Korneichuk-Stechkin Lemma, Ostrowski and Landau Inequalities, and Optimal Recovery Problems for $L$-Space Valued Functions

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Abstract

We prove an analogue of the Korneichuk–Stechkin lemma for functions with values in $L$-spaces. As applications, we obtain sharp Ostrowski type inequalities and solve problems of optimal recovery of identity and convexifying operators, as well as the problem of integral recovery on the classes of $L$-space valued functions with given majorant of modulus of continuity. The recovery is done based on $n$ mean values of the functions over intervals. Moreover, on the classes of functions with given majorant of modulus of continuity of their Hukuhara type derivative, we solve the problem of optimal recovery of the function and the Hukuhara type derivative. The recovery is done based on $n$ values of the function. We obtain sharp Landau type inequalities and solve an analogue of the Stechkin problem about approximation of unbounded operators by bounded ones and the problem of optimal recovery of an unbounded operator on a class of elements, known with error. Consideration of $L$-space valued functions gives a unified approach to solution of the mentioned above extremal problems for the classes of multi- and fuzzy-valued functions, and for the classes of functions with values in Banach spaces, in particular random processes, and many other classes of functions.

1. Introduction

Let $\omega$ be a modulus of continuity i.e. a non-decreasing continuous semi-additive function such that $\omega(0) = 0$. For a segment $[a, b] \subset \mathbb{R}$ denote by $H^\omega([a, b], \mathbb{R})$ the class of functions $f: [a, b] \to \mathbb{R}$ such that $|f(t) - f(s)| \leq \omega(|t - s|)$ for all $t, s \in [a, b]$. The moduli of continuity $\omega(\cdot)$ as independent functions with mentioned above properties, the classes $H^\omega([a, b], \mathbb{R})$, as well as the classes $W^r H^\omega([a, b], \mathbb{R})$, were introduced by Nikol’skii in [1]. For two positive almost everywhere and integrable functions $\psi_1: [a, a'] \to \mathbb{R}_+$ and $\psi_2: [b', b] \to \mathbb{R}_+$,
$a < a' \leq b' < b$, such that
\[
\int_{a}^{a'} \psi_1(t) dt = \int_{b'}^{b} \psi_2(t) dt,
\]
the Korneichuk–Stechkin lemma, see [2, Section 7.1], gives an estimate for the functional
\[
(\psi_1, \psi_2) \mapsto \sup_{f \in H^\omega([a,b], \mathbb{R})} \left| \int_{a}^{a'} f(t) \psi_1(t) dt - \int_{b'}^{b} f(t) \psi_2(t) dt \right|,
\]
which is sharp in the case of concave modulus of continuity $\omega$. This lemma was published in [3, 4], (see also a remark in [3]) for the classes $H^\omega([a, b], \mathbb{R})$ with $\omega(t) = t^a$, $0 < \alpha \leq 1$, and was generalized to the case of arbitrary modulus of continuity in [5]. The Korneichuk–Stechkin lemma played an important role in the solution of many extremal problems of approximation theory, see [2, Chapter 7] and references therein. Some of its generalizations and more applications can be found in [6, 7].

The theory of Banach space valued, multi-valued and fuzzy-valued functions was actively developed over the last several decades (see [8–10]), in particular, due to its applications in optimization theory, approximation theory, mathematical economics, numerical analysis and other branches of applied mathematics. Some results on approximation of multi- and fuzzy-valued functions can be found in [11, 12].

Banach spaces, spaces of sets and spaces of fuzzy sets belong to the class of so-called $L$-spaces (i.e., semi-linear metric spaces $(X, h_X)$ with two additional axioms, which connect the metric $h_X$ with the algebraic operations). The notion of an $L$-space was introduced in [13], see also [14]. In Section 2 we present necessary definitions and facts related to $L$-spaces. In particular, for the sake of completeness, we present the definition and some properties of the Lebesgue integral for bounded $L$-space valued functions.

In Section 3 we generalize the Korneichuk–Stechkin lemma to the case of $L$-space valued functions. Let $(X, h_X)$ be an $L$-space and $H^\omega([a, b], X)$ be the class of functions $f: [a, b] \to X$ such that $h_X(f(t'), f(t'')) \leq \omega(|t' - t''|)$ for all $t', t'' \in [a, b]$. Let also $\psi_1: [a, a'] \to \mathbb{R}$ and $\psi_2: [b', b] \to \mathbb{R}$, $a < a' \leq b' < b$, be positive almost everywhere, measurable, and bounded functions such that equality (1) holds. We obtain, see Lemma 8, an estimate for the functional
\[
S(\psi_1, \psi_2) := \sup_{f \in H^\omega([a,b], X)} h_X \left( \int_{a}^{a'} f(t) \psi_1(t) dt, \int_{b'}^{b} f(t) \psi_2(t) dt \right),
\]
which is sharp in the case of concave modulus of continuity $\omega$. In a series of applications that we consider in this article, we show that our generalization may be an important tool for solution of extremal problems involving $L$-space valued functions.

In Section 4 we obtain a general estimate of the functional (3) for rather arbitrary functions $\psi_1$ and $\psi_2$ in terms of the Korneichuk $\Sigma$-rearrangement
of the function $\Psi(t) = \int_a^t (\psi_1(u) - \psi_2(u)) du$. This estimate generalizes the estimate for functional (2), obtained by Korneichuk in [15], see also [2, Theorem 7.1.9].

In 1937 Ostrowski [16] proved a sharp inequality that estimates the deviation of a value of a function from its mean value using the uniform norm of the function’s derivative. Such inequalities have been intensively studied, see [17] for a survey of the obtained results. It is worth noting that the general estimate for functional (2), which was obtained by Korneichuk some 50 years ago, essentially contains a series of results on the Ostrowski type inequalities, which were obtained much later. In particular, from this estimate, one can easily obtain the main result from [18] and one of the results in [19]. In [20–28] such type of inequalities are investigated for non-real-valued functions. The obtained in this article estimate for functional (3) implies some of the main results in papers [20, 21, 23, 28].

An important part of approximation theory and optimal algorithms theory is the theory of optimal recovery of operators. Statements of the problems of this theory, many results and further references can be found in monographs [29, 30]. We consider the optimal recovery problem in the following setting.

Let a metric space $(X, h_X)$, sets $Z, Y$, a class of elements $W \subset Z$, as well as mappings $\Lambda: Z \to X$ and $I: W \to Y$ be given. We call an arbitrary mapping $\Phi: Y \to X$ a method of recovery of the mapping $\Lambda$ on the class $W$ based on the information given by the mapping $I$. The error of recovery of the mapping $\Lambda$ on the class $W$ by the method $\Phi$ based on the information given by the mapping $I$ is given by the formula

$$E(\Lambda, W, I, \Phi, X) = \sup_{z \in W} h_X(\Lambda(z), \Phi(I(z))).$$

The quantity

$$E(\Lambda, W, I, X) = \inf_{\Phi} E(\Lambda, W, I, \Phi, X)$$

is called the optimal error of recovery of the mapping $\Lambda$ on the class $W$ based on the information given by the mapping $I$. The problem of optimal recovery of the mapping $\Lambda$ on the class $W$ with the information given by $I$ in the metric of the space $X$ is to find quantity (4) and a method $\Phi^* \ (\text{if such a method exists})$ for which the infimum on the right-hand side of (4) is attained. If $I$ is some class of information operators, then it is also of interest to find the quantity

$$E(\Lambda, W, I, X) = \inf_{I \in I} E(\Lambda, W, I, X)$$

and the best information operator.

In Section 5 we consider the problem of optimal recovery of the convexifying operator (see Section 2 for definitions) and of the integral on the class $H^\omega((a, b], X)$. Under some additional assumptions (which hold in particular in the case of Banach space valued functions), the convexifying operator turns into the identity operator. For real-valued functions, these problems are well
studied when the informational operator maps a function from the class to its values at \( n \) points of the segment \([a, b]\). Regarding recovery of a function we refer to [31, Chapters 5 and 6]; regarding the recovery of the integral we refer to [32]. In [33] the problem of optimization of approximate integration was solved for the class of multi-valued functions. Here as informational operators, we use the ones that map functions from the class to their mean values on \( n \in \mathbb{N} \) intervals belonging to \([a, b]\). This kind of information operators is of interest, since analog measuring devices give such mean values of the measured functions. Moreover, the results on optimal recovery given such type of information, easily imply corresponding results on optimal recovery for the case, when the information operators map functions to their values at \( n \) points of the interval \([a, b]\). It seems that the results regarding optimal recovery of the identity operator using this kind of information is new even in the case \( X = \mathbb{R} \). The problem of optimization of approximate integration given the "interval" information for the functions from the class \( H^\omega([a, b], \mathbb{R}) \) was solved in [34]. Since a random process can be viewed as a function into a Banach space of random variables, our results can be applied to recovery problems for random processes. Some results in this direction can be found in [35–37]. Results on optimal recovery problems for \( L \)-space valued functions can be found in [38, 39].

In Section 6 we consider the problem of optimal recovery of the identity operator and the operator \( D_H \) of Hukuhara type derivative on the class \( W^1H^\omega([a, b], X) \) (see Section 6 for the definitions). In these problems the recovery is done based on the information operator that maps a function to its values at \( n \) points of the interval \([a, b]\). We again refer to [31, Chapters 5 and 6] for the results on optimal recovery of functions on the class \( W^1H^\omega([a, b], \mathbb{R}) \) and its periodic analog, as well as on optimal recovery of the derivative of the functions on these classes.

In Section 7, we obtain sharp inequalities of Landau type for divided differences of Hukuhara type as well as for derivatives of Hukuhara type of the functions from the classes \( \overline{W}^1H^\omega([a, b], X) = \bigcup_{k>0} k \cdot W^1H^\omega([a, b], X) \). Many known results on Landau and Landau–Kolmogorov type inequalities can be found in [40–42]. For the functions with values in Banach spaces, some inequalities were obtained in [28, 43]; for the \( L \)-spaces valued functions defined on \( \mathbb{R} \) or \( \mathbb{R}_+ \) — in [44].

Inequalities of such type are intimately connected to the Stechkin problem about approximation of an operator by the ones with smaller norm, in particular approximation of unbounded operators by bounded ones. The problem was first stated in [45], where the first results on the solution of the problem were obtained. Information on further results can be found in [40, 42]. In [44], a generalization of the Stechkin problem for the case of unbounded operators acting in \( L \)-spaces was proposed; some results about approximation of Hukuhara type derivatives by Lipschitz operators on the classes \( W^1H^\omega(J, X) \), where \( J = \mathbb{R} \) or \( J = \mathbb{R}_+ \) were obtained. Here we consider this problem for the operator...
of Hukuhara type divided difference and the Hukuhara type derivative. We also consider the problem of optimal recovery of the operator $D_H$ on the class $W^1H^\omega([a, b], X)$ in the case, when the elements of the class are known with error. Known results and further references can be found in [40, 42, 44].

2. L-spaces

2.1. Definitions

Definition 1. A set $X$ is called a semilinear space, if operations of addition of elements and their multiplication on real numbers are defined in $X$, and the following conditions are satisfied for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$:

\[
\begin{align*}
    x + y &= y + x; \\
    x + (y + z) &= (x + y) + z; \\
    \exists \theta \in X : x + \theta &= x; \\
    \alpha(x + y) &= \alpha x + \alpha y; \\
    \alpha(\beta x) &= (\alpha \beta)x; \\
    1 \cdot x &= x, \ 0 \cdot x &= \theta.
\end{align*}
\]

Definition 2. We call an element $x \in X$ convex, if for all $\alpha, \beta \geq 0$, $(\alpha + \beta)x = \alpha x + \beta x$. Denote by $X^c$ the subspace of all convex elements of the space $X$.

Remark 1. Some authors (see e.g., [9]) include into the axioms of a semi-linear space the requirement $X = X^c$.

Definition 3. A semilinear space $X$, endowed with a metric $h_X$, is called an L-space, if it is complete and separable and for all $x, y, z \in X$, and $\alpha \in \mathbb{R}$

\[
    h_X(\alpha x, \alpha y) = |\alpha|h_X(x, y);
\]

\[
    h_X(x + z, y + z) \leq h_X(x, y). \quad (5)
\]

Remark 2. It follows from the triangle inequality and (5) that

\[
    \forall x, y, z, w \in X \quad h_X(x + z, y + w) \leq h_X(x, y) + h_X(z, w).
\]

Definition 4. An L-space $X$ is called isotropic, if inequality (5) turns into equality for all $x, y, z \in X$.

Next we list some of the examples of L-spaces. More details can be found in [38, 39, 46]. Arbitrary separable Banach space and arbitrary complete and separable quasilinear normed space (see [14]) are L-spaces. The space $\Omega(X)$ of non-empty compact subsets of a separable Banach space $X$ endowed with usual Hausdorff metric, the space $\Omega_{conv}(X)$ of convex elements from $\Omega(X)$, and
spaces of fuzzy sets (see e.g., [10]) are also examples of $L$-spaces. All $L$-spaces mentioned above are isotropic.

An example of a non-isotropic $L$-space can be built as follows. Let $X = [0, \infty)$, for $\lambda \in \mathbb{R}$, $x, y \in X$ set $x \oplus y = \max\{x, y\}$, $\lambda \odot x = |\lambda|x$. Then $(X, \oplus, \odot)$ with the metric $h_X(x, y) = |x - y|$, $x, y \in X$, is a non-isotropic $L$-space.

A function $f: [a, b] \to X$ is said to be measurable, if for any element $x \in X$ the real-valued function $h_X(f(t), x)$ is measurable. For $[a, b] \subset \mathbb{R}$ and an $L$-space $(X, h_X)$, denote by $C([a, b], X)$ and $B([a, b], X)$ the spaces of continuous (resp. bounded and measurable) functions $f: [a, b] \to X$ with the metrics

$$h_{C([a, b], X)}(f, g) := \max_{t \in [a, b]} h_X(f(t), g(t)) \quad \text{and}$$

$$h_{B([a, b], X)}(f, g) := \sup_{t \in [a, b]} h_X(f(t), g(t)).$$

### 2.2. Hukuhara type derivative

The notion of the Hukuhara difference of two sets was introduced in [47].

**Definition 5.** Let $X$ be an $L$-space. We say that $z \in X$ is the Hukuhara type difference of $x, y \in X$, if $x = y + z$. We denote this difference by $z = x -_H y$.

Note that in an isotropic $L$-space the Hukuharu difference $x -_H y$ is unique, provided it exists. On the other hand, in a non-isotropic $L$-space, uniqueness is not guaranteed. For example, in the space $(X, \oplus, \odot)$, for arbitrary $x \in X$ the difference $x -_H x$ exists and is not unique for each $x \neq 0$. Everywhere below, when we consider Hukuhara differences, we assume that the $L$-space is isotropic.

**Definition 6.** If $t \in (a, b)$, and for all small enough $\gamma > 0$ there exist differences $f(t + \gamma) -_H f(t)$ and $f(t) -_H f(t - \gamma)$, and both limits $\lim_{\gamma \to +0} \gamma^{-1}(f(t + \gamma) -_H f(t))$ and $\lim_{\gamma \to +0} \gamma^{-1}(f(t) -_H f(t - \gamma))$ exist and are equal to each other, then the function $f: [a, b] \to X$ has a Hukuhara type derivative $\mathcal{D}_H f(t)$ at the point $t$ (if $t = a$ or $t = b$ then there exists only one limit) and

$$\mathcal{D}_H f(t) := \lim_{\gamma \to +0} \gamma^{-1}(f(t + \gamma) -_H f(t)).$$

One can find properties of Hukuhara type differences and elements of calculus based on Hukuhara type difference and derivative in $L$-spaces in [46].

### 2.3. Integration in $L$-spaces

For completeness we present the definition and some of the properties of the Lebesgue integral for the functions $f \in B([a, b], X)$, where $X$ is an $L$-space (see [13] and [14, Section 5]). First, we recall the definition of a convexifying operator.
Definition 7. A surjective operator $P: X \to X^c$ is called convexifying, if

$$h_X(P(x), P(y)) \leq h_X(x, y) \text{ for all } x, y \in X;$$

$$P \circ P = P,$$

$$P(\alpha x + \beta y) = \alpha P(x) + \beta P(y) \text{ for all } x, y \in X \text{ and } \alpha, \beta \in \mathbb{R}.$$ 

The operator $\text{conv}: \Omega(\mathbb{R}^m) \to \Omega(\mathbb{R}^m)$ that maps each $x \in \Omega(\mathbb{R}^m)$ to its convex hull $\text{conv}x$ is an example of a convexifying operator.

A mapping $f$ is called simple, if it has a finite number of values $\{f_k\}_{k=1}^n$ on pairwise disjoint measurable sets $\{T_k\}_{k=1}^n, n \in \mathbb{N}$. The Lebesgue integral of a simple mapping $f$ is by definition

$$\int_a^b f(s)ds := \sum_{i=1}^n P(f_i)\mu(T_i),$$

where $\mu$ is the Lebesgue measure. The following properties hold for simple $f, g$.

1. For all $\alpha, \beta \in \mathbb{R}$

$$\int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt.$$

2. The function $t \mapsto h_X(f(t), g(t))$ is integrable and

$$h_X\left(\int_a^b f(t)dt, \int_a^b g(t)dt\right) \leq \int_a^b h_X(f(t), g(t))dt.$$

3. The function $P(f(\cdot))$ is integrable and

$$\int_a^b f(t)dt = P\left(\int_a^b f(t)dt\right) = \int_a^b P(f(t))dt.$$

4. For disjoint measurable sets $T_1$ and $T_2$ such that $[a, b] = T_1 \cup T_2$

$$\int_a^b f(t)dt = \int_{T_1} f(t)dt + \int_{T_2} f(t)dt.$$

In this article we are interested in integrals $\int_a^b f(x)dx$, where $f(t) = \psi(t) \cdot g(t)$, $t \in [a, b], \psi \in B([a, b], \mathbb{R})$ and $g \in C([a, b], X)$. Any such function $f$ is a uniform limit of a sequence $\{f^k\}$ of simple functions. Using standard arguments, one can prove that the sequence $\left\{\int_a^b f^k(t)dt\right\}_{k \in \mathbb{N}}$ is a Cauchy sequence. By definition, the integral $\int_a^b f(t)dt$ of the function $f$ is set to be the limit of this sequence.

It is clear that properties 1–4 of the Lebesgue integral for simple functions are preserved after the limiting process. Moreover, if $\rho$ is an absolutely continuous strictly monotone function from $[c, d] \subset \mathbb{R}$ onto $[a, b] \subset \mathbb{R}$, then

$$\int_a^b f(t)dt = \int_c^d f(\rho(s))\rho'(s)ds.$$
Indeed, from Properties 1–4 and the possibility to change variables in the integral for real-valued functions, it follows that this property holds for the case, when \( f \) is simple. The general case can be obtained using the limiting procedure.

Note that in the case of \( X \) being a Banach space, the integral becomes the Bochner integral, see [48, Sections 3.7–3.8]; in the case \( X = \Omega(\mathbb{R}^m) \), the integral coincides with the Aumann integral, see [14, Theorem 12].

### 2.4. Some properties of L-spaces

**Definition 8.** We say that an element \( x \in X \) is invertible, if there exists an element \( x' \in X \) such that \( x + x' = \theta \). In this case the element \( x' \) is called the inverse to \( x \). Denote by \( X^{\text{inv}} \) the set of all invertible elements of the space \( X \).

**Assumption 1.** In what follows we assume that \( X^{\text{inv}} \cap X^c \neq \{\theta\} \).

In the space \( \Omega(X) \) any element of the form \( \{x\}, x \in X \), is convex and invertible. We need the following lemmas, see [44].

**Lemma 1.** If \( x \in X^{\text{inv}} \), then its inverse element \( x' \) is unique.

**Lemma 2.** If \( x \in X^{\text{inv}} \cap X^c \), then \( x' \in X^c \).

**Lemma 3.** For all \( x \in X^c \) and \( \alpha, \beta \in \mathbb{R} \), \( h_X(\alpha x, \beta x) \leq |\alpha - \beta| \cdot h_X(x, \theta) \). If \( X \) is isotropic and \( \alpha \cdot \beta \geq 0 \), then the inequality becomes equality.

In addition, we also need the following lemmas. We omit their elementary proofs.

**Lemma 4.** Let \( X \) be an isotropic L-space. Then for any \( x \in X^c \cap X^{\text{inv}} \),

\[
h_X(x, x') = h_X(x + x, \theta) = 2h_X(x, \theta).
\]

**Lemma 5.** For any \( x \in X^{\text{inv}} \cap X^c \), \( h_X(x', \theta) = h_X(x, \theta) \).

### 2.5. Auxiliary results

**Definition 9.** For \( f : [a, b] \to \mathbb{R} \) and \( x \in X^{\text{inv}} \) define the function \( f_x : [a, b] \to X \),

\[
f_x(t) = f_+(t) \cdot x + f_-(t) \cdot x',
\]

where for real \( \xi \), \( \xi_\pm := \max(\pm \xi, 0) \).
Lemma 6. Let $X$ be an isotropic L-space and $f \in H^\omega([a, b], \mathbb{R})$. If $x \in X^{\text{inv}} \cap X^c$ is such that $h_X(x, \theta) = 1$, then $f_x \in H^\omega([a, b], X)$ and

$$
\int_a^b f_x(t) dt = \int_a^b f_+(t) dt \cdot x + \int_a^b f_-(t) dt \cdot x'.
$$

(8)

Proof If $s, t \in [a, b]$ are such that $f(s), f(t) \geq 0$, then due to Lemma 3,

$$
h_X(f_x(s), f_x(t)) = h_X(f(s) \cdot x, f(t) \cdot x) \leq \omega(|s - t|).
$$

Analogously, due to Lemma 5, in the case, when $f(s), f(t) \leq 0$. When $f(s) \geq 0 \geq f(t)$,

$$
h_X(f_x(s), f_x(t)) = h_X(f_+(s) \cdot x, f_-(t) \cdot x') = h_X(f_+(s) \cdot x + f_-(t) \cdot x, \theta)
$$

$$
= |f(s) - f(t)| h_X(x, \theta) \leq \omega(|t - s|).
$$

Hence $f_x \in H^\omega([a, b], X)$. Equality (8) follows from (7) and convexity of $x$ and $x'$. Indeed, let for definiteness $\int_a^b f_+(t) dt \geq \int_a^b f_-(t) dt$. Then

$$
\int_a^b f_x(t) dt = \left( \int_a^b f_+(t) dt \right) \cdot x + \left( \int_a^b f_-(t) dt \right) \cdot x'
$$

$$
= \left( \int_a^b f_+(t) dt \right) \cdot x = \left( \int_a^b f(t) dt \right) +
$$

$$
= \left( \int_a^b f_-(t) dt \right) \cdot x + \left( \int_a^b f(t) dt \right) \cdot x'.
$$

Lemma 7. Let $X$ be an isotropic L-space, $x \in X^c \cap X^{\text{inv}}$, and $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Then the derivative $D_H f_x(t)$ exists at each point $t \in [a, b]$ and $D_H f_x(t) = (f'(t))_x$.

Proof First of all note that if $t, t + \gamma \in [a, b]$, then $f_x(t + \gamma) - H f_x(t) = (f(t + \gamma) - f(t))_x$. Let $\gamma > 0$. If $f'(t) > 0$, then for all small enough $\gamma, f(t + \gamma) > f(t)$, hence

$$
\lim_{\gamma \to +0} \gamma^{-1} (f_x(t + \gamma) - H f_x(t))
$$

$$
= \lim_{\gamma \to +0} \gamma^{-1} (f(t + \gamma) - f(t)) x = f'(t) \cdot x = (f'(t))_x.
$$
Analogously in the case \( f'(t) < 0 \). Finally, if \( f'(t) = 0 \), then, due to Lemma 5,
\[
\begin{align*}
  h_X \left( \gamma^{-1}(f_X(t + \gamma) - f_X(t), \theta) \right) \\
  = \left| \gamma^{-1}(f(t + \gamma) - f(t)) \right| h_X(x, \theta) \to 0, \gamma \to +0.
\end{align*}
\]
Analogously for the quantity \( \lim_{\gamma \to +0} \gamma^{-1}(f_X(t) - f_X(t - \gamma)) \). The lemma is proved.

3. On the Korneichuk–Stechkin lemma

Lemma 8. Let positive almost everywhere functions \( \psi_1 \in B([a, a'], \mathbb{R}), \psi_2 \in B([b', b], \mathbb{R}), \ a < a' \leq b' < b \) such that equality (1) holds be given. Let also \( \omega \) be a modulus of continuity and the function \( \rho: [a, c] \to [c, b], c = (a' + b')/2 \), be defined by the relations
\[
\int_a^s \psi_1(t)dt = \int_{\rho(s)}^b \psi_2(t)dt, \text{ if } s \in [a, a'], \quad \text{and} \quad \rho(s) = a' + b' - s, \text{ if } s \in [a', c].
\tag{9}
\]
Then for an L-space \( X \) and the functional \( S(\psi_1, \psi_2) \) defined in (3)
\[
S(\psi_1, \psi_2) \leq \int_a^{a'} \psi_1(s)\omega(\rho(s) - s)ds = \int_{b'}^b \psi_2(t)\omega(t - \rho^{-1}(t))dt. \tag{10}
\]
If \( \omega \) is a concave modulus of continuity and \( X \) is isotropic, then (10) turns into equality. In this case, the supremum is attained on the functions \((\pm g + \alpha)_x(\cdot) + y \in H^\omega([a, b], X), \) where \( \alpha \in \mathbb{R}, x \in X, x \in X^c \cap X^{inv}, h_X(x, \theta) = 1, \) and
\[
g(t) = \begin{cases} 
-\int_c^t \omega'(\rho(s) - s)ds, & a \leq t \leq c, \\
\int_t^c \omega'(s - \rho^{-1}(s))ds, & c \leq t \leq b. 
\end{cases} \tag{11}
\]
Proof. Differentiating equality (9), we get \( \psi_1(s) = -\psi_2(\rho(s))\rho'(s) \) for all \( s \in [a, a'] \). After a substitution \( t = \rho(s) \), we obtain that
\[
\int_{b'}^b \psi_2(t)f(t)dt = -\int_a^{a'} \psi_2(\rho(s))\rho'(s)f(\rho(s))ds = \int_a^{a'} \psi_1(s)f(\rho(s))ds.
\]
Hence
\[
\begin{align*}
  h_X \left( \int_a^{a'} \psi_1(t)f(t)dt, \int_{b'}^b \psi_2(t)f(t)dt \right) \\
  = h_X \left( \int_a^{a'} \psi_1(t)f(t)dt, \int_a^{a'} \psi_1(t)f(\rho(t))dt \right) \\
  \leq \int_a^{a'} h_X \left( \psi_1(t)f(t), \psi_1(t)f(\rho(t)) \right) dt \\
  = \int_a^{a'} \psi_1(t)h_X(f(t), f(\rho(t)))dt \leq \int_a^{a'} \psi_1(t)\omega(\rho(t) - t)dt
\end{align*}
\]
The function $f$ is called the distribution function of $G$ i.e. for $hX$ and since $X$ is isotropic, we obtain

$$h_X \left( \int_a^{a'} \psi_1(t) f(t) dt, \int_{b'}^b \psi_2(t) f(t) dt \right) = h_X \left( \int_a^{a'} \psi_1(t) f(t) dt + \int_{a}^{a'} \psi_1(t) y dt, \int_{b'}^b \psi_2(t) f(t) dt + \int_{b'}^b \psi_2(t) y dt \right) = h_X \left( \int_a^{a'} \psi_1(t) (f(t) + y) dt, \int_{b'}^b \psi_2(t) (f(t) + y) dt \right)$$

and hence if the supremum in (10) is attained on some function $f$, then it is attained on all functions $f(\cdot) + y$, $y \in X$. Let $G(\cdot) = g(\cdot) + \alpha$, where $g$ is defined in (11) and $\alpha \in \mathbb{R}$. It is known (see [2, § 7.1]) that the function $G$ belongs to $H^\omega([a, b], \mathbb{R})$ and is extremal in (10) for real-valued functions i.e. for $I = \int_a^{a'} \psi_1(t) G(t) dt$ and $J = \int_{b'}^b \psi_2(t) G(t) dt$ the equality $|I - J| = \int_a^{a'} \psi_1(t) \omega(\rho(t) - t) dt$ holds. By (11), the function $g$ is non-decreasing. Hence there are three possibilities 1) $I \geq 0$, $J \geq 0$, 2) $I \leq 0$, $J \geq 0$ and 3) $I \leq 0$, $J \leq 0$. Considering each of them and taking into account Lemmas 3, 5, and 6 we obtain

$$h_X \left( \int_a^{a'} \psi_1(t) G_X(t) dt, \int_{b'}^b \psi_2(t) G_X(t) dt \right) = |I - J|h_X(x, \theta)$$

$$= \int_a^{a'} \psi_1(t) \omega(\rho(t) - t) dt.$$

We elaborate the proof in the case $I \leq 0$ and $J \geq 0$:

$$h_X \left( \int_a^{a'} \psi_1(t) G_X(t) dt, \int_{b'}^b \psi_2(t) G_X(t) dt \right) = h_X(I_- \cdot x', J_+ \cdot x)$$

$$= h_X(I_- \cdot x' + I_- \cdot x, I_- \cdot x + J_+ \cdot x) = h_X(\theta, (-I + J) \cdot x) = |I - J|h(x, \theta).$$

The case $G(\cdot) = -g(\cdot) + \alpha$, can be considered analogously. The lemma is proved.

Recall that for a measurable non-negative function $f: [a, b] \to \mathbb{R}$ the function

$$m(f, y) := \text{meas}\{t \in [a, b]: f(t) > y\}, y \in \mathbb{R}$$

is called the distribution function of $f$, where $\text{meas}$ means the Lebesgue measure. The function

$$r(f, t) := \inf\{y: m(f, y) \leq t\}, t \in [0, b - a]$$
is called the non-increasing (or Hardy’s) rearrangement of \( f \). \( r(f, \cdot) \) is a non-increasing on \([0, b - a]\), and equimeasurable with \( f \) function.

**Remark 3.** For a concave modulus of continuity \( \omega \), and an isotropic \( L \)-space \( X \), the statement of **Lemma 8** can be rewritten as follows:

\[
S(\psi_1, \psi_2) = \int_0^{b-a} r'(\Psi, s)\omega(s)ds = \int_0^{b-a} r(\Psi, s)\omega'(s)ds, \tag{12}
\]

where \( \Psi(s) = \int_a^s (\psi_1(u) - \psi_2(u))du, s \in [a, b] \).

The equality of the quantities in the right-hand sides of inequalities (10) and (12) for concave \( \omega \) was proved by Korneichuk, see e.g. [2, Lemma 7.1.2].

### 4. Estimate for the functional \( S(\psi_1, \psi_2) \) and Ostrowski type inequalities

#### 4.1. General estimate for the functional \( S(\psi_1, \psi_2) \)

**Definition 10.** A function \( \varphi \in C([a, b], \mathbb{R}) \) is called a hat-function, if

1. \( \varphi(a) = \varphi(b) = 0, |\varphi(t)| > 0 \) for \( a < t < b \), and
2. \( \forall y \in (0, \max_{t \in [a,b]} |\varphi(t)|) \) the equation \( |\varphi(t)| = y \) has exactly two roots on \((a, b)\).

Each hat-function \( \varphi \) is continued to the whole line by \( \varphi(t) = 0, t \notin (a, b) \).

Denote by \( D[a,b] \) the set of functions \( \psi : [a, b] \rightarrow \mathbb{R} \) that have finite one-sided limits \( \psi(t + 0) \) and \( \psi(t - 0) \) for all \( t \in (a, b) \), and finite limits \( \psi(a + 0) \) and \( \psi(b - 0) \). Set \( D_0[a,b] = \{ \psi \in D[a,b] : \int_a^b \psi(t)dt = 0 \} \) and \( D_0^1[a,b] = \{ f(t) = \int_a^t \psi(u)du : \psi \in D_0[a,b] \} \). As it is known (see e.g., [15] and [2, Chapter 7]), each function \( f \in D_0^1[a,b] \) can be represented as a finite or countable sum of hat-functions

\[
f(t) = \sum_k \varphi_k(t). \tag{13}
\]

This equality is called the \( \Sigma \)-representation of \( f \). The following properties are satisfied (see [2, Chapter 7]). 1) \( |f(t)| = \sum_k |\varphi_k(t)|, \ a \leq t \leq b \); 2) The intervals \((\alpha_k, \alpha_k')\), \((\beta_k', \beta_k)\), on which the functions \( \varphi_k(t) \) are strictly monotone, are disjoint and on every one of them \( \varphi_k(t) = f(t) + c_k, c_k \in \mathbb{R} \), and hence \( f'(t) = \sum_k \varphi_k'(t) \) almost everywhere on \([a, b] \); 3) \( \int_a^b |f(t)|dt = \sum_k \int_{\alpha_k}^{\beta_k} |\varphi_k(t)|dt \); 4) \( \int_a^b f = \sum_k \int_{\alpha_k}^{\beta_k} \varphi_k \).

**Definition 11.** For a function \( f \in D_0^1[a,b] \) with \( \Sigma \)-representation (13), the Korneichuk \( \Sigma \)-rearrangement of \( f \) is defined by equality

\[
R(f; t) = \sum_k r(|\varphi_k|, t), \ 0 \leq t \leq b - a.
\]
**Theorem 1.** Let \( \omega \) be a concave modulus of continuity and \( \psi_1, \psi_2 \in B([a, b], \mathbb{R}) \) be such that \( \psi_1 - \psi_2 \in D_0[a, b] \). Set \( \Psi(t) = \int_0^t [\psi_1(u) - \psi_2(u)]du \). Then

\[
S(\psi_1, \psi_2) \leq \int_0^{b-a} |R(\Psi; t)|\omega(t)dt. \tag{14}
\]

**Proof** Set \( E_\pm = \{ t \in [a, b]: \pm \psi_1(t) \geq \pm \psi_2(t) \} \). If \( P \) is the convexifying operator (see Def. 7), then \( \psi_1(s)P(f(s)) = (\psi_1(s) - \psi_2(s))P(f(s)) + \psi_2(s)P(f(s)) \) for all \( s \in E_+ \) and \( \psi_2(s)P(f(s)) = (\psi_2(s) - \psi_1(s))P(f(s)) + \psi_1(s)P(f(s)) \) for all \( s \in E_- \). Hence for any function \( f \in H^\omega([a, b], X) \) we obtain

\[
h_X \left( \int_a^b f(t)(\psi_1(t) - \psi_2(t))dt, \int_a^b f(t)(\psi_1(t) - \psi_2(t))dt \right)
= h_X \left( \int_{E_+} f(t)(\psi_1(t) - \psi_2(t))dt + \int_{E_-} f(t)(\psi_1(t) - \psi_2(t))dt + \int_{E_+} f(t)(\psi_2(t) - \psi_1(t))dt \right)
\]

Moreover, the inequality in the above chain becomes equality in the case of isotropic space \( X \). Let \( \Psi(t) = \sum_k \varphi_k(t) \) be the \( \Sigma \)-representation of the function \( \Psi \). Since \( \Psi' = \psi_1 - \psi_2 = \sum_k \varphi_k' \), due to the mentioned above properties of \( \Sigma \)-representations,

\[
(\psi_1 - \psi_2)_+ = \sum_k (\varphi_k')_+, \quad (\psi_1 - \psi_2)_- = \sum_k (\varphi_k')_-.
\]

Moreover, for each \( k \) the pair of functions \( (\varphi_k')_+ \) and \( (\varphi_k')_- \) satisfies the assumptions of Lemma 8 on the segments \([\alpha_k, \alpha_k']\) and \([\beta_k', \beta_k]\) of strict monotonicity of \( \varphi_k \). Applying Lemma 8 (more precisely, equality (12)), we obtain

\[
h_X \left( \int_a^b f(t)(\psi_1(t) - \psi_2(t))_+dt, \int_a^b f(t)(\psi_1(t) - \psi_2(t))_-dt \right)
= h_X \left( \int_a^b f(t) \sum_k (\varphi_k')_+dt, \int_a^b f(t) \sum_k (\varphi_k')_-dt \right)
\]

Finally, notice that \( R(\Psi, \cdot) \) is a non-increasing absolutely continuous on \([0, b-a]\) function, \( R(\Psi, b-a) = 0 \) (see e.g., [2, p. 310]), and hence integration by parts
gives
\[ \int_0^{b-a} R(\Psi, t) \omega'(t) \, dt = \int_0^{b-a} |R'(\Psi; t)| \omega(t) \, dt. \]

\[ \square \]

**Remark 4.** In the case of isotropic \( X \), estimate (14) is sharp, provided the extremal in Lemma 8 functions for \( \psi_1 = (\varphi'_+)_+ \) and \( \psi_2 = (\varphi'_-)_- \) can be “glued” so that the obtained function belongs to \( H^\omega([a, b], X) \).

Assume that the \( \Sigma \)-representation of the function \( \Psi \) is \( \Psi(t) = \sum_{k=1}^n \varphi_k(t) \), if \([\alpha_k, \beta_k]\) is the support of the hat-function \( \varphi_k \), \( k = 1, \ldots, n \), then
\[ \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_n < \beta_n, \]
and on the segments \([\alpha_k, \beta_k]\) and \([\alpha_{k+1}, \beta_{k+1}]\) the functions \( \varphi_k \) and \( \varphi_{k+1} \) have opposite signs, \( k = 1, \ldots, n-1 \). Below we sketch the procedure of gluing. We start with the case \( X = \mathbb{R} \). Let \( g_k \in H^\omega([\alpha_k, \beta_k], \mathbb{R}) \) be an extremal for the functional \( S(\varphi'_+, \varphi'_-) \) function. On the set \( \bigcup_{k=1}^n [\alpha_k, \beta_k] \) define the function \( g \), setting \( g(t) = g_k(t) + c_k, t \in [\alpha_k, \beta_k] \), where \( c_k \) are such that \( g(\beta_k) = g(\alpha_{k+1}), k = 1, \ldots, n-1 \). Next, we continue \( g \) to the whole segment \([a, b]\) setting \( g(t) = g(\alpha_1) \), if \( t \leq \alpha_1 \), \( g(t) = g(\beta_1) \), if \( t \in (\beta_1, \alpha_{k+1}) \), \( k = 1, \ldots, n-1 \), and \( g(t) = g(\beta_n) \), if \( t \geq \beta_n \). Lemma 4.1 from [7] contains a criteria for \( g \) to belong to \( H^\omega([a, b], \mathbb{R}) \). In particular, this is true, if for some \( m, 1 \leq m \leq n \),
\[ \beta_1 - \alpha_1 \leq \beta_2 - \alpha_2 \leq \cdots \leq \beta_m - \alpha_m \text{ and } \beta_m - \alpha_m \geq \beta_{m+1} - \alpha_{m+1} \geq \cdots \geq \beta_n - \alpha_n. \]

If \( g \in H^\omega([a, b], \mathbb{R}) \), then the function \( (g + \gamma)_x(\cdot) + \gamma \) with \( \gamma \in \mathbb{R} \), \( \gamma \in X \) and \( x \in X^c \cap X^{inv} \), \( h_X(x, \theta) = 1 \), is a glued extremal for (14).

### 4.2. Ostrowski type inequalities

The following theorem is a wide generalization of [18, Theorem 2], in which \( X = \mathbb{R} \), \( \omega(t) = t \), and \([c, d] \subset [a, b]\).

**Theorem 2.** Let two segments \([a, b]\) and \([c, d]\) be given. Set \( M = \max\{b-a, d-c\} \), \( m = \min\{b-a, d-c\} \), for \( \alpha, \beta \geq 0 \) set \( I(\alpha, \beta) = \int_0^\beta \omega(s) \, ds \) and assume for definiteness that \( a \leq c \). Then for all \( f \in H^\omega([a, \max\{b, d\}], X) \)

\[ h_X \left( \frac{1}{b-a} \int_a^b f(t) \, dt, \frac{1}{d-c} \int_c^d f(t) \, dt \right) \leq \begin{cases} \frac{M-m}{M^2} I \left( 0, \frac{M(c-a)}{M-m} \right) + I \left( 0, \frac{M(b-d)}{M-m} \right), & [c, d] \subset [a, b], \\ \frac{1}{M+m} I \left( \frac{M(b-c)}{m}, d-a \right) + \frac{M-m}{M^2} I \left( 0, \frac{M(b-c)}{m} \right), & b \in [c, d], \\ \frac{1}{M+m} I(c-b, d-a), & c \geq b. \end{cases} \]

If \( X \) is isotropic and \( \omega \) is concave, then the inequality is sharp.
The theorem follows from Theorem 1 applied to the functions \( \psi_1 = \frac{1}{b-a} \chi_{[a,b]} \) and \( \psi_2 = \frac{1}{d-c} \chi_{[c,d]} \). The extremal function can be obtained using the described above procedure of gluing. Below we use Theorem 2 only for the case \([c, d] \subset [a, b]\). Some of the properties of extremal functions for this case when \( X = \mathbb{R} \) are given by the following lemma. We give the proofs of Lemma 9 and Theorem 2 for this case in Appendix A. We omit the technical details for the other two cases.

**Lemma 9.** Let \( X = \mathbb{R} \), \([c, d] \subset [a, b]\) and \( \omega \) be a concave modulus of continuity. Then one of the extremals for (15) functions \( G \) satisfies the following properties:

1. \( G \) is non-decreasing on \([a, \gamma]\), and is non-increasing on \([\gamma, b]\), where \( \gamma := \frac{bc - ad}{b + c - a - d}; \) (16)
2. \( G(\gamma) - G(a) = \omega(\gamma - a) \), \( G(\gamma) - G(b) = \omega(b - \gamma) \);
3. If \( c + d \leq a + b \), then \( b - \gamma \geq \gamma - a \).

If \( G \) is an extremal for (15) function, then \( \pm G + C, C \in \mathbb{R}, \) is also extremal.

Direct computations show that Theorem 2 implies the following result.

**Corollary 1.** If in Theorem 2 additionally \( c + d = a + b \), i.e. the midpoints of the intervals \([a, b]\) and \([c, d]\) coincide, then

\[
h_X \left( \int_a^b f(t) dt, \frac{b-a}{d-c} \int_c^d f(t) dt \right) \leq \frac{4(c-a)}{b-a} \int_0^{(b-a)/2} \omega(t) dt.
\]

If \( X \) is isotropic and \( \omega \) is concave, then the inequality is sharp.

Applying Theorem 2 to a segment that contains \( t \), and the segment \([c, d]\) (while both are contained in \([a, b]\)) and then shrinking the first one into a point, we obtain

**Corollary 2.** Let \( t \in [a, b], [c, d] \subset [a, b]\) and \( \omega(\cdot) \) be an arbitrary modulus of continuity. If \( f \in H^\omega([a,b], X) \) and \( P \) is the convexifying operator, then

\[
h_X \left( P(f(t)), \frac{1}{d-c} \int_c^d f(u) du \right) \leq \frac{1}{d-c} \int_c^d \omega(|s-t|) ds.
\] (17)

If \( X \) is isotropic, then the inequality is sharp. An extremal function is \((\omega(\cdot - t))x, \) where \( x \in X^c, h_X(x, \theta) = 1 \).

Let a segment \([a, b]\) and numbers \( t, h \) such that \( a \leq t < t + h \leq (a + b)/2 \) be given. Applying Theorem 1 to \( \psi_1 = \frac{1}{b-a} \chi_{[a,b]} \) and \( \psi_2 = \frac{1}{2h} \left( \chi_{[t,t+h]} + \chi_{[a+b-t-h,a+b-t]} \right) \), and passing to the limit as \( h \to 0 \), one can obtain the following generalization of [19, Theorem 2]. We omit the technical details of the proof, since we do not need this result in the sequel.
Corollary 3. For arbitrary \( f \in H^\omega([a, b], X) \) and \( t \in [a, (a + b)/2) \)
\[
 h_X \left( \frac{1}{2} (P(f(t)) + P(f(a + b - t))), \frac{1}{b-a} \int_a^b f(u) du \right) \\
\leq \frac{2}{b-a} \left[ \int_0^{t-a} \omega(u) du + \int_0^{(a+b-2t)/2} \omega(u) du \right]. \quad (18)
\]

If \( X \) is isotropic, then inequality (18) becomes equality for \( f(u) = \min\{\omega(|u - t|), \omega(|u + t - a - b|)\} \cdot x, x \in X^c, h_X(x, \theta) = 1. \)

Remark 5. Inequalities (17) and (18) can easily be proved directly.

5. On optimal recovery problems on the class \( H^\omega([a, b], X) \)

In this section we consider the problems of recovery of the convexifying operator \( P \) and the integral \( \text{Int}(f) = \int_a^b f(t) dt \) on the class \( H^\omega([a, b], X) \), given the information operator \( I_t(f) = \left( \frac{1}{2h} \int_{t-h}^{t+h} f(t) dt, \ldots, \frac{1}{2h} \int_{t_n-h}^{t_n+h} f(t) dt \right) \), where \( n \in \mathbb{N}, h > 0 \) and \( t := (t_1, \ldots, t_n) \) are such that
\[
a \leq t_1 - h < t_1 + h < t_2 - h < \cdots < t_n + h \leq b, \quad (19)
\]
using arbitrary method of recovery \( \Phi: X^n \to B([a, b], X) \) and \( \Phi: X^n \to X \) respectively. Define a vector \( \tau = \tau(t) \) with components
\[
\tau_1 = a, \tau_i = \frac{1}{2}(t_{i-1} + t_i), i = 2, \ldots, n, \tau_{n+1} = b \quad (20)
\]
and set
\[
t^* = (t_1^*, t_2^*, \ldots, t_n^*) = \left( a + \frac{b-a}{2n}, a + \frac{3(b-a)}{2n}, \ldots, a + \frac{(2n-1)(b-a)}{2n} \right). \quad (21)
\]

We need the following well known estimate for the value of the optimal recovery (4). It follows e.g. from Theorem 1 and Lemma 2 in [30, Section 1.2]. We give its short proof for completeness.

Lemma 10. If \( f, g \in W \) are such that \( I(f) = I(g) \), then
\[
\mathcal{E}(\Lambda, W, I, X) \geq \frac{1}{2} h_X(\Lambda(f), \Lambda(g)).
\]

Proof We have
\[
\sup_{z \in W} h_X(\Lambda(z), \Phi(I(z))) \geq \max \left\{ h_X(\Lambda(f), \Phi(I(f))), h_X(\Lambda(g), \Phi(I(g))) \right\}
\]
\[
\geq \frac{1}{2} \left( h_X(\Lambda(f), \Phi(I(f))) + h_X(\Lambda(g), \Phi(I(f))) \right) \geq \frac{1}{2} h_X(\Lambda(f), \Lambda(g)).
\]
\[\square\]
5.1. Real-valued extremal functions

In what follows we need the following simple statement, which allows to build new functions from the class $H^\omega([a, b], \mathbb{R})$ based on the old ones.

**Lemma 11.** If $\omega$ is an arbitrary modulus of continuity, then

1. $f \in H^\omega([0, a], \mathbb{R})$ implies $f(| \cdot |) \in H^\omega([-a, a], \mathbb{R})$.
2. $f \in H^\omega([a, b], \mathbb{R})$ implies $f(\cdot - c) \in H^\omega([a + c, b + c], \mathbb{R})$.
3. If $[c, d] \subset [a, b]$ and $f \in H^\omega([c, d], \mathbb{R})$, then continuation of $f$ by the formula $f(x) = f(d)$ for $x > d$, $f(x) = f(c)$ for $x < c$ belongs to $H^\omega([a, b], \mathbb{R})$.
4. If $f, g \in H^\omega([a, b], \mathbb{R})$, then $\max\{f, g\}, \min\{f, g\} \in H^\omega([a, b], \mathbb{R})$.
5. If $\omega$ is concave, then the function $f(t) = \frac{\sgn t}{2} \omega(2|t|)$ belongs to $H^\omega([-a, a], \mathbb{R})$ for all $a \in \mathbb{R}$.

For given $n \in \mathbb{N}$, $h > 0$ and $t$ that satisfy (19), denote by $H^h_t$ the class of functions $y \in H^\omega([a, b], \mathbb{R})$ such that $\int_{t_i-h}^{t_i+h} y(t) dt = 0$ for all $i = 1, \ldots, n$. Note that for arbitrary $f \in H^h_t$, $x \in X^c \cap X^{inv}$, due to Lemma 6, $\int_{t_i-h}^{t_i+h} f_x(t) dt = \theta$, $i = 1, \ldots, n$. Proofs of the following two lemmas will be given in Appendix B.

**Lemma 12.** Let numbers $n \in \mathbb{N}$, $h > 0$ and $t := (t_1, \ldots, t_n)$ that satisfy (19) be given. For arbitrary modulus of continuity $\omega$ there exists a function $f_t \in H^h_t$ such that

$$\max_{t \in [a, b]} |f_t(t)| \geq \frac{1}{2h} \int_{(b-a)/(2n)-h}^{(b-a)/(2n)+h} \omega(u) du.$$

**Lemma 13.** Let numbers $n \in \mathbb{N}$, $h > 0$ and $t := (t_1, \ldots, t_n)$ that satisfy (19) be given. Let $\omega$ be a concave modulus of continuity. Then there exists a function $f_t \in H^h_t$ such that

$$\int_{a}^{b} f_t(t) dt \geq 2n \left(1 - \frac{2nh}{b-a}\right) \int_{0}^{(b-a)/(2n)} \omega(t) dt.$$

5.2. Optimal recovery of the convexifying operator

**Theorem 3.** Let numbers $n \in \mathbb{N}$, $h > 0$ and $t := (t_1, \ldots, t_n)$ that satisfy (19) be given. For the convexifying operator $P$, isotropic space $X$ and arbitrary modulus of continuity $\omega$

$$\inf_{t} \mathcal{E}(P, H^\omega([a, b], X), I_t, B([a, b], X)) = \frac{1}{2h} \int_{(b-a)/(2n)-h}^{(b-a)/(2n)+h} \omega(u) du.$$

The optimal informational operator is $I^*_t$ and the optimal recovery method is

$$\Phi^*(I^*_t(f))(u) = \frac{1}{2h} \int_{t_{k-h}^{*}}^{t_{k+h}^{*}} f(t) dt, \ u \in [\tau_k(t^*), \tau_{k+1}(t^*)].$$
where the vectors $t^*$ and $\tau = \tau(t^*)$ are defined in (21) and (20), respectively.

Proof

For $f \in H^\omega([a, b], X)$, $t \in [\tau_i(t^*), \tau_{i+1}(t^*)]$, and $i \in \{1, \ldots, n\}$, Corollary 2 implies

$$h_X(P(f(t)), \Phi^*(I_t^*)(t)) = h_X\left(P(f(t)), \frac{1}{2h} \int_{t_i^*-h}^{t_i^*+h} f(u) du\right) \leq \frac{1}{2h} \int_{t_i^*-h}^{t_i^*+h} \omega(|u - t|) du$$

Choose $x \in X^c \cap X^{inv}$, $h_X(x, \theta) = 1$, and for the function $f_t$ from Lemma 12 set

$$E_n = (f_t)_x \text{ and } \overline{E}_n = (f_t)_x.$$  \hspace{1cm} (22)

Note that the functions $E_n$ and $\overline{E}_n$ are convex-valued and by Lemma 6 we have $I(\overline{E}_n) = I(E_n) = (\theta, \ldots, \theta)$. Using Lemma 10, we obtain

$$E(P, H^\omega([a, b], X), I_t, B([a, b], X)) \geq \frac{1}{2} \max_{t \in [a, b]} h_X(\overline{E}_n(t), E_n(t))$$

$$= \max_{t \in [a, b]} |f_t(t)| \geq \frac{1}{2h} \int_{(b-a)/(2n)-h}^{(b-a)/(2n)+h} \omega(u) du.$$

The theorem is proved.

5.3. Optimal recovery of the integral

Theorem 4. Let numbers $n \in \mathbb{N}$, $h > 0$, and $t := (t_1, \ldots, t_n)$ that satisfy (19) be given. For an isotropic space $X$ and a concave modulus of continuity $\omega$,

$$\inf_t E(\text{Int}, H^\omega([a, b], X), I_t, X) = 2n \left(1 - \frac{2nh}{b-a}\right) \int_0^{(b-a)/(2n)} \omega(t) dt.$$  \hspace{1cm} (20)

The optimal informational operator is $I_t^*$ and the optimal recovery method is

$$\Phi^*(I_t^*(f)) = \frac{b-a}{n} \sum_{k=1}^n \frac{1}{2h} \int_{t_k^*-h}^{t_k^*+h} f(t) dt,$$

where the vector $t^*$ is defined in (21).
Lemma 14. Let \( \omega \) be a concave modulus of continuity and a partition \( t \) be given. Then there exists a function \( f_t \in \mathcal{W}^1H^\omega([a, b], \mathbb{R}) \) such that \( f_t(t_i) = 0 \), \( i = 0, \ldots, n \).
\[ i = 0, 1, \ldots, n, \text{ and} \]
\[ \max_{t \in [a,b]} |f(t)| \geq \frac{1}{4} \int_0^{(b-a)/n} \omega(u)du. \tag{24} \]

If the partition \( t \) is uniform, then inequality (24) becomes equality.

The proof of the lemma is given in Appendix B.

Using the definition of the class \( W^1 H^\omega([a, b], X) \), for an isotropic \( L \)-space \( X, f \in W^1 H^\omega([a, b], X) \) and \( t \in [a, b] \), applying Theorem 2 to \( D_H f \in H^\omega([a, b], X) \), one has

\[
h_X \left( \frac{b - t}{b - a} f(a) + \frac{t - a}{b - a} f(b) \right) = h_X \left( f(a) + \int_a^t D_H f(u)du, \right.
\]
\[
\left. \frac{b - t}{b - a} f(a) + \frac{t - a}{b - a} f(b) + \frac{t - a}{b - a} \int_a^b D_H f(u)du \right)
\]
\[
= h_X \left( \int_a^t D_H f(u)du, \frac{t - a}{b - a} \int_a^b D_H f(u)du \right) \leq \frac{(b - t)(t - a)}{(b - a)^2} \int_0^{b-a} \omega(u)du. \tag{25} \]

Next we apply the obtained inequality to prove an estimate of the deviation of a function \( f \in W^1 H^\omega([a, b], X) \) at a fixed point \( t \in [a, b] \) from the interpolating polygonal function. Let a partition \( t \) as in (23) be given. The interpolating polygonal function is

\[ l_f(t; t) = \frac{t_{k+1} - t}{t_{k+1} - t_k} f(t_k) + \frac{t - t_k}{t_{k+1} - t_k} f(t_{k+1}), t \in [t_k, t_{k+1}]. \tag{26} \]

Applying (25), we obtain that for \( t \in [t_k, t_{k+1}] \)

\[ h_X(f(t), l_f(t; t)) \leq \frac{(t_{k+1} - t)(t - t_k)}{(t_{k+1} - t_k)^2} \int_0^{t_{k+1} - t_k} \omega(u)du. \]

Therefore for the uniform partition \( t^* \) of the segment \([a, b] \) the following generalization of a result by Malozemov \[49\] holds: for each \( f \in W^1 H^\omega([a, b], X) \)

\[ \max_{t \in [a,b]} h_X(f(t), l_f(t^*; t)) \leq \frac{1}{4} \int_0^{(b-a)/n} \omega(u)du. \tag{27} \]

**Theorem 5.** If \( \omega \) is a concave modulus of continuity, \( t \) is a partition of \([a, b], I_t(f) = (f(t_0), f(t_1), \ldots, f(t_n)) \) is the information operator and \( \text{Id} \) is the identity operator, then for an isotropic \( L \)-space \( X \)

\[ \inf_t E(\text{Id}, W^1 H^\omega([a, b], X), I_t, C([a, b], X)) = \frac{1}{4} \int_0^{(b-a)/n} \omega(t)dt. \]

The optimal information operator is \( I_{t^*} \) where \( t^* \) is the uniform partition, and the optimal method of recovery is \( \Phi(I_{t^*}(f)) = l_f(t^*) \), where \( l_f(t) \) is defined by (26).
Proof For arbitrary partition $t$ let $f_t$ be the function from Lemma 14, and $x \in X_c \cap X^{\text{inv}}, h_X(x, \theta) = 1$. Using Lemmas 10 and 4, and isotropness of $X$, we obtain

$$
\mathcal{E}(\text{Id}, W^1 H^\omega([a, b], X), I_t, (C[a, b], X)) \geq \frac{1}{2} \max_{t \in [a, b]} h_X ((f_t)_x(t), (f_t)_x'(t))
$$

$$
= \frac{1}{2} \max_{t \in [a, b]} h_X (2(f_t)_+(t) \cdot x, 2(f_t)_-(t) \cdot x) = \max_{t \in [a, b]} |f_t(t)| \geq \frac{1}{4} \int_0^{(b-a)/n} \omega(t)dt.
$$

It follows from (27) that in the case of the uniform partition we have equalities in the above inequalities. The theorem is proved. \qed

6.2. Recovery of the derivative

Consider the problem about the deviation of the Hukuhara type derivative of a function $f \in W^1 H^\omega([a, b], X)$ from the derivative of its interpolating polygonal function.

The Hukuhara type derivative of the interpolating at the points of the partition $t$ polygonal function $l_f(t)$ for $t \in (t_k, t_{k+1})$ is equal to

$$
\mathcal{D}_H l_f(t; t) = \frac{f(t_{k+1}) - H f(t_k)}{t_{k+1} - t_k} = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathcal{D}_H f(u)du, \; k = 0, \ldots, n - 1.
$$

We define it at the points $t_k$, setting

$$
\mathcal{D}_H l_f(t; t_k) = \begin{cases} 
(f(t_{k+1}) - H f(t_k))/(t_{k+1} - t_k), & \text{if } k = 0, 1, \ldots, n - 1, \\
(f(t_n) - H f(t_{n-1}))/ (t_n - t_{n-1}), & \text{if } k = n.
\end{cases}
$$

For $t \in [t_k, t_{k+1}]$ we obtain, using Corollary 2

$$
h_X (\mathcal{D}_H f(t), \mathcal{D}_H l_f(t; t_k)) \leq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \omega(|s - t|)ds. \tag{28}
$$

The following theorem generalizes the results from [50].

Theorem 6. Let $\omega$ be an arbitrary modulus of continuity and $t^* = (t_0^*, \ldots, t_n^*)$ be the uniform partition of the segment $[a, b]$. Then

$$
\mathcal{E}(\mathcal{D}_H, W^1 H^\omega([a, b], X), I_{t^*}, B([a, b], X)) = \frac{n}{b - a} \int_0^{(b-a)/n} \omega(u)du.
$$

The optimal method of recovery is

$$
\Phi(f(t_0^*), f(t_1^*), \ldots, f(t_n^*)) = \mathcal{D}_H l_f(t^*).
$$

Proof From (28) it follows that

$$
\sup_{f \in W^1 H^\omega([a, b], X)} \sup_{t \in [a, b]} h_X (\mathcal{D}_H f(t), \mathcal{D}_H l_f(t^*), t) \leq \frac{n}{b - a} \int_0^{(b-a)/n} \omega(u)du.
$$
An extremal function is built as follows. Set \( g_0(t) = \min_{k : 2k \leq n} \omega(|t - t_{2k}^*|) \) and
\[
g(t) = g_0(t) - \frac{1}{b - a} \int_a^b g_0(u) du.
\]
The function \( f^*_t(t) := \int_a^t g(u) du \) belongs to \( W^1 H^\omega([a, b], \mathbb{R}) \). Moreover, since \( t^* \)
is the uniform partition, \( f^*_t(t_k) = 0, k = 0, \ldots, n \), and hence \( I_f(t^*) \equiv 0 \). Finally, applying Lemma 10 to functions \( f^*_t(x) \) and \( (f^*_t)_x \) \((x \in X^c \cap X^{\text{inv}}, h_X(x, \theta) = 1)\) we obtain
\[
E(D_H, W^1 H^\omega([a, b], X), I^*, B([a, b], X)) \geq \frac{1}{2} h_X(D_H(f^*_t)_x(a), D_H(f^*_t)_x(a)) \geq \frac{|f^*_t'(a)|}{b - a} \int_a^b g_0(u) du = \frac{n}{b - a} \int_0^{(b-a)/n} \omega(u) du.
\]
The theorem is proved.

\[ \square \]

7. On Inequalities of Landau type and Stechkin’s problem for Hukuhara type divided differences and derivatives

7.1. Deviations of Hukuhara type divided differences and derivatives

Let \( t \in [a, b] \) and non-negative numbers \( \gamma_1, \gamma_2, h_1, h_2 \) such that
\[
\gamma_1 + \gamma_2 > 0, h_1 + h_2 > 0, \text{ and } [t - \gamma_1, t + \gamma_2] \subset [t - h_1, t + h_2] \subset [a, b]
\]
be given. For a function \( f \in W^1 H^\omega([a, b], X) \) set
\[
\Delta^H_{\gamma_1, \gamma_2} f(t) = \frac{f(t + \gamma_2) - H f(t - \gamma_1)}{\gamma_1 + \gamma_2}.
\]
Applying Theorem 2 to the segments \([t - \gamma_1, t + \gamma_2]\) and \([t - h_1, t + h_2]\), and writing \( I(\alpha) \) instead of \( I(0, \alpha) \), we obtain
\[
\begin{align*}
&h_X(\Delta^H_{\gamma_1, \gamma_2} f(t), \Delta^H_{h_1, h_2} f(t)) \\
&\quad = h_X\left(\frac{1}{\gamma_1 + \gamma_2} \int_{t-\gamma_1}^{t+\gamma_2} D_H f(u) du, \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} D_H f(u) du\right) \\
&\quad \leq \frac{(h_1 - \gamma_1) + (h_2 - \gamma_2)}{(h_1 + h_2)^2} \left\{ I\left(\frac{(h_1 + h_2)(h_1 - \gamma_1)}{(h_1 - \gamma_1) + (h_2 - \gamma_2)}\right) + I\left(\frac{(h_1 + h_2)(h_2 - \gamma_2)}{(h_1 - \gamma_1) + (h_2 - \gamma_2)}\right)\right\} \\
&\quad =: K(\gamma_1, \gamma_2; h_1, h_2).
\end{align*}
\]
If \( \omega \) is a concave modulus of continuity and \( X \) is isotropic, then the estimate
\[
h_X(\Delta^H_{\gamma_1, \gamma_2} f(t), \Delta^H_{h_1, h_2} f(t)) \leq K(\gamma_1, \gamma_2; h_1, h_2)
\]
(30)
is sharp. Extremal functions can be built as follows. Start with the extremal function \( g \) from Theorem 2 for the segments \([t - \gamma_1, t + \gamma_2]\) and \([t - h_1, t + h_2]\). Continue it setting

\[
g(u) = g(t - h_1) \text{ for } u \leq t - h_1 \text{ and } g(u) = g(t + h_2) \text{ for } t \geq t + h_2. \tag{31}
\]

Inequality (30) becomes equality on the functions \( f(u) = \int_a^u g(s)ds + \gamma, \quad u \in [a, b], \gamma \in X^c \). Shrinking the segment \([t - \gamma_1, t + \gamma_2]\) into the point \( t \), we obtain

\[
h_X(D_Hf(t), \Delta^H_{h_1, h_2} f(t)) \leq \frac{I(h_1) + I(h_2)}{h_1 + h_2}. \tag{32}
\]

In an isotropic space \( X \) this inequality is sharp for arbitrary modulus of continuity \( \omega \). Extremal functions can be built analogously to the extremal ones for (30), except we need to start from the extremal function from Corollary 2.

### 7.2. Landau type inequalities

Below for brevity we write \( \bar{W}^1H^\omega([a, b], X) := \bigcup_{k \geq 0} k \cdot W^1H^\omega([a, b], X) \),

\[
\|x\|_X = h_X(x, \theta), \quad \|f\|_{\omega, X} = \sup_{t', t'' \in [a, b], t' \neq t''} \frac{h_X(f(t'), f(t''))}{\omega(|t' - t''|)},
\]

\[
\|f\|_{C([a, b], X)} = \sup_{t \in [a, b]} \|f(t)\|_X.
\]

**Theorem 7.** Let \( \omega \) be a modulus of continuity, and \( X \) be an isotropic \( L \)-space. For all \( t \in [a, b] \), non-negative \( \gamma_1, \gamma_2, h_1, h_2 \) that satisfy (29), and \( f \in \bar{W}^1H^\omega([a, b], X) \),

\[
\|\Delta^H_{\gamma_1, \gamma_2} f(t)\|_X \leq K(\gamma_1, \gamma_2, h_1, h_2)\|D_Hf\|_{\omega, X} + \|\Delta^H_{h_1, h_2} f(t)\|_X, \tag{33}
\]

\[
\|D_Hf(t)\|_X \leq \frac{I(h_1) + I(h_2)}{h_1 + h_2} \|D_Hf\|_{\omega, X} + \|\Delta^H_{h_1, h_2} f(t)\|_X. \tag{34}
\]

Inequality (33) is sharp for concave \( \omega \). Inequality (34) is sharp for arbitrary \( \omega \).

**Proof** Inequalities (33) and (34) follow from (30) and (32), respectively. An extremal function for (33) can be built as follows. Let \( g \) be a non-negative extremal function in Theorem 2 for the case \( X = \mathbb{R} \) and the segments \([t - h_1, t + h_2], [t - \gamma_1, t + \gamma_2]\). Continue it to the whole segment \([a, b]\) by (31). Note that due to Lemma 9 there exists \( \gamma \in (t - \gamma_1, t + \gamma_2) \) such that \( g \) increases on \((t - h_1, \gamma)\) and decreases on \((\gamma, t + h_2)\). Hence

\[
\frac{1}{\gamma_1 + \gamma_2} \int_{t - \gamma_1}^{t + \gamma_2} g(u)du \geq \frac{1}{h_1 + h_2} \int_{t - h_1}^{t + h_2} g(u)du,
\]
and the function \( f(u) = \int_a^u g(s)ds \) turns inequality (33) into equality in the case \( X = \mathbb{R} \). Indeed,

\[
\Delta^H_{\gamma_1, \gamma_2} f(t) = \left( \frac{1}{\gamma_1 + \gamma_2} \int_{t-\gamma_1}^{t+\gamma_2} g(u)du - \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} g(u)du \right) + \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} g(u)du
\]

\[
= K(\gamma_1, \gamma_2, h_1, h_2) + \Delta^H_{h_1, h_2} f(t).
\]

In general case, the function \( f_X \) with \( x \in X^c, \|x\|_X = 1 \) is extremal for inequality (33).

An extremal function for (34) can be built analogously to the one in (33), but we need to start from a non-negative extremal for Corollary 2 for the point \( t \) and the segment \([t - h_1, t + h_2]\).

**Theorem 8.** Under the conditions of Theorem 7 for any \( f \in \overline{W}^1 H^\omega([a, b], X) \),

\[
\|\Delta^H_{\gamma_1, \gamma_2} f(t)\|_X \leq K(\gamma_1, \gamma_2; h_1, h_2)\|D_H f\|_{\omega, X} + \frac{2}{h_1 + h_2}\|f\|_{C([a, b], X)},
\]

(35)

\[
\|D_H f(t)\|_X \leq \frac{I(h_1) + I(h_2)}{h_1 + h_2}\|D_H f\|_{\omega, X} + \frac{2}{h_1 + h_2}\|f\|_{C([a, b], X)}.
\]

(36)

If for given \( t \in [a, b] \) and \( h > \gamma > 0 \)

\[
\gamma_1 = \min\{\gamma, t-a\}, \quad \gamma_2 = \min\{\gamma, b-t\}, \quad h_1 = \min\{h, t-a\}, \quad h_2 = \min\{h, b-t\}
\]

(37)

and \( \omega \) is concave, then inequality (35) is sharp. If for \( t \in [a, b] \) and \( h > 0 \)

\[
h_1 = \min\{h, t-a\}, \quad h_2 = \min\{h, b-t\},
\]

(38)

and \( \omega \) is an arbitrary modulus of continuity, then inequality (36) is sharp.

**Proof** Inequalities (35) and (36) follow from (33) and (34), since

\[
\|\Delta^H_{h_1, h_2} f(t)\|_X \leq \frac{2}{h_1 + h_2}\|f\|_{C([a, b], X)}.
\]

We prove their sharpness under the above conditions on the numbers \( t, \gamma_1, \gamma_2, h_1 \) and \( h_2 \). Let for definiteness \( t \leq (a+b)/2 \). Then \( h_2 \geq h_1 \) and \( (t + \gamma_2) + (t - \gamma_1) = (t + h_2) + (t - h_1) \). For inequality (35) as a function \( g \) we take the non-negative extremal function in Theorem 2 for the segments \([t - \gamma_1, t + \gamma_2] \) and \([t - h_1, t + h_2] \) and \( X = \mathbb{R} \) such that \( g(t + h_2) = 0 \). Continue it to the segment \([a, b] \) setting \( g(u) = 0, u \notin [t - h_1, t + h_2] \). Lemma 9 justifies existence of such function \( g \) and continuity of its extension by the rule above. For inequality (36) we take \( g(s) = (\omega(h_2) - \omega(|s - t|))_+, s \in [a, b] \). Both functions belong to \( H^\omega([a, b], \mathbb{R}) \).
It is clear that if \( \xi \in [t - h_1, t + h_2] \) so that \( \int_{t-h_1}^{t} g(u) du = \int_{t-h_2}^{t+h_2} g(u) du \). The function

\[
    f(u) = \left( \int_{\xi}^{u} g(s) ds \right)_x, \quad x \in X^c \cap X^{inv}, \quad \|x\|_X = 1
\]

is extremal. The theorem is proved. \( \square \)

Note that for the extremal in inequality (36) function one has

\[
    \|f\|_{C([a,b],X)} = \frac{1}{2} \int_{t-h_1}^{t+h_2} [\omega(h_2) - \omega(|s-t|)] ds = \frac{h_1 + h_2}{2} \omega(h_2) - \frac{I(h_1) + I(h_2)}{2} = \frac{h_1 + h_2}{2} \max\{\omega(h_1), \omega(h_2)\} - \frac{I(h_1) + I(h_2)}{2}. \quad (40)
\]

### 7.3. Approximation of operators by the ones with smaller norms

In the space \( C([a,b], X) \) consider the cone \( C^H([a,b], X) \) that consists of functions \( f \) such that for all \( t \in [a,b] \) and \( \gamma_1, \gamma_2 > 0 \) such that \( [t - \gamma_1, t + \gamma_2] \subset [a,b] \), the difference \( \Delta_{\gamma_1,\gamma_2}^H f(t) \) exists. We call a positively homogeneous operator \( T: C^H([a,b], X) \to X \) bounded, if

\[
    \|T\| = \sup \{\|Tf\|_X : f \in C^H([a,b], X), \quad \|f\|_{C([a,b],X)} \leq 1\} < \infty.
\]

Assume that an operator \( A: \overline{W}^1 H^\omega([a,b], X) \to X \), a number \( N > 0 \) and an operator \( T: C^H([a,b], X) \to X \) such that \( \|T\| \leq N \) are given. Set

\[
    U(A, T) = \sup_{f \in \overline{W}^1 H^\omega([a,b], X)} h_X(Af, Tf).
\]

The quantity

\[
    E(A, N) = \inf_{\|T\| \leq N} U(A, T)
\]

is called the best approximation of the operator \( A \) by operators \( T \) with \( \|T\| \leq N \). It is clear that if \( A \) is defined on \( C^H([a,b], X) \), is bounded, and \( N \geq \|A\|, \) then \( E(A, N) = 0 \).

For \( t \in [a,b] \) denote by \( \Delta_{\gamma_1,\gamma_2}(t) \) and \( D_H(t) \) the operators that act by the formulae

\[
    \Delta_{\gamma_1,\gamma_2}(t)f = \Delta_{\gamma_1,\gamma_2}^H f(t) \quad \text{and} \quad D_H(t)f = D_H f(t).
\]

**Theorem 9.** Let \( \omega \) be a modulus of continuity, \( X \) be an isotropic \( L \)-space, \( t \in [a,b] \), and numbers \( h > \gamma > 0 \) be given. Let also numbers \( \gamma_1, \gamma_2, h_1, h_2 \) be defined by (37). If \( \omega \) is concave, then

\[
    E\left( \Delta_{\gamma_1,\gamma_2}(t), \frac{2}{h_1 + h_2} \right) = U\left( \Delta_{\gamma_1,\gamma_2}(t), \Delta_{h_1, h_2}(t) \right) = K(\gamma_1, \gamma_2; h_1, h_2), \quad (41)
\]
and for arbitrary \( \omega \)

\[
E \left( \mathcal{D}_H(t), \frac{2}{h_1 + h_2} \right) = U \left( \mathcal{D}_H(t), \Delta_{h_1,h_2}(t) \right) = \frac{I(h_1) + I(h_2)}{h_1 + h_2}. \tag{42}
\]

**Proof** It is clear that \( \| \Delta_{h_1,h_2}(t) \| \leq \frac{2}{h_1 + h_2} \). Due to (30) and (32) we have

\[
E \left( \Delta_{\gamma_1,\gamma_2}(t), \frac{2}{h_1 + h_2} \right) \leq U(\Delta_{\gamma_1,\gamma_2}(t), \Delta_{h_1,h_2}(t)) \leq K(\gamma_1, \gamma_2; h_1, h_2)
\]

and

\[
E \left( \mathcal{D}_H(t), \frac{2}{h_1 + h_2} \right) \leq U(\mathcal{D}_H(t), \Delta_{h_1,h_2}(t)) \leq \frac{I(h_1) + I(h_2)}{h_1 + h_2}.
\]

We also proved that there exist functions \( f_1, f_2 \in W^1 H^\omega([a, b], X) \) such that

\[
\| \Delta_{\gamma_1,\gamma_2}^H f_1(t) \| = K(\gamma_1, \gamma_2; h_1, h_2) + \frac{2}{h_1 + h_2} \| f_1 \|_{C([a, b], X)} \tag{43}
\]

and

\[
\| \mathcal{D}_H f_2(t) \| = \frac{I(h_1) + I(h_2)}{h_1 + h_2} + \frac{2}{h_1 + h_2} \| f_2 \|_{C([a, b], X)}.
\]

To prove (41), assume there exists an operator \( T, \| T \| \leq \frac{2}{h_1 + h_2} \) such that

\[
U(\Delta_{\gamma_1,\gamma_2}(t), T) < K(\gamma_1, \gamma_2; h_1, h_2).
\]

Then for the function \( f_1 \) we get a strict inequality

\[
\| \Delta_{\gamma_1,\gamma_2}^H f_1(t) \| < K(\gamma_1, \gamma_2; h_1, h_2) + \frac{2}{h_1 + h_2} \| f_1 \|_{C([a, b], X)},
\]

which contradicts to (43). Equality (42) can be proved similarly. \( \square \)

### 7.4. Recovery of an operator given inexact data

Finally, we consider the problem of optimal recovery of an operator \( A \) on the elements of the class \( W^1 H^\omega([a, b], X) \) known with error. For an operator \( A \), bounded operator \( T \) and a number \( \delta > 0 \) set

\[
U_\delta(A, T) = \sup \{ h_X(Af, Tg) : f \in W^1 H^\omega([a, b], X), \quad g \in C([a, b], X), h_{C([a, b], X)}(f, g) \leq \delta \}.
\]

The problem is to find the quantity

\[
\mathcal{E}_\delta(A) = \inf_T U_\delta(A, T)
\]

and the operator \( T^* \) on which the infimum on the right-hand side of the equality is attained.

**Theorem 10.** Let \( \omega \) be a modulus of continuity, \( t \in [a, b], h > 0, \) and \( \mathcal{D}_H(t)f = \mathcal{D}_H f(t) \) for \( f \in W^1 H^\omega([a, b], X) \). If the numbers \( h_1, h_2 \) are defined by (38), and

\[
\delta = \frac{h_1 + h_2}{2} \max \{ \omega(h_1), \omega(h_2) \} - \frac{I(h_1) + I(h_2)}{2},
\]

then...
then for the operator $\Delta_{h_1, h_2}(t)f = \Delta^H_{h_1, h_2}f(t)$ we have
\[
E_\delta(D_H(t)) = U_\delta(D_H(t), \Delta_{h_1, h_2}(t)) = \max\{\omega(h_1), \omega(h_2)\}.
\]

**Proof.** For each $f \in W^1_H([a, b], X), g \in C([a, b], X)$ such that $h_{C([a,b],X)}(f, g) \leq \delta$, due to (32),
\[
h_X(D_Hf(t), \Delta^H_{h_1, h_2}g(t)) \leq h_X(D_Hf(t), \Delta^H_{h_1, h_2}f(t)) + h_X(\Delta^H_{h_1, h_2}f(t), \Delta^H_{h_1, h_2}g(t)) \leq \frac{I(h_1) + I(h_2)}{h_1 + h_2} + \frac{2}{h_1 + h_2}\delta = \max\{\omega(h_1), \omega(h_2)\}.
\]
Hence, $E_\delta(D_H(t)) \leq \max\{\omega(h_1), \omega(h_2)\}$. On the other hand, for the function $f$ defined by (39), due to (40),
\[
E_\delta(D_H(t)) \geq \|D_Hf(t)\|_X = \frac{I(h_1) + I(h_2)}{h_1 + h_2} + \frac{2}{h_1 + h_2}\|f\|_{C([a,b],X)} = \max\{\omega(h_1), \omega(h_2)\}
\]
and the theorem is proved. \hfill \square

**References**

[1] Nikol’skii, S. M. (1946). Fourier series of functions with a given modulus of continuity. *Dokl. Akad. Nauk SSSR* 52(3):191–194.

[2] Korneichuk, N. P. (1991). *Exact Constants in Approximation Theory. Encyclopedia of Mathematics and its Applications.* Cambridge: Cambridge University Press.

[3] Korneichuk, N. P. (1959). Approximation of periodic functions satisfying Lipschitz’s condition by Bernstein-Rogosinski’s sums. *Dokl. Akad. Nauk SSSR* 125:258–261.

[4] Korneichuk, N. P. (1961). On the degree of approximation of functions of class $H^{(a)}$ by means of trigonometric polynomials. In: *Studies of Modern Problems of Constructive Theory of Functions.* Moscow: Fizmathgiz, pp. 148–154.

[5] Korneichuk, N. P. (1962). On extremal properties of periodic functions. *Dopovidi Akad. Nauk Ukrain. RSR* 8:993–998.

[6] Bagdasarov, S. (1998). *Chebyshev Splines and Kolmogorov Inequalities.* Operator Theory: Advances and Applications. Basel: Springer.

[7] Stepanets, A. I. (2018). *Uniform Approximations by Trigonometric Polynomials.* Berlin: De Gruyter.

[8] Aubin, J. P., Frankowska, H. (1990). *Set-Valued Analysis.* Boston: Birkhäuser.

[9] Borisovich, Y. G., Gel’man, B. D., Myshkis, A. D., Obukhovskii, V. V. (1984). Multivalued mappings. *J. Sov. Math.* 24(6):719–791.

[10] Diamond, P., Kloeden, P. (1994). *Metric Spaces of Fuzzy Sets: Theory and Applications.* Singapore: World Scientific Publishing Company.

[11] Dyn, N., Farkhi, E., Mokhov, A. (2014). *Approximation Of Set-valued Functions: Adaptation of Classical Approximation Operators.* Singapore: World Scientific Publishing Company.

[12] Anastassiou, G. (2010). *Fuzzy Mathematics: Approximation Theory, Studies in Fuzziness and Soft Computing.* Berlin: Springer.
[13] Vahrameev, S. A. (1980). *Applied Mathematics and Mathematical Software of Computers*. Moscow: MSU Publisher (in Russian).

[14] Aseev, S. M. (1986). Quasilinear operators and their application in the theory of multivalued mappings. *Proc. Steklov Inst. Math.* 167:23–52.

[15] Korneichuk, N. P. (1971). Extremal values of functionals and the best approximation on classes of periodic functions. *Math. USSR-Izv.* 1971(1):97–129.

[16] Ostrowski, A. (1937). Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert. *Comment. Math. Hel.* 10:226–227.

[17] Dragomir, S. S. (2017). Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Australian J. Math. Anal. Appl.* 14(1):1–287.

[18] Barnet, N. S., Cerone, P., Dragomir, A. M., Fink, A. M. (2002). Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty[a, b]$ and applications. *Comput. Math. Appl.* 44(1):241–251.

[19] Guessab, A., Schmeisser, G. (2002). Sharp integral inequalities of the Hermite–Hadamard type. *J. Approx. Theory* 115(2):260–288.

[20] Barnett, N. S., Buse, C., Cerone, P., Dragomir, S. S. (2002). Ostrowski’s inequality for vector-valued functions and applications. *Comput. Math. Appl.* 44(5–6):559–572.

[21] Chalco-Cano, Y., Flores-Franulic, A., Roman-Flores, H. (2012). Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* 31(2):457–472.

[22] Chalco-Cano, Y., Lodwick, W. A. (2015). Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* 19:3293–3300.

[23] Anastassiou, G. A. (2003). Fuzzy Ostrowski type inequalities. *Comput. Appl. Math.* 22(2):279–292.

[24] Roman-Flores, H., Chalco-Cano, Y., Lodwick, W. A. (2018). Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* 37:1306–1318.

[25] Zhao, D., An, T., Ye, G., Liu, W. (2020). Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* 396:82–101. DOI: 10.1016/j.fss.2019.10.006.

[26] Barnett, N. S., Buse, C., Cerone, P., Dragomir, S. S. (2002). On weighted Ostrowski type inequalities for operators and vector-valued functions. *J. Inequal. Pure Appl. Math.* 3(1):1–21.

[27] Dragomir, S. S. (2003). A weighted Ostrowski type inequality for functions with values in Hilbert spaces and applications. *J. Korean Math. Soc.* 40(2):207–224.

[28] Anastassiou, G. A. (2012). Ostrowski and Landau inequalities for Banach space valued functions. *Math. Comput. Model.* 55(3):312–329.

[29] Traub, J. F., Woźniakowski, H. (1980). *A General Theory of Optimal Algorithms*. New York: Academic Press.

[30] Zhensykbaev, A. A. (2003). *Problems of Recovery of Operators*. Moscow-Izhevsk: Institute of Computer Studies.

[31] Korneichuk, N. P. (1984). *Splines in Approximation Theory*. Moscow: Nauka (in Russian).

[32] Korneichuk, N. P. (1968). Best cubature formulas for some classes of functions of many variables. *Math. Notes Acad. Sci. USSR* 3:360–367.

[33] Babenko, V. F., Babenko, V. V., Polischuk, M. V. (2016). On the optimal recovery of integrals of set-valued functions. *Ukrainian Math. J.* 67(9):1306–1315.
A. Proof of Theorem 2

A.1. Case $d = b$

In the case, when $d = b$, Theorem 2 follows from the following lemma.
Lemma 15. Let $a < c < b$, and a modulus of continuity $\omega$ be given. Then for all $f \in H^\omega([a, b], X)$
\[
h_X \left( \int_a^b f(t) dt, \frac{b-a}{b-c} \int_c^b f(t) dt \right) \leq \frac{c-a}{b-a} \int_0^b \omega(t) dt. \tag{A.1}
\]
If $\omega$ is concave, then inequality (A.1) turns into equality for the functions $g, y \in H^\omega([a, b], X)$, where $y \in X, x \in X^c \cap X^{inv}$, $h_X(x, \theta) = 1$ and
\[
g(t) = g(a, b, c; t) = \begin{cases} -\frac{c-a}{b-a} \omega \left( \frac{b-a}{c-a} (c - t) \right), & t \in [a, c], \\ \frac{b-c}{b-a} \omega \left( \frac{b-a}{b-c} (t - c) \right), & t \in [c, b]. \end{cases} \tag{A.2}
\]

Proof Consider the functions $\psi_1 : [a, c] \to \mathbb{R}, \psi_1 \equiv 1$ and $\psi_2 : [c, b] \to \mathbb{R}, \psi_2 \equiv \frac{c-a}{b-c}$. They satisfy the conditions of Lemma 8. For the function $\rho$ from (9), one has $\rho(t) = b - \frac{c-a}{c-a} (t - a), \rho(t) = t - \frac{b-a}{c-a} (c - t), \rho^{-1}(t) = a + \frac{c-a}{b-c} (b - t)$ and $t - \rho^{-1}(t) = \frac{b-a}{b-c} (c - t)$. By Lemma 8 for all $f \in H^\omega([a, b], X)$
\[
h_X \left( \int_a^b f(t) dt, \frac{b-a}{b-c} \int_c^b f(t) dt \right) \leq \int_a^c \omega \left( \frac{b-a}{c-a} (c - t) \right) dt = \frac{c-a}{b-a} \int_0^b \omega(t) dt.
\]
Moreover, the extremal functions (11) become (A.2). The lemma is proved. \qed

A.2. Case $d < b$

Direct computations show that for $\gamma$ defined in (16), $\frac{\gamma-a}{\gamma-c} = \frac{b-a}{d-c} = \frac{b-a}{d-c}$. For all functions $f \in H^\omega([a, b], X)$
\[
h_X \left( \int_a^b f(t) dt, \frac{b-a}{d-c} \int_c^d f(t) dt \right) \leq h_X \left( \int_a^\gamma f(s) ds, \frac{\gamma-a}{\gamma-c} \int_c^\gamma f(s) ds \right) + h_X \left( \int_\gamma^b f(s) ds, \frac{b-a}{d-\gamma} \int_\gamma^d f(s) ds \right).
\]
Using Lemma 15 for the function $f$ to estimate the first summand, and for the function $f(b + \gamma - \cdot)$ to estimate the second summand, one obtains (15).

To finish the proof of the theorem we need to prove the sharpness of inequality (15) in the case, when $\omega$ is concave. Consider the function
\[
G(t) = \begin{cases} g(a, \gamma, c; t), & t \in [a, \gamma], \\ g(\gamma, b, d; b + \gamma - t) + C, & t \in [\gamma, b], \end{cases} \tag{A.3}
\]
where the function $g$ is defined by (A.2) and $C = g(a, \gamma, c; \gamma) - g(\gamma, b, d; b)$, so that the function $g$ is continuous at point $\gamma$, and hence on the whole segment
A.3. Proof of Lemma 9

We prove that the function defined in (A.3) satisfies the stated conditions. The first two properties follow from formulae (A.3) and (A.2). Direct computations show that \( b - \gamma = \frac{(b - d)(b - a)}{b + c - a - d} \) and \( \gamma - a = \frac{(b - a)(c - a)}{b + c - a - d} \), which implies the third property. The last statement of the lemma is obvious.

B. Constructions of real-valued extremal functions

B.1. Proof of Lemma 12

Proof Among \(2n\) segments \([\tau_i, t_i] \) and \([t_i, \tau_{i+1}] \), \(i = 1, \ldots, n\), there exists at least one with length at least \(\frac{b - a}{2n}\). Let for definitness it be the segment \([\tau_{i^*}, t_{i^*}]\), \(i^* \in \{1, \ldots, n\}\). We define a functions \(f_i\) on the segment \([\tau_{i^*}, t_{i^*} + h] = [a, t_1 + h]\), if \(i^* = 1\), or on the segment \([t_{i^*-1} + h, t_{i^*} + h]\), if \(i^* > 1\), by the formula

\[
 f_i(u) = \frac{1}{2h} \int_{t_{i^*} - h}^{t_{i^*} + h} \omega(|s - \tau_{i^*}|)ds - \omega(|u - \tau_{i^*}|).
\]

Next we continue this function to the whole segment \([a, b]\) as follows. We set \(f_i(u) = f_i(t_{i^*} + h)\) on \([t_{i^*} + h, t_{i^* + 1} - h]\); \(f_i(u) = f_i(t_{i^*} + t_{i^* + 1} - u)\) on \([t_{i^* + 1} - h, t_{i^* + 1} + h]\); \(f_i(u) = f_i(t_{i^* + 1} + h)\) on \([t_{i^* + 1} + h, t_{i^* + 2} - h]\); \(f_i(u) = f_i(t_{i^* + 1} + t_{i^* + 2} - u)\) on \([t_{i^* + 2} - h, t_{i^* + 2} + h]\) and so on. In the case \(i^* > 1\) the process goes analogously for \(u < t_{i^*-1} - h\).

From the definition it follows that \(f_i \in H^h_t\) and

\[
 \max_{t \in [a, b]} |f_i(t)| \geq f_i(\tau_{i^*}) = \frac{1}{2h} \int_{t_{i^*} - h}^{t_{i^*} + h} \omega(|u - \tau_{i^*}|)du \geq \frac{1}{2h} \int_{(b - a)/(2n) - h}^{(b - a)/(2n) + h} \omega(u)du.
\]

B.2. Proof of Lemma 13

Proof Consider the even function \(y_0\), defined on \([0, \infty)\) by the following equation.

\[
 y_0(t) = \begin{cases} 
 -\frac{2nh}{b-a}\omega\left(\frac{b-a}{2nh}(h-t)\right), & t \in [0, h), \\
 \frac{b-a-2nh}{b-a}\omega\left(\frac{b-a}{b-a-2nh}(t-h)\right), & t \in [h, \frac{b-a}{2n}], \\
 y_0\left(\frac{b-a}{2n}\right), & t > \frac{b-a}{2n}.
\end{cases}
\]
Note that the restriction of the function \( y_0 \) to the segment \([0, (b - a)/(2n)]\) coincides with the function \( g \) built according to formula (11) of Lemma 8 with \( \psi_1 = \frac{1}{h} \chi_{[0,h]} \) and \( \psi_2 = \frac{2n}{b-a-2mh} \chi_{[h,(b-a)/(2n)]} \). Hence \( y_0(t) \in H^\omega([0, (b - a)/(2n)], \mathbb{R}) \), since \( \omega \) is a concave modulus of continuity. Set
\[
y_1(t) := \min\{y_0(t - t_1), y_0(t - t_2), \ldots, y_0(t - t_n)\}, \quad t \in \mathbb{R}.
\]
Then \( y_1(t) \in H^\omega([a, b], \mathbb{R}) \), and using notations from (20), \( y_1(t) = y_0(t - t_k) \), \( t \in [\tau_k, \tau_{k+1}] \), \( k = 1, \ldots, n \). Set \( y(t) := y_1(t) + C \), where the constant \( C \) is chosen in such a way that \( \int_{-h}^h (y_0(t) + C)dt = 0 \). This implies
\[
\int_{tk-h}^{tk+h} y(t)dt = 0, \quad k = 1, \ldots, n. \tag{B.2}
\]
Hence \( y \in H^\omega_b \). We estimate the integral \( \int_a^b y(t)dt \) from below. The function \( y_0 \) is even and \( J(t) := \int_0^t y_0(s)ds \) is convex, since \( y_0 \) is non-decreasing on \([0, \infty)\). Hence
\[
\int_{a}^{b} y(t)dt = C(b - a) + \int_{a}^{b} y_1(t)dt = C(b - a) + \sum_{k=1}^{n} \int_{\tau_k}^{\tau_{k+1}} y_0(t - t_k)dt
\]
\[
= C(b - a)
\]
\[
+ \sum_{k=1}^{n} \int_{\tau_k-t_k}^{\tau_{k+1}-t_k} y_0(t)dt = C(b - a) + \sum_{k=1}^{n} \left[ J(\tau_{k+1} - t_k) + J(t_k - \tau_k) \right]
\]
\[
\geq C(b - a)
\]
\[
+ 2n J\left( \frac{1}{2n} \sum_{k=1}^{n} (\tau_{k+1} - t_k) + \frac{1}{2n} \sum_{k=1}^{n} (t_k - \tau_k) \right)
\]
\[
= C(b - a) + 2n J\left( \frac{b - a}{2n} \right).
\]
Using (B.1) and (B.2) to compute the right-hand side of the latter inequality, we obtain
\[
C(b - a) + 2n J\left( \frac{b - a}{2n} \right) = 2n \left( 1 - \frac{2nh}{b - a} \right) \int_0^{(b - a)/(2n)} \omega(t)dt
\]
and the lemma is proved. \( \square \)

**B.3. Proof of Lemma 14**

**Proof** In the space \( \mathbb{R}^{n+1} \) consider the sphere
\[
S^n = \left\{ \xi = (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} |\xi_i| = b - a \right\}.
\]
Each \( \xi \in S^n \) generates a set of points on the segment \([a, b]\)
\[
\eta_0(\xi) = a, \quad \eta_1(\xi) = a + |\xi_1|, \quad \eta_2(\xi) = \eta_1(\xi) + |\xi_2|, \ldots, \eta_n(\xi)
\]
\[
= \eta_{n-1}(\xi) + |\xi_n|, \quad \eta_{n+1}(\xi) = b.
\]
Let \( h(t) = \frac{1}{2} \omega(2|t|), t \in \mathbb{R} \). For \( \xi \in S^n \), set \( h_\xi(t) = \min_{k=1,\ldots,n} h(t-\eta_k(\xi)) \) and \( g_\xi(t) = h_\xi(t) \cdot \text{sgn} \xi_i \) for \( t \in [\eta_{i-1}(\xi), \eta_i(\xi)] \), \( i = 1, \ldots, n + 1 \) (here \( \text{sgn} \) is the odd function equal to 1 for all positive arguments). Then, due to concavity of \( \omega \), \( g_\xi \in H^\omega([a, b], \mathbb{R}) \). Set

\[
G_\xi(t) = \int_a^t g_\xi(u)du
\]

and define the vector field on \( S^n \), by the formula \( \xi \mapsto (G_\xi(t_1), \ldots, G_\xi(t_n)) \). It is easy to see that this field is continuous and odd. The Borsuk theorem [51] implies that there exists \( \xi^* = \xi^*(t) = (\xi_1^*, \ldots, \xi_{n+1}^*) \in S^n \) such that \( G_{\xi^*}(t_1) = G_{\xi^*}(t_2) = \cdots = G_{\xi^*}(t_n) = 0 \). Moreover, \( G_{\xi^*}(a) = 0 \). Hence the function \( G_{\xi^*} \) has at least \( n + 1 \) zeros on \([a, b]\) and thus \( g_{\xi^*} = G_{\xi^*}' \) has at least \( n \) changes of sign. Since \( g_{\xi^*} \) can change its sign only at the points \( \eta_1(\xi^*), \ldots, \eta_n(\xi^*) \), all these points are distinct, \( g_{\xi^*} \) has exactly \( n \) sign changes on \([a, b]\), and \( \eta_i(\xi^*) \) is the unique point of local extremum of \( G_{\xi^*} \) inside the segment \([t_{i-1}, t_i], i = 1, \ldots, n \). Since \( \omega \) is non-decreasing, the function \( u \mapsto \int_0^u \omega(t)dt \) is convex, hence applying the Jensen inequality we obtain

\[
\sqrt{b-a} G_{\xi^*} = \int_a^b |g_{\xi^*}(u)| \, du = \frac{1}{2} \int_0^{2|\xi_1^*|} \omega(2u)du + \frac{1}{2} \sum_{i=2}^n |\xi_i^*|/2 \omega(2u)du + \frac{1}{2} \int_0^{(|\xi_1^*|+|\xi_{n+1}^*|)/2} \omega(2u)du \geq \int_0^{(|\xi_1^*|+|\xi_{n+1}^*|)/2} \omega(2u)du + \frac{1}{2} \sum_{i=2}^n |\xi_i^*|/2 \omega(2u)du + \frac{1}{2} \int_0^{(|\xi_1^*|+|\xi_{n+1}^*|)/2} \omega(2u)du = \frac{1}{2} \int_0^{(|\xi_1^*|+|\xi_{n+1}^*|)/2} \omega(u)du + \frac{1}{2} \sum_{i=2}^n \int_0^{|\xi_i^*|/2} \omega(u)du \geq \frac{n}{2} \int_0^{(b-a)/n} \omega(u)du.
\]

The function \( G_{\xi^*} \) is monotone on the segments \([a, \eta_1(\xi^*)], [\eta_i(\xi^*), \eta_{i+1}(\xi^*)], i = 1, \ldots, n-1, [\eta_n(\xi^*), b] \) and \( G_{\xi^*}(0) = G_{\xi^*}(b) = 0 \). Hence

\[
2n \max_i |G_{\xi^*}(\eta_i(\xi^*)))| \geq 2 \sum_{i=1}^n |G_{\xi^*}(\eta_i(\xi^*)))| = \sqrt{b-a} G_{\xi^*} \geq \frac{n}{2} \int_0^{(b-a)/n} \omega(u)du,
\]

which implies (24) for \( f_t = G_{\xi^*}(t) \). If \( t \) is the uniform partition, then \( |\xi_1^*| = |\xi_{n+1}^*| = b-a/2n \) and \( |\xi_k^*| = b-a/k \) for \( k = 2, \ldots, n \), so that \( \eta_k(\xi^*) = a + 2k-1(b-a), k = 1, \ldots, n \), all inequalities above become equalities, and hence (24) also becomes equality. The lemma is proved. \( \square \)

For the uniform partition, the obtained function \( G_{\xi^*} \) is well known and realizes many extremal properties for the class \( W^1H^\omega([a, b], \mathbb{R}) \), see e.g. [2, Chapter 7.1].