Annealed scaling relations for Voronoi percolation

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Abstract

We prove annealed scaling relations for planar Voronoi percolation. To our knowledge, this is the first result of this kind for a continuum percolation model. We are mostly inspired by the proof of scaling relations for Bernoulli percolation by Kesten [Kes87]. Along the way, we show an annealed quasi-multiplicativity property by relying on the quenched box-crossing property proved by Ahlberg, Griffiths, Morris and Tassion [AGMT16]. Intermediate results also include the study of quenched and annealed notions of pivotal events and the extension of the quenched box-crossing property of [AGMT16] to the near-critical regime.

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1 The model and the main result
1.1 Percolation on planar lattices

Consider bond percolation on the square lattice $\mathbb{Z}^2$ or site percolation on the planar triangular lattice $\mathbb{T}$. In these models, each edge or site is open (respectively closed) with probability $p$ (respectively $1 - p$) independently of the others. Let $\theta(p)$ be the probability that there is an infinite open path starting from 0. It is well known (see for instance [Gri99, BR06b]) that there exists a critical point $p_c \in (0,1)$ such that:

i) $\forall p \in [0, p_c), \theta(p) = 0$,
ii) $\forall p \in (p_c, 1], \theta(p) > 0$.

It is a theorem by Kesten [Kes80] that $p_c = 1/2$ for these two models. Moreover, it has been proved by Harris [Har60] that $\theta(1/2) = 0$.

Let us say a little more about the behaviour of this model at and near the critical point: Thanks to the Russo-Seymour-Welsh (RSW) theory and the study of interfaces between open and dual paths, one can obtain the so-called quasi-multiplicativity property of arm events and derive estimates on “pivotal events” (see [Kes87, Wer07, Nol08, SS10, Man12]). Both are important tools in order to:

(a) obtain the scaling relations proved by Kesten (see [Kes87, Wer07, Nol08]),
(b) study dynamical percolation and noise sensitivity of percolation, see [BKS99, SS10, GPS10, BGS13, GS14, HPS15, GV18]
(c) study the scaling limits of percolation, near-critical percolation, and dynamical percolation, see [SS11, GPS13a, GPS13b].

The goal of this paper is twofold: On the one hand, we prove the quasi-multiplicativity property (and some estimates on “pivotal events”) for planar Voronoi percolation, which is a continuum percolation model. The proof of the quasi-multiplicativity property is the most delicate part of the paper, mostly because we have to deal with the spatial dependencies of the model. We also prove some scaling relations for this continuum percolation model. The recent anealed and quenched box-crossing properties for Voronoi percolation proved by Tassion [Tas16] and by Ahlberg-Griffiths-Morris-Tassion [AGMT16] will be crucial for us.

Before recalling the definition of Voronoi percolation, let us note that the authors of [AGMT16] and [AB17] prove noise sensitivity results for Voronoi percolation by following ideas from [BKS99, SS10, ABGM14]. We also see the present paper as a first step to be able to apply the more quantitative noise sensitivity methods from [GPS10]. Indeed, to apply methods from [GPS10], one needs to have good controls on the probabilities of arm and pivotal events.
1.2 Planar Voronoi percolation: box-crossing estimates and the quasi-multiplicativity property

In this subsection, we introduce the model of Voronoi percolation. We refer to Section 8.3 of [BR06b] for more details.

A. Voronoi percolation. The model of (planar) Voronoi percolation is defined as follows: Let \( \eta \) be a (homogeneous) Poisson process of intensity 1 in \( \mathbb{R}^2 \) and construct its Voronoi tiling as follows: for each point \( x \in \eta \), the Voronoi cell of \( x \), denoted by \( C(x) \), is the set of all points \( u \in \mathbb{R}^2 \) such that for all \( x' \in \eta \), \( \| u - x \|_2 \leq \| u - x' \|_2 \). We say that \( x \) is the center of \( C(x) \). Also, we say that two points of \( \eta \) are adjacent if their cells intersect each other. It is not difficult to see that a.s. all the cells are bounded convex polygons.

Now, consider some parameter \( p \in [0,1] \) and, given \( \eta \), declare each \( x \in \eta \) open (we will choose to say that we “color the point \( \text{black} \)” with probability \( p \) and closed (\( \text{white} \)) with probability \( 1 - p \), independently of the other points of \( \eta \). The coloured configuration we obtain is denoted by \( \omega \in \{-1,1\}^\eta \) (where 1 means black and -1 means white).\(^1\)

We will always write \( \eta \) for the point process and \( \omega \) for the coloured point process. The distribution of \( \eta \) will be denoted by \( \mathbb{P} \) and the distribution of \( \omega \) will be denoted by \( \mathbb{P}_p \).

Given the configuration \( \omega \), we define a colouring of the plane as follows: each point \( u \in \mathbb{R}^2 \) is coloured black if it is contained in the cell of a black point \( x \in \eta \) and is coloured white if it is contained in the cell of a white point \( x \in \eta \) (note that the points at the boundary of the cells may be coloured both black and white but this fact will be of no importance). Moreover, we will call black (respectively white) path a continuous path included in the black (respectively white) region of the plane. Note that these paths are continuous objects that do not necessarily live on the lattice induced by the Voronoi tiling.

B. The critical point. Let \( \{0 \leftrightarrow +\infty\} \) be the event that there is a black path from the origin to infinity and let \( \theta^{an}(p) = \mathbb{P}_p[0 \leftrightarrow \infty] \) denote the (annealed) percolation function. The critical point \( p_c \) is defined as follows:

\[
p_c := \inf \{ p \in [0,1] : \theta^{an}(p) > 0 \}.
\]

It has been proved by Zvavitch [Zva96] that \( \theta^{an}(1/2) = 0 \) - hence \( p_c \geq 1/2 \) - and it is a result of Bollobás and Riordan [BR06a] that \( p_c = 1/2 \). A crucial fact for this result is the so-called self-duality property of the model: a.s., a rectangle is crossed lengthwise by a black path if and only if it is not crossed widthwise by a white path (see for instance Lemma 12 in Chapter 8 of [BR06b]). An important step to show the result of Bollobás and Riordan is the proof of a weak box-crossing property. A stronger version has more recently been proved by Tassion [Tas16] and has led to the derivation of quenched crossing estimates in [AGMT16] that will be crucial in the present paper. Before stating these box-crossing results, let us note that an alternative proof of \( p_c = 1/2 \) can be found in the recent paper [DCRT17]. In the said article, Duminil-Copin, Raoufi and Tassion prove the exponential decay of connection probabilities for subcritical Voronoi percolation in any dimension.

C. Box-crossing properties. We first need two definitions/notations:

**Definition 1.1.** Given \( \eta \), we will write \( P_p^\eta \) for the conditional distribution of \( \omega \) given \( \eta \) (which is simply the product law \( (p\delta_1 + (1 - p)\delta_{-1})^\otimes \)). More generally, if \( E \) is a countable set, we will write \( P_p^E = (p\delta_1 + (1 - p)\delta_{-1})^\otimes \).

**Definition 1.2.** For any \( \rho_1, \rho_2 > 0 \), \( \text{Cross}(\rho_1, \rho_2) \) (respectively \( \text{Cross}^*(\rho_1, \rho_2) \)) will denote the event that there is a black (respectively a white) path included in \([-\rho_1, \rho_1] \times [-\rho_2, \rho_2] \) that connects the left side of this rectangle to its right side.

\(^1\)There is no problem of measurability here: \( \omega \) can be seen for instance as a point process with values in \( \mathbb{R}^2 \times \{-1,1\} \) whose intensity is \( \text{Leb}_{\mathbb{R}^2} \otimes (p\delta_1 + (1 - p)\delta_{-1}) \), where \( \text{Leb}_{\mathbb{R}^2} \) is the Lebesgue measure on the plane.
Now, we can state the annealed box-crossing property obtained by Tassion and the quenched box-crossing property obtained by Ahlberg, Griffiths, Morris and Tassion. An important step in the present paper will be the extension of these results to the “near-critical regime”, see Subsection 5.1.

**Theorem 1.3** (Theorem 3 of [Tas16]). Let $\rho > 0$. There exists a constant $C = C(\rho) \in (0,1)$ such that, for every $R \in (0, +\infty)$:

$$c \leq \mathbb{P}_{1/2}[\text{Cross}(\rho R, R)] \leq 1 - c.$$  

**Theorem 1.4** (Theorem 1.4 of [AGMT16] and the paragraph below it. See also our Appendix B where we recall the main ingredients of the proof of this theorem.\(^2\)). Let $\rho > 0$.

i) There exists an absolute constant $C = C(\rho) < +\infty$ such that, for every $R \in (0, +\infty)$:

$$\mathbb{Var}\left(\mathbb{P}_{1/2}[\text{Cross}(\rho R, R)]\right) \leq C R^{-\epsilon}.$$  

This implies the following estimate:

ii) For every $\gamma \in (0, +\infty)$, there exists a positive constant $c = c(\rho, \gamma) \in (0,1)$ such that, for every $R \in (0, +\infty)$:

$$\mathbb{P}\left[c \leq \mathbb{P}_{1/2}[\text{Cross}(\rho R, R)] \leq 1 - c\right] \geq 1 - R^{-\gamma}.$$  

**D. Arm events.** Once we have such crossing properties, a natural goal is to study arm events. Here, we define these events:

**Definition 1.5** ($j$-arm events). Let $j \in \mathbb{N}^*$ and $0 \leq r \leq R$. The $j$-arm event between scales $r$ and $R$ is the event that there exist $j$ paths of alternating colors in the annulus $[-R, R]^2 \setminus [-r, r]^2$ from $\partial[-r, r]^2$ to $\partial[-R, R]^2$ (if $j$ is odd, we ask that there are: (a) $j - 1$ paths of alternating color, and: (b) one additional black path such that there is no Voronoi cell intersected by both this additional path and one of the $j - 1$ other paths). Let $A_j(r, R)$ denote this event. We write as follows the **annealed** probability of this event:

$$\alpha_{j, \rho}^{an}(r, R) := \mathbb{P}_\rho[A_j(r, R)].$$

We write $\alpha_{j}^{an}(r, R) := \alpha_{j, \rho}^{an}(1, R)$ for any $j \in \mathbb{N}^*$. If $r > R$, we choose that $\alpha_{j, \rho}^{an}(r, R) = 1$. Also, we will often use the following simplified notation:

$$\alpha_{j}^{an}(r, R) := \alpha_{j, 1/2}^{an}(r, R).$$

An important property of the quantities $\alpha_{j}^{an}(r, R)$ is that they decay polynomially fast: There exists a constant $C = C(j) \in [1, +\infty)$ such that, for every $1 \leq r \leq R$:

$$\frac{1}{C} \left(\frac{r}{R}\right)^C \leq \alpha_{j, 1/2}^{an}(r, R) \leq C \left(\frac{r}{R}\right)^{1/C}.$$  

(1.1)

The right-hand-inequality is proved in [Tas16] (Item 2 of Theorem 3) and we prove the left-hand-inequality in Subsection 3.1. In the present paper, we prove the **annealed quasi-multiplicativity property** for the quantities $\alpha_{j, 1/2}^{an}(r, R)$. This is the most delicate part of the paper. Even if this is an annealed result, the quenched box-crossing property Theorem 1.4 will be a very crucial ingredient in the proof.

**Proposition 1.6** (Annealed quasi-multiplicativity property). Let $j \in \mathbb{N}^*$. There exists a constant $C = C(j) \in [1, +\infty)$ such that, for all $1 \leq r_1 \leq r_2 \leq r_3$:

$$\frac{1}{C} \alpha_{j, 1/2}^{an}(r_1, r_3) \leq \alpha_{j, 1/2}^{an}(r_1, r_2) \alpha_{j, 1/2}^{an}(r_2, r_3) \leq C \alpha_{j, 1/2}^{an}(r_1, r_3).$$  

(1.2)

\(^2\)Actually, in Appendix B we will modify a little the proof of [AGMT16] so that this proof will be easier to adapt to the near-critical phase.
Remark 1.7. In Proposition 1.6, the case $j = 1$ is easier. More precisely, the right-hand inequality is a direct consequence of the box-crossing property Theorem 1.3 and the (annealed) FKG-Harris inequality (stated in Subsection 2.2), and the proof of the left-hand-inequality is written in Subsection 3.1.

Remark 1.8. Our choice to impose that the radii $r_i$ are at least 1 is arbitrary. In fact, we could have chosen any $a > 0$ and rather asked that $a \leq r_1 \leq r_2 \leq r_3$. We would have obtained the same result with some constant $C = C(j, a)$.

Figure 1: In this paper, we deal with degenerate events: the $j$-arm events $A_{j}(r, R)$. It is not clear that, when conditioning on $A_{j}(r, R)$ with $r \ll R$, the random environment at scale $r$ is not typically degenerate. An example of a degenerate environment is illustrated in the figure: the Voronoi tiling is extremely dense in some regions and not dense at all in some other regions. In the region where the Voronoi tiling is extremely dense, it might be very costly to extend the arms to other scales. The biggest issue comes from the regions where the Voronoi tiling is not dense at all: in these regions, there are a lot of spatial dependences, which could even imply interactions between scale $r$ and some other scales.

The main difficulty in the study of arm events (compared box crossing events for instance) is that they are degenerate events. As a result, it could a priori be that, if $r \ll R$ and if we condition on $A_{j}(r, R)$, then with high probability the point process $\eta$ is very degenerate at scale $r$, see Figure 1. We refer to Subsection 2.3 for some key properties and some tools developed to overcome this difficulty (see in particular Propositions 2.4 and 2.5).

Let us now state the main result of our paper.

1.3 The main result: annealed scaling relations for Voronoi percolation

It is believed that, for a wide class of percolation models, the evolutions as $p$ goes to $p_c = 1/2$ of some key quantities are determined by the some universal critical exponents. Such quantities are for instance the percolation function, the correlation length and the probabilities of arm events. The famous scaling relations proved by Kesten [Kes87] are simple relations between these exponents. More precisely, Kesten proved that, for bond percolation on the
square lattice (and site percolation on the triangular lattice): i) if we assume that these key quantities are indeed described by exponents, then these exponents satisfy the relations predicted by theoretical physicists in the 70’s (we refer to [Kes87] for precise references concerning these predictions), and ii) even if we do not assume that these exponents exist, the corresponding relations between the percolation function, the correlation length etc hold. There is only one planar percolation model for which it is known that such exponents exist: site percolation on the triangular lattice, which is the only model for which conformal invariance has been proved, see the proof by Smirnov [Smi01]. These exponents have even been computed thanks to the theory of SLE’s (Schramm Loewner Evolution) of Schramm, see [SW01, LSW02, Wer07].

Let us go back to Voronoi percolation. For this model, the existence of these exponents is not known (conformal invariance is not proved for this model even if a first important step has been made in this direction by Benjamini and Schramm [BS98]). To state our main result, let us define the annealed correlation length.

**Definition 1.9.** Let $\epsilon_0 \in (0, 1)$ be sufficiently small$^3$ and let $p \in (1/2, 1]$. The **annealed correlation length** at parameter $p$, denoted by $L^{an}(p) = L^{an,\epsilon_0}(p)$, is defined as follows:

$$L^{an}(p) = \inf \{ R \geq 1 : \mathbb{P}_p[\text{Cross}(2R, R)] \geq 1 - \epsilon_0 \}.$$  

An important property is that, for every $p > 1/2$, $L^{an}(p) < +\infty$, see Lemma 5.1. The idea behind the definition of the correlation length is that this is the larger scale such that the percolation configuration at this scale “looks critical”. See Subsection ?? for the statement of results in this spirit. In particular, we are going to prove that the annealed and quenched box-crossing properties Theorems 1.3 and 1.4 are also true for $p > 1/2$ as soon as we work under the correlation length (i.e., as soon as we work in the “near-critical phase”). In Section 6, we will prove the following result:

**Proposition 1.10.** Let $p \in (1/2, 3/4]$ and let $\epsilon_0$ be the parameter of Definition 1.9. Also, let $j \in \mathbb{N}^*$. There exists a constant $C = C(\epsilon_0, j) \in [1, +\infty)$ such that, for every $1 \leq r \leq R \leq L^{an}(p)$:

$$\frac{1}{C} \alpha_{j,1/2}^{an}(r, R) \leq \alpha_{j,p}^{an}(r, R) \leq C \alpha_{j,1/2}^{an}(r, R).$$

In the present paper, we focus on the following exponents: It is believed that there exist $\nu \in (0, +\infty)$, $\beta \in (0, +\infty)$ and $\zeta_j \in (0, +\infty)$ such that:

$$\forall p \in (1/2, 1), \theta^{an}(p) = (p - 1/2)^{\beta + o(1)},$$

$$\forall p \in (1/2, 1), \nu^{an}(p) = (p - 1/2)^{-\nu + o(1)},$$

$$\forall j \in \mathbb{N}^* \text{ and } \forall 1 \leq r \leq R, \alpha^{an}_{j,1/2}(r, R) = \left(r \frac{\epsilon_j}{R} \right)^{-\zeta_j + o(1)}.$$

where $o(1)$ goes to 0 as $p$ goes to 1/2 (respectively as $r/R$ goes to 0). Moreover, it is believed that the following relations hold between these exponents:

$$\beta = \nu \zeta_1;$$

$$\nu = \frac{1}{2 - \zeta_4}.$$  

The main results of the present paper is that, if these exponents exist, then these two scaling relations hold. As in [Kes87], we also prove that, even if we do not assume that the exponents exist, then the corresponding relations between the percolation function, the correlation length and the arm events probability hold. More precisely, we obtain the following:

$^3$More precisely, we need that $1 - \epsilon_0 > \mathbb{P}_{1/2}[\text{Cross}(2R, R)]$ for every $R \geq 1$ - which is possible thanks to Theorem 1.3 - and that $\epsilon_0$ is sufficiently small so that a Peierls argument can be used - see the proof of Lemma 6.2 for more about this second condition.

$^4$The number $3/4$ does not have to be taken seriously, we consider $p \in (1/2, 3/4]$ only to avoid problems with $p$ close to 1.
Theorem 1.11. Let $p \in \{1/2, 3/4\}$ and let $\epsilon_0$ be the parameter of Definition 1.9. There exists a constant $C = C(\epsilon_0) \in [1, +\infty)$ such that:

$$\frac{1}{C} \alpha_{1,1/2}^n(L_n^\theta(p)) \leq \theta^n(p) \leq C \alpha_{1,1/2}^n(L_n^\theta(p)),$$

and:

$$\frac{1}{C} \frac{1}{(p - 1/2)} \leq L_n^\theta(p)^2 \alpha_{1,1/2}^n(L_n^\theta(p)) \leq C \frac{1}{p - 1/2}.$$

Proposition 1.10 and Theorem 1.11 are proved in Section 6 by relying on all the other sections.

Remark 1.12. Note that (1.3) (together with the quasi-multiplicativity property and (1.1)) implies that for every $\epsilon_0 > \epsilon_0 > 0$ sufficiently small, there exist $c = c(\epsilon_0, \epsilon_0) > 0$ such that, for every $p \in (1/2; 3/4]$:

$$cL_n^{\epsilon_0}(p) \leq L_n^{\epsilon_0}(p) \leq L_n^{\epsilon_0}(p).$$

In [Kes87], Kesten also proves other scaling relations. We believe that with the results of the present paper, analogues of these other scaling relations can also be proved but we have restricted ourself to the two scaling relations (1.3) and (1.4).

1.4 Estimates on the 4-arm event, $\theta^n(p)$, and $L_n^\theta(p)$

In the present paper, we prove some estimates on arm events. In particular, we will obtain the following estimates on the 4-arm events in Subsections 4.1 and 4.2:

Proposition 1.13. There exists an absolute constant $\epsilon > 0$ such that:

i) For every $R \in [1, +\infty)$:

$$\alpha_{1,1/2}^n(R) \leq \frac{1}{\epsilon} R^{-1+\epsilon}. $$

ii) For every $1 \leq r \leq R$:

$$\alpha_{1,1/2}^n(r, R) \geq \epsilon \left( \frac{r}{R} \right)^{2-\epsilon}. $$

If we apply the first part of Proposition 1.13 to the scaling relation (1.4) of Theorem 1.11, then we obtain that:

$$L_n^\theta(p) \geq \epsilon (p - 1/2)^{-1+\epsilon},$$

for some $\epsilon > 0$. If we rather use the second part of Proposition 1.13, then we obtain that:

$$L_n^\theta(p) \leq C(p - 1/2)^{-C},$$

for some $C < +\infty$. As a result, if the exponent $\nu$ exists, then $\nu \in (1, +\infty)$ (which is exactly - as far as we know - what is known for Bernoulli percolation on $\mathbb{Z}^2$, see [Kes87]). By using the polynomial decay property (1.1) and the scaling relation (1.3) of Theorem 1.11, we deduce that:

$$\epsilon (p - 1/2)^{C} \leq \theta^n(p) \leq C (p - 1/2)^{\epsilon}, $$

for some $C < +\infty$ and $\epsilon > 0$. In [KZ87], Kesten and Zhang proved the following for Bernoulli percolation on $\mathbb{Z}^2$:

$$\theta(p) \geq \epsilon (p - 1/2)^{1-\epsilon}. $$

In the case of Bernoulli percolation on the triangular lattice, it is known (see [LSW02] and [SW01]) that:

$$L(p) = (p - 1/2)^{-4/3+o(1)} $$

and:

$$\theta(p) = (p - 1/2)^{5/36+o(1)},$$

where $o(1) \to 0$ as $p \searrow 1/2$.

The estimate (1.5) is strengthened in the two following papers:
• In [Van18], we prove that $\theta^{an}(p) \geq \epsilon(p - 1/2)^{1-\epsilon}$ (by relying a lot on the present paper and in particular on Appendix D).

• In the recent work [DCRT17] Duminil-Copin, Raoufi and Tassion use the OSSS inequality to prove that, for Voronoi percolation in any dimension $d \geq 2$, there exists $c = c(d) > 0$ such that, for any $p > p_c$:

$$\theta^{an}(p) \geq c(p - p_c).$$

1.5 Quenched or annealed results?

In the present paper, our main goal is to prove annealed properties. The most important ones are the annealed scaling relations from Theorem 1.11 and the annealed quasi-multiplicativity property Proposition 1.6. However, the quenched property Theorem 1.4 will be one of our main tools. The multiple passages from quenched to annealed will be rather technical, see in particular Section 7. This is why it seems at first sight that it would be easier to prove quenched properties. We indeed believe that one could use Theorem 1.4 to prove a quenched quasi-multiplicativity property with a less technical proof than the proof of the annealed quasi-multiplicativity property. However:

• Proving scaling relations at the quenched level seems much more complicated since the classical ideas (that we follow in the present paper) deeply use translation invariance properties.

• In order to prove annealed scaling relations, we need the annealed quasi-multiplicativity Proposition 1.6 and we did not manage to deduce this annealed quasi-multiplicativity from a quenched quasi-multiplicativity property.

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2 Strategy and organization of the paper

2.1 Some notations

Before stating the main intermediate results and explaining the global strategy, let us introduce some notations.

Boxes, annuli and quads. In all the paper, we will write $B_R = [-R, R]^2$ and we will write $A(r, R) = [-R, R]^2 \setminus (-r, r)^2$. Also, for every $y \in \mathbb{R}^2$, we will write $B_y = B_r = y + B_r$ and $A(y; r, R) = y + A(r, R)$.

A quad $Q$ is a topological rectangle in the plane with two distinguished opposite sides. Also, a black (respectively white) path included in $Q$ that joins one distinguished side to the other is called a crossing (respectively dual crossing). The event that $Q$ is crossed (respectively dual crossed) will be written $\text{Cross}(Q)$ (respectively $\text{Cross}^*(Q)$).

Other notations. In all the paper, we will use the following notations: (a) $O(1)$ is a positive bounded function, (b) $\Omega(1)$ is a positive function bounded away from 0 and (c) if $f$ and $g$ are two non-negative functions, then $f \simeq g$ means $\Omega(1)f \leq g \leq O(1)f$.

We will also use the following notations: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\mathcal{G} \subseteq \mathcal{F}$ is a $\sigma$-field, $B$ is some event such that $\mathbb{P}[B] > 0$, and $A$ is some event, then:

$$\mathbb{P}[A \mid B, \mathcal{G}] := \frac{\mathbb{P}[A \cap B \mid \mathcal{G}]}{\mathbb{P}[B \mid \mathcal{G}]} 1_{\{\mathbb{P}[B \mid \mathcal{G}] > 0\}}.$$  

Note that, $\mathbb{P}[\cdot \mid B]$-a.s., we have: $\mathbb{P}[A \mid B, \mathcal{G}]$ is the conditional expectation of $A$ with respect to $\mathcal{G}$ and under $\mathbb{P}[\cdot \mid B]$. 

8
2.2 Correlation inequalities for Voronoi percolation

In this subsection, we recall two very useful families of correlation inequalities: the FKG-Harris inequalities and the BK inequalities, which are inequalities for increasing events. First, let us define what is an increasing event in our context. Since we work in random environment, it is interesting to consider quenched and annealed notions of increasing events.

**Definition 2.1.**

i) First, we recall the classical notion of increasing events: Let $E$ be a countable set. An event $A$ of the product $\sigma$-algebra on $\{-1,1\}^E$ is increasing if for any $\omega, \omega' \in \{-1,1\}^E$ such that $\omega \leq \omega'$ and $\omega \in A$, we have $\omega' \in A$.

ii) An event $A$ measurable with respect to the coloured configuration $\omega$ is quenched-increasing if, for every point configuration of the plane $\eta$, every $\omega \in \{-1,1\}^\eta$, and every $\omega' \in \{-1,1\}^\eta$ such that $\omega \in A$ and $\omega \leq \omega'$, we have $\omega' \in A$.

iii) An event $A$ is annealed-increasing if, for any coloured configuration $\omega \in A$ and any $\omega'$ obtained from $\omega$ by adding black points or deleting white points, we have $\omega' \in A$.

Note that, if $A$ is annealed-increasing, then $A$ is quenched-increasing.

**The FKG-Harris inequalities.**

i) The classical FKG-Harris inequality is the following (see [Gri99, BR06b]): Let $E$ be a countable set. Remember that we write $P^E_p := (p\delta_1 + (1-p)\delta_{-1})^\otimes E$. Let $A$ and $B$ be increasing events. Then, for every $p$ we have:

$$P^E_p [A \cap B] \geq P^E_p [A] \cdot P^E_p [B].$$

ii) In the quenched case, the FKG-Harris inequality is a direct consequence of the above inequality and can be stated as follows: Let $A$ and $B$ be two quenched-increasing events. Then, for every point configuration of the plane $\eta$ and every $p$ we have:

$$P^\eta_p [A \cap B] \geq P^\eta_p [A] \cdot P^\eta_p [B].$$

iii) In the annealed case, we have: Let $A$ and $B$ be two annealed-increasing events. Then, for every $p$:

$$P_p [A \cap B] \geq P_p [A] \cdot P_p [B].$$

See Lemma 14 in Chapter 8 of [BR06b] for the proof of this inequality. (Note that this does not hold in general for quenched-increasing events; indeed if $A$ depends only on $\eta$ and if $\mathbb{P}[A] \in ]0,1[$ then $A$ and $A'$ are quenched-increasing and $0 = \mathbb{P}[A \cap A'] < \mathbb{P}[A] \mathbb{P}[A'].$)

**The BK inequalities.** Let $A$ and $B$ be two quenched increasing events measurable with respect to $\omega$ restricted to a bounded domain. Define the disjoint occurrence of $A$ and $B$ as follows (where, for every coloured configuration $\omega$, we write $\eta(\omega)$ for the underlying (non-coloured) point configuration):

$$A \square B = \{ \omega \in \Omega : \exists I_1, I_2 \text{ finite disjoint subsets of } \eta(\omega), \omega^{I_1} \in A \text{ and } \omega^{I_2} \in B \},$$

where $\Omega$ is the set of all coloured configurations and, if $I \subseteq \eta(\omega)$, $\omega^I \subseteq \{-1,1\}^{\eta(\omega)}$ is the set of all $\omega'$ such that $\omega'_i = \omega_i$ for every $i \in I$.

We will use the following quenched BK inequality which is a direct consequence of the classical BK inequality (see for instance [Gri99] or [BR06b]): For every $\eta$ and every $p$ we have:

$$P^\eta_p [A \square B] \leq P^\eta_p [A] \cdot P^\eta_p [B].$$

Unfortunately, the annealed-version of the BK-inequality is only known for $p = 1/2$ (and it seems actually not clear whether or not it should be true for $p \neq 1/2$). This will be a difficulty when we want to extend some results (for instance Theorem 1.4) to the near-critical phase, see Section 5.
Proposition 2.2 (Lemma 3.4 of [AGMT16],[Joo12] - both refer to van den Berg). Let $A$ and $B$ be two annealed increasing events measurable with respect to the coloured configuration $\omega$ restricted to a bounded domain. Then:
\[
\mathbb{P}_{1/2}[A \sqcap B] \leq \mathbb{P}_{1/2}[A] \cdot \mathbb{P}_{1/2}[B].
\]

2.3 Consequences of the annealed quasi-multiplicativity property

In this subsection, we discuss important consequences of the (annealed) quasi-multiplicativity property Proposition 1.6. As mentioned in Subsection 1.2, Proposition 1.6 is the most technical part of the paper. For this reason, we have chosen to postpone its proof to the final technical part of the paper. For this reason, we have chosen to postpone its proof to the final section: Section 7. We will use this property in most of the other sections of the paper. For more about which section depends on which sections, see the beginning of Subsection 2.5.

Let us now state some results that will be useful all along the paper and which are consequences of the quasi-multiplicativity property (and of intermediate results from Section 7). These results are essentially useful to overcome the spatial dependencies of Voronoi percolation. Note also that a.s. $A_j(r,R) \subseteq \hat{A}_j(r,R)$ (i.e. $\mathbb{P}[A_j(r,R) \setminus \hat{A}_j(r,R)] = 0$). The following result will be proved in Subsection 7.2.

Proposition 2.4. Let $j \in \mathbb{N}^*$, let $1 \leq r \leq R$, and write:
\[
f_j(r,R) = f_{j,1/2}(r,R) := \mathbb{P}_{1/2}[\hat{A}_j(r,R)].
\]
There exists a constant $C = C(j) < +\infty$ such that:
\[
\alpha_{j,1/2}^{an}(r,R) \leq f_j(r,R) \leq C \alpha_{j,1/2}^{an}(r,R).
\]

The following is a consequence of Proposition 2.4 and illustrates how this last proposition can help us to overcome spatial dependency problems.

Proposition 2.5. Let $j \in \mathbb{N}^*$. For every $h \in (0,1)$, there exists a constant $\epsilon = \epsilon(j,h) \in (0,1)$ such that, for every $1 \leq r \leq R$ and for every event $G$ which is measurable with respect to $\omega \setminus A(2r,R/2)$ and satisfies $\mathbb{P}_{1/2}[G] \geq 1 - \epsilon$, we have:
\[
\mathbb{P}_{1/2}[A_j(r,R) \cap G] \geq (1 - h) \alpha_{j,1/2}^{an}(r,R).
\]

Proof. We have:
\[
\mathbb{P}_{1/2}[A_j(r,R) \setminus G] \leq \mathbb{P}_{1/2}[\hat{A}_j(2r,R/2) \setminus G] = f_j(2r,R/2) \cdot \mathbb{P}_{1/2}[\neg G],
\]
by spatial independence. Proposition 2.4 implies that $f_j(2r,R/2) \asymp \alpha_{j}^{an}(r/2,2R)$. Moreover, the quasi-multiplicativity property and the lower-bound of (1.1) imply that $\mathbb{P}_{1/2}[A_j(r,R)] = \alpha_{j,1/2}^{an}(r/2,2R) \asymp \alpha_{j,1/2}^{an}(r,R)$, which ends the proof. \hfill \Box

To prove this, use for instance the following result with $X = 1_{A_j(r,R)}$ and $\mathcal{G} = \sigma(\omega \cap A(r,R))$: Let $X$ be a non-negative random variable and let $\mathcal{G}$ be a sub-$\sigma$-field of the underlying $\sigma$-field. Then, a.s. we have: $\mathbb{E}[X | \mathcal{G}] = 0 \Rightarrow X = 0$. 

\footnote{To prove this, use for instance the following result with $X = 1_{A_j(r,R)}$ and $\mathcal{G} = \sigma(\omega \cap A(r,R))$: Let $X$ be a non-negative random variable and let $\mathcal{G}$ be a sub-$\sigma$-field of the underlying $\sigma$-field. Then, a.s. we have: $\mathbb{E}[X | \mathcal{G}] = 0 \Rightarrow X = 0$.}
Remark 2.6. Note that, with essentially the same proof, we obtain the following result: Let \( j \in \mathbb{N}^* \). For every \( h \in (0, 1) \), there exists a constant \( \epsilon_0 = \epsilon_0(j, h) \in (0, 1) \) such that, for every \( 1 \leq r \leq \rho \leq R \) and for every event \( G \) which is measurable with respect to \( \omega \setminus (A(2r, \rho/2) \cup A(2\rho, R/2)) \) and satisfies \( \mathbb{P}_{1/2}[G] \geq 1 - \epsilon \), we have:

\[
\mathbb{P}_{1/2}[A_1(r, R) \cap G] \geq (1 - h)\alpha_{1/2}^{\text{an}}(r, R).
\]

In Section 7, we also use the quasi-multiplicativity property to compute universal arm exponents. Let us first introduce some notations. For every \( 1 \leq j \leq n \), let \( \omega_j \) denote the probability of the \( j \)-event in the half plane (i.e. the event that there are \( j \) paths of alternating color from \( \partial B_r \) to \( \partial B_R \) that live in the upper half-plane). See Subsection 7.3: the quasi-multiplicativity property can also be proved for these quantities. We have the following:

Proposition 2.7. The computation of the universal arm-exponents by Aizenman holds for Voronoi percolation: Let \( 1 \leq r \leq R \), we have:

i) \( \alpha_{1/2}^{\text{an}}(r, R) \asymp \frac{r}{R} \).

ii) \( \alpha_{3,1/2}^{\text{an}}(r, R) \asymp \frac{(r/R)^2}{2} \).

iii) \( \alpha_{5,1/2}^{\text{an}}(r, R) \asymp \frac{(r/R)^2}{2} \).

Items i) and ii) of Proposition 2.7 will be proved in Subsection 7.3 while Item iii) will be proved in Subsection 7.4.

2.4 Some important events: the pivotal events and the “good” events

2.4.1 Pivotal events

A crucial step in the proof of the scaling relations is the study of pivotal events for crossing and arm events. In the present work, we introduce a quenched and an annealed definitions for pivotal events. Let us begin with a classical definition: Let \( E \) be a countable set and let \( A \) be an event of the product \( \sigma \)-algebra on \( \{-1, 1\}^E \). A point \( i \in E \) is pivotal for a configuration \( \omega \in \{-1, 1\}^E \) and the event \( A \) if changing the value of \( \omega_i \) changes the value of \( \mathbb{1}_A(\omega) \). We will write \( \text{Piv}_i^E(A) \) for the event that \( i \) is pivotal for \( A \) (if \( E = \{1, \ldots, n\} \), we will use the notation \( \text{Piv}_i^n(A) \)). More generally, if \( I \) is a finite subset of \( E \), we say that \( I \) is pivotal for \( \omega \) and \( A \) if there exists \( \omega' \in \{-1, 1\}^E \) such that \( \omega \) and \( \omega' \) coincide outside of \( I \) and \( \mathbb{1}_A(\omega') \neq \mathbb{1}_A(\omega) \). The corresponding event will be denoted by \( \text{Piv}_I^E(A) \). Let us now introduce a quenched and an annealed notions of pivotal sets. The quenched version is very similar to the above notion:

Definition 2.8. Let \( A \) be an event measurable with respect to the coloured configuration \( \omega \) and let \( \eta \) be the underlying (non-coloured) point configuration. A bounded Borel set \( D \) is quenched-pivotal for \( \omega \) and \( A \) if there exists \( \omega' \in \{-1, 1\}^\eta \) (note that \( \omega' \) has the same underlying point configuration as \( \omega \) such that \( \omega \) and \( \omega' \) coincide on \( \eta \cap D' \) and \( \mathbb{1}_A(\omega') \neq \mathbb{1}_A(\omega) \)). We will write \( \text{Piv}_D^\eta(A) \) for the event that \( D \) is quenched-pivotal for \( A \).

We will also use the following terminology: if \( x \in \eta \), we say that \( x \) is quenched-pivotal for \( A \) if changing the color of \( x \) modifies the value of \( \mathbb{1}_A \). If we work conditionally on \( \eta \) and if \( x \in \eta \), then \( \{x \} \) is quenched-pivotal for \( A \) is an event of the product space \( \{-1, 1\}^\eta \), and we will denote this event by \( \text{Piv}_x^\eta(A) \).

Definition 2.9. A bounded Borel set \( D \) is annealed-pivotal for some coloured configuration \( \omega \) and some event \( A \) if both \( \mathbb{P}_p[A \mid \omega \setminus D] \) and \( \mathbb{P}_p[-A \mid \omega \setminus D] \) are positive. We will write \( \text{Piv}_D(A) \) for the event that \( D \) is annealed-pivotal for \( A \) (note that we omit the parameter \( p \) in the notation - actually, as far as \( p \in (0, 1) \) and since \( D \) is bounded, the event \( \text{Piv}_D(A) \) does not depend on \( p \).
We have the following link between annealed and quenched pivotal events: Let $p \in (0, 1)$, let $D$ be a bounded Borel set, and let $A$ be an event measurable with respect to the coloured configuration $\omega$. Then, a.s. we have $\Pr^\omega[D \subset \Pr^\omega(D(A), i.e. $\Pr^\omega[\Pr^\omega_\delta(A) \setminus \Pr^\omega(D(A)) = 0$. This is an easy consequence of the fact that, if $D$ is quenched-pivotal for $A$, then a.s. (since $\eta \cap D$ is finite) $\Pr[A | \omega \setminus D, \eta \cap D]$ and $\Pr[-A | \omega \setminus D, \eta \cap D]$ are positive.

Let $Q$ be a quad. The event that some box is (quenched or annealed) pivotal for the event $\text{Cross}(Q)$ is closely related to arm events and particularly to the 4-arm events. We prove estimates in this spirit in Subsections 4.1 and 4.3.

2.4.2 The events Dense, QBC and GI

In this subsection, we define three "good" events that we will use all along the paper. Their introduction is motivated by the three following observations: i) As explained for instance in Subsection 2.6.1, we will often condition on the event that the point configuration in some region is sufficiently dense in order to overcome spatial dependence difficulties. ii) It is often interesting to condition on $\eta$ since the conditional measure is the product measure $\Pr^\omega_\eta = (p\delta_1 + (1-p)\delta_{12})^{\otimes \eta}$. To apply geometric arguments under this quenched measure, we will often also condition on the event that $\Pr^\omega[\text{Cross}(Q)]$ is non-negligible for a huge family of quads $Q$. iii) When we study arm-event, it can be very useful to condition on the event that the arms are well-separated.

Definition 2.10. Let $D$ be a bounded subset of the plane and let $\delta \in (0, 1)$. We denote by $\text{Dense}_\delta(D)$ the event that, for every point $u \in D$, there exists $x \in \eta \cap D$ such that $||x - u||_2 \leq \delta \cdot \text{diam}(D)$.

Lemma 2.11. Let $R \geq 1$ and $\delta \in (0, 1)$. We have:

$$\Pr[\text{Dense}_\delta(B_R)] \geq 1 - O(1) \delta^{-2} \exp\left(-\frac{(\delta \cdot R)^2}{2}\right).$$

Proof. This lemma can be obtained by covering $B_R$ by a family $(S_i)_{1 \leq i \leq N}$ of $N \approx \delta^{-2}$ squares of side-length $\delta \cdot R/\sqrt{2}$ and by observing that:

$$\text{Dense}_\delta(B_R) \subseteq \{\forall i, \eta \cap S_i \neq \emptyset\}$$

and:

$$\forall i, \Pr[\eta \cap S_i = \emptyset] = \exp\left(-\frac{(\delta \cdot R)^2}{2}\right).$$

See Lemma 18 in Chapter 8 of [BR06b] for the proof of a similar result.

In the following, we restrict ourselves to the case $p = 1/2$. See Subsection 5.2 for the extension of these results to the near-critical phase.

Definition 2.12. Let $D$ be a subset of the plane and let $\delta \in (0, 1)$. We denote by $Q_\delta(D)$ the set of all quads $Q \subseteq D$ which are drawn on the grid $(\delta \text{diam}(D)) \cdot \mathbb{Z}^2$ (i.e. whose sides are included in the edges of $(\delta \text{diam}(D)) \cdot \mathbb{Z}^2$ and whose corners are vertices of $(\delta \text{diam}(D)) \cdot \mathbb{Z}^2$). Also, we denote by $Q_\delta(D)'$ the set of all quads $Q \subseteq D$ such that there exists a quad $Q' \in Q_\delta(D)$ such that $\text{Cross}(Q') \subseteq \text{Cross}(Q)$.

The following result will be proved in Subsection 3.2 by using Theorem 1.4.

\begin{footnote}
Use for instance the following result with $X = 1_A, G_1 = \sigma(\omega \setminus D)$ and $G_2 = \sigma(\eta, \omega \setminus D)$: Let $X$ be a non-negative random variable and let $G_1 \subseteq G_2$ be two sub-$\sigma$-fields of the underlying $\sigma$-field. Then, a.s. we have: $E[X | G_1] = 0 \Rightarrow E[X | G_2] = 0$.
\end{footnote}
Proposition 2.13. There is an absolute constant $C < +\infty$ such that the following holds: Let $\delta \in (0,1)$ and $\gamma \in (0, +\infty)$. There exists a constant $c = c(\delta, \gamma) \in (0,1)$ such that, for every bounded subset of the plane $D$ such that $\text{diam}(D) \geq \delta^{-2}/100$, we have:

$$\mathbb{P}[\text{QBC}_\delta^\gamma(D)] \geq 1 - C \text{diam}(D)^{-\gamma},$$

where:

$$\text{QBC}_\delta^\gamma(D) = \left\{ Q \in Q_\delta(D), \; \mathbb{P}_{1/2}^Q \left[ \text{Cross}(Q) \right] \geq c(\delta, \gamma) \right\}.$$

The notation QBC means “Quenched Box-Crossing property”.

Let us end this subsection by defining quantities related to the well-separateness of interfaces. Let $\delta \in (0,1)$, let $1 \leq r \leq R$, and let $\beta_1, \cdots, \beta_k$ be the interfaces from $\partial B_r$ to $\partial B_R$ (an interface is a continuous path $\beta$ drawn on the edges of the Voronoi tiling and such that one side of $\beta$ is black and its other side is white). Also, let $z^\text{ext}_{i}(\partial B_r)$ (respectively $z^\text{int}_{i}(\partial B_r)$) denote the endpoint on $\partial B_r$ (respectively on $\partial B_r$) of $\beta_i$, and let $s^\text{ext}(r, R)$ (resp. $s^\text{int}(r, R)$) be the least distance between $z^\text{ext}_{i}(\partial B_r)$ (respectively $z^\text{int}_{i}(\partial B_r)$) and $\cup_{j \neq i} \beta_j$.

Let $\text{GI}^\text{ext}_\delta(R)$ (for “Good Interfaces”) be the event that there does not exist $y \in \partial B_R$ such that the 3-arm event in $A(y;10\delta R, R/4) \cap B_R$ holds. Note that, if $r \leq 3R/4$, then:

$$\text{GI}^\text{ext}_\delta(R) \subseteq \left\{ s^\text{ext}(r, R) \geq 10\delta R \right\}.$$

This inclusion will be very useful. Note that we do not have:

$$\left\{ s^\text{ext}(r, R) \geq 10\delta R \right\} \subseteq \left\{ s^\text{ext}(3R/4, R) \geq 10\delta R \right\},$$

this is why we have introduced the event $\text{GI}^\text{ext}_\delta(R)$: what will be useful is that $\text{GI}^\text{ext}_\delta(R)$ “depends only on the crossings in $A(3R/4, R)$” and that this event contains $\{ s^\text{ext}(r, R) \geq 10\delta R \}$ (for $r \leq 3R/4$).

Similarly, let $\text{GI}^\text{int}_\delta(r)$ be the event that there does not exist $y \in \partial B_r$ such that the 3-arm event in $A(y;10\delta r, r/2) \setminus B_r$ holds. Note that, if $R \geq 3r/2$:

$$\text{GI}^\text{int}_\delta(r) \subseteq \left\{ s^\text{int}(r, R) \geq 10\delta r \right\}.$$

We will prove the following lemma in Subsection 3.3:

Lemma 2.14. Let $\delta \in (0,1)$ and let $r, R \geq 100\delta$. There exist absolute constants $C < +\infty$ and $\epsilon > 0$ such that:

$$\mathbb{P}_{1/2}^{\text{GI}^\text{ext}_\delta(R)} \geq 1 - C \delta,$$

and:

$$\mathbb{P}_{1/2}^{\text{GI}^\text{int}_\delta(r)} \geq 1 - C \delta^\epsilon.$$

2.5 Organization of the paper and interdependence of the sections

As explained in Subsection 2.3, we postpone the proof of the quasi-multiplicativity property to the final section: Section 7. We summarize the interdependence of the sections of the paper in Figures 2 and 3.

2.6 Some ideas of proof

Let us end Section 2 by giving a few more details about our strategies of proofs.

2.6.1 The quasi-multiplicativity property (at $p = 1/2$)

In this subsection, we work at the parameter $p = 1/2$, and we explain ideas behind the proof of the quasi-multiplicativity property Proposition 1.6 in the case $j \geq 2$ (see Subsection 3.1 for the proof of the easier case $j = 1$). The proof is written in Section 7. We begin with two observations that illustrate the new difficulties compared to Bernoulli percolation on a deterministic lattice.
Preliminary results: Subs. 3.1 and 3.2
Quasi-multiplicativity property for $j$ even: Subs. 7.1 and 7.2
Quasi-multiplicativity for events in the half-plane and universal arm exponents in the half-plane: Subs. 7.3

Figure 2: Interdependence between Sections 3 and 7

Extension of the box-crossing properties to near-criticality: Subs. 5.1
Pivotal events: Sect. 4
Extension of all the results to near-criticality: Subs. 5.2
Kesten’s scaling relations: Sect. 6

(a) The first observation is that, for Voronoi percolation, the following result does not seem easier to prove than the quasi-multiplicativity property itself: There exists a constant $C = C(j)$ such that, for every $R \geq 100$:

$$\alpha_{\frac{1}{2}, 100}(100, R) \leq C \alpha_{\frac{1}{2}, 1}(1, R).$$

A first idea to prove the above would be to condition on the event $A_{\frac{1}{2}, 100}(100, R)$ and on the coloured configuration outside of $B_{100}$ and then extend the arms to $\partial B_{10}$ “by hands”.

The problem is that $A_{\frac{1}{2}, 100}(100, R)$ is a degenerated event and thus, as already suggested in Figure 1, it does not seem obvious at all that the following does not happen: “If we condition on $A_{\frac{1}{2}, 100}(100, R)$, then, with high probability, the point configuration on the neighbourhood of $\partial B_{100}$ is very dense”. This may be a problem since it is difficult to extend the arms “by hands” when the point configuration is very dense.

(b) For Bernoulli percolation on a deterministic lattice, the left-hand-inequality of the quasi-multiplicativity property is very easy since it is the consequence of independence on disjoint sets. For Voronoi percolation, even the following does not seem easy to prove, because of the spatial dependencies of the model: There exists $C = C(j) < +\infty$ such that, for every $1 \leq r_1 \leq r_2 \leq r_3/2$: $\alpha^{an}_{\frac{1}{2}}(r_1, r_2) \leq C \alpha^{an}_{\frac{1}{2}}(r_1, r_2) \alpha^{an}_{\frac{1}{2}}(2r_2, r_3)$. However, one can note that the left-hand-inequality of the quasi-multiplicativity property is a direct consequence of Proposition 2.4 (which enables to use spatial independence properties). Actually, our strategy will be the following: we will first prove Lemma 7.5 and Corollary 7.9 which are results analogous to Proposition 2.4. Then we will prove the quasi-multiplicativity property, and finally we will prove Proposition 2.4.

Now, let us be a little more precise about the proof of the quasi-multiplicativity property.

In the spirit of [Kes87, Wer07, Nol08, SS10], we will prove that:
i) If $A_j(r, R)$ holds and if the configuration “looks good” near $\partial B_R$, then we can extend the arms at larger scale, see Lemma 7.5.

ii) If we condition on $A_j(r, R)$, then the configuration near $\partial B_R$ “looks good” with high probability, see Lemma 7.8.

A difference with the case of Bernoulli percolation on a deterministic lattice is that, in the notion of “looking good”, we will have to ask that both the random tiling and the random colouring look good. Concerning the random colouring: As in the case of Bernoulli percolation on $\mathbb{Z}^2$ or on $\mathbb{T}$, we will ask that the interfaces between black and white crossings are well-separated so that we can use box-crossing estimates. Concerning the random tiling: (1) To avoid spatial dependence problems, we will ask that $\text{Dense}_3(A(R/2, 2R))$ (see Definitions 2.10) holds for some $\delta$. (2) In order to use box-crossing estimates even if we condition on $\eta$, we will ask that $\text{QBC}_j(A(R))$ (see Proposition 2.13) holds for some annulus $A(R)$ at scale $R$ and some $\delta$. The idea is that, if conditions (1) and (2) are satisfied and if we condition on $\eta$ and on well-separated interfaces, then we can extend these interfaces by using box-crossing techniques and the (quenched) Harris-FKG. As we will see in Section 7, we will have to consider events a little more complicated because we will want our events to be measurable with respect to $\omega \cap A(R/2, 2R)$ to avoid other spatial dependency problems.

Also, we will see that, for technical reasons, we will have to consider different notions of well-separateness of interfaces. More precisely, we will first prove the quasi-multiplicativity property in the case $j$ even (in Subsection 7.1) and with the following definition of well-separateness: two interfaces are well separated if their end-points are well-separated. By following the same proof, we will also obtain the quasi-multiplicativity property for $j$-arms events in the half-plane (with $j$ either even or odd). Thanks to this last property, we will be able to compute the universal exponent of the 3-arm event in the half-plane. Then, it will be possible (by using our knowledge on $\alpha_{3,1/2}(r, R)$) to deal with the following slightly different definition of well-separateness of interfaces: two interfaces are well-separated if each end-point is far enough from the union of the other interfaces (and not only far enough from the other end-points). This other notion of well-separateness is the one defined in Subsection 2.4.2, and we will need this notion to prove the quasi-multiplicativity property when $j$ is odd (see Subsection 7.4).

Once Theorems 1.3 and 1.4 are extended to the near-critical phase, it will not be difficult to extend our results on arm events and pivotal events to this phase.

2.6.2 The annealed scaling relations

Once we have proved the quasi-multiplicativity property and all the results stated in Section 2, the ideas for the proof of the annealed scaling relations are the same as in the original paper of Kesten [Kes87] (see also [Wer07, Nol08]). The only difference is that we will need to combine quenched and annealed notions of pivotal events.

3 Preliminary results

In this section, we only work at the parameter $p = 1/2$, hence we intentionally forget the subscript $p$ in the notations.

3.1 Warm-up: proof of (1.1) and of the quasi-multiplicativity property for $j = 1$

In this subsection, we prove that the probabilities of arm events decay polynomially fast, i.e. we prove (1.1) (this can be seen as an illustration of how we use the events Dense($\cdot$) from Definition 2.10). We also prove the quasi-multiplicativity property in the case $j = 1$ (this can be seen as an illustration of how we use the events Dense($\cdot$) and events of the kind
To prove these inequalities, we do not rely on any of the other results proved in this paper but only on the (annealed) FKG property and on the (annealed) box-crossing property Theorem 1.3.

**Proof of (1.1).** As explained below (1.1), the upper-bound is proved in [Tas16]. Let us prove the lower-bound. First, note that we can choose a constant $M = M(j) \in [10, +\infty)$ such that we can define $j$ sets of $2^n \approx \log(R/r)$ rectangles: $\{Q^i_1, \ldots, Q^i_{2n}\}$, $i = 0, \ldots, j-1$ that satisfy:

(a) For every $i \in \{0, \ldots, j-1\}$ and every $l \in \{1, \ldots, n\}$, $Q^i_{2l-1}$ and $Q^i_{2l}$ are $(2^l r) \times (2^l - M r)$ rectangles;

(b) For all $i \neq i' \in \{0, \ldots, j-1\}$ and all $l, l' \in \{1, \ldots, 2n\}$, $Q^i_l$ is at distance at least $\max(2^{l-M} r, 2^{l'-M} r)$ from $Q^i_{l'}$;

(c) If for every $l \in \{1, \ldots, 2n\}$ and every $i \in \{0, \ldots, j-1\}$ odd (respectively even) the rectangle $Q^i_l$ is crossed lengthwise (respectively dual-crossed lengthwise), then $A_j(r, R)$ holds. (See Figure 4.)

Figure 4: The rectangles $Q^i_l$ for some $i$.

Let $i \in \{0, \ldots, j-1\}$ odd (respectively even), and let $\text{Dense}(Q^i_l)$ be the event that, for any $u \in Q^i_l$, there exists a black (respectively white) point $x \in \eta \cap Q^i_l$ at Euclidean distance less than $2^l - 2M r$ from $x$. Note that the event $\text{Dense}(Q^i_l)$ is slightly different from the event $\text{Dense}(Q^i_l)$ of Definition 2.10; in particular, they are annealed increasing (respectively annealed decreasing) if $i$ is odd (respectively even). We have:

$$\alpha^{an}_j(r, R) \geq \mathbb{P} \left[ \bigcap_{i, l} \text{Cross}(Q^i_l) \right] \geq \mathbb{P} \left[ \bigcap_{i, l} \text{Cross}(Q^i_l) \cap \text{Dense}(Q^i_l) \right].$$

Next, note that the $j$ families $\left(\text{Cross}(Q^i_l) \cap \overline{\text{Dense}}(Q^i_l)\right)_{1 \leq l \leq 2n}$ are independent since $\overline{\text{Dense}}(Q^i_l)$ depends only on $\eta \cap Q^i_l$ and since, if $l, l' \in \{1, \ldots, 2n\}$, $i \neq i' \in \{0, \ldots, j-1\}$ and $\text{Dense}(Q^i_l) \cap \text{Dense}(Q^{i'}_{l'})$ holds, then no Voronoi cell can intersect both $Q^i_l$ and $Q^{i'}_{l'}$. As a result the above equals:

$$\prod_{i=0}^{j-1} \mathbb{P} \left[ \bigcap_{l=1}^{2n} \text{Cross}(Q^i_l) \cap \overline{\text{Dense}}(Q^i_l) \right].$$

We can now use the (annealed) FKG-Harris inequality. Indeed, for every $i$ odd (respectively even) and every $l \in \{1, \ldots, 2n\}$, the event $\text{Cross}(Q^i_l) \cap \overline{\text{Dense}}(Q^i_l)$ is annealed increasing. We
thus obtain that the above is at most:

\[ \prod_{i=0}^{j-1} \prod_{l=1}^{2n} \mathbb{P}[\text{Cross}(Q_i^l) \cap \text{Dense}(Q_i^l)]. \]

By the same proof as Lemma 2.11, we have: \( \mathbb{P}[\text{Dense}(Q_i^l)] \geq 1 - C \exp(-c2^l), \) for some \( C = C(M) < +\infty \) and \( c = c(M) > 0. \) By using this estimate and the box-crossing property Theorem 1.3, we obtain that there exists a constant \( c = c(M) > 0 \) such that, for every \( l \) large enough (larger than some \( l_0(M), \) say) and every \( i, \) \( \mathbb{P}_{1/2}[\text{Cross}(Q_i^l) \cap \text{Dense}(Q_i^l)] \geq c. \) Moreover, it is easy to see that, for every \( i \in \{0, \ldots, j-1\} \) and every \( l \in \{1, \ldots, l_0\}, \) we have \( \mathbb{P}_{1/2}[\text{Cross}(Q_i^l) \cap \text{Dense}(Q_i^l)] \geq c' \) for some \( c' = c'(M, l_0) > 0. \) Thus, we have:

\[ \alpha_j^{an}(r, R) \geq (\min\{c, c'\})^{2n_j}, \]

which ends the proof. \( \square \)

**Proof of the quasi-multiplicativity property in the case \( j = 1. \)** As pointed out in Remark 1.7, if \( j = 1 \) then the right-hand-inequality of the quasi-multiplicativity property is a direct consequence of the annealed FKG-Harris inequality and of the annealed box-crossing result Theorem 1.3. Here, we prove the left-hand-inequality (by relying on the right-hand-inequality). The main difficulty is the lack of spatial independence. To overcome it, we work with the following events and quantities (analogous to those introduced in Subsection 7.2). Let \( 1 \leq r \leq R \) and:

\[ \begin{align*}
\hat{A}_1^{\text{ext}}(r, R) &:= \left\{ \mathbb{P}[A_1(r, R) \mid \omega \cap B_R > 0] \right\}, \\
\hat{A}_1^{\text{int}}(r, R) &:= \left\{ \mathbb{P}[A_1(r, R) \mid \omega \backslash B_r > 0] \right\}, \\
f_1^{\text{ext}}(r, R) &:= \mathbb{P}[\hat{A}_1^{\text{ext}}(r, R)], \\
f_1^{\text{int}}(r, R) &:= \mathbb{P}[\hat{A}_1^{\text{int}}(r, R)].
\end{align*} \]

Note that \( \alpha_1^{an}(r, R) \leq f_1^{\text{ext}}(r, R) \) and \( \alpha_1^{an}(r, R) \leq f_1^{\text{int}}(r, R). \) What is interesting with these events is that, if \( 1 \leq r_1 \leq r_2 \leq r_3, \) then \( \hat{A}_1^{\text{ext}}(r_1; r_2) \) and \( \hat{A}_1^{\text{int}}(r_2; r_3) \) are independent (indeed, the first one is measurable with respect to \( \omega \cap B_{r_2} \) while the second one is measurable with respect to \( \omega \backslash B_{r_2} \)). Hence we have:

\[ \alpha_1^{an}(r_1, r_3) \leq \mathbb{P}[\hat{A}_1^{\text{ext}}(r_1, r_2) \cap \hat{A}_1^{\text{int}}(r_2, r_3)] = f_1^{\text{ext}}(r_1, r_2) f_1^{\text{int}}(r_2, r_3). \]

As a result, it is sufficient to prove that \( f_1^{\text{ext}}(r, R), f_1^{\text{int}}(r, R) \leq O(1) \alpha_1^{an}(r, R). \) We will prove this only for \( f_1^{\text{int}}(r, R) \) since the proof for \( f_1^{\text{ext}}(r, R) \) is the same.

Let \( \text{Dense}(r) := \text{Dense}_{1/100}(A(r/2, 2r)) \) where \( \text{Dense}_a(D) \) is defined in Definition 2.10. With the same proof as Lemma 2.11, we have: \( \mathbb{P}[\text{Dense}(r)] \geq 1 - O(1) \exp(-\Omega(1) r^2). \) If \( \text{Dense}(r) \) holds, then we have the following: if \( x \in \eta \) is such that the Voronoi cell of \( x \) intersects \( A(2r, R), \) then \( x \notin B_r. \) As a result, \( \hat{A}_1^{\text{int}}(r, R) \cap \text{Dense}(r) \subseteq A_1(2r, R). \) So:

\[ f_1^{\text{int}}(r, R) \leq \alpha_1^{an}(2r, R) + \mathbb{P} \left[ \hat{A}_1^{\text{int}}(r, R) \backslash \text{Dense}(r) \right]. \]

Moreover, by using the fact that \( \text{Dense}(r) \) and \( \hat{A}_1^{\text{int}}(2r, R) \) are independent (the first one is measurable with respect to \( \eta \cap A(r/2, 2r) \) while the second one is measurable with respect
to $\omega \setminus B_{2r}$, we obtain that $f_1^{\text{int}}(r, R)$ is at most:

$$
\alpha_1^{an}(2r, R) + \mathbb{P}[\tilde{A}_1^{\text{int}}(r, R) \setminus \text{Dense}(r)] \
\leq \alpha_1^{an}(2r, R) + \mathbb{P}[\tilde{A}_1^{\text{int}}(2r, R) \setminus \text{Dense}(r)] \
= \alpha_1^{an}(2r, R) + f_1^{\text{int}}(2r, R) (1 - \mathbb{P}[\text{Dense}(r)]) \
\leq \alpha_1^{an}(2r, R) + f_1^{\text{int}}(2r, R) O(1) \exp\left(-\Omega(1) r^2\right) .
$$

By iterating the above inequality, we obtain that:

$$
f_1^{\text{int}}(r, R) \leq \alpha_1^{an}(2r, R) + O(1) \sum_{i=0}^{\lfloor \log_2(R/r) \rfloor - 2} \left( \alpha_1^{an}(2^{i+2}r, R) \exp\left(-\Omega(1) (2^i r)^2\right) \right) \
+ O(1) \exp\left(-\Omega(1) (2^{\lfloor \log_2(R/r) \rfloor - 1} r)^2\right) .
$$

We now use the right-hand-inequality of the quasi-multiplicativity property and (1.1), which imply that there exists a constant $C_1 < \infty$ such that, for every $i \in \{1, \cdots, \lfloor \log_2(R/r) \rfloor + 1\}$, we have:

$$
\alpha_1^{an}(2^i r, R) \leq C_1^i \alpha_1^{an}(r, R) .
$$

We finally obtain:

$$
f_1^{\text{int}}(r, R) \leq O(1) \alpha_1^{an}(r, R) \times \left( C_1 + \sum_{i=0}^{\lfloor \log_2(R/r) \rfloor - 2} (C_1^{i+2} \exp\left(-\Omega(1) (2^i r)^2\right)) \right) \
+ C_1^{\lfloor \log_2(R/r) \rfloor + 1} \exp\left(-\Omega(1) (2^{\lfloor \log_2(R/r) \rfloor - 1} r)^2\right) .
$$

This ends the proof since the quantity between parentheses can be bounded by some absolute constant. \[\square\]

### 3.2 A generalization of Theorem 1.4 to a family of quads

The fact that we can choose any $\gamma > 0$ in the quenched box-crossing property Theorem 1.4 is crucial for us. In particular, this implies that, the quenched box crossing property is true for a lot of quads simultaneously with high probability. In this subsection, we do not use any of the other results proved in this paper. We use the notations from Definition 2.12 and Proposition 2.13 and we prove Proposition 2.13.

**Proof of Proposition 2.13.** Let $(Q_i)_{i \in \{1, \cdots, N(D, \delta)\}}$ be an enumeration of all $2\delta \text{diam}(D) \times \delta \text{diam}(D)$ rectangles that intersect $D$ and that are drawn on the grid $(\delta \text{diam}(D)) \cdot \mathbb{Z}^2$. Note that, if $Q \in Q_\delta(D)$, then:

$$
\bigcap_{i=1}^{N(D, \delta)} \{Q_i \text{ is crossed lengthwise} \} \subseteq \text{Cross}(Q) .
$$

The (quenched) FKG-Harris inequality implies that, for every $Q \in Q_\delta(D)$ and for every $\eta$ we have:

$$
\prod_{i \in \{1, \cdots, N(D, \delta)\}} \mathbb{P}^\eta [Q_i \text{ is crossed lengthwise} \leq \mathbb{P}^\eta [\text{Cross}(Q)] . \hspace{1cm} (3.1)
$$

Now, let $\gamma' > 0$ to be fixed later. Theorem 1.4 implies that there exists a constant $c_0 = c_0(\gamma') \in (0, 1)$ such that:

$$
\forall i, \mathbb{P}[\mathbb{P}^\eta [Q_i \text{ is crossed lengthwise} \geq c_0] \geq 1 - (\delta \text{diam}(D))^{-\gamma'} .
$$
By a union bound we obtain that:
\[
P \left[ \forall i, \mathbf{P}^\eta [Q_i \text{ is crossed lengthwise}] \geq c_0 \right] \geq 1 - O(1) N(D, \delta) (\delta \text{diam}(D))^{-\gamma'} \\
\geq 1 - O(1) \delta^{-2} (\delta \text{diam}(D))^{-\gamma'}.
\]
Together with (3.1), this implies that:
\[
P \left[ \forall Q \in \mathcal{Q}_3(D), \mathbf{P}^\eta [\text{Cross}(Q)] \geq c_0^{N(D, \delta)} \right] \geq 1 - O(1) \delta^{-2} (\delta \text{diam}(D))^{-\gamma'}.
\]
We now use the fact that \(\text{diam}(D) \geq \delta^{-2}/100\) and we choose \(\gamma' = 2\gamma + 2\). We have:
\[
\delta^{-2} (\delta \text{diam}(D))^{-\gamma'} = \delta^{-2-\gamma'} \text{diam}(D)^{-\gamma'} \\
\leq (\text{diam}(D))^{1-\gamma'}/2 = \text{diam}(D)^{-\gamma'}.
\]
Hence, the result holds by choosing \(c = c_0^\sup_{D} N(D, \delta) (= c_0^{O(1)\delta^{-2}})\).

We will need to work with the following family of quads in the proof of Lemma 7.4:

**Definition 3.1.** Let \(\hat{\mathcal{Q}}_3'(D)\) be the set of all quads \(Q \subseteq D\) such that there exists \(k \in \mathbb{N}\) such that \(Q\) is drawn on the grid \((2^k \delta \text{diam}(D)) \cdot \mathbb{Z}^2\) and the length of each side of \(Q\) is less than \(100 \cdot 2^k \delta \text{diam}(D)\). Also, let \(\mathcal{Q}_3(D)\) be the set of all quads \(Q \subseteq D\) such that there exists a quad \(Q' \in \hat{\mathcal{Q}}_3'(D)\) such that \(\text{Cross}(Q') \subseteq \text{Cross}(Q)\).

**Proposition 3.2.** Let \(\delta \in (0,1)\) and \(\gamma \in (0, +\infty)\). There exists \(\tilde{c} = \tilde{c}(\gamma) \in (0,1)\) such that,\(^7\) for every bounded subset of the plane \(D\) such that \(\text{diam}(D) \geq \delta^{-2}/100\), we have:
\[
P \left[ \forall Q \in \hat{\mathcal{Q}}_3'(D), \mathbf{P}^\eta [\text{Cross}(Q)] \geq \tilde{c} \right] \geq 1 - O(1) \text{diam}(D)^{-\gamma},
\]
where the constants in \(O(1)\) are absolute constants.

**Remark 3.3.** One could use Proposition 3.2 and gluing arguments to prove Proposition 2.13 (with \(c(\delta, \gamma) = \tilde{c}(\gamma)^{O(1)\delta^{-2}}\)) but since we will essentially use Proposition 2.13 we have chosen to write the proof of this proposition and then mimic the proof in order to obtain Proposition 3.2.

**Proof of Proposition 3.2.** First, we work with the following set of quads: Let \(\hat{\mathcal{Q}}_3(D) \subseteq \mathcal{Q}_3(D)\) be the set of all quads \(Q \subseteq D\) drawn on the grid \((\delta \text{diam}(D)) \cdot \mathbb{Z}^2\) such that the length of each side of \(Q\) is less than \(100 \cdot \delta \text{diam}(D)\). We have:
\[
\hat{\mathcal{Q}}_3(D) = \bigcup_{k=0}^{+\infty} \hat{\mathcal{Q}}_{2^k\delta}(D).
\]
By following the proof of Proposition 2.13 we obtain that, with the constant \(c_0\) of this proof:
\[
P \left[ \forall Q \in \hat{\mathcal{Q}}_3(D), \mathbf{P}^\eta [\text{Cross}(Q)] \geq c_0 \right] \geq 1 - O(1) \text{diam}(D)^{-\gamma}.
\] \hfill (3.2)

The fact that we have \(c_0^{O(1)}\) instead of \(c_0^{O(1)\delta^{-2}}\) comes from the fact that we only consider quads of side length \(\leq 100 \cdot \delta \text{diam}(D)\). Now, note that the sets \(\hat{\mathcal{Q}}_{2^k\delta}(D)\) are empty when \(k \geq \log_2(2\text{diam}(D)/\delta)\), hence:
\[
\hat{\mathcal{Q}}_3(D) = \bigcup_{k=0}^{\log_2(2\text{diam}(D)/\delta)} \hat{\mathcal{Q}}_{2^k\delta}(D).
\]
\(^7\)The fact that \(\tilde{c}\) does not depend on \(\delta\) will be crucial.
Note also that \( \log_2(2\text{diam}(D)/\delta) \leq \log_2\left(O(1) \sqrt{\text{diam}(D)}\right) \) since \( \text{diam}(D) \geq \delta^{-2}/100 \). Now, we apply \((3.2)\) to \( \hat{Q}_{2k}(D) \) for every \( k \in \{0, \cdots \} \), \( \log_2(2\text{diam}(D)/\delta) \) and with \( \gamma + 1 \) instead of \( \gamma \). A union bound then implies that there exists a constant \( \tilde{c} = \tilde{c}(\gamma) > 0 \) such that:

\[
P \left[ \forall Q \in \hat{Q}_3(D), \ P^0[\text{Cross}(Q)] \geq \tilde{c} \right] \geq 1 - O(1) \log_2 \left(O(1) \sqrt{\text{diam}(D)}\right) \text{diam}(D)^{-\gamma-1} \geq 1 - O(1) \text{diam}(D)^{-\gamma}.
\]

\[\square\]

### 3.3 “Strong” well-separateness of interfaces

In this subsection, we prove Lemma 2.14 i.e. we prove that the interfaces are well-separated with high probability. Subsections 3.1 and 3.2 do not depend on the other subsections of the paper but this is not the case of the present subsection. Indeed, we are going to rely on the results of Subsections 7.1, 7.2 and 7.3 where the quasi-multiplicativity property is proved in the case of an even number of arms and in the case of arm events in the half-plane, and where the exponent of the 3-arm event in the half-plane is computed.

**Remark 3.4.** In Subsection 7.1 we will prove another “well-separateness of interfaces lemma”: Lemma 7.4; but the notion of well-separateness of interfaces Lemma 7.4 is weaker than the one in Lemma 2.14. Lemma 7.4 is actually enough when we deal with an even number of arms or with arm events restricted to a wedge, but not when we deal with an odd number of arms.

**Proof of Lemma 2.14.** First, note that there exist \( \asymp \delta^{-1} \) points \( y \in \partial B_R \) such that, if the event \( GI^\text{int}_3(R) \) holds, then the 3-arm event in \( A(y; 20\delta r, R/4) \cap B_R \) holds. Note also that, if \( y \in \partial B_R \), then \( A(y; 20\delta r, R/4) \cap B_R \) is included in a half-plane whose boundary contains \( y \). Together with Item ii) of Proposition 2.7, this implies that:

\[
P[GI^\text{int}_3(R)] \leq O(1) \delta^{-1} \left( \frac{\delta R}{R} \right)^2 = O(1) \delta.
\]

Now, let us study \( GI^\text{int}_3(r) \). As above, there exist \( \asymp r/\delta \) points \( y \in \partial B_r \) such that, if the event \( GI^\text{int}_3(r) \) holds, then the 3-arm event in \( A(y; 20\delta r, r/4) \setminus B_r \) holds. However, it is not true that, for every \( y \in \partial B_r \), \( A(y; 20\delta r, r/2) \setminus B_r \) is included in a half-plane whose boundary contains \( y \) (there are problems at the corners of \( B_r \)). This is why we need the following result:

**Claim 3.5.** Let \( y \in \partial B_r \) and let \( \rho \in [20\delta r, r/2] \) such that \( y \) is at distance at least \( \rho \) from the corners of \( B_r \). Then, there exists a constant \( \epsilon > 0 \) such that:

\[
P[3\text{-arm event in } A(y; 20\delta r, r/4) \setminus B_r] \leq O(1) \left( \frac{\delta r}{\rho} \right)^2 \left( \frac{\rho}{r} \right)^\epsilon.
\]

**Proof.** Write \( A_3^+(y; 20\delta r, \rho) \) for the 3-arm event in \( A(y; 20\delta r, \rho) \setminus B_r \) and let:

\[
\hat{A}_3^+(y; 20\delta r, \rho) := \left\{ P \left[ A_3^+(y; 20\delta r, \rho) \right] \cap A(y; 20\delta r, \rho) > 0 \right\}.
\]

Let \( y_0 \) be the corner of \( B_r \) closest to \( y \), let \( A_3(y_0; 2\rho, r/2) \) be the 3-arm event translated by \( y_0 \), and let:

\[
\hat{A}_3(y_0; \rho, r/2) := \left\{ P \left[ A_3(y_0; 2\rho, r/2) \cap A(y_0; 2\rho, r/2) > 0 \right] \right\}.
\]

The events \( \hat{A}_3^+(y; 20\delta r, \rho) \) and \( \hat{A}_3(y_0; 2\rho, r/2) \) are independent and are a.s. included\(^*\) in the 3-arm event in \( A(y; 20\delta r, r/4) \). Remember that in the present subsection we rely on

\(^*\)We say that \( A \) is a.s. included in \( B \) if \( P[A \setminus B] = 0 \).
the results of Subsections 7.1, 7.2 and 7.3 where the quasi-multiplicativity property and its consequences (e.g. Proposition 2.4) are proved for \( j \) odd and also for arm events in the half plane for any \( j \). We apply Proposition 2.4 for the 2-arm event in the whole plane and for the 3-arm event in the half-plane, which enables to obtain that:

\[
P \left[ \hat{A}_3(y_0; 2\rho, r/2) \right] \leq P \left[ \hat{A}_2(y_0; 2\rho, r/2) \right] \asymp \alpha_3^n(2\rho, r/2)
\]

and:

\[
P \left[ \hat{A}_3^+(y; 20\delta r, \rho) \right] \asymp \alpha_3^{n_+}(20\delta r, \rho).
\]

If we combine these estimates with (1.1) and with the computation of the 3-arm event in the half-plane (Item (ii) of Proposition 2.7), we obtain that:

\[
P \left[ \hat{A}_3(y_0; 2\rho, r/2) \right] \leq O(1) \left( \frac{\delta r}{\rho} \right)^{\Omega(1)}
\]

and:

\[
P \left[ \hat{A}_3^+(y; 20\delta r, \rho) \right] \asymp \left( \frac{\delta r}{\rho} \right)^{\Omega(1)}.
\]

Finally:

\[
P \left[ \text{3-arm event in } A(y; 20\delta r, r/4) \setminus B_r \right] \leq O(1) \left( \frac{\delta r}{\rho} \right)^{\Omega(1)}
\]

which ends the proof.

We can (and do) assume that the constant \( \epsilon \) of the claim is in \((0, 1)\). Now, note that there exists \( N(\delta) = \log_2(\delta^{-1}) \) finite subsets of \( \partial B_r \): \( Y_1, \cdots, Y_{N(\delta)} \) such that: (a) \( |Y_i| \approx 2^i \), (b) for every \( y \in Y_i \), there exists a corner of \( B_r \) at distance \( \approx 2^i \delta r \) from \( y \) and (c) if \( GI^+_\delta(r) \) holds, then there exists \( y \in \cup_{i=1}^{N(\delta)} Y_i \) such that the 3-arm event in \( A(y; 20\delta r, r/4) \setminus B_r \) holds.

Combined with the claim, these observations imply that:

\[
P \left[ GI^+_\delta(r) \right] \leq O(1) \sum_{i=1}^{N(\delta)} 2^i \left( \frac{\delta r}{2^i \delta r} \right)^2 \left( \frac{2^i \delta r}{2^i \delta r} \right)^\varepsilon
\]

\[
\leq O(1) \delta^\varepsilon \sum_{i=1}^{N(\delta)} 2^{(c-1)}
\]

\[
= O(1) \delta^\varepsilon.
\]



\section{Pivotal events and some estimates on arm events}

In this section, we only work at the parameter \( p = 1/2 \), hence we intentionally forget the subscript \( p \) in the notations. We will rely on the quasi-multiplicativity property (proved in Section 7), on its consequences Propositions 2.4 and 2.5, and on the preliminary results from Section 3. Our goal is to estimate pivotal events. We refer to Subsection 2.4.1 for the notations we use for these events. Our main goal is to prove the following result:

\textbf{Proposition 4.1.} Let \( \rho \geq 1 \) and \( R \geq 100\rho \). We have:\footnote{The constant 2 in \( \approx 2\rho Z \) does not have to be taken seriously. The reason why we look at grids of mesh \( \geq 2 \) is only that we have stated the quasi-multiplicativity property Proposition 1.6 for \( 1 \leq r_1 \leq r_2 \leq r_3 \) i.e. for arm events around boxes of side length at least 2.}

\[
\sum_{S \text{ square of the grid } (2\rho Z)^2} P_1/2 \left[ \text{Piv}_S(\text{Cross}(R, 2R)) \right] \asymp \left( \frac{R}{\rho} \right)^2 \alpha_{4,1/2}^n(\rho, R).
\]
The event $\text{Piv}_S(\text{Cross}(R, 2R))$ is an annealed-pivotal event. We will also prove similar bounds for quenched-pivotal event, see Lemma 4.6. Let us make two observations in order to illustrate the difficulties that will arise in the proof of Proposition 4.1. Let $S$ be a square of the grid $2\rho \mathbb{Z}^2$.

i) Even when $S$ is far-away from $[-2R, 2R] \times [-R, R]$, we have $\mathbb{P}_{1/2} [\text{Piv}_S(\text{Cross}(2R, R))] \neq 0$ (this is the case if $\eta$ does not intersect a wide region that contains $[-2R, 2R] \times [-R, R]$).

ii) Assume that $S \subseteq [-2R, 2R] \times [-R, R]$ and let $A_4(S, R)$ denote the event that there are two black arms included in $[-2R, 2R] \times [-R, R] \setminus S$ from $\partial S$ to the left and right sides of $[-2R, 2R] \times [-R, R]$ and two white arms included in $[-2R, 2R] \times [-R, R] \setminus S$ from $\partial S$ to the top and bottom sides of $B_R$. The events $\text{Piv}_S(\text{Cross}(2R, R))$ and $A_4(S, R)$ are closely related. However, we do not have $A_4(S, R) = \text{Piv}_S(\text{Cross}(2R, R))$ (contrary to Bernoulli percolation on $\mathbb{Z}^2$).

**Remark 4.2.** Proposition 4.1 is stated for the crossing event $\text{Cross}(2R, R)$ since we will apply this result to $2R \times R$ rectangles, but of course the proof works for any shape of rectangle.

### 4.1 The case of the bulk

Let $1 \leq \rho \leq R/10 \leq R$, let $y$ be a point of the plane and let $S = B_{\rho}(y)$ be the square of size $2\rho$ around $y$. In this subsection, we use the quasi-multiplicativity property and its consequences to estimate the probability of $\text{Piv}_S(\text{Cross}(2R, R))$ when $S$ is “in the bulk”.

We start with the following lemma:

**Lemma 4.3.** Let $\rho$, $R$ and $S$ be as above, and assume that $S$ is at distance at least $R/3$ from the sides of the square $[-2R, 2R] \times [-R, R]$. Also, let $A_4(S, R)$ be the event that there are two black arms in $[-2R, 2R] \times [-R, R] \setminus S$ from $\partial S$ to the left and right sides of $[-2R, 2R] \times [-R, R]$ and two white arms in $[-2R, 2R] \times [-R, R] \setminus S$ from $\partial S$ to the top and bottom sides of $[-2R, 2R] \times [-R, R]$. Then:

$$
\mathbb{P} \left[ A_4(S, R) \right] \asymp \alpha_4(\rho, R),
$$

where the constants in $\asymp$ are absolute constants.

**Proof.** The proof of the inequality $\mathbb{P} \left[ A_4(S, R) \right] \leq O(1) \alpha_4(\rho, R)$ is a direct consequence of the quasi-multiplicativity property (and of (1.1)). Let us prove the other inequality. We write the proof only for $y = 0$ (i.e. for $S = B_\rho$) since the proof for other values of $y$ is the same. Note that it is sufficient to prove the result for $R$ sufficiently large. Let $\delta \in (0, 1)$ to be determined later and assume that $R \geq \delta^{-2}$. Consider the following events (see Definition 2.10 and Proposition 2.13):

$$
\text{Dense}_{\delta}(R) := \text{Dense}_{\delta/100} (A(R/4, 2R)),
$$

$$
\text{QBC}_{\delta}(R) := \text{QBC}_{\delta} (A(3R/8, 2R)),
$$

and let $\text{GI}^{\text{ext}}(R/2)$ be defined as in Subsection 2.4.2.

Note that the event $\text{Dense}_{\delta}(R) \cap \text{QBC}_{\delta}(R) \cap \text{GI}^{\text{ext}}(R/2)$ is measurable with respect to $\omega \setminus B_{R/4}$. With exactly the same proof as Lemma 2.11, we obtain that $\mathbb{P} \left[ \text{Dense}_{\delta}(R) \right] \geq 1 - O(1) \delta^{-2} \exp(-O(1) (\delta \cdot R)^2) \geq 1 - O(1) \exp(-O(1) \delta^{-1})$ (since $R \geq \delta^{-2}$). Moreover, Proposition 2.13 implies that $\mathbb{P} \left[ \text{QBC}_{\delta}(R) \right] \geq 1 - O(1) R^{-1} \geq 1 - O(1) \delta^{-2}$ and Lemma 2.14 implies that $\mathbb{P} \left[ \text{GI}^{\text{ext}}(R/2) \right] \geq 1 - O(1) \delta$. Therefore, $\mathbb{P} \left[ \text{Dense}_{\delta}(R) \cap \text{QBC}_{\delta}(R) \cap \text{GI}^{\text{ext}}(R/2) \right]$ can be made as close to 1 as we want provided that we take $\delta$ sufficiently small. Hence, we can use Proposition 2.5 (which is the key result here) to say that, if $\delta$ is sufficiently small, then:

$$
\mathbb{P} \left[ A_4(\rho, R/2) \cap \text{Dense}_{\delta}(R) \cap \text{QBC}_{\delta}(R) \cap \text{GI}^{\text{ext}}(R/2) \right] \geq \alpha_4(\rho, R/2)/2 \geq \alpha_4(\rho, R/2) \cdot \alpha_4(\rho, R).
$$

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See Subsection 2.4.2: we have \( s^{\text{ext}}(\rho, R/2) \geq 50 \delta R = \{ s^{\text{ext}}(\rho, R/2) \geq 100 \delta R/2 \} \supseteq \text{GI}^{\text{ext}}_\delta(R) \). Hence, we also have:

\[
P_\delta \left[ A_4(\rho, R/2) \cap \text{Dense}_3(R) \cap \text{QBC}_\delta(R) \cap \{ s^{\text{ext}}(r, R/2) \geq 50 \delta R \} \right] \geq \alpha_4^{\text{qn}}(\rho, R)/2. \tag{4.1}
\]

Let \( \eta \in \text{Dense}_3(R) \cap \text{QBC}_\delta(R) \) be such that \( P^{\eta} \left[ A_4(r, R/2) \cap \{ s^{\text{ext}}(\rho, R/2) \geq 50 \delta R \} \right] > 0 \) and write \( \beta_0, \cdots, \beta_{k-1} \) for the interfaces that cross \( A(r, R) \) (in counter-clockwise order and the right-hand-side of \( \beta_0 \)- when going from \( \partial B_r \) to \( \partial B_R \) - assumed to be black, say). First, we work under the following conditional probability measure:

\[
\nu^{\eta}_{\rho, R, (\beta_j)} := P^{\eta} \left[ \cdot \bigg| A_4(\rho, R/2) \cap \text{Dense}_3(R) \cap \text{QBC}_\delta(R) \cap \{ s^{\text{ext}}(\rho, R/2) \geq 50 \delta R \}, \beta_0, \cdots, \beta_{k-1} \right].
\]

Thanks to (4.1), it is sufficient to prove that there exists a constant \( c = c(\delta) > 0 \) such that:

\[
\nu^{\eta}_{\rho, R, (\beta_j)} \left[ A_4^\square(S, R) \right] \geq c.
\]

Since \( \eta \in \text{Dense}_3(R) \cap \{ s^{\text{ext}}(\rho, R/2) \geq 50 \delta R \} \), we can choose four quads \( Q(\beta_j), j \in \{0, \cdots, 3\} \) such that:

(a) For every \( j \in \{0, \cdots, 3\} \), \( Q(\beta_j) \in \mathcal{Q}_\delta(A(3R/8, 2R)) \);

(b) For every \( j \in \{0, \cdots, 3\} \), one of the distinguished sides of \( Q(\beta_j) \) is included in \( \beta_j \);

(c) The other distinguished side of \( Q(\beta_j) \) (respectively \( Q(\beta_0) \), \( Q(\beta_2) \) and \( Q(\beta_3) \)) is included in the right (respectively top, left and bottom) side of \([-2R, 2R] \times [-R, R] \);

(d) For every \( j \in \{0, \cdots, 3\} \), \( Q(\beta_j) \) is included in the region between \( \beta_j \) and \( \beta_{j-1} \) (where \( \beta_{-1} := \beta_{k-1} \));

(e) If \( 0 \leq i \neq j \leq 3 \), then there is no Voronoi cell that intersects both \( Q(\beta_i) \) and \( Q(\beta_j) \).

See Figure 5. Let \( F \) be the event that, for every \( j \in \{0, \cdots, 3\} \), \( Q(\beta_j) \) is crossed (respectively dual-crossed) when \( j \) is even (respectively odd). Note that conditioning on \( (\beta_j) \), affects the percolation process as follows: if \( j \) is even (respectively odd) then there is a black (respectively white) crossing from \( \beta_j \) to \( \beta_{j-1} \). Hence, by using the fact that \( \eta \in \text{QBC}_\delta(R) \) and by applying the (quenched) Harris-FKG inequality, we obtain that there exists \( c = c(\delta) > 0 \) such that:

\[
\nu^{\eta}_{\rho, R, (\beta_j)} \left[ F \right] \geq c > 0.
\]

This ends the proof since \( F \subseteq A_4^\square(r, R) \).

---

**Figure 5:** The quads \( Q(\beta_1) \) and \( Q(\beta_2) \).
We have the following strengthening of Lemma 4.3, in the spirit of Proposition 2.5:

**Corollary 4.4.** There exists an absolute constant $\epsilon \in (0, 1)$ such that, for every event $G$ measurable with respect to $\omega \setminus A(y; 2\rho, R/6)$ that satisfies $P[G] \geq 1 - \epsilon$, we have:

$$P \left[ A^\square_1(S, R) \cap G \right] \geq \epsilon \alpha^\square_1(\rho, R).$$

**Proof.** The proof is exactly the same as the similar result Proposition 2.5. More precisely, this is a direct consequence of Lemma 4.3 and Proposition 2.4.

Now, let us prove the following result:

**Lemma 4.5.** Let $\rho$, $R$ and $S$ as above and assume that $S$ is at distance at least $R/2$ from the sides of the square $[-2R, 2R] \times [-R, R]$. Then:

$$P \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \geq \alpha^\square_1(\rho, R).$$

**Proof.** The fact that $P \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \geq \Omega(1) \alpha^\square_1(\rho, R)$ is a direct consequence of Lemma 4.3. Indeed, (except on a zero probability set) we have:\n
\begin{equation}
A^\square_1(S, R) \subseteq \text{Piv}_S(\text{Cross}(2R, R)).
\end{equation}

Now, let us prove that $P \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \leq O(1) \alpha^\square_1(\rho, R)$. We write the proof in the case $y = 0$. For every $\rho' > 0$, let $\text{Dense}(\rho') := \text{Dense}_{1/100}(A(\rho', 2\rho'))$ (remember Definition 2.10). Note that we have:

$$\text{Piv}_S(\text{Cross}(2R, R)) \subseteq A_4(2\rho, R/2) \cap (\text{Piv}_S(\text{Cross}(2R, R)) \setminus \text{Dense}(\rho)).$$

More generally, for all $k \in \{0, \cdots, \lfloor \log_2(\frac{R}{\rho}) \rfloor \} := k_0$ we have:

$$\text{Piv}_S(\text{Cross}(2R, R)) \subseteq A_4(2^{k+1}\rho, R/2) \cup (\text{Piv}_S(\text{Cross}(2R, R)) \setminus \text{Dense}(2^k \rho)),$$

which implies that $\text{Piv}_S(\text{Cross}(2R, R))$ is included in:

$$A_4(2\rho, R/2) \cup \left( \bigcup_{k=0}^{k_0} A_4(2^{k+2}\rho, R/2) \setminus \text{Dense}(2^k \rho) \right) \cup \neg \text{Dense}(2^{k_0+1} \rho)
\subseteq \tilde{A}_4(2\rho, R/2) \cup \left( \bigcup_{k=0}^{k_0} \tilde{A}_4(2^{k+2}\rho, R/2) \setminus \text{Dense}(2^k \rho) \right) \cup \neg \text{Dense}(2^{k_0+1} \rho),$$

where $\tilde{A}_4(2\rho, R/2)$ is defined in Definition 2.3. By using Proposition 2.4 and the fact that $\tilde{A}_4(2^{k+1}\rho, R/3)$ and $\text{Dense}(2^k \rho)$ are independent, we obtain that, for each $k \in \{0, \cdots, k_0\}$:

$$P \left[ \tilde{A}_4(2^{k+2}\rho, R/3) \setminus \text{Dense}(2^k \rho) \right] \leq \alpha^\square_1(2^{k+2} \rho, R/2) P \left[ \neg \text{Dense}(2^k \rho) \right].$$

With the same proof as Lemma 2.11, we obtain that:

$$P \left[ \neg \text{Dense}(2^k \rho) \right] \leq O(1) \left(2^k \rho\right)^2 e^{-\Omega(1)(2^k \rho)^2} \leq O(1) e^{-\Omega(1)(2^k \rho)^2}. \quad (4.4)$$

Note also that the quasi-multiplicativity property and (1.1) imply that:

$$\alpha^\square_1(2^{k+2} \rho, R/2) \leq O(1) 2^{O(1)k} \alpha^\square_1(\rho, R).$$

Therefore:

$$P \left[ \tilde{A}_4(2^{k+2}\rho, R/2) \setminus \text{Dense}(2^k \rho) \right] \leq O(1) \alpha^\square_1(\rho, R) 2^{O(1)k} e^{-\Omega(1)(2^k \rho)^2}. \quad (4.5)$$

---

\footnote{Consider a configuration for which $\mathcal{A}^\square_1(S, R)$ holds. If we replace the configuration restricted to $S$ by a sufficiently dense set of black (respectively white) points then $\text{Cross}(2R, R)$ is satisfied (respectively not satisfied).}
Similarly, $1 \leq O(1) \alpha_4^n(\rho, 2k_0) 2^{O(1)k_0}$, hence:

$$
\mathbb{P} \left[ \text{Dense}(2k_0 + 1, \rho) \right] \leq O(1) \alpha_4^n(\rho, 2k_0) 2^{O(1)k_0} e^{-\Omega(1)(2^{k_0})^2} \leq O(1) \alpha_4^n(\rho, R) 2^{O(1)k_0} e^{-\Omega(1)(2^{k_0})^2}. \quad (4.6)
$$

Now, we can conclude by doing an union-bound on (4.3) and by using the inequalities (4.5) and (4.6).

We still consider the case where $S$ is in the bulk. We end this subsection by showing another result which will be useful in the proof of the annealed scaling relations. The difference with Lemma 4.5 is that we study \textit{quenched} pivotal events (see Subsection 2.4.1 for the definition of these pivotal events). We first need the following definition: Let $S$ be as in Lemma 4.3 and remember the definition of $A_{\square}^4(R, S)$ in the statement of this lemma. We define the following analogous event: $	ilde{A}_{\square}^4(S, R)$ is the event that there are two black paths $\gamma_0$ and $\gamma_2$ and two white arms $\gamma_1$ and $\gamma_3$ as in the definition of $A_{\square}^4(R, S)$ with the further property that, for every $i \in \{0, \cdots, 3\}$, $\gamma_i \cap A(\rho, 2\rho) \subseteq Q_i$ where the rectangles $Q_i = Q_i(y, \rho)$ are defined on Figure 6.

![Figure 6: The rectangles $Q_i$ and the rectangles $\tilde{Q}_i$.](image)

**Lemma 4.6.** Let $R$ and $S$ be as above, and assume that $\rho = 1$ (i.e. $S$ is a $2 \times 2$ square). Also, assume that $S$ is at distance at least $R/3$ from the sides of the rectangle $[-2R, 2R] \times [-R, R]$. We have:

$$
\mathbb{P} \left[ \left\{ |\eta \cap S| = 1 \right\} \cap \text{Piv}_{A_{\square}^4(R, S)}(\text{Cross}(2R, R)) \right] \geq \Omega(1) \alpha_4^n(R).
$$

Before proving Lemma 4.6, let us note this lemma together with results from [AGMT16] implies that $\alpha_4^n(R) \leq O(1) R^{-(1+\epsilon)}$ for some $\epsilon > 0$, which is the first part of Proposition 1.13:

**Proof of the first part of Proposition 1.13.** By [AGMT16], if we let $S_1$ be the set of all squares of $2Z^2$ included in $[-2R, 2R] \times [-R, R]$ and at distance at least $R/3$ from the sides of $[-2R, 2R] \times [-R, R]$, then:

$$
\mathbb{E} \left[ \sum_{S \in S_1} \sum_{x \in \partial \Omega \cap S} \mathbb{P}^{\tau} \left[ \text{Piv}_{\tilde{A}_{\square}^4}(\text{Cross}(2R, R)) \right]^2 \right] \leq O(1) R^{-\Omega(1)}. \quad (4.7)
$$

See the end of Appendix B where we recall how the authors of [AGMT16] have obtained this estimate. (The definition of $S_1$ is not the same as in Appendix B but the proof is exactly
the same with the present definition.) The left-hand-side of (4.7) is at least:

$$\sum_{S \text{ square of the grid } \mathbb{Z}^2} \mathbb{E} \left[ \mathbb{P}^\eta \left( \text{Piv}_{\frac{\sqrt{2}}{2}} (\text{Cross}(R, R))^2 1_{|\eta \cap S| = 1} \right) \right]$$

$$\geq \sum_{S \text{ square of the grid } \mathbb{Z}^2} \mathbb{P} \left[ \{|\eta \cap S| = 1\} \cap \text{Piv}_{\frac{\sqrt{2}}{2}} (\text{Cross}(2R, R))^2 \right] (\text{by Jensen}).$$

We conclude by restricting the sum to the squares $S$ at least less than $R/3$ from the sides of $[-2R, 2R] \times [-R, R]$ and by using Lemma 4.6. □

The difficulty in the proof of Lemma 4.6 is that it is not obvious that, if $A_4(100, R)$ holds (for instance), then we can easily extend the arms until scale 1. We overcome this difficulty by considering the event that the Voronoi tiling near 0 “looks like the hexagonal lattice”.

Proof of Lemma 4.6. We write the proof in the case $y = 0$ (i.e. $S = B_\rho = B_1$). The strategy is illustrated by Figure 7. Note that it is sufficient to prove the result for $R$ larger than some constant. Let $r_1 \geq 1000$ to be determined later and assume that $R \geq r_1$. With the same proof as Lemma 4.3 (except that we work both at scale $r$ and at scale $R$ instead of working only at scale $R$) we obtain that, if $r$ is sufficiently large, then:

$$\mathbb{P} \left[ \tilde{A}_4(\emptyset, R) \right] \geq \Omega(1) \alpha_{4}^{\omega}(r, R),$$

where the constants in $\Omega(1)$ are absolute constants and the event $\tilde{A}_4(\emptyset, \cdot)$ is defined above Lemma 4.6.

As in Corollary 4.4, we actually have the following better result: There exists $r_0 \geq 1$ and $\epsilon \in (0, 1)$ such that, for every $r \geq r_0$ and for every event $G$ measurable with respect to
\( \omega \setminus A(2r, R/2) \) that satisfies \( \mathbb{P}[G] \geq 1 - \epsilon \), we have:

\[
\mathbb{P}\left[ \tilde{A}_4^{\square} (B_r, R) \cap G \right] \geq \epsilon \alpha_4^{an}(r, R). \tag{4.8}
\]

Now, for any \( r \geq 1 \) and any \( N \in \mathbb{N} \), write Dense\(^N(r)\) for the event that Dense\(_{1/100}(A(r/2, 2r))\) holds and that \( |\eta \cap A(r/2, 2r)| \leq N \). This event is a little different from the other events “Dense” that we study in this paper since this is an event that \( \eta \) is sufficiently dense but not too much. Let \( \epsilon > 0 \) as above and note that there exists \( r_1 \geq r_0 \) and \( N \in \mathbb{N} \) (with \( N \) depending on \( r_1 \)) such that \( \mathbb{P}[\text{Dense}\(^N(r_1)\)] \geq 1 - \epsilon \). Fix such \( r_1 \) and an \( N \). By the above, we have:

\[
\mathbb{P}\left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \right] \geq \Omega(1) \alpha_4^{an}(r_1, R) \geq \Omega(1)\alpha_4^{an}(R). \tag{4.9}
\]

The event Dense\(^N(r_1)\) provides sufficiently spatial independence so that, given a coloured configuration that satisfies \( \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \), one can extend the four arms until scale 1 with probability larger than some constant independent of \( R \). This can be done for instance as follows:

Let \( \text{Color}(r) \) denote the event that each point of \( \eta \cap \tilde{Q}_i \) is black (respectively white) if \( i \) is even (respectively odd), where the \( \tilde{Q}_i = \tilde{Q}(r) \)'s are the rectangle defined in Figure 6. Note that:

i) By the quenched FKG-Harris inequality (for the first inequality) and (4.9) (for the second inequality), we have:

\[
\mathbb{P}\left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \cap \text{Color}(r_1/2) \right] = \mathbb{E}\left[ \mathbb{P}^\eta \left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Color}(r_1/2) \right] \mathbb{1}_{\text{Dense}\(^N(r_1)\)} \right] \geq \frac{1}{2N} \mathbb{E}\left[ \mathbb{P}^\eta \left[ \tilde{A}_4^{\square} (B_{r_1}, R) \right] \mathbb{1}_{\text{Dense}\(^N(r_1)\)} \right] = \frac{1}{2N} \mathbb{P}\left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \right] \geq c_1 \alpha_4^{an}(R),
\]

where \( c_1 > 0 \) is a constant that depends only on \( r_1 \) and \( N \).

ii) The event:

\( \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \cap \text{Color}(r_1/2) \)

is independent of \( \omega \cap B_{r_1/2} \).

As a result, for any event \( A \) measurable with respect to \( \omega \cap B_{r_1/2} \) we have:

\[
\mathbb{P}\left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \cap \text{Color}(r_1/2) \cap A \right] \geq c_1 \mathbb{P}[A] \alpha_4^{an}(R),
\]

so it is sufficient for our purpose to find an event \( A \) measurable with respect to \( \omega \cap B_{r_1/2} \) such that \( \mathbb{P}[A] > 0 \) and:

\[
\mathbb{P}\left[ [\eta \cap S = 1] \cap \text{Piv}_2^{\mathbb{T}} (\text{Cross}(2R, R)) \right] \geq \Omega(1) \mathbb{P}\left[ \tilde{A}_4^{\square} (B_{r_1}, R) \cap \text{Dense}\(^N(r_1)\) \cap \text{Color}(r_1/2) \cap A \right], \tag{4.10}
\]

where the constants in \( \Omega(1) \) only depend on \( r_1 \). We choose \( A = \text{Hex}(r_1/2) \) where \( \text{Hex}(r) \) is the event (measurable with respect to \( \omega \cap B_r \)) that the Voronoi diagram “looks like the hexagonal lattice” in \( B_r \). More precisely, we let \( \mathbb{T} \) denote the triangular lattice of mesh size 2 and we define \( \text{Hex}(r) \) as the event that there exists a bijection \( f : \mathbb{T} \cap B_r \to \eta \cap B_r \) such
that \(|f(y) - y| \leq 1/100\) for every \(y\). On the event \(\text{Hex}(r_1/2)\), we have \(|\eta \cap S| = 1\). It is easy to see that \(\mathbb{P} [\text{Hex}(r_1/2)] > 0\) and that, if we condition on the event \(\mathbb{A}_\eta^\vartriangle (B_{r_1}, R) \cap \text{Dense}^N(r_1) \cap \text{Color}(r_1/2) \cap \text{Hex}(r_1/2)\), then we can extend the four arms “by hand” until the Voronoi cell of \(f(0)\) (where \(f\) is the above bijection) with probability larger than some constant that depends only on \(r_1\) (see Figure 7). Hence, (4.10) holds and we are done. \(\square\)

### 4.2 An estimate on the 4-arm event

Thanks to Proposition 2.7 (whose proof is written in Section 7), we have the following: Let \(1 \leq r \leq R\), then:

\[
\alpha_3^{a,n,+}(r, R) \leq \alpha_3^{a,n}(r, R) \leq \alpha_3^{a}(r, R). \tag{4.11}
\]

We now prove that \(\alpha_3^{a,n}(r, R) \geq \Omega(1)(r/R)^{2-\epsilon}\) for some \(\epsilon > 0\) (which strengthens the above inequality) i.e. we prove the second part of Proposition 1.13. In the case of percolation on \(\mathbb{Z}^2\) or on the triangular lattice, the analogue of this proposition is a direct consequence of Reimer’s inequality ([Rei00]). In the context of Voronoi percolation, it seems a priori natural to try to prove the following annealed Reimer’s inequality: Let \(A\) and \(B\) be two events measurable with respect to \(\omega\) restricted to a bounded domain, and define the disjoint occurrence of \(A\) and \(B\) as in (2.1); then, \(\mathbb{P} [\{A \vartriangle B\} \leq \mathbb{P} [A] \mathbb{P} [B]\). Unfortunately, this inequality is not true in general since it is not true as soon as \(A = B\), \(A\) depends only on \(\eta\), and \(\mathbb{P} [A] \in [0, 1]\).

**Proof of the second part of Proposition 1.13.** Let \(M \in [100, +\infty)\) to be determined later. We are inspired by the proof (by Vincent Beffara) of Proposition A.1 of [GPS10]. For any \(\rho \geq M\), let \(\text{Dense}(\rho) := \text{Dense}_{1/100}(A(\rho, 2\rho))\) (remember Definition 2.10). Also, let \(\text{Circ}(r_1, r_2)\) denote the event that there is a black circuit in \(A(r_1, r_2)\) and write, for any \(c \in (0, 1)\):

\[
\text{QAC}_c(\rho) = \{\text{GP}(A(\rho, 2\rho)) \geq c\}
\]

(for “Quenched Annulus Crossings”). Theorem 1.4 (applied for instance to four rectangles that surround the origin) and the (quenched) Harris-FKG inequality implies that there exists a constant \(c \in (0, 1)\) such that, for every \(\rho:\)

\[
\mathbb{P} [\text{QAC}_c(\rho)] \geq 1 - \rho^{-3}. \tag{4.12}
\]

Fix such a constant \(c\). Now, let \(\text{GP}(\rho, M)\) (for “Good Point configuration”) be the following event:

\[
\bigcap_{k=1}^{\log_2(M)} \text{Dense}(5^k \rho) \cap \text{QAC}_{c}(5^k \rho).
\]

If we use (a direct analogue of) Lemma 2.11 and (4.12), we obtain that:

\[
\mathbb{P} [\text{GP}(\rho, M)] \geq 1 - \sum_{k=1}^{\log_2(M)} \left( O(1) e^{-\Omega(1)(5^k \rho)^2} + (5^k \rho)^{-3} \right) \geq 1 - O(1) \rho^{-3}.
\]

Now, let \(\eta \in \text{GP}(\rho, M)\) be such that \(\mathbb{P}^\eta [\mathbb{A}_\beta(\rho, M, \rho)] > 0\). Also, let \(\beta_0, \beta_1, \beta_2\) be three simple paths drawn in the Voronoi grid, included in \(A(\rho, M\rho)\), that go from \(\partial B_{\rho}\) to \(\partial B_{3\rho}\) and that can arise as three consecutive interfaces (in counter-clockwise order, say). Write \(S_\beta\) for the region between \(\beta_1\) and \(\beta_3\). Write \(\mathbb{A}_\beta\) for the event that \(\beta_0, \beta_1, \beta_2\) are indeed consecutive interfaces, and write \(B_\beta\) for the event that \(\mathbb{A}_\beta, \beta_1, \beta_2\) holds and that there is an additional (i.e. disjoint from the Voronoi cells adjacent to \(\beta_1 \cup \beta_2 \cup \beta_3\)) black path in \(S_\beta\). Observe that, since \(\eta \in \text{Dense}(2\rho)\), the \([\log_2(M)]\) events

\[
\{ \exists \text{ a black path in } S_\beta \cap A(5^k \rho, 2 \times 5^k \rho) \text{ from a cell adjacent to } \beta_1 \text{ to a cell adjacent to } \beta_3 \},
\]

are

for \( k = 1, \cdots, \lfloor \log_2(M) \rfloor \), are independent under \( P^0 \). By classical circuit arguments, we obtain:

\[
P^0 \left[ A_5(\rho, M\rho) \mid A_{3\rho, \gamma_1, \gamma_2} \right] = P^0 \left[ \mathcal{B}_3 \mid A_{3\rho, \gamma_1, \gamma_2} \right] \leq (1 - c)^{\lfloor \log_2(M) \rfloor}.
\]

Hence:

\[
P^0 \left[ A_5(\rho, M\rho) \right] \leq (1 - c)^{\lfloor \log_2(M) \rfloor} E^0 \left[ X^3 1_{Y \geq 4} \right],
\]

where \( Y = Y(\rho, M) \) is the number of interfaces from \( \partial B_\rho \) to \( \partial B_{M\rho} \). By taking the expectation, we obtain that:

\[
\alpha_5^{an}(\rho, M\rho) \leq (1 - c)^{\lfloor \log_2(M) \rfloor} \mathbb{E} \left[ Y^3 1_{Y \geq 4} \right] + \mathbb{P}[-\text{QAC}(\rho)] \\
\leq (1 - c)^{\lfloor \log_2(M) \rfloor} \mathbb{E} \left[ Y^3 1_{Y \geq 4} \right] + O(1) \rho^{-3} \quad (\text{by (4.12)}).
\]

Now, we use the annealed BK inequality Proposition 2.2. Remember that this is an inequality for annealed-increasing events that depends on a \( \omega \) only in a bounded domain. We use the fact that \( A_{2j}(\rho, \rho M) \) is included in the \( j \)-disjoint occurrence of \( A_1(\rho, \rho M) \) to obtain that:\footnote{The event \( A_1(\rho, \rho M) \) do not depend on \( \omega \) in a finite domain but can be approximated for instance by the event “there is a black arm that uses only cells whose center \( x \in \eta \) belongs to \( B_{R_0} \)” for some very large \( R_0 \).}

\[
\mathbb{P} \left[ Y \geq 2j + 1 \right] = \mathbb{P} \left[ Y \geq 2j \right] = \alpha_2^{an}(r_1, r_2) \leq \alpha_1^{an}(\rho, \rho M)^j.
\]

The above together with (1.1) imply that \( \mathbb{P} \left[ Y \geq j \right] \leq O(1) M^{-ja} \) for some \( a > 0 \). Moreover, \( \mathbb{P} \left[ Y \geq 4 \right] = \alpha_4^{an}(\rho, M\rho) \geq \Omega(1) M^{-b} \) for some \( b < +\infty \). Hence:

\[
\mathbb{E} \left[ Y^3 1_{Y \geq 4} \right] \leq O(1) \alpha_4^{an}(\rho, M\rho).
\]

Therefore:

\[
\alpha_5^{an}(\rho, M\rho) \leq O(1) (1 - c)^{\lfloor \log_2(M) \rfloor} \alpha_4^{an}(\rho, M\rho) + O(1) \rho^{-3}.
\]

Remember that we have made the assumption that \( \rho \geq M \). By using the fact that the exponent of the 5-arm event is 2 (see Proposition 2.7), we obtain that, if \( M \) is sufficiently large, then

\[
\alpha_5^{an}(\rho, M\rho) - O(1) \rho^{-3} \geq \Omega(1) M^{-2} - O(1) M^{-3} \geq \Omega(1) M^{-2}.
\]

Hence, if \( M \) is sufficiently large then for every \( \rho \geq M \) we have:

\[
M^{-2} \leq O(1) (1 - c)^{\lfloor \log_2(M) \rfloor} \alpha_5^{an}(\rho, M\rho) \leq M^{-2} \epsilon \alpha_4^{an}(\rho, M\rho), \quad (4.13)
\]

for some \( \epsilon > 0 \).

Let us end the proof. Let \( C = C(j = 4) \) be the constant that appears in the statement of the quasi-multiplicativity property Proposition 1.6 and fix \( M \geq 100 \) sufficiently large so that (4.13) holds and so that \( M^c \geq C \). First, note that it is sufficient to prove the result for quantities of the form \( \alpha_5^{an}(M^i, M^j) \), where \( j \geq i \) are positive integers. Next, note that the quasi-multiplicativity property implies that:

\[
\alpha_4^{an}(M^i, M^j) \geq C^{-j-i} \prod_{k=1}^{j-1} \alpha_4^{an}(M^k, M^{k+1}).
\]

If we use (4.13), we obtain that:

\[
\alpha_4^{an}(M^i, M^j) \geq C^{-j-i} M^{(-2+2\epsilon)(j-i)},
\]

which is at least \( M^{(-2+\epsilon)(j-i)} \) since \( M^c \geq C \). This ends the proof. \( \square \)
4.3 Pivotal events for crossing events and arm events

In this subsection, we prove Proposition 4.1. Note that, if we only sum on the squares \( S \) in the “bulk” of the rectangle \([-2R, 2R] \times [-R, R]\) then Proposition 4.1 is a direct consequence of Lemma 4.5. We now have to deal with all the other squares, included those which are outside of \([-2R, 2R] \times [-R, R]\). This is essentially technical so the reader can skip this whole subsection in first reading and only keep in mind that we also prove the following analogue of Proposition 4.1 for arm events:

**Proposition 4.7.** Let \( \rho \geq 1 \) and let \( R \geq 100\rho \). Also, let \( j \in \mathbb{N}^* \). Then:

\[
\sum_{\text{S square of the grid } (2\rho \mathbb{Z})^2} \mathbb{P}[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(\rho, R) + O(1) \alpha_j^{an}(R) \left( \frac{R}{\rho} \right)^2 \alpha_4^{an}(\rho, R),
\]

where the constants in the \( O(1) \)'s may only depend on \( j \). If \( \rho = 1 \) (and since \( R^2 \alpha_4^{an}(R) \geq \Omega(1) \) by (4.11)), we have the following simpler formula:

\[
\sum_{\text{S square of the grid } (2\rho \mathbb{Z})^2} \mathbb{P}[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(R) R^2 \alpha_4^{an}(R).
\]

In order to prove Proposition 4.1, we first pursue the analysis of Subsection 4.1. To deal with the spatial dependencies of the model, we first need to introduce the notation \( \text{Piv}^E_D(A) \) which in words denotes the event that, conditionally on the coloured configuration in the set \( E \), the probability that the set \( D \) is (annealed) pivotal for \( A \) is positive. We introduce this quantity since it is measurable with respect to \( E \). We will often let \( E \) be an annulus which surrounds the set \( D \). Let \( D \) be a bounded Borel set, let \( A \) be an event measurable with respect to the coloured configuration \( \omega \) and let \( E \) be a Borel set. We write:

\[
\text{Piv}^E_D(A) := \left\{ \mathbb{P}[\text{Piv}_D(A) \mid \omega \cap E] > 0 \right\}.
\]

Let \( 1 \leq \rho \leq R/10 \leq R \), let \( y \) be a point of the plane and let \( S = B_\rho(y) \) be the square of size length \( 2\rho \) centered at \( y \).

**Lemma 4.8.** Let \( y, \rho, R \) and \( S = B_\rho(y) \) be as above. Let \( \rho_1 \in [\rho, +\infty) \) and \( \rho_2 \in [\rho_1, +\infty) \) and assume that \( S \) is included in the bounded connected component of \( A(y; \rho_1, \rho_2) \) and that \( A(y; \rho_1, \rho_2) \subseteq [-2R, 2R] \times [-R, R] \) (in particular, \( y \in [-2R, 2R] \times [-R, R] \)). Then:

\[
\mathbb{P}\left[ \text{Piv}_S^{A(y; \rho_1, \rho_2)}(\text{Cross}(2R, R)) \right] \leq O(1) \alpha_4^{an}(\rho_1, \rho_2).
\]

**Proof.** We write the proof for \( y = 0 \), since the proof in the other cases is the same. The proof is very similar to the proof of the inequality \( \mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))] \leq O(1) \alpha_3^{an}(\rho, R) \) of Lemma 4.5. Hence, we choose to indicate which are the results analogous to (4.2) and (4.4) (that the two key estimates in the proof of Lemma 4.5), and we omit the rest of the proof.

(a) The following inclusion is analogous to (4.2): For \( 0 < \rho' \leq \rho'' \), let:

\[
\text{Dense}\left( \rho', \rho'' \right) := \text{Dense}_{1/100}(A(\rho', 2\rho')) \cap \text{Dense}_{1/100}(A(\rho'', 2\rho'')).
\]

Then, for every \( k \in \{0, \cdots, \lfloor \log_2(\rho_2/(4\rho_1)) \rfloor \} \), \( \text{Piv}_S^{A(\rho_1, \rho_2)}(\text{Cross}(2R, R)) \) is included in:

\[
A_4(2^{k+1} \rho_1, \rho_2/2) \cup \left( \text{Piv}_S^{A(\rho_1, \rho_2)}(\text{Cross}(2R, R)) \setminus \text{Dense}(2^k \rho_1, \rho_2/2) \right).
\]

(b) The following inequalities are analogous to (4.4):

\[
\mathbb{P}\left[ \text{Dense}(2^k \rho_1, \rho_2/2) \right] \leq O(1) \exp(-\Omega(1)(2^k \rho_1)^2) + O(1) \exp(-\Omega(1)\rho_2^2) \leq O(1) \exp(-\Omega(1)(2^k \rho_1)^2).
\]

\( \square \)
We now use Lemma 4.8 to estimate the quantity $\mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))]$ when $S$ intersects the rectangle $[-2R, 2R] \times [-R, R]$ (for instance when $S$ is included in this rectangle). We first need the following notations: Let $d_0 = d_0(S)$ be the distance between $S$ and the closest side of $[-2R, 2R] \times [-R, R]$ and let $y_0$ be the orthogonal projection of $y$ on this side. Also, let $d_1 = d_1(S) \geq d_0$ be the distance between $y_0$ and the closest corner of $[-2R, 2R] \times [-R, R]$ and let $y_1$ be this corner. Write $\alpha_j^{a_1, \ldots, a_j}(\cdot, \cdot)$ for the probability of the $j$-arm event in the quarter plane. The following lemma is a generalization of Lemma 4.5.

**Lemma 4.9.** Let $y, \rho, R$ and $S = B_\rho(y)$ be as above. Remember that we have assumed that $\rho \leq R/10$. We have:

$$\mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))] \leq O(1) \alpha_2^{a_1, \ldots, a_j}(d_1 + \rho, R) \alpha_3^{a_1, \ldots, a_j}(d_0 + \rho, d_1) \alpha_4^{a_1, \ldots, a_j}(\rho, d_0).$$

**Proof.** We use the notations $S_0 = B_{10(d_0 + \rho)}(y_0)$ and $S_1 = B_{10(d_1 + \rho)}(y_1)$, see Figure 8 for an illustration. We also consider the annuli $A(y; \rho, (\rho + d_0)/10)$ and $A(y; 10(d_0 + \rho), d_1)$ (note that these annuli may be empty). Since $S \subseteq S_0 \subseteq S_1$, $\text{Piv}_S(\text{Cross}(2R, R))$ is included in the following intersection of three events:

\[
\text{Piv}_{S_1}(\text{Cross}(2R, R)) \cap \text{Piv}_{S_0}^{A(y_0; 10(d_0 + \rho), d_1)}(\text{Cross}(2R, R)) \\
\cap \text{Piv}_{S}^{A(y; \rho, (\rho + d_0)/10)}(\text{Cross}(2R, R)).
\]

Note furthermore that: $S$ is the inner square of $A(y; \rho, (\rho + d_0)/10)$, ii) $A(y; \rho, (\rho + d_0)/10)$ is included in $S_0$, iii) $S_0$ is included in the inner square of $A(y; 10(d_0 + \rho), d_1)$ and iv) $A(y; 10(d_0 + \rho), d_1)$ is included in $S_1$. Now note that:

i) $\text{Piv}_{S}^{A(y; \rho, (\rho + d_0)/10)}(\text{Cross}(2R, R))$ is measurable with respect to $\omega \cap A(y; \rho, (\rho + d_0)/10)$,

ii) $\text{Piv}_{S_0}^{A(y_0; 10(d_0 + \rho), d_1)}(\text{Cross}(2R, R))$ is measurable with respect to $\omega \cap A(y_0; 10(d_0 + \rho), d_1)$,

iii) $\text{Piv}_{S_1}(\text{Cross}(2R, R))$ is measurable with respect to $\omega \setminus S_1$.

![Figure 8: The points y, y0 and y1, the boxes S, S0 and S1, and some annuli centered at y or y0.](image-url)
Hence, by spatial independence, we deduce that $\mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))]$ is at most:

$$\mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))] \times \mathbb{P}\left[\text{Piv}_{S_0}^{(y_0, 10(d_0 + \rho), d_1)}(\text{Cross}(2R, R))\right]$$

$$\times \mathbb{P}\left[\text{Piv}_S^{A(y, \rho, (\rho + d_0)/10)}(\text{Cross}(2R, R))\right].$$

Lemma 4.8 implies that:

$$\mathbb{P}\left[\text{Piv}_S^{A(y, \rho, (\rho + d_0)/10)}(\text{Cross}(2R, R))\right] \leq O(1) \alpha_4^{an}(\rho, \rho + d_0).$$

Next, note that, by the quasi-multiplicativity property and (1.1), $\alpha_4^{an}(\rho, \rho + d_0) \leq O(1) \alpha_4^{an}(\rho, d_0)$.

It is a classical fact that when we want to estimate the probability of boundary events for boxes close to the boundary of the rectangle, the 3-arm event in the half-plane and the 2-arm event in the quarter plane play an important role. Actually, with exactly the same proof as Lemma 4.8 but applied to the 3-arm event in the half-plane, we have:

$$\mathbb{P}\left[\text{Piv}_{S_0}^{A(y_0, 10(d_0 + \rho), d_1)}(\text{Cross}(2R, R))\right] \leq O(1) \alpha_3^{an,+}(d_0 + \rho, d_1).$$

Once again by exactly the same proof as Lemma 4.8 we have:

$$\mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))] \leq O(1) \alpha_2^{an,+}(d_1 + \rho, R),$$

which ends the proof.

Let us now estimate the probability that boxes outside of $[-2R, 2R] \times [-R, R]$ are pivotal. Roughly speaking, this estimate implies that, if we want to bound

$$\sum_{S \text{ square of the grid } 2\rho \mathbb{Z}^2 \text{ not included in } [-2R, 2R] \times [-R, R]} \mathbb{P}[\text{Piv}_S(\text{Cross}(2R, R))]$$

thanks to the estimate of the present subsection, then it is enough to control the sum over the squares $S$ that are intersect $\partial([-2R, 2R] \times [-R, R])$, which will help us a lot.

**Lemma 4.10.** Let $\rho \geq 1$ and let $R \geq 100\rho$. Also, let $S$ be a square of the grid $2\rho \mathbb{Z}^2$ that intersects $\partial([-2R, 2R] \times [-R, R])$. Moreover, let $S'$ be the set of all squares of the grid $2\rho \mathbb{Z}^2$ that do not intersect $[-2R, 2R] \times [-R, R]$ and are such that, for any $S' \in S$, $S$ is the argmin of $S'' \rightarrow \text{dist}(S'', S')$ where $S''$ ranges over the set of squares of the grid $2\rho \mathbb{Z}^2$ that intersect $\partial([-2R, 2R] \times [-R, R])$. Then:

$$\sum_{S' \in S} \mathbb{P}[\text{Piv}_{S'}(\text{Cross}(2R, R))] \leq O(1) \alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0),$$

where $d_0 = d_0(S)$ and $d_1 = d_1(S)$ are the distances defined above Lemma 4.9.

**Proof.** For any $S' \in S$, we write $d' = \text{dist}(S', [-2R, 2R] \times [-R, R])$. We first observe that, if we sum only on the squares $S'$ that are distance at least $\log(R)$ from $[-2R, 2R] \times [-R, R]$ then the result is easy since $\text{Piv}_{S'}(\text{Cross}(2R, R))$ implies that there is no point of the Poisson process in a set of size of order $(d')^2$, which is an event of probability less than $O(1) \exp(-\Omega(1)/(d')^2)$. Thus, the sum over such squares $S'$ is less than $O(1) \exp(-\Omega(1) \log^2(R))$ which is much less than $\alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0)$ by (1.1).

Now, let $S' \in S$ be such that $d' \leq \log(R)$ and let $S'' := \text{B}_{\rho + 2d'}(y)$. Note that $S''$ has the same center as $S$ and that $S'' \supseteq S, S'$. In particular, $\text{Piv}_{S'}(\text{Cross}(2R, R)) \subseteq \text{Piv}_{S''}(\text{Cross}(2R, R))$. Let $\rho'' := \rho + 2d'$. Since $\rho'' \leq R/10$, we can apply Lemma 4.9 to $S''$ and we obtain that:

$$\mathbb{P}[\text{Piv}_{S''}(\text{Cross}(2R, R))] \leq O(1) \alpha_2^{an,+}(d''_1 + \rho'', R) \alpha_3^{an,+}(d''_0 + \rho'', d'') \alpha_4^{an}(\rho'', d''), \quad (4.14)$$

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where \( d_0'' := d_0(S'') \) and \( d_1'' := d_1(S'') \). Note that \( d_0'' \) and \( d_1'' \) satisfy \(|d_0 - d_0''| \leq 2d'\) and \(|d_1'' - d_1| \leq 2d'\).

We now distinguish between the two cases \( d' \leq 2\rho \) and \( d' \in [2\rho, \log(R)]\):

- If \( d' \leq 2\rho \), then \(|\rho'' - \rho|, |d_1'' - d_1| \) and \(|d_0'' - d_0|\) are less than or equal to \( 2\rho \). As a result, the quasi-multiplicativity property and (4.14) imply that:

\[
\mathbb{P} \left[ \text{Piv}_{S'}(\text{Cross}(2R, R)) \right] \leq O(1) \alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0) .
\]

Since there are \( O(1) \) squares \( S' \in S \) such that \( d' \leq 2\rho \), the proof is over in this case.

- So, let \( S' \in S \) be such that \( d' \in [2\rho, \log(R)] \), and observe that \( \text{Piv}_{S'}(\text{Cross}(2R, R)) \) is included in the intersection of the two independent events \( \text{Piv}_{S''}(\text{Cross}(2R, R)) \) and \( \neg \text{Dense}_{1/100}(S'' \setminus S) \). By using (4.14) and the fact that \( \mathbb{P} \left[ \neg \text{Dense}_{1/100}(S'' \setminus S) \right] \leq O(1) \exp(-\Omega(1)(d')^2) \), we obtain that \( \mathbb{P} \left[ \text{Piv}_{S'}(\text{Cross}(2R, R)) \right] \) is at most:

\[
O(1) \exp(-\Omega(1)(d')^2) \alpha_2^{an,+}(d_1 + \rho'', R) \alpha_3^{an,+}(d_0 + \rho'', d_1) \alpha_4^{an}(\rho'', d_0) .
\]

By using the quasi-multiplicativity property and the fact that \( \exp(-\Omega(1)(d')^2) \) decays super-polynomially fast, we obtain that this is also at most:

\[
O(1) \alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0) .
\]

Let us now sum over each \( S' \) such that \( d' \in [2\rho, \log(R)] \). Since, for each integer \( k \in [\log_2(\rho), \log_2(\log(R))] \), there exists at most \( O(1) \) squares \( S' \) such that \( d' \in [2^k, 2^{k+1}] \), the sum is at most:

\[
O(1) \sum_{k=\log_2(\rho)}^{\log_2(\log(R))} 2^{2k} \exp(-\Omega(1)2^{2k}) \alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0) \
\leq O(1) \alpha_2^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0) .
\]

This ends the proof. \( \square \)

Now, we can prove Proposition 4.1.

**Proof of Proposition 4.1.** Let \( S_1 \) be the set of the squares of the grid \((2\rho\mathbb{Z})^2\) that are included in \([-2R, 2R] \times [-R, R]\) and are at distance at most \( R/2 \) from the sides of this rectangles, and let \( S_2 \supseteq S_1 \) be the set of the squares of the grid \((2\rho\mathbb{Z})^2\) that satisfy the hypotheses of Lemma 4.9. First, note that if we use Lemma 4.5, we obtain that:

\[
\sum_{S \in S_1} \mathbb{P} \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] = O(1) \left( \frac{R}{\rho} \right)^2 \alpha_4^{an}(\rho, R) .
\]

Hence, it is sufficient to prove that:

\[
\sum_{S \text{ square of the grid } (2\rho\mathbb{Z})^2} \mathbb{P} \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \leq O(1) \left( \frac{R}{\rho} \right)^2 \alpha_4^{an}(\rho, R) .
\]

Moreover, by Lemma 4.10, it is sufficient to prove the estimate by summing only on \( S_2 \). Let \( S \in S_2 \). By using Lemma 4.9 combined with the estimates (4.11) (to control \( \alpha_3^{an,+}(\cdot, \cdot) \)) and (1.1) (to control \( \alpha_2^{an,+}(\cdot, \cdot) \)), we obtain that there exists an exponent \( a > 0 \) such that:

\[
\mathbb{P} \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \leq O(1) \left( \frac{d_1 + \rho}{R} \right)^a \alpha_4^{an}(\rho, d_0) \alpha_4^{an}(d_0 + \rho, d_1) .
\]

The quasi-multiplicativity property (together with (1.1)) implies that:

\[
\alpha_4^{an}(\rho, d_0) \alpha_4^{an}(d_0 + \rho, d_1) \leq O(1) \alpha_4^{an}(\rho, d_0 + \rho) \alpha_4^{an}(d_0 + \rho, d_1 + \rho) \leq O(1) \alpha_4^{an}(\rho, d_1 + \rho) .
\]
If we use the quasi-multiplicativity property once again together with the estimate \((4.11)\), we obtain that:

\[
\Pr[\text{Piv}_S(\text{Cross}(2R, R))] \leq O(1) \left( \frac{d_1 + \rho}{R} \right)^a \left( \frac{R}{d_1 + \rho} \right)^2 \alpha_4^{an}(\rho, R).
\]

Now, note that the numbers of squares \(S \in S_2\) such that \(d_1 + \rho \in [(2^k - 1)\rho, 2^{k+1}\rho]\) is 0 if \(k \geq \log_2(R/\rho)\) and is at most \(O(1) 2^{3k}\) otherwise. Hence we obtain that:

\[
\sum_{S \in S_2} \Pr[\text{Piv}_S(\text{Cross}(2R, R))] \leq O(1) \alpha_4^{an}(r, R) \sum_{k=0}^{\lfloor \log_2(R/\rho) \rfloor} 2^{3k} \left( \frac{2^k \rho}{R} \right)^{a-2} \alpha_4^{an}(\rho, R).
\]

which is the estimate we want.

Now, let us discuss the same kind of questions for arm events instead of crossing events, i.e. let us prove Proposition 4.7. The main difference is that we will have to use Proposition 2.13 instead of just \((4.11)\) to estimate \(\alpha_4^{an}(\cdot, \cdot)\). As previously, let \(y\) be a point of the plane, let \(\rho \geq 1\) let \(S := B_\rho(y)\) and let \(R \in [10\rho, \infty)\). Also, let \(j \in \mathbb{N}^*\). We will need the following lemmas, similar to Lemmas 4.5, 4.9 and 4.10.

**Lemma 4.11.** Let \(y, \rho, R\) and \(S = B_\rho(y)\) as above and assume that \(S \subseteq A(R/4, R/2)\). Then:

\[
\Pr[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(R) \alpha_4^{an}(\rho, R).
\]

The following is a generalization of Lemma 4.11.

**Lemma 4.12.** Let \(y, \rho, R\) and \(S = B_\rho(y)\) as above and assume that \(S \subseteq B_{R/2}\). Also, let \(d\) be the distance between \(y\) and \(0\). Then:

\[
\Pr[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(R) \alpha_4^{an}(\rho, d) \quad \text{if } d \geq 2\rho,
\]

and:

\[
\Pr[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(\rho, R) \quad \text{otherwise}.
\]

Let \(d_0 = d_0(S)\) and \(d_1 = d_1(S)\) be defined as in the study of \(\text{Cross}(2R, R)\), except that we consider distances to the box \(B_\rho\) instead of the rectangle \([-2R, 2R] \times [-R, R]\).

**Lemma 4.13.** Let \(y, \rho, R\) and \(S = B_\rho(y)\) as above and assume that \(S \cap A(R/2, R) \neq \emptyset\). Then:

\[
\Pr[\text{Piv}_S(A_j(1, R))] \leq O(1) \alpha_j^{an}(R) \alpha_3^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1, \alpha_4^{an}(\rho, d_0)).
\]

The following lemma is the analogue of Lemma 4.10:

**Lemma 4.14.** Let \(\rho \geq 1\) and let \(R \geq 100\rho\). Also, let \(S\) be a square of the grid \(2\rho \mathbb{Z}^2\) that intersects \(\partial B_{2R}\). Moreover, let \(S\) be the set of all squares of the grid \(2\rho \mathbb{Z}^2\) that do not intersect \(B_{2R}\) and are such that, for any \(S' \in S\), \(S\) is the argmin of \(S' \mapsto \text{dist}(S', S)\) where \(S'\) ranges over the squares of the grid \(2\rho \mathbb{Z}^2\) that intersect \(\partial B_{2R}\). Then:

\[
\sum_{S' \in S} \Pr[\text{Piv}_{S'}(\text{Cross}(2R, R))] \leq O(1) \alpha_j(R) \alpha_3^{an,+}(d_1 + \rho, R) \alpha_3^{an,+}(d_0 + \rho, d_1) \alpha_4^{an}(\rho, d_0).
\]

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Proof of Lemmas 4.11, 4.12, 4.13 and 4.14. The proofs of the above lemmas are very similar to the proofs of the analogous results for the crossing events, i.e. Lemmas 4.5, 4.9 and 4.10. Actually, there are two differences. The first one is that we do not have to consider the 2-arm event in the quarter plane but rather the 3-arm event in the quarter plane when we study the contribution of the boxes close to the corners. This difference does not add any difficulty. There is a second difference only when \( j \) is odd and larger than 1. This difference adds some new difficulties which are only technical. More precisely, if some box in the bulk is pivotal (and if \( \eta \) is sufficiently dense around this box) then there is a 4-arm around this box if \( j \) is even and there is either a 4-arm around or a 6-arm event if \( j \) is odd. For more details, see Appendix C. See also [Nol08] (see for instance Figure 12 therein) where Nolin deals with the same problem for the model of Bernoulli site percolation on the triangular lattice. \( \square \)

Proof of Proposition 4.7. Let \( S_1 \) be the set of the squares of the grid \((2\rho\mathbb{Z})^2\) that intersect \( B^\epsilon_{2\rho} \) (i.e. the squares whose contribution is studied in Lemmas 4.13 and 4.14. By using exactly the same arguments as in the proof of Proposition 4.1, we obtain that:

\[
\sum_{S \in S_1} \mathbb{P} \left[ \Pi_{S}(A_j(1,R)) \right] \leq O(1) \alpha_j^{an}(R) \left( \frac{R}{\rho} \right)^2 \alpha_4^{an}(\rho,R).
\]

Let \( S_2 \) be the set of the squares of the grid \((2\rho\mathbb{Z})^2\) that satisfy the hypothesis of Lemma 4.12. Thanks to this lemma, we obtain that:

\[
\sum_{S \in S_2} \mathbb{P} \left[ \Pi_{S}(A_j(1,R)) \right] \leq O(1) \alpha_j^{an}(\rho,R) + O(1) \alpha_j^{an}(R) \sum_{k=0}^{\lfloor \log_2(\frac{R}{\rho}) \rfloor} 2^{2k} \alpha_4^{an}(\rho,2^k \rho).
\]

The quasi-multiplicativity property and the fact that \( \alpha_4^{an}(2^k \rho,R) \geq \Omega(1) \left( \frac{2^k \rho}{R} \right)^{2-\varepsilon} \) (see Item (ii) of Proposition 1.13) imply that:

\[
\alpha_4^{an}(\rho,2^k \rho) \leq O(1) \alpha_4^{an}(\rho,R) \left( \frac{R}{2^k \rho} \right)^{2-\varepsilon}.
\]

Therefore, \( \sum_{S \in S_2} \mathbb{P} \left[ \Pi_{S}(A_j(1,R)) \right] \) is less than or equal to:

\[
O(1) \alpha_j^{an}(\rho,R) + O(1) \alpha_j^{an}(R) \alpha_4^{an}(\rho,R) \sum_{k=0}^{\lfloor \log_2(\frac{R}{\rho}) \rfloor} 2^{2k} \left( \frac{R}{2^k \rho} \right)^{2-\varepsilon} \\
= O(1) \alpha_j^{an}(\rho,R) + O(1) \alpha_j^{an}(R) \alpha_4^{an}(\rho,R) \left( \frac{R}{\rho} \right)^2.
\]

We are done since \( S_1 \cup S_2 = \{ \text{squares of the grid (2}\rho\mathbb{Z})^2\}. \) \( \square \)

5 Extension of the results to the near-critical phase

In this section, we extend the results of other sections to the near-critical phase. Remember the definition of the correlation length \( L^{an}(p) \) from Definition 1.9. Let us first prove the following result.

**Lemma 5.1.** For every \( p > 1/2, L^{an}(p) < +\infty. \)

**Proof.** This is a simple consequence of the exponential decay property Theorem 2 of [BR06a] or Theorem 1 of [DCRT17]: for every \( p < 1/2, \) there exists a constant \( c = c(p) > 0 \) such that:

\[
\alpha_4^{an}(R) \leq \exp(-c(p) R).
\]

Indeed, by duality, this implies that for every \( p > 1/2 \) there exists a constant \( c' = c'(p) > 0 \) such that:

\[
\mathbb{P}_p \left[ \text{Cross}(2R, R) \right] \geq 1 - \exp(-c'(p)R).
\]

\( \square \)
5.1 Extension of the annealed and quenched box-crossing properties

Let us use the idea of Lemma 4.17 of [ATT16] in order to extend the annealed box-crossing property to the near-critical regime.

Proposition 5.2. Let \( p \in (1/2, 3/4) \) and let \( \rho > 0 \). There exists a constant \( c = c(p) \in (0, 1) \) such that, for every \( R \in (0, L^{an}(p)] \):

\[
c \leq \mathbb{P}_p [\text{Cross}(\rho R, R)] \leq 1 - c.
\]

The constants \( C \) and \( c \) may also depend on \( \epsilon_0 \) in the definition of \( L^{an}(p) \).

Proof. The left-hand-inequality is a direct consequence of Theorem 1.3 (and is true for any \( R \in (0, +\infty) \)). Let us prove the right-hand-inequality. Let \( \text{Circ}(r_1, r_2) \) be the event that there is a black circuit in the annulus \( A(r_1, r_2) \). Note that this event holds if and only if there is no white path from \( \partial B_{r_1} \) to \( \partial B_{r_2} \). Thanks to (1.1) (and by using the fact that Voronoi percolation of parameter 1/2 is self-dual), we know that there exists \( h > 0 \) such that

\[
\mathbb{P}_{1/2} [\text{Circ}(\rho, M \rho/2)] \geq 1 - \frac{1}{h} M^{-h} \text{ for any } \rho \geq 1 \text{ and } M \geq 1.
\]

Let \( N \in \mathbb{N}^* \) be such that:

\[
\left(1 - \frac{1}{h} N^{-h}\right)^3 \leq (1 - \epsilon_0)^{1/2}.
\]

Next, fix some constant \( \tau \in (0, 1) \) sufficiently small so that:

\[
\left(1 - \frac{1}{h (2N)^{3/2}}\right)^4 \geq (1 - \epsilon_0)^{1/2},
\]

where \( \epsilon_0 \) is the (small) constant used to define \( L^{an}(p) \).

Thanks to the (annealed) FKG-Harris inequality, it is sufficient to prove that for every \( r \in [N, L^{an}(p)] \) we have:

\[
\mathbb{P}_p [\text{Cross}(2r, r)] \leq 1 - \tau.
\]

Assume (for a contradiction) that there exists \( r \in [N, L^{an}(p)] \) such that \( \mathbb{P}_p [\text{Cross}(2r, r)] \geq 1 - \tau \). By the standard square-root trick, we have the existence of a segment \( I_r \) included in the left side of \([0, r] \times [0, 2r]\) and a segment \( I'_r \) included in the right side of \([0, r] \times [0, 2r]\) such that: (a) the length of \( I_r \) and \( I'_r \) is \( r/N \) and (b) the probability (under \( \mathbb{P}_p \)) that there is black path in \([0, r] \times [0, 2r]\) from \( I_r \) to \( I'_r \) is at least \( 1 - \frac{1}{h} (2N)^{3/2} \). Let \( \text{Cross}(I_r, I'_r) \) denote this last event.

Now, note that there exist four events obtained by translation and symmetry axis from \( \text{Cross}(I_r, I'_r) \) and three events obtained by translation of \( \text{Circ}(r/N, r/2) \) such that, if these three event hold, then \( \text{Cross}(4r, 2r) \) hold. See Figure 9 for more precisions. By applying the (annealed) FKG-Harris inequality, we obtain that:

\[
\mathbb{P}_p [\text{Cross}(4r, 2r)] \geq \left(1 - \frac{1}{h (2N)^{3/2}}\right)^4 \left(1 - \frac{1}{h} N^{-h}\right)^3 \geq (1 - \epsilon_0)^{1/2} (1 - \epsilon_0)^{1/2} = 1 - \epsilon_0,
\]

which is a contradiction since \( 2r \leq L^{an}(p) \). Note that we have used that (since \( p > 1/2 \)):

\[
\mathbb{P}_p [\text{Circ}(r/N, r/2)] \geq \mathbb{P}_{1/2} [\text{Circ}(r/N, r/2)] = 1 - \epsilon_0.
\]

\[\square\]

Now, we extend the quenched box-crossing result Theorem 1.4.

Proposition 5.3. Let \( \rho > 0 \). We have the following:
Figure 9: Seven events to obtain $\text{Cross}(2r, 4r)$.

i) There exists an absolute constant $\epsilon > 0$ and a constant $C = C(\rho) < +\infty$ such that, for every $p \in (1/2, 3/4]$ and every $R \in (0, +\infty)$ we have:

$$\text{Var} \left( P^\eta_p [\text{Cross}(\rho R, R)] \right) \leq C R^{-\epsilon}.$$  

This implies the following estimate:

ii) For every $\gamma \in (0, +\infty)$, there exists a positive constant $c = c(\rho, \gamma) \in (0, 1)$ such that, for every $p \in (1/2, 3/4]$ and every $R \in (0, L^{an}(p)]$:

$$P \left[ c \leq P^\eta_p [\text{Cross}(\rho R, R)] \leq 1 - c \right] \geq 1 - R^{-\gamma}.$$  

The constants $C$ and $c$ may also depend on $\epsilon_0$ in the definition of $L^{an}(p)$.

Proof. The way we obtain Item ii) from Item i) is exactly the same as for Theorem 1.4 (see [AGMT16]) except that we use Proposition 5.2 instead of Theorem 1.3. So, let us prove Item i). To this purpose, we rely on Appendix B where we have recalled the main steps of the proof of Theorem 1.4. In the case $p > 1/2$, the first step is exactly the same and we obtain that:

$$\text{Var} \left( P^\eta_p [\text{Cross}(\rho R, R)] \right) \leq \mathbb{E} \left[ \sum_{x \in \eta} P^\eta_p [\text{Piv}_2(\text{Cross}(\rho R, R))]^2 \right].$$  

(5.1)

For the second step, we cannot use the BK inequality in the case $p > 1/2$ since we do not know whether this inequality is true or not. So we prove the result corresponding to this step for $p > 1/2$ by using the (already known) analogous result for $p = 1/2$. More precisely, since $p > 1/2$, we have the following:

$$P \left[ P^\eta_p \left[ A^\text{cell}_1(1, R) \right] \geq R^{-\epsilon} \right] \leq P \left[ P^\eta_{1/2} \left[ A^\text{cell}_1(1, R) \right] \geq R^{-\epsilon} \right].$$  

(5.2)

where $A^\text{cell}_1(S, R)$ is defined in the paragraph above (B.3). Thanks to (5.2), the following is a direct consequence of (B.3): For every $\gamma > 0$, there exists $\epsilon > 0$ such that the following holds:

$$P \left[ P^\eta_p \left[ A^\text{cell}_1(1, R) \right] \geq R^{-\epsilon} \right] \leq \frac{1}{\epsilon} R^{-\gamma}.$$  

To conclude, we do exactly as in the last step of the proof of Theorem 1.4 written in Appendix A. We can apply this strategy even if $p > 1/2$ since the Schramm-Steif results discussed in Appendix A (and more precisely Corollary A.4 which is the inequality that we need) hold for every $p$. The only dependence on $p$ in Corollary A.4 is a factor $\frac{1}{2^{\sqrt{p(1-p)}}}$ factor, but this is not problem since we have restricted ourself to the case $p \in [1/4, 3/4]$. □
Remark 5.4. Now, we can explain the reason why, in Appendix B, we have (slightly) changed the algorithm used to estimate the sum $\sum_{x\in I} \mathbb{P}_p^\uparrow \left[ \text{Piv}_x^\uparrow (\text{Cross}(\rho R, R)) \right]^2$. The reason is that we wanted to be able to bound the revealment of the algorithm with a quantity that involves only white arms (so that we can use the obvious inequality 5.2), which is not possible with the algorithm chosen in [AGMT16]. Note that we also had to (slightly) extend Schramm-Steif’s theorem, see Appendix A.

5.2 Extension of the results of Sections 3, 4 and 7

Before proving the annealed scaling relations, we need to prove that the results of Sections 3, 4 and 7 are also true in the near-critical phase. More precisely, we are going to explain why every result of Sections 3, 4 and 7 is also true for $p \in (1/2, 3/4]$, providing that we assume that every length is less than or equal to the correlation length $L^{an}(p)$. For instance, we will obtain that the quasi-multiplicativity property Proposition 1.6 is also true for $p \in (1/2, 3/4]$ if we assume that $1 \leq r_1 \leq r_2 \leq r_3 \leq L^{an}(p)$. An important fact is that the different constants (e.g. the constant $C = C(j)$ from Proposition 1.6) will not depend on $p$ (but only on the parameter $c_0$ from the definition of $L^{an}(p) = L^{an,c_0}(p)$).

In Subsection 5.1, we have proved that the annealed and box crossing estimates also hold in the near-critical phase. Now, what are the properties specific to $p = 1/2$ that we use in Sections 3, 4 and 7? There are two such properties:

- The annealed BK inequality (see Subsection 2.2). The only place where we have used this inequality is in the proof of the second part of Proposition 1.13 (see Subsection 4.2). In this proof, we have used this inequality to prove that:

$$\alpha_{2j,1/2}^{an}(r_1, r_2) \leq \left( \alpha_{1,1/2}^{an}(r_1, r_2) \right)^j,$$

since $A_{2j}(r_1, r_2)$ is included in the $j$-disjoint occurrence of $A_1(r_1, r_2)$. Let $A_1^j(r_1, r_2)$ denote the event that there is a white path from $\partial B_{1/2}$ to $\partial B_{1/2}$. Note that $A_{2j}(r_1, r_2)$ is also included in the $j$-disjoint occurrence of $A_1^j(r_1, r_2)$, hence:

$$\alpha_{2j,p}^{an}(r_1, r_2) \leq \mathbb{P}_p \left[ A_1^j(r_1, r_2)^\uparrow \right] \leq \mathbb{P}_{1/2} \left[ A_1^j(r_1, r_2)^\uparrow \right] \leq \mathbb{P}_{1/2} \left[ A_1^j(r_1, r_2)^\uparrow \right],$$

by the annealed BK inequality. Hence, there exists $a > 0$ such that $\alpha_{2j,p}^{an}(r_1, r_2) \leq O(1) (r_1/r_2)^a$, which is exactly what we needed in the proof of the second part of Proposition 1.13.

- The fact that the model is self-dual. We have actually used this property implicitly all along Sections 4 and 7. More precisely: If $Q$ is a quad, let $\text{Cross}^*(Q)$ be the event that there is a crossing of $Q$ by a white path. We have used a lot Proposition 2.13 both for the event $\text{Cross}(Q)$ and $\text{Cross}^*(Q)$ although we have proved these results only for the event $\text{Cross}(Q)$. When $p = 1/2$ this is not a problem since $\text{Cross}(Q)$ and $\text{Cross}^*(Q)$ have the same $\mathbb{P}_{1/2}$ and $\mathbb{P}_{1/2}$-probabilities; but when $p > 1/2$ we need to prove that Proposition 2.13 also holds with $\text{Cross}^*(Q)$ instead of $\text{Cross}(Q)$ as soon as we consider a domain $D$ such that $\text{diam}(D) \leq L^{an}(p)$. The proof is actually the same except that we use Proposition 5.3 instead of Theorem 1.4.

6 Proof of the annealed scaling relations

In this section, we prove our main result Theorem 1.11 by using the results of all the other sections.

We first prove Proposition 1.10 and the scaling relation (1.4) of Theorem 1.11. We follow the classical strategy for Bernoulli percolation on a deterministic lattice by Kesten [Kes87], see also [Wer07, Nol08]. The only difference is that we will have to deal both with the
annealed and the quenched notions of pivotal events. We refer to Subsection 2.4.1 for these two notions of pivotal events. Let us recall that, as explained in Section 5, all our results on arm and pivotal events also hold for \( p \in (1/2, 3/4) \) (with constant that do not depend on \( p \)) as soon as we work under the correlation length \( L^{an}(p) \).

**Proof of Proposition 1.10 and of (1.4) from Theorem 1.11.** We will need the following lemma:

**Lemma 6.1.** Let \( R \in [1, L^{an}(p)] \). We have:

\[
\frac{d}{dp} \mathbb{P}^p_c [\text{Cross}(2R, R)] \asymp R^2 \alpha_{4,n}^a(R),
\]

\( \forall j \in \mathbb{N}^* \), \( \left| \frac{d}{dp} \log(\alpha_{j,n}^a(R)) \right| \leq O(1) R^2 \alpha_{4,n}^a(R). \quad (6.2) \)

The constants in the \( \asymp \) and \( O(1) \) may only depend on the choice of \( \epsilon_0 \) in Definition 1.9 (and on \( j \) for the \( O(1) \)).

Before proving Lemma 6.1, let us explain why this lemma implies Proposition 1.10 and (1.4). If we integrate (6.1) from 1/2 to \( p \), we obtain that:

\[
\int_{1/2}^p R^2 \alpha_{4,n}^a(R)\, du \asymp \mathbb{P}_p [\text{Cross}(2R, R)] - \mathbb{P}_{1/2} [\text{Cross}(2R, R)] \leq 1,
\]

Moreover, if we integrate (6.2) from 1/2 to \( p \), we obtain that:

\[
\left| \log(\alpha_{j,n}^a(R)) - \log(\alpha_{j,1/2}^a(R)) \right| \leq O(1) \int_{1/2}^p R^2 \alpha_{4,n}^a(R)\, du.
\]

Hence:

\[
\left| \log(\alpha_{j,p}^a(R)) - \log(\alpha_{j,1/2}^a(R)) \right| \leq O(1) \quad \text{for } j_{max},
\]

Together with the quasi-multiplicativity property, this implies Proposition 1.10.

Now, let us integrate (6.1) from 1/2 to \( p \) with the choice \( R = L^{an}(p) \). If we use Proposition 1.10 with \( j = 4 \), we obtain that:

\[
(p - 1/2) L^{an}(p) \alpha_{4,1/2}^n(L^{an}(p)) \asymp \mathbb{P}_p [\text{Cross}(2L^{an}(p), L^{an}(p))] - \mathbb{P}_{1/2} [\text{Cross}(2L^{an}(p), L^{an}(p))] \in [1 - \epsilon_0 - \mathbb{P}_{1/2} [\text{Cross}(2L^{an}(p), L^{an}(p))], 1],
\]

which implies the scaling relation (1.4) from Theorem 1.11 since \( \epsilon_0 \) is sufficiently small.

**Proof of Lemma 6.1.** Kesten’s idea is to use Russo’s differential formula (see for instance Theorem 2.25 in [Gri99]) that can be stated as follows: Let \( n \in \mathbb{N}^* \) and let \( A \subseteq \{-1, 1\}^n \) be an increasing event. Also, let \( P^\alpha_p = p\delta_1 + (1 - p)\delta_{-1} \). Then:

\[
\frac{d}{dp} P^\alpha_p [A] = \sum_{i=1}^n P^\alpha_p [\text{Piv}^\alpha_p (A)].
\]

(See Subsection 2.4.1 for our notations for pivotal points and pivotal sets.) To use this formula, we have to work at a quenched level. Note that, for every \( \eta \), the number of \( x \in \eta \) whose cell intersects \( \text{Cross}(2R, R) \) is finite hence, if we condition on \( \eta \), the event \( \text{Cross}(2R, R) \) depends on the colors of only finitely many points. So, we can use Russo’s formula and we obtain that for every \( \eta \):

\[
\frac{d}{dp} P^\alpha_p [\text{Cross}(2R, R)] = \sum_{x \in \eta} P^\alpha_p [\text{Piv}^\alpha_p (\text{Cross}(2R, R))].
\]

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Now, let $R \leq L^{an}(p)$ and let $(S_i)_{i \in \mathbb{N}}$ be an enumeration of the squares of the $\mathbb{Z}^2$ grid. For all $i$, write $N_i(\text{Cross}(2R, R))$ for the number of points $x \in \eta$ which are quenched-pivotal for $\text{Cross}(2R, R)$. Note that, if one fixes $R$ and let $M > 0$ then:

$$\text{Card}\{x : P_p^n[\text{Piv}_n^q(\text{Cross}(2R, R))] > 0\}.$$ 

is larger than $M$ with probability less than $O(1) e^{-\Omega(1)M^2}$. This implies that one can exchange the expectation with the derivative and obtain that:

$$\frac{d}{dp} P_p[\text{Cross}(2R, R)] = \sum_{i \in \mathbb{N}} E_p[N_i(\text{Cross}(2R, R))]. \quad (6.3)$$

By using the fact that a.s. $\text{Piv}_n^q(\text{Cross}(2R, R)) \subseteq \text{Piv}_n^q(\text{Cross}(2R, R))$ and that $\text{Piv}_n^q(\text{Cross}(2R, R))$ is independent of the configuration in $S_i$ we obtain that:

$$E_p[N_i(\text{Cross}(2R, R))] \leq E_p[|\{x \in \eta \cap S_i\}| P_p[\text{Piv}_{S_i}(\text{Cross}(2R, R))]] \leq O(1) P_p[\text{Piv}_{S_i}(\text{Cross}(2R, R))].$$

If we combine the above with (6.3) and if we use Proposition 4.1, we obtain that:

$$\frac{d}{dp} P_p[\text{Cross}(2R, R)] \leq O(1) \sum_{i \in \mathbb{N}} P_p[\text{Piv}_{S_i}(\text{Cross}(2R, R))] = O(1) R^2 \alpha_{4,p}^{an}(R),$$

which is the upper-bound of (6.1). The lower-bound is a simple consequence of Lemma 4.6. Indeed, this lemma implies that:

$$E_p[N_i(\text{Cross}(2R, R))] \geq P_p[|\{x \in \eta \cap S_i\}| P_p[\text{Piv}_{S_i}(\text{Cross}(2R, R))]] \geq \Omega(1) \alpha_{4,p}^{an}(R).$$

Now, let us prove (6.2) for $j = 1$. Since $A_1(R)$ is also an increasing event, by the same techniques as above, we obtain that:

$$\frac{d}{dp} P_p[A_1(R)] = E_p[N(A_1(R))] \leq O(1) \sum_{i \in \mathbb{N}} P_p[\text{Piv}_{S_i}(A_1(R))].$$

Thanks to Proposition 4.7, we have obtained (6.2) for $j = 1$.

We now prove (6.2) with $j \geq 2$. This is a little different since the events $A_j(R)$ are not monotonic any more. To overcome this issue, we need an extension of Russo’s formula to non-monotonic events. Actually, the following holds (and the proof is exactly the same as for classical Russo’s formula):

Let $n \in \mathbb{N}^*$ and let $A \subseteq \{-1, 1\}$ be any event. Then:

$$\left| \frac{d}{dp} P_p^n[A] \right| \leq \sum_{i=1}^n P_p^n[\text{Piv}_n^q(A)].$$

Now, the proof of (6.2) is the same as for $j = 1$ and we are done. 

This ends the proof of Proposition 1.10 and of the scaling relation (1.4) from Theorem 1.11.

What remains to prove is the scaling relation (1.3) from Theorem 1.11. Thanks to Proposition 1.10 applied to $j = 1$, it is sufficient to prove the following lemma:
Lemma 6.2. Let \( p \in (1/2, 3/4] \). We have:

\[
\theta^n(p) \simeq \alpha_{1,p}^n (L^n(p)) ,
\]

where the constants in \( \simeq \) may only depend on the \( \epsilon_0 \) in the definition of \( L^n(p) \).

Proof. First note that it is sufficient to prove the result for \( p \in (1/2, p_0) \) for some \( p_0 \in (1/2, 3/4] \). Remember the definition of \( \text{Dens}_1(D) \) in Subsection 2.4.2. By Lemma 2.11, if \( R \) is sufficiently large (\( R \geq R_0 \geq 1000 \), say), then \( \mathbb{P} \left[ \text{Dens}_{1/100}([0,2R] \times [0,R]) \right] \geq 1 - \epsilon_0 \). Let \( p_0 \) be the infimum over the \( p \)'s in \( [1/2, 3/4] \) such that \( L^n(p) \geq R_0 \vee 1000 \). Note that \( p_0 > 1/2 \). Consider some parameter \( p \in (1/2, p_0) \). We now apply a Peierls argument. If \( Q \) is a \( 4L^n(p) \times 2L^n(p) \) rectangle, then we say that \( Q \) is good if: i) \( Q \) is crossed lengthwise, ii) the two \( 2L^n(p) \times 2L^n(p) \) squares whose union is \( Q \) are crossed from top to bottom and from left to right and iii) \( \text{Dens}_{1/100}(Q) \) holds. Note that \( \mathbb{P} \left[ Q \right] \) is good \( \geq 1 - 4\epsilon_0 \) by definition of the correlation length.

Now, let \( L(p) \) be the square lattice scaled by a factor \( 2L^n(p) \). We say that an edge \( e = \{x,y\} \) of this lattice is good if the \( 4L^n(p) \times 2L^n(p) \) rectangle which is the union of the two \( 2L^n(p) \times 2L^n(p) \) squares centered at \( x \) and \( y \) is good. Note that we have defined a 2-dependent percolation model on \( L(p) \) with parameter at least \( 1 - 4\epsilon_0 \). A standard Peierls-type argument implies that, since we have chosen \( \epsilon_0 \) small enough, the probability that there is an infinite good path starting \( 0 \) is positive.

Now, note that if (a) \( A_j(1,3L^n(p)) \) holds, (b) the twelve edges of \( L(p) \) closest to \( 0 \) are good, (c) there is an infinite path made of good edges starting from \( 0 \) and (d) \( B_j \) - the ball of radius \( 1 \) - is entirely colored black, then \( \{0 \leftrightarrow \infty\} \) holds. This observation and the above paragraph (together with the annealed FKG-Harris inequality) imply the lemma. \( \square \)

7 The quasi-multiplicativity property

In this section, we only work at \( p = 1/2 \), hence we forget the subscript \( p \) in the notations. In Subsections 7.1, 7.2 and 7.3, we only use the preliminary results of Subsections 3.1 and 3.2. In Subsections 7.4, we use the results of Subsection 3.3 which are consequences of Subsections 7.1, 7.2 and 7.3.

7.1 The case \( j \) even

In this subsection, we prove the quasi-multiplicativity property Proposition 1.6 in the case \( j \) even:

Proposition 7.1. Let \( j \in \mathbb{N}^* \) even. There exists a constant \( C = C(j) \in [1, +\infty) \) such that, for all \( 1 \leq r_1 \leq r_2 \leq r_3 \):

\[
\frac{1}{C} \alpha_j^{an}(r_1, r_3) \leq \alpha_j^{an}(r_1, r_2) \alpha_j^{an}(r_2, r_3) \leq C \alpha_j^{an}(r_1, r_3) .
\]

Remark 7.2. As we will see in Subsection 7.3, the same proof will imply the quasi-multiplicativity for the quantities \( \alpha_{j,n,+}^{an}(\cdot, \cdot) \), for any \( j \).

As explained in Subsection 2.6.1, we first need to define what is a “good percolation configuration” (i.e. a configuration for which it is not difficult to extend the \( j \) arms).

We write the proof of Proposition 7.1 for \( j = 4 \) since the proof for other even integers is the same.

7.1.1 What does “looking good” means for the Voronoi percolation configurations

The point configuration. We consider \( \delta \in (0, 1/1000) \) and \( R \in \left[ \delta^{-2}, +\infty \right) \). In the proof of Proposition 7.1, we use the following notations (where the notations from the right-
Next, we define the two following events:

\[
\begin{align*}
\text{QBC}_\delta^\text{ext}(R) & := \text{QBC}_\delta^\text{ext}(A(R, 3R/4, 3R/2)) , \\
\text{QBC}_\delta^\text{int}(R) & := \text{QBC}_\delta^\text{int}(A(R, 4R)) .
\end{align*}
\]

As a result we have:

\[
\text{GP}_\delta^\text{ext}(R) := \mathbb{P}\left[\text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \text{QBC}_\delta^\text{ext}(R) \mid \eta \cap A(R/2, 2R) \geq 3/4\right] \quad (7.2)
\]

and:

\[
\text{GP}_\delta^\text{int}(R) := \mathbb{P}\left[\text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \text{QBC}_\delta^\text{int}(R) \mid \eta \cap A(R/2, 2R) \geq 3/4\right] \quad (7.3)
\]

(for “Good Point configuration”). In other words, \(\text{GP}_\delta^\text{ext}(R)\) (respectively \(\text{GP}_\delta^\text{int}(R)\)) is the event that, conditionally on \(\eta \cap A(R/2, 2R)\), the probability that \(\text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \text{QBC}_\delta^\text{ext}(R)\) (respectively \(\text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \text{QBC}_\delta^\text{int}(R)\)) holds at least 3/4. Note that, if \(\text{Dense}_\delta(R)\) holds and if the Voronoi cell of some \(x \in \eta\) intersects \(A(3R/4, 3R/2)\), then \(x \in A(R/2, 2R)\). Hence, \(\text{Dense}_\delta(R) \cap \text{QBC}_\delta(R)\) is measurable with respect to \(\eta \cap A(R/2, 2R)\).

As a result we have:

\[
\text{GP}_\delta^\text{ext}(R) = \text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \left\{\mathbb{P}\left[\text{QBC}_\delta^\text{ext}(R) \mid \eta \cap A(R/2, 2R) \geq 3/4\right] \right\},
\]

and the analogous result for \(\text{GP}_\delta^\text{int}(R)\). The reason why we do not choose to define \(\text{GP}_\delta^\text{ext}(R) = \text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \cap \text{QBC}_\delta^\text{ext}(R)\) is that we want \(\text{GP}_\delta^\text{ext}(R)\) to be measurable with respect to \(\eta \cap A(R/2, 2R)\) (and similarly for \(\text{GP}_\delta^\text{int}(R)\)). This will be crucial in the whole proof.

**The interfaces.** In Subsection 3.3, we have estimated the events \(\text{GI}_\delta^\text{ext}(R)\) and \(\text{GI}_\delta^\text{int}(R)\) in order to describe the structure of the interfaces. In particular, we have proved Lemma 2.14. However, the main tool in the proof of Lemma 2.14 was the computation of the exponent of the 3-arm event in the half-plane. As we will see in Subsection 7.3, the quasi-multiplicativity is a crucial ingredient in the computation of this exponent. Consequently, we cannot use Lemma 2.14 in the present proof and what we choose is to consider a variant of the quantities \(s^\text{ext}(r, R)\) and \(s^\text{int}(r, R)\). More precisely:

We still consider \(\delta \in (0, 1/1000)\) and \(R \in [\delta^{-1}, +\infty)\). We also consider \(r \in [1, R]\). Following the Appendix of [SS10], we define \(s^\text{ext}(r, R)\) to be the least distance between any two pairs of endpoints on \(\partial B_R\) of two interfaces in \(A(r, R)\) that go from \(\partial B_r\) to \(\partial B_R\). We write \(\text{GI}_\delta^\text{ext}(R) = \{s^\text{ext}(3R/4, R) \geq 100R\}\) and:

\[
G_\delta^\text{ext}(R) = \text{GP}_\delta^\text{ext}(R) \cap \text{GI}_\delta^\text{ext}(R).
\]

Since we have assumed that \(R \geq \delta^{-2}\), the event \(\text{Dense}_\delta(R) \cap \text{GI}_\delta^\text{ext}(R)\) is measurable with respect to \(\omega \cap A(R/2, 2R)\). Therefore, \(G_\delta^\text{ext}(R)\) is measurable with respect to \(\omega \cap A(R/2, 2R)\).

Similarly, let \(s^\text{int}(r, R)\) be the least distance between any two pairs of endpoints on \(\partial B_r\) of two interfaces in \(A(r, R)\) that go from \(\partial B_R\) to \(\partial B_r\) and we write \(\text{GI}_\delta^\text{int}(R) = \{s^\text{int}(R, 3R/2) \geq 100R\}\). We write \(G_\delta^\text{int}(R) = \text{GP}_\delta^\text{int}(R) \cap \text{GI}_\delta^\text{int}(R)\). The event \(G_\delta^\text{int}(R)\) is measurable with respect to \(\omega \cap A(R/2, 2R)\).

**Remark 7.3.** As noted above, conditioning on \(\text{Dense}_\delta(R)\) implies nice spatial independence properties. In what follows, we will often work with quads \(Q\) and \(Q'\) at distance more than \(\delta R\) from each other and we will often use implicitly that, when we condition on \(\text{Dense}_\delta(R)\), there is no Voronoi cell that intersects the two quads. This implies in particular that the events of \(\text{Cross}(Q)\) and \(\text{Cross}^*(Q')\) are (conditionally) independent.
Lemma 7.4. There exists $\epsilon > 0$ such that, for every $\delta \in (0, 1/1000)$ and every $R \in [\delta^{-2}, +\infty)$ we have:

$$\mathbb{P} \left[ G_{\delta}^{ext}(R) \right] \geq 1 - \frac{1}{\epsilon} \delta^\epsilon$$

(7.4)

and:

$$\mathbb{P} \left[ G_{\delta}^{int}(R) \right] \geq 1 - \frac{1}{\epsilon} \delta^\epsilon.$$  

(7.5)

Proof. We write only the proof of (7.4) since the proof of (7.5) is exactly the same. With the same proof as for Lemma 2.11 and thanks to Proposition 2.13 we have:

We have the following estimates:

$$\mathbb{P} \left[ \text{Dense}_\delta(R) \cap \text{QBC}_\delta(R) \right] \geq 1 - O(1) \left( \left( \frac{R}{\delta} \right)^2 \exp \left( -\Omega(1)(\delta R^2) \right) + R^{-1} \right).$$

Remember that $R \geq \delta^{-2}$, hence the above is at least:

$$1 - O(1) \left( \delta^{-6} \exp \left( \delta^{-2} \right) + \delta^2 \right) \geq 1 - O(1) \delta^{-2}.$$

If we apply Proposition 2.13 once again, we obtain that $\mathbb{P} \left[ \text{QBC}_{\delta}^{ext}(R) \right] \geq 1 - O(1) R^{-1}$. Therefore, conditionally on $\eta \cap A(\bar{R}/2, R)$ (and in fact conditionally on any $\sigma$-field), with probability at least $1 - O(1) R^{-1}$ (with possibly a bigger constant in the $O(1)$), $\text{QBC}_{\delta}^{ext}(R)$ holds with probability at least $3/4$. As a result:

$$\mathbb{P} \left[ \text{GP}_{\delta}^{ext}(R) \right] \geq 1 - O(1) R^{-1} \geq 1 - O(1) \delta^{-2}.$$

What remains to check is that $\mathbb{P} \left[ \text{GI}_{\delta}^{ext}(R) \right] \geq 1 - O(1) \delta^\epsilon$ for some $\epsilon > 0$. To this purpose, we follow the proof of Lemma A.2 in [SS10] (which is written for site percolation on $\mathbb{T}$).

First, we choose $\alpha \subseteq \partial B_R$ an arc of diameter $R/8$ and we let $Y$ be the set of points in $B_R$ at distance at most $R/8$ from $\alpha$. Let $\alpha_1$ be one of the two arcs in $\partial Y \cap \partial B_R$. Let $k$ be the number of interfaces crossing from $\partial Y \setminus \partial B_R$ to $\alpha$ and let $\beta_1, \ldots, \beta_k$ be these interfaces ordered so that, for $i_1 < i_2 \leq k$, the interface $\beta_{i_1}$ separates $\alpha_1$ from $\beta_{i_2}$ in $Y$ (we will choose to say that “$\beta_{i_2}$ is on the right-hand-side of $\beta_{i_1}$” and we will write $Y_i$ for the component of $Y \setminus \beta_i$ separated from $\alpha_1$ by $\beta_i$). Let $z_i$ denote the endpoint of $\beta_i$ on $\alpha$. We want to prove that there exists an absolute constant $\epsilon > 0$ such that:

$$\mathbb{P} \left[ \forall i \in \{1, \ldots, k-1\}, |z_i - z_{i+1}| \leq 10\delta |R| \right] \geq 1 - O(1) \delta^\epsilon.$$  

(7.6)

The strategy in [SS10] is to condition on $i \leq k$ and on $\delta_1$, use the fact that the percolation configuration on the right-hand-side of $\beta_i$ remains unbiased and finally conclude thanks to the box-crossing property. The fact that the (conditioned) configuration on the right-hand-side of $\beta_i$ is unbiased is not true in the case of Voronoi percolation since it gives information about the structure of the random lattice.

The strategy we choose is to condition on some $\eta$ such that $\text{Dense}_\delta(R) \cap \widetilde{\text{QBC}}_\delta(R)$ holds where:

$$\widetilde{\text{QBC}}_\delta(R) := \{ \forall Q \in \widetilde{Q}_\delta(A(R/2, 2R)), \mathbb{P}^\eta[\text{Cross}(Q)] \geq \widetilde{c}(1) \}$$

(see Definition 3.1 for the set of quads $\widetilde{Q}_\delta(D)$; the constant $\widetilde{c}(1)$ is the constant that comes from Proposition 3.2).

Now, since $\eta$ is fixed, if we condition on $i \leq k$ and on $\beta_i$, then the (conditioned) configuration on the right-hand-side of $\beta_i$ remains unbiased. Moreover, the fact that $\text{QBC}_\delta(R)$ holds implies that we can use the box-crossing properties that are used in the proof of Lemma A.2 of [SS10]. Finally and (we refer to [SS10] for more details), we obtain that, for some absolute constant $\epsilon > 0$ (the fact that $\epsilon$ does not depend on $\delta$ is crucial and comes from the fact that $\widetilde{c}(1)$ does not depend on $\delta$) and for any $\eta$ such that $\text{Dense}_\delta(R) \cap \widetilde{\text{QBC}}_\delta(R)$ holds:

$$\mathbb{P}^\eta \left[ \forall i \in \{1, \ldots, k-1\}, |z_i - z_{i+1}| \leq 10\delta |R| \right] \geq 1 - O(1) \delta^\epsilon.$$
Next, note that with the same proof as for Lemma 2.11 and thanks to Proposition 3.2 we have:

\[
\mathbb{P} \left[ \text{Dense}_3(R) \cap Q\overline{B}C_4(R) \right] \geq 1 - O(1) \left( \left( \frac{R}{\delta} \right)^2 \exp \left( -\Omega(1)(\delta R)^2 \right) + R^{-1} \right) \\
\geq 1 - O(1) \delta^{-2}.
\]

So, we have obtained (7.6) (with \( \epsilon = \tilde{c} \wedge 2 \)). It is not difficult to see (by choosing an appropriate covering of \( \partial B_R \) by \( O(1) \) such arcs \( a^n \)) that it implies that:

\[
\mathbb{P} \left[ \tilde{G}^{ext}_R \right] = \mathbb{P} \left[ \tilde{s}^{ext}(3R/4, R) \geq 10 \delta R \right] \geq 1 - O(1) \delta^c.
\]

\( \square \)

### 7.1.2 Extension of the arms when the point configuration and the coloring are good

The following lemma is the analogue of Lemma A.3 in [SS10] and roughly says that if the 4-arm event holds at some scale and if the configuration looks good then with positive probability the 4-arm event extends to a larger scale and the configuration looks good at this larger scale. If \( \delta \in (0, 1/1000), R \in [\delta^{-2}, +\infty) \) and \( r \in [1, R] \) write:

\[
g^{ext}_{4, R}(r, R) = \mathbb{P} \left[ A_4(r, R) \cap G^{ext}_R \right].
\]

Similarly, if \( \delta \in (0, 1/1000), r \in [\delta^{-2}, +\infty) \) and \( R \in [r, +\infty) \) write:

\[
g^{int}_{4, R}(r) = \mathbb{P} \left[ A_4(r, R) \cap \mathbb{G} \right].
\]

**Lemma 7.5.** There exists \( \delta \in (0, 1/1000) \) such that, for any \( \delta \in (0, 1/1000) \), there is some constant \( a = a(\delta) \in (0, 1) \) such that:

1. For every \( R \in [\delta^{-2} \vee \delta^{-2}, +\infty) \) and every \( r \in [1, R/4] \) we have:

\[
g^{ext}_{4, R}(r, 4R) \geq a g^{ext}_{4, R}(r, R). \tag{7.7}
\]

2. For every \( r \in [4(\delta^{-2} \vee \delta^{-2})^2, +\infty) \) and every \( R \in [4r, +\infty) \) we have:

\[
g^{int}_{4, R}(r/4, R) \geq a g^{int}_{4, R}(r, R). \tag{7.8}
\]

**Proof of Lemma 7.5.** The proof is very similar to the proof of Lemma 4.3, though a little more technical.

Let us first prove (7.7). Let \( \delta \in (0, 1/1000) \) to be determined later and consider \( R, r \) and \( \delta \) as in the statement of the lemma. We write \( \mathbb{P}^\eta_{B_{2R}} \) for the probability measure \( \mathbb{P} \) conditioned on \( \eta \cap B_{2R} \). Note that this is the probability measure defined as follows: colour each point of \( \eta \cap B_{2R} \) independently (in black with probability 1/2 and in white with probability 1/2) and sample (independently of the colouring of \( \eta \cap B_{2R} \)) a coloured Poisson point process in \( \mathbb{R}^2 \setminus B_{2R} \).

Fix some \( \eta \in \mathbb{G}^{ext}_\delta(R) \) such that \( \mathbb{P}^\eta_{B_{2R}} \left[ A_4(r, R) \cap \{ \tilde{s}^{ext}(r, R) \geq 100 \delta R \} \right] > 0 \) and write \( \beta_0, \ldots, \beta_{k-1} \) for the interfaces that cross \( A(r, R) \) in counter-clockwise order. We assume that the right-hand-side of \( \beta_0 \) (if one goes from \( \partial B_r \) to \( \partial B_R \)) is black. First, we work under the following conditional probability measure:

\[
\nu_{r, R, (\beta_0)} := \mathbb{P}^\eta_{B_{2R}} \left[ A_4(r, R) \cap \mathbb{G}^{ext}_\delta(R) \cap \{ \tilde{s}^{ext}(r, R) \geq 10 \delta R \}, \beta_0, \ldots, \beta_{k-1} \right].
\]
Figure 10: The quads $Q_{\text{ext}}(R, 0), \cdots, Q_{\text{ext}}(R, 3)$.

We keep such an $\eta$ fixed until we explicitly say that we take the expectation under $\mathbb{P}$ (see below (7.13)). Let us define four rectangles $Q_{\text{ext}}(R, 0), \cdots, Q_{\text{ext}}(R, 3)$ (which belong to the set of quads $Q_{1/100}(A(R, 4R))$ from Definition 2.12) in Figure 10.

It is not difficult to see that we can choose four quads $Q(\beta_i) \in Q_5(A(3R/4, 3R/2))$, $i \in \{0, \cdots, 3\}$, such that: (a) the intersection of $Q(\beta_i)$ and $\beta_i \cup \beta_{i+1}$ is one of the two distinguished arcs of $Q(\gamma_i)$, (b) $Q(\beta_i)$ lies on the right-hand side of $\beta_i$ and on the left-hand side of $\beta_{i+1}$, (c) if $Q(\beta_i)$ is crossed, then $Q_{\text{ext}}(R, i)$ is crossed widthwise, (d) if $0 \leq i \neq j \leq 3$, then there is no Voronoi cell that intersects both $Q(\beta_i)$ and $Q_{\text{ext}}(R, j)$, (e) if $0 \leq i \neq j \leq 3$, then there is no Voronoi cell that intersects both $Q(\beta_i)$ and $Q(\beta_j)$. (See Figure 11.)

Figure 11: The quads $Q(\beta_1)$ and $Q(\beta_2)$.

Note that, since the quads $Q(\beta_i)$ belong to the set of quads $Q_{5}(A(3R/4, 3R/2))$ from Definition 2.12, then the probability under $\nu_{\eta_{R, \{\beta_i\}}}$ that $Q(\beta_i)$ is crossed is at least $c(1)$, where $c(1)$ is the constant of Proposition 2.13. (Note that we have implicitly used the (quenched) Harris-FKG inequality since conditioning on $(\beta_i)_j$ affects the percolation process.
as follows: if the right-hand-side of $\beta_j$ is black (respectively white) then there is a black (respectively white) crossing from $\beta_j$ to $\beta_{j+1}$.

Now, let $F = F(\beta_0, \ldots, \beta_{k-1})$ denote the event that there are black crossings in $Q(\beta_1)$ and $Q(\beta_3)$ and white crossings in $Q(\beta_0)$ and $Q(\beta_2)$. We have:

$$\nu^\eta_{r,R,(\beta_j)_j}[F] \geq c(\delta,1)^4.$$  \hfill (7.9)

Our next goal is to prove the following:

$$\nu^\eta_{r,R,(\beta_j)_j}[\text{QBC}^{\text{ext}}(R) \cap F] \geq 3/4.$$  \hfill (7.10)

To this purpose, we need the following elementary lemma whose proof is left to the reader:

**Lemma 7.6.** Let $X$ be a random variable, $\mathcal{F}$ and $\mathcal{G}$ two independent sub-$\sigma$-algebras and $A \in \mathcal{F}$ such that $X \mathbb{1}_A$ is independent of $\mathcal{G}$. Then: 

$$\{\mathbb{E}[X | F] \geq 3/4\} \cap A = \{\mathbb{E}[X | F \vee \mathcal{G}] \geq 3/4\} \cap A.$$ (We even have: $\mathbb{E}[X | F] = \mathbb{E}[X | F \vee \mathcal{G}] = \mathbb{E}[X | F \vee \mathcal{G}] \mathbb{1}_A$.)

Let us use Lemma 7.6 to prove (7.10). To this purpose, note that, since we have assumed that $\eta \in \text{Dense}_3(R)$, then (under $\nu^\eta_{r,R,(\beta_j)_j}$), the event $F$ is measurable with respect to $\omega \cap B_{2R}$. Also, remember that $\text{GP}^{\text{ext}}_3(R)$ is the intersection of $\text{Dense}_3(R)$ and $\text{QBC}_3(R)$ and $\{\mathbb{P}[\text{QBC}^{\text{ext}}(R) \cap A(2/2R)] \geq 3/4\}$.

If we apply Lemma 7.6 to $\mathcal{F} = \sigma(\eta \cap A(2/2R))$, $\mathcal{G} = \sigma(\eta \cap B_{2R})$, $X = \mathbb{1}_{\text{QBC}^{\text{ext}}(R)}$ and $A = \text{Dense}_3(R)$ then we obtain that we can replace $\{\mathbb{P}[\text{QBC}^{\text{ext}}(R) | \eta \cap A(2/2R)] \geq 3/4\}$ by $\{\mathbb{P}_{B_{2R}}[\text{QBC}^{\text{ext}}(R)] \geq 3/4\}$ and still get the same event $\text{GP}^{\text{ext}}_3(R)$. As a result, the configuration $\eta$ that we have fixed satisfies the event $\{\mathbb{P}_{B_{2R}}[\text{QBC}^{\text{ext}}(R)] \geq 3/4\}$. Moreover, $\nu^\eta_{r,R,(\beta_j)_j}[F]$ is the probability measure $\mathbb{P}_{B_{2R}}$ conditioned on events which are measurable with respect to $\omega \cap B_{2R}$. Since (under $\mathbb{P}_{B_{2R}}$), $\text{QBC}^{\text{ext}}(R)$ is independent of $\omega \cap B_{2R}$, this implies (7.10).

We are now in shape to extend the arms to the scale $4R$ since $\text{QBC}^{\text{ext}}(R)$ gives quenched box crossing estimates for quads in $A(R,4R)$. More precisely, if $F$ holds and if there are black crossings of $Q^{\text{ext}}(R,1)$, $Q^{\text{ext}}(R,3)$ and white crossings of $Q^{\text{ext}}(R,0)$ and $Q^{\text{ext}}(R,2)$, then $A_4(r,4R)$ holds. Thanks to the (quenched) FKG-Harris inequality, this implies that there exists an absolute constant $c' > 0$ such that:

$$\nu^\eta_{r,R,(\beta_j)_j}[A_4(r,4R) \cap \text{QBC}^{\text{ext}}(R), \eta] \geq c',$$

hence:

$$\nu^\eta_{r,R,(\beta_j)_j}[A_4(r,4R) | \text{QBC}^{\text{ext}}(R)] \geq c'. \hfill (7.11)$$

Now, we use once again that (under $\nu^\eta_{r,R,(\beta_j)_j}$) the events $G^{\text{ext}}_3(R)$ and $F$ are measurable with respect to $\omega \cap B_{2R}$. Moreover, $G^{\text{ext}}_3(4R)$ is measurable with respect to $\omega \setminus B_{2R}$. Together with Lemma 7.4, this implies that:

$$\nu^\eta_{r,R,(\beta_j)_j}[G^{\text{ext}}_3(4R) | F] \geq 1 - \frac{1}{\epsilon}. \hfill (7.12)$$

We now combine (7.10), (7.11) and (7.12) and we obtain:

$$\nu^\eta_{r,R,(\beta_j)_j}[A_4(r,4R) \cap G^{\text{ext}}_3(4R) | F] \geq 3c'/4 - \frac{1}{\epsilon}. \hfill (7.13)$$

We choose $\delta$ sufficiently small so that $3c'/4 - \frac{1}{\epsilon} \geq c'/2$ (here the fact that $c'$ does not depend on $\delta$ is crucial). Now, by combining the above inequality with (7.9) we obtain that:

$$\nu^\eta_{r,R,(\beta_j)_j}[A_4(r,4R) \cap G^{\text{ext}}_3(4R)] \geq \frac{c(\delta,1)c'}{2}. \hfill (7.13)$$

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If we take the expectation under $\mathbf{P}_{B_{2R}}^\eta$ and then under $\mathbf{P}$, we obtain that:

$$g_{4,\delta}^{\text{ext}}(r, 4R) = \mathbb{P}[A_4(r, 4R) \cap G_{\delta}^{\text{ext}}(4R)] \geq 
\frac{c(\delta, 1) c'}{2} \mathbb{P}\left[ A_4(r, R) \cap \text{GP}_\delta^{\text{ext}}(R) \cap \{ \tilde{x}^{\text{ext}}(r, R) \geq 10 \delta R \}\right].$$

Note that, if $1 \leq r_1 \leq r_2 \leq r_3$, then $\tilde{x}^{\text{ext}}(r_1, r_3) \geq \tilde{x}^{\text{ext}}(r_2, r_3)$, hence:

$$\mathbb{P}\left[ A_4(r, R) \cap \text{GP}_\delta^{\text{ext}}(R) \cap \{ \tilde{x}^{\text{ext}}(r, R) \geq 10 \delta R \}\right] \geq g_{4,\delta}^{\text{ext}}(r, R).$$

Finally, we have obtained (7.7) (with $a = a(\delta) = c(\delta, 1) c'/2$).

Note that we have obtained the following more precise result: Let $\tilde{A}_4^{\text{ext}}(r, R)$ denote the event that there are four arms of alternating colors $\gamma_0, \cdots, \gamma_3$ from $\partial B_r$ to $\partial B_R$ such that $\gamma_i \cap A(R/2, R) \subseteq Q^{\text{ext}}(R/4)$. Let $\delta, r$ and $R$ be as in the statement of item 1 of Lemma 7.5. Then:

$$\mathbb{P}\left[ \tilde{A}_4^{\text{ext}}(r, 4R) \cap G_{\delta}^{\text{ext}}(4R) \right] \geq a g_{4,\delta}^{\text{ext}}(r, R).$$

Actually, if we follow the proof we can see that we have also obtained the following: Let $F_R$ be an event measurable with respect to $\omega \setminus B_{2R}$ such that $\mathbb{P}[F_R] \geq 1 - c'/4$. Then:

$$\mathbb{P}\left[ \tilde{A}_4^{\text{ext}}(r, 4R) \cap G_{\delta}^{\text{ext}}(4R) \cap F_R \right] \geq \frac{a}{2} g_{4,\delta}^{\text{ext}}(r, R).$$

The proof of (7.8) is essentially the same. As in the case of (7.7), we can also obtain a more precise result: Let $Q^{\text{int}}(r, 0), \cdots, Q^{\text{int}}(\rho, 3)$ be the four rectangles defined on Figure 12 (which are the analogues of the rectangles $Q^{\text{ext}}(R, i)$ of the proof of Lemma 7.5) and write $\tilde{A}_4^{\text{int}}(r, R)$ for the event that there are four arms of alternating color $\gamma_0, \cdots, \gamma_3$ from $\partial B_r$ to $\partial B_R$ such that $\gamma_i \cap A(r, 2r) \subseteq Q^{\text{int}}(4r, i)$. If $\delta$ is sufficiently small and if $\delta, r$ and $R$ are as in item 2 of Lemma 7.5, then the following holds: There exists $a = a(\delta) > 0$ and $c' > 0$ such that, for every event $F_r$ measurable with respect to $\omega \cap B_{r/2}$ that satisfies $\mathbb{P}[F_r] \geq 1 - c'$, we have:

$$\mathbb{P}\left[ \tilde{A}_4^{\text{int}}(r/4, R) \cap G_{\delta}^{\text{int}}(r/4) \cap F_r \right] \geq a g_{4,\delta}^{\text{int}}(r, R). \quad (7.14)$$

Figure 12: The quads $Q^{\text{int}}(r, 0), \cdots, Q^{\text{int}}(r, 3)$.
Remark 7.7. Note that (1.1) implies that there exists a constant \( c' > 0 \) such that, for all \( \rho_1 \leq \rho_2 \) sufficiently large and satisfying \( \rho_1 \geq \rho_2/4^4 \), we have \( \alpha_{4,n}'(\rho_1, \rho_2) \geq c' \). Together with Lemma 7.4, this implies that, if \( \delta \in (0, 1/1000) \) is sufficiently small, then for all \( R \in [\delta^{-2}, +\infty) \) and every \( r \in [R/4^3, R] \) we have:

\[
g_{4,\delta}^{\text{ext}}(r, R) \geq c'/2.
\]

Similarly, if \( \delta \in (0, 1/1000) \) is sufficiently small, then for all \( r \in [\delta^{-2}, +\infty) \) and all \( R \in [r, 4^4 r] \) we have:

\[
g_{4,\delta}^{\text{int}}(r, R) \geq c'/2.
\]

7.1.3 The probability to look good if the 4-arm event holds

We now consider the events:

\[
\hat{A}_4^{\text{ext}}(r, R) = \left\{ \mathbb{P} \left[ A_4(r, R) \mid \omega \cap B_{R} \right] > 0 \right\},
\]

\[
\hat{A}_4^{\text{int}}(r, R) = \left\{ \mathbb{P} \left[ A_4(r, R) \mid \omega \setminus B_{r} \right] > 0 \right\}
\]

and the two following quantities:

\[
f_4^{\text{ext}}(r, R) = \mathbb{P} \left[ \hat{A}_4^{\text{ext}}(r, R) \right] ;
\]

\[
f_4^{\text{int}}(r, R) = \mathbb{P} \left[ \hat{A}_4^{\text{int}}(r, R) \right].
\]

What we want to prove is that the quantities \( \alpha_{4,n}^{\text{int}}(r, R), g_{4,\delta}^{\text{ext}}, g_{4,\delta}^{\text{int}}, f_4^{\text{ext}}(r, R) \) and \( f_4^{\text{int}}(r, R) \) are of the same order. We have the following result:

**Lemma 7.8.** There exist \( C_1 \in [1, +\infty) \) and \( \tau \in [\delta^{-2}, +\infty) \) such that, for every \( r \in [\tau, +\infty) \) and \( R \in [16r, +\infty) \):

\[
g_{4,\delta}^{\text{ext}}(r, R) \geq f_4^{\text{ext}}(r, R)/C_1, \tag{7.15}
\]

and:

\[
g_{4,\delta}^{\text{int}}(r, R) \geq f_4^{\text{int}}(r, R)/C_1. \tag{7.16}
\]

We have the following corollary (which is a direct consequence of Lemma 7.8 and Remark 7.7):

**Corollary 7.9.** There exists a constant \( C_2 < +\infty \) such that, for every \( r \in [\tau, +\infty) \) and every \( R \in [r, +\infty) \):

\[
g_{4,\delta}^{\text{int}}(r, R) \leq \alpha_{4,n}^{\text{int}}(r, R) \leq f_4^{\text{int}}(r, R) \leq C_2 g_{4,\delta}^{\text{int}}(r, R),
\]

and:

\[
g_{4,\delta}^{\text{ext}}(r, R) \leq \alpha_{4,n}^{\text{ext}}(r, R) \leq f_4^{\text{ext}}(r, R) \leq C_2 g_{4,\delta}^{\text{ext}}(r, R).
\]

**Proof of Lemma 7.8.** We only prove (7.15) since the the proof of (7.16) is essentially the same. Let \( \delta \in (0, \delta_0) \) to be chosen later, let \( \tau = 4^4 \delta^{-2} \) and let \( r \) and \( R \) be as in the statement of the lemma.

First, note that, if Dense_{4}(R/4) holds then, for every \( x \in \eta \), if the Voronoi cell of \( x \) intersects \( A(r, R/4) \), then \( x \in B_{R} \). Hence, \( \hat{A}_4^{\text{ext}}(r, R) \cap \text{Dense}_{4}(R/4) \subseteq A_4(r, R/4) \). Remember that Dense_{4}(R/4) \( \subseteq G_{\delta}^{\text{ext}}(R/4) \). We deduce that:

\[
\hat{A}_4^{\text{ext}}(r, R) \subseteq \left( A_4(r, R/4) \cap G_{\delta}^{\text{ext}}(R/4) \right) \cup \left( \hat{A}_4^{\text{ext}}(r, R) \setminus G_{\delta}^{\text{ext}}(R/4) \right)
\]

\[
\subseteq \left( A_4(r, R/4) \cap G_{\delta}^{\text{ext}}(R/4) \right) \cup \left( \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_{\delta}^{\text{ext}}(R/4) \right).
\]

As a result, \( f_4^{\text{ext}}(r, R) \) is smaller than or equal to:

\[
\mathbb{P} \left[ A_4(r, R/4) \cap G_{\delta}^{\text{ext}}(R/4) \right] + \mathbb{P} \left[ \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_{\delta}^{\text{ext}}(R/4) \right]
\]

\[
= g_{4,\delta}^{\text{ext}}(r, R/4) + \mathbb{P} \left[ \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_{\delta}^{\text{ext}}(R/4) \right].
\]

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Remember that $G^\text{ext}_{\delta}(R/4)$ is measurable with respect to $\omega \cap A(R/8, R/2)$ and that $\hat{A}^\text{ext}_4(r, R/16)$ is measurable with respect to $\omega \cap B(R/16)$. Hence, we can use the spatial independence properties of Voronoi percolation to say that the above equals:

$$g^\text{ext}_{4, \delta}(r, R/4) + f^\text{ext}_4(r, R/16) \cdot \mathbb{P}[-G^\text{ext}_{\delta}(R/4)].$$  \hspace{1cm} (7.17)

So, thanks to Lemma 7.4 we know that, since $R/4 \geq \delta^{-2}$ we have:

$$f^\text{ext}_4(r, R) \leq g^\text{ext}_{4, \delta}(r, R/4) + \frac{1}{\epsilon} \delta^2 f^\text{ext}_4(r, R/16).$$

By repeating the above argument, we obtain that $f^\text{ext}_4(r, R)$ is at most:

$$\sum_{i=0}^{l-1} \left( \frac{1}{\epsilon} \delta^i \right) g^\text{ext}_{4, \delta} \left( r, \frac{R}{4 \cdot 16^i} \right),$$

where $l = \lceil \log_4 (R/r) \rceil$. Thanks to Lemma 7.5 we obtain that the above is at most:

$$\sum_{i=0}^{l-1} \left( \frac{1}{\epsilon} \delta^i \right) a(\delta)^{-1} a(\delta)^{-(2i+1)} g^\text{ext}_{4, \delta}(r, R) + \left( \frac{1}{\epsilon} \delta^i \right) l. \hspace{1cm} (7.18)$$

Lemma 7.5 also implies the following inequality:

$$g^\text{ext}_{4, \delta}(r, R) \geq a(\delta)^{2i-1} g^\text{ext}_{4, \delta} \left( r, \frac{R}{4 \cdot 16^{i-1}} \right). \hspace{1cm} (7.19)$$

Remember Remark 7.7: there exists a universal constant $c' > 0$ such that:

$$g^\text{ext}_{4, \delta} \left( r, \frac{R}{4 \cdot 16^{i-1}} \right) \geq c'. \hspace{1cm} (7.20)$$

Let us end the proof: We choose $\delta \in (1, \delta)$ small enough so that $\frac{1}{2} \delta a(\delta)^{-2} \leq 1/2$. If we combine (7.18), (7.19) and (7.20) we obtain that:

$$f^\text{ext}_4(r, R) \leq \left( \sum_{i=0}^{\infty} \left( \frac{1}{\epsilon} \delta^i \right) a(\delta)^{-1} a(\delta)^{-(2i+1)} \right) + \left( \frac{1}{\epsilon} \delta^i \right) \frac{1}{a(\delta)^{2i-1} \cdot b(\delta)} g^\text{ext}_{4, \delta}(r, R) \leq \left( \frac{2}{a(\delta) \cdot b(\delta)} + \frac{1}{2 \cdot a(\delta) \cdot b(\delta)} \right) g^\text{ext}_{4, \delta}(r, R).$$

And the proof is over (with $C_{1-2} = \frac{2}{a(\delta) a(\delta)} + \frac{1}{2 a(\delta) a(\delta)}$).

**7.1.4 Proof of the quasi-multiplicativity property**

We are now in shape to prove Proposition 7.1. We first prove it for $r_1$ sufficiently large and we prove separately the left-hand and right-hand inequalities. Below, $\tau$ is the constant of Lemma 7.8.

**Proof of the left-hand-inequality of Proposition 7.1 in the case $r_1 \geq \tau$.** We have:

$$a^\text{ext}_4(r_1, r_3) \leq \mathbb{P}[A_4(r_1, r_2) \cap A_4(r_2, r_3)] \leq \mathbb{P}[\hat{A}^\text{ext}_4(r_1, r_2) \cap \hat{A}_4^\text{int}(r_2, r_3)] = \mathbb{P}[\hat{A}^\text{ext}_4(r_1, r_2)] \cdot \mathbb{P}[\hat{A}_4^\text{int}(r_2, r_3)],$$

by spatial independence. The above inequality can be rewritten:

$$a^\text{ext}_4(r_1, r_3) \leq f^\text{ext}_4(r_1, r_2) f^\text{int}_4(r_2, r_3),$$

and we are done thanks to Corollary 7.9.
Proof of the right-hand-inequality of Proposition 7.1 in the case $r_1 \geq 4\pi$. We distinguish between four cases:

1. Assume that $r_1 \geq r_2/6$ and $r_2 \geq r_3/6$. Then, this is a direct consequence of (1.1).

2. Assume that $r_1 \geq r_2/6$ and $r_2 \leq r_3/6$. By using Corollary 7.9, we obtain that:

$$a_4^{an}(r_1, r_2) a_4^{an}(r_2, r_3) \leq a_4^{an}(r_2, r_3) \leq O(1) g_{4,5}^{int}(r_2, r_3).$$

By applying Lemma 7.5 (here, we use that $r_2 \geq 4\pi$ since $r_1 \geq 4\pi$), we obtain that the above is at most $O(1) a_4^{an}(r_2/4, r_3)$ (which is at most $O(1) a_4^{an}(r_1, r_3)$ since $r_1 \geq r_2/4$).

3. The case “$r_1 \leq r_2/4$ and $r_2 \geq r_3/4$” is treated similarly.

4. Now, we treat the case $r_1 \leq r_2/4$ and $r_2 \leq r_3/4$. First, we write the simple inequality:

$$a_4^{an}(r_1, r_2) a_4^{an}(r_2, r_3) \leq a_4^{an}(r_1, \frac{r_2}{3}) a_4^{an}(3r_2, r_3).$$

Corollary 7.9 implies that:

$$a_4^{an}(r_1, \frac{r_2}{3}) a_4^{an}(3r_2, r_3) \leq O(1) g_{4,5}^{ext}(r_1, \frac{r_2}{3}) g_{4,5}^{int}(3r_2, r_3).$$

(7.21)

Now the idea of the proof is very similar to the proof of Lemma 7.5. However, we have to be a little more careful since we work with more conditionings. First, as it is explained in the paragraph below (7.10), we can do some little changes in our conditionings without modifying the definition of the events we work with. More precisely, we have:

$$\text{GP}_{\bar{T}}^{ext}(r_2/3) = \text{Dense}_{\bar{T}}(r_2/3) \cap \text{QBC}_{\bar{T}}(r_2/3) \cap \left\{ \mathbb{P} \left[ \text{QBC}^{ext}(r_2/3) \left| \eta \cap B_{2r_2/3} \right. \right] \geq 3/4 \right\}$$

and similarly:

$$\text{GP}_{\bar{T}}^{int}(3r_2) = \text{Dense}_{\bar{T}}(3r_2) \cap \text{QBC}_{\bar{T}}(3r_2) \cap \left\{ \mathbb{P} \left[ \text{QBC}^{int}(3r_2) \left| \eta \setminus B_{3r_2/2} \right. \right] \geq 3/4 \right\}.$$  

(7.23)

Since QBC$^{ext}(\cdot)$ and QBC$^{int}(\cdot)$ do not depend on the colouring, we actually have:

$$\mathbb{P} \left[ \text{QBC}^{ext}(r_2/3) \left| \eta \cap B_{2r_2/3} \right. \right] = \mathbb{P} \left[ \text{QBC}^{ext}(r_2/3) \left| \omega \cap B_{2r_2/3} \right. \right]$$

and

$$\mathbb{P} \left[ \text{QBC}^{int}(3r_2) \left| \eta \setminus B_{3r_2/2} \right. \right] = \mathbb{P} \left[ \text{QBC}^{int}(3r_2) \left| \omega \setminus B_{3r_2/2} \right. \right].$$

As a result, (7.21) can be rewritten as follows:

$$a_4^{an}(r_1, \frac{r_2}{3}) a_4^{an}(3r_2, r_3) \leq \mathbb{P} \left[ \text{A}_4(r_1, r_2/3) \cap \text{GI}_{\bar{T}}^{ext}(r_2/3) \cap \text{Dense}_{\bar{T}}(r_2/3) \right.$$  

$$\left. \cap \left\{ \mathbb{P} \left[ \text{QBC}^{ext}(r_2/3) \left| \omega \cap B_{2r_2/3} \right. \right] \geq 3/4 \right\} \right]$$

$$\times \mathbb{P} \left[ \text{A}_4(3r_2, r_3) \cap \text{GI}_{\bar{T}}^{int}(3r_2) \cap \text{Dense}(3r_2) \cap \text{QBC}_{\bar{T}}(3r_2) \right.$$  

$$\left. \cap \left\{ \mathbb{P} \left[ \text{QBC}^{int}(3r_2) \left| \omega \setminus B_{3r_2/2} \right. \right] \geq 3/4 \right\} \right].$$

Remember that QBC$^{ext}(R) = \text{QBC}_{1/100}^1(A(R, 4R))$ and QBC$^{int}(R) = \text{QBC}_{1/100}^1(A(R/4, R)).$

We need the following lemma:
Lemma 7.10. Let $\mathcal{F}$ and $\mathcal{G}$ be two independent sub-$\sigma$-algebras, $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{G}$, $B_1$ and $B_2$ two events such that $A_1 \cap B_1 \in \mathcal{G}$ and $A_2 \cap B_2 \in \mathcal{F}$. Then:

$$\mathbb{P}[A_1 \cap B_1 \cap A_2 \cap B_2] \geq \frac{9}{16} \mathbb{P}[A_1 \cap \{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] \mathbb{P}[A_2 \cap \{\mathbb{P}[B_2 | \mathcal{G}] \geq 3/4\}] .$$

Proof. By independence, we have:

$$\mathbb{P}[A_1 \cap B_1 \cap A_2 \cap B_2] \geq \mathbb{P}[A_1 \cap B_1 \cap \{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] \cap A_2 \cap B_2 \cap \{\mathbb{P}[B_2 | \mathcal{G}] \geq 3/4\}] = \mathbb{P}[A_1 \cap B_1 \cap \{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] \cdot \mathbb{P}[A_2 \cap B_2 \cap \{\mathbb{P}[B_2 | \mathcal{G}] \geq 3/4\}] .$$

Now, note that:

$$\mathbb{P}[A_1 \cap B_1 \cap \{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] = \mathbb{E} \left[ 1_{A_1} \mathbb{P}[B_1 | \mathcal{F}] 1_{\{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] \right] \geq \frac{3}{4} \mathbb{P}[A_1 \cap \{\mathbb{P}[B_1 | \mathcal{F}] \geq 3/4\}] .$$

If we estimate $\mathbb{P}[A_2 \cap B_2 \cap \{\mathbb{P}[B_2 | \mathcal{G}] \geq 3/4\}]$ similarly, we obtain the lemma. \qed

If we apply this lemma to $\mathcal{F} = \sigma(\omega \cap B_{2r_2/3})$, $\mathcal{G} = \sigma(\omega \setminus B_{3r_2/2})$, $A_1 = A_2 = \mathcal{A}_4(r_1, r_2/3) \cap \mathcal{G}_{1\sigma}^{\text{ext}}(r_2/3) \cap \text{Dense}_{\sigma}(r_2/3) \cap \text{QBC}_{\sigma}(r_2/2)$, and $B_1 = \mathcal{A}_4(3r_2, r_3) \cap \mathcal{G}_{1\sigma}^{\text{int}}(3r_2) \cap \text{Dense}_{\sigma}(3r_2) \cap \text{QBC}_{\sigma}(3r_2)$, $B_2 = \text{QBC}_{\sigma}^{\text{ext}}(3r_2)$ and $B_2 = \text{QBC}_{\sigma}^{\text{int}}(3r_2)$, we obtain that:

$$\alpha_4^{an}(r_1, r_2/3) \alpha_4^{an}(3r_2, r_3) \leq \frac{16}{9} \mathbb{P}[\mathcal{A}_4(r_1, r_2/3) \cap \mathcal{G}_{1\sigma}^{\text{ext}}(r_2/3) \cap \text{Dense}_{\sigma}(r_2/3) \cap \text{QBC}_{\sigma}(r_2/3) \cap \text{QBC}_{\sigma}^{\text{ext}}(r_2/3) \cap \mathcal{A}_4(3r_2, r_3) \cap \mathcal{G}_{1\sigma}^{\text{int}}(3r_2) \cap \text{Dense}_{\sigma}(3r_2) \cap \text{QBC}_{\sigma}(3r_2) \cap \text{QBC}_{\sigma}^{\text{int}}(3r_2) .$$

Now, we can condition on $\eta$ and on the interfaces and conclude, with arguments similar to the proof of Lemma 7.5) that the above is at most $O(1) \alpha_4^{an}(r_1, r_3)$ since in the event $\text{QBC}_{\sigma}(r_2/3) \cap \text{QBC}_{\sigma}^{\text{ext}}(r_2/3) \cap \text{QBC}_{\sigma}(3r_2) \cap \text{QBC}_{\sigma}^{\text{int}}(3r_2)$ give estimates about the crossing of sufficiently many quads to glue together the interfaces from $r_1$ to $r_2/3$ and the interfaces from $3r_2$ to $r_3$.

We have obtained the quasi-multiplicativity property for $r_1 \geq 4\sigma$: there exists a constant $C' \in [1, +\infty)$ such that, for every $4\sigma \leq r_2 \leq r_3$:

$$\frac{1}{C'} \alpha_4^{an}(r_1, r_3) \leq \alpha_4^{an}(r_1, r_2) \alpha_4^{an}(r_2, r_3) \leq C' \alpha_4^{an}(r_1, r_3) .$$

In order to obtain the full result, we need the following lemma:

Lemma 7.11. For every $\overline{\sigma}$ sufficiently large, there exists a constant $C_3 = C_3(\overline{\sigma}) < +\infty$ such that, for every $r \in [1, \overline{\sigma}]$ and every $R \in [r, +\infty)$, we have:

$$\alpha_4^{an}(\overline{\sigma}, R) \leq C_3 \alpha_4^{an}(r, R) .$$

Before proving this lemma, let us explain why this enables us to conclude the proof of Proposition 7.1. Fix a quantity $\overline{\sigma} \geq 4\sigma$ sufficiently large so that Lemma 7.11 holds. Let $r_1 \leq \overline{\sigma}$, $r_2 \in [r_1, +\infty)$ and $r_3 \in [r_2, +\infty)$. We distinguish between three cases:

1. If $r_3 \leq \overline{\sigma}$ we are done thanks to (1.1).
2. If $r_3 \geq \bar{r} \geq r_2$ then we can use Lemma 7.11 to obtain that:

$$
\alpha_4^{an}(r_1, r_3) = \alpha_4^{an}(r_2, r_3) \leq \frac{1}{\alpha_4^{an}(1, \bar{r})} \alpha_4^{an}(r_1, r_2) \alpha_4^{an}(r_2, r_3) \\
\leq \frac{1}{\alpha_4^{an}(1, \bar{r})} \alpha_4^{an}(r_2, r_3) \\
\leq \frac{1}{\alpha_4^{an}(1, \bar{r})} \alpha_4^{an}(\bar{r}, r_3) \\
\leq C \alpha_4^{an}(1, \bar{r}) \alpha_4^{an}(r_1, r_3). 
$$

3. If $r_2 \geq \bar{r}$, we can use Lemma 7.11 and (7.24) to obtain that:

$$
\alpha_4^{an}(r_1, r_3) \leq \alpha_4^{an}(\bar{r}, r_3) \\
\leq C' \alpha_4^{an}(\bar{r}, 2) \alpha_4^{an}(r_2, r_3) \\
\leq C' C \alpha_4^{an}(r_1, r_2) \alpha_4^{an}(r_2, r_3) \\
\leq C' C \alpha_4^{an}(\bar{r}, r_2) \alpha_4^{an}(\bar{r}, r_3) \\
\leq C' C' \alpha_4^{an}(\bar{r}, r_3) \\
\leq C' C' C' \alpha_4^{an}(r_1, r_3), 
$$

and we are done.

**Sketch of proof of Lemma 7.11.** The proof is very similar to the proof of Lemma 4.6. Therefore, we only sketch it. Let $\text{Dense}^N(r)$ be the event defined in the proof of Lemma 4.6. Corollary 7.9 and the inequality (7.14) imply that we can find $r$ and $N$ (that depends on $r$) such that, for every $R \geq 4r$:

$$
P\left[\tilde{A}_4^{int}(r, R) \cap \text{Dense}^N(r)\right] \geq \Omega(1)\alpha_4^{an}(r, R),
$$

where $\tilde{A}_4^{int}(r, R)$ is the event defined in (7.14). Now, if we follow the proof of Lemma 4.6, we obtain that we can extend the four arms with probability larger than some constant that depends only on $r$ and $N$. More precisely, we obtain that:

$$
\alpha_4^{an}(R) \geq \Omega(1)P\left[\tilde{A}_4^{int}(r, R) \cap \text{Dense}^N(r)\right],
$$

where $\Omega(1)$ depends only on $r$ and $N$. This ends the proof. \hfill \Box

### 7.2 A consequence of the quasi-multiplicativity property

In this subsection, we prove Proposition 2.4 (where, instead of the quantities $\tilde{A}_j^{ext}(r, R)$ and $\tilde{A}_j^{int}(r, R)$ studied in Subsection 7.1, we consider the analogous quantity $\tilde{A}_j(r, R) = \{P_{\tilde{A}_j(r, R)}[A_j(r, R)] > 0\}$). We prove it only in the case $j$ even. See Subsection 7.4 for the extension to $j$ odd.

**Proof of Proposition 2.4 in the case $j$ even.** We write the proof for $j = 4$ since the proof for any $j \in \mathbb{N}^*$ even is essentially the same. Let $\text{Dense}(R) := \text{Dense}_{1/100}(A(R/2, 2R))$. We have:

$$
\tilde{A}_4(r, R) \subseteq \tilde{A}_4^{int}(r, R/2) \cup (\tilde{A}_4(r, R) \setminus \text{Dense}(R)).
$$
Hence:

\[ f_4(r, R) \leq f_{4n}^4(r, R/2) + \mathbb{P}[-\text{Dense}(R)]. \]

By Corollary 7.9, if \( r \) is sufficiently large, then \( f_{4n}^4(r, R/2) \approx \alpha_4^an(r, R/2) \). Moreover, thanks to the quasi-multiplicativity property, \( \alpha_4^an(r, R/2) \approx \alpha_4^an(r, R) \). Furthermore, \( \mathbb{P}[-\text{Dense}(R)] \leq O(1) \exp(-\Omega(1)R^2) \) while \( \alpha_4^an(r, R) \) decays polynomially fast in \( r/R \geq 1/R \). Hence, if \( r \) sufficiently large \( (r \geq r_0, \text{say}) \) and if \( R \in [r, +\infty) \), then:

\[ f_4(r, R) \leq O(1) \alpha_4^an(r, R), \]

which is what we want. If \( 1 \leq r \leq r_0 \), then we have:

\[ f_4(r, R) \leq f_4(r_0, R) \leq O(1) \alpha_4^an(r_0, R) \leq O(1) \alpha^an(r, R), \]

where the last inequality follows from the quasi-multiplicativity property. This ends the proof. \( \square \)

### 7.3 Arm events in the half-plane

In this subsection, we study \( j \)-arm events in the half-plane for any \( j \in \mathbb{N}^* \).

**Remark 7.12.** In Subsection 7.1, we have restricted ourselves to the case \( j \) even since we wanted to deal with arms of alternating colors. In the case of the half-plane, whatever \( j \) is odd or even, the arms are of alternating colors. As a result, if we follow the arguments of Subsection 7.1, we obtain the quasi-multiplicativity property for \( j \)-arm events in the half-plane for any \( j \in \mathbb{N}^* \). (Of course, the proof also works for arm events in any wedge, for instance in the quarter-plane.) We also obtain the analogues of Propositions 2.4 and 2.5.

We now use the quasi-multiplicativity property for arm events in the half-plane to compute the exponents of the 2- and 3-arm events in the half-plane.

**Proof of Items i) and ii) of Proposition 2.7.** (We follow [Wer07], first exercise sheet.) First, note that thanks to the quasi-multiplicativity property, it is sufficient to prove the result for \( r = 1 \) and for \( R \geq 1 \) sufficiently large. We first define the two following events (where \( H \) is the upper half-plane):

1. For every \( j \in \mathbb{Z} \), let \( I_j = [j, j+1] \times \{0\} \) and write \( F_{j}^{2+}(R) \) for the event that there exists \( y \in I_j \) and \( \gamma_1, \gamma_2 \) two paths such that: (a) \( \gamma_1 \) and \( \gamma_2 \) belong to \( B_R(y) \cap H \), (b) \( \gamma_1 \) and \( \gamma_2 \) join \( y \) to \( \partial B_R(y) \), (c) \( \gamma_1 \) is black and \( \gamma_2 \) is white and (d) \( \gamma_1 \) is on the right-hand-side of \( \gamma_2 \). (Note that this implies in particular that \( y \) belongs to the intersection of two Voronoi cells.)

2. Let \( S \) be a \( 1 \times 1 \) square of the grid \( \mathbb{Z}^2 \) and write \( F_{S}^{3+}(R) \) for the event that there exists \( y \in S \) which is the lowest point in \( B_R(y) \) of a black connected component which intersects \( \partial B_R(y) \).

Let \( \eta \in \text{Dense}_{1/100}(B_R) \cap \text{QBC}_{1/100}^5(B_R) \) (see Subsection 2.4.2 for the definition of these events). If we use the quenched box-crossing property Theorem 1.4 and if we follow the first exercise sheet of [Wer07] (where one has to use the BK-inequality, that we can use since we work at the quenched level), we obtain that there exists an absolute constant \( C \in [1, +\infty) \) such that:

\[ \frac{1}{C} \leq \sum_{j \in I_{j} \cap \partial B_{R/2} \neq \emptyset} \mathbb{P}^\eta \left[ F_{j}^{2+}(R) \right] \leq C, \]

and:

\[ \frac{1}{C} \leq \sum_{S : S \cap \partial B_{R/2} \neq \emptyset} \mathbb{P}^\eta \left[ F_{S}^{3+}(R) \right] \leq C. \]  \hspace{1cm} (7.25)

\(^{12}\text{Actually for the 3-arm event this is even easier than in the case of Bernoulli percolation treated in [Wer07] since, for Voronoi percolation, a.s. a cluster cannot have two lowest points.} \]
With the same proof as Lemma 2.11 and thanks to Proposition 2.13 we have:

\[
\mathbb{P} \left[ \text{Dense}_{1/100}(B_R) \cap \text{QBC}_{1/100}^2(B_R) \right] \geq 1 - \left( O(1) e^{-\Omega(1)} R^2 + O(1) R^{-5} \right) \geq 1 - O(1) R^{-5}.
\]

Let us conclude the proof in the case of the 3-arm event (the case of the 2-arm event is treated similarly). If we combine the above estimate with (7.25), we obtain that:

\[
\frac{1}{C} \leq \sum_{S : S \cap B_{R/2} \neq \emptyset} \left( \mathbb{P} \left[ F_{S}^{3,+}(R) \right] - O(1) R^{-5} \right)
\]

and:

\[
\sum_{S : S \cap B_{R/2} \neq \emptyset} \left( \mathbb{P} \left[ F_{S}^{3,+}(R) \right] - O(1) R^{-5} \right) \leq C.
\]

We now use the fact that the (annealed) model is translation invariant and we obtain that for every \( S \):

\[
\frac{1}{C} R^{-2} - O(1) R^{-3} \leq \mathbb{P} \left[ F_{S}^{3,+}(R) \right] \leq C R^{-2} + O(1) R^{-3}.
\]

Hence, for every \( R \) sufficiently large we have:

\[
\mathbb{P} \left[ F_{S}^{3,+}(R) \right] \asymp R^{-2}.
\]

Remember that our goal is to prove that \( \alpha_{3}^{an,+n}(R) \asymp R^{-2} \) for \( R \) sufficiently large. Therefore, it remains to prove that, for every \( R \) sufficiently large:

\[
\alpha_{3}^{an}(R) \asymp \mathbb{P} \left[ F_{S}^{3,+}(R) \right].
\]

This is the purpose of what follows.

i) Proof that \( \mathbb{P} \left[ F_{S}^{3,+}(R) \right] \lesssim O(1) \alpha_{3}^{an,+}(R) \). Let \( y \) be the center of \( S \). We use the notation \( A(y; \rho_1, \rho_2) = y + A(\rho_1, \rho_2) \). Also, we write Dense\((y, \rho) = \text{Dense}_{1/100}(A(y; \rho_1, 2\rho_1)) \) and we let \( A_{3}^{+}(y; \rho_1, \rho_2) \) be the 3-arm event in the half-plane translated by \( y \). We also write:

\[
\hat{A}_{3}^{+}(\rho_1, \rho_2) = \left\{ \mathbb{P} \left[ A_{3}^{+}(y; \rho_1, \rho_2) \big| \omega \cap A(y; \rho_1, \rho_2) \right] > 0 \right\}.
\]

See Remark 7.12: we can use the following result analogous to Proposition 2.4:

\[
\alpha_{3}^{an,+}(\rho_1, \rho_2) \asymp \mathbb{P} \left[ \hat{A}_{3}^{+}(\rho_1, \rho_2) \right]. \tag{7.26}
\]

Now, the proof is essentially the same as the proof of the inequality \( \mathbb{P} \left[ A_{4}^{\square}(S, R) \right] \lesssim O(1) \alpha_{3}^{an}(\rho, R) \) of Proposition 4.3. Hence, we leave it to the reader.

ii) Proof that \( \mathbb{P} \left[ F_{i}^{3,+}(R) \right] \geq \Omega(1) \alpha_{3}^{an,+}(R) \). We also leave this proof to the reader since it can be done by extending the arms “by hands” exactly as in the proof of Lemma 7.11.

\[\square\]

### 7.4 The case \( j \) odd

The purpose of the following proposition is to generalize Proposition 7.1 to an odd number of arms.

**Proof of Proposition 1.6 in the case \( j \) odd.** To deal with an odd number of arms, it is not sufficient to work with the quantities \( \overline{s}^{xt}(r, R) \) and \( \overline{s}^{nt}(r, R) \) that we have studied for the case \( j \) even. More precisely, in order to extend the two consecutive arms of the same color, we need to work with a configuration of interfaces that satisfy the following stronger condition:
the endpoints are far away from the other interfaces” (and not only “the endpoints are far away from each other”). In other words, we want to work with a configuration of interfaces that satisfy the event \(GI_\delta^ext(R)\) (or \(GI_\delta^int(R)\)) of lemma 2.14. What saves us is that we have already obtained the quasi-multiplicativity property for a lot of quantities, and especially that this has enabled us to compute exponent of the 3-arm event in the half-plane. Therefore, we can use Lemma 2.14 which estimates the quantities \(\mathbb{P}[GP_\delta^ext(R)]\) and \(\mathbb{P}[GP_\delta^int(R)]\).

So, let us modify the event \(G_\delta^ext(R) = GP_\delta^ext(R) \cap \tilde{GI}_\delta^ext(R)\) (respectively \(G_\delta^int(R) = GP_\delta^int(R) \cap GI_\delta^int(R)\)) used throughout Subsection 7.1 into:

\[GP_\delta^ext(R) \cap GI_\delta^ext(R)\]

(respectively \(GP_\delta^int(R) \cap GI_\delta^int(R)\)). Now, the proof of the quasi-multiplicativity property for \(j\) odd is the same as for \(j\) even. \(\square\)

We end this subsection by noting that: (a) Now, the proof of Proposition 2.4 for \(j\) odd is the same as the proof for \(j\) even and (b) we can compute the universal arm-exponent for the 5-arm event. Let us be a little more precise about the computation of this exponent:

**Proof of item iii) of Proposition 2.7.** We work with the following event:

Let \(S\) be a \(1 \times 1\) square of the grid \(\mathbb{Z}^2\) and write \(F_5^S(R)\) for the event that there exists a point \(x \in \eta \cap S\) such that: (a) the cell of \(x\) is white, (b) there exist five paths \(\gamma_1, \cdots, \gamma_5\) (in counter-clockwise order, say) that join the cell of \(x\) to \(\partial B_R(x)\), (c) \(\gamma_i\) is white (respectively black) if \(i\) is odd (respectively even) and (d) if \(i \neq j\) then there is no Voronoi cell that is intersected by \(\gamma_i\) and \(\gamma_j\) (except at the boundary of the cell of \(x\)).

If we follow the proof of items i) and ii) of Proposition 2.7 (see Subsection 7.3), we obtain that it is sufficient to prove that for every \(R \geq 1\) sufficiently large we have:

\[\alpha^5_\delta(R) \asymp \mathbb{P}[F_5^S(R)].\]

The proof of this estimate is also the same as the proof of the corresponding estimate for items i) and ii) of Proposition 2.7. \(\square\)

### A An extension of Schramm-Steif algorithm theorem

In this appendix, we state an extension of the Schramm-Steif randomized algorithm theorem that has been proved by Roberts and Sengul in [RS16].

We first need the following definition: Let \(n \in \mathbb{N}\) and let \(f : \{-1, 1\}^n \to \mathbb{R}\). An **algorithm** that determines \(f\) is a procedure that asks the values of the bits step by step where at each step the algorithm can ask for the value of one or several bits and the choice of the new bit(s) to discover is determined by the value of the bits previously queried. We also ask that the algorithm stops once \(f\) is determined (i.e. once the values of the undiscovered bits have no influence on the value of \(f\)).

We denote by \(\mathbb{P}^n_p\) the probability measure on \(\Omega^n := \{-1, 1\}^n\) defined by

\[\mathbb{P}^n_p = (p\delta_1 + (1 - p)\delta_{-1})^\otimes n.\]

A crucial quantity is the revealment of a (randomized) algorithm. This is defined as follows:

\[\delta^p_{\mathcal{A}} = \max_{i \in \{1, \cdots, n\}} \mathbb{P}_p[i \text{ is queried by } \mathcal{A}].\]

\[\delta^p_{\mathcal{A}} = \max_{i \in \{1, \cdots, n\}} \mathbb{P}_p[i \text{ is queried by } \mathcal{A}].\]

Here, we use the fact that we have obtained the quasi-multiplicativity for \(j = 5\), which was not the case in Subsection 7.3, that is why we have written the proofs of items i) and ii) and of item iii) separately.
To state Schramm-Steif result, we also need to introduce the notion of discrete Fourier decomposition: Let $S \subseteq \{1, \cdots, n\}$ and let $\omega \in \Omega^n$. We write:

$$\chi^p_S(\omega) = \prod_{i \in S} \left( \frac{1-p}{p} \mathbb{1}_{\omega_i=1} - \frac{p}{1-p} \mathbb{1}_{\omega_i=-1} \right).$$

Note that $(\chi^p_S)_{S \subseteq \{1, \cdots, n\}}$ is an orthonormal family of $L^2(\Omega^n, \mathbb{P}^p)$, thus we can define $(\hat{f}^p_S)_S$, the Fourier coefficients of $f : \Omega^n \rightarrow \mathbb{R}$ at level $p$, as the unique coefficients such that:

$$f = \sum_{S \subseteq \{1, \cdots, n\}} \hat{f}^p(S) \chi^p_S.$$

The result by Schramm and Steif is the following (they proved it for $p = 1/2$ but the proof for any $p$ is the same):

**Theorem A.1** (Theorem 1.8 of [SS10]). For every $f : \Omega^n \rightarrow \mathbb{R}$, every algorithm $A$ that determines $f$ and every $k \in \mathbb{N}^*$ we have:

$$\sum_{S \subseteq \{1, \cdots, n\} : \vert S \vert = k} \hat{f}^p(S)^2 \leq \delta^p_A k \mathbb{E}_p [f^2].$$

The extension that we will need is the following: Let $I \subseteq \{1, \cdots, n\}$ and, if $A$ is some algorithm, write:

$$\delta^p_A(I) = \max_{i \in I} \mathbb{P}_p [i \text{ is queried by } A].$$

**Proposition A.2** (Theorem 2.3 of [RS16]). For every $f : \Omega^n \rightarrow \mathbb{R}$, every algorithm $A$ that determines $f$, every $I \subseteq \{1, \cdots, n\}$ and every $k \in \mathbb{N}^*$ we have:

$$\sum_{S \subseteq I : \vert S \vert = k} \hat{f}^p(S)^2 \leq \delta^p_A(I) k \mathbb{E}_p [f^2].$$

**Proof.** The proof is very close to the proof in [SS10], Theorem 1.8, except that we need to (slightly) change the definition of their function $g$. More precisely, we need to work with:

$$g : \omega \mapsto \sum_{S \subseteq I : \vert S \vert = k} \hat{f}^p(S) \chi^p_S(\omega).$$

See [RS16] for more details. \qed

The reason why we are interested in the above theorem is the following property (see Subsection 2.4.1 for the definition of the pivotal event $\text{Piv}^p_I(A)$):

**Proposition A.3.** Let $A \subseteq \Omega^n$ be an increasing event. Also, let $f$ be the $\pm 1$-indicator function of $A$ (i.e. $f = 2\mathbb{1}_A - 1$). Then, for every $i \in \{1, \cdots, n\}$, we have:

$$\hat{f}^p(\{i\}) = 2\sqrt{p(1-p)} \mathbb{P}^p_\delta [\text{Piv}^p_I(A)].$$

**Proof.** The proof is exactly the same as in the case $p = 1/2$, which can be found for instance in [GS14], Proposition 4.5. \qed

The two above propositions imply the following corollary, which is the result that we will use:

**Corollary A.4.** Let $A \subseteq \Omega^n$ be an increasing event. Then, for every algorithm $A$ that determines $\mathbb{1}_A$ and every $I \subseteq \{1, \cdots, n\}$, we have:

$$\sum_{i \in I} \mathbb{P}^p_\delta [\text{Piv}^p_I(A)]^2 \leq \frac{1}{2\sqrt{p(1-p)}} \delta^p_A(I).$$
B The proof of the quenched box-crossing property in [AGMT16]

In this section, we recall the main steps of the proof of Theorem 1.4 by Ahlberg, Griffiths, Morris and Tassion (which is Theorem 1.4 in [AGMT16]). There are two reasons why we recall this proof here: i) In [AGMT16], the theorem is proved for the analogous model in which $\eta$ is a family of $n$ points sampled uniformly and independently in some fixed rectangle. As pointed out in [AGMT16] (below the statement of their Theorem 1.4) the proof in the case we are interested in (i.e. for which $\eta$ is a Poisson process in the whole plane) is essentially the same. We explain briefly why it works. ii) In order to extend this result to $p > 1/2$ (in Subsection 5.1) we have to change a little the end of the proof.

Let us first say that Item ii) of Theorem 1.4 is an easy consequence of Item i) as it explained at the end of the paper [AGMT16]. As a result, we only explain the strategy to obtain Item i).

A martingale estimate. First, the authors of [AGMT16] prove a martingale estimate:

Let $\rho, R > 0$. Let $N \in \mathbb{N}$ and consider $\eta_N$ a configuration of $4N^2$ points sampled uniformly in $[-N, N]^2$, independently of each other. Remember the notations used for pivotal points in Subsection 2.4.1. The following is not exactly Theorem 2.1 in [AGMT16]) but the proof is the same:

$$\operatorname{Var} (\mathbb{P}^{\eta_N}[\text{Cross}(\rho R,R)]) \leq \mathbb{E} \left[ \sum_{x \in \eta_N} \mathbb{P}^{\eta_N} [\text{Piv}^2_x(\text{Cross}(\rho R,R))]^2 \right], \quad (B.1)$$

where $\mathbb{P}^{\eta_N} := \left( \frac{\delta_1}{2} + \frac{\delta_{-1}}{2} \right)^{\eta_N}$ and where the definition of the pivotal event can be find in Subsection 2.4.1. (Note that the point process $\eta_N$ is a.s. finite, hence we have only finitely many Voronoi cells.)

Now, note that we can couple $\eta_N$ with a Poisson process of intensity 1 (denoted as usual by $\eta$) so that, with probability going to 1 as $N$ goes to $+\infty$, we have:

$$\eta \cap [-N^{1/4}, N^{1/4}] = \eta_N \cap [-N^{1/4}, N^{1/4}].$$

Since the event $\text{Cross}(\rho R,R)$ depends only of the color of the points of $\eta \cap [-N^{1/4}, N^{1/4}]$ with probability going to 1 as $N$ goes to $+\infty$ superpolynomially fast in $N$, the above together with (B.1) implies that:

$$\operatorname{Var} (\mathbb{P}^{\eta}[\text{Cross}(\rho R,R)]) \leq \mathbb{E} \left[ \sum_{x \in \eta} \mathbb{P}^{\eta} [\text{Piv}^2_x(\text{Cross}(\rho R,R))]^2 \right]. \quad (B.2)$$

(See Subsection 2.4.1 for the definition of $\text{Piv}^2_x(\cdot)$.)

An estimate on the 1-arm event. Next, the authors prove a weak version of the quenched boxing property in their Theorem 3.1 (they actually prove this theorem in the case where $\eta$ is a Poisson process in the whole plane which is what we need) and then prove a quenched estimate on arm events in their Proposition 3.11. This proposition is proved in the case where $\eta$ is a set of $n$ independent points sampled uniformly in a rectangle but with exactly the same proof we obtain the following analogous result:

\[ \sum_{k=0}^{+\infty} \left| \mathbb{P} [\eta_N \cap [-N^{1/4}, N^{1/4}] = k] - \mathbb{P} [\eta \cap [-N^{1/4}, N^{1/4}] = k] \right| \leq 2 \times (2N^2) \times \left( \frac{4\sqrt{N}}{4N^2} \right)^2 = \frac{8}{N}. \]
Let $S$ be the $1 \times 1$ square centered at 0 and let $A^{\text{cell}}_1(S, r)$ be the event that there exists a point $x \in \eta \cap S$ such that there is a white path from the cell of $x$ to $S + \partial B_r$ (note that the cell of $x$ is not necessarily white). For every $\gamma > 0$, there exists $\epsilon > 0$ such that the following holds:

$$\mathbb{P} \left[ \mathbb{P}^\eta \left[ A^{\text{cell}}_1(S, r) \right] \geq r^{-\epsilon} \right] \leq \frac{1}{\epsilon} r^{-\gamma}. \quad (B.3)$$

(The reason why we study white arms instead of black arms is that it will be more suitable for us when we deal with the case $p > 1/2$, see Subsection 5.1.)

**A Schramm-Steif algorithm method.** The last step of the proof uses a Schramm-Steif algorithm method from [SS10]. In our case, we are going to use the extension of Schramm-Steif result discussed in Appendix A. Let $S_1$ (respectively $S_2$) be the subset of all the $1 \times 1$ squares of the grid $\mathbb{Z}^2$ that are below (respectively above) the line $\mathbb{R} \times \{R/2\}$ and that are at distance at most $(\rho + 1)R$ from the rectangle $[-\rho R, \rho R] \times [-R, R]$ (we include the squares that intersect $\mathbb{R} \times \{R/2\}$ in both $S_1$ and $S_2$). Also, let $S_3$ be all the remaining $1 \times 1$ squares of the grid $\mathbb{Z}^2$. The following is a direct consequence of (B.2) (and of the symmetries of the model):

$$\mathbb{V} \mathbb{a} \mathbb{r} \left( \mathbb{P}^\eta \left[ \text{Cross}(\rho R, R) \right] \right)$$

$$\leq 3 \sum_{k=1}^{3} \mathbb{E} \left[ \sum_{S \in S_k} \sum_{x \in \eta \cap S} \mathbb{P}^\eta \left[ \text{Piv}^3_x(\text{Cross}(\rho R, R)) \right]^2 \right]$$

$$= 2 \mathbb{E} \left[ \sum_{S \in S_1} \sum_{x \in \eta \cap S} \mathbb{P}^\eta \left[ \text{Piv}^3_x(\text{Cross}(R)) \right]^2 \right] + \mathbb{E} \left[ \sum_{S \in S_2} \sum_{x \in \eta \cap S} \mathbb{P}^\eta \left[ \text{Piv}^3_x(\text{Cross}(\rho R, R)) \right]^2 \right].$$

Let us first deal with the sum over $S_3$. This sum is less than the expectation of the number of points which are at distance at least $(\rho + 1)R$ from the rectangle $[-\rho R, \rho R] \times [-R, R]$ but whose cell intersects this rectangle. It is not difficult to see that this quantity is less than $O(1) e^{-\Omega(1)R^2}$ (where the constants in $O(1)$ and $\Omega(1)$ may depend on $\rho$).

Now, let us bound the sum over $S_1$. Here, we follow the ideas of [AGMT16] but we use a slightly different algorithm, which can be defined as follows: (We use the same notations as [ABGM14] where the authors use this kind of algorithm to study the Boolean model): Let $Q_0$ denote the set of all $x \in \eta$ whose cell intersects the set $([R \times \{R/2\}) \cap ([-\rho R, \rho R] \times [-R, R])$. Also, let $A_k$ be the set of all $x \in Q_0$ which are white. For each $k \in \mathbb{N}^*$, we define $A_k$ and $Q_k$ (for “active” and “queried” sets) inductively as follows:

i) Let $Q_k$ be the set of all points $x \in \eta$ such that: (a) the cell of $x$ is adjacent to the cell of some $y \in A_{k-1}$ and (b) the cell of $x$ intersects the rectangle $[-\rho R, \rho R] \times [-R, R]$.

Reveal the color of each point of $Q_k$.

ii) Let $A_k$ be the set of all $x \in Q_k$ which are white.

iii) Stop if $A_k = A_{k-1}$.

Note that $A_R$ determines the event that there is a white top-bottom crossing of $[-\rho R, \rho R] \times [-R, R]$ which is the complement of the event $\text{Cross}(\rho R, R)$. As a result, this algorithm determines $\text{Cross}(\rho R, R)$. Now, we can use Corollary A.4 and we obtain that:

$$\mathbb{E} \left[ \sum_{S \in S_1} \sum_{x \in \eta \cap S} \mathbb{P}^\eta \left[ \text{Piv}^3_x(\text{Cross}(\rho R, R)) \right]^2 \right] \leq \delta_{A_R}(S_1),$$

where:

$$\delta_{A_R}(S_1) = \max_{S \in S_1} \max_{x \in \eta} \mathbb{P}^\eta \left[ x \text{ is queried by } A_R \right].$$
What remains to show is that this last quantity is at least polynomially small in $R$. This can be done as follows:

$$\max_{S \in \mathcal{S}_i} \max_{x \in \eta} P^\eta [x \text{ is queried by } A_R] \leq \max_{S \in \mathcal{S}_i} P^\eta \left[A^*_{\text{cell}}(S, \frac{R}{2} - 1)\right].$$

The fact that the above quantity is at least polynomially small in $R$ is an easy consequence of (B.3) (for instance with $\gamma = 3$). We refer to [AGMT16] for more details.

## C Pivotal events for $A_j(1, R)$ when $j$ is odd

In this appendix, we prove Lemmas 4.11, 4.12, 4.13 and 4.14 in the case $j$ odd. We do not need it in order to prove our main result Theorem 1.11. However, we need it in order to prove that Propositions 4.7 and 1.10 also hold when $j$ is odd. Let $S \subseteq A(R/4, R/2)$ and let $\text{Dense}(y, \rho') := \text{Dense}_{1/100}(A(y, \rho', 2\rho'))$ and assume that $\text{Piv}_S(A_j(r, R)) \cap \text{Dense}(y, 2^k \rho)$ holds for some $k \in \{0, \ldots, \lfloor \log_2 \left(\frac{R}{\rho} \right)\rfloor\} =: k_0$. This implies that $A_j(1, R/8)$ holds. In the case $j$ even, this also implies that the 4-arm event $A_4(y, 2^{k+1} \rho, 2^k \rho)$ holds. If $j$ is odd, this rather implies that the following more complicated event holds:

$$\bigcup_{l=k}^{k_0-1} \left(\tilde{A}_4(y, 2^{k+1} \rho, 2^l \rho) \cap \tilde{A}_5(y, 2^{l+2} \rho, 2^k \rho)\right),$$

where: i) $\tilde{A}_4(y, \rho', \rho'')$ is the event that there is a point $x \in \eta$ such that: (a) $C(x)$ (the Voronoi cell of $x$) intersects $A(y, \rho', 2\rho'')$ and (b) there is a 4-arm event in $A(y, \rho', 2\rho'')$ from $\partial B_{\rho'}(y)$ to $\partial B_{\rho'}(y) \cup \partial C(x)$ (the term $\partial C(x)$ is useful only if $C(x)$ intersects $B_{\rho'}(y)$) and ii) $\tilde{A}_4(y, \rho', \rho'')$ is the event that there is a point $x \in \eta$ such that: (a) $C(x)$ intersects $A(y, \rho'/2, \rho')$ and (b) there is a 5-arm event in $A(y, \rho'/2, \rho')$ from $\partial B_{\rho'}(y) \cup \partial C(x)$ to $\partial B_{\rho'}(y)$. (Actually, instead of the 5-arm event, we could have asked that a 6-arm event with colors following the order $(B, B, W, B, B, W)$ holds, where $B$ = black and $W$ = white.) See [Nol08] (for instance Figure 12 therein) for a similar observation in the case of Bernoulli percolation on the triangular lattice. Now, write:

$$\tilde{A}_4(y, \rho', \rho'') = \left\{\mathbb{P} \left[\tilde{A}_4(y, \rho', \rho'') \mid \omega \cap A(y, \rho', \rho')\right] > 0\right\},$$

and:

$$\tilde{A}_5(y, \rho', \rho'') = \left\{\mathbb{P} \left[\tilde{A}_5(y, \rho', \rho'') \mid \omega \cap A(y, \rho', \rho')\right] > 0\right\}.$$

By spatial independence and by a union-bound, we obtain that:

$$\mathbb{P} \left[\bigcup_{l=k}^{k_0-1} \left(\tilde{A}_4(y, 2^{k+1} \rho, 2^l \rho) \cap \tilde{A}_5(y, 2^{l+2} \rho, 2^k \rho)\right)\right] = \sum_{l=k}^{k_0} \mathbb{P} \left[\tilde{A}_4(y, 2^{k+1} \rho, 2^l \rho)\right] \cdot \mathbb{P} \left[\tilde{A}_5(y, 2^{l+2} \rho, 2^k \rho)\right].$$

By using arguments very similar to the arguments of the proof of $\mathbb{P} [\text{Piv}_S(\text{Cross}(2R, R)) \leq O(1) \alpha^2(\rho, R)]$ in Lemma 4.5 and of the proof of Lemma 4.8, we obtain that:

$$\mathbb{P} \left[\tilde{A}_4(y, \rho', \rho'')\right] \leq O(1) \alpha^4(\rho', \rho'),$$

and:

$$\mathbb{P} \left[\tilde{A}_5(y, \rho', \rho'')\right] \leq O(1) \alpha^5(\rho', \rho').$$
Proposition 1.13 and item iii) of Proposition 2.7 imply that:
\[
\alpha_n^a(\rho_1, \rho_2) \leq O(1) \left( \frac{\rho_1}{\rho_2} \right)^\epsilon \alpha_n^a(\rho_1, \rho_2). \tag{C.1}
\]
Together with the above results and the quasi-multiplicativity property, this implies that:
\[
\sum_{k=0}^{n-1} \mathbb{P} \left[ \hat{A}_4(y; 2^{k+1} \rho, 2^{k+1} \rho) \right] \cdot \mathbb{P} \left[ \hat{A}_5(y; 2^{k} \rho, 2^{k} \rho) \right] \leq O(1) \alpha_n^a(\rho, R).
\]
Finally, if we had said that the event \( \text{Piv}_S(\mathbb{A}_j(r, R)) \cap \text{Dense}(y; 2^k \rho) \) implied that the 4-arm event \( \mathbb{A}_4(y; 2^k \rho, R/8) \) held, then it would have given the right estimate. Now, the proof of Lemma 4.11 is very similar to the proof of the inequality \( \mathbb{P} \left[ \text{Piv}_S(\text{Cross}(2R, R)) \right] \leq O(1) \alpha_n^a(\rho, R) \) of Lemma 4.5. To obtain the other lemmas, we need to make similar observations for arm events near the boundaries of the annulus \( \mathbb{A}(r, R) \). The only difference is that, instead of using the estimate (C.1), we will need to use the following similar results (whose proofs are exactly the same as the proof of the second part of Proposition 1.13):
\[
\alpha_n^a(\rho', \rho'') \leq O(1) \left( \frac{\rho'}{\rho''} \right)^\epsilon \alpha_n^a(\rho', \rho''),
\]
and:
\[
\alpha_n^a(\rho', \rho'') \leq O(1) \left( \frac{\rho'}{\rho''} \right)^\epsilon \alpha_n^a(\rho', \rho'').
\]
We leave the details to the reader.

\section{The quantities \( \mathbb{E} \left[ \mathbb{P}^\eta \left[ \mathbb{A}_j(r, R) \right]^2 \right] \)}

In this appendix, we only work at \( p = 1/2 \), hence we forget the subscript \( p \) in the notations. This appendix is devoted to the study of the quantities:
\[
\tilde{\alpha}_j(r, R) := \sqrt{\mathbb{E} \left[ \mathbb{P}^\eta \left[ \mathbb{A}_j(r, R) \right]^2 \right]}.
\]
More precisely, we prove that some of the results that we have proved for the quantities \( \alpha_n^a(r, R) \) are also true for the \( \tilde{\alpha}_j(r, R) \)'s. We will actually do not use the results of Appendix D in the present paper but we include them since their will be crucial for the paper [Van18] where we prove in particular that:
\[
\tilde{\alpha}_j(r, R) \asymp \alpha_n^a(r, R). \tag{D.1}
\]
We refer to [Van18] for the motivations behind (D.1). Let us only say that we will use (D.1) in order to obtain a strict inequality for the exponent of the annealed percolation function.

Let us start the study of the quantities \( \tilde{\alpha}_j(r, R) \). First note that, by Jensen’s inequality, we have:
\[
\tilde{\alpha}_j(r, R)^2 \leq \alpha_n^a(r, R) \leq \alpha_n^a(r, R). \tag{D.2}
\]
As a result, the polynomial decay property:
\[
\frac{1}{C} \left( \frac{r}{R} \right)^C \leq \tilde{\alpha}_j(r, R) \leq C \left( \frac{r}{R} \right)^{1/C}. \tag{D.3}
\]
is a direct consequence of (1.1).
D.1 The quasi-multiplicativity property

In this subsection, we explain how the proof of the quasi-multiplicativity of Section 7 can be adapted in order to prove the following:

**Proposition D.1.** The quasi-multiplicativity property also holds for the quantities \( \tilde{\alpha}_j(r, R) \) i.e. there exists \( C = C(j) \in [1, +\infty) \) such that, for every \( 1 \leq r_1 \leq r_2 \leq r_3 \),

\[
\frac{1}{C} \tilde{\alpha}_j(r_1, r_3) \leq \tilde{\alpha}_j(r_1, r_2) \tilde{\alpha}_j(r_2, r_3) \leq C \tilde{\alpha}_j(r_1, r_3).
\]

**Proof.** The proof is very close to the proof of Proposition 1.6. To simplify the notations, we write the proof in the case \( j = 4 \). The proof for any other even integer is the same and the proof for any odd integer requires the same modification as in Subsection 7.4. We use the same notations as in Subsection 7.1 (remember in particular the definition of the events \( G^{ext}_\delta(R) \) and \( G^{int}_\delta(r) \) in the beginning of this section).

Let us first state and prove an analogue of Lemma 7.5. We need the following notation:

If \( \delta \in (0, 1/1000) \), \( R \in [\delta^{-2}, +\infty) \) and \( r \in [1, R] \) we write:

\[
\tilde{g}^{\text{ext}}_{4, \delta}(r, R) = \sqrt{\mathbb{E} \left[ \mathbb{P}^\eta \left[ A_4(r, R) \cap G^{\text{ext}}_\delta(R) \right] \right]^2}.
\]

Similarly, if \( \delta \in (0, 1/1000) \), \( r \in [\delta^{-2}, +\infty) \) and \( R \in [r, +\infty) \) we write:

\[
\tilde{g}^{\text{int}}_{4, \delta}(r, R) = \sqrt{\mathbb{E} \left[ \mathbb{P}^\eta \left[ A_4(r, R) \cap G^{\text{int}}_\delta(r) \right] \right]^2}.
\]

**Lemma D.2.** There exists \( \delta \in (0, 1/1000) \) such that, for any \( \delta \in (0, 1/1000) \), there is some constant \( a = a(\delta) \in (0, 1) \) such that:

1. For every \( R \in [\delta^{-2} \vee \delta^{-2}, +\infty) \) and every \( r \in [1, R/4] \) we have:

\[
\tilde{g}^{\text{ext}}_{4, \delta}(r, 4R) \geq a \tilde{g}^{\text{ext}}_{4, \delta}(r, R).
\]

(D.4)

2. For every \( r \in [4(\delta^{-2} \vee \delta^{-2}), +\infty) \) and every \( R \in [4r, +\infty) \) we have:

\[
\tilde{g}^{\text{int}}_{4, \delta}(r/4, R) \geq a \tilde{g}^{\text{int}}_{4, \delta}(r, R).
\]

(D.5)

**Proof.** We write only the proof of (D.4) since the proof of (D.5) is the same. We use exactly the same notations as in the proof of Lemma 7.5. By (7.13) we have:

\[
\nu^\eta_{r, R, (\beta_3)} \left[ A_4(r, 4R) \cap G^{\text{ext}}_\delta(4R) \right] \geq c,
\]

for some constant \( c = c(\delta) > 0 \) and if \( \delta \) is sufficiently small. If we take the expectation under \( \mathbb{P}^\eta_{B_{2n}} \), we obtain that:

\[
\mathbb{P}^\eta_{B_{2n}} \left[ A_4(r, 4R) \cap G^{\text{ext}}_\delta(4R) \right] \geq \mathbb{P}^\eta_{B_{2n}} \left[ A_4(r, R) \cap GP^{\text{ext}}_\delta(R) \cap \{ \tilde{z}^{\text{ext}}(r, R) \geq 10\delta R \} \right] \geq \mathbb{P}^\eta_{B_{2n}} \left[ A_4(r, R) \cap G^{\text{ext}}_\delta(R) \right].
\]

We then conclude by using both the following martingale inequality:

\[
\tilde{g}^{\text{ext}}_{4, \delta}(r, 4R)^2 = \mathbb{E} \left[ \mathbb{P}^\eta \left[ A_4(r, 4R) \cap G^{\text{ext}}_\delta(4R) \right] \right] \geq \mathbb{E} \left[ \mathbb{P}^\eta_{B_{2n}} \left[ A_4(r, 4R) \cap G^{\text{ext}}_\delta(4R) \right] \right],
\]

and the following pointwise equality:

\[
\mathbb{P}^\eta \left[ A_4(r, R) \cap G^{\text{ext}}_\delta(R) \right] = \mathbb{P}^\eta_{B_{2n}} \left[ A_4(r, R) \cap G^{\text{ext}}_\delta(R) \right].
\]

\( \square \)
Remark D.3. Note that Remark 7.7 and Jensen’s inequality imply the following: There exists \( c' > 0 \) such that: if \( \delta \in (0, 1/1000) \) is sufficiently small, then for all \( R \in [\delta^{-2}, +\infty) \) and every \( r \in [R/4^3, R] \) we have:

\[
\tilde{g}_{4,\delta}^{\text{ext}}(r, R) \geq c'.
\]

Similarly, if \( \delta \in (0, 1/1000) \) is sufficiently small, then for all \( r \in [\delta^{-2}, +\infty) \) and all \( R \in [r, 4^3 r] \) we have:

\[
\tilde{g}_{4,\delta}^{\text{int}}(r, R) \geq c'.
\]

We now state and prove an analogue of Lemma 7.8. We first need the two following notations:

\[
\tilde{f}_4^{\text{ext}}(r, R) = \sqrt{E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{ext}}(r, R) \right]^2 \right]} ; \tilde{f}_4^{\text{int}}(r, R) = \sqrt{E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{int}}(r, R) \right]^2 \right]}.
\]

Lemma D.4. There exist \( C_1 \in [1, +\infty) \) and \( \tau \in (\beta^{-2}, +\infty) \) such that, for every \( r \in [\tau, +\infty) \) and \( R \in [16 r, +\infty) \):

\[
\tilde{g}_{4,\delta}^{\text{ext}}(r, R) \geq \tilde{f}_4^{\text{ext}}(r, R)/C_1 ,
\]

and:

\[
\tilde{g}_{4,\delta}^{\text{int}}(r, R) \geq \tilde{f}_4^{\text{int}}(r, R)/C_1 .
\]

We have the following corollary (which is a direct consequence of Lemma D.4 and Remark D.3):

Corollary D.5. There exists a constant \( C_2 < +\infty \) such that, for every \( r \in [\tau, +\infty) \) and every \( R \in [r, +\infty) \):

\[
\tilde{g}_{4,\delta}^{\text{ext}}(r, R) \leq \tilde{\alpha}_4(r, R) \leq \tilde{f}_4^{\text{ext}}(r, R) \leq C_2 \tilde{g}_{4,\delta}^{\text{ext}}(r, R) ,
\]

and:

\[
\tilde{g}_{4,\delta}^{\text{int}}(r, R) \leq \tilde{\alpha}_4(r, R) \leq \tilde{f}_4^{\text{int}}(r, R) \leq C_2 \tilde{g}_{4,\delta}^{\text{int}}(r, R) .
\]

Proof of Lemma 7.8. Let us prove (7.15) (the proof of (7.16) is essentially the same). As noted in the proof of Lemma 7.8, we have:

\[
\hat{A}_4^{\text{ext}}(r, R) \subseteq \left( A_4(r, R/4) \cap G_8^{\text{ext}}(R/4) \right) \cup \left( \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_8^{\text{ext}}(R/4) \right).
\]

This implies that \( \tilde{f}_4^{\text{ext}}(r, R)^2 \) is smaller or equal to:

\[
E \left[ \left( \mathbb{P}^\eta \left[ A_4(r, R/4) \cap G_8^{\text{ext}}(R/4) \right] \right)^2 \right] + 2E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{ext}}(r, R/16) \cap G_8^{\text{ext}}(R/4) \right] \right] 
\]

\[
\leq g_{4,\delta}^{\text{ext}}(r, R)^2 + 2E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_8^{\text{ext}}(R/4) \right] \right] 
\]

\[
+ E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_8^{\text{ext}}(R/4) \right] \right] 
\]

\[
\leq g_{4,\delta}^{\text{ext}}(r, R)^2 + 3E \left[ \mathbb{P}^\eta \left[ \hat{A}_4^{\text{ext}}(r, R/16) \setminus G_8^{\text{ext}}(R/4) \right] \right] .
\]

Remember that \( G_8^{\text{ext}}(R/4) \) is measurable with respect to \( \omega \cap A(R/8, R/2) \) and that \( \hat{A}_4^{\text{ext}}(r, R/16) \) is measurable with respect to \( \omega \cap B(R/16) \). Hence, we can use the spatial independence properties of Voronoi percolation to say that the above equals:

\[
\tilde{g}_{4,\delta}^{\text{ext}}(r, R)^2 + 3\tilde{f}_4^{\text{ext}}(r, R/16)^2 \cdot \mathbb{P} \left[ G_8^{\text{ext}}(R/4) \right] .
\]

This inequality is the analogue of (7.17) in the proof of Lemma 7.8. Now, the proof is exactly the same as the proof of Lemma 7.8. \( \square \)
We are now in shape to prove Proposition D.1. We first prove it for \( r_1 \) sufficiently large.

**Proof of the left-hand-inequality in the case \( r_1 \geq r \).** Thanks to Corollary D.5, the proof is the same as in Subsection 7.1.

**Proof of the right-hand-inequality in the case \( r_1 \geq 4r \).** If we do not have both \( r_1 \leq r_2/6 \) and \( r_2 \leq r_3/6 \) then the proof is exactly the same as in Subsection 7.1, so let us assume that \( r_1 \leq r_2/6 \) and \( r_2 \leq r_3/6 \). By Corollary D.5, we have:

\[
\bar{\alpha}_4(r_1, \frac{r_2}{3}) \bar{\alpha}_4(3r_2, r_3) \leq \bar{g}^\text{ext}_{4,3}(r_1, \frac{r_2}{3}) \bar{g}^\text{int}_{4,3}(r_1, 3r_2).
\]

By (7.22) and (7.23), \( \left( \bar{g}^\text{ext}_{4,3}(r_1, \frac{r_2}{3}) \bar{g}^\text{int}_{4,3}(r_1, 3r_2) \right)^2 \) equals:

\[
\mathbb{E} \left[ \mathbb{P}^n \left[ A_4(r_1, r_2/3) \cap \bar{G}^\text{ext}_{4,3} (r_2/3) \cap \text{Dense}_{\mathcal{F}}(r_2/3) \cap \text{QBC}_{\mathcal{F}}(r_2/3) \right] \right.
\]
\[
\quad \times \mathbb{P}^n \left[ A_4(3r_2, r_3) \cap \bar{G}^\text{int}_{\mathcal{F}}(3r_2) \cap \text{Dense}_{\mathcal{F}}(3r_2) \cap \text{QBC}_{\mathcal{F}}(3r_2) \right] \left( \mathbb{P} \left[ B_{2r_2/3} \right] \right)^2 \bigg]\bigg] \right)^2.
\]

Write \( X_1 = \mathbb{P}^n \left[ A_4(r_1, r_2/3) \cap \bar{G}^\text{ext}_{\mathcal{F}}(r_2/3) \right], \ A_1 = \text{Dense}_{\mathcal{F}}(r_2/3) \cap \text{QBC}_{\mathcal{F}}(r_2/3), \ B_1 = \text{QBC}^\text{ext}(r_2/3), \ X_2 = \mathbb{P}^n \left[ A_4(3r_2, r_3) \cap \bar{G}^\text{int}_{\mathcal{F}}(3r_2) \right], \ A_2 = \text{Dense}_{\mathcal{F}}(3r_2) \cap \text{QBC}_{\mathcal{F}}(3r_2) \) and \( B_2 = \text{QBC}^\text{int}(3r_2) \). Then, the above equals:

\[
\mathbb{E} \left[ X_1^2 I_{A_1} I_{\{B_1 \cap B_{r_2/3} \geq 3/4\}} \right] \mathbb{E} \left[ X_2^2 I_{A_2} I_{\{B_2 \cap B_{3r_2/2} \geq 3/4\}} \right] .
\]

Let us use the following lemma whose proof is the same as Lemma 7.10.

**Lemma D.6.** Let \( \mathcal{F} \) and \( \mathcal{G} \) be two independent sub-\( \sigma \)-algebras, \( A_1 \in \mathcal{F} \), \( A_2 \in \mathcal{G} \), \( B_1 \) and \( B_2 \) two events such that \( A_1 \cap B_1 \parallel \mathcal{G} \) and \( A_2 \cap B_2 \parallel \mathcal{F} \). Also, let \( X_1 \) and \( X_2 \) be two random variables such that \( X_1 I_A \) is measurable with respect to \( \mathcal{F} \) and \( X_2 I_{A_2} \) is measurable with respect to \( \mathcal{G} \). Then:

\[
\mathbb{E} \left[ (X_1 X_2)^2 I_{A_1 \cap B_1 \cap A_2 \cap B_2} \right] \geq \frac{9}{16} \left( \mathbb{E} \left[ X_1^2 I_{A_1} I_{\{B_1 \cap \mathcal{F} \geq 3/4\}} \right] \cdot \mathbb{E} \left[ X_2^2 I_{A_2} I_{\{B_2 \cap \mathcal{G} \geq 3/4\}} \right] \right).
\]

If we apply this lemma to \( \mathcal{F} = \sigma(\eta \cap B_{2r_2/3}) \) and \( \mathcal{G} = \sigma(\eta \setminus B_{3r_2/2}) \), we obtain that \( \bar{\alpha}_4(r_1, \frac{r_2}{3}) \bar{\alpha}_4(3r_2, r_3) \) is less than or equal to:

\[
\frac{16}{9} \mathbb{E} \left[ \mathbb{P}^n \left[ A_4(r_1, r_2/3) \cap \bar{G}^\text{ext}_{\mathcal{F}}(r_2/3) \right] \cdot \mathbb{P}^n \left[ A_4(2r_2, r_3) \cap \bar{G}^\text{int}_{\mathcal{F}}(3r_2) \right] \right] .
\]

\[
\mathbb{E} \left[ \mathbb{P}^n \left[ A_4(r_1, r_2/3) \cap \bar{G}^\text{ext}_{\mathcal{F}}(r_2/3) \cap \text{Dense}_{\mathcal{F}}(r_2/3) \cap \text{QBC}_{\mathcal{F}}(r_2/3) \cap \text{QBC}^\text{ext}(r_2/3) \right] \right.
\]
\[
\quad \times \mathbb{P}^n \left[ A_4(2r_2, r_3) \cap \bar{G}^\text{int}_{\mathcal{F}}(3r_2) \cap \text{Dense}_{\mathcal{F}}(3r_2) \cap \text{QBC}_{\mathcal{F}}(3r_2) \cap \text{QBC}^\text{int}(3r_2) \right] \left( \mathbb{P} \left[ B_{2r_2/3} \right] \right)^2 \bigg].
\]

Now, by “ghning” arguments, the above is at most \( O(1) \bar{\alpha}_4(r_1, r_3)^2 \) and we are done. \( \square \)

We have obtained the quasi-multiplicativity property for \( r_1 \geq 4r \), so (as in Subsection 7.1) it only remains to prove the following lemma, which is the analogue of Lemma 7.11.

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Lemma D.7. For every $\tilde{r}$ sufficiently large, there exists a constant $C_3 = C_3(\tilde{r}) < +\infty$ such that, for every $r \in [1, \tilde{r}]$ and every $R \in [r, +\infty)$, we have:

$$\tilde{\alpha}_4(\tilde{r}, R) \leq C_3 \tilde{\alpha}_4(r, R).$$

Sketch of proof of Lemma D.7. First note that, by making similar observations as at the end of the proof of Lemma 7.5 and by following the proof of Lemma D.2, we obtain the following result: Let $\tilde{A}_4^{\text{int}}(r, R)$ be the event defined at the end of the proof of Lemma 7.5. Then, if $\delta$ is sufficiently small and if $\delta$, $r$ and $R$ are as in Item 2 of Lemma D.2, then the following holds: there exists $a = a(\delta) > 0$ and $c > 0$ such that, for every event $F_r$ measurable with respect to $\omega \cap B_{r/2}$ such that $\mathbb{P}[F_r] \geq 1 - c$:

$$\mathbb{E} \left[ \mathbb{P}^\eta \left[ \tilde{A}_4^{\text{int}}(r/4, R) \cap C_4^{\text{int}}(r/4) \cap F_r \right]^2 \right] \geq a \tilde{\alpha}_4(r, R)^2.$$

Let $\text{Dense}^N(r)$ be the event defined in the proof of Lemma 4.6. Corollary D.5 and the above inequality imply that, if $r$ is sufficiently large and if $N$ (that depends on $r$) is sufficiently large, then, for every $R \geq 4r$:

$$\mathbb{E} \left[ \mathbb{P}^\eta \left[ \tilde{A}_4^{\text{int}}(r, R) \cap \text{Dense}^N(r) \right]^2 \right] \geq \Omega(1) \tilde{\alpha}_4(r, R)^2.$$

where $\tilde{A}_4^{\text{int}}(r, R)$ is the event defined in (7.14). Now, if we follow the proof of Lemma 4.6, we obtain that we can extend the four arms and that:

$$\tilde{\alpha}_4(R)^2 \geq \Omega(1) \mathbb{E} \left[ \mathbb{P}^\eta \left[ \tilde{A}_4^{\text{int}}(r, R) \cap \text{Dense}^N(r) \right]^2 \right],$$

where $\Omega(1)$ depends only on $r$ and $N$, which ends the proof. \hfill \Box

This ends the proof of Proposition D.1. \hfill \Box

We also have the following analogues of Propositions 2.4 and 2.5 (with the same proofs):

Proposition D.8. Let $j \in \mathbb{N}^*$, let $1 \leq r \leq R$, and write:

$$\tilde{f}_j(r, R) = \sqrt{\mathbb{E} \left[ \mathbb{P}^\eta \left[ \tilde{A}_j(r, R) \right]^2 \right]}.$$  \hfill (D.8)

There exists a constant $C = C(j) < +\infty$ such that:

$$\tilde{\alpha}_j(r, R) \leq \tilde{f}_j(r, R) \leq C \tilde{\alpha}_j(r, R).$$

Proposition D.9. Let $j \in \mathbb{N}^*$. For every $h \in (0, 1)$, there exists a constant $\epsilon = \epsilon(j, h) \in (0, 1)$ such that, for every $1 \leq r \leq R$ and for every event $G$ which is measurable with respect to $\omega \setminus A(2r, R/2)$ and that satisfies $\mathbb{P}[G] \geq 1 - \epsilon$, we have:

$$\mathbb{E} \left[ \mathbb{P}^\eta \left[ \tilde{A}_j(r, R) \cap G \right]^2 \right] \geq (1 - h) \tilde{\alpha}_j(r, R)^2.$$

D.2 Pivotal events

Let us first prove the following lemma which is the analogue of Lemma 4.6.

Lemma D.10. Let $R \geq 1$ and let $S$ be a $2 \times 2$ square included in $B_R$ and at distance at least $R/3$ from the sides of $B_R$. Then:

$$\mathbb{E} \left[ \mathbb{P}^\eta \left[ \text{Piv}_S^\eta(\text{Cross}(R, R)) \right] \mathbb{I}_{|\eta \cap S| = 1} \right] \geq \Omega(1) \tilde{\alpha}_4(R)^2.$$

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Proof. With exactly the same proof as in Subsection 4.1 (but by using Proposition D.1 instead of Proposition 1.6, Proposition D.8 instead of Proposition D.8, and Proposition D.9 instead of Proposition 2.5), we have the following: There exists $r_0 \geq 1$ and $\epsilon \in (0, 1)$ such that, for every $r \geq r_0$ and for every event $G$ measurable with respect to $\omega \setminus A(2r, R/2)$ that satisfies $P[G] \geq 1 - \epsilon$, we have:
\[
E \left[ P^n \left( \tilde{A}_4 (B_r, R) \cap G \right) \right] \geq \epsilon \tilde{\alpha}_4 (r, R)^2,
\]
where $\tilde{A}_4 (B_r, R)$ is the event defined above Lemma 4.6. Now, the proof is exactly the same as the proof of Lemma 4.6.

The following is a direct consequence of Lemma D.10 and of the main intermediate results from [AGMT16]:

**Corollary D.11.** There exists $\epsilon > 0$ such that, for every $R \in (0, +\infty)$:
\[
\tilde{\alpha}_4 (R) \leq \frac{1}{\epsilon} R^{1+\epsilon}.
\]

**Proof.** In the proof of the first part of Proposition 1.13, we have explained how to prove the analogous result for $\alpha_5^{\text{an}} (R)$. In this proof (written in Subsection 4.1), we have used the following result from [AGMT16]:
\[
\sum_{S \text{ square of the grid } \mathbb{Z}^2} E \left[ P^n \left( \text{Piv}_5 (\text{Cross}(R, R))^2 \mathds{1}_{|y \cap S| = 1} \right) \right] \leq O(1) R^{-\Omega(1)}.
\]
We have then used Jensen’s inequality and Lemma 4.6. If we do not use Jensen’s inequality and if we use Lemma D.10 instead of Lemma 4.6, we obtain the desired result.

As in Subsection 4.2, we prove an estimate between the 4 and 5-arm events. Note that we know that $\alpha_5^{\text{an}} (r, R) \approx \left( \frac{r}{R} \right)^2$ but this does not imply that $\tilde{\alpha}_5 (r, R) \approx (r/R)^2$ (we will prove this last estimate in [Van18]). We have the following:

**Proposition D.12.** There exists an absolute constant $\epsilon > 0$ such that, for every $1 \leq r \leq R$:
\[
\tilde{\alpha}_5 (r, R) \leq \frac{1}{\epsilon} \tilde{\alpha}_4 (r, R) \left( \frac{r}{R} \right)^\epsilon.
\]

**Proof.** First, let $N < +\infty$ be sufficiently large so that:
\[
\tilde{\alpha}_5 (\rho, \rho')^2 \geq \frac{1}{N} \left( \frac{\rho}{\rho'} \right)^N.
\]

Let $M \geq 100$ and let $\rho \geq M$, let $GP(\rho, M)$ be defined as follows:
\[
GP(\rho, M) = \bigcap_{k=1}^{\lfloor \log_5 (M) \rfloor} \text{Dense}(5^k \rho) \cap \text{QBC} \left( (A(5^k \rho, 10 \cdot 5^k \rho)) \right),
\]
where $c > 0$ is chosen sufficiently small so that $P[GP(\rho, M)] \geq 1 - O(1) \rho^{-N}$. As in the proof of the second part of Proposition 1.13 (in Subsection 4.2), we have:
\[
P^n \left[ A_5 (\rho, M \rho) \right] \leq (1 - a)^{2 \lfloor \log_5 (M) \rfloor} \mathbb{E}^n \left[ Y^4 \mathds{1}_{Y \geq 4} \right],
\]
where $Y$ is the number of interfaces from $\partial B_\rho$ to $\partial B_{M \rho}$ and where $a \in (0, 1)$ depends only on the constant $c$ above. Hence:
\[
\tilde{\alpha}_5 (\rho, M \rho)^2 \leq (1 - a)^{2 \lfloor \log_5 (M) \rfloor} \mathbb{E} \left[ \mathbb{E}^n \left[ Y^4 \mathds{1}_{Y \geq 4} \right]^2 \right] + O(1) \rho^{-N}.
\]
Lemma D.13. Define proof of Lemma 4.5 and of the different lemmas of Subsection 4.3. In particular, we will need the following lemma whose proof is very closed to the event). In particular, we will need the following lemma whose proof is very closed to the 

\[ \mathbf{E}^\eta [ Y^3 \mathbf{1}_{ Y \geq 4 } ] \leq d \mathbf{P}^\eta [ A_4 ( \rho, M \rho ) ] , \]

where \( d < +\infty \) depends only on \( c \). Finally:

\[ \tilde{\alpha}_4 ( \rho, M \rho )^2 \leq O(1) (1 - a)^2 \log_3 ( M ) \tilde{\alpha}_4 ( \rho, M \rho )^2 + O(1) \rho^{-N} . \]

Now, the proof is essentially the same as the proof of the second part of Proposition 1.13 (except that we use Proposition D.1 instead of Proposition 1.6).

In [Van18], we will need results similar to the results of Subsections 4.1 and 4.3 but for the quantity \( \mathbf{E} \left[ \mathbf{P}^\eta [ \mathbf{Piv}_S ( A ) ]^2 \right] \) instead of \( \mathbf{P} [ \mathbf{Piv}_S ( A ) ] \) (where \( A \) is a crossing or an arm event). In particular, we will need the following lemma whose proof is very closed to the proof of Lemma 4.5 and of the different lemmas of Subsection 4.3.

**Lemma D.13.** Define \( \mathbf{Piv}^E_S ( A ) \) as in the beginning of Subsection 4.3. Let \( \rho, r, R \in [1, +\infty] \) such that \( \rho \leq r/10 \) and \( r \leq R/2 \), let \( y \) be a point of the plane and let \( S = B_\rho ( y ) \) be the square of size length \( 2r \) centered at \( y \). We have the following:

i) Let \( \rho_1 \in [\rho, +\infty) \) and \( \rho_2 \in [\rho_1, +\infty) \) and assume that \( S \) is included in the bounded connected component of \( A(y, \rho_1, \rho_2) \) and that \( B_{\rho_2} ( y ) \subseteq A(r, R) \). Then:

\[ \mathbf{E} \left[ \mathbf{P}^\eta [ \mathbf{Piv}_S^A ( y, \rho_1, \rho_2 ) ( A_j ( r, R ) ) ]^2 \right] \leq O(1) \tilde{\alpha}_4 ( \rho_1, \rho_2 )^2 . \]

ii) Let \( \rho_1, \rho_2 \in [r, R] \) such that \( \rho_1 \leq \rho_2 \) and assume that \( S \subseteq A(\rho_1, \rho_2) \). Then:

\[ \mathbf{E} \left[ \mathbf{P}^\eta [ \mathbf{Piv}_S^A ( r, \rho_1 ) \cup A(\rho_2, R) ( A_j ( r, R ) ) ]^2 \right] \leq O(1) \tilde{\alpha}_j ( r, R )^2 . \]

**Proof.** We prove only i) since the proof of ii) is essentially the same (actually, in the case \( j \) odd, the proof of i) is slightly more technical). We first prove i) in the case \( j \) even and then in the case \( j \) odd. We need the following notations, where \( 0 < \rho' \leq \rho'' \):

\[ \text{Dense}(\rho', \rho'') = \text{Dense}_{1/100}(A(y, \rho', 2\rho')) \cap \text{Dense}_{1/100}(A(y, \rho'', 2\rho'')) . \]

Since \( j \) is even, then for any \( k \in \{0, \cdots, \lfloor \log_2 ( \rho_2 / (4 \rho_1) ) \rfloor = k_0 \}, \mathbf{Piv}_S^A ( y, \rho_1, \rho_2 ) ( A_j ( r, R ) ) \) is included in:

\[ A_4 ( y, 2^{k+1} \rho_1, \rho_2 / 2 ) \cup \left( \mathbf{Piv}_S^A ( y, \rho_1, \rho_2 ) \left( \text{Cross}(2R, R) \setminus \text{Dense}(2^{k} \rho_1, \rho_2 / 2 ) \right) \right) , \]

where \( A_4 ( y, \cdot, \cdot ) \) is the 4-arm event translated by \( y \). As a result, \( \mathbf{Piv}_S^A ( y, \rho_1, \rho_2 ) ( A_j ( r, R ) ) \) is also included in:

\[ A_4 ( y, 2 \rho_1, \rho_2 / 2 ) \cup \left( \bigcup_{k=0}^{k_0} A_4 ( y, 2^{k+2} \rho_1, \rho_2 / 2 ) \setminus \text{Dense}(2^k \rho_1, \rho_2 / 2 ) \bigcup \neg \text{Dense}(2^{k+1} \rho_1, \rho_2 / 2 ) \right) \]

\[ \subseteq \hat{A}_4 ( y, 2 \rho_1, \rho_2 / 2 ) \left( \bigcup_{k=0}^{k_0} \hat{A}_4 ( y, 2^{k+2} \rho_1, \rho_2 / 2 ) \setminus \text{Dense}(2^k \rho_1, \rho_2 ) \bigcup \neg \text{Dense}(2^{k+1} \rho_1, \rho_2 / 2 ) \right) . \]

By \( \sigma \)-additivity, \( \mathbf{P}^\eta [ \mathbf{Piv}_S^A ( y, \rho_1, \rho_2 ) ( A_j ( r, R ) ) ] \) is less than or equal to:

\[ \mathbf{P}^\eta [ \hat{A}_4 ( y, 2 \rho_1, \rho_2 / 2 ) ] + \sum_{k=0}^{k_0} \mathbf{P}^\eta [ \hat{A}_4 ( y, 2^{k+2} \rho_1, \rho_2 / 2 ) ] \mathbf{1}_{ \neg \text{Dense}(2^k \rho_1, \rho_2 / 2 ) } + \mathbf{1}_{ \neg \text{Dense}(2^{k+1} \rho_1, \rho_2 / 2 ) } . \]

Now, note that, for every \( k \in \{0, \cdots, k_0\} : \)
\[ \mathbb{P}^\eta \left[ \hat{A}_4(y; 2^{k+2} \rho_1, \rho_2) \right] \text{ is independent of } 1_{-\text{Dense}(2^k \rho_1, \rho_2)}. \]

\[ \mathbb{E} \left[ \mathbb{P}^\eta \left[ \hat{A}_4(y; 2^{k+2} \rho_1, \rho_2) \right]^2 \right] \leq C (2^k)^C \tilde{\alpha}_4(y; \rho_1, \rho_2) \text{ for some } C < +\infty \text{ by Proposition D.8, Proposition D.1 and (D.3)}. \]

\[ \mathbb{P} \left[ \text{Dense}(2^k \rho_1, \rho_2/2) \right] \leq O(1) \exp(-\Omega(1)(2^k \rho_1)^2). \]

Note also that:

\[ \mathbb{E} \left[ \mathbb{P}^\eta \left[ \hat{A}_4(y; 2 \rho_1, \rho_2) \right]^2 \right] \leq C_{\tilde{\alpha}_4}(\rho_1, \rho_2) \text{ and:} \]

\[ \mathbb{P} \left[ \text{Dense}(2^{k+1} \rho_1, \rho_2/2) \right] \leq O(1) \exp(-\Omega(1)(2^k \rho_1)^2). \]

Let us end the proof by showing that:

\[ \mathbb{E} \left[ \left( \mathbb{P}^\eta \left[ \hat{A}_4(y; 2 \rho_1, \rho_2) \right] + \sum_{k=0}^{k_0} \mathbb{P}^\eta \left[ \hat{A}_4(y; 2^{k+2} \rho_1, \rho_2/2) 1_{-\text{Dense}(2^k \rho_1, \rho_2/2)} \right] + 1_{-\text{Dense}(2^{k+1} \rho_1, \rho_2/2)} \right)^2 \right] \]

is at most \( O(1) \tilde{\alpha}_4(y; \rho_1, \rho_2)^2 \). If we expand the above square, apply the Cauchy-Schwarz inequality to each of the \( 2^{k_0+3} \) terms and use the five items above, then we obtain the desired result since:

\[ \sum_{k,l} \sqrt{(2^k)^C \exp(-\Omega(1)(2^k \rho_1)^2)(2^l)^C \exp(-\Omega(1)(2^l \rho_1)^2)} < +\infty. \]

Let us now assume that \( j \) is odd. In this case, as explained in Appendix C, the pivotal event cannot be well described only by using the 4-arm event but can be described by using the 4-arm and the 5-arm events. As in Appendix C, the result is obtained by using a comparison estimate between the 4-arm and the 5-arm event. The only difference is that in the present case we use Proposition D.12 instead of the analogous result for \( \alpha^{an}_4(\cdot, \cdot) \) and \( \alpha^{an}_5(\cdot, \cdot) \).

\[ \Box \]

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