LIMITING MEASURES OF SUPERSINGULARITIES

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Abstract. Conditional on some combinatorial conjectures, we prove that the measures of supersingularities of level \( Np \) oldforms tend to the zero measure on the interval \( \left( \frac{1}{p+1}, \frac{p}{p+1} \right) \) when \( p \) is coprime to \( 6N \) and \( \Gamma_0(N) \)-regular. We do this by asymptotically sharpening a local theorem by Berger, Li, and Zhu for primes bigger than 3 and weights not congruent to 1 modulo \( p+1 \).

1. Introduction

Let \( p > 3 \) be a prime number and \( k \in \mathbb{Z}_{\geq 2} \) be a positive integer. The goal of this article is to extend the computations in [Ars] and as a result compute the modulo \( p \) representations \( V_{k,a} \) for \( a \in \mathbb{Z}_p \) with “large” valuation. To avoid reintroducing cumbersome definitions, we use the same notation as in sections 1, 2, 3, 4, and 6 of [Ars], the only difference being that we do not assume that \( \nu < \frac{p-1}{2} \)—instead we consider slopes that are large with respect to the weight \( k \). To be more specific, throughout the article we fix \( a \in \mathbb{Z}_p \) such that \( v_p(a) > \left\lfloor \frac{k-1}{p+1} \right\rfloor \). The following is the main theorem in [BLZ04] (which is true for the primes 2 and 3 as well).

**Theorem 1** (Berger, Li, Zhu). If \( v_p(a) > \left\lfloor \frac{k-2}{p-1} \right\rfloor \) then

\[
V_{k,a} \cong V_{k,0} \cong \begin{cases} \text{ind}(\omega_{k-1}^0) & \text{if } p+1 \nmid k-1, \\ (\mu_{\omega_-,\omega^{(k-1)/(p+1)}}_{\omega^{(k-1)/(p+1)}}) & \text{if } p+1 \mid k-1. \end{cases}
\]

(1)

Conditional on the combinatorial conjectures [A[F we prove the following theorem (recall that we assume \( p > 3 \)).

**Theorem 2.** If \( v_p(a) > \left\lfloor \frac{k-1}{p-1} \right\rfloor + \left\lfloor \log_p k \right\rfloor \) then

\[
V_{k,a} \cong \begin{cases} \text{ind}(\omega_{k-1}^0) & \text{if } p+1 \mid k-1, \\ \Omega_{k,a} & \text{if } p+1 \nmid k-1. \end{cases}
\]

(2)

where either \( \Omega_{k,a} \cong \text{ind}(\omega_2^{p-1}) \) or \( \Omega_{k,a} \cong \mu_{\lambda} \oplus \mu_{\lambda-1} \) for some \( \lambda \in \mathbb{F}_p \).

Theorem 2 says very little about \( V_{k,a} \) when \( p+1 \mid k-1 \), and of course we expect that \( \Omega_{k,a} \cong \mu_{\sqrt{-1}} \oplus \mu_{-\sqrt{-1}} \) and we do not expect that the error term \( \left\lfloor \log_p k \right\rfloor \) be optimal. We immediately deduce the following corollary (again conditional on the combinatorial conjectures [A[F]).

**Corollary 3.** Let \( N \) be a positive integer coprime to \( p \) such that \( p \) is \( \Gamma_0(N) \)-regular, and for \( l \in \mathbb{Z}_{\geq 2} \) let \( \mu_l \) be the discrete measure of supersingularities of weight \( l \), level \( \Gamma_0(Np) \) oldforms. Then the sequence of restrictions of \( \mu_l \) to the interval \( \left( \frac{1}{p+1}, \frac{p}{p+1} \right) \) tends to the zero measure on \( \left( \frac{1}{p+1}, \frac{p}{p+1} \right) \) as \( l \to \infty \).
See [Gou01] and conjecture 2.1.1 in [BG16] for a stronger question.

**Proof that theorem 2 implies corollary 3.** As in subsection 4.2 of [BLZ04] we can use theorem 2 to conclude that
\[ \mu_l \text{ is supported on } \left[ 0, \frac{1}{p+1} + \frac{\log{l}}{l-1}, 1 \right] \cup \left[ \frac{P}{p+1} - \frac{\log{l}}{l-1}, 1 \right], \]
and that completes the proof because
\[ \frac{\log{l}}{l-1} \to 0 \]
as \( l \to \infty \)—we omit the details.

2. Setup

Let \( q = \left\lfloor \frac{k-1}{p+1} \right\rfloor \). In light of theorem 1, let us assume that
\[ \left\lfloor \frac{k-2}{p+1} \right\rfloor \geq v_p(a) > q + \left\lfloor \log{p} \right\rfloor. \]
In particular,
\[ k \geq \frac{1}{2}(p+1)^2. \]

Instead of computing the modulo \( p \) representation at the weight \( k \), we use known local constancy results to equate it with a corresponding representation at a suitably close weight \( k + \epsilon \approx k \) and compute the latter representation instead. This avoids problems arising from the dimension of the associated \( \text{GL}_2(\mathbb{Q}_p) \)-representation (via the local Langlands correspondence) being too small. Together with local constancy results by Chenevier and Colmez, equation (5) implies that
\[ V_{k,a} \sim V_{t+2,a}. \]

This article is a continuation of [Ars], so to avoid repeating ourselves we refer to [Ars] for all the definitions. Let us just mention that there is the bijectively associated module
\[ \mathfrak{G}_{t+2,a} \cong \text{ind}^G \mathfrak{S}_t/\mathcal{I}, \]
and \( \mathcal{S} \) contains the reduction modulo \( p \) of any integral element in the image of \( T - a \), where \( T \in \text{End}_{G}(\text{ind}_{G}^{\overline{\mathbb{Q}}} \Sigma) \) corresponds to the double coset of \((\ell \; 1)\) and

\[
T(\gamma \cdot \varphi_{p} v) = \sum_{\ell \in \mathbb{F}_{p}} \gamma(\ell \; \varphi_{p}) \cdot \varphi_{p} \left( (\ell \; 1) \cdot v \right) + \gamma(\ell \; 0) \cdot \varphi_{p} \left( (\ell \; 0) \cdot v \right).
\]  

(12)

Let us also note that \( \varphi_{p} + 2 \cdot a \) is a subquotient of a module which has a series whose factors are subquotients of certain \( \widetilde{N}_{0}, \ldots, \widetilde{N}_{\nu - 1} \), and for \( \alpha \in \{0, \ldots, \nu - 1\} \) we similarly define \( \text{sub}(\alpha) \), \( \text{quot}(\alpha) \), \( T_{\alpha, \alpha} \), \( T_{\alpha} \). For \( \alpha \in \mathbb{Z}_{\geq 0} \), let

\[
h_{\alpha} = \sum_{j=0}^{n} (-1)^{j} \theta^{j+n+1} r - np - \alpha - (p - 1) \in \Sigma_{\alpha},
\]

\[
h_{\alpha}^{s} = (\ell \; \ell)h_{\alpha}.
\]  

(13)

For \( \alpha, \beta, R \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{C}_{R}(\alpha, \beta) \) be such that

\[
\sum_{\beta = -R}^{\alpha} \mathcal{C}_{R}(\alpha, \beta) \left( \frac{R - 1}{R} \right)^{\alpha - \beta} = \left( \frac{R - 1}{R} \right)^{\alpha - 1} \in \mathbb{Q}_{p}[X].
\]  

(14)

Note that both sides of equation \((14)\) are polynomials in \( X \) over \( \mathbb{Q}_{p} \) of degree \( R \).

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3. Combinatorial conjectures

Computer calculations suggest that all of the following combinatorial statements are true, and we expect to prove them at some point. Let

\[
q' = q - \left\lfloor \frac{r - 2 \alpha}{p - 1} \right\rfloor = \begin{cases} 
q & \text{if } q \leq \frac{r - 2 \alpha}{p - 1}, \\
q - 1 & \text{if } q > \frac{r - 2 \alpha}{p - 1}.
\end{cases}
\]

(15)

**Conjecture A.** If \( \alpha \in \{q + 1, \ldots, r - q - 1\} \) and \( l \in [1, \frac{2 \alpha}{p - 1}] \cap \mathbb{Z} \) then

\[
v_{p} \left( \binom{q'}{\alpha} \right) < v_{p} \left( \binom{r}{\alpha - l(p - 1)} \left( \frac{q'}{q'} \right) \right) + l(p - 1).
\]

(16)

**Conjecture B.** If \( \alpha \in \{q + 1, \ldots, r - q - 1\} \) and \( l \in [q', \frac{2 \alpha}{p - 1}] \cap \mathbb{Z} \) then

\[
v_{p} \left( \binom{q'}{\alpha} \right) < v_{p} \left( \binom{r}{\alpha - l(p - 1) - \alpha} \left( \frac{q'}{q'} \right) \right) + l(p - 1) + 2 \alpha - r.
\]

(17)

**Conjecture C.** If \( \alpha \in \{q + 1, \ldots, r - q - 1\} \) and \( \beta \in \{0, \ldots, q'\} \) then

\[
v_{p} \left( \binom{q'}{\alpha} \right) < v_{p} \left( \binom{q'}{\beta} \right) + v_{p} \left( \mathcal{C}_{q'}(\alpha, \alpha - \beta) \right) + \alpha - \beta.
\]

(18)

**Conjecture D.** If \( r = q(p + 1) + 1 \) and \( l \in [1, \frac{q + 1}{p - 1}] \cap \mathbb{Z} \) then

\[
v_{p} \left( \binom{q'}{q} \right) < v_{p} \left( \binom{r}{q - l(p - 1) - \alpha} \left( \frac{q'}{q} \right) \right) + l(p - 1).
\]

(19)

**Conjecture E.** If \( r = q(p + 1) + 1 \) and \( l \in [q + 1, q + \frac{q + 1}{p - 1}] \cap \mathbb{Z} \) then

\[
v_{p} \left( \binom{q'}{q} \right) < v_{p} \left( \binom{r}{l(p - 1) + q} \left( \frac{q'}{q} \right) \right) + (l - q) \alpha - (p - 1) - 1.
\]

(20)

**Conjecture F.** If \( r = q(p + 1) + 1 \) and \( \beta \in \{0, \ldots, q - 1\} \) then

\[
v_{p} \left( \binom{q'}{q} \right) < v_{p} \left( \binom{q'}{\beta} \right) + v_{p} \left( \mathcal{C}_{q}(q, q - \beta) \right) + q - \beta.
\]

(21)
4. Lemmas

In this section we prove some technical lemmas. Where there are similarities to lemmas in [Ars] we only provide sketches of the proofs and refer to [Ars] for the details.

**Lemma 4.** If \( \alpha \in \{0, \ldots, n\} \) then
\[
\alpha \bullet \mathfrak{P} h_{\alpha} \equiv 3 \, p^\alpha \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \bullet \mathfrak{P} x^\alpha y^{l-\alpha} + O(p^n). \tag{22}
\]
If \( \alpha \in \{0, \ldots, n\} \), \( \beta \in \{\alpha, \ldots, n\} \), and \((C_i)_{i \in \mathbb{Z}}\) is a family of elements of \( \mathbb{Z}_p \) then
\[
\sum_{i} \left( \sum_{\alpha=\beta} \left( (-\alpha+1) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \sum_{\mu \in \mathbb{F}_p} [\mu]^{-l(\begin{smallmatrix} p & \mu \\ 0 & 1 \end{smallmatrix})} \right) \bullet \mathfrak{P} x^{\alpha} y^{l-\alpha} \right) = 3 \sum_{\alpha=\beta} \left( (-\alpha+1) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \sum_{\mu \in \mathbb{F}_p} [\mu]^{-l(\begin{smallmatrix} p & \mu \\ 0 & 1 \end{smallmatrix})} \right) \bullet \mathfrak{P} h_{\alpha-\beta} + O(p^n).
\tag{23}
\]

**Proof.** The proof is similar to the proof of lemma 13 in [Ars]. We have
\[
\alpha \bullet \mathfrak{P} h_{\alpha} \equiv 3 \, T \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \bullet \mathfrak{P}, \tag{24}
\]
where, due to the explicit equation for \( T \) and part (5) of lemma 6 in [Ars],
\[
A_{\mu} = \sum_{\xi \geq 0} \sum_{n \geq 0} (-1)^{i+j} \left( \begin{smallmatrix} i+j & n \end{smallmatrix} \right) (-\alpha+1)^{i+j} \left( \begin{smallmatrix} i+j & n \end{smallmatrix} \right) \left( \begin{smallmatrix} p & \mu \\ 0 & 1 \end{smallmatrix} \right) \bullet \mathfrak{P} x^{\alpha} y^{l-\alpha} + O(p^n), \tag{25}
\]
and
\[
A = \sum_{j=0}^{n} \mathfrak{P}^{\alpha} \left( \begin{smallmatrix} i+j & n \end{smallmatrix} \right) \left( \begin{smallmatrix} j & \mu \end{smallmatrix} \right) \sum_{\mu \in \mathbb{F}_p} [\mu]^{-l(\begin{smallmatrix} p & \mu \\ 0 & 1 \end{smallmatrix})} \bullet \mathfrak{P} x^{\alpha} y^{l-\alpha} + O(p^n). \tag{26}
\]

Equations (24), (25), and (26) imply equation (22). Equation (22) implies that
\[
\frac{ap-\alpha}{p-1} \sum_{l=\alpha-\beta} \sum_{\mu \in \mathbb{F}_p} [\mu]^{-l(\begin{smallmatrix} p & \mu \\ 0 & 1 \end{smallmatrix})} \bullet \mathfrak{P} x^{\alpha} y^{l-\alpha} + O(p^n) \tag{27}
\]
which implies equation (23). \qed

**Lemma 5.** Let \( \alpha \in \{0, \ldots, \delta\} \) and \( v \in \mathbb{Q} \) and the family \((D_i)_{i \in \mathbb{Z}}\) of elements of \( \mathbb{Z}_p \) be such that
\[
D_i = 0 \text{ for } i \notin \left[ \frac{-\alpha}{p-1}, \frac{\alpha}{p-1} \right],
\]
\[
v \leq v_p(\theta_w(D_\alpha)) \text{ for } \alpha \leq w \leq 2\delta,
\]
\[
v < v_p(\theta_w(D_\alpha)) \text{ for } 0 \leq w < \alpha. \tag{28}
\]
For $j \in \mathbb{Z}$, let
\[ \Delta_j = (-1)^{j-n}(1-p)^{-\alpha}(\frac{\alpha}{j-n})\vartheta_\alpha(D_\star), \]
so that $(\Delta_j)_{j \in \mathbb{Z}}$ is supported on the set of indices $\{n, \ldots, \alpha + n\}$ and therefore $\vartheta_\alpha(D_\star)$ is properly defined for $0 \leqslant w < \alpha$. Then $v \leqslant v_p(\vartheta_\alpha(D_\star)) \leqslant v_p(\Delta_j)$ for all $j \in \mathbb{Z}$, and
\[
\sum_i (\Delta_i - D_i) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \\
\equiv_3 - \sum_{i \leq \delta + v_p(a) - \alpha}/(p-1) D_i \bullet \overline{\vartheta}_p h_i(p-1)+\alpha \\
- \sum_{i \geq (t-\alpha'-\delta-v_p(a))/(p-1)} D_i \bullet \overline{\vartheta}_p h_{t-i(p-1)-\alpha} \\
+ E \bullet \overline{\vartheta}_p \theta^{\alpha+1} h + F \bullet \overline{\vartheta}_p h' + O(p^\delta),
\]
for some polynomials $h, h'$ and some $E, F \in \mathbb{Z}_p$ with $v_p(E) \geq v$ and $v_p(F) > v$.

**Proof.** The proof is similar to the proof of lemma 14 in [178]. We have
\[
\sum_i (\Delta_i - D_i) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \\
\equiv_3 a^{-1} T \left( \sum_i (\Delta_i - D_i) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \right) \\
\equiv_3 a^{-1} \sum_i (\Delta_i - D_i) \sum_{\lambda_0 \in \mathbb{P}_p} (\frac{p}{\lambda_0}) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} (-[\lambda]x + py)^{-i(p-1)-\alpha} \\
+ a^{-1} \sum_i (\Delta_i - D_i) (p^{i(p-1)-\alpha}(\frac{p}{\lambda_0}) + p^i(p-1)+\alpha(\frac{1}{\lambda_0})) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \\
\equiv_3 a^{-1} \sum_i (\Delta_i - D_i) \sum_{\lambda_0 \in \mathbb{P}_p} (\frac{p}{\lambda_0}) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} (-[\lambda]x + py)^{-i(p-1)-\alpha} \\
- \sum_{i \leq \delta + v_p(a) - \alpha}/(p-1) D_i \bullet \overline{\vartheta}_p h_i(p-1)+\alpha \\
- \sum_{i \geq (t-\alpha'-\delta-v_p(a))/(p-1)} D_i \bullet \overline{\vartheta}_p h_{t-i(p-1)-\alpha} + O(p^\delta). \tag{31}
\]
The third congruence follows from lemma 4. We also have
\[
\sum_{\lambda_0 \in \mathbb{P}_p} (\frac{p}{\lambda_0}) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} (-[\lambda]x + py)^{-i(p-1)-\alpha} \\
\equiv_3 \sum_{\xi=0}^{2\delta} \left( \sum_{\epsilon=\alpha}^{\delta} \sum_{\lambda_0 \in \mathbb{P}_p} [-\lambda]^{t-i(p-1)-\alpha} \left( \frac{p}{\lambda_0} \right) \bullet \overline{\vartheta}_p x^{t-i(p-1)-\alpha} y^\xi + O(p^{2\delta}) \right) \\
\equiv_3 a \sum_{\xi=0}^{2\delta} \left( \sum_{\epsilon=\alpha}^{\delta} \sum_{\lambda_0 \in \mathbb{P}_p} [-\lambda]^{t-i(p-1)-\alpha} \left( \frac{1}{\lambda_0} \right) \bullet \overline{\vartheta}_p x^{t-i(p-1)-\alpha} h^\xi \right) + O(p^{2\delta}). \tag{32}
\]
The second congruence follows from lemma 4. By assumption, if
\[
X_\xi = \sum_i (\Delta_i - D_i) \left( \sum_{\lambda_0 \in \mathbb{P}_p} (\frac{p}{\lambda_0}) \bullet \overline{\vartheta}_p x^{i(p-1)+\alpha} (-[\lambda]x + py)^{-i(p-1)-\alpha} \right), \tag{33}
\]
then $v_p(X_\xi) > v$ for $\xi \in \{0, \ldots, \alpha\}$, and $v_p(X_\xi) \geq v$ for $\xi \in \{\alpha + 1, \ldots, 2\delta\}$. This means that equation (32) implies that
\[
\sum_{\xi=0}^{2\delta} X_\xi \bullet \overline{\vartheta}_p \sum_{\lambda_0 \in \mathbb{P}_p} [-\lambda]^{t-i(p-1)-\alpha} \left( \frac{1}{\lambda_0} \right) h^\xi \right) + O(p^{2\delta-\alpha}). \tag{34}
\]
which together with equation (31) implies equation (30) with
\[ E^\theta_{\alpha+1} = \sum_{\xi=0}^{2\delta} X_\xi \sum_{\lambda \in F_p^*} \mathcal{K} \left[ -\lambda \right]^{\alpha-\xi} (1^{(\lambda)}) h^* \xi. \]
\[ F h' = \sum_{\xi=0}^{\delta} X_\xi \sum_{\lambda \in F_p^*} \mathcal{K} \left[ -\lambda \right]^{\alpha-\xi} (1^{(\lambda)}) h^* \xi. \] (35)

**Lemma 6.** Let \((C_i)_{i \in \mathbb{Z}}\) be any family of elements of \(\mathbb{Z}_p\). Suppose that \(\alpha \in \{0, \ldots, \delta\} \) and \(\beta \in \{\alpha, \ldots, n\} \) and \(v \in \mathbb{Q}\) and
\[ D_i = [i \in \left(\frac{-\alpha}{p-1}, \frac{-\beta}{p-1}\right)] D_i + \sum_{i=\alpha-\beta}^{\alpha} C_i (p^{-\beta+\alpha}) \] (36)
satisfy
\[ v \leq v_p (\partial_w (D_*)) \] (37)
\[ v < v_p (\partial_w (D_*)) \] (38)
\[ \text{Note that } (D_i)_{i \in \mathbb{Z}} \text{ is supported on the finite set of indices } \left\{ \frac{-\alpha}{p-1}, \ldots, \frac{-\beta}{p-1} \right\}. \]

Then
\[ (1 - p)^{-\alpha} \left[ \right. \sum_{\alpha} \sum_{\beta} C_i p^l \sum_{\mu \in F_p^*} \mathcal{K} \left[ \mu \right] \left( 0^{(\mu)} \right) \sum_{\nu} \mathcal{K} h_{\alpha-\beta} + O(p^\delta) \]
(39)

for some polynomials \(h, h'\) and some \(E, F \in \mathbb{Z}_p\) with \(v_p (E) \geq v\) and \(v_p (F) > v\).

**Proof.** Lemma 4 implies that
\[ \sum_i (D_i - D_i') \sum_{\nu} x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \]
(40)
Equation (39) together with lemma 5 implies that
\[ \sum_i \mathcal{K} \sum_{\nu} x^{i(p-1)+\alpha} y^{-i(p-1)-\alpha} \]
(41)
Lemma 7. If $\alpha \in \mathbb{Z}_{\geq 1}$ and $\beta \in \{0, \ldots, \alpha\}$ then

$$v_p\left(\binom{\alpha}{\beta}\right) \leq \lfloor \log p \alpha \rfloor.$$  \hfill (42)

Proof. A theorem by Kummer says that

$$v_p\left(\binom{\alpha}{\beta}\right)$$

is the number of times one carries over a digit when adding $\beta$ and $\alpha - \beta$, and is therefore strictly less than the number $\lfloor \log p \alpha \rfloor + 1$ of digits of $\alpha$.

$$\blacksquare$$

5. Proof of theorem [2]

The goal of this section is to compute $\mathfrak{T}_{t+2,a}$. Let $Q$ be an infinite-dimensional factor of $\mathfrak{T}_{t+2,a}$. The first two subsections imply the following two facts about $Q$.

1. $Q$ is not a factor of $\hat{N}_\alpha$ for $\alpha \in \{0, \ldots, \nu - 1\}$.
2. $Q$ is not a factor of $\hat{N}_\alpha$ for $\alpha \in \{\nu + 1, \ldots, \nu - 1\}$.

From these two facts we can conclude that either $Q$ is a factor of $\text{ind}^G\text{sub}(\varrho)$ or it is a factor of $\text{ind}^G\text{quot}(\varrho)$. In light of Berger’s modulo $p$ local Langlands correspondence and as in section 10 of [Ars] this means that

$$\mathfrak{T}_{k,a} \simeq \mathfrak{T}_{t+2,a} \simeq \text{ind}^G(\omega^\varrho_{r+1}) \otimes \omega^\varrho \simeq \text{ind}^G(\omega^{r+1}) \simeq \mathfrak{T}_{k,0}$$

for $r - \varrho(p + 1) \in \{-1, \ldots, p - 1\} \setminus \{-1, 1, p - 2\}$. The next two subsections prove the following facts about two of the remaining cases $r - \varrho(p + 1) \in \{1, p - 2\}$.

3. If

$$r - \varrho(p + 1) = p - 2$$

(and therefore $\text{sub}(\varrho) \simeq \sigma_{p-2}(\varrho)$) then $Q$ is a factor of $\text{ind}^G\text{quot}(\varrho)$.

4. If

$$r - \varrho(p + 1) = 1$$

(and therefore $\text{quot}(\varrho) \simeq \sigma_{p-2}(\varrho + 1)$) then $Q$ is a factor of $\text{of ind}^G\text{quot}(\varrho)/T_{q,\varrho}$.

These four claims together complete the proof as in [Ars]—we omit the details.

5.1. No infinite-dimensional factor of $\mathfrak{T}_{t+2,a}$ is a subquotient of $\hat{N}_\alpha$ for $\alpha \in \{0, \ldots, \nu - 1\}$. Let $\alpha \in \{0, \ldots, \nu - 1\}$. Let

$$M^{(r)}_\alpha = \left(\binom{r-\alpha+j}{i(p-1)+j}\right)_{\{i \mid i(p-1)+\alpha \in \varrho(r-\varrho), \alpha - \varrho \leq j \leq \alpha\}}.$$

Then $M^{(r)}_\alpha$ has

$$R \leq \left\lfloor \frac{r-2p+1}{p-1} \right\rfloor \leq \varrho + 1$$

rows and $C = \varrho + 1$ columns. Let

$$M^{(r)\prime}_\alpha = \left(\binom{r-\alpha+j}{i(p-1)+j}\right)_{\{i \mid i(p-1)+\alpha \in \varrho(r-\varrho), \alpha - R \leq j \leq \alpha\}}$$

\hfill (46)
be the right $R \times R$ submatrix of $M^{(r)}_\alpha$. Lemma [7] and the equation
\[
(r - \alpha + j)_{i(p-1)+j} = (i(p-1) + \alpha)_{i-j}^{-1}
\] (47)
imply that the $\mathbb{Z}_p$-module determined by the image of $M^{(r)\prime}_\alpha$ contains $p^{[\log_p k]} \times$
the $\mathbb{Z}_p$-module determined by the image of the matrix
\[
M^{(r)\prime \prime}_\alpha = \left( (\begin{smallmatrix} i(p-1)+\alpha \\ \alpha-j \end{smallmatrix}) \right)_{0 \leq i,j < R}
\] (48)
with corresponding entries $\left( (\begin{smallmatrix} i(p-1)+\alpha \\ \alpha-j \end{smallmatrix}) \right)_{0 \leq i,j < R}$.
For some $\gamma \in \mathbb{Z}_{\geq 0}$, $M^{(r)\prime \prime}_\alpha$ is obtained from
\[
M^{(r)\prime \prime \prime}_\alpha = \left( (\begin{smallmatrix} (i(p-1)+\gamma) \\ j \end{smallmatrix}) \right)_{0 \leq i,j < R}
\] (49)
by permuting the rows. By Vandermonde’s convolution formula,
\[
M^{(r)\prime \prime \prime}_\alpha = \left( (\begin{smallmatrix} (i(p-1)) \\ j \end{smallmatrix}) \right)_{0 \leq i,j < R} \cdot \left( (\begin{smallmatrix} \gamma \\ j-i \end{smallmatrix}) \right)_{0 \leq i,j < R}.
\] (50)
Since
\[
\left( (\begin{smallmatrix} \gamma \\ j-i \end{smallmatrix}) \right)_{0 \leq i,j < R}
\] is upper triangular with 1’s on the diagonal and
\[
\det \left( (\begin{smallmatrix} (i(p-1)) \\ j \end{smallmatrix}) \right)_{0 \leq i,j < R} = (p-1)^R \det \left( (\begin{smallmatrix} 1 \\ j \end{smallmatrix}) \right)_{0 \leq i,j < R} = (p-1)^R
\] (51)
by a variant of Vandermonde’s determinant identity, the reduction modulo $p$ of $M^{(r)\prime \prime \prime}_\alpha$ has full rank (in characteristic $p$). Therefore, for each $u$ such that
\[
u(p-1) + \alpha \in [\rho + 1, r - \rho - 1],
\] there exist constants $C_\alpha(r, u), \ldots, C_\alpha - r(r, u)$ such that
\[
\sum_{i=\alpha-r}^{\alpha} C_i(r, u)_{i(p-1)+j} \equiv [i = u]p^{[\log_p k]}
\] (52)
for all $i$ such that
\[
i(p-1) + \alpha \in [\rho + 1, r - \rho - 1].
\] By adding linear combinations of equation (52) for varying $u$, we get that
\[
\sum_{i(p-1)+\alpha \in [\rho + 1, r - \rho - 1]} \sum_{i=\alpha-r}^{\alpha} C_i_{i(p-1)+l} x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha}
+ \sum_{i(p-1)+\alpha \in [0, \rho]} \sum_{i=\alpha-r}^{\alpha} D_i_{i(p-1)+l} x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha}
\equiv p^{[\log_p k]} \varrho^\alpha x^{p-1} y^{r-\alpha(p+1)+p+1} \in \mathbb{Z}_p
\] (53)
for some $C_i, D_i$. Let $D_i(r)$ be the coefficient of $x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha}$ on the left side of equation (53). Then
\[
\varrho w(D_{\bullet}(r)) = \sum_i D_i(r)_{i(p-1)+j} \equiv [i = w] p^{[\log_p k]}
\] (54)
is zero for $0 \leq w < \alpha$, and has valuation that is greater than or equal to $[\log_p k]$ for $w \geq \alpha$, with equality for $w = \alpha$. Let $D_i$ be the coefficient of $x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha}$ in
\[
\sum_{i(p-1)+\alpha \in [\rho + 1, r - \rho - 1]} \sum_{i=\alpha-r}^{\alpha} C_i_{i(p-1)+l} x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha}
+ \sum_{i(p-1)+\alpha \in [0, \rho]} \sum_{i=\alpha-r}^{\alpha} D_i_{i(p-1)+l} x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha},
\] (55)
where $D_i'' = D_i'$ and $D_i'' = D_i' - D_i$ for $i(p-1) + \alpha \in [0, \rho]$. Since
\[
\nu_p((i(p-1) + \alpha)! \leq \nu_p(g!) \leq k
\] (56)
for $i(p-1)+\alpha \in [0,\varrho]$, it is easy to show by using lemma 5 in [Ars] that
\[ \vartheta_w(D_{\bullet}) = \vartheta_w(D_{\bullet}(r)) + O(ep^{-k-W}) \] (57)
for all $0 \leq w \leq W$. In particular, we can apply lemma 6 to the constants $(D_i)_{i \in \mathbb{Z}}$ and to $v = [\log_p k]$, and as a result get that
\[ (1-p)^{-\alpha} \vartheta(D_{\bullet}) \bullet \sum_{\varrho \in \mathbb{Q}_p, \varrho \vDash \theta^\alpha x^{n(p-1)}y^t-\alpha(p+1)-n(p-1)} \]
\[ = \sum_{\varrho \in \mathbb{Q}_p} \sum_{i=\alpha}^{\alpha} C \varrho^\alpha \sum_{\mu \in \mathbb{F}_q} [\mu]^{-t}(\alpha,1) \bullet \varrho \theta^{t-\alpha-\mu} h_{\alpha-t} \]
\[ + \sum_{i(\varrho-1)+\alpha \in [0,\varrho]} \varrho_{\varrho-1} + \alpha \theta^{t-\alpha-\mu} h_{\alpha-t} \]
\[ - \sum_{i \neq (\varrho-1) \alpha \in [0,\varrho]} \varrho_{\varrho-1} + \alpha \theta^{t-\alpha-\mu} h_{\alpha-t} \]
\[ + E \bullet \varrho \theta^{t-\alpha-\mu} h + F \bullet \varrho \theta^{t-\alpha-\mu} h + O(p^\delta), \] (58)
for some $h,h'$ and some $E,F \in \mathbb{Z}_p$ with $v_p(E) \geq [\log_p k]$ and $v_p(F) > [\log_p k]$. Here
\[ D_i'' = D_i'' - \sum_{\varrho \in \mathbb{Q}_p} [\mu]^{-t}(\alpha,1) \bullet \varrho \theta^{t-\alpha-\mu} h_{\alpha-t} \]
(59)
for all $i$ such that
\[ i(p-1)+\alpha \in [0,\varrho] \cup [t-\varrho,\varrho]. \]
The left side of equation (58) is $p^{[\log_p k]} \psi$, where $\psi$ is an integral element whose reduction modulo $p$ represents a generator of $\hat{N}_\alpha$. We can use lemma 4 to get that the first and second lines on the right side of equation (58) are
\[ O(p^{v_p(\varrho-\alpha)}) = O(p^{[\log_p k]}). \] (60)
We can also use equation (53) to get that
\[ D_i = O(p^{[\log_p k]}). \] (61)
for all $i$ such that
\[ i(p-1)+\alpha \in [0,\varrho] \cup [t-\varrho,\varrho]. \]
This is because
\[ D_i = D_i(r) + O(ep^{-v_p(\delta)}) \] (62)
for all $i$ such that $i(p-1)+\alpha \in [0,\varrho]$, and
\[ D_{i-1} = D_{i-1}(r) + O(ep^{-v_p(\delta)}) \] (63)
for all $i$ such that $i(p-1)+\alpha \in (t-\varrho,\varrho]$. We also have, due to equation (53),
\[ D_w = O(ep^{-v_p(\delta)}) \] (64)
for all $i$ and all
\[ w \in \mathbb{Z}_{<0} \cup \mathbb{Z}_{\geq 0}. \]
This, together with lemma 4, implies that the sum of the third and fourth lines on the right side of equation (58) is $p^{[\log_p k]} \times$ an integral element whose reduction modulo $p$ represents the trivial element of $\hat{N}_\alpha$. Finally, the fifth line of on the right side of equation (58) is evidently $p^{[\log_p k]} \times$ an integral element whose reduction modulo $p$ represents the trivial element of $\hat{N}_\alpha$. So, similarly as in the
main proof in [Ars], we can conclude that no infinite-dimensional factor of $\Theta_{t+2,a}$ is a subquotient of $\tilde{N}_\alpha$—again we omit the details.

5.2. No infinite-dimensional factor of $\Theta_{t+2,a}$ is a subquotient of $\tilde{N}_\alpha$ for $\alpha \in \{g + 1, \ldots, \nu - 1\}$. Let $\alpha \in \{g + 1, \ldots, \nu - 1\}$. Let
\begin{equation}
C_l = \mathcal{C}_{\theta}(\alpha, l) \left(\frac{r}{r-l}\right)
\end{equation}
for $l \in \{\alpha - g', \ldots, \alpha\}$. As in subsection 5.1 we can conclude that
\begin{equation}
\sum_{i(p-1)+\alpha \in (\alpha, g' + \alpha)} \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{r-\alpha+l}{i(p-1)+l}\right) x^{i(p-1)+\alpha} y^{r-i(p-1)-\alpha} = 0 \in \tilde{\Sigma}_r.
\end{equation}
Let $D_i$ be the coefficient of $x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha}$ in
\begin{equation}
\sum_{i(p-1)+\alpha \in (\alpha, t-r + g' + \alpha)} \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{t-\alpha+l}{i(p-1)+l}\right) x^{i(p-1)+\alpha} y^{t-i(p-1)-\alpha}.
\end{equation}
Then it is easy to show by using lemma 5 in [Ars] that
\begin{equation}
\vartheta_w(D_\ast) = O(ep^{-k-2\delta})
\end{equation}
for all $0 \leq w \leq 2\delta$. In particular, we can apply lemma 6 to the constants $(D_i)_{i \in \mathbb{Z}}$ and to $v = \delta$, and as a result get that
\begin{equation}
\sum_{p^{-\alpha}}^{ap^{-\alpha}} \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{p(l)}{p^{l+1}}\right) \sum_{\mu \in \mathcal{P}_p}[\mu]^{-t} \left(\begin{smallmatrix} p & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{\tilde{G}}_{\mu} h_{\alpha-l}
\end{equation}
\begin{equation}
= \sum_{\alpha \in [0, \alpha] \cup (t-r+g'+\alpha)}\sum_{l=\alpha-g'}^{\alpha} D_i \left(\begin{smallmatrix} p & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{\tilde{G}}_{\mu} h_{\alpha-l} + O(p^\delta),
\end{equation}
where
\begin{equation}
D_i = \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{t-\alpha+l}{i(p-1)+l}\right)
\end{equation}
for all $i$ such that
\begin{equation}
i(p-1) + \alpha \in [0, \alpha] \cup (t-r+g'+\alpha) + \alpha, t].
\end{equation}
By approximating $D_i$ with $D_i (r) + O(ep^{-v}(D_i))$ as in subsection 5.1 we can show that the third and fourth lines of equation (69) are in
\begin{equation}
O(p^{\alpha-2v(p)}(\alpha)) = O(ap^{-\alpha+(k-3v_p(a)-p+3)}) = O(ap^{-\alpha+(p-3)(k/(p-1)-1)}) = O(ap^{-\alpha+2[\log_p k]}),
\end{equation}
Consequently we get that
\begin{equation}
\sum_{i(p-1)+\alpha \in [0, \alpha] \cup (t-r+g'+\alpha) + \alpha, t} \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{t-\alpha+l}{i(p-1)+l}\right) \mathcal{\tilde{G}}_{\mu} h_{\alpha-l} + O(ap^{-2[\log_p k]}).
\end{equation}
Lemma 4 and the definition of $(C_l)_{\alpha-g' \leq l \leq \alpha}$ then imply that
\begin{equation}
\sum_{i(p-1)+\alpha \in [0, \alpha]} X_i \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \sum_{\mu \in \mathcal{P}_p}[\mu]^{-t} \left(\begin{smallmatrix} p & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{\tilde{G}}_{\mu} h_{\alpha-l} + O(p^{2[\log_p k]}),
\end{equation}
\begin{equation}
= \sum_{i(p-1)+\alpha \in [0, \alpha]} \sum_{l=\alpha-g'}^{\alpha} C_l \left(\frac{t-\alpha+l}{i(p-1)+l}\right) \sum_{\mu \in \mathcal{P}_p}[\mu]^{-t} \left(\begin{smallmatrix} p & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mathcal{\tilde{G}}_{\mu} h_{\alpha-l} + O(p^{2[\log_p k]}),
\end{equation}
\begin{equation}

\end{equation}
where

\[
X_i = p^{-i(p-1)} {t \choose i(p-1)+\alpha} (t' - i), \\
X_i^* = p^{i(p-1)+2t} {t \choose i(p-1)+\alpha} (t' - i). 
\]

(74)

By conjecture [A],

\[
v_p(X_0) = v_p \left( {\alpha \choose \alpha} \right) < v_p \left( p^{-i(p-1)} {t \choose i(p-1)+\alpha} (t' - i) \right) = v_p(X_i) 
\]
for all \( i \) such that \( i(p-1) + \alpha \in [0, \alpha) \). By conjecture [B]

\[
v_p(X_0) = v_p \left( {\alpha \choose \alpha} \right) < v_p \left( p^{i(p-1)+2t} {t \choose i(p-1)+\alpha} (t' - i) \right) = v_p(X_i^*) 
\]
for all \( i \) such that \( i(p-1) + \alpha \in (t - r + \rho(p-1) + \alpha, t] \). By conjecture [C]

\[
v_p(X_0) = v_p \left( {\alpha \choose \alpha} \right) < v_p \left( \mathcal{C}_p(\alpha, l) \right)= v_p \left( C_l p^l \right) = v_p(C_l p^l) 
\]
for all \( l \in \{\alpha - \rho', \ldots, \alpha\} \). Moreover, by lemma [7]

\[
v_p(X_0) = v_p \left( {\alpha \choose \alpha} \right) < v_p \left( \frac{p}{\pi} \right) < 2 \left[ \log_p k \right]. 
\]

(78)

So if we divide both sides of equation (73) by \( \frac{p}{\pi} \) we get an integral element, and if we reduce that integral element modulo \( p \) then the only contributing term to the result is the “\( i = 0 \)” term in the first line of equation (73). Therefore we can conclude that \( \mathcal{S} \) contains

\[
\left( \frac{p}{\pi} \right) \bullet \mathbb{F}_p h_\alpha, 
\]

which represents a generator of \( \hat{\mathcal{N}}_\alpha \), and we can conclude the desired result.

5.3. If \( r - \rho(p+1) = p - 2 \) then no infinite-dimensional factor of \( 0_{t+2,a} \) is a subquotient of \( \text{ind}^G \text{sub}(q) \). Let \( r - \rho(p+1) = p - 2 \), so that

\[
\text{sub}(q) \cong \sigma_{p-2}(q). 
\]

(80)

The proof in this case is very similar to the proof in subsection 5.1, so we just give a rough sketch. We let

\[
M^{(r)} = \left( \frac{r}{i(p-1)+\rho} \right)_{\{i \mid i(p-1)+\rho \in [\rho, r-r \rho)\}}. 
\]

(81)

As in subsection 5.1 we can prove that the image of a certain lattice under the right square submatrix of \( M^{(r)} \) (seen as an endomorphism) contains \( p^{[\log_p k]} \times \) that lattice. We can conclude the following analogous equation to equation (53).

\[
\sum_{i(p-1)+\rho \in [\rho+1, r-r \rho-1]} \sum_{j=0}^p C_i \left( \frac{r-p+\rho}{i(p-1)+\rho} \right) \sum_{e \in [\rho, r-r \rho] \cap [\rho, r-\rho]} \sum_{f : e \times p-1 \rho \times e \times (p-1)-p+1} D^{(r+1)+\rho} \left( e \times (p-1)+\rho \right) 
\]

\[
= p^{[\log_p k]} \theta^e y^{r-\rho(p+1)} 
\]

(82)

for some integers \( \sum_{j=0}^p D^{(r)} \). The main difference is that we must write \( \theta^e y^{r-\rho(p+1)} \) instead of \( \theta^e x^{p-1} y^{r-\rho(p+1) - p+1} \). This then means that we can only conclude that

\[
D_{w} = O(e p^{-v_p(\delta)}) 
\]

(83)

for all \( w \in \mathbb{Z} \cup \mathbb{Z} (-\frac{1}{2}) \).
(rather than for all \( w \in \mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{> t - 2p} \)) in the equation
\[
(1 - p)^{e} \vartheta_{e}(D_{\ast}) \bullet \varpi_{p} \theta_{p} e_{p}; x^{n(p - 1)} y^{t - e(p + 1) - n(p - 1)} \\
= \sum_{i \geq 0}^{n(p - 1)} C_{i} P_{\ast}^{\prime} \sum_{\mu \in \mathbb{F}_{p}} [\mu]^{-t} \left( \frac{\mu}{\mu} \right) \bullet \varpi_{p} h_{0} - l \\
+ \sum_{i \in \mathbb{F}_{p}} D_{i} \bullet \varpi_{p} h_{i(p - 1) + e} \\
+ \sum_{i \in \mathbb{F}_{p}} D_{i} \bullet \varpi_{p} h_{i(p - 1) + e} \\
+ E \bullet \varpi_{p} \theta_{p}^{t+1} h + F \bullet \varpi_{p} h! + O(p^{\delta}),
\]
for some \( h, h' \) and some \( E, F \in \mathbb{Z}_{p} \) with \( v_{p}(E) \geq [\log_{p} k] \) and \( v_{p}(F) \geq [\log_{p} k] \), which is the analogous equation to equation (57). In other words, the difference is that \( D_{0} \) is not negligible, and instead
\[
D_{0} = \vartheta_{e}(D_{\ast}) + O(p^{\delta}).
\]
So upon dividing equation (84) by \( \vartheta_{e}(D_{\ast}) \) and reducing modulo \( p \) we get that \( \mathcal{I} \) contains
\[
1 \bullet \varpi_{p} \left( \theta_{p} e_{p}; x^{n(p - 1)} y^{t - e(p + 1) - n(p - 1)} + h_{0} \right).
\]
It is easy to show that his represents a generator of \( \text{ind}^{G} \text{sub}(\varrho) \) (but is trivial in \( \text{ind}^{G} \text{quot}(\varrho) \)), which finishes the proof of the desired result as in subsection 5.1.

5.4. **If** \( r - \varrho(p + 1) = 1 \) **then** each infinite-dimensional factor of \( \mathfrak{S}_{t+2,a} \) **is a subquotient of** \( \text{ind}^{G} \text{quot}(\varrho)/T_{q,e} \). **Let** \( r - \varrho(p + 1) = 1 \), so that
\[
\text{quot}(\varrho) \cong \sigma_{p-2}(\varrho + 1).
\]
Just as in in subsection 5.3, we can conclude that \( \mathcal{I} \) contains a representative of a generator of \( \text{ind}^{G} \text{sub}(\varrho) \). So it remains to show that \( \mathcal{I} \) contains a representative of a generator of
\[
T_{q,e} \left( \text{ind}^{G} \text{quot}(\varrho) \right).
\]
Let
\[
C_{i} = \mathfrak{C}_{\varrho}(\varrho, t) \left( \frac{t}{\varrho} \right)
\]
for \( l \in \{0, \ldots, p\} \). As in subsection 5.2 we can conclude that
\[
\sum_{i \in \mathbb{F}_{p}} \sum_{\mu \in \mathbb{F}_{p}} [\mu]^{-t} \left( \frac{\mu}{\mu} \right) \bullet \varpi_{p} h_{0} - l \\
\cong \sum_{i = 0}^{p-1} C_{i} P_{\ast}^{\prime} \sum_{\mu \in \mathbb{F}_{p}} [\mu]^{-t} \left( \frac{\mu}{\mu} \right) \bullet \varpi_{p} h_{0} - l + O(p^{2[\log_{p} k]}),
\]
where
\[
X_{i} = p^{-i} \left( \frac{t}{\varrho} \right) \left( \frac{e-i}{\varrho} \right),
\]
\[
X_{i}^{*} = p^{-i} \left( \frac{t}{\varrho} \right) \left( \frac{e-i}{\varrho} \right).
\]
Again, by conjecture [D]
\[
v_{p}(X_{0}) = v_{p} \left( \left( \frac{t}{\varrho} \right) \right) < v_{p} \left( p^{-i} \left( \frac{t}{\varrho} \right) \left( \frac{e-i}{\varrho} \right) \right) = v_{p}(X_{i})
\]
for all $i$ such that $i(p - 1) + \varrho \in [0, \varrho]$. By conjecture E,
\[ v_p(X_0) = v_p \left( \binom{t}{\varrho} \right) < v_p \left( p^{(i\varrho)(p-1)-1} \binom{t}{i(p-1)+\varrho} \binom{\varrho-i}{\varrho} \right) = v_p(X_1) \] (93)
for all $i$ such that $i(p - 1) + \varrho \in (t - r + \varrho, t]$. By conjecture F,
\[ v_p(X_0) = v_p \left( \binom{t}{\varrho} \right) < v_p \left( C_\varrho \left( t \binom{\varrho}{l} p^l \right) \right) = v_p(C lp^l) \] (94)
for all $l \in \{1, \ldots, \varrho\}$. And, by lemma 7,
\[ v_p(X_0) = v_p \left( \binom{t}{\varrho} \right) = v_p \left( \binom{t}{\varrho} \right) < 2 \lfloor \log_p k \rfloor. \] (95)
This means that if we divide both sides of equation (90) by $\binom{t}{\varrho}$ and reduce the resulting integral element modulo $p$, the two contributing terms are the “$i = 0$” term in the first line of equation (90) and the “$l = 0$” term in the third line of equation (90). Therefore $\mathcal{S}$ contains
\[ \sum_{\mu \in \mathbb{F}_p} \left( \binom{\mu}{\varrho} \right) \in \mathbb{F}_p, \] (96)
which is a representative of a generator of $T_{q, \varrho} \left( \text{ind}_G \text{quot}(\varrho) \right)$, (97)
and that completes the proof. ■

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