Quasi-solutions of genuinely nonlinear forward-backward ultra-parabolic equations

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Abstract. In the present paper we have proved the existence of quasi-solutions of genuinely nonlinear forward-backward ultra-parabolic equations. Quasi-solutions are obtained with the help of the vanishing anisotropic temporal diffusion method. Moreover, at the present stage of our research we assume that various choices of temporal artificial diffusion coefficients lead to entropy solutions or to quasi-solutions. The latter assumption is the subject of our further scientific research.

Key words: entropy solution, forward-backward ultra-parabolic equation, genuine nonlinearity condition, kinetic solution

1. Introduction

Ultra-parabolic equations are used in the theory of boundary layers and in the mathematical models of Brownian motion [1–3]. Entropy solutions of nonlinear ultra-parabolic equations were studied in [4–7], see references therein. It is important to note that entropy solutions were firstly obtained in [8] for hyperbolic differential equations and later extended to various types of partial differential equations.

The research on the well-posedness of nonlinear forward-backward parabolic equations was started in [9–14]. Moreover, the history of the research on the linear case was described in [15].

Here we apply the vanishing anisotropic temporal diffusion (viscosity) method [16]. The technique of elliptic regularization was invented in [17] during the study of degenerate parabolic equations, and subsequently adapted for the Navier-Stokes equations [18], and for forward-backward parabolic equations [9, 11]. Independently, the elliptic regularization of hyperbolic nonlinear equations was studied in [19, 20]. Furthermore, singular limits of anisotropic elliptic perturbations were obtained in [21, 22]. Also, for hyperbolic equations the vanishing viscosity method with gradient dependent viscosity was applied in [23, 24].

In this paper we have obtained quasi-solutions of forward-backward ultra-parabolic equations. In comparison with [25] \((p_1 = p_2 = 2)\), we have not obtained entropy solutions when \(|p_1−2|+|p_2−2|\neq 0.\) It is important to mention that quasi-solutions for hyperbolic equations were studied in [26–30] and, correspondingly, for ultra-parabolic equations [31]. In [27] it was shown that several types of regularization can lead to quasi-solutions but not to entropy solutions.

We have obtained a quasi-solution with the help of a corresponding sequence of weak solutions \(\{u_\varepsilon\}_{\varepsilon>0}\) to problem \(\Pi_\varepsilon\) as \(\varepsilon \to 0+\). The main difference from previous results on quasi-solutions is that only the right-hand sides of kinetic boundary conditions (4.7b)–(4.7e) are not
sign-defined. A similar type of quasi-solutions for forward-backward p-parabolic equations was obtained in [32]. In the present paper we have used the vanishing diffusion method with the help of anisotropic $p$-Laplacian, $p = (2, \ldots, 2, p_1, p_2) \in \mathbb{R}^{d+2}$, $p_1, p_2 > 1$. The physical meaning of the vanishing anisotropic diffusion method is that we take into account fast and slow diffusive regimes when $|p_1 - 2| + |p_2 - 2| \neq 0$, see [33, Chapter 5].

This paper is organized as follows. In section 2 we formulate the non-homogeneous Dirichlet problem $\Pi_0$. Since this problem is ill-posed, in section 3 we formulate problem $\Pi_\varepsilon$. Quasi-solutions to problem $\Pi_0$ are obtained as singular limits of weak solutions to problem $\Pi_\varepsilon$ as $\varepsilon \to 0+$, see sections 4 and 5. We have not proved yet that singular limits of weak solutions to problem $\Pi_\varepsilon$ are also entropy solutions to problem $\Pi_0$ even if $|p_1 - 2| + |p_2 - 2| \neq 0$. So, it is still an open question (see Remark 4).

2. Genuinely nonlinear forward-backward ultra-parabolic equation

Let scalar functions $a(z), b(z)$ and vector function $\varphi(z)$ satisfy the following conditions.

Conditions on $a, b, \varphi$. Let $a, b \in C^2(\mathbb{R})$, $a(0) = b(0) = 0$, $\varphi(z) = (\varphi_1(z), \ldots, \varphi_d(z))$, $z \in \mathbb{R}$, $\varphi_j \in C^2(\mathbb{R})$, $j = 1, \ldots, d$, $\varphi(0) = 0$. Function $a$ is non-monotonic. Moreover, $a'$ and $b'$ satisfy the genuine nonlinearity condition

$$\text{mes}\{\lambda \in \mathbb{R} : a'(\lambda)\theta + b'(\lambda)\vartheta = 0\} = 0$$

for every $(\theta, \vartheta) \in \mathbb{S}^1$.

Under Conditions on $a, b, \varphi$, we are going to formulate boundary value problem $\Pi_0$. Let $G_{T,S} = \Omega \times (0,T) \times (0,S)$, $\Gamma_0 = \overline{\Omega} \times \{t = 0\} \times [0,S]$, $\Gamma_T = \overline{\Omega} \times \{t = T\} \times [0,S]$, $\Xi_0 = \overline{\Omega} \times [0,T] \times \{s = 0\}$, $\Xi_S = \overline{\Omega} \times [0,T] \times \{s = S\}$, $\Gamma_i = \partial G_{T,S}\setminus(\Gamma_0 \cup \Gamma_T \cup \Xi_0 \cup \Xi_S)$, a bounded domain $\Omega \subset \mathbb{R}^d$ (mes $\Omega < \infty$) has $C^1$-smooth boundary $\partial \Omega$. Here we deal with anisotropic Sobolev spaces, see, for example, [34]. The anisotropic Sobolev space $W_0^{1,p}(G_{T,S})$ is equipped with the norm

$$\|u\|_{W_0^{1,p}(G_{T,S})} = \sum_{i=1}^{d} \|\partial_{x_i} u\|_{L^2(G_{T,S})} + \|\partial_t u\|_{L^p(G_{T,S})} + \|\partial_{\varphi_i} u\|_{L^p(G_{T,S})} + \|u\|_{L^1(G_{T,S})},$$

where $p = (2, \ldots, 2, p_1, p_2) \in \mathbb{R}^{d+2}$, $p_1, p_2 > 1$.

Problem $\Pi_0$. For arbitrary initial and final conditions $u_0^\Gamma, u_T^\Gamma \in C_0^{1,\alpha}(\Omega \times (0,S))$, $u_0^S, u_S^S \in C_0^{1,\alpha}(\Omega \times (0,T))$, $\alpha \in (0,1)$, the unknown function $u : G_{T,S} \to \mathbb{R}$ satisfies

$$\partial_t a(u) + \partial_s b(u) + \text{div}_x \varphi(u) = \Delta_x u, \quad (x, t, s) \in G_{T,S},$$

$$u|_{\Gamma_0} \approx u_0^\Gamma(x, s), \quad (x, s) \in \Omega \times (0,S),$$

$$u|_{\Gamma_T} \approx u_T^\Gamma(x, s), \quad (x, s) \in \Omega \times (0,S),$$

$$u|_{\Xi_0} \approx u_0^\Sigma(x, t), \quad (x, t) \in \Omega \times (0,T),$$

$$u|_{\Xi_S} \approx u_S^\Sigma(x, t), \quad (x, t) \in \Omega \times (0,T),$$

$$u|_{\Gamma_i} = 0,$$

in the form given in Definition 3 from section 4. The sign $\approx$ means the equality only on a part of the boundary.

Remark 1. We formulate equation (2.2a) in the sense of distributions. Since function $a(z)$ is non-monotonic on $\mathbb{R}$, equation (2.2a) is a forward-backward ultra-parabolic equation. Moreover, a quasi-solution $u$ can deviate from initial and final data $u_0^\Gamma, u_T^\Gamma, u_0^\Sigma$ and $u_S^\Sigma$, see Remark 5. Therefore, the difficulty of problem $\Pi_0$ is that equation (2.2a) and initial and final conditions (2.2b)–(2.2e) should be reformulated in the form of kinetic equalities given in Definition 3.
3. Anisotropic elliptic regularization

We are going to construct a quasi-solution as a singular limit of weak solutions $u_\varepsilon$ to the non-homogeneous Dirichlet problem $\Pi_\varepsilon$ as $\varepsilon \to 0+$.

**Problem $\Pi_\varepsilon$.** For arbitrary initial and final conditions $u_0^\varepsilon, u_T^\varepsilon \in C^{1,\alpha}_0(\Omega \times (0, S))$, $u_0, u_S^\varepsilon \in C^{1,\alpha}_0(\Omega \times (0, T))$, $\alpha \in (0, 1)$, the unknown function $u_\varepsilon$ satisfies the boundary value problem

$$
\begin{align*}
\partial t u_\varepsilon + \partial_s b(u_\varepsilon) + \text{div}_x \varphi(u_\varepsilon) &= \Delta_x u_\varepsilon + \varepsilon \partial_t (|\partial_t u_\varepsilon|^{p_1 - 2} \partial_t u_\varepsilon) \\
+ \varepsilon \partial_s (|\partial_s u_\varepsilon|^{p_2 - 2} \partial_s u_\varepsilon), & \quad (x, t, s) \in G_{T,S},
\end{align*}
$$

(3.3a)

$$
\begin{align*}
u_\varepsilon|_{\Gamma_0} = u_0^\varepsilon, \quad \nu_\varepsilon|_{\Gamma_T} = u_T^\varepsilon, \quad \nu_\varepsilon|_{\Gamma_S} = u_S^\varepsilon, \quad \nu_\varepsilon|_{\Gamma_t} = 0,
\end{align*}
$$

(3.3b)

in a weak sense, see Definition 1.

We assume here that $\varepsilon \in (0, 1]$, $p_1, p_2 > 1$. Let $V^{1,p}(G_{T,S}) = \{v \in W^{1,p}(G_{T,S}) : v|_{\Gamma_t} = 0\}$.

**Definition 1.** Function $u_\varepsilon \in L^\infty(G_{T,S}) \cap V^{1,p}(G_{T,S})$ is called a weak solution to problem $\Pi_\varepsilon$ if the following demands hold:

1. Let $\tilde{u} \in L^\infty(G_{T,S}) \cap V^{1,p}(G_{T,S})$ be an extension of functions $u_0^\varepsilon, u_T^\varepsilon, u_0^\varepsilon$ and $u_S^\varepsilon$ into $G_{T,S}$ such that $u_\varepsilon - \tilde{u} \in L^\infty(G_{T,S}) \cap W^{1,p}(G_{T,S})$.

2. The following equality holds

$$
\int_{G_{T,S}} (-a(u_\varepsilon)\partial_t \phi - b(u_\varepsilon)\partial_s \phi - \varphi(u_\varepsilon) \cdot \nabla_x \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon |\partial_t u_\varepsilon|^{p_1 - 2} \partial_t u_\varepsilon \partial_t \phi \\
+ \varepsilon |\partial_s u_\varepsilon|^{p_2 - 2} \partial_s u_\varepsilon \partial_s \phi) \, dx \, dt \, ds = 0
$$

(3.4a)

for every $\phi \in L^\infty(G_{T,S}) \cap W^{1,p}(G_{T,S})$.

**Remark 2.** We can reformulate (3.4a) in the equivalent way:

$$
\int_{G_{T,S}} (\partial_t a(u_\varepsilon) \phi + \partial_s b(u_\varepsilon) \phi + \text{div}_x \varphi(u_\varepsilon) \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon |\partial_t u_\varepsilon|^{p_1 - 2} \partial_t u_\varepsilon \partial_t \phi \\
+ \varepsilon |\partial_s u_\varepsilon|^{p_2 - 2} \partial_s u_\varepsilon \partial_s \phi) \, dx \, dt \, ds = 0
$$

(3.4b)

**Remark 3.** We assume that extension $\tilde{u}$ into $G_{T,S}$ exists if $u_0^\varepsilon, u_T^\varepsilon, u_0^\varepsilon$ and $u_S^\varepsilon$ are from $C^{1,\alpha}_0$, $0 < \alpha < 1$. In the case $p_1 = p_2 = 2$ we deal with entropy solutions (see Remark 4) and with the help of [25, Theorem 1] we can decrease the smoothness of initial and final data: $u_0^\varepsilon, u_T^\varepsilon \in L^\infty(\Omega \times (0, S))$, $u_0, u_S^\varepsilon \in L^\infty(\Omega \times (0, T))$.

**Proposition 1.** Under Conditions on $a, b \& \varphi$, problem $\Pi_\varepsilon$ has at least one weak solution $u_\varepsilon$ for all $u_0^\varepsilon, u_T^\varepsilon \in C^{1,\alpha}_0(\Omega \times (0, S))$, $u_0, u_S^\varepsilon \in C^{1,\alpha}_0(\Omega \times (0, T))$, $\alpha \in (0, 1)$. Moreover, the maximum principle

$$
\|u_\varepsilon\|_{L^\infty(G_{T,S})} \leq M = \max \left(\|u_0^\varepsilon\|_{L^\infty(\Omega \times (0, S))}, \|u_T^\varepsilon\|_{L^\infty(\Omega \times (0, S))}, \|u_0\|_{L^\infty(\Omega \times (0, T))}, \|u_S^\varepsilon\|_{L^\infty(\Omega \times (0, T))}\right),
$$

(3.5)

and the energy estimate

$$
\int_{G_{T,S}} (|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_t u_\varepsilon|^{p_1} + \varepsilon |\partial_s u_\varepsilon|^{p_2}) \, dx \, dt \, ds < C
$$

(3.6)

hold. The constant $C$ does not depend on $\varepsilon \in (0, 1]$.

This proposition can be proved with the help of results formulated in [25, 35].
4. Kinetic formulation of forward-backward ultra-parabolic equations

In this section we deal with the kinetic formulation of forward-backward ultra-parabolic equations. Here we use methods developed in [25–27,31,36–41].

Consider the function $\chi$ which is defined in the following way

$$
\chi(\lambda; v) = \begin{cases} 
+1, & \text{if } 0 < \lambda < v, \\
-1, & \text{if } v < \lambda < 0, \\
0, & \text{elsewhere.}
\end{cases}
$$

**Definition 2.** Let $N$ be a positive integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and the function $h \in L^\infty(\mathcal{O} \times (-L, L))$ satisfying $0 \leq h(z, \lambda)\text{sgn}(\lambda) \leq 1$ for almost every $(z, \lambda) \in \mathbb{R}^{N+1}$. It is said that $h$ is a $\chi$-function if there exists a function $v \in L^\infty(\mathcal{O})$ such that

$$
h(z, \lambda) = \chi(\lambda; v(z))
$$

for a.e. $z \in \mathcal{O}$. Note that $v(z) = \int_{-L}^{L} h(z, \lambda) \, d\lambda = \int_{-L}^{L} \chi(\lambda; v(z)) \, d\lambda$.

The following lemma formulated and proved in [40] guarantees the link between sequences of $\chi$-functions and their limits.

**Lemma 1.** Let $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and $h_n \in L^\infty(\mathcal{O} \times (-L, L))$ be a sequence of $\chi$-functions converging weakly to $h \in L^\infty(\mathcal{O} \times (-L, L))$. We define $v_n(\cdot) = \int_{-L}^{L} h_n(\cdot, \lambda) \, d\lambda$ and $v(\cdot) = \int_{-L}^{L} h(\cdot, \lambda) \, d\lambda$. Then the three assertions are equivalent:

- $h_n$ converges strongly to $h$ in $L^1_{\text{loc}}(\mathcal{O} \times (-L, L))$,
- $v_n$ converges strongly to $v$ in $L^1_{\text{loc}}(\mathcal{O})$,
- $h$ is a $\chi$-function.

**Definition 3.** Function $u : G_{T,S} \to \mathbb{R}$ is called a quasi-solution of problem $\Pi_0$ if it satisfies the following equations:

1. (Kinetic equation)

$$
a'(\lambda)\partial_t \chi(\lambda; u) + b'(\lambda)\partial_s \chi(\lambda; u) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u) = \Delta_x \chi(\lambda; u) + \partial_x (m(x, t, s, \lambda) + n(x, t, s, \lambda)),
$$

2. (Kinetic boundary conditions)

$$
a'(\lambda) \left( \chi(\lambda; u_0^{\tau} \chi(x, s)) - \chi(\lambda; u_0^{\tau} \chi(x, s)) \right) \\
- \delta_{(\lambda = u_0^{\tau} \chi(x, s))} \left( a(u_0^{\tau} \chi(x, s)) - a(u_0^{\tau} \chi(x, s)) \right) = \partial_s \mu_0^{\tau}(x, s, \lambda),
$$

$$
a'(\lambda) \left( \chi(\lambda; u_T^{\tau} \chi(x, s)) - \chi(\lambda; u_T^{\tau} \chi(x, s)) \right) \\
- \delta_{(\lambda = u_T^{\tau} \chi(x, s))} \left( a(u_T^{\tau} \chi(x, s)) - a(u_T^{\tau} \chi(x, s)) \right) = -\partial_s \mu_T^{\tau}(x, s, \lambda),
$$

$$
b'(\lambda) \left( \chi(\lambda; u_0^{\tau} \chi(x, t)) - \chi(\lambda; u_0^{\tau} \chi(x, t)) \right) \\
- \delta_{(\lambda = u_0^{\tau} \chi(x, t))} \left( b(u_0^{\tau} \chi(x, t)) - b(u_0^{\tau} \chi(x, t)) \right) = \partial_t \mu_0^{\tau}(x, t, \lambda),
$$

$$
b'(\lambda) \left( \chi(\lambda; u_S^{\tau} \chi(x, t)) - \chi(\lambda; u_S^{\tau} \chi(x, t)) \right) \\
- \delta_{(\lambda = u_S^{\tau} \chi(x, t))} \left( b(u_S^{\tau} \chi(x, t)) - b(u_S^{\tau} \chi(x, t)) \right) = -\partial_t \mu_S^{\tau}(x, t, \lambda),
$$

$$
(4.7a)

$$
(4.7b)

$$
(4.7c)

$$
(4.7d)

$$
(4.7e)
where \( \mu^0_G, \mu^T_G \in \mathcal{M}(\Omega \times (0, S) \times (-M, M)) \), \( \mu^0_S, \mu^T_S \in \mathcal{M}(\Omega \times (0, T) \times (-M, M)) \), \( n, m \in \mathcal{M}^+(G_{T,S} \times (-M, M)) \), \( n = \delta_{(\lambda=0)}|\nabla_x u|^2 \)

**Remark 4.** Kinetic boundary conditions (4.7b)–(4.7e) are deduced with help of methods proposed in [36, 38]. When \( \mu^0_G, \mu^T_G, \mu^0_S \) and \( \mu^T_S \) are positive measures, Definition 3 corresponds to an entropy solution to problem \( \Pi_0 \), see [25]. Moreover, in the case \( p_1 = p_2 = 2 \) an entropy solution is a singular limit of weak solutions to problem \( \Pi_0 \) as \( \varepsilon \to 0+ \). When \( |p_1 - 2| + |p_2 - 2| \neq 0 \), we can only prove that \( u = \lim \limits_{\varepsilon \to 0+} u_\varepsilon \) is a quasi-solution to problem \( \Pi_0 \).

**Remark 5.** In Definition 3 we have used the fact that a quasi-solution to problem \( \Pi_0 \) has traces \( u^0_\varepsilon, u^T_\varepsilon, u^S_\varepsilon \), and \( u^0_\varepsilon, u^T_\varepsilon, u^S_\varepsilon \) in the \( L^1 \) sense, see [25, 28, 29, 31].

The main result of the present paper is the following theorem.

**Theorem 1.** Under Conditions \( a, b, c, \varphi \), problem \( \Pi_0 \) has at least one quasi-solution.

When \( p_1 = p_2 = 2 \), it was proved in [25] that \( u_\varepsilon \) converges in \( L^1(G_{T,S}) \) to the unique entropy solution of problem \( \Pi_0 \). When \( |p_1 - 2| + |p_2 - 2| \neq 0 \), the form of a quasi-solution to problem \( \Pi_0 \) implies its nonuniqueness.

5. Brief scheme of the proof of Theorem 1

In order to prove Theorem 1, we partly repeat results from [25, 37]. Using kinetic formulation of (2.2a), the genuine nonlinearity condition (2.1), the precompactness of \( \int_{-M}^M \chi(\lambda; u_\varepsilon) \, d\lambda \) (see [41]) and Lemma 1, we prove that \( \{u_\varepsilon\}_{\varepsilon > 0} \) is a precompact set in \( L^1(G_{T,S}) \).

We need to introduce the convex entropy flux pair \( (\eta, q) \):

\[
q'_\varepsilon(z) = \varphi'(z)\eta'(z), \quad q'_a(z) = a'(z)\eta'(z), \quad q'_b(z) = b'(z)\eta'(z), \quad \eta''(z) \geq 0, \\
q(z) = (q_a(z), q_b(z)), \quad z \in \mathbb{R}.
\]

The entropy inequality

\[
\partial_t q_a(u) + \partial_s q_b(u) + \text{div}_x q_a(u) - \Delta_x \eta(u) \leq -\eta''(u)|\nabla_x u|^2 \tag{5.8}
\]

is valid for every convex entropy flux pair \( (\eta, q) \). We get this inequality from

\[
\partial_t q_a(u_\varepsilon) + \partial_s q_b(u_\varepsilon) + \text{div}_x q_a(u_\varepsilon) = \Delta_x \eta(u_\varepsilon) + \varepsilon \partial_t(|\partial_t u_\varepsilon|^{p_1-2}\partial_t \eta(u_\varepsilon)) \\
+ \varepsilon \partial_s(|\partial_s u_\varepsilon|^{p_2-2}\partial_s \eta(u_\varepsilon)) - \eta''(u_\varepsilon)(|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_t u_\varepsilon|^{p_1} + \varepsilon |\partial_s u_\varepsilon|^{p_2}),
\]

which is deduced by putting \( \phi = \eta'(u_\varepsilon) \gamma \) in (3.4b), where \( \gamma \) is an arbitrary nonnegative finite test function in \( G_{T,S} \).

Using the kinetic formulation, from (5.8) we obtain

\[
\int_{-M}^M \eta'(\lambda)(a'(\lambda)\partial_t \chi(\lambda; u) + b'(\lambda)\partial_s \chi(\lambda; u) + \varphi'(\lambda) \cdot \nabla_x \chi(\lambda; u) - \Delta_x \chi(\lambda; u)) \, d\lambda = \\
\int_{-M}^M \eta''(\lambda)(m + n) \, d\lambda = -\int_{-M}^M \eta''(\lambda)(m + n) \, d\lambda \leq 0. \tag{5.9}
\]

Kinetic equation (4.7e) follows from (5.9) due to arbitrariness of \( \eta' \).
Furthermore, kinetic boundary conditions (4.7b)–(4.7e) are deduced from the following equalities

\[ q_\alpha(u_0^\tau(x, s)) - q_\alpha(u_0^\tau(x, s)) - \eta'(u_0^\tau(x, s))(a(u_0^\tau(x, s)) - a(u_0^\tau(x, s))) = \]

\[ \int_{-M}^{M} \eta'(\lambda)(a'(\lambda)(\chi(\lambda; u_0^\tau(x, s))) - \chi(\lambda; u_0^\tau(x, s))) - a(u_0^\tau(x, s)) \delta(\lambda = u_0^\tau(x, s)) d\lambda = \]

\[ - \int_{-M}^{M} \eta'(\lambda) \partial_\lambda \mu_0^\tau(x, s, \lambda) d\lambda = \int_{-M}^{M} \eta''(\lambda) \mu_0^\tau(x, s, \lambda) d\lambda, \]

\[ q_\beta(u_0^\tau(x, s)) - q_\beta(u_0^\tau(x, s)) - \eta'(u_0^\tau(x, s))(a(u_0^\tau(x, s)) - a(u_0^\tau(x, s))) = \]

\[ \int_{-M}^{M} \eta'(\lambda)(a'(\lambda)(\chi(\lambda; u_0^\tau(x, s))) - \chi(\lambda; u_0^\tau(x, s))) - a(u_0^\tau(x, s)) \delta(\lambda = u_0^\tau(x, s)) d\lambda = \]

\[ - \int_{-M}^{M} \eta'(\lambda) \partial_\lambda \mu_0^\tau(x, s, \lambda) d\lambda = \int_{-M}^{M} \eta''(\lambda) \mu_0^\tau(x, s, \lambda) d\lambda, \]

\[ q_\delta(u_0^\tau(x, t)) - q_\delta(u_0^\tau(x, t)) - \eta'(u_0^\tau(x, t))(b(u_0^\tau(x, t)) - b(u_0^\tau(x, t))) = \]

\[ \int_{-M}^{M} \eta'(\lambda)(b'(\lambda)(\chi(\lambda; u_0^\tau(x, t))) - \chi(\lambda; u_0^\tau(x, t))) - b(u_0^\tau(x, t)) \delta(\lambda = u_0^\tau(x, t)) d\lambda = \]

\[ - \int_{-M}^{M} \eta'(\lambda) \partial_\lambda \mu_0^\tau(x, t, \lambda) d\lambda = \int_{-M}^{M} \eta''(\lambda) \mu_0^\tau(x, t, \lambda) d\lambda, \]

\[ q_\gamma(u_0^\tau(x, t)) - q_\gamma(u_0^\tau(x, t)) - \eta'(u_0^\tau(x, t))(b(u_0^\tau(x, t)) - b(u_0^\tau(x, t))) = \]

\[ \int_{-M}^{M} \eta'(\lambda)(b'(\lambda)(\chi(\lambda; u_0^\tau(x, t))) - \chi(\lambda; u_0^\tau(x, t))) - b(u_0^\tau(x, t)) \delta(\lambda = u_0^\tau(x, t)) d\lambda = \]

\[ - \int_{-M}^{M} \eta'(\lambda) \partial_\lambda \mu_0^\tau(x, t, \lambda) d\lambda = \int_{-M}^{M} \eta''(\lambda) \mu_0^\tau(x, t, \lambda) d\lambda, \]

which are valid for every convex entropy flux pair \((\eta, q)\), where \(\mu_0^\tau, \mu_\tau \in \mathcal{M}(\Omega \times (0, S) \times (-M, M)), \mu_0^\tau, \mu_\tau \in \mathcal{M}(\Omega \times (0, T) \times (-M, M)). \) When \(p_1 = p_2 = 2\), it was shown in [25] that \(\mu_0^\tau, \mu_\tau, \mu_0^\tau \) and \(\mu_\tau\) are positive measures: \(\mu_0^\tau, \mu_\tau \in \mathcal{M}^+(\Omega \times (0, S) \times (-M, M)), \mu_0^\tau, \mu_\tau \in \mathcal{M}^+(\Omega \times (0, T) \times (-M, M)). \)

There is still an open question about the convergence of \(u_\varepsilon\) to entropy solution \(u\) of problem \(\Pi_0\) as \(\varepsilon \to +0\) when \(|p_1 - 2| + |p_2 - 2| \neq 0\), see Remark 4.

6. Conclusion
In the present paper we have enriched results presented in [25] in the case when temporal artificial diffusion coefficients depend on partial derivatives of \(u_\varepsilon\) in \(t\) and \(s\) variables. Namely, we have shown that various choices of temporal artificial diffusion coefficients lead to quasi-solutions or to entropy solutions of problem \(\Pi_0\). The later assumption is still under discussion.

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