Differentiability of evolution operators for dynamical systems with hysteresis

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Abstract. Hysteresis is a collective name for strongly nonlinear phenomena which occur in engineering, physical and economic systems. The wording ‘strongly nonlinear’ means that linearization cannot encapsulate the observed effects. In particular, standard linearization techniques are not applicable in the analysis of dynamical systems with hysteresis nonlinearities. Nevertheless, evolution operators of such dynamical systems typically have derivatives at many “important” points, which is enough to develop efficient modifications of gradient methods, linear stability analysis methods, etc. In this paper, we suggest a linearization algorithm for dynamical systems with the Preisach hysteresis operator. We present and justify an algorithm for calculation of the derivative of the Poincare map for scalar equations with a periodic solution of a simple form. We also suggest a version of this algorithm for a general class of non-periodic solutions of multi-dimensional equations with the Preisach nonlinearity.

1. Introduction
The mathematical theory of systems with hysteresis is a new chapter of applied mathematics, which is being actively developed worldwide by synergized efforts of mathematicians and engineers on the basis of methods of nonlinear analysis, dynamical systems and control theory, see [1, 2, 3, 4, 5, 7, 10]. A review of the current state of the art and further references may be found in the fundamental three volume set [11].

Following the paradigm suggested in [4], hysteresis nonlinearities are treated as nonlinear operators in functional spaces. An important feature of all hysteresis nonlinearities is that they are intrinsically non-smooth: there always exist points at which the operators do not have a derivative in a natural (e.g. Fréchet) sense.

There is a rapidly growing interest in the theoretical analysis and computer modeling of closed loop dynamical systems with hysteresis nonlinearities. The aforementioned non-smoothness of hysteresis operators causes non-smoothness of the corresponding evolution operators at least at some points of the phase space. Thus, most efficient methods (standard in other problems) based on linear approximations, such as gradient methods for solving boundary value problems, linear stability analysis, etc., are not yet developed for systems with hysteresis nonlinearities.

There are strong indications that the evolution operators generated by dynamical systems with hysteresis are in fact linearizable on a subset of the phase space of the dynamical system,
and the “size” of this subset is sufficient to develop efficient modifications of classical linearization
techniques for a range of boundary-value and stability problems. Moreover, the derivatives of
the evolution operator can be found from differential-operator equations which are analogues to
classical first variational equations.

The dynamical systems with hysteresis studied in this paper have the form of the equation
\[
\frac{dz}{dt} = f(t, z, y)
\] (1)
coupled with the hysteresis relation between the variables \(x = \ell(z)\) and \(y\), the scalar input
and output, respectively, of the Preisach hysteresis operator with the internal state \(\omega\). Here
\(x = \ell(z)\) is a scalar component of the vector \(z\) in an appropriate basis. The evolution of
the system is thus described by the evolutions of the phase variable \(z\) and the state \(\omega\) of the hysteresis
nonlinearity, while the output \(y\) of this nonlinearity is a given function of \(\omega\). Correspondingly, the
evolution operator for any time interval \(\tau\) is a mapping in the space of couples \((z, \omega)\). Equations
(1), and their further modifications, are important in various application areas, for example,
ferromagnetic oscillators, control systems, power electronics and hydrology. The famous Preisach
operator \([4, 5]\), relating the variables \(y\) and \(x\), is used as a model for many real world phenomena,
see \([11, 12, 13]\) and the references therein.

This paper is organized as follows. In section 2 we consider a simple example: a scalar
differential equation (1), which has a periodic solution of a simple form. We formulate an
algorithm for the calculation of the derivative of the Poincare map for this equation at the
initial value of the periodic solution and discuss the application of this algorithm to the stability
analysis of such solutions. The results of section 2 are proved in section 3. In section 4 we
consider a more general class of non-periodic solutions of vector equations (1) and suggest a
generalization of the linearization algorithm of section 2 for this class. However, we formulate
this algorithm as a conjecture without proving it: we suggest a set of conditions which we believe
ensure that our linearization algorithm is correct for the setting of section 4.

2. Stability of a periodic solution

2.1. Preisach model

In applications the Preisach model naturally arises as a superposition of simple two-state
hysteresis nonlinearities, non-ideal relays (switches), with a common input. We use an equivalent
alternative definition of the model due to \([4]\). The output of the model is a scalar function defined
by the formula
\[
y(t) = \text{mes}_\mu \Omega(t) := \int_{\Omega(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta
\] (2)
where \(\mu : \Pi \to \mathbb{R}\) is a nonnegative continuously differentiable integrable function, \(\Pi = \{(\alpha, \beta) : 0 \leq \beta - \alpha \leq d\}\) is a strip in the \((\alpha, \beta)\) plane, and the domain \(\Omega(t) \subset \Pi\) of integration
changes in time. For each \(t\), this domain has the form
\[
\Omega(t) = \{(\alpha, \beta) \in \Pi : \alpha + \beta \leq 2x(t) + \omega(t; \beta - \alpha)\},
\] (3)
where \(x(t)\) is a scalar continuous input of the model and the function \(\omega(t; \cdot) : [0, d] \to \mathbb{R}\), called
the state of the model, satisfies
\[
\omega(t; 0) = 0; \quad |\omega(t; \xi_1) - \omega(t; \xi_2)| \leq |\xi_1 - \xi_2|, \quad 0 \leq \xi_1, \xi_2 \leq d,
\]
for all \(t\). The evolution of the state \(\omega = \omega(t; \cdot)\) (and hence, the evolution of \(\Omega\)) is determined
by the input with the following simple rules. If the continuous input is monotone on a segment
Now, a standard two-step procedure is used to define the state and output for any continuous input \( x : \mathbb{R}_+ \to \mathbb{R} \) (see, e.g., [4, 5, 6]). First, formula (4) is combined with the semigroup identity to define the state \( \omega(t; \cdot) \) at every moment \( t \geq 0 \) for any continuous piecewise monotone input \( x : \mathbb{R}_+ \to \mathbb{R} \) and any initial state \( \omega(0; \cdot) = \omega_0(\cdot) \), while formulas (2), (3) define the corresponding output. We denote the output by

\[
y(t) = \mathcal{P}[\omega_0]x(t), \quad t \geq 0,
\]

reflecting the fact that it depends both on input and initial state, and remark that \( \omega = \omega(t; \xi) \) is continuous in both arguments due to (4), hence the output \( y : \mathbb{R}_+ \to \mathbb{R} \) of the Preisach model is also continuous. As the second step, one extends the input-output operator (5) to the whole class \( C([\mathbb{R}_+]) \) of continuous inputs from the subset of piecewise monotone ones by continuity, based on the Lipschitz estimate that holds for every \( t \geq 0 \):

\[
\| \mathcal{P}[\omega_0^1]x_1 - \mathcal{P}[\omega_0^2]x_2 \|_{C([0, t])} \leq M(\| \omega_0^1 - \omega_0^2 \|_C + \| x_1 - x_2 \|_{C([0, t])}),
\]

where \( \| \omega_0^1 - \omega_0^2 \|_C = \max_{\xi \in [0, d]} | \omega_0^1(\xi) - \omega_0^2(\xi) | \) measures the distance between initial states and the uniform norm in the space of inputs and outputs is used.

In section 4 we continue the discussion of the evolution of states of the Preisach model.

2.2. Assumptions
We consider the equation

\[
\begin{align*}
\dot{x}(t) &= f(t, x, y), \\
y(t) &= \mathcal{P}[\omega_0]x(t),
\end{align*}
\]

where \( x, y \) are the scalar input and scalar output of the Preisach nonlinearity, and \( \omega_0 \) is its initial state. The function \( f \) is assumed to be \( T \)-periodic in the variable \( t \):

\[
f(t, x, y) = f(t + T, x, y),
\]

and continuously differentiable with respect to the set of its arguments.

Solutions of (6) are pairs \((x(t), \omega(t)), t \geq 0\), where \( \omega(t) \) is the state of the Preisach nonlinearity. Differentiability of \( f \) ensures that any initial data

\[
x(0) = x_0, \quad \omega(0) = \omega_0
\]

define a unique solution

\[
(x(t), \omega(t)) = (x(t; x_0, \omega_0), \omega(t; x_0, \omega_0))
\]

of (6), at least on some interval \( 0 \leq t < \delta \) (see, e.g., [1]).

In this section we suppose that system (6) has a periodic solution \((x_*(t), \omega_*(t))\) such that the following conditions hold (see figure 1):

1. The function \( x_*(t) \) has one local maximum and one local minimum in the segment \([0, T]\), the minimum \( x_*^{\text{min}} \) is reached at the moment \( t_1 > 0 \), and the maximum \( x_*^{\text{max}} \) is reached at the moment \( t_2, t_1 < t_2 < T \).
2. The derivative of the function $x_*(t)$ is strictly negative on the intervals $[0, t_1)$ and $(t_2, T]$, and strictly positive on the interval $(t_1, t_2)$.
3. The relations $x''_*(t_1) > 0, x''_*(t_2) < 0$ hold.
4. The point $(x_*(0), x^M_*)$ lies outside the strip $\Pi$ on the Preisach plane $(\alpha, \beta)$.

These conditions imply that the variable state $\omega_*(t; \cdot)$ is a *vertical segment* for all $t$ from the interval $[0, t_1]$ and a *horizontal segment* at the moment $t_2$, i.e.,

$$\omega_*(t) = \omega_v \text{ for } t \in [0, t_1]; \quad \omega_*(t_2) = \omega_h$$

with

$$\omega_v(\xi) = \xi, \quad \omega_h(\xi) = -\xi, \quad \xi \in [0, d].$$

### 2.3. Main result

Denote

$$\varphi(x,v) = \text{mes}_\mu\{(\alpha, \beta) \in \Pi: \alpha \leq v\} + \text{mes}_\mu\{(\alpha, \beta) \in \Pi: v \leq \alpha \leq \beta \leq x\}, \quad x \geq v,$$

$$\psi(x,v) = \text{mes}_\mu\{(\alpha, \beta) \in \Pi: \beta \leq v\} - \text{mes}_\mu\{(\alpha, \beta) \in \Pi: x \leq \alpha \leq \beta \leq v\}, \quad x \leq v,$$

(see figure 2), and

$$\begin{align*}
J^x_\varphi(x,v) &= \frac{\partial \varphi}{\partial x}(x,v) = \int^x_{\max\{v, x-d\}} \mu(\alpha, x) \, d\alpha, \quad J^\alpha_\varphi(x,v) = \frac{\partial \varphi}{\partial \alpha}(x,v) = \int^\alpha_{\min\{x, v+d\}} \mu(v, \beta) \, d\beta, \\
J^x_\psi(x,v) &= \frac{\partial \psi}{\partial x}(x,v) = \int^x_{\min\{x, x+d\}} \mu(x, \beta) \, d\beta, \quad J^\alpha_\psi(x,v) = \frac{\partial \psi}{\partial \alpha}(x,v) = \int^\alpha_{\min\{v-d, x\}} \mu(\alpha, x) \, d\alpha.
\end{align*}$$

Consider the following two auxiliary initial value problems

$$\begin{align*}
x' &= f(t, x, \varphi(x,v)), \quad x(t_1) = v, \quad (10) \\
x' &= f(t, x, \psi(x,v)), \quad x(t_2) = v, \quad (11)
\end{align*}$$
Figure 2. Functions $\varphi(x, v)$ and $\psi(x, v)$ are the measures of the shaded sets

with a scalar parameter $v$.

From Conditions 1–4 above it follows that $x_*(t)$ is a solution of problem (10) on the interval $[t_1, t_2]$ with $v = x_*^m$, and a solution of problem (11) on the interval $[t_2, t_1 + T]$ with $v = x_*^M$.

Denote by $y_*$ the output of the Preisach nonlinearity for the periodic solution: $y_*(t) = \mathcal{P}[\omega_*(0)]x_*(t)$. Consider the linearization (first variational equation) of problem (10) in a vicinity of its solution $x_*(t)$:

$$\ddot{x}_1(t) = F_1(t)\dot{x}_1(t) + G_1(t), \quad \dot{x}_1(\theta) = 1$$

(12)

with $\theta = t_1$, and

$$F_1(t) = \frac{\partial f}{\partial x}(t, x_*(t), y_*(t)) + \frac{\partial f}{\partial y}(t, x_*(t), y_*(t))J^\varphi_{\varphi}(x_*(t), x_*^m),$$

$$G_1(t) = \frac{\partial f}{\partial y}(t, x_*(t), y_*(t))J^\varphi_{\varphi}(x_*(t), x_*^m),$$

(13)

where $\ddot{x}_1 = \frac{\partial x}{\partial \theta}$. Similarly, consider the linearization of problem (11):

$$\ddot{x}_2(t) = F_2(t)\dot{x}_2(t) + G_2(t), \quad \dot{x}_2(\theta) = 1,$$

(14)

with $\theta = t_2$, and

$$F_2(t) = \frac{\partial f}{\partial x}(t, x_*(t), y_*(t)) + \frac{\partial f}{\partial y}(t, x_*(t), y_*(t))J^\psi_{\psi}(x_*(t), x_*^M),$$

$$G_2(t) = \frac{\partial f}{\partial y}(t, x_*(t), y_*(t))J^\psi_{\psi}(x_*(t), x_*^M).$$

(15)

We denote the solutions of (12) and (14) by $\ddot{x}_1(t; \theta)$ and $\ddot{x}_2(t; \theta)$.

Note that if $|x_0 - x_*(0)|$ is sufficiently small, and the initial state of the Preisach nonlinearity is the vertical segment $\omega_v$, then the solution $(x(t), \omega(t))$ of equation (6) with initial values $(x_0, \omega_v)$ is defined on the segment $[0, T]$. 

Theorem 1. Let $|x_0 - x_*(0)|$ be sufficiently small, and let the initial state of the Preisach nonlinearity be the vertical segment $\omega_v$. Then the function $x(T; x_0, \omega_v)$ of the argument $x_0$ is differentiable at the point $x_0 = x_*(0)$, and its derivative is equal to

$$
\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v) = \tilde{x}_1(t_2; t_1)\tilde{x}_2(t_1 + T; t_2),
$$

(16)

where $\tilde{x}_1(t; \theta)$ and $\tilde{x}_2(t; \theta)$ are solutions of problems (12) and (14).

2.4. Remarks

a. If initial data (7) are close to the initial values $x_*(0)$, $\omega_v(0) = \omega_v$, of the periodic solution (i.e., $|x_0 - x_*(0)|$, $\|w_0 - w_v\|_C$ are sufficiently small), then Condition 4 implies that $\omega(T) = \omega_v$ due to the continuous dependence of solution of (6) on initial data. Therefore, Theorem 1 implies that the periodic solution $(x_*(t), \omega_*(t))$ is exponentially stable if $|\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v)| < 1$, and unstable if $|\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v)| > 1$, where the derivative is given by (16). More precisely, if $|\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v)| < 1$, then for any initial data (7) that are sufficiently close to $x_*(0)$, $\omega_v(0)$ solution (8) is defined for all $t \geq 0$ and

$$
|x(t) - x_*(t)| + |y(t) - y_*(t)| + \|w(t) - w_*(t)\|_C \leq Ce^{-\gamma t}(|x_0 - x_*(0)| + \|w_0 - w_*(0)\|_C),
$$

for each $t$ with some $C, \gamma > 0$ independent of $x_0, \omega_0$. If $|\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v)| > 1$, then $x(t)$ leaves a fixed vicinity of $x_*(t)$ at some moment $t > 0$ for any initial data (7) close enough to, but different from, $x_*(0)$, $\omega_v(0)$.

b. Condition 4 can be replaced with the condition $(x^m_*, x^M_*) \not\in \Pi$ (we chose to use Condition 4 in order to simplify the proof). If the point $(x^m_*, x^M_*)$ lies in the interior of the strip $\Pi$, then generically equation (6) has a connected continuum of periodic solutions whenever it has one such solution (see [2]), consequently periodic solutions are not asymptotically stable in this case.

c. Statements similar to Theorem 1 are valid for multidimensional systems with the Preisach nonlinearity and for periodic solutions with the $x_*$-component having more than two local extrema on the period. Although we do not consider them here, a straightforward modification of the above linearization algorithm works for such more complicated solutions if the local extrema of $x_*(t)$ satisfy some additional non-degeneracy conditions. These are either a modification of Condition 4 or, alternatively, another non-degeneracy condition, which is considered in the next section for non-periodic solutions.

Analogues of Theorem 1 can be formulated also for equations with the Preisach operator on the left hand side, like for example $y' = f(t, x)$, $y(t) = P[\omega_0]x(t)$. For these, Condition 4 is unnecessary. Such equations naturally arise in hydrological models [8, 9]; further examples of equations with the time derivative of the Preisach operator can be found in [10].

d. Formula (16) can be easily understood for an equation (6) where $f$ has jumps in $t$. More precisely, assume that $f$ is discontinuous at $t = t_1, t_2$ and has finite left and right limits on these lines. Suppose the component $x_*$ of a periodic solution satisfies Conditions 1, 2 and 4 above and the following assumption, which replaces Condition 3:

3*. The left and right derivatives of the function $x_*$ at its extremum points $t_1, t_2$ are non-zero.

Then Lemma 1 of the next section implies that the $x$-component of a solution $(x(t), \omega(t))$ with initial values $x(0) = x_0$, $\omega(0) = \omega_v$, is $C^1$-close to $x_*$ on each of the segments $[0, t_1]$, $[t_1, t_2]$ and $[t_2, T]$ whenever $x_0$ is close to $x_*(0)$. Consequently, $x$ satisfies the same Conditions 1, 2, 3* and 4. Hence both $x_*$ and $x$ strictly decrease on the intervals $[0, t_1]$, $[t_2, T]$, strictly
increase on \([t_1, t_2]\), and reach minimum and maximum values on the segment \([0, T]\) at the same points \(t_1\) and \(t_2\), respectively; also, the point \((x(0), x^M)\) lies outside the strip \(\Pi\) on the Preisach plane \((\alpha, \beta)\), like the point \((x_*(0), x_*^M)\), where we use the notation \(x^m = x(t_1)\), \(x^M = x(t_2)\) for the minimum and maximum of \(x\).

Now, from the definition of the Preisach nonlinearity it follows that because \(x\) decreases on \([0, t_1]\) and the initial state is \(\omega(0) = w_v\), the state of the Preisach nonlinearity at every moment \(t \in [0, t_1]\) is \(w_v\), and the output \(y\) on the segment \([0, t_1]\) equals

\[
y(t) = \mathcal{P}[\omega_v]x(t) = \chi(x(t)) \quad \text{with} \quad \chi(v) := \operatorname{mes}_\mu\{(\alpha, \beta) \in \Pi: \alpha \leq v\}.
\]

By the same argument, \(y_*(t) = \chi(x_*(t))\) for \(t \in [0, t_1]\) and hence on this time segment \(x\) and \(x_*\) are solutions of the same ordinary differential equation

\[
x' = f(t, x, \chi(x)). \tag{17}
\]

Similarly, the relations \(\omega(t_1) = \omega_*(t_1) = \omega_v\) and the fact that \(x\) and \(x_*\) increase on the segment \([t_1, t_2]\) imply the equalities

\[
y(t) = \varphi(x(t), x(t_1)), \quad y_*(t) = \varphi(x_*(t), x_*(t_1)) \quad \text{for} \quad t \in [t_1, t_2]
\]

and one infers that on the segment \([t_1, t_2]\) the function \(x\) solves problem (10) for the parameter value \(v = x(t_1) = x^m\), while \(x_*\) solves the same problem for the parameter value \(v = x_*(t_1) = x_*^m\). Finally, because \(\omega(t_2) = \omega_*(t_2) = \omega_h\) (which follows from the relations \((x^m, x^M) \not\in \Pi\), \((x_*^m, x_*^M) \not\in \Pi\)) and \(x\) and \(x_*\) decrease on the segment \([t_2, T]\), one concludes that

\[
y(t) = \psi(x(t), x(t_2)), \quad y_*(t) = \psi(x_*(t), x_*(t_2)) \quad \text{for} \quad t \in [t_2, T]
\]

and consequently \(x\) solves problem (11) on the segment \([t_2, T]\) for the parameter value \(v = x(t_2) = x^M\), and \(x_*\) solves this problem on the same segment for the parameter value \(v = x_*(t_2) = x_*^M\).

Thus, for the initial values \(x_0\) close to \(x_0(0)\) and \(\omega(0) = \omega_v\), system (6) on the segment \([0, T]\) is equivalent to the sequence of three initial value problems for ordinary differential equations on the time intervals \([0, t_1]\), \([t_1, t_2]\) and \([t_2, T]\). Therefore, one calculates the derivative \(\partial x / \partial x_0(T; x_*(0), \omega_v)\) of the Poincare map of (6) simply by the standard linearization of ODEs. It should be just noted that the value \(x(t_1)\) of a solution of equation (17) at the end point of the first segment \([0, t_1]\) is used both as an initial value and as a value of the parameter \(v\) in the right-hand side for problem (10) on the second segment \([t_1, t_2]\) (and similarly the end value of the solution of (10) is used as an initial value and the parameter value for problem (11) on the last segment \([t_2, T]\)). Therefore, one uses theorems on differentiation of solutions of ODE both with respect to initial values and parameters. In this manner, one arrives at the formula

\[
\frac{\partial x}{\partial x_0}(T; x_*(0), \omega_v) = \tilde{x}_3(t_1; 0)\tilde{x}_1(t_2; t_1)\tilde{x}_2(T; t_2), \tag{18}
\]

where \(\tilde{x}_3(\cdot; 0)\) is the solution of the linearization

\[
\tilde{x}'_3(t) = D(t)\tilde{x}_3(t), \quad D(t) = \frac{\partial f}{\partial x}(t, x_*(t), y_*(t)) + \frac{\partial f}{\partial y}(t, x_*(t), y_*(t))\frac{\partial \chi}{\partial x}(x_*(t))
\]

\[
\tilde{x}_3(0) = 1,
\]

of (17) around its solution \(x_*\), and \(\tilde{x}_1, \tilde{x}_2\) are as in Theorem 1. It is easy to see then that Condition 4 implies \(\tilde{x}_3(t_1; 0) = \tilde{x}_2(t_1; 0)\) and that \(\tilde{x}_2(t_1; 0)\tilde{x}_2(T; t_2) = \tilde{x}_2(T + t_1; t_2)\) (details are in the next section), hence (18) is equivalent to formula (16) of Theorem 1.
The proof of Theorem 1 below is somewhat more involved than the above argument. The reason is that under the Conditions 1–4 a solution $x$ close to $x_*$ can have more than one minimum and more than one maximum in the interval $[0, T]$; also generally, the perturbed solution $x$ reaches its global minimum and maximum not precisely at the same moments $t_1$ and $t_2$ as $x_*$ does. However, all local extrema of $x$ are located close to the extremum points $t_1, t_2$ of $x_*$ for small perturbations of the periodic solution of (6). As we show in the next section, this leads to the same formula (16) under the conditions of Theorem 1 as in the simpler case of Condition 3*.

**Example.**

Consider the equation

$$x'(t) = x(t) + \mathcal{P}[\omega_0]x(t) + a + b \sin \frac{2\pi t}{T},$$

(19)

where the Preisach operator $\mathcal{P}$ has a uniform measure $\mu(\alpha, \beta) = \mu$ defined on the set

$$\{(\alpha, \beta): |\alpha|, |\beta| \leq \gamma \text{ and } 0 \leq \beta - \alpha \leq d\}.$$ 

Note that we can always consider a measure with a bounded support, as long as the solution of the system does not leave the region where the measure support is located.

For the purposes of this example we choose the constants $a, b, T, d, \mu$ and $\gamma$ such that (19) has a periodic solution that has an explicit form. Let

$$a = -\frac{1}{2}, \quad b = \frac{27}{28}, \quad T = \frac{2}{5}(4 + 7 \ln 5), \quad d = 1, \quad \mu = \frac{2}{7}, \quad \gamma = 2.$$ 

Hence, the functions $\varphi(x, v)$ and $\psi(x, v)$ take the form

$$\varphi(x, v) = \begin{cases} \frac{1}{7}(3 + 2x), & x > 1 + v \text{ and } v \leq 1, \\ \frac{1}{7}(3 - 2v(x - 2) + x^2), & v > 1, \\ \frac{1}{7}(4 + x^2 + v^2 - 2v(x - 1)), & \text{otherwise}; \end{cases}$$

$$\psi(x, v) = \begin{cases} \frac{2}{7}(2 + x), & x < v - 1 \text{ and } v \geq -1, \\ \frac{2}{7}(2 + 2v - x)(2 + x), & v < -1, \\ \frac{2}{7}(3 - x^2 - v^2 + 2v(x + 1)), & \text{otherwise}. \end{cases}$$

Consider the initial value problem for (19) with $x_*(0) = -1$ and $\omega_0 = \omega_v$. The solution $x_*(t)$ of this problem is $T$-periodic, reaches its maximum $x^*_M = 1$ at the moment $\frac{1}{2}T$ and minimum $x^*_m = -1$ at the moment $T$, and has the following explicit form on the interval $[0, T]$:

$$x_*(t) = \begin{cases} \frac{5t - 4}{2(t^2 + 2)}, & 0 \leq t \leq \frac{4}{5}, \\ \frac{4}{5}(1 - e^{\frac{1}{5}(4 - 5t)}), & \frac{4}{5} < t \leq \frac{1}{2}T, \\ -\frac{5}{2}(5t - 8 - 7\ln 5), & \frac{1}{2}T < t \leq \frac{1}{2}T + \frac{4}{5}, \\ \frac{5}{4}(1 - e^{\frac{1}{5}(8 - 5t)}), & \frac{1}{2}T + \frac{4}{5} < t \leq T. \end{cases}$$

To apply Theorem 1 we need to solve the problems (12) and (14), which take the form

$$\ddot{x}_1(t) = -\frac{5}{7}\dot{x}_1(t) - \min\left\{0, \frac{2}{7}x_*(t)\right\}(1 - \ddot{x}_1(t)),$$

$$\ddot{x}_2(t) = -\frac{5}{7}\dot{x}_2(t) + \max\left\{0, \frac{2}{7}x_*(t)\right\}(1 - \ddot{x}_2(t)).$$
These problems can also be solved explicitly. With straightforward calculations, for the equality (16) we obtain
\[ \frac{\partial x}{\partial x_0}(T; -1, \omega_v) = \frac{351649}{26471025} \approx 0.01328. \]

Hence, the periodic solution of (19) is asymptotically stable.

3. Proof of Theorem 1
3.1. Auxiliary lemmas
We start with a rough estimation of the deviation of the perturbed solution \((x(t), \omega(t))\) with initial values \(x(0) = x_0, \omega(0) = \omega_v\) from the periodic solution. Denote
\[ r(t) = x(t) - x_*(t), \quad r_0 = r(0) = x_0 - x_*(0). \]

Lemma 1. If \(|r_0|\) is sufficiently small, then
\[ |r(t)| \leq e^{L_1 t}|r_0|, \quad 0 \leq t \leq T. \]

Proof. Fix a vicinity \(U\) of the solution \(x_*(t)\). In this vicinity function \(f\) is a Lipschitz function:
\[ |f(x_2, y_2) - f(x_1, y_1)| \leq c_1|x_2 - x_1| + c_2|y_2 - y_1|. \]

If \(x_0\) is close to \(x_*(0)\), then \(x(t) \in U\) for \(t \in [0, T]\) by the theorem on continuous dependence of solutions on initial conditions. Therefore, the Lipschitz estimate implies
\[ |x'(t) - x'_*(t)| \leq c_1|x(t) - x_*(t)| + c_2|y(t) - y_*(t)|. \]

Now, from the Lipschitz continuity of the Preisach operator in \(C[0, T]\) it follows that
\[ |x'(t) - x'_*(t)| \leq L_1\|x(t) - x_*(t)\|_{C[0, T]}. \] (20)

Denote
\[ u(t) = |r_0| + \int_0^t |r'(s)| ds = |x_0 - x_*(0)| + \int_0^t |x'(s) - x'_*(s)| ds. \]

Then (20) implies \(u'(t) \leq L_1 u(t)\), which implies
\[ u(t) \leq u(0)e^{L_1 t}, \]
and the conclusion of the Lemma follows from \(|r(t)| \leq u(t), r(0) = u(0)\).\qed

Lemma 1 together with relation (20) implies that for sufficiently small \(|r_0|\)
\[ ||r(t)||_{C[0, T]} \leq L|r_0|. \] (21)

Let \(k = \min\{|x''_*(t_1)|, |x''_*(t_2)|\}\) and \(K = \max\{|x''_*(t_1)|, |x''_*(t_2)|\}\). The continuity of \(x''_*\) implies that there exist \(\Delta_1 > 0\) and \(\Delta_2 > 0\) such that
\[ 2K \geq x''_*(\xi_1) \geq \frac{k}{2} \text{ for } \xi_1 \in [t_1 - \Delta_1, t_1 + \Delta_1], \]
\[ -2K \leq x''_*(\xi_2) \leq -\frac{k}{2} \text{ for } \xi_2 \in [t_2 - \Delta_2, t_2 + \Delta_2]. \]
Let \( \Delta = \min\{\Delta_1, \Delta_2\} \). Then, because \( x'_*(t_1) = x'_*(t_2) = 0 \), we have the following relations:

\[
2K(t - t_1) \leq x'_*(t) \leq \frac{k}{2}(t - t_1) \quad \text{for} \quad t \in [t_1 - \Delta, t_1],
\]

\[
2K(t - t_1) \geq x'_*(t) \geq \frac{k}{2}(t - t_1) \quad \text{for} \quad t \in [t_1, t_1 + \Delta],
\]

\[
2K(t_2 - t) \geq x'_*(t) \geq \frac{k}{2}(t_2 - t) \quad \text{for} \quad t \in [t_1 - \Delta, t_2],
\]

\[
2K(t_2 - t) \leq x'_*(t) \leq \frac{k}{2}(t_2 - t) \quad \text{for} \quad t \in [t_2, t_2 + \Delta].
\]

(22a) \hspace{1cm} (22b) \hspace{1cm} (22c) \hspace{1cm} (22d)

Now consider the derivative \( x'_*(t) \) on the intervals \([0, t_1 - \Delta]\), \([t_1 + \Delta, t_2 - \Delta]\), and \([t_2 + \Delta, T]\). According to Condition 2, \( x'_*(t) \) is strictly negative on the first and the third of these intervals, and strictly positive on the second one. Therefore there exists a constant \( a > 0 \) such that \( |x'_*(t)| \geq a \) on all the three intervals.

Let \( \tau = r_0 \frac{4L}{K} \), \( \tau_1 = 20\tau \frac{K}{k} \),

(23)

where \( L \) is defined in (21). Consider a sufficiently small \( r_0 \) such that (21) holds, and

\[
\tau < \Delta, \quad \tau < t_1, \quad 2\tau + \tau_1 < t_2 - t_1, \quad \tau + \tau_1 < T - t_2, \quad \frac{k\tau}{2} < a.
\]

(24)

Consider the solutions \( x(t) \) and \( x_*(t) \) on the intervals \([0, t_1 - \tau]\), \([t_1 - \tau, t_1 + \tau]\), \([t_1 + \tau, t_2 - \tau]\), \([t_2 - \tau, t_2 + \tau]\) and \([t_2 + \tau, T]\). Conditions 1 and 2 imply that \( x'_*(t) < 0 \) on the interval \([0, t_1 - \tau]\), then on the interval \([t_1 - \tau, t_1 + \tau]\) the function \( x_*(t) \) reaches its minimum \( x_\text{m}^* \) at the moment \( t_1 \), and \( x'_*(t) > 0 \) on the interval \([t_1 + \tau, t_2 - \tau]\). The following lemma states that the perturbed solution \( x(t) \) is strictly monotone outside the two small intervals around the points \( t_1, t_2 \) (see figure 3) and gives a lower estimate of \( |x'(t)| \).

**Figure 3.** Solutions \( x_*(t) \) and \( x(t) \)
Thus the function $x(t)$ strictly increases on the interval $[t_1 + \tau, t_2 - \tau]$ and strictly decreases on the intervals $[0, t_1 - \tau]$ and $[t_2 + \tau, T]$. Moreover, on these intervals

$$|x'(t)| \geq \frac{k}{2} \tau, \quad |x'(t)| \geq \frac{k}{4} \tau. \quad (25)$$

**Proof.** First consider the interval $[0, t_1 - \tau]$. Relation (22a) implies that on $[t_1 - \Delta, t_1 - \tau]$

$$x'(t) \leq \frac{k}{2} (t - t_1) \leq -\frac{k\tau}{2}. \quad (26)$$

According to (24), $x'(t) \leq -a \leq -k\tau/2$ on $[0, t_1 - \Delta]$. Hence,

$$x'(t) \leq -\frac{k\tau}{2} \text{ for } t \in [0, t_1 - \tau].$$

Therefore, (21) and (23) imply that on the interval $[0, t_1 - \tau]$

$$x'(t) \leq x'_a(t) + |r'(t)| \leq -\frac{k\tau}{2} + Lr_0 = -\frac{k\tau}{4} < 0. \quad (26)$$

Thus the function $x(t)$ decreases on the interval $[0, t_1 - \tau]$, and the derivatives $x'_a(t)$ and $x'(t)$ on this interval satisfy (25). The other intervals can be considered in the same way. \qed

### 3.2. Calculation of the derivative of evolution map

The conclusion of Theorem 1 follows from the three lemmas below.

**Lemma 3.** Denote by $x^m_1$ the absolute minimum of the function $x(t)$ on the interval $[t_1 - \tau, t_1 + \tau]$, and by $x^M_1$ the absolute maximum of $x$ on the same interval. Then the following relations are valid:

$$r(t_1 + \tau + \tau_1) = \tilde{x}_2(t_1; 0) r_0 + o(r_0), \quad (27)$$

$$x^m_1 - x^m_1 = \tilde{x}_2(t_1; 0) r_0 + o(r_0), \quad (28)$$

and the output of the Preisach operator for the perturbed solution at the moment $t_1 + \tau + \tau_1$ equals

$$y(t_1 + \tau + \tau_1) = \varphi(x(t_1 + \tau + \tau_1), x^M_1). \quad (29)$$

**Lemma 4.** Denote by $x^m_2$ the absolute minimum of function $x(t)$ on the interval $[t_2 - \tau, t_2 + \tau]$, and by $x^M_2$ the absolute maximum of $x$ on the same interval. Then the following relations hold:

$$r(t_2 + \tau + \tau_1) = \tilde{x}_2(t_1; 0) \tilde{x}_1(t_2; t_1) r_0 + o(r_0), \quad (30)$$

$$x^m_2 - x^m_2 = \tilde{x}_2(t_1; 0) \tilde{x}_1(t_2; t_1) r_0 + o(r_0), \quad (31)$$

and the output of the Preisach operator for the perturbed solution at the moment $t_2 + \tau + \tau_1$ equals

$$y(t_2 + \tau + \tau_1) = \psi(x(t_2 + \tau + \tau_1), x^M_2).$$

**Lemma 5.** The following equality holds:

$$r(T) = \tilde{x}_2(t_1; 0) \tilde{x}_1(t_2; t_1) \tilde{x}_2(T; t_2) r_0 + o(r_0). \quad (32)$$
Lemmas 3 and 4 are proved in the next two subsections. The proof of Lemma 5 is similar to that of Lemma 4 and we omit it.

To complete the proof, it remains to show that
\[ \tilde{x}_2(t_1; 0)\tilde{x}_2(T; t_2) = \tilde{x}_2(T + t_1; t_2), \]
then (32) implies (16). First of all, because the right-hand side of (14) is \( T \)-periodic in \( t \), \( \tilde{x}_2(t_1; 0) = \tilde{x}_2(T + t_1; T) \). As we show in the proof of Lemma 3 below, \( \frac{\partial \psi}{\partial y}(x_\ast(t), x_\ast^M) = 0 \) on the interval \([T; T + t_1]\), hence on this interval \( \tilde{x}_2 \) is a solution of the homogeneous linear equation \( \tilde{x}'_2(t) = F_2(t)\tilde{x}_2(t) \). This implies that if we denote by \( \tilde{X}_2(t; \theta; \tilde{x}_0) \) the solution of (14) with an initial value \( \tilde{X}_2(\theta; \theta; \tilde{x}_0) = \tilde{x}_0 \), then
\[ \tilde{X}_2(T + t_1; T; \tilde{x}_0) = \tilde{x}_0 \cdot \tilde{X}_2(T + t_1; T; 1) = \tilde{x}_0 \tilde{x}_2(T + t_1; T), \]
since \( \tilde{X}_2(\cdot, \cdot; 1) = \tilde{x}_2(\cdot, \cdot) \) by definition. Finally, according to the semi-group property of solutions,
\[ \tilde{X}_2(T + t_1; t_2; 1) = \tilde{X}_2(T + t_1; T; \tilde{X}_2(T; t_2; 1)) = \tilde{X}_2(T; t_2; 1)\tilde{X}_2(T + t_1; T; 1), \]
i.e.
\[ \tilde{x}_2(T + t_1; t_2) = \tilde{x}_2(T; t_2)\tilde{x}_2(T + t_1; T), \]
which is equivalent to (33). Thus, (32) is equivalent to
\[ r(T) = \tilde{x}_2(T + t_1; t_2)\tilde{x}_1(t_2; t_1)r_0 + o(r_0), \]
and after dividing by \( r_0 \) and passing to the limit as \( r_0 \to 0 \), we get equation (16), so the theorem is proved. \( \square \)

3.3. Proof of Lemma 3
The functions \( x \) and \( x_\ast \) are both decreasing on the interval \([0, t_1 - \tau]\), therefore the outputs \( y \) and \( y_\ast \) of the Preisach nonlinearity on this interval are determined by the equalities
\[ y = \psi(x(t), x_\ast^M), \quad y_\ast = \psi(x_\ast(t), x_\ast^M). \]
Hence, on the interval \([0, t_1 - \tau]\) the functions \( x \) and \( x_\ast \) are solutions of the equation
\[ x' = f(t, x, \psi(x, x_\ast^M)) \]
with initial values \( x_0 \) and \( x_\ast(0) \), respectively. Therefore, to estimate \( r(t_1 - \tau) \) we use the linearization of this equation along the solution \( x_\ast(t) \):
\[ \tilde{x}'_2(t) = F_2(t)\tilde{x}_2(t), \quad \tilde{x}_2(0) = 1, \]
where \( F_2(t) \) is defined in (15). Note that \( J^\psi_\ast(x_\ast(t), x_\ast^M) = \frac{\partial \psi}{\partial y}(x_\ast(t), x_\ast^M) = 0 \) on the interval \([0, t_1]\), because \( (x_\ast(t), x_\ast^M) \notin \Pi \) on this interval. Therefore, the solution \( \tilde{x}_2 \) of (34) on this interval is equal to the solution \( \tilde{x}_2(t; 0) \) of (14). Hence
\[ r(t_1 - \tau) = \tilde{x}_2(t_1 - \tau; 0)r_0 + o(r_0). \]
Furthermore,
\[ r(t_1 + \tau + \tau_1) = r(t_1 - \tau) + r'(\xi)(2\tau + \tau_1) \]
for some $\xi \in [t_1 - \tau, t_1 + \tau + \tau_1]$. Combining the last two equations and taking into account that $\|r'\|_{C[0,T]}$, $\tau, \tau_1 = O(r_0)$, we obtain

$$r(t_1 + \tau + \tau_1) = \tilde{x}_2(t_1 - \tau; 0)r_0 + o(r_0).$$

To prove (27) it remains to note that $\tau = O(r_0)$ implies

$$\tilde{x}_2(t_1 - \tau; 0) = \tilde{x}_2(t_1; 0) + O(r_0).$$

(36)

Now consider the absolute maximum $x^M_1$ of the function $x(t)$ on the interval $[t_1 - \tau, t_1 + \tau]$. Equations (22b) and (22c) imply that $|r'_k(t)| \leq 2K \tau$ on this interval, therefore

$$|x'(t)| \leq |x'_*(t)| + |r'(t)| \leq 2K \tau + Lr_0 \leq \left(2K + \frac{k}{2}\right) \tau \leq \frac{5K}{2} \tau.$$

(37)

Hence,

$$x^M_1 - x(t_1 + \tau) \leq |x'(\xi)| \cdot 2 \tau \leq 5K \tau^2.$$

Similarly to (26), we get from (25):

$$x'(t) \geq x'_*(t) - |r'(t)| \geq \frac{k\tau}{2} - Lr_0 \geq \frac{k\tau}{4} > 0$$

for $t \in [t_1 + \tau, t_2 - \tau]$, so $x(t)$ increases on this interval. Then, taking into account that $\tau_1 = 20rK/k$,

$$x(t_1 + \tau + \tau_1) - x(t_1 + \tau) \geq \frac{k\tau}{4} \tau_1 = 5K \tau^2 \geq x^M_1 - x(t_1 + \tau),$$

which implies $x(t_1 + \tau + \tau_1) \geq x^M_1$. Therefore, by the wiping out property of the Preisach nonlinearity, the state $\omega(t_1 + \tau + \tau_1)$ consists of two segments, which implies that $y(t_1 + \tau + \tau_1) = \varphi(x(t_1 + \tau + \tau_1), x^m_1)$ and thus (29) holds.

The absolute minimum $x^m_1$ is estimated as follows:

$$x^m_1 - x'_* = (x^m_1 - x(t_1 - \tau)) + (x(t_1 - \tau) - x'_*(t_1 - \tau)) + (x'_*(t_1 - \tau) - x^m_1)$$

$$= o(r_0) + r(t_1 - \tau) + o(r_0) = \tilde{x}_2(t_1 - \tau; 0)r_0 + o(r_0),$$

where we use (35) and (37). This together with (36) implies (28). \qed

3.4. Proof of Lemma 4

Consider the interval $[t_1 + \tau + \tau_1, t_2 - \tau]$. The functions $x_*$ and $x$ are both increasing on this interval. Therefore $x_*$ solves the equation

$$x' = f(t, x, \varphi(x, v))$$

(38)

for the parameter value $v = x^m_1$; the function $x$ solves the same equation for $v = x^m_1$. The initial values $x(t_1 + \tau + \tau_1)$ and $x_*(t_1 + \tau + \tau_1)$ are related by (27), while the parameter values $x^m_1$ and $x^m_*$ are related by (28). Differentiating (38) with respect to the initial value of $x$ and the parameter $v$ along the solution $x_*$, we obtain

$$r(t_2 - \tau) = \alpha(t_2 - \tau)r(t_1 + \tau + \tau_1) + \beta(t_2 - \tau)(x^m - x^m_*)$$

$$+ o(r(t_1 + \tau + \tau_1) + |x^m - x^m_*|),$$
where $\alpha$ and $\beta$ solve the problems

$$
\alpha'(t) = F_1(t)\alpha(t), \quad \alpha(t_1 + \tau + \tau_1) = 1,
$$

$$
\beta'(t) = F_1(t)\beta(t) + G_1(t), \quad \beta(t_1 + \tau + \tau_1) = 0,
$$

with $F_1(t)$ and $G_1(t)$ defined by (13). Now, applying (27) and (28) and noticing that $\alpha(t) + \beta(t) = \tilde{x}_1(t; t_1 + \tau + \tau_1)$ solves the problem (12), we obtain

$$
r(t_2 - \tau) = \tilde{x}_1(t_2 - \tau; t_1 + \tau + \tau_1)\tilde{x}_2(t_1; 0)r_0 + o(r_0).
$$

Combining this equation with

$$
r(t_2 + \tau + \tau_1) = r(t_2 - \tau) + o(r_0),
$$

$$
\tilde{x}_1(t_2 - \tau; t_1 + \tau + \tau_1) = \tilde{x}_1(t_2; t_1) + O(r_0),
$$

we arrive at (30). The other conclusions of the Lemma can be proved in the same way as that of Lemma 3.

\begin{proof}

\end{proof}

4. General linearization algorithm

4.1. Step-like states of the Preisach operator

In this section we consider a Preisach operator with the domain $\Omega(t)$, see (3), restricted to the sets $S \in \Pi$ bounded by lines $L$ with an infinite horizontal link, and with a finite number of vertical and horizontal links, see figure 4.

We denote the height of the infinite horizontal link by $w^{(1)}$. Let $w^{(2k)}$, $1 < 2k \leq K$, be the ascending sequence of the horizontal coordinates of the vertical links. That is, the vertical links belong to the straight lines $L_{2k} = (w^{(2k)}, \beta)$, $1 < 2k \leq K$. Similarly, let $w^{(2k+1)}$, $1 < 2k + 1 \leq K$,
be the descending sequence of the vertical coordinates of the finite horizontal links. By definition, the numbers \( w^{(k)}, 1 \leq k \leq K \), satisfy the inequalities
\[
\begin{align*}
  w^{(1)} > \ldots > w^{(2k_0-1)} > w^{(2k_0)} > \ldots > w^{(2)} & \quad \text{if } K = 2k_0, \\
  w^{(1)} > \ldots > w^{(2k_0+1)} > w^{(2k_0)} > \ldots > w^{(2)} & \quad \text{if } K = 2k_0 + 1. \\
\end{align*}
\]
(39)

The totality \( W \) of all states \( \omega \) described by such step-like curves may be naturally identified with the family of all finite vectors
\[ w = \{ w^{(k)} \}_{k=1}^{K} \]

satisfying the inequalities (39). The dimension of such a vector will be denoted by \( \# w = K \).

The totality of all \( K \)-dimensional vectors will be denoted by \( W^K \). It is convenient to define also \( w^{(0)} = -\infty \), \( w^{(-1)} = \infty \).

We will identify the variable state \( \omega(t) \) of the Preisach operator with a variable vector \( w(t) \) of variable dimension \( K' \). In these notations, the variable state \( \omega(t; \xi) \) of the Preisach model is given by the following equalities:
\[
\omega(t; \xi) = \begin{cases} 
(-1)^{K'}, \xi, & \xi < (-1)^{K'} (w^{(K'-1)} - w^{(K')}), \\
2(w^{(j)} - w^{(K')}) + (-1)^{j} \xi, & 2 \leq j \leq K' - 1, \\
(-1)^{j} (w^{(j+1)} - w^{(j)}) \leq \xi < (-1)^{j} (w^{(j-1)} - w^{(j)}), \\
2(w^{(1)} - w^{(K)}) - \xi, & \xi \geq w^{(1)} - w^{(2)}. 
\end{cases}
\]
(40)

We also introduce semi-closed intervals \( I_k = I_k(w) = [w^{(k-1)}, w^{(k+1)}], 0 \leq k < K \), and an open interval \( I_K = I_K(w) = (w^{(K)}, w^{(K-1)}) \), which form a partition of the whole real line. For any sequence \( w \in W^K \) we denote by \( J_k(w) \), \( i = 1, \ldots, K \), the following expressions:
\[
\begin{align*}
  J_{2j+1}(w) &= \int_{I_{2j+1}(w)} \mu(\alpha, w_{2j+1}) \, d\alpha, & 1 \leq 2j + 1 \leq K, \\
  J_{2j}(w) &= \int_{I_{2j}(w)} \mu(w_{2j}, \beta) \, d\beta, & 1 < 2j \leq K. 
\end{align*}
\]
(41)

In other words, (41) are ‘\( \mu \)-weights’ of the horizontal and vertical links of \( L \), and \( I_k \) are their projections on the vertical and horizontal axes, see figure 4. Comparing this notation with (9), we note that \( J_K(w) = J^e_{K}(w^{(K)}, w^{(K-1)}) \) for an odd \( K \), and \( J_K(w) = J^e_{K}(w^{(K)}, w^{(K-1)}) \) for an even \( K \).

We also need the following ‘complementary’ notation:
\[
\begin{align*}
  \overline{J}_K(w) &= \int_{w^{(K-1)}}^{w^{(K)}} \mu(\alpha, w^{(K)}) \, d\beta, & \text{if } K \text{ is odd}, \\
  \overline{J}_K(w) &= - \int_{w^{(K-1)}}^{w^{(K)}} \mu(\alpha, w^{(K)}) \, d\alpha, & \text{if } K \text{ is even}. 
\end{align*}
\]
(42)

4.2. Assumptions

Consider the system
\[
\begin{align*}
  z'(t) &= f(t, z(t), y(t)), & z \in \mathbb{R}^n, \\
  y(t) &= \mathcal{P}[\omega_0] x(t), \\
  x(t) &= \langle c, z(t) \rangle. 
\end{align*}
\]
(43)

Here the function \( f : \mathbb{R}^{n+2} \to \mathbb{R}^n \) is continuously differentiable with respect to the set of its arguments, \( y(t) \) denotes the output of the Preisach operator with the input \( x(t) \), \( c \in \mathbb{R}^n \) is a constant vector, and \( \langle \cdot, \cdot \rangle \) is a scalar product in \( \mathbb{R}^n \).
Let \((z(t), \omega(t)), 0 \leq t \leq T\), be a solution of system (43). Let the following three assumptions hold:

(i) The initial state \(\omega_0 = w_0\) belongs to \(W^{K_0}\) and the following natural condition holds:
\[
x(0) = \langle c, z_0 \rangle = w_0(K_0),
\]
where \(K_0 = \#w_0\).

(ii) The function \(x(t) = \langle c, z(t) \rangle, 0 \leq t \leq T\) is piecewise monotone with the extremum points \(\sigma_i\):
\[
0 < \sigma_1 < \ldots < \sigma_m < T.
\]
Moreover, \(x'(t) \neq 0\) within each interval \((\sigma_{i-1}, \sigma_i), i = 1, \ldots, m\) with \(\sigma_0 = 0\). We denote the set of points \(\sigma_i, i = 1, \ldots, m\), by \(\mathcal{S}\).

(iii) All numbers \(x_j = x(\sigma_j)\) are pairwise different, and also different from the numbers \(w_0^{(i)}\), \(i = 1, \ldots, K_0\).

Assumptions (i) and (ii) ensure that \(\omega(t) \in W\) for each \(t\). Moreover, for every \(t\)
\[
x(t) = \langle c, z(t) \rangle = w(K_0)(t).
\]
To avoid some clumsy notation we also assume that either
\[
K_0 \text{ is odd, and } x'(0) = \langle c, f(0, z_0, y_0) \rangle > 0,
\]
or
\[
K_0 \text{ is even, and } x'(0) = \langle c, f(0, z_0, y_0) \rangle < 0,
\]
where \(y_0 = y(0)\).

We denote by \(T\) the set of numbers \(\tau\) such that \(x(\tau)\) equals for the first time either \(w_0^{(i)}\) for some \(1 \leq i < K_0\), or some \(x(\sigma_i)\) with \(\sigma_i < \tau\), see figure 5. Namely,
\[
T = \bigcup_{i=1}^{K_0-1} \{ \inf \{ t : t > 0 \text{ and } x(t) = w_0^{(i)} \} \} \cup \bigcup_{i=1}^{m} \{ \inf \{ t : t > \sigma_i \text{ and } x(t) = x(\sigma_i) \} \}.
\]
4.3. Evolution operator and its derivative

Denote by $\mathcal{E}_{[0,t]}$ the evolution operator along the trajectory of a solution of (43), which acts on the subset of $\mathbb{R}^n \times W$ of pairs $(z, w)$ satisfying $\langle c, z \rangle = w^{(K)}$:

$$\mathcal{E}_{[0,t]}(z_0, w_0) := (z(t), w(t)).$$

Here $(z(t), w(t))$ is a solution of (43) with initial conditions $(z_0, w_0)$.

We will be interested in small perturbations of initial conditions $(z_0, w_0)$, for which the perturbed Preisach state $w^{(K)}_0$ belongs to the same class $W^{K_0} \subset W$ as $w_0$ does. Since the admissible initial conditions are pairs $(z, w) \in \mathbb{R}^n \times W^{K_0}$, which satisfy the restriction $x = \langle c, z \rangle = w^{(K)}$, they can be naturally identified with vectors

$$(z, \text{Ch} w) = (z_1, \ldots, z_n, w^{(1)}, \ldots, w^{(K_0-1)}),$$

where $z = (z_1, \ldots, z_n)$, $\text{Ch} w = (w^{(1)}, \ldots, w^{(K_1-1)})$, and $w^{(i)}$ satisfy (39). The totality of such vectors is denoted by

$$\mathcal{H}(K_0) \subset \mathbb{R}^n \times W^{K_0-1} = \mathbb{R}^{n+K_0-1}.$$

Thus we can locally interpret the evolution operator as a mapping from $\mathcal{H}(K_0)$ to $\mathcal{H}(K^t)$.

We now consider the question about a derivative of the evolution operator with respect to initial data. Let assumptions (i)–(iii) hold. Consider a perturbation $(z_0^\delta, w_0^\delta) \in \mathbb{R}^n \times W^{K_0}$ of the initial data $(z_0, w_0) \in \mathbb{R}^n \times W^{K_0}$ of the form

$$z_0^\delta = z_0 + \delta \tilde{z}_0, \quad w_0^\delta = w_0 + \delta \tilde{w}_0,$$

(44)

where $\tilde{z}_0 \in \mathbb{R}^n$ and $\tilde{w}_0 \in W^{K_0}$ are given vectors satisfying $w_0^{(K_0)} = \langle c, \tilde{z}_0 \rangle$, and $\delta$ is a small positive parameter.

The next proposition follows from the continuous dependence of solutions of system (43) on initial data in $C^1$-norm.

**Proposition 1.** For any closed set $\mathcal{R} \subset [0,T]$ such that $\mathcal{R} \cap (S \cup T) = \emptyset$ and any pair $(\tilde{z}_0, \tilde{w}_0) \in \mathbb{R}^n \times W^{K_0}$ with $w_0^{(K_0)} = \langle c, \tilde{z}_0 \rangle$ there exists a $\bar{\delta} > 0$ such that the perturbed solution $(z^\delta(t), w^\delta(t))$ with the initial data (44) and $0 < \delta < \bar{\delta}$ is defined on $[0,T]$, and the dimension $\#w^\delta(t)$ of $w^\delta(t)$ is equal to the dimension $\#w(t)$ of $w(t)$ for each $t \in \mathcal{R}$.

Proposition 1 implies that for any $t \notin S \cup T$ there exists a sufficiently small $\bar{\delta}(t) > 0$ such that the difference

$$r(t, \delta) = (z^\delta(t), \text{Ch} w^\delta(t)) - (z(t), \text{Ch} w(t)) = \mathcal{E}_{[0,t]}(z_0^\delta, \text{Ch} w_0^\delta) - \mathcal{E}_{[0,t]}(z_0, \text{Ch} w_0)$$

is well defined for any $0 < \delta < \bar{\delta}(t)$.

We would like to construct a linear mapping $A_t : \mathcal{H}(K_0) \to \mathcal{H}(K^t)$ for $t \notin S \cup T$ such that for $0 < \delta < \bar{\delta}(t)$ the following relationship holds:

$$\mathcal{E}_{[0,t]}(z_0^\delta, \text{Ch} w_0^\delta) - \mathcal{E}_{[0,t]}(z_0, \text{Ch} w_0) = \delta A_t(\tilde{z}_0, \text{Ch} \tilde{w}_0) + o(\delta).$$

(45)

In coordinate form, it means that we are going to construct a matrix $A_t$ of variable size $(n + K_0 - 1) \times (n + K^t - 1)$.

Below we construct the functions $\tilde{z}(t)$ and $\tilde{w}(t)$ such that

$$r(t, \delta) = \delta(\tilde{z}(t), \text{Ch} \tilde{w}(t)) + o(\delta), \quad t \notin S \cup T.$$

(46)

Let $0 < t_1 < \ldots < t_p < T$ be points from the set $S \cup T$, and denote $t_0 = 0$, $t_{p+1} = T$. Then the following iterative procedure is used to construct $\tilde{z}(t)$ and $\tilde{w}(t)$:
1. Let $\tilde{z}(0) = \tilde{z}_0$, $\tilde{w}(0) = \tilde{w}_0$, and $i = 0$.
2. Consider the interval $(t_i, t_{i+1})$. On this interval let

$$
\text{Ch} \, \tilde{w}(t) = \text{Ch} \, \tilde{w}(t_i).
$$

Hence, the vector $\text{Ch} \, \tilde{w}(t)$ is constant on each interval $(t_i, t_{i+1})$. Note that the interval $(t_i, t_{i+1})$ does not contain points from $S \cup T$, thus the vector $\text{Ch} \, w(t)$ is also constant on this interval due to the properties of the Preisach operator, and $\# \tilde{w}(t) = \# w(t) = K_{ti}$ for $t \in (t_i, t_{i+1})$. The last component equals $\tilde{w}^{(K_{ti})}(c, \tilde{z}(t))$ for every $t$.

3. The function $\tilde{z}(t)$ is defined on this interval from the system of linear differential equations

$$
\tilde{z}' = f_\tilde{z}(t, z(t), y(t)) \tilde{z}
+ f_y(t, z(t), y(t))H(\text{Ch} \, w(t); \text{Ch} \, \tilde{w}(t))
+ f_y(t, z(t), y(t)) \left(-w_{K_{ti-1}} \mathbf{J} \mathbf{J}_K(\mathbf{w}(t)) + \langle c, \tilde{z} \rangle J_K(\mathbf{w}(t)) \right),
$$

where

$$
H(\text{Ch} \, w(t); \text{Ch} \, \tilde{w}(t)) = \sum_{i=1}^{K_{ti}-1} \tilde{w}^{(i)}(t) J_i(\text{Ch} \, w(t)).
$$

4. At the moment $t_{i+1}$ the dimension and the components of the vector $\tilde{w}(t)$ are updated according to the following switch rules:

   If $t_{i+1} \in T$, then
   $$
   K_{ti+1} = K_{ti} - 2,
   \tilde{w}^{(k)}(t_{i+1}) = \tilde{w}^{(k)}(t_i), \quad 1 \leq k < K_{ti+1}.
   $$
   (R1)

   If $t_{i+1} \in S$, then
   $$
   K_{ti+1} = K_{ti} + 1,
   \tilde{w}^{(k)}(t_{i+1}) = \tilde{w}^{(k)}(t_i), \quad 1 \leq k < K_{ti+1} - 1,
   \tilde{w}^{(K_{ti+1}-1)}(t_{i+1}) = \langle c, \tilde{z}(t_{i+1}) \rangle.
   $$
   (R2)

   If $t_{i+1} = T$, then the procedure ends.

5. Let $i := i + 1$, and go to step number 2.

**Hypothesis 1.** Suppose that $x^n(\sigma_i) \neq 0$, $i = 1, \ldots, m$. Let $\tilde{z}(t), \text{Ch} \, \tilde{w}(t)$ be the functions constructed by repeating steps 1–5 above, satisfying system (47), switch rules (R1)–(R2) and the initial conditions $\tilde{z}(0) = \tilde{z}_0$, Ch $\tilde{w}(0) = \text{Ch} \, \tilde{w}_0$. Then equation (46) holds for $t \notin S \cup T$ and sufficiently small $\delta > 0$.

Now we can define the linear mapping $A_t$. Equation (47) may be rewritten equivalently as

$$
\tilde{z}' = \left( f_\tilde{z}(t, z(t), y(t)) \right. + J_K(x(t)) f_y(t, z(t), y(t)) c^T \left. \right) \tilde{z}
+ f_y(t, z(t), y(t)) \left( H(\text{Ch} \, w(t); \text{Ch} \, \tilde{w}(t)) - \tilde{w}_{K_{ti}-1} \mathbf{J} \mathbf{J}_K(\mathbf{w}(t)) \right),
$$

where the vectors are treated as columns, and $c^T$ is the transposition of $c$. Equation (48) is a linear nonhomogeneous system in $\tilde{z}$. Note that the nonhomogeneous term

$$
f_y(t, z(t), y(t)) \left( H(\text{Ch} \, w(t); \text{Ch} \, \tilde{w}(t)) - \tilde{w}_{K_{ti}-1} \mathbf{J} \mathbf{J}_K(\mathbf{w}(t)) \right)
$$

is linear with respect to $\text{Ch} \, \tilde{w}$. Also the switch rule (R1) is linear with respect $\text{Ch} \, \tilde{w}$, and the switch rule (R2) is linear with respect to $\tilde{z}$ and $\tilde{w}$. Thus, we conclude that the functions $\tilde{z}(t)$ and $\text{Ch} \, \tilde{w}(t)$ depend linearly on $(\tilde{z}_0, \text{Ch} \, \tilde{w}_0)$. Hence, the mapping

$$
A_t : (\tilde{z}_0, \text{Ch} \, \tilde{w}_0) \rightarrow (\tilde{z}(t), \text{Ch} \, \tilde{w}(t))
$$

(49)
is linear. Hypothesis 1 implies that (45) holds for $t \notin S \cup T$ and sufficiently small $\delta > 0$. Therefore, the following corollary holds.

**Corollary 1.** Suppose that $x''(\sigma_i) \neq 0$, $i = 1, \ldots, m$. Then for each $t \notin S \cup T$ the evolution operator $E_{[0,t]}$ is differentiable at the point $(z_0, Ch w_0)$, and its derivative is defined by (49).

### 4.4. Discontinuous measure densities

The weight function $\mu(\alpha, \beta)$ in the definition of the output of the Preisach nonlinearity was assumed to be smooth everywhere above. In this final section we consider the weight functions that have some discontinuities (which are natural in many examples and applications [1, 7, 8, 9]). Let us restrict ourselves to the situation where the only possible discontinuities are along the straight lines $\alpha = \alpha_s$ and $\beta = \beta_s$, and moreover, let the limits $\mu(\alpha_s, v \pm 0)$ and $\mu(v \pm 0, \beta_s)$ be finite. Denote

$$
\gamma_-(v) = \mu(\alpha_s, v + 0) - \mu(\alpha_s, v - 0),
$$

$$
\gamma_+(v) = \mu(v + 0, \beta_s) - \mu(v - 0, \beta_s)
$$

and

$$
\Gamma_-(v) = \int_{\alpha_s}^{v} \gamma_-(s) \, ds, \quad \Gamma_+(v) = \int_{v}^{\beta_s} \gamma_+(s) \, ds.
$$

Let us introduce the sets

$$
U_+ = \{ t \geq 0 : x(t) = \alpha_s \text{ and } x'(t) < 0 \},
$$

$$
U_- = \{ t \geq 0 : x(t) = \beta_s \text{ and } x'(t) > 0 \},
$$

$$
U_0 = \{ t \geq 0 : (x(t) = \alpha_s \text{ or } x(t) = \beta_s) \text{ and } x'(t) = 0 \}.
$$

We suppose that the set $U_0$ is empty, and the sets $U_\pm$ are finite (or empty), and that they do not intersect with $S \cup T$.

Let the evolution of $(\tilde{z}, \tilde{w})$ be defined by (47), (R1), (R2) and the additional switch rule: if $t_{i+1} \in U_+ \cup U_-$, then

$$
K^{t_{i+1}} = K^{t_i},
$$

$$
\tilde{w}(t_{i+1}) = \tilde{w}(t_i),
$$

$$
\tilde{z}(t_{i+1}) = \tilde{z}(t_{i+1} - 0) + \langle c, \tilde{z}(t_{i+1} - 0) \rangle \Gamma_{\pm} \left( w(K^{t_{i+1} - 1}(t_{i+1})) f_y(t_{i+1}, z(t_{i+1}), y(t_{i+1})) \right) x'(t_{i+1}) \tag{R3}
$$

with $x(t) = \langle c, z(t) \rangle$ and $x'(t) = \langle c, z'(t) \rangle = \langle c, f(t, z(t), y(t)) \rangle$, where $t_i$ include points from $U_\pm$, and $\Gamma_+$ is used if $t_{i+1} \in U_+$, and $\Gamma_-$ if $t_{i+1} \in U_-$. We note that (R3) may be rewritten equivalently as

$$
\tilde{z}(t_{i+1}) = \left( I + \Gamma_{\pm} \left( \frac{w(K^{t_{i+1} - 1})(t_{i+1})}{x'(t_{i+1})} f_y(t_{i+1}, z(t_{i+1}), y(t_{i+1})) \right) c^T \right) \tilde{z}(t_{i+1} - 0),
$$

where $I$ denotes the identity $n \times n$ matrix. Hence, the switch rule (R3) is linear with respect to $\tilde{z}$.

**Hypothesis 2.** Let $\mu(\alpha, \beta)$ be smooth, except for discontinuities at the straight lines $\alpha = \alpha_s$ and $\beta = \beta_s$. Suppose that $x''(\sigma_i) \neq 0$, $i = 1, \ldots, m$. Let $(\tilde{z}, Ch \tilde{w})$ be the functions constructed by repeating steps 1–5 from section 4.3, satisfying system (47), switch rules (R1)–(R3) and the initial conditions $\tilde{z}(0) = \tilde{z}_0$, $Ch \tilde{w}(0) = Ch \tilde{w}_0$. Then for $t \notin S \cup T \cup U_+ \cup U_-$ the equality

$$
r(t, \delta) = \delta(\tilde{z}(t), Ch \tilde{w}(t)) + o(\delta) \tag{52}
$$

holds for sufficiently small $\delta > 0$. 
Consider the linear mapping

$$A_t : (z_0, Ch \tilde{w}_0) \rightarrow (\tilde{z}(t), Ch \tilde{w}(t)),$$

with functions $\tilde{z}$ and $Ch \tilde{w}$ defined as in Hypothesis 2.

**Corollary 2.** Let $\mu(\alpha, \beta)$ be smooth, except for discontinuities at the straight lines $\alpha = \alpha_*$ and $\beta = \beta_*$. Suppose that $x''(\sigma_i) \neq 0$, $i = 1, \ldots, m$. Then for each $t \notin S \cup T \cup U_+ \cup U_-$ the evolution operator $E_{[0,t]}$ is differentiable at the point $(z_0, Ch w_0)$, and its derivative is defined by (53).

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