Derivative using complex variable conjugate approach for analytic signal of magnetic field anomaly due to 2D finite prism

A Manan¹, L Hamimu¹ and R Chahyani²

¹ Department of Geophysical Engineering, Universitas Halu Oleo, Kampus Hijau Bumi Tridharma, Anduonohu, Kota Kendari 93232, Indonesia
² LPPM IAIN Kendari, Jln. Sultan Qaimuddin No. 17, Baruga, Kota Kendari 93116, Indonesia

*corresponding e-mail : amanan.geophysics@gmail.com

Abstract. The complex variable conjugate approach has been derived analytically for derivative computation. Computational results are then used in calculating the amplitude of analytic signal. It is the square root of the square of the total magnetic field anomaly derivative. The total magnetic fields are generated by the upper and lower parts of a 2D finite prism, and subtraction of both parts yields the magnetic field anomaly. While the approach is obtained by truncating the Taylor series expansion of the total magnetic field function in argument of the complex conjugate in the \( h^2 \) order-term. Truncating the series does not significantly affect the computational results. This is because when step-size \( h \) get smaller, and at \( h \leq 10^{-2} \), then the errors due to the series truncation became 0 \( (K \rightarrow 0) \). For derivative computation, the approach has precision on the order of \( 10^{-17} \) to \( 10^{-12} \) towards analytical settlement. On the order of \( h \leq 10^{-2} \), the approach is insensitive to the selection of step-size \( h \) for small numbers, so it can be done arbitrarily without any particular treatment or requires a complicated combination of numbers. The computational results of the analytic signal amplitude show that the positive and negative polarity on the magnetic profile is transformed into a positive profile only. This can facilitate the interpretation of actual magnetic data, especially in determining the causative source position of anomalies.

Keywords: Derivative, Complex Variable Conjugate, Analytic Signal, 2D Finite Prism

1. Introduction
In geophysics, magnetic method is one method that utilizes potential field data that is measured on the surface. This method is widely used in mineral exploration, preliminary surveys of oil and gas prospect areas, archeology and environmental surveys because it is relatively simple and easy to do, and is also relatively economically inexpensive.

In general, magnetic data show positive and negative polarities which are due to the dipolar nature of the anomalous source magnetic field [1], so that it often creates difficulties in data interpretation. The problem of magnetic data polarity can be solved by using analytic signal. Analytic signal transforms magnetic anomalies into positive anomalies only. Signal analytic is a complex function with the real and imaginary components being the horizontal and vertical derivatives respectively which can be proven that the imaginary component is a Hilbert transform of the real one [2,3]. So, it is the sum between the original signal and its Hilbert transform. Analytic
signals for interpretation of magnetic data were first applied by [2], and are currently generally used in the form of amplitude. The amplitude is the absolute value of the analytic signal.

In this study, derivative in computational analytic signal utilizes conjugate of complex number or variable. This approach is relatively new because so far the completion of derivatives is not in the form of a conjugate. A conjugate of complex variable is a complex variable containing the imaginary part in the negative form. The use of complex variables for derivative estimate was first introduced by [4,5]. Furthermore [6] developed a simple form of Complex Variable Approximation by using the theory of previous researchers for functions that are real, continuous and analytic in complex plane.

Estimates of derivative using the complex variable conjugate approach, and hereinafter called the Complex Conjugate, were arranged in Matlab® programming language using simple codes. This approach was carried out by first truncating the Taylor series of the function in argument of the complex conjugate in the $h^2$ order-term. The codes that had been made were then used in computing the analytic signal of the magnetic field anomaly generated by a causative source in shape of 2D finite prism.

2. Analytic Signal
The analytic signal $a(x)$ of the function $f(x)$ is a complex variable defined by equation [3]

$$a(x) = f(x) - iH[f(x)]$$

(1)

where $H(x)$ is the Hilbert transform of $f(x)$. $H[f(x)]$ itself is

$$H[f(x)] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x')}{{x-x'}} dx'$$

(2)

and for the inverse

$$f(x') = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H[f(x)]}{{x-x'}} dx$$

(3)

Analytic signals have been widely applied, including by [7-10] for two-dimensional analytic signal. For a potential field $M$ generated by a 2D source aligned parallel to the y-axis, and measured along the x-axis, then equation (1) can be written [2]

$$a(x, z) = \frac{\partial M}{\partial x} + i \frac{\partial M}{\partial z}$$

(4)

which satisfies the Cauchy-Riemann conditions [3]. Equations (1) and (4) apply to relationships

$$f(x) = \frac{\partial M}{\partial x} \quad \text{and} \quad H[f(x)] = -\frac{\partial M}{\partial z}$$

(5)

The absolute value of equation (4) yields the analytic signal amplitude as follows

$$|a(x, z)| = \sqrt{\left(\frac{\partial M}{\partial x}\right)^2 + \left(\frac{\partial M}{\partial z}\right)^2}$$

(6)

For magnetic fields generated by a causative object whose position does not change vertically or are at a certain height, then $\frac{\partial M}{\partial z} = 0$. Therefore, the amplitude in equation (6) can be written as

$$|a(x, z = \text{konstan})| = \sqrt{\left(\frac{\partial M}{\partial x}\right)^2}$$

(7)

In equation (7), the amplitude is only determined by a magnetic field $M$ at the measurement points $x$ on the surface.

3. Derivative Using Complex Conjugate
The approach to the derivative of a function, $f'(x)$, using complex conjugate is conducted by truncating the Taylor series expansion of the function $f(x)$ in argument of $z^* = x - ih$, that is $f(z^*)$. If $f$ is a real and analytic function, then the series expansion can be expressed

$$f(z^*) = f(x - ih) = f(x) - ih \frac{f'(x)}{1!} - h^2 \frac{f''(x)}{2!} + ih^3 \frac{f'''(x)}{3!} + h^4 \frac{f^{(4)}(x)}{4!} - ih^5 \frac{f^{(5)}(x)}{5!} - ...$$

(8)
If what is taken in equation (8) is the imaginary part and multiplies it by $1/h$, then the complex conjugate approach for the first derivative is obtained as follows

$$f'(x) = -\frac{\text{Im}\{f(x - ih)\}}{h} + h^2 \frac{f''(x)}{2!} - \ldots$$  \hspace{1cm} (9)

Whereas if taking only the real one of both sides, then yields the complex conjugate approach for the value of the function

$$f(x) = \text{Re}\{f(x - ih)\} + h^2 \frac{f''(x)}{2!} - \ldots$$  \hspace{1cm} (10)

The $h^2$ and higher order terms can be ignored since the step-size $h$ can be selected arbitrarily for sufficiently small numbers, so that the completion of complex conjugate in equation (9) and (10) becomes

$$f'(x) = -\frac{\text{Im}\{f(x - ih)\}}{h} + E_T$$  \hspace{1cm} (11)

$$f(x) = \text{Re}\{f(x - ih)\} + E_T$$  \hspace{1cm} (12)

with $E_T$ is the truncation error. Equations (11) and (12) are analogous to the original form of the complex variable approach that has been formulated by [6].

Function in argument of the complex conjugate function $f(z')$ can also be derived from function having argument of the complex variable $f(z)$. Equation (8) is rearranged into

$$f(z') = \left\{ f(x) - h^2 \frac{f'(x)}{2!} + h^4 \frac{f''(x)}{4!} - \ldots \right\} - \left\{ h \frac{f'(x)}{1!} - h^3 \frac{f'''(x)}{3!} + \ldots \right\}$$

$$f(z') = \text{Re}\{f(z)\} - i\text{Im}\{f(z)\}$$  \hspace{1cm} (13)

which shows the relationship between the complex conjugate ($z'$) and the complex variable ($z$). The complex variable approach has been widely used for estimates of the derivative, among which can be seen in [4,5,6,11].

![Figure 1. 2D vertical prism with infinite cross section](image1)

![Figure 2. 2D vertical prism with finite cross section](image2)

**4. Causative Object of Magnetic Field**

Model $2D$ for the causative source of a total magnetic field in the form of a vertical prism with an infinite cross section is shown in Figure 1. The magnitude of the causative total magnetic field is [12]

$$F = kF \left\{ \sin 2I \sin \beta \left[ \ln \left( \frac{r_i + L}{r_i} + L \right) - \ln \left( \frac{r_i + L}{r_i} + L \right) - \ln \left( \frac{r_i + L}{r_i} + L \right) - \ln \left( \frac{r_i + L}{r_i} + L \right) \right] \right. $$

$$- \left( \cos^2 I \sin \beta - \sin^2 I \right) \left[ \tan^{-1} \left( \frac{L}{x} \right) - \tan^{-1} \left( \frac{L}{x - b} \right) - \tan^{-1} \left( \frac{Ld}{x(r_i^2 + L^2)} \right) + \tan^{-1} \left( \frac{Ld}{(x - b)(r_i^2 + L^2)} \right) \right] \right\}$$  \hspace{1cm} (14)
Whereas for prism with finite cross section such as Figure 2
\[ F = 2kF_{e}\ln\left(\frac{r_{2}r_{1}}{r_{a}r_{1}}\right)\sin(2I)\sin\beta + \left(\cos^{2}I\sin^{2}\beta - \sin^{2}I\left(\phi_{1} - \phi_{2} - \phi_{3} + \phi_{4}\right)\right) \]  
where \( F \) is the measured total field on the surface, \( k \) the magnetic susceptibility, \( F_{e} \) the intensity of the earth's magnetic field, \( I \) the inclination of the earth's magnetic field, \( \beta \) the prism strike angle relative to the magnetic north, \( b \) the prism width, \( d \) the prism depth, \( L \) the half-strike length of the prism, \( r \) and \( \phi \) with \( i=1,2,3 \) and 4 are distances and angles that establish the prism geometry.

In general, a magnetic object can be considered as 2D object when its strike length is at least 10 times its width \((2L\geq10b)\) [12], and the strike perpendicular to the profile [13]. Therefore, a 2D prism model with a finite cross-section can be approximated from \( 2^{\frac{1}{2}}D \) model by following the definition [13], and assuming that the prism composed of many very small square prisms.

A single square prism is illustrated as Figure 2. This prism generates a magnetic field anomaly that can be calculated by first calculating each of the total magnetic fields caused by the Upper Prism and the Lower Prism by using equation (14), then subtracting the results.

5. Synthetic Magnetic Field Source
The total magnetic field measured on the surface at the measurement points \((-100\leq x\leq100) \) km with spaces 0.5 km was designed because it was generated by a causative object in shape of 2D finite prism using equation (14) and following [12-14]. The used parameter values are \( F_{e}=45.000 \) nT, \( k=7\times10^{3} \) SI (chromite), \( I=70^{\circ}, \beta=30^{\circ}, b=35 \) km, \( L=8b \) km, and \( d=10 \) m for Upper Prism and 30 m for Lower Prism. Whereas the magnitude of \( r \) and \( \phi \) can be seen in Table 1 in the form of parametric equations with the dip angle \( \xi=90^{\circ} \).

| Table 1: Parametric equations of distance and angle |
|---|---|---|
| Index \( i \) | Distance \( r_{i} \) | Angle \( \phi \) |
| 1 | \( d^{2}+(x+d\cot\xi)^{2} \) | \( \tan^{-1}\left[\frac{d}{(x+d\cot\xi)}\right] \) |
| 2 | \( D^{2}+(x+D\cot\xi)^{2} \) | \( \tan^{-1}\left[\frac{D}{(x+D\cot\xi)}\right] \) |
| 3 | \( d^{2}+(x+d\cot\xi-b)^{2} \) | \( \tan^{-1}\left[\frac{d}{(x+d\cot\xi-b)}\right] \) |
| 4 | \( D^{2}+(x+D\cot\xi-b)^{2} \) | \( \tan^{-1}\left[\frac{D}{(x+D\cot\xi-b)}\right] \) |
6. Result and Discussion

6.1. Magnetic field anomaly due to a 2D finite prism

Figure 3a displays the total magnetic field profile $F$ caused by the Upper and Lower parts of the Prism which is calculated analytically. While the magnetic field anomaly $\Delta F$ which is calculated by subtracting the total magnetic fields of the Upper and Lower parts of the Prism shown in Figure 3b. Magnetic profile in the direction of N-S with strike in the direction of E 30°W.

The profiles in Figure 3 shows the presence of positive and negative polarities. These polarities are present because of the dipolar nature of the magnetic field of the 2D finite prism. In case of the field data, polarity can cause difficulties in interpreting magnetic data, especially in determining the location and depth of the source.

![Figure 3. Magnetic field profile in the direction of N-S with strike in the direction of E 30°W.](image)

6.2. Analytical derivative of magnetic field anomaly

The results of the derivative of the total magnetic field due to the Upper and Lower parts of the Prism are analytically shown in Figure 4a. As for the derivative of the magnetic field anomaly shown in Figure 4b.

Profile resulting from the total magnetic field derivatives $\frac{\partial F}{\partial x}$ in Figure 4a for each of the Upper and Lower parts of the Prism still shows polarity like the total field profile $F$. Similarly for the magnetic field anomaly derivative $\frac{\partial \Delta F}{\partial x}$ in the Figure 4b also shows the same thing as $\Delta F$.

![Figure 4. Analytical derivative profiles of (a) the total magnetic field due to the Upper and Lower parts of the prism, and (b) the magnetic field anomaly](image)

![Figure 5. The complex Conjugate of (a) the magnetic field anomaly, and (b) The derivative of the magnetic field anomaly](image)
6.3. The complex conjugate

Derivative of the magnetic field anomaly using the complex conjugate is calculated by deriving subtraction results of the total magnetic field caused by the Upper and the Lower parts of the Prism ($\Delta F$) with respect to measurement points ($x$) where each total magnetic field is also obtained using the complex conjugate. Furthermore compared to the analytical solution obtained in Section 6.2, and its results can be seen in Figure 5. The complex conjugate approach (red line) and the analytical solution (blue line) are almost exact for all measurement points ($-100 \leq x \leq 100$). The red line coincides with the blue line. To find out the precision of computational works, relative error of the complex conjugate $E_r$ is calculated towards the analytical solution using formula

$$E_r = \frac{\Delta F_{\text{complex conjugate}} - \Delta F_{\text{analytic}}}{\Delta F_{\text{analytic}}}$$

for the magnetic field anomaly, and

$$E_r = \frac{\partial \Delta F_{\text{complex conjugate}}/\partial x - \partial \Delta F_{\text{analytic}}/\partial x}{\partial \Delta F_{\text{analytic}}/\partial x}$$

for approach to the derivative. Figure 6 shows the $E_r$ values for both approaches in equations (16) and (17).

Based on Figure 6, the use of the complex conjugate in estimating a function and derivative exhibits highly accuracy. Accuracy can be identified from a very small Relative Error value. Figure 6a displays $E_r$ for estimates of the magnetic field anomaly function $\Delta F$, which it is between $7.16341336035101 \times 10^{-20}$ and $1.185821058201096 \times 10^{-13}$. While Figure 6b displays $E_r$ for the approach to the derivative of the magnetic field anomaly $\partial \Delta F/\partial x$ which $E_r$ is obtained between $3.159607284251312 \times 10^{-12}$ and $7.079603533496899 \times 10^{-12}$.

![Figure 6](image)

**Figure 6.** Relative errors of the approach of the complex conjugate towards the analytical solution for (a) the magnetic field anomaly, and (b) the magnetic field anomaly derivative

![Figure 7](image)

**Figure 7.** Truncation errors as a function of step-size $h$ ($x = 100$ km) (a) the total magnetic field function $F(x)$, and (b) the total magnetic field derivative $\partial F/\partial x$.

If the values $E_r$ of the approach of the complex conjugate in Figure 6 are compared with machine epsilon which is $2.220446049250313 \times 10^{-16}$, then it can be said that the approach for the magnetic field anomaly and its derivative has high precision. This is because the truncation of the series in the $h^2$ order-term in equation (9) and (10) does not have any significant effect on the computational results. The magnitude of the error due to the series truncation in equation (10) can be written
(18) for estimates of the total magnetic field function, and
(19) for approach to the derivative in equation (9). Both of these equations are similar to the equations that had been used by [11] to tested how precise the codes were made in Fortran® for derivative computation.

Truncation of the Taylor series of the total magnetic field function \( F(x) \) in argument of the complex conjugate parsed like equation (8) in the \( h^2 \) order-term does not have any significant effect on the computational results. Figure 7 shows the errors due to the series truncation is getting smaller with increasing the small step-size \( h \) (\( 10^0 \rightarrow 10^{-10} \)). At \( h \leq 10^{-2} \), the \( E_T \) value has become and is stable at 0. Therefore, to reduce or eliminate the truncation effect of the complex conjugate series on the computational results, it can be done by minimizing the step-size \( h \). The choice of a small \( h \) will result in minimal truncation errors. The choice of \( h \) can be done in such a manner without requiring certain treatment.

In its implementation, derivative computation using the complex conjugate approach does not present a possible subtraction of two possible numbers which are the same as seen in the mathematical form of equation (9), thus eliminating the possibility of cancelation errors. Its implementation in the Matlab® programming language is relatively easy to do just by creating simple codes.

6.4. Analytic signal amplitude
The amplitude of the analytic signal from the magnetic field anomaly in Figure 8b is made by implementing equation (7) in the Matlab® language codes with derivative using the complex conjugate resulted in Figure 5b.

![Figure 8. Profiles of (a) The magnetic field anomaly derivative using the complex conjugate, and (b) Analytic Signal Amplitude](image)

The computational result in Figure 8b shows that the Analytic Signal Amplitude transforms the profile of the magnetic field anomaly derivative with positive and negative polarities being positive anomaly without further data processing. This is because it does not depend on other parameters, and only on the magnetic field anomaly or 2D finite prism total magnetic field is measured on the surface (-100 ≤ \( x \) ≤ 100 km). In Nabighian’s paper [2] explained that the amplitude was invariant with respect to object magnetization vector direction and local geomagnetic field. Besides that, it’s also independent to induced field vector [7]. Therefore, in interpretation of actual magnetic data, this
transformation will be very useful and makes it easy especially to determine the location of the causative source of anomalous objects.

7. Conclusion
The approach of the complex conjugate to estimating derivatives is extremely easy to implement in the Matlab programming language by only designing simple codes, easy to run and highly accurate. Accurate precision is on the order of $10^{-17}$ to $10^{-13}$ for derivative computation of the total magnetic field or magnetic field anomaly. The effect of the Taylor series truncation of the total magnetic field function mathematically in the complex conjugate argument in the $h^2$ order-term does not affect the computational results because for a small step-size $h \leq 10^{-2}$, then the truncation errors become 0. Because of the complex conjugate is insensitive to the selection of $h$ of the complex conjugate argument for $h \leq 10^{-2}$, then the selection of $h$ for small numbers can be done arbitrarily without requiring certain treatments. Besides that, the complex conjugate equations does not allow the presence of a subtraction in the same two numbers, so that it is free from the cancelation operation. The resulting derivatives are then used in computing the Analytic Signal Amplitude of the magnetic field anomaly of the 2D finite prism-shaped object. Computational results show the loss of negative polarities from the magnetic profiles. So, the amplitude is able to transform negative and positive polarities on magnetic data into positive polarity only without further data processing.

References
[1] Young C T 2004 Basic magnetic processing and display in MATLAB http://pages.mtu.edu/~ctyoung/somepdfs/sageep2004.pdf accessed on August 02, 2018
[2] Nabighian M N 1972 The analytic signal of two-dimensional magnetic bodies with polygonal cross-section: its properties and use for automated anomaly interpretation Geophysics 37 (3) 507-517
[3] Blakely R J 1995 Potential Theory in Gravity and Magnetic Applications (New York: Cambridge University Press) pp 350-352
[4] Lyness J N and Moler C B 1967 Numerical differentiation of analytic functions SIAM J NumAnal 4 202-210
[5] Lyness J N 1967 Numerical algorithms based on the theory of complex variables Proc. of the 22nd Nat. Conf. ACM 1 124-134
[6] Squire W and Trapp G 1998 Using complex variables to estimate derivatives of real functions SIAM Rev 40 110-112
[7] Bournas N and Baker H A 2001 Interpretation of magnetic anomalies using the horizontal gradient analytic signal ANNAI DI GEOFISICA 44 (3) 505-526
[8] Pilkington M and Keating P 2006 Short Note: The relationship between local wavenumber and analytic signal in magnetic interpretation Geophysics 71 (1) L1-L3
[9] Tuma S L and Mendonça C A 2007 Stepped inversion of magnetic data Geophysics 73 (3) L21-L30
[10] Ansari A H and Alamdar K 2009 Reduction to the pole of magnetic anomalies using analytic signal World Applied Sciences Journal 7 (4) 405-409
[11] Martins J R R A Sturza P and Alonso J 2003 The complex derivative approximation ACM Trans. Math. Softw. 29 245-262
[12] Telford W M Gedart L P and Sheriff R E 2001 Applied Geophysics 2nd Ed (New York: Cambridge University Press) pp 92-95
[13] Stocco S Godi A and Sambuelli L 2009 Modelling and compact inversion of magnetic data: A Matlab code Computers & Geosciences 35 2111-2118
[14] Bhattacharyya B K 1964 Magnetic anomalies due to prism-shaped bodies with arbitrary polarization Geophysics 29 (4) 517-531