ON THE GENERALISED SPRINGER CORRESPONDENCE FOR GROUPS OF TYPE $E_8$

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Abstract. We complete the determination of the generalised Springer correspondence for connected reductive algebraic groups, by proving a conjecture of Lusztig on the last open cases which occur for groups of type $E_8$.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraic closure $k$ of the finite field $\mathbb{F}_p$ with $p$ elements, where $p$ is a prime. Let $W$ be the Weyl group of $G$ and $\mathcal{N}_G$ be the set of all pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O} \subseteq G$ is a unipotent conjugacy class and $\mathcal{E}$ is an irreducible local system on $\mathcal{O}$ (taken up to isomorphism) which is equivariant for the conjugation action of $G$. The Springer correspondence (originally defined in [38] for $p$ not too small; for arbitrary $p$ see [15]) defines an injective map $\iota_G : \text{Irr}(W) \hookrightarrow \mathcal{N}_G$ which plays a crucial role, for example, in the determination of the values of the Deligne–Lusztig Green functions of [5]; for recent surveys see [7, Chap. 13] and [10, §2.8]. However, the map $\iota_G$ is not surjective in general. In order to understand the missing pairs in $\mathcal{N}_G$, Lusztig [17] developed a generalisation of Springer’s correspondence. This in turn constitutes a substantial part of the general problem of computing the complete character tables of finite groups of Lie type.

With very few exceptions, the problem of determining explicitly the generalised Springer correspondence has been solved in [17], [32], [37] (see also the further references there). The exceptions occur for $G$ of type $E_6$ with $p \neq 3$ and $E_8$ with $p = 3$. Recently, Lusztig [31] settled the case where $G$ is of type $E_6$ and stated a conjecture concerning the last open cases in type $E_8$. It is the purpose of this paper to prove that conjecture, thus completing the determination of the generalised Springer correspondence in all cases.

The paper is organised as follows. In Section 2, we recall the definition of the generalised Springer correspondence, due to [17]. In Section 3, we explain two parametrisations of unipotent characters and unipotent character sheaves, one in terms of Lusztig’s Fourier transform matrices associated to families in the Weyl group, and the other one in terms of Harish-Chandra series, for simple groups of adjoint type with a split rational structure. In the last two sections, we focus on the specific case where $G$ is simple of type $E_8$ in characteristic $p = 3$: Section 4 contains the description of the generalised Springer

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correspondence for this group, so that we can formulate Lusztig’s conjecture on the last open cases as indicated above. Finally, the proof of this conjecture is given in Section 5. It is based on considering the Hecke algebra associated to the finite group $E_8(q)$ (where $q = 3^f$ for some $f \geq 1$) and its natural $(B,N)$-pair, exploiting a well-known formula relating characters of this Hecke algebra with the unipotent principal series characters of $E_8(q)$.

**Notation.** As soon as a prime $p$ is fixed in a given setting, we denote by $k = \overline{\mathbb{F}}_p$ an algebraic closure of the finite field $\mathbb{F}_p$ with $p$ elements; furthermore, we tacitly assume to have fixed any other prime $\ell \neq p$, as well as an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of $\ell$-adic numbers. It will be convenient to assume the existence of an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ and to fix such an isomorphism once and for all\(^1\), so that we can speak of “complex” conjugation or absolute values for elements of $\overline{\mathbb{Q}}_\ell$ using this isomorphism. In this way, we will also identify the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$ as subsets of $\overline{\mathbb{Q}}_\ell$ and just write $\mathbb{Q} \subseteq \mathbb{R} \subseteq \overline{\mathbb{Q}}_\ell$. In several places below, we will need to fix a square root of $p$ (or of powers of $p$), so we do this right away:

\[
\text{From now on we fix, once and for all, a square root } \sqrt{p} \text{ of } p \text{ in } \overline{\mathbb{Q}}_\ell.
\]

Furthermore, whenever $q = p^e$ ($e \geq 1$), we set $\sqrt{q} := (\sqrt{p})^e$.

For any finite group $\Gamma$, we denote by $\text{CF}(\Gamma)$ the set of class functions $\Gamma \to \overline{\mathbb{Q}}_\ell$ and by $\text{Irr}(\Gamma) \subseteq \text{CF}(\Gamma)$ the subset consisting of irreducible characters of $\Gamma$ over $\overline{\mathbb{Q}}_\ell$. Thus, $\text{Irr}(\Gamma)$ is an orthonormal basis of $\text{CF}(\Gamma)$ with respect to the scalar product

\[
\langle f, f' \rangle_\Gamma := |\Gamma|^{-1} \sum_{g \in \Gamma} f(g)\overline{f'(g)} \quad \text{(for } f, f' \in \text{CF}(\Gamma)).
\]

Now let $p$ be a prime and $G$ be a connected reductive group over $k = \overline{\mathbb{F}}_p$, defined over the finite subfield $\mathbb{F}_q \subseteq k$ where $q = p^f$ for some $f \geq 1$, with corresponding Frobenius map $F: G \to G$. With respect to this $F$, we fix a maximally split torus $T_0 \subseteq G$ and an $F$-stable Borel subgroup $B_0 \subseteq G$ such that $T_0 \subseteq B_0$. Let $\Phi$ be the set of roots of $G$ with respect to $T_0$, and let $\Pi \subseteq \Phi$ be the subset of simple roots determined by $T_0 \subseteq B_0$.

We denote by $W = N_G(T_0)/T_0$ the Weyl group of $G$ (relative to $T_0$). Given any closed subgroup $H \subseteq G$ (including the case where $H = G$), we denote by $H^\circ \subseteq H$ its identity component, by $Z(H) \subseteq H$ the centre of $H$ and by $H_{\text{uni}} \subseteq H$ the (closed) subvariety consisting of all unipotent elements of $H$. If $F(H) = H$, we set

\[
H^F := \{ h \in H \mid F(h) = h \} \subseteq H,
\]

a finite subgroup of $H$. In particular, $G^F$ is the finite group of Lie type associated to $(G, F)$.

\(^1\)Strictly speaking, the existence of such an isomorphism requires the axiom of choice. However, what we really need is an isomorphism between algebraic closures of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$, and such an isomorphism is known to exist without reference to the axiom of choice, cf. [3, Remark 1.2.11].
2. THE GENERALISED SPRINGER CORRESPONDENCE

Let $p$ be a prime, and let $G$ be a connected reductive group over $k = \mathbb{F}_p$, with Frobenius map $F: G \to G$ providing $G$ with an $\mathbb{F}_q$-rational structure where $q = p^f$ for some $f \geq 1$ (and with the further notation as in the introduction). In this section, we give the definition of the generalised Springer correspondence for $G$, due to Lusztig [17].

2.1. We start by briefly introducing some of the most important notions of Lusztig’s theory of character sheaves [24] and the underlying theory of perverse sheaves [1]. For the details we refer to [17], [18]–[22]. Recall that $\ell \neq p$ is a fixed prime. Let $D_G$ be the bounded derived category of constructible $\mathbb{Q}_\ell$-sheaves on $G$ in the sense of Beilinson–Bernstein–Deligne [1]. To each $K \in D_G$ and each $i \in \mathbb{Z}$ is associated the $i$th cohomology sheaf $H^i(K)$, whose stalks $H^i_g(K)$ (for $g \in G$) are finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

The support of such a $K$ is defined as

$$\text{supp } K := \{ g \in G | H^i_g(K) \neq 0 \text{ for some } i \in \mathbb{Z} \} \subseteq G$$

(where the bar stands for the Zariski closure, here in $G$). We define a sign

$$\hat{\epsilon}_K := (-1)^{\dim G - \dim \text{supp } K}.$$  

Let $\mathcal{M}_G$ be the full subcategory of $D_G$ consisting of the perverse sheaves on $G$; the category $\mathcal{M}_G$ is abelian. Assume that $X$ is a locally closed subvariety of $G$ and that $\mathcal{L}$ be an irreducible $\mathbb{Q}_\ell$-local system on $X$. (We will just speak of a “local system” when we mean a $\mathbb{Q}_\ell$-local system from now on.) There is a unique extension of $\mathcal{L}$ to $\overline{X}$, namely, the intersection cohomology complex $IC(\overline{X}, \mathcal{L})[\dim X]$ (where $[]$ denotes shift), due to Deligne–Goresky–MacPherson (see [12], [1]). The following notation will be convenient: For $X$ and $\mathcal{L}$ as above, and for any closed subvariety $Y \subseteq G$ such that $X \subseteq Y$, we denote by

$$IC(\overline{X}, \mathcal{L})[\dim X]_{\#}^Y \in D_Y$$

the extension of $IC(\overline{X}, \mathcal{L})[\dim X]$ to $Y$, by $0$ on $Y \setminus \overline{X}$. Now let us bring the Frobenius endomorphism $F: G \to G$ into the picture. Let $F^*: D_G \to D_G$ be its inverse image functor. Let $K \in D_G$ be such that $F^*K \cong K$ in $D_G$, and let $\varphi: F^*K \cong K$ be an isomorphism. For each $i \in \mathbb{Z}$ and $g \in G^F$, $\varphi$ induces linear maps

$$\varphi_{i,g} : H^i_g(K) \to H^i_g(K).$$

Since only finitely many of the $H^i_g(K)$ ($g \in G^F$, $i \in \mathbb{Z}$) are non-zero, one can define a characteristic function [19, 8.4]

$$\chi_{K,\varphi} : G^F \to \mathbb{Q}_\ell, \quad g \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \text{Trace}(\varphi_{i,g}, H^i_g(K)).$$

In [18 §2], Lusztig defines the character sheaves on $G$ as certain simple objects of $\mathcal{M}_G$ which are equivariant for the conjugation action of $G$ on itself. We denote by $\hat{G}$ a set of representatives for the isomorphism classes of character sheaves on $G$. An important subset of $\hat{G}$ is the one consisting of cuspidal character sheaves, as defined in [18, 3.10]. We denote by $G^c \subseteq \hat{G}$ a set of representatives for the isomorphism classes of cuspidal
character sheaves on $G$. The inverse image functor $F^*$ may also be regarded as a functor $F^*: \mathcal{M}G \to \mathcal{M}G$, and we have $F^*(\hat{G}) = \hat{G}$. Thus, we can consider the subset

$$\hat{G}^F := \{ A \in \hat{G} \mid F^* A \cong A \} \subseteq \hat{G}$$

consisting of the $F$-stable character sheaves in $\hat{G}$. Of particular relevance for our purposes will be the set $\hat{G}^{un} \subseteq \hat{G}$ of (representatives for the isomorphism classes of) unipotent character sheaves. These are, by definition, the simple constituents of a perverse cohomology sheaf $\mathcal{P}^i(K^w_\phi) \in \mathcal{M}G$ for some $w \in W$ and some $i \in \mathbb{Z}$, where $K^w_\phi \in \mathcal{D}G$ is as defined in [13, 2.4], with respect to the trivial local system $L_0 = \bigcup \ell$ on the torus $T_0$.

2.2. Let $\mathcal{N}_G$ be the set of all pairs $(O, E)$ where $O \subseteq G$ is a unipotent conjugacy class and $E$ is an irreducible local system on $O$ (up to isomorphism) which is equivariant for the conjugation action of $G$ on $O$. We define another set $\mathcal{M}_G$, as follows: Consider a triple $(L, O_0, E_0)$ consisting of a Levi complement $L$ of some parabolic subgroup of $G$, a unipotent class $O_0$ of $L$ and an irreducible local system $E_0$ on $O_0$ (up to isomorphism) which is equivariant for the conjugation action of $L$ on $O_0$; furthermore, assume that $(\mathbb{Z}(L)^\circ, O_0, 1 \boxtimes E_0)$ is a cuspidal pair for $L$ in the sense of [17, 2.4], where $1 \boxtimes E_0$ denotes the inverse image of $E_0$ under the canonical map $\mathbb{Z}(L)^\circ \cdot L_0 \to O_0$. The group $G$ acts naturally on the set of all such triples by means of $G$-conjugacy:

$$g(L, O_0, E_0) := (gLg^{-1}, gO_0g^{-1}, \text{Int}(g^{-1})^* E_0) \quad \text{for } g \in G,$$

where $\text{Int}(g^{-1}) : G \to G$ is the inner automorphism given by conjugation with $g^{-1}$. Let $\mathcal{M}_G$ be the set of equivalence classes of triples $(L, O_0, E_0)$ as above under this action. By a slight abuse of notation, we will just write $(L, O_0, E_0) \in \mathcal{M}_G$ rather than $[(L, O_0, E_0)] \in \mathcal{M}_G$ or the like. We typically write $i, i', ...$ for elements of $\mathcal{N}_G$ and $j, j', ...$ for elements of $\mathcal{M}_G$. Following [17, §6], any $j \in \mathcal{M}_G$ gives rise to a certain perverse sheaf $K_j \in \mathcal{M}G$ whose endomorphism algebra $A_j := \text{End}_{\mathcal{M}G}(K_j)$ is isomorphic to the group algebra of the relative Weyl group $\mathcal{W}_j := W_G(L) = N_G(L)/L$, see [17, Theorem 9.2]. Thus, the isomorphism classes of the simple direct summands of $K_j$ are naturally parametrised by $\text{Irr}(\mathcal{W}_j)$, so we have

$$(2.2.1) \quad K_j \cong \bigoplus_{\phi \in \text{Irr}(\mathcal{W}_j)} (A_\phi \otimes V_\phi)$$

where $A_\phi$ is the simple direct summand of $K_j$ corresponding to $\phi \in \text{Irr}(\mathcal{W}_j)$, and where $V_\phi = \text{Hom}_{\mathcal{M}G}(A_\phi, K_j)$. Then, for any $\phi \in \text{Irr}(\mathcal{W}_j)$, there exists a unique $(O, E) \in \mathcal{N}_G$ for which

$$(2.2.2) \quad A_\phi|_{\text{Guni}} \cong \text{IC}(O, E)[\dim \mathbb{Z}(L)^\circ + \dim O]^{\# \text{Guni}},$$

and the isomorphism class of $A_\phi$ is uniquely determined by this property among the simple perverse sheaves which are constituents of $K_j$, see [22, 24.1]. So for each $j \in \mathcal{M}_G$, the above procedure gives rise to an injective map

$$\text{Irr}(\mathcal{W}_j) \hookrightarrow \mathcal{N}_G.$$ 

Conversely, given any $(O, E) \in \mathcal{N}_G$, there exists a unique $j \in \mathcal{M}_G$ such that $(O, E)$ is in the image of the map $\text{Irr}(\mathcal{W}_j) \hookrightarrow \mathcal{N}_G$ just defined. So there is an associated surjective
map
\[ \tau : N_G \to M_G, \]
whose fibres are called the blocks of \( N_G \). For any \( j = (L, \mathcal{O}_0, \mathcal{E}_0) \in M_G \), the elements in the block \( \tau^{-1}(j) \subseteq N_G \) are thus parametrised by the irreducible characters of \( \mathcal{W}_j = W_G(L) \).

The collection of the bijections
\[ (2.2.3) \quad \text{Irr}(\mathcal{W}_j) \cong \tau^{-1}(j) \quad \text{(for } j \in M_G \text{)} \]
is called the generalised Springer correspondence. If \( i = (\mathcal{O}, \mathcal{E}) \in N_G \) and if \( \phi \in \text{Irr}(\mathcal{W}_\tau(i)) \) corresponds to \( i \) under \( (2.2.3) \), we will write \( A_i := A_\phi \).

If we only consider the element \( (T_0, \{1\}, \mathcal{O}_G) \in M_G \), the map \( (2.2.3) \) defines an injection
\[ (2.2.4) \quad \text{Irr}(W) \hookrightarrow N_G, \]
which is called the (ordinary) Springer correspondence. The problem of determining the generalised Springer correspondence (that is, explicitly describing the bijections \( (2.2.3) \) for all \( j \in M_G \)) can be reduced to considering simple algebraic groups \( G \) of simply connected type, thus can be approached by means of a case-by-case analysis. This has been accomplished for almost all such \( G \), thanks to the work of Lusztig [17], Lusztig–Spaltenstein [32], Spaltenstein [37] (see also the references there for earlier results concerning the ordinary Springer correspondence), and again Lusztig [31], the only remaining open problems occur for \( G \) of type \( E_8 \) in characteristic \( p = 3 \) (for which a conjecture is made in [31, §6]). In particular, the ordinary Springer correspondence \( (2.2.4) \) is explicitly known in complete generality.

2.3. We keep the setting of 2.2 and consider the Frobenius map \( F : G \to G \). It defines actions on \( N_G \) and \( M_G \), given by
\[ N_G \to N_G, \quad (\mathcal{O}, \mathcal{E}) \mapsto (F^{-1}(\mathcal{O}), F^*\mathcal{E}), \]
and
\[ M_G \to M_G, \quad (L, \mathcal{O}_0, \mathcal{E}_0) \mapsto (F^{-1}(L), F^{-1}(\mathcal{O}_0), F^*\mathcal{E}_0). \]

Let \( N_G^F \), \( M_G^F \) be the respective sets of fixed points under these actions where, in terms of the local systems, this is only meant up to isomorphism, and for the triples in \( M_G \) in addition only up to \( G \)-conjugacy. The map \( \tau \) commutes with the action of \( F \) on \( N_G, M_G \), so it gives rise to a surjective map \( N_G^F \to M_G^F \). Furthermore, the generalised Springer correspondence \( (2.2.3) \) induces bijections
\[ \text{Irr}(\mathcal{W}_j)^F \cong \tau^{-1}(j) \cap N_G^F \quad \text{(for } j \in M_G^F \text{)}. \]
(Here, \( \text{Irr}(\mathcal{W}_j)^F \) denotes the set of all characters in \( \text{Irr}(\mathcal{W}_j) \) which are invariant under the automorphism of \( \mathcal{W}_j \) induced by \( F \).) Let \( j = (L, \mathcal{O}_0, \mathcal{E}_0) \in M_G^F \) and \( \phi \in \text{Irr}(\mathcal{W}_j)^F \), and assume that \( i = (\mathcal{O}, \mathcal{E}) \) is the corresponding element of \( \tau^{-1}(j) \cap N_G^F \). Choosing an isomorphism \( \varphi_0 : F^*\mathcal{E}_0 \cong \mathcal{E}_0 \) which induces a map of finite order at the stalk of \( \mathcal{E}_0 \) at any element of \( \mathcal{O}_0^F \) allows the definition of a unique isomorphism \( \varphi_i : F^*A_i \cong A_i \) (see [22]...
and also \[3.11\] below). We set
\[
\begin{align*}
    a_i &:= -\dim \mathcal{O} - \dim Z(L)^\circ, \\
    b_i &:= \dim \text{supp } A_i, \\
    d_i &:= \frac{1}{2}(a_i + b_i).
\end{align*}
\]
We then have
\[
\mathcal{H}^a(A_i)|_\mathcal{O} \cong \begin{cases} 
    \mathcal{E} & \text{if } a = a_i, \\
    0 & \text{if } a \neq a_i,
\end{cases}
\]
so we can define an isomorphism \(\psi_1: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}\) by the requirement that \(q^d\psi_1\) is equal to the isomorphism \(F^*\mathcal{H}^a(A_i)|_\mathcal{O} \xrightarrow{\sim} \mathcal{H}^a(A_i)|_\mathcal{O}\) induced by \(\varphi_i\). It is shown in [22, (24.2.4)] that for any \(u \in \mathcal{O}^F\), the induced map \(\psi_{1,u}: \mathcal{E}_u \to \mathcal{E}_u\) on the stalk of \(\mathcal{E}\) at \(u\) is of finite order. Now we define two functions
\[
X_i, \ Y_i: G_{\text{uni}}^F \to \mathbb{Q}_\ell
\]
by
\[
X_i(u) := (-1)^a q^{-d_i} \chi_{A_i, \varphi_i}(u)
\]
and
\[
Y_i(u) := \begin{cases} 
    \text{Trace}(\psi_{1,u}, \mathcal{E}_u) & \text{if } u \in \mathcal{O}^F, \\
    0 & \text{if } u \not\in \mathcal{O}^F,
\end{cases}
\]
for \(u \in G_{\text{uni}}^F\). Both \(X_i\) and \(Y_i\) are invariant under the conjugation action of \(G^F\) on \(G_{\text{uni}}^F\).

**Theorem 2.4** (Lusztig [22, §24]). In the setting of 2.3, the following hold.

(a) The functions \(Y_i, i \in N_G^F\), form a basis of the vector space consisting of all functions \(G_{\text{uni}}^F \to \mathbb{Q}_\ell\) which are invariant under \(G^F\)-conjugacy.

(b) There is a system of equations
\[
X_i = \sum_{\nu, \iota \in N_G^F} p_{\nu, \iota} Y_{\iota} \quad (i \in N_G^F),
\]
for some uniquely determined \(p_{\nu, \iota} \in \mathbb{Z}\).

**Proof.** See [22, (24.2.7) and (24.2.9)]. Note that the restrictions [22] (23.0.1)] on the characteristic \(p\) of \(k\) can be removed thanks to the remarks in [28, 3.10].

\[\square\]

2.5. As an immediate consequence of Theorem 2.4 (and of 2.3), we see that:

(i) We have \(p_{\iota, \iota} = 1\) for all \(i \in N_G^F\);

(ii) If \(\iota' = (O', \mathcal{E}') \neq i = (O, \mathcal{E})\), then \(p_{\iota', \iota} \neq 0\) implies \(O' \neq O\) and \(O' \subseteq \overline{O}\);

(iii) If \(\iota', i \in N_G^F\) belong to different blocks, we have \(p_{\iota', \iota} = 0\).

So if we define a total order \(\leq\) on \(N_G^F\) in such a way that for \(i = (O, \mathcal{E})\), \(\iota' = (O', \mathcal{E}') \in N_G^F\) we have
\[
\iota' \leq i \quad \text{whenever } \ O' \subseteq \overline{O},
\]
(note that the latter defines a partial order on the set of unipotent classes of \(G\)), the matrix \((p_{\iota', \iota})_{\iota', \iota \in N_G^F}\) has upper unitriangular shape. In [22, §24], Lusztig provides an algorithm for
computing this matrix \((p_{i,i})\) which entirely relies on combinatorial data. This algorithm is implemented in Michel’s development version of CHEVIE [33] and is accessible via the functions UnipotentClasses and ICTTable.

Remark 2.6. (a) Let \(O \subset G\) be a unipotent conjugacy class, and let us fix an element \(u \in O\). Given a \(G\)-equivariant irreducible local system \(E\) on \(O\), the stalk \(E_u\) is in a natural way an irreducible module for the group \(A_G(u) := CG(u)/C_G(u)\), and the assignment \(E \mapsto E_u\) defines a bijection between the set of isomorphism classes of \(G\)-equivariant irreducible local systems on \(O\) and the set of isomorphism classes of irreducible modules for \(A_G(u)\) (see [31, 3.5] or [25, 19.7]). Using this identification, it will sometimes be convenient to write \((u, \zeta) \in N_G\) instead of \((O, E)\) when \(\zeta \in \text{Irr}(A_G(u))\) describes the local system \(E\) and, similarly, \((L, u, \zeta) \in M_G\) instead of \((L, O_0, E_0)\) in case \(u \in O_0\) and \(E_0\) corresponds to \(\zeta \in \text{Irr}(A_L(u))\).

(b) We will also use the following notation: For \(u \in G_{\text{uni}}\), we write \(\overline{\pi} \in A_G(u)\) for the image of \(u\) under the canonical map \(CG(u) \to A_G(u)\). If \(u \in G_{\text{uni}}\) is fixed and \(A_G(u)\) is a cyclic group generated by \(\overline{\pi}\), we will denote the irreducible characters of \(A_G(u)\) just by their values at \(\overline{\pi}\).

(c) Finally, assume that \(O \subset G\) is a unipotent conjugacy class so that \(A_G(u) = (\overline{\pi})\) for \(u \in O\). If \(E\) is an irreducible \(G\)-equivariant local system on \(O\) corresponding to the irreducible character of \(A_G(u)\) which takes the value \(\zeta \in \mathbb{Q}_l^\times\) at \(\overline{\pi}\), we will write \((O, \zeta) := (O, E)\). Similarly, if \((L, O_0, E_0) \in M_G\) and \(A_G(u) = (\overline{\pi})\) for \(u \in O_0\), we set \((L, O_0, \zeta) := (L, O_0, E_0)\) if \(E_0\) is parametrised by the irreducible character of \(A_L(u)\) which takes the value \(\zeta\) at \(\overline{\pi}\). (Note that this is independent of the choice of \(u\) in either case, so we do not have to refer to a specific choice of \(u \in O\), \(u \in O_0\), respectively.)

3. Parametrisations of unipotent characters and unipotent character sheaves

Throughout this section, we assume that \(G\) is a simple algebraic group of adjoint type over \(k = \mathbb{F}_p\) (for a prime \(p\)). We also assume that \(F: G \to G\) is a Frobenius map with respect to an \(\mathbb{F}_q\)-rational structure on \(G\) (where \(q\) is a power of \(p\)) such that \((G, F)\) is of split type. The further notation is as in the introduction. We will summarise Lusztig’s results showing that there are two “parallel” parametrisations of the unipotent characters of \(G^F\) and the unipotent character sheaves on \(G\), one in terms of Lusztig’s families in the Weyl group as introduced in [16, §4], the other one via Harish-Chandra series. We mostly follow [29, §3]. Note that this is independent of the chosen \(F\) (at least as long as \((G, F)\) is of split type); see [30] for a conceptual explanation for this fact. At the end of this section, we provide precise descriptions of the characteristic functions of unipotent character sheaves and of the characteristic functions as they appear in [23], due to Lusztig [22, §25], [22, §24], respectively. This will be an essential ingredient in the proof of Lusztig’s conjecture, see Section 5 below.

3.1. For \(w \in W\), let \(R_w \in \text{CF}(G^F)\) be the virtual character of Deligne–Lusztig, as defined in [3 §1]. We denote by \(\text{Uch}(G^F) := \{\rho \in \text{Irr}(G^F) \mid \langle \rho, R_w \rangle_{G^F} \neq 0 \text{ for some } w \in W\} \subseteq \text{Irr}(G^F)\).
the set of unipotent characters of $G^F$. Furthermore, given $\phi \in \text{Irr}(W)$, let

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w \in \text{CF}(G^F)$$

be the unipotent almost character corresponding to $\phi$. In [16 §4], Lusztig introduces parameter sets for the irreducible characters of $G^F$. Since we assumed that $F$ defines a split rational structure on $G$ and since we will only be concerned with unipotent characters, this description simplifies considerably, so let us indicate it here (without providing the exact definitions). There is a set $X(W)$, together with a pairing

$$\{ , \} : X(W) \times X(W) \to \overline{\mathbb{Q}}_\ell,$$

as well as an explicit embedding

$$\text{Irr}(W) \hookrightarrow X(W), \quad \phi \mapsto x_\phi,$$

all of which only depend on $W$ (and not on $p$ or $q$). By [16 Main Theorem 4.23], there is a bijection

$$X(W) \xrightarrow{\sim} \text{Uch}(G^F), \quad x \mapsto \rho_x,$$

satisfying

$$\langle \rho_x, R_\phi \rangle_{G^F} = \Delta(x) \{x, x_\phi\} \quad \text{for any } x \in X(W), \phi \in \text{Irr}(W).$$

(Here, $\Delta(x)$ is a certain sign attached to $x \in X(W)$, see [16 4.14].) Furthermore, for any $x \in X(W)$, Lusztig sets

$$R_x := \sum_{y \in X(W)} \{y, x\} \Delta(y) \rho_y \in \text{CF}(G^F),$$

which we will refer to as the unipotent almost character corresponding to $x \in X(W)$. We have $R_{x_\phi} = R_\phi$ for all $\phi \in \text{Irr}(W)$. Since the “Fourier matrix” $(\{x, y\})_{x, y \in X(W)}$ is hermitian (in fact, it is symmetric with our assumptions) and of order 2, we get, for any $x \in X(W)$:

$$\rho_x = \Delta(x) \sum_{y \in X(W)} \{y, x\} R_y.$$

Thus, with respect to the scalar product $\langle , \rangle_{G^F}$, the almost characters $R_x, x \in X(W)$, form an orthonormal basis of the subspace of $\text{CF}(G^F)$ generated by $\text{Uch}(G^F)$, and knowing the values of the $R_x$ is equivalent to knowing the values of the unipotent characters of $G^F$.

### 3.2. Recall the notions on character sheaves introduced in 2.1

In addition to those, we need the following. For $\phi \in \text{Irr}(W)$, let

$$R_{\phi,0} := \frac{1}{|W|} \sum_{w \in W} \phi(w) \sum_{i \in \mathbb{Z}} (-1)^{i+\dim G^P} \nu^H i (K_w^{\phi,0}),$$

see [20 14.10]. For any $\phi \in \text{Irr}(W)$ and $A \in \hat{G}^{\text{un}}$, let $(A : R_{\phi,0}^\phi)$ be the multiplicity of $A$ in $R_{\phi,0}^\phi$ in the subgroup of the Grothendieck group of $\mathcal{M}G$ spanned by the character sheaves
The set $\hat{G}^{un}$ can be parametrised by means of a scheme analogous to that for the unipotent characters in \[3.1\]. There is a parametrisation

\[ (3.2.1) \quad X(W) \xrightarrow{\sim} \hat{G}^{un}, \quad x \mapsto A_x, \]

such that

\[ (3.2.2) \quad (A_x : R^{L_\phi}_x) = \varepsilon_{A_x} \{ x, x_\phi \} \quad \text{for any } x \in X(W) \text{ and any } \phi \in \text{Irr}(W). \]

3.3. The conditions (3.1.3), (3.2.2) do not completely determine the parametrisations (3.2.1), respectively, so one needs further information to uniquely specify a labelling of $\text{Uch}(G^F)$ and $\hat{G}^{un}$. It will be convenient for our purposes to follow the description in [29, §3]. Let $W$ be an irreducible Weyl group (or the trivial group \{1\}) with simple reflections $S \subseteq W$. With each such $(W, S)$ is associated a small (possibly empty) set $\mathcal{S}_W$ which in most cases may be regarded as a subset of the set of roots of unity in $\mathcal{O}_k$, see [29, 3.1]. For $J \subseteq S$, let $W_I = \langle J \rangle$ be the Coxeter group with Coxeter generators $J$. The definition of the sets $\mathcal{S}_W$ (with $W$ irreducible or \{1\}) shows that, if $\mathcal{S}_{W_J} \neq \emptyset$, then $(W_J, \langle J \rangle)$ is necessarily simple or \{1\}. Furthermore (still assuming that $\mathcal{S}_{W_J} \neq \emptyset$), by [13, Theorem 5.9], one obtains another Weyl group $W^{S/J}$, with simple reflections corresponding to the elements in $S \setminus J$. We have $W^{S/\emptyset} = W$ and $W^{S/S} = \{1\}$. Let

\[ \mathcal{S}_W := \{ (J, \phi, \zeta) \mid J \subseteq S, \phi \in \text{Irr}(W^{S/J}), \zeta \in \mathcal{S}_{W_J} \}. \]

The sets $\mathcal{S}_W$ and $\text{Irr}(W)$ canonically embed into $\mathcal{S}_W$, via

\[ \mathcal{S}_W \hookrightarrow \mathcal{S}_W, \quad \zeta \mapsto (S, 1, \zeta), \]

and

\[ (3.3.1) \quad \text{Irr}(W) \hookrightarrow \mathcal{S}_W, \quad \phi \mapsto (\emptyset, \phi, 1), \]

respectively. (We have $\mathcal{S}_W^{\{1\}} = \{1\}$.)

3.4. In the setting of 3.3, let us now take $W = W$. We will typically denote by $J$ a subset of the simple roots $\Pi$ rather than the corresponding subset of the simple reflections in $W$; the definition of $W_J$, $W^{\Pi/J}$, ... is analogous to that in 3.3. For any $J \subseteq \Pi$, let $P_J = B_0 W_J B_0$ be the standard parabolic subgroup of $G$, and let $L_J \subseteq P_J$ be the unique Levi complement of $P_J$ which contains $T_0$. Let us denote by $\text{Uch}(L^F_J) = \text{Uch}(L^F_J)$ the set of cuspidal unipotent characters of $G^F$. The definition of $\mathcal{S}_{W_J}$ is designed to be such that it coincides with the set of eigenvalues of Frobenius $\lambda_\rho$ for $\rho \in \text{Uch}(L^F_J) = \text{Uch}(L^F_J)$, as defined in [16, Chap. 11]. More precisely, up to a few exceptions (which we shall not explicitly be concerned with here), a cuspidal unipotent character on $L^F_J$ is uniquely determined by its eigenvalue of Frobenius, and this gives the bijection

\[ \text{Uch}(L^F_J) \xrightarrow{\sim} \mathcal{S}_{W_J} \]

(see [29, 3.2, 3.3] for the details). For any $J \subseteq \Pi$, let

\[ R^H_{L_J} : \text{CF}(L^F_J) \to \text{CF}(G^F) \]
be the Harish-Chandra induction (obtained by inflating class functions on $L^F_j$ to $P^F_j$ via the canonical map $P^F_j \to L^F_j$ followed by inducing them from $P^F_j$ to $G^F$). For any (cuspidal) unipotent character $\rho_0$ of $L^F_j$, the irreducible characters of $G^F$ which appear as constituents of $R^G_{L_j}(\rho_0)$ are in Uch($G^F$); conversely, given any $\rho \in$ Uch($G^F$), there exists a unique $J \subseteq \Pi$ and a unique $\rho_0 \in$ Uch($L^F_j$) such that $\rho$ is a constituent of $R^G_{L_j}(\rho_0)$, see [14]. Thus, Uch($G^F$) is partitioned into Harish-Chandra series, one for each pair $(L_j, \rho_0)$ with $J \subseteq \Pi$ such that $\tilde{\mathfrak{S}}_{\mathfrak{W}_j} \neq \emptyset$, and with $\rho_0 \in$ Uch($L^F_j$). As soon as a square root of $p$ (and of $q$) is fixed, as in (1.0.1), the irreducible characters of $W^{\Pi/S}$ naturally parametrise the unipotent characters of $G^F$ in the series with respect to $(L_j, \rho_0)$, see [16] 8.6, 8.7. Hence, we obtain a bijection

$$Uch(G^F) \cong \tilde{\mathfrak{S}}_{\mathfrak{W}}.$$

Remark 3.5. As mentioned in §3.3, the bijection $X(W) \cong Uch(G^F)$, $x \mapsto \rho_x$ in (3.1.2) is in general not completely determined by the requirement (3.1.3). However, in view of [6] Proposition 6.4, it can be chosen in such a way that it satisfies the following two additional conditions, and then it is uniquely determined:

(i) We have $\lambda_{\rho_x} = \tilde{\lambda}_x$ for all $x \in X(W)$, where $\lambda_{\rho_x} \in \mathbb{Q}_\ell^X$ is the eigenvalue of Frobenius of $\rho_x$ as defined in [16] Chap. 11, and where $\tilde{\lambda}_x$ is defined in a way analogous to $\lambda(\mathfrak{f})$ in [6] p. 135.

(ii) We have $\rho_{x,\phi} = \rho_\phi$ for all $\phi \in Irr(W)$, where $x, \phi \in X(W)$ is the image of $\phi$ under (3.1.1), and where $\rho_\phi \in$ Uch($G^F$) denotes the image of $\phi$ under (3.3.1) with $W = \mathfrak{W}$. (Thus, $\rho_\phi$ is the unipotent principal series character parametrised by $\phi$.) In fact, up to a few “exceptional” $\phi \in Irr(W)$, (ii) is automatically implied by (3.1.3), see [16] Proposition 12.6. The “exceptional” $\phi$ are taken care of by (i) and the definition of the $\tilde{\lambda}_x$. Finally, (i) removes all other possible ambiguities.

3.6. Just as for the unipotent characters, there is also a parametrisation of the unipotent character sheaves $\hat{G}^{un}$ in terms of “Harish-Chandra series”, as follows. First of all, to any $A \in \hat{G}$ is associated a root of unity $\lambda_A \in \mathbb{Q}_\ell$ which may be defined as in [29] 3.6 (see also [8] or [35] Theorem 3.3). Now let $P \subseteq G$ be any parabolic subgroup, and let $L \subseteq P$ be any Levi complement of $P$. There is an induction functor

$$(3.6.1) \quad \text{ind}^G_{L \subseteq P} : \{L\text{-equivariant perverse sheaves on } L\} \to \mathcal{D}G.$$

see [18] 4.1. Let $A_0 \in \hat{L}^{un}$ be a cuspidal unipotent character sheaf. By the results of [18] §4, the induced complex $\text{ind}^G_{L \subseteq P}(A_0) \in \mathcal{M}G$ is a semisimple perverse sheaf which is a direct sum of unipotent character sheaves on $G$. Furthermore, the definition of $\text{ind}^G_{L \subseteq P}(A_0)$ turns out to be independent of $L$ [17] Proposition 4.5, so we may (and will) just denote it as $\text{ind}^G_{L}(A_0)$ from now on. On the other hand, any $A \in \hat{G}^{un}$ is a constituent of $\text{ind}^G_{L_j}(A_0)$ for some $J \subseteq \Pi$ and some $A_0 \in \hat{L}^{un}$. By [17] §3 and [35] Lemma 5.9, the endomorphism algebra of any such $\text{ind}^G_{L}(A_0)$ is canonically isomorphic to the group algebra of $W_G(L_j) \cong W^{\Pi/J}$ (again assuming the fixed choices of $\sqrt{p}$, $\sqrt{q}$ in (1.0.1)). This gives rise to a parametrisation of $\hat{G}^{un}$ analogous to that of Uch($G^F$) described in §3.4. For
any $J \subseteq \Pi$ such that $\mathcal{E}_{W_J}$ is non-empty, there is a bijection

$$\hat{L}_j^{o,un} \sim \mathcal{E}_{W_J},$$

which in most cases is defined by identifying a given $A \in \hat{L}_j^{o,un}$ with the root of unity $\lambda_A$. (There are a few exceptions where $A \in \hat{L}_j^{o,un}$ is not uniquely determined by $\lambda_A$ [29, Theorem 3.7], but we will not be explicitly concerned with those.) In this way, the unipotent character sheaves on $G$ are parametrised via

$$\hat{G}^{un} \sim \mathcal{E}_W.$$

**Corollary 3.7.** Let $X(W) \sim Uch(G^F), x \mapsto \rho_x$ be the unique bijection which satisfies (3.1.3) and the conditions (i) and (ii) in Remark 3.5. Then, with the notation of 3.4, 3.6, there exists a unique bijection

$$X(W) \sim \hat{G}^{un}, x \mapsto A_x,$$

which satisfies the property (3.2.2) and such that in addition the following diagram commutes (where $\text{Irr}(W) \hookrightarrow X(W)$ is the embedding (3.1.1), and $\text{Irr}(W) \hookrightarrow \mathcal{E}_W$ is the embedding (3.3.1) with $W = W$):

$$\begin{array}{ccc}
X(W) & \sim & \hat{G}^{un} \\
\downarrow & & \downarrow \\
Uch(G^F) & \sim & \mathcal{E}_W \\
\downarrow & & \downarrow \\
\text{Irr}(W) & \sim & \hat{G}^{un}
\end{array}$$

**Proof.** Clearly, the requirement on the commutativity of the diagram printed in the corollary uniquely specifies the bijection. The commutativity of the diagram with respect to $\text{Irr}(W)$ follows from Remark 3.5. The fact that the bijection $X(W) \sim \hat{G}^{un}$ thus defined satisfies (3.2.2) is noted in [29, 3.10].

**Remark 3.8.** The relevance of Corollary 3.7 for our purposes is that the parametrisation of $\hat{G}^{un}$ in terms of $X(W)$ can now directly be read off from the parametrisations of unipotent characters of $G^F$ in terms of $\mathcal{E}_W$ (that is, Harish-Chandra series) and $X(W)$. Thus, we can just use the tables provided in the appendix of [16] when referring to the parametrisation of unipotent character sheaves on $G$.

From now on, we assume that the bijections $X(W) \sim Uch(G^F), x \mapsto \rho_x,$ and $X(W) \sim \hat{G}^{un}, x \mapsto A_x,$ are chosen according to Corollary 3.7. In particular, for a given $x \in X(W)$, whenever we write $\rho_x \in Uch(G^F)$ or $A_x \in \hat{G}^{un}$, we tacitly refer to this convention.

**Remark 3.9.** Recall the notation of [2.2]. Let us consider an element $j = (L, O_0, E_0) \in \mathcal{M}_G$. Since the elements of $\mathcal{M}_G$ are defined only up to $G$-conjugacy, we may assume that
\[ L = L_J \] is the standard Levi subgroup of the standard parabolic subgroup \( P_J \subseteq G \) for some \( J \subseteq \Pi \). Let \( \Sigma := Z(L_J)^G \mathcal{O}_0 \) and \( \mathcal{E} := 1 \otimes \mathcal{E}_0 \), so that \( (\Sigma, \mathcal{E}) \) is a cuspidal pair for \( L_J \) in the sense of [17, 2.4]. We also set
\[
A_0 := \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma]^{#L_J}.
\]
By [19, (7.1.4)] combined with [22, Theorem 23.1] and the results of [28], \( A_0 \) is a cuspidal character sheaf on \( L_J \). Furthermore, by [17, Proposition 4.5], we have a canonical isomorphism
\[
K_j \cong \text{ind}_{L_J}^G(A_0).
\]
Recall (2.2) that the \( A_i \) for \( i \in \tau^{-1}(j) \) are defined as simple direct summands of \( K_j \). On the other hand, by [18, Proposition 4.8], the simple direct summands of \( \text{ind}_{L_j}^G(A_0) \) are character sheaves on \( G \), and \( A_0 \) is a unipotent character sheaf on \( L_J \) if and only if one (or, equivalently, any) of these simple direct summands is a unipotent character sheaf on \( G \), so the analogous statement holds for the \( A_i \) with \( i \in \tau^{-1}(j) \). Let us now assume that \( A_0 \) is a unipotent character sheaf on \( L_J \). Hence, for any \( i \in \tau^{-1}(j) \), \( A_i \) is a unipotent character sheaf on \( G \), so there exists some \( x \in X(W) \) such that \( A_i \cong A_x \). Comparing the descriptions in [22, 3.6] for the parametrisations of the simple constituents of \( K_j \), \( \text{ind}_{L_J}^G(A_0) \), respectively, we see that they are entirely analogous. Thus, if \( i \) is the image of \( \phi \in \text{Irr}(W_j) \) under (2.2.3), then \( A_x \) corresponds to \( (J, \phi, \lambda_{A_0}) \in \mathfrak{G}_W \) in [3.6] with the canonical identification \( W_j \cong W^{j/J} \). In other words, considering the block \( \tau^{-1}(j) \) with \( j \) as above (so that the associated \( A_0 \) is unipotent), the problem of explicitly determining the bijection
\[
\text{Irr}(W_j) \xrightarrow{\sim} \tau^{-1}(j)
\]
is equivalent to solving the following problem:
\[ (\diamondsuit) \] For any \( i \in \tau^{-1}(j) \), find the \( x \in X(W) \) such that \( A_x \cong A_i \).

3.10. Let \( x \in X(W) \), and let us consider the corresponding unipotent character sheaf \( A_x \in \mathcal{G}^\text{un} \). Since we are assuming that \( (G, F) \) is of split type, it is known that \( A_x \) is automatically \( F \)-stable. (This follows, for example, from [29, §3] or [30]; it is also implicitly contained in [33, §5], [36, §4].) Let \( \varphi : F^*A_x \xrightarrow{\sim} A_x \) be an isomorphism. As \( A_x \) is \( G \)-equivariant for the conjugation action, the resulting characteristic function \( \chi_{A_x, \varphi} \) lies in \( \text{CF}(G^F) \). Here, \( \varphi \) (and, hence, \( \chi_{A_x, \varphi} \)) is uniquely defined up to multiplication with a scalar in \( \mathbb{T}_F^\bullet \); however, explicitly specifying an isomorphism \( \varphi \) appears to be a very difficult problem. In [22, 25.1], Lusztig formulates a condition which determines an isomorphism \( F^*A_x \xrightarrow{\sim} A_x \) up to multiplication with a root of unity. So let us assume that \( \varphi_x : F^*A_x \xrightarrow{\sim} A_x \) satisfies this requirement. Then, by the results of [22, §25], the corresponding characteristic function \( \chi_x := \chi_{A_x, \varphi_x} : G^F \rightarrow \mathbb{T}_F^\bullet \) has norm 1 with respect to the scalar product \( \langle \cdot, \cdot \rangle_{G^F} \); moreover, assuming that for any \( x \in X(W) \) a choice for \( \varphi_x \) is made as above, the set of functions \( \{ \chi_x \mid x \in X(W) \} \) is an orthonormal basis of the subspace of \( \text{CF}(G^F) \) that it generates.

3.11. Let us consider an element \( i \in \mathcal{N}_G^F \). We describe the isomorphism \( \varphi_i : F^*A_i \xrightarrow{\sim} A_i \) in [23, 25.1] following [22, 24.2]. As in Remark 3.9, we can represent \( j = \tau(i) \in \mathcal{M}_G^F \) by a triple
(L_j, O_0, E_0) for some J \subseteq \Pi. Thus, we have F(L_j) = L_j, and then also F(O_0) = O_0. Furthermore, F acts trivially on W = W_G(L_j). By [17, §9], there is a canonical basis \theta_w (w \in W) for the endomorphism algebra \mathcal{A}_i = \text{End}_{\mathcal{W}_G}(K_j), and we have an isomorphism 
 \mathcal{O}_w[\mathcal{W}] \cong \mathcal{A}_i, \quad w \mapsto \theta_w.

Now V_A := \text{Hom}_{\mathcal{W}_G}(A_j, K_i) is an irreducible left \mathcal{A}_i-module via the obvious composition of morphisms. By [22, 24.2], we can choose an isomorphism \varphi_0 : F^*E_0 \sim E_0 which induces a map of finite order at the stalk of \mathcal{E}_0 at any element of \mathcal{O}_0^\circ, and such a choice determines an isomorphism \varphi_1 : F^*A_i \sim A_i. Let us first consider any isomorphism \varphi_1 : F^*A_i \sim A_i. This gives rise to a bijective linear map 
 \sigma_{\varphi_1} : V_A \rightarrow V_A, \quad v \mapsto \varphi_1 \circ F^*(v) \circ \tilde{\varphi}_1^{-1}

which satisfies \tilde{\sigma}_{A_i} \circ \theta_w = \theta_w \circ \tilde{\sigma}_{A_i} for all w \in W. (Here, for v : A_i \rightarrow K_i in \mathcal{M}_G, F^*(v) denotes the induced map \text{F}^*A_i \rightarrow \text{F}^*K_i.) By Schur’s lemma, there exists some \zeta_i \in \mathcal{O}_0^\circ such that \tilde{\sigma}_{\varphi_1} = \zeta_i \cdot \text{id}_{V_A}. Thus, setting \varphi_1 := \zeta_i \tilde{\varphi}_1, we have \sigma_{\varphi_1} = \text{id}_{V_A}, and this requirement uniquely determines the isomorphism \varphi_1 : F^*A_i \sim A_i.

**Remark 3.12.** In the setting of [3.11] we see that the isomorphism \varphi_1 : F^*A_i \sim A_i is uniquely determined as soon as \varphi_0 : F^*E_0 \sim E_0 is chosen. So it will be convenient for the further discussion to be able to refer to (any) fixed choices for these isomorphisms:

\begin{center}
For any j = (L_j, O_0, E_0) \in \mathcal{M}_G^\circ, we assume to have fixed an isomorphism \varphi_0 : F^*E_0 \sim E_0 which induces a map of finite order at the stalk of \mathcal{E}_0 at any element of \mathcal{O}_0^\circ.
\end{center}

Now let us assume that A_i is a unipotent character sheaf on G, so that there exists some x \in X(W) for which A_x \cong A_i. Then the isomorphism \varphi_x in [3.10] does in general not coincide with the isomorphism \varphi_1 defined in [3.11] (not even by multiplication with a root of unity)! Hence, the same holds for the associated characteristic functions.

**Corollary 3.13.** Let i = (O, E) \in \mathcal{N}_G^\circ, and assume that A_i is a unipotent character sheaf on G. Let x \in X(W) be such that A_x is isomorphic to A_i. Let j = \tau(i) = (L_i, O_0, E_0) \in \mathcal{M}_G^\circ. Then there exists a unique isomorphism \varphi_x : F^*A_x \sim A_x as in [3.11], such that for any u \in G^F_{\text{uni}}, we have 

\[ X_{A_x \varphi_x}(u) = q^e X_i(u), \]

where 

\[ e_i = \frac{1}{2}(\dim G - \dim O - \dim Z(L)^0). \]

**Proof.** We start with any isomorphism \varphi_x : F^*A_x \sim A_x as in [3.10]. We have to compare such \varphi_x with the isomorphism \varphi_1 : F^*A_i \sim A_i defined in [3.11]. Recall from [3.11] that the fixed choice of \varphi_0 determines an isomorphism \varphi_1 : F^*K_1 \sim K_1. As we have seen in Remark 3.9, A_x \cong A_i is a simple direct summand of K_j, so V_{A_x} = \text{Hom}_{\mathcal{W}_G}(A_j, K_i) \cong V_{A_i} is an irreducible left \mathcal{A}_i-module. Just as we did for \varphi_{A_i}, we may thus consider the bijective linear map 

\[ \sigma_{\varphi_x} : V_{A_x} \rightarrow V_{A_x}, \quad v \mapsto \varphi_1 \circ F^*(v) \circ \tilde{\varphi}_x^{-1}. \]
As noted in [22, p. 153], $\sigma_\varphi_x$ is equal to $q^{-(\dim G - \dim \text{supp } A_x)/2}$ times a map of finite order. We also know that there exists some $\varphi_x \in \mathbb{Q}_l^x$ for which $\varphi_x = \varphi_x \varphi_1$, so we get

$$\sigma_\varphi_x = \varphi_x^{-1} \sigma_\varphi_1 = \varphi_x^{-1} \text{id}_{V_{\lambda_1}}.$$ 

Hence, $\delta_x := q^{(\dim G - \dim \text{supp } A_x)/2} \varphi_x^{-1}$ must be a root of unity. So we get, for any $u \in G_{\text{uni}}$:

$$\chi_{A_x, \varphi_x}(u) = \varphi_x \chi_{A_x, \varphi_1}(u) = (-1)^q \delta_x^{-1} q^{(\dim G - \dim \mathcal{O} - \dim Z(L_\varphi))/2} \chi_{\lambda_1}(u).$$

Then $\varphi_x := (-1)^q \delta_x \varphi_x : F^* A_x \cong A_x$ is the unique isomorphism for which the corollary holds. (Note that, as $(-1)^q \delta_x \in \mathbb{Q}_l^x$ is a root of unity, the choice of $\varphi_x$ is in accordance with [3.10]).

**Remark 3.14.** Let $x \in X(W)$ and $i \in N^F$ be such that $A_x \cong A_i$. Note that the only choice that is involved to specify the isomorphisms $\varphi_1$ and $\varphi_x$ by means of [3.11] and Corollary 3.13 is the one of an isomorphism $F^* E_0 \cong E_0$ as in Remark 3.12. While both $\varphi_1$ and $\varphi_x$ will indeed change if we vary the choice of $\varphi_0 : F^* E_0 \cong E_0$, the interrelation between $\varphi_1$ and $\varphi_x$ via [3.11] and Corollary 3.13 will not.

### 4. Groups of type $E_8$ in characteristic 3 and Lusztig’s conjecture

From now until the end of the paper, we assume that $G$ is the simple group of type $E_8$ over $k = \mathbb{F}_3$ and that $F : G \to G$ is the Frobenius map with respect to an $F_q$-rational structure on $G$, where $q = 3^f$ for some $f \geq 1$. Thus, $G^F = E_8(q)$ is necessarily the (untwisted) group $E_8(3^f)$. The further notational conventions for this $G$ and $F$ are as in the introduction; in addition, we use the following: We write $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_8\} \subseteq \Phi$, labelled in such a way that the Dynkin diagram of $G$ is given by

![Dynkin Diagram of E_8](image)

Viewing $W$ as a reflection group associated to the root system $\Phi$, we denote by $s_i \in W$ the reflection corresponding to the simple root $\alpha_i \in \Pi$. Thus, $W$ is a Coxeter group with Coxeter generators $s_1, s_2, \ldots, s_8$, so that the products $s_1 s_3, s_2 s_4, s_2 s_5, s_3 s_5, s_3 s_6, s_4 s_7, s_7 s_8$ have order 3 while the product of any other two different Coxeter generators is of order 2. Our notation for the unipotent conjugacy classes of $G$ is the same as the one used by Spaltenstein [37].

#### 4.1. We recall the notation of [3.4] By the table in [16, pp. 366–370] or [2, pp. 484–488], we have $|X(W)| = |\mathcal{S}_W| = 166$. The set $\mathcal{S}^\circ_W$ is non-empty for the following subsets $J \subseteq \Pi$: $J = \emptyset$, $J = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$, $J = \{\alpha_1, \alpha_2, \ldots, \alpha_7\}$ and $J = \Pi$. Thus, the 166 elements of $\mathcal{S}_W$ fall into the following Harish-Chandra series.

(a) For $J = \emptyset$, we have 112 elements in the principal series, that is, they are in the image of the embedding $\text{Irr}(W) \hookrightarrow \mathcal{S}_W$, $\phi \mapsto (\emptyset, \phi, 1)$.

(b) Let $J = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subseteq \Pi$, so that the group $L_j/\mathbb{Z}(L_j)\circ$ is simple of type $\text{D}_4$. We have $\mathcal{S}_W^J = \{-1\}$, and the relative Weyl group $W^{\Pi/J} = W_G(L_j)$ is
isomorphic to $W(F_4)$. Thus, $W^{II/J}$ has 25 irreducible characters, so there are 25 elements in $\mathcal{S}_W$ of the form $(J, \phi, -1)$, $\phi \in \text{Irr}(W^{II/J})$.

(c) Let $J = \{\alpha_1, \ldots, \alpha_6\} \subseteq \Pi$, so that the group $L_J / Z(L_J)$ is simple of type $E_6$. We have $\mathcal{S}_W = \{\zeta_3, \zeta_3^2\}$ where $\zeta_3 \in \overline{\mathbb{Q}}^\times$ is a primitive 3rd root of unity. The relative Weyl group $W^{II/J} = W_G(L_J)$ is isomorphic to $W(G_2) \cong D_{12}$, the dihedral group of order 12. Thus, $W^{II/J}$ has 6 irreducible characters, so there are 6 elements in $\mathcal{S}_W$ of the form $(J, \phi, \zeta_3^i)$, and 6 elements of the form $(J, \phi, \zeta_3^j)$ (where $\phi \in \text{Irr}(W^{II/J})$), in total 12 elements.

(d) Let $J = \{\alpha_1, \ldots, \alpha_7\} \subseteq \Pi$, so that the group $L_J / Z(L_J)$ is simple of type $E_7$. We have $\mathcal{S}_W = \{i, -i\}$ where $i \in \overline{\mathbb{Q}}^\times$ is a primitive 4th root of unity. The relative Weyl group $W^{II/J} = W_G(L_J)$ is isomorphic to $W(A_1) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, $W^{II/J}$ has 2 irreducible characters $\pm 1$, and $(J, \pm 1, \pm i)$ give rise to 4 elements of $\mathcal{S}_W$.

(e) For $J = \Pi$, the set $\mathcal{S}_W = \mathcal{S}_G$ consists of 13 elements, that is, $G^F$ has 13 cuspidal unipotent characters (and $G$ has 13 cuspidal unipotent character sheaves).

### 4.2.

We now describe the set $\mathcal{N}_G$, following Lusztig [17, 15.3]. As the elements of $M_G$ are defined only up to $G$-conjugacy, we can (and will) always represent $j \in M_G$ as an element of the form $j = (L_J, O_0, E_0)$ for some $J \subseteq \Pi$. Note that, in contrast to the description of $\mathcal{S}_W$ in 4.1, the set $\mathcal{N}_G$ depends on the characteristic of $k$! With our assumption that $p = 3$, we have $|\mathcal{N}_G| = 127$, and the set $\mathcal{N}_G$ is partitioned into the following blocks $\tau^{-1}(j)$, $j \in M_G$.

(a) Let $J = \emptyset$. There is a unique cuspidal pair on $T_0$ in the sense of [17, 2.4], namely, $(T_0, \mathcal{U}_0)$. Thus, $j = (T_0, \{1\}, 1)$ is uniquely determined by $J$, and it gives rise to the “Springer block” $\tau^{-1}(j)$. We have $\mathcal{V}_j = W_G(T_0) = W$, and this group has 112 irreducible characters. In view of (2.2.4), we have 112 elements in $\tau^{-1}(j)$ as well.

(b) Let $J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$. According to [17, §15], there are two possibilities for $(O_0, E_0)$ such that $(L_J, O_0, E_0) \in M_G$. The relative Weyl group $\mathcal{W}_j \cong W(G_2)$ is the dihedral group of order 12, so it has 6 irreducible characters. Thus, each of the two $j$ above gives rise to 6 elements of $\tau^{-1}(j)$, in total we get 12 such elements of $\mathcal{N}_G$.

(c) Let $J = \Pi$. There are 3 elements in $M_G$ of the form $(G, O_0, E_0)$; these are just the ones where $(O_0, E_0)$ is a cuspidal pair for $G$ itself, and we obtain 3 elements of $\mathcal{N}_G$ in this way.

Note that in (a), the cuspidal character sheaf $A_0$ on $T_0$ associated to $(T_0, \{1\}, 1)$ in the setting of Remark 3.9 is just the constant local system $L_0 = \mathcal{U}_0$ on $T_0$, so it is in $T_0^{un}$. The discussion in Remark 3.9 thus shows that we have

$$\{A_i \mid i \in \tau^{-1}(j)\} = \{A_x \mid x = x_\phi \text{ for some } \phi \in \text{Irr}(W)\}.$$
So the set \(\tau^{-1}(j) = \{(O_0, E_0)\} \subseteq \mathcal{N}_G\) is a singleton, the group \(\mathcal{W}_j\) is the trivial one, and, of course, determining the generalised Springer correspondence for these \(j\) is evident. So it remains to consider the two cases where \(J\) is as in (c), and this is what the rest of this paper will be concerned with.

**Corollary 4.3.** For any \(i \in \mathcal{N}_G\), the character sheaf \(A_i\) is unipotent, that is, there exists some \(x \in X(W)\) such that \(A_x\) is isomorphic to \(A_i\). In particular, we have \(\mathcal{N}_G = \mathcal{N}_G^\circ\).

**Proof.** By the remarks in 4.2, the set of the 112 different \(A_i\) for \(i \in \tau^{-1}((T_0, \{1\}, 1))\) considered in 4.2(a) coincides with the set of the \(A_{x_\phi}\) with \(\phi \in \text{Irr}(W)\) considered in 4.1(a). For \(j = (G, O_0, E_0)\) as in 4.2(c), we have noted that \(i = (O_0, E_0)\) is the unique element in its block \(\tau^{-1}(j)\), and \(A_i \cong K_1\) is a cuspidal character sheaf on \(G\). But any one of the 13 cuspidal character sheaves on \(G\) is unipotent (see the proof of [36, Proposition 5.3]), so \(A_i\) is isomorphic to \(A_x\) for some \(x \in X(W)\). It remains to consider the case where \(j = (L_J, O_0, E_0)\) is an element as in 4.2(c), so \(J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}\). In view of Remark 3.9 we have to show that the cuspidal character sheaf \(A_0\) on \(L_J\) associated to \(j\) is unipotent.

Now \(L_J/\mathbb{Z}(L_J)\) is the simple (adjoint) group of type \(E_6\) in characteristic 3, and the proof of [21, Proposition 20.3] shows that this group has exactly 2 cuspidal character sheaves, both of which are unipotent. Using [21, 17.10], we conclude that \(A_0 \in \hat{L}_J^0\) must be unipotent as well, as desired. Since all the unipotent character sheaves on \(G\) are \(F\)-stable (cf. [3.10], \(\mathcal{N}_G = \mathcal{N}_G^\circ\) follows from (2.2.2). \(\square\)

**Remark 4.4.** Corollary 4.3 allows a “global” reformulation of \((\bigodot)\) in Remark 3.9 for \(G\) of type \(E_8\) in characteristic 3:

\((\bigodot)\) For any \(i \in \mathcal{N}_G\), find the \(x \in X(W)\) such that \(A_x \cong A_i\).

Thus, solving \((\bigodot)\) is equivalent to determining the generalised Springer correspondence (2.2.3) (with respect to all \(j \in \mathcal{M}_G\)) for \(G\).

**4.5.** We turn to the case 4.2(c). So let \(J := \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \subseteq \Pi\), and let \((O_0, E_0)\) be one of the two pairs such that \(j = (L_J, O_0, E_0) \in \mathcal{M}_G\). From the proof of [21, Proposition 20.3], we see that the two cuspidal (unipotent) character sheaves for simple groups of type \(E_6\) in characteristic 3 are supported by the unipotent variety. In view of [21, 17.10], the two cuspidal (unipotent) character sheaves on \(L_J\) are thus supported by \(\mathbb{Z}(L_J)^0\), so \(O_0\) must be the regular unipotent class of \(L_J\). For \(u_0 \in O_0\), we have \(A_{L_J}(u_0) = (\mathfrak{o}_0) \cong \mathbb{Z}/3\mathbb{Z}\).

Hence, using the conventions in Remark 2.6 we can write \(j = (L_J, O_0, E_0) = (L_J, O_0, \zeta)\) where \(\zeta \in \overline{\mathbb{Q}}_l^\times\) is a 3rd root of unity. Recall from Remark 3.9 that \(K_1\) is isomorphic to \(\text{ind}_{L_J}^G(A_0)\) where

\[A_0 := \text{IC}(\mathbb{Z}(L_J)^0.\mathfrak{O}_0, 1 \boxtimes E_0)[\dim \mathbb{Z}(L_J)^0 + \dim O_0] \# L_J \subseteq \hat{L}_J^{0, \text{un}}.\]

We also recall (see [36]) that the isomorphism classes of the simple constituents of \(\text{ind}_{L_J}^G(A_0)\) are parametrised by triples \((J, \phi, \lambda_A)\) in \(\mathfrak{S}_W\) where \(\phi\) runs over the irreducible characters of \(W^{B/J}\). The definition of \(\lambda_A\) in [29, 3.6] then shows that we have \(\lambda_A = \zeta\), and this is a primitive 3rd root of unity by the definition of \(\mathfrak{S}_W^\circ\) in [29, 3.1]. Now by (2.2.3), we have a bijection

\[\text{Irr}(\mathcal{W}_j) \cong \tau^{-1}(j)\].
Here, \( \mathcal{W}_i \cong W^{\Pi/J} \cong W(G_2) \) is the dihedral group of order 12, so it has 6 irreducible characters: four linear characters and two irreducible characters of degree 2. While Spaltenstein \([37]\) has shown to which elements of \( \tau^{-1}(j) \) the four linear characters of \( \mathcal{W}_i \) are mapped, this is not known for the two other irreducible characters! Put differently, using the characterisation \([33]\) in Remark 3.9. For any of the two \( x \in X(W) \) corresponding to a triple \((J, \phi, \lambda_0) \in G_W \) with \( \phi \in \text{Irr}(W^{\Pi/J}) \) of degree 2, the task is to find the \( i \in N_G \) such that \( A_i \cong A_1 \). (The two possibilities for those \( i \) are \((E_7 + A_1, \lambda_{A_i})\) and \((E_7, \lambda_{A_i})\).)

This ambiguity, for either of the two \( A_0 \) considered above, is exactly the last one that needs to be resolved to complete the generalised Springer correspondence. In order to describe it more precisely, we now proceed by fixing our notation for the relative Weyl group \( \mathcal{W}_i \cong W^{\Pi/J} \cong W(G_2) \) and its irreducible characters.

4.6. As before, let \( J := \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \subseteq \Pi \). Let us give the concrete description of \( W^{\Pi/J} \), following \([29, 3.1]\). The group \( W^{\Pi/J} \) is defined as the subgroup of \( W \) generated by the two elements

\[
\sigma_i := w_0^{J \cup \{\alpha_i\}} w_0^J = w_0^J w_0^{J \cup \{\alpha_i\}} \quad \text{for } i = 7, 8
\]

(where, for \( J' \subseteq \Pi \), \( w_0^{J'} \) denotes the longest element of the Coxeter group \( W_{J'} \), the length function on \( W_{J'} \) being defined with respect to \( J' \subseteq \Pi \)). The elements \( \sigma_7, \sigma_8 \) have order 2 and generate \( W^{\Pi/J} \) as a Coxeter group; more precisely, \( W^{\Pi/J} \) is a Weyl group coming from a natural root system induced from that in \( G \): This induced root system is of type \( G_2 \), with \( \sigma_7 \) being the reflection with respect to a short simple root and \( \sigma_8 \) the reflection with respect to a long simple root, so the associated Dynkin diagram is as follows:

\[ G_2 \]

\[ \sigma_7 \quad \sigma_8 \]

Note that this is compatible with \([17, \text{Theorem 9.2(a)}]\): We have a canonical isomorphism \( W^{\Pi/J} \cong W_G(L_J) \) under which \( \sigma_i \) corresponds to the unique non-trivial element of \( N_{L,J \cup \{\alpha_i\}}(L_J)/L_J \) for \( i = 7, 8 \) (cf. \([31, 2.1]\)). We now give our names for the irreducible characters of \( W^{\Pi/J} \cong W_G(L_J) \), which will be similar to those used in \([31, 5.1]\) (except that we write \( \text{sgn} \) instead of \( s \)): Thus, 1 denotes the trivial character, \( \text{sgn} \) the sign character; \( \epsilon, \epsilon' \) are the two remaining irreducible characters of degree 1 such that

\[
\epsilon(\sigma_7) = \epsilon'(\sigma_8) = 1, \quad \epsilon(\sigma_8) = \epsilon'(\sigma_7) = -1.
\]

Furthermore, \( \rho \) is the character of the reflection representation and \( \rho' \) the other irreducible character of degree 2.

4.7. We still assume that \( J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \subseteq \Pi \). Since we will refer to different sources in several places below, we give the explicit conversion between our notation concerning the relative Weyl group \( W^{\Pi/J} \cong W_G(L_J) \) in \([16, \text{p. 361}]\), Carter \([2, \text{p. 412}]\) (which coincides with the one in Michel's \textsc{CHEVIE} \([33]\)) and Spaltenstein \([37, \text{p. 327}]\), respectively. First of all, it is easy to compare our notation with the one in Lusztig's book as he explicitly states his conventions in \([16, \text{p. 361}]\). The fact that our short (respectively, long) roots coincide with the short (respectively, long) roots as indicated by Carter \([2, \text{p. 412}]\) implicitly follows from comparing his table \([p. 487]\) describing the
Harish-Chandra series of unipotent characters with Lusztig’s [16, p. 367]. However, one must be extremely careful when translating Spaltenstein’s conventions to our notation as far as the characters $\epsilon, \epsilon'$ are concerned: Note that Spaltenstein’s declarations for the long and short roots [37, p. 327] are opposite to ours (hence also to Lusztig’s and Carter’s)! Thus, forgetting the underlying root system and just regarding $W_{\Pi/J}$ as a Coxeter group with Coxeter generators $\sigma_7, \sigma_8$ for a moment: In order to match with the setting of [37], we must identify $\sigma_7 \leftrightarrow s_7$ and $\sigma_8 \leftrightarrow s_c$, and then we have $\epsilon \leftrightarrow \epsilon_c, \epsilon' \leftrightarrow \epsilon_l$. (The other irreducible characters are not affected by the distribution of long or short roots.) The conversion scheme between our notation and that of Lusztig, Carter (thus also CHEVIE) and Spaltenstein as described above is given in Table 1. (Whenever we refer to [37], we will still explicitly provide the relevant results with our notation to avoid any possible confusion, most notably the table in [37, p. 329], see Table 2 below.) Note that Spaltenstein does not distinguish the characters $\theta', \theta''$ (which is the last ambiguity in the generalised Springer correspondence that remains to be resolved), so we only know that $\{\theta', \theta''\} \leftrightarrow \{\rho, \rho'\}$ at this point.

| Our notation | Lusztig [16, p. 361] | Carter [2, p. 412] | Spaltenstein [37, p. 327] |
|--------------|----------------------|---------------------|---------------------------|
| 1            | 1                    | $\phi_{1,0}$        | 1                         |
| sgn          | $\epsilon$           | $\phi_{1,6}$        | $\epsilon$                |
| $\epsilon$   | $\epsilon'$          | $\phi'_{1,3}$       | $\epsilon_c$              |
| $\epsilon'$  | $\epsilon''$         | $\phi''_{1,3}$      | $\epsilon_l$              |
| $\rho$       | $r$                  | $\phi_{2,1}$        | $\theta'$ or $\theta''$   |
| $\rho'$      | $r'$                 | $\phi_{2,2}$        | $\theta''$ or $\theta'$   |

Table 1. Conversion scheme between our notation for the irreducible characters of $W_{\Pi/J} \cong W(G_2)$ and the ones used in [2, p. 412], [16, p. 361] and [37, p. 327]

4.8. Let $J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$. Continuing our discussion in 4.5 we can now state Lusztig’s conjecture in a precise way. So let $j$ be one of the two elements of $\mathcal{M}_G$ of the form $(L_J, \mathcal{O}_0, \zeta_3)\in \mathcal{Q}_L^\times$ is a primitive 3rd root of unity. With the notation of 4.6 the elements in the block $\tau^{-1}(j) \subseteq N_G$ are parametrised by

$$\text{Irr}(W_G(L_J)) = \text{Irr}(W_{\Pi/J}) = \{1, \text{sgn}, \epsilon, \epsilon', \rho, \rho'\}.$$ 

As already mentioned before, Spaltenstein [37] has shown to which element $i \in \tau^{-1}(j)$ any of the characters $1, \text{sgn}, \epsilon, \epsilon'$ belongs, but he has not given the correspondence as far as $\rho, \rho'$ are concerned. We will for now use the notation in [37] for the last two characters, that is, we just write $\{\theta', \theta''\} = \{\rho, \rho'\}$ without yet distinguishing between $\theta'$ and $\theta''$. With this ambiguity, the generalised Springer correspondence for the two blocks is provided in the table in [37, p. 329]. The conversion to our notation is given in Table 2; this table must be understood as follows: The first column contains the unipotent classes $\mathcal{O}$ for which
there exists a local system $E$ on $O$ such that $(O, E) \in \tau^{-1}(j)$. The second column gives the group $A_G(u) = C_G(u)/C_{G_0}(u)$ for $u \in O$. Here, we have $A_G(u) = \langle \overline{u} \rangle$ in all cases, so the local system on $O$ can be described by the value of the corresponding irreducible character of $A_G(u)$ at $\overline{u}$ in the third column; this value is in fact equal to $\zeta_3$ for any one of the unipotent classes considered. The last column gives the name of the irreducible character of $W_G(L_J) \cong \mathcal{W}^{\Pi/J}$ (notation of [4.6]) which is mapped to $(O, \zeta_3)$ under the generalised Springer correspondence. With the distribution of $\theta'$, $\theta''$ as in Table 2, Lusztig’s conjecture [31, \S 6] states:

$(\heartsuit)$ We have $\theta' = \rho$ and $\theta'' = \rho'$, for any one of the two $j \in \mathcal{M}_G$ as above.

| Class of $u$ | $A_G(u)$ | $\zeta$ | $\rho^G_{u,\bar{u}}$ |
|--------------|-----------|---------|----------------------|
| $E_8$        | $\mathbb{Z}/3\mathbb{Z}$ | $\zeta_3$ | 1                     |
| $E_8(a_1)$   | $\mathbb{Z}/3\mathbb{Z}$ | $\zeta_3$ | $\epsilon$           |
| $E_7 + A_1$  | $\mathbb{Z}/6\mathbb{Z}$ | $\zeta_3$ | $\theta'$            |
| $E_7$        | $\mathbb{Z}/3\mathbb{Z}$ | $\zeta_3$ | $\theta''$           |
| $E_6 + A_1$  | $\mathbb{Z}/3\mathbb{Z}$ | $\zeta_3$ | $\epsilon'$          |
| $E_6$        | $\mathbb{Z}/3\mathbb{Z}$ | $\zeta_3$ | $\text{sgn}$         |

Table 2. The (incomplete) generalised Springer correspondence in $E_8$, $p = 3$, with respect to the blocks $\tau^{-1}(j)$, $j = (L_J, O_0, \zeta_3)$, where $O_0 \subseteq L_J$ is the regular unipotent class and $\zeta_3 \in \overline{\mathbb{Q}_l}$ is a primitive 3rd root of unity, due to Spaltenstein [37, p. 329].

5. THE PROOF OF LUSZTIG’S CONJECTURE

The setting and notation is exactly the same as in the previous section. (In particular, $G$ is simple of type $E_8$, $p = 3$ and $G^F = E_8(3^r)$, $r \geq 1$.) In addition, the names for the irreducible characters of $W$ are as in [16, 4.13]. From now until the end of the paper, we use the following convention (see Section 3):

For any $j = (L_J, O_0, E_0) \in \mathcal{M}_G^F$, we assume to have fixed an isomorphism $\varphi_0 : F^*E_0 \cong E_0$ which induces a map of finite order at the stalk of $E_0$ at any element of $O_0^F$. Then, for any $i \in \tau^{-1}(j) \subseteq \mathcal{N}_G^F = N_G$, with corresponding $x \in X(W)$ such that $A_x \cong A_i$ (see Corollary 4.3), we set $\chi_i := \chi_{A_i, \varphi_i}$ and $\chi_x := \chi_{A_x, \varphi_x}$, where $\varphi_i : F^*A_i \cong A_i$ and $\varphi_x : F^*A_x \cong A_x$ are the unique isomorphisms defined through [3.11] and [3.13] respectively.
5.1. Let us consider the Hecke algebra associated to $G^F$ and its $(B, N)$-pair $(B_0^F, N_G(T_0)^F)$, that is, the endomorphism algebra

$$
\mathcal{H}_q := \text{End}_{G^F}(\mathbb{Q}_F[G^F/B_0^F])^{\text{opp}}.
$$

(Here, “opp” stands for the opposite algebra.) $\mathcal{H}_q$ has a $\mathbb{Q}_F$-basis $\{T_w \mid w \in W\}$ where

$$
T_w : \mathbb{Q}_F[G^F/B_0^F] \to \mathbb{Q}_F[G^F/B_0^F], \quad xB_0^F \mapsto \sum_{gB_0^F \in G^F/B_0^F, \, x^{-1}gB_0^F \in B_0^F} yB_0^F,
$$

for $w \in W$. (Here and in what follows, whenever $w \in W$ is given, we denote by $\hat{w}$ a representative of $w$ in $N_G(T_0)^F$.) Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function of $W$ with respect to the Coxeter generators $S = \{s_1, \ldots, s_8\}$. Then the multiplication in $\mathcal{H}_q$ is determined by the following equations:

$$
T_s \cdot T_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\
qT_{sw} + (q-1)T_w & \text{if } \ell(sw) = \ell(w) - 1
\end{cases} \quad (\text{for } s \in S, \ w \in W).
$$

For any $g \in G^F$ and any $w \in W$, we set

$$
m(g, w) := \frac{|O_g \cap B_0^F \hat{w}B_0^F| : |C_G(g)|}{|B_0^F|}
$$

where $O_g \subseteq G^F$ is the $G^F$-conjugacy class of $g$. The irreducible characters of $W$ naturally parametrise the isomorphism classes of irreducible modules of $\mathcal{H}_q$, see [11, Cor. 8.7] (recall our fixed choice of $\sqrt{q}$ in [1, (1.0.1)]). Given $\phi \in \text{Irr}(W)$, let $\psi_0$ be the corresponding module of $\mathcal{H}_q$, and let $\rho_\phi \in \text{Uch}(G^F)$ be the image of $\phi$ under the map $\text{Irr}(W) \hookrightarrow \mathfrak{S}_W \twoheadrightarrow \text{Uch}(G^F)$, see [3.4]. By [2, Remark 3.6(b)] and [3, §11D], we have

$$
m(g, w) = \sum_{\phi \in \text{Irr}(W)} \rho_\phi(g) \text{Trace}(T_w, V_\phi) \quad \text{for any } g \in G^F, \ w \in W,
$$

where $\text{Trace}(T_w, V_\phi)$ is the trace of the endomorphism of $V_\phi$ defined by $T_w$. It is shown in [20, Corollary 14.14] (see also [23, 3.6]) that the unipotent principal series characters $\rho_\phi, \phi \in \text{Irr}(W)$, can be expressed as linear combinations of characteristic functions of unipotent character sheaves: There exists a root of unity $\xi_x \in \mathbb{Q}_F^\times$ such that

$$
\rho_\phi = (-1)^{\dim G} \sum_{x \in X(W)} \xi_x(A_x : R_0^x)\chi_x \quad \text{for } \phi \in \text{Irr}(W).
$$

(Note that, in general, $\xi_x$ depends on the choice of the isomorphism $F^*A_x \cong A_x$, as well as on the one of a square root of $q$ in $\mathbb{Q}_F$; but as we have fixed those in the beginning of this section and in [1, (1.0.1)], respectively, we just write $\xi_x$ here. Also recall that our choice of $\varphi_x$ meets the requirement in [22, 25.1], which is the same as [20, (13.8.1)].) Using [3.2.2], we get

$$
\rho_\phi = (-1)^{\dim G} \sum_{x \in X(W)} \xi_x\hat{\varepsilon}_x(x, x_0)\chi_x \quad \text{for } \phi \in \text{Irr}(W),
$$

where $\varepsilon_x$
where \( \xi_x := \xi_{A_x} \in \{ \pm 1 \} \). For any \( g \in G^F \) and any \( w \in W \), we thus obtain

\[
(\dagger) \quad m(g, w) = \sum_{x \in X(W)} \left( \sum_{\phi \in \text{Int}(W)} \{x, x_\phi\} \text{Trace}(T_w, V_\phi) \right) \nu_x \cdot \chi_x(g),
\]

with \( \nu_x := (-1)^{\dim G} \xi_x \xi_x \in \overline{Q}_{\ell}^\times \) (a root of unity).

The character values \( \text{Trace}(T_w, V_\phi) \) of the Hecke algebra \( H_q \) have been determined by Geck and Michel in \cite{GeckMichel}; they are (just as the entries \( \{x, x_\phi\} \) of the Fourier matrix) readily available through CHEVIE \cite{CHEVIE}, using the function \texttt{HeckeCharValues}. Hence, the coefficients of \( \nu_x \chi_x(g) \) in (\dagger) can be computed explicitly.

5.2. Let \( i \in N_G^F = N_G \). Thus, by Corollary \ref{cor:chi_x} whenever we know the \( x \in X(W) \) corresponding to \( i \) under (\( \check{\chi} \)) in Remark \ref{rem:chi_x}, we can express \( \chi_x|_{G^F_{\text{uni}}} \) as an explicit scalar multiple of \( X_i \). There are in total 4 different \( i \in N_G \) for which the solution to (\( \check{\chi} \)) is not known, see \ref{rem:chi_x}. The idea to prove (\( \check{\chi} \)) in \ref{rem:chi_x} consists in evaluating the equation (\( \dagger \)) in 5.1 with a suitable \( w \in W \) (and \( u \in G^F_{\text{uni}} \)) and obtaining contradictions whenever (\( \check{\chi} \)) does not hold. (We will not have to consider such \( x \in X(W) \) which do not correspond to an element of \( N_G \) as above, see Lemma 5.3 below.) Let us set

\[
(5.2.1) \quad w_{49} := s_2 s_3 s_4 s_5 s_6 s_7 s_8 \in W
\]

(i.e., the 49th element in \texttt{ChevieClassInfo(W).classtext} in CHEVIE \cite{CHEVIE}). Then \( w_{49} \) is an element of minimal length in its conjugacy class \( C_{49} \subseteq W \). In \cite{Lusztig}, Lusztig defined a surjective map from \( W \) to the set of unipotent classes of \( G \), and \( C_{49} \) is sent to \( E_7 + A_1 \) under this map. By the results of \cite{Lusztig} and \ref{rem:chi_x}, we have \( \mathcal{O} \cap B_0 w_{49} B_0 = \emptyset \) unless \( \mathcal{O} \in \{ E_8, E_8(a_1), E_8(a_2), E_7 + A_1 \} \). In particular, we know that \( O_u \cap B_0^F w_{49} B_0^F = \emptyset \) for \( u \in E_7^F \), so we have

\[
m(u, w_{49}) = 0 \quad \text{for any } u \in E_7^F.
\]

Now let us consider the right hand side of (\dagger), for \( u \) and \( w_{49} \) as above. To do this, we thus need a detailed information on the values \( \chi_x(u) \) for \( x \in X(W) \) and \( u \in E_7^F \).

**Lemma 5.3.** Let \( x \in X(W) \), and assume that \( \chi_x(u) \neq 0 \) for some \( u \in G^F_{\text{uni}} \). Then there exists some \( i \in N_G \) such that \( A_x \) is isomorphic to \( A_i \).

**Proof.** By the description in \ref{rem:chi_x}, \( A_x \) is parametrised by some triple \((J, \phi, \zeta) \in \mathfrak{C}_{WJ} \), that is, \( A_x \) is (isomorphic to) a simple direct summand of \( \text{ind}_{L_J}^{G_J}(A_0) \) where \( A_0 \) corresponds to \( \zeta \in \mathfrak{C}_{WJ} \). Since \( A_0 \in L_J^0 \), there exists some cuspidal pair \((\Sigma, \mathcal{E})\) for \( L_J \) such that

\[
A_0 \cong \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma]^{\#L_J}
\]

(see \cite[Proposition 3.12]{Lusztig}, \cite[§2]{Lusztig}). In view of the cleanliness of cuspidal character sheaves (\cite[Theorem 23.1]{Lusztig}, \cite{Lusztig}), we know that \( A_0 \) is 0 outside of \( \Sigma \). Now we have

\[
\text{supp } A_x = \bigcup_{g \in G} g(\text{supp } A_0) R_0(\mathcal{P}_J) g^{-1},
\]

see, e.g., \cite[2.9]{Lusztig}. Thus, our assumption that \( \chi_x(u) \neq 0 \) for some \( u \in G^F_{\text{uni}} \) implies that \( \Sigma \cap (L_J)_{\text{uni}} \neq \emptyset \). By the definition of cuspidal pairs \cite[§2]{Lusztig}, \( \Sigma \) is of the form \( Z(L_J)^0 \mathcal{C} \) for
Then, for any \( u^0 \) is given in Table 3, with the following conventions: Let us consider a fixed line, and the associated perverse sheaf \( K_{\rho} \) we do not yet know which of \( \rho \) in \( \mathbf{L}_J \), \( \mathcal{O}_0 \rightarrow \mathcal{O}_0 \), that is, we have \( j := (\mathbf{L}_J, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G \). The associated perverse sheaf \( K_{\rho} \) (see [22]) is isomorphic to \( \text{ind}^G_{\mathbf{L}_J}(A_0) \) by [17, Prop. 4.5], so \( A_x \) must be isomorphic to \( A_i \) for some \( i \in \tau^{-1}(j) \subseteq \mathcal{N}_G \).

**Lemma 5.4.** Let \( i = (\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G = \mathcal{N}_G^F \) be such that \( \mathcal{O} \notin \{E_8, E_8(a_1), E_8(a_2), E_7 + A_1, E_7 \} \). Then, for any \( u \in E_7^F \), we have
\[
X_i(u) = 0.
\]

**Proof.** Let us assume that \( X_i(u) \neq 0 \). By Theorem 2.4 (and the definition of the \( Y_i \) for \( i' \in \mathcal{N}_G^F \)), there must exist some \( i' = (\mathcal{O}', \mathcal{E}') \in \mathcal{N}_G^F \) with \( \mathcal{O}' = E_7 \) and with \( p_{i',i} \neq 0 \) in Theorem 2.4(b). It then follows from 2.5 that we must have \( E_7 \subseteq \mathcal{O} \), which happens precisely for \( \mathcal{O} \in \{E_8, E_8(a_1), E_8(a_2), E_7 + A_1, E_7 \} \).

**5.5.** We can now proceed with the evaluation of the right hand side of the equation (\( \bigcirc \)) in [5.3] for any \( g = u \in E_7^F \) and with \( w = w_{40} \) as in [5.2]. In view of Lemma 5.3 and Lemma 5.4, we are reduced to considering those \( x \in X(W) \) for which \( A_x \cong A_i \) where \( i = (\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G = \mathcal{N}_G^F \) is such that \( \mathcal{O} \in \{E_8, E_8(a_1), E_8(a_2), E_7 + A_1, E_7 \} \). For such an \( i \), let \( j = \tau(i) = (\mathbf{L}_J, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G^F \), with \( J \subseteq \Pi \). Recall from [12] that there are three possibilities for \( J \), namely, \( J = \emptyset, J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \) or \( J = \Pi \). In the case where \( J = \Pi \), we have \( A_i \cong K_1 \), and this is a cuspidal character sheaf on \( G \). We necessarily have \( \mathcal{O} = \mathcal{O}_0 = E_7 + A_1 \), see [37, p. 337]. By the cleaness of cuspidal character sheaves (23.1), [28], we already know that \( X_i(u) = 0 \) (thus also \( \chi_x(u) = 0 \)) for \( u \in E_7^F \). So it remains to consider the following two cases (labelled as in 4.2):

(a) \( J = \emptyset \), that is, \( i \) is in the image of the ordinary Springer correspondence (2.2.4), so it is parametrised by some explicitly known \( \phi \in \text{Irr}(W) \); see 5.6 below.

(c) \( J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \); see 5.7 below.

A detailed description of the generalised Springer correspondence with respect to (a) and (c) is given in Table 3 with the following conventions: Let us consider a fixed line, and let \( \mathcal{O} \subseteq G \) be the unipotent class appearing in the first column in that line. We assume to have fixed a representative \( u \in \mathcal{O} \), and then \( d_u \) is the dimension of the variety of all Borel subgroups of \( G \) which contain \( u \). As for the notation \( a : b \) in the last two columns, \( a \) denotes the irreducible character of \( A_G(u) \) describing the local system on \( \mathcal{O} \), and \( b \) is the irreducible character of the relative Weyl group \( W^{I/J} \) such that \( b \) is mapped to \( (\mathcal{O}, a) \) under the generalised Springer correspondence. In the last column, we do not yet know which of \( \rho \) belongs to which place, only that \( \omega : \rho \) and \( \omega^2 : \rho \) both appear exactly once (and analogously for \( \rho' \)). We also provide the position of the corresponding unipotent character in CHEVIE [35] via (\( \bigodot \)) in Remark 4.4 as it appears in UnipotentCharacters(W), with the convention \( \omega = E_3 \).

**5.6.** Let us assume that \( i = (\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G^F \) is as in [5.3](a), so that \( j = \tau(i) = (T_0, \{1\}, 1) \). Let \( x \in X(W) \) be such that \( A_x \cong A_i \). By 2.3(iii), we know that whenever \( p_{i',i} \neq 0 \) for some \( i' \in \mathcal{N}_G^F \), we must have \( \tau(i') = j \). The only such \( i' = (\mathcal{O}', \mathcal{E}') \) with \( \mathcal{O}' = E_7 \) is \( i' = (E_7, 1) \). So for \( u \in E_7^F \), in order to express \( X_i(u) \) in terms of the \( Y_i(u) \), we
only need to know \( p_{i',i} \) with \( i' = (E_7,1) \). This is easily obtained using the CHEVIE \cite{CHEVIE} functions \texttt{UnipotentClasses} and \texttt{ICCTable}. (For any \( i \) as above, we have \( p_{i',i} \in \{0,1\} \).) In combination with Corollary \ref{cor:coefficients}, we can give the coefficient of \( \chi_x \) at \( Y_{i'} \); see Table \ref{table:coefficients}.

| \( \phi \in \text{Irr}(W) \) so that \( x = x_{\phi} \) \cite{Lusztig, Carter} | CHEVIE \cite{CHEVIE} | Coefficient of \( \chi_x \) at \( Y_{i'} \) |
|-------------------------|-----------------|-----------------|
| \( 1_x \) \hspace{1cm} \( \phi_{1,0} \) \hspace{1cm} \#1 \hspace{1cm} 1 \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |
| \( 8_z \) \hspace{1cm} \( \phi_{8,1} \) \hspace{1cm} \#68 \hspace{1cm} \( q \) \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |
| \( 35_x \) \hspace{1cm} \( \phi_{35,2} \) \hspace{1cm} \#5 \hspace{1cm} \( q^2 \) \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |
| \( 28_x \) \hspace{1cm} \( \phi_{28,8} \) \hspace{1cm} \#3 \hspace{1cm} \( 0 \) \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |
| \( 112_z \) \hspace{1cm} \( \phi_{112,3} \) \hspace{1cm} \#72 \hspace{1cm} \( q^3 \) \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |
| \( 84_x \) \hspace{1cm} \( \phi_{84,4} \) \hspace{1cm} \#10 \hspace{1cm} \( q^4 \) \hspace{1cm} | \hspace{1cm} \hspace{1cm} | \hspace{1cm} \hspace{1cm} |

Table 4. Coefficients of \( \chi_x \) at \( Y_{i'} \) for the relevant \( x = x_{\phi} \in X(W) \), where \( i' = (E_7,1) \). The CHEVIE column refers to the position of the corresponding unipotent principal series character in \texttt{UnipotentCharacters}(W).

5.7. Now let us consider the case (c) in \ref{section:5.5}. So let \( J = \{ \alpha_1, \alpha_2, \ldots, \alpha_6 \} \subseteq \Pi \), and let \( j = (L_J, O_0, E_0) \) be one of the two associated elements of \( \mathcal{M}_G^F \). Thus, \( L_J/Z(L_J)^{\circ} \) is the
simple (adjoint) group of type $E_6$, and $W'_j = W_G(L_j) \cong W^{\Pi/J} \cong W(G_2)$ is the dihedral group of order 12. We use the notation of $\mathbf{L}/\mathbf{G}$ for the irreducible characters of $W^{\Pi/J}$.

From now on we fix a primitive 3rd root of unity $\omega \in \mathbb{U}^\times$. Recall from $\mathbf{L}/\mathbf{G}$ that the above two elements $j$ of $\mathscr{M}_G$ are $(L_j, \mathcal{O}_0, \omega)$ and $(L_j, \mathcal{O}_0, \omega^2)$ (with the conventions in Remark 2.6), where $\mathcal{O}_0 \subseteq L_j$ is the regular unipotent class. The block of $N^F_G$ corresponding to $j = (L_j, \mathcal{O}_0, \omega)$ is given by

$$\tau^{-1}(j) = \{(E_8, \omega), (E_8(a_1), \omega), (E_7 + A_1, \omega), (E_7, \omega), (E_6 + A_1, \omega), (E_6, \omega)\}.$$ 

In view of Table 2, the generalised Springer correspondence with respect to this block is

$$1 \leftrightarrow (E_8, \omega), \quad \epsilon \leftrightarrow (E_8(a_1), \omega), \quad \epsilon' \leftrightarrow (E_6 + A_1, \omega), \quad \text{sgn} \leftrightarrow (E_6, \omega),$$

$$\theta' \leftrightarrow (E_7 + A_1, \omega), \quad \theta'' \leftrightarrow (E_7, \omega),$$

where $\{\theta', \theta''\} = \{\rho, \rho'\}$. So we get the following possibilities with respect to $j = (L_j, \mathcal{O}_0, \omega)$:

- $(\omega a) \rho \leftrightarrow (E_7 + A_1, \omega)$ and $\rho' \leftrightarrow (E_7, \omega)$, or
- $(\omega b) \rho \leftrightarrow (E_7, \omega)$ and $\rho' \leftrightarrow (E_7 + A_1, \omega)$.

Analogously, replacing $\omega$ by $\omega^2$ in the above discussion, one of the following holds with respect to $j = (L_j, \mathcal{O}_0, \omega^2)$:

- $(\omega^2 a) \rho \leftrightarrow (E_7 + A_1, \omega^2)$ and $\rho' \leftrightarrow (E_7, \omega^2)$, or
- $(\omega^2 b) \rho \leftrightarrow (E_7, \omega^2)$ and $\rho' \leftrightarrow (E_7 + A_1, \omega^2)$.

In order to prove Lusztig’s conjecture $\mathbb{L}$ (see $\mathbf{L}/\mathbf{G}$), we thus have to show that $(\omega a)$ and $(\omega^2 a)$ hold. We do this by excluding $(\omega b)$ and $(\omega^2 b)$.

5.8. We keep the setting and notation of 5.7. Thus, we have $j = (L_j, \mathcal{O}_0, \zeta_3) \in \mathcal{M}_G^F$ with $J = \{a_1, a_2, \ldots, a_6\}$ and $\zeta_3 \in \{\omega, \omega^2\}$. For $i \in \tau^{-1}(j)$, we want to evaluate $X_i$ at $u \in E^F$, so we need to know the coefficient $p_{\nu,i}$ in Theorem 2.4(b) where $i' = (E_7, \zeta_3) \in \tau^{-1}(j)$. But note that $p_{\nu,i}$ depends on whether $(\zeta_3 a)$ or $(\zeta_3 b)$ in 5.7 holds. Since the corresponding coefficient matrices $P_{(\zeta_3 a)}$, $P_{(\zeta_3 b)}$ (indexed according to whether $(\zeta_3 a)$ or $(\zeta_3 b)$ holds) only depend on $\gamma'_j \cong W(G_2)$ (and not on $G$ or $L_j$), we can just take them from [31 5.2]. They are given by

$$P_{(\zeta_3 a)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
q^2 & 0 & 1 & 0 & 0 & 0 \\
q^2 & q^2 & 1 & 1 & 0 & 0 \\
q^6 & 0 & q^4 & q^4 & 1 & 0 \\
q^6 & q^8 & q^4 + q^8 & q^4 + q^6 & 1 & 1
\end{pmatrix} \quad \leftrightarrow \quad (1, \epsilon, \rho, \rho', \epsilon', \text{sgn})$$
and

\[
P_{(\zeta,b)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & q & 1 & 0 & 0 & 0 \\
q^3 & q & 1 & 1 & 0 & 0 \\
q^6 & 0 & q^5 & q^3 & 1 & 0 \\
q^8 & q^5 + q^7 & q^6 & q^3 + q^7 & 1 & 1
\end{pmatrix} \quad \leftrightarrow \quad (1, \epsilon, \rho', \rho, \epsilon', \text{sgn}),
\]

where the vector on the right describes the ordering of the irreducible characters of \(W(G_2)\) corresponding to the rows and columns of \(P_{(\zeta,a)}\), \(P_{(\zeta,b)}\), respectively. Hence, for \(i' = (E_7, \zeta_3) \in \tau^{-1}(j)\) as above, the coefficients of the various \(X_i\) at \(Y_{i'}\) are the entries in the 4th line of either of the matrices \(P_{(\zeta,a)}\), \(P_{(\zeta,b)}\). Recall (3.9, 4.4) that the \(x \in X(W)\) corresponding to a given \(i \in \{(E_7 + A_1, \zeta_3), (E_7, \zeta_3)\}\) depends on whether \((\zeta_3,a)\) or \((\zeta_3,b)\) holds. Using Corollary 3.13 and the matrices \(P_{(\zeta,a)}, P_{(\zeta,b)}\), we find the coefficient of \(Y_{i'}\) in the relevant \(\chi_x\). These are given in Table 5 with the following conventions: In the first column, we describe \(x \in X(W)\) by giving the corresponding element of \(\mathcal{G}_W\) via Corollary 3.14 here, we write \(E_6\) instead of \(J = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}\). The second column contains the \textsc{chevie} number of the unipotent character \(\rho_x\). In the third column, the index of \(\mathcal{F}\) specifies that family by being the special character in it \([16\text{ (4.1.4)}]\), and we also give the label of \(\rho_x\) in this family. The \(\zeta_3\) in the last two columns is always meant to be the unique element in \(\{\omega, \omega^2\}\) which appears in the line considered.

| \(x \in X(W)\) via \(X(W) \cong \mathcal{G}_W\) | CHEVIE | Family:Label | Coefficient of \(\chi_x\) at \(Y_{i'}\) | Case \((\zeta_3,a)\) | Case \((\zeta_3,b)\) |
|---|---|---|---|---|---|
| \(E_6, 1, \omega\) | #142 | \(\mathcal{F}_{1400_2} : (g_3, \omega)\) | \(q^5\) | \(q^6\) |
| \(E_6, 1, \omega^2\) | #148 | \(\mathcal{F}_{1400_2} : (g_3, \omega^2)\) | \(q^5\) | \(q^6\) |
| \(E_6, \epsilon, \omega\) | #143 | \(\mathcal{F}_{1400_2} : (g_3, \omega)\) | \(q^5\) | \(q^6\) |
| \(E_6, \epsilon, \omega^2\) | #149 | \(\mathcal{F}_{1400_2} : (g_3, \omega^2)\) | \(q^5\) | \(q^6\) |
| \(E_6, \rho, \omega\) | #152 | \(\mathcal{F}_{4480_2} : (g_3, \omega)\) | \(q^6\) | \(q^7\) |
| \(E_6, \rho, \omega^2\) | #146 | \(\mathcal{F}_{4480_2} : (g_3, \omega^2)\) | \(q^6\) | \(q^7\) |
| \(E_6, \rho', \omega\) | #153 | \(\mathcal{F}_{4480_2} : (g_6, \omega)\) | \(q^7\) | \(q^6\) |
| \(E_6, \rho', \omega^2\) | #147 | \(\mathcal{F}_{4480_2} : (g_6, \omega^2)\) | \(q^7\) | \(q^6\) |

Table 5. Coefficients of \(\chi_x\) at \(Y_{i'}\) for the relevant \(x \in X(W)\), where \(i' = (E_7, \zeta_3)\) with \(\zeta_3 \in \{\omega, \omega^2\}\) the same as in the first and third column.

**Theorem 5.9.** In the setting of \(\mathcal{F}_7\) (\(\omega a\)) and (\(\omega^2 a\)) hold, that is, Lusztig’s conjecture \(\checkmark\) (see 4.8) is true.
Proof. For $x \in X(W)$ and $w \in W$, let us set
\[ c_x(w) := \sum_{\phi \in \text{Irr}(W)} \{x, x_{\phi}\} \text{Trace}(T_w, V_{\phi}). \]
Hence, with $w_{49} \in W$ as defined in (5.2.1), (♠) gives
\[ 0 = m(u, w_{49}) = \sum_{x \in X(W)} c_x(w_{49}) \nu_x \chi_x(u) \] for any $u \in E_F$.

In view of Lemma 5.4 and Table 3, we only need to look at the values $c_x(w_{49})$ for $x \in X(W)$ as in Table 4 and Table 5. The $c_x(w_{49})$ for those $x$ are given in Table 6.

| $x \in X(W)$ via $X(W) \cong G_W$ | CHEVIE | $c_x(w_{49})$ |
|---------------------------------|--------|-------------|
| $1_x$                           | #1     | $q^{14}$    |
| $8_x$                           | #68    | $q^{12}$    |
| $35_x$                          | #5     | $-q^{11}$   |
| $28_x$                          | #3     | 0           |
| $112_x$                         | #72    | 0           |
| $84_x$                          | #10    | $-q^{10}$   |
| $(E_6, 1, \omega)$              | #142   | $q^9$       |
| $(E_6, 1, \omega^2)$            | #148   | $q^9$       |
| $(E_6, \epsilon, \omega)$      | #143   | $-q^8$      |
| $(E_6, \epsilon, \omega^2)$    | #149   | $-q^8$      |
| $(E_6, \rho, \omega)$          | #152   | 0           |
| $(E_6, \rho, \omega^2)$        | #146   | 0           |
| $(E_6, \rho', \omega)$         | #153   | 0           |
| $(E_6, \rho', \omega^2)$       | #147   | 0           |

Table 6. $c_x(w_{49})$ for the relevant $x \in X(W)$

Now let us consider the elements $i' = (O', E') \in N_G^F$ with $O' = E_7$. For $u_0 \in E_7$, we have $A_G(u_0) = \langle \overline{m}_0 \rangle \cong \mathbb{Z}/3\mathbb{Z}$. Thus, with our conventions in Remark 2.6, there are three $i'$ as above: $(E_7, 1)$, $(E_7, \omega)$ and $(E_7, \omega^2)$. For $\varsigma \in \text{Irr}(A_G(u_0))$, let us set
\[ \gamma_\varsigma := Y_{(E_7, \varsigma)}|_{E_7^F} : E_7^F \to \overline{\mathbb{Q}}_\ell. \]

By Theorem 2.4 (and since, by definition, the $Y_{(E_7, \varsigma)}$ vanish outside of $E_7^F$), $\gamma_1$, $\gamma_\omega$ and $\gamma_{\omega^2}$ are linearly independent functions $E_7^F \to \overline{\mathbb{Q}}_\ell$. Using Tables 4, 5, 6 to evaluate (5.9.1) for the various $u \in E_7^F$ (depending on which two of $(\omega a)$, $(\omega b)$, $(\omega^2 a)$, $(\omega^2 b)$ hold), we obtain an expression of the zero function $E_7^F \to \overline{\mathbb{Q}}_\ell$ as a linear combination of $\gamma_1$, $\gamma_\omega$ and $\gamma_{\omega^2}$, as
their coefficients are independent of the chosen \( u \in E^F_7 \). Due to the linear independence of \( \gamma_1, \gamma_\omega, \gamma_\omega^2 \), we know that any of their coefficients must be 0. If \( n \in \{1, 2, \ldots, 166\} \) is the CHEVIE number of \( x \in X(W) \) as in Table 6, it will be convenient to write \( \chi_n, \nu_n \) instead of \( \chi_x, \nu_x \), respectively. Now let us assume, if possible, that \((\omega b)\) holds. Then the coefficient of \( \gamma_\omega \) in (5.9.1) is \( q^{15}\nu_{142} - q^{13}\nu_{143} \), and this must be 0. However, \( \nu_{142} \) and \( \nu_{143} \) are roots of unity and \( q \geq 3 \), a contradiction. So \((\omega a)\) must hold. Similarly, if we assume that \((\omega_2 b)\) holds, the coefficient of \( \gamma_\omega^2 \) in (5.9.1) is \( q^{15}\nu_{148} - q^{13}\nu_{149} = 0 \) which contradicts the fact that both \( \nu_{148} \) and \( \nu_{149} \) are roots of unity. So \((\omega_2 a)\) holds as well, as claimed. □

**Remark 5.10.** (a) Instead of evaluating the equation \((\star)\) in 5.1 at elements of \( E^F_7 \), we could have also considered the class \( E_7 + A_1 \subseteq G \) and, e.g., taken \( w \) as a Coxeter element of \( W \). This leads to a similar contradiction when Lusztig’s conjecture is assumed to be false.

(b) In contrast, if we evaluate \((\star)\) with the correct assumptions \((\omega a)\) and \((\omega_2 a)\) in 5.7 for \( u \in G_{\text{uni}}^F \) and \( w \in W \) such that \( m(u, w) = 0 \), we see that the coefficient of any fixed \( \gamma_\varsigma q^e \) \((\varsigma \in \text{Irr}(AG(u_0)), e \geq 0)\) on the right hand side of \((\star)\) will never consist of exactly one root of unity \( \nu_x \), \( x \in X(W) \), so we will (of course) not run into a contradiction of the type as in the proof of Theorem 5.9.

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