A BISHOP TYPE INEQUALITY ON METRIC MEASURE SPACES WITH RICCI CURVATURE BOUNDED BELOW

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Abstract. We define a Bishop-type inequality on metric measure spaces with Riemannian curvature-dimension condition. The main result in this short article is that any \(\text{RCD}^*\) spaces with the Bishop-type inequalities possess only one regular set in not only the measure theoretical sense but also the set theoretical one. As a corollary, the Hausdorff dimension of such \(\text{RCD}^*\) spaces are exactly \(N\). We also prove that every tangent cone at any point on such \(\text{RCD}^*\) spaces is a metric cone.

1. Introduction

Ricci limit spaces are the Gromov-Hausdorff limits of sequences of Riemannian manifolds with Ricci curvature bounded from below. They are roughly classified into two classes, collapsing and non-collapsing. For simplicity, we only consider closed Riemannian manifolds in this section. Let \(\{(M_n, g_n)\}_{n \in \mathbb{N}}\) be a sequence of \(N\)-dimensional closed Riemannian manifolds with Ricci curvature bounded from below and uniformly bounded diameters. By the Gromov compactness theorem, we are able to find a convergent subsequence \(\{(M_{n_k}, g_{n_k})\}_{k}\). We denote the Riemannian volume measure of \((M_n, g_n)\) by \(\text{vol}_n\). Let \((Y, \nu)\) be the limit space of such subsequence \(\{(M_{n_k}, g_{n_k}, \text{vol}_{n_k})\}\), more precisely, of \(\{(M_{n_k}, g_{n_k}, \text{vol}_{n_k})\}_k\) where \(\text{vol}\) is the normalized volume measure \(\text{vol}/\text{vol}(M)\). Then we call \((Y, \nu)\) a non-collapsing Ricci limit space if

\[
\lim_{n \to \infty} \text{vol}_n(M_n) > 0,
\]

and we call \((Y, \nu)\) a collapsing Ricci limit space if \(\lim_{n \to \infty} \text{vol}_n(M_n) = 0\). On the series of papers of Colding and Cheeger-Colding \([8, 10, 11]\), it is proven that there are many nice properties on non-collapsing Ricci limit spaces.

Theorem 1.1 \([8, 10, 11]\). Let \((Y, \nu)\) be a non-collapsing Ricci limit space of a sequence of \(N\)-dimensional Riemannian manifolds. Then

- \(\dim_H Y = N\),
- \(\nu\)-almost every point has the unique tangent cone that is isometric to \(\mathbb{R}^N\),
- \(\nu\) is absolutely continuous to \(\mathcal{H}^N\) on some nice regular set of full measure and vice versa,
- all the tangent cones are metric cones,
- \((Y, \nu)\) is rectifiable in the sense of Cheeger-Colding (see Definition 5.3 in \([10]\)).

Compare with the collapsing and non-collapsing Ricci limit spaces.

Theorem 1.2 \([8, 10, 12]\). Let \((Y, \nu)\) be a collapsing Ricci limit space of a sequence of \(N\)-dimensional Riemannian manifolds. Then

- \(\dim_H Y \leq N - 1\),
- there exists an integer \(k\) with \(1 \leq k < N\) such that \(\nu\)-almost every points has the unique tangent cone that is isometric to \(\mathbb{R}^k\),
$\nu$ is absolutely continuous to $H^k$ on some nice regular set of full measure and vice versa.

$(Y, \mu)$ is rectifiable in the sense of Cheeger-Colding.

It is known that there is an example of collapsing Ricci limit space on which there exists a point whose tangent cones are not metric cones (Example 8.95 in [8]).

On the other hand, without using the Gromov-Hausdorff approximation by Riemannian manifolds, a concept of a metric measure space with lower Ricci curvature bound is also defined. Lott-Villani, Sturm have introduced the curvature-dimension condition $CD(K, N)$ independently for the case $K \in \mathbb{R}$, $N = \infty$ and $K = 0$, $1 < N < \infty$ by [20], for any case by [24, 25]. They prove that any Ricci limit space with finite total mass satisfy $CD$ conditions. However Finsler manifolds with $N$-Ricci curvature bounded below from $K$ satisfies $CD(K, N)$ condition though generic such manifolds are not Ricci limit spaces (see [22] for the definition of $N$-Ricci curvature and the relation with $CD$ condition, and see [8] for the relation of Ricci limit spaces and Finsler manifolds). In order to rule out Finsler manifolds, Ambrosio-Gigli-Savàre introduce the Riemannian curvature-dimension condition $RCD(K, \infty)$ for $K \in \mathbb{R}$ on metric measure space with finite mass [3] and with $\sigma$-finite measure [1]. Later Erbar-Kuwada-Sturm [13] and Ambrosio-Mondino-Savàre [4] define $RCD^*(K, N)$ condition for finite $N \in (1, \infty)$ independently, where $^*$ means the reduced curvature-dimension condition that is defined by Bacher-Sturm [6] for the tensorization property and the local-to-global property. Recently, Mondino-Naber prove the nice local structures on finite dimensional $RCD$ spaces [21].

**Theorem 1.3** ([21],[25]). Let $(X, d, m)$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume supp $m = X$. Then

- $\dim_h(X, d) \in (0, N]$,
- for $m$-almost every point $x \in X$, there exists an integer $k = k_x$ with $1 \leq k \leq N$ such that the tangent cones are unique and isomorphic to $\mathbb{R}^k$,
- $(X, d, m)$ is rectifiable in the sense weaker than Cheeger-Colding’s one.

We denote by $R_k$ the set of all points whose tangent cones are isomorphic to $k$-dimensional Euclidean space, and by $R = \bigcup_k R_k$ (see Definition 2.4).

Since $RCD$ spaces are not defined as limits of sequences of Riemannian manifolds, neither collapsing nor non-collapsing makes sense. In this article, we prove the $RCD$ spaces with Bishop-like inequalities behave like non-collapsing spaces. More precisely, our main result is as follows.

**Theorem 1.4** (Theorem 4.4, Corollary 4.5, Corollary 4.6, and Theorem 5.2). Let $(X, d, m)$ be an $RCD^*(K, N)$ space with the generalized Bishop inequality $BI(K, N)$. Assume supp $m = X$. Then

1. $R = R_N$,
2. on $R_N$, the measure $m$ is absolutely continuous to $H^N$ and vice versa,
3. $\dim_h(X, d) = N$,
4. All tangent cones are metric cones.

**Remark 1.5.** The family of $RCD^*(K, N)$ spaces with the generalized Bishop inequality $BI(K, N)$ includes the family of non-collapsing Ricci limit spaces of appropriate curvature and dimension bounds. However these two classes do not coincide. See Remark 5.3.

2. Preliminaries

A triplet $(X, d, m)$ consisting of a complete separable metric space $(X, d)$ and a locally finite $\sigma$-compact positive Borel measure $m$ on $X$ is called a metric measure
space. Two metric measure spaces \((X, d, m)\) and \((Y, r, \nu)\) are isomorphic if there exists an isometry \(f: \text{supp} m \rightarrow \text{supp} \nu\) with \(f_* m = \nu\). A continuous curve \(\gamma: [0, 1] \rightarrow X\) is absolutely continuous if there exists an \(L^1(0, 1)\) function \(g\) such that
\[
d(\gamma_s, \gamma_t) \leq \int_s^t g(r) \, dr, \quad \text{for any } s \leq t \in [0, 1].
\]
For a continuous curve \(\gamma: [0, 1] \rightarrow X\), the metric derivative \(|\dot{\gamma}|\) is defined by
\[
|\dot{\gamma}_t| := \lim_{s \rightarrow t} \frac{d(\gamma_t, \gamma_s)}{|s - t|}
\]
as long as the right-hand side makes sense. It is known that every absolutely continuous curve has the metric derivative for almost every point \([2]\). We call an absolutely continuous curve \(\gamma: [0, 1] \rightarrow X\) a geodesic if \(|\dot{\gamma}_t| = d(\gamma_t, \gamma_1)\) for almost every \(t \in [0, 1]\). A metric space \((X, d)\) is called a geodesic space if for any two points, there exists a geodesic connecting them.

We denote the set of all Lipschitz functions on \(X\) by \(\text{LIP}(X)\). For \(f \in \text{LIP}(X)\), the local Lipschitz constant at \(x\), \(|\nabla f|(x)\), is defined as
\[
|\nabla f|(x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}
\]
if \(x\) is not isolated, otherwise \(|\nabla f|(x) = \infty\). For \(f \in L^2(X, m)\), we define the Cheeger energy \(\text{Ch}(f)\) as
\[
\text{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{n \rightarrow \infty} \int_X |\nabla f_n|^2 \, dm : f_n \in \text{LIP}(X), f_n \rightarrow f \text{ in } L^2(X, m) \right\}.
\]
Set \(D(\text{Ch}) := \{ f \in L^2(X, m) : \text{Ch}(f) < \infty \}\). We define the Sobolev space \(W^{1,2}(X, d, m) := L^2(X, m) \cap D(\text{Ch})\) equipped with the norm \(\|f\|_{1,2}^2 := \|f\|_2^2 + 2\text{Ch}(f)\). It is known that \(W^{1,2}\) is a Banach space. We say that \((X, d, m)\) is infinitesimally Hilbertian if \(W^{1,2}\) is a Hilbert space.

We denote the set of all Borel probability measures on \(X\) by \(\mathcal{P}(X)\). We define \(\mathcal{P}_2(X)\) as the set of all Borel probability measures with finite second moment, that is, \(\mu \in \mathcal{P}_2(X)\) if and only if \(\mu \in \mathcal{P}(X)\) and there exists a point \(o \in X\) such that \(\int_X d(x, o)^2 \, d\mu(x) < \infty\). We call a measure \(q \in \mathcal{P}(X \times X)\) a coupling between \(\mu\) and \(\nu\) if \((p_1)_* q = \mu\) and \((p_2)_* q = \nu\), where \(p_i : X \times X \rightarrow X\) are natural projections for \(i = 1, 2\). For two probability measures \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\), we define the \(L^2\)-Wasserstein distance between \(\mu_0\) and \(\mu_1\) as
\[
W_2(\mu_0, \mu_1) := \inf \left\{ \int_{X \times X} d(x, y)^2 \, dq(x, y) : q \in \mathcal{Cp}(\mu_0, \mu_1) \right\}^{1/2},
\]
where \(\mathcal{Cp}(\mu_0, \mu_1)\) is the set of all couplings of \(\mu_0\) and \(\mu_1\). The pair \((\mathcal{P}_2(X), W_2)\) is called the \(L^2\)-Wasserstein space, which is a complete separable geodesic metric space if so is \((X, d)\). We explain how geodesics in \(X\) relates to those in \(\mathcal{P}_2(X)\). We denote the space of all geodesics in \(X\) by \(\text{Geo}(X)\), equipped with the sup distance. Define the evaluation map \(e_t : \text{Geo}(X) \rightarrow X\) for \(t \in [0, 1]\) by \(e_t(\gamma) = \gamma_t\). Let \((\mu_1) \in \text{Geo}(\mathcal{P}_2(X))\) be a geodesic connecting \(\mu_0, \mu_1\) in \(\mathcal{P}_2(X)\). Then there exists a probability measure \(\pi \in \mathcal{P}(	ext{Geo}(X))\) such that \((e_t)_* \pi = \mu_t\), by which we say that the geodesic \((\mu_t)_t\) can be lifted to \(\pi\).
2.1. The curvature-dimension condition. For given $K \in \mathbb{R}$ and $N \in (1, \infty)$, we define the distortion coefficients, $\sigma_{K,N}^{(t)}$, for $t \in [0, 1]$, by

$$
\sigma_{K,N}^{(t)}(\theta) := \begin{cases} 
\infty & \text{if } K\theta^2 \geq N\pi^2, \\
\frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\
\frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 = 0, \\
\frac{\sinh(\theta\sqrt{-K/N})}{\sin(\theta\sqrt{-K/N})} & \text{if } K\theta^2 < 0.
\end{cases}
$$

Let $(Y, d)$ be a geodesic metric space and $f : Y \to \mathbb{R} \cup \{\pm \infty\}$ a function on $Y$.

**Definition 2.1** ([13]). A function $f : Y \to \mathbb{R} \cup \{\pm \infty\}$ is said to be $(K, N)$-convex for $K \in \mathbb{R}$ and $N \in (1, \infty)$ if for any two points $y_0, y_1 \in Y$, there exists a geodesic $(y_t)_t$ connecting them such that

$$
\exp \left( -\frac{1}{N} f(y_t) \right) \geq \sigma_{K,N}^{(1-t)}(d(y_0, y_1)) \exp \left( -\frac{1}{N} f(y_0) \right) + \sigma_{K,N}^{(t)}(d(y_0, y_1)) \exp \left( -\frac{1}{N} f(y_1) \right)
$$

holds for any $t \in [0, 1]$.

Let $(X, d, m)$ be a geodesic metric measure space. Consider $\mu = \rho m \ll m$ a probability measure that is absolutely continuous with respect to $m$ and its Radon-Nikodym derivative being $\rho$. We define the relative entropy functional $\text{Ent}_m(\mu)$ by

$$
\text{Ent}_m(\mu) := \int_{\{\rho > 0\}} \rho \log \rho \, dm,
$$

whenever $(\rho \log \rho)_+$ is integrable, otherwise we define $\text{Ent}_m(\mu) = \infty$.

**Definition 2.2** ([13], cf. [3]). Let $(X, d, m)$ be a geodesic metric measure space. We say that $(X, d, m)$ satisfies the entropic curvature-dimension condition $CD^e(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty)$ if the relative entropy functional $\text{Ent}_m$ is $(K, N)$-convex. Moreover if $(X, d, m)$ is infinitesimally Hilbertian, $(X, d, m)$ is called an $RCD^*(K, N)$ space.

Under the infinitesimal Hilbertianity condition, $CD^e(K, N)$ is equivalent to $CD^*(K, N)$.

2.2. Tangent cones and regular sets on $RCD$ spaces. Let $(X, d, m)$ be a metric measure space. Take a point $x_0 \in \text{supp} \, m$ and fix it. We call a quadruple $(X, d, m, x_0)$ a pointed metric measure space. We say that a pointed metric measure space $(X, d, m, x_0)$ is normalized if

$$
\int_{B_1(x_0)} 1 - d(x_0, \cdot) \, dm = 1.
$$

For $r \in (0, 1)$, define $d_r := d/r$ and

$$
m_r^x := \left( \int_{B_r(x)} 1 - d_r(x, \cdot) \, dm \right)^{-1} m.
$$

Note that the pointed metric measure space $(X, d_r, m_r^x, x)$ is normalized.

Let $C(\cdot) : [0, \infty) \to [1, \infty)$ be a nondecreasing function. Define $\mathcal{M}_{C(\cdot)}$ the family of pointed metric measure spaces $(X, d, m, x)$ that satisfy

$$
m(B_{2r}(x)) \leq C(R)m(B_r(x))
$$
for any \( x \in \text{supp} \, m \), and any \( 0 < r \leq R < \infty \). Gigli, Mondino, and Savaré have proven that there exists a distance function \( D_{C(\cdot)} : \mathcal{M}_{C(\cdot)} \times \mathcal{M}_{C(\cdot)} \to [0, \infty] \), which induces the same topology as the Gromov-Hausdorff one on \( \mathcal{M}_{C(\cdot)} \) \(^{(10)} \). It is known that every \( \text{RCD}^\ast (K,N) \) space for given \( K \in \mathbb{R} \), \( N \in (1,\infty) \) belongs to \( \mathcal{M}_{C(\cdot)} \) for a common function \( C(\cdot) : (0,\infty) \to [1,\infty) \) \((\text{see} \ [25])\), more precisely, they satisfy

\[
\frac{m(B_R(x))}{m(B_r(x))} \leq \int_0^R \frac{S_{K,N}^{-1}(t)}{S_{K,N}(t)} dt
\]

for any \( x \in \text{supp} \, m \), \( 0 < r \leq R \), where

\[
S_{K,N}(t) := \begin{cases} \sin \left( t \sqrt{\frac{K}{N}} \right) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ \sinh \left( t \sqrt{\frac{K}{N}} \right) & \text{if } K < 0. 
\end{cases}
\]

Let \((X,d,m)\) be an \( \text{RCD}^\ast (K,N) \) space for \( K \in \mathbb{R} \) and \( N \in (1,\infty) \). By a simple calculation, we have \((X,d_r,m_r^x)\) for some \( x \in \text{supp} \, m \) being an \( \text{RCD}^\ast (r^2K,N) \) space. Take a point \( x \in \text{supp} \, m \) and fix it. Consider the family of normalized metric measure spaces \( \{(X,d_r,m_r^x)\}_{r \in (0,1)} \). The following comes from a generalization of Gromov’s compactness theorem.

**Theorem 2.3.** The family of normalized metric measure spaces \( \{(X,d_r,m_r^x)\}_{r \in (0,1)} \) is compact with respect to the pointed measured Gromov-Hausdorff topology. Moreover every limit space \((X,d_r,m_r^x,x) \to (Y,d_Y,m_Y,y)\) is a normalized \( \text{RCD}^\ast (0,N) \) space for the non-increasing sequence \( \{r_n\}_n \) with \( r_n \to 0 \).

We define the tangent cone at a point \( x \in \text{supp} \, m \) by

\[
\text{Tan}(X,d,m,x) := \{(Y,d_Y,m_Y,y) : (X,d_{r_n},m_{r_n}^x,x) \to (Y,d_Y,m_Y,y)\},
\]

where \( \{r_n\}_n \) is a non-increasing sequence converging to 0. For simplicity, we just denote by \( \text{Tan}(X,x) \) instead of \( \text{Tan}(X,d,m,x) \) if there is no confusion.

**Definition 2.4.** Let \((X,d,m)\) be an \( \text{RCD}^\ast (K,N) \) space for \( K \in \mathbb{R} \) and \( N \in (1,\infty) \). We call a point \( x \in \text{supp} \, m \) a \( k \)-regular point if \( \text{Tan}(X,x) = \{(\mathbb{R}^k,d_E,L_k^0,0)\} \), where \( L_k^0 \) is the normalized measure at 0. We denote the set of \( k \)-regular points by \( \mathcal{R}_k \).

Mondino-Naber proved the following \( [21] \).

**Theorem 2.5** \(([21] \text{ Corollary 1.2})\). Let \((X,d,m)\) be an \( \text{RCD}^\ast (K,N) \) space for \( K \in \mathbb{R} \) and \( N \in (1,\infty) \). Then

\[
m \left( X \setminus \bigcup_{1 \leq k \leq N} \mathcal{R}_k \right) = 0.
\]

Note that by **Theorem 2.5** we know neither the uniqueness of \( \mathcal{R}_k \) nor \( m \) being absolutely continuous to the \( k \)-dimensional Hausdorff measure \( \mathcal{H}_k \) on \( \mathcal{R}_k \). And on Ricci limit spaces, we know \( m(X \setminus \mathcal{R}_l) = 0 \) for some \( 1 \leq l \leq N \) \((\text{see} \ [13] \ [12])\).

### 3. Possible volume growth on regular sets

In this section we prove a behavior of measure of small balls on regular sets under some assumption, which is crucial in the later section.

We say a metric measure space \((Y,r,\nu)\) nontrivial if \( \text{Diam} \, (Y,r) > 0 \) and \( \nu(Y) \in (0,\infty] \).
Definition 3.1. Let $(X, d, m)$ be a metric measure space. We define the following weakly regular set.

$W_k := \{ x \in X : \text{There exists a nontrivial } X' \text{ with } X' \times \mathbb{R}^k \in \text{Tan}(X, x) \}.$

The author and Lakzian proved the following lemma, which is independently proven by Kell [17].

Lemma 3.2 ([19 Corollary 5.5], [17 Lemma 7]). Let $(X, d, m)$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}, N \in (1, \infty)$. Assume $\text{supp } m = X$ and $X$ is nontrivial. Fix a point $y \in X$. Then for any $R > 0$, there exists a constant $C = C(R, y)$ such that

$m(B_r(x)) \leq Cs$

for each $x \in B_R(y)$ and any $s \in (0, 1]$.

Proposition 3.3. Let $(X, d, m)$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$. Take a point $x \in W_{k - 1}$. Suppose that

\[
\limsup_{r \to 0} \frac{m(B_r(x))}{r^\alpha} =: A < \infty
\]

for some $\alpha < k$. Then

\[
\liminf_{r \to 0} \frac{m(B_r(x))}{r^\alpha} = 0.
\]

Proof. Let \{\{r_i\}_{i \in \mathbb{N}} be a sequence that realizes $X_{r_i} \to X' \times \mathbb{R}^{k - 1} \in \text{Tan}(X, x)$, where $(X', d', m')$ is nontrivial. Since $(X', d', m')$ is an $RCD^*(0, N - k + 1)$ space by the splitting theorem [14], $m'(B_r(x')) \leq C r$ for small $r > 0$ holds by Lemma 3.2. Thus $L^{k - 1} \times m'(B_r(0, x')) \leq L^{k - 1}(B_r(0)) m'(B_r(x')) \leq C^2 r^k$ holds for some constant $C_0 > 0$. For any $\epsilon > 0$, we obtain

\[
\liminf_{r \to 0} \frac{m(B_r(x))}{r^\alpha} \leq \liminf_{i \to \infty} \frac{m(B_{r_i}(x))}{(r_i)^\alpha}
\]

\[
= \liminf_{i \to \infty} \frac{m_{r_i}^* (B_{r_i}^{d_i}(x))}{(r_i)^\alpha} \cdot \frac{\int_{B_{r_i}(x)} 1 - \frac{1}{r_i} d(x, -) dm}{r_i^\alpha}
\]

\[
\leq \liminf_{i \to \infty} \frac{m_{r_i}^* (B_{r_i}^{d_i}(x))}{(r_i)^\alpha} \cdot \frac{m(B_{r_i}(x))}{r_i^\alpha}
\]

\[
\leq (A + 1) \lim_{i \to \infty} \frac{m_{r_i}^* (B_{r_i}^{d_i}(x))}{(r_i)^\alpha}
\]

\[
\leq (A + 1) L^{k - 1} \times m'(B_r(0, x'))
\]

\[
\leq C_0 (A + 1) r^{k - \alpha}.
\]

Since $\epsilon > 0$ is arbitrary small, we have the conclusion. \hfill \Box

Proposition 3.4. Let $(X, d, m)$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$. Take a point $x \in R_k$. Suppose that

\[
\liminf_{r \to 0} \frac{m(B_r(x))}{r^\beta} =: D > 0
\]

holds for some $\beta > k$. Then we obtain

\[
\limsup_{r \to 0} \frac{m(B_r(x))}{r^\beta} = \infty.
\]

Before proving Proposition 3.4, we show the following lemma.
Lemma 3.5. Take $r \in (0,1)$ and define the sequence $\{r_n\}_{n \in \mathbb{N}}$ by $r_n := r^n$. Assume $D \in (0,\infty)$. Then for sufficiently large $n$,

$$\int_{B_{r_n}(x)} 1 - \frac{1}{r_n} d(x, \cdot) \, dm \geq \frac{D}{2} \frac{1 - r}{r} \frac{r^{\beta + 1}}{1 - r^{\beta + 1}} \frac{1}{(r^n)^{\beta}}$$

holds.

Proof. This lemma is just a calculation. For sufficiently large $n$, we have $m(B_{r_n}(x))/r_n^\beta \geq D/2$. For simplicity, we denote $B_r(x)$ by $B_r$ and $d(x, \cdot)$ by $d_x$. Then

$$\int_{B_{r_n}} 1 - \frac{1}{r_n} d_x \, dm = \int_{B_{r_n} \setminus B_{r_n+1}} 1 - \frac{1}{r_n} d_x \, dm + \int_{B_{r_n+1}} 1 - \frac{1}{r_n} d_x \, dm$$

$$\geq \int_{B_{r_n+1} \setminus B_{r_n+2}} 1 - \frac{1}{r_n} d_x \, dm + \int_{B_{r_n+2}} 1 - \frac{1}{r_n} d_x \, dm$$

$$\geq \left(1 - \frac{r_{n+1}}{r_n}\right) m(B_{r_{n+1}} \setminus B_{r_{n+2}}) + \int_{B_{r_{n+2}} \setminus B_{r_{n+3}}} 1 - \frac{1}{r_n} d_x \, dm + \int_{B_{r_{n+3}}} 1 - \frac{1}{r_n} d_x \, dm$$

$$\geq (1 - r) \sum_{i=1}^\infty r^{i-1} m(B_{r_{n+1}})$$

$$\geq (1 - r) \frac{D}{2} \sum_{i=1}^\infty r^{i-1} \frac{1}{r^{n+i}}$$

$$= \frac{D}{2} \frac{1 - r}{r} \frac{r^{\beta + 1}}{1 - r^{\beta + 1}} \sum_{i=1}^\infty r \beta^{i}.$$ 

Nothing to prove if $D = \infty$, so assume $D \in (0,\infty)$. Since $D < \infty$, by Lemma 3.5, we obtain

$$\int_{B_{r_n}(x)} 1 - \frac{1}{r_n} d(x, \cdot) \, dm \geq \frac{D}{2} \frac{1 - r}{r} \frac{r^{\beta + 1}}{1 - r^{\beta + 1}} \frac{1}{(r^n)^{\beta}}$$

holds. Note that since $x \in \mathcal{R}_k$, for any $\epsilon > 0$, $m^x(B^{d_{r_n}}_{\epsilon r_n}(x)) \rightarrow C_1 \epsilon^k$ for some constant $C_1$, which is independent of the choice of the sequence $\{r_n\}_n$. We obtain

$$\limsup_{r \to 0} \frac{m(B_r(x))}{r^\beta} \geq \liminf_{n \to \infty} \frac{m(B_{r_n}(x))}{(r^n)^{\beta}}$$

$$= \liminf_{n \to \infty} \frac{m^x(B^{d_{r_n}}_{\epsilon r_n}(x))}{\epsilon^k} \frac{1}{r_n} \int_{B_{r_n}(x)} 1 - \frac{1}{r_n} d(x, \cdot) \, dm$$

$$\geq \liminf_{n \to \infty} \frac{m^x(B^{d_{r_n}}_{\epsilon r_n}(x))}{\epsilon^k} \frac{D}{2} \frac{1 - r}{r} \frac{r^{\beta + 1}}{1 - r^{\beta + 1}} \frac{1}{(r^n)^{\beta}}$$

$$= \frac{C_1 D}{2} \frac{1 - r}{r} \frac{r^{\beta + 1}}{1 - r^{\beta + 1}} \epsilon^{(-\beta - k)}.$$ 

Since $\epsilon > 0$ is arbitrary, we have the conclusion.

4. Bishop inequality and regular sets on RCD spaces

From now, we always assume that every metric measure space $(X, d, m)$ satisfies $\text{supp } m = X$. In this section we define variants of the Bishop inequalities on $\text{RCD}^*(K,N)$ spaces. We prove every $\text{RCD}^*(K,N)$ space satisfies the weak generalized Bishop inequality. For any $K' \in \mathbb{R}$ and $N' \in \mathbb{N}$, define $\text{V}_{K',N'}$ as the
normalized volume measure on the space form of curvature $K'$ and of dimension $N'$, where normalized means in the sense of [21].

**Definition 4.1.** Let $(X, d, m)$ be a metric measure space. For given $K' \in \mathbb{R}$ and $N' \in \mathbb{N}$, we say that $(X, d, m)$ satisfies the generalized Bishop inequality $BI(K', N')$, if for any $\epsilon > 0$, and any $x \in X$, there exists a small number $r_{\epsilon, x} > 0$ such that

$$m^\epsilon_x(B_{r}^d(x)) \leq (1 + \epsilon)\nu_{r^2K', N'}(s) \quad \text{holds for any } r \in (0, r_{\epsilon, x}) \text{ and any } s \in (0, \min\{\text{Diam } X, 1\}).$$

**Proof.** Suppose $(X, d, m)$ does not satisfy the above property. Then there exist $\epsilon > 0, x \in R_l$ for some $1 \leq l \leq N$, and $s \in (0, 1)$ such that

$$m^\epsilon_{x,s}(B_{r}^d(x)) > (1 + \epsilon)\nu_{s^2K', s}(s)$$

holds for some decreasing sequence $\{r_n\}_{n \in \mathbb{N}}$ with $r_n \to 0$. Letting $n \to \infty$ leads

$$\nu_{l}(B_s(0)) \geq (1 + \epsilon)\nu_{0,l}(s) = (1 + \epsilon)\nu_{l}(B_s(0)).$$

This is a contradiction. \hfill \Box

The following is the main result in this paper.

**Theorem 4.4.** Let $(X, d, m)$ be a $RCD^*(K, N)$ space. Then $(X, d, m)$ satisfies the following: for any $\epsilon > 0, x \in R_l$ for $1 \leq l \leq N$, and $s \in (0, 1)$, there exists a small number $r_{\epsilon, x, s} > 0$ such that

$$m^\epsilon_{x,s}(B_{r}^d(x)) \leq (1 + \epsilon)\nu_{r^2K, s}(s)$$

holds for any $r \in (0, r_{\epsilon, x, s})$. The following is the main result in this paper.
as \( s \to 0 \) for some \( C' := C'(r', N) > 0 \). Here \( s \to 0 \) is equivalent to \( r \to 0 \). Therefore

\[
\limsup_{r \to 0} \frac{m(B_r(x))}{r^N} < \infty.
\]

This contradicts Proposition 4.3. Hence \( \mathcal{R}_l = \emptyset \) if \( l < N \). Hence combining this fact and Theorem 2.5 leads the consequence. \( \Box \)

The following is a direct consequence of the proof of Theorem 4.4.

**Corollary 4.5.** Let \((X, d, m)\) be an \( RCD^*(K, N) \) space for \( K \in \mathbb{R}, N \in \mathbb{N} \) and assume \((X, d, m)\) satisfies the generalized Bishop inequality. Then

\[
0 < \liminf_{r \to 0} \frac{m(B_r(x))}{r^N} \leq \limsup_{r \to 0} \frac{m(B_r(x))}{r^N} < \infty
\]

holds for any \( x \in X \). In particular \( \text{(4.2)} \) holds for any \( x \in \mathcal{R}_N \).

For given two Borel measures \( \mu, \nu \) on \( X \), we say \( \mu \) and \( \nu \) are equivalent if \( \mu \) is absolutely continuous with respect to \( \nu \), \( \mu \ll \nu \), and also \( \nu \ll \mu \).

**Corollary 4.6.** Let \((X, d, m)\) be the same as Corollary 4.5. Then \( m \) is equivalent to the \( N \)-dimensional Hausdorff measure \( \mathcal{H}^N \) on \( \mathcal{R}_N \). Moreover the Hausdorff dimension of \((X, d)\) is \( N \).

**Proof.** We define the set \( C_i, i = 1, 2, \ldots \), by

\[
C_i := \left\{ x \in X : \frac{1}{i} < \liminf_{r \to 0} \frac{m(B_r(x))}{r^N} < i, \text{ for } r \in (0, 1) \right\}.
\]

By Proposition 4.3 in Appendix A, we claim that \( C_i \) is a Borel set for each \( i \in \mathbb{N} \). By Corollary 4.5, \( \mathcal{R}_N \subset \bigcup_{i \in \mathbb{N}} C_i \). By Theorem 2.4.3 in [5], \( m|_{C_i} \) is equivalent to \( \mathcal{H}^N \) on \( C_i \). Since \( m(X \setminus \mathcal{R}_N) = 0 \), there exist a number \( i \in \mathbb{N} \) and a measurable subset \( A \subset C_i \) such that \( 0 < c_i^{-1}m(A) \leq \mathcal{H}^N(A) \leq c_i m(A) < \infty \) hold for some constant \( c_i > 0 \). Thus \( \dim \mathcal{H} X \geq N \). On the other hand, by Corollary 2.5 in [25], we have \( \dim \mathcal{H} X \leq N \) if \( \sup m = X \). Thus we have the conclusion. \( \Box \)

5. Tangent cones

In this section, we consider tangent cones at any points on \( RCD^*(K, N) \) spaces with the generalized Bishop inequality \( BI(K, N) \). For noncollapsing Ricci limit spaces, all tangent cones are metric cones. Here we prove the same result in our setting. The following result plays a key role in our proof.

**Theorem 5.1** ([5] Theorem 1.1). Let \((X, d, m)\) be an \( RCD^*(0, N) \) space with \( \sup m = X \), \( o \in X \), and \( R > r > 0 \) such that

\[
m(B_R(o)) = \left( \frac{R}{r} \right)^N m(B_r(o)).
\]

Then exactly one of the following holds:

1. \( S_{R/2}(o) \) contains only one point. In this case \((X, o)\) is pointed isometric to \(([0, \text{Diam}(X, d)], 0)\) and via the isometry, \( m|_{B_t(o)} \) can be seen as \( c_2x^{N-1} dx \) for \( c := Nm(B_R(o)) \).

2. \( S_{R/2}(o) \) contains two points. In this case \((X, d)\) is a 1-dimensional Riemannian manifold possibly with boundary, and there is a bijective local isometry from \( B_2(o) \) to \((-R, R)\) sending \( o \) to \( 0 \) and the measure \( m|_{B_t(o)} \) to the measure \( c|x|^{N-1} dx \) for \( c := Nm(B_R(o))/2 \). Moreover such local isometry is an isometry when restricted to \( \overline{B}_{R/2}(o) \).
(3) $S_{R/2}(o)$ contains more than two points. In this case $N \geq 2$ and there exists an $RCD^*(N-2,N-1)$ space $(Z,d_Z,m_Z)$ with $\text{Diam}(Z,d_Z) \leq \pi$ such that the ball $B_R(o)$ is locally isometric to the ball $B_R(o_Y)$ of the cone built over $Z$. Moreover such local isometry is an isometry when restricted to $\overline{B}_{R/2}(o)$.

**Theorem 5.2.** Let $(X,d,m)$ be an $RCD^*(K,N)$ space for $K \in \mathbb{R}$ and $N \in (1,\infty)$. Assume the generalized Bishop inequality $BI(K,N)$. Then tangent cones are metric cones. Moreover, every tangent cone $(Y,d_Y,m_Y)$ is isometric to a cone over an $RCD^*(N-2,N-1)$ space $(Z,d_Z,m_Z)$.

**Proof.** Take an arbitrary point $x \in X$ and fix it. By the Corollary 4.3 we have

$$0 < \limsup_{r \to 0} \frac{m(B_r(x))}{r^N} < \infty.$$ 

And by the Bishop-Gromov inequality, we have the monotonicity of $r \mapsto m(B_r(x))/V_{K,N}(r)$. Hence we obtain the limit

$$C_1 := \lim_{r \to 0} \frac{m(B_r(x))}{r^N}.$$ 

Take a tangent cone $(Y,d_Y,m_Y,y) \in \text{Tan}(X,x)$. Denote by $\{r_n\}_{n \in \mathbb{N}}$ the decreasing sequence that realized the convergent sequence $\{(X,d_{r_n},m_{r_n}^n,x)\}_{n \in \mathbb{N}}$ to $Y$. Since

$$\frac{r^N}{m(B_r(x))} \leq \frac{r^N}{\int_{B_r(x)} 1 - d_r(x,\cdot) \, dm} \leq 2^{N+1} \left( \frac{r^N}{m(B_r/2(x))} \right)^{-1} \int_{B_{r/2}(x)} 1 - d_r(x,\cdot) \, dm$$

holds, we are able to take a subsequence $\{r_{n_k}\}_{k \in \mathbb{N}}$ such that $r_{n_k}^N / \int_{B_{r_{n_k}}(x)} 1 - d_{r_{n_k}}(x,\cdot) \, dm$ converges to a positive and finite constant $C_2$. For simplicity, we just write $\{r_k\}$ instead of $\{r_{n_k}\}$. Then we obtain

$$m_Y(B_s(y)) = \lim_{n \to \infty} m_{r_n}^n(B_{r_n}d^n(x)) = \lim_{k \to \infty} m_{r_k}^k(B_{r_k}d^k(x)) = s^N \lim_{k \to \infty} \frac{r_k^N}{m_{r_k}^k(B_{r_k}(x))} \frac{r_k^N}{\int_{B_{r_k}(x)} 1 - d_{r_k}(x,\cdot) \, dm} = C_1C_2s^N.$$ 

Therefore the equality

$$\frac{m_Y(B_{s_2}(y))}{m_Y(B_{s_1}(y))} = \left( \frac{s_2}{s_1} \right)^N$$

holds for any $s_1,s_2 > 0$. Since $N > 1$ and $\mathcal{R} = \mathcal{R}_N$, $(Y,d_Y,m_Y)$ has no 1-dimensional regular set by Theorem 1.1 in [8]. Therefore, by Theorem 5.2 $(Y,d_Y,m_Y)$ has to be a metric cone built over an $RCD^*(N-2,N-1)$ space $(Z,d_Z,m_Z)$. □

**Remark 5.3.** By Theorem 4.4, Corollary 4.5, Theorem 5.2, it seems that $RCD$ spaces with the generalized Bishop inequality are able to be called “Non-collapsing $RCD$ spaces”. However the following example holds. Consider the closed convex domain $U$ in the $N$-dimensional Euclidean space. Since non-collapsing Ricci limit space do not have boundaries (see [8]), such a domain $U$ is not a non-collapsing Ricci limit space. However, $U$ with the standard Euclidean metric and the Lebesgue measure satisfies $RCD(0,N)$ condition, moreover, it satisfies the generalized Bishop inequality. Still, as far as the author knows, whether $U$ is a collapsing Ricci limit space or not is unknown.
Appendix A. Continuity of the volume of balls on metric measure spaces

Let \((X, d, m)\) be a geodesic metric measure space with \(\text{supp} \ m = X\). We say that \((X, d, m)\) satisfies the \(BG(K, N)\) condition for \(K \in \mathbb{R}, N \in (1, \infty)\) if

\[
m(B_R(x)) \leq \frac{\int_0^R S_{K,N}(t)^{N-1} \, dt}{\int_0^R S_{K,N}(t)^{N-1} \, dt} m(B_r(x))
\]

holds for any \(x \in X, 0 < r \leq R\). Note that, as already stated in Section 2 all \(RCD^*(K, N)\) spaces satisfy the \(BG(K, N)\) condition. Define the function \(F(r) := \int_0^r S_{K,N}^{-1}(t) \, dt\). In [18], we have the following estimate.

Lemma A.1. Let \((X, d, m)\) be a geodesic metric measure space satisfying \(BG(K, N)\) condition for \(K \in \mathbb{R}, N \in (1, \infty)\). For given \(r > 0\) we take two points \(x, y \in X\) such that \(d(x, y) < r\). Then we obtain

\[
\frac{m(B_r(x) \setminus B_r(y))}{m(B_r(x))} \leq \frac{F'(r-d(x, y)/2)}{F(r+d(x, y)/2)} d(x, y) + o(d(x, y))
\]
as \(d(x, y) \to 0\).

In the same spirit as [18], we prove the local Lipschitz continuity of the volume of balls.

Lemma A.2. Let \((X, d, m)\) be an \(RCD^*(K, N)\) space for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). Take a point \(x_0 \in X\) and fix it. Let \(R > 0\) be a positive number. Then there exists a positive number \(C_2 := C_2(x_0, R) > 0\), such that

\[
|m(B_r(x)) - m(B_r(y))| \leq C_2 d(x, y)
\]
for any \(x, y \in B_R(x_0)\) and any \(r \in (0, 1)\).

Proof. By Lemma 5.2 there exists \(C' := C'(x_0, R) > 0\) such that \(m(B_r(x)) \leq C' r\) for any \(x \in B_R(x_0)\) and any \(r \in (0, 1)\). Take two points \(u, v \in B_R(x_0)\) with \(d(u, v) < r\). For simplicity, we write \(d := d(u, v)\). Then by Lemma A.1

\[
|m(B_r(u)) - m(B_r(v))| \leq m(B_r(u) \setminus B_r(v)) + m(B_r(v) \setminus B_r(u))
\]

\[
\leq 2 \left\{ \frac{F'(r-d/2)}{F(r+d/2)} d + o(d) \right\} (m(B_r(u)) + m(B_r(v)))
\]

(A.1)

Note that the curly bracket in (A.1) satisfies

\[
\lim_{r \to 0} \lim_{d(x, y) \to 0} \left\{ 2C' \left( \frac{F'(r-d/2)}{F(r+d/2)} + \frac{o(d)}{d} \right) \right\} =: \hat{C} < \infty.
\]

Hence for sufficiently small \(d\), (A.1) is bounded above. Again take arbitrary distinct points \(x, y \in B_R(x_0)\). Let \(\gamma : [0, 1] \to X\) be a geodesic connecting them. Divide the interval \([0, 1]\) into \([0, 1] = \bigcup [t_i, t_{i+1}], i = 1, \ldots, n\), so that \(d(\gamma_{t_i}, \gamma_{t_{i+1}}) < r\). By (A.1), we have

\[
|m(B_r(x)) - m(B_r(y))| \leq \sum_{i=1}^n |m(B_r(\gamma_{t_i})) - m(B_r(\gamma_{t_{i+1}}))|
\]

\[
\leq 2C' \sum_{i=1}^n \left\{ \frac{F'(r-d(\gamma_{t_i}, \gamma_{t_{i+1}})/2)}{F(r+d(\gamma_{t_i}, \gamma_{t_{i+1}})/2)} + \frac{o(d(\gamma_{t_i}, \gamma_{t_{i+1}}))}{d(\gamma_{t_i}, \gamma_{t_{i+1}})} \right\} d(\gamma_{t_i}, \gamma_{t_{i+1}}).
\]
Since we are able to divide $[0, 1]$ so that $d(\gamma_t, \gamma_{t+1})$ takes arbitrary small value, we obtain

$$|m(B_r(x)) - m(B_r(y))| \leq \tilde{C}d(x, y).$$

Putting $C_2 := \tilde{C}$ leads the consequence.

**Proposition A.3.** Let $(X, d, m)$ be a geodesic metric measure space that satisfies $BG(K, N)$ condition for $K \in \mathbb{R}$, $N \in (1, \infty)$. For given $a, b \in (0, \infty)$ with $a < b$, define the set $C_{a,b}$ by

$$C_{a,b} := \left\{ x \in X ; a \leq \liminf_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \leq \limsup_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \leq b \right\}. $$

Then $C_{a,b}$ is Borel measurable.

**Proof.** Since

$$C_{a,b} = \left\{ x \in X ; a \leq \liminf_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \right\} \cap \left\{ x \in X ; \limsup_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \leq b \right\},$$

and since the case for the limit supremum are able to be proven by the same way below, we only prove that

$$A_a := \left\{ x \in X ; a \leq \liminf_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \right\}$$

is a Borel set. Let $\{r_i\}_{i \in \mathbb{N}}$ be a decreasing sequence that satisfies $r_i/r_{i+1} \to 1$ as $i \to \infty$, for instance take $r_i = i^{-1}$. Consider a family of continuous functions $f_{r_i}(x) := m(B_{r_i}(x))/r_i^N$, where the continuity of each $f_{r_i}$ for $i \in \mathbb{N}$ is guaranteed by Lemma [4,2]. Hence $\liminf_{i} f_{r_i}$ is a Borel function. Take an arbitrary $x \in X$ and fix it. Define $\{l_j\}_{j \in \mathbb{N}}$ such that $m(B_{l_j}(x))/l_j^N \to \liminf_{r \downarrow 0} m(B_r(x))/r^N$. For each $j \in \mathbb{N}$, there exists $i(j) \in \mathbb{N}$ such that $r_{i(j)+1} \leq l_j < r_{i(j)}$. Then

$$\left( \frac{r_{i(j)+1}}{r_{i(j)}} \right)^N \cdot \frac{m(B_{i(j)+1}(x))}{r_{i(j)+1}^N} \leq \left( \frac{r_{i(j)+1}}{l_j} \right)^N \cdot \frac{m(B_{i(j)+1}(x))}{l_j^N} \leq \frac{m(B_{r_{i(j)}}(x))}{r_{i(j)}^N} \leq \frac{m(B_{l_j}(x))}{l_j^N} \left( \frac{r_{i(j)}}{r_{i(j)+1}} \right)^N \leq \frac{m(B_{r_{i(j)}}(x))}{r_{i(j)}^N} \left( \frac{r_{i(j)}}{l_j} \right)^N$$

holds for any $j \in \mathbb{N}$. By the assumption of $\{r_i\}_{i \in \mathbb{N}}$, we obtain

$$\liminf_{i \to \infty} f_{r_i}(x) \leq \liminf_{j \to \infty} \frac{m(B_{l_j}(x))}{l_j^N} = \liminf_{r \downarrow 0} \frac{m(B_r(x))}{r^N} \leq \liminf_{i \to \infty} f_{r_i}(x).$$

Therefore $A_a$ is Borel measurable. \hfill $\square$

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