REMARKS ON GLOBAL WEAK SOLUTIONS TO A TWO-FLUID TYPE MODEL

HUANYAO WEN
School of Mathematics, South China University of Technology
Guangzhou 510641, China

CHANGJIANG ZHU
School of Mathematics, South China University of Technology
Guangzhou 510641, China

Dedicated to Professor Shuxing Chen on the occasion of his 80th birthday

Abstract. The present paper aims to give a review of a two-fluid type model mostly on large-data solutions. Some derivations of the model arising in different physical background will be introduced. In addition, we will sketch the proof of global existence of weak solutions to the Dirichlet problem for the model in one dimension with more general pressure law which can be non-monotone, in the context of allowing unconstrained transition to single-phase flow.

1. Introduction. In this paper we consider the following two-fluid type model:

\[
\begin{align*}
  m_t + (mu)_x &= 0, \\
  n_t + (nu)_x &= 0, \\
  \left[ (m+n)u \right]_t + \left[ (m+n)u^2 \right]_x + [P(m,n)]_x &= \mu u_{xx}, \quad \text{in} \quad (0,1) \times (0,\infty),
\end{align*}
\]

where the physical meanings of the unknowns will be presented later. This model is quite relevant with some physical models such as viscous liquid-gas flow model, two-fluid model, Vlasov-Fokker-Planck/compressible Navier-Stokes system and compressible MHD system. The main differences of (1.1) deriving from different models mainly focus on the pressure function $P$. The present paper aims to give a review of the model mostly on large-data solutions and to prove that the Dirichlet problem for (1.1) with a more general pressure which can be non-monotone admits a global weak solution with large initial data and allows unconstrained transition to single-phase flow. The latter part of this work is an extension of our previous work [21] where the corresponding result was obtained for a special monotone pressure.

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* Corresponding author.
For Pressure I.

\[ P(m, n) = C^0 \left( -b(m, n) + \sqrt{b(m, n)^2 + c(n)} \right), \]  
with \( b(m, n) = k_0 - m - a_0 n, \) \( c(n) = 4k_0a_0n \) for some known positive constants \( C^0, k_0 \) and \( a_0. \)

Pressure I corresponds to the viscous liquid-gas flow model which plays the role as a useful tool for engineers to obtain insight into gas-liquid flow dynamics in macroscopic flow systems where insight into volume averaged quantities is sufficient \([18, 32, 36]\). In this case, \( m = \alpha_l \rho_l \) and \( n = \alpha_g \rho_g \) where the unknown variables \( \alpha_l \) and \( \alpha_g \in [0, 1] \) denote volume fractions for each fluids in the mixture satisfying \( \alpha_l + \alpha_g = 1 \), and \( \rho_l \) and \( \rho_g \) denote the densities; \( P \) is the common pressure, \( u \) is the equal velocity of two fluids and \( \mu \) is the viscosity. For the viscous liquid-gas flow model, one of the common pressure functions is given by \((1.2)\)

\[
\begin{cases}
\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}, & \rho_g = \frac{P}{a_g^2}, \\
\alpha_l + \alpha_g = 1,
\end{cases}
\]  

where \( a_l, a_g \) are sonic speeds, respectively, in the liquid and gas, and \( P_{l,0} \) and \( \rho_{l,0} \) are the reference pressure and density given as constants. Further, an explicit pressure function is given by \((1.2)\).

An early work on this model was achieved by Evje and Karlsen \([18]\) where they obtained the global existence of weak solutions without transition to single-phase flow in one dimension, i.e., there are positive lower limits for each mass. Later on, a blowup criterion in terms of the concentration of mass and global existence of weak solutions with small data in two dimensions were obtained (please refer to \([59, 60]\)). The initial assumptions in \([59, 60]\) indicated that transition to single-phase flow is not allowed. In the framework of Besov spaces, Hao and Li \([27]\) obtained a global well-posedness result under some smallness assumptions for Cauchy problem in multi-dimensions. When the initial data may vanish in an open set of the domain, Guo, Yang, and Yao \([26]\) proved the global existence of strong solutions to the Cauchy problem in three dimensions with small initial energy where an additional restriction is imposed in the sense that that

\[ 0 \leq \underline{\rho}_0 m_0 \leq n_0 \leq \overline{\rho}_0 m_0 \]  

(1.4)

where \( \underline{\rho}_0, \overline{\rho}_0 \) are positive constants. It demonstrates from \((1.4)\) that if the liquid phase (the gas phase) vanishes at some points in space, then the gas phase (the liquid phase) is also forced to vanish at the same points. The additional restriction \((1.4)\) is used in view of possible singular effect of the pressure gradient. Another aspect reflecting the use of \((1.4)\) is the energy functional \( G(m, n) \) of the form

\[ G(m, n) = m \int_{\tilde{m}}^{\tilde{m}} \frac{P(s, \frac{n}{m} s) - P(\tilde{m}, \tilde{n})}{s^2} ds + \frac{m}{\tilde{m}} P(\tilde{m}, \tilde{n}) - \frac{m}{\tilde{m}} P(\tilde{m}, \frac{n}{m} \tilde{m}), \]  

(1.5)

which was first introduced in \([18]\) and also used in \([26]\). Wen, Yao, and Zhu \([57]\) proved the local existence and uniqueness of strong solutions and established a blow-up criterion in terms of the concentration of mass to the model in a smooth bounded domain of \( \mathbb{R}^3 \) with vacuum and \((1.4)\). However, it is still unknown whether the boundedness of mass can be derived for large data in three dimensions. Thus that the global existence and uniqueness of strong large solutions is still open. Under
some weak assumptions on the initial data, which can be discontinuous and large as well as involve transition to pure single-phase points or regions, Evje, Wen, and Zhu [21] obtained the existence of global bounded weak solutions. The proof in [21] relies on a decomposition of the pressure term appearing in the mixture momentum equation into two components yielding a different energy functional from (1.5), i.e.,

\[
\tilde{G}(m, n) = n \int_{\tilde{\rho}_g}^{\rho_g} \frac{P_2(s) - P_2(\tilde{\rho}_g)}{s^2} ds + m \int_{\tilde{\rho}_l}^{\rho_l} \frac{P_1(s) - P_1(\tilde{\rho}_l)}{s^2} ds,
\]

(1.6)

where \( \tilde{\rho}_l \) and \( \tilde{\rho}_g \) are given positive constants, and

\[
\rho_g = P(m, n) a^2_g, \quad \rho_l = P(m, n) + 2C_0k_0 a^2_l,
\]

\[
P_1(\rho_l) = P_{l,0} + a^2_l(\rho_l - \rho_{l,0}), \quad P_2(\rho_g) = a^2_g \rho_g,
\]

\[
P_1(\rho_l) = P_2(\rho_g) = P(m, n), \quad P_{l}(\tilde{\rho}_l) = P_{2}(\tilde{\rho}_g).
\]

Note that one will not need (1.4) to bound (1.6) compared with (1.5). In addition, [21] relies also on the Zlotnik inequality employed to derive the bounds of the masses where the monotonicity of the pressure with respect to each component which can be obtained from that

\[
P_m = C^0 \left\{ 1 - \frac{b(m, n)}{\sqrt{b^2(m, n) + c(n)}} \right\} > 0,
\]

\[
P_n = C^0 \left\{ a_0 + \frac{a_0}{\sqrt{b^2(m, n) + c(n)}} (m + a_0 n + k_0) \right\} > 0
\]

is used. For study of the viscous liquid-gas flow model with a singular pressure, please refer for instance to [17, 19, 25, 61, 62] and references therein.

**For Pressure II.**

- Pressure II.

\[
P(m, n) = Am^\gamma + n^\alpha,
\]

(1.7)

for some constants \( A > 0, \gamma > 1 \) and \( \alpha \geq 1 \).

When \( \alpha = 1 \), (1.1) with Pressure II corresponds to the asymptotic limit of a coupled system of the compressible Navier-Stokes equations with a Vlasov-Fokker-Planck equation ([9, 44]), i.e.,

\[
\begin{cases}
\rho_t + \text{div}_x (\rho u) = 0, \\
(\rho u)_t + \text{div}_x (\rho u u - \mu \triangle u - (\mu + \lambda) \nabla \text{div} u + \nabla p) = 1/\epsilon \int_{\mathbb{R}^3} (v - u) F dv, \\
\end{cases}
\]

(1.8)

which yields (1.1) as \( \epsilon \to 0 \) with

\[
P(m, n) = p(m) + n^\alpha
\]

for \( \alpha = 1 \). Here \( (m, n, u) \) is the limit of \( (\rho, \int F dv, u) \) in some sense, denoting density of the fluid, density of the particle, and velocity of the fluid and particle, respectively. In fact, the model (1.8) is also called the fluid-particle interaction model which arises in a lot of industrial applications such as the analysis of sedimentation phenomenon with applications in medicine and chemical engineering (see [4, 48, 49]), and the modeling of aerosols and sprays in the study of Diesel engines.
Without the viscosity terms, i.e., $\mu = \lambda = 0$, (1.1) was derived formally from (1.8) as $\epsilon \to 0$ by Carrillo and Goudon [9] where they introduced the flowing regime and the bubbling regime with respect to two different scalings and investigated the stability and asymptotic limits. The system (1.1) is associated to the flowing regime\(^1\). For any fixed $\epsilon > 0$, the global existence of weak solutions was obtained by Mellet and Vasseur in [43]. Then they (see [44]) gave a rigorous derivation of (1.1) from (1.8) when $\epsilon \to 0$ as long as the regular solution to (1.1) exists on the time interval $[0, T]$. The proof in [44] relies on a relative entropy method similar to the “weak-strong” uniqueness principle established by Dafermos (see [13]) for multidimensional systems of hyperbolic conservation laws. However, the global existence of the solution to (1.1) with arbitrarily large initial data was not given in [44].

The case $\alpha = 2$ is associated to the compressible isentropic MHD system ([34, 39]), where $n$ represents vertical magnetic field, respectively. More specifically, the compressible isentropic MHD system is stated as follows:

$$
\begin{align*}
\rho_t + \text{div}(\rho v) &= 0, \\
(\rho v)_t + \text{div}(\rho v \otimes v) + \nabla p(\rho) - \mu \Delta v - (\mu + \lambda)\text{div}v &= (\nabla \times H) \times H, \\
H_t - \nabla \times (v \times H) &= -\nabla \times (\nu \nabla \times H), \\
\text{div}H &= 0,
\end{align*}
$$

(1.9)

where $\rho, v, H, \mu, \lambda$ denote density, velocity, magnetic field and the coefficients of viscosity, respectively, and $\nu \geq 0$ is the resistivity coefficient denoting magnetic diffusion of the magnetic field. The existence of global-in-time weak solution to (1.9) was obtained by Hu and Wang for isentropic flow (see [28]) and by Ducomet and Feireisl for the full system (see [15]). For the global existence and uniqueness results with smallness assumptions, please refer to [37, 47]. The non-resistive case for (1.9), i.e., $\nu = 0$, is more difficult to study and is more related to the two-fluid type model (1.1). In the one-dimensional case, global existence and uniqueness of regular solutions with large initial data was obtained by Jiang and Zhang ([34]). The global existence and uniqueness of small solutions for Cauchy problem in two dimensions was obtained by Hu ([29]) and by Wu and Wu ([58]). For the three-dimensional case with boundary, Tan and Wang obtained the global existence and uniqueness result with small initial data ([50]). It is still unknown for the existence of global-in-time solutions to (1.9) without resistivity in multi-dimensions. For a special case as in [39], letting $\nu = 0$, $\rho = \rho(x_1, x_2, t)$, $v = (u_1, u_2, 0) =: \left( u(x_1, x_2, t), 0 \right)$, and $H = (0, 0, n(x_1, x_2, t))$, then

$$
\begin{align*}
(\nabla \times H) \times H &= \left(-\nabla\left(\frac{n_2}{2}\right), 0\right), \\
H_t - \nabla \times (v \times H) &= \left(0, 0, n_t + \text{div}(nu)\right), \\
\text{div}H &= 0,
\end{align*}
$$

\(^1\)The system (1.1) with $n_t + (nu)_x = 0$ replaced by $n_t + (nu)_x = n_x x$ ("$n_t + \nabla \cdot \left( nu \right) = \Delta n$" in multi-dimensions) is associated to the bubbling regime. See [2, 10, 1, 31] for the global existence of weakly dissipative solutions and weak-strong uniqueness and low Mach number limits in high dimensions, respectively. For the well-posedness results, please refer to [11, 14, 16, 30].
which combined with (1.9) yields
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(pu)_t + \text{div}(pu \otimes u) + \nabla p = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u,
\end{cases}
\]
(1.10)

which was derived by Barrett, Lu and Süli ([3]) as a macroscopic closure of the compressible Navier–Stokes–Fokker–Planck equations:
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(pu)_t + \text{div}(pu \otimes u) + \nabla p = \mu \Delta u - (\mu + \nu) \nabla \text{div} u, \\
T_t + \text{div}(uT) - (\nabla uT + T \nabla T u) = \varepsilon \Delta T + \frac{kA_0}{2\lambda} nI - \frac{A_0 T}{\lambda},
\end{cases}
\]
(1.11)

which was derived by Barrett, Lu and Süli ([3]) as a macroscopic closure of the compressible diffusive Oldroyd-B type model, i.e.,
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(pu)_t + \text{div}(pu \otimes u) + \nabla P = \mu \Delta u - (\mu + \lambda) \nabla \text{div} u, \\
\alpha_t + \text{div}(\alpha u) = 0, \\
\n, \\
\text{where } n(x,t) := \int_{\mathbb{R}^3} \psi(x,t,q)dq \text{ represents polymer number density and the extra stress tensor } T(x,t) := k \int_{\mathbb{R}^3} \psi(x,t,q)qq^T dq. \text{ In [3], Barrett, Lu and Süli also showed the existence of global-in-time weak solutions to (1.11) in two dimensional setting. In [42], Lu and Zhang proved the weak-strong uniqueness, and gave a refined blow-up criterion and a conditional regularity result in two dimensional setting. Very recently, Wang and Wen ([51]) showed the global-in-time existence of strong solutions and derived some associated optimal time-decay estimates. In the framework of Besov spaces, please refer to [63].}

For Pressure III. • Pressure III. P is determined implicitly by
\[
\begin{cases}
A_+(\rho_+)^\gamma = A_-(\rho_-)^{\alpha}, \\
m\rho_+ + n\rho_- = \rho_+\rho_-,
\end{cases}
\]
(1.12)

for constants $A_+, A_- > 0$ and $\gamma, \alpha > 1$, where $m = a_+\rho_+, n = a_-\rho_-, \alpha_+ \geq 0$, $\alpha_- \geq 0$ and $\alpha_+ + \alpha_- = 1$.

The system (1.1) with pressure III is corresponding to a special case for the following two-fluid model:
\[
\begin{cases}
\alpha^+ + \alpha^- = 1, \\
\partial_t (\alpha^+\rho^+) + \text{div}(\alpha^+\rho^+u^+) = 0, \\
\partial_t (\alpha^+\rho^+u^+) - \partial_t (\alpha^+\rho^+u^+) + \text{div}(\alpha^+\rho^+u^+ \otimes u^+) + \alpha^+\nabla P^+(\rho^+) = \text{div}(\alpha^+\tau^+),
\end{cases}
\]
(1.13)
where the unknown variable \((\alpha^\pm, p^\pm, u^\pm, P^\pm, \tau^\pm)\) denotes volume fraction, density, velocity, pressure and viscous stress tensor for each fluid. For more information about the modelling of (1.13), please refer for instance to [5, 32, 36]. As a matter of fact, letting \(u^+ = u^− \equiv u, P^+ = A_+(\rho^+)\), \(P^− = A_−(\rho^−)\), \(\tau^+ = \tau^− = \mu[\nabla u + (\nabla u)^T] + \lambda \text{div}uN\), and denoting \(m = \alpha^+\rho^+, n = \alpha^−\rho^−\), one can derive (1.1) with pressure III from (1.13) where the sum of the two momentum equations in (1.13) produces (1.1)3. The model (1.13) is more realistic since velocities of two fluids can be unequal. However from mathematical points of view, there will be more challenges such as the non-conservative pressure which makes the defined weak solution need more regularity. Until now, it is still unknown whether global-in-time solutions exist for arbitrarily large initial data in multi-dimensions, although it has been achieved when the pressure functions for each fluids are equal, i.e., \(P^+ = P^−\), and the viscosity coefficients are chosen as a special case depending on density in one dimension (see [6]). For singular pressure in one dimension, we refer to [22]. When the pressure functions are equal and that a third-order derivative of \(\alpha^\pm\rho^\pm\) of so-called Korteweg type is included in the momentum equations of (1.13), Cui, Wang, Yao, Zhu ([12]) proved the existence and uniqueness of global strong solutions with small initial data for Cauchy problem in three dimensions, and obtained some associated time-decay estimates. The Korteweg term plays an essential role in the analysis. Evje, Wang and Wen ([20]) found that the unequal pressure has stability effect for the Cauchy problem in three dimensions such that the strong solution with small initial data exists globally in time and decays algebraically. Very recently, Evje-Wang-Wen’s results were extended by Wang, Wen and Yao ([55]) to the Dirichlet problem.

When \(n = 0\), the system (1.1) reduces to compressible Navier-Stokes equations for isentropic flow. In this case, some seminal works on the global existence of weak solutions with large initial data have been achieved. More specifically, Lions [38] obtained the global existence of weak solution with large initial data in multi-dimensions, where \(p(n) = Rn^n\) for some positive constant \(R\) and any given \(\gamma \geq \frac{9}{5}\) for three dimensions. The constraint for \(\gamma\) was relaxed to \(\gamma > 1\) by Jiang-Zhang [33] for spherically symmetric weak solutions and to \(\gamma > \frac{3}{2}\) in three dimensions by Feireisl-Novotný-Petzeltová [24]. The pressure function in [24, 33, 38] is monotone, which plays important role in their proof. Later on, Feireisl [23] extended their result in (24) to the case for more general pressure \(P(\rho)\) of monotonicity for \(\rho \geq \rho_−\). Very recently, Bresch and Jabin [7] developed a new compactness method for the density which does not rely on the monotonicity assumptions on the pressure.

The Lions theory has been extended recently by Vasseur-Wen-Yu [54] to (1.1) with pressure laws depending on two phases, i.e., (1.7), where \(\gamma > \frac{9}{5}, \alpha \geq 1\) and (1.4) was needed or alternatively \(\gamma > \frac{9}{5}\) and \(\alpha > \frac{9}{5}\) close enough. It was extended to the case that both \(\gamma\) and \(\alpha\) can touch \(\frac{9}{5}\) by Novotný and Pokorný [45]. It should be mentioned that [45] covered more general pressure laws including the cases of (1.7) and (1.12) and some non-monotone pressure functions. The condition for two densities, i.e., (1.4), plays essential role in [45, 54]. For the extension of Novotný-Pokorný’s work to a two-fluid model of Baer-Nunziato type, we refer to [46] (see also the weak-strong uniqueness result [35]). Without the assumption (1.4), Bresch, Mucha, Zatorska [8] obtained the global existence of weak solutions with large initial data for the two-fluid Stokes equations\(^2\) on the d-dimensional torus \(\mathbb{T}^d\) for \(d = 2, 3,\)

\(^2\)The acceleration in the momentum equation, i.e., the first two terms on the left-hand side of the momentum equation, is ignored.
where the pressure is given by (1.12). The compactness of the density in [8] relies on Breschi-Jabin’s new compactness tools for compressible Navier-Stokes equations. Very recently, Wen [56] obtained the global existence of weak solutions for the model (1.1) with the pressure (1.7) and (1.12) for $\gamma \geq \frac{5}{3}$ and $\alpha \geq \frac{1}{5}$, where the additional assumption (1.4) is not needed. Note that for Pressure II and III (besides Pressure I), $P$ is monotone with respect to each components, since

for Pressure II: $P_m = A\gamma m^{\gamma-1} \geq 0, \quad P_n = \alpha n^{\alpha-1} \geq 0$;

for Pressure III: $P_m = A+\gamma\rho^{\gamma-1} \frac{\partial \rho^+}{\partial m} = A+\gamma\rho^{\gamma-1} \frac{(\frac{A}{A+})^{\frac{1}{\gamma}}\rho^{1-\frac{2}{\gamma}}}{\frac{\alpha}{\gamma}(1-\alpha_+) + \alpha_+} \geq 0$,

$P_n = A+\gamma\rho^{\gamma-1} \frac{\partial \rho^+}{\partial n} = A+\gamma\rho^{\gamma-1} \frac{1}{\frac{\alpha}{\gamma}(1-\alpha_+) + \alpha_+} \geq 0$,

where we have used

$$\begin{align*}
\frac{\partial \rho^+ (m,n)}{\partial m} &= \frac{(A+)^{\frac{1}{\gamma}}\rho^{1-\frac{2}{\gamma}}}{\frac{\alpha}{\gamma}(1-\alpha_+) + \alpha_+}, \\
\frac{\partial \rho^+ (m,n)}{\partial n} &= \frac{1}{\frac{\alpha}{\gamma}(1-\alpha_+) + \alpha_+}.
\end{align*}$$

For global existence theory of (1.1) with Pressure II ($\alpha = 2$) in two dimensions, we refer to [39] (see [28] for the full compressible MHD system with resistivity). For more results about the large time behavior and some other related works with large initial data, we refer to [40, 41].

In this work, we consider the model (1.1) equipped with the initial-boundary conditions:

$m(x,0) = m_0(x), \quad n(x,0) = n_0(x), \quad (m+n)u(x,0) = M_0(x), \quad \text{for} \quad x \in [0,1], \quad (1.15)$

and

$u(0,t) = u(1,t) = 0, \quad \text{for} \quad t \geq 0, \quad (1.16)$

where we consider a more general pressure function which satisfies

$$\begin{align*}
P &\in C([0, +\infty) \times [0, +\infty)) \cap C^1((0, +\infty) \times (0, +\infty)), \\
c_0(m^\gamma + n^\alpha - 1) \leq P(m,n) \leq C_0(m^\gamma + n^\alpha + 1), \\
|P(m_1,n_1) - P(m_2,n_2)|^2 &\leq C_1 \left( \frac{|m_1-m_2|^2}{m_1+m_2} + \frac{|n_1-n_2|^2}{n_1+n_2} \right)
\end{align*}$$

for any given $\gamma, \alpha \geq 1$, where $m_i, n_i \in [0, M]$ and $m_i+n_i > 0$ for $i=1, 2$, and $c_0, C_0$ and $C_1$ are known positive constants. The Helmholtz free energy function $H(m,n)$ corresponding to $P$ is a solution of the partial differential equation of the first order:

$$P(m,n) = m \frac{\partial H(m,n)}{\partial m} + n \frac{\partial H(m,n)}{\partial n} - H(m,n), \quad (1.18)$$

which is not unique and is usually chosen as

$$H(m,n) = m \int_1^n \frac{P(s, s \frac{n}{m})}{s^2 \frac{m}{s}} ds, \quad \text{if} \quad m > 0, \quad H(0,0) = 0.$$

However, this option generally requires that $n \leq Cm$ for some positive constant $C$.

In this work, we choose a Helmholtz free energy function $H(m,n)$ which satisfies

$$C(m^\gamma + n^\alpha - 1) \leq H(m,n) \leq \tilde{C}(m^{\gamma+1} + n^{\alpha+1} + 1) \quad (1.19)$$
for some known positive constants $C$ and $\hat{C}$, allowing unconstrained transition to single-phase flow, i.e., removing the constraints $n \leq \hat{C}m$ or $m \leq \hat{C}n$. Some examples satisfying (1.17)-(1.19) are stated as below:

- **Pressure I** which corresponds to the case $\gamma = \alpha = 1$ in (1.17) and (1.19).

\[ P(m, n) = C^0 \left( -b(m, n) + \sqrt{b(m, n)^2 + c(m)^2} \right), \]

with $b(m, n) = k_0 - m - a_0 n, c(n) = 4k_0a_0 n$ for some known positive constants $C^0$, $k_0$ and $a_0$; The corresponding Helmholtz free energy function $H(m, n)$ is given by

\[ H(m, n) = \hat{G}(m, n) - P_1(\hat{\rho}_l) + m + n \]

\[ = n \int_{\hat{\rho}_l}^{\rho_l} \frac{P_2(s) - P_2(\hat{\rho}_l)}{s^2} \, ds + m \int_{\hat{\rho}_l}^{\rho_l} \frac{P_1(s) - P_1(\hat{\rho}_l)}{s^2} \, ds - P_1(\hat{\rho}_l) + m + n. \]

- **Pressure II.** $P(m, n) = Am^\gamma + n^\alpha$ for $\gamma > 1, \alpha \geq 1$;

\[ H(m, n) = \begin{cases} 
\frac{Am^\gamma}{\gamma - 1} + \frac{n^\alpha}{\alpha - 1}, & \text{for } \alpha > 1, \\
\frac{Am^\gamma}{\gamma - 1} + n \log n, & \text{for } \alpha = 1,
\end{cases} \]

where $n \log n \geq n - 1$ for any $n \geq 0$.

- **Pressure III.** $P$ is determined implicitly by

\[ A_+ (\rho_+)^\gamma = A_- (\rho_-)^\alpha, \]

\[ m \rho_+ - n \rho_- = \rho_+ \rho_- , \]

for constants $A_+, A_- > 0$ and $\gamma, \alpha > 1$, where $m = \alpha_+ \rho_+, n = \alpha_- \rho_-, \alpha_+ \geq 0, \alpha_- \geq 0$ and $\alpha_+ + \alpha_- = 1$; $H(m, n) = P(m, n)(\frac{2m}{\alpha_+} + \frac{2n}{\alpha_-})$.

- An academic non-monotone pressure, i.e., $P(m, n) = m^2 + n^2 - mn$, and $H(m, n) = m^2 + n^2 - mn$.

The details for checking the above examples will be given in the Appendix later.

We give a definition of the global weak solutions as in [21].

**Definition 1.1.** We call $(m, n, u)$ a global weak solution of (1.1), (1.1.5) and (1.16) if for any $0 < T < +\infty$,

1. $m, n \in L^\infty(Q_T), (m + n)u^2 \in L^\infty(0, T; L^1(0, 1)), m, n \geq 0$ a.e. in $Q_T, u \in L^2(0, T; H^0_0(0, 1))$;

2. $(m, n, u)$ satisfy the system (1.1) in the sense of distribution, i.e., in $\mathcal{D}'(Q_T)$;

3. $(m, n, (m + n)u)(x, 0) = (m_0(x), n_0(x), M_0(x)), \text{weak } * \text{ in } L^\infty(0, 1) \times L^\infty(0, 1)$

\[ \times L^2(0, 1), \]

where $Q_T = (0, 1) \times (0, T)$.

The main result in the rest of the paper is stated as follows:

**Theorem 1.2.** If $\inf_{x \in (0, 1)} m_0, \inf_{x \in (0, 1)} n_0 \geq 0, m_0, n_0 \in L^\infty(0, 1)$, and $\frac{M_0}{\sqrt{m_0 + n_0}} \in L^2(0, 1)$, then there exists a global weak solution $(m, n, u)$ to (1.1), (1.1.5) and (1.16) with pressure function satisfying (1.17).

2. Global weak solutions for more general pressure law. In this section, we sketch the proof of Theorem 1.2. The size of the viscosity coefficient is not essential in the proof and we assume $\mu = 1$ without loss of generality.
Construction of approximate sequences. As in [21], we mollify the initial data and give a positive lower bound of each mass as follows:

\[ \begin{cases} m_{0\delta} = \eta_\delta \ast \hat{m}_0 + \delta, \\
0_{0\delta} = \eta_\delta \ast \hat{n}_0 + \delta, \\
u_{0\delta} = \frac{\varphi_\delta}{\sqrt{m_{0\delta} + n_{0\delta}}} \eta_\delta \ast \left( \frac{M_0}{\sqrt{m_{0\delta} + n_{0\delta}}} \right), \end{cases} \]

where \( \delta \in (0, 1) \), \( \varphi_\delta \in C^\infty_0((0, 1)) \), \( 0 \leq \varphi_\delta \leq 1 \) in \([0,1]\) and \( \varphi_\delta \equiv 1 \) in \((\delta, 1 - \delta)\). Then \( m_{0\delta} \geq \delta > 0 \), \( n_{0\delta} \geq \delta > 0 \), \( (m_{0\delta}, n_{0\delta}) \in C^\infty([0,1]) \) and \( u_{0\delta} \in C^\infty_0((0,1)) \) for \( 0 < \alpha < 1 \), such that

\[ \begin{align*}
&\sqrt{m_{0\delta} + n_{0\delta}} u_{0\delta} \to \frac{M_0}{\sqrt{m_0 + n_0}} \text{ in } L^2(I) \text{ as } \delta \to 0, \\
&m_{0\delta} \to m_0, \quad n_{0\delta} \to n_0, \quad n_{0\delta} \to n_0 \text{ a.e., in } I, \text{ as } \delta \to 0, \\
&\|m_{0\delta}\|_{L^\infty} \leq \|m_0\|_{L^\infty} + 1, \quad \|n_{0\delta}\|_{L^\infty} \leq \|n_0\|_{L^\infty} + 1.
\end{align*} \]

By [18, 60], it is easy to verify that there exists a sequence of global strong solutions \((m^\delta, n^\delta, u^\delta)\) to

\[ \begin{align*}
m^\delta + (m^\delta u^\delta)_x &= 0, \\
n^\delta + (n^\delta u^\delta)_x &= 0, \\
\left( [m^\delta + n^\delta] u^\delta \right)_t + \left[ (m^\delta + n^\delta) (u^\delta)^2 \right]_x + \left[ P(m^\delta, n^\delta) \right]_x &= u^\delta, \quad \text{in } (0,1) \times (0, \infty), \\
\end{align*} \]

with the initial and boundary conditions

\[ \begin{align*}
m^\delta(x, 0) &= m_{0\delta}(x), \\
n^\delta(x, 0) &= n_{0\delta}(x), \\
u^\delta(x, 0) &= u_{0\delta}(x), \quad \text{for } x \in [0, 1], \\
u^\delta(0, t) &= u^\delta(1, t) = 0, \quad t \geq 0,
\end{align*} \]

where \( m^\delta > 0 \) and \( n^\delta > 0 \) on \([0, 1] \times [0, \infty)\).

**Some a priori estimates.**

**Lemma 2.1.** Under the conditions of Theorem 1.2, it holds that

\[ \int_0^1 \left[ \frac{(m^\delta + n^\delta)(u^\delta)^2}{2} + H(m^\delta, n^\delta) \right] + \int_0^t \int_0^1 (u^\delta)^2 dx dt \]

\[ \geq \int_0^1 \left[ \frac{(m^\delta_0 + n^\delta_0)(u^\delta_0)^2}{2} + H(m^\delta_0, n^\delta_0) \right]. \]

**Proof.** Multiplying (2.1)_3 by \( u^\delta \), and using (2.1) and (2.1)\_2, we get

\[ \left[ \frac{1}{2} (m^\delta + n^\delta) |u^\delta|^2 \right]_t + \left[ \frac{1}{2} (m^\delta + n^\delta) u^\delta |u^\delta|^2 \right]_x + \left[ P(m^\delta, n^\delta) u^\delta \right]_x 
- \left[ u^\delta |u^\delta|^2 \right]_x - P(m^\delta, n^\delta) u^\delta + |u^\delta|^2 = 0, \]

where

\[ P(m^\delta, n^\delta) u^\delta = m^\delta u^\delta \frac{\partial H(m^\delta, n^\delta)}{\partial m} + n^\delta u^\delta \frac{\partial H(m^\delta, n^\delta)}{\partial n} - H(m^\delta, n^\delta) u^\delta 
= - m^\delta \frac{\partial H(m^\delta, n^\delta)}{\partial m} + n^\delta \frac{\partial H(m^\delta, n^\delta)}{\partial n} - H(m^\delta, n^\delta) u^\delta 
= - \left[ H(m^\delta, n^\delta)_t + u^\delta H(m^\delta, n^\delta)_x \right] - H(m^\delta, n^\delta) u^\delta 
= - H(m^\delta, n^\delta)_t - [u^\delta H(m^\delta, n^\delta)]_x. \]
Hence we have

\[
\left[ \frac{1}{2}(m^\delta + n^\delta)|u^\delta|^2 \right]_t + \left[ \frac{1}{2}(m^\delta + n^\delta)u^\delta |u^\delta|^2 \right]_x + \left[ P(m^\delta, n^\delta)u^\delta \right]_x - \left[ u^\delta u^\delta \right]_x + H(m^\delta, n^\delta)_t + [u^\delta H(m^\delta, n^\delta)]_x + |u^\delta|^2 = 0.
\]

Integrating (2.4) over \((0, 1) \times (0, t)\), and using the boundary condition, we have

\[
\int_0^1 \left[ \frac{1}{2}(m^\delta + n^\delta)|u^\delta|^2 + H(m^\delta, n^\delta) \right] dx + \int_0^t \int_0^1 |u^\delta|^2 dx dt = \int_0^1 \left[ \frac{1}{2}(m^\delta_0 + n^\delta_0)|u^\delta_0|^2 + H(m^\delta_0, n^\delta_0) \right] dx.
\]

The proof of Lemma 2.1 is complete.

With Lemma 2.1 and (1.19), it is easy to get the following corollary.

**Corollary 1.** Under the conditions of Theorem 1.2, it holds that

\[
\int_0^1 \left[ \frac{(m^\delta + n^\delta)(u^\delta)^2}{2} + (m^\delta)^\gamma + (n^\delta)^\gamma \right] + \int_0^t \int_0^1 (u^\delta_x)^2 \leq C_2,
\]

where \(C_2\) is a known positive constant independent of \(t\) and \(\delta\).

**Lemma 2.2.** Under the conditions of Theorem 1.2, it holds that

\[
\begin{cases}
0 < m^\delta \leq \exp\{C_3(1 + T)\}, \\
0 < n^\delta \leq \exp\{C_4(1 + T)\},
\end{cases}
\]

for some positive constants \(C_3\) and \(C_4\) which are independent of \(t\) and \(\delta\).

**Remark 1.** In our previous work [21], the boundedness of \(m^\delta\) and \(n^\delta\) is uniform for \(T\) by virtue of Zlotnik inequality for the case of Pressure I. The monotonicity of the pressure plays essential role in the proof. However, for the more general pressure satisfying (1.17) which can be non-monotone, we have to modify the proof. It turns out that the upper bounds in (2.7) depend on \(T\).

**Proof.** As in [21], the derivative of velocity can be written by virtue of the momentum equation as below:

\[
u^\delta = \int_0^x \left( (m^\delta + n^\delta)u^\delta \right)_t - \int_0^1 \int_0^y \left( (m^\delta + n^\delta)u^\delta \right)_t + (m^\delta + n^\delta)(u^\delta)^2
\]

\[- \int_0^1 (m^\delta + n^\delta)(u^\delta)^2 + P(m^\delta, n^\delta) - \int_0^1 P(m^\delta, n^\delta).
\]

Multiplying (2.1) by \(\frac{1}{m^\delta}\), and substituting (2.8) into the resulting equation, we have

\[
d \left[ \log m^\delta(X^\delta(t; \tilde{x}, s), t) \right] = -d \int \left[ (m^\delta + n^\delta)(u^\delta) \right] + \frac{d}{dt} \int_0^1 \left[ (m^\delta + n^\delta)(u^\delta) \right]
\]

\[+ \int_0^1 (m^\delta + n^\delta)(u^\delta)^2 - P(m^\delta, n^\delta) + \int_0^1 P(m^\delta, n^\delta),
\]

(2.9)
where $X^\delta$ satisfies
\[
\begin{align*}
\frac{dX^\delta(t; \tilde{x}, s)}{dt} &= u^\delta(X^\delta(t; \tilde{x}, s), t), \quad 0 \leq t < s, \\
X^\delta(s; \tilde{x}, s) &= \tilde{x},
\end{align*}
\]
for any given $(\tilde{x}, s) \in [0, 1] \times [0, T]$.

Integrating (2.9) with respect to $t$ over $(0, s)$, we have
\[
\log m^{\delta}(\tilde{x}, s)
\]
\[
= \log m^{\delta}_{0}(X^{\delta}(0; \tilde{x}, s)) - \int_{0}^{\tilde{x}} \left( (m^{\delta} + n^{\delta})u^{\delta} \right) + \int_{0}^{X^{\delta}(0; \tilde{x}, s)} \left( (m^{\delta}_{0} + n^{\delta}_{0})u^{\delta}_{0} \right)
\]
\[
+ \int_{0}^{1} \int_{0}^{y} \left( (m^{\delta} + n^{\delta})u^{\delta} \right) - \int_{0}^{1} \int_{0}^{y} \left( (m^{\delta}_{0} + n^{\delta}_{0})u^{\delta}_{0} \right) + \int_{0}^{s} \int_{0}^{1} (m^{\delta} + n^{\delta})(u^{\delta})^{2}
\]
\[
- \int_{0}^{s} P(m^{\delta}, n^{\delta}) + \int_{0}^{s} \int_{0}^{1} P(m^{\delta}, n^{\delta}).
\]

In view of (1.17), we have
\[
- \int_{0}^{s} P(m^{\delta}, n^{\delta}) \leq -c_{0} \int_{0}^{s} (m^{\gamma} + n^{\alpha} - 1) \leq c_{0} T.
\]

Hence we have
\[
\log m^{\delta}(\tilde{x}, s)
\]
\[
\leq \log m^{\delta}_{0}(X^{\delta}(0; \tilde{x}, s)) - \int_{0}^{\tilde{x}} \left( (m^{\delta} + n^{\delta})u^{\delta} \right) + \int_{0}^{X^{\delta}(0; \tilde{x}, s)} \left( (m^{\delta}_{0} + n^{\delta}_{0})u^{\delta}_{0} \right)
\]
\[
+ \int_{0}^{1} \int_{0}^{y} \left( (m^{\delta} + n^{\delta})u^{\delta} \right) - \int_{0}^{1} \int_{0}^{y} \left( (m^{\delta}_{0} + n^{\delta}_{0})u^{\delta}_{0} \right) + \int_{0}^{s} \int_{0}^{1} (m^{\delta} + n^{\delta})(u^{\delta})^{2}
\]
\[
+ c_{0} T + \int_{0}^{s} \int_{0}^{1} P(m^{\delta}, n^{\delta}).
\] (2.10)

By virtue of Corollary 1, conservation of mass, Hölder inequality and Sobolev inequality, the terms on the right-hand side of (2.10) except the last two terms can be bounded uniformly for $t$ and $\delta$. Namely,
\[
\log m^{\delta}(\tilde{x}, s) \leq \tilde{C}_{3}(1 + T) + \int_{0}^{s} \int_{0}^{1} P(m^{\delta}, n^{\delta}),
\]

where $\tilde{C}_{3}$ is independent of $t$ and $\delta$. Recalling that $P(m^{\delta}, n^{\delta}) \leq C_{0}[(m^{\delta})^{\gamma} + (n^{\delta})^{\alpha} + 1]$ by (1.17), and using Corollary 1, we have
\[
\log m^{\delta}(\tilde{x}, s) \leq \tilde{C}_{3}(1 + T) + C_{0} \int_{0}^{s} \int_{0}^{1} [(m^{\delta})^{\gamma} + (n^{\delta})^{\alpha} + 1] \leq \tilde{C}_{3}(1 + T),
\]

for some known positive $C_{3}$ independent of $t$ and $\delta$. Thus we get
\[
m^{\delta} \leq \exp\{C_{3}(1 + T)\}.
\] (2.11)

Similarly, there exists a positive constant $C_{4}$ which is independent of $t$ and $\delta$, such that
\[
n^{\delta} \leq \exp\{C_{4}(1 + T)\}.
\]

The proof of Lemma 2.2 is complete. \qed
Lemma 2.3. It follows from Corollary 1 and Lemma 2.2 that there exist \((m, n, u)\) and a subsequence of \((m^\delta, n^\delta, u^\delta)\) still denoted by themselves, such that

\[
(m^\delta, n^\delta) \to (m, n) \text{ weak* in } L^\infty(Q_T),
\]

\[
P(m^\delta, n^\delta) \to P(m, n) \text{ weak* in } L^\infty(Q_T),
\]

\[
u^\delta \to u \text{ weakly in } L^2(0, T; H^1_0(I)),
\]

as \(\delta \to 0\). By virtue of the same analysis as that in [21], the limits satisfy

\[
\begin{align*}
&\left\{ \begin{array}{l}
    m_t + (mu)_x = 0 \text{ and } n_t + (nu)_x = 0 \text{ in } \mathcal{D}'(Q_T), \\
    (m + n)u + (m + n)^2 + \left( P(m, n) \right)_x = u_{xx} \text{ in } \mathcal{D}'(Q_T), \\
    (m, n, (m + n)u)(x, 0) = (m_0, n_0, M_0)(x) \text{ for a.e. } x \in I, \\
    m, n \geq 0 \text{ a.e. in } Q_T.
\end{array} \right.
\end{align*}
\]

(2.15)

To prove Theorem 1.2, it suffices to verify that \(\overline{P(m, n)} = P(m, n)\). This can be done by achieving the following two lemmas.

**Lemma 2.3.** Under the conditions of Theorem 1.2, it holds that

\[
\lim_{t \to \infty} \limsup_{\delta \to 0} \int_0^t \left[ \Phi(m^{\delta} + \frac{1}{T}) - \Phi(m + \frac{1}{T}) + \Phi(n^{\delta} + \frac{1}{T}) - \Phi(n + \frac{1}{T}) \right] (t) dx \leq 0,
\]

(2.16)

for a.e. \(t \geq 0\), where \(\Phi(\xi) = \xi \log \xi\).

**Proof.** As in [21], we have

\[
\begin{align*}
&\int_0^t \left( \Phi(m^{\delta} + \frac{1}{T}) - \Phi(m + \frac{1}{T}) + \Phi(n^{\delta} + \frac{1}{T}) - \Phi(n + \frac{1}{T}) \right) (t) \\
= &\int_{Q_t} (m+n)u_x - \int_{Q_t} (m^{\delta} + n^{\delta})u^{\delta}_x + \int_0^t \left[ \Phi(m_0^{\delta} + \frac{1}{T}) - \Phi(m_0 + \frac{1}{T}) + \Phi(n_0^{\delta} + \frac{1}{T}) - \Phi(n_0 + \frac{1}{T}) \right] \\
&+ \frac{1}{T} \int_{Q_t} u^{\delta}_x \left( \log(m^{\delta} + \frac{1}{T}) + \log(n^{\delta} + \frac{1}{T}) \right) - \frac{1}{T} \int_{Q_t} u_x \left( \log(m + \frac{1}{T}) + \log(n + \frac{1}{T}) \right) \\
\triangleq & I_1^{\delta} - \int_{Q_t} (m^{\delta} - m + n^{\delta} - n) P(m^{\delta}, n^{\delta}),
\end{align*}
\]

(2.17)

where

\[
\limsup_{\delta \to 0} I_1^{\delta} \to 0, \quad \text{as } l \to \infty.
\]

Denote

\[
I_2^{\delta} = - \int_{Q_t} (m^{\delta} - m + n^{\delta} - n) P(m^{\delta}, n^{\delta}).
\]

Then we divide it into two parts:

\[
I_2^{\delta} = - \int_{Q_t} (m^{\delta} - m + n^{\delta} - n) P^l(m, n) - \int_{Q_t} (m^{\delta} - m + n^{\delta} - n) \left[ P(m^{\delta}, n^{\delta}) - P^l(m, n) \right]
\]

\[
= I_1^{\delta, l} + I_2^{\delta, l},
\]

(2.18)

where \(P^l(m, n) = P(m + \frac{1}{T}, n + \frac{1}{T})\).

Noticing from (1.17) that

\[
|P(m_1, n_1) - P(m_2, n_2)|^2 \leq C_1 \left( \frac{|m_1 - m_2|^2}{m_1 + m_2} + \frac{|n_1 - n_2|^2}{n_1 + n_2} \right),
\]

(2.19)
Similarly, $\delta$ constants but independent of $\delta l$, where we have the boundedness of $m^\delta, n^\delta, m, n$.

By virtue of Taylor’s theorem, we have

$$
\Phi(m^\delta+\frac{1}{l}) - \Phi(m+\frac{1}{l}) = \Phi'(m+\frac{1}{l}) (m^\delta - m) + \Phi''(m+\frac{1}{l}+\theta(m^\delta - m)) (m^\delta - m)^2
$$

for some $\theta \in (0, 1)$. Since $m^\delta$ and $m$ have a uniform upper bound for $\delta$ and that $m+\frac{1}{l} + \theta(m^\delta - m) \leq (1-\theta)m+\frac{1}{l} + \theta m^\delta \leq m^\delta + m + \frac{1}{l} \leq M_1$,

where the known constant $M_1$ may depend on $T$, we have

$$
\frac{(m^\delta - m)^2}{M_1} \leq \frac{(m^\delta - m)^2}{m^\delta + m + \frac{1}{l}} \leq \Phi(m^\delta+\frac{1}{l}) - \Phi(m+\frac{1}{l}) - \Phi'(m+\frac{1}{l}) (m^\delta - m).
$$

Similarly,

$$
\frac{(n^\delta - n)^2}{M_2} \leq \frac{(n^\delta - n)^2}{n^\delta + n + \frac{1}{l}} = \Phi(n^\delta+\frac{1}{l}) - \Phi(n+\frac{1}{l}) - \Phi'(n+\frac{1}{l}) (n^\delta - n),
$$

for some positive constant $M_2$ that may depend on $T$.

Then we have

$$
II_2^{\delta,l} \leq \frac{1}{l} \hat{C}(T) + C \int_{Q_t} \left[ \frac{|m^\delta - m|^2}{m^\delta + m + \frac{1}{l}} + \frac{|n^\delta - n|^2}{n^\delta + n + \frac{1}{l}} \right]
$$

$$
\leq \frac{1}{l} \hat{C}(T) + C \int_{Q_t} \left[ \Phi(m^\delta+\frac{1}{l}) - \Phi(m+\frac{1}{l}) + \Phi(n^\delta+\frac{1}{l}) - \Phi(n+\frac{1}{l}) \right] - C \int_{Q_t} \left[ \Phi'(m+\frac{1}{l})(m^\delta - m) + \Phi'(n+\frac{1}{l})(n^\delta - n) \right].
$$

Putting (2.18) and (2.21) into (2.17), we have

$$
\int_0^t \left( \Phi(m^\delta+\frac{1}{l}) - \Phi(m+\frac{1}{l}) + \Phi(n^\delta+\frac{1}{l}) - \Phi(n+\frac{1}{l}) \right) dt
$$

$$
\leq I_1^{\delta,l} + I_1^{\delta,l} + II_2^{\delta,l}
$$

$$
\leq I_1^{\delta,l} + I_1^{\delta,l} + \frac{1}{l} \hat{C}(T) + C \int_{Q_t} \left[ \Phi(m^\delta+\frac{1}{l}) - \Phi(m+\frac{1}{l}) + \Phi(n^\delta+\frac{1}{l}) - \Phi(n+\frac{1}{l}) \right] - C \int_{Q_t} \left[ \Phi'(m+\frac{1}{l})(m^\delta - m) + \Phi'(n+\frac{1}{l})(n^\delta - n) \right].
$$
\( = III^{\delta,l} + C \int_{Q_t} \left[ \Phi(m^\delta + \frac{1}{l}) - \Phi(m + \frac{1}{l}) + \Phi(n^\delta + \frac{1}{l}) - \Phi(n + \frac{1}{l}) \right] \), (2.22)

where

\[ III^{\delta,l} = I_1^{\delta,l} + II_1^{\delta,l} + \frac{1}{l} C(T) - C \int_{Q_t} \left[ \Phi'(m + \frac{1}{l})(m^\delta - m) + \Phi'(n + \frac{1}{l})(n^\delta - n) \right]. \]

It is not difficult to verify that

\[ \limsup_{\delta \to 0} III^{\delta,l} \to 0, \quad \text{as} \ l \to \infty. \]

Then using Gronwall inequality, we finish the proof of the lemma.

With (2.16), (2.19) and (2.20), it immediately yields the following corollary.

**Corollary 2.** Under the conditions of Theorem 1.2, it holds that

\[ \| (m^\delta - m)(t) \|_{L^2} + \| (n^\delta - n)(t) \|_{L^2} \to 0, \]

for a.e. \( t \geq 0 \), as \( \delta \to 0 \).

With the strong convergence of \( m^\delta \) and \( n^\delta \) as in (2.23), the continuity of \( P \), Egorov's Theorem and the boundedness of the \( (m^\delta, n^\delta, m, n) \), one can verify that

\[ P(m,n) = P(m,n). \]

This completes the proof of Theorem 1.2.

3. **Concluding remarks.** From the assumptions on the pressure, i.e., (1.17), it concludes that Theorem 1.2 is valid for these models in one dimension, namely, for the viscous liquid-gas flow model, two-fluid model with equal velocity, the model from a hydrodynamic limit of Vlasov-Fokker-Planck/compressible Navier-Stokes system, and compressible MHD system. In addition, Theorem 1.2 holds as well for the model (1.1) with some academic pressure functions which can be non-monotone, allowing unconstrained transition to single-phase flow. In multidimensions, some results on the global weak solutions to the model with pressures II and III for some \( \gamma \) and \( \alpha \) have been achieved in the context of allowing transition to single-phase flow. However, it is still open for the cases that the pressure functions themselves or gradient have singularities such as Pressure I.

4. **Appendix.**

**Lemma 4.1.**

- For Pressure I, the Helmholtz free energy function \( H(m,n) \) can be chosen as

\[ H(m,n) = \widetilde{G}(m,n) - P_1(\tilde{\rho}_1) + m + n \]

\[ = n \int_{\tilde{\rho}_0}^{\rho_0} \frac{P_2(s) - P_2(\tilde{\rho}_0)}{s^2} \ ds + m \int_{\tilde{\rho}_0}^{\rho_1} \frac{P_1(s) - P_1(\tilde{\rho}_1)}{s^2} \ ds - P_1(\tilde{\rho}_1) + m + n. \]

- For Pressure II, \( H(m,n) \) can be chosen as

\[ H(m,n) = \begin{cases} \frac{A m^\gamma}{\gamma - 1} + \frac{n^\alpha}{\alpha - 1}, & \text{for} \ \alpha > 1, \\ \frac{A m^\gamma}{\gamma - 1} + n \log n, & \text{for} \ \alpha = 1. \end{cases} \]

- For Pressure III, \( H(m,n) \) can be chosen as

\[ H(m,n) = P(m,n) \left( \frac{\alpha_+}{\gamma - 1} + \frac{\alpha_-}{\alpha - 1} \right). \]
An academic non-monotone pressure, i.e., \( P(m, n) = m^2 + n^2 - mn \), and \( H(m, n) = m^2 + n^2 - mn \).

**Remark 2.** It is not difficult to verify that the four examples satisfy (1.17) and (1.19).

**Proof.** For Pressure I: Denote \( H(m, n) = \hat{G}(m, n) - P_1(\hat{\rho}_1) + m + n \), i.e.,

\[
H(m, n) = n \int_{\tilde{\rho}_1}^{\rho_1} \frac{P_2(s) - P_2(\tilde{\rho}_1)}{s^2} ds + m \int_{\tilde{\rho}_1}^{\rho_1} \frac{P_1(s) - P_1(\tilde{\rho}_1)}{s^2} ds - P_1(\tilde{\rho}_1) + m + n.
\]

Then we have

\[
mH_m + nH_n - H = m^2 \frac{P_1(\rho_1) - P_1(\hat{\rho}_1)}{\rho_1^2} \frac{\partial \rho_1}{\partial m} + mn \frac{P_2(\rho_g) - P_2(\hat{\rho}_g)}{\rho_g^2} \frac{\partial \rho_g}{\partial m} + n^2 \frac{P_2(\rho_g) - P_2(\hat{\rho}_g)}{\rho_g^2} \frac{\partial \rho_g}{\partial n} + mn \frac{P_1(\rho_1) - P_1(\hat{\rho}_1)}{\rho_1^2} \frac{\partial \rho_1}{\partial m} + P_1(\hat{\rho}_1).
\]

Since

\[
a_1^2(\rho_1)_m = a_g^2(\rho_g)_m, \quad a_1^2(\rho_1)_n = a_g^2(\rho_g)_n,
\]

\[
(\rho_g)_m = \frac{a_1^2 \rho_g + a_g^2 \rho_1 - a_1^2 m - a_g^2 n}{a_1^2 \rho_g + a_g^2 \rho_1 - a_1^2 m - a_g^2 n},
\]

we have

\[
mH_m + nH_n - H = [P_1(\rho_1) - P_1(\hat{\rho}_1)] \cdot \frac{m^2 a_1^2 (\rho_g)_m + mn a_g^2 (\rho_g)_n + mn a_1^2 (\rho_g)_n + P_1(\hat{\rho}_1)}{a_1^2 \rho_g + a_1^2 \rho_1 - a_1^2 m - a_1^2 n}.
\]

Thus the first example satisfies (1.18). One can verify the cases for Pressures II and III similarly.

For the non-monotone pressure, i.e., \( P(m, n) = m^2 + n^2 - mn \), we compute that

\[
mH_m + nH_n - H = m(2m - n) + n(2n - m) - m^2 - n^2 + mn = m^2 + n^2 - mn = P(m, n),
\]

which satisfies (1.18). \(\square\)

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E-mail address: mahywen@scut.edu.cn
E-mail address: machjzhu@scut.edu.cn