We derive the analytic expression of the two one-loop dipole contributions to the 
elastic 4-gluon amplitude in QCD for arbitrary transverse momentum. The first 
one corresponds to the double QCD pomeron exchange, the other to an order \( \alpha^2 \) 
correction to one-pomeron exchange.

1 Aims

It is well-known that the bare pomeron singularity in QCD is violating the 
Froissart bound. The computation in the QCD framework of unitarity corrections 
to the bare pomeron is thus required. The first order correction implies 
the computation of the two-pomeron contribution to the elastic gluon-gluon 
amplitude. The aim of our paper is to give the first complete derivation of 
the analytical expression of the one-loop dipole contribution to the elastic 
amplitude in the QCD-dipole picture of BFKL dynamics. The solution for the 
forward amplitude has been already derived in ref. and the non-forward expression conjectured on the basis of assuming conformal invariance. Here we 
give the full proof of the result and thus of the conformal invariant property. 
Note that the QCD dipole formulation is known to be equivalent at tree level 
to the derivation of the BFKL amplitude in terms of Feynman graphs. The idea of our derivation is to use this formalism as an effective theory defining 
the propagation and interaction vertices of two QCD Pomerons, which are colorless compound states of reggeized gluons in the BFKL representation.

We first introduce the \( SL(2,C) \)-invariant formalism for the 4-gluon elastic 
amplitude \( A_Q(k, k'; Y) \) in the BFKL derivation. The solution of the BFKL 
equation is more easily expressed in terms of the Fourier transformed amplitude \( f_Q(\rho, \rho'; Y) \) given by the relation

\[
A_Q(k, k'; Y) = \frac{1}{(2\pi)^3} \int d^2 k d^2 k' \ e^{i\rho(k-k')-i\rho'(k'-k)} f_Q(\rho, \rho'; Y). 
\tag{1}
\]

Using the \( SL(2,C) \)-invariant formalism, the solution of the BFKL equation reads

\[
f_Q(\rho, \rho'; Y) = \alpha^2 \left| \frac{\rho' \rho}{16} \right| \int dh \ E^h_Q(\rho') E^h_Q(\rho) \ d(h) e^{i\omega(h) Y}, 
\tag{2}
\]
where the factor $\alpha^2$ comes from the coupling to incident dipoles.

In equation (2), the symbolic notation $\int dh \equiv \sum_{n=-\infty}^{\infty} \int d\nu$ corresponds to the integration over the $SL(2,C)$ quantum numbers with $h = i\nu + \frac{1-n}{2}$. $E_Q^\hbar(\rho)$ and $\omega(h)$ are, respectively, the $SL(2,C)$ Eigenfunctions and Eigenvalues of the BFKL kernel. The Eigenvalues read

$$\omega(h) = \frac{\bar{\alpha}N_c}{\pi} \chi(h) \equiv \frac{\bar{\alpha}N_c}{\pi} 2 \left\{ \Psi(1) - \Re \left[ \Psi \left( \frac{1+|n|}{2} + i\nu \right) \right] \right\}, \quad (3)$$

where $\Psi \equiv (\log \Gamma)'$. The $SL(2,C)$ Eigenvectors are defined by

$$E_Q^\hbar(\rho) = \frac{2\pi^2}{|\rho| \bar{b}(h)} \int d^2 b \ e^{iQb} E^h \left( b - \frac{\rho}{2}, b + \frac{\rho}{2} \right), \quad (4)$$

with

$$E^h \left( b - \frac{\rho}{2}, b + \frac{\rho}{2} \right) = (-)^{h-\bar{h}} \left( \frac{\rho}{b^2 - \frac{\nu^2}{4}} \right)^h \left( \frac{\bar{\rho}}{b^2 - \frac{\nu^2}{4}} \right)^{\bar{h}}, \quad (5)$$

where $\bar{h} = 1 - \bar{h}$, $b$ is the 2-d impact-parameter, and

$$d(h) = \left\{ \nu^2 + \frac{(n-1)^2}{4} \right\}^{\frac{1}{2}} \left\{ \nu^2 + \frac{(n+1)^2}{4} \right\}^{-1}, \quad b(h) = \frac{\pi^3 4^{h+\bar{h}-1}}{3 - h} \frac{\gamma(1-h)}{\gamma(1/2-h)}, \quad (6)$$

where, by definition,

$$\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}.$$

Note that an analytic expression of the Eigenvectors $E_Q^\hbar(\rho)$ in the mixed representation has been provided in terms of a combination of products of two Bessel functions. For simplicity, we did not include the impact factors. Note also that the leading contribution to the amplitude (2) is the $n = 0$ component which corresponds to the BFKL Pomeron.

### 2 Formulation

The formulation of the general one-loop amplitude in the QCD dipole model can be written:

$$f_{\text{(one-loop)}}(\rho_0 \rho_1; \rho_0' \rho_1'|Y = y + y') = \frac{1}{2!(2\pi)^8} \times$$

$$\int^y_0 dy \int^y_0 dy' \int \frac{d^2 \rho_{\alpha_0} d^2 \rho_{\alpha_1} d^2 \rho_{\beta_0} d^2 \rho_{\beta_1}}{|\rho_{\alpha_0}|^2 |\rho_{\beta_0}|^2} \frac{d^2 \rho_{\alpha_1} d^2 \rho_{\beta_1}}{|\rho_{\alpha_1}|^2 |\rho_{\beta_1}|^2}$$

$$\int \frac{d^2 \rho_{\alpha_0} d^2 \rho_{\alpha_1} d^2 \rho_{\beta_0} d^2 \rho_{\beta_1}}{|\rho_{\alpha_0}|^2 |\rho_{\beta_0}|^2} \frac{d^2 \rho_{\alpha_1} d^2 \rho_{\beta_1}}{|\rho_{\alpha_1}|^2 |\rho_{\beta_1}|^2}$$

$$\int \frac{d^2 \rho_{\alpha_0} d^2 \rho_{\alpha_1} d^2 \rho_{\beta_0} d^2 \rho_{\beta_1}}{|\rho_{\alpha_0}|^2 |\rho_{\beta_0}|^2} \frac{d^2 \rho_{\alpha_1} d^2 \rho_{\beta_1}}{|\rho_{\alpha_1}|^2 |\rho_{\beta_1}|^2}$$
\[ \times n_2 (\rho_0 \rho_1; \rho_{a_0} \rho_{a_1}, \rho_{b_0} \rho_{b_1} | y - \bar{y}, \bar{y}) \tilde{n}_2 (\rho'_0 \rho'_1; \rho'_{a_0} \rho'_{a_1}, \rho'_{b_0} \rho'_{b_1} | y' - \bar{y}', \bar{y}') T(\rho_{a_0} \rho_{a_1}, \rho_{a_0}; \rho_{b_0} \rho_{b_1}), \]  

where \( \rho_0 \rho_1 \) are the transverse coordinates of one of the initially colliding dipoles (resp. \( \rho'_0 \rho'_1 \) for the second one), \( \rho_{a_0} \rho_{a_1} \) and \( \rho_{b_0} \rho_{b_1} \), the two interacting dipoles emerging from the dipole \( \rho_0 \rho_1 \) after evolution in rapidity (resp. \( \rho_i \to \rho_{i'}, \) for the second one). It is important to notice that one has to introduce the probability distributions \( n_2(\cdots | y - \bar{y}, \bar{y}) \) of producing two dipoles after a mixed rapidity evolution, namely with a rapidity \( y - \bar{y} \) with one-Pomeron type of evolution and a rapidity \( \bar{y} \) with two-Pomeron type of evolution and then one has to integrate over \( \bar{y} \). The interaction amplitudes \( T(\rho_{a_0} \rho_{a_1}, \rho_{a_0}; \rho_{b_0} \rho_{b_1}) \) and \( T(\rho_{b_0} \rho_{b_1}, \rho_{a_0}; \rho_{b_0} \rho_{b_1}) \) are the elementary two-gluon exchange amplitudes between two colorless dipoles, namely

\[ T(\rho_{a_0} \rho_{a_1}, \rho_{a_0}; \rho_{b_0} \rho_{b_1}) = \int d^2q e^{i\frac{2\pi}{\pi}} (\rho_{a_0} \rho_{a_1}, \rho_{a_0}; \rho_{b_0} \rho_{b_1}) \int d\rho \rho_{a_0} \rho_{b_0} \rho_{b_1}. \]  

\[ \text{Eq.}(8) \]  

\[ n(\cdots) \]  

results from the solution of an evolution equation. The solution is a mere extension to the mixed evolution of the one formulated in ref.\[ \text{Eq.}(9) \]  

\[ \tilde{n}(\cdots) \]  

where

\[ n_2 (\rho_0 \rho_1; \rho_{a_0} \rho_{a_1}, \rho_{b_0} \rho_{b_1} | y - \bar{y}, \bar{y}) = \frac{\tilde{N}_C}{\pi} \int \frac{d\rho \rho_{a_0} \rho_{b_0} \rho_{b_1}}{|\rho_{a_0} \rho_{b_0} \rho_{b_1}|} \times \]  

\[ \int d\omega \frac{e^{\omega \bar{y}}}{\omega - \omega (h)} \int d\omega \frac{e^{\omega y}}{\omega (h_a) + \omega (h_b) - \omega} n_2 (\rho_0 \rho_1; \rho_{a_0} \rho_{a_1}, \rho_{b_0} \rho_{b_1}) \]  

\[ \text{Eq.}(9) \]  

with

\[ E_{1\gamma} (\rho_{a_0} \rho_{a_1}, \rho_{b_0} \rho_{b_1}) \]  

\[ \text{Eq.}(10) \]  

\[ R_{1\gamma}^{h_1, h_2} \]  

\[ \text{Eq.}(11) \]  

\[ \text{Eq.}(12) \]  

where \( \rho = \rho_0 - \rho_1, \rho_{a_0} = \rho_{a_0} - \rho_{a_1}, \rho_{b_0} = \rho_{b_0} - \rho_{b_1}. \)
where \( g_{3P}(h, h_a, h_b) \) is the celebrated triple Pomeron coupling as obtained in
the QCD dipole model, namely:

\[
g_{3P} = \int \frac{d^2r_0 d^2r_1 d^2r_2}{|r_01 r_02 r_12|^2} \frac{[r_{02}]}{r_0 r_2} \left[ \frac{r_{12}}{(1 - r_1)(1 - r_2)} \right]^{h_a} \left[ \frac{\bar{r}_{01}}{\bar{r}_{02} r_12} \right]^{h_b}.
\]

(13)

Considering the Fourier transforms (3) of the \( SL(2, C) \) Eigenvectors, one
writes

\[
n_{2}^{h, h_a, h_b}(\rho_0 \rho_1; \rho_{aa} \rho_{a1}, \rho_{bb} \rho_{b1}) = g_{3P} \int d^2 q_a d^2 q_b d^2 Q \ e^{-i(q_a h_a + q_b h_b + Q b)} \times
\]

\[
\times E^h_Q(\rho) E^{h_a}_q(\rho_a) E^{h_b}_q(\rho_b) \ \delta^{(2)}(Q + q_a + q_b) \int d^2v d^2w \ e^{-i(q_a - q_b + Q b)} \bar{Q} e^{i(2)}
\]

\[
\left\{ \left[ v -1 + h - h_a - h_b \left[ \frac{w - v}{2} \right]^{-1 + h - h_a + h_b} \left[ \frac{w + v}{2} \right]^{-1 + h + h_a - h_b} \right] \right\} \times \{ a.b. \}, \ (14)
\]

where \( \rho_{ab} = v, \rho_{aa} + \rho_{bb}, w, 2h_a = \rho_{aa} + \rho_{a1}, 2h_b = \rho_{bb} + \rho_1 \) and \( b \) is the
overall impact parameter. The notation \{a.h.\} indicates the anti-holomorphic
part of the bracketed term in the integrand for which the integration variables
are complex conjugates and the exponents are replaced by their tilde.

Equivalently, the distribution \( n_2 \) for the lower vertex is given by the same
equation (14) by using prime indices.

3 Calculation

Inserting these equations and the definition (3) in Eqn.(7), the integration over
intermediate states and variables yields a drastic simplification due to the
appearance of quite a few \( \delta \)-functions. First, integrating over impact parameters,
one gets:

\[
\delta(q_a - q^a) \delta(q_b - q^b) \delta(q_a^b - q^b) \delta(q_b^b - q^b).
\]

Then, integrating over the intermediate dipole sizes, one finds

\[
\delta^{(2)}(h_a, h^a) \delta^{(2)}(h_b, h^b) \delta^{(2)}(h^a_a, 1 - h^a_a) \delta^{(2)}(h^b_b, 1 - h^b_b),
\]

where \( \delta^{(2)}(h, h') \equiv \delta_{nn'} \delta(\nu - \nu'). \) The integration over \( q_a - q_b \) finally gives
\( \delta(\nu - \nu'). \)
Plugging in the general formula (1) the results obtained in formulae (2) to (4) and integrating over $\delta$-functions, one gets:

$$f_Q(\rho, \rho'; Y) = \alpha^4 \left( \frac{\alpha N_C}{\pi} \right)^2 \int dh'dh_a dh_b \ g_{3P}(h, h_a, h_b) \ g_{3P}(h', h_a, h_b)$$

$$\times \left[ E_Q^h(\rho) \ E_Q^{h'}(\rho') \right] \ H_Q(h, h')$$

$$\times \int_0^y dy \int_0^{Y-y} dy' \int d\omega d\omega' d\omega'_1 \frac{e^{\omega y + \omega_1(y-y')}}{(\omega(h_a) + \omega(h_b) - \omega)(\omega_1 - \omega(h))}$$

where, after the change of variables $w = v(1 - 2t) \ w' = v(1 - 2t')$,

$$H_Q(h, h') = \int d^2 v d^2 t d^2 t' \ e^{iQv(t-t')} \times$$

$$\{ v^{h-h'-1} t^{-1+h+h_a} (1 - t)^{-1+h-h_b+h_a t^{h_a-h_b-h'} (1 - t')^{-h_a+h_b-h'}} \times \{ a.h. \}. \ (15)$$

After integration over $v$ one obtains (16)

$$\{ t^{-1+h+h_b-h_a} (1 - t)^{-1+h-h_a+h_a t^{h_a-h_b-h'} (1 - t')^{-h_a+h_b-h'}} \times \{ a.h. \}. \ (17)$$

The remaining integral is of a type which has already been met (11). Following the method of paper (14) the result can be expressed as follows

$$H_Q(h, h') = \frac{1 + h'}{h - h'} \ h' + h \ e^{i\pi(h-h')} \gamma(h - h_a + h_b) \gamma(1 - h' + h_a - h_b). \ (18)$$

The integral over $h'$ can now easily be performed. The remaining poles at $h' = h$ and $h' = 1 - h$ give twice the same contribution due to the over completeness relation of the $E_Q^h$ generators (1). One finally obtains:

$$f_Q(\rho, \rho'; Y) = \alpha^4 \left( \frac{\alpha N_C}{\pi} \right)^2 \int dh'dh_a dh_b \ \left| g_{3P}(h, h_a, h_b) \right|^2 \left[ E_Q^h(\rho) \ E_Q^{h'}(\rho') \right]$$

$$\times \int_0^y dy \int_0^{Y-y} dy' \int d\omega d\omega' d\omega'_1 \frac{e^{\omega y + \omega_1(y-y')}}{(\omega(h_a) + \omega(h_b) - \omega)(\omega_1 - \omega(h))}$$
tree-level BFKL 4-gluon amplitude, since the only scale-dependence present in the conformal Eigenvectors $E_Q^b(\rho) E_Q^\dagger(\rho')$. It is worthwhile to notice that we recover at $Q = 0$ the forward one-loop amplitude which has been computed by a simpler method in paper [1]. Conformal invariance at one-dipole loop level, which has been assumed in [2], is thus now fully proven.

4 Result

The integration over rapidity variables yields two different contributions depending on the sign of the quantity $\omega(h_a) + \omega(h_b) - \omega(h)$. Indeed for $\omega(h_a) + \omega(h_b) < \omega(h)$, the relevant poles are situated at $\omega = \omega_1 = \omega' = \omega_1' = \omega(h)$, leading to expression (20), see further on, which is associated with the single Pomeron dependence $e^{\omega(h)Y}$. In the opposite case, namely $\omega(h_a) + \omega(h_b) > \omega(h)$, the relevant poles are situated at $\omega = \omega_1 = \omega' = \omega_1' = \omega(h_a) + \omega(h_b)$. The resulting amplitude is given by (21) which corresponds to the double-Pomeron energy behaviour $e^{(\omega(h_a) + \omega(h_b))Y}$. Notice that either expression depends only on the sum $Y = y + y'$, as it should from longitudinal boost invariance.

The final expressions read:

$$f_Q^{(P)}(\rho, \rho'; Y) = \alpha^4 \int dh E_Q^b(\rho') E_Q^h(\rho) \int dh_a dh_b \left| \frac{g_{3P}(h_a, h_b)}{b(h_a)b(h_b)} \right|^2 e^{\omega(h)Y} \frac{\left(\chi(h) - \chi(h_a) - \chi(h_b)\right)^2}{(\chi(h) - \chi(h_a) - \chi(h_b))^2},$$

(20)

$$f_Q^{(P\otimes P)}(\rho, \rho'; Y) = \alpha^4 \int dh E_Q^b(\rho') E_Q^h(\rho) \int dh_a dh_b \left| \frac{g_{3P}(h_a, h_a, h_b)}{b(h_a)b(h_b)b(h)} \right|^2 e^{(\omega(h_a) + \omega(h))Y} \frac{\left(\chi(h) - \chi(h_a) - \chi(h_b)\right)^2}{(\chi(h) - \chi(h_a) - \chi(h_b))^2},$$

(21)

where $g_{3P}$ is the triple-QCD-Pomeron vertex (e.g. 1 $\rightarrow$ 2 dipole vertex in the $1/N_C$ limit) which has been recently derived [12] and evaluated [13].

The two contributions correspond respectively to the one dipole loop correction to the BFKL Pomeron (20) and to the two Pomeron exchange (21). Indeed, the energy dependence of these contributions is fixed by the asymptotic behaviour in $Y$ of formulae (20,21), which has been shown [1] to be, respectively, in the vicinity of the one-Pomeron and two-Pomeron intercepts. Notice that
the forward amplitudes \((Q = 0)\) are obtained by replacing \(E_Q^b(\rho) \rightarrow (\rho)^{1-b} \). Formulae (20,21) are the main results of this paper.

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