second-rank \(\text{SO}(4)\) tensor, \(A_{IJ} = -A_{JI}\), its counterpart with isospinor indices, \(A^{\bar{i}j}_{\bar{i}j} = -A^{\bar{i}j}_{\bar{j}i} = A_{IJ} (\tau^I)^{\bar{i}i} (\tau^J)^{\bar{j}j}\) can be decomposed as
\[
A^{\bar{i}j}_{\bar{i}j} = \varepsilon_{ij} A^{\bar{i}j}_{\bar{i}j} + \varepsilon_{ij} A_{ij} \rightarrow A^{\bar{i}j}_{\bar{i}j} = -\varepsilon^{ij} A^{\bar{i}j}_{\bar{j}i} - \varepsilon^{ij} A^{\bar{i}j}_{\bar{i}j}, \quad A_{ij} = A_{ji}, \quad A_{ij} = A_{ji}. \quad (6.3)
\]
Here the two independent symmetric isospinors $A_{ij}$ and $\overline{A}_{ij}$ represent the self-dual and anti-self-dual parts of the antisymmetric tensor $A_{IJ}$. Given another antisymmetric second-rank $\text{SO}(4)$ tensor, $B_{IJ} = -B_{JI}$, and the corresponding isospinor counterparts $B_{ij}$ and $\overline{B}_{ij}$, one can check that

$$\frac{1}{2}A^{IJ}B_{IJ} = A^{ij}B_{ij} + A^{\overline{i}\overline{j}}B_{\overline{i}\overline{j}}.$$  

(6.4)

Finally, consider the completely antisymmetric fourth-rank tensor $\varepsilon_{IJKL}$ normalized by $\varepsilon_{1234} = 1$. Its isospinor counterpart is

$$\varepsilon_{\overline{i}\overline{j}k\overline{l}} := \varepsilon_{IJKL}(\tau^I)_{\overline{i}l}(\tau^J)_{\overline{j}l}(\tau^K)_{k\overline{l}}(\tau^L)_{l\overline{i}} = \left(\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{\overline{i}\overline{j}}\varepsilon_{\overline{k}\overline{l}} - \varepsilon_{il}\varepsilon_{jk}\varepsilon_{\overline{i}\overline{j}}\varepsilon_{\overline{k}\overline{l}}\right).$$  

(6.5)

We are now prepared to specify the $\mathcal{N}$-extended supergravity algebra, which was derived in section 2, to the case $\mathcal{N} = 4$ and rewrite it using the isospinor notation introduced. The covariant derivatives are

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i) = E_A + \Omega_A + \Phi_A,$$  

(6.6)

where the original $\text{SO}(4)$ connection $\Phi_A$ now turns into a sum of two $\text{SU}(2)$ connections, the left $(\Phi_L)_A$ and the right $(\Phi_R)_A$ ones,

$$\Phi_A = (\Phi_L)_A + (\Phi_R)_A, \quad (\Phi_L)_A = \Phi_A^{kl}L_{kl}, \quad (\Phi_R)_A = \Phi_A^{\overline{k}\overline{l}}R_{\overline{k}\overline{l}}.$$

(6.7)

Here $L_{kl}$ are the generators of $\text{SU}(2)_L$ and $R_{\overline{k}\overline{l}}$ the generators of $\text{SU}(2)_R$. They are related to the $\text{SO}(4)$ generators $\mathcal{N}_{KL}$ as

$$\mathcal{N}_{KL} \rightarrow \mathcal{N}_{k\overline{l}} = \varepsilon_{k\overline{l}}L_{kl} + \varepsilon_{k\overline{l}}R_{\overline{k}\overline{l}}.$$  

(6.8)

The same decomposition into left and right sectors takes place for the $\text{SO}(4)$ curvature and for the $\text{SO}(4)$ gauge parameters. The two sets of $\text{SU}(2)$ generators act on the spinor covariant derivatives $\mathcal{D}_a^\overline{i} := \mathcal{D}_a^l(\tau^l)^\overline{i}$ as follows:

$$[L_{kl}, \mathcal{D}_a^\overline{i}] = \varepsilon^{\overline{i}(k}\mathcal{D}_a^\overline{j}\varepsilon^{j\overline{l}}), \quad [R_{\overline{k}\overline{l}}, \mathcal{D}_a^\overline{i}] = \varepsilon^{\overline{i}(k}\mathcal{D}_a^\overline{j}\varepsilon^{j\overline{l}}).$$  

(6.9)

As shown in section 2, in $\mathcal{N}$-extended curved superspace the torsion and the curvature of dimension 1 are given in terms of the three tensor superfields: $X^{IJKL}$, $C_a^{IJ}$ and $S^{IJ}$. We recall that the completely antisymmetric curvature $X^{IJKL}$ does not occur for $\mathcal{N} < 4$. In the $\mathcal{N} = 4$ case, these superfields take the form:

$$X^{IJKL} \rightarrow X^{\overline{i}\overline{j}k\overline{l}} = \varepsilon^{\overline{i}\overline{j}\overline{k}\overline{l}}X = \left(\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{\overline{i}\overline{j}}\varepsilon_{\overline{k}\overline{l}} + \varepsilon_{il}\varepsilon_{jk}\varepsilon_{\overline{i}\overline{j}}\varepsilon_{\overline{k}\overline{l}}\right)X,$$  

(6.10)

$$C_a^{IJ} \rightarrow C_a^{\overline{i}\overline{j}} = -\varepsilon^{\overline{i}}B_a^{\overline{j}} - \varepsilon^{\overline{j}}C_a^{\overline{i}}, \quad B_a^{ij} = B_a^{\overline{j}i}, \quad C_a^{\overline{i}\overline{j}} = C_a^{\overline{j}\overline{i}},$$  

(6.11)

$$S^{IJ} \rightarrow S^{\overline{i}\overline{j}} + \varepsilon^{\overline{i}}\varepsilon^{\overline{j}}S, \quad S^{\overline{i}\overline{j}} = S^{\overline{j}\overline{i}} = S^{\overline{i}\overline{j}}.$$  

(6.12)
The algebra of spinor covariant derivatives becomes
\[ \{ D^i_{\alpha}, D^j_{\beta} \} = 2i \varepsilon^{ij} (\gamma^c)_{\alpha\beta} D_c + 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} (2S + X)L^ij - 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} S^{kij}L_{kl} + 4i C_{\alpha\beta} \bar{S}^{ij}L_{ij} \]
\[ + 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} (2S - X)R^ij - 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} S^{ijkl}R_{kl} + 4i B_{\alpha\beta} \bar{S}^{ij} \]
\[ + 2i \varepsilon_{\alpha\beta} (\varepsilon^{ij} R^\delta ij + \varepsilon^{ij} C^{\gamma\delta ij})M_{\gamma\delta} - 4i (S^{ij})^\gamma + i \varepsilon^{ij} S)M_{\alpha\beta} . \]  

(6.13)

It can be shown that the dimension-3/2 Bianchi identities take the form:
\[ D^i_{\alpha} S^{jk\bar{k}} = 2T^{i,jk\bar{k}}_{\alpha} - 2\varepsilon^{ij} (T^{k\bar{k}}_{\alpha} - \varepsilon^{ij} S^a_{\alpha}) (\bar{S}^i_{\alpha} \bar{S}^{j\bar{k}}_{\alpha}) , \]  

(6.14a)
\[ D^i_{\alpha} B^{\beta\gamma} \bar{j} \bar{k} = -\varepsilon^{i,j} (A_{\alpha\beta} \bar{S}^{k\bar{j}i}_{\alpha} - C_{\alpha\beta} \bar{k}^{\bar{j}i}_{\alpha}) - \frac{2}{3} \varepsilon_{\alpha\beta\gamma} \varepsilon^{i,j} (D^{k\bar{i}i}_{\gamma}) (2S - X) + 2\varepsilon_{\alpha\beta T_{\gamma}}^{i,jk\bar{k}} , \]  

(6.14b)
\[ D^i_{\alpha} C_{\beta\gamma} \bar{j} \bar{k} = -(A_{\alpha\beta\gamma} i\bar{\bar{j}} + C_{\alpha\beta\gamma} i\bar{\bar{j}}) \varepsilon^{k\bar{i}i}_{\alpha} + \frac{2}{3} \varepsilon_{\alpha\beta\gamma} (D^{i\bar{k}i}_{\gamma}) (2S + X) \varepsilon^{k\bar{i}i}_{\alpha} + 2\varepsilon_{\alpha\beta T_{\gamma}}^{i,jk\bar{k}} . \]  

(6.14c)

Here the superfields appearing in the right-hand sides have the following algebraic properties:
\[ T^{k\bar{j}i}_{\alpha} = T^{k\bar{j}i}_{\alpha} , \quad T_{\alpha}^{ijk} = -T^{ijk}_{\alpha} , \quad C_{\alpha\beta} \bar{\bar{i}} = C_{\alpha\beta\gamma} \bar{\bar{i}} , \quad A_{\alpha\beta\gamma} \bar{\bar{i}} = A_{\alpha\beta\gamma} \bar{\bar{i}} . \]  

(6.15)

These superfields are related to those introduced in eqs. (2.18a)-(2.18c) as follows:
\[ T^{1JK}_{\alpha} \to T^{i\bar{i}jk\bar{k}}_{\alpha} = -\varepsilon^{i\bar{i}} T_{\alpha}^{ijk\bar{k}} - \varepsilon^{ij} T_{\alpha}^{k\bar{i}i} , \]  

(6.16a)
\[ C_{\alpha\beta\gamma}^{1JK} \to C_{\gamma\bar{i}\bar{k}}^{\bar{i}jk\bar{k}} = \varepsilon^{i\bar{i}} A_{\alpha\beta\gamma} \bar{k}^{i\bar{i}} - \varepsilon^{jk} A_{\alpha\beta} \bar{\bar{k}}^{\bar{i}}\bar{i} \]  

(6.16b)
\[ C_{\alpha}^{1JK} \to C_{\alpha}^{i\bar{i}jk\bar{k}} = -\varepsilon^{i\bar{i}} (D_{\alpha}^{i\bar{k}i} X) \varepsilon^{\bar{i}j} + \varepsilon^{jk} (D_{\alpha}^{i\bar{k}i} X) \varepsilon^{\bar{i}i} . \]  

(6.16c)

An important property of the \( N = 4 \) curved superspace geometry is its invariance under the discrete transformation
\[ \mathcal{M} : SU(2)_{L} \leftrightarrow SU(2)_{R} \]  

(6.17)

which changes the tensor types of superfields as \( D^{(p/2)}_{L} \otimes D^{(q/2)}_{R} \to D^{(q/2)}_{L} \otimes D^{(p/2)}_{R} \), where \( D^{(p/2)} \) denotes the spin-\( p \) representation of \( SU(2) \). In this rigid supersymmetric case, this transformation is an outer automorphism of the \( N = 4 \) super-Poincaré algebra, which underlies mirror symmetry in 3D \( N = 4 \) Abelian gauge theories [69]. It has been studied by Zupnik [45, 46] within the 3D \( N = 4 \) rigid harmonic superspace [14]. Following [45, 46], we call \( \mathcal{M} \) the mirror map.

The various geometric objects behave differently under the mirror map:
\[ \mathcal{M} \cdot S = S , \quad \mathcal{M} \cdot S^{ij\bar{i}} = S^{ij\bar{i}} , \quad \mathcal{M} \cdot X = -X , \]  

(6.18a)
\[ \mathcal{M} \cdot C_{\alpha}^{\bar{j}} = B_{\alpha}^{\bar{j}} , \quad \mathcal{M} \cdot B_{\alpha}^{ij} = C_{\alpha}^{\bar{j}} ; \]  

(6.18b)
\[ \mathcal{M} \cdot S_{\alpha}^{\bar{j}i} = S_{\alpha}^{\bar{j}i} , \quad \mathcal{M} \cdot A_{\alpha\beta}^{\bar{i}i} = -A_{\alpha\beta}^{\bar{i}i} , \quad \mathcal{M} \cdot C_{\alpha\beta}^{\bar{i}i} = C_{\alpha\beta}^{\bar{i}i} , \]  

(6.18c)
\[ \mathcal{M} \cdot T_{\alpha}^{i\bar{j}k\bar{i}} = T_{\alpha}^{i\bar{j}k\bar{i}} , \quad \mathcal{M} \cdot T_{\alpha}^{i\bar{j}k\bar{i}} = T_{\alpha}^{i\bar{j}k\bar{i}} . \]  

(6.18d)
We conclude by giving the super-Weyl transformation in the isospin notation:

\[
\delta_{\sigma} D^{\alpha}_{\dot{\alpha}} = \frac{1}{2} \sigma D^{\alpha}_{\dot{\alpha}} + (D^{\beta\dot{\alpha}}_{\mu} \sigma) M_{\mu\beta} - (D^{\dot{\alpha}}_{\mu\dot{\alpha}} \sigma) L^ij - (D_{\mu\dot{\alpha}} \sigma) R_{\dot{\alpha}ij} ,
\]

\[
\delta_{\sigma} D_a = \sigma D_a + \frac{i}{2} (\gamma_a) \gamma^\delta (D^{\delta\dot{\alpha}}_{\gamma} \sigma) M_{a\dot{\alpha}} + \varepsilon_{abc} (D^b \sigma) M^c
\]

\[
+ \frac{i}{16} (\gamma_a) \gamma^\delta (D^{k\dot{\alpha}}_{\gamma} \sigma) L_{kl} + \frac{i}{16} (\gamma_a) \gamma^\delta (D^{\delta\dot{\alpha}}_{\gamma} \sigma) R_{\dot{\alpha}kl} .
\]

The dimension-1 torsion and curvature superfields transform as follows:

\[
\delta_{\sigma} S^{ij}_{\dot{i}\dot{j}} = \sigma S^{ij}_{\dot{i}\dot{j}} - \frac{i}{8} [D^i_{(i}, D^{j)}_{\gamma)} \sigma] , \quad \delta_{\sigma} S = \sigma S - \frac{i}{32} [D^{k\dot{k}}_{\gamma} , D^\gamma_{\dot{k}k}] \sigma ,
\]

\[
\delta_{\sigma} B_a^{ij} = \sigma B_a^{ij} - \frac{i}{16} (\gamma_a) \gamma^\delta (D^{i\dot{\alpha}}_{\gamma} , D^{j}_{\dot{\alpha}}) \sigma ,
\]

\[
\delta_{\sigma} C_a^{\dot{i}\dot{j}} = \sigma C_a^{\dot{i}\dot{j}} - \frac{i}{16} (\gamma_a) \gamma^\delta (D^{k\dot{i}}_{\gamma} , D^{\dot{j}}_{\dot{\alpha}k}) \sigma ,
\]

\[
\delta_{\sigma} X = \sigma X .
\]

### 6.2 Covariant projective multiplets

In this section we introduce a curved-superspace extension of the \( \mathcal{N} = 4 \) superconformal projective multiplets [33]. As in the \( \mathcal{N} = 3 \) case, it is natural to start our analysis with a more detailed look at the properties of the \( \mathcal{N} = 4 \) vector multiplet in conformal supergravity, and then turn to more general supermultiplets.

In accordance with the consideration of subsection 6.1, the vector-multiplet field strength \( W^{IJ} = -W^{JI} \) is equivalently described by two symmetric second-rank isospinors, \( W^{ij} \) and \( W^{\dot{i}\dot{j}} \), which are defined as

\[
W^{IJ} \rightarrow W^{i\dot{i}j\dot{j}} = -\varepsilon^{\dot{i}j} W^{ij} - \varepsilon^{i\dot{i}} W^{\dot{i}\dot{j}} , \quad W^{ij} = W^{ji} , \quad W^{i\dot{i}j\dot{j}} = W^{\dot{i}\dot{j}i} \]

and transform under the local groups SU(2)\(_L\) and SU(2)\(_R\), respectively. The dimension-3/2 Bianchi identity (2.35) turns into the two independent analyticity constraints

\[
D^{i\dot{i}}_{a} W^{k\dot{k}} = 0 ,
\]

\[
D^{i\dot{i}}_{a} W^{k\dot{k}} = 0 .
\]

As a result, the field strengths \( W^{ij} \) and \( W^{i\dot{i}j\dot{j}} \) are completely independent of each other. Therefore, the \( \mathcal{N} = 4 \) supermultiplet described by \( W^{IJ} \) is reducible and is, in fact, a superposition of two inequivalent off-shell \( \mathcal{N} = 4 \) vector multiplets. One of them is characterized by the condition \( W^{ij} = 0 \), while for the other vector multiplet \( W^{i\dot{i}j\dot{j}} = 0 \).
The existence of two inequivalent off-shell $\mathcal{N} = 4$ vector multiplets in three dimensions was discovered by Brooks and Gates [70] (see also [71] where the results of [70] were recast in terms of $\mathcal{N} = 2$ superfields).

A superfield $W^{ij}$ under the constraint (6.21a) will be called a left linear multiplet. Similarly, eq. (6.21b) defines a right linear multiplet. These multiplets are 3D analogues of the 4D $\mathcal{N} = 2$ linear multiplet [72, 73].

The constraints (6.21a) and (6.21b) can be rewritten as generalized chirality conditions. This can be achieved, as in the $\mathcal{N} = 3$ case studied earlier, by allowing for auxiliary bosonic dimensions. Specifically, let us introduce left and right isospinor variables, $v_L := v^i \in \mathbb{C}^2 \setminus \{0\}$ and $v_R := v^\dagger \in \mathbb{C}^2 \setminus \{0\}$, and use them to define two different subsets, $\mathcal{D}_{\alpha}^{(1)i}$ and $\mathcal{D}_{\alpha}^{(1)i}$, in the set of spinor covariant derivatives $\mathcal{D}_{\alpha}^{\bar{i}\bar{i}}$.

\[
\mathcal{D}_{\alpha}^{(1)i} := v_i \mathcal{D}_{\alpha}^{\bar{i}} , \quad \mathcal{D}_{\alpha}^{(1)i} := v_i \mathcal{D}_{\alpha}^{\bar{i}} , \quad (6.22)
\]
as well as the index-free superfields

\[
W_L^{(2)} := v_i v_j W^{ij} \equiv W^{(2)} , \quad W_R^{(2)} := v_i v_j W^{\bar{i}\bar{j}} \equiv W^{(2)} \quad (6.23)
\]
associated with the left and the right linear multiplets, respectively. Now, the constraints (6.21a) and (6.21b) become

\[
\mathcal{D}_{\alpha}^{(1)i} W_L^{(2)} = 0 \quad , \quad (6.24a) \\
\mathcal{D}_{\alpha}^{(1)i} W_R^{(2)} = 0 \quad . \quad (6.24b)
\]
The parenthesized superscripts attached to $\mathcal{D}_{\alpha}^{(1)i}$ and $W_L^{(2)}$ indicate the degree of homogeneity in the left isospinor $v_i$. The same convention is used for the right objects $\mathcal{D}_{\alpha}^{(1)i}$ and $W_R^{(2)}$, but with $v_i \rightarrow v_i$. In complete analogy with the $\mathcal{N} = 3$ case, both $v_i$ and $v_i$ are chosen to be inert under the local SU(2)$_L$ and SU(2)$_R$ transformations.

Since the right linear multiplet, $W^{\bar{i}\bar{j}}$, can be obtained from the left one, $W^{ij}$, by applying the mirror map, it suffices to restrict our analysis to the latter. Consider an infinitesimal supergravity gauge transformation

\[
\delta_K \mathcal{D}_A = [K, \mathcal{D}_A] , \quad K = K^C \mathcal{D}_C + \frac{1}{2} K^{cd} \mathcal{M}_{cd} + K^{kl} \mathcal{L}_{kl} + K^{k\bar{l}} \mathcal{R}_{k\bar{l}} . \quad (6.25)
\]
It acts on $W^{ij}$ as

\[
\delta_K W^{ij} = K^C \mathcal{D}_C W^{ij} + 2 W^{l(j} K^{i)}l . \quad (6.26)
\]
In terms of $W_L^{(2)}$ this transformation law takes the form

$$
\delta_K W_L^{(2)} = \left( K^C \mathcal{D}_C + K^{ij} L_{ij} \right) W_L^{(2)},
$$

$$
K^{ij} L_{ij} W_L^{(2)} = - \left( K_L^{(2)} \partial_L^{(-2)} - 2 K_L^{(0)} \right) W_L^{(2)},
$$

where we have used the notations:

$$
K_L^{(2)} = K^{ij} v_i v_j, \quad K_L^{(0)} = \frac{v_i u_j}{(v_L, u_L)} K^{ij}, \quad (v_L, u_L) := v^i u_i.
$$

The differential operator $\partial_L^{(-2)}$ is defined as

$$
\partial_L^{(-2)} := \frac{1}{(v_L, u_L)} u^i \frac{\partial}{\partial v^i}.
$$

Here we have introduced a second left isospinor variable $u_L := u^i$ which is restricted to be linearly independent of $v_L$, that is $(v_L, u_L) \neq 0$. Thus $v^i$ and $u^i$ can be used to define a new basis for the left isospinor indices, with the aid of the completeness relation

$$
\delta_j^i = \frac{1}{(v_L, u_L)} \left( v^i u_j - v_j u^i \right),
$$

in complete analogy with our previous consideration for $\mathcal{N} = 3$ supergravity, see eq. (5.31). For example, the generators $L^{ij}$ of the group $SU(2)_L$ turn into

$$
L^{(2)} := v_i v_j L^{ij}, \quad L^{(0)} := \frac{1}{(v_L, u_L)} v_i u_j L^{ij}, \quad L^{(-2)} := \frac{1}{(v_L, u_L)^2} u_i u_j L^{ij}.
$$

Then it follows from (6.27b) that

$$
L^{(2)} W_L^{(2)} = 0.
$$

This identity is crucial for the consistency of the constraints (6.24a). Indeed, the spinor covariant derivatives $\mathcal{D}_\alpha^{(1)i}$ obey the anticommutation relations

$$
\{ \mathcal{D}_\alpha^{(1)i}, \mathcal{D}_\beta^{(1)j} \} = \left[ 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} (2S + X) L^{(2)} + 4i C_{\alpha \beta}^{\gamma} L^{(2)} - 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} S^{(2)kl} R_{kl} + 4i B_{\alpha \beta}^{(2)} R^{ij} \right. \\
\left. + 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} B^{(2)\gamma} M_{\gamma \delta} - 4i S^{(2)ij} M_{\alpha \beta} \right],
$$

where we have defined

$$
B_{\alpha \beta}^{(2)} := B_{\alpha \beta}^{ij} v_i v_j, \quad S^{(2)ij} := S^{ij} v_i v_j.
$$

Since $W_L^{(2)}$ is inert under both the Lorentz and $SU(2)_R$ transformations, eq. (6.32) guarantees that the requirement $\{ \mathcal{D}_\alpha^{(1)i}, \mathcal{D}_\beta^{(1)j} \} W_L^{(2)} = 0$ holds.
The properties of $W^{(2)}_L$, which we have just described, are analogous to those of the $O(2)$ multiplet in 4D $\mathcal{N} = 2$ supergravity [19, 21]. To comply with the four-dimensional terminology, $W^{(2)}_L$ and $W^{(2)}_R$ will be called left and right $O(2)$ multiplets, respectively.

The super-Weyl transformation of $W^{(2)}_L$ is

$$\delta_{\sigma} W^{(2)}_L = \sigma W^{(2)}_L.$$  \hspace{1cm} (6.35)

We are now prepared to introduce a large family of off-shell supermultiplets with properties similar to those of $W^{(2)}_L$. A covariant left projective multiplet of weight $n$, $Q^{(n)}_L(z, v_L)$, is defined to be a Lorentz and SU(2)$_R$ scalar superfield that lives on the curved $\mathcal{N} = 4$ superspace $\mathcal{M}^{3|8}$, is holomorphic with respect to the isospinor variables $v^i$ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(i) it obeys the covariant analyticity constraints

$$D^{(1)i}_{\alpha} Q^{(n)}_L = 0 ;$$  \hspace{1cm} (6.36)

(ii) it is a homogeneous function of $v_L$ of degree $n$, that is,

$$Q^{(n)}_L(c v_L) = c^n Q^{(n)}_L(v_L) , \quad c \in \mathbb{C}^* ;$$  \hspace{1cm} (6.37)

(iii) the supergravity gauge transformations act on $Q^{(n)}_L$ as follows:

$$\delta_K Q^{(n)}_L = \left( K^C D_C + K^{ij} L_{ij} \right) Q^{(n)}_L ,$$

$$K^{ij} L_{ij} Q^{(n)}_L = - \left( K^{(2)}(2) \partial^{(-2)}_L - n K^{(0)}_L \right) Q^{(n)}_L .$$  \hspace{1cm} (6.38)

By construction, $Q^{(n)}_L$ is independent of $u_L$. One can see that $\delta_K Q^{(n)}_L$ is also independent of the isospinor $u_L$, due to \(6.37\).

It is important to note that

$$L^{(2)} Q^{(n)}_L = 0 , \quad L^{(0)} Q^{(n)}_L = - \frac{n}{2} Q^{(n)}_L ,$$  \hspace{1cm} (6.39)

as a consequence of \(6.38\). Since $Q^{(n)}_L$ is invariant under the Lorentz and SU(2)$_R$ transformations, the first relation in \(6.39\) guarantees that the covariant analyticity constraints \(6.36\) are indeed consistent.

As is clear from the above consideration, the isospinor $v^i \in \mathbb{C}^2 \setminus \{0\}$ is defined modulo the equivalence relation $v^i \sim c v^i$, with $c \in \mathbb{C}^*$, hence it parametrizes $\mathbb{C}P^1$. Therefore, the covariant left projective multiplets live in curved projective superspace, $\mathcal{M}^{3|8} \times \mathbb{C}P^1$. 44
Let $Q^{(n)}_L(v_L)$ be a left projective supermultiplet of weight $n$. Assuming that it varies homogeneously under the super-Weyl transformations, the analyticity constraints (6.36) uniquely fix its transformation law to be

$$\delta_\sigma Q^{(n)}_L = \frac{n}{2} \sigma Q^{(n)}_L .$$

(6.40)

This relation can be derived by noticing that the transformation rules of the $D^{(1)}_{\alpha \bar{i}}$ derivatives under super-Weyl transformations are

$$\delta_\sigma D^{(1)}_{\alpha \bar{i}} = \frac{1}{2} \sigma D^{(1)}_{\alpha \bar{i}} + (D^{(1)}_{\alpha \beta}) \sigma M_{\alpha \beta} - (D^{(1)}_{\alpha \bar{i}} \sigma) R^{\bar{i}} + (D^{(0)}_{\alpha \bar{i}} \sigma)L^{(0)} - (D^{(-1)}_{\alpha \bar{i}} \sigma)L^{(2)},$$

(6.41)

where

$$D^{(-1)}_{\alpha \bar{i}} := \frac{1}{(v_L, u_L)} u_i D^{\bar{i}}_{\alpha} .$$

(6.42)

We conclude this subsection with two comments. Firstly, for any integer $n$, the space of left weight-$n$ projective superfields can be endowed with a real structure. Associated with $Q^{(n)}_L(v_L)$ is its smile-conjugate $\tilde{Q}^{(n)}_L(v_L)$ which is defined according to eq. (5.38) with obvious modifications. The important property (5.39) also extends to the $\mathcal{N} = 4$ left projective multiplets. Thus, if $n$ is even, we can consistently define real left projective superfields.

Our second comment is that applying the mirror map to $Q^{(n)}_L(v_L)$ gives a covariant right projective multiplet of weight $n$, $Q^{(n)}_R(v_R)$. The entire consideration of this section naturally extends to the right projective multiplets. In what follows, for the left and right projective multiplets we often use two alternative types of notation, specifically

$$Q^{(n)}_L \equiv Q^{(n)} , \quad Q^{(n)}_R \equiv \tilde{Q}^{(n)} .$$

(6.43)

### 6.3 Hybrid projective multiplets

The definitions and properties of the left projective multiplets, which we presented in subsection 6.2 are completely analogous to those given in [19, 21] for the 4D $\mathcal{N} = 2$ covariant projective multiplets. A nontrivial new aspect of the 3D case is that there exist two types of $\mathcal{N} = 4$ covariant projective multiplets, the left and the right ones. Moreover, in three dimensions we can define hybrid projective multiplets of the form

$$Q^{(n,m)}(v_L, v_R) := \sum_{Q_L, Q_R} Q^{(n)}_L(v_L)Q^{(m)}_R(v_R) .$$

(6.44)
They obey the following analyticity constraint

\[ D^{(1,1)}_\alpha Q^{(n,m)}(v_L, v_R) = 0 \, , \quad D^{(1,1)}_{\bar{\alpha}} := \bar{D}^{\bar{\alpha}}_i v_i v_i \]  

(6.45)

and are characterized by the algebraic properties

\[ L^{(2)} Q^{(n,m)} = R^{(2)} Q^{(n,m)} = 0 \, . \]  

(6.46)

The analyticity constraint is consistent, since the operators \( D^{(1,1)}_\alpha \) satisfy the anticommutation relations:

\[ \{ D^{(1,1)}_\alpha , D^{(1,1)}_\beta \} = 4i C^{(2)}_{\alpha \beta} L^{(2)} + 4i B^{(2)}_{\alpha \beta} R^{(2)} - 4i S^{(2,2)} M_{\alpha \beta} \, . \]  

(6.47)

It should be remarked that \( S^{(2,2)} = v_i v_j v_i v_j S^{ij\bar{ij}} \) is a hybrid projective multiplet,

\[ D^{(1,1)}_\alpha S^{(2,2)} = 0 \, . \]  

(6.48)

The explicit representation (6.44) can be formalized. A hybrid projective multiplets \( Q^{(n,m)}(v_L, v_R) \), is defined to be a scalar superfield that lives on the curved \( \mathcal{N} = 4 \) superspace \( \mathcal{M}^{3|8} \), is holomorphic with respect to the isospinor variables \( v^i, \bar{v}^i \) on an open domain of \( \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\} \), and is characterized by the following conditions:

(i) it obeys the covariant analyticity constraint (6.45);
(ii) it is a homogeneous function of degree \( n \) in \( v_L \) and of degree \( m \) in \( v_R \), that is,

\[ Q^{(n,m)}(c_L v_L, c_R v_R) = c^n_L c^m_R Q^{(n,m)}(v_L, v_R) \, , \quad c_L, c_R \in \mathbb{C}^* \]  

(6.49)

(iii) under the supergravity gauge group, \( Q^{(n,m)} \) transforms as follows:

\[ \delta_K Q^{(n,m)} = \left( K^C D_C + K^{ij} L_{ij} + K^{ij} R_{ij} \right) Q^{(n,m)} \, , \]  

(6.50a)

\[ K^{ij} L_{ij} Q^{(n,m)} = - \left( K^{(2)}_L \partial^{(-2)}_L - n K^{(0)}_L \right) Q^{(n,m)} \, , \]  

(6.50b)

\[ K^{ij} R_{ij} Q^{(n,m)} = - \left( K^{(2)}_R \partial^{(-2)}_R - m K^{(0)}_R \right) Q^{(n,m)} \, . \]  

(6.50c)

If \( Q^{(n,m)} \) has a homogeneous super-Weyl transformation law, \( \delta_\sigma Q^{(n,m)} \propto \sigma Q^{(n,m)} \), then it proves to have the unique form:

\[ \delta_\sigma Q^{(n,m)} = \frac{1}{2} (n + m) \sigma Q^{(n,m)} \, . \]  

(6.51)

There exist hybrid projective multiplets with inhomogeneous super-Weyl transformation laws. For example, the torsion \( S^{(2,2)} \) transforms as

\[ \delta_\sigma S^{(2,2)} = \sigma S^{(2,2)} - i \frac{1}{4} D^{(2,2)} \sigma \, , \quad D^{(2,2)} := D^{(1,1)} D^{(1,1)} \, . \]  

(6.52)
The results of this subsection are consistent with the interpretation that the isospinors $v^i, v^\bar{i} \in \mathbb{C}^2 \setminus \{0\}$ are defined modulo the equivalence relations $v^i \sim c_L v^i, v^\bar{i} \sim c_R v^\bar{i}$, with $c_L, c_R \in \mathbb{C}^*$, hence $(v^i, v^\bar{i})$ parametrizes $\mathbb{C}P^1 \times \mathbb{C}P^1$. Therefore, the hybrid projective multiplets live in \textit{curved bi-projective superspace $M^{3|8} \times \mathbb{C}P^1 \times \mathbb{C}P^1$}.

Hybrid projective multiplets can naturally be defined in rigid $\mathcal{N} = 4$ bi-projective superspace $\mathbb{R}^{3|8} \times \mathbb{C}P^1 \times \mathbb{C}P^1$, but this possibility has not been considered in \cite{33}. Let us dimensionally reduce this superspace to two dimensions. The result is the 2D $\mathcal{N} = (4, 4)$ bi-projective superspace $\mathbb{R}^{2|8} \times \mathbb{C}P^1 \times \mathbb{C}P^1$ which was introduced more than twenty years ago by Buscher, Lindström and Roček \cite{75} and further studied in \cite{76, 77}. Its local version has been developed in \cite{22}.

### 6.4 Covariant projection operators

In this subsection we develop techniques to engineer covariant left/right and hybrid projective multiplets. For this we have to introduce a new superfield type – \textit{isotwistor multiplets} of arbitrary weight $(n, m)$, with $n, m$ integers. Such a superfield $T^{(n,m)}(v_L, v_R)$ has the same properties as the hybrid projective multiplet $Q^{(n,m)}(v_L, v_R)$ except for the analyticity condition (6.45). More specifically, the properties (6.49) and (6.50a)–(6.50c) are required to hold for $T^{(n,m)}(v_L, v_R)$. However, no analyticity constraint is imposed on $T^{(n,m)}(v_L, v_R)$. As a result, $T^{(n,m)}(v_L, v_R)$ may transform as a tensor field with respect to the local Lorentz group (its Lorentz indices are suppressed). \textit{Left and right isotwistor multiplets} correspond to special cases of isotwistor superfields:

\begin{align}
T^{(n)}_L(v_L) &:= T^{(n,0)}(v_L, v_R) \ , \quad \frac{\partial}{\partial v_R} T^{(n,0)} = 0 \ ; \\
T^{(m)}_R(v_R) &:= T^{(0,m)}(v_L, v_R) \ , \quad \frac{\partial}{\partial v_L} T^{(0,m)} = 0 \ .
\end{align}

Consider a covariant left projective multiplet $Q^{(n)}_L(v_L)$ of weight $n$. It can be proved that there exists a left isotwistor superfield $T^{(n-4)}_L(v_L)$ such that

\[ Q^{(n)}_L = \Delta^{(4)}_L T^{(n-4)}_L , \]

where $\Delta^{(4)}_L$ denotes the following fourth-order operator:

\begin{align}
\Delta^{(4)}_L &= \frac{1}{96} \left( (D^{(2)\bar{k}\bar{l}} - 16iS^{(2)\bar{k}\bar{l}}) D^{(2)}_{\bar{k}\bar{l}} - (D^{(2)\alpha\beta} - 16iB^{(2)\alpha\beta}) D^{(2)}_{\alpha\beta} \right) \\
&= \frac{1}{96} \left( D^{(2)\bar{k}\bar{l}}(D^{(2)}_{\bar{k}\bar{l}} - 16iS^{(2)\bar{k}\bar{l}}) - D^{(2)\alpha\beta}(D^{(2)}_{\alpha\beta} - 16iB^{(2)}_{\alpha\beta}) \right) ,
\end{align}

47
with

\[ \mathcal{D}_{ij}^{(2)} := \mathcal{D}_{i}^{(1)\gamma} \mathcal{D}_{ij}^{(1)}, \quad \mathcal{D}_{\alpha\beta}^{(2)} := \mathcal{D}_{(\alpha}^{(1)\bar{k}} \mathcal{D}_{\beta)k}^{(1)}. \]  

(6.56)

The opposite statement also holds. Given an arbitrary left isotwistor superfield \( T_{L}^{(n-4)}(v_{L}) \), the superfield \( Q_{L}^{(n)} \) defined by eq. (6.54) can be shown to satisfy the constraint

\[ \mathcal{D}_{\alpha}^{(1)\bar{i}} Q_{L}^{(n)} = 0. \]  

(6.57)

We will call \( \Delta_{L}^{(4)} \) the left projection operator. The derivation of \( \Delta_{L}^{(4)} \) and the proof of (6.57) are given in Appendix B.

It should be pointed out that the fourth-order operators that appear in the right-hand sides of (6.55a) and (6.55b) are related to each other as follows:

\[ \mathcal{D}_{j}^{(2)\bar{k}} \mathcal{D}_{kl}^{(2)} = -\mathcal{D}_{(\alpha}^{(2)\beta} \mathcal{D}_{\alpha\beta}^{(2)} - 8iS_{(2)\bar{k}}^{(2)\bar{l}} \mathcal{D}_{kl}^{(2)} - 8iB_{(2)\alpha\beta}^{(2)} \mathcal{D}_{\alpha\beta}^{(2)} - 16i(\mathcal{D}_{k}^{(1)\alpha} S_{\bar{k}l}^{(2)}) \mathcal{D}_{\alpha\beta}^{(1)}. \]  

(6.58)

This relation may be rewritten in a slightly different form using the identity

\[ \mathcal{D}_{\bar{i}}^{(1)\alpha} S_{\bar{k}l}^{(2)} = \mathcal{D}_{\alpha}^{(1)\bar{k}} B_{\alpha\beta}^{(2)}. \]  

(6.59)

Suppose that the left isotwistor superfield \( T_{L}^{(n-4)} \) in (6.54) has the super-Weyl transformation law

\[ \delta_{\sigma} T_{L}^{(n-4)} = \frac{n - 4}{2} \sigma T_{L}^{(n-4)}. \]  

(6.60)

Then it can be shown that \( Q_{L}^{(n)} = \Delta_{L}^{(4)} T_{L}^{(n-4)} \) also transforms homogeneously as

\[ \delta_{\sigma} Q_{L}^{(n)} = \frac{n}{2} \sigma Q_{L}^{(n)}, \]  

(6.61)

which is the unique homogeneous transformation law compatible with the analyticity of \( Q_{L}^{(n)} \) (in accordance with our discussion in the previous subsection).

A simple application of the construction (6.54) is to choose an ordinary \((v_{L}\text{-independent})\) superfield \( P \) in the role of \( T_{L}^{(0)} \). Then, \( \Sigma_{L}^{(4)} := \Delta_{L}^{(4)} P \) is a covariant \( O(4) \) multiplet. If \( P \) is invariant under the super-Weyl transformations, \( \delta_{\sigma} P = 0 \), then \( \Sigma_{L}^{(4)} \) transforms as \( \delta_{\sigma} \Sigma_{L}^{(4)} = 2\sigma \Sigma_{L}^{(4)} \).

The above consideration can be extended to the space of right projective multiplets by making use of the mirror map. The right projection operator \( \Delta_{R}^{(4)} \) proves to be

\[ \Delta_{R}^{(4)} = \frac{1}{96} \left( \mathcal{D}_{k}^{(2)\alpha} - 16iS_{k}^{(2)\alpha} \frac{\mathcal{D}_{kl}^{(2)}}{\mathcal{D}_{k}^{(2)}} - (\mathcal{D}_{\alpha}^{(2)\beta} - 16iC_{\alpha}^{(2)\beta} \mathcal{D}_{\alpha\beta}^{(2)}) \right), \]  

(6.62a)

\[ = \frac{1}{96} \left( \mathcal{D}_{k}^{(2)\alpha} \left( \mathcal{D}_{k}^{(2)} - 16iS_{k}^{(2)\alpha} \right) - \mathcal{D}_{\alpha}^{(2)\beta} \left( \mathcal{D}_{\alpha\beta}^{(2)} - 16iC_{\alpha\beta}^{(2)} \right) \right), \]  

(6.62b)
with
\[ D^{(2)}_{ij} := D^{(1)\gamma}_i D^{(1)}_{\gamma j}, \quad D^{(2)}_{\alpha \beta} := D^{(1)k}_\alpha D^{(1)}_{\beta k}. \] (6.63)

It can be shown that the fourth-order operators which appear in the right-hand sides of (6.62a) and (6.62b) are related to each other as follows:
\[ D^{(2)}_{kl} D^{(2)}_{kl} = -D^{(2)}_{\alpha \beta} D^{(2)}_{\alpha \beta} - 8i S^{(2)}_{kl} D^{(2)}_{kl} - 8i C^{(2)}_{\alpha \beta} D^{(2)}_{\alpha \beta} - 16i (D^{(1)k}_{\alpha} S^{(2)}_{kl}) D^{(1)}_{\alpha l}. \] (6.64)

This relation may be rewritten in a slightly different form using the identity
\[ D^{(1)\alpha} S^{(2)}_{kl} = D^{(1)k}_{\beta} C^{(2)}_{\alpha \beta}. \] (6.65)

Finally, we can construct a hybrid projection operator. Let \( T^{(n-2,m-2)}(v_L, v_R) \) be a Lorentz-scalar isotwistor superfield of weight \((n-2, m-2)\). We introduce the second-order differential operator
\[ \Delta^{(2,2)} := \frac{i}{4} \left( D^{(2,2)} - 4i S^{(2,2)} \right). \] (6.66)

It is not difficult to verify that
\[ Q^{(n,m)} := \Delta^{(2,2)} T^{(n-2,m-2)} \] (6.67)
satisfies (6.45), and thus \( \Delta^{(2,2)} \) maps any isotwistor superfield into a hybrid projective one. Therefore \( \Delta^{(2,2)} \) is the hybrid projection operator.

Suppose that the isotwistor superfield \( T^{(n-2,m-2)} \) in (6.67) has the super-Weyl transformation law
\[ \delta_\sigma T^{(n-2,m-2)} = \frac{1}{2} (n + m - 2) \sigma T^{(n-2,m-2)}. \] (6.68)

Then, it can be shown that the super-Weyl transformation of the hybrid projective multiplet \( Q^{(n,m)} := \Delta^{(2,2)} T^{(n-2,m-2)} \) is given by eq. (6.51).

A simple application of the construction (6.67) is to choose an ordinary (i.e., independent of \( v_L \) and \( v_R \)) superfield \( P \), with the super-Weyl transformation \( \delta_\sigma P = \sigma P \), in the role of \( T^{(0,0)} \). Then, \( Q^{(2,2)} = \Delta^{(2,2)} P = Q^{ij} v_i v_j \bar{v}_i v_j \) is hybrid projective.

The careful reader could have noticed that the left and right projection operators \( \Delta^{(4)}_L \) and \( \Delta^{(4)}_R \) have a structure which is formally equivalent to the chiral projector of 4D \( \mathcal{N} = 2 \) supergravity [79]. This property is not accidental and will be used in appendix B. Recently, in the projective superspace approach to 4D \( \mathcal{N} = 2 \) supergravity, a new
powerful representation of the chiral projector has been derived \cite{61}. It is interesting that this recent result similarly holds for $\Delta_L^{(4)}$ and $\Delta_R^{(4)}$. In particular, it turns out that in terms of isotwistor superfields one can obtain alternative representations for $\Delta_L^{(4)}$ and $\Delta_R^{(4)}$. These are

\begin{align}
\Delta_L^{(4)} \oint (v_R, dv_R) T^{(n,-2)} &= -\frac{i}{4} \oint (v_R, dv_R) \left( D^{(2,-2)} - 4iS^{(2,-2)} \right) \Delta^{(2,2)} T^{(n,-2)}, \quad (6.69a) \\
\Delta_R^{(4)} \oint (v_L, dv_L) T^{(-2,m)} &= -\frac{i}{4} \oint (v_L, dv_L) \left( D^{(-2,2)} - 4iS^{(-2,2)} \right) \Delta^{(2,2)} T^{(-2,m)}, \quad (6.69b)
\end{align}

with $T^{(n,-2)}$ and $T^{(-2,m)}$ isotwistor superfields of weight $(n, -2)$ and weight $(-2, m)$ respectively, and

\begin{align}
D^{(2,-2)} &= D^{(1,-1)}D^{(1,-1)}, \quad D^{(-2,2)} := D^{(1,-1)}D^{(-1,1)}, \quad (6.70a) \\
D^{(1,-1)} &= \frac{1}{(v_R, u_R)v_iu_iD_{\alpha}^{\hat{i}}}, \quad D^{(-1,1)} := \frac{1}{(v_L, u_\perp)u_iv_iD^{\hat{\alpha}}}, \quad (6.70b) \\
S^{(2,-2)} &= \frac{1}{(v_R, u_R)^2v_iv_ju_iu_jS^{\hat{i}\hat{j}}}, \quad S^{(-2,2)} := \frac{1}{(v_L, u_\perp)^2u_iv_jv_iS^{\hat{i}\hat{j}}} \quad (6.70c)
\end{align}

Note that the right-hand side of (6.69a) has the following properties: (i) it is independent of the constant isospinors $u_R = u^\perp$ constrained by the only conditions $(v_R, u_R) \neq 0$; and (ii) it obeys the left analyticity constraint (6.36). The proof of these statement are given in appendix B. The mirrored results hold for the right-hand side of (6.69b).

Let us conclude by pointing out that the representations (6.69a) and (6.69b) are useful for applications. The point is that any weight-$n$ left $T_L^{(n)}(v_L)$ and weight-$m$ right $T_R^{(m)}(v_R)$ isotwistor superfields can be represented in the following integral form:

\begin{align}
T_L^{(n)}(v_L) = \oint \frac{(v_R, dv_R)}{2\pi} T_{L}^{(n,-2)}(v_L, v_R), \quad T_R^{(m)}(v_R) = \oint \frac{(v_L, dv_L)}{2\pi} T_{R}^{(-2,m)}(v_L, v_R), \quad (6.71)
\end{align}

for some isotwistor superfields $T_L^{(n,-2)}(v_L, v_R)$ and $T_R^{(-2,m)}(v_L, v_R)$ of weights $(n, -2)$ and $(-2, m)$ respectively.

### 6.5 Locally supersymmetric actions

A remarkable feature of $\mathcal{N} = 4$ supergravity is that it allows three types of locally supersymmetric and super-Weyl invariant actions, for which the measure involves integration over four or six Grassmann variables only.

We introduce three types of real Lagrangians: (i) a left projective superfield $\mathcal{L}_L^{(2)}(z, v_L)$; (ii) a right projective superfield $\mathcal{L}_R^{(2)}(z, v_R)$; and (iii) a hybrid multiplet $\mathcal{L}^{(0,0)}(z, v_L, v_R)$. 

50
All the Lagrangians are required to be real with respect to the smile-conjugation. With the standard notation $E^{-1} = \text{Ber}(E_A^M)$, our locally supersymmetric and super-Weyl invariant action principle is given by

$$S = S_{\text{left}} + S_{\text{right}} + S_{\text{hybrid}},$$

(6.72a)

$$S_{\text{left}}(\mathcal{L}_L^{(2)}) = \frac{1}{2\pi} \oint (v_L, dv_L) \int d^3x d^8\theta E C_L^{(-4)} \mathcal{L}_L^{(2)},$$

(6.72b)

$$S_{\text{right}}(\mathcal{L}_R^{(2)}) = \frac{1}{2\pi} \oint (v_R, dv_R) \int d^3x d^8\theta E C_R^{(-4)} \mathcal{L}_R^{(2)},$$

(6.72c)

$$S_{\text{hybrid}}(\mathcal{L}^{(0,0)}) = \frac{1}{(2\pi)^2} \oint (v_L, dv_L) \oint (v_R, dv_R) \int d^3x d^8\theta E C^{(-2,-2)} \mathcal{L}^{(0,0)}.$$  

(6.72d)

The action involves some model-independent Lorentz-scalar isotwistor superfields $C_L^{(-4)}$, $C_R^{(-4)}$ and $C^{(-2,-2)}$, of which $C_L^{(-4)}$ and $C_R^{(-4)}$ are left and right respectively. These superfields are required to be real with respect the smile-conjugation, to have definite super-Weyl transformation laws and obey special differential equations:

$$\delta_\sigma C_L^{(-4)} = -2\sigma C_L^{(-4)} , \quad \Delta_L^{(4)} C_L^{(-4)} = 1 ;$$

(6.73a)

$$\delta_\sigma C_R^{(-4)} = -2\sigma C_R^{(-4)} , \quad \Delta_R^{(4)} C_R^{(-4)} = 1 ;$$

(6.73b)

$$\delta_\sigma C^{(-2,-2)} = -\sigma C^{(-2,-2)} , \quad \Delta^{(2,2)} C^{(-2,-2)} = 1 .$$

(6.73c)

All the Lagrangians are required to possess uniquely defined homogeneous super-Weyl transformations

$$\delta_\sigma \mathcal{L}_L^{(2)} = \sigma \mathcal{L}_L^{(2)} , \quad \delta_\sigma \mathcal{L}_R^{(2)} = \sigma \mathcal{L}_R^{(2)} , \quad \delta_\sigma \mathcal{L}^{(0,0)} = 0 .$$

(6.74)

The super-Weyl invariance of the action follows from the above transformation laws in conjunction with

$$\delta_\sigma E = \sigma E .$$

(6.75)

The invariance of the action under the supergravity gauge transformations can be proved using the same considerations as in the 4D $\mathcal{N} = 2$ case \cite{19,21}.

It turns out that the action does not depend on the kinematic isotwistor superfields $C_L^{(-4)}$, $C_R^{(-4)}$ and $C^{(-2,-2)}$, provided the corresponding Lagrangians are independent. To prove this claim, it suffices to consider the left sector of the action, eq. (6.72b). Let us represent the corresponding Lagrangian in the form $\mathcal{L}_L^{(2)} = \Delta_L^{(4)} \mathcal{T}_L^{(-2)}$, for some left isotwistor superfield $\mathcal{T}_L^{(-2)}$. We can now use the fact that $\Delta_L^{(4)}$ is symmetric, that is for any left isotwistor superfields $\Psi^{(-n)}$ and $\Phi^{(n-6)}$ it holds that

$$\int d^3x d^8\theta E \oint (v_L, dv_L) \left\{ \Psi^{(-n)} \Delta_L^{(4)} \Phi^{(n-6)} - \Phi^{(n-6)} \Delta_L^{(4)} \Psi^{(-n)} \right\} = 0 ,$$

(6.76)
as a consequence of the representations (6.55a) and (6.55b). Using this observation and
the representation \( L^{(2)}_\text{L} = \Delta^{(4)}_\text{L} T^{(-2)}_\text{L} \) introduced above, the action (6.72b) can be brought
to the form

\[
S_{\text{left}} = \frac{1}{2\pi} \oint (v_\text{L}, dv_\text{L}) \int d^3x d^8\theta E T^{(-2)}_\text{L},
\]

which makes manifest the fact that \( S_{\text{left}} \) does not depend on \( C^\text{(-4)}_\text{L} \).

There is a freedom in the choice of \( C^\text{(-4)}_\text{L} \), \( C^\text{(-4)}_\text{R} \) and \( C^\text{(-2,-2)} \). For instance, given a
real left weight-\( m \) isotwistor superfield \( \Gamma^{(m)}_\text{L} \), a real right weight-\( n \) isotwistor superfield \( \Gamma^{(n)}_\text{R} \) and a real hybrid weight-\( (p,q) \) isotwistor superfield \( \Gamma^{(p,q)} \), we may define \( C^\text{(-4)}_\text{L} \), \( C^\text{(-4)}_\text{R} \) and \( C^\text{(-2,-2)} \) as

\[
C^\text{(-4)}_\text{L} = \frac{\Gamma^{(m)}_\text{L}}{\Delta^{(4)}_\text{L} \Gamma^{(m)}_\text{L}} , \quad C^\text{(-4)}_\text{R} = \frac{\Gamma^{(n)}_\text{R}}{\Delta^{(4)}_\text{R} \Gamma^{(n)}_\text{R}} , \quad C^\text{(-2,-2)} = \frac{\Gamma^{(p,q)}}{\Delta^{(2,2)} \Gamma^{(p,q)}} .
\]

Then the differential equations in (6.73a)–(6.73b) are satisfied. To respect the super-Weyl
transformation laws in (6.73a)–(6.73c), the superfields \( \Gamma^{(m)}_\text{L} \), \( \Gamma^{(n)}_\text{R} \) and \( \Gamma^{(p,q)} \) should trans-
form as \( \delta_\sigma \Gamma^{(m)}_\text{L} = (m/2)\sigma \Gamma^{(m)}_\text{L} \), \( \delta_\sigma \Gamma^{(n)}_\text{R} = (n/2)\sigma \Gamma^{(n)}_\text{R} \) and \( \delta_\sigma \Gamma^{(p,q)} = [(p + q + 2)/2]\sigma \Gamma^{(p,q)} \).

It is natural to put forward an additional requirement that the action be invariant
under the mirror transformation. It is satisfied under the following conditions: (i) the
Lagrangians \( L^{(2)}_\text{L} \) and \( L^{(2)}_\text{R} \) are the mirror images of each other; (ii) the Lagrangian \( L^{(0,0)} \)
is mirror invariant; (iii) \( C^\text{(-4)}_\text{L} \) and \( C^\text{(-4)}_\text{R} \) are the mirror images of each other; (iv) \( C^\text{(-2,-2)} \)
is mirror invariant. If the kinematic factors are chosen as in (6.78), the conditions (iii)
and (iv) imply \( m = n \) and \( p = q \).

The simplest way to generate \( C^\text{(-4)}_\text{L} \) and \( C^\text{(-4)}_\text{R} \) is to use ordinary real scalar superfields
\( P_\text{L}(z) \), \( P_\text{R}(z) \) and \( P(z) \) and choose

\[
C^\text{(-4)}_\text{L} = \frac{P_\text{L}}{\Delta^{(4)}_\text{L} P_\text{L}} , \quad C^\text{(-4)}_\text{R} = \frac{P_\text{R}}{\Delta^{(4)}_\text{R} P_\text{R}} , \quad C^\text{(-2,-2)} = \frac{P}{\Delta^{(2,2)} P} .
\]

In order to guarantee the fulfillment of the super-Weyl transformation laws in (6.73a)–
(6.73c), the superfields \( P_\text{L} \), \( P_\text{R} \) and \( P \) must transform as

\[
\delta_\sigma P_\text{L} = \delta_\sigma P_\text{R} = 0 , \quad \delta_\sigma P = \sigma P .
\]

The transformation of \( P \) is similar to that appearing in the \( \mathcal{N} = 3 \) case. If the action is
chosen to be mirror invariant, then \( P_\text{L} = P_\text{R} \).

In complete analogy with our four-dimensional analysis given in [19], it is of interest to
give flat superspace versions of the actions (6.72b)–(6.72d). In the flat superspace limit,
the dependence on the compensating superfields \( C_L^{(-4)}, C_R^{(-4)} \) and \( C^{(-2,-2)} \) can be seen to drop out. The actions (6.72b) and (6.72c) reduce to

\[
S_{\text{left}}(L^{(2)}_L) = \frac{1}{2\pi} \oint (v_L, dv_L) \int d^3x \, D^{(-4)}_L L^{(2)}_L \bigg|_{\theta=0}, \quad (6.81a)
\]

\[
S_{\text{right}}(L^{(2)}_R) = \frac{1}{2\pi} \oint (v_R, dv_R) \int d^3x \, D^{(-4)}_R L^{(2)}_R \bigg|_{\theta=0}, \quad (6.81b)
\]

with \( L^{(2)}_L, L^{(2)}_R \) and \( L^{(0,0)} \) the flat-superspace versions of the Lagrangians in (6.72b)–(6.72d). Here we have introduced two fourth-order operators, \( D^{(-4)}_L \) and \( D^{(-4)}_R \), defined in terms of the flat covariant derivatives \( D^{\tilde{\alpha}}_i \), specifically

\[
D^{(-4)}_L := \frac{1}{48} D^{(-2)kl} D_{kl}^{(-2)} , \quad D^{(-4)}_R := \frac{1}{48} D^{(-2)ij} D_{ij}^{(-2)} , \quad D^{(-1)\gamma}_i := \frac{u_i}{(v_L, u_L)} D^{\tilde{\alpha}}_i , \quad (6.82a)
\]

\[
D^{(-4)}_R := \frac{1}{48} D^{(-2)ij} D_{ij}^{(-2)} , \quad D^{(-2)} := D^{(-1)\gamma}_i D^{(1,1)}_g \gamma_{ij} , \quad D^{(-1)\tilde{\alpha}}_i := \frac{u_i}{(v_R, u_R)} D^{\tilde{\alpha}}_i . \quad (6.82b)
\]

The functional (6.81a) and (6.81b) are the 3D versions \( [33] \) of the 4D projective-superspace action \( [23] \). The flat-superspace limit of the hybrid action (6.72a) is

\[
S_{\text{hybrid}}(L^{(0,0)}) = \frac{1}{(2\pi)^2} \oint (v_L, dv_L) \int (v_R, dv_R) \int d^3x \, D^{(-2,-2)}_H L^{(0,0)} \bigg|_{\theta=0} . \quad (6.83)
\]

where

\[
D^{(-2,-2)}_H := \frac{i}{64} (D^{\alpha(-1,-1)}_\alpha D^{(-1,-1)}_\alpha)(D^{(1,-1)}_\beta D^{(1,-1)}_\gamma)(D^{(0,-1)}_\gamma D^{(-1,1)}_\gamma) ,
\]

\[
D^{(-1,-1)}_\alpha := \frac{u_i v_i}{(v_L, u_L)(v_R, u_R)} D^{\tilde{\alpha}}_\alpha , \quad D^{(-1,1)}_\alpha := \frac{u_i v_i}{(v_L, u_L)} D^{\tilde{\alpha}}_\alpha . \quad (6.84)
\]

The hybrid action (6.83) proves to be invariant under two types of projective transformations, left and right ones. The left transformations have the form:

\[
(u_L, v_L) \rightarrow (u_L, v_L) F , \quad F = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{GL}(2, \mathbb{C}) . \quad (6.85)
\]

The right projective transformations are defined similarly. Since \( \{ D^{\tilde{\alpha}}_\alpha, D^{\tilde{\beta}}_\beta \} \propto \partial_{\alpha\beta} \), the left/right actions (6.81a) and (6.81b) generate two derivatives at the component level, while the hybrid action (6.83) gives rise to three derivatives. To the best of our knowledge, the hybrid projective action has been presented here for the first time.

The flat-superspace hybrid action (6.83) can also be rewritten in the following forms:

\[
S_{\text{hybrid}}(L^{(0,0)}) = S_{\text{left}}(\mathcal{Q}^{(2)}_L) , \quad \mathcal{Q}^{(2)}_L := \frac{i}{8\pi} \oint (v_R, dv_R) D^{(1,-1)}_\alpha D^{(1,-1)}_\alpha L^{(0,0)} ; \quad (6.86a)
\]

\[
S_{\text{hybrid}}(L^{(0,0)}) = S_{\text{right}}(\mathcal{Q}^{(2)}_R) , \quad \mathcal{Q}^{(2)}_R := \frac{i}{8\pi} \oint (v_L, dv_L) D^{(0,-1)}_\alpha D^{(-1,1)}_\alpha L^{(0,0)} . \quad (6.86b)
\]
6.6 Vector multiplet prepotentials

The field strengths of two inequivalent vector multiplets are described by left and right linear multiplets, $W_L^{(2)}$ and $W_R^{(2)}$ subject to the constraints (6.24a) and (6.24b). These constraints can be solved in terms of covariant weight-zero tropical multiplets. It suffices to restrict our analysis to the right linear multiplets $W_R^{(2)} = \bar{W}_{ij} v_i v_j$.

A general solution to the constraint (6.24b) is

$$W_R^{(2)}(v_R) = {i \over 4} \oint \left( D^{(2)ij} - 4iS^{(2)ij} \right) \frac{(v_L, dv_L)}{2\pi (v_L, u_L)^2} V_L(v_L),$$

where $V_L(v_L)$ is a left tropical multiplet of weight zero. The right-hand side of (6.87) involves a constant isospinor $u_L = u^i$ constrained by the only condition $(v_L, u_L) \neq 0$. It can be shown that (6.87) is invariant under an arbitrary infinitesimal variation of $u_L$, that is $\delta u_L = \alpha u_L + \beta v_L$, with $\alpha, \beta \in \mathbb{C}$. Thus $W_R^{(2)}(v_R)$ is independent of $u_L$.

The relation (6.87) demonstrates a remarkable interplay between left and right projective multiplets. The left-hand side of (6.87) is the right $O(2)$ multiplet $W_R^{(2)}$, while the right-hand side is given in term of the left tropical prepotential $V_L$. The above result can be represented in a slightly different form:

$$W_{ij} = {i \over 4} \oint \frac{(v_L, dv_L)}{2\pi} \left( D^{(-2)ij} - 4iS^{(-2)ij} \right) V_L(v_L).$$

This representation can be used to show that $W_{ij}$ is invariant under gauge transformations

$$\delta V_L = \lambda_L + \dot{\lambda}_L,$$

where the gauge parameter $\lambda_L$ is an arbitrary left arctic multiplet of weight zero.

Let us represent $V_L$ in terms of an unconstrained left isotwistor superfield $T_L^{(-4)}(v_L)$,

$$V_L(v_L) = \Delta_L^{(4)} T_L^{(-4)}(v_L),$$

with the super-Weyl transformation law

$$\delta_\sigma T_L^{(-4)} = -2\sigma T_L^{(-4)}.$$

As remarked at the end of subsection 6.4, a left isotwistor superfield $T_L^{(-4)}(v_L)$ can be represented in terms of a weight-$(−4, −2)$ isotwistor superfield $T_L^{(-4,−2)}(v_L, v_R)$ through the integral equation (6.71). It then appears that (6.87) is equivalent to

$$W_R^{(2)}(v_R) \equiv \Delta_R^{(4)} \oint \frac{(v_L, dv_L)}{2\pi} \oint \frac{(v_R, dv_R)}{2\pi (v_R, \hat{v}_R)^2} \Delta_L^{(2,2)} T_L^{(-4,−2)}(v_L, \hat{v}_R).$$
with the operator $\Delta^{(2,2)}$ given by
\[
\Delta^{(2,2)} := \frac{i}{4} v_i v_j \hat{v}_i \hat{v}_j \left(D^{\alpha \bar{\alpha}}D_{\alpha}^{\bar{\alpha}} - 4i S^{ijij} \right). \tag{6.93}
\]
In the form of equation (6.92) it becomes manifest that $W_R^{(2)}(v_R)$, originally defined by (6.87), satisfies the right analyticity constraint.

The proof of (6.92) is achieved in few steps. By using (6.90), (6.71) and (6.69a), the relation (6.87) can be equivalently written as
\[
W_R^{(2)}(v_R) = - \oint (v_L, d v_L) \frac{1}{2\pi} \oint (\hat{v}_R, d \hat{v}_R) \frac{1}{2\pi} \Delta^{-(-2,2)} \Delta^{(2,2)} \Delta^{(2,-2)} T^{(-4,-2)}_L(v_L, \hat{v}_R). \tag{6.94}
\]
Here we have introduced the operators
\[
\Delta^{-(-2,2)} := \frac{i}{4} u_i u_j \left(D^{(2)ij} - 4i S^{(2)ij} \right), \quad \Delta^{(2,-2)} := \frac{i}{4} \hat{u}_i \hat{u}_j \left(D^{(2)ij} - 4i S^{(2)ij} \right). \tag{6.95}
\]
In (6.94) we have the freedom to choose \(^{15}\hat{u}_R = v_R\) and obtain
\[
W_R^{(2)}(v_R) = - \oint (v_L, d v_L) \frac{1}{2\pi} \Delta^{-(-2,2)} \Delta^{(2,2)} \oint (\hat{v}_R, d \hat{v}_R) \frac{1}{2\pi(\hat{v}_R, v_R)} \Delta^{(2,2)} T^{(-4,-2)}_L(v_L, \hat{v}_R). \tag{6.96}
\]
Now, making use of eq. (6.69b), we readily arrive at (6.92).

6.7 Poincaré supergravity

To describe $\mathcal{N} = 4$ Poincaré, we need two compensators coupled to conformal supergravity. In the role of compensators we can choose a left linear multiplet $W^{ij}$ and a right linear multiplet $W_{\bar{i}\bar{j}}$ such that
\[
W_L := \sqrt{W^{ij} W_{ij}} \neq 0, \quad W_R := \sqrt{W_{\bar{i}\bar{j}} W^{\bar{i}\bar{j}}} \neq 0. \tag{6.97}
\]
These scalar superfields are characterized by the super-Weyl transformation laws
\[
\delta_\sigma W_L = \sigma W_L, \quad \delta_\sigma W_R = \sigma W_R. \tag{6.98}
\]
These scalars turn out to have interesting properties. The superfield $W_L$ satisfies the equation
\[
(D^{k(i} \bar{D}_{\bar{j}k}) - 4i C^{ij}_{\alpha \beta})(W_L)^{-1} = 0, \tag{6.99}
\]
\(^{15}\)We assume that the contour integral in the isotwistor variable $\hat{v}_R$ is such that $(\hat{v}_R, v_R) \neq 0.$
which can be derived using the identity

\[
\begin{align*}
D_{\alpha k}^{(i} D_{\beta l}^{j)} W_{ij} &= -\frac{1}{6} \varepsilon_{\alpha \beta} \varepsilon_{k(i} \varepsilon_{j)l} D_{\gamma l}^{(k} D_{\gamma k}^{l)} W_{ipq} - 4i C_{\alpha \beta} W_{k(i} \varepsilon_{j)l} + 2i \varepsilon_{\alpha \beta} S_{kl}^{ikl} W_{ij} \\
&- 2i \varepsilon_{\alpha \beta} S_{ij}^{kkl} W_{kl} - 2i \varepsilon_{\alpha \beta} \varepsilon_{kl} S_{i\beta k}^{pkl} W_{jp}.
\end{align*}
\]

Similarly one can derive the equation

\[
\left( D_{\alpha}^{(i} D_{\beta k)} - 4i B_{\alpha}^{ij} \right) (W_R)^{-1} = 0.
\]

Poincaré supergravity is described by two Lagrangians, left and right ones, which can be chosen as

\[
\begin{align*}
L_{\text{SUGRA, left}}^{(2)} &= \frac{1}{\kappa^2} W_L^{(2)} \ln \frac{W_L^{(2)}}{\alpha_L^{(1)}} + \frac{\xi_L}{\kappa^2} V_L W_L^{(2)} , \\
L_{\text{SUGRA, right}}^{(2)} &= \frac{1}{\kappa^2} W_R^{(2)} \ln \frac{W_R^{(2)}}{\alpha_R^{(1)}} + \frac{\xi_R}{\kappa^2} V_R W_R^{(2)} ,
\end{align*}
\]

Here \( V_L \) is the tropical prepotential for \( W_L^{(2)} \), see equation (6.87), while \( V_R \) is the tropical prepotential for \( W_R^{(2)} \), in particular

\[
W_L^{(2)}(v_L) = \frac{i}{4} \left( D_{i}^{(2)} \bar{W}_{ij}^{(2)} \right) \oint \frac{(v_R, dv_R)}{2\pi} \frac{u_i u_j}{(v_R, u_R)^2} V_R(v_R) .
\]

The action is invariant under left and right gauge transformations, the left one being given by eq. (6.89). The cosmological term is described by two BF-couplings. Using the representation (6.92), it can be shown that the action does not change if the BF coupling constants are modified as

\[
\begin{align*}
\xi_L &\rightarrow \xi_L + a , & \xi_R &\rightarrow \xi_R - a ,
\end{align*}
\]

for any real constant \( a \). Moreover, using eq. (6.92), integration by parts and the relations (6.69a)–(6.69b), the reader can prove the following important results:

\[
S_{\text{left}}(V_L W_L^{(2)}) = S_{\text{right}}(V_R W_R^{(2)}) = -S_{\text{hybrid}}(V_L V_R) .
\]

Note that the freedom (6.104) is absent if the theory is required to be mirror invariant, for then \( \xi_L = \xi_R \equiv \xi/2 \).

There exists a dual off-shell formulation for Poincaré supergravity with two compensators, a vector multiplet and a hypermultiplet. Let us use the freedom (6.104) to set
\( \xi_R = 0 , \)

\[
\mathcal{L}^{(2)}_{\text{SUGRA, left}} = \frac{1}{\kappa^2} W_L^{(2)} \ln \left( \frac{W_L^{(2)}}{i \Upsilon_L^{(1)} \Upsilon_L^{(1)}} \right) + \frac{\xi}{\kappa^2} V_L W_L^{(2)} = \frac{1}{\kappa^2} W_L^{(2)} \ln \left( \frac{W_L^{(2)}}{i \Upsilon_L^{(1)} e^{-\xi V_L} \Upsilon_L^{(1)}} \right), \tag{6.106a}
\]

\[
\mathcal{L}^{(2)}_{\text{SUGRA, right}} = \frac{1}{\kappa^2} W_R^{(2)} \ln \left( \frac{W_R^{(2)}}{i \Upsilon_R^{(1)} \Upsilon_R^{(1)}} \right). \tag{6.106b}
\]

The left model (6.106a) can now be dualized in the same fashion as it was done in subsection 5.6. As a result, we arrive at the following formulation

\[
\mathcal{L}^{(2)}_{\text{SUGRA, left}} = -\frac{i}{\kappa^2} \dot{\Upsilon}_L^{(1)} e^{-\xi V_L} \Upsilon_L^{(1)} , \tag{6.107a}
\]

\[
\mathcal{L}^{(2)}_{\text{SUGRA, right}} = \frac{1}{\kappa^2} W_R^{(2)} \ln \left( \frac{W_R^{(2)}}{i \Upsilon_R^{(1)} \Upsilon_R^{(1)}} \right). \tag{6.107b}
\]

The theory is invariant under the gauge transformations (6.89) provided the hypermultiplet transforms as

\[
\delta \Upsilon_L^{(1)} = \xi \lambda \Upsilon_L^{(1)} . \tag{6.108}
\]

In the case of supergravity without cosmological term, \( \xi = 0 \), we can also dualize \( W_R^{(2)} \) into a right weight-one arctic multiplet \( \Upsilon_R^{(1)} \) and its conjugate \( \dot{\Upsilon}_R^{(1)} \).

It is instructive to see explicitly how the compensators can be used to obtain Poincaré supergravity from the conformal one by a process known as “de-gauging” \[49\] (or, equivalently, fixing the conformal gauge). We will use the formulation with two vector multiplets, left and right ones, as the compensators. First of all, we note that the super-Weyl freedom can be completely fixed by choosing the gauge condition

\[
W_L = 1 . \tag{6.109}
\]

Let \( w_{ij} \) denote the field strength \( W_{ij} \) in this gauge. An important observation is that, because the superfield \( w^{ij} \) is analytic \( \mathcal{D}^{(\tilde{w}, w^{kl})} = 0 \), from \( \mathcal{D}^{(\tilde{w}, w^{kl})} = 0 \) one can obtain that \( w_{ij} \) is annihilated by the spinor covariant derivatives,

\[
\mathcal{D}^{(\tilde{w}, w^{kl})} = 0 . \tag{6.110}
\]

This condition implies nontrivial constraints on the geometry. In particular, the consistency condition

\[
0 = \{ \mathcal{D}^{(\tilde{w}, w^{kl})} \} w^{kl} = 2 i \varepsilon^{ij} \varepsilon^{\tilde{ij}} (\gamma^c)_{\alpha \beta} \mathcal{D}_c w^{kl} + 2 i \varepsilon_{\alpha \beta \gamma} \varepsilon^{\tilde{ij}} (2 S + X)(\varepsilon^{k(i w^j)l} + \varepsilon^{l(i w^j)k})
- 4 i \varepsilon_{\alpha \beta \gamma} \varepsilon^{\tilde{ij}} S^{(k} w_p^{j)p} + 4 i C_{\alpha \beta} \varepsilon^{\tilde{ij}} (\varepsilon^{k(i w^j)l} + \varepsilon^{l(i w^j)k}) \tag{6.111}
\]
is equivalent to

$$D_a w^{kl} = 0, \quad X = -2S, \quad S^{ij} = w^{ij} S^{ij}, \quad C_{\alpha\beta}^{ij} = 0,$$

for some right $O(2)$ multiplet $S^{ij}$,

$$\mathcal{D}^{i(k} S^{j)l} = 0.$$  \hspace{1cm} (6.113)

As a result, $w_{ij}$ is covariantly constant in the super-Weyl gauge (6.109). All the relations in (6.112) and (6.113) are artifacts of the same super-Weyl gauge fixing. The algebra of covariant derivatives reduces to

$$\{D_{A}^{\hat{i}}, D_{B}^{\hat{j}}\} = 2i \varepsilon^{ij} \varepsilon^{\hat{i} \hat{j}} D_{\alpha\beta} - 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} S^{ ij} w^{kl} L_{kl} + 8i \varepsilon_{\alpha\beta} \varepsilon^{ij} S R^{ij} - 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} w^{ij} S^{kl} R_{kl} + 4i B_{\alpha\beta}^{ij} R_{ij} + 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} B^{\gamma\delta ij} M_{ij} - 4i w^{ij} S^{ij} M_{\alpha\beta} - 4i \varepsilon_{\alpha\beta} \varepsilon^{ij} S M_{\alpha\beta}.$$  \hspace{1cm} (6.114)

It is clear that the super-Weyl gauge condition has broken the mirror symmetry. The structure group is still $SL(2, \mathbb{R}) \times SU(2)_L \times SU(2)_R$. However, the $SU(2)_L$ curvature can be seen to take its values in a one-dimensional subalgebra of $su(2)$ generated by $w^{kl} L_{kl}$. Therefore, the $SU(2)_L$ gauge freedom can be partially fixed by choosing the $SU(2)_L$ connection as

$$(\Phi_L)_A = \Phi_A L, \quad L := w^{kl} L_{kl}.$$  \hspace{1cm} (6.115)

As a result, in the left sector we stay with a residual gauge group $U(1)_L \subset SU(2)_L$ generated by $L$. The condition of covariant constancy, $D_A w^{ij} = 0$, now means that $w^{ij}$ is constant, $\partial_M w^{ij} = 0$.

Using the second compensator, $W^{ij}$, allows us to partially fix the gauge group $SU(2)_R$ by imposing a condition $W^{ij} \propto \delta^{ij}$, in complete analogy with 4D $\mathcal{N} = 2$ supergravity [32]. In this gauge, we stay with a residual local group $U(1)_R \subset SU(2)_R$.

### 6.8 Dynamical systems

All the $\mathcal{N} = 3$ locally supersymmetric sigma-models considered in subsection 5.7 can be readily generalized to off-shell $\mathcal{N} = 4$ theories described by either left projective multiplets or right ones. The nontrivial new feature of $\mathcal{N} = 4$ supersymmetry is that it allows off-shell couplings that mix left and right projective multiplets. To illustrate this idea, it suffices to consider dynamical systems involving left and right vector multiplets.

Consider several vector multiplets described by right tropical prepotentials $V_I^{(0)}$ and left tropical prepotential $V_I^{(0)}$, and let $W_I^{(2)}$ and $W_I^{(2)}$ be the corresponding left and
right $\mathcal{O}(2)$ field strengths. A gauge-invariant action functional is generated by three Lagrangians (left, right and hybrid) of the form:

$$
\mathcal{L}^{(2)}_L = \mathcal{F}_L(W^I, m^{I\bar{J}} V_{\bar{J}}), \quad m^{I\bar{J}} = (m^{I\bar{J}})^* = \text{const} \quad (6.116a)
$$

$$
\mathcal{L}^{(2)}_R = \mathcal{F}_R(W_{\bar{I}}, m^{I\bar{J}} W^I_{\bar{J}}), \quad m^{I\bar{J}} = (m^{I\bar{J}})^* = \text{const} \quad (6.116b)
$$

$$
\mathcal{L}^{(0,0)} = \mathcal{H}(W^I, W_{\bar{J}}) + \mu^{I\bar{J}} V_I V_{\bar{J}}, \quad \mu^{I\bar{J}} = (\mu^{I\bar{J}})^* = \text{const} \quad (6.116c)
$$

The kinetic terms should obey the following homogeneity conditions:

$$
W^I \frac{\partial}{\partial W^I} \mathcal{F}_L = \mathcal{F}_L, \quad (6.117a)
$$

$$
W_{\bar{I}} \frac{\partial}{\partial W_{\bar{I}}} \mathcal{F}_R = \mathcal{F}_R, \quad (6.117b)
$$

$$
\left( W^I \frac{\partial}{\partial W^I} + W_{\bar{I}} \frac{\partial}{\partial W_{\bar{I}}} \right) \mathcal{H} = 0. \quad (6.117c)
$$

The $m$- and $\mu$-terms in (6.116a)–(6.116c) are three different forms of the BF couplings.

## 7 Conclusion

As is well known, off-shell supergravity-matter couplings in diverse dimensions may be conveniently derived starting from a superconformal perspective. In this paper we have developed the superspace geometry of $\mathcal{N}$-extended conformal supergravity in three space-time dimensions. Using this geometric setup, we have constructed general off-shell supergravity-matter couplings for $\mathcal{N} \leq 4$. In the most interesting and previously unexplored cases $\mathcal{N} = 3$ and $\mathcal{N} = 4$, we have proposed new off-shell supermultiplets coupled to conformal supergravity, in terms of which both the supergravity and matter actions are given.

It should be emphasized that the conventional constraints on $\mathcal{N}$-extended superspace geometry, eqs. (2.13a)–(2.13c), were introduced fifteen years ago in [48]. However, the corresponding Bianchi identities were not been solved by Howe et al. Moreover, the issue of constructing supergravity-matter couplings or even a superfield supergravity action in the case $\mathcal{N} = 3, 4$ was not addressed in [48].

Our approach to the three-dimensional $\mathcal{N} = 3, 4$ supergravity theories is a natural extension of the projective-superspace formulations for general 5D $\mathcal{N} = 1$ and 4D $\mathcal{N} = 2$ supergravity-matter theories which were developed in [17, 18, 19, 20, 21]. More specifically,
this is true for $\mathcal{N} = 3$ supergravity. In the $\mathcal{N} = 4$ case, however, we have discovered a new theoretical phenomenon as compared with the situation in higher dimensions. It is the existence of three types of covariant off-shell projective supermultiplets (left, right and hybrid ones) in terms of which the general matter couplings are constructed.

In this paper, the supergravity-matter couplings are formulated using superspace and superfields. Of course, many applications require a reformulation in terms of component fields. In four dimensions, techniques have been developed [61, 68, 78] to reduce the 4D $\mathcal{N} = 2$ supergravity-matter actions of [19, 20, 21] to components. Similar techniques can be developed in three dimensions for the theories constructed above. This issue of component reduction will be addressed in a separate publication.

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**Note added in proof:** After this paper had been accepted for publication, one of us (UL) was informed of a paper by Howe and Sezgin [82] in which some aspects of 3D $\mathcal{N} = 8$ superconformal geometry were elaborated following [48]. Even in this special case, our results obtained in section 2 are more complete.

**A Notation and conventions**

Our conventions for spinors in three space-time dimensions (3D) are compatible with the 4D two-component spinor formalism used in [80, 81]. More specifically, the starting point for setting up our 3D spinor formalism is the 4D sigma-matrices

\[
(\sigma_{m})^{\alpha\beta} := (\mathbb{1}, \vec{\sigma}), \quad (\bar{\sigma}_{m})^{\dot{\alpha}\dot{\beta}} := (\mathbb{1}, -\vec{\sigma}), \quad m = 0, 1, 2, 3 \quad (A.1)
\]
where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. By deleting the matrices with space index $m = 2$ we obtain the 3D gamma-matrices

\[
\begin{align*}
(\sigma_m)_{\alpha\beta} & \rightarrow (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha} = (1, \sigma_1, \sigma_3), \\
(\tilde{\sigma}_m)_{\alpha\beta} & \rightarrow (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\gamma_m)_{\gamma\delta},
\end{align*}
\]

where the spinor indices are raised and lowered using the $\text{SL}(2, \mathbb{R})$ invariant tensors

\[
\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta
\]

as follows:

\[
\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta.
\]

By construction, the matrices $(\gamma_m)_{\alpha\beta}$ and $(\gamma_m)^{\alpha\beta}$ are real and symmetric. Using the properties of the 4D sigma-matrices, we can immediately read off the properties of the 3D gamma-matrices. In particular, for the matrices

\[
\gamma_m := (\gamma_m)_{\alpha\beta} = \varepsilon^\beta\gamma(\gamma_m)_{\alpha\gamma}
\]

we readily obtain the relations

\[
\begin{align*}
\{\gamma_m, \gamma_n\} = 2\eta_{mn}\mathbb{1}, \\
\gamma_m\gamma_n = \eta_{mn}\mathbb{1} + \varepsilon_{mnp}\gamma^p,
\end{align*}
\]

where the 3D Minkowski metric is $\eta_{mn} = \eta^{mn} = \text{diag}(-1, 1, 1)$, and the Levi-Civita tensor is normalized as $\varepsilon_{012} = -\varepsilon^{012} = -1$. As usual, the 3D vector indices are labeled by values $m = 0, 1, 2$. Some useful relations involving $\gamma$-matrices are

\[
\begin{align*}
(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta} = 2\varepsilon_{\alpha(\gamma}\varepsilon_{\delta)\beta}, \\
\varepsilon^{abc}(\gamma^b)_{\alpha\beta}(\gamma^c)_{\gamma\delta} = \varepsilon_{\alpha(\gamma}\varepsilon_{\beta\delta)\gamma} + \varepsilon_{\beta(\gamma}\varepsilon_{\alpha\delta)\gamma}, \\
\text{tr}[(\gamma_a)_{\alpha\beta}(\gamma_b)_{\gamma\delta}] = 2\eta_{ac}\eta_{bd} - 2\eta_{ac}\eta_{bc} + 2\eta_{ad}\eta_{bc}.
\end{align*}
\]

Given a three-vector $V_m$, it can equivalently be realized as a symmetric spinor $V_{\alpha\beta} = V_{\beta\alpha}$. The relationship between $V_m$ and $V_{\alpha\beta}$ is as follows:

\[
V_{\alpha\beta} := (\gamma^a)_{\alpha\beta}V_a = V_{\beta\alpha}, \quad V_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}V_{\alpha\beta}.
\]
In three-dimensions an antisymmetric tensor $F_{ab} = -F_{ba}$ is Hodge-dual to a three-vector $F_a$, specifically

$$F_a = \frac{1}{2} \varepsilon_{abc} F^{bc}, \quad F_{ab} = -\varepsilon_{abc} F^c .$$  \hspace{1cm} (A.9)

Then, the symmetric spinor $F_{\alpha \beta} = F_{\beta \alpha}$, which is associated with $F_a$, can equivalently be defined in terms of $F_{ab}$:

$$F_{\alpha \beta} := (\gamma^a)_{\alpha \beta} F_a = \frac{1}{2} (\gamma^a)_{\alpha \beta} \varepsilon_{abc} F^{bc} .$$  \hspace{1cm} (A.10)

These three algebraic objects, $F_a$, $F_{ab}$ and $F_{\alpha \beta}$, are in one-to-one correspondence to each other, $F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha \beta}$. The corresponding inner products are related to each other as follows:

$$-F^a G_a = \frac{1}{2} F^{ab} G_{ab} = \frac{1}{2} F^{\alpha \beta} G_{\alpha \beta} .$$  \hspace{1cm} (A.11)

Let $\mathcal{M}_{ab} = -\mathcal{M}_{ba}$ be the Lorentz generators. They act on a vector $V_a$ as

$$\mathcal{M}_{ab} V_c = 2 \eta_{[a} V_{b]} ,$$  \hspace{1cm} (A.12)

and on a spinor $\psi_{\alpha}$ as

$$\mathcal{M}_{ab} \psi_{\alpha} = \frac{1}{2} \varepsilon_{abc} (\gamma^c)_{\alpha \beta} \psi_{\beta} .$$  \hspace{1cm} (A.13)

In accordance with (A.8)–(A.10), the Lorentz generators can also be realized as the vector $\mathcal{M}_{a}$ or the symmetric spinor $\mathcal{M}_{\alpha \beta}$ such that

$$\mathcal{M}_{a} \psi_{\alpha} = -\frac{1}{2} (\gamma^a)_{\alpha \beta} \psi_{\beta} , \quad \mathcal{M}_{\alpha \beta} \psi_{\gamma} = \varepsilon_{\gamma (\alpha} \psi_{\beta)} .$$  \hspace{1cm} (A.14)

As is clear from the explicit form of the $\gamma$-matrices, we are using a Majorana representation in which all the $\gamma$-matrices are real, and any Majorana spinor $\psi_{\alpha}$ is real,

$$(\psi^a)^* = \psi^a , \quad (\psi_{\alpha})^* = \psi_{\alpha} .$$  \hspace{1cm} (A.15)

In this paper we often make use of the group isomorphisms $\text{SO}(3) \cong \text{SU}(2)/\mathbb{Z}_2$ and $\text{SO}(4) \cong (\text{SU}(2)_L \times \text{SU}(2)_R)/\mathbb{Z}_2$ in order to convert each $\text{SO}(3)$ and $\text{SO}(4)$ vector index into a pair of $\text{SU}(2)$ ones. In the case of $\text{SO}(4)$, the $R$-symmetry group of $\mathcal{N} = 4$ supersymmetry, we first introduce the following $\Sigma$-matrices

$$(\Sigma_I)_{\tilde{i}i} = (1, i\sigma_1, i\sigma_2, i\sigma_3) , \quad I = 1, \cdots, 4 , \quad i = 1, 2 , \quad \tilde{i} = \tilde{1}, \tilde{2}$$  \hspace{1cm} (A.16)
The corresponding $\Sigma$-matrices are \( (\Sigma_I)_{ij} = (1, i\sigma_1, i\sigma_3) = (\Sigma_I)_{ji} \), \( I = 1, 2, 3 \), \( i,j = 1,2 \). (A.23)

They are obtained from the SO(4) $\Sigma$-matrices, eq. (A.16), by removing $\Sigma_3$. The remaining matrices $(\Sigma_I)_{ij}$ are symmetric in $i,j$. The symmetric $\tau$-matrices are defined as $\tau_I := \frac{1}{\sqrt{2}} (\Sigma_I)_{ij} = (\tau_I)_{ji}$. Their properties are

\[
(\tau_I)_{ij} (\tau^I)_{kl} = -\epsilon_{ijkl} , \quad (\tau_I)_{ij} (\tau_I)_{ij} = \delta_{IJ} .
\] (A.24)

More relations involving $\mathcal{N} = 3, 4$ isospinors are described in the main body of the paper.
B Derivation of the left projection operator

In this Appendix we derive the left covariant projection operator $\Delta_L^{(4)}$ used in subsection 6.4. The expression for the right covariant projection operator $\Delta_R^{(4)}$ follows using the mirror map.

The covariant projector operator is a fourth order differential operator $\Delta_L^{(4)}$ such that given any weight-$(n-4)$ left isotwistor superfield $U_{\alpha_1...\gamma_p i_1...i_q}^{(n-4)}$,

$$Q_{\alpha_1...\gamma_p}^{(n)} = \Delta_L^{(4)} U_{\alpha_1...\gamma_p i_1...i_q}^{(n-4)}$$  \hspace{1cm} (B.1)

is a weight-$n$ projective superfield. The projector will be of the form

$$\Delta_L^{(4)} = \frac{1}{48}(D^{(2)i\bar{j}j} + \ldots), \hspace{1cm} D^{(2)i\bar{j}j} := D^{(1)\gamma_{\alpha}} D_{(1)\bar{j}j}^{\gamma_{\alpha}} ,$$  \hspace{1cm} (B.2)

with the first term being the flat superspace limit and the dots denoting curvature dependent terms. A systematic, albeit time consuming, way to construct the full projector is to act with $D^{(1)i\bar{j}j}$ on $D^{(2)i\bar{j}j}$; the result is a function of curvature terms and covariant derivatives which vanish in the flat limit. One then iteratively adds curvature dependent terms to complete $D^{(2)i\bar{j}j}$ to the full $\Delta_L^{(4)}$. Instead, we use a short cut and derive the projector using known results together with some simple observations. Let us list the steps:

(i) A crucial observation is that, when acting on weight-$n$ left isotwistor superfields, eventually carrying also Lorentz and SU$(2)_R$ indices, the algebra of $D^{(1)i\bar{j}j}$ derivatives (6.13) becomes as follows

$$\{D^{(1)i\bar{j}j}, D^{(1)i\bar{j}j}\} U_{\gamma_1...\gamma_p k_1...k_q}^{(n)} = \left( -2i\varepsilon_{\alpha\beta}\varepsilon^{ij} S^{(2)kj} R_{kl} + 4i B^{(2)i\bar{j}j}_\alpha R_{\bar{i}j} - 4i S^{(2)i\bar{j}j} M_{\alpha\beta} + 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} B^{(2)\gamma\delta} M_{\gamma\delta} \right) U_{\gamma_1...\gamma_p k_1...k_q}^{(n)}. \hspace{1cm} (B.3)$$

(ii) We next note an analogue to the superspace geometry of 4D, $\mathcal{N} = 2$ conformal supergravity as formulated in [49], where the structure group is SL$(2,\mathbb{C}) \times U(2)$. When acting on a superfield $U_{\alpha_1...\alpha_p i_1...i_q}$ with $p$ undotted spinor indices and $q$ SU$(2)$ indices, the undotted spinor covariant derivatives algebra reduces to\textsuperscript{16}

$$\{D^{i}_\alpha, D^{i}_\beta\} U_{\alpha_1...\alpha_p i_1...i_q} = \left( 2\varepsilon_{\alpha\beta} \varepsilon^{ij} S_{kl} J_{kj} + 4Y_{\alpha\beta} J^{ij} + 4S^{ij} M_{\alpha\beta} + 2\varepsilon_{\alpha\beta} \varepsilon^{ij} Y_{\gamma\delta} M_{\gamma\delta} \right) U_{\alpha_1...\alpha_p i_1...i_q}. \hspace{1cm} (B.4)$$

\textsuperscript{16}We use the 4D notations and the algebra of [21].
Here, $M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}}$ are the 4D Lorentz generators in spinor notations and $J^{ij}$ is the SU(2) generator. Some relevant dimension-3/2 Bianchi identities are

\[
\mathcal{D}^{i}S^{jk} = \mathcal{D}^{i}_{(\alpha} Y_{\beta\gamma)} = 0 , \quad \mathcal{D}^{i}_{\alpha} S_{ij} + \mathcal{D}^{i}_{j} Y_{\beta\alpha} = 0 .
\] (B.5)

(iii) Since the action on spinor and isospinor indices of the 3D generators $M_{\alpha\beta}$ and $R_{ij}$, see eqs. (2.8b) and (6.9), are formally equivalent to the ones of $M_{\alpha\beta}$ and $J^{ij}$, the 3D \{\mathcal{D}^{(1)i\dot{\alpha}}, \mathcal{D}^{(1)j\dot{\beta}}\} algebra in (B.3) becomes equivalent to the \{\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}\} algebra in (B.4) if we identify

\[
\mathcal{D}_{\alpha} \leftrightarrow \mathcal{D}^{(1)i\dot{\alpha}}, \quad S_{ij} \leftrightarrow -iS^{(2)i\dot{j}}, \quad Y_{\alpha\beta} \leftrightarrow iB^{(2)i\dot{j}} .
\] (B.6)

This correspondence holds also at higher mass-dimensions due to the 3D dimension-3/2 Bianchi identities

\[
\mathcal{D}^{(1)i\dot{\alpha}}S^{(2)j\dot{k}} = \mathcal{D}^{(1)i\dot{\alpha}}B^{(2)i\dot{j}} = 0 , \quad -\mathcal{D}^{(1)i\dot{\alpha}}S^{(2)i\dot{j}} + \mathcal{D}^{(1)\dot{j}i\alpha}B^{(2)i\dot{j}} = 0 .
\] (B.7)

(iv) The antichiral projector in 4D $\mathcal{N} = 2$ supergravity [79, 61] is

\[
\Delta^{4} = \frac{1}{96} \left( (\mathcal{D}^{ij} + 16S^{ij})\mathcal{D}_{ij} - (\mathcal{D}^{\alpha\beta} - 16Y^{\alpha\beta})\mathcal{D}_{\alpha\beta} \right) ,
\] (B.8a)

\[
= \frac{1}{96} \left( \mathcal{D}^{ij}(\mathcal{D}_{ij} + 16S_{ij}) - \mathcal{D}^{\alpha\beta}(\mathcal{D}_{\alpha\beta} - 16Y_{\alpha\beta}) \right) ,
\] (B.8b)

with

\[
\mathcal{D}_{ij} := \mathcal{D}^{\gamma}_{(i} \mathcal{D}^{\gamma j)} , \quad \mathcal{D}_{\alpha\beta} := \mathcal{D}^{k}_{(\alpha} \mathcal{D}^{k}_{\beta)} .
\] (B.9)

From the previous discussion it follows that we may now find the left projection operator (6.55a), (6.55b) using the identifications (B.6) in (B.8a), (B.8b).

Another important property of the left projection operator is (6.60)–(6.61). To show those equations we use the super-Weyl transformation rules of the $\mathcal{D}^{(1)i\dot{\alpha}}$ derivatives (6.41) along with those of the dimension-1 superfields $S^{(2)i\dot{j}}$ and $B^{(2)i\dot{j}}$, as well as the transformation rules of the following dimension-3/2 superfield $T^{(3)i\dot{j}}$

\[
T^{(3)i\dot{j}} := v_{i}v_{j}v_{k}T^{ij\dot{k}} = -\frac{1}{3} \mathcal{D}^{(1)i\dot{\alpha}}S^{(2)i\dot{j}} = \frac{1}{3} \mathcal{D}^{(1)i\dot{\alpha}}B^{(2)i\dot{j}} ,
\] (B.10a)

\[
\mathcal{D}^{(1)i\dot{\alpha}}S^{(2)j\dot{k}} = 2T^{(3)i\dot{j}j\dot{k}} , \quad \mathcal{D}^{(1)i\dot{\alpha}}B^{(2)i\dot{j}} = 2\epsilon_{\alpha(\beta}T^{(3)i\dot{\gamma}\dot{j}} .
\] (B.10b)

\[\text{Note that for this sector of the geometry the U(1) generator never appears and we can forget about it in our considerations.}\]
We have
\[ \delta \sigma S_{ij}^{(2)} = \sigma S_{ij}^{(2)} - \frac{i}{4} (D_{ij}^{(2)} \sigma), \quad \delta \sigma B_{\alpha\beta}^{(2)} = \sigma B_{\alpha\beta}^{(2)} + \frac{i}{4} (D_{\alpha\beta}^{(2)} \sigma), \quad (B.11) \]
and
\[ \delta \sigma T_{\alpha}^{(3)i} = \frac{3}{2} \sigma T_{\alpha}^{(3)i} + \frac{i}{12} (D_{i}^{(1)} D_{ij}^{(2)} \sigma) - \frac{2}{3} s_{ij}^{(2)} (D_{ij}^{(1)} \sigma), \quad (B.12a) \]
\[ = \frac{3}{2} \sigma T_{\alpha}^{(3)i} + \frac{i}{12} (D_{i}^{(1)} D_{ij}^{(2)} \sigma) - \frac{2}{3} b_{\alpha\beta}^{(2)} (D_{ij}^{(1)} \sigma). \quad (B.12b) \]

To check eq. (6.61), the reader may also use the following equation
\[ D_{i}^{(1)} D_{ij}^{(2)} U_{L}^{(n-4)} = \left( - D_{i}^{(1)} D_{ij}^{(2)} - 8i s_{ij}^{(2)} D_{ij}^{(1)\bar{j}} - 8i b_{ij}^{(2)} D_{ij}^{(1)\bar{j}} \right) U_{L}^{(n-4)}, \quad (B.13) \]
together with (6.58).

We conclude this appendix by proving that the right-hand side of (6.69a) is: (i) independent of the isospinors \( u_{\bar{R}} = u_{\bar{i}} \); and (ii) obeys the left analyticity constraint (6.36). The derivation is completely analogous to the 4D \( N = 2 \) analysis given in appendix C of [61]. It is instructive, however, to repeat the computation in the 3D \( N = 4 \) case.

To prove the independence of (6.69a) from \( u_{\bar{i}} \) it is sufficient to prove its invariance under infinitesimal projective transformations of the form
\[ u_{\bar{i}} \rightarrow u_{\bar{i}} + \delta u_{\bar{i}} , \quad \delta u_{\bar{i}} = \alpha(t) u_{\bar{i}} + \beta(t) v_{\bar{i}}(t). \quad (B.14) \]

Here the time \( t \) is the integration variable of the contour integral. Since both \( u_{\bar{i}} \) and \( \delta u_{\bar{i}} \) are required to be time-independent, the transformation parameters should obey the equations:
\[ \dot{\alpha} = \beta \left( \frac{\dot{v}_{R}, u_{R}}{v_{R}, v_{R}} \right), \quad \dot{\beta} = -\beta \left( \frac{\dot{v}_{R}, u_{R}}{v_{R}, v_{R}} \right). \quad (B.15) \]

Equation (6.69a) is manifestly invariant under the \( \alpha \)-transformations. It remains to check invariance under \( \beta \)-transformations (B.14). Applying the \( \beta \)-transformation gives
\[ \delta \left( D^{(2,-2)} - 4i s^{(2,-2)} \right) \Delta^{(2,2)} U^{(n,-2)} = -\frac{16\beta}{v_{R}, u_{R}} \partial_{R}^{(-2)} s^{(2,2)} \Delta^{(2,2)} U^{(n,-2)}. \quad (B.16) \]
From [61]
\[ \beta \left( \frac{\dot{v}_{R}, v_{R}}{v_{R}, u_{R}} \right) \partial_{R}^{(-2)} V^{(n+4,2)} = -\frac{d}{dt} \left( \frac{b}{v_{R}, u_{R}} \right) V^{(n+4,2)}. \quad (B.17) \]
for any isotwistor superfield $V^{(n+4,2)}$ of weight $(n + 4, 2)$, such as $(S^{(2)} \Delta^{(2)} U^{(n,-2)})$ appearing in (B.16). Using this we find that the right hand side of (6.69a) is independent of $u_R$.

Now let us prove that the right hand side of (6.69a) obeys the left analyticity constraint (6.36). First of all, consider a weight-$(n + 2, 0)$ hybrid superfield $P^{(n+2,0)}(z, v_L, v_R)$, as for example the superfield $\Delta^{(2)} U^{(n,-2)}$. Using the identities

\[
\mathcal{D}_{\alpha}^{(1,-1)} \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) P^{(n+2,0)} = 4i (v_R, u_R) B_{\alpha \beta}^{(2)} \partial_{\alpha}^{(1,-1)} \partial_{\beta}^{(1,-1)} P^{(n+2,0)} \\
+ 2i (v_R, u_R) (D_{\alpha}^{(1,1)} S^{(2,-2)}) \partial_{\alpha}^{(1,-1)} P^{(n+2,0)}, \quad (B.18a)
\]

\[
[D_{\alpha}^{(1,1)}, D^{(-2,2)}] P^{(n+2,0)} = \left( - 4i B_{\alpha \beta}^{(2)} D_{\beta}^{(1,-1)} - 4i (D_{\alpha}^{(1,-1)} S^{(2,0)}) \right) \\
+ \partial_{\alpha}^{(1,-1)} \left( 4i S^{(2,2)} D_{\alpha}^{(1,1)} + 2i (D_{\alpha}^{(1,-1)} S^{(2,2)}) \right) \right) P^{(m+2,0)}, \quad (B.18b)
\]

one can show that

\[
\mathcal{D}_{\alpha k}^{(1)} \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) P^{(n+2,0)} = - \frac{u_k}{(v_R, u_R)} D_{\alpha}^{(1,1)} \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) P^{(n+2,0)} \\
+ v_k \left( D_{\alpha}^{(1,-1)} \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) P^{(n+2,0)} \right) \\
\mathcal{D}_{\alpha k}^{(1)} \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) P^{(n+2,0)} = 2i \partial_{\alpha}^{(1,-1)} \left[ \frac{u_k}{(v_R, u_R)} \left( - 2 S^{(2,2)} D_{\alpha}^{(1,1)} - (D_{\alpha}^{(1,-1)} S^{(2,2)}) \right) \right] \\
+ v_k \left( 2 B_{\alpha \beta}^{(2)} D_{\beta}^{(1,-1)} + (D_{\alpha}^{(1,1)} S^{(2,-2)}) \right) \right) P^{(n+2,0)}. \quad (B.19)
\]

It is shown in [61] that the right-hand side of the last equation is zero when integrated over a closed contour. Thus we have shown that

\[
\mathcal{D}_{\alpha k}^{(1)} \left( v_R, dv_R \right) \left( \mathcal{D}^{(2,-2)} - 4i S^{(2,-2)} \right) \Delta^{(2,2)} U^{(m,0)} = 0. \quad (B.20)
\]

As a result, the right hand side of (6.69a) indeed obeys the left analyticity constraint (6.36).

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