NONLINEAR ELLIPTIC EQUATIONS WITH INTEGRO-DIFFERENTIAL DIVERGENCE FORM OPERATORS AND MEASURE DATA UNDER SIGN CONDITION ON THE NONLINEARITY

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Abstract. We study existence problem for semilinear equations with Borel measure data and operator generated by a symmetric Markov semigroup. We assume merely that the nonlinear part satisfies the so-called sign condition. Using the method of sub and super-solutions we show the existence of maximal measure for which there exists a solution to the problem (the so-called reduced measure introduced by H. Brezis, M. Marcus and A.C. Ponce).

1. Introduction

Formulation of the problem. Let $E$ be a locally compact separable metric space and $m$ be a full support positive Radon measure on $E$. Consider a self-adjoint operator $(A,D(A))$ on $L^2(E;m)$ generating a Markov semigroup $(T_t)$, a Caratheodory function $f : E \times \mathbb{R} \to \mathbb{R}$ satisfying the sign condition:

$$f(x,u) \cdot u \leq 0, \quad u \in \mathbb{R}, \quad m\text{-a.e. } x \in E,$$

and Borel measure $\mu$ on $E$ which obeys $\int_E \rho \, d\mu < \infty$ for a strictly positive weight $\rho : E \to \mathbb{R}$ (class of such measures shall be denoted by $\mathcal{M}_\rho$). In the present paper, we investigate the existence problem for the following equation:

$$-Au = f(\cdot,u) + \mu. \quad (1.2)$$

Throughout the paper, we assume that there exists the Green function $G$ for $A$ (see Section 2.2), and $T_t \rho \leq \rho \, m\text{-a.e.}$ for any $t \geq 0$ (e.g. $\rho \equiv 1$ or $\rho$ being the principal eigenfunction for $-A$ satisfies the last requirement). The model example of purely non-local operator, which fits into our framework is

$$Au(x) = \lim_{r \searrow 0} \int_{\mathbb{R}^d \setminus B(x,r)} \frac{a(x,y)(u(x) - u(y))}{|x-y|^d \varphi(|x-y|)} \, dy, \quad (1.3)$$

with symmetric $a$ bounded between two strictly positive constants and $\varphi$ satisfying some standard growth assumptions (see Section 3). In particular, taking $a \equiv \text{const.}$, $\varphi(r) = r^{2\alpha}$ gives the fractional Laplacian (i.e. $A = -(\Delta)^\alpha$), and taking $1/\varphi(r) = \int_0^1 r^{-2\alpha} \nu(d\alpha)$, with $\nu$ being a positive finite Borel measure compactly supported in $(0,1)$, gives a mixed relativistic symmetric stable operator (see [20]).

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A model example of a local operator fitting into our framework is

$$Au(x) = \sum_{i,j=1}^{d} (a_{i,j}(x)u_{x_i})x_j,$$

(1.4)

where $a = [a_{i,j}]_{i,j=1}^{d}$ is a symmetric matrix with Borel measurable locally integrable entries satisfying ellipticity condition:

$$w(x)|\xi|^2 \leq \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \leq v(x)|\xi|^2, \quad \xi, x \in \mathbb{R}^d,$$

where $v, w$ are suitable positive functions (see e.g. [14]). For other examples of noteworthy classes of operators, which the theory of the present paper is applicable to, see Section 3.

By a solution to (1.2), we mean a Borel function $u$ on $E$ (finite $m$-a.e.) such that $f(\cdot, u) \in L^1(E; \rho \cdot m)$, and

$$u(x) = \int_E G(x,y)f(y,u(y))m(dy) + \int_E G(x,y)\mu(dy), \quad m\text{-a.e. } x \in E.$$  

(1.5)

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form generated by $(A, D(A))$:

$$\mathcal{E}(w, v) := (\sqrt{-A}w, \sqrt{-A}v), \quad w, v \in D(\mathcal{E}) := D(\sqrt{-A}),$$

and $\text{Cap}_A$ be a capacity generated by $A$: for open $U \subset E$,

$$\text{Cap}_A(U) := \inf \mathcal{E}(w, w) : w \geq 1_U \text{ m-a.e.}, \quad w \in D(\mathcal{E}),$$

(1.6)

and for arbitrary $B \subset E$, $\text{Cap}_A(B) = \inf \{\text{Cap}_A(V) : B \subset V, \ V \text{ is open}\}$. Using this subadditive set function, we may consider a unique decomposition:

$$\mu = \mu_d + \mu_c$$

of any measure $\mu \in \mathcal{M}_p$, where $\mu_c \perp \text{Cap}_A$ and $\mu_d \ll \text{Cap}_A$. It is well known (see [30]) that if $\mu \ll \text{Cap}_A$, then there exists a solution to (1.2). However, if $\mu_c$ is non-trivial, then by [9, Theorem 4.14], one can find a function $f$ (independent of $x \in E$) satisfying the above conditions such that there is no solution to (1.2) with $A = \Delta$ on a bounded domain $D \subset \mathbb{R}^d$ with zero Dirichlet boundary condition. This shows that the presence of the non-trivial concentrated part $\mu_c$ of measure $\mu$ changes the picture completely. Our goal is to study the existence and non-existence mechanism for (1.2) hidden in the relation between operator $A$, right-hand side $f$ and concentrated measure $\mu_c$.

It is worth mentioning here that, by [26, Proposition 4.3, Theorem 4.9], if $u$ is a solution to (1.2) with $\mu \in \mathcal{M}_1$, then it is a renormalized solution to (1.2), i.e.

(i) $f(\cdot, u) \in L^1(E; m)$, $T_k(u) := \max\{-k, \min\{u, k\}\} \in D_{e}(\mathcal{E})$, $k \geq 0$,

(ii) for any $k \geq 0$ there exists a bounded measure $\lambda_k$ such that $\lambda_k \ll \text{Cap}_A$ and for any bounded $\eta \in D_{e}(\mathcal{E})$

$$\mathcal{E}(T_k(u), \eta) = \int_E f(\cdot, u)\eta dm + \int_E \tilde{\eta} d\mu_d + \int_E \tilde{\eta} d\lambda_k,$$

(iii) $\int_E \xi d\lambda_k \rightarrow \int_E \xi d\mu_c$ as $k \rightarrow \infty$ for any $\xi \in C_0(E)$ (i.e. narrowly).

Here $\tilde{\eta}$ is a quasi-continuous $m$-version of $\eta$, and $(\mathcal{E}, D_{e}(\mathcal{E}))$ is the extended Dirichlet form (see Section 2). The converse is also true - renormalized solution solves (1.2) - if we assume additionally e.g., that the resolvent of $(A, D(A))$ maps $B_0(E)$ into $C_0(E)$ (see [26, Theorem 4.9]).

Main results of the paper. Fix a strictly positive bounded Borel function $\varrho \in L^1(E; m) \cap L^2(E; m)$ such that

$$R\varrho := \int_E G(\cdot, y)\varrho(y) m(dy) \leq \rho \quad m\text{-a.e.},$$
and \( R_0 \) is bounded. For the existence results, we need one more assumption on \( f \), namely that for any \( \underline{u}, \overline{u} \in L^1(\Omega; g \cdot m) \) such that \( f(\cdot, \underline{u}), f(\cdot, \overline{u}) \in L^1(\Omega; \rho \cdot m) \) we have
\[
\underline{u} \leq x \leq \sup_{y \in [\underline{u}(x) \overline{u}(x)]} |f(x, y)| \leq \overline{u} \in L^1(\Omega; \rho \cdot m).
\]

The above condition is satisfied, supposing e.g. that \( f \) is non-increasing with respect to \( y \) or there exists an increasing function \( g \) on \( \mathbb{R} \) such that \( c_1 g(y) \leq |f(x, y)| \leq c_2 g(y), \ x \in E, y \in \mathbb{R} \) for some \( c_1, c_2 > 0 \). In Section 4, we extend Perron’s method of supersolutions and subsolutions and prove the following result (Theorem 4.10).

**Theorem 1.** Let \( A, f, \mu \) be as in the foregoing. Assume that there exists a subsolution \( \underline{u} \) to (1.2) and a supersolution \( \overline{u} \) to (1.2) such that \( \underline{u} \leq \overline{u}, m \)-a.e. Then, there exists a maximal solution \( u \) to (1.2) such that \( \underline{u} \leq u \leq \overline{u}, m \)-a.e.

The above result, but for \( A = \Delta \), was proved in [38]. To go further in Section 5, we extend the theory of *reduced measures* introduced by Brezis, Marcus and Ponce in [8, 9] for the classical Dirichlet Laplacian and monotone \( f \). Let \( \mathcal{G}(f) \) be a class of measures \( \mu \in \mathcal{M}_\rho \) for which there exists a solution to (1.2), and for given \( \nu \in \mathcal{M}_\rho \) denote by \( \mathcal{G}_\nu(f) \) (resp. \( \mathcal{G}_2(f) \)) the class of measures \( \mu \in \mathcal{G}(f) \) satisfying \( \mu \leq \nu \) (resp. \( \mu \geq \nu \)). The measures belonging to \( \mathcal{G}(f) \) are called *good measures*. As we mentioned before in general \( \mathcal{G}(f) \subseteq \mathcal{M}_\rho \). It appears that for any \( \mu \in \mathcal{M}_\rho \) such that \( \mathcal{G}_\nu(f) \neq \emptyset \), we can always find the biggest measure \( \mu^{* \cdot f} \) less than \( \mu \) such that \( \mu^{* \cdot f} \in \mathcal{G}(f) \) - such measure is called a *reduced measure*. This is, in particular, the content of our second main result (Theorem 5.2).

**Theorem 2.** Let \( A, f, \mu \) be as in the foregoing. Assume that \( \mathcal{G}_\mu(f) \neq \emptyset \).

1. There exists \( \mu^{* \cdot f} \in \mathcal{G}_\mu(f) \) such that
\[
\mu^{* \cdot f} = \max \mathcal{G}_\mu(f).
\]

2. Let \( \phi \) be a strictly positive function in \( L^1(\Omega, \rho \cdot m) \). For any \( n \geq 1 \) there exists a maximal solution \( u_n \) to
\[
-Au = \max \{-n \phi, f\}(\cdot, u) + \mu.
\]
Moreover, \( u_n \searrow u^{* \cdot f} \), where \( u^{* \cdot f} \) is a maximal solution to
\[
-Au = f(\cdot, u) + \mu^{* \cdot f}.
\]

3. \( \mu^{* \cdot f} = \mu_d - \mu_e^* + \nu \) for a Borel measure \( \nu \) on \( E \) satisfying \( 0 \leq \nu \leq \mu_e^* \).

Clearly, when \( \mu \) is a good measure, then \( \mu = \mu^{* \cdot f} \) and so, by assertion (2) of the above theorem, for any \( \mu \in \mathcal{G}(f) \) there exists a maximal solution to (1.2). For brevity, and when there is no risk for confusion, we mostly omit the superscript \( \cdot \) on \( \mu^{* \cdot f} \) and \( u^{* \cdot f} \). In case \( \mathcal{G}_\mu(f) \neq \emptyset \), we may also consider the problem of the existence of the smallest good measure greater than \( \mu \). By a simple calculation we find that

\[
\mu_{* \cdot f} := -(-\mu)^{* \cdot f},
\]
where \( \tilde{f}(x, y) := -f(x, -y), x \in E, y \in \mathbb{R} \), is a solution to the latter problem. Therefore, by Theorem 2 provided \( \mathcal{G}_\mu(f) \neq \emptyset \), we get the existence of \( \mu_{* \cdot f} \in \mathcal{G}(f) \) such that

\[
\mu_{* \cdot f} = \min \mathcal{G}_\mu(f).
\]

Using the notion of reduced measures and the results of Theorem 1 and Theorem 2, we easily get the following result (we follow the idea of A.C. Ponce).

**Theorem 3.** Let \( A, f, \mu \) be as in the foregoing. Assume that there exists a subsolution and a supersolution to (1.2). Then there exists a maximal solution to (1.2).
Observe that contrary to Theorem 1, we do not demand in Theorem 3 that the subsolution be less than or equal to the supersolution m.a.e.

In Section 6, we prove a series of results concerning the properties of the set $\mathcal{G}(f)$ and the reduction operator $\mu \mapsto \mu^*$. They exhibit that the properties of these two mathematical objects, proved before in the literature for Dirichlet Laplacian and non-increasing $f$, extend to the general framework considered here with an exception that the reduction operator is no longer Lipschitz continuous. Therefore, the main concern of Section 6 will be continuity of the reduction operator. We also observe that certain mapping defined via the reduction operator is a continuous metric projection onto $\mathcal{G}(f)$.

**Theorem 4.** Let $A,f,\mu$ be as in the foregoing. The set $\mathcal{G}(f)$ is a convex and closed subset of $\mathcal{M}_\rho$ with total variation norm $\|\mu\|_\rho := \int_E \rho d|\mu|$. Moreover, the mapping

$$\Pi_f : \mathcal{M}_\rho \to \mathcal{G}(f)$$

defined as $\Pi_f(\mu) := (\mu^*)^* + (-\mu^-)_*$ is a continuous metric projection onto $\mathcal{G}(f)$, i.e.

$$\|\Pi_f(\mu) - \mu\|_\rho = \inf_{\nu \in \mathcal{G}(f)} \|\nu - \mu\|_\rho.$$ 

Moreover, if $Q : \mathcal{M}_\rho \to \mathcal{G}(f)$ is a metric projection onto $\mathcal{G}(f)$, with a property that for any orthogonal $\mu, \nu \in \mathcal{M}_\rho$,

$$Q(\mu + \nu) = Q(\mu) + Q(\nu),$$

then $Q = \Pi_f$.

One of the illustrative results concerning the structure of the set $\mathcal{G}(f)$, easily following from the results of Section 6, is the following equality

$$\mathcal{G}(f) = \mathcal{A}(f) + L^1(E; \rho \cdot m),$$

where $\mathcal{A}(f)$ is the class of *admissible measures*: it consists of measures $\mu \in \mathcal{M}_\rho$ such that $|f(\cdot, R\mu)| \in L^1(E; \rho \cdot m)$. In Section 7, we prove much stronger result. Let $B_{L^1}(0,r) := \{u \in L^1(E; \rho \cdot m) : \|u\|_{L^1(E; \rho \cdot m)} \leq r\}$.

**Theorem 5.** Let $A,f,\mu$ be as in the foregoing. Moreover, assume that $\rho$ is bounded and there exists $\varepsilon > 0$ such that $\sup_{|y| \leq \varepsilon} |f(\cdot, y)| \in L^1(E; \rho \cdot m)$.

1. For any $r > 0$

$$\mathcal{G}(f) = \mathcal{A}(f) + B_{L^1}(0,r).$$

2. Let $\text{cl}$ denote the closure operator in $(\mathcal{M}_\rho, \|\cdot\|_\rho)$. Then

$$\mathcal{G}(f) = \text{cl} \mathcal{A}(f).$$

The last assertion implies that for any $g$, satisfying the same conditions as $f$, if

$$f(u) \sim g(u), \quad |u| \to \infty,$$

then $\mu^* f = \mu^{*g}$ for positive $\mu \in \mathcal{M}_\rho$ (see Corollary 7.5). In other words, the reduction operator and the class of good measures, for $f$ independent of the state variable, depend only on the behavior of $f$ at infinity.

**Some comments on the literature related to the problem.** Concerning the existence results for (1.2) with $\mu \ll \text{Cap}_A$, we mention the paper by H. Brezis and W. Strauss [11], and by Y. Konishi [34], where $f$ is assumed to be non-increasing and independent of $x \in E$ and $\mu \in L^1(E; m)$, the paper by T. Klimsiak and A. Rozkosz [32], where $f$ is non-increasing, and their another paper [29], where $f$ merely satisfies the sign condition.

As to the existence results for (1.2) with general measure data, above all, Bénilan and Brezis’ paper [4] should be mentioned. It was published in 2004, however it summarizes, among other things, the existence and non-existence results on the problem (1.2), with
A = Δ and non-increasing f, achieved in the period 1975-2004 (see Appendix A in [4]). Most part of the said paper is concerned with variational problems related to the Thomas-Fermi energy functional:

\[ J_{TF}(\eta) := \frac{1}{2} \int_E R\eta \cdot \eta dm + \int_E (j(\cdot, \eta) - R\mu) dm, \]

where \( j : E \times \mathbb{R} \to [0, \infty) \) is a function satisfying

\begin{itemize}
  \item \( j(x, 0) = 0 \) m.a.e., \( j(x, r) = \infty, r < 0 \) m.a.e., \( j(x, r) < \infty, r > 0 \) m.a.e.
  \item \( r \mapsto j(x, r) \) is convex and l.s.c. m.a.e.,
\end{itemize}

and the domain of \( J_{TF} \) is as follows

\[ D(J_{TF}) := \{ \eta \in L^1(E; m) : \eta \geq 0, \int_E R\eta \cdot \eta dm < \infty, j(\cdot, \eta) - R\mu \in L^1(E; m) \}. \]

However, in [4] it is given an interesting result which relates problem (1.2) with Thomas-Fermi functional. Namely, under some additional assumptions on \( R \), it is proved in [4, Theorem 1] that if a strictly positive \( \eta_0 \in D(J_{TF}) \) (\( I := \int_E \eta_0 dm \)) satisfies

\[ J_{TF}(\eta_0) \leq J_{TF}(\eta), \quad \eta \in D(J_{TF}), \quad \int_E \eta dm = I, \]

then there exists \( \lambda \in \mathbb{R} \) such that

\[ -Au = f_\lambda(\cdot, u) + \mu, \quad (1.8) \]

with \( u := R\eta_0 - R\mu \) and \( f_\lambda(x, y) := -(\partial j)^{-1}(x, y - \lambda) \), where \( \partial j \) is the subdifferential of \( j \) with respect to the second variable. Although the paper [4] is focused on the minimization problem (1.7), which leads to (1.8) in the special case of \( \eta_0 \) being strictly positive, it is worth mentioning that for the existence of \( \eta_0 \) in (1.7) it is always assumed in [4] that, up to translation, \( \mu \in A(f_0) \) (see conditions (H), (H'), (3.18) in [4]).

In 2004 Brezis, Marcus and Ponce [8, 9] introduced the notion of reduced measures for (1.2) with \( A = \Delta \) and \( f \) being non-increasing. Since then the research on equations of the form (1.2) has flourished once more, mostly with \( A \) being the Dirichlet Laplacian or Dirichlet (rarely regional) fractional Laplacian. We limit ourselves to mentioning [3, 15, 22, 38, 43, 45] in case of Dirichlet Laplacian or divergence form diffusion operators and [16, 17, 18, 35, 37] in case of the fractional Laplacian.

The theory of reduced measures was generalized by the author of the present paper in [27] to a class of Dirichlet operators and with \( f \) being non-increasing. The goal of the present paper is to analyze equation (1.2) for the same class of operators as considered in [27] but under the assumption that \( f \) merely satisfies the sign condition (1.1). Some results, however, are new even for monotone \( f \), e.g. Theorem 5.

2. Notation, basic notions and standing assumptions

As it was said in the introduction, \( I \) is assumed to be a self-adjoint operator on \( L^2(E; m) \) generating a strongly continuous Markov semigroup \((T_t)_{t\geq0}\) on \( L^2(E; m) \) - so called Dirichlet operators. Thus, \((T_t)_{t\geq0}\) is a contraction on \( L^2(E; m) \) and, as a result, \((0, \infty)\) is a subset of the resolvent set for \((T_t)_{t\geq0}\). By \((J_\alpha)_{\alpha>0}\) we denote the resolvent family for \((T_t)_{t\geq0}\) on \( L^2(E; m) \). Throughout the paper, we assume that \((T_t)_{t\geq0}\) is transient, i.e. there exists a strictly positive function \( g \in L^2(E; m) \) such that

\[ J_0 g := \text{ess sup}_{n \geq 1} \int_0^n T_t g \, dt < \infty \quad m\text{-a.e.} \]

Moreover, we assume that there exists the Green function \( G \) for \(-A\). The precise meaning of this condition shall be explained in Section 3.
In what follows we fix a strictly positive excessive function (see Section 2.2) \( \rho \) and strictly positive bounded Borel measurable function \( \varrho \in L^1(E; m) \cap L^2(E; m) \) such that \( J_0 \varrho \) is bounded and
\[
J_0 \varrho \leq \rho \quad \text{m.a.e.} \tag{2.1}
\]
For the existence of \( \varrho \) see e.g. [33, Lemma 6.1]. Observe that if \( m(E) < \infty \) and \( E \) is Green bounded, i.e.
\[
\sup_{x \in E} \int_E G(x, y) m(dy) < \infty,
\]
then we may take \( \varrho \equiv \rho \equiv \text{const.} \). We also consider the following condition
\[(A1) \quad \text{for any } u, \varpi \in L^1(E; \varrho \cdot m) \text{ such that } u \leq \varpi \text{ and } f(\cdot, u), f(\cdot, \varpi) \in L^1(E; \rho \cdot m) \text{ we have}
\[
x \mapsto \sup_{y \in [u(x), \varpi(x)]} |f(x, y)| \in L^1(E; \rho \cdot m).
\]
Observe that (A1) is easily verified provided \( f \) is non-increasing with respect to \( y \) or there exists an increasing function \( g : \mathbb{R} \to \mathbb{R} \) such that \( c_1 g(y) \leq |f(x, y)| \leq c_2 g(y) \), \( x \in E, y \in \mathbb{R} \) for some \( c_1, c_2 > 0 \). By \( \mathcal{M}_\rho \) we denote the set of Borel measures on \( E \) such that
\[
\| \mu \|_\rho := \int_E \rho \, d|\mu| < \infty.
\]
\( \mathcal{B}(E) \) stands for the set of Borel measurable function on \( E \), and \( \mathcal{B}_b(E) \) (resp. \( \mathcal{B}^+(E) \)) is a subset of \( \mathcal{B}(E) \) consisting of bounded (resp. positive) functions. For given Borel measure \( \mu \) on \( E \) and \( \eta \in \mathcal{B}(E) \) such that \( \int_E |\eta| \, d|\mu| < \infty \) we let
\[
\langle \eta, \mu \rangle := \int_E \eta \, d\mu.
\]
We also denote by \( \eta \cdot \mu \) a Borel measure on \( E \) defined as follows
\[
(\xi, \eta \cdot \mu) := \langle \xi \eta, \mu \rangle, \quad \xi \in \mathcal{B}_b(E). \tag{1.6}
\]
By \((\mathcal{E}, D(\mathcal{E}))\), we denote a symmetric Dirichlet form on \( L^2(D; m) \) generated by \((A, D(A))\) defined as follows: \( D(\mathcal{E}) := D(\sqrt{-A}) \), and \( \mathcal{E}(u, v) := \langle \sqrt{-A} u, \sqrt{-A} v \rangle \), \( u, v \in D(\mathcal{E}) \). Throughout the paper, we assume that \((\mathcal{E}, D(\mathcal{E}))\) is regular, i.e. \( C_c(E) \cap D(\mathcal{E}) \) is dense in \( D(\mathcal{E}) \) with the norm \( \| \cdot \|_{\mathcal{E}} := \langle (\cdot, \cdot) + (\cdot, L^2(E; m)) \rangle^{1/2} \) and in \( C_c(E) \) with the uniform convergence norm. Self-adjoint Dirichlet operators with regular associated form \((\mathcal{E}, D(\mathcal{E}))\) shall be called regular.

2.1. Elements of potential theory. Let us remind that \( \text{Cap} \) is a set function defined by (1.6). We say that a property \( P \) holds quasi-everywhere (q.e. for short) on \( E \) if it holds except a set \( B \subset E \) such that \( \text{Cap}(B) = 0 \). An increasing sequence \( \{ F_n \} \) of closed subsets of \( E \) is called a nest iff \( \text{Cap}(E \setminus F_n) \to 0, n \to \infty \). A function \( u \) on \( E \) is called quasi-continuous iff for any \( \varepsilon > 0 \) there exists closed set \( F_\varepsilon \subset E \) such that \( \text{Cap}(E \setminus F_\varepsilon) \leq \varepsilon \) and \( u_{|_{F_\varepsilon}} \) is continuous. By [24, Theorem 2.1.2], \( u \) is quasi-continuous if and only if there exists a nest \( \{ F_n \} \) such that for any \( n \geq 1, u_{|_{F_n}} \) is continuous. An increasing sequence \( \{ F_n \} \) of closed subsets of \( E \) is called a generalized nest iff for every compact \( K \subset E, \text{Cap}(K \setminus F_n) \to 0, n \to \infty \). A Borel measure \( \mu \) on \( E \) is called smooth iff it is absolutely continuous with respect to \( \text{Cap} \), and there exists a generalized nest \( \{ F_n \} \) such that \( |\mu|(F_n) < \infty, n \geq 1 \). \( \mathcal{M}_0^0 \) stands for a subset of \( \mathcal{M}_\rho \) consisting of smooth measures. We say that a measurable function \( u \) on \( E \) is quasi-integrable iff for every \( \varepsilon > 0 \) there exists a closed set \( F_\varepsilon \subset E \) such that \( \text{Cap}(E \setminus F_\varepsilon) \leq \varepsilon \) and \( 1_{F_\varepsilon} u \in L^1(E; m) \). We say that a measurable function \( u \) on \( E \) is locally quasi-integrable iff for every compact \( K \subset E, 1_K u \) is quasi-integrable.

Proposition 2.1. A measurable function \( u \) on \( E \) is locally quasi-integrable iff there exists a generalized nest \( \{ F_n \} \) such that \( 1_{F_n} u \in L^1(E; m), \; n \geq 1 \).
**Proof.** Sufficiency. Consider a compact set $K \subset E$. We shall show that $1_K u$ is quasi-integrable. Fix $\varepsilon > 0$. Let $V$ be a relatively compact open set such that $K \subset V$. By the assumption there exists a closed set $F_n$ such that $\text{Cap}(V \setminus F_n) \leq \varepsilon$ and $1_{F_n} u \in L^1(E;m)$. Set $F_\varepsilon = V^c \cup F_n$. Then $1_{F_\varepsilon} \cap K u \in L^1(E;m)$ and

$$\text{Cap}(E \setminus F_\varepsilon) = \text{Cap}(V \setminus F_n) \leq \text{Cap}(V \setminus F_\varepsilon) \leq \varepsilon.$$  

Necessity. Let $\{E_n\}$ be an increasing sequence of relatively compact open sets such that $\bigcup_{n \geq 1} E_n = E$. By the assumption for every $n \geq 1$ there exists closed $F_n \subset E_n$ such that $\text{Cap}(E_n \setminus F_n) \leq \frac{1}{n}$ and $1_{F_n} u \in L^1(E;m)$. Set

$$F_n = \bigcup_{j=1}^n F_{kj},$$

Clearly $\{F_n\}$ is an increasing sequence of closed sets and $1_{F_n} u \in L^1(E;m)$, $n \geq 1$. Let $K \subset E$ be a compact set. Then there exists $n_0 \geq 1$ such that $K \subset E_{n_0}, n \geq n_0$. For $n \geq n_0$,

$$\text{Cap}(K \setminus F_n) \leq \text{Cap}(E_{n_0} \setminus F_n) \leq \frac{1}{n}.$$  

Therefore, $\{F_n\}$ is a generalized nest.

Since we assumed that $(T_t)$ is transient, there exists a strictly positive function $g$ on $E$ such that

$$\int_E |u| g \, dm \leq (\mathcal{E}(u,u))^{1/2}, \quad u \in D(\mathcal{E}).$$

Therefore, there exists the extension $D_\varepsilon(\mathcal{E})$ of $D(\mathcal{E})$ such that: $D_\varepsilon(\mathcal{E}) \subset L^1(E; g \cdot m)$, $D(\mathcal{E}) = D_\varepsilon(\mathcal{E}) \cap L^2(E;m)$, and for any $u \in D_\varepsilon(\mathcal{E})$, there exists a sequence $\{u_n\} \subset D(\mathcal{E})$ that is a Cauchy sequence in the norm $\| \cdot \|_{\mathcal{E}}$ and satisfies $u_n \to u$ in $L^1(E; g \cdot m)$ and $\mathcal{E}(u_n, u_n) \to \mathcal{E}(u, u)$ (see [24, Theorem 1.5.1, Theorem 1.5.2]). Clearly, $(\mathcal{E}, D_\varepsilon(\mathcal{E}))$ is a Hilbert space. By [24, Theorem 2.1.7], any $u \in D_\varepsilon(\mathcal{E})$ possesses an $m$-version which is quasi-continuous. In what follows for $u \in D_\varepsilon(\mathcal{E})$ we denote by $\bar{u}$ quasi-continuous $m$-version of $u$.

### 2.2. Probabilistic potential theory

Let $\partial$ be either an isolated point added to $E$ provided $E$ is compact - or a one-point compactification of $E$ provided $E$ is not compact. We let $E_\partial := E \cup \{\partial\}$. Throughout the paper, we adopt the convention that whenever $f$ is a function defined on $B$, then it is automatically extended to $B \cup \{\partial\}$ by putting $f(\partial) = 0$.

Let

$$\Omega := \{\omega : [0, \infty) \to E_\partial : \omega \text{ is cádlág, and } \omega(s) = \partial, s \geq t \text{ whenever } \omega(t) = \partial\}.$$  

Recall that a function $\omega : [0, \infty) \to E_\partial$ is called cádlág if it is right-continuous on $[0, \infty)$ and left-limited on $(0, \infty)$. We endow $\Omega$ with the Skorohod topology $d$. Then $(\Omega, d)$ is a separable metric space (see [5, Section 12]). We also consider *shift operators* $(\theta_t)_{t \geq 0}$:

$$\theta_t : \Omega \to \Omega, \quad \theta_t(\omega)(s) := \omega(s+t), \quad s, t \geq 0,$$

and a family of *projection operators* (also called the *canonical process*) $X_t : \Omega \to E_\partial$, $t \geq 0$, $X_t(\omega) := \omega(t), \omega \in \Omega$. Let $\mathcal{F}_t^0 := \sigma(X_s^1(B) : s \leq t, B \in \mathcal{B}(E_\partial))$. By [24, Theorem 7.2.1], there exists a family $(P_x)_{x \in E_\partial}$ of Borel probability measures on $\Omega$ and a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ on $\Omega$ such that $\mathcal{X} := ((P_x)_{x \in E_\partial}, (\mathcal{F}_t)_{t \geq 0})$ is a *Hunt process* on $E_\partial$ associated with $(A, D(A))$, i.e. for any $f \in \mathcal{B}(E) \cap L^2(E;m)$,

$$T_t f(x) = \int_{\Omega} f(X_t(\omega)) \, P_x(d\omega), \quad t \geq 0, \text{ m-a.e.}$$

By the very definition of a Hunt process, $\mathcal{F}_t^0 \subset \mathcal{F}_t, t \geq 0$. The question of uniqueness of $\mathcal{X}$ is treated in [24, Theorem 4.2.8]. Let $\zeta$ stand for the lifetime of process $\mathcal{X}$, i.e.

$$\zeta(\omega) := \inf\{t \geq 0 : X_t(\omega) = \partial\}.$$
We let $\mathcal{F}_\infty := \sigma(\mathcal{F}_t : t \geq 0)$. For $t \in [0, \infty]$, we denote by $b\mathcal{F}_t$ a set of bounded real valued $\mathcal{F}_t$ measurable functions. In what follows, we consider the following notation

$$\mathbb{E}_x F := \int_\Omega F(\omega) P_x(\,d\omega), \quad F \in b\mathcal{F}_\infty$$

We define for any $f \in B^+(E)$,

$$P_t f(x) := \mathbb{E}_x f(X_t), \quad R_\alpha f(x) := \mathbb{E}_x \int_0^\infty e^{-\alpha s} f(X_s) \,ds, \quad t \geq 0, \alpha \geq 0, x \in E.$$ 

We let $R := R_0$. We say that a property $P$ holds almost surely (a.s.) (resp. quasi almost surely (q.a.s.)) on $\Omega$ if it holds $P_x$-a.s. for any $x \in E$ (resp. for q.e. $x \in E$). A Borel measurable positive function $f$ on $E$ is called $\alpha$-excessive, where $\alpha \geq 0$, if

$$\sup_{t>0} e^{-\alpha t} P_t f(x) = f(x), \quad x \in E.$$ 

In case $\alpha = 0$, we just say that $f$ is excessive.

Recall that a family $(A_t)_{t \geq 0}$ of $\mathbb{R} \cup \{+\infty\}$ valued functions on $\Omega$ is called an additive functional of $X$ if there exist $\Lambda \in \mathcal{F}_\infty$ and $N \subset E$ such that

1. $\theta_t(\Lambda) \subset \Lambda$, $t \geq 0$, $Cap(N) = 0$, $P_x(\Lambda) = 1$, $x \in E \setminus N$,

furthermore, for any $\omega \in \Lambda$,

1. $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega)$, $s, t \geq 0$,
2. $|A_t(\omega)| < \infty$, $t \in [0, \zeta(\omega))$,
3. $t \mapsto A_t(\omega)$ is càdlàg on $[0, \zeta(\omega))$,
4. $A_t(\omega) = A_{\zeta(\omega)}(\omega)$, $t \geq \zeta(\omega)$,
5. $A_t(\omega)$ is càdlàg on $[0, \zeta(\omega))$,
6. $A_t(\omega)$ is càdlàg on $[0, \zeta(\omega))$.

$\Lambda$ is called a defining set of $(A_t)$, and $N$ is called an exceptional set of $(A_t)$. An additive functional (AF for short) $(A_t)$ of $X$ is said to be positive if $A_t(\omega) \geq 0$, $t \geq 0$, $\omega \in \Lambda$. We say that an AF $(A_t)$ of $X$ is continuous if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty)$ for any $\omega \in \Lambda$.

In what follows, we frequently use the notion of positive continuous additive functionals (PCAF for short) of $X$. We say that $(A_t)$ is a martingale additive functional (MAF for short) of $X$ if it is an AF of $X$ and an $(\mathcal{F}_t)$-martingale under measure $P_x$ for any $x \in E \setminus N$. Analogously, we say that $(A_t)$ is a local MAF of $X$ if it is an AF of $X$, and it is a local $(\mathcal{F}_t)$-martingale under measure $P_x$ for any $x \in E \setminus N$. By [24, Theorem 5.1.4] there is a ono-to-one correspondence between PCAs of $X$ and positive smooth measures - so called Revuz duality. PCAF $(A_t)$ of $X$ and positive smooth measure $\nu$ on $E$ are in Revuz duality if for any positive $f \in B(E)$,

$$\mathbb{E}_x \int_0^\infty f(X_r) \,dA_r = \int_E G(x, y) f(y) \nu(dy), \quad x \in E \setminus N.$$ 

By [24, Theorem 5.1.3, Theorem 5.1.4], there exists a unique PCAF of $X$ satisfying the above identity for any $f \in B^+(E)$. We shall denote it by $A^\nu$. We say that $\nu$ is strictly smooth if the exceptional set $N$ of $A^\nu$ is empty.

**Proposition 2.2.** A positive Borel measure $\mu$ on $E$ is smooth iff it is absolutely continuous with respect to $\text{Cap}$ and there exists a strictly positive quasi-continuous function $u$ on $E$ such that $\int_E u \,d\mu < \infty$.

**Proof.** Sufficiency. Since $u$ is quasi-continuous there exists a nest $\{F_n\}$ such that $u|_{F_n}$ is quasi-continuous for every $n \geq 1$. Let $\{E_n\}$ be an increasing sequence of relatively compact open sets such that $\bigcup_{n \geq 1} E_n = E$. Set $\tilde{F}_n = \overline{E_n} \cap F_n$. Obviously, $\{\tilde{F}_n\}$ is a generalized nest. Moreover, by continuity of $u|_{F_n}$ and compactness of $\overline{E_n}$, $\inf_{\tilde{F}_n} u = c_n > 0$. Thus

$$\mu(\tilde{F}_n) \leq \frac{1}{c_n} \int_E u \,d\mu < \infty, \quad n \geq 1.$$
Necessity. Let \( \phi \) be a strictly positive bounded function such that \( R\phi \) is bounded (see [39, Corollary 1.3.6]) and \( \phi \in L^1(E; m) \). Set
\[
\eta(x) := \mathbb{E}_x \int_0^\zeta \phi(X_r) e^{-At} \, dr, \quad x \in E.
\]
By [24, Lemma 5.1.5], \( \eta, R\phi \) are quasi-continuous, and
\[
\mathbb{E}_x \int_0^\zeta \eta(X_r) \, dA^\mu_t \leq R\phi(x), \quad \text{q.e.}
\]
From this and [24, Theorem 5.1.3],
\[
\int_E \eta R\phi \, d\mu = \int_E \left( \int_0^\zeta \eta(X_r) \, dA^\mu_t \right) \phi(x) \, m(dx) \leq \langle R\phi, \phi \rangle \leq \|R\phi\|_\infty \|\phi\|_{L^1(E; m)}.
\]
Function \( u = \eta R\phi \) fulfills the requirements.

**Corollary 2.3.** Let \( u \) be a measurable function on \( E \). Then \( u \) is locally quasi-integrable iff
\[
P_x \left( \int_0^t |u(X_r)| \, dr < \infty, \, t < \zeta \right) = 1, \quad \text{q.e.}
\]

**Proof.** Follows from [24, Theorem 5.1.4] and Proposition 2.1.

\[\square\]

### 3. Green’s functions

We say that a Borel measurable function \( G : E \times E \to \mathbb{R}^+ \cup \{+\infty\} \) is the Green function for \(-A\) if
\[
Rf(x) = \int_E G(x,y) f(y) \, m(dy), \quad x \in E, \ f \in \mathcal{B}^+(E),
\]
and \( G(\cdot, \cdot) \) are excessive for any \( x, y \in E \). By [24, Theorem 4.2.4], there exists the Green function for \(-A\) if and only if \( P_t f(x) = 0 \) for any \( x \in E \) and \( t > 0 \) provided \( f \in \mathcal{B}^+(E) \) and \( \int_E f \, dm = 0 \). Furthermore, by [24, Lemma 4.2.4], there exists the Green function for \(-A\) if and only if \( R_\alpha f(x) = 0 \) for any \( x \in E \) and \( \alpha > 0 \) provided \( f \in \mathcal{B}^+(E) \) and \( \int_E f \, dm = 0 \).

At this point, we would like to formulate a general condition guaranteeing the existence of Green’s function. For \( f \in \mathcal{B}(E) \), we let \( |f|_\infty := \sup_{x \in E} |f(x)| \).

**Proposition 3.1.** Assume that \( E \) is complete. Furthermore, suppose that

(i) \( T_t(C_{\mathcal{C}}(E)) \subset C_b(E), \ t > 0; \)

(ii) \( J_\alpha \alpha > 0 \) is strongly Feller: for some \( \alpha > 0 \) (hence for any \( \alpha > 0 \)) \( J_\alpha (L^2(E; m) \cap L^\infty(E; m)) \subset C_b(E); \)

(iii) there exists a set \( \mathcal{C} \subset \{ f \in D(A) \cap C_b(E) : Af \text{ is bounded} \} \) such that any function in \( C_{\mathcal{C}}(E) \) is the limit in the supremum norm of functions from \( \mathcal{C} \).

Then there exists the Green function for \(-A\).

**Proof.** For \( f \in C_{\mathcal{C}}(E) \), we set
\[
S_t f(x) := \overline{T_t f}(x), \quad x \in E,
\]
where \( \overline{T_t} f \) is continuous \( m \)-version of \( T_t f \). Clearly, \( S_t \) is a linear operator, and for any positive \( f \in C_{\mathcal{C}}(E) \), we have \( S_t f(x) \geq 0, \ x \in E \). By the Riesz theorem, there exists a Borel measure \( \sigma(t, x, dy) \) such that for any \( f \in C_{\mathcal{C}}(E) \),
\[
S_t f(x) = \int_E f(y) \sigma(t, x, dy), \quad x \in E, \ t > 0.
\]
Thus, we may extend operator \( S_t \) to \( S_t : \mathcal{B}(E) \to \mathcal{B}(E) \). By a standard semigroup identity, we find that for any \( f \in \mathcal{C} \),
\[
|S_t f(x) - f(x)| \leq \int_0^t \int_E \sigma(r, x, dy) |Af(y)| \, dr \leq t \|Af\|_{L^\infty(E; m)}, \quad t > 0, \ x \in E.
\]
Now, let $f \in C_c(E)$ and let $(f_n) \subset \mathcal{C}$ converge uniformly to $f$. Then
\[
|S_t f(x) - f(x)| \leq |S_t f(x) - S_t f_n(x)| + |S_t f_n(x) - f_n(x)| + |f_n(x) - f(x)|
\leq |S_t f_n(x) - f_n(x)| + 2|f_n - f|_\infty.
\]
Letting $t \to 0^+$ and then $n \to \infty$, we obtain that for any $f \in C_c(E)$,
\[
|S_t f - f|_\infty \to 0 \quad \text{as} \quad t \to 0^+.
\]
By [6, Lemma 2.8, page 181], for any $f \in C_b(E)$ and any compact $K \subset E$,
\[
\sup_{x \in K} |S_t f(x) - f(x)| \to 0 \quad \text{as} \quad t \to 0^+.
\] (3.2)
By [44, Theorem 2.9, page 150], there exists a Hunt process $((Q_x)_{x \in E}, (H_t)_{t \geq 0})$ associated with the semigroup kernel $(S_t)$. Since $S_t(C_b(E)) \subset C_b(E)$, $t > 0$, we easily deduce that for any $\alpha > 0$ and $f \in C_b(E)$,
\[
U_\alpha f(x) := \int_0^\infty e^{-\alpha t} S_t f(x) \, dt, \quad x \in E
\]
belongs to $C_b(E)$. From this and (3.1), we deduce that for any $f \in C_c(E)$, $R_\alpha f(x) = J_\alpha f(x)$, $x \in E$, where $J_\alpha f$ is the continuous $m$-version of $J_\alpha f$. Fix $x \in E$. By the Riesz theorem, there exists a measure $v_\alpha(x, dy)$ such that $U_\alpha f(x) = \int_E f(y) v_\alpha(x, dy)$, $\alpha > 0$, $f \in C_b(E)$. As a result, $U_\alpha$ may be extended as the mapping $U_\alpha : \mathcal{B}_b(E) \to \mathcal{B}_b(E)$. We easily find that for any $f \in \mathcal{B}_b(E) \cap L^2(E; m)$
\[
R_\alpha f(x) = J_\alpha f(x), \quad x \in E.
\]
Consequently, we obtain that $v_\alpha(x, dy) = u_\alpha(x, y)m(dy)$, $x \in E$, $\alpha > 0$.

For sufficient conditions ensuring condition (i) of the above proposition see e.g. [2, 20, 23]
Condition (ii) of the above proposition is satisfied, for example, if for any $t > 0$ there exists $c_t > 0$ such that
\[
\|T_t f\|_{L^\infty(E; m)} \leq c_t \|f\|_{L^1(E; m)} \cap L^1(E; m), \quad f \in L^2(E; m) \cap L^1(E; m),
\]
(3.3)
and furthermore, there exists a dense set $\mathcal{C} \subset L^1(E; m)$ such that $T_t(\mathcal{C}) \subset C_b(E)$, $t > 0$ (the last condition holds whenever (i) is satisfied). Condition (3.3) is satisfied provided there exist $\delta \geq 0$ and $\nu > 0$ such that
\[
\|u\|^{2+4/\nu}_{L^2(E; m)} \leq C[\mathcal{E}(u, u) + \delta \|u\|^2_{L^2(E; m)}] \|u\|^{4/\nu}_{L^1(E; m)}, \quad u \in D(\mathcal{E}) \cap L^1(E; m).
\]
(see [13, Theorem 2.1]). The above condition is satisfied for (1.3) (see e.g. [2, Theorem 1.2]).

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function with $\varphi(0) = 0$. Now, we focus on the operator (1.3). Consider the following conditions:
(A) there exist $c_1, c_2 > 0$ such that
\[
c_1 \leq a(x, y) \leq c_2, \quad x, y \in \mathbb{R}^d,
\]
(B) there exists $c_3 > 0$ such that
\[
\int_0^r \frac{s}{\varphi(s)} \, ds \leq c_3 \frac{r^2}{\varphi(r)}, \quad r > 0,
\]
(C) there exist $c_4, c_5, \delta_1, \delta_2 > 0$ such that
\[
c_4 \left(\frac{R}{r}\right)^{\delta_1} \leq \frac{\varphi(R)}{\varphi(r)} \leq c_5 \left(\frac{R}{r}\right)^{\delta_2}, \quad 0 < r \leq R.
\]
By [20, Theorem 1.2], we get the following result.
Proposition 3.2. Let $A$ be the operator of the form (1.3) satisfying (A)–(C). Then there exists the Green function for $-A$.

**Part of $A$ on an open set** $D \subset E$. For a given self-adjoint Dirichlet operator $A$ and an open set $D \subset E$, we may define self-adjoint Dirichlet operator $A_{|D}$ as follows: let

$$
E_{|D}(u, v) := \mathcal{E}(u, v), \quad u, v \in D(E_{|D}) := \{ w \in D(\mathcal{E}) : \tilde{w} = 0 \text{ q.e. on } E \setminus D \}.
$$

By [24, Theorem 4.4.3], $E_{|D}$ is a Dirichlet form, and if $\mathcal{E}$ is regular, then $E_{|D}$ is regular too. Therefore, by [24, Theorem 1.3.1, Theorem 1.4.1], there exists a unique self-adjoint Dirichlet operator $(B, D(B))$ on $L^2(D;m) \subset L^2(E;m)$ such that $D(B) \subset D(E_{|D})$, and

$$
(-Bu, v)_{L^2(D;m)} = E_{|D}(u, v), \quad u \in D(B), \quad v \in D(E_{|D}).
$$

We let $A_{|D} := B$. The operator $A_{|D}$ is called a *part of $A$ on $D$* (or restriction of $A$ to $D$). This operation on a Dirichlet operator $A$ is often used when approaching the Dirichlet problem on $D$ for $A$.

**Perturbation of $A$ by a smooth measure.** Let $\nu$ be a positive smooth measure on $E$, and $A$ be a regular self-adjoint Dirichlet operator on $L^2(E;m)$. Define

$$
E_\nu(u, v) := \mathcal{E}(u, v) + \int_E \tilde{w} \tilde{v} \, d\nu, \quad u, v \in D(E_\nu) := \{ w \in D(\mathcal{E}) : \tilde{w} \in L^2(E;\nu) \}.
$$

By [36, Theorem IV.4.4], $E_\nu$ is a symmetric Dirichlet form on $L^2(E;m)$. Therefore, there exists a unique self-adjoint operator $B$ on $L^2(E;\nu)$ such that $D(B) \subset D(E_\nu)$, and

$$
(-Bu, v)_{L^2(E;m)} = E_\nu(u, v), \quad u \in D(B), \quad v \in D(E_\nu).
$$

We set $A_\nu := B$. Formally, $-A_\nu = -A + \nu$, so $-A_\nu$ may be called a perturbation of $-A$ be the measure $\nu$.

**Resurrected (regional) operator.** Let $A$ be a regular self-adjoint Dirichlet operator on $L^2(E;m)$ with regular symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. By the Beurling-Deny formulæ (see [24, Theorem 3.2.1]) for any $u, v \in C_c(E) \cap D(\mathcal{E})$,

$$
\mathcal{E}(u, v) = \mathcal{E}(u, v) + \int_{E \times E \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) \, J(dx, dy) + \int_E u \, v \, d\kappa, \quad (3.4)
$$

where $\mathcal{E}(c)$ is a symmetric form, with domain $D(\mathcal{E}(c)) = D(\mathcal{E}) \cap C_c(E)$, which satisfies the strong local property:

$$
\mathcal{E}(c)(u, v) = 0 \quad \text{for } u, v \in D(\mathcal{E}(c))
$$

such that $v$ is constant on a neighbourhood of $\text{supp}[u]$. $J$ is a symmetric positive Radon measure on $E \times E \setminus \text{diag}$ and $\kappa$ is a positive Radon measure on $E$. Such $\mathcal{E}(c), J, \kappa$ are uniquely determined by $(\mathcal{E}, D(\mathcal{E}))$. Let $\mathcal{E}^{\text{res}}(u, v) := \mathcal{E}(u, v) - \int_E u \, v \, d\kappa$. By [19, Theorem 5.2.17], $(\mathcal{E}^{\text{res}}, D(\mathcal{E}^{\text{res}}))$ is a regular symmetric Dirichlet form on $L^2(E;m)$ with $D(\mathcal{E}^{\text{res}})$ described in [19, (5.2.25)]. Therefore, there exists a unique self-adjoint Dirichlet operator $(B, D(B))$ such that $D(B) \subset D(\mathcal{E}^{\text{res}})$ and

$$
(-Bu, v) = \mathcal{E}^{\text{res}}(u, v), \quad u \in D(B), \quad v \in D(\mathcal{E}^{\text{res}}).
$$

We let $A^{\text{res}} := B$. Operator $A^{\text{res}}$ is very useful for interpretation of the Neumann problem on $D$ for purely jumping Dirichlet operators $A$ (see e.g. [1]). Indeed, one takes $(A_{|D})^{\text{res}}$ (for the fractional Laplacian, we then derive so called regional fractional Laplacian).

**Proposition 3.3.** Let $(A, D(A))$ be a regular self-adjoint Dirichlet operator on $L^2(E;m)$. Suppose that there exists the Green function $G$ for $-A$.

1. If $D$ is an open subset of $E$, then there exists the Green function for $-A_{|D}$.
2. If $\nu$ is a strictly smooth positive measure on $E$, then there exists the Green function for $-A_{\nu}$.
If the killing measure \( \kappa \) from decomposition (3.4) is strictly smooth, then there exists the Green function for \( -A^{\text{res}} \).

Proof. (1) It follows from [24, Theorem 4.4.2]. For (2), see [24, Theorem A.2.12]. Ad (3).

Let \( \phi(x) = 1, x \in E \). Clearly, \( \phi(X_t) = 1_{\{t < \zeta\}} \). By the comment preceding Lemma 5.3.3 in [24],

\[
\phi(X_t) - \phi(X_0) = -A^\kappa_t + M_t, \quad t \geq 0
\]

is the Doob-Meyer decomposition of supermartingale \( \phi(X) \), where \( A^\kappa \) is a PCAF associated with killing measure \( \kappa \) appearing in the Beurling-Deny formulæ (3.4), and \( M \) is a martingale additive functional of \( X \). By [42, Theorem 62.19] a family of measures \( (Q_x) \) given by

\[
Q_x(F1_{\{T < \zeta\}}) := P_x(Fm_T), \quad F \in \mathcal{F}_T, x \in E, T \geq 0,
\]

constitutes a Right Markov process on \( E \), where

\[
m_t := \frac{\phi(X_t)}{\phi(X_0)} e^{A^\kappa_t} = \phi(X_t) e^{A^\kappa_t} = 1_{\{t < \zeta\}} e^{A^\kappa_t}, \quad t \geq 0.
\]

By [19, Theorem 5.2.17], \( ((Q_x), X) \) is a Hunt process associated with \( E^{\text{res}} \). By (3.5), \( Q_x \) is equivalent to \( P_x \) for any \( x \in E \). Thus, there exists the Green function for \( -A^{\text{res}} \). \( \square \)

4. Method of sub- and supersolutions

In what follows \( f : E \times \mathbb{R} \to \mathbb{R} \). Consider the following conditions.

Car) \( f \) is a Carathéodory function, i.e.

- \( x \mapsto f(x,y) \) is Borel measurable for any \( y \in \mathbb{R} \),
- \( y \mapsto f(x,y) \) is continuous for \( m \)-a.e. \( x \in E \);

Sig) for any \( u \in \mathbb{R} \), \( f(x,u)u \leq 0 \) \( m \)-a.e. \( x \in E \);

Int) for any \( \underline{u}, \overline{u} \in L^1(E; \rho \cdot m) \) such that \( f(\cdot, \underline{u}), f(\cdot, \overline{u}) \in L^1(E; \rho \cdot m) \) we have

\[
x \longmapsto \sup_{y \in [\underline{u}(x), \overline{u}(x)]} |f(x,y)| \in L^1(E; \rho \cdot m);
\]

qM) for any \( M > 0 \) the mapping \( E \ni x \longmapsto \sup_{|y| \leq M} |f(x,y)| \) is locally quasi-integrable.

M) for any \( M > 0 \) the mapping \( E \ni x \longmapsto \sup_{|y| \leq M} |f(x,y)| \in L^1(E; \rho \cdot m) \).

For any \( \mu \in \mathcal{M}_\rho \), we let \( \mu_d \) denote the part of \( \mu \), which is absolutely continuous with respect to \( Cap \), and by \( \mu_c \) we denote the part of \( \mu \), which is orthogonal to \( Cap \). Observe that \( \mu_d \) is a smooth measure. Indeed, by the very definition \( \mu_d \ll Cap \). Now, let \( g \in L^2(E; m) \cap \mathcal{B}(E) \) be strictly positive. We have

\[
\int_E R_1(\rho \land g) d|\mu_d| \leq \int_E \rho d|\mu| < \infty.
\]

We used here the fact that \( \rho \) is excessive. Clearly, \( \eta := R_1(\rho \land g) \) is strictly positive. By [24, Theorem 4.2.3], \( \eta \) is quasi-continuous. Therefore, by Proposition 2.2, \( \mu_d \) is smooth.

Remark 4.1. Throughout the paper, we frequently use without special mentioning the following facts.

(a) If \( \mu \in \mathcal{M}_\rho \), then \( R|\mu| < \infty \) q.e. and, as a result, \( R|\mu| \) is quasi-continuous. The last assertion follows from [27, Theorem 3.1]. At the same time,

\[
\int_E R|\mu| \varrho dm = \int_E R \varrho d|\mu| \leq \int_E \rho d|\mu| < \infty.
\]

Thus, since \( \varrho \) is strictly positive, we obtain that \( R|\mu| < \infty \) m.a.e. By [19, Theorem A.2.13(v)], we have that, in fact, \( R|\mu| < \infty \) q.e. (see [24, Lemma 2.1.4]).

(b) If \( u_1, u_2 \) are quasi-continuous functions on \( E \), then \( u_1 \leq u_2 \) m.a.e. if and only if \( u_1 \leq u_2 \) q.e.
(c) Let $B \in \mathcal{B}(E)$. If $\text{Cap}(B) = 0$, then $P_x(\exists_{t \geq 0} : X_t \in B) = 0$ q.e. (see [24, Theorem 4.2.1]).

Recall that a measurable function $\tau : \Omega \to [0, \infty]$ is called a stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. We say that a non-decreasing sequence $\{\tau_k\}$ of stopping times is a reducing sequence for a measurable function $u$ on $E$ if $\tau_k \wedge \zeta \to \zeta$ q.a.s., and

$$E_x \sup_{t \leq \tau_k} |u(X_t)| < \infty, \quad k \geq 0 \text{ q.e.}$$

**Lemma 4.2.** Let $\mu \in \mathcal{M}_\rho$. Set $w := R|\mu|$, $u(x) := R\mu(x)$, $x \in E \setminus N$ and zero on $N$, where $N = \{x \in E : w(x) = \infty\}$. Then, there exists a local MAF $\mathcal{M}$ such that, for any $x \in E \setminus N$,

$$u(X_t) = u(X_0) - \int_0^t dA^\mu_t + \int_0^t dM_t, \quad t \geq 0 \text{ } P_x\text{-a.s.} \tag{4.1}$$

Moreover, for $\tau_k := \inf\{t > 0 : w(X_t) \geq k\} \wedge k$, we have

$$E_x \sup_{t \leq \tau_k} |u(X_t)| + E_x \sup_{t \leq \tau_k} |M_t| < \infty, \quad k \geq 0, \ x \in E \setminus N. \tag{4.2}$$

In particular, $\{\tau_k\}$ is a reducing sequence for $u$.

**Proof.** By [27, Theorem 3.7], we obtain (4.1). From the fact that $w(X)$ is a positive supermartingale under the measure $P_x$, for any $x \in E \setminus N$ (see e.g. [7, Theorem II.2.12]) and from [42, Theorem 51.1], we infer (4.2). The last assertion is obvious. □

**Definition 4.3.** Let $\mu \in \mathcal{M}_\rho$. We say that a measurable function $u$ is a solution to (1.2) if $f(\cdot, u) \in L^1(E; \rho \cdot m)$ and

$$u(x) = \int_E G(x,y)f(y,u(y))\,m(dy) + \int_E G(x,y)\mu(dy), \quad m\text{-a.e.} \tag{4.3}$$

**Remark 4.4.** By Remark 4.1, $R[f(\cdot, u)] < \infty$ q.e., $R|\mu| < \infty$ q.e., and $Rf(\cdot, u)$, $R\mu$ are quasi-continuous. As a result, one sees that a solution to (1.2) has always a quasi-continuous $m$-version. Observe also that by (2.1), $u \in L^1(E; \rho \cdot m)$.

**Proposition 4.5.** Assume that $\mu \in \mathcal{M}_1$, and $R_1(\mathcal{B}_0(E)) \subset C_b(E)$ or $u \in L^1(E; m)$, $R_1(C_b(E)) \subset C_b(E)$. Then $u$ is a solution to (1.2) if and only if $u$ is a renormalized solution to (1.2):

(i) $T_k(u) := \min\{k, \max\{u, -k\}\} \in D_u(E)$, $k > 0$;
(ii) $f(\cdot, u) \in L^1(E; m)$;
(iii) There exists a family $\{\nu_k\}_{k \geq 0} \subset \mathcal{M}_1$ of smooth measures such that $\nu_k \to \mu_c$ in the narrow topology, as $k \to \infty$;
(iv) For any bounded $\eta \in D(E)$,

$$\mathcal{E}(T_k(u), \eta) = \langle f(\cdot, u), \eta \rangle + \langle \mu_d, \hat{\eta} \rangle + \langle \nu_k, \hat{\eta} \rangle.$$

**Proof.** It follows from [26, Theorem 4.9]. □

**Definition 4.6.** Let $\mu \in \mathcal{M}_\rho$. We say that a measurable function $u$ is a subsolution (resp. supersolution) to (1.2) if $f(\cdot, u) \in L^1(E; \rho \cdot m)$ and there exists a positive measure $\nu \in \mathcal{M}_\rho$ such that

$$u(x) = \int_E G(x,y)f(y,u(y))\,m(dy) + \int_E G(x,y)\mu(dy) - \int_E G(x,y)\nu(dy), \quad m\text{-a.e.} \tag{4.4}$$

(resp.)

$$u(x) = \int_E G(x,y)f(y,u(y))\,m(dy) + \int_E G(x,y)\mu(dy) + \int_E G(x,y)\nu(dy), \quad m\text{-a.e.} \tag{4.5}$$

Throughout the paper, unless stated otherwise, we always consider quasi-continuous $m$-versions of solutions, supersolutions and subsolutions to (1.2). These versions may be defined as right-hand sides of (4.3), (4.4) or (4.5), where finite, and zero otherwise.
**Proposition 4.7.** Let $u, w$ be subsolutions to (1.2). Then $u \lor w$ is a subsolution to (1.2).

**Proof.** By the definitions of sub- and supersolution to (1.2) there exist positive $\nu^1, \nu^2 \in \mathcal{M}_\mu$ such that

$$u = Rf(\cdot, u) + R\mu - R\nu^1, \quad w = Rf(\cdot, u) + R\mu - R\nu^2, \quad \text{q.e.} \quad (4.6)$$

By Lemma 4.2 there exist local MAFs $M, N$ such that

$$u(X_t) = u(X_0) - \int_0^t f(X_r, u(X_r)) \, dr - \int_0^t dA^\mu_r + \int_0^t dA^\nu^1_r + \int_0^t dM_r, \quad t \geq 0,$$

$$w(X_t) = w(X_0) - \int_0^t f(X_r, w(X_r)) \, dr - \int_0^t dA^\mu_r + \int_0^t dA^\nu^2_r + \int_0^t dN_r, \quad t \geq 0,$$

q.a.s. By the Tanaka-Meyer formula (see, e.g., [41, IV. Theorem 70]) there exists an increasing càdlàg process $C$, with $C_0 = 0$, such that

$$(u \lor w)(X_t) = (u \lor w)(X_0) - \int_0^t 1_{\{u \geq w\}}(X_r) f(X_r, u(X_r)) \, dr - \int_0^t 1_{\{u \geq w\}}(X_r) dA^\mu_r$$

$$+ \int_0^t 1_{\{u \geq w\}}(X_r) dA^{\nu^1}_r + \int_0^t 1_{\{u \geq w\}}(X_r) dM_r$$

$$+ \int_0^t 1_{\{u > w\}}(X_r) f(X_r, w(X_r)) \, dr - \int_0^t 1_{\{u > w\}}(X_r) dA^\mu_r$$

$$+ \int_0^t 1_{\{u > w\}}(X_r) dA^{\nu^2}_r + \int_0^t 1_{\{u > w\}}(X_r) dN_r + \int_0^t dC_r, \quad t \geq 0, \quad \text{q.a.s.} \quad (4.7)$$

Hence,

$$(u \lor w)(X_t) = (u \lor w)(X_0) - \int_0^t f(X_r, (u \lor w)(X_r)) \, dr - \int_0^t dA^\mu_r$$

$$+ \int_0^t 1_{\{u \geq w\}}(X_r) dA^{\nu^1}_r + \int_0^t 1_{\{u > w\}}(X_r) dA^{\nu^2}_r + \int_0^t dC_r$$

$$+ \int_0^t 1_{\{u \geq w\}}(X_r) dM_r + \int_0^t 1_{\{u > w\}}(X_r) dN_r, \quad t \geq 0, \quad \text{q.a.s.} \quad (4.7)$$

From the above formula, we deduce that $C$ is an additive functional of $X$. Thus, $C$ is a positive AF of $X$. Moreover, since $X$ is a Hunt process and $u, w$ are quasi-continuous, $u(X), w(X), M, N$ have only totally inaccessible jumps (see [24, Theorem A.2.1, Theorem A.3.6]). Therefore, dual predictable projection $\tilde{C}$ of $C$ is continuous. By [24, Theorem A.3.16], $\tilde{C}$ is a PCAF. Consequently, by [24, Theorem 5.1.4], there exists a positive smooth measure $\beta$ such that $\tilde{C} = A^\beta$. Let $\{\tau_k\}$ be a reducing sequence for $u \lor w$. Then, by (4.7),

$$(u \lor w)(x) = \mathbb{E}_x(u \lor w)(X_{\tau_k}) + \mathbb{E}_x \int_0^{\tau_k} f(X_r, (u \lor w)(X_r)) \, dr + \mathbb{E}_x \int_0^{\tau_k} dA^\mu_r$$

$$- \mathbb{E}_x \int_0^{\tau_k} 1_{\{u \geq w\}}(X_r) dA^{\nu^1}_r - \mathbb{E}_x \int_0^{\tau_k} 1_{\{u > w\}}(X_r) dA^{\nu^2}_r - \mathbb{E}_x \int_0^{\tau_k} dA^\beta_r, \quad \text{q.e.} \quad (4.8)$$

By [27, Theorem 3.7],

$$\mathbb{E}_x u(X_{\tau_k}) \rightarrow R\mu_c(x) - R\nu^1_c(x), \quad \mathbb{E}_x w(X_{\tau_k}) \rightarrow R\mu_c(x) - R\nu^2_c(x), \quad \text{q.e.}$$

Moreover, by [27, Theorem 3.7, Theorem 6.3],

$$\mathbb{E}_x |u(X_{\tau_k}) - w(X_{\tau_k})| \rightarrow R|\nu^1_c - \nu^2_c|(x), \quad \text{q.e.}$$

Thus,

$$\lim_{k \to \infty} \mathbb{E}_x (u \lor w)(X_{\tau_k}) = \lim_{k \to \infty} \frac{1}{2} \left( \mathbb{E}_x u(X_{\tau_k}) + \mathbb{E}_x w(X_{\tau_k}) + \mathbb{E}_x |u(X_{\tau_k}) - w(X_{\tau_k})| \right)$$

$$= R\mu_c(x) - \frac{1}{2} R(\nu^1_c + \nu^2_c - |\nu^1_c - \nu^2_c|)(x) = R\mu_c(x) - R(\nu^1_c \lor \nu^2_c)(x), \quad \text{q.e.}$$
Therefore, letting $k \to \infty$ in (4.8), we get
\[ u \cup w = Rf(\cdot, u \cup w) + R\mu - R(1_{[u \leq w]} \cdot \nu^1_d) - R(1_{[w > u]} \cdot \nu^2_d) - R\beta - R(\nu^1_c \land \nu^2_c), \quad \text{q.e.} \]
From the fact that $u \leq u \cup w$ and [27, Lemma 4.6], we infer that $\beta \in \mathcal{M}_\rho$. Therefore, $u \cup w$ is a subsolution to (1.2).

**Proposition 4.8.** Assume Car. Let $\mu \in \mathcal{M}_\rho$. Suppose that there exist positive $g \in L^1(E; \rho \cdot m)$ and $c > 0$ such that
\[ |f(x, y)| \leq cg(x), \quad x \in E, \ y \in \mathbb{R}. \tag{4.9} \]
Then there exists a solution to (1.2).

**Proof.** Set $r := c\|g\|_{L^1(E; \rho \cdot m)} + \|\mu\|_{\mathcal{M}_\rho}$. For $u \in L^1(E; \rho \cdot m)$, we let
\[ \Phi(u) = Rf(\cdot, u) + R\mu. \]
Observe that by (4.9)
\[ |\Phi(u)| \leq crg + r|\mu|. \tag{4.10} \]
Hence
\[ \|\Phi(u)\|_{L^1(E; \rho \cdot m)} \leq r. \]
By (4.9) and Car, we get easily that $\Phi$ is continuous. Let $\{u_n\} \subset L^1(E; \rho \cdot m)$. Observe that
\[ \Phi(u_n) = Rf^+(\cdot, u_n) - Rf^-(\cdot, u_n) + R\mu \]
By [21, Lemma 94, page 306], $\{Rf^+(\cdot, u_n)\}, \{Rf^-(\cdot, u_n)\}$ have subsequence (still denoted by $(n)$) convergent $m$-a.e. Thus, up to subsequence, $\{\Phi(u_n)\}$ is convergent $m$-a.e. By (4.10) the Lebesgue dominated convergence theorem is applicable, and so, $\{\Phi(u_n)\}$ is convergent in $L^1(E; \rho \cdot m)$. By Schauder’s fixed point theorem, we get the result. \hfill \Box

**Proposition 4.9.** Assume Car. Let $u$ be a subsolution to (1.2) and $w$ be a supersolution to (1.2). Then
\begin{enumerate}
  \item $u^+ + Rf^-(\cdot, u) \leq R(1_{[u \geq 0]} \cdot \mu^+_d) + R\mu^+_c$, q.e., and
  \[ \|f^-(\cdot, u)\|_{L^1(E; \rho \cdot m)} \leq \|1_{[u \geq 0]} \cdot \mu^+_d\|_{\mathcal{M}_\rho} + \|\mu^+_c\|_{\mathcal{M}_\rho}, \]
  \item $w^- + Rf^+(\cdot, w) \leq R(1_{[w \leq 0]} \cdot \mu^-_d) + R\mu^-_c$, q.e., and
  \[ \|f^+(\cdot, w)\|_{L^1(E; \rho \cdot m)} \leq \|1_{[w \leq 0]} \cdot \mu^-_d\|_{\mathcal{M}_\rho} + \|\mu^-_c\|_{\mathcal{M}_\rho}. \]
\end{enumerate}

**Proof.** The proofs of (1) and (2) are analogous. We shall give the proof of (1). Since $u$ is a subsolution, there exists a positive $\nu \in \mathcal{M}_\rho$ such that
\[ u = Rf(\cdot, u) + R\mu - R\nu, \quad \text{q.e.} \]
By Lemma 4.2 and the Tanaka-Meyer formula
\[ u^+(x) - \mathbb{E}_x \int_0^{\tau_k} 1_{[u \geq 0]}(X_r)f(X_r, u(X_r)) \, dr \leq \mathbb{E}_x u^+(X_{\tau_k}) + \mathbb{E}_x \int_0^{\tau_k} 1_{[u \geq 0]}(X_r) \, dA^\mu_d - \mathbb{E}_x \int_0^{\tau_k} 1_{[u \geq 0]}(X_r) \, dA^\mu_c, \quad \text{q.e.}, \]
where $\{\tau_k\}$ is a reducing sequence for $u$. From this and Sig, we infer that
\[ u^+(x) + \mathbb{E}_x \int_0^{\tau_k} f^-(X_r, u(X_r)) \, dr \leq \mathbb{E}_x u^+(X_{\tau_k}) + \mathbb{E}_x \int_0^{\tau_k} 1_{[u \geq 0]}(X_r) \, dA^\mu_d, \quad \text{q.e.} \]
Letting $k \to \infty$ and using [27, Theorem 3.7, Theorem 6.3] yields
\[ u^+ + Rf^-(\cdot, u) \leq R\mu^+_c + R(1_{[u \geq 0]} \cdot \mu_d), \quad \text{q.e.} \]
By [27, Lemma 4.6], we get (1). \hfill \Box

**Theorem 4.10.** Let $\mu \in \mathcal{M}_\rho$. Assume Car, Int.}
(1) Let \( \psi \in L^1(E; \varrho \cdot m) \) be such that \( f(\cdot, \psi) \in L^1(E; \rho \cdot m) \). Suppose that there exists a subsolution \( \underline{u} \) to (1.2) such that \( \underline{u} \leq \psi \) m.a.e. Then there exists a maximal subsolution \( u^* \) to (1.2) such that \( u^* \leq \psi \) m.a.e.

(2) Assume that there exists a subsolution \( \underline{u} \) to (1.2) and a supersolution \( \overline{u} \) to (1.2) such that \( \underline{u} \leq \overline{u} \), m.a.e. Then there exists a maximal solution \( u \) to (1.2) such that \( \underline{u} \leq u \leq \overline{u} \), m.a.e. The maximal solution \( u \) is at the same time a maximal subsolution lying between \( \underline{u} \) and \( \overline{u} \), m.a.e.

(3) Assume \( \text{Sig} \). For any subsolution \( \underline{u} \) to (1.2) we have \( \underline{u} \leq Ry^+ \) m.a.e.

(4) Assume \( \text{Sig} \). Let \( \psi : E \to \mathbb{R} \cup \{+\infty\} \) be a Borel measurable function. Suppose that there exists a subsolution \( \underline{u} \) to (1.2) such that \( \underline{u} \leq \psi \) m.a.e. Then there exists a maximal subsolution \( u^* \) to (1.2) such that \( u^* \leq \psi \) m.a.e.

**Proof.** Ad (1). Set

\[
S_\psi = \{v : v is a subsolution to (1.2) and v \leq \psi \text{ m.a.e.}\}
\]

By assumptions \( S_\psi \) is nonempty. By Remark 4.4, \( S_\psi \subset L^1(E; \varrho \cdot m) \). Let

\[
\alpha = \sup_{v \in S_\psi} \int_E v \varrho \, dm.
\]

By the assumptions made on \( \psi \), \( \alpha < \infty \). Let \( \{v_n\} \subset S_\psi \) be such that \( \int_E v_n \varrho \, dm \not\to \alpha \). Set

\[
u_n = \max\{v_1, \ldots, v_n\}, \quad n \geq 1.
\]

By Proposition 4.7, \( \{u_n\} \subset S_\psi \). Set \( u^* = \sup_{n \geq 1} u_n \). Clearly, \( \underline{u} \leq u^* \leq \psi \) m.a.e. By the assumptions made on \( \psi \) and Int), we have \( f(\cdot, u^*) \in L^1(E; \rho \cdot m) \). Since \( u_n \) is a subsolution to (1.2) there exists a positive \( \nu_n \in \mathcal{M}_\rho \) such that

\[
u_n = Rf(\cdot, u_n) + R\mu - R\nu_n \text{ q.e.} (4.11)
\]

Clearly,

\[
u_1 \leq u_n \leq u^*, \quad n \geq 1 \text{ q.e.} (4.12)
\]

Hence

\[
|f(\cdot, u_n)| \leq \sup_{y \in [u_1, u^*]} |f(\cdot, y)| = g_1.
\]

By Int), \( g_1 \in L^1(E; \rho \cdot m) \). Therefore, by the Lebesgue dominated convergence theorem,

\[
\int_E |Rf(\cdot, u_n) - Rf(\cdot, u^*)| \varrho \, dm \leq \int_E |f(\cdot, u_n) - f(\cdot, u^*)| \rho \, dm \to 0, \quad n \to \infty.
\]

Thus, up to subsequence, \( \lim_{n \to \infty} Rf(\cdot, u_n) = Rf(\cdot, u^*) \) m.a.e. This in turn implies that \( (R\nu_n)_{n \geq 1} \) is convergent m.a.e. Let \( \eta := \lim_{n \to \infty} R\nu_n \) m.a.e. By [21, Lemma 94, page 306], \( \eta \) has an \( m \)-version (still denoted by \( \eta \)) such that \( \eta \) is an excessive function. By (4.11),(4.12), we have

\[
R\nu_n \leq R|f(\cdot, u_n)| + R|f(\cdot, u_1)| + R|\mu| + R|\nu_1| \text{ q.e.}
\]

Letting \( n \to \infty \) and using [7, page 197] yields

\[
\eta \leq R|f(\cdot, u^*)| + R|f(\cdot, u_1)| + R|\mu| + R|\nu_1|.
\]

By [25, Proposition 3.9], there exists a positive Borel measure \( \beta \) on \( E \) such that \( \eta = R\beta \). From the above inequality and [27, Lemma 4.6], we conclude that \( \beta \in \mathcal{M}_\rho \). Going back to (4.11) and letting \( n \to \infty \), we deduce from what has been already proven that

\[
u^* = Rf(\cdot, u^*) + R\mu - R\beta \text{ q.e.}
\]

Thus, \( u^* \in S_\psi \). What is left is to show that \( u^* \) is maximal. Let \( v \in S_\psi \). Clearly \( v \cup u_n \not\to v \cup u^* \).

By Proposition 4.7, \( v \cup u_n \in S_\psi \). Thus,

\[
\alpha = \lim_{n \to \infty} \int_E v_n \varrho \, dm \leq \lim_{n \to \infty} \int_E u_n \varrho \, dm \leq \lim_{n \to \infty} \int_E v \cup u_n \varrho \, dm \leq \alpha.
\]
By the Lebesgue monotone convergence theorem
\[ \int_E u^* \varrho \, dm = \int_E v \vee u^* \varrho \, dm = \alpha. \]
Therefore,
\[ \int_E (v \vee u^* - u^*) \varrho \, dm = 0. \]
Hence, \( u^* \leq v \vee u^* \) m.a.e., which implies that \( v \leq u^* \) m.a.e.

Ad (2). Set
\[ \hat{f}(x, y) = f(x, (y \wedge \mathbf{\pi}(x)) \vee \underline{u}(x)), \quad x \in E, \; y \in \mathbb{R}. \]
By Int), \( \hat{f} \) satisfies (4.9) with \( g(x) := \sup_{y \in [\underline{u}(x), \mathbf{\pi}(x)]} |f(x, y)|, \; x \in E. \) Therefore, there exists a solution \( \hat{u} \) to (1.2) with \( f \) replaced by \( \hat{f} \). Since \( \underline{u} \) is a subsolution to (1.2), there exists a positive measure \( \nu \in \mathcal{M}_p \) such that
\[
\underline{u} = Rf(\cdot, \underline{u}) + R\mu - R\nu \quad \text{q.e.}
\]
By Lemma 4.2 and the Tanaka-Meyer formula
\[
(u(x) - \hat{u}(x))^+ \leq \mathbb{E}_x((u(X_{r_k}) - \hat{u}(X_{r_k}))^+ + \mathbb{E}_x \int_0^{r_k} 1_{[\underline{u} \mathbf{\pi}]}(X_r)(f(X_r, u(X_{r_k})) - \hat{f}(X_r, \hat{u}(X_{r_k}))) \, dr
\]
\[- \mathbb{E}_x \int_0^{r_k} 1_{[\underline{u} \mathbf{\pi}]}(X_r) \, dA^\nu \quad \text{q.e.}
\]
Observe that, by the definition of \( \hat{f}, 1_{[\underline{u} \mathbf{\pi}]}(f(\cdot, \underline{u}) - \hat{f}(\cdot, \hat{u})) \leq 0. \) Thus,
\[
(u(x) - \hat{u}(x))^+ \leq \mathbb{E}_x((u(X_{r_k}) - \hat{u}(X_{r_k}))^+ \quad \text{q.e.}
\]
Letting \( k \to \infty \) and using [27, Theorem 3.6, Theorem 6.3] yields
\[
(u(x) - \hat{u}(x))^+ \leq R(-\nu_c)^+ = 0 \quad \text{q.e.,}
\]
and so \( \underline{u} \leq \hat{u} \) m.a.e. Analogous reasoning for \( \hat{u}, \mathbf{\pi} \) shows that \( \hat{u} \leq \mathbf{\pi} \) m.a.e. Consequently, \( \hat{f}(\cdot, \hat{u}) = f(\cdot, \hat{u}) \) m.a.e. Therefore, \( \hat{u} \) is, in fact, a solution to (1.2) and \( \underline{u} \leq \hat{u} \leq \mathbf{\pi} \) m.a.e. Now, we shall show the existence of a maximal solution to (1.2) lying between \( \underline{u}, \mathbf{\pi} \). Applying (1) with \( \psi = \mathbf{\pi} \) gives the existence of a maximal subsolution \( u^* \) to (1.2) such that \( \underline{u} \leq u^* \leq \mathbf{\pi} \) m.a.e. By what has been already proven, there exists a solution \( \hat{u} \) to (1.2) such that \( u^* \leq \hat{u} \leq \mathbf{\pi} \) m.a.e. On the other hand, since \( \hat{u} \) also is a subsolution to (1.2), \( \hat{u} \leq u^* \) m.a.e. Thus, \( \hat{u} = u^* \) m.a.e. Now, we easily deduce that \( u^* \) is a maximal solution to (1.2).

Ad (3) Let \( w = R\mu^+, \) and \( v \) be a subsolution to (1.2). By the definition of a subsolution to (1.2) there exists a positive \( \nu \in \mathcal{M}_p \) such that
\[
v = Rf(\cdot, v) + R\mu - R\nu \quad \text{q.e.}
\]
By Lemma 4.2 and the Tanaka-Meyer formula
\[
(v(x) - w(x))^+ \leq \mathbb{E}_x((v(X_{r_k}) - w(X_{r_k}))^+ + \mathbb{E}_x \int_0^{r_k} 1_{[v > w]}(X_r)f(X_r, v(X_{r_k})) \, dr
\]
\[- \mathbb{E}_x \int_0^{r_k} 1_{[v > w]}(X_r) \, dA^\nu - \mathbb{E}_x \int_0^{r_k} 1_{[w > v]}(X_r) \, dA^\nu,
\]
where \( \{r_k\} \) is a reducing sequence for \( v - w. \) Since \( w \) is positive, we have, by Sig), that
\[
1_{[v > w]}f(\cdot, v) \leq 0. \]
Consequently,
\[
(v(x) - w(x))^+ \leq \mathbb{E}_x((v(X_{r_k}) - w(X_{r_k}))^+ \quad \text{q.e.}
\]
By [27, Theorem 3.7, Theorem 6.3],
\[
\lim_{k \to \infty} \mathbb{E}_x((v(X_{r_k}) - w(X_{r_k}))^+ = R(\mu_c - \nu_c - \mu_c^+)^+(x) = 0 \quad \text{q.e.}
\]
Therefore, \( v \leq w \) m.a.e.
Ad (4). We maintain the notation of the proof of (1). The proof of (4) runs exactly the same lines as the proof of (1) but with different justification of the facts that $\alpha < \infty$ and $f(\cdot, u^*) \in L^1(E; \rho \cdot m)$. The first property is a consequence of (3). For the second one, observe that, by Sig),

$$[f(\cdot, u^*)] = f^-(\cdot, u^*) + f^-(\cdot, (u^*)^+).$$

(4.13)

By Proposition 4.9,

$$\|f^-(\cdot, u_n)\|_{L^1(E; \rho \cdot m)} \leq \|\mu\|_{\mathcal{M}_\rho}$$

By Fatou’s lemma $f^-(\cdot, u^*) \in L^1(E; \rho \cdot m)$. At the same time, we have

$$-u_1^* \leq -(u^*)^- \leq 0.$$ 

Therefore by Int), $f(\cdot, (u^*)^-) \in L^1(E; \rho \cdot m)$. Consequently, by (4.13), $f(\cdot, u^*) \in L^1(E; \rho \cdot m)$. 

\[\square\]

**Proposition 4.11. Assume Car), Int).** Let $\underline{u}$ (resp. $\overline{u}$) be a subsolution (resp. supersolution) to (1.2). Let $\bar{\mu} \in \mathcal{M}_\rho$, $\bar{f}$ be a measurable function on $E \times \mathbb{R}$, and $\bar{u}$ be a solution to (1.2) with $f, \mu$ replaced by $\bar{f}, \bar{\mu}$ such that $\underline{u} \leq \bar{u} \leq \overline{u} \ \text{m.a.e.}$. Let $u$ be a maximal solution to (1.2) such that $u \leq u \leq \overline{u}$ m.a.e. Assume that $\bar{f}(\cdot, \bar{u}) \leq f(\cdot, \bar{u})$ m.a.e. and $\bar{\mu} \leq \mu$. Then $\bar{u} \leq u \ \text{m.a.e.}$

**Proof.** Observe that

$$-Au = f(\cdot, \bar{u}) + \mu - (f(\cdot, \bar{u}) - \bar{f}(\cdot, \bar{u})) - (\mu - \bar{\mu}).$$

By Int), $f(\cdot, \bar{u}) \in L^1(E; \rho \cdot m)$. Therefore, $\bar{u}$ is a subsolution to

$$-Au = f(\cdot, u) + \mu.$$

By Theorem 4.10(2), $u$ is a maximal subsolution to the above problem lying between $\underline{u}$ and $\overline{u}$ m.a.e. Thus, $\bar{u} \leq u \ \text{m.a.e.}$ 

\[\square\]

5. **Existence of maximal and minimal good measure**

**Standing assumption:** In the remainder of the paper, we assume that conditions Car), Sig), Int) and qM) are in force.

We begin with the following lemma which will be crucial in our proof techniques when passing to the limit in variety of semilinear equations.

**Lemma 5.1.** Assume that $\{u_n\}$ is a sequence of quasi-continuous functions on $E$, $u \in \mathcal{B}(E)$ and $\{\tau_k\}$ is a non-decreasing sequence of stopping times such that $\tau_k \to \zeta$ a.s. Suppose that for some $p > 0$ and any $k \geq 1$,

$$\mathbb{E}_x \sup_{0 \leq t \leq \tau_k} |u_n(X_t) - u(X_t)|^p \to 0, \quad \text{as } n \to \infty \text{ q.e.}$$

(5.1)

Then $u$ is quasi-continuous.

**Proof.** By [24, Theorem 4.2.2] process $u_n(X)$ is right-continuous on $[0, \infty)$ q.a.s. Therefore, by (5.1), $u$ shares this property too. Consequently, by [24, Theorem 4.6.1, Theorem A.2.7], $u$ is quasi-continuous. 

\[\square\]

We say that $\mu \in \mathcal{M}_\rho$ is a good measure iff there exists a solution to (1.2). We let $\mathcal{G}(f)$ denote the set of all good measures. It is clear that $\mathcal{G}(f)$ also depends on $A$. 


Theorem 5.2. Assume that there exists a subsolution \( u \) to (1.2). Let \( u^* \) be a maximal subsolution to (1.2) (cf. Theorem 4.10(4)). Set
\[
\mu^* := -Au^* - f(\cdot, u^*)
\] (5.2)
in the sense that \( \mu^* = \mu - \nu \), where \( \nu \) comes from the definition of a subsolution to (1.2) applied to \( u^* \) (see Definition 4.6). Then \( (\mu^*)_d = \mu_d, \mu^* \) is the largest measure less then \( \mu \) such that (1.2) has a solution with \( \mu \) replaced by \( \mu^* \), and \( u^* \) is a maximal solution to
\[
-Au = f(\cdot, v) + \mu^*.
\] (5.3)
Moreover, for any \( n \geq 1 \) and strictly positive \( \phi \in L^1(E, \rho \cdot m) \), there exists a maximal solution \( u_n \) to
\[
-Au = \max\{-n\phi, f(\cdot, v) + \mu, 0\},
\] (5.4)
and \( u_n \prec u^* \text{ m-a.e.} \). Furthermore, for any solution \( u \) to (5.3), and any reducing sequence \( \{\tau_k\} \) for \( u \), we have
\[
\mathbb{E}_x u(X_{\tau_k}) \rightarrow R[(\mu^*)_c](x), \text{ q.e.}
\] (5.5)
Proof. Let \( \phi \) be a strictly positive bounded Borel function on \( E \) such that \( \phi \in L^1(E, \rho \cdot m) \). Set
\[
f_n(x, y) = f(x, y) \vee (-n\phi), \quad x \in E, \ y \in \mathbb{R}.
\]
Clearly, \( f_n \) satisfies Sig) and \( f_n \geq f_{n+1} \geq f, \ n \geq 1 \). Let \( w := R\mu^* \). Observe that
\[
0 \leq f_n(\cdot, w) = f_n^*(\cdot, w) \leq n\phi.
\]
Thus, \( f_n(\cdot, w) \in L^1(E, \rho \cdot m) \). Moreover,
\[
-Aw = f_n(\cdot, w) + \mu + (-f_n(\cdot, w) + \mu^*).
\]
Consequently, \( w \) is a supersolution to (5.4). Set \( \pi := w \). Since \( \pi \) is a subsolution to (1.2), there exists a positive \( \nu \in \mathcal{M}_\rho \) such that
\[
-A\pi = f(\cdot, \pi) + \mu - \nu.
\]
Therefore,
\[
-A\pi = f_n(\cdot, \pi) + \mu - f_n(\cdot, \pi) - f(\cdot, \pi) - \nu.
\]
Hence, \( \pi \) is a subsolution to (5.4). By Theorem 4.10(3), \( \pi \leq \pi \text{ m-a.e.} \). Consequently, by Theorem 4.10, there exists a maximal solution \( u_n \) to (5.4) such that \( \pi \leq u_n \leq \pi \text{ m-a.e.} \). By Proposition 4.11, \( u_n \geq u_{n+1}, \ n \geq 1 \) q.e. Put \( u = \inf_{n \geq 1} u_n \) q.e. By Proposition 4.9,
\[
|u_n| + R|f_n(\cdot, u_n)| \leq R|\mu|, \text{ q.e.,} \quad \|f_n(\cdot, u_n)\|_{L^1(E, \rho \cdot m)} \leq \|\mu\|_{\mathcal{M}_\rho}.
\] (5.6)
Therefore, by Fatou’s lemma and Car),
\[
|u| + R|f(\cdot, u)| \leq R|\mu|, \text{ q.e.,} \quad \|f(\cdot, u)\|_{L^1(E, \rho \cdot m)} \leq \|\mu\|_{\mathcal{M}_\rho}.
\] (5.7)
Let \( \{\delta_k\} \) be a common reducing sequence for \( \pi, \pi \). Let
\[
\sigma_k = \inf\{t \geq 0 : |\pi(X_t)| \geq k\} \land k, \quad \delta_{k,j} = \inf\{t \geq 0 : \sup_{\|y\| \leq k} |f(X_t, y)| dr \geq j\}.
\]
By the definition of a reducing sequence \( \lim_{k \rightarrow \infty} \delta_k \land \zeta = \zeta \). Since \( \pi, \pi \) are quasi-continuous, \( \lim_{k \rightarrow \infty} \sigma_k \land \zeta = \zeta \) (see [24, Theorem 4.22]). Finally, by qM) and Corollary 2.3, \( \lim_{j \rightarrow \infty} \delta_{k,j} \land \zeta = \zeta \). Let \( \tau_{k,j} := \delta_k \land \sigma_k \land \delta_{k,j} \). Observe that
\[
\tau_{k,j} \prec \tau_k := \delta_k \land \sigma_k, \quad j \rightarrow \infty.
\] (5.8)
By Lemma 4.2,
\[
u_n(X_{t \land \tau_{k,j}}) = \mathbb{E}_x \left[ u_n(X_{\tau_{k,j}}) + \int_{t \land \tau_{k,j}}^{\tau_{k,j}} f_n(x_t, u_n(x_t)) dt + \int_{t \land \tau_{k,j}}^{\tau_{k,j} \land \tau_k} dA_{\mu^*}[F_{t \land \tau_{k,j}}] \right] \quad \text{q.a.s.} \] (5.9)
By [12, Lemma 6.1], for any \( q \in (0,1) \), there exists \( c_q > 0 \) such that
\[
\left( \mathbb{E}_x \sup_{0 \leq t \leq \tau_{k,j}} |u_n(X_t) - u_l(X_t)|^q \right)^{1/q} \leq c_q \mathbb{E}_x \left[ \left| u_n(X_{\tau_{k,j}}) - u_l(X_{\tau_{k,j}}) \right| \right]
\]
\[
+ \int_0^{T_{k,j}} \left| f_n(x, u_n(x)) - f_l(x, u_l(x)) \right| \, dx \quad \text{q.e.}
\]
Due to the choice of \( \{\tau_{k,j}\} \) and Remark 4.1(c), the right-hand side of the above equation tends to zero as \( n, l \to \infty \). Consequently, by Lemma 5.1, \( u \) is quasi-continuous. Taking \( t = 0 \) in (5.9), we get
\[
u_n(x) = \mathbb{E}_x u_n(X_{\tau_{k,j}}) + \mathbb{E}_x \int_0^{T_{k,j}} f_n(x, u_n(x)) \, dx + \mathbb{E}_x \int_0^{T_{k,j}} dA^{\mu_d} \quad \text{q.e.} 
\]
(5.10)
Letting \( n \to \infty \) and using properties of \( \{\tau_{k,j}\} \) and Remark 4.1(c) yields
\[
u(x) = \mathbb{E}_x u(X_{\tau_{k,j}}) + \mathbb{E}_x \int_0^{T_{k,j}} f(x, u(x)) \, dx + \mathbb{E}_x \int_0^{T_{k,j}} dA^{\mu_d} \quad \text{q.e.}
\]
(5.11)
Applying (5.7), (5.8) and the fact that \( \{\tau_k\} \) is a reducing sequence for \( u \), we find, by letting \( j \to \infty \) in (5.11), that
\[
nu(x) = \mathbb{E}_x u(X_{\tau_k}) + \mathbb{E}_x \int_0^{T_k} f(x, u(x)) \, dx + \mathbb{E}_x \int_0^{T_k} dA^{\mu_d} \quad \text{q.e.}
\]
(5.12)
Now, we shall show that there exists \( \beta \in \mathcal{M}_\rho \) such that \( u = R\beta \). For this, observe that
\[
nu_n = v_n - w_n \quad \text{q.e.,}
\]
(5.13)
where \( v_n = Rf_n^+(\cdot, u_n) + R\mu^+ \) and \( w_n = Rf_n^-(\cdot, u_n) + R\mu^- \). Clearly, \( v_n, w_n \) are excessive functions. Moreover, by (5.6)
\[
v_n \leq 2R|\mu|, \quad w_n \leq 2R|\mu| \quad \text{q.e.}
\]
(5.14)
Therefore, by [21, Lemma 94, page 306] there exists a subsequence (still denoted by \( (n) \)) and excessive functions \( v, w \) such that \( v_n \to v \) and \( w_n \to w \) \( m \)-a.e. By [25, Proposition 3.9], there exist positive Borel measures \( \beta_1, \beta_2 \) such that \( v = R\beta_1, w = R\beta_2 \). By (5.14) and [27, Lemma 4.6], \( \beta_1, \beta_2 \in \mathcal{M}_\rho \). Set \( \beta = \beta_1 - \beta_2 \). Due to (5.13), and the fact that \( u, R\beta \) are quasi-continuous, see Remark 4.1, we get \( u = R\beta \) \( q.e. \). Consequently, by [27, Theorem 3.7]
\[
\mathbb{E}_x u(X_{\tau_k}) \to R\beta_c(x), \quad \text{q.e.}
\]
Therefore, letting \( k \to \infty \) in (5.12) and using (5.7) yields
\[
u = Rf(\cdot, u) + R\mu_d + R\beta_c \quad \text{q.e.}
\]
(5.15)
Since \( u \leq u_n \), we get, by the inverse maximum principle (see [27, Theorem 6.1]) that \( \beta_c \leq \mu_c \).
Observe that
\[
-Au = f(\cdot, u) + \mu - (\mu_c - \beta_c).
\]
Thus, \( u \) is a subsolution to (1.2), which in turn implies that \( u \leq u^* \). On the other hand, by Proposition 4.11, \( u^* \leq u_n, n \geq 1 \). Thus \( u = u^* \). Consequently, \( \mu^* = \mu_d + \beta_c \), and so \( (\mu^*)_d = \mu_d \).
What is left is to show that \( \mu^* \) is the maximal measure less then \( \mu \) for which there exists a solution to (1.2) with \( \mu \) replaced by \( \mu^* \). Let \( \gamma \in \mathcal{M}_\rho, \gamma \leq \mu \) and \( v \) be a solution to
\[
-Av = f(\cdot, v) + \gamma.
\]
Since \( \gamma \leq \mu \), we have that \( \gamma_d \leq \mu_d \) and \( \gamma_c \leq \mu_c \). We have already proved that \( (\mu^*)_d = \mu_d \). So that, we only have to prove that \( \gamma_c \leq (\mu^*)_c \). Since \( \gamma \leq \mu \), \( v \) is a subsolution to (1.2). Thus, \( v \leq u^* \). By the inverse maximum principle \( \gamma_c \leq (\mu^*)_c \). 
\[\square\]
Analogous reasoning, but for supersolutions, leads to the following result.
Theorem 5.3. Assume that there exists a supersolution $\overline{u}$ to (1.2). Let $u_*$ be a minimal supersolution to (1.2). Set
$$\mu_* := -Au_* - f(\cdot, u_*).$$
Then $(\mu_*)_d = \mu_*$ is the smallest measure greater than $\mu$ such that (1.2) has a solution with $\mu$ replaced by $\mu_*$, and $u_*$ is a minimal solution to
$$-Av = f(\cdot, v) + \mu_*.$$  
Moreover, for any $n \geq 1$ and strictly positive $\phi \in L^1(E, \rho \cdot m)$, there exists a minimal solution $u_n$ to
$$-Av = \min\{n\phi, f\}(\cdot, v) + \mu,$$
and $u_n \nearrow u_*$. 

Remark 5.4. By Theorems 5.2, 5.3 if $\mu \in G(f)$, then $u^*$ is a maximal solution (subsolution) to (1.2) and $u_*$ is a minimal solution (supersolution) to (1.2).

Proposition 5.5. Let $\mu_1, \mu_2 \in M_\rho$ and $f_1, f_2$ satisfy Car), Sig), Int) and qM). Assume that $\mu_1 \leq \mu_2$ and $f_1 \leq f_2$. Let $u_1, u_2$ be solutions to
$$-Av = f_1(\cdot, v) + \mu_1, \quad -Av = f_2(\cdot, v) + \mu_2,$$
respectively.

(1) If $u_2$ is maximal and $f_2(\cdot, h) \in L^1(E; \rho \cdot m)$ for some $h \in L^1(E; \rho \cdot m)$ that satisfies $h \leq u_1$ a.e., then $u_1 \leq u_2$ a.e.

(2) If $u_1$ is minimal and $f_1(\cdot, h) \in L^1(E; \rho \cdot m)$ for some $h \in L^1(E; \rho \cdot m)$ that satisfies $u_2 \leq h$ a.e., then $u_1 \leq u_2$ a.e.

Proof. The proofs of both results are analogous, so that we only give the the proof of (1).

Step 1. Suppose that $f_2(\cdot, u_1) \in L^1(E; \rho \cdot m)$. Observe that
$$-Au_1 = f_2(\cdot, u_1) + \mu_2 + (f_1(\cdot, u_1) - f_2(\cdot, u_1)) + (\mu_1 - \mu_2).$$
Therefore, $u_1$ is a subsolution to $-Av = f_2(\cdot, v) + \mu_2$. By Remark 5.4, $u_1 \leq u_2$ a.e.

Step 2. The general case. By Theorem 5.2, there exists a maximal solution $u_1^*$ to $-Av = f_1(\cdot, v) + \mu_1$. By the same theorem, there exist sequences $\{u_1^n\}, \{u_2^n\}$ such that $u_1^n$ is a maximal solution to $-Av = f_1^n(\cdot, v) + \mu_1$, $u_2^n$ is a maximal solution to $-Av = f_2^n(\cdot, v) + \mu_2$, and $u_1^n \nearrow u_1^*, u_2^n \nearrow u_2$ a.e. Here $f_1^n(x, y) = \max\{-n\phi(x), f_1(x, y)\}, x \in E, y \in \mathbb{R}, i = 1, 2$, and $\phi$ is as in Theorem 5.2. By Theorem 4.10(3), $h \leq u_1 \leq u_1^n \leq R\mu_1$ a.e. By the assumptions made on $h$, and by Int), $f_2^n(\cdot, u_1^n) \in L^1(E; \rho \cdot m)$. By Step 1, $u_1^n \leq u_2^n$ a.e. Hence, $u_1 \leq u_2$ a.e. \qed

6. The class of good measures and the reduction operator

In this section we shall investigate the class of good measures. Our goal is to provide some properties of the set $G(f)$ and the mapping $\mu \mapsto \mu^*$. The main results of this section are Theorem 6.4, in which, by applying some basic properties of the mentioned objects, we prove an existence result for (1.2), and Theorems 6.20, 6.23 devoted to continuity of the operator $\mu \mapsto \mu^*$ and built up from it metric projection onto $G(f)$. Proposition 6.9 also deserves attention. It is the first result concerning the structure of $G(f)$. In the next section we considerably strengthen this result (Theorem 7.2).

In what follows we set for given $\mu \in M_\rho$,
$$G_{\leq \mu}(f) = \{\nu \in G(f) : \nu \leq \mu\} \quad G_{\geq \mu}(f) = \{\nu \in G(f) : \nu \geq \mu\}.$$
Let us note that by Theorems 5.2, 5.3, $\mu^* , \mu_*$ are well defined iff $\mathcal{G}_{\mathcal{M}}(f) \neq \emptyset , \mathcal{G}_{2\mathcal{M}}(f) \neq \emptyset$, respectively, and then

$$
\mu^* = \sup \mathcal{G}_{\mathcal{M}}(f), \quad \mu_* = \inf \mathcal{G}_{2\mathcal{M}}(f).
$$

Let $\tilde{\mathcal{M}}_\rho := \{ \mu \in \mathcal{M}_\rho : \mathcal{G}_{\mathcal{M}}(f) \neq \emptyset \}$. We call the mapping

$$
\tilde{\mathcal{M}}_\rho \ni \mu \mapsto \mu^* \in \mathcal{G}(f)
$$

the reduction operator.

**Remark 6.1.** Notice that if there exists a positive $g \in L^1(E; \rho \cdot m)$ and $M \leq 0$ such that $f(x, y) \leq g(x), \ x \in E, y \leq M$, then $\mathcal{M}_\rho = \tilde{\mathcal{M}}_\rho$. Indeed, observe that $\underline{w} := -R\mu^*$ (resp. $\bar{u} := 0$) is a subsolution (resp. supersolution) to

$$
-Au = f(\cdot, u) - \mu^*.
$$

Therefore, by Theorem 4.10, there exists a solution $w$ to the above equation. Thus, $-\mu^* \in \mathcal{G}(f)$. Clearly, $-\mu^* \leq \mu$.

### 6.1. Basic properties of good measures and the reduction operator: application to the existence problem

**Proposition 6.2.** We have the following.

1. $\mathcal{M}_\rho^0 \subset \mathcal{G}(f)$.
2. Suppose that $\nu \in \mathcal{M}_\rho$, $\mu \in \tilde{\mathcal{M}}_\rho$, and $\mu \leq \nu$. Then $\mu^* \leq \nu^*$.
3. If $\mu, \nu \in \mathcal{G}(f)$, then $\mu \lor \nu, \mu \land \nu \in \mathcal{G}(f)$.
4. If $\mu$ is a positive measure, then $\mu^* \geq 0$.

**Proof.** Ad (1). It follows from the existence result proved in [30]. Ad (2). By the assumptions there exist $\mu^*, \nu^*$. By Theorem 5.2, $\mu^* \leq \mu$, and so $\mu^* \leq \nu$. By Theorem 5.2 again, $\mu^* \leq \nu^*$. Ad (3). Since $\mu$ is a good measure, there exists a solution $u$ to (1.2). Observe that $u$ is also a subsolution to (1.2) with $\mu$ replaced by $\mu \lor \nu$. Thus, there exists $(\mu \lor \nu)^*$. By Theorem 5.2, $(\mu \lor \nu)^* \leq \mu \lor \nu$. On the other hand, by (2)

$$
\mu = \mu^* \leq (\mu \lor \nu)^*, \quad \nu = \nu^* \leq (\mu \lor \nu)^*.
$$

So that $(\mu \lor \nu)^* = \mu \lor \nu$. Thus, $\mu \lor \nu \in \mathcal{G}(f)$. Analogous reasoning for the minimum gives that $\mu \land \nu \in \mathcal{G}(f)$. Ad (4). By (1), $0 \in \mathcal{G}(f)$. We assumed that $0 \leq \mu$. Therefore by (2), $0 = 0^* \leq \mu^*$.

**Proposition 6.3.** Let $\mu, \nu \in \mathcal{M}_\rho$, and $\mu \land \nu \in \tilde{\mathcal{M}}_\rho$. Then $(\mu \land \nu)^* = \mu^* \land \nu^*$.

**Proof.** First observe that $\mu \land \nu \in \tilde{\mathcal{M}}_\rho$. By Theorem 5.2, $\mu^* \leq \mu, \nu^* \leq \nu$. Hence $\mu^* \land \nu^* \leq \mu \land \nu$. By (2) and (3) of Proposition 6.2, $\mu^* \land \nu^* \leq (\mu \land \nu)^*$. On the other hand, by Proposition 6.2(2), $\mu^* \geq (\mu \land \nu)^*$ and $\nu^* \geq (\mu \land \nu)^*$. Thus, $\mu^* \land \nu^* \geq (\mu \land \nu)^*$.

**Theorem 6.4.** Assume that there exists a subsolution and a supersolution to (1.2). Then there exists a solution to (1.2).

**Proof.** Thanks to the assumptions made, there exist a subsolution $\underline{w}$ and a supersolution $\bar{w}$ to (1.2). Therefore, according to the definition of these objects, there exist two positive measures $\nu_1, \nu_2 \in \mathcal{M}_\rho$ such that

$$
-A\nu = f(\cdot, \nu) + \mu - \nu_1, \quad -A\bar{w} = f(\cdot, \bar{w}) + \mu + \nu_2.
$$

In particular, $\mu - \nu_1 \in \mathcal{G}_{2\mathcal{M}}(f), \mu + \nu_2 \in \mathcal{G}_{\mathcal{M}}(f)$. Thus, there exist $\mu_*, \mu^*$. Obviously, $\mu_*, \mu^* \in \mathcal{G}(f)$. Let $\bar{w}$ be a maximal solution to $-A\nu = f(\cdot, \nu) + \mu_*$ (see Remark 5.4). By Theorems 5.2, 5.3, $\mu^* \leq \mu \leq \mu_*$. Therefore, by Proposition 5.5, $u^* \leq \bar{w}$. Observe that $u_*$ is a subsolution to (1.2) and $\bar{w}$ is a supersolution to (1.2). By Theorem 4.10, there exists a solution to (1.2).
As a corollary to the above result, we obtain the following useful properties of the set $\mathcal{G}(f)$.

**Proposition 6.5.** Assume that $\mu_1, \mu_2 \in \mathcal{G}(f)$, $\mu \in \mathcal{M}_\rho$ and $\mu_1 \leq \mu \leq \mu_2$. Then $\mu \in \mathcal{G}(f)$.

**Proof.** Since $\mu_1, \mu_2 \in \mathcal{G}(f)$. There exists a solution $u_1$ to (1.2) with $\mu$ replaced by $\mu_1$, and a solution $u_2$ to (1.2) with $\mu$ replaced by $\mu_2$. Since $\mu_1 \leq \mu \leq \mu_2$, $u_1$ is a subsolution to (1.2) and $u_2$ is a supersolution to (1.2). By Theorem 6.4, there exists a solution to (1.2). So, $\mu \in \mathcal{G}(f)$. \hfill $\Box$

**Corollary 6.6.** Let $\mu \in \hat{\mathcal{M}}_\rho$. Then $\mu \in \mathcal{G}(f)$ if and only if $\mu^+ \in \mathcal{G}(f)$.

**Proof.** If $\mu \in \mathcal{G}(f)$, then by Proposition 6.2.(3), $\mu^+ \in \mathcal{G}(f)$. Suppose that $\mu^+ \in \mathcal{G}(f)$. We have $\mu \leq \mu^+$. At the same time, since $\mu \in \hat{\mathcal{M}}_\rho$, there exists $\nu \in \mathcal{G}(f)$ such that $\nu \leq \mu$. Therefore, by Proposition 6.5, $\mu \in \mathcal{G}(f)$. \hfill $\Box$

**Proposition 6.7.** We have $\mathcal{G}(f) + \mathcal{M}^0_\rho \subset \mathcal{G}(f)$.

**Proof.** Let $\gamma \in \mathcal{G}(f)$ and $\beta \in \mathcal{M}^0_\rho$. Write

\[ \gamma = (\gamma + \beta) + (-\beta). \]

Once we show that for any $\mu \in \mathcal{M}_\rho, \nu \in \mathcal{M}^0_\rho$ such that $\mu + \nu \in \mathcal{G}(f)$ we have that $\mu \in \mathcal{G}(f)$, then we conclude the result by taking $\mu = \gamma + \beta, \nu = -\beta$. So, let $\mu \in \mathcal{M}_\rho, \nu \in \mathcal{M}^0_\rho$ and $\mu + \nu \in \mathcal{G}(f)$. Set

\[ \overline{\sigma} = \max\{\mu + \nu, \mu_d\}, \quad \underline{\sigma} = \min\{\mu + \nu, \mu_d\}. \]

Since $\mu + \nu \in \mathcal{G}(f)$, we get by Proposition 6.2 (1),(3) that $\overline{\sigma}, \underline{\sigma} \in \mathcal{G}(f)$. Observe that

\[ \overline{\sigma} = \mu_d + \max\{\mu_c + \nu, 0\} = \mu_d + (\mu_c + \nu)^+ = \mu_d + \mu_c^+ + \nu^+ \geq \mu, \]

and

\[ \underline{\sigma} = \mu_d + \min\{\mu_c + \nu, 0\} = \mu_d - (\mu_c + \nu)^- = \mu_d - \mu_c^- - \nu^- \leq \mu. \]

Thus, $\underline{\sigma} \leq \mu \leq \overline{\sigma}$. By Proposition 6.5, $\mu \in \mathcal{G}(f)$. \hfill $\Box$

**Corollary 6.8.** We have that $\mu \in \mathcal{G}(f)$ if and only if $\mu_c \in \mathcal{G}(f)$.

6.2. Further properties of the reduction operator and good measures - the class of admissible measures. We let

\[ \mathcal{A}(f) = \{ \mu \in \mathcal{M}_\rho : f(\cdot, R\mu) \in L^1(E; \rho \cdot m) \}. \]

Elements of $\mathcal{A}(f)$ shall be called admissible measures.

**Proposition 6.9.** We have $\mathcal{A}(f) + L^1(E; \rho \cdot m) = \mathcal{G}(f)$.

**Proof.** The inclusion $" \subset "$ follows directly from Proposition 6.7. Suppose that $\mu \in \mathcal{G}(f)$. Therefore, there exists a solution $u$ to (1.2). Set $\nu := \mu + f(\cdot, u)$. Then $\nu \in \mathcal{A}(f)$ since $R\nu = u$ and by the very definition of a solution to (1.2), we have $f(\cdot, u) \in L^1(E; \rho \cdot m)$. Thus, $\mu = \nu - f(\cdot, u) \in \mathcal{A}(f) + L^1(E; \rho \cdot m)$. \hfill $\Box$

**Proposition 6.10.** The set $\mathcal{G}(f)$ is closed in $(\mathcal{M}_\rho, \| \cdot \|_{\mathcal{M}_\rho})$.

**Proof.** Let $(\mu_n)_{n \geq 1} \subset \mathcal{G}(f)$. Let $\mu \in \mathcal{M}_\rho$ and $\| \mu_n - \mu \|_{\mathcal{M}_\rho} \to 0$, $n \to \infty$. Set $\mu_0 = 0$. We may assume that $\sum_{n \geq 1} \| \mu_{n+1} - \mu_n \|_{\mathcal{M}_\rho} < \infty$. Then $\mu = \sum_{n \geq 0} (\mu_{n+1} - \mu_n)$, where the limit is understood in the norm $\| \cdot \|_{\mathcal{M}_\rho}$. Observe that

\[ \underline{\sigma} := - \sum_{n \geq 0} (\mu_{n+1} - \mu_n)^- \leq \mu_n \leq \sum_{n \geq 0} (\mu_{n+1} - \mu_n)^+ =: \overline{\sigma}. \]

Since $\mu_n \in \mathcal{G}(f)$, there exist $\sigma^+, \sigma^-$ and by Proposition 6.2(2), $\sigma^+ \leq \mu_n \leq \sigma^-$. Letting $n \to \infty$ yields $\sigma^+ \leq \mu \leq \sigma^-$. By Proposition 6.5, $\mu \in \mathcal{G}(f)$. \hfill $\Box$
Proposition 6.11. Assume that \( \mu \in \tilde{M}_\rho \). Then \( |\mu^*| \leq |\mu| \).

Proof. By Theorem 4.10(3), \( u^* \leq w \), where \( w = R \mu^* \), and \( v \leq u^* \), where \( v = -R \mu^- \). By the inverse maximum principle \( -\mu^- \leq (\mu^*)^c \leq \mu^- \). Hence \( |(\mu^*)^c| \leq |\mu| \). Since \( (\mu^*)^d = \mu_d \), we get the result. \( \square \)

Corollary 6.12. Assume that \( \mu, \nu \in \tilde{M}_\rho \), and \( \mu \perp \nu \). Then \( \mu^* \perp \nu^* \).

Proposition 6.13. Let \( \mu, \nu \in \tilde{M}_\rho \), and \( \mu \perp \nu \). Then \( (\mu + \nu)^* = \mu^* + \nu^* \).

Proof. First we show that \( (\mu + \nu)^* \) is well defined and \( \mu^* + \nu^* \leq (\mu + \nu)^* \). Clearly, \( \mu^* + \nu^* \leq \mu + \nu \). So, it is enough to prove that \( \mu^* + \nu^* \in \mathcal{G}(f) \) since then \( \mathcal{G}_{\mu + \nu}(f) \neq \emptyset \), and, by Proposition 6.6(2), \( \mu^* + \nu^* \leq (\mu + \nu)^* \). By Proposition 6.11, \( \mu^* \perp \nu^* \). Thus
\[ \mu^* \perp \nu^* = (\mu^* + \nu^*)^+ \quad \mu^* \perp \nu^* = -(\mu^* + \nu^*)^-- \leq \mu^* + \nu^* \leq \mu + \nu. \]
Therefore, by Proposition 6.2(3), \( \mathcal{G}_{\mu + \nu}(f) \neq \emptyset \) and \( (\mu^* + \nu^*)^+ \in \mathcal{G}(f) \). Hence, by Corollary 6.6, \( \mu^* + \nu^* \in \mathcal{G}(f) \). For the inequality \( \mu^* + \nu^* \geq (\mu + \nu)^* \) observe that
\[ (\mu + \nu)^* = s_{\mu} \cdot \mu + s_{\nu} \cdot \nu, \tag{6.1} \]
where
\[ s_{\mu} = \frac{d(\mu + \nu)^*}{d|\mu|} \quad \frac{d|\mu|}{d\mu}, \quad s_{\nu} = \frac{d(\mu + \nu)^*}{d|\nu|} \quad \frac{d|\nu|}{d\nu}. \]

By Proposition 6.11, \( s_{\mu}, s_{\nu} \) are well defined and \( |s_{\mu}| \leq 1, |s_{\nu}| \leq 1 \). Therefore, from (6.1) and the fact that \( \mu \perp \nu \), we infer that
\[ -[\mu^* + \nu^*] \leq s_{\mu} \cdot \mu \leq [(\mu + \nu)^*], \quad -[\mu^* + \nu^*] \leq s_{\nu} \cdot \nu \leq [(\mu + \nu)^*]. \]

By Proposition 6.2(3), \( [\mu^* + \nu^*] \), \([\mu^* + \nu^*] \) is well defined and \( \mathcal{G}(f) \). Therefore, by Proposition 6.5, \( s_{\mu} \cdot \mu + s_{\nu} \cdot \nu \in \mathcal{G}(f) \). By (6.1), \( s_{\mu} \cdot \mu + s_{\nu} \cdot \nu \leq \mu + \nu \). From this and the fact that \( \mu \perp \nu \), we conclude that \( s_{\mu} \cdot \mu \leq \mu, s_{\nu} \cdot \nu \leq \nu \). Consequently, since \( s_{\mu} \cdot \mu, s_{\nu} \cdot \nu \in \mathcal{G}(f) \), we have \( s_{\mu} \cdot \mu \leq \mu^*, s_{\nu} \cdot \nu \leq \nu^* \). This combined with (6.1) gives \( (\mu + \nu)^* \leq \mu^* + \nu^* \). \( \square \)

Corollary 6.14. Let \( \mu \in \tilde{M}_\rho \) and \( A \in \mathcal{B}(E) \). Then \( (\mu|_A)^* = \mu^*_A \).

Proof. First we show that \( \mathcal{G}_{\mu|_A}(f) \neq \emptyset \). For this it is enough to prove that \( \nu \in \mathcal{G}_{\mu|_A}(f) \) for any \( \nu \in \mathcal{G}_{\mu}(f) \). But this follows easily from Proposition 6.2(3) and Proposition 6.5 since \( -\nu \leq \nu \leq \nu^* \). By Proposition 6.13,
\[ (\mu^*)|_A + (\nu^*)|_A \leq (\mu|_A)^* + (\nu|_A)^* \]
Applying Proposition 6.11 yields \( |(\mu^*)|_A|, |(\mu|_A)^*| \leq |\mu|_A \) and \( |(\nu^*)|_A|, |(\nu|_A)^*| \leq |\nu|_A \). From this and the above equality, we get the result. \( \square \)

Corollary 6.15. Let \( \mu, \nu \in \tilde{M}_\rho \). Then \( (\mu \vee \nu)^* = \mu^* \vee \nu^* \).

Proof. It is enough to repeat step by step the proof of [9, Theorem 4.9]. \( \square \)

Corollary 6.16. Let \( \mu \in \tilde{M}_\rho \) and \( \nu \in \tilde{M}_\rho^0 \). Then \( (\mu + \nu)^* = \mu^* + \nu^* \).

Proof. Observe that by Proposition 6.2(1), \( \mathcal{G}_{\mu + \nu}(f) \neq \emptyset \). Next, by Proposition 6.13 and Proposition 6.2(1),
\[ (\mu + \nu)^* = (\mu^c)^* + (\mu^d + \nu)^* = (\mu^c)^* + \nu^* + \mu^d + \nu = (\mu^c + \mu^d)^* + \nu = \mu^* + \nu. \]

Corollary 6.17. Let \( \mu \in \tilde{M}_\rho \). Then \( (\mu^c)^* = (\mu^*)^c \).
Proof. Let \( \beta \in \mathcal{G}_{\mathcal{L}p}(f) \). Then, by Corollary 6.8, \( \beta_c \in \mathcal{G}_{\mathcal{L}p}(f) \). Thus, \( \mathcal{G}_{\mathcal{L}p}(f) \neq \emptyset \). By Corollary 6.16 and Theorem 5.2,
\[
(\mu^*)_c = \mu^* - (\mu^*)_d = \mu^* - \mu_d = (\mu_c + \mu_d)^* - \mu_d = (\mu_c)^*.
\]
\(\square\)

From now on for every \( \mu \in \mathcal{M}_p \) without ambiguity we may write \( \mu^*_c \).

**Proposition 6.18.** Let \( \mu \in \mathcal{N}_p \). Then
\[
\mu^* = \mu_d - \mu_c^* + (\mu^*_c)^*.
\]

**Proof.** Let \( \beta \in \mathcal{G}_{\mathcal{L}p}(f) \). Then \( -\beta_c^* \leq -\mu_c^* \). By Proposition 6.2.(3) and Corollary 6.8, \( -\beta_c^* \in \mathcal{G}(f) \). Thus, \( \mathcal{G}_{\mathcal{L}p}(f) \neq \emptyset \). By Proposition 6.5, \( -(\mu_c^*)^* = -\mu_c^* \). From this and Corollary 6.16, we get
\[
\mu^* = \mu_d + (\mu_c^*)^* + (\mu_c^*)^* = \mu_d - \mu_c^* + (\mu_c^*)^*.
\]
\(\square\)

### 6.3. Continuity of the Reduction Operator

The main result of the present subsection is continuity of the reduction operator with respect to the norm \( \| \cdot \|_{\mathcal{M}_p} \). Note that if we assume additionally that \( f \) is non-increasing with respect to the second variable, then the reduction operator is even Lipschitz continuous (see [27, Theorem 5.10]).

**Lemma 6.19.** Let \( \mu_n \in \mathcal{M}_p, n \geq 1, \mu_n \in \mathcal{N}_p \), and \( \mu \leq \mu_{n+1} \leq \mu_n, n \geq 1 \). Assume that \( \mu_n \rightarrow \mu \) in the norm \( \| \cdot \|_{\mathcal{M}_p} \). Then \( \mu_n^* \rightarrow \mu^* \) in the norm \( \| \cdot \|_{\mathcal{M}_p} \).

**Proof.** By Proposition 6.2, \( \{\mu_n^*\} \) is a nondecreasing sequence and
\[
\mu^* \leq \mu_n^* \leq \mu_n, \quad n \geq 1.
\]
Since \( \{\mu_n^*\} \) is nondecreasing, we may set \( \beta = \lim_{n \rightarrow \infty} \mu_n^* \), where the limit is understood in the norm \( \| \cdot \|_{\mathcal{M}_p} \). Letting \( n \rightarrow \infty \) in the above inequality yields \( \mu^* \leq \beta \leq \mu \). Since \( \mu^* \leq \beta \leq \mu^*_1 \), we have by Proposition 6.5 that \( \beta \in \mathcal{G}_{\mathcal{L}p}(f) \). Thus, \( \beta = \mu^* \).

**Theorem 6.20.** Let \( \mu, \mu_n \in \mathcal{M}_p, n \geq 1 \). Assume that \( \mathcal{G}_{\mathcal{L}p}(f) \neq \emptyset, \mathcal{G}_{\mathcal{L}p}(f) \neq \emptyset, n \geq 1, \) and \( \mu_n \rightarrow \mu \) in the norm \( \| \cdot \|_{\mathcal{M}_p} \). Then \( \mu_n^* \rightarrow \mu^* \) in the norm \( \| \cdot \|_{\mathcal{M}_p} \).

**Proof.** All the convergences of measures considered in the proof below will be understood in the norm \( \| \cdot \|_{\mathcal{M}_p} \).

**Step 1.** We assume additionally that \( 0 \leq \mu \leq \mu_n, n \geq 1 \). Let \( (n_k) \) be a subsequence of \( (n) \). By [40, Proposition 4.2.4], there exists a further subsequence \( (n_{k_l}) \), and positive \( \beta \in \mathcal{M}_p \) such that
\[
|\mu_{n_{k_l}} - \mu| \leq \frac{1}{l} \beta, \quad l \geq 1.
\]
Thus,
\[
\mu \leq \mu_{n_{k_l}} \leq \mu + \frac{1}{l} \beta, \quad l \geq 1.
\]
By Proposition 6.2,
\[
\mu^* \leq \mu_{n_{k_l}}^* \leq (\mu + \frac{1}{l} \beta)^*, \quad l \geq 1.
\]
By Lemma 6.19, \( (\mu + \frac{1}{l} \beta)^* \rightarrow \mu^* \). From this and the above inequality \( \mu_{n_{k_l}}^* \rightarrow \mu^* \). Since \( (n_k) \) was an arbitrary subsequence of \( (n) \), we conclude that \( \mu_{n}^* \rightarrow \mu^* \).

**Step 2.** We assume additionally that \( 0 \leq \mu_n \leq \mu, n \geq 1 \). Then \( \mu_n = s_n \cdot \mu \) for some Borel function \( s_n \) on \( E \) such that \( 0 \leq s_n \leq 1 \). Since \( \mu_n \rightarrow \mu \), we have that \( s_n \rightarrow 1 \) in \( L^1(E; \mu) \). Let \( a \in (0, 1) \). Observe that for any positive \( \nu \in \mathcal{M}_p \),
\[
av^* \leq (av)^*.
\] (6.2)
Set $A_n = \{s_n \geq a\}$. Then
\[ a\mu|_{A_n} \leq \mu_n \leq \mu, \quad n \geq 1. \]

Moreover,
\[ \mu(A_n^c) = \mu([1-s_n] > 1-a) \leq \frac{1}{1-a} \int_E |1-s_n|d\mu \to 0. \]

By (6.2), (6.3), Proposition 6.14
\[ a(\mu^*)|_{A_n} \leq \mu_n^* \leq \mu^*, \quad n \geq 1. \]

From this and (6.4), we get
\[ a\mu^* \leq \liminf_{n \to \infty} \mu_n^* \leq \limsup_{n \to \infty} \mu_n^* \leq \mu^*. \]

Since $a \in (0,1)$ was arbitrary, we get $\mu_n^* \to \mu^*$.

**Step 3.** We assume additionally that $\mu_n \geq 0$, $n \geq 1$. Then observe that
\[ 0 \leq \mu \land \mu_n \leq \mu \lor \mu_n, \quad n \geq 1. \]

Clearly, $\mu \land \mu_n \to \mu$ and $\mu \lor \mu_n \to \mu$. Therefore, by Step 1 and Step 2, $(\mu \land \mu_n)^* \to \mu^*$ and $(\mu \lor \mu_n)^* \to \mu^*$. By (6.5) and Proposition 6.2,
\[ (\mu \land \mu_n)^* \leq \mu_n^* \leq (\mu \lor \mu_n)^*, \quad n \geq 1. \]

Thus, $\mu_n^* \to \mu^*$.

**Step 4.** The general case. By Proposition 6.18,
\[ \mu_n^* = (\mu_n)_d - (\mu_n)_c + (\mu_n^*)_c, \quad \mu^* = \mu_d - \mu_c + (\mu_c^*)_. \]

Since $\mu_n \to \mu$, we have $(\mu_n)_d \to \mu_d$, $(\mu_n)_c \to \mu_c$, $(\mu_n^*)_c \to \mu_c^*$. By Step 3, $(\mu_n^*)_c \to (\mu_c^*)_c$. As a result, by (6.6), $\mu_n^* \to \mu^*$.

**6.4. Existence and continuity of the metric projection onto good measures.** In what follows, in order to emphasize the dependence of the reduction operator on the nonlinearity $f$, we shall denote by $\mu^{*,f}$ and $\mu_{*,f}$ the measure $\mu^*$ and $\mu_*$ appearing in the assertions of Theorem 5.2 and Theorem 5.3, respectively.

We let
\[ \Pi_f(\mu) := \mu^{*,f}, \quad \mu \in \tilde{M}_\rho. \]

We also let $\tilde{M}_\rho := \{\mu \in M_\rho : \mathcal{G}_{2\mu} \neq \emptyset\}$. By Proposition 6.5, $\tilde{M}_\rho \cap \tilde{M}_\rho = \mathcal{G}(f)$. Thus, we may extend operator $\Pi_f$:
\[ \Pi_f(\mu) = \begin{cases} \mu^{*,f}, & \mu \in \tilde{M}_\rho \\ \mu_{*,f}, & \mu \in \tilde{M}_\rho. \end{cases} \]

Since for $\mu \in \tilde{M}_\rho \cap \tilde{M}_\rho$, we have $\mu^{*,f} = \mu_{*,f} = \mu$, the operator $\Pi_f$ is well defined on $\tilde{M}_\rho \cup \tilde{M}_\rho$.

We denote
\[ \tilde{f}(x,y) := -f(x,-y), \quad x \in E, y \in \mathbb{R}. \]

We get at once that if $f$ satisfies one of the conditions (Int), (Car), (qM), (Sig), then $\tilde{f}$ satisfies it too.

**Remark 6.21.** Observe that by Theorems 5.2, 5.3 for any $\mu \in \tilde{M}_\rho$,
\[ -\Pi_f(-\mu) = -(-\mu)^{*,f} = \mu_{*,f}. \]
Proposition 6.22. The mapping
\[
\Pi_f : \mathcal{M}_\rho \cup \mathcal{M}_\rho \to \mathcal{G}(f)
\]
is the metric projection onto \(\mathcal{G}(f)\). Moreover,
\[
|\mu - \Pi_f(\mu)| \leq (\mu - \nu)^+, \quad \mu \in \mathcal{M}_\rho, \nu \in \mathcal{G}(f),
\]
and
\[
|\mu - \Pi_f(\mu)| \leq (\nu - \mu)^+, \quad \mu \in \mathcal{M}_\rho, \nu \in \mathcal{G}(f).
\]
Furthermore, for any \(\mu \in \mathcal{M}_\rho \cup \mathcal{M}_\rho\), the measure \(\Pi_f(\mu)\) is the only element of \(\mathcal{G}(f)\) satisfying
\[
\|\mu - \Pi_f(\mu)\|_{\mathcal{M}_\rho} = \inf_{\nu \in \mathcal{G}(f)} \|\mu - \nu\|_{\mathcal{M}_\rho}.
\]

Proof. Let \(\nu \in \mathcal{G}(f)\) and \(\mu \in \mathcal{M}_\rho\). The last relation implies that there exists \(\beta \in \mathcal{G}(f)\) such that \(\beta \leq \mu\). Thus, \(\beta \wedge \mu \leq \mu \wedge \nu \leq \nu\). By Propositions 6.2 and Proposition 6.5, \(\mu \wedge \nu \in \mathcal{G}(f)\). This in turn implies that \(\mu \wedge \nu \leq (\mu \wedge \nu)^*f = \mu^*f \wedge \nu\) (see Proposition 6.3). Consequently,
\[
|\mu - \Pi_f(\mu)| = |\mu - \mu^*f| = |\mu - \mu^*f| = |\mu \wedge \nu| \leq |\mu - \nu| \leq |\mu - \nu|.
\]
Therefore,
\[
\|\mu - \Pi_f(\mu)\|_{\mathcal{M}_\rho} \leq |\mu - \nu|_{\mathcal{M}_\rho} \quad \text{for any } \nu \in \mathcal{G}(f).
\]
This completes the proof of case \(\mu \in \mathcal{M}_\rho\). Now, let \(\mu \in \mathcal{M}_\rho\). Observe that \(-\mu \in \mathcal{M}_\rho\). Therefore, by Remark 6.21
\[
\|\mu - \Pi_f(\mu)\|_{\mathcal{M}_\rho} = \|\mu - (-\Pi_f(\mu))\|_{\mathcal{M}_\rho} = \inf_{\nu \in \mathcal{G}(f)} \|\nu - (-\mu)\|_{\mathcal{M}_\rho} = \inf_{\nu \in \mathcal{G}(f)} \|\nu + \mu\|_{\mathcal{M}_\rho}.
\]
Applying (6.10) with \(-\mu\) in place of \(\mu\) yields (6.9). For the proof of the last assertion of the proposition suppose that \(\mu \in \mathcal{M}_\rho\) (the proof of the second case is analogous) and suppose that \(\nu \in \mathcal{G}(f)\) realizes the distance between \(\mu\) and \(\mathcal{G}(f)\). Notice that
\[
|\mu - \mu \wedge \nu|_{\mathcal{M}_\rho} = (\mu - \nu)^+_{\mathcal{M}_\rho}.
\]
Since \(\mu \in \mathcal{M}_\rho\), there exists \(\gamma \in \mathcal{G}(f)\) such that \(\gamma \leq \mu\). Hence, \(\gamma \wedge \nu \leq \mu \wedge \nu \leq \nu\). By Proposition 6.5, \(\mu \wedge \nu \in \mathcal{G}(f)\). Therefore, since \(\nu\) realizes the distance between \(\mu\) and \(\mathcal{G}(f)\), we have
\[
\|\mu - \nu\|_{\mathcal{M}_\rho} \leq |\mu - \nu|_{\mathcal{M}_\rho}.
\]
This combined with (6.11) yields \((\mu - \nu)^+ = 0\), so that \(\nu \leq \mu\). The last inequality is crucial since it implies that \(\nu \leq \mu^*f\). We also have, \(\Pi_f(\mu) = \mu^*f \leq \mu\). On the other hand, since \(\Pi_f(\mu)\) and \(\nu\) realize the distance between \(\mu\) and \(\mathcal{G}(f)\), we have
\[
\|\mu - \Pi_f(\mu)\|_{\mathcal{M}_\rho} = |\mu - \nu|_{\mathcal{M}_\rho}.
\]
Therefore, we deduce at once that \(\Pi_f(\mu) = \nu\).

Finally, we define the operator
\[
\Pi_f : \mathcal{M}_\rho \to \mathcal{G}(f)
\]
as follows:
\[
\Pi_f(\mu) := \Pi_f(\mu^+) + \Pi_f(-\mu^-)
\]
\[
= \Pi_f(\mu^+) - \Pi_f(-\mu^-) = (\mu^+)^*f - (\mu^-)^*f = (\mu^+)^*f + (-\mu^-)^*f.
\]
By Propositions 6.11, 6.13, \(\Pi_f(\mu) \in \mathcal{G}(f)\) for any \(\mu \in \mathcal{M}_\rho\).

Theorem 6.23. \(\Pi_f : \mathcal{M}_\rho \to \mathcal{G}(f)\) is a continuous metric projection onto \(\mathcal{G}(f)\). Moreover,
\[
|\mu - \Pi_f(\mu)| \leq |\mu - \nu|, \quad \mu \in \mathcal{M}_\rho, \nu \in \mathcal{G}(f).
\]
Furthermore, if \(Q : \mathcal{M}_\rho \to \mathcal{G}(f)\) is a metric projection, with the property: \(\mu \perp \nu\) implies \(Q(\mu + \nu) = Q(\mu) + Q(\nu)\), then \(Q = \Pi_f\).
Proof. Continuity of \( \Pi_f \) follows from Theorem 6.20. Let \( \nu \in \mathcal{G}(f) \) and \( \mu \in \mathcal{M}_p \). Clearly, \( \Pi_f(\mu) \in \mathcal{G}(f) \). Moreover, by Proposition 6.22, for any \( \nu_1 \in \mathcal{G}(f) \), \( \nu_2 \in \mathcal{G}(\tilde{f}) = -\mathcal{G}(f) \)
\[
|\mu - \Pi_f(\mu)| = |\mu^+ - \Pi_f(\mu^+)| + |\mu^- - \Pi_f(\mu^-)| \leq (\mu^+ - \nu_1)^+ + (\mu^- - \nu_2)^+.
\]

Let \( \nu \in \mathcal{G}(f) \). Then \( \nu^+ \in \mathcal{G}(f) \) and \( \nu^- \in -\mathcal{G}(f) \). Therefore,
\[
|\mu - \Pi_f(\mu)| \leq (\mu^+ - \nu^+)^+ + (\mu^- - \nu^-)^+ \leq |\mu - \nu|.
\]

This implies the inequality asserted in the theorem, and the fact that \( \Pi_f \) is the metric projection onto \( \mathcal{G}(f) \). For the last assertion of the theorem, observe that operator \( \Pi_f \) shares additivity property formulated in the assertion of the theorem for \( Q \). Therefore, if \( \Pi_f = Q \) on \( \mathcal{M}_p \cup \mathcal{M}_p \), then \( \Pi_f = Q \) on \( \mathcal{M}_p \). The fact that \( \Pi_f = Q \) on \( \mathcal{M}_p \cup \mathcal{M}_p \) follows easily from Proposition 6.22.

7. Characterization of the class of good measures

Proposition 7.1. Assume that \( \{\mu_n\} \) is a sequence of positive Borel measures such that \( \sup_{n \geq 1} R\mu_n < \infty \) q.e. Suppose that \( R\mu_n \to 0 \) m-a.e. Then there exists a subsequence (not relabeled) such that \( R\mu_n \to 0 \) q.e.

Proof. By [28, Lemma 5.1], there exists a subsequence (not relabeled) such that \( k \wedge R\mu_n \to 0 \) q.e. for any \( k \geq 1 \). As a result, since \( \sup_{n \geq 1} R\mu_n < \infty \) q.e., we infer from this convergence that up to subsequence \( R\mu_n \to 0 \) q.e.

Theorem 7.2. Let \( \mu \in \mathcal{M}_p \).

(1) \( \mu \in \mathcal{G}(f) \) if and only if there exists a sequence \( \{g_n\} \subset L^1(E; \rho \cdot m) \) such that
   (i) \( g_n + \mu \in A(f) \), \( n \geq 1 \).
   (ii) \( g_n \to 0 \) in \( L^1(E; \rho \cdot m) \).

(2) Assume that \( \rho \) is bounded and there exists \( \varepsilon > 0 \) such that \( \sup_{|y| \leq \varepsilon} |f(\cdot, y)| \in L^1(E; \rho \cdot m) \), then \( \mu \in \mathcal{G}(f) \) if and only if there exists a sequence \( \{g_n\} \subset L^1(E; \rho \cdot m) \) such that condition (i) and the following one
   (ii') \( g_n \to 0 \) in \( L^1(E; \rho \cdot m) \)
hold.

Furthermore, in both cases ((1) and (2)), if \( \mu \) is positive (resp. negative) then \( g_n \) may be taken negative (resp. positive).

Proof. Sufficiency (in both cases) follows from Corollary 6.14 and Proposition 6.10. Let \( \mu \in \mathcal{G}(f) \). Set
\[
f_{n,m}(x, y) := \frac{1}{m} f^+(x, y) - \frac{1}{m} f^-(x, y), \quad x \in E, y \in \mathbb{R}.
\]
Since \( \mu \in \mathcal{G}(f) \), there exists a solution \( u \) to \( -Au = f(\cdot, u) + \mu \). Hence
\[
-Au = f_{n,m}(\cdot, u) + (\mu - f_{n,m}(\cdot, u) + f(\cdot, u)).
\]
Clearly, \( f_{n,m}(\cdot, u) \in L^1(E; \rho \cdot m) \). Therefore, from the above equation, \( \mu - f_{n,m}(\cdot, u) + f(\cdot, u) \in \mathcal{G}(f_{n,m}) \). Thus, by Proposition 6.7, \( \mu \in \mathcal{G}(f_{n,m}) \). Consequently, by Theorem 5.2, there exists a maximal solution \( u_{n,m} \) to
\[
-Au = f_{n,m}(\cdot, v) + \mu.
\]
By Proposition 5.5, \( u_{n,m} \leq u_{n+1,m}, u_{n,m} \geq u_{n,m+1}, n, m \geq 1 \) q.e. Set
\[
w_n := \lim_{m \to \infty} u_{n,m} = \inf_{m \geq 1} u_{n,m}, \quad z_m := \lim_{n \to \infty} u_{n,m} = \sup_{n \geq 1} u_{n,m} \quad \text{q.e.}
\]
and
\[
w := \lim_{n \to \infty} w_n = \sup_{n \geq 1} w_n, \quad z := \lim_{m \to \infty} z_m = \inf_{m \geq 1} z_m \quad \text{q.e.}
\]
Thus, by \( \text{Int} \),

\[
f^-(\cdot, u_{n,m}) = |f(\cdot, u_{n,m}^+)| \leq \sup_{0 \leq y \leq u_{n,1}^+} |f(\cdot, y)| \in L^1(E; \rho \cdot m).
\]

From this, we conclude that, up to subsequence,

\[
Rf^-(\cdot, u_{n,m}) \to Rf^-(\cdot, w_n) \quad m\text{-a.e.}
\]

By Proposition 4.9,

\[
|u_{n,m}| + R|f_n(\cdot, u_{n,m})| \leq R|\mu| \quad \text{q.e.} \tag{7.1}
\]

By [21, Lemma 94, page 306], up to subsequence, \( \frac{1}{m} Rf^+(\cdot, u_{n,m}) \to e_n, \ m \to \infty, \ m\text{-a.e.} \) for some excessive function \( e_n \). By [25, Proposition 3.9], there exists a positive Borel measure \( \beta_n \) such that \( e_n = R \beta_n \). Therefore,

\[
w_n = R \beta_n - \frac{1}{n} Rf^-(\cdot, w_n) + R \mu \quad m\text{-a.e.} \tag{7.2}
\]

Now, we shall show that \( w_n \) is quasi-continuous and \( \beta_n \perp \text{Cap} \). Set \( h := R|\mu| \). By Remark 4.1(a), \( h \) is quasi-continuous. Set

\[
\tau_k := \inf\{t \geq 0 : h(X_t) \geq k\} \land k, \quad \delta_{k,j} := \inf\{t \geq 0 : \int_0^t \sup_{y \leq j} |f(X_r, y)| \, dr \geq j\}.
\]

Since \( h \) is quasi-continuous, \( \lim_{k \to \infty} \tau_k = \zeta \). By q(M) and Corollary 2.3, \( \lim_{j \to \infty} \delta_{k,j} = \zeta \) for any \( k \geq 1 \). Thus, \( \lim_{j \to \infty} \lim_{j \to \infty} \delta_{k,j} = \zeta \), where \( \tau_{k,j} := \tau_k \land \delta_{k,j} \). By Lemma 4.2, \( \{\sigma_{k}\} \) is a reducing sequence for \( h \), hence \( \{\tau_{k,j}\} \) is a reducing sequence for \( h \) fixed \( j \geq 1 \). Therefore, by (7.1), the last sentence also is in force with \( h \) replace by any of the following functions:

\[
u_{n,m}, w_n, z_n, w, z.
\]

By Lemma 4.2,

\[
u_{n,m}(X_{t \land \tau_{k,j}}) = \mathbb{E}_x\left[u_{n,m}(X_{\tau_{k,j}}) + \frac{1}{m} \int_{t \land \tau_{k,j}}^{\tau_{k,j}} f^+(\cdot, u_{n,m})(X_r) \, dr - \frac{1}{n} \int_{t \land \tau_{k,j}}^{\tau_{k,j}} f^-(\cdot, u_{n,m})(X_r) \, dr + \int_{t \land \tau_{k,j}}^{\tau_{k,j}} dA^\mu_r(\mathcal{F}_{t \land \tau_{k,j}})\right] \quad \text{q.a.s.} \tag{7.3}
\]

By [12, Lemma 6.1], for any \( q \in (0,1) \) there exists \( c_q > 0 \) such that

\[
\left( \mathbb{E}_x \sup_{t \leq \tau_{k,j}} |u_{n,m}(X_t) - u_{n,l}(X_t)|^q\right)^{1/q} \leq c_q \mathbb{E}_x \left[ |u_{n,m}(X_{\tau_{k,j}}) - u_{n,l}(X_{\tau_{k,j}})| \right]
+ \frac{1}{m} \int_0^{\tau_{k,j}} f^+(\cdot, u_{n,m})(X_r) \, dr + \frac{1}{t} \int_0^{\tau_{k,j}} f^+(\cdot, u_{n,l})(X_r) \, dr \tag{7.4}
\]

\[+ \frac{1}{n} \int_0^{\tau_{k,j}} f^-(\cdot, u_{n,m})(X_r) - f^-(\cdot, u_{n,l})(X_r) \, dr \quad \text{q.e.} \tag{7.5}
\]

By Remark 4.1(c) and the choice of \( \{\tau_{k,j}\} \), we obtain that the right-hand side of the above inequality tends to zero as \( m, l \to \infty \). Consequently, by Lemma 5.1, \( w_n \) is quasi-continuous. Taking \( t = 0 \) in (7.3), we get

\[
u_{n,m}(x) = \mathbb{E}_x u_{n,m}(X_{\tau_{k,j}}) + \frac{1}{m} \mathbb{E}_x \int_0^{\tau_{k,j}} f^+(\cdot, u_{n,m})(X_r) \, dr - \frac{1}{n} \mathbb{E}_x \int_0^{\tau_{k,j}} f^-(\cdot, u_{n,m})(X_r) \, dr + \mathbb{E}_x \int_0^{\tau_{k,j}} dA^\mu_r(\mathcal{F}_{t \land \tau_{k,j}}) \quad \text{q.e.} \tag{7.6}
\]

Letting \( m \to \infty \) and using the definition of \( \tau_{k,j} \) and Remark 4.1(c) yields

\[
w_n(x) = \mathbb{E}_x w_n(X_{\tau_{k,j}}) - \frac{1}{n} \mathbb{E}_x \int_0^{\tau_{k,j}} f^-(\cdot, w_n)(X_r) \, dr + \mathbb{E}_x \int_0^{\tau_{k,j}} dA^\mu_r(\mathcal{F}_{t \land \tau_{k,j}}) \quad \text{q.e.}
\]
Now, letting $j \to \infty$, and using quasi-continuity of $w_n$ and the fact that $(\tau_k)$ is a reducing sequence for $w_n$ (and $\tau_{k,j} \leq \tau_k$), we find that

$$w_n(x) = \mathbb{E}_x w_n(X_{\tau_k}) - \frac{1}{n} \mathbb{E}_x \int_0^{\tau_k} f^-(\cdot, w_n)(X_r) \, dr + \mathbb{E}_x \int_0^{\tau_k} dA_{\tau_k}^\mu \quad \text{q.e.}$$

On the other hand, since $w_n$ is quasi-continuous, we have by Remark 4.1(a)–(b) that, in fact, (7.2) holds q.e. Therefore, by [27, Theorem 3.7], letting $k \to \infty$ in the above equation gives

$$w_n = R(\mu + \beta_n)c - \frac{1}{n} R f^-(\cdot, w_n) + R\mu_d \quad \text{q.e.}$$

Thus,

$$w_n = R(\beta_n)c - \frac{1}{n} R f^-(\cdot, w_n) + R\mu \quad \text{q.e.}$$

From this and (7.2), we conclude that $(\beta_n)c = \beta_n$. Since $u_{n,m} \leq u_{n,1}$, $n, m \geq 1$ q.e., we have $w_n \leq u_{n,1}$, $n \geq 1$ q.e. Therefore, by the inverse maximum principle (see [27, Theorem 6.1]), $\beta_n + \mu_c \leq \mu_c$. Hence, $\beta_n = 0$. Consequently,

$$w_n = -\frac{1}{n} R f^-(\cdot, w_n) + R\mu \quad \text{q.e.} \quad \text{(7.7)}$$

Repeating the reasoning (7.1)–(7.7), with $u_{n,m}$ replaced by $w_n$ (and this time letting $n \to \infty$) and with $w_n$ replaced by $w$, we find that $w = R\mu$ q.e. Analogous reasoning shows that $z = R\lambda$ q.e. Set $u_n = u_{n,n}$, $f_n = f_{n,n}$, then

$$-Au_n = f_n(\cdot, u_n) + \mu.$$ 

Observe that $w_n \leq u_n \leq z_n$ q.e. Thus, $u_n \to R\mu$ q.e. By (7.1), [21, Lemma 94, page 306] and Proposition 7.1, there exist excessive functions $e_1, e_2$ such that, up to subsequence,

$$e_1^n := \frac{1}{n} R f^+(\cdot, u_n) \to e_1, \quad e_2^n := \frac{1}{n} R f^-(\cdot, u_n) \to e_2 \quad \text{q.e.}$$

By [25, Proposition 3.9] and once again (7.1), there exist positive Borel measures $\beta_1, \beta_2$ such that $e_1 = R\beta_1$, $e_2 = R\beta_2$. At the same time, since $u_n \to u$ q.e., we have $\frac{1}{n} R f^-(\cdot, u_n) \to 0$ q.e. Thus, $e_1 = e_2$, and so $\beta_1 = \beta_2$. By Lemma 4.2

$$e_1^n(x) = \mathbb{E}_x e_1^n(X_{\tau_k}) + \frac{1}{n} \mathbb{E}_x \int_0^{\tau_k} f^+(\cdot, u_n) \quad \text{q.e.}$$

By the choice of $\{\tau_k\}$ and Remark 4.1(c), we obtain, by letting $n \to \infty$, that $e_1(x) = \mathbb{E}_x e_1(X_{\tau_k})$ q.e. Thus, by [27, Theorem 3.7], $e_1 = R(\beta_1)c$. Consequently, $R(\beta_1)c = R\beta_1$, so that $(\beta_1)c = \beta_1$. By [26, Proposition 3.7], there exists positive smooth measures $\lambda_n, \lambda \in \mathcal{M}_\rho$ such that

$$u_n^+ = -e_2^n + R\mu^+ - R\lambda_n, \quad u^+ = R\mu^+ - R\lambda.$$ 

Letting $n \to \infty$ and using (7.1) and [25, Proposition 3.9], we deduce that there exists a positive measure $\lambda^0 \in \mathcal{M}_\rho$ such that

$$u^+ = -R\beta_1 + R\mu^+ - R\lambda^0, \quad u^+ = R\mu^+ - R\lambda.$$ 

Thus, $\lambda^0 + \beta_1 = \lambda$. Since $\lambda$ is smooth and $(\beta_1)c = \beta_1$, we conclude that $\beta_1 = 0$. Consequently, $e_1 = e_2 = 0$. As a result, we obtain that

$$R[f_n(\cdot, u_n)] \to 0 \quad \text{q.e.}$$

From this and (7.1) we infer that for any positive smooth measure $\nu$ such that $R\nu \leq \rho$ we have

$$\int_E |f_n(\cdot, u_n)| R\nu \, dm \to 0 \quad \text{as} \quad n \to \infty. \quad \text{(7.8)}$$

Set $F := \{R|\mu| > \varepsilon\}$ and $h := \rho \cdot 1_F$. Let $e_h$ be the smallest excessive function less than or equal to $h$. Clearly, $e_h \leq |\rho|_\infty R|\mu| \wedge \rho$. Therefore, by [25, Proposition 3.9], there exists a
positive measure $\nu \in \mathcal{M}_\rho$ such that $e_h = R\nu$. Since $e_h$ is bounded, $\nu$ is a smooth measure. Consequently, \eqref{eq:7.8} holds. We have
\[
|f_n(\cdot,u_n)|\rho \leq |f_n(\cdot,u_n)|\rho + 1_{F^c}|f_n(\cdot,u_n)|\rho \\
\leq |f_n(\cdot,u_n)|\epsilon_h + \sup_{|y| \leq \epsilon} |f_n(\cdot,y)|\rho = |f_n(\cdot,u_n)|R\nu + \frac{1}{\epsilon} \sup_{|y| \leq \epsilon} |f(\cdot,y)|\rho.
\]
By \eqref{eq:7.8} and the assumptions made on $f$, we get the result. The last assertion of the theorem is obvious from the construction.

We let $B_{L^1}(0,r) := \{u \in L^1(E;\rho \cdot m) : \|u\|_{L^1(E;\rho \cdot m)} \leq r\}$.

**Corollary 7.3.** Under the notation and assumptions of Theorem 7.2(2), we have
\[
(i) \text{ for any } r > 0, \mathcal{A}(f) + B_{L^1}(0,r) = \mathcal{G}(f), \\
(ii) \text{ cl}\mathcal{A}(f) = \mathcal{G}(f), \text{ where cl denotes closure in the total variation norm } \|\cdot\|_{\rho}.
\]

**Proof.** It follows directly from Theorem 7.2. □

**Remark 7.4.** Assume that $g$ is a function satisfying $\text{Car}(\cdot,\text{Sig}(\cdot),\text{Int})$. Furthermore, assume that $f,g$ satisfy M). Suppose that there exist constants $c_1,c_2,r > 0$ such that
\[
c_1 \leq \frac{|f(x,y)|}{|g(x,y)|} \leq c_2, \text{ m-a.e., } |y| \geq r. \tag{7.9}
\]
Then $\mathcal{G}(f) = \mathcal{G}(g)$.

**Proof.** By \eqref{eq:7.9}, we easily get that $\mathcal{A}(f) = \mathcal{A}(g)$. Therefore, by Corollary 7.3, $\mathcal{G}(f) = \mathcal{G}(g)$. □

**Corollary 7.5.** Under assumptions of Remark 7.4, we have $\Pi_f = \Pi_g$. In particular, for positive $\mu \in \mathcal{M}_\rho$, $\mu^+f = \mu^+g$.

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