Spurious poles of the axial gauge propagators and dynamics of the interacting fields

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The origin of the spurious poles of the gauge field propagators in the temporal axial and the null-plane gauges is discussed. The conclusion is that these poles do not require any special prescription. They are a manifestation of the fact that the gauge field acquires a static configuration.

I. INTRODUCTION

The issue of spurious poles of the propagators in axial and null-plane gauges has been controversial for almost two decades. The absence of ghosts in these physical gauges makes them very attractive despite the loss of the manifest Lorentz covariance, which latter is usually recovered in calculations of observables. Uncertainty in the treatment of spurious poles is probably the predominate reason why many theorists prefer to use covariant gauges (and enjoy the relativistic invariance during the intermediate stages of calculations). Much effort has been spent in order to find a universal prescription to handle the spurious poles. The principal value prescription and the one of Leibbrandt and Mandelstam were tested in various calculations of the Wilson loop up to fourth order perturbation theory. The total score of gains and losses looks approximately equal. Solutions to the problem were also looked for along the lines of path-dependent formulation; its connection with the problem of the residual symmetry was also understood. It may look surprising, that the object of the controversy, the propagator of the perturbation theory, is so simple, and that the problem is not specific for non-abelian gauge fields, but exists in the same form in QED as well. All of the above studies attempted to solve the problem of spurious poles in the general context of gauge field theory.

Spurious poles are safe in practical calculations like, for example, in computing cross sections. They either cancel with the traces in the numerators of the matrix elements or are unaccessible for kinematic reasons. However, the problem does appear in the less–standard calculations, where the choice of the prescription may affect the physical results.

In this paper we discuss the problem of spurious poles keeping in mind an environment where the Lorentz and/or translational symmetry is already broken by the actual geometry of the physical process, as in, for example, deep inelastic electron-proton scattering or in central collisions of hadrons or heavy ions. For these processes, one has a natural choice of the axis for the gauge condition. Moreover, use of the covariant gauges is highly undesirable in these cases. Expecting the creation of the statistical system one immediately is faced with the problem of the unphysical ghosts distribution.

Instead of an examination of the consequences of different prescriptions for quantum field theory calculations, this paper concentrates on the different ways to derive the propagator in the temporal axial gauge \( A^0 = 0 \) and the null-plane gauge \( A^+ = 0 \). Since the object is primitive, the focus will be on the classical aspects of the derivation. There is no difference between the Green functions of the quantum perturbation theory and the singular (fundamental) solutions of the system of Maxwell equations which describe the interaction of the gauge field with the classical current. Causality and the influence of initial and boundary conditions will be taken as the main guidelines in the derivation of the Green functions.

This investigation of the nature of spurious poles is undertaken as a part of a general study of field dynamics in deeply inelastic high energy processes. This analysis is conceived as a prototype for a calculation of the gluon Green functions in the “wedge dynamic”. The final conclusion of this work is that contributions of spurious poles correspond to the classical character of the gauge field. These poles do not need special treatment, but they serve as important indicators of the static configuration of the gauge field.

II. GAUGE FIELD CORRELATORS IN THE GAUGE \( A^0 = 0 \)

In this section we briefly illustrate our approach for obtaining the fundamental solutions of Maxwell equations using a technically more simple example of the temporal axial gauge \( A^0 = 0 \). This gauge has the known problem of the
infinite growth of the propagator which manifests itself as an additional pole of the polarization sum and uncertainty of the treatment of this pole. However, because of the manifest translational invariance (the gauge condition singles out a direction and not the point!), one may use a simple Fourier analysis and avoid specific problems due to nonlocality of the propagator in the gauge $A^\tau = 0$. We shall not attempt to invert the matrix differential operator, either algebraically or by means of the path integral formalism. Instead, we shall obtain various propagators of the perturbation theory via their expansion over a set of classical solutions. Analysis of these solutions sheds light upon the meaning of the different prescriptions to handle the gauge poles.

The gauge invariant action of the Abelian theory is as follows,

$$\mathcal{S} = \int \mathcal{L}(x) d^4x = \int \left[-\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - j^\mu A_\mu\right]\sqrt{-g} d^4x.$$  \hspace{1cm} (2.1)

Its variation with respect to the gauge field yields the Lagrangian equations of motion,

$$\partial_\mu F^{\mu\nu} = j^\nu.$$  \hspace{1cm} (2.2)

With the gauge condition $A^0 = 0$, the system of Maxwell equations with the classical current $j^\mu$ can be written down in terms of the 3-dimensional Fourier components as follows:

$$[(\partial_i^2 + k_i^2)\delta^{ij} - k_i k_j] A^j(k, t) = j^i(k, t)$$  \hspace{1cm} (2.3)

$$C(x) = \partial_x E_x + \partial_y E_y + \partial_z E_z - j^0 = 0,$$  \hspace{1cm} (2.4)

where the indices $i$ and $j$ numerate the spacial coordinates and $E_i = \hat{A}_i$ is the strength of the electric field. Equation (2.4) is a constraint corresponding to Gauss’ law.

As mentioned above, the set of the ordered field correlators used in the quantum theory,

$$D^{\mu\nu}_{00}(x, y) = -i(0\langle T(A^\mu(x) A^\nu(y))|0\rangle), \quad D^{\mu\nu}_{01}(x, y) = -i(0\langle A^\nu(y)A^\mu(x)|0\rangle),$$

$$D^{\mu\nu}_{01}(x, y) = -i(0\langle T(A^\mu(x) A^\nu(y))|0\rangle), \quad D^{\mu\nu}_{11}(x, y) = -i(0\langle T(A^\mu(x) A^\nu(y))|0\rangle),$$  \hspace{1cm} (2.5)

along with the retarded and advanced Green functions

$$D_{ret}^{\mu\nu}(x, y) = D^{\mu\nu}_{00}(x, y) - D^{\mu\nu}_{01}(x, y) = -i\theta(x^0 - y^0)\langle 0| [A^\mu(x), A^\nu(y)]|0\rangle,$$

$$D_{adv}^{\mu\nu}(x, y) = D^{\mu\nu}_{00}(x, y) - D^{\mu\nu}_{10}(x, y) = i\theta(y^0 - x^0)\langle 0| [A^\mu(x), A^\nu(y)]|0\rangle$$  \hspace{1cm} (2.6)

coincide with various singular or fundamental solutions of the classical equations and can be studied regardless their quantum nature. The quantum content reveals itself only when the Fock space is constructed and the bilinear expansion of these correlators over the eigenfunctions is obtained as a quantum average over the vacuum state. It is also important that except for the simple linear relations between various correlators we have also dispersion relations which reflect causal properties of the theory,

$$\text{Re}D_{ret, adv}^{\mu\nu}(k^0, k) = \frac{1}{\pi \text{i}} \int_{-\infty}^{\infty} \frac{\text{Im}D_{ret, adv}^{\mu\nu}(\omega, k)}{\omega - k^0} d\omega.$$  \hspace{1cm} (2.7)

They can be formally derived from (2.6) and the relation, $D_0 = D_{ret} - D_{adv} = 2\text{Im}D_{ret}$.

A. Wightman functions of the free gauge field

Our first goal is to find solutions of the homogeneous system (2.3) of Maxwell equations (with $j_i(k, t) = 0$). The solutions of the homogeneous equations will be looked for in terms of the auxiliary functions,

$$\Phi = \partial_x A_x + \partial_y A_y, \quad \Psi = \partial_y A_x - \partial_x A_y, \quad \text{and} \quad A = A_z,$$  \hspace{1cm} (2.8)

for which one obtains a system of equations,

$$(\partial_t^2 + k_1^2 + k_2^2)\Psi(k, t) = 0,$$  \hspace{1cm} (2.9)

$$(\partial_t^2 + k_2^2)\Phi(k, t) - k_1^2 k_2 A(k, t) = 0.$$  \hspace{1cm} (2.10)
\[ (\partial_t^2 + k_\perp^2)A(k, t) - k_z^2 \Phi(k, t) = 0 . \] (2.11)

This nonsymmetric form is consciously chosen in order to mimic the physical asymmetry of the gauge \( A^\tau = 0 \). Eq. (2.9) has solutions \( e^{-i|k||t}\) and \( e^{i|k||t}\). The system of equations (2.10) and (2.11) has the first integral,
\[ \partial_t(\Phi(k, t) + k_z A(k, t)) \equiv -i\rho(k) = 0 , \] (2.12)

which expresses the conservation of the constraint with time but does not coincide with Gauss’ law. To solve this system it is necessary to differentiate the equations once more, thus converting them into the wave equations for the electric field strength. The third order of these independent equations agrees with the number of non-vanishing components of the vector potential. Using of Eq. (2.12) immediately leads to the two equations of third order for the functions \( \Phi \) and \( A \):
\[ (\partial_t^3 + k_\perp^2 \partial_z)\Phi(k, t) = -ik_\perp^2 \rho(k) , \quad (\partial_t^3 + k_\perp^2 \partial_z)A(k, t) = -ik_z \rho(k) , \] (2.13)

which have as set of solutions,
\[ \int dt' e^{-i|k||t'} , \quad \int dt' e^{i|k||t'} , \quad \text{and} \quad \rho(k) t - \rho_1(k) t_0 . \]

The \( t \)-independent term in the third solution is a pure gauge and can be omitted. After a short exercise in algebra and normalization according to
\[ \int d^3r V_{k,i}^{(\lambda)}(r, t) i \partial_j V_{k,j}^{(\lambda)}(r, t) = \delta_{\lambda\lambda'} , \delta(k - k') \] (2.14)

one obtains three modes: two orthogonal radiation modes,
\[ V_{k,1}^{(1)}(x) = \frac{1}{(2\pi)^{3/2}(|k|)^{1/2}} \begin{pmatrix} k_y/|k| \\ -k_x/|k| \\ 0 \end{pmatrix} e^{-ikx} , \quad V_{k,2}^{(2)}(x) = \frac{1}{(2\pi)^{3/2}(|k|)^{1/2}} \begin{pmatrix} k_x k_z/|k| \perp^2 \\ k_y k_z/|k| \perp^2 \\ -1 \end{pmatrix} e^{-ikx} , \] (2.15)

which obey Gauss law without the charge, and a Coulomb mode,
\[ V_{k,3}^{(3)}(x) = \frac{ik_\perp \rho(k)}{k^2} e^{ikr} t . \] (2.16)

The norm of the Coulomb mode, (as defined by Eq. (2.14)), equals zero, and this mode is orthogonal to \( V^{(1)} \) and \( V^{(2)} \). One can easily write down the coordinate form of this solution,
\[ V^{(3)}(r, t) = -t \frac{\partial}{\partial x^i} \int \frac{\rho(r')}{|r - r'|} dr' , \] (2.17)

which is negative of the Coulomb field strength times \( t \). Though this solution obeys the equations of motion without the current, it does not obey Gauss’ law without a charge. Therefore, it should be discarded in the decomposition of the radiation field. However, it should have been kept when or if the radiation field in the presence of the static source \( j^0(k) = \rho(k) \) is considered.

With the two radiation modes in hand, we can obtain the field correlators of the radiation field. The sum over the two physical polarizations invokes a transverse projector
\[ d^{ij}(k) = \sum_\lambda \epsilon^{i}_{(\lambda)}(k) \epsilon^{j*}_{(\lambda)}(k) = \delta^{ij} - \frac{k^i k^j}{k^2} . \] (2.18)

We thus see that an artificial asymmetry of the structure of the modes is washed out. The final non-covariant expressions for the Wightman functions are as follows:
\[ D^{ij}_{10}(x, y) = -2\pi i \int \frac{d^3k}{2|k| (2\pi)^3} d^{ij}(k) e^{-i|k|t + ikr} , \quad D^{ij}_{00}(x, y) = -2\pi i \int \frac{d^3k}{2|k| (2\pi)^3} d^{ij}(k) e^{+i|k|t + ikr} , \] (2.19)
and we shall delay their rewriting in the formal covariant form until the propagators are found.
B. Green functions of the gauge field

We now consider the interaction of the gauge field with the classical source. The first way to obtain the Green function is more or less formal and not general since it relies on the Fourier analysis in terms of plane waves. Projecting the field and the current vectors in the system \( \text{(2.3)} \) onto the two “orthogonal” directions, \textit{i.e.}

\[
A_i^{(tr)} = (\delta_{ij} - k_i k_j/k^2)A_j, \quad A_i^{(L)} = (k_i k_j/k^2)A_j,
\]

we arrive at two different ordinary differential equations for the radiation and the longitudinal fields:

\[
[(\partial_t^2 + k^2)A_i^{(tr)}(k, t)] = j_i^{(tr)}(k, t),
\]

and

\[
\partial_t^2 A_i^{(L)}(k, t) = j_i^{(L)}(k, t).
\]

The fundamental solutions of these equations corresponding to the retarded solutions are known,

\[
A_i^{(tr)}(k, t) = \int_{-\infty}^{+\infty} \theta(t-t') \frac{\sin|k|(t-t')}{|k|} j_i^{(tr)}(k, t') dt',
\]

\[
A_i^{(L)}(k, t) = \int_{-\infty}^{+\infty} \theta(t-t') (t-t') j_i^{(L)}(k, t') dt',
\]

After the Fourier transformation over time, \textit{i.e.} in the full energy-momentum representation, we find that

\[
A_i^{(tr)}(k, \omega) = \frac{1}{(\omega + i0)^2 - k^2} (\delta^{ij} - \frac{k^i k^j}{k^2}) j^j(k, \omega),
\]

\[
A_i^{(L)}(k, \omega) = \frac{1}{(\omega + i0)^2} \frac{k^i k^j}{k^2} j^j(k, \omega).
\]

We have thus obtained the \((\omega + i0)^0\) prescription for the poles for both transverse and longitudinal modes which guarantees the retarded character of the response. Correspondingly, for the poles of the advanced propagator we now must obtain the \((\omega - i0)\) prescription. Combining Eqs. \text{(2.22)} and \text{(2.23)} together we find a familiar form of the axial gauge propagator,

\[
D_{ret}^{ij}(k, \omega) = \frac{d^{ij}(k)}{(\omega + i0)^2 - k^2} + \frac{k^i k^j}{(\omega + i0)^2 k^2} = \frac{1}{(\omega + i0)^2 - k^2} \left( \delta^{ij} - \frac{k^i k^j}{(\omega + i0)^2} \right).
\]

The full covariant form is obtainable according to the following rule of replacements:

\[
k^i \rightarrow k^\mu - u^\mu (ku), \quad \delta^{ij} \rightarrow -g^{\mu\nu} + u^\mu u^\nu, \quad \omega \rightarrow ku, \quad k^2 \rightarrow (ku)^2 - k^2.
\]

Then we immediately get:

\[
d_{ret}^{\mu\nu}(k) = \frac{d^{\mu\nu}(k)}{(\omega + i0)^2 - k^2},
\]

\[
d^{\mu\nu}(k) = -\frac{\delta^{\mu\nu} + \frac{k^\mu u^\nu + u^\mu k^\nu}{ku} - \frac{k^\mu k^\nu}{(ku)^2}}{k^2}.
\]

Considering this fully covariant form of the propagator as a given, we immediately run into several problems which manifest themselves through the poles of the projector \( d^{\mu\nu}(k) \).

(1) The projector \( d^{\mu\nu}(k) \) has no first order poles at \( ku = 0 \). Unlike the two first order poles at \( \omega = \pm |k| \) in the transverse modes, we have one second order pole in the longitudinal mode. The residue in this pole is therefore given by the derivative of the integrand. Consequently, one obtains the term with the linear time dependence of the longitudinal constituent of the vector potential, corresponding to the static configuration of the electric field, which is not a subject of quantum dynamics.
(2) Disposition of the poles of the transverse modes at \( \omega = \pm |k| - i0 \) eventually leads to the propagator which explicitly exhibits the proper light-cone behavior, including the Lorentz invariant definitions of “before” and “after.” The absence of the \(|k|^2 \) (Laplacian) in Eq. (2.22) means that the longitudinal field is not propagating. Consequently, the \((\omega \pm i0)\)-prescriptions are not Lorentz invariant for these modes and even for the retarded and advanced propagators, they are misleading.

(3) Consider now the difference \( 2D_0 = D_{ret} - D_{adv} \), which obeys the homogeneous equation and, therefore, cannot contain the longitudinal modes in its decomposition. However,

\[
\frac{1}{(\omega + i0)^2} - \frac{1}{(\omega - i0)^2} = 2\pi i \delta'(\omega) \neq 0 .
\]

Therefore, the commutator of the free field acquires an unphysical contribution from the longitudinal field which is a remnant of the improper handling the gauge poles. From this point of view and in order to get agreement with the dispersion relations, the principal value prescription looks the most attractive, because the Coulomb part of the propagator will not contribute to its imaginary part. However, it is not a physical solution of the problem.

(4) Since projections of Eq. (2.3) onto Eqs. (2.21) and (2.22) are orthogonal, we may use current conservation and rewrite Eq. (2.22) as

\[
\partial_t \text{div} E + \text{div} j = \partial_t (\text{div} E - j^0) = 0 .
\]

This is now in the form of the equation of the constraint conservation. Therefore, there is no reason to integrate it in the “retarded” or “advanced” manner. Moreover, there is no physically motivated prescription for the integration which recovers the potential \( A \) via the electric field \( E = \dot{A} \).

All these problems appear if we obtain the propagator formally inverting the differential operator of the initial system Eq. (2.3). These problems cannot be resolved until the origin of every pole is traced.

C. Straightforward integration of the field equations

Solution of the Cauchy problem for the free radiation field meets with no difficulties. Therefore we must find a way of obtaining the solution of the inhomogeneous equations which would properly treat the static solutions as a certain limit of the full emission problem. For this purpose, we shall rewrite the system of Maxwell equations in the following form,

\[
(\partial_t^2 + k_\perp^2 + k_z^2)\Psi(k, t) = i[k_x j_y(k, t) - k_y j_x(k, t)] \equiv j_\psi(k, t) ,
\]

\[
(\partial_t^2 + k_\perp^2)\Phi(k, t) - ik_\perp^2 k_z A(k, t) = i[k_x j_y(k, t) + k_y j_x(k, t)] \equiv j_\phi(k, t) ,
\]

\[
(\partial_t^2 + k_\perp^2)A(k, t) + ik_\perp \Phi(k, t) = j_z(k, t) ,
\]

and attempt to find a partial solution by means of the “variation of parameters” method. As in the case of the homogeneous system in section II A it is expedient to differentiate Eqs. (2.31) and (2.32) once more. The difference of the resulting equations gives an equation of the constraint conservation. The equation of constraint can be integrated as follows,

\[
\dot{\Phi}(k, t) + ik_\perp \dot{A}(k, t)) = [j^0(k, t) - \rho(k)] ,
\]

where \( \rho(k) \) is a constant of integration and, until it is set equal to zero, the equation of Gauss’ law is not explicitly used. Eq. (2.33) allows one to obtain two independent equations instead the system (2.31)–(2.32):

\[
(\partial_t^2 + k^2)\dot{\Phi}(k, t) = k_\perp^2 [j^0(k, t) - \rho(k)] + \partial_t j_\phi(k, t) \equiv f_\phi(k, t) ,
\]

\[
(\partial_t^2 + k^2)\dot{A}(k, t) = -ik_\perp [j^0(k, t) - \rho(k)] - \partial_t j_z(k, t) \equiv f_z(k, t) .
\]

Varying the constants in the decomposition of the partial solution we find that

\[
\mathcal{F} = \frac{i}{2|k|} \left( e^{-i|k|t} \int_{-\infty}^{t} d t' f(t')e^{i|k|t'} - e^{i|k|t} \int_{-\infty}^{t} d t' f(t')e^{-i|k|t'} \right) ,
\]

(2.36)
where $\Phi$ stands for $\Psi$, $\dot{\Phi}$ or $\dot{A}$ and $f$ stands for the R.H.S. of either (2.34) or (2.32).

Starting from this point, one may wish to take a short cut and find the solution by means of the symbolic Fourier calculus. Using the Fourier representation for the source, $f$, and the time-independent integration constant $\rho$,

$$f(k, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(k, \omega)e^{-i\omega t}, \quad \rho(k) = \int_{-\infty}^{\infty} d\omega \rho(k)\delta(\omega)e^{-i\omega t},$$

one obtains for the magnetic field $\Psi$ of the transverse mode

$$\Psi(k, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} j_{\Psi}(k, \omega) - \frac{\rho(k)}{\omega} e^{-i\omega t},$$

and for the components of the electric field,

$$\dot{\Phi}(k, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{j^0(k, \omega) - 2\pi\delta(\omega)\rho(k)}{(\omega + i\delta)^2 - k^2} e^{-i\omega t},$$

and

$$\dot{A}(k, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-ik_z[j^0(k, \omega) - 2\pi\delta(\omega)\rho(k)]}{(\omega + i\delta)^2 - k^2} e^{-i\omega t}.$$  

The integration which has led to Eqs. (2.38)-(2.40) gives the electric and magnetic fields and is thus retarded. These equations do not have poles at $\omega = 0$. However, we still have to integrate these equations once more in order to find the vector potential of the gauge field. This results in an $\omega$ appearing in the denominator and yields the final answer,

$$A_t(k, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega + i\delta)^2 - k^2}[k_t j^0(k, \omega) - 2\pi\delta(\omega)\rho(k)] + j_z(k, \omega),$$

again, without any motivated prescription for the new pole. In order to obtain the already known expression for the propagator we must set the constant of integration $\rho(k)$ equal to zero, thus explicitly incorporating Gauss’ law. Next, it is expedient to use current conservation and to rewrite the Fourier component of the charge density as $j^0(k, \omega) = -k_t j_z(k, \omega)/\omega$. In this way, we immediately obtain Eqs. (2.27)-(2.28), but without any physical handle on the second order pole at $\omega = 0$. However, if $j^0(k, \omega) \sim \delta(\omega)$ then $k_t j_z(k, \omega) = 0$, and a formal usage of current conservation in the Fourier representation becomes ambiguous. Formally, the problem manifests itself through the ambiguity of the function $\omega^{-1}\delta(\omega)$. In this case, one should return to the Eqs. (2.33)-(2.41), perform the $\omega$-integration using the $\delta(\omega)$, and end up with the linear dependence of the vector potential on time $t$. The loss of continuity in the description of the limit static case which shows up here is very unlikely. Unless we are dealing with the canonical scattering problem, the physics associated with the static fields is indeed important. The amount of mathematical ambiguities that have appeared in the last few lines is more than sufficient to show that it is better not to take the short cut by using the Fourier picture. Let us proceed more gradually and continue the integration of the time variables.

Eq. (2.36) with $F = \Psi$ and $f = j_{\Psi}$ already gives the solution in quadrature form. To obtain $\Phi$ and $A$, one should integrate twice, e.g.,

$$A(k, t_1) = \int_{-\infty}^{t_1} dt \int_{-\infty}^{t'} dt' e^{-i|k||t|} f_z(t_2) = \int_{-\infty}^{t_1} dt \int_{-\infty}^{t'} dt' e^{-i|k||t|} \int_{-\infty}^{t_2} dt_2 e^{-i|k||t_2|} f_z(t_2).$$

(2.42)

Of the two integrations here, the first one, $dt_2$, recovers the electric field, $\dot{A}$, via the source $f_z$, while the second integration, $dt'$, is used to find the potential $A$. It is easy to see that this integration is held to the limits $t_2 < t' < t_1$ and thus can be done first;

$$A(k, t_1) = \frac{-1}{|k|} \int_{-\infty}^{t_1} dt \int_{-\infty}^{t_1} dt_2 \left[ (\cos[k|(t_1 - t_2)| - 1] \frac{1}{|k|}) [ik_z j^0(k, t_2) - \partial_z j_z(k, t_2)] \right].$$

(2.43)

where the constant $\rho(k)$ of the constraint integration is temporarily buried into $j^0(k, t)$. In fact, the function $\theta(t_1 - t_2)|k|^{-1}[\cos[k|(t_1 - t_2)| - 1]$ is exactly the retarded Green function of the ordinary differential equations (2.34) and (2.33). Thus, it could be used immediately to obtain the solution. In this way, one cannot trace the origin of the spurious pole.
Every term on the RHS of Eq. (2.44) should be integrated by parts with the assumption that the sources vanish as \( t \to -\infty \). After the time derivatives of \( j^0 \) are replaced by the divergence of the current, \( \partial_t j^0(k, t) = -i k_l j^l(k, t) \), and the same calculations are repeated for the function \( \Phi \), we arrive at the final answer,

\[
A_i(k, t_1) = \int_{-\infty}^{t_1} \sin \left( \frac{|k|(t_1 - t_2)}{|k|} \right) \left( \delta_{11} - \frac{k_i k_l}{k^2} \right) j^l(k, t_2) dt_2 - \int_{-\infty}^{t_1} t_2 \frac{k_i k_l}{k^2} j^l(k, t_2) dt_2 - \frac{k_i}{ik^2} t_1 j^0(k, t_1), \tag{2.44}
\]

Recalling the expression (2.16) for the field of a static charge, we see that the last term in Eq. (2.44) represents an instantaneous distribution of the potential at the moment \( t_1 \), corresponding to the charge density taken at that same moment. Remembering that the charge density \( j^0(k, t_1) \) in Eq. (2.44) still includes the arbitrary constant \( \rho(k) \), we see that imposing the constraint indeed affects only the potential of static charge distribution and puts it in agreement with Gauss’ law. Although the proper status of the static field in the Green function has been recovered, the last two terms of Eq. (2.44) have lost an explicit translational invariance. To restore it, one should rewrite the last term as

\[
\frac{k_i}{ik^2} t_1 \int_{-\infty}^{t_1} dt_2 \frac{dj^0(k, t_2)}{dt_2} = \frac{k_i k_l}{k^2} t_1 \int_{-\infty}^{t_1} j^l(k, t_2),
\]

which is meaningful only if the source is not entirely static. After that, one obtains propagator in the form given earlier by the Eqs. (2.23) and (2.24). The subsequent Fourier transformation leads to the second order pole at \( \omega = 0 \) in the longitudinal part of the propagator.

Two points from the above discussion are the most essential. First, the desired prescription is not found even for the retarded and advanced propagators, even in the case when we may rely on the most powerful arguments coming from analyticity and causality. Second, the Wightman functions (solutions of the homogeneous Maxwell equations) are built entirely from the free radiation fields and do not have these poles at all. Thus, the spurious poles cannot acquire any prescription in the \( T \)-ordered (Feynman) Green functions either. Actually, these poles are the price we pay for the loss of control over the dynamics of the longitudinal field when it approaches the static limit.

### III. PROPAGATOR OF THE NULL-PLANE GAUGE

The null-plane gauge \( A^+ = A^0 + A^3 = 0 \) is a constituent part of the null-plane dynamics which uses the light-like direction \( x^+ = x^0 + x^3 \) as the direction of the dynamical evolution. Physically, one should think of this gauge as the limit of the temporal axial gauge \( uA = 0 \), when the 4-vector \( u^\mu \) is of the form: \( u^\mu = (\cosh y, 0, \sinh y) \). The vector \( u^\mu \) can be thought of as the velocity of the proton and it is normal to the space-like hyperplane where the observables identifying the proton are defined. Geometrically, this plane is almost parallel to the null-plane in the limit of the potential, \( y \to \pm \infty \). In this limit, the normal and tangential directions become almost degenerate; however, we shall keep this difference in mind. If \( y \to +\infty \), then the Lorentz contracted proton is confined in the \( xy \) plane which moves with rapidity \( y \) in the positive \( z \)-direction. The gauge condition becomes \( uA \approx e^y(A^0 - A^3)/2 = 0, A^- \approx 0 \). If \( y \to -\infty \), then the proton is moving in the negative \( z \)-direction and the gauge condition changes to \( uA \approx e^{-y}(A^0 + A^3)/2 = 0, A^+ \approx 0 \). Thus, we obtain two null-plane gauges as the limit of the temporal axial gauge; however, in a correspondence which is opposite to what is naively expected. This important fact finds a natural physical explanation in the scope of the wedge form of Hamiltonian dynamics [34]. Some auxiliary arguments are submitted at the end of this section. For the present discussion, it is important that the spurious pole of the polarization sum of the null-plane gauges, \( d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{kn} \), (3.1)

as it was in the case of temporal axial gauge, originates from the longitudinal constituent of the gauge field. Here, \( n^\mu \) has only one non-zero component, either \( n^- \) or \( n^+ \).

Mathematically, the light-like limit of the time-like direction is always singular. Thus, for the sake of safety, it is expedient to start from the very beginning. The metric tensor has the following nonvanishing components, \( g_{++} = g_{--} = 1/2, g^{+-} = g^{-+} = 2, g_{rs} = g^{rs} = -\delta_{rs} \). Here, \( r, s = 1, 2 \) label the \( x- \) and \( y- \)coordinates. Tree components of the potential, \( A_+ = A^-/2, A_r, \) are canonical coordinates, and have the electric fields, \( E^- = -F^{++}/2 = -2\partial_- A_+ \) and \( E^r = -F^{r+}/2 = 2\partial_+ A_r \), as the canonical momenta. Maxwell equations can be conveniently rewritten in terms of three functions,

\[
\Phi = \partial_x A_x + \partial_y A_y, \quad \Psi = \partial_y A_x - \partial_x A_y, \quad \text{and} \quad A_+ \equiv 2A^- \tag{3.2}
\]

with the sources,
\[ j^\phi = \partial_x j^z + \partial_y j^y, \quad j^\psi = \partial_y j^x - \partial_z j^y, \quad j^- = 2j_+ \quad \text{and} \quad j^+ = 2j_- \tag{3.3} \]
onumber

on the right hand side:

\[ (4\partial_+ \partial_- - \nabla_{\perp}^2)\Psi = -j^\psi, \tag{3.4} \]

\[ 4\partial_+ \partial_- \Phi - 2\nabla_{\perp}^2 \partial_- A_+ = -j^\phi, \tag{3.5} \]

\[ 4\partial_+ \partial_- A_+ - 2\nabla_{\perp}^2 A_+ + 2\partial_+ \Phi = j^-, \tag{3.6} \]

\[ -4\partial_-^2 A_+ + 2\partial_- \Phi = j^+. \tag{3.7} \]

The last of these equations has no “time” derivative \( \partial_+ \) and is a constraint equation which expresses Gauss’ law with the charge density \( j^+ \). One can easily transform the system of dynamical equations (3.4)-(3.5) into three independent equations, Eq. (3.3) and

\[ (4\partial_+ \partial_- - \nabla_{\perp}^2)(\partial_- \Phi) = (\partial_+ \partial_- - \frac{1}{2} \nabla_{\perp}^2)j^\phi + \partial_- j^- = -\partial_+ \partial_- j^\phi - \frac{1}{2} \nabla_{\perp}^2 \partial_+ j^+, \tag{3.8} \]

and

\[ (4\partial_+ \partial_- - \nabla_{\perp}^2)(\partial_- A_+) = \frac{1}{2} j^\phi + \partial_- j^- = \frac{1}{2} \partial_+ j^+ + \frac{1}{2} \partial_- j^- . \tag{3.9} \]

Here, the right hand side is given in two forms, the original one, and after its transformation, the one that accounts for charge conservation,

\[ j^\phi + \partial_- j^- + \partial_+ j^+ = 0 . \tag{3.10} \]

The second form is very useful since it helps clarify the structure of the field created by the external source.

For the static source with \( \partial_+ j^+(x) = 0 \) (or \( j^+(k) \sim \delta(k^-) \)) the derivative \( \partial_- \) can be easily removed from both sides of the Eqs. (3.8) and (3.9) and thus, no pole \((k^+)^{-1}\) can appear. In this case, the equations of motion lead to the diagonal retarded propagator,

\[ A_i(k) = -\frac{j_i(k)}{k^+(k^- - i\epsilon) - k_{\perp}^2}, \quad i = 1, 2, + . \tag{3.11} \]

while the static field has to be recovered via Gauss’ law (3.7) and has only two transverse components:

\[ A_r^{(\text{stat})}(k) = \frac{k_r}{k^+ k_{\perp}^2} j^+(k^+, k_{\perp}) , \tag{3.12} \]

and without any prescription for the pole \((k^+)^{-1}\). The source, independent on \( x^+ \), can depend on \( x^0 \) and \( x^3 \) in a single combination, \( x^- = x^0 - x^3 \), and therefore, should propagate without longitudinal dispersion at the speed of light in the \( x^+ \)-direction. Expression (3.12) is nothing but the Williams-Weiszacker field of this source. If \( \partial_+ j^+(x) \neq 0 \) then the system of equations (3.4), (3.8) and (3.9) can be explicitly integrated to

\[ A_i(k) = \frac{1}{k^+(k^- - i\epsilon) - k_{\perp}^2} \left( -\frac{j_i(k)}{k_r} + \frac{k_r}{k^+} j^+(k^+, k_{\perp}) \right), \quad i = 1, 2, + . \tag{3.13} \]

and, as before, the pole \((k^+)^{-1}\) is due to the integration \( dx^- \) that recovers the vector potential via the electric field. No prescription can be justified for this integration. If we assume that the static source is propagating at the speed of light and take \( j^+(k) \sim \delta(k^-) \), then we recover the Williams-Weiszacker formula (3.12) as the limit of the full propagator.

The pole \((k^+)^{-1}\) encounters various types of physical singularities. One of them is the source with no \( x^+ \) dependence propagating in the \( x^+ \)-direction. The second one emerges as the field pattern corresponding to the residue in the pole \((k^+)^{-1}\), providing it is accessible in the calculations. In configuration space, this pattern is independent of \( x^- \) and therefore, propagates at the speed of light without longitudinal dispersion in the \( x^- \) direction. This field, since it is off-mass-shell, corresponds to the smallest Feynman \( x \) and consequently, to the negative rapidities in configuration.
space. One more option is that this is a proper field of the back-scattered ultrarelativistic charge. In any of these cases, we are dealing with the bounded systems of charge and its static field, propagating in the null-plane direction which should be renormalized to their physical parameters.

This simple observation explains the source of controversy which was found in Refs. 10,11. Evaluation of the gluon propagator in the gauge $A^+ = 0$ has led to the propagator of the gauge $A^− = 0$. In fact, these two gauges are complementary. As long as the theory is supposed to describe the process of measurement, it unavoidably deals with the physical singularities on two null-planes. This is a clear manifestation of the strong localization of the entire process at its initial moment which points to the wedge form of dynamics as a picture which incorporates this most important physical feature of any measurement at extremely high energies.

IV. PHYSICAL DISCUSSIONS OF THE SPURIOUS POLES

It was already mentioned in the Introduction that the spurious poles are mostly safe in calculations of the observables like cross sections. In exceptional cases when they are mathematically dangerous, the gauge poles of the propagators were shown to have a clear physical meaning; the gauge field approaches a static configuration. The gauge poles are entirely due to the longitudinal constituent of the gauge field which is not a dynamical variable. Therefore, the real problem is to trace the physical origin of the static configuration and identify the physical object it belongs to. Then the natural remedy may be renormalization, i.e., the brute force identification of the dangerous element with the physical object of known properties.

Let us begin with the two popular examples from QED when this kind of strategy has proved to be fruitful. If an electron emits a long wave photon, then in the limit of $\omega \to 0$, the photon is inseparable from the proper field of the electron. In this case, the formal divergent perturbation series is assembled (renormalized) to form the classical field of the electron [12]. The reaction of radiation becomes negligible and the electric current $j$ should be treated as a $\epsilon$-number rather than an operator.

When the electronic or muonic pair is created with low relative momenta the whole series which describes the soft emission should be summed to form the Coulomb field of the charged pair. In this case, the study ends up by replacing the plane waves of the final state with the states of scattering in the Coulomb field [13].

More examples may be found in QCD calculations of the deep inelastic e–p scattering or hadronic collisions. In these cases, it is common to use the infinite momentum frame where the proton has only $P^+$ component of the momentum and to connect the gauge condition with the axis $x^+$: $A^+ = 0$. The poles like $1/k^+$ of the null-plane polarization sum eventually enter the splitting kernels of the DGLAP equations [14]. They are due to different processes and are treated differently. The poles $(1−z)^{−1}$ $(z = p^+/k^+)$ which appear in the kernels $P_{gg}$ and $P_{gq}$ come from the final state gluons. They are treated according to the so-called $(+)$-prescription.

$$\frac{f(z)}{1−z} \to \left( \frac{f(z)}{1−z} \right)_{+} + c\delta(1−z) = \frac{f(z)−f(1)}{1−z} + c\delta(1−z) \, .$$

The principal value treatment of the pole shields the collinear singularity, telling us that the emitted forward gluon is a part of the proper field of the proton. The $\delta$-counterterm indicates that this “emission” does not change the quantum numbers of the proton. Overall, this procedure is really a kind of renormalization of the proton wave function in the environment of the strongly localized interaction with the electron. The second type of pole, $1/z$, is treated in different way. This pole appears in the kernels $P_{gg}$ and $P_{gq}$ and is due to the retarded tree propagators between consecutive emissions. It also reflects the dynamics of the longitudinal gluon field. Unless we discuss the problem of unitarity, there is no need to screen it. Including the fusion process into the equations of the QCD evolution naturally leads to the saturation of the evolution rate. The QFK approach [6] describes the QCD evolution as a (virtual) sequential in real temporal scale process [7]: every act of emission has the preceding and the subsequent configurations of the longitudinal field as the boundary conditions. Therefore, a proper treatment of the longitudinal fields dynamics at the intermediate stage of the deeply inelastic process is imperative. In fact, this dynamics is responsible for the low-$x$ enhancement of the structure functions.

V. CONCLUSION

One may view the spurious poles of the gauges $A^0 = 0$ and $A^+ = 0$ as an artifact of the global choice of the gauge. In the class of pure scattering problems this point of view seems to be, though narrow, but quite appropriate. If the intermediate dynamics of the system is a subject of physical analysis, the choice of the gauge condition still cannot affect the observables, but it explicitly affects the definition of the physical states of the gauge field. In this case,
a specific gauge may be profitable as long as it can help single out the physically important details. The temporal axial gauge explicitly reveals the static field configurations by its spurious poles. The poles of the null-plane gauge correspond to the proper fields of the ultrarelativistic charged particles.

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