Quantum key distribution based on orthogonal states

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Quantum key distribution (QKD) is one of the most significant areas in quantum information theory. For nearly four decades, substantial QKD protocols and cryptographic methods are developed. In early years, the security of QKD protocols is depend on switching different bases, which, in fact, is based on non-orthogonal states. The most famous example is the BB84 protocol. Later, other techniques were developed for orthogonal states cryptography. Representations of such protocols include the GV protocol and order-rearrangement protocols. It might be harder to implement protocols based on orthogonal states since they require extra techniques to obtain the security. In this paper, we present two QKD protocols based on orthogonal states. One of them needs not to employ order-rearrangement techniques while the other needs. We give analyses of their security and efficiency. Also, anti-noisy discussions would be given, namely, we modify the protocols such that they could be implemented in noisy channels as in noiseless ones without errors. Our protocols are highly efficient when considering consumptions of both qubits and classical bits while they are robust over several noisy channels. Moreover, the requirement of maximally entangled states could be less than previous protocols and so the efficiency of measurements could be increased.

Keywords: Quantum key distribution; Order-rearrangement; Orthogonal states; Noise; Qubit.

I. INTRODUCTION

In information theory, cryptography is always one of the most important fields. Unfortunately, the most useful cryptosystem nowadays, the RSA system, is not secure in quantum era[1], for the essential reason that its security is depend on the low capacity of classical computations. To obtain the unconditional security, cryptographic schemes of which the security is only depend on physical laws are needed. The only classical cryptography which has been proven to be secure is encoding messages with an one-time pad. However, transmitting an one-time pad by classical channels could be totally insecure since classical messages can be cloned without being detected. On the other hand, quantum effects can provide possibilities of transmitting an one-time pad, of which the security is only depend on physical laws and can be proven mathematically. Such a task is called a quantum key distribution, and QKD for short.

In nearly four decades, great developments have been made since the first QKD protocol was proposed in 1984[2], which obtains the security by switching two mutual unbiased bases and of which the security has been proven[3]. After the BB84 protocol, several BB84-like protocols were proposed, such as Ekert’s protocol[4], BBM92 protocol[5], six-state protocol[6] and others[3][7]. These protocols obtain the security by employing non-orthogonal states. It was not until 1995 that the first cryptographic protocol based on orthogonal states was published[8]. The idea of the protocol is sending states with a time delay such that eavesdroppers can never get entire states without being detected. Another technique can be employed to implement a QKD protocol based on orthogonal states is the order-rearrangement. Protocols employing this technique can be found in [9][10][11][12][13]. Other protocols include [14][15][16][17][18][19][20].

Besides designing QKD protocols, there is another problem. In practical, Channels employed to implement a QKD protocol are always noisy. Therefore, the robustness of a protocol over noisy channels has to be investigated. Previous works include [21][22][23][24] for collective noises, [25][26][27][28] for Pauli noises and [29][30][31][32][33][34] for amplitude damping(AD) and phase damping(PD) noises. There are also other researches on noises[35][36][37][38][39][40], for examples.

In this paper, two QKD protocols based on orthogonal states are proposed in section II while efficient analyses are given in section III, comparing with several protocols. The discussions of the security are given in section IV and implementing protocols over noisy environments would be argued in section V. The last section, section VI, is devoted to conclusions. Comparing with certain previous protocols, our protocols, on one hand, are highly efficient when considering consumptions of both qubits and classical bits, and on the other hand, are robust over several noisy channels, namely, they could be implemented in noisy channels as in noiseless ones without errors after modifications. Moreover, the requirement of maximally entangled states could be less and so the efficiency of measurements could be increased.

II. TWO PROTOCOLS

Let us propose the protocols in this section, which are stated as follow.

Protocol I:
Step 1: Alice and Bob agree to encode 00, 11, 01, 10 by states \(|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}(01 - 10)\rangle_{AB}, |\varphi'\rangle_{AB} = \frac{1}{\sqrt{2}}(01 + 10)\rangle_{AB}\), respectively, in \(C^2 \otimes C^2\).

Step 2: To share a N-2-bit key string, Alice creates a string of 2N states chosen randomly in \(S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}\), which is only known by her.

Step 3: Alice creates \(\frac{N}{2}\) decoy states, all be \(|+\rangle = \frac{1}{\sqrt{2}}|0 + 1\rangle_B\) and inserts them in the states string in step 2 randomly. Now Alice has a states string with 2.5N states and she records the positions of decoy states by a 2.5N-bit string \(r = r_1r_2, ..., r_{2.5N}\). In more details, \(r_i = 1\) if the i-th state is a decoy state and \(r_i = 0\), otherwise.

Step 4: Alice sends the partita B of the states string to Bob.

Step 5: After receiving the particles, Bob publicly announces this fact.

Step 6: After Alice receives Bob’s receipt, she sends the partita A of the states string to Bob.

Step 7: Bob receives the states string and the string \(r\). He then measures decoy states via basis \(|+\rangle = \frac{1}{\sqrt{2}}|0 + 1\rangle_B\) and \(|-\rangle = \frac{1}{\sqrt{2}}|0 + 1\rangle_B\), and other states via basis \(S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}\).

Step 8: Alice and Bob run the checking procedure as follow. Bob publishes all his outcomes on decoy states and the outcomes on half of other states (let us called them checking states) chosen randomly. Bob publishes his outcomes on checking states together with the positions of them. Alice verifies whether the checking states are agreed with what she created while Bob verifies whether the decoy states are \(|+\rangle\). They calculate the error rates.

Step 9: If the error rates are acceptable on both decoy states and checking states, Alice and Bob agree a secret key by the outcomes of the remaining N states, which are neither decoy states nor checking states.

Step 10: Alice and Bob repeat the above procedure until they share a sufficiently long secret key and run error correcting or private amplified procedures if needed.

Protocol II:

Step 1: Alice and Bob agree to encode 00, 11, 01, 10 by states \(|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}(01 - 10)\rangle_{AB}, |\varphi'\rangle_{AB} = \frac{1}{\sqrt{2}}(01 + 10)\rangle_{AB}\), respectively, in \(C^2 \otimes C^2\).

Step 2: To share a N-2-bit key string, Alice creates a string of 2N states chosen randomly in \(S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}\) which is only known by her.

Step 3: Alice divides the 2N states into N blocks such that each block includes two adjacent states. Alice chooses a random N-bit string \(s = s_1, s_2, ..., s_N\) and exchanges the order of the B partita of the first state and the A partita of the second state in the i-th block if \(s_i = 1\).

Step 4: Alice sends all partite of the states string to Bob.

Step 5: After receiving the particles, Bob publicly announces this fact.

Step 6: After Alice receives Bob’s receipt, she sends the string s to Bob.

Step 7: Now, Bob has the states string and the string s. He then reordered the states by information of the string s, recovering them and then measures via basis S.

Step 8: Alice and Bob run a checking procedure as follow. Bob publishes the positions and outcomes on half of states (let us called them checking states) chosen randomly. Alice verifies whether the checking states are agreed with what she created and calculates the error rate.

Step 9: If the error rate is acceptable, Alice and Bob agree a secret key by the outcomes of the remaining N states, which are not employed as checking states.

Step 10: Alice and Bob repeat the above procedure until they share a sufficiently long secret key and run error correcting or private amplified procedures if needed.

III. ANALYSES OF THE PROTOCOLS

Before providing the security proof, let us give analyses of the protocols.

A. Efficiency

In protocol I, to generate a N-2-bit key string, Alice and Bob consume 2N states in \(C^2 \otimes C^2\) and \(\frac{N}{2}\) single qubit states. Totally, they consume 4.5N qubits. And on the other hand, Alice and Bob consume 2.5N classical bits for publishing the string r, 2N classical bits for publishing the positions of checking states and 2N classical bits for publishing the measurement outcomes of N checking states. In more details, for example, Bob sends a string \(b = b_1b_2, ..., b_{2N}\) to Alice with \(b_i = 0\) represents that the i-th state is not a checking state, and \(b_i = 1\), otherwise. And for the N checking states, Bob sends a string \(c = c_1c_2, ..., c_N\) to Alice with \(c_j = 0, 1, 2, 3\) represents that the measurement outcome of the j-th state is \(|00\rangle, |11\rangle, |\varphi\rangle, |\varphi'\rangle\), respectively. Of course, they have to consume another three classical bits including Bob declares his receipt in step 5. Alice and Bob publish their errors checking procedures are passed. The total classical bits needed (for classical communications) in such a protocol are nearly 6.5N. Equivalently, they consume nearly 2.25N qubits and 3.25N classical bits for a N-bit key string.

In protocol II, to generate a N-2-bit key string, Alice and Bob consume 2N states in \(C^2 \otimes C^2\). Totally, they consume 4N qubits. And on the other hand, Alice and Bob consume N classical bits for publishing the string s, 2N classical bits for publishing the positions of
checking states and 2N classical bits for publishing the outcomes of N checking states. In more details, for example, Bob sends a string \( d = d_1d_2,...,d_{2N} \) to Alice with \( d_i = 0 \) represents that the i-th state is not a checking state, and \( d_i = 1 \), otherwise. And for the N checking states, Bob sends a string \( e = e_1e_2,...,e_N \) to Alice with \( e_j = 0,1,2,3 \) represents that the outcome of the j-th state is \( |00\rangle,|11\rangle,|\varphi\rangle,|\varphi'\rangle \), respectively. Of course, they have to consume another two classical bits including Bob declares his receipt in step 5, Alice publishes whether the errors checking procedure is passed. The total classical bits (for classical communications) needed in such a protocol are nearly 5N. Equivalently, they consume nearly 2N qubits and 2.5N classical bits for a N-bit key string.

IV. SECURITY

Let us focus on the security of the protocols. Assume that there is an eavesdropper, says Eye, who wants to steal the secret key of Alice and Bob. We would assume that Alice and Bob hold authenticated classical channels which might not be private, and quantum channels without any further assumption. Namely, Eve might eavesdrop the classical communications but with no abilities to forge messages or pretend to be one of the legitimated parties, while she can do anything under physical laws in quantum channels. The security is guaranteed in the sense that Eve can not get enough information on the secret key or she would create errors which are detectable by the checking procedure. We also assume that Eve provides a collective attack, namely, Eve attacks each state Alice sent by the same method.

For Eve, she has three kinds of attacks. She might intercept a state sent by Alice and implement one of the three actions. Firstly, she might add auxiliary partite (her partite) then do a transformation and resend the state to Bob (let us call this a purified attack). Secondly, she might take the state herself and send another state created by her to Bob, instead (let us call this a substituted attack). Thirdly, she might measure the state and resend it to Bob (let us call this a measure-resend attack). Let us also assume that in protocol I, Eve can implement different attacks in step 4 and step 6.

A. purified attack

Let us analyse purified attacks firstly. For such attacks, the decoy states in the protocol I (step 3) can be aborted, and so the efficiency can be increased.

1. Eve purifies via single qubits

If Eve purifies states sent by Alice via single qubits, the security of the protocols (both protocol I and protocol II) can correspond to previous protocols, such as [17] or [9]. For example, if Eve purifies states sent by Alice via basis \( \{|j\rangle | j = 0,1\} \), which would change states \( |00\rangle,|11\rangle,|\varphi\rangle,|\varphi'\rangle \) into \( |00\rangle_{AB}|00\rangle_{EE'},|11\rangle_{AB}|11\rangle_{EE'},|\varphi'_p\rangle_{ABEE'} = \frac{1}{\sqrt{2}}(|0101 - 1010\rangle_{ABEE'},|\varphi'_p\rangle_{ABEE'}) \), respectively, where E and E' are partite of Eve. However, Alice and Bob can detect Eve on checking states \( |\varphi\rangle \) or \( |\varphi'\rangle \). In more details, \( |\varphi'_p\rangle_{ABEE'} = \frac{1}{\sqrt{2}}(|0101 - 1010\rangle_{ABEE'},|\varphi'_p\rangle_{ABEE'}) \). When Bob measures the state via basis S on partite A and B, he will get an outcome \( |\varphi\rangle \) or \( |\varphi'\rangle \) with equal probabilities. The calculation of \( |\varphi'_p\rangle \) is essentially same. The error rate created by Eve and detectable by Alice and Bob is now \( \frac{1}{4} \), for Alice and Bob choose an entangled state for checking with a probability \( \frac{1}{2} \) and get an error outcome with
a probability $\frac{1}{4}$, if so.

If Eve purifies states via other bases, the arguments are similar. We note that if so, errors could occur on all checking states. For example, Eve purifies states via basis $\{ |+\rangle = \frac{1}{\sqrt{2}} (0 + 1) , |-\rangle = \frac{1}{\sqrt{2}} (0 - 1) \}$, then an error occurs with a probability $\frac{1}{2}$ for each checking state and so the error rate is $\frac{1}{2}$.

2. Eve purifies via two-qubit states

Now assume that Eve attacks by purifying states via $S$. This attack is not suitable for protocol I since in protocol I, Eve can only obtain one partita of states in the same time, if she tries to employ such an attack. Let us analyse for protocol II.

Since Alice changes the order of two partite of two states, Eve guesses the right order with a probability $\frac{1}{2}$. If she guesses the order wrong, she will purify the states wrong and create errors and can be detected with probabilities. In more details, let a block of two states be $|b\rangle_{xy} = |x\rangle_{12} |y\rangle_{34}$, where $|x\rangle$ and $|y\rangle$ are states in $S$. If Eve guesses the order of states wrong, she will purify states via $S$ on partite 1, 3, and partite 2, 4, respectively. $|b\rangle_{xy}$ can be one of

\begin{equation}
|b\rangle_{xy} = |\phi\rangle_{12} |\phi\rangle_{34}, b_{\phi\phi'} = |\phi'\rangle_{12} |\phi\rangle_{34}, b_{00} = |00\rangle_{12} |00\rangle_{34}, b_{11} = |11\rangle_{12} |11\rangle_{34},
\end{equation}

with equal probabilities. Let us calculate the error rate for each case. Assume that $|b\rangle_{xy}$ becomes $|b\rangle_{xy'}$ after being purified via $S$ on partite 1, 3 and partite 2, 4, respectively. And let Eve’s partite be $E_1, E_2, E_3, E_4$.

\begin{equation}
\begin{aligned}
b_{\phi\phi'} &= |\phi'\rangle_{12} |\phi\rangle_{34} = \frac{1}{2} (0101 + 0110 + 1001 + 1010)_{1234} = \frac{1}{2} (0011 + 0110 + 1001 + 1100)_{1324}, \\
b_{\phi\phi'} &= \frac{1}{2} (0011 + \phi' \phi' - \phi \phi + 1100)_{1324}, \\
b_{\phi\phi'} &= \frac{1}{2} (00110011 + \phi' \phi' \phi' - \phi \phi \phi \phi + 11001100)_{1324 E_1 E_2 E_3 E_4}, \\
&= \frac{1}{4} (|0011\rangle |\phi' \phi' - \phi \phi\rangle + |1100\rangle |\phi' \phi' - \phi \phi\rangle + |\phi \phi\rangle |0011 + 1100 - \phi' \phi' - \phi \phi\rangle) \\
&+ |\phi' \phi'} |0011 + 1100 + \phi' \phi' + \phi \phi\rangle + |\phi \phi'\rangle |0011 - 1100\rangle + |\phi' \phi\rangle |0011 - 1100\rangle )_{1234 E_1 E_2 E_3 E_4}.
\end{aligned}
\end{equation}

The probability of getting the correct outcome by measuring via $S$ on partite 1, 2 is $\frac{3}{10}$ and so the error rate is $\frac{7}{10}$. Similarly, error rates for $b_{\phi\phi'}$, $b_{\phi\phi'}$ and $b_{\phi'\phi'}$ are all

\begin{equation}
\begin{aligned}
b_{\phi0} &= |\phi\rangle_{12} |00\rangle_{34} = \frac{1}{\sqrt{2}} (0100 - 1000)_{1234} = \frac{1}{2} |0000\rangle - 00\phi - \phi' 00 + \phi 00)_{1324}, \\
b_{\phi00} &= \frac{1}{2} (00\phi' 00 \phi' - 00\phi 00 \phi' - \phi' 00 \phi' 00 + \phi 00 \phi 00)_{1324 E_1 E_2 E_3 E_4}, \\
&= \frac{1}{4} (|00\rangle |\phi + \phi'\rangle |00\rangle |\phi' - \phi\rangle - |00\rangle |\phi' - \phi\rangle |\phi' - \phi\rangle |00\rangle \\
&+ |\phi') |00\rangle |00\phi' + 00\phi + \phi' 00 + \phi 00\rangle \\
&+ |\phi') |00\rangle |00\phi' + 00\phi - \phi' 00 - \phi 00\rangle )_{1234 E_1 E_2 E_3 E_4}.
\end{aligned}
\end{equation}

The probability of getting the correct outcome (that is $|\phi\rangle$) by measuring via $S$ on partite 1, 2 is $\frac{1}{4}$ and so the
error rate is $\frac{4}{7}$. Similarly, error rates for $b_{φ'1p}$, $b_{φ1p}$ and $b_{φ1p}$ are all $\frac{4}{7}$. And on the other hand, the probability of getting the correct outcome by measuring via $S$ on partite 3, 4 (that is $|00⟩$) is $\frac{1}{2}$ and so the error rate is $\frac{1}{2}$. Similarly, error rates for $b_{0pφ'}$, $b_{1pφ'}$ and $b_{1pφ}$ are all $\frac{1}{2}$.

$$b_{01} = |00⟩_{12}|11⟩_{34} = |0011⟩_{1234} = |0101⟩_{1234} = \frac{1}{2}|ϕφ + φφ' + φ'φ + φ'φ'⟩_{1324}.$$  

$$b_{01p} = \frac{1}{2}|ϕϕϕφ + φφ'φφ' + φ'φ'φ'φ + φ'φ'φ'φ⟩_{1324}E_1E_2E_3E_4$$

$$= \frac{1}{4}(|0011⟩|φ + φ')|φ + φ') + |1100⟩|φ' - φ⟩|φ' - φ⟩$$

$$+ |φ⟩|φφφφ - φφ'φφ' - φφ'φφ' + φφ'φφ'⟩$$

$$+ |φ'⟩|φφ'φφ' + φφ'φφ' - φφ'φφ' + φφ'φφ'⟩⟩_{1324}E_1E_2E_3E_4.$$

The probability of getting the correct outcome (that is $|00⟩$) by measuring via $S$ on partite 1, 2 is $\frac{3}{4}$ and so the error rate is $\frac{4}{7}$. Similarly, the error rate for $b_{10p}$ is $\frac{4}{7}$.

$$b_{00} = |00⟩_{12}|00⟩_{34} = |0000⟩_{1234} = |0000⟩_{1324}.$$  

$$b_{00p} = |00000000⟩_{1324}E_1E_2E_3E_4 = |00000000⟩_{1234}E_1E_2E_3E_4.$$  

The probability of getting the correct outcome (that is $|00⟩$) by measuring via $S$ on partite 1, 2 is $\frac{1}{2}$ and so the error rate is $0$. Similarly, the error rate for $b_{11p}$ is $0$.

Now, the average error rate for Eve guesses the order wrong is $\frac{15}{16}(4 \times \frac{7}{10} + 4 \times \frac{1}{2} + 4 \times \frac{1}{4} + 2 \times \frac{1}{2} + 2 \times 0) = \frac{93}{160}$ and so the whole error rate is $\frac{93}{160}$ which is larger than $\frac{4}{7}$, the error rate of attacking the BB84 protocol by purification.

### B. Substituted attack

Let us focus on substituted attacks. Step 3 of protocol I is not needed in such case and so the efficiency or security can be increased. Eve might take a state sent by Alice herself, measuring it or keeping it until she obtains more information. However, Alice and Bob will not continue the procedure until Bob receives a state. Thus, Eve has to send another state to Bob, instead. Eve could get enough information after stealing the A partita in protocol I or after Alice publishes the order string $s$ in protocol II. She can measure the state via basis $S$ and know exactly what Alice sent. But Bob can detect the attack, for the reason Eve might substitute a product state by an entangled state or substitute an entangled state by a product state. In both cases, Bob’s measurement outcome could be incorrect.

In more details, in protocol I, if Eve steals the B partita of the state and sends one partita of her state, instead. Since, at this time, Eve is not able to discriminate whether the state is separated or entangled, she can not send a state to Bob and transform it into the state she needs after stealing the A partita, since local transformations (local unitary operations) of a state can not generate or break entanglement. For example, assume that Eve sends a partita of a product state, says the partita $E_1$ of $|0⟩_{E_1}|0⟩_{E_2}$ to Bob and keeps the state sent by Alice in step 4. If after stealing the partita $A$ of the state and finding that the state is entangled, for example, be $|ϕ⟩$, she can not prevent her being detectable. For now, no matter what state she sends to Bob, Bob will get a product state and there is at least with a probability $\frac{1}{2}$ he will obtain an error outcome when measuring via basis $S$. The same argument is suitable if Eve sends a maximally entangled state, instead. In such case, if she finds that Alice sent a product state, she can do nothing to decrease the error rate of Bob less than $\frac{1}{2}$. Hence, the error rate is not less than $\frac{1}{2}$, for Eve wrongly guesses the state sent by Alice is entangled or separated and then Bob’s measurement outcome is incorrect, if so.

These arguments are also held for protocol II. Since Eve can not know the order of the states before sending her states to Bob, she might send a product state instead of an entangled state or conversely. If so, the error rate of Bob’s checking procedure will not less than $\frac{1}{2}$ and the whole error rate will not less than $\frac{4}{7}$.

It is worth remarking that protocol I without step 3 can not employ the four Bell states instead of $S$. The four Bell states can be transformed into each other via local transformations. For this reason, Eve can only steal the partita $B$ of states sent by Alice and send a partita of maximally entangled states of her in step 4. Then she can send the other partita with transformations depending
on outcomes of her Bell measurements after stealing the partita A.

C. Measure-resend attack

The security analyses for such attacks are similar to above. If Eve measures via single qubits and resends the states to Bob, then Bob will receive product states. Thus, Bob can detect Eve by the outcomes of entangled states sent by Alice. The probability would be at least \( \frac{1}{4} \) for half of checking states be entangled and with a probability \( \frac{1}{2} \) being incorrect for those states. To against such an attack, step 3 of protocol I is not needed, similar to above.

Eve might choose to measure via two-qubit states. Such an attack can only happen in protocol II, since Eve can never obtain both partite of states in protocol I when employing a measure-resend attack only. In protocol II, since Eve can not know the orders of states until she finishes her measurement and resends states to Bob, she might change correlations of states. For example, let Eve measure states sent by Alice via basis S, the only basis for Eve might gather the secret key without increasing errors. If she guesses the order incorrect, she will measure partite 1, 3 and partite 2, 4, respectively. If partite 1, 2 are entangled, after Eve’s interaction, they become separated. Hence, the probability for Bob obtains an error for such a state is at least \( \frac{1}{2} \). If the partite are separated, without loss generality, the state be \(|00\rangle\), there are three cases including Alice sends \(|00\rangle_{12}|00\rangle_{34}, |00\rangle_{12}|11\rangle_{34} \) or \(|00\rangle_{12}|\varphi\rangle_{34}. Eve will not provide any error in the first case while Eve provides an error with a probability \( \frac{1}{2} \) in the second and \( \frac{1}{2} \) in the third. The arrange error rate is not less than \( \frac{1}{4} \).

D. Two stages attack

Two stages attacks only suitable for protocol I, since in protocol II, Alice sends states to Bob in only one stage. Let us assume that Eve attacks protocol I with different strategies, in step 4 and step 6, when Alice sends different partite to Bob. If Eve employs a substituted attack in step 4, she might substitute product states instead of entangled states or substitute entangled states instead of product states, which would increase error rates as in substituted attacks only. If Eve employs measure-resend attacks in step 4, she might break entanglement of states, and errors would be occurred as in measure-resend attacks only.

The only case left is that Eve purifies states in step 4, and measures states in step 6. For example, Eve might purify states via basis \(|j\rangle\) \( j = 0, 1 \) in step 4, twice. The states become \(|0000\rangle_{ABEE'} + |1111\rangle_{ABEE'}, |\varphi\rangle_{ABE} = \frac{1}{\sqrt{2}} (|\varphi'\rangle - |\varphi\rangle)_{ABE'}, |\varphi'\rangle_{ABE} = \frac{1}{\sqrt{2}} (|\varphi\rangle - |\varphi'\rangle)_{ABE'}\). After Eve operates bit-flip gates on the partita A', respectively and corresponding to \(|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\). Eve steals the partita A, she measures the partita AE via basis S. If, for a state, the outcome is \(|00\rangle \) or \(|11\rangle \), she knows the state is \(|00\rangle \) or \(|11\rangle \), respectively and she sends the partita A' to Bob. If the outcome is \(|\varphi\rangle\), she operates nothing and sends the partita A' to Bob while if the outcome is \(|\varphi'\rangle\), she operates the phase-flip gate on the partita A' and sends the partita A' to Bob. Here, the attack is undetectable without step 3 and Eve obtains all information on product states.

V. IMPLEMENT PROTOCOLS OVER NOISY CHANNELS

We shall give discussions of implementing the protocols over noisy channels including collective dephasing (CD) noises, collective rotation (CR) noises, Pauli noises, amplitude damping (AD) and phase damping (PD) noises and mixtures of them. Instead of discussing fidelities or estimating noises of the channels, we modify the protocols such that they could be implemented in noisy channels as in noiseless ones without errors.

A. Collective dephasing (CD)

Collective dephasing noises assume that the whole protocol is implemented in a same-time cycle and so the noise affect each qubit equivalently via \( \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \) under the computational basis, where \( \phi \) is the parameter depending on the noise. Previous results for such noises include references [21, 22, 23, 24, 40].

For such noises, we shall modify protocol I, step 3, substituting decoy states \(|+\rangle\) by \(|\varphi\rangle\), and protocol I step 7, measuring via S on decoy states, instead. Protocol II needs not to be modified. Now, Protocol I and protocol II completely immune such noises, since all states in the protocols are changed nothing but global phases when affecting by the noises. The consumption of states in protocol I now becomes \( 2.5N \) qubits for a N-bit key string while the consumption of states in protocol II remains unchanged.

B. Collective rotation (CR)

Collective rotation noises assume that the whole protocol is implemented in a same-time cycle and so the noise
affect each qubit equivalently via \[
\begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{bmatrix}
\]
under the computational basis, where \( \theta \) is the parameter depending on the noise and evolving upon time. Previous results for such noises include \([21, 22, 23, 24, 40]\).

We shall modify protocol I and II both. Firstly, we substitute \( S \) by \( S' \) as \( |\varphi\rangle_{AB}, |\varphi'\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB} \) in both protocol I and protocol II. Then in protocol I, decoy states in step 3 are substituted by \( |\varphi\rangle_B \). And in step 7, Bob measures via \( S' \), instead. The reason for employing \( S' \) instead of \( S \) is that states in \( S' \) are not changed when affecting by CR while states in \( S \) might not. The consumption of states now becomes 5N qubits for protocol I and 4N qubits for protocol II for a N-bit key string.

C. Pauli noises

Pauli noises act on each qubit via Pauli operators, \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ ZX = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \). under the computational basis, with probabilities \( p_I, p_Z, p_X, p_{ZX} \), summing to 1. Previous works include \([25, 26, 27, 28, 40]\).

1. One Pauli channel

In one Pauli channels, states suffer two of the four Pauli operators with one of which is \( I \).

Let us assume that states suffer I with a probability \( p \) and \( Z \) with a probability \( 1-p \).

As for protocol I, to deal with the noise, Alice prefers to send states assisted with auxiliary partite. She sends two partite together and assumes that they suffer the same effects. In more details, Alice employs states \( |00\rangle + |11\rangle \rangle_{AB}, |11\rangle + |00\rangle \rangle_{AB}, |\varphi\rangle_{AB} + |\varphi'\rangle_{AB}, |\varphi'\rangle_{AB} + |\varphi''\rangle_{AB} \), instead of \( |00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi\rangle_{AB} \), respectively. Alice always sends partite A, A' together and B, B' together and assumes that partite A, A' always suffer the same effects while partite B, B' always suffer the same effects. The same method and assumption are also applied for decoy states in step 3, that is, Alice employs \( |++, \rangle_{BB}, |,, \rangle_{BB}, |00\rangle_{BB}, |11\rangle_{BB}, |\varphi\rangle_{AB}, |\varphi\rangle_{AB} \), respectively. Alice always sends partite A, A' together and B, B' together and assumes that partite A, A' always suffer the same effects while partite B, B' always suffer the same effects. The same method and assumption are also applied for decoy states in step 3, that is, Alice employs \( |++, \rangle_{BB}, |,, \rangle_{BB}, |00\rangle_{BB}, |11\rangle_{BB}, |\varphi\rangle_{AB}, |\varphi\rangle_{AB} \), respectively.

For protocol II, Alice does nothing but always sends \( \text{H} \rangle \) on all partite he obtains before measuring. And for states suffering I with a probability \( p \) and \( ZX \) with a probability \( 1-p \), they provide operator \( H' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) instead of \( H \).

2. Two Pauli channel

In two Pauli channels, \( p_I, p_Z, p_X, p_{ZX} \) might all be non-zero, and so the final states might be completely mixed. We follow the above method to deal with such noises.

For protocol I, Alice employs states

\[
\begin{align*}
|00\rangle + |00\rangle_{ABA'B''A''B''}, & \quad |11\rangle + |00\rangle_{ABA'B''A''B''}, \\
|\varphi\rangle + |00\rangle_{ABA'B''A''B''} = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle + |10\rangle - |01\rangle)_{ABA'B''}, \\
|\varphi'\rangle + |00\rangle_{ABA'B''A''B''} = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle + |10\rangle + |01\rangle)_{ABA'B''},
\end{align*}
\]

instead of \( |00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi\rangle_{AB} \), respectively. Alice always sends partite A, A', B together and B', B' together and assumes that partite A, A' always suffer the same effects while partite B, B' always suffer the same effects. The same manipulations are also applied for partite B, B' and basis \( |0\rangle, |1\rangle \) for A' and A'' for A', B'' for B'. If the outcomes on A, A' are \( |+, \rangle \) and \( |0\rangle \), he does nothing, while he transforms the partita A via the operator Z, X, or ZX if the outcomes are \( |-, \rangle \) and \( |0\rangle \), \( |+, \rangle \) and \( |1\rangle \), or \( |-, \rangle \) and \( |1\rangle \) on A' and A'', respectively. If the outcomes on A, A' are \( |+, \rangle \) and \( |0\rangle \), he does nothing, while he transforms the partita A via the operator Z, X, or ZX if the outcomes are \( |-, \rangle \) and \( |0\rangle \), \( |+, \rangle \) and \( |1\rangle \), or \( |-, \rangle \) and \( |1\rangle \) on A' and A'', respectively.
D. Phase damping (PD) and Amplitude damping (AD)

Phase damping noises have Kraus operators $E_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$, representing that a state sent by the channel has a probability 1-p remains unchanged and a probability p suffers errors.

Amplitude damping noises have Kraus operators $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$, representing that a state sent by the channel has a probability 1-p suffers $E_0$ and a probability $p$ suffers $E_1$. Previous works include [29-30][31][32][33][34][35][36][37][38][39][40].

For such noises, Alice can employ states
\[
|0101\rangle_{AA'}BB', \quad |1010\rangle_{AA'}BB',
\]
\[
|\psi\rangle_{AA'}BB' = \sqrt{\frac{1}{2}} (|0110\rangle - |1001\rangle)_{AA'}BB',
\]
\[
|\psi'\rangle_{AA'}BB' = \sqrt{\frac{1}{2}} (|0110\rangle + |1001\rangle)_{AA'}BB',
\]
instead of $|00\rangle_{AB}, |11\rangle_{AB}, |\psi\rangle_{AB}, |\psi'\rangle_{AB}$, respectively, and decoy states $|\phi\rangle_{BB}'$ instead of $|+\rangle_B$. That is, Alice employs an auxiliary partita for each partita, setting ordinary states of auxiliary partite be $|1\rangle$, then provides C-NOT gates on auxiliary partite. Alice always sends partite A and A', partite B and B' together to assume that partite A and A' always suffer the same effects of noises and so do partite B and B'. For Bob, after receiving states, he firstly discards states with partite lost. The states left are those not affected by noises. Bob then provides C-NOT gates on those states then continue step 7. The above arguments are suitable for both protocol I and protocol II.

E. Mixture of noises

To deal with mixtures of noises, one might employ a combination of the above strategies or consider a technique called decoherence-free subspace. We also mention that another obstacle in implementing such protocols, including the two protocols, other order-rearrangement protocols or even the BB84 protocol with a Harramard gate, is that one needs to employ a quantum memory. Therefore, investigations on the quantum memories might be significant.

VI. CONCLUSION

In this paper, we presented two quantum key distribution protocols based on orthogonal states. Protocol I consumes more but needs not to employ an order-rearrangement technique while protocol II consumes less with an order-rearrangement technique. Both protocols employ the same set of coding states, of which half are maximally entangled and the other half are separated. We demonstrated the advantages of the protocols, comparing with certain previous protocols and we provided security proofs for them. Arguments of implementing the protocols over noisy channels were also proposed.

Our protocols, on one hand, are highly efficient. In protocol II, all states are employed for key states except those for checking, which are always assumed to be half of the states like in the BB84 protocol, while in protocol I, another 12.5 percent of states are employed for decoy states but needs not to employ order-rearrangement techniques. Our protocols need not to employ a full Bell measurement which might be lowly efficient. The states we employ are half maximally entangled and half separated. On the other hand, our protocols can be modified for noisy environments and more robust comparing with several previous protocols. For example, collective daphasing noises affect nothing in protocol II while protocol I only needs to be modified a bit to against such noises.

As for the remaining problems, implementing such protocols over other noisy channels is considerable. Another direction could be experimental realizations of the protocols. We also mention that another obstacle in implementing such protocols, including the two protocols, other order-rearrangement protocols or even the BB84 protocol with a Harramard gate, is that one needs to employ a quantum memory. Therefore, investigations on the quantum memories might be significant.

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