Smarandache Isotopy Theory Of Smarandache: Quasigroups And Loops *†

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Abstract

The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache $f, g$ principal isotopes and G-Smarandache loops.

1 Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [11], [1], [3], [4], [5] and [12]. In [12], Page 102, the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14] [15] [16] [17] [18] [19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

*2000 Mathematics Subject Classification. Primary 20N05; Secondary 08A05.
†Keywords and Phrases: Smarandache: groupoids; quasigroups; loops; $f,g$ principal isotopes
2 Definitions and Notations

Definition 2.1 Let \( L \) be a non-empty set. Define a binary operation \((\cdot)\) on \( L \): If \( x \cdot y \in L \) \( \forall x, y \in L \), \((L,\cdot)\) is called a groupoid. If the system of equations \( a \cdot x = b \) and \( y \cdot a = b \) have unique solutions for \( x \) and \( y \) respectively, then \((L,\cdot)\) is called a quasigroup. Furthermore, if there exists a unique element \( e \in L \) called the identity element such that \( \forall x \in L, x \cdot e = e \cdot x = x \), \((L,\cdot)\) is called a loop.

If there exists at least a non-empty and non-trivial subset \( M \) of a groupoid(quasigroup or semigroup or loop) \( L \) such that \((M,\cdot)\) is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of \((L,\cdot)\), then \( L \) is called a Smarandache: groupoid(S-groupoid)(quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)) with Smarandache: subsemigroup(S-subsemigroup)(subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)) \( M \).

Let \((G,\cdot)\) be a quasigroup(loop). The bijection \( L_x : G \to G \) defined as \( yL_x = x \cdot y \forall x, y \in G \) is called a left translation(multiplication) of \( G \) while the bijection \( R_x : G \to G \) defined as \( yR_x = y \cdot x \forall x, y \in G \) is called a right translation(multiplication) of \( G \).

The set \( SYM(L,\cdot) = SYM(L) \) of all bijections in a groupoid \((L,\cdot)\) forms a group called the permutation(symmetric) group of the groupoid \((L,\cdot)\).

Definition 2.2 If \((L,\cdot)\) and \((G,\circ)\) are two distinct groupoids, then the triple \((U,V,W) : (L,\cdot) \to (G,\circ)\) such that \( U, V, W : L \to G \) are bijections is called an isotopism if and only if

\[
xU \circ yV = (x \cdot y)W \forall x, y \in L.
\]

So we call \( L \) and \( G \) groupoid isotopes. If \( L = G \) and \( W = I \)(identity mapping) then \((U,V,I)\) is called a principal isotopism, so we call \( G \) a principal isotope of \( L \). But if in addition \( G \) is a quasigroup such that for some \( f,g \in G \), \( U = R_g \) and \( V = L_f \), then \((R_g,L_f,\cdot) : (G,\cdot) \to (G,\circ)\) is called an \( f,g \)-principal isotopism while \((G,\cdot)\) and \((G,\circ)\) are called quasigroup isotopes.

If \( U = V = W \), then \( U \) is called an isomorphism, hence we write \((L,\cdot) \cong (G,\circ)\). A loop \((L,\cdot)\) is called a G-loop if and only if \((L,\cdot) \cong (G,\circ)\) for all loop isotopes \((G,\circ)\) of \((L,\cdot)\).

Now, if \((L,\cdot)\) and \((G,\circ)\) are S-groupoids with S-subsemigroups \( L' \) and \( G' \) respectively such that \((G')A = L'\), where \( A \in \{U,V,W\}\), then the isotopism \((U,V,W) : (L,\cdot) \to (G,\circ)\) is called a Smarandache isotopism(S-isotopism). Consequently, if \( W = I \) the triple \((U,V,I)\) is called a Smarandache principal isotopism. But if in addition \( G \) is a S-quasigroup with S-subgroup \( H' \) such that for some \( f,g \in H \), \( U = R_g \) and \( V = L_f \), and \((R_g,L_f,\cdot) : (G,\cdot) \to (G,\circ)\) is an isotopism, then the triple is called a Smarandache \( f,g \)-principal isotopism while \( f \) and \( g \) are called Smarandache elements(S-elements).

Thus, if \( U = V = W \), then \( U \) is called a Smarandache isomorphism, hence we write \((L,\cdot) \simeq (G,\circ)\). An S-loop \((L,\cdot)\) is called a G-Smarandache loop(GS-loop) if and only if \((L,\cdot) \simeq (G,\circ)\) for all loop isotopes(or particularly all S-loop isotopes) \((G,\circ)\) of \((L,\cdot)\).
Example 2.1 The systems \((L, \cdot)\) and \((L, \ast)\), \(L = \{0, 1, 2, 3, 4\}\) with the multiplication tables below are S-quasigroups with S-subgroups \((L', \cdot)\) and \((L'', \ast)\) respectively, \(L' = \{0, 1\}\) and \(L'' = \{1, 2\}\). \((L, \cdot)\) is taken from Example 2.2 of [10]. The triple \((U,V,W)\) such that 

\[
U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}
\]

are permutations on \(L\), is an S-isotopism of \((L, \cdot)\) onto \((L, \ast)\). Notice that \(A(L') = L''\) for all \(A \in \{U,V,W\}\) and \(U,V,W : L' \to L''\) are all bijections.

Example 2.2 According to Example 4.2.2 of [15], the system \((\mathbb{Z}_6, \times)\) i.e. the set \(L = \mathbb{Z}_6\) under multiplication modulo 6 is an S-semigroup with S-subgroups \((L', \times)\) and \((L'', \times)\), \(L' = \{2, 4\}\) and \(L'' = \{1, 5\}\). This can be deduced from its multiplication table, below. The triple \((U,V,W)\) such that 

\[
U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix}
\]

are permutations on \(L\), is an S-isotopism of \((\mathbb{Z}_6, \times)\) unto an S-semigroup \((\mathbb{Z}_6, \ast)\) with S-subgroups \((L''', \ast)\) and \((L''', \ast)\), \(L''' = \{2, 5\}\) and \(L'''' = \{0, 3\}\) as shown in the second table below. Notice that \(A(L') = L'''\) and \(A(L'') = L''''\) for all \(A \in \{U,V,W\}\) and \(U,V,W : L' \to L'''\) and \(U,V,W : L'' \to L''''\) are all bijections.

Remark 2.1 Taking careful look at Definition 2.2 and comparing it with Definition 4.4.1, [15], it will be observed that the author did not allow the component bijections \(U,V\) and \(W\) in \((U,V,W)\) to act on the whole S-loop \(L\) but only on the S-subloop(S-subgroup) \(L'\). We feel this is necessary to adjust here so that the set \(L - L'\) is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be
explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations (isotopism and isomorphism) on the whole domain into the whole co-domain but not those of Vasantha Kandasamy [12] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.

3 Smarandache Isotopy and Isomorphy Classes

**Theorem 3.1** Let $\mathfrak{G} = \left\{ (G_\omega, o_\omega) \right\}_{\omega \in \Omega}$ be a set of distinct S-groupoids with a corresponding set of S-subsemigroups $\mathfrak{H} = \left\{ (H_\omega, o_\omega) \right\}_{\omega \in \Omega}$. Define a relation $\sim$ on $\mathfrak{G}$ such that for all $(G_{\omega_i}, o_{\omega_i}), (G_{\omega_j}, o_{\omega_j}) \in \mathfrak{G}$, where $\omega_i, \omega_j \in \Omega$,

$$(G_{\omega_i}, o_{\omega_i}) \sim (G_{\omega_j}, o_{\omega_j}) \iff (G_{\omega_i}, o_{\omega_i}) \text{ and } (G_{\omega_j}, o_{\omega_j}) \text{ are S-isotopic.}$$

Then $\sim$ is an equivalence relation on $\mathfrak{G}$.

**Proof**

Let $(G_{\omega_i}, o_{\omega_i}), (G_{\omega_j}, o_{\omega_j}), (G_{\omega_k}, o_{\omega_k}) \in \mathfrak{G}$, where $\omega_i, \omega_j, \omega_k \in \Omega$.

**Reflexivity** If $I : G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$x I o_{\omega_i} y I = (x o_{\omega_i} y) I \forall x, y \in G_{\omega_i} \implies \text{the triple } (I, I, I) : (G_{\omega_i}, o_{\omega_i}) \to (G_{\omega_i}, o_{\omega_i})$$

is an S-isotopism since $(H_{\omega_i}) I = H_{\omega_i} \forall \omega_i \in \Omega$. In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

**Symmetry** Let $(G_{\omega_i}, o_{\omega_i}) \sim (G_{\omega_j}, o_{\omega_j})$. Then there exist bijections

$$U, V, W : (G_{\omega_i}, o_{\omega_i}) \to (G_{\omega_j}, o_{\omega_j})$$

such that $(H_{\omega_i}) A = H_{\omega_j} \forall A \in \{U, V, W\}$

so that the triple

$$\alpha = (U, V, W) : (G_{\omega_i}, o_{\omega_i}) \to (G_{\omega_j}, o_{\omega_j})$$

is an isotopism. Since each of $U, V, W$ is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} : (G_{\omega_j}, o_{\omega_j}) \to (G_{\omega_i}, o_{\omega_i})$$

are bijective. In fact, $(H_{\omega_j}) A^{-1} = H_{\omega_i} \forall A \in \{U, V, W\}$ since $A$ is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) : (G_{\omega_j}, o_{\omega_j}) \to (G_{\omega_i}, o_{\omega_i})$$

is an isotopism. Thus, $(G_{\omega_j}, o_{\omega_j}) \sim (G_{\omega_i}, o_{\omega_i})$. 

4
Transitivity Let \((G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j})\) and \((G_{\omega_j}, \circ_{\omega_j}) \sim (G_{\omega_k}, \circ_{\omega_k})\). Then there exist bijections

\[ U_1, V_1, W_1 : (G_{\omega_i}, \circ_{\omega_i}) \rightarrow (G_{\omega_j}, \circ_{\omega_j}) \text{ and } U_2, V_2, W_2 : (G_{\omega_j}, \circ_{\omega_j}) \rightarrow (G_{\omega_k}, \circ_{\omega_k}) \]

such that \((H_{\omega_i}) A = H_{\omega_j} \forall A \in \{U_1, V_1, W_1\}\)

and \((H_{\omega_j}) B = H_{\omega_k} \forall B \in \{U_2, V_2, W_2\}\) so that the triples

\[ \alpha_1 = (U_1, V_1, W_1) : (G_{\omega_i}, \circ_{\omega_i}) \rightarrow (G_{\omega_j}, \circ_{\omega_j}) \]

and

\[ \alpha_2 = (U_2, V_2, W_2) : (G_{\omega_j}, \circ_{\omega_j}) \rightarrow (G_{\omega_k}, \circ_{\omega_k}) \]

are isotopisms. Since each of \(U_i, V_i, W_i, i = 1, 2, \) is bijective, then

\[ U_3 = U_1 U_2, V_3 = V_1 V_2, W_3 = W_1 W_2 : (G_{\omega_i}, \circ_{\omega_i}) \rightarrow (G_{\omega_k}, \circ_{\omega_k}) \]

are bijections such that \((H_{\omega_i}) A_3 = (H_{\omega_i}) A_1 A_2 = (H_{\omega_j}) A_2 = H_{\omega_k}\) so that the triple

\[ \alpha_3 = \alpha_1 \alpha_2 = (U_3, V_3, W_3) : (G_{\omega_i}, \circ_{\omega_i}) \rightarrow (G_{\omega_k}, \circ_{\omega_k}) \]

is an isotopism. Thus, \((G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_k}, \circ_{\omega_k})\).

Remark 3.1 As a follow up to Theorem 3.1, the elements of the set \(\mathcal{G}/ \sim\) will be referred to as Smarandache isotopy classes (S-isotopy classes). Similarly, if \(\sim\) meant "S-isomorphism" in Theorem 3.1, then the elements of \(\mathcal{G}/ \sim\) will be referred to as Smarandache isomorphism classes (S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.

Corollary 3.1 Let \(\mathcal{L}_n, S\mathcal{L}_n\) and \(N\mathcal{S}\mathcal{L}_n\) be the sets of; all finite loops of order \(n\); all finite S-loops of order \(n\) and all finite non S-loops of order \(n\) respectively.

1. If \(A^n\) and \(B^n\) represent the isomorphy class of \(\mathcal{L}_n\) and the S-isomorphy class of \(S\mathcal{L}_n\) respectively, then

\[ (a) \ |S\mathcal{L}_n| + |N\mathcal{S}\mathcal{L}_n| = |\mathcal{L}_n|; \]

\[ (i) \ |S\mathcal{L}_5| + |N\mathcal{S}\mathcal{L}_5| = 56, \]

\[ (ii) \ |S\mathcal{L}_6| + |N\mathcal{S}\mathcal{L}_6| = 9,408 \text{ and} \]

\[ (iii) \ |S\mathcal{L}_7| + |N\mathcal{S}\mathcal{L}_7| = 16,942,080. \]

(b) \(|N\mathcal{S}\mathcal{L}_n| = \sum_{i=1} \left| A^n_i \right| - \sum_{i=1} \left| B^n_i \right| ;\)

\[ (i) \ |N\mathcal{S}\mathcal{L}_5| = \sum_{i=1}^6 \left| A^5_i \right| - \sum_{i=1} \left| B^5_i \right| , \]

\[ (ii) \ |N\mathcal{S}\mathcal{L}_6| = \sum_{i=1}^{109} \left| A^6_i \right| - \sum_{i=1} \left| B^6_i \right| \text{ and} \]

\[ (iii) \ |N\mathcal{S}\mathcal{L}_7| = \sum_{i=1}^{23,746} \left| A^7_i \right| - \sum_{i=1} \left| B^7_i \right| . \]
2. If $\mathcal{A}_n^i$ and $\mathcal{B}_n^i$ represent the isotopy class of $\mathcal{L}_n$ and the $S$-isotopy class of $\mathcal{S}\mathcal{L}_n$ respectively, then

$$|\mathcal{N}\mathcal{S}\mathcal{L}_n| = \sum_{i=1}^2 |\mathcal{A}_n^i| - \sum_{i=1}^2 |\mathcal{B}_n^i|;$$

(i) $|\mathcal{N}\mathcal{S}\mathcal{L}_5| = \sum_{i=1}^2 |\mathcal{A}_5^i| - \sum_{i=1}^2 |\mathcal{B}_5^i|$, 

(ii) $|\mathcal{N}\mathcal{S}\mathcal{L}_6| = \sum_{i=1}^{22} |\mathcal{A}_6^i| - \sum_{i=1}^2 |\mathcal{B}_6^i|$ and

(iii) $|\mathcal{N}\mathcal{S}\mathcal{L}_7| = \sum_{i=1}^{564} |\mathcal{A}_7^i| - \sum_{i=1}^2 |\mathcal{B}_7^i|$.

Proof

An S-loop is an S-groupoid. Thus by Theorem 3.1, we have S-isomorphy classes and S-isotopy classes. Recall that $|\mathcal{L}_n| = |\mathcal{S}\mathcal{L}_n| + |\mathcal{N}\mathcal{S}\mathcal{L}_n| - |\mathcal{S}\mathcal{L}_n \cap \mathcal{N}\mathcal{S}\mathcal{L}_n|$ but $\mathcal{S}\mathcal{L}_n \cap \mathcal{N}\mathcal{S}\mathcal{L}_n = \emptyset$ so $|\mathcal{L}_n| = |\mathcal{S}\mathcal{L}_n| + |\mathcal{N}\mathcal{S}\mathcal{L}_n|$. As stated and shown in [11], [5], [2] and [9], the facts in Table 1 are true where $n$ is the order of a finite loop. Hence the claims follow.

| $n$ | 5 | 6 | 7 |
|-----|---|---|---|
| $|\mathcal{L}_n|$ | 56 | 9, 408 | 16, 942, 080 |
| $\{\mathcal{A}_n^i\}_{i=1}^k$ | $k = 6$ | $k = 109$ | $k = 23, 746$ |
| $\{\mathcal{B}_n^i\}_{i=1}^m$ | $m = 2$ | $m = 22$ | $m = 564$ |

Table 1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

Question 3.1 How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{S}\mathcal{L}_n|$ or $|\mathcal{N}\mathcal{S}\mathcal{L}_n|$.

Theorem 3.2 Let $(G, \cdot)$ be a finite S-groupoid of order $n$ with a finite S-subsemigroup $(H, \cdot)$ of order $m$. Also, let

$$\text{ISOT}(G, \cdot), \text{SISOT}(G, \cdot) \text{ and } \text{NSISOT}(G, \cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G, \cdot)$. Then,

$$\text{ISOT}(G, \cdot) \text{ is a group and } \text{SISOT}(G, \cdot) \leq \text{ISOT}(G, \cdot).$$

Furthermore:

1. $|\text{ISOT}(G, \cdot)| = (n!)^3$;
2. $|\text{SISOT}(G, \cdot)| = (m!)^3$;
3. $|\text{NSISOT}(G, \cdot)| = (n!)^3 - (m!)^3$.

Proof
1. This has been shown to be true in [Theorem 4.1.1, [3]].

2. An S-isotopism is an isotopism. So, $SISOT(G, \cdot) \subset ISOT(G, \cdot)$. Thus, we need to just verify the axioms of a group to show that $SISOT(G, \cdot) \leq ISOT(G, \cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in SISOT(G, \cdot)$$

such that $\alpha = (U, V, W) : (G, \cdot) \rightarrow (G, \circ)$,

where $(G, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subgroups $(H, \cdot)$ and $(K, \circ)$ respectively, we can set

$$U' := U|_H, \ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K \forall A \in \{U, V, W\},$$

so that $SISOT(H, \cdot) = \{(U', V', W')\}$. This is possible because of the following arguments.

Let

$$X = \left\{ f' := f|_H \bigg| f : G \rightarrow G, \ f : H \rightarrow K \text{ is bijective and } f(H) = K \right\}.$$  

Let

$$SYM(H, K) = \{\text{bijections from } H \text{ unto } K\}.$$  

By definition, it is easy to see that $X \subseteq SYM(H, K)$. Now, for all $U \in SYM(H, K)$, define $U : H^c \rightarrow K^c$ so that $U : G \rightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H, K) \subseteq X$ so that $SYM(H, K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|ISOT(H, \cdot)| = (m!)^3 \text{ so that } |SISOT(G, \cdot)| = (m!)^3 \text{ since } SYM(H, K) = X.$$  

3. 

$$NSISOT(G, \cdot) = (SISOT(G, \cdot))^c.$$  

So, the identity isotopism

$$(I, I, I) \notin NSISOT(G, \cdot), \text{ hence } NSISOT(G, \cdot) \nsubseteq ISOT(G, \cdot).$$  

Furthermore,

$$|NSISOT(G, \cdot)| = (n!)^3 - (m!)^3.$$  

Corollary 3.2 Let $(G, \cdot)$ be a finite S-groupoid of order $n$ with an S-subsemigroup $(H, \cdot)$. If $ISOT(G, \cdot)$ is the group of all isotopisms of $(G, \cdot)$ and $S_n$ is the symmetric group of degree $n$, then

$$ISOT(G, \cdot) \supseteq S_n \times S_n \times S_n.$$  

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Thus, we need \( G \), \( G \) prove the claim of this theorem, it suffices to produce a closed binary operation \( ' \) because \( (G, \cdot) \) of all principal isotopisms on \( (G, \cdot) \). \( \text{PISOT}(G, \cdot) \) is an S-subgroup in \( \text{ISOT}(G, \cdot) \) while \( S_n \times S_n \times \{I\} \) is an S-subgroup in \( S_n \times S_n \times S_n \). If

\[
\Upsilon : \text{ISOT}(G, \cdot) \rightarrow S_n \times S_n \times S_n \text{ is defined as}
\]

\[
\Upsilon((A, B, I)) = < A, B, I > \forall (A, B, I) \in \text{ISOT}(G, \cdot),
\]

then

\[
\Upsilon(\text{PISOT}(G, \cdot)) = S_n \times S_n \times \{I\}. \quad \therefore \text{ISOT}(G, \cdot) \ni S_n \times S_n \times S_n.
\]

4 Smarandache \( f, g \)-Isotopes of Smarandache Loops

**Theorem 4.1** Let \( (G, \cdot) \) and \( (H, \ast) \) be S-groupoids. If \( (G, \cdot) \) and \( (H, \ast) \) are S-isotopic, then \( (H, \ast) \) is S-isomorphic to some Smarandache principal isotope \( (G, \circ) \) of \( (G, \cdot) \).

**Proof**

Since \( (G, \cdot) \) and \( (H, \ast) \) are S-isotopic S-groupoids with S-subsemigroups \( (G_1, \cdot) \) and \( (H_1, \ast) \), then there exist bijections \( U, V, W : (G, \cdot) \rightarrow (H, \ast) \) such that the triple \( \alpha = (U, V, W) : (G, \cdot) \rightarrow (H, \ast) \) is an isotopism and \( (G_1)A = H_1 \forall A \in \{U, V, W\} \). To prove the claim of this theorem, it suffices to produce a closed binary operation \( \ast \) on \( G \), bijections \( X, Y : G \rightarrow G \), and bijection \( Z : G \rightarrow H \) so that

- the triple \( \beta = (X, Y, I) : (G, \cdot) \rightarrow (G, \circ) \) is a Smarandache principal isotopism and
- \( Z : (G, \circ) \rightarrow (H, \ast) \) is an S-isomorphism or the triple \( \gamma = (Z, Z, Z) : (G, \circ) \rightarrow (H, \ast) \) is an S-isotopism.

Thus, we need \( (G, \circ) \) so that the commutative diagram below is true:

\[
(G, \cdot) \xrightarrow{\alpha} (H, \ast) \xrightarrow{\beta} \text{isotopism} \quad \beta \quad \gamma \quad \text{isomorphism}
\]

\[
(G, \circ) \quad \text{principal isotopism}
\]

because following the proof of transitivity in Theorem 3.1 \( \alpha = \beta \gamma \) which implies \( (U, V, W) = (XZ, YZ, Z) \) and so we can make the choices; \( Z = W, Y = VW^{-1}, \) and \( X = UW^{-1} \) and consequently,

\[
x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \forall x, y \in G.
\]

Hence, \( (G, \circ) \) is a groupoid principal isotope of \( (G, \cdot) \) and \( (H, \ast) \) is an isomorph of \( (G, \circ) \). It remains to show that these two relationships are Smarandache.

Note that \( ((H_1)Z^{-1}, \circ) = (G_1, \circ) \) is a non-trivial subsemigroup in \( (G, \circ) \). Thus, \( (G, \circ) \) is an S-groupoid. So \( (G, \circ) \ni (H, \ast) \). \( (G, \cdot) \) and \( (G, \circ) \) are Smarandache principal isotopes because \( (G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1 \) and \( (G_1)WV^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1 \).

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Corollary 4.1 Let \((G,\cdot)\) be an S-groupoid with an arbitrary groupoid isotope \((H,\ast)\). Any such groupoid \((H,\ast)\) is an S-groupoid if and only if all the principal isotopes of \((G,\cdot)\) are S-groupoids.

**Proof**

By classical result in principal isotopy \([III,\; III.1.4\; Theorem]\), if \((G,\cdot)\) and \((H,\ast)\) are isotopic groupoids, then \((H,\ast)\) is isomorphic to some principal isotope \((G,\circ)\) of \((G,\cdot)\). Assuming \((H,\ast)\) is an S-groupoid then since \((H,\ast)\cong (G,\circ)\), \((G,\circ)\) is an S-groupoid. Conversely, let us assume all the principal isotopes of \((G,\cdot)\) are S-groupoids. Since \((H,\ast)\cong (G,\circ)\), then \((H,\ast)\) is an S-groupoid.

**Theorem 4.2** Let \((G,\cdot)\) be an S-quasigroup. If \((H,\ast)\) is an S-loop which is S-isotopic to \((G,\cdot)\), then there exist S-elements \(f\) and \(g\) so that \((H,\ast)\) is S-isomorphic to a Smarandache \(f,g\) principal isotope \((G,\circ)\) of \((G,\cdot)\).

**Proof**

An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1, \((H,\ast)\) is S-isomorphic to a Smarandache principal isotope \((G,\circ)\) of \((G,\cdot)\). Let \(\alpha = (U,V,I)\) be the Smarandache principal isotopism of \((G,\cdot)\) onto \((G,\circ)\). Since \((H,\ast)\) is a S-loop and \((G,\circ)\cong (H,\ast)\) implies that \((G,\circ)\cong (H,\cdot)\), then \((G,\circ)\) is necessarily an S-loop and consequently, \((G,\circ)\) has a two-sided identity element say \(e\) and an S-subgroup \((G_2,\circ)\). Let \(\alpha = (U,V,I)\) be the Smarandache principal isotopism of \((G,\cdot)\) onto \((G,\circ)\). Then,

\[xU \circ yV = x \cdot y \; \forall \; x,y \in G \iff x \circ y = xU^{-1} \cdot yV^{-1} \; \forall \; x,y \in G.\]

So,

\[y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1} L_e U^{-1} \; \forall \; y \in G \text{ and } x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1} R_e V^{-1} \; \forall \; x \in G.\]

Assign \(f = eU^{-1}, g = eV^{-1} \in G_2\). This assignments are well defined and hence \(V = L_f \) and \(U = R_g\). So that \(\alpha = (R_g, L_f, I)\) is a Smarandache \(f,g\) principal isotopism of \((G,\circ)\) onto \((G,\cdot)\). This completes the proof.

**Corollary 4.2** Let \((G,\cdot)\) be an S-quasigroup(S-loop) with an arbitrary groupoid isotope \((H,\ast)\). Any such groupoid \((H,\ast)\) is an S-quasigroup(S-loop) if and only if all the principal isotopes of \((G,\cdot)\) are S-quasigroups(S-loops).

**Proof**

This follows immediately from Corollary 4.1 since an S-quasigroup and an S-loop are S-groupoids.

**Corollary 4.3** If \((G,\cdot)\) and \((H,\ast)\) are S-loops which are S-isotopic, then there exist S-elements \(f\) and \(g\) so that \((H,\ast)\) is S-isomorphic to a Smarandache \(f,g\) principal isotope \((G,\circ)\) of \((G,\cdot)\).

**Proof**

An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.
5 G-Smarandache Loops

Lemma 5.1 Let \((G, \cdot)\) and \((H, \ast)\) be S-isotopic S-loops. If \((G, \cdot)\) is a group, then \((G, \cdot)\) and \((H, \ast)\) are S-isomorphic groups.

Proof
By Corollary 4.3, there exist S-elements \(f\) and \(g\) in \((G, \cdot)\) so that \((H, \ast) \supseteq (G, \circ)\) such that \((G, \circ)\) is a Smarandache \(f, g\) principal isotope of \((G, \cdot)\).

Let us set the mapping \(\psi := R_{f \cdot g} = R_{fg} : G \to G\). This mapping is bijective.

Now, let us consider when \(\psi := R_{fg} : (G, \cdot) \to (G, \circ)\). Since \((G, \cdot)\) is associative and \(x \circ y = xR_g^{-1} \cdot yL_f^{-1} \forall x, y \in G\), the following arguments are true.

\[
x \circ y = xR_g^{-1} \cdot yL_f^{-1} = xR_gR_g^{-1} \cdot yR_gL_f^{-1} = x \cdot f g \cdot g^{-1} \cdot f^{-1} \cdot y \cdot f g = x \cdot y \cdot f g = (x \cdot y)R_{fg} = (x \cdot y)\psi \forall x, y \in G.
\]

So, \((G, \cdot) \cong (G, \circ)\). Thus, \((G, \circ)\) is a group. If \((G_1, \cdot)\) and \((G_1, \circ)\) are the S-subgroups in \((G, \cdot)\) and \((G, \circ)\), then \(((G_1, \cdot))R_{fg} = (G_1, \circ)\). Hence, \((G, \cdot) \supseteq (G, \circ)\).

\[
\therefore (G, \cdot) \supseteq (H, \ast)\) and \((H, \ast)\) is a group.
\]

Corollary 5.1 Every group which is an S-loop is a GS-loop.

Proof
This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

Corollary 5.2 An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache \(f, g\) principal isotopes.

Proof
Let \((G, \cdot)\) be an S-loop with arbitrary S-isotope \((H, \ast)\). Let us assume that \((G, \cdot) \supseteq (H, \ast)\). From Corollary 4.3, for any arbitrary S-isotope \((H, \ast)\) of \((G, \cdot)\), there exists a Smarandache \(f, g\) principal isotope \((G, \circ)\) of \((G, \cdot)\) such that \((H, \ast) \supseteq (G, \circ)\). So, \((G, \cdot) \supseteq (G, \circ)\).

Conversely, let \((G, \cdot) \supseteq (G, \circ)\), using the fact in Corollary 4.3 again, for any arbitrary S-isotope \((H, \ast)\) of \((G, \cdot)\), there exists a Smarandache \(f, g\) principal isotope \((G, \circ)\) of \((G, \cdot)\) such that \((G, \circ) \supseteq (H, \ast)\). Therefore, \((G, \cdot) \supseteq (H, \ast)\).

Corollary 5.3 A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache \(f, g\) principal isotopes.

Proof
This follows by the definition of a GS-loop and Corollary 5.2.

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