EXTENDING CANTOR’S PARADOX
A CRITIQUE OF INFINITY AND SELFREFERENCE

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Abstract. The inconsistencies involved in the foundation of set theory were invariably caused by infinity and self-reference; and only with the opportune axiomatic restrictions could them be obviated. Throughout history, both concepts have proved to be an inexhaustible source of paradoxes and contradictions. It seems therefore legitimate to pose some questions concerning their formal consistency. This is just the objective of this paper. Starting from an extension of Cantor Paradox that suggests the inconsistency of the actual infinite, the paper makes a short review of its controversial history and proposes a new way of criticism based on $\omega$-order. Self-reference is also examined from a critique perspective which includes syntactic and semantic considerations. The critique affects the formal sentence involved in Gödel’s first incompleteness theorem and its ordinary language interpretation.

The most relevant problems of contemporary philosophy were already posed by Presocratic thinkers as early as the VII century b.C. (some of them perhaps suggested or directly taken from the cultural precedents developed in the Neolithic Fluvial Cultures.) Among those problems, three deserve special consideration: the problem of change; infinity; and self-reference. The first of them is surely the most difficult, and at the same time most relevant, problem ever posed by man. For that reason, it is surprising what little attention we currently pay to that fascinating question, specially when compared to the attention we pay to the other two. After more than twenty seven centuries, the problem of change remains unsolved. In spite of its apparent obviousness, no one has been capable of explaining, for instance, how one leaves a place to occupy another. Physics, the science of change (or, as Maxwell said, the science of the regular succession of events) seems to have forgotten its most fundamental problem. In their turn, some philosophers as Hegel declared it as an inconsistent process! while others,
as McTaggart, came to the same conclusion as Parmenides on the impossibility of change.\(^5\)

Contrarily to the problem of change, which has an immediate reflection in the agitated evolution of the Universe, both infinity and self-reference are pure theoretical devices unrelated to the natural world. Cantor and Gödel (the princes of infinity and self-reference respectively) were two platonist fundamentalists of scarce devotion to natural sciences and enormous influence in contemporary mathematics.\(^6\)

Twenty seven centuries of discussions were not sufficient to prove the consistency of the actual infinity, which finally had to be legitimated by the expeditious way of axioms (Axiom of Infinity). Although the consistency of self-reference has been less discussed, the paradoxes it originates have been, and continue to be, the source of interminable discussions. One of them, the Liar Paradox, led (via Richard Paradox) to the celebrated Gödel’s first incompleteness theorem.

1. ON PARADOXES AND INCONSISTENCIES

It is *paradoxical* that formal literature be not more exigent in the use of the term ’paradox,’ frequently used in the place of ’contradiction’ or ’inconsistency.’ When, for instance, we write:

\[ H \Rightarrow (c \land \neg c) \]  

we say \( H \) is inconsistent because it is the immediate cause of contradiction \((c \land \neg c)\). But when the cause of the contradiction is not so immediate and well known it is relatively frequent to use the term paradox instead of contradiction. So if we write:

\[ p \Rightarrow \neg p \]  
\[ \neg p \Rightarrow p \]  

we say \( p \) is a paradox, as is the case of the Liar paradox:

\[ p = \neg p \] is not true  

But in these cases we must admit a detail that is frequently ignored: the First Law of Logic (Principle of Identity) does not hold for this type of sentences. Otherwise we would have:

\[ p \Rightarrow p \]  

while according to the own proposition \( p \):

\[ p \Rightarrow \neg p \]  

\(^5\)[81].

\(^6\)For the case of Cantor see [31], [83], [25, pag. 141]; for that of Gödel [48, pags. 235-236], [50, pag. 359], [41], [32] [88], [60], [51].
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Whence:

\[ p \Rightarrow (p \land \neg p) \]

which is a contradiction. We should therefore determine if the sentences as \( p \)
have the privilege of existing out of the Laws of Logic or if they are simple
inconsistencies.

The paradoxes of Set Theory we will examine here are surely inconsistencies
derived from both the actual infinity and self-reference. Burali-Forti made use of
his famous paradox as an inconsistency to prove that the law of trichotomy does
not hold for all transfinite ordinals.\(^7\) In his turn, Cantor did not hesitate in using
the expression 'inconsistent totalities'\(^8\) to refer to certain sets we will immediately
deal with. In any case, it is not the purpose of this work to state when and how
we should use the terms paradox, antinomy or inconsistency. In this sense we
will be so informal as the formal literature related to mathematical formalism.

2. Extending Cantor Paradox

Although Burali-Forti was the first in publishing\(^9\) an inconsistency related to
transfinite sets, Cantor was the first to discover a paradox in the nascent set
theory: I am referring to the maximum cardinal paradox.\(^10\) There is no agree-
ment regarding the date Cantor discovered his paradox\(^11\) (the proposed dates
range from 1883\(^12\) to 1896.\(^13\)) Burali-Forti Paradox on the set of all ordinals and
Cantor Paradox on the set of all cardinals are both related to the size of the
considered totalities,\(^14\) perhaps too big as to be consistent according to Cantor.
From a finitist perspective it appears ironic that a set may be inconsistent just
because of its excessive size. One can be infinite but only within certain limits.
Obviously the simplest explanation for both paradoxes is that they are incon-
sistencies derived from the actual infinity, i.e. from assuming the existence of
complete infinite totalities. But nobody has dared to analyze this alternative. It
was finally accepted that some infinite totalities (as the set of the real numbers)
do exist while others (as the totality of cardinals or the totality of ordinals) do
not because they lead to contradictions.

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\(^7\)[21].

\(^8\)Letter to Dedekind quoted in [31, pag. 245].

\(^9\)[20], [44].

\(^10\)[44], [31].

\(^11\)[44].

\(^12\)[104].

\(^13\)[52].

\(^14\)Later we will deal with Russell Paradox, although this paradox is related to self-reference
rather than to cardinality. As we will see, Burali-Forti and Cantor Paradoxes are also related
to self-reference.
A common way of presenting Cantor Paradox is as follows: Let $U$ be the set of all sets, the so called universal set\(^\text{15}\) and $P(U)$ its power set, the set of all its subsets. As Cantor did, let us denote by $\overline{U}$ and $\overline{P(U)}$ their respective cardinals. Being $U$ the set of all sets, we can write:

$$\overline{U} \geq \overline{P(U)} \quad (8)$$

On the other hand, and according to Cantor theorem on the power set,\(^\text{16}\) it holds:

$$\overline{U} < \overline{P(U)} \quad (9)$$

which contradicts (8). This is the famous Cantor inconsistency or paradox. As is well known, Cantor gave no importance to it\(^\text{17}\) and solved the question by assuming the existence of two types of infinite totalities, the consistent and the inconsistent ones\(^\text{18}\). In Cantor opinion, the inconsistency of the last would surely due to their excessive size. We would be in the face of the mother of all infinities, the absolute infinity which directly leads to God.\(^\text{19}\)

As we will immediately see, it is possible to extend Cantor Paradox to other sets much more modest than the set of all sets. But neither Cantor nor his successors considered such a possibility. We will do it here. This is just the objective of the discussion that follows. A discussion that will take place within the framework of Cantor set theory (naive set theory).

Since the elements of a (naive) set can be sets, sets of sets, sets of sets of sets and so on, we will begin by defining the following relation $R$ between two sets: we will say that two sets $A$ and $B$ are $R$-related, written $A \, R \, B$, if $B$ contains at least one element which forms part of the definition of at least one element of $A$. For instance, if:

$$A = \{ \{\{a, \{b\}\}\}, \{c\}, d, \{\{\{e\}\}\} \ldots \} \quad (10)$$

$$B = \{1, 2, b\} \quad (11)$$

then $A$ is $R$-related to $B$ because the element $b$ of $B$ forms part of the definition of the element $\{\{a, \{b\}\}\}$ of $A$. In these conditions let $X$ be any set such that $X > 1$, and let $Y$ be any non empty subset of $X$. From $Y$ we define the set $T_\overline{Y}$ according to:

$$T_\overline{Y} = \{Z \mid Z \cap Y = \emptyset \land \neg \exists V (V \cap Y \neq \emptyset \land Z \, R \, V)\} \quad (12)$$

\(^{15}\)Cantor’s naive set theory admits sets as the universal set $U$.

\(^{16}\)[23].

\(^{17}\)[66].

\(^{18}\)[22].

\(^{19}\)[22].
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$T_Y$ is, therefore, the set of all sets that do not contain elements of $Y$ nor are $\mathbf{R}$-related to any set containing elements of $Y$. Let us now consider the set $P(T_Y)$, the power set of $T_Y$. The elements of $P(T_Y)$ are all of them subsets of $T_Y$ and therefore sets of sets that do not contain elements of $Y$ nor $\mathbf{R}$-related to sets containing elements of $Y$. That is to say:

$$\forall D \in P(T_Y) : D \cap Y = \emptyset \land \neg \exists V (V \cap Y \neq \emptyset \land D \mathbf{R} V)$$

(13)

Consequently, it holds:

$$\forall D \in P(T_Y) : D \in T_Y$$

(14)

And then:

$$P(T_Y) \subset T_Y$$

(15)

Accordingly, we can write:

$$\overline{P(T_Y)} \leq \overline{T_Y}$$

(16)

On the other hand, and in accordance with Cantor’s theorem we have:

$$\overline{P(T_Y)} > \overline{T_Y}$$

(17)

Again a contradiction. But $X$ is any set of cardinal greater than 1 and $Y$ any one of its nonempty subsets. We can therefore state that every set of cardinal $c$ gives rise to at least $2^c - 1$ inconsistent totalities.

The above argument not only proves the number of inconsistent infinite totalities is inconceivably greater than the number of consistent ones, it also suggests the excessive size of the sets could not be the cause of the inconsistency. Consider, for example, the set $X$ of all sets whose elements are exclusively defined by means of the natural number 1:

$$X = \{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}, \{1, \{1\}\} \ldots \}$$

(18)

and the set $Y$ defined by $Y = U - X$. According to the above argumentation the set $T_Y$ defined by:

$$T_Y = \{Z \mid Z \cap Y = \emptyset \land \neg \exists V (V \cap Y \neq \emptyset \land Z \mathbf{R} V)\}$$

(19)

is an inconsistent totality, but compared to the universal set it is an insignificant totality: it lacks practically all sets of numerical content$^{20}$ or related to numbers (integers, rationals, reals, complexes, hyperreals, etc.); and all of non numerical content. $T_Y$ is in fact an insignificant subset of the universal set, it lacks almost all its elements. But it is also inconsistent. What both sets have in common,

$^{20}$Recall, for instance, that between any two real numbers an uncountable infinity ($2^{\aleph_0}$) of other different reals numbers do exist. What, as Wittgenstein would surely say, makes one feel dizzy [132].
from the infinitist perspective, is that they are complete infinite totalities; i.e. totalities whose infinitely many elements exist all at once.

Had we know the existence of so many inconsistent infinite totalities, and not necessarily so greater as the absolute infinity, and perhaps Cantor transfinite set theory would have been received in a different way. Perhaps the very notion of the actual infinity would have been put into question; and perhaps we would have discovered the way of proving its inconsistency: just the $\omega$-order. But this was not the case. The history of the reception of set theory and the way of dealing with its inconsistencies -all of them promoted by the actual infinity and self-reference- is well known. From the beginnings of the XX century a great deal of effort has been carried out to found set theory on a consistent background free of inconsistencies. Although the objective could only be reached with the aid of the appropriate axiomatic patching. At least half a dozen axiomatic set theories have been developed ever since.\textsuperscript{21} Some hundred pages are needed to explain all axioms of contemporary axiomatic set theories. Just the contrary one expects on the foundations of a formal science as mathematics.

As was suggested at the beginning of this section, the simplest explanation of Cantor and Burali-Forti paradoxes is that rather than paradoxes they are true contradictions derived from the inconsistency of the actual infinity, contradictions derived from assuming the existence of complete infinite totalities. The same applies to the set of all sets, and to the set of all sets that are not member of themselves (Russell paradox), although in this case there is an additional cause of inconsistency related to the very definition of set. All sets involved in the paradoxes of naive set theory were finally removed from the theory by the opportune axiomatic restrictions. Nobody dared to suggest the possibility that those paradoxes were in fact contradictions derived from the hypothesis of the actual infinity; i.e. from assuming the existence of infinite sets as complete totalities. What is true is that Cantor set of all cardinals, Burali-Forti set of all ordinals, the set of all sets, and Russell set of all sets that are not members of themselves, are all of them inconsistent totalities when considered from the perspective of the actual infinity. Even Turing’s famous halting problem is related to the actual infinity because it is also assumed here the existence of all pairs (programs, inputs) as a complete infinite totality.\textsuperscript{22} Under the hypothesis of the potential infinity, on the other hand, none of those totalities makes sense because from this perspective only finite totalities can be considered, indefinitely extensible, but always finite.

\textsuperscript{21}There are also some contemporary attempts to recover naive set theory [63].

\textsuperscript{22}[125].
3. CRITICISM OF THE ACTUAL INFINITY

The history of infinity is one of an interminable discussion between its supporters and its opponents. A discussion in which one of the alternatives always dominated the other, although the dominant alternative was not always the same. Fortunately excellent and abundant literature on the history of infinity and its controversies is available.\textsuperscript{23} Here we will limit ourselves to recall the most significant details of that history, from Zeno of Elea to the actual paradise. Apart from the absolute hegemony of infinitism and infinitists (supporters of the actual infinity) in contemporary mathematics, the lack of criticism of the actual infinity is surprising. And that in spite of the paradoxes and extravagances its assumption gives rise to, its cognitive sterility in experimental sciences and the tremendous difficulties it poses in certain areas of physics (as renormalization in physics of elementary particles.\textsuperscript{24})

The (known) history of the mathematical infinity began with Zeno of Elea and his famous paradoxes. Although Zeno’s intentions were surely more related to the problem of Change\textsuperscript{25} than to the mathematical infinity. Scarcely 300 words from the original Zeno’s texts have survived until the present day.\textsuperscript{26} It is by his doxographers\textsuperscript{27} that we know Zeno’s original works. Thus, it seems proved he wrote a book with more than forty arguments in defense of Parmenides’ thesis. We also know the book was available in Plato Academy and that it was read by the young Aristotle\textsuperscript{28}. By way of example let us recall the following original Zeno’s argument:\textsuperscript{29}

What moves, don’t move neither in the place it is nor in that it is not.

The famous race between Achilles and the tortoise and Achilles passing over the so called $Z$-points\textsuperscript{30} (Dichotomy I) and $Z^*$-points\textsuperscript{31} (Dichotomy II) posed very difficult problems in which the actual infinity was already involved.\textsuperscript{32} Aristotle

\textsuperscript{23}See for instance: [136], [75], [116], [15], [108], [29], [72], [84], [86], [68], [69], [1], [85], [28], [128] etc.
\textsuperscript{24} [64], [115], [67].
\textsuperscript{25} Parmenides and Zeno defended the impossibility of change with convincing arguments.
\textsuperscript{26} [13].
\textsuperscript{27} Among them Plato [103], Aristotle [7] and Simplicius [119].
\textsuperscript{28} [30].
\textsuperscript{29} [13, pag. 177].
\textsuperscript{30} The $\omega$-ordered sequence of points $1/2, 3/4, 7/8, 15/16, \ldots$ Achilles traverses in his race from point 0 to 1 [126].
\textsuperscript{31} The $\omega^*$-ordered sequence of points $\ldots, 1/16, 1/8, 1/4, 1/2$ Achilles traverses in his race from point 0 to 1.
\textsuperscript{32} [112], [85], [78].
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suggested a solution to both dichotomies: the one to one correspondence between the successive Z-points (or Z*-points) and the successive instants at which Achilles passes over them.33 But Aristotle was not convinced of the legitimacy of that correspondence34 and suggested other radically different solution: the consideration of two types of infinities, namely the actual and the potential infinity. According to the first, infinite totalities would be complete totalities. In accordance with the second, infinite totalities would always be incomplete so that complete totalities could only be of a finite size. Aristotle preferred the second type of infinitude, the potential infinity. And until the end of the XIX century that was the dominant alternative on the infinity, although the other always maintained an active affiliation. In its turn, Zeno’s Paradoxes still remain unsolved, in spite of certain proposed solutions and pseudo-solutions.35

Galileo famous paradox36 on the one to one correspondence between the set of natural numbers and the set of their squares37 introduced a significant novelty in the controversy on the mathematical infinity: the properties of the infinite numbers could be different from those of the finite numbers. That is at least the only explanation Galileo found to the paradoxical fact that existing more natural numbers than perfect squares both sets can be put into a one to one correspondence \( f(n) = n^2, \forall n \in \mathbb{N} \). In fact, Galileo paradox only arises when both sets are considered as complete totalities (hypothesis of the actual infinity). From the perspective of the potential infinite Galileo paradox does not take place because from that perspective both sets are incomplete totalities and in these circumstances we can only pair finite totalities with the same number of elements. As large as we wish but always finite and incomplete. If in the place of the perfect squares Galileo would have considered the expofactorials38 perhaps he would not

33[7].
34[7].
35Most of those solutions needed the development of new areas of mathematics, as Cantor’s transfinite arithmetics, topology, or measure theory [53], [54], [135], [55], [57], [56], and more recently internal set theory [80], [79]. It is also remarkable the solutions posed by P. Lynds in the framework of quantum mechanics [73], [74]. Some of those solutions, however, have been seriously contested [93], [3], and in most cases the proposed solutions do not explain where Zeno arguments fail [93], [99]. In addition, the proposed solution have given rise to new problems so exciting as Zeno’s original ones [112], [65] [117].
36Un example of the so called paradoxes of reflexivity, in which a whole is put into a one to one correspondence with one of its proper parts [116], [34]. This type of paradoxes had already posed by many other authors as Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. de Rimini, G. de Ockham etc. [116].
37[43].
38When I began to use this type of numbers I ignored they had already been defined by C. A. Pickover ([101] cited in [130]) with the name of superfactorials (n$), the same name used by
have come to the same conclusion. The expofactorial of a number \( n \), written \( n^! \), is the factorial \( n! \) raised \( n! \) times to the power of \( n! \). So, while the expofactorial of 2 is 16, the expofactorial of 3 is:

\[
3^! = 6^{6^{6^{6^{6^{6^{6^{6^{-1}}}}}}}} = 6^{6^{6^{6^{6^{6^{6^{6^{-1}}}}}}}} = 6^{6^{6^{6^{6^{6^{6^{6^{-1}}}}}}}} = 6^{6^{6^{6^{6^{6^{6^{6^{-1}}}}}}}} = \ldots
\]  

(20)

where the incomplete exponent of the last expression has nothing less than 36306 digits (roughly ten pages of standard text). Modest \( 3^! \) is a number so large that no modern (nor presumably future) computer can calculate it. Imagine, for instance \( 100^! \). The set \( E \) of expofactorials lacks of \textit{practically all} natural numbers and, however, it can also be put into a one to one correspondence with the set of natural numbers (through the bijection \( f(n) = n^! \)). Naturally, there is no problem in pairing off the firsts \( n \) natural numbers with the firsts \( n \) expofactorials, being the \( n \)-th expofactorial inconceivably greater than the \( n \)-th natural. Other thing to claim that both sets exist as complete totalities with the same number of elements. And things may get worse with the \( n \)-expofactorials recursively defined from expofactorials so that the 2-expofactorial of \( n \), denoted by \( n^{12} \), is the expofactorial \( n^! \) raised \( n^! \) times to the power of \( n^! \), the 3-expofactorial \( n^{13} \) of \( n \) is the 2-expofactorial \( n^{12} \) raised \( n^{12} \) times to the power of \( n^{12} \), the 4-expofactorial \( n^{14} \) of \( n \) is the 3-expofactorial \( n^{13} \) raised \( n^{13} \) times to the power of \( n^{13} \) and so on and on. Could you imagine \( 9^{19} \)? Simply terrifying.

The euclidian Axiom of the Whole and the Part\textsuperscript{39} controlled the infinitist impulses of authors as al-Harrani\textsuperscript{40} Leibniz\textsuperscript{41} or Bolzano.\textsuperscript{42} Among the most famous bijections between a whole and one of its parts are the classical bijective correspondences defined by Bolzano between a real interval and one of its proper subintervals, for instance the correspondence:

\[
[0, 5] \xrightarrow{f} [0, 12]
\]  

(21)

defined by:

\[
y = \frac{12}{5} x, \ \forall x \in [0, 5]
\]  

(22)

\footnote{Sloane and Plouffe to define \( \pi = \prod_{k=1}^{n} k! \). So I maintain the term expofactorial and the notation \( n^! \). On the other hand, and as far as I know, the recursive definition of \( n \)-expofactorials appears for the first time in this paper.}

\textsuperscript{39}The whole is greater than the part. Common Notion 5. Book 1 of Euclid’s Elements \[40\].

\textsuperscript{40}[71].

\textsuperscript{41}Leibniz position on the infinity is more ambiguous \[85\], \[8\] than could be expected from his famous declaration of supporter of the actual infinity \[70, pag. 416\].

\textsuperscript{42}[18].
In spite of which, Bolzano did not believe they were a sufficient condition to prove the equipotence of the paired intervals. Dedekind took the next step by assuming that exhaustive injections (bijections or one to one correspondences) were in fact a sufficient condition to prove the equipotence of the paired sets. This implies to assume that two sets $A$ and $B$ have the same number of elements if they can be put into a one to one correspondence. Or in other words, if after pairing every element of a set $A$ with a different element of a set $B$, no element of $B$ results unpaired (exhaustive injection), then the sets $A$ and $B$ have the same number of elements. This is a reasonable conclusion. So reasonable as to conclude that if after pairing every element of a set $A$ with a different element of a set $B$, one or more elements of $B$ remains unpaired (non-exhaustive injection), then $A$ and $B$ have not the same number of elements. We are then faced with a dilemma: if exhaustive injections and non-exhaustive injections have or do not have the same level of legitimacy, when used as instruments to compare the number of elements of two infinite set, as is the case with finite sets. If they have, then the actual infinity is inconsistent because in these conditions it can be immediately proved that every infinite set $X$ has and does not have the same number of elements as one of its proper subsets $Y$. In fact, by definition, a one to one correspondence (exhaustive injection) $f$ exists between $X$ and $Y$. Consequently $X$ and $Y$ have the same cardinality. On the other hand, the non-exhaustive injection $g$ from $Y$ to $X$ defined by $g(y) = y$, $\forall y$ in $Y$ proves they have not the same cardinality.

Exhaustive and non exhaustive injections make use of the same basic method of pairing the elements of two sets. And only of it. There is no reason (except reasons of infinitist convenience) to assume that non-exhaustive injections are less legitimate than exhaustive ones to state if two infinite sets have, or not, the same number of elements. Unless we explain the reason for which $X$ and $Y$ have the same number of elements in spite of the fact that after having paired each element of $Y$ with a different element of $X$, at least one element of $X$ remains unpaired. And the reason cannot be the exhaustive injection between $X$ and $Y$ just because the non-exhaustive injection between both sets is so legitimate and conclusive as the exhaustive one. If we were faced with a contradiction we could not use one side of the contradiction to arbitrarily deny the other. Thus modern infinitism should explicitly declare, by the appropriate ad hoc axiom, a new singularity of infinite sets, namely that the number of their elements cannot be compared by means of non-exhaustive injections. But no step has been taken in this direction. On the contrary, infinitists define infinite set just as those that can be put into an injective and exhaustive correspondence with one of its proper subsets, ignoring that this
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subset can also be put into an injective and non-exhaustive correspondence with
the same infinite superset. As Dedekind proposed, infinitist definition of infinite
sets is entirely based on the violation of the old euclidian Axiom of the Whole
and the Part. And committed the violation, the infinitist orgy don’t make us
wait: Cantor inaugurated the actual infinitist paradise just at the end of the XIX
century. Paradise that for others authors as Brouwer, Poincaré o Wittgenstein,
was rather a nightmare, a malady or even a joke.\textsuperscript{45} From the beginnning of the
XX century, infinitism become absolutely dominant in mathematics. Although
finitist has not disappeared its presence in contemporary mathematics is almost
irrelevant. In addition, the critique of infinity is practically non-existent and
the scarce efforts carried out have been systematically rejected in behalf of the
supposed differences between the infinite and the finite numbers.\textsuperscript{46}

At the beginning of the XXI century a new way of criticism was opened. It was
based on the notion of \(\omega\)-order (see below), a formal consequence of the actual
infinity as Cantor himself proved.\textsuperscript{47} Although the new way of criticism has its
roots in certain discussions that took place at the beginning of the second half
of the XX century. In fact, in the year 1954 J. F. Thomson introduced the term
supertask\textsuperscript{48} in a famous argument regarding a lamp that is turned on and off
infinitely many times. Thomson defended the impossibility of performing such a
supertask.\textsuperscript{49} His argument was motivated by other similar arguments defended
by M. Black\textsuperscript{50} with its infinite machines, and criticized by R. Taylor\textsuperscript{51} and J.
Watling.\textsuperscript{52} The argument of the Thomson lamp was in turn severaly criticized
by P. Benacerraf\textsuperscript{53} and that criticism gave rise to a new infinitist theory at the end
of the XX century: supertask theory.\textsuperscript{54} Over recent years supertask theory has
extended its theoretical scenario to the physical world. The possibility to actually
perform a supertask have been discussed.\textsuperscript{55} As could be expected the actual
performance of such supertasks would imply the most extravagant pathologies in
the physical world, both newtonian and relativist. But what is really surprising is

\begin{footnotes}
\item[45]\textsuperscript{133}
\item[46]See for instance \[109\].
\item[47]\textsuperscript{24}, pages 110-118.
\item[48]The notion of supertask is much more older. J. Gregory used it in the XVII century \[85, page 53\].
\item[49]\textsuperscript{124}.
\item[50]\textsuperscript{16}.
\item[51]\textsuperscript{123}.
\item[52]\textsuperscript{129}.
\item[53]\textsuperscript{10}.
\item[54]See, for instance, \[19\], \[27\], \[99\].
\item[55]\textsuperscript{102}, \[95\], \[99\], \[112\], \[55\], \[57\], \[56\] \[95\], \[96\], \[97\], \[38\], \[98\], \[91\], \[4\], \[5\], \[100\] \[131\], \[62\],
\[36\], \[37\], \[91\], \[35\], \[117\].
\end{footnotes}
that in the place of considering such unbelievable pathologies as clair indicatives of the inconsistent nature of supertasks (and then of the actual infinity), its defenders prefer to accept those pathologies before questioning the consistency of the pathogene, ultimately the actual infinity.

Supertasks are $\omega$-ordered sequences of actions. A sequence (or a set) is $\omega$-ordered if it has a first element and every element has an immediate successor. The set $\mathbb{N}$ of natural numbers with its natural order of precedence is a clair example of $\omega$-ordered set. From the point of view of the actual infinity, an $\omega$-ordered sequence (or a set) is a complete totality, although no last element completes it. Thus, an $\omega$-ordered list is one that is simultaneously complete (as the actual infinity requires) and uncompletable (because no last element completes it). The existence of lists that are simultaneously complete and uncompletable is another extravagance of the actual infinity, although less popular than others (perhaps because being simultaneously complete and uncompletable seems contradictory even to infinitists). Still less known is the following asymmetry also derived from $\omega$-order. Imagine a straight line segment of 30000 millions light-years (within the scale of the supposed size of the Universe) and denote its two extremes by $A$ and $B$. Assume $AB$ is divided into an $\omega$-ordered sequence of adjacent intervals (an $\omega$-ordered partition as is usually called) whose first interval begins just at point $A$. Let $C$ be a point on $AB$ placed at a distance of $B$ far less than Planck distance ($10^{-33}$ cm). $\omega$-Order makes it inevitable that between $A$ and $C$ only a finite number of intervals will always lie, while an infinite number of them (i.e. practically all of them) must necessarily lie between $C$ and $B$. No matter how close is $C$ from $B$, between $C$ and $B$ will always be an infinite number of intervals while between $A$ and $C$ this number will always be finite.

The same asymmetry applies to supertasks: if $t_b$ if the first instant at which all tasks of a supertask have been performed (i.e the first instant at which no task is performed), then at any instant prior to $t_b$, whatsoever it be, only a finite number of tasks will have been carried out and infinitely many of them must still be performed. There is no way that only a finite number $n$ of tasks remain to be performed (in spite of the fact that they are performed successively, one by one, one after the other) simply because they would be the impossible last $n$ tasks of an $\omega$-ordered sequence of tasks.\footnote{See ‘The Aleph-Zero or Zero Dichotomy’ at http://www.interciencia.es.} Thus while a supertask is being performed only a finite number of tasks will have been performed and an infinite number of them will always remain to be performed. By way of example, imagine that God decides to count all natural numbers in its natural order of precedence (which,
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as St. Augustine\textsuperscript{57} and other infinitist saints believed, He can). Although a time far less than Planck time ($10^{-43}$s) remains to finish, He will have counted only a finite number of number and an infinite number of numbers (practically of all them) remain still to be counted. Or in other words, while He is counting He will have counted only a finite number of numbers and an infinitude of them remain still to be counted. It is impossible to avoid this huge and unaesthetic asymmetry. As Aristotle claimed, perhaps it is impossible to traverse the untraversable\textsuperscript{58} or to complete the uncompletable. A sign of inconsistency?

The criticism of $\omega$-order I have just referred to, analyzes the assumption that it is possible to complete an uncompletable sequence of actions, be they the tasks of a supertask or the definitions of a recursive $\omega$-ordered sequence of definitions. Benacerraf was right in that we cannot infer consequences on the final state of a supermachine from the successive performed actions with that supermachine\textsuperscript{[10]}. But Benacerraf did not consider the consequences of performing a supertask on the supermachine that performs the supertask. The formal development of this line of argument leads to many contradictions involving $\omega$-order and then the actual infinity.\textsuperscript{59} None of those arguments have been refuted. Although some recalcitrant infinitists defend arguments so picturesque as, for instance, that Thomson’s lamp may be turned on or off for reasons unknown; others pretend to invalidate an argument by confronting its conclusions with the conclusions of other independent arguments; or defend the idea that the inconsistencies of $\omega$-order found in supertask theory do not apply to $\omega$-ordered sets. But $\omega$-order is $\omega$-order in all circumstances, and consists of being a complete totality that has a first element (be it the element of a set or the task of a supertask) and such that every element has an immediate successor; i.e. a totality that is both complete and uncompletable. To be complete and uncompletable not only seems contradictory, the criticism of $\omega$-order I am referring to, actually proves that this is in fact the case.

It is even possible to develop arguments of this type in set theoretical terms. For instance, in each step of the recursive definition of an strictly increasing $\omega$-ordered sequence of finite nested sets, a real variable $x$ is defined as the cardinal of the set just defined at each step. It can then be proved that $x$ is and is not defined as a finite cardinal.$^{60}$ We could also consider the set $\mathbb{Q}^+$ of positive rational numbers and prove an inconsistency derived from the fact that $\mathbb{Q}^+$ may be densely ordered (between any two rationals infinitely many other rationals do exist) and $\omega$-ordered.

\textsuperscript{57}[2].

\textsuperscript{58}[6, page 291].

\textsuperscript{59}Take a glance at http://www.interciencia.es.

\textsuperscript{60}See ‘Recursive definitions’ at http://www.interciencia.es.
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(between any two successive rationals no other rational exists). In effect, being \( \mathbb{Q}^+ \) denumerable there exists a bijection \( f \) between \( \mathbb{Q}^+ \) and the set \( \mathbb{N} \) of natural numbers so that \( \mathbb{Q}^+ \) can be written as \( \{ f(1), f(2), f(3), \ldots \} \). If \( x \) is a rational variable whose initial value is 1, and \( \langle d_i \rangle_{i \in \mathbb{N}} \) a sequence of positive rationals, both defined according to:

\[
\begin{align*}
    i &= 1, 2, 3, \ldots \\
    d_i &= | f(i + 1) - f(1) | \\
    d_i < x &\Rightarrow x = d_i
\end{align*}
\]  

(23)

where \( | f(i + 1) - f(1) | \) is the absolute value of \( f(i + 1) - f(1) \); and ‘<’ represents the natural order of \( \mathbb{Q} \), then it can be proved that \( f(1) + x \) is and is not the less rational greater than \( f(1) \).\(^{61}\)

4. Criticism of self-reference

The other great obstruction in the history of human thought is semantic self-reference. Also of Presocratic origins (paradoxes of Epimenides and Eubulides) it has been, and continues to be, an inexhaustible source of discussions.\(^{62}\) To say that a sentence is self-referent is, at the very least, an equivocal way of speaking because sentences have not autonomous existence, they are only human instruments of reference created exclusively by human beings. So that beyond any sentence there always is a man or a woman trying to say something. Beyond the celebrated liar sentence:

\[
\text{This sentence is false} \quad (24)
\]

there is a human trying to say something. He exactly says:

\[
\text{This I say is false} \quad (25)
\]

If in the place of saying (25) our human says

\[
\text{This I tread on is yellow} \quad (26)
\]

we would look at his feet and would identify the object he is referring to. But what object is being referred to when he says (25)? As far as I’m concerned, I must recognize I don’t know of what I’m predicating falseness when I say (25). I even have the impression of saying nothing when I say (25). Furthermore, since the subject of a self-referent sentence is the whole sentence, we will not know the subject of the sentence until the whole sentence has been completely uttered; but at this moment we have already predicated the subject. This is as to count one’s chickens before they are hatched. We are saying that something is false before to know of what thing are we predicating its falseness. As we will see later, a similar

\(^{61}\)See ‘An inconsistency in the field of rational numbers’ at http://www.interciencia.es.

\(^{62}\)See, for instance, [76], [9], [61], [118], [39], [120], etc.
situation appears in Russell’s paradox of the set of all sets which are not members of themselves. Up to a certain point, Russell’s paradox is a set theoretical version of the canonical liar paradox.

5. SYNTACTIC CRITIQUE OF SELF-REFERENCE

Let the world be all that can be considered and consider the set $M$ of all monadic declarative sentences of the form:

$$S \text{ is } P$$ (27)

where $S$ is any subject (any part of the world, including sentences), and $is \ P$ is any single assertion about $S$ composed of a verb ($is$) and a single predicate ($P$). A self-referent sentence is one that refers to itself. Consequently, and according to (27), all self-referent sentences in $M$ can be written as:

This sentence is $P$ (28)

Two well known examples of self-referent sentences (28) are the Liar Paradox:

This sentence is not true (29)

and ordinary language interpretation of the sentence involved in the first Gödel’s Incompleteness Theorem:

This sentence is not $P$-demonstrable (30)

where $P$ is the formalized calculus used by Gödel in his seminal paper of 1931.63

According to the meaning and the syntactic role of the pronoun ‘this’, the reference

This sentence (31)

refers to a sentence of which it forms part as such reference. So the sentence referred to by:

This sentence (32)

in:

This sentence is $P$ (33)

has to be the whole sentence:

This sentence is $P$ (34)

Consequently the sentence that is $P$ is just:

This sentence is $P$ (35)

63[46].
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which, on the other hand, is what is usually assumed on this affair. But if we replace the reference [This sentence] by the referred object [This sentence is P] we get:

This sentence is P is P \hspace{1cm} (36)

and for the same reasons above, the reference:

This sentence \hspace{1cm} (37)

in:

This sentence is P is P \hspace{1cm} (38)

refers to other different sentence, namely to:

This sentence is P is P \hspace{1cm} (39)

Thus, by replacing the reference with the referred object we have changed the referred object! But this is impossible in sentences syntactically well formed. We must therefore conclude that monadic self-referent sentences are syntactically impossible, and then nonsensical. By similar reasons, authors as Wittgenstein suggested the impossibility of self-reference.\textsuperscript{64}

3.332. No proposition can make a statement about itself, since a propositional sign cannot be contained in itself (that is the whole of the ‘theory of types’).

The apparent meaning of monadic self-referent sentences is a semantic illusion probably produced by a double (and almost simultaneous) mental processing of the sentence, the first to state the subject, the second to predicate it. Accordingly, the sentences:

This sentence is not true \hspace{1cm} (40)

This sentence is not P-demonstrable \hspace{1cm} (41)

are nonsensical. To say (40) or (41) is so absurd as to say:

This sentence is anthropophagous \hspace{1cm} (42)

Self-reference may be obscured in different ways, for instance by two or more successive sentences that refers cyclically to each other, as for instance:

P1: P2 is true \hspace{1cm} (43)

P2: P1 is false \hspace{1cm} (44)

\textsuperscript{64}[134, 3.332, p. 43].
But the appropriate replacements lead immediately to

\[ P2: \text{(P2 is true) is false} \]  \hspace{1cm} (45) \\
\[ P2: \text{P2 is false} \]  \hspace{1cm} (46) \\
\[ \text{This sentence is false} \]  \hspace{1cm} (47) \\

When, on the other hand, we write:

\[ A = \text{A is false} \]  \hspace{1cm} (48)

we are stating (right side of the equation) that a certain sentence named A is false; but, according to the left side of the equation, the sentence named A is just the sentence that states the falseness of A. It is therefore the same sentence (40). Other purely syntactic way of expressing self-reference is the following Quine's sentence:65

"yields a sentence with property P when appended to its own quotation" yields a sentence with property P when appended to its own quotation.

which refers to itself because it just consists of a sentence appended to its own quotation. It is a devious version of (28). The above syntactic critique should suffice to rule out all self-reference sentences from both ordinary and formal languages. Inoffensive as may seem the sentence:

\[ \text{This sentence has five words} \]  \hspace{1cm} (49)

is as nonsensical as any other self-referent sentence. It is only a semantic illusion.

Apart from being syntactically impossible, some self-referent sentences are also paradoxical. In these cases we found some significant similarities in all of them:

1. They are negative sentences of the form: \text{This sentence is not P}. Where not P may be, for instance: not true (false); not autologic (heterologic); not not-richardian (richardian).66
2. The predicate P is sensitive to the double negation: not (not true) = true; not (not autologic) = autologic; not (not richardian) = richardian
3. The sentences are not empirically verifiable (as it is for instance (49)) so we have to speculate on if they are P or they are not.
4. The speculation on if it is P or it is not P leads to the (contradiction) paradox because the selfreference activates the effects of the double negation when we ask if it is not P.

65[105].
66[39], [61].
67[107].
(5) If we ask if it is P it results that it is not P; but if we ask if it is not P then it result that, as a consequence of the double negation, it is P.  

(6) And mainly: the First Law of Logic does not hold for these sentences, otherwise they would be inconsistent. According to this law we would have:

\[ p \Rightarrow p \]  \hspace{1cm} (50)  

while according to the own proposition:

\[ p \Rightarrow \neg p \]  \hspace{1cm} (51)  

And then:

\[ p \Rightarrow (p \land \neg p) \]  \hspace{1cm} (52)  

which is a contradiction. We are then forced to concede to these type of sentences the privilege of existing out of the Laws of Logic.

6. GÖDEL’S INCOMPLETENESS THEOREM

A well known precedent that motivates Gödel’s work on the incompleteness of formal systems was the so called Richard’s paradox\(^{68}\) which consists of the following (fallacious) argument: consider the list of all arithmetical properties of natural numbers (to be even, or prime, or perfect square etc.) Assume now we order that list according to a certain criterion, for instance by its number of letters and alphabetically for those with the same number of letters. We would have an \(\omega\)-ordered list of arithmetical properties each of whose item can, therefore, be indexed by a natural number. Assume the property ‘to be an even number’ is indexed by number 201; in such a case we would say 201 is richardian because it is not described by the property it indexes (it is not an even number); on the contrary if the property ‘to be a perfect square’ is indexed by number 100 then this number is not richardian because it is described by the property it indexes (it is a perfect square). Now assume that the property ‘to be richardian’ is indexed by number 92, will 92 be richardian? It is clear that 92 is richardian if and only if it not richardian. Voila Richard’s paradox.

We will now focus our attention on Gödel’s work on self-reference and incompleteness. As we know, Gödel proved his famous incompleteness theorems in a seminal paper published in 1931.\(^{69}\) The first of those theorems states the existence of true (metaarithmetic) sentences that cannot be proved nor refuted in a certain formal calculus (\(P\)) which includes the complete arithmetic of Russell and Whitehead’s

\(^{68}\)[107].  
\(^{69}\)[46].
Principia Mathematica\textsuperscript{70} and other related systems. At the end of the introductory section 1 of his work, Gödel recognizes the analogy of his argument with both the Liar argument and Richard antinomy, particularly whit the last one. But while Richard’s argument is fallacious, Gödel’s one is not. The fallacy in Richard’s argument is due to the inclusion of the property of being richardian in the list of the arithmetic properties. But to be or not richardian depends on the particular way of indexing that list rather than on arithmetic operations. To be richardian is, at best, a metaarithmetic property, and metaarithmetic properties were not indexed in the list. Gödel solved this problem by including certain metaarithmetic sentences in his formal calculus $P$. In fact, with the aid of the successive powers of the successive prime numbers, Gödel succeeded in representing each element (constant signs, variables, formulas, proofs) of his formal calculus by a unique an exclusive natural number. This Gödel numbering allows the inclusion of certain metaarithmetic formulas of his formal calculus, particularly the concepts of \textit{formula}, \textit{deduction} and \textit{deducible formula}.\textsuperscript{71} Metaarithmetic is then partially arithmetized in the sense that certain relations between formulas can be translated into certain exclusive arithmetical relations between natural numbers. In additions these metaarithmetic sentences can be mirrored in ordinary language by appropriately interpreting the meanings of its signs.\textsuperscript{72}

After eighteen dense pages (Spanish version of Gödel’s paper) of rigorous formalism Gödel was able to define a sentence, let us call G, whose natural interpretation asserts its own unprovability in Gödel’s formal system $P$. In the following page Gödel proves that sentence is undecidable by speculating on the demonstrability of both $G$ and $\neg G$: if $G$ were demonstrable then $\neg G$ would also be demonstrable; and if $\neg G$ were demonstrable so would be $G$. Both conclusions are evidently impossible if $P$ is consistent. $G$ is therefore undecidable.

It is worth noting $G$ is a self-referent sentence in negative form for which the First Law of Logic does not hold. In fact, according to Gödel if $G$ were $P$-demonstrable then $\neg G$ would also be $P$-demonstrable, which in a consistent system as $P$ implies that $G$ is not $P$-demonstrable. Accordingly we can write:

$$G \Rightarrow \neg G \quad (53)$$

On the other hand, if the First Law of Logic could be applied to $G$ we would have

$$G \Rightarrow G \quad (54)$$

\textsuperscript{70}[111].
\textsuperscript{71}[49, page 55].
\textsuperscript{72}[49, page 55].
And then from (53) and (54) we would have:

\[ G \Rightarrow (G \land \neg G) \]  

(55)

As in the above self-referent paradoxes, we have to decide if \( G \) is an inconsistent sentence or a sentence with the privilege of existing out of the Laws of Logic. In addition, \( G \) is a formal self-referent P-sentence whose natural interpretation in ordinary language is a self-referent sentence subjected to the above syntactic criticism. It is, then, a sentence syntactically impossible.

Since 1931 other proofs of Gödel’s first incompleteness theorem have been obtained, but invariably all of them imply -in one way or another- self-reference.\(^73\) Gödel’s incompleteness theorems have been used (and abused) for the most picturesque purposes\(^74\) and, as is well known, it is far from being considered only as a simple theorem of logic.\(^75\) But what the so considered most important theorem of the history of human knowledge really proves is the existence of a formal undecidable sentence for which the First Law of logic does not hold and whose natural interpretation in ordinary languages is a self-referent sentence. And all self-referent sentences are syntactically impossible and then nonsensical in ordinary languages. The natural interpretation of Gödel’s sentence \( G \) is but a semantic illusion. And semantic illusions are neither true nor false.

The inclusion of Gödel sentence \( G \) in the set of primitive formula (axioms) of \( P \) does not solve the problem of its incompleteness. But what if we remove self-reference from both ordinary and formal languages? (this removal had to be axiomatically stated in set theory in order to avoid inconsistencies). Or what if we remove from ordinary and formal languages all sentences that do not satisfy the First Law of Logic?

7. **Russell paradox**

Russell discovered his famous paradox when he was analyzing Cantor’s paradox on the set of all cardinals.\(^76\) Russell defined its own universal set but only with regular sets (sets that are not member of themselves). Thus, Russell set \( R \) is the set of all sets that do not belong to themselves, and only of them. Or in other words, the set of all regular sets and only of them. According to its definition, it is immediate to prove that \( R \) belongs to itself whenever it does not belong to itself.

This is Russell’s paradox. Although in this case it is quite clear the sentence:

\[ R \text{ is the set of all regular sets, and only of them} \]  

(56)

\(^{73}\)[45], [61], [121], [51].

\(^{74}\)[42].

\(^{75}\)[61], [88], [89], [51] etc.

\(^{76}\)[20].
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is inconsistent because from it we immediately infer:

\[ R \text{ belongs to itself} \quad (57) \]

otherwise there would be a regular set (the set \( R \)) which is not in \( R \), and then \( R \) would not be the set of all regular sets, and only of them. But from (56) we can also derive:

\[ R \text{ does not belong to itself} \quad (58) \]

otherwise \( R \) would contain a set which is not regular (the set \( R \)) and then \( R \) would not be the set of all regular sets, and only of them. Accordingly, if we term (56) as \( p \) and (57) as \( q \) we can write:

\[ p \Rightarrow q \land \neg q \quad (59) \]

On the other hand, the set \( R \) as element of the set \( R \) is not defined until the set \( R \) be defined as a set, which in turn is impossible until \( R \) be defined as an element. As in semantic self-reference, this is as to try to count one’s chickens before they are hatched. In effect, all elements of a set should have been defined before grouping them as a set. That is the only way to know which elements are we grouping (in the same way we should know the subject of a predicate before we can predicate it). Shortly before the discovering of Russell’s paradox, Charles Dogson, better known as Lewis Carroll, proposed the following definition of Class: ([26], p. 31):

Classification or the formation of Classes is a Mental Process, in which we imagine that we have put together, in a group, certain things. Such a group is called a "Class".

Carroll’s proposal is clearly non platonic. Classes are mental constructions, theoretical objects resulting from our mental activity. Carroll’s definition could be rewritten as:

**Definition 1.** A set is the theoretical object that results from a mental process of grouping arbitrary objects previously defined.

Constructive Definition 1 is, on the one hand, not circular, and on the other incompatible with self-referent sets because before defining the set, all its elements have to be previously defined, either by enumeration or by comprehension. And this excludes the set being defined, simply because the set cannot be defined as element before it be defined as set; and if it is not previously defined as element it cannot be grouped to form the set. In addition, it seems reasonable to require that all elements to be grouped be previously defined, particularly if we pretend to know what are we grouping. Being self-referent, Russell’s set \( R \) is not a set according to Definition 1. And for the same reasons of self-reference neither the
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set of all cardinals nor the set of all ordinals could be sets according to the same
definition. Thus, to assume the above non platonic definition of set suffices to
solve Cantor, Burali-Forti and Russell paradoxes. But modern set theories have
never considered this non platonic option. Self-reference had finally to be removed
from set theory, but by less elegant means as the Axiom of Regularity, the Theory
of Types or the Theory of Classes.\footnote{[90], [137], [110], [14], [47].}

The axiomatic removal of self-reference from set theory should serve us to reflect
on the future of semantic self-reference. Sets and monadic sentences are more
related to each other than it may seem. When we say:

\[ S \text{ is } P \] (60)

we assert the subject \(S\) has the predicate or property \(P\). The elements of a set
behave as the subject of the above declarative monadic sentence because all of
them are predicated with the same predicate: its membership to the set. A set
that belong to itself is as a self-referent sentence: it also states a predicate of
itself, in this case its membership to the set, i.e. the membership to itself. Now
then, if self-reference is not appropriate for set theory and we had to remove it
in order to avoid inconsistencies, why not to remove semantic self-reference from
both formal and ordinary languages?

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