Abstract—This paper considers the consensusability of multi-agent systems with delay and packet dropout. By proposing a kind of predictor-like protocol, sufficient and necessary conditions are given for the mean-square consensusability in terms of system matrices, time delay, communication graph and the packet drop probability. Moreover, sufficient and necessary conditions are also obtained for the formationability of multi-agent systems.

Index Terms—Consensusability, Delay, Packet Dropout, Predictor-like protocol, Formationable, Multi-agent system.

I. INTRODUCTION

Multi-agent systems have attracted much attention in various scientific communities due to their broad applications in many areas including distributed computation [1], formation control [2], distributed sensor networks [3]. Consensus is the most fundamental control problem in multi-agent systems. Due to the fact that each individual agent lacks global knowledge of the whole system and can only interact with its neighbors, one key issue of consensus is to study conditions under which the consensus can be achieved under a given protocol and other is the design of a consensus protocol. Numerous results have been reported in the literature for the design of distributed consensus protocols for multi-agent systems. See [6], [8] and references therein. For the consensusability problem, [4] and [5] gave a necessary and sufficient condition for the continuous-time and discrete-time multi-agent systems in the deterministic case respectively. [24] studied the case with multiplicative noise and time delay.

Time delays are unavoidable in information acquisition and transmission of practical multi-agent systems and should be taken into account in designing the consensus protocol. An initial study is given in [8] which provides a necessary and sufficient condition on the upper bound of time delays under the assumption that all the delays are equal and time-invariant. Sufficient conditions have been given in [9] for average consensus with constant, time varying and nonuniform time delays. [10] studied the output consensus for multi-agent systems with different types of time delays including communication delay, identical self-delay and different self-delay. [11] considered discrete-time multi-agent systems with dynamically changing topologies and time-varying communication delays.

On the other hand, random link failures or transmission noises exist widely in networked multi-agent systems, which motivates the study of stochastic consensus problem. In the literature, [12] provided two kinds of average consensus protocols which are biased compensation method and balanced compensation method in the presence of random link failures. It was shown in [13] that the consensus value will diverge when the traditional consensus algorithms are applied in the presence of noises. Under a fixed topology, necessary and sufficient conditions were given in [14] for mean square average consensus. [15] derived a sufficient condition for the switching topologies case. For the multiplicative-noise case, [16] revealed that multiplicative noises may enhance the almost sure consensus, but may have damaging effect on the mean square consensus. [17] studied the mean square consensus for linear discrete-time systems by solving a modified algebraic Riccati equation. [18] considered the stochastic consensus conditions. [24] gave the stochastic consentability analysis of linear multi-agent systems with time delays and multiplicative noises. Though plenty works have been done for multi-agent systems with either time delay or multiplicative noise, there is little progress for discrete-time multi-agent systems with both input delay and packet dropout. The consensus problem for the latter remains challenging. Note that the optimal control problem for the single agent system case was only solved recently by [27].

In this paper, we will study the consensusability problem of multi-agent systems with delay and packet dropout. Different from the consensus protocols in the literature where the protocol is mostly in the feedback form of the current state or the delayed state and there exists a maximum delay within which consensus can be achieved, a new kind of predictor-like consensus protocol is proposed in this paper to deal with the delay. Sufficient and necessary conditions are given for the mean-square consensusability in terms of system matrix, time delay, communication graph and the packet dropout probability under the predictor-like protocol. It will be shown that the derived results can be reduced to the deterministic case obtained in the literature. Moreover, sufficient and necessary conditions are obtained for the formationability of multi-agent systems.

The remainder of the paper is organized as follows. Section II presents some preliminary knowledge about algebraic graph theory. Problem formulation is given in Section III. Section IV
shows preliminaries on modified Riccati equation. Main results are stated in Section V. Some concluding remarks are given in the last section. Related theorems and proofs are given in Appendix.

The following notation will be used throughout this paper: \( R^n \) denotes the family of \( n \)-dimensional vectors; \( x' \) denotes the transpose of \( x \); a symmetric matrix \( M > 0 \) means that \( M \) is strictly positive-definite (positive semi-definite). \( \dot{x}(k|t) \triangleq E[x(k)|\mathcal{F}_{t-1}] \) denotes the conditional expectation with respect to the filtration \( \mathcal{F}_{t-1} \). \( \lambda_i(A) \) means the \( i \)th eigenvalue of matrix \( A \).

II. ALGEBRAIC GRAPH THEORY

In this paper, the information exchange among agents is modeled by an undirected graph. Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) be a diagraph with the set of vertices \( \mathcal{E} = \{1, \ldots, N\} \), the set of edges \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \), and the weighted adjacency matrix \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N} \) is symmetric. In \( \mathcal{G} \), the \( i \)-th vertex represents the \( i \)-th agent. Let \( a_{ij} > 0 \) if and only if \((i, j) \in \mathcal{E}\), i.e., there is a communication link between agents \( i \) and \( j \). Undirected graph \( \mathcal{G} \) is connected if any two distinct agents of \( \mathcal{G} \) can be connected via a path that follows the edges of \( \mathcal{G} \). For agent \( i \), the degree is defined as \( d_i \triangleq \sum_{j=1}^{N} a_{ij} \).

Diagonal matrix \( \mathcal{D} \triangleq \text{diag}\{d_1, \ldots, d_N\} \) is used to denote the degree matrix of diagraph \( \mathcal{G} \). Denote the Laplacian matrix by \( \mathcal{L}_G = \mathcal{D} - \mathcal{A} \). The eigenvalues of \( \mathcal{L}_G \) are denoted by \( \lambda_i(\mathcal{L}_G) \in \mathbb{R}, i = 1, \ldots, N \), and an ascending order in magnitude is written as \( 0 = \lambda_1(\mathcal{L}_G) \leq \cdots \leq \lambda_N(\mathcal{L}_G) \), that is, the Laplacian matrix \( \mathcal{L}_G \) of an undirected graph has at least one zero eigenvalue and all the nonzero eigenvalues are in the open right half plane. Furthermore, \( \mathcal{L}_G \) has exactly one zero eigenvalue if and only if \( \mathcal{G} \) is connected [21].

III. PROBLEM FORMULATION

Consider a multi-agent system as depicted in Fig. 1 where the dynamic is given by

\[
x_i(k+1) = Ax_i(k) + \gamma(k)Bu_i(k-d),
\]

while \( x_i \in \mathbb{R}^n \) is the state of the \( i \)th agent, \( u_i \in \mathbb{R}^m \) is the control input of the \( i \)th agent, \( A, B \) are constant matrices with appropriate dimensions. \( d \) represents the input delay. \( \gamma(k) = 1 \) denotes that the data packet has been successfully delivered to the plant, and \( \gamma(k) = 0 \) signifies the dropout of the data packet. Without loss of generality, the random process \( \{\gamma(k), k \geq 0\} \) is modeled as an independent and identically distributed (i.i.d.) Bernoulli process with probability distribution \( P(\gamma(k) = 0) = p \) and \( P(\gamma(k) = 1) = 1 - p \), where \( p \in (0, 1) \) is said to be the packet dropout rate. The initial values are given by \( x_i(0), u_i(-1), \cdots, u_i(-d) \). Note that the channel fading and time delay occur simultaneously due to the unreliable network placed in the path from the controller to the agent \( i \). Moreover, the information exchange between the controllers of agent \( i \) and \( j \) happens in the controller processor.

Remark 1: Noting that the random process \( \gamma \) is identical for which we will derive some necessary and sufficient conditions for consensus usability of multi-agent systems with both delay and packet dropout. The derived results will provide insights into the interplay among system dynamic, delay and network topology and demonstrate the advantage of the predictor-like consensus protocol. They could also shed some light on resolving the non-identical \( \gamma \) case which is interesting and is left for our future study.

We further make the following general assumption.

Assumption 1: All the eigenvalues of \( A \) are either on or outside the unit circle, \( B \) has full column rank.

Assumption 2: System \((A, B, 0, A^dB)\) is mean-square stabilizable, that is, for the system

\[
x(k+1) = Ax(k) + Bu(k) + \nu(k)A^dBu(k)
\]

where \( \nu(k) \) is a sequence of white noise with zero mean and unit covariance, there exists a feedback controller \( u(k) = Kx(k) \) where \( K \) is a time-invariant matrix such that the closed-loop system is mean-square stable, i.e. \( \lim_{k \to \infty} E\|x(k)\|^2 = 0 \).

Assumption 3: The undirected graph is connected.

Denote \( w(k) = \gamma(k) - E[\gamma(k)] \), then system [1] is reformulated as

\[
x_i(k+1) = Ax_i(k) + (1 - p)Bu_i(k-d) + w(k)Bu_i(k-d), \quad i = 1, \cdots, N,
\]

where \( \{w(k), k \in \mathbb{N}\} \) is a sequence of random variables defined on \((\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_k)\) with \( E[w(k)] = 0 \) and \( E[w(k)w(s)] = p(1-p)\delta_{k,s} \). We simply denote \( \mu = 1-p \) and \( \sigma^2 = p(1-p) \).

In the literature [2], [8], the relative state \( x_j(k) - x_i(k) \) between agents is used to design the consensus protocol like \( u_i(k) = K \sum_{j \in N_i} [x_j(k) - x_i(k)] \). Differently in this paper, we firstly calculate the following predictor using each agent’s own state and historical inputs for \( k \geq d \) in the way that

\[
\dot{x}_i(k|k-d) = E[x_i(k)|\mathcal{F}_{k-d-1}]
\]

\[
= A^d x_i(k-d) + \mu \sum_{j=1}^{d} A^{d-j}Bu_i(k-d-j).
\]

Fig. 1. Multi-agent system with unreliable networks
Then the relative predictor $\hat{x}_j(k|k-1) - \hat{x}_i(k|k-1)$ is applied
to design the consensus protocol. To be specific, the distributed
protocol for $k \geq d$ is described as

$$u_i(k - d) = K \sum_{j \in N_i} [\hat{x}_j(k|k-d) - \hat{x}_i(k|k-d)].$$ (4)

The aim is to find sufficient and necessary conditions for the
mean-square consensusability of multi-agent system (2) under protocol (4) where the definition on the mean square
consensusability is given below.

**Definition 1:** The discrete-time multi-agent system (2) with
a fixed undirected graph is said to be mean-square consen-
susable under protocol (4) if for any finite initial values $x_i(0), u_i(-d), \ldots, u_i(-1)$, there exists a control gain $K$ such that the controller (4) enforces consensus, i.e.,
$$\lim_{k \to \infty} E\|x_j(k) - x_i(k)\|^2 = 0, \forall i, j = 1, \ldots, N.$$grey

By substituting (4) into (2), the closed-loop multi-agent system becomes
$$x_i(k + 1) = Ax_i(k) + \mu BK \sum_{j \in N_i} [\hat{x}_j(k|k-d) - \hat{x}_i(k|k-d)] + BK \sum_{j \in N_i} [\hat{x}_j(k|k-d) - \hat{x}_i(k|k-d)]u(k), k \geq d.$$ (5)

Let $X(k) = [x_1(k) \cdots x_N(k)]'$, $\hat{X}(k|k-d) = [\hat{x}_1(k|k-d) \cdots \hat{x}_N(k|k-d)]'$, then (5) can be reformulated as
$$X(k + 1) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(K_G \otimes BK)\hat{X}(k|k-d), k \geq d.$$ (6)

Denote $\bar{X}(k) = \frac{1}{N} \sum_{i=1}^{N} X_i(k)$, then
$$\bar{X}(k + 1) = \frac{1}{N} (1_N \otimes I) X(k + 1) = A\bar{X}(k) - \mu(N) (1_N L_G \otimes BK) \bar{X}(k|k-d)$$
$$-w_k(N) (1_N L_G \otimes BK) \hat{X}(k|k-d) = A\bar{X}(k),$$ (7)

where $1_N L_G = 0$ has been used in the derivation of the last
equality. Given the initial condition $\bar{X}(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$ and equation (7), it yields that $\bar{X}(k)$ is deterministic. This
further implies that $\bar{X}(k|s) = E[\bar{X}(k)|F_{s-1}] = \bar{X}(k)$ for any
positive integer $s$. We now present the dynamic equation of $\bar{X}(k + 1) = \bar{X}(k + 1) - (1_N \otimes I) \bar{X}(k + 1)$ with $\bar{X}(k|k-d) = \bar{X}(k|k-d) - (1_N \otimes I) \bar{X}(k|k-d) - (1_N \otimes I) \bar{X}(k).$ It is obtained by subtracting (7) from (6) that
$$\delta(k + 1) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(K_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$
$$\times (1_N \otimes I) \bar{X}(k) = (I_N \otimes A)X(k) - \mu L_G BK \hat{X}(k|k-d) - w_k(L_G \otimes BK)\hat{X}(k|k-d) - (I_N \otimes A)$$

IV. PRELIMINARIES ON MODIFIED RICCATI EQUATION

Based on Theorem 1 the simultaneous stabilizability of the systems in (2) is necessary for consensusability. To this end, we shall present some results with respect to the stabilizability criterion and further investigate a corresponding modified algebraic Riccati equation. Firstly, the following equivalent conditions have been given in [26].

**Lemma 1:** The following statements are equivalent.

1) System
$$x(k + 1) = Ax(k) + \mu Bu(k - d) + w_k Bu(k - d)$$ (10)
is mean-square stable under the controller $u(k - d) = K \hat{x}(k|k-d)$.

2) System
$$x(k + 1) = Ax(k) + \mu Bu(k) + w_k A^d Bu(k)$$ (11)
is mean-square stabilizable under the controller $u(k) = K \hat{x}(k)$.

3) For any $Q > 0$, there exist matrices $K$ and $P > 0$ satisfying the following equation:
$$P = Q + (A + \mu BK)P(A + \mu BK)$$
$$+ \sigma^2 K^B(A^d)^d P A^d BK.$$ (12)
4) There exist matrices $K$ and $P > 0$ satisfying the following equation:

$$P \succ (A + \mu BK)'P(A + \mu BK) + \sigma^2 K'B'(A')^dPA'BK.$$

In particular, it has also been shown in [26] that the existence of a unique positive definite solution to the algebraic Riccati equation

$$P = A'PA + Q - \gamma A'PB\left[R + \mu^2 B'PB\right]$$

$$+ \sigma^2 B'(A')^dPA'B]^{-1} B'PA$$

is necessary and sufficient for the mean-square stabilizability of system [10] with $Q > 0$. Motivated by the results in [26], we define the parameterized algebraic Riccati equation (PARE)

$$P = A'PA + Q - \gamma A'PB\left[R + B'PB\right]$$

$$+ \sigma^2 B'(A')^dPA'B\left[R + B'PB\right]^{-1} B'PA$$

and denote

$$g_{\gamma}(P) = A'PA + Q - \gamma A'PB\left[R + B'PB\right]$$

$$+ \sigma^2 B'(A')^dPA'B\left[R + B'PB\right]^{-1} B'PA, (15)$$

$$\Phi(K, P) = (1 - \gamma)A'PA + Q + \gamma(F_1'PF_1 + F_2'PF_2 + K'RK + Q), (16)$$

$$\Psi(K, P) = F_1'PF_1 + F_2'PF_2 + K'RK + Q, (17)$$

where $F_1 = A + BK, F_2 = A'BK$.

**Theorem 2:** Consider the PARE [14]. Let $A$ be unstable, $(A, B, 0, A^dB)$ is mean-square stabilizable and $Q > 0, R > 0$. Then the following hold.

1) The PARE has a unique strictly positive definite solution if and only if $\gamma > \gamma_c$, where $\gamma_c$ is the critical value defined as

$$\gamma_c = \inf\{\gamma \in [0, 1] | P = g_{\gamma}(P), P > 0\}.$$

2) The critical value $\gamma_c$ satisfies the following analytical bounds:

$$\underline{\gamma} \leq \gamma_c \leq \overline{\gamma}$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are defined by

$$\underline{\gamma} = \arg\min\{\exists S | (1 - \gamma)A'SA + Q = S, S \geq 0\}$$

$$\overline{\gamma} = \arg\min\{\exists(K, P) | P > \Phi(K, P)\}$$

3) The critical value can be numerically computed by the solution of the following quasiconvex LMI optimization problem

$$\gamma_c = \arg\min_{\gamma} \Delta_{\gamma}(Y, Z) > 0, 0 \leq Y \leq I$$

$$\Delta_{\gamma}(Y, Z) =$$

$$\begin{bmatrix}
Y & Y & \sqrt{\gamma}ZR^\frac{1}{2} \\
Y & Q^{-1} & 0 \\
\sqrt{\gamma}R^\frac{1}{2}Z' & 0 & I \\
\sqrt{\gamma}(AY + BZ') & 0 & Y \\
\sqrt{\gamma}A'BZ' & 0 & 0 \\
\sqrt{1 - \gamma}AY & 0 & 0 \\
0 & 0 & 0 \\
0 & Y & 0 \\
0 & 0 & Y
\end{bmatrix}$$

Proof. Based on Theorem [6][7][8] and [9] in Appendix, the results follow by using similar proof to that of Lemma 5.4 in [22].

**V. MEAN-SQUARE CONSENSUSABILITY**

Denote for $i = 2, \cdots, N$,

$$\gamma_i = \frac{\mu^2}{\mu^2 + \sigma^2} \frac{4(\lambda_i(L_G)[\lambda_2(L_G) + \lambda_N(L_G)] - \lambda_i^2(L_G))}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}.$$

It is noted that

$$\gamma_2 = \frac{\mu^2}{\mu^2 + \sigma^2} \frac{4\lambda_2(L_G)\lambda_N(L_G)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}$$

$$= \frac{\mu^2}{\mu^2 + \sigma^2} \left[1 - \left(\frac{\lambda_N(L_G) - \lambda_2(L_G)}{\lambda_N(L_G) + \lambda_2(L_G)}\right)^2\right].$$

We now present the main result of the mean-square consensusability for multi-agent system [3].

**Theorem 3:** Let Assumption [1][3] hold. If $\gamma_2 > \gamma_c$ where $\gamma_c$ is given in Theorem 2 then the multi-agent system is mean-square consensusable under protocol [4].

**Proof.** Consider the Riccati equation

$$P = A'PA + Q - \gamma_i A'PB\left[R + B'PB\right]$$

$$+ \sigma^2 B'(A')^dPA'B\left[R + B'PB\right]^{-1} B'PA. (18)$$

Since

$$\frac{4(\lambda_i(L_G)[\lambda_2(L_G) + \lambda_N(L_G)] - \lambda_i^2(L_G))}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}$$

$$- \frac{4\lambda_2(L_G)\lambda_N(L_G)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}$$

$$= \frac{4(\lambda_2(L_G) - \lambda_i(L_G))(\lambda_i(L_G) - \lambda_N(L_G))}{[\lambda_N(L_G) + \lambda_2(L_G)]^2} \geq 0,$$

then it follows that $\gamma_i \geq \gamma_2 > \gamma_c$ for $i > 2$. Using Theorem 2 the Riccati equation (15) admits a solution $P > 0$. Since $K$ has a full column rank, then $B'PB + B'(A')^dPA'B > 0.$
Using the fact that $M^{-1} < N^{-1}$ when $M > N > 0$ and $R > 0$, $Q > 0$, we have
\[
P > A'P - \gamma_i A'P B B'PA - \gamma_i B' A' d^2 P A d^2 B B'PA.
\]
From $p \in (0, 1)$, one has $\mu > 0$ and $\sigma^2 > 0$ which yields that $\mu^2 B' P B > 0$. Thus (19) further implies that
\[
P > A'P - \gamma_i (L_G) A'P B B'PA + \mu^2 B'PA
\]
\[+ \sigma^2 B'(A') d^2 P A d^2 B B'PA,
\]
where $\gamma_i = \frac{\mu^2}{\gamma^2 (L_G)} \left[\lambda_n (L_G) \bar{\gamma}_i - \lambda_n (L_G) - \lambda_i (L_G)\right]$. By letting the feedback gain matrix
\[
K = \frac{2 \mu}{\lambda_2 (L_G) + \lambda_n (L_G)} \left[\gamma_i L_G (B') (A') d^2 P A d^2 B B'PA.
\]
the Riccati equation (20) is equivalently rewritten as
\[
P > [A - \lambda_i (L_G) \mu B K]^T P [A - \lambda_i (L_G) \mu B K]
\]
\[+ \gamma_i L_G (B') (A') d^2 P A d^2 B B'PA.
\]
Combining with Lemma 1, system 9 is mean-square stabilizable. This yields that the multi-agent system 2 is mean-square consensusable. The proof is now completed.

Remark 2: Noting that $\mu = 1 - p$ and $\sigma^2 = p(1 - p)$, the condition $\gamma_2 > \gamma_c$ in Theorem 3 becomes $(1 - p) \left[1 - \left(\frac{\lambda_n (L_G) - \lambda_2 (L_G)}{\lambda_n (L_G) + \lambda_2 (L_G)}\right)^2\right] > \gamma_c$.

Remark 3: When delay time is reduced to 0, the sufficient condition
\[
\frac{\mu^2}{\mu^2 + \sigma^2} \left[1 - \left(\frac{\lambda_n (L_G) - \lambda_2 (L_G)}{\lambda_n (L_G) + \lambda_2 (L_G)}\right)^2\right] > \gamma_c
\]
is consistent with the result obtained in [2] for the consensusability of discrete-time linear multi-agent systems over analog fading networks where $\mu$ and $\sigma^2$ are corresponding to the expectation and the covariance of identical channel fading.

We next give a necessary condition for the mean-square consensusability of multi-agent system 2.

Theorem 4: Under Assumption 1, 3, and $\text{Rank}(B) = 1$, the multi-agent system 2 is mean-square consensusable under protocol 4 if only if
\[
\Pi_i |\lambda_i^y (A)|^2 < \left[1 + \frac{\lambda_2 (L_G)}{\gamma_i L_G (B')} (\lambda_2 (L_G) - \lambda_i (L_G))\right] \left[1 - \lambda_2 (L_G) / \gamma_i L_G (B')\right],
\]
where $\lambda_i^y (A)$ denotes the unstable eigenvalue of matrix $A$.

Proof. Using Theorem 1, systems 9 are mean-square stabilizable simultaneously for all $i = 2, \ldots, N$. By applying Lemma 1 the following systems
\[
\delta_i (k + 1) = A \delta_i (k) - \lambda_i \mu B K \delta_i (k) - \tilde{w}_i \lambda_i A' d^2 B K \delta_i (k)
\]
are mean-square stable for all $i = 2, \ldots, N$. Combining with the fact that $
\lim_{k \to \infty} E[\delta_i (k)] = 0$ implies that $\lim_{k \to \infty} E[\delta_i (k)] = 0$, it yields that $A - \lambda_i \mu B K$ is Schur stable, i.e. all the eigenvalues of $A - \lambda_i \mu B K$ are within the unit disk. The result then follows from [5].

Remark 4: Consider the case of $\text{Rank}(B) = 1$. When the communication is delay free and packets can be perfectly delivered, that is, $d = 0$ and $p = 0$, $\gamma_c = 1 - \frac{1}{\Pi_i |\lambda_i (A)|^2}$ which has been obtained in [23]. From Theorem 3, $\gamma_2 > \gamma_c$ is reduced to (23). Together with Theorem 4 (23) is necessary and sufficient for the consensusability of multi-agent systems 2 under protocol 4. This is consistent with Theorem 3.1 in [5] for the deterministic linear multi-agent systems under
\[
u_i (k) = K \sum_{j \in N_i} [x_j (k) - x_i (k)].
\]
We then study the scalar multi-agent systems. It shall be shown that $\gamma_2 > \gamma_c$ in Theorem 5 is necessary and sufficient for the consensusability.

Theorem 5: Let $A = a \geq 1, B = b > 0$ be constants, the multi-agent system 2 is mean-square consensusable by the control protocol 4 if and only if
\[
\frac{\mu^2}{(\mu^2 + a^2 d^2 \sigma^2)} \left[1 - \frac{\lambda_n (L_G) - \lambda_2 (L_G)}{\lambda_2 (L_G) + \lambda_n (L_G)}\right] > 1 - \frac{1}{a^2}
\]
\[\mu^2 > 2 \lambda_1 (L_G) a b k^2 - 2 \lambda_1 (L_G) a b k^2 + a^2 d^2 \lambda_1 (L_G) b^2 k^2 \leq 1,
\]
[26]
\[\mu > \frac{\sqrt{(a^2 + a^2 d^2 \sigma^2) (a^2 - 1)}}{\lambda_2 (L_G) (\mu^2 + a^2 d^2 \sigma^2)}\]

Thus, we obtain that $\bigcup_{i=2}^N \left\{ \frac{\mu - \sqrt{(a^2 + a^2 d^2 \sigma^2) (a^2 - 1)}}{\lambda_2 (L_G) (\mu^2 + a^2 d^2 \sigma^2)} \left| \frac{\mu - \sqrt{(a^2 + a^2 d^2 \sigma^2) (a^2 - 1)}}{\lambda_2 (L_G) (\mu^2 + a^2 d^2 \sigma^2)} \right| \neq 0 \right\}$. Using $\lambda_2 (L_G) < \lambda_1 (L_G) < \lambda_n (L_G)$, it is further derived that
\[
\frac{\mu}{\lambda_2 (L_G)} - \frac{\lambda_2 (L_G)}{\lambda_2 (L_G) + \lambda_n (L_G)} \left[1 - \frac{1}{a^2}\right]
\]

By applying some algebraic transformations, we have
\[
\left[\lambda_n (L_G) - \lambda_2 (L_G)\right]^2 \left[\lambda_2 (L_G) + \lambda_n (L_G)\right]^2 - 1 \mu^2 \leq - (\mu^2 + a^2 d^2) \left(1 - \frac{1}{a^2}\right).
\]
Thus, (24) follows.

“Sufficiency” From (24), it yields that
\[
\frac{\mu^2}{(\mu^2 + a^2 d^2 \sigma^2)} \left[1 - \frac{\lambda_n (L_G) - \lambda_2 (L_G)}{\lambda_2 (L_G) + \lambda_n (L_G)}\right] > 1 - \frac{1}{a^2}.
\]
Selecting the feedback gain in the form of \[ k = \frac{2\mu}{\mu^2 + \sigma^2} \] which gives that \( k = \frac{2\mu}{\mu^2 + \sigma^2} \). Then (25) follows. Thus, system (2) is mean-square consensuable. The proof is now completed.

Remark 5: For system (2) with delay and \( p = 0 \), the advantage of using the predictor-like protocol (4) is that the allowable delay for consensus can be arbitrarily large. However, when using the protocol without delay compensation, there exists a maximum delay margin within which consensus can be achieved (23). Take the case of \( \text{Rank}(B) = 1 \) for example, by combining Theorem 3, Theorem 4 with Lemma 5.4 in [22], the equivalent condition for consensus of system (2) is \( \Pi_1[|\lambda|^2(A)] < \left(1 + \frac{\lambda_2}{\lambda_N}\right)^2 \). This is exactly the necessary and sufficient condition to ensure the consensus for system (2) without delay obtained in [5]. This indicates that system (2) is consensuable for any large delay under the basic assumption. Furthermore, recalling Theorem 3 in [29], for scalar system with input delay, when \( 1 + \frac{\lambda_2}{\lambda_N} < A < 1 + \frac{\lambda_2}{\lambda_N} \) or \( -1 < A < 1 \), no delay is allowed for consensuability via relative feedback protocols. This illustrates the advantage of using predictor-like protocol (4) which can tolerate any large delay.

As an important application, the result on consensuability is extended to study formationability of the discrete-time multi-agent systems (2). In particular, given a formation vector \( H = [H_1, \ldots, H_N] \), the following control protocol is adopted to study the formation problem of the discrete-time multi-agent systems:

\[
u_i(k - d) = K \sum_{j \in N_i} \left( [\hat{x}_j(k - d) - H_j] - [\hat{x}_i(k - d) - H_i] \right),
\]

where \( H_i - H_j \) is the desired formation vector between agent \( i \) and agent \( j \). Noting that the common knowledge of the directions of reference axes is required for all the agents, the protocol \( \nu_i(k) = K \sum_{j \in N_i} \left( [\hat{x}_j(k) - H_j] - [\hat{x}_i(k) - H_i] \right) \) has been widely adopted in formation control [5] and references therein, we now apply the predictor-like protocol (4) to the formationable problem.

Definition 2: The discrete-time multi-agent system (2) is said to be formationable under protocol (27) if for any finite \( x_j(0), u_i(-d), \ldots, u_i(-1) \), there exists a control gain \( K \) in (27) such that \( \lim_{k \to \infty} E\|x_j(k) - H_j\| = 0, \forall i, j = 1, \ldots, N \).

Based on Theorem 3, sufficient and necessary conditions on formationability of the discrete-time multi-agent systems is stated as follows.

Corollary 1: Assume that Assumption 1 holds and \( A(H_i - H_j) = (H_i - H_j) \), \( \forall i, j = 1, \ldots, N \). If \( \gamma_2 > \gamma_c \) where \( \gamma_c \) is given in Theorem 3, then the following statements hold:

1) If \( \gamma_2 > \gamma_c \), the multi-agent system (2) is mean-square formationable under protocol (27).

2) Let \( \text{Rank}(B) = 1 \), the multi-agent system (2) is mean-square consensuable under protocol (27) if (27) holds.

3) Let \( A = a \geq 1, B = b > 0 \), the multi-agent system (2) is mean-square formationable under protocol (27) if

\[
\frac{\mu^2}{\mu^2 + \sigma^2} \left[ 1 - \frac{\lambda_2}{\lambda_N} \right] > 0.
\]

Proof. Denote \( \delta_i(k) = [x_i(k) - H_i] - [\hat{X}(k) - \hat{H}] \) where \( \hat{X}(k) = \frac{1}{N} \sum_{i=1}^{N} x_i(k) \), \( \hat{H} = \frac{1}{N} \sum_{i=1}^{N} H_i \). Then mean-square formationability is equivalent to that \( \lim_{k \to \infty} E\|\delta_i(k)\|^2 = 0 \). By stacking \( \delta_i \) into a column vector \( \delta(k) \), the following dynamical equation is in force:

\[
\delta(k + 1) = (I_N \otimes A)\delta(k) - \mu(L_G \otimes BK)\delta(k - d) - w_k(L_G \otimes BK)\hat{\delta}(k - d) + [I_N \otimes (A - I_N)] \begin{bmatrix} H_1 - \hat{H} \\ H_2 - \hat{H} \\ \vdots \\ H_N - \hat{H} \end{bmatrix}.
\]

Together with \( A(H_i - H_j) = (H_i - H_j) \), it follows that \( (A - I_N)(H_i - \hat{H}) = 0 \). The above equation is thus reformulated as

\[
\delta(k + 1) = (I_N \otimes A)\delta(k) - \mu(L_G \otimes BK)\delta(k - d) - w_k(L_G \otimes BK)\hat{\delta}(k - d) + [I_N \otimes (A - I_N)] \begin{bmatrix} H_1 - \hat{H} \\ H_2 - \hat{H} \\ \vdots \\ H_N - \hat{H} \end{bmatrix}.
\]

The remainder of the proof follows from Theorem 1, 3, 5 and 6. The proof is now completed.

VI. CONCLUSIONS

In this paper, we studied the consensuability of multi-agent systems with delay and packet dropout. By proposing a kind of predictor-like protocol, sufficient and necessary conditions have been given for the mean-square consensuability in terms of system matrices, time delay, communication graph and the packetdrop probability. It has been shown that the derived results are exactly the necessary and sufficient condition obtained in [5] for the delay and packet drop free. Moreover, sufficient and necessary conditions have been obtained for the formationability of multi-agent systems.

APPENDIX

The following results can be obtained by similar discussions as in [23]. We give some brief proofs for the completion of the work.

Lemma 2: Assume that \( P \in \{S \in R^{n \times n}, S \geq 0, R > 0\} \). Then the following statements hold.

1) With \( K_P = -[R + B'PB + B'(A')^dPA^dB]^{-1}B'PA \), \( g_r(P) = \Phi(K_P, P) \).

2) \( g_r(P) = \min_g \Phi(K, P) \leq \Phi(K, P) \).

3) If \( P_1 \leq P_2 \), then \( g_r(P_1) \leq g_r(P_2) \).

4) If \( \gamma_1 \leq \gamma_2 \), then \( g_{r_1}(P) \geq g_{r_2}(P) \).

5) If \( \alpha \in [0, 1] \), then \( g_r(\alpha P_1 + (1 - \alpha)P_2) \geq \alpha g_r(P_1) + (1 - \alpha)g_r(P_2) \).
6) \( g_\gamma(P) \geq (1 - \gamma)A'PA + Q \).
7) Provided that the equation \((1 - \gamma)A'XA + Q = X\) has a solution \(X > 0\). If \( \bar{P} \geq g_\gamma(\bar{P}) \), then \( \bar{P} > 0 \)

**Proof.**

1) Using the definition of \( K_\rho \), we have
   \[
   \Phi(K_\rho, P) = A'PA + Q - \gamma A'PB \left[ R + B'PB \right]^{-1} B'PA
   \]
   \[+ B'(A')^dPA^dB]^{-1} B'PA = g_\gamma(\bar{P}). \]
   2) By using the definitions of \( \Phi(K, P) \) and \( \Psi(K, P) \), it holds that \( \min_K \Phi(K, P) = \min_K \Psi(K, P) \). Combining with the fact that \( P \geq 0, R > 0 \), the minimum of \( K \) can be found by using \( \frac{\partial \Phi(K, P)}{\partial K} = 0 \), that is \( 0 = B'P(A + BK) + B'(A')^dRA^dBK + RK \). This implies that \( K = -\left[R + B'PB + B'(A')^dPA^dB\right]^{-1} B'PA \).
   Together with from fact 1), the result follows.
   3) If \( P_1 \leq P_2 \), we have by using the above two facts
   \[ g_\gamma(P_1) = \Phi(K_{P_1}, P_1) \leq \Phi(K_{P_2}, P_1) \leq \Phi(K_{P_2}, P_2) = g_\gamma(P_2). \]
   4) Noting that \( A'PB \left[ R + B'PB + B'(A')^dPA^dB\right]^{-1} B'PA \geq 0 \), the fact follows directly.
   5) Let \( Z = \alpha P_1 + (1 - \alpha)P_2 \), then
   \[ g_\gamma(Z) = (1 - \gamma)(A'ZA + Q) + \gamma \Psi(K_Z, Z) \]
   Further rewriting \( \Psi(K_Z, Z) \) yields that
   \[ \Psi(K_Z, Z) = \alpha \Psi(K_Z, P_1) + (1 - \alpha) \Psi(K_Z, P_2) \geq \alpha \Psi(K_{P_1}, P_1) + (1 - \alpha) \Psi(K_{P_2}, P_2). \]
   Thus
   \[ g_\gamma(Z) \geq (1 - \gamma)(A'ZA + Q) + \gamma \Psi(K_{P_1}, P_1) + \gamma(1 - \alpha) \Psi(K_{P_2}, P_2) = \alpha g_\gamma(P_1) + (1 - \alpha) g_\gamma(P_2). \]
   6) By using the facts that \( F_1'PF_1 \geq 0, F_2'PF_2 \geq 0, K'RK \geq 0 \), the result is straightforward.
   7) Using the above fact, it follows that \( \tilde{P} \geq g_\gamma(\tilde{P}) \geq (1 - \gamma)A'PA + Q \).
   Combining with \((1 - \gamma)A'XA + Q = X\), there holds that \( P - X \geq (1 - \gamma)A'(P - X)A \), which gives \( P - X \geq 0 \). Since \( X > 0 \), it is thus obtained that \( \tilde{P} > 0 \).

**Theorem 6:** Suppose there exists a matrix \( \tilde{K} \) and a positive-definite matrix \( \bar{P} \) such that \( \bar{P} > \Phi(\tilde{K}, \bar{P}) \).

1) for any initial condition \( P_0 \), the MARE converges, and the limit is independent of the initial condition
   \[ \lim_{t \to \infty} P_t = \lim_{t \to \infty} g_\gamma(F_0) = \bar{P}. \]
2) \( \bar{P} \) is the unique positive-semidefinite fixed point of the MARE.

**Proof.**

1) We first let the initial condition be \( Q_0 = 0 \). Let \( Q_k = g_\gamma(Q_0) \). Since \( 0 = Q_0 \leq Q_1 = Q \), from 3) of Lemma 2, it follows that \( Q_1 = g_\gamma(Q_0) \leq g_\gamma(Q_1) = Q_2 \). By induction, it is obtained that \( Q_t \leq Q_{t+1} \) for \( t \geq 0 \). We show the sequence has an upper bound. Define the linear operator \( \mathcal{L}(Y) = (1 - \gamma)A'YA + \gamma(F_1'YF_1 + F_2'YF_2) \).
   Noting that \( \bar{P} > \Phi(\bar{K}, \bar{P}) = \mathcal{L}(\bar{P}) + Q + \gamma K'RK \geq \mathcal{L}(\bar{P}) \). On the other hand, we have \( Q_{t+1} = g_\gamma(Q_t) \leq \Phi(K_{Q_t}, Q_t) = \mathcal{L}(Q_t) + Q + \gamma K_{Q_t}'RK_{Q_t} \). In view of \( Q + \gamma K_{Q_t}'RK_{Q_t} \geq 0 \) and using Lemma 3 in [23], we conclude that there exists \( M_\gamma \) such that \( Q_t \leq M_\gamma \), for \( t \geq 0 \).
   Accordingly, the sequence converges, i.e., \( \lim_{t \to \infty} Q_t = \bar{P} \).
   We next consider the case that the initial condition is selected as \( R_0 \geq \bar{P} \). First, define \( \bar{K} = -R + B'PB + B'(A')^dPA^dB \)
   \[ \bar{P} = \mathcal{L}(\bar{Y}) + Q + \bar{K}'RK > \mathcal{L}(\bar{Y}) \]
   where \( \bar{Q} > 0 \) has been used in the derivation of last inequality. Using again Lemma 3 in [23], we have that \( \lim_{t \to \infty} \mathcal{L}(\bar{Y}) = 0 \) for all \( Y \geq 0 \). Since \( R_0 \geq \bar{P} \), then \( R_1 = g_\gamma(R_0) \geq g_\gamma(\bar{P}) = \bar{P} \). By induction, it follows that \( R_t \geq \bar{P} \geq \bar{P} \geq 0 \). Noting that
   \[ 0 \leq R_{t+1} - \bar{P} = \bar{P} - R_t = \Phi(K_{R_t}, R_t) - \Phi(K_\rho, \bar{P}) \leq \Phi(K_{R_t}, R_t) - \Phi(K_\rho, \bar{P}) = \gamma(1 - \gamma)(R_t - \bar{P})A + \gamma F_1'(R_t - \bar{P})F_1 + 2 \gamma F_2'(R_t - \bar{P})F_2 + \gamma \mathcal{L}(R_t - \bar{P}) \]
   \[ \to 0, t \to \infty, \]
   which gives that \( \lim_{t \to \infty} R_{t+1} = \bar{P} \).
   We now prove that the Riccati iteration converges to \( \bar{P} \) for all initial values \( P_0 \geq 0 \). Let \( Q_0 = 0 \) and \( R_0 = P_0 \), it is obvious that \( Q_0 \leq P_0 \leq R_0 \).
   Consider the Riccati iterations initialized at \( Q_0, P_0 \) and \( R_0 \). Then it follows that \( \bar{Q}_t \leq P_t \leq R_t \forall t \geq 0 \). Based on the above discussions, it has already been obtained that \( \lim_{t \to \infty} R_t = \bar{P} \)
   for all \( R_t \). Based on the above discussions, it has already been obtained that \( \lim_{t \to \infty} R_t = \bar{P} \).
   2) It is now claimed that the solution is unique. Otherwise, let \( \bar{P} \) be another solution, i.e., \( \bar{P} = g_\gamma(\bar{P}) \) and let the initial value be \( \bar{P} \). Thus we have a constant sequence with \( \bar{P} \).
   Using the above prove, we have that the constant sequence also converges to \( \bar{P} \). Thus \( \bar{P} = \bar{P} \).
   The proof is now completed.

**Theorem 7:** If \( (A, B, 0, A'B) \) is mean-square stabilizable and \( A \) is unstable. Then there exists a \( \gamma \in (0, 1) \) such that
   \[ \lim_{t \to \infty} P_t = +\infty, \] for \( 0 \leq \gamma \leq \lambda_c \) and \( \exists P_0 \geq 0 \)
   \[ P_t \leq M_{P_0} \forall t, \] for \( \lambda_c < \gamma \leq 1 \) and \( \forall P_0 \geq 0 \)
   where \( M_{P_0} > 0 \) depends on the initial condition \( P_0 \geq 0 \).
   **Proof.** If \( \lambda = 1 \), the Riccati difference equation becomes the delay-dependent Riccati equation in [26] and [27] which has been shown to converge to a unique positive definite solution under the mean-square stabilizability of \( (A, B, 0, A'B) \) for the zero initial value. Based on similar discussions in Theorem 6, the Riccati iteration converges to a fixed point for any initial values \( P_0 \geq 0 \). Hence, \( P_t \) is always bounded for any initial
values $P_0 \geq 0$. If $\lambda = 0$, the equation is reduced to $P_{t+1} = A'P_tA + Q$. If $A$ is unstable, there always exists one initial value $P_0 \geq 0$ such that $P_t$ is unbounded. Accordingly, the critical value $\lambda_c \in [0, 1)$ exists. We now prove there exists a single critical value. In fact, for any $\lambda > \lambda_c$, it is obtained that $P_{t+1} = g_\lambda(P_t) \leq g_\lambda(P_t)$ which is bounded. This completes the proof.

**Theorem 8:** If $(A, B, 0, A^dB)$ is mean-square stabilizable and $A$ is unstable. Then the critical value satisfies $\gamma \leq \gamma_c \leq \bar{\gamma}$ where

$$\gamma_c = \arg\min_{\gamma} \{3S((1 - \lambda)A'SA + Q = S, S \geq 0)\}$$

$$\bar{\gamma} = \arg\min_{\gamma} \{3S((K, P)|P > \Phi(K, P))\}$$

**Proof.** Consider $S_{t+1} = (1 - \gamma)A'S_tA + Q$ with $S_0 = 0$, it is obtained that $\lim_{t \to \infty} S_t = \infty$ for $\gamma > \gamma$ in the proof of Theorem 3 in [23]. Noting that the initial value $P_0 \geq 0$, i.e. $P_0 \geq S_0$. Assume that $P_t \geq S_t$. From 6) of Lemma 2 it holds that $P_{t+1} \geq (1 - \gamma)A'P_tA + Q \geq (1 - \gamma)A'S_tA + Q = S_{t+1}$. By induction, we have that $P_t \geq S_t, \forall t \geq 0, \forall P_0 \geq 0$. This implies that $\lim_{t \to \infty} P_t \geq \lim_{t \to \infty} S_t = \infty$. That is, $P_t$ is unbounded for any $\gamma < \gamma_c$ and any initial values $P_0 \geq 0$. Therefore, $\gamma_c \geq \bar{\gamma}$. On the other hand, when $\gamma > \bar{\gamma}$, there exists $X$ such that $X > \Phi(K, X) \geq g_\gamma(X)$. Using 7) of Lemma 2 it yields that $X > 0$. Using Lemma 3 of [23], $P_t$ is bounded. That is, $\gamma_c \leq \bar{\gamma}$.

**Theorem 9:** If $(A, B, 0, A^dB)$ is mean-square stabilizable, then the following statements are equivalent.

1. $\exists X$ such that $X > g_\gamma(X)$.
2. $\exists K, X > 0$ such that $X > \Phi(K, X)$.
3. $\exists Z$ and $0 \leq Y \leq I$ such that

$$\Gamma_\gamma(Y, Z) = \begin{bmatrix}
Y & \sqrt{\gamma}(AY + BZ)' \\
\sqrt{\gamma}(AY + BZ) & Y \\
\sqrt{\gamma}A'Z & Y \\
\sqrt{\gamma}(A'Z)' & \sqrt{1 - \gamma}(AY)' \\
0 & 0 \\
0 & Y \\
0 & Y
\end{bmatrix} > 0.$$

**Proof.** Using facts 1) and 2) in Lemma 2 the equivalence between 1) and 2) follows. We now establish the equivalence between 2) and 3). Let $F = A + BK$, then $X > \Phi(K, X)$ is in fact $X > (1 - \gamma)A'XA + F'XF + \gamma K'B'(A')dA^dBK + \gamma K'RKQ + Q$. By using Schur complement, the inequality is equivalent to

$$X - (1 - \gamma)A'XA + \lambda K'K(B'(A')dA^dBK + \sqrt{\gamma}F'X^{-1}) > 0.$$
[28] H. Hu, Z. Lin, Consensus of a class of discrete-time nonlinear multi-agent systems in the presence of communication delays, *ISA transactions*, 71: 10-20, 2017.

[29] J. Xu, H. Zhang, L. Xie, Input delay margin for consensusability of multi-agent systems, *Automatica*, 49: 1816-1820, 2013.

[30] G. Gu, L. Marinovici, F. L. Lewis, Consensusability of discrete-time dynamic multiagent systems, *IEEE Transactions on Automatic Control*, 57(8): 2085-2089, 2012.

[31] Y. Zhang, Y. Tian, Maximum allowable loss probability for consensus of multi-agent systems over random weighted lossy networks, *IEEE Transactions on Automatic Control*, 57(8): 2127-2132, 2012.