Block Sequential Decoding Techniques for Polar Subcodes

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Abstract

A reduced complexity sequential decoding algorithm for polar subcodes is described. The proposed approach relies on a decomposition of the polar (sub)code into a number of outer codes, and on-demand construction of codewords of these codes in the descending order of their probability. The proposed algorithm can be also used for decoding of polar codes with CRC and short extended BCH codes. It has lower average decoding complexity compared to the existing decoding algorithms for the corresponding codes.

Index Terms

Polar codes, polar subcodes, sequential decoding, Plotkin construction.

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I. INTRODUCTION

Polar codes is a family of capacity-achieving codes with low-complexity construction, encoding and decoding algorithms [1]. However, their finite-length performance is quite poor. Improved code constructions, like polar codes with CRC [2] and polar subcodes [3], [4], were shown to outperform state-of-the-art LDPC and turbo-codes. However, list decoding techniques are needed in order to perform near-maximum likelihood decoding of such codes [2].

The complexity of the Tal-Vardy list decoding algorithm turns out to be rather high. It can be reduced by employing block decoding techniques, i.e. by joint processing of sufficiently large blocks of information symbols [5]. The complexity of this method can be further reduced by constructing unrolled decoders, which avoid as much as possible flow control logic. An alternative approach is to utilize stack decoding [6], and, in particular, the sequential decoding method introduced in [7], [8]. The latter approach avoids construction of many useless low-probability paths in the code tree. For sufficiently high SNR, its complexity approaches that of the successive cancellation (SC) decoding algorithm with the performance close to that of the list SC method.

Many of the implementation tricks considered in [5] in the context of list decoding are applicable in the case of sequential decoding as well. We show that by combining them (as well as some new ones) with the principle of sequential decoding, one can obtain the performance close to that of the SC list decoder with large list size with complexity approaching (at high SNR) that of the unrolled SC decoder. The proposed approach can be used both for polar subcodes and polar codes with CRC. Furthermore, we show that, by exploiting the representation of a linear code via a system of dynamic freezing constraints, the proposed approach can be used for decoding of other error correcting codes. In particular, we show that for a $(128, 64, 22)$ extended BCH code the proposed algorithm provides better performance and lower complexity compared to a recent trellis-based sequential-type algorithm [9].
The paper is organized as follows. The background on polar codes is presented in Section II. The block sequential decoding algorithm is introduced in Section III. Implementation issues are discussed in Section V. Complexity analysis is provided in Section VI. Simulation results are presented in Section VII.

II. BACKGROUND

A. Polar codes and polar subcodes

A polar code $((n = 2^m, k)$ over $\mathbb{F}_2$ is a linear block code generated by $k$ rows of matrix $A_m = F^\otimes m$, where $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\otimes m$ denotes $m$-times Kronecker product of the matrix with itself. Hence, a codeword of a classical polar code is obtained as $c_{n-1} = u_{n-1} A_m$, where $a^t_s = (a_s, \ldots, a_t)$, $u_i = 0, i \in \mathcal{F}$, $\mathcal{F} \subset \{0, \ldots, n-1\}$ is the set of $n-k$ frozen symbol indices, and the remaining symbols are set to the data symbols being encoded.

It was suggested in [3] to set frozen symbols $u_i, i \in \mathcal{F}$ not to zero, but to linear combinations of some other symbols, i.e.

$$u_i = \sum_{s=0}^{i-1} V_{j_i,s} u_s,$$

where $V$ is a $(n-k) \times n$ binary matrix, such that its rows end in distinct columns, and $j_i$ is the index of row with the last non-zero element in column $i$. Such symbols with non-trivial right hand side expressions are called dynamic frozen, and the corresponding codes are referred to as polar subcodes. Decoding of such codes can be implemented by a straightforward generalization of the successive cancellation algorithm and its derivatives.

Properly constructed polar subcodes may have higher minimum distance than classical polar codes. This results in substantially better performance [3], [10] under the list SC algorithm and its derivatives. Polar codes with CRC [2] can be considered as a special case of polar subcodes.

1Polar codes are typically defined with the bit-reversal permutation matrix. However, it is convenient here to omit it, since this results in a simpler description of the proposed decoding algorithm.
A system of dynamic freezing constraints may be constructed for any linear code of length $2^m$, by setting $V = QH A^T_m$, where $H$ is the check matrix and $Q$ is a suitable invertible matrix. This enables one to decode such code with the list SC algorithm. Extended primitive narrow-sense BCH codes were shown to admit near-ML decoding with relatively small list size [3].

B. Generalized Plotkin decomposition

**Theorem 1** ([3]). Any linear $(2n, k, d)$ code $C$ has a generator matrix given by $G = \begin{pmatrix} I_{k_1} & 0 & \tilde{I} \\ 0 & I_{k_2} & 0 \end{pmatrix} \begin{pmatrix} G_0 & 0 \\ G_1 & G_1 \\ G_2 & G_2 \end{pmatrix}$, where $I_l$ is a $l \times l$ identity matrix, $G_i, 0 \leq i < 3$, are $k_i \times n$ matrices, $k = k_0 + k_1$, $\tilde{I}$ is obtained by stacking a $(k_0 - k_2) \times k_2$ zero matrix and $I_{k_2}$, and $k_2 \leq k_0$.

This theorem enables one to represent any linear block code $C$ of even length in a way similar to classical Plotkin concatenation of two codes, and use the corresponding low complexity decoding algorithms. This will be referred to as generalized Plotkin decomposition (GPD) of $C$, i.e. $C$ is decomposed into codes $C_0$ and $C_1$ generated by $G_0$ and $G_1$, respectively, with correction matrix $G_2$.

**Example 1.** Consider a $(16, 6, 6)$ code generated by $G = \begin{pmatrix} 10010110 & 10010110 \\ 01010101 & 01010101 \\ 00110011 & 00110011 \\ 00001111 & 00001111 \\ 10000010 & 11011000 \\ 11111111 & 00000000 \end{pmatrix}$. Its
GPD is given by $G_0 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, G_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$. Code $C_0$ generated by $G_0$ can be further decomposed into $(4, 1)$ codes $C_{01}$ and $C_{11}$ with correction matrix $G_{02} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$.

The GPD may be further extended to obtain the construction of interlinked generalized concatenated codes (IGCC) [3]. IGCC encodes the subvector $u^{(i)} \in \mathbb{F}_2^{K_i}$ of the data vector not with the outer code $C_i$, as in the classical generalized concatenated codes [11], but with its coset given by $C_i + \left( \sum_{s=0}^{i-1} u^{(i)} M^{(s,i)} \right)$, where $M^{(s,i)} \in \mathbb{F}_2^{K_i \times N}$ are some matrices, as shown in Figure 1. This results in a linear block code of length $Nn$ and dimension $\sum_{i=0}^{n-1} K_i$.

IGCC can be decoded using the multistage decoding algorithm, which was introduced originally for the case of multilevel/generalized concatenated codes [11], [12]. However, one needs to perform decoding not in outer codes, but in their cosets. This can be done with any decoder for $C_i$, provided that the signs of its input log-likelihood ratios (LLRs) are appropriately adjusted.

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**Fig. 1:** Encoder of an interlinked generalized concatenated code
C. Sequential decoding of polar codes

Let \( u_{n-1} \) be the vector of input symbols of the polarizing transformation used by the transmitter. Given a received noisy vector \( y_{n-1} \), the sequential decoding algorithm constructs a number of partial candidate vectors \( v_{0}^{\phi-1} \in \mathbb{F}_2^\phi, \phi \leq n \), evaluates how close their continuations \( v_{0}^{n-1} \) may be to the received sequence, and eventually produces a single codeword, being a solution of the decoding problem.

The algorithm makes use of a priority queue (PQ). A PQ is a data structure, which contains tuples \( (M, v_{0}^{\phi-1}) \), where \( M = M(v_{0}^{\phi-1}, y_{n-1}) \) is the score of vector \( v_{0}^{\phi-1} \), and provides efficient algorithms for the following operations [13]:

- push a tuple into the PQ;
- pop a tuple \( (M, v_{0}^{\phi-1}) \) (or just \( v_{0}^{\phi-1} \)) with the highest \( M \);
- remove a tuple from the PQ.

We assume here that the PQ may contain at most \( D \) elements.

Formally, the stack decoding algorithm for polar codes operates as follows [14]:

1) Push into the PQ a zero-length vector with score 0. Let \( q_{0}^{n-1} = 0 \), where \( q_{\phi} \) is the counter for the number of visits to phase \( \phi \).

2) Extract from the PQ a node \( v_{0}^{\phi-1} \) with the highest score. Let \( q_{\phi} \leftarrow q_{\phi} + 1 \).

3) If \( \phi = n \), return codeword \( v_{0}^{n-1}A_m \) and terminate the algorithm.

4) If the number of valid children of node \( v_{0}^{\phi-1} \) exceeds the amount of free space in the PQ, remove from it the element with the smallest score.

5) Compute the scores \( M(v_{0}^{\phi}, y_{0}^{n-1}) \) of valid children \( v_{0}^{\phi} \) of the extracted node, and push them into the PQ.

6) If \( q_{\phi} \geq L \), remove from PQ all nodes \( v_{0}^{j-1}, j \leq \phi \).

7) Go to step 2.

In what follows, by iteration we mean one pass of the above algorithm over steps 2–7.
A score function \( M(v_0^{\phi-1}, y_0^{n-1}) \) can be obtained as a generalization of the Fano metric, which was introduced originally for sequential decoding of convolutional codes \[15\]. In the context of polar codes, this function after some approximations becomes [8]

\[
M(v_0^{\phi-1}, y_0^{n-1}) = \sum_{i=0}^{\phi-1} \tau(S_m^{(i)}(v_i^{-1}|y_0^{n-1}), v_i) - \Psi(\phi),
\]

(2)

where \( \Psi(\phi) = E_{y_0^{n-1}}[R(u_0^{\phi-1}|Y_0^{n-1})] \) is the bias function, which can be pre-computed offline, \( Y_0^{n-1} \) are the random variables corresponding to the received vector,

\[
\tau(S, v) = \begin{cases} 
0, & \text{sgn}(S) = (-1)^v \\
-|S|, & \text{otherwise}.
\end{cases}
\]

is the penalty function, and \( S_m^{(i)}(v_i^{-1}, y_0^{n-1}) \) are the modified log-likelihood ratios, which are given by

\[
S^{(2i)}(v_0^{2i-1}, y_0^{2\lambda-1}) = Q(a, b) = \text{sgn}(a) \text{sgn}(b) \min(|a|, |b|),
\]

(3)

\[
S^{(2i+1)}(v_0^{2i}, y_0^{2\lambda-1}) = P(v_2i, a, b) = (-1)^v_2i a + b,
\]

(4)

where \( a = S^{(i)}(v_0^{2i-1} \oplus v_0^{2i-1}, y_0^{2\lambda-1}) \) and \( b = S^{(i)}(v_0^{2i-1}, y_0^{2\lambda-1}) \).

The first term of (2) is the total penalty of path \( v_0^{\phi-1} \) for its deviation from the one given by hard decisions based on LLRs \( S_m^{(i)}(v_i^{-1}|y_0^{n-1}) \). These deviations may be required either to satisfy freezing constraints, or to compensate possibly incorrect decisions on non-frozen symbols. The first term appears to be exactly the path score used in the min-sum versions [16] of the Tal-Vardy list decoding algorithm and Niu-Chen stack algorithm. The second term is the expected value of the first term under the assumption that \( u_0^{\phi-1} = v_0^{\phi-1} \). It was shown in [8] that introduction of the bias term results in significant reduction of the average number of iterations performed by the stack algorithm.

Similarly to the case of sequential decoding of convolutional codes, the above described algorithm does not necessarily implement maximum likelihood decoding, even for \( L = \infty \).
III. BLOCK SEQUENTIAL DECODING

We propose to reduce the complexity of sequential decoding by joint processing of blocks of input symbols of the polarizing transformation. Similar approach was suggested in [5] in the context of list decoding. However, we show that in the context of sequential decoding this idea provides some additional benefits. Namely, one does not need to construct immediately $L$ most probable codewords for each block. Instead, these codewords can be constructed on-demand, and in many cases just one codeword is sufficient. Such codeword can be obtained using an appropriate maximum-likelihood decoding algorithm (or even by taking a hard-decision vector) with much lower complexity compared to straightforward (block) SC list decoding.

A. Recursive decomposition of polar subcodes

Let us consider decoding of an $(n = 2^l, k)$ code $C$. We propose to recursively apply to it the generalized Plotkin decomposition (see Theorem I) until one obtains codes, which admit efficient decoding. This results in a decomposition tree similar to that introduced in [17].

Each non-leaf node of this tree corresponds to a code $C_i$, and two its children correspond to codes $C_{i0}$ and $C_{i1}$ obtained from its GPD. Each node in this tree is identified by some index $i \in \mathbb{B}$, where $\mathbb{B} = \bigcup_{j=0}^{l} \{0, 1\}^j$. Codes corresponding to the leaves of this tree will be referred to as outer codes. Let $\mathcal{L} \subset \mathbb{B}$ be the set of indices $i$ of leaves. Let $I$ be the array of leaf indices $i$ arranged lexicographically in the ascending order. Let $V$ be the number of leaves in the tree. Figure 2 presents the GPD tree for $(16, 6)$ code considered in Example 1.

Essentially, the list and sequential SC algorithms recursively decompose $(n, k)$ polar code $C$, until codes of length 1 are obtained. Each of these codes correspond to some $u_\phi$, $0 \leq \phi < n$, where $\phi$ is the phase number. We propose to arrange symbols $u_\phi$ into a number of blocks, which correspond to the $(n_i, k_i, d_i)$ codes $C_i, i \in \mathcal{L}$ (or their cosets), obtained via the GPD. The $i$-th block starts at phase $\phi_i - n_i + 1$ and ends at phase $\phi_i$, so that $\phi_{i1} = \phi_i$, $\phi_{i0} = \phi_i - n_i/2$, where
\( n_i = 2^{m_i} \) for some \( m_i \leq m \), and \( \phi = n - 1 \). Symbols within the same block are processed jointly. This processing reduces to list decoding of outer codes \( C, i \in L \).

It remains to transform the path score function given by (2) into a form suitable for use with list decoders of outer codes. Let

\[
E(c_0^{n-1}, S_0^{m-1}) = -\sum_{i=0}^{n-1} \tau(S_i, c_i)
\]

be the ellipsoidal weight (also known as correlation discrepancy) of vector \( c_0^{n-1} \in \mathbb{F}_2^n \) with respect to LLR vector \( S_0^{m-1} \) [18], [19].

**Lemma 1.** For any \( c_0^{2n-1} \in \mathbb{F}_2^{2n-1} \) one has \( E(c_0^{2n-1}, S_0^{2m-1}) = E(c_0^{n-1} \oplus c_n^{2n-1}, \bar{S}_0^{n-1}) + E(c_n^{2n-1}, \bar{S}_0^{m-1}) \), where \( \bar{S}_i = Q(S_i, S_{i+n}), \bar{S}_i = P(c_i \oplus c_{i+n}, S_i, S_{i+n}) \).

**Proof:** Observe that \( E(c_0^{2n-1}, S_0^{2m-1}) = \sum_{i=0}^{n-1} E((c_i, c_{i+n}), (S_i, S_{i+n})) \). Hence, it is sufficient to prove the statement for \( n = 1 \). It can be seen that \( E(c_0^1, S_0^1) = \gamma = E((0, 0), (S_0', S_1')) \), where \( S_i' = (-1)^{c_i} S_i \). Hence, it is sufficient to consider the case of \( c_i = 0 \).

For the case \( Q(S_0', S_1') > 0 \) one has \( \gamma = -\tau(0, S_0') - \tau(0, S_1') = -\tau(0, S_0' + S_1') \), while for \( Q(S_0', S_1') < 0 \) one has \( \gamma = -\min(|S_0'|, |S_1'|) - \tau(0, S_0' + S_1') \). The latter equality follows by considering the cases of \( S_0' + S_1' > 0 \) and \( S_0' + S_1' \leq 0 \).

**Theorem 2.** \( E(u_0^{2m-1} A_m, S) = -\sum_{i=0}^{2m-1} \tau(S_m^{(i)}(u_0^{i-1}, y_0^{2m-1}), u_i), \) where \( S = \)
\((S_0^{(0)}(y_0), \ldots, S_0^{(0)}(y_{2^m-1}))\).

**Proof:** For \(m = 0\), the statement is obvious. Let us assume that it is valid for some \(m \geq 0\). Then, from Lemma 1 one obtains
\[
E(u_0^{2^{m+1}-1} A_{m+1}, S) = E(u_0^{2^{m-1}} A_m, \tilde{S}) + E(u_2^{2^m-1} A_m, \bar{S}),
\]
where \(\tilde{S}_i = S_1^{(0)}(y_i, y_{i+2^m}), 0 \leq i < 2^m\), and \(\bar{S}_i = S_1^{(1)}((u_0^{2^{m-1}} A_m)_i, (y_i, y_{i+2^m}))\). Then the result follows from the inductive assumption.

Theorem 2 implies that the sum of some terms in (2), which correspond to the same block in the GPD tree, can be obtained by construction of a codeword of the corresponding code \(C_i\), and computing its ellipsoidal weight. In some cases this can be more efficient than performing iterations of the above described sequential decoding algorithm.

**B. The algorithm**

The main idea of the proposed approach is to combine the steps of the above described sequential decoding algorithm, which correspond to the same block in the GPD tree. Each combined step reduces to list decoding of the corresponding code \(C_i\) (or its coset), and may produce at most \(2^{k_i}\) codewords. However, some simplifications are possible:

1) The codewords of outer codes \(C_i, i \in \mathcal{L}\), should be constructed in the ascending order of their ellipsoidal weight. Furthermore, these codewords should be constructed on-demand, i.e. only if they have chances to be a part of a path with high \(M(u_0^{d-1}, y_0^{n-1})\).

2) In many cases the hard decision vector corresponding to some intermediate LLR vector \(S\) is error free, i.e. is a codeword of \(C_i\). In this case one should avoid invoking a relatively complex soft-decision decoding algorithm, unless non-ML codewords of the corresponding outer code is needed.

For the sake of simplicity, we replace indices \(i \in \mathcal{L}\) of outer codes by their position \(j\) in the \(I\) array (see Section III-A), \(j \in \{0, 1, \ldots, \mathcal{V} - 1\}\), where \(\mathcal{V}\) is the number of outer codes, considered by the decoder. Note that for any path the \(j\)-th outer decoder is invoked only after
Algorithm 1: Block sequential decoding algorithm

**Decoding**($S_{n-1}^0, L, D$)

1. $l \leftarrow \text{AssignInitialPath}()$; $\text{PushPath}(0, l)$
2. $q_0^{-1} = 0; \psi_l = 0; R_l = 0; w_l = 0; B_l = 0$
3. $s = \text{GetArrayPointers}_W(l, 0)$
4. $s[i] = S_i, 0 \leq i < n$
5. while $\text{HasPaths}()$
6. do $(M, l) \leftarrow \text{PopMax}()$
7. if $\lceil \phi_{\psi_l-1}/2^{m_{\psi_l-1}} \rceil$ is odd
8. then $\text{IterativelyUpdateC}()$
9. if $\psi_l = V$
10. then return $\text{GetArrayPointers}_C R(l, 0, 0)$
11. if $B_l$
12. then $\text{RemoveBadPaths}(D)$
13. $\text{BackwardPass}(l)$
14. $\text{IterativelyCalcS}(l, m - m_{\psi_l}, \lceil \phi_{\psi_l}/2^{m_{\psi_l}} \rceil)$
15. $\text{ForwardPass}(l)$
16. $q_{\psi_l} \leftarrow q_{\psi_l} + 1$
17. if $q_{\psi_l} \geq L$
18. then for All paths $l'$ stored in PQ
19. do if $\psi_{l'} \leq \psi_l$
20. then $\text{KillPath}(l')$
21. Remove $l'$ from the PQ

(a) The algorithm

| Variable | Description |
|----------|-------------|
| $l$      | index of a path $v_0^{\phi_{\psi_l-1}}$ |
| $V$      | Number of leaves in the GPD tree |
| $q_i$    | Number of invocations of the $i$-th outer decoder |
| $\psi_l$ | The index of outer decoder to be invoked for the $l$-th path |
| $\phi_i$ | The last phase of the $i$-th block |
| $B_l$    | True if the $l$-th path should be cloned |
| $m_j$    | $= \log_2 n_j$, where $n_j$ is the length of outer code $C_j$ |
| $d_j$    | Minimum distance of outer code $C_j$ |
| $H_j$    | Check matrix of $C_j$ |
| $S_i$    | Log-likelihood ratio |
| $Z_l$    | Continuation status of the $l$-th path |
| $\tilde R_l$ | Accumulated penalty $R(v_0^{\phi_{\psi_l-1}} | y_0^{n-1})$ for the $l$-th path |
| $\tilde Z_l$ | Saved state for the last outer decoder used for the $l$-th path |
| $M$      | Score of a path |
| $\hat c$ | A pointer to an array containing a codeword of $C_j$ |
| $w_l$    | A vector of dynamic frozen symbols intermediate values |

(b) Variables used in the algorithm

Fig. 3: Block sequential decoding algorithm
(j − 1)-th, so that the decisions of decoders 0, . . . , j − 1 are used both to compute the input LLRs and select a coset representative for the j-th decoder.

We assume that for each outer code C_j there are Preprocess(C_j, S, Z) and GetNextCodeword(C_j, Z, ˇc) algorithms. The first one performs some code-dependent preprocessing of LLR vector S, and saves its results in a state variable Z. The second algorithm uses Z to construct the next most probable codeword in the list, which is stored in the array given by pointer ˇc, and returns tuple [e, b], where b is a boolean value, which indicates if more codewords can be obtained by the subsequent calls, and e = E(ˇc, S). Structure Z includes the following fields:

- p — a coset representative, which enables decoding of codes with non-trivial dynamic freezing constraints (1), as described in Section V-3.
- S — vector of LLRs.
- Any additional data needed for efficient recovery of codewords of C_j for given S.

Figure 3a illustrates the proposed decoding algorithm. Table 3b presents the description of its internal variables. The algorithm is based on the Tal-Vardy list decoder infrastructure [20] with some modifications, which are discussed below. The input arguments for the algorithm are the log-likelihood ratios S_i = \log \frac{W(y_i|0)}{W(y_i|1)}, where y_i is the result of transmission of codeword symbol c_i over a memoryless output-symmetric channel, maximal number of times L the decoder is allowed to pass via any phase or block, and maximal total number of paths Θ.

The function AssignInitialPath performs the appropriate initialization operations, and returns an identifier of the initial path. The corresponding entry is pushed into the priority queue. A writable pointer s to an array of values S^{(i)}_0 is obtained, and the input LLRs are copied into this array. The following loop is performed until either a codeword is obtained, or no paths remain in the priority queue.

At line 6 path l with the highest score M is extracted from the priority queue. Here ψ_l ∈ {0, 1, . . . , V − 1} denotes the number of the outer code to be decoded on the current iteration for
path $l$, $\phi_j$ denotes the index of the last input symbol corresponding to the $j$-th outer code, while $n_j = 2^{m_j}, j \in \{0, 1, \ldots, V - 1\}$ is its length. If necessary, partial sums of the input symbols $v_i$ of the polarizing transformation, which are needed for computing of $S_m^{(i)}(v^{i-1}_0, y^{n-1}_0)$, are updated at line 8. If the last outer code has been processed, a read-only pointer to the codeword is returned at line 10, and decoding terminates.

The boolean variable $B_l$ is set to true iff one more codeword of code $C_{\psi_l-1}$ can be obtained by the corresponding outer decoder. In this case the decoder ensures at line 12 that there are at most $D - 2$ entries in the priority queue (if not, the paths with the lowest score are killed), and makes a call to BackwardPass function, which constructs the next most probable codeword of $C_{\psi_l-1}$. This variable is set in the ForwardPass and BackwardPass functions.

At line 14 the vector of log-likelihood ratios $S$ is computed. The decoder makes a call to the ForwardPass algorithm, which constructs the most probable continuation of the $l$-th path, i.e. performs (near) maximum likelihood decoding of vector $S$ in an appropriate coset of an outer code. If the number of times $q_{\psi_l}$ the decoder has visited the $\psi_l$-th block exceeds $L$, then paths shorter than $\phi_{\psi_l}$ are removed in line 18.

The first steps of ForwardPass algorithm (see Figure 4a) are to obtain writable pointers to the array $S$ of log-likelihood ratios $S_m^{(i)}(u^{i-1}_0, y^{n-1}_0), \phi_{\psi_l-1} < i \leq \phi_{\psi_l}$, computed by IterativelyCalcS, where $\phi_j$ is the phase of the last symbol corresponding to the $j$-th block, and to the array $\hat{c}$, which is used to store the most probable continuation of the $l$-th path. At line 3 a coset representative of the outer code is obtained as described in Section V-B and the signs of the LLRs are appropriately adjusted at line 4. If the product of a check matrix $H_{\psi_l}$ of the corresponding outer code $C_{\psi_l}$ and the hard-decision vector corresponding to $S$ is non-zero, then an appropriate pre-processing algorithm for $C_{\psi_l}$ is invoked (see Section II for details), and the most probable codeword is constructed. Variable $e$ is assigned to the ellipsoidal weight of this codeword, while $b$ is set to true iff less probable codewords can be obtained.

If the hard decision vector $\hat{c}$ appears to be a valid codeword, $S$ is saved in the state variable
(a) Preprocessing and construction of the most probable codeword of outer codes
(b) Obtaining the next most probable codeword of outer codes

Fig. 4: On-demand construction of codewords of outer codes
\[ Z_l, \text{ so that computationally expensive pre-processing can be done later if other codewords of } C_{\psi_l}, \text{ besides } \hat{c}, \text{ are needed for the decoder, and the corresponding flag } Z_l \text{ is set.} \]

At line 12 the coset representative is added to the obtained codeword, as required by the GPD or, equivalently, dynamic freezing constraints. At line 13 some preprocessing is performed to enable calculation of the coset representatives in the subsequent blocks. The details of this operation are provided in Section V-B. Finally, the value \[ R_l = \hat{R}(v_{\phi_{\psi_l}}^{\phi_{\psi_l}-1} | y_{n-1}^n), \] where \[ v_{\phi_{\psi_l}-1}^{\phi_{\psi_l}} = \hat{c}A_{m_j}, \] is updated according to Theorem 2, and the path is pushed to the priority queue. The previous value of \[ R_l \] is saved in \[ \tilde{R}_l \], so that it can be used later to obtain the score of less probable continuations of this path.

Figure 4b illustrates the algorithm, which is used to obtain less probable codewords of outer codes in the increasing order of their ellipsoidal weight. At line 11 the path is cloned. If the hard decision vector obtained during the previous call to the ForwardPass was a valid codeword of the corresponding outer code, i.e. if \( Z_l = 1 \), then it is very likely that the less probable codewords will not be needed. Hence, it is possible to skip construction of such a codeword. However, occasionally such codewords may be needed, and some provision needs to be done in order to recover them later. It can be easily seen that the ellipsoidal weight of any such codeword cannot be less than \[ d_{\psi_l-1} \min_i |Z_l S_i|, \] where \( d_{\psi_l-1} \) is the minimum distance of \( C_{\psi_l-1} \). We propose to use this value for computing an estimate of the log-likelihood \( R_l' \) of the less probable path \( l' \). If this path is later selected by the decoder for further processing, the corresponding codeword should be actually constructed. Therefore we set \( Z_l' = 2 \).

If \( Z_l = 0 \), then a writable pointer to the destination array for storing the codeword is obtained at line 15, and an appropriate codeword of the outer code is stored in this array. The control variables \( B_{l'} \) and \( Z_{l'} \) for the cloned path are initialized, and the coset representative is added to the obtained codeword at line 18.

If a path with \( Z_l = 2 \) is obtained, this means that one has to actually perform preprocessing, which was skipped during a previous call to ForwardPass. Furthermore, one should compute
the less probable codeword of the outer code together with its exact ellipsoidal weight, which
was substituted with an estimate during the previous call to BackwardPass. These operations
are performed in lines 2–10. Observe that at line 4 one would obtain exactly the same codeword
as the hard decision vector constructed in a call to ForwardPass. This codeword is now useless,
and this operation can be skipped, provided that this does not affect operation of the subsequent
call to GetNextCodeword.

The details of low-level functions used in the proposed block sequential decoding algorithm
are discussed in Section V.

Example 2. Consider decoding of the \((16, 6)\) code introduced in Example 1 for the
case of AWGN channel at \(E_b/N_0 = 2\) dB. We need the values of bias function
\(\Psi_b(3) \approx -1.1, \Psi_b(7) \approx -1.8, \Psi_b(15) \approx -2.4\). Let the input LLRs \(S_0^{(0)}\) be equal
\((0.7, 3.2, 1.1, 3.7, 3.5, 0.7, 1.9, 3.3, 1.3, 3.3, 2.2, 1.2, -3.7, 3.8, 3.7, -1.4)\). Let \(l = 0\) be the index
of the initial path. At the first iteration at line 14 the decoder computes the vector of LLRs \(S_2^{(0)}\),
which is equal \((-0.7, 0.7, 1.1, -1.2)\). The syndrome for the corresponding hard-decision vector
with respect to a check matrix of \(C_{00}\) is non-zero. ForwardPass function obtains codeword
\((1, 1, 1, 1) \in C_{00}\) with the ellipsoidal weight \(e = 1.8\). Hence, at line 15 of the ForwardPass
function a path with score \(-0.7\) is pushed to the PQ, and \(B_0, Z_0\) are set to 1 and 0 for the
corresponding path, respectively.

This path is extracted from the PQ at the next iteration of the Decode algorithm. BackwardPass function obtains codeword \((0, 0, 0, 0)\) with ellipsoidal weight \(e = 1.9\). The
path is cloned (let the ID of the cloned path be \(l' = 1\)), and an entry with score \(-0.8\) and
\(B_1 = 0\) (since all two codewords of \((4, 1)\) code \(C_{00}\) have been explored) is pushed to the PQ.

The vector of LLRs \(S_2^{(1)}\), given by \((-4.2, -2.5, 0.8, -2.6)\), is obtained at line 14 for path 0. Now we have to decode it in the coset of code \(C_{01}\) given by offset vector \(Z_{l', p} = (1, 0, 1, 0)\). Hence, the Preprocess function for this code is applied to vector \((4.2, -2.5, -0.8, -2.6)\).
one obtains codeword \((1,1,1,1)\) with \(e = 4.2\), and path 0 is pushed to the PQ with score 
\[-1.8 - 4.2 + 1.8 = -4.2.\]

At the next iteration of the decoder path \(l = 1\) is extracted from the PQ. The vector of LLRs
\(S_{2}^{(1)}\), given by \((-2.8,3.9,3.0,-0.2)\), is obtained at line 14. The corresponding offset vector is 
\(Z_{l,p} = (0,0,0,0)\). Hence, all-zero codeword of code \(C_{01}\) is obtained with \(e = 3.0\), and path 1 is
pushed to the PQ with score 
\[-1.9 - 3.0 + 1.8 = -3.1.\]

This path is extracted at the next iteration from the PQ. The LLRs \(S_{1}^{(1)}\) are equal to
\((2.0,6.5,3.3,4.9,-0.2,4.5,5.6,1.9)\). These values are preprocessed by the decoder for code
\(C_{1}\), and the all-zero codeword with \(e = 0.2\) is obtained at line 8 of the ForwardPass function.
Hence, path 1 is pushed to the PQ with score 
\[-1.9 - 3.0 - 0.2 + 2.4 = -2.7.\]

This path is extracted at the next iteration of the decoder, and, since all leaf nodes in the
GPD tree have been visited, the decoder terminates returning the all-zero codeword.

The proposed algorithm can be tailored to implement decoding of polar codes with CRC. To
do this, one should add CRC validation to line 9 of the algorithm shown in Figure 3a, so that
iterations are performed until either a correct codeword is found, or no more paths remain in
the PQ.

The proposed algorithm is not guaranteed to provide the same performance as the original
sequential decoding algorithm. In some cases its performance may be better, since the decoders
for outer codes may avoid some errors of the sequential decoder. However, in some cases
performance degradation may occur, if it happens that for an incorrect path \(v_{0}^{n-1}\), where \(u_{0}^{\phi_{i}} = v_{0}^{\phi_{i}}\)
for some \(i, \forall j > i : M(v_{0}^{\phi_{j}},y_{0}^{n-1}) > M(u_{0}^{\phi_{i}},y_{0}^{n-1})\), and \(\exists \tau \in (\phi_{i},\phi_{j}) : \forall s \geq 0 M(v_{0}^{\tau},y_{0}^{n-1}) < M(u_{0}^{\phi_{i} + s},y_{0}^{n-1})\). However, simulation results presented below show that the impact of this
problem is negligible.
IV. DECODING OF OUTER CODES

As described in Section III-A, GPD is applied recursively until one obtains outer codes, which allow efficient ML decoding. Consider some outer code $C$. We need to construct a decoder, which can find the codewords $c^{(i)} \in C$ in the increasing order of their ellipsoidal weight $E(c^{(i)}, S_{0}^{N-1})$, where $S_{0}^{N-1}$ is the vector of LLRs. Some of the techniques presented below resemble those suggested in [5], but we present also the algorithms for some additional outer codes, most importantly first-order Reed-Muller and extended Hamming codes.

A. Low rate codes

Decoding of $(N,0)$, $(N,1)$ and $(N,2)$ codes is performed by exhaustive enumeration of their codewords $c^{(i)}$, computing the corresponding ellipsoidal weight $E(c^{(i)}, S_{0}^{N-1})$ for each codeword, and sorting them in the ascending order of $E(c^{(i)}, S_{0}^{N-1})$.

B. First order Reed-Muller and related codes

The first order Reed-Muller code $RM(1, \mu)$ is obtained as a polar code with the set of frozen symbol indices $\hat{F} = \{0, \ldots, 2^\mu - 1\} \setminus (\{0\} \cup \{2^i | 0 \leq i < \mu\})$. Decoding of such codes can be implemented using the fast Hadamard transform (FHT) with complexity $O(N \log N)$ [21]. FHT computes correlations $Z(c^{(i)}, S_{0}^{N-1}) = \sum_{j=0}^{N-1} (-1)^{c^{(i)} S_j}$ for $N$ codewords of the corresponding codes. The correlations for the remaining codewords are given by $Z(c^{(i+N)}, S_{0}^{N-1}) = -Z(c^{(i)}, S_{0}^{N-1})$, and $c^{(i+N)} = c^{(i)} + 1$, where 1 is a vector of 1’s. The ellipsoidal weight of a codeword is related to its correlation by

$$E(c^{(i)}, L_{0}^{N-1}) = \frac{1}{2} \left( \sum_{j=0}^{N-1} |S_j| - Z(c^{(i)}, S_{0}^{N-1}) \right)$$

Another type of outer codes, commonly arising in the GPD of polar codes, is a concatenation of a first order Reed-Muller code $RM(1, \mu - t)$ and a $(2^t, 1, 2^t)$ repetition code. Obviously, such codes may be also decoded using the FHT of order $2^{\mu-t}$. 
We propose also to use FHT-based decoder for the case of codes given by a union of at most 4 cosets of a first order Reed-Muller code $R$, i.e. $C = R \cup (R + c')$, and $C = R \cup (R + c') \cup (R + c'') \cup (R + c' + c'')$, where $c', c'' \notin R$. This turns out to be more efficient in practice than performing additional steps of GPD.

C. Rate-1 code

For $(N, N)$ codes we propose to find 4 most probable codewords, which can be obtained by identifying 2 smallest values $|S_j|$, $0 \leq j < N$, and flipping the corresponding bits of the hard decision vector. Simulations show that finding just 4 (out of $2^N$) most probable codewords of $(N, N)$ code does not result in any noticeable performance loss.

D. Single parity check code

We propose to perform decoding of $(N, N - 1, 2)$ codes by testing a few pre-defined error patterns $E^{(i)}$. First, the codeword symbols are arranged in the increasing order of their reliabilities, so that $|S_{t[0]}| \leq |S_{t[1]}| \leq \ldots |S_{t[N-1]}|$. Second, a hard decision vector $\hat{c}$ is constructed, and its parity $p$ is calculated. Then the codewords are constructed as $c^{(i)} = \hat{c} + e^{(i)}$, where $e^{(i)}$ is the vector containing 1’s on positions $t[\epsilon_{i, j}]$ and 0’s elsewhere, for all $E^{(i)} = \{\epsilon_{i, 0}, \ldots, \epsilon_{i, w_i}\} \in T^{(p)}$. The set of test error patterns $T^{(p)}$ can be constructed either analytically using the expressions derived in [22], or by simulations. It turns out that the same set of test error patterns can be used for decoding of codes of arbitrary length without any noticeable performance loss compared to the optimal decoder. In most cases it is sufficient to identify the positions $t[0], t[1]$ of only two least reliable symbols. This can be done using the tournament algorithm.

E. Double parity check codes

A $(N, N - 2, 2)$ polar code with the set of frozen symbol indices $F = \{0, 1\}$ can be obtained by interleaving two $(N/2, N/2 - 1, 2)$ codes. This enables one to decode such codes using a combination of two decoders for a single parity check code.
F. $(16,10,4)$, $(16,11,4)$ and $(16,12,2)$ codes

These codes, obtained by Plotkin concatenation of $(8,4,4)$ or $(8,3,4)$ codes and $(8,7,2)$ or $(8,8,1)$ codes, commonly arise in the GPD of polar codes. Decoding of these codes can be implemented using the approach introduced in [23].

V. LOW-LEVEL ALGORITHMS

A. Data structures and basic procedures

The proposed decoding algorithm can be implemented using the techniques suggested in [20]. However, several simplifications are possible. Let $l, \lambda, \phi, \beta$ denote the path, layer, phase and branch number, respectively. Each path is associated with arrays of intermediate LLRs $S_{l,\lambda}[\beta], 0 \leq l < \Theta, 0 \leq \lambda \leq m - \mu, 0 \leq \beta < 2^{m-\lambda}$, where $\Theta$ is the maximal number of paths considered by the decoder (i.e. the maximal size of the priority queue), and $2^\mu, 0 \leq \mu < m$, is the length of the shortest outer code, which has an efficient decoder implementation. Each path is also associated with value $R_l$, which contains values $\hat{R}(u^\phi_0|y_0^{n-1})$, similarly to [16], [24].

It was suggested in [20] to store the arrays of partial sum tuples $C_{l,\lambda}[\beta][\phi \mod 2]$. We propose to rename these arrays to $C_{l,\lambda,\phi \mod 2}[\beta]$. By examining the RecursivelyUpdateC algorithm presented in [20], one can see that $C_{l,\lambda,1}[\beta]$ is just copied to $C_{l,\lambda-1,\psi}[2\beta+1]$ for some $\psi \in \{0,1\}$, and this copy operation terminates on some layer $\lambda'$. Observe that $\lambda - \lambda'$ is equal to the maximal integer $d$, such that $\phi + 1$ is divisible by $2^d$. Therefore, we propose to co-locate $C_{l,\lambda,1}[\beta]$ with $C_{l,\lambda_0,0}[\beta]$. If bit reversal permutation is not used, this means that the corresponding pointers are given by $C_{l,\lambda,1} = C_{l,\lambda_0,0} + 2^{m-\lambda}(2^{\lambda-\lambda'} - 1)$. This not only results in the reduction of the amount of data stored by a factor of two, but also enables one to avoid ”copy on write” operation (see line 6 of Algorithm 9 in [20]). Therefore, the last index will be omitted in what follows.

We use the array pointer mechanism suggested in [20] to avoid data copying. However, we distinguish the case of read and write data access. Retrieving read-only pointers is performed by functions GetArrayPointerC_R$(l, \lambda)$ and GetArrayPointerS_R$(l, \lambda)$ shown in Figure
IterativelyCalcS(l, λ, φ)

1. \( d \leftarrow \max \{0 \leq d' \leq \lambda - 1 | \phi \text{ is divisible by } 2^{d'} \} \)
2. \( \lambda' \leftarrow \lambda - d \)
3. \( S' \leftarrow \text{GetArrayPointerS_R}(l, \lambda' - 1) \)
4. \( N \leftarrow 2^{m - \lambda'} \)
5. if \( \phi 2^{-d} \) is odd
   then \( \tilde{C} \leftarrow \text{GetArrayPointerC_R}(l, \lambda') \)
6. \( S'' \leftarrow \text{GetArrayPointerS_W}(l, \lambda') \)
7. \( S''[\beta] \leftarrow P(\tilde{C}[\beta], S'[\beta], S'[\beta + N]), 0 \leq \beta < N \)
8. \( S' \leftarrow S''; \lambda' \leftarrow \lambda' + 1; N = N/2 \)
9. while \( \lambda' \leq \lambda \)
10. do \( S'' \leftarrow \text{GetArrayPointerS_W}(l, \lambda') \)
11. \( S''[\beta] \leftarrow Q(S'[\beta + N], S'[\beta]), 0 \leq \beta < N \)
12. \( S' \leftarrow S''; \lambda' \leftarrow \lambda' + 1; N \leftarrow N/2 \)

(a) Computing \( S_{\lambda}^{(\phi)} (v_0^{\delta - 1}, y_0^{N - 1}) \)

IterativelyUpdateC(l, λ, φ)

1. \( \delta \leftarrow \max \{d|\phi + 1 \text{ is divisible by } 2^{d} \} \)
2. \( \tilde{C} \leftarrow \text{GetArrayPointerC_W}(l, \lambda - \delta, 0) \)
3. \( N \leftarrow 2^{m - \lambda}; \tilde{C} = \tilde{C} + N(2^\delta - 2); C'' \leftarrow \tilde{C} + N; \)
4. \( \lambda' \leftarrow \lambda - \delta \)
5. while \( \lambda > \lambda' \)
6. do \( C' \leftarrow \text{GetArrayPointerC_R}(l, \lambda) \)
7. \( \tilde{C}[\beta] \leftarrow C'[\beta] \oplus C''[\beta], 0 \leq \beta < N \)
8. \( N \leftarrow 2N; C'' \leftarrow \tilde{C}; \tilde{C} \leftarrow \tilde{C} - N \)
9. \( \lambda \leftarrow \lambda - 1 \)

(b) Updating \( C \) arrays

Fig. 5: Computing LLRs and partial sums

Retrieving writable pointers is performed by function \( \text{GetArrayPointerW}(T, l, \lambda) \), where \( T \in \{C', S'\} \) shown in Figure 7. This function implements reference counting mechanism similar to that described in [20]. It is discussed in more details in Section V-C.

Figures 5a and 5b present iterative algorithms for computing \( S_{l,\lambda}[\beta] \) and \( C_{l,\lambda}[\beta] \). These algorithms resemble the recursive ones given in [20]. However, the proposed implementation avoids costly array dereferencing operations.
B. Processing of dynamic frozen symbols

Decoding of polar subcodes requires one to be able to compute the values of dynamic frozen symbols, i.e. some linear combinations of symbols \( v_i \) for any path \( v_{0}^{\phi_{\psi_{l}}} \). The Tal-Vardy list decoding algorithm does not store these values explicitly. It is possible to express their values from the content of arrays \( C_{l,\lambda} \). However, we employ an alternative approach, which is more efficient in practice. In most cases, polar subcodes have only a few non-trivial dynamic frozen symbols, which depend on a small number of other symbols. Let \( f \) be the number of non-trivial equations (1) for the considered code. It can be assumed without loss of generality that these equations correspond to \( f \) initial rows of matrix \( V \). Let \( i_{s}, \in \mathcal{F}, 0 \leq s < f \), be the indices of the corresponding dynamic frozen symbols. Let \( \mathcal{P} = \{ j | V_{s,j} = 1, 0 \leq j < i_{s}, 0 \leq s < f \} \) be the set of indices of symbols participating in any of the dynamic freezing constraints.

We propose to allocate boolean variables \( w_{l,s}, 0 \leq l < D \) for each path, initialize them to 0 at decoder startup, and flip the value of \( w_{l,s} \) at each phase \( j < i_{s} \), such that \( V_{s,j} = 1 \) and \( v_{j} = 1 \), where \( v_{j} \) is the value of the \( j \)-th symbol on the \( l \)-th path. Then at phase \( i_{s} \) the value of \( v_{l,s} \) is exactly the value of the \( s \)-th dynamic frozen symbol for the corresponding path.

However, the above described block sequential decoding algorithm does not compute explicitly the values \( v_{j} \). But one can obtain these values as \( v_{j} = (\hat{c} \otimes m_{\psi_{l}})_{j} \mod 2^{m_{\psi_{l}}} \), where \( \hat{c} \) is a codeword of an outer code obtained for path \( l \) at block \( \psi_{l} \). This approach is illustrated in Figure 6. Observe that the operations at lines 2 and 4–5 of this algorithm can be efficiently implemented via bit mask manipulation techniques.

If there is a non-trivial dynamic frozen symbol in some block \( \psi_{l} \), i.e. \( \phi_{\psi_{l}} - 2^{m_{\psi_{l}}} < i_{s} \leq \phi_{\psi_{l}} \) for some \( s \), and \( v_{l,s} = 1 \) when the decoder reaches this block, then one should perform decoding in a non-trivial coset of the corresponding outer code. The coset representative is given by

\[
p_{s} = \sum_{i=\phi_{\psi_{l}}-2^{m_{\psi_{l}}}+1}^{\phi_{\psi_{l}}} (F \otimes m_{\psi_{l}})_{i} \mod 2^{m_{\psi_{l}}},
\]

(5)
Fig. 6: Accumulating the values of dynamic frozen symbols

where \( A_{i,-} \) denotes the \( i \)-th row of matrix \( A \). Algorithm GetCoset, called on line 3 of the ForwardPass algorithm, returns \( p = \sum_{s: \phi_{\psi_l} - 2^{m_{\psi_l}} s < i \leq \phi_{\psi_l}} w_{l,s} p_s \). The vectors \( p_s \) can be pre-computed.

Algorithms PrepareForDFEvaluation and GetCoset jointly implement multiplication by matrices \( M^{(s,t)} \) shown in Figure 11.

C. Memory management

Many paths considered by the proposed decoding algorithm share common values of \( S_{l,\lambda}[\beta] \) and \( C_{l,\lambda}[\beta] \), similarly to the case of Tal-Vardy list decoding algorithm. In order to avoid duplicate calculations one can use the same shared memory data structures. That is, for each path \( l \) and for each layer \( \lambda \) we store the index of the array containing the corresponding values \( S_{l,\lambda}[\beta] \) and \( C_{l,\lambda}[\beta] \). This index is given by \( p = \text{PathIndex2ArrayIndex}[l, \lambda] \), so that the corresponding data can be accessed as \( \text{ArrayPointer}[T][p], T \in \{S', C'\} \). Furthermore, for each integer \( p \) we maintain the number of references to this array \( \text{ArrayReferenceCount}[p] \). If the decoder needs to write the data into an array, which is referenced by more than one path, a new array needs to be allocated. Observe that there is no need to copy anything into this array, since it will be immediately overwritten. This is an important advantage with respect to the implementation.
def GetArrayPointerW(T, l, λ):
    p ← PathIndex2ArrayIndex[l, λ]
    if p = -1
        then p ← ALLOCATE(λ)
    else if ArrayReferenceCount[p] > 1
        then ArrayReferenceCount[p]−−
        p ← ALLOCATE(λ)
    return ArrayPointer[T][p]

Fig. 7: Write access to the data

def GetArrayPointerS_W(l, λ)
    return GetArrayPointerW('S', l, λ)

def GetArrayPointerC_W(l, λ, φ)
    δ ← max{d|φ + 1 is divisible by 2d }
    C ← GetArrayPointerW('C', l, λ − δ)
    if φ ≡ 1 mod 2
        then C ← C + 2m−λ(2δ − 1)
    return C

Fig. 8: Read-only access to the data

described in [2]. However, the sequence of array read/write and stack push/pop operations still satisfies the validity assumptions introduced in [2], so the proposed algorithm can be shown to be well-defined by exactly the same reasoning as the original Tal-Vardy algorithm.

Only one path considered by the decoder is constructed fully. Most of the paths are accessed only a few times and quickly abandoned. Hence, one does not need to provide the memory needed to accommodate all D paths. Therefore, we propose to create common memory pools for arrays C and S, denoted PoolC and PoolS, respectively. If a new array needs to be provisioned, a part of memory pool is assigned to it. Arrays C and S are provisioned simultaneously. Let Φ denote the amount of memory consumed from these pools. If Φ exceeds the size of the memory pools Λ, then decoding needs to be terminated. For sufficiently large Λ this typically occurs after the
ALLOCATE($\lambda$)

1. $[t, q] \leftarrow \text{Pop}(\text{InactiveArrayIndices}[\lambda]); t \leftarrow t(m + 1) + \lambda; \text{ArrayReferenceCount}[t] = 1$

2. If $q = 1$
   3. Then if $\Phi > \Lambda$
      4. Then ABORT

5. $\text{ArrayPointer}[S'][t] = \text{PoolS} + \Phi; \text{ArrayPointer}[C'][t] = \text{PoolC} + \Phi$

6. $\Phi_{+} = 2^{m-\lambda}$

7. Return $t$

Fig. 9: Adaptive memory allocation

correct path has been killed by the decoder. If the number of references to some array drops to 0, then the index of the array is saved in a stack of unused arrays, similarly to [20], so that it can be re-used later. The indices of unused arrays corresponding to different layers are stored in different stacks $\text{InactiveArrayIndices}[\lambda]$, since these arrays have different sizes. Allocation from the common pools occurs only if the corresponding index is extracted for the first time ($q = 1$ in Figure 9 which illustrates the proposed approach).

VI. COMPLEXITY ANALYSIS

The worst-case complexity of the proposed decoding algorithm corresponds to the case when exactly $LV$ iterations are performed, i.e. $q_i = L, 0 \leq i < V$. In this case the number of operations performed by the decoder is given by

$$C \leq L \sum_{i=0}^{V-1} \left( C'_i + C''_{i-1} + \Lambda(m - m_i, \lceil \phi_i / 2^{m_i} \rceil) \right),$$

(6)

where $C'_i$ is the complexity of a call to $\text{Preprocess}$ and $\text{GetNextCodeword}$ (see $\text{ForwardPass}$ function) for outer code $C_i$, $C''_{i}$ is the complexity of subsequent calls$^2$ to $\text{GetNextCodeword}$

$^2$We assume $C''_{-1} = 0.$
(see BackwardPass). Here \( \Lambda(\lambda, \phi) = 2^{m-\lambda}(2^{d+1} - 1) \) is the complexity of computing \( S^{(\phi)}_\lambda \) via function IterativelyCalcS, where \( d = d(\phi) < \lambda \) is the maximal integer, such that \( 2^{d(\phi)}|\phi \).

Application of the proposed approach makes sense only if Preprocess and GetNextCodeword functions provide a simpler way to obtain \( L \) most probable codewords of \( C_i \) compared to the Tal-Vardy algorithm\(^3\) with list size \( L \). Hence, the worst-case complexity of the proposed approach can be upper-bounded by considering the case (this corresponds to the algorithm presented in [8]) of \( m_i = 0 \). In this case one has \( C'_i = C''_i = 0, 0 \leq i < V = n \), and \( C \leq L \sum_{\phi=0}^{n-1} \Lambda(m, \phi) = \sum_{\phi=0}^{n-1}(2^{d(\phi)+1} - 1) \). For any \( d < m - 1 \) there are \( 2^{m-d-1} \) integers \( \phi < 2^m \) divisible by \( 2^d \) (and 2 of them for \( d = m - 1 \), but not divisible by \( 2^{d+1} \). Hence, one obtains

\[
C \leq L(2^m - 1 + \sum_{d=0}^{m-1} 2^{m-d-1}(2^{d+1} - 1)) = Lm2^m, \tag{7}
\]

which is identical to the complexity of the Tal-Vardy list decoding algorithm. The best case complexity corresponds to the case when the decoder visits each block exactly once, so it is given by (6) and (7) with \( L = 1 \).

There are additional costs associated with PQ operations. With appropriate implementation [13], [25], their complexity is upper bounded by \( O(DV) \).

### VII. NUMERIC RESULTS

Figure 10a illustrates the performance of the proposed block sequential decoding (BSD) algorithm. Simulations were run for the case of AWGN channel, BPSK modulation and randomized polar subcode [4]. For comparison, we report also the performance of the Tal-Vardy [2] sequential [8] and min-sum stack [14] decoding algorithms for the same code, and the CCSDS LDPC code under belief propagation decoding. It can be seen that the proposed algorithm provides essentially the same performance as the sequential and Tal-Vardy algorithms.

\(^3L \) must be sufficiently large to ensure that the Tal-Vardy algorithm always finds \( L \) most probable codewords.
Furthermore, for $L = 32$ its performance is close to that of the LDPC code with at most 200 decoder iterations. Even better performance is obtained for $L = 128$.

Figure 10 illustrates the average number of summation and comparison operations performed by the considered algorithms. It can be seen that the complexity of the sequential algorithm is much lower compared to the original stack algorithm (which corresponds to $\Psi(\phi) = 0, 0 \leq \phi < n$). Furthermore, the average complexity of the block sequential algorithm converges quickly to a value slightly less than $n \log_2 n$, the complexity of the SC algorithm. The complexity of the proposed algorithm is 1.5–2 times lower compared to that of the sequential decoder, and substantially lower compared to $L n \log_2 n$, the complexity of the Tal-Vardy list decoding algorithm, and the average complexity of the min-sum stack decoding algorithm. It is also substantially lower compared to the complexity of the BP decoder for the LDPC code. Observe that reducing the maximal number of iterations for the BP algorithm results in a noticeable performance degradation without significant complexity reduction for $FER < 0.1$.

By exploiting a representation of linear block code via the system of dynamic freezing constraints (1), the proposed approach can be used for decoding of other linear block codes,
besides polar subcodes. Extended primitive narrow-sense BCH (eBCH) codes were shown to have sufficiently low SC decoding error probability \[4\], and are therefore well-suited for decoding using the proposed algorithm. Figure 11 illustrates performance and complexity of the proposed approach for the case of \((128, 64, 22)\) eBCH code. For comparison, we report also the results for the Chen-Chen-Lin-Chang algorithm (a sequential-type trellis-based decoding method), reproduced from \[9\]. It can be seen that the proposed approach again provides the best performance and lowest decoding complexity.

Figure 12 illustrates the performance and throughput of the software implementation of the proposed block sequential decoding algorithm, as well as fast list and adaptive list decoding algorithms introduced in \[5\], for the case of polar subcodes and polar codes with CRC-8. Simulations were performed on Intel Core i7-2600K CPU running at 3.4 GHz with maximum turbo frequency 3.8 GHz. SIMD techniques introduced in \[5\], based on single-precision floating point arithmetic, were used to implement LLR computation in the proposed algorithm. Throughput results for the fast and adaptive list decoding algorithms are reproduced from \[5\]. The performance of polar codes with CRC under the proposed block sequential decoding algorithm...
is very close to that of the list decoder with the same $L$, and is therefore not shown. As it may be expected, polar subcodes provide better performance than polar codes with CRC, and increasing list size $L$ results in better performance. One can see that for polar subcodes at sufficiently high SNR the proposed block sequential decoding algorithm even for $L = 32$ provides the same or even better average throughput as the fast list decoding algorithm introduced in [5] for polar codes with CRC and $L = 2$. Furthermore, at high SNR the throughput of block sequential decoding algorithm for polar codes with CRC exceeds that of the fast list decoding algorithm. Observe that the algorithm presented in [5] relies on unrolling to eliminate redundant calculations, i.e. the decoder is specific for each code. The proposed block sequential decoding algorithm is generic, but still provides higher throughput despite of much more sophisticated flow control structure.

It can be also seen that for $E_b/N_0 < 4.2$ dB the block sequential decoding algorithm for a $(2048, 1723)$ polar subcode provides higher throughput and substantially better performance compared to the adaptive list decoding algorithm [5] for a polar code with CRC-32. However, for higher values of $E_b/N_0$ the throughput of the adaptive list decoding algorithm becomes much higher. The reason for this is that in this case with high probability the decoding is successful.
### TABLE I: Decoder memory requirements

| L  | D  | Ξ, KB |
|----|----|-------|
| 8  | 70 | 385   |
| 32 | 240| 1457  |
| 256| 1620|11254 |

| L  | D  | Ξ, KB |
|----|----|-------|
| 8  | 100| 786   |
| 32 | 370| 3071  |
| 256| 3020|24215 |

| L  | D  | Ξ, KB |
|----|----|-------|
| 8  | 250| 4764  |
| 32 | 900| 18617 |
| 256| 8230|160791|

| L  | D  | Ξ, KB |
|----|----|-------|
| 8  | 100| 681   |
| 32 | 400| 2686  |
| 256| 2450|20859 |

already with $L = 1$ (i.e. with plain SC decoding), and this can be easily verified by CRC. Hence, the highly complex list decoder is almost not used. It is, however, not clear how to extend the idea of adaptive list decoding to the case of polar subcodes, which provide much better performance.

Table II presents the amount of memory used by the decoder while decoding some codes. The value of $Ξ$ is the maximal amount of memory, which was consumed by the decoder for storing arrays $S, C$, and outer decoder state variables $Z$ from the common memory pools, described in Section V-C during the simulations. The values of parameters $L, D$ were selected to minimize overall memory usage during block sequential decoding, while ensuring that the performance does not degrade with respect to the case of $D = Lk$, which corresponds to maximal possible memory footprint. Minimization for each code was carried out for FER at $10^{-3}$.

### VIII. Conclusions

In this paper the block sequential decoding algorithm was introduced. It was shown that the input symbols of the polarizing transformation can be processed blockwise, and the processing
operation reduces to on-demand construction of codewords of the codes arising in the GPD of the code being decoded. A set of such codes was identified, which admit low complexity list decoding.

It was shown that the proposed algorithm has lower complexity than the sequential, stack and list decoding algorithms, while having approximately the same performance. At sufficiently high SNR, the throughput of the software implementation of the proposed algorithm exceeds the throughput of the fast list decoder with much smaller list size, i.e. the proposed algorithm provides better performance and lower decoding complexity compared to the list decoding algorithm by Sarkis et al \cite{5}. The proposed algorithm can be used for decoding of polar (sub)codes, polar codes with CRC and short extended BCH codes.

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