A scheme for unconventional geometric quantum computation in cavity QED

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We present a scheme for implementing the unconventional geometric two-qubit phase gate with nonzero dynamical phase by using the two-channel Raman interaction of two atoms in a cavity. We show that the dynamical phase acquired in a cyclic evolution is proportional to the geometric phase acquired in the same cyclic evolution, hence the total phase possesses the same geometric features as the geometric phase. In our scheme the atomic excited state is adiabatically eliminated and the operation of the proposed logic gate involves only in the metastable states of the atom and hence is not affected by spontaneous emission.

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I. INTRODUCTION

Quantum computation employs the principle of coherent superposition and quantum entanglement to solve certain problems, such as factoring large integers and searching data in an array, much faster than a classical computer. The basic building blocks of a quantum computer are quantum logic gates. It was shown that any quantum computation can be reduced to a sequence of two classes of quantum gates, namely, universal two-qubit logic gates and one-qubit local operations. The standard paradigm of quantum computation is the dynamical one where the local interactions between the qubits are controlled in such a way that one can enact a sequence of quantum gates. On the other hand, it has been recognized that the quantum gate operations can also be implemented through the geometric effects on the wave function of the systems, this is the so-called geometric quantum computation. Compared with the dynamical gates, the geometric quantum computation possesses practical advantages. It is well known that geometric phases depend only on some global geometric features, and do not depend on the details of the path, the time spent, the driving Hamiltonian, and the initial and final states of the evolution. Therefore the geometric quantum computation is largely insensitive to local inaccuracies and fluctuations, and thus provides us a possible way to achieve fault-tolerant quantum gates.

In the implementation of geometric quantum computation, one practical question we usually meet is how to remove or avoid the dynamical phases since geometric phases are generally accompanied by dynamical ones which are not robust against local inaccuracies and fluctuations. To this end one simple method is to choose the dark states as qubit space, thus the dynamical phase is always zero. Another general method is to let the evolution be dragged by the Hamiltonian along several special closed loops, then the dynamical phases accumulated in different loops may be canceled, with the geometric phases being added. This is the so-called multi-loop scheme.

The geometric quantum computation which is based on the cancelation of dynamical phases is referred to as conventional geometric quantum computation. Correspondingly several schemes have been presented recently to realize the so-called unconventional geometric quantum computation. The central idea of the unconventional geometric quantum computation is that for certain quantum evolution of a quantum system of interest one can implement fault-tolerant quantum computation by using the total phase accumulated in the evolution if it depends only on global geometric features of the evolution. In comparison with conventional geometric gates, unconventional geometric gates do not require additional operations to cancel the dynamical phases and thus simplify the realization operations. Schemes for implementing the unconventional geometric gate have been proposed in trapped ion systems and in cavity QED systems. In the schemes of cavity QED, the excited states are utilized as the computational bases, thus the spontaneous emission cannot be avoided in such schemes.

In this paper we make use of the two-channel Raman interaction in cavity QED to realize the unconventional geometric gate. In our scheme the atomic excited states are adiabatically eliminated and never excited during the quantum gate construction, therefore atomic spontaneous emission can be avoided in our scheme.

II. THEORETICAL MODEL OF TWO-CHANNEL RAMAN COUPLING IN A CAVITY

We consider two identical three-level atoms in Λ-configuration placed in a high-Q cavity. The level structure of the atoms is shown in Fig. where |e⟩, |g⟩,
\[ H(t) = \sum_{j=1}^{2} \left[ r(t)\sigma_j^+ + g^*(t)\alpha^j \right] + \sum_{j=1}^{2} \left[ g(t)\alpha \sigma_j^+ + g^*(t)\alpha^j \right], \]

where \( \sigma^+ = |e\rangle \langle g| \) and \( \sigma^- = |g\rangle \langle e| \) are atomic operators, \( r(t) \) and \( g(t) \) are respectively the effective classical and quantum couplings and they take the following form:

\[ r(t) = -\frac{\langle e| \vec{d}| g \rangle \cdot E_{p}(t)}{\delta_1}, \]
\[ g(t) = -\frac{\langle e| \vec{d}| g \rangle \cdot E_{g}(t)}{\delta_2}. \]

Here \( \langle i| \vec{d}| j \rangle \) \((i, j = g, e, c)\) denote the atomic dipole matrix elements. From above equations it is easy to note that the effective coupling parameters \( r(t) \) and \( g(t) \) can be controlled by adjusting the driving light fields. Based on such a feature of the Hamiltonian, a scheme to generate arbitrary quantum states of the cavity fields was proposed.

Now we further make the following transformation on the Hamiltonian:

\[ H_I = \exp(\imath H_0 t) \sum_{j=1}^{2} \left[ g(t)\alpha \sigma_j^+ + g^*(t)\alpha^j \right] \exp(-\imath H_0 t), \]

where \( H_0 = \sum_{j=1}^{2} \left[ r(t)\sigma_j^+ + r^*(t)\sigma_j^- \right]. \) For simplicity, we assume \( r(t) \) is real, then \( H_0 = r(t) \sum_{j=1}^{2} [\sigma_j^+ + \sigma_j^-]. \) After a simple calculation we obtain:

\[ H_I = \frac{1}{2} \sum_{j=1}^{2} \left[ |\pm\rangle_j \langle \pm| - |\mp\rangle_j \langle \mp| + e^{2\imath r(t)} |\pm\rangle_j \langle \pm| - e^{-2\imath r(t)} |\mp\rangle_j \langle \mp| \right] g(t)\alpha + H.c., \]

where \( |\pm\rangle_j = (|g\rangle_j \pm |e\rangle_j)/\sqrt{2} \) are eigenstates of \( \sigma_j^\pm = \sigma_j^x \pm \sigma_j^y \) with eigenvalues \( \pm 1 \), respectively. In the strong effective classical driving regime \( r(t) \gg |g| \), the terms in Eq. (5) which oscillate with high frequencies can be eliminated in the rotating-wave approximation, and Eq. (5) can thus be simplified as:

\[ H_{\text{eff}} = \frac{1}{2} \sum_{j=1}^{2} \left[ |\pm\rangle_j \langle \pm| - |\mp\rangle_j \langle \mp| \right] \left[ g(t)\alpha + g^*(t)\alpha^j \right], \]

\[ = \frac{1}{2} \left[ g(t)\alpha + g^*(t)\alpha^j \right] (\sigma_1^x + \sigma_2^y). \]

Similar Hamiltonians have been derived in the strongly driving Jaynes-Cummings model and the two-channel Raman interaction in cavity QED. In comparison with the trapped-ion model proposed to realize the unconventional geometric quantum computation, the Hamiltonian in our model is of a similar form to that of the trapped-ion model. In the next section we will show how to construct an unconventional geometric two-qubit phase gate based on the above Hamiltonian.

### III. UNCONVENTIONAL GEOMETRIC TWO-QUBIT PHASE GATE

We choose the eigenstates of \( \sigma_j^z (j = 1, 2) \), that is, \( |\pm\rangle_j = (|g\rangle_j \pm |e\rangle_j)/\sqrt{2} \), as the computational basis, so that the Hamiltonian will not give rise to any population changes in such a computational basis when the system is governed by the Hamiltonian. In the computational basis \( \{|+\rangle_1|+\rangle_2, |+\rangle_1|−\rangle_2, |−\rangle_1|+\rangle_2, |−\rangle_1|−\rangle_2\} \) the Hamiltonian is diagonal and takes the form:

\[ H_{\text{eff}} = \frac{1}{2} \left[ g(t)\alpha + g^*(t)\alpha^j \right] \times \text{diag}[\lambda_1++, \lambda_1+, \lambda_1-, \lambda_1--], \]

where

\[ \delta_1 = \frac{\langle e| \vec{d}| g \rangle \cdot E_{p}(t)}{\imath \omega_g}, \quad \delta_2 = \frac{\langle e| \vec{d}| g \rangle \cdot E_{g}(t)}{\imath \omega_g}. \]
where \(\lambda_{kl}(k, l = +, -)\) are the eigenvalues of \((\sigma_1^2 + \sigma_2^2)\) and \(\lambda_{\pm} = -\frac{1}{2}, \lambda_{+} = \lambda_{-} = 0\). The time evolution matrix \(U(t)\) is thus diagonal

\[
U(t) = \text{diag}[U_{++}(t), 1, 1, U_{--}(t)],
\]

where the diagonal matrix elements \(U_{kl}(t)\) can be derived from Eq. (7),

\[
U_{kl}(t) = \tilde{T} e^{-i\frac{1}{2} \lambda_{kl} \int_0^t [g(t)\alpha + g^*(t)\alpha^\dagger]dt},
\]

\[
= \lim_{N \to \infty} \prod_{n=1}^N e^{-i\frac{1}{2} \lambda_{kl} \int_0^t [g(t_n)\alpha + g^*(t_n)\alpha^\dagger]dt},
\]

\[
= \lim_{N \to \infty} \prod_{n=1}^N D[\Delta\alpha_{kl}(t_n)],
\]

where \(\tilde{T}\) is the time ordering operator, \(\Delta t = t/N\) is the time interval, and \(\Delta\alpha_{kl}(t_n) = -i\frac{1}{2} \lambda_{kl} g^*(t_n)\Delta t\). \(D(\alpha)\) is the displacement operator which takes the form \(D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a]\). The displacement operators satisfy the following relation

\[
D(\alpha)D(\beta) = e^{i\text{Im}(\alpha^* \beta)} D(\alpha + \beta).
\]

Based on the above formula, the Eq. (10) can be further simplified as

\[
U_{kl}(t) = e^{i\gamma_{kl}} D(\int_c \alpha_{kl}),
\]

with \(\gamma_{kl} = \text{Im}(\int_c \alpha_{kl}^\dagger d\alpha_{kl})\) and

\[
d\alpha_{kl} = -i\frac{1}{2} \lambda_{kl} g^*(t)dt.
\]

For a closed path \(c\), \(U_{kl}(t) = e^{i\gamma_{kl}} D(0) = e^{i\gamma_{kl}}\). Here \(\gamma_{kl}\) is the total phase acquired by the state \(|k\rangle_1|l\rangle_2\) \((k, l = +, -)\) in the cyclic evolution from \(t = 0\) to \(t = T\). The total phase \(\gamma_{kl}\) consists of two parts, one part is geometric phase \(\gamma_{kl}^g\), and the other part is the dynamical phase \(\gamma_{kl}^d\). According to the coherent-state path integral methods [10, 16, 17], the geometric phase \(\gamma_{kl}^g\) and the dynamical phase \(\gamma_{kl}^d\) can be calculated in the following way

\[
\gamma_{kl}^g = \frac{i}{2} \int_0^T (\alpha_{kl}^\dagger \dot{\alpha}_{kl} - \dot{\alpha}_{kl}^\dagger \alpha_{kl}) dt,
\]

\[
\gamma_{kl}^d = -\int_0^T H_{kl}(\alpha_{kl}^*, \alpha_{kl}; t) dt,
\]

with

\[
H_{kl}(\alpha_{kl}^*, \alpha_{kl}; t) = \langle \alpha_{kl}(t) | H_{kl}(t) | \alpha_{kl}(t) \rangle.
\]

From Eq. (14) we obtain

\[
\alpha_{kl}(t) = -i\frac{1}{2} \lambda_{kl} \int_0^t g^*(\tau) d\tau.
\]

Substituting Eq. (7) and Eq. (15) into Eq. (14), we get

\[
H_{kl}(\alpha_{kl}^*, \alpha_{kl}; t) = -\frac{i}{4} \lambda_{kl}^2 \left[ g(t) \int_0^t g^*(\tau) d\tau - g^*(t) \int_0^t g(\tau) d\tau \right],
\]

where for the sake of simplicity we have set \(G(t) = g(t) \int_0^t g^*(\tau) d\tau - g^*(t) \int_0^t g(\tau) d\tau\). With Eq. (15) and Eq. (16), the geometric phase \(\gamma_{kl}^g\) and the dynamical phase \(\gamma_{kl}^d\) can be calculated according to the formulas (12) and (13),

\[
\gamma_{kl}^g = \frac{i}{8} \lambda_{kl}^2 \int_0^T G(t) dt,
\]

\[
\gamma_{kl}^d = \frac{i}{4} \lambda_{kl}^2 \int_0^T G(t) dt,
\]

and the total phase is given by

\[
\gamma_{kl} = \gamma_{kl}^g + \gamma_{kl}^d = \frac{i}{8} \lambda_{kl}^2 \int_0^T G(t) dt.
\]

Comparing the above three equations, we have,

\[
\gamma_{kl} = \frac{1}{2} \gamma_{kl}^d = -\gamma_{kl}^g.
\]

The relations between the total phase \(\gamma_{kl}\), the dynamical phase \(\gamma_{kl}^d\) and the geometric phase \(\gamma_{kl}^g\) indicate that in the system examined here the total phase \(\gamma_{kl}\) and the dynamical phase \(\gamma_{kl}^d\) possess the global geometric features as the geometric phase \(\gamma_{kl}^g\) does. Therefore the cyclic evolution

\[
U(T) = \text{diag} [e^{i\gamma}, 1, 1, e^{i\gamma}]
\]

with \(\gamma = \frac{1}{4} \int_0^T G(t) dt\) is a two-qubit phase gate operation which is robust against some local inaccuracies and fluctuations, this gate is nontrivial if \(\gamma \neq 2n\pi\). As described in the preceding section, the effective coupling parameter \(g(t)\) can be controlled by adjusting the driving light field, so that the cyclic evolution and certain total phase \(\gamma = \frac{1}{4} \int_0^T G(t) dt\) can be achieved.

It is worth noting that in our scheme the atomic excited state is adiabatically eliminated and never populated. The quantum phase gate operation only involves atomic metastable states, therefore the effect of the spontaneous emission can be ignored.

**IV. CONCLUSIONS**

In this paper, we present a scheme for implementing the unconventional geometric two-qubit phase gate with nonzero dynamical phase by using the two-channel Raman interaction of two atoms in a cavity. We show that
the dynamical phase acquired in a cyclic evolution is propor-
tional to the geometric phase acquired in the same cyclic evolution, hence the total phase possesses the same geometric features as the geometric phase does. In our scheme the atomic excited state is adiabatically eliminated and the operation of the proposed logic gate involves only in the metastable states of the atom and hence is not affected by spontaneous emission.

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