Orbifold Physics and de Sitter Spacetime

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ABSTRACT

It is generally believed the way to resolve the black hole information paradox in string theory is to embed the black hole in anti-deSitter spacetime — without of course claiming that Schwarzschild-AdS is a realistic spacetime. Here we propose that, similarly, the best way to study topologically non-trivial versions of de Sitter spacetime from a stringy point of view is to embed them in an anti-de Sitter orbifold bulk, again without claiming that this is literally how de Sitter arises in string theory. Our results indicate that string theory may rule out the more complex spacetime topologies which are compatible with local de Sitter geometry, while still allowing the simplest versions.
1. AdS Embedding as a Toolbox

It is well known that it is difficult to obtain realistic curved spacetimes from string theory. This is not necessarily a drawback in itself, but it does impede the analysis of important questions which string theory might be expected to answer. Ingenious ways of answering these questions have been found, without actually exhibiting the desired spacetimes explicitly.

An outstanding example of this is the black hole information paradox. We do not yet know how to follow the detailed evolution of a realistic evaporating black hole in string theory (but see [1] and its references for some ideas), but it is generally believed that string theory predicts that there is no information loss. The strongest argument in this direction is due to Maldacena [2]: one simply embeds the black hole in AdS$_4$, using the Schwarzschild-AdS$_4$ metric, and shows that a dual field theory can be constructed in the familiar AdS/CFT style. Since the evolution of the field theory is unitary, it is clear that the same should be true of the black hole evaporation process. Notice that no claim is made that Schwarzschild-AdS$_4$ is a realistic spacetime; in fact of course a real black hole spacetime is utterly different. Nevertheless one believes that AdS is so well-understood in string theory that one can reasonably expect to use it to learn something about generic string black hole evaporation in this manner.

In a similar way, we do not yet know exactly how accelerating spacetimes can arise within string theory (see [3] for a review of the current situation). But it is now clear that our Universe is accelerating [4], and so string theory should help us to answer some of the many questions that arise regarding the quantum nature of an accelerating cosmos. By analogy with the black hole information paradox, we might hope that some clues — if not complete answers — can be found by formally embedding a candidate accelerating cosmology in a suitable version of AdS$_5$. As in the black hole case, we shall not claim that accelerating spacetimes can be literally obtained as a brane-world in string theory; as in the black hole case, we accept that a real derivation of de Sitter spacetime from string theory will certainly be far more complex than that. We regard the AdS embedding as a tool which may allow us to make progress pending a truly “stringy” derivation of cosmic acceleration. (For the more literal interpretation, not necessarily in the string context, see [5][6][7].)

In fact, dS$_4$ embeds in AdS$_5$ in an extremely natural way: we do not even have to modify the local geometry of AdS$_5$, as we do in the black hole case. This is particularly clear in the Euclidean formulation, since the five-dimensional hyperbolic space $H^5$ (Euclidean AdS$_5$) metric with curvature $-1/L^2$ is

$$g(H^5) = dr \otimes dr + sinh^2(r/L)g(S^4),$$

where $g(S^4)$ is the metric of the 4-sphere (“Euclidean deSitter”) of curvature $+1/L^2$. (In an attempt to avoid confusion, we shall throughout this work indicate the dimensions of Euclidean spaces by superscripts, and those of Lorentzian spaces by subscripts.) When we embed dS$_4$ or related spacetimes in versions of AdS$_5$ in this way, we shall refer to a de Sitter slice of anti-de Sitter, avoiding the term brane to emphasise that we are not committed to a literal brane-world scenario.

What kinds of questions about accelerating universes might we hope to answer by means of a formal AdS$_5$ embedding? One example, which has in fact already been analysed
in exactly this way, is the question of the entropy of de Sitter spacetime. Related examples might be the class of problems (see for example [9]) which arise when one tries to extend the holographic principle to de Sitter spacetime [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21]. Since holography works so well [22] for AdS$_5$, it is natural to try to investigate de Sitter holography by embedding dS$_4$ in AdS$_5$, and one of our objectives here is to set the stage for this. In the Lorentzian case, the conformal boundary of the embedded dS$_4$ actually lies on the conformal boundary of the ambient AdS$_5$. In view of the AdS/CFT correspondence, this is clearly an important point. (It appears in [6] — see figure 2 of that reference — but is not further analysed.)

Another question which can be addressed by means of a formal AdS$_5$ embedding is that of the topology of de Sitter spacetime. It is not generally appreciated that there are in fact many different topological spaces which can accept the de Sitter metric locally. We have no reason to believe that the version of de Sitter spacetime which may emerge, possibly in truncated or metastable form (see [23]) from string theory, will necessarily be the most familiar version with symmetry group O(1,4). In fact, just the opposite is true, since a truncated or otherwise mutilated spacetime will not be maximally symmetric. Furthermore, Witten [10] has stressed that there is every reason to doubt that quantum gravity effects related to the acceleration of the universe are symmetrical under the full de Sitter group O(1,4). This doubt will be confirmed if, as has been suggested [24], the entropy of de Sitter spacetime arises ultimately from a finite-dimensional Hilbert space of states, since the full de Sitter group has no finite-dimensional unitary representations. This suggests that we should study the less symmetrical versions of de Sitter spacetime, many of which have compact symmetry groups. There are various possible kinds of “less-symmetrical” de Sitter spacetime; the version [25] [26] which differs least from “ordinary” de Sitter spacetime is obtained by taking the spatial sections to be copies of the real projective space $\mathbb{R}P^3$ instead of the three-sphere S$^3$, while the time axis is left untouched. We call this dS($\mathbb{R}P^3$); it is a four-dimensional spacetime with only six Killing vectors. (We call the spherical version dS(S$^3$), and use dS$_4$ when we do not wish to be specific.) A much more radical departure from conventional cosmology is to pass to “elliptic” de Sitter (see [27] [28]); in this case, the identification involves both space and time, which entails a loss of time-orientability. There are still other possibilities, which we shall introduce in this work. The question then is: what are the physical consequences of modifying the topology of de Sitter spacetime? As we shall see, this is a question on which a formal AdS$_5$ embedding can shed much light.

In accordance with our general “AdS toolbox” philosophy, we advocate that these variants of dS$_4$ should be studied by embedding them in a suitable version of AdS$_5$. We shall argue that the modifications of AdS$_5$ which are necessary to achieve this often (though not always) lead to serious instabilities. In this way, the AdS$_5$ embedding strongly suggests that string theory severely constrains the topology of an accelerating universe.

We start with a discussion of the ambiguities in the global structures of de Sitter and anti-de Sitter spacetimes, beginning with the maximally symmetric versions. This is followed by an explanation of how it is possible for the boundary of dS$_4$ to “touch” that of AdS$_5$. Then we shall explain the consequences of taking the quotients of both AdS$_5$ and dS$_4$ by various small, finite groups of isometries, so that the symmetry groups are reduced. Next we discuss the embedding of various versions of “de Sitter spacetime”
in suitable versions of “anti-de Sitter spacetime”. Finally, recent results on tachyonic instabilities in AdS orbifolds, combined with a study of the breaking of supersymmetry, allow us to constrain the global geometries of both. Inter alia we gain some insights into de Sitter holography.

2. Maximally Symmetric Versions of [Anti]de Sitter

As is well known, there are many spaces with the local geometry of anti-de Sitter spacetime. These are obtained by factoring some maximally symmetric version by a discrete group of symmetries. The reference for this material is [29]; see also [30][31][32][33][34] for early work on AdS quotients, and [35][36][37] for more recent applications of related ideas.

The construction of five-dimensional anti-de Sitter space begins with the locus

\[ -A^2 - B^2 + w^2 + x^2 + y^2 + z^2 = -L^2, \]  

defined in a flat six-dimensional space of signature (2,4). It is clear that \( A^2 + B^2 \geq L^2 \) always, and so circles in this direction cannot be contracted to a point; on the other hand, there is no such restriction on the other directions, and we conclude that the topology of this manifold is \( S^1 \times \mathbb{R}^4 \). To see what this implies, we choose coordinates defined such that the time direction is perpendicular to spatial slices. Such coordinates are given by

\[
\begin{align*}
A &= L \sin(T/L) \\
B &= L \cos(T/L) \cosh(R/L) \\
w &= L \cos(T/L) \sinh(R/L) \cos(\chi) \\
z &= L \cos(T/L) \sinh(R/L) \sin(\chi) \cos(\theta) \\
y &= L \cos(T/L) \sinh(R/L) \sin(\chi) \sin(\theta) \cos(\phi) \\
x &= L \cos(T/L) \sinh(R/L) \sin(\chi) \sin(\theta) \sin(\phi).
\end{align*}
\]  

The induced metric is then

\[
g(\text{AdS}_5) = -dT \otimes dT + \cos^2(T/L)[dR \otimes dR + L^2 \sinh^2(R/L)[d\chi \otimes d\chi \\
+ \sin^2(\chi)[d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi]].
\]  

With this induced metric, this is a Lorentzian space of negative curvature \(-1/L^2\). We see that the spatial sections are copies of the four-dimensional hyperbolic space \( \mathbb{H}^4 \), with topology \( \mathbb{R}^4 \). Thus the circle in \( S^1 \times \mathbb{R}^4 \) is timelike, parametrised by T. The metric resembles a FRW metric with spatial sections of negative curvature. The apparent singularities (at intervals of \( \pi L \)) are coordinate singularities: they occur because all of the timelike geodesics perpendicular to the spatial surface at \( T = \) constant intersect periodically. Beginning at \( T = -\pi L \), a given collection of timelike geodesics contracts towards each other, intersecting at \( T = -\pi L/2 \) and \( T = +\pi L/2 \), and the whole cycle repeats after \( T = \pi L \) is reached: the period is \( 2\pi L \).

It is evident from the formula for the coordinate A that these coordinates do not cover the entire manifold. Nevertheless, they do faithfully represent the behaviour of timelike
geodesics in AdS$_5$, and, as Gibbons [29] emphasises, the periodic intersections of those geodesics is a geometric fact which cannot be abolished merely by claiming to pass to the universal cover — a step which is often said to be necessary to rid the spacetime of its closed timelike worldlines. In fact, it is difficult to see how an inertial observer in pure anti-de Sitter spacetime could determine whether or not the temporal circles had been “unwrapped”. Time, for him, is periodic as measured by the structures available for his inspection [29].

Gibbons advocates a pragmatic attitude: if we are considering some physical system in anti-de Sitter spacetime which is such that the natural periodicity of this spacetime is not observable — for example, if the (locally measured) period is vast even by cosmological standards — then we need not pass to the universal cover. The choice should be determined by the physical circumstances. In the context of cosmology, spacetimes with negative cosmological constants typically display spacelike singularities — see [38] for a discussion of this. In cosmology, therefore, one can question whether “unwrapping AdS” (to its universal cover) is really appropriate. We can think of the periodicity as a device to avoid pathologies which may arise in the long run if we introduce matter into anti-de Sitter spacetime and break time translation invariance.

In our applications, where we regard AdS$_5$ as a tool to study the physics of an embedded dS$_4$ slice, it is possible to argue that the existence of a cyclic time coordinate on AdS$_5$ is of no physical concern provided that it can be reconciled with a non-cyclic time coordinate on the de Sitter slice. In fact, as we shall see, the de Sitter slice plays out its infinite history within much less than one “Great Year” of the ambient anti-de Sitter spacetime, so here the cyclic version of AdS$_5$ is physically appropriate in Gibbons’ sense. For if we “unwrap” the ambient space, most of it will lie “before or after the infinite past or future” for de Sitter — and this seems physically irrelevant. Furthermore, the proper time of the static observers in anti-de Sitter spacetime is periodic, but not with period $2\pi L$ in general. In global coordinates (see equation 13 below) the proper time for such an observer (at a constant value of $r$) is periodic with period $2\pi L \cosh(r/L)$. This implies that, for static observers in the region where the de Sitter slice resides, the period of each cycle is enormous. Thus neither de Sitter observers, nor static bulk observers nearby, would necessarily be aware of the supposed periodicity of anti-de Sitter spacetime.

Our attitude will be a conservative one: we take it that the periodic identification in the timelike direction of the locus given by equation 2 is just a mathematical device which helps us to focus on a finite interval of anti-de Sitter (global) time — which is all that we shall need. However, it should be mentioned that some authors are willing to interpret the periodicity literally, for physical reasons. For example, this has been discussed by Allen and Jacobson [39] and recently by Li [40]. The latter is concerned with a simple de Sitter brane model devised to explain the smallness of the observed cosmological constant, and the scheme works best precisely when the anti-de Sitter bulk is not unwrapped to its universal cover. (The fact that quantum field theory makes sense on anti-de Sitter spacetime, even when time is cyclic, was established in [11].) Furthermore, the existence of closed timelike worldlines in AdS-like “Gödel” spacetimes has attracted considerable attention recently in connection with string dualities [42].

For the sake of definiteness, we shall therefore continue to define “AdS$_5$” to be precisely the locus given above, with topology $S^1 \times \mathbb{R}^4$. This “wrapped” version of anti-de Sitter
is in fact the one being referred to when it is claimed — as it usually is — that the symmetry group is the orthogonal group \(O(2,4)\). (See equation 2.) For if we unwrapped this spacetime, then the symmetry group would have to include a group of time translations with structure \(\mathbb{R}\), not the \(O(2)\) contained in \(O(2,4)\). We stress, however, that even the “wrapped” version being considered here is maximally symmetric: one sees that \(O(2,4)\) has the maximal possible dimension (fifteen) for a five-dimensional semi-Riemannian manifold. (In fact, it is well known that, in string theory, orientation-reversing isometries of \(AdS_5\) are not symmetries of the theory, so the precise symmetry is \(SO(2,4)\) rather than \(O(2,4)\). In order to avoid confusion, we shall always take “symmetry group” to mean the full geometric symmetry group. It is easy to make the necessary adjustments to preserve the relevant volume form in each case.)

There is still another version of anti-de Sitter spacetime which is maximally symmetric in this sense, namely the elliptic anti-de Sitter spacetime. The significance of this version can be understood by means of the AdS/CFT correspondence, as follows.

In the AdS_5/CFT_4 correspondence, the CFT does not inhabit Minkowski space, for the conformal group has no natural action there — a fact emphasised in [43]. Instead it inhabits conformally compactified Minkowski space \(CCM_4\), defined as the locus (in a flat space of the appropriate signature) given by coordinates \((A,B,w,x,y,z)\), not all zero, satisfying

\[
-A^2 - B^2 + w^2 + x^2 + y^2 + z^2 = 0,
\]

subject to the identification \((A,B,w,x,y,z) = (sA,sB,sw,sx,sy,sz)\), where \(s\) is any real non-zero number. Following [13], we can think of this space as being obtained from \(AdS_5\) by taking the locus in equation 2 and imposing scaling invariance. By using a positive scaling factor, we can send all of \(A,B,w,x,y,z\) to infinity, which is tantamount to letting \(L^2\) tend to zero. Notice however that, having done this, we can still scale \((A,B,w,x,y,z)\) to \((-A,-B,-w,-x,-y,-z)\), and so these are identified on conformally compactified Minkowski space, though not on \(AdS_5\) itself.

The symmetry group of this space is the conformal group \(PO(2,4)\). Here the \(P\) refers to the fact that there are two elements of the orthogonal group \(O(2,4)\) which leave unmoved every point of \(CCM_4\), namely the identity \(I_6\) and \(-I_6\) (since \((A,B,w,x,y,z) = (-A,-B,-w,-x,-y,-z)\)). Thus the symmetry group is \(O(2,4)/\{I_6, -I_6\}\), and this by definition is the projective orthogonal group \(PO(2,4)\). Similar considerations show that the topology of \(CCM_4\) is that of \([S^1 \times S^3]/\mathbb{Z}_2\), which is very different indeed to the \(\mathbb{R}^4\) topology of Minkowski space. A beautiful interpretation [29] of the difference is given by regarding Minkowski space as the vector space of \(2 \times 2\) hermitian matrices — that is, as the (non-compact) Lie algebra of the “electroweak” group \(U(2)\) — while \(CCM_4\) is the compact Lie group \(U(2)\) itself. Note that \(U(2)\) is isomorphic to \([U(1) \times SU(2)]/\mathbb{Z}_2\), not to \(U(1) \times SU(2)\). Thus, \(CCM_4\) is the “matrix exponential of Minkowski space”, and the CFT lives on this matrix exponential. Note that while it is often claimed that the conformal field theory of the AdS/CFT correspondence actually inhabits the universal cover of \(CCM_4\), that universal cover does not have \(PO(2,4)\) as its symmetry group. If we are really considering \(PO(2,4)\) (which is usually called “\(SO(2,4)\)” in the literature) to be “the conformal group” in the AdS/CFT correspondence, then we are committing ourselves to a conformal field theory defined on \([S^1 \times S^3]/\mathbb{Z}_2\).

The similarity of equation 5 to equation 2 is apparent, and it is often claimed on this
basis that AdS$_5$ and CCM$_4$ have the same symmetry group. That is not the case, however: the isometry group of AdS$_5$ is, as mentioned earlier, O(2,4), which is not at all the same as the PO(2,4) symmetry group of CCM$_4$. In fact, O(2,4) is the double cover of PO(2,4), in just the same way that SU(2) is the double cover of SO(3). It therefore follows that, if indeed AdS$_5$ is the correct version of anti-de Sitter spacetime, then some states in AdS$_5$ will be dual to states on CCM$_4$ which are “spinorial”, in the sense that they will behave non-trivially under apparently trivial PO(2,4) transformations. (Of course, O(2,4) and PO(2,4) are identical locally, just as are SU(2) and SO(3). See [44] for a strictly local but explicit derivation of this local isomorphism between the bulk isometry group and the boundary conformal group.)

We can be more precise about this if we factor AdS$_5$ by the cyclic group of order two defined by the antipodal map,

$$\Omega : (A, B, w, x, y, z) \rightarrow (-A, -B, -w, -x, -y, -z).$$

Call this group $\mathbb{Z}_2^3$. The fixed point of $\Omega$ does not lie on AdS$_5$, so the quotient AdS$_5/\mathbb{Z}_2^3$, which is called EllAdS$_5$, the elliptic anti-de Sitter space, is then non-singular and has exactly the same symmetry group as CCM$_4$, namely O(2,4)/{I$_6$, -I$_6$} = PO(2,4). Thus we see that the CFT in the AdS/CFT correspondence lives not on the boundary of AdS$_5$ but rather on the boundary of EllAdS$_5$. Indeed, it is EllAdS$_5$ that has CCM$_4$ as its conformal boundary, as can be seen by observing that the topology of EllAdS$_5$ is $[S^1 \times \mathbb{R}^4]/\mathbb{Z}_2$, while as we saw the topology of CCM$_4$ is $[S^1 \times S^3]/\mathbb{Z}_2$. It does not, however, follow from this that EllAdS$_5$ is the “correct” version of anti-de Sitter spacetime. It simply means that, to reach the CFT from AdS$_5$, we have to proceed to the conformal boundary and then project down to CCM$_4$. Since the antipodal map is trivial as far as the “vector” states on CCM$_4$ are concerned, the CFT can only be aware of it through “spinorial” states which transform according to O(2,4), not PO(2,4). On the other hand, if in fact EllAdS$_5$ is the “correct” version of anti-de Sitter spacetime, then such “spinorial” states will not exist.

The physical meaning of the antipodal map will be discussed further below. For the present let us note an agreeable property of EllAdS$_5$: it is both time and space orientable. In fact, EllAdS$_n$ is always time-orientable [29], but it is space-orientable only when $n$ is odd, as is the case here. This is in contrast to the more familiar [27] [28] elliptic de Sitter space, which is never time-orientable. Notice that PO(2,4) has the maximum possible dimension for an isometry group of a five-dimensional manifold, namely 15. Hence EllAdS$_5$ has as much right to be considered “the maximally symmetric five-dimensional spacetime of constant negative curvature” as AdS$_5$.

Thus we see that there are two maximally symmetric versions of “wrapped” anti-de Sitter spacetime: AdS$_5$ and EllAdS$_5$. Conformally compactified Minkowski space, CCM$_4$, is the conformal boundary of EllAdS$_5$, and it is reached from AdS$_5$ simply by proceeding to the boundary and then projecting down to CCM$_4$. Both arrangements are of course compatible with the AdS/CFT philosophy. If we are willing to break the maximal symmetry, then there are many other versions as well; but these all descend from AdS$_5$ and EllAdS$_5$, and these descendants will be considered in a later section. First however let us consider some of the possible ambiguities of de Sitter spacetime.

The simply connected version of four-dimensional de Sitter spacetime, which we call
dS(S^3), is given by the locus, in a space of signature (1,4), defined by

\[-A^2 + w^2 + x^2 + y^2 + z^2 = +L^2.\]  

(7)

It is easy to see that the topology is \(\mathbb{R} \times S^3\). The induced metric is maximally symmetric, with ten-dimensional isometry group \(O(1,4)\), and has constant positive curvature \(1/L^2\).

The elliptic de Sitter spacetime \(\text{Ell}dS_4\), defined in the obvious way by antipodal identification, has an isometry group \(\text{PO}(1,4) = O(1,4)/\mathbb{Z}_2\). (This happens to be isomorphic to \(\text{SO}(1,4)\), which of course is a subgroup of \(O(1,4)\), but the reader should not be misled by this: the fact that the quotient is isomorphic to a subgroup is a peculiarity of this case. More typically, \(\text{PO}(2,4)\) is certainly not isomorphic to \(\text{SO}(2,4)\).) Of course, \(\text{Ell}dS_4\) is maximally symmetric, so we have the same kind of ambiguity here as in the anti-de Sitter case. However, when we require a de Sitter spacetime to fit inside anti-de Sitter spacetime, the two ambiguities may be said to clash, in the following sense.

The point is that, contrary to what one might expect, \(\text{Ell}AdS_5\) does not contain elliptic de Sitter space, but rather the usual \(dS(S^3)\). To see this, take the above equation defining \(AdS_5\) and write it as follows:

\[-A^2 + w^2 + x^2 + y^2 + z^2 = B^2 - L^2.\]  

(8)

Comparing this with the equation defining de Sitter spacetime, we see at once that for each positive constant \(B_0 > L\) there is a pair of copies of \(dS(S^3)\) (with the same cosmological constant) embedded in \(AdS_5\), one at \(B = B_0\), the other at \(B = -B_0\). The effect of factoring out \(\mathbb{Z}_2^3\) to obtain \(\text{Ell}AdS_5\) is therefore to identify these two copies of \(dS(S^3)\) with each other: \(\mathbb{Z}_2^3\) does not map either copy of \(dS(S^3)\) into itself. Hence the pair of copies of \(dS(S^3)\) in \(AdS_5\) becomes one copy of \(dS(S^3)\) in \(\text{Ell}AdS_5\), which we take to be the one at \(B = B_0\).

Thus we see that the version of a de Sitter spacetime obtained by embedding it in \(\text{either} AdS_5\) or \(\text{Ell}AdS_5\) is the simply connected de Sitter spacetime \(dS(S^3)\), \textit{not} elliptic de Sitter. However, this is not to say that the embedding picture rules out elliptic de Sitter: for we have yet to consider the consequences of symmetry breaking. We shall return to this below.

Before leaving this discussion, we stress the following point. In contrast to the Euclidean case, in which there is a copy of \(S^4\) passing through every point of \(H^5\) except the origin, the Lorentzian manifolds \(AdS_5\) and \(\text{Ell}AdS_5\) are not completely foliated by copies of \(dS(S^3)\); this only works in the region \(|B| > L\). For simplicity, we shall henceforth mainly focus on this region of \(AdS_5\) and \(\text{Ell}AdS_5\).

There is still another ambiguity associated with de Sitter spacetime; in this case it is related to the conformal compactification.

It is well known that the de Sitter metric may be written using conformal time as

\[
g(dS_4) = \frac{L^2}{\sin^2(\eta)}[-d\eta \otimes d\eta + d\chi \otimes d\chi + \sin^2(\chi)\{d\theta \otimes d\theta + \sin^2(\theta)d\phi \otimes d\phi\}],\]  

\[\text{(9)}\]

where \(L^2\) is as in equation 7 and where \(\eta\) takes its values in the open interval \((0, \pi)\). We can therefore compactify by extending conformal de Sitter time to the compact \textit{closed} interval \([0, \pi]\). This is the usual picture. What is not generally appreciated, however, is that \textit{this is a choice}: there is another interpretation, which in some ways is more natural.
To see this, let us replicate the procedure we used to construct \( \text{CCM}_4 \). We use a positive scaling factor to send all of the coordinates in equation 7 to infinity, which effectively sends \( L^2 \) to zero. The resulting locus

\[- A^2 + w^2 + x^2 + y^2 + z^2 = 0, \tag{10}\]

with an overall scaling of the coordinates, represents one copy of \( S^3 \). To see why this is so, notice that we can use an overall scaling by a positive factor to obtain from \( 10 \)

\[A^2 = 1 = w^2 + x^2 + y^2 + z^2; \tag{11}\]

which represents two unit three-spheres, one at \( A = 1 \), the other at \( A = -1 \). But the remaining freedom of scaling by \(-1\) identifies these two. Thus there is a natural sense in which the conformal manifold associated with de Sitter spacetime is one copy of \( S^3 \). Actually, the conformal representation of \( S^3 \) given by equation \( 10 \) is the conformal boundary of elliptic de Sitter spacetime \( \text{Ell}_{dS}^4 \). Exactly as in the anti-de Sitter case, to reach the conformal space which a CFT might be expected to inhabit, one proceeds to the boundary and then projects. The great difference, of course, is that in the de Sitter case the boundary is disconnected, and the projection converts two distinct copies of \( S^3 \) into one.

We can actually implement this identification without losing time-orientability — that is, without accepting elliptic de Sitter as the “correct” version of de Sitter spacetime — in the following novel way. Notice that dropping the conformal factor from \( g(\text{dS}_4) \) in equation \( 9 \) has two effects: firstly of course it makes the extension to \([0, \pi]\) possible, but secondly it removes all dependence on \( \eta \). Thus translations along the conformal time axis are symmetries as far as the conformally deformed metric is concerned, and, in particular, the map \( \eta \to \eta + \pi \) is now a symmetry. (Nothing of this sort happens in the anti-de Sitter case: see equation \( 14 \) below.) We may therefore assume that the conformal time coordinate \( \eta \) values 0 and \( \pi \) are actually identified with each other, so that de Sitter space is regarded as an open submanifold of a compact space with topology \( S^1 \times S^3 \). We immediately stress that this identification only affects the unphysical, compactified spacetime, of which de Sitter spacetime is a proper submanifold; there are now closed timelike worldlines in the unphysical space, but not in de Sitter spacetime itself. The situation is clarified by a glance at the Penrose diagram, given in Figure 1.

Here we have assumed the conventional topology, \( S^3 \), for the spatial sections of \( \text{dS}_4 \). The letters at top and bottom indicate the identification. It is clear that the identification cannot be detected in a finite amount of proper time by any observer. In view of this, one might well ask whether there is anything to be gained from this interpretation of the conformal compactification of de Sitter spacetime.

The answer is that, classically, there is indeed nothing to be gained. But it is otherwise when we try, following Witten and Strominger \( \text{[10][11]} \), to take a holographic view of de Sitter spacetime. The matching of the bulk and boundary symmetry groups is an important aspect of the AdS/CFT correspondence, one which we in fact used above. Contrary to what is sometimes asserted, however, the isometry group of de Sitter spacetime does not at all match the conformal group of the boundary, in the usual interpretation of the Penrose diagram. For it is clear that the boundary in that case is disconnected, consisting of two copies of the three-sphere \( S^3 \). The conformal group of \( S^3 \) is \( \text{SO}(1,4) \) (see equation \( 9 \).
Figure 1: Cyclic Compactification of dS(S$^3$).

Recall that $O(1,4)/\mathbb{Z}_2 = SO(1,4)$; thus the full conformal group of the $S^3 \cup S^3$ boundary is the semi-direct product $[SO(1,4) \times SO(1,4)] \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ corresponds to the map which exchanges the two copies of $S^3$. This is of course far larger than $O(1,4)$, the isometry group of dS(S$^3$).

Now Witten and Strominger observe, in discussing the possible existence of a “dS/CFT duality”, that a scalar field correlator between a point on the sphere in the infinite past and the antipodal point (in space and time) on the sphere in the infinite future is singular. This is to be expected, because the two points are causally connected by a null geodesic. Strominger argues that, since the Green functions “know” about this causal connection, they only transform simply under one copy of the conformal group $SO(1,4)$; which suggests that the dS/CFT duality should involve a conformal field theory defined on one copy of $S^3$. An interesting but somewhat drastic way to implement this suggestion is to modify the geometry of de Sitter space itself so that there is only one boundary component. This leads to elliptic de Sitter spacetime [27][28], and consequently to the loss of time orientability; this is a very serious drawback, for it entails all manner of interpretational questions which have not yet been fully resolved, and it is hard to see how such a spacetime can be related to conventional FRW models.

A simpler and less drastic alternative, however, is to modify the compactification instead of the spacetime, and this is what we did above (Figure 1). The interior of the diagram is entirely unaffected, so time orientability is not lost; nor is causality affected; nor do we lose the maximal symmetry. (The Witten-Strominger past-future correlator singularity now becomes a singularity for an antipodal correlator on $S^3$, necessitating a further topological identification which will be discussed below.)

The conformal group of the “boundary” — it is no longer a boundary in the strict topological sense — is now $SO(1,4)$ instead of $[SO(1,4) \times SO(1,4)] \rtimes \mathbb{Z}_2$, while the isometry group of the bulk remains as $O(1,4)$. These are not exactly the same, just as the
symmetry groups of AdS$_5$ and CCM$_4$ are not exactly the same. As in the previous case, this means that, if there is some kind of dS/CFT “correspondence” or “relationship”, then there may be spinorial states on S$^3$ to account for states on de Sitter spacetime which are not trivial under the antipodal map — recall that O(1,4)/$\mathbb{Z}_2^2$ = SO(1,4). Alternatively, if elliptic de Sitter is the correct version, then these spinorial states will not exist. (The “missing” $\mathbb{Z}_2^2$ is the one which exchanges future with past. This is an isometry of all these versions of de Sitter space, but of course it is not a symmetry which the “boundary” can be expected to detect after we have identified future infinity with past infinity.)

Thus, in the de Sitter case, we encounter yet another ambiguity in the definition of the spacetime, or, rather, of its conformal compactification. The idea that the de Sitter boundary should “really” or “holographically” have one connected component has been disputed [15], and continues to be debated [16,17], but we shall not enter into this question here; we shall return to it below. The point is that there is indeed a question, one which we hope to settle by embedding de Sitter in anti-de Sitter.

Let us summarize the results of this section. Both anti-de Sitter and de Sitter spacetimes can be interpreted in many ways. In this section, we have confined ourselves to versions which are maximally symmetric; even so, there are still several possibilities. For simplicity, we follow Gibbons [29] and restrict attention to versions of anti-de Sitter spacetime with cyclic time, since these seem to be the most relevant versions for dealing with embedded de Sitter spacetimes. Even then, there are two maximally symmetric versions: AdS$_5$, given by equation 2, and its elliptic form, AdS$_5$/Z$_2^Ω$. On the other hand, we found three maximally symmetric versions of de Sitter spacetime, namely dS(S$^3$), its elliptic version, and its cyclic version (which is really a new interpretation of the conformal compactification rather than of the spacetime itself). In the maximally symmetric cases we have been considering here, it is relatively straightforward to determine the relationships between all of these spacetimes and to fix their symmetry groups. When we wish to break these symmetries, however, the situation becomes sufficiently complex that we need a more detailed understanding of the ways in which these spacetimes and their symmetries fit together. The next two sections are devoted to the relevant techniques.

3. Touching Infinities

In this section we shall explore the relationship between an anti-de Sitter spacetime and an embedded de Sitter spacetime in more detail.

We can introduce a useful global coordinate system for AdS$_5$ and EllAdS$_5$ as follows. Referring to equation 2, define coordinates (t,r,χ,θ,φ) by

\[
\begin{align*}
A &= L \sin(t/L) \cosh(r/L) \\
B &= L \cos(t/L) \cosh(r/L) \\
w &= L \sinh(r/L) \cos(\chi) \\
z &= L \sinh(r/L) \sin(\chi) \cos(\theta) \\
y &= L \sinh(r/L) \sin(\chi) \sin(\theta) \cos(\phi) \\
x &= L \sinh(r/L) \sin(\chi) \sin(\theta) \sin(\phi). \\
\end{align*}
\]  (12)
The metric on either $\text{AdS}_5$ or $\text{EllAdS}_5$ in these coordinates is

$$
\begin{align*}
g(\text{AdS}_5) &= -\cosh^2(r/L) \, dt \otimes dt + dr \otimes dr + L^2 \sinh^2(r/L) \left[ d\chi \otimes d\chi \\
&\quad + \sin^2(\chi) \{ d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi \} \right],
\end{align*}
$$

(13)

and this form of the metric is globally valid. Therefore it is this form of the metric which must be used to determine the nature of conformal infinity: evidently the latter lies at “$r = \infty$”. Notice that the spacelike part of the metric is just the metric on hyperbolic space, $H^4$ (see equation 1). As usual we can therefore assume that $r \geq 0$. On $\text{AdS}_5$, the time coordinate $t$ runs from $-\pi L$ to $+\pi L$, after which it repeats itself. (Compare the formula for $A$ in equations 12 with the formula for $A$ in equations 3.) No timelike curve can be closed unless $t$ increases by at least $2\pi$ along it (though of course the proper time elapsed will not in general be equal to $2\pi L$). On $\text{EllAdS}_5$, where the points with coordinates $(t,r,\chi,\theta,\phi)$ and $(\pi L - t,r,\pi - \chi,\pi - \theta,\pi + \phi)$ are identified, it is possible for $t$ to increase by less than $2\pi L$ along a closed timelike curve — the curve $A = L \sin(t/L)$, $B = L \cos(t/L)$, $w = z = y = x = 0$, is a closed timelike curve along which $t$ (which is proper time in this case) only increases by $\pi L$ for each circuit. Notice however that this special curve does not lie in the region in which we are interested, $B > L$, which is foliated by copies of de Sitter. Thus we arrive at the useful conclusion that in both $\text{AdS}_5$ and $\text{EllAdS}_5$, $t$ must increase by at least $\pi L$ along a closed timelike curve; in fact, it must increase by at least $2\pi L$ in the region $|B| > L$.

The form of the metric given in equation 13 is useful because it allows us to correlate the angular coordinates in the anti-de Sitter bulk with those on the conformal boundary. For if we write

$$
\begin{align*}
g(\text{AdS}_5) &= -\cosh^2(r/L) \left[ dt \otimes dt + \text{sech}^2(r/L) \, dr \otimes dr + L^2 \tanh^2(r/L) \{ d\chi \otimes d\chi \\
&\quad + \sin^2(\chi) \{ d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi \} \right] \tag{14}
\end{align*}
$$

then we see that the metric at “$r = \infty$” is

$$
\begin{align*}
g(\text{AdS}_5, \infty) &= -dt \otimes dt + L^2 \{ d\chi \otimes d\chi + \sin^2(\chi) \{ d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi \} \},
\end{align*}
$$

(15)

which is just the standard representative of the conformal structure on $S^1 \times S^3$ (for $\text{AdS}_5$) or on $\text{CCM}_4 = [S^1 \times S^3]/\mathbb{Z}_2$ (for $\text{EllAdS}_5$). Clearly the angular coordinates $\chi, \theta, \phi$ are the same in the bulk and on the boundary.

We saw that the region $B > L$ of $\text{AdS}_5$ or $\text{EllAdS}_5$ can be foliated by copies of $dS(S^3)$, and we can make this explicit by choosing new local time and radial coordinates, $\tau$ and $\rho$, as follows. (We retain the angular coordinates $\chi, \theta, \phi$.)

$$
\begin{align*}
A &= L \sinh(\tau/L) \sinh(\rho/L) \\
B &= L \cosh(\rho/L) \\
w &= L \sinh(\rho/L) \cosh(\tau/L) \cos(\chi) \\
z &= L \sinh(\rho/L) \cosh(\tau/L) \sin(\chi) \cos(\theta) \\
y &= L \sinh(\rho/L) \cosh(\tau/L) \sin(\chi) \sin(\theta) \cos(\phi) \\
x &= L \sinh(\rho/L) \cosh(\tau/L) \sin(\chi) \sin(\theta) \sin(\phi). \tag{16}
\end{align*}
$$
Notice that the equation for $B$ enforces $B > L$ if $\rho > 0$. The anti-de Sitter metric now becomes
\begin{equation}
 g(AdS_5) = d\rho \otimes d\rho + \sinh^2(\rho/L) \left[-d\tau \otimes d\tau + L^2 \cosh^2(\tau/L) \{d\chi \otimes d\chi + \sin^2(\chi) \{d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi\}\}\right];
\end{equation}
(17)
or
\begin{equation}
 g(AdS_5) = d\rho \otimes d\rho + \sinh^2(\rho/L) g(dS_4),
\end{equation}
(18)
where of course
\begin{equation}
 g(dS_4) = -d\tau \otimes d\tau + L^2 \cosh^2(\tau/L) \{d\chi \otimes d\chi + \sin^2(\chi) \{d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi\}\}\right).
\end{equation}
(19)
is the global de Sitter metric. Clearly there is a copy of $dS(S^3)$ at any fixed value of $\rho > 0$ in $AdS_5$ or $EllAdS_5$. The cosmological constant of such a de Sitter slice at $\rho = c$, a constant, is $\Lambda_{dS} = +3/[L^2 \sinh^2(c/L)]$, which can be made very small for suitable choices of $c$ and $L$; notice that the embedded de Sitter spacetime need not have the same (magnitude) cosmological constant as the ambient anti-de Sitter spacetime.

The ratio $\Lambda/B$ can be computed in both coordinate systems introduced in this section, and so we obtain at once
\begin{equation}
 \tan(t/L) = \sinh(\tau/L) \tanh(c/L).
\end{equation}
(20)
From (18) and (19) we see that $\tau \sinh(c/L)$ is proper time for de Sitter; as always in de Sitter spacetime, it extends from $-\infty$ to $+\infty$, and hence so must $\tau$. But this corresponds to the interval $(-\pi L/2, +\pi L/2)$ for $t$, which means that de Sitter observers are entirely unaware of the cyclic nature of time in $AdS_5$ and $EllAdS_5$. (Such cycles require $t$ to increase by $2\pi L$ in this region.) Furthermore, comparing the expressions for $w$ in equations (12) and (16) we see that
\begin{equation}
 \sinh(r/L) = \sinh(c/L) \cosh(\tau/L),
\end{equation}
(21)
which implies that the minimum value of $r$ on the de Sitter slice is precisely $c$. An anti-de Sitter bulk observer who stays in the region $r > c$ will always experience a time dilation factor of at least $\cosh(c/L)$ (see equation (13)), and so the period of the proper time experienced by a bulk observer in the vicinity of the de Sitter slice is at least $2\pi \sqrt{L^2 + (3/\Lambda_{dS})}$, which is of course a huge number compared to $2\pi L$. That is, the possibly cyclic nature of AdS time is far from apparent even to bulk observers in the neighbourhood of the de Sitter slice.

Following the philosophy advocated by Gibbons [29], we conclude that the cyclic version of anti-de Sitter space is the appropriate one here. Indeed, if we were to unwind $AdS_5$ or $EllAdS_5$ to its universal cover, then the de Sitter slice would be repeated endlessly, and all of the other copies would lie either “after the infinite future” or “before the infinite past” of a given de Sitter slice, which seems physically meaningless. Actually we can embed two copies of de Sitter spacetime within the $2\pi L$ cycle of the global anti-de Sitter time coordinate $t$; one of these lies in the centre of the range of $t$, and the other is (apparently) split into two halves, one “above” the slice we are considering here, the other “below”. (In fact of course the periodicity of $t$ means that these two halves are...
joined.) If we follow our philosophy to its logical conclusion, we should try to eliminate this superfluous copy, since it too lies “beyond infinite time” for a given de Sitter observer. This will be important later when we discuss taking the quotient of anti-de Sitter spacetime by an isometry which shrinks \( t \) by a factor of two.

A still more important point now is this. In equation 1, the boundary of \( H^5 \) is at “\( r = \infty \)”, and fixing \( r \) therefore severs all contact between infinity and the copy of \( S^4 \) at \( r = c \), since \( S^4 \) is finite (compact). But in equation 18 it is not clear that fixing \( \rho \) completely severs contact between anti-de Sitter infinity and the Lorentzian de Sitter slice, since the latter is itself infinitely large in the time direction. From which it follows that for fixed \( \rho = c \), the consequence of letting \( \tau \) tend to infinity is that \( r \) must tend to infinity. Thus the infinite nature of de Sitter proper time means that the conformal infinity of the de Sitter slice (\( \tau \to \pm \infty \)) actually lies on the conformal infinity of AdS$_5$ or EllAdS$_5$ (\( r \to \infty \)). Thus, the Euclidean and Lorentzian pictures of this situation are fundamentally different.

Now of course the AdS/CFT philosophy is that CFT physics on the boundary gives a complete account of the interior. One might be tempted to claim, in view of the fact that the dS boundary lies on the AdS boundary while the dS bulk inhabits the AdS bulk, that AdS/CFT imposes a similar holographic equivalence between the bulk and the boundary of a de Sitter spacetime embedded in anti-de Sitter spacetime. However, it is not at all clear that we obtain an exact equivalence in this way; in fact we would not expect to do so, since the de Sitter slice is not the part of the anti-de Sitter bulk which is in direct causal contact with this part of the anti-de Sitter boundary. This may be related to the difficulties besetting attempts to establish a full “dS/CFT correspondence”. In fact, the de Sitter slice most closely resembles the relevant AdS$_5$ null cones when the constant \( c \) is very small compared to \( L \); but in view of the relation \( \Lambda_{dS} = +3/[L^2 \sinh^2(c/L)] \), this would mean that the de Sitter cosmological constant is very large, which of course is not the physical case. That is, we expect the dS/CFT relationship to hold exactly only in the limit of a large \( \Lambda_{dS} \). These remarks may well be related to the claim, in [19], that dS/CFT cannot probe the interior of any given static patch in de Sitter spacetime.

Thus we have our first lesson from embedding dS$_4$ as a slice in AdS$_5$: some kind of dS/CFT correspondence may well be valid, but unfortunately it is probably only precise in the unphysical limit of large cosmological constant.

We now turn to another application of de Sitter embedding.

### 4. Breaking Symmetries with Topology

AdS$_5$, EllAdS$_5$ and their conformal boundaries have very large (15-dimensional) groups of symmetries, and of course one is interested in breaking these symmetries in some cases. This can be done in the traditional way by means of vacuum expectation values of scalar fields [48]. The question of breaking the specifically conformal symmetries on the boundary is also of much interest [49]. But there is another approach to symmetry breaking.

String theory has taught us that one of the most interesting and subtle forms of symmetry breaking arises when one takes the quotient of a manifold by a discrete group [50]. The prototype here is “Wilson loop symmetry breaking”, which arises on Calabi-Yau compactification manifolds which are not simply connected — that is, they have been
obtained from simply connected Calabi-Yau manifolds by taking a quotient by a finite
group of holomorphic isometries. It turns out that the existence of non-contractible loops
on the quotient space allows one to break gauge symmetries. One might call this general
phenomenon “topological symmetry breaking”, since it only works on the non-simply-
connected version of the Calabi-Yau manifold. In a similar (but subtly different) way,
taking the quotient by a discrete group normally breaks some of the geometric symmetries
of a space. (This is not an issue for Calabi-Yau spaces because their symmetry groups
are always very small (finite) in any case.)

It is possible to take the quotient of anti-de Sitter spacetimes by discrete groups; there are several motivations for doing so [30][31][32][33][34][35][36]. The idea of taking quotients of de Sitter spacetime has attracted much less attention, but there are strong indications that this will be necessary, as we shall soon argue. Before doing so, however, let us clarify the precise way in which taking quotients by discrete groups breaks geometric symmetries. As we shall see, there is a subtle difference between this kind of symmetry breaking and the usual kind.

Suppose that one has a manifold M admitting a group G(M) of diffeomorphisms (such
as isometries, conformal symmetries, and so on). Let \( \Gamma \) be a subgroup of G(M) (which
need not act without fixed points on M) and let N(\( \Gamma \)) be the normalizer of \( \Gamma \) in G(M).
That is,

\[
N(\Gamma) = \{ g \in G(M) \mid g\gamma g^{-1} \in \Gamma \ \forall \ \gamma \in \Gamma \}.
\]  

(22)

(In this work, \( \Gamma \) will almost always be either \( \mathbb{Z}_2 \) or a product of copies of \( \mathbb{Z}_2 \). Clearly, the normalizer of \( \mathbb{Z}_2 \) in any larger group will consist of all those elements of the larger group which commute with the generator of \( \mathbb{Z}_2 \). It turns out, though the argument is less straightforward, that the same is true for a product of copies of \( \mathbb{Z}_2 \) (and also for \( \mathbb{Z}_4 \)) in the cases we shall consider. Thus, the reader can interpret “normalizer” as “centralizer” in this work.)

Now N(\( \Gamma \)) contains all those elements of G(M) which descend to well-defined diffeo-
morphisms of \( M/\Gamma \); for if \( m\Gamma \) is any element of the latter, and \( g \) is any element of G(M),
then the definition

\[
(m\Gamma)g = mg\Gamma
\]  

(23)

makes sense if and only if \( g \) is an element of N(\( \Gamma \)). But notice that, with this definition,
every element of \( \Gamma \) itself has no effect on each element of \( M/\Gamma \). Thus the symmetry group
of \( M/\Gamma \), which we denote by G(M/\( \Gamma \)), is not N(\( \Gamma \)) (as is sometimes said) but rather the
quotient N(\( \Gamma \))/\( \Gamma \):

\[
G(M/\Gamma) = N(\Gamma)/\Gamma.
\]  

(24)

(Of course, \( \Gamma \) is a normal subgroup of N(\( \Gamma \)), so this quotient is always a group.) Clearly,
G(M/\( \Gamma \)) will in general be substantially “smaller” than G(M), and so we can say that
factoring by \( \Gamma \) has “broken” G(M) to G(M/\( \Gamma \)). Notice that nothing we have said here
requires G to be a group of isometries or \( \Gamma \) to act freely. (Notice too that G(M/\( \Gamma \)) is not
in general naturally isomorphic to a subgroup of G(M), so this kind of symmetry breaking
is not quite the same as the usual kind, as we mentioned above.)

For example, the normalizer of \( \mathbb{Z}_2^\Omega \) (equation [6]) in O(2,4) is the entire group, O(2,4)
itself, and so the isometry group of the elliptic anti-de Sitter space \( \text{AdS}_5/\mathbb{Z}_2^\Omega \) is precisely
O(2,4)/\( \mathbb{Z}_2 \) or PO(2,4), as we saw. Similarly, the isometry group of S^3, namely O(4),
contains the antipodal map in the form of the matrix diag\((-1,-1,-1,-1)\), which is normalized by the whole group; so the isometry group of the real projective space $\mathbb{RP}^3$ is the projective orthogonal group $\text{PO}(4) = \mathbb{O}(4)/\mathbb{Z}_2$.

For an example involving conformal rather than isometric symmetries, consider the space $\text{CCM}_4$ discussed in section 2. If we wish to consider, as in \cite{49}, non-conformal versions of AdS/CFT, then of course we should try to break the specifically conformal symmetries of the boundary (that is, the symmetries other than the ordinary isometries). Let us show how to do this. Recall that $\text{CCM}_4$ has the structure $[S^1 \times S^4]/\mathbb{Z}_2$, where $S^n$ is the n-sphere. This space has the semi-Riemannian structure given by equation \ref{13} this is the Einstein static universe metric, with a seven-dimensional isometry group given by

$$
\text{Isom}(\text{CCM}_4) = [\mathbb{O}(2) \times \mathbb{O}(4)]/\mathbb{Z}_2. 
$$

This group is of course a (small) compact subgroup of the full (conformal) symmetry group, $\text{Conf}(\text{CCM}_4) = \mathbb{PO}(2,4)$. Now $\text{CCM}_4$ admits an isometry defined by

$$
\mathbb{R} : (A, B, w, x, y, z) \rightarrow (A, B, -w, -x, -y, -z),
$$

corresponding to the $[\mathbb{O}(2) \times \mathbb{O}(4)]/\mathbb{Z}_2$ element diag\((1,1,-1,-1,-1,-1)\). It is easy to see from equation \ref{5} that $\mathbb{R}$ has no fixed point on $\text{CCM}_4$. (Remember that, by definition, the point \((0, 0, 0, 0, 0, 0)\) does not lie on $\text{CCM}_4$.) Recalling that \((-A, -B, w, x, y, z) = (A, B, -w, -x, -y, -z)\), we see that if $\mathbb{R}$ generates $\mathbb{Z}_2^\mathbb{R}$, then the quotient manifold has the structure

$$
\text{CCM}_4/\mathbb{Z}_2^\mathbb{R} = S^{1/2} \times \mathbb{RP}^3.
$$

Here $S^{1/2}$ denotes a circle half the circumference of the original; that is, a circle modulo the action $\phi \rightarrow \phi + \pi$. (This is not the usual “$S^1/\mathbb{Z}_2$”, which is just a closed interval.) The normalizer of $\mathbb{Z}_2^\mathbb{R}$ in the isometry group $[\mathbb{O}(2) \times \mathbb{O}(4)]/\mathbb{Z}_2$ is the whole group, and so, by equation \ref{24} the isometry group of $\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}$ is just

$$
\text{Isom}(\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}) = \{[\mathbb{O}(2) \times \mathbb{O}(4)]/\mathbb{Z}_2\}/\mathbb{Z}_2^\mathbb{R} = \mathbb{PO}(2) \times \mathbb{PO}(4) = \mathbb{O}(2) \times \mathbb{O}(4),
$$

where we have used the fact that $\mathbb{PO}(2) = \mathbb{O}(2)$. Since $\mathbb{PO}(4) = [\mathbb{SO}(3) \times \mathbb{SO}(3)] \lt \mathbb{Z}_2$ is the isometry group of $\mathbb{RP}^3$, this result is not very surprising in view of equation \ref{27}. But now let us ask what happens to the conformal symmetry group of $\text{CCM}_4$ when we factor by $\mathbb{Z}_2^\mathbb{R}$. The normalizer of $\mathbb{Z}_2^\mathbb{R}$ in $\mathbb{PO}(2,4)$ is in fact exactly the isometry group of $\text{CCM}_4$, $[\mathbb{O}(2) \times \mathbb{O}(4)]/\mathbb{Z}_2$; that is precisely why we are interested in $\mathbb{Z}_2^\mathbb{R}$. For it follows, again from \ref{24} that

$$
\text{Conf}(\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}) = \mathbb{O}(2) \times \mathbb{PO}(4) = \text{Isom}(\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}).
$$

Thus, while $\text{CCM}_4$ has a conformal group $\mathbb{PO}(2,4)$ with eight generators beyond those of the isometry group (equation \ref{24}), we now see that $\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}$ has no conformal symmetries other than its isometries: the specifically conformal symmetries of $\text{CCM}_4$ have all been broken. Thus we might hope that $\text{CCM}_4/\mathbb{Z}_2^\mathbb{R}$ will play a role in the study of non-conformal bulk/boundary duality, leading to a different approach to that of \cite{49}.

With regard to these examples, we stress again that $\mathbb{PO}(2,4)$ is not a subgroup of $\mathbb{O}(2,4)$, that $\mathbb{Z}_2 \times [\mathbb{SO}(3) \times \mathbb{SO}(3)] \lt \mathbb{Z}_2$ is not a subgroup of $\mathbb{O}(1,4)$, and so on; in
each case, the relevant group is related to the final symmetry group in the same way that SU(2) is related to SO(3). As in that case, the consequence may be that certain matter fields may transform “spinorially” after the quotient is taken. We saw examples of this earlier, in discussing the elliptic versions of both anti-de Sitter and de Sitter spacetimes. The existence or non-existence of such spinorial states could provide an intrinsic way of distinguishing such spacetimes from their quotients. In a later section, however, we shall consider a more decisive way of doing so.

Now that we have a simple technique for deciding how much symmetry a quotient space possesses, let us turn to the study of the “less symmetric” versions of anti-de Sitter and de Sitter spacetimes, obtained by taking such quotients. We begin with the de Sitter-like cases, and return later to the anti-de Sitter cases.

5. Less-Symmetric Versions of de Sitter

The four-dimensional, simply connected version of de Sitter spacetime, dS(S³), has the maximal number of Killing vectors (ten), but it is becoming clear that this large group is probably not entirely physical. Witten [10] emphasises that if, as has been suggested [24], the Hilbert space of de Sitter quantum gravity is finite-dimensional, then quantum gravity must break the de Sitter group, O(1,4), to some much smaller group which (unlike O(1,4)) has finite-dimensional unitary representations. This is compatible with the fact that de Sitter spacetime has no spatial infinity at which de Sitter gauge charges might be evaluated. *The de Sitter group is not the symmetry group of quantum gravity in an accelerating universe*. It follows that ordinary de Sitter spacetime is not the right background for investigating the true nature of the acceleration.

We have suggested elsewhere [26] that the correct version of “de Sitter spacetime” from this point of view is obtained simply by taking the spatial sections to be copies of IRP³ — the antipodally identified version of S³ — instead of S³. Classically, there is no basis whatever for preferring dS(S³) to dS(IRP³). For our purposes here, however, there is an important geometric distinction, namely the fact that the isometry groups are very different. In fact, the isometry group of dS(IRP³) is, as we shall show in detail below, the six-dimensional compact group Z₂ × PO(4), where as usual PO(n) denotes the projective orthogonal group. (It may be more useful to express this as Z₂ × [(SO(3) × SO(3)) ∋ Z₂], as was done in [26].) Thus, merely by changing the interpretation of de Sitter spacetime in a way which is classically harmless, we reduce the relevant symmetry group from one which has no finite-dimensional unitary representations to one which does. Physically, the effect of replacing S³ with IRP³ is simply to reduce the multiplicity of mutually boosted families of de Sitter observers to one family, since boosts do not commute with the antipodal map on the spatial sections. This is precisely what normally happens in cosmology: in generic FRW cosmologies there is, by construction, a family of observers who are distinguished by being the observers to whom the Universe appears to be isotropic. Corresponding to this, the symmetry group of a generic FRW cosmology is six-dimensional, not ten-dimensional: there are no boost (or time translation) symmetries in cosmology. Thus, one could regard the non-trivial topology of the spatial sections of dS(IRP³) as a simple way of connecting de Sitter spacetime with more realistic cosmologies.

The claim is that quantum gravity reduces the size of the symmetry group (from ten
Killing vectors to six), and that this is implemented formally by the non-trivial topology of the spatial sections of dS(\(\mathbb{RP}^3\)). One can think of this as a way of mediating between “observer complementarity” (see for example \[51\]) and ordinary FRW cosmologies. In the former, the de Sitter group is reduced to the small group of rotational and time translation symmetries seen by one single “static” observer, and it is said that this is the way in which O(1,4) is replaced by a subgroup which can describe a finite number of physical states. However, there is a puzzle here: what has become of the spatial translation symmetries of de Sitter spacetime, which must still exist? The answer is that there must be another, complementary description in which the spatial translations are manifest but the time translation symmetry (seen by one observer) is not. The \(\mathbb{RP}^3\) de Sitter spacetime allows us to reduce O(1,4) not to the “static” symmetry group but to the compact subgroup corresponding to a single family of distinguished (isotropic but not static) observers. This is closer to the conventional procedures of physical cosmology, in the sense that in cosmology we normally distinguish families of observers whose worldlines fill the entire spacetime, not individual observers. Some such “complementarity” seems to be necessary to give a complete account of the symmetries of quantum de Sitter spacetime.

Closely related arguments in favour of replacing \(S^3\) with \(\mathbb{RP}^3\) come from other studies of de Sitter entropy. For example, an ingenious attempt \[52\] to derive the entropy as entanglement entropy founders precisely because of the high degree of symmetry of dS(\(S^3\)). From yet another point of view, de Sitter entropy is traditionally \[53\] derived not in de Sitter spacetime itself, but rather in Schwarzschild-de Sitter spacetime, SdS(\(S^3\)), which has a Penrose diagram given in Figure 2. The left and right sides are topologically identified,

![Penrose diagram of SdS(\(S^3\))](image)

Figure 2: Penrose diagram of SdS(\(S^3\))

as shown. An attempt to derive the de Sitter entropy formula in terms of entanglement entropy in this geometry would begin with a pair of independent systems (coupled so as to produce a pure state), one in each of the diamond-shaped regions in the diagram. But the “independence” of those two regions is compromised by the fact that their future and past infinities are identical, due to the topological identifications. One cannot really regard them as independent if any kind of dS/CFT correspondence is valid, and we have argued that some kind of de Sitter holography must hold for de Sitter spacetime, even if only very approximately. The solution to this problem is to note \[26\] that the strange
The structure of Figure 2 arises from the assumption that the spatial sections of "de Sitter spacetime" have the topology of $S^3$. If we replace $S^3$ with $\mathbb{R}P^3$, then $\text{SdS}(S^3)$ is replaced by $\text{SdS}(\mathbb{R}P^3)$, and it is shown in [26] how this splits the conformal infinity and restores the independence of the two systems.

In view of all this, we shall assume henceforth that the version of de Sitter spacetime which is most relevant to current theoretical concerns is one of those which are not maximally symmetric. Let us consider the consequences of replacing $S^3$ by $\mathbb{R}P^3$ in each of the maximally symmetric versions of de Sitter spacetime studied earlier.

First, take $\text{dS}(S^3)$, given by equation 7, and consider the map

$$\mathcal{R} : (A, w, x, y, z) \rightarrow (A, -w, -x, -y, -z).$$  

(We shall systematically abuse notation and denote by $\mathcal{R}$ any of the maps which reverse the signs of $w, x, y,$ and $z$ but not $A$ or $B$.) The fixed points of this map do not lie on the locus, and so the quotient of $\text{dS}(S^3)$ by the $\mathbb{Z}_2$ generated by $\mathcal{R}$ is non-singular. Clearly we are just performing an antipodal identification of the spatial sections, leaving time untouched: by definition, this is the completely non-singular spacetime $\text{dS}(\mathbb{R}P^3)$, whose virtues we have just been describing. The Penrose diagram of $\text{dS}(\mathbb{R}P^3)$ has the form shown in Figure 3. The stars represent copies of $\mathbb{R}P^2$. The detailed interpretation is given in [26].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{penrose.png}
\caption{Penrose diagram of $\text{dS}(\mathbb{R}P^3)$.}
\end{figure}

The isometry group of $\text{dS}(S^3)$ is $O(1,4)$, and the element of $O(1,4)$ corresponding to $\mathcal{R}$ is $\text{diag}(1, -1, -1, -1, -1)$. This generates a $\mathbb{Z}_2$, denoted $\mathbb{Z}_2^{\mathcal{R}}$, which is normalized by the subgroup $\mathbb{Z}_2 \rtimes O(4)$, where this $\mathbb{Z}_2$ is generated by $\text{diag}(-1, 1, 1, 1, 1)$. Factoring by $\mathbb{Z}_2^{\mathcal{R}}$, as required by equation 24, we find that the isometry group of $\text{dS}(\mathbb{R}P^3)$ is just $\mathbb{Z}_2 \rtimes \text{PO}(4)$, which may be written as $\mathbb{Z}_2 \rtimes [\text{SO}(3) \times \text{SO}(3)] \leq \mathbb{Z}_2$, where the final product is semi-direct. ($\mathbb{Z}_2$ acts by switching the two $\text{SO}(3)$ factors.) Thus topological symmetry breaking can reduce the size of a symmetry group quite substantially — in
this instance, from the 10-dimensional non-compact group $O(1,4)$ to the 6-dimensional compact group $\mathbb{Z}_2 \times [SO(3) \times SO(3)] \triangleleft \mathbb{Z}_2$.

Next, recall that we argued (see Figure 1) that it is actually quite natural to identify the future conformal infinity of $dS(S^3)$ with its past conformal infinity — bear in mind that this just affects the compactification, not the spacetime. However, if we do that, then the Witten-Strominger past-future correlator singularity becomes a singularity for an antipodal correlator on $S^3$. This is telling us that we ought to perform a further, purely spatial identification: in other words, the cyclic interpretation of the Penrose diagram only makes sense physically if, once again, we systematically replace $S^3$ with $\mathbb{RP}^3$. The effect is that the compactification of $dS(\mathbb{RP}^3)$ has one boundary at infinity instead of two, as shown in Figure 4.

As we have repeatedly stressed, this is a modification of the conformal compactification, not of $dS(\mathbb{RP}^3)$, so the isometry group of the spacetime has not changed — it is still $\mathbb{Z}_2 \times \text{PO}(4)$. However, the conformal group of the boundary has certainly changed. As we saw when discussing Figure 1, the conformal group of $S^3$ is $SO(1,4)$. By means of another application of equation 24, one finds that the conformal group of $\mathbb{RP}^3$ is the same as its isometry group — it is the projective orthogonal group $\text{PO}(4)$. Since both connected components of the conformal boundary of $dS(\mathbb{RP}^3)$ have the structure of $\mathbb{RP}^3$, we see that the conformal group of the two-component boundary of $dS(\mathbb{RP}^3)$ is $[\text{PO}(4) \times \text{PO}(4)] \triangleleft \mathbb{Z}_2$, which is not isomorphic to the isometry group of $dS(\mathbb{RP}^3)$. However, the conformal group of the “boundary” of the cyclic compactification is just $\text{PO}(4)$, which is of course far closer to $\mathbb{Z}_2 \times \text{PO}(4)$. (They are still not exactly isomorphic — as in the case of the cyclic compactification of $dS(S^3)$, we cannot expect the “boundary” to recognise the past/future symmetry of $dS(\mathbb{RP}^3)$.)

Finally we turn to the $\mathbb{RP}^3$ version of elliptic de Sitter spacetime. Note that the full antipodal map on $dS(S^3)$ generates a $\mathbb{Z}_2$ which is the diagonal subgroup of the group

Figure 4: Cyclic Compactification of $dS(\mathbb{RP}^3)$
\[ \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \text{ generated by } \mathcal{K} \text{ and the map } \]

\[ \alpha : (A, w, x, y, z) \to (-A, w, x, y, z). \]  

(31)

Since the global de Sitter proper time \( \tau \) in equation [19] is related to the coordinate \( A \) in equation [7] by \( A = L \sinh(\tau/L) \), we see that this map just corresponds to \( \tau \to -\tau \). Extending \( \mathcal{K} \) to \( \text{ElldS}_4 \) in the obvious way, we see that the quotient of \( \text{ElldS}_4 \) by \( \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \) is the same as the quotient \( \text{dS}(S^3)/[\mathbb{Z}_2^3 \times \mathbb{Z}_2^3] \), which simply means that we identify according to \( \tau \to -\tau \) in \( \text{dS}(\mathbb{R}P^3) \). Clearly \( \tau \to -\tau \) has a fixed point set \( \tau = 0 \), which is the \( \mathbb{R}P^3 \) of minimum size in \( \text{dS}(\mathbb{R}P^3) \). This is an orbifold singularity in the quotient. We can think of it as a “spacelike brane” [54], which occurs at a finite proper time prior to any point in the spacetime, and which cuts off the spacetime towards the past. (Past-directed curves reaching this brane simply terminate there.)

Thus we see that the \( \mathbb{R}P^3 \) version of elliptic de Sitter space — let us call it \( \text{ElldS}(\mathbb{R}P^3) \) — differs very greatly from ordinary \( \text{ElldS}_4 \): the latter is non-singular but non-time-orientable, while \( \text{ElldS}(\mathbb{R}P^3) \) is (orbifold) singular, but it is time orientable. For whereas in \( \text{ElldS}_4 \) it is not possible to decide globally whether an inextensible timelike geodesic is future-directed or past-directed, this can be done on \( \text{ElldS}(\mathbb{R}P^3) \). The reader can confirm this by following a timelike curve in Figure 5, the Penrose diagram for this spacetime, as it crosses the diagonal line shown, and comparing this with the behaviour of a covering curve in elliptic de Sitter spacetime. (The stars denote copies of \( \mathbb{R}P^2 \) as usual.) In elliptic de Sitter, the covering curve changes character from future-directed to past-directed or vice versa, but here it does not do so. In this cosmology, the Universe never contracts: it begins at the orbifold hypersurface \( \tau = 0 \) (corresponding to conformal time \( \eta = \pi/2 \)) and expands indefinitely from there, each spatial section being of course a copy of \( \mathbb{R}P^3 \). There is only one boundary at infinity, as in elliptic de Sitter itself; unlike the latter, however, \( \text{ElldS}(\mathbb{R}P^3) \) has \( \mathbb{R}P^3 \) as its conformal boundary. The normalizer of \( \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \) in the \( \text{dS}(S^3) \)}
isometry group $O(1,4)$ is just the same as the normalizer of $\mathbb{Z}_2^4$, namely $\mathbb{Z}_2 \times O(4)$, and so using equation 24 as usual we find that the isometry group of $Ell\mathbb{d}S(\mathbb{R}\mathbb{P}^3)$ is just $PO(4)$, which is of course precisely the conformal group of the $\mathbb{R}\mathbb{P}^3$ conformal boundary. The reason for this precise agreement is that $Ell\mathbb{d}S(\mathbb{R}\mathbb{P}^3)$ does not have the past/future symmetry of the other versions of de Sitter we are considering here.

Despite its strange character, this cosmology has several attractive features: the natural absence of curvature singularities and of any “bounce” should be noted. Another kind of expanding cosmology without a big bang is studied in [55]. One could perhaps construct a realistic version of $Ell\mathbb{d}S(\mathbb{R}\mathbb{P}^3)$ along similar lines, though of course it would be a challenge to produce the correct “initial conditions” from the still poorly-understood physics of spacelike branes. However, in view of the growing suspicion that the “de Sitter phase” of the universe may not be eternal [23], we may have to learn to deal with spacetimes of precisely this kind. Notice that this cosmology is compatible with the very interesting “final boundary condition” advocated by Lasenby and Doran [56]. In short, this kind of cosmology should not be rejected out of hand. Later we shall argue that it should be rejected, but not for reasons that are obvious at this point.

None of these “less-symmetric” versions of de Sitter spacetime can be accommodated in either $AdS_5$ or $EllAdS_5$ — recall that we saw that the relevant region of the latter was foliated by $dS(S^3)$. Hence we must modify our versions of anti-de Sitter spacetime if we wish them to contain a de Sitter slice of one of these kinds. We shall now show how this is done.

6. Less-Symmetric Versions of Anti-de Sitter

The map $\mathfrak{N}$ defined by 30 can be extended in the obvious way from $dS(S^3)$ to $AdS_5$, via the embedding given by equation 8. (This is formally the same as in equation 26.) Alternatively, recall that we have taken care to use the same spherical polar coordinates for anti-de Sitter space and the de Sitter slice, so $\mathfrak{N}$ can be extended in that way, as the map which sends $(\chi, \theta, \phi)$ to $(\pi - \chi, \pi - \theta, \pi + \phi)$. The obvious way to obtain $dS(\mathbb{R}\mathbb{P}^3)$ as an $AdS$ slice is to perform this extension and take the quotient of $AdS_5$ by the $\mathbb{Z}_2$ generated by the extension. Let us see how this works.

While $\mathfrak{N}$ has no fixed point on $dS(S^3)$, it does have fixed points on $AdS_5$ — it fixes every point on the circle $A^2 + B^2 = L^2$, which is the timelike geodesic given in global $AdS_5$ coordinates by $r = 0$. In fact this is the worldline of the origin of these coordinates in the hyperbolic space $H^4$, which, as we saw (see equation 13), gives the geometry of the spatial sections of $AdS_5$ in global coordinates. Thus the extension of $\mathfrak{N}$ from $dS(S^3)$ to $AdS_5$ has fixed points, because the action of $\mathfrak{N}$ on $H^4$ has a fixed point — an important fact to which we shall return later. The quotient spacetime has the structure

$$AdS_5/\mathbb{Z}_2^8 = S^1 \times [H^4/\mathbb{Z}_2^8].$$

(32)

It is (orbifold) singular and contains the non-singular spacetime $dS(\mathbb{R}\mathbb{P}^3)$ as a slice (Figure 3). Using equation 24, one finds that its isometry group is $O(2) \times PO(4)$. From our discussion of $CCM_4/\mathbb{Z}_2^4$ in section 5, we know that the relevant conformal group is exactly the same, $O(2) \times PO(4)$ (see equation 29).
There is in fact something rather unsatisfactory about this procedure, however. We mentioned earlier that each temporal cycle of AdS₅ contains two de Sitter slices — this can be seen by comparing the formulae for B in equations 12 and 16. (The point is that the coordinates in 16, which describe one de Sitter slice, require B to be positive, which, from 12, means that t is restricted to lie between $-\pi L/2$ and $+\pi L/2$. The remaining range of t allows for another de Sitter slice within the $2\pi L$ range of t.) The same comments carry over to the quotients we are considering here: we have in fact embedded dS(IRP³) in the (quotient version of) anti-de Sitter twice over. This unappealing feature can be eliminated as follows.

Define an isometry of AdS₅ by

$$\sqrt{\aleph} : (A, B, w, x, y, z) \rightarrow (-A, -B, -x, w, -z, y).$$

(33)

This is of order four, and of course $(\sqrt{\aleph})^2 = \aleph$. Clearly $\sqrt{\aleph}$ does not map the de Sitter slice in AdS₅ into itself; instead, it maps the slice at $-B_0$ to the one at $B_0$ (see equation 3). Having done this, its square, $\aleph$, converts the S³ sections to IRP³. Thus we obtain one dS(IRP³) slice in the orbifold quotient AdS₅/ZZ√ℵ. (This last is indeed an orbifold, because although $\sqrt{\aleph}$ has no fixed point, its square does.) This gives us another and more elegant embedding of dS(IRP³) as a slice. A somewhat intricate calculation using the usual techniques shows that the isometry group of AdS₅/ZZ√ℵ is isomorphic to O(2)×SO(2)×SO(3); rather oddly, the bulk has fewer Killing vectors than the slice. (There is of course no contradiction in this.)

Turning now to the version of de Sitter pictured in Figure 1 in “anti-de Sitter” space–time, we observe that AdS₅ admits an isometric action by the ZZ² defined by Ξ, given by

$$\Xi : (A, B, w, x, y, z) \rightarrow (-A, -B, w, x, y, z).$$

(34)

Unlike $\aleph$, the isometry Ξ has no fixed point on AdS₅, so the quotient AdS₅/ZZ² is actually non-singular. Its structure is that of $S^{1/2} \times R^4$, where $S^{1/2}$ denotes as usual the circle with half the circumference of the original one. The shrinking of the circle breaks the O(2,4) symmetry group of AdS₅ down to O(2)×O(4); the corresponding conformal group is the conformal group of CCM₄/ZZ², which is the same as CCM₄/ZZ²; hence this group is O(2)×PO(4).

Like $\sqrt{\aleph}$, Ξ does not map the dS(S³) embedded in AdS₅ into itself; instead it maps the dS(S³) at $-B_0$ to the one at $B_0$, without changing it. The effect of factoring out Ξ is to identify these two slices with each other, leaving just one de Sitter slice in each cycle of anti-de Sitter global time.

Now the shrinking of the timelike circle means that the global anti-de Sitter time coordinate t effectively runs from $-\pi L/2$ to $+\pi L/2$, instead of from $-\pi L$ to $+\pi L$. However, we saw earlier (see equation 20) that the entire infinite history of the de Sitter slice extends precisely from $t = -\pi L/2$ to $t = +\pi L/2$, where the interval can be regarded as closed provided that we include the conformal infinities of de Sitter spacetime. So in fact the periodicity of global time in AdS₅/ZZ² actually forces the “time” of the de Sitter slice to be periodic — provided of course that this “time” extends to the conformal boundary. In other words, it is the conformal de Sitter time $\eta$ (see equation 20) which is forced to be periodic, not the global proper time. This is of course precisely the situation portrayed in Figure 1.
The picture of the de Sitter slice thus obtained is rather attractive, since one copy of the de Sitter slice sits neatly in the non-singular “anti-de Sitter” spacetime $\text{AdS}_5/\mathbb{Z}^2$, and the problems associated with the disconnectedness of the the conformal boundary of de Sitter spacetime are resolved by the periodicity of anti-de Sitter time. However, we motivated the picture of the de Sitter compactification given in Figure 1 by means of the observation [10][11] that a scalar field correlator between a point on the sphere in the infinite past and the antipodal point (in space and time) on the sphere in the infinite future is singular. The idea of Figure 1 was to provide a geometric formulation of Strominger’s argument that the relevant Green functions only transform simply under one copy of the conformal group $SO(1,4)$; for now there is only one sphere at infinity. The problem is now to explain the correlator singularity between antipodal points on this sphere. The obvious step is of course to replace $S^3$ by $\mathbb{RP}^3$, which is how we obtained Figure 4. In the present case, the way to obtain the space in Figure 4 as an AdS slice is simply to extend the definition of $\mathcal{H}$ to $\text{AdS}_5/\mathbb{Z}^2$ (interpreting equation 26 appropriately). The resulting quotient can also be regarded as a quotient of elliptic anti-de Sitter space by $\mathbb{Z}^2$. The structure is

$$\text{EllAdS}_5/\mathbb{Z}^2 = \text{AdS}_5/[\mathbb{Z}^2 \times \mathbb{Z}^2] = S^{1/2} \times [H^4/\mathbb{Z}^2],$$

and it is of course orbifold singular, unlike $\text{AdS}_5/\mathbb{Z}^2$. Its isometry group is $O(2) \times \text{PO}(4)$, which is the same as the conformal symmetry group of its boundary, $\text{CCM}_4/\mathbb{Z}^2$, which was discussed as an example in section 4.

Finally, we can realise the version of de Sitter pictured in Figure 5 as an AdS slice by extending the map $\alpha$, defined by equation 31, to $\text{AdS}_5/\mathbb{Z}^2$ by means of the embedding given by equation 8. The effect of course is just to reverse the anti-de Sitter time coordinates $T$ (equations 3) and $t$ (equations 12), just as $\alpha$ reverses the sign of de Sitter time $\tau$. If we take the quotient $\text{AdS}_5/[\mathbb{Z}^2 \times \mathbb{Z}^2]$, then of course we are cutting off $T$ and $t$ at the fixed points, namely zero and $\pi L$, and placing spacelike branes at the orbifold singularities there, just as we did in Figure 5. Thinking in terms of global coordinates, $t$ now extends from zero to $\pi L$, just as de Sitter global time $\tau$ extends from zero to infinity in Figure 5. The space pictured in Figure 5 has now been realised as a slice in $\text{AdS}_5/[\mathbb{Z}^2 \times \mathbb{Z}^2]$,

$$\text{AdS}_5/[\mathbb{Z}^2 \times \mathbb{Z}^2] = [0, \pi L] \times [H^4/\mathbb{Z}^2],$$

where $[0, \pi L]$ is a closed interval. The isometry group of the latter spacetime is computed as follows: the normalizer of the $\mathbb{Z}^2 \times \mathbb{Z}^2$ group in $O(2,4)$ is $\mathbb{Z}^2 \times \mathbb{Z}^2 \times O(4)$, where $\mathbb{Z}^2$ is generated by the matrix $\text{diag}(1,-1,1,1,1)$. Therefore the isometry group is $\mathbb{Z}^2 \times \text{PO}(4)$. (There are two copies of the slice for each $B_0 > L$, one at $B = B_0$ and one at $-B_0$, and the effect of $\mathbb{Z}^2$ is just to exchange them. The “other” de Sitter slice is the time-reverse of the one at $B_0$, that is, it always contracts; in the Penrose diagram it sits “on top of” the given slice. We could try to eliminate this unwelcome slice in the same way as before, but we shall not do so here.) The relevant boundary conformal group is precisely the same. Notice that time is not cyclic, and that there is no continuous time translation (or “time rotation”) symmetry in this version of anti-de Sitter spacetime.

In this section and the previous one, we have developed a straightforward, systematic way of analysing some of the “less-symmetric” versions of [anti-]de Sitter spacetimes and
their relationships. There are of course many others, but the ones considered here provide a useful, physically interesting sample. We shall now turn to the problem of using physical arguments to eliminate some of these spacetimes from contention.

7. A Topological Selection Criterion

We have argued strongly, here and in [26], that dS($\mathbb{RP}^3$) (or one of its relatives discussed in the previous section) is the right version of de Sitter space for investigations of quantum gravity in an accelerating universe. In the preceding section, however, we saw that the dS($S^3$) isometry $\mathbb{R}$ has fixed points in anti-de Sitter spacetime when it is extended into the bulk, so that dS($\mathbb{RP}^3$) has to be embedded as a slice in an AdS orbifold. The physical importance of this observation will now be explained.

Adams, Polchinski, and Silverstein [57] have conjectured that the condensation of closed string tachyons coming from the twisted sector of a non-supersymmetric orbifold would tend to resolve the orbifold singularity and restore supersymmetry. This is implemented by means of a dilaton pulse which expands outward at the speed of light, ultimately restoring the geometry to its pre-orbifold state. Strong evidence in favour of this conjecture has recently been obtained by studying both the late-time structure [58] [59] and the internal consistency of the proposed mechanism [60] [61]. While this work applies directly to the flat case, it has been argued by Horowitz and Jacobson [62] that a similar phenomenon can be expected in non-supersymmetric orbifolds of AdS. Indeed, the AdS/CFT correspondence suggests that matter configurations on slices of such orbifolds are unstable.

Thus we have an addition to our “AdS toolbox”: if we are obliged to embed a spacetime as a slice in a non-supersymmetric orbifold version of AdS, then this is evidence that the object represented by the slice will not be stable in string theory. As we have seen, topologically non-trivial versions of de Sitter spacetime do embed in AdS orbifolds, so the survival of supersymmetry in such orbifolds gives us a criterion for the acceptability of variant versions of dS$_4$.

Now in fact the orbifold singularities of quotients of AdS$_5$ have been extensively studied, precisely from the point of view of supersymmetry breaking. It is convenient to do this by embedding AdS$_5$ in a three-dimensional complex flat space $\mathbb{C}^3$, as follows. Define complex coordinates $Z_1$, $Z_2$, and $Z_3$ in terms of $(A,B,w,x,y,z)$ by

$$
Z_1 = A + iB,
Z_2 = w + ix,
Z_3 = y + iz.
$$

(37)

Then AdS$_5$ is defined as the locus in $\mathbb{C}^3$ given by

$$
-Z_1 \overline{Z}_1 + Z_2 \overline{Z}_2 + Z_3 \overline{Z}_3 = -L^2.
$$

(38)

The actions of $\mathbb{R}$, $\sqrt{\mathbb{R}}$, $\alpha$, and $\Xi$ can be extended from AdS$_5$ to $\mathbb{C}^3$ by

$$
\mathbb{R} : (Z_1, Z_2, Z_3) \rightarrow (Z_1, -Z_2, -Z_3)
$$

(39)
It is immediately clear that the extended action of $\alpha$ on $\mathbb{C}^3$ is not holomorphic, and we therefore expect that all supersymmetries are broken in the projection from AdS$_5$ to AdS$_5/\mathbb{Z}_2^\alpha$. Therefore, the further quotient AdS$_5/[\mathbb{Z}_2^\alpha \times \mathbb{Z}_2^\beta]$ (see equation 36) has no supersymmetries either. One can in fact prove this directly simply by recalling our computation of the isometry group of AdS$_5/[\mathbb{Z}_2^\alpha \times \mathbb{Z}_2^\beta]$, which is $\mathbb{Z}_2^3 \rtimes \text{PO}(4)$. This group contains no continuous timelike symmetries: that is, the spacetime has no global timelike Killing vector fields. As is well known from the de Sitter case (see for example [10]), this means that the spacetime cannot be supersymmetric.

The other three cases are more subtle because $\Xi$, $\aleph$, and $\sqrt{\aleph}$ do act holomorphically on $\mathbb{C}^3$. Such actions were analysed in [63], where the effects of holomorphic maps on the Killing spinors of AdS$_5$ were exhibited explicitly. The results were as follows. (Note that these authors use the same definition of AdS$_5$ as we use here, the version (equations 2 and 38) with cyclic time.)

Let $\mathbb{Z}_n$ act on $\mathbb{C}^3$ as follows: if $\gamma$ is a primitive $n$th root of unity, set

$$(Z_1, Z_2, Z_3) \to (\gamma^d Z_1, \gamma^a Z_2, \gamma^b Z_3). \quad (40)$$

Then Ghosh and Mukhi show that the effect on a general AdS$_5$ Killing spinor is that of a matrix with eigenvalues

$$(a + b - d), -(a + b + d), (a - b + d), -(a - b - d). \quad (41)$$

For $\Xi$, we have $\gamma = -1$, $d = 1$, $a = b = 0$, and so we see that none of the eigenvalues is zero; thus neither AdS$_5/\mathbb{Z}_2^\alpha$ nor any quotient of it, in particular AdS$_5/[\mathbb{Z}_2^\alpha \times \mathbb{Z}_2^\beta]$, has any supersymmetry. For $\aleph$, by contrast, we have $\gamma = -1$, $d = 0$, $a = b = 1$, so that precisely two of the eigenvalues vanish, and we conclude that AdS$_5/\mathbb{Z}_2^\alpha$ has half the supersymmetry of AdS$_5$ itself. Finally and most interestingly, for $\sqrt{\aleph}$ we have $\gamma = i$, $d = 2$, $a = b = 1$, so AdS$_5/\mathbb{Z}_2^{\sqrt{\aleph}}$ is quarter-supersymmetric: it too is a supersymmetric orbifold.

To summarize: of the four versions of anti-de Sitter spacetime we are considering here, two are supersymmetric (one half, one quarter) and the other two are not. This means that the version of de Sitter spacetime pictured in Figure 3 embeds as a slice in the supersymmetric anti-de Sitter orbifolds AdS$_5/\mathbb{Z}_2^\aleph$ and AdS$_5/\mathbb{Z}_4^{\sqrt{\aleph}}$, while the version in Figure 4 embeds in the non-supersymmetric orbifold AdS$_5/[\mathbb{Z}_2^\alpha \times \mathbb{Z}_2^\beta]$, and similarly the version of de Sitter pictured in Figure 5 embeds as a slice in the non-supersymmetric orbifold AdS$_5/[\mathbb{Z}_2^\beta \times \mathbb{Z}_2^\alpha]$. Before drawing any conclusions from this, we should ask whether the orbifold singularities here are generic, in the following sense. An accelerating universe does not of course have the exact de Sitter metric: there will be perturbations. The same is true of the bulk space. Might these perturbations themselves actually remove the singularities?

We know that the spatial sections of AdS$_5$ in global coordinates are just copies of the hyperbolic space $\mathbb{H}^4$ (equation 13), which has a very special structure: the sectional
curvatures are all \textit{exactly the same}, independent of both direction and position. This property would not survive even a small perturbation. Must a finite group have fixed points on a perturbed version of \(H^4\)? Surprisingly, there is a very precise answer to this difficult question, given by a classical theorem of Cartan (see \cite{64}, page 111):

\textbf{THEOREM} (Cartan, 1929): Let \(M\) be a geodesically complete, simply connected Riemannian manifold of non-positive sectional curvature, and let \(G\) be a compact group of isometries of \(M\). Then there is a point in \(M\) which is fixed by every element of \(G\).

The key point here is that the theorem does \textit{not} require that the sectional curvatures should all be the same in all directions or at all points: it only requires that they should be non-positive. A small perturbation of \(H^4\) will not preserve the constancy of the curvature, but nor will it change a negative sectional curvature to one which is positive. Thus, all of the finite (hence of course compact) groups of isometries we are considering here, acting on a mildly perturbed version of \(H^4\), will still have a fixed point. The singularity can only be resolved if the disturbance of the geometry is so large that at least one sectional curvature reaches a positive value. This has two consequences: first, the orbifold singularities are indeed generic, not a result of the highly symmetric geometry of (exact) \(\text{AdS}_5\). Second, if indeed the Adams-Polchinski-Silverstein process \cite{57} does resolve the singularities, it can only do so by means of major disturbances of the geometry, which we can interpret as stringy instabilities for the relevant versions of de Sitter spacetime, embedded as slices in the orbifold.

We conclude that the versions of de Sitter spacetime pictured in Figures 4 and 5 are unstable in string theory, because the corresponding anti-de Sitter bulk spacetimes are non-supersymmetric orbifolds. (Before finally abandoning \(\text{AdS}_5/\mathbb{Z}_2^2\), however, we note that, while it has no supersymmetries, nor does it have any orbifold singularities. As the relevant boundary theory has no conformal symmetries other than its isometries (both groups are isomorphic to \(O(2) \times \text{PO}(4)\)), this version of \(\text{AdS}_5\) may be of interest for other purposes, in the study of supersymmetry and conformal symmetry breaking.)

The simplest non-maximally-symmetric version of de Sitter, \(\text{dS}(\mathbb{R}P^3)\), is however still a candidate, for the corresponding AdS orbifolds are supersymmetric. The examples chosen for discussion here do not exhaust the list of possibilities, but precisely the same methods apply in other cases. For example, it is not hard to show that \textit{elliptic} de Sitter spacetime embeds in a non-supersymmetric \(\text{AdS}_5\) orbifold. We therefore predict that it is not stable in string theory. In fact, \textit{all versions of de Sitter spacetime with only one boundary component are ruled out.} Thus it seems that we must accept that de Sitter spacetime is \textit{not} “really” or “holographically” dual to a single CFT inhabiting one boundary space. This supports the version of de Sitter holography put forward in \cite{45}, with its two independent but entangled CFTs.

The only survivors now are \(\text{dS}(\mathbb{R}P^3)\) and some of the generalisations of it obtained by replacing \(\mathbb{Z}_2\) by some larger finite subgroup of the isometry group of \(S^3\), that is, \(O(4)\). All of these can be obtained as slices in \(\text{AdS}_5\) orbifolds; most cannot be obtained as slices in \textit{supersymmetric} \(\text{AdS}_5\) orbifolds, however, so many candidates can be winnowed out; for example, it can be shown that the recently proposed cosmology with “dodecahedral” spatial geometry \cite{65} would be unstable in string theory. This is a striking example of
the use of string theory to constrain spacetime topology. Note that it has been claimed that observations do indeed rule out the “dodecahedral” model.

In fact it is possible to prove that the only topologically non-trivial versions of de Sitter spacetime which do survive our criterion consists of the versions with $S^3$ replaced by $S^3/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts in such a way that the quotient is homogeneous (that is, it has a transitive group of isometries). These are obtained by defining the action of $\mathbb{Z}_n$ by means of the map $0$, where $\gamma$ is a primitive $n$th root of unity, and $d = 0, a = b = 1$. This of course includes $dS(\mathbb{CP}^3)$ as a special case; all of the corresponding $AdS_5/\mathbb{Z}_n$ quotients are half-supersymmetric. These cosmologies, with $n > 2$, are distinguished from $dS(\mathbb{CP}^3)$ in two ways. First, we saw that $dS(\mathbb{CP}^3)$ can be obtained in a particularly satisfactory way as a slice in $AdS_5/\mathbb{Z}_2^{\sqrt{5}}$; recall that this embedding dispenses with the physically meaningless “second slice”. It is not hard to see that no such construction is possible for $n > 2$ (because the relevant isometry of $AdS_5$ does not exchange the two slices). Secondly, it is well known (see for example [26]) that $dS(\mathbb{CP}^3)$ is the only non-trivial spatial quotient of $dS(S^3)$ with spatial sections which are globally isotropic. Currently there is no evidence for any topologically-induced anisotropies in our universe. (Notice too that anisotropic quotients considerably reduce the rotation group seen by the static observers, whose observations are so crucial for “observer complementarity” [51]; for the specific cosmologies we are considering here, of the form $dS(S^3)/\mathbb{Z}_n$, $n > 2$, they will only see a two-parameter group of symmetries, one for time and one for rotations about an axis.) We conclude very tentatively that, both theoretically and observationally, $dS(\mathbb{CP}^3)$ is the favoured version of de Sitter spacetime. The globally anisotropic but homogeneous versions should however be investigated more thoroughly.

As far as anti-de Sitter spacetime is concerned, we have found that the favoured versions are $AdS_5/\mathbb{Z}_2^N$, and, perhaps even more so, $AdS_5/\mathbb{Z}_4^{\sqrt{N}}$. The possible importance of such quotients of $AdS_5$ was in fact suggested by Ghosh and Mukhi [63] on entirely different grounds, namely an analogy between the $AdS_5/\mathbb{Z}_2^N$ orbifold and the one obtained by taking the quotient $S^5/\mathbb{Z}_2$, which is constructed by replacing the obvious $S^3$ submanifolds by copies of $\mathbb{RP}^3$. This $S^5/\mathbb{Z}_2$ orbifold is of interest because blowing up its circle of fixed points is a relevant deformation in the AdS/CFT context. As suggested in [63], this indicates that $AdS_5/\mathbb{Z}_2^N$ may have some special role to play, independently of its role as the bulk corresponding to $dS(\mathbb{RP}^3)$. Similar remarks apply to $AdS_5/\mathbb{Z}_4^{\sqrt{N}}$. The special “light states” arising from flat gauge connections on these versions of $AdS_5$ [62] will undoubtedly be important in understanding this more completely.

8. Conclusion

Anti-de Sitter spacetime arises so naturally in string theory that it seems puzzling that the cosmological constant of our world is positive rather than negative. Perhaps the real role of AdS, however, is as a tool which can be used to extract answers to questions which cannot yet be approached using string theory directly. This is how it has been used [2] to investigate the stringy status of the black hole information paradox. As Gibbons observes [29], one of the most fundamental questions in quantum gravity is that of how to translate non-trivial spacetime geometry and topology into the quantum-mechanical context. We have suggested here that, once again, embedding in a version of AdS is the way to bring
string theory to bear on this problem.

Quantum gravity in de Sitter spacetime suggests that the relevant version of de Sitter spacetime is one of the many versions which are not maximally symmetric, and we interpret this to mean that the physical version is topologically non-trivial. The question is then: how complex can the topology of “de Sitter spacetime” become? The answer we have proposed here, using the “AdS toolbox”, is, “not very.” We saw that \( \text{dS}(\mathbb{R}P^3) \), the version advocated by de Sitter himself, explored in \([25]\), and discussed in this context in \([26]\), seems to be the natural candidate for the “true” form of de Sitter spacetime. However, more complex quotients of \( S^3 \) are still allowed: these are the versions of de Sitter spacetime which are homogeneous but only \textit{locally} isotropic. Since \( \mathbb{R}P^3 \) is perfectly isotropic, even globally — unlike any other quotient of \( S^3 \) — recent results \([66]\), which are consistent with an absence of topologically-induced cosmic anisotropies, may be said to support this candidate. But much remains to be done in the effort to understand whether and why \( \text{dS}(\mathbb{R}P^3) \) is really preferred to all other spatially homogeneous versions of de Sitter spacetime.

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