Second Order Twist Contributions to the Balitsky-Kovchegov Equation at small-x: Deterministic and Stochastic Pictures

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I study the second order twist corrections to a toy model of a dipole-dipole interaction in the context of a both deterministic and stochastic effects. This work is done in the high \( N_c \) limit in the Bjorken picture. I show that the correction to the second twist terms of the stochastic picture suggest additional importance of the second twist correction in the stochastic model as compared to the deterministic model.

I. INTRODUCTION

There has been a large body of work in the field of high energy QCD related to the scattering of virtual photons on bound states \([1, 2, 3, 4, 5]\). Important results in this field include the parton model of Deep Inelastic scattering, first given by Feynman \([6]\) and later modified by Bjorken and Paschos \([7]\), and the DGLAP evolution equations \([8, 9, 10]\), both cornerstones of perturbative QCD. This work was originally performed in Lorentz-covariant Feynman diagrams techniques \([11, 12]\), however, one may also proceed with these calculations using light cone perturbation theory as set forth by Lepage and Brodsky \([13]\), in their paper following the work of Bjorken, Kogut and Soper \([14]\). This method allows the analysis to proceed via the light cone wave function, which allows for the description of the Fock state of a hadron as a function of gluon and quark numbers. Further, as outlined in \([15]\), the use of the light cone gauge, which simplifies the gluon field to two independent components—the transverse components—and a single dependent component. Such work often is done using large \( N_c \) perturbation theory and the small-x regime \([15]\), where the amplitudes may be described using the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \([16, 17]\). In these pictures, one must consider the equations via careful choice of approximations and frame. To this end, one may consider the following simplifications: take a target that is ultra-relativistic, thus defining the "Infinite Momentum Frame" or Bjorken Frame \([4]\) and setting the momentum of the target much larger than the mass and the center of mass energy is high (thus the Bjorken-x, which is in exact analogy to the Feynman-x, is small); one ignores the evolution of the quark distributions of the hadron in this frame as the distribution functions of such run significantly slower than the gluon distribution; one also must assume that the coupling constant is fixed (or at least \( \alpha_s \ll 1 \)). These assumptions give rise to the double logarithmic approximation of the DGLAP evolution equations \([3]\).

One may consider DIS in the rest-frame of the nucleon. In this frame, the virtual photon fluctuates into a quark-antiquark pair, that, in the large-\( N_c \) limit may develop into a cascade of gluons which may be described by the Mueller dipole model \([18]\). When resummation is performed on this cascade, one reaches the Balitsky-Kovchegov (BK) evolution equation which is unitary and does not diffuse into the IR, thus distinguishing it from earlier equations and generates a saturation scale that grows with energy, justifying the use of perturbative QCD \([19, 20]\). As described by Balitsky, the relevant logarithms in this equation may arise from the expansion of non-local operators which gives rise to a twist interpretation of the equation, where higher twists represent the original dipole branching and further scattering off of the nucleon \([20]\). In the derivation of this equation, the use of light cone perturbation theory allows for the factorization of the relevant diagrams \([19]\) which often allows for the diagrams to be represented in the form of Muller Vertices \([11]\).

However, this equation assumes a linear scaling in the bulk of the phase space, which allows for unrestrained growth of density of the dipoles in this equation. To correct for this, one argues for a modeling of the non-linearity using recombination in analogy to branching-diffusion models. This process is captured using stochastic corrections of tip fluctuations, which may not be modeled analytically, and front fluctuations, which will be considered in this paper \([2]\). Such corrections are motivated by their applicability to the Fischer-Kolmogorov-Petrivsky-Piscounov equation and its mapping to the BK equation. Thus, for the purposes of this paper, I shall consider the equations without such corrections to be the "Deterministic equations" and those with such to be the "Stochastic equations".

This paper aims to connect the work done on these fluctuations on the second order twist corrections with the operator language of Balitsky \([20]\).

II. CALCULATING THE SECOND ORDER TWIST

To investigate the dynamics of the second order twist contribution in the stochastic picture, I must first evaluate the deterministic picture. In order to do this, I utilize a simplified picture where the probe maybe one or more dipoles prepared in some manner that is irrelevant to this discussion and the scattering target is an individual dipole. This toy model will allow us to consider the
relevant evolution equations without needing to concern ourselves with the physical compilations of true Deep Inelastic Scattering or the process by which an additional dipole is produced from the original. Then, define the contribution from each twist to the forward scattering amplitude, defined as

$$T = 1 - S$$

$$T = T_d + T_{dd} + T_{ddd} + ...$$  \hspace{1cm} (1)

With $T_d$ being the contribution from a single dipole (leading twist), $T_{dd}$ from two dipoles (higher twist) and so forth. The first twist scattering amplitude of a dipole with transverse size $x_\perp$ and Bjorken variable $x$ is given as [21].

Which gives

$$T_d(x, \frac{1}{x_\perp}) = \frac{1}{2\pi P^+} \int d\xi e^{-iP^+\xi^-} \times \langle P|F^{a}_+(0, \xi^-, x_\perp)\gamma^+ W_{ab} F^{b}_+(0)|P\rangle$$ \hspace{1cm} (2)

FIG. 1. LoIst Twist Gluon Distribution.

$$T_d = \frac{\alpha_s \pi^2}{2NC} x^2 G(x, \frac{1}{x_\perp})$$ \hspace{1cm} (3)

I require the gluons to form a color singlet, as the dipole must also be a color singlet. By setting the scattering off of a dipole, rather than utilizing a Muller vertex, I escape the divergences that necessitate renormalization at scale $Q$, but this scale is identified with $\frac{1}{\perp} = Q^2$ as in [3]. W is the "gauge link" or Wilson operator, which by choice of the gauge $A_\perp = 0$, $\gamma^+ W_{ab}$ may be set to 1. This process is illustrated in figure [1] with the relevant integral to be calculated. In this gauge,

$$F^{a}_+ = \partial_\perp A^{a}_\perp$$

For the next-order twist, I consider the case of a state of two dipoles scattering off of the target proton. For the purposes of this discussion, I shall consider the method of preparation of this system to be irrelevant, but I require that the impact parameter of both dipoles be equal to zero, and that the dipoles have transverse size $x_{1\perp}$ and $x_{2\perp}$ respectively. This results in a system where a set of two dipoles each scatter off of the target once as illustrated in figure [2]. This yields the factorized equation, This yields in the mean field picture, following similar analysis to that for the quark distribution in [11], and noting a $C_F$ arises from the splitting of the state into two dipoles as can be seen in equation (74) of [2],

$$T_{dd}(x, x_1, \frac{1}{x_{1\perp}}, \frac{1}{x_{2\perp}}) = \frac{1}{P^+} \int d\xi d\xi' e^{iP^+\xi^- - iP^+\xi_1^-} \times \langle P|F^{a}_+(0, \xi, x_\perp) F^{b}_+(0, \xi', x_\perp)|P\rangle$$ \hspace{1cm} (4)

FIG. 2. Second order Twist

One notes that by starting with the dipoles independent of the method of preparation of the double dipole system, one suppresses the anomalous dimension arising in the Operator product expansion Holver this is irrelevant for this discussion as I am simply calculating the interaction of a double dipole interaction to analyze stochastic effects, which are independent of the quadratic terms.

III. DETERMINISTIC EQUATIONS IN THE SCALING REGION

In [3] the equation for $T$ in the scaling region ($x_\perp << 1/QS_0$) becomes

$$T_d = (x_\perp QS)^{1+2iv_\perp} C(\alpha_s)e^{\gamma_0 y(0, v_{sp})}$$ \hspace{1cm} (7)

Where $y$ is the rapidity and is equal to $\ln(x^2 E^2)$ where $E$ is the center of mass energy for the dipoles, $\alpha_s = \frac{\alpha_s N_c}{\pi}$ and $\gamma(0, v_{sp})$ is an eigenvalue of the BK Kernel. Using [2] to further refine this equation as is done in equation (118), and noting that the logarithmic term comes from the integral arising in the BK equation as in equation (120). I then write, with $\gamma_0 = \frac{1}{2} + iv_{sp}$, and $C(\alpha_s)$ being some constant of the form $\alpha_s \times K$

$$T_d(x_\perp, y) = C(\alpha_s) \ln(x^2 QS(y))^{(x_\perp QS(y))^2} \times e^{-\frac{\ln(x^2 QS(y)^2)}{\alpha_s x^2 K(\gamma_0 y)}}$$ \hspace{1cm} (8)
First Order Twist Deterministic Equation in the Scaling Region ($\gamma = 1$)

First Order Twist Deterministic Equation in the Scaling Region ($\gamma = 3$)

Stochastic version of $T_d$ which will be noted as $T_d^S$ for clarity henceforth. $P(\delta) = e^{-\gamma \delta}$ is the probability of having a front delayed by $\delta$ [2].

$$T_d^S \propto \int_0^{\ln(\frac{1}{x_1 Q^2 S})} d\delta P(\delta) \left[ \ln(x_1 Q^2 S(y)^2) - \delta \right]$$

IV. STOCHASTIC EFFECTS IN THE SCALING REGION

From Munier [2], the equation for scattering amplitude accounting for front fluctuations ($T_d^S$ stands for the
FIG. 5. Ratio of Stochastic to Deterministic Equations in the First Order Twist

FIG. 6. Ratio of Stochastic to Deterministic Equations in the Second Order Twist

FIG. 7. Ratio of Stochastic Equations in the Scaling Region

This discrepancy accounts for the failure of factorization to completely capture the interaction between the two dipoles. Figures 5, 6 represent ratios of stochastic equations, with the max of each ratio in the relevant ranges—keeping with those set in figure 4—set to 1. Figure 5 is the ratios of the first order twist equations, and figure 6 is the same for the second order twist. One can see that the ratios of terms grow substantially more quickly in the second order twist. As expected, the stochastic terms contribute larger corrections as $xQ$ grows. Figure 7 represents the ratios of stochastic corrections (Second order divided by first order).

V. SUMMARY AND CONCLUSIONS

By implementing the stochastic corrections to the second-order twist of a dipole-dipole interaction, I see that the paradigm of factorization fails to properly describe the process. This work has translated the work previously done on the fluctuations of higher twists to the operator representation of the gluon distribution. Further, this work shows, through this connection, that the corrections to the scattering amplitude from the second order twist stochastic component is inversely proportional to the gluon distribution of the target dipole.

In this work, I have presented a proof-of-concept of an approach that may be applied to later studies to solve the problem of a Stochastic version of the BK evolution equations. Additionally this work may be implemented in the study of proper Deep-Inelastic scattering rather than in the toy model presented in this paper.

Further work in this vein would include a mechanism by which to measure this effect as well as including effects of nuclear fluctuations on small nuclei. Such work would require similar analysis of the JIMWLK equation.

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[1] A. H. Mueller and S. Munier, “Phenomenological picture of fluctuations in branching random walks,” Phys. Rev. E, vol. 90, p. 042143, Oct 2014.
[2] S. Munier, “Lecture notes on” quantum chromodynamics and statistical physics”, arXiv preprint arXiv:1410.6478, 2014.
[3] Y. V. Kovchegov and E. Levin, Quantum Chromodynamics at High Energy. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology, Cambridge University Press, 2012.
[4] J. D. Bjorken, “Asymptotic sum rules at infinite momentum,” Physical Review, vol. 179, no. 5, p. 1547, 1969.
[5] D. E. Kharzeev and J. Raufeisen, “High energy nuclear interactions and qcd: an introduction,” in AIP Conference Proceedings, vol. 631, pp. 27–69, American Institute of Physics, 2002.
[6] R. P. Feynman, “Very high-energy collisions of hadrons,” Phys. Rev. Lett., vol. 23, pp. 1415–1417, Dec 1969.
[7] J. Bjorken and E. Paschos, “Inelastic electron-proton and \( y \)-proton scattering and the structure of the nucleon,” PHYSICAL REVIEW, vol. 185, no. 5, 1960.
[8] G. Altarelli and G. Parisi, “Asymptotic freedom in parton language,” Nuclear Physics B, vol. 126, no. 2, pp. 298–318, 1977.
[9] V. N. Gribov and L. N. Lipatov, “Deep inelastic ep-scattering in a perturbation theory,” tech. rep., Inst. of Nuclear Physics, Leningrad, 1972.
[10] Y. L. Dokshitzer, “Calculation of the structure functions for deep inelastic scattering and e+ e- annihilation by perturbation theory in quantum chromodynamics,” Zh. Eksp. Teor. Fiz, vol. 73, p. 1216, 1977.
[11] G. Sterman, An Introduction to Quantum Field Theory. Aug. 1993.
[12] M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory. Reading, USA: Addison-Wesley, 1995.
[13] G. P. Lepage and S. J. Brodsky, “Exclusive processes in perturbative quantum chromodynamics,” Physical Review D, vol. 22, no. 9, p. 2157, 1980.
[14] J. D. Bjorken, J. B. Kogut, and D. E. Soper, “Quantum electrodynamics at infinite momentum: Scattering from an external field,” Phys. Rev. D, vol. 3, pp. 1382–1399, Mar 1971.
[15] C. Marquet and C. Royon, “Small-x qcd effects in forward-jet and mueller–navelet jet production,” Nuclear Physics B, vol. 739, p. 131–155, Apr 2006. mainly background work about jet production.
[16] E. A. Kuraev, L. Lipatov, and V. S. Fadin, “Pomeranchuk singularity in non-abelian gauge theories,” Zhurnal Eksperimental’noj i Teoreticheskoi Fiziki, vol. 72, no. 2, pp. 377–389, 1977.
[17] L. Lipatov, “Reggeization of the vector meson and the vacuum singularity in nonabelian gauge theories,” 1976.
[18] A. H. Mueller, “Small-x physics, high parton densities and parton saturation in qcd,” in Particle Production Spanning MeV and TeV Energies, pp. 71–99, Springer, 2000.
[19] Y. V. Kovchegov, “Unitarization of the bfkl pomeron on a nucleus,” Physical Review D, vol. 61, no. 7, p. 074018, 2000.
[20] I. Balitsky, “Operator expansion for high-energy scattering,” Nuclear Physics B, vol. 463, no. 1, pp. 99–157, 1996.
[21] F. Dominguez, Unintegrated gluon distributions at small-x. PhD thesis, Columbia University, 2011.