Budget Pacing in Repeated Auctions: Regret and Efficiency without Convergence*

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Abstract

We study the aggregate welfare and individual regret guarantees of dynamic pacing algorithms in the context of repeated auctions with budgets. Such algorithms are commonly used as bidding agents in Internet advertising platforms, adaptively learning to shade bids by a tunable linear multiplier in order to match a specified budget. We show that when agents simultaneously apply a natural form of gradient-based pacing, the liquid welfare obtained over the course of the learning dynamics is at least half the optimal expected liquid welfare obtainable by any allocation rule. Crucially, this result holds without requiring convergence of the dynamics, allowing us to circumvent known complexity-theoretic obstacles of finding equilibria. This result is also robust to the correlation structure between agent valuations and holds for any core auction, a broad class of auctions that includes first-price, second-price, and generalized second-price auctions as special cases. For individual guarantees, we further show such pacing algorithms enjoy dynamic regret bounds for individual value maximization, with respect to the sequence of budget-pacing bids, for any auction satisfying a monotone bang-for-buck property. To complement our theoretical findings, we provide semi-synthetic numerical simulations based on auction data from the Bing Advertising platform.

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1 Introduction

Online advertising increasingly dominates the marketing landscape, accounting for 54.2% of total media ad spending in the US in 2019 ( $129 billion) [29]. Such ads are predominantly allocated by auction: advertisers submit bids to an Internet platform to determine whether they will be displayed as part of a given page view and at what price. A typical advertiser participates in many thousands of auctions each day, across a variety of possible ad sizes and formats, payment options (pay per impression, per click, per conversion, etc.), and bids tailored to a variety of signals about user demographics and intent. To further complicate the decision-making process from the advertiser’s perspective, these many auction instances are strategically linked through a budget constraint, the total amount of money that can be allocated to advertising. An advertiser therefore faces the daunting task of choosing how to appropriately allocate a global budget across a complex landscape of advertising opportunities, and then convert that intent into a bidding strategy.

To help address this difficulty, all major online platforms provide automated budget management services that adjust campaign parameters on an advertiser’s behalf. This is commonly achieved via budget-pacing: an advertiser specifies a global budget target and a maximum willingness to pay for different types of advertising opportunities, and these maximums are then scaled down (or “paced”) by a multiplier so that the realized daily spend matches the target budget. An algorithmic bidding agent learns, online, how best to pace the advertiser’s bids as it observes auction outcomes. This campaign management service lowers the barrier to entry into the online advertising ecosystem and removes the need for the advertiser to constantly adjust their campaign in the face of changing market conditions. Moreover, the platform is often better positioned to manage the budget since they have direct access to detailed market statistics.

Bidding agents are now near-universally adopted across all mature advertising platforms, but this success raises some pressing questions about the whole-market view. What can we say about the aggregate market outcomes when nearly all advertiser spend is controlled by automated bidding agents that are simultaneously learning to pace their bids? And to what extent does this depend on the details of the underlying auction?

Central to our question is the interplay between individual learning and aggregate market efficiency, each of which has been studied on its own. For example, when each advertising opportunity is sold by a second-price auction, it is known that linear bidding strategies (i.e., mappings from maximal willingness-to-pay to a bid) are in fact optimal over all possible bidding strategies for both utility-maximizing and value-maximizing agents, and that gradient-based methods can be used by an agent to achieve vanishing regret relative to the best bidding strategy in hindsight [18, 12]. On the other hand, when multiple bidding agents participating in second-price auctions choose pacing factors that form a pure Nash equilibrium, the resulting outcome is known to be approximately efficient (in the sense of maximizing expected liquid welfare; more on this below) [1, 7]. At first glance this combination of results seems to address the question of aggregate performance of bidding agents in second-price auctions. But convergence of online learning algorithms to a Nash equilibrium, let alone a pure Nash equilibrium, is notoriously difficult to guarantee and should not be taken for granted. Moreover, finding a pure Nash equilibrium of the pacing game is PPAD-hard for second-price auctions [20], and hence we should not assume that bidding agents employing polynomial-time online learning strategies will efficiently converge to a pure Nash equilibrium in the full generality of second-price auctions. So the question remains: if bidding agents do not converge, what happens to overall market performance?

1.1 Our Contributions

We provide (classes of) bidding algorithms that simultaneously admit good aggregate guarantees in terms of overall market efficiency, without relying on convergence, while still providing good individual guarantees as online learning algorithms that benefit a particular advertiser. Closely related are three literatures: (i) on aggregate outcomes in single-shot budget-constrained ad auctions, without regard to bidding dynamics, (ii)
on online learning with budget constraints, without regard to the aggregate performance, and (iii) conditions under which learning agents converge to equilibrium in repeated games. With this perspective in mind, we match a state-of-art aggregate guarantee from (i), while being qualitatively on par with state-of-art individual guarantees in (ii), without relying on convergence and thereby side-stepping conditions from (iii). We accomplish this with bidding algorithms that are arguably quite natural and for a broad class of auctions.

In our model there are \( T \) rounds, each corresponding to an auction instance. All of our results apply to multiple auction rules, including first-price, second-price, and GSP auctions. The private values (i.e., maximum willingness to pay) observed by the agents are randomly drawn in each round and can be arbitrarily correlated with each other, capturing scenarios where the willingness to pay of different advertisers is correlated through characteristics of the impression. In each round the bidding agents place bids on behalf of their respective advertisers. The agents operate independently of each other, interacting only through the feedback they receive from the auction.

We focus on a gradient-based pacing algorithm (Algorithm 1) that was first introduced by Balseiro and Gur [12] in the context of second-price auctions. They derive this algorithm via the Lagrangian dual of the (quasilinear) utility maximization problem. We directly extend their algorithm to a richer class of allocation problems and auction formats. Underlying this extension is a modified interpretation of this algorithm as stochastic gradient descent on a certain artificial objective that applies even beyond second-price auctions.

**First Result: Aggregate Market Performance.** We prove that when the bidding agents employ Algorithm 1 to tune their pacing multipliers, the resulting market outcome over the full time horizon achieves at least half of the optimal expected liquid welfare. Crucially, this guarantee does not depend on the convergence of the algorithms’ actions to an equilibrium of the bidding game. Nevertheless, it matches the best possible guarantee even for a pure Nash equilibrium in a static truthful auction [1, 7].

Liquid welfare is the maximum amount that the agents are jointly willing to pay for a given allocation. Put differently, it is the maximum revenue that can be extracted for this allocation by an all-knowing auctioneer. Liquid welfare coincides with compensating variation when specialized to our setting. Conveniently, it is an appropriate welfare measure for agents who seek to maximize value (e.g., number of clicks or impressions received) subject to constraints, rather than a monetary utility objective. Utilitarian welfare would be reasonable aggregate objective. However, strong impossibility results are known even in a single-shot (non-repeated) setting for a single good [28]. Thus, liquid welfare is a meaningful notion of social surplus in budgeted environments, and it specializes to utilitarian welfare when budgets are infinite. It has become a standard objective in the analysis of budget-constrained auctions [28, 6, 1, 7].

While our discussion so far has focused mainly on second-price single-item auctions, our approximation result actually holds for a far richer set of allocation problems and auction formats, including those used in real-world advertising platforms. We allow arbitrary downward-closed constraints on the set of feasible allocations of a single divisible good, which captures single-item auctions as well as complex settings such as sponsored search auctions with multiple slots and separable click rates. Further, our result applies even when the underlying mechanism is not truthful. We accommodate any core auction: an auction that

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1 For non-truthful auctions the restriction to linear bidding strategies is not without loss, so our individual guarantees are with respect to the best linear bidding strategy. See our description of the individual regret guarantees later in this Section for further discussion.

2 An impression is the industry term for an “atomic” advertising opportunity: a specific slot on a specific webpage when this webpage is rendered for a specific user. Impression characteristics depend on the slot, the page, and the user.

3 Compensating variation, a standard notion in economics, is the amount that agents would need to pay (or be paid) to return to their original utility levels after some change (typically a change in prices). For us, the change being considered is the allocation received.

4 When the agents’ objective can be expressed in dollars, such as the objective of maximizing advertiser utility subject to the budget constraints

5 I.e., a good that can be divided fractionally among the agents. Any item can be interpreted as divisible via probabilistic allocation.
generates outcomes in the core, meaning that no coalition of agents could improve their joint utility by renegotiating the outcome with the auctioneer [5]. This is a well-studied class of auctions that includes both first and second-price auctions, as well as the generalized second price (GSP) auction, and has previously been studied in the context of advertising auctions [31, 35]. We emphasize, however, that the problem remains non-trivial (and almost as challenging) in a much simpler model with a repeated single-item auction and constant private values.

**Second Result: Individual Regret Guarantees.** We have analyzed aggregate market performance for a broad class of (possibly non-truthful) auctions including first-price and GSP auctions, but is gradient-based pacing an effective learning method in those settings? Regret guarantees are known when the underlying auction is truthful [12, 16], but what about non-truthful auctions? To address this question, we bound the regret obtained by an individual bidding agent participating in any auction format that satisfies a monotone bang-per-buck condition, which implies that the marginal value obtained per dollar spent weakly decreases in an agent’s bid. For example, first and second-price auctions satisfy this condition, as does the GSP auction. We show that the total value obtained by the bidding agent has regret $O(T^{3/4})$ relative to the best pacing multiplier in hindsight, in the stochastic environment where the profile of opposing bids is drawn independently from the same distribution in each round.

When the underlying auction is truthful, the best fixed pacing multiplier in hindsight is known to be optimal over the class of all possible bidding strategies (i.e., mappings from value to bid, which may not be linear), for both utility-maximizing and value-maximizing agents [24, 12, 11]. This means that, for truthful auctions such as the second-price auction, our benchmark for regret is actually the optimal bidding policy in hindsight. But even beyond truthful auctions, we argue that the best linear policy (i.e., best pacing multiplier) is an appropriate benchmark. First, while we abstractly model agent values as a willingness to pay per impression, in practice the variation in values is primarily driven by click rate estimates that are internal to the platform. In such an environment, an advertiser whose bidding algorithm is external to the platform would necessarily be limited to a linear policy. Indeed, if the algorithm cannot access the platform’s click rate estimates, then a bidding “strategy” simply reduces to a single real-valued bid that would be (linearly) multiplied by the click rate; see Appendix F for a more formal discussion. The benchmark therefore tracks the best performance that one could achieve in hindsight with an externally provided bid. Second, linear pacing is commonly used in practice as an algorithmic bidding policy even for non-truthful auctions [23], so from a practical perspective it is useful to focus attention on linear pacing policies. Third, linear pacing is reasonable from the online learning perspective: it is typical to choose a subset of policies as a hypothesis class (even if this class is not known to contain an optimal policy), and the set of linear policies is a common and natural class to consider.

While the discussion above is framed in a stochastic environment, we prove an even stronger individual guarantee by permitting the opposing bids to change adaptively and adversarially based on the auction history. (Indeed, realistic auction environments are not necessarily stochastic from the individual bidder’s perspective, because the other agents’ bidding algorithms may be revising their bids.) In such an environment, we show that gradient-based pacing achieves vanishing regret relative to the perfect pacing sequence, which is the sequence of pacing multipliers such that the expected spend in each round is precisely the per-round budget.

In a stochastic environment, this perfect pacing sequence is precisely the single best fixed pacing multiplier in hindsight. More generally, this sequence may not be uniform and is not necessarily the sequence that maximizes expected value subject to the budget constraint. Achieving low regret against the latter objective in an adversarial environment is essentially hopeless (more on this in Section 1.2). Therefore, we suggest the perfect pacing sequence as a reasonable and tractable benchmark for this problem, and

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6 A typical goal in regret minimization is regret $\tilde{O}(T^\gamma)$ for some constant $\gamma \in [\frac{1}{2}, 1)$. As a baseline, regret $O(\sqrt{T})$ is the best possible in the worst case, even in a stochastic environment with only two possible actions [3].

7 This guarantee is parameterized by the total round-to-round change in the “perfect” pacing multipliers.
Numerical Simulations. To complement our theoretical findings, we provide semi-synthetic numerical simulations of Algorithm 1 based on auction and campaign data from the Bing Advertising platform. We focus on regret relative to the standard benchmark: the best fixed pacing multiplier in hindsight. Motivated by the impossibility of low regret guarantees against this benchmark in adversarial environments, we simulate the progression of a multi-player environment in which the competing bidding agents engage in simultaneous learning. We find numerically that simultaneous execution of Algorithm 1 yields vanishing regret in our simulations, with regret rate less than $T^{3/5}$.

1.2 Related Work

Pacing in Ad auctions. Budget-pacing is a popular approach for repeated bidding under budget constraints, both in practice and in theory [11]. Balseiro and Gur [12] attain convergence guarantees in repeated second-price auctions, under strong convexity-like assumptions. Borgs et al. [18] attain a similar result for first-price auctions, without convexity assumptions (via a different algorithm). Our emphasis on welfare guarantees without requiring convergence appears novel, and possibly necessary given the aforementioned PPAD-hardness result [20].

Balseiro and Gur [12] establish individual regret guarantees for repeated second-price auctions, for both stochastic and adversarial environments, under various convexity assumptions. Balseiro et al. [16] extend similar guarantees to repeated truthful auctions, without convexity assumptions. In particular, both papers obtain regret-optimal guarantees for the stochastic environment. Balseiro et al. [16] obtain bounds on dynamic regret for an adversarial environment that does not deviate too much from a stochastic one. Our individual guarantee is different from theirs in several respects: (i) it applies to a much wider family of auctions, (ii) the problem instance can deviate arbitrarily far from any fixed stochastic instance, but (iii) the benchmark is the perfect pacing sequence rather than the best outcome in hindsight.

Our individual regret guarantees hold under a bandit feedback model, where the agent learns only the allocation and payment obtained by their chosen bid. We are not aware of any published guarantees on individual regret for non-truthful auctions, even in the stochastic environment and even under the more relaxed full-feedback model.

A static (single-shot) game between budget-constrained bidders who tune their pacing multipliers, a.k.a. the pacing game, along with the appropriate equilibrium concept was studied in Conitzer et al. [24, 23], in the context of first- and second-price auctions with quasilinear utilities, and then extended to more general payment constraints [1] and utility measures [7]. In particular, any pure Nash equilibrium of this game achieves at least half of the optimal liquid welfare when the underlying auction is truthful. In a related contextual auction setting and simultaneously with our work, Balseiro et al. [15] establishes a similar bound on liquid welfare at any (possibly non-linear) Bayes-Nash equilibrium for i.i.d. bidders, for a class of standard auctions that includes first-price and second-price auctions. In contrast to these results, our efficiency result does not rely on convergence to equilibrium, and applies to all core auctions.

A growing line of work in mechanism design targets bidding agents that maximize value or utility under spending constraints and are assumed to reach equilibrium [11, 13, 27]. In contrast, our emphasis is not on mechanism design: we take the auction specification as exogenous and focus on the learning dynamics.

Throttling (a.k.a. probabilistic pacing) is an alternative approach: instead of pacing their bids, agents participate in only a fraction of the auctions [11]. Very recently, Chen et al. [21] proved that throttling converges to a Nash equilibrium in the first-price auctions (albeit without any stated implications on welfare or liquid welfare). On the other hand, no such convergence is possible for second-price auctions since a
Nash equilibrium is PPAD-hard to compute [20]. The equilibria obtained by throttling and pacing dynamics in the first-price auction can differ in revenue by at most a factor of 2 [21].

**Learning theory.** Repeated bidding with a budget is a special case of multi-armed bandit problems with global constraints, a.k.a. *bandits with knapsacks* (BwK) [9][2][37] (see Chapter 10 in [45] for a survey). BwK problems in adversarial environments do not admit regret bounds: instead, one is doomed to approximation ratios, even against a time-invariant benchmark and even in relatively simple examples [37]. A similar impossibility result is derived in [12] specifically for repeated budget-constrained bidding in second-price auctions. The essential reason for this impossibility is the *spend-or-save dilemma* [37], whereby the algorithm does not know whether to spend the budget now or save it for the future.

Several known algorithms for BwK are potentially applicable to our problem, and the corresponding individual guarantees on regret may be within reach for the stochastic environment, but haven’t yet been published. However, it is unclear how to derive aggregate guarantees for such algorithms. The algorithm analyzed in the present work is based upon stochastic gradient descent, which is a standard, well-understood algorithm in online convex optimization [36].

A long line of work targets convergence of learning algorithms in repeated games (not specifically focusing on ad auctions). When algorithms achieve low regret (in terms of cumulative payoffs), the *average play* (time-averaged distribution over chosen actions) converges to a (coarse) correlated equilibrium [4][40][34], and this implies welfare bounds for various auction formats in the absence of budget constraints [43]. In contrast, we show in Appendix E that for repeated auctions with budgets, low individual regret on its own does not imply any bounded approximation for liquid welfare. Convergence in the last iterate is more challenging: strong negative results are known even for two-player zero-sum games [10][39][22]. Yet, a recent line of work (starting from Daskalakis et al. [26], see [25][33][46] and references therein) achieves last-iterate convergence under full feedback and substantial convexity-like assumptions, using two specific regret-minimizing algorithms.

To the best of our understanding, these positive results do not apply to the setting of repeated auctions with budgets.

**Follow-up work.** Two closely related follow-up papers appeared after the initial publication of our paper on arxiv.org. Fikioris and Tardos [30] focus on the special case of repeated first-price auctions and achieve liquid welfare guarantees as long as each bidding algorithm satisfies a particular individual guarantee. Specifically, they assume multiplicative $\gamma \geq 1$ approximation relative to the best fixed pacing multiplier, and obtain multiplicative approximation ratio $R = \gamma + O(1)$ on liquid welfare. They achieve $\gamma = T/B$ and $R \approx T/B + 1/2$ plugging in a recent result on bandits with knapsacks [19]. Further, they achieve $R \approx 2.4$ if $\gamma = 1$. However, it is currently not known how to achieve $\gamma < T/B$, let alone $\gamma = 1$, with non-trivial budget-constrained bidding algorithms, e.g., such as those guaranteed to achieve vanishing regret in a stochastic environment.

Lucier et al. [38] extend our results to bidders that face return-on-investment constraints in addition to budget constraints. They achieve the same 2-approximation guarantee on liquid welfare and similar individual guarantees. Their algorithm coincides with ours when specialized to budget constraints. Their results hold for single-item allocation problems and any auction format in which the single item is sold to the highest bidder and the payment lies between the highest and second-highest bids.

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8One could run a BwK algorithm on bids discretized as multiples of some $\epsilon > 0$. Individual guarantees would depend on bounding the discretization error which is a known challenge in BwK [9][8]. We are only aware of one such result for BwK when one has contexts and a “continuous” action set (in our case, resp., private values and bids). This result concerns a different problem: *dynamic pricing*, where actions correspond to posted prices, and only achieves regret $O(T^{4/5})$ in the stochastic environment [8].
2 Our Model and Preliminaries

The Allocation Problem. Our setting is a repeated auction with budgets. There is one seller (the platform) and \( n \) bidding agents. In each time \( t = 1, \ldots, T \), the seller has a single unit of a good available for sale. We will sometimes refer to the good as a (divisible) item. An allocation profile is a vector \( x = (x_1, \ldots, x_n) \in [0, 1]^n \) where \( x_k \) is the quantity of the good allocated to agent \( k \). There is a convex and closed set \( \mathcal{X} \subseteq [0, 1]^n \) of feasible allocation profiles, which is assumed to be downward closed. An allocation profile is feasible if \( x \in \mathcal{X} \). An allocation sequence is a sequence of allocation profiles \((x_1, \ldots, x_T)\) where \( x_t = (x_{1t}, \ldots, x_{nt}) \) is the allocation profile at time \( t \).

At each time \( t \), each agent \( k \) has a value \( v_{k,t} \in [0, \bar{v}] \) per unit of the item received. By scaling values, we will assume that \( \bar{v} \geq 1 \). The value profile \( v_t = (v_{1,t}, \ldots, v_{n,t}) \) at time \( t \) is drawn from a distribution function \( F \) independently across different time periods. We emphasize that \( F \) is not necessarily a product distribution, so the values held by different agents can be arbitrarily correlated within each round.

Two special cases of our model are of particular interest in the context of ad auctions. In a single-slot ad auction, the “item” for sale is an opportunity to display one ad to the current user. A Multi-slot ad auction is a natural generalization in which multiple ads can be displayed in each round. The formal details are standard, we provide them in Appendix A.1.

Auctions and Budgets. At each time \( t \) the good is allocated using an auction mechanism that we now describe. In round \( t \), each agent \( k \in [n] \) first observes her value \( v_{k,t} \geq 0 \). After all agents have observed their values, each agent \( k \) then submits a bid \( b_{k,t} \geq 0 \) to the auction. All agents submit bids simultaneously. The auction is defined by an allocation rule \( x \) and a payment rule \( p_k \), where \( x(b) \in \mathcal{X} \) is the allocation profile generated under a bid profile \( b \), and \( p_k(b) \geq 0 \) is the payment made by agent \( k \). All of the auction formats we consider will have weakly monotone allocation and payment rules, so we will assume this for the remainder of the paper. That is, for any \( k \) and any bids of the other agents \( b_{-k} \), both \( x_k(b, b_{-k}) \) and \( p_k(b_k, b_{-k}) \) are weakly increasing in \( b_k \).

Each agent \( k \) has a fixed budget \( B_k \) that can be spent over the \( T \) time periods. Once an agent has run out of budget she can no longer bid in future rounds. We emphasize that this budget constraint binds ex post, and must be satisfied on every realization of value sequences. An important quantity in our analysis is the target expenditure rate for agent \( k \) which is defined by \( \rho_k = B_k / T \).

Each agent \( k \) applies a dynamic bidding strategy that, in each round \( t \), maps the observed history of auction outcomes and the realized valuation \( v_{k,t} \) to a bid \( b_{k,t} \). In a slight abuse of notation we will tend to write \( b(v) \) for the sequence of bid profiles that are generated when the sequence of value profiles is \( v \). In this paper we will focus on a particular class of bidding strategies (gradient-based pacing) described later in this section.

Given an execution of the auction over \( T \) rounds, we will typically write \( x_{k,t} \) for the realized allocation obtained by agent \( k \) in time \( t \), and \( z_{k,t} \) for the realized payment of agent \( k \) in time \( t \). This notation omits the dependency on the agents’ bids when this dependency is clear from the context.

Core Auctions. Our results apply to Core Auctions, a wide class of auction mechanisms which includes standard auctions such as first-price auctions, second-price auctions for single-slot allocations, and generalized second-price (GSP) auctions for multi-slot allocations. Roughly speaking, no subset of players (which may or may not include the seller) could jointly benefit by renegotiating the auction outcome among themselves. Stated more formally in our context, a core auction satisfies the following two properties:

\[ \text{The set } \mathcal{X} \subseteq [0, 1]^n \text{ is downward closed if, for any } x, x' \in [0, 1]^n \text{ such that } x \in \mathcal{X} \text{ and } x' \leq x_k \text{ for all } k, \text{ we have } x' \in \mathcal{X}. \]

\[ \text{All of the auctions we consider satisfy the property that each agent’s payment will be at most her bid, so it is always possible to bid in such a way that does not overspend one’s remaining budget.} \]

\[ \text{For the sake of completeness, these special cases are defined in Appendix A.2.} \]
1. The auction is individually rational (IR): each agent’s payment does not exceed her declared welfare for the allocation received. That is, \( p_k(b) \leq b_k x_k(b) \) for every agent \( k \) and every bidding profile \( b \).

2. The seller and any subset of agents \( S \subseteq [n] \) could not strictly benefit by jointly abandoning the auction and deviating to another outcome. That is, for any bidding profile \( b \) and any allocation profile \( y \in X \),
\[
\sum_{k \in S} p_k(b) + \sum_{k \in S} b_k x_k(b) \geq \sum_{k \in S} b_k y_k.
\] (2.1)

The left-hand (resp., right-hand) side of Eq. (2.1) is the total welfare obtained by the seller and the agents in \( S \) under the auction (resp., under a deviation to allocation \( y \)). Note that on both sides, the summation over agents in \( S \) accounts not only for the agent utilities, but also those agents’ payments to the seller.

The second property implies that a core auction must always maximize declared welfare. Indeed, taking \( S = [n] \), Eq. (2.1) states that \( \sum_k b_k x_k(b) \geq \sum_k b_k y_k \) for all feasible allocation profiles \( y \). Also, the first property is simply a restatement of the core condition that no subset of buyers could jointly benefit by renegotiating an auction outcome that does not include the seller, i.e. no buyer (and hence no set of buyers) would strictly prefer to switch to the null outcome in which no goods are allocated and no payments are made.

**Monotone Bang-Per-Buck.** Our guarantees on individual regret require an additional property of the auction mechanism called monotone bang-per-buck (MBB). Roughly speaking, if increasing one’s bid leads to an increased allocation, then the increase in payment per unit of new allocation is at least the minimum bid needed for the increase. Formally, an auction mechanism satisfies MBB if for any agent \( k \), any profile of bids \( b_{-k} \) of the other agents, and any bids \( b_k \leq b_k' \) of agent \( k \), we have
\[
p_k(b_k', b_{-k}) - p_k(b_k, b_{-k}) \geq b_k \times (x_k(b_k', b_{-k}) - x_k(b_k, b_{-k})).
\] (2.2)

One can readily check that the MBB property is satisfied by many common auction formats, such as first-price and second-price auctions as well as generalized second-price auctions (see Appendix A for further discussion). The name MBB is motivated by a simple observation that in a MBB auction each buyer’s expected payment per expected unit of allocation received is weakly increasing in their bid, for any distribution of competing bids. See, for example, inequality (4.9) in Section 4.3.

**Agent Objective.** To state individual regret guarantees, we must define an objective for each individual agent. Two natural, well-motivated candidates are (constrained) value maximization and utility maximization. We focus on the value maximization in this paper (which is increasingly standard in online advertising markets with automated bidding algorithms [14]), but also discuss utility maximization below. Our aggregate guarantees are independent of the agent objective.

A value-maximizing agent’s objective is to maximize the total value received subject to budget and maximum bid constraints. More specifically, a value-maximizing agent aims to maximize the total value of the allocation obtained over all rounds, \( \sum_t v_{k,t} x_{k,t} \), subject to the following two constraints. First, the total payment across all rounds must not exceed the budget: \( \sum_t z_{k,t} \leq B_k \) for all \( k \). Second, the bid placed in round \( t \) must not exceed the value of the good in round \( t \): \( b_{k,t} \leq v_{k,t} \) for all \( k \) and \( t \). This (constrained) value-optimization objective is particularly appealing as a model of automated bidding, in which the constraints and mandate to maximize value are interpreted as instructions to a bidding algorithm [5].
(Such instructions typically need to be relatively simple, as opposed to the actual advertiser’s objective, which may be more complex.)

In contrast, a utility-maximizing agent aims to maximize total quasi-linear utility, \( \sum t v_{k,t} x_{k,t} - z_{k,t} \), subject to the budget constraint \( \sum t z_{k,t} \leq B_k \). Implicitly, this formulation assumes that values are expressible in units of money. This objective is especially relevant in scenarios where the agents are the advertisers themselves.

We note that for any auction that is truthful for quasi-linear bidders within a single round, such as the second-price auction, the utility maximization problem reduces to the value maximization problem described above. Indeed, the ex-post optimal strategy for a utility-maximizing agent is to bid \( v_{k,t} \times \alpha \) in each round for the largest \( \alpha \in [0,1] \) such that the budget constraint is satisfied \([12]\), which also maximizes the total value received. So, our individual guarantees apply also to the objective of utility maximization whenever the underlying single-round auction mechanism is truthful.

**Liquid Welfare.** Liquid welfare is a measure of welfare that accounts for non-quasi-linear agent utilities. Intuitively, an agent’s liquid welfare for an allocation sequence is the agent’s maximum willingness to pay for the allocation. This generalizes the common notion of welfare in quasi-linear environments, and motivates the choice to measure welfare in transferable units of money. In our setting with a budget constraint that binds across rounds, liquid welfare is defined as follows.

**Definition 2.1.** Given a sequence of value profiles \( \nu = (v_{k,t}) \in [0,\bar{\nu}]^{nT} \) and any sequence of feasible allocations \( x = (x_{k,t}) \in X^T \), the liquid value obtained by agent \( k \) is \( W_k(x) = \min \{ B_k, \sum_{t=1}^T x_{k,t} v_{k,t} \} \).

The liquid welfare of allocation sequence \( x \in X^T \) is \( W(x) = \sum_{k=1}^n W_k(x) \).

We emphasize that liquid welfare depends on the allocations, but not on the agents’ payments.

Our objective of interest is the expected liquid welfare obtained by the platform over any randomness in the valuation sequence and the agents’ bidding strategies (which induces randomness in the allocation sequence). Since the bid placed in one round can depend on allocations obtained in the previous rounds, we define a mapping from the entire sequence of \( T \) value profiles to an allocation sequence. An allocation sequence rule is a function \( x: [0,\bar{\nu}]^{nT} \to X^T \), where \( x_{k,t}(\nu_1, \ldots, \nu_T) \) is the allocation obtained by agent \( k \) in round \( t \). Then the expected liquid welfare of allocation sequence rule \( x \) is \( \mathbb{E}_{\nu_1, \ldots, \nu_T \sim F}[W(x(\nu_1, \ldots, \nu_T))] \).

**Pacing Algorithm.** We use a budget-pacing algorithm motivated by stochastic gradient descent that was introduced and analyzed in \([12]\) in the context of second-price auctions. See \([Algorithm 1]\) Each bidder \( k \) maintains a pacing multiplier \( \mu_{k,t} \in [0,\bar{\mu}] \) for each round \( t \). Multiplier \( \mu_{k,t} \) is determined by the algorithm before the value \( v_{k,t} \) is revealed. The bid is set to \( v_{k,t}/(1 + \mu_{k,t}) \), or the remaining budget \( B_{k,t} \) if the latter is smaller. Once the round’s outcome is revealed, the multiplier is updated as per Line \([5]\), where \( P_{[a,b]} \) denotes the projection onto the interval \([a,b]\). Intuitively, the agent’s goal is to keep expenditures near the expenditure rate \( \rho_k \). Hence, if \( \rho_k \) is above (resp., below) the current expenditure rate \( z_{k,t} \), the agent decreases (resp., increases) her multiplier, by the amount proportional to the current “deviation” from the target expenditure rate \( \rho_k \).

Parameters \( \bar{\mu} \) and \( \epsilon_k \) will be specified later. While the upper bound \( \bar{\mu} \) could in general depend on the agent \( k \), as in \( \bar{\mu} = \bar{\mu}_k \), we suppress this dependence for the sake of clarity. Our bounds depend on \( \max_k \bar{\mu}_k \).

A convenient property of \([Algorithm 1]\) is that it does not run out of budget too early\(^{[14]}\).

\(^{[14]}\)This result is proven in \([12]\) (Eq. (A-4) of Theorem 3.3) for second-price auctions, but an inspection of their proof shows this holds surely for any sequence of valuations and competing prices so long the price is at most the bid if the agent wins.
Theorem 3.3. Fix any core auction and any distribution \( F \) over agent value profiles. Suppose that each agent \( k \) employs a generalized pacing algorithm to bid, possibly with a different step-size \( \epsilon_k \). Write \( x : [0, \overline{v}]^n \rightarrow X^T \) for the corresponding allocation sequence rule. Then for any allocation rule \( y : [0, \overline{v}]^n \rightarrow X \) we have

\[
\mathbb{E}_{\mathbf{v}_1, \ldots, \mathbf{v}_T \sim F} [W(x(\mathbf{v}_1, \ldots, \mathbf{v}_T))] \geq \frac{W(y, F)}{2} - O \left( \sqrt{T \log (\overline{v}^2 nT)} \right).
\] (3.1)
Remark 3.4. In the proof, it is not necessary for each advertiser \( k \) to employ a fixed step-size \( \epsilon_k \) throughout the entire time horizon: it suffices if \( \epsilon_k \) is constant within each epoch. This is needed for Lemma 3.8.

Remark 3.5. It is natural to ask whether this analysis can apply to other autobidding algorithms beyond generalized pacing. Our dependence on the algorithm’s details is mostly captured by Lemma 3.8, which is itself a corollary of Eq. (3.2): that the amount spent over a sequence of rounds with non-trivial pacing depends only on the starting and ending bids, but otherwise not the learning path between. Thus, while we believe that liquid welfare guarantees can be obtained for many autobidding algorithms, we suspect that extending our analysis to learning methods that do not satisfy such a path-independence property would require substantial new ideas.

3.1 Proof Intuition

Before proving Theorem 3.3, let us sketch the proof and provide intuition. One easy observation is that since \( B_k \) is an upper bound on the liquid welfare obtainable by agent \( k \), any agents who are (approximately) exhausting their budgets in our dynamics are achieving optimal liquid welfare. We therefore focus on agents who do not exhaust their budgets. We’d like to argue that such agents are often bidding very high, frequently choosing pacing multipliers equal to 0 (i.e., bidding their values). This would be helpful because our auction is assumed to be a core auction, which implies that either the high-bidding agents are winning (and generating high liquid welfare) or other, budget-exhausting agents are generating high revenue for the seller (which likewise implies high liquid welfare).

Why should agents who are underspending their budget be placing high bids in many rounds? While it’s true that the pacing dynamics increases the next-round bid whenever spend is below the per-round target, this is only a local adjustment and does not depend on total spend. We make no convergence assumptions about the dynamics and, as we show in Appendix 3, regret bounds do not directly imply a bound on liquid welfare. So how do we analyze bidding patterns in aggregate across rounds?

It is here where we use the fact that agents use generalized pacing algorithms. For each \( k \), the evolution of the multipliers \( \mu_{k,t} \) has the following convenient property: over any contiguous range of time steps \([t_1, t_2]\) such that the multiplier is never 0 or \( \overline{\mu} \), the total amount spent from time \( t_1 \) to time \( t_2 \) is determined by \( \mu_{k,t_2} - \mu_{k,t_1} \). More precisely,

\[
    
    t_2 \sum_{t = t_1}^{t_2-1} z_{k,t} = (t_2 - t_1 - 1) \rho_k + \frac{1}{\epsilon_k}(\mu_{k,t_2} - \mu_{k,t_1})
\]

(3.2)

which follows immediately from the update rule of \( \mu_{k,t} \) on line 5 of Algorithm 1: if the multipliers are never 0 or \( \overline{\mu} \) then the projection operator is never invoked, so we have \( \mu_{k,t+1} = \mu_{k,t} - \epsilon_k(\rho_k - z_{k,t}) \) and (3.2) follows.

Motivated by this observation, we introduce the notion of an epoch: essentially, a maximal contiguous sequence of rounds in which an agent’s pacing multiplier is strictly greater than 0. In Lemma 3.8 we show that an agent \( k \)’s total spend over an epoch of length \( t \) must be (approximately) \( t \rho_k \). In other words, the average spend over an epoch approximately matches the target per-round spend. This is a direct implication of the update rule for the pacing multiplier: since multiplier increases and decreases balance out (approximately) over the course of an epoch, the budget deficits and surpluses must balance out as well. An immediate implication is that an agent whose total spend is much less than \( T \rho_k \) must often have her pacing multiplier set equal to 0.

To summarize, each agent either spends most of her budget by time \( T \) or spends many rounds bidding her value. It might seem that by combining these two cases we should obtain a constant approximation factor, and not just in expectation but for every realized value sequence \( v \). But this is too good to be true. Indeed, this proof sketch misses an important subtlety: whether an agent exhausts her budget or not depends on the
realization of the value sequence, which is also correlated with the benchmark allocation $y$. For example, what if an agent under-spends her budget precisely on those value sequences where it would have been optimal (according to the benchmark $y$) for her liquid welfare to equal her budget? This could result in a situation where, conditional on $\mu_{k,t} = 0$, the value obtained by the benchmark is much higher than expected and cannot be approximated. To compare against the liquid welfare of $y$ we must control the extent of this correlation. To this end we employ a variation of the Azuma-Hoeffding inequality (Lemma 3.9) to argue that since value realizations are independent across time, the pacing sequence and the benchmark allocation cannot be too heavily correlated.

### 3.2 Proof Preliminaries

Before getting into the details of the proof of Theorem 3.3 we first introduce some notation and define an epoch more formally. Write $\mu_{k,t} = b$ if at time $t$, the agent’s algorithm has stopped, i.e., if the agent is out of money or if $t = T + 1$. For $t_1 \leq t_2 \in \mathbb{N}$, we slightly abuse notation and write $[t_1, t_2] \triangleq \{t_1, \ldots, t_2\}$ to be the set of integers between them, inclusive, when the meaning is clear. Similarly, we write $[t_1, t_2)$ for the half-open set of integers between them and analogously for $(t_1, t_2]$ and $(t_1, t_2)$.

**Definition 3.6.** Fix an agent $k$, time horizon $T$, and a sequence of pacing multipliers $\mu_{k,1}, \ldots, \mu_{k,T}$. A half-open interval $[t_1, t_2)$ is an epoch with respect to these multipliers if it holds that $\mu_{t_1} = 0$ and $\mu_t > 0$ for each $t_1 < t < t_2$ and $t_2$ is maximal with this property.

**Remark 3.7.** Because we initialized any generalized pacing algorithm at $\mu_{k,1} = 0$ for all $k \in [n]$, the epochs completely partition the set of times the agent is bidding. Moreover, if the pacing multipliers at times $t_1$ and $t_1 + 1$ are both zero, then $[t_1, t_1 + 1)$ is an epoch; we refer to this as a trivial epoch.

The following lemma shows that an agent’s total spend over a maximal epoch can be bounded from below by an amount roughly equal to the target spend rate $\rho_k$ times the epoch length, plus an adjustment for the first round of the epoch. This lemma is crucial in proving Theorem 3.3 and is proved in Appendix B.

**Lemma 3.8.** Fix an agent $k$, fix any choice of core auction and any sequence of bids of other agents $(b_{-k,1}, \ldots, b_{-k,T})$. Fix any realization of values $(v_{k,1}, \ldots, v_{k,T})$ for agent $k$ and suppose that $\mu_{k,1}, \ldots, \mu_{k,T}$ is the sequence of multipliers generated by the generalized pacing algorithm given the auction format and the other bids. Then for any epoch $[t_1, t_2)$ where $\mu_{k,t_2} \neq b$, we have

\[
\sum_{t=t_1}^{t_2-1} x_{k,t}v_{k,t} \geq x_{k,t_1}v_{k,t_1} - z_{k,t_1} + \rho_k \cdot (t_2 - t_1 - 1). \tag{3.3}
\]

Next, motivated by the intuition in Section 3.1, we introduce a concentration inequality that will be helpful for our analysis. Roughly speaking, we will use this lemma to show that the sequence of values obtained by agent $k$ in the benchmark on rounds in which $\mu_{k,t} = 0$ are not “far from expectation,” in the sense that the total expected value obtained over such rounds is not much greater than $\rho_k$ per round.

**Lemma 3.9.** Let $Y_1, \ldots, Y_T$ be random variables and $\mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_T$ be a filtration such that:

1. $0 \leq Y_t \leq \overline{y}$ with probability 1 for some parameter $\overline{y} \geq 0$ for all $t$.
2. $\mathbb{E}[Y_t] \leq \rho$ for some parameter $\rho \geq 0$ for all $t$.
3. For all $t$, $Y_t$ is $\mathcal{F}_t$-measurable but is independent of $\mathcal{F}_{t-1}$.

Suppose that $X_1, \ldots, X_n \in [0, 1]$ are random variables such that $X_t$ is $\mathcal{F}_{t-1}$-measurable. Then

\[
\Pr \left( \sum_{t=1}^{T} X_t Y_t + (1 - X_t) \rho \geq \rho \cdot T + \theta \right) \leq \exp \left( -\frac{2\theta^2}{T \rho^2} \right). \tag{3.4}
\]

The proof of Lemma 3.9 appears in Appendix B. We are now ready to prove Theorem 3.3.
3.3 Proof of Theorem 3.3

We first introduce some notations. Fix the generalized pacing dynamics algorithm used by each agent. We will then write \( x = \{x_{k,t}\}_{k \in [n], t \in [T]} \) for the random variable corresponding to the allocation obtained under these bidding dynamics, given the values \( \{v_{k,t}\} \). We also write \( \mu_{k,t} \) for the pacing multiplier of agent \( k \) in round \( t \), and \( z_{k,t} \) for the realized spend of agent \( k \) in round \( t \), which are likewise random variables. For notational convenience we will write \( \text{WEL}_{\text{GPD}}(v) \) for the liquid welfare (where GPD stands for “Generalized Pacing Dynamics”) given valuation sequence \( v \). That is,

\[
\text{WEL}_{\text{GPD}}(v) \triangleq \sum_{k=1}^{n} \min \left\{ B_k, \sum_{t=1}^{T} x_{k,t} v_{k,t} \right\}.
\]  

We also write \( \text{WEL}_{k,GPD}(v) \) for the liquid welfare obtained by agent \( k \), and \( \text{WEL}_{GPD}(F) \) for the total expected liquid welfare \( \mathbb{E}_{v \sim F} \left[ \text{WEL}_{GPD}(v) \right] \).

We claim that to prove Theorem 3.3, it is sufficient to show that inequality (3.1) holds for any allocation rule \( y \) such that

\[
\mathbb{E}_{v \sim F}[y_{k}(v)v_{k}] \leq \rho_{k} \quad \text{and} \quad \mathbb{W}_{k}(y, F) = T \cdot \mathbb{E}_{v \sim F}[y_{k}(v)v_{k}] \leq \rho_{k} \cdot T \quad \forall k \in [n].
\]  

This is sufficient because if one of the above conditions is violated, one can always decrease the allocation for agent \( k \), which maintains the feasibility (since we assume that the set of feasible allocations \( X \) is downward closed) without affecting the ex ante liquid welfare \( \mathbb{W}(y, F) \). We will therefore assume without loss that \( y \) satisfies (3.6).

Preliminaries completed, we now prove Theorem 3.3 in three steps. First, we will define a “good” event in which the benchmark allocations are not too heavily correlated with the pacing multipliers of this value is replaced by the target spend \( \rho_{k} \).

We would like to apply Lemma 3.9 to bound \( R_{k}(v) \). Some notation: we will write \( Y_{t} = y_{k}(v_{t})v_{k,t} \) and \( X_{t} = 1\{\mu_{k,t} = 0\} \) with \( F_{t} = \sigma(v_{1}, \ldots, v_{t}) \). Then because \( \mu_{k,t} \) is \( F_{t-1} \) measurable and the sequence \( \{v_{j}\} \) is a sequence of independent random variables, Lemma 3.9 implies that with probability at least \( 1 - \frac{1}{(\bar{v}nT)^{2}} \), we have

\[
\sum_{t=1}^{T} \left[ 1\{\mu_{k,t} = 0\}y_{k}(v)v_{k,t} + 1\{\mu_{k,t} \neq 0\}\rho_{k} \right] \leq \rho_{k} \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}.
\]

Taking a union bound over \( k \in [n] \), with probability at least \( 1 - \frac{1}{(\bar{v}nT)^{2}} \) over the randomness in the sequence \( v = (v_{1}, \ldots, v_{T}) \), we have that

\[
R_{k}(v) \leq \rho_{k} \cdot T + \bar{v} \sqrt{T \log(\bar{v}nT)}, \quad \forall k \in [n].
\]  

We will write \( E_{\text{GOOD}} \) for the event in which (3.7) holds. Going back to the intuition provided before the statement of Lemma 3.9, \( E_{\text{GOOD}} \) is the event that the value each agent obtains in the benchmark allocation \( y \) is not “too high” on rounds in which their pacing multipliers are 0.
Step 2: A Bound on Liquid Welfare For “Good” Value Realizations. Fix any realized sequence \( v_1, \ldots, v_T \) such that (3.7) holds. We will now proceed to derive a lower bound on the liquid welfare of the agents (under allocation \( x \)) by considering the two different possible cases for \( \text{WEL}_{k,GPD}(v) \). Recall that the liquid welfare \( \text{WEL}_{k,GPD}(v) \) of any agent \( k \) is either \( B_k = \rho_k \cdot T \) or is \( \sum_{t=1}^{T} x_{k,t}v_{k,t} \). For any \( k \) such that \( \text{WEL}_{k,GPD}(v) = B_k \), we obtain via (3.7) that

\[
\text{WEL}_{k,GPD}(v) = B_k \geq R_k(v) - \overline{v} \sqrt{T \log(\overline{v}nT)}.
\] (3.8)

We now consider all of the remaining agents. Let \( A \subseteq [n] \) be the set of agents \( k \) such that \( \sum_{t=1}^{T} x_{k,t}v_{k,t} < B_k \) on this realized sequence \( v \), and hence \( \text{WEL}_{k,GPD}(v) = \sum_{t=1}^{T} x_{k,t}v_{k,t} \). That is, each of their contributions to the liquid welfare on this realized sequence is uniquely determined by their true value for winning the items in the auction. Note that this further implies that no agent in \( A \) runs out of budget early because the generalized pacing algorithm does not allow overbidding (i.e., bidding above the value). Thus for all \( k \in A, \mu_k,T \neq b \) and \( \mu_k,t \geq 0 \) surely for all \( t \). We claim that

\[
\sum_{k \in A} \text{WEL}_{k,GPD}(v) = \sum_{k \in A} \sum_{t=1}^{T} x_{k,t}v_{k,t} \geq \sum_{k \in A} R_k(v) - \sum_{k \in [n]} \sum_{t=1}^{T} z_{k,t}.
\] (3.9)

To show that the inequality holds, we partition the interval \([1, T]\) into maximal epochs for each agent \( k \) and bound the value obtained by agent \( k \) on each maximal epoch separately. Fix any agent \( k \in A \) and suppose that \([t_1, t_2]\) is a maximal epoch inside \([1, T]\). Since agent \( k \) does not run out of budget on the entire interval \([1, T]\), she does not run out of budget on this epoch in particular. We can therefore apply Lemma 3.8 to obtain

\[
\sum_{t=t_1}^{t_2-1} x_{k,t}v_{k,t} \geq x_{k,t_1}v_{k,t_1} - z_{k,t_1} + \rho_k \cdot (t_2 - t_1 - 1).
\]

Since \([1, T]\) can be partitioned into maximal epochs for each agent \( k \in A \), we can sum over all time periods and apply the definition of a maximal epoch to conclude that

\[
\sum_{t=1}^{T} x_{k,t}v_{k,t} \geq \sum_{t=1}^{T} [1\{\mu_k,t = 0\}(x_{k,t}v_{k,t} - z_{k,t}) + 1\{\mu_k,t \neq 0\} \cdot \rho_k].
\]

Summing over all \( k \in A \) and changing the order of the summations implies that

\[
\sum_{k \in A} \sum_{t=1}^{T} x_{k,t}v_{k,t} \geq \sum_{t=1}^{T} \sum_{k \in A} [1\{\mu_k,t = 0\}(x_{k,t}v_{k,t} - z_{k,t})] + \sum_{k \in A} \sum_{t=1}^{T} 1\{\mu_k,t \neq 0\} \cdot \rho_k.
\] (3.10)

We will now use the assumption that the auction is a core auction. For any \( t \in [1, T] \), let \( S \subseteq A \) be the set of agents in \( A \) for which \( \mu_k,t = 0 \). We have that

\[
\sum_{k \in A} [1\{\mu_k,t = 0\}(x_{k,t}v_{k,t} - z_{k,t})] = \sum_{k \in S}(x_{k,t}v_{k,t} - z_{k,t})
\]

\[
\geq \sum_{k \in S} y_k(v_t)v_{k,t} - \sum_{k=1}^{n} z_{k,t} = \sum_{k \in A} 1\{\mu_k,t = 0\}y_k(v_t)v_{k,t} - \sum_{k=1}^{n} z_{k,t}.
\]

The inequality follows from the definition of a core auction and the no unnecessary pacing condition (which implies \( b_{k,t} = v_{k,t} \) for all \( k \in S \)), by considering the deviation in which the agents in \( S \) jointly switch to allocation \( \{y_k(v_t)\} \). Substituting the above inequality into (3.10) and rearranging yields (3.9).

Summing over (3.8) for each \( k \notin A \) and combining it with (3.9) we obtain that as long as inequality (3.7) holds (i.e., event \( E_{GOOD} \) occurs), we have

\[
\sum_{k \in [n]} \text{WEL}_{k,GPD}(v) \geq \sum_{k \in [n]} R_k(v) - \sum_{k \in [n]} \sum_{t \in [T]} z_{k,t} - n\overline{v} \sqrt{T \log(\overline{v}nT)}.
\] (3.11)
Step 3: A Bound on Expected Liquid Welfare. Since the liquid welfare is nonnegative, we can take expectations over \(v_1, \ldots, v_T\) to conclude from (3.11) that

\[
E \left[ \sum_{k=1}^{n} WEL_{K, GPD}(v) \right] \geq E \left[ 1 \{ \text{GOOD} \} \cdot \sum_{k=1}^{n} WEL_{K, GPD}(v) \right] \\
\geq E \left[ 1 \{ \text{GOOD} \} \cdot \sum_{k=1}^{n} R_k(v) \right] - E \left[ \sum_{k=1}^{n} \sum_{t=1}^{T} z_{k,t} \right] - n\bar{v}\sqrt{T\log(\bar{v}nT)}, \tag{3.12}
\]

where the last inequality holds via (3.11).

It remains to analyze the expectations on the right side of the inequality. First, note that

\[
E \left[ 1 \{ \text{GOOD} \} \cdot \sum_{k=1}^{n} R_k(v) \right] = E \left[ \sum_{k=1}^{n} R_k(v) \right] - E \left[ (1 - 1 \{ \text{GOOD} \}) \sum_{k=1}^{n} R_k(v) \right] \\
\geq E \left[ \sum_{k=1}^{n} R_k(v) \right] - \frac{n\bar{v}}{\bar{v}nT} = E \left[ \sum_{k=1}^{n} R_k(v) \right] - 1/T.
\]

The inequality holds due to the fact that \(R_k(v) \leq \bar{v}T\) as well as our bound on the probability that event (3.7) does not hold. Let \(q_{k,t}\) be the unconditional probability that \(\mu_{k,t} = 0\). Then

\[
E \left[ R_k(v) \right] = \sum_{t=1}^{T} E \left[ 1 \{ \mu_{k,t} = 0 \} \cdot E[y_k(v_{k,t})v_{k,t} | F_{t-1}] + 1 \{ \mu_{k,t} \neq 0 \} \cdot \rho_k \right] \\
= \sum_{t=1}^{T} [q_{k,t} E[y_k(v_k)v_k] + (1 - q_{k,t}) \rho_k] \geq \sum_{t=1}^{T} E[y_k(v_k)v_k] = \overline{W}_k(y, F).
\]

The first inequality uses the conditional independence of \(y_k(v_{k,t})v_{k,t}\) on \(\mu_{k,t}\), as this is already determined by time \(t - 1\). The inequality holds since \(E[y_k(v_k)v_k] \leq \rho_k\) according to our assumption on \(y\). Substituting the inequalities into (3.12) we obtain that

\[
WEL_{GPD}(F) \geq \sum_{k=1}^{n} \overline{W}_k(y, F) - E \left[ \sum_{k=1}^{n} \sum_{t=1}^{T} z_{k,t} \right] - n\bar{v}\sqrt{T\log(\bar{v}nT)} - 1/T. \tag{3.13}
\]

Recall that

\[
E \left[ \sum_{k=1}^{n} \sum_{t=1}^{T} z_{k,t} \right] = \sum_{k=1}^{n} \sum_{t=1}^{T} E[z_{k,t}] = \sum_{k=1}^{n} E[P_k] \leq WEL_{GPD}(F), \tag{3.14}
\]

where \(P_k\) is the total expenditure of agent \(k\), and this is upper bounded by the liquid value they obtain. Substituting into (3.13) and rearranging the terms, then noting that \(WEL_{GPD}(F)\) is precisely the left-hand side of (3.1), we conclude that inequality (3.1) holds.

4 Individual Guarantee: Vanishing Regret for MBB Auctions

In this section, we supplement our aggregate welfare guarantees with bounds on the individual performance of Algorithm 1 when used in any auction that satisfies the monotone bang-for-buck (MBB) condition. We focus on a particular single agent \(k\), henceforth called simply the agent. Recall that the agent faces an online bidding problem such that the objective is value maximization subject to the budget constraint. The action set consists of pacing multipliers \(\mu \geq 0\) like in Algorithm 1. That is, the agents are restricted to linear values-to-bids policies without overbidding.

We make no assumptions on the behavior of the other agents. Their bidding strategies \(b_{-k}\) can depend arbitrarily on the realized values and the observed history. In each round \(t\), let \(G_t\) be the joint distribution of value \(v_{k,t}\) and other agents’ bids \(b_{-k,t}\) given the history in the previous rounds. Thus, one can think of pair \((v_{k,t}, b_{-k,t})\) as being drawn from a distribution \(G_t\) that could depend on the history up to time \(t\).

The dependence between the agent’s problem on \(G_t\) is captured via the following notation. Let \(V_t(\mu_{k,t})\) and \(Z_t(\mu_{k,t})\) be, respectively, the agent’s expected value and expected payment in round \(t\) for a particular
multiplier $\mu_{k,t}$, where the expectation is taken over the distribution $G_t$. More formally, for a given auction defined by an allocation rule $x$ and a payment rule $p$ (see Section 2) and a given distribution $G_t$ we have

$$Z_t(\mu_{k,t}) = \mathbb{E} \left[ p_k \left( b_{k,t}, b_{-k,t} \right) \right] \quad \text{and} \quad V_t(\mu_{k,t}) = \mathbb{E} \left[ v_{k,t} \cdot x_k \left( b_{k,t}, b_{-k,t} \right) \right],$$

(4.1)

where the bid is $b_{k,t} = v_{k,t}/(1 + \mu_{k,t})$ and the expectations are over $(v_{k,t}, b_{-k,t}) \sim G_t$.

We obtain regret relative to a non-standard benchmark called the perfect pacing sequence: essentially, a sequence $(\mu^*_1, \ldots, \mu^*_T)$ such that the expected spend $Z_t(\mu_t^*)$ is exactly $\rho$ in each round $t$ (the precise definition is below). Our guarantee holds as long as the environment does not change too quickly, as quantified by the path-length $\sum_{t=1}^{T-1} |\mu_{t+1}^* - \mu_t^*|$. In the important special case of stochastic environment — when the distribution $G_t$ is the same in all rounds — we obtain regret relative to the best fixed pacing multiplier in hindsight.

Remark 4.1. The perfect pacing sequence $\mu^*$ does not necessarily optimize value in hindsight, and this is by design. More formally, after the auction is run for $T$ rounds, let $G_1, \ldots, G_T$ be the resulting sequence of distributions. Fixing these distributions, let $V(\mu) = \sum_{t \in [T]} V_t(\mu_{k,t})$ be the expected value collected by agent $k$ with a particular sequence of pacing multipliers $\mu = (\mu_{k,1}, \ldots, \mu_{k,T})$. Then $\mu^*$ does not necessarily optimize $V(\mu)$ among all $\mu$. However, one should not hope to obtain regret against $\sup_{\mu} V(\mu)$ because of the spend-or-save dilemma, as discussed in Section 1.2. While inevitable, this situation is unsatisfying. One interpretation is that the standard benchmark of the best-in-hindsight multiplier is “too hard” in the adversarial environment, and a more “fair” alternative benchmark is needed. The reason our benchmark admits vanishing regret is precisely that it gives up on solving the spend-or-save dilemma, and instead optimizes for each round separately.

Now let us state our results more precisely. We will drop the dependence on the agent’s index $k$ from our notation. For each round $t$, we write $v_t, \mu_t$, for the agent’s value and multiplier, and $b_t = v_t/(1 + \mu_t)$ for its bid.

Consider $V_t(\cdot)$ and $Z_t(\cdot)$ as functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Both functions are monotonically non-decreasing, by monotonicity of the auction allocation and payment rules. It will also be convenient for us to assume that the expenditure function $Z_t$ almost surely satisfies the following smoothness assumptions:

**Assumption 4.2.** There exists $\lambda \geq 0$ such that $Z_t$ is $\lambda$-Lipschitz for all $t$.

**Assumption 4.3.** There exists $\delta > 0$ such that $Z_t(0) \geq \delta$ for all $t$. As our bounds will depend inversely on $\delta$, we assume without loss of generality that $\delta \leq \rho$.

**Remark 4.4.** We argue that these assumptions are mild in the context of advertising auctions. Indeed, the variation in impression types, click rate estimates, etc., and possibly also the randomness in the auction and/or the bidding algorithms would likely introduce some smoothness into the expected payment $Z_t(\cdot)$, and ensure that the maximum allowable bid would result in a non-trivial payment. In fact, one could eliminate the need for these assumptions by adding a small amount of noise to the auction allocation rule, such as by perturbing bids slightly, and/or a small but positive reserve price, at the cost of a similarly small loss of welfare. Alternatively, the assumptions hold when all agents use bounded pacing multipliers, the joint value profile distribution is sufficiently smooth (for Assumption 4.2) and with probability at least $\epsilon' > 0$, agent $k$ has the highest value while some other agent bids at least $\delta' > 0$ (for Assumption 4.3).

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15 We only require the weaker condition: $Z_t(0) \geq \delta \times \frac{V_t(0)}{\lambda}$ for all $t$. In particular, this allows $Z_t(0) = 0$ if $V_t(0) = 0$.

16 If the pricing rule is non-decreasing in the agent’s bid given the other bids (which is the typical case), Assumption 4.2 holds (for some $\lambda > 0$) the agent’s conditional valuation given the other bids almost surely admits a density that is bounded pointwise by some absolute constant $\sigma > 0$. More generally, Assumption 4.2 holds for any bounded pricing rule if the joint valuation distribution has a Lipschitz density, since then the induced density on joint bids is nicely behaved as a function of the agent’s multiplier.
We next formalize our notion of the perfect pacing sequence.

**Definition 4.5.** Fix some $\overline{\mu} \geq \overline{\nu}/\rho$, where $\overline{\nu}$ is an upper bound on the values. For each round $t$, the perfect pacing multiplier is any $\mu^*_t \in [0, \overline{\mu}]$ such that $Z_t(\mu^*_t) = \rho$, or $\mu^*_t = 0$ if $Z_t(0) < \rho$. A perfect pacing sequence is $(\mu^*_1, \ldots, \mu^*_T)$, where each $\mu^*_t$ is a perfect pacing multiplier.

**Remark 4.6.** To see that $\mu^*_t$ is well-defined, note that $Z_t(\overline{\mu}) \leq \overline{\nu}/(1 + \overline{\mu}) \leq \rho$. The Lipschitzness and monotonicity of $Z_t$ therefore imply that if there does not exist any $\mu$ such that $Z_t(\mu) = \rho$, then it must be that $Z_t(0) < \rho$ (and hence $Z_t(\mu) < \rho$ for all $\mu \in [0, \overline{\mu}]$). This latter case occurs if even bidding the true value $v_t$ generates an expected spend less than $\rho$. In this case we take $\mu^*_t = 0$, which corresponds to the most that the agent can bid without violating the mandate to not bid more than the value $v_t$.

We are interested in regret relative to the perfect pacing sequence, called pacing regret:

$$R_{\text{pace}}(T) \triangleq \sum_{t \in [T]} \mathbb{E}[V_t(\mu^*_t)] - \sum_{t \in [T]} \mathbb{E}[V_t(\mu_t)].$$  \hfill (4.2)

With these definitions in place, we are ready to state our main result of this section: an upper bound on the agent’s pacing regret.

**Theorem 4.7.** Consider a repeated auction that is individually rational (IR) and satisfies monotone bang-per-buck (MBB) property. Posit Assumptions 4.2 and 4.3. Suppose the path-length $P^* = \sum_{t=1}^{T-1} |\mu^*_{t+1} - \mu^*_t|$ is upper-bounded by some number $P$ with probability 1. Then, Algorithm 1 with step-size $\epsilon$ has pacing regret

$$R_{\text{pace}}(T) < O \left( \frac{\overline{\nu}}{\delta} \sqrt{2\lambda T \cdot \text{REG}_1} + \frac{\overline{\mu} \sqrt{T}}{\rho} \right), \quad \text{where } \text{REG}_1 \triangleq \frac{P+1}{\epsilon} \cdot \overline{\mu}^2 + \epsilon T \cdot (\rho + \overline{\mu})^2. \hfill (4.3)$$

**Corollary 4.8.** In the setting of Theorem 4.7 assume that parameters $(\overline{\nu}, \overline{\mu}, \lambda, \delta)$ are absolute constants. Then:

(a) $R_{\text{pace}}(T) < O \left( \sqrt{T \left( \frac{P+1}{\epsilon} + \epsilon T \right)} \right)$.

(b) Taking step-size $\epsilon = 1/\sqrt{T}$ (not knowing $P$), we obtain $R_{\text{pace}}(T) < \sqrt{P+1} \cdot O \left( T^{3/4} \right)$.

(c) If $P$ is known, taking $\epsilon = \sqrt{\frac{P+1}{T}}$ yields an improved bound, $R_{\text{pace}}(T) < (P+1)^{1/4} \cdot O \left( T^{3/4} \right)$.

### 4.1 Discussion, Corollaries and Extensions

**Theorem 4.7** treats the auction environment as a black box that produces a $(v_{k,t}, b_{-k,t})$ pair for each round $t$ according to some distribution $G_t$. The theorem makes no explicit assumptions on the other agents’ algorithms or their aggregate behavior. This approach is desirable since the agents may be reluctant to follow a particular bidding algorithm, and even when they do, the aggregate behavior is not well-understood.

We emphasize that the environment, as expressed by distribution $G_t$, can change over time. The dependence on this change is summarized with a uniform upper bound $P$ on path-length $P^* = \sum_{t=1}^{T-1} |\mu^*_{t+1} - \mu^*_t|$. We obtain a non-trivial guarantee for an arbitrary non-stationary environment with $P = o(T)$ if $P$ is known, and $P = o(\sqrt{T})$ otherwise. Note that path-length is a rather weak notion of change: in particular, distribution $G_t$ can change a lot from one round to another while $\mu^*_t$ stays the same.

Our guarantees go beyond slight perturbations of the same stochastic environment: in fact, our environment may be very far from any one stochastic environment. To make this point concrete, consider a “piecewise stochastic” environment, when the sequence of distributions $G_1, \ldots, G_T$ has a few “switches” (changes from one round to another) and stays the same in between.
Corollary 4.9. If the sequence \( G_1, \ldots, G_T \) has at most \( N \) switches, the Theorem 4.7 holds with \( P = \overline{\mu} N \).

Our guarantees extend beyond the original model: the marginal distribution of the agent’s value and the auction itself can change over time, as long as the assumptions hold.

Extension 4.10. Theorem 4.7 holds as written if the distribution from which the value profile \( v_t \) is sampled and the auction itself can arbitrarily change from one round to another, possibly depending on the previous rounds.

For the stochastic environment, the perfect pacing multiplier is close to the value-optimizing fixed multiplier, and we obtain a regret bound against the latter. Formally, suppose distribution \( G_t \) does not change over time almost surely. Then neither does the perfect pacing multiplier or the \( V_t(\cdot) \) function, so \( \mu_t^* = \mu^* \) and \( V_t = V \). Let \( Y(\mu) \) be the expected value of the bidding strategy that uses the same pacing multiplier \( \mu \) for each round \( t \), bidding \( b_t = v_t/(1 + \mu) \), until running out of budget. Our benchmark is the pacing multiplier that maximizes \( Y(\cdot) \). Thus, our result is as follows:

Corollary 4.11. In the setting of Theorem 4.7, suppose distribution \( G_t \) does not change over time almost surely and assume that parameters \( (\overline{\tau}, \overline{\mu}, \lambda, \delta) \) are absolute constants. Taking step-size \( \epsilon = 1/\sqrt{T} \), we obtain

\[
\sum_{t \in [T]} \mathbb{E}[V_t(\mu_t)] \geq \sup_{\mu \in [0, \overline{\mu}]} Y(\mu) - O(T^{3/4}).
\]

Corollary 4.11 is proved by relating \( \sup_\mu Y(\mu) \) and \( V(\mu^*) \) (this is spelled out in Appendix D.1) and invoking Corollary 4.8(b) with \( P = 0 \).

Taking a step back, what types of results are desirable for individual guarantees? The stochastic environment is commonly understood as a “minimal” desiderata, as a relatively simple special case for which vanishing regret is feasible. A common motivation is that a “small” bidder enters a “large” market in which the bidding dynamics has already converged to a (possibly randomized) stationary state. Of course, the downside is that realistic environments are neither stationary nor guaranteed to converge. The other extreme is to treat the auction environment as a “black box”, like we do in Theorem 4.7 and possibly obtain improved guarantees for some “nice” special cases. In particular, path-length \( P^* \) can quantify convergence to a “well-behaved” environment in which the perfect pacing multiplier does not change over time, so that our guarantee is strong when the convergence is sufficiently fast. However, such “black-box” guarantees for budget-constrained bidders tend to be weak or vacuous in the worst case.

A third approach would be to explicitly take advantage of the fact that all agents are controlled by bidding algorithms with particular properties, e.g., by instances of the same bidding algorithm. In particular, if all other agents use our algorithm with step size \( \epsilon' \) (and parameters \( (\overline{\tau}, \overline{\mu}, \lambda, \delta) \) are absolute constants), one can show that the perfect pacing multiplier changes by at most \( O(\epsilon') \) in each round, so that the path-length \( P^* \) can be upper-bounded as \( O(\epsilon'T) \). Plugging this back into Corollary 4.8(b), we obtain vanishing regret with step-size \( \epsilon' = O(1/\sqrt{T}) \).

Corollary 4.12. Consider a repeated auction that is individually rational (IR) and satisfies monotone bang-per-buck (MBB) property. Assume that parameters \( \overline{\tau}, \overline{\mu} \) are absolute constants. Suppose all agents use Algorithm 1 so that agent \( k \) has step-size \( \epsilon_k = 1/\sqrt{T} \) and all other agents have step-size \( \epsilon' = f(T)/\sqrt{T} \), where \( f(T) \to 0 \). Posit that Assumptions 4.2 and 4.3 are satisfied (see Remark 4.4) for some absolute-constant \( \lambda \) and \( \delta \). Then for agent \( k \), the path-length \( P^* \) is at most \( O(\epsilon'T) \) with probability 1, so the right-hand side of (4.3) is at most \( O \left( T \cdot \sqrt{f(T)} \right) \).

Note that this analysis yields vanishing regret only for a particular agent \( k \), whereas for all other agents the guarantee that comes out of our analysis is vacuous. And if all agents use the same parameter \( \epsilon = 1/\sqrt{T} \), then we can only guarantee path-length \( P^* = O(\sqrt{T}) \), which does not suffice to guarantee vanishing regret. We leave open the question of whether it is possible to obtain vanishing regret simultaneously for all agents (with any bidding algorithms).
4.2 Stochastic Gradient Descent (SGD) interpretation

To prove Theorem 4.7 it will be helpful to interpret Algorithm 1 as using stochastic gradient descent (SGD), a standard algorithm in online convex optimization (see Appendix C). Here, SGD uses pacing multipliers \( \mu \) as actions, and optimizes an appropriately-defined artificial objective \( H_t(\cdot) \) in each round \( t \), where

\[
H_t(\mu) = \rho \cdot \mu - \int_0^\infty Z_t(x) dx.
\]

The per-round spend \( z_t \) provides a stochastic signal that in expectation equals the gradient of \( H_t \):

\[
H_t'(\mu_t) = \rho - Z_t(\mu_t) = \rho - \mathbb{E}_{G_t}[z_t(\mu_t)].
\]

The SGD machinery applies because the function \( H_t \) is convex and \((\bar{\tau} + \rho)\)-Lipschitz (see Appendix D.2).

This interpretation provides a concrete intuition for what Algorithm 1 actually does: for better or worse, it optimizes the aggregate artificial objective \( \sum_{t \in [T]} H_t(\mu_t) \), whose optimum is precisely \( \sum_t H_t(\mu^*_t) \). The analysis simplifies accordingly, as we can directly invoke the known guarantees for SGD and conclude that the artificial objective comes close to this target optimum. Much of the proof of Theorem 4.7 involves relating this artificial objective to value optimization.

Remark 4.13. The fact that Algorithm 1 optimizes an artificial objective whose optimum is attained by the perfect pacing sequence suggests the latter as an appropriate benchmark for analyzing this algorithm.

A similar (but technically different) interpretation appeared in \cite{12} in the context of second-price auctions. Their interpretation relies on Lagrangian duality and does not appear to extend beyond second-price auctions.

Remark 4.14. Our analysis only uses two properties of the dynamics. First, that the dynamics obtain low regret with respect to an adversarially generated sequence of convex functions \( H_t \) (we inherit this from SGD). Second, that the dynamics do not terminate too early to bound any possible loss on these periods. The proof of Theorem 4.7 generalizes almost immediately to any algorithm satisfying these properties (Lemma 2.2).

4.3 Proof of Theorem 4.7

Our analysis proceeds in three steps. First, we invoke the SGD machinery to show that the sequence of chosen multipliers \( (\mu_t) \) approximates the target optimum \( \sum_t H_t(\mu^*_t) \). Second, we relate the \( H_t \) functions to the per-round payments to show that the expected payment is not too far from \( \rho \) in each round. Finally, we use the MBB property of the auctions to bound the total loss in value.

The following technical lemma will enable us to meaningfully relate the \( Z_t \) and \( H_t \) functions whose proof is deferred to the Appendix:

Lemma 4.15. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be an increasing, \( \lambda \)-Lipschitz function such that \( f(0) = 0 \). Let \( R = \int_0^x f(y) dy \) for some \( x \in \mathbb{R} \). Then \( |f(x)| \leq \sqrt{2 \lambda R} \).

To begin, observe that Algorithm 1 is equivalent to running (projected) stochastic gradient descent on the sequence of \( H_t \) functions from (4.4), where the maximum norm of the gradient is at most \( \bar{\tau} \) using the assumption that the payment is at most the bid, which in turn is at most the maximum possible value. Note that these functions are indeed convex and Lipschitz (see Appendix D.2 for a proof) and indeed, \( H_t'(\mu) = \rho - Z_t(\mu) = \rho - \mathbb{E}_{G_t}[z_t(\mu)] \) by construction of \( H_t \). We now wish to bound the sequence regret, \( \sum_t \mathbb{E}[H_t(\mu_t) - H_t(\mu^*_t)] \). While the dynamics may end before time \( T \), we may upper bound the regret by continuing to time \( T \), as each term is nonnegative. Standard analysis of SGD (see Proposition C.1 and Appendix D.2 for details) then implies

\[
\text{REG}_1 \triangleq \sum_{t=1}^T \mathbb{E} \left[ H_t(\mu_t) - H_t(\mu^*_t) \right] \leq \frac{\rho + 1}{\varepsilon} \cdot \bar{\tau}^2 + \epsilon T \cdot (\rho + \bar{\tau})^2.
\]
Now let us translate (4.6) into a bound on values. From Lemma 2.2, we may assume that the dynamics continue until time $T$ without stopping at the cost of $O(\sqrt{T}/p)$ loss in value which we will account for at the end. Define the function $W_t(\mu)$ by

$$W_t(\mu) = \begin{cases} V_t(\mu) & \text{if } Z_t(\mu) < \rho \\ V_t(\mu) \times \frac{\rho}{Z_t(\mu)} & \text{if } Z_t(\mu) \geq \rho. \end{cases}$$

Observe that $W_t(\mu) \leq V_t(\mu)$ by construction; it can be viewed as (approximately) the per-round value obtained by bidding with pacing multiplier $\mu$ — if the environment at time $t$ persisted for all rounds — until the budget is expected to terminate due to overspending. We further observe that $V_t(\mu_t) - V_t(\mu_t^*) \geq W_t(\mu_t) - W_t(\mu_t^*)$ for all $t$. To see this, note that $V_t(\mu_t^*) = W_t(\mu_t^*)$ since $Z_t(\mu_t^*) \leq \rho$, and moreover, the first term on the right side equals the first term of the left side, which is non-negative, with some scaling factor at most 1. Therefore,

$$\sum_{t=1}^{T} (V_t(\mu_t^*) - V_t(\mu_t)) \leq \sum_{t=1}^{T} (W_t(\mu_t^*) - W_t(\mu_t)), \quad (4.7)$$

so it will suffice to show that the right side is not too large.

For any round $t$, we now consider two cases depending on the value of $\mu_t$ and derive a lower bound on the difference in $W_t$ values:

1. **Case 1:** $\mu_t < \mu_t^*$. Note that in this case we must have $\mu_t^* > 0$ and hence $Z_t(\mu_t) \geq Z_t(\mu_t^*) = \rho$, and therefore

$$W_t(\mu_t) = V_t(\mu_t) \times \frac{\rho}{Z_t(\mu_t)} \geq W_t(\mu_t^*) \times \frac{\rho}{Z_t(\mu_t)}.$$

By rearranging and adding $W_t(\mu_t^*)$ to each side of the last inequality, it follows that

$$W_t(\mu_t^*) - W_t(\mu_t) \leq \frac{Z_t(\mu_t) - \rho}{Z_t(\mu_t)} \cdot W_t(\mu_t^*) \leq (Z_t(\mu_t) - \rho) \cdot \frac{\sigma}{\rho}, \quad (4.8)$$

where we use the fact that $Z_t(\mu_t) \geq \rho$ as well as $W_t(\mu_t^*) \leq \sigma$.

2. **Case 2:** $\mu_t \geq \mu_t^*$. In this case, we have $Z_t(\mu_t) \leq Z_t(\mu_t^*) \leq \rho$ by assumption, and hence $W_t(\mu_t^*) = V_t(\mu_t^*)$ and $W_t(\mu_t) = V_t(\mu_t)$. We now claim that the MBB property of the auction implies

$$\frac{W_t(\mu_t^*)}{W_t(\mu_t)} = \frac{V_t(\mu_t^*)}{V_t(\mu_t)} \leq \frac{Z_t(\mu_t^*)}{Z_t(\mu_t)} \quad (4.9)$$

To prove (4.9), define $\gamma = Z_t(\mu_t^*)/V_t(\mu_t)$ and note that since the auction is individually rational, it must be that an agent bidding $v_t/(1 + \mu_t)$ pays at most $v_t/(1 + \mu_t)$ per unit allocated, and hence $\gamma \leq \frac{1}{1 + \mu_t}$. If we write $Z_t(\mu_t | v_t)$ for the expected value of $Z_t(\mu_t)$ conditional on the realization of $v_t$ (recalling that $v_t$ can be correlated with the competing bids), and similarly for $V_t(\mu_t | v_t)$, then by MBB and linearity of expectation

$$Z_t(\mu_t^* | v_t) - Z_t(\mu_t | v_t) \geq \frac{1}{1 + \mu_t} \left( V_t(\mu_t^* | v_t) - V_t(\mu_t | v_t) \right)$$

where we used the fact that the bid given $v_t$ and $\mu_t$ is $v_t/(1 + \mu_t)$, and $V_t(\cdot | v_t)$ is precisely $v_t$ times the agent’s expected allocation. Taking expectations over $v_t$ then implies $Z_t(\mu_t^*) - Z_t(\mu_t) \geq \frac{1}{1 + \mu_t} \left( V_t(\mu_t^*) - V_t(\mu_t) \right)$. Rearranging and recalling that $Z_t(\mu_t) = \gamma V_t(\mu_t)$, we conclude

$$Z_t(\mu_t^*) \geq \gamma V_t(\mu_t) + \frac{1}{1 + \mu_t} \left( V_t(\mu_t^*) - V_t(\mu_t) \right) \geq \gamma V_t(\mu_t^*)$$
which implies (4.9). A rearrangement of (4.9) then implies
\[
W_t(\mu_t) - W_t(\mu_\ast) \leq \frac{V_t(\mu_t)}{Z_t(\mu_t)} \times (Z_t(\mu_t) - Z_t(\mu_\ast)) \leq \frac{\tau}{\min\{\rho, \delta\}} (Z_t(\mu_t) - Z_t(\mu_\ast)).
\] (4.10)

The second inequality comes from considering the cases that \(\mu_\ast = 0\) (and applying Assumption 4.3) and \(\mu_\ast > 0\).

In either case, (4.8) and (4.10) implies that (recalling the assumption \(\delta \leq \rho\))
\[
W_t(\mu_\ast) - W_t(\mu_t) \leq \frac{\tau}{\delta} |Z_t(\mu_\ast) - Z_t(\mu_t)|.
\] (4.11)

We now relate this bound to the \(H_t\) functions. Observe that
\[
H_t(\mu_t) - H_t(\mu_\ast) = \int_{\mu_t}^{\mu_\ast} [\rho - Z_t(x)] \, dx = \int_0^{\mu_\ast - \mu_t} [\rho - Z_t(\mu_\ast + x)] \, dx,
\]
and moreover that the integrand \(f(x) = \rho - Z_t(\mu_\ast + x)\) is increasing and \(\lambda\)-Lipschitz by the assumption on \(Z_t\). In general, \(f(0) \geq 0\) (with equality if \(Z_t(\mu_\ast) = \rho\)). If we consider \(g(x) = f(x) - f(0)\) and apply Lemma 4.15 we obtain
\[
|Z_t(\mu_t) - Z_t(\mu_\ast)| = |f(\mu_t - \mu_\ast) - f(0)| \leq \sqrt{2\lambda(H_t(\mu_t) - H_t(\mu_\ast) - (\mu_t - \mu_\ast)(\rho - Z_t(\mu_\ast)))}.
\]

Note that if \(\mu_t < \mu_\ast\), then \(\rho = Z_t(\mu_\ast)\) so the last subtracted term is zero, while if \(\mu_t \geq \mu_\ast\), then the last subtracted term is nonnegative. In either case, we deduce that
\[
|Z_t(\mu_t) - Z_t(\mu_\ast)| \leq \sqrt{2\lambda(H_t(\mu_t) - H_t(\mu_\ast))}
\] (4.12)

By combining these inequalities, we obtain:
\[
\mathbb{E} \left[ \sum_{t=1}^{T} V_t(\mu_\ast) - V(\mu_t) \right] \leq \mathbb{E} \left[ \sum_{t=1}^{T} W_t(\mu_\ast) - W_t(\mu_t) \right] \quad \text{(by (4.7))}
\]
\[
\leq \frac{\tau}{\delta} \mathbb{E} \left[ \sum_{t=1}^{T} |Z_t(\mu_\ast) - Z_t(\mu_t)| \right] \quad \text{(by (4.11))}
\]
\[
\leq \sqrt{2\lambda T \frac{\tau}{\delta}} \mathbb{E} \left[ \sum_{t=1}^{T} \sqrt{H_t(\mu_t) - H_t(\mu_\ast)} \right] \quad \text{(by (4.12))}
\]
\[
\leq \sqrt{2\lambda T \frac{\tau}{\delta}} \left( \mathbb{E} \left[ \sum_{t=1}^{T} H_t(\mu_t) - H_t(\mu_\ast) \right] \right) \quad \text{(by Cauchy-Schwarz)}
\]
\[
\leq \sqrt{2\lambda T \frac{\tau}{\delta}} \mathbb{E} \left[ \sum_{t=1}^{T} H_t(\mu_t) - H_t(\mu_\ast) \right] \quad \text{(by Jensen’s inequality)}
\]
\[
\leq \frac{\tau}{\delta} \sqrt{2\lambda T \cdot \text{REG}_1} \quad \text{(by (4.6)).}
\]

Accounting for the \(O(\pi \tau \sqrt{T}/\rho)\) loss from possibly terminating early gives the claim.
5 Numerical Evaluation

In this Section we complement our theoretical findings with a numerical simulation study of Algorithm 1. We consider the multi-player environment with simultaneous learning by competing bidders. Our simulations are semi-synthetic, based on campaign and bidding data collected from the Bing Advertising platform.

We focus on regret relative to the standard benchmark: the best fixed pacing multiplier in hindsight. Recall that, relative to this benchmark, vanishing regret is achievable for a stochastic environment but is provably impossible for an adversarial environment. The achievability of vanishing regret in a multi-player environment remains unknown and is currently an open question.

This gap motivates our simulation study, which focuses on this multi-player environment with simultaneous learning. Our simulations suggest that simultaneous execution of Algorithm 1 yields vanishing regret, with regret rate $< T^{3/5}$.

5.1 Data and Algorithm Description

Our data consists of campaign and auction data collected over a 7-day period in April 2022. The dataset contains daily budget targets and other campaign parameters (such as maximum bid, when specified) for campaigns from a North American advertising segment. It also contains, for a random subsample of $N \approx 2.4M$ auction instances over this period, the list of participating campaigns, click probability predictions for each participant, realized bids from each participant, and auction outcomes (including auction winner and payment).

Given this dataset, we simulate a joint execution of Algorithm 1 as follows. For any campaign that participates in fewer than $\theta = 1000$ auction instances, we maintain the bid that they originally placed in the dataset. For all other campaigns, we generate new bids by simulating the execution of Algorithm 1 for each participant of each auction instance. The target spend rate for campaign $k$ is calculated by taking $T_k$ to be the average number of per-day auction instances for each campaign, and per-impression values are taken to be proportional to platform estimated click rates. The step size $\epsilon_k$ for each agent $k$ is chosen to be proportional to $1/\sqrt{T_k}$. We provide a warm start by initializing each campaign’s multiplier to be proportional to $1/\rho_k$.

Given the bids for all auction participants, we simulate the auction outcome using estimated click rates and a simplified auction rule. Each bidding agent’s bid is calculated as in Algorithm 1 using the estimated click rate as the value. We take there to be only a single winner in each auction instance, corresponding to the highest total bid. Payments are calculated according to a specified payment rule; we consider both first-price and second-price payment rules in our experiments.

5.2 Regret Analysis

Following each execution of the simulation, we calculate the total regret for each bidding agent by determining the single best linear policy multiplier in hindsight. This is done via binary search, up to an error of $10^{-9}$. We run $n = 100$ simulations for each scenario, each using a random subsample of 95% of auction instances. This provides an estimate of the regret of each agent $k$.

To estimate the rate of regret growth for each agent over time, we simulate longer time periods by iterating over the collection of auction instances in our dataset multiple times. We simulated $K$ iterations over the dataset, where $K \in \{1, 2, 5, 10, 20, 50, 100\}$. To reduce the potential impact of cyclic data patterns, each of the $K$ iterations includes only a subsample of the impression auctions drawn independently at random, where each impression is included in each iteration with probability $\beta = 1/2$. We then calculate regret for each agent, as described above, as the number of auction instances grows.
Figure 1: Illustration of estimated regret in repeated auction simulation. For each of 100 randomly-sampled campaigns, we trace out the evolution of regret as the time horizon (the number of auction instances) is amplified, illustrated in log-log scale. Each line corresponds to a single campaign. Results are shown for (a) second-price auctions and (b) first-price auctions. Instances with negative regret are excluded.

5.3 Results

Figure 1 illustrates the outcome of this simulation for a random sample of 100 of the simulated campaigns. The evolution of regret is plotted against the increase in the number of auction instances, in log-log scale. Both first-price and second-price simulation results are shown. We note that regret increases approximately linearly in log-log scale for many advertisers, informally suggesting a polynomial regret rate.

Hypothesizing a regret rate of the form $O(T^\alpha)$, we estimate $\alpha$ using log-log regression. Figure 2 illustrates the outcome of log-log regression for all campaigns and simulation treatments, separately for first-price and second-price auction formats. We obtain an estimated slope of $\alpha = 0.58$ for second-price auctions (standard error 0.0026) and $\alpha = 0.55$ for first-price auctions (standard error 0.0028).

Taken together, these results provide evidence that simultaneous execution of Algorithm 1 yields vanishing regret, with regret rate $< T^{3/5}$.

6 Conclusions and Open Questions

We have shown that a natural budget-pacing algorithm achieves good aggregate and individual performance guarantees for a wide class of auctions and arbitrarily correlated private values, without relying on convergence to an equilibrium.

The main questions left open by our work concern simultaneously achieving similar individual and aggregate guarantees. Can such results be extended to other (classes of) algorithms, perhaps bringing to bear the “generalized pacing” notion from Definition 3.1? More concretely, are there other “generalized pacing” algorithms that satisfy compelling individual guarantees, e.g., vanishing regret in the stochastic environment? Can one improve the individual guarantees while keeping the same liquid welfare guarantee?

The key goal for individual guarantees is vanishing regret for all agents. Previously this goal seemed hopeless for budget-constrained bidders, given the strong impossibility results known for adversarial environments. We provide a plausible framework in which this goal might be achievable: regret bounds relative to the perfect pacing sequence. Recall that our specific guarantee falls just short: Theorem 4.7 requires path-
length $P^* = o(\sqrt{T})$, but we can only guarantee $P^* = O(\sqrt{T})$ when all agents run our algorithm. Finally, our semi-synthetic numerical simulations suggest that our algorithm achieves vanishing regret against the standard benchmark (best pacing multiplier in hindsight) in simultaneous learning environments. We leave open the question of whether vanishing regret under simultaneous learning can be proven analytically, but conjecture that this is the case.

While optimizing the guarantees on individual regret is not our goal per se, it would be interesting to improve regret bounds for repeated non-truthful auctions, particularly for repeated first-price auctions, even if they are not accompanied by aggregate guarantees. Improved regret bounds for the stochastic environment are especially interesting when combined with some guarantees for the adversarial environment. As linear policies (mappings from values to bids) are not necessarily value-optimizing for non-truthful auctions, an important sub-question is which class(es) of policies one would want to optimize over.
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A Examples of Auctions

In this section we provide the details of the ad-auctions we mentioned in Section 2.

A.1 Single-slot and Multiple-slot Ad Auctions

- **Single-Slot Ad Auctions**: A round corresponds to an ad impression, and there is a single ad slot available for each impression. An impression has a type \( \theta_t \in \Theta \); this type might describe, for example, a keyword being searched for, user demographics, intent prediction, etc. In each round \( t \) the impression type \( \theta_t \) is drawn independently from a distribution over types. Each agent \( k \) has a fixed value function \( v_k : \Theta \rightarrow \mathbb{R}_{\geq 0} \) that maps each impression type to a value for being displayed. The value profile in round \( t \) is then \( \mathbf{v}_t = (v_1(\theta_t), \ldots, v_n(\theta_t)) \). A single ad can be displayed each round. We then interpret \( \mathbf{x}_{k,t} \in [0, 1] \) as the probability that advertiser \( k \) is allocated the ad slot, and an allocation profile \( \mathbf{x}_t \) is feasible if and only if \( \sum_k \mathbf{x}_{k,t} \leq 1 \).

- **Multiple-Slot Pay-per-click Ad Auctions**: We can generalize the previous example to allow multiple ad slots, using a polymatroid formulation due to [32]. Impression types and agent values are as before, but we now think of there as being \( m \geq 1 \) slots available in each round, with click rates \( 1 \geq \alpha_1 \geq \ldots \geq \alpha_m \geq 0 \). If ad \( k \) is placed in slot \( i \), the value to agent \( k \) is \( v_k(\theta_t) \times \alpha_i \). That is, we think of \( v_k(\theta_t) \) as representing both the advertiser-specific click rate (which can depend on the impression type) as well as the advertiser's value for a click. In this case we would take \( v_{k,t} = v_k(\theta_t) \) and \( x_{k,t} = \alpha_i \). The set of feasible allocation profiles \( X \subseteq [0, 1]^n \) is then a polymatroid: \( x_t \in X \) if and only if, for each \( \ell \leq m \) and each \( S \subseteq [n] \) with \( |S| = \ell \), \( \sum_{k \in S} x_{k,t} \leq \sum_{i=1}^{k} \alpha_i \).

A.2 Examples of Core Auctions

The following are some notable examples of core auctions. Note that all of the following auction formats satisfy the monotone bang-per-buck property.

- **First-Price Auction** chooses a welfare-maximizing allocation \( \mathbf{x}(\mathbf{b}) \in \arg \max \{\sum_k b_kx_k\} \). Each agent \( k \) then pays her bid for the allocation obtained: \( p_k(\mathbf{b}) = b_kx_k(\mathbf{b}) \). To see this auction is MBB, first note that \( x_k(\mathbf{b}) \) is nondecreasing in \( b_k \). Then observe that if \( b_{-k} \) is fixed, and \( b_k < b'_k \), then setting \( \mathbf{b}' = (b'_k, b_{-k}) \), we clearly have

  \[
  p_k(\mathbf{b}') - p_k(\mathbf{b}) = b'_kx_k(\mathbf{b}') - b_kx_k(\mathbf{b}) \geq b_k(x_k(\mathbf{b}') - x_k(\mathbf{b})),
  \]

  as needed.

- **Second-Price Auction for a Single Slot**. Consider the single-slot environment described earlier; recall that feasible allocations \( \mathbf{x} \) satisfy \( \sum_k x_k \leq 1 \). The second-price auction chooses a welfare-maximizing allocation \( \mathbf{x}(\mathbf{b}) \in \arg \max \{\sum_k b_kx_k\} \), then each agent \( k \) pays \( x_k \) times the second-highest bid.

- **Generalized Second-Price (GSP) Auction**. Consider the multi-slot environment described earlier. In the GSP auction, slots are allocated greedily by the respective bid, and each agent pays a price per unit equal to the next-highest bid. Formally, given the bids \( \mathbf{b} \), we let \( \pi \) be a permutation of the agents so that \( b_{\pi(1)} \geq b_{\pi(2)} \geq \ldots \geq b_{\pi(n)} \). That is, \( \pi(1) \) is the highest-bidding agent, then \( \pi(2) \), etc. Agent \( \pi(k) \) is then allocated to slot \( k \) for each \( k \leq m \). That is, \( x_{\pi(k)} = \alpha_k \) for each \( k \leq m \) and \( x_{\pi(k)} = 0 \) for each \( k > m \). The payments are set so that \( p_{\pi(k)} = x_{\pi(k)}b_{\pi(k+1)} \) for all \( k < n \), and \( p_{\pi(n)} = 0 \).\(^{17}\)

\(^{17}\)Note that \( p_{\pi(k)} = 0 \) whenever \( x_{\pi(k)} = 0 \).
note that GSP is a core auction; see Appendix \[G\] for a proof. To see that the MBB property holds, let \( b_k < b'_k \) and suppose that with bid profiles \( b \), agent \( k \) is assigned the \( j \)th slot, while under \( b' \), the agent is assigned the \( \ell \leq j \)th slot (where we allow the value of a slot to be 0 if the agent is not in the top \( m \) bids). If \( j = \ell \), then it is easy to see that the price and allocation of agent \( k \) is unchanged, trivially confirming that the MBB condition holds. If instead \( j > \ell \), then it is easy to see that \( b_{\pi'((\ell+1)} \geq b_k \), with \( \pi' \) the permutation under \( b' \), and hence

\[
p_k(b') - p_k(b) \geq \alpha_k b_k - \alpha_k b_{\pi(j+1)} \geq b_k(\alpha_\ell - \alpha_j) = b_k(x_k(b') - x_k(b)).
\]

This implies that the MBB property holds for single-slot auctions.

## B Omitted Proofs from Section 3

### B.1 Motivating ex ante liquid welfare

Our definition of ex ante liquid welfare assumes that the same allocation rule \( y \) is used in every round. We now show that this is without loss of generality. The following lemma shows that given any allocation sequence rule there is a single-round allocation rule with the same ex ante liquid welfare.

**Lemma B.1.** Let \( \tilde{y} : [0, \bar{\nu}]^n T \to X_T^n \) be an allocation sequence rule that takes in the entire sequence \( \nu_1, \ldots, \nu_T \) and allocates \( \tilde{y}_{k,t}(\nu_1, \ldots, \nu_T) \) units to agent \( k \) at time \( t \). Then there exists a (single-round) allocation rule \( \hat{y} : [0, \bar{\nu}]^n \to X \) such that

\[
\hat{W}(\hat{y}, F) \triangleq \sum_{k=1}^n \min\left\{ B_{k}, \mathbb{E}_{v_1, \ldots, v_T \sim F} \left[ \sum_{t=1}^T \tilde{y}_{k,t}(v_1, \ldots, v_T) v_{k,t} \right] \right\} = \sum_{k=1}^n T \cdot \min\left\{ \rho_k, \mathbb{E}_{v \sim F} \left[ y_k(v) v_k \right] \right\} = \mathbb{W}(y, F).
\]

**Proof.** For each \( t \), by slightly abusing notation, we define an allocation rule \( \hat{y}_t : [0, \bar{\nu}]^n \to [0, 1]^n \) by

\[
\hat{y}_{k,t}(v_t) \triangleq \mathbb{E}_{v_{-t} \sim F^t \to [0,1]} \left[ y_{k,t}(v_1, \ldots, v_T) | v_t \right],
\]

Note that this is a feasible allocation rule as the set of feasible allocations is convex and closed. We have

\[
\mathbb{E}_{v_1, \ldots, v_T \sim F} \left[ \tilde{y}_{k,t}(v_1, \ldots, v_T) v_{k,t} \right] = \mathbb{E}_{v_t} \left[ \mathbb{E}_{v_{-t} \sim F} \left[ \tilde{y}_{k,t}(v_1, \ldots, v_T) v_{k,t} | v_t \right] \right] = \mathbb{E}_{v_t} \left[ \hat{y}_{k,t}(v_t) v_{k,t} \right] = \mathbb{E}_{v \sim F} \left[ \hat{y}_{k,t}(v) v_k \right].
\]

Now we define the allocation rule \( \tilde{y} \) by setting \( \tilde{y}_k = \frac{1}{T} \sum_{t=1}^T \hat{y}_{k,t} \) for each \( k \in [n] \), which is again feasible because the set of feasible allocations is convex. By the linearity of the expectations operator, we have

\[
\mathbb{E}_{v_1, \ldots, v_T \sim F} \left[ \sum_{t=1}^T \tilde{y}_{k,t}(v_1, \ldots, v_T) v_{k,t} \right] = \sum_{t=1}^T \mathbb{E}_{v_1, \ldots, v_T \sim F} \left[ \hat{y}_{k,t}(v_1, \ldots, v_T) v_{k,t} \right] = \sum_{t=1}^T \mathbb{E}_{v \sim F} \left[ \hat{y}_{k,t}(v) v_k \right] = T \cdot \mathbb{E}_{v \sim F} \left[ \hat{y}_k(v) v_k \right] = \mathbb{W}(y, F).
\]

This proves the Lemma. \( \square \)
B.2 Proof of Lemma 3.8

Proof. The assumption that $\mu_{k,t_2} \neq b$ means that the agent participates in all periods of $[t_1, t_2)$. Moreover, if $t_2 = t_1 + 1$, then (3.3) is trivial as the third term on the right hand side of inequality (3.3) is zero and $z_{k,t_1} \geq 0$. Therefore, we may assume $t_2 \geq t_1 + 2$.

By the definition of an epoch, there is no negative projection in the dynamics on the epoch until possibly time $t_2$. I.e., $\mu_{k,t} > 0$ for all $t_1 \leq t < t_2$. The pacing recurrence condition implies that

$$0 < \mu_{k,t_2-1} = P_1(\mu_{k,t_2-2} + \epsilon(z_{k,t_2-2} - \rho_k)) \leq \mu_{k,t_2-2} + \epsilon(z_{k,t_2-2} - \rho_k) \leq \epsilon \sum_{t=t_1}^{t_2-2} (z_{k,t} - \rho_k) = \epsilon \sum_{t=t_1}^{t_2-2} (z_{k,t} - (t_2 - t_1 - 1)\rho_k).$$

The first and second inequalities follow because the multipliers are positive during an epoch. The third inequality follows from applying the second inequality repeatedly. Note that the inequality holds even if there is a positive projection during the epoch (i.e., if $\mu_{k,t} = \pi$ for some $t \in [t_1, t_2)$).

Thus, the expenditure of agent $k$ on this epoch is at least $\sum_{t=t_1}^{t_2-2} z_{k,t} \geq (t_2 - t_1 - 1)\rho_k$. Let us now consider the value obtained by the agent on this epoch. Because agents never overbid in the pacing algorithm and because payments are always lower than the value in a core auction, the value obtained by the agent on the epoch is at least the expenditure, which we just lower bounded. To get a slight sharpening of this, note that on the first period of the epoch, the agent actually receives $x_{k,t_1} v_{k,t_1}$ value and pays $z_{k,t_1}$, which we know is at most $x_{k,t_1} v_{k,t_1}$ value. Therefore, we can trade $z_{k,t_1}$ expenditure for $x_{k,t_1} v_{k,t_1}$ value. It follows from the above bound and this observation that

$$\sum_{t=t_1}^{t_2-1} x_{k,t} v_{k,t} \geq x_{k,t_1} v_{k,t_1} - z_{k,t_1} + \sum_{t=t_1}^{t_2-2} z_{k,t} \geq x_{k,t_1} v_{k,t_1} - z_{k,t_1} + (t_2 - t_1 - 1)\rho_k.$$

\[\square\]

B.3 Proof of Lemma 3.9

Proof. Let $\Delta_t = X_t Y_t + (1 - X_t)\rho - (E[X_t Y_t + (1 - X_t)\rho | F_{t-1}]$. Clearly, the sequence $\Delta_t$ forms a $F_t$-martingale difference sequence by construction. Moreover, we observe that

$$E[X_t Y_t + (1 - X_t)\rho | F_{t-1}] = X_t E[Y_t | F_{t-1}] + (1 - X_t)\rho = X_t E[Y_t] + (1 - X_t)\rho,$$

where we use the facts that $X_t$ is $F_{t-1}$-measurable and $Y_t$ is independent of $F_{t-1}$. It follows that $\Delta_t \in [-E[Y_t], \varpi - E[Y_t]]$. As an immediate consequence of the Azuma-Hoeffding inequality, we obtain

$$\Pr \left( \sum_{t=1}^{T} \Delta_t \geq \theta \right) \leq \exp \left( \frac{-2\theta^2}{T\varpi^2} \right).$$

(B.1)

But observe that

$$\left\{ \sum_{t=1}^{T} \Delta_t \geq \theta \right\} = \left\{ \sum_{t=1}^{T} [X_t Y_t + (1 - X_t)\rho] \geq \theta + \sum_{t=1}^{T} [X_t E[Y_t] + (1 - X_t)\rho] \right\}$$

$$\geq \left\{ \sum_{t=1}^{T} [X_t Y_t + (1 - X_t)\rho] \geq \theta + T \cdot \rho \right\},$$

using the assumption $E[Y_t] \leq \rho$. This inclusion together with (B.1) yields (3.4) \[\square\]
Recall that our general approach for establishing individual guarantees is through a reduction to regret bounds for projected stochastic gradient descent. The general goal of stochastic gradient descent is to obtain a sequence of points $x_1, \ldots, x_T$ in an online fashion that achieve low-regret with respect to the loss of a sequence of convex functions $f_1, \ldots, f_T$ compared to the best fixed point in some convex set $\mathcal{K}$. Moreover, at each time $t$, the agent receives noisy feedback via a noisy, but unbiased estimate of $\nabla f_t(x_t)$.

The general form of the algorithm is given below:

**Algorithm 2: Stochastic Gradient Descent**

**Input:** $\mathcal{K}$ convex set, $T$ time horizon, step-size $\epsilon > 0$

**Output:** Sequence $x_1, \ldots, x_T \in \mathcal{K}$

1. Initialize $x_1 \in \mathcal{K}$.
2. for $t = 1, \ldots, T$ do
   3. Observe $\tilde{\nabla}_t$, an unbiased estimator of $\nabla f_t(x_t)$, where $f_t$ is a (adversarially chosen) convex function (i.e. $\mathbb{E}[\tilde{\nabla}_t] = \nabla f_t(x_t)$ where the expectation is over any randomness in the dynamics after $x_t$ and $f_t$ are chosen);
   4. Update $x_{t+1} = P_{\mathcal{K}}(x_t - \epsilon \tilde{\nabla}_t)$.

It is well-known that Algorithm 2 attains vanishing regret with respect to the loss of the best fixed action in $\mathcal{K}$ on the sequence of function $f_1, \ldots, f_t$. For completeness, we give the following guarantee on path-length (dynamic) guarantees for stochastic gradient descent that is immediately obtained by combining standard regret bounds.

**Proposition C.1.** Let $\mathcal{K}$ be a convex set with diameter $D$, and suppose that $f_1, \ldots, f_T$ is a sequence of convex functions, where $f_t$ can be adversarially chosen depending on the previous $f_1, \ldots, f_{t-1}$ as well as $x_1, \ldots, x_t$. Suppose that the sequence $x_1, \ldots, x_T$ is obtained by running Algorithm 2 and it further holds that $\|\tilde{\nabla}_t\| \leq G$ with probability 1. Then for any (not necessarily fixed) sequence $u_1, \ldots, u_T \in \mathcal{K}$ satisfying $\sum_{t=1}^{T-1} \|u_{t+1} - u_t\|^2 + 1 \leq P$ with probability one, then

$$
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \right] \leq O \left( \frac{D^2 P}{\epsilon} + \epsilon G^2 T \right). \quad (C.1)
$$

**Proof.** The proof is an immediate consequence of combining standard analyses of stochastic gradient descent and dynamic regret. We have deterministically by convexity that

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq \sum_{t=1}^{T} \nabla f_t(x_t)^T (x_t - u_t)
$$

$$
= \mathbb{E} \left[ \sum_{t=1}^{T} \tilde{\nabla}_t^T (x_t - u_t) \right],
$$

where the expectation is over all random choices in the dynamics using the assumption on the gradient estimates. At this point, the dynamics follow online gradient descent with cost functions $\tilde{f}_t(x) = \tilde{\nabla}_t^T x$.  


Hence we may invoke Theorem 10.1.1 of [36] to deduce that

\[
E \left[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \right] \leq E \left[ \sum_{t=1}^{T} \nabla_t^T (x_t - u_t) \right] \leq O \left( \frac{D^2 P}{\epsilon} + \epsilon G^2 T \right),
\]

where we use the assumption that $\|\nabla_t\| \leq G$ almost surely as well as the almost sure path-length assumption.

\[ \square \]

## D Omitted Proofs from Section 5

### D.1 Stochastic environment: Proof of Corollary 4.11

**Proposition D.1** (Stochastic Environment). Suppose that $G_t$ does not change over time almost surely. Define $\mu^*$, $V(\cdot)$ and $Y(\cdot)$ as in Corollary 4.11. Assume that the auction satisfies MBB and posit Assumption 4.2. Then

\[
Y(\mu) \leq T \cdot V(\mu^*) + \bar{\pi}^2 / \rho \quad \forall \mu \in [0, \bar{\mu}].
\]

**Proof.** Fix $\mu \in [0, \bar{\mu}]$. Define the random process $M_k = \sum_{t=1}^{k} \left[ x_t v_t - V(\mu) \right]$, where $x_t$ and $v_t$ are the (random) allocation and valuations at each time $t$ when bidding with multiplier $\mu$. Notice that $M_k$ is a martingale by construction and $E[M_1] = 0$. Define $\tau_\mu$ to be first time that such a bidding algorithm runs out of money at the end of the period, or $T$ if this does not occur. Because $M_k$ is a martingale, by the Optional Stopping Theorem,

\[
Y(\mu) \leq E \left[ \sum_{t=1}^{\tau_\mu} x_t v_t \right] = E \left[ \sum_{t=1}^{\tau_\mu} V(\mu) \right] = E[\tau_\mu] \cdot V(\mu), \tag{D.1}
\]

where the inequality occurs because the stopping rule counts the value obtained on the period where the agent may (strictly) exceed the budget, and then using Wald’s identity in the last step as the function $V(\mu)$ is constant over time by assumption so independent of $\tau_\mu$.

We now derive an upper bound on the expectation of the stopping time. On the one hand, we certainly have $E[\tau_\mu] \leq T$ as $\tau_\mu \leq T$ by definition. An analogous martingale argument with $N_k = \sum_{t=1}^{k} \left[ z_t - Z(\mu) \right]$ (where $z_t$ is the (random) expenditure at time $t$) similarly implies that

\[
E[\tau_\mu] \cdot Z(\mu) = E \left[ \sum_{t=1}^{\tau_\mu} z_t \right] \leq B + \bar{\pi}, \tag{D.2}
\]

where the extra term arises in the analysis because the agent may spend $\bar{\pi}$ on the final round before she exceeds her budget. Combining (D.1) and (D.2) implies that $Y(\mu) \leq V(\mu) \cdot \min \left\{ T, \frac{B + \bar{\pi}}{Z(\mu)} \right\}$.

Suppose first that $\mu \geq \mu^*$ so that $V(\mu) \leq V(\mu^*)$. As we have seen, this implies that $Y(\mu) \leq V(\mu^*) \cdot T \leq V(\mu^*) \cdot T$ as claimed. If instead $\mu < \mu^*$, note that by definition this implies that $Z(\mu^*) = \rho$, and our inequality implies that $Y(\mu) \leq (B + \bar{\pi}) \cdot V(\mu)/Z(\mu^*) \leq (B + \bar{\pi}) \cdot V(\mu^*)/Z(\mu^*)$ by the monotone bang-for-buck property (see the proof of Theorem 4.7). This yields $Y(\mu) \leq V(\mu^*) \cdot (T + \bar{\pi}/\rho) \leq V(\mu^*) \cdot T + \bar{\pi}^2 / \rho$. In either case, we get the obtained bound. \[ \square \]
D.2 Artificial objective $H_t$

Lemma D.2. The function $H_t$ is convex and is $(\sigma + \rho)$-Lipschitz.

Proof. The convexity is immediate from the fundamental theorem of calculus and monotonicity assumption on $Z_t$, as the expected expenditure is a weakly decreasing function of the multiplier. To show that the Lipschitz condition holds note that

$$|H_t(y) - H_t(x)| \leq \rho |y - x| + \int_x^y Z_t(s) \, ds \leq (\rho + Z_t(0)) |y - x|.$$ 

The proof of the Lipschitz condition follows from the fact the expected expenditure function is at most the expected valuation. \hfill $\Box$

D.3 Proof of Lemma 4.15

Proof. The conclusion is trivial if $\lambda = 0$, so we assume $\lambda > 0$. Moreover, note that we always have $R \geq 0$, and by considering the function $-f(-y)$ if necessary, we may assume without loss of generality that $x \geq 0$ and thus the same for $f(x)$.

For a contradiction, suppose that $f(x) > \sqrt{2\lambda R}$. Then by the assumption that $f$ is $\lambda$-Lipschitz, it follows that $f(y) > \sqrt{2\lambda R} - \lambda(x - y)$ for all $y \in [x - \sqrt{2R/\lambda}, x]$ (note $f(y) > 0$ on this region so this region is contained in $[0, x]$, where $f(y) \geq 0$). Hence, we have

$$\int_0^x f(y) \, dy \geq \int_{x-\sqrt{2R/\lambda}}^x f(y) \, dy > \int_{x-\sqrt{2R/\lambda}}^x \left[\sqrt{2\lambda R} - \lambda(x - y)\right] \, dy.$$ 

This latter integral gives the area of a right triangle with height $\sqrt{2\lambda R}$ and base $\sqrt{2R/\lambda}$, which is clearly equal to $R$. This contradicts the assumption that $R = \int_0^x f(y) \, dy$, proving the lemma. \hfill $\Box$

E Vanishing Regret Does Not Imply Approximately Optimal Liquid Welfare

In this work we construct bidding algorithms that simultaneously achieve low individual regret and an approximation to the optimal aggregate liquid welfare. This combination is reminiscent of similar results from the analysis of smooth games, which include many auction games. No-regret learning algorithms converge (in distribution) to coarse correlated equilibria (CCE), and for smooth games it is known that the allocations obtained at CCE approximately optimize the aggregate welfare [17, 42, 44]. Thus, for auction games that satisfy smoothness conditions, achieving low regret directly implies (on a per-instance basis) an aggregate welfare approximation.

These results do not directly apply in our case: our class of games includes non-smooth games, and our welfare metric is different. But one might still wonder whether a similar direct implication applies. In this section we present an example showing that this direct implication does not hold in our setting with budgets and liquid welfare, even for single-item second-price auctions. We provide an auction environment and construct a coarse correlated equilibrium for the agents. The fact that agents employ bidding strategies that forms a coarse correlated equilibrium means, in particular, that all agents achieve zero regret. Nevertheless, in our example, the allocation that results from this bidding equilibrium has an unbounded approximation factor with respect to expected liquid welfare.

Proposition E.1. There exists an instance of a second-price auction for a single good and two agents, described by a distribution over valuations, and a coarse correlated equilibrium of the auction (i.e., a pair
of bidding strategies under which each agent has regret zero) such that the resulting expected liquid welfare is arbitrarily small compared to the optimal liquid welfare.

Proof. The per-round auction in our example is a second-price auction for a single good. There are two agents. The distribution $F$ over value profiles is such that $(v_1, v_2) = (2, 1)$ with probability 1. The target per-round spend rates for the agents are $(\rho_1, \rho_2) = (1/(1+\pi), 1)$, where $\pi$ is some arbitrarily large constant independent of $T$.

Consider the following bidding strategies for the agents. Agent 1 bids value 2 for all periods and agent 2 bids 0 for all periods. Under this strategy profile, agent 1 receives all the items over the $T$ rounds, and both agents pay nothing.

Under these strategies, agent 1 has zero regret as she obtains the maximum possible value. Agent 2 likewise has zero regret, since no choice of bid less than $v_2$ can cause her to win in any round. Note that the agents would still have zero regret if their objective were changed to maximizing value minus (any scalar multiple $\lambda \in [0, 1]$ times) payments.

The liquid welfare of this equilibrium is $T/(\pi + 1)$, the total budget of agent 1. However, allocating all goods to agent 2 achieves a liquid welfare of $T$. Since $\pi$ is an arbitrarily large constant, this approximation factor is unbounded.

\[\square\]

F Linear Pacing and Advertiser-Feasible Bidding Strategies

In our analysis we restricted our attention to bidding strategies that are linear in the following sense. In each round $t$ an agent $k$ first chooses a pacing factor $\mu_{k,t} \geq 0$, then $v_{k,t}$ is revealed and the agent bids $b_{k,t} = v_{k,t}/(\mu_{k,t} + 1)$. The important restriction is that $\mu_{k,t}$ is chosen independently of $v_{k,t}$. This choice of $\mu_{k,t}$ can therefore be interpreted as a linear mapping from $v_{k,t}$ to $b_{k,t}$.

When the underlying auction is truthful, this restriction to linear policies is known to be without loss. However, for non-truthful auctions (such as a first-price auction) a value-maximizing or utility-maximizing agent might be able to strictly improve their outcome with a non-linear mapping from value to bid. For example, in a first-price auction where the highest competing bid is known to be exactly 1, an agent with value 2 and an agent with value 3 would both optimize their quasi-linear utility by bidding (slightly more than) 1 even though this is not implementable with a linear bidding strategy.

One motivation for our restriction to linear policies is that they capture bidding strategies that are implementable by an external agent (i.e., an advertiser or third-party bid optimizer) who does not have visibility into the precise value estimates of the platform, such as click-rate estimates.

We now make this intuition more precise by adding click rates to our auction model. We will associate each round $t$ with a potential ad impression to be shown to a user. The auction in round $t$ determines which ad will be shown; the user may or may not click on the ad. Each agent $k$ has a value $v_k$ that is obtained only if their advertisement is clicked. The advertising platform has access to an estimated click rate $c_{k,t} \in [0, 1]$ that describes the likelihood that the user will click on agent $k$’s advertisement if shown. Thus the expected value to agent $k$ of winning the auction in round $t$ is $v_k c_{k,t}$. We will write $v_k = v_k c_{k,t}$.

In each round, the agent can place a bid $\beta_{k,t}$ that is interpreted as a willingness to pay per click. Once all agents have placed bids, the mechanism multiplies these by the corresponding click rate estimates to determine the effective bids for winning the auction. These are denoted $b_{k,t} = \beta_{k,t} c_{k,t}$. The auction mechanism in round $t$ is then resolved using the effective bids $b_{k,t}$.

We note that if the click rates $c_{k,t}$ are visible to the agents then this formulation is equivalent to our model from Section 2, as it simply expresses the values $v_{k,t}$ and the bids $b_{k,t}$ in a different way. However, for an advertiser that is external to the platform, the bid $\beta_{k,t}$ must be placed without observing the realization of the estimated click rate $c_{k,t}$. Equivalently, the estimated click rate $c_{k,t}$ is realized after the bid $\beta_{k,t}$ is fixed.
We can therefore write $\mu_{k,t} = \frac{v_{k}}{\beta_{k,t}} - 1$ which is independent of $c_{k,t}$. We then have that for every possible realization of $v_{k,t}$,

$$b_{k,t} = \beta_{k,t}c_{k,t} = \frac{1}{\mu_{k,t} + 1} v_{k}c_{k,t} = \frac{1}{\mu_{k,t} + 1} v_{k,t}$$

and hence the advertiser’s effective bidding strategy will be linear.

\[G\] Generalized Second Price is a Core Auction

In this section we show that the GSP auction for sponsored search allocation with separable click rates is a core auction. Recall the definition of a GSP auction. There are $m \geq 1$ slots with click rates $1 \geq \alpha_1 \geq \ldots \geq \alpha_m \geq 0$. There are $n$ bidders, each bidder placing a bid $b_i \geq 0$. We will reindex agents in order of bid, so that $b_1 \geq b_2 \geq \ldots \geq b_n$. Without loss of generality, we will assume $m = n$ (by removing extra bidders with bid 0 or extra slots with click rate 0), and we will define $b_{n+1} = \alpha_{m+1} = 0$ for convenience.

In the GSP auction, slots are allocated greedily by bid, and each agent pays a price per unit equal to the next-highest bid. That is, agent $i$ receives slot $j$ for a declared value of $b_i \alpha_i$, and pays $b_{i+1} \alpha_i$.

We claim that the GSP auction is a core auction. First, since $b_{i+1} \leq b_i$ for all $i$, we have that $b_{i+1} \alpha_i \leq b_i \alpha_i$, and hence each bidder pays at most her declared welfare for the allocation received.

It remains to show the second property of a core auction. Choose any subset of bidders $S \subseteq [n]$. The allocation $\gamma$ to agents in $S$ that maximizes declared welfare is the one that allocates greedily in index order. More formally, for each $i \in S$, let $\sigma(i)$ be 1 plus the number of elements of $S$ with index less than $i$. For example, if $S = \{2, 6, 7\}$, then $\sigma(2) = 1$, $\sigma(6) = 2$, and $\sigma(7) = 3$. Then the declared-welfare-maximizing allocation $\gamma$ to agents in $S$ is such that $y_i = \alpha_{\sigma(i)}$ for each $i$, for a total declared welfare of $\sum_{i \in S} b_i \alpha_{\sigma(i)}$.

The core auction property on subset of bidders $S$ therefore reduces to showing that

$$\sum_{i \notin S} b_{i+1} \alpha_i + \sum_{i \in S} b_i \alpha_i \geq \sum_{i \in S} b_i \alpha_{\sigma(i)}. \quad (G.1)$$

To establish inequality $(G.1)$, we first note that

$$\sum_{i \in S} b_i \alpha_{\sigma(i)} - \sum_{i \in S} b_i \alpha_i = \sum_{i \in S} b_i (\alpha_{\sigma(i)} - \alpha_i)$$

$$= \sum_{i \in S} \sum_{j = \sigma(i)}^i b_i (\alpha_j - \alpha_{j+1})$$

$$\leq \sum_{i \in S} \sum_{j = \sigma(i)}^i b_{j+1} (\alpha_j - \alpha_{j+1}).$$

This final double summation contains an instance of the term $b_{j+1} (\alpha_j - \alpha_{j+1})$ for each $i \in S$ such that $\sigma(i) \leq j < i$. But for each $j$ and each $i$ such that $\sigma(i) \leq j < i$ (i.e., such that the “new” allocation to agent $i$ under $\gamma$ is slot $j$ or better, and the “old” allocation is worse than slot $j$), there must be some $k \leq j$ such that $k \notin S$. The number of such $i$ is therefore at most the number of agents at index $j$ or less that are not in $S$. More precisely, observe that $j \geq \{i \in S : i \leq j\} + \{i \in S : \sigma(i) \leq j < i\}$. This holds as $\sigma(i) \leq i$ for all $i$ and is injective, so the sets on the right hand side are evidently disjoint and $\sigma$ maps each such element to a unique index in $[j]$. But $j = \{i \in S : i \leq j\} + \{k \notin S : k \leq j\}$, so cancelling terms shows that $|\{k \notin S : k \leq j\}| \geq \{|i \in S : \sigma(i) \leq j < i\}|$. Thus, by rearranging the order of summation, we have
where the final inequality is a telescoping sum. We therefore conclude that

\[
\sum_{i \in S} b_{i \sigma(i)} a_{\sigma(i)} - \sum_{i \in S} b_{i} a_{i} \leq \sum_{k \in S} b_{k+1} a_{k}
\]

and rearranging yields the desired inequality (G.1).