Lyapunov-based saturated continuous twisting algorithm

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Abstract
In this article, a second-order system, which is affected by disturbances and uncertainties, with a saturating actuator is considered. A novel robust feedback control law is designed based on the sliding mode technique. The twisting and the continuous twisting algorithms are incorporated into the design, which is based on level curves of a Lyapunov function. The performance of the standard continuous twisting algorithm is greatly improved in the case that the initial condition of the system is far away from the origin. A parameter setting for the controller is given by establishing global finite-time stability properties of the closed-loop system origin. Feasibility and effectiveness of the proposed approach are indicated in a real-world application as well as numerical simulation.

KEYWORDS
higher-order sliding mode control, Lyapunov function approach, saturating actuator

1 | INTRODUCTION

The continuous twisting algorithm (CTA) is a well-known higher-order sliding mode control method that has been successfully applied to systems affected by perturbations, which are Lipschitz continuous with respect to time. For systems of order two, similar to the twisting algorithm, no sliding function has to be defined and both of the states converge to zero in a finite time by applying this approach. However, unlike the other algorithms such as the twisting algorithm, the control signal introduced into the system is continuous. It also enjoys the advantage that, under discrete-time measurements, the accuracy of the sliding variables is improved in the case that the sampling interval is small enough.\textsuperscript{1,2} Moreover, as it is analyzed in Pérez-Ventura and Fridman’s work,\textsuperscript{3} it can be shown that the chattering effect is reduced comparing to the twisting algorithm if the actuator dynamics are fast enough.

In the presence of saturating actuators, the high-order sliding mode controllers such as the conventional supertwisting and the standard continuous twisting controllers generate the signals that may exceed the saturation limits. As a result, due to the discontinuous integral actions of the controllers, the windup effect is produced. For a first-order system, a modification to the suboptimal second-order sliding mode controller has been made by Ferrara and Rubagotti\textsuperscript{4} in order to guarantee the finite-time convergence of the sliding variable in the case of such a system with a saturating actuator. Several versions of the saturated supertwisting algorithm have been recorded in the literature, see, for example, the works done by Castillo et al.,\textsuperscript{5} Golkani et al.,\textsuperscript{6} and Seeber and Reichhartinger.\textsuperscript{7} A domain of attraction for a system under the conventional supertwisting control has been computed by Behera et al.\textsuperscript{8} It is proved that in the case the initial condition
of the closed-loop system is within this domain, the control signal does not exceed the limits, that is, windup does not occur. The aforementioned concepts need to be modified in order to be applicable to a system of order more than one.

For a second-order system, having adopted the twisting algorithm for the design of an observer-based control law, a Lipschitz continuous control signal, which remains within the saturation bounds, is introduced in the paper given by Golkanietal. A sliding function needs to be defined therein such that the relative degree of the system with respect to this function is one. Hence, the system states tend to the origin asymptotically. Since the control law contains estimated information of the time derivative of the sliding variable, this sliding mode control law is redundant for a perturbed double integrator system. This is due to the fact that the controller as well as the estimator reconstruct disturbances. Furthermore, a compact saturated CTA is proposed in the work of Golkanietal. However, the modification makes a fairly restrictive assumption on the bound and class of addressed perturbations.

In this article, the concept proposed by Castillo et al. for the first-order system is modified to be applicable to a second-order system. The main contribution of this article is to introduce a saturated higher-order sliding mode control strategy, in which the properties of the standard CTA are maintained. The rest of the article is organized as follows: the problem and the objective are explained in Section 2. The proposed control law design is described in Section 3. The stability analyses of the closed-loop system are carried out in Section 4. Simulation results comparing the proposed approach with the existing saturated and standard continuous twisting algorithms are illustrated in Section 5. The approach is applied in course of an experiment in Section 6 followed by a conclusion given in Section 7.

2 PROBLEM STATEMENT

Consider a second-order system described by

\[
\begin{align*}
\frac{dz_1}{dt} &= z_2, \\
\frac{dz_2}{dt} &= \text{sat}_\rho(u) + a(t),
\end{align*}
\]

where $z_1, z_2 \in \mathbb{R}$ are the state variables. It is assumed that both of them are available for measurement. $u$ is the scalar control input and the actuator is saturated if $|u| > \rho$, where $\rho$ is a known constant. This saturation is realized from the definition

\[
\text{sat}_\rho(\chi) = \begin{cases} 
\chi & \text{for } |\chi| \leq \eta, \\
\eta \left\lfloor \frac{\chi}{\eta} \right\rfloor & \text{for } |\chi| > \eta,
\end{cases}
\]

where $\chi \in \mathbb{R}$ and the notation $\left\lfloor \chi \right\rfloor^0 = \text{sgn}(\chi)$ as a particular case of $\left\lfloor \chi \right\rfloor^q = |\chi|^q \text{sgn}(\chi)$ is used. The effect of perturbations and uncertainties is represented by the function $a$.

**Assumption 1.** The function $a$ is globally bounded and Lipschitz continuous with respect to time, that is,

\[
|a(t)| \leq a_M < \rho \quad \text{and} \quad \left| \frac{da}{dt} \right| \leq L, \quad \forall t > 0,
\]

where $a_M$ and $L$ are some known constants.

**Remark 1.** The inequality $a_M < \rho$ has to be satisfied in order to be able in principle to steer the system states to zero through the saturated control signal (otherwise, $z_2 = 0$ cannot be an equilibrium state of system (1) for every admissible function $a$).

**Remark 2.** It is worth mentioning that $a$ can be a globally bounded function of $t$ and $z_2$. In that case, $\frac{da}{dt} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial z_2} \frac{dz_2}{dt}$ and hence, the upper bound $L$ is equal to $L_t + (\rho + a_M)L_{z_2}$, where the constants $L_t$ and $L_{z_2}$ denote upper bounds as

\[
\left| \frac{\partial a}{\partial t} \right| \leq L_t \quad \text{and} \quad \left| \frac{\partial a}{\partial z_2} \right| \leq L_{z_2}, \quad \forall t > 0.
\]
These constants are also assumed to be known. Since $z_2$ can be regarded as the velocity of a mechanical system, addressing uncertainties depending on this state is important in many practical applications. Uncertainties depending on $z_1$ are not considered in the class of system uncertainties here. The reason is that if $a$ is a function of $z_1$, then $z_2$ appears in the time derivative of $a$ and therefore, bounds of $z_2$ need to be known. Such bounds cannot be obtained from the saturated control input.

The objective is to design a feedback control law for system (1) such that

- both of the system states $z_1$ and $z_2$ tend to the origin in a finite time despite the presence of disturbances and uncertainties;
- the absolute value of the control signal $u$ is confined to the saturation limit $\rho$ for any initial condition $z_1(t = 0) = z_{1,0} \in \mathbb{R}$ and $z_2(t = 0) = z_{2,0} \in \mathbb{R}$, that is, $u(t) \in [-\rho, \rho]$ holds for all $t > 0$.

In order to achieve that, a new version of the saturated CTA is introduced in the next section.

### 3 | PROPOSED SCHEME

In order to achieve the finite-time convergence of the states of a perturbed double integrator system, the standard continuous twisting controller is introduced in the paper given by Torres-González et al.\textsuperscript{1} as

$$u = -k_1 \left\lceil z_1 \right\rceil \frac{1}{3} - k_2 \left\lceil z_2 \right\rceil \frac{1}{2} + \nu,$$

$$\frac{d\nu}{dt} = -k_3 \left\lceil z_1 \right\rceil^0 - k_4 \left\lceil z_2 \right\rceil^0,$$

where the positive control parameters $k_1$, $k_2$, $k_3$, and $k_4$ need to be chosen appropriately ($k_4 = 0$ is a possible selection). However, the saturating actuator is not taken into consideration therein. The Lyapunov-based design of the saturated supertwisting algorithm presented in the work of Castillo et al.\textsuperscript{5} is adopted in this article to introduce a novel Lyapunov-based saturated continuous twisting algorithm. The control law for system (1) is designed as

$$\begin{bmatrix} u \\ \frac{dv}{dt} \end{bmatrix} = \begin{cases} \begin{bmatrix} -\frac{2}{3} \rho \left( \left\lceil z_1 \right\rceil^0 + \frac{1}{2} \left\lceil z_2 \right\rceil^0 \right) \\ 0 \\ -k_1 \left\lceil z_1 \right\rceil^\frac{1}{3} - k_2 \left\lceil z_2 \right\rceil^\frac{1}{2} + \nu \\ -k_3 \left\lceil z_1 \right\rceil^0 \end{bmatrix} & \text{if } t < T \\
\begin{bmatrix} \nu \\ 0 \end{bmatrix} & \text{if } t \geq T, \end{cases}$$

where $\nu(t = 0) = 0$ is assumed and $T$ is

$$T = \inf \left\{ t \left| \left\lceil z_1(t) \right\rceil^\frac{1}{3} + \left\lceil z_2(t) \right\rceil^\frac{1}{2} \leq \gamma \right. \right\}$$

with a nonnegative constant $\gamma$. Here, the relay controller based on the twisting algorithm is applied up to the time instant $T$ and thereafter, a continuous twisting controller is employed. Thus, there is at most one switch between these two sliding mode algorithms. It is shown later that the actuator saturation does not happen for any $t \geq T$. A block diagram of the proposed approach is depicted in Figure 1. It is outlined later how to choose the switching level parameter $\gamma$. Furthermore, an appropriate choice of the positive gains $k_1$, $k_2$, and $k_3$ is made exemplarily, whose values can be scaled depending on the disturbances bounds in (3). It is noted that, comparing to the saturated CTA presented in the paper of Golkani et al.\textsuperscript{10} the proposed control scheme enjoys the advantage that the continuous twisting algorithm is recovered close to the origin. This contributes significantly to an increase of the allowable disturbances bound. Moreover, the restriction on the class of disturbances is relaxed. These are detailed in the next section, where the stability of the closed-loop system origin is guaranteed by considering level sets of the Lyapunov function used by Torres-González et al.\textsuperscript{1}
FIGURE 1  Block diagram of proposed control law (6). It is noted that the output of the comparison block is one if its input is less than or equal to the constant $\gamma$ and zero otherwise. The output of OR is initialized with zero and the switches are in the upper positions when this output is one.

TABLE 1  Permissible values of $L_1$ for different values of $\mu$ in the case that the control parameters are chosen as given in (10)

| $\frac{1}{3} \leq \mu \leq 1$ | 0.21 |
| $\frac{1}{6} \leq \mu < \frac{1}{4}$ | 0.27 |
| $\frac{1}{12} \leq \mu < \frac{1}{6}$ | 0.39 |
| $0 \leq \mu < \frac{1}{12}$ | 0.72 |

4  STABILITY ANALYSIS

Investigation into global finite-time stability properties of the closed-loop system origin is conducted in the following.

Theorem 1. Suppose that the inequalities given in (3) hold and let $\mu \leq \frac{1}{3}$ be a given nonnegative constant. Then there exist constants $\gamma, \bar{k}_1, \bar{k}_2, \bar{k}_3, L_1,$ and $L_2$ such that for all

$$a_M \in [0, \mu \rho], \quad L \in [0, L_1(\rho - a_M)],$$

the choice of the control parameters as

$$k_1 = \frac{L^2}{\bar{k}_1}, \quad k_2 = \frac{L^2}{\bar{k}_2}, \quad k_3 = \frac{L}{\bar{k}_3}$$

with

$$\bar{L} = \max(L, L_2(\rho + a_M))$$

guarantees that the origin of the closed-loop system (plant (1) under control law (6)) is globally finite-time stable.

Remark 3. By solving the sum-of-squares problem formulated within the following proof using, for example, YALMIP$^{11}$ and SeDuMi$^{12}$ the values of the constants $L_1$ and $L_2$ as well as $\gamma, \bar{k}_1, \bar{k}_2,$ and $\bar{k}_3$ can be determined numerically. A valid choice of the control parameters is

$$\gamma = 0, \quad \bar{k}_1 = 10, \quad \bar{k}_2 = 5, \quad \bar{k}_3 = 1.1.$$  (10)

Furthermore, $L_2 = 0.09$ and different values of $L_1$ as listed in Table 1 are obtained. Please note that $L_1$ increases while $\mu$ decreases.
Proof. For \( t < T \), the closed-loop dynamics is written as

\[
\frac{dz_1}{dt} = z_2, \quad \frac{dz_2}{dt} = -\frac{2}{3} \rho \left( \left\lfloor z_1 \right\rfloor^0 + \frac{1}{2} \left\lfloor z_2 \right\rfloor^0 \right) + a(t). \tag{11a}
\]

It becomes evident that the finite-time convergence of the states toward zero is ensured through the twisting algorithm since \( a_M < \xi \) is satisfied.\textsuperscript{13,14} Hence, the switching condition given in (7) is met eventually (this is true for any \( \gamma \geq 0 \); the specific value of \( \gamma \) is discussed later).

For \( t \geq T \), the closed-loop system without control input saturation, that is, with \( \frac{dz_2}{dt} = u + a(t) \) instead of (1b), can be represented by

\[
\frac{d\tilde{x}_1}{dt} = \tilde{x}_2, \quad \frac{d\tilde{x}_2}{dt} = -k_1 \left\lfloor \tilde{x}_1 \right\rfloor^1 - k_2 \left\lfloor \tilde{x}_2 \right\rfloor^1 + \tilde{x}_3, \quad \frac{d\tilde{x}_3}{dt} = -k_3 \left\lfloor \tilde{x}_1 \right\rfloor^0 + \frac{1}{L} \left( \frac{da}{dt} \right), \tag{12b}
\]

where the state vector is defined as

\[
\tilde{x} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{bmatrix}^T = \begin{bmatrix} z_1 & z_2 & \nu + a \end{bmatrix}^T. \tag{13}
\]

In the course of the proof it is shown that the control input \( u \) then satisfies \( |u(t)| \leq \rho \) for all \( t \geq T \). Thus, the control signal remains within the limits for all time and (12) equivalently governs the closed-loop system also with control input saturation.

The control gains \( k_1, k_2, \) and \( k_3 \) are scaled for all \( L \) as given in (9a). This is justified by changing the coordinates in (12) as

\[
\tilde{x} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{bmatrix}^T = L \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T. \tag{14}
\]

The transformation yields

\[
\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -k_1 \left\lfloor x_1 \right\rfloor^1 - k_2 \left\lfloor x_2 \right\rfloor^1 + x_3, \quad \frac{dx_3}{dt} = -k_3 \left\lfloor x_1 \right\rfloor^0 + \frac{1}{L} \left( \frac{da}{dt} \right). \tag{15c}
\]

Please note that \( \frac{1}{L} \left| \frac{da}{dt} \right| \leq 1 \) holds according to (3) and (9b).

In order to guarantee the stability of the combination of the aforementioned closed loops, it needs to be ensured that the control signal obtained from the trajectories of (12) remains within the given saturation bounds for all \( t \geq T \). To achieve that, consider level sets

\[
\Omega_c = \{ x \in \mathbb{R}^3 \mid V(x) \leq c \}, \tag{16}
\]

of a Lyapunov function \( V \), which is designed later. First, a level \( c_1 \) is computed such that \( |u| \leq \rho \) holds in the entire level set \( \Omega_{c_1} \). Then, a level \( c_2 \) is calculated such that the trajectory is contained in \( \Omega_{c_2} \) at \( t = T \). Hence, if the condition

\[
c_2 \leq c_1 \tag{17}
\]

is fulfilled, that is, if \( \Omega_{c_2} \subseteq \Omega_{c_1} \), then the trajectory remains within \( \Omega_{c_2} \) and consequently, \( |u(t)| \leq \rho \) holds for all \( t \geq T \). To complete the proof, it only remains to actually compute the constants \( c_1 \) and \( c_2 \) and to construct a Lyapunov function such that these constants satisfy (17).
To compute $c_1$, it is noted that $\Omega_{c_1}$ is an invariant set in the space $\left( x_1, x_2, \frac{v}{L} \right)$. Therein, the function $\frac{a}{L}$ acts as an offset for the third coordinate and a displacement of the equilibrium point with the maximum amplitude of $\frac{a}{L}$ is made along the $\frac{v}{L}$-axis. On the planes, where the control signal is on the border of the saturation limit, $\frac{v}{L}$ can be represented by $\pm \frac{a}{L} + k_1 [x_1] \hat{1} + k_2 [x_2] \hat{2}$. The largest level $c_1$ such that $|u| \leq \rho$ holds in the entire set $\Omega_{c_1}$ can be then computed as

$$c_1 = \min_{|a| \leq \rho \omega \mu} \min_{x} \left\{ V(x) | x_3 = \frac{\rho}{L} + \tilde{k}_1 [x_1] \hat{1} + \tilde{k}_2 [x_2] \hat{2} + \frac{a}{L} \right\}.$$  

(18)

Please note that due to symmetry reasons, only $u = \rho$ needs to be considered. In order to calculate $c_2$, the definition (7) of $T$ along with the fact that $v(T) = 0$ can be rewritten in terms of $x_1, x_2$ to obtain

$$c_2 = \max_{|a| \leq \rho \omega \mu} \left\{ V \left( x_1, x_2, \frac{a}{L} \right) | \frac{1}{L} | x_1 \hat{1} + \frac{1}{L} | x_2 \hat{2} \leq \gamma \right\}.$$  

(19)

It is worth mentioning that $c_1$ and $c_2$ themselves are not computed; instead, a sum-of-squares problem is formulated to ensure that (17) is satisfied with the Lyapunov function $V$ designed in the following.

The Lyapunov function candidate considered by Torres-González et al.\(^1\) as

$$V(x) = a_1 |x_1|^3 + a_2 x_1 x_2 + a_3 |x_2|^3 + a_4 |x_1|^2 - a_5 x_2 x_3^2 + a_6 |x_3|^5$$  

(20)

can be employed. It is differentiable and homogeneous\(^{15}\) of degree five with the weights $r = [3 \quad 2 \quad 1]^T$. The coefficient vector is defined as $\alpha = [a_1 \ldots a_6]^T \in \mathbb{R}^6$. Having taken the time derivative of $V$ along the trajectories of system (15),

$$\frac{dV}{dt} = -(W_1 + W_2)$$  

(21)

is derived, where the functions $W_1$ and $W_2$ are written as

$$W_1(x) = \beta_1 [x_1] \hat{1} + \beta_2 x_1 x_2 \hat{1} - \beta_5 [x_1] \hat{2} - \beta_6 x_1 x_3 \hat{2} + \beta_4 |x_1|^3 \hat{1} - \beta_7 |x_1|^2 \hat{2} + \beta_5 |x_2|^2 - \beta_8 |x_2|^3$$

$$- \beta_9 x_1 x_3 + \beta_{10} x_1 x_2 \hat{3} + \beta_6 x_1 x_3 - \beta_{11} x_1^0 x_2^0 x_3^0 + \beta_{12} x_1^0 x_2 |x_3| + \beta_{13} x_1^0 x_2^0 |x_3| + \beta_{14} x_1^0 x_2^0 |x_3|^4$$  

(22a)

and

$$W_2(x) = -\beta_{15} x_1 |x_3| + \beta_{16} x_2 |x_3|^2 - \beta_{17} |x_3|^4.$$  

(22b)

Their coefficients are

$$\beta_1 = a_2 \tilde{k}_1, \quad \beta_2 = a_2 \tilde{k}_2, \quad \beta_3 = \frac{5}{3} a_1, \quad \beta_4 = \frac{5}{2} a_3 \tilde{k}_1, \quad \beta_5 = \frac{5}{2} a_3 \tilde{k}_2 - a_2,$$

$$\beta_6 = a_2, \quad \beta_7 = 2 a_4 \tilde{k}_3, \quad \beta_8 = a_5 \tilde{k}_1, \quad \beta_9 = 5 a_6 \tilde{k}_3, \quad \beta_{10} = \frac{5}{2} a_3,$$

$$\beta_{11} = a_4, \quad \beta_{12} = a_5 \tilde{k}_2, \quad \beta_{13} = 3 a_5 \tilde{k}_3, \beta_{14} = a_5,$$  

(23a)

and

$$\left[ \begin{array}{ccc} \beta_{15} & \beta_{16} & \beta_{17} \end{array} \right]^T = \left[ \begin{array}{ccc} 2 a_4 & 3 a_5 & 5 a_6 \end{array} \right]^T \left( \frac{da}{dt} \right)$$  

(23b)

Conditions of the control parameters and the coefficients $\alpha$ are determined using a sum-of-squares optimization such that the positive definiteness of $V$ and $W_1 + W_2$ and the aforementioned inequality (17) obtained from the level sets of the Lyapunov function $V$ are satisfied. It is noted that the coefficients of $W_2$ are bounded since $\frac{1}{L} \left| \frac{da}{dt} \right|$
bounded by one as mentioned above. The extrema of the coefficients $\beta_{15}$, $\beta_{16}$, and $\beta_{17}$ are denoted by the overlined letters as

$$
\begin{bmatrix}
\beta_{15} \\
\beta_{16} \\
\beta_{17}
\end{bmatrix}^T = \pm \begin{bmatrix}
2a_4 \\
3a_5 \\
5a_6
\end{bmatrix}^T.
$$

(24)

Similar to the analysis conducted in the literature, a quadrant analysis is carried out in the Appendix to formulate a sum-of-squares program in order to find allowable values of $\gamma$, $k_1$, $k_2$, $k_5$, $L_1$, and $L_2$ for given values of $\mu$ such that $V$ and $W_1 + W_2$ are positive definite and inequality (17) holds for every $L \geq 0$. It is noted that $\gamma = 0$ can be chosen there since the twisting algorithm can make that the states tend to the origin in a finite time.

Remark 4. It is worth mentioning that the proposed saturated continuous twisting algorithm does not work with every relay controller, that is, a suitable one needs to be incorporated into the control law design. For instance, having applied the control algorithm with prescribed convergence law as

$$
u = -\rho \left[ z_2 + \kappa \frac{\left[ z_1 \right]}{\lambda} \right]^0,$$

(25)

where $\kappa$ is a positive constant chosen such that $\kappa^2 < 2(\rho - a_M)$ is fulfilled, the system trajectory may reach the sliding manifold $z_2 + \kappa \frac{\left[ z_1 \right]}{\lambda} = 0$ before switching to the continuous twisting algorithm. This produces the chattering effect and thus, such a relay controller cannot be used. For the case when $\kappa^2 > 2(\rho - a_M)$ holds, as mentioned in the work of Fridman et al., the twisting behavior is recovered and therefore, it can be employed. Alternatively, the so-called quasi-continuous control algorithm can be applied by changing the proof accordingly. That algorithm may conceivably be used in order to extend the proposed Lyapunov-based saturated continuous twisting algorithm to systems of order more than two. In this case, the continuous twisting algorithm for the higher-order systems (see the papers given by Mendoza-Avila et al.) can be employed for $t \geq T$ since a family of Lyapunov functions has been proposed for it.

5 SIMULATION EXAMPLE

In this section, the result obtained in simulation through the saturated as well as the standard continuous twisting algorithm recorded in the literature is compared with the achieved closed-loop performance of the algorithm introduced in this article. The saturated CTA proposed in the work of Golkaniet al. for system (1) to retain the control input within the bounds $\pm \rho$ reads as

$$
u = -k_1 \text{sat}_{\epsilon_1} \left( \left[ z_1 \right]^\frac{1}{2} \right) - k_2 \text{sat}_{\epsilon_2} \left( \left[ z_2 \right]^\frac{1}{2} \right) + \nu,$$

(26a)

$$
\frac{dv}{dt} = -k_3 \left[ z_1 \right]^0 \lambda \nu, \quad |\nu_0| \leq \frac{k_3}{\lambda},
$$

(26b)

where the sat function is defined as given in (2). The initial value $\nu(t = 0) = \nu_0$ as well as the positive constants $k_1$, $k_2$, and $k_3$ have to be selected appropriately. Furthermore, the positive gains $\epsilon_1$, $\epsilon_2$, and $\lambda$ are chosen such that

$$
\frac{1}{2} k_1 \epsilon_1 = k_2 \epsilon_2 > \frac{k_3}{\lambda} + a_M,$$

(27a)

$$
\rho \geq k_1 \epsilon_1 + k_2 \epsilon_2 + \frac{k_3}{\lambda},
$$

(27b)

hold. It is inferred from the stability analysis of the closed-loop system carried out therein that

$$
k_3 > \lambda a_M + L
$$

(28)

also needs to be satisfied. Thus, $\frac{k_3}{\lambda}$ needs to be greater than $a_M$. Having considered (27), it is deduced that $k_2 \epsilon_2$ is greater than $2a_M$ and therefore, only perturbations with $a_M < \frac{\epsilon}{t}$ can be addressed.
Suppose that the actuator is saturated with $\rho = 10$ and the system is affected by perturbations

$$a(t) = 2.5 + 0.4 \sin(t) + 0.2 \sin(4t).$$

(29)

It is noted that $a$ and its time derivative are bounded by $a_M = 3.1$ and $L = 1.2$, respectively. Hence, the conditions laid down in Theorem 1 are met and the proposed Lyapunov-based saturated CTA is able to handle these disturbances, whereas the abovementioned saturated CTA cannot deal with them. As assigned in the work of Torres-González et al., the scaled parameters $k_1 = 14.68$, $k_2 = 8.22$, $k_3 = 2.76$, and $k_4 = 1.32$ of the standard CTA given in (5) are chosen based on $L$.

In order to assess the effectiveness of proposed control law (6), the above constants $k_1$, $k_2$, and $k_3$ are left unchanged in its implementation. The switching level $\gamma = 0.05$ is selected. For controller (26), there is no chance to tune $\epsilon_1$, $\epsilon_2$, and $\lambda$ such that all inequalities given in (27) and (28) are satisfied. They are therefore assigned as $\epsilon_1 = 0.39$, $\epsilon_2 = 0.34$, and $\lambda = 2$ to confine the control signal to $\pm \rho$. The simulation results are obtained through MATLAB/Simulink employing the forward Euler method with the sampling interval of 1 ms and the initial values $z_1,0 = 230$ and $z_2,0 = -20$.

As it can be seen in their performance depicted in Figure 2, the proposed approach contributes greatly to the alleviation of the windup effect. In the zoomed portions of the upper and middle plots, it is revealed that the states of the system under control law (26) do not converge with the same precision as those obtained through the other algorithms. This is expected since (28) is not met. The evolution of the control signals is shown in the lower plot. These are the signals realized through the saturating actuator. Please note that for controller (5), this is different from the produced control signal $u$, which does not remain within the saturation bounds. As a result, large overshoots and undershoots as well as a long settling time can be seen in the performance of the standard CTA.

**Experimental Implementation**

In this section, a differential cylinder, whose layout is illustrated in Figure 3, is taken into consideration to test the proposed algorithm in a real-world application. The position control system of the piston rod $y$ can be developed in two steps, see, for example, Koch and Reichhartinger's paper. In the first step, having conducted an exact linearization of the valve dynamics, a controller for the nonlinear valve system is designed. In the second step, an outer-loop controller is made for the moving piston. Since this subsystem is subject to an unknown load force $F_{\text{ext}}$, a robust control technique should be adopted.
The cylinder moves at two different velocities at a constant flow rate \( Q_A \) and \( Q_B \), respectively. The flows \( Q_A \) and \( Q_B \) are regulated by a servo valve. A hydraulic pump provides the valve with a constant pressure, which is assumed to be independent of \( F_{\text{ext}} \). The pressure and the volume in the chambers A and B are denoted by \( p_A, p_B \) and \( V_A, V_B \), respectively. A mathematical model describing the dynamics of the piston movement is derived by applying Newton’s second law of motion as

\[
m \frac{d^2 y}{dt^2} = F_h - F_r - F_{\text{ext}},
\]

where \( m \) is the total moving mass, that is, the sum of the piston mass and the mass of the hydraulic medium, and \( F_r \) represents the friction force. The hydraulic force reads as

\[
F_h = (p_A - \alpha_k p_B)A_k,
\]

where \( A_k \) is the so-called piston ring surface and \( \alpha_k \) denotes the ratio between the piston rod cross-section and the piston ring surface.

It is assumed that the valve is controlled and the closed-loop dynamics is described by an integrator, that is,

\[
\frac{dF_h}{dt} = u_h,
\]

where \( u_h \) is considered as the input of the system. This input is obtained through

\[
u_h = -k_0 (F_h - F_{h,d}),
\]

where \( F_{h,d} \) denotes the desired piston force and \( k_0 \) is a positive constant. Hence, an inner force control loop with closed-loop dynamics

\[
\frac{1}{k_0} \cdot \frac{dF_h}{dt} + F_h = F_{h,d}
\]

is established. The choice \( k_0 > 0 \) ensures that the hydraulic force \( F_h \) asymptotically tracks a constant desired force \( F_{h,d} \).

Having applied the Lyapunov-based saturated continuous twisting control law, a bounded desired piston force is introduced as

\[
F_{h,d} = u_{\text{CTA}} + F_t,
\]

where \( u_{\text{CTA}} \) is identical to \( u \) given in (6) with \( z_1 = y - y_d \) and \( z_2 = \frac{dy}{dt} - \frac{dy_d}{dt} \). It yields an outer feedback loop for reference trajectory tracking of the piston position and velocity. In the outer closed loop, \( \tilde{x}_3 \) reads as

\[
\tilde{x}_3 = v - F_{\text{ext}} - m \frac{d^2 y_d}{dt^2}.
\]

Please note that this control strategy does not require any information on the piston acceleration and external load. However, implementation of the outer control loop requires a model based estimation of the friction force \( F_r \) as well as estimation of the piston velocity \( \frac{dy}{dt} \). In order to achieve a perfect tracking in principle, a smooth function of time \( t \) to be
twice differentiable and slow enough such that its second time derivative is negligible is used as the reference position $y_d$ (see Figure 4). As a consequence, $-F_{ext}$ is regarded as perturbations $a$ considered in (1). It is noted that for assessment purposes, a force sensor, which provides real-time measurements of the external load force, is installed at the test rig.

In practice, changes in the reference signal can drive the control signal into saturation even after the switching time instant $T$. To take this into account, the Lyapunov-based saturated continuous twisting algorithm is reinitialized, that is, restarted as if $t$ were equal to zero in (6)–(7), whenever the control signal enters the saturation region, i.e. when $|u(t)| > \rho$ for some $t > T$. Following the theoretical analysis, this can only happen due to a large change in the reference and therefore, the number of switches is still limited by the number of such changes in the reference signal. This modification of the proposed strategy allows to prevent the windup effect in practice even with repeatedly changing reference signals.

Having set the saturation bound $\rho = 500 \text{ N}$, the closed-loop performance of the standard CTA as well as the algorithm proposed here is demonstrated in Figure 4. The parameters of control law (5) are well tuned as $k_1 = 1000$, $k_2 = 800$, $k_3 = 1500$, and $k_4 = 0$. It is noted that eliminating the term $\lceil z_2 \rceil^0$ of the standard CTA, which is not necessary for the stability of the closed-loop system origin (see, e.g., the work of Mendoza-Avila et al.18,19), reduces the windup effect to some extent. However, this cannot prevent the actuator from being saturated and hence, cannot remove the overshoot as shown in the corresponding response curves of the control input and piston position. Similar to the previous section, the above selected gains $k_1, k_2,$ and $k_3$ are left unchanged in the implementation of the Lyapunov-based saturated CTA. Furthermore, $0.4$ is assigned to $\gamma$. It is revealed that the generated control signal remains within the saturation limits and therefore, the windup effect is mitigated. Please note that for this practical application, the saturated controller is reinitialized in the implementation when the first step in the reference position occurs, which drives the control signal into saturation. In the results shown in the right plots of Figure 4, the saturated controller is thus reinitialized after the second two. It is indicated in the middle plots that both of the controllers can reconstruct the external force satisfactorily. In the case of standard CTA, an abrupt increase in the external force can be seen. This is due to that the piston rod gets to the end possible position as a result of the windup effect. This does not happen in the course of Lyapunov-based saturated CTA since the piston rod moves within the permissible range.

7  |  CONCLUSION

This article presents a Lyapunov-based saturated continuous twisting algorithm. Its principle consists in switching from a twisting controller to a continuous twisting controller once a certain switching condition is met. Having applied this control approach to a perturbed second-order system with a saturating actuator, the finite-time convergence of the states of the closed-loop system is ensured by means of level sets of a Lyapunov function. Since the continuous twisting algorithm
is recovered after a finite time, the bound and class of addressed disturbances are enlarged comparing to the saturated continuous twisting algorithm. This is confirmed through a numerical simulation. In experimental studies on a differential cylinder as well as in simulation, the performance of the proposed control technique illustrates the counteraction of the windup effect.

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CONFLICT OF INTEREST
The authors declare no potential conflict of interest.

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APPENDIX A. QUADRANT ANALYSIS

Since the functions $V$ defined in (20) and $W_1$ and $W_2$ given in (22) are symmetric with respect to the origin, the following four sets out of eight sets only need to be considered:

\[
\mathcal{M}_1 = \{ x_1, x_2, x_3 \geq 0 \},
\]

\[
\mathcal{M}_2 = \{ x_2, x_3 \geq 0, x_1 \leq 0 \},
\]

\[
\mathcal{M}_3 = \{ x_1, x_3 \geq 0, x_2 \leq 0 \},
\]

\[
\mathcal{M}_4 = \{ x_1, x_2 \geq 0, x_3 \leq 0 \}.
\]

The coordinates are changed to $\theta = [\theta_1 \theta_2 \theta_3]^T$ as

\[
|x_1| = \delta_1^6, \quad |x_2| = \delta_2^4, \quad |x_3| = \delta_3^2.
\]

Due to that the function $a$ and its time derivative play a role in computing the levels $c_1$ and $c_2$, it is also required to make a transformation as

\[
\frac{\rho + a}{L} = \delta_4^2.
\]

It is noted that the even numbers are assigned to the exponents in order to satisfy the octant constraints. For the aforementioned sets $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3,$ and $\mathcal{M}_4$, $V(\theta)$ is given as

\[
V_1(\theta) = \alpha_1 \theta_1^{10} + \alpha_2 \theta_1^6 \theta_2^4 + \alpha_3 \theta_1^6 \theta_2 + \alpha_4 \theta_1^6 \theta_3^4 - \alpha_5 \theta_1^6 \theta_3^2 + \alpha_6 \theta_1^{10},
\]

\[
V_2(\theta) = \alpha_1 \theta_1^{10} - \alpha_2 \theta_1^6 \theta_2^4 + \alpha_3 \theta_1^6 \theta_2 + \alpha_4 \theta_1^6 \theta_3^4 - \alpha_5 \theta_1^6 \theta_3^2 + \alpha_6 \theta_1^{10},
\]

\[
V_3(\theta) = \alpha_1 \theta_1^{10} - \alpha_2 \theta_1^6 \theta_2^4 + \alpha_3 \theta_1^6 \theta_2 + \alpha_4 \theta_1^6 \theta_3^4 + \alpha_5 \theta_1^6 \theta_3^2 + \alpha_6 \theta_1^{10},
\]

\[
V_4(\theta) = \alpha_1 \theta_1^{10} + \alpha_2 \theta_1^6 \theta_2^4 + \alpha_3 \theta_1^6 \theta_2 + \alpha_4 \theta_1^6 \theta_3^4 + \alpha_5 \theta_1^6 \theta_3^2 + \alpha_6 \theta_1^{10},
\]

respectively. For these sets, $W_1(\theta)$ is written as

\[
W_{11}(\theta) = \beta_1 \theta_1^6 - \beta_2 \theta_1^6 \theta_2^2 - \beta_3 \theta_1^4 \theta_2^4 + \beta_4 \theta_1^4 \theta_2^6 + \beta_5 \theta_2^8 - (\beta_6 - \beta_7) \theta_2^6 \theta_3^2
\]

\[
- \beta_4 \theta_1^2 \theta_3^8 - \beta_6 \theta_1 \theta_2^2 \theta_3^6 + (\beta_4 + \beta_5) \theta_2^4 \theta_3^6 + \theta_3^8 + \theta_1^6 \theta_3^4 - \theta_1 \theta_2 \theta_3^4 - \theta_2^2 \theta_3^2 - \theta_1 \theta_3^4 - \theta_2 \theta_3^2 - \theta_3^4
\]

\[
W_{12}(\theta) = \beta_1 \theta_1^6 - \beta_2 \theta_1^6 \theta_2^2 + \beta_3 \theta_1^4 \theta_2^4 + \beta_4 \theta_1^4 \theta_2^6 + \beta_5 \theta_2^8 - (\beta_6 - \beta_7) \theta_2^6 \theta_3^2
\]

\[
+ \beta_4 \theta_1^2 \theta_3^8 - \beta_6 \theta_1 \theta_2^2 \theta_3^6 + (\beta_4 + \beta_5) \theta_2^4 \theta_3^6 + \theta_3^8 + \theta_1^6 \theta_3^4 - \theta_1 \theta_2 \theta_3^4 - \theta_2^2 \theta_3^2 + \theta_1 \theta_3^4 + \theta_2 \theta_3^2 + \theta_3^4
\]

\[
W_{13}(\theta) = \beta_1 \theta_1^6 - \beta_2 \theta_1^6 \theta_2^2 - \beta_3 \theta_1^4 \theta_2^4 + \beta_4 \theta_1^4 \theta_2^6 + \beta_5 \theta_2^8 - (\beta_6 - \beta_7) \theta_2^6 \theta_3^2
\]

\[
- \beta_4 \theta_1^2 \theta_3^8 + \beta_6 \theta_1 \theta_2^2 \theta_3^6 + (\beta_4 + \beta_5) \theta_2^4 \theta_3^6 + \theta_3^8 + \theta_1^6 \theta_3^4 + \theta_1 \theta_2 \theta_3^4 + \theta_2^2 \theta_3^2 + \theta_1 \theta_3^4 + \theta_2 \theta_3^2 + \theta_3^4
\]

\[
W_{14}(\theta) = \beta_1 \theta_1^6 + \beta_2 \theta_1^6 \theta_2^2 + \beta_3 \theta_1^4 \theta_2^4 + \beta_4 \theta_1^4 \theta_2^6 + \beta_5 \theta_2^8 - (\beta_6 - \beta_7) \theta_2^6 \theta_3^2
\]

\[
- \beta_4 \theta_1^2 \theta_3^8 + \beta_6 \theta_1 \theta_2^2 \theta_3^6 + (\beta_4 + \beta_5) \theta_2^4 \theta_3^6 + \theta_3^8 + \theta_1^6 \theta_3^4 + \theta_1 \theta_2 \theta_3^4 + \theta_2^2 \theta_3^2 + \theta_1 \theta_3^4 + \theta_2 \theta_3^2 + \theta_3^4.
\]
respectively. It can be seen that thus far to guarantee that $V$ and $W_1 + W_2$ are positive definite, four polynomials for the Lyapunov function and eight polynomials for its time derivative need to be sums of squares at the same time. Please note that the positive definiteness of a function is ensured if the corresponding polynomials, after subtraction of positive polynomials with the same degree, are sums of squares (e.g., $V$ is positive definite if $V_1(\theta) - Y$, $V_2(\theta) - Y$, $V_3(\theta) - Y$, and $V_4(\theta) - Y$ are sums of squares, where $Y$ is $\epsilon(\theta_0^{10} + \theta_0^{10} + \theta_0^{10})$ with a small positive real constant $\epsilon$). In the following, another eleven polynomials are introduced, which also need to be sums of squares in order to ensure that the constants $c_1$ and $c_2$ satisfy (17). This guarantees that the control signal is confined to the saturation limits for all $t \geq T$.

As given in (18), the largest level $c_1$ is realized in the case $x_3 = \frac{a}{L} + k_1 [x_1] \frac{1}{i} + k_2 [x_2] \frac{1}{i} + \frac{a}{L}$ holds. Therein, the function $V(x)$ is represented by $V_{\rho}(x_1, x_2, \frac{a}{L})$ as

$$V_{\rho}(x_1, x_2, \frac{a}{L}) = a_1 [x_1] \frac{2}{i} + a_2 x_1 x_2 + a_3 [x_2] \frac{1}{i} + a_4 \left[ \frac{\rho}{L} + \bar{k}_1 [x_1] \frac{1}{i} + \bar{k}_2 [x_2] \frac{1}{i} + \frac{a}{L} \right]^2 - a_5 x_2 \left( \frac{\rho}{L} + \bar{k}_1 [x_1] \frac{1}{i} + \bar{k}_2 [x_2] \frac{1}{i} + \frac{a}{L} \right)^3 + a_6 \left[ \frac{\rho}{L} + \bar{k}_1 [x_1] \frac{1}{i} + \bar{k}_2 [x_2] \frac{1}{i} + \frac{a}{L} \right]^5. \quad (A7)$$

In the first step, as it is given in the following, this function is expressed using polynomials. After that, the lower bound $c_1$ is established by insuring that these polynomials, after subtraction of $c_1$, are sums of squares. Having made the transformation of $x_1$, $x_2$, and $\frac{a}{L}$ as given in (A2) and (A3), $V_{\rho}(\theta_1, \theta_2, \theta_3)$ for the set $M_1$ is written as

$$V_{\rho,1}(\theta_1, \theta_2, \theta_3) = a_1 \theta_1^{10} + a_2 \theta_1^6 \theta_2^4 + a_3 \theta_1^{10} + a_4 \theta_1^6 (k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^2 - a_5 \theta_2^2 (k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^3 + a_6 (k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^5. \quad (A8)$$

In each of the sets $M_2$ and $M_3$, two polynomials based on the sign of $\frac{a}{L} + k_1 [x_1] \frac{1}{i} + k_2 [x_2] \frac{1}{i} + \frac{a}{L}$ are derived. For the set $M_2$, they are

$$V_{\rho,21}(\theta_1, \theta_2, \theta_3) = a_1 \theta_1^{10} - a_2 \theta_1^6 \theta_2^4 + a_3 \theta_1^{10} - a_4 \theta_1^6 (-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^2 - a_5 \theta_2^2 (-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^3$$

$$+ a_6 (-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^5 - (-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)H_{21}(\theta_1, \theta_2, \theta_3). \quad (A9a)$$

$$V_{\rho,22}(\theta_1, \theta_2, \theta_3) = a_1 \theta_1^{10} - a_2 \theta_1^6 \theta_2^4 + a_3 \theta_1^{10} + a_4 \theta_1^6 (k_1 \theta_1^2 - k_2 \theta_2^2 - \theta_2^2)^2 - a_5 \theta_2^2 (-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2)^3$$

$$+ a_6 (-k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)^5 - (k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)H_{22}(\theta_1, \theta_2, \theta_3). \quad (A9b)$$

Please note that $\text{(A9a)}$ corresponds to the case $-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2 \geq 0$ and $\text{(A9b)}$ to $-k_1 \theta_1^2 + k_2 \theta_2^2 + \theta_2^2 < 0$. Therein, $H_{21}$ and $H_{22}$ are arbitrary sum-of-squares polynomials. These are integrated to avoid infeasibility of the resulting sum-of-squares problem since the exponent of either $-k_1 \theta_1^2$ or $-k_2 \theta_2^2 - \theta_2^2$ is five. In the set $M_3$, the two polynomials read as

$$V_{\rho,31}(\theta_1, \theta_2, \theta_3) = a_1 \theta_1^{10} - a_2 \theta_1^6 \theta_2^4 + a_3 \theta_1^{10} + a_4 \theta_1^6 (k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)^2 + a_5 \theta_2^2 (k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)^3$$

$$+ a_6 (k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)^5 - (k_1 \theta_1^2 - k_2 \theta_2^2 + \theta_2^2)H_{31}(\theta_1, \theta_2, \theta_3). \quad (A10a)$$
\[ V_{p,32} (\theta_1, \theta_2, \theta_4) = \alpha_1 \theta_1^{10} - \alpha_2 \theta_1^6 \theta_2^4 + \alpha_3 \theta_1^{10} - \alpha_4 \theta_1^6(-k_1 \theta_1^2 + \overline{k_2} \theta_2^2 - \theta_4^2)^2 + \alpha_5 \theta_1^4(\overline{k_1} \theta_1^2 - \overline{k_2} \theta_2^2 + \theta_4^2)^3 \]

\[ + \alpha_6(-k_1 \theta_1^2 + \overline{k_2} \theta_2^2 - \theta_4^2)^5 - (-k_1 \theta_1^2 + \overline{k_2} \theta_2^2 - \theta_4^2)H_{31}(\theta_1, \theta_2, \theta_4). \]  

(A10b)

Here the polynomials \( H_{31} \) and \( H_{32} \), which are also sums of squares, are incorporated to distinguish the two possible signs of \(-k_1 \theta_1^2 + \overline{k_2} \theta_2^2 - \theta_4^2\). It is noted that the set \( M_4 \) is disregarded in this case since \( x_1 \) is greater than or equal to zero for \( x_1, x_2 \geq 0 \) and \( u = \rho \) (the upper bound of \(|a|\) is a portion of \( \rho \) as given in (8)). According to (A3), (8), and (9b), \( \theta_4^2 \) is constrained as

\[ \theta_4^2 \in \left[ \frac{\rho - a_M}{L_1(\rho - a_M)}, \frac{\rho + a_M}{L_2(\rho + a_M)} \right] = \left[ \frac{1}{L_1}, \frac{1}{L_2} \right]. \]  

(A11)

The sum-of-squares problem is formulated here such that the polynomials

\[ V_{p,1} - S_1 - S_2 - c_1, \ V_{p,21} - S_1 - S_2 - c_1, \ V_{p,22} - S_1 - S_2 - c_1, \ V_{p,31} - S_1 - S_2 - c_1, \ V_{p,32} - S_1 - S_2 - c_1 \]  

(A12)

are sums of squares, where \( S_1 \) and \( S_2 \) are

\[ S_1(\theta_1, \theta_2, \theta_4) = \left( \frac{\theta_4^2}{L_1} \right) G_1(\theta_1, \theta_2, \theta_4). \]  

(A13a)

\[ S_2(\theta_1, \theta_2, \theta_4) = \left( \frac{1}{L_2} - \frac{\theta_4^2}{L_1} \right) G_2(\theta_1, \theta_2, \theta_4). \]  

(A13b)

Therein, the arbitrary polynomials \( G_1 \) and \( G_2 \) are sums of squares themselves.

It can be considered that \( \max \left( \frac{L_1}{L_1} |x_1|^2, \frac{L_1}{L_2} |x_2|^2 \right) \leq \gamma \) holds in the case the switching condition \( \frac{L_1}{L_1} |x_1|^2 + \frac{L_2}{L_2} |x_2|^2 \leq \gamma \) is met. Therefore, the smallest level \( c_2 \) presented in (19) can be computed through maximizing \( V_j(a L) \) written as

\[ V_j \left( \frac{a}{L} \right) = a \gamma^5 \left[ \frac{L^2}{L_1} |x_1|^2 |x_2|^2 \right]^2 - a_5 \gamma^5 \left[ \frac{L^2}{L_2} |x_2|^2 \right]^2 - a_6 \gamma^5 \left[ \frac{L^2}{L_2} |x_1|^2 \right]^2 \]  

(A14)

It can be seen that by setting \( \gamma = 0 \), \( V_j \) decreases to its lowest value and hence, the maximum upper bound of \( a_M \) for the given saturation limit \( \rho \) is realized. Based on (8) and (9b), it is derived that

\[ c_2 \geq a_6 \left( \frac{a_M}{L} \right)^5 \leq a_6 \left( \frac{\mu}{(1 + \mu)L_2} \right)^5. \]  

(A15)

To guarantee that inequality (17) is satisfied, the constraint

\[ c_1 \geq a_6 \left( \frac{\mu}{(1 + \mu)L_2} \right)^5 \]  

(A16)

is also integrated into the sum-of-squares problem. Alternatively, nonzero values of \( \gamma \) can be also considered by noting that

\[ \tilde{c}_2 = a_1 \gamma^5 \left[ \frac{L^2}{L_1} |x_1|^2 |x_2|^2 \right]^2 - a_5 \gamma^5 \left[ \frac{L^2}{L_2} |x_2|^2 \right]^2 - a_6 \gamma^5 \left[ \frac{L^2}{L_2} |x_1|^2 \right]^2 \]  

(A17)

is an upper bound for \( c_2 \), that is, \( c_2 \leq \tilde{c}_2 \) holds (note that \( a_4 \) is negative). Hence, if the constraint

\[ c_1 \geq a_1 \gamma^5 \left[ \frac{L^2}{L_1} |x_1|^2 |x_2|^2 \right]^2 - a_5 \gamma^5 \left[ \frac{L^2}{L_2} |x_2|^2 \right]^2 - a_6 \gamma^5 \left[ \frac{L^2}{L_2} |x_1|^2 \right]^2 \]  

(A18)
is added to the sum-of-squares problem instead of (A16) and the problem is still feasible, then the corresponding value of \( \gamma \) can be also chosen. Permissible values of \( L_1 \) and \( L_2 \) for different values of \( \mu \) are investigated such that feasible solutions for the coefficients \( \alpha \) and coefficients of \( H_{21}, H_{22}, H_{31}, H_{32}, G_1, \) and \( G_2 \) are achieved. A numerical solution yields the values listed in Table 1. Exemplarily, for \( \mu = 0.33 \), having chosen the constants given in (10), the coefficients \( \alpha \) obtained through the aforementioned toolbox with \( \varepsilon = 0.0008, L_1 = 0.21, \) and \( L_2 = 0.09 \) are

\[
\begin{align*}
\alpha_1 &= 11,217, \quad \alpha_2 = 1793, \quad \alpha_3 = 618, \\
\alpha_4 &= -224, \quad \alpha_5 = 2.36, \quad \alpha_6 = 0.2.
\end{align*}
\]