VALUE DISTRIBUTION FOR THE DERIVATIVES OF THE LOGARITHM OF $L$-FUNCTIONS FROM THE SELBERG CLASS IN THE HALF-PLANE OF ABSOLUTE CONVERGENCE

TAKASHI NAKAMURA AND ŁUKASZ PAŃKOWSKI

Abstract. In the present paper, we show that, for every $\delta > 0$, the function $(\log L(s))^{(m)}$, where $m \in \mathbb{N} \cup \{0\}$ and $L(s) := \sum_{n=1}^{\infty} a(n)n^{-s}$ is an element of the Selberg class $S$ takes any value infinitely often in any strip $1 < \Re(s) < 1 + \delta$, provided $\sum_{p \leq x} |a(p)|^2 \sim \kappa \pi(x)$ for some $\kappa > 0$. In particular, $L(s)$ takes any non-zero value infinitely often in the strip $1 < \Re(s) < 1 + \delta$, and the first derivative of $L(s)$ vanishes infinitely often.

1. Introduction and statement of main results

Let $S_A$ be the set of functions defined, for $\sigma := \Re(s) > 1$, as

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \exp \left( \sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \right),$$

where $a(n) \ll n^\varepsilon$ for any $\varepsilon > 0$ and $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$. Then it is well known that both the Dirichlet series and the Euler product converge absolutely when $\Re(s) := \sigma > 1$ and $a(p) = b(p)$ for every prime $p$ (e.g. [25, p. 112]). Moreover, the set $S_A$ includes the Selberg class $S$ (for the definition we refer to [13] or [25, Section 6]), which contains a lot of $L$-functions from number theory. As mentioned in [13, Section 2.1], the Riemann zeta function $\zeta(s)$, Dirichlet $L$-functions $L(s + i\theta, \chi)$ with $\theta \in \mathbb{R}$ and $\chi$ is a primitive character, $L$-functions associated with a holomorphic newforms of a congruence subgroup of $\text{SL}_2(\mathbb{Z})$ (after some normalization) are elements of the Selberg class. It should be noted that in fact $S \subsetneq S_A$, since for example $\zeta(s)/\zeta(2s) \in S_A$ but $\zeta(s)/\zeta(2s) \notin S$ by the fact that $\zeta(s)/\zeta(2s)$ has poles on the line $\Re(s) = 1/4$. Moreover, we can see that $S_A$ makes an abelian group structure (see Lemma 2.7).

Many mathematicians have been studying the distribution of the logarithmic derivative of the Riemann zeta function (see eg. [11]). For instance it is known that there are some relationships between mean value of products of logarithmic derivatives of $\zeta(s)$ near the critical line, correlations of the zeros of $\zeta(s)$ and the distribution of integers representable as a product of a fixed number of prime powers (see [10] and [11]). Moreover, Stopple investigated recently zeros of the second derivative of the logarithm of the Riemann zeta function in [26]. He proved that $(\log \zeta(s))''$ appears in the pair correlation for the zeros of $\zeta(s)$ (see for example [5]). In the present paper, we show the following result on value distribution of the $m$-th derivative of the logarithm of $L$-function from $S_A$.

2010 Mathematics Subject Classification. Primary 11M06, 11M26.

Key words and phrases. Derivatives of the logarithm of $L$-functions, Selberg class, value-distribution, zeros.
Theorem 1.1. Let $m \in \mathbb{N} \cup \{0\}$, $z \in \mathbb{C}$ and $L(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \in S_{\Lambda}$ satisfies
\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa
\] for some $\kappa > 0$. Then, for any $\delta > 0$, we have
\[
\# \{s : 1 < \text{Re}(s) < 1 + \delta, \ \text{Im}(s) \in [0, T] \text{ and } (\log L(s))^{(m)} = z\} \gg T
\] for sufficiently large $T$.

**Remark 1.2.** The condition (1.2) is closely related to the well-known Selberg conjecture
\[
\sum_{p \leq x} \frac{|a(p)|^2}{p} = \kappa \log \log x + O(1), \quad (\kappa > 0)
\] (1.4)

Obviously, by partial summation, it is implied by (1.2), however, it is a slightly weaker assumption than (1.2), since, in order to deduce (1.2) we need to assume that the error term in (1.4) is $C_1 + C_2/\log x + O((\log x)^{-2})$ for $C_1, C_2 \geq 0$.

**Remark 1.3.** As we show in Lemma 2.2 the assumption (1.2) implies that the abscissa of absolute convergence of $L(s)$ is equal to 1, which is also a necessary condition for (1.3).

The main reason, why the assumption that the abscissa of absolute convergence is 1 is not enough in our case, is the fact that we need to estimate the number of primes $p$ for which $a(p)$ is not too close to 0. Hence, if $|a(p)| > c$ for every prime $p$ and some constant $c > 0$, then (1.3) is equivalent to the fact that the abscissa of absolute convergence is 1.

As an immediate consequence of Theorem 1.1 we obtain the following.

**Corollary 1.4.** Let $z \in \mathbb{C} \setminus \{0\}$ and $L(s) \in S_{\Lambda}$ satisfies (1.2). Then, for any $\delta > 0$, we have
\[
\# \{s : 1 < \text{Re}(s) < 1 + \delta, \ \text{Im}(s) \in [0, T] \text{ and } L(s) = z\} \gg T
\] for sufficiently large $T$.

When $L(s) = \zeta(s)$, Bohr [3] proved that one has (1.3) (see also Remark 2.6). It is expected that the assertion (1.3) is true for the zeta functions defined as
\[
L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p} \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \quad \sigma > 1,
\] where the $\alpha_j(p)$ are complex numbers with $|\alpha_j(p)| \leq 1$ (see [25, p. 188, l. 12–13]). Note that the coefficients $a(n)$ appeared in $L(s)$ satisfy $a(n) \ll n^\varepsilon$ for any $\varepsilon > 0$ by [25, Lemma 2.2]. Hence we obtain $L(s) \in S_{\Lambda}$. Therefore, we have (1.3) for not only $L(s)$ given above but also $L(s) \in S_{\Lambda}$ by Corollary 1.4 Related to the $c$-values theorem above, the following uniqueness theorem proved by Li [15, Theorem 1] should be mentioned. If two $L$-functions $L_1$ and $L_2$ (without the Euler product) satisfy the same functional equation, $a(1) = 1$, and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ for two distinct complex numbers $c_1$ and $c_2$, then one has $L_1 = L_2$. Furthermore, Ki showed in [14, Theorem 1] that if two functions $L_1$ and $L_2$ in the extended Selberg class $S^\#$ (see for instance [13, p. 160] or [25, p. 217]) satisfy the same functional equation with positive degree, if $a(1) = 1$ and $L_1^{-1}(c) = L_2^{-1}(c)$ for a nonzero complex number $c$, then we have $L_1 = L_2$. Now let $L(s) \in S_{\Lambda}$ satisfy the all
assumptions of Corollary 1.4. Then from Corollary 1.4 we can see that for any \( c \in \mathbb{C} \setminus \{0\} \) and sufficiently large \( T \), it holds that

\[
\# L^{-1}(c) \geq \# \{ s \in \mathbb{C} : L(s) = c, \ Re(s) > 1, \ \Im(s) \in [0, T] \} \gg T.
\]

Next, since \( L(s) \) has no zeros in the half-plane of absolute convergence and \((\log L(s))' = L'(s)/L(s)\), we obtain immediately the following result by using Theorem 1.1 for \( m = 1 \) and \( z = 0 \).

**Corollary 1.5.** Let \( L(s) \in \mathcal{S}_A \) satisfies (1.2). Then for any \( \delta > 0 \), it holds that

\[
\# \{ s : 1 < \Re(s) < 1 + \delta, \ \Im(s) \in [0, T] \ \text{and} \ L'(s) = 0 \} \gg T
\]

for sufficiently large \( T \).

It is well-known that the first derivative of the Riemann zeta function has an infinite number of zeros in the region of absolute convergence \( \sigma > 1 \) (see [27, Theorem 11.5 (B)]). Corollary 1.5 is a generalization of this result. It should be mentioned that there are a lot of papers on zeros of the derivatives of the Riemann zeta function (see for instance [2], [16], [24] and articles which cite them). On the other hand, there are few papers treat zeros of the derivatives of other zeta or \( L \)-functions. However, it is worth writing the following fact proved in [28, Theorem 2]. Let \( \chi \) be a Dirichlet character to the modulus \( q \) and \( m \) be the smallest prime that does not divide \( q \). Then the \( k \)-th derivatives of the Dirichlet \( L \)-function \( L^{(k)}(s, \chi) \) does not vanish for the half-plane \( \sigma > 1 + \frac{m}{2} \left( 1 + \sqrt{1 + \frac{4k^2}{m \log m}} \right) \), \( k \in \mathbb{N} \).

As an application of Corollary 1.4 we show the following.

**Corollary 1.6.** Let \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \) and \( L_j(s) := \sum_{n=1}^{\infty} a_j(n)n^{-s} \in \mathcal{S}_A \) for \( j = 1, 2 \). Assume

\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a_1(p) - a_2(p)|^2 = \kappa, \qquad (\kappa > 0).
\]

Then for any \( \delta > 0 \), one has

\[
\# \{ s : 1 < \Re(s) < 1 + \delta, \ \Im(s) \in [0, T] \ \text{and} \ c_1L_1(s) + c_2L_2(s) = 0 \} \gg T
\]

for sufficiently large \( T \).

Now we mention earlier works related to zeros of zeta functions in the half-plane \( \sigma > 1 \). Davenport and Heilbronn [9] showed that if \( 0 < \alpha \neq 1/2, 1 \) is rational or transcendental, the Hurwitz zeta-function \( \zeta(s, \alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s} \) has infinitely many zeros in the region \( \Re(s) > 1 \). They also proved an analogue for the degree 2 Epstein zeta functions. Cassels [7] showed that \( \zeta(s, \alpha) \) has the same property when \( \alpha \) is algebraic and irrational. Saias and Weingartner [23] showed that a Dirichlet series with periodic coefficients \( F(s) \) does not vanish in the half-plane \( \sigma > 1 \) is equivalent to \( F(s) = P(s)L(s, \chi) \), where \( P(s) \) is a Dirichlet polynomial that does not vanish in \( \sigma > 1 \). Afterwards, Booker and Thorne [6], and very recently Righetti [22] generalized the work of Saias and Weingartner into general \( L \)-functions with bounded coefficients at primes.
By using Corollary 1.6, we obtain that the Euler-Zagier double zeta-function \( \zeta_2(s, s) = (\zeta^2(s) - \zeta(2s))/2 \) has zeros for \( \sigma > 1 \). Moreover, we can prove that the zeta-functions associated to symmetric matrices treated by Ibukiyama and Saito in [12, Theorem 1.2] vanish infinitely many times in the region of absolute convergence. In addition, some Epstein zeta functions, for example,

\[
\zeta(s; I_6) = -4(\zeta(s)L(s-2, \chi_4) - 4\zeta(s-2)L(s, \chi_4)),
\]

\[
\zeta(s; \mathcal{L}_{24}) = \frac{65520}{691}(\zeta(s)\zeta(s-11) - L(s; \Delta)),
\]

have infinitely many zeros for \( \sigma > 3 \) and \( \sigma > 12 \), respectively. It is known that \( \zeta_2(s, s) \) and \( \zeta(s; \mathcal{L}_{24}) \) vanish in the half-plane \( \sigma > 1 \) and \( \sigma > 12 \) from the numerical computations [18, Figure 1] and [21, Fig. 1]. Note that the examples above are mentioned in neither [6] nor [22]. Furthermore, we have to remark that these zeta functions mentioned above have infinitely many zeros outside of the region of absolute convergence (see [19, Main Theorem 1] and [20, Theorem 3.1]).

In Sections 2, we prove Theorem 1.1 and its corollaries. Some topics related to almost periodicity are discussed in Section 3. More precisely, we prove that for any \( \Re(\eta) > 0 \), the function \( \zeta(s) \pm \zeta(s + \eta) \) has zeros when \( \sigma > 1 \) (see Corollary 3.1) but for any \( \delta > 0 \), there exists \( \theta \in \mathbb{R} \setminus \{0\} \) such that the function \( \zeta(s) + \zeta(s + i\theta) \) does not vanish in the region \( \sigma \geq 1 + \delta \) (see Proposition 3.2).

2. Proofs of Theorem 1.1 and its corollaries

**Lemma 2.1.** Let \( r_1, \ldots, r_n \in \mathbb{C} \) be such that \( 0 < |r_1| \leq |r_2| \leq \cdots \leq |r_n| \) and \( R_0 = 0 \), \( R_j = |r_1| + \cdots + |r_j| \). Then

\[
\left\{ \sum_{j=1}^n c_j r_j : |c_j| = 1, \ c_j \in \mathbb{C} \right\} = \{ z \in \mathbb{C} : T_n \leq z \leq R_n \},
\]

where

\[
T_n = \begin{cases} 
|r_n| - R_{n-1} & \text{if } R_{n-1} \leq |r_n|, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** From [8, Proposition 3.3] every complex number \( z \) with \( T_n \leq |z| \leq R_n \) can be written as

\[
z = \sum_{j=1}^n c_j' r_j, \quad |c_j'| = 1.
\]

Hence, taking \( c_j = c_j' |r_j| / r_j \) completes the proof. \( \square \)

**Lemma 2.2.** Let \( L(s) = \sum_p \sum_{k \geq 1} b(p^k)p^{-ks} \) for \( \sigma > 1 \) be such that \( b(p^k) \ll p^{k\theta} \) for some \( \theta < 1/2 \), \( b(p) \ll p^\varepsilon \) for every \( \varepsilon > 0 \) and

\[
\lim_{x \to \infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p \leq x} |b(p)|^2 = \kappa
\]

for some \( \kappa > 0 \) and a non-negative integer \( m \). Then the abscissa of absolute convergence of \( \log L(s) \) is \( 1 \).
Proof. Assume that the abscissa of absolute convergence is smaller than 1. Then for some \( \theta + 1/2 < \sigma < 1 \) we have \( \sum_p \sum_{k \geq 1} |b(p^k)| p^{-k\sigma} < \infty \), and hence \( \sum_{p \leq x} |b(p)| p^{-\sigma} = O(1) \). Therefore, by Cauchy-Schwarz inequality we get for sufficiently small \( \varepsilon > 0 \) that

\[
\left( \sum_{p \leq x} \frac{|b(p)|^2}{p} \right)^2 \leq \sum_{p \leq x} \frac{|b(p)|}{p^\sigma} \sum_{p \leq x} \frac{|b(p)|^3}{p^{2-\sigma}} \ll \sum_{p \leq x} p^{3\varepsilon+\sigma-2} = o \left( \sum_{p \leq x} p^{-1} \right) = o(\log \log x).
\]

On the other hand, by partial summation and (2.1), we obtain that

\[
\sum_{p \leq x} \frac{|b(p)|^2}{p} \gg \log \log x,
\]

and hence we get a contradiction. \( \square \)

**Lemma 2.3.** Let \( b(p) \) be a sequence of complex numbers indexed by primes. Assume that \( b(p) \ll p^\varepsilon \) for every \( \varepsilon > 0 \) and

\[
\lim_{x \to \infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p \leq x} |b(p)|^2 = \kappa
\]

for some \( \kappa > 0 \) and a non-negative integer \( m \). Then for any \( c > 1, \eta > 0 \) and \( \varepsilon > 0 \) we have

\[
\sum_{x < p \leq cx} 1 \gg x^{1-\varepsilon}.
\]

Proof. One can easily get that

\[
\sum_{x < p \leq cx} |b(p)|^2 \ll \sum_{x < p \leq cx} 1 + x^{-2\eta} \sum_{x < p \leq cx} 1 \ll \sum_{x < p \leq cx} 1 + \frac{x^{1-2\eta}}{\log x}.
\]

On the other hand, we have

\[
\sum_{x < p \leq cx} |b(p)|^2 \gg x (\log x)^{l-1}.
\]

Hence the proof is complete. \( \square \)

**Lemma 2.4.** Let \( L(s) = \sum_p \sum_{k \geq 1} b(p^k) p^{-ks} \) for \( \sigma > 1 \) be such that \( b(p^k) \ll p^{k\theta} \) for some \( \theta < 1/2 \), \( b(p) \ll p^\varepsilon \) for every \( \varepsilon > 0 \) and

\[
\lim_{x \to \infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p \leq x} |b(p)|^2 = \kappa \tag{2.2}
\]

for some \( \kappa > 0 \) and a non-negative integer \( m \). Then, for every complex \( z \) and \( \delta > 0 \) there exist \( 1 < \sigma < 1 + \delta \) and a sequence \( \chi(p) \) of complex number indexed by primes such that \( |\chi(p)| = 1 \) and

\[
\sum_p \sum_{k \geq 1} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} = z.
\]
Proof. We follow the idea introduced by Cassels in [7].
Assume that $N_1$ is a positive integer, $\varepsilon > 0$ and $c_0 > 0$; we precise these parameters later. Put $M_j = [c_0 N_j]$ and $N_{j+1} = N_j + M_j$. We shall show that there exist $\sigma \in (1, 1 + \delta)$ and a sequence $\chi(p)$ with $|\chi(p)| = 1$ such that

$$
\left| \sum_{(p,k): p^k \leq N_j} \chi(p)^k b(p^k) \frac{p^{\sigma k}}{p^{k\sigma}} - z + \sum_{(p,k): \delta(p) \leq p^{-\varepsilon}} \frac{b(p^k)}{p^{k\sigma}} \right| \leq 10^{-2} \sum_{(p,k): p^k > N_j} |b(p^k)| \frac{p^{k\sigma}}{p^{k\sigma}},
$$

(2.3)

where $\sum^*$ denotes the double sum over $(p, k)$ satisfying $|\delta(p)| > p^{-\varepsilon}$, $p$ is prime and $k \in \mathbb{N}$. Let us note that for every $\sigma \in (1, 1 + \delta)$ we have

$$
\sum_{(p,k): \delta(p) \leq p^{-\varepsilon}} \frac{|b(p^k)|}{p^{k\sigma}} \leq \sum_{p: \delta(p) \leq p^{-\varepsilon}} \frac{1}{p^{1+\varepsilon}} + \sum_{p} \sum_{k \geq 2} \frac{|b(p^k)|}{p^k} =: S_0 < \infty.
$$

By (2.2) and Lemma 2.2, the abscissa of convergence of $\sum_p \sum_{k \geq 1} |b(p^k)| p^{-k\sigma}$ is 1, then by Landau’s theorem, this series has a pole at $\sigma = 1$, which implies that

$$
\sum_{(p,k)} |b(p^k)| p^{-k\sigma} \to \infty \quad \text{as} \quad \sigma \to 1^+.
$$

(2.4)

Therefore, we can find $\sigma \in (1, 1 + \delta)$ such that

$$
\sum_{(p,k): p^k \leq N_1} |b(p^k)| p^{-k\sigma} + |z| + S_0 \leq 10^{-2} \sum_{(p,k): p^k > N_1} |b(p^k)| p^{-k\sigma},
$$

and hence (2.3) holds for $j = 1$ and arbitrary $\chi(p)$’s with $p \leq N_1$.

Now, let us assume that complex numbers $\chi(p)$ are chosen for all $p \leq N_j$. We shall find $\chi(p)$ with $N_j < p \leq N_{j+1}$ and $|b(p)| > p^{-\varepsilon}$ such that (2.3) holds with $j + 1$ instead of $j$.

Let $\mathfrak{A}$ denote the set of pairs $(p, 1)$ satisfying $p \in (N_j, N_{j+1}]$ is a prime number and $|b(p)| > p^{-\varepsilon}$. Moreover, define

$$
\mathfrak{B} = \{(p, k) : p^k \in (N_j, N_{j+1}], \text{ p is prime, } k \geq 2, \ |b(p)| > p^{-\varepsilon}\}.
$$

Note that $\chi(p)^k$’s are already defined for $(p, k) \in \mathfrak{B}$, since for suitable $N_1$ and $c_0$ we have $p \leq \sqrt{N_{j+1}} < N_{j}$ if $(p, k) \in \mathfrak{B}$.

Using Lemma 2.3 gives that

$$
|\mathfrak{A}| \gg N_j^{1-\varepsilon}
$$

and since $k \geq 2$ for every $(p, k) \in \mathfrak{B}$ we have

$$
|\mathfrak{B}| \ll N_j^{\frac{1}{2}}.
$$

Moreover, note that for every $p_1, p_2$ satisfying $(p_1, 1), (p_2, 1) \in \mathfrak{A}$, by Ramanujan’s conjecture, we have

$$
\frac{|b(p_1)|}{b(p_2)} \ll N_j^{2\varepsilon} \quad \text{and} \quad \left( \frac{p_2}{p_1} \right)^{\sigma} \leq \left( \frac{N_{j+1}}{N_j} \right)^{\sigma} \leq (c_0 + 1)^{1+\delta},
$$

so

$$
\frac{|b(p_2)|}{p_2^{\sigma}} \gg N_j^{-2\varepsilon} \frac{|b(p_1)|}{p_1^{\sigma}}.
$$
Hence, using Lemma 2.1 with the sequence $b(p)p^{-\sigma}$, where $(p, 1) \in \mathfrak{A}$, we obtain that

$$\sum_{(p, 1) \in \mathfrak{A}}^* \frac{b(p)\chi(p)}{p^\sigma}, \quad |\chi(p)| = 1,$$

takes all values $z_0$ with $|z_0| \leq \sum_{(p, 1) \in \mathfrak{A}}^* |b(p)p^{-\sigma}| =: S_3$, since for sufficiently large $N_1$ and arbitrary $p_0$ satisfying $(p_0, 1) \in \mathfrak{A}$, we have

$$\sum_{(p_0, 1) \neq (p, 1) \in \mathfrak{A}}^* \frac{|b(p)|}{p^\sigma} \gg N_j^{1-3\varepsilon} \frac{|b(p_0)|}{p_0^\sigma} > \frac{|b(p_0)|}{p_0^\sigma},$$

so the inner radius $T[\mathfrak{A}]$ in Lemma 2.1 is 0.

Write

$$\Lambda := \sum_{(p, k): p^k \leq N_j}^* \frac{\chi(p)k b(p^k)}{p^{k\sigma}} - z + \sum_{(p, k): |b(p)| \leq p^{\varepsilon}}^* \frac{b(p^k)}{p^{k\sigma}} + \sum_{(p, k) \in \mathfrak{B}}^* \frac{\chi(p)k b(p^k)}{p^{k\sigma}}$$

and put

$$z_0 = \begin{cases} -\Lambda & \text{if } 0 < |\Lambda| \leq S_3, \\ -S_3\Lambda/|\Lambda| & \text{if } |\Lambda| > S_3, \\ 0 & \text{if } \Lambda = 0. \end{cases}$$

Then, from Lemma 2.1 we can choose $\chi(p)$ for $(p, 1) \in \mathfrak{A}$ such that

$$\left| \sum_{(p, k): p^k \leq N_j + M_j}^* \frac{\chi(p)k b(p^k)}{p^{k\sigma}} - z + \sum_{(p, k): |b(p)| \leq p^{\varepsilon}}^* \frac{b(p^k)}{p^{k\sigma}} \right| = \left| \Lambda + \sum_{(p, 1) \in \mathfrak{A}}^* \frac{b(p)\chi(p)}{p^\sigma} \right| \leq \max(0, S_1 + S_2 - S_3),$$

where

$$S_1 := \left| \sum_{(p, k): p^k \leq N_j}^* \frac{\chi(p)k b(p^k)}{p^{k\sigma}} - z + \sum_{(p, k): |b(p)| \leq p^{\varepsilon}}^* \frac{b(p^k)}{p^{k\sigma}} \right|$$

and

$$S_2 := \sum_{(p, k) \in \mathfrak{B}}^* \frac{|b(p^k)|}{p^{k\sigma}},$$

so $|\Lambda| \leq S_1 + S_2$.

Now, let us notice that

$$\frac{S_3}{S_2} \geq \frac{N_j^{\sigma-\theta} |\mathfrak{A}|}{N_j^{\sigma+\varepsilon} |\mathfrak{B}|} \gg N_j^{1/2-\theta-2\varepsilon} \geq \frac{101}{99}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large $N_1$. Hence

$$S_2 - S_3 \leq -10^{-2}(S_2 + S_3).$$

Moreover, from (2.3) we have

$$S_1 \leq 10^{-2}(S_2 + S_3 + S_4),$$
where
\[ S_4 := \sum_{(p,k) : p^k > N_{j+1}} |b(p^k)| \frac{1}{p^{k\sigma}}. \]
Thus \( S_1 + S_2 - S_3 < 10^{-2}S_4 \) and, by induction, (2.3) holds for all \( j \in \mathbb{N} \). So letting \( N_j \to \infty \) completes the proof. \( \square \)

Kronecker’s approximation theorem (see for example [25, Lemma 1.8]) plays an important role in the proof of the following lemma.

**Lemma 2.5.** Let \( L(s) = \sum_p \sum_{k \geq 1} b(p^k)p^{-ks} \) for \( \sigma > 1 \) be such that \( b(p^k) \ll p^{k\theta} \) for some \( \theta < 1/2 \), \( b(p) \ll p^z \) for every \( \varepsilon > 0 \) and
\[ \lim_{x \to \infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p \leq x} |b(p)|^2 = \kappa \]
for some \( \kappa > 0 \) and a non-negative integer \( m \). Then, for every \( z \) and \( \delta > 0 \), the set of real \( \tau \) satisfying
\[ L(s + i\tau) = z \quad \text{for some } 1 < \text{Re}(s) < 1 + \delta, \]
has a positive lower density. In particular, \( \text{the Lebesgue measure of } \tau \in [0, T] \text{ satisfying the above equation is greater than } CT, \) where \( C \) is a some positive constant and \( T \) is sufficiently large.

**Proof.** By Lemma 2.4 we choose \( \sigma \in (1, 1 + \delta) \) and a sequence \( \chi(p) \) with \( |\chi(p)| = 1 \) such that
\[ \sum_p \sum_{k \geq 1} \chi(p)^k b(p^k) \frac{1}{p^{k\sigma}} = z. \]
Next, since \( F(s) = \sum_p \sum_{k \geq 1} \chi(p)^k b(p^k)p^{-ks} \) is analytic in the half-plane \( \text{Re}(s) > 1 \), we can find \( r \) with \( 0 < r < \sigma - 1 \) such that \( F(s) - z \neq 0 \) if \( |s - \sigma| = r \). Then we put \( \varepsilon := \min_{|s - z| = r} |F(s) - z| \).

Since the series \( \sum_p \sum_{k=1}^\infty |b(p^k)|p^{-k(\sigma - r)} \) converges absolutely, we can take a positive integer \( M \) such that
\[ \sum_{p \leq M} \sum_{k \geq M} |b(p^k)| \frac{1}{p^{k(\sigma - r)}} + \sum_{p > M} \sum_{k=1}^\infty |b(p^k)| \frac{1}{p^{k(\sigma - r)}} < \frac{\varepsilon}{4}. \quad (2.5) \]

Moreover, if we assume that
\[ \max_{p \leq M} |p^{-ir} - \chi(p)| < \varepsilon_1 \quad (2.6) \]
for \( \varepsilon_1 > 0 \), then
\[ |p^{-ikr} - \chi(p)^k| = |p^{-ir} - \chi(p)||p^{-i(k-1)r} + p^{-i(k-2)r} \chi(p) + \cdots + p^{-ir} \chi(p)^{k-2} + \chi(p)^{k-1}| \]
\[ < k\varepsilon_1 \leq M\varepsilon_1, \quad 1 \leq k \leq M. \]

Therefore, for sufficiently small \( \varepsilon_1 \) and \( s \) satisfying \( |s - \sigma| = r \), we obtain
\[ \left| \sum_{p \leq M} \sum_{k=1}^M \frac{b(p^k)}{p^{k(s+ir)}} - \sum_{p \leq M} \sum_{k=1}^M \frac{b(p^k)\chi(p)^k}{p^{ks}} \right| < M\varepsilon_1 \sum_{p \leq M} \sum_{k=1}^M |b(p^k)| \frac{\varepsilon}{2}, \]
and 
\[ |L(s + i\tau) - z - (F(s) - z)| = |L(s + i\tau) - F(s)| < \varepsilon \leq |F(s) - z|, \]
provided (2.6) holds.

Thus, by Rouché’s theorem (see for example [25, Theorem 8.1]), for every \( \tau \) satisfying (2.6) there is a complex number \( s \) with \( |s - \sigma| \leq r \) such that \( L(s + i\tau) = z \). But, by the classical Kronecker approximation theorem, the set of \( \tau \) satisfying (2.6) has a positive density, so the number of solutions of the equation \( L(s + i\tau) = z \) with \( 1 < \text{Re}(s) < 1 + \delta \) and \( \tau \in [0, T] \) is \( \gg T \) for sufficiently large \( T > 0 \). \hfill \Box

Now we are in a position to show Theorem 1.1.

**Proof of Theorem 1.1.** Obviously, the case \( m = 0 \) follows immediately from Lemma 2.5 since \( a(p) = b(p) \) for every prime \( p \). Thus it suffices to show that for every \( m \geq 1 \) the function \((\log \mathcal{L}(s))^{(m)}\) satisfies the assumption of Lemma 2.5.

Note that
\[
(-1)^m (\log \mathcal{L}(s))^{(m)} = \sum_p \sum_{k=1}^{\infty} \frac{b(p^k)(k \log p)^m}{p^{ks}}, \quad \sigma > 1.
\]

Obviously, one has \( b(p)(\log p)^m = a(p)(\log p)^m \ll p^\varepsilon \) for every \( \varepsilon > 0 \), and \( b(p^k)(k \log p)^m \ll p^{\theta k} \) for some \( \theta \) with \( \theta < \theta_1 < 1/2 \) by the assumption \( b(p^k) \ll p^{\theta k} \) for some \( \theta < 1/2 \).

Moreover, by partial summation and (1.2), we get
\[
\sum_{p \leq x} |b(p)|^2 (\log p)^{2m} = \sum_{p \leq x} |a(p)|^2 (\log p)^{2m} = \kappa(\log x)^{2m} \pi(x)(1 + o(1)),
\]
which completes the proof. \hfill \Box

**Remark 2.6.** In Bohr’s proof of Corollary 1.4 for \( \mathcal{L}(s) = \zeta(s) \), the convexity of
\[ -\log(1 - p^{-s}) = \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} \]
plays a crucial role (see also [25, Theorem 1.3] and [27, Theorem 11.6 (B)]). However, we prove Corollary 1.4 without using the convexity since the closed curve described by \( \sum_{k=1}^{\infty} b(p^k)p^{-ks} \) is not always convex when \( t \) runs through the whole \( \mathbb{R} \) (see also [17]).

In order to prove Corollary 1.6, we show the following lemma. It should be mentioned that one has \( \mathcal{L}_1\mathcal{L}_2 \in \mathcal{S}_A \) when \( \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}_A \) as well as in the case of the Selberg class \( \mathcal{S} \).

**Lemma 2.7.** Let \( \mathcal{L}(s) \in \mathcal{S}_A \). Then we have \( 1/\mathcal{L}(s) \in \mathcal{S}_A \).

**Proof.** Suppose that \( \mathcal{L}(s) \in \mathcal{S}_A \) is expressed as (1.1). It is known that \( a(1) = 1 \), by (1.1) for \( s \rightarrow \infty \). Then we have
\[
\frac{1}{\mathcal{L}(s)} = \sum_{n=1}^{\infty} \frac{a^{-1}(n)}{n^s} = \prod_p \exp \left( \sum_{k=1}^{\infty} \frac{-b(p^k)}{p^{ks}} \right),
\]
where \( a^{-1}(n) \) is the Dirichlet inverse of \( a(n) \) given by
\[
a^{-1}(1) = \frac{1}{a(1)} = 1, \quad a^{-1}(n) = -\sum_{d|n, d<n} a(n/d)a^{-1}(d), \quad n > 1
\]
(see for instance [1, Theorem 2.8 and Example 2 in Section 11.4]). By (1.1) and the assumption \( L(s) \in S_A \), we can see that \(-b(p^k) \ll p^k\theta \) for some \( \theta < 1/2 \) and the Euler product of \( 1/L(s) \) converges absolutely when \( \sigma > 1 \). Hence we only have to show \( a^{-1}(n) \ll n^\varepsilon \) for any \( \varepsilon > 0 \). Suppose \( a^{-1}(d) \ll d^\varepsilon \) for all divisors \( d < n \). From the expression of \( a^{-1}(n) \) and the assumption \( a(n) \ll n^\varepsilon \), it holds that

\[
a^{-1}(n) \ll n^\varepsilon \sum_{d|n, d<n} a^{-1}(d) \ll n^\varepsilon \sum_{d|n, d<n} 1 \ll n^\varepsilon d(n),
\]

where \( d(n) \) is the divisor function. On the other hand, it is well-known that \( d(n) \ll n^\varepsilon \) (see for example [1, Theorem 13.12]). Therefore we have Lemma 2.7. \( \square \)

**Proof of Corollary 1.6.** Obviously, the statement \( c_1L_1(s) + c_2L_2(s) = 0 \) is equivalent to \( L_1(s)/L_2(s) = -c_2/c_1 \) when \( L_1(s), L_2(s) \in S_A \) and \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Furthermore, if \( L_1, L_2 \in S_A \), then one has \( L_1/L_2 \in S_A \) from Lemma 2.7. Therefore, we obtain Corollary 1.6 from Corollary 1.4 since (1.6) means exactly that (1.2) holds for the function \( L_1/L_2 \). \( \square \)

**Remark 2.8.** Note that

\[
\sum_{p \leq x} |a_1(p) - a_2(p)|^2 = \sum_{p \leq x} |a_1(p)|^2 + \sum_{p \leq x} |a_2(p)|^2 - 2 \text{Re} \sum_{p \leq x} a_1(p)\overline{a_2(p)}.
\]  

(2.7)

Therefore, if the abscissa of absolute convergence for both \( L \)-functions \( L_1 \) and \( L_2 \) is 1, then the assumption (1.6) in Corollary 1.6 can be replaced by Selberg’s orthonormality conjecture in the following stronger form

\[
\forall j=1,2 \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a_j(p)|^2 = \kappa_j, \quad \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} a_1(p)\overline{a_2(p)} = 0,
\]

for some \( \kappa_1, \kappa_2 > 0 \).

On the other hand, if the abscissa of absolute convergence of one of them, say \( L_2 \), is less than 1, then as in the proof of Lemma 2.2, we get

\[
\sum_{p \leq x} |a_2(p)|^2 \leq \sqrt{\sum_{p \leq x} \frac{|a_2(p)|}{p^{\sigma_0}}} \sqrt{\sum_{p \leq x} |a_2(p)|^3 p^{\sigma_0} \ll x^{1/2+\sigma_0/2+\varepsilon}}
\]

for some \( \sigma_0 < 1 \) and every \( \varepsilon > 0 \). Moreover, by Cauchy-Schwarz inequality, we have

\[
\text{Re} \sum_{p \leq x} a_1(p)\overline{a_2(p)} \leq \sqrt{\sum_{p \leq x} |a_1(p)|^2} \sqrt{\sum_{p \leq x} |a_2(p)|^2} \ll x^{3/4+\sigma_0/4+\varepsilon}
\]

for every \( \varepsilon > 0 \).

Therefore, by (2.7), we obtain

\[
\sum_{p \leq x} |a_1(p) - a_2(p)|^2 = \sum_{p \leq x} |a_1(p)|^2 + O(x^{3/4+\sigma_0/4+\varepsilon}),
\]

and assuming (1.2) for \( L_1 \) implies Corollary 1.6.
3. Almost periodicity and Corollary 1.6

We quote the notion of almost periodicity from [25, Section 9.5]. In 1922, Bohr [4] proved that every Dirichlet series $f(s)$, having a finite abscissa of absolute convergence $\sigma_a$ is almost periodic in the half-plane $\sigma > \sigma_a$. Namely, for any given $\delta > 0$ and $\varepsilon > 0$, there exists a length $l := l(f, \delta, \varepsilon)$ such that every interval of length $m$ contains a number $\tau$ for which

$$|f(\sigma + it + ir) - f(\sigma + it)| < \varepsilon$$

holds for any $\sigma \geq \sigma_a + \delta$ and for all $t \in \mathbb{R}$. From the Dirichlet series expression, the zeta function $\mathcal{L}(s) \in \mathcal{S}_A$ is almost periodic when $\sigma > 1$. By using Corollary 1.6, we have the following corollary as a kind of analogue of the almost periodicity.

**Corollary 3.1.** Let $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \in \mathcal{S}_A$ satisfies (1.2). Suppose $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $\text{Re}(\eta) > 0$. Then one has

$$\# \{s : \text{Re}(s) > 1, \ \text{Im}(s) \in [0, T] \text{ and } c_1\mathcal{L}(s) + c_2\mathcal{L}(s + \eta) = 0\} \gg T$$

for sufficiently large $T$.

**Proof.** The corollary follows from Remark 2.8 since the abscissa of absolute convergence of $\mathcal{L}(s + \eta)$ is smaller than 1. \hfill $\square$

On the contrary, we have the following proposition when $\text{Re}(\eta) = 0$.

**Proposition 3.2.** Let $\mathcal{L}(s) \in \mathcal{S}_A$. Then for any $\delta > 0$, there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that the function

$$\mathcal{L}(s) + \mathcal{L}(s + i\theta)$$

does not vanish in the region $\sigma \geq 1 + \delta$.

**Proof.** For any $\varepsilon > 0$, we can find $\theta \in \mathbb{R} \setminus \{0\}$ which satisfies

$$|\mathcal{L}(s) - \mathcal{L}(s + i\theta)| < \varepsilon, \quad \text{Re}(s) \geq 1 + \delta$$

from almost periodicity of $\mathcal{L}(s) \in \mathcal{S}_A$. Hence we have

$$|\mathcal{L}(s) + \mathcal{L}(s + i\theta)| = |2\mathcal{L}(s) + \mathcal{L}(s + i\theta) - \mathcal{L}(s)| \geq |2\mathcal{L}(s)| - |\mathcal{L}(s) - \mathcal{L}(s + i\theta)|$$

$$> 2 \prod_p \exp\left(-\sum_{k=1}^{\infty} \frac{|b(p^k)|}{p^{k(1+\delta)}}\right) - \varepsilon, \quad \text{Re}(s) \geq 1 + \delta.$$ 

From the assumption for $\mathcal{L}(s) \in \mathcal{S}_A$, the sum $\sum_p \sum_{k=1}^{\infty} |b(p^k)| p^{-k(1+\delta)}$ converges absolutely when $\delta > 0$. Hence, by taking suitable $\varepsilon > 0$, we have

$$|\mathcal{L}(s) + \mathcal{L}(s + i\theta)| > 0, \quad \text{Re}(s) \geq 1 + \delta.$$ 

This inequality implies Proposition 3.2. \hfill $\square$

**Remark 3.3.** Proposition 3.2 should be compared with the following fact. Let $\theta \in \mathbb{R} \setminus \{0\}$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then the function

$$c_1\zeta(s) + c_2\zeta(s + i\theta)$$

vanishes in the strip $1/2 < \sigma < 1$. This is an easy consequence of [25, Theorem 10.7].
Hence, for any \( \delta > 0 \), there exist \( \theta \in \mathbb{R} \setminus \{0\} \) such that the function
\[
\zeta(s) + \zeta(s + i\theta)
\]
does not vanish in the half-plane \( \sigma \geq 1 + \delta \), but has infinitely many zeros in the vertical strip \( 1/2 < \sigma < 1 \).

**Acknowledgments.** The first author was partially supported by JSPS grant 24740029. The second author was partially supported by (JSPS) KAKENHI grant no. 26004317 and the grant no. 2013/11/B/ST1/02799 from the National Science Centre.

**References**

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] B. C. Berndt, *The number of zeros for \( \zeta^{(k)}(s) \)*, J. London Math. Soc. 2 (1970), 577–580.

[3] H. Bohr, *Über das Verhalten von \( \zeta(s) > 1 \) in der Halbebene \( \sigma > 1 \)*, Nachr. Akad. Wiss. Göttingen II Math. Phys. Kl. (1911), 409–428.

[4] H. Bohr, *Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen \( L \)-Funktionen*, Math. Ann. 85 (1922), no. 1, 115–122.

[5] E. B. Bogomolny and J. P. Keating, *Gutzwiller’s trace formula and spectral statistics: beyond the diagonal approximation*, Phys. Rev. Lett. 77 (1996), no. 8, 1472–1475.

[6] A. Booker and F. Thorne, *Zeros of \( L \)-functions outside the critical strip*, Algebra & Number Theory (2014), no. 9, 2027–2042. (arXiv:1306.6362).

[7] J. W. S. Cassels, *Footnote to a note of Davenport and Heilbronn*. J. London Math. Soc. 36 (1961) 177-184.

[8] T. Chatterjee, S. Gun, *On the zeros of generalized Hurwitz zeta functions*, J. Number Theory 145 (2014), 352–361.

[9] H. Davenport and H. Heilbronn, *On the zeros of certain Dirichlet series I, II*, J. London Math. Soc. 11 (1936), 181-185, 307-312.

[10] D. W. Farmer, S. M. Gonek, Y. Lee and S. J. Lester, *Mean values of \( \zeta'/\zeta(s) \), correlations of zeros and the distribution of almost primes*, Q. J. Math. 64 (2013), no. 4, 1057–1089.

[11] D. A. Goldston, S. M. Gonek and H. L. Montgomery, *Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals*, J. Reine Angew. Math. 537 (2001), 105–126.

[12] T. Ibukiyama and H. Saito, *On zeta functions associated to symmetric matrices, I. An explicit form of zeta functions*, Amer. J. Math. 117 (1995), no. 5, 1097–1155.

[13] J. Kaczorowski, *Axiomatic theory of \( L \)-functions: the Selberg class*. Analytic number theory, 133–209, Lecture Notes in Math., 1891, Springer, Berlin, 2006.

[14] H. Ki, *A remark on the uniqueness of the Dirichlet series with a Riemann-type function equation*, Adv. Math. 231 (2012), no. 5, 2484–2490.

[15] B. Q. Li, *A uniqueness theorem for Dirichlet series satisfying a Riemann type functional equation*, Adv. Math. 226 (2011), no. 5, 4198–4211.

[16] N. Levinson and H. L. Montgomery, *Zeros of the derivatives of the Riemann zeta function*, Acta Math. 133 (1974), 49–65.

[17] K. Matsumoto, *Probabilistic value-distribution theory of zeta-functions*, Sugaku 53 (2001), 279–296 (in Japanese); English Transl.: Sugaku Expositions 17 (2004), 51–71.

[18] K. Matsumoto and M. Shōji, *Numerical computations on the zeros of the Euler double zeta-function I*, arXiv:1403.3765.

[19] T. Nakamura and L. Pańkowski, *On complex zeros off the critical line for non-monomial polynomial of zeta-functions*, arXiv:1212.5890.
[20] T. Nakamura and L. Pańkowski, *On zeros and c-values of Epstein zeta-functions*, Šiauliai Mathematical Seminar (Special volume celebrating the 65th birthday of Professor Antanas Laurinčikas) 8 (2013), 181–196.

[21] N. V. Proskurin, *On the zeros of the zeta function of the Leech lattice*, J. Math. Sci. (N. Y.) 193 (2013), no. 1, 124–128.

[22] M. Righetti, *Zeros of combinations of Euler products for \( \sigma > 1 \)*, arXiv:1412.6331.

[23] E. Saias and A. Weingartner, *Zeros of Dirichlet series with periodic coefficients*, Acta Arith. 140 (2009), no. 4, 335–344.

[24] A. Speiser, *Geometrisches zur Riemannschen Zetafunktion*, Math. Ann. 110 (1935), no. 1, 514–521.

[25] J. Steuding, *Value-Distribution of L-functions*, Lecture Notes in Mathematics, 1877, Springer, Berlin, 2007.

[26] J. Stopple, *Notes on \( \log(\zeta(s))' \)*, arXiv:1311.5465 (to appear in the Rocky Mountain Journal of Mathematics).

[27] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.

[28] C. Y. Yildirim, *Zeros of derivatives of Dirichlet L-functions*, Turk. J. of Mathematics 20, (1996), 521–534.

(T. Nakamura) **DEPARTMENT OF LIBERAL ARTS, FACULTY OF SCIENCE AND TECHNOLOGY, TOKYO UNIVERSITY OF SCIENCE, 2641 YAMAZAKI, NODA-SHI, CHIBA-KEN, 278-8510, JAPAN**

*E-mail address*: nakumuratakashis@rs.tus.ac.jp

*URL*: https://sites.google.com/site/takashinakumurazeta/

(L. Pańkowski) **FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, ADAM MICKIEWICZ UNIVERSITY, UMULTOWSKA 87, 61-614 POZNAŃ, POLAND, AND GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN**

*E-mail address*: lpan@amu.edu.pl