Complete Calabi-Yau metrics from Kahler metrics in D=4

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Abstract
In the present work, a family of Calabi-Yau manifolds with a local Hamiltonian Killing
vector is described in terms of a non linear equation whose solutions determine the local
form of the geometries. The main assumptions are that the complex (3,0)-form is of the
form $e^{ik} \Psi$, where $\Psi$ is preserved by the Killing vector, and that the space of the orbits
of the Killing vector is, for fixed value of the momentum map coordinate, a complex 4-
manifold, in such a way that the complex structure of the 4-manifold is part of the complex
structure of the complex 3-fold. The family considered here include the ones considered in
[26]-[28] as a particular case. We also present an explicit example with holonomy exactly
SU(3) by use of the linearization introduced in [26], which was considered in the context
of D6 branes wrapping a complex 1-cycle in a hyperkahler 2-fold.

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1. Introduction
The development of the subject of Calabi-Yau (CY) manifolds is an illustrative example of
the interplay between algebraic geometry and string theory. On the one hand, CY spaces are
interpreted as internal spaces of string and M-theory giving supersymmetric field theories after
compactification. In fact, CY 3-folds may provide compactifications which are more realistic than the ones corresponding to other Ricci-flat manifolds such as $G_2$ holonomy spaces, for which the generation of chiral matter and non abelian gauge symmetries seems harder (but not impossible) to achieve. On the other hand, string theory compactifications stimulated several new trends in the algebro-geometrical aspects of CY spaces, an example is the subject of mirror symmetry.

By definition a CY manifold is a compact Kahler $n$-dimensional manifold with vanishing first Chern class. The Yau proof of the Calabi conjecture implies that these manifolds admit a Ricci-flat metric and their holonomy is reduced from $SO(2n)$ to $SU(n)$ [1]. Although compact Ricci-flat metrics exist, no explicit expressions have been found. The main technical problem for that is that a compact Ricci-flat metric does not admit globally defined Killing vectors (leaving aside the possibility to have trivial flat $U(1)$ factors), and the absence of continuous symmetries makes the task of solving the Einstein equations explicitly really hard. For the non compact case, the definition usually adopted is that a CY manifold is a Ricci-flat Kahler manifold, which also implies that the holonomy is reduced to $SU(n)$ or to a smaller subgroup. In this case several Calabi-Yau metrics with isometries have been found in [2]-[8], and [15]-[25]. Some of these metrics posses conical singularities but in some cases these singularities have been resolved to give complete metrics.

Although non compact Calabi-Yau metrics are not suitable for studying compactification in string theory, they have several applications in mathematical and theoretical physics. For instance, the localization techniques pioneered by Kontsevich [38]-[39] to calculate Gromov-Witten invariants is more easy to implement in the non compact case and sometimes these invariants may be calculated for arbitrary genus. Also, it was conjectured in [40] that Chern-Simmons on $S^3$ is equivalent to topological strings on the resolved conifold $T^*S^3$, which is Calabi-Yau. These has been generalized in [41] where it is shown that for some three dimensional manifold $M$ the space $T^*M$, is Calabi-Yau and it was conjectured that Chern-Simmons on $M$ is dual to topological strings propagating in $T^*M$ (See [42] for a nice review).

In view of the above discussion, to find general methods for constructing non compact CY metrics with isometries is a task of interest. An step in that direction was initiated by Fayyazuddin in [26] where the supergravity backgrounds corresponding to D6 branes wrapping a complex submanifold inside a 4-dimensional hyperkahler space were characterized in terms of a single linear equation. It was also shown in that reference that the uplift to eleven dimensions results in a purely geometrical background of the form $M_{1,4} \times Y_6$ where $Y_6$ is a Calabi-Yau space. The Ricci-flat Kahler metric on $Y_6$ is therefore determined by this linear equation, which is expressed in term of the laplacian over the curved hyperkahler space the branes wrap. For all these geometries there is a $U(1)$ isometry preserving the whole $SU(3)$ structure (which is in particular hamiltonian and therefore it defines a momentum map local coordinate) such that space formed by the orbits of the Killing vector is, for fixed values of the momentum map coordinate, a Kahler manifold. The Fayyazuddin construction was reconsidered in [27] where it was shown that the assumption that the quantities defining the geometry vary over a complex submanifold may be relaxed without violating the Calabi-Yau condition. The resulting geometries were described in terms of a non linear equation, which reduce to the Fayyazuddin one if the quantities describing the geometry vary over a complex submanifold. The non linear operator is defined in terms of the metric of the hyperkahler space, in fact, this method can be interpreted as a solution generating technique which starts with a hyperkahler metric and gives a non compact Calabi-Yau metric as outcome.

The two approaches mentioned above have been used to find non-trivial Calabi-Yau metrics.
with holonomy exactly SU(3). Nevertheless, none of these examples were complete metrics. This situation was substantially improved in [28] where isometries which do not preserve the SU(3) structure, but just the metric \( g_6 \) and the Kahler two form \( \omega_6 \), were considered. These authors showed that one may start with a hyperkähler structure as well and construct complete Calabi-Yau metrics. In particular, the resolution of the \( Y^{p,q} \) cone found in [30]-[32] was rediscovered in these terms. The calculations made in [28] are impressive, but there is a striking fact there that motivates the present note, which is the following. The best results obtained in [28] are obtained in terms of the flat hyperkähler structure on \( \mathbb{R}^4 \), in particular, the resolution of the Ricci-flat cone over \( Y^{p,q} \). Instead, for a curved hyperkähler structure, the resulting equations seem harder to solve and more restricted solutions are found, or even no solutions at all. One may wonder if a method for constructing Calabi-Yau metrics without the use of initial hyperkähler structures may be developed, which may allow us to avoid this kind of problems. In the present such a method will be presented and family of Calabi-Yau geometries characterized by a single non linear equation which is not necessarily related to a hyperkähler metric. It should be emphasized that there is nothing wrong with the use of hyperkähler structures as initial input. What the present letter shows is that this is just optional.

The organization of the present work is as follows. In section 2.1 generalities about SU(3) structures are reviewed. In section 2.2 the SU(3) structures with a Hamiltonian Killing vector, that is, a Killing vector preserving also the Kahler form are characterized. In section 2.3 a family of Calabi-Yau metrics of this type is presented, for which the complex \((3,0)\) form is of the form \( \Psi = e^{ik} \Psi \) in such a way that \( \Psi \) is preserved by the Killing vector but \( \Psi \) may not be preserved due to the phase factor. In section 3.1 and 3.2 it is explained that the metrics considered in [27] and [26] belong to the family of section 2.3. In section 3.3 an example where the Fayyazuddin linearization [26] works properly is worked out explicitly and a non-trivial Calabi-Yau metric is obtained as outcome. In section 3.4 we also show that the results of [28] belong to the family constructed here. Section 4 contains the discussion of the results obtained.

2. Calabi-Yau metrics with Hamiltonian isometries

2.1 The general form of the \( SU(3) \) structure

In this subsection a large family of Calabi-Yau (CY) manifolds in dimension 6 with an isometry group with orbits of codimension one will be characterized. It will be assumed that the Killing vector \( V \) corresponding to this isometry preserve not only the metric, but the full Kahler two form \( \omega_6 \). It will be convenient to give an operative definition of CY manifolds in six dimensions first, for more details see for instance [14]. Roughly speaking, a Calabi-Yau manifold \( M_6 \) is Kahler manifold, thus complex sympletic, which in addition admits a Ricci-flat metric \( g_6 \). This definition means that there exist an endomorphism of the tangent space \( J : TM_6 \rightarrow TM_6 \) such that \( J^2 = -I_d \) and for which \( g_6(X,JY) = -g_6(JX,Y) \) being \( X \) and \( Y \) arbitrary vector fields. It is commonly said that the metric \( g_6 \) is hermitian with respect to \( J \) and the tensor \( (g_6)^{\mu\alpha}_\nu J^\alpha_\nu \) is skew symmetric, therefore locally it defines a 2-form

\[
\omega_6 = \frac{1}{2} (g_6)^{\mu\alpha}_\nu J^\alpha_\nu dx^\mu \wedge dx^\nu.
\]  

(2.1)

Here \( x^\mu \) is a local choice of coordinates for \( M_6 \). The endomorphism \( J \) it is called an almost complex structure. If the Nijenhuis tensor

\[
N(X,Y) = [X,Y] + J [X, JY] + J [JX, Y] - [JX, JY],
\]


vanishes identically then the tensor $J$ will be called a complex structure and $M_6$ a complex manifold. This is the case for any CY manifold. The Newlander-Niremberg theorem states that there is an atlas of charts for $M_6$ which are open subsets in $C^n$, in such a way that the transition maps are holomorphic functions. These local charts are parameterized by complex coordinates $(z_i, \bar{z}_i)$ with $i = 1, 2, 3$ for which the complex structure looks like

$$J^i_j = -J^j_i = i \delta^i_j, \quad J^j_j = J^j_j = 0,$$

(2.2)

and for which the metric and the 2-form (2.1) are expressed as follows

$$g_6 = (g_6)^{ij} dz_i \otimes d\bar{z}_j,$$

(2.3)

$$\omega_6 = \frac{i}{2} (g_6)^{ij} dz_i \wedge d\bar{z}_j.$$

(2.4)

The form (2.4) is called of type $(1,1)$ with respect to $J$, while a generic 2-form containing only terms of the form $(dz_i \wedge d\bar{z}_j)$ or $(d\bar{z}_i \wedge d\bar{z}_j)$ will be called of type $(2,0)$ or $(0,2)$, respectively.

A complex manifold which is sympletic with respect to (2.1) is known as a Kahler manifold, thus CY spaces are all Kahler. The Kahler condition itself implies that the holonomy is reduced from $SO(6)$ to $U(3)$. Furthermore, the fact that $g_6$ is Ricci-flat is equivalent to the existence of a 3-form

$$\Psi = \psi_+ + i \psi_-,$$

(2.5)

of type $(3,0)$ with respect to $J$, satisfying the compatibility conditions [9]

$$\omega_6 \wedge \psi_+ = 0, \quad \psi_+ \wedge \psi_- = \frac{2}{3} \quad \omega_6 \wedge \omega_6 \wedge \omega_6 \simeq dV(g_6),$$

(2.6)

and which is closed, i.e,

$$d\psi_+ = d\psi_- = 0.$$

(2.7)

The relations (2.6) can be expressed in more compact way as

$$\omega_6 \wedge \Psi = 0, \quad \Psi \wedge \overline{\Psi} = \frac{1}{3} \omega_6 \wedge \omega_6 \wedge \omega_6 \simeq dV(g_6).$$

(2.8)

In the formula (2.8) $dV(g_6)$ denote the volume form of $g_6$. In the situations described in (2.7) the holonomy is further reduced from $U(3)$ to $SU(3)$, thus CY manifolds are of $SU(3)$ holonomy. The converse of these statements are also true, that is, for any Ricci-flat Kahler metric in D=6 there will exist an SU(3) structure $(\omega_6, \Psi)$ satisfying (2.8) and also

$$d\omega_6 = d\Psi = 0.$$ 

(2.9)

The knowledge SU(3) structure determine univocally metric $g_6$. In fact, the task to find complex coordinates for a given CY manifold may be not simple, but there always exists a tetrad basis $e^a$ with $a = 1,..,6$ for which the SU(3) structure is expressed as

$$\omega_6 = \frac{i}{2} (E_1 \wedge \overline{E}_1 + E_2 \wedge \overline{E}_2 + E_3 \wedge \overline{E}_3, )$$

(2.10)

$$\Psi = E_1 \wedge E_2 \wedge E_3,$$

(2.11)

where $E_i \equiv e_j + i e_{j+1}$ ($j = 1, 3, 5$), and for which the metric is

$$g_6 = E_1 \otimes \overline{E}_1 + E_2 \otimes \overline{E}_2 + E_3 \otimes \overline{E}_3$$

(2.12)

Note that the multiplication by a phase factor $E_i \rightarrow e^{i k} E_i$ does not change the metric and induce the transformation $\Psi \rightarrow e^{3i k} \Psi$ on the $(3,0)$ form. This phase transformation does not alter the conditions (2.8), this fact will be important in the following.
2.2 Kahler structures with Hamiltonian isometries

The description given above just collects general facts about CY manifolds. In the following we will assume that our CY manifold $M_6$ is equipped with a metric $g_6$ in such a way that there is a Killing vector $V$ preserving $g_6$ and the Kahler form $\omega_6$. In this situation there exists a local coordinate system $(\alpha, x^i)$ with $i = 1, \ldots, 5$ for which $V = \partial_{\alpha}$ and for which the metric tensor $g_6$ takes the following form

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + H g_5,$$  \hfill (2.13)

where the function $H$, the one form $A$ and the metric tensor $g_5$ are independent on the coordinate $\alpha$. Thus these objects live in a 5-dimensional space which we denote $M_5$. The metric $g_5$ appearing in (2.13) can be expressed as $g_5 = e^a \otimes e^a$ with $a = 1, \ldots, 5$ for some basis of $\alpha$-independent 1-forms $e^a$. Then, if $V$ also preserves the Kahler form $\omega_6$ (as we are assuming), one has the decomposition

$$\omega_6 = \omega_4 + \frac{1}{\sqrt{H}} e^5 \wedge (d\alpha + A).$$  \hfill (2.14)

Here the 1-form $e^5$ is by definition

$$e^5 = \sqrt{H} i_{\partial_{\alpha}} \omega_6,$$  \hfill (2.15)

$i_V$ denoting the contraction with the vector field $V$. The elementary formula in differential geometry

$$d_5(i_{\partial_{\alpha}} \omega_6) = \mathcal{L}_{\partial_{\alpha}} \omega_6 - i_{\partial_{\alpha}} d\omega_6,$$  \hfill (2.16)

together with (2.15) implies that

$$d_5(\sqrt{H} e^5) = \mathcal{L}_{\partial_{\alpha}} \omega_6 - i_{\partial_{\alpha}} d\omega_6.$$  \hfill (2.17)

Here $d_5 = \partial_i \ dx^i$ and $\mathcal{L}_{\partial_{\alpha}}$ is the Lie derivate along the vector $\partial_{\alpha}$. But the vector $\partial_{\alpha}$, by assumption, preserves $\omega_6$ and $\omega_6$ is closed, thus the right hand side of (2.17) vanishes and

$$d_5(\sqrt{H} e^5) = 0.$$  \hfill (2.18)

The last relation can be integrated, at least locally, to obtain that

$$e^5 = \sqrt{H} dy,$$  \hfill (2.19)

$y$ being some function of the coordinates $x^i$ parameterizing $M_5$, which is known as the momentum map of the isometry. At least locally, one can take the function $y$ defined in (2.19) as one of the coordinates, which leads to the decomposition $M_5 = M_4 \times R_y$ and $d_5 = d_4 + \partial_y \ dy$. The metric (2.13) in this coordinates becomes

$$g_6 = \frac{(d\alpha + A)^2}{H^2} + H^2 dy^2 + H \ g_4(y),$$  \hfill (2.20)

where the tensor $H \ g_4(y)$ will be determined below under certain additional assumptions. The Kahler form is

$$\omega_6 = \omega_4(y) + dy \wedge (d\alpha + A).$$  \hfill (2.21)

The next task will be to find specific examples of this type of structures.
2.3 Calabi-Yau metrics with Hamiltonian isometries

In this subsection, the generic Kahler structure described above will be extended to an specific family of Calabi-Yau structures. The main assumption will be that, for fixed values of the coordinates \((\alpha, y)\), the resulting 4-manifold is complex, and that the two form \(\omega_4\) appearing in (2.21) is of type \((1,1)\) with respect a complex coordinate system for this manifold. This may be paraphrased by saying that the complex structure of the complex 4-manifold is part of the complex structure of the Ricci-flat Kahler 6-manifold. By denoting the complex coordinates as \((z_1, z_2, \bar{z}_1, \bar{z}_2)\), the main assumption implies that (2.20) may be expressed as

\[
g_6 = \frac{(d\alpha + A)^2}{H^2} + H^2 dy^2 + H g_4(y)_{z_1 \bar{z}_1, dz_i \otimes d\bar{z}_j},
\tag{2.22}
\]

and the dependence on the coordinate \(y\) is only as a parameter.

In order to extend the Kahler structure given above to an \(SU(3)\) structure, an anzatz for the form \(\Psi\) of (2.5) is needed, in such a way that the compatibility conditions (2.8) are identically satisfied. By analogy with the choice \([28]\) we propose the following form for \(\Psi\)

\[
\Psi = e^{iK} \Omega_4 \wedge [H dy + i(d\alpha + A/H)],
\tag{2.23}
\]

\(K\) being a function that may depend \(\alpha\) and varying over \(M_5\). The remaining quantities appearing in 2.23 are assumed to be \(\alpha\)-independent. The compatibility conditions (2.8) are then satisfied if and only if

\[
2 \omega_4 \wedge \omega_4 = \Omega_4 \wedge \overline{\Omega}_4 = 4 \det(H g_4) dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2,
\tag{2.24}
\]

This relation is, for fixed value of the coordinate \(y\), the same as the compatibility condition for \(SU(2)\) structures. It is a standard fact that if there is complex coordinate system for which \(\omega_4\) is of type \((1,1)\), then \(\Omega_4\) is of type \((2,0)\) with respect to it. This means that

\[
\Omega_4 = H f \ dz_1 \wedge dz_2,
\]

\(f\) being a function independent on \(\alpha\) and varying over \(M_5\) and the factor \(H\) in front is just by convenience. The compatibility condition (2.24) implies that

\[
2 \omega_4 \wedge \omega_4 = \Omega_4 \wedge \overline{\Omega}_4 = H^2 f^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,
\tag{2.25}
\]

and by comparing (2.24) with (2.25) one obtains

\[
H^2 f^2 = 4 \det(H g_4).
\tag{2.26}
\]

Taking into account all these relations and (2.23) it follows easily that

\[
\Psi = e^{iK} H^2 f \ dz_1 \wedge dz_2 \wedge [dy + i(d\alpha + A/H^2)].
\tag{2.27}
\]

The next task is to fix the unknown quantities \(A, H, f\) and \(K\) by the Calabi-Yau condition (2.9). The first one applied to (2.21) gives

\[
d_4 \omega_4(y) = 0,
\tag{2.28}
\]
and

\[ d_4 A = \partial_y \omega_4, \quad (2.29) \]

Note that the equation (2.28) imply, for fixed value of \( y \), that \( H g_4 \) is not only complex but also Kahler. The second (2.9) gives several equations, corresponding to the vanishing of each component of \( d\Psi \). The vanishing of the terms with \((dz_1 \wedge dz_2 \wedge dy \wedge d\alpha)\) imply that

\[ K_y = 0, \quad (2.30) \]

\[ H^2 f \partial_\alpha K - f_y = 0. \quad (2.31) \]

The second equation implies that \( K = K_0 + \alpha K_1 \), with \( K_0 \) and \( K_1 \) independent of \( y \). By combining this with the first one it is obtained that

\[ H^2 f K_1 = f_y. \quad (2.32) \]

The terms of the form \((dz_1 \wedge dz_2 \wedge d\alpha \wedge d\bar{z}_i)\) vanish if and only if

\[ \bar{\partial} K_1 = 0, \quad (2.33) \]

\[ -f \bar{\partial} K_0 + i \bar{\partial} f + f K_1 \bar{A} = 0. \quad (2.34) \]

Since \( K_1 \) is real and \( y \)-independent, the first of these equations imply that it is a constant, which can be set to 0, 1 without losing generality. The case \( K_1 = 0 \) correspond to a Killing vector preserving the whole SU(3) structure, which is the case considered in [27]. But for the moment we focus in the case \( K = 1 \). In this case the last equation implies that

\[ d_4^* f = d_4 K_0 - K_1 f A. \quad (2.35) \]

For these cases the terms with \((dz_1 \wedge dz_2 \wedge dy \wedge d\bar{z}_i)\) vanish when

\[ d_4^* f_y = -K_1 \partial_y(f A). \quad (2.36) \]

An immediate consequence the last two equation is

\[ d_4 K_0 = 0. \quad (2.37) \]

Inserting this relation into (2.35) gives

\[ d_4^*(\log f) = -A. \quad (2.38) \]

By taking \( d_4 \) in both sides of the last equation and using (2.29) it is seen that

\[ d_4 d_4^*(\log f) = -\partial_y \omega_4. \quad (2.39) \]

But the condition (2.28) implies that the complex 4-dimensional manifold \( M_4 \) is also a Kahler manifold, with \( \omega_4 \) being the Kahler form. Therefore \( \omega_4 \) has a Kahler potential \( G \), that is, \( \omega_4 = d_4 d_4^* G \). The equation (2.39) imply that

\[ f = U(z_1, z_2) e^{-G_y}, \quad (2.40) \]

with \( U(z_1, z_2) \) an arbitrary holomorphic function. In addition, equation (2.32) gives that \( H^2 = G_{yy} \), and by combining this with (2.25) and (2.40) it is obtained that

\[ U(z_1, z_2)(e^{-2G_y})_y = 32 (G_{17}G_{27} - G_{12}G_{27}), \quad (2.41) \]
and that \( H^2 = G_{yy} \), with \( G_{ij} = \partial_i \partial_j G \). But the holomorphic function can be absorbed by a holomorphic coordinate change \( z_i' = f_i(z_1, z_2) \), thus there exist always a local coordinate system such that (2.41) takes the form
\[
(e^{-2G})_y = 32 (G_{1\bar{1}} G_{2\bar{2}} - G_{1\bar{2}} G_{2\bar{1}}),
\]
In this way, all the quantities appearing in the six dimensional metric are expressed in terms of \( G \). Explicitly, the Calabi-Yau metric is
\[
g_6 = \frac{(d\alpha + d\bar{z}_i G_{ij})^2}{G_{yy}} + G_{yy} dy^2 + 2 G_{i\bar{j}} dz_i \otimes d\bar{z}_j.
\]
(2.43)

For \( K_1 = 0 \), a calculation completely analogous to the one given above shows that the metric is still (2.43) but in this case \( G \) is given by
\[
G_{yy} = 8(G_{1\bar{1}} G_{2\bar{2}} - G_{1\bar{2}} G_{2\bar{1}}).
\]
(2.44)

Note that in both cases \( K_1 = 0, 1 \) the metric is determined in terms of a single function \( G \).

It should be mentioned that the method described by (2.42) or (2.44) may be generalized to arbitrary complex dimensions in straightforward manner. The resulting metrics will be described by (2.43) but the function \( G \) will depend on \( n \)-complex coordinates \( z_i \) with \( i = 1, ..., n \) and will be the solution of
\[
(e^{-2G})_y = 2^{2n+1} \det(G_{ij}),
\]
for \( K_1 = 1 \) and of
\[
G_{yy} = 2^{n+1} \det(G_{ij}),
\]
(2.46)
for \( K_1 = 0 \), \( \det(G_{ij}) \) being the determinant of the matrix whose entries are the second derivatives of \( G \) of type \((1,1)\). The resulting metric (2.43) will have \((n+1)\) complex dimensions but in the following we will keep considering the case \( n = 2 \).

3. Solutions related to hyperkahler structures

In the following sections, the connection between the solution generating technique given by (2.42)-(2.44) and the known ones given in [26]-[28] will be detailed. The assumptions for obtaining the CY metrics (2.42)-(2.44) were the following: there is an isometry preserving the CY metric and the Kahler two form; the complex 3-form has the generic expression (2.23); the manifold obtained for fixed values of \( y \) and \( \alpha \) is complex, in such a way that the metric is of the form (2.22) and such that the two form \( \omega_4 \) appearing in (2.21) is of type \((1,1)\) with respect to the complex coordinates. The last assumption automatically implies that the complex \((3,0)\) form is given by (2.27). These, together with the Calabi-Yau condition, determined completely the local form of the Calabi-Yau metric (2.42)-(2.44). The task is now to show that the metrics [26]-[28] are under these hypothesis and therefore they are a particular case of (2.42)-(2.44).

3.1 Examples with isometries preserving the whole SU(3) structure

In this subsection the results of [26]-[27] are briefly reported, for more details about the proofs we refer the reader to the original references. The solution generating techniques of [26]-[27]
start with a hyperkahler structure $\tilde{\omega}_i$ with $i = 1, 2, 3$ and one of these closed two forms, say $\tilde{\omega}_i$ is deformed to a new $y$-dependent two form

$$\omega_4(y) = \tilde{\omega}_1 - d_4 d_1^c G,$$

(3.47)

while $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are kept intact. This 2-form plays the role of $\omega_4(y)$ in (2.21). Here the operator $d^c = J_1 d$ is constructed in terms of the complex structure $J_1$ which is defined by $\tilde{\omega}_1$ and the hyperkahler metric by the relation (2.3). In the expression (3.47) $G$ denotes an unknown function which varies on $M_4$ and which, in a generic situation, may depend also on the coordinate $y$. If there is a Killing vector preserving the whole $SU(3)$ structure, which corresponds to the case $K_1 = 0$ in (2.23), then the $SU(3)$ structure (2.21) and (2.27) take the following form

$$\omega_6 = \tilde{\omega}_1 - d_4 d_1^c G + dy \wedge (d\alpha + A),$$

$$\psi_+ = H^2 \tilde{\omega}_3 \wedge dy + \tilde{\omega}_2 \wedge (d\alpha + A),$$

$$\psi_- = -H^2 \tilde{\omega}_2 \wedge dy + \tilde{\omega}_3 \wedge (d\alpha + A),$$

with $\Psi = \psi_- + i\psi_+$. Given the deformed structure (3.47), the compatibility condition (2.6) imply that

$$(\tilde{\omega}_1 - d_4 d_1^c G) \wedge (\tilde{\omega}_1 - d_4 d_1^c G) = H^2 \tilde{\omega}_2 \wedge \tilde{\omega}_2,$$

(3.49)

and, as the wedge products appearing in the last equality are all proportional to the volume form $dV(g_4)$ of the initial hyperkahler metric $g_4$, the relation

$$(\tilde{\omega}_1 - d_4 d_1^c G) \wedge (\tilde{\omega}_1 - d_4 d_1^c G) = M(G) \tilde{\omega}_1 \wedge \tilde{\omega}_1,$$

(3.50)

defines a non-linear expression $M(G)$ involving $G$. The CY condition (2.9) applied to (3.48) impose further constraints, which are explained in detail in [27] and which we will not reproduce here. The result is that the geometry is described in terms of a non-linear differential equation determining the function $G$ and which involves the operator $M(G)$, these equation is \footnote{This equation strongly resembles the one found in [13] for the G2 holonomy case.}

$$G_{yy} = M(G).$$

(3.51)

In addition the explicit expression for the $SU(3)$ structure is completely determined in terms of $G$ as

$$\omega_6 = \tilde{\omega}_1 - d_4 d_1^c G + dy \wedge (d\alpha - d_1^c G y),$$

$$\psi_+ = G_{yy} \tilde{\omega}_3 \wedge dy + \tilde{\omega}_2 \wedge (d\alpha - d_1^c G y),$$

$$\psi_- = -G_{yy} \tilde{\omega}_2 \wedge dy + \tilde{\omega}_3 \wedge (d\alpha - d_1^c G y).$$

(3.52)

The generic form of the 6-dimensional Calabi-Yau metric corresponding to this structure is given by

$$g_6 = g_4(y) + G_{yy} dy^2 + \frac{(d\alpha - d_1^c G y)^2}{G_{yy}},$$

(3.53)

where $g_4(y)$ is the Kahler 4-dimensional metric corresponding to the deformed Kahler structure $\omega_1(y) = \tilde{\omega}_1 - d_4 d_1^c G$.

It is important to remark that the metrics of this subsection are under the hypothesis leading to (2.42)-(2.44). First of all, the two form $\omega_1(y)$ introduced in (3.47) is of type (1, 1) with respect to the complex coordinates which diagonalize $J_1$, this follows from the fact that $\omega_1$
is of type (1,1) with respect to these coordinates, and the term \( d_4 d_4^i G \) is also of this type. The form \( \tilde{\omega}_2 + i \tilde{\omega}_3 \) is kept intact and, for a closed hyperkahler structure, is of type (2,0). Moreover (3.47) is closed with respect to \( d_4 \), which leads immediately to the condition (2.28). In addition (3.48) is of the type (2.44). All this imply that the metrics (3.52)-(3.53) are a subcase of the family of Calabi-Yau metrics described in section 2.3.

### 3.2 The Fayyazuddin linearization

The family of SU(3) structures (3.52) and (3.53) found above are completely determined in terms of a single function \( G \) which is a solution of (3.51). This is a non-linear equation and the general solution is not known, but it can be solved in some specific examples. The source of the non-linearity of the operator \( M(G) \) defined in (3.50) and (3.51) is the quadratic term

\[
Q(G) = d_4 d_4^i G \wedge d_4 d_4^i G, \quad (3.54)
\]

therefore the operator \( M(G) \) will reduce to a linear one if \( Q(G) \) vanish. This will be the case when the function \( G \) is of the form \( G = G(w, \overline{w}) \) where \( w = f(z_1, z_2) \) is an holomorphic function of the coordinates \( (z_1, z_2) \) which diagonalize the complex structure \( J_1 \) [10]. This affirmation may be justified as follows. By use of the simple expression

\[
\text{dd}^c G = 2i G_{i\overline{j}} dz_i \wedge d\overline{z}_j, \quad (3.55)
\]

the quadratic term (3.54) may be rewritten as

\[
Q(G) = -4 \left( G_1 \overline{G}_{2\overline{2}} - G_{1\overline{2}} G_{2\overline{1}} \right) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2. \quad (3.56)
\]

But the functional dependence \( G = G(w, \overline{w}) \) imply that

\[
G_{i\overline{j}} = w_i \overline{w}_j G_{w\overline{w}},
\]

and by inserting this into (3.56) gives \( Q(G) = 0 \). This result may be paraphrased as follows. If the function \( G \) depends only on two complex coordinates \( (w, \overline{w}) \) then the quantity \( d_4 d_4^i G \) is essentially a 2-form in two dimensions, therefore the wedge product (3.54) vanish identically.

The situation described above is essentially the one considered by Fayyazuddin in the reference [26] and, if suitable boundary conditions are imposed, the resulting metrics give a dual description of D6 branes wrapping a complex submanifold in a hyperkahler manifold. A simple example is obtained when the initial hyperkahler structure is the flat metric on \( R^4 \) and \( G_{yy} \) varies over an arbitrary set of 2-dimensional hyperplanes inside \( R^4 \). There it was shown in [26] that the resulting metrics are the direct sum of the flat metric in \( R^2 \approx C \) and a general Gibbons-Hawking metric in dimension four [43]. These metrics are of holonomy SU(2), which is a subgroup of SU(3). Our aim in the following is to improve this situation and find Calabi-Yau metrics with holonomy exactly SU(3) by use of this linearization.

### 3.3 Calabi-Yau extensions of the 4-dimensional BKTY metrics

In the present subsection Fayyazuddin linearization explained above will be illustrated with an explicit example. This linearization is performed in terms of an initial hyperkahler structure and the one considered in references [5]-[6] will be chosen by simplicity, namely the distance element

\[
g_4 = z \, dz^2 + z \left( dx^2 + du^2 \right) + \frac{1}{z} (dt - x \, du)^2. \quad (3.57)
\]
By denoting $V = z$ and $A = -xdy$ it is seen that (3.57) takes the usual Gibbons-Hawking form [43], which means that it is hyperkahler and with a tri-holomorphic Killing vector $K = \partial_t$. The solution (3.57) corresponds to a superposition of 6-branes, which results in a linearly growing potential independent on the coordinates $(x,u)$. In fact $V = z$ is the electric potential for an infinite plane with constant density charge at $z = 0$, for which the electric field is constant.

The first difficulty for (3.57) is that crossing the plane $z = 0$ implies a change in its signature. Something similar happens for instance, for the Taub-Nut metric with negative mass parameter. The last one corresponds to a potential $V = 1 - 1/r$ and has a change of signature when crossing the region $r = 1$. For the Taub-Nut metric the explanation is that it is asymptotic to the Atiyah-Hitchin metric, which is complete and regular. The change of the signature is an indication the the Taub-Nut approximation breaks down for $r > 1$. It is plausible to think that something similar happens for (3.57). In fact, there have been several approaches to interpret its meaning. The authors of [6] proposed to replace $z$ by $|z|$ in (3.57). They justify this procedure by interpreting the region $z = 0$ as a source plane and the regions $z > 0$ and $z < 0$ are the sides of a domain wall. The problem is that the metric in the surface $z = 0$ is singular. Another idea was introduced in [5]. In that reference the authors were able to identify an exact hyperkahler metric for which (3.57) is the asymptotic form. These authors observed that the coordinates $(x,u)$ may parameterize a torus $T^2$ by making the coordinate $t$ periodic such that the periods satisfy

$$n = \frac{T_x T_u}{T_t},$$

being $n$ an integer. The resulting manifold is a nilmanifold for which the curvature of the connection pulled back to the $T^2$ satisfy the Dirac quantization condition

$$\frac{1}{T_t} \int_{T^2} F = n.$$

By defining the "proper time" $w = 2z^{3/2}/3$ one can write (3.57) as

$$g_4 = dw^2 + \left(\frac{3w}{2}\right)^{-\frac{2}{3}}(\sigma^3) + \left(\frac{3w}{2}\right)^{\frac{2}{3}}((\sigma^1)^2 + (\sigma^2)^2)$$

(3.58)

where $\sigma_k$ are left invariant forms on the Heisenberg group. The metric (3.58) for $n = 2$ is in fact of the Gibbons-Hawking form, and it was conjectured in [5] that they describe the asymptotic form of some specific CY metrics found in [7]-[8] by Bando, Kobayashi, Tian and Yau (BTKY metrics). These metrics arise as a degenerate limit of a K3 surfaces. The point is that K3 surfaces has 58 parameter moduli space and as one moves to the boundary of the moduli space the metric may decompactify while remaining complete and non singular. The metric (3.58) is believed to describe the asymptotic metric of a K3 surface in one of those limits of the parameters.

Our task is now to extend (3.57) to a CY six metric. We do not expect the resulting metric to be complete as the initial hyperkahler is just valid as an asymptotic expression. But this example is illustrates clearly how the Fayyazuddin linearization applies in a generic case. In order to use the linearization a complex coordinate system for (3.57) should be found. A Kahler 2-form for this metric is

$$\omega = dt \wedge du - z dz \wedge dx,$$

(3.59)

and the corresponding complex structure has the following non zero components

$$J^t_t = \frac{x}{z}, \quad J^u_t = \frac{1}{z}, \quad J^z_x = -1,$$
\[ J' = \frac{z^2 + x^2}{z}, \quad J^u = \frac{x}{z}, \quad J^z = 1. \] (3.60)

A complex coordinate system \( z_i \) with \( i = 1, 2 \) is then any choice for which the components of the complex structure take the form \( \bar{J}_j^i = -J^i_j = \delta^i_j \). This amounts to find a coordinate change for which
\[
\frac{\partial x^a}{\partial z^i} j_b \frac{\partial z^j}{\partial x^b} = \delta^i_j,
\]
and the last equation is equivalent following the following system
\[
(J^u - i\delta^u_b) \partial_b z^i = 0, \quad i = 1, 2 \] (3.61)

It can be checked from (3.60) that the equations (3.61) are equivalent to the two following independent equations
\[
\partial_z z^i = -i\partial_x z^i, \quad i\partial_t z^i = (z - ix)\partial_t z^i.
\]

Two independent solutions of the last system are given by \( z^1 = -x + iz \) and \( z^2 = iu(z - ix) + t \).

Now let us suppose that the function \( G \) in (3.51) is of the form \( G = u^2 + U(w, \bar{w}, u) \) and we choose \( w = z_1 \). Let us denote \( U_{uu} = H^2 \). If we further assume that \( U \) does not depend on \( x \) then by taking the derivative of (3.51) with respect to \( u \) twice gives an equation for \( H^2 \), namely
\[
\left( \frac{1}{z} \partial_z^2 + \partial_u^2 \right) H^2 = 0, \quad (3.62)
\]
with solution
\[
H^2 = 1 + \frac{m}{(4z^3 + 9u^2)^{\frac{3}{2}}}. \quad (3.63)
\]

By integrating twice with respect to the variable \( u \) and remembering that \( U_{uu} = H^2 \) it follows that
\[
G = u^2 - \frac{(\sqrt{3})^5}{15} m \leq z^3 \leq \left[ -1 + \left( 1 + \frac{9u^2}{4z^3} \right)^{\frac{5}{6}} - \frac{15u^2}{4z^3} 2F_1 \left[ \left( \frac{1}{6}, \frac{1}{2} \right), \left( \frac{3}{2} \right), -\frac{9u^2}{4z^3} \right] \right] \quad (3.64)
\]
where \( 2F_1 \) denote a generalized hypergeometric function. Now a simple calculation shows that
\[
A = \frac{m}{2} u \leq (9u^2 + 4z^3)^{\frac{3}{2}} \leq \left[ -3 + 2^4 \left( 1 + \frac{9u^2}{4z^3} \right)^{\frac{5}{6}} 2F_1 \left[ \left( \frac{1}{6}, \frac{1}{2} \right), \left( \frac{3}{2} \right), -\frac{9u^2}{4z^3} \right] \right] dx. \quad (3.65)
\]

Also a simple calculation shows that (3.47) is in this case
\[
\omega_1 (u) = \omega_1 - dd^c G = \omega_1 - G_{11} dz_1 \land d\bar{z}_1 = \omega_1 + G_{zz} dz_1 \land d\bar{z}_1 \quad (3.66)
\]
\( \omega_1 \) given in (3.59) and in the last step we took into account that \( z = i\bar{z}_1 - iz_1 \). The explicit expression of (3.66) is obtained from (3.64), the result is
\[
\omega_1 (u) = \omega_1 + \frac{m}{2} \left( \frac{9u^2}{4z^3} \right)^{\frac{5}{6}} \left[ 2 \ 2^2 2^2 \left[ \left( \frac{1}{6}, \frac{1}{2} \right), \left( \frac{3}{2} \right), -\frac{9u^2}{4z^3} \right] \right] d\bar{z}_1 \land d\bar{z}_1. \quad (3.67)
\]
The metric \( g_4(u) \) in (3.53) is the one that correspond to the modified Kahler potential (3.67) namely
\[
g_4(u) = \frac{1}{z} (dt - xdu)^2 + z (du^2 + dz^2 + dx^2)
\]
By collecting the results (3.63)-(3.68) it follows that the Calabi-Yau extension (3.53) of the BTKY metric is

\[ g_6 = \left(1 + \frac{m}{(4z^3 + 9u^2)^2} \right)^{-1} (d\alpha + A)^2 + \left(1 + \frac{m}{(4z^3 + 9u^2)^2} \right) du^2 + \frac{1}{z}(dt - xdu)^2 + z \, du^2 + \left\{ \frac{m}{2(4 + 9u^2)^2} z^2 (9u^2 + 4z^3)^2 \right\} \left[ 2 \left( \frac{9}{2} - \frac{9u^2}{z^3} \right) z^3 + z \right] (dz^2 + dx^2). \]  

with $A$ given in (3.65). This example is a non-trivial Ricci-flat and Kahler metric in six dimensions, with holonomy exactly SU(3). Nevertheless in the region near $z = 0$ we do not expect our solution to be valid, as the approximation (3.57) breaks down.

### 3.4 Complete examples with Hamiltonian isometries

The explicit examples presented in the previous sections do possess isometries preserving the full SU(3) structure, in other words, they correspond to the case $K_1 = 0$ of section 3.1. The remaining case $K_1 = 1$ was considered in [28]. These authors propose an anzatz which is given in terms of an initial hyperkahler structure which is deformed as in (3.47). In addition they propose a sympletic form $\omega_0$ of the form (3.48). The unique difference with the case considered in section 3.1 is that the complex three form is now $\alpha$-dependent and is given by $\Psi = e^{i\alpha}(\psi_+ + i\psi_-)$, with $\psi_\pm$ given (3.48). This imply that the isometry preserves the Kahler 2-form but not $\Psi$. The compatibility and the Calabi-Yau conditions were worked out explicitly in [28] and the outcome is again that the metric and the SU(3) structure is completely determined by $G$, which is now a solution of the equation

\[ (e^{-\frac{i}{2}G_\psi})_y = M(G), \]  

$M(G)$ being the non-linear operator defined by (3.50). The CY metric is again given by (3.53) but now $G$ is a solution of (3.70). It has been shown in [28] that complete metrics may be obtained when the initial hyperkahler structure is the flat one. In this case (3.70) becomes

\[ (e^{-\frac{i}{2}G_\psi})_y = 2(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}). \]  

By parameterizing

\[ z_1 = r \cos \frac{\theta}{2} \exp \left( \frac{i(\psi + \phi)}{2} \right), \quad z_2 = r \sin \frac{\theta}{2} \exp \left( \frac{i(\psi - \phi)}{2} \right), \]  

and assuming that $G$ is a function of $r$ and $y$ the equation (3.72) reduce to

\[ (e^{-\frac{i}{2}G_\psi})_y = \frac{1}{2r^3} \partial_r \left[ r^4 \left(1 + \frac{1}{2r^3} \partial_r G \right)^2 \right], \]  

which is the equation (61) of reference [28]. Particular solutions of this equation has been found in that reference and which, after appropriate coordinate transformations and different rescalings, give the resulting family of Calabi-Yau metrics [28]

\[ g_6 = \frac{dy^2}{W} + \frac{1}{4} Wy^2(d\alpha - s^2\sigma_3)^2 + y^2 \left( \frac{ds^2}{V} + \frac{1}{4} V s^2 \sigma_3^2 + \frac{1}{4} s^2 (\sigma_1^2 + \sigma_2^2) \right) \]  

(3.74)
with
\[ W = 1 - \frac{a}{y^6} \quad V = 1 - s^2 - \frac{b}{s^4}. \]
The metric with \( b = 0 \) describes a higher dimensional generalization of Eguchi-Hanson instanton [11]-[12], with \( R^2 \times CP^2 \) topology and an asymptotic \( R^6/Z_3 \) [29]. For \( a = 0 \), the metric is a cone of \( Y^{p,q} \). The general solution describes a resolution of the \( Y^{p,q} \) cone, and the detail global analysis can be found in [30]-[32]. More details of this calculation can be found in the original reference [28].

It is important to remark that the equation (3.71), which corresponds to the flat metric, is completely equivalent to (2.42). This may be seen by making the redefinition \( G_{i\bar{j}} \rightarrow \delta_{i\bar{j}} + G_{i\bar{j}} \) in (2.42), which gives (3.71) as a result, and vice versa. In addition the complex coordinates \( z_i \) appearing in (2.43) are locally given by (3.72). But although the starting point is the flat hyperkahler structure, it is not necessarily true that (3.72) parameterize \( R^4 \) globally, in fact there may appear singularities in the resulting Calabi-Yau metric which can be avoided by changing the periodicity of the angular variables or the range of the radial coordinate. In any case, the above reasoning shows that metrics (3.74) are special solutions of (2.42) - (2.43).

A priori, it may be expected that the use of curved hyperkahler backgrounds will enhance the number of solutions of (3.70). In particular, it may sound plausible that if one starts with a gravitational instanton admitting a flat limit (such as the Taub-Nut one), then the resulting Calabi-Yau metrics obtained by solving (3.70) will contain the ones arising from (3.71) as a particular case and moreover, the families described by (3.71) such as (3.74) will be reobtained by taking the corresponding flat limit. As (3.71) is equivalent to (2.42) this reasoning will imply that (3.70) describe a more general family that (2.42). But what the results of the present work are showing is that the situation is the opposite, that is, any Calabi-Yau metric found in terms of a curved hyperkahler space by solving (3.70) can be obtained from solutions of the "flat" equation (2.42) as well. Thus the number of solutions of (3.70) are less or equal to the solutions of (2.42). The arguments showing this are the same than in the section 3.1 namely, that all the metrics described by (3.70) are under the hypothesis giving the equation (2.42). Although this conclusion may sound a bit odd, there is further evidence for that, which is the following. If one starts with a curved hyperkahler metric with tri-axial symmetry instead of the flat one, then Calabi-Yau metrics resulting from (3.70) are the one with \( R^2 \times CP^2 \) topology and an asymptotic \( R^6/Z_3 \) together with the resolution of the cone over \( T^{1,1}/Z_2 \) [28]. But \( T^{1,1}/Z_2 \) is a particular case of the \( Y^{p,q} \) Einstein-Sasaki manifolds thus, the solutions obtained with the tri-axial metrics are an special subcase of (3.74). For other curved manifolds the system becomes harder to solve and no new solutions have been found. Although that formally there is nothing wrong with the use of hyperkahler structures to guess new solutions, it may be the case the use of curved geometries complicates the task instead of helping to solve it. For this reason it is perhaps convenient to find a formalism which avoid this problem, and the one developed here in (2.42)-(2.44) possess these advantages, as these equations do not make any reference to any vielbein of a curved hyperkahler metric.

\[ \text{\footnotesize See the last paragraph of section 3.1, in fact it is not difficult to see that the } \alpha \text{-dependent phase does not change these arguments at all.} \]
4. Discussion

In the present work, a family of Calabi-Yau manifolds with a local Hamiltonian Killing vector, i.e., a Killing vector which preserve the metric together with the Kahler form was characterized. It was assumed that the complex (3, 0)-form is of the form \( e^{ik} \Psi \), where \( \Psi \) is preserved by the Killing vector as well, and that the space of the orbits of the Killing vector is, for fixed value of the momentum map coordinate, a complex manifold, in such a way that the complex structure of the 2-fold is part of the complex structure of the 3-fold. Under these assumptions, it was shown that the local form of the geometry is completely determined in terms of a function \( G \) satisfying the non-linear equation (2.42) if the phase \( k \) is non-trivial or (2.44) if the phase \( k \) is zero. It has been also pointed out that the constructions given in [26], [27] and [28] are included in this family.

The advantages of this method are that, unlike the ones presented in [26], [27] and [28], it does not require a hyperkahler structure as initial input. As it was discussed in section 2, it is only required that the 4 dimensional manifold defined by the orbits of the Killing vector for fixed momentum map coordinate is a complex 2-fold, and the Calabi-Yau conditions imply automatically that it is Kahler. In fact, the equations (2.42) and (2.44) for the function \( G \) defining the six dimensional metric does not contain any reference to the vielbein of the complex 2-fold. In this form one may avoid the complications in the calculation of the local form of the geometry due to the non-trivial curvature of an initial hyperkahler geometry.

It is perhaps better to compare this situation with known results in four dimensions. Consider a 4-dimensional Calabi-Yau (hyperkahler) space, such that the Killing vector preserve the Kahler form \( \omega_4 \) but not \( \Omega_4 \). As is well known, the general local form of the Ricci-flat Kahler 4-metric is [33]-[34]

\[
g_4 = u_z [e^u (dx^2 + dy^2) + dz^2] + u_z^{-1} [dt + (u_x dy - u_y dx)]^2,
\]

where \( u \) is the solution of the equation

\[
(e^u)_{zz} + u_{yy} + u_{xx} = 0. \tag{4.76}
\]

Equation (4.76) is known as the continuum limit of the sl(n) Toda equation and is called SU(\( \infty \)) Toda equation. The three dimensional base metric, namely

\[
g_3 = e^u (dx^2 + dy^2) + dz^2;
\]

is Einstein-Weyl [35]-[37]. But the general Einstein-Weyl equation is not related to a Toda system, so these base metrics are Einstein-Weyl spaces of restricted type. One may try to find solutions of (4.76) by perturbing around a solution related to a known Einstein-Weyl structure. This is not wrong, but optional. In the same way the 4-dimensional metric (2.43) is Kahler with Kahler potential \( G \), but \( G \) is of restricted type, given by solutions of (2.42) or (2.44). One may try to find a solution to these equations by perturbing around a known hyperkahler one, as it was done in [26]-[28], but this is optional as well.

We also presented in section 3.3 an example which is obtained by means of the Fayyazuddin linearization. This example has holonomy exactly SU(3), but it is not complete. It may be interesting to see if it is possible to find complete metrics by means of this linearization, which will correspond to D6 branes wrapping a complex 1-cycle inside a hyperkahler. We hope to answer this question in the near future.
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