Automorphisms of Weighted Complete Intersections

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Abstract—We show that smooth well-formed weighted complete intersections have finite automorphism groups, with several obvious exceptions.

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1. INTRODUCTION

When studying algebraic varieties, it is important to understand their automorphism groups. In some particular cases these groups have an especially nice structure. For instance, recall the following classical result due to H. Matsumura and P. Monsky (cf. [17, Lemma 14.2]).

Theorem 1.1 (see [20, Theorems 1, 2]). Let \( X \subset \mathbb{P}^N, \; N \geq 3, \) be a smooth hypersurface of degree \( d \geq 3. \) Suppose that \( (N,d) \neq (3,4). \) Then the group \( \text{Aut}(X) \) is finite.

The following beautiful generalization of Theorem 1.1 was proved by O. Benoist.

Theorem 1.2 [2, Theorem 3.1]. Let \( X \) be a smooth complete intersection of dimension at least 2 in \( \mathbb{P}^N \) that is not contained in a hyperplane. Suppose that \( X \) does not coincide with \( \mathbb{P}^N, \) is not a quadric hypersurface in \( \mathbb{P}^N, \) and is not a K3 surface. Then the group \( \text{Aut}(X) \) is finite.

The goal of this paper is to generalize Theorems 1.1 and 1.2 to the case of smooth weighted complete intersections. We refer the reader to [6, 15] (see also Section 2 below) for definitions and basic properties of weighted projective spaces and complete intersections therein. Our main result is as follows.

Theorem 1.3. Let \( X \) be a smooth well-formed weighted complete intersection of dimension \( n. \) Suppose that either \( n \geq 3 \) or \( K_X \neq 0. \) Then the group \( \text{Aut}(X) \) is finite unless \( X \) is isomorphic either to \( \mathbb{P}^n \) or to a quadric hypersurface in \( \mathbb{P}^{n+1}. \)

Under a minor additional assumption (cf. Definition 2.2 below), one can make the assertion of Theorem 1.3 more precise.

Corollary 1.4. Let \( X \subset \mathbb{P} \) be a smooth well-formed weighted complete intersection of dimension \( n \) that is not an intersection with a linear cone. Suppose that either \( n \geq 3 \) or \( K_X \neq 0. \) Then the group \( \text{Aut}(X) \) is finite unless either \( X = \mathbb{P} \cong \mathbb{P}^n \) or \( X \) is a quadric hypersurface in \( \mathbb{P} \cong \mathbb{P}^{n+1}. \)

Note that if \( X \) is not an intersection with a linear cone, then the hypothesis of Theorem 1.3 is equivalent to the requirement that \( X \) is none of the weighted complete intersections listed in Table 2 below.

We refer the reader to [12; 18, Theorem 1.1.2; 26, Theorem 1.2] for other results concerning finiteness of automorphism groups.

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Theorem 1.3 is mostly implied by the results of [8] (see Theorem 4.5 below). However, some cases are not covered by [8] and have to be classified and treated separately (see Proposition 3.7(iv) and Lemma 4.8).

To deduce Corollary 1.4 from Theorem 1.3, we need the following statement; it is well known to experts, but we could not find an appropriate reference (while consider this statement interesting in itself). We will say that a weighted complete intersection $X \subset \mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$ of multidegree $(d_1, \ldots, d_k)$ is normalized if the inequalities $a_0 \leq \ldots \leq a_N$ and $d_1 \leq \ldots \leq d_k$ hold.

**Proposition 1.5** (cf. [15, Lemma 18.3]). Let $X \subset \mathbb{P}(a_0, \ldots, a_N)$ and $X' \subset \mathbb{P}(a'_0, \ldots, a'_N)$ be normalized quasismooth well-formed weighted complete intersections of multidegrees $(d_1, \ldots, d_k)$ and $(d'_1, \ldots, d'_k)$, respectively, such that $X$ and $X'$ are not intersections with linear cones. Suppose that $X \cong X'$ and $\dim X \geq 3$. Then $N = N'$, $k = k'$, $a_i = a'_i$ for every $0 \leq i \leq N$, and $d_j = d'_j$ for every $1 \leq j \leq k$.

After the first draft of this paper had been completed, A. Massarenti informed us that a result essentially similar to Theorem 1.3 was proved earlier in [1, Proposition 5.7]. Note however that in [1, Sect. 5] the authors work with smooth weighted complete intersections subject to certain strong additional assumptions (see [1, Assumptions 5.2] for details).

Throughout the paper we work over an algebraically closed field $k$ of characteristic zero.

The plan of the paper is as follows. In Section 2 we recall some preliminary facts on weighted complete intersections. In Section 3 we show the uniqueness of presentation of a variety as a weighted complete intersection, that is, we prove Proposition 1.5. Finally, in Section 4 we prove Theorem 1.3 and Corollary 1.4.

## 2. PRELIMINARIES

In this section we recall the basic properties of weighted complete intersections. We refer the reader to [6, 15] for more details. Some properties of smooth weighted complete intersections can also be found in the earlier paper [22].

Let $a_0, \ldots, a_N$ be positive integers. Consider the graded algebra $k[x_0, \ldots, x_N]$, where the grading is defined by assigning the weights $a_i$ to the variables $x_i$. Put

$$\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N) = \text{Proj } k[x_0, \ldots, x_N].$$

The weighted projective space $\mathbb{P}$ is said to be well-formed if the greatest common divisor of any $N$ weights $a_i$ is $1$. Every weighted projective space is isomorphic to a well-formed one (see [6, Sect. 1.3.1]). A subvariety $X \subset \mathbb{P}$ is said to be well-formed if $\mathbb{P}$ is well-formed and

$$\text{codim}_X(X \cap \text{Sing } \mathbb{P}) \geq 2,$$

where the dimension of the empty set is defined to be $-1$.

We say that a subvariety $X \subset \mathbb{P}$ of codimension $k$ is a weighted complete intersection of multidegree $(d_1, \ldots, d_k)$ if its weighted homogeneous ideal in $k[x_0, \ldots, x_N]$ is generated by a regular sequence of $k$ homogeneous elements of degrees $d_1, \ldots, d_k$. This is equivalent to requiring that the codimension (of every irreducible component) of $X$ equals the (minimum) number of generators of the weighted homogeneous ideal of $X$ (cf. [14, Ch. II, Theorem 8.21A(e)]). Note that $\mathbb{P}$ can be thought of as a weighted complete intersection of codimension $0$ in itself; this gives us a smooth Fano variety if and only if $\mathbb{P} \cong \mathbb{P}^N$.

**Definition 2.1** (see [15, Definition 6.3]). Let $p : \mathbb{A}^{N+1} \setminus \{0\} \to \mathbb{P}$ be the natural projection to the weighted projective space. A subvariety $X \subset \mathbb{P}$ is said to be quasismooth if $p^{-1}(X)$ is smooth.

Note that a smooth well-formed weighted complete intersection is always quasismooth (see [27, Corollary 2.14]).
Table 1. Fano weighted complete intersections in dimensions 1 and 2

| Dimension | Family | $\mathbb{P}$ | Degrees | Dimension | Family | $\mathbb{P}$ | Degrees |
|-----------|--------|---------------|---------|-----------|--------|---------------|---------|
| 1         | 1.1    | $\mathbb{P}^2$ | 2       | 1         | 1.2    | $\mathbb{P}^1$ | $\emptyset$ |
| 2         | 2.1    | $\mathbb{P}(1^2, 2, 3)$ | 6       | 2         | 2.2    | $\mathbb{P}(1^3, 2)$ | 4       |
| 2         | 2.3    | $\mathbb{P}^3$ | 3       | 2         | 2.4    | $\mathbb{P}^4$ | 2, 2    |
| 2         | 2.5    | $\mathbb{P}^3$ | 2       | 2         | 2.6    | $\mathbb{P}^2$ | $\emptyset$ |

The following definition describes weighted complete intersections that are to some extent analogous to complete intersections in an ordinary projective space that are contained in a hyperplane.

Definition 2.2 (cf. [15, Definition 6.5]). A weighted complete intersection $X \subset \mathbb{P}$ is said to be an intersection with a linear cone if one has $d_j = a_i$ for some $i$ and $j$.

Remark 2.3. A general quasismooth well-formed weighted complete intersection is isomorphic to a quasismooth well-formed weighted complete intersection that is not an intersection with a linear cone (cf. [27, Remark 5.2]). Note however that this does not hold without the generality assumption. For instance, a general weighted complete intersection of bidegree $(2, 4)$ in $\mathbb{P}(1^{n+2}, 2)$ is isomorphic to a quartic hypersurface in $\mathbb{P}^{n+1}$, while certain weighted complete intersections of this type are isomorphic to double covers of an $n$-dimensional quadric branched over an intersection with a quartic.

Given a subvariety $X \subset \mathbb{P}$, we denote by $O_X(1)$ the restriction of the sheaf $O_{\mathbb{P}}(1)$ to $X$ (see [6, Sect. 1.4.1]). Note that the sheaf $O_{\mathbb{P}}(1)$ may not be invertible. However, if $X$ is well-formed, then $O_X(1)$ is a well-defined divisorial sheaf on $X$. Furthermore, if $X$ is well-formed and smooth, then $O_X(1)$ is a line bundle on $X$.

Lemma 2.4 [24, Remark 4.2; 25, Proposition 2.3] (cf. [22, Theorem 3.7]). Let $X$ be a quasismooth well-formed weighted complete intersection of dimension at least 3. Then the class of the divisorial sheaf $O_X(1)$ generates the group $\text{Cl}(X)$ of classes of Weil divisors on $X$. In particular, under the additional assumption that $X$ is smooth, the class of the line bundle $O_X(1)$ generates the group $\text{Pic}(X)$.

One can describe the canonical class of a weighted complete intersection. For a weighted complete intersection $X$ of multidegree $(d_1, \ldots, d_k)$ in $\mathbb{P}$, define

$$i_X = \sum a_j - \sum d_i.$$ 

Let $\omega_X$ be the dualizing sheaf on $X$.

Theorem 2.5 (see [6, Theorem 3.3.4; 15, Sect. 6.14]). Let $X$ be a quasismooth well-formed weighted complete intersection. Then

$$\omega_X = O_X(-i_X).$$

Using the bounds on numerical invariants of smooth weighted complete intersections found in [5, Theorem 1.3; 27, Theorem 1.1; 25, Corollary 5.3(i)], one can easily obtain the classically known lists of all smooth Fano weighted complete intersections of small dimensions. Namely, we have the following.

Lemma 2.6. Let $X$ be a smooth well-formed Fano weighted complete intersection of dimension at most 2 in $\mathbb{P}$ that is not an intersection with a linear cone. Then $X$ is one of the varieties listed in Table 1.

Remark 2.7. Let $X$ be a smooth well-formed Fano weighted complete intersection of dimension 2. If we do not assume that $X$ is not an intersection with a linear cone, we cannot use the
Table 2. Calabi–Yau weighted complete intersections in dimensions 1 and 2

| Dimension | Family | P          | Degrees | Dimension | Family | P          | Degrees |
|-----------|--------|------------|---------|-----------|--------|------------|---------|
| 1         | 1.1    | \(\mathbb{P}(1,2,3)\) | 6       | 2         | 2.1    | \(\mathbb{P}(1^3,3)\) | 6       |
| 1         | 1.2    | \(\mathbb{P}(1,1,2)\) | 4       | 2         | 2.2    | \(\mathbb{P}^3\) | 4       |
| 1         | 1.3    | \(\mathbb{P}^2\) | 3       | 2         | 2.3    | \(\mathbb{P}^4\) | 2, 3    |
| 1         | 1.4    | \(\mathbb{P}^3\) | 2, 2    | 2         | 2.4    | \(\mathbb{P}^5\) | 2, 2, 2 |

classification provided by Lemma 2.6. However, Lemma 2.6 applied together with Remark 2.3 shows that if \(i_X = 1\), then \(X\) is a del Pezzo surface of (anticanonical) degree at most 4.

Recall that the Fano index of a Fano variety \(X\) is defined as the maximum integer \(m\) such that the canonical class \(K_X\) is divisible by \(m\) in the Picard group of \(X\). Theorem 2.5 and Lemmas 2.4 and 2.6 imply the following.

**Corollary 2.8.** Let \(X\) be a smooth Fano well-formed weighted complete intersection of dimension at least 2. Then the Fano index of \(X\) equals \(i_X\).

**Proof.** Suppose that \(\dim X = 2\). Note that the Fano index is constant in the family of smooth weighted complete intersections of a given multidegree in a given weighted projective space. Similarly, \(i_X\) is constant in such a family. Thus, by Remark 2.3 we may assume that \(X\) is not an intersection with a linear cone. Now the assertion follows from the classification provided in Lemma 2.6.

If \(\dim X \geq 3\), then we apply Theorem 2.5 together with Lemma 2.4. \(\square\)

Note that the assertion of Corollary 2.8 fails in dimension 1: if \(X\) is a conic in \(\mathbb{P}^2\), then \(i_X = 1\), while the Fano index of \(X\) is 2.

**Lemma 2.9.** Let \(X \subset \mathbb{P}(a_0,\ldots,a_N)\) be a smooth well-formed weighted complete intersection of multidegree \((d_1,\ldots,d_k)\). Then a general weighted complete intersection \(X'\) of multidegree \((d_1,\ldots,d_k)\) in \(\mathbb{P}(1,a_0,\ldots,a_N)\) is smooth and well-formed, and \(i_{X'} = i_X + 1\).

**Proof.** Straightforward. \(\square\)

For the converse of Lemma 2.9, see [25, Theorem 1.2].

Similarly to Lemma 2.6, we can classify three-dimensional smooth well-formed Fano weighted complete intersections that are not intersections with a linear cone (see [28, Table 2]). This together with Lemma 2.9 allows us to classify smooth well-formed weighted complete intersections of dimension up to 2 with trivial canonical class.

**Lemma 2.10.** Let \(X\) be a smooth well-formed weighted complete intersection of dimension at most 2 in \(\mathbb{P}\) that is not an intersection with a linear cone. Suppose that \(K_X = 0\). Then \(X\) is one of the varieties listed in Table 2.

**Remark 2.11.** Note that each of the four families of elliptic curves listed in Table 2 in fact contains all elliptic curves up to isomorphism. This is not the case for K3 surfaces. For instance, a general member of family 2.2 in Table 2 does not appear in family 2.1 for degree reasons.

### 3. UNIQUENESS OF EMBEDDINGS

In this section we prove Proposition 1.5. We begin with two well-known facts that can be proved in approximately the same way as their analogs for complete intersections in the ordinary projective space. However, we provide their proofs for the reader’s convenience.

The proof of the following statement was communicated to us by A. Kuznetsov.
Lemma 3.1. Let $X \subset \mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$ be a subvariety. Let $C^*_X$ be the affine cone over $X$ with vertex removed, and let $p: C^*_X \to X$ be the projection. Then one has
\[ p_* (\mathcal{O}_{C^*_X}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m). \]

Proof. Denote the polynomial ring $k[x_0, \ldots, x_N]$ by $R$ and let
\[ U = (\text{Spec } R) \setminus \{0\} \cong \mathbb{A}^{N+1} \setminus \{0\}. \]

Let
\[ Y = \text{Spec}_\mathbb{P} \left( \bigoplus_{m \geq 0} \mathcal{O}_\mathbb{P}(m) \right) \]
be the relative spectrum. Since $R \cong \Gamma(\bigoplus_{m \geq 0} \mathcal{O}_\mathbb{P}(m))$, one obtains the map
\[ Y \to \text{Spec } R \cong \mathbb{A}^{N+1} \]
which is a weighted blow-up of the origin (with weights $a_0, \ldots, a_N$). Removing the origin, one gets the isomorphism
\[ U \cong \text{Spec}_\mathbb{P} \left( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_\mathbb{P}(m) \right). \]

Consider the fibered product
\[ C^*_X \cong X \times_\mathbb{P} U. \]
Taking into account that $\mathcal{O}_\mathbb{P}(m)|_X = \mathcal{O}_X(m)$ by definition, we obtain an isomorphism
\[ C^*_X \cong \text{Spec}_X \left( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m) \right), \]
and the assertion of the lemma follows. \hfill \square

For a subvariety $X \subset \mathbb{P}(a_0, \ldots, a_N)$, we define the Poincaré series
\[ P_X(t) = \sum_{m \geq 0} h^0(X, \mathcal{O}_X(m)) t^m. \]

The proof of the following fact was kindly communicated to us by T. Sano.

Proposition 3.2 (see [6, Theorems 3.4.4, 3.2.4(iii); 25, Lemma 2.4]). Let $X \subset \mathbb{P}(a_0, \ldots, a_N)$ be a weighted complete intersection of multidegree $(d_1, \ldots, d_k)$. Then
\begin{enumerate}
  \item one has
  \[ P_X(t) = \prod_{j=1}^k \frac{(1 - t^{d_j})}{(1 - t^{a_j})}; \]
  \item one has $H^i(X, \mathcal{O}_X(m)) = 0$ for all $m \in \mathbb{Z}$ and all $0 < i < \dim X$.
\end{enumerate}

Proof. Denote the graded polynomial ring $k[x_0, \ldots, x_N]$ by $R$, denote the weighted homogeneous ideal $(f_1, \ldots, f_k)$ that defines $X$ by $I$, and put $S = R/I$, so that $X \cong \text{Proj } S$. Since the sequence $f_1, \ldots, f_k$ is regular, it easily follows that
\[ \sum_{m \geq 0} \dim(S_m) t^m = \prod_{j=1}^k \frac{(1 - t^{d_j})}{(1 - t^{a_j})}, \]  
where $S_m$ is the $m$th graded component of $S$. 

Denote the affine cone
\[ \text{Spec } S \subseteq \text{Spec } R \cong \mathbb{A}^{N+1} \]
over \( X \) by \( C_X \). By construction, one has an isomorphism of graded algebras
\[ S \cong H^0(C_X, \mathcal{O}_{C_X}). \quad (3.2) \]

Following Lemma 3.1, denote by \( C_X^* \) the cone \( C_X \) with its vertex \( P \) removed. Consider the local cohomology groups \( H^*_P(C_X, \mathcal{O}_{C_X}) \) (see, for instance, [13, p. 2, definition]). We have the exact sequence
\[ \ldots \to H^i_P(C_X, \mathcal{O}_{C_X}) \to H^i(C_X, \mathcal{O}_{C_X}) \to H^i(C_X^*, \mathcal{O}_{C_X^*}) \to H^{i+1}_P(C_X, \mathcal{O}_{C_X}) \to \ldots \]
(see [13, Corollary 1.9]). Since \( C_X \) is a complete intersection in the affine space, it is a Cohen–Macaulay variety. Therefore, by [13, Proposition 3.7, Theorem 3.8] one has
\[ H^i_P(C_X, \mathcal{O}_{C_X}) = 0 \]
for all \( i < \dim C_X = \dim X + 1 \). Hence, we obtain an isomorphism
\[ H^i(C_X, \mathcal{O}_{C_X}) \cong H^i(C_X^*, \mathcal{O}_{C_X^*}) \quad (3.3) \]
for all \( i < \dim X \).

Finally, denote the natural projection \( C_X^* \to X \) by \( p \). Then
\[ H^i(C_X^*, \mathcal{O}_{C_X^*}) \cong H^i(X, p_*\mathcal{O}_{C_X^*}) \quad (3.4) \]
for all \( i \). On the other hand, by Lemma 3.1 we have
\[ p_*(\mathcal{O}_{C_X^*}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m). \quad (3.5) \]

For \( i = 0 \), we combine the isomorphisms (3.2)–(3.5) to obtain an isomorphism of graded algebras
\[ S \cong \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m)). \]
This together with (3.1) gives assertion (i).

For \( 0 < i < \dim X \), we combine the isomorphisms (3.3)–(3.5) to obtain an isomorphism
\[ H^i(C_X, \mathcal{O}_{C_X}) \cong \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{O}_X(m)). \]
Since \( C_X \) is an affine variety, we have \( H^i(C_X, \mathcal{O}_{C_X}) = 0 \) for all \( i > 0 \) (see, for instance, [14, Ch. III, Theorem 3.5]). This proves assertion (ii). □

Proposition 3.2(i) implies the following property, which can be viewed as an analog of linear normality for ordinary complete intersections.

**Corollary 3.3.** Let \( X \subseteq \mathbb{P} \) be a quasismooth well-formed weighted complete intersection. Then the restriction map
\[ H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(m)) \to H^0(X, \mathcal{O}_X(m)) \]
is surjective for every \( m \in \mathbb{Z} \).

**Proof.** The dimension of the image of the restriction map is computed by the coefficient in the Poincaré series (3.1). On the other hand, by Proposition 3.2(i) this coefficient also equals the dimension of \( H^0(X, \mathcal{O}_X(m)) \). □
To proceed, we will need the following elementary observation.

**Lemma 3.4** (see [15, Lemma 18.3]). Let $N$ and $N'$ be positive integers, and let $k$ and $k'$ be nonnegative integers. Let $a_0 \leq \ldots \leq a_N$, $a'_0 \leq \ldots \leq a'_N$, $d_1 \leq \ldots \leq d_k$, and $d'_1 \leq \ldots \leq d'_k$ be positive integers. Suppose that

$$
\prod_{j=1}^k (1 - t^{d_j}) = \prod_{j'=1}^{k'} (1 - t^{d'_j})
$$

as rational functions of the variable $t$. Suppose that $a_i \neq d_j$ for all $i$ and $j$ and $a'_i \neq d'_j$, for all $i'$ and $j'$. Then $N = N'$, $k = k'$, $a_i = a'_i$ for every $0 \leq i \leq N$, and $d_j = d'_j$ for every $1 \leq j \leq k$.

**Proof.** Note that the numerators and denominators of the rational functions on the left- and the right-hand sides of (3.6) may have common divisors (for instance, if some $d_j$ is divisible by some $a_i$, or the other way around). To prove the assertion, we will keep track of the numbers that are roots of either the numerator or the denominator, but not both of them.

Observe that equality (3.6) is equivalent to the equality obtained from (3.6) by interchanging the sets $\{a_i\}$ and $\{a'_i\}$ with $\{d_j\}$ and $\{d'_j\}$, respectively. Thus we may assume that $d_k$ is the maximum number among $a_N$, $a'_N$, $d_k$, and $d'_k$. By assumption we know that $a_N < d_k$. Let $\zeta$ be a primitive $d_k$th root of unity. Then $\zeta$ is a root of the numerator of the left-hand side of (3.6) but not the root of its denominator. Hence $\zeta$ is a root of the numerator $\nu(t)$ of the right-hand side of (3.6) as well. Since $d'_j \leq d_k$ for all $j'$, we see that $\nu(t)$ is divisible by $1 - t^{d_k}$. Canceling the factor $1 - t^{d_k}$ in (3.6), we complete the proof of the lemma by induction. □

Now we prove the main result of this section.

**Proof of Proposition 1.5.** By Lemma 2.4, the group $\text{Cl}(X)$ is generated by the class of the line bundle $\mathcal{O}_X(1)$, while the group $\text{Cl}(X')$ is generated by the class of the line bundle $\mathcal{O}_{X'}(1)$. Therefore, we see from Proposition 3.2(i) that

$$
\prod_{i=0}^N (1 - t^{a_i}) = P_X(t) = P_{X'}(t) = \frac{\prod_{j'=1}^{k'} (1 - t^{d'_j})}{\prod_{i'=0}^{N'} (1 - t^{a'_{i'}})}.
$$

Since neither $X$ nor $X'$ is an intersection with a linear cone, the required assertion follows from Lemma 3.4. □

**Remark 3.5.** The assertion of Proposition 1.5 also holds for smooth Fano weighted complete intersections of dimension 2. This follows from their explicit classification (see Lemma 2.6). Note however that the assertion fails in dimension 1. Indeed, a conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ (which can be viewed as a complete intersection of codimension 0 in itself).

**Remark 3.6.** We point out that the assumption that $X$ is a Fano variety is essential for the validity of Proposition 1.5 in dimension 2. For instance, there exist smooth quartics in $\mathbb{P}^3$ that also have a structure of a double cover of $\mathbb{P}^2$ branched in a sextic curve (see, e.g., [20, proof of Theorem 4]). Similarly, by Remark 2.11 the assertion of Proposition 1.5 fails for elliptic curves.

We do not know whether the assertion of Proposition 1.5 holds in dimension 2 in the case when $X$ and $X'$ are quasismooth del Pezzo surfaces. We point out that the assumption that $X$ alone is quasismooth is not enough for this. Indeed, the weighted projective plane $\mathbb{P}(1, 1, 2)$ (which can be viewed as a quasismooth well-formed weighted complete intersection of codimension 0 in itself) can be embedded as a quadratic cone in $\mathbb{P}^3$ (which is not quasismooth).

In the proof of Theorem 1.3, we will need a classification of weighted complete intersections of large Fano index.
Proposition 3.7 (cf. [28, Theorem 2.7]). Let $X \subset \mathbb{P}$ be a smooth well-formed Fano weighted complete intersection of dimension $n \geq 2$. Then

(i) one has $i_X \leq n + 1$;
(ii) if $i_X = n + 1$, then $X \cong \mathbb{P}^n$;
(iii) if $i_X = n$, then $X$ is isomorphic to a quadric in $\mathbb{P}^{n+1}$;
(iv) if $i_X = n-1$ and $n \geq 3$, then $X$ is isomorphic either to a hypersurface of degree 6 in $\mathbb{P} = \mathbb{P}(1^n, 2, 3)$, or to a hypersurface of degree 4 in $\mathbb{P} = \mathbb{P}(1^{n+1}, 2)$, or to a cubic hypersurface in $\mathbb{P} = \mathbb{P}^{n+1}$, or to an intersection of two quadrics in $\mathbb{P} = \mathbb{P}^{n+2}$.

Proof. Recall that $i_X$ equals the Fano index of $X$ by Corollary 2.8. By [16, Corollary 3.1.15], we know that $i_X \leq n + 1$; if $i_X = n + 1$, then $X$ is isomorphic to $\mathbb{P}^n$, and if $i_X = n$, then $X$ is isomorphic to a quadric in $\mathbb{P}^{n+1}$. This proves assertions (i)–(iii).

Now suppose that $i_X = n-1$ and $n \geq 3$. Note that $\text{Pic}(X) \cong \mathbb{Z}$ by Lemma 2.4. Thus it follows from the classification of smooth Fano varieties of Fano index $n-1$ (see [9–11] or [16, Theorem 3.2.5]) that $X$ is isomorphic either to one of the weighted complete intersections listed in assertion (iv) or to a linear section of the Grassmannian $\text{Gr}(2, 5)$ in its Plücker embedding.

It remains to show that a weighted complete intersection $X$ cannot be isomorphic to a linear section of the Grassmannian $\text{Gr}(2, 5)$. Suppose that $X$ is isomorphic to such a variety. Then $n \leq 6$. If $n = 3$, then the Fano index of $X$ is 2 and its anticanonical degree is 40. If $n = 4$, then the Fano index of $X$ is 3 and its anticanonical degree is 405. In both cases we see from Remark 2.3 that there exists a smooth weighted complete intersection $X$ with the same $n$ and $i_X$ that is not an intersection with a linear cone. This is impossible according to the classification of smooth Fano weighted complete intersections of dimensions 3 and 4 (see [28, Table 2] and [27, Table 1], respectively; cf. also [16, §12]).

Therefore, we see that $5 \leq n \leq 6$. Recall that

$$H^4(\text{Gr}(2, 5), \mathbb{Z}) \cong \mathbb{Z}^2.$$ 

Thus, the Lefschetz hyperplane section theorem implies that

$$H^4(X, \mathbb{Z}) \cong \mathbb{Z}^2.$$ 

On the other hand, since $X$ is a weighted complete intersection of dimension greater than 4, one has $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ by the Lefschetz-type theorem for complete intersections in toric varieties (see [19, Proposition 1.4]). The obtained contradiction completes the proof of assertion (iv). 

Proposition 1.5 allows us to prove more precise classification results concerning Fano weighted complete intersections (which we will not use directly in our further proofs).

Corollary 3.8. Let $X \subset \mathbb{P}$ be a smooth well-formed Fano weighted complete intersection of dimension $n \geq 2$ that is not an intersection with a linear cone. In this case,

(i) if $i_X = n + 1$, then $X = \mathbb{P} = \mathbb{P}^n$;
(ii) if $i_X = n$, then $X$ is a quadric in $\mathbb{P} = \mathbb{P}^{n+1}$;
(iii) if $i_X = n - 1$, then $X$ is either a hypersurface of degree 6 in $\mathbb{P} = \mathbb{P}(1^n, 2, 3)$, or a hypersurface of degree 4 in $\mathbb{P} = \mathbb{P}(1^{n+1}, 2)$, or a cubic hypersurface in $\mathbb{P} = \mathbb{P}^{n+1}$, or an intersection of two quadrics in $\mathbb{P} = \mathbb{P}^{n+2}$.

Proof. Assertions (i) and (ii) follow from assertions (ii) and (iii) of Proposition 3.7, respectively, combined with Proposition 1.5. If $n = 2$, then assertion (iii) follows from Lemma 2.6. If $n \geq 3$, then assertion (iii) follows from Proposition 3.7(iv) and Proposition 1.5. 


Remark 3.9. An alternative way to prove Corollary 3.8 (which in turn can be used to deduce Proposition 3.7) is by induction on dimension using the classification of smooth well-formed Fano weighted complete intersections of low dimension (say, the one provided by Lemma 2.6) together with [25, Theorem 1.2].

4. AUTOMORPHISMS

In this section we prove Theorem 1.3.

Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$ be a weighted projective space. For any subvariety $X \subset \mathbb{P}$, we denote by $\text{Aut}(\mathbb{P}; X)$ the stabilizer of $X$ in $\text{Aut}(\mathbb{P})$. We denote by $\text{Aut}_\mathbb{P}(X)$ the image of $\text{Aut}(\mathbb{P}; X)$ under the restriction map to $\text{Aut}(X)$. In other words, the group $\text{Aut}_\mathbb{P}(X)$ consists of the automorphisms of $X$ induced by those of $\mathbb{P}$.

We start with a general result that is well known to experts (see, for instance, [18, Lemma 3.1.2]) and that was pointed out to us by A. Massarenti.

Lemma 4.1. Let $X$ be a normal variety, let $A$ be a very ample Weil divisor on $X$, and let $[A]$ be the class of $A$ in $\text{Cl}(X)$. Denote by $\text{Aut}(X; [A])$ the stabilizer of $[A]$ in $\text{Aut}(X)$. Then $\text{Aut}(X; [A])$ is a linear algebraic group.

Corollary 4.2. Let $X$ be a quasismooth well-formed weighted complete intersection of dimension $n$. Suppose that either $n \geq 3$ or $K_X \neq 0$. Then $\text{Aut}(X)$ is a linear algebraic group.

Proof. Note that the divisor class $K_X$ is $\text{Aut}(X)$-invariant. Moreover, if $K_X \neq 0$, then either $K_X$ or $-K_X$ is ample by Theorem 2.5. On the other hand, if $n \geq 3$, then $\text{Cl}(X) \cong \mathbb{Z}$ by Lemma 2.4, so that an ample generator of $\text{Cl}(X)$ is $\text{Aut}(X)$-invariant. In both cases we see that $\text{Aut}(X)$ preserves some ample (and thus also some very ample) divisor class on $X$. Hence $\text{Aut}(X)$ is a linear algebraic group by Lemma 4.1. \hfill \Box

The following lemma will not be used in the proof of Theorem 1.3 but will allow us to prove a (weaker) analog of it that applies to a slightly wider class of smooth weighted complete intersections (see Corollary 4.7(ii) below).

Lemma 4.3. Let $X$ be a subvariety of $\mathbb{P}$. Then $\text{Aut}_\mathbb{P}(X)$ is a linear algebraic group.

Proof. The group $\text{Aut}(\mathbb{P})$ is obviously a linear algebraic group. The stabilizer $\text{Aut}(\mathbb{P}; X)$ of $X$ in $\text{Aut}(\mathbb{P})$ and the kernel $\text{Aut}(\mathbb{P}; X)_{\text{id}}$ of its action on $X$ are cut out in $\text{Aut}(\mathbb{P})$ by algebraic equations, so they are linear algebraic groups. The group $\text{Aut}(\mathbb{P}; X)_{\text{id}}$ is a normal subgroup of $\text{Aut}(\mathbb{P}; X)$. Therefore, the group

$$\text{Aut}_\mathbb{P}(X) \cong \text{Aut}(\mathbb{P}; X)/\text{Aut}(\mathbb{P}; X)_{\text{id}}$$

is a linear algebraic group as well (see [3, Theorem 6.8]). \hfill \Box

Corollary 4.4. Let $X$ be a smooth irreducible subvariety of $\mathbb{P}$. Suppose that $K_X$ is numerically effective. Then the group $\text{Aut}_\mathbb{P}(X)$ is finite.

Proof. By Lemma 4.3, the group $\text{Aut}_\mathbb{P}(X)$ is a linear algebraic group. Therefore, if $\text{Aut}_\mathbb{P}(X)$ is infinite, then it contains a subgroup isomorphic either to $\mathbb{K}^\times$ or to $\mathbb{K}^+$, which implies that $X$ is covered by rational curves. On the other hand, since $K_X$ is numerically effective, $X$ cannot be covered by rational curves (see [21, Theorem 1]). \hfill \Box

The main tool we use in the proof of Theorem 1.3 is the following result from [8].

Theorem 4.5 (see [8, Theorem 8.11(c)]). Let $X$ be a smooth weighted complete intersection of dimension $n \geq 2$. Then

$$H^n(X, \Omega^1_X \otimes \mathcal{O}_X(-i)) = 0$$

for every integer $i \leq n - 2$. 

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Remark 4.6. In fact, the assertion of [8, Theorem 8.11(c)] gives more vanishing results and holds under the weaker assumption that \( X \) is quasismooth. However, we do not want to go into details of the definition of the sheaf \( \Omega^1_X \) here, and in any case we will need the smoothness of \( X \) at the next step.

Theorem 4.5 allows us to prove the finiteness of various automorphism groups.

Corollary 4.7. Let \( X \) be a smooth well-formed weighted complete intersection of dimension \( n \geq 2 \). Suppose that \( i_X \leq n - 2 \). Then

(i) one has \( H^0(X, T_X) = 0 \);

(ii) the group \( \text{Aut}_\mathbb{F}(X) \) is finite;

(iii) if either \( \dim X \geq 3 \) or \( K_X \neq 0 \), then the group \( \text{Aut}(X) \) is finite.

Proof. By Theorem 4.5 we have

\[
H^n(X, \Omega_X^1 \otimes \mathcal{O}_X^{-i_X}) = 0.
\]

Recall that \( \omega_X = \mathcal{O}_X(-i_X) \) by Theorem 2.5. Thus assertion (i) follows from the Serre duality. Assertion (ii) follows from assertion (i), because \( \text{Aut}_\mathbb{F}(X) \) is a linear algebraic group by Lemma 4.3. Similarly, assertion (iii) follows from assertion (i), because the automorphism group of any variety subject to the above assumptions is a linear algebraic group by Corollary 4.2. \( \square \)

Recall that the smooth Fano threefold \( V_5 \) of Fano index \( \dim V_5 - 1 = 2 \) is defined as an intersection of the Grassmannian \( \text{Gr}(2, 5) \subset \mathbb{P}^9 \) in its Plücker embedding with a linear section of codimension 3 has an infinite automorphism group \( \text{Aut}(V_5) \cong \text{PGL}_2(k) \) (see [23, Proposition 4.4] or [4, Proposition 7.1.10]). The next lemma shows that such a situation is impossible for smooth weighted complete intersections.

Lemma 4.8. Let \( X \subset \mathbb{P} \) be a smooth well-formed weighted complete intersection of dimension \( n \geq 2 \). Suppose that \( i_X = n - 1 \). Then the group \( \text{Aut}(X) \) is finite.

Proof. If \( n = 2 \), the assertion follows from Remark 2.7 and the properties of the automorphism groups of smooth del Pezzo surfaces (see, for instance, [7, Corollary 8.2.40]). Thus, we assume that \( n \geq 3 \) and use the classification provided by Proposition 3.7(iv). If \( X \) is isomorphic to an intersection of two quadrics in \( \mathbb{P} = \mathbb{P}^{n+2} \) or to a cubic hypersurface in \( \mathbb{P} = \mathbb{P}^{n+1} \), then the assertion follows from Theorem 1.2 (in the latter case one can also use Theorem 1.1).

Now suppose that \( X \) is isomorphic either to a hypersurface of degree 4 in \( \mathbb{P} = \mathbb{P}(1^n+1,2) \) or to a hypersurface of degree 6 in \( \mathbb{P} = \mathbb{P}(1^n,2,3) \). The argument in these cases is similar to that in the proof of [18, Lemma 4.4.1]. Denote by \( H \) the ample divisor such that \(-K_X \sim (n-1)H \). Then there exists an \( \text{Aut}(X) \)-equivariant double cover \( \phi : X \to Y \), where in the former case \( Y \cong \mathbb{P}^n \) and \( \phi \) is given by the linear system \( |H| \), while in the latter case \( Y \cong \mathbb{P}(1^n,2) \) and \( \phi \) is given by the linear system \( |2H| \). Let \( H' \) be the ample Weil divisor generating the group \( \text{Cl}(Y) \cong \mathbb{Z} \), and let \( B \subset Y \) be the branch divisor of \( \phi \). In the former case one has \( B \sim 4H' \), and in the latter case one has \( B \sim 6H' \). Note that in the latter case \( H' \) is not a Cartier divisor, but \( 2H' \) is; note also that in this case \( \phi \) is branched over the singular point of \( Y \) as well. In both cases \( B \) is smooth. Furthermore, it follows from the adjunction formula that either \( K_B \) is ample, or \( K_B \sim 0 \), or \( B \) is a (smooth well-formed) Fano weighted hypersurface of dimension \( n - 1 \geq 3 \) and Fano index \( i_B \leq n - 3 \).

Since the double cover \( \phi \) is \( \text{Aut}(X) \)-equivariant, we see that the quotient of the group \( \text{Aut}(X) \) by its normal subgroup of order 2 generated by the Galois involution of \( \phi \) is isomorphic to a subgroup of the stabilizer \( \text{Aut}(Y;B) \) of \( B \) in \( \text{Aut}(Y) \). Since \( B \) is not contained in any divisor linearly equivalent to the very ample divisor \( 2H' \), we conclude that \( \text{Aut}(Y;B) \) acts faithfully on \( B \) (see, for instance, [26, Lemma 2.1]). Hence

\[
\text{Aut}(Y;B) \cong \text{Aut}_Y(B).
\]
On the other hand, the group $\text{Aut}_Y(B)$ is finite by Corollary 4.7(ii); alternatively, one can apply Corollaries 4.4 and 4.7(iii). This means that the group $\text{Aut}(X)$ is finite as well. □

Now we prove our main results.

**Proof of Theorem 1.3.** First suppose that $n = 1$. We can assume that $K_X$ is ample. In this case the finiteness of $\text{Aut}(X)$ is well known (see, for instance, [14, Ch. IV, Exercise 5.2]).

Now suppose that $n \geq 2$. If $i_X \leq n - 2$, then the group $\text{Aut}(X)$ is finite by Corollary 4.7(iii). If $i_X = n - 1$, then the group $\text{Aut}(X)$ is finite by Lemma 4.8. Finally, if $i_X \geq n$, then we know from Proposition 3.7 that $X$ is isomorphic either to $\mathbb{P}^n$ or to a quadric hypersurface in $\mathbb{P}^{n+1}$. □

Corollary 1.4 immediately follows from Theorem 1.3 and Proposition 1.5.

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