UNIQUENESS OF EQUIVARIANT HARMONIC MAPS TO
SYMMETRIC SPACES AND BUILDINGS

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Abstract. We prove uniqueness of equivariant harmonic maps into irreducible symmetric spaces of non-compact type and Euclidean buildings associated to isometric actions by Zariski dense subgroups.

1. Introduction

Assume that $M$ and $N$ are Riemannian manifolds, $M$ has finite volume and $N$ has non-positive sectional curvature. Hartman [Ha] proved the following uniqueness result for harmonic maps: Let $u : M \to N$ be a finite energy harmonic map of rank greater at 1 at some point $p \in M$. If $N$ has negative sectional curvature at $u(p)$, then $u$ is the only harmonic map in its homotopy class (cf. [Ha, Corollary following (H)]). The second author [Me] generalized Hartman’s uniqueness result to the case when the target space is a geodesic metric space $\tilde{X}$ with curvature $< 0$ in the sense of Alexandrov. On the other hand, if there exists a 2-plane in $T_{u(p)}N$ with sectional curvature 0 for all $p \in M$, then uniqueness fails. For example in the extreme case, when $N$ is a flat torus, then there exists a family of harmonic maps obtained by translations of a given harmonic map.

Analogous uniqueness statements hold for equivariant harmonic maps. More precisely, let $\rho : \pi_1(M) \to \text{Isom}(\tilde{X})$ be a homomorphism into the isometry group of an NPC space $\tilde{X}$ and $f$ be a $\rho$-equivariant map (cf. Definition 2.10). Using the same principle as in the homotopy problem, a finite energy $\rho$-equivariant harmonic map $\tilde{u} : \tilde{M} \to \tilde{X}$ is unique provided $\tilde{u}$ has rank greater than 1 at some point $p \in \tilde{M}$ and $\tilde{X}$ has negative curvature at $\tilde{u}(p)$.

In this note, we study uniqueness for equivariant harmonic maps into irreducible symmetric spaces of non-compact type and Euclidean buildings. We will assume that Euclidean buildings are locally finite simplicial complexes. However, we conjecture that a similar uniqueness result holds in the case of non-locally finite thick Euclidean buildings (cf. Remark 3.6). The importance of the latter case is that limits of symmetric spaces of non-compact type are thick Euclidean buildings (cf. [KL1]). This is important in the study of the compactification of character varieties and higher Teichmüller theory.

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Symmetric spaces of non-compact type (resp. Euclidean buildings) are examples of Riemannian manifolds of non-positive sectional curvature (resp. NPC spaces or complete CAT(0) spaces). Harmonic maps into Riemannian manifolds of non-positive sectional curvature and NPC spaces have been important in the study of geometric rigidity problems (e.g. [Si], [Co1], [GS], [JY], [MSY], [DMV] among many others). The uniqueness of harmonic maps into symmetric spaces (resp. Euclidean buildings) does not follow from [Ha] (resp. [Me]) unless \( \tilde{X} \) has rank 1 (resp. \( \tilde{X} \) is a \( \mathbb{R} \)-tree). Indeed, every point \( P \) in a rank \( n \) symmetric space \( X \) (resp. \( n \)-dimensional Euclidean building) is contained in a convex, isometric embedding of \( \mathbb{R}^n \). The novelty of this paper is that the uniqueness is proven, not with the assumption on the curvature bound as in [Ha] and [Me], but with an assumption on the homomorphism \( \rho : \pi_1(M) \to \text{Isom}(\tilde{X}) \).

The main theorem of this paper is the following:

**Theorem 1.1 (Existence and Uniqueness).** Let \( M \) be a Riemannian manifold with finite volume, \( \tilde{X} \) be an irreducible symmetric space of non-compact type, and \( \rho : \pi_1(M) \to \text{Isom}(\tilde{X}) \) a homomorphism. Assume:

(i) The subgroup \( \rho(\pi_1(M)) \) does not fix a point at infinity.

(ii) There exists a finite energy \( \rho \)-equivariant map \( \tilde{f} : \tilde{M} \to \tilde{X} \).

Then there exists a unique finite energy \( \rho \)-equivariant harmonic map \( \tilde{u} : \tilde{M} \to \tilde{X} \).

The same conclusion holds if \( \tilde{X} \) is a locally finite Euclidean building with the additional assumption that the action of \( \rho(\pi_1(M)) \) does not fix a non-empty closed convex strict subset of \( \tilde{X} \).

The existence results for harmonic maps is contained in (e.g. [L], [Do], [Co1], [GS], [J], [KS2], [KS3]). Thus, the goal of this paper is to prove the uniqueness assertion in Theorem 1.1.

The assumptions on the subgroup \( \rho(\pi_1(M)) \) in Theorem 1.1 are related to the notion of Zariski dense. Indeed, in either the case when \( \tilde{X} \) is a symmetric space of non-compact type or an Euclidean building, if the action of the subgroup \( \Gamma \) of \( \text{Isom}(\tilde{X}) \) neither fixes a point at infinity nor a non-empty closed convex strict subset, then \( \Gamma \) is Zariski dense (cf. [CaMo, Proposition 2.8]). The converse also holds if \( \tilde{X} \) is a symmetric spaces of non-compact type and \( \text{rank}(\tilde{X}) \geq 2 \) (cf. [KL2, Theorem 4.1]), but there exist Zariski dense subgroups that fixes a non-empty closed convex strict subset if \( \text{rank}(\tilde{X}) = 1 \) (cf. [Ca, Section 4]).

**Remark 1.2.** For the case when \( \tilde{X} = G/K \) is a symmetric space, Theorem 1.1 may be deduced from the gauge theoretic approach due to Donaldson [Do] and Corlette [Co2]. Indeed, harmonic maps to symmetric spaces can be thought of as a solution to Hitchin’s equations and uniqueness follows along the lines of [Co2, Proposition 2.3]. The point of this paper is to provide a simple geometric proof of the uniqueness of harmonic maps that works for Euclidean buildings as well.

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2. Preliminaries

We start with some definitions. We will assume that $\tilde{X}$ is a complete metric space.

**Definition 2.1.** A geodesic $\sigma : I \to \tilde{X}$ is a map from an interval $I \subset \mathbb{R}$ such that $d(\sigma(s), \sigma(s + t)) = |t|$ for all $s, t \in I$. A geodesic line, geodesic ray and geodesic segment are geodesics with domain $\mathbb{R}$, $[0, \infty)$ and closed interval $[a, b]$ respectively.

**Definition 2.2.** Geodesics $\sigma : I \to \tilde{X}$ and $\hat{\sigma} : I \to \tilde{X}$ are said to be parallel if there exists a constant $C > 0$ such that
$$d(\sigma(s), \hat{\sigma}(s)) = C, \forall s \in I.$$  

**Remark 2.3.** A geodesic rays $\sigma : [0, \infty) \to \tilde{X}$ and $\bar{\sigma} : [0, \infty) \to \tilde{X}$ are asymptotic if there exists a constant $C > 0$ such that
$$d(\sigma(s), \bar{\sigma}(s)) \leq C, \forall s \in \mathbb{R} \ (\text{resp. } \forall s \in [0, \infty)).$$

By [BH, II.2.13], the terms parallel geodesic rays and asymptotic geodesic rays are equivalent.

**Definition 2.4.** A point at infinity is an asymptotic class of geodesic rays. We denote by $[\sigma]$ the asymptotic class containing the geodesic ray $\sigma$.

**Definition 2.5.** A symmetric space $\tilde{X}$ is a Riemannian manifold such that, for any $P \in \tilde{X}$, there exists $S_P \in \text{Isom}(\tilde{X})$ such that $P$ is an isolated fixed point of $S_P$ and $S_P \circ S_P$ is the identity map. The isometry $S_P$ is called an inversion symmetry at $P$.

**Definition 2.6.** Given a geodesic line $\sigma : \mathbb{R} \to \tilde{X}$ and $s \in \mathbb{R}$, the composition $T_s = S_{\sigma(s)} \circ S_{\sigma(0)}$ is called a transvection along $\sigma$. We have that
$$T_{s+s'} = T_s \circ T_{s'}$$
and $\{T_s\}$ forms a one-parameter subgroup of $\text{Isom}(\tilde{X})$ that act as parallel transports along $\sigma$ (cf. [Eb, 2.1.1]).

**Definition 2.7** (cf. [BH] Definition 10A.1). A Euclidean building of dimension $n$ is a piecewise Euclidean simplicial complex $\tilde{X}$ such that:

1. $\tilde{X}$ is the union of a collection $\mathcal{A}$ of subcomplexes $A$, called apartments, such that the intrinsic metric $d_A$ on $A$ makes $(A, d_A)$ isometric to the Euclidean space $\mathbb{R}^n$ and induces the given Euclidean metric on each simplex.

2. Any two simplices $B$ and $B'$ of $X$ are contained in at least one apartment.

3. Given two apartments $A$ and $A'$ containing both simplices $B$ and $B'$, there is a simplicial isometry from $(A, d_A)$ to $(A', d_{A'})$ which leaves both $B$ and $B'$ pointwise fixed.

Furthermore, will assume
(4) \( \tilde{X} \) is locally finite.

**Definition 2.8.** A symmetric space of non-compact type \( \tilde{X} \) (resp. a Euclidean building) is said to be irreducible if it is not isometric to a non-trivial product \( \tilde{X}_1 \times \tilde{X}_2 \) of two symmetric spaces of non-compact type (resp. Euclidean buildings).

**Notation 2.9.** Given \( P, Q \in \tilde{X} \) and \( s \in \mathbb{R} \), we denote 
\[
(1 - s)P + sQ
\]
to be the geodesic interpolation between \( P \) and \( Q \); i.e. 
\[
(1 - s)P + sQ = \tilde{\sigma}(\delta s) \text{ where } \delta = d(P, Q) \text{ and } \tilde{\sigma} : [0, \delta] \to \tilde{X} \text{ is a geodesic segment with } \tilde{\sigma}(0) = P \text{ and } \tilde{\sigma}(\delta) = Q.
\]

**Definition 2.10.** Let \( \text{Isom}(\tilde{X}) \) be the group of isometries of \( \tilde{X} \) and \( \rho : \pi_1(M) \to \text{Isom}(\tilde{X}) \) be a homomorphism from the fundamental group of a Riemannian manifold \( M \). Let \( \pi_1(M) \) act on the universal cover \( \tilde{M} \) of \( M \) by deck transformations. A map \( \tilde{u} : \tilde{M} \to \tilde{X} \) is said to be \( \rho \)-equivariant if 
\[
\rho(\gamma)\tilde{u}(p) = \tilde{u}(\gamma p), \quad \forall \gamma \in \pi_1(M), \ p \in \tilde{M}
\]
where we write \( gP \) for \( g \in \text{Is}(\tilde{X}) \) and \( P \in \tilde{X} \) instead of \( g(P) \) for simplicity.

If \( \tilde{X} \) is a Riemannian manifold, then \( |d\tilde{f}|^2 \) is the norm of the differential \( d\tilde{f} : T\tilde{M} \to T\tilde{X} \). If \( \tilde{X} \) is a NPC space, then \( |d\tilde{f}|^2 \) is the energy density function in the sense of [KS1]. Either way, if \( \tilde{f} \) is \( \rho \)-equivariant, then \( |d\tilde{f}|^2 \) is invariant under the action of \( \rho(\gamma) \) for any \( \gamma \in \pi_1(M) \), and the energy of \( \tilde{f} \) is defined to be 
\[
E_{\tilde{f}} = \int_M |d\tilde{f}|^2 d\text{vol}_M.
\]

**3. Proof of Theorem 1.1**

The existence results for harmonic maps is contained in (e.g. [L], [Do], [Co1], [GS], [J], [KS2], [KS3]). Thus, we need to only prove the uniqueness assertion.

3.1. Geodesic interpolation. We assume on the contrary that there exist two distinct \( \rho \)-equivariant harmonic maps 
\[
\tilde{u}_0 : \tilde{M} \to \tilde{X} \text{ and } \tilde{u}_1 : \tilde{M} \to \tilde{X}.
\]
Using Notation 2.9, define the geodesic interpolation of \( \tilde{u}_0 \) and \( \tilde{u}_1 \); i.e. 
\[
\tilde{u}_s : \tilde{M} \to \tilde{X}, \quad \tilde{u}_s(q) = (1 - s)\tilde{u}_0(q) + s\tilde{u}_1(q).
\]
Since \( \tilde{u}_0 \) and \( \tilde{u}_1 \) are \( \rho \)-equivariant, \( \tilde{u}_s \) is also \( \rho \)-equivariant. By the convexity of energy (cf. [KS1, (2.2vi)]), 
\[
E_{\tilde{u}_s} \leq (1 - s)E_{\tilde{u}_0} + sE_{\tilde{u}_1} - s(1 - s)\int_M |\nabla d(\tilde{u}_0, \tilde{u}_1)|^2 d\text{vol}_M
\]
Lemma 3.1. Scaling if necessary, assume $d(\tilde{u}_0(p_0), \tilde{u}_1(p_0)) = 1$ for some point $p_0 \in \tilde{M}$. Then, for $\tilde{u}_s$ defined above, we have the following:

- $d(\tilde{u}_s(p), \tilde{u}_1(p)) = s$, $\forall p \in \tilde{M}$
- $|\tilde{u}_s(V)|^2(p) = |\tilde{u}_0(V)|^2(p)$, for a.e. $s \in [0, 1]$, $p \in \tilde{M}$, $V \in T_p\tilde{M}$.

Proof. Since $\tilde{u}_0$ and $\tilde{u}_1$ are energy minimizing, we conclude

$$0 = \int_{\tilde{M}} |\nabla d(\tilde{u}_0, \tilde{u}_1)|^2 d\text{vol}_M$$

(3.1)

$$E^{\tilde{u}_s} = E^{\tilde{u}_0}, \forall s \in [0, 1]$$

(3.2)

First, (3.1) implies that $\nabla d(\tilde{u}_0, \tilde{u}_1) = 0$ a.e. in $\tilde{M}$. Hence, $d(\tilde{u}_0, \tilde{u}_1)$ is constant; i.e.

$$d(\tilde{u}_0, \tilde{u}_1) \equiv 1.$$ (3.3)

For each $q \in \tilde{M}$, define the geodesic segment

$$\tilde{\sigma}_q : [0, 1] \to \tilde{X}, \quad \tilde{\sigma}_q(s) = \tilde{u}_s(q).$$

(3.4)

Note that equality (3.2) implies that

$$|(\tilde{u}_s)_*(V)|^2(p) = |(\tilde{u}_0)_*(V)|^2(p), \quad \text{for a.e. } s \in [0, 1], \; p \in \tilde{M}, \; V \in T_p\tilde{M}.$$ (3.5)

Indeed, for $\{P, Q, R, S\} \subset \tilde{X}$, the quadrilateral comparison for NPC spaces implies

$$d^2(P_s, Q_s) \leq (1-s)d^2(P, Q) + sd^2(R, S)$$

where $P_s = (1-s)P + sR$ and $Q_s = (1-s)Q + sR$. Applying the above inequality with $P = \tilde{u}_0(p), \; S = \tilde{u}_1(p), \; R = \tilde{u}_1(\exp_p(tV))$ and $Q = \tilde{u}_0(\exp_p(tV))$ where $t > 0$ and $V \in T_p\tilde{M}$, dividing by $t^2$ and letting $t \to 0$, we obtain (cf. [KS1, Theorem 1.9.6])

$$|(\tilde{u}_s)_*(V)|^2(p) \leq (1-s)|\tilde{u}_0(V)|^2(p) + s|\tilde{u}_1(V)|^2(p), \quad \text{a.e. } p \in \tilde{M}, \; V \in T_p\tilde{M}.$$ (3.5)

Integrating the above over all unit vectors $V \in T_p\tilde{M}$ and then over $p \in F$, we obtain

$$E^{\tilde{u}_s} \leq (1-s)E^{\tilde{u}_0} + sE^{\tilde{u}_1}.$$ (3.5)

Combining this with (3.2) implies (3.5). □

Note that up to this point, we have only used the fact that $\tilde{X}$ is an NPC space. We will now specialize to the two cases: (i) $\tilde{X}$ is an irreducible symmetric space of non-compact type and (ii) $\tilde{X}$ is an irreducible Euclidean building.
3.2. Symmetric spaces. Throughout this subsection \( \tilde{X} \) is an irreducible symmetric space of non-compact type. For each \( q \in \tilde{M} \), extend the geodesic segment \( \tilde{\sigma}_q \) of (3.4) to a geodesic line

\[
\sigma_q : \mathbb{R} \to \tilde{X}.
\]

Let

\[
F : \tilde{M} \times \mathbb{R} \to \tilde{X}, \quad F(q, s) = \sigma_q(s).
\]

Let \( \nabla^{F^{-1}} \) be the induced connection on the vector bundle

\[
(T^* (M \times \mathbb{R}))^\otimes k \otimes F^{-1} T\tilde{X} \to \tilde{M} \times \mathbb{R}.
\]

For each \( s \in [0, 1] \), let \( \nabla^{\tilde{u}^{-1}} \) be the induced connection on the vector bundle

\[
(T^* M)^\otimes k \otimes \tilde{u}_s^{-1} T\tilde{X} \to \tilde{M}.
\]

Use the inclusion \( \tilde{M} \to \tilde{M} \times \{s\} \) and the identity \( F(\cdot, s) = \tilde{u}_s(\cdot) \) to identify (3.8) as a subbundle of (3.7).

Let \( s, \partial_s \) denote the standard coordinate and coordinate vector on \( \mathbb{R} \). Let \((E_1, \ldots, E_n)\) denote a local orthonormal frame of \( \tilde{M} \). Set

\[
V = dF (\partial_s), \quad X_\alpha = dF (E_\alpha) \text{ for } \alpha = 1, \ldots, n
\]
as sections of \( F^{-1} T\tilde{X} \). Applying the usual second variation formula of the energy (e.g. [ES], [Sc]), we obtain

\[
\left. \frac{d^2}{ds^2} \right|_{s=t} E_{\tilde{u}_s}(r) = 2 \int_{\tilde{M}} \sum_{\alpha=1}^n \left( \| \nabla^{F^{-1}} V \|^2 - \left\langle R^{\tilde{X}} (V, X_\alpha) V, X_\alpha \right\rangle \right) \left. d\text{vol}_{\tilde{M}} \right|_{s=t}
\]
for any \( t \in [0, 1] \) where \( R^{\tilde{X}} \) is the Riemannian curvature operator of \( \tilde{X} \). By (3.2), the left hand side of the above equality is equal to 0. The integrand on the right hand side is non-positive by the assumption of non-positive curvature. Thus, we conclude that for any \( \alpha = 1, \ldots, n \),

\[
\nabla^{F^{-1}} V \equiv 0
\]

\[
\left\langle R^{\tilde{X}} (V, X_\alpha) V, X_\alpha \right\rangle \equiv 0.
\]

From the above, we conclude

\[
\nabla^{F^{-1}} \left( d\tilde{u}_s(E_\alpha) \right) = \nabla^{F^{-1}} X_\alpha = 0
\]

and

\[
\nabla^{\tilde{u}^{-1}} \nabla^{F^{-1}} \nabla^{\tilde{u}^{-1}} = \nabla^{\tilde{u}^{-1}} \nabla^{F^{-1}}, \quad \forall \beta = 1, \ldots, n.
\]

Since \( \nabla_{\partial_s} E_\alpha = \nabla_{\partial_s} E_\beta = 0 \), we have

\[
\left( \nabla^{F^{-1}} \nabla^{\tilde{u}^{-1}} d\tilde{u}_s \right) (E_\alpha, E_\beta)
\]
Furthermore, define (3.12)

\[ \nabla_{\partial_s}^{-1} \left( \nabla_{\partial_s}^{-1} d\tilde{u}_s(E_\alpha, E_\beta) \right) - \nabla_{\partial_s}^{-1} d\tilde{u}_s(\nabla_{\partial_s} E_\alpha, E_\beta) - \nabla_{\partial_s}^{-1} d\tilde{u}_s(E_\alpha, \nabla_{\partial_s} E_\beta) \]

More generally, we can inductively use (3.11) multiple times to switch the order of differentiation and apply (3.10) to conclude

\[ \nabla_{\partial_s}^{-1} \left( \nabla_{\partial_s}^{-1} \cdots \nabla_{\partial_s}^{-1} d\tilde{u}_s \right) = 0. \]

**Claim 3.2.** Fix a point \( p \in \tilde{M} \) and let \( T_s \in \text{Isom}(\tilde{X}) \) be the transvection along \( \sigma_p \) as in Definition 2.6. Then

\[ \tilde{u}_s = T_s \tilde{u}_0, \quad \forall s \in [0, 1]. \]

**Proof.** For \( s \in [0, 1] \), define a harmonic map

\[ \tilde{v}_s : \tilde{M} \to \tilde{X}, \quad \tilde{v}_s = T_s \tilde{u}_0. \]

Define

\[ \Phi : \tilde{M} \times [0, 1] \to \tilde{X}, \quad \Phi(q, s) = \tilde{v}_s(q). \]

Since \( T_s \) is a transvection along the geodesic \( \sigma_p \),

\[ F(p, s) = \tilde{u}_s(p) = \sigma_p(s) = T_s \sigma_p(0) = T_s \tilde{u}_0(p) = \tilde{v}_s(p) = \Phi(p, s). \]

Furthermore, \( T_s \) defines a parallel transport along \( \sigma_p \), and thus

\[ \nabla_{\partial_s}^{\Phi^{-1}} (d\tilde{v}_s(E_\alpha)) = \nabla_{\partial_s}^{\Phi^{-1}} (dT_s(d\tilde{u}_0(E_\alpha))) = 0 \text{ at } (p, s), \quad \forall s \in (0, 1). \]

By (3.10) and (3.13), the vector fields \( d\tilde{u}_s(E_\alpha) \) and \( d\tilde{v}_s(E_\alpha) \) are both parallel along \( \sigma_p(s) \). Since \( d\tilde{u}_0(E_\alpha) = d\tilde{v}_0(E_\alpha) \) at \( p \), we conclude

\[ d\tilde{u}_s(E_\alpha) = d\tilde{v}_s(E_\alpha) \text{ at } p \in \tilde{M}, \quad \forall s \in [0, 1]. \]

Next, since \( T_s \) is an isometry,

\[ \nabla_{E_\alpha}^{\tilde{v}_s^{-1}} (d\tilde{v}_s(E_\beta)) = \nabla_{E_\alpha}^{\tilde{v}_s^{-1}} (dT_s \circ d\tilde{u}_0(E_\beta)) \]

Thus,

\[ \nabla_{E_\alpha}^{\tilde{v}_s^{-1}} d\tilde{v}_s(E_\alpha, E_\beta) = \nabla_{E_\alpha}^{\tilde{v}_s^{-1}} (d\tilde{v}_s(E_\beta)) - d\tilde{v}_s \left( \nabla_{E_\alpha}^{\tilde{M}} E_\beta \right) \]

\[ = dT_s \left( \nabla_{E_\alpha}^{\tilde{u}_0^{-1}} d\tilde{u}_0(E_\beta) \right) - dT_s \left( d\tilde{u}_0 \left( \nabla_{E_\alpha}^{\tilde{M}} E_\beta \right) \right). \]
Since $T_s$ defines a parallel transport along $\tilde{\sigma}_p(s)$, both vector fields on the right hand side are parallel along $\sigma_p(s)$. Thus,
\[
\nabla_{\partial_s}^{-1} \left( \nabla_{\tilde{v}}^{-1} d\tilde{v}(E_\alpha, E_\beta) \right) = 0.
\]
Continuing inductively, we can prove
\[
\nabla_{\partial_s}^{-1} \left( \nabla_{\tilde{v}}^{-1} \cdots \nabla_{\tilde{v}}^{-1} d\tilde{v} \right) = 0.
\]
Combined with (3.12) and the fact that $\tilde{u}_0 = \tilde{v}_0$, we conclude
\[
\nabla_{\tilde{v}}^{-1} \cdots \nabla_{\tilde{v}}^{-1} d\tilde{v} = \nabla_{\tilde{v}}^{-1} \cdots \nabla_{\tilde{v}}^{-1} d\tilde{v} \text{ at } p, \forall s \in [0, 1].
\]
In other words, $\tilde{u}_s$ and $\tilde{v}_s$ agree up to infinitely high order at $p$ which in turn implies that $\tilde{u}_s = \tilde{v}_s = T_s \tilde{u}_0$ by [Sa, Theorem 1].

**Claim 3.3.** Let $p$ be the point fixed in Claim 3.2 and let the geodesic ray $\sigma_q : [0, \infty) \to \tilde{X}$ be the restriction of the geodesic line defined in (3.6). Then
\[
d(\sigma_q(s), \sigma_p(s)) = \delta_{p,q}, \quad \forall q \in \tilde{M}, \ s \in [0, \infty)
\]
where $\delta_{p,q} := d(\sigma_q(0), \sigma_p(0))$. In particular, $\sigma_q$ is the unique geodesic ray parallel to $\sigma_p$ with value at $s = 0$ equal to $\tilde{u}_0(q)$.

**Proof.** As above, let $T_s$ be the transvection along $\sigma_p$. By Claim 3.2, $\tilde{u}_s(q) = T_s \tilde{u}_0(q)$ for $s \in [0, 1]$. Since
\[
T_s \tilde{u}_0(q) = (T_{\frac{s}{2}} \circ T_{s-\frac{1}{2}}) \tilde{u}_0(q) = T_{\frac{s}{2}} \sigma_q(s - \frac{1}{2}), \quad \forall s \in \left[ \frac{1}{2}, \frac{3}{2} \right],
\]
the restriction of $s \mapsto T_s \tilde{u}_0(q)$ to $\left[ \frac{1}{2}, \frac{3}{2} \right]$ is a geodesic segment. Using an analogous argument, we can inductively show that for any $n \in \mathbb{N}$, the restriction to $\left[ \frac{n}{2}, \frac{n+1}{2} \right]$ of the map $s \mapsto T_s \tilde{u}_0(q) = (T_{\frac{s}{2}} \circ T_{s-\frac{1}{2}}) \tilde{u}_0(q)$ is a geodesic segment. Thus, we conclude that $s \mapsto T_s \tilde{u}_0(q)$ is a geodesic ray with $T_s \tilde{u}_0(q) = \sigma_q(s)$ for $s \in [0, 1]$. Since the two geodesic rays $s \mapsto T_s \tilde{u}_0(q)$ and $s \mapsto \sigma_q(s)$ agree on $[0, 1]$, they are the same geodesic ray. Since $T_s$ is an isometry,
\[
d(\sigma_q(s), \sigma_p(s)) = d(T_s \tilde{u}_0(q), T_s \tilde{u}_0(p)) = d(\tilde{u}_0(q), \tilde{u}_0(p)) = d(\sigma_0(q), \sigma_0(p)).
\]

For $Q = \tilde{u}_0(q)$, let $\sigma^Q = \sigma_q$. By Claim 3.3, there exists map from $\tilde{u}_0(M)$ to a family of pairwise parallel geodesic lines given by
\[
Q \mapsto \sigma^Q.
\]
Since $\sigma^Q = \sigma_q$ and $\sigma^{p(\gamma)Q} = \sigma_{\gamma q}$ are extensions of $\sigma_q$ and $\sigma_{\gamma q}$ and
\[
\rho(\gamma) \sigma_q(s) = \rho(\gamma) \tilde{u}_s(q) = \tilde{u}_s(\gamma q) = \sigma_{\gamma q}(s), \quad \forall s \in [0, 1],
\]
we have
\[
\rho(\gamma) \sigma^Q = \sigma^{p(\gamma)Q}, \quad \forall \gamma \in \pi_1(M).
\]
Since $\sigma^Q$ and $\sigma^{\rho(\gamma)Q}$ are parallel geodesic rays, we conclude that that
\[
\rho(\gamma)[\sigma_q] = [\sigma_q], \quad \gamma \in \pi_1(M).
\]
In other words, $\rho(\pi_1(M))$ fixes a point at infinity, contradicting assumption (i).

### 3.3. Euclidean buildings.
Throughout this subsection, $\bar{X}$ is an irreducible locally finite Euclidean building of dimension $n$. An open $n$-dimensional simplex of $\bar{X}$ will be referred to as a chamber. An apartment of $\bar{X}$ is a convex isometric embedding of $\mathbb{R}^n$ in $\bar{X}$.

Let $\tilde{u}_s$ be the geodesic interpolation maps defined in §3.1. The regular set $\mathcal{R}(\tilde{u}_s)$ is the set of all points $q \in M$ with the following property: There exists a neighborhood $\mathcal{U}_q$ of $q$ such that $\tilde{u}_s(\mathcal{U}_q)$ is contained in an apartment $A_q$ of $\bar{X}$.

**Theorem 3.4** ([GS] Theorem 6.4). The singular set $\mathcal{S}(\tilde{u}_s)$, i.e. the complement of $\mathcal{R}(\tilde{u}_s)$, is a closed set of Hausdorff codimension at least 2.

For each $q \in \mathcal{M}$, let $\mathcal{R}^*(\tilde{u}_s)$ be the set of points in $\mathcal{R}(\tilde{u}_s)$ such that
\[
\exists \epsilon > 0 \text{ and a chamber } C^* \text{ such that } \tilde{u}_s(B_q(\epsilon)) \subset \mathcal{C}^*.
\]
After identifying $A \simeq \mathbb{R}^n$, $\tilde{u}_s|_{\mathcal{U}_q}$ is a harmonic map into Euclidean space, and it follows that the set $\mathcal{U}_q \setminus \mathcal{R}^*(\tilde{u}_s)$ is a closed set of codimension at least 1. Thus, $\mathcal{R}^*(u_q)$ is a closed set of codimension 1.

For each $q \in \mathcal{M}$, let $\tilde{\sigma}_q(s) = \tilde{u}_s(q)$ (cf. (3.4)) and denote by $R_q$ the set of all points $s \in [0, 1]$ such that
\[
\exists \epsilon > 0 \text{ and a chamber } C \text{ such that } \tilde{\sigma}_q((s - \epsilon, s + \epsilon)) \subset C.
\]

The complement of $R_q$ in $[0, 1]$ is a finite set. Thus, the complement of $\mathcal{R}^{**} = \{(q, s) : q \in \mathcal{R}^*(\tilde{u}_s), s \in R_q\}$ in $\mathcal{M} \times [0, 1]$ is an closed set of measure 0.

Fix $(q, s) \in \mathcal{R}^{**}$ and let $C, C^*$ be the chamber as in (3.14), (3.15) respectively. Let $(x^1, \ldots, x^n)$ be local coordinates in a neighborhood of $q$ with coordinate vector fields $(\partial_1, \ldots, \partial_n)$. Let $A$ be the apartment containing chambers $C$ and $C^*$ as in (3.14) and (3.15). After isometrically identifying $A$ with $\mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$ be the usual inner product defined on $A \simeq \mathbb{R}^n$. Thus, (3.5) with $V = \partial_{\alpha}$, implies at $q_0$
\[
s \mapsto \left\langle \frac{\partial \tilde{u}_s}{\partial x^\alpha}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle = \text{constant in } (s_0 - \epsilon, s_0 + \epsilon).
\]

We can differentiate this twice with respect to $s$ to obtain
\[
0 = \frac{\partial^2}{\partial s^2} \left\langle \frac{\partial \tilde{u}_s}{\partial x^\alpha}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle = 2 \left\langle \frac{\partial}{\partial x^\alpha} \frac{\partial^2 \tilde{u}_s}{\partial s^2}, \frac{\partial \tilde{u}_s}{\partial x^\alpha} \right\rangle + 2 \left\langle \frac{\partial}{\partial x^\alpha} \frac{\partial \tilde{u}_s}{\partial s}, \frac{\partial^2 \tilde{u}_s}{\partial x^\alpha} \right\rangle.
\]

Since $\tilde{\sigma}_{q_0}$ is a geodesic, $\frac{\partial^2 \tilde{u}_{q_0}}{\partial x^\alpha^2}(q_0) = \tilde{\sigma}_{q_0}''(s) = 0$. Thus,
\[
\frac{\partial \tilde{\sigma}_q'(s)}{\partial x^\alpha} \bigg|_{q=q_0, s=s_0} = \frac{\partial}{\partial x^\alpha} \frac{\partial \tilde{u}_s}{\partial s}(q_0) = 0.
\]
Since the choice of \( \alpha \in \{1, \ldots, n\} \) is arbitrary and \( R^* \) is of full measure in \( \tilde{M} \times [0,1] \), we conclude that geodesic segments \( \bar{\sigma}_p \) and \( \bar{\sigma}_q \) are parallel for any \( p, q \in \tilde{M} \); i.e.

\[
d(\bar{\sigma}_p(s), \bar{\sigma}_q(s)) =: \delta_{p,q}, \quad \forall s \in [0,1]
\]

where \( \delta_{p,q} := d(\sigma_q(0), \sigma_p(0)) = d(\tilde{u}_0(q), \tilde{u}_0(p)) \).

Next note the following:

- (Existence of a geodesic extension) Given a geodesic segment, property (2) of Definition 2.7 implies that there exists an apartment containing its endpoints and hence its image. We can thus extend the geodesic segment to a geodesic line in this apartment.

- (Non-uniqueness of a geodesic extension) Unlike symmetric spaces, the geodesic extensions are not necessarily unique in a Euclidean building. Indeed, there may be many apartments containing the endpoints of a given geodesic segment.

Because of the non-uniqueness of geodesic extensions, the proof for the building case is slightly different from the symmetric space case as we see below.

We define the sets

\[
C_0, C_1, \ldots, C_n
\]

inductively follows. First, let \( C_0 = \tilde{u}_0(\tilde{M}) \), and then let \( C_n \) be the union of the images of all geodesic segments connecting points of \( C_{n-1} \). The \( \rho(\pi_1(M)) \)-invariance of \( C_0 \) implies the \( \rho(\pi_1(M)) \)-invariance of \( C_n \).

To each \( Q = \tilde{u}_0(q) \in C_0 \), we assign a geodesic segment \( \bar{\sigma}^Q = \bar{\sigma}_q \) (cf. (3.4)). By (3.16), \( \{\bar{\sigma}^Q\}_{Q \in C_0} \) is a family of pairwise parallel of geodesic segments. Since \( \tilde{u}_s \) is \( \rho \)-equivariant, the assignment \( Q \mapsto \bar{\sigma}^Q \) is \( \rho(\pi_1(M)) \)-equivariant; i.e. \( \rho(\gamma)\bar{\sigma}^Q = \bar{\sigma}^{\rho(\gamma)Q} \) for any \( Q \in C_0 \) and \( \gamma \in \rho(\pi_1(M)) \).

For \( n \in \mathbb{N} \), we inductively a \( \rho(\pi_1(M)) \)-equivariant map from \( C_n \) to a family of pairwise parallel geodesic segments as follows: For any pair of points \( Q_0, Q_1 \in C_{n-1} \), apply the Flat Quadrilateral Theorem (cf. [BH, 2.11]) with vertices \( Q_0, Q_1, P_1 := \bar{\sigma}^{Q_1}(1), P_0 := \sigma^{Q_0}(1) \) to define a one-parameter family of parallel geodesic segments \( \bar{\sigma}^{Q_t} : [0,1] \to \tilde{X} \) with initial point \( Q_t = (1-t)Q_0 + tQ_1 \) and terminal point \( P_t = (1-t)P_0 + tP_1 \) (cf. (2.9)). The inductive hypothesis implies that the map \( Q \mapsto \bar{\sigma}^Q \) defined on \( C_n \) is also \( \rho(\pi_1(M)) \)-equivariant. The above construction defines a \( \rho(\pi_1(M)) \)-equivariant map

\[
Q \mapsto \bar{\sigma}^Q
\]

from \( \tilde{X} \) to a family of pairwise geodesic segments. Indeed, we are assuming that the action of \( \rho(\pi_1(M)) \) does not fix a non-empty closed convex strict subset of \( \tilde{X} \). Thus,

\[
\tilde{X} = \bigcup_{n=0}^{\infty} C_n
\]
since the right hand side is the convex hull of $C_0 = \tilde{u}_0(\tilde{M})$ and each $C_n$ is invariant under the action of $\rho(\pi_1(M))$.

**Claim 3.5.** There exists a $\rho(\pi_1(M))$-equivariant map

$$Q \mapsto \sigma^Q : [0, \infty) \to \tilde{X}$$

from $\tilde{X}$ into a family of pairwise parallel rays; i.e. $\rho(\gamma)\sigma^Q = \sigma^{\rho(\gamma)Q}$ for all $Q \in \tilde{X}$, $\gamma \in \pi_1(M)$ and $d(\sigma_p(s), \sigma_q(s)) = \delta_{p,q}$ for all $s \in [0, \infty)$.

**Proof.** For $Q \in \tilde{X}$, we inductively construct a sequence $\{Q_i\}$ of points in $\tilde{X}$ by first setting $Q_0 = Q$ and then defining $Q_i = \bar{\sigma}^{Q_{i-1}}(3/4)$. Next, let

$$L^Q = \bigcup_{i=0}^{\infty} I^{Q_i}$$

where $I^{Q_i} = \bar{\sigma}^{Q_i}([0,1])$. Therefore, $L^Q$ is a union of pairwise parallel geodesic segments. Thus, $\{L^Q\}_{Q \in \tilde{X}}$ is a family of pairwise parallel geodesic rays. Moreover, the $\rho(\pi_1(M))$-equivariance of the map $Q \mapsto \bar{\sigma}^Q$ implies $\rho(\gamma)\bar{\sigma}^{Q_{i-1}}(3/4) = \bar{\sigma}^{\rho(\gamma)Q_{i-1}}(3/4)$. Thus, if $\{Q_i\}$ is the sequence constructed starting with $Q_0 = Q$, then $\{\rho(\gamma)Q_i\}$ is the sequence constructed starting with $\rho(\gamma)Q_0 = \rho(\gamma)Q$. We thus conclude

$$\rho(\gamma)L^Q = \bigcup_{i=-\infty}^{\infty} \rho(\gamma)I^{Q_i} = \bigcup_{i=-\infty}^{\infty} I^{\rho(\gamma)Q_i} = L^{\rho(\gamma)Q}.$$ 

We are done by letting the geodesic ray $\sigma^Q : [0, \infty) \to \tilde{X}$ be the extension of the geodesic segment $\bar{\sigma}^Q : [0,1] \to \tilde{X}$ parameterizing $L^Q$. □

Claim 3.5 implies that $\rho(\pi_1(M))$ fixes the point $[\sigma^Q]$ at infinity. This contradicts assumption (i) and completes the proof.

**Remark 3.6** (Generalization to thick Euclidean buildings). Conjecturally, a harmonic map into a thick Euclidean building has locally finite image and the Gromov-Schoen regularity result holds in this case. If this is the case, then Theorem 1.1 holds when $\tilde{X}$ is assumed to be a thick Euclidean building. Indeed, the only place where the assumption that the Euclidean building is locally finite is used is in the application of Theorem 3.4. (More precisely, the target space is assumed to be locally finite in the regularity result of Gromov-Schoen [GS, Theorem 6.4].) The rest of the proof of Theorem 1.1 goes through exactly as in the locally finite Euclidean building case.

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