Quantization of static space-times

Yongge Ma∗†

Center for Gravitational Physics and Geometry
The Pennsylvania State University
University Park, PA 16802, USA.

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Abstract

A 4-dimensional Lorentzian static space-time is equivalent to 3-dimensional Euclidean gravity coupled to a massless Klein-Gordon field. By quantizing canonically the coupling model in the framework of loop quantum gravity, we obtain a quantum theory which actually describes quantized static space-times. The Kinematical Hilbert space is the product of the Hilbert space of gravity with that of imaginary scalar fields. It turns out that the Hamiltonian constraint of the 2+1 model corresponds to a densely defined operator in the underlying Hilbert space, and hence it is finite without renormalization. As a new point of view, this quantized model might shed some light on a few physical problems concerning quantum gravity.

∗E-mail: ma@gravity.phys.psu.edu
†New address: Dept of Physics, Beijing Normal University, Beijing 100875, China.
1 Introduction

Since the Ashtekar variables was proposed in 1986 [1], considerable progress has been made in non-perturbative canonical quantum gravity, namely, loop quantum gravity or quantum geometry [2, 3]. The kinematics of this theory has been rigorously defined [4, 5]. Certain geometric operators corresponding to the measurement of length, area, volume, and the integrated norm of any smooth one forms are shown to have discrete spectra [6, 7, 8, 9, 10, 11]. The classical limit of the quantum theory is currently under investigations [12, 13, 14, 15, 16]. Moreover, the fundamental discreteness in loop quantum gravity is crucially used to make much new progress such as: the derivation of black hole entropy from loop quantum gravity [17, 18, 19], the resolution of the big-bang singularity in loop quantum cosmology [20], and the proof of the finiteness in the path integral, namely spin-foam, approach to quantum gravity [21, 22].

Despite these achievements, some important elements in this approach are yet to be understood. Although the Barbero-Immirzi parameter [23] could be crucially selected in the calculations of black hole entropy in order to match the semi-classical result of Bekenstein and Hawking, some aesthetic criticisms are raised against the real connection formulation of Lorentz gravity [24]. Despite the systematic efforts toward constructing Hamiltonian operators in the underlying Hilbert space [25, 26], the dynamics of the quantum theory has not been fully understood, especially if one wants to consider asymptotically non-flat cases. The primary goal of this paper is to test the loop quantization techniques by considering a kind of simplified model, which are static space-times. Since static solutions to the vacuum Einstein equation are equivalent to 3-dimensional Euclidean gravity coupled to massless Klein-Gordon fields, the “Immirzi ambiguity” would be avoided in the model. It is known that 3+1 Lorentz gravity coupled to Higgs scalar fields could be quantized in the framework of loop quantum gravity [27]. While, since that construction of the Hamiltonian constraint operator, which reflects the dynamics, depends crucially on the detail structure of 3+1 dimension, it is not obvious if a similar construction is still available in 2+1 dimension. Meanwhile, the canonical treatment of the Euclidean 3-space could provide a possibility to address the quantization of surface terms arising from the 3+1 Lorentz gravity. Moreover, in this framework one could expect to calculate black hole entropy by counting the number of the physical states on an apparent horizon.

In Section 2, paying attention to the necessary background for further expositions, we briefly
review the main framework of canonical quantization of 2+1 Euclidean gravity as well as Higgs fields coupled to gravity, which is developed by Thiemann [28, 27]. We show in Section 3 that 4-dimensional Lorentz static spacetimes are equivalent to 3-dimensional Euclidean gravity coupled to massless Klein-Gordon fields. Static space-times are then quantized in section 4 by canonically quantizing the model of Euclidean 3-gravity coupled to the scalar fields. The kinematical Hilbert space of the model is the product of the Hilbert space of gravity and that of imaginary scalar fields, \( \mathcal{H}_E \otimes \mathcal{H}_S \). The Hamiltonian constraint is successfully quantized as a densely-defined operator. Similar to the case in 3+1 dimension, it is completely finite without renormalization. Section 5 addresses a few directions for future investigations. The operator corresponding to the area of any oriented 2-manifold is constructed under certain circumstances, and its self-adjointness is proved. This operator is supposed to be useful for future applications of this framework, including a possible quantization of the surface terms in the gravitational Hamiltonian.

2 2+1 Euclidean quantum gravity and Higgs field coupling

2.1 3-dimensional Euclidean canonical gravity

3-dimensional Euclidean canonical gravity in the Ashtekar formalism is defined over an oriented 2-manifold \( S \) which is a foliation of the 3-manifold \( \Sigma = S \times \mathbb{R} \) [4, 28]. The basic variables are \( SU(2) \) connections \( A^i_a \) and conjugate electric fields \( E^b_j \) where \( a, b, \ldots = 1, 2 \) for spatial indices and \( i, j, \ldots = 1, 2, 3 \) for internal \( SU(2) \) indices, \( \epsilon^{ab} \) is the 2-dimensional Levi-Civita tensor density of weight 1, and \( e^j_a \) denotes the pull-back of the co-triads \( ^{(3)}e^i_a \) on \( S \). The 2-metric on \( S \) reads \( q_{ab} = e^i_a e^i_b \). The symplectic structure is given by \( \{ A^i_a(x), E^j_b(y) \} = \delta^i_j \delta^a_b \delta^2(x, y) \), where we set the usual gravitational constant \( k_G = 1 \). Besides the Gauss and diffeomorphism constraints, the dynamics is reflected by the Ashtekar’s Hamiltonian constraint

\[
H(N) := \frac{1}{2} \int_S d^2 x N \epsilon^{ijk} F^a_{bb} \frac{E^a_j E^b_k}{\sqrt{q}},
\]

where \( F_{ab} \) is the curvature of \( A^i_a \), \( q \) denotes the determinant of \( q_{ab} \), and the smear function \( N \) is some scalar field.

Given any graph \( \Gamma \) with \( n \) edges \( e_I, I = 1, \ldots, n \), and \( m \) vertices \( v_\beta, \beta = 1, \ldots, m \), embedded in the 2-manifold \( S \), the holonomy of the \( SU(2) \) connection \( A^i_a \) along any edge \( e_I \) gives an element of \( SU(2) \) as: \( h[A, e_I] = \mathcal{P} \exp \int_{e_I} ds e_I(s) A^i_a(e_I(s)) \tau_i \), where \( \mathcal{P} \) denotes path ordering and \( \tau_i \) are the \( SU(2) \) generators in the fundamental representation. Given a function \( f_n : [SU(2)]^n \rightarrow \mathbb{C} \), the
cylindrical function with respect to $\Gamma$ is defined as: $\Psi_{\Gamma, f_n}(A) := f_n(h[e_1], \ldots, h[e_n])$. Since any two cylindrical functions based on different graphs can always be viewed as being defined on the same graph which is just constructed as the union of the original ones, it is straightforward to define a scalar product for them by:

$$<\Psi_{\Gamma, f_n} | \Psi_{\Gamma, g_n}> := \int_{[SU(2)]^n} dh_1 \cdots dh_n f_n^* (h_1, \ldots, h_n) g_n (h_1, \ldots, h_n),$$  

(2)

where $dh_1 \cdots dh_n$ is the Haar measure of $[SU(2)]^n$ which is naturally induced by that of $SU(2)$.

The kinematical Hilbert space, $H_\mathcal{E} = L^2(A, d\mu_H)$, is obtained by completing the space of all finite linear combinations of cylindrical functions in the norm induced by the quadratic form (2) on a cylindrical function.

The formal expression of the momentum operator is some functional derivative with respect to $A^i_a$, i.e., $\hat{E}^a_i(x) = -i\hbar(\delta / \delta A^i_a (x))$. Its action on a cylindrical function yields

$$\hat{E}^a_i(x) \circ \Psi_{\Gamma, f_n} = -il_p \sum_{I=1}^{n} \int ds \dot{e}^I_a(s) \delta^2(x, e_I(s)) X^I_i(t) \circ \Psi,$$

(3)

where $l_p = k_G \hbar$ is the Planck length and $X^I_i(t) \equiv Tr(h_I[0, t] \tau_i h_I[t, 1] \partial / \partial h_I)$. Classically, let

$$E^i \equiv \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} E^a_j E^b_k,$$

(4)

the area of any bounded region $B \subset \mathcal{S}$ reads

$$V_B = \int_B d^2 x \sqrt{q} = \int_B d^2 x \sqrt{E^i E_i},$$

(5)

Note that $\hat{E}^a_i(x)$ is an operator-valued distribution rather than a genuine operator. In order to construct a well-defined operator $\hat{V}_B$ corresponding to the classical area $V_B$, some regularization procedure is necessary. The final expression of the area operator reads

$$\hat{V}_B = \sum_{v_{\beta} \in B} \hat{V}_{v_{\beta}},$$

(6)

where

$$\hat{V}_{v_{\alpha}} = \frac{l_p^2}{8} \sqrt{\sum_{I, J} 2 sgn(e_I, e_J) X^I_i X^J_j X^I_i X^J_j},$$

(7)

here $X^I_i(0) = Tr(\tau_i h_I(\partial / \partial h_I))$ is the right invariant vector field on $SU(2)$ evaluated at $h_I$ and $sgn(e_I, e_J)$ is the sign of $\epsilon_{ab} \dot{e}^I_a(0) \dot{e}^J_b(0)$. Note that, for convenience, each edge is subdivided into two parts and equipped each part with an orientation that is outgoing from the vertex.
The regularization of the Hamiltonian constraint operator is rather complicated. By choosing the triangulation $T$ adapted to the graph $\Gamma$, a densely defined operator corresponding to $H(N)$ could be constructed as:

$$\hat{H}(N) = \frac{2}{\hbar} \sum_{v_\beta \in \Gamma} \sum_{\triangle, \triangle' \in T} \epsilon^{ij} \epsilon^{kl} N(v) \text{Tr} \left( h_{a_{ij}(\triangle)} h_{s_k(\triangle)} h^{-1}_{s_k(\triangle)}, \sqrt{V_v} h_{s_l(\triangle)} h^{-1}_{s_l(\triangle)} \sqrt{V_v} \right).$$  (8)

We refer to [28] for details.

A complete orthonormal basis in $H_E$ is the (non-gauge-invariant) spin network states. Moreover, the gauge invariant spin network states, $\Psi_S(A)$, form a complete orthonormal basis in the $SU(2)$ gauge invariant Hilbert space $H_E^0 = L_2(\mathbb{A}/\mathbb{G}, d\mu_H)$ [29, 5, 4]. Let $\Phi$ denote the space of finite linear combinations of functions $\Psi_S(A)$, the distributional dual, $\Phi'$, is the set of all continuous linear functionals on $\Phi$. We thus have the inclusion $\Phi \subset H_E^0 \subset \Phi'$. A diffeomorphism invariant distribution on $\Phi$ is defined by

$$[\Psi_S] := \sum_{\Psi \in \{\Psi_S\}} \Psi,$$  (9)

where $\{\Psi_S\} \equiv \{\hat{U}(\varphi) \circ \Psi_S, \varphi \in Diff(S)\}$ is the orbit of $\Psi_S$ under the diffeomorphism group on $S$. Therefore every diffeomorphism invariant state is a linear combination of the distributions $[\Psi_S]$ and belongs to $\Phi'$. This space of the solutions to the diffeomorphism constraint is denoted by $\Phi'_{Diff}$ [4, 28].

2.2 Higgs field coupled to gravity

Suppose Higgs scalar fields $\phi(x)$ are valued in the Lie algebra, $Lie(G)$, of some compact group $G$. The quantization of $\phi = \phi_A \tau^A$, where $\tau^A$ are the generators of $G$, could be included into the formalism of loop quantum gravity by introducing the concept of point holonomies [27]. Analogously to the holonomy associated to an edge $e_I$, a point holonomy associated with a point $v$ is a $G$-valued function on the space, $\mathcal{U}$, of Higgs fields given by

$$h_v(\phi) := \exp[\phi_A(v) \tau^A].$$  (10)

A function $\Psi(\phi)$ is said to be cylindrical with respect to a vertex set $\{v_\beta\}$ if it depends only on the finite number of point holonomies, i.e.,

$$\Psi_{v,f} := f[h_{v_1}(\phi), \ldots, h_{v_m}(\phi)],$$  (11)
where $f$ is a complex-valued function on $G^m$.

In quite analogy with Eq.(2), an inner product could be defined on the space of cylindrical functions $\Psi_{v,f}$ with the aid of Haar measure, and the Hilbert space, $\mathcal{H}_S = L_2(\mathcal{U},d\mu_H)$, of Higgs fields is obtained by completing the space of all finite linear combinations of cylindrical functions in the norm induced by this inner product. Moreover, one could mimic the concept of spin networks to define vertex functions on $\mathcal{U}$ by assigning each vertex $v_\beta$ an irreducible representations of $G$. It turns out that the vertex functions provide an orthonormal basis for $\mathcal{H}_S$. See [27] for details.

The formal expression of the momentum operator corresponding to the conjugate momentum, $p^A$, of the Higgs fields $\phi_A$ reads $\hat{p}^A(x) = -i\hbar\delta/\delta\phi_A(x)$. In the case of 2+1 space-times introduced in the last subsection, we can smear it on two-surface $S$ and get a densely well-defined operator $\hat{p}^A(S) \equiv \int_S d^2x\hat{p}^A(x)$. Its action on a cylindrical function yields as same as the result in 3+1 dimension [27],

$$\hat{p}^A(S) \circ \Psi_{v,f} = -i\hbar \sum_{v_\beta} X^A_{v_\beta} \circ \Psi_{v,f},$$

(12)

where $X^A_{v_\beta} \equiv (1/2)[X^A_R(h_{v_\beta}) + X^A_L(h_{v_\beta})]$, here $X^A_R(h)$ and $X^A_L(h)$ are respectively the right and left invariant vector fields on $G$.

In the framework of Higgs scalar fields coupled to gravity, the elementary excitations of Higgs fields are at the vertexes of a given graph, while those of gravity are along the edges. The total Hilbert space is given by the product $\mathcal{H}_E \otimes \mathcal{H}_S$. Moreover, in the case of 3+1 Lorentz gravity coupled to Higgs scalar fields, through suitable regularization a densely well-defined operator corresponding to the Hamiltonian of the system could be constructed, which means the Hamiltonian is completely finite without renormalization [27].

3 Dimension reduction

It is well known that 3+1 gravity with a hypersurface-orthogonal killing vector field is equivalent to 2+1 gravity coupled to a massless scalar field, since the early work of Geroch [30]. While that work concerned only the equations of motion, We now start with the variation principle to study static spacetimes. Consider the Einstein-Hilbert action for 4-dimensional Lorentz gravity, $S_{EH}[g^{ab}] = \int_M \mathcal{L}_G = \int \sqrt{-g^{(4)}}R$, defined on a 4-manifold $M = \Sigma \times \mathbb{R}$. The Lagrangian density can also be written as a version depending only on the geometrical quantities on the hypersurface $\Sigma$ as

$$\mathcal{L}_G = \sqrt{h}\mathcal{N}(R + K_{ab}K^{ab} - K^2),$$

(13)
where \( h \) denotes the determinant of the induced 3-metric \( h_{ab} \) on \( \Sigma \), \( R \) and \( K_{ab} \) are respectively the scalar curvature of \( h_{ab} \) and extrinsic curvature of \( \Sigma \), \( K \equiv K^a_a \), and \( \mathcal{N} \) is the lapse function. If we only consider the configurations of static space-times, it is convenient to choose the hypersurface \( \Sigma \) orthogonal to the timelike killing vector field \( \xi^a \). Then the extrinsic curvature of \( \Sigma \) vanishes, and Eq. (13) is reduced to \( \mathcal{L}_G = \mathcal{N} \sqrt{h} R \).

We now conformally transform \( h_{ab} \) as \( \bar{h}_{ab} = \Omega^{-2} h_{ab} \), and hence obtain \( \Omega^{-2} [ \bar{R} - 4 \bar{h}^{ab} \bar{\nabla}_a \ln \Omega - 2 \bar{h}^{ab} (\nabla_a \ln \Omega) \nabla_b \ln \Omega] \), where \( \bar{R} \) is the scalar curvature of \( \bar{h}_{ab} \). Taking account of \( \bar{h} = \Omega^{-6} h \), we let \( \Omega = \mathcal{N}^{-1} \). The Lagrangian density then becomes

\[
\mathcal{L}_G = \sqrt{\bar{h}} [ \bar{R} - 4 \bar{h}^{ab} \nabla_a \nabla_b \ln \Omega - 2 \bar{h}^{ab} (\nabla_a \ln \Omega) \nabla_b \ln \Omega].
\]

Note that the second term in Eq. (15) is a total divergence term, since \( \nabla_a \) is compatible with \( \bar{h}^{bc} \). Let \( \Lambda \equiv \sqrt{2} \ln \Omega \), straightforward calculations show that the Lagrangian (15) gives the same equations of motion as those of Euclidean 3-gravity \( \bar{h}^{ab} \) coupled to a massless Klein-Gordon field \( \Lambda \), which is defined by the coupled action

\[
S_E + S_{KG} = \int_{\Sigma} \sqrt{\bar{h}} [ \bar{R} - \bar{h}^{ab} (\partial_a \Lambda) (\partial_b \Lambda)].
\]

Therefore, a static 4-dimensional space-time is "conformally" equivalent to 3-dimensional Euclidean gravity coupled to a massless scalar field.

The above dimensional reduction motivates us to quantize the model of 3-dimensional Euclidean gravity coupled to massless Klein-Gordon fields as an equivalent description of quantized static space-times. In order to apply the canonical quantization framework of loop quantum gravity, we "imaginarize" the scalar field as \( \phi = i \Lambda \), and write the gravitational action in Eq. (16) in Palatini formalism. The total action is then defined as

\[
S_T = S_P + \frac{1}{2} S_{KG} = \frac{1}{2} \int_{\Sigma} [ \epsilon^{abc}(3) \bar{e}_{ai}(3) F_{bc}^i + \sqrt{h} \bar{h}^{ab} (\partial_a \phi) (\partial_b \phi)],
\]

where \( \epsilon_{abc} F_{bc}^i \) is the curvature of the \( SU(2) \) connection 1-form, \( A^i_a \), on \( \Sigma \). In complete analogy with the Palatiti formalism coupled to real Klein-Gordon fields [32], the variation of action (17) gives the same equations of motion as those of action (16). Hence, the two actions give the same classical theory. Suppose the 3-manifold admit a foliation \( \Sigma = S \times R \). In the corresponding
Hamiltonian formalism the above imaginarization is just a canonical transformation on the phase space. Through 2+1 decomposition one can obtain the Hamiltonian of the system, which is just the linear combination of following first class constraints:

\[ G(\Lambda^i) := \int_S d^2x \Lambda^i D_a \bar{E}_a^i, \tag{18} \]

\[ V(N^a) := - \int_S N^a (\bar{E}^b_i F^i_{ab} + p \partial_a \phi), \tag{19} \]

\[ H(N) := \frac{1}{2} \int_S d^2x \frac{N}{\sqrt{q}} \left[ \epsilon^{ijk} F_{abi} \bar{E}^a_j \bar{E}^b_k - \bar{E}^a_i \bar{E}^b_i (\partial_a \phi \partial_b \phi - p^2) \right], \tag{20} \]

where \( p \) is the conjugate momentum of \( \phi \). They are the Gauss, vector, and Hamiltonian constraints. The former two constraints reflect the symmetries of internal \( SU(2) \) and the diffeomorphisms on \( S \). The latter one reflects dynamics. Note that the 2-metric \( \bar{q}_{ab} \) induced from \( \bar{h}_{ab} \) is related to that from \( h_{ab} \) by

\[ \bar{q}_{ab} = \Omega^{-2} q_{ab}. \tag{21} \]

4 Canonical quantization

As purely imaginary numbers, the scalar fields \( \phi \) defined in last section are valued in the Lie algebra of \( U(1) \). We are now ready to apply the canonical quantization framework outlined in last section to quantize the model. Although here \( \phi \) originally are not Higgs fields, we may still suppose them to be located at the vertexes of a graph, as the same treatment appears in the study of quantum fields on a lattice \[33\]. Thus, the method to quantize Higgs fields could be naturally borrowed. The kinematical Hilbert space is still given by the product \( \mathcal{H}_E \otimes \mathcal{H}_S \), while the cylindrical functions in \( \mathcal{H}_S \) are defined on \( U(1)^m \) which is the product of point holonomies associated to the vertexes \( v_\beta \). The Gauss and diffeomorphism constraints could be solved by exactly the same procedure employed for pure gravity. The non-trivial task would be how to construct a well-defined operator corresponding to Eq.\( (20) \). It turns out that by introducing properly the triangulation \( T \) of \( S \) adapting to the given graph \( \Gamma \), for example according to Ref.\[28\], the Hamiltonian constraint can be regulated in a consistent strategy and promoted to a densely defined operator. While the gravitational term has been expressed as Eq.\( (8) \), we now regulate the other two terms involving the scalar fields. From Eq.\( (8) \), we have \( \bar{q} = \bar{E}^i \bar{E}_i \) and \[28\]

\[ \bar{E}_i = \frac{1}{2} \epsilon^{a b} \epsilon_{i j k} \{ A^j \bar{V}, \{ A^k \bar{V} \}, \tag{22} \]
where $\tilde{V}$ is the area of $S$ measured by $\tilde{q}_{ab}$. Let $\tilde{V}(x,\epsilon) \equiv \int d^2 y \theta_{\epsilon}(x,y)\sqrt{\tilde{q}}$, where $\theta_{\epsilon}(x,y)$ is the characteristic function of a box of coordinate size $\epsilon^2$ and center $x$. Consider the following regulated point-splitting of the term in Eq.(20) which involves the momentum $p$,

$$H_{KG,p}(N) = \frac{1}{2} \int d^2 x N(x)p(x) \int d^2 y p(y)\theta_{\epsilon}(x,y)\theta_{\epsilon}(u,x)\theta_{\epsilon}(w,y) \int d^2 u \left[ \tilde{E}^i \right] \int d^2 w \left[ \tilde{E}^i \right]$$

$$= \frac{1}{4} \int d^2 x N(x)p(x) \int d^2 y p(y)\theta_{\epsilon}(x,y) \int d^2 u \int d^2 w \theta_{\epsilon}(u,x)\theta_{\epsilon}(w,y)$$

$$\epsilon^{ab}(u)\{A^j_{a}(u), \sqrt{\tilde{V}(u,\epsilon)}\}\{A^k_{b}(u), \sqrt{\tilde{V}(u,\epsilon)}\}$$

$$\epsilon^{cd}(w)\{A_{ij}(w), \sqrt{\tilde{V}(w,\epsilon)}\}\{A_{ik}(w), \sqrt{\tilde{V}(w,\epsilon)}\}$$

$$= \int d^2 x N(x)p(x) \int d^2 y p(y)\theta_{\epsilon}(x,y) \int d^2 u \int d^2 w \theta_{\epsilon}(u,x)\theta_{\epsilon}(w,y)$$

$$\epsilon^{ab}(u) Tr\{\{A^j_{a}(u), \sqrt{\tilde{V}(u,\epsilon)}\}\{A^k_{b}(u), \sqrt{\tilde{V}(w,\epsilon)}\}\{A^d_{a}(u), \sqrt{\tilde{V}(w,\epsilon)}\}\{A_{ik}(w), \sqrt{\tilde{V}(w,\epsilon)}\}\}$$

$$\epsilon^{cd}(w) Tr\{\{A^j_{a}(u), \sqrt{\tilde{V}(u,\epsilon)}\}\{A^k_{b}(u), \sqrt{\tilde{V}(w,\epsilon)}\}\{A_{ik}(w), \sqrt{\tilde{V}(w,\epsilon)}\}\{A^d_{a}(u), \sqrt{\tilde{V}(w,\epsilon)}\}\}.$$  \hspace{1cm} (23)

where $A_a \equiv A^j_{a}^i \tau_i$ and the equation $Tr(\tau_i \tau_j) = -\delta_{ij}/2$ is used. Let $\Delta$ be a triangle of the triangulation $T$ adapted to the graph $\Gamma$ and its basepoint be a vertex $v(\Delta)$ of $\Gamma$. As the image of $[0,\delta]$, where $\delta$ is a small parameter, the two edges $s_I(\Delta), I = 1,2$ incident at $v(\Delta)$ coincide with the segments of two edges of $\Gamma$. In the light of the observation in Ref.[27], we have

$$\theta_{\epsilon}(x,y)\epsilon^{IJ} h_{s_I(\Delta)}\{h^{-1}_{s_I(\Delta)}, \sqrt{\tilde{V}(\Delta,\epsilon)}\} h_{s_J(\Delta)}\{h^{-1}_{s_J(\Delta)}, \sqrt{\tilde{V}(\Delta,\epsilon)}\}$$

$$= 2 \int_{\Delta} d^2 y \theta_{\epsilon}(x,y)\epsilon^{ab}(y)\{A^a_{a}(y), \sqrt{\tilde{V}(y,\epsilon)}\}\{A^b_{b}(y), \sqrt{\tilde{V}(y,\epsilon)}\} + O(\delta^3).$$ \hspace{1cm} (24)

Thus, up to order $\delta$ which vanishes in the limit as we remove the triangulation, Eq.(23) can be expressed as

$$H_{KG,p}^{T,\epsilon}(N) = \int d^2 x N(x)p(x) \int d^2 y p(y)\theta_{\epsilon}(x,y) \sum_{\Delta,\Delta' \in T} \frac{1}{4} \theta_{\epsilon}(v(\Delta),x)\theta_{\epsilon}(v(\Delta'),y)$$

$$\epsilon^{IJ} Tr\{h_{s_I(\Delta)}\{h^{-1}_{s_I(\Delta)}, \sqrt{\tilde{V}(\Delta,\epsilon)}\} h_{s_J(\Delta')}\{h^{-1}_{s_J(\Delta')}, \sqrt{\tilde{V}(\Delta',\epsilon)}\}\}$$

$$\epsilon^{KLT} Tr\{h_{s_J(\Delta)}\{h^{-1}_{s_J(\Delta)}, \sqrt{\tilde{V}(\Delta,\epsilon)}\} h_{s_L(\Delta')}\{h^{-1}_{s_L(\Delta')}, \sqrt{\tilde{V}(\Delta',\epsilon)}\}\}.$$ \hspace{1cm} (25)

Eq.(12) implies $\dot{p}(S) = -i\hbar \sum_{v,\beta} X_v \beta$, where $X_v \equiv 1/2 [X_R(h_v) + X_L(h_v)]$, here $X_R(h_v)$ and $X_L(h_v)$ are respectively the right and left invariant vector fields at $h_v \in U(1)$. Now we replace $\int d^2 x p(x)$
and $\hat{V}$ by their corresponding operators, replace Poisson brackets by commutators times $1/(i\hbar)$, and take $\epsilon$ to zero. The result reads

$$
\hat{H}_{KG,\phi}(N) = \frac{1}{4\hbar^2} \sum_{v_{\beta}} N(v_{\beta}) X_{v_{\beta}} X_{v_{\beta}} \sum_{\Delta(v_{\beta}), \Delta'(v_{\beta}) \in T} \epsilon^{IJ} \epsilon^{KL} \\
\text{Tr} \left( h_{s_I(\Delta)} \hat{h}^{-1}_{s_I(\Delta)} \hat{V}(v) \right) \right) \\
\text{Tr} \left( h_{s_J(\Delta')} \hat{h}^{-1}_{s_J(\Delta')} \hat{V}(v) \right) \\
\epsilon^{KL} \hat{h}^{-1}(v) \left( h_{s_K(v(\Delta'))} \hat{h}(v) \right) \theta_{\epsilon}(v, v(\Delta')) \\
\exp \left( \hat{h}(v) + \delta s^a(0) \partial_a \phi(v) + O(\delta^2) \right) - \hat{h}(v) \\
\delta s^a(0) \partial_a \phi(v). 
$$

which is a densely well-defined operator. Note that $\hat{V}(v)$ is expressed as the same as Eq. (21). Now we turn to the term, $H_{KG,\phi}$, involving the derivatives of $\phi$ and regulate it as

$$
H_{KG,\phi}(N) = \frac{1}{2} \int d^2 x N(x) \int d^2 y \theta_{\epsilon}(x, y) e^{ab}(x)(\partial_a \phi(x)) \frac{\epsilon^{cd}(y)(\partial_c \phi(y))}{\sqrt{V(x, \epsilon)}} \\
\frac{\epsilon^{d}(x)}{\sqrt{V(y, \epsilon)}} \\
= \int d^2 x N(x) \int d^2 y \theta_{\epsilon}(x, y) e^{ab}(x)(\partial_a \phi(x)) e^{cd}(y)(\partial_c \phi(y)) \\
\text{Tr} \left( \{ A_{b}(x), \sqrt{V(x, \epsilon)} \} \{ A_{d}(y), \sqrt{V(y, \epsilon)} \} \right). 
$$

Notice that classically we have, on an edge $s_I$ incident at a vertex $v = s(0)$,

$$
\hat{h}^{-1}(v)[\hat{h}(s(\delta)) - \hat{h}(v)] = \hat{h}^{-1}(v) \left[ \exp \left( \phi(v) + \delta s^a(0) \partial_a \phi(v) + O(\delta^2) \right) - \hat{h}(v) \right] \\
= \delta s^a(0) \partial_a \phi(v). 
$$

Hence on the triangulation $T$, Eq. (27) can be expressed as

$$
H^{T\epsilon}_{KG,\phi}(N) = \frac{1}{4} \sum_{\Delta, \Delta' \in T} N(v(\Delta)) \epsilon^{IJ} \hat{h}^{-1}(v(\Delta)) \left( h_{s_I(v(\Delta))} - h(v(\Delta)) \right) \\
\epsilon^{KL} \hat{h}^{-1}(v(\Delta')) \left( h_{s_K(v(\Delta'))} - h(v(\Delta')) \right) \theta_{\epsilon}(v(\Delta), v(\Delta')) \\
\exp \left( \hat{h}(v) + \delta s^a(0) \partial_a \phi(v) + O(\delta^2) \right) - \hat{h}(v) \\
\delta s^a(0) \partial_a \phi(v). 
$$

In the limit $\epsilon \to 0$, the operator version of Eq. (23) reads

$$
\hat{H}_{KG,\phi}(N) = \frac{1}{4\hbar^2} \sum_{v_{\beta}} N(v_{\beta}) \hat{h}^{-2}(v) \sum_{\Delta(v_{\beta}), \Delta'(v_{\beta}) \in T} \left( h_{s_I(v(\Delta))} - h(v) \right) \left( h_{s_K(v(\Delta'))} - h(v) \right) \\
\epsilon^{IJ} \epsilon^{KL} \text{Tr} \left( h_{s_{I}(\Delta)} \hat{h}^{-1}_{s_{I}(\Delta)} \sqrt{\hat{V}(v)} h_{s_{K}(\Delta')} \hat{h}^{-1}_{s_{K}(\Delta')} \sqrt{\hat{V}(v)} \right), 
$$

which is also densely well defined. In conclusion, the Hamiltonian constraint (23) has been quantized as a densely defined operator in $\mathcal{H}_E \otimes \mathcal{H}_S$. 

10
5 Future directions: area operator and beyond

The consistency of the Hamiltonian operator constructed in last section is left for future investigations. While, from the structure of this operator, it is reasonable to expect that it should share the advantages of its analogue in 3+1 dimension [27], namely cylindrical consistency, diffeomorphism covariance, and anomaly-freeness. Also, the complete set of solutions to the all constraints can be characterized following the procedure of Ref. [27].

We now discuss the construction of an area operator. It should be noted that the sum of the area operator associated to a vertex \( \hat{V}_v \) used in last section does not correspond to the area measured by the physical 2-metric \( q_{ab} \). Classically, from Eqs. (5) and (21) the physical area is

\[
V_B = \int_B d^2x \Omega^2 \sqrt{q} = \int_B d^2x \exp \left(-i \sqrt{2} \phi \right) \sqrt{q}.
\]

(31)

Let \( \chi \equiv i \sqrt{2} \phi \) and assume \( \Omega \leq 1 \) (the opposite case is yet to be studied), we then have the following Fourier transform

\[
e^{-\chi} = \frac{1}{\pi} \int_0^\infty d\eta \left( \frac{1}{1 + \eta^2} \right) \left( e^{-i \chi} + e^{i \chi} \right) = \frac{1}{\pi} \int_0^\infty d\eta \left( \frac{1}{1 + \eta^2} \right) \left( e^{\sqrt{2} \eta \phi} + e^{-\sqrt{2} \eta \phi} \right) \equiv \frac{1}{\pi} \int_0^\infty d\eta \left( \frac{1}{1 + \eta^2} \right) \left( h_x(\eta) + h_x^{-1}(\eta) \right),
\]

(32)

where \( h_x(\eta) \in U(1) \) depends on the parameter \( \eta \). Thus, taking account of Eq. (3) and transform (32) it is straightforward to define an operator corresponding to the area \( V_B \) as

\[
\hat{V}_B = \sum_{\nu, \beta} \frac{1}{\pi} \int_0^\infty d\eta \left( \frac{1}{1 + \eta^2} \right) \left( \hat{h}_\nu(\eta) + \hat{h}_\nu^{-1}(\eta) \right) \hat{V}(v_\beta).
\]

(33)

This operator is not only densely defined but also essentially self-adjoint and positive semi-definite. To see the latter, notice that \( \hat{h}_\nu(\eta) \) is a family of unitary matrix and the adjointness relation implemented in \( \mathcal{H}_S \) is \( \hat{h}_\nu^\dagger = \hat{h}_\nu^{-1} \), and hence

\[
\left( \hat{h}_\nu(\eta) + \hat{h}_\nu^{-1}(\eta) \right)^\dagger = \left( \hat{h}_\nu^{-1}(\eta) + \hat{h}_\nu(\eta) \right).
\]

(34)

With a well-defined area operator at hand, it is possible to address some physical problems.

As argued in Ref. [34], the surface term arising from the gravitational action of (13) could be taken as the definition of the total energy even for space-times that are not asymptotically flat.
The derivation of the surface term depends crucially on a reference background space-time, which is supposed to be static. It is essentially expressed as

\[ E = -\int_{S^\infty_t} \sqrt{q} N (\langle^{(2)} K \rangle - \langle^{(2)} K_0 \rangle), \]  

where \(\langle^{(2)} K\rangle\) and \(\langle^{(2)} K_0\rangle\) are the traces of the 2-dimensional extrinsic curvature of \(S^\infty_t\) in \(\Sigma_t\) corresponding respectively to the field metric and the background metric; here the 2-surface \(S^\infty_t\) is the intersection of \(\Sigma_t\) and a boundary near infinity. If we only consider static space-times, it seems naturally to apply our framework of quantization and consider the Hilbert space \(\mathcal{H}_E \otimes \mathcal{H}_S\) defined on \(S^\infty_t\). A proper regularization of Eq.(35) is necessary before its quantization, and the area operator (33) is supposed to play a key role [35]. Another appealing topic which deserves investigating is to calculate black hole entropy. Our framework provides a possibility to count the numbers of quantum states in the physical Hilbert space associated to the apparent horizons of static black holes. Two essential factors are needed for this consideration. First, we need a local definition of apparent horizons, i.e., to define the horizon by the intrinsic geometry of the 2-surface itself. This is in quite analogy with the definition of isolated horizon which is a generalization of event horizon [36]. Second, to solve exactly all the quantum constraints. It would be amazing, if one could find that the number of physical states becomes finite on an apparent horizon while it is infinite on other non-horizon surfaces.

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