Distant total irregularity strength of graphs via random vertex ordering

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Abstract

Let \( c : V \cup E \to \{1, 2, \ldots, k\} \) be a (not necessarily proper) total colouring of a graph \( G = (V, E) \) with maximum degree \( \Delta \). Two vertices \( u, v \in V \) are \textit{sum distinguished} if they differ with respect to sums of their incident colours, i.e. \( c(u) + \sum_{e \ni u} c(e) \neq c(v) + \sum_{e \ni v} c(e) \). The least integer \( k \) admitting such colouring \( c \) under which every \( u, v \in V \) at distance \( 1 \leq d(u, v) \leq r \) in \( G \) are sum distinguished is denoted by \( ts_r(G) \).

Such graph invariants link the concept of the total vertex irregularity strength of graphs with so called 1-2-Conjecture, whose concern is the case of \( r = 1 \). Within this paper we combine probabilistic approach with purely combinatorial one in order to prove that \( ts_r(G) \leq (2 + o(1))\Delta^{r-1} \) for every integer \( r \geq 2 \) and each graph \( G \), thus improving the previously best result: \( ts_r(G) \leq 3\Delta^{r-1} \).

Keywords: total vertex irregularity strength of a graph, 1–2 Conjecture, \( r \)-distant total irregularity strength of a graph

1. Introduction

The cornerstone of the field of vertex distinguishing graph colourings is the graph invariant called \textit{irregularity strength}. For a graph \( G = (V, E) \) it is usually denoted by \( s(G) \) and can be defined as the least integer \( k \) so that we may construct an irregular multigraph, i.e. a multigraph with pairwise distinct degrees of all vertices, of \( G \) by multiplying its edges, each at most \( k \) times (including the original one), see [8]. This study thus originated from the basic fact that no graph \( G \) with more than one vertex is irregular itself and related research on possible alternative definitions of an irregular graph, see e.g. [3]. Equivalently, \( s(G) \) is also defined as the least \( k \) so that there exists an edge colouring \( c : E \to \{1, 2, \ldots, k\} \) such that for every pair \( u, v \in E \), \( u \neq v \), the sum of colours incident with \( u \) is distinct from the sum of colours incident with \( v \). Note that \( s(G) \) exists only for graphs without isolated edges and with at most one isolated
was proved that for every graph $G$ which there exists such colouring with total vertex irregularity strength the bound of 2 should make the optimal general upper bound in both cases, see [27, 28, 31]. Many interesting results, concepts and open problems concerning this graph invariant can also be found e.g. in [6, 9, 10, 11, 12, 16, 18, 25], and many others.

In [5], Baˇ ca et al. introduced a total version of the concept above. Given any graph $G = (V, E)$ and a (not necessarily proper) total colouring $c : V ∪ E → \{1, 2, . . . , k\}$, let
denote the weight of any vertex $v ∈ V$, which shall also be called the sum at $v$ and denoted simply by $w(v)$ in cases when $c$ is unambiguous from context. The least $k$ for which there exists such colouring with $w(u) ≠ w(v)$ for every $u, v ∈ V$, $u ≠ v$, is called the total vertex irregularity strength of $G$ and denoted by $\text{tvs}(G)$. In [5], among others, it was proved that for every graph $G$ with $n$ vertices, $\frac{2n}{\ell + 1} ≤ \text{tvs}(G) ≤ n + \Delta - 2\delta + 1$. Up to know the best upper bounds (for graphs with $\delta > 1$, see [1, 21]. This tight upper bound can however be improved in the case of graphs with maximum degree $\Delta ≥ \Delta_0$, see [3], and $\text{tvs}(G) ≤ (2 + o(1))\frac{\Delta}{\ell} + 4$ for $\delta ≥ n^{0.5}\ln n$, see [20]. Many other results e.g. for particular graph families can also be found in [4, 22, 26, 29] and other papers.

In this article we consider a distant generalization of $\text{tvs}(G)$ from [23], motivated among others by the study on distant chromatic numbers, see e.g. [17] for a survey concerning these. For any positive integer $r$, two distinct vertices at distance at most $r$ in $G$ shall be called r-neighbours. We denote by $N^r(v)$ the set of all $r$-neighbours of any $v ∈ V$ in $G$, and set $d^r(v) = |N^r(v)|$. The least integer $k$ for which there exists a total colouring $c : V ∪ E → \{1, 2, . . . , k\}$ such that there are no $r$-neighbours $u, v$ in $G$ which are in conflict, i.e. with $w(u) = w(v)$ (cf. [11]), we call the $r$-distant total irregularity strength of $G$, and denote by $\text{ts}_r(G)$. It is known that $\text{ts}_r(G) ≤ 3\Delta^{r-1}$ for every graph $G$, see [23], also for a comment implying that a general upper bound for $\text{ts}_r(G)$ cannot be (much) smaller than $\Delta^{r-1}$. In this paper we combine the probabilistic method with algorithmic approach similar to those in e.g. [3, 15, 20, 23] to prove that in fact $\text{ts}_r(G) ≤ (2 + o(1))\Delta^{r-1}$ (for $r ≥ 2$).

**Theorem 1.** For every integer $r ≥ 2$ there exists a constant $\Delta_0$ such that for each graph $G$ with maximum degree $\Delta ≥ \Delta_0$,

$$\text{ts}_r(G) ≤ 2\Delta^{r-1} + 3\Delta^{r-\frac{1}{2}}\ln^2 \Delta + 4,$$

hence

$$\text{ts}_r(G) ≤ (2 + o(1))\Delta^{r-1}$$

for all graphs.

It is also worth mentioning that the case of $r = 1$ was introduced and considered separately in [27], where the well known 1-2-Conjecture concerning this invariant was introduced. It is known that $\text{ts}_1(G) ≤ 3$ for all graphs, see Theorem 2.8 in [15], even in case of a natural list generalization of the problem, see [31], though it is believed that the upper bound of 2 should make the optimal general upper bound in both cases, see [27, 28, 31].
We also refer a reader to \[24\] to see an improvement of a similar probabilistic flavor for the upper bound from \[23\] on the correspondent of \(t_s(G)\) concerning the case of edge colourings exclusively.

2. Probabilistic Tools

We shall use probabilistic approach in the first part of the proof of Theorem 1, basing on the Lovász Local Lemma, see e.g. \[2\], combined with the Chernoff Bound, see e.g. \[14\] (Th. 2.1, page 26). We recall these below.

**Theorem 2 (The Local Lemma).** Let \(A_1, A_2, \ldots, A_n\) be events in an arbitrary probability space. Suppose that each event \(A_i\) is mutually independent of a set of all the other events \(A_j\) but at most \(D\), and that \(\Pr(A_i) \leq p\) for all \(1 \leq i \leq n\). If 
\[ ep(D + 1) \leq 1, \]
then \(\Pr(\bigcap_{i=1}^{n} \overline{A_i}) > 0\).

**Theorem 3 (Chernoff Bound).** For any \(0 \leq t \leq np\),
\[ \Pr(\text{BIN}(n, p) > np + t) < e^{-\frac{t^2}{2np}}, \quad \text{and} \quad \Pr(\text{BIN}(n, p) < np - t) < e^{-\frac{t^2}{2np}} \leq e^{-\frac{t^2}{3np}} \]
where \(\text{BIN}(n, p)\) is the sum of \(n\) independent Bernoulli variables, each equal to 1 with probability \(p\) and 0 otherwise.

Note that if \(X\) is a random variable with binomial distribution \(\text{BIN}(n, p)\) where \(n \leq k\), then we may still apply the Chernoff Bound above, even if we do not know the exact value of \(n\), to prove that \(\Pr(X > kp + t) < e^{-\frac{t^2}{3kp}}\) (for \(t \leq \lfloor kp \rfloor\)).

3. Proof of Theorem \[1\]

Fix an integer \(r \geq 2\). Within our proof we shall not specify \(\Delta_0\). Instead, we shall assume that \(G = (V, E)\) is a graph with sufficiently large maximum degree \(\Delta\), i.e. large enough so that all inequalities below are fulfilled.

We first partition \(V\) into a subset of vertices with relatively small degrees and a subset of those with big degrees:
\[ S = \left\{ u \in V : d(u) \leq \Delta^\frac{2}{3} \right\}; \]
\[ B = \left\{ u \in V : d(u) > \Delta^\frac{2}{3} \right\}. \]
Moreover, for every \(v \in V\), we denote: \(S(v) = N(v) \cap S, \ s(v) = |S(v)|, \ B(v) = N(v) \cap B, \ b(v) = |B(v)|\).

Now we randomly order the vertices of \(V\) into a sequence. For this goal, associate with every vertex \(v \in V\) a random variable \(X_v \sim U[0, 1]\) having the uniform distribution on \([0, 1]\) where all these random variables \(X_v\), \(v \in V\) are independent, or in other words pick a (real) number uniformly at random from the interval \([0, 1]\) and associate it with \(v\) for every \(v \in V\). Note that with probability one all these numbers are pairwise distinct.
In such a case, these independent random variables uniquely define a natural ordering $v_1, v_2, \ldots, v_n$ of the vertices in $V$ where $X_{v_i} < X_{v_j}$ if and only if $1 \leq i < j \leq n$.

For every vertex $v \in V$, any its neighbour or $r$-neighbour $u$ which precedes $v$ in the obtained ordering of the elements of $V$ shall be called a backward neighbour or $r$-neighbour, resp., of $v$. Analogously, the remaining ones shall be called forward neighbours or $r$-neighbours, resp., of $v$, while the edges joining $v$ with its forward or backward neighbours shall be referred to as forward or backward edges, resp., as well. Also, for any subset $W \subset V$, let $N_+(v)$, $N_-(v)$, $N_B(v)$ denote the sets of all backward neighbours, forward $r$-neighbours and $r$-neighbours in $W$ of $v$, respectively. Set $d_-(v) = |N_-(v)|$, $d_B(v) = |N_B(v)|$, and let $b_-(v)$ denote the number of backward neighbours of $v$ which belong to $B(v)$.

Denote $D(v) = \sum_{u \in N(v)} d(u)$ and note that $d_r(v) \leq D(v) \Delta^{-2} \leq d(v) \Delta^{-1}$.

Let us also partition $V$ into a subset $I$ consisting of initial vertices of the obtained sequence and the remaining part $R$:

$$I = \left\{ v : X_v < \frac{\ln \Delta}{\Delta^\tau} \right\},$$

$$R = \left\{ v : X_v \geq \frac{\ln \Delta}{\Delta^\tau} \right\}.$$

**Lemma 4.** With positive probability, the obtained ordering has the following features for every vertex $v$ in $G$ with $b(v) \geq \Delta^\tau \ln \Delta$:

**F1:** $d_+(v) \leq 2d(v) \Delta^{-\frac{\tau}{2}} \ln \Delta$;

**F2:** if $v \in R$, then: $b_-(v) \geq X_v b(v) - \sqrt{X_v b(v) \ln \Delta}$;

**F3:** if $v \in R$, then: $d_-(v) \leq X_v D(v) \Delta^{-2} + \sqrt{X_v D(v) \Delta^{-2} \ln \Delta}$.

**Proof.** For every vertex $v \in V$ of degree $d$ in $G$ and with $b(v) \geq \Delta^\tau \ln \Delta$ (hence also $d \geq \Delta^\tau \ln \Delta$), let $A_{v,1}$ denote the event that $d_+(v) > 2d \Delta^{-\frac{\tau}{2}} \ln \Delta$, let $A_{v,2}$ be the event that $v$ belongs to $R$ and $b_-(v) < X_v b(v) - \sqrt{X_v b(v) \ln \Delta}$, and let $A_{v,3}$ denote the event that $v$ belongs to $R$ and $d_-(v) > X_v D(v) \Delta^{-2} + \sqrt{X_v D(v) \Delta^{-2} \ln \Delta}$.

As $|N_+(v)| \leq d \Delta^{-1}$ and for each $u \in N_+(v)$, the probability that $u$ belongs to $I$ equals $\frac{\ln \Delta}{\Delta^\tau}$, then by the Chernoff Bound (and the comment below it),

$$\Pr(A_{v,1}) \leq \Pr\left( d_+(v) > d \Delta^{-\frac{\tau}{2}} \ln \Delta + \sqrt{d \Delta^{-\frac{\tau}{2}} \ln \Delta} \ln \Delta \right)$$

$$< e^{-\frac{d \Delta^{-\frac{\tau}{2}} \ln \Delta}{2d \Delta^{-\frac{\tau}{2}} \ln \Delta}} = \Delta^{-\frac{\ln \Delta}{\Delta^\tau}} < \frac{1}{\Delta^\tau}.$$ (3)

Subsequently note that for any $x \in [0, 1]$:

$$\Pr(b_-(v) < X_v b(v) - \sqrt{X_v b(v) \ln \Delta} | X_v = x)$$

$$= \Pr(\text{BIN}(b(v), x) < x b(v) - \sqrt{x b(v) \ln \Delta})$$

$$< \frac{1}{\Delta^\tau},$$

4
where the last inequality follows by the Chernoff Bound if $\sqrt{xb(v)} \ln \Delta \leq xb(v)$, while it is trivial otherwise. Hence,

$$\Pr(A_{v,2}) \leq \Pr(b_{-}(v) < X_v b(v) - \sqrt{X_v b(v)} \ln \Delta) \leq \int_{0}^{1} \frac{1}{\Delta^{3r}} dx = \frac{1}{\Delta^{3r}}. \quad (4)$$

For the sake of analyzing $A_{v,3}$, note now first that for $x \in [0, \frac{\ln \Delta}{\Delta^r}]$,

$$\Pr(d_{-}^{c}(v) > X_v D(v) \Delta^{r-2} + \sqrt{X_v D(v) \Delta^{r-2}} \ln \Delta \wedge v \in R|X_v = x) = 0. \quad (5)$$

On the other hand, analogously as above, for $x \in [\frac{\ln \Delta}{\Delta^r}, 1]$:

$$\Pr(d_{-}^{c}(v) > X_v D(v) \Delta^{r-2} + \sqrt{X_v D(v) \Delta^{r-2}} \ln \Delta \wedge v \in R|X_v = x) \leq \Pr(\text{BIN}(D(v) \Delta^{r-2}, x) > x D(v) \Delta^{r-2} + \sqrt{x D(v) \Delta^{r-2}} \ln \Delta) \leq \frac{1}{\Delta^{3r}}, \quad (6)$$

where the last inequality follows by the Chernoff Bound, as $x \geq \frac{\ln \Delta}{\Delta^r}$ and $b(v) \geq \Delta^\frac{r}{2} \ln \Delta$ (where $D(v) \geq b(v) \Delta^\frac{r}{2}$) imply that $\sqrt{x D(v) \Delta^{r-2}} \ln \Delta \leq x D(v) \Delta^{r-2}$. Hence, by (5) and (6),

$$\Pr(A_{v,3}) \leq \int_{0}^{1} \frac{1}{\Delta^{3r}} dx = \frac{1}{\Delta^{3r}}. \quad (7)$$

Note that each event $A_{v,i}$ is mutually independent of all other events except those $A_{u,j}$ with $u$ at distance at most $2r$ from $v$, $i, j \in \{1, 2, 3\}$, i.e., at most $3 \Delta^{2r} + 2$ events. Thus, as by (3), (4) and (7), the probability of each such event is bounded from above by $\Delta^{-3r}$, by the Lovász Local Lemma, with positive probability none of the events $A_{v,i}$ with $v \in V$ (and $b(v) \geq \Delta^\frac{r}{2} \ln \Delta$) and $i \in \{1, 2, 3\}$ appears. \hfill \square

Let $v_1, v_2, \ldots, v_n$ be the ordering of the vertices of $V$ guaranteed by Lemma 4. Set

$$K = \Delta^{r-1} + \lceil \Delta^{r-\frac{r}{2}} \ln^2 \Delta \rceil \quad \text{and} \quad k = \lceil \Delta^{r-\frac{r}{2}} \ln^2 \Delta \rceil,$$

and assign initial colour $1$ to all the vertices and initial colour $K + 1$ to all the edges of $G$. We shall construct our final colouring $f : V \cup E \to \{1, 2, \ldots, 2K + k + 1\}$ using an algorithm within which we shall be analyzing the consecutive vertices in the ordering (starting from $v_1$). Denote by $c_t(a)$ the contemporary colour of every $a \in V \cup E$ at every stage of the ongoing algorithm (hence initially $c_t(v) = 1$ and $c_t(e) = K + 1$ for every $v \in V$ and $e \in E$). The final target sum of every vertex $v \in V$, $w_f(v)$, shall be chosen the moment $v$ is analyzed. For every $v \in V$, ever since $w_f(v)$ is chosen, we shall require so that

$$0 \leq w_f(v) - w_{v_1}(v) \leq K. \quad (8)$$

We shall admit at most two alterations of the colour for every edge in $E$ - only when any of its ends is being analyzed (vertex colours shall be adjusted at the end of the algorithm). For every currently analyzed vertex $v$ and its neighbour $u \in N(v)$, we admit the following alterations of the colour of $e = uv$ (the moment $v$ is analyzed):
Let $v$ be a vertex and $e = uv$ be an edge incident with $v$. If $e$ is a forward edge of $v$, then adding $0, 1, \ldots, K - 1$ to the number of available options for $w_{c_i}(v)$, $i \in \{1, 2, \ldots, 2K + 1\}$, guarantees that there are at least $|N^-_e(v)|$ available options for $w_{c_i}(v)$.

If $e$ is a backward edge of $v$, then adding $0, 1, \ldots, K - 1$ to the number of available options for $w_{c_i}(v)$, $i \in \{1, 2, \ldots, 2K + 1\}$, guarantees that there are at least $|N^-_e(v)|$ available options for $w_{c_i}(v)$.

If $e$ is a forward edge of $v$, then adding $-K, -K + 1, \ldots, K - 1$ to the number of available options for $w_{c_i}(v)$, $i \in \{1, 2, \ldots, 2K + 1\}$, guarantees that there are at least $|N^-_e(v)|$ available options for $w_{c_i}(v)$.

If $e$ is a backward edge of $v$, then adding $-k, -k + 1, \ldots, k - 1$ to the number of available options for $w_{c_i}(v)$, $i \in \{1, 2, \ldots, 2K + 1\}$, guarantees that there are at least $|N^-_e(v)|$ available options for $w_{c_i}(v)$.

Note that the admitted alterations guarantee that $c_i(e) \in \{1, 2, \ldots, 2K + 1\}$ for every $e \in E$ at each stage of the construction.

Suppose we are about to analyze a vertex $v = vi$, $i \in \{1, 2, \ldots, n\}$, and thus far all our requirements have been fulfilled. We shall show that in every case the admitted alterations on the edges incident with $v$ provide us more options for $w_{c_i}(v)$ than there are backward $r$-neighbours of $v$, and hence one of these options can be fixed as $w_f(v)$ so that this value is distinct from every $w_f(u)$ already fixed for any $u \in N^-_e(v)$. Denote the degree of $v$ by $d$, and assume that $d > 0$ (otherwise, we set $w_f(v) = 1$):

- If $v \not\in V$ and $b(v) \geq \Delta^+ \ln \Delta$, then the admitted alterations provide at least $dk \geq d\Delta^{-\frac{1}{2}} \ln^2 \Delta$ available options for $w_{c_i}(v)$. As by $F_1$ (from Lemma 3), $|N^{-}_e(v)| \leq d^+(v) \leq 2d\Delta^{-\frac{1}{2}} \ln \Delta < d\Delta^{-\frac{1}{2}} \ln^2 \Delta$, at least one of these available options is distinct from all $w_f(u)$ with $u \in N^{-}_e(v)$.

- If $v \in S$, then the admitted alterations provide at least $s(v)K + b(v)K \geq s(v)\Delta^{-\frac{1}{2}} \ln^2 \Delta + b(v)(\Delta^{-1} + \Delta^{-\frac{1}{2}} \ln^2 \Delta)$ available options for $w_{c_i}(v)$. On the other hand, $|N^{-}_e(v)| \leq d^+(v) \leq D(v)\Delta^{-2} \leq (s(v)\Delta^{-\frac{3}{2}} + b(v)\Delta^{-2})\Delta^{-2}$, hence at least one of these available options is distinct from all $w_f(u)$ with $u \in N^{-}_e(v)$.

- If $v \in B$ and $b(v) < \Delta^+ \ln \Delta$, then the admitted alterations provide at least $dk \geq d\Delta^{-\frac{1}{2}} \ln^2 \Delta$ available options for $w_{c_i}(v)$. On the other hand, analogously as in the case above, $|N^{-}_e(v)| \leq d^+(v) \leq s(v)\Delta^{-\frac{1}{2}} \Delta^{-2} + b(v)\Delta^{-1} \leq d\Delta^{-\frac{1}{2}} + \Delta^{-\frac{1}{2}} \ln \Delta < d\Delta^{-\frac{1}{2}} \ln^2 \Delta$, as $v \in B$ implies that $d \geq \Delta^+$. We thus have at least one option available for $v$ distinct from all $w_f(u)$ with $u \in N^{-}_e(v)$.

- If $v \in R$, $v \in B$ and $b(v) \geq \Delta^+ \ln \Delta$, then by $F_2$ the number of available options for $w_{c_i}(v)$ via admitted alterations of colours of the edges incident with $v$ is not smaller than:

$$
\begin{align*}
\text{b}_-(v)K + (d - \text{b}_-(v))k & \geq \text{b}_-(v)\Delta^{-1} + d\Delta^{-\frac{1}{2}} \ln^2 \Delta \\
& \geq (X, b(v) - \sqrt{X, b(v)\ln \Delta})\Delta^{-1} + d\Delta^{-\frac{1}{2}} \ln^2 \Delta \\
& \geq X, b(v)\Delta^{-1} - \sqrt{d\Delta^{-1} \ln \Delta} + d\Delta^{-\frac{1}{2}} \ln^2 \Delta \\
& \geq X, b(v)\Delta^{-1} + d\Delta^{-\frac{1}{2}} \ln^2 \Delta - d\Delta^{-\frac{1}{2}} \ln \Delta
\end{align*}
$$

(where the last inequality follows by the fact that $d \geq \Delta^+\frac{1}{2}$). This number is however greater than the number of backward $r$-neighbours of $v$, as by $F_3$,

$$
|N^{-}_e(v)| \leq X, D(v)\Delta^{-2} + \sqrt{X, D(v)\Delta^{-2} \ln \Delta} \\
\leq X, (b(v)\Delta + s(v)\Delta^+\Delta^{-2})\Delta^{-2} + \sqrt{d\Delta^{-1} \ln \Delta} \\
\leq X, b(v)\Delta^{-1} + d\Delta^{-\frac{1}{2}} + d\Delta^{-\frac{1}{2}} \ln \Delta.
$$
Thus in all cases there is at least one available sum, say $w^*$, for $v$ which is distinct from all $w_f(u)$ with $u \in N_r(v)$. We then set $w_f(v) = w^*$ and perform the admitted alterations on the edges incident with $v$ so that $w_c(v) = w^*$ afterwards.

By our construction, after analyzing $v_n$, all $w_f(v_i)$ are fixed for $i = 1, \ldots, n$ so that $w_f(u) \neq w_f(v)$ whenever $u$ and $v$ are $r$-neighbours in $G$ and (5) holds for every $v \in V$. We then modify (if necessary) the colour of every vertex $v$ by adding to it the integer $w(v) - w_f(v)$, completing the construction of the desired total colouring $f$ of $G$ (by setting $f(a) = c_i(a)$ for every $a \in V \cup E$ afterwards). Note that $1 \leq f(e) \leq 2K + k + 1$ for every $e \in E$ and, by (5), $1 \leq f(v) \leq K + 1$ for every $v \in V$, hence the thesis follows.

4. Remarks

We have put an effort to optimize the second order term from the upper bound in (2), up to a constant and a power in the logarithmic factor, which could still be slightly improved (at the cost of the clarity of presentation). Nevertheless, some multiplicative poly-logarithmic (in $\Delta$) factor seems unavoidable in this term within our approach.

We conclude by posing a conjecture, which to our believes expresses a true asymptotically optimal upper bound for the investigated parameters.

**Conjecture 5.** For every integer $r \geq 2$ and each graph $G$ with maximum degree $\Delta$, \[ts_r(G) \leq (1 + o(1))\Delta^{r-1}.\]

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