Research Article
Recurrence and Chaos of Local Dendrite Maps

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Let $X$ be a local dendrite, and let $f$ be a continuous self-mapping of $X$. Let $E(X)$ represent the subset of endpoints of $X$. Let $AP(f)$ denote the subset of almost periodic points of $f$, $R(f)$ be the subset of recurrent points of $f$, and $P(f)$ be the subset of periodic points of $f$. In this work, it is shown that $R(f) = AP(f)$ if and only if $E(X)$ is countable. Also, we show that if $E(X)$ is countable, then $R(f) = X$ (respectively, $R(f) = X$) if and only if either $X = S^1$, and $f$ is a homeomorphism topologically conjugate to an irrational rotation, or $P(f) = X$ (respectively, $P(f) = X$). In this setting, we derive that if $E(X)$ is countable, then, on local dendrites $\neq S^1$, transitivity $\Rightarrow$ chaos.

1. Introduction

A metric space that is compact and connected is called a continuum. An arc is any space homeomorphic to the compact interval $[0, 1]$. A topological space is arcwise connected if any two of its points can be joined by an arc. A circle is any space homeomorphic to a simple closed curve.

A continuum that is locally connected and uniquely arcwise connected is called a dendrite. In this setting, every two distinct points $x$ and $y$ can be joined by a single arc denoted by $[x, y]$. Note that, in a dendrite, any subcontinuum is also a dendrite ([1], Corollary 10.6).

Let $x$ be a point of a dendrite $D$. We consider the order of $x$ in $D$ as defined in [1], Definition 9.3, p. 141. If the order of $x$ in $D$ is finite, then it is equal to the number of connected components of $D\setminus\{x\}$. If the order of $x$ in $D$ is infinite, then it is countable and the diameters of connected components of $D\setminus\{x\}$ tend to zero ([2], (2.6), p. 92). By an endpoint of $D$, we mean a point of order 1. We denote $E(D)$ the subset of all endpoints of $D$. A point $x \in D\setminus E(D)$ is called a cut point (see ([1], Theorem 10.7, p. 168)). A branching point of $D$ is a point of order $\geq 3$. The set of all branching points will be denoted by $B(D)$. According to [1], Theorem 10.23, $B(D)$ is at most countable.

A local dendrite is a continuum such that every point of it has a dendrite neighborhood. We consider that $X$ is a local dendrite. The point $x \in X$ is a branching point if it has a closed neighborhood $U$, which is a dendrite, and $x$ is a branching point of $U$. The set of all branching points of $X$ will be denoted by $B(X)$. According to Theorem 10.23 of [1] and Theorem 4 of [3], $B(X)$ is at most countable. A point $a \in X$ is an endpoint of $X$ if it admits an arc neighborhood $U$ in $X$ and $U\setminus\{a\}$ is connected. $E(X)$ is the set of endpoints of $X$.

Recall that a graph is a continuum, which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints (i.e., it is a one-dimensional compact connected polyhedron).

Let $\mathbb{N}$ be the set of positive integers. A map is a continuous function. If $X$ is a topological space, then a self-mapping is a map from $X$ to itself.

Let $f$ denote a self-mapping of a continuum $X$ and $x \in X$. The orbit of the point $x$ is $O(x, f) = \{f^n(x): n \in \mathbb{N}\}$. The point $x$ is periodic if $f^n(x) = x$ for some nonzero integer $n$. The subset of all periodic points of $f$ will be represented by $P(f)$. The point $x$ is recurrent if there exists an increasing sequence $(m_n) \in \mathbb{N}$ verifying $f^{m_n}(x)$ converges to $x$. Note that every iterate of a recurrent point is also
The subset of all recurrent points of $f$ will be represented by $R(f)$. The point $x$ is almost periodic if there exists a set $U$ containing $x$ and an integer $N$ such that, for all $n \in \mathbb{N}$, $\{f^{n+i}(x) : i = 0, 1, \ldots, N\} \cap U \neq \emptyset$. The subset of all almost periodic points of $f$ will be represented by $AP(f)$. By definition, we get the following inclusions:

$$P(f) \subset AP(f) \subset R(f).$$

Such inclusions are not reversible. We say that $f$ is as follows:

(i) Pointwise periodic if $P(f) = X$
(ii) Pointwise recurrent if $R(f) = X$
(iii) Relatively recurrent if $R(f) = X$

A compact metric space $X$ has the APR property if, for every continuous self-mapping $f$ of $X$, we get the equality $AP(f) = R(f)$ (see ([4], Definition 1.1)).

In [5], Lemma 3.1, it was shown that the graph has the APR property. A dendrite $D$ has the APR property if and only if $E(D)$ is countable ([16], Theorem 1.2).

In this study, we show that a local dendrite map $f$ has the APR property if and only if $E(X)$ is countable (see Theorem 4).

Many works studied maps such that $P(f)$ or $R(f)$ verifies additional properties. For example, [7–9] studied homeomorphisms satisfying $P(f) = X$. In [10], it is proved that, for interval map, if $P(f)$ is closed then $P(f)$ is equal to the set non-wandering points $\Omega(f)$. For graph maps, Mai proved in [11], Theorem 4.4, the following theorem.

**Theorem 1.** $R(f)$ is the whole graph $X$ if and only if either $X = S^1$, and $f$ is a homeomorphism topologically conjugate to an irrational rotation, or $f$ is a periodic homeomorphism.

Also, for graph maps, Hattab proved in [5], Main Theorem, the following theorem.

**Theorem 2.** $R(f)$ is the whole graph $X$ if and only if exactly one of the following properties holds:

(a) $X = S^1$, and $f$ is a homeomorphism topologically conjugate to an irrational rotation

(b) $P(f) = X$

For a dendrite with countable set of endpoints, Naghmouchi ([12], Theorem 1.6) proved that a self-mapping is pointwise recurrent if and only if it is a pointwise periodic homeomorphism.

For monotone local dendrite maps, in [4], Theorems 5.1 and 5.4, it was shown that $R(f) = X$ if and only if $R(f) = X$. The following result is an immediate of [4], Theorems 5.1 and 5.4.

**Theorem 3.** $R(f)$ is the whole local dendrite $X$ if and only if exactly one of the following properties holds:

(a) $X = S^1$, and $f$ is a homeomorphism topologically conjugate to an irrational rotation

(b) Every cut point is periodic

By item (b) of Theorem 3 and the fact that the set of cut points is dense, $P(f) = X$.

In this study, we show that for a local dendrite map $f$: $X \rightarrow X$, such that $E(X)$ is countable, the condition that $f$ is monotone is not essential in Theorem 3 (see Theorem 6).

Let $f$ be a self-mapping of a compact metric space without isolated points $(X, d)$. We say that $f$ is a transitive map if there is $x_0 \in X$ such that $\overline{f}(x_0, f) = X$. The map $f$ is sensitive if there is a positive number $a$ with $\forall x \in X$, and for every neighborhood $U$ of $x$, there exist $y \in U$ and an integer $n$ satisfying $d(f^n(x), f^n(y)) > a$. The map $f$ is chaotic in the sense of Devaney if $f$ is transitive, $P(f) = X$, and $f$ is sensitive [13]. To simplify notation, in this study chaotic means chaotic in the sense of Devaney. By [14], the transitivity and the density of periodic points imply the sensitivity condition. Also, it is easy to see that a transitive map satisfies $R(f) = X$. Thus, on all compact metric space without isolated points satisfying the PR property: $P(f) = R(f)$ (intervals, trees, dendrites, Warsaw circle), and transitivity $\Rightarrow$ chaos ([15, 16]).

For a graph $X \neq S^1$, by item (b) of Theorem 2, a self-mapping $f$ of $X$ is chaotic if and only if $f$ is transitive.

For a local dendrite $X \neq S^1$, by item (b) of Theorem 3, a monotone self-mapping $f$ of $X$ is chaotic if and only if $f$ is transitive.

In this study, it is shown that on local dendrites $\neq S^1$, transitivity $\Rightarrow$ chaos (see Theorem 7).

### 2. Recurrence and Almost Periodicity on Local Dendrites

Throughout this section, the letter $X$ denotes a local dendrite. In [3], Theorem 4, p. 303, it is proved that the following conditions are equivalent:

(a) $X$ is a local dendrite.

(b) $X$ is a locally connected continuum, which contains at most a finite number of circles.

(c) $X$ is a continuum and there exist a finite number of dendrites $D_1$, $D_2$, $D_k$ such that $X = D_1 \cup D_2 \cup \cdots \cup D_k$, and for every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, the set $D_i \cap D_j$ is finite.

If $S_1$, $S_2$, $S_3$, are the circles in the local dendrite $X$, then $\Gamma(X)$ is the intersection of all the subgraphs in $X$ containing the union of $S_i$, $S_i$. Therefore, $\Gamma(X)$ is the smallest graph containing all circles of the local dendrite $X$.

**Lemma 1.** Let $X$ be a local dendrite. Then, for any connected component $C$ of $X \setminus \Gamma(X)$, $C \cap \Gamma(X)$ is reduced to a branching point.

**Proof.** By [17], Lemma 2.9, any arc in $X$ with endpoints $x, y \in \Gamma(X)$ is included in $\Gamma(X)$. Thus, by [17], Lemma 2.11, for any connected component $C$ of $X \setminus \Gamma(X)$, $C \cap \Gamma(X)$ is reduced to a point if $C \neq B(\Gamma(X))$. Then there exists a unique arc $[z, y]$ in $\Gamma(X)$ emanating from $z$ and $y \in B(\Gamma(X))$. 

Therefore, $\Gamma(X) \setminus (a, b]$ is a graph of $X$, which contains all circles and smaller than $\Gamma(X)$, a contradiction. This ends the proof.

Set $X \setminus \Gamma(X) = \cup_{x \in S} C_x$, where $C_x$ is the connected components of $X \setminus \Gamma(X)$. By Lemma 1, for any $i \in \mathcal{A}$, $C_x \cap \Gamma(X)$ is reduced to a branching point $z_i$. Since $B(X)$ is at most countable and the order in $X$ is at most countable, the set $\mathcal{A}_k$ is at most countable. Let $\mathcal{A}_k$ be a subset of $\mathcal{A}$ such that, for each $i \in \mathcal{A}_k$, $C_x \cap \Gamma(X) = \{z_i\}$. Put $\mathcal{C}_k = \cup_{i \in \mathcal{A}_k} C_i$. Since $\Gamma(X)$ contains all circles of $X$, by [17], Lemma 2.9, we obtain that $(\mathcal{C}_k)_k$ is pairwise disjoint sub-dendrites of $X$. Suppose now that $X \notin \Gamma(X)$. Let $A$ be an arc contained in $X \setminus \Gamma(X)$. The symbols $\mathcal{A}_k$, $A$, and $\partial A$ denote the closure, the interior, and the endpoints of $A$ as an arc, respectively.

We define the mapping $r_A: X \rightarrow A$ as follows.

Since $A$ is an arc contained in $X \setminus \Gamma(X)$, there exists $i \in \mathcal{A}_k$ such that $A \subset C_i$. Since $C_i$ is a dendrite, by [1], Lemma 10.25, there exists a retraction $r_i: C_i \rightarrow A$ such that for all $p \in C_i$:

$$r'(p) = \text{the single point } a \in A \text{ with } [a, p) \cap A = [a] .$$

The map $r_i: X \rightarrow C_i$ such that the restriction of $r_i$ to $C_i$ is the identity and $r(X \setminus C_i) = \{z_i\}$ is a retraction. We put $r_A = r' \circ r_i$.

It is easy to see that $r_A$ is a retraction from $X$ onto $A$. Suppose that $r_A(z_i) \in \partial A$.

Note that any arc $A$ is ordered by a standard total order $\prec$. Consequently, we get a preorder in $X$ defined by $p \prec q$ if and only if $r_A(p) < r_A(q)$.

The main achievements of this section are to prove the following theorem.

**Theorem 4.** A local dendrite $X$ has the APR property if and only if $E(X)$ is countable.

If $X = \Gamma(X)$, then, by [5], Lemma 3.1, $X$ has the APR property. If $X \notin \Gamma(X)$, then we need the following lemmas.

**Lemma 2.** Let $X$ be a local dendrite and $f$ denote a continuous self-map of $X$. If there exists $p \in R(f) \setminus \overline{f(\mathbb{F})}$ such that $O(p, f) \cap (X \setminus \Gamma(X)) \neq \emptyset$, then $E(X)$ is uncountable.

Note that the proof of Lemma 2 extends [18], Lemmas 2–4, and the proof of [18], Theorem 2, shown originally for dendrites, to the case of local dendrites.

**Proof.** Let $X$ be a local dendrite. Assume that $A \subset X \setminus \Gamma(X)$ is an arc. Let $i \in \mathcal{A}_k$, and let $C_i$ be a connected component of $X \setminus \Gamma(X)$ such that $A \subset C_i$ and $C_i \cap \Gamma(X) = [z_i]$. Consider the retraction $r_A = r' \circ r_i$ from $X$ onto $A$, and recall that $C_i$ is a dendrite.

We start by the following claims.

**Claim 1.** Let $\varepsilon > 0$ and pick $p \in A$. Thus, there is $\delta > 0$ satisfying $r_A^{-1}(A \cap B(\delta, p) - \{p\})$ contained in $B(\varepsilon, p)$, where $B(\delta, p)$ is the open ball of radius $\delta$ centered at $p$.

Let $A = [x, y]$. Let $p \in A$, and let $\varepsilon > 0$. Since $C_i$ is a dendrite, by [18], Lemma 2, there is $\alpha > 0$ with $(r')^{-1}((A \cap B(\alpha, p) - \{p\}))$ contained in $B(\varepsilon, p)$. For $\delta = \inf\{a, (d(p, y)/3), (d(p, x)/3)\}$, we obtain

$$r_A^{-1}(A \cap B(\delta, p) - \{p\}) \subset (r')^{-1}(A \cap B(\alpha, p) - \{p\}) \subset B(\varepsilon, p).$$

This proves Claim 1.

**Claim 2.** Let $p_0 < q_0$ denotes two points in $A$ such that $p_0 < r_A(f(p_0))$ and $r_A(f(q_0)) < q_0$. Hence, there is a fixed point $z$ of $f$ satisfying $p_0 < r_A(z) < q_0$.

Let $f_i: C_i \rightarrow C_i$ be the restriction of $r_i \circ f$ to $C_i$. Recall that $r_A = r' \circ r_i$. We claim that

$$r'(f_i(p_0)) < p_0,$$

and indeed,

$$r'(f_i(q_0)) < q_0.$$
Consider \( B = [a, b] \subset X \setminus \Gamma (X) \) an arc. Let \( i \in \mathcal{A} \), and let \( C_i \) be a connected component of \( X \setminus \Gamma (X) \) such that \( B \subset C_i \) and \( \overline{C_i} \cap \Gamma (X) = \{ z_i \} \). Suppose that \( B \cap S \) contains a point, which is not isolated. Consequently, \( B \cap S \neq \emptyset \). Pick \( x = f^n (p) \in B \cap S \).

**Case 1.** We suppose that \( r_B (z_i) \) is endpoint of \( B \).

The fact that \( \mathcal{S} \subset R (f) \setminus P (f) \) implies that there is \( \varepsilon > 0 \) with \( B (x, \varepsilon) \cap P (f) = \emptyset \). By Claim 1, there is \( 0 < \alpha = \inf \{(d (x, a))/3, (d (x, b))/3\} \) satisfying \( r_B (B \cap B (x, \alpha))/\{x\}) \subset B (x, \varepsilon) \).

Since \( f^{n} (p) \) is a non-isolated point of \( B \cap S \), we can pick \( m > n \) verifying \( f^{m} (p) \in (B \cap B (x, \alpha))/\{x\}) \) and the arc \( \{x, f^{m} (p)\} \subset B (x, \alpha) \) (note that local dendrites are uniformly locally arcwise connected). Assume, for example, that \( x < f^{m} (p) \). Let \( g = f^{m-n} \). Note that \( g \colon X \longrightarrow X, g (x) \in B \), and \( x < g (x) \).

Using an inductive argument, over \( k \), we will show that

\[
y \leq r_B (g^k (y)), \quad \text{for each } y \in [x, g (x)) \text{ and every } k \in N (\ast). \quad (8)
\]

To prove the initialization of \( (\ast) \), we argue by contradiction; indeed, we assume that there is \( y \in [x, g (x)) \) satisfying \( r_B (g^k (y)) < y \). Since \( x < g (x) = r_B (g (x)) \) and \( x \neq y \), by Claim 2, there is a fixed point \( z \) of \( g \) satisfying \( x < r_B (z) < y \leq g (x) \). Thus, \( z \in r_B (B \cap B (x, \alpha)/\{x\}) \subset B (x, \varepsilon) \). Hence, \( z = f^{m-n} (z) \) implies that \( z \in P (f) \). This leads to a contradiction with the selection of \( \varepsilon \in (B (x, \varepsilon) \cap P (f) = \emptyset) \) and ends the initialization step of the proof of \( (\ast) \).

Let us now assume that \( y \leq r_B (g^k (y)) \), for all \( y \in [x, g (x)) \), particularly \( g (x) \leq r_B (g^{k+1} (x)) \). Since \( x < g (x) \), we have \( x < r_B (g^{k+1} (x)) \). If there is \( y \in [x, g (x)) \) such that \( r_B (g^{k+1} (y)) < y \), then by the above, \( x < r_B (g^{k+1} (x)) \), we have \( x < y \). By applying Claim 2 to \( g^{k+1} \), there is a fixed point \( z \) of \( g^{k+1} \) satisfying \( x < r_B (z) < y < g (x) \). Since \( g^{k+1} (z) = z \), \( f^{(k+1)(m-n)} (z) = z \). Consequently, \( z \in P (f) \), which leads to a contradiction. Thus, \( y \leq r_B (g^{k+1} (y)) \) for all \( y \in [x, g (x)) \). This ends the proof of \( (\ast) \).

Consequently, \( g (x) \leq r_B (g^k (x)) \) for every \( k \in N \). Therefore, \( r_B^{-1} \{w \in B \colon w < g (x)\} \) is an open neighborhood of \( x \) disjoint from the subset \( \{g^k (x) \colon k \in N\} \). Thus, \( x \notin R (g) = R (f^{m-n}) \), but according to [20], Theorem 1, \( R (f) = R (f^{m-n}) \). Therefore, \( x \notin R (f) \). This leads to a contradiction, since \( x \in S \subset R (f) \). Therefore, for every arc \( B \) in \( X \setminus \Gamma (X) \) such that, for all \( i \in \mathcal{A} \), \( r_B (z_i) \in \partial B \), the set \( B \cap S \) is discrete.

**Case 2.** We suppose now that \( r_B (z_i) = c \in \overline{B} \). Since \( B \cap S \) contains a non-isolated point, \( B \cap S \) is infinite. Therefore, we can fix \( x = f^n (p) \) in the open arcs \( (a, c) \) or \( (c, b) \) and we proceed as the Case 1 to lead to a contradiction.

Thus, for all arc \( B \subset X \setminus \Gamma (X) \), the set \( B \cap S \) is discrete. Finally, a direct application of Claim 3 implies that \( E (X) \) is uncountable.

**Lemma 3.** Retractions preserve the APR property; i.e., let \( Y \) denote a subcontinuum of a continuum \( X \), which is a retract of \( X \). If \( X \) has the APR property, then \( Y \) has also the APR property.

**Proof.** Let \( g \) denote a self-mapping of \( Y \), and \( r \colon X \longrightarrow Y \) be a retraction. The composition \( f = g \circ r \) is a self-mapping of \( Y \). Thus, by [21], Lemma 3.1, for all \( n \in N \), \( f^n = g^n \circ r \).

Therefore, by the proof of [21], Proposition 3.2, \( R (f) = R (g) \). If \( x \in X \setminus Y \), then \( f^n (x) = g^n \circ r (x) \in Y \), and since \( X \setminus Y \) is open, \( x \notin AP (f) \). Thus, \( AP (f) \subset Y \). If \( x \in Y \), then, for all \( n \in N \), \( f^n (x) = g^n (x) \). Therefore, \( AP (f) = AP (g) \). Consequently, if the continuum \( X \) has the APR property, then \( AP (f) = R (f) \), which implies that \( AP (g) = R (g) \). Thus, \( Y \) has the APR property.

The following result is due to [22], Lemma 2.4. In this study, we complete its proof. \( \square \)

**Lemma 4.** Let \( X \) be a graph, and \( C \subset X \) be a Cantor set. Then, there is a self-mapping \( g \) of \( X \) such that \( C \) is the closure of an orbit of \( g \) and \( X \setminus C \) is \( g \)-invariant.

**Proof.** Since \( C \) is homeomorphic to the Cantor ternary set, \( C \) will be totally ordered. For convenience, we may assume that \( 0 = \min C \) and \( 1 = \max C \). By [23], Theorem 2.1, there exists a minimal homeomorphism \( \psi \colon C \{0, 1\} \longrightarrow C \{0, 1\} \) defined on the non-compact locally compact and totally disconnected set \( C \{0, 1\} \). It is easy to see that there exists a unique homeomorphism \( \psi \) of \( C \) whose restriction to \( C \{0, 1\} \) is \( \psi (0) = 0 \) and \( \psi (1) = 1 \). Put \( BE (X) = \{x \in X \colon x \) is a branch point or an end point of \( X \)\}. We define \( h \colon C \cup BE (X) \longrightarrow X \) as follows: \( h (x) = \psi (x) \) if \( x \in C \) and \( h (x) = x \) if \( x \in BE (X) \). It is easy to see that \( h \) is continuous. Therefore, \( X \setminus (C \cup BE (X)) \) is a union of countably many open arcs \( (A_n = (a_{n1}, a_{n2})) \) whose endpoints belong to \( C \cup BE (X) \). By an open arc, we mean here and below always an arc without its endpoints, which is simultaneously an open set in \( X \), and its endpoints belong to \( C \cup BE (X) \), which is disjoint with \( C \cup BE (X) \). Define \( g \colon X \longrightarrow X \) as follows: \( g \). There are only finitely many such arcs. Let \( B_n \) be one with minimal diameter. Let \( r \colon B_n \longrightarrow B_n \) be a homeomorphism sending \( a_{n1} \) to \( g (a_{n1}) \) and \( a_{n2} \) to the other endpoint of \( B_n \). Finally, put \( g (x) = r (x) \) on \( A_n \).

We need to prove that \( g \) is continuous. The proof of the continuity of \( g \) in \( F \cup BE (X) \) is the same as in [22], Lemma 2.4. Now let \( x \in X \setminus (F \cup BE (X)) \). Let \( (x_n) \) be a sequence of \( X \), which converges to \( x \). Let \( B \) be an open arc with endpoints \( a, b \in F \cup BE (X) \) containing \( x \). There exists \( N \in N \) such that, for all \( n \geq N \), \( x_n \in A \). If \( g (a) = g (b) \), then, for all \( n \geq N \), \( g (x_n) = g (a) = g (x) \). If \( g (a) \neq g (b) \), then, for all \( n \geq N \), \( g (x_n) = r (x_n) \). Since \( r \) is a homeomorphism, \( (r (x_n)) \) converges to \( r (x) \). In both cases, \( g \) is continuous. \( \square \)
Proof. Let $X$ be a local dendrite such that $X \setminus \Gamma(X) \neq \emptyset$.

Suppose that $X$ has an uncountable subset of endpoints $E(X)$. Since $E(X) = \cup_{i \in \mathcal{I}} (E(C_i) \cap \Gamma(x_i))$ and $\mathcal{I}$ is countable, there exists $i \in \mathcal{I}$ such that $E(C_i)$ is uncountable. By [6], Theorem 1.2, $C_i$ does not have the APR property. Since $r_i: X \to C_i$ is a retraction, by Lemma 3, $X$ does not have the APR property. Therefore, if $X$ has the APR property, then the set $E(X)$ is countable.

Conversely, suppose that $E(X)$ is countable. Let $f$ be a self-mapping of $X$. If $R(f) \subset AP(f)$, then $R(f) = AP(f)$ and we are done. Assume then that $R(f) \setminus AP(f)$ is nonempty.

Case 1. $(R(f) \setminus AP(f)) \cap (X \setminus \Gamma(X)) \neq \emptyset$

Since $R(f) \setminus AP(f) \subset R(f) \setminus P(f)$, $(R(f) \setminus AP(f)) \cap (X \setminus \Gamma(X)) \neq \emptyset$, and consequently, by Lemma 2, $E(X)$ is uncountable, which leads to a contradiction.

Case 2. $(R(f) \setminus AP(f)) \subset \Gamma(X)$

Let $w \in (R(f) \setminus AP(f))$. Since $R(f)$ and $AP(f)$ are invariant (see [24], Propositions 4.1.2 and 4.2.4)), $O(w, f) \subset (R(f) \setminus AP(f)) \subset \Gamma(X)$. Since $O(w, f)$ is not a minimal set (see [24], Theorem 4.2.2)), there is a nonempty closed $f$-invariant subset $M$ such that $M \subset O(w, f)$. Since $\Gamma(X)$ is closed, the set $F = O(w, f)$ is a closed $f$-invariant subset of $\Gamma(X)$. By Lemma 4, there exists a continuous map $g: \Gamma(X) \to \Gamma(X)$ such that $g$ is an extension of $f: F \to \Gamma(X)$. It is easy to see that $R(f) \subset R(g)$; consequently, $w \in R(g)$. $M$ is also a nonempty closed $g$-invariant subset such that $M \subset O(w, g)$. Thus, $O(w, g)$ is not a minimal set of $g$; consequently, by [24], Theorem 4.2.2, $g \notin AP(g)$.

If $w \in AP(g) \setminus AP(f)$, then there exists a minimal set $M \subset O(w, g)$. We distinguish two subcases.

Case 2.1. $M$ is finite

Let $(a, b)$ be a connected component of $\Gamma(X) \setminus M \cup BE(\Gamma(X))$ containing $w$. Since $M \cup BE(\Gamma(X))$ is a finite subset, the restriction of $g$ to $M$ is a homeomorphism.

Case 2.1. $M$ is infinite

We recall the construction of $g$ given in the proof of [22], Lemma 4. Let $h$ be the function defined by $h(x) = f(x)$ if $x \in F$ and $h(x) = x$ if $x \in BE(\Gamma(X)) \setminus F$, where $BE(\Gamma(X))$ is the set of branching points or endpoints of $\Gamma(X)$. It is easy to see that $h$ is continuous. The set $\Gamma(X) \setminus F \cup BE(\Gamma(X))$ is a union of countably many open arcs $(I_n)_{n \in \mathbb{N}}$ in $\Gamma(X)$ with boundary $\partial I_n$ belonging to $F \cup BE(\Gamma(X))$ (note that $A$ is an open set in $\Gamma(X)$). Define $g: \Gamma(X) \to \Gamma(X)$ as follows. If $x \in F \cup BE(\Gamma(X))$, put $g(x) = h(x)$. Consequently, $F \cup BE(\Gamma(X))$ is $g$-invariant. If $x \in \Gamma(X) \setminus F \cup BE(\Gamma(X))$, there is an open arc $A$ with endpoints $a, b \in F \cup BE(\Gamma(X))$ such that $x \in A$. Consider all arcs in $\Gamma(X) \setminus F \cup BE(\Gamma(X))$ whose endpoints are $g(a)$ and $g(b)$. There are only finitely many such arcs. Let $B$ be one with minimal diameter. Denote by $r$ a homeomorphism $A \to B$ sending $a$ to $g(a)$ and $b$ to $g(b)$.

Finally, put $g(x) = r(x)$ on $A$. It is easy to see that $\Gamma(X) \setminus F \cup BE(\Gamma(X)) = g$-invariant.

If $w \notin AP(g)$, then one of the connected components of $\Gamma(X) \setminus F \cup BE(\Gamma(X))$, say $A$, contains a point $p$ of $AP(g)$ and has $w$ as an endpoint, and the other endpoint is $w \in F \cup BE(\Gamma(X))$. Since $A$ is an open set, there exists $n \in \mathbb{N}$ such that $g^n(p) \in A$. Since $\Gamma(X) \setminus F \cup BE(\Gamma(X))$ is $g$-invariant, $g^n(A)$ is a connected subset of $\Gamma(X) \setminus F \cup BE(\Gamma(X))$, which intersects $A$. Consequently, $g^n(A) \subset A$. Since $A \cap F \cup BE(\Gamma(X)) = \emptyset$, $g^n(A) = A$. Since $g$ is continuous, $g^n(\overline{A}) \subset g^n(A) = A$. Consequently, $g^n(\{w, w\}) \subset A$.

In both cases, we get $R(f) = AP(f)$. \qed

3. Pointwise Recurrent Local Dendrite Map

It was shown, in [4], Theorems 5.1 and 5.4, that for a monotone local dendrite map $f: X \to X$ we get the following equivalence:

$$R(f) = X, \quad \text{if and only if } R(f) = X(\ast).$$

(9)

Note that the equivalence ($\ast$) is false for an arbitrary local dendrite mapping. Indeed, on the interval, the tent map is a transitive nonrecurrent map [25]. Also, by Theorem 5.1 of [26], there exists a transitive nonrecurrent dendrite map. The main results of this section are the following theorems.

Theorem 5. Let $f$ be a self-continuous mapping of a local dendrite $X$ such that $E(X)$ is countable. $R(f) = X$ if and only if exactly one of the following properties holds:

(a) $X = S^1$, and $f$ is a homeomorphism topologically conjugate to an irrational rotation.
Theorem 6. Let \( f \) be a self-continuous mapping of a local dendrite \( X \) such that \( E(X) \) is countable. \( \overline{f}(f) = X \) if and only if exactly one of the following properties holds:

(a) \( X = S^1 \), and \( f \) is a homeomorphism topologically conjugate to an irrational rotation

(b) \( \overline{P}(f) = X \)

We need the following results.

**Proposition 1.** Let \( X \) be a local dendrite, which is not a circle. Then, there exists no minimal self-map of \( X \).

**Proof.** If \( X \not= \mathbb{S}^1 \), then \( X \) has a cut point; by [27], Theorem 4.8, \( X \) does not have a minimal map. \( \square \)

**Proposition 2.** Let \( f : X \rightarrow X \) be a pointwise recurrent local dendrite map, and let \( (p_n)_{n \in \mathbb{N}} \subset X \) be a sequence of periodic points of \( f \) such that \( p_{n+1} \) is in an arc with endpoints \( p_n \) and \( p_{n+2} \) for all \( n \in \mathbb{N} \) and \( \lim_{n \rightarrow \infty} p_n = p_\infty \). Then, \( p_\infty \) is an almost periodic point.

**Proof.** Let \( U \) be a dendrite neighborhood of \( p_\infty \). We may assume without loss of generality that \( U \) contains all points of \( \{p_n\}_{n \in \mathbb{N}} \). Let \( (p_n, p_{n+2}) \subset U \) be the arc in \( U \) with endpoints \( p_n \) and \( p_{n+2} \). Consequently, \( p_{n+1} \in (p_n, p_{n+2}) \subset U \). By [12], Lemma 2.2, the connected component \( U_n \) of \( U \setminus \{p_n, p_\infty\} \) that contains the open arc \( (p_n, p_\infty) \subset U \) satisfies \( \lim_{n \rightarrow \infty} \text{diam}(U_n) = 0 \). Let \( V_n = U_n \cup \{p_n, p_\infty\} \). Let \( N_n \) be the period of the point \( p_n \). By a similar proof of [12], Lemma 2.7, the orbit of the point \( p_\infty \) under the map \( f^{N_n+1} \) is contained in the set \( V_n \). Since \( \lim_{n \rightarrow \infty} \text{diam}(V_n) = \lim_{n \rightarrow \infty} \text{diam}(U_n) = 0 \), \( p_\infty \) will be regularly recurrent, which implies that \( p_\infty \) is almost periodic. \( \square \)

**Proposition 3.** Let \( X \) be a local dendrite with a countable set of endpoints. Then, every sub-local dendrite of \( X \) has a countable set of endpoints.

**Proof.** Let \( Y \) be a sub-local dendrite of \( X \) such that \( E(Y) \) is uncountable. Set \( X \setminus \Gamma(Y) = \bigcup_{j \in \mathcal{A}} C_j \) and \( Y \setminus \Gamma(Y) = \bigcup_{j \in \mathcal{A}} \overline{C}_j \), where \( C_j \) and \( \overline{C}_j \) are the connected components of \( X \setminus \Gamma(Y) \) and \( Y \setminus \Gamma(Y) \), respectively. It is easy to see that \( \Gamma(Y) \subset \Gamma(X) \) and every connected component of \( Y \setminus \Gamma(Y) \) is contained in a connected component of \( X \setminus \Gamma(Y) \). Since \( E(Y) = \bigcup_{j \in \mathcal{A}} (E(\overline{C}_j) \setminus \{z_j\}) \), where \( \{z_j\} = \overline{C}_j \cap \Gamma(Y) \), there exists \( j \in \mathcal{A} \) such that \( E(\overline{C}_j) \) is uncountable. Since there exists \( k \in \mathcal{A} \) such that \( \overline{C}_j \subset \overline{C}_k \), by [12], Lemma 4.4, \( E(\overline{C}_k) \) is uncountable. Since \( E(X) = \bigcup_{j \in \mathcal{A}} (E(\overline{C}_j) \setminus \{z_j\}) \), \( E(X) \) is uncountable. \( \square \)

**Proposition 4.** Let \( X \) be a local dendrite such that \( E(X) \) is countable, \( f : X \rightarrow X \) be a pointwise recurrent continuous map, and let \( W \) be a minimal set of \( f \). If \( W \neq X \), then \( W \) is a periodic orbit of \( f \).

**Proof.** Since \( W \neq X \), \( X \setminus W \) is a nonempty open set. Let \( U \) be a connected component of \( X \setminus W \). Then, \( U \cup \overline{W} \neq X \). It follows from \( U \subset R(f) \) that there exists \( n \in \mathbb{N} \) such that \( f^n(U) \cap U \neq \emptyset \). Consequently, \( f^n(U) \cup U \) is connected, and \( f^n(U) \cap W = \emptyset \) (if a point of \( U \) gets mapped into \( W \), then this point is not recurrent). Therefore, \( f^n(U) \subset U \), and hence, \( f^n(U) \subset W \). This implies \( f^n(U) \subset W \), \( W \subset U \), and \( U \) contains a pointwise non-wandering circle map without periodic points. Therefore, \( f \) is topologically conjugate to an irrational rotation.

(\( \square \))

**Proposition 5.** Let \( X \neq \mathbb{S}^1 \) be a local dendrite, and \( f : X \rightarrow X \) be a pointwise recurrent continuous map. If \( X \) contains a nonperiodic point, then it contains a sub-dendrite with uncountable set of endpoints.

**Proof.** Since \( R(f) = X \), by Theorem 4 and Proposition 4, \( P(f) = X \). Then, by Propositions 2–4 and an analogous proof of [12], Theorem 1.6, we get Proposition 5. \( \square \)

**Theorem 7.** Let \( X \neq \mathbb{S}^1 \) be a local dendrite, and \( f : X \rightarrow X \) be a continuous mapping. If \( E(X) \) is countable, then \( f \) is chaotic in the sense of Devaney if and only if \( f \) is transitive.
Data Availability

There are no data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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