A NEW TYPE OF NON-LANDING EXPONENTIAL RAYS

JIANXUN FU
Center for Mathematical Sciences, Huazhong University of Science and Technology
Wuhan 430074, China

SONG ZHANG∗
Academy of Mathematics and Systems Science, Chinese Academy of Sciences
Beijing 100190, China

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Abstract. In this paper, we will construct a new type of non-landing exponential rays, each of whose accumulation sets is bounded, disjoint from the ray and homeomorphic to the closed topologist’s sine curve.

1. Introduction. Let \( E_\lambda(z) = \lambda \cdot e^z \) (\( \lambda \neq 0 \)). An escaping point is by definition a point \( z \) such that the iterates \( E_\lambda^n(z) \to \infty \) as \( n \to \infty \). It is known that all the escaping points of \( E_\lambda \) are organized in rays, one end of which are toward infinity [16]. This result has been extended to transcendental entire functions in the Eremenko-Lyubich class with finite order [15]. For these rays, an interesting topic is the landing problem. On one hand, as an analogue of Douady-Hubbard’s theorem on external rays in polynomial dynamics [7, Theorem 18.10], all periodic rays land for exponential maps with non-escaping singular value [11, 17]. On the other hand, people are also interested in the non-landing rays of exponential maps. For example, the authors in [3, 4, 5, 14] constructed some non-landing rays, each of whose accumulation sets is unbounded and an indecomposable continuum containing some ray in the extended complex plane. Recently, the first author and G. Zhang constructed some non-landing rays, each of whose accumulation sets is bounded and can be an indecomposable continuum containing part of the ray, an indecomposable continuum disjoint from the ray or a Jordan arc [6].

One common feature of the non-landing exponential rays constructed above is that each of the corresponding accumulation sets is an indecomposable continuum or a Jordan arc. Hence, it is natural to ask whether there are some non-landing exponential rays with accumulation sets whose complexities of topology are “between the two extremal cases”, such as the closed topologist’s sine curve

\[ \{(x, y) \mid 0 < x \leq 2/\pi, y = \sin(1/x)\} \cup \{(0, y) \mid -1 \leq y \leq 1\}, \]

which is decomposable but not locally connected [8, 9].

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∗ Corresponding author: Song Zhang.
In this paper, we will give a positive answer to this question by considering the case of post-singularly finite exponential maps. For simplicity, rather than deal with the general case, we shall deal only with the case where $\lambda = 2\pi i$.

**Theorem 1.1.** For post-singularly finite exponential map $2\pi i \cdot e^z$, there exist non-landing rays each of whose accumulation set is bounded, disjoint from the ray and homeomorphic to the closed topologist’s sine curve.

We would like to mention that for transcendental entire functions of disjoint type, Rempe in [12] proved that for each function with bounded slope, the Julia continuum is arc-like. Conversely, they also proved that each arc-like continuum with at least one terminal point can be realised as a Julia continuum. In particular, the closed topologist’s sine curve can occur as Julia continuum of a disjoint type entire function. While we consider the case of post-singularly finite exponential maps, which is not of disjoint type. In fact, in this case, the Julia set is the whole complex plane, which can be regarded as the closure of all the escaping rays. Theorem 1.1 says that the closed topologist’s sine curve can occur as the accumulation set of one of the escaping rays.

The organization of the paper is as follows. In §2, we recall some basic knowledge on the escaping rays of exponential maps, hyperbolic expansion lemma and Bounded-wiggling lemma appeared in [6]. In §3, we show how to determine the itineraries of the non-landing rays whose accumulation sets will be homeomorphic to the closed topologist’s sine curve and define the folding points of the rays. In §4, we study several properties of the non-landing rays determined in §3 and prove Theorem 1.1.

2. Preliminaries. In this section, we first briefly sketch some basic knowledge on the escaping rays of exponential maps (see [2, 13] for a thorough background in this aspect). Then, we recall the hyperbolic expansion lemma and the Bounded-wiggling lemma.

2.1. The escaping rays of exponential maps. Here, the construction of the non-landing rays described in Theorem 1.1 rely on the parameter $\lambda = 2\pi i$. In fact, the construction can be adapted to all post-singularly finite exponential maps.

Let

$$E(z) = 2\pi i \cdot e^z.$$ 

Following [1] or [5], there is a unique ray $\gamma_{2\pi i}$ of $E$ landing at $2\pi i$ with $E(\gamma_{2\pi i}) = \gamma_{2\pi i}$. Let $\eta_{2\pi i}$ be the pre-image of $\gamma_{2\pi i}$ which lands at the origin. Then $E^{-1}(\eta_{2\pi i})$ consists of infinitely many disjoint curves $t_j$, $j \in \mathbb{Z}$, with $t_{j+1} = t_j + 2\pi i$ (we assume that 0 lies in the strip bounded by $t_1$ and $t_0$). Note that every $t_j$ extends from the left to right across the entire plane. Let $T_j$, $j \in \mathbb{Z}$, be the union of $t_j$ and the open strip bounded above by $t_{j+1}$ and below by $t_j$. Then the *itinerary* of $z \in \mathbb{C}$ is defined to be the sequence of integers $\underline{s}(z) = (s_0 s_1 s_2 \cdots)$, where $E^{j} (z) \in T_{s_j}$ for all $j \geq 0$.

In our case, the itineraries we involved always satisfy that $|s_j|$ is uniformly bounded. For $\zeta > 0$ large enough and $M > 0$, set

$$H_\zeta = \left( \bigcup_{|j| \leq M} T_j \right) \cap \{z \mid \Re(z) > \zeta \}.$$ 

Denote by $\omega_\zeta(\zeta)$ the set of escaping points $z \in \mathbb{C}$ with orbits in $H_\zeta$ satisfying $\underline{s}(z) = \underline{s}$ and $\Re(E^{n+1}(z)) > \Re(E^n(z))$ for all $n \geq 0$. We call it a *tail* associated
to itinerary \( s \). In fact, the following lemma shows that \( \omega_2(\zeta) \) is nonempty and a continuous curve toward infinity.

**Lemma 2.1** (Cf. [1, 4, 5]). For \( s \in \{ (s_0s_1s_2 \cdots) ||s_j|| \leq M \) for each \( j \}, \) there exists \( \zeta \in \mathbb{R}^+ \) depending only on \( M \) such that the tail \( \omega_2 = \omega_2(\zeta) \) is a continuous curve \( h_2 \) whose parameterization can be written as the form \( (t, h_2(t)) \) with \( \zeta \leq t < \infty \).

Suppose \( L_j : \mathbb{C} \setminus \eta_{2\pi} \rightarrow T_j \) \((j \in \mathbb{Z})\) are the branches of the inverse of \( E \). Then for given \( s = (s_0s_1s_2 \cdots) \), the set

\[
\gamma_2 = \bigcup_{n=1}^{\infty} L_{s_0} \circ \cdots \circ L_{s_{n-1}}(\omega_{2n}(s))
\]

is called the escaping ray with itinerary \( s \), where \( \sigma(s) = (s_1s_2 \cdots) \). Note that the ray \( \gamma_2 \) can be parameterized as a continuous curve \( \gamma_2(t) \) \((0 < t < \infty)\) such that \( \gamma_2(t) = \omega_2(t) \) for \( t \geq \zeta \). The set \( A_2 = \bigcap_{t>0} \gamma_2((0, t)) \) is called the accumulation set of \( \gamma_2 \). If the set \( A_2 \) consists of a single point, then we say \( \gamma_2 \) is landing; otherwise, \( \gamma_2 \) is non-landing.

As the starting point of our proof, we now recall the “folding phenomena” which was observed by authors in [3, 4, 5]. That is, if a ray is close enough to the origin, then the preimages of the ray extend far away to the negative infinity, and the further preimages will fold somewhere. In our case, suppose \( \omega \) is a tail in the strip \( T_1 \). We pull back \( \omega \) by \( L_1^k \) with \( k \) large enough, so that the endpoint of \( L_1^k(\omega) \) \((\subset T_1)\) is close to \( 2\pi i \). Then pulling back \( L_1^k(\omega) \) by \( L_0 \), we get that the endpoint of \( L_0 \circ L_1^k(\omega) \) \((\subset T_0)\) is close to the origin. Next pulling back \( L_0 \circ L_1^k(\omega) \) by \( L_{1 \pm 1} \), we get the endpoint of \( L_{1 \pm 1} \circ L_0 \circ L_1^k(\omega) \) \((\subset T_{1 \pm 1})\) extends to the far left. At last, we pull back \( L_{1 \pm 1} \circ L_0 \circ L_1^k(\omega) \) by \( L_1 \) and set

\[
G_k^\pm = L_1 \circ L_{1 \pm 1} \circ L_0 \circ L_1^k.
\]

As a result, the curve \( G_k^\pm(\omega)(\subset T_1) \) will fold at some point contained in a bounded subset of the plane independent of \( k \). Moreover, if \( k \) is large, the endpoint can reach the right half-plane as far as we wanted. It is worth pointing out that for the folding direction, \( G_k^+ \) folds \( \omega \) up and \( G_k^- \) folds \( \omega \) down. For later use, we sometimes denote \( G_k^\pm \) by \( G_k \) for simplicity.

Denote by \( I_k^j \) the corresponding itinerary block \((1, \pm 1, 0, 1)^k \) of \( G_k^\pm \), where \( 1_k \) is the block of 1 of length \( k \). For our purpose, we will be only concerned with the itineraries of the following form

\[
(I_{k_1}^-; I_{k_2}^+; I_{k_3}^-; I_{k_4}^+; \cdots; I_{k_{2n-1}}^-; I_{k_{2n}}^+; \cdots).
\]

### 2.2. Hyperbolic expansion and bounded-wigging Lemma

Let us recall the following hyperbolic expansion lemma. Denote by \( X \) the hyperbolic Riemann surface \( \mathbb{C} \setminus \{0, 2\pi i\} \) and \( \rho_X \) the corresponding hyperbolic density function on \( X \).

**Lemma 2.2** (Cf. [6]). There is a \( \delta_0 \in (0, 1) \) such that for all \( k \geq 1 \) and \( z \in X \), we have

\[
|G_k(z)| \cdot \rho_X(G_k(z)) < \delta_0 \cdot \rho_X(z),
\]

where \( G_k = L_1 \circ L_{1 \pm 1} \circ L_0 \circ L_1^k \).

**Notations.** In the remainder of the paper, we fix some notations for simplicity. First, we always use hyperbolic metric in \( X = \mathbb{C} \setminus \{0, 2\pi i\} \) rather than Euclidean metric in \( \mathbb{C} \) unless otherwise stated. In particular, we use \( \text{dist}(\cdot, \cdot) \) and \( \text{diam}(\cdot) \) to denote respectively the hyperbolic distance and the hyperbolic diameter in \( X \). For
any \( R > 0 \) and a set \( A \), we denote by \( U_R(A) \) the \( R \)-neighborhood of \( A \) with respect to hyperbolic metric in \( X \). In addition, the symbol \( \delta_0 \) always represent the number which is defined in Lemma 2.2.

In the following, we give the concept of folding point and recall the Bounded-wiggling Lemma.

For \( r_0 > 0 \), let

\[
\Omega = \{ z \mid |z - 2\pi i| > r_0 \} \cap \{ z \mid \text{Re}(z) > -1/r_0 \}.
\]

Let \( \Gamma_{a,b} \subset T_1 \cap \Omega \) be an injective curve with two endpoints \( a \) and \( b \). We say that the map \( G_k^\pm \) folds \( \Gamma_{a,b} \) into two pieces with respect to a positive number \( \zeta \) if \( \min\{\text{Re}(G_k^\pm(a)), \text{Re}(G_k^\pm(b))\} > \zeta \) and there is a point \( c \in G_k^\pm(\Gamma_{a,b}) \) such that \( \text{Re}(c) \leq \zeta \). Note that if one endpoint of \( \Gamma_{a,b} \) is at \( \infty \), then we put \( G_k^\pm(\infty) = \infty \). The point \( c \), together with the endpoints \( G_k^\pm(a) \) and \( G_k^\pm(b) \), is called the folding points of \( G_k^\pm(\Gamma_{a,b}) \).

Now we claim that for \( r_0 \) small enough, if \( \Gamma_{a,b} \subset T_1 \cap \Omega \), then \( G_k^\pm(\Gamma_{a,b}) \subset T_1 \cap \Omega \). In fact, for any \( z \in T_1 \), \( L_0 \circ L_i^k(z) \in T_0 \) is bounded away from \( 2\pi i \) by some constant which is independent of \( k \) and \( r_0 \). Note that \( E(2\pi i) = 2\pi i \). This implies that \( G_k^\pm(z) = L_1 \circ L_{i+1} \circ L_0 \circ L_i^k(z) \) is also bounded away from \( 2\pi i \) by some constant which is independent of \( k \) and \( r_0 \). On the other hand, if \( r_0 \) is small enough, it is clear that for any \( z \in T_1 \), \( \text{Re}(G_k^\pm(z)) > -1/r_0 \). This completes the claim.

For integers \( k_1, \ldots, k_n \ (n \geq 2) \), let

\[
f_i = E^{k_i+3} \quad \text{and} \quad g_i^\pm = G_k^\pm _{k_i},
\]

where \( 1 \leq i \leq n \). In the following, we will denote \( g_i^\pm \) by \( g_i \) for simplicity.

For an injective curve \( \Gamma \subset T_1 \cap \Omega \) (finite or infinite), let us define the folding points of \( g_1 \circ \cdots \circ g_n(\Gamma) \) by induction. First, the folding points of \( g_n(\Gamma) \) are defined to be the union of the point \( c \) (if it exists) and the endpoints of \( g_n(\Gamma) \) as above. Suppose the folding points of \( g_1 \circ \cdots \circ g_n(\Gamma) \) have been defined. We say that \( z \) is a folding point of \( g_1 \circ \cdots \circ g_n(\Gamma) \) if either of the following two conditions holds:

1. There is a folding point \( w \) of \( g_1 \circ \cdots \circ g_n(\Gamma) \) such that \( z = g_i(w) \).
2. There is a subarc \( \Lambda \) of \( g_1 \circ \cdots \circ g_n(\Gamma) \) with the endpoints being two adjacent folding points of \( g_1 \circ \cdots \circ g_n(\Gamma) \) such that \( z \) is the folding point of \( g_i(\Lambda) \).

For any two points \( x,y \in \Gamma \), let \( \Gamma_{x,y} \subset \Gamma \) be the subarc within \( x \) and \( y \). The wiggling of \( \Gamma \) is defined by

\[
W(\Gamma) = \sup \text{diam}(\Gamma_{x,y}),
\]

where the sup is taken over all \( \Gamma_{x,y} \) with \( x,y \in \Gamma \) and \( \text{Re}(x) = \text{Re}(y) \).

**Lemma 2.3** (Bounded-wiggling, cf. [6]). Let \( \Gamma \subset T_1 \cap \Omega \) be an injective curve with \( W(\Gamma) \leq K_0 \). Then there exists a \( K \) depending only on \( K_0 \) such that for any arc \( \Lambda \) between two adjacent folding points of \( g_1 \circ \cdots \circ g_n(\Gamma) \), we have

\[
W(\Lambda) \leq K,
\]

where \( i,n \) are integers with \( 0 < i < n < \infty \).

3. **Realization of the prescribed ray.** The goal of this section is to pick out the itineraries elaborately from the following form

\[
\mathbf{s}^\dagger = (I_{k_1}^\pm; I_{k_2}^\pm; I_{k_3}^\pm; \cdots; I_{k_{2n-1}}^\pm; I_{k_{2n}}^\pm; \cdots),
\]

so that the corresponding accumulation sets of non-landing rays are homeomorphic to the closed topologist’s sine curve.
3.1. The choice of \( k_n \). For any itinerary \( \gamma \) and integer \( n > 0 \), let \( \gamma^0_n(\gamma) \) be the tail \( \omega_1 \) and \( z^0_n(\gamma) \) be its endpoint (note that \( \text{Re}(z^0_n(\gamma)) = \zeta \)). For \( 0 \leq i \leq n - 1 \), denote the tail of \( g_{n+1} \circ \cdots \circ g_n(\omega_1) \) by \( \gamma^0_i(\gamma) \) and the endpoint of \( \gamma^0_i(\gamma) \) by \( z^0_i(\gamma) \). Then for \( 0 \leq i \leq n \) and \( i - n \leq j \leq i \), we define \( \gamma^0_i(\gamma) \) and \( z^0_i(\gamma) \) by

\[
g_i(\gamma^0_i(\gamma)) = z^0_{i-1}(\gamma) \quad \text{and} \quad g_i(z^0_i(\gamma)) = z^0_{i-1}(\gamma).
\]

Let \( \gamma_i(\gamma) \) be the escaping ray containing \( \gamma^0_i(\gamma) \) and \( [z^0_i(\gamma), z^0_{i+1}(\gamma)] \) be the subarc of \( \gamma^0_i(\gamma) \) within \( z^0_i(\gamma) \) and \( z^0_{i+1}(\gamma) \), if it is well defined. When \( \gamma = (1, 1, \ldots, 1, \ldots) \), we denote \( z^0_i(\gamma) \) (cor. \( \gamma^0_i(\gamma) \) and \( [z^0_i(\gamma), z^0_{i+1}(\gamma)] \)) by \( z^0_i \) (cor. \( \gamma^0_i \) and \( [z^0_i, z^0_{i+1}] \)) for simplicity. See Figure 1 for a sketch of the above definitions.

Now let us discuss how to choose the sequence \( \{k_n\} \). We first take \( k_1 \geq 1 \) large enough. For an integer \( n > 0 \), suppose \( k_i (1 \leq i \leq n - 1) \) have been defined. Let us define \( k_n \). Note that \( \gamma_{n-1}^{-1} = g_n(\gamma_n^0) \) extends \( \gamma_n^0 \) and folds at some point with bounded hyperbolic distance from \( z^0_{n-1} \). Moreover, the endpoint \( z^0_{n-1} \) can reach to the right half-plane as far as wanted provided that \( k_n \) is large enough. On the other hand, we claim that \( [z^0_{n-1}, z^0_n] \) is independent of \( k_n \). In fact, \( [z^0_{n-1}, z^0_n] \) belongs to the tail associated to the itinerary \( (1, -1, 0, 1, 1, \ldots) \). This tail is unique since \( \gamma_{2n} \) is unique as we mentioned above. Then we show that the point \( z^0_{n-1} \) is independent of \( k_n \). Note that \( z^0_{n-1} = f_{k-1} \circ \cdots \circ f_1(z^0_0) \). Since the point \( z^0_0 \) has itinerary \( (f_{k_1}; f_{k_2}; \ldots; 1, (-1)^{n-1}, 0, 1, \ldots; 1, (-1)^n, 0, 1, 1, \ldots) \), this finishes the claim. Therefore, if we take \( \xi_n \in [z^0_{n-1}, z^0_n] \), then \( k_n \) can be defined as follows.

\((*)\) Let \( k_n \) be the least integer such that \( \text{Re}(z^0_{n-1}) \geq \text{Re}(\xi_{n-1}) \).

3.2. The choice of \( \xi_n \). In this subsection, we will choose the sequence \( \{\xi_n\} \) to realize the non-landing rays whose accumulation sets are homeomorphic to the closed topologist’s sine curve.

**Lemma 3.1.** Let \( n \geq 1 \). Then for any itinerary \( \gamma_n(\gamma) \subset T_1 \), there is a homeomorphism

\[
\chi : [z^0_{n-1}, z^0_n] \to [z^0_{n-1}(\gamma), z^0_n(\gamma)],
\]

such that

(1) \( \chi(z^0_{n-1}) = z^0_{n-1}(\gamma) \) for all \( j = 0, 1, \ldots, n - 1 \); and

(2) for any \( \xi \in [z^0_{n-1}, z^0_n] \), we have

\[
\text{dist}(\xi, \xi(\gamma)) < D_1,
\]

where \( D_1 \) is a universal constant.

**Proof.** According to Lemma 5.1 of \[6\], there is a universal constant \( D_2 > 0 \) such that for all \( \gamma^0_n(\gamma) \subset T_1 \) and all \( 0 \leq l \leq n - 1 \), dist\( (z^l_n, z^l_{n-1}(\gamma)) < D_2 \). Note that \( [z^0_{n-1}, z^0_n] \) and \( [z^0_{n-1}(\gamma), z^0_n(\gamma)] \) are contained in the tail of \( \gamma_i(\gamma) \). This implies that there is a homeomorphism

\[
\chi_{n-1} : [z^0_{n-1}, z^0_{n-1}+1] \to [z^0_{n-1}(\gamma), z^0_{n-1}+1(\gamma)],
\]

\[
\xi \mapsto \xi(\gamma)
\]

such that for any \( \xi \in [z^0_{n-1}, z^0_{n-1}+1] \), dist\( (\xi, \xi(\gamma)) < D_1 \) for all \( n \geq 1, 1 \leq j \leq n - 2 \), where \( D_1 \) is a constant only depending on \( D_2 \). Hence, we can define the
homeomorphism $\chi : [s_{n-1}, s_{n-1}^{-1}] \rightarrow [z_{n-1}^0(s), z_{n-1}^{-1}(s)]$ by

$$\chi(\xi) = \begin{cases} 
\chi_{n-1}^0(\xi), & \xi \in [s_{n-1}, s_{n-1}^1]; \\
\chi_{n-1}^1(\xi), & \xi \in [s_{n-1}, s_{n-1}^2]; \\
\vdots & \\
\chi_{n-2}^n(\xi), & \xi \in (s_{n-1}, s_{n-1}].
\end{cases}$$

Lemma 3.2. There exists a $D_3 > 0$ such that for any $n \geq 2$, there is a homeomorphism $\phi_{n-1}$ from $[z_{n-1}^0(s), \xi(s)]$ to $[z_{n-1}^0(s), z_{n-1}^{-1}(s)]$ satisfies the following: if $w \in [z_{n-1}^0(s), \xi(s)]$, then

$$\text{dist}(w, \phi_{n-1}(w)) < D_3;$$

and

$$\text{dist}(g_i \circ \cdots \circ g_{n-1}(w), g_i \circ \cdots \circ g_{n-1}(\phi_{n-1}(w))) < \delta_{n-i}^n \cdot D_3$$

for any $1 \leq i \leq n - 1$.

Proof. By Lemma 3.1 and a similar argument as in the proof of Lemma 5.3 in [6], we get that there is a universal constant $D_4$ such that $\text{dist}(z_{n-1}^{-1}(s)), \xi(s)) < D_4$. By Lemma 2.3, it follows that the wiggling of both $[z_{n-1}^0(s), \xi(s)]$ and $[z_{n-1}^0(s), z_{n-1}^{-1}(s)]$ is bounded by some universal constant. This implies that the existence of $\tilde{\phi}_{n-1}$ so that (1) holds for some universal constant $D_3$. The inequality (2) then follows from (1) and Lemma 2.2.

Recall that $k_1 \geq 1$ be an integer large enough. Let $z_0^1 \in \gamma_0^1$ satisfy that

$$\text{Re}(z_0^1) = \text{Re}(z_0^{-1}).$$

Then we define the sequence $\{\xi_n\}$ as follows. Let

$$\xi_0 = z_0^1,$$
$$\xi_1 = z_1^1,$$
$$\xi_2 = z_2^1,$$

and

$$\xi_3 = z_3^1.$$ 

For $s \in \mathbb{Z}^+$, suppose for $n = 4s - i$ ($i = 1, 2, 3, 4$), $\xi_n$ has been defined. Then for $n = 4s + i$ ($i = 0, 1, 2, 3$), we define

$$\xi_{4s} = (f_{4s} \circ \tilde{\phi}_{4s-1} \circ f_{4s} \circ \tilde{\phi}_{4s-2})(z_{4s-2}^1),$$

$$\xi_{4s+1} = z_{4s+1}^1,$$

and

$$\xi_{4s+3} = z_{4s+3}^1.$$ 

where $f_i = E^{k_i+3}$. Thus, the itinerary $s^*$ of the prescribed ray can be determined by (*) appeared in the last subsection. In particular, for $\xi_i$ ($i = 0, 1, \cdots, 5$) defined above, see Figure 1 for an explanation of how $\gamma_0^0$ is folded by $g_1^0 \circ g_2^0 \circ \cdots \circ g_5^0 \circ g_6^0$. 
3.3. The folding points of $\gamma_{g^*}$. Recall the definition of the folding point $c$ appeared in §2.2, one can also take any point $c_0$ to be the folding point provided that the diameter of the subarc between $c$ and $c_0$ is bounded by some universal constant. From this, we can also define the set of folding points of $\gamma_{g^*}$ by the following inductive way, and denote it by $Z_i$.

\[ \gamma_{g^*}^{i-n} = g_{i+1} \circ \cdots \circ g_n(\gamma_0) \]

Let $n \geq 1$ be fixed and $Z_n = \{z_0\}$. For any $0 \leq i \leq n-1$, suppose $Z_{i+1}$ has been defined. Let us define $Z_i$ as follows. Let $W_0 = \{z_0\}$. Suppose $W_j$ has been defined for $j \geq 0$. Let $W_{j+1}$ be the set consisting of the well defined points of

\[ \{z, g_{i+1} \circ \ldots \circ g_n \circ \ldots \circ \phi_{n-1} \circ \ldots \circ \phi_{i+1}(z)\} \]

where $z \in W_j$ and $\phi_l (i+1 \leq l \leq n-1)$ are the maps defined in Lemma 3.2. Define

\[ Z_i = g_{i+1}(Z_{i+1}) \cup W_{n-i-1}. \] (4)

In fact, the set of folding points is defined in accordance with geometric visualization, and hence is readily comprehensible.
4. **The accumulation set of the prescribed ray.** Let $\gamma_0 = \gamma_2^*$ be the escaping ray with itinerary $\mathbf{s}^*$ obtained in § 3. In this section, we will prove that the accumulation set of $\gamma_0$ is bounded, disjoint from $\gamma_0$ and homeomorphic to the closed topologist’s sine curve.

Let $i \geq 0$ and $j \leq i$. From now on, we redefine the notations $\gamma_i, \gamma_i^j$ and $z_i^j$ as follows. Let $\gamma_i = f_1 \circ \cdots \circ f_1(\gamma_0)$. We denote the tail of $\gamma_i$ by $\gamma_i^0$ and the endpoint of $\gamma_i^0$ by $z_i^0$. Then we define $\gamma_i^j$ and $z_i^j$ by

$$g_i(\gamma_i^j) = \gamma_i^{j-1} 	ext{ and } g_i(z_i^j) = z_i^{j-1}.$$

In addition, let $[z_i^{j1}, z_i^{j2}]$ be the subarc of $\gamma_i$ within $z_i^{j1}$ and $z_i^{j2}$, if it is well defined.

4.1. **The accumulation set of $\gamma_0$ is bounded.** In this subsection, we prove

**Proposition 4.1.** The accumulation set of $\gamma_0$ is bounded in the plane.

**Proof.** For $k \geq 1$, let $I_k = [\nu_0^{-k+1}, \nu_0^{-k}]$. It is sufficient to prove that $\gamma_0 \backslash \gamma_0^0$ is contained in $D_3$-neighborhood $U_{D_3}(I_1)$ of $I_1$ with respect to hyperbolic metric in $X$ for some constant $D_3 > 0$ defined later. For any compact subsets $A, B \subset X$, recall that the semi-Hausdorff distance is defined by

$$\hat{d}_H(A, B) = \sup \inf \text{dist}(x, y).$$

Since $\xi_{n-1} \in (\nu_0^{-n+1}, \nu_0^{-n})$, Lemma 2.2 together with Lemma 3.2 implies that

$$\begin{align*}
\hat{d}_H(I_n, I_1 \cup \cdots \cup I_{n-1}) &\leq \hat{d}_H(g_1 \circ \cdots \circ g_{n-1}([\nu_0^{-n+1}, \nu_0^{-1}]), g_1 \circ \cdots \circ g_{n-1}([\nu_0^{-1}, \xi_{n-1}])) \\
&\leq \delta_0^{n-1} \hat{d}_H([\nu_0^{-n+1}, \nu_0^{-1}], [\nu_0^{-1}, \xi_{n-1}]) \\
&< \delta_0^{n-1} \cdot D_3.
\end{align*}$$

Then, we have

$$\begin{align*}
\hat{d}_H(I_n, I_1) &\leq \hat{d}_H(I_n, I_1 \cup \cdots \cup I_{n-1}) + \hat{d}_H(I_{n-1}, I_1 \cup \cdots \cup I_{n-2}) + \cdots + \hat{d}_H(I_2, I_1) \\
&\leq \frac{D_3}{1 - \delta_0}.
\end{align*}$$

holds for all $n \geq 1$. Hence, the proof of the proposition can be finished by taking $D_3 = D_3/(1 - \delta_0)$ provided that the first inequality of (5) holds.

Now we prove the first inequality of (5) by induction. We first prove the case $n = 3$, that is

$$\hat{d}_H(I_3, I_1) \leq \hat{d}_H(I_3, I_1 \cup I_2) + \hat{d}_H(I_2, I_1).$$

Since $\hat{d}_H(I_3, I_1 \cup I_2) = \min\{\hat{d}_H(I_3, I_1), \hat{d}_H(I_3, I_2)\}$ and $\hat{d}_H(I_3, I_1) \leq \hat{d}_H(I_3, I_1) + \hat{d}_H(I_2, I_1)$, it suffices to prove that

$$\hat{d}_H(I_3, I_1) \leq \hat{d}_H(I_3, I_2) + \hat{d}_H(I_2, I_1).$$

By

$$\text{dist}(x_3, x_1) \leq \text{dist}(x_3, x_2) + \text{dist}(x_2, x_1)$$

for any $x_j \in I_j$ ($j = 1, 2, 3$), we get

$$\inf_{x_1 \in I_1} \text{dist}(x_3, x_1) \leq \text{dist}(x_3, x_2) + \inf_{x_1 \in I_1} \text{dist}(x_2, x_1).$$
This implies that
\[
\inf_{x_1 \in I_1} \inf_{x_2 \in I_2} \text{dist}(x_3, x_1) \leq \inf_{x_2 \in I_2} (\text{dist}(x_3, x_2) + \inf_{x_1 \in I_1} \text{dist}(x_2, x_1)) \\
\leq \inf_{x_2 \in I_2} \text{dist}(x_3, x_2) + \sup_{x_2 \in I_2} \inf_{x_1 \in I_1} \text{dist}(x_2, x_1).
\]

Therefore, we can get that
\[
\sup_{x_3 \in I_3} \inf_{x_1 \in I_1} \text{dist}(x_3, x_1) \leq \sup_{x_3 \in I_3} \inf_{x_2 \in I_2} \text{dist}(x_3, x_2) + \inf_{x_2 \in I_2} \sup_{x_1 \in I_1} \text{dist}(x_2, x_1),
\]
that is,
\[
\tilde{d}_H(I_1, I_1) \leq \tilde{d}_H(I_2, I_1) + \tilde{d}_H(I_3, I_3).
\]

Suppose the first inequality of (5) holds for \( n = k \). Then we have
\[
\tilde{d}_H(I_{k+1}, I_1 \cup I_2) \leq \tilde{d}_H(I_{k+1}, (I_1 \cup I_2) \cup \cdots \cup I_k) + \tilde{d}_H(I_k, (I_1 \cup I_2) \cup \cdots \cup I_{k-1}) \\
+ \cdots + \tilde{d}_H(I_3, I_1 \cup I_2).
\]

Let \( n = k + 1 \). By (6), we have
\[
\tilde{d}_H(I_{k+1}, I_1) \leq \tilde{d}_H(I_{k+1}, I_1 \cup I_2) + \tilde{d}_H(I_2, I_1).
\]

Combining with the two inequalities above, we complete the proof of the first inequality of (5) for \( n = k + 1 \). \( \square \)

4.2. The topology of accumulation set of \( \gamma_0 \). For any \( k \geq 1 \), set
\[
\tilde{f}_k = f_k \circ \cdots \circ f_1,
\]
\[
\tilde{g}_k = g_1 \circ \cdots \circ g_k
\]
and
\[
\phi_k = \tilde{g}_k \circ \tilde{f}_k,
\]
where \( \tilde{f}_k \) is the map defined by Lemma 3.2.

To get the accumulation set of \( \gamma_0 \), we first give a parameterization of \( I_k = [z_0^{-k+1}, z_0^k] \) for \( k \geq 1 \). In fact, according to the choice of \( \xi_{k-1} \), we can parameterize \( I_k \) inductively as follows (see Figure 2). Let \( \omega_1(t) (0 \leq t \leq 1/2) \) be any parameterization of \( I_1 \) with \( \omega_1(0) = z_0^0 \) and \( \omega_1(1/2) = z_0^{-1} \). We parameterize \( I_2 = \omega_2(t) (1/2 \leq t \leq 1) \) by
\[
\omega_2(1-t) = \phi_1(\omega_1(t)), \quad 0 \leq t \leq 1/2.
\]
Then we can define \( I_3 = \omega_3(t) (0 \leq t \leq 1) \) by
\[
\omega_3(t) = \begin{cases} 
\phi_2(\omega_1(t)), & t \in [0, 1/2] \\
\phi_2(\omega_2(t)), & t \in (1/2, 1] 
\end{cases}.
\]

Next, for \( s \geq 1 \), we can parameterize \( I_{4s} = \omega_{4s}(t) (0 \leq t \leq s), I_{4s+1} = \omega_{4s+1}(t) (s \leq t \leq s+1/2), I_{4s+2} = \omega_{4s+2}(s+1/2 \leq t \leq s+1) \) and \( I_{4s+3} = \omega_{4s+3}(t) (0 \leq t \leq s+1) \) respectively as follows. Let
\[
\omega_{4s}(t) = \phi_{4s-1} \circ \omega_{4s-1}(t) (0 \leq t \leq s).
\]
For \( t \in [0, 1/2] \), the parameterization \( \omega_{4s+1} \) and \( \omega_{4s+2} \) are defined by
\[
\omega_{4s+1}(s+t) = \phi_{4s} \circ \omega_{4s}(s-t)
\]
and
\[
\omega_{4s+2}(s+1-t) = \phi_{4s+1} \circ \omega_{4s+1}(s+t).
\]
Then we define
\[
\omega_{4s+3}(t) = \begin{cases} 
\phi_{4s+2}(\omega_{4s}(t)), & t \in [0, s]; \\
\phi_{4s+2}(\omega_{4s+1}(t)), & t \in (s, s + 1/2]; \\
\phi_{4s+2}(\omega_{4s+2}(t)), & t \in (s + 1/2, s + 1]. 
\end{cases}
\]

This completes the parameterization of $I_k$ for all $k \geq 1$.

**Figure 2.** Here is the sketch of $\gamma_{0^{-12}}$. The black polylines with different-sized bold show the idea of how the limit curve $\eta$ is produced in the accumulation set of $\gamma_0$.

**Lemma 4.2.** For any integer $s_0 \geq 1$, \{\omega_{4s-1}\}_{s \geq s_0} converges uniformly on $[0, s_0]$.

**Proof.** Note that for any $t \in [0, s_0]$, we have
\[
\omega_{4s-1}(t) = \tilde{g}_{4s+2} \circ f_{4s+2} \circ f_{4s+1} \circ f_{4s} \circ \tilde{f}_{4s-1} \circ \omega_{4s-1}(t)
\]
and
\[
\omega_{4s+3}(t) = \phi_{4s+2} \circ \phi_{4s-1} \circ \omega_{4s-1}(t) \\
= \tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2} \circ f_{4s+2} \circ f_{4s+1} \circ f_{4s} \circ \tilde{\phi}_{4s-1} \circ \tilde{f}_{4s-1} \circ \omega_{4s-1}(t).
\]
If we set
\[
\omega_s(t) = \tilde{g}_{4s+2} \circ f_{4s+2} \circ f_{4s+1} \circ f_{4s} \circ \tilde{\phi}_{4s-1} \circ \tilde{f}_{4s-1} \circ \omega_{4s-1}(t),
\]
\[
\omega_{4s+3}(t) = \phi_{4s+2} \circ \phi_{4s-1} \circ \omega_{4s-1}(t). 
\]
then
\[\text{dist}(\omega_{4s-1}(t), \omega_{4s+3}(t)) \leq \text{dist}(\omega_{4s-1}(t), \omega_s(t)) + \text{dist}(\omega_s(t), \omega_{4s+3}(t))\]
by (3), we have
\[\text{dist}(\omega_{4s-1}(t), \omega_s(t)) + \text{dist}(\omega_s(t), \omega_{4s+3}(t)) < (\delta_0^{4s-1} + \delta_0^{4s+2}) \cdot D_3\] (7)
by Lemma 3.2. This proves the lemma. \(\blacksquare\)

See Figure 2 for an illustration of Lemma 4.2.

**Definition** (limit curve). We define the limit curve \(\eta(t)\) \((0 \leq t < \infty)\) by
\[\eta(t) = \lim_{s \to \infty} \omega_{4s-1}(t), \quad t \in [0, s_0],\]
for any \(s_0 \geq 1\).

**Lemma 4.3.** By reparameterizing \(\omega_{4s-1}\) if necessary, the curve \(\eta\) is one-to-one.

**Proof.** We first claim that \(\eta\) is not a single point. By taking \(k_1\) large enough, we can get that \(\eta(0) \neq \eta(1/2)\). In fact, (7) implies that
\[\text{dist}(\eta(0), \omega_3(0)) \leq 2/(1 - \delta_0^4) \quad \text{and} \quad \text{dist}(\eta(1/2), \omega_3(1/2)) \leq 2/(1 - \delta_0^4)\]
after a simple calculation. Moreover, according to Lemma 3.2, \(\text{dist}(\omega_3(0), z_0^0)\) and \(\text{dist}(\omega_3(1/2), z_0^1)\) are bounded by some constant depending only on \(D_4\). In addition, by (3), we have \(\text{dist}(z_0^0, z_0^1) \to \infty\) as \(k_1 \to \infty\). This implies that \(\text{dist}(\eta(0), \eta(1/2)) > 0\) provided \(k_1\) is large enough. Hence, \(\eta\) is not a single point.

Suppose \(\eta\) is not one-to-one. Then there exist \(0 \leq t_1 < t_2 < \infty\) and \(d > 0\) such that
\[\eta(t_1) = \eta(t_2) \quad \text{and} \quad \text{diam}\{\{\eta(t) \mid t_1 \leq t \leq t_2\}\} > d.\]
Note that we can reparameterize \(\omega_1\) to exclude the case \(\text{diam}\{\{\eta(t) \mid t_1 \leq t \leq t_2\}\} = 0\). Let \(D > K\) be an arbitrarily large number, where \(K\) is defined in Lemma 2.3. Take \(l \geq 1\) such that
\[l \equiv -1 \, (\text{mod} \, 4) \quad \text{and} \quad d > \delta_0^0 \cdot D.\]
Let \(k \geq 1\) be an integer with \(t_1, t_2 \in [0, k]\) and \(r = \max\{(l+1)/4, k\}\). Then
\[\hat{f}_{4r-1} \circ \omega_{4s-1}(t), \quad s \geq k,\]
converges uniformly to \(\hat{f}_{4r-1} \circ \eta(t)\) on \([0, k]\) as \(s \to +\infty\).

**Claim.** For any \(t_1 < t < t_2\), \(\hat{f}_{4r-1} \circ \omega_{4s-1}(t)\) \((s \geq k)\) is not a folding point.

Suppose \(s \leq r\). Note that \(\{\omega_{4s-1}(t) \mid t_1 \leq t \leq t_2\} \subset I_{4s-1}\) and \(\hat{f}_{4s-1}(I_{4s-1}) = \{z_0^0, z_1^0\}\) is contained in the tail of \(\gamma_{4s-1}\). This implies that \(\hat{f}_{4r-1} \circ \omega_{4s-1}(t)\) \(t_1 \leq t \leq t_2\) \(\subset \hat{f}_{4r-1}(I_{4s-1})\) contains no folding points.

Suppose \(s > r\). We now prove the claim by induction.

For \(s = r+1\), we have
\[\hat{f}_{4r-1} \circ \omega_{4r+3}(t) = \hat{f}_{4r-1} \circ \phi_{4r+2} \circ \phi_{4r-1} \circ \omega_{4r-1}(t) = g_{4r} \circ g_{4r+1} \circ g_{4r+2} \circ \hat{f}_{4r+2} \circ f_{4r+2} \circ f_{4r+1} \circ f_{4r} \circ \phi_{4r-1} \circ \hat{f}_{4r-1} \circ \omega_{4r-1}(t).\]
Since we have proved that \(\hat{f}_{4r-1} \circ \omega_{4r-1}(t) \ (t_1 < t < t_2)\) is not a folding point, then so does \(\phi_{4r-1} \circ \hat{f}_{4r-1} \circ \omega_{4r-1}(t)\) by (4). If we set \(w = f_{4r+2} \circ f_{4r+1} \circ f_{4r} \circ \phi_{4r-1} \circ f_{4r-1} \circ \omega_{4r-1}(t)\), then also by (4), \(\phi_{4r-1} \circ \hat{f}_{4r-1} \circ \omega_{4r-1}(t) = g_{4r} \circ g_{4r+1} \circ g_{4r+2}(w)\)
is not a folding point implies that $g_{4r} \circ g_{4r+1} \circ g_{4r+2} \circ \hat{\phi}_{4r+2}(w) = \hat{f}_{4r-1} \circ \omega_{4r+3}(t)$ is also not a folding point.

Suppose we have proved that for some $s \geq r+1$, $\hat{f}_{4r-1} \circ \omega_{4s-1}(t)$ is not a folding point. To prove $\hat{f}_{4r-1} \circ \omega_{4s+3}(t)$ is not a folding point, we first note that

\[
\hat{f}_{4r-1} \circ \omega_{4s+3}(t) \\
= \hat{f}_{4r-1} \circ \phi_{4s+2} \circ \phi_{4s-1} \circ \omega_{4s-1}(t) \\
= g_{4r} \circ \cdots \circ g_{4s+2} \circ \hat{\phi}_{4s+2} \circ f_{4s+2} \circ f_{4s+1} \circ f_{4s} \circ \\
\hat{\phi}_{4s-1} \circ f_{4s-1} \circ \cdots \circ f_{4r} \circ \hat{f}_{4r-1} \circ \omega_{4s-1}(t).
\]

Let $w_1 = f_{4s-1} \circ \cdots \circ f_{4r} \circ \hat{f}_{4r-1} \circ \omega_{4s-1}(t)$ and $w_2 = f_{4s-2} \circ f_{4s+1} \circ f_{4s} \circ \hat{\phi}_{4s-1}(w_1)$. Since $\hat{f}_{4r-1} \circ \omega_{4s-1}(t) = g_{4r} \circ \cdots \circ g_{4s-1}(w_1)$ is not a folding point, then by (4), $g_{4r} \circ \cdots \circ g_{4s-1} \circ \hat{\phi}_{4s-1}(w_1)$ is also not a folding point. Note that $g_{4r} \circ \cdots \circ g_{4s-1} \circ \hat{\phi}_{4s-1}(w_1) = g_{4r} \circ \cdots \circ g_{4s+2}(w_2)$. Hence, by (4) we have $g_{4r} \circ \cdots \circ g_{4s+2} \circ \hat{\phi}_{4s+2}(w_2) = \hat{f}_{4r-1} \circ \omega_{4s+3}(t)$ is not a folding point. This completes the claim.

On the other hand, by Lemma 2.2, we have

$$\text{diam} \{ \hat{f}_{4r-1} \circ \eta(t) \mid t_1 \leq t \leq t_2 \} > \text{diam} \{ \{ \eta(t) \mid t_1 \leq t \leq t_2 \} / \delta_0 > d / \delta_0 > D. $$

Note that $\hat{f}_{4r-1} \circ \omega_{4s-1}(t)$ converges uniformly to $\hat{f}_{4r-1} \circ \eta(t)$ on $[t_1, t_2]$ and $\eta(t_1) = \eta(t_2)$. Then for any $\epsilon > 0$, by taking $s$ large enough, we can get that

$$\text{diam} \{ \hat{f}_{4r-1} \circ \omega_{4s-1}(t) \mid t_1 \leq t \leq t_2 \} > D$$

and

$$\text{dist} \{ \hat{f}_{4r-1} \circ \omega_{4s-1}(t_1), \hat{f}_{4r-1} \circ \omega_{4s-1}(t_2) \} < \epsilon.$$ 

Since $\{ \hat{f}_{4r-1} \circ \omega_{4s-1}(t) \mid t_1 \leq t \leq t_2 \}$ is contained in $T_1$, by taking $D$ large enough, we must have $W(\{ \hat{f}_{4r-1} \circ \omega_{4s-1}(t) \mid t_1 \leq t \leq t_2 \}) > K$. However, $\{ \hat{f}_{4r-1} \circ \omega_{4s-1}(t) \mid t_1 \leq t \leq t_2 \}$ contains no folding points of $\gamma_{4r-1}$ in its interior, this contradicts with Lemma 2.3. Therefore, $\eta$ is one-to-one.

**Lemma 4.4.** The two sequences $\{ \eta(s-t) \}_{s \geq 1}$ and $\{ \eta(s+t) \}_{s \geq 1}$ are both uniformly convergent to the same arc $\alpha(t)$ for $t \in [0, 1/2]$ as $s \to \infty$. In particular, we have

$$\bigcap_{n>0} \text{Cl}(\eta(n, \infty)) = \alpha.$$

**Proof.** We first prove that $\{ \eta(s-t) \}_{s \geq 1}$ is uniformly convergent to an arc $\alpha(t)$ on $t \in [0, 1/2]$. Let

$$w_1 = f_{4s+2} \circ \hat{\phi}_{4s+1} \circ f_{4s+1} \circ \hat{\phi}_{4s} \circ \hat{f}_{4s} \circ \omega_{4s}(s-t),$$

$$w_2 = f_{4s+1} \circ \hat{\phi}_{4s} \circ \hat{f}_{4s} \circ \omega_{4s}(s-t),$$

$$w_3 = f_{4s} \circ \omega_{4s}(s-t)$$

and

$$w_4 = \hat{f}_{4s+2} \circ \omega_{4s}(s-t).$$

Note that

$$\tilde{g}_{4s+2}(w_1) = \tilde{g}_{4s+1} \circ \hat{\phi}_{4s+1}(w_2)$$

$$\tilde{g}_{4s+1}(w_2) = \tilde{g}_{4s} \circ \hat{\phi}_{4s}(w_3)$$

and

$$\tilde{g}_{4s}(w_3) = \tilde{g}_{4s+2}(w_4).$$
Then by Lemma 3.2, we get that
\[ \text{dist}(\omega_{4s+3}(s+1-t), \omega_{4s+3}(s-t)) \]
\[ = \text{dist}(\phi_{4s+2} \circ \omega_{4s+2}(s+1-t), \phi_{4s+2} \circ \omega_{4s}(s-t)) \]
\[ = \text{dist}(\tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2} \circ f_{4s+2} \circ \tilde{\phi}_{4s+1} \circ f_{4s+1} \circ \tilde{\phi}_{4s} \circ f_{4s} \circ \omega_{4s}(s-t), \]
\[ \tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2} \circ \tilde{f}_{4s+2} \circ \omega_{4s}(s-t)) \]
\[ \leq \text{dist}(\tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2}(w_1), \tilde{g}_{4s+2}(w_1)) + \text{dist}(\tilde{g}_{4s+1} \circ \tilde{\phi}_{4s+1}(w_2), \tilde{g}_{4s+1}(w_2)) \]
\[ + \text{dist}(\tilde{g}_{4s} \circ \tilde{\phi}_{4s}(w_3), \tilde{g}_{4s}(w_3)) + \text{dist}(\tilde{g}_{4s+2}(w_4), \tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2}(w_4)) \]
\[ \leq (\delta_{0}^{4s+2} + \delta_{0}^{4s+1} + \delta_{0}^{4s} + \delta_{0}^{4s+2}) \cdot D_3. \]

Therefore, we have
\[ \text{dist}(\omega_{4s+3}(s+1-t), \omega_{4s+3}(s-t)) \xrightarrow{\text{as } s \to \infty} 0. \quad (8) \]

Here and subsequently, the symbol “⇒” means uniform convergence. In addition, Lemma 4.2 implies that for any $\epsilon > 0$ and $s_0 \geq 1$, there exists an $s^*$ depending only on $\epsilon$ such that for all $s \geq \max\{s^*, s_0\}$, we have
\[ \text{dist}(\omega_{4s+3}(t), \eta(t)) < \epsilon, \quad t \in [0, s_0]. \]

Then for the above $\epsilon > 0$, we obtain that for all $s \geq s^*$,
\[ \text{dist}(\omega_{4s+3}(s-t), \eta(s-t)) < \epsilon, \quad \text{dist}(\omega_{4s+3}(s+1-t), \eta(s+1-t)) < \epsilon. \quad (9) \]

Hence, by (8) and (9),
\[ \text{dist}(\eta(s+1-t), \eta(s-t)) \xrightarrow{\text{as } s \to \infty} 0. \]

We denote the limit arc by $\alpha(t)$ ($0 \leq t \leq 1/2$).

To prove that $\{\eta(s+t)\}_{s \geq 1}$ is also uniformly convergent to the limit arc $\alpha(t)$, it is sufficient to prove that $\text{dist}(\eta(s+t), \eta(s-t)) \xrightarrow{\text{as } t \to 1/2} 0$ for $t \in [0, 1/2]$ as $s \to \infty$. According to Lemma 3.2, we can get
\[ \text{dist}(\omega_{4s+3}(s-t), \omega_{4s+3}(s+t)) \]
\[ = \text{dist}(\phi_{4s+2} \circ \omega_{4s}(s-t), \phi_{4s+2} \circ \omega_{4s+1}(s+t)) \]
\[ = \text{dist}(\phi_{4s+2} \circ \omega_{4s}(s-t), \phi_{4s+2} \circ \phi_{4s} \circ \omega_{4s}(s-t)) \]
\[ = \text{dist}(\tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2}(w_4), \tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2} \circ f_{4s+2}(w_2)) \]
\[ \leq \text{dist}(\tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2}(w_4), \tilde{g}_{4s+2}(w_4)) + \text{dist}(\tilde{g}_{4s}(w_3), \tilde{g}_{4s} \circ \tilde{\phi}_{4s}(w_3)) \]
\[ + \text{dist}(\tilde{g}_{4s+2} \circ f_{4s+2}(w_2), \tilde{g}_{4s+2} \circ \tilde{\phi}_{4s+2} \circ f_{4s+2}(w_2)) \]
\[ \leq (\delta_{0}^{4s+2} + \delta_{0}^{4s} + \delta_{0}^{4s+2}) \cdot D_3. \]

Therefore,
\[ \text{dist}(\omega_{4s+3}(s-t), \omega_{4s+3}(s+t)) \xrightarrow{\text{as } s \to \infty} 0. \quad (10) \]

Combining with (9) and (10), we obtain
\[ \text{dist}(\eta(s-t), \eta(s+t)) \]
\[ \leq \text{dist}(\eta(s-t), \omega_{4s+3}(s-t)) + \text{dist}(\omega_{4s+3}(s-t), \omega_{4s+3}(s+t)) + \text{dist}(\omega_{4s+3}(s+t), \eta(s+t)) \]
\[ \xrightarrow{\text{as } s \to \infty} 0. \]
Finally, we prove \( \bigcap_{n>0} \eta((n,\infty)) = \alpha \). Let \( \{t_n\} \) be a sequence with \( t_n \to \infty \) as \( n \to \infty \). Then for any subsequence \( \{t_{n_k}\} \) of \( \{t_n\} \) such that \( \{\eta(t_{n_k})\} \) converges to a point \( z \), it suffices to prove that \( z \in \alpha \). Suppose \( z \notin \alpha \). Since \( \alpha \) is compact, we get that \( \text{dist}(z,\alpha) > 0 \). It is worth pointing out that for any \( t_{n_k} \), there exist an \( x_k \in [0,1/2] \) and a positive integer \( N_k \) such that \( t_{n_k} = N_k + x_k \). Then by the conclusion proved above, we get \( \text{dist}(\eta(t_{n_k}),\alpha(x_k)) \to 0 \) as \( k \to \infty \). This implies that \( \text{dist}(z,\alpha) \to 0 \), a contradiction.

The proof of the lemma is finished. \( \square \)

**Lemma 4.5.** The arc \( \alpha \) defined in Lemma 4.4 is a Jordan arc, and moreover, \( \overline{\eta \setminus \eta} = \alpha \).

**Proof.** First, we prove that the arc \( \alpha \) is one-to-one. From Lemma 4.4 and the proof of Lemma 4.2, we get that for \( t \in [0,1/2] \),

\[
\text{dist}(\omega_{4s-1}(s-t),\alpha(t)) \leq \text{dist}(\omega_{4s-1}(s-t),\eta(s-t)) + \text{dist}(\eta(s-t),\alpha(t)) = 0,
\]

as \( s \to \infty \). This means that for \( t \in [0,1/2] \), we have

\[
\omega_{4s-1}(s-t) \ni \alpha(t) \quad \text{as} \quad s \to \infty.
\]

(11)

Note that for any \( s \geq 1 \), the set of the folding points of \( \omega_{4s-1} \) is \( \{\omega_{4s-1}(m/2) | 0 \leq m \leq 2s, m \in \mathbb{Z}\} \) by (4). We get that for any \( s \geq 1 \), \( \omega_{4s-1}(s-t) \) \( (0 < t < 1/2) \) contains no folding point, so does \( \tilde{f}_l \circ \omega_{4s-1}(s-t) \) \((0 < t < 1/2)\) for any \( l \geq 1 \). This combining with (11) implies that the arc \( \alpha \) is one-to-one by the same reasoning as in the proof of Lemma 4.3. Hence, \( \alpha \) is a Jordan arc.

Then, we prove \( \overline{\eta \setminus \eta} = \eta \cup \alpha \). By Lemma 4.4, we get that \( \eta \cup \alpha \subset \overline{\eta \setminus \eta} \). For the other direction, suppose \( \overline{\eta \setminus \eta} = \emptyset \), the proof is obvious. Otherwise, suppose \( \overline{\eta \setminus \eta} \neq \emptyset \). We only need to prove that for any \( z \in \overline{\eta \setminus \eta} \), we have \( z \in \alpha \). Since \( z \in \overline{\eta \setminus \eta} \), we get that \( \eta(t_n) \to z \) as \( n \to \infty \) for some sequence \( \{t_n\} \). This implies \( t_n \to \infty \). In fact, if there existed a bounded subsequence \( \{t_{n_k}\} \) of \( \{t_n\} \) such that \( \eta(t_{n_k}) \to z \), then by the fact that \( \eta(t) \) \((0 \leq t \leq T)\) is closed for any \( 0 < T < \infty \), we get \( z \in \eta \). This contradicts with the fact that \( z \in \overline{\eta \setminus \eta} \). Then Lemma 4.4 implies that there is a \( t_0 \in [0,1/2] \) such that \( \eta(t_0) \to \alpha(t_0) \) as \( n \to \infty \). Thus, we get \( z \in \alpha \).

Finally, we prove that \( \eta \cap \alpha = \emptyset \). Otherwise, suppose \( \eta(t_0) \in \eta \cap \alpha \) for some \( t_0 \in [0,\infty) \). Since \( \eta([t_0,\infty)) \) is a ray while \( \alpha \) is a Jordan arc, there are \( t_1, t_2 \in (t_0, \infty) \) such that \( \eta([t_1, t_2]) \notin \eta([t_0, \infty)) \cap \alpha \) (or else \( \alpha \) would contain a dense subset of \( \eta([t_0, \infty)) \)), and hence contain \( \eta([t_0, \infty)) \), which contradicts the fact that \( \alpha \) is a Jordan arc. Let \( A_1 = \eta([t_1, t_2]) \) and \( A_2 = \overline{\eta \setminus \eta}([t_1, t_2]) \). Then the fact \( \overline{\eta \setminus \eta} = \eta \cup \alpha \) and Lemma 4.4 imply that \( A_1 \) and \( A_2 \) are closed connected subsets of \( \mathbb{C} \) whose intersection consists of precisely two points \( \eta(t_1) \) and \( \eta(t_2) \). According to [8, Theorem 61.4] or [10, Theorem 11.5], we get that \( \overline{\eta} = A_1 \cup A_2 \) separates \( \mathbb{C} \). Since \( \overline{\eta} \) is contained in the accumulation set of \( \gamma_0 \), Proposition 4.1 implies that \( \overline{\eta} \) is bounded. Therefore, \( \overline{\eta} \) separates the plane. However, we can also prove that \( \overline{\eta} \) does not separate the plane. In fact, suppose this were not the case. Let \( U \) be one of the bounded components of \( \mathbb{C} \setminus \overline{\eta} \). Then, we have \( \mathbb{C} = E^n(U) \subset T_{-1} \cup T_0 \cup T_1 \) for any \( n \geq 0 \). This means that \( U \) is contained in the Fatou set of \( E \) by Montel Theorem. This contradicts the fact that the Fatou set of \( E \) is empty. Hence, we get \( \eta \cap \alpha = \emptyset \). Combining with the conclusion that \( \eta = \eta \cup \alpha \), we get \( \eta \setminus \eta = \alpha \).

The proof of the lemma is finished. \( \square \)

**Proposition 4.6.** The accumulation set \( A_{x_0}^* \) of \( \gamma_0 \) is exactly the set \( \eta \).
Proof. Note that $\omega_{4s-1} \subseteq \gamma_0$ for any $s \geq 1$, then the definition of $\eta$ implies that $\eta \subseteq A_{2\eta}$. For the other direction, suppose we have proved the following facts:

1. $\text{dist}(\omega_{4s-3}(s + 1 - t), \omega_{4s+2}(s + 1 - t)) = 0$ on $t \in [0, 1/2]$;
2. $\text{dist}(\omega_{4s+2}(s + 1 - t), \omega_{4s+1}(s + t)) = 0$ on $t \in [0, 1/2]$;
3. For any $s_0 \geq 1$, we have $\text{dist}(\omega_{4s+3}(t), \omega_{4s+4}(t)) = 0$ on $t \in [0, s_0]$.

as $s \to \infty$. Then, these imply that $\gamma_0$ has no accumulation points other than $\eta$. That is, $A_{2\eta} \subseteq \eta$. Hence, $A_{2\eta} = \eta$.

Now we prove the above items (1)-(3). For the proof of item (1), Lemma 3.2 implies that for $t \in [0, 1/2]$, we have

$$\text{dist}(\omega_{4s+3}(s + 1 - t), \omega_{4s+2}(s + 1 - t)) = \text{dist}(\phi_{4s+3} \circ \omega_{4s+2}(s + 1 - t), \omega_{4s+2}(s + 1 - t))$$

$$= \text{dist}((\tilde{g}_{4s+3} \circ \phi_{4s+2} \circ \tilde{f}_{4s+2} \circ \omega_{4s+2}(s + 1 - t)), \tilde{g}_{4s+2} \circ \tilde{f}_{4s+2} \circ \omega_{4s+2}(s + 1 - t))$$

$$\leq \delta_{4s+2} D_3.$$ 

Since the same proof works for items (2)-(3), we omit them for avoiding too much repetition. This completes the proof.

**Proposition 4.7.** The ray $\gamma_0$ is disjoint from $\eta$.

**Proof.** It suffices to prove that for any $k \geq 0$, $\eta \cap \gamma_0^k = \emptyset$.

**Claim.** There is a universal constant $D_6$ such that for any $l \geq k + 3$, we have

$$\gamma_l \setminus \gamma_l^{-k} \subseteq U_{D_6}([z^0_l, z_l^{l-k}])$$

and hence,

$$\tilde{f}_l(\eta) \subseteq U_{D_6}([z^0_l, z_l^{l-k}]).$$

(12)

For $i \geq 1$, let $I^l_i = [z_{i}^{-i+1}, z_{i}^{-i}]$. Note that

$$\gamma_l \setminus \gamma_l^{-k} = (\bigcup_{i=1}^{\infty} I^l_i) \cup [z^0_l, z_l^{l-k}].$$

Hence, we only need to prove that for any $i \geq 1$,

$$\tilde{d}_H(I^l_i, [z^0_l, z_l^{l-k}]) \leq D_6$$

for some constant $D_6 > 0$.

In fact, we can get the following inequalities:

$$\tilde{d}_H(I^l_i, [z^0_l, z_l^{l-k}])$$

$$\leq \tilde{d}_H(I^l_i, [z^0_l, z_l^{l-k}]) + \tilde{d}_H(I^l_{i-1}, [z^0_l, z_l^{l-k}]) + \tilde{d}_H(I^l_{i-2}, [z^0_l, z_l^{l-k}]) + \cdots + \tilde{d}_H(I^l_2, [z^0_l, z_l^{l-k}]) + \tilde{d}_H(I^l_1, [z^0_l, z_l^{l-k}])$$

$$\leq (\delta_{i+1} + \delta_{i+2} + \cdots + \delta_{i+3}) D_3$$

$$< D_3/(1 - \delta_0).$$

The first inequality holds by the same reasoning as in the proof of (5). For the second inequality, we prove it as follows. For any $1 \leq j \leq i$, let $A = [z_{j+i}^{-j+i-1}, z_{j+i}^{j+i-1}]$ and $B = [z_{j+i}^{j+i-1}, z_{j+i}^{j+i-1}]$. Since $j + l - k \geq 4$, then according to the definition of $\{\xi_n\}$, we have $[z_{j+i}^{j+i-1}, z_{j+i}^{j+i-1}] \supseteq [z_{j+i}^{j+i-1}, \xi_j + l - 1]$, and hence $\tilde{d}_H(B, A) < D_3$ by Lemma 3.2. Note that

$$g_{j+i} \circ \cdots \circ g_{j+i-1}(A) = [z^0_l, z_l^{l-k}] \cup I^l_1 \cup \cdots \cup I^l_{j+i-1}$$

and

$$g_{j+i} \circ \cdots \circ g_{j+i-1}(B) = I^l_j.$$
Therefore, combining with the fact that \([z_i^0, z_i^{l-k}] \supseteq [z_i^0, \xi_i] (l-k \geq 3)\) and Lemma 3.2, we get
\[
\tilde{d}_H(I_j^1, [z_i^0, z_i^{l-k}] \cup I_1^1 \cup \cdots \cup I_{j-1}^1) \leq \delta_0^{-1} \tilde{d}_H(B, A) < \delta_0^{-1} D_3.
\]
This completes the claim by taking \(D_0 = D_3/(1-\delta_0)\).

Since \([z_i^0, z_i^{l-k}]\) and \(\gamma_i^{l-k}\) are contained in the tail \(\gamma_i^0\) which can be written as a continuous function with respect to \(\text{Re}(z)\) and \([z_i^0, z_i^{l-k}] \cap \gamma_i^{l-k} = z_i^{l-k}\), there is a constant \(D_7 > 0\) depending only on \(D_0\) such that
\[
U_{D_0}([z_i^0, z_i^{l-k}]) \cap \gamma_i^{l-k} \subset U_{D_7}(z_i^{l-k}).
\]
From (12) and (13), we can obtain
\[
\tilde{\eta} \cap \gamma_0^{-k} = \tilde{g}_k(\tilde{f}_l(\tilde{\eta}) \cap \gamma_l^{l-k}) \subset U_{\epsilon}(z_0^{-k})
\]
with \(\epsilon = D_7 \cdot \delta_0\) by Lemma 2.2. Let \(l \to \infty\), we get that for any \(k \geq 0\), the set \(\tilde{\eta} \cap \gamma_0^{-k}\) contains at most one point \(z_0^{-k}\). This implies that the ray \(\gamma_0\) is disjoint from \(\tilde{\eta}\).

**Proof of the Theorem 1.1.** Let \(S = \{(x, y)|0 < x \leq 2/\pi, y = \sin(1/x)\}\) and \(T = \{(0, y)|-1 \leq y \leq 1\}\). According to Proposition 4.1, Proposition 4.6 and Proposition 4.7, it suffices to construct a homeomorphism \(\varphi\) from \(\tilde{\eta}\) to the closed topologist’s sine curve \(\mathcal{S} = S \cup T\).

For an integer \(i \geq 1\), let \(\eta_i = \{\eta(t)|t \in [(i-1)/2, i/2]\}\) and
\[
S_i = \{(x, \sin(1/x))|x \in [2/(2i+1)\pi), 2/(2i-1)\pi]\}.
\]
Then
\[
\eta = \bigcup_{i \geq 1} \eta_i \quad \text{and} \quad S = \bigcup_{i \geq 1} S_i.
\]
Note that we also denote by \(\eta_i(t)\) and \(S_i(x)\) the corresponding map without confusion. For any \((x, y) \in \mathbb{R}^2\), let \(\tau_1\) and \(\tau_2\) be the projections of \(\mathbb{R}^2\) onto its first and second coordinates, i.e., \(\tau_1(x, y) = x\) and \(\tau_2(x, y) = y\).

Then, we can define \(\varphi : \tilde{\eta} \to \mathcal{S}\) as follows. Let \(\varphi_1 : \eta_1 \to S_1\) be a homeomorphism with \(\varphi_1(\eta_1(0)) = (2/\pi, 1)\) and \(\varphi_1(\eta_1(1/2)) = (2/(3\pi), -1)\). Then for odd integer \(i > 0\), we define \(\varphi_i : \eta_i \to S_i\) by
\[
\eta_i(t) \to S_i(x_i(t)/(1 + (i-1)\pi \cdot x_i(t))),
\]
where \(x_i(t) = \tau_1 \circ \varphi_1 \circ \eta_1(t - (i-1)/2)\); and for even integer \(i > 0\), we define \(\varphi_i : \eta_i \to S_i\) by
\[
\eta_i(t) \to S_i(\tilde{x}_i(t)/(1 + (i+1)\pi \cdot \tilde{x}_i(t)-1)),
\]
where \(\tilde{x}_i(t) = \tau_1 \circ \varphi_1 \circ \eta_1(i/2-t)\). This implies that for any \(s \geq 1\) and \(t \in [0, 1/2]\), we have
\[
\tau_2 \circ \varphi_{2s+1} \circ \eta_{2s+1}(s+t) = \tau_2 \circ \varphi_{2s} \circ \eta_{2s}(s-t) = \tau_2 \circ \varphi_1 \circ \eta_1(t) = \sin(1/x_1(t)).
\]
In fact,
\[
\varphi_{2s+1}(\eta_{2s+1}(s+t)) = S_{2s+1}\left(\frac{x_{2s+1}(s+t)}{1 + 2s\pi x_{2s+1}(s+t)}\right) = \left(\frac{x_1(t)}{1 + 2s\pi x_1(t)} \cdot \sin\left(\frac{1}{x_1(t)}\right)\right).
\]
and

\[ \varphi_{2s}(\eta_{2s}(s-t)) = S_{2s}\left(\frac{x_{2s}(s-t)}{(2s+1)\pi x_{2s}(s-t)-1}, \sin\left(\frac{1}{x_{1}(t)}\right)\right). \]

Note that

\[ \lim_{s \to \infty} \varphi_{2s+1}(\eta_{2s+1}(s+t)) = \lim_{s \to \infty} \varphi_{2s}(\eta_{2s}(s-t)) = (0, \sin(1/x_1(t))). \]

Then, we define \( \varphi_{\infty} : \alpha \to T \) by \( \varphi_{\infty}(\alpha(t)) = (0, \sin(1/x_1(t))) \). Hence, according to Lemma 4.3 and Lemma 4.5, we can define the map \( \varphi : \eta \to \overline{S} \) by

\[ \varphi(z) = \begin{cases} 
\varphi_1(z), & z \in \eta_1; \\
\varphi_2(z), & z \in \eta_2; \\
\vdots & \vdots \\
\varphi_n(z), & z \in \eta_n; \\
\varphi_{\infty}(z), & z \in \alpha.
\end{cases} \]

Note that for any \( 0 \leq n \leq \infty, \varphi_n(z) \) is a homeomorphism.

Now we prove that \( \varphi : \eta \to \overline{S} \) is a homeomorphism. By Lemma 4.4 and Lemma 4.5, it is clear that \( \varphi \) is homeomorphic on \( \eta \). Since \( \varphi \) is a bijection and \( \eta \) is compact, it suffice to prove that \( \varphi \) is continuous on \( \alpha \). Suppose \( z_0 = \alpha(t_0) \) is a point of \( \alpha \). In the following, we prove that for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ \varphi(B_{\delta}(z_0) \cap \eta) \subseteq B_{\epsilon}(\varphi(z_0)) \cap \overline{S}, \]

where \( B_r(z) \) denotes the Euclidean disk with center \( z \in \mathbb{C} \) and radius \( r > 0 \).

By the definition of \( \varphi \), we get that there are \( s_0 > 0 \) and \( \delta_1 > 0 \) such that if \( s \geq s_0 \) and \( t_0 - \delta_1 \leq t \leq t_0 + \delta_1 \), then the Euclidean distance

\[ d(\varphi(z_0), \varphi_{2s+1}(\eta_{2s+1}(s+t))) = d((0, \sin(1/x_1(t_0))), (x_1(t)/\pi x_1(t), \sin(1/x_1(t)))) \leq \epsilon, \]

\[ d(\varphi(z_0), \varphi_{2s}(\eta_{2s}(s-t))) = d((0, \sin(1/x_1(t_0))), (x_1(t)/(2s+1)\pi x_1(t)-1, \sin(1/x_1(t)))) \leq \epsilon, \]

and

\[ d(\varphi(z_0), \varphi_{\infty}(\alpha(t))) = d((0, \sin(1/x_1(t_0))), (0, \sin(1/x_1(t)))) < \epsilon. \]

Let

\[ A_{2s+1}(\delta_1) = \{ \eta_{2s+1}(s+t) | s > s_0, t_0 - \delta_1 < t < t_0 + \delta_1 \}, \]

\[ A_{2s}(\delta_1) = \{ \eta_{2s}(s-t) | s > s_0, t_0 - \delta_1 < t < t_0 + \delta_1 \}, \]

\[ A_{\infty}(\delta_1) = \{ \alpha(t) | t_0 - \delta_1 < t < t_0 + \delta_1 \} \]

and

\[ A(\delta_1) = A_{2s+1}(\delta_1) \cup A_{2s}(\delta_1) \cup A_{\infty}(\delta_1). \]
Then by Lemma 4.4 and Lemma 4.5, we get that
\[ A_{2s+1}(\delta_1) \cup A_{2s}(\delta_1) \cup A_\infty(\delta_1) = A(\delta_1) \]
are both compact. Thus, there is a constant \( D_8 \) such that
\[ d(A(\delta_1/2), \eta \setminus A(\delta_1)) > D_8. \]
Since \( z_0 = \alpha(t_0) \in A(\delta_1/2) \), then (14)-(16) imply that \( \varphi(B_\delta(z_0) \cap \eta) \subseteq B_\epsilon(\varphi(z_0)) \cap \mathcal{S} \) by taking \( \delta = D_8 \). Therefore, \( \varphi \) is continuous on \( \alpha \).

This completes the proof of Theorem 1.1. \( \square \)

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E-mail address: jianxunf@hust.edu.cn
E-mail address: zhangsong1989724@163.com