A Grothendieck-Witt space for stable infinity categories with duality

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Abstract

We construct a Grothendieck-Witt space for any stable infinity category with duality. If we apply our construction to perfect complexes over a commutative ring in which 2 is invertible we recover the classical Grothendieck-Witt space. Our Grothendieck-Witt space is a grouplike $E_\infty$-space which is part of a genuine $C_2$-spectrum, the connective real $K$-theory spectrum.

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1 Introduction

In this paper we carry over the hermitian $S_\bullet$-construction which can be found e.g. in [15] to the $\infty$-categorical setting. The input of our construction is an $\infty$-category with duality in the sense of [10] whose underlying $\infty$-category is stable. For such an $\infty$-category $C$ we define a Grothendieck-Witt space $GW(C)$ which has the structure of a grouplike $E_\infty$-space, so gives rise to a connective spectrum. We show in the last section that this spectrum is in fact part of a genuine $C_2$-spectrum $KR(C)$, the connective real $K$-theory spectrum of $C$.

This way we obtain for example for any $E_\infty$-ring spectrum $R$ and tensor invertible object $L \in Perf(R)$ (here $Perf(R)$ denotes the stable $\infty$-category of perfect $R$-modules) a real $K$-theory spectrum $KR(R,L)$, in particular spectra
KR$(R, R[n])$ for any $n \in \mathbb{Z}$, by considering the $L$-twisted duality on Perf$(R)$ (see [10] §8).

To justify our constructions we prove in section 4 that our Grothendieck-Witt space is equivalent to the classical Grothendieck-Witt space (as defined e.g. in [13]) in the case that $R$ is a discrete ring in which 2 is invertible and $L$ a shifted invertible (in the discrete sense) $R$-module.

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## 2 Recollections and preliminaries

We use the same conventions as in [10]. In particular Cat$^h_{\infty}$ is the $\infty$-category of small $\infty$-categories with duality. We denote be Cat$^h_{\infty}$ the subcategory of Cat$_{\infty}$ of stable $\infty$-categories and exact functors between them.

It follows from [10, Proposition 2.2] that the induced functor

$$(\text{Cat}^h_{\infty})^{t_{C_2}} \to \text{Cat}^h_{\infty}$$

(on the left we use the induced $C_2$-action) is a monomorphism in $\text{Cat}^h_{\infty}$ whose good image consists of those $\infty$-categories with duality whose underlying $\infty$-category is stable and those functors between $\infty$-categories with duality whose underlying functor is exact. We write for this good image $(\text{Cat}^h_{\infty})^{st}$.

Usually we will not distinguish between a category and its nerve viewed as an $\infty$-category.

We will frequently see objects $[n] \in \Delta$ as categories. For a category $C$ we write Ar$(C)$ for the arrow category, i.e. the functor category Fun$(\Delta, C)$.

**Proposition 2.1.** Let $n \in \mathbb{N}$ and $C_1, \ldots, C_n$ be stable $\infty$-categories. Let $D$ be an $\infty$-category which admits finite limits and denote by Sp$(D)$ the stabilization of $D$.

Let

$$\text{Fun}'(C_1 \times \cdots \times C_n, D) \subset \text{Fun}(C_1 \times \cdots \times C_n, D)$$

be the full subcategory on those functors which preserve finite limits separately in each variable, and let

$$\text{Fun}'(C_1 \times \cdots \times C_n, \text{Sp}(D)) \subset \text{Fun}(C_1 \times \cdots \times C_n, \text{Sp}(D))$$

be the full subcategory on those functors which are exact separately in each variable. Then composition with the functor $\Omega^{\infty}: \text{Sp}(D) \to D$ induces an equivalence

$$\text{Fun}'(C_1 \times \cdots \times C_n, \text{Sp}(D)) \to \text{Fun}'(C_1 \times \cdots \times C_n, D)$$

of $\infty$-categories. If $C_1 = \cdots = C_n$ then this equivalence respects the $\Sigma_n$-actions.

**Proof.** This follows from [13, Corollary 1.4.2.23.].

Let $C \in \text{Cat}_{\infty}$. To give a duality on $C$ (or equivalently on $C^{\text{op}}$) is the same as to give a $C_2$-homotopy fixed point of Fun$(C \times C, \text{Spc})$ (or equivalently of
Remark 3.2. The space \( \Omega_B \) is informally given by which is nondegenerate representable in the sense of [11]. The bilinear functor \( E \) is denoted by \( \divides.alt0 \).

Definition 3.1. Let \( C \in \Cat_{\infty} \). Building on Waldhausen’s definition we define for any \( n \in \mathbb{N} \) the \( \infty \)-category \( S_n(C) \) to be the full subcategory of the functor category \( \Fun(Ar([n]), C) \) on those functors \( A \) such that for any \( 0 \leq i \leq n \) the object \( A_{i,i} \) is a zero object in \( C \) and such that for any \( 0 \leq i \leq j \leq k \leq n \) the square

\[
\begin{array}{ccc}
A_{i,j} & \longrightarrow & A_{i,k} \\
\downarrow & & \downarrow \\
A_{j,j} & \longrightarrow & A_{j,k}
\end{array}
\]

is exact in \( C \). These properties are preserved by the suspension and loop functors on \( C \), thus the \( \infty \)-categories \( S_n(C) \) are stable. The simplicial \( \infty \)-category \( \Fun(Ar([\bullet]), C) \) restricts to a simplicial \( \infty \)-category \( S_\bullet(C) \). Taking levelwise core groupoids yields the simplicial object \( S_\bullet(C) \) in spaces whose realization we denote by \( |S_\bullet(C)| \).

Definition 3.1. The \( K \)-theory space \( K(C) \) of the stable \( \infty \)-category \( C \) is defined to be the loop space \( \Omega |S_\bullet(C)| \), where we take a zero object of \( C \) as base point.

Remark 3.2. The space \( K(C) \) has the natural structure of a grouplike \( E_\infty \)-space (see also the discussion at the end of this section for the case of the Grothendieck-Witt space).

Our \( \infty \)-categorical definition of the Grothendieck-Witt space of a stable \( \infty \)-category with duality is modelled on the hermitian \( S_\bullet \)-construction given for example in [15]. This uses the edgewise subdivision of a simplicial object which we introduce now.

Definition 3.3. Let \( X : \Delta^{op} \to C, [n] \mapsto X_n, \) be a simplicial object in an \( \infty \)-category \( C \). Then the edgewise subdivision \( E(X) \) is defined to be the simplicial object \( X \circ \iota^{op} \), where \( \iota : \Delta \to \Delta \) is the endofunctor defined by \( [n] \mapsto [n]^{op} \ast [n] \).

Thus we have \( E(X)_n = X_{2n+1} \). The inclusions \( [n] \mapsto [n]^{op} \ast [n] \) define a natural transformation from the identity functor on \( \Delta \) to \( \iota \) and thus we are at the disposal of a natural map of simplicial objects \( E(X) \to X \).

For \( C \) a stable \( \infty \)-category we let \( S^n_\bullet(C) := E(S^n_\bullet(C)) \), and likewise \( S^{n-\ast}_\bullet(C) := E(S^{n-\ast}_\bullet(C)) \).

For each \( n \in \mathbb{N} \) the category \( [n] \) has a unique structure of a category with strict duality and the assignment \( [n] \mapsto [n]^{op} \ast [n] \) can be viewed as a functor.
from $\Delta$ to the category of categories with strict duality $\text{CD}$, therefore the same holds for the assignment $[n] \mapsto \text{Ar}([n]^{\text{op}} \ast [n])$.

In [10, §11] a functor $\epsilon: \text{CD} \to \text{Cat}_{\infty}^{hC_2}$ is constructed. Moreover $\text{Cat}_{\infty}^{hC_2}$ is cartesian closed, and the internal hom commutes with the forgetful functor $\text{Cat}_{\infty}^{hC_2} \to \text{Cat}_{\infty}$.

Thus for an $\infty$-category $C$ with duality $\text{Fun}(\text{Ar}([n]^{\text{op}} \ast [n]), C)$ is an object of $\text{Cat}_{\infty}^{hC_2}$ functorial in $[n]$.

If now $C$ is a stable $\infty$-category with duality then for any $n \in \mathbb{N}$ the full subcategory $\mathcal{S}_n(C)$ of $\text{Fun}(\text{Ar}([n]), C)$ is preserved by the duality (since the dual of an exact square is again an exact square), thus $\mathcal{S}_n^\ast(C)$ can be viewed as a simplicial object in $\text{Cat}_{\infty}^{hC_2}$, and $\mathcal{S}_n^\ast(\ast)\cdot(C)$ can be viewed as a simplicial object in $\text{Spc}_{\infty}^{hC_2} = \text{Spc}[C_2]$.

Taking levelwise the homotopy $C_2$-fixed points of the latter object defines the simplicial space $(\mathcal{S}_n^\ast(\ast)\cdot(C))_n$.

**Definition 3.4.** For $C$ a stable $\infty$-category with duality we let the Grothendieck-Witt space $\text{GW}(C)$ of $C$ be the homotopy fiber of the composition

$$\text{GW}(C) \to \text{SymMonCat}_{\infty}^{hC_2} \to \text{Mon}_{\infty}(\text{Spc})[C_2].$$

We now equip $\text{GW}(C)$ with an $E_{\infty}$-structure which will turn out to be grouplike (i.e. an infinite loop space structure), see Proposition 5.8. The monomorphisms

$$\text{Cat}_{\infty}^{\ast} \to \text{Cat}_{\infty}^{\text{preadd}} \to \text{Cat}_{\infty}$$

as well as the full embedding

$$\text{Cat}_{\infty}^{\text{preadd}} \to \text{SymMonCat}_{\infty}^{hC_2}$$

of $\infty$-categories carry $C_2$-actions (see [10, §6]), hence we have an induced composition

$$s: (\text{Cat}_{\infty}^{hC_2})^{\ast} \to (\text{Cat}_{\infty}^{\ast})^{hC_2} \to (\text{Cat}_{\infty}^{\text{preadd}})^{hC_2} \to \text{SymMonCat}_{\infty}^{hC_2}.$$
4 The comparison

We denote the (hermitian) $S_\ast$-construction used in [15] for an exact category with weak equivalences (and duality) $\mathcal{E}$ by the same symbols as we used in the $\infty$-categorical situation except that we write $S^{\text{str}}$ instead of $S$. Thus for example if $\mathcal{E}$ has a duality then $S^{\text{str},e}(\mathcal{E})$ is a simplicial exact category with weak equivalences and duality and $S^{\text{str},e,\sim}(\mathcal{E})$ denotes the simplicial subcategory of weak equivalences.

We denote by $\|\|: \text{Cat}^1 \to \text{Spc}$ the natural functor from the 1-category of small categories $\text{Cat}^1$ to the $\infty$-category of spaces which takes the realization of the nerve. Thus if $\mathcal{C}_\ast$ is for example a simplicial category then $\|\mathcal{C}_\ast\|$ will be a simplicial object in $\text{Spc}$. The realization of this simplicial object is denoted by $\|\mathcal{C}_\ast\|_r$.

The Grothendieck-Witt space $GW(\mathcal{E})$ is then defined to be the homotopy fiber of the natural map $\|S^{\text{str},e,\sim}(\mathcal{E})\|_h \to \|S^{\text{str},\sim}(\mathcal{E})\|_r$.

As in the $\infty$-categorical case we can equip $GW(\mathcal{E})$ with a natural $E_\infty$-structure (use that the functor $H^{\text{lax}}$ (see [10], §11) is symmetric monoidal for the cartesian symmetric monoidal structures since it is a right adjoint).

For a ring $R$ we denote by $\mathcal{P}_R$ the category of finitely generated projective $R$-modules and by $\text{Cpx}^b(\mathcal{P}_R)$ the exact category with weak equivalences of bounded complexes with values in $\mathcal{P}_R$. We denote by $\text{Perf}(R)$ the stable $\infty$-category of perfect $R$-modules.

Note that we exhibit a natural functor $\text{Cpx}^b(\mathcal{P}_R) \to \text{Perf}(R)$ which is a localization at the quasi isomorphisms.

We now assume that $R$ is commutative, fix for the whole section an integer $N \in \mathbb{Z}$ and an invertible $R$-module $L$ and equip $\text{Cpx}^b(\mathcal{P}_R)$ with the strong duality $X \mapsto \text{Hom}(X,L[N])$.

[10] Corollary 8.5] equips $\text{Perf}(R)$ with the duality given by the object $L[N] \in \text{Pic}(\text{Perf}(R))$, and the naturality of the construction of loc. cit. shows that the functor $\text{Cpx}^b(\mathcal{P}_R) \to \text{Perf}(R)$ preserves the dualities.

To emphasize the dependence on the duality we denote the corresponding Grothendieck-Witt spaces by $GW(\text{Cpx}^b(\mathcal{P}_R), N, L)$ and $GW(\text{Perf}(R), N, L)$.

For any $n \in \mathbb{N}$ we obtain a functor

$$\text{Fun}(\text{Ar}([n]), \text{Cpx}^b(\mathcal{P}_R)) \to \text{Fun}(\text{Ar}([n]), \text{Perf}(R))$$

between $\infty$-categories with duality.

The restriction to the full subcategory $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$ of this functor factors through $S_n(\text{Perf}(R))$ yielding functors

$$S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)) \to S_n(\text{Perf}(R))$$

$$S_n^{\text{str},\sim}(\text{Cpx}^b(\mathcal{P}_R)) \to S_\ast\sim(\text{Perf}(R))$$

between $\infty$-categories with duality.
The functors
\[ \mathcal{H}^{\text{dax}}(S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))) \to \mathcal{H}^{\text{dax}}(S_n^{\text{perf}}(\text{Perf}(R))) \]
induced on lax hermitian objects (see [10, §11]) by the latter functors together with the equivalence
\[ \mathcal{H}(S_n^{\text{perf}}(\text{Perf}(R))) \simeq \mathcal{H}^{\text{dax}}(S_n^{\text{perf}}(\text{Perf}(R))) \]
yields functors
\[ (S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)))_h \to (S_n^{\text{perf}}(\text{Perf}(R)))_h \]
(see also [10, Proposition 11.8]).
Every map in \( S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)) \) is sent to an equivalence under the functor \(\text{Id} \), thus we obtain maps
\[ |(S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)))_h| \to (S_n^{\text{perf}}(\text{Perf}(R)))_h \]
in \( \text{Spc}[C_2] \). Also every map in \( (S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)))_h \) is sent to an equivalence under the functor \(\text{Id} \), thus we obtain maps
\[ |(S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)))_h| \to (S_n^{\text{perf}}(\text{Perf}(R)))_h \]
in \( \text{Spc} \).

After edgewise subdivision we get a map
\[ |(S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)))_h| \to (S_n^{\text{perf}}(\text{Perf}(R)))_h \]
between simplicial objects in \( \text{Spc} \).

Altogether we arrive at a commutative square
\[ \begin{array}{ccc}
|S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))| & \to & S_n^{\text{perf}}(\text{Perf}(R)) \\
\downarrow & & \downarrow \\
(S_n^{\text{str}}(\text{Perf}(R)))_h & \to & S_n^{\text{perf}}(\text{Perf}(R))
\end{array} \]
of simplicial objects in \( \text{Spc} \), which yields after taking realizations and fibers of the horizontal induced maps the comparison map
\[ \text{GW}(\text{Cpx}^b(\mathcal{P}_R), N, L) \to \text{GW}(\text{Perf}(R), N, L). \]

**Remark 4.1.** The comparison map (5) can be made compatible with the \( E_{\infty} \)-structures on both sides. We leave the details to the interested reader.

**Theorem 4.2.** If \( 2 \) is invertible in \( R \) the comparison map (5) is an equivalence.

**Proof.** Combine the next two Lemmas.

**Lemma 4.3.** The maps (3) are equivalences.

**Proof.** This is standard.

The main input to our comparison statement is

**Lemma 4.4.** If \( 2 \) is invertible in \( R \) then the maps (4) are equivalences.
Proof. We have a commutative diagram

\[
\begin{array}{c}
\left| S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)) \right| \\
\downarrow \\
\left| S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)) \right|
\end{array}
\xymatrix{
S_n(\text{Perf}(R))_h \\
S_n(\text{Perf}(R))
}
\tag{6}
\]  

in $\text{Spc}$. We want to show that the upper horizontal map is an equivalence. By Lemma 4.3 the lower horizontal map is an equivalence. We will show that for any $X \in S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))$ the space of paths $\text{map}(X, X^\vee)$ in $|S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))|$ (or equivalently in $S_n(\text{Perf}(R))$) carries a natural $C_2$-action, that the homotopy fibers of the vertical maps in the diagram over $X$ (resp. the image of $X$) are canonically identified with $\text{map}(X, X^\vee)^{hC_2}$ and that the induced map (by the commutative square) on these fibers respect these identifications. From this the claim follows.

We first apply [15, Lemma 4] to the exact category $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$ with duality (considering only the isomorphisms as weak equivalences) to obtain a category $C = S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))^{\text{str}}$ with a strict duality which is equivalent to $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$ as category with (strong) duality. We denote by $C^\sim$ the subcategory of $C$ of weak equivalences (which correspond to the objectwise quasi isomorphisms in $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$).

For a category $D$ we let $\text{Tw}(D)$ be the twisted arrow category of $D$ whose objects are the morphisms of $D$, and a map from $f:A \to B$ to $g:C \to D$ is a commutative square

\[
\begin{array}{c}
A \\
\downarrow f \\
B
\end{array}
\xymatrix{
\quad \\
\quad C \\
\downarrow g \\
\quad D
}
\]

in $D$. If $D$ has a strict duality then the assignment $f \mapsto f^\vee$ defines a (strict) $C_2$-action on $\text{Tw}(D)$ whose (strict) $C_2$-fixed points is the category of hermitian objects of $D$.

Similarly for an $\infty$-category $D$ the assignment

\[
\Delta^{op} \ni [n] \mapsto \text{map}([n] \star [n]^{op}, D)
\]

defines a complete Segal space whose associated $\infty$-category is defined to be the twisted arrow category $\text{Tw}(D)$ of $D$ (this is compatible with the 1-categorical definition). If $D$ has a duality then the above assignment has values in $\text{Spc}[C_2]$, thus $\text{Tw}(D)$ has a $C_2$-action. Moreover by the construction in [10, §11] we have a canonical equivalence

\[
\text{Tw}(D)^{hC_2} \simeq \text{H}^{\text{las}}(D).
\]

The canonical map $\text{Tw}(D) \to D \times D^{op}$ is $C_2$-equivariant, where the action on $D \times D^{op}$ is given by $(X, Y) \mapsto (Y^\vee, X^\vee)$, and we have $(D \times D^{op})^{hC_2} \simeq D$.

The right vertical map of diagram (6) can thus be identified with the map

\[
\text{Tw}(S_n(\text{Perf}(R)))_{hC_2} \to (S_n(\text{Perf}(R)) \times S_n(\text{Perf}(R))^{\text{op}})^{hC_2}.
\]
We have a commutative diagram

\[
\begin{array}{ccc}
\text{Tw}(C^-) & \longrightarrow & \text{Tw}(S_n^-(\text{Perf}(R))) \\
\downarrow & & \downarrow \\
C^- \times (C^-)^{\text{op}} & \longrightarrow & S_n^-(\text{Perf}(R)) \times S_n^-(\text{Perf}(R))^{\text{op}}
\end{array}
\]

in $\text{Cat}_\infty[C_2]$. In the induced diagram

\[
\begin{array}{ccc}
|\text{Tw}(C^-)| & \longrightarrow & |\text{Tw}(S_n^-(\text{Perf}(R)))| \\
|C^- \times (C^-)^{\text{op}}| & \longrightarrow & |S_n^-(\text{Perf}(R)) \times S_n^-(\text{Perf}(R))^{\text{op}}|
\end{array}
\]

in $\text{Spc}[C_2]$ the horizontal maps are equivalences. Thus diagram (6) can be identified with the diagram

\[
\begin{array}{ccc}
|\text{Tw}(C^-)^{C_2}| & \longrightarrow & |\text{Tw}(C^-)^{hC_2}| \\
\downarrow & & \downarrow \\
|C^- \times (C^-)^{\text{op}})^{C_2}| & \longrightarrow & |C^- \times (C^-)^{\text{op}}^{hC_2}|
\end{array}
\]

whose lower entries can be identified with $|C^-|$.

For $X \in C$ we let $P_X$ be defined by the (strict) pullback diagram

\[
\begin{array}{ccc}
P_X & \longrightarrow & \text{Tw}(C^-) \\
\downarrow & & \downarrow \\
(C^- \times (C^-)^{\text{op}})/(X, X^\vee) & \longrightarrow & C^- \times (C^-)^{\text{op}}
\end{array}
\]

of categories. Since $(X, X^\vee) \in C^- \times (C^-)^{\text{op}}$ is a fixed point with respect to the $C_2$-action $P_X$ inherits a $C_2$-action and this diagram becomes $C_2$-equivariant. Taking $C_2$-fixed points of this diagram gives a diagram canonically isomorphic to the pullback diagram

\[
\begin{array}{ccc}
P_{C_2}^X & \longrightarrow & C^-_h \\
\downarrow & & \downarrow \\
C^-/X & \longrightarrow & C^-.
\end{array}
\]

For a map $X \to Y$ in $C^-$ we have an induced $C_2$-equivariant map $P_X \to P_Y$.

Claim 1: For any map $X \to Y$ in $C^-$ the map $|P_X| \to |P_Y|$ is an equivalence.

Claim 2: For any $X \in C$ the map $|P_{C_2}^X| \to |P_{X}^{hC_2}|$ is an equivalence.

Claim 3: For any map $X \to Y$ in $C^-$ the map $|P_{C_2}^X| \to |P_{C_2}^Y|$ is an equivalence.

Claim 3 follows from Claims 1 and 2.

It follows from Claim 1 and Quillen’s Theorem B (dual of [8, Theorem 5.6]) that the realization of diagram (8) is a pullback diagram, similarly it follows
from Claim 3 and Quillen’s Theorem B that the realization of diagram (9) is a pullback diagram.

Thus the induced map on the homotopy fibers over an \( X \in C \) of the vertical maps in diagram (7) can be identified with the map \( |P_X^{C_2}| \rightarrow |P_X|^{hC_2} \) (for the second fiber note that homotopy fixed points preserve fiber sequences) which is an equivalence by Claim 2. So we see that if we prove Claims 1 and 2 the proof is finished.

**Proof of Claim 1:** It follows from [4, Propositions 6.2 and 8.2] and the correction [\ref{5}] that \( |P_X| \) is canonically equivalent to the mapping space map(\( X, X' \)) in \( |C'| \). The category \( P_X \) is naturally isomorphic to the category denoted \( C'(X, X')_{\text{Hom-tw}} \) in \( \text{[3]} \), and thus a collection of connected components of \( C(X, X')_{\text{Hom-tw}} \). The factorizations necessary for these arguments are given by cylinder constructions. Thus the claim follows.

**Proof of Claim 2:** Let \( X \in C \) and \( X' \) be the image of \( X \) in \( S_n^{\text{str}}(\text{Cpx}^k(\mathcal{P}_R)) \). Let \( N\mathbb{Z}[\Delta^\bullet] \) be the cosimplicial object in \( \text{Cpx}^k(\mathcal{P}_R) \) which assigns to \([n]\) the complex corresponding to the simplicial abelian group \( \mathbb{Z}[\Delta^\bullet] \) under the Dold-Kan correspondence. Thus it is Reedy cofibrant, and the cosimplicial object \( N\mathbb{Z}[\Delta^\bullet] \otimes X' \) in \( S_n^{\text{str}}(\text{Cpx}^k(\mathcal{P}_R)) \) is a special cosimplicial resolution of \( X' \) in the sense of \( \text{[3]} \). We let \( X^* \) be the image of this cosimplicial object in \( C \). Also let \((C'/X)_f \) be the full subcategory of \( C'/X \) on those maps \( Y \rightarrow X \) which are surjections. Then by \( \text{[5, Proposition 6.12]} \) the functor \( \varphi: \Delta \rightarrow (C'/X)_f \) which sends \([n]\) to \( X'^n \rightarrow X \) is left cofinal.

Let the categories \( R \) and \( S \) be defined by the pullback diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\psi} & R \\
\downarrow & & \downarrow \iota
\end{array}
\xymatrix{
\Delta \ar[r]^-{\varphi} & (C'/X)_f \ar[r] & C'/X.
}
\]

(10)

Let \( r \in R \), so \( r \) consists of a surjection \( Y \rightarrow X \) in \( C' \) together with a hermitian structure on \( Y \). Since for any map \( Z \rightarrow Y \) in \( C' \) there exists a unique hermitian structure on \( Z \) compatible with the one on \( Y \) it follows that the natural functor

\[\psi/r \rightarrow \varphi/(Y \rightarrow X)\]

is an isomorphism, hence \( \psi \) is also left cofinal.

The vertical functors in diagram (10) are right fibrations, and the fiber over an object \( Y \rightarrow X \) is \( \text{Hom}_{C'}(Y, Y')^{C_2} \) (i.e. the set of hermitian structures on \( Y \)). Thus for a vertical map \( A \rightarrow B \) in this diagram we exhibit a functor \( j_B: B^{op} \rightarrow \text{Set} \rightarrow \text{sSet} \), and \( |A| \) is naturally equivalent to \( \text{hocolim} j_B \). It follows that \( |S| \rightarrow |R| \) is an equivalence.

Let \( K \) be the simplicial set defined by \([n] \mapsto \text{Hom}_{C'}(X^n, (X^n)^*) \) (\( K \) has then in fact the structure of a simplicial \( R \)-module) and \( K^- \) the subsimplicial set on those simplices which are in \( C' \). The simplicial set \( K \) has a natural \( C_2 \)-action and \( K^- \) is stable under this action.

It follows from the above considerations that \( |S| \) is naturally equivalent to \((K^-)^{C_2}\).

Note that the natural map

\[K^{C_2} \rightarrow K^{hC_2}\]
is an weak homotopy equivalence since 2 is invertible in \( R \), thus, since \( K^- \) consists of certain connected components of \( K \), the same follows for the map 

\[(K^-)^{C_2} \rightarrow (K^-)^{hC_2} .\]

For a map \( f : Y \rightarrow Z \) in \( S_n^{\text{eff}}(Cpx^b(P_R)) \) denote by \( c(f) \) the construction \([16, 1.5.5]\) applied to the map \( f^{\text{op}} \) in \( S_n^{\text{eff}}(Cpx^b(P_R))^{\text{op}} \). We therefore obtain a factorization \( Y \rightarrow c(f) \rightarrow Z \) of \( f \) into an inclusion which is a quasi isomorphism followed by a surjection, and moreover there is a retraction \( c(f) \rightarrow Y \) of the first map. This construction is functorial in \( f \). If \( Y \) has a hermitian structure then the retraction induces a hermitian structure on \( c(f) \). These constructions can be transported to \( C \).

For an object \( a \in P_{C_2}^X \) with underlying object \( f : Y \rightarrow X \) in \( C^-/X \) the above factorization applied to \( f \) yields an object \( (c(f) \rightarrow X) \in (C^-/X)_f \) and also an object in \( R \) (using the induced hermitian structure on \( c(f) \) ). This assignment defines a functor \( p : P_{C_2}^X \rightarrow R \) together with natural transformations \( \text{id} \rightarrow i \circ p \) and \( \text{id} \rightarrow p \circ i \). It follows that the realization of \( i \) is an equivalence.

Hence we have seen that the natural map \( |S| \rightarrow |P_{C_2}^X| \) in \( \text{Spc} \) is an equivalence, and that in the commutative square

\[
\begin{array}{ccc}
(K^-)^{C_2} & \rightarrow & |P_{C_2}^X| \\
\downarrow & & \downarrow \\
(K^-)^{hC_2} & \rightarrow & |P_X|^{hC_2}
\end{array}
\]

in \( \text{Spc} \) the upper horizontal and the left vertical maps are equivalences. Also the lower horizontal map is an equivalence. Hence Claim 2 and thus the Lemma are proved.

5 The zeroth Grothendieck-Witt group

Let \( C \in \text{Cat}_{\infty}^{hC_2} \). Then the right fibration

\[ p : Tw(C) \rightarrow C \times C^{\text{op}} \]

inherits a \( C_2 \)-action (see the proof of Lemma \([14]\) ). Thus for \( X \in C \) the space \( \text{map}(X, X^\vee) \) has a natural \( C_2 \)-action, since it arises as the homotopy fiber of \( p \) over a homotopy fixed point for the \( C_2 \)-action.

Using the equivalence \( Tw(C)^{hC_2} \cong \mathcal{H}^{\text{lax}}(C) \) we see that the fiber over \( X \) of the functor \( \mathcal{H}^{\text{lax}}(C) \rightarrow C \) is the \( \infty \)-groupoid map \( (X, X^\vee)^{hC_2} \), so to give a lax hermitian structure on \( X \) is the same as to give a \( C_2 \)-homotopy fixed point of map \( (X, X^\vee) \).

On the other hand the symmetric functor \( C^{\text{op}} \times C^{\text{op}} \rightarrow \text{Spc} \) corresponding to the duality on \( C \) can be viewed as a map in \( \text{Cat}_{\infty}[C_2] \) (where the \( C_2 \)-action on the source is the switch map and on the target the trivial action), and taking homotopy fixed points yields a functor

\[ C^{\text{op}} \rightarrow \text{Spc}[C_2] \]

which is informally given by \( X \mapsto \text{map}(X, X^\vee) \). This way \( \text{map}(X, X^\vee) \) also inherits a \( C_2 \)-action which can be seen to be naturally equivalent to the action from above.
If $C$ is stable the same argument as above yields a functor

$$C^{\text{op}} \to \text{Sp} [C_2]$$

which is informally given by $X \mapsto \text{map}^\oplus (X, X^\vee)$. Composing with

$$\Omega^\infty : \text{Sp} \to \text{Spc}$$

yields the functor above. Let $Q : C^{\text{op}} \to \text{Sp}$ be given by $X \mapsto \text{map}^\oplus (X, X^\vee)^{hC_2}$. We see that a lax hermitian structure on an $X \in C$ is the same as a quadratic object structure of $(C, Q)$ on $X$ in the sense of [12], and a hermitian structure on $X$ is same as a Poincare object structure on $X$.

**Lemma 5.1.** Let $I \in \text{Cat}_\infty$ and $C \in \text{Cat}^{hC_2}_\infty$. Then there is a natural functor

$$\mathcal{H}(\text{Fun}(G^{lax}(I), C)) \to \text{Fun}(I, \mathcal{H}^{lax}(C))$$

inducing an equivalence on core groupoids.

**Proof.** Functorially in $J \in \text{Cat}_\infty$ we have a chain of maps

$$\text{map}(J, \mathcal{H}(\text{Fun}(G^{lax}(I), C))) \simeq \text{map}(G(J), \text{Fun}(G^{lax}(I), C))$$

$$\simeq \text{map}(G(J) \times G^{lax}(I), C) \to \text{map}(G^{lax}(J \times I), C) \simeq \text{map}(J \times I, \mathcal{H}^{lax}(C)) \simeq \text{map}(J, \text{Fun}(I, \mathcal{H}^{lax}(C)))$$

in $\text{Spc}$ defining the functor in question. Taking core groupoids the functor reduces to the equivalence

$$\text{map}(G^{lax}(I), C) \simeq \text{map}(I, \mathcal{H}^{lax}(C)).$$

$\square$

**Remark 5.2.** In general the functor in Lemma 5.1 is not an equivalence, since in general the map $G^{lax}(J \times I) \to G(J) \times G^{lax}(I)$ is not an equivalence. We always have an equivalence

$$\mathcal{H}(\text{Fun}(G(I), C)) \simeq \text{Fun}(I, \mathcal{H}(C))$$

of $\infty$-categories.

Let $C \in \text{Cat}^{hC_2}_\infty$ and $\alpha \in \mathcal{H}(\text{Fun}([1], C))$ (so $\alpha$ can be identified with a lax hermitian object of $C$). Let $F_\alpha$ be fiber over $\alpha$ of the functor

$$\mathcal{H}(\text{Fun}([3], C)) \to \mathcal{H}(\text{Fun}([1], C))$$

induced by the duality preserving functor $[1] \to [3]$ which sends 0 to 1 and 1 to 2.

Because of Lemma 5.1 the core groupoid $F_\alpha$ can then be identified with $(\mathcal{H}^{lax}(C))_\alpha^- \simeq \text{Fun}([0], \mathcal{H}^{lax}(C))$. Since $\mathcal{H}^{lax}(C) \to C$ is a right fibration the latter category can be identified with $(C_{\mathcal{H}(C)}^-)_\alpha^-$. The duality preserving functor $[3] \to [2]$ given by $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2$ exhibits $\text{Fun}([2], C)$ as the full subcategory of $\text{Fun}([3], C)$ of those functors for
which the middle induced map is an equivalence. Therefore we also get a full embedding
\[ \mathcal{H}(\text{Fun}([2], C)) \to \mathcal{H}(\text{Fun}([3], C)). \]
The considerations above show that the core groupoid of the fiber of the functor
\[ \mathcal{H}(\text{Fun}([2], C)) \to \mathcal{H}(C) \]
induced by \([0] \to [2], 0 \mapsto 1\), over a hermitian object \(X\) is naturally equivalent to \((C/X)^\sim\) (here \(X\) also denotes the object underlying the hermitian object).

**Proposition 5.3.** Let \(C \in (\text{Cat}_\text{hC}_2)^{\text{st}}\) and \(X \in \mathcal{H}(C)\). Let \(F\) be the fiber over \(X\) of the functor
\[ \mathcal{H}(S_2(C)) \to \mathcal{H}(C) \]
induced by the inclusion \([0] \to \text{Ar}([2]), 0 \mapsto \text{id}_2\). Then the fiber of the natural map
\[ F^\sim \to (C/X)^\sim \]
over a map \(f: Y \to X\) is naturally equivalent to the subspace of the space of paths in \(\text{map}_C(Y, Y^\sim)\) from the zero map to the map
\[ Y \overset{f}{\to} X \simeq X^\sim \overset{f^\sim}{\to} Y^\sim \]
on those connected components which exhibit the resulting commutative square
\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Y^\sim
\end{array}
\]
as an exact square in \(C\).

**Proof.** This follows from the above considerations together with the next two Lemmas.

**Lemma 5.4.** Let \(D\) be the category associated to the 1-skeleton of \([1] \times [1]\) (so \(D\) is Joyal-equivalent to two \(\Lambda^2_1\)'s glued together along their endpoints) and \(i: D \to [1] \times [1]\) the natural map. Then for an \(\infty\)-category \(C\) the fiber of the functor
\[ i^*\text{Fun}([1] \times [1], C) \to \text{Fun}(D, C) \]
over an object \(\alpha: D \to C\) in the target is naturally equivalent to the space of paths in \(\text{map}(X, Y)\) (\(X\) being \(\alpha((0, 0))\) and \(Y\) being \(\alpha((1, 1))\)) from the composition of one composable pair of maps in \(D\) to the composition of the other composable pair.

**Proof.** This follows from the fact that there is a pushout square
\[
\begin{array}{ccc}
E & \longrightarrow & D \\
\downarrow & & \downarrow \\
[1] & \longrightarrow & [1] \times [1],
\end{array}
\]
where \(E\) is obtained by gluing two copies of \([1]\) together along their endpoints.
**Lemma 5.5.** Let $C$ be an $\infty$-category which has a zero object. Let $i : [2] \to [1] \times [1]$ be the map which sends $0$ to $(0, 0)$, $1$ to $(0, 1)$ and $2$ to $(1, 1)$. Let

$$\text{Fun}'([1] \times [1], C) \subset \text{Fun}([1] \times [1], C)$$

be the full subcategory on those squares such that the entry in spot $(1, 0)$ is a zero object. Then the fiber of the functor

$$\text{Fun}'([1] \times [1], C) \to \text{Fun}([2], C)$$

over an object $\alpha : [2] \to C$ is naturally equivalent to the space of paths in the mapping space $\text{map}(\alpha(0), \alpha(2))$ from the zero map to $\alpha(0 \to 2)$.

If $C$ has a duality and $\alpha \in \mathcal{H}(\text{Fun}([2], C))$, then $\text{map}(\alpha(0), \alpha(2))$ has a natural $C_2$-action, the map $\alpha(0 \to 2)$ naturally lies in $\text{map}(\alpha(0), \alpha(2))^{hC_2}$ and the fiber of the fiber

$$\mathcal{H}(\text{Fun}'([1] \times [1], C) \to \mathcal{H}(\text{Fun}([2], C)))$$

is naturally equivalent to the space of paths in $\text{map}(\alpha(0), \alpha(2))^{hC_2}$ from the zero map to $\alpha(0 \to 2)$.

**Proof.** The first part follows from Lemma 5.4, the second by taking hermitian objects. \qed

**Corollary 5.6.** Let $C \in (\text{Cat}^hC_2)^{st}$ and $X \in \mathcal{H}(C)$. Let $F$ be the fiber over $X$ of the functor

$$\mathcal{H}(S_2(C)) \to \mathcal{H}(C)$$

induced by the inclusion $[0] \to \text{Ar}([2])$, $0 \mapsto \text{id}_2$. Then giving a point in $F$ is the same as giving a Lagrangian of $X$ in the sense of [12 Example 7].

**Corollary 5.7.** Let $C \in (\text{Cat}^hC_2)^{st}$, $X \in \mathcal{H}(C)$ and $\varphi : X \to X^\vee$ the corresponding map in

$$\text{map}(X, X^\vee)^{hC_2} \simeq \Omega^\infty \text{map}^S(X, X^\vee)^{hC_2}.$$

Choose an inverse $-\varphi$ of $\varphi$ with respect to the infinite loop space structure on $\text{map}(X, X^\vee)^{hC_2}$. Then there is an object of $\mathcal{H}(S_2(C))$ whose underlying exact triangle in $C$ has the form

$$\begin{array}{c}
X \\
\Downarrow\text{diag} \\
0 \\
\Downarrow\varphi(-\varphi) \\
X^\vee
\end{array}$$

and where the hermitian structure on $X \oplus X$ is given by $\varphi \oplus (-\varphi)$.

**Proof.** This follows now from the proof of [12 Proposition 11]. \qed

**Proposition 5.8.** For $C \in (\text{Cat}^hC_2)^{st}$ the $E_\infty$-structure on $\text{GW}(C)$ defined in section 3 is grouplike.
Proof. There is a coequalizer diagram

\[ \pi_0(S^c\vDash(C)_h) \rightrightarrows \pi_0(S^c_0\vDash(C)_h) \rightarrow \pi_0(|S^c\vDash(C)_h|) \]

in \text{Set}. Let a point in \( \pi_0(S^c_0\vDash(C)_h) \) be represented by a hermitian object \( X \in H(C) \). Let \( \varphi \in \text{map}(X,X^\vee h\mathbb{C}) \) be the corresponding map with a choice of an inverse \( -\varphi \). Let \( W' \in \mathcal{H}(S_2(C)) \) be the object described in Corollary 5.7 and \( W \) be the image of \( W' \) under the functor \( \mathcal{H}(S_2(C)) \rightarrow \mathcal{H}(S_3(C)) \) induced by the map \( [3] \rightarrow [2] \) given by \( 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2 \). The object \( W \) determines an element of \( \pi_0(S^c_1\vDash(C)_h) \) which is sent under the two maps above to 0 resp. \( (X \oplus X, \varphi \oplus (-\varphi)) \). So we see that in \( \pi_0(|S^c\vDash(C)_h|) \) an inverse of the image of \( (X,\varphi) \) is given by \( (X,-\varphi) \), in particular the \( E_\infty \)-space \( |S^c\vDash(C)_h| \) is grouplike.

It follows that also \( GW(C) \) is grouplike.

Definition 5.9. For \( n \in \mathbb{N} \) the abelian group \( GW_n(C) \colon = \pi_n GW(C) \) is called the \( n \)-th Grothendieck-Witt group of the stable \( \infty \)-category \( C \) with duality. In particular the group \( GW_0(C) \) is called the Grothendieck-Witt group of \( C \).

6 Hyperbolic categories

We denote a right adjoint of the forgetful functor

\[ \text{Cat}^{h\mathbb{C}}_\infty \rightarrow \text{Cat}_\infty \]

by \( \text{Hyp} \) and call \( \text{Hyp}(C) \) the hyperbolic category associated to \( C \in \text{Cat}_\infty \). The underlying category of \( \text{Hyp}(C) \) is equivalent to \( C \times C^{\text{op}} \), and the duality is informally given by \( C \times C^{\text{op}} \ni (X,Y) \mapsto (Y,X) \).

Lemma 6.1. Let \( C \in \text{Cat}_\infty \). Then there is a natural equivalence

\[ \mathcal{H}^{\text{lax}}(\text{Hyp}(C)) \simeq \text{Tw}(C) \]

of \( \infty \)-categories.

Proof. The \( \infty \)-category \( \mathcal{H}^{\text{lax}}(\text{Hyp}(C)) \) is given as the complete Segal space

\[ [n] \mapsto \text{map}_{\text{Cat}^{h\mathbb{C}}_\infty}([n] *[n]^{\text{op}}, \text{Hyp}(C)), \]

which by adjunction is equivalent to

\[ [n] \mapsto \text{map}_{\text{Cat}_\infty}([n] *[n]^{\text{op}}, C). \]

But this is a possible definition of the twisted arrow category \( \text{Tw}(C) \).

Corollary 6.2. There is a natural equivalence

\[ \mathcal{H}(\text{Hyp}(C)) \simeq C^\sim \]

in \( \text{Spc} \) for \( C \in \text{Cat}_\infty \).

Proof. The core groupoid of the full subcategory of \( \text{Tw}(C) \) on the equivalences is naturally equivalent to \( C^\sim \).
Lemma 6.3. There is a natural equivalence
\[ \text{Fun}(I, \text{Hyp}(C)) \cong \text{Hyp}(\text{Fun}(I, C)) \]
in $\text{Cat}^{hC_2}_\infty$ for $I \in \text{Cat}^{hC_2}_\infty$ and $C \in \text{Cat}_\infty$.

Proof. By adjunction a map
\[ \text{Fun}(I, \text{Hyp}(C)) \rightarrow \text{Hyp}(\text{Fun}(I, C)) \]
is the same as a map
\[ \text{Fun}(I, \text{Hyp}(C)) \rightarrow \text{Fun}(I, C) \]
in $\text{Cat}_\infty$, and such a map is induced by the counit $\text{Hyp}(C) \rightarrow C$. One checks that the resulting map in $\text{Cat}^{hC_2}_\infty$ is an equivalence.

Corollary 6.4. There is a natural equivalence
\[ \mathcal{H}(\text{Fun}(I, \text{Hyp}(C))) \cong \text{Fun}(I, C) \]
in $\text{Cat}_\infty$ for $I \in \text{Cat}^{hC_2}_\infty$ and $C \in \text{Cat}_\infty$.

It follows

Lemma 6.5. For $C \in \text{Cat}^s_\infty$ the simplicial object $S^{s,\infty}_\ast(\text{Hyp}(C))_h$ in $\text{Spc}$ is naturally equivalent to the simplicial object $S^{s,\infty}_\ast(C)$.

Proposition 6.6. For $C \in \text{Cat}^s_\infty$ there is a natural equivalence $\text{GW}(\text{Hyp}(C)) \simeq K(C)$ of grouplike $E_\infty$-spaces.

Proof. Note first that for any simplicial space $X$ the natural map $E(X) \rightarrow X$ induces an equivalence $[E(X)] \rightarrow [X]$ (this follows from [15, Lemma 1]). Thus by Lemma 6.5 $\text{GW}(\text{Hyp}(C))$ is given as the fiber of the map
\[ |S^{s,\infty}_\ast(C)| \rightarrow |S^{s,\infty}_\ast(C)| \times |S^{s,\infty}_\ast(C^{\text{op}})| \simeq |S^{s,\infty}_\ast(C)| \times |S^{s,\infty}_\ast(C)| \]
which is naturally equivalent to the diagonal. The claim follows.

7 A connective real $K$-theory spectrum

Recall from [10, §6] the left adjoints
\[ \text{Cat} \xrightarrow{f} \text{Cat}^\Sigma \xrightarrow{l} \text{Cat}_{\text{preadd}}, \]
where $l$ is a localization. The functor $f$ sends a small $\infty$-category $C$ to the full subcategory of $\mathcal{P}(C)$ which contains the essential image of $C$ under the Yoneda embedding $C \rightarrow \mathcal{P}(C)$ and is closed under finite coproducts, see the proof of [14, Proposition 5.3.6.2].

For $C \in \text{Cat}^\Sigma_\infty$ we let $\mathcal{P}_\Sigma(C) := \text{Fun}(C^{\text{op}}, \text{Spc})$, see [14, §5.5.8]. For $C \in \text{Cat}_\infty$ we let
\[ \mathcal{P}^\oplus(C) := \text{Fun}(C^{\text{op}}, E_\infty(\text{Spc})) \simeq E_\infty(\mathcal{P}(C)) \]
(for the last equivalence see [13, Remark 2.1.3.4]), and for $C \in \text{Cat}_\Sigma^\infty$ we set

$$\mathcal{P}_\Sigma^\oplus(C) := \text{Fun}_\Pi(C^{\text{op}}, E_\infty(\text{Spc})) \simeq E_\infty(\mathcal{P}_\Sigma(C)).$$

If $C \in \text{Cat}_\infty^{\text{preadd}}$ then $\mathcal{P}_\Sigma^\oplus(C) \simeq \mathcal{P}_\Sigma(C)$, see [8, Corollary 2.5 (iii)].

For $C \in \text{Cat}_\infty^\Sigma$ we have a canonical left adjoint $\mathcal{P}(C) \to \mathcal{P}^\oplus(C)$.

For $C \in \text{Cat}_\Sigma^\infty$ we have a natural square of left adjoints

$$\begin{array}{ccc}
\mathcal{P}(C) & \to & \mathcal{P}_\Sigma(C) \\
\downarrow & & \downarrow \\
\mathcal{P}^\oplus(C) & \to & \mathcal{P}_\Sigma^\oplus(C)
\end{array}$$

which commutes up to a natural equivalence since the corresponding right adjoints do. In particular we exhibit a natural functor $C \to \mathcal{P}_\Sigma^\oplus(C)$.

For $C \in \text{Cat}_\infty^\Sigma$ let $F^\oplus(C)$ be the smallest full subcategory of $\mathcal{P}^\oplus(C)$ that contains the essential image of $C \to \mathcal{P}^\oplus(C)$ and is closed under finite coproducts. Note $F^\oplus(C)$ is preadditive. Similarly for $C \in \text{Cat}_\Sigma^\infty$ let $F^\oplus_\Sigma(C)$ be the essential image of $C \to \mathcal{P}^\oplus_\Sigma(C)$. $F^\oplus_\Sigma(C)$ is also preadditive.

**Proposition 7.1.** i) There is a natural equivalence of functors $l \circ f \simeq F^\oplus$, so for $C \in \text{Cat}_\infty$ the category $F^\oplus(C)$ is the free preadditive category on the $\infty$-category $C$.

ii) There is a natural equivalence of functors $l \simeq F^\oplus_\Sigma$, so for $C \in \text{Cat}_\Sigma^\infty$ the category $F^\oplus_\Sigma(C)$ is the free preadditive category on the $\infty$-category $C$ with finite coproducts.

**Proof.** The first point is [7, Proposition 2.8], the second point follows similarly. $lacksquare$

For a small $\infty$-category $C$ which has pullbacks we denote by $\text{Span}(C)$ the $\infty$-category of spans in $C$. That is the $\infty$-category denoted $A^{\text{eff}}(C)$ in [1, Definition 3.6].

The assignment $C \mapsto \text{Span}(C)$ can be viewed as a functor

$$\text{Span} : \text{Cat}_{\infty}^{\text{lex}} \to \text{Cat}_\infty,$$

where $\text{Cat}_{\infty}^{\text{lex}}$ is the subcategory of $\text{Cat}_\infty$ of $\infty$-categories with all finite limits and left exact functors between them (see loc. cit.).

We denote by $\text{Fin}$ the $\infty$-category of finite sets.

**Proposition 7.2.** The $\infty$-category $\text{Span}(\text{Fin})$ is preadditive, and the natural functor $F^\oplus(\ast) \to \text{Span}(\text{Fin})$ sending the point to the one element set is an equivalence.

**Proof.** This follows by comparing mapping spaces. $lacksquare$

We recall the

**Theorem 7.3.** Let $G$ be a finite group. Then the symmetric monoidal $\infty$-category of genuine $G$-spectra is equivalent to the full subcategory

$$\text{Fun}_U(\text{Span}(\text{Fin}[G]), \text{Sp}) \subset \text{Fun}(\text{Span}(\text{Fin}[G]), \text{Sp})$$

of finite coproduct preserving functors equipped with the symmetric monoidal structure which is induced by the Day convolution product on the functor category.
Proof. The equivalence is \cite{2}, Example B.6, the symmetric monoidal structure is (partially) discussed in \cite{2} §3.

Lemma 7.4. Let \( C \) be a cocomplete \( \infty \)-category. Then the functor \( F: C \to \text{Fun}(\text{Spc}, C) \) which sends \( X \in C \) to the functor \( \text{Spc} \ni K \mapsto K \otimes C \) is left adjoint to the functor \( \text{Fun}(\text{Spc}, C) \ni \varphi \mapsto \varphi(*) \).

Proof. This follows from the fact that for any \( X \in C \) the functor \( F(X) \) is the left Kan extension of the functor \( * \to C \) which sends the unique object of \( * \) to \( X \) along the inclusion \( * \to \text{Spc} \) which sends the object of \( * \) to \( * \).

Let \( C \in \text{Cat}_{\text{lex}}^{\text{op}} \) and \( \varphi: \text{Spc} \to \text{Cat}_{\text{op}}^{\text{op}} \) the functor \( K \mapsto \text{Fun}(\text{Spc}(K), C) \) (note that \( K \mapsto \text{Fun}(K, C) \) is a functor \( \text{Spc} \to (\text{Cat}_{\text{lex}}^{\text{op}})^{\text{op}} \)). Then \( \text{Span}(C) \cong \varphi(*) \), and this equivalence is by Lemma \( \ref{K} \) adjoint to a map \( F \to \varphi \) in \( \text{Fun}(\text{Spc}, \text{Cat}_{\text{op}}^{\text{op}}) \), where \( F \) is defined by \( \text{Spc} \ni K \mapsto \text{Fun}(K, \text{Span}(C)) \).

In particular this exhibits for any \( K \in \text{Spc} \) a natural functor

\[
\mathfrak{f}^K_C \circ \text{Span}(\text{Fun}(K, C)) \to \text{Fun}(K, \text{Span}(C)).
\]

For a finite group \( G \) we set \( g_G := \mathfrak{f}^{BG}_C \), so we have

\[
g_G: \text{Span}(\text{Fin}(G)) \to \text{Span}(\text{Fin}[G]).
\]

Lemma 7.5. The \( \infty \)-category \( (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}} \) is preadditive.

So by Proposition \( \ref{C_2} \) for \( C \in (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}} \) we exhibit a natural functor

\[
\text{Span}(\text{Fin}) \to (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}}
\]
sending the generator to \( C \), which induces the second map in the composition

\[
\text{Span}(\text{Fin}[C_2]) \xrightarrow{\varphi_C} \text{Span}(\text{Fin})[C_2] \to (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}}[C_2] \to (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}} \xrightarrow{\text{GW}} \text{Sp},
\]

whereas the third map is the map \( t_{\text{Cat}_{\infty}^{\text{ht}}} \) described in \cite{10} §9 (and \( \text{GW} \) takes values in grouplike \( E_{\infty} \)-spaces aka connective spectra).

By Theorem \( \ref{K} \) we obtain a genuine \( C_2 \)-spectrum which we denote by \( \text{KR}(C) \) and call the connective real \( K \)-theory spectrum of \( C \).

Proposition 7.6. For \( C \in (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}} \) the underlying spectrum of \( \text{KR}(C) \) is naturally equivalent to the \( K \)-theory spectrum \( K(C) \) (which thus inherits a natural \( C_2 \)-action), and the \( C_2 \)-fixed points of \( \text{KR}(C) \) are canonically equivalent to \( \text{GW}(C) \).

Proof. In the defining composition of \( \text{KR}(C) \) the \( C \)-set \( * \) is sent to the \( C_2 \)-fixed points. But the image of \( * \) in \( (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}}[C_2] \) is \( C \) with the trivial \( C_2 \)-action yielding the second claim.

The underlying spectrum of \( \text{KR}(C) \) is the image of the \( C_2 \)-orbit \( C_2 \). Its image in \( (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}}[C_2] \) is \( C \times C \) with the \( C_2 \)-action which switches the two factors. The resulting object \( t_{\text{Cat}_{\infty}^{\text{ht}}}(C \times C) \) of \( (\text{Cat}_{\infty}^{\text{ht}})^{\text{st}} \) is then seen to be naturally equivalent to \( \text{Hyp}(C) \), so the first claim follows from Proposition \( \ref{1} \).
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