CRITICAL WAITING TIME PROCESSES IN INFINITE ERGODIC THEORY

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ABSTRACT. We study limit laws for return time processes defined on infinite conservative ergodic measure preserving dynamical systems. Especially for the critical cases with purely atomic limiting distribution we derive distorted processes possessing non-degenerated limits. For these processes also large deviation asymptotics are stated. The Farey map is used as an illustrating example giving new insights into the metric number theory of continued fractions.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper \((X, T, A, \mu)\) will denote a conservative ergodic measure preserving dynamical systems where \(\mu\) is an infinite \(\sigma\)-finite measure. Kac’s Theorem implies that in this situation the mean return time to sets of finite positive measure is infinite. In terms of Markov chains this corresponds to the null recurrent setting. Hence, new probabilistic properties of such dynamical systems lead to interesting results for null recurrent Markov chains, whereas known results for these Markov chains sometimes allow analog statements for infinite measure preserving transformations.

In this paper we are going to study the critical cases of the generalized Thaler’s Dynkin-Lamperti laws describing the asymptotic behaviour of the following processes.

- \(Z_n(x) := \max\{k \leq n : T^k(x) \in A\}, \ x \in A\) where \(A_n := \bigcup_{k=0}^n T^{-k} A, \)
- \(Y_n(x) := \min\{k > n : T^k(x) \in A\}, \ x \in X\)
- \(V_n(x) := Y_n - Z_n, \)

Namely, it is shown in [Tha98] that \(Z_n/n, Y_n/n, V_n/N\) all converge strongly in distribution to certain random variables depending only on the exponent \((1 - \alpha)\) of the wandering rate (cf. (T), Subsection 2). For certain values of \(\alpha\) these random variables turn out to be atomic. In order to derive non-degenerated results also for these cases we consider distorted processes, i.e.

\[
\frac{F(X_n)}{F(n)} \quad \text{and} \quad \frac{F(n - X_n)}{F(n)},
\]

where \(F\) is a regularly varying function and \((X_n)\) denotes any of the sequences \((Y_n), (V_n)\).

In particular we introduce the processes

\[
\Lambda_n := \frac{\sum_{k=0}^n \mu(A \cap \{\varphi > k\})}{\mu(A_n)}, \quad \Gamma_n := \frac{n \sum_{k=0}^n \mu(A \cap \{\varphi > k\})}{\mu(A_n) V_n},
\]

\[
\Delta_n := \frac{\sum_{k=0}^{n-1} \mu(A \cap \{\varphi > k\})}{\mu(A_n)}, \quad \Theta_n := \frac{n \sum_{k=0}^n \mu(A \cap \{\varphi > k\})}{\mu(A_n) Y_n}.
\]

We call \(\Lambda_n\) the distorted total waiting time process and \(\Delta_n\) the distorted residual waiting time process. In here,

\[
\varphi(x) = \inf\{n \geq 1 : T^n(x) \in A\}, \ x \in X, \quad (1.1)
\]
denotes the first return time to the set $A$.

We remark that the analog questions for $(Z_n)$ are already treated in [KS05b]. In [KS05a] some limit theorems for $(Z_n)$ have been applied to the Farey interval map deriving new number theoretical results for continued fractions. In the last section of this paper we also develop some consequences of the main theorems for continued fractions.

Finally, we would like to point out that other related results can be found in [TZ06], [Zwe03].

1.1. Infinite ergodic theory. A characterization of $(X, T, A, \mu)$ being a conservative ergodic measure preserving dynamical system where $\mu$ is an infinite $\sigma$-finite measure as used in this paper will be given in terms of the transfer operator below. For further definitions and details we refer the reader to [Aar97].

Let

$$P_\mu := \{ \nu : \nu \text{ probability measure on } A \text{ with } \nu \ll \mu \}$$

denote the set of probability measures on $A$ which are absolutely continuous with respect to $\mu$. The measures from $P_\mu$ represent the admissible initial distributions for the processes under consideration. With $P_\mu$ we will sometimes also denote the set of the corresponding densities.

Let us recall the notion of the wandering rate. For a fixed set $A \in A$ with $0 < \mu(A) < \infty$ we set

$$A_n := \bigcup_{k=0}^n T^{-k}A \quad \text{and} \quad W_n := W_n(A) := \mu(A_n), \quad n \geq 0,$$

and call the sequence $(W_n(A))$ the wandering rate of $A$. The following identities hold

$$W_n(A) = \sum_{k=0}^n \mu(A \cap \{ \varphi > k \}) = \int_A \min(\varphi, n + 1) \, d\mu, \quad n \geq 0.$$

Since $T$ is conservative and ergodic, for all $\nu \in P_\mu$, we have

$$\nu(\{ \varphi < \infty \}) = 1, \quad \nu(\{ Y_n < \infty \}) = 1 \quad \text{for all } n \geq 1, \quad \text{and} \quad \lim_{n \to \infty} \nu(A_n) = 1.$$

To explore the stochastic properties of a non-singular transformation of a $\sigma$–finite measure space it is often useful to study the long-term behaviour of the iterates of its transfer operator

$$\hat{T} : L_1(\mu) \to L_1(\mu), \quad f \mapsto \hat{T}(f) := d(\nu_f \circ T^{-1}) \frac{d\mu}{\nu_f},$$

where $\nu_f$ denote the measure with density $f$ with respect to $\mu$. Clearly, $\hat{T}$ is a positive linear operator characterized by

$$\int_B \hat{T}(f) \, d\mu = \int_{T^{-1}(B)} f \, d\mu, \quad f \in L_1(\mu), \quad B \in A.$$

An approximation argument shows that equivalently for all $f \in L_1(\mu)$ and $g \in L_\infty(\mu)$

$$\int_X \hat{T}(f) \cdot g \, d\mu = \int_X f \cdot g \circ T \, d\mu.$$

The ergodic properties of $(X, T, A, \mu)$ can be characterized in terms of the transfer operator in the following way (cf. [Aar97]). A system is conservative and ergodic if and only if for
all \( f \in L_1^+ (\mu) := \{ f \in L_1 (\mu) : f \geq 0 \text{ and } \int_X f \, d\mu > 0 \} \) we have \( \mu \)-a.e. 
\[
\sum_{n \geq 0} \hat{T}^n (f) = \infty.
\]

Note that invariance of \( \mu \) under \( T \) means \( \hat{T} (1) = 1 \).

The following two definitions are in many situation crucial within infinite ergodic theory.

- A set \( A \in \mathcal{A} \) with \( 0 < \mu (A) < \infty \) is called \textit{uniform} for \( f \in \mathcal{P}_\mu \) if there exists a sequence \((b_n)\) of positive reals such that
\[
\frac{1}{d_n} \sum_{k=0}^{n-1} \hat{T}^k (f) \to 1 \quad \mu \text{-a.e. uniformly on } A
\]
(i.e. uniform convergence in \( L_\infty (\mu|_{A\cap A}) \)).

- The set \( A \) is called a \textit{uniform} set if it is uniform for some \( f \in \mathcal{P}_\mu \).

Note that from [Aar97], Proposition 3.8.7, we know, that \((b_n)\) is regularly varying with exponent \( \alpha \) (for the definition of this property see Subsection 3.1) if and only if \((W_n)\) is regularly varying with exponent \( (1 - \alpha) \). In this case \( \alpha \) lies in the interval \([0, 1]\) and
\[
d_n W_n \sim \frac{n}{\Gamma (1 + \alpha) \Gamma (2 - \alpha)} ~ (n \to \infty). \tag{1.2}
\]

In here, \( c_n \sim b_n \) for some sequences \((c_n)\) and \((b_n)\) means that \( b_n \neq 0 \) has only finitely many exceptions and \( \lim_{n \to \infty} \frac{b_n}{c_n} = 1 \).

Next, we recall the notion of uniformly returning sets, which will be used to state the conditions in Theorem 2.4 (cf. [KS05b], Subsection 1.2)

- A set \( A \in \mathcal{A} \) with \( 0 < \mu (A) < \infty \) is called \textit{uniformly returning} for \( f \in \mathcal{P}_\mu \) if there exists a positive increasing sequence \((b_n)\) such that
\[
b_n \hat{T}^n (f) \to 1 \quad \mu \text{-a.e. uniformly on } A.
\]

- The set \( A \) is called \textit{uniformly returning} if it is uniformly returning for some \( f \in \mathcal{P}_\mu \).

From [KS05b], Proposition 1.2, we know that for \( \beta \in [0, 1) \) we have that \((b_n)\) is regularly varying with exponent \( \beta \) if and only if \((W_n)\) is regularly varying with the same exponent. In this case,
\[
b_n \sim W_n \Gamma (1 - \beta) \Gamma (1 + \beta) \quad (n \to \infty).
\]

Also (see [KS05b], Proposition 1.1) we have that every uniformly returning set is uniform. In the proof of this fact it is shown that there exists a function \( f \in \mathcal{P}_\mu \) such that \( A \) is uniform as well as uniformly returning for \( f \). This observation will be relevant in Theorem 2.5. Under some extra conditions the reverse implication is also true (cf. [KS05a], Proposition 2.6).

**Example.** Let \( T : [0, 1] \to [0, 1] \) be an interval map with two increasing full branches and an indifferent fixed point at 0 satisfying Thaler’s conditions in [Tha00]. Then any set \( A \in \mathcal{B}_{[0,1]} \) with positive distance from the indifferent fixed point 0 and positive Lebesgue measure \( \lambda (A) \) is uniformly returning.

Sometimes the limiting behaviour of processes defined in terms of a non-singular transformation does not depend on the initial distribution. This is formalized as follows.

Let \( \nu \) be a probability measure on the measurable space \((X, \mathcal{A})\) and \((R_n)_{n \geq 1}\) be a sequence of measurable real functions on \( X \). Then distributional convergence of \((R_n)_{n \geq 1}\)
w.r.t. \( \nu \) to some random variable \( R \) with values in \([−\infty, \infty]\) will be denoted by \( R_n \overset{\nu}{\Rightarrow} R \). Strong distributional convergence abbreviated by \( R_n \overset{\mu}{\Rightarrow} R \) on the \( \sigma \)-finite measures space \((X, A, \mu)\) means that \( R_n \overset{\nu}{\Rightarrow} R \) for all \( \nu \in \mathcal{P}_\mu \). In particular for \( c \in [−\infty, \infty] \),

\[
R_n \overset{\nu}{\Rightarrow} c \iff R_n \overset{\mu}{\rightarrow} c
\]

locally in measure,

which we also denote by \( R_n \overset{\mu}{\rightarrow} c \).

### 2. Statements of Main Results

We begin this section with recalling the following interesting limit laws for the processes \( Y_n \) and \( V_n \) which are due to Thaler [Tha98].

(T) **Thaler’s Dynkin-Lamperti Theorem.** Let \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \) be a uniform set. If the wandering rate \((W_n(A))\) is regularly varying with exponent \( 1-\alpha \) for \( \alpha \in [0, 1] \), then we have

\( Y_n \overset{\mu}{\Rightarrow} \varphi_\alpha \),

where \( \varphi_\alpha, \alpha \in (0, 1) \), denotes the random variable on \([1, \infty)\) with density

\[
f_{\varphi_\alpha}(x) = \sin \frac{\pi \alpha}{x} \frac{1}{x(x-1)^\alpha}, \quad x > 1,
\]

and \( \varphi_0 = \infty, \varphi_1 = 1 \).

\( V_n \overset{\mu}{\Rightarrow} \eta_\alpha \),

where \( \eta_\alpha, \alpha \in (0, 1) \), denotes the random variable on \([0, \infty)\) with density

\[
f_{\eta_\alpha}(x) = \sin \frac{\pi \alpha}{1+\alpha} \frac{1-(\max\{1-x,0\})^\alpha}{x^\alpha}, \quad x > 0,
\]

and \( \eta_0 = \infty, \eta_1 = 0 \).

To apply (T) to the distorted processes we need the following proposition from [KS05a]. Its first part was independently proved by Thaler in [Tha05].

**Proposition 2.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \( Y_n : \Omega \rightarrow [0, \infty] \) be measurable \((n \geq 1)\), and let \( Y \) be a random variable with values in \([0, \infty]\). Then

1. If \( \mathbb{P}(Y = 0) = 0 = \mathbb{P}(Y = \infty) \) and \( F \) is a regularly varying function with exponent \( \beta \in \mathbb{R} \), then

\[
\frac{Y_n}{n} \overset{\mathbb{P}}{\Rightarrow} Y \quad \iff \quad \frac{F(Y_n)}{F(n)} \overset{\mathbb{P}}{\Rightarrow} Y^\beta.
\]

2. If \( Y = 0 \) and \( F \) is a regularly varying function with exponent \( \beta \in \mathbb{R} \setminus \{0\} \) then

\[
\frac{Y_n}{n} \overset{\mathbb{P}}{\Rightarrow} 0 \quad \iff \quad \frac{F(Y_n)}{F(n)} \overset{\mathbb{P}}{\Rightarrow} \left\{ \begin{array}{ll}
0 & \text{for } \beta > 0 \\
\infty & \text{for } \beta < 0
\end{array} \right..
\]

3. If \( Y = \infty \) and \( F \) is a regularly varying function with exponent \( \beta \in \mathbb{R} \setminus \{0\} \) then

\[
\frac{Y_n}{n} \overset{\mathbb{P}}{\Rightarrow} \infty \quad \iff \quad \frac{F(Y_n)}{F(n)} \overset{\mathbb{P}}{\Rightarrow} \left\{ \begin{array}{ll}
\infty & \text{for } \beta > 0 \\
0 & \text{for } \beta < 0
\end{array} \right..
\]
The following corollary is a direct consequence of (T), Proposition 2.1, and the fact that
\[ \Lambda_n = \frac{F(V_n)}{F(n)}, \quad \Gamma_n := \frac{G(V_n)}{G(n)}, \quad \Delta_n = \frac{F(Y_n - n)}{F(n)}, \quad \Theta_n = \frac{G(Y_n)}{G(n)}, \]
with \( F(n) := W_n \) and \( G(n) := W_n/n \).

**Corollary 2.2.** Let \( A \in A \) with \( 0 < \mu(A) < \infty \) be a uniform set such that the wandering rate \( (W_n) \) is regularly varying with exponent \( 1 - \alpha \).

1. If \( 0 \leq \alpha < 1 \), then we have
   \[ \Lambda_n \overset{\mathcal{L}(\mu)}{\longrightarrow} \lambda_\alpha, \]
   where \( \lambda_\alpha \) denotes the random variable on \([0, \infty)\) with density
   \[ f_{\lambda_\alpha}(x) = \frac{1}{1 - \alpha} \sin \frac{\pi \alpha}{\pi} \left(1 - \frac{1}{x^{1/\alpha}} \right)^\alpha, \quad \alpha \in (0, 1), \]
   and \( \lambda_0 = \infty \) (cf. Fig. 2.1).

2. If \( 0 < \alpha \leq 1 \), then we have
   \[ \Gamma_n \overset{\mathcal{L}(\mu)}{\longrightarrow} \gamma_\alpha, \]
   where \( \gamma_\alpha \) denotes the random variable on \([0, \infty)\) with density
   \[ f_{\gamma_\alpha}(x) = \frac{\sin \frac{\pi \alpha}{\pi}}{\alpha \pi} \left(1 - \frac{1}{x^{1/\alpha}} \right)^\alpha, \quad \alpha \in (0, 1), \]
   and \( \gamma_1 = \infty \) (cf. Fig. 2.2).

3. If \( 0 \leq \alpha < 1 \), then we have
   \[ \Delta_n \overset{\mathcal{L}(\mu)}{\longrightarrow} \delta_\alpha, \]
   where \( \gamma_\alpha \) denotes the random variable on \([0, \infty)\) with density
   \[ f_{\delta_\alpha}(x) = \frac{1}{1 - \alpha} \sin \frac{\pi \alpha}{\pi} \left(1 + \frac{1}{x^{1/\alpha}} \right)^\alpha, \quad \alpha \in (0, 1), \]
   and \( \delta_0 = \infty \) (cf. Fig. 2.3).
Figure 2.2. The densities $f_{\gamma_\alpha}$ of the limiting distribution of the normalized Kac process for different values of $\alpha \in (0, 1)$. The extreme case is given by $\gamma_1 = \infty$.

Figure 2.3. The densities $f_{\delta_\alpha}$ of the limiting distribution of the normalized Kac process for different values of $\alpha \in (0, 1)$. The extreme case is given by $\delta_0 = \infty$.

(4) If $0 < \alpha \leq 1$, then we have

$$
\Theta_n \overset{\mathcal{L}(\mu)}{\longrightarrow} \theta_\alpha,
$$

where $\chi_\alpha$ denotes the random variable on $[0, 1]$ with density

$$
 f_{\theta_\alpha} (x) = \frac{\sin \pi \alpha}{\pi \alpha} \frac{1}{(1 - x^{1/\alpha})^\alpha}, \quad \alpha \in (0, 1),
$$

and $\theta_1 = 1$ (cf. Fig. 2.4).
Remark. For $\alpha \in (0, 1)$ we have

$$
\lambda_\alpha \overset{\text{dist.}}{=} (\eta_\alpha)^{1-\alpha}, \quad \gamma_\alpha \overset{\text{dist.}}{=} (\eta_\alpha)^{-\alpha}, \quad \delta_\alpha \overset{\text{dist.}}{=} (\varphi_\alpha - 1)^{1-\alpha}, \quad \text{and} \quad \theta_\alpha \overset{\text{dist.}}{=} (\varphi_\alpha)^{-\alpha}.
$$

Note, that in particular $\theta_{1/2}$ obeys the arc-sine law, i.e. it has density

$$
f_{\theta_{1/2}}(x) = \frac{2}{\pi \sqrt{1-x^2}}, \quad 0 < x < 1,
$$

and $\delta_{1/2}$ obeys the Cauchy law, i.e. it has density

$$
f_{\delta_{1/2}}(x) = \frac{2}{\pi (1+x^2)}, \quad x > 0.
$$

The following two theorems treat the four cases with $\alpha \in \{0, 1\}$ not covered by Corollary 2.2.

**Theorem 2.3.** Let $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ be a uniform set. If the wandering rate $(W_n)$ is regularly varying with exponent 1 then we have

$$
\Gamma_n \overset{\mathcal{L}(\mu)}{\to} U \quad \text{and} \quad \Theta_n \overset{\mathcal{L}(\mu)}{\to} U,
$$

(2.1)

where $U$ denotes the random variable uniformly distributed on the unit interval.

**Example.** Let $f(0) = 0, f(x) = x + x^2 e^{-x}, \quad x > 0$, and let $a \in (0, 1)$ be determined by $f(a) = 1$. Define $T : [0, 1] \to [0, 1]$ by

$$
T(x) := \begin{cases} 
  f(x), & x \in [0, a], \\
  \frac{x-a}{1-a}, & x \in (a, 1].
\end{cases}
$$

Then the map $T$ satisfies Thaler’s conditions (T1)–(T4) in [Tha95]. Any set $A \in \mathcal{B}_{[0,1]}$ with $\lambda(A) > 0$ which is bounded away from the indifferent fixed points is a uniform set for $T$. Furthermore, we have

$$
W_n \sim \text{const} \cdot \frac{n}{\log(n)} \quad (n \to \infty).
$$
Hence,
\[
\frac{\log (n) \zeta(\mu)}{\log (V_n)} \xrightarrow{\mathcal{L}(\mu)} U \quad \text{and} \quad \frac{\log (n) \zeta(\mu)}{\log (Y_n)} \xrightarrow{\mathcal{L}(\mu)} U.
\]

**Theorem 2.4.** Let \( A \in \mathcal{A} \) with \( 0 < \mu (A) < \infty \) be a uniformly returning set. If the wandering rate \( (W_n) \) is slowly varying, then we have
\[
\Lambda_n \xrightarrow{\mathcal{L}(\mu)} U \quad \text{and} \quad \Delta_n \xrightarrow{\mathcal{L}(\mu)} U,
\]
where \( U \) denotes the random variable uniformly distributed on the unit interval.

**Example.** We consider the Lasota–Yorke map \( T : [0, 1] \to [0, 1] \), defined by
\[
T(x) = \begin{cases} 
x - x^2, & x \in [0, \frac{1}{2}], \\
2x - 1, & x \in (\frac{1}{2}, 1].
\end{cases}
\]
This map satisfies the Thaler’s conditions (i)–(iv) in [Tha00]. Any compact subset \( A \) of \( (0, 1) \) with \( \lambda (A) > 0 \) is a uniformly returning set and we have
\[
W_n \sim \log(n) \quad \text{as} \quad n \to \infty.
\]
Hence,
\[
\frac{\log (Y_n - n)}{\log (n)} \xrightarrow{\mathcal{L}(\mu)} U \quad \text{and} \quad \frac{\log (V_n)}{\log (n)} \xrightarrow{\mathcal{L}(\mu)} U.
\]

Another application of the above theorem will be given in the last section on continued fractions.

**Remark.** The processes considered in [KS05b], Theorem 1.5 and Theorem 1.6, can be expressed in terms of \( F \) and \( G \) by \( G(n)/G(Z_n) \) for \( \alpha = 0 \) and \( F(n - Z_n)/F(n) \) for \( \alpha = 1 \), respectively. Hence, taking the earlier result from [KS05b] into account we have developed non-degenerated results for all critical cases for the processes \( Z_n, Y_n \), and \( V_n \).

Finally, we give a common large deviation asymptotic for the two processes \( Z_n \) and \( Y_n \), as well as a large deviation asymptotic for the process \( V_n \). An application of this theorem will be given in the last section on continued fractions.

**Theorem 2.5.** Let \( A \) be both uniformly returning and uniform for \( f \in \mathcal{P}_\mu \) and let \( \nu \) denote the probability measure with density \( f \). We suppose that \( \mu (A \cap \{ \varphi > n \}) \) satisfies the following asymptotic
\[
\mu (A \cap \{ \varphi > n \}) \sim n^{-1} L(n) \quad \text{as} \quad n \to \infty,
\]
where \( L \) a slowly varying function.

- For \( 0 \leq x < 1 \) and \( y \geq 0 \) with \( x + y \neq 0 \) we have
  \[
  \nu \left( \frac{n - Z_n}{n} \geq x, \frac{Y_n - n}{n} > y \right) \sim \log \left( \frac{1 + y}{x + y} \right) \cdot \frac{L(n)}{W_n} \quad \text{as} \quad n \to \infty.
  \]
- For \( x > 0 \) we have
  \[
  \nu \left( \frac{V_n}{n} > x \right) \sim H(x) \frac{L(n)}{W_n} \quad \text{as} \quad n \to \infty,
  \]
where
\[
H(x) := \begin{cases} 
1 - \log(x) & \text{for} \ x \in (0, 1), \\
1/x & \text{for} \ x \geq 1.
\end{cases}
\]
3. PROOFS

3.1. Some facts from regular variation. We recall the concepts of regularly varying functions and sequences (see also [BGT89] for a comprehensive account). Throughout we use the convention that for two sequences \((a_n), (b_n)\) we write \(a_n = o(b_n)\) if \(b_n \neq 0\) fails only for finitely many \(n\) and \(\lim_{n \to \infty} a_n/b_n = 0\).

A measurable function \(R : \mathbb{R}^+ \to \mathbb{R}\) with \(R > 0\) on \((a, \infty)\) for some \(a > 0\) is called regularly varying at \(\infty\) with exponent \(\rho \in \mathbb{R}\) if

\[
\lim_{t \to \infty} R(\lambda t)/R(t) = \lambda^\rho \quad \text{for all } \lambda > 0.
\]

A regularly varying function \(L\) with exponent \(\rho = 0\) is called slowly varying at \(\infty\), i.e.

\[
\lim_{t \to \infty} L(\lambda t)/L(t) = 1 \quad \text{for all } \lambda > 0.
\]

Clearly, a function \(R : \mathbb{R}^+ \to \mathbb{R}\) is regularly varying at \(\infty\) with exponent \(\rho \in \mathbb{R}\) if and only if

\[
R(t) = t^\rho L(t), \quad t \in \mathbb{R}^+,
\]

for \(L\) slowly varying at \(\infty\).

A function \(R\) is said to be regularly varying at \(0\) if \(t \mapsto R(t^{-1})\) is regularly varying at \(\infty\).

A sequence \((u_n)\) is regularly varying with exponent \(\rho\) if \(u_n = R(n), \ n \geq 1\), for \(R : \mathbb{R}^+ \to \mathbb{R}\) regularly varying at \(\infty\) with exponent \(\rho\).

The following facts will be needed in the proofs of the preparatory lemmas and propositions of this sections, as well as for the main theorems.

(KL) Karamata’s Lemma ([Fel71], [Kar33]). If \((a_n)\) is a regularly varying sequence with exponent \(\rho\) and if \(p \geq -\rho - 1\), then

\[
\lim_{n \to \infty} \frac{n^{p+1} a_n}{\sum_{k \leq n} k^p a_k} = p + \rho + 1.
\]

(UA) Uniform asymptotics ([Sen76]) Let \((p_n)\) and \((q_n)\) be two positive sequences with \(p_n \to \infty\) and \(\frac{q_n}{p_n} \in [1/K, K]\), \(K \geq 1\) for \(n\) large enough. Then for every slowly varying function \(L\) we have

\[
\lim_{n \to \infty} \frac{L(p_n)}{L(q_n)} = 1.
\]

(EL) Erickson Lemma ([Eri70]) Let \(L \nearrow \infty\) be a monotone increasing continuous slowly varying function. Let \(a_t(x)\) be defined by \(a_t(x) := L^{-1}(xL(t))\) with \(x \in (0, \infty)\), where \(L^{-1}(\cdot)\) denoting the inverse function of \(L(\cdot)\). Then we have for every fixed \(x \in (0, \infty)\)

\[
a_t(x) \to \infty \quad (t \to \infty)
\]

and for \(0 < x < y\)

\[
a_t(x)/a_t(y) \to 0 \quad (t \to \infty).
\]
3.2. **Compactness results.** Under the assumption that $T$ is a nonsingular ergodic transformation on $(X, \mathcal{A}, \mu)$ the compactness theorem in [Aar97, Section 3.6], gives the following implication.

- If $R_n \circ T - R_n \overset{\mu}{\to} 0$ and $R_n \overset{\nu}{\to} R$ for some $\nu \in \mathcal{P}_\mu$ then $R_n \overset{\mathcal{L}(\mu)}{\to} R$.

Hence, before proving the main theorems we show the following two lemmas.

**Lemma 3.1.** Let $A \in \mathcal{A}$ be a set of positive finite measure $\mu(A)$ and $\tilde{L}(t) \to \infty, t \to \infty$, be a slowly varying function such that

$$
\tilde{L}(x) = C \exp \left( \int_B^x \frac{\zeta(t)}{t} \, dt \right) \quad \text{for all } x \geq B,
$$

where $C \in (0, \infty)$ and $\zeta$ a continuous function on $[B, \infty)$ with

$$
\zeta(x) \to 0 \quad (x \to \infty).
$$

Then we have

$$
\frac{1}{L(n)} \left( \tilde{L}(Y_n \circ T) - \tilde{L}(Y_n) \right) \overset{\mu}{\to} 0 \quad (3.1)
$$

and

$$
\frac{1}{L(n)} \left( \tilde{L}(Y_n \circ T) - \tilde{L}(Y_n) \right) \overset{\mu}{\to} 0. \quad (3.2)
$$

**Proof.** Without loss of generality we assume that there exists $\delta \in (0, 1)$ such that

$$
|\zeta(t)| < \delta \quad \text{for all } t \geq B.
$$

For $\varepsilon > 0$ we define

$$
K_{\varepsilon,n} := \left\{ Y_n < \infty \wedge \frac{1}{L(n)} \left| \tilde{L}(Y_n \circ T) - \tilde{L}(Y_n) \right| \geq \varepsilon \right\} \quad (n \in \mathbb{N}).
$$

Since

$$
Y_n(T(x)) = \begin{cases}
Y_n(x) - 1, & x \in \{Y_n < \infty\} \cap T^{-(n+1)}A^c, \\
n + \varphi\left(T^{n+1}(x)\right), & x \in T^{-(n+1)}A,
\end{cases} \quad (3.3)
$$

we conclude

$$
K_{\varepsilon,n} \subset \left( \{Y_n < \infty\} \cap T^{-(n+1)}A^c \cap \left\{ \frac{1}{L(n)} \left( \tilde{L}(Y_n) - \tilde{L}(Y_n - 1) \right) \geq \varepsilon \right\} \right)
\cup \left( T^{-(n+1)}A \cap \left\{ \frac{1}{L(n)} \left( \tilde{L}(n + \varphi(T^{n+1}(\omega))) - \tilde{L}(n + 1) \right) \geq \varepsilon \right\} \right).
$$

For $n \geq B$ large enough such that $C\delta B^{-\delta} (n - 1)^{\delta-1} \leq \varepsilon$ we have

$$
\tilde{L}(Y_n(\omega)) - \tilde{L}(Y_n(\omega) - 1) = C \exp \left( \int_B^{Y_n(\omega)} \frac{\zeta(t)}{t} \, dt \right) \times
$$

$$
\exp \left( \int_{Y_n(\omega)-1}^{Y_n(\omega)} \frac{\zeta(t)}{t} \, dt \right) - 1.
$$

Since $|\zeta(t)| < \delta$ on $[B, \infty)$ we have by the Mean-Value Theorem

$$
\tilde{L}(Y_n(\omega)) - \tilde{L}(Y_n(\omega) - 1) \leq \frac{C}{B^{\delta}} \left( Y_n(\omega) \right)^{\delta} - \left( Y_n(\omega) - 1 \right)^{\delta}
$$

$$
\leq \frac{C\delta}{B^{\delta}} (n - 1)^{\delta-1}.
$$
Now choose \( n \geq B \) large enough such that \( \frac{C(n-1)^{\delta-1}}{B^n L(n)} < \varepsilon \). This implies

\[
K_{\varepsilon, n} \subset \left( T^{-n+1} A \cap \left\{ \frac{1}{L(n)} \left( L(n + \varphi(T^{n+1}(\omega))) - L(n + 1) \right) \geq \varepsilon \right\} \right).
\]

Similarly as above, we obtain for sufficiently large \( n \)

\[
L(n + \varphi(T^{n+1}(\omega))) - L(n + 1) = C \exp \left( \int_B^{n+1} \frac{\zeta(t)}{t} dt \right) \times \left[ \exp \left( \int_{n+1}^{n+\varphi(T^{n+1}(\omega))} \frac{\zeta(t)}{t} dt \right) - 1 \right].
\]

Since \( |\zeta(t)| < \delta \) on \( [B, \infty) \), there exists a constant \( C_\delta \), such that

\[
L(n + \varphi(T^{n+1}(\omega))) - L(n + 1) \leq C_\delta \left( (n + \varphi(T^{n+1}(\omega)))^\delta - (n + 1)^\delta \right) =: E.
\]

By the Mean-Value Theorem, we have

\[
E \leq \delta C_\delta n^{\delta-1} \left( \varphi(T^{n+1}(\omega)) - 1 \right).
\]

Hence,

\[
T^{-(n+1)} A \cap \left\{ \frac{L(n + \varphi(T^{n+1}(\omega))) - L(n + 1)}{L(n)} \geq \varepsilon \right\}
\]

\[
\subset T^{-(n+1)} \left( A \cap \left\{ \varphi \geq \frac{n^{\delta-1} L(n)}{\delta C_\delta} \varepsilon + 1 \right\} \right).
\]

Using the invariance of \( \mu \) and the fact that by choice of \( \delta \) we have \( n^{1-\delta} L(n) \to \infty \), we obtain

\[
\mu(K_{\varepsilon, n}) \leq \mu \left( A \cap \left\{ \varphi \geq \frac{n^{1-\delta} L(n)}{\delta C_\delta} \varepsilon + 1 \right\} \right) \to 0 \quad \text{for} \quad n \to \infty.
\]

This implies

\[
\lim_{n \to \infty} \nu(K_{\varepsilon, n}) = 0 \quad \text{for all} \quad \nu \in \mathcal{P}_\mu.
\]

Using this and the fact \( \{Y_n < \infty\} = X \) modulo a set of \( \mu \)-measure 0, we finally conclude for all \( \nu \in \mathcal{P}_\mu \)

\[
\lim_{n \to \infty} \nu \left( \left\{ \frac{L(Y_n \circ T - L(Y_n))}{L(n)} \geq \varepsilon \right\} \right) = 0.
\]

The second assertion follows analogously by using the first part of the lemma and [KS05b], Lemma 3.1.

\( \square \)

**Lemma 3.2.** Let \( A \in \mathcal{A} \) be a set of positive finite measure \( \mu(A) \), then

\[
\Delta_n \circ T - \Delta_n \xrightarrow{\mu} 0 \quad \text{and} \quad \Lambda_n \circ T - \Lambda_n \xrightarrow{\mu} 0.
\]

**Proof.** Let \( \varepsilon > 0 \) be given, and let

\[
K_{\varepsilon, n} := \{Y_n < \infty \land |\Delta_n \circ T - \Delta_n| \geq \varepsilon\}.
\]
choose \( n \) large enough such that \( \frac{\mu(A)}{W_n} < \varepsilon \). By (3.3) we have
\[
K_{\varepsilon,n} \subset T^{-(n+1)}A \cap \left\{ \varphi \left( T^{n+1}(\omega) \right) - 1 \geq \varepsilon \frac{W_n}{\mu(A)} \right\}
\]
this implies
\[
\mu(K_{\varepsilon,n}) \to 0 \quad \text{as} \quad n \to \infty.
\]
Thus,
\[
\lim_{n \to \infty} \nu(K_{\varepsilon,n}) = 0 \quad \text{for all} \quad \nu \in \mathcal{P}_\mu.
\]
From the fact that \( \{ Y_n < \infty \} = X \) modulo a set of \( \mu \)-measure 0 the first assertion of the lemma follows.

The second assertion follows analogously by using the first part of the lemma and [KS05b], Lemma 3.2.

3.3. Proofs of main theorems.

Proof. (First part of Theorem 2.3) For \( \alpha = 1 \) we have \( W_n/n \sim 1/L(n) \) for a slowly varying function \( L \) with \( L(n) \to \infty \). Due to the Representation Theorem for slowly varying functions (cf. [Sen76]) there exists a slowly varying function \( \tilde{L} \) with the same properties as in Lemma 3.1 such that \( L(x) \sim \tilde{L}(x) \) as \( x \to \infty \).

Therefore, to prove the first part it suffices to show
\[
\lim_{n \to \infty} \nu(\tilde{L}(n)) \frac{\nu(\tilde{L}(V_n))}{\tilde{L}(V_n)} \sim u.
\]

Let \( A \) be a uniform set for some \( f \in \mathcal{P}_\mu \). First, for every \( x \in (0, 1] \), we have
\[
\nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) = \sum_{k=0}^{n} \nu(\varphi > k + \left\lfloor a_n(x^{-1}) \right\rfloor, Z_n = k)
\]
\[
= \int A \sum_{k=0}^{n} \hat{f}^k(f) \cdot 1_{A \cap \{ \varphi > \left\lfloor a_n(x^{-1}) \right\rfloor \}} \, d\mu,
\]
where \( \nu \) denotes the probability measure with density \( f \in \mathcal{P}_\mu \) and \( a_n(x^{-1}) = \tilde{L}^{-1}(x^{-1}\tilde{L}(n)) \).

Note, by (EL) we have
\[
a_n(x^{-1}) \to \infty \quad \text{and} \quad \frac{n}{a_n(x^{-1})} \to 0 \quad \text{for} \quad n \to \infty.
\]

By the asymptotic in (1.24) and (KL) we obtain on the one hand that for \( \varepsilon \in (0, 1) \) and sufficiently large \( n \)
\[
\nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \leq (1 + \varepsilon) \mu(A \cap \{ \varphi > \left\lfloor a_n(x^{-1}) \right\rfloor \}) \cdot \tilde{L}(n)
\]
\[
\sim (1 + \varepsilon) x.
\]

This implies
\[
\limsup \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \leq x.
\]
On the other hand, we similarly obtain
\[ \liminf \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \geq x. \]

Both inequalities give
\[ \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \to x. \]

Now let \( x > 1 \). Then we have
\[ \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \leq \nu \left( \frac{\tilde{L}(n)}{L(Y_n)} < x \right) \leq 1. \]

From this it follows that
\[ \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \to 1. \]

Hence, we obtain for all \( x \in (0, \infty) \)
\[ \nu \left( \frac{\tilde{L}(n)}{L(V_n)} < x \right) \to 1 - \max (1 - x, 0). \]

Using this and Lemma 3.1, the convergence in (3.4) follows by the compactness theorem. Finally, since \( V_n \to \infty \) in probability, it is clear that the slowly varying function \( \tilde{L} \) may be replaced by any function \( L \) with \( L(n) \sim C \cdot \tilde{L}(n) \), \( C > 0 \), as \( n \to \infty \). This finishes the proof of the first part. \( \square \)

**Proof.** (Second part of Theorem 2.3) Let \( f \) and \( \tilde{L} \) be given as in the first part of the proof of this theorem. Since
\[ \{Z_n \leq k\} = \{Y_k > n\} \quad \text{for } 1 \leq k \leq n, \]
we have for every \( x \in (0, 1) \)
\[ \nu \left( \frac{\tilde{L}(n)}{L(Y_n)} < x \right) = \nu \left( Y_n > \left| a_n \left( x^{-1} \right) \right| \right) = \nu \left( Z_{|a_n(x^{-1})|} \leq n \right) = \int_A \sum_{k=0}^n \tilde{T}^k(f) \cdot 1_{A \cap \{\varphi > |a_n(x^{-1})| - k\}} \, d\mu, \]

where \( \nu \) denotes the probability measure with density \( f \in P_\mu \) and \( a_n \left( x^{-1} \right) = \tilde{L}^{-1} \left( x^{-1} \tilde{L}(n) \right) \).

Using the monotonicity of the sequence \( \{1_{A \cap \{\varphi > n\}}\} \) we obtain by the asymptotic in (1.2) on the one hand that
\[ \nu \left( \frac{\tilde{L}(n)}{L(Y_n)} < x \right) \leq \int_A 1_{A \cap \{\varphi > |a_n(x^{-1})| - n\}} \cdot \sum_{k=0}^n \tilde{T}^k(f) \, d\mu \sim \mu \left( A \cap \{\varphi > |a_n(x^{-1})| - n\} \right) \cdot \tilde{L}(n). \]

This together with (KL) and (UA) implies
\[ \limsup \nu \left( \frac{\tilde{L}(n)}{L(Y_n)} < x \right) \leq x. \]
On the other hand, we derive in a similar way
\[ \nu \left( \frac{L(n)}{L(Y_n)} < x \right) \geq \int_A 1_{A \cap \{ \varphi > a_n(x-1) \}} \cdot \sum_{k=0}^{n} \hat{T}^k(f) \ d\mu \]
\[ \sim \mu (A \cap \{ \varphi > a_n(x-1) \}) \cdot \overline{L}(n) . \]
This gives the opposite inequality
\[ \lim \inf \nu \left( \frac{L(n)}{L(Y_n)} < x \right) \geq x. \]
Hence, we obtain
\[ \overline{L}(n) \to U. \]
Using this and Lemma 3.1, the second assertion of the theorem follows by the compactness theorem and the fact that \( Y_n \to \infty \) in probability. \( \Box \)

The proof of Theorem 2.4 splits into two parts.

Proof. (First part of Theorem 2.4) Let \( A \) be a uniformly returning set for some \( f \in P_\mu \). Let \( W_n \sim L(n) \) as \( n \to \infty \), without loss of generality we may assume that \( L \) is monotone increasing and continuous. We have for every fixed \( x \in (0,1) \)
\[ \nu \left( \frac{L(V_n)}{L(n)} > x \right) = \nu (V_n > a_n(x)) \]
\[ = \sum_{k=0}^{n} \nu (Y_n > k + a_n(x), Z_n = k) \]
\[ = \sum_{k=0}^{n} \cdots + \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \int_A 1_{A \cap \{ \varphi > n-k \}} \cdot \hat{T}^k(f) \ d\mu \]
\[ =: I(n) + J(n). \]
First, we have to prove that
\[ I(n) \to 1 - x \text{ as } n \to \infty. \] (3.5)
In fact, we have
\[ I(n) = \sum_{k=0}^{n} \int_A 1_{A \cap \{ \varphi > n-k \}} \cdot \hat{T}^k(f) \ d\mu = \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \sum_{k=n-a_n(x)}^{\cdots} \int_A 1_{A \cap \{ \varphi > n-k \}} \cdot \hat{T}^k(f) \ d\mu \]
\[ =: \nu (A_n) - \overline{I}(n) . \]
we first note, that
\[ \lim_{n \to \infty} \nu (A_n) = 1. \]
By a similar argument as in the proof of Theorem 2.4 we obtain for all \( \varepsilon \in (0, 1) \) and sufficiently large \( n \) on the one hand that
\[
\tilde{I}(n) \leq (1 + \varepsilon)^2 \frac{1}{W_n} W_{\lfloor a_n(x) \rfloor + 2} \sim (1 + \varepsilon)^2 x.
\]
On the other hand we have
\[
x (1 - \varepsilon) \sim (1 - \varepsilon) \frac{1}{W_n} W_{\lfloor a_n(x) \rfloor + 2} \leq \tilde{I}(n).
\]
Both inequalities give
\[
\tilde{I}(n) \to x \quad \text{as} \quad n \to \infty,
\]
and consequently (3.5) holds.

Now we prove that
\[
J(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}
\]
In fact, we have for all \( \varepsilon \in (0, 1) \) and \( n \) sufficiently large
\[
J(n) \leq (1 + \varepsilon)^2 \frac{\lfloor a_n(x) \rfloor + 1}{W_n} \cdot \mu (A \cap \{ \varphi > \lfloor a_n(x) \rfloor \})
\sim x (1 + \varepsilon)^2 \frac{\lfloor a_n(x) \rfloor \cdot \mu (A \cap \{ \varphi > \lfloor a_n(x) \rfloor \})}{W_{\lfloor a_n(x) \rfloor}} \to 0.
\]
This gives (3.6). From this and (3.5) it follows that
\[
\nu \left( \frac{L(Y_n)}{L(n)} > x \right) \to 1 - x \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x \in (0, 1).
\]
Now let \( x \in [1, \infty) \). Then we have for \( x \geq 1 \) that
\[
\nu \left( \frac{L(Y_n)}{L(n)} > x \right) \leq \nu \left( \frac{L(Y_n)}{L(n)} > 1 \right)
\leq \nu \left( \frac{V_n}{n} > 1 \right) \to 0 \quad \text{as} \quad n \to \infty.
\]
Combining the above we get for all \( x \in (0, \infty) \)
\[
\nu \left( \frac{L(Y_n)}{L(n)} \leq x \right) \to 1 - \max \{1 - x, 0\} \quad \text{as} \quad n \to \infty.
\]
Finally, since \( V_n \to \infty \) in probability, it is clear that the slowly varying function \( L \) may be replaced by any function \( L_1 \) with \( L_1(n) \sim C \cdot L(n), \ C > 0, \) as \( n \to \infty \). Hence, by Lemma 3.2 and the compactness result the theorem follows. \( \square \)

Proof. (Second part of Theorem 2.4) Let \( f \) and \( L \) be given as in the first part of the proof of this theorem. Then for every fixed \( x \in (0, 1) \) we have
\[
\nu \left( \frac{L(Y_n - n)}{L(n)} \leq x \right) = \nu (Y_n \leq \lfloor a_n(x) \rfloor + n)
= \nu (Z_n + \lfloor a_n(x) \rfloor > n)
= \int_A \sum_{k=n+1}^n 1_{A \cap \{ \varphi > n + \lfloor a_n(x) \rfloor - k \}} \cdot \tilde{T}^k(f) \, d\mu,
\]
where \( \nu \) denotes the probability measure with density \( f \in P_\mu \) and \( a_n (x) = L^{-1} (xL(n)) \).

Note, by (EL) we have
\[
a_n (x) \to \infty \quad \text{and} \quad \frac{a_n (x)}{n} \to 0 \quad \text{for } n \to \infty.
\]

A similar arguments as in [KS05b], Lemma 3.3, show that for all \( \varepsilon \in (0, 1) \) there exists \( n_0 \) such that for all \( n \geq n_0 \) and \( k \in [n, n + a_n (x)] \) we have uniformly on \( A \)
\[
(1 - \varepsilon) \frac{1}{W_n} \leq \hat{T}^k (f) \leq (1 + \varepsilon)^2 \frac{1}{W_n}.
\]

From this it follows on the one hand that, for \( n \) sufficiently large,
\[
\nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right) \leq (1 + \varepsilon)^2 \frac{1}{W_n} W_{[a_n(x)]-1} \sim (1 + \varepsilon)^2 x.
\]

Similarly for \( n \) sufficiently large,
\[
x (1 - \varepsilon) \sim (1 - \varepsilon) \frac{1}{W_n} W_{[a_n(x)]-1} \leq \nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right).
\]

Combining these inequalities we get
\[
x (1 - \varepsilon) \leq \liminf \nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right) \leq \limsup \nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right) \leq (1 + \varepsilon)^2 x.
\]

Since \( \varepsilon \) was arbitrary, we conclude
\[
\nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right) \to x \quad \text{as } n \to \infty \quad \text{for all } x \in (0, 1). \tag{3.7}
\]

Now let \( x \in [1, \infty) \). Then we have
\[
\nu \left( \frac{L(y_n - n)}{L(n)} > x \right) \leq \nu \left( \frac{L(y_n - n)}{L(n)} > 1 \right) \leq \nu \left( \frac{y_n}{n} - 1 > 1 \right) \to 0 \quad \text{as } n \to \infty.
\]

From this and (3.7) it follows that
\[
\nu \left( \frac{L(y_n - n)}{L(n)} \leq x \right) \to 1 - \max (1 - x, 0) \quad \text{as } n \to \infty \quad \text{for all } x \in (0, \infty).
\]

Finally, since \( y_n - n \to \infty \) in probability, it is clear that the slowly varying function \( L \) may be replaced by any function \( L_1 \) with \( L_1 (n) \sim C \cdot L(n) \), \( C > 0 \), as \( n \to \infty \). From this, Lemma 3.2 and the compactness result the theorem follows.

Finally, we prove the large deviation asymptotic stated in Theorem 2.5.

**Proof.** (First part of Theorem 2.5) Let \( 0 \leq x < 1 \) and \( y \geq 0 \) be fixed with \( x + y \neq 0 \). We have
\[

\nu \left( \frac{n - Z_n}{n} \geq x, \frac{Y_n - n}{n} > y \right) = \nu \left( Z_n \leq \lfloor n (1 - x) \rfloor, Y_n > \lfloor n (1 + y) \rfloor \right)

= \nu \left( Z_{\lfloor n (1 + y) \rfloor} \leq \lfloor n (1 - x) \rfloor \right)

= \int_A \sum_{k=0}^{\lfloor n (1-x) \rfloor} 1_{A \cap \{ \varphi > \lfloor n (1+y) \rfloor - k \}} \cdot \hat{T}^k (f) \, d\mu.

\]
For $\delta \in (0, 1-x)$ and $\varepsilon \in (0, 1)$ fixed but arbitrary we divide the above sum into two parts as follows.

$$\nu\left(\frac{n - Z_n}{n} \geq x, \frac{Y_n - n}{n} > y\right) = \sum_{k=0}^{[n\delta]-1} \cdots + \sum_{k=[n\delta]}^{[n(1-x)]} \cdots =: I(n) + J(n).$$

By monotonicity of $(A \cap \{\varphi > n\})$ we first have

$$I(n) \leq \int_A 1_{A \cap \{\varphi > [n(1+y)]-[n\delta]+1\}} \cdot \sum_{k=0}^{[n\delta]-1} \hat{T}^k(f) \, d\mu.$$  

Using (1.2) and the fact that $A$ is uniform for $f$, we obtain for sufficiently large $n$

$$I(n) \leq (1 + \varepsilon)^2 \frac{[n\delta] - 1}{[n(1+y)]-[n\delta]+1} \cdot \frac{L([n(1+y)]-[n\delta]+1)}{W_{[n\delta]-1}} \delta L(n) \frac{y}{y+1-\delta} W_n \text{ as } n \to \infty.$$

Thus,

$$\limsup_{n \to \infty} \frac{W_n}{L(n)} \cdot I(n) \leq (1 + \varepsilon)^2 \frac{\delta}{y+1-\delta}.$$  

Letting $\delta \to 0$, we observe

$$I(n) = o\left(\frac{L(n)}{W_n}\right), \text{ as } n \to \infty. \quad (3.8)$$

For the second part of the sum we have to show that

$$J(n) \sim \frac{L(n)}{W_n} \cdot \log \left(\frac{1+y}{x+y}\right) \text{ as } n \to \infty. \quad (3.9)$$

A similarly argument as in [KSD05], Lemma 3.3, shows that for all $n$ sufficiently large and $k \in [[n\delta], [n(1-x)]]$ we have uniformly on $A$

$$(1 - \varepsilon) \frac{1}{W_n} \leq \hat{T}^k(f) \leq (1 + \varepsilon)^2 \frac{1}{W_n}. \quad (3.10)$$

Hence, using the right-hand side of (3.10) and (UA), we obtain for $n$ sufficiently large

$$J(n) \leq \frac{(1 + \varepsilon)^2}{W_n} \cdot \frac{[n(1+y)]-[n\delta]}{k=[n(1+y)]-[n(1-x)]} \mu(A \cap \{\varphi > k\}) \leq \frac{(1 + \varepsilon)^2}{W_n} \cdot \frac{L(n)}{W_n} \cdot \log \left(\frac{1+y-\delta}{x+y}\right) \text{ as } n \to \infty.$$

This implies

$$\limsup_{n \to \infty} \frac{W_n}{L(n)} \cdot J(n) \leq (1 + \varepsilon)^3 \log \left(\frac{1+y-\delta}{x+y}\right).$$

Similarly, using the left-hand side of (3.10), we get

$$\liminf_{n \to \infty} \frac{W_n}{L(n)} \cdot J(n) \geq (1 + \varepsilon)^2 \log \left(\frac{1+y-\delta}{x+y}\right).$$
Since \( \varepsilon \) and \( \delta \) were arbitrary, (3.9) holds. Combining (3.8) and (3.9) proves then the claim of the theorem.

Proof. (Second part of Theorem 2.5) First, let \( x \in (0, 1) \). We have

\[
\nu \left( \frac{V_n}{n} > x \right) = \sum_{k=0}^{n} \nu \left( Y_n > k + \lfloor nx \rfloor, Z_n = k \right)
\]

\[
= \sum_{k=0}^{n-\lfloor nx \rfloor - 1} \int_{A} 1_{A \cap \{ \varphi > n-k \}} \cdot \hat{T}^k (f) \ d\mu
\]

\[
+ \sum_{k=\lfloor nx \rfloor}^{n} \int_{A} 1_{A \cap \{ \varphi > \lfloor nx \rfloor \}} \cdot \hat{T}^k (f) \ d\mu
\]

\[
=: I (n) + J (n).
\]

Let \( \delta \in (0, 1-x) \) and \( \varepsilon \in (0, 1) \) be fixed but arbitrary. First, we prove that

\[
J (n) \sim L \left( \frac{n}{W_n} \right), \quad \text{as } n \to \infty.
\] (3.11)

In fact, we have for sufficiently large \( n \)

\[
J (n) \leq (1 + \varepsilon)^2 \frac{\lfloor nx \rfloor - 1}{W_n} \cdot \mu (A \cap \{ \varphi > \lfloor nx \rfloor \})
\]

\[
\sim (1 + \varepsilon)^2 \frac{L (n)}{W_n} \quad \text{as } n \to \infty.
\]

Similarly we get

\[
J (n) \geq (1 - \varepsilon) \frac{\lfloor nx \rfloor - 1}{W_n} \cdot \mu (A \cap \{ \varphi > \lfloor nx \rfloor \})
\]

\[
\sim (1 - \varepsilon) \frac{L \mu (n)}{W_n} \quad \text{as } n \to \infty.
\]

Combining both inequality (3.11) follows.

Now we have to prove that

\[
I (n) \sim - \log (x) \cdot \frac{L (n)}{W_n} \quad \text{as } n \to \infty.
\] (3.12)

divide \( I (n) \) into two parts as follows

\[
I (n) = \sum_{k=0}^{n} \cdots + \sum_{k=\lfloor n \delta \rfloor}^{n-\lfloor nx \rfloor - 1} \cdots =: I_1 (n) + I_2 (n).
\]

Using the monotonicity of \( 1_{A \cap \{ \varphi > n \}} \), the fact that \( A \) is uniformly for \( f \), and (1.2) we obtain, for \( n \) sufficiently large,

\[
I_1 (n) \leq \int_{A} 1_{A \cap \{ \varphi > n-\lfloor n \delta \rfloor + 1 \}} \cdot \sum_{k=0}^{\lfloor n \delta \rfloor - 1} \hat{T}^k (f) \ d\mu
\]

\[
\leq (1 + \varepsilon)^2 \frac{\lfloor n \delta \rfloor - 1}{n - \lfloor n \delta \rfloor + 1} \cdot \frac{L \mu \left( n - \lfloor n \delta \rfloor + 1 \right)}{W_{\lfloor n \delta \rfloor - 1}}
\]

\[
\sim (1 + \varepsilon)^2 \frac{\delta \cdot L \mu (n)}{1 - \delta \cdot W_n} \quad \text{as } n \to \infty.
\]
Consequently,
\[ I_1 (n) = o \left( \frac{L_\mu (n)}{W_n} \right), \quad \text{as } n \to \infty. \] (3.13)

Now using the fact that \( A \) is uniformly returning for \( f \) we have, for \( n \) sufficiently large,
\[ I_2 (n) \leq \frac{(1 + \epsilon)^2}{W_n} \cdot \sum_{k=\left\lfloor nx \right\rfloor + 1}^{n - \lfloor nx \rfloor} \mu (A \cap \{ \varphi > k \}) \]
\[ \sim (1 + \epsilon)^2 \frac{L_\mu (n)}{W_n} \cdot \log \left( \frac{1 - \delta}{x} \right) \quad \text{as } n \to \infty. \]

This implies
\[ \limsup_{n \to \infty} \frac{W_n}{L_\mu (n)} \cdot I_2 (n) \leq (1 + \epsilon)^3 \log \left( \frac{1 - \delta}{x} \right). \]
Similarly, we get
\[ \liminf_{n \to \infty} \frac{W_n}{L_\mu (n)} \cdot I_2 (n) \geq (1 + \epsilon)^2 \log \left( \frac{1 - \delta}{x} \right). \]

Since \( \epsilon \) and \( \delta \) were arbitrary, we have
\[ I_2 (n) \sim - \log (x) \cdot \frac{L(n)}{W_n}, \quad \text{as } n \to \infty. \] (3.14)

The asymptotics (3.13) and (3.14) prove (3.12). Combining (3.11) and (3.12) proves the second part of the theorem for \( x \in (0, 1) \).

Now we consider the case \( x \geq 1 \). Since
\[ \nu \left( \frac{V_n}{n} > x \right) = \sum_{k=0}^{\left\lfloor \frac{1}{x} \right\rfloor} \int_A 1_{A \cap \{ \varphi > \left\lfloor \frac{nx}{x} \right\rfloor \}} \cdot \hat{T}_k (f) \, d\mu \]
we have, for \( n \) sufficiently large,
\[ \nu \left( \frac{V_n}{n} > x \right) \leq \mu (A \cap \{ \varphi > \left\lfloor nx \right\}) \cdot \frac{n}{W_n} \cdot (1 + \epsilon) \]
\[ \sim \frac{L(n)}{xW_n} (1 + \epsilon). \]

Similarly, we obtain the reverse inequality proving the statement in the theorem for \( x \geq 1 \). \hfill \Box \Box

4. APPLICATION TO CONTINUED FRACTION

Any irrational number \( x \in I := [0, 1] \setminus \mathbb{Q} \) has a simple infinite continued fraction expansion
\[ x = \frac{1}{\kappa_1 (x) + \frac{1}{\kappa_2 (x) + \cdots}}, \]
where the unique continued fraction digits \( \kappa_n (x) \) are from the positive integers \( \mathbb{N} \). The Gauss transformation \( G : I \to I \) is given by
\[ G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \]
where \([x]\) denotes the greatest integer not exceeding \(x \in \mathbb{R}\). Write \(G^n\) for the \(n\)-th iterate of \(G\), \(n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\) with \(G^0 = \text{id}\). It is then well known that for all \(n \in \mathbb{N}\), we have

\[
\kappa_n(x) = \left\lfloor \frac{1}{G^{n-1}x} \right\rfloor.
\]

Clearly, the \(\kappa_n\), \(n \in \mathbb{N}\), define random variables on the measure space \((\mathbb{I}, \mathcal{B}, \mathbb{P})\), where \(\mathcal{B}\) denotes the Borel \(\sigma\)-algebra of \(\mathbb{I}\) and \(\mathbb{P}\) some probability measure on \(\mathcal{B}\). Then each \(\kappa_n\) has infinite expectation with respect to the Lebesgue measure on \([0, 1]\), which we will denote by \(\lambda\).

Given \(n \geq 1\), we define the Process

\[
\psi_n(x) := \max\left\{ p \in \mathbb{N}_0 : \sum_{i=1}^p \kappa_i(x) \leq n \right\}, \quad x \in \mathbb{I},
\]

and we consider the Process

\[
\sigma_n(x) := \kappa_{\psi_n(x)+1}, \quad x \in \mathbb{I}.
\]

In this paper we want to demonstrate how infinite ergodic theory can be employed to derive new insights into the stochastic structure of the Process \((\sigma_n)\). The underlying dynamical system will be given by the Farey map.

This process turns out to be related to the total waiting time processes considered in the first part of this paper. This allows us to derive the following main theorem. Its proof will be postponed to the end of Subsection 4.2.

**Theorem 4.1.** Let \(\sigma_n\) be the process given in (4.1). Then the following holds.

1. We have

\[
\frac{\log(\sigma_n)}{\log(n)} \xrightarrow{\mathcal{L}(\mu)} U,
\]

where the random variable \(U\) is uniformly distributed on the unit interval.

2. For any \(\nu \in \mathcal{D}\) and \(x \in (0, 1)\) we have

\[
\nu\left(\frac{\sigma_n}{n} > x\right) \sim \frac{H(x)}{\log(n)} \quad \text{as} \quad n \to \infty,
\]

where

\[
H(x) := \begin{cases} 
1 - \log(x) & \text{for } x \in (0, 1), \\
1/x & \text{for } x \geq 1.
\end{cases}
\]

**4.1 Farey vs. Gauss map.** We consider the Farey map \(T : [0, 1] \to [0, 1]\), defined by

\[
T(x) := \begin{cases} 
T_0(x), & x \in \left[0, \frac{1}{2}\right], \\
T_1(x), & x \in \left[\frac{1}{2}, 1\right],
\end{cases}
\]

where

\[
T_0(x) := \frac{x}{1-x} \quad \text{and} \quad T_1(x) := \frac{1}{x} - 1.
\]

It is known that \(([0, 1], T, \mathcal{B}, \mu)\) defines a conservative ergodic measure preserving dynamical system, where \(\mu\) denotes the \(\sigma\)-finite invariant measure with density \(h(x) := \frac{dx}{d\lambda}(x) = \frac{1}{x}\). Also any Borel set \(A \in \mathcal{B}\) with \(\lambda(A) > 0\) which is bounded away from the indifferent fixed point 0 is a uniform set. Furthermore, from [KS05a], Lemma 3.3, we know that the set \(K_1 := \left(\frac{1}{2}, 1\right]\) is uniformly returning for any \(f \in \mathcal{D}\), where

\[
\mathcal{D} := \{ f \in \mathcal{P}_\mu : f \in C^2((0, 1)) \text{ with } f' > 0 \text{ and } f'' \leq 0 \}.
\]
For the wandering rate we have
\[ W_n := W_n(K_1) = \int_0^1 \frac{1}{x} \, dx = \log(n + 2) \sim \log(n) \quad (n \to \infty). \]

The inverse branches of the Farey map are
\[
\begin{align*}
  u_0(x) &:= (T_0)^{-1}(x) = \frac{x}{1 + x}, \\
  u_1(x) &:= (T_1)^{-1}(x) = \frac{1}{1 + x}.
\end{align*}
\]

For \( x \neq 0 \) the map \( u_0(x) \) is conjugated to the right translation \( x \mapsto F(x) := x + 1 \), i.e.
\[ u_0 = J \circ F \circ J \quad \text{with} \quad J(x) = J^{-1}(x) = \frac{1}{x}. \]

This shows that for the \( n \)-th iterate we have
\[ u_0^n(x) = J \circ F^n \circ J(x) = \frac{x}{1 + nx}. \quad (4.4) \]

Moreover, we have \( u_1(x) = J \circ F(x) \).

Let \( \mathcal{F} = \{K_n\}_{n \geq 1} \) be the countable collection of pairwise disjoint subintervals of \([0, 1]\) given by \( K_n := \left(\frac{1}{n+1}, \frac{1}{n}\right] \). Setting \( A_0 = [0, 1) \), it is easy to check that \( T(K_n) = K_{n+1} \) for all \( n \geq 1 \). The first entry time \( e : \mathbb{I} \to \mathbb{N} \) into the interval \( K_1 \) is defined as
\[ e(x) := \min \{k \geq 0 : T^k(x) \in K_1\}. \]

Then the first entry time is connected to the first digit in the continued fraction expansion by
\[ \kappa_1(x) = 1 + e(x) \quad \text{and} \quad \varphi(x) = \kappa_1 \circ T(x), \quad x \in \mathbb{I}. \]

We now consider the induced map \( S : \mathbb{I} \to \mathbb{I} \) defined by
\[ S(x) := T^{e(x)+1}(x). \]

Since for all \( n \geq 1 \)
\[ \{x \in \mathbb{I} : e(x) = n - 1\} = K_n \cap \mathbb{I}, \]
we have by (4.4) for any \( x \in K_n \cap \mathbb{I} \)
\[ S(x) = T^n(x) = T_1 \circ T_0^{n-1}(x) = \frac{1}{x} - n = \frac{1}{x} - \kappa_1(x). \]

This implies that the induced transformation \( S \) coincides with Gauss map \( G \) on \( \mathbb{I} \).

4.2. Renewal theory for continued fractions. In the next lemma we connect the number theoretical process \( \sigma_n \) defined in (4.1) with the total waiting time process \( V_n \) defined with respect to the Farey map. Let \( (\tau_n)_{n \in \mathbb{N}} \) be the sequence of return times, i.e. integer valued positive random variables defined recursively by
\[
\begin{align*}
  \tau_1(x) &:= \varphi(x) = \inf\{p \geq 1 : T^p(x) \in K_1\}, \quad x \in X, \\
  \tau_n(x) &:= \inf\{p \geq 1 : T^{p+\sum_{k=1}^{n-1} \tau_k(x)}(x) \in K_1\}, \quad x \in X.
\end{align*}
\]

The renewal process is then given by
\[ N_n(x) := \max\{k \leq n : S_k(x) \leq n\}, \quad x \in A_n = \bigcup_{k=0}^{n} T^{-k}K_1, \]
else,
where
\[ S_0 := 0, \quad S_n := \sum_{k=1}^{n} \tau_k, \quad n \in \mathbb{N}. \]

**Lemma 4.2.** for all \( x \in \mathbb{I} \) and \( n \geq 1 \) we have \( \kappa_n := \sum_{k=0}^{n} T^{-k} K_1 \).
Then for the process \( \sigma_n \) defined in (4.1) we have for all \( x \in \mathbb{I} \) and \( n \geq 1 \)
\[ \sigma_n(x) = \begin{cases} V_{n-1}(x), & x \in A_{n-1}, \\ 1 + Y_{n-1}(x), & \text{else.} \end{cases} \]

**Proof.** As a consequence of the observations in Subsection 4.1 we will argue as follows. For \( x \in \mathbb{I} \cap A_{n-1} \) we have that \( \kappa_1(x) > n \) implies \( \psi_n(x) = 0 \). For \( x \in \mathbb{I} \cap A_{n-1} \) we distinguish two cases. Either the process starts in \( x \in K_1 \), then we have \( \kappa_1(x) = 1 \) and inductively for \( n \geq 2 \)
\[ \kappa_n(x) = \tau_{n-1}(x), \]
or the process starts in \( x \in K_1^c \), then we have \( \kappa_1(x) = 1 + \tau_1(x) \) and inductively for \( n \geq 2 \)
\[ \kappa_n(x) = \tau_n(x). \]
This implies that
\[ \psi_n(x) = \begin{cases} N_{n-1}(x) + 1, & x \in K_1, \\ N_{n-1}(x), & \text{else.} \end{cases} \]
Hence, we have for \( x \in \mathbb{I} \cap A_{n-1} \)
\[ \kappa_{\psi_n(x)+1} = \tau_{N_{n-1}(x)+1} \]
and for \( x \in \mathbb{I} \cap A_{n-1} \)
\[ \kappa_{\psi_n(x)+1} = \kappa_1(x) = 1 + \tau_1(x). \]
From this the assertion follows.

After these preparations we are now in the position to give the proof of Theorem 4.1.

**Proof.** (Theorem 4.1) The asymptotic (4.2) is an immediate consequence of Lemma 4.2, the second part of Theorem 2.4 and the fact that \( W_n \sim \log(n) \).

Finally, (4.3) follows from the second part of Theorem 2.5 by observing that \( K_1 := \left( \frac{1}{2}, 1 \right] \) is uniformly returning for any \( f \in D \) and that
\[ \mu(A_1 \cap \{ \varphi > n \}) = \int_{n+1}^{n+2} \frac{1}{x} \, dx \sim \frac{1}{n} \quad \text{as} \quad n \to \infty. \]

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