A Finsler Geodesic Flow On $T^2$ With Positive Metric Entropy

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Abstract

We use a theorem of P. Berger and D. Turaev to construct an example of a Finsler geodesic flow on the 2-torus with a transverse section, such that its Poincaré return map has positive metric entropy. The Finsler metric generating the flow can be chosen to be arbitrarily $C^\infty$-close to a flat metric.

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1 Introduction

We give an example of a Finsler geodesic flow on the 2-torus that exhibits aspects of both chaotic and integrable dynamics. The flow exhibits integrable behaviour in the sense that a large region of the unit tangent bundle is foliated by invariant tori, on each of which the geodesic flow is a linear flow. It is chaotic in the sense that its return map of a certain transverse section contains a stochastic island, i.e. a region of positive metric entropy.

In section 2 we use a result of P. Berger and D. Turaev to obtain a perturbation of the standard shear map of the cylinder with positive metric entropy. In section 3 this map is embedded as a return map of the geodesic flow on the unit tangent bundle $ST^2$.

2 A twist map $\hat{f}$ with positive metric entropy

Let $(M, \omega)$ be a surface with a smooth area form. Let $f : M \to M$ be a diffeomorphism. The maximal Lyapunov exponent of $x \in M$ is given by

$$\lambda(x) = \limsup_{n \to \infty} \frac{1}{n} \log \|Df^n(x)\|$$

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If \( f \) preserves \( \omega \) then we have that the metric entropy of \( f \) is given by

\[
h_\omega(f) := \int_M \lambda(x) d\mu_\omega
\]

From [1] we have the following theorem

**Theorem 2.1.** (Berger, Turaev ’19) Let \( (M, \omega) \) be a surface with a smooth area form. If \( f : M \to M \) is a smooth area-preserving diffeomorphism with a non-hyperbolic periodic point, then there is an arbitrarily \( C^\infty \)-small perturbation of \( f \), such that the perturbed map \( \hat{f} \) is a smooth, area-preserving diffeomorphism and has positive metric entropy \( h_\omega(\hat{f}) > 0 \).

**Remark 2.2.** The theorem is proved in [1] by perturbing the diffeomorphism \( f \) locally along the orbit of the periodic point, such that one obtains an invariant domain \( I \subset M \) of positive measure with positive maximal Lyapunov exponent at every point in \( I \). Since the perturbation is local the perturbed map \( \hat{f} \) agrees with \( f \) away from the orbit of the periodic point. The boundary of \( I \) consists of finitely many \( C^0 \)-embedded circles that lie in the union of the stable and unstable manifolds of a set of hyperbolic periodic points. In the spirit of [1] we call such a set \( I \) a stochastic island.

Let \( Z = S^1 \times \mathbb{R} \) be the cylinder equipped with the standard symplectic form \( dx \wedge dy \). The shear map \( f_1 : Z \to Z \) with \( (x, y) \mapsto (x + y \mod 1, y) \) is the simplest example of a twist map of the cylinder. Note that \( f_1 \) has a non-hyperbolic fixed point at \((0, 0)\).

From theorem 2.1 we obtain an area-preserving (i.e. symplectic because of dimension two) \( C^\infty \) diffeomorphism \( \hat{f} : Z \to Z \) with a stochastic island \( I \), which is arbitrarily close to \( f_1 \) in \( C^\infty \). The twist of \( f_1 \) is uniformly equal to 1. Thus, if \( \hat{f} \) is \( C^1 \)-close enough to \( f_1 \) it has bounded twist away from zero. Note that if \( \hat{f} \) is \( C^3 \)-close enough to \( f_1 \) then KAM-theory guarantees that \( \hat{f} \) possesses invariant essential circles and consequently has zero flux (see [2]). Consequently, \( \hat{f} \) is a twist map of the cylinder if it is close enough to \( f_1 \) in \( C^r \) for \( r > 3 \).

As mentioned in remark 2.2 there is a \( K > 0 \), such that \( \hat{f}(x, y) = f_1(x, y) \) for every \( |y| > K \).

### 3 Embedding \( \hat{f} \) into the geodesic flow

We identify the tangent bundle \( TT^2 \) with \( T^2 \times \mathbb{R}^2 \). A Finsler metric on \( T^2 \) is a map

\[
F : TT^2 \to [0, \infty)
\]

with the following properties

1. (Regularity) \( F \) is \( C^\infty \) on \( TT^2 - 0 \)
2. (Positive homogeneity) \( F(x, \lambda y) = \lambda \cdot F(x, y) \) for \( \lambda > 0 \)
3. (Strong convexity) The hessian

\[
(g_{ij}(x, v)) = \left( \partial_{v_i} v_j \frac{1}{2} F(x, v)^2 \right)
\]

is positive-definite for every \((x, v) \in TT^2 - 0\).
A Finsler metric is called reversible if \( F(x, v) = F(x, -v) \) for every \((x, v) \in TT^2\). The unit tangent bundle \( ST^2 \cong T^2 \times S^1 \) is given by \( ST^2 = F^{-1}(\{1\}) \).

The geodesic flow \( \phi^t : ST^2 \to ST^2 \) is the restriction of the Euler-Lagrange flow of the Lagrangian \( L_F \), with
\[
L_F = \frac{1}{2} F^2
\]
to the unit tangent bundle. A geodesic we call either a trajectory of the Euler-Lagrange flow, or its projection to \( T^2 \), i.e. a geodesic is a curve \( t \mapsto c(t) \subset T^2 \) satisfying the Euler-Lagrange equation
\[
\partial_x L_F(c, \dot{c}) - \partial_t (\partial_v L_F(c, \dot{c})) = 0
\]
To embed the map \( \hat{f} \) into the geodesic flow we use a theorem of Moser [3] to express \( \hat{f} \) as the time-1 map of a strictly convex, time-periodic Hamiltonian on \( S^1 \).

**Theorem 3.1.** (J. Moser ‘86) Given a \( C^\infty \) twist map \( f : Z \to Z \) with \( f(x, y) = (x + c \cdot y, y) \) for large \(|y|\), there exists a strictly convex, time-periodic Hamiltonian \( H \) on \( S^1 \), such that the time-1 map \( \psi_H^{(1)} : Z \to Z \) agrees with \( f \).

**Remark 3.2.** Moser’s original theorem is formulated for twist maps \( f \) on the closed annulus \( A = S^1 \times [0, 1] \). In the proof in [3] the map \( f \) is extended to a twist map on the cylinder \( Z \) with \( f(x, y) = (x + c \cdot y, y) \) for \(|y| > D\) for a positive constant \( D \). There exist constants \( D_+, D_- \in \mathbb{R} \), such that the Hamiltonian \( H \) is equal to \( \frac{1}{2} y^2 + D_{\pm} \) for large values of \(|y|\), depending on whether \( y > D \) or \( y < -D \).

Let \( H \) be the Hamiltonian obtained from theorem 3.1 generating the previously constructed twist map \( \hat{f} \). We lift \( H \) to obtain a \( \mathbb{Z} \cdot \mathbb{T} \)-periodic Hamiltonian \( \hat{H} \) on \( \mathbb{R}^2 \). The Legendre transformation \( \mathcal{L}_t : \mathbb{R}^2 \to \mathbb{R}^2 \) is a global diffeomorphism and agrees with the identity for large values of \(|y|\). Thus, we obtain an associated Lagrangian \( \hat{L} : S^1 \times \mathbb{R}^2 \to \mathbb{R} \) with
\[
\hat{L}(t, x, y) = \frac{1}{2} y^2 - D_{\pm} \quad \text{for} \quad |y| > D
\]
Observe that the time-dependent Euler-Lagrange flow of \( \hat{L} \) is complete (i.e. the Euler-Lagrange solutions exist for all times) because the sets \( \{y = \text{const.}\} \) are invariant for large \(|y|\). To embed \( \hat{L} \) into a Finsler metric we need to perturb it for large values of \(|y|\). Let \( h_+, h_- : \mathbb{R} \to \mathbb{R} \) be smooth functions with
\[
h_+(y) = \begin{cases} 
\frac{1}{2} y^2 - D_+ & \text{if} \ y < D + 1 \\
\frac{\sqrt{A + By^2}}{Y} & \text{if} \ y > D + 2
\end{cases}
\]
and
\[
h_-(y) = \begin{cases} 
\sqrt{A + By^2} & \text{if} \ y < -D - 2 \\
\frac{1}{2} y^2 - D_- & \text{if} \ y > -D - 1
\end{cases}
\]
With constants \( A, B > 0 \) chosen in such a way that it is possible to choose \( h_+, h_- \) with \( h_+'' > 0 \). We define a Lagrangian \( L \) via
\[
L(t, x, y) = \begin{cases} 
h_-(y) & \text{if} \ y < -D \\
\hat{L}(t, x, y) & \text{if} \ -D \leq y \leq D \\
h_+(y) & \text{if} \ y > D
\end{cases}
\]
Observe that the time-dependent flow of \( \hat{L} \) is the same as the time-dependent flow of \( L \) since \( \hat{L} \) and \( L \) only differ where they are both only dependent on \( y \). Let \( F_0 \) be a Finsler metric on \( T^2 \) given by
\[
F_0(t, x, v_1, v_2) = \sqrt{A v_1^2 + B v_2^2}
\]
The Lagrangian $L$ is chosen in such a way that $L(t, x, y) = F_0(t, x, 1, y)$ for large values of $|y|$. We define a map $F$ on $T^2$ via

$$F(t, x, v_1, v_2) = \begin{cases} v_1 \cdot L(t, x, \frac{v_2}{v_1}) & \text{if } v_1 > 0 \\ F_0(t, x, v_1, v_2) & \text{if } v_1 \leq 0 \end{cases}$$

**Remark 3.3.** From the proof of Moser’s theorem 3.1 it follows that if $\hat{f}$ is chosen $C^\infty$-close to the shear map then the obtained Hamiltonian will be $C^\infty$-close to $H = \frac{1}{2}y^2$. Consequently, the above obtained Lagrangian $L$ can be chosen to be $C^\infty$-close to a function $h$ only dependent on $y$ with

$$h(y) = \begin{cases} \frac{1}{2}y^2 & \text{if } |y| < D + 1 \\ \frac{1}{\sqrt{A + By^2}} & \text{if } |y| > D + 2 \end{cases}$$

Thus, for any compact subset $K$ of $T^2$ we can find a sequence $\hat{f}_i$ of twist maps converging to $f_1$ and do the above construction, such that the resulting Finsler metrics $F_i$ become arbitrarily $C^\infty$-close to the flat metric

$$\hat{F}(t, x, v_1, v_2) = \begin{cases} v_1 \cdot h(\frac{v_2}{v_1}) & \text{if } v_1 > 0 \\ F_0(t, x, v_1, v_2) & \text{if } v_1 \leq 0 \end{cases}$$

on the set $K$.

The following two propositions are due to J.P. Schröder [4]. We include their proofs for completeness.

**Proposition 3.4.** $F$ defines a $C^\infty$ Finsler metric on $T^2$.

*Proof.* Regularity and Positive homogeneity follow directly from the definition of $F$.

We check strict convexity in each fiber. Let $(t, x) \in T^2$ be fixed and define $f : \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{R}$ via

$$f(v_1, v_2) := v_1 \cdot l(v_2/v_1)$$

where $l(y) := L(t, x, y)$ for every $y \in \mathbb{R}$. We compute the derivatives

$$\partial_1 f(u_1, u_2) = l\left(\frac{u_2}{u_1}\right) - \frac{u_2}{u_1} \cdot l'\left(\frac{u_2}{u_1}\right)$$

$$\partial_2 f(u_1, u_2) = l'\left(\frac{u_2}{u_1}\right)$$

second derivatives

$$\partial_{11} f(u_1, u_2) = \frac{u_2^2}{u_1} \cdot l''\left(\frac{u_2}{u_1}\right)$$

$$\partial_{12} f(u_1, u_2) = \partial_{21} f(u_1, u_2) = -\frac{u_2}{u_1} \cdot l''\left(\frac{u_2}{u_1}\right)$$

$$\partial_{22} f(u_1, u_2) = \frac{1}{u_1} \cdot l''\left(\frac{u_2}{u_1}\right)$$

Consequently, we have for $u = (u_1, u_2) \in \mathbb{R}_{>0} \times \mathbb{R}$ and $v = (v_1, v_2) \in \mathbb{R}^2$

$$\langle v, Hess f(u) v \rangle = \left(v_1 \cdot \frac{u_2}{u_1} - v_2\right)^2 \cdot l''\left(\frac{u_2}{u_1}\right)$$
To see that $F$ is strictly convex we have to check that fiberwise the Hessian of $L_F = \frac{1}{2} F^2$ is positive definite. For $(t, x) \in T^2$ fixed let $L_F : T_{(t, x)}T^2 \to \mathbb{R}$ be given by $L_F(u) = \frac{1}{2} F(t, x, u)^2$. For $u \in \mathbb{R}_{>0} \times \mathbb{R}$ we then have

$$L_F(u) = \frac{1}{2} (u_1 \cdot l(u_2/u_1))^2$$

We compute partial derivatives

$$\partial_1 L_F(u) = f(u) \cdot \partial_1 f(u)$$

$$\partial_2 L_F(u) = f(u) \cdot \partial_2 f(u)$$

$$\partial_{11} L_F(u) = (\partial_1 f(u))^2 + f(u) \cdot \partial_{11} f(u)$$

$$\partial_{12} L_F(u) = \partial_1 f(u) \cdot \partial_2 f(u) + f(u) \cdot \partial_{12} f(u)$$

$$\partial_{22} L_F(u) = (\partial_2 f(u))^2 + f(u) \cdot \partial_{22} f(u)$$

from this we get that

$$Hess L_F(u) = A + f(u) \cdot Hess f(u)$$

where

$$A = \begin{pmatrix}
(\partial_1 f(u))^2 & \partial_1 f(u) \cdot \partial_2 f(u) \\
\partial_1 f(u) \cdot \partial_2 f(u) & (\partial_2 f(u))^2
\end{pmatrix}$$

For $v \in \mathbb{R}^2$ we compute

$$v^T Av = v_1^2 (\partial_1 f(u))^2 + 2v_1 v_2 \partial_1 f(u) \cdot \partial_2 f(u) + v_2^2 (\partial_2 f(u))^2$$

$$= (v_1 \partial_1 f(u) + v_2 \partial_2 f(u))^2$$

$$= (Df(u)v)^2$$

Hence we have

$$v^T Hess L_F(u) v = v^T (A + f(u) Hess f(u)) v$$

$$= v^T Av + f(u) v^T Hess f(u) v$$

$$= (Df(u)v)^2 + f(u) \left( v_1 \frac{u_2}{u_1} - v_2 \right) \frac{2 l''(u_2/u_1)}{u_1}$$

Observe that $a$ and $b$ are each $\geq 0$. Assume now, that $b = 0$. Since $f, l''$ and $u_1$ are $> 0$ it follows that $v_1 \frac{u_2}{u_1} - v_2 = 0$. From this it follows that $v = \lambda \cdot u$ are linearly dependent. In this case we have $Df(u)v = \lambda \cdot Df(u)u = f(v) > 0$. Hence the Hessian $Hess L_F(x, u)$ is positive definite for $u \in \mathbb{R}_{>0} \times \mathbb{R}$. It is also positive definite for $u \in \mathbb{R}_{\leq 0} \times \mathbb{R} - \{0\}$ since $F$ coincides there with the Finsler metric $F_0$.

\[ \square \]

**Proposition 3.5.** Let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the curve given by

$$\gamma(t) = (t, \theta(t))$$

Then $\gamma$ is a reparametrization of a lifted $F$-geodesic if and only if $\theta$ is an Euler-Lagrange solution of $L$ (seen as a Lagrangian lifted to $\mathbb{R}$).
Proof. Observe that we have the following relation between the Lagrangian action $A_L$ and the Finsler length $l_F$.

$$A_L(\theta|_{[a,b]}) = \int_a^b L(t, \theta(t), \theta'(t))dt$$

$$= \int_a^b F(t, \theta(t), 1, \theta'(t))dt$$

$$= \int_a^b F(\gamma(t), \dot{\gamma}(t))dt$$

$$= l_F(\gamma|_{[a,b]})$$

Assume now that $\gamma: [a, b] \to \mathbb{R}^2$ is a reparametrization of an $F$-geodesic, i.e. $\partial_{s=0}l_F(\gamma_s) = 0$ for any proper variation of $\gamma$. Let $\theta_s: [a, b] \to \mathbb{R}$ be a proper variation of $\theta_s$ and $\gamma_s$ the pair of vectors $\{\gamma(t), e_2\}$ always forms a basis of $\mathbb{R}^2$. Thus, we can rewrite the vector field $X$ as

$$X(t) = \lambda(t)\dot{\gamma}(t) + \mu(t)e_2 \overline{A(t)B(t)}$$

for functions $\lambda, \mu$ with $\lambda(a) = \lambda(b) = \mu(a) = \mu(b) = 0$. Let $\gamma_s$ be a proper variation of $\gamma$ corresponding to the variational vector field $X$ and $\beta_s$ a proper variation of $\gamma$ corresponding to $B$, i.e.

$$\partial_{s|s=0} \gamma_s(t) = X(t) \quad \text{and} \quad \partial_{s|s=0} \beta_s(t) = B(t)$$

Observe that for small $|s|$ the curve $\eta_s: t \mapsto \gamma(t + s\lambda(t))$ is a reparametrization of $\gamma$ and hence has length independent of $s$. Thus

$$0 = \partial_{s|s=0}l_F(\eta_s)$$

$$= \partial_{s|s=0} \int_a^b F(\eta_s(t), \partial_t \eta_s(t))dt$$

$$= \int_a^b \partial_{s|s=0}F(\eta_s(t), \partial_t \eta_s(t))dt$$

$$= \int_a^b \partial_1 F(\eta_0(t), \partial_t \eta_0(t))\partial_{s|s=0}\eta_s(t) + \partial_2 F(\eta_0(t), \partial_t \eta_0(t))\partial_{s|s=0}\partial_t \eta_s(t)dt$$

$$= \int_a^b \partial_1 F(\gamma(t), \partial_t \gamma(t))\partial_{s|s=0}\eta_s(t) + \partial_2 F(\gamma(t), \partial_t \gamma(t))\partial_t \partial_{s|s=0}\eta_s(t)dt$$

$$= \int_a^b \partial_1 F(\gamma(t), \partial_t \gamma(t))A(t) + \partial_2 F(\gamma(t), \partial_t \gamma(t))\dot{A}(t)dt$$
Proposition 3.6. The return map

\[ \partial_s|_{s=0} l_F(\gamma_s) = \partial_s|_{s=0} \int_a^b F(\gamma_s(t), \dot{\gamma}_s(t)) dt \]

\[ = \int_a^b \partial_1 F(\gamma_0(t), \dot{\gamma}_0(t)) \partial_s|_{s=0} \gamma_s(t) + \partial_2 F(\gamma_0(t), \dot{\gamma}_0(t)) \partial_s|_{s=0} \dot{\gamma}_s(t) \]

\[ = \int_a^b \partial_1 F(\gamma(t), \dot{\gamma}(t)) \partial_s|_{s=0} \gamma_s(t) + \partial_2 F(\gamma(t), \dot{\gamma}(t)) \partial_s|_{s=0} \dot{\gamma}_s(t) \]

\[ = \int_a^b \partial_1 F(\gamma(t), \dot{\gamma}(t)) X(t) + \partial_2 F(\gamma(t), \dot{\gamma}(t)) \dot{X}(t) \]

\[ = \int_a^b \partial_1 F(\gamma(t), \dot{\gamma}(t)) (A(t) + B(t)) + \partial_2 F(\gamma(t), \dot{\gamma}(t)) (\dot{A}(t) + \dot{B}(t)) \]

\[ = \int_a^b \partial_1 F(\gamma(t), \dot{\gamma}(t)) B(t) + \partial_2 F(\gamma(t), \dot{\gamma}(t)) \dot{B}(t) \]

\[ = \partial_s|_{s=0} l_F(\beta_s) \]

Consequently, the curve \( \gamma \) is critical with respect to the Finsler length if

\[ \partial_s|_{s=0} l_F(\beta_s) = 0 \]

for every proper variation \( \beta_s \), which varies \( \gamma \) only in \( e_2 \) direction, i.e. \( \beta_s \) is of the form

\[ \beta_s(t) = (t, \theta_s(t)) \]

For those variations we have already computed that if \( \theta_s \) is critical with respect to the \( L \)-action then \( \beta_s \) is critical with respect to the Finsler length.

We define sets \( V, V_0 \subset ST^2 \) via

\[ V = \{(x, v) \in ST^2 \mid x \in T^2, v_1 > 0\}, \quad V_0 = \{(0, h, v) \in ST^2 \mid h \in S^1, v_1 > 0\} \]

From the completeness of the time-dependent Euler-Lagrange flow of \( L \) and proposition 3.5 it follows that the lifts to \( \mathbb{R}^2 \) of geodesics \( c \), with \( v_1 > 0 \) are graphs over the euclidean line \( \mathbb{R} \cdot e_1 \subset \mathbb{R} \) and pass through the section \( V_0 \) after finite time. Thus, the first return map \( R : V_0 \rightarrow V_0 \) of the geodesic flow is well-defined.

Proposition 3.6. The return map \( R : V_0 \rightarrow V_0 \) is conjugated via a diffeomorphism to the twist map \( \hat{f} \).

Proof. To see that the return map \( R \) is conjugated to \( \hat{f} \) observe that \( R \) is given by

\[ R(0, h, v_1, v_2) = \left( 0, \theta(1), \frac{(1, \theta'(1))}{F(0, \theta(1), 1, \theta'(1))} \right) \quad (1) \]

where \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) is the Euler-Lagrange solution of the lifted Lagrangian \( L \) with \( \theta(0) = h \) and \( \theta'(0) = \frac{v_2}{v_1} \). This is true because after proposition 3.5 the curve \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \) with \( \gamma(t) = (t, \theta(t)) \) is a reparametrized lift of the geodesic \( c : \mathbb{R} \rightarrow T^2 \) with initial values \( c(0) = (0, h) \) and \( c(0) = (v_1, v_2) \). The reparametrized lift \( \gamma \) passes through a translate \( V_0 + e_1 \) of \( V_0 \) again for the first time at time \( t = 1 \) and thus the return map \( R \) maps...
(0, h, v_1, v_2) to \((\gamma(1), \frac{\dot{\gamma}(1)}{F(\gamma(1), \dot{\gamma}(1))})\), which is equal to the expression in equation (1). The diffeomorphism \(g : V_0 \rightarrow Z\) conjugating \(R\) and \(\hat{f}\) is given by
\[
g(0, h, v_1, v_2) = \left( h, \frac{v_2}{v_1} \right)
\]
with inverse
\[
g^{-1}(x, y) = \left( 0, x, \frac{(1, y)}{F(0, x, 1, y)} \right)
\]

Proposition 3.7. The first return map \(R\) has positive metric entropy.

Proof. Let \(g : V_0 \rightarrow Z\) be the diffeomorphism conjugating \(R\) to \(\hat{f}\). Let \(I \subset Z\) denote the stochastic island for \(\hat{f}\), i.e. every point \(x \in I\) has positive maximal Lyapunov exponent and \(\text{Area}(I) = \int_I |dx \wedge dy| > 0\). From the conjugacy of \(\hat{f}\) and \(R\) it follows that the maximal Lyapunov exponent of every \(v \in g^{-1}(I)\) remains positive. Let \(\omega\) denote the area form of \(V_0\) obtained by restricting the differential \(d\lambda\) of the standard Liouville form \(\lambda\) to \(V_0\). The return map \(R\) preserves \(\omega\). Since the pullback \(g^*(dx \wedge dy)\) and \(\omega\) are both area forms there is a positive function \(j : V_0 \rightarrow \mathbb{R}_{>0}\), such that
\[
j \cdot \omega = g^*(dx \wedge dy)
\]
Since \(j\) is positive we have that \(\int_{g^{-1}(I)} j \cdot \omega \neq 0\) if and only if \(\int_{g^{-1}(I)} j \cdot \omega \neq 0\). And since
\[
\int_{g^{-1}(I)} j \cdot \omega = \int_{g^{-1}(I)} g^*(dx \wedge dy) = \int_I dx \wedge dy
\]
we have that \(\text{Area}(g^{-1}(I)) > 0\). Consequently, \(R\) has positive metric entropy.

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