On the quotient of projective frame space and the Desargues theorem

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Abstract

We consider an \( n \)-dimensional projective space \( \mathbb{P}_n \) \((n \geq 2)\) and a fixed point \( A \) on it. Let \( F(\mathbb{P}_n) \) be the manifold of all the projective frames of \( \mathbb{P}_n \) having \( A \) as their first vertex. We define the action of \( G = St_A \subset GP(n) \) on \( F(\mathbb{P}_n) \) in a natural way. The Lie group epimorphism \( \beta: G \to GL(V) \) acts as follows \( g \mapsto d_A g \) where \( V = T_A \mathbb{P}_n \). We study the geometry of orbit space \( \Phi(\mathbb{P}_n) \) of projective frame space \( F(\mathbb{P}_n) \) under the action of the kernel \( H \) of this epimorphism \( \beta \). By applying some \( n \)-dimensional version of the Desargues theorem we could get a purely geometrical description of such \( H \)-orbits.

1 Introduction

According to the prolongations and scopes method [7] of studying geometrical structures on submanifolds immersed into homogeneous spaces (\( \mathbb{P}_n \) in our case) frame bundles over such submanifolds and quotients of these bundles are considered (see, e.g., [1], [3], [9], [10]). Such principal bundles and their structure groups are studied purely in terms of structure equations of their Maurer – Cartan forms [6] because it is sufficient for obtaining local results considered in these papers. But there is a natural question about the explicit description of such quotient bundles in purely geometrical terms.

In the paper we restrict our attention to one of the simplest cases. Namely, let’s consider a projective \( n \)-dimensional space \( \mathbb{P}_n \) together with its distinguished point \( A \in \mathbb{P}_n \). Let \( G \) be the stabilizer of \( A \) in the projective transformation group of \( \mathbb{P}_n \). The structure equations of \( G \) are the following.

\[
d\omega^i_j = \omega^j_k \wedge \omega^i_k, \quad d\omega_i = \omega^k_i \wedge \omega_k,
\]
where $\omega^i_j, \omega_i$ are the Maurer – Cartan forms of $G$. From these equations it follows that there exists a quotient group $\hat{G}$ of $G$ isomorphic (at least locally) to $GL(n)$. But it is not easy to describe its action on geometric images (i.e. points, frames etc.). In particular one can ask the following question: let $H$ be the corresponding normal subgroup, how to describe $H$-orbits of projective frames? Its importance comes from the following facts. Firstly, as we will see further, these $H$-orbits can be identified with bases of some vector space. And, secondly, any action of elements of $G$ sends $H$-orbit as a whole object to another $H$-orbit.

2 Basic concepts and claims

According to the approach proposed in [4] we start with the following definition.

Definition 1 Let $V_{n+1}$ be $(n + 1)$-dimensional vector space and let $\sim$ be the collinearity relation on $V^0_{n+1} = V_{n+1}\{\vec{0}\}$. Then an n-dimensional projective space is $\mathbb{P}_n$ is the quotient space $\mathbb{P}_n = V^0_{n+1}/\sim$. The vector space $V_{n+1}$ is said to be associated with $\mathbb{P}_n$. The canonical surjection $\pi: V^0_{n+1} \rightarrow \mathbb{P}_n$ sends each vector $\vec{a} \in V^0_{n+1}$ to its equivalence class $[\vec{a}]$.

Definition 2 A projective frame in $\mathbb{P}_n$ is an ordered set $\mathcal{R}$ of $n + 2$ points $A_0, A_1, \ldots, A_n, E \in \mathbb{P}_n$ such that any $n + 1$ points of $\mathcal{R}$ are in generic position:

$$\mathcal{R} = \{A_0, A_1, \ldots, A_n, E\}.$$  

Definition 3 Say a basis

$$\vec{\mathcal{R}} = \{\vec{A}_0, \vec{A}_1, \ldots, \vec{A}_n\}$$

of $V_{n+1}$ generates the frame (1) iff

$$\pi(\vec{A}_0) = A_0, \pi(\vec{A}_1) = A_1, \ldots, \pi(\vec{A}_n) = A_n,$$

$$\pi(\vec{A}_0 + \vec{A}_1 + \ldots + \vec{A}_n) = E.$$ 

Let us distinguish some point $A \in \mathbb{P}_n$ and call it a center of $\mathbb{P}_n$.

Definition 4 We say a projective frame $\mathcal{R} = \{A_0, A_1, \ldots, A_n, E\}$ to be adapted (or, equivalently, centroprojective) if its first vertex $A_0$ coincides with the center, i.e. $A_0 = A$. 

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Definition 5  Homogeneous coordinates of a point $M$ w.r.t. a frame $\mathcal{R}$ are the coordinates of some vector $\vec{M} \in \pi^{-1}(M)$ w.r.t. a basis $\vec{\mathcal{R}}$ generating the frame $\mathcal{R}$:

$$M(x^0: x^1: \ldots : x^n)_{\mathcal{R}} \iff \vec{M} = x^0 \vec{A}_0 + x^1 \vec{A}_1 + \ldots + x^n \vec{A}_n.$$  

By $U_{\mathcal{R}}$ denote the open subset of $\mathbb{P}_n$ given by $x^0 \neq 0$. Affine (non-homogeneous) coordinates of the point $M \in U_{\mathcal{R}}$ w.r.t. the frame $\mathcal{R}$ are the following quotients:

$$X^i = \frac{x^i}{x^0}, \quad i = 1, n.$$  

Remark. The point $A$ is given by $(x^0 : 0 : \ldots : 0), x^0 \neq 0$, for any adapted frame $\mathcal{R}$, so $U_{\mathcal{R}}$ is an open neighborhood of $A$.

Definition 6  By $\varphi_{\mathcal{R}}$ we denote the affine chart on $U_{\mathcal{R}}$, i.e. the mapping sending each point $M \in U_{\mathcal{R}}$ to its non-homogeneous coordinates:

$$\varphi_{\mathcal{R}} : U_{\mathcal{R}} \to \mathbb{R}^n, \quad M \mapsto (X^1, \ldots, X^n).$$  

Proposition 1  Let $\mathcal{R}$ and $\mathcal{R}'$ be some frames such that $\mathcal{R}$ is adapted. Then the following conditions are equivalent:

1) $\mathcal{R}'$ is adapted also;

2) for any bases $\vec{\mathcal{R}}$ and $\vec{\mathcal{R}}'$ generating these frames there exist coefficients $a_0^0, a_i^j, a_j^i (i, j = 1, n)$ such that

$$\vec{A}_0' = a_0^0 \vec{A}_0, \quad \vec{A}_i' = a_i^0 \vec{A}_0 + a_i^j \vec{A}_j, \quad a_0^0 \neq 0, \quad \det(a_i^j) \neq 0;$$

3) there exist unique coefficients $\alpha_i, \alpha_j^i$ such that for any point $M$ its affine coordinates change under the law

$$X^i = \frac{\alpha_j^i X^j}{1 + \alpha_j X^j}. \tag{2}$$

Definition 7  The equations (2) are called transition equations of a pair $(\mathcal{R}, \mathcal{R}')$.

Definition 8  The mapping $f : \mathbb{P}_n \to \mathbb{P}_n$ is called a projective transformation $\iff$ there exists a non-degenerate linear operator $u : V_{n+1} \to V_{n+1}$ such that the following diagram commutes:

$$\begin{array}{ccc}
V_{n+1}^0 & \xrightarrow{u} & V_{n+1}^0 \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{P}_n & \xrightarrow{f} & \mathbb{P}_n
\end{array}$$
Remark. $u$ is determined by $f$ up to a non-zero scalar multiple.

We introduce the following notation:

| Notation            | Description                                           |
|---------------------|-------------------------------------------------------|
| $F(P_n)$            | the set of all the adapted projective frames          |
| $V$                 | the tangent space $T_A(\mathbb{P}_n)$ to $\mathbb{P}_n$ at $A$ |
| $F(V)$              | the set of all the linear frames (i.e. bases) of $V$  |
| $GP(n)$             | the projective transformation group of $\mathbb{P}_n$ |
| $G$                 | the stabilizer of $A$ in $GP(n)$                     |
| $GL(V)$             | the group of non-degenerate linear operators in $V$   |

Let $R \in F(P_n)$ and $\varepsilon \in F(V)$, where

$$R = \{A_0, A_1, \ldots, A_n, E\}, \quad \varepsilon = \{\vec{e}_1, \ldots, \vec{e}_n\}.$$

We denote

$$g \cdot R = \{g(A_0), g(A_1), \ldots, g(A_n), g(E)\}, \quad g \in G;$$

$$\psi \cdot \varepsilon = \{\psi(\vec{e}_1), \ldots, \psi(\vec{e}_n)\}, \quad \psi \in GL(V).$$

**Proposition 2**  
1) $g \cdot R \in F(\mathbb{P}_n)$ for any $g \in G, R \in F(P_n)$.  
2) For any two frames $R, R' \in F(\mathbb{P}_n)$ there exists a unique $g \in G$ such that $R' = g \cdot R$.

**Definition 9** Let $g \in G, R \in F(\mathbb{P}_n)$. Equations of $g$ w.r.t. $R$ are the transition equations of the pair $(R, g \cdot R)$.

**Definition 10** A frame action of $G$ on $F(P_n)$ is the action

$$q_P : G \times F(P_n) \to F(P_n)$$

defined as follows:

$$(f, R) \mapsto f \cdot R, \quad f \in G, \quad R \in F(P_n).$$

**Definition 11** A frame action of $GL(V)$ on $F(V)$ is the action

$$q_V : G \times F(V) \to F(V)$$

defined as follows:

$$(\psi, \varepsilon) \mapsto \psi \cdot \varepsilon, \quad f \in G, \quad R \in F(P_n).$$

**Proposition 3** The mappings $q_P$ and $q_V$ are well-defined free transitive smooth actions of the Lie groups $G$ and $GL(V)$ on $F(P_n)$ and $F(V)$ respectively.
3  \(H\)-orbits and quotient frame space

**Definition 12** A linearizing mapping is the mapping \(\beta: G \rightarrow GL(V)\) acting as follows
\[
\beta: f \mapsto d_A f.
\]

**Lemma 1** (see [5]) Let \(\varphi: G \rightarrow G'\) be a continuous homomorphism of Lie groups and let \(H = \ker \varphi\). Then \(H\) is a properly embedded, normal Lie subgroup of \(G\), \(G/H\) is canonically a Lie group, and the induced map \(\bar{\varphi}: G/H \rightarrow G'\) is an injective immersion of this Lie group as a Lie subgroup of \(G'\).

**Proposition 4** 1) \(\beta\) is a Lie group epimorphism.
2) The kernel \(H := \ker \beta\) is a closed normal subgroup of \(G\).
3) \(\bar{\beta}: fH \mapsto d_A f\) is a canonical Lie group isomorphism between \(G/H\) and \(GL(V)\).

**Remark.** This proposition allows us not to distinguish the Lie groups \(G/H\) and \(GL(V)\).

**Definition 13** We say the group \(\bar{G} := G/H = GL(V)\) is a linear quotient group of \(G\).

**Proposition 5** Let \(g \in G\), \(R \in F(\mathbb{P}_n)\), and let (2) be the equations of \(g\) w.r.t. \(R\). Then
\[
g \in H \iff \alpha_j^i = \delta_j^i \iff X^i = \frac{\tilde{X}^i}{1 + \alpha_j X^j}.
\]

**Definition 14** Two frames \(R\) and \(R'\) are said to be equivalent \((R \sim R')\) if they belong to the same \(H\)-orbit. We denote by \([R]\) the equivalence class of the frame \(R\), by \(\Phi(\mathbb{P}_n)\) the set of all such equivalence classes, and by \(p: F(\mathbb{P}_n) \rightarrow \Phi(\mathbb{P}_n)\) denote the canonical projection \(R \mapsto [R]\).

**Proposition 6** \(p\) is a surjective submersion.

**Proposition 7** Every action of \(G\) on \(F(\mathbb{P}_n)\) sends \(H\)-orbits to \(H\)-orbits, i.e.
\[
(\forall f \in G)(\forall R, R' \in F(\mathbb{P}_n))(R \sim R' \Rightarrow f \cdot R \sim f \cdot R').
\]
Definition 15 A quotient action of $G$ on $\Phi(P_n)$ is the action
\[ \gamma: G \times \Phi(P_n) \to \Phi(P_n) \]
defined as follows:
\[ f \cdot [R] = [f \cdot R], \quad f \in G, \ R \in F(P_n). \]

Proposition 8 Any two elements $f, g \in G$ belonging to the same $H$-coset act on $\Phi(P_n)$ by the same way i.e.
\[ (\forall f, g \in G)(\forall R \in F(P_n))(f \in gH \Rightarrow f \cdot [R] = g \cdot [R]). \]

Definition 16 A quotient action of $\bar{G}$ on $\Phi(P_n)$ is the action $\bar{\gamma}: \bar{G} \times \Phi(P_n) \to \Phi(P_n)$ defined as follows:
\[ gH \cdot [R] = [g \cdot R], \quad g \in G, \ R \in F(P_n). \]

Proposition 9 The following diagram commutes for any $g \in G$:
\[
\begin{array}{c}
F(P_n) \xrightarrow{g} F(P_n) \\
p \downarrow \quad \downarrow p \\
\Phi(P_n) \xrightarrow{gH} \Phi(P_n)
\end{array}
\]

Proof follows immediately from Definition 16. □

Proposition 10 $\bar{\gamma}$ is free, transitive and smooth.

4 Isomorphism of $\bar{G}$-spaces $\Phi(\mathbb{P}_n)$ and $F(V)$

Definition 17 Let $\alpha: F(\mathbb{P}_n) \to F(V)$ be the map acting as follows:
\[ \alpha: R \mapsto \varepsilon(\varphi_R) \]
where $\varepsilon(\varphi_R)$ is the natural basis of the space $V$ generated by $\varphi_R$. 
Remark. The mapping $\alpha$ is a surjective submersion.

**Proposition 11** The following diagram commutes for any $g \in G$:

\[
\begin{array}{ccc}
F(\mathbb{P}_n) & \xrightarrow{g} & F(\mathbb{P}_n) \\
\downarrow \alpha & & \downarrow \alpha \\
F(V) & \xrightarrow{d_A g} & F(V)
\end{array}
\]

**Proposition 12** Equivalence classes of frames are exactly preimages under the map $\alpha$, i.e.

\[
(\forall R, R' \in F(\mathbb{P}_n))(R \sim R' \iff \alpha(R) = \alpha(R')).
\]

**Definition 18** Let $\bar{\alpha}: \Phi(\mathbb{P}_n) \to F(V)$ be the map acting as follows:

\[
\bar{\alpha}: [\mathcal{R}] \mapsto \varepsilon(\varphi_{\mathcal{R}}).
\]

**Proposition 13** The following diagram commutes:

\[
\begin{array}{ccc}
F(\mathbb{P}_n) & \xrightarrow{\alpha} & F(V) \\
p & & \downarrow \bar{\alpha} \\
\Phi(\mathbb{P}_n)
\end{array}
\]

Proof follows immediately from Definitions 14, 17 and 18. □

**Proposition 14** $\bar{\alpha}$ is a diffeomorphism.

**Proposition 15** The following diagram commutes for any $g \in G$:

\[
\begin{array}{ccc}
\Phi(\mathbb{P}_n) & \xrightarrow{g H} & \Phi(\mathbb{P}_n) \\
\downarrow p & & \downarrow p \\
F(\mathbb{P}_n) & \xrightarrow{\alpha} & F(V) \\
\downarrow \alpha & & \downarrow \alpha \\
F(V) & \xrightarrow{d_A g} & F(V)
\end{array}
\]

**Theorem 1** $\bar{\alpha}: \Phi(\mathbb{P}_n) \to F(V)$ is an isomorphism of $\bar{G}$-spaces.

Proof follows immediately from Propositions 15, 6 and 14. □
5 Perspectivity and the Desargues theorem

Further on we restrict ourselves to the case \( n \geq 2 \). Let \( \mathcal{R}, \mathcal{R}' \in F(P_n) \), where
\[
\mathcal{R} = \{A_0, A_1, \ldots, A_n, E\}, \quad \mathcal{R}' = \{A'_0, A'_1, \ldots, A'_n, E'\}.
\]
Recall that \( A_0 = A'_0 = A \). Consider two bases \( \mathcal{R} \) and \( \mathcal{R}' \) generating the frames \( \mathcal{R} \) and \( \mathcal{R}' \) respectively:
\[
\mathcal{R} = \{\vec{A}_0, \vec{A}_1, \ldots, \vec{A}_n\}, \quad \mathcal{R}' = \{\vec{A}'_0, \vec{A}'_1, \ldots, \vec{A}'_n\}.
\]

**Definition 19** Two frames \( \mathcal{R} \) and \( \mathcal{R}' \) are said to be in perspective if
\[
A'_i \in A_i A_0 \quad (i = 1, n), \quad E' \in EA_0.
\]

**Proposition 16** Let \( \mathcal{R} \) and \( \mathcal{R}' \) be any adapted frames. Then the following conditions are equivalent:
1) \( \mathcal{R} \) and \( \mathcal{R}' \) are in perspective;
2) for any bases \( \mathcal{R} \) and \( \mathcal{R}' \) generating these frames there exist coefficients \( b_0^0, b_i^0, c_0^0 \) such that
\[
\vec{A}'_0 = b_0^0 \vec{A}_0, \quad \vec{A}'_i = b_i^0 \vec{A}_0 + c_0^0 \vec{A}_i, \quad b_0^0 \neq 0, \quad c_0^0 \neq 0, \quad i = 1, n.
\]
3) for any basis \( \mathcal{R} \) generating \( \mathcal{R} \) there are unique numbers \( a_1, \ldots, a_n, h \neq 0 \) and a unique basis \( \mathcal{R}'' \) generating \( \mathcal{R}' \) such that the following equalities are hold:
\[
\vec{A}''_0 = \vec{A}_0, \quad \vec{A}''_i = a_i \vec{A}_0 + h \vec{A}_i, \quad i = 1, n.
\]
4) there exist unique coefficients \( h \neq 0, a_1, \ldots, a_n \) such that for any point \( M \) its affine coordinates change under the law
\[
X^i = \frac{h X^i}{1 + a_j X^j}.
\]

**Definition 20** Transform coefficients of a pair \( (\mathcal{R}, \mathcal{R}') \) of frames in perspective are the numbers \( a_1, \ldots, a_n, h \) determined in Proposition 16.
Remark 1. Proposition 16 implies that the transform coefficients don’t depend on the choice of \( \mathcal{R} \). So they are completely determined by the pair \((\mathcal{R}, \mathcal{R}')\).

Remark 2. In a particular case \( a_i = 0 \) for all \( i = \overline{1, n} \) the transformation (6) is homothetic in the affine chart \((\varphi_{\mathcal{R}}, U_{\mathcal{R}})\).

Proposition 17 Let \( \mathcal{R} \) and \( \mathcal{R}' \) be any adapted frames. Then the following conditions are equivalent:

1) \( \mathcal{R} \sim \mathcal{R}' \);

2) for any bases \( \mathcal{R} \) and \( \mathcal{R}' \) generating these frames there exist coefficients \( a_0^0, a_i^0 \) such that

\[
\tilde{A}_0 = a_0^0 \bar{A}_0, \quad \tilde{A}_i = a_i^0 \bar{A}_0 + a_0^0 \tilde{A}_i, \quad a_0^0 \neq 0.
\]

3) for any basis \( \bar{\mathcal{R}} \) generating \( \mathcal{R} \) there are unique numbers \( a_1, \ldots, a_n \), and a unique basis \( \bar{\mathcal{R}}'' \) generating \( \mathcal{R}' \) such that the following equalities are hold:

\[
\tilde{A}_0'' = \tilde{A}_0, \quad \tilde{A}_i'' = a_i \tilde{A}_0 + \tilde{A}_i, \quad i = \overline{1, n};
\]

4) there exist unique coefficients \( a_1, \ldots, a_n \) such that for any point \( M \) its affine coordinates change under the law

\[
X^i = \frac{\tilde{X}^i}{1 + a_j X^j}.
\]

Theorem 2 \( \mathcal{R} \sim \mathcal{R}' \Leftrightarrow \mathcal{R} \) and \( \mathcal{R}' \) are in perspective & the canonical coefficients of the pair \((\mathcal{R}, \mathcal{R}')\) satisfy the following condition:

\[
h = 1.
\]

Proof follows immediately from (6) and (7). \( \square \)

Definition 21 We say the frames \( \mathcal{R} \) and \( \mathcal{R}' \) to be in strict perspective if they are in perspective and their corresponding points are not coincide, i.e. \( A'_i \neq A_i \) \((i = \overline{1, n})\), \( E' \neq E. \)

For any two frames \( \mathcal{R} \) and \( \mathcal{R}' \) in strict perspective the minimal subspace (with respect to inclusion) \( \mathcal{L}_{(\mathcal{R}, \mathcal{R}')} \subset P_n \) containing the set of points \( B_{ij}, B_i \) is defined, where

\[
B_{ij} = A_i A_j \cap A_i' A_j', \quad B_i = A_i E \cap A_i' E', \quad 1 \leq i < j \leq n.
\]
Lemma 2 (the Desargues theorem, classical version) Let $A_1B_1$, $A_2B_2$ and $A_3B_3$ be three concurrent lines on $\mathbb{P}_2$. Then the points

$$C_{12} = A_1A_2 \cap B_1B_2, \quad C_{13} = A_1A_3 \cap B_1B_3, \quad C_{23} = A_2A_3 \cap B_2B_3$$

are lying on a straight line.

Proof see, e.g., in [8]. □

Lemma 3 (the Desargues theorem, frame version) Let $\mathcal{R}, \mathcal{R}' \in \mathbb{P}_2$ be two frames in strict perspective. Then $L(\mathcal{R}, \mathcal{R}')$ is a straight line.

Theorem 3 $L(\mathcal{R}, \mathcal{R}') \subset \mathbb{P}_n$ is a hyperplane in $\mathbb{P}_n$ for any two frames $\mathcal{R}$ and $\mathcal{R}'$ in strict perspective.

Proof is based on [2]. In the case $n = 2$ the statement is just Lemma 3. Further on, we shall assume that $n > 2$. For any collection of subsets $Y_1, \ldots, Y_s \subset \mathbb{P}_n$ we denote by $\langle Y_1, \ldots, Y_s \rangle$ the minimal plane in $\mathbb{P}_n$ containing all of them. Then for $1 \leq i < j \leq n$ we have

$$B_{ij} \in A_iA_j \subset \langle A_1, \ldots, A_n \rangle =: \mathcal{M},$$

$$B_{ij} \in A'_iA'_j \subset \langle A'_1, \ldots, A'_n \rangle =: \mathcal{M'}.$$

Let $\mathcal{N} = \mathcal{M} \cap \mathcal{M}'$. Then $B_{ij} \in \mathcal{N}$ and dim $\mathcal{N} = n - 2$ due to the conditions of the theorem. Consider 3-plane $L_{ij} := \langle A_0A_i, A_0A_j, A_0E \rangle$. The 2-planes $\langle A_i, A_j, E \rangle$ and $\langle A'_i, A'_j, E' \rangle$ are lying on it. Their intersection is a line containing the points $B_{ij}, B_i$ and $B_j$. So, the point $B_{ij}$ is lying on the line $B_iB_j$ for any $i < j$. Obviously $B_i \notin \mathcal{N}$, and therefore dim $S = n - 1$ where $S := \langle B_1, \mathcal{N} \rangle$, and for any $j > 1$ we have $B_j \in S$. So, for any $i, j$ such that $1 \leq i < j \leq n$ we have $B_{ij} \in S$, $B_i \in S$. Thus, $L(\mathcal{R}, \mathcal{R'}) \subset S$. Obviously, the opposite inclusion is also hold. □

Definition 22 We say $L(\mathcal{R}, \mathcal{R'})$ to be the Desargues hyperplane generated by $\mathcal{R}$ and $\mathcal{R'}$.

6 Geometrical description of $H$-orbits

Theorem 4 Let $\mathcal{R}$ and $\mathcal{R}'$ be in strict perspective. Then they are equivalent iff the Desargues hyperplane $L(\mathcal{R}, \mathcal{R'})$ is passing through $A$.  

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**Proof.** Let \( R \) and \( R' \) be in strict perspective. Then they are in perspective, and according to Proposition 16 there exist bases \( \mathbf{R} \) and \( \mathbf{R}' \) generating these frames such that for some numbers \( a_1, \ldots, a_n, h \) (\( h \neq 0 \)) the equalities (5) hold. Let \( (x^0 : x^1 : \ldots : x^n) \) be the homogeneous coordinates on \( \mathbb{P}_n \) with respect to \( R \). In these coordinates hyperplanes \( L \) and \( L' \) are given by the following equations:

\[
L: x^0 = 0, \quad L': a_i x^i - h x^0 = 0.
\]

Thus, the equation of the bunch \( S \) of hyperplanes \( S(\lambda: \mu) \) passing through \( L \cap L' \) one can present as follows

\[
S(\lambda: \mu): \lambda x^0 + \mu a_i x^i = 0, \quad \lambda^2 + \mu^2 \neq 0.
\]

For the point \( B_1 \) there exists a vector \( \bar{B}_1 \in \pi^{-1}(B_1) \) such that

\[
\bar{B}_1 = -\frac{e}{a_1} \bar{A}_1 + \bar{E}, \quad e := a_1 + \ldots + a_n + 1 - h.
\]

Hyperplane \( L(R,R') \) is distinguished from the bunch by the condition \( B_1 \in L(R,R') \). The latter imposes the relation on \( \lambda \) and \( \mu \):

\[
\lambda = (1 - h)\mu.
\]

We substitute this into (9) and obtain the equation of \( L(R,R') \):

\[
L(R,R'): (1 - h)x^0 + a_i x^i = 0, \quad \lambda^2 + \mu^2 \neq 0.
\]

Therefore

\[
A_0 \in L(R,R') \quad \text{if and only if} \quad h = 1 \quad \text{and} \quad \mathcal{R} \sim \mathcal{R}'. \quad \square
\]

### 7 Conclusion

The theorem 4 shows the existence of surprising relation between the Desargues theorem and the geometry of \( H \)-orbits.

We can distinguish some applications of the results above:

- explicit construction of quotient bundles of the adapted frame bundle over a submanifold \( S \subset \mathbb{P}_n \);

- geometrical description of linear connections on a submanifold \( S \subset \mathbb{P}_n \).

One of the further directions of research is studying relations between other classical theorems of projective geometry and the representations group theory.
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