Mixtures of Hidden Truncation Hyperbolic Factor Analyzers

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Abstract
The mixture of factor analyzers model was first introduced over 20 years ago and, in the meantime, has been extended to several non-Gaussian analogs. In general, these analogs account for situations with heavy tailed and/or skewed clusters. An approach is introduced that unifies many of these approaches into one very general model: the mixture of hidden truncation hyperbolic factor analyzers (MHTHFA) model. In the process of doing this, a hidden truncation hyperbolic factor analysis model is also introduced. The MHTHFA model is illustrated for clustering as well as semi-supervised classification using two real datasets.

Keywords Hidden truncation hyperbolic distribution · Hidden truncation hyperbolic factor analysis · MHTHFA · Mixture of factor analyzers · Mixture of hidden truncation hyperbolic factor analyzers

1 Introduction

Model-based clustering is an effective tool for identifying homogeneous subpopulations within a heterogeneous population. Although often employed for cluster analysis, mixture modeling approaches are traditionally ill-suited to modeling high-dimensional data sets due to the prohibitively large number of model parameters that must be estimated. Given that modern technology allows us to collect and store vast amounts of data with ease, mixture models must be adapted to handle high-dimensional data. The mixture of factor analyzers (MFA) model (Ghahramani and Hinton 1997; McLachlan and Peel 2000b) reduces the number of parameters to be estimated by introducing latent factors. A number of other models have been developed based on the MFA model and place additional restrictions

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on the component covariance parameters. These include the mixture of probabilistic principal component analyzers model (Tipping and Bishop 1999) and the family of parsimonious Gaussian mixture models (PGMM) of McNicholas and Murphy (2008, 2010). Note that all of the aforementioned models are developed based on Gaussian mixtures. These approaches, as well as some others, are covered within the excellent review of work on model-based clustering of high-dimensional data given by Bouveyron and Brunet-Saumard (2014).

Some work has been carried out extending the MFA to $t$-mixtures (e.g., Peel and McLachlan 2000; Andrews and McNicholas 2011a, b; Steane et al. 2012; Lin et al. 2014) and, in recent years, there has been a surge in interest in using non-Gaussian distributions to develop mixture models capable of detecting asymmetric clusters. This includes the mixture of shifted asymmetric Laplace (SAL) distributions (Franczak et al. 2014), mixture of skew-normal distributions (Lin 2009), mixture of skew-$t$ distributions (Vrbik and McNicholas 2012, 2014; Lee and McLachlan 2014), the mixture of normal inverse Gaussian (NIG) distributions (Karlis and Santourian 2009; Subedi and McNicholas 2014), mixture of variance-gamma distributions (McNicholas et al. 2017), mixture of generalized hyperbolic distributions (Browne and McNicholas 2015), mixture of joint generalized hyperbolic distributions (Tang et al. 2018), and a mixture of coalesced generalized hyperbolic distributions (Tortora et al. 2019) among others. A recent review of work in model-based clustering, including some coverage of non-Gaussian mixture models that have appeared in the literature to date, is given by McNicholas (2016b).

When one considers the various formulations of skewed distributions that have appeared in the model-based clustering literature, it is notable that there are different ways of imposing skewness within the component densities. For example, skewness can be introduced based on a mean-variance normal mixture or via hidden truncation. Franczak et al. (2014) use a scale mixture of normals to develop a mixture of SAL distributions. Murray et al. (2014a, b) use a mean-variance mixture of normal distributions to develop a mixture of skew-$t$ distributions, Browne and McNicholas (2015) follow a similar approach for a mixture of generalized hyperbolic distributions, as do Karlis and Santourian (2009) for a mixture of NIG distributions and McNicholas et al. (2017) for a mixture of variance-gamma distributions. Lin (2009) and Lin (2010) develop a multivariate skew-normal mixture model based on the truncated-normal distribution and a multivariate skew-$t$ mixture model based on the truncated $t$-distribution, respectively. Lee and McLachlan (2016) use a mixture of canonical fundamental skew-$t$ (CFUST) distributions (see Arellano-Valle and Genton 2005, for details on the CFUST distribution). Murray et al. (2017a) introduce the hidden truncation hyperbolic (HTH) distribution. The HTH approach is based on a truncated hyperbolic random variable, and Murray et al. (2017a) demonstrate the effectiveness of a mixture of HTH distributions for clustering. The HTH distribution contains certain formulations of skew-$t$ and skew-normal distributions as limiting cases. In particular, the HTH distribution includes the CFUST distribution as a limiting case and, as a consequence, the canonical fundamental skew-normal distribution as a limiting case.

Some work has been done on exploiting the aforementioned distributions to extend the MFA model. In fact, the work of Murray et al. (2014a, b) and McNicholas et al. (2017) is in this direction and Tortora et al. (2016) develop a mixture of generalized hyperbolic factor analyzers (MGHFA) model along analogous lines. Lin et al. (2016) develop a mixture of factor analyzers using the multivariate skew-normal distribution used in Lin (2009) and Murray et al. (2017b) develop a mixture of factor models using the formulation of the skew-$t$ distribution employed in Lin (2010). The logical conclusion to this vein of research is a mixture of HTH factor analyzers (MHTHFA) model, which is developed herein.
2 Background

2.1 Mixture Model-Based Classification

Suppose we observe \( p \)-dimensional \( x_1, \ldots, x_n \) from a \( G \)-component finite mixture model. First, suppose all \( n \) are unlabelled, i.e., a clustering scenario. Then, the model-based clustering likelihood can be written

\[
L(\vartheta) = \prod_{i=1}^{n} \sum_{g=1}^{G} \pi_g f(x_i | \theta_g),
\]

where \( \pi_g > 0 \), such that \( \sum_{g=1}^{G} \pi_g = 1 \) is the \( g \)th mixing proportion, \( f(x_i | \theta_g) \) is the \( g \)th component density, and \( \vartheta = (\pi_1, \ldots, \pi_G, \theta_1, \ldots, \theta_G) \). Now, suppose \( k \) of the \( n \) are unlabelled and we want to use all \( n \) to find labels for the \( k \) unlabelled observations, i.e., semi-supervised classification. Following McNicholas (2010), suppose it is the first \( k \) observations that are unlabelled. Then, the likelihood can be written

\[
L(\vartheta) = \prod_{i=1}^{k} \prod_{g=1}^{G} \pi_g f(x_i | \theta_g) z_{ig} \prod_{j=k+1}^{n} \sum_{h=1}^{H} \pi_g f(x_j | \theta_h),
\]

for \( H \geq G \) (\( H = G \) is commonly assumed), where \( z_{ig} = 1 \) if \( x_i \) belongs to component \( g \) and \( z_{ig} = 0 \) otherwise. This semi-supervised classification paradigm is also referred to as partial classification (see McLachlan 1992). Further details on model-based clustering and classification are given in the monographs by McLachlan and Peel (2000a) and McNicholas (2016a).

2.2 Hidden Truncation Hyperbolic Distribution

We generate a \( p \)-dimensional random variable \( X \) following the HTH distribution through the stochastic representation

\[
X = \mu + \sqrt{W} Y,
\]

where \( Y \sim \text{SN}_p(0, \Sigma, \Lambda) \), \( W \sim \text{GIG}(\omega, \omega, \lambda) \), and \( W \) and \( Y \) are independent. Here \( Y \sim \text{SN}_p(0, \Sigma, \Lambda) \) denotes that \( Y \) follows the skew-normal distribution of Sahu et al. (2003) with density

\[
f_{\text{SN}}(y | \mu, \Sigma, \Lambda) = 2^q \phi_p(y | \mu, \Omega) \Phi_r(\Lambda' \Omega^{-1}(y - \mu) | \Delta),
\]

with location vector \( \mu \), scale matrix \( \Sigma, p \times r \) skewness matrix \( \Lambda, \Omega = \Sigma + \Lambda\Lambda' \), \( \Delta = I_q - \Lambda' \Omega^{-1} \Lambda \) and where \( \phi_p(\cdot | \mu, \Sigma) \) and \( \Phi_r(\cdot | \mu, \Sigma) \) are the density and cumulative distribution function, respectively, of the multivariate normal distribution. Furthermore, \( W \sim \text{GIG}(\psi, \chi, \lambda) \) denotes that \( W \) follows the generalized inverse Gaussian distribution with density

\[
p(w | \psi, \chi, \lambda) = \frac{(\psi/\chi)^{\lambda/2} w^{\lambda-1}}{2 K_\lambda(\sqrt{\psi/\chi})} \exp\left\{ -\frac{\psi w + \chi/w}{2} \right\},
\]

for \( w > 0 \) with \( (\psi, \chi) \in \mathbb{R}^2_{++}, \lambda \in \mathbb{R}, \) and \( K_\lambda(\cdot) \) is the modified Bessel function of the third kind with index \( \lambda \). The HTH density is

\[
f_{\text{HTH}}(x | \theta) = 2^q h_p(x | \mu, \Omega, \lambda, \omega, \omega) \\
\times H_q \left( \Lambda' \Omega^{-1}(x - \mu) \left( \frac{\omega}{\omega + \delta(x | \mu, \Omega)} \right)^{1/4} \right) \left| 0, \Delta, \lambda - (p/2), \gamma, \gamma \right).
\]
where $\theta = (\mu, \Sigma, \Lambda, \lambda, \omega), \gamma = \omega \sqrt{1 + \delta(x, \mu \mid \Omega)/\omega}$,

$$h_p(\cdot \mid \mu, \Sigma, \lambda, \omega, \omega) = \left[\frac{1}{\pi^{p/2} |\Sigma|^{1/2}} \right]^{(\lambda - p)/2} \sqrt{\psi(\chi^2 + \delta(x, \mu \mid \Sigma))},$$

$\delta(x, \mu \mid \Sigma) = (x - \mu) \Sigma^{-1} (x - \mu)$ is the squared Mahalanobis distance between $x$ and $\mu$. $h_p(\cdot \mid \mu, \Sigma, \lambda, \omega, \omega)$ is the density of a $p$-dimensional symmetric hyperbolic random variable and $H_q(\cdot \mid \mu, \Sigma, \lambda, \omega, \omega)$ is the $q$-dimensional CDF of a symmetric hyperbolic random variable. Note that $\Lambda$ is a $p \times r$ skewness matrix, where $1 \leq r \leq p$. Refer to Murray et al. (2017a, 2019) for extensive details on the HTH distribution.

### 3 HTH Factor Analysis Model

Consider $n$ independent $p$-dimensional random variables $X_1, \ldots, X_n$. The HTH factor analysis model is written

$$X_i = \mu + BU_i + \epsilon_i,$$

for $i = 1, \ldots, n$, where $\mu$ is a $p$-dimensional location parameter, $B$ is a $p \times q$ matrix of factor loadings, $U_i$ is a $q$-dimensional vector of latent factors, and $\epsilon_i$ is a $p$-dimensional error vector. Note that the $X_i$ are independently distributed, as are the $U_i$, the $X_i$ and $U_i$ are independent, and $q < p$. We also have that

$$\begin{bmatrix} U_i \\ \epsilon_i \end{bmatrix} \sim \text{HTH}_{q+p} \left( \begin{bmatrix} -A^{-1/2} \Lambda a_\lambda \\ 0_p \end{bmatrix}, \begin{bmatrix} A^{-1} 0_{q \times p} \\ 0_p \times D \end{bmatrix}, \begin{bmatrix} A^{-1/2} \Lambda \\ 0_p \end{bmatrix}, \omega, \omega, \lambda \right),$$

where $\Lambda$ is a $q \times r$ skewness matrix,

$$\Lambda = I_q + \left[1 - a_\lambda^t a_\lambda / K_{\lambda+1}(\omega)\right] \Lambda \Lambda', a_\lambda = \sqrt{2 / \pi} K_{\lambda+1/2}(\omega) - 1_r,$$

$1_r$ denotes an $r$-dimensional vector of $1$s, and $D$ is a diagonal matrix with positive diagonal elements. Note this model requires that $1 \leq r \leq q$ and the value of $q$ must satisfy

$$(p - q)^2 > p + q. \quad (1)$$

See Lawley and Maxwell (1962) for details about (1). It follows that $X_i \mid w_i \sim \text{SN}_p(\mu - \alpha a_\lambda, \Sigma, \alpha, \lambda, \omega, \omega)$ and $X_i \sim \text{HTH}_p(\mu - \alpha a_\lambda, \Sigma, \alpha, \lambda, \omega, \omega)$, where $W \sim \text{GIG}(\omega, \omega, \lambda), \Sigma = BA^{-1}B' + D$ and $\alpha = BA^{-1} A$.

The density of the HTH factor analysis model is

$$f_{\text{HTHFA}}(x \mid \theta) = 2^f h_p(x \mid \mu - \alpha a_\lambda, \Omega, \lambda, \omega, \omega) \times$$

$$H_r \left( \alpha' \Omega^{-1}(x - \mu + \alpha a_\lambda) \left[ \frac{\omega}{\omega^2 + \delta(x \mid \mu - \alpha a_\lambda, \Omega)} \right]^{1/4} \right)^{\omega} \mid 0, \Delta, \lambda - \frac{p}{2}, \gamma, \gamma \right),$$

where $\theta = (\mu, B, D, \Lambda, \omega, \lambda), \Omega = \Sigma + \alpha \alpha', \Delta = I_r - \alpha' \Omega^{-1} \alpha$. For notational convenience, hereafter let $\tilde{B} = BA^{-1/2}, \tilde{U} = A^{1/2}U$, and $r = \mu - \alpha a_\lambda$. The following hierarchical representation exists for the HTHFA model:

$$X \mid (\tilde{u}, v, w) \sim \mathcal{N}_p(\mu + \tilde{B}U, wD),$$

$$\tilde{U} \mid v, w \sim \mathcal{N}_q \left( A(v - a_\lambda), wI_q \right),$$

$$V \mid w \sim \text{TN}_p(0, wI_p),$$

$$W \sim \text{GIG}(\omega, \omega, \lambda).$$
4 Mixtures of HTH Factor Analyzers

4.1 The Model

We develop a MHTHFA to model high-dimensional heterogenous data. Consider \( n \) independent \( p \)-dimensional random variables \( X_1, \ldots, X_n \). The MHTHFA model is given by

\[
X_i = \mu_g + B_g U_{ig} + \epsilon_{ig}
\]

with probability \( \pi_g \), for \( i = 1, \ldots, n \) and \( g = 1, \ldots, G \), where \( \pi_g > 0 \), \( \sum_{g=1}^{G} \pi_g = 1 \), \( \mu_g \) is a \( p \)-dimensional location parameter, \( B_g \) is a \( p \times q \) matrix of factor loadings, \( U_{ig} \) is a \( q \)-dimensional vector of latent factors, and \( \epsilon_{ig} \) is a \( p \)-dimensional error vector. Analogous independence relations hold for and between the \( X_i \) and \( U_i \) as for the HTH factor analysis model (Section 3). We also have that

\[
\left[\begin{array}{c}
U_{ig} \\
\epsilon_{ig}
\end{array}\right] \sim \text{HTH}_{q+p} \left( \begin{array}{c}
-A_g^{-1/2} \Lambda_g \alpha_g \\
\mathbf{0}_p
\end{array}\right), \left[\begin{array}{c}
A_g^{-1} \mathbf{0}_{p \times q} \\
D_g
\end{array}\right], \left[\begin{array}{c}
A_g^{-1/2} \mathbf{0}_p \\
\omega_g, \omega_g, \lambda_g
\end{array}\right),
\]

where \( A_g \) is a \( q \times r \) skewness matrix,

\[
A_g = \mathbf{I}_q + \left[1 - a_{\lambda_g} a_{\lambda_g} \frac{G_{\lambda_g+1}(\omega_g)}{G_{\lambda_g}(\omega_g)} \right] \Lambda_g \Lambda_g', \quad a_{\lambda_g} = \sqrt{\frac{2}{\pi} \frac{G_{\lambda_g+1/2}(\omega_g)}{G_{\lambda_g}(\omega_g)}} \mathbf{1}_r,
\]

\( \mathbf{1}_r \) denotes an \( r \)-dimensional vector of 1s, and \( D_g \) is a diagonal matrix with positive diagonal elements. Again, we require that \( 1 \leq r \leq q \) and the value of \( q \) must satisfy (1). The density of the MHTHFA model is

\[
g(x | \vartheta) = \sum_{g=1}^{G} \pi_g f_{\text{HTHFA}}(x | \theta_g), \quad (3)
\]

where \( \vartheta = (\pi_1, \ldots, \pi_G, \theta_1, \ldots, \theta_G) \), \( \pi_g \) is the \( g \)th mixing proportion, \( \theta_g = (\mu_g, \Sigma_g, A_g, \lambda_g, \omega_g, \omega_g) \) and \( f_{\text{HTHFA}}(x | \theta_g) \) is as defined in Eq. 2.

4.2 ECM Algorithm for MHTHFA

We employ an expectation conditional-maximization (ECM) algorithm (Meng and Rubin 1993) for model fitting and parameter estimation. The complete-data log-likelihood for the HTHFA model is

\[
l_c = C + \sum_{g=1}^{G} \left[ (\lambda_g - 1) \sum_{i=1}^{n} \log w_{ig} - n \log K_{\lambda_g}(\omega_g) - \frac{\omega_g}{2} \sum_{i=1}^{n} \left( w_{ig} + \frac{1}{w_{ig}} \right) \right. \\
- \frac{1}{2} \sum_{i=1}^{n} \frac{1}{w_{ig}} \left[ \bar{u}_{ig} \bar{u}_{ig} - \bar{u}_{ig} A_g (v_{ig} - a_{\lambda_g}) - (v_{ig} - a_{\lambda_g})' A_g' \bar{u}_{ig} \\
+ (v_{ig} - a_{\lambda_g})' A_g (v_{ig} - a_{\lambda_g}) \right] - \frac{n}{2} \log |D_g| - \frac{1}{2} \text{tr} \left[ D_g^{-1} \sum_{i=1}^{N} \tilde{\Psi}_{ig} \right]
\]

where

\[
\tilde{\Psi}_{ig} = \frac{1}{w_{ig}} (x_i - \mu_g - \tilde{B}_g \tilde{u}_{ig} - \tilde{B}_g \bar{u}_{ig}),
\]

\( C \) is a constant with respect to the model parameters, and \( z_{ig} \) is as defined before. The algorithm alternates between a expectation (E) step and a conditional-maximization (CM) step. On the E-step, we compute the expected value of the complete-data log-likelihood.
conditional on the current parameter estimates. For the MHHTHFA model, this involves the following conditional expectations:

\[
\begin{align*}
\mathbb{E} [W_{ig} \mid x_i, z_{ig} = 1] &= \hat{a}_{ig}, \\
\mathbb{E} [1/W_{ig} \mid x_i, z_{ig} = 1] &= \hat{b}_{ig} \\
\mathbb{E} [\log W_{ig} \mid x_i, z_{ig} = 1] &= \hat{c}_{ig}, \\
\mathbb{E} [(1/W_{ig}) \hat{U}_{ig} \mid x_i, z_{ig} = 1] &= \hat{s}_{i1g}, \\
\mathbb{E} [(1/W_{ig}) \hat{V}_{ig} \mid x_i, z_{ig} = 1] &= \hat{s}_{i2g}, \\
\mathbb{E} [(1/W_{ig}) \hat{V}_{ig} \mid x_i, z_{ig} = 1] &= \hat{s}_{i3g}, \\
\mathbb{E} [(1/W_{ig}) \hat{V}_{ig} \mid x_i, z_{ig} = 1] &= \hat{s}_{i4g}, \\
\mathbb{E} [(1/W_{ig}) \hat{V}_{ig} \mid x_i, z_{ig} = 1] &= \hat{s}_{i5g}.
\end{align*}
\]

Details on these expectations are given in the Appendix.

At each CM-step, the following model parameters are updated sequentially and conditionally on the other parameters. The mixing proportions and location parameters are updated via

\[ \hat{f}_g = \frac{n_g}{n} \quad \text{and} \quad \hat{\mu}_g = \frac{\sum_{i=1}^n \hat{z}_{ig}(\hat{b}_{ig} x_i - \hat{B}_g \hat{s}_{i1g})}{\sum_{i=1}^n \hat{z}_{ig}} \]  

respectively, where \( n_g = \sum_{i=1}^n \hat{z}_{ig} \). We update \( \hat{B}_g \) by

\[ \hat{B}_g = \left[ \sum_{i=1}^n \hat{z}_{ig} (x_i - \hat{\mu}_g)' \hat{s}_{i1g} \right] \left[ \sum_{i=1}^n \hat{z}_{ig} \hat{s}_{i2g} \right]^{-1} \]

and the update for \( D_g \) is

\[ \hat{D}_g = \frac{1}{n_g} \text{diag} \left\{ \sum_{i=1}^n \hat{z}_{ig} \hat{\Psi}_{ig} \right\} . \]

The update for the \( A_g \) is

\[ \hat{A}_g = \sum_{i=1}^n \left( \hat{s}_{i5g} - \hat{s}_{i1g} \hat{a}_{i5g} \right) \left( \hat{s}_{i4g} - \hat{a}_{i5g} \hat{s}_{i3ig} - \hat{s}_{i3ig} \hat{a}_{i5g}' - \hat{b}_{ig} \hat{a}_{i5g}' \hat{a}_{i5g} \right)^{-1}, \]

and the update for \( \omega_g \) is given by

\[ \hat{\omega}_g = \omega_g + \frac{\partial}{\partial \omega} t_g(\omega, \hat{\omega}_g) |_{\omega = \hat{\omega}_g} + \frac{\partial^2}{\partial \omega^2} t_g(\omega, \hat{\omega}_g) |_{\omega = \hat{\omega}_g}, \]

where

\[ t_g(\omega_g, \hat{\omega}_g) = -\log K_{\lambda_g} (\omega_g) + (\lambda_g - 1) \bar{c}_g - \frac{\omega_g}{2} \left( \bar{a}_g + \bar{b}_g \right) , \]

\[ \bar{a}_g = \sum_{i=1}^n \hat{z}_{ig} \hat{a}_{ig} / n_g, \quad \bar{b}_g = \sum_{i=1}^n \hat{z}_{ig} \hat{b}_{ig} / n_g, \quad \text{and} \quad \bar{c}_g = \sum_{i=1}^n \hat{z}_{ig} \hat{c}_{ig} / n_g. \]

We update \( \lambda_g \) by \( \lambda_g = \hat{c}_g \hat{\lambda}_g / m_g \), where

\[ m_g = \frac{\partial}{\partial \lambda} \log K_{\lambda} (\hat{\omega}_g) |_{\lambda = \hat{\lambda}_g} + \frac{1}{2} \frac{\partial}{\partial \lambda} \frac{K_{\lambda + 1/2}(\hat{\omega}_g)}{K_{\lambda}(\hat{\omega}_g)} |_{\lambda = \hat{\lambda}_g} \left[ \sum_{i=1}^n \text{tr} \left\{ \hat{s}_{i1ig} \hat{1}_r \right\} \right]^{1/2} \]

\[ - \frac{1}{2} \frac{\partial}{\partial \hat{\lambda}_g} \left( \frac{K_{\lambda + 1/2}(\hat{\omega}_g)}{K_{\lambda}(\hat{\omega}_g)} \right)^2 |_{\lambda = \hat{\lambda}_g} \sum_{i=1}^n \text{tr} \left\{ \hat{b}_{ig} \hat{1}_r \hat{s}_{i3ig} \right\} \left( \hat{\lambda}_g \hat{A}_g \right)^\top \hat{\lambda}_g \hat{A}_g. \]
4.3 Model Selection

The values of $q$ and $r$ need to be selected as well as, perhaps, the number of components $G$. The Bayesian information criterion (BIC; Schwarz 1978) is the most popular model selection criterion for model-based clustering and semi-supervised classification. The BIC can be written

$$\text{BIC} = 2l_{\text{obs}}(\hat{\theta}) - \rho \log n,$$

where $l_{\text{obs}}(\hat{\theta})$ is the maximized observed likelihood and $\rho$ is the number of free parameters in the model. Note that, for our MHTHFA model,

$$\rho = G - 1 + G\left[p + qr + 2 + pq + p - \frac{q}{2}(q - 1)\right].$$

4.4 Initialization and Convergence

In our ECM algorithms, the group memberships $\hat{z}_{ig}$ are initialized using $k$-means clustering. For each ECM algorithm, five sets of starting values are obtained by performing $k$-means clustering and our ECM algorithm is initialized using the set of values that corresponds to the largest log-likelihood value for the MHTHFA model in question.

The parameters $\hat{\mu}_g$ and $\hat{\Sigma}_g$ are initialized using a weighted mean and covariance matrix, respectively. The matrix $\hat{\Lambda}_g$ is initialized using values randomly generated from a normal distribution with mean 0 and standard deviation 1, and $\hat{\omega}_g$ and $\hat{\lambda}_g$ are each initialized as 1. We initialize the matrix of factor loadings $\hat{B}_g$ following the approach outlined by McNicholas and Murphy (2008).

We determine convergence of the ECM algorithm using a criterion based on the Aitken acceleration (Aitken 1926). The Aitken acceleration at iteration $k$ is

$$a^{(k)} = \frac{l^{(k+1)} - l^{(k)}}{l^{(k)} - l^{(k-1)}},$$

where $l^{(k)}$ is the log-likelihood at iteration $k$. An asymptotic estimate of the log-likelihood at iteration $k$ is

$$l^{(k)}_{\infty} = l^{(k-1)} + \frac{1}{1 - a^{(k-1)}}(l^{(k)} - l^{(k-1)}).$$

Following Lindsay (1995), we stop the algorithm when $l^{(k)}_{\infty} - l^{(k)} < \epsilon$, with $\epsilon = 0.01$, for the analyses herein (Section 5).

5 Illustrations

5.1 Overview

The MHTHFA model is illustrated for clustering (Section 5.2) and semi-supervised classification (Section 5.3) using two well-known datasets. Because the true classes of the points that are treated as unlabelled are actually known, performance can be assessed using the adjusted Rand index (ARI; Hubert and Arabie 1985). The ARI takes a value 1 for perfect class agreement and has expected value 0 under random classification. Negative values of
the ARI are possible and reflect classification performance that is, in some sense, worse than guessing. Extensive details on the ARI are given by Steinley (2004).

5.2 Australian Institute of Sport Data

We consider the Australian Institute of Sport (AIS) data which contains 11 continuous variables for 100 female and 102 male athletes. These data are available in the UCI Machine Learning Repository (Lichman 2013). The MHTHFA model is fitted for $G = 2$, $q = 1, \ldots, 6$, $r = 1, 2, 3$, and $r \leq q$. For comparison, we also fit the MGHFA and PGMM models for $G = 2$ and $q = 1, \ldots, 6$. The BIC is used to select the best model in each case and the results are summarized in Table 1. Although all models obtain very good clustering results, the selected MHTHFA model outperforms the MGHFA and PGMM approaches in terms of the ARI.

5.3 Sonar Data

Gorman and Sejnowski (1988) report data on patterns obtained by bouncing sonar signals off a metal cylinder and rocks, respectively, at various angles and under various conditions. These data are available from the UCI machine learning repository. In all, 111 signals are obtained by bouncing sonar signals off a metal cylinder and 97 are obtained by bouncing sonar signals off rocks. To illustrate the MHTHFA approach for semi-supervised classification, 104 of the 208 patterns are randomly selected to be treated as unlabelled. This results in 53 of the metal signals and 51 of the rock signals being treated as unlabelled. The MHTHFA, MGHFA, and PGMM approaches are fitted to these data, for semi-supervised classification, for $G = 2$, $q = 1, \ldots, 8$ and, for MHTHFA, $r = 1, 2$. The classification results, based on the best model selected by the BIC in each case, are given in Table 2. The MHTHFA model substantially outperforms both the MGHFA and PGMM approaches (Tables 2 and 3).

6 Discussion

The MFA model has been extended using the HTH distribution that was developed by Murray et al. (2017a). The HTH distribution contains many of the non-Gaussian distributions used in the model-based clustering literature as special and limiting cases. The resulting

| Model  | $q$ | $r$ | BIC    | ARI |
|--------|-----|-----|--------|-----|
| MHTHFA | 4   | 2   | -2369.3| 0.92|
| MGHFA  | 4   |     | -2266.6| 0.90|
| PGMM   | 4   |     | -2394.8| 0.81|

Table 1 Results from the best MHTHFA, MGHFA, and PGMM models fit to the AIS data for $G = 2$, as selected by the BIC

| Model    | A | B |
|----------|---|---|
| MHTHFA   | 48| 5 |
| MGHFA    | 49| 4 |
| PGMM     | 42| 11|

| Class    | A | B |
|----------|---|---|
| Metal    | 7 | 44|
| Rock     | 31| 20|

Table 2 Cross-tabulations of true (metal, rock) against predicted (A, B) classes for the selected MHTHFA, MGHFA, and PGMM models on the sonar data
MHHTFA model retains the flexibility of the HTH mixture model with the added advantage of being able to model high-dimensional data. This work can be viewed as completing a line of research on extensions to the MFA model using non-Gaussian distributions based on hidden truncation. Illustrations on well-known datasets demonstrate that the MHHTFA model is effective for clustering and semi-supervised classification; in fact, it outperforms both the MGHFA and PGMM approaches. Given the flexibility of this model and the inherent ability to capture various distributions as special cases, this model is particularly useful for high-dimensional clustering applications where the underlying distribution is unknown.

Future work will focus on decreasing the computation time required to fit the MHHTFA model. The R programming language (R Core Team 2018) has been used for all model implementation to date but developing parallel code using Python or Julia will reduce the overall computation time and increase the practical applicability of this model. Other work could focus on developing a parsimonious family of MHHTFA models analogous to the PGMM models; however, more efficient implementation is essentially a pre-requisite for such developments. An analogous approach to that presented herein could be taken to extending the mixture of common factor analyzers model (Yoshida et al. 2004; Baek et al. 2010) to the mixture of HTH distributions; the resulting approach might be effective for clustering higher dimensional data. The same may be true of an MHHTFA analog of the LASSO-penalized approach of Bhattacharya and McNicholas (2014). Consideration of how the MHTHFA model works within the fractionally supervised paradigm may also be of interest (see Vrbik and McNicholas 2015, Gallaugher and McNicholas 2019a). Finally, a matrix variate analog of the HTH distribution and MHTHFA model will be considered and may follow somewhat similar lines to the non-Gaussian matrix variate mixture work of Gallaugher and McNicholas (2017, 2018, 2019b).

### Appendix: E-Step Calculations

Herein, we present the expectations required for the E-step of the ECM algorithm for the mixtures of HTH factor analyzers model.

#### A.1 $\mathbb{E}[W_{ig} | x_i, z_{ig} = 1]$ and $\mathbb{E}[1/W_{ig} | x_i, z_{ig} = 1]$ 

To derive the expectations $\mathbb{E}[W_{ig} | x_i, z_{ig} = 1]$ and $\mathbb{E}[1/W_{ig} | x_i, z_{ig} = 1]$ as well as $\mathbb{E}[\log W_{ig} | x_i, z_{ig} = 1]$ in the following section, first note that

$$f(w_{ig} | x_i, z_{ig} = 1) = \frac{w_{ig}^{\lambda_{ig} - p/2 - 1}}{2\lambda_{ig} - p/2} \left( \omega_g + \delta(x_i | \mu_g, \Sigma_g) \right)^{\omega_g + \delta(x_i | \mu_g, \Sigma_g)} \Phi \left( a' g \Omega_g^{-1} (x_i - \mu_g) / \sqrt{w_{ig}} | \Delta_g \right)^{(\lambda_{ig} - p)/2}$$

$$\times \exp \left\{ \omega_g w_{ig} + \frac{\omega_g + \delta(x_i | \mu_g, \Sigma_g)}{w_{ig}} \right\} \mathbb{E} \left[ a' g \Omega_g^{-1} (x_i - \mu_g) / \sqrt{w_{ig}} | \Delta_g \right]$$

$$\div H_r \left( a' g \Omega_g^{-1} (x_i - \mu_g) / \sqrt{w_{ig}} | \Delta_g , \lambda_{ig} - (p/2), \gamma_g, \gamma_g \right).$$

---

**Table 3** Results from the best MHTHFA, MGHFA, and PGMM models fit to the sonar data for $G = 2$, as selected by the BIC

| Model      | $q$ | $r$ | BIC    | ARI |
|------------|-----|-----|--------|-----|
| MHTHFA     | 7   | 1   | -27258.8 | 0.59|
| MGHFA      | 6   |     | -27627.6 | 0.05|
| PGMM       | 7   |     | -29279.9 | 0.10|
Therefore,
\[
\begin{align*}
\mathbb{E} \left[ W_{ig} \mid x_i, z_{ig} = 1 \right] &= \int_0^\infty f(x_i, z_{ig}) \frac{\omega_g^{\lambda_g - p/2} + \omega_g}{\omega_g + \delta(x_i | r_g, \Omega_g)} \left[ \frac{\omega_g}{\omega_g + \delta(x_i | r_g, \Omega_g)} \right]^{(\lambda_g - p/2)/2} \times \exp \left\{ \omega_g w + \frac{\omega_g + \delta(x_i | r_g, \Omega_g)}{\omega_g} \right\} \Phi \left( \alpha_g \Omega_g^{-1} (x_i - r_g) / \sqrt{\omega_g} \right) \, dw \\
&= K_{\lambda_g - p/2 - 1} \left( \frac{\omega_g + \delta(x_i | r_g, \Omega_g)}{\omega_g} \right)^{-1/2} K_{\lambda_g - p/2} \left( \frac{\omega_g + \delta(x_i | r_g, \Omega_g)}{\omega_g} \right) \frac{\omega_g + \delta(x_i | r_g, \Omega_g)}{\omega_g} \frac{\omega_g}{\omega_g + \delta(x_i | r_g, \Omega_g)} \left[ \frac{\omega_g}{\omega_g + \delta(x_i | r_g, \Omega_g)} \right]^{(\lambda_g - p/2)/2} \times \exp \left\{ \omega_g w + \frac{\omega_g + \delta(x_i | r_g, \Omega_g)}{\omega_g} \right\} \Phi \left( \alpha_g \Omega_g^{-1} (x_i - r_g) / \sqrt{\omega_g} \right) \, dw \\
&= \frac{\lambda_g - p/2 - 1}{\lambda_g - p/2} \mathbb{E} \left[ W_{ig} \mid x_i, z_{ig} = 1 \right] = \frac{\lambda_g - p/2 - 1}{\lambda_g - p/2} \mathbb{E} \left[ W_{ig} \mid x_i, z_{ig} = 1 \right].
\end{align*}
\]

A.2 $\mathbb{E} [\log W_{ig} \mid x_i, z_{ig} = 1]$ 

To update $\mathbb{E} [\log W_{ig} \mid x_i, z_{ig} = 1]$, where $W_{ig} \sim \text{GIG}(\psi_g, \chi_g, \lambda_g)$, first note that
\[
\mathbb{E} \left[ \log W_{ig} \mid z_{ig} = 1 \right] = \frac{d}{d\lambda} \log K_{\lambda} \left( \sqrt{\chi_g} \psi_g \right) + \log \left( \sqrt{\chi_g} \psi_g \right).
\]

We can show that
\[
W_{ig} \mid x_i, v_{ig}, z_{ig} = 1 \sim \text{GIG}(\omega_g, \omega_g + (v_{ig} - k_g)'/\Lambda_g^{-1}(v_{ig} - k_g) + \delta(x_i | r_g, \Omega_g), \lambda_g - (p + r)/2),
\]
where $r_g = \mu_g - \alpha_g a_g$ and $k_g = \Lambda_g^{-1}(x_i - \mu_g)$. Therefore,
\[
\mathbb{E} \left[ \log W_{ig} \mid x_i, v_{ig}, z_{ig} = 1 \right] = \frac{d}{d\tau} \log K_{\tau} \left( \sqrt{\chi^*} \psi_g \right) + \log \left( \sqrt{\chi^*} \psi_g \right).
\]

Let
\[
\xi_{ig} = \sqrt{1 + \frac{\delta(x_i | \mu_g, \Sigma_g) + (v_{ig} - k_g)'\Delta_g^{-1}(v_{ig} - k_g)}{\omega_g}},
\]
then $\xi_{ig} \geq 1$ and $W_{ig} \mid x_i, v_{ig}, z_{ig} = 1 \sim \text{GIG}(\omega_g, \omega_g \xi_{ig}^2, \tau)$. Consequently,
\[
\mathbb{E} \left[ \log W_{ig} \mid x_i, v_{ig}, z_{ig} = 1 \right] = \frac{d}{d\tau} \log K_{\tau} \left( \omega_g \xi_{ig} \right) + \log \xi_{ig}.
\]
The reader is directed to the supplementary material in Murray et al. (2017a) for details on a method for estimating this expectation via a series expansion.

A.3 $\mathbb{E}[(1/W_{ig})V_{ig} \mid x_i, z_{ig} = 1]$ and $\mathbb{E}[(1/W_{ig})V_{ig}' \mid x_i, z_{ig} = 1]$

Recall that $V_{ig} \mid w_{ig}, z_{ig} = 1 \sim HN_r(w_{ig} I_r)$. We can show that

$$f(v_{ig} \mid x_i, z_{ig} = 1) = \frac{1}{c_\lambda} h_r\left(v_{ig} \mid k_g, \sqrt{\frac{\omega_g + \delta(x_i \mid r_g, \Omega_g)}{\omega_g}} \Delta_g, \lambda_g - \frac{p}{2}, \gamma_g, \gamma_g\right),$$

(9)

where the support of $V_{ig}$ is $\mathbb{R}^+_r$, i.e., the positive plane of $\mathbb{R}_r$ and

$$c_\lambda = H_r\left(k\left(\frac{\omega}{\omega + \delta(x_i \mid r_g, \Omega_g)\Delta_g, \lambda_g - \frac{p}{2}, \gamma_g, \gamma_g}\right)^{1/4} \mid 0, \Delta_g, \lambda_g - \frac{p}{2}, \gamma_g, \gamma_g\right).$$

It follows that

$$V_{ig} \mid w_{ig}, x_i, z_{ig} = 1 \sim TH_r\left(k_g, \sqrt{\frac{\omega_g + \delta(x_i \mid r_g, \Omega_g)}{\omega_g}} \Delta_g, \lambda_g - \frac{p}{2}, \gamma_g, \gamma_g; \mathbb{R}^+_r\right).$$

Here, $TH_r(\mu, \Sigma, \lambda, \psi, \chi; \mathbb{R}^+_r)$ denotes the $r$-dimensional symmetric truncated hyperbolic distribution with density

$$f_{TH}(v \mid \mu, \Sigma, \lambda, \psi, \chi; \mathbb{R}^+_r) = \frac{h_r(v \mid \mu, \Sigma, \lambda, \psi, \chi)}{\int_0^\infty \ldots \int_0^\infty h_r(v \mid \mu, \Sigma, \lambda, \psi, \chi) dV},$$

and $\mathbb{I}_{\mathbb{R}^+_r}(u) = 1$ when $u \in \mathbb{R}^+_r$ and 0 otherwise. In this way, the symmetric hyperbolic distribution is truncated to exist only within with region $\mathbb{R}^+_r$. To update $\mathbb{E}[(1/W_{ig})V_{ig} \mid x_i, z_{ig} = 1]$ and $\mathbb{E}[(1/W_{ig})V_{ig}V_{ig}' \mid x_i, z_{ig} = 1]$, we can make use of the fact that

$$\mathbb{E}[(1/W_{ig})V_{ig} \mid x_i, z_{ig} = 1] = \mathbb{E}[(1/W_{ig}) \mid x_i, z_{ig} = 1] \mathbb{E}[V_{ig} \mid x_i, z_{ig} = 1]$$

and

$$\mathbb{E}[(1/W_{ig})V_{ig}V_{ig}' \mid x_i, z_{ig} = 1] = \mathbb{E}[(1/W_{ig}) \mid x_i, z_{ig} = 1] \mathbb{E}[Y_{ig}Y_{ig}' \mid x_i, z_{ig} = 1],$$

where

$$Y_{ig} \mid w_{ig}, x_i, z_{ig} = 1 \sim TH_r\left(k_g, \sqrt{\frac{\omega_g + \delta(x_i \mid r_g, \Omega_g)}{\omega_g}} \Delta_g, \lambda_g - \frac{p}{2} - 1, \gamma_g, \gamma_g; \mathbb{R}^+_r\right).$$

The expectations $\mathbb{E}[Y_{ig} \mid x_i, z_{ig} = 1]$ and $\mathbb{E}[Y_{ig}Y_{ig}' \mid x_i, z_{ig} = 1]$ can easily be estimated using the moments of the truncated symmetric hyperbolic distribution defined in Murray et al. (2017a).

A.4 $\mathbb{E}[(1/W_{ig})\tilde{U}_{ig} \mid x_i, z_{ig} = 1]$ and $\mathbb{E}[(1/W_{ig})\tilde{U}_{ig}\tilde{U}_{ig}' \mid x_i, z_{ig} = 1]$

Note that $\tilde{U}_{ig} \mid x_i, v_{ig}, w_{ig}, z_{ig} = 1 \sim \mathcal{N}_q(q, w_{ig} C)$ where $q = C[d + \Lambda_g(V_{ig} - a_{z_g})]$, $d = \tilde{B}_g^{-1}(X_i - \mu_g)$, and $C = (I_g + \tilde{B}_g^{-1}D_g^{-1})^{-1}$. We can show

$$\mathbb{E}[\tilde{U}_{ig} \mid x_i, z_{ig} = 1] = \mathbb{E}[\mathbb{E}[\tilde{U}_{ig} \mid x_i, v_{ig}, w_{ig}, z_{ig} = 1] \mid x_i, z_{ig} = 1] = \mathbb{E}[C[d + \Lambda_g(\mathbb{E}[V_{ig} \mid x_i, z_{ig} = 1] - a_{z_g})]].$$

$\square$ Springer
Therefore, it follows that

\[
\mathbb{E} \left[ \frac{1}{W_{ig}} \tilde{U}_{ig} \mid x_i, z_{ig} = 1 \right] = \mathbb{E} \left[ \frac{1}{W_{ig}} \tilde{U}_{ig} \mid x_i, \nu_{ig}, w_{ig}, z_{ig} = 1 \mid x_i, z_{ig} = 1 \right] = E \left[ \frac{1}{W_{ig}} \left( \mathbf{Cd} + \mathbf{C} \Lambda_g \left( \mathbf{V}_{ig} - \mathbf{a}_{x_i} \right) \right) \mid x_i, z_{ig} = 1 \right] = \mathbf{C} \left[ \mathbb{E} \left[ \frac{1}{W_{ig}} \mid x_i, z_{ig} = 1 \right] + \Lambda_g \mathbb{E} \left[ \frac{1}{W_{ig}} \mathbf{V}_{ig} \mid x_i, z_{ig} = 1 \right] - \mathbf{a}_{x_i} \mathbb{E} \left[ \frac{1}{W_{ig}} \mid x_i, z_{ig} = 1 \right] \right]
\]

and

\[
\mathbb{E} \left[ \frac{1}{W_{ig}} \tilde{U}_{ig} \tilde{U}_{ig}^\prime \mid x_i, z_{ig} = 1 \right] = \mathbb{E} \left[ \frac{1}{W_{ig}} \mathbb{E} \left[ \tilde{U}_{ig} \tilde{U}_{ig}^\prime \mid x_i, \nu_{ig}, w_{ig}, z_{ig} = 1 \mid x_i, z_{ig} = 1 \right] \right] = \mathbb{E} \left[ \frac{1}{W_{ig}} \left( \mathbf{Cd} + \mathbf{C} \Lambda_g \left( \mathbf{V}_{ig} - \mathbf{a}_{x_i} \right) \right)^2 + \mathbf{C} \mid x_i, z_{ig} = 1 \right] = \left( \mathbb{E} \left[ \frac{1}{W_{ig}} \mathbf{V}_{ig} \tilde{U}_{ig} \mid x_i, z_{ig} = 1 \right] - \mathbf{a}_{x_i} \mathbb{E} \left[ \frac{1}{W_{ig}} \tilde{U}_{ig} \mid x_i, z_{ig} = 1 \right] \right) \Lambda_g^\prime + \mathbb{E} \left[ \frac{1}{W_{ig}} \tilde{U}_{ig} \tilde{U}_{ig}^\prime \mid x_i, z_{ig} = 1 \right] \mathbf{d}^\prime + \mathbf{I}_q \mathbf{C}.
\]

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