Field on Poincaré group and quantum description of orientable objects

D.M. Gitman\textsuperscript{a}\textsuperscript{*} and A.L. Shelepin\textsuperscript{b}\textsuperscript{†}

\textsuperscript{a}Instituto de Física, Universidade de São Paulo, 
Caixa Postal 66318-CEP, 05315-970 São Paulo, S.P., Brazil
\textsuperscript{b}Moscow Institute of Radio Engineering, Electronics and Automation, 
Prospect Vernadskogo, 78, 117454, Moscow, Russia

February 14, 2009

Abstract

We propose an approach to the quantum-mechanical description of relativistic orientable objects. It generalizes Wigner’s ideas concerning the treatment of nonrelativistic orientable objects (in particular, a nonrelativistic rotator) with the help of two reference frames (space-fixed and body-fixed). A technical realization of this generalization (for instance, in 3 + 1 dimensions) amounts to introducing wave functions that depend on elements of the Poincaré group $G$. A complete set of transformations that test the symmetries of an orientable object and of the embedding space belongs to the group $\Pi = G \times G$. All such transformations can be studied by considering a generalized regular representation of $G$ in the space of scalar functions on the group, $f(x, z)$, that depend on the Minkowski space points $x \in G/\text{Spin}(3,1)$ as well as on the orientation variables given by the elements $z$ of a matrix $Z \in \text{Spin}(3,1)$. In particular, the field $f(x, z)$ is a generating function of usual spin-tensor multicomponent fields. In the theory under consideration, there are four different types of spinors, and an orientable object is characterized by ten quantum numbers. We study the corresponding relativistic wave equations and their symmetry properties.

Introduction

The problem of a quantum-mechanical description of orientable objects is not one of the issues that are frequently encountered in the literature, although the general approach to such a description cannot be considered as being completely formulated. It is well-known that for a quantum-mechanical description of a point-like spinless particle in an $n$-dimensional (pseudo)Euclidean space it is sufficient to use one wave function that depends on space-time coordinates $x^\mu$ alone. A complete description of orientable objects

\textsuperscript{*}E-mail: gitman@dfn.if.usp.br
\textsuperscript{†}E-mail: alex@shelepin.msk.ru
requires some additional coordinates. For example, in order to determine the exact localization of a rigid body in a 3-dimensional space one needs to assign three coordinates that determine the position of its mass center, as well as three angles that determine the orientation. It is natural to consider a quantum-mechanical description of such orientable objects as being achieved by an introduction of wave functions depending not only on the \( n \) coordinates of the mass center, but also on some auxiliary variables that describe the orientation. In the known examples (a spinning particle and a non-relativistic rigid rotator), the orientation is usually taken into account by an introduction of multi-component wave functions depending on the space-time coordinates \( x^\mu \). As to the first example, the construction and classification of such functions is largely due to the theory of representations of the Poincaré and Lorentz groups, rather than due to the well-formulated theory that describes spinning particles as particles that possess orientation.

Until recently, the only example of a well-developed physical theory in which wave functions depend on orientation (and only on orientation) has been the theory of the above mentioned rigid rotator, constructed by Wigner, Casimir and Eckart back in the 1930’s (see [1] for references and historical remarks). One reference frame (laboratory, space-fixed, s.r.f. \textit{in what follows}) is assigned with the surrounding objects, while another one (molecular, body-fixed, b.r.f. \textit{in what follows}) is assigned with the rotating body. Correspondingly, there are two sets of operators of angular momentum: those of s.r.f. (left generators of the rotation group \( \hat{J}^L_k \)), and those of the b.r.f. (right generators of the rotation group \( \hat{J}^R_k \)). The interpretation of \( \hat{J}^R_k \) as projections of momentum in the b.r.f. belongs to Wigner and Casimir (1931), and basically lays the foundation of the theory of molecular spectra. Mathematically, the construction of a theory of a non-relativistic rigid rotator is a construction of a filed on the group \( G = SO(3) \sim SU(2) \), see below.

It should be noted that the description of relativistic spinning particles can be reformulated in terms of one scalar field depending on the space-time coordinates \( x^\mu \) as well as on some auxiliary continuous variables that describe spin. Such a reformulation has a long history. At the end of 1940’s and the beginning of 1950’s, independently by various authors [2, 3, 4, 5], mainly in connection with a construction of relativistic wave equations (RWE), were introduced some fields depending not only on \( x^\mu \) but also on a certain set of spinning variables. A systematic treatment of these fields as fields on homogenous spaces of the Poincaré group was made by Finkelstein [6] in 1955. He also presented a classification and explicit constructions of homogenous spaces of the Poincaré group which contain the Minkowski space that is a homogenous space of the latter group. In 1964, Lurçat [7] suggested to construct a quantum theory on the whole Poincaré group, instead of the Minkowski space.

In 70-90, the ideas of constructing fields in various homogenous spaces of the Poincaré group, gained a certain development, in particular, in the papers [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Properties of various spaces were considered, as well as some possibilities for introducing interactions in the spin phase-space and possibilities for constructing Lagrangian formulations. Restrictions were studied to be imposed on scalar fields by using a choice of a homogenous space. Thus, the authors of [8] arrived at the conclusion that the minimal dimension of a homogenous space suitable for a description of integer and semi-integer spins equals to eight. In the papers [16, 17], we have developed a general approach to constructing fields on groups of motions in Euclidean and pseudo-Euclidean spaces, elaborating in detail the cases of 2, 3- and 4-dimensions. In that approach, a scalar field on
the Poincaré group is a generating function of usual multicomponent fields. In particular, it has been demonstrated that, as distinct from the case of scalar fields on homogenous spaces, a field on a group as a whole is closed with respect to discrete transformations. The task of constructing RWE in that approach looks especially natural, since it is intimately related with a classification of scalar functions on a group.

However, until now there has been no clear understanding of a connection between the approach that describes nonrelativistic orientable objects (à la Wigner) and the one that describes spinning particles in terms of multicomponent functions. In the present article, we look at both problems from a viewpoint that suggests a universal approach to a quantum-mechanical description of orientable objects on a basis of the theory of representations of spatial groups, using, at the same time, some physical concepts that bear their origin in the intuitively obvious nonrelativistic theory of orientable objects. Such a treatment of relativistic spinning particles allows a natural appearance and interpretation of constructions that have been frequently introduced by hand, in a heuristic manner.

The first observation that plays a key role in the suggested approach is the following (we further make a reference to the easily imaginable 3-dimensional Euclidean case): a description of an orientable object is completely determined by the positions of the corresponding b.r.f. with respect to some s.r.f., which means that there exists a technical possibility to describe these positions by elements of the corresponding transformation group (namely, the (pseudo)Euclidean group of motions $G$); see examples below. Thus, we believe that quantum mechanical description of orientable objects can be performed by scalar wave functions $f(g)$ on the corresponding group of motions, $g \in G$.

The following step is a realization of the fact that testing the symmetry of a space that contains an orientable object is related to the behavior of wave functions with respect to the transformations of the s.r.f., which we further call left transformations, $g_l \in G$, and which belong to the same group $G$, whereas testing the symmetry properties of the orientable object itself is related to the behavior of wave functions with respect to the transformations of the b.r.f., which we further call right transformations, $g_r \in G$, and, which, once again, belong to $G$. We thus arrive at the necessity of studying the behavior of the wave function describing an orientable object under the action of transformations being the direct product of left and right transformations, i.e., under the action of the group $\Pi = G \times G$.

Finally, an issue that concludes the construction is the fact that a complete classification of the wave functions $f(g)$ of an orientable object under the action of transformations that belong to $\Pi$ can be extracted from the study of the so-called generalized regular representation (GRR)

$$\mathbb{T}(g_l; g_r)f(g) = f(g_l^{-1}g_lg_r),$$

while the right and left generators of the group $G$ in this representation are interpreted as ones related to the above-defined right and left transformations.

The present article begins with a consideration of some simple cases that describe orientable objects in 2 and 3 dimensions. In these cases, the use of visually obvious concepts allows one to present a convincing demonstration of the possibility to realize the above-declared program. We next present our approach in its general setting, and finally proceed to a detailed treatment of the physically most interesting description of orientable objects in the Minkowski space, in which a description of the position of an orientable object requires to indicate, besides the four coordinates $x_\mu$, also its orientation.
with respect to the laboratory reference frame. The latter is given by a pseudoothogonal matrix \( V \in SO(3,1) \). Thus, the orientable object in the Minkowski space is described by a pair \((x,V)\). An element of the group of motions \( M_0(3,1) = T(4) \otimes SO(3,1) \) is given by the same pair \((a,\Lambda)\), where \( a \) is a translation, and \( \Lambda \) is a rotation. It is easy to verify that under changes of the laboratory and body-fixed reference frames, respectively, we have

\[
(x',V') = (a,\Lambda)^{-1}(x,V), \quad (x',V') = (x,V)(a,\Lambda).
\]

In what follows, we consider a generalized regular representation being a representation in the space of functions on the Poincaré group. An important (not only technically) question is a parametrization of the matrices \( V \). Using the homomorphism \( SL(2,C) = Spin(3,1) \sim SO(3,1) \), we consider functions \( f(x,z) \) of space-time coordinates \( x \) and complex-valued spinor variables \( z \) being the elements of a matrix \( Z \in SL(2,C) \).

The field \( f(x,z) \) is a generating function of the usual spin-tensor multicomponent fields (they appear as coefficients of the power expansion with respect to \( z \)) and admits a number of symmetry operations.

The maximal set of commuting operators (their number being equal to that of the group parameters) in the space of functions consists of the Casimir operators and of the (equal in number) left and right generators. Functions \( f(x,z) \) on the 3+1 Poincaré group depend on ten parameters.

In the conventional description of relativistic particles (in 3 + 1 dim.) in terms of spin-tensor fields, based, in particular, on a classification of Poincaré and Lorentz group representations, there appear 8 particle characteristics (quantum numbers): 2 numbers \( j_1 \) and \( j_2 \) that label Lorentz group representations and 6 numbers related to the Poincaré group, those are the mass \( m \), spin \( s \) (Casimirs of the group), and 4 numbers, which are eigenvalues of some combinations of left generators of the group. In particular, the latter 4 numbers can be some components of the momentum and a spin projection. The proposed description of relativistic orientable objects is based on a classification of group representations of transformation of both s.r.f. and b.r.f., and the orientable object is characterized already by 10 quantum numbers, which are related to the maximal set of commuting operators.

We introduce the concept of a symmetry of a field on a group and classify the symmetries of a field on the Poincaré group. In particular, it turns out that the discrete transformations \( (C,P,T) \) correspond to involutive automorphisms of the group and are reduced to a complex conjugation and to a change of arguments of scalar functions \( f(x,z) \).

The orientation variables \( z \), as well as the corresponding decomposition coefficients, have two types of indices, related to the above-mentioned left and right transformations. The “left” indices related to changes of the s.r.f. (Lorentz transformations) are the usual (vector or spinor) indices, whereas “right” indices are related to changes of the b.r.f. Consequently, in contrast with nonrelativistic theory (in which there is only one type of spinors) and usual relativistic theory (in which there are two types of spinors), in a relativistic theory of oriented objects there are four types of spinors with different rules of transformations.

Using left-invariant (i.e., Lorentz invariant) differential operators of first order with respect to the variables \( x,z \), in the corresponding eigenvalue problems, we arrive to RWE in the theory under consideration. Reducing the general RWE to the space of \( f(x,z) \) being polynomials of a fixed order \( 2s \) in \( z \), we may obtain, in particular, all the known
types of RWE for spin $s$ in spin-tensor representation. We study RWE and their symmetries with respect to the symmetries of functions on the Poincaré group (left and right transformations, outer automorphisms). In such a way, there appear different types of RWE (reducible from the point of view of left transformations) which play important role in physics and which often have to be constructed “by hands”. For example, here we naturally obtain eight-component relativistic wave equation for particle of spin $1/2$, which was derived in course of a canonical quantization from Berezin-Marinov action (see [18]), and which allows one to avoid difficulties with infinite number of negative energy levels to construct a consistent version of relativistic quantum mechanics of noninteracting spinning particles.

1 Two dimensional Euclidean case

Let us consider a two-dimensional Euclidean space with a Cartesian reference frame, given by an orthonormalized basis $e_i$, $i = 1, 2$ (space-fixed reference frame, or s.r.f.). Suppose we would like to describe some orientable object in the s.r.f. (as such an object, one can imagine a two-dimensional solid body). To this end, we attach to the orientable object an additional Cartesian frame (body-fixed reference frame, or b.r.f.), given by an orthonormalized basis $\xi_i$, $i = 1, 2$. Then, the orientable object is described by a position $x = (x^i, i = 1, 2)$ of the b.r.f. origin with respect to the s.r.f., and by an angle $\theta$ between the corresponding axis of the s.r.f. and b.r.f. Therefore, as coordinates of the orientable object, we choose the pair $x, \theta$.

We will now look to two types of transformations. Translations of the s.r.f. origin by a vector $a$ with its rotations by an angle $\phi$ form the first type; in what follows, we call them left transformations. Translations of the b.r.f. origin by a vector $b$ with its rotations by an angle $\psi$ form another type; in what follows, we call them right transformations.

The coordinates $x, \theta$ of the orientable object are changed to $x', \theta'$ under the left transformations:

$$
\begin{align*}
x'^1 &= (x^1 - a^1) \cos \phi + (x^2 - a^2) \sin \phi, \\
x'^2 &= (x^2 - a^2) \cos \phi - (x^1 - a^1) \sin \phi, \\
\theta' &= \theta - \phi,
\end{align*}
$$

where $a^1$ and $a^2$ are components of the translation $a$ in the s.r.f.. The left transformations form the group $M(2) = T(2) \rtimes SO(2)$ of motions of the two-dimensional Euclidean space.
Under the right transformations, the coordinates \( x, \theta \) are changed as follows:

\[
\begin{aligned}
x^{1'} &= x^1 + b^1 \cos \theta - b^2 \sin \theta , \\
x^{2'} &= x^2 + b^2 \cos \theta + b^1 \sin \theta , \\
\theta' &= \theta + \psi ,
\end{aligned}
\]

where \( b^1 \) and \( b^2 \) are components of the translation \( \mathbf{b} \) in the b.r.f.. The right transformations also form the group \( M(2) = T(2) \rtimes SO(2) \).

Consider now transformations realized as combinations of all possible right and left transformations. The general transformations form a group, which is the direct product \( \Pi = M(2) \times M(2) \). The general transformations act on the coordinates \( x, \theta \) as follows:

\[
\begin{aligned}
x^{1'} &= (x^1 - a^1 + b^1 \cos \theta - b^2 \sin \theta) \cos \phi + (x^2 - a^2 + b^2 \cos \theta + b^1 \sin \theta) \sin \phi , \\
x^{2'} &= (x^2 - a^2 + b^2 \cos \theta + b^1 \sin \theta) \cos \phi - (x^1 - a^1 + b^1 \cos \theta - b^2 \sin \theta) \sin \phi , \\
\theta' &= \theta - \phi + \psi .
\end{aligned}
\]

The left, right, and general transformations can be described, in a convenient manner, with the help of a \( 3 \times 3 \)-matrix function \( g(\mathbf{r}, \varphi) \) of the variables \( \mathbf{r} \) and \( \varphi \). This matrix function has the form

\[
g(\mathbf{r}, \varphi) = \begin{pmatrix} V(\varphi) & \mathbf{r} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix},
\]

\[
V(\varphi) = \|\mathbf{v}_k(\varphi)\| = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \end{pmatrix} .
\]

Let the initial position of the orientable object be \( \mathbf{x}, \theta \). Then, the mutual orientation of the bases \( \mathbf{e}_i \) and \( \xi_i \) is given by a \( 2 \times 2 \) matrix \( V(\theta) \),

\[
\xi_k = \mathbf{e}_i \mathbf{v}'_k(\theta) , \quad V(\theta) = \|\mathbf{v}_k(\theta)\| = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} .
\]

One can verify that the left transformations \( \Pi \), the right transformations \( \Pi \), and the general transformations \( \Pi \) respectively, can be written as

\[
g(\mathbf{x}', \theta') = g^{-1}(\mathbf{a}, \phi) g(\mathbf{x}, \theta) ,
\]

\[
g(\mathbf{x}', \theta') = g(\mathbf{x}, \theta) g(\mathbf{b}, \psi) ,
\]

\[
g(\mathbf{x}', \theta') = g^{-1}(\mathbf{a}, \phi) g(\mathbf{x}, \theta) g(\mathbf{b}, \psi) .
\]

Relation (\ref{eq:general}) implies

\[
\begin{aligned}
g_{am}(\mathbf{x}', \theta') &= \sum_{\beta, c} g^{-1}_{\alpha\beta}(\mathbf{a}, \phi) g_{\beta c}(\mathbf{x}, \theta) g_{cm}(\mathbf{b}, \psi) = \sum_{\beta, c} t_{am|\beta c}(\mathbf{a}, \phi; \mathbf{b}, \psi) g_{\beta c}(\mathbf{x}, \theta) ,
\end{aligned}
\]

where

\[
t_{am|\beta c}(\mathbf{a}, \phi; \mathbf{b}, \psi) = g^{-1}_{\alpha \beta}(\mathbf{a}, \phi) g_{cm}(\mathbf{b}, \psi) ,
\]

or, introducing the composed and ordered indices \( A = (\alpha m) \), \( B = (\beta c) \), we have

\[
g_A(\mathbf{x}', \theta') = \sum_B t_{A|B}(\mathbf{a}, \phi; \mathbf{b}, \psi) g_B(\mathbf{x}, \theta) .
\]
One can consider (7) as a vector representation of the group Π in the space of “vector functions” $g_B(x, \theta)$. For $b = \psi = 0$, this is a left vector representation of the group $M(2)$, and for $a = \phi = 0$ this is a right vector representation of the same group.

Let us define another representation of the group Π that acts on scalar functions $f(g(x, \theta))$ of the vectors $g_B(x, \theta)$.

$$T(a, \phi; b, \psi)f(g(x, \theta)) = f(t(a, \phi; b, \psi)g(x, \theta)) = f(g^{-1}(a, \phi)g(x, \theta)g(b, \psi)).$$  \hspace{1cm} (9)

One can verify that (9) is actually a representation. In what follows, it is called the regular representation. For this representation, the left (for $b = 0, \psi = 0$) and right (for $a = 0, \phi = 0$) regular representations of the group $M(2)$ are subrepresentations. Thus, the generators of the group Π in the regular representation consist of all the generators of the right and left regular representations of the group $M(2)$.

An expansion of the left or right regular representation contains (with accuracy up to an equivalence) all irreps of the group $M(2)$.

For the left and right generators (corresponding to the parameters $a_k, \phi$ and $b_k, \psi$), we have

$$\hat{p}_k = -i\partial_k, \quad \hat{J} = \hat{L} + \hat{S},$$

$$\hat{p}_k^R = iV_k^i\partial_i, \quad \hat{J}^R = -\hat{S},$$  \hspace{1cm} (10)

$$\hat{L} = i(x^1\partial_2 - x^2\partial_1), \quad \hat{S} = -i\frac{\partial}{\partial \theta}.$$  \hspace{1cm} (11)

An invariant measure on the group has the form $d\mu(x, \theta) = (4\pi)^{-1}dx_1dx_2d\theta$, where integration is taken over the manifold $R^2 \times S^1$.

The group $M(2)$ is a three-parameter one; therefore, a maximal set of commuting operators includes three operators. In accordance with the general theory, this set is formed by the Casimir operators plus the equal number of left and right generators. We can select this set as the Casimir operator $\hat{p}_L^2 = \hat{p}_R^2$ plus the left generator $\hat{J}$ and the right generator $\hat{S}$, i.e., the operators of squared momentum, total momentum and intrinsic momentum.

Let us return to relation (9) that determines the action of the group $M(2) \times M(2)$. In the general case, irreps of this group are characterized by two (generally) different numbers. In the case under consideration, the representation $T_{\Pi}(g, h)$ acts on functions of three variables only, and in this space one can only construct a part of irreps of $M(2) \times M(2)$, the Casimir operators of the subgroups $M(2)$ being identical, $\hat{p}_L^2 = \hat{p}_R^2$. At the same time, the projections $J_L$ and $J_R$ may take various values.

Note that there is a certain inequivalence of left and right transformations in a given interpretation. If we consider a point like object without an orientation, we cannot relate it to any definite b.r.f., and, consequently, we cannot determine right transformations. In this case, the intrinsic momentum equals to zero, so it is sufficient to examine only the left transformations, the related two quantum numbers $(p_1, p_2$ or $p_2, J = L)$ completely characterize a non-orientable object.

Two dimensions, however, do not allow one to perceive, in full measure, the peculiarity inherent in the description of orientable objects, because the subgroup of rotations $SO(2)$
yields $\xi$ b.r.f.. Index being "external", the second being "internal") under the rotations of the s.r.f. and matrix, whereas they are vector components with respect to internal transformations.

It is obvious that in the first case the quantities $v_k$ fixed reference frames. It is obvious that in the first case the quantities $v_k$ satisfy the condition $v_k^iv_l^i = \delta_{kl}$, and, therefore, the elements of the matrix $V = ||v_k^i||$ satisfy the condition $v_k^iv_l^i = \delta_{kl}$, i.e., the matrix $V$ is orthogonal, $V^T = V^{-1}$.

Thus, the orientation of a three-dimensional rotator is determined by a $3 \times 3$ orthogonal matrix, $V \in O(3)$, composed of the coefficients of a re-decomposition of the bases (s.r.f. and b.r.f.). If both systems $\{e_i\}$ and $\{\xi_k\}$ are right or left, the matrix $V \in SO(3)$ depends on three real-valued parameters, which can be chosen as the Euler angles:

$$V = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \phi \cos \psi \cos \theta - \sin \phi \sin \psi & -\sin \phi \cos \psi - \cos \phi \sin \psi \cos \theta & \cos \phi \sin \theta \\ \sin \phi \cos \psi \cos \theta + \cos \phi \sin \psi & -\sin \phi \sin \psi \cos \theta & \cos \phi \sin \theta \\ -\cos \phi \sin \theta & \sin \phi \sin \psi & \cos \theta \end{pmatrix}. \quad (13)$$

We now examine two kinds of transformations: rotations of the space-fixed and body-fixed reference frames. It is obvious that in the first case the quantities $\{e_i\}$ transform as vector components, whereas in the second case they remain intact, i.e., they are scalars with respect to internal transformations (rotations of the body). On the contrary, the quantities $\{\xi_k\}$ are scalars with respect to external transformations (rotations of s.r.f.), whereas they are vector components with respect to internal transformations.

The elements $v_k^i$ specify the body orientation with respect to s.r.f., and we can consider $v_k^i$ as coordinate set of rotator. Let us describe the transformations of the set $v_k^i$ (the first index being "external", the second being "internal") under the rotations of the s.r.f. and b.r.f.

In the matrix notations, we can present (12) as $\xi = eV$. A rotation of the s.r.f. $e' = e\Lambda$ yields $\xi = eV = e'\Lambda^{-1}V = e'V'$, whence

$$V' = \Lambda^{-1}V. \quad (14)$$

2 Non-relativistic three-dimensional rotator

2.1 General

To describe a rigid rotator (gyroscope), we will use two orthonormalized reference frames: s.r.f. $\{e_i, \ i = 1, 2, 3\}$ and b.r.f. $\{\xi_k, \ k = 1, 2, 3\}$,

$$\xi_k = e_i v_k^i. \quad (12)$$

The scalar product $(e_i, e_j) = \delta_{ij}$, and, therefore, the elements of the matrix $V = ||v_k^i||$ depend on three real-valued parameters, which can be chosen as the Euler angles:

$$V = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \phi \cos \psi \cos \theta - \sin \phi \sin \psi & -\sin \phi \cos \psi - \cos \phi \sin \psi \cos \theta & \cos \phi \sin \theta \\ \sin \phi \cos \psi \cos \theta + \cos \phi \sin \psi & -\sin \phi \sin \psi \cos \theta & \cos \phi \sin \theta \\ -\cos \phi \sin \theta & \sin \phi \sin \psi & \cos \theta \end{pmatrix}. \quad (13)$$

We now examine two kinds of transformations: rotations of the space-fixed and body-fixed reference frames. It is obvious that in the first case the quantities $\{e_i\}$ transform as vector components, whereas in the second case they remain intact, i.e., they are scalars with respect to internal transformations (rotations of the body). On the contrary, the quantities $\{\xi_k\}$ are scalars with respect to external transformations (rotations of s.r.f.), whereas they are vector components with respect to internal transformations.

The elements $v_k^i$ specify the body orientation with respect to s.r.f., and we can consider $v_k^i$ as coordinate set of rotator. Let us describe the transformations of the set $v_k^i$ (the first index being "external", the second being "internal") under the rotations of the s.r.f. and b.r.f.

In the matrix notations, we can present (12) as $\xi = eV$. A rotation of the s.r.f. $e' = e\Lambda$ yields $\xi = eV = e'\Lambda^{-1}V = e'V'$, whence

$$V' = \Lambda^{-1}V. \quad (14)$$
A rotation of the b.r.f. $\xi' = \xi \Lambda$ yields $\xi = \xi' \Lambda^{-1} = eV$, whence

$$V' = V \Lambda,$$

(15)

and, therefore, external transformations correspond to left-multiplication, whereas internal ones correspond to right-multiplication.

The general transformations have the form

$$V' = \Lambda^{-1}V \Lambda,$$

(16)

By analogy with the matrix $V$ that determines the orientation, the rotation matrices $\Lambda$ and $\Lambda^{-1}$ are parameterized by three Euler angles. In the representation (16), generators are given by the standard $3 \times 3$ matrices. In addition, the matrices of generators of transformations (14), (15) have the same form (however, their action is different, being related to left- and right-multiplication).

To find generators of an arbitrary irrep of $SO(3)$, one has to examine representations in the space of functions on the group, i.e., functions $f(\phi, \psi, \theta)$ of the rotator orientation.

The left regular representation $T_L(g)$ acts in the space of functions $f(q), q = q(\phi, \psi, \theta) \in SO(3)$, on the group as follows:

$$T_L(g)f(q) = f'(q) = f(g^{-1}q), \quad g \in G,$$

(17)

which corresponds to a change of the s.r.f.; see (14); whereas the right regular representation $T_R(h)$ acts in the same space as follows:

$$T_R(h)f(q) = f'(q) = f(qh), \quad h \in G,$$

(18)

which corresponds to a change of the b.r.f.; see (15). The decomposition of the left (and right) regular representation contains any irrep of the group.

Each set of the left and right transformations forms the group $SO(3)$. Since these two transformation sets commute with each other, we can consider them as the direct product $\Pi = SO(3) \times SO(3)$. The transformations from $\Pi$ act in the space of functions depending on three parameters (on the rotator orientation) as follows:

$$T_{\Pi}(g, h)f(q) = f(g^{-1}qh) = f'(q).$$

(19)

It is obvious that the generators of $\Pi$ in this representation consist of the generators of the subgroups $SO(3)$ (17) and (18).

For generators that correspond to the one-parameter subgroup $\omega(t)$ in the left and right regular representations, we have

$$\hat{J}_\omega f(q) = -i \lim_{t \to 0} \frac{f(\omega^{-1}(t)q) - f(q)}{t}, \quad \hat{I}_\omega f(q) = -i \lim_{t \to 0} \frac{f(q\omega(t)) - f(q)}{t},$$

(20)

the multiplier $i$ provides the hermiticity of the generators. Accordingly, the operators of finite transformations corresponding to these one-parameter subgroups are given by

$$T_L(\omega(t)) = \exp(i\hat{J}_\omega t), \quad T_R(\omega(t)) = \exp(i\hat{I}_\omega t).$$
Let us denote by $\phi(t), \psi(t),$ and $\theta(t)$ the Euler angles of an element $\omega^{-1}(t)q$. Then,

$$\hat{J}_\omega f(q) = -i \frac{d(f(\omega^{-1}(t)q))}{dt} \bigg|_{t=0} = -i \left( \frac{df}{d\phi}\phi'(0) + \frac{df}{d\theta}\theta'(0) + \frac{df}{d\psi}\psi'(0) \right).$$

Having denoted by $\phi(t), \psi(t),$ and $\theta(t)$ the Euler angles of the element $q\omega(t)$, we obtain an analogous formula for the right generators.

Let us choose the one-parameter subgroups as follows:

$$\omega_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformations $\omega_k^{-1}(t)q$ correspond to rotations about the axes $e_k$, whereas $q\omega_k(t)$ correspond to rotations about $\xi_k$. Direct calculations yield the following expressions for the generators of the s.r.f. rotations

$$\hat{J}_1 = -i \left( \cos \phi \frac{\partial}{\partial \phi} - \sin \phi \frac{\partial}{\partial \theta} - \cos \phi \cot \theta \frac{\partial}{\partial \phi} \right),$$

$$\hat{J}_2 = -i \left( \sin \phi \frac{\partial}{\partial \phi} + \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \right),$$

$$\hat{J}_3 = -i \frac{\partial}{\partial \phi}, \quad (21)$$

and b.r.f. rotations

$$\hat{I}_1 = -i \left( \cos \psi \frac{\partial}{\partial \phi} - \sin \psi \frac{\partial}{\partial \theta} - \cos \psi \cot \theta \frac{\partial}{\partial \psi} \right),$$

$$\hat{I}_2 = i \left( \sin \psi \frac{\partial}{\partial \phi} + \cos \psi \frac{\partial}{\partial \theta} - \sin \psi \cot \theta \frac{\partial}{\partial \psi} \right),$$

$$\hat{I}_3 = i \frac{\partial}{\partial \psi}. \quad (22)$$

It is easy to see that all the right generators commute with all the left generators,

$$[\hat{J}_i, \hat{I}_k] = 0, \quad [\hat{J}_i, \hat{J}_k] = i\epsilon^{ikl} \hat{J}_l, \quad [\hat{I}_i, \hat{I}_k] = i\epsilon^{ikl} \hat{I}_l.$$

This follows from the associativity of the group multiplication: in the product $g^{-1}qh$ the result does not depend on whether one multiplies first from the right or from the left.

The quantities $\hat{I}_k$ remain the same with a change of the s.r.f., and, therefore, they are three “external” (coordinate) scalars; however, with a change of b.r.f. they transform as vector components. That is, $\hat{J}_k$ and $e_k$ are “external” vectors and “internal” scalars; $\hat{I}_k$ and $\xi_k$ are “external” scalars and “internal” vectors. The quantities $v^i_k$ possess one “external” and one “internal” index.

Let us construct a minimal set of commuting operators in the space of functions $f(\phi, \psi, \theta)$. The algebra of operators $\hat{I}_k$ has the same commutation relations as the algebra of operators $\hat{J}_k$, and therefore the standard results of the angular momentum theory are immediately valid for them. We obtain rotation multiplets of dimension $2I + 1$, where
$I$ is the integer or half-integer maximal value of projection $K = I_3$ to the fixed axis $\xi_3$, the squared value of the momentum being $\hat{I}^2 = I(I + 1)$. The value of total momentum does not depend on the choice of axes (which can be verified by using the explicit form of generators),

$$J(J + 1) = \hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 = \hat{I}_1^2 + \hat{I}_2^2 + \hat{I}_3^2 = \hat{I}^2 = I(I + 1),$$

and, therefore, the quantum numbers $I$ and $J$ must be identical. The set of quantities that can be measured simultaneously (commuting operators) and characterize the states of a rotator is

$$\hat{J}_3, \quad \hat{I}_3.$$  

(23)

Eigenvalues of these operators are $J(J + 1), M = -J, -J + 1, \ldots, J$, and $K = -J, -J + 1, \ldots, J$. Thus, quantum states of a rotator $|JMK\rangle$ are uniquely determined by the momentum $J$ and its two projections: $M$ to a space-fixed axis, and $K$ to a body-fixed axis. The dimension of the multiplet for a given value of $J$ obviously equals to $(2J + 1)^2$.

An explicit form of the states $|JMK\rangle$ is given by the so-called Wigner D-functions, being the matrix elements of the irreps $T_J(g)$ of the group $SO(3)$.

Wave functions that do not depend of $\psi$ are eigenfunctions of $\hat{I}_3$ with the eigenvalue $K = 0$. In addition, the operators $\hat{J}^k$ (21) acquire the form of the “usual” operators of intrinsic momentum for a non-orientable point particle, which depend only on the two angles $\theta$ and $\phi$. Such states are $|JM\rangle = |JM0\rangle$.

For a given $J$, the action of operators $\hat{J}_k$ on $(2J+1)$ states $|JMK\rangle$ for a fixed $K$ yields a linear combination of states with the same $K$; a similar fact is valid with respect to the action of $\hat{I}_k$ with a fixed projection $M$. The action of $\hat{J}_k$ and $\hat{I}_k$ on a matrix composed of $|JMK\rangle$ can be, therefore, represented as the (respectively, left- and right-) multiplication by $(2J + 1) \times (2J + 1)$ matrices of generators in this representation.

### 2.2 Symmetries

The Hamiltonian of a rotator, $\hat{H}$, due to spatial isotropy, cannot depend explicitly on the orientation of the rotator, and is, therefore, an “external” scalar. That is, the only combination of left generators $\hat{J}_k$, on which it can depend, is the Casimir operator $\hat{J}^2$. However, it can also be a function of the operators $\hat{I}_k$, which are “external” scalars as well.

Consider stationary states of a rotator with a given momentum, restricting ourselves, for simplicity, to quadratic Hamiltonians. Aligning the basis vectors $\xi_k$ with the axes of inertia of the rotator, we present the Hamiltonian as

$$\hat{H} = \sum A_k(\hat{I}_k)^2.$$  

(24)

where $A_k$ are inertia momenta. In the simplest case of a completely symmetric rotator, $A_1 = A_2 = A_3 = A$, the spectrum consists of $(2J + 1)^2$-times degenerate multiplets $|JMK\rangle$ with the energy

$$E_J = AJ(J + 1).$$

A simple solution takes place in the case of axial symmetry. Let the internal axis $\xi_3$ be the axis of symmetry, $A_1 = A_2 = A_\perp$. Then, $\hat{I}_3 = K$ is conserved; however, the multiplet
\[ |J M K \rangle \] is split in \( |K \rangle \) (of course, there remains a \( 2(2J + 1) \)-times degeneration in \( M \) and \( \text{sign} K \)):

\[
E_{JK} = \langle J M K | \hat{A}_\perp \left( (\hat{I}_1)^2 + (\hat{I}_2)^2 \right) + A_3 (\hat{I}_3)^2 | J M K \rangle \\
= A_\perp (J(J + 1) - K^2) + A_3 K^2 = A_\perp J(J + 1) + (A_3 - A_\perp) K^2.
\]

A rotator with an arbitrary relation of inertia momenta requires a more detailed analysis \[19\]. In the basis \( |J M K \rangle \), the Hamiltonian \(24\) has non-vanishing matrix elements, \( H_{KK'} \), with \( \Delta K = 0, \pm 2 \), i.e., only a mixture of states with a definite parity \( K \) is admitted,

\[
\langle J M K | \hat{H} | J M K \rangle = \frac{1}{2} (A_1 + A_2) [J(J + 1) - K^2] + A_3 K^2, \\
\langle J M K | \hat{H} | J M K + 2 \rangle = \frac{1}{4} (A_1 - A_2) [(J - K)(J - K - 1)(J + K + 1)(J + K + 2)]^{1/2},
\]

whereas the degeneration of energy levels in \( \text{sign} K \) is removed.

Thus, the quantum number \( K \), being an eigenvalue of the right generator \( \hat{I}_3 \) of the rotation group, plays an important role in the description of a quantum rotator, and, correspondingly, in molecular and nuclear spectroscopy.

For a completely symmetric rotator, not only the left transformations, but also each of the right transformations, are symmetry transformations of the Hamiltonian \(24\), the symmetry group being \( SO(3) \times SO(3) \). In the case of axial symmetry, only the right rotation, with the generator \( \hat{I}_3 \), around the axis \( \xi_3 \) is a symmetry of the body, the symmetry group being \( SO(3) \times SO(2) \). This symmetry corresponds to an additive quantum number \( K \). Finally, in the case of three different momenta of inertia, the body is not symmetric, and, accordingly, the right transformations with generators \( \hat{I}_k \) are not its symmetries, the symmetry group being \( SO(3) \).

Consequently, whereas the symmetry with respect to the left transformations (changes of the s.r.f.) is interpreted as symmetries of the embedding space, in which the object is contained (in this case, a rigid body), or as external symmetries, the symmetry with respect to the right transformations (changes of the b.r.f.) is interpreted as symmetries of the object itself, or as internal symmetries.

In terms of the Euler angles, the expressions for the generators \(21\)–\(22\) look quite complicated, as well as the composition law. In many cases, it is more convenient to use, instead of the Euler angles \( \psi, \theta, \phi \), complex-valued Cayley–Klein parameters

\[
z^1 = \cos(\theta/2)e^{i(-\phi+\psi)/2}, \quad z^2 = i \sin(\theta/2)e^{i(\phi+\psi)/2}, \quad (25)
\]

which are transformed by a spinor representation of the group \(SU(2) \sim SO(3)\). By introducing \( 2 \times 2 \) matrices, \( E = \sigma^i e_i \) and \( \Xi = \sigma^k \xi_k \), formula \(12\), that determines the relation between s.r.f. and b.r.f., can be presented in the form

\[
\Xi = Z^\dagger EZ, \quad Z^\dagger = Z^{-1}, \quad Z = \begin{pmatrix} z_{11}^1 & z_{12}^1 \\ z_{21}^1 & z_{22}^1 \end{pmatrix} = \begin{pmatrix} z_1^1 & -z_2^2 \\ z_2^2 & z_1^2 \end{pmatrix} \in SU(2). \quad (26)
\]

Rotations of s.r.f. \((14)\) and and b.r.f. \((15)\) correspond to transformations in terms of unitary matrices \( U \) and \( \overline{U} \),

\[
Z' = U^\dagger Z \overline{U}, \quad U, \overline{U} \in SU(2), \quad (27)
\]
and, therefore, the elements of the matrix $Z$, according to (26), have two kinds of spinor indices: the first one being left (external), the second one being right (internal).

The coordinates of the vector $x = x^i e_i$ under the rotations of the reference frame $\{e_i\}$ change as follows:

$$X' = U^\dagger X U, \quad X = \sigma_i x^i,$$

where the matrices $U$ and $-U$ correspond to the same transformation.

Using (28) and the relation $\tilde{U} = \sigma_2 U \sigma_2$, it is easy to see that $\sigma_k^{\alpha \beta} = (\sigma_k)^{\alpha \beta}$ is an invariant tensor. A consequence of the unimodularity of the $2 \times 2$ matrix $U$ is the presence of an invariant antisymmetric tensor $\varepsilon^{\alpha \beta} = -\varepsilon^{\beta \alpha}$, $\varepsilon^{12} = \varepsilon_{21} = 1$. This allows one to lower and rise the spinor indices, $z_\alpha = \varepsilon_{\alpha \beta} z^\beta$, $z^\alpha = \varepsilon^{\alpha \beta} z_\beta$.

In terms of the variables $z^\alpha_a$ and derivatives $\partial^a \alpha = \partial / \partial z^\alpha_a$, the generators take the form

$$\hat{J}_k = \frac{1}{2} (\sigma_k)^{a \beta} z_\beta a \partial^a \alpha, \quad \hat{I}_k = \frac{1}{2} (\sigma_k)^{a \beta} z^\beta a \partial \alpha_a,$$

(29)

An explicit form of the states $|JM_K\rangle$ is given by polynomials of $2J$-th degree placed in the following tables

$$J = 1/2: \quad M \backslash K \quad 1/2 \quad -1/2 \quad M \backslash K \quad 1 \quad 0 \quad -1$$

$$\begin{array}{ccc}
1/2 & z^1 & z^2 \\
-1/2 & \tilde{z}^1 & \tilde{z}^2 \\
\end{array}
\begin{array}{ccc}
1/2 & z^1 & z^2 \\
1/2 & \tilde{z}^1 & \tilde{z}^2 \\
\end{array}
J = 1: \quad M \backslash K \quad 1 \quad 0 \quad -1$$

$$\begin{array}{ccc}
-1 & (z^1)^2 & z^1 \tilde{z}^2 \\
0 & z^1 z^2 & z^1 \tilde{z}^1 - z^2 \tilde{z}^2 \\
1 & (z^2)^2 & \tilde{z}^1 z^2 \\
0 & \tilde{z}^1 \tilde{z}^2 & \tilde{z}^2 \z^1 \\
\end{array}
(30)

The polynomial of second degree $(-1/2) z^\beta a z_\beta a = z^1 \tilde{z}^1 + z^2 \tilde{z}^2 = 1$, being absent from (30), is a group invariant. A scalar product, defined by integration with the invariant measure $d\mu(z)$ on the group $SU(2)$,

$$\int f_1(z) f_2(z) d\mu(z), \quad d\mu(z) = \frac{1}{8\pi^2} \delta(|z^1|^2 + |z^2|^2 - 1) d^2z^1 d^2z^2 = \frac{1}{8\pi^2} \sin \theta d\theta d\phi d\psi,$$

allows one to verify the orthogonality of the states (30) and obtain the normalization coefficients.

As mentioned above, a simultaneous consideration of left and right transformations implies an analysis of representations of the direct product $SU(2) \times SU(2)$. The irreps of $SU(2) \times SU(2)$ are characterized by eigenvalues of two different Casimir operators (the operators of squared total momentum) $\hat{J}^2$ and $\hat{I}^2$. However, in the case under consideration $\hat{J}^2 = \hat{I}^2$, and the states are characterized only by three numbers: the total momentum $J$ and the two projections $M, K$. This is a consequence of the fact that in the case under consideration the commuting sets act in a space of functions depending merely on three parameters. In this space, one can only construct a part of representations of the direct product.
For the sake of clarity, the figure shows the weight diagrams of representations with $J = 1/2$ and $J = 1$. For the left transformations, one mixes the states horizontally, and for the right transformations, vertically. In particular, at $J = 1$, considering only the left or only the right transformations (respectively, at fixed eigenvalues $\hat{I}_3$ and $\hat{J}_3$), we obtain two different sets of three equivalent irreps (in the general case, the number of equivalent irreps in the expansion will be obviously equal to the dimension of this irrep). However, if one examines both kinds of transformations at the same time, then all the nine states with $M, K = -1, 0, 1$ turn out to be related by the rising and lowering operators $\hat{J}_\pm, \hat{I}_\pm$. That is, the diagram of states of a rotator with a fixed total momentum $J$ coincides with the weight diagram of the representation $T_{J,J}$ of the direct product $SU(2) \times SU(2)$.

In the above theory of a non-relativistic rotator, the wave functions $f(z)$ depend only on its orientation. Extending the consideration to translations implies a transition from functions $f(z)$ on the group $\text{Spin}(3) = SU(2)$ to functions $f(x, z)$ on $M(3) = T(3) \otimes \text{Spin}(3)$ – the group of motions of a three-dimensional Euclidean space. The position (coordinates $x_i$) and orientation are then determined by two matrices $(X, Z)$, whereas the transformations $M(3) –$ translations and rotations – by two matrices $(A, U)$.

For a translation and a consequent rotation of the reference frame $\{e_k\}$ by a vector $a$, we have

$$X' = U^\dagger(X - A)U, \quad Z' = U^{-1}Z, \quad A = \sigma_i a^i,$$

where $a^i$ are the components of $a$ in the reference frame $\{\xi_k\}$, whereas for a translation and consequent rotation of the reference frame $\{\xi_k\}$ by a vector $b$, we have

$$X' = X + ZBZ^\dagger, \quad Z' = ZU, \quad B = \sigma_i b^i,$$

where $b^i$ are the components of $b$ in the reference frame $\{\xi_k\}$.

The orientations of b.r.f. and s.r.f. are still related by formula (26). However, for the generators of rotations, instead of (29), we obtain the formulas (similar to (10), (11) in the two-dimensional case)

$$\hat{J}_k = \hat{L}_k + \hat{S}_k^L = -i \varepsilon_{ijk} x_i \partial / \partial x_j + \frac{1}{2}(\sigma_k)^{a}_b z^\beta \vartheta^{a}_\alpha, \quad \hat{I}_k = \hat{S}_k^R = -i (\sigma_k)^{a}_b z^\beta \vartheta^{a}_\alpha, \quad \hat{S}_k = \frac{1}{2} (\sigma_k)^{a}_b z^\beta \vartheta^{a}_\alpha,$$ (33)

that is, the right generators still contain only the intrinsic momentum in b.r.f., whereas the left ones are the sum of the orbital $\hat{L}_k$ and intrinsic $\hat{S}_k^L$ momenta.
3 Description of orientable objects and changes of reference frames

The position of a point-like object in a \(d\)-dimensional Euclidean space is described by space coordinates, \(x^k, k = 1, \ldots, d\) (respectively, by space-time coordinates \(x^\mu, \mu = 0, 1, \ldots, d-1\), in a pseudo-Euclidean space).

For an orientable object, nevertheless, be it a rigid body or a spinning elementary particle, such a treatment is obviously incomplete.

Whereas for a description of non-orientable objects it is sufficient to use one s.r.f., for a description of orientable objects it is convenient to use two orthonormalized reference frames: a s.r.f. \(\{e_i\}\) and a b.r.f. \(\{\xi_k\}\) reference frame,

\[
\xi_k = v^i_k e_i. \tag{34}
\]

For Euclidean spaces, \((e_i, e_j) = \delta_{ij}\), and the matrix \(V = \|v^i_k\|\), composed of the coefficients of a re-decomposition of the bases (s.r.f. and b.r.f.), is orthogonal, \(V^{-1} = V^T\). For pseudo-Euclidean spaces (and, in particular, the four-dimensional Minkowski space) the matrix \(V\) is pseudo-orthogonal, \(V^{-1} = \eta V^T \eta, \eta = \text{diag}(1, -1, \ldots, -1)\).

That is, besides \(d\) spatial coordinates \(x^\mu\), a description of an orientable object requires to use \(d(d-1)/2\) parameters that determine a \(d \times d\) (pseudo)orthogonal matrix \(V\). Thus, the complete set of its coordinates is \((x^i, v^i_k)\).

Let us consider the transformation law for \((x^i, v^i_k)\) at the changes of s.r.f. and b.r.f. Consider first rotations. In the matrix notation \((34)\) reads \(\xi = eV\). A rotation of the s.r.f. \(e' = e\Lambda\), where \(\Lambda\) is the matrix of rotations, yields \(\xi = eV = e'\Lambda^{-1}V\), whence \(V' = \Lambda^{-1}V\).

A rotation of the b.r.f. \(\xi' = \xi\Lambda\) yields \(\xi = eV = \xi'\Lambda^{-1}, \text{ whence } V' = V\Lambda\). Both for rotations and translations, simple calculations lead to the following results.

For rotations and translations, simple calculations lead to the following results.

A rotation of the s.r.f. with a consequent translation:

\[
x' = \Lambda^{-1}(x - a), \quad V' = \Lambda^{-1}V, \tag{35}
\]

where the column \(a\) is composed of the coordinates \(a_\mu\) of the translation vector in the s.r.f.. A translation of the b.r.f. with a consequent rotation:

\[
x' = x + Vb, \quad V' = V\Lambda, \tag{36}
\]

where the column \(b\) is composed of the coordinates of the translation vector in the b.r.f.; \(Vb\) are the coordinates of the same vector in the s.r.f.. It is easy to see that rotations of the s.r.f. do not affect the coordinates of the body \(x\), but only its orientation \(V\).

Let us now present these transformations in terms of the group of motions. The group of motions of a Euclidean space is a group of transformations that preserves the distance

\[
r^2 = \delta_{ik}(x^i - y^i)(x^k - y^k) \tag{37}
\]

between two points. The group of motions of a pseudo-Euclidean space preserves the interval

\[
s^2 = \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu), \tag{38}
\]

15
where \( \eta = \text{diag}(1, -1, \ldots, -1) \) is the metric tensor of the Minkowski space.

The transformations consist of translations and rotations (for a pseudo-Euclidean space also of boosts (hyperbolic rotations)). If \( x \) are the coordinates of a point in a basis \( \{ e_i \} \), then, as a result of a rotation and translation, we obtain a point with the coordinates

\[
x' = \Lambda x + a. \tag{39}
\]

Each element \( g \) of the group of motions corresponds to a pair of matrices, \( g = (a, \Lambda) \), where \( a \) is a column of the elements \( a^i \) or \( a^\mu \), corresponding to translations; \( \Lambda \) is a matrix of \( SO(d) \) or \( SO(d - 1, 1) \).

Those transformations \( (39) \) that can be continuously connected with the identity form a Lie group called the proper Poincaré group \( M_0(d - 1, 1) \). The corresponding homogenous transformations \( (a = 0) \) form the proper Lorentz group \( SO_0(d - 1, 1) \). In the Euclidean space, we have \( M_0(d) \) and \( SO_0(d) \), respectively. The law of composition and the inverse element of the Poincaré group have the form

\[
(g_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1), \quad g^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}), \tag{40}
\]

whence it follows that the groups \( M_0(d - 1, 1) \) and \( M_0(d) \) are semi-direct products:

\[
M_0(d - 1, 1) = T(d) \otimes SO_0(d - 1, 1), \quad M_0(D) = T(d) \otimes SO_0(d),
\]

where \( T(d) \) is the group of \( d \)-dimensional translations.

Note that an element \( g \) of the group \( M_0(d) \) or \( M_0(d - 1, 1) \) can also be associated with one \( (d + 1) \times (d + 1) \) matrix, whereas the transformation \( (39) \) takes the form

\[
g \Longleftrightarrow \left( \begin{array}{cc} \Lambda & a \\ 0 & 1 \end{array} \right), \quad \left( \begin{array}{c} x' \\ 1 \end{array} \right) = \left( \begin{array}{cc} \Lambda & a \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} x \\ 1 \end{array} \right).
\]

A pair \( (x, V) \) uniquely determines the position and orientation of the b.r.f. with respect to the s.r.f.. That is, the manifold of the Poincaré group (as observed in \([7, 12]\)) is isomorphic to the space of all b.r.f. This pair can be associated with an element of the group \( q \), \( q \leftrightarrow (x, V) \). It is easy to see that a change of the s.r.f. \( (35) \) corresponds to left-multiplication by group elements \( g^{-1} \):

\[
q' = g^{-1} q \leftrightarrow (x', V') = (a, \Lambda)^{-1}(x, V) = (\Lambda^{-1}(x - a), \Lambda^{-1}V), \tag{41}
\]

whereas a change of the b.r.f. \( (36) \) corresponds to right-multiplication by group elements \( h \):

\[
q' = qh \leftrightarrow (x', V') = (x, V)(b, \Lambda) = (x + Vb, V\Lambda). \tag{42}
\]

In the matrix form, the general transformation \( q' = g^{-1}qh \) is given by

\[
\left( \begin{array}{cc} V' & x' \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \Lambda & a \\ 0 & 1 \end{array} \right)^{-1} \left( \begin{array}{cc} V & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \Lambda & b \\ 0 & 1 \end{array} \right). \tag{43}
\]

As to the choice of variables that describe the orientation of an object, one can mention the following:
As such variables, one can select \( d(d-1)/2 \) independent variables (angles); however, the law of composition presented in terms of these angles is sufficiently involved even in the case of \( d = 3 \), not to mention the higher dimensions.

One can select \( d^2 \) real-valued parameters, being the elements \( v^i_k \) of a (pseudo)orthogonal matrix \( V \), which is the matrix of the vector representation of the group \( SO_0(d) \) or \( SO_0(d-1,1) \).

Having in mind the different transformation laws (41) and (42), we are going to denote the elements of the matrix \( V \) as \( v^\mu_n \), with the Greek indices \( \mu, \nu, \ldots \) for the left transformations of the Poincaré group and with the underlined Latin indices \( \underline{\mu}, \underline{n}, \ldots \) for the right transformations. Note that in 3+1 dimensions \( v^\mu_n \) are given by tetrads, i.e., objects that transform as vectors (in the first index \( \mu \)) under the change of a laboratory (space-fixed) reference frame and as vectors (in the second index \( n \)) under the change of a local (body-fixed) reference frame.

On the other hand, one can select elements of the spinor representation for spaces of three and four dimensions: these are the elements of a complex-valued \( 2 \times 2 \) matrix \( Z = ||z^\alpha_2|| \). The composition law in this case has an especially simple form; besides, such a parameterization allows one to describe half-integer spins. In other words, having at one’s disposal \( z^\alpha_2 \), one can obtain \( v^\mu_n \), whereas the inverse operation is two-fold. It is this parameterization that will be used in what follows. We underline “right-hand” indices in order to avoid misunderstanding, since we shall examine quantities with fixed values of indices (for instance, spinors \( z^\alpha_1 \) and \( z^\alpha_2 \)).

### 4 Parameterization of the Poincaré group

First we recall that there exists a well-known one-to-one correspondence between the 4-vectors \( x \) and \( 2 \times 2 \) Hermitian matrices \( X \) \(^1\).

\[
X = x^\mu \sigma_\mu = \begin{pmatrix}
x^0 + x^3 & x^1 - ix^2 \\
x^1 + ix^2 & x^0 - x^3
\end{pmatrix}, \quad \det X = x^\mu x^\mu, \quad x^\mu = \frac{1}{2} \text{Tr}(X \bar{\sigma}^\mu).
\] (44)

One ought to say that this correspondence plays an important role in twistor theory \([20,21,22]\). If \( x \) is subject to the transformation

\[
x' = gx, \quad x'^\nu = \Lambda^\nu_\mu x^\mu + a^\nu,
\]

then \( X \) transforms (see, for example, \([20,23,24]\)) as follows:

\[
X' = gX = UXU^\dagger + A,
\] (45)

where \( A = a^\mu \sigma_\mu \), and the complex matrices \( U \) obey the conditions

\[
\sigma_\nu \Lambda^\nu_\mu = U \sigma_\mu U^\dagger.
\] (46)

\(^1\)We use two sets of \( 2 \times 2 \) matrices \( \sigma_\mu = (\sigma_0, \sigma_k) \) and \( \bar{\sigma}_\mu = (\sigma_0, -\sigma_k) \),

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
The representation of the Poincaré transformations in the form (45) is closely related to a representation of finite rotations in \( \mathbb{R}^d \) in terms of the Clifford algebra. In higher dimensions the transformation law has the same form, where \( A \) is a vector element and \( U \) corresponds to an invertible element (spinor element) of the Clifford algebra \([25]\).

Using (46), we get \( \det U = e^{i\phi} \). Matrices \( U \) that differ only by a phase factor correspond to the same \( \Lambda \), and we can fix this arbitrariness by imposing the condition \( \det U = 1 \). However, even after this two matrices \( (U, -U) \) correspond to one \( \Lambda \). Considering both \( U \) and \( -U \) as representatives for \( \Lambda \), we, in fact, go over from \( SO_0(3,1) \) to its double covering group \( \text{Spin}(3,1) = SL(2,\mathbb{C}) \), \( U = \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \in SL(2,\mathbb{C}) \), \( u_1^1 u_2^2 - u_1^2 u_2^1 = 1 \), (47) and from \( M_0(3,1) = T(4) \rtimes SO(3,1) \) to \( M(3,1) = T(4) \rtimes \text{Spin}(3,1) \).

As is known, this allows one to avoid double-valued representations for half-integer spins. Thus, there exists a one-to-one correspondence between elements \( g \) of \( M(3,1) \) and two \( 2 \times 2 \) matrices, \( g \leftrightarrow (A, U) \). The first one, \( A \), corresponds to translations and the second one, \( U \), corresponds to rotations. Eq. (45) describes the action of \( M(3,1) \) in the Minkowski space (the latter is the coset space \( M(3,1)/\text{Spin}(3,1) \)).

As a consequence of (45), one can obtain the composition law and the inverse element:

\[
(A_2, U_2)(A_1, U_1) = (U_2A_1 U_2^\dagger + A_2, U_2 U_1), \quad g^{-1} = (-U^{-1} A(U^{-1})^\dagger, U^{-1}).
\]

(48)

The matrices \( U \) satisfy the identity

\[
\sigma_2 U \sigma_2 = (U^T)^{-1}.
\]

(49)

An equivalent picture arises in terms of the matrices \( \overline{X} = \sigma^\mu \sigma_\mu \). Using the relation \( \overline{X} = \sigma_2 X^T \sigma_2 \), the transformation law for \( X \) (45), and identity (49), one can get

\[
\overline{X}' = (U^\dagger)^{-1} X U^{-1} + \overline{A},
\]

(50)

Thus, \( \overline{X} \) are transformed by means of the elements \( (\overline{A}, (U^\dagger)^{-1}) \). The relation \( (A, U) \rightarrow (\overline{A}, (U^\dagger)^{-1}) \) determines an automorphism of the group \( M(3,1) \). In the Euclidean case, the matrices \( U \) are unitary, and the latter relation is reduced to \( (A, U) \rightarrow (-A, U) \).

5 Regular representation and spin-coordinate space

In the above chosen representation, the position and the translation parameters are given by a Hermitian \( 2 \times 2 \) matrix \( X \), whereas the orientation and rotations are given by a complex-valued \( 2 \times 2 \) matrix, which we denote as \( Z, Z \in SL(2,\mathbb{C}) \). Using the \( 2 \times 2 \) matrices \( E = \sigma^\mu e_\mu \) and \( \Xi = \sigma^2 \xi_2 \), we can rewrite formula (33), which determines the mutual orientation of the laboratory and body-fixed reference frames, in the form

\[
\Xi = Z^\dagger E Z.
\]

(51)
Making a comparison of (34) and (51), we obtain
\[
\sigma^\mu_{\underline{\alpha}} \varepsilon_\mu = Z^\dagger \sigma^\mu_{\underline{\alpha}} Z,
\]
or, in terms of the components,
\[
(\sigma^\mu_{\underline{\mu}})_\mu = z^\beta_{\underline{\beta}} (\sigma^\mu_{\underline{\beta}})_{\underline{\alpha}} z^\alpha_{\underline{\alpha}}.
\]
Multiplying this by \((\bar{\sigma}^\mu)^{\underline{\alpha}}_{\underline{\beta}}\) and using the identity \(\text{Tr} \sigma^\mu_{\underline{\alpha}} \bar{\sigma}^\nu_{\underline{\beta}} = (\sigma^\mu_{\underline{\alpha}})^{\underline{\beta}}_{\underline{\alpha}} = 2\eta^\mu_{\nu}\), we find
\[
v^\mu_{\underline{\mu}} = \frac{1}{2} (\sigma^\mu_{\underline{\alpha}})^{\underline{\beta}}_{\underline{\alpha}} (\bar{\sigma}^\nu_{\underline{\beta}})_{\underline{\alpha}} z^\alpha_{\underline{\alpha}}.
\]
where \(v^\mu_{\underline{\mu}}\) are elements of pseudoorthogonal matrix \(V \in SO(3,1), V^{-1} = \eta V^T \eta\), and obey the orthogonality conditions
\[
v^\mu_{\underline{\mu}} v^\nu_{\underline{\nu}} = \delta^\mu_{\nu}, \quad v^\mu_{\underline{\mu}} v^\nu_{\underline{\nu}} = \delta^\nu_{\mu}.
\]
Thus, the pair \((X, Z)\) uniquely identifies the position and orientation of the b.r.f. with respect to the s.r.f.; besides, a change of the s.r.f. corresponds to the left-multiplication by \((A, U)^{-1}\), while a change of the b.r.f. corresponds to the right-multiplication by \((A, U)\):
\[
(X', Z') = (A, U)^{-1}(X, Z)(A, U) = (U^(-1)(X - A)(U^\dagger)^{-1} + ZAZ^\dagger, U^{-1}ZU).\]
Let us now examine functions of the coordinates and orientation, i.e., functions defined on the Poincaré group, \(f(q), q \in M(3,1)\). The action of the group \(M(3,1) \times M(3,1)\) in the space of functions \(f(q)\) is given by
\[
T(g, h)f(q) = f'(q) = f(g^{-1}qh),
\]
\[
q \leftrightarrow (X, Z), \quad g \leftrightarrow (A, U), \quad h \leftrightarrow (A, U).
\]
As a consequence of (55), we have
\[
f'(q') = f(q), \quad q' = gqh^{-1}.
\]
In view of the relations
\[
X = x^\mu \sigma_\mu, \quad Z = \begin{pmatrix} z^1_{\underline{1}} & z^1_{\underline{2}} \\ z^2_{\underline{1}} & z^2_{\underline{2}} \end{pmatrix} \in SL(2, C),
\]
the mapping \(q \leftrightarrow (X, Z)\) leads to the correspondence
\[
q \leftrightarrow (x, z), \quad \text{where} \quad x = (x^\mu), \quad z = (z^\alpha_{\underline{b}}),
\]
\[
\mu = 0, 1, 2, 3, \quad \alpha, b = 1, 2, \quad z^1_{\underline{1}}z^2_{\underline{2}} - z^2_{\underline{1}}z^1_{\underline{2}} = 1,
\]
and relation (57) takes the form
\[
f'(x', z') = f(x, z), \quad (x', z') \leftrightarrow q' = gqh^{-1}.
\]
The action of the left GRR \(T_L(g)\) in the space of functions \(f(q)\) on the group
\[
T_L(g)f(q) = f'(q) = f(g^{-1}q),
\]
is related to the change of the s.r.f.; see (41); and, correspondingly,
\[
f'(q') = f(q), \quad q' = gq.
\]
On the other hand, we have the correspondence \( q' \leftrightarrow (x', z') \),
\[
q' = gq \leftrightarrow (X', Z') = (A, U)(X, Z) = (UXU^+ + A, UZ) \leftrightarrow (x', z'),
\]
\[
X' = UXU^+ + A \quad \Rightarrow \quad x'^\mu = \Lambda^{\mu}_\nu x^\nu + a^\mu,
\]
\[
Z' = UZ \quad \Rightarrow \quad z'^{\alpha}_{\beta} = U^\alpha_\beta z^{\beta}_{\beta}, \quad U = (U^\alpha_\beta),
\]
Then, relation (62) takes the form
\[
f'(x', z') = f(x, z), \tag{63}
\]
\[
x'^\mu = \Lambda^{\mu}_\nu x^\nu + a^\mu, \quad \Lambda \in SO_0(3, 1) \leftrightarrow U \in SL(2, C), \tag{64}
\]
\[
z'^{\alpha}_{\beta} = U^\alpha_\beta z^{\beta}_{\beta}. \tag{65}
\]
Relations (63)–(65) admit a remarkable interpretation. We may treat \( x \) and \( x' \) in these relations as position coordinates in the Minkowski space \( M(3, 1)/SL(2, C) \) (in different Lorentz reference frames) related by the proper Poincaré transformations, and the sets \( z \) and \( z' \) (coordinates on \( SL(2, C) \)) may be treated as spin coordinates in such frames. Carrying two-dimensional spinor representation of the Lorentz group, the variables \( z^{\alpha}_{\beta} \) and \( z'^{\alpha}_{\beta} \) are invariant under translations as one can expect for spin degrees of freedom.

Therefore, we may treat the sets \( (x, z) \) as points in the position-spin space with the transformation law (64), (65) under the change from one Lorentz reference frame to another. In this case, equations (63)–(65) present the transformation law for scalar functions in the position-spin space.

On the other hand, as has been demonstrated, the sets \( (x, z) \) are in one-to-one correspondence to group \( M(3, 1) \) elements. Thus, the functions (60) are still functions on this group. For this reason, we often call them scalar functions on the group as well, recalling that the term “scalar” originates from the above interpretation.

Generally speaking, the functions \( f(x, z) \) are not analytical functions of the complex variables \( z^{\alpha}_{\beta} \), i.e., they depend on both \( z^{\alpha}_{\beta} \) and the complex-conjugate \( \bar{z}^{\dot{\alpha}}_{\dot{\beta}} \). Correspondingly, we shall later regard the functions \( f(x, z) \) as functions of the variables \( x^\mu, z^{\alpha}_{\beta}, \bar{z}^{\dot{\alpha}}_{\dot{\beta}} \).

Consider now the right GRR \( T_R(h) \). This representation is defined in the space of functions \( f(q), q \in M(3, 1) \) as follows:
\[
T_R(h)f(q) = f'(q) = f(qh), \tag{66}
\]
The action of a right GRR corresponds to a change of the b.r.f.; see (42). As a consequence of relation (66), one can write
\[
f'(q') = f(q), \quad q' = qh^{-1} \leftrightarrow (X', Z') = (X - ZU^{-1}A(U^{-1})^\dagger Z^\dagger, ZU^{-1}), \tag{67}
\]
whence
\[
f'(x', z') = f(x, z), \quad x'^\mu = x^\mu - \nu^\mu_m (\Lambda^{-1})^\mu_\nu a^\nu, \quad \Lambda \in SO_0(3, 1), \tag{68}
\]
\[
z'^{\alpha}_{\beta} = U^\alpha_\beta z^{\beta}_{\beta}, \tag{69}
\]
where \( \nu^\mu_m \), which determine the orientation of the b.r.f., are expressed in terms of \( z \); see (52).
These transformations are essentially different from the Lorentz transformations (64), (65) in the extended spin-coordinate space. For the parameters of right translations \( a^\mu = 0 \), according to (68), we have \( x'^\mu = x^\mu \), i.e., the right rotations lead only to a change of orientation, and, as distinct from the left rotations, do not affect the space-time coordinates \( x^\mu \). On the other hand, the right translations (68), as distinct from the left ones (64), create a "mixture" of the space coordinates \( x^\mu \) with the spin (orientation) coordinates \( z^i \).

Besides, it is easy to see that, whereas the left transformations of the group \( M(3,1) \) never affect the interval,
\[
g^{-1}X - g^{-1}Y = X' - Y' = U^{-1}(X - Y)(U^{-1})^\dagger,  
\]
\[
s'^2 = \det(X' - Y') = \det(X - Y) = s^2;  
\]
the right transformations at \( A \neq 0 \) do not affect the interval only on condition that \( Z_1 = Z_2 \), which corresponds to an equal orientation of b.r.f.:
\[
Xh - Yh = X'' - Y'' = X - Y + (Z_1 - Z_2)A(Z_1^\dagger - Z_2^\dagger),  
\]
\[
s''^2 = \det[X - Y + (Z_1 - Z_2)A(Z_1^\dagger - Z_2^\dagger)].  
\]

The generators of the left GRR and of the right GRR are applied for a classification of scalar functions on the Poincaré group, because they enter the maximal set of commuting operators on the group.

Let us consider the space of functions on the Poincaré group. An invariant measure on the group has the form \( d\mu(x,z) = d^4xd\mu(z) \), where \( d\mu(z) = (i/2)^3d^2z_1^2d^2z_2^2d^2z_3^2 |z_1|^{-2} \) is the measure on the Lorentz group. If a GRR acts in the space of all functions on the group \( G \), then a regular representation acts in the space of functions \( L^2(G,\mu) \), such that the norm
\[
\int \hat{f}(g)f(g)d\mu(g)  
\]
is finite, where \( d\mu(g) \) is an invariant measure on the group [26, 27]. The regular representation is unitary, as it follows from (72), as well as from the invariance of the measure.

On the other hand, polynomials in \( z \) and \( \bar{z} \) carry finite-dimensional nonunitary representations of the Lorentz group and therefore integral (72) diverges in this case. Thus, we use different spaces of functions on the Poincaré group: \( L^2(G,\mu) \) for unitary representations of principal series (corresponding to infinite-component fields in Majorana type equations) and a space of polynomials in \( z, \bar{z} \) for finite-component representations (corresponding to spin-tensor fields) of the Lorentz group.

6 A field on the Poincaré group and spin-tensor fields

We shall now discuss a relation between the description of orientable objects, in particular, higher spins, in terms of scalar functions \( f(x, z) \) on the Poincaré group, with the standard description in terms of multi-component fields.

\[ ^2 \text{A decomposition of the regular representation does not include unitary irreps of the auxiliary series, characterized by the nonlocal scalar product } \int \hat{f}(\hat{g}_1)f(\hat{g}_2)I(\hat{g}_1, \hat{g}_2)d\mu(\hat{g}_1)d\mu(\hat{g}_2), \text{ where the kernel } I(\hat{g}_1, \hat{g}_2) \text{ is invariant with respect to group transformations, } \hat{g}_k \in G/H, H \subset G. \text{ However, such representations have not been applied in physics so far.} \]
Spin-tensor fields that describe particles of different spins are defined by a transformation law corresponding to a change of s.r.f. These fields are related to multi-component functions on the Minkowski space (i.e., functions depending not only on \(x\) but also on a certain discrete parameter).

The relation \(f'(q') = f(q), q' = gq\), connected with the left GRR \((61)\), also determines a law of field transformation with the change of s.r.f.; however, this transformations act in the extended (spin-coordinate) space. Scalar functions on the Poincaré group \(f(q), q = (x, z)\), depend not only on \(x\) but also on the set of the variables \(z\):

\[
f'(x', z') = f(x, z), \quad x' = gx = \Lambda x + a \leftrightarrow UXU^\dagger + A, \quad z' = g z \leftrightarrow UZ. \tag{73}
\]

In contrast to a scalar field in the Minkowski space, this field is reducible not only with respect to mass, but with respect to spin as well. It is well known \([23, 27, 26]\) that any irrep of the group \(G\) is contained (up to an equivalence) in the decomposition of the left (or right) GRR. Thus, the task of classification of \(M(3, 1)\) irreps reduces to the task of classification of scalar functions \((73)\).

In this section, we examine only left transformations; besides, for the sake of brevity, we omit the second ("right") index of \(z^\alpha_2\), because the elements of the first and second columns \(z^\alpha_1, z^\alpha_2\) of the matrix \((58)\) transform under the action of the left GRR of \(M(3, 1)\) in the same way.

According to \((73)\) and \((50)\), one can write the transformation law of \(x^\mu, z_\alpha, \bar{z}_\dot{\alpha}\) in component-wise form

\[
x^\nu \sigma_{\mu\alpha\dot{\alpha}} = U^\alpha_\beta x^\mu \sigma_{\mu\beta\dot{\beta}} U^\dot{\beta}_\dot{\alpha} + a^\mu \sigma_{\mu\alpha\dot{\alpha}}, \quad x^\nu \bar{\sigma}^{\dot{\alpha}\dot{\beta}} = (U^{-1})^\dot{\alpha}_{\dot{\beta}} x^\mu \bar{\sigma}^{\dot{\alpha}_{\dot{\beta}}} (U^{-1})^\beta_\alpha + a^\mu \bar{\sigma}^{\dot{\alpha}_{\dot{\beta}}}, \tag{74}
\]

\[
z'_\alpha = U^\alpha_\beta z_\beta, \quad \bar{z}'_\dot{\alpha} = U^\dot{\alpha}_{\dot{\beta}} \bar{z}_\beta, \quad z'^{\alpha} = (U^{-1})^\alpha_\beta z^\beta, \quad \bar{z}'^{\dot{\alpha}} = (U^{-1})^\dot{\alpha}_{\dot{\beta}} \bar{z}^{\dot{\beta}}. \tag{75}
\]

It is easy to see from \((74)\) that the tensors

\[
\sigma_{\mu\alpha\dot{\alpha}} = (\sigma_{\mu})_{\alpha\dot{\alpha}}, \quad \bar{\sigma}^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}_{\dot{\mu}})^{\dot{\alpha}\dot{\beta}}, \tag{76}
\]

are invariant. These tensors are usually applied to convert the vector indices into spinor ones and vice versa, or to construct vector from two spinors of different types:

\[
x_{\alpha\dot{\alpha}} = (X)_{\alpha\dot{\alpha}} = \sigma_{\mu\alpha\dot{\alpha}} x^\mu, \quad x^\mu = \frac{1}{2} \bar{\sigma}^{\mu\dot{\alpha}} x_{\alpha\dot{\alpha}}, \quad q^\mu = \frac{1}{2} \bar{\sigma}^{\mu\dot{\alpha}} z^{\alpha\dot{\alpha}}. \tag{77}
\]

In consequence of the unimodularity of \(2 \times 2\) matrices \(U\) there exist invariant antisymmetric tensors \(\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\beta}\dot{\alpha}}, \varepsilon^{12} = \varepsilon^{12} = 1, \epsilon_{12} = \varepsilon_{12} = -1\). Now spinor indices are lowered and raised according to the rules

\[
z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^{\alpha} = \varepsilon^{\alpha\beta} z_\beta, \tag{78}
\]

and in particular one can get \(\bar{\sigma}_{\mu\alpha\dot{\alpha}} \equiv \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon^{\alpha\beta} \bar{\sigma}_{\dot{\mu}}^{\dot{\beta}} = \sigma_{\mu\alpha\dot{\alpha}}\). Below we will also use the notations

\[
\partial_\alpha = \partial/\partial z^\alpha, \quad \partial^{\dot{\alpha}} = \partial/\partial z^{\dot{\alpha}}, \quad \bar{\partial}^\alpha = \varepsilon^{\alpha\beta} \partial_\beta = -\partial/\partial z_\alpha, \quad \partial_{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\beta}} = -\partial/\partial z^{\dot{\alpha}}. \tag{79}
\]
In the framework of the present approach, a standard spin description in terms of multicomponent functions arises under the separation of space and spin variables.

Since \( z \) is invariant under translations, any function \( \phi(z) \) carries a representation of the Lorentz group. Let a function \( f(x, z) \) allow the representation

\[
\begin{align*}
  f(x, z) &= \phi^n(z)\psi_n(x),
\end{align*}
\]

where \( \phi^n(z) \) form a basis in the representation space of the Lorentz group. The latter means that one can decompose the functions \( \phi^n(z') \) of the transformed argument \( z' = g z \) in terms of the functions \( \phi^n(z) \):

\[
\phi^n(z') = \phi^l(z)L^n_l(U).
\]

An action of the Poincaré group on a line \( \phi(z) \) composed of \( \phi^n(z) \) is reduced to the multiplication by a matrix \( L(U) \), where \( U \in \text{Spin}(3,1), \phi(z') = \phi(z)L(U) \).

As one compares the decompositions of the function \( f'(x', z') = f(x, z) \) over the transformed basis \( \phi(z') \) and over the initial basis \( \phi(z) \),

\[
\begin{align*}
  f'(x', z') &= \phi(z')\psi'(x') = \phi(z)L(U)\psi'(x') = \phi(z)\psi(x),
\end{align*}
\]

where \( \psi(x) \) is a column with components \( \psi_n(x) \), one obtains

\[
\psi'(x') = L(U^{-1})\psi(x),
\]

i.e., the transformation law of a spin-tensor field in Minkowski space. This law corresponds to the representation of the Poincaré group acting in a linear space of tensor fields as follows

\[
T(g)\psi(x) = L(U^{-1})\psi(L^{-1}(x - a)).
\]

According to (81) and (82), the functions \( \phi(z) \) and \( \psi(x) \) transform according to contragradient representations of the Lorentz group (we recall that the representation \( [T(g^{-1})]^T \) is called contragradient to \( T(g) \)).

For example, let us consider scalar functions on the Poincaré group \( f_1(x, z) = \psi_\alpha(x)z^\alpha \) and \( f_2(x, z) = \bar{\psi}_\alpha(x)z^\alpha \), which correspond to spinor representations of Lorentz group. According to (80) and (82)

\[
\psi'_\alpha(x') = U_\alpha^\beta\psi_\beta(x), \quad \bar{\psi'}_\alpha(x') = \bar{U}_\alpha^\beta\bar{\psi}_\beta(x).
\]

The product \( \psi_\alpha(x)\bar{\psi}^{*\alpha}(x) \) is Poincaré-invariant.

Thus, tensor fields of all spins are contained in the decomposition of the field (78) on the Poincaré group, and the problems of their classification and construction of explicit realizations are reduced to the problem of a decomposition of the left GRR.

The field \( f(x, z) \) itself may be regarded as the generating function of usual multicomponent spin-tensor fields; the latter arise as the coefficients of a series in the powers of the orientation (spin) variables \( z \). Below we write out generating function (147) for spin 1/2 and generating function (165) for spin 1 (see also [16] where some other examples for 2+1 and 3+1 dimensions are contained).

Notice that we have rejected the phase transformations \( U = e^{i\phi} \). These transformations of the \( U(1) \) group do not change the space-time coordinates \( x \), but change the phase of \( z \). According to (81) and (82), this leads to a phase transformation of the tensor field components \( \psi_n(x) \). Taking account of this transformations implies a consideration of functions on the group \( T(4) \times \text{Spin}(3,1) \times U(1) \).
7 The maximal set of commuting operators

Let us construct the maximal set of commuting operators, which will be used afterwards to classify fields on the group. Using the above parameterization, we obtain

\[ T_L(g)f(x, z) = f(g^{-1}x, g^{-1}z), \quad g^{-1}x \leftrightarrow U^{-1}(X - A)(U^{-1})^\dagger, \quad g^{-1}z \leftrightarrow U^{-1}Z, \quad (84) \]
\[ T_R(g)f(x, z) = f(xg, zg), \quad xg \leftrightarrow X + ZAZ^\dagger, \quad zg \leftrightarrow ZU. \quad (85) \]

In view of (84), \( x \) transforms according to the vector representation of the Lorentz group, whereas \( z \) transforms according to the spinor representation. If we restrict the consideration to functions that do not depend on \( z \), then (84) reduces to the transformations of the left quasi-regular representation, that corresponds to the case of a usual scalar field \( f'(x') = f(x) \). If, however, we restrict the consideration to functions that do not depend on \( x \), then (84) reduces to the transformations of the left GRR of the Lorentz group.

The generators that correspond to translations and rotations have the form

\[ \hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu}, \quad (86) \]
\[ \hat{p}^R = -v^\nu_m \hat{p}_\nu, \quad \hat{J}_{mn} = \hat{S}_{mn}, \quad (87) \]

where \( \hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \) are the operators of projections of the orbital momentum; \( \hat{S}_{\mu\nu} \) are the operators of spin projections, whereas \( v^\nu_m \in SO_0(3, 1) \) is expressed in terms of \( z \); see (52). The operators of right translations can also be expressed in the form \( \hat{P}^R = -Z \hat{P}Z^\dagger \); the operators \( \hat{S}_{\mu\nu} \) and \( \hat{S}_{mn} \) are he left and right generators of \( SL(2, C) = \text{Spin}(3, 1) \) and depend only on \( z \) and \( \partial/\partial z \),

\[ \hat{S}_{\mu\nu} = i \left( (\sigma_{\mu\nu})_\alpha^\beta \bar{z}^\alpha_\beta \partial_\alpha \bar{z}^\beta_\alpha \right), \quad \hat{S}_{mn} = i \left( (\sigma_{mn})_\alpha^\beta \bar{z}^\alpha_\beta \partial_\alpha \bar{z}^\beta_\alpha \right), \quad (88) \]
\[ \hat{S}_{mn} = i \left( (\sigma_{mn})_\alpha^\beta \bar{z}^\alpha_\beta \partial_\alpha \bar{z}^\beta_\alpha \right), \quad (89) \]

where

\[ (\sigma_{\mu\nu})_\alpha^\beta = \frac{1}{4}(\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu)_\alpha^\beta, \quad (\sigma_{\mu\nu})_\alpha^\beta = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)_\alpha^\beta. \quad (90) \]

To present the spin operators of the group \( M(3, 1) \), it is also useful to apply the three-dimensional vector notation \( \hat{S}_k = \frac{1}{2} \epsilon_{ijk} \hat{S}^i_j, \hat{B}_k = \hat{S}_{k0}, \)

\[ \hat{S}_k = \frac{1}{2} \left( z_\alpha \sigma_k \partial_\alpha - z_\alpha^* \sigma_k \partial_\alpha^* \right), \quad (91) \]
\[ \hat{B}_k = i \left( z_\alpha \sigma_k \partial_\alpha + z_\alpha^* \sigma_k \partial_\alpha^* \right), \quad z_\alpha = (z_1^\alpha z_2^\alpha), \quad \partial_\alpha = (\partial/\partial z_1^\alpha \partial/\partial z_2^\alpha)^T; \quad (92) \]
\[ \hat{S}^R_k = -\frac{1}{2} \left( z_\alpha \sigma_k \partial_\alpha - z_\alpha^* \sigma_k \partial_\alpha^* \right), \quad (93) \]
\[ \hat{B}^R_k = -\frac{i}{2} \left( z_\alpha \sigma_k \partial_\alpha + z_\alpha^* \sigma_k \partial_\alpha^* \right), \quad z_\alpha = (z_1^\alpha z_2^\alpha), \quad \partial_\alpha = (\partial/\partial z_1^\alpha \partial/\partial z_2^\alpha)^T. \quad (94) \]

The algebra of generators (86) has the form

\[ [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{J}_{\mu\nu}, \hat{p}_\rho] = i(\eta_{\rho\nu} \hat{p}_\mu - \eta_{\rho\mu} \hat{p}_\nu), \]
\[ [\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\eta_{\rho\nu} \hat{J}_{\mu\sigma} - i\eta_{\rho\sigma} \hat{J}_{\mu\nu} - i\eta_{\mu\sigma} \hat{J}_{\nu\rho} + i\eta_{\mu\rho} \hat{J}_{\nu\sigma}. \quad (95) \]
The right generators (87) obey the same commutation relations. Since the group multiplication is associative, \((g_1^{-1}hg_2) = g_1^{-1}(hg_2)\), each of the right generators (87) commutes with each of the left generators (86).

In the space of Fourier transforms

\[
\varphi(p, z) = (2\pi)^{-d/2} \int f(x, z)e^{ipx}dx,
\]

is not difficult to obtain analogies of formulas (84), (85) in the momenta representation:

\[
T_L(g)\varphi(p, z) = e^{iap}\varphi(p', g^{-1}z), \quad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^+, \quad P = p_\mu\sigma^\mu, \quad (97)
\]

\[
T_R(g)\varphi(p, z) = e^{-i4p}\varphi(p, zg), \quad a' \leftrightarrow A' = ZAZ^+. \quad (98)
\]

Note that \(\det Z\) and \(\det P = m^2\) are invariant with respect to transformations (97), (98). Here, \(m^2\) is an eigenvalue of the Casimir operator \(\hat{p}^2 = \eta^{\mu\nu}p_\mu p_\nu\).

In the Euclidean case (group \(M(3)\)), one has two kinds of representations, depending on the value of the squared momentum \(\det P = p^2\): 1) \(p^2 \neq 0\) (moving particles); 2) \(p^2 = 0\) (particles at rest); in this case \(p_i = 0\), and the irreps are labeled by the eigenvalues of the Casimir operator of the rotation subgroup.

For the group \(M(3,1)\), there exist four kinds of transformations, depending on the eigenvalues of \(m^2\) of the Casimir operator: 1) \(m^2 > 0\) (massive particles); 2) \(m^2 = 0\) (massless particles); 3) \(m^2 < 0\) (tachyons); 4) \(m^2 = 0\); the irreps are labeled by the eigenvalues of the Casimir operator for the Lorentz subgroup; besides, the corresponding functions are independent of \(x\).

To classify functions \(f(x, z)\) on the Poincaré group, we use the maximal set of commuting operators. In accordance with the theory of harmonic analysis on Lie groups \([27, 24]\), there exists a maximal set of commuting operators, which includes Casimir operators, a set of the left generators and a set of the right generators (both sets contain the same number of generators). The total number of commuting operators is equal to the number of parameters of the group. In a decomposition of the left GRR (84), the nonequivalent representations are distinguished by eigenvalues of the Casimir operators, equivalent representations are distinguished by eigenvalues of the right generators, and the states inside the irrep are distinguished by eigenvalues of the left generators. And, vice versa, in an expansion of the right GRR (85), the states inside an irrep are distinguished by the right generators, while equivalent irreps are distinguished by the left generators.

Notice that some aspects of the theory of harmonic analysis on the 3+1 and 2+1 Poincaré groups have been considered in \([28, 29, 30, 31]\) respectively.

Functions on the Poincaré group \(M(3,1)\) depend on 10 parameters, and correspondingly, there exist 10 commuting operators (two Casimir operators and two sets of four operators, constructed from the left (86) and right (87) generators).

The Poincaré group and the (spinorial) Lorentz subgroup have two Casimir operators each:

\[
\hat{p}^2 = p_\mu\hat{p}^\mu, \quad \hat{W}^2 = \hat{W}_\mu\hat{W}^\mu, \quad \text{where} \quad \hat{W}^\mu = \frac{1}{2}e^{\mu\rho\sigma}(p_\nu\hat{J}_{\rho\sigma} + \frac{1}{2}e^{\nu\rho\sigma}p_\nu\hat{S}_{\rho\sigma}), \quad (99)
\]

\[
\frac{1}{2}\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} = \frac{1}{2}\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} = \hat{S}^2 - \hat{B}^2, \quad \frac{1}{2}e^{\mu\rho\sigma}\hat{S}_{\mu\nu}\hat{S}_{\rho\sigma} = \frac{1}{16}e^{\mu\rho\sigma}\hat{S}_{\mu\nu}\hat{S}_{\rho\sigma} = \hat{S}\hat{B}. \quad (100)
\]
we have 4 numbers which are eigenvalues of some combinations of right generators of the Poincaré group. Here, in addition to the 2 Casimirs and 4 left generators of the group, commuting operators (101) that are combinations of the left and right generators of the group are characterized already by 10 quantum numbers. These numbers are related to a set of group representations of transformation of both s.r.f. and b.r.f., an oriented object can be some components of the momentum and a spin projection.

Including four left generators \( \hat{p}_\mu \) (the eigenvalue of the Casimir operator \( \hat{p}^2 \) obviously is expressed in terms of their eigenvalues), the Lubanski–Pauli operator \( \hat{W}^2 \), and the operator of helicity \( \hat{p}\hat{S} \), expressed in terms of the left generators. Besides the Casimir operators and functions of the left generators, the set also includes four functions of the right generators: \( \hat{S}^2 - \hat{B}^2 \), \( \hat{S}\hat{B} \), \( \hat{S}_3^R \), \( \hat{B}_3^R \). The first two of them are the Casimir operators of the Lorentz group and determine the characteristics \( j_1, j_2 \) of the irreps \( T_{j_1j_2} \) of this group. Here, \( \hat{S}^2 = \hat{S}^k\hat{S}_k = \hat{S}_R^k\hat{S}_R^k \), \( \hat{B}^2 = \hat{B}^k\hat{B}_k = \hat{B}_R^k\hat{B}_R^k \).

It is known that \( \hat{S}^k\hat{S}_k \) and \( \hat{B}_k \) can be linearly combined into \( \hat{M}_k \) and \( \hat{\bar{M}}_k \),

\[
\hat{M}_k = \frac{1}{2}(\hat{S}_k - i\hat{B}_k) = \frac{1}{2}z^2\sigma_k\partial_2, \\
\hat{\bar{M}}_k = -\frac{1}{2}(\hat{S}_k + i\hat{B}_k) = \frac{1}{2}z^2*\sigma_k\partial_2, 
\]

so that \([\hat{M}_k, \hat{\bar{M}}_k] = 0\); in addition, unitary representations of the Lorentz group, according to the condition \( \hat{S}_k^\dagger = \hat{S}_k, \hat{B}_k^\dagger = \hat{B}_k \), must satisfy the relation \( \hat{M}_k^\dagger = \hat{\bar{M}}_k \) (for finite-dimensional non-unitary representations \( \hat{S}_k^\dagger = \hat{S}_k, \hat{B}_k^\dagger = -\hat{\bar{B}}_k \) and \( \hat{M}_k^\dagger = -\hat{\bar{M}}_k \)).

Taking into account the fact that \( \hat{M}_k \) and \( \hat{\bar{M}}_k \) obey the commutation relations of the algebra \( \mathfrak{su}(2) \), we find the spectrum of the Casimir operators of the Lorentz subgroup as follows:

\[
\hat{\bar{S}}^2 - \hat{\bar{B}}^2 = 2(\hat{M}^2 + \hat{\bar{M}}^2) = 2j_1(j_1 + 1) + 2j_2(j_2 + 1) = -\frac{1}{2}(k^2 - \rho^2 - 4), \\
\hat{\bar{S}}\hat{\bar{B}} = -i(\hat{\bar{M}}^2 - \hat{\bar{M}}^2) = -i(j_1(j_1 + 1) - j_2(j_2 + 1)) = k\rho,
\]

where \( \rho = -i(j_1 + j_2 + 1), \quad k = j_1 - j_2 \).

Therefore, irreps of the Lorentz group \( SL(2, C) \) are labeled by a pair of numbers, \([j_1, j_2]\). It is convenient to label unitary infinite-dimensional irreps by pairs of numbers \((k, \rho)\); besides the irreps \((k, \rho)\) and \((-k, -\rho)\) are equivalent [24, 33].

In the conventional description of relativistic particles (in \( 3 + 1 \) dim.) in terms of spin-tensor fields, based, in particular, on a classification of Poincaré and Lorentz group representations, there appear 8 particle characteristics (quantum numbers): 2 numbers \( j_1 \) and \( j_2 \) that label Lorentz group representations and 6 numbers related to the Poincaré group, those are the mass \( m \), spin \( s \) (Casimirs of the group), and 4 numbers, which are eigenvalues of some combinations of left generators of the group. In particular, the latter 4 numbers can be some components of the momentum and a spin projection.

In the above proposed description of relativistic oriented objects, based on a classification of group representations of transformation of both s.r.f. and b.r.f., an oriented object is characterized already by 10 quantum numbers. These numbers are related to a set of commuting operators (101) that are combinations of the left and right generators of the Poincaré group. Here, in addition to the 2 Casimirs and 4 left generators of the group, we have 4 numbers which are eigenvalues of some combinations of right generators of the
Poincaré group. Among the latter, we have 2 Casimirs of the (right) Lorentz subgroup that fix $j_1$ and $j_2$.

Thus, as compared with the conventional description, in the proposed one, an oriented object has two more characteristics. Their physical interpretation is a subject of an individual investigation.

In addition to the set of commuting operators (101), we use the generator of phase transformations of $Z$ (117), which are also symmetry transformations of field on the Poincaré group,

$$\hat{\Phi} = -i\partial/\partial\phi = l\hat{\Gamma}^5,$$

where $l$ labels the irrep of $U(1)$ that governs the transformation of $Z$, the chirality operator being

$$\hat{\Gamma}^5 = \frac{1}{2} \left( \zeta^\alpha \partial^\alpha - \zeta^\beta \partial^\beta \right).$$

In the irreps of the Lorentz group $T_{[j_1,j_2]}$ its eigenvalue is $\Gamma^5 = j_1 - j_2$. It anticommutes with the operator of space reflection $P$, and, therefore, its eigenvalues change their sign under the action of $P$.

8 Fields on the Poincaré group. Symmetries

In the general case, symmetry transformations of a certain object are given by some transformations that leave this object invariant. For physical systems, such transformations are usually regarded as symmetries of a Lagrangian (or those of a Hamiltonian; in this case one separately examines dynamical symmetries, including transitions between the states with different values of energy) of this system and of the corresponding equations.

In a broad sense, a symmetry operator of an equation is an arbitrary operator $\hat{B}$ that transforms solutions $\psi$ of an equation $\hat{L}\psi = 0$ into solutions $\hat{B}\psi$ of the same equation, i.e, an operator that obeys the condition

$$\hat{L}(\hat{B}\psi) = 0$$

for each $\psi$ that belong to the totality of solutions of this equation (see, for example, [34]).

In what follows, it is expedient to specify the class of symmetry operators under consideration (e.g., linear, non-linear, integral, integro-differential, and so on). Of special interest are symmetry operators that belong to the class of linear differential operators of first order, and which may be considered as generators of continuous groups of transformations. These are so-called Lie, or classical, symmetries; “higher” (non-classical) and non-local symmetries are a subject of a separate study (see [34] and references therein).

Free relativistic wave equations are invariant with respect to the group of relativistic symmetry, the Poincaré group. In essence, they are conditions that the corresponding field $\psi$ should belong to a certain representation of this group. For instance, the Dirac equation can be obtained as a condition of belonging to an irrep of the extended (due to discrete transformations) Poincaré group, which is characterized, in particular, by the sign of the product of parity, charge, and energy (an exact formulation is given in [17]). Symmetry transformations of an equation, evidently, map a field carrying some representation into a field carrying the same representation. This observation indicates the possibility to define the notion of field symmetry without appealing to any relativistic wave equations.
So, let a field $\psi$ be transformed according to the representation $T(g)$ of the group $G$. Let us call the operator $\hat{B}$ a symmetry operator of this field (in a broad sense), in case the field $\hat{B}\psi$ is transformed according to the same representation.

Consider symmetry transformations using the example of a scalar field on a certain homogenous space of the group $G$. The representation in the space of scalar functions on the homogenous space $G/K$, $K \in G$ (the so-called left quasi-regular representation) is given by

$$T(g)f(y) = f'(y) = f(g^{-1}y), \quad g \in G, \quad y \in G/K,$$

or $f'(y') = f(y)$, where $y' = gy$. Let us denote the space of scalar functions in $G/K$ as $V_{G/K}$.

If the subgroup $K$ is trivial, we have an important particular case. In this case, the coset space is identical with the group $G$ itself and we deal with the GRR (61), and, correspondingly, with a scalar field on the entire group (62).

According to the above definition, symmetry transformations of the scalar field on $G/K$ are mappings of a field into itself:

$$f(y) \in V_{G/K} \rightarrow \hat{B}[f(y)] \in V_{G/K}.$$  \hspace{1cm} (108)

We shall impose the following natural restrictions on $\hat{B}$:

1) the operator $\hat{B}$ is invertible;
2) the transformation $\hat{B}[\psi(y)]$ reduces to a change of function arguments on a homogenous space (change of coordinates) $\hat{B}(f(y)) = f(\tilde{y})$.

That is, $\tilde{y} = \hat{B}y$, where $\hat{B}$ acts on $G/K$, and, due to the existence of $\hat{B}^{-1}$, the mapping $y \rightarrow \tilde{y}$ is bijective.

Acting by the operator $\hat{B}$ on (107), we obtain

$$\hat{B}T(g)\hat{B}^{-1}\hat{B}\psi(y) = \hat{B}\psi(g^{-1}y),$$

$$\hat{B}T(g)\hat{B}^{-1}\psi(\tilde{y}) = \psi(\hat{B}g^{-1}\hat{B}^{-1}\tilde{y}), \quad \hat{B}T(g)\hat{B}^{-1} = T(\tilde{g}),$$

where $T(\tilde{g})f(\tilde{y}) = f(\tilde{g}^{-1}\tilde{y})$, and the mapping $g \rightarrow \tilde{g} = \hat{B}g\hat{B}^{-1}$ preserves the law of group composition, and, therefore, determines an automorphism of the group $G$.

One can say that the symmetry transformations of the field $f(y) \rightarrow f(\tilde{y})$ are generated by these automorphisms.

Let us consider the Poincaré group $M(3,1)$. From the physical point of view, it is interesting to consider the special case of spaces that include the Minkowski space as their subspace. Notice that there exist 11 spaces of this kind [6].

First of all, let us examine a scalar field $f(x)$ in the Minkowski space, $x \in M(3,1)/\text{Spin}(3,1)$. The inner automorphisms

$$g \rightarrow g_0gg_0^{-1}, \quad x \rightarrow g_0x$$  \hspace{1cm} (110)

correspond to the left finite transformations of the Poincaré group (proper Lorentz transformations), i.e., to changes of the s.r.f.. The outer automorphisms of the Poincaré group

$$g \rightarrow \hat{B}g\hat{B}^{-1}, \quad x \rightarrow \hat{B}x$$  \hspace{1cm} (111)

correspond to space- and time-reflections, as well as to scale transformations (dilatations).

A scalar field $f(q)$ on the Poincaré group, $q \in M(3,1)$, is a special case. In comparison with the case of homogenous spaces $G/K$ with a nontrivial subgroup $K$, in this case there
exist a larger variety of symmetries. Namely, one can multiply $q$ by an element of the group not only from the left but also from the right, and, therefore, we consider a representation $\mathbb{T}(g, h)$ of the direct product $M(3, 1) \times M(3, 1)$, $\mathbb{T}(g, h)f(q) = f(g^{-1}qh)$.

Symmetry transformations of the general form correspond to the automorphisms 

$$(g, h) \rightarrow (\tilde{g} = B_1 g B_1^{-1}, \tilde{h} = B_2 h B_2^{-1})$$

of the group $M(3, 1) \times M(3, 1)$. They are generated by three kinds of transformations of $q$:

\begin{align*}
&g \rightarrow g_0gg_0^{-1}, \quad h \rightarrow h, \quad q \rightarrow g_0q \quad \text{(proper Lorentz transformations);} \\
&g \rightarrow g, \quad h \rightarrow h_0hh_0^{-1}, \quad q \rightarrow qh_0^{-1} \quad \text{(right transformations);} \\
&g \rightarrow \hat{B}g\hat{B}^{-1}, \quad h \rightarrow \hat{B}h\hat{B}^{-1}, \quad q \rightarrow \hat{B}q\hat{B}^{-1} \quad \text{(outer automorphisms).}
\end{align*}

In comparison with the symmetries of a scalar field in the Minkowski space (110), (111), we have additional symmetries (113).

Instead of the right transformations (113) corresponding to a change of the s.r.f., we can consider inner automorphisms being a composition of left and right transformations,

$$g \rightarrow h_0gh_0^{-1}, \quad h \rightarrow h_0hh_0^{-1}, \quad q \rightarrow h_0qh_0^{-1} \quad \text{(inner automorphisms).}$$

Therefore, a scalar field on the group, besides the symmetry with respect to the proper Lorentz transformations, also possesses nontrivial symmetries related to inner $q \rightarrow h_0qh_0^{-1}$ and outer (space and time reflections $I_S, I_T$, as well as scale transformations $(A, U) \rightarrow (cA, U)$, $(X, Z) \rightarrow (cX, Z)$) automorphisms. Notice that, unlike the proper Lorentz transformations, the field symmetries corresponding to automorphisms, generally do not preserve the interval (in particular, this is valid for the scale transformations).

The symmetries (112) correspond to a change of the s.r.f., whereas (113) correspond to a change of the b.r.f. Outer automorphisms (114), however, act simultaneously on both reference frames. It is necessary to obtain a group element $\tilde{q} \in G$ as a result of the transformation. Let us explain this using the example of a rotator, where the matrix $V \in SO(3)$ relates s.r.f. and b.r.f., see (12), (13). As a result of a space reflection of one of the frames, we find that they are now related by a matrix $V'$, $\det V' = -1$, and, therefore, $V' \notin SO(3)$.

If one explicitly indicates $i$ as an argument of functions on the group, then complex conjugation

$$f(x, z, i) \rightarrow f(x, \bar{z}, -i)$$

(116)

can also be considered as a change of arguments of a function. As will be seen below, this corresponds to charge conjugation (see also [16, 17]).

The phase transformations $Z' = e^{il\varphi}Z$ that do not affect $x^\mu$ can also be considered as symmetry transformations of a field; however, on a larger space, i.e., fields on the group $M(3, 1) \times U(1)$ (instead of the unimodularity condition $\det Z = 1$, one has $| \det Z | = 1$),

$$z \rightarrow ze^{il\varphi}.$$  

(117)

A real-valued number $l$ determines one-dimensional irreps of the group $U(1)$. These transformations commute with transformations from $M(3, 1)$, so that the order of multiplication is not important, and, therefore, these transformations can be treated as both (112) and
However, in view of the fact that, as distinct from the Lorentz transformations (112), they do not affect $x^\mu$, it is natural to consider them as a particular case of the right transformations (113), where $g_0 \in M(3,1) \times U(1)$.

A scalar field on the Poincaré group $f(x,z)$ has a larger variety of symmetries. Indeed, let us examine the action of right transformations on a usual scalar field $f(x)$. Right rotations and boosts, in accordance with (68), do not affect $x$, i.e., they are reduced to the identity transformation. Right translations are not symmetry transformations, since, in accordance with (68), they map functions $f(x)$ on a homogenous space into functions $f(x,z)$ on the entire group. Further, in the case of fields carrying a fixed nonzero mass, symmetries do not include scale transformations which are a particular case of automorphisms (114), and so on.

The physical sense of the right transformations (113) can be clarified by the example of the compact subgroup of rotations $SO(3) \sim SU(2)$, which describes a rotator. According to the above consideration, in the case of a nonrelativistic rigid rotator left transformations (changes of s.r.f.) correspond to external symmetries (symmetries of the embedding space), whereas right transformations (changes of b.r.f.) correspond to symmetries of the rotating body itself. However, if the body is not symmetric, then only one part of the right symmetries of a field on the group $SU(2)$ corresponds to the symmetries of the rotator Hamiltonian (24), while the remaining part of the right symmetries are violated. In a similar way, the transformations (112) of a scalar field on the Poincaré group, i.e., the proper Lorentz transformations, correspond to external symmetries, and one can suppose that (113) corresponds to internal symmetries. Below, we present a more detailed analysis.

### 9 Discrete symmetries

A particular case of symmetries of a scalar field on the Poincaré group is given by discrete symmetry transformations, which cannot be related continuously with the identity transformation. These transformations are generated by two outer automorphisms of the type (114): the space $I_s$ (also denoted by $P$, as the transformation of space parity) and the time $I_t$ reflections, as well as the complex conjugation of functions on the group (116), which corresponds to charge conjugation $C$.

In fact, one has to consider a larger class of transformations, namely compositions of the above-mentioned outer automorphisms and internal, or equivalently, some right transformations (namely, right rotations) of the Poincaré group. It is precisely these combinations that present the symmetry transformations of various important Hamiltonians and relativistic wave equations. In other words, outer automorphisms and right transformations must be examined simultaneously.

Therefore, it is expedient to approach the definition of discrete symmetries from a different point of view. Let us define them as such symmetry transformations of a scalar field on the Poincaré group that satisfy the following conditions:

(i) A squared discrete transformation equals to the identity transformation;

(ii) The totality of discrete transformations forms a group.

From (ii), it follows that a product of discrete transformations is also a discrete transformation and its square equals to the identity operator. Hence follows that discrete transformations commute with each other.
Among the proper Lorentz transformations (112), condition (i) is satisfied only by the rotation by the angle $2\pi$. Condition (i) is also met by transformations generated by involutive (both outer (114) and inner (115)) automorphisms of the group. However, it is only the subset of the indicated automorphisms that satisfy condition (ii). Finally, the above definition is also met by the complex conjugation (116) of functions on the group.

In [17], it was shown that a field on $M(3,1)$ has only six independent transformations that obey the above definition, namely, the rotation by $2\pi$, the complex conjugation of functions on the group, two outer automorphisms (the space reflection $I_s$ and the time reflection $I_t$), and two inner automorphisms.

Automorphisms determine transformations of the space-time and orientation coordinates $(x,z)$. A substitution of the transformed coordinates into functions $f(x,z)$ (or generators (86), (87)) leads to a change of the sign of some physical quantities. (Notice that a simultaneous change of variables in the expressions for the generators and functions $f(x,z)$ leaves the signs unchanged.)

Let us start from outer automorphisms. In the case of space reflection $I_s$:

$$e_\mu \to -(-1)^{\delta_{0\mu}} e_\mu, \quad \xi_m \to -(-1)^{\delta_{0m}} \xi_m,$$

or in terms of the $2 \times 2$ matrices:

$$E = \sigma^\mu e_\mu \to \sigma^2 E \sigma^2, \quad \Xi = \sigma^m \xi_m \to \sigma^2 \Xi \sigma^2.$$

Hence, for coordinates of a vector $x = x^\mu e_\mu$ we have $x^\mu \to -(-1)^{\delta_{0\mu}} x^\mu$, or $X \to \overline{X}$, whereas the orientation variables, according to (49) and (51), are found in the form $Z \to \sigma^2 Z \sigma^2 = (Z^\dagger)^{-1}$. Thus, the space reflection corresponds to an outer automorphism of the Poincaré group:

$$I_s : \quad (X,Z) \to (\overline{X},(Z^\dagger)^{-1}). \quad (118)$$

In the case of space reflection, $x$ and $z$ have to be replaced in all of the constructions, according to (118). In particular, the momentum is found to be $P \to \overline{P}$, where $\overline{P} = p_\mu \sigma^\mu$. The generators of space rotations remain unchanged, whereas the generators of boosts do change their sign.

In a similar way, in the case of time reflection, $e_\mu \to (-1)^{\delta_{0\mu}} e_\mu, \xi_m \to (-1)^{\delta_{0m}} \xi_m$, we find

$$I_t : \quad (X,Z) \to (-\overline{X},(Z^\dagger)^{-1}). \quad (119)$$

The inversion $I_x = I_s I_t$ corresponds to an automorphism

$$I_x : \quad (X,Z) \to (-X,Z). \quad (120)$$

One can show that, in the framework of characteristics determined by the Poincaré group, complex conjugation (the change $i \to -i$)

$$C : \quad f(x,z) \to f^*(x,z), \quad (121)$$

corresponds to the charge conjugation. Indeed, both (121) and the charge conjugation change signs of all the generators, $\hat{p}_\mu \to -\hat{p}_\mu, \hat{L}_{\mu\nu} \to -\hat{L}_{\mu\nu}, \hat{S}_{\mu\nu} \to -\hat{S}_{\mu\nu}$. The study of relativistic wave equations shows [16] that transformation (121) also changes the sign of the current $j^\mu$. 

31
Time reversal $T$, as well as time reflection $I_t$, is determined by the relation $X \rightarrow -X$, which, however, is valid under the subsidiary condition of the preservation of the sign of energy, which corresponds to $P \rightarrow P$. In consequence, there hold the relations

$$\hat{p}_\mu \rightarrow -(-1)^{\delta_{0\mu}} \hat{p}_\mu, \quad \hat{L}_{\mu\nu} \rightarrow -(-1)^{\delta_{0\mu} + \delta_{0\nu}} \hat{L}_{\mu\nu}, \quad \hat{S}_{\mu\nu} \rightarrow -(-1)^{\delta_{0\mu} + \delta_{0\nu}} \hat{S}_{\mu\nu}.$$

These conditions are met by the transformation $CI_I$.

However, it is known [35, 36] that it is possible to give two distinct definitions of time-reversal transformation obeying the above-mentioned conditions. Wigner’s time reversal $T_w$ leaves the total charge (and, correspondingly, $j^0$) unaltered, and reverses the direction of the current $j^k$. Schrödinger’s time reversal $T_{sch}$ leaves the current $j^k$ invariant and reverses the charge [37]. The transformation $C I_I$ changes the sign of $j^0$, and, therefore, can be identified with Schrödinger’s time reversal, $T_{sch} = CI_I$.

Wigner’s time reversal $T$ and the $CPT$-transformation can be defined by considering both outer and inner automorphisms of the proper Poincaré group [17]. Namely, $CPT = I_x I_z$, where $I_z$ is defined as follows:

$$I_z : \quad (X, Z) \rightarrow (X, Z(-i\sigma_2)), \quad (122)$$

and is a composition of the inner automorphism $(X, Z) \rightarrow (X^T, (Z^T)^{-1})$ (which, in its turn, can be presented as a product of automorphisms [118] and $(X, Z) \rightarrow (X, Z)$) and of a rotation by the angle $\pi$. Wigner’s time reversal is a composition of the above-considered transformations:

$$T = I_x T_{sch} = CI_I I_t. \quad (123)$$

(Discrete transformations $I_z$ and $T$ are compositions of a transformation with a unity square and of a rotation by $\pi$ that changes the signs of two spatial axes. Consequently, instead of (i), we use, in this case, a weaker condition, namely, that the square of a discrete transformation be equal to the identity transformation or the rotation by $2\pi$. This definition does not change anything conceptually; however, it is more convenient technically.)

One can see that $C^2 = P^2 = I_z^2 = 1$. The operators $I_z$ and $I_3$, the latter acts as $(X, Z) \rightarrow (X, Z(-i\sigma_3))$, are products of the involutive inner automorphism and the rotation by $\pi$. Correspondingly, $I_z^2 = I_3^2 = T^2 = R_{2\pi}$, where $R_{2\pi}$ is the operator of a rotation by $2\pi$, which changes the sign of $z$.

Therefore, charge conjugation corresponds to complex conjugation of scalar functions on the group, whereas the remaining five independent transformations correspond to a change of the arguments of scalar functions on the group:

$$\begin{array}{|c|c|c|c|c|}
\hline
 & x^0 & x & \bar{z}^0 & \bar{z}_\alpha & \bar{z}^\alpha & \bar{z}_\alpha^* \\
\hline
R_{2\pi} & x^0 & x & -\bar{z}^0 & -\bar{z}_\alpha & -\bar{z}^\alpha & -\bar{z}_\alpha^* \\
P = I_z & x^0 & -x & \bar{z}^0 & \bar{z}_\alpha & \bar{z}^\alpha & \bar{z}_\alpha^* \\
I_x & -x^0 & -x & \bar{z}^0 & \bar{z}_\alpha & \bar{z}^\alpha & \bar{z}_\alpha^* \\
I_z & x^0 & x & \bar{z}^0 & \bar{z}_\alpha & \bar{z}^\alpha & \bar{z}_\alpha^* \\
I_3 & x^0 & x & -i\bar{z}^0 & i\bar{z}_\alpha & i\bar{z}^\alpha & -i\bar{z}_\alpha^* \\
\hline
\end{array} \quad (124)$$

32
For the sake of clarity, we have used the notation that we applied in [16], $z^\alpha = z_1^\alpha$, $z^\dot{\alpha} = z_2^\dot{\alpha}$, $z^\alpha = z_2^\alpha$, $z^\dot{\alpha} = z_1^\dot{\alpha}$ (the dot over an index duplicates the sign of the complex conjugation of $z$).

The formulas for the transformations $P, C, T$ of four kinds of spinors, denoted usually (see, [38]) by $\xi^\alpha$, $\xi^\dot{\alpha}$, $\eta_\alpha$, $\eta_\dot{\alpha}$, are identical (on condition that $P^2 = 1$) with those in the case of $z^\alpha$, $z^\dot{\alpha}$, $z^\alpha$, $z^\dot{\alpha}$.

Expanding $f(x, z)$ in the powers of $z$, one can obtain the transformation laws for spin-tensors of an arbitrary rank, without any reference to the Dirac equation or some other RWE [17].

In the general case, it is only a part of discrete transformations that presents a symmetry transformation of RWE, since some RWE fix certain characteristics that label representations of the extended (by discrete transformations) Poincaré group. In particular, discrete symmetries of the Dirac and Weyl equations are generated by two unmatched sets of three operators, respectively, $P, C, T$ and $PC, I_x, T$.

10 Classification of scalar functions and equivalent representations

Among functions on the group, there are functions that transform identically under the action of a left GRR (for instance, in the case of $M(3,1)$ these are the functions $f(x^\mu, z_1^\alpha)$ and $f(x^\mu, z_2^\alpha)$). A natural extension of functions that transform “identically” is given by functions that transform by equivalent representations.

Let us recall that representations $T_1(g)$ and $T_2(g)$ acting in linear spaces $L_1$ and $L_2$, respectively, are equivalent in case there exists an nonsingular linear operator $A : L_1 \rightarrow L_2$ such that

$$\hat{A}T_1(g) = T_2(g)\hat{A}. \tag{125}$$

If $L_1$ and $L_2$ are subspaces of the space of functions in $M(3,1)$, then $\hat{A}T_1(g)f_1(x, z) = T_2(g)\hat{A}f_1(x, z)$, and, respectively,

$$f_2(x, z) = (\hat{A}f_1)(x, z),$$

where $f_1(x, z) \in L_1$ and $f_2(x, z) \in L_2$.

In particular, the left and the right GRR of a Lie group $G$ are equivalent. The operator

$$\hat{A}(f)(q) = f(q^{-1}) \tag{126}$$

realizes the equivalence [27, 23]. This is quite natural, since the left and right transformations are two representations of the same abstract Lie group. However, in the case under consideration the left and right transformations have a different geometrical and physical sense, as transformations of s.r.f. and b.r.f., which are transformed one into another by the operator $\hat{A}$. Thus, the former retain the distance (37) or the interval (38) unchanged, as distinct from the latter.

Transformation (126) implies that, starting from the space of functions $f(q), q \leftrightarrow (X, Z)$ that depend on the coordinates of b.r.f. in s.r.f., one goes over to another space,
namely, that of functions $f(q^{-1})$, $q^{-1} \leftrightarrow (-Z^{-1}X(Z^{-1})^\dagger, Z^{-1})$ depending on the coordinates of a s.r.f. in b.r.f.

Let us now consider functions from two different subspaces in the space of functions on the group, which transform identically under the action $T_L(g)$ (e.g., the above-mentioned functions $f(x^\mu, z_\alpha^1)$ and $f(x^\mu, z_\alpha^2)$). A question arises if they describe equal states, and, if they do describe different states, which are physical characteristics that distinguish these states.

Functions on the Poincaré group depend on the variables $z^\alpha_b$ having two kinds of indices: one of them is related to the transformations of a s.r.f., the other is related to the transformations of b.r.f.. Fields corresponding to equivalent subrepresentations of the left GRR transform equally under the proper Lorentz transformations (changes of s.r.f.) and differently under a change of b.r.f. or under automorphisms, in particular, under discrete transformations (space reflection, time reversal, charge conjugation). They can be classified with the help of the right generators of the group, which are contained in the maximal set of commuting operators. Thus, different equivalent subrepresentations of the left GRR describe, in the general sense, different physical situations.

Below, we shall use, whenever necessary, the notation $SL(2,\mathbb{C})_{\text{left}}$ (Lorentz transformations) and $SL(2,\mathbb{C})_{\text{right}}$ for the action from the left and from the right.

### 11 Four kinds of spinors

It is known that the nonrelativistic spin is described by the group $SU(2)$ and there exists only one kind of spinors; in the relativistic theory there exist two kinds of spinors (left and right spinors, transforming differently by boosts and distinguished usually with the help of dotted and undotted spinor indices). A manifest construction of the extended Poincaré group shows that in the relativistic theory with discrete transformations there exist four kinds of spinors, transforming differently by discrete transformations; see (124). Namely, besides the dotted and undotted spinors that transform into each other by the space reflection $P$, it is necessary to distinguish the spinors $z^\alpha_1$ and $z^\alpha_2$, that transform into each other by the CPT-transformation.

The same four kinds of spinors $z^\alpha_1, z^\alpha_2, \dot{z}^\alpha_2, \dot{z}^\alpha_1$, or, using the notation from (16), $z^\alpha, \dot{z}^\alpha, \bar{z}_\dot{\alpha}, \bar{z}_\dot{\alpha}$, appear, in the course of a simultaneous consideration of left and right finite transformations of the Lorentz and Poincaré groups, as columns and rows of the matrices $Z, \dot{Z} \in SL(2,\mathbb{C})$,

$$Z = \begin{pmatrix} z^1_1 & z^1_2 \\ z^2_1 & z^2_2 \end{pmatrix}, \quad \dot{Z} = \begin{pmatrix} \dot{z}^1_1 & \dot{z}^1_2 \\ \dot{z}^2_1 & \dot{z}^2_2 \end{pmatrix},$$

$$Z = \begin{pmatrix} z^1_1 & \dot{z}^1_2 \\ \dot{z}^2_1 & z^2_2 \end{pmatrix}, \quad \dot{Z} = \begin{pmatrix} \dot{z}^1_1 & z^1_2 \\ z^2_1 & \dot{z}^2_2 \end{pmatrix}.$$

(127) (128)

Note that $z^\alpha_2 \dot{z}^\alpha_1 / 2 = \det Z = 1$.

A pair of spinors $(z^\alpha_1, \dot{z}^\alpha_1)$ or differential operators $\partial_\dot{\alpha} = \partial / \partial z^\alpha_1, \partial_\dot{\alpha} = \partial / \partial \dot{z}^\alpha_1$ allows one to construct not only quantities with vector indices of the same kind (e.g., the left
\( S_{\mu\nu} \) (88) and right \( S_{mn} \) (89) generators), but also quantities with indices of two different types, one being right, the other left.

First of all, these are the tetrads \( v_{\mu n} \) (52), the operators
\[
\hat{V}_{12}^{\mu n} = \frac{1}{2} \sigma_{\mu \alpha} \bar{z}_{\alpha}^{\hat{\alpha}} \partial_{\hat{\beta}}^{\beta}, \quad \hat{V}_{21}^{\mu n} = \frac{1}{2} \sigma_{\mu \beta} \bar{z}_{\beta}^{\hat{\alpha}} z_{\alpha}^{\hat{\alpha}} \partial_{\hat{\alpha}}^{\hat{\beta}},
\]
related by the space reflection, and the operator
\[
\hat{V}_{22}^{\mu n} = \frac{1}{2} \sigma_{\mu \beta} \bar{z}_{\beta}^{\hat{\alpha}} \partial_{\hat{\alpha}}^{\hat{\beta}}.
\]
Along with the tetrads, these operators can be used to construct relativistic wave equations. They connect irreps \( T_{[j_1,j_2]} \) of the Lorentz group with different \( j_1, j_2 \); the operators \( \hat{V}_{12}^{\mu n} \) and \( \hat{V}_{21}^{\mu n} \) preserve \( j_1 + j_2 \), while the operators \( \hat{V}_{11}^{\mu n} = v_{\mu n} \) and \( \hat{V}_{22}^{\mu n} \) preserve \( j_1 - j_2 \).

It should also be noted that the subspaces of functions \( f(x^{\mu}, z^{\alpha}, \bar{z}_{\alpha}^{\hat{\alpha}}) \), \( f(x^{\mu}, z^{\alpha}, \bar{z}_{\alpha}^{\hat{\alpha}}) \) are invariant for the operators \( \hat{V}_{12}^{\mu n} \) and \( \hat{V}_{21}^{\mu n} \). Let us denote these subspaces by \( V_- \) and \( V_+ \). The polynomials of degree \( 2s \) in the subspaces \( V_- \) and \( V_+ \) are eigenfunctions of the operator of right rotations \( \hat{S}_R^3 \) with the eigenvalues \( s \) and \(-s \) (since \( z^{\alpha}_1, \bar{z}_{\alpha}^{\hat{\alpha}_1} \) and \( z^{\alpha}_2, \bar{z}_{\alpha}^{\hat{\alpha}_2} \) are eigenvectors for \( \hat{S}_R^3 \) with the respective eigenvalues \( 1/2 \) and \(-1/2 \)).

### 12 Left-invariant relativistic wave equations

Consider, at first, equations that determine the eigenvalues of the Casimir operators \( \hat{p}^2 = \hat{p}_R^2 \) and \( \hat{W}^2 = \hat{W}_R^2 \) for the Poincaré group, those are the Klein-Gordon equation and the Lubanski-Pauli equation:
\[
\hat{p}^2 f(x, z) = m^2 f(x, z), \quad \hat{W}^2 f(x, z) = -m^2 s(s + 1) f(x, z).
\]

They must be satisfied by any free fields with a definite mass \( m \) and spin \( s \). These equations are invariant with respect to the left \( (112) \) and right \( (113) \) transformations, and also with respect to some of the outer automorphisms \( (114) \) (namely, the involutive automorphisms) and the phase transformations \( (117) \). Amongst all the symmetries of a field on the Poincaré group, they do not possess only the symmetry with respect to the part of the outer automorphisms \( (114) \) (namely, scale transformations).

As known, free relativistic particles are usually described on the basis of such RWE as Dirac, Weyl, Duffin-Kemmer, etc. Making a comparison of the latter equations with equations \( (131) \), \( (132) \), we note two aspects closely related to each other. In the first place, some equations of first order contain additional information in comparison with \( (131) \) and \( (132) \) (Lorentz characteristics \( j_1, j_2 \), chirality or inner parity, charge). For example, the Dirac equation entails the existence of a pair of particles related through charge conjugation. Secondly, these equations possess only some of the symmetries (in particular, discrete) of equations \( (131) \), \( (132) \).
The point is that the conventional approach to relativistic wave equations takes into account only the characteristics determined by the left generators of the Poincaré group (these can be used to construct only six commuting operators: the 4-momentum $\hat{p}_\mu$, the operator $\hat{W}^2$, that determines spin $s$, and the spin-projection $\hat{S}^3$).

If we, however, take into account not only the left but also the right transformations, then, instead of two (both dotted and undotted), we now have four kinds of spinors (two pairs, related by charge conjugation), Lorentz characteristics being $j_1, j_2$; altogether, there are four additional quantum numbers (in total, the maximal set of commuting operators contains ten operators, whose number is equal to that of the parameters of the Poincaré group).

Correspondingly, equations containing additional information possess non-trivial transformation properties with respect to the right transformations and can be consistently deduced from group-theory conditions only by taking into account the “right” characteristics, or, at any rate, the characteristics of the Poincaré group being extended by discrete transformations.

It should be noted that the situation with RWE is analogous to the case of a non-relativistic rotator, whose Hamiltonian is invariant with respect to the left transformations (external symmetries), being, however, generally non-invariant under some (or even all) of the right transformations, depending on the degree of violation of an internal symmetry. RWE and the corresponding Lagrangians must be invariant under the left transformations of the Poincaré group (Lorentz transformations), but may be non-invariant under some of the right transformations, which corresponds to the violation of an internal symmetry.

Thus, one may attempt to obtain equation with a broken right (that is, internal) symmetry acting by analogy with the case of a non-relativistic rotator. The Hamiltonian of a non-relativistic rotator is constructed from the right generators $\hat{I}_k$ of the group of rotations $SU(2)$; usually the Hamiltonian $H = H(\hat{I}_k)$ is considered as the sum of squared right generators with various coefficients, see (24).

A Lagrangian for the relativistic rotator can be chosen as a function of right generators of the Lorentz group $SL(2, C)$, $L = L(\hat{S}^R_{mn})$. However, our purpose here is to take into account not only rotational but also translational movement, so we must take account not only of the orientation coordinates $z$, which are used to construct the operators $\hat{S}^R_{mn}$, but also of the space-time coordinates $x^\mu$.

The simplest opportunity is to consider equations for the eigenvalues of the generators of right translations (87),

$$\hat{p}^R_m f(x, z) = \kappa_m f(x, z). \quad (133)$$

However, the generators $\hat{p}^R_m = \hat{p}_\mu v^\mu_m$ do not commute with the Casimir operators (103) of the Lorentz group $SL(2, C)_{\text{right}}$, and therefore the corresponding functions $f(x, z)$ cannot be characterized by any fixed Lorentz characteristics $j_1, j_2$. Furthermore, the explicit form of the operators $\hat{p}^R_m$ implies that the functions $f(x, z)$ must contain arbitrarily large powers in $z$ and the corresponding representation of the Lorentz group must be infinite-dimensional. The above-said is also valid for the left-invariant operators

$$\hat{p}_\mu \hat{S}^{\mu\nu} v^\nu_m. \quad (134)$$

Nevertheless, there exist another possible approach, which is based on the use of the
\[ \hat{\Gamma}^{\mu \nu} = \hat{V}_{12}^{\mu \nu} + \hat{V}_{21}^{\mu \nu} = \frac{1}{2} \left( \sigma^{\mu \alpha \beta} \sigma_{\alpha \beta}^{\nu} \hat{a}_{\beta} \hat{b} \right) + \sigma^{\mu \alpha} \sigma_{\beta}^{\nu} \hat{a}_{\beta} \hat{b} \]  
(135)

\[ \hat{\Gamma}^{\mu \nu} = \hat{V}_{12}^{\mu \nu} - \hat{V}_{21}^{\mu \nu} = \frac{1}{2} \left( \sigma^{\mu \alpha \beta} \sigma_{\alpha \beta}^{\nu} \hat{a}_{\beta} \hat{b} - \sigma^{\mu \alpha} \sigma_{\beta}^{\nu} \hat{a}_{\beta} \hat{b} \right) \]  
(136)

(with one external (left) and one internal (right) index) satisfying the commutation relations

\[ [\hat{\Gamma}^{\mu \nu}, \hat{\Gamma}^{\rho \sigma}] = -i(\hat{S}^{\mu \nu \rho \sigma} + \hat{\eta}^{\mu \nu \rho \sigma}), \]  
(137)

\[ [\hat{S}^{\mu \nu}, \hat{\Gamma}^{\rho \sigma}] = i(\eta^{\rho \sigma \mu \nu} \hat{\eta}^{\rho \sigma} - \eta^{\rho \sigma \mu \nu} \hat{\eta}^{\rho \sigma}), \]  
(138)

\[ [\hat{\Gamma}^{\mu \nu}, \hat{\Gamma}^{\rho \sigma}] = i(\hat{S}^{\mu \nu \rho \sigma} + \hat{\eta}^{\mu \nu \rho \sigma}), \]  
(139)

\[ [\hat{S}^{\mu \nu}, \hat{\Gamma}^{\rho \sigma}] = i(\eta^{\rho \sigma \mu \nu} \hat{\eta}^{\rho \sigma} - \eta^{\rho \sigma \mu \nu} \hat{\eta}^{\rho \sigma}), \]  
(140)

\[ [\hat{\Gamma}^{5}, \hat{\Gamma}^{\mu \nu}] = -\hat{\Gamma}^{\mu \nu}, \quad [\hat{\Gamma}^{5}, \hat{\Gamma}^{\mu \nu}] = \hat{\Gamma}^{\mu \nu}. \]  
(141)

For space reflection, in accordance with (124), we have

\[ P : \quad \hat{\Gamma}^{\mu \nu} \rightarrow (-1)^{\mu + \nu} \hat{\Gamma}^{\mu \nu}, \quad \hat{\Gamma}^{\mu \nu} \rightarrow (-1)^{\mu + \nu} \hat{\Gamma}^{\mu \nu}. \]  
(142)

The operators \( \hat{\Gamma}^{\mu \nu} \) and \( \hat{\Gamma}^{\mu \nu} \) relate the irreps of the Lorentz group \( T_{[j_1, j_2]} \) with the irreps \( T_{[j_1 + j_2 - 1]} \) and \( T_{[j_1 - j_2 + 1]} \). In the irreps \( T_{[j_1, j_2]} \) the eigenvalue of the chirality operator \( \hat{\Gamma}^{5} \) is \( \Gamma^{5} = j_1 - j_2 \), and therefore they join states of different chirality and with a fixed sum \( j_1 + j_2 \), \( |\Gamma^{5}| \leq j_1 + j_2 \) into one multiplet.

Using contractions with respect to the left index \( \hat{p}_\mu \hat{\Gamma}^{\mu \nu} \) and \( \hat{p}_\mu \hat{\Gamma}^{\mu \nu} \), we have 8 left-invariant equations:

\[ (\hat{p}_\mu \hat{\Gamma}^{\mu \nu} - x^\nu) f(x, z) = 0, \]  
(143)

\[ (\hat{p}_\mu \hat{\Gamma}^{\mu \nu} - x^\nu) f(x, z) = 0. \]  
(144)

In accordance with (142), it is only the operators \( \hat{p}_\mu \hat{\Gamma}^{\mu \nu} \) and \( \hat{p}_\mu \hat{\Gamma}^{\mu \nu} \), \( i = 1, 2, 3 \) that are invariant under space reflection, and the associated four equations possess solutions with a definite inner parity.

The commutation relations (138) and (140) entail that the left-invariant equations under consideration also possess some of the right symmetries. If the equations related to the operators \( \hat{\Gamma}^{\mu \nu} \) and \( \hat{\Gamma}^{\mu \nu} \) possess a symmetry with respect to \( SU(2) = \text{Spin}(3) \in SL(2, C)_{\text{right}} \), then the equations related to the operators \( \hat{\Gamma}^{\mu \nu} \) and \( \hat{\Gamma}^{\mu \nu} \), \( i = 1, 2, 3 \), possess a symmetry with respect to three different subgroups \( SU(1, 1) = \text{Spin}(2, 1) \in SL(2, C)_{\text{right}} \).

Let us make a more detailed analysis of the equation

\[ (\hat{p}_\mu \hat{\Gamma}^{\mu \nu} - x^0) f(x, z) = 0, \]  
(145)

\[ ^3 \text{A construction of the basic types of RWE as equations for the eigenvalues of certain sets of commuting operators acting in the space of scalar functions on the Poincaré group was carried out in our paper [16]. However, we did not present a systematic consideration of the properties of these equations with respect to the right transformations.} \]
related to the temporal component of the right 4-vector \( \hat{p}_\mu \hat{\Gamma}^{\mu\alpha} \),

\[
2\hat{\Gamma}^{\mu\alpha} = \bar{\sigma}^{\mu\alpha\beta}(z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta} + z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta}) + \sigma^{\mu\beta\alpha}(z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta} + z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta}) \\
= (\bar{\sigma}^{\mu\alpha\beta} z^\alpha_\beta \frac{\partial}{\partial z^\alpha_\beta} + \sigma^{\mu\beta\alpha} z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta}) + (\bar{\sigma}^{\mu\alpha\beta} z^\alpha_\beta \frac{\partial}{\partial z^\alpha_\beta} + \sigma^{\mu\beta\alpha} z^\beta_\alpha \frac{\partial}{\partial z^\alpha_\beta}).
\] (146)

It is easy to see that the subspaces \( V_- \) (the functions \( f(x^\mu, z^\alpha_1, z^\alpha_2) \)) and \( V_+ \) (the functions \( f(x^\mu, z^\alpha_2, z^\alpha_1) \)) are invariant not only with respect to the operators \( \hat{p}_\mu \hat{\Gamma}^{\mu\alpha} \), but also for the operators of parity \( P \) and time reversal \( T \). The action of the operators \( \hat{\Gamma}^{\mu\alpha} \) on a row \((z^1 z^2 z^3 z^4)\) amounts to the multiplication of the latter by the 4x4 matrices \( \gamma^\mu/2 \),

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}.
\]

Substituting the \( z \)-linear functions corresponding to spin 1/2 from the subspace \( V_- \),

\[
f_D(x, z) = \chi_\alpha(x) z^\alpha + \psi^{\alpha'}(x) z^{\alpha'}_\alpha = Z_D \Psi_D(x), \quad Z_D = (z^\alpha_\alpha \bar{z}^\alpha_\alpha), \quad \Psi_D(x) = \begin{pmatrix} \chi_\alpha(x) \\ \psi^{\alpha'}(x) \end{pmatrix},
\] (147)

into equation (145) and making a comparison of the coefficients at \( z^\alpha \) and \( \bar{z}^{\alpha'} \) in the left- and right-hand parts, we obtain the Dirac equation

\[
(\hat{p}_\mu \gamma^\mu - \kappa) \Psi_D(x) = 0,
\] (148)

where \( \kappa = 2z^\alpha_\alpha \). The action of the chirality operator (105) amounts to the multiplication by \( \gamma^5/2 \), \( \gamma^5 = \text{diag} \{\sigma^0, -\sigma^0\} \):

\[
\hat{\Gamma}^5 f_D(x, z) = \frac{1}{2} Z_D \gamma^5 \Psi_D(x).
\]

For the states with a definite momentum, (148) is a set of homogenous equations \( (p_\mu \gamma^\mu - \kappa) \Psi_D(x) = 0 \), the existence of whose non-trivial solutions demands that its determinant, equal to \( (p^2 - \kappa^2)^2 \), must turn to zero, which implies \( \kappa = \varepsilon m \), where \( \varepsilon = \pm 1 \).

A charge-conjugate state corresponds to a complex-conjugate function from the subspace \( V_+ \),

\[
f_D(x, z) = -\psi_\alpha(x) z^\alpha - \chi^{\alpha'}(x) z^{\alpha'}_\alpha,
\]

(the minus sign arises due to spinor anticommutation, \( \psi_\alpha z^\alpha = -z_\alpha \psi^\alpha \)) or, equivalently, in the matrix form

\[
Z_D \Psi_D(x) \rightarrow \bar{Z}^*_D \bar{\Psi}_D(x) = Z_D \bar{\Psi}_D(x), \quad \Psi_D(x) = -\begin{pmatrix} \psi_\alpha(x) \\ \chi^{\alpha'}(x) \end{pmatrix} = i \alpha^2 \begin{pmatrix} \psi^\alpha(x) \\ -\chi^{\alpha'}(x) \end{pmatrix},
\] (149)

where \( Z_D = (z^\alpha_\alpha, \bar{z}^{\alpha'}_\alpha) \) and \( Z_D \) have the same transformation law with respect to \( SL(2, C)_{\text{left}} \). We have thus obtained various scalar functions \( f(x, z) \) for the description of particles and antiparticles, and correspondingly, two Dirac equations at the same time, for both signs of the charge. This is in good agreement with the results of the article [18], which shows that a consistent quantization of a classical model of a spinning particle entails as a result exactly the same (charge-symmetrical) quantum mechanics. It is completely equivalent to the one-particle sector of the corresponding quantum field theory.
Consequently, in the case of $z$-linear functions of general form,

$$f(x, z) = \chi(x)^\alpha z^\alpha + \psi(x)^\beta z^\beta = (z^\alpha \frac{1}{\overline{\chi}^\alpha} z^\overline{\alpha} \frac{1}{\chi^\overline{\alpha}}) (\chi^\alpha \frac{1}{\overline{\psi}^\alpha} \chi^\overline{\alpha} \frac{1}{\psi^{\overline{\alpha}}})^T,$$

equation (145) can be represented in the matrix form

$$p_\mu \gamma^{\mu 0} \Psi(x) = \gamma(z \Psi(x), \gamma = \text{diag}(\gamma^\mu, \gamma^\mu),$$

where $\Psi(x) = (\chi(z \frac{1}{\overline{\chi}^\alpha} \chi(z \frac{1}{\psi^\alpha} \psi(z \chi^\alpha)))^T$ is a column of 8 coefficients at $z^\alpha \overline{\chi}^\alpha, z^\alpha \chi^\overline{\alpha}, z^\alpha \overline{\psi}^\alpha, z^\overline{\alpha} \psi^\alpha$, and, therefore, in the case of $z$-linear functions it is equivalent to a pair of Dirac equations with the same mass, that transform into each another at charge conjugation.

This is an eight-component equation and the corresponding Lagrangian

$$\mathcal{L} = \frac{i}{\sqrt{\gamma}} \gamma^{\mu 0} \partial_\mu \Psi - m \overline{\Psi} \Psi,$$

is invariant under right rotations, as distinct from right boosts; the operators of right rotations, owing to (138), commute with $\hat{\Gamma}^{\mu 0}$. However, each of these two Dirac equations is separately invariant only with respect to right rotations generated by $\hat{S}_R^3$, since in the case of the other two generators ($\hat{S}_R^1$ and $\hat{S}_R^2$) the spaces of functions $V_-$ and $V_+$ are not invariant. This means that in the series – equation (131) for the Casimir operator $\hat{p}_2^2$ – equation (145) – Dirac equation (148) – there occurs a reduction of the right (inner) symmetries:

$$\text{SL}(2, C)_{\text{right}} \times U(1) \rightarrow SU(2) \times U(1) \rightarrow U(1) \times U(1).$$

Let us emphasize the fact the introduction of the “Dirac” mass with the help of the left-invariant equation (145) is related with a violation of the symmetry with respect to right boosts. The functions that satisfy the equation are characterized by a definite parity and present a superposition of states with different chirality. A priori, judging from a purely group-theoretical viewpoint, there is no obstacle for an introduction of mass without having to impose equation (145), which would entail the existence of massive chiral fermions, satisfying only the equations (131), (132) for the Casimir operators. In this case, the right symmetry remains unbroken.

Let us now consider equations (143) at $n = 1, 2, 3$, related to the spatial components of the right 4-vector $\hat{p}_\mu \hat{\Gamma}^{\mu 0}$.

The action of the operators $\hat{\Gamma}^{\mu 3}$,

$$2\hat{\Gamma}^{\mu 3} = \sigma^{\mu \beta \overline{\alpha}} \frac{1}{\overline{\chi}^\beta} \partial_\beta \frac{1}{\chi^\overline{\alpha}} - \sigma^{\mu \beta \overline{\alpha}} \frac{1}{\overline{\chi}^\beta} \partial_\beta \frac{1}{\chi^\overline{\alpha}} - \sigma^{\mu \beta \overline{\alpha}} \frac{1}{\overline{\psi}^\beta} \partial_\beta \frac{1}{\psi^\overline{\alpha}} - \sigma^{\mu \beta \overline{\alpha}} \frac{1}{\overline{\psi}^\beta} \partial_\beta \frac{1}{\psi^\overline{\alpha}},$$

on a row $(z^1 z^2 z^3 z^4)$ amounts to its multiplication by the 4x4 matrices

$$\tilde{\sigma}^\mu = \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}.$$ (155)

Substituting $f(x, z) = \chi(x)^\alpha z^\alpha + \psi(x)^\beta z^\beta = (z^\alpha \frac{1}{\overline{\chi}^\alpha} z^\overline{\alpha} \frac{1}{\chi^\overline{\alpha}}) \Psi(x)$ into equation (143) at $n = 3$, we obtain $(\hat{p}_\mu \tilde{\sigma}^\mu - \kappa) \Psi(x) = 0$, where $\kappa = 2 \kappa^3$. 
Multiplying \((\hat{p}_\mu \tilde{\gamma}^\mu - \mathbf{x})\) by \((\hat{p}_\mu \tilde{\gamma}^\mu + \mathbf{x})\), we find that \(\hat{p}_\mu^2 \Psi(x) = -\mathbf{x}^2 \Psi(x)\), whence one can deduce that the corresponding equations are related to tachyons. A similar conclusion can be reached by considering states with a definite momentum. In this case, we have a set of homogenous equations \((p_\mu \tilde{\gamma}^\mu - \mathbf{x})\Psi(x) = 0\), the existence of whose non-trivial solutions demands that its determinant, equal to \((p^2 + \mathbf{x}^2)^2\), must turn to zero.

For the operators \(\hat{\Gamma}^{\mu\mu} \) and \(\hat{\Gamma}^{\mu2} \)

\[
2\hat{\Gamma}^{\mu\mu} = \tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta) -\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta),
\]

\[
2\hat{\Gamma}^{\mu2} = i\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta) - i\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta). \tag{156}
\]

the invariant subspaces are given by the functions \(f(x, \tilde{z}, \tilde{z})\) and \(f(x, z, \tilde{z})\).

For \(z\)-linear functions \(f(x, z)\), the left-invariant equations, related to the spatial components of the right 4-vector \(\hat{p}_\mu \hat{\Gamma}^{\mu\mu} \), can be presented in the matrix form as follows:

\[
\hat{p}_\mu \gamma^{\mu\alpha} \Psi(x) = 2z \Psi(x), \tag{158}
\]

\[
\gamma^{\mu\downarrow} = \begin{pmatrix} 0 & \tilde{\gamma}^\mu \\ \tilde{\gamma}^\mu & 0 \end{pmatrix}, \quad \gamma^{\mu2} = \begin{pmatrix} 0 & -i\tilde{\gamma}^\mu \\ i\tilde{\gamma}^\mu & 0 \end{pmatrix}, \quad \gamma^{\mu3} = \begin{pmatrix} \tilde{\gamma}^\mu & 0 \\ 0 & -\tilde{\gamma}^\mu \end{pmatrix},
\]

where \(\Psi^{\mu}(x)\) is an 8-component column being composed of the coefficients at \(z, \tilde{z}, \tilde{z}, \tilde{z}\) in the power expansion of the function \(f(x, z)\). These equations describe tachyons. The matrices \(\gamma^{\mu\mu}\) obey the conditions

\[
[\gamma^{\mu\mu}, \gamma^{\mu\nu}] = 2\eta^{\mu\nu} \gamma^{\mu\nu}, \quad (\gamma^{\mu\mu})^\dagger = (-1)^{\delta_{\mu0} + \delta_{\mu0}} \gamma^{\mu\mu}. \tag{159}
\]

Let us now present the expressions for the operators \(\hat{\Gamma}^{\mu\mu} \) that enter equations \((154)\), namely:

\[
2\hat{\Gamma}^{\mu\mu} = \tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta) -\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta),
\]

\[
2\hat{\Gamma}^{\mu2} = i\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta) - i\tilde{\sigma}^{\mu\beta\alpha}(\tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta + \tilde{z}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\beta). \tag{160}
\]

In the matrix form, equations \((144)\) for the \(z\)-linear functions are written as follows:

\[
\hat{p}_\mu \gamma^{\mu\alpha} \Psi^{\alpha}(x) = 2z \Psi^{\alpha}(x), \tag{161}
\]

\[
\gamma^{\mu\downarrow} = \begin{pmatrix} 0 & \tilde{\gamma}^\mu \\ \tilde{\gamma}^\mu & 0 \end{pmatrix}, \quad \gamma^{\mu2} = \begin{pmatrix} 0 & -i\tilde{\gamma}^\mu \\ i\tilde{\gamma}^\mu & 0 \end{pmatrix}, \quad \gamma^{\mu3} = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}.
\]

40
The matrices $\gamma^{\mu\nu}$ obey the conditions
\[
[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = -\eta^{\mu\sigma}\eta^{\rho\nu}, \quad (\gamma^{\mu\nu})^\dagger = -(-1)^{\delta_{\mu\nu} + \delta_{\rho\sigma}} \gamma^{\mu\nu}.
\] (162)

It is easy to see that the equation related to $\gamma^0$ describes tachyons and the remaining three equations – usual particles (bradyons) of spin 1/2. For the latter equations, the Hermitian-conjugate ones (161), owing to (162), we present in the form
\[
\overline{\Psi} (-i\gamma^\mu \partial_\mu + 2\kappa) = 0, \quad \overline{\Psi} = \Psi^0 k,
\] (163)

where $2\kappa = \pm m$, which allows us to present the corresponding currents $j^\mu = \overline{\Psi} \gamma^\mu \Psi$ and Lagrangians,
\[
\mathcal{L}^k = i\overline{\Psi} \gamma^\mu \partial_\mu \Psi - 2\kappa \overline{\Psi} \Psi, \quad \overline{\Psi} = \Psi^0 k.
\] (164)

Notice that the eight-component equation related to $\gamma^3$ splits into two Dirac equations with opposite signs of the mass term. As the equation (161), it is invariant under rotations generated by $\hat{S}_1^3$, it is, however, not invariant under rotations generated by $\hat{S}_2^1$ and $\hat{S}_2^2$.

A special role played by the Dirac equation in relativistic theory can be largely explained by the fact that it actually contains all the four types of spinors: two of them in a manifest form and the other two are obtained as a result of the complex conjugation of the equation, as well as by the consideration of both signs of the mass term. The study of the properties of the Dirac equation allows one to establish, in particular, the properties of discrete transformations, as well as to introduce chirality, being the quantum number corresponding to the operator $\gamma^5$.

One can also do the contrary, by examining both the left and right generators of the Poincaré group, as well as the corresponding quantum numbers. This allows one not only to give a consistent description of discrete transformations, chirality, particles and antiparticles without the aid of RWE, but also to give an exact group-theory formulation of the conditions for which the equation is valid (in other words, to give a group-theory deduction of the Dirac equation, as was done in [17]). Such a formulation enables one, in particular, to establish the relation of the mass-term sign with the characteristics of the extended Poincaré group.

Let us return to the equation (165) for the operator $\hat{\Gamma}^{\mu\nu}$, for $z$-linear functions from the subspaces; this equation is equivalent to a pair of Dirac equations. Polynomials of degree $2s$ from the subspaces $V_-$ and $V_+$ are eigenfunctions of the operator $\hat{S}_R^2$ with the eigenvalues $s$ and $-s$.

In the case of $z$-quadratic functions from the subspace $V_-$,

\[
f(x, z) = \chi_{\alpha\beta}(x) z^\alpha z^\beta + \phi_{\alpha} (x) z^\alpha \bar{z}^\beta + \psi^{\alpha\beta}(x) \bar{z}^\alpha z^\beta = \Phi_\mu(x) q^\mu + \frac{1}{2} F_{\mu\nu}(x) q^{\mu\nu},
\] (165)

\[
q^\mu = \frac{1}{2} \sigma^\mu_{\alpha\beta} z^\alpha \bar{z}^\beta, \quad q_\mu q^\mu = 0, \quad q_{\mu\nu} = -q_{\nu\mu} = \frac{1}{2} \left( (\sigma_{\mu\nu})_{\alpha\beta} z^\alpha z^\beta + (\bar{\sigma}_{\mu\nu})_{\alpha\beta} \bar{z}^\alpha \bar{z}^\beta \right),
\]

\[
\Phi_\mu(x) = -\bar{\sigma}_\mu \bar{\phi}_{\alpha\beta}(x), \quad F_{\mu\nu}(x) = -2 \left( (\sigma_{\mu\nu})_{\alpha\beta} \chi^{\alpha\beta}(x) + (\bar{\sigma}_{\mu\nu})_{\alpha\beta} \psi^{\alpha\beta}(x) \right),
\]

or in the case of their conjugates from $V_+$, a similar analysis leads to the 10-component equations of Duffin-Kemmer for spin 1 [16].
Substituting polynomials of degree $2s$ from the subspaces $V_-$ or $V_+$ into (145), one can obtain a matrix equation of the form

$$(\alpha^\mu \hat{p}_\mu - \kappa) \psi = 0, \quad [S^{\lambda\mu}, \alpha^\nu] = i(\eta^{\mu\lambda} \alpha^\lambda - \eta^{\lambda\nu} \alpha^\mu), \quad [\alpha^\mu, \alpha^\nu] = S^{\mu\nu}. \quad (166)$$

The commutation relations for the matrices follow from the commutation relations for differential operators, according to which $\hat{S}^{\mu\nu}$ and $\hat{\Gamma}^{\nu\alpha}$ obey the commutation relations of $SO(3, 2)$. Finite-component equations of the form (166) are known as the Bhabha equations, although for the first time they were systematically considered by Lubanski. These equations are classified according to the finite-dimensional irreps of $SO(3, 2)$. The case of $f(x, z) \in V_-$ (or $V_+$) corresponds to the symmetric irreps $T^{[2s]}_0$ of $SO(3, 2)$.

It can be demonstrated that in case the eigenvalue $\hat{p}_\mu \hat{\Gamma}^{\nu\alpha}$ equals to $\pm ms$, where $2s$ is a (fixed by $\hat{S}^3_R$) polynomial degree, the eigenvalue of the Lubanski-Pauli operator $\hat{W}^2$ corresponds to spin $s$ [16]. Consequently, in the case of functions from subspaces $V_-$ and $V_+$ we arrive at the equation

$$(\hat{p}_\mu \hat{\Gamma}^{\nu\alpha} - \varepsilon ms)f(x, z) = 0, \quad (167)$$

where $s$ is spin, $\varepsilon = \pm 1$. As has been already observed, this equation is invariant with respect to space reflection. A consideration of the rest frame shows [17] that in equation (167), as well as in the Dirac equation in (148),

$$\varepsilon = \eta \text{sign} p_0 \text{sign} S^R_3,$$

i.e. the mass-term sign is the product of the signs of parity, energy and projection $S^R_3$.

### Concluding remarks

We have examined a field $f(x, z)$ on the Poincaré group, depending on ten real-valued parameters. Four of the parameters $x^\mu$ correspond to position, while six parameters correspond to orientation, which is convenient to describe by the elements of a complex matrix $Z \in SL(2, \mathbb{C}) = \text{Spin}(3, 1)$. As distinct from fields in homogenous spaces, in particular, the Minkowski space $M(3, 1)/\text{Spin}(3, 1)$, a field on the entire group admits two types of transformations – left and right ones, corresponding to a change of the laboratory (space-fixed) and local (body-fixed) reference frames. These transformations commute with each other. Correspondingly, the variables $z = \{z^a_{\underline{\mu}}, \bar{z}^{\dot{a}}_{\underline{\mu}}\}$, describing the orientation, have two kinds of indices – the usual ones (left, laboratory, indices) and underlined ones (right, local, indices).

For an analysis of physical aspects, we use a close mathematical analogy with the theory of a three-dimensional non-relativistic rotator, constructed in the space of functions depending only on three-dimensional orientation (i.e., functions on the group $SO(3)$), which provides one with a common and intuitively clear interpretation.

The scalar field $f(x, z)$ contains the fields of all spins and presents a generating function of usual spin-tensor fields, the latter being the coefficients of expansion in the powers of the complex-valued variables $z^a_{\underline{\mu}}, \bar{z}^{\dot{a}}_{\underline{\mu}}$. Field classification is based on the use of the maximal set of commuting operators (two Casimir operators and four left and right generators each, the total being ten, equal to the number of group parameters).
We repeat ones more that in the conventional description of relativistic particles (in $3+1$ dim.) in terms of spin-tensor fields, based, in particular, on a classification of Poincaré and Lorentz group representations, there appear 8 particle characteristics (quantum numbers): 2 numbers $j_1$ and $j_2$ that label Lorentz group representations and 6 numbers related to the Poincaré group, those are the mass $m$, spin $s$ (Casimirs of the group), and 4 numbers, which are eigenvalues of some combinations of left generators of the group. In particular, the latter 4 numbers can be some components of the momentum and a spin projection.

The proposed description of relativistic orientable objects is based on a classification of group representations of transformation of both s.r.f. and b.r.f., and the orientable object is characterized already by 10 quantum numbers. These numbers are related to a maximal set of commuting operators (101). Here, in addition to the 2 Casimirs and 4 left generators of the group, we have 4 numbers which are eigenvalues of some combinations of right generators of the Poincaré group. Among the latter, we have 2 Casimirs of the (right) Lorentz subgroup that fix $j_1$ and $j_2$. Thus, as compared with the conventional description, in the proposed one, an oriented object has two more characteristics. Their physical interpretation is a subject of an individual investigation.

Field symmetries, defined as transformations of a field into itself, are divided, from the mathematical viewpoint, into three types: the left transformations of the Poincaré group (Lorentz transformations), right transformations, and outer automorphisms.

The first type of transformations is regarded as external symmetries (the symmetries of the embedding space), the second type is regarded as internal symmetries. Outer automorphisms (reflections $I_s, I_t$, being involutive automorphisms, and scale transformations) affect both (space-fixed and body-fixed) reference frames. Charge conjugation corresponds to the complex conjugation of functions $f(x, z)$.

A scalar field on the Poincaré group provides an adequate language for a construction and analysis of the symmetries of RWE. It can be regarded as a field depending on a 4-vector $x^\mu$ and on four types of spinors, two pairs being related by charge conjugation. These pairs have equal transformation properties with respect to $SL(2, C)_{left}$ and different transformation properties with respect to $SL(2, C)_{right}$ and outer automorphisms. As a consequence of the presence of four types of spinors for spin $1/2$, there arise eight-component equations corresponding to both signs of charge, i.e., we thus reproduce the one-particle sector of quantum field theory.

The equations related to the Casimir operators (Klein-Gordon and Pauli-Lubanski equations) are invariant with respect not only to the left but also to the right transformations; however, this is not the case for other equations. As in the rotator theory, the right (internal) symmetries may be broken, and the analysis shows that the basic RWE (Dirac, Duffin-Kemmer equations) possess only some of the right symmetries, so that only some of the potential right symmetries are realized in Nature.

Acknowledgement

The authors are grateful to Profs. S. N. Solodukhin and I. V. Tyutin for useful discussions. D.M.G. acknowledges the permanent support of FAPESP and CNPq.
References

[1] L.S. Biedenharn and J.D. Louck. *Angular Momentum in Quantum Physics*. Addison-Wesley, Reading, Massachusetts, 1981.

[2] V.L. Ginzburg and I.E. Tamm. *On the theory of spin*. Zh. Ehksp. Teor. Fiz., 17:227–237, 1947.

[3] V. Bargmann and E.P. Wigner. *Group theoretical discussion of relativistic wave equations*. Proc. Nat. Acad. USA, 34:211–223, 1948.

[4] H. Yukawa. *Quantum theory of non-local fields. I. Free fields*. Phys. Rev., 77(2):219–226, 1950.

[5] Yu.M. Shirokov. *Relativistic theory of spin*. Zh. Ehksp. Teor. Fiz., 21(6):748–760, 1951.

[6] D. Finkelstein. *Internal structure of spinning particles*. Phys. Rev., 100(3):924–931, 1955.

[7] F. Lurçat. *Quantum field theory and the dinamical role of spin*. Physics, 1:95, 1964.

[8] H. Bacry and A. Kihlberg. *Wavefunctions on homogeneous spaces*. J. Math. Phys., 10(12):2132–2141, 1969.

[9] A. Kihlberg. *Fields on a homogeneous space of the Poincare group*. Ann. Inst. Henri Poincaré, 13(1):57–76, 1970.

[10] C.P. Boyer and G.N. Fleming. *Quantum field theory on a seven-dimensional homogeneous space of the Poincaré group*. J. Math. Phys., 15(7):1007–1024, 1974.

[11] H. Arodz. *Metric tensors, Lagrangian formalism and Abelian gauge field on the Poincaré group*. Acta Phys. Pol., Ser. B, 7(3):177–190, 1976.

[12] M. Toller. *Classical field theory in the space of reference frames*. Nuovo Cimento B, 44(1):67–98, 1978.

[13] M. Toller. *Free quantum fields on the Poincaré group*. J. Math. Phys., 37(6):2694–2730, 1996.

[14] W. Drechsler. *Geometro-stohastically quantized fields with internal spin variables*. J. Math. Phys., 38(11):5531–5558, 1997.

[15] L. Hannibal. *Relativisyc spin on the Poincaré group*. Found. Phys., 27(1):43–56, 1997.

[16] D.M. Gitman and A.L. Shelepin. *Fields on the Poincaré group: Arbitrary spin description and relativistic wave equations*. Int. J. Theor. Phys., 40:603–684, 2001. arXiv:hep-th/0003146.
[17] I.L. Buchbinder, D.M. Gitman, and A.L. Shelepin. Discrete symmetries as automorphisms of the proper Poincaré group. Int. J. Theor. Phys., 41(4):753–790, 2002. arXiv:hep-th/0010035.

[18] S.P. Gavrilov, D.M. Gitman, Quantization of Point-Like Particles and Consisitent Relativistic Quantum Mechanics, Int. J. Mod. Phys. A15 (2000) 4499-4538; Quantization of the Relativistic Particle, Class.Quant.Grav. 17 issue 19 (2000) L133-L139; S.P. Gavrilov, D.M. Gitman, Quantization of a spinning particle in an arbitrary background, Class.Quant.Grav. 18 (2001) 2989-2998.

[19] R.N. Zare. Angular Momentum. Understanding Spatial Aspects in Chemistry and Physics. Wiley, New York, 1988.

[20] R. Penrose. Structure of space-time. Benjamin, New York, 1968.

[21] R. Penrose, M.A.H. MacCallum Twistor theory: approach to the quantization of fields and space-time. Phys. Rep., 6(4):241–316, 1972.

[22] R.O. Wells. Complex manifolds and mathematical physics. Bull. Amer. Math. Soc., 1(2):296–336, 1979.

[23] N.Ya. Vilenkin. Special Functions and the Theory of Group Representations. AMS, Providence, 1968.

[24] A.O. Barut and R. Raczka. Theory of Group Representations and Applications. PWN, Warszawa, 1977.

[25] I.M. Benn and R.W. Tucker. An Introduction to Spinors and Geometry with Applications in Physics. Adam Hilger, Bristol, 1988.

[26] N.Ja. Vilenkin and A.U. Klimyk. Representations of Lie Groups and Special Functions, volume 1. Kluwer Acad. Publ., Dordrecht, 1991.

[27] D.P. Zhelobenko and A.I. Schtern. Representations of Lie Groups. Nauka, Moscow, 1983.

[28] G. Rideau. On the reduction of the regular representation of the Poincaré group. Commun. Math. Phys., 3:218–227, 1966.

[29] N.X. Hai. Harmonic analysis on the Poincaré group, I. Generalized matrix elements. Commun. Math. Phys., 12:331–350, 1969; N.X. Hai. Harmonic analysis on the Poincaré group, II. The Fourier transform. Commun. Math. Phys., 22:301–320, 1971.

[30] V.V. Varlamov. Towards the quantum electrodynamics on the Poincare group. arXiv: hepth/0403070.

[31] V.V. Varlamov. General solutions of relativistic wave equations II: Arbitrary spin chains. Int. J. Theor. Phys., 46:741–805, 2007. arXiv:math-ph/0503058.
[32] D.M. Gitman and A.L. Shelepin. *Poincaré group and relativistic wave equations in 2+1 dimensions*. J. Phys. A, 30:6093–6121, 1997.

[33] I.M. Gel’fand, M.I. Graev, and N.Ya. Vilenkin. *Generalized Functions*, volume 5. Academic Press, New York, 1966.

[34] W.I. Fushchich and A.G. Nikitin. *Symmetry of equations of quantum mechanics*. Allerton Press, New York, 1994.

[35] H. Umezava, S. Kamefuchi, and S. Tanaka. *On the time reversal in the quantized field theory*. Prog. Theor. Phys., 12(3):383–400, 1954.

[36] N. Kemmer, J.C. Polkinghorn, and D.L. Pursey. *Invariance in elementary particle physics*. Rep. Progr. Phys., 22:368–432, 1959.

[37] J. Schwinger. *The theory of quantized fields. I*. Phys. Rev., 82(6):914–927, 1951.

[38] E.M. Lifshitz, V.B. Berestetskii, and L.P. Pitaevskii. *Quantum Electrodynamics*. Pergamon, Oxford, 1982.