GLOBAL PRYM-TORELLI THEOREM FOR DOUBLE COVERINGS OF ELLIPTIC CURVES

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Abstract. The Prym variety for a branched double covering of a nonsingular projective curve is defined as a polarized abelian variety. We prove that any double covering of an elliptic curve which has more than 4 branch points is recovered from its Prym variety.

1. Introduction

Let C and C’ be nonsingular projective curves, and let \( \phi : C \to C’ \) be a double covering branched at 2n points. In [14] the Prym variety \( P(\phi) \) for the double covering \( \phi \) is defined as a polarized abelian variety of dimension \( d = g’ - 1 + n \), where \( g’ \) is the genus of \( C’ \). Let \( R = R_{g’,2n} \) be the moduli space of such coverings, and let \( A = A_d \) be the moduli space of polarized abelian varieties of dimension \( d \). Then the construction of the Prym variety defines the Prym map \( P : R \to A \), and the Prym-Torelli problem asks whether the Prym map is injective. If \( g’ = 0 \), then it is injective by the classical Torelli theorem for hyperelliptic curves. We consider the case \( g’ > 0 \) and \( \dim R < \dim A \), where we note that \( \dim R = 3g’ - 3 + 2n \) and \( \dim A = \frac{(g’-1+n)(g’+n)}{2} \). The generically injectivity for the Prym map has been proved in most cases.

Theorem 1.1. The Prym map is generically injective in the following cases:

1. (Friedman and Smith [7], Kanev [9]) \( n = 0 \) and \( \dim R < \dim A \),
2. (Marcucci and Pirola [11]) \( g’ > 1 \), \( n > 0 \) and \( \dim R < \dim A - 1 \),
3. (Naranjo and Ortega [17]) \( g’ > 1 \), \( n > 0 \) and \( \dim R = \dim A - 1 \),
4. (Marcucci and Naranjo [10]) \( g’ = 1 \), \( n > 0 \) and \( \dim R \leq \dim A \).

The Prym varieties for unramified coverings have been intensively studied because they are principally polarized abelian varieties. For ramified coverings, Nagaraj and Ramanan [15] proved the above Theorem 1.1 (2) for \( n = 2 \), and then Marcucci and Pirola [11] proved it for any \( n > 0 \). When \( g’ > 1 \) and \( \dim R = \dim A \), there are only two cases \( (g’, n) = (6,0), (3,2) \). If \( (g’, n) = (6,0) \) then the Prym map is generically finite of degree 27 ([6]), and if \( (g’, n) = (3,2) \) then it is generically finite of degree 3 ([15], [4]).

Although the Prym map is not injective for many cases in Theorem 1.1 ([5], [16], [15], [18]), we prove the injectivity when \( g’ = 1 \). The following is the main result of this paper, which improves Theorem 1.1 (4).
Theorem 1.2 (Theorem 3.1). If \( g' = 1, \ n > 0 \) and \( \dim \mathcal{R} \leq \dim \mathcal{A} \), then the Prym map is injective.

To prove this theorem we use the Gauss map for the polarization divisor, which is a standard approach to Torelli problems. Let \( \mathcal{L} \) be an ample invertible sheaf which represents the polarization of the Prym variety \( P = P(\phi) \). For a member \( D \in |\mathcal{L}| \), we consider the Gauss map

\[
\Psi_D : D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{d-1} = \text{Grass} (d - 1, H^0(P, \Omega_P^1)^\vee).
\]

It is not difficult to show that there exists a member \( D_0 \in |\mathcal{L}| \) such that the branch divisor of \( \Psi_D \) recovers the original covering \( \phi : C \rightarrow C' \) in a similar way as Andreotti’s proof [1] of Torelli theorem for hyperelliptic curves. The essential part of our proof is to distinguish the special member \( D_0 \in |\mathcal{L}| \). We study the restriction \( \Psi_{D, |Bs|} : Bs|\mathcal{L}| \setminus D_{\text{sing}} \rightarrow \mathbf{P}^{d-1} \) of the Gauss map to the base locus of the linear system \( |\mathcal{L}| \). Although \( \Psi_D \) is difficult to compute, the restriction \( \Psi_{D, |Bs|} \) is rather simple for any member \( D \in |\mathcal{L}| \). By using the image of \( \Psi_{D, |Bs|} \) and the branch divisor of \( \Psi_{D, |Bs|} \), we can specify the member \( D_0 \in |\mathcal{L}| \) which has the desired property.

In Section 2 we summarize some basic properties of bielliptic curves and their Prym varieties. In Section 3 we explain the strategy of the proof of Theorem 1.2 by using the key Propositions in Section 6. In Section 4 we explicitly describe the base locus of the linear system of polarization divisors. In Section 5 we show that the restricted Gauss map \( \Psi_{D, |Bs|} \) is the same map as the restriction of the Gauss map for the theta divisor on Jacobian variety of \( C \). By giving a simple description for \( \Psi_{D, |Bs|} \), we prove some properties on the branch divisor of \( \Psi_{D, |Bs|} \). In Section 6 we present key Propositions, which are consequences of the results in Section 5.

In this paper, we work over an algebraically closed field \( k \) of characteristic \( \neq 2 \).

2. Properties of bielliptic curves and Prym varieties

Let \( C \) be a nonsingular projective curve of genus \( g \) over \( k \), and let \( \sigma \) be an involution on \( C \). In this paper, we call the pair \( (C, \sigma) \) a bielliptic curve of genus \( g \), if \( g > 1 \) and the quotient \( E = C/\sigma \) is a nonsingular curve of genus 1. We denote by \( \phi : C \rightarrow E \) the quotient morphism. First we note the following.

Lemma 2.1 ([16] (3.3)). Let \( (C, \sigma) \) be a bielliptic curve of genus \( g \). If \( g > 3 \), then \( C \) is not a hyperelliptic curve.

Let \( N : J(C) \rightarrow J(E) \) be the norm map of \( \phi \), which is a homomorphism on their Jacobian varieties.

Lemma 2.2 ([14]). Let \( \phi : C \rightarrow E \) be the covering defined from a bielliptic curve \( (C, \sigma) \).

1. \( \phi^* : \text{Pic}^0(E) \rightarrow \text{Pic}^0(C) \) is injective.
2. The kernel \( P \) of the norm map \( N : J(C) \rightarrow J(E) \) is reduced and connected.
By Lemma 2.2, the kernel $P$ of the norm map $N$ is an abelian variety of dimension $n = g - 1$. Let $P^v$ be the dual abelian variety of $P$, and let $\lambda_P : P \to P^v$ be the polarization isogeny which is defined as the restriction of the principal polarization on the Jacobian variety $J(C)$. Then the polarized abelian variety $(P, \lambda_P)$ is called the Prym variety for the covering $\phi : C \to E$. We denote by $K(P) \subset P$ the kernel of the polarization $\lambda_P : P \to P^v$. An ample invertible sheaf $\mathcal{L}$ on $P$ represents the the polarization isogeny $\lambda_P$ if the polarization isogeny $\lambda_P$ is given by

$$
\lambda_P : P(k) \longrightarrow P^v(k) = \text{Pic}^0(P); \ x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^v,
$$

where $t_x : P \to P$ denotes the translation by $x \in P(k)$.

**Lemma 2.3 ([14]).** Let $(P, \lambda_P)$ be the Prym variety defined from a bielliptic curve $(C, \sigma)$, and let $\mathcal{L}$ be an ample invertible sheaf which represents $\lambda_P$.

1. $K(P) = \phi^*J(E)_2 \subset J(C)$, where $J(E)_2$ denotes the set of points of order 2 on $J(E)$.
2. $\deg \lambda_P = 4$ and $h^0(P, \mathcal{L}) = 2$.

### 3. Proof of Main theorem

The main result of this paper is the following.

**Theorem 3.1.** If $g > 3$, then the isomorphism class of a bielliptic curve of genus $g$ is determined by the isomorphism class of its Prym variety.

Let $(P, \lambda_P)$ be the Prym variety of dimension $n \geq 3$ defined from a bielliptic curve $(C, \sigma)$ of genus $g = n + 1$. We will recover the data $(E, e_1 + \cdots + e_{2n}, \eta)$ from the polarized abelian variety $(P, \lambda_P)$, where $E = C/\sigma$ is the quotient curve, $e_1 + \cdots + e_{2n}$ is the branch divisor of the covering $\phi : C \to E$, and $\eta \in \text{Pic}(E)$ is the invertible sheaf with $\phi^*\eta \cong \Omega_E$. We remark that $\eta^\otimes 2 \cong \mathcal{O}_E(e_1 + \cdots + e_{2n})$, and $\eta$ is the invertible sheaf which determines the double covering with the branch divisor $e_1 + \cdots + e_{2n}$.

**Proof of Theorem 3.1** Let $\mathcal{L}$ be an ample invertible sheaf on $P$ which represents the polarization $\lambda_P$. We denote by $K(P)$ the Kernel of $\lambda_P : P \to P^v$. By Lemma 2.3 we have $\sharp K(P) = 4$ and $h^0(P, \mathcal{L}) = 2$. We define the subset $\Pi_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$
\Pi_{\mathcal{L}} = \{ D \in |\mathcal{L}| \mid t_x(D) = D \subset P \text{ for some } x \in K(P) \setminus \{0\},
$$

where $t_x$ is the translation by $x \in P(k)$. By Lemma 4.8, $\Pi_{\mathcal{L}}$ is a set of 6 members for any representative $\mathcal{L}$ of the polarization $\lambda_P$. For a member $D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, we consider the Gauss map

$$
\Psi_D : D \setminus D_{\text{sing}} \longrightarrow \mathbb{P}^{n-1} = \text{Grass}(n - 1, H^0(P, \Omega^1_P)^v),
$$

where $\Psi_D(x)$ is defined by the inclusion $T_x(D) \subset T_x(P) \cong H^0(P, \Omega^1_P)^v$ of the tangent spaces at the point $x \in D \setminus D_{\text{sing}} \subset P$. We set $U_D = \text{Bs}|\mathcal{L}| \setminus D_{\text{sing}}$, where $\text{Bs}|\mathcal{L}| \subset P$ denotes the set of base points of the pencil $|\mathcal{L}|$. Let $X_D = \Psi_D(U_D) \subset \mathbb{P}^{n-1}$ be the Zariski closure of $\Psi_D(U_D) \subset \mathbb{P}^{n-1}$, and let $\nu_D : X_D \to \mathbb{P}^{n-1}$ be the natural projection. We denote by $\tilde{X}_D$ the blowing up of $\mathbb{P}^{n-1}$ at points $x \in \nu_D^{-1}(U_D)$.

By Lemma 2.2, the fiber $\pi^{-1}(x)$ is isomorphic to $\mathbb{P}^{n-1}$, and the fiber $\pi^{-1}(y)$ is isomorphic to $\mathbb{P}^{n-1}$ for $y \in \nu_D^{-1}(U_D) \setminus \{x\}$. We denote by $\pi' : \tilde{X}_D \longrightarrow X_D$ the blow-up map. The fiber $\pi'^{-1}(y)$ is isomorphic to $\mathbb{P}^{n-1}$ for $y \in \nu_D^{-1}(U_D)$. We denote by $\tilde{\pi} : \tilde{X}_D \longrightarrow X_D$ the blow-up map. The fiber $\tilde{\pi}^{-1}(x)$ is isomorphic to $\mathbb{P}^{n-1}$ for $x \in \nu_D^{-1}(U_D)$. We denote by $\tilde{\pi}' : \tilde{X}_D \longrightarrow X_D$ the blow-up map. The fiber $\tilde{\pi}'^{-1}(y)$ is isomorphic to $\mathbb{P}^{n-1}$ for $y \in \nu_D^{-1}(U_D) \setminus \{x\}$.
There is a unique morphism \( \psi_D : U_D \to X_D \) such that \( \Psi_D|_{U_D} = \nu_D \circ \psi_D \). We consider the closed subset \( Z_D = \psi_D(\text{Ram}(\psi_D)) \subset X_D \), where \( \text{Ram}(\psi_D) \subset U_D \) denotes the ramification divisor of \( \psi_D \). By Proposition 6.2, \( Z_D \) has a canonical decomposition \( Z_D = \bigcup_{j=1}^{2n} Z_{D,i,j} \), and there is a unique hyperplane \( H_{D,i} \subset \mathbb{P}^{n-1} \) such that \( \nu_D(Z_{D,i,j}) \subset H_{D,i} \) for any \( 1 \leq i \leq 2n \). Then the effective divisor \( \nu_D^* H_{D,i} - Z_{D,i,j} \) on \( X_D \) has 2 irreducible components for general \( D \in |\mathcal{L}| \cap \mathcal{L}_\nu \), and these components coincide for special \( D \in |\mathcal{L}| \cap \mathcal{L}_\nu \). We define the subset \( \Pi'_\mathcal{L} \) in the linear pencil \( |\mathcal{L}| \) by

\[
\Pi'_\mathcal{L} = \{ D \in |\mathcal{L}| \cap \mathcal{L}_\nu \mid \nu_D^* H_{D,i} - Z_{D,i,j} \text{ is irreducible for } 1 \leq i \leq 2n \}.
\]

By Lemma 6.3, \( \Pi'_\mathcal{L} \) is a set of 4 members for any representative \( \mathcal{L} \) of the polarization \( \lambda_D \). For a member \( D \in \Pi'_\mathcal{L} \), we consider the dual variety \( (X_D')^\vee \subset (\mathbb{P}^{n-1})^\vee \) of \( X_D \subset \mathbb{P}^{n-1} \) and the dual variety \( H_{D,i}' \subset (\mathbb{P}^{n-1})^\vee \) of \( H_{D,i} \subset \mathbb{P}^{n-1} \). By Proposition 6.4, \( H_{D,i}' \) is a point on \( (X_D')^\vee \), and we have an isomorphism

\[
(E, e_1 + \cdots + e_{2n}, \eta) \cong ((X_D')^\vee, H_{D,1}' + \cdots + H_{D,2n}', \mathcal{O}_{(\mathbb{P}^{n-1})^\vee}(1)|_{(X_D')^\vee}).
\]

4. Pencil of polarization divisors

Let \( (C, \sigma) \) be a bielliptic curve of genus \( g = n + 1 > 3 \). For \( \delta \in \text{Pic}^n(C) \), we set the divisor \( W_\delta \subset J(C) \) by

\[
W_\delta(k) = \{ L \in \text{Pic}^0(C) = J(C)(k) \mid h^0(C, L \otimes \delta) > 0 \}.
\]

We remark that the singular locus of \( W_\delta \) is given by

\[
W_\delta,\text{sing}(k) = \{ L \in \text{Pic}^0(C) \mid h^0(C, L \otimes \delta) > 1 \},
\]

and \( \dim W_\delta,\text{sing} = n - 3 \) (2, Proposition 8), because \( C \) is not a hyperelliptic curve by Lemma 2.1. Let \( \lambda_C : J(C) \to J(C)^\vee \) be the homomorphism defined by

\[
\lambda_C : J(C)(k) \to J(C)^\vee(k) = \text{Pic}^0(J(C)); x \mapsto [t_x^* \mathcal{O}_C(W_\delta) \otimes \mathcal{O}_C(-W_\delta)],
\]

which does not depend on the choice of \( \delta \in \text{Pic}^n(C) \). Let \( \iota_q : C \to J(C) \) be the morphism defined by

\[
\iota_q : C(k) \to \text{Pic}^0(C) = J(C)(k); q' \mapsto [\mathcal{O}_C(q' - q)].
\]

for \( q \in C(k) \).

Lemma 4.1.

\[
x = \iota_q^*[\mathcal{O}_{J(C)}(W_\delta - W_{\delta + x})] \in \text{Pic}^0(C)
\]

for any \( q \in C(k) \) and \( x \in \text{Pic}^0(C) \).

Proof. The statement means that \((-1) \circ \lambda_C \) is the inverse of the homomorphism \( \iota_q^* : J(C)^\vee \to J(C) \) defined by the pull-back \( \iota_q^* : \text{Pic}^0(J(C)) \to \text{Pic}^0(C) \) of invertible sheaves. It is well-known (13, Lemma 6.9). \( \square \)
Let $P$ be the kernel of the Norm map $N : J(C) \to J(E)$, and let $D_{\delta} \subset P$ the fiber of the restriction of the norm map $N|_{W_{\delta}} : W_{\delta} \to J(E)$ at $0 \in J(E)$. We denote by $L_{\delta} = \mathcal{O}_P(D_{\delta}) = \mathcal{O}_{J(C)}(W_{\delta})|_P$ the restriction of $\mathcal{O}_{J(C)}(W_{\delta})$ to $P$. Since $W_{\delta}$ is the theta divisor of $J(C)$, the ample invertible sheaf $L_{\delta}$ represents the polarization $\lambda_P$.

**Lemma 4.2.** $D_{\delta+p^*s} \subset P$ is a member of the linear system $|L_{\delta}|$ for any $s \in \text{Pic}^0(E)$.

**Proof.** By Lemma 4.1,

$$\phi^*s = \iota_q^*[\mathcal{O}_{J(C)}(W_{\delta} - W_{\delta+p^*s})] \in \text{Pic}^0(C)$$

for $s \in \text{Pic}^0(E)$ and $q \in C(k)$. We set $s' = \text{Pic}^0(J(E))$ by $s = \iota_{\phi(q)}^*s'$, where $\iota_{\phi(q)} : E \to J(E)$ is the isomorphism determined by $\iota_{\phi(q)}(\phi(q)) = 0$. Then we have

$$N^*s' = [\mathcal{O}_{J(C)}(W_{\delta} - W_{\delta+p^*s})] \in \text{Pic}^0(J(C)),$$

because $\phi^*s = \iota_{q}^*N^*s'$ and $\iota_{q}^* : \text{Pic}^0(J(C)) \to \text{Pic}^0(C)$ is an isomorphism. Since $(N^*s')|_P = 0 \in \text{Pic}^0(P)$, we have

$$\mathcal{O}_P(D_{\delta+p^*s}) \cong \mathcal{O}_{J(C)}(W_{\delta+p^*s})|_P \cong \mathcal{O}_{J(C)}(W_{\delta})|_P = L_{\delta}. \quad \square$$

We denote by $C^{(i)}$ the $i$-th symmetric products of $C$. For $\delta \in \text{Pic}^n(C)$, we define the morphism $\beta_{\delta}^i : C^{(n-2i)} \times E^{(i)} \to J(C)$ by

$$\beta_{\delta}^i : C^{(n-2i)}(k) \times E^{(i)}(k) \to J(C)(k) = \text{Pic}^0(C);$$

$$(q_1 + \cdots + q_{n-2i}, p_1 + \cdots + p_i) \longmapsto \mathcal{O}_C\left(\sum_{j=1}^{n-2i} q_j\right) \otimes \phi^*\mathcal{O}_E\left(\sum_{j=1}^{i} p_j\right) \otimes \delta'.'$$

We remark that $W_{\delta} = \text{Image}(\beta_{\delta}^0)$, and we set

$$B_{\delta}^i = \begin{cases} \text{Image}(\beta_{\delta}^i) & (1 \leq 2i \leq n), \\ \emptyset & (2i > n). \end{cases}$$

**Lemma 4.3.** $B_{\delta}^1 \setminus W_{\delta,\text{sing}} \neq \emptyset$ and $B_{\delta}^2 \subset W_{\delta,\text{sing}}$.

**Proof.** Let $B^1$ be the image of the morphism $\beta^1 : C^{(n-2)} \times E \to C^{(n)}$ defined by

$$\beta^1 : C^{(n-2)}(k) \times E(k) \to C^{(n)}(k); \ (q_1 + \cdots + q_{n-2}, \phi(q)) \longmapsto q_1 + \cdots + q_{n-2} + q + \sigma(q).$$

Since $C$ is not a hyperelliptic curve, we have $\dim(\beta_{\delta}^0)^{-1}(W_{\delta,\text{sing}}) = n - 2 < n - 1 = \dim B^1$, hence $B_{\delta}^1 = \beta_{\delta}^0(B^1) \notin W_{\delta,\text{sing}}$.

To prove the second statement, we assume that $n \geq 4$, because $B_{\delta}^2 = \emptyset$ for $n = 3$. Let $F \subset C^{(n-4)} \times E^{(2)}$ be the fiber of the composition

$$C^{(n-4)} \times E^{(2)} \xrightarrow{\beta_{\delta}^2} J(C) \xrightarrow{N} J(E)$$

at $\eta - N(\delta) \in J(E)(k)$, where $\eta \in \text{Pic}^n(E)$ denotes the unique invertible sheaf on $E$ with $\phi^*\eta \cong \Omega_C^1$. We set $U = (C^{(n-4)} \times E^{(2)}) \setminus (F \cup (C^{(n-4)} \times \Delta_E))$, where $\Delta_E \subset E^{(2)}$.
denotes the image of the diagonal in $E \times E$. For $y = (q_1 + \cdots + q_{n-4}, \phi(q) + \phi(r)) \in U(k)$, there are points $q', r' \in C(k)$ such that
\[
\begin{align*}
\mathcal{O}_E(\phi(q')) &\cong \eta \otimes \mathcal{O}_E(-\phi(q_1) - \cdots - \phi(q_{n-4}) - \phi(q) - 2\phi(r)), \\
\mathcal{O}_E(\phi(r')) &\cong \eta \otimes \mathcal{O}_E(-\phi(q_1) - \cdots - \phi(q_{n-4}) - 2\phi(q) - \phi(r)).
\end{align*}
\]
Then we have
\[
\begin{align*}
\Omega^1_C(-q_1 - \cdots - q_{n-4} - q - \sigma(q) - r - \sigma(r)) \\
\cong \phi^*\eta \otimes \mathcal{O}_C(-q_1 - \cdots - q_{n-4} - q - \sigma(q) - r - \sigma(r)) \\
\cong \mathcal{O}_C(\sigma(q_1) + \cdots + \sigma(q_{n-4}) + q' + \sigma(q') + r + \sigma(r)) \\
\cong \mathcal{O}_C(\sigma(q_1) + \cdots + \sigma(q_{n-4}) + q + \sigma(q) + r' + \sigma(r')).
\end{align*}
\]
We remark that $\phi(q) \neq \phi(q')$ and $\phi(q) \neq \phi(r)$, because $y \notin F$ and $y \notin C^{(n-4)} \times \Delta_E$. Hence we have $h^0(C, \Omega^1_C(-q_1 - \cdots - q_{n-4} - q - \sigma(q) - r - \sigma(r))) > 1$ and $\beta^2_\delta(y) \in W_{\delta,sing}$. Since $U \subset (\beta^2_\delta)^{-1}(W_{\delta,sing})$ is a dense subset of $C^{(n-4)} \times E(2)$, we have $C^{(n-4)} \times E(2) = (\beta^2_\delta)^{-1}(W_{\delta,sing})$. \hfill \qed

**Lemma 4.4.** For $s \in \text{Pic}^0(E)$, $D_\delta = D_{\delta + \phi^*s} \subset P$ if and only if $s = 0$ or $s = \eta - N(\delta)$.

**Proof.** For $L \in P(k) \subset \text{Pic}^0(C)$, we have $0 = \phi^*N(L) = L + \sigma^*L \in \text{Pic}^0(C)$. Hence we have $D_\delta = D_{\delta + \phi^*(\eta - N(\delta))}$, because $L \in D_\delta(k) \iff h^0(C, L \otimes \delta) > 0 \iff h^0(C, \sigma^*L \otimes \sigma^*\delta) > 0 \iff h^0(C, L^\vee \otimes \sigma^*\delta) > 0 \iff h^0(C, \Omega^1_C \otimes L \otimes \sigma^*\delta^\vee) > 0 \iff L \in D_{[\Omega^1_C \otimes L \otimes \sigma^*\delta^\vee]}(k) = D_{\delta + \phi^*(\eta - N(\delta))}(k)$.

We assume that $D_\delta = D_{\delta + \phi^*s}$ for $s \neq 0 \in \text{Pic}^0(E)$. Let $\alpha_{\delta + \phi^*s} : C^{(n-2)} \times C \to J(C)$ be the morphism defined by
\[
\begin{align*}
\alpha_{\delta + \phi^*s} : C^{(n-2)}(k) \times C(k) &\to \text{Pic}^0(C) = J(C)(k); \\
(q_1 + \cdots + q_{n-2}, q) &\mapsto \mathcal{O}_C(q_1 + \cdots + q_{n-2} + 2q) \otimes \delta^\vee \otimes \phi^*s^\vee.
\end{align*}
\]
Then the set $D_\delta \setminus (W_{\delta,sing} \cup B^1_\delta \cup \text{Image}(\alpha_{\delta + \phi^*s}))$ is not empty, because $\dim D_\delta \cap (W_{\delta,sing} \cup B^1_{\delta} \cup \text{Image}(\alpha_{\delta + \phi^*s})) < n - 1 = \dim D_\delta$.

For $L \in D_\delta(k) \setminus (W_{\delta,sing} \cup B^1_\delta \cup \text{Image}(\alpha_{\delta + \phi^*s}))$, there is $r_1 + \cdots + r_n \in C^{(n)}(k)$ such that $L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(r_1 + \cdots + r_n)$, because $L \in D_\delta(k) = D_{\delta + \phi^*s}(k) \subset W_{\delta,sing}(K)$. Since $L \in W_{\delta}(k) \setminus W_{\delta,sing}(k)$, we have $h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee) = h^0(C, L \otimes \delta) = 1$. Let $q_1 + \cdots + q_n \in C^{(n)}(k)$ and $q'_1 + \cdots + q'_n \in C^{(n)}(k)$ be the effective divisors defined by
\[
L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_n), \quad \Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(q'_1 + \cdots + q'_n).
\]
Let $\phi(u_i) \in E(k)$ be the point determined by $L = [\mathcal{O}_E(\phi(r_i) - \phi(u_i))]$. Then
\[
L \otimes \delta \otimes \mathcal{O}_C(\sigma(r_i)) \cong \mathcal{O}_C(r_1 + \cdots + r_n - r_i + u_i + \sigma(u_i)).
\]
If $\sigma(r_i) \notin \{q'_1, \ldots, q'_n\}$, then
\[
h^0(C, L \otimes \delta \otimes \mathcal{O}_C(\sigma(r_i))) = h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee \otimes \mathcal{O}_C(\sigma(r_i))) + 1 = 1.
\]
hence
\[ q_1 + \cdots + q_n + \sigma(r_i) = r_1 + \cdots + r_n - r_i + u_i + \sigma(u_i). \]
Since \( s \neq 0 \), we have \( \sigma(r_i) = r_i \) for some \( j \neq i \), and \( L \in B_1^0(k) \). It is a contradiction to \( L \notin B_1^0(k) \), hence \( \sigma(r_i) \in \{q'_1, \ldots, q'_n\} \) for any \( 1 \leq i \leq n \). Here the condition \( L \notin \text{Image}(\alpha_{\delta+\phi^*s}) \) implies that \( \sharp \{r_1, \ldots, r_n\} = n \) and
\[ L^\vee \otimes \sigma^*\delta \otimes \phi^*s \cong \mathcal{O}_C(\sigma(r_1) + \cdots + \sigma(r_n)) = \mathcal{O}_C(q'_1 + \cdots + q'_n) \cong \Omega^1_C \otimes L^\vee \otimes \delta^\vee. \]
Hence we have \( \phi^*s = [\Omega^1_C] - \delta - \sigma^*\delta = \phi^*(\eta - N(\delta)) \), and \( s = \eta - N(\delta) \) by Lemma 2.2.

Let \( B_\delta \subset J(C) \) be the subset
\[
B_\delta = \bigcap_{s \in \text{Pic}^0(E)} W_{\delta+\phi^*s}.
\]

**Lemma 4.5.** \( B_\delta \setminus W_{\delta,\text{sing}} = B_1^0 \setminus W_{\delta,\text{sing}} \).

**Proof.** If \( L \in B_1^0(k) \), then \( L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q + \sigma(q)) \) for some \( q_1, \ldots, q_{n-2}, q \in C(k) \). For \( s \in \text{Pic}^0(E) \), there is a point \( q' \in C(k) \) such that \( s = [\mathcal{O}_E(\phi(q') - \phi(q))] \). Since \( L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q' + \sigma(q')) \), we have \( h^0(C, L \otimes \delta \otimes \phi^*s) > 0 \) and \( L \in W_{\delta+\phi^*s}(k) \). Hence the inclusion \( B_1^0 \subset B_\delta \) holds.

For \( L \in B_\delta(k) \setminus W_{\delta,\text{sing}}(k) \), there is a unique \( r_1 + \cdots + r_n \in C^{(n)}(k) \) such that \( \Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(r_1 + \cdots + r_n) \), because \( h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee) = h^0(C, L \otimes \delta) = 1 \). Let \( \Sigma \subset \text{Pic}^0(E) \) be the finite subset defined by
\[
\Sigma = \{ s \in \text{Pic}^0(E) \mid L \otimes \delta \otimes \phi^*s \in \beta^0_\delta(\Delta^{(n)}) \},
\]
where \( \Delta = \{ \sigma(r_1), \ldots, \sigma(r_n) \} \subset C \) and \( \Delta^{(n)} \subset C^{(n)} \). For \( s \in \text{Pic}^0(E) \setminus (\Sigma \cup \{0\}) \), there is a divisor \( q_1 + \cdots + q_n \in C^{(n)}(k) \) such that \( L \otimes \delta \otimes \phi^*s \cong \mathcal{O}_C(q_1 + \cdots + q_n) \), because \( L \in B_\delta(k) \subset W_{\delta+\phi^*s}(k) \). Since \( s \notin \Sigma \), we may assume that \( q_n \notin \Delta \). The condition \( \sigma(q_n) \notin \{r_1, \ldots, r_n\} \) implies that \( h^0(C, \Omega^1_C \otimes L^\vee \otimes \delta^\vee \otimes \mathcal{O}_C(-\sigma(q_n))) = 0 \) and \( h^0(C, L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n))) = 1 \). Let \( \phi(q') \in E(k) \) be the point determined by \( s = [\mathcal{O}_E(\phi(q_n) - \phi(q'))] \). Then
\[
L \otimes \delta \otimes \mathcal{O}_C(\sigma(q_n)) \cong \mathcal{O}_C(q_1 + \cdots + q_{n-1} + q' + \sigma(q')).
\]
Since \( s \neq 0 \), we have \( \sigma(q_n) \in \{q_1, \ldots, q_{n-1}\} \) and \( L \in B_1^0(k) \). \( \square \)

**Lemma 4.6.** The map
\[
\text{Pic}^0(E) \longrightarrow |\mathcal{L}_\delta|; \ s \mapsto D_{\delta+\phi^*s}
\]
is a double covering, and the base locus \( \text{Bs}|\mathcal{L}_\delta| \) of the linear system \( |\mathcal{L}_\delta| \) is \( B_\delta \cap P \), which is of dimension \( n - 2 \).

**Proof.** The map is well-defined by Lemma 4.2. Since \( \dim |\mathcal{L}_\delta| = 1 \), it is a double covering by Lemma 4.4. Hence we have
\[
\text{Bs}|\mathcal{L}_\delta| = \bigcup_{s \in \text{Pic}^0(E)} D_{\delta+\phi^*s} = B_\delta \cap P.
\]
By Lemma 4.3, $B^1_\delta$ is irreducible of dimension $n - 1$. Since the restriction of the Norm map $N|_{B^1_\delta}: B^1_\delta \to J(E)$ is surjective, we have $\dim B^1_\delta \cap P = n - 2$, hence $\dim B_\delta \cap P = n - 2$ by Lemma 4.5.

**Lemma 4.7.** Let $\mathcal{L}$ be an ample invertible sheaf which represents the polarization $\lambda_P$ on $P$, then there is $\delta \in \text{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$.

**Proof.** For any $\delta' \in \text{Pic}^n(C)$, we have $\mathcal{L} \otimes \mathcal{L}_{\delta'}^\vee \in \text{Pic}^0(P)$, because $\mathcal{L}_{\delta'}$ gives the same polarization as $\lambda_P$. Then $\mathcal{L} \cong t_\ast \mathcal{L}_{\delta'} \cong \mathcal{L}_{\delta' + x}$ for some $x \in P(k)$. Let $s \in \text{Pic}^0(E)$ be a point with $2s = \eta - N(\delta' + x)$. For $\delta = \delta' + x + \phi^s s$, we have $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$.

For an ample invertible sheaf $\mathcal{L}$ which represents the polarization $\lambda_P$, we set a subset in the linear system $|\mathcal{L}|$ by

$$\Pi_\mathcal{L} = \{D \in |\mathcal{L}| \mid t_x(D) = D \text{ for some } x \in K(P) \setminus \{0\},$$

where $K(P)$ is the kernel of the polarization $\lambda_P$.

**Lemma 4.8.** $\sharp \Pi_\mathcal{L} = 6$.

**Proof.** By Lemma 4.7, there is $\delta \in \text{Pic}^n(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_\delta$. For any $D \in |\mathcal{L}_{\delta}|$, by Lemma 4.1, there is $s \in \text{Pic}^0(E)$ such that $D = D_{\delta + \phi^s s}$. If $D_{\delta + \phi^s s} \in \Pi_{\mathcal{L}_{\delta}}$, then by Lemma 2.3, there is $t \in J(E)_2 \setminus \{0\}$ such that $t_{\phi^s}(D_{\delta + \phi^s s}) = D_{\delta + \phi^s s}$. Since $t_{\phi^s}(D_{\delta + \phi^s s}) = D_{\delta + \phi^s (s-t)}$ and $t \not= 0$, by Lemma 4.3, we have

$$\delta + \phi^s (s-t) = \delta + \phi^s s + \phi^s(\eta - N(\delta + \phi^s s)) = \delta - \phi^s s,$$

hence $t = 2s$ by Lemma 2.2. It means that

$$\Pi_{\mathcal{L}_{\delta}} = \{D_{\delta + \phi^s s} \in |\mathcal{L}_{\delta}| \mid s \in J(E)_4 \setminus J(E)_2\}.$$

Since $\sharp(J(E)_4 \setminus J(E)_2) = 12$ and $D_{\delta + \phi^s s} = D_{\delta - \phi^s s}$, we have $\sharp \Pi_\mathcal{L} = 6$.

5. **Gauss maps**

5.1. **Gauss map for Jacobian and Gauss map for Prym.** Let

$$\Psi_{J(C),\delta}: W_\delta \setminus W_{\delta,\text{sing}} \to \text{Grass}(n, H^0(C, \Omega^1_C)^\vee) = \text{Grass}(n, H^0(C, \Omega^1_C)^\vee)$$

be the Gauss map for the subvariety $W_\delta \subset J(C)$. For $L \in W_\delta(k) \setminus W_{\delta,\text{sing}}(k)$, the tangent space $T_L(W_\delta)$ at $W_\delta$ defines the image $\Psi_{J(C),\delta}(L)$ by the natural identifications

$$T_L(W_\delta) \subset T_L(J(C)) \cong (\Omega^1_{J(C)}(L))^\vee \cong H^0(J(C), \Omega^1_{J(C)(L)})^\vee \cong H^0(C, \Omega^1_C)^\vee.$$ 

**Lemma 5.1.** For $L \in W_\delta(k) \setminus W_{\delta,\text{sing}}(k)$, the image $\Psi_{J(C),\delta}(L)$ of the Gauss map is identified with the canonical divisor

$$q_1 + \cdots + q_n + q'_1 + \cdots + q'_n \in |\Omega^1_C| = \text{Grass}(1, H^0(C, \Omega^1_C)) \cong \text{Grass}(1, H^0(C, \Omega^1_C)),$$

where the effective divisors $q_1 + \cdots + q_n$ and $q'_1 + \cdots + q'_n$ are uniquely determined by $L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_n)$ and $\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(q'_1 + \cdots + q'_n)$.

**Proof.** It is a special case of Proposition (4.2) in [3] Chapter IV].
Lemma 5.2. Let \( K \in |\Omega_C^1| \) be an effective canonical divisor. If \( q_1 + \sigma(q_1) \leq K \) for some \( q_1 \in C(k) \), then \( K = \sum_{i=1}^{n}(q_i + \sigma(q_i)) \) for some \( q_2, \ldots, q_n \in C(k) \).

Proof. When

\[
K = \sum_{i=1}^{m}(q_i + \sigma(q_i)) + q + \sum_{j=1}^{2n-2m-1} r_j
\]

for \( 1 \leq m \leq n-1 \), we show that \( \sigma(q) \in \{ r_1, \ldots, r_{2n-2m-1} \} \). First we assume that \( 1 \leq m \leq n-2 \). Since \( C \) is not a hyperelliptic curve, by Clifford’s theorem, we have

\[
m + 2 > h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)) + q + \sigma(q))) \geq h^0(E, \mathcal{O}_E(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) = m + 1
\]

and

\[
m + 1 > h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) \geq h^0(E, \mathcal{O}_E(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) = m,
\]

hence \( h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i))) + q + \sigma(q))) = m + 1 \) and \( h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) = m \). Since \( \sigma(q) \) is not a base point of \( |\mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)) + q + \sigma(q)| = \phi^*|\mathcal{O}_E(\sum_{i=1}^{m}(q_i))| \), we have

\[
m = h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i)))) < h^0(C, \mathcal{O}_C(\sum_{i=1}^{m}(q_i + \sigma(q_i))) + q + \sigma(q))) = m + 1,
\]

hence

\[
h^0(C, \mathcal{O}_C(\sum_{j=1}^{2n-2m-1} r_j - \sigma(q))) = h^0(C, \mathcal{O}_C(\sum_{j=1}^{2n-2m-1} r_j)) = n - m - 1.
\]

It implies that \( \sigma(q) \leq \sum_{j=1}^{2n-2m-1} r_j \). We consider the case \( m = n - 1 \). Let \( \eta \in \text{Pic}^n(E) \) be the invertible sheaf with \( \phi^*\eta \cong \Omega_C^1 \). There is a point \( q' \in C(k) \) such that \( \sum_{j=1}^{n-1}(q_i + \phi(q')) \in |\eta| \). Then \( \mathcal{O}_C(q + r_1) \cong \mathcal{O}_C(q' + \sigma(q')) \). Since \( C \) is not a hyperelliptic curve, we have \( q + r_1 = q' + \sigma(q') \) and \( \sigma(q) = r_1 \).

By the injective homomorphism

\[
H^0(E, \eta) \longrightarrow H^0(C, \phi^*\eta) \cong H^0(C, \Omega_C^1),
\]

we have the closed immersion

\[
\iota : \mathbb{P}(H^0(E, \eta)^\vee) \longrightarrow \mathbb{P}(H^0(C, \Omega_C^1)^\vee).
\]

Lemma 5.3. For \( L \in B_\delta(k) \setminus W_{\delta_{\text{sing}}}(k) \),

\[
\Psi_{J(C),\delta}(L) = \iota(\mathbb{P}(H^0(E, \eta)^\vee)) \subset \mathbb{P}(H^0(C, \Omega_C^1)^\vee).
\]

Proof. For \( L \in B_\delta(k) \setminus W_{\delta_{\text{sing}}}(k) \), by Lemma 5.1, the image \( \Psi_{J(C),\delta}(L) \) of the Gauss map is given by

\[
q_1 + \cdots + q_n + q'_1 + \cdots + q'_n \in |\Omega_C^1| \cong \mathbb{P}(H^0(C, \Omega_C^1)^\vee),
\]
where the effective divisors \(q_1 + \cdots + q_n\) and \(q'_1 + \cdots + q'_n\) are uniquely determined by \(L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_n)\) and \(\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(q'_1 + \cdots + q'_n)\). Since \(L \in B_\delta(k)\), by Lemma \ref{lem:1.5}, \(\sigma(q_i) = q_j\) for some \(i \neq j\). By Lemma \ref{lem:5.2} we have \(\Psi_{J(C), \delta}(L) \in \iota(\mathbb{P}(H^0(E, \eta)^\vee))\).

Let

\[
\Psi_{P, \delta} : D_{\delta, \text{sing}} = W_{\delta, \text{sing}} = (W_{\delta, \text{sing}} \cup \Sigma_{\delta}) \cap D_{\delta}.
\]

By Lemma \ref{lem:4.5}, \((\phi^*H^0(E, \Omega^1_C))^\perp \subset H^0(C, \Omega^1_C)^\vee \cong T_L(J(C))\)

to \(\phi^*H^0(E, \Omega^1_E) \subset H^0(C, \Omega^1_C)^\vee\), and it corresponds to the ramification divisor \(\text{Ram}(\phi) \in [\Omega^1_C]\) of the covering \(\phi : C \to E\). We define the finite set \(\Sigma_{\delta}\) by

\[
\Sigma_{\delta} = \{ \beta^0_\delta(r_1 + \cdots + r_n) \in J(C) \mid r_1 + \cdots + r_n \leq \text{Ram}(\phi) \}.
\]

**Lemma 5.4.** \(D_{\delta, \text{sing}} = (W_{\delta, \text{sing}} \cup \Sigma_{\delta}) \cap D_{\delta}\).

**Proof.** If \(L \in D_{\delta}(k) \cap W_{\delta, \text{sing}}(k)\), then \(L \in D_{\delta, \text{sing}}(k)\). If \(L \in D_{\delta}(k) \setminus W_{\delta, \text{sing}}(k)\), then by Lemma \ref{lem:5.1}

\[
L \in D_{\delta, \text{sing}}(k) \iff T_L(W_{\delta}) = T_L(P) \subset T_L(J(C)) \iff L \in \Sigma_{\delta}.
\]

**Lemma 5.5.** \((B_{\delta} \cap P) \setminus W_{\delta, \text{sing}} = (B_{\delta} \cap P) \setminus D_{\delta, \text{sing}}\).

**Proof.** By Lemma \ref{lem:1.5} \((B_{\delta} \setminus W_{\delta, \text{sing}}) \cap \Sigma_{\delta} = (B_{\delta} \setminus W_{\delta, \text{sing}}) \cap \Sigma_{\delta}\), and it is empty because \(\text{Ram}(\phi)\) is reduced. Hence by Lemma \ref{lem:5.4}

\[
W_{\delta, \text{sing}} \cap B_{\delta} \cap P = W_{\delta, \text{sing}} \cap B_{\delta} \cap D_{\delta} = D_{\delta, \text{sing}} \cap B_{\delta} = D_{\delta, \text{sing}} \cap B_{\delta} \cap P.
\]

We denote by

\[
\pi : \mathbb{P}(H^0(C, \Omega^1_C)^\vee) \setminus \{V_P\} \to \mathbb{P}(H^0(P, \Omega^1_P)^\vee);
\]

\[
[V \subset H^0(C, \Omega^1_C)^\vee] \to [V \cap V_P \subset V_P \cong H^0(P, \Omega^1_P)^\vee]
\]

the projection, where \(V_P = (\phi^*H^0(E, \Omega^1_E))^\perp \subset H^0(C, \Omega^1_C)^\vee\) is the image of the dual of the restriction

\[
H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)}) \to H^0(P, \Omega^1_P).
\]

**Lemma 5.6.** \(\Psi_{P, \delta}(L) = \pi \circ \Psi_{J(C), \delta}(L)\) for \(L \in D_{\delta}(k) \setminus D_{\delta, \text{sing}}(k)\).
Proof. For \( L \in D_\delta(k) \setminus D_{\delta,\text{sing}}(k) \), the tangent spaces at \( L \) satisfies
\[
T_L(P) \cap T_L(W_\delta) = T_L(D_\delta) \subset T_L(J(C)) ,
\]
because \( P \cap W_\delta = D_\delta \subset J(C) \). Since \( T_L(P) \subset T_L(J(C)) \) is identified with \( V_P \subset \Omega^0(C, \Omega^1_C) \) by \( T_L(J(C)) \cong \Omega^0(C, \Omega^1_C) \), we have \( \Psi_P,\delta(L) = \pi \circ \Psi_{J(C),\delta}(L). \)

By Lemma 5.3, we have the morphism
\[
\Psi_{J(C),\delta}^B : B_\delta \setminus W_{\delta,\text{sing}} \to \text{P}(H^0(E, \eta)^\vee)
\]
satisfying \( \iota \circ \Psi_{J(C),\delta}^B \).

Lemma 5.7. The restriction \( \Psi_{P,\delta}|_{(B_\delta \cap P) \setminus D_{\delta,\text{sing}}} \) of the Gauss map \( \Psi_{P,\delta} \) is identified with the restriction \( \Psi_{J(C),\delta}^B|_{(B_\delta \cap P) \setminus D_{\delta,\text{sing}}} \) of \( \Psi_{J(C),\delta}^B \) by the isomorphism
\[
\pi \circ \iota : \text{P}(H^0(E, \eta)^\vee) \sim \text{P}(H^0(P, \Omega^1_P)^\vee).
\]

Proof. Since the composition
\[
H^0(E, \eta) \to H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)}) \to H^0(P, \Omega^1_P),
\]
is an isomorphism, it is a consequence of Lemma 5.3 and Lemma 5.6.

5.2. Description for the restricted Gauss maps. Let \( \gamma_\delta : E^{(n-2)} \times E \to J(E) \) be the morphism defined by
\[
\gamma_\delta : E^{(n-2)}(k) \times E(k) \to \text{Pic}^0(E);
\]
\[
(p_1 + \cdots + p_{n-2}, p) \mapsto \mathcal{O}_E(p_1 + \cdots + p_{n-2} + 2p) \otimes (N(\delta))^\vee.
\]
Let \( X_\delta \subset E^{(n-2)} \times E \) be the fiber of \( \gamma_\delta \) at \( 0 \in J(E) \), and let \( Y_\delta \subset C^{(n-2)} \times E \) be the fiber of the composition
\[
C^{(n-2)} \times E \xrightarrow{\phi^{(n-2)} \times \text{id}_E} E^{(n-2)} \times E \xrightarrow{\gamma_\delta} J(E)
\]
at \( 0 \in J(E) \). We denote by \( \psi_\delta : Y_\delta \to X_\delta \) the induced morphism by \( \phi^{(n-2)} \times \text{id}_E \).

Let \( \nu_\delta : X_\delta \to |\eta| \cong \text{P}(H^0(E, \eta)^\vee) \) be the morphism defined by
\[
\nu_\delta : X_\delta(k) \to |\eta|; (p_1 + \cdots + p_{n-2}, p) \mapsto p_1 + \cdots + p_{n-2} + p + t_{\eta-N(\delta)}(p),
\]
where \( t_{\eta-N(\delta)}(p) \in E(k) \) is the point determined by
\[
[\mathcal{O}_E(t_{\eta-N(\delta)}(p))] = [\mathcal{O}_E(p)] + \eta - N(\delta) \in \text{Pic}(E).
\]

We remark that \( \beta^1_\delta(Y_\delta) = B_\delta \cap P \subset J(C) \), and we set
\[
Y_\delta^\circ = (\beta^1_\delta)^{-1}((B_\delta \cap P) \setminus D_{\delta,\text{sing}}) = (\beta^1_\delta)^{-1}((B_\delta \cap P) \setminus D_{\delta,\text{sing}}).
\]

Lemma 5.8. The diagram
\[
\begin{array}{ccc}
Y_\delta^\circ & \xrightarrow{\beta^1_\delta} & (B_\delta \cap P) \setminus D_{\delta,\text{sing}} \\
\psi_\delta \downarrow & & \downarrow \Psi_{J(C),\delta}^B \\
X_\delta & \xrightarrow{\nu_\delta} & \text{P}(H^0(E, \eta)^\vee)
\end{array}
\]
is commutative.
Proof. Let $L \in \text{Pic}^0(C)$ be the invertible sheaf which represents the point $\beta_\delta^1(y) \in J(C)$ for $y = (q_1 + \cdots + q_{n-2}, \phi(q)) \in Y_\delta^0(k)$. Then $q_1 + \cdots + q_{n-2} + q + \sigma(q) \in C^{n}(k)$ is the unique effective divisor with $L \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + q + \sigma(q)) \otimes \delta^\vee$. Since

$$\nu_\delta \circ \psi_\delta(y) = \phi(q_1) + \cdots + \phi(q_{n-2}) + \phi(q) + t_{\eta - N(\delta)}(\phi(q)) \in |\eta|,$$

we have

$$q_1 + \sigma(q_1) + \cdots + q_{n-2} + \sigma(q_{n-2}) + q + \sigma(q) + r + \sigma(r) \in |\Omega^1_\delta|,$$

where $r \in C(k)$ is given by $\phi(r) = t_{\eta - N(\delta)}(\phi(q))$. Then $\sigma(q_1) + \cdots + \sigma(q_{n-2}) + r + \sigma(r) \in C^{n}(k)$ is the unique effective divisor with $\mathcal{O}_C(\sigma(q_1) + \cdots + \sigma(q_{n-2}) + r + \sigma(r)) \cong \Omega^1_\delta \otimes L^\vee \otimes \delta^\vee$, and by Lemma 5.1, $\Phi_{J(C),\delta}^B(L)$ is equal to $\nu_\delta \circ \psi_\delta(y)$. □

**Lemma 5.9.** $X_\delta$ and $Y_\delta$ are nonsingular projective varieties.

Proof. We fix a point $p_0 \in E(k)$. Let $\gamma_{\delta,p_0} : E^{(n-2)} \to J(E)$ be the morphism defined by

$$\gamma_{\delta,p_0} : E^{(n-2)}(k) \to \text{Pic}^0(E) = J(E)(k);$$

$$p_1 + \cdots + p_{n-2} \mapsto \mathcal{O}_E(p_1 + \cdots + p_{n-2} + 2p_0) \otimes (N(\delta))^\vee.$$

Then $\psi_\delta : Y_\delta \to X_\delta$ is the base change of $\phi^{(n-2)} : C^{(n-2)} \to E^{(n-2)}$ by the étale covering of degree 4;

$$Y_\delta \xrightarrow{\psi_\delta} X_\delta \xrightarrow{pr_2} E \xrightarrow{pr_1} C^{(n-2)} \xrightarrow{\phi^{(n-2)}} E^{(n-2)} \xrightarrow{\gamma_{\delta,p_0}} J(E),$$

where $(-2)_{J(E)} \circ t_{p_0} : E \to J(E)$ is given by

$$(-2)_{J(E)} \circ t_{p_0} : E(k) \to \text{Pic}^0(E) = J(E)(k); p \mapsto \mathcal{O}_E(2p_0 + 2p).$$

□

**Lemma 5.10.** $\beta_\delta^1|_{Y_\delta^0} : Y_\delta^0 \to (B_\delta \cap P) \setminus D_{\delta,\text{sing}}$ is an isomorphism. In particular, $(B_\delta \cap P) \setminus D_{\delta,\text{sing}}$ is a nonsingular variety.

Proof. By Lemma 5.3, the image $\beta_\delta^1|_{Y_\delta^0} : Y_\delta^0 \to (B_\delta \cap P) \setminus D_{\delta,\text{sing}}$ is a closed subset in $W_\delta \setminus W_{\delta,\text{sing}}$. We show that $\beta_\delta^1|_{Y_\delta^0} : Y_\delta^0 \to W_\delta \setminus W_{\delta,\text{sing}}$ is a closed immersion. Let $\beta^1 : C^{(n-2)} \times E \to C^{(n)}$ be the morphism given in the proof of Lemma 4.3. Since $\beta_\delta^1 = \beta_\delta^0 \circ \beta^1$ and $\beta_\delta^0 : C^{(n)} \to J(C)$ induces the isomorphism

$$\beta_\delta^0 : C^{(n)} \setminus (\beta_\delta^0)^{-1}(W_{\delta,\text{sing}}) \xrightarrow{\cong} W_\delta \setminus W_{\delta,\text{sing}},$$

it is enough to show that the finite morphism

$$\beta^1 : (C^{(n-2)} \times E) \setminus (\beta_\delta^1)^{-1}(W_{\delta,\text{sing}}) \to C^{(n)} \setminus (\beta_\delta^0)^{-1}(W_{\delta,\text{sing}})$$

is a closed immersion. We remark that it is injective by Lemma 4.3. For $y = (q_1 + \cdots + q_{n-2}, \phi(q_0)) \in (C^{(n-2)} \times E) \setminus (\beta_\delta^1)^{-1}(W_{\delta,\text{sing}})$, we prove that the homomorphism

$$T_y(C^{(n-2)} \times E) \to T_{\beta^1(y)}(C^{(n)})$$

is a closed immersion.
on the tangent spaces is injective. If \( \phi(q_0) \notin \{ \phi(q_1), \ldots, \phi(q_{n-2}) \} \), then the point
\[ y' = (q_1 + \cdots + q_{n-2} + q_0 + \sigma(q_0)) \in C^{(n-2)}(k) \times C^{(2)}(k) \]
is not contained in the ramification divisor of the natural covering \( C^{(n-2)} \times C^{(2)} \to C^{(n)} \). Since the morphism
\[ E(k) \to C^{(2)}(k); \phi(q) \longmapsto q + \sigma(q) \]
is a closed immersion, the homomorphism
\[ T_y(C^{(n-2)} \times E) \hookrightarrow T_{y'}(C^{(n-2)} \times C^{(2)}) \cong T_{\beta^1(y)}(C^{(n)}) \]
is injective. We consider the case when \( y = (q_1 + \cdots + q_{n-2}, q_0) \) and \( q_0 \notin \{ q_1, \ldots, q_{n-2-i} \} \) for some \( i \geq 1 \). First we assume that \( \sigma(q_0) = q_0 \). Then by Lemma 4.3 we have \( i = 1 \). The point \( \tilde{y} = (q_1 + \cdots + q_{n-3}, q_0, \phi(q_0)) \in C^{(n-3)} \times C \times E \) is not contained in the ramification divisor of the natural covering \( C^{(n-3)} \times C \times E \to C^{(n-2)} \times E \), and the point \( \tilde{y}' = (q_1 + \cdots + q_{n-3}, 3q_0) \in C^{(n-3)} \times C^{(3)} \) is not contained in the ramification divisor of the natural covering \( C^{(n-3)} \times C^{(3)} \to C^{(n)} \). Since the morphism
\[ C(k) \times E(k) \to C^{(3)}(k); (q', \phi(q)) \longmapsto q' + q + \sigma(q) \]
is a closed immersion, the homomorphism
\[ T_y(C^{(n-2)} \times E) \cong T_{y'}(C^{(n-3)} \times C \times E) = T_{y'}(C^{(n-3)} \times C^{(3)}) \cong T_{\beta^1(y)}(C^{(n)}) \]
is injective. We assume that \( \sigma(q_0) \neq q_0 \). Then by Lemma 4.3 we have \( \sigma(q) \notin \{ q_1, \ldots, q_{n-2-i} \} \). The point \( \tilde{y} = (q_1 + \cdots + q_{n-2-i} + iq_0, q_0) \in C^{(n-2)} \times C \) is not contained in the ramification divisor of the covering \( \pi : C^{(n-2)} \times C \to C^{(n-2)} \times E \), and the point \( \tilde{y}' = (q_1 + \cdots + q_{n-2-i} + (i+1)q_0, \sigma(q_0)) \in C^{(n-1)} \times C \) is not contained in the ramification divisor of the natural covering \( C^{(n-1)} \times C \to C^{(n)} \). Since the morphism
\[ C^{(n-2)}(k) \times C(k) \to C^{(n-1)}(k) \times C(k); (q_1 + \cdots + q_{n-2}, q) \longmapsto (q_1 + \cdots + q_{n-2} + q, \sigma(q)) \]
is a closed immersion, the homomorphism
\[ T_y(C^{(n-2)} \times E) \cong T_{y'}(C^{(n-2)} \times C) \hookrightarrow T_{y'}(C^{(n-1)} \times C) \cong T_{\beta^1(y)}(C^{(n)}) \]
is injective.

Let \( X'_\delta = \overline{\Psi^B_{J(C),\delta}(B_\delta \cap P \setminus D_{\delta,\text{sing}})} \) be the Zariski closure of the image of the restricted Gauss map \( \Psi^B_{J(C),\delta}(B_\delta \cap P \setminus D_{\delta,\text{sing}}) \in \mathbf{P}(H^0(E, \eta)^\vee) \).

**Lemma 5.11.** \( \nu_\delta(X'_\delta) = X'_\delta \).

**Proof.** By Lemma 5.8 we have
\[ \overline{\Psi^B_{J(C),\delta}(B_\delta \cap P \setminus D_{\delta,\text{sing}})} = \nu_\delta(\psi_\delta(Y_\delta^{\circ})) \subseteq \nu_\delta(X_\delta), \]
hence \( X'_\delta = \overline{\nu_\delta(\psi_\delta(Y_\delta^{\circ}))} \subseteq \nu_\delta(X_\delta) \) and \( Y_\delta^{\circ} \subseteq (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta) \). Since \( Y_\delta^{\circ} \) is dense in \( Y_\delta \), we have \( Y_\delta \subseteq (\nu_\delta \circ \psi_\delta)^{-1}(X'_\delta) \) and \( \nu_\delta(X_\delta) = (\nu_\delta \circ \psi_\delta)(Y_\delta) \subseteq X'_\delta \).

**Lemma 5.12.** If \( N(\delta) - \eta \notin J(E)_2 \setminus \{ 0 \} \), then \( \nu_\delta : X_\delta \to X'_\delta \) is the normalization of \( X'_\delta \).
Proof. We set morphisms $\alpha^\pm_\delta : E^{(n-3)} \times E \to E^{(n)}$, $\mu_\delta : E^{(n-3)} \times E \to E^{(n)}$ and $\nu^2_\delta : E^{(n-4)} \times E^{(2)} \to E^{(n)}$ by

$$\alpha^+_\delta : E^{(n-3)}(k) \times E(k) \to E^{(n)}(k);$$

$$\alpha^-_\delta : E^{(n-3)}(k) \times E(k) \to E^{(n)}(k);$$

$$\mu_\delta : E^{(n-3)}(k) \times E(k) \to E^{(n)}(k);$$

and

$$\nu^2_\delta : E^{(n-4)}(k) \times E^{(2)}(k) \to E^{(n)}(k);$$

$$(p_1 + \cdots + p_{n-3}, p) \mapsto p_1 + \cdots + p_{n-3} + 2p + t_{\eta-N(\delta)}(p),$$

$$(p_1 + \cdots + p_{n-3}, p) \mapsto p_1 + \cdots + p_{n-3} + 2p + t_{N(\delta)-\eta}(p),$$

$$(p_1 + \cdots + p_{n-3}, p) \mapsto p_1 + \cdots + p_{n-3} + p + t_{\eta-N(\delta)}(p) + t_{N(\delta)-\eta}(p)$$

By the natural inclusion $X_\delta' \subset \eta \subset E^{(n)}$, the subset

$$U = X_\delta' \setminus \{ \text{Image } (\alpha^+_\delta) \cup \text{Image } (\alpha^-_\delta) \cup \text{Image } (\mu_\delta) \cup \text{Image } (\nu^2_\delta) \},$$

is open dense in $X_\delta'$, where we consider as $\text{Image } (\nu^2_\delta) = \emptyset$ if $n = 3$. We show that the morphism

$$\nu_\delta : \nu^{-1}_\delta(U) \to E^{(n)} \setminus \{ \text{Image } (\alpha^+_\delta) \cup \text{Image } (\alpha^-_\delta) \cup \text{Image } (\mu_\delta) \cup \text{Image } (\nu^2_\delta) \}$$

is a closed immersion. For $u = p_1 + \cdots + p_n \in U(k)$, we assume that

$$p + t_{\eta-N(\delta)}(p) \leq p_1 + \cdots + p_n \quad \text{and} \quad p' + t_{\eta-N(\delta)}(p') \leq p_1 + \cdots + p_n$$

for some $p \neq p' \in E(k)$. Since $u \notin \text{Image } (\nu^2_\delta)$, we have

$$t_{\eta-N(\delta)}(p) = p' \quad \text{or} \quad p = t_{\eta-N(\delta)}(p'),$$

and furthermore $u \notin \text{Image } (\mu_\delta)$ implies that

$$t_{\eta-N(\delta)}(p) = p' \quad \text{and} \quad p = t_{\eta-N(\delta)}(p'),$$

hence $N(\delta) - \eta \in J(E)_2 \setminus \{0\}$. This means that $\nu_\delta : \nu^{-1}_\delta(U) \to U$ is bijective if $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$. In the following, we prove that the homomorphism

$$T_x(E^{(n-2)} \times E) \to T_{\nu_\delta(x)}(E^{(n)})$$

on the tangent spaces is injective for $x \in \nu^{-1}_\delta(U)$. Let $\tilde{\nu}_\delta : E^{(n-2)} \times E \to E^{(n-2)} \times E^{(2)}$ be the morphism defined by

$$\tilde{\nu}_\delta : E^{(n-2)}(k) \times E(k) \to E^{(n-2)}(k) \times E^{(2)}(k);$$

$$(p_1 + \cdots + p_{n-2}, p) \mapsto (p_1 + \cdots + p_{n-2}; p + t_{\eta-N(\delta)}(p)).$$

If $N(\delta) - \eta \notin J(E)_2 \setminus \{0\}$, then the morphism $\tilde{\nu}_\delta$ is a closed immersion. For $x \in \nu^{-1}_\delta(U)$, the image $\tilde{\nu}_\delta(x)$ is not contained in the ramification divisor of the natural covering $E^{(n-2)} \times E^{(2)} \to E^{(n)}$, because $\nu_\delta(x) \notin \text{Image } (\alpha^+_\delta) \cup \text{Image } (\alpha^-_\delta)$. Hence the homomorphism

$$T_x(E^{(n-2)} \times E) \hookrightarrow T_{\tilde{\nu}_\delta(x)}(E^{(n)} \times E^{(2)}) \cong T_{\nu_\delta(x)}(E^{(n)})$$
is invective. By Lemma 5.9, the finite birational morphism \( \nu_\delta : X_\delta \to X'_\delta \) gives the normalization of \( X'_\delta \).

Remark 5.13. If \( N(\delta) - \eta \in J(E)_2 \setminus \{0\} \), then \( \nu_\delta : X_\delta \to X'_\delta \) is a covering of degree 2.

5.3. The branch locus of the restricted Gauss maps. Let \( R_\delta \subset Y_\delta \) be the divisor defined by

\[
R_\delta(k) = \{(q_1 + \cdots + q_{n-2}, p) \in Y_\delta(k) \mid p_i = \sigma(p_j) \text{ for some } i \neq j\}.
\]

Lemma 5.14. \( \beta^1_\delta(R_\delta) \subset W_{\delta, \text{sing}} \).

Proof. It is a consequence of Lemma 4.3 because \( \beta^1_\delta(R_\delta) \subset B^2_\delta \).

Let \( S_{\delta,r} \subset Y_\delta \) be the divisor defined by

\[
S_{\delta,r}(k) = \{(q_1 + \cdots + q_{n-2}, p) \in Y_\delta(k) \mid q_1 + \cdots + q_{n-2} \geq r\}
\]

for \( r \in \text{Ram}(\phi) \). Then the ramification divisor of \( \psi_\delta : Y_\delta \to X_\delta \) is

\[
\text{Ram}(\psi_\delta) = R_\delta \cup \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r}.
\]

Lemma 5.15. \( \beta^1_\delta(S_{\delta,r}) \notin W_{\delta, \text{sing}} \), and moreover \( \beta^1_\delta(S_{\delta,r}) \cap W_{\delta, \text{sing}} = \emptyset \) if \( n = 3 \).

Proof. Let \( W^1_{\delta,r} \subset J(C) \) be the subvariety defined by

\[
W^1_{\delta,r}(k) = \{L \in \text{Pic}^0(C) \mid h^0(C, \mathcal{L} \otimes \mathcal{O}_C(-r) \otimes \delta) > 1\},
\]

and let \( T_{\delta,r} \subset J(C) \) be the image of the morphism

\[
\begin{align*}
C^{(n-3)}(k) \times E(k) &\longrightarrow \text{Pic}^0(C) = J(C)(k); \\
(q_1 + \cdots + q_{n-3}, \phi(q)) &\longmapsto \mathcal{O}_C(q_1 + \cdots + q_{n-3} + r + q + \sigma(q)) \otimes \delta^\vee.
\end{align*}
\]

Since \( C \) is not a hyperelliptic curve, by Martens’ theorem [12, Theorem 1], we have \( \dim W^1_{\delta,r} \leq n - 4 \) and \( T_{\delta,r} \nsubseteq W^1_{\delta,r} \), hence \( \dim T_{\delta,r} = n - 2 \). We remark that \( W^1_{\delta,r} = \emptyset \) in the case when \( n = 3 \). Since \( \beta^1_\delta(S_{\delta,r}) = T_{\delta,r} \cap P \) is the fiber of the composition

\[
T_{\delta,r} \subset J(C) \to J(E)
\]

at \( 0 \in J(E) \), we have \( \dim \beta^1_\delta(S_{\delta,r}) = n - 3 \). Let \( T_{\delta,2r} \subset T_{\delta,r} \) be the image of the morphism

\[
\begin{align*}
C^{(n-4)}(k) \times E(k) &\longrightarrow \text{Pic}^0(C) = J(C)(k); \\
(q_1 + \cdots + q_{n-4}, \phi(q)) &\longmapsto \mathcal{O}_C(q_1 + \cdots + q_{n-4} + 2r + q + \sigma(q)) \otimes \delta^\vee.
\end{align*}
\]

Since \( 2r = r + \sigma(r) \), we have \( T_{\delta,2r} \subset B^2_\delta \subset W_{\delta, \text{sing}} \) by Lemma 4.3. For \( L \in (T_{\delta,r}(k) \cap W_{\delta, \text{sing}}(k)) \setminus T_{\delta,2r}(k) \), there is \( (q_1 + \cdots + q_{n-3}, \phi(q)) \in C^{(n-3)}(k) \times E(k) \) such that \( L = \mathcal{O}_C(q_1 + \cdots + q_{n-3} + r + q + \sigma(q)) \otimes \delta^\vee \) and \( r \notin \{q_1, \ldots, q_{n-3}\} \). If \( q'_1 + \cdots + q'_{n-i} + ir \in |\Omega^1_C(-q_1 - \cdots - q_{n-3} - r - q - \sigma(q))| = |\Omega^1_C \otimes (L \otimes \delta)^\vee| \) and \( r \notin \{q'_1, \ldots, q'_{n-i}\} \), then by Lemma 5.2 the number \( i \) is odd. By the same way, any member in the linear system \( |\Omega^1_C(-q'_1 - \cdots - q'_{n-i} - ir)| = |L \otimes \delta| \) has an odd
multiplicity at \( r \). It implies that \( h^0(C, L \otimes \mathcal{O}_C(-r) \otimes \delta) = h^0(C, L \otimes \delta) > 1 \). Hence we have
\[
T_{\delta,r} \cap W_{\delta,\text{sing}} = T_{\delta,2r} \cup (T_{\delta,r} \cap W_{\delta,1}^{1}).
\]
When \( n = 3 \), it implies that \( T_{\delta,r} \cap W_{\delta,\text{sing}} = \emptyset \). When \( n \geq 4 \),
\[
\beta_\delta^{1}(S_{\delta,r}) \cap W_{\delta,\text{sing}} = (T_{\delta,2r} \cap P) \cup (\beta_\delta^{1}(S_{\delta,r}) \cap W_{\delta,1}^{1})
\]
is a proper closed subset of \( \beta_\delta^{1}(S_{\delta,r}) = T_{\delta,r} \cap P \), because \( \dim (T_{\delta,2r} \cap P) \leq n - 4 \) and \( \dim W_{\delta,1}^{1} \leq n - 4 \).

Let \( Z_{\delta,r} = \psi_\delta(S_{\delta,r}) \) be the image of \( S_{\delta,r} \) by \( \psi_\delta : Y_\delta \to X_\delta \). Then
\[
Z_{\delta,r}(k) = \{(p_1 + \cdots + p_{n-2}, p) \in X_\delta \mid \phi(r) = p_1 + \cdots + p_{n-2}\}.
\]

**Lemma 5.16.** If \( n \geq 4 \), then \( Z_{\delta,r} \) is irreducible. If \( n = 3 \), then \( X_\delta \cong E \), and \( Z_{\delta,r} \subset X_\delta \) is a \( J(X_\delta)_{2}\)-orbit by the natural action of \( J(X_\delta) \) on the curve \( X_\delta \) of genus 1.

**Proof.** Let \( \gamma_{\delta, p_0} : E^{(n-2)} \to J(E) \) be the morphism given in the proof of Lemma 5.9 for fixed \( p_0 \in E(k) \), and let \( i_r : E^{(n-3)} \to E^{(n-2)} \) be the morphism defined by
\[
i_r : E^{(n-3)}(k) \longrightarrow E^{(n-2)}(k); p_1 + \cdots + p_{n-3} \longmapsto p_1 + \cdots + p_{n-3} + \phi(r).
\]
If \( n \geq 4 \), then \( Z_{\delta,r} \) is a \( P^{n-4} \)-bundle over \( E \) by the base change
\[
\begin{array}{c c c c c c}
Z_{\delta,r} & \longrightarrow & X_\delta & \overset{\text{pr}_2}{\longrightarrow} & E \\
\downarrow & & \downarrow & & \downarrow \\
E^{(n-3)} & \longrightarrow & E^{(n-2)} & \overset{i_r}{\longrightarrow} & J(E) & \overset{\gamma_{\delta, p_0}}{\longrightarrow} & J(E)
\end{array}
\]
of the \( P^{n-4} \)-bundle \( \gamma_{\delta, p_0} \circ i_r : E^{(n-3)} \to J(E) \), hence \( Z_{\delta,r} \) is irreducible. If \( n = 3 \), then \( \text{pr}_2 : X_\delta \to E \) is an isomorphism, and
\[
Z_{\delta,r} \cong \{p \in E(k) \mid \mathcal{O}_E(\phi(r) + 2p) \cong N(\delta)\}
\]
is an orbit of \( J(E)_{2}\)-action.

We denote by \( \text{Ram}(\psi_\delta) \subset Y_\delta^0 \) the ramification divisor of \( \psi_\delta^0 = \psi_\delta|_{Y_\delta^0} : Y_\delta^0 \to X_\delta \).

**Lemma 5.17.**
\[
\overline{\psi_\delta^0(\text{Ram}(\psi_\delta^0))} = \bigcup_{r \in \text{Ram}(\phi)} Z_{\delta,r}.
\]

**Proof.** Since \( \text{Ram}(\psi_\delta) = R_\delta \cup \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r}, \) by Lemma 5.14, we have \( \text{Ram}(\psi_\delta^0) = \bigcup_{r \in \text{Ram}(\phi)} S_{\delta,r} \cap Y_\delta^0 \). By Lemma 5.13, \( S_{\delta,r} \cap Y_\delta^0 \neq \emptyset \) for \( n \geq 3 \), and \( S_{\delta,r} \cap Y_\delta^0 = S_{\delta,r} \) for \( n = 3 \). Since \( \psi_\delta \) is a finite morphism, \( \psi_\delta^0(S_{\delta,r} \cap Y_\delta^0) \) is of dimension \( n - 3 \). By Lemma 5.16, we have \( \overline{\psi_\delta^0(S_{\delta,r} \cap Y_\delta^0)} = \psi_\delta(S_{\delta,r}) = Z_{\delta,r}. \)

Let \( H_r \subset P(H^0(E, \eta)^\vee) \) be the hyperplane corresponding to the subspace
\[
H^0(E, \eta \otimes \mathcal{O}_E(-\phi(\eta))) \subset H^0(E, \eta).
\]

**Lemma 5.18.** \( H_r \) is the unique hyperplane with the property \( \nu_\delta(Z_{\delta,r}) \subset H_r. \).
Proof. The inclusion \( \nu_{\delta}(Z_{\delta,r}) \subset H_r \) is obvious. We prove the uniqueness of the hyperplane \( H_r \). Let \( z = (p_1 + \cdots + p_{n-3} + \phi(r), p) \in Z_{\delta,r}(k) \) be satisfying \( p_i \neq t_{\eta-N(\delta)}(p_j) \) for \( i \neq j \). We take a point \( p' \in E(k) \setminus \{ p \} \) such that \( \mathcal{O}_E(2p') \cong \mathcal{O}_E(2p) \) and \( \mathcal{O}_E(p' - p) \not\cong \eta \otimes N(\delta)^{\vee} \). Then \( z' = (p_1 + \cdots + p_{n-3} + \phi(r), p') \) is contained in \( Z_{\delta,r}(k) \), and \( \nu_{\delta}(z) \neq \nu_{\delta}(z') \). It implies the uniqueness in the case when \( n = 3 \).

When \( n \geq 4 \), we show that \( \nu_{\delta}(Z_{\delta,r}) \subset H_r \) is a non-linear hypersurface in \( H_r \). Let \( l \subset H_r \) be the line containing the two points \( \nu_{\delta}(z), \nu_{\delta}(z') \in \nu_{\delta}(Z_{\delta,r}) \). Then the line \( l \subset \mathbb{P}(H^0(E, \eta)^{\vee}) \) corresponds to the linear pencil

\[
|\eta(-p_1 - \cdots - p_{n-3} - \phi(r))| \subset |\eta| \cong \mathbb{P}(H^0(E, \eta)^{\vee}).
\]

For a point \( p_0 \in E(k) \), there is a unique point \( p'_0 \in E(k) \) such that \( p_0 + p'_0 \in |\eta(-p_1 - \cdots - p_{n-3} - \phi(r))| \). If \( \mathcal{O}_E(2p_0) \not\cong \mathcal{O}_E(2p), \mathcal{O}_E(2p'_0) \not\cong \mathcal{O}_E(2p) \) and

\[
p_0, p'_0 \notin \{ t_{\eta-N(\delta)}(p_1), \ldots, t_{\eta-N(\delta)}(p_{n-3}), t_{N(\delta)-\eta}(p_1), \ldots, t_{N(\delta)-\eta}(p_{n-3}) \},
\]

then the point \( p_1 + \cdots + p_{n-3} + \phi(r) + p_0 + p'_0 \in |\eta| \) on the line \( l \) is not contained in \( \nu_{\delta}(Z_{\delta,r}) \). \( \square \)

Lemma 5.19. The pull-back of the divisor \( H_r \) by \( \nu_{\delta} : X_{\delta} \to \mathbb{P}(H^0(E, \eta)^{\vee}) \) is

\[
\nu_{\delta}^* H_r = Z_{\delta,r} + M_{\delta,r} + M'_{\delta,r},
\]

where \( M_{\delta,y} \) and \( M'_{\delta,y} \) are irreducible divisors on \( X_{\delta} \) defined by

\[
M_{\delta,r}(k) = \{ (p_1 + \cdots + p_{n-2}, p) \in X_{\delta}(k) \mid p = \phi(r) \},
\]

\[
M'_{\delta,r}(k) = \{ (p_1 + \cdots + p_{n-2}, p) \in X_{\delta}(k) \mid p = t_{N(\delta)-\eta}(\phi(r)) \}.
\]

Proof. Let \( I_r \) be an irreducible divisor on \( E^{(n)} \) defined by

\[
I_r(k) = \{ p_1 + \cdots + p_n \in E^{(n)}(k) \mid p_1 + \cdots + p_n \geq \phi(r) \},
\]

and let \( Z_r, M_r, M'_r \) be irreducible divisors on \( E^{(n-2)} \otimes E \) defined by

\[
Z_r(k) = \{ (p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \otimes E(k) \mid p_1 + \cdots + p_{n-2} \geq \phi(r) \},
\]

\[
M_r(k) = \{ (p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \otimes E(k) \mid p = \phi(r) \},
\]

\[
M'_r(k) = \{ (p_1 + \cdots + p_{n-2}, p) \in E^{(n-2)}(k) \otimes E(k) \mid p = t_{N(\delta)-\eta}(\phi(r)) \}.
\]

Then the pull-back of the divisor \( I_r \) by the morphism

\[
E^{(n-2)}(k) \otimes E(k) \to E^{(n)}(k);
\]

\[
(p_1 + \cdots + p_{n-2}, p) \mapsto p_1 + \cdots + p_{n-2} + p + t_{\eta-N(\delta)}(p)
\]

is the divisor \( Z_r + M_r + M'_r \) on \( E^{(n-2)} \otimes E \). Since the restriction of \( I_r \) to \( |\eta| \subset E^{(n)} \) is the divisor \( H_r \) on \( \mathbb{P}(H^0(E, \eta)^{\vee}) \cong |\eta| \), the pull-back \( \nu_{\delta}^* H_r \) is the restriction of \( Z_r + M_r + M'_r \) to \( X_{\delta} \). \( \square \)

Corollary 5.20. \( \nu_{\delta}^* H_r - Z_{\delta,r} \) is an irreducible divisor on \( X_{\delta} \) if and only if \( N(\delta) = \eta \).

We consider the dual variety \( (\Phi_{[\eta]}(E))^\vee \subset \mathbb{P}(H^0(E, \eta)^{\vee}) \) of the image of the closed immersion \( \Phi_{[\eta]} : E \to \mathbb{P}(H^0(E, \eta)) \).
Lemma 5.21. The projective curve $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta))$ is reflexive. In particular, $\Phi_{|\eta|}(E) = ((\Phi_{|\eta|}(E))^\vee)^\vee \subset \mathbf{P}(H^0(E, \eta))$.

Proof. If $1 \leq i < n$, then $h^0(E, \eta \otimes \mathcal{O}_E(-ip)) = n - i$ for any $p \in E(k)$. If $n = 3$, then $h^0(E, \eta \otimes \mathcal{O}_E(-3p)) = 0$ for general $p \in E(k)$. Hence $h^0(E, \eta \otimes \mathcal{O}_E(-2p)) > h^0(E, \eta \otimes \mathcal{O}_E(-3p))$ for general $p \in E(k)$. Then there is a hyperplane $H \subset \mathbf{P}(H^0(E, \eta))$ which intersects $\Phi_{|\eta|}(E)$ at $\Phi_{|\eta|}(p)$ with the multiplicity 2. By [3 (3.5)], $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta))$ is reflexive, because the characteristic of the base field $k$ is not equal to 2. □

Lemma 5.22. If $N(\delta) = \eta$, then the dual variety of $X^t_\delta \subset \mathbf{P}(H^0(E, \eta)^\vee)$ is $\Phi_{|\eta|}(E) \subset \mathbf{P}(H^0(E, \eta))$.

Proof. By Lemma 5.21 we show that the dual variety of $\Phi_{|\eta|}(E)$ is $X^t_\delta$. For $L \in (B_\delta \cap P) \setminus D_{\delta, \text{sing}} \subset \text{Pic}^0(C)$, there is a unique effective divisor $q_1 + \cdots + q_{n-2} + \sigma(q) \in C^{(\nu)}(k)$ such that $L \otimes \delta \cong \mathcal{O}_C(q_1 + \cdots + q_{n-2} + \sigma(q))$.

Since $L \in P(k)$, we have

$$\eta = N(\delta) = [\mathcal{O}_E(\phi(q_1) + \cdots + \phi(q_{n-2}) + 2\phi(q))],$$

hence

$$\Omega^1_C \otimes L^\vee \otimes \delta^\vee \cong \phi^* \eta \otimes L^\vee \otimes \delta^\vee \cong \mathcal{O}_C(\sigma(q_1) + \cdots + \sigma(q_{n-2}) + \sigma(q) + q)$$

and $\Psi^B_{J(C), \delta}(L) \in \mathbf{P}(H^0(E, \eta)^\vee)$ is defined by the effective divisor

$$\phi(q_1) + \cdots + \phi(q_{n-2}) + 2\phi(q) \in |\eta| \cong \mathbf{P}(H^0(E, \eta)^\vee).$$

It means that the hyperplane in $\mathbf{P}(H^0(E, \eta))$ corresponding $\Psi^B_{J(C), \delta}(L)$ is tangent to the image $\Phi_{|\eta|}(E)$. Hence we have

$$\Psi^B_{J(C), \delta}((B_\delta \cap P) \setminus D_{\delta, \text{sing}}) \subset (\Phi_{|\eta|}(E))^\vee.$$

Since $(\Phi_{|\eta|}(E))^\vee$ and $\Psi^B_{J(C), \delta}((B_\delta \cap P) \setminus D_{\delta, \text{sing}})$ are irreducible hypersurfaces in $\mathbf{P}(H^0(E, \eta)^\vee)$, we have $X^t_\delta = (\Phi_{|\eta|}(E))^\vee$. □

6. Key Propositions

Let $\mathcal{L}$ be an ample invertible sheaf on $P$ which represents the the polarization $\lambda_P$.

Lemma 6.1. $U_D = Bs |\mathcal{L}| \setminus D_{\text{sing}}$ is nonsingular for any $D \in |\mathcal{L}|$.

Proof. Since $D = D_\delta$ for some $\delta \in \text{Pic}^n(C)$, it is a consequence of Lemma 4.6 and Lemma 5.10. □

Let

$$\Psi_D : D \setminus D_{\text{sing}} \longrightarrow \mathbf{P}^{n-1} = \text{Grass}(n - 1, H^0(P, \Omega_1^1)^\vee)$$

be the Gauss map for $D \in |\mathcal{L}|$, and let $\nu_D : X_D \rightarrow X^t_D$ be the normalization of $X^t_D = \overline{\Psi_D(U_D)} \subset \mathbf{P}^{n-1}$. Then by Lemma 6.1 there is a unique morphism
Let $s$ be a member in $\mathcal{L}\setminus \Pi_{\mathcal{L}}$, where $\Pi_{\mathcal{L}} \subset \mathcal{L}$ is the subset in Lemma 4.8.

Proposition 6.2. Let $D \subset P$ be a member in $|\mathcal{L}| \setminus \Pi_{\mathcal{L}}$, where $\Pi_{\mathcal{L}} \subset |\mathcal{L}|$ is the subset in Lemma 4.8.

(1) If $n = 3$, then $X_D$ is a nonsingular projective curve of genus 1, and $Z_D$ is a disjoint union of 6 orbits $Z_{D,1}, \ldots, Z_{D,6}$ by the $J(X_D)_2$-action.

(2) If $n \geq 4$, then $Z_D$ has $2n$ irreducible components $Z_{D,1}, \ldots, Z_{D,2n}$.

(3) For any subset $Z_{D,i} \subset Z_D$ in (1) and (2), there is a unique hyperplane $H_{D,i} \subset \mathbb{P}^{n-1}$ such that $\nu_D(Z_{D,i}) \subset H_{D,i}$.

Proof. By Lemma 4.7, there is $\delta \in \text{Pic}^0(C)$ such that $N(\delta) = \eta$ and $\mathcal{L} \cong \mathcal{L}_{\delta}$.

By Lemma 4.6, there is $s \in \text{Pic}^0(E)$ such that $D = D_{\delta + s}$. By the proof of Lemma 4.8, $D \notin \Pi_{\mathcal{L}}$ implies $s \notin J(E)_4 \setminus J(E)_2$. By Lemma 5.7, the Gauss map $\Psi_D : U_D \to \mathbb{P}^{n-1}$ is identified with $\Psi^0_{(C),\delta + s} : U_D \to \mathbb{P}(H^0(E, \eta)^\vee)$.

Since $N(\delta + \phi^*s) - \eta = 2s \notin J(E)_2 \setminus \{0\}$, by Lemma 5.11 and Lemma 5.12, the normalization of $X_{\delta + \phi^*s} = X_D'$ is given by $\nu_{\delta + \phi^*s} : X_{\delta + \phi^*s} \to X_{\delta + \phi^*s}$, and by Lemma 5.8 and Lemma 5.10, $\psi_D : U_D \to \mathbb{P}^{n-1}$ is identified with $\psi_{\delta + \phi^*s} : Y_{\delta + \phi^*s} \to X_{\delta + \phi^*s}$. Hence the statements (1), (2) and (3) are consequence of Lemma 5.16, Lemma 5.17 and Lemma 5.18.

We define the subset $\Pi'_{\mathcal{L}}$ in the linear pencil $|\mathcal{L}|$ by

$$\Pi'_{\mathcal{L}} = \{D \in |\mathcal{L}| \setminus \Pi_{\mathcal{L}} \mid \nu_D H_{D,i} - Z_{D,i} \text{ is irreducible for } 1 \leq i \leq 2n\}.$$  

Lemma 6.3. $\sharp \Pi'_{\mathcal{L}} = 4$.

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. Then by Corollary 5.20, $D = D_{\delta + s} \in \Pi'_{\mathcal{L}} \iff N(\delta + \phi^*s) = \eta \iff s \in J(E)_2$,

and by Lemma 4.4, we have $\sharp \Pi'_{\mathcal{L}} = \sharp J(C)_2 = 4$.  

Let $e_1 + \cdots + e_{2n}$ be the branch divisor of the original covering $\phi : C \to E$, and let $\eta \in \text{Pic}(E)$ be the invertible sheaf with $\phi^*\eta \cong \Omega^1_E$.

Proposition 6.4. For any member $D \in \Pi'_{\mathcal{L}}$, there is an isomorphism

$$(E, e_1 + \cdots + e_{2n}, \eta) \cong ((X_D')^\vee, H_{D,1}^\vee + \cdots + H_{D,2n}^\vee, O_{(\mathbb{P}^{n-1})^\vee}(1)|_{(X_D')^\vee}),$$

where $H_{D,i}^\vee \in (\mathbb{P}^{n-1})^\vee$ is the point corresponding to the hyperplane $H_{D,i}$, and $(X_D')^\vee \subset (\mathbb{P}^{n-1})^\vee$ is the dual variety of $X_D' \subset \mathbb{P}^{n-1}$.

Proof. We use the same identification for Gauss maps as in the proof of Proposition 6.2. When $D \in \Pi'_{\mathcal{L}}$, we may assume that $D = D_{\delta}$ and $N(\delta) = \eta$ by Corollary 5.20. Then the point $H_{D,i}^\vee$ is identified with the point $H_{\phi(r)}^\vee = \Phi_{\phi(r)}(\phi(r))$ for $r \in \text{Ram}(\phi)$, and $(X_D')^\vee$ is identified with $(X_{\delta}')^\vee$, which coincides with $\Phi_{\phi(r)}(E) \subset \mathbb{P}(H^0(E, \eta))$ by Lemma 5.22.
Remark 6.5. For a member $D \in \Pi'_{\mathcal{L}}$ the Gauss map $\Psi_D : D \setminus D_{\text{sing}} \to \mathbb{P}^{n-1}$ is of degree $2^n$, and $X'_D + \sum_{i=1}^{2n} H_{D,i}$ is the branch divisor of $\Psi_D$. But for $D \notin \Pi'_{\mathcal{L}}$ the Gauss map $\Psi_D$ is not easy to compute.

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