ON THE VERLINDE FORMULAS FOR SO(3)-BUNDLES

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Abstract. This paper computes the quantization of the moduli space of flat SO(3)-bundles over an oriented surface with boundary, with prescribed holonomies around the boundary circles. The result agrees with the generalized Verlinde formula conjectured by Fuchs and Schweigert.

1. Introduction

Let $G$ be a compact, connected Lie group, $\Sigma$ a compact oriented surface of genus $h$ with $r$ boundary components. Given conjugacy classes $C_1, \ldots, C_r \subset G$, denote by

$$\mathcal{M}(\Sigma, C_1, \ldots, C_r)$$

the moduli space of flat $G$-bundles over $\Sigma$, with boundary holonomies in prescribed conjugacy classes $C_j$. The choice of an invariant inner product on $\mathfrak{g}$ defines a symplectic structure on the moduli space. Under suitable integrality conditions the moduli space carries a pre-quantum line bundle $L$, and one can define the quantization

$$\mathcal{Q}(\mathcal{M}(\Sigma, C_1, \ldots, C_r)) \in \mathbb{Z}$$

as the index of the Spin$_c$-Dirac operator with coefficients in $L$. (It may be necessary to use a partial desingularization as in [16].) Choosing a complex structure on $\Sigma$ further defines a Kahler structure on the moduli space. If $G$ is simply connected, Kodaira vanishing results [20] show that the above index coincides with the dimension of the space of holomorphic sections of $L$. It is given by the celebrated Verlinde formula [22, 21, 7, 19, 5]. For symplectic approaches to the Verlinde formulas, much in the spirit of the present paper, see [11, 10, 12, 6, 2].

Much less is known for non-simply connected groups. For surfaces without boundary ($r = 0$), and taking $G = \text{PU}(n)$, Verlinde-type formulas were obtained by Pantev [17] in the case $n = 2$ and by Beauville [4] for $n$ prime. For more general compact, semi-simple connected Lie groups, Fuchs and Schweigert [9] conjectured a generalization of the Verlinde formula, expressed in terms of orbit Lie algebras. Partial results on these conjectures were obtained in [2].

In this article, we will establish Fuchs-Schweigert formulas for the index (2) for the simplest case $G = \text{SO}(3)$. We will use the recently developed quantization procedure [15, 14] for quasi-Hamiltonian actions with group-valued moment map [1]. In order to apply these techniques, we present the moduli spaces (1) as symplectic quotients of quasi-Hamiltonian $\tilde{G}$-spaces for the universal cover $\tilde{G} = \text{SU}(2)$. In more detail, let $D_i \subset \text{SU}(2)$ be conjugacy classes, and consider the quasi-Hamiltonian $\text{SU}(2)$-space

$$\tilde{M} = D_1 \times \cdots \times D_s \times \text{SU}(2)^{2h}$$
with moment map the product of holonomies,
\[
\tilde{\Phi}(d_1, \ldots, d_s, a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^s d_i \prod_{j=1}^h (a_j b_j^{-1} a_j^{-1} b_j^{-1}).
\]

Put \(M = \tilde{M}/\Gamma\), where \(\Gamma \subset Z^{s+2h}\) is the subgroup preserving \(\tilde{M} \subset SU(2)\) and \(\tilde{\Phi}\). Then \(M\) is a quasi-Hamiltonian \(SU(2)\)-space, and all connected components of \(\tilde{\Phi}\) are symplectic quotients \(\tilde{M}/SU(2)\) for suitable choices of \(D_j\) (see Section 2.3). Our first main result gives necessary and sufficient conditions under which the space \(M\) admits a level \(k\) pre-quantization [13]. Using localization, we then compute the corresponding quantization \(Q(M) \in \mathbb{R}^k(SU(2))\), an element of the level \(k\) fusion ring (Verlinde ring). These results are summarized in Theorem 3.7. We reformulate the result as an equivariant version of the Fuchs-Schweigert formula (Theorem 4.1); the non-equivariant formula (see (16) in Section 4) is then obtained from a ‘quantization commutes with reduction’ principle.

Using the results of [14], it is also possible to compute quantizations of moduli spaces for non-simply connected groups of higher rank. However, the determination of the pre-quantization conditions and the evaluation of the fixed point contributions becomes more involved. We will return to these questions in a forthcoming paper; see also the author’s abstracts in Oberwolfach Report No. 2011/09.

2. Preliminaries

The following notation, consistent with [15], will be used in this paper. For the Lie group \(SU(2)\) let \(T\) be the maximal torus given as the image of 
\[
j : U(1) \to SU(2), \quad j(z) = (z \ 0 \ 0 \ \bar{z}).
\]
Let \(\Lambda = \ker \exp_T \subset t\) denote the integral lattice and \(\Lambda^* \subset t^*\) its dual, the (real) weight lattice. Let \(\rho \in \Lambda^*\) be the generator dual to the generator \(d_j(2\pi i) \in \Lambda\). We will use the basic inner product 
\[
\langle \xi, \xi' \rangle := \frac{1}{4\pi^2} \text{tr}(\xi^\dagger \xi'), \quad \xi, \xi' \in \mathfrak{su}(2)
\]
to identify \(\mathfrak{su}(2) \cong \mathfrak{su}(2)^*\). Under this identification, \(\|\rho\|^2 = \frac{1}{2}\), and \(\Lambda = 2\Lambda^*\) with generator \(2\rho\). The following two elements of \(SU(2)\) will play a special role in this paper:
\[
u_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_* = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]
Observe that \(t_* = \exp(\rho/2)\), with square \(c = \exp \rho\) the non-trivial element in the center \(Z := Z(SU(2)) \cong \mathbb{Z}_2\). The element \(u_* \in N(T)\) represents the non-trivial element of the Weyl group \(W = N(T)/T \cong \mathbb{Z}_2\). Both \(u_*, t_*\) are contained in the conjugacy class \(D_* \subset SU(2)\) of elements of trace 0. Note that \(D_*\) is the unique conjugacy class in \(SU(2)\) that is invariant under multiplication by \(Z\). The quotient \(C_* = D_*/Z \cong \mathbb{R}P(2)\) is the conjugacy class in \(SO(3)\) consisting of rotations by \(\pi\).

2.1. The fusion ring \(R_k(SU(2))\). We view the representation ring \(R(SU(2))\) as the subring of \(C^\infty(SU(2))\) generated by characters of \(SU(2)\)-representations. As a \(\mathbb{Z}\)-module, it is free with basis \(\chi_0, \chi_1, \chi_2, \ldots\), where \(\chi_m\) is the character of the irreducible \(SU(2)\)-representation on the \(m\)-th symmetric power \(S^m(\mathbb{C}^2)\). The ring structure is determined by the formula
\[
\chi_m \chi_{m'} = \chi_{m+m'} + \chi_{m+m'-2} + \cdots + \chi_{|m-m'|}.
\]
For $k = 0, 1, 2, \ldots$ let $I_k(\text{SU}(2))$ be the ideal generated by $\chi_{k+1}$ and let
\[ R_k(\text{SU}(2)) = R(\text{SU}(2))/I_k(\text{SU}(2)) \]
be the level $k$ fusion ring (or Verlinde ring). As a $\mathbb{Z}$-module, $R_k(\text{SU}(2))$ is free, with basis $\tau_0, \tau_1, \ldots, \tau_k$ the images of $\chi_0, \chi_1, \ldots, \chi_k$ under the quotient homomorphism. Let $q = e^{\frac{i\pi}{k+2}}$ be the $2k + 4$-th root of unity, and define special points
\[ t_l = j(q^{l + 1}), \quad l = 0, \ldots, k. \]
Then $I_k(\text{SU}(2)) \subset R(\text{SU}(2))$ has another description as the ideal of characters vanishing at all special points (3). Hence, the evaluation of characters at the special points descends to evaluations $R_k(\text{SU}(2)) \to \mathbb{C}$, $t \mapsto \tau(t_l)$.

The product in the complexified fusion ring $R_k(\text{SU}(2)) \otimes_{\mathbb{Z}} \mathbb{C}$ can be diagonalized using the $S$-matrix, given by the Kac-Peterson formula
\[ S_{m,l} = (\frac{k}{2} + 1)^{-\frac{1}{2}} \sin \left( \frac{\pi(l+1)(m+1)}{k+2} \right), \]
for $l, m = 0, 1, \ldots, k$. The $S$-matrix is orthogonal, and the alternative basis elements
\[ \tilde{\tau}_l = \sum_m S_{0,l} S_{m,l} \tau_m \]
satisfy $\tilde{\tau}_m(t_l) = \delta_{m,l}$, hence
\[ \tilde{\tau}_m \tilde{\tau}_{m'} = \delta_{m,m'} \tilde{\tau}_m. \]
The basis elements $\{\tau_0, \ldots, \tau_k\}$ are expressed in terms of the alternative basis as $\tau_m = \sum_l S_{0,l} S_{m,l} \tilde{\tau}_l$.

### 2.2. Quasi-Hamiltonian $G$-spaces

We recall some basic definitions and facts from [1]. Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, equipped with an invariant inner product, denoted by a dot $\cdot$. Let $\theta^L, \theta^R$ denote the left-invariant, right-invariant Maurer-Cartan forms on $G$, and let $\eta = \frac{1}{2\pi}[\theta^L, \theta^R]$ denote the Cartan 3-form on $G$. For a $G$-manifold $M$, and $\xi \in \mathfrak{g}$, let $\xi^t$ denote the generating vector field, defined in terms of the action on functions $f \in \mathcal{C}^\infty(M)$ by $(\xi^t f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-t\xi)x)$. The Lie group $G$ is itself viewed as a $G$-manifold for the conjugation action.

**Definition 2.1.** A quasi-Hamiltonian $G$-space is a triple $(M, \omega, \Phi)$ consisting of a $G$-manifold $M$, a $G$-invariant 2-form $\omega$ on $M$, and an equivariant map $\Phi : M \to G$, called the moment map, satisfying:

1. $d\omega + \Phi^*\eta = 0$,
2. $i_{\xi^t}\omega + \frac{1}{2}\Phi^*(([\theta^L, \theta^R] \cdot \xi)) = 0$ for all $\xi \in \mathfrak{g}$,
3. at every point $x \in M$, $\ker \omega_x \cap \ker d\Phi_x = \{0\}$.

The fusion product of two quasi-Hamiltonian $G$-spaces $(M_1, \omega_1, \Phi_1)$ and $(M_2, \omega_2, \Phi_2)$ is the product $M_1 \times M_2$, with the diagonal $G$-action, 2-form
\[ \omega = pr_1^*\omega_1 + pr_2^*\omega_2 + \frac{1}{2}pr_1^*\Phi_1^*\theta^L \cdot pr_2^*\Phi_2^*\theta^R, \]
and moment map $\Phi = \Phi_1 \Phi_2$.

The symplectic quotient of a quasi-Hamiltonian $G$-space is the symplectic space $M/G = \Phi^{-1}(e)/G$. Similar to the theory of Hamiltonian group actions, the group unit $e$ is a regular value of $\Phi$ if and only if $G$ acts locally freely on the level set $\Phi^{-1}(e)$, and in this case the pull-back of the 2-form to the level set descends to a
symplectic 2-form on the orbifold $\Phi^{-1}(e)/G$. If $e$ is a singular value, then $M//G$ is a singular symplectic space as defined in [18].

The conjugacy classes $C \subset G$ are basic examples of quasi-Hamiltonian $G$-spaces. The moment map is the inclusion into $G$, and the 2-form $\omega$ is given on generating vector fields by the formula

$$\omega_g(\zeta^\sharp(g), \xi^\sharp(g)) = \frac{1}{2}(\xi \cdot \text{Ad}_g\zeta - \zeta \cdot \text{Ad}_g\xi).$$

Together with the double $D(G) = G \times G$, equipped with diagonal $G$-action and moment map $\Phi(g, h) = g h g^{-1} h^{-1}$, these are the building blocks of the main example appearing in this paper. As shown in [1], the moduli space of flat $G$-bundles over a compact, oriented surface $\Sigma$ of genus $h$ with $s$ boundary components, with boundary holonomies in prescribed conjugacy classes $C_j$, $j = 1, \ldots, s$, is a symplectic quotient of a fusion product:

$$M(\Sigma, C_1, \ldots, C_s) = C_1 \times \cdots \times C_s \times D(G)^h//G.$$

If the group $G$ is simply connected, then the fibers of the moment map for any compact, connected quasi-Hamiltonian $G$-space are connected. In particular, (7) is connected in that case. If $G$ is non-simply connected, the space (7) may have several components.

To clarify the decomposition into components, we use the following construction. Suppose $p: \tilde{G} \to G$ is a homomorphism of compact, connected Lie groups, with finite kernel $Z$. Then $Z$ is a subgroup of the center of $\tilde{G}$, and $G = \tilde{G}/Z$. For any quasi-Hamiltonian $G$-space $(N, \omega, \Phi)$, let $\tilde{N}$ denote the fiber product defined by the pull-back square

$$\begin{array}{ccc}
\tilde{N} & \xrightarrow{\Phi} & \tilde{G} \\
p \downarrow & & \downarrow p \\
N & \xrightarrow{\Phi} & G
\end{array}$$

Then $(\tilde{N}, \tilde{\omega}, \tilde{\Phi})$ is a quasi-Hamiltonian $\tilde{G}$-space, for the diagonal $\tilde{G}$-action on $\tilde{N} \subset N \times \tilde{G}$, and with the 2-form $\tilde{\omega} = p^*_N \omega$. Simple properties of this construction are:

**Proposition 2.2.** (i) We have a canonical identification of symplectic quotients

$$\tilde{N}/\tilde{G} \cong N/G.$$

(ii) For a fusion product $N = N_1 \times \cdots \times N_r$ of quasi-Hamiltonian $G$-spaces, the space $\tilde{N}$ is a quotient of $\tilde{N}_1 \times \cdots \times \tilde{N}_r$ by the group $\{(c_1, \ldots, c_r) \in Z^r \mid \prod_{j=1}^r c_j = e\}$.

(iii) If $\Phi: N \to G$ lifts to a moment map $\Phi': N \to \tilde{G}$, thus turning $N$ into a quasi-Hamiltonian $\tilde{G}$-space $(N, \omega, \Phi')$, then $\tilde{N} = N \times Z$ as a fusion product of quasi-Hamiltonian $\tilde{G}$-spaces. Here $Z$ is viewed as a quasi-Hamiltonian $\tilde{G}$-space, with trivial action and with moment map the inclusion to $\tilde{G}$.

**Proof.** (i) By definition of $\tilde{N}$, the level sets $\Phi^{-1}(\tilde{e})$ and $\Phi^{-1}(e)$ are identified, and the pull-backs of the 2-forms to the level sets coincide. Since central elements in $\tilde{G}$ act trivially on $\tilde{N}$, the orbit spaces $\Phi^{-1}(\tilde{e})/\tilde{G}$ and $\Phi^{-1}(e)/G$ are identified as well.
(ii) Think of the spaces $\tilde{N}_i$ as submanifolds of $N_i \times \tilde{G}$. The canonical map

$$\tilde{N}_1 \times \cdots \times \tilde{N}_r \to \tilde{N}, \ (x_1, g_1, x_2, g_2, \ldots, x_r, g_r) \mapsto (x_1, \ldots, x_r, g_1, \ldots, g_r)$$

is exactly the quotient map by \{ $(c_1, \ldots, c_r) \in \mathbb{Z}^r \mid \prod_{j=1}^r c_j = e$ \}, and it preserves the $\tilde{G}$-actions and 2-forms.

(iii) The map $N \times \mathbb{Z} \to \tilde{N}$, $(x, c) \mapsto (x, \Phi'(x)c)$ is the desired diffeomorphism. □

2.3. The moduli space example. Our main interest is the moduli space of flat SO(3)-bundles with prescribed boundary holonomies, i.e. (7) with $G = SO(3)$. In the notation of the previous Section, we will describe the quasi-Hamiltonian SU(2)-space $\tilde{N}$ associated to the quasi-Hamiltonian SO(3)-space $N = C_1 \times \cdots \times C_s \times D(SO(3))^h$.

Choose conjugacy classes $D_j \in SU(2)$ with $p(D_j) = C_j$, and define a quasi-Hamiltonian SU(2)-space

$$\tilde{M} = D_1 \times \cdots \times D_s \times D(SU(2))^h.$$  

Put

$$M = \tilde{M}/\Gamma,$$

where $\Gamma \subset \mathbb{Z}^{s+2h}$ consists of $\gamma = (\gamma_1, \ldots, \gamma_{s+2h})$ with the properties $\prod_{j=1}^s \gamma_j = e$ and $\gamma_j D_j = D_j$ for $j \leq s$. (Equivalently, $\gamma_j = e$ for all $D_j \neq D_*$.) The conditions guarantee that $\gamma$ acts on $\tilde{M}$, preserving the 2-form and moment map which hence descend to $M = \tilde{M}/\Gamma$. Let $C_* \cong \mathbb{R}P(2)$ be the SO(3)-conjugacy class consisting of rotations by $\pi$. It is the unique SO(3)-conjugacy class whose pre-image in SU(2) is connected. This pre-image is the SU(2)-conjugacy class $D_* \cong S^2$ of matrices of trace 0.

Lemma 2.3. With $N$ as above, we have

$$\tilde{N} \cong \begin{cases} 
M & \text{if } \exists j : C_j = C_* \\
M \times Z & \text{if } \forall j : C_j \neq C_*.
\end{cases}$$

Proof. The moment map $D(SU(2)) \to SU(2)$ (given by Lie group commutator) is invariant under the action of $Z \times Z$, hence it descends to a lift $D(SO(3)) \to SU(2)$ of the commutator map for SO(3). Thus

$$\tilde{D}(SO(3)) = D(SO(3)) \times Z.$$

If $C_j \neq C_*$, the map $D_j \to C_j$ is a diffeomorphism, and defines a lift of the moment map $C_j \to SO(3)$. Hence

$$\tilde{C}_j = D_j \times Z$$

in that case. On the other hand, the conjugacy class $C_*$ satisfies

$$\tilde{C}_* = D_*.$$

With these ingredients, the claim follows from Proposition 2.2. □

We may choose the labeling of the conjugacy classes $C_1, \ldots, C_s$ in such a way that $C_j = C_*$ for $j \leq r$ and $C_j \neq C_*$ for $j > r$. The space (10) is then a fusion product

$$M = M' \times D_{r+1} \times \cdots \times D_s \times D(SO(3))^h,$$

(11)
where $\mathbf{D}(\text{SO}(3))$ is viewed as a quasi-Hamiltonian SU(2)-space (using the canonical lift of the SO(3) moment map, as in the proof of Lemma 2.3), and where

$$M' = (\mathcal{D}_s \times \cdots \times \mathcal{D}_s) / \Gamma'$$

with $r$ factors, and with $\Gamma' = \{(\gamma_1, \ldots, \gamma_r) \in Z^r | \prod \gamma_j = e\}$. Let us describe the 2-form $\omega'$ of the space $M'$, in terms of its pull-back $\tilde{\omega}'$ to the universal cover $\tilde{M} = \mathcal{D}_s \times \cdots \times \mathcal{D}_s$. Since the 2-form on $\mathcal{D}_s$ is just zero, only the fusion terms contribute. By iterative use of the formula (5) for the fusion product, one obtains

$$\tilde{\omega}' = \frac{1}{2} \sum_{i<j} g_i^* \theta^L \cdot \text{Ad}_{g_{i+1} \cdots g_{j-1}}(g_j^* \theta^R),$$

where $g_i: \tilde{M} \to \mathcal{D}_s \subset \text{SU}(2)$ denotes projection onto the $i$-th factor.

3. Quantization of the moduli space of flat $\text{SO}(3)$-bundles

In this section we use localization to compute the quantization of the space $M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\text{SU}(2))^h) / \Gamma$, as an element of the level $k$ fusion ring $R_k(\text{SU}(2))$.

3.1. Pre-quantization. Recall that we fix the inner product $\cdot$ on $\mathfrak{su}(2)$ to be the basic inner product. Then $\eta \in \Omega^3(\text{SU}(2))$ is integral, and represents a generator $x \in H^3(\text{SU}(2); \mathbb{Z}) \cong \mathbb{Z}$. The condition $d\omega + \Phi^* \eta = 0$ from the definition of a quasi-Hamiltonian space says that the pair $(\omega, \eta)$ defines a relative cocycle in $\Omega^3(\Phi)$, the algebraic mapping cone of the pull-back map $\Phi^*: \Omega^*(G) \to \Omega^*(M)$. Let $k \in \mathbb{N}$.

**Definition 3.1.** [13, 15] A level $k$ pre-quantization of a quasi-Hamiltonian SU(2)-space $(M, \omega, \Phi)$ is an integral lift $\alpha \in H^3(\Phi; \mathbb{Z})$ of the class $k[(\omega, \eta)] \in H^3(\Phi; \mathbb{R})$.

A necessary and sufficient condition for the existence of a level $k$ pre-quantization is that for all smooth singular 2-cycles $\Sigma \in Z_2(M)$, and all smooth singular 3-chains $C \in C_3(G)$ such that $\partial C = \Phi(\Sigma)$,

$$k(\int_{\Sigma} \omega + \int_{C} \eta) \in \mathbb{Z}.$$ 

We list some basic properties and examples of level $k$ pre-quantizations.

(a) The set of level $k$ pre-quantizations is a torsor under the torsion group $\text{Tor}(H^2(M; \mathbb{Z}))$ of isomorphism classes of flat line bundles.

(b) The level $k$ pre-quantized conjugacy classes of SU(2) are exactly those of the elements $\exp(\frac{2\pi}{k} \rho)$ with $m = 0, \ldots, k$ [15, Proposition 7.3].

(c) The double $\mathbf{D}(\text{SO}(3))$ (viewed as a quasi-Hamiltonian SU(2)-space) admits a level $k$ pre-quantization if and only if $k$ is even [15, Proposition 7.4].

(d) If $M_1$ and $M_2$ are pre-quantized quasi-Hamiltonian SU(2)-spaces at level $k$, then their fusion product $M_1 \times M_2$ inherits a pre-quantization at level $k$. Conversely, a pre-quantization of the product induces pre-quantizations of the factors. See [13, Proposition 3.8].

(e) A level $k$ pre-quantization of $M$ induces a pre-quantization of the symplectic quotient $M//\text{SU}(2)$, equipped with the $k$-th multiple of the symplectic form.

(f) The long exact sequence in relative cohomology gives a necessary condition $k\Phi^*(x) = 0$ for the existence of a level $k$ pre-quantization. If $H^2(M; \mathbb{R}) = 0$, this condition is also sufficient [13, Proposition 4.2].
if and only if the following conditions are satisfied:

3.2. Pre-quantization of \( M, \omega, \Phi \) from \([15]\) produces a pre-quantization \( \mathbb{Q}(M) \in R_k(\text{SU}(2)) \), an element of the level \( k \) fusion ring. It is obtained as a push-forward in twisted equivariant \( K \)-homology, using the Freed-Hopkins-Teleman theorem \([8]\) to identify \( R_k(\text{SU}(2)) \) with the equivariant twisted \( K \)-homology of \( \text{SU}(2) \) at level \( k+2 \). This is the quasi-Hamiltonian counterpart of the \( \text{Spin}_c \) quantization of an ordinary compact Hamiltonian \( \text{SU}(2) \)-space, which produces an element of \( R(\text{SU}(2)) \) as the equivariant index of a \( \text{Spin}_c \)-Dirac operator with coefficients in an equivariant pre-quantum line bundle. The quantization procedure for quasi-Hamiltonian \( G \)-spaces satisfies properties similar to its Hamiltonian analog. These include

1. compatibility with products, \( \mathbb{Q}(M_1 \times M_2) = \mathbb{Q}(M_1) \mathbb{Q}(M_2) \); and
2. the ‘quantization commutes with reduction’ principle, \( \mathbb{Q}(M/\!/G) = \mathbb{Q}(M)^G \).

Here \( R_k(G) \rightarrow \mathbb{Z} \), \( \tau \rightarrow \tau^G \) is the trace defined by \( \tau^G_m = \delta_m,0 \).

3.2. Pre-quantization of \( M \). Let us now consider level \( k \) pre-quantizations of the quasi-Hamiltonian \( \text{SU}(2) \)-space

\[
M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathcal{D}(\text{SU}(2))^h)/\Gamma
\]

from (10).

**Theorem 3.2.** The quasi-Hamiltonian \( \text{SU}(2) \)-space \( M \) carries a level \( k \) pre-quantization if and only if the following conditions are satisfied:

(i) The conjugacy classes \( \mathcal{D}_j \) are of the form \( \text{SU}(2). \exp(m_j \rho) \) with \( m_j \in \{0, \ldots, k\} \),

(ii) if \( h \geq 1 \), then \( k \in 2\mathbb{N} \),

(iii) if the number of \( \mathcal{D}_r \)-factors is \( r \geq 3 \), then \( k \in 4\mathbb{N} \).

Note that if at least one \( \mathcal{D}_s \)-factor appears, then the first condition requires that \( k \in 2\mathbb{N} \) since \( \mathcal{D}_s = \text{SU}(2). \exp(\frac{1}{2} \rho) \).

**Proof.** Since a level \( k \) pre-quantization of \( M \) induces a level \( k \) pre-quantization of the universal cover \( \tilde{M} \), it is a necessary condition that all \( \mathcal{D}_j \) be pre-quantizable. That is, \( \mathcal{D}_j = \text{SU}(2). \exp(m_j \rho) \) with \( m_j \in \{0, \ldots, k\} \).

Let us enumerate the conjugacy classes in such a way that \( \mathcal{D}_1 = \cdots = \mathcal{D}_r = \mathcal{D}_s \). Using the decomposition (11) and the known pre-quantization conditions (b),(c) for the conjugacy classes \( \mathcal{D}_j \) and the double \( \mathcal{D}(\text{SO}(3)) \), together with the fusion property (d), the proof is reduced to the case \( h = 0, s = r \). We may thus assume \( M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_r)/\Gamma \) with \( r \) factors. If \( r = 1 \) then \( M = \mathcal{D}_s \), which is pre-quantized at level \( k \) if and only if \( k \) is even. Suppose \( r > 1 \). The non-trivial element \( c \in \mathbb{Z} \) acts on \( H^2(\mathcal{D}_s; \mathbb{R}) \cong \mathbb{R} \) as multiplication by \(-1 \). Hence, \( \Gamma \) acts on \( H^2(M; \mathbb{R}) \cong \mathbb{R}^r \) by componentwise sign changes. In particular, the \( \Gamma \)-invariant part is trivial. Since \( \Gamma \) acts freely, it follows that

\[
H^2(M; \mathbb{R}) \cong H^2(\tilde{M}; \mathbb{R})^\Gamma = 0.
\]
Hence, by Property (f), a level $k$ pre-quantization exists if and only if $k\Phi^*(x) = 0$. If $r = 2$, so that $M = (D_s \times D_s)/\mathbb{Z}_2$, Poincaré duality gives that $H^3(M; \mathbb{Z}) \cong \mathbb{Z}_2$; therefore $2\Phi^*(x) = 0$. Hence the condition $k \in 2\mathbb{N}$ is also sufficient if $r = 2$.

It remains to consider the case $r \geq 3$. By Property (g), the condition $k \in 4\mathbb{N}$ is sufficient. Let us show that it is also necessary. Observe that the non-identity component of the normalizer, the circle $Tu_* = N(T) - T$, is a single conjugacy class inside $N(T)$. Since $u_* \in D_s$, it follows that $Tu_* \subset D_s$. Let $X \subset M = D_s \times \cdots \times D_s$ be the 2-torus given as the image of the map

$$T \times T \to \tilde{M}, \quad (h_1, h_2) \mapsto (h_1 u_*, h_2 u_*, h_1 h_2 u_*, u_*, \ldots, u_*),$$

and denote by $X$ its image in $M$. Let $\hat{\omega}_X, \omega_X$ be the pull-backs of the quasi-Hamiltonian 2-forms on $\tilde{X}, X$. Since $Tu_* = u_* T$, we have $\hat{\Phi}(X) = \Phi(X) \subset Tu_*^T$. Since the generator $x \in H^3(SU(2), \mathbb{Z})$ pulls back to zero on this circle (for dimension reasons), the existence of a level $k$ pre-quantization of $M$ requires that $k \int_X \omega_X \in \mathbb{Z}$. Since the projection $\tilde{X} \to X$ is a 4-fold covering, $\int_X \omega_X = \frac{1}{4} \int_{\tilde{X}} \hat{\omega}_X$. Hence it is necessary that $k \int_{\tilde{X}} \hat{\omega}_X \in 4\mathbb{Z}$.

Let $\theta \in \Omega^1(T; t)$ be the Maurer-Cartan form for $T$. From the general formula (12), and using $(hu_*)^\theta = -h^\theta, (hu_*)^\theta = h^\theta$, we obtain

$$\hat{\omega}_X = \frac{1}{2}( -h_1^\theta \wedge h_2^\theta + h_1^\theta \wedge (h_1 h_2)^\theta - h_2^\theta \wedge (h_1 h_2)^\theta) = \frac{1}{2} h_1^\theta \wedge h_2^\theta.$$

Writing elements of $T$ in the form $h = j(e^{2\pi i v})$, we may take $v \in [0, 1]$ as the coordinate on $T \cong \mathbb{R}/\mathbb{Z}$. Since the lattice $\Lambda$ is generated by $2\rho$, we find $h_1^\theta = 2dv_1 \otimes \rho$, hence

$$\hat{\omega}_X = 2\|\rho\|^2 \ dv_1 \wedge dv_2 = dv_1 \wedge dv_2$$

integrates to 1. This gives the condition $k \in 4\mathbb{N}$. \hfill \Box

### 3.3. Fixed point components.

Suppose $M$ is a level $k$ pre-quantized quasi-Hamiltonian SU(2)-space, and let $Q(M) \in R_k(SU(2))$ be its quantization. By [15, Theorem 9.5], the numbers $Q(M)(t)$ with $t = t_l, \ l = 0, \ldots, k$ are given as a sum of contributions from the fixed point manifolds of $t$:

\begin{equation}
Q(M)(t) = \sum_{F \subset M^t} \int_F \frac{\hat{A}(F)}{D_{\mathbb{R}}(\nu_F, t)} \ Ch(\mathbb{L}_F, t)^{1/2}.
\end{equation}

The ingredients of the right hand side will be described below, and explicitly computed in the context of our main example (10). The quantizations of SU(2)-conjugacy classes and of the double $D(SO(3))$ (viewed as a quasi-Hamiltonian SU(2)-space) were computed in [15].

For the remainder of this section, we therefore focus on the case $h = 0$, $s = r \geq 2$, i.e. $M = (D_s \times \cdots \times D_s)/T$.

#### 3.3.1. Fixed point sets of $M$.

We need to determine the components $F \subset M^t$ of the fixed point manifold for $t = t_l, \ l = 0, \ldots, k$, and describe various aspects of $F$ and its normal bundle $\nu_F$. Consider first a general regular element $t \in T^{reg}$. Define the following two submanifolds of $D_s$, labeled by the elements of the center $Z = \{e, c\}$ as follows:

$$Y^{(c)} = D_s \cap T = \{t_*, t_*^{-1}\}, \ Y^{(c)} = Tu_*.$$
Thus $Y^{(c)}$ is the fixed point set of $\text{Ad}(t_*)$, while $Y^{(c)}$ consists of elements satisfying $\text{Ad}(t_*)(g) = cg$. Note that both are $Z$-invariant. For $\gamma = (\gamma_1, \ldots, \gamma_r) \in \Gamma$, consider the $\Gamma$-invariant submanifold
\[
\tilde{F}(\gamma) = Y^{(\gamma_1)} \times \cdots \times Y^{(\gamma_r)}.
\]
and put $F^{(\gamma)} = \tilde{F}(\gamma)/\Gamma$. Let $l(\gamma)$ be the number of $\gamma_i$'s that are equal to $c$. Then $\tilde{F}(\gamma)$ is a disjoint union of $2^{r-l(\gamma)}$ tori of dimension $l(\gamma)$. Let $\varepsilon = (e, \ldots, e)$ denote the group unit in $\Gamma$. If $\gamma \neq \varepsilon$, then $\Gamma$ acts transitively on the set of components of $\tilde{F}(\gamma)$. Hence $F^{(\gamma)}$ is a (connected) torus, and since $|\Gamma| = 2^{r-1}$, it follows that the projection restricts to a $2^{l(\gamma)-1}$-fold covering on each component of $F^{(\gamma)}$. If $\gamma = \varepsilon$, $F^{(\varepsilon)}$ consists of $2^r$ points, and hence $F^{(\varepsilon)}$ consists of two points.

**Proposition 3.3.** The fixed point set of $t \in T_{\text{reg}}$ in $M$ is
\[
M^t = \begin{cases} 
F^{(\varepsilon)} & \text{if } t \notin \{t_*, t_*^{-1}\}, \\
\prod_{\gamma \in \Gamma} F^{(\gamma)} & \text{if } t \in \{t_*, t_*^{-1}\}.
\end{cases}
\]

**Proof.** An element $(g_1, \ldots, g_r) \in \tilde{M}$ maps to a point in $M^t$ if and only if there exists $\gamma = (\gamma_1, \ldots, \gamma_r) \in \Gamma$ with $\text{Ad}(t)g_i = g_i\gamma_i$, for $i = 1, \ldots, r$. If $\gamma_i = e$, this condition gives $g_i = t$, since $t$ is regular. If $\gamma_i = c$, the condition says that $\text{Ad}(g_i^{-1})(t) = \gamma_i t$. Since $t$ is regular, this happens if and only if $t \in \{t_*, t_*^{-1}\}$, with $g_i \in N(T)$ representing the non-trivial Weyl group element. \qed

**3.3.2. The symplectic volume of the components of the fixed point set.** Each $F^{(\gamma)} \subset M^t$ is a quasi-Hamiltonian $T$-space, with moment map the restriction of $\Phi$. (See e.g. [14, Proposition 3.1].) In particular, they are symplectic.

**Lemma 3.4.** The symplectic volume of each component of $F^{(\gamma)}$ is equal to 1. Thus $\text{vol}(F^{(\gamma)}) = 2^{1-l(\gamma)}$.

**Proof.** The construction from [3] associates to any quasi-Hamiltonian $G$-space (with $G$ compact, but possibly disconnected) a Liouville volume, in such a way that the volume of a fusion product is the product of the volumes. If $G = T$, so that the space is symplectic, the Liouville volume coincides with the symplectic volume. For a $G$-conjugacy class $C \cong G/G_g$, the Liouville volume is given by the formula [3, Proposition 3.6]
\[
\text{vol}C = |\det_{g_g}(1 - \text{Ad}_g)|^{1/2} \frac{\text{vol}(G)}{\text{vol}(G_g)},
\]

involving the Riemannian volumes of $G$ and of the stabilizer group $G_g$. The spaces $Y^{(z)}$ for $z \in Z$ can be viewed as conjugacy classes for the group $N(T)$, of elements $t_*$ if $z = e$ and $u_*$ if $z = c$. Application of the formula gives
\[
\text{vol}(Y^{(z)}) = \begin{cases} 
2 & \text{if } z = e, \\
1 & \text{if } z = c.
\end{cases}
\]

This is obvious for $z = e$, while for $z = c$ (so that $g = u_*$, $N(T)_g = \mathbb{Z}_4$) we have $|\det_t(1 - \text{Ad}_{u_*})|^{1/2} = \sqrt{2}$ (since $\text{Ad}_{u_*}$ acts as $-1$ on $t$), $\text{vol}(N(T)) = 2\text{vol}(T) = 2||\alpha|| = 2\sqrt{2}$, and $\text{vol}(N(T)_g) = 4$. It follows that
\[
\text{vol}(\tilde{F}(\gamma)) = \prod_{i=1}^{r} \text{vol}(Y^{(\gamma_i)}) = 2^{r-l(\gamma)}.
\]
Since the moment map for the quasi-Hamiltonian $N(T)$-space $\tilde{F}(\gamma)$ takes values in $T$, this coincides with the symplectic volume. Since $2^{r-l} (\gamma)$ is also the number of components of $\tilde{F}(\gamma)$, it follows that each component has volume 1.

$\square$

### 3.4. Fixed point contributions.

In this Section, we assume that $M = (D_\ast \times \cdots \times D_\ast)/\Gamma$ carries a level $k$ pre-quantization. Thus $k \in 2\mathbb{N}$ if $r = 2$ and $k \in 4\mathbb{N}$ if $r > 2$.

Our aim is to compute the fixed point contributions to $Q(M)(t)$, as described in formula (13), for $t = t_\ast, l = 0, \ldots, k$.

If $t \neq t_\ast$, Proposition 3.3 shows that $M^t = F(\varepsilon)$ consists of just two points, covered by the set $\hat{M}^t = \tilde{F}(\varepsilon)$ (consisting of $2^r$ points). The fixed point contribution of $F(\varepsilon)$ is just that for $\tilde{F}(\varepsilon)$, divided by $|\Gamma| = 2^{r-1}$. Hence

$$Q(M)(t) = 2^{1-r} Q(M^t) = 2^{1-r} Q(D_\ast)^r(t),$$

with $Q(D_\ast) = \tau_{k/2}$ [15, Proposition 11.2].

If $t = t_\ast$, $Q(M)(t_\ast)$ is a sum over the contributions from all $F(\gamma)$, $\gamma \in \Gamma$. The contribution from $F(\gamma)$ is $2^{1-r} (Q(D_\ast)(t_\ast))^r$, as before. Calculation of the contributions from $F = F(\gamma)$, $\gamma \neq \varepsilon$ requires more work:

**Proposition 3.5.** The contribution of the fixed point manifold $F = F(\gamma)$, $\gamma \neq \varepsilon$ to $Q(M)(t_\ast)$ is

$$\int_F \frac{\hat{A}(F)}{D_y(v_F, t_\ast)} \frac{\text{Ch}(L_F, t_\ast)^{1/2}}{\text{Ch}(L_F, t_\ast)^{1/2}} = 2^{1-r} \left(\frac{k}{2} + 1\right)^{(\gamma)/2} \varphi(\gamma),$$

where the scalar $\varphi(\gamma) = \mu_{F(\gamma)}(t_\ast) \in U(1)$ is the action of $t_\ast$ on the pre-quantum line bundle over $F(\gamma)$.

**Proof.** Since $F = F(\gamma)$ is a torus, $\hat{A}(F) = 1$. To compute the $D_y$-class, note that the normal bundle of $Tu_\ast$ in $D_\ast$ is an orientable real line bundle, hence it is trivializable. Consequently, the normal bundle $v_{\tilde{F}(\gamma)}$ to $\tilde{F}(\gamma)$ in $M$ is trivializable, and thus the normal bundle $v_F = v_{\tilde{F}(\gamma)}/\Gamma$ to $F$ in $M$ is a flat Euclidean vector bundle of rank $2r - l(\gamma)$. The element $t_\ast$ acts by multiplication by $-1$ on the fibers of $v_F$, since $\text{Ad}(t_\ast)$ has order 2 and cannot act trivially. By definition of the $D_y$-class (see [2, Section 2.3] or [14, Section 5.3]), it follows that

$$D_y(v_F, t_\ast) = i^{\text{rank}(v_F)/2} \text{det}^{1/2} (1 - (-1)) = (2t)^{-l(\gamma)/2}.$$  

By [15, Proposition 9.3], the restriction $TM|_F$ inherits a distinguished Spin-c-structure (depending on the choice of level $k$ pre-quantization), equivariant for the action of $t_\ast$. The line bundle $L_F \to F$ is the Spin-c-line bundle associated to this Spin-c-structure, and

$$\text{Ch}(L_F, t_\ast)^{1/2} = \sigma(L_F)(t_\ast)^{1/2} \exp\left(\frac{1}{2} \partial_1(L_F)\right)$$

is the square root of its equivariant Chern character, with $\sigma(L_F)(t_\ast) \in U(1)$ the action of $t_\ast$ the Spin-c-line bundle. As discussed in [2, Section 2.3] (see also [14, Section 5.3]), the sign of the square root is determined as follows. Since $\Phi$ restricts to a surjective map $F \to T$, the fixed point set $F$ meets $\Phi^{-1}(e)$. Pick any $x \in F \cap \Phi^{-1}(e)$. Observe that $\omega$ is non-degenerate at points of $\Phi^{-1}(e)$, and choose a $t_\ast$-invariant compatible complex structure to view $T_x M$ as a Hermitian vector space. Let $A \in U(T_x M)$ be the transformation defined by $t_\ast$ and $A^{1/2}$ its unique square root for which all eigenvalues are of the form $e^{iu}$ with $0 \leq u < \pi$. Then

$$\sigma(L_F)(t_\ast)^{1/2} = \varphi(\gamma) \text{det}_{C}(A^{1/2}).$$

Since \( t_* \) acts trivially on \( T_mF \) and as \(-1\) on the normal bundle, the transformation \( A^{1/2} \) acts trivially on \( T_nF \) and as \( i \) on the normal bundle. Thus \( \det_C(A^{1/2}) = i^{r-l(\gamma)/2} \), which cancels a similar factor in the expression for the \( D_2 \)-class.

It remains to find the integral \( \int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F)) \). To this end, we interpret \( \mathcal{L}_F \) as a pre-quantum line bundle. By the same argument as in Property (g) of Section 3.1, (see also [15, Section 11.1]), the level \( k \) pre-quantization and the canonical twisted Spin\(_c\)-structure on \( M \) combine to give an element of \( H^3(\Phi; \mathbb{Z}) \) at level \( 2k+4 \). Since \( H^2(M; \mathbb{R}) = 0 \), this element defines a pre-quantization at level \( 2k+4 \). Pull-back to \( \hat{F} \) defines a level \( 2k+4 \) pre-quantization of \( F \), with \( \mathcal{L}_F \) as the pre-quantum line bundle. Hence \( c_1(\mathcal{L}_F) \) is the \( 2k+4 \)-th multiple of the class of the symplectic form on \( F \). It follows that

\[
\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F)) = (k+2)^{l(\gamma)} \text{vol}(F) = 2^{1-l(\gamma)/2} \left( \frac{4}{2} + 1 \right)^{l(\gamma)/2}
\]

where we have used Lemma 3.4.

The phase factors \( \varphi^{(\gamma)} \) depend on the choice of pre-quantization. Recall again that the set of pre-quantizations of a quasi-Hamiltonian \( \text{SU}(2) \)-space is a torsor under the group of isomorphism classes of flat line bundles. In our case this is the group

\[
\text{Tor}(H^2(M; \mathbb{Z})) \cong \text{Hom}(\Gamma, U(1)).
\]

The homomorphism \( \psi: \Gamma \to U(1) \) defines the flat line bundle \( \hat{M} \times_{\Gamma} C_\psi \), where \( C_\psi \) is the 1-dimensional \( \Gamma \)-representation defined by \( \psi \). Changing the pre-quantization by such a flat line bundle changes \( \varphi^{(\gamma)} \) for \( F = F^{(\gamma)} \) to \( \psi(\gamma)\varphi^{(\gamma)} \).

**Lemma 3.5.** \( \psi: \Gamma \to U(1) \) is a distinguished pre-quantization if and only if \( \psi(\gamma) = 1 \) for all \( \gamma \in \text{Hom}(\Gamma, U(1)) \).

**Lemma 3.6.** If \( r \geq 3 \) and \( k \in 4\mathbb{N} \), the phase factor for the pre-quantization labeled by \( \psi \in \text{Hom}(\Gamma, U(1)) \) is given by

\[
\varphi^{(\gamma)} = (-1)^{\frac{k}{r}} \left( r-l(\gamma)/2 \right) \psi(\gamma).
\]

**Proof.** The phase factor \( \varphi^{(\gamma)} \) for the distinguished pre-quantization at level 4 is given by \( \det_C(A) = (-1)^{-l(\gamma)/2} \), in the notation from the proof of Proposition 3.5. For the distinguished pre-quantization at level \( k \in 4\mathbb{N} \), we have to take the \( \frac{k}{r} \)-th power of this number, and changing the pre-quantization by \( \psi \) we have to multiply by \( \psi(\gamma) \).

If \( r = 2 \), there are \( |\Gamma| = 2 \) distinct pre-quantizations at all even levels \( k \in 2\mathbb{N} \), related by elements \( \psi \in \text{Hom}(\Gamma, U(1)) \). Aside from the discrete fixed point set \( F^{(c)} \), there is a single non-discrete fixed point component \( F^{(\gamma)} \) of \( t_* \), given by \( \gamma = (c, c) \). The non-trivial homomorphism \( \psi \in \text{Hom}(\Gamma, U(1)) \cong \mathbb{Z}_2 \) satisfies \( \psi(c, c) = -1 \), hence the weight \( \varphi^{(\gamma)} \) is equal to 1 for one of the pre-quantizations and \(-1\) for the other.

**3.5. Quantization of \( M \).** We are now ready to summarize our computation of \( \mathcal{Q}(M) \) for \( M = (D_* \times \cdots \times D_*)/\Gamma \). Assuming that \( k \) is even, recall that \( D_* \) has a unique pre-quantization at level \( k \), and \( \mathcal{Q}(D_*) = \tau_{k/2} \). Define an element

\[
\chi = \tau_0 - \tau_2 + \tau_4 - \cdots + (-1)^{k/2} \tau_k \in R_k(\text{SU}(2)).
\]
By the orthogonality relations for $R_k(SU(2))$, this element satisfies $\chi(t) = (\frac{k}{2} + 1)$ and $\chi(t) = 0$ for $t = t_l$, $l \neq k/2$. Hence we may write the sum over the fixed point contributions as follows:

$$Q(M)(t) = 2^{1-r}\left(\tau_{k/2}(t)^r + \chi(t) \sum_{\gamma \in \Gamma \setminus \{\varepsilon\}} (\frac{k}{2} + 1)^{l(\gamma)/2 - 1}\varphi(\gamma)\right)$$

**Theorem 3.7.** Consider the quasi-Hamiltonian $SU(2)$-space $M = (D_1 \times \cdots \times D_4)/\Gamma$ with $r \geq 2$ factors, where $\Gamma \subset Z^r$ consists of all $\gamma = (\gamma_1, \ldots, \gamma_r)$ with $\prod_{i=1}^r \gamma_i = e$.

1. If $r \geq 2$, the space $M$ is pre-quantized at level $k$ if and only if $k \in 4\mathbb{N}$. The different pre-quantizations are indexed by the elements $\psi \in Hom(\Gamma, U(1))$, and the corresponding level $k$ quantization is given by the formula,

$$Q_{\psi}(M) = 2^{1-r}\left((\tau_{k/2})^r + \chi \sum_{\gamma \in \Gamma \setminus \{\varepsilon\}} \psi(\gamma)(\frac{k}{2} + 1)^{l(\gamma)/2 - 1}(-1)^{k/2} = \frac{k}{4}(r-\frac{r}{2})\right).$$

2. If $r = 2$, the space $M$ is pre-quantized at level $k$ if and only if $k \in 2\mathbb{N}$.

At any such level, there are two distinct pre-quantizations indexed by the action $\pm 1$ of $t_*$ on the pre-quantum line bundle over $F(\gamma)$, for $\gamma = (c,e)$. The corresponding level $k$ quantizations of $M$ are

$$Q_{\pm}(M) = \frac{1}{2}((\tau_{k/2})^2 \pm \chi).$$

### 3.6. Multiplicity computations.

Being elements of $R_k(SU(2))$, the coefficients of $Q(M)$ in its decomposition with respect to the basis $\tau_0, \ldots, \tau_k$ must be integers. In this Section, we will compute these multiplicities for small $r$.

#### 3.6.1. $r = 2$ factors.

Assume $k \in 2\mathbb{N}$, and let $Q_{\pm}(M)$ be the quantizations corresponding to the pre-quantizations labeled by $\pm 1$. The multiplication rules for level $k$ characters give

$$(\tau_{k/2})^2 = \tau_0 + \tau_2 + \ldots + \tau_k.$$ 

Hence, if $k \in 4\mathbb{N}$ we obtain

$$Q_{+}(M) = \tau_0 + \tau_4 + \ldots + \tau_k,$$

$$Q_{-}(M) = \tau_2 + \tau_6 + \ldots + \tau_{k-2},$$

while for $k \in 4\mathbb{N} - 2$,

$$Q_{+}(M) = \tau_0 + \tau_4 + \ldots + \tau_{k-2},$$

$$Q_{-}(M) = \tau_2 + \tau_6 + \ldots + \tau_k.$$ 

#### 3.6.2. $r = 3$ factors.

Let $Q_{\psi}(M)$ denote the level $k \in 4\mathbb{N}$ pre-quantization indexed by $\psi \in Hom(\Gamma, U(1))$. Since $r = 3$, $l(\gamma) = 2$ for any $\gamma \neq \varepsilon$ and the quantization formula simplifies to:

$$Q_{\psi}(M) = \frac{1}{4}\left(\tau_{2m}^3 + \chi \sum_{\gamma \neq \varepsilon} \psi(\gamma)\right).$$

For the trivial homomorphism $\psi = 1$, we have $\sum_{\gamma \neq \varepsilon} \psi(\gamma) = 3$, while for a non-trivial homomorphism $\psi \neq 1$, $\sum_{\gamma \neq \varepsilon} \psi(\gamma) = -1$. We have,

$$(\tau_{k/2})^3 = \tau_0 + 3\tau_2 + \ldots + (\frac{k}{2} + 1)\tau_{k/2} + \ldots + 3\tau_{k-2} + \tau_k.$$
We therefore obtain
\[ Q_\psi(M) = (\tau_0 + 2\tau_4 + 3\tau_8 + \ldots + 3\tau_{k-8} + 2\tau_{k-4} + \tau_k) \]
\[ + (\tau_6 + 2\tau_{10} + \ldots + 2\tau_{k-10} + \tau_{k-6}) \]
if \( \psi = 1 \),
\[ Q_\psi(M) = (\tau_0 + 4\tau_4 + 6\tau_8 + \ldots + 3\tau_{k-8} + 2\tau_{k-4} + \tau_k) \]
\[ + (\tau_2 + 2\tau_6 + 3\tau_{10} + \ldots + 3\tau_{k-10} + 2\tau_{k-6} + \tau_{k-2}) \]
if \( \psi \neq 1 \).

Note that the coefficients are symmetric about the midpoint \( \frac{k}{2} \) of the interval \([0, k]\).

In closed form, \( Q_\psi(M) = \sum_{j=0}^{k/2} a_{2j} \tau_{2j} \), where
\[ a_{2j} = \begin{cases} \frac{1}{4}(2j + 1 + 4\delta_j, 1)(-1)^j & : 2j \leq k/2, \\ \frac{1}{4}(k - 2j + 1 + 4\delta_j, 1)(-1)^j & : 2j \geq k/2. \end{cases} \]

3.6.3. \( r = 4 \) factors. If \( r = 4 \) we have \( |\Gamma| = 8 \). There is a unique element \( \gamma' \in \Gamma \) with \( l(\gamma') = 4 \), and \( l(\gamma) = 2 \) for \( \gamma \neq \gamma', \varepsilon \). Hence we may write the quantization formula for levels \( k \in 4\mathbb{N} \) as:
\[ Q_\psi(M) = \frac{1}{8} \left( \tau_{k/2}^4 + \left( \psi(\gamma) \left( \frac{k}{2} + 1 \right) + (-1)^{k/4} \sum_{l(\gamma) = 2} \psi(\gamma) \chi \right) \right). \]

One finds that there are 4 homomorphisms \( \psi \) with \( \sum_{l(\gamma) = 2} \psi(\gamma) = 0 \), \( \psi(\gamma') = -1 \) and 3 homomorphisms with \( \sum_{l(\gamma) = 2} \psi(\gamma) = 2 \). Of course, \( \sum_{l(\gamma) = 2} \psi(\gamma) = 6 \), \( \psi(\gamma') = 1 \) for \( \psi = 1 \). Therefore, we have
\[ Q_\psi(M) = \begin{cases} \frac{1}{8} \left( \tau_{k/2}^4 + \left( 6(-1)^{k/4} + \left( \frac{k}{2} + 1 \right) \right) \chi \right) & : \psi = 1 \\ \frac{1}{8} \left( \tau_{k/2}^4 - \left( \frac{k}{2} + 1 \right) \chi \right) & : \sum_{l(\gamma) = 2} \psi(\gamma) = 0 \\ \frac{1}{8} \left( \tau_{k/2}^4 + (2(-1)^{k/4+1} + \left( \frac{k}{2} + 1 \right) \right) \chi & : \sum_{l(\gamma) = 2} \psi(\gamma) = -2 \end{cases} \]

with
\[ (\tau_{k/2})^4 = \sum_{j=0}^{k/2} (\frac{k}{2} + 1 - 2j^2 + jk) \tau_{2j}. \]

One may verify that the multiplicities of \( \tau_{2j} \) in \( Q_\psi(M) \) are integers, as required.

4. Fuchs-Schweigert

The formulas appearing in Theorem 3.7 may be rewritten in terms of the so-called S-matrix. For \( z \in Z \), define \( S(z) \) by
\[ S(z) = \begin{cases} 1 & \text{if } z = c \\ \tilde{S}(z) & \text{if } z = e. \end{cases} \]

In the terminology of [9], \( S(z) \) is the S-matrix of the orbit Lie algebra associated to the central element \( z \). (This interpretation may seem obscure for SU(2), but becomes natural for higher rank groups.) Consider once again the space \( M = M/\Gamma \) from (10). Recall that \( \Gamma \) consists of elements \( \gamma = (\gamma_1, \ldots, \gamma_{s+2h}) \in Z^{s+2h} \) such that \( \prod_{j=1}^s \gamma_j = e \), and \( \gamma_j = e \) for all \( j \leq s \) with \( C_j \neq C_e \). In particular \( |\Gamma| = 2^{2h+r-1} \) if \( r \geq 1 \), while \( |\Gamma| = 2^{2h} \) if \( r = 0 \). To write the Fuchs-Schweigert formula, it is convenient to use the following notation. For \( \gamma \in \Gamma \), let \( \sum_l^{(\gamma)} \) denote the full sum \( \sum_{l=0}^{k} \gamma_l \) if all \( \gamma_l = e \), and consisting of the single term \( l = \frac{k}{2} \) if at least one \( \gamma_l \neq e \). (For higher rank groups, this becomes a sum over level \( k \) weights that are fixed...
under the action of all $\gamma_i \in \mathbb{Z}$ on the set of level $k$ weights.) We will prove the following equivariant analogue to the Fuchs-Schweigert formula:

**Theorem 4.1.** Suppose the quasi-Hamiltonian $\text{SU}(2)$-space

$$M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathcal{D}(\text{SU}(2))^h) / \Gamma$$

is pre-quantized at level $k$. Then

$$Q(M) = \frac{1}{|\Gamma|} \sum_{\gamma_1, \ldots, \gamma_s} \varphi'(\gamma) \sum_{l} \tilde{S}_{m_1,l}^{(\gamma_1)} \cdots \tilde{S}_{m_s,l}^{(\gamma_s)} \tilde{\tau}_l,$$

where $\varphi'(\gamma) \in \mathbb{U}(1)$ are phase factors depending on the choice of pre-quantization, with $\varphi'(\varepsilon) = 1$.

An explicit description of the phase factors $\varphi'(\gamma)$ will be given during the course of the proof.

**Proof of Theorem 4.1.** The space $M$ is a fusion product of the space $(\mathcal{C}_x)\hat{\tau}$, conjugacy classes $\mathcal{D}_j \neq \mathcal{D}_i$, and $h$ factors of $\mathcal{D}(\text{SO}(3))$ (viewed as a quasi-Hamiltonian $\text{SU}(2)$-space). Since the fusion product in the basis $\tilde{\tau}_m$ is diagonalized, we may verify the formula separately for factors of these three types.

We begin with the case $h = 0, s = r$, with $r \geq 3$ (thus necessarily $k \in 4\mathbb{N}$). We re-write the right hand side of (14), separating the term $\gamma = \varepsilon$ from the sum over terms $\gamma \neq \varepsilon$. The right hand side of (14) becomes

$$Q(M) = \frac{1}{|\Gamma|} \left( \varphi'(\varepsilon) \sum_{l} (S_{k/2,l})^r \tilde{\tau}_l + \sum_{\gamma \neq \varepsilon} \varphi'(\gamma) \frac{(S_{k/2,k/2})^r - \tilde{\tau}_l}{(S_{0,k/2})^r} \tilde{\tau}_{k/2} \right).$$

The sum over $l$ is just $(\tau_{k/2})^r$. The element $\chi \in R_k(\text{SU}(2))$ considered in Section 3.5 satisfies $\chi(t_l) = (\left. \frac{r}{2} \right) + 1)\delta_{l,k/2}$ for $l = 0, \ldots, k$, hence

$$\tilde{\tau}_{k/2} = (\left. \frac{r}{2} \right) + 1)^{-1} \chi.$$

Furthermore, by definition of the $S$-matrix,

$$S_{0,k/2} = (\left. \frac{r}{2} \right) + 1)^{-\frac{r}{2}}, \quad S_{k/2,k/2} = (\left. \frac{r}{2} \right) + 1)^{-\frac{r}{2}}(1)^{-1}.$$

Equation (15) becomes

$$Q(M) = \frac{1}{2^{r-1}} \left( \varphi'(\varepsilon) (\tau_{k/2})^r + \sum_{\gamma \neq \varepsilon} \varphi'(\gamma) (-1)^{\left( \frac{r}{2} - l(\gamma) \right)} \left( (\left. \frac{r}{2} \right) + 1 \right)^{-1} \chi \right)$$

which agrees with Theorem 3.7 for $\varphi'(\gamma) = \psi(\gamma)(-1)^{\frac{k(k-1)}{2}}$.

The calculation is similar for the case $h = 0, s = r = 2, k \in 2\mathbb{N}$. Here, $|\Gamma| = 2$, and the generator $\gamma = (c, e) \in \Gamma$ has $l(\gamma) = 2$. We hence obtain

$$Q(M) = \frac{1}{2} \left( \varphi'(\varepsilon, \varepsilon) \tau_{k/2}^r + \varphi'(c, c) \chi \right)$$

which agrees with Theorem 3.7 if we put $\varphi'(\varepsilon, \varepsilon) = 1$, and $\varphi'(c, c) = \pm 1$. If $h = 0$ and $s = r = 1, k \in 2\mathbb{N}$, then $\Gamma = \{\varepsilon\}$, and the formula becomes $Q(M) = \varphi'(\varepsilon)\tau_{k/2}$, which is the correct expression for $Q(\mathcal{D}_s)$ for $\varphi'(\varepsilon) = 1$. Similarly, if $h = r = 0, s = 1$ so that $M$ is a conjugacy class $\mathcal{D}_j \neq \mathcal{D}_s$, the formula reduces to $Q(M) = \tau_m = Q(\mathcal{D}_s)$.

Consider finally the case $h = 1, s = 0$ so that $M = \mathcal{D}(\text{SO}(3))$. Pre-quantizability of this space requires $k \in 2\mathbb{N}$, and as shown in [15] the distinct pre-quantizations
are indexed by $\varphi \in \text{Hom}(\Gamma, U(1))$, with $\Gamma = \mathbb{Z} \times \mathbb{Z}$. Separating off the term $(e, e)$, (14) becomes
\[
Q(M) = \frac{1}{4} \left( \varphi'(\epsilon) \sum_l \frac{1}{S_{0,l}^2} \bar{\tau}_l + \sum_{\gamma \neq (e, e)} \varphi'(\gamma) \frac{1}{S_{0,k/2}^2} \bar{\tau}_{k/2} \right).
\]
We have $\frac{1}{S_{0,k/2}^2} \bar{\tau}_{k/2} = \chi$, and
\[
Q(D(SU(2))) = \sum_m \tau_m^2 = \sum_{l,m} S_{m,l}^2 \bar{\tau}_l = \sum_l \frac{1}{S_{0,l}^2} \bar{\tau}_l,
\]
where we use the symmetry and orthogonality of the $S$-matrix. Thus the formula may be re-written
\[
Q(M) = \frac{1}{4} \left( \varphi'(\epsilon) Q(D(SU(2))) + \sum_{\gamma \neq (e, e)} \varphi'(\gamma) \chi \right).
\]
This agrees with the formula for $Q(D(SO(3))$ given in [15, Section 11.4] if one puts $\varphi'(e, e) = 1$ and $\varphi'(\gamma) = (-1)^{k/2} \varphi(\gamma)$ for $\gamma \neq (e, e)$.

By combining this result with the ‘quantization commutes with reduction’ theorem for quasi-Hamiltonian spaces [15, Theorem 10.1], and since the coefficient of $\tau_0$ in $\bar{\tau}_l$ is $S_{0,l}^2$, we obtain the Fuchs-Schweigert formula [9] for the $SO(3)$ moduli space $\mathcal{M} (\Sigma, C_1, \ldots, C_s)$, where $\Sigma$ is of genus $h$ with $s$ boundary components. Recall that this moduli space has up two 2 connected components, of the form $\mathcal{M} / \! / SU(2)$ for suitable choice of lifts $D_j$. We have,
\[
(16) \quad Q(M / \! / SU(2)) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \varphi'(\gamma) \sum_l \frac{S^{(\gamma_1)}_{m_1,l} \cdots S^{(\gamma_s)}_{m_s,l}}{(S_{0,l})^{s+2h-2}}.
\]

Remark 4.2. The above Fuchs-Schweigert type formula computes the quantization of the moduli space of $SO(3)$-bundles interpreted as the index of a pre-quantum line bundle, while the original conjecture in [9] concerns the dimension of the space of conformal blocks. It is expected that, just as in the case of simply-connected groups, the space of conformal blocks can be re-interpreted as the space of holomorphic sections, and that a Kodaira vanishing result can further identify its dimension with the index considered here. We are not aware of a reference addressing such questions in generality for non-simply connected groups.

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