THE REALIZATION SPACE OF AN UNSTABLE COALGEBRA

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Abstract. Unstable coalgebras over the Steenrod algebra form the natural target category for singular homology with prime field coefficients. The realization problem asks whether an unstable coalgebra is isomorphic to the homology of a topological space. We study the moduli space of such realizations and describe a decomposition in terms of cohomological invariants of the unstable coalgebra. This is accomplished by a thorough comparative study of the homotopy theories of cosimplicial unstable coalgebras and cosimplicial spaces.

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1. Introduction

Let $p$ be a prime and $C$ an unstable (co)algebra over the Steenrod algebra $A_p$. The realization problem asks whether $C$ is isomorphic to the (co)homology of a topological space $X$. In case the answer turns out to be affirmative, one may further ask how many such spaces $X$ there are up to homology equivalence. The purpose of this work is to study a moduli space of topological realizations associated with a given unstable coalgebra $C$. We describe a decomposition of it in terms of cohomological invariants of $C$. In fact, the same setup works for rational coefficients. We thus achieve a harmonization of the picture in positive and zero characteristic. As a consequence, this decomposition gives obstruction theories for the existence and uniqueness of topological realizations where the obstructions are defined by

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André-Quillen cohomology classes. These obstruction theories recover and sharpen results of Blanc \cite{Blanc}. A comparison with Bousfield’s paper \cite{Bousfield} is work in progress.

Let us start by briefly putting the problem into historical perspective. In the rational context, at least if one restricts to simply connected objects, such realizations always exist by celebrated theorems of Quillen \cite{Quillen} and Sullivan \cite{Sullivan}. In contrast, there are many deep non-realization theorems known in finite characteristic: some of the most notable ones are by Adams \cite{Adams}, Liulevicius \cite{Liulevicius}, Ravenel \cite{Ravenel}, Hill/Hopkins/Ravenel \cite{HHR}, Schwartz \cite{Schwartz} and Gaudens/Schwartz \cite{Gaudens}.

Moduli spaces parametrizing homotopy types with a given cohomology algebra or homotopy Lie algebra were first constructed in rational homotopy theory. The case of cohomology was treated by Félix \cite{Felix}, Lemaire-Sigrist \cite{Lemaire-Sigrist}, and Schlessinger-Stasheff \cite{Schlessinger-Stasheff}. All these authors relied on the obstruction theory developed by Halperin-Stasheff \cite{Halperin-Stasheff}. The (moduli) set of equivalence classes of realizations was identified with the quotient of a rational variety by the action of a unipotent algebraic group. This description is also familiar from the study of moduli problems in algebraic geometry by the method of geometric invariant theory \cite{GIT}. Moreover, already in the unpublished paper \cite{Schlessinger-Stasheff}, Schlessinger and Stasheff associated to a graded algebra $A$ a differential graded coalgebra $C_A$, whose set of components, defined in terms of an algebraic notion of homotopy, parametrizes the different realizations of $A$. This coalgebra represents a moduli space for $A$ which encodes higher order information.

An obstruction theory for unstable coalgebras was developed by Blanc in \cite{Blanc}. He defined obstruction classes for a coalgebra $C$ and proved that the vanishing of these classes is necessary and sufficient for the existence of a realization. He went on and defined difference classes which distinguish two given realizations. Both kinds of classes live in certain André-Quillen cohomology groups associated to $C$. Earlier, Bousfield \cite{Bousfield} developed an obstruction theory for realizing maps with obstruction classes in the unstable Adams spectral sequence. For very nice unstable algebras, obstruction theories using the Massey-Peterson machinery were also introduced by Harper \cite{Harper} and McCleary \cite{McCleary} already in the late seventies.

In a landmark paper, Blanc, Dwyer and Goerss \cite{Blanc-Dwyer-Goerss} studied the moduli space of realizations of a given Π-algebra over the integers. Again, the components of this moduli space correspond to different realizations. By means of simplicial ($E_2$-) resolutions, the authors showed a decomposition of this moduli space into a tower of fibrations, thus obtaining obstruction theories for the realization and uniqueness problems for Π-algebras. Their work relied on earlier work of Dwyer, Kan and Stover \cite{Dwyer-Kan-Stover} and the use of resolution model categories generalized by Bousfield \cite{Bousfield} later on. Along the same lines, Goerss and Hopkins in \cite{Goerss-Hopkins}, \cite{Goerss-Hopkins2}, and \cite{Goerss-Hopkins3}, treat the following realization problem: given a Hopf algebroid and a commutative algebra in comodules over it, describe the moduli space of $E_\infty$-algebra in spectra whose homology is the former. These results gave rise to some profound applications in stable homotopy theory \cite{Goerss-Hopkins4}.

In this paper, we consider an unstable coalgebra $C$ over the Steenrod algebra and we study the (possibly empty) moduli space of realizations $\mathcal{M}_{\text{Top}}(C)$. We show there exists a decomposition of $\mathcal{M}_{\text{Top}}(C)$ into a tower of fibrations. These fibrations are then classified in terms of spaces whose homotopy groups are related
to André-Quillen cohomology groups of $C$. Our results apply not only to unstable coalgebras at any given prime but also to the rational case.

Following and dualizing the blueprint [7] the decomposition of $\mathcal{M}_{\text{Top}}(C)$ is achieved by means of cosimplicial resolutions and their Postnikov decompositions in the cosimplicial direction. At several steps this translation requires non-trivial input. This accounts in part for the length of the article. Another reason is that the article is essentially self-contained.

We prefer unstable coalgebras and homology to unstable algebras and cohomology because one avoids in this way all kinds of (non-)finiteness issues. In the presence of suitable finiteness assumptions, the two viewpoints can be translated into each other.

Applications, computations and examples will be the subject of a later paper. We also intend to include a comparison between the Schlessinger-Stasheff theory and ours in future work.

We will now describe our main results. Following the work of Dwyer and Kan on moduli spaces in homotopy theory, a moduli space of objects in a certain homotopy theory is defined as the classifying space of a category of weak equivalences. Under favorable homotopical assumptions, it turns out that this space encodes the homotopical information of the spaces of homotopy automorphisms of these objects. We refer to Appendix D for the necessary background and a detailed list of references.

Specifically, fix a prime field $\mathbb{F}$ and an unstable coalgebra $C$ over the corresponding Steenrod algebra. Consider the category whose objects are all spaces with $\mathbb{F}$-homology isomorphic to $C$, as unstable coalgebras, and whose morphisms are the $\mathbb{F}$-homology equivalences. By Dwyer and Kan’s definition space of realizations $\mathcal{M}_{\text{Top}}(C)$ is the nerve of this category. In order to decompose this space into pieces and allow an inductive reconstruction and obstruction theories, the spaces are replaced by cosimplicial resolutions. These resolutions are given by products of Eilenberg-MacLane spaces of type $\mathbb{F}$ in each cosimplicial degree. More precisely, they are fibrant replacements in the resolution model category structure on cosimplicial spaces $cS^G$ with respect to the set of Eilenberg-MacLane spaces. There is a notion of a Postnikov decomposition for such resolutions defined with respect to the cosimplicial direction. In this cosimplicial context, it is given dually by a skeletal filtration and its terms are characterized by connectivity and vanishing conditions which are similar to the classical case but are defined with respect to the (external) simplicial structure of the category of cosimplicial spaces.

A potential $n$-stage for $C$ is then defined to be a cosimplicial space which satisfies these conditions so that it may potentially be the appropriate skeleton of an actual realization. The definition makes sense also for $n = \infty$ giving rise to the notion of an $\infty$-stage. The category whose objects are potential $n$-stages of $C$ defines a moduli space $\mathcal{M}_n(C)$ in $cS^G$ and the skeleton functor induces a map

$$\text{sk}_n : \mathcal{M}_n(C) \to \mathcal{M}_{n-1}(C).$$

The following theorem addresses the relation between the moduli spaces of $\infty$-stages and genuine realizations of $C$:

**Theorem 6.13** Suppose that the homology spectral sequence of the cosimplicial $\infty$-stage for $C$ converges. Then the totalization functor induces a weak equivalence

$$\mathcal{M}_{\infty}(C) \simeq \mathcal{M}_{\text{Top}}(C).$$
This is the case if the unstable coalgebra $C$ is simply-connected, i.e. $C_1 = 0$ and $C_0 = F$.

As a further step towards a decomposition of $\mathcal{M}_\mathcal{X}(C)$ we show:

**Theorem 6.11** Let $C$ be an unstable coalgebra. Then there is a weak equivalence

$$\mathcal{M}_\mathcal{X}(C) \simeq \holim_n \mathcal{M}_n(C).$$

The difference between $\mathcal{M}_n(C)$ and $\mathcal{M}_{n-1}(C)$ is explained in the next theorem. First, one associates to an abelian unstable coalgebra $M$, which is simultaneously a $C$-comodule, a cosimplicial unstable coalgebra $K_C(M, n)$ of “Eilenberg-MacLane type”, as explained in Section 4. This object co-represents André-Quillen cohomology. Then we define André-Quillen spaces to be the (derived) mapping spaces in the model category of cosimplicial unstable coalgebras under $C$:

$$\mathcal{A}Q^n_C(C; M) \coloneqq \map^{\text{der}}_{C(C, C)}(K_C(M, n), cC),$$

where $cC$ denotes the constant cosimplicial object defined by $C$. The homotopy groups of this mapping space yield the André-Quillen cohomology of $C$ with coefficients in $M$. Let us write $C[n]$ for the $C$-comodule and unstable module obtained by shifting $n$ degrees up with respect to its internal grading.

**Theorem 6.8** For every $n \geq 1$, there is a homotopy pullback square

$$\begin{array}{ccc}
\mathcal{M}_n(C) & \rightarrow & B\text{Aut}(C) \\
\downarrow^{\text{sk}_n} & & \downarrow^\Delta \\
\mathcal{M}_{n-1}(C) & \rightarrow & \mathcal{A}Q^{n+2}_C(C; C[n]).
\end{array}$$

As a consequence, we obtain:

**Corollary 6.10** Let $X^\bullet$ be a potential $n$-stage for an unstable coalgebra $C$. Then there is a homotopy pullback square

$$\begin{array}{ccc}
\mathcal{A}Q^{n+1}_C(C; C[n]) & \rightarrow & \mathcal{M}_n(C) \\
\downarrow^{\text{sk}_n X^\bullet} & & \downarrow^{\text{sk}_n} \\
\mathcal{M}_{n-1}(C) & \rightarrow & \mathcal{M}_{n-1}(C).
\end{array}$$

Finally, the bottom of the tower of moduli spaces can be identified as follows.

**Theorem 6.6** There is a weak equivalence $\mathcal{M}_0(C) \simeq B\text{Aut}(C)$.

Before proving these results, we need to develop two main technical tools. The first tool is a cosimplicial version of the spiral spectral sequence, which first appeared in the study of $\Pi$-algebras and pointed simplicial spaces by Dwyer/Kan/Stover [22]. In particular, its structure as a sequence of modules over its zeroth term will be important. The spiral exact sequence establishes a fundamental link between the homotopy theories of cosimplicial spaces and cosimplicial unstable coalgebras.

We give a detailed account of the homotopy theory of cosimplicial unstable coalgebras. This culminates in a homotopy excision theorem for cosimplicial unstable
coalgebras. It leads to the following homotopy excision theorem in $cS^G$, the resolution model category of cosimplicial spaces with respect to Eilenberg-MacLane spaces. It is a new theorem and our second main tool.

**Theorem 5.12 (Homotopy Excision for cosimplicial spaces)** Let

\[ E^0 \longrightarrow X^0 \]

\[ Y^0 \longrightarrow Z^0 \]

\[ f \]

\[ g \]

be a homotopy pullback square in $cS^G$ where $f$ is $m$-connected and $g$ is $n$-connected. Then the square is homotopy $(m+n)$-cocartesian.

A detailed account of this resolution model category will be given in Section 5, but let us simply mention here that a map of cosimplicial spaces is a weak equivalence in $cS^G$ if the induced map on homology is a weak equivalence of cosimplicial unstable coalgebras. The relevant connectivity notion of the theorem can be defined in terms of the cohomotopy groups of cosimplicial unstable coalgebras.

The paper is structured as follows.

Section 2 contains a review of facts about the categories of unstable right modules and unstable coalgebras.

Resolution model categories of cosimplicial objects with respect to injective models $G$ are the subject of Section 3. After setting the scene with a brief review of Bousfield’s work, we introduce the natural and the bigraded homotopy groups of a cosimplicial object with respect to $G$. Several long exact sequences including the all-important spiral exact sequence (Theorem 3.13) are explained here. Then we define (quasi-) $G$-cofree maps and prove some useful factorization lemmas. Finally, we discuss the notion of cosimplicial connectivity with respect to $G$.

Section 4 is concerned with the homotopy theory of cosimplicial unstable coalgebras. The associated model structure is essentially due to Quillen [45] and it is also an example of a resolution model category. After an excursion to cosimplicial comodules over a cosimplicial ring, we discuss the Künneth spectral sequences in this setting (Theorem 4.13), which are dual to the ones shown in [45]. Using these spectral sequences, we obtain a homotopy excision theorem for cosimplicial unstable coalgebras (Theorem 4.17).

Then we recall the definition of André-Quillen cohomology of unstable coalgebras following [45]. We construct certain twisted “Eilenberg-MacLane” objects $K_C(M,n)$ and observe that they co-represent André-Quillen cohomology. Furthermore, we determine the moduli spaces of these objects which will later be needed for the proof of Theorem 5.8. Postnikov decompositions of cosimplicial objects given by their skeletal filtration are considered. We prove, using the excision theorem, that each map in this skeletal tower is a “principal cofibration”, i.e., it is a pushout of an attachment defined by a map from a twisted Eilenberg-MacLane object (see Proposition 4.29). This yields a dual version of $k$-invariants which enter into the obstruction theory. The arguments are greatly simplified in this case given that the maps of the filtration are at least 1-connected; a final subsection analyzes in detail some subtle points of the case where the maps are only 0-connected which may also be of independent interest.
Section 5 is devoted to the resolution model category of cosimplicial spaces with respect to the class of GEMs (generalized Eilenberg-MacLane spaces) of type $\mathbb{F}$. The Künneth theorem for singular homology with field coefficients provides an important link between the topological and the algebraic homotopy theories and its consequences are explained. As an example, the homotopy excision theorem for cosimplicial spaces is deduced from the one for cosimplicial unstable coalgebras (Theorem 5.12). We prove the existence of twisted “Eilenberg-MacLane” objects $L_{C}(M, n)$ in cosimplicial spaces and study their relation with the objects of type $K_{C}(M, n)$ (Proposition 5.24). These new objects appear as layers of Postnikov decompositions of cosimplicial spaces (see Proposition 5.30). Although the homology of an $L$-object is not a $K$-object, these two types of objects are nevertheless similar with respect to their functions in each homotopy theory. Ultimately it is this fact that allows a reduction from spaces to unstable coalgebras.

Section 6 starts with a discussion of potential $n$-stages defined as cosimplicial spaces that approximate an honest realization. A key result which determines when a potential $n$-stage admits an extension to a potential $(n + 1)$-stage is easily deduced from previous results (Theorem 6.4). Then a synthesis of the results obtained so far, in particular, on Postnikov decompositions and the relation between $K$- and $L$-objects, produces our main results about the moduli space of realizations of an unstable coalgebra as listed above.

The paper ends with four Appendices that contain results which are interesting in their own right.

Facts and definitions on cosimplicial objects in Reedy and resolution model categories used throughout the paper are reviewed in Appendix A.

The spiral exact sequence is constructed in Appendix B. This is done in a cosimplicial unpointed resolution model category. It relates the two sorts of homotopy groups which one associates to a cosimplicial object and a class of injectives. Its algebraic structure, an action of its zeroth term, is explored in detail. Finally, the associated spiral spectral sequence is discussed.

In Appendix C we prove a universal algebra description of unstable algebras. The result is used to translate the abstract study of the spiral exact sequence in Appendix B to our concrete setting. After discussing some bad properties of rational cohomology in degree 0 we manage anyhow to define unstable $\mathbb{Q}$-algebras as an algebraic theory analogous to the mod $p$ case. In a final subsection we identify the spiral spectral sequence as the vector space dual of the homology spectral sequence of a cosimplicial space.

In Appendix D, we provide a brief survey of some necessary background material on moduli spaces in homotopy theory. The material is mainly drawn from several papers of Dwyer and Kan with some slight modifications and generalizations on a few occasions in order to fit our purposes in this paper.

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2. Unstable coalgebras

In this section we recall some basic definitions and facts about unstable modules and coalgebras over the Steenrod algebra. For more details see Schwartz [51] and Lannes [36].

2.1. Preliminaries. Fix \( p \in \text{Spec}(\mathbb{Z}) \). Let \( \mathbb{F} = \mathbb{F}_p \) denote the prime field of characteristic \( p \) with the convention that \( \mathbb{F}_0 = \mathbb{Q} \). The category of \( \mathbb{F} \)-vector spaces graded over the non-negative integers will be denoted by \( \text{Vec} \). The degree of a homogenous element \( x \) of a graded vector space \( V \) will be indicated by \( |x| \). We write

\[
\Sigma : \text{Vec} \to \text{Vec}
\]

for the functor which moves up the grading, i.e. \( (\Sigma V)_q = V_{q-1} \). The graded tensor product of graded vector spaces \( V \) and \( W \), defined by

\[
(V \otimes W)_q = \bigoplus_{k+l=q} V_k \otimes W_l,
\]

is a symmetric monoidal pairing on \( \text{Vec} \). Moreover, this pairing is closed in the sense that there exist internal Hom-objects defined as follows

\[
\text{Hom}_{\text{Vec}}(V, W)_q = \text{Hom}_{\text{Vec}}(\Sigma^q V, W).
\]

Let \( A = A_p \) be the Steenrod algebra at the prime \( p \) with the convention that \( A_0 = \mathbb{Q} \), concentrated in degree 0. Since we are going to work with homology instead of cohomology, we reverse the grading on \( A \) and let \( A' \), which consists of cohomology operations raising the cohomological degree by \( i \), sit in degree \( -i \).

We recall the definition of an unstable \( A \)-module.

**Definition 2.1.** An unstable right \( A \)-module consists of

(a) a graded \( \mathbb{F} \)-vector space \( M \in \text{Vec} \),
(b) a homomorphism

\[
M \otimes A \to M
\]

which in addition to the usual module properties also satisfies the following instability conditions:

(i) \( xSq^n = 0 \) for \( p = 2 \) and \( |x| < 2n \)

(ii) \( xP^n = 0 \) for \( p \) odd and \( |x| < 2pn \)

(iii) \( x\beta P^n = 0 \) for \( p \) odd and \( |x| = 2pn + 1 \).

We write \( \mathcal{U} \) for the category of unstable right \( A \)-modules. It is easy to see that \( \mathcal{U} \) is an abelian category. In the rational case \( \mathcal{U} \) is just the category \( \text{Vec} \) of graded rational vector spaces.

For \( M, N \in \mathcal{U} \), the tensor product \( M \otimes N \) of the underlying graded vector spaces can be given an \( A \)-module structure via the Cartan formula. Moreover, the resulting \( A \)-module is an object of \( \mathcal{U} \), i.e. it satisfies the instability conditions.

The categories \( \text{Vec} \) and \( \mathcal{U} \) are connected via an adjunction

\[
\mathcal{U} \dashv \text{Vec}
\]

where the left adjoint is the forgetful functor and the right adjoint sends a graded vector space \( V \) to the maximal unstable submodule of the \( A \)-module \( \text{Hom}_{\text{Vec}}(A, V) \),
with \(A\)-action given by \((f \cdot a)(a') = f(a \cdot a')\). Every object in \(U\) of the form \(F(V)\) is injective and thus it follows that the abelian category \(U\) has enough injectives.

Let \(\text{Coalg}_F\) denote the category of cocommutative, coassociative, counital coalgebras over \(F\). There is a functor

\[
F : \text{Sets} \to \text{Coalg}_F,
\]

which sends a set \(S\) to the free \(F\)-module \(F(S)\) generated by \(S\) with comultiplication determined by \(\Delta(s) = s \otimes s\) and counit by \(\epsilon(s) = 1\) where \(s \in S\). The functor \(F\) admits a right adjoint \(\pi_0\) which sends a coalgebra \((C, \Delta_C, \epsilon)\) to the set of set-like elements, i.e. the elements \(c \in C\) which satisfy \(\Delta_C(c) = c \otimes c\). An object \(C \in \text{Coalg}_F\) is called set-like if it is isomorphic to \(F(S)\) for some set \(S\). The names discrete \([12]\) and group-like \([55]\) are also used in the literature for this concept. Note that there is a canonical isomorphism \(\pi_0 F(S) \cong S\) whence it follows that the category of set-like coalgebras is equivalent to the category of sets.

Let \(\text{Coalg}_{\text{vec}}\) denote the category of cocommutative, coassociative, counital coalgebras in \(\text{Vec}\) with the respect to the graded tensor product. We recall the definition of an unstable coalgebra.

**Definition 2.3.** An unstable coalgebra over the mod \(p\) Steenrod algebra \(A\) consists of an unstable right \(A\)-module \(C\) whose underlying graded vector space is also a cocommutative coalgebra \((C, \Delta_C, \epsilon)\) in \(\text{Coalg}_{\text{vec}}\) such that the two structures satisfy the following compatibility conditions:

1. The comultiplication \(\Delta_C : C \to C \otimes C\) is a morphism in \(U\),
2. The \(p\)-th root map (dual to Frobenius) \(\xi : C_{pn} \to C_n\) satisfies

\[
\xi(x) = x Sq^n \quad \text{for } p = 2 \text{ and } |x| = 2n
\]

\[
\xi(x) = x P^{n/2} \quad \text{for } p > 2, n \text{ even and } |x| = pn.
\]

**Remark 2.4.** The 0-th degree of an unstable coalgebra is set-like as an \(F\)-coalgebra. This is a consequence of the fact that the \(p\)-th root map is the identity in degree 0 by \([23]\) applied for \(n = 0\). A proof in the dual context of \(p\)-Boolean algebras can be found in \([41]\) Appendix. Indeed, this implies the result for finite dimensional \(F\)-coalgebras, by duality, which then extends to all \(F\)-coalgebras since each \(F\)-coalgebra is the filtered colimit of its finite dimensional subcoalgebras.

For \(F = Q\), condition (b) above does not make sense. We have the following definition in this case.

**Definition 2.5.** An unstable coalgebra over \(Q\) is a cocommutative and counital graded \(Q\)-coalgebra \(C_\bullet = \{C_n\}_{n \geq 0} \in \text{Coalg}_{\text{vec}}\) such that \(C_0 \in \text{Coalg}_Q\) is set-like.

Let \(\mathcal{C}A\) be the category of unstable coalgebras over the Steenrod algebra \(A\). The category \(\mathcal{C}A\) is complete and cocomplete and has a generating set given by the finite (\(=\) finite dimensional in finitely many degrees) coalgebras. As these objects are also finitely presentable, the category \(\mathcal{C}A\) is finitely presentable. The categorical product of a pair of objects in \(\mathcal{C}A\) is given by the tensor product of unstable modules endowed with the canonically induced coalgebra structure. Note that the terminal object is given by the unit \(\underline{F}\) of the monoidal pairing, i.e. the unstable module which is concentrated in degree 0 where it is the field \(F\) with the canonical coalgebra structure.

A basepoint of an unstable coalgebra \(C\) is given by a section \(\sigma : \underline{F} \to C\) of the counit map \(\epsilon : C \to \underline{F}\). Evidently, a basepoint is specified by a set-like element of
C. Let $\mathcal{C}A_\ast = \mathbb{F}/\mathcal{C}A$ denote the category of pointed unstable coalgebras. More generally, for a given unstable coalgebra $C$, we denote as usual the associated under-category by $C/\mathcal{C}A$. There is a standard adjunction

$$\mathcal{C}A \xrightarrow{\sim} C/\mathcal{C}A$$

where the left adjoint is defined by taking the coproduct with $C$, i.e. $D \mapsto D \oplus C$ for $D \in \mathcal{C}A$, and the right adjoint is the forgetful functor.

The categories $\mathcal{U}$ and $\mathcal{C}A$ are connected by an adjunction

$$\mathcal{C}A \xrightarrow{\sim} \mathcal{U}$$

where the right adjoint defines the cofree unstable coalgebra of an unstable $A$-module and the left adjoint is the forgetful functor. The composite of the adjunctions $\mathcal{U}$ and $\mathcal{C}A$ are connected by an adjunction

$$J : \mathcal{C}A \xrightarrow{\sim} \text{Vec} : G$$

where $J$ is the forgetful functor and the right adjoint $G$ defines the cofree unstable coalgebra associated with a graded vector space. In general, an unstable coalgebra is called injective if it is a retract of an object of the form $G(V)$. More generally, for $C \in \mathcal{C}A$ there is an induced adjunction between the under-categories

$$J : \mathcal{C}A \xrightarrow{\sim} \text{Vec} : G$$

Next we recall a description of the functor $G$ in terms of Eilenberg-MacLane spaces given in equation (2.12). This provides a fundamental link in the comparison between $\mathcal{C}A$ and the homotopy category of spaces.

### 2.2. Homology and GEMs

Let $\mathcal{S}$ be the category of simplicial sets. The term “space” will always refer to a simplicial set. Let $K(\mathbb{F}, m)$ denote the (fibrant model of an) Eilenberg-MacLane space representing the singular cohomology functor $X \mapsto H^m(X, \mathbb{F})$. We denote the singular homology functor with coefficients in $\mathbb{F}$ by

$$H_\ast : \mathcal{S} \rightarrow \text{Vec}, X \mapsto \bigoplus_{m \geq 0} H_m(X; \mathbb{F}) = H_\ast(X).$$

The graded homology $H_\ast(X)$ of a simplicial set $X$ is naturally an unstable coalgebra where the comultiplication is induced by the diagonal map $X \rightarrow X \times X$ combined with the Künneth isomorphism. Thus $H_\ast$ factors through the forgetful functor $J : \mathcal{C}A \rightarrow \text{Vec},$

$$H_\ast : \mathcal{S} \xrightarrow{\sim} \mathcal{C}A.$$

We note that there are also pointed versions of the above where we consider the unreduced homology of a pointed space as a pointed unstable coalgebra.

The classical Künneth theorem can be stated in the following way.

**Theorem 2.11.** The homology functor $H_\ast : \mathcal{S} \rightarrow \mathcal{C}A$ preserves finite products.

An arbitrary product of spaces of type $K(\mathbb{F}, m)$, for possibly different $m$ but fixed $\mathbb{F}$, is called a generalized Eilenberg-MacLane space or GEM for brevity. Given a graded vector space $V$, we denote the associated GEM by

$$K(V) := \prod_{n \geq 0} K(V_n, n).$$
By classical results of Cartan [14] and Serre, see e.g. Bousfield [12], the unstable coalgebra $H_*(K(V))$ is the cofree unstable coalgebra associated with $V$, i.e.
\[(2.12)\] $G(V) \cong H_*(K(V))$,
and therefore for any $C \in CA$, we have a natural isomorphism
\[(2.13)\] $\text{Hom}_{CA}(C, H_*(K(V))) \cong \text{Hom}_{\text{vec}}(J(C), V)$.
For any GEM $G$ and space $X$ we obtain a natural isomorphism
\[(2.14)\] $[X, G] \cong \text{Hom}_{CA}(H_*(X), H_*(G))$.
Combining the last two isomorphisms, we note that the functor $K : \text{Vec} \to \text{Ho}Sq$ is a right adjoint to the homology functor $H_* : \text{Ho}Sq \to \text{Vec}$. Another direct consequence is the following proposition.

**Proposition 2.15.** The set of objects $\{H_*(K(\mathbb{F}, m))| m \geq 0\}$ is a cogenerating set for the category $CA$. In particular, the following statements for a map $f : C \to D$ in $CA$ are equivalent:

1. $f$ is an isomorphism (respectively, monomorphism).
2. The induced map $\text{Hom}_{CA}(D, H_*(K(\mathbb{F}, m))) \to \text{Hom}_{CA}(C, H_*(K(\mathbb{F}, m)))$ is an isomorphism (respectively, epimorphism) for every $m \geq 0$.

### 2.3. Coabelian objects

We denote by $\mathcal{V}$ the full subcategory of all unstable right $\mathcal{A}$-modules $M$ such that
\[
px^n = 0 \text{ for } |x| \leq 2pn \text{ and } p \text{ odd, or } \\
qx^n = 0 \text{ for } |x| \leq 2n \text{ and } p = 2.
\]
This subcategory is equivalent to the category of coabelian cogroup objects in $CA_*$, i.e. the unstable coalgebras $C$ for which the diagonal $\mathcal{U}$-homomorphism $C \to C \otimes C$ is a morphism in $CA_*$. This condition implies that the coalgebra structure must in fact be trivial. The above additional conditions are forced by Definition 2.3(b) so that the trivial coalgebra structure on $M \otimes \mathbb{F}$ is unstable. These instability conditions imply in particular that $M$ is trivial in non-positive degrees. The category $\mathcal{V}$ is an abelian subcategory of $U$ which has enough injectives (cf. [12, 8.5]).

Given an unstable coalgebra $C$, we denote by $UC$ the category of $C$-comodules in $U$, that is, objects $M \in U$ with a $C$-comodule structure, i.e. a map in $U$
\[
\Delta_M : M \to C \otimes M
\]
which satisfies the obvious comodule properties. Let $VC$ be the subcategory of the $C$-comodules which are in $\mathcal{V}$. This subcategory $VC$ is equivalent to the category of coabelian cogroup objects in the category $C/CA$ and consequently there is an adjunction as follows
\[
\iota_C : VC \rightleftarrows C/CA : A_{BC}.
\]
The left adjoint is defined by $\iota_C(M) = C \oplus M$ whose comultiplication is specified by the comultiplication of $C$ and by
\[
\Delta_M + \tau\Delta_M : M \to (C \otimes M) \oplus (M \otimes C)
\]
where $\tau$ denotes the twist map. This combined comultiplication can also be expressed in terms of derivations: for $M \in VC$ and $D \in C/CA$, there is a natural isomorphism
\[(2.16)\] $\text{Der}_{CA}(M, D) \cong \text{Hom}_{C/CA}(\iota_C(M), D)$.
where the $D$-comodule structure on $M$ is defined by the given map $f : C \to D$.

The right adjoint $Ab_C$ is called the coabelianization functor and carries an object $f : C \to D$ to the kernel of the following map in $UC$:

$$\text{Id} \otimes \Delta_D - (\text{Id} \otimes f \otimes \text{Id})(\Delta_C \otimes \text{Id}) - (\text{Id} \otimes \tau)(\text{Id} \otimes f \otimes \text{Id})(\Delta_C \otimes \text{Id}) : C \otimes D \to C \otimes D \otimes D.$$

This construction generalizes the notion of primitive elements in the relative setting. We recall the definition.

**Definition 2.17.** Let $C \in \text{Coalg}_{\text{Vec}}$ be a pointed graded coalgebra. The sub-comodule of primitive elements is defined to be

$$\text{Prim}(C) = \{x \in C | \Delta(x) = 1 \otimes x + x \otimes 1\}$$

where $1$ denotes the image of the basepoint of $C$ at $1 \in F$.

In the case where $C = F$, the coabelianization functor $Ab_F : CA \to V$ takes a pointed unstable coalgebra $D$ to the sub-comodule of primitives $\text{Prim}(D)$ with the induced unstable $A$-module structure.

3. Resolution model categories

We review the theory of resolution model structures due to Dwyer/Kan/Stover [21] and Bousfield [13]. We are especially interested in the cosimplicial and un-pointed version from [13] Section 12. The introductory Subsection 3.1 provides a short recollection.

In Subsection 3.2 we discuss the properties of natural homotopy groups of cosimplicial objects and their connection with weak equivalences in a resolution model structure via the spiral exact sequence. These tools were introduced in [21, 22] in the “dual” setting of simplicial spaces.

In Subsection 3.3, we investigate properties of cofree maps between cosimplicial objects with respect to a class of objects $G$. The slightly weaker notion of quasi-$G$-cofree maps is introduced. The definition of these maps can be regarded as formally dual to the definition of relative cell complexes and will be of great technical convenience later on.

In the last Subsection 3.4 cosimplicial connectivity is defined and an important factorization of maps of cosimplicial objects is proved in Proposition 3.30.

3.1. Cosimplicial resolutions. This subsection recalls briefly the relevant notions from [13]. (Especially relevant is section 12 of that article where the unpointed theory is outlined.)

Let $M$ be a left proper model category with terminal object $\ast$, and let $M_\ast$ denote the associated pointed model category whose weak equivalences, cofibrations, and fibrations are obtained by forgetting the basepoints. Let $\text{Ho}(M)$ and $\text{Ho}(M_\ast)$ be the respective homotopy categories. Let $\text{Ho}(M)_\ast$ be the slice category $[\ast] \downarrow \text{Ho}(M)$. In [13] Lemma 12.1 it is explained that for left proper model categories the ordinary (derived) loop functor $\Omega : \text{Ho}(M_\ast) \to \text{Ho}(M_\ast)$ yields objects $\Omega^nY$ in $\text{Ho}(M)_\ast$ defined up to isomorphism for each object $Y$ in $\text{Ho}(M)_\ast$. The object $\Omega^nY$ admits a group object structure in $\text{Ho}(M)$ for $n \geq 1$, which is abelian for $n \geq 2$. For $X$ in $\text{Ho}(M)$, we write

$$[X, Y]_n = [X, \Omega^nY] = \text{Hom}_{\text{Ho}(M)}(X, \Omega^nY).$$

Let $G$ be a class of group objects in $\text{Ho}(M)$. Then each $G \in G$, with its unit map, represents an object of $\text{Ho}(M)_\ast$ and thus has an $n$-fold loop object $\Omega^nG$ in
Ho(\mathcal{M}) giving an associated homotopy functor \([-,G]_n\) on Ho(\mathcal{M}) for \(n \geq 0\). A map \(i: A \rightarrow B\) in Ho(\mathcal{M}) is called \(G\)-monic when
\[
i^*: [B,G]_n \rightarrow [A,G]_n
\]
is surjective for each \(G \in \mathcal{G}\) and \(n \geq 0\), and an object \(Y\) in Ho(\mathcal{M}) is called \(G\)-injective when
\[
i^*: [B,Y] \rightarrow [A,Y]
\]
is surjective for each \(G\)-monic map \(i: A \rightarrow B\) in Ho(\mathcal{M}). We denote the class of \(G\)-injective objects by \(G\)-inj. The homotopy category Ho(\mathcal{M}) has enough \(G\)-injectives when every object is the domain of a \(G\)-monic morphism to a \(G\)-injective target.

**Definition 3.1.** A class \(\mathcal{G}\) of group objects in Ho(\mathcal{M}) is called a class of injective models if Ho(\mathcal{M}) has enough \(G\)-injectives and \(G\)-injective covers can be chosen functorially for every object, see [13, 4.2].

We remark immediately that we are going to make stronger assumptions on the objects in \(\mathcal{G}\), see Assumption 3.3.

A morphism (resp. object) in \(\mathcal{M}\) is \(G\)-monic (resp. \(G\)-injective) if its image in Ho(\mathcal{M}) is so. A fibration in \(\mathcal{M}\) is \(G\)-injective if it has the right lifting property with respect to the \(G\)-monic cofibrations in \(\mathcal{M}\). Note that \([X^\bullet,G]\) is a simplicial group for any \(X^\bullet\) in \(c\mathcal{M}\) and \(G \in \mathcal{G}\). A map \(X^\bullet \rightarrow Y^\bullet\) in \(c\mathcal{M}\) is called:

(i) a \(G\)-weak equivalence if for each \(G \in \mathcal{G}\) the induced map
\[
[Y^\bullet,G] \rightarrow [X^\bullet,G]
\]
is a weak equivalence of simplicial groups.

(ii) a \(G\)-cofibration if it is a Reedy cofibration and for each \(G \in \mathcal{G}\) the induced map
\[
[Y^\bullet,G] \rightarrow [X^\bullet,G]
\]
is a fibration of simplicial groups.

(iii) a \(G\)-fibration if for each \(s \geq 0\) the induced map
\[
X^s \rightarrow Y^s \times_{M^s,Y^s} M^s X^s
\]
is a \(G\)-injective fibration.

**Theorem 3.2** (Dwyer-Kan-Stover [21], Bousfield [13]). Let \(\mathcal{M}\) be a left proper model category. If \(\mathcal{G}\) is a class of injective models for Ho(\mathcal{M}), the category \(c\mathcal{M}\) admits a left proper simplicial model structure given by \(G\)-weak equivalences, \(G\)-fibrations, \(G\)-cofibrations and with respect to the external simplicial structure.

For a description of the external simplicial structure we refer to Appendix A.

This is called the \(G\)-resolution model category and will be denoted by \(c\mathcal{M}^G\). We note that if \(\mathcal{G}\) is a set, then it defines a class of injective models ([13, 4.5]) and so the resolution model category exists. A useful observation is that an object is \(G\)-cofibrant if and only if it is Reedy cofibrant.

Theorem 3.2 may be regarded as a vast generalization of a theorem of Quillen [45, II.4] which treats the dual statement in the case of a discrete model category \(\mathcal{M}\) with a set \(\mathcal{G}\) of small projective generators.
3.2. Natural homotopy groups and the spiral exact sequence. We are going to use freely notation and results from Appendices B.

Assumption 3.3. We let $\mathcal{M}$ be a left proper simplicial model category and $\mathcal{G}$ a class of injective models. The objects $G$ in $\mathcal{G}$ are assumed to be abelian group objects in $\text{Ho}(\mathcal{M})$. Each one possesses a unit map $\ast \to G$ in $\text{Ho}(\mathcal{M})$ and we assume that it is given by a map in $\mathcal{M}_\ast$. In other words, we want to be able to regard the objects in $\mathcal{G}$ as objects of $\mathcal{M}_\ast$. Finally, we want all $G$ to be fibrant in $\mathcal{M}$ and $\mathcal{G}$ to be closed under $\Omega$.

Remark 3.4. Some remarks on these assumptions are in order.

(1) The assumption that all $G \in \mathcal{G}$ are fibrant is cosmetic and simplifies notation. We do not want to invoke derived mapping spaces.

(2) The $\mathcal{G}$-resolution model structure is determined by the $\mathcal{G}$-injective objects (13 4.1). If $G$ is $\mathcal{G}$-injective then $\Omega G$ is also $\mathcal{G}$-injective. Hence, the assumption that the set $\mathcal{G}$ is closed under internal loops is not necessary. It is convenient because it avoids going back and forth between $\mathcal{G}$ and the set of $\mathcal{G}$-injective objects.

(3) The assumptions that all $G \in \mathcal{G}$ are strictly pointed and homotopy abelian are technically convenient in the construction of the spiral exact sequence, compare Theorem B.16 and Appendices B. It is likely that they can be relaxed.

(4) A simplicial structure on $\mathcal{M}$ is used in the construction of the spiral exact sequence in Appendix B.

In our case $\mathcal{M}$ will be the category of unpointed simplicial sets and $\mathcal{G}$ will consist of the Eilenberg-MacLane spaces $K(\mathbb{F}, n)$ for $n \geq 0$, where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$.

Given a Reedy cofibrant object $X^\bullet$ in $c\mathcal{M}$ and $G$ in $\mathcal{G}$, we can consider morphisms in the homotopy category of $\mathcal{M}$ degreewise to obtain a simplicial set $[X^\bullet, G]$. It is in fact a simplicial abelian group. For every $s \geq 0$ we have a functor

$$
\begin{align*}
(\text{Ho}(\mathcal{M}^\mathcal{G}))^\text{op} \times \mathcal{G} & \to \text{Ab} \\
(X^\bullet, G) & \mapsto \pi_s[X^\bullet, G].
\end{align*}
$$

These groups detect $\mathcal{G}$-equivalences by definition. Following [21] we introduce another set of invariants. For every $s \geq 0$ we have a functor

$$
\begin{align*}
(\text{Ho}(\mathcal{M}^\mathcal{G}))^\text{op} \times \mathcal{G} & \to \text{Ab} \\
(X^\bullet, G) & \mapsto [X^\bullet, \Omega^s_{\text{ext}}CG]_{c\mathcal{M}^\mathcal{G}} =: \pi_s^G(X^\bullet, G)
\end{align*}
$$

which we call the natural homotopy groups of $X^\bullet$.

Remark 3.7. Cosimplicial objects admit Postnikov decompositions with respect to the natural homotopy groups. This is one reason for introducing them. Let $X^\bullet$ be a Reedy cofibrant object of $c\mathcal{M}$ and $G \in \mathcal{G}$. There is a natural isomorphism

$$
\text{map}(\text{sk}_n X^\bullet, cG) \cong \text{cosk}_n \text{map}(X^\bullet, cG),
$$

which shows that

$$
\pi^G_s(\text{sk}_n X^\bullet, G) \cong \begin{cases} 
\pi^G_s(X^\bullet, G), & \text{if } s < n \\
0, & \text{else.}
\end{cases}
$$

Here, we use that, for a Kan complex $K$, $\text{cosk}_{n+1} K$ is a model for the $n$-th Postnikov section. Thus, the skeletal filtration

$$
\text{sk}_1 X^\bullet \to \cdots \to \text{sk}_n X^\bullet \to \cdots \to X^\bullet
$$
gives a Postnikov filtration with respect to the natural homotopy groups of $X^\bullet$. The skeletal inclusions $\text{sk}_n X^\bullet \to \text{sk}_{n+1} X^\bullet$ are Reedy cofibrations by Lemma A.6. Note, however, that they are usually not $G$-cofibrations. This can be seen by comparing the cofiber with the $G$-homotopy cofiber of the map $i_n: \text{sk}_n X^\bullet \to X^\bullet$. Since $\text{sk}^n X^\bullet$ is a model for the cosimplicial $(n-1)$-Postnikov section, the $G$-homotopy cofiber of $i_n$ has trivial natural homotopy groups in all degrees $< n$. However, the cofiber $C^\bullet$ of $i_n$ has $C^k = \ast$ for all $0 \leq k \leq n$. The contradiction is in degree $n$.

Thanks to Assumption 3.3 we can supply the simplicial set $\text{Hom}_{\mathcal{M}}(X^\bullet, G)$ with a canonical basepoint $X^0 \to \ast \to G$. We then obtain a natural isomorphism
\begin{equation}
\text{Hom}_{\mathcal{M}}(X^\bullet, G) \cong \text{map}^\text{ext}(X^\bullet, cG).
\end{equation}
If $X^\bullet$ is Reedy cofibrant and $G$ is fibrant then this simplicial set is fibrant and there is another natural isomorphism:
\begin{equation}
\pi^G_\ast(X^\bullet, G) = [X^\bullet, \Omega^G\text{map}^\text{ext}]_{c\mathcal{M}} \cong \pi_s \text{map}^\text{ext}(X^\bullet, cG).
\end{equation}

There are various long exact sequences associated with these homotopy groups. First, $G$-homotopy cofiber sequences give rise to a long exact sequence of natural homotopy groups as follows:
\begin{equation}
\pi^G_\ast(Y^\bullet, X^\bullet), G) = \pi_s \text{Hom}_{\mathcal{M}}((Y^\bullet, X^\bullet), (G, *)) \cong \pi_s \text{Hom}_{\mathcal{M}}((C^\bullet, *), (G, *))
\end{equation}
where $C^\bullet$ denotes the canonically pointed cofiber of $i$. We emphasize that here we consider morphisms of pairs degreewise, so for example, the $n$-simplices of $\text{Hom}_{\mathcal{M}}((Y^\bullet, X^\bullet), (G, *))$ are the morphisms $Y^n \to G$ which send $X^n$ to the basepoint of $G$.

**Proposition 3.11.** Let $X^\bullet \to Y^\bullet$ be a $G$-cofibration. Then, for every $G \in \mathcal{G}$, there is a long exact sequence of natural homotopy groups
\begin{align*}
\cdots \to \pi^G_\ast &((Y^\bullet, X^\bullet), G) \to \pi^G_\ast(Y^\bullet, G) \to \pi^G_\ast(X^\bullet, G) \to \pi^G_{\ast-1}((Y^\bullet, X^\bullet), G) \to \cdots \\
\cdots \to \pi^G_0 &((Y^\bullet, X^\bullet), G) \to \pi^G_0(Y^\bullet, G) \to \pi^G_0(X^\bullet, G).
\end{align*}

**Proof.** The induced map $\text{map}(Y^\bullet, cG) \to \text{map}(X^\bullet, cG)$ is a Kan fibration with fiber $\text{Hom}((Y^\bullet, X^\bullet), (G, *))$ at the basepoint. So there is a long exact sequence of homotopy groups as required. \hfill \square

Often there are also long exact sequences for the homotopy groups $\pi^G_\ast[\ast, G]$. The relative groups of $(Y^\bullet, X^\bullet)$ in this case are defined to be the homotopy groups of the simplicial group $[(Y^\bullet, X^\bullet), (G, *)]$ whose $n$-simplices are the homotopy classes of morphisms of pairs $(Y^n, X^n) \to (G, *)$ in the model category of morphisms of $\mathcal{M}$. If $C^\bullet$ is the pointed cofiber of the $G$-cofibration $X^\bullet \to Y^\bullet$ and since $\mathcal{M}$ is left proper by assumption (otherwise simply assume that $X^\bullet$ is degreewise cofibrant), then we have an isomorphism
\begin{equation}
[(Y^\bullet, X^\bullet), (G, *)] \cong [(C^\bullet, *), (G, *)].
\end{equation}

**Proposition 3.12.** Let $X^\bullet \to Y^\bullet$ be a $G$-cofibration. Suppose that
\begin{equation}
[Y^\bullet, G] \to [X^\bullet, G]
\end{equation}
is an epimorphism for every $G \in \mathcal{G}$. Then for every $G \in \mathcal{G}$, there is a long exact sequence
\begin{align*}
\cdots \to \pi^G_\ast &[(Y^\bullet, X^\bullet), (G, *)] \to \pi^G_\ast(Y^\bullet, G) \to \pi^G_\ast(X^\bullet, G) \to \pi^G_{\ast-1}[(Y^\bullet, X^\bullet), (G, *)] \to \cdots
\end{align*}
\[ \cdots \to \pi_0([Y^\bullet, X^\bullet], (G, *)) \to \pi_0[Y^\bullet, G] \to \pi_0[X^\bullet, G] \to 0. \]

**Proof.** Since \( X^\bullet \to Y^\bullet \) is a \( G \)-cofibration, the induced map of simplicial groups

\[ [Y^\bullet, G] \to [X^\bullet, G] \]

is a fibration for every \( G \in \mathcal{G} \). Hence it suffices to identify the fiber of this simplicial map with the simplicial group \([Y^\bullet, X^\bullet], (G, *)\]. There is a long exact sequence of simplicial groups

\[ \cdots [X^\bullet, \Omega G] \to [(Y^\bullet, X^\bullet), (G, *)] \to [Y^\bullet, G] \to [X^\bullet, G] \]

which, by the assumption, simplifies to the required short exact sequences of simplicial groups

\[ 0 \to [(Y^\bullet, X^\bullet), (G, *)] \to [Y^\bullet, G] \to [X^\bullet, G] \to 0. \]

\[ \square \]

One central tool in the context of resolution model structures is the next long exact sequence relating both types of homotopy groups. The notion of \( \mathcal{H} \)-algebra is defined in [B.18] \( \pi_0 \)-modules are defined in [B.21]

**Theorem B.13** (The spiral exact sequence). There is an isomorphism

\[ \pi_0^S(X^\bullet, G) \cong \pi_0[X^\bullet, G] =: \pi_0 \]

of \( \mathcal{H} \)-algebras and a long exact sequence of \( \mathcal{H} \)-algebras and \( \pi_0 \)-modules

\[ \cdots \to \pi_{k-1}^S(X^\bullet, \Omega G) \to \pi_k^S(X^\bullet, G) \to \pi_*[X^\bullet, G] \to \pi_{k-2}^S(X^\bullet, \Omega G) \to \cdots \]

\[ \cdots \to \pi_2[X^\bullet, G] \to \pi_0^S(X^\bullet, \Omega G) \to \pi_1^S(X^\bullet, G) \to \pi_1[X^\bullet, G] \to 0, \]

where \( \Omega \) is the internal loop functor from \( \mathcal{M} \).

The construction of this spiral exact sequence will be deferred to Appendix B. The \( \pi_0 \)-module structure is proved in Theorem B.16. The \( \pi_0 \)-module structure over \( \pi_0 \) on the terms \( \pi_*[X^\bullet, G] \) corresponds to the usual action of the unstable algebra \( \pi_0 H^*(X^\bullet) \) on \( \pi_* H^*(X^\bullet) \) by Corollary C.13. We now examine consequences of the existence of the spiral exact sequence.

**Corollary 3.14.** A map \( X^\bullet \to Y^\bullet \) is a \( G \)-equivalence if and only if it induces isomorphisms \( \pi_s^S(Y^\bullet, G) \to \pi_s^S(X^\bullet, G) \) for all \( s \geq 0 \) and \( G \in \mathcal{G} \).

**Proof.** This follows immediately from the spiral exact sequence by induction in the degree of connectivity over the whole class \( \mathcal{G} \) and using the five lemma. \( \square \)

### 3.3. (Quasi-)\( G \)-cofree maps

Let \( \mathcal{M} \) be a model category and \( \mathcal{G} \) a class of group objects in \( \text{Ho}(\mathcal{M}) \). The class of \( \mathcal{G} \)-cofree maps can be viewed as dual analogues of relative cell complexes where objects of \( \mathcal{G} \) are externally looped and then co-attached to the relative coskeleton. Quasi-\( \mathcal{G} \)-cofree maps are \( \mathcal{G} \)-cofree up to Reedy equivalence. We give here their definitions and immediate consequences. The class of \( \mathcal{G} \)-fibrations admits a useful description as retracts of quasi-\( \mathcal{G} \)-cofree maps.

Let \( \Delta_{\text{id}} \subset \Delta_{\text{surj}} \subset \Delta \) be the subcategories of identity and surjective maps respectively. The restriction functor

\[ \text{Fun}(\Delta_{\text{surj}}, \mathcal{M}) \to \text{Fun}(\Delta_{\text{id}}, \mathcal{M}) \]
has a right adjoint $R$ which is defined objectwise by $(R\{X^i\}_{i \geq 0})_j = \prod_{i \geq j} X^i$ for some sequence $\{X^i\}_{i \geq 0}$ of objects in $\mathcal{M}$.

**Definition 3.15.** A cosimplicial object $X^\bullet$ in $c\mathcal{M}$ is called codegeneracy cofree if its restriction to $\Delta_{\text{surj}}$ is of the form $R(\{M^i\}_{i \geq 0})$ for some $\{M^i\}_{i \geq 0}: \Delta_{\text{id}} \to \mathcal{M}$. A map $f: X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is called codegeneracy cofree if there is a codegeneracy cofree object $F^\bullet$ such that when we regard $f$ as a map in $\text{Fun}(\Delta_{\text{surj}}, \mathcal{M})$ it can be factored into an isomorphism $X^\bullet \cong Y^\bullet \times F^\bullet$ followed by the projection $Y^\bullet \times F^\bullet \to Y^\bullet$.

**Definition 3.16.** An object in $c\mathcal{M}$ is called $G$-cofree if it is codegeneracy cofree on a sequence $(G_i)_{i \geq 0}$ of fibrant $G$-injective objects. A map $f: X^\bullet \to Y^\bullet$ is called $G$-cofree if it is a codegeneracy cofree map, as defined in Definition 3.15, where $F^\bullet$ is a $G$-cofree object.

In [40] Miller calls these maps almost $G$-cofree. The class of $G$-cofree maps is stable under pullbacks. An object $X^\bullet$ is $G$-cofree if and only if the map $X^\bullet \to \ast$ is $G$-cofree. Therefore, $G$-cofree objects are stable under products. The following proposition explains the analogy to relative cell complexes.

**Proposition 3.17.** Suppose $f: X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is $G$-cofree on $(G_s)_{s \geq 0}$. Then for all $s \geq 0$ there are pullback diagrams in $c\mathcal{M}$:

\[
\begin{array}{ccc}
\cosk_s(f) & \to & \hom(\Delta^s, cG_s) \\
\downarrow_{\gamma_{s-1}(f)} & & \downarrow \\
\cosk_{s-1}(f) & \to & \hom(\partial\Delta^s, cG_s)
\end{array}
\]

**Proof.** This is the dual of [29 VII.1.14].

**Proposition 3.18.** Every $G$-cofree map is a $G$-fibration.

**Proof.** For a $G$-cofree map $X^\bullet \to Y^\bullet$ there exists a fibrant $G$-injective object $G_n$ such that the map $X^n \to Y^n$ is isomorphic to the projection $Y^n \times G_n \to Y^n$ for every $n \geq 0$. These are obviously $G$-injective fibrations. It follows easily that all maps $X^n \to Y^n \times_{M=X^\bullet} M^nX^\bullet$ are $G$-injective fibrations.

We follow a standard method by Miller [40, p. 57] to produce a functorial factorization of any map into a $G$-equivariance followed by a $G$-cofree map under some assumptions on $\mathcal{M}$ and $G$.

Consider a monad (or triple) $\Gamma: \mathcal{M} \to \mathcal{M}$ with natural transformations $e: \text{Id} \to \Gamma(X)$ and $\mu: \Gamma^2 \to \Gamma$. For every $X \in \mathcal{M}$, there is an augmented cosimplicial object in $\mathcal{M}$ by setting $(\Gamma^\bullet X)^n = \Gamma^{n+1}X$ and

\[d^i = T_i e T^{n-i}: (\Gamma^\bullet X)^n \to (\Gamma^\bullet X)^{n+1}, \quad s^i = T^i \mu T^{n-i}: (\Gamma^\bullet X)^{n+1} \to (\Gamma^\bullet X)^n\]

with augmentation given by $e$. Applying $\Gamma$ once again yields an augmented cosimplicial object

\[\Gamma(X) \to \Gamma(\Gamma^\bullet(X))\]

with a cosimplicial contraction.

Given an object $Y$ in $\mathcal{M}$ we let $\mathcal{M} \downarrow Y$ denote the category of objects over $Y$. We can obtain a relative version of the monad $\Gamma$ on this slice category. The forgetful functor $\mathcal{M} \downarrow Y \to \mathcal{M}$ has a right adjoint given by $X \mapsto (X \times Y \overset{pr_Y}{\to} Y)$. The monad
Γ can be conjugated by this adjunction and yields a monad in the category \( \mathcal{M} \downarrow Y \). For an object \( f : X \to Y \) in \( \mathcal{M} \downarrow Y \) this is explicitly given by

\[
\Gamma_Y(X \xrightarrow{f} Y) = (\Gamma(X) \times Y \xrightarrow{pr_Y} Y)
\]

We abbreviate \( \Gamma_Y(X \xrightarrow{f} Y) \) by \( \Gamma_Y(X) \). The map \( f \) factors in \( \mathcal{M} \) as

\[
X \xrightarrow{(c,f)} \Gamma(X) \times Y = \Gamma_Y(X) \xrightarrow{pr_Y} Y.
\]

As above, one obtains a cosimplicial object. In particular, from the factorization of the map \( f \) we obtain morphisms

\[
(cX \xrightarrow{c} \Gamma_Y^\bullet(X) \xrightarrow{p} cY)
\]

of cosimplicial objects in \( \mathcal{M} \).

**Assumption 3.20.** We assume that there are functors \( U : \mathcal{M} \to \mathcal{A} \) and \( G : \mathcal{A} \to \mathcal{M} \) such that

1. \( \mathcal{A} \) is an abelian category,
2. the composite \( GU : \Gamma \) is a monad which preserves weak equivalences,
3. for any \( A \) in \( \mathcal{A} \) the object \( G(A) \) is fibrant \( \mathcal{G} \)-injective, and
4. \( G \) preserves products.

**Proposition 3.21.** With Assumption 3.20 every map in \( c\mathcal{M} \) can be factored functorially into a \( \mathcal{G} \)-equivalence followed by a \( \mathcal{G} \)-cofree map.

**Proof.** Prolong the monad \( \Gamma_Y \) to cosimplicial objects by applying it degreewise. Let \( cX^\bullet \) denote the vertically constant bicosimplicial object over \( \mathcal{M} \) with

\[
(cX^\bullet)^{n,m} = X^n
\]

for all \( n, m \geq 0 \). For a map \( f : X^\bullet \to Y^\bullet \) we obtain a version of (3.19) with extra cosimplicial degree

\[
cX^\bullet \xrightarrow{c(f)} \Gamma_Y^\bullet(X^\bullet) \xrightarrow{p(f)} cY^\bullet,
\]

where \( \Gamma_Y^\bullet(X^\bullet) \) is a bicosimplicial object with

\[
\Gamma_Y^\bullet(X^\bullet)^{m,n} = \Gamma_Y^{m,n}(X^n).
\]

Taking the diagonal we reach a functorial factorization

\[
X^\bullet \cong \text{diag } cX^\bullet \xrightarrow{\text{diag } \tilde{c}(f)} \text{diag } \Gamma_Y^\bullet(X^\bullet) \xrightarrow{\text{diag } \tilde{p}(f)} \text{diag } cY^\bullet \cong Y^\bullet.
\]

To prove that \( \tilde{p}(f) \) is \( \mathcal{G} \)-cofree, we use that \( \Gamma = GU \) factors over an abelian category and the observation that any cosimplicial object over an abelian category is codegeneracy cofree. This fact is well known, see e.g. [29, Ex. VII.5.8]. One also needs to check that the diagonal of a bicosimplicial object, which is codegeneracy cofree both horizontally and vertically, is codegeneracy cofree. It follows that \( \tilde{p}(f) \) is \( \mathcal{G} \)-cofree. Finally, one uses that \( G \) preserves products.

It remains to show that the map \( \tilde{c}(f) \) is a \( \mathcal{G} \)-equivalence. We first assume that \( f \) is a map between cosimplicially constant objects. Then we are in the case of (3.19). The induced map of simplicial groups

\[
[\text{diag } \Gamma_Y^\bullet(X^\bullet), G] \to [X^\bullet, G]
\]

has a simplicial contraction and is a simplicial homotopy equivalence for all \( G \in \mathcal{G} \) (see [13, Lemma 7.2]).
For a general map $f$ in $cM$, the statement about $\tilde{\ell}(f)$ follows from the previous case and [13, Lemma 6.9].

This method produces a useful factorization but it has the drawback that it requires additional strong technical assumptions. These assumptions are satisfied, however, in many cases including those we are interested in. Another drawback is that we do not see a way to prove Proposition [3.30] using just $G$-cofree maps. This leads us to weakening the notion of $G$-cofree to the notion of quasi-$G$-cofree maps introduced below. We then proceed by proving a similar factorization statement (Proposition [3.26]) as above assuming only right properness instead.

**Definition 3.22.** A map $f: X' \to Y'$ in $cM$ is called quasi-$G$-cofree if there is a sequence of fibrant $G$-injective objects $(G_s)_{s \geq 0}$ such that for all $s \geq 0$, there are homotopy pullback diagrams

$$
\begin{align*}
\cosk_s(f) & \longrightarrow \hom(\Delta^s, cG_s) \\
\gamma_{s-1}(f) & \downarrow \quad \downarrow \\
\cosk_{s-1}(f) & \longrightarrow \hom(\Delta^s, cG_s)
\end{align*}
$$

in $cM$ equipped with the Reedy model structure.

Quasi-$G$-cofree maps are useful because usually one exploits the fact that the square is a homotopy pullback rather than a pullback. The notion seems more appropriate than the notion of $G$-cofree when $M$ has a more complicated model structure than the discrete one because it is invariant under Reedy equivalences of Reedy fibrant objects, see e.g. the proof of Proposition [3.30]. If the model structure is discrete both notions coincide.

**Lemma 3.23.** Let $M$ be a right proper model category and $f: X^\bullet \to Y^\bullet$ a Reedy fibration in $cM$. Suppose that for each $s \geq 0$ there are fibrant $G$-injective objects $G_s$ together with a factorization

$$
X^s \xrightarrow{=} Y^s \xrightarrow{\pi} Y^s \times_{M^sY^\bullet} M^sX^\bullet
$$

Then $f$ is a quasi-$G$-cofree map.

**Proof.** For every map $f: X^\bullet \to Y^\bullet$ set

$$
M^sf = Y^s \times_{M^sY^\bullet} M^sX^\bullet.
$$

Let us abbreviate $\hom^{\text{ext}}(K, -)$ by $(-)^K$. Then there are pullback squares:

$$
\begin{align*}
\cosk_s(f) & \longrightarrow (cX^s)^{\Delta^s} \\
\gamma_{s-1}(f) & \downarrow \quad \downarrow \\
\cosk_{s-1}(f) & \longrightarrow (cM^sf)^{\Delta^s} \times (cM^sf)^{\Delta^s} (cX^s)^{\Delta^s}
\end{align*}
$$

(3.24)

If $f$ is a Reedy fibration, then so is the map $cX^s \to cM^s(f)$ by Lemma [A.3]. It follows by the partial compatibility of the Reedy model structure with the external simplicial enrichment in Lemma [A.13] that the vertical map on the right is a Reedy
fibration. Since $\mathcal{M}$ is right proper, this pullback square is a homotopy pullback. In a right proper model category homotopy pullbacks are homotopy invariant. Thus, by replacing $X^*$ with the factorization required in the statement we retain a homotopy pullback square. At the same time we achieve simplifications on the right hand side of diagram (3.24):

$$
\begin{array}{ccc}
(cM^s f)^{\Delta^s} \times_{(cM^s f)^{\Delta^s}} (cX^s)^{\hat{\Delta}^s} & \cong & (cM^s f)^{\Delta^s} \times_{(cM^s f)^{\hat{\Delta}^s}} (c(M^s f \times G_s))^\hat{\Delta}^s \\
\cong & (cM^s f)^{\Delta^s} \times (cG_s)^{\hat{\Delta}^s} &
\end{array}
$$

Projecting away from the term $(cM^s f)^{\Delta^s}$ one obtains a homotopy pullback square:

$$
\begin{array}{ccc}
(cX^s)^{\Delta^s} & \rightarrow & (cG_s)^{\Delta^s} \\
\downarrow & & \downarrow \\
(cM^s f)^{\Delta^s} \times_{(cM^s f)^{\hat{\Delta}^s}} (cX^s)^{\hat{\Delta}^s} & \rightarrow & (cG_s)^{\hat{\Delta}^s}
\end{array}
$$

Concatenation with diagram (3.24) yields the desired homotopy pullback. □

**Corollary 3.25.** Let $\mathcal{M}$ be a proper model category and $\mathcal{G}$ a class of injective objects. The various notions are connected as follows:

1. A composition of a trivial Reedy fibration with a $\mathcal{G}$-cofree map is quasi-$\mathcal{G}$-cofree.
2. A $\mathcal{G}$-cofree map is quasi-$\mathcal{G}$-cofree.
3. A Reedy fibration which is quasi-$\mathcal{G}$-cofree is a $\mathcal{G}$-fibration.

**Proof.** Assertion (1) follows directly from Lemma 3.23. Part (2) is a special case of (1). For part (3) note that the map

$$(cG_s)^{\Delta^s} \rightarrow (cG_s)^{\hat{\Delta}^s}$$

is a $\mathcal{G}$-fibration by Lemma A.4 and compatibility of the external simplicial structure with the $\mathcal{G}$-resolution model structure. Let $f$ be quasi-$\mathcal{G}$-cofree. This means that in the diagram of Definition 3.22 the map

$$\gamma_{s-1}(f) : \cosk_s(f) \rightarrow \cosk_{s-1}(f)$$

factors into a Reedy equivalence $\alpha : \cosk_s(f) \rightarrow P^*$ followed by a $\mathcal{G}$-fibration $\beta : P^* \rightarrow \cosk_{s-1}(f)$. We factor $\alpha$ further into a trivial Reedy cofibration $\delta$ followed by a trivial Reedy fibration $\varphi$. But $f$ is supposed to be a Reedy fibration. So one easily finds that $\gamma_{s-1}(f)$ is a retract of $\varphi \circ \beta$ using the dotted lift in the following diagram:

$$
\begin{array}{ccc}
\cosk_s(f) & \rightarrow & \cosk_s(f) \\
\downarrow & \delta & \downarrow \\
Z^* & \rightarrow & \cosk_{s-1}(f) \\
\varphi \circ \beta & \gamma_{s-1}(f)
\end{array}
$$

Since $\varphi \circ \beta$ is a $\mathcal{G}$-fibration, so is $\gamma_{s-1}(f)$ and, by induction, $f$. □

Here is the promised second method to factor maps.

**Proposition 3.26.** Let $\mathcal{M}$ be a proper model category and $\mathcal{G}$ a class of injective objects. Every map in $c\mathcal{M}$ can be factored functorially into a trivial $\mathcal{G}$-cofibration followed by a quasi-$\mathcal{G}$-cofree map.
Let \( f : X^\bullet \to Y^\bullet \) be a map. We construct a cosimplicial object \( Z^\bullet \) and a factorization

\[
X^\bullet \xrightarrow{j} Z^\bullet \xrightarrow{q} Y^\bullet
\]

by induction on the cosimplicial degree. Let \( \gamma : X^0 \to G_0 \) be a functorial \( G \)-monic map into a fibrant \( G \)-injective object. Factorize the map \( (f^0, \gamma) \) in \( \mathcal{M} \)

\[
X^0 \to Z^0 \sim Y^0 \times G_0
\]

into a cofibration followed by a trivial fibration. Because \( X^0 \to G_0 \) is \( G \)-monic it follows that \( X^0 \to Y^0 \times G_0 \) and, in turn, \( X^0 \to Z^0 \) is \( G \)-monic. Suppose we have constructed \( Z^\bullet \) up to cosimplicial degree \( s \). For the inductive step, we need to find a suitable factorization

\[
L^s Z^\bullet \cup_{L^s X^\bullet} X^s \to Z^s \to Y^s \times_{M^s Y^\bullet} M^s Z^\bullet.
\]

Let \( L^s Z^\bullet \cup_{L^s X^\bullet} X^s \to G_s \) be a functorial \( G \)-monic map into a fibrant \( G \)-injective object. Choose a factorization in \( \mathcal{M} \) as before,

\[
L^s Z^\bullet \cup_{L^s X^\bullet} X^s \to Z^s \sim (Y^s \times_{M^s Y^\bullet} M^s Z^\bullet) \times G_s
\]

into a \( G \)-monic cofibration followed by a trivial fibration. This completes the inductive definition of \( Z^\bullet \) and the factorization shown above. By construction, the resulting map \( j : X^\bullet \to Z^\bullet \) has the property that all maps

\[
L^s(j) \to Z^s
\]

are \( G \)-monic cofibrations in \( \mathcal{M} \). This implies that \( j \) is a trivial \( G \)-cofibration [13, Corollary 3.15]. The map \( q : Z^\bullet \to Y^\bullet \) is a Reedy fibration and satisfies the assumptions of Lemma [3.23]. So it is quasi-\( G \)-cofree, as required. 

**Corollary 3.27.** Let \( \mathcal{M} \) and \( G \) be as in Proposition [3.26]. Then every \( G \)-fibration is a retract of a quasi-\( G \)-cofree Reedy fibration. If the model structure on \( \mathcal{M} \) is discrete then every \( G \)-fibration is a retract of a \( G \)-cofree map.

**Proof.** This follows directly from [3.26] \( \square \)

### 3.4. Cosimplicial connectivity

The following generalizes the notion of connectivity of maps to general resolution model categories. The main result of this subsection establishes yet another factorization property which takes into account the cosimplicial connectivity of a map and gives a more controlled way of replacing it by a cofree map.

**Definition 3.28.** Let \( c\mathcal{M}^G \) be a resolution model category and \( n \geq 0 \).

1. A morphism \( f : X^\bullet \to Y^\bullet \) is called **cosimplicially \( n \)-connected** if the induced map

\[
\pi_s^G(Y^\bullet, G) \to \pi_s^G(X^\bullet, G)
\]

is an isomorphism for \( 0 \leq s < n \) and an epimorphism for \( s = n \) for all \( G \in \mathcal{G} \).

2. An object \( C^\bullet \) of \( c\mathcal{M}^G \) is called **cosimplicially \( n \)-connected** if the canonical map \( C^\bullet \to \ast \) is cosimplicially \( n \)-connected.

Note that if a map \( \ast \to C^\bullet \) is cosimplicially \( n \)-connected, then \( C^\bullet \) is cosimplicially \( (n-1) \)-connected, and the converse also holds for pointed objects. In this case, we say that the pointed object \( (C^\bullet, \ast) \) is **cosimplicially \( n \)-connected**. Every pointed object is cosimplicially 0-connected.
Cosimplicial connectivity has to be clearly distinguished and is completely different from connectivity notions internal to the category $\mathcal{M}$. When it is clear from the context that a connectivity statement refers to the cosimplicial direction, rather than an internal direction, we will drop the word “cosimplicially” from the phrase. Obviously, the notion also depends on the class $\mathcal{G}$ which will also be suppressed from the notation. We emphasize that the notion of cosimplicial connectivity is invariant under $\mathcal{G}$-equivalences.

Cosimplicial connectivity can be formulated in several equivalent ways. We state some of them in the following lemma - the restriction to $\mathcal{G}$-cofibrations is, of course, simply for technical convenience.

**Lemma 3.29.** Let $f : X^* \to Y^*$ be a $\mathcal{G}$-cofibration in $c\mathcal{M}^\mathcal{G}$ and $(C^*, *)$ its pointed cofiber. Then the following assertions are equivalent:

(i) The map $f$ is $n$-connected.

(ii) The map $f$ is $0$-connected and $(C^*, *)$ is $n$-connected.

(iii) The map $f$ is $0$-connected and the induced map

$$\pi_s[C^*, G]_\mathcal{M} \to \pi_s[* , G]_\mathcal{M} \cong \begin{cases} G, & \text{for } s = 0 \\ 0, & \text{else} \end{cases}$$

is an isomorphism for $0 \leq s < n$ and an epimorphism for $s = n$ for all $G \in \mathcal{G}$.

(iv) The induced map

$$\pi_s[Y^*, G]_\mathcal{M} \to \pi_s[X^*, G]_\mathcal{M}$$

is an isomorphism for $0 \leq s < n$ and an epimorphism for $s = n$ for all $G \in \mathcal{G}$.

**Proof.** Then consider the cofiber sequence

$$X^* \xrightarrow{f} Y^* \to C^*$$

This gives rise to a homotopy fiber sequence (cf. Proposition 3.11)

$$\text{Hom}_{\mathcal{M}}((C^*, *),(G, *)) \to \text{map}(Y^*, cG) \to \text{map}(X^*, cG)$$

By Proposition 3.11, (i) is equivalent to

(ii)’ The map $f$ is $0$-connected and $\pi_s\text{Hom}_{\mathcal{M}}((C^*, *),(G, *)) = 0$ for all $0 \leq s < n$ and $G \in \mathcal{G}$.

which is clearly equivalent to (ii), by another application of Proposition 3.11 to the map $* \to C^*$. Thus (i) and (ii) are equivalent. The equivalence between (ii) and (iii) follows from comparing the spiral exact sequences of $C^*$ and $*$ and applying the five-lemma. There is a homotopy fiber sequence (cf. Proposition 3.12)

$$[(C^*, o),(G, o)] \to [Y^*, G] \to [X^*, G].$$

By Proposition 3.12, (iv) is equivalent to

(iii)’ The map $f$ is $0$-connected and $\pi_s[(C^*, o),(G, o)] = 0$ for all $0 \leq s < n$ and $G \in \mathcal{G}$.

which is clearly equivalent to (iii), by another application of Proposition 3.12 to the map $* \to C^*$. □
Proposition 3.30. Let $M$ be a proper model category and $G$ a class of injective objects. Then every cosimplicially $n$-connected map $f$ can be written functorially as a composition $qj$ where $j$ is a Reedy cofibration and $q$ is a quasi-$G$-cofree map on objects $(G_s)_{s \geq 0}$ such that $G_s = *$ for $s \leq n$.

The proof will be a modification of the proof of Proposition 3.26.

Proof. Let $f : X^* \to Y^*$ be $n$-connected. Replace $X^*$ Reedy cofibrantly and factor the resulting map to $Y^*$ into a Reedy cofibration $\varphi$ followed by a trivial Reedy fibration $\rho$. Once we have succeeded in factoring $\varphi$ according to the claim we obtain the required factorization for $f$ by taking a pushout:

\[
\begin{array}{ccc}
X^* & \xrightarrow{j} & Z^* & \xrightarrow{q} & Y^* \\
\downarrow \cong & & \downarrow \cong & & \downarrow \rho \\
\aleph X^* & \xrightarrow{\varphi} & \aleph Z^* & \xrightarrow{\kappa} & \aleph Y^*
\end{array}
\]

Since $\alpha$ is a Reedy cofibration and $M$ is left proper the middle vertical map is a Reedy equivalence. Hence, by 2-out-of-3, $j$ is a $G$-equivalence since $\alpha$ is one. The map $q$ is Reedy equivalent to the quasi-$G$-cofree map $\kappa$. Hence, it is itself a quasi-$G$-cofree on the same sequence of objects. Thus, we reduce to the case where $f$ is a Reedy cofibration between Reedy cofibrant objects.

In this special case we will produce a factorization

\[
X^* \xrightarrow{j} Z^* \xrightarrow{q} Y^*
\]

of $f$, as claimed, such that

1. $j$ is a Reedy cofibration and a $G$-equivalence and
2. $q$ is quasi-$G$-cofree on $(G_s)_{s \geq 0}$ such that $G_s = *$ for $s \leq n$.

For $0 \leq s \leq n$ we set $Z^s = Y^s$ and $f^s = j^s$. Suppose we have constructed $Z^s$ up to cosimplicial degree $s - 1 \geq n$. Then we choose functorially a $G$-monic cofibration

\[
\alpha_s : X^s \cup_{L^s X^*} L^s Z^s \to G_s
\]

into a fibrant $G$-injective object and factor the canonical map

\[
\beta_s : X^s \cup_{L^s X^*} L^s Z^s \to (Y^s \times_{M^s Y^*} M^s Z^*) \times G_s
\]

into a cofibration

\[
L^s j : X^s \cup_{L^s X^*} L^s Z^s \to Z^s
\]

followed by a trivial fibration

\[
\gamma_s : Z^s \to (Y^s \times_{M^s Y^*} M^s Z^*) \times G_s.
\]

Since $\alpha_s$ is $G$-monic, so are $\beta_s$ and $L^s j$. We take

\[
M^s q : Z^s \to Y^s \times_{M^s Y^*} M^s Z^*
\]

to be $\text{pr}_{G_s} \circ \gamma_s$. This process yields the factorization of $f$.

The resulting map $q : Z^* \to Y^*$ is quasi-$G$-cofree by Lemma 3.33. Here, we use the right properness of $M$. By construction, it is quasi-$G$-cofree on a sequence $(G_s)_{s \geq 0}$ with $G_s = *$ for $0 \leq s \leq n$. The $m$-th latching map of $j$ is a cofibration for all $0 \leq m \leq n$ because $f$ is a Reedy cofibration and for $m > n$ by construction. So the map $j$ is a Reedy cofibration.
It remains to show that $j$ is a $G$-equivalence. By the connectivity assumption
and Lemma 3.29 there are isomorphisms
\[
\pi_s^j(Y^*, G) = \pi_s^j(Z^*, G) \cong \pi_s^j(X^*, G),
\]
for $0 \leq s < n$ and an epimorphism for $s = n$, because these groups only depend on $Z^m = Y^m$ for $0 \leq m < n$. For every $s > n$ the map $L^s j$ is a $G$-monic cofibration. Thus, for all fibrant $G \in \mathcal{G}$ the induced map
\[
\text{Hom}(Z^*, G) \to \text{Hom}(X^* \cup_{L^s X^*} L^s Z^*, G)
\]
is surjective. This means that every diagram of the form
\[
\begin{array}{ccc}
\partial \Delta^s & \to & \text{Hom}(Z^*, G) = \text{map}(Z^*, cG) \\
\downarrow & & \downarrow \\
\Delta^s & \to & \text{Hom}(X^*, G) = \text{map}(X^*, cG)
\end{array}
\]
admits a dotted lift. In the special case here, where $X^*$ is Reedy cofibrant, $Z^*$ is also Reedy cofibrant and the mapping spaces on the right in the diagram above are Kan complexes. Therefore, the map
\[
\pi_s^j (j): \pi_s^j(Z^*, G) \to \pi_s^j(X^*, G)
\]
is injective for all $s \geq n$. For $s > n$ it is also surjective because in the diagram
\[
\begin{array}{ccc}
\partial \Delta^s & \to & \text{map}(Z^*, cG) \\
\downarrow & & \downarrow \\
\Delta^s & \to & \Delta^s/\partial \Delta^s \to \text{map}(X^*, cG)
\end{array}
\]
and the lift $h_1$ yields a lift $h_2$. This proves the claim that $\pi_s^j (j)$ is an isomorphism for all $s \geq 0$. \hfill \Box

4. Cosimplicial unstable coalgebras

In this section we are going to study the homotopy theory of cosimplicial unstable algebras. First we discuss the resolution model structure on $cCA$ and then a closely related model structure on cosimplicial comodules over a cosimplicial coalgebra. This will be used to prove Künneth spectral sequences dual to the ones shown by Quillen [45, II.6, Theorem 6]. As a consequence, we obtain a homotopy excision result for cosimplicial unstable coalgebras (Theorem 4.17). This is the main result needed for Proposition 4.29 which is the analogue of the “difference construction” for $\Pi$-algebras in the sense of Blanc/Dwyer/Goerss [7].

We also discuss André-Quillen cohomology in the context of unstable coalgebras. Then we construct objects of type $K_{C}(M, n)$, which play the role of twisted Eilenberg-MacLane spaces in this context and also represent André-Quillen cohomology, and end with a careful study of the moduli spaces they define.

4.1. Cosimplicial resolutions of unstable coalgebras. We may regard $CA$ as a proper model category endowed with the discrete model structure. Consider
\[
\mathcal{E} = \{ H_\cdot K(\mathbb{F}, m) | m \geq 0 \}
\]
as a set of injective objects. By Proposition 4.15 a map in $CA$ is $\mathcal{E}$-monic if and only if it is injective. Moreover, every unstable coalgebra can be embedded into a
product of objects from $\mathcal{E}$. An unstable coalgebra is $\mathcal{E}$-injective if and only if it is a retract of a product of objects from $\mathcal{E}$ (cf. [13, Lemma 3.10]).

**Theorem 4.2.** There is a proper simplicial model structure $\mathcal{C}A^E$ on the category of cosimplicial unstable coalgebras, whose weak equivalences are the $\mathcal{E}$-weak equivalences, cofibrations are the $\mathcal{E}$-cofibrations and fibrations are the $\mathcal{E}$-fibrations.

**Proof.** This is an application of Theorem 3.2 or the dual of [45, II.4, Theorem 4]. Right properness will be shown in Corollary [13, 4.4]. □

Some terms from Section 3 can be described more explicitly in $\mathcal{C}A$.

**Remark 4.3.** The underlying model structure on $\mathcal{C}A$ is discrete. This has nice consequences:

1. For all $s \geq 0$ and any $H \in \mathcal{E}$ the Hurewicz maps
   \[ \pi^s_n(C^\bullet, H) \rightarrow \pi^s_n(C^\bullet, H) \]
   are natural isomorphisms. Further, there are natural isomorphisms
   \[ \pi^s_n(C^\bullet, H_\ast K(F, m)) \cong (\pi^s_n(C^\bullet)_m)^\vee \]
   where $m$ on the left denotes the internal grading and $(-)^\vee$ is the dual vector space. This shows that in $\mathcal{C}A$ we deal essentially with only one invariant, namely $\pi^s_n(C^\bullet)$.

2. All objects in $\mathcal{C}A$ are Reedy and hence $\mathcal{E}$-cofibrant.

The cohomotopy groups of a cosimplicial (unstable) coalgebra $C^\bullet$ carry additional structure which is induced from the comultiplication. The graded vector space $\pi^0_n(C^\bullet)$ of a cosimplicial (unstable) coalgebra $C^\bullet$ is again an (unstable) coalgebra. The higher cohomotopy groups $\pi^s_n(C^\bullet)$, for $s > 0$, are $\pi^0_n(C^\bullet)$-comodules where the comodule structure is induced from the unique degeneracy map $C^n \rightarrow C^0$. Moreover, these are coabelian in the sense that, as unstable modules, they are also objects of $\mathcal{V}$ (see the Appendix of [8] for the dual statement).

The normalization functor is an equivalence from $cVec$ to the category $Ch^+(Vec)$ of cochain complexes in $Vec$. It can be used to lift a model structure from the injective model structure on $Ch^+(Vec)$. We recall that the normalization of a cocommutative coalgebra $C^\bullet$ in $cVec$ carries the structure of a cocommutative coalgebra in $Ch^+(Vec)$ which is induced by the comultiplication of $C^\bullet$ and the shuffle map (cf. [45, II, 6.6-6.7]). The model category $Ch^+(Vec)$ has the cohomology equivalences as weak equivalences, the monomorphisms in positive degrees as cofibrations, and the epimorphisms as fibrations. The resulting model structure on $cVec$ is an example of a resolution model category as explained in [13, 4.4]. With respect to this model structure, the adjunction (2.8)

\[ J : \mathcal{C}A^E \rightleftarrows cVec : G \]

is a Quillen adjunction. One derives the following characterizations.

**Proposition 4.4.** A map $f : C^\bullet \rightarrow D^\bullet$ in $\mathcal{C}A$ is

1. an $\mathcal{E}$-equivalence if and only if it induces isomorphisms
   \[ \pi^n(f) : \pi^n(C^\bullet) \cong \pi^n(D^\bullet) \]
   for all $n$, or equivalently, if the induced map between the associated normalized cochain complexes of graded vector spaces is a quasi-isomorphism.
(2) an $\mathcal{E}$-cofibration if and only if the induced map

$$NF : NC^* \to ND^*$$

between the associated cochain complexes is injective in positive degrees.

4.2. Cosimplicial comodules over a cosimplicial coalgebra. In this subsection, we discuss the homotopy theory of cosimplicial comodules over a cosimplicial coalgebra $C^*$ over $\mathbb{F}$. The results here are mostly dual to results about simplicial modules over a simplicial ring due to Quillen [45]. We will mainly be interested in the Künneth spectral sequences obtained in Theorem 4.9.

Recall that Vec is the category of graded $\mathbb{F}$-vector spaces and a cosimplicial coalgebra $C^*$ is a cococommutative, counital coalgebra object in Vec. In particular, every $C^*$ carries a cosimplicial and an internal grading.

Definition 4.5. A cosimplicial $C^*$-comodule $M^*$ is a cosimplicial graded $\mathbb{F}$-vector space together with maps $M^n \to M^n \otimes C^n$ which are compatible with the cosimplicial structure maps and supply $M^n$ with right $C^n$-comodule structure. We denote the category of cosimplicial $C^*$-comodules by Comod$_{C^*}$.

The category Comod$_{C^*}$ is an abelian category whose objects come equipped with a cosimplicial and an internal grading. The forgetful functor $U$ admits a right adjoint

$$U : \text{Comod}_{C^*} \rightleftharpoons \text{Vec} : - \otimes C^*$$

which is comonadic: Comod$_{C^*}$ is a category of coalgebras with respect to a comonad on Vec.

We want to promote this adjunction to a Quillen adjunction by transferring the model structure from Vec to Comod$_{C^*}$. The standard methods for transferring a (cofibrantly generated) model structure along an adjunction do not apply to this case because here the transfer is in the other direction, from right to left along the adjunction. Moreover, Comod$_{C^*}$ is not a category of cosimplicial objects in a model category, so the resulting model category is not, strictly speaking, an example of a resolution model category. We give a direct proof for the existence of the induced model structure on Comod$_{C^*}$, essentially by dualizing arguments of [45].

A map $f : X^* \to Y^*$ in Comod$_{C^*}$ is:

- a weak equivalence (resp. cofibration) if $U(f)$ is a weak equivalence (resp. cofibration) in Vec.
- a cofree map if it is the transfinite composition of an inverse diagram $F : \lambda^p \to \text{Comod}_{C^*}$ for some ordinal $\lambda$,

$$X^* \to \cdots \to F(\alpha + 1) \overset{p_{\alpha}}{\to} F(\alpha) \to \cdots \to F(1) \overset{p_0}{\to} F(0) = Y^*$$

such that for each $\alpha < \lambda$ there is a fibration $q_\alpha : V^*_\alpha \to W^*_\alpha$ in Vec and a pullback square

$$\begin{array}{ccc}
F(\alpha + 1) & \to & V^*_\alpha \otimes C^* \\
\downarrow p_\alpha & & \downarrow q_\alpha \otimes C^* \\
F(\alpha) & \to & W^*_\alpha \otimes C^*
\end{array}$$

and for each limit ordinal $\alpha \leq \lambda$, $F(\alpha) \to \lim_{\kappa < \lambda} F(\kappa)$ is an isomorphism.

(This is equivalent to saying that $f$ is cellular in $(\text{Comod}_{C^*})^\mathrm{op}$ with respect to the class of maps of the form $q \otimes C^*$ where $q$ is a fibration in Vec.)
• a fibration if it is a retract of a cofree map.

**Theorem 4.6.** These classes of weak equivalences, cofibrations and fibrations define a proper simplicial model structure on $\text{Comod}_C$

**Proof.** It is clear that $\text{Comod}_C$ has colimits - they are lifted from $\text{cVec}$. Following [13], it can be shown that $\text{Comod}_C$ is locally presentable and therefore also complete. It can be checked directly that finite limits are lifted from finite limits in $\text{cVec}$ due to the exactness of the graded tensor product. The “2-out-of-3” and retracts axioms are obvious. The two factorization axioms are proved in Lemmas 4.7 and 4.8 below. In both factorizations the second map is actually a cofree map. For the lifting axiom in the case of a trivial cofibration and a fibration, it suffices to consider a lifting problem as follows

$$
\begin{array}{ccc}
A^* & \longrightarrow & V^* \otimes C^* \\
\downarrow j & & \downarrow q \otimes C^*
\end{array}
$$

where $j$ is a cofibration and a weak equivalence and $q$ a fibration in $\text{cVec}$. This square admits a lift because the adjoint diagram in the model category $\text{cVec}$ admits a lift

$$
\begin{array}{ccc}
U(A^*) & \longrightarrow & V^* \\
\downarrow U(j) & & \downarrow q \\
U(B^*) & \longrightarrow & W^* \otimes C^*
\end{array}
$$

For the other half of the lifting axiom, consider a commutative square

$$
\begin{array}{ccc}
X^* & \longrightarrow & E^* \\
\downarrow i & & \downarrow p \\
D^* & \longrightarrow & Y^*
\end{array}
$$

where $i$ is a cofibration and $p$ a fibration and a weak equivalence. By Lemma 4.7 there is a factorization $p = p'j$ into a cofibration and a trivial fibration as follows

$$
E^* \simeq Y^* \oplus (Z^* \otimes C^*) \sim Y^*
$$

where $Z^*$ is weakly trivial in $\text{cVec}$. By the “2-out-of-3” property, $j$ is a weak equivalence. The commutative square

$$
\begin{array}{ccc}
X^* & \longrightarrow & Y^* \oplus (Z^* \otimes C^*) \\
\downarrow j & & \downarrow p' \\
D^* & \longrightarrow & Y^*
\end{array}
$$

admits a lift $h : D^* \rightarrow Y^* \oplus (Z^* \otimes C^*)$ which is induced by an extension of $U(X^*) \rightarrow U(Z^*)$ to $U(D^*)$ in $\text{cVec}$. Moreover, given the half of the lifting axiom
that is already proved, the commutative square

\[
\begin{array}{ccc}
E^\bullet & \xrightarrow{j} & E^\bullet \\
\downarrow p & & \downarrow p \\
Y^\bullet \oplus (Z^\bullet \otimes C^\bullet) & \xrightarrow{p'} & Y^\bullet
\end{array}
\]

admits a lift \( h' : Y^\bullet \oplus (Z^\bullet \otimes C^\bullet) \to E^\bullet \) - which shows that \( p \) is a retract of \( p' \). The composition \( h'' := h' \circ h : D^\bullet \to E^\bullet \) is a required lift for the original square.

Properness is inherited from \( cVec \). The simplicial structure is the external one and is defined dually to the one in [45], see Appendix A. The compatibility of the simplicial structure with the model structure can easily be deduced from the corresponding compatibility in \( cVec \). \qed

**Lemma 4.7.** Every map in \( Comod_{C^\bullet} \) can be factored functorially into a cofibration followed by a cofree map which is also a weak equivalence.

**Proof.** Let \( f : X^\bullet \to Y^\bullet \) be a map in \( Comod_{C^\bullet} \). Choose a functorial factorization in \( cVec \) of the zero map \( U(X^\bullet) \xrightarrow{j} Z^\bullet \xrightarrow{q} 0 \), into a cofibration \( j \) and a trivial fibration \( q \). Consider the adjoint of \( j \), \( i : X^\bullet \to Z^\bullet \otimes C^\bullet \). Then \( f \) factors as

\[
X^\bullet \xrightarrow{(f,i)} Y^\bullet \oplus (Z^\bullet \otimes C^\bullet) \xrightarrow{p} Y^\bullet
\]

where \( p \) is the projection. \( p \) is obviously cofree. It is also a weak equivalence since \( Z^\bullet \otimes C^\bullet \) is weakly trivial. \qed

**Lemma 4.8.** Every map in \( Comod_{C^\bullet} \) can be factored functorially into a trivial cofibration followed by a cofree map.

**Proof.** Let \( f : X^\bullet \to Y^\bullet \) be a morphism in \( Comod_{C^\bullet} \). First, we show that we can factor \( f \) functorially into a cofibration which is injective on cohomotopy groups followed by a cofree map. There is a functorial factorization of \( U(f) \) in \( cVec \)

\[
U(X^\bullet) \xrightarrow{(f,i')} U(Y^\bullet) \oplus W^\bullet \xrightarrow{q} U(Y^\bullet)
\]

where \( i' \) is a cofibration and induces a monomorphism on cohomotopy groups. The map \( q \) is the projection. Then the adjoint map \( i' \) of \( i' \) in \( Comod_{C^\bullet} \) gives the desired factorization:

\[
X^\bullet \xrightarrow{(f,i)} Z^\bullet := Y^\bullet \oplus (W^\bullet \otimes C^\bullet) \xrightarrow{p} Y^\bullet
\]

where \( p \) denotes the projection away from \( W^\bullet \otimes C^\bullet \). Since \( i' \) factors through \( i \), it follows that \( i \) also induces a monomorphism on cohomotopy groups.

Thus it suffices to show that a cofibration \( f : X^\bullet \to Y^\bullet \) which is injective on cohomotopy groups can be factored functorially into a trivial cofibration followed by a cofree map. We proceed with an inductive construction. Set \( i_0 = f \). For the inductive step, suppose that we are given a cofibration \( i_n : X^\bullet \to Z^\bullet_n \) which is injective on cohomotopy in all degrees and surjective in degrees less than \( n \). We claim that there is a functorial factorization of \( i_n \) into a cofibration \( i_{n+1} \), whose connectivity is improved by one, followed by a cofree map.

Let \( W^\bullet \) be the cofiber of \( i_n \) in \( Comod_{C^\bullet} \) (or \( cVec \)) and \( q : Z^\bullet_n \to W^\bullet \) the canonical map. By assumption, we have that \( \pi^s(W^\bullet) = 0 \) for \( s < n \) and \( \pi^s(Z^\bullet_n) \to \pi^s(W^\bullet) \) is surjective with kernel \( \pi^s(X^\bullet) \) for all \( s \geq 0 \). Let \( 0 \xrightarrow{\sim} P(W^\bullet) \to U(W^\bullet) \)
be a functorial (trivial cofibration, fibration)-factorization in $cVec$. Consider the pullback in $Comod_{C^\bullet}$,

$$
\begin{array}{c}
Z_{n+1} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
W^* \\
\end{array} \quad \begin{array}{c}
\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \\
p_n \\
\end{array} \quad \begin{array}{c}
P(W^*) \otimes C^\bullet \\
\end{array}
$$

There is a canonical factorization of $i_n$ through $Z_{n+1}^*$ since the bottom composition is the zero map. We claim that this factorization $i_n = p_n i_{n+1}$ has the desired properties. Obviously $p_n$ is cofree and $i_{n+1}$ is injective on cohomotopy in all degrees and surjective in degrees less than $n$. But since $P(W^*) \otimes C^\bullet$ is weakly trivial and $W^*(n-1)$-connected, there is an exact sequence

$$0 \to \pi^n(Z_{n+1}^*) \to \pi^n(Z_n^*) \to \pi^n(W^*)$$

from which it follows that the monomorphism $\pi^n(X^*) \to \pi^n(Z_{n+1}^*)$ is actually an isomorphism.

By this procedure, we obtain inductively a sequence of maps $(i_{n+1}, p_n)$, $n \geq 0$, as above, such that $i_n = p_n i_{n+1}$. The desired factorization of $f$ is obtained by first passing to the limit

$$\begin{array}{c}
\lim_{\leftarrow n} Z_n \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \\
\end{array} \quad \begin{array}{c}
P \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \\
\end{array} \quad \begin{array}{c}
Y^* \\
\end{array}
$$

and then applying Lemma 1.7 to factor $i = p' j$ into a cofibration $j$ followed by $p'$ which is a cofree map and a weak equivalence. By construction, the map $p$ is cofree and therefore so is $pp'$. The inverse system induced on cohomotopy by the tower of objects $(Z_n^*)$ in $Comod_{C^\bullet}$ satisfies the Mittag-Leffler condition since the maps become isomorphisms eventually. Therefore $\pi^s(\lim_{\leftarrow n} Z_n^*) = \lim_{\leftarrow n} \pi^s(Z_n^*)$, so $i$ is a weak equivalence, and therefore so is $j$. Thus the factorization $f = (pp')j$ has the required properties.

Let $C$ be a cocommutative coalgebra over $F$, $M$ and $N$ two $C$-comodules. The cotensor product is the equalizer of the diagram

$$M \Box_C N \rightarrow M \otimes N \Rightarrow M \otimes C \otimes N$$

where the maps on the right are defined using the two comodule structure maps. A $C$-comodule $M$ is called cofree if it is of the form $V \otimes C$ for some $F$-vector space $V$. $M$ is called injective if it is a retract of a cofree $C$-comodule. We have $(V \otimes C) \Box_C N \cong V \otimes N$. For background material and the general properties of the cotensor product, we refer to Milnor/Moore [11], Eilenberg/Moore [23], and Neisendorfer [43], and Doi [15].

The definition of the cotensor product extends pointwise to the context of a cosimplicial cocommutative coalgebra $C^\bullet$ in $cVec$ with $C^\bullet$-comodules $B^\bullet$ and $A^\bullet$. The cotensor product with a $C^\bullet$-comodule $A^\bullet$,

$$- \Box_C A^\bullet : Comod_{C^\bullet} \rightarrow cVec,$$

is left exact and therefore admits right derived functors denoted $\text{Cotor}_{p, C^\bullet}(-, A^\bullet)$. Note that the functor Cotor comes with a trigradation, i.e. $\text{Cotor}_{p, C^\bullet}(-, A^\bullet)$ is bigraded for all $p \geq 0$. As usual, we will usually suppress from the notation the internal grading.
Here we will be interested in the derived cotensor product and its relation with \( \text{Cotor}_{C^*}^\bullet(\ast, A^\ast) \) via a coalgebraic version of the Künneth spectral sequence. We define the derived cotensor product

\[
B^\ast \square^R_{C^*} A^\ast := I^\ast \square_{C^*} J^\ast
\]

where \( B^\ast \sim I^\ast \) and \( A^\ast \sim J^\ast \) are functorial fibrant replacements in \( \text{Comod}_{C^*} \). It can be shown by standard homotopical algebra arguments that the derived cotensor product is invariant under weak equivalences in both variables. Indeed, any two fibrant replacements are (cochain) homotopy equivalent and such equivalences are preserved by the cotensor product.

The following theorem is the coalgebraic version of [45, II.6, Theorem 6]. We only give a sketch of the proof since the arguments are dual to the ones by Quillen.

**Theorem 4.9** (Künneth spectral sequences). Let \( C^\ast \) be a cocommutative cosimplicial coalgebra in \( \text{cVec} \) and \( A^\ast, B^\ast \) two \( C^\ast \)-comodules. Then there are first quadrant spectral sequences

\[
\begin{align*}
(a) \quad & E_2^{p,q} = \pi^p(\text{Cotor}_{C^*}^q(B^\ast, A^\ast)) \Rightarrow \pi^{p+q}(B^\ast \square^R_{C^*} A^\ast) \\
(b) \quad & E_2^{p,q} = \pi^p(\pi^q(B^\ast), \pi^q(A^\ast)) \Rightarrow \pi^{p+q}(B^\ast \square^R_{C^*} A^\ast) \\
(c) \quad & E_2^{p,q} = \pi^p(\pi^q(B^\ast), A^\ast) \Rightarrow \pi^{p+q}(B^\ast \square^R_{C^*} A^\ast) \\
(d) \quad & E_2^{p,q} = \pi^p(B^\ast \square^R_{C^*} \pi^q(A^\ast)) \Rightarrow \pi^{p+q}(B^\ast \square^R_{C^*} A^\ast)
\end{align*}
\]

which are natural in \( A^\ast, B^\ast \) and \( C^\ast \).

The edge homomorphism of (a) is \( \pi^*(B^\ast \square^R_{C^*} A^\ast) \rightarrow \pi^*(B^\ast \square^R_{C^*} A^\ast) \) and is induced by the canonical map \( B^\ast \square^R_{C^*} A^\ast \rightarrow B^\ast \square^R_{C^*} A^\ast \). This canonical map is a weak equivalence if \( \text{Cotor}_{C^*}^{p,n}(B_n, A_n) = 0 \) for \( p > 0 \) and \( n \geq 0 \).

The notation in (b) and (c) requires some explanation. In (b), \( \pi^*(B^\ast) \), a graded coalgebra in \( \text{Vec} \), becomes a \( \pi^*(C^\ast) \)-comodule using the \( C^\ast \)-comodule structure of \( B^\ast \) and the shuffle map. In (c), \( \pi^*(B^\ast) \) denotes the constant object \( \pi^*(B^\ast) \in \text{cVec} \). The \( C^\ast \)-comodule structure and the shuffle map makes this a comodule over the constant cosimplicial coalgebra \( \pi^0(C^\ast) \). We then regard as it as \( C^\ast \)-comodule via the canonical coalgebra map \( \pi^0(C^\ast) \rightarrow C^\ast \).

**Proof.** (Sketch) (a) One constructs inductively an exact sequence in \( \text{Comod}_{C^*} \):

\[
B^\ast \rightarrow I_0^\ast \rightarrow I_1^\ast \rightarrow \ldots
\]

by setting \( B_0^\ast = B^\ast \), \( B_p^\ast \sim I_p^\ast \) a fibrant replacement of \( B_p^\ast \) in \( \text{Comod}_{C^*} \) and

\[
B_{p+1}^\ast = \text{coker}(B_p^\ast \rightarrow I_p^\ast).
\]

Then \( I_p^\ast \) is weakly trivial for all \( p > 0 \), i.e. \( \pi^s(I_p^\ast) = 0 \) for all \( s \geq 0 \) and \( p > 0 \). Since \( I_p^\ast \) is fibrant, it follows that \( I_p^\ast \square^R_{C^*} A^\ast \) is also weakly trivial, i.e. \( \pi^s(I_p^\ast \square^R_{C^*} A^\ast) = 0 \) for all \( p > 0 \). Apply the normalization functor with respect to the cosimplicial direction of the complex (4.10) to obtain a double complex

\[
N^\ast(I_p^\ast \square_{C^*} A^\ast)
\]

in which the vertical direction corresponds to the cosimplicial direction. Since \( I_p^\ast \) is an injective \( C^n \)-comodule for all \( n \), by the description of the fibrant objects in
Comod$_G^*$, $I^*_*$ is an injective resolution of the $C^n$-comodule $B^n$. Hence,

$$H^qN^p_q(I^*_\otimes C^\bullet A^\bullet) = N^pCotor^p_{C^\bullet}(B^p, A^p).$$

So there is a spectral sequence associated to this double complex with $E_2$-term

$$E^{pq}_2 = H^p_kH^q_kN^p_q(I^*_\otimes C^\bullet A^\bullet) = \pi^p(Cotor^p_{C^\bullet}(B^p, A^p))$$

where the subscripts $v, h$ denote the vertical and horizontal directions respectively. To see that the spectral sequence converges towards $\pi^{p+q}(I^*_0 \otimes C^\bullet A^\bullet)$, it suffices to note that

$$H^p_kH^q_kN^p_q(I^*_\otimes C^\bullet A^\bullet) = \begin{cases} 0, & \text{if } q > 0 \\ \pi^p(I^*_0 \otimes C^\bullet A^\bullet), & \text{if } q = 0 \end{cases}$$

Resolving $A^\bullet \to J^*_0$ instead of $B^\bullet$, we obtain a similar spectral sequence. This new spectral sequence degenerates if we replace $B^\bullet$ with $J^*_0$. This means that the map $A^\bullet \otimes C^\bullet I^*_0 \to J^*_0 \otimes C^\bullet I^*_0$ is a weak equivalence and therefore $B^\bullet \otimes^R_{C^\bullet} A^\bullet \cong I^*_0 \otimes C^\bullet A^\bullet$. This completes the construction of the spectral sequence and proves the last claim (cf. [45, II.6, 6.10]).

(b) Consider the category $cComod_G^*$ of cosimplicial objects over $C^\bullet$-comodules endowed with the resolution model structure with respect to the class $G$ of cofree $C^\bullet$-comodules. We choose a $G$-fibrant replacement $B^\bullet \to I^*_0$, which we may in addition assume to be cofree in $cComod_G^*$. By (a), we have

$$\text{Tot}(I^*_0 \otimes C^\bullet A^\bullet) = \text{diag}(I^*_0 \otimes C^\bullet A^\bullet) \cong B^\bullet \otimes^R_{C^\bullet} A^\bullet.\tag{4.12}$$

Consider the spectral sequence with $E^{pq}_2 = \pi^p\pi^q_2$ of the associated double complex where the original cosimplicial direction (the $\bullet$-direction) is viewed as the vertical one. Since the fibrant replacement is cofree, the normalization of $\pi^*(I^*_0)$, along the horizontal $\bullet$-direction, defines an injective $\pi^*(C^\bullet)$-resolution of $\pi^*(B^\bullet)$ (see the proof of [45, II.6, Lemma 1]). Then

$$E^{pq}_2 = \pi^p(\pi^*(I^*_0) \otimes_{C^\bullet} \pi^*(A^\bullet))^q$$

by arguments dual to [45, II.6, Lemma 1]. As a consequence, the spectral sequence has the required $E_2$-term and it converges towards the derived cotensor product by (4.12).

(c) For a $C^\bullet$-comodule $D^\bullet$, we consider a truncation functor defined by

$$T(D^\bullet) = D^\bullet/c\pi^0(D^\bullet).$$

We define cone and suspension functors, $C$ and $\Sigma$, using the simplicial structure as follows,

$$C(D^\bullet) = D^\bullet \otimes \Delta^1/D^\bullet \otimes \Delta^0$$

$$\Sigma(D^\bullet) = D^\bullet \otimes \Delta^1/D^\bullet \otimes \partial\Delta^1.$$

Note that $C(D^\bullet)$ is weakly trivial for each $D^\bullet$. There is a natural exact sequence of cosimplicial $C^\bullet$-comodules

$$c\pi^0(D^\bullet) \to D^\bullet \to CTD^\bullet \to \Sigma TD^\bullet$$

For $k \geq 1$, we have natural isomorphisms $\pi^k(TD^\bullet) \cong \pi^k(D^\bullet)$ and $\pi^{k-1}(\Sigma D^\bullet) \cong \pi^k(D^\bullet)$. Consequently, we obtain exact sequences for each $k \geq 0$:

$$c\pi^k(B^\bullet) \to \Sigma^k B^\bullet \to C\Sigma^k B^\bullet \to \Sigma C\Sigma^k B^\bullet \cong \Sigma^{k+1} B^\bullet.$$
Let $A^* \to I^*$ be a fibrant replacement. Since the functor $-\Box_C I^*$ is exact, we get long exact sequences for $k \geq 0$:

$$\ldots \to \pi^*(\Sigma^{k+1} B^* \Box_C I^*) \to \pi^*(c\pi^k(B^*) \Box_C I^*) \to \pi^*(\Sigma^k B^* \Box_A I^*)$$

$$\to \pi^{n+1}(\Sigma^{k+1} B^* \Box_C I^*) \to \ldots$$

which can be spliced together to obtain an exact couple with

$$D_{2}^{p,q} = \pi^p(\Sigma^q B^* \Box_C I^*)$$

and

$$E_{2}^{p,q} = \pi^p(\pi^q B^* \Box_C I^*).$$

The spectral sequence defined by the exact couple has the required $E_2$-term and converges towards $\pi^*(B^* \Box_C I^*)$ by (a) above.

Since cotensor products in $\text{Comod}_C$ are related to pullbacks in $\text{CAE}$, we obtain the following

**Corollary 4.13.** The model category $\text{CAE}$ is right proper.

**Proof.** Consider a pullback squares in $\text{CAE}$

$$\begin{array}{ccc}
E^* & \xrightarrow{\tilde{g}} & A^* \\
\downarrow & & \downarrow f \\
B^* & \xrightarrow{g} & C^*
\end{array}$$

where $g$ is a weak equivalence and $f$ a fibration in $\text{CAE}$. We claim that $\tilde{g}$ is also a weak equivalence. By the characterization of fibrations (Proposition 3.27), it suffices to consider the case where $f$ comes from a pullback square as follows

$$\begin{array}{ccc}
A^* & \to & (cG(V))^\Delta^e \\
\downarrow f & & \downarrow \\
C^* & \xrightarrow{h} & (cG(V))^\Delta^e
\end{array}$$

for some map $h$ and $V \in \text{Vec}$. Regarding $B^*$ and $A^*$ as $C^*$-comodules, we observe that $A^n$ is cofree as $C^n$-comodule and $E^* = B^* \Box_C A^*$. By Theorem 4.9(a), it follows that $E^* \simeq B^* \Box_C A^*$ and the result follows.

### 4.3. Homotopy excision

In this subsection, we prove a homotopy excision theorem in $\text{CAE}$ using the results on the homotopy theory of $C^*$-comodules. The homotopy excision theorem says that a homotopy pullback square in $\text{CAE}$ is also a homotopy pushout in a cosimplicial range depending on the connectivities of the maps involved. In a way familiar from classical obstruction theory of spaces, this theorem will be essential in later sections in identifying obstructions to extending maps and in describing moduli spaces of such extensions. First we recall the following notion.

**Definition 4.14.** A commutative square in $\text{CAE}$

$$\begin{array}{ccc}
E^* & \longrightarrow & A^* \\
\downarrow & & \downarrow \\
B^* & \longrightarrow & C^*
\end{array}$$

is called
(a) *homotopy n-cocartesian* if the canonical map
\[
\text{hocolim} \left( A^* \leftarrow E^* \to B^* \right) \to C^*
\]
is cosimplicially n-connected.

(b) *homotopy n-cartesian* if the canonical map
\[
E^* \to \text{holim} \left( A^* \to C^* \leftarrow B^* \right)
\]
is cosimplicially n-connected.

Similarly we define homotopy n-(co)cartesian squares in Comod\(_C^\bullet\) or cVec by considering instead the homotopy (co)limits in the respective categories.

The definition of “cosimplicially n-connected” is given in Definition 3.28. In this particular case, a map \( f : D^* \to C^* \) in c\( C\mathcal{A}^E \) (or cVec) is n-connected if and only if it induces isomorphisms on cohomotopy groups in degrees \( < n \) and a monomorphism in degree \( n \) (cf. Remark 1.3 and Proposition 1.4).

We have the following elementary lemma.

**Lemma 4.15.** Consider a homotopy pullback square in c\( C\mathcal{A}^E \) (or cVec),

\[
\begin{array}{ccc}
E^* & \longrightarrow & A^* \\
\downarrow g & & \downarrow f \\
B^* & \longrightarrow & C^*
\end{array}
\]

If \( f \) is n-connected, then so is \( g \).

**Proof.** We may assume that \( f \) is a \( \mathcal{E} \)-cofree map and \( E^* = B^* \Box_{C^*} A^* \). Then the result follows easily from Theorem 1.9(b). The case of cVec is easier. \( \square \)

**Remark 4.16.** Let \( p : E^* \to B^* \) be an \( \mathcal{E} \)-cofree map in c\( C\mathcal{A}^E \). Then we have a (homotopy) pullback square

\[
\begin{array}{ccc}
\text{cosk}_s(p) & \longrightarrow & \text{hom}(\Delta^s, cG_s) \\
\gamma_{s-1}(p) & & \downarrow \\
\text{cosk}_{s-1}(p) & \longrightarrow & \text{hom}(\partial\Delta^s, cG_s)
\end{array}
\]

By inspection, the right vertical map is \((s - 1)\)-connected. Hence, or otherwise, the left vertical map is \((s - 1)\)-connected.

There is a forgetful functor
\[
V : c\mathcal{A}/C^\bullet \to \text{Comod}_C^\bullet
\]
which sends \( f : D^* \to C^* \) to \( D^* \) regarded as a cosimplicial \( C^\bullet \)-comodule with structure map \((\text{id}, f) : D^* \to D^* \Box C^* \) where \(- \Box -\) is the graded tensor product applied pointwise. The functor \( V \) admits a right adjoint but we will not need this fact here. Given that homotopy pullbacks in Comod\(_C^\bullet\) are also homotopy pushouts, and \( V \) preserves homotopy pushouts, the homotopy excision property in c\( C\mathcal{A}^E \) will first be expressed as a comparison between taking homotopy pullbacks in c\( C\mathcal{A}^E \) and Comod\(_C^\bullet\) respectively.
Theorem 4.17 (Homotopy excision for cosimplicial unstable coalgebras). Let

\[
\begin{array}{c}
E^* \\
\downarrow^f \\
B^* \xrightarrow{g} C^*
\end{array}
\]

be a homotopy pullback square in \(\mathcal{CA}^E\) in which \(f\) is \(n\)-connected and \(g\) is \(m\)-connected. Then

(a) the square is homotopy \((n+m+1)\)-cartesian in \(\text{Comod}_{C^*}\),

(b) the square is homotopy \((n+m)\)-cocartesian in \(\mathcal{CA}^E\).

Proof. (a) The idea of the proof is to compare \(E^*\) with the homotopy pullback in \(\text{Comod}_{C^*}\). By Proposition 3.30 we may assume that \(f\) and \(g\) are \(E\)-cofree maps of the form

\[
f : A^* \rightarrow \ldots \rightarrow \cosk_{m+1}(f) \xrightarrow{f_m} \cosk_m(f) = C^*
\]

\[
g : B^* \rightarrow \ldots \rightarrow \cosk_{n+1}(g) \xrightarrow{g_n} \cosk_n(f) = C^*
\]

and \(E^*\) is the strict pullback of \(f\) and \(g\). Therefore, \(E^* = B^* \square_{C^*} A^*\) as \(C^*\)-comodules.

We first consider the case where each of the maps \(f\) and \(g\) is defined by a single coattachment. This means that we have pullbacks squares in \(\mathcal{CA}\) as follows

\[
\begin{array}{ccc}
E^* & \rightarrow & A^* \\
\downarrow & & \downarrow^f \\
B^* \xrightarrow{g} C^* & \rightarrow & cG(V)^\Delta^k
\end{array}
\]

\[
\begin{array}{ccc}
cG(W)^\Delta^l & \rightarrow & cG(W)^{\tilde{c}\Delta^l}
\end{array}
\]

where \(k \geq m + 1\) and \(l \geq n + 1\). Let \(\hat{E}^* := A^* \otimes_{C^*} B^*\) denote the pullback of \(f\) and \(g\) in \(C^*\)-comodules. Then it suffices to show that the canonical map

\[
c : E^* \rightarrow \hat{E}^*
\]

is \((n+m+1)\)-connected, i.e. the induced map \(\pi^s E^* \rightarrow \pi^s \hat{E}^*\) is an isomorphism for \(s \leq m + n\) and injective for \(s = m + n + 1\). To show this we determine the connectivity of the kernel of \(c\) in cosimplicial graded vector spaces. Let

\[
\overline{A}^* = \ker \{ f : A^* \rightarrow C^* \}
\]

\[
\overline{B}^* = \ker \{ g : B^* \rightarrow C^* \}
\]

These cosimplicial (graded) vector spaces are \(C^*\)-comodules where the comodule structure is defined canonically by \(f\) and \(g\) respectively. We claim that

\[
(4.18) \quad B^* \square_{c \cdot C^*} \overline{A}^* = \ker \{ c : B^* \square_{c \cdot C^*} A^* \rightarrow A^* \otimes_{C^*} B^* \}.
\]

The inclusion of the cotensor on the left in the kernel of \(c\) is obvious. To see the other inclusion, and to determine the connectivity of this kernel, we note that in each cosimplicial degree \(s\), there are isomorphisms of graded vector spaces

\[
A^* \cong C^* \otimes (\Omega^k cG(V))^s
\]
and
\[ B^s \cong C^s \otimes (\Omega^1 cG(W))^s \]
such that both \( f \) and \( g \) are isomorphic to the projections onto \( C^s \). Thus, for all \( s \geq 0 \), there are isomorphisms of graded vector spaces
\[
E^s \cong C^s \otimes (\Omega^k cG(V))^s \otimes (\Omega^1 cG(W))^s
\]
\[
( \cong (C^s \otimes (\Omega^k G(V))^s) \boxtimes (C^s \otimes (\Omega^1 G(W))^s) ).
\]
Let \( \Omega^k cG(V) = \ker \left[ \Omega^k cG(V) \to \Omega^k \right] \). Then we have isomorphisms of graded vector spaces, for all \( s \geq 0 \),
\[
\hat{E}^s \cong C^s \oplus \left( C^s \otimes (\Omega^k cG(V))^s \right) \oplus \left( C^s \otimes (\Omega^k cG(W))^s \right).
\]
Using these identifications of \( E^s \) and \( \hat{E}^s \), it is easy to see that the map \( c \) is given by the canonical projection. As a consequence, we have isomorphisms:
\[
(\ker c)^s \cong C^s \otimes (\Omega^k G(V))^s \otimes (\Omega^k G(W))^s
\]
\[
\cong \left( C^s \otimes (\Omega^k G(V))^s \right) \boxtimes \left( C^s \otimes (\Omega^k G(W))^s \right).
\]
Therefore the kernel of \( c \) is identified with the cotensor product as claimed in (4.18).
We estimate the connectivity of this cotensor product \( [1.18] \) in \( \text{Comod}_{C^*} \) using the Künneth spectral sequence from Theorem [1.9] b). We have
\[
E_2^{p,q} = \text{Cotor}_{\pi^*(C^*)}^p(\pi^*(B^*), \pi^*(A^*))^q
\]
and the two arguments vanish in degrees less than or equal to \( m \) and \( n \) respectively. It follows that \( E_2^{p,q} = 0 \) for \( p + q \leq m + n + 1 \), and therefore
\[
\pi^*(\ker (c)) = 0
\]
for \( * \leq n + m + 1 \). Since \( \ker (c) \to E^* \to \hat{E}^* \) is a fiber sequence of cosimplicial \( C^* \)-comodules, we conclude that the map \( c \colon E^* \to \hat{E}^* \) is \((m + n + 1)\)-connected.

It is easy to see that homotopy \( n \)-cartesian squares are closed under composition. Then the general case follows by induction and Lemma [4.15].

For Part (b) we note that homotopy colimits in \( cC\mathcal{A} \) can be computed in underlying cosimplicial graded vector spaces (or in \( \text{Comod}_{C^*} \)) since the forgetful functor from \( cC\mathcal{A} \) to \( c\text{Vec} \) (or in \( \text{Comod}_{C^*} \)) is left Quillen. The statement now follows from (a) by comparing the long exact sequences of cohomotopy groups.

\[ \square \]

4.4. André-Quillen cohomology. Following the homotopical approach to homology initiated by Quillen [15], the André-Quillen cohomology groups of an unstable coalgebra are defined as the right derived functors of the coabelianization functor. For an unstable coalgebra \( C \), recall from Section 2 that we have an adjunction
\[
\iota_C : \mathcal{V}C \rightleftarrows C/\mathcal{C}A : \text{Ab}_C,
\]
which is given essentially by the inclusion of the full subcategory of coabelian cogroup objects in \( C/\mathcal{C}A \). Passing to the respective categories of cosimplicial objects, this can be extended to a Quillen adjunction:
\[
\iota_C : c\mathcal{V}C \rightleftarrows c(C/\mathcal{C}A) : \text{Ab}_C.
\]
The category on the right is the under-category \( cC/c\mathcal{C}A \) and is endowed with the model structure induced by \( c\mathcal{C}A \). The model category on the left is the standard
pointed model category of cosimplicial objects in an abelian category (cf. [13, 4.4]). As remarked in [45], for a given object $D \in C/\mathcal{A}$, we may regard the derived coabelianization

$$(\mathbb{R}Ab_C)(D)$$

as the cohomology of $D$. Then, given an object $M \in \mathcal{V}C$, the 0-th André-Quillen cohomology group of $D$ with coefficients in $M$ is defined to be

$$AQ^0_C(D; M) = [cM, (\mathbb{R}Ab_C)(cD)].$$

In general, the $n$-th André-Quillen cohomology group is

$$AQ^n_C(D; M) = [\Omega^n(cM), (\mathbb{R}Ab_C)(cD)] \cong \pi^n\text{Hom}_{\mathcal{V}C}(M, (\mathbb{R}Ab_C)(cD))$$

where $\Omega$ denotes the derived loop functor in the pointed model category $\mathcal{V}C$. These cohomology groups are also the non-additive right derived functors of the functor

$$\text{Hom}_{\mathcal{V}C}(M, Ab_C(-)) : C/\mathcal{A} \to \text{Vec}.$$  

We recall (2.16) that this is the same as the functor of derivations

$$\text{Der}_{C/\mathcal{A}}(M, D) : C/\mathcal{A} \to \text{Vec}.$$  

For an object $D \in C/\mathcal{A}$ and fibrant replacement $D \to H^\bullet$ in $c(C/\mathcal{A})$, the $n$-th derived functor is defined to be (cf. [13, 5.5])

$$(R^n\text{Hom}_{\mathcal{V}C}(M, Ab_C(-)))(D) = \pi^n\text{Hom}_{\mathcal{V}C}(M, Ab_C(H^\bullet)).$$

It follows from standard homotopical algebra arguments that this is independent of the choice of fibrant replacement up to natural isomorphism. A particularly convenient choice of fibrant replacement comes from the cosimplicial resolution defined by the monad on $C/\mathcal{A}$ associated to the adjunction $J : C/\mathcal{A} \to J(C)/\text{Vec} : G$.

So André-Quillen cohomology can also be regarded as monadic (or triple) cohomology where $M$ is the choice of coabelian coefficients.

4.5. **Objects of type $K_C(M, n)$**. The adjoint of (4.19) gives a natural isomorphism

$$AQ^n_C(D; M) \cong [(L\iota_C)(\Omega^n(cM)), cD]$$

which can be regarded as a representability theorem for André-Quillen cohomology. Since the canonical natural transformation in $\text{Ho}(\mathcal{V}C)$:

$$\Sigma \Omega \to \text{Id}$$

is a natural isomorphism, there are also natural isomorphisms

$$(4.20) \quad AQ^n_C(D; M) \cong [(L\iota_C)(\Sigma^n\Omega^{n+k}(cM)), cD].$$

The representing object $(L\iota_C)(\Omega^n(cM))$ is described explicitly as follows. First, the normalized cochain complex associated to $\Omega^n(cM)$ is quasi-isomorphic to the complex which has $M$ in degree $n$ and is trivial everywhere else. By the Dold-Kan correspondence, this means that $\Omega^n(cM)$ is given up to weak equivalence by a sum of copies of $M$ in each cosimplicial degree. Since every object in $\mathcal{V}C$ is cofibrant, the functor $\iota_C$ preserves the weak equivalences, and so we may choose $\iota_C(\Omega^n(cM))$
for the value of the derived functor at \( \Omega^n(cM) \). Explicitly, this forms the semi-direct product of copies of \( M \) with the coalgebra \( C \) in each cosimplicial degree. The cohomotopy groups of the resulting object are

\[
\pi^s(t_C(\Omega^n(cM))) \cong \begin{cases} 
M & \text{if } s = n \\
C & \text{if } s = 0 \\
0 & \text{otherwise}
\end{cases}
\]

for \( n > 0 \). For \( n = 0 \), clearly \( \pi^0(t_C(cM)) = c(t_C(M)) \) with vanishing higher cohomotopy groups. These are isomorphisms of unstable coalgebras and \( C \)-comodules respectively. Since \( n \) for \( \text{Definition } 4.21 \).

Definition 4.21. Let \( C \) be an unstable coalgebra. An object \( D^\bullet \in \mathcal{C}A \) is said to be of type \( K(C, 0) \) if it weakly equivalent to the constant cosimplicial object \( cC \).

Definition 4.22. Let \( C \in \mathcal{C}A, M \in \mathcal{V}C \) and \( n \geq 1 \). An object \( D^\bullet \) of \( \mathcal{C}A \) is said to be of type \( K(C, M, n) \) if the following are satisfied:

(a) there are isomorphisms of coalgebras and \( C \)-comodules respectively:

\[
\pi^sD^\bullet \cong \begin{cases} 
C & \text{if } s = 0 \\
M & \text{if } s = n \\
0 & \text{otherwise},
\end{cases}
\]

(b) there is a map \( D^\bullet \rightarrow D^\bullet_0 \) to an object of type \( K(C, 0) \) such that the composite \( \text{sk}_1(D^\bullet) \rightarrow D^\bullet \rightarrow D^\bullet_0 \) is a weak equivalence.

Occasionally, we will also use the notation \( K_C(M, n) \) to denote an object of type \( K(t_C(M), 0) \).

It is clear from the definition that an object of type \( K(C, M, n) \) can be roughly regarded up to weak equivalence both as an object under and over \( cC \), but the choices involved will be non-canonical. It will often be necessary to include such a choice to the structure and view the resulting object as an object of a slice category instead.

Definition 4.23. Let \( C \in \mathcal{C}A \) and \( M \in \mathcal{V}C \).

(a) A pointed object of type \( K_C(M, n) \), \( n \geq 1 \), is a pair \( (D^\bullet, i) \) where \( D^\bullet \) is an object of type \( K_C(M, n) \) and \( i : cC \rightarrow D^\bullet \) is a map which induces an isomorphism on \( \pi^0 \)-groups.

(b) A structured object of type \( K_C(M, n) \), \( n \geq 1 \), is a pair \( (D^\bullet, \eta) \) where \( D^\bullet \) is an object of type \( K_C(M, n) \) and \( \eta : D^\bullet \rightarrow D^\bullet_0 \) is a map as in (b) above.

(c) A pointed object of type \( K_C(M, 0) \) is a pair \( (D^\bullet, i) \) where \( D^\bullet \) is an object of type \( K(t_C(M), 0) \) and \( i : cC \rightarrow D^\bullet \) is a map such that \( \pi^0(i) \) can be identified up to isomorphism with the standard inclusion \( C \rightarrow t_C(M) \).

(d) A structured object of type \( K_C(M, 0) \) is a pair \( (D^\bullet, \eta) \) where \( D^\bullet \) is an object of type \( K(t_C(M), 0) \) and \( \eta : D^\bullet \rightarrow D^\bullet_0 \) is a map to an object of type \( K(C, 0) \) such that \( \pi^0(\eta) \) can be identified up to isomorphism with the canonical projection \( t_C(M) \rightarrow C \).
Remark 4.24. Let \((D^\bullet, \eta)\) be a structured object of type \(K_C(M, n)\) and \(n > 0\). Then the homotopy pushout of the maps
\[
D_0^\bullet \xleftarrow{\eta} D^\bullet \xrightarrow{\eta} D_0^\bullet
\]
is a structured object of type \(K_C(M, n - 1)\).

Remark 4.25. As explained above, the objects \(\iota_C(\Omega^n(cM))\) are pointed and structured objects of type \(K_C(M, n)\). We sketch a different construction of objects of type \(K_C(M, n)\) based on the homotopy excision theorem. The construction is by induction. For \(n = 0\), the constant coalgebra \(cC\) is an object of type \(K(C, 0)\) by definition. Assume that an object \(D^\bullet\) of type \(K_C(M, n)\) has been constructed for \(n > 0\). Consider the homotopy pullback square
\[
\begin{array}{ccc}
E^\bullet & \xrightarrow{\eta} & \sk_1(D^\bullet) \\
\downarrow & & \downarrow \\
\sk_1(D^\bullet) & \xrightarrow{\eta} & D^\bullet
\end{array}
\]
where \(\sk_1(D^\bullet)\) is an object of type \(K(C, 0)\). Then, by homotopy excision \([4.17]\), \(\sk_{n+2}(E^\bullet)\) is an object of type \(K_C(M, n + 1)\). We can always regard this as structured by making a choice among one of the maps to \(\sk_1(D^\bullet)\). The case \(n = 1\) is somewhat special. We can consider the following homotopy pullback square
\[
\begin{array}{ccc}
E^\bullet & \xrightarrow{} & cC \\
\downarrow & & \downarrow \\
cC & \xrightarrow{} & c(\iota_C(M))
\end{array}
\]
and an application of homotopy excision shows that \(\sk_2(E^\bullet)\) is of type \(K_C(M, 1)\).

To be more precise, one applies Theorem \([4.33]\) and Corollary \([4.39]\) below and uses the fact that \(C \cup_{\iota_C(M)} C = C\).

Next we show the homotopical uniqueness of \(K\)-objects and moreover, determine the homotopy type of the moduli space of structured \(K\)-objects. For \(C\) an unstable coalgebra, \(M\) a \(C\)-comodule in \(\mc{V}C\) and \(n \geq 0\), we denote this moduli space by
\[
\mathcal{M}(K_C(M, n) \rightarrow K(C, 0)).
\]
This is an example of a moduli space for maps where the arrow \(\rightarrow\) indicates that these maps satisfy an additional property. It is the classifying space of the category \(\mathcal{W}(K_C(M, n) \rightarrow K(C, 0))\) whose objects are structured objects of type \(K_C(M, n)\):
\[
D^\bullet \xrightarrow{\eta} D_0^\bullet
\]
and morphisms are square diagrams as follows
\[
\begin{array}{ccc}
D^\bullet & \xrightarrow{\eta} & D_0^\bullet \\
\downarrow & \sim & \downarrow \\
E^\bullet & \xrightarrow{\eta'} & E_0^\bullet
\end{array}
\]
where the vertical arrows are weak equivalences in \(cC\mathcal{A}^F\). Note that we have also allowed the case \(n = 0\). We will also use the notation \(\mathcal{W}(K(C, 0))\) and \(\mathcal{M}(K(C, 0))\).
to denote respectively the category and moduli space for objects weakly equivalent $cC$.

An automorphism of $\iota_C(M)$ is an isomorphism $\phi : \iota_C(M) \to \iota_C(M)$ which is compatible with an isomorphism $\phi_0$ of $C$ along the projection map:

$$
\begin{array}{c}
\iota_C(M) \\
\downarrow \phi \\
C
\end{array}
\xrightarrow{\cong}

\begin{array}{c}
\iota_C(M) \\
\downarrow \phi_0 \\
C
\end{array}
$$

This group of automorphisms will be denoted by $\operatorname{Aut}_C(M)$.

**Proposition 4.26.** Let $C \in \mathcal{C}$, $M \in \mathcal{V}C$ and $n \geq 0$. Then:

(a) There is a weak equivalence $\mathcal{M}(K(C,0)) \simeq B\operatorname{Aut}(C)$.

(b) There is a weak equivalence $\mathcal{M}(K_C(M,0) \to K(C,0)) \simeq B\operatorname{Aut}_C(M)$.

(c) There is a weak equivalence

$$
\mathcal{M}(K_C(M,n + 1) \to K(C,0)) \simeq \mathcal{M}(K_C(M,n) \to K(C,0)).
$$

**Proof.** (a) Let $\mathcal{W}(C)$ denote the subcategory of unstable coalgebras isomorphic to $C$ and isomorphisms between them. This is a connected groupoid, so its classifying space is weakly equivalent to $B\operatorname{Aut}(C)$. There is a pair of functors

$$
c : \mathcal{W}(C) \rightleftharpoons \mathcal{W}(K(C,0)) : \pi^0.
$$

and natural weak equivalences

$$
\text{Id} \to \pi^0 \circ c \quad \text{and} \quad c \circ \pi^0 \to \text{Id}
$$

which show that this (adjoint) pair of functors induces an inverse pair of homotopy equivalences $\mathcal{M}(K(C,0)) \simeq B(\mathcal{W}(C))$ and the result follows.

(b) Let $\mathcal{W}(C; M)$ denote the subcategory of $\mathcal{C}\mathcal{A}^{-}$ whose objects are isomorphic to the canonical projection

$$
\iota_C(M) \to C
$$

and morphisms are isomorphisms between such objects. Since this is a connected groupoid, we have

$$
B(\mathcal{W}(C; M)) \simeq B\operatorname{Aut}_C(M).
$$

There is a pair of functors

$$
c : \mathcal{W}(C; M) \rightleftharpoons \mathcal{W}(K_C(M,0) \to K(C,0)) : \pi^0
$$

and the two composites are naturally weakly equivalent to the respective identity functors as before. The required result follows similarly.

(c) There is a pair of functors

$$
F : \mathcal{W}(K_C(M,n + 1) \to K(C,0)) \rightleftharpoons \mathcal{W}(K_C(M,n) \to K(C,0)) : G
$$

defined as follows:

(i) $F$ sends an object $\eta : D^* \to D^*_0$ to the homotopy pushout of the diagram

$$
D^*_0 \xrightarrow{\eta} D^* \xrightarrow{\eta} D^*_0
$$

as structured object of type $K_C(M,n)$, see Remark 4.23.
(ii) Let \( \eta : D^* \to D^* \) be an object of \( \mathcal{W}(K_C(M, n) \to K(C, 0)) \). Form the homotopy pullback

\[
\begin{array}{ccc}
E^* & \to & \text{sk}_1(D^*) \\
\downarrow \eta & & \downarrow i \\
\text{sk}_1(D^*) & \to & D^*
\end{array}
\]

where \( i \) is the natural inclusion map. \( G \) sends the object \( \eta \) to the natural map

\[
\text{sk}_{n+2}(E^*) \to E^* \xrightarrow{\eta} \text{sk}_1(D^*)
\]

which is an object of \( \mathcal{W}(K_C(M, n + 1) \to K(C, 0)) \) by Theorem 4.17 (or Proposition 4.29 below).

There are zigzags of natural weak equivalences connecting the composite functors \( F \circ G \) and \( G \circ F \) to the respective identity functors. Thus this pair of functors induces a pair of inverse homotopy equivalences between the classifying spaces as required. \( \square \)

As a consequence, we have the following

**Corollary 4.27.** Let \( C \in \mathcal{C}, M \in \mathcal{V} \mathcal{C} \) and \( n \geq 0 \). Then there is a weak equivalence

\[
\mathcal{M}(K_C(M, n) \to K(C, 0)) \simeq B\text{Aut}_C(M).
\]

In particular, the moduli spaces of structured \( K \)-objects are connected.

Since objects of type \( K_C(M, n) \) are homotopically unique, we will denote by \( K_C(M, n) \) a choice of such an object, even though it will be non-canonical, whenever we are only interested in the actual homotopy type.

**Theorem 4.28.** Let \( C \in \mathcal{C}, M \in \mathcal{V} \mathcal{C}, D \in \mathcal{C}/\mathcal{C} \mathcal{A} \) and \( n, k \geq 0 \). For every structured pointed object of type \( K_C(M, n + k) \), there are isomorphisms

\[
\text{AQ}^n_{\mathcal{C}}(D; M) \cong \pi_0\text{map}^\text{der}_{\mathcal{C}/\mathcal{C} \mathcal{A}}(K_C(M, n), cD) \\
\cong \pi_k\text{map}^\text{der}_{\mathcal{C}/\mathcal{C} \mathcal{A}}(K_C(M, n + k), cD).
\]

Here \( \text{map}^\text{der}(-, -) \) denotes the derived mapping space.

**Proof.** Without loss of generality we may choose \( \iota_C(\Omega^{n+k}(cM)) \) to be the model for an object of type \( K_C(M, n + k) \). Then the result follows from (4.20) and the last remark. We note that in general the mapping space is given a (non-canonical) basepoint by the structure map of \( K_C(M, n + k) \) to an object of type \( K(C, 0) \). \( \square \)

### 4.6. Postnikov decompositions.

The skeletal filtration of a cosimplicial unstable coalgebra is formally analogous to the Postnikov tower of a space. In this subsection we prove that this filtration is principal, i.e it can be defined in terms of attaching maps. This is the analogue of the “difference construction” from [2, Proposition 6.3]. It is a consequence of the following proposition which in turn is an immediate application of Theorem 4.17

**Proposition 4.29.** Let \( f : A^* \to C^* \) be an \( n \)-connected map in \( \mathcal{C} \mathcal{A}^F \), \( n \geq 1 \), and \( C := \pi^0(C^*) \). Let \( D^* \) denote the homotopy cofiber of \( f \) and \( M = \pi^n(D^*) \). Consider
the homotopy pullback square

\[
\begin{array}{ccc}
E^\bullet & \longrightarrow & A^\bullet \\
\downarrow & & \downarrow f \\
\text{sk}_1(C^\bullet) & \longrightarrow & C^\bullet
\end{array}
\]

Then:

(a) \(M\) is a \(C\)-comodule and \(\text{sk}_{n+2}(E^\bullet)\) together with the canonical map

\[
\text{sk}_{n+2}(E^\bullet) \to \text{sk}_1(C^\bullet)
\]

define a structured object of type \(K_C(M, n+1)\).

(b) If the cohomotopy of the homotopy cofiber \(D^\bullet\) is concentrated in degree \(n\), then the diagram

\[
\begin{array}{ccc}
\text{sk}_{n+2}(E^\bullet) & \longrightarrow & A^\bullet \\
\downarrow & & \downarrow f \\
\text{sk}_1(C^\bullet) & \longrightarrow & C^\bullet
\end{array}
\]

is a homotopy pushout.

As a consequence of (4.29) for every \(C^\bullet \in \mathcal{C}\mathcal{A}^\mathcal{E}\), the skeletal filtration

\[
\text{sk}_1(C^\bullet) \to \text{sk}_2(C^\bullet) \to \cdots \to \text{sk}_n(C^\bullet) \to \cdots \to C^\bullet
\]

is defined by homotopy pushouts as follows

\[
\begin{array}{ccc}
K_C(M, n+1) & \xrightarrow{w_n} & \text{sk}_n(C^\bullet) \\
\downarrow & & \downarrow \\
K(C, 0) & \longrightarrow & \text{sk}_{n+1}(C^\bullet)
\end{array}
\]

where \(C = \pi^0(C^\bullet), M = \pi^n(C^\bullet)\) and the collection of maps \(w_n\) may be regarded as analogues of the Postnikov \(k\)-invariants in this context.

The next proposition reformulates and generalizes Proposition (4.29) in terms of moduli spaces. For \(C^\bullet \in \mathcal{C}\mathcal{A}^\mathcal{E}\) such that \(\text{sk}_n(C^\bullet) \simeq C^\bullet\), for some \(n \geq 1\), and \(M\) a coabelian \(\pi^0(C^\bullet)\)-comodule, let

\[
\mathcal{W}(C^\bullet + (M, n))
\]

be the subcategory of cosimplicial unstable coalgebras \(D^\bullet\) such that \(\text{sk}_n(D^\bullet)\) is weakly equivalent to \(C^\bullet\) and \(\pi^n(D^\bullet)\) is isomorphic to \(M\) as comodules. The morphisms are given by weak equivalences between such objects. The classifying space of this category, denoted by

\[
\mathcal{M}(C^\bullet + (M, n)),
\]

is the moduli space of \((n+1)\)-skeletal extensions of the \(n\)-skeletal object \(C^\bullet\) by the \(\pi^0(C^\bullet)\)-comodule \(M\).

**Proposition 4.30.** Let \(n \geq 1\) and suppose that \(C^\bullet\) is an object of \(\mathcal{C}\mathcal{A}^\mathcal{E}\) such that \(\text{sk}_n C^\bullet \Rightarrow C^\bullet\). Let \(M\) be an object of \(\mathcal{V}\pi^0(C^\bullet)\). Then there is a natural weak equivalence

\[
\mathcal{M}(C^\bullet + (M, n)) \simeq \mathcal{M}(K(C, 0) \leftarrow K_C(M, n+1) \rightsquigarrow C^\bullet).
\]
Proof. There is a pair of functors

\[ F : \mathcal{W}(K(C, 0) \leftarrow K_C(M, n + 1) \hookrightarrow C^*) \rightleftharpoons \mathcal{W}(C^* + (M, n)) : G \]

where:

(i) \( F \) is defined to be the homotopy pushout of the diagram. By Proposition 4.29 or otherwise, this object is in \( \mathcal{W}(C^* + (M, n)) \).

(ii) Given an object \( D^* \in \mathcal{W}(C^* + (M, n)) \), form the homotopy pullback

\[
\begin{array}{c}
\text{sk}_{n+2}(E^*) \\
\downarrow \\
\text{sk}_n(D^*) \\
\downarrow \\
C \\
\end{array} \] \quad \学业{1} \quad \begin{array}{c}
\text{sk}_1(D^*) \\
\downarrow \\
D^* \\
\end{array}
\]

The functor \( G \) sends \( D^* \) to the diagram

\[
\begin{array}{c}
\text{sk}_1(D^*) \\
\downarrow \\
C \\
\end{array} \] \quad \学业{2} \quad \begin{array}{c}
\text{sk}_{n+2}(E^*) \\
\downarrow \\
\text{sk}_n(D^*) \\
\end{array}
\]

which, by Proposition 4.29, is a diagram of the required type.

The composites \( F \circ G \) and \( G \circ F \) are connected to the respective identity functors via natural zigzags of natural weak equivalences. Hence these functors induce a pair of inverse homotopy equivalences as required. \( \square \)

4.7. An extension of Proposition 4.29 We discuss a generalization of Proposition 4.29 to the case of maps which are 0-connected but do not necessarily induce isomorphisms on \( \pi^0 \).

Recall that \( \text{Cog}_{\text{Vec}} \) denotes the category of cocommutative, counital graded coalgebras over \( \mathbb{F} \), that is, the category of coalgebras in \( \text{Vec} \) with respect to the graded tensor product.

Definition 4.31. Let \( K \rightarrow C \leftarrow L \) be inclusions of coalgebras in \( \text{Cog}_{\text{Vec}} \). We obtain a pushout of \( P \)-comodules

\[
\begin{array}{c}
C \\
\downarrow \\
C/L \\
\end{array} \quad \begin{array}{c}
C/K \\
\downarrow \\
C/(K \oplus L) - \phi = C/K \boxdot_C C/L \\
\end{array} \quad \begin{array}{c}
C/K \boxdot_C C/L \\
\downarrow \\
C \otimes_C C/L \\
\end{array}
\]

where the \( C \)-comodule structures and the outer maps are defined by the comultiplication of \( C \). Thus we obtain a canonical dotted arrow \( \phi : C/(K \oplus L) \rightarrow C/K \boxdot_C C/L \) in \( \text{Vec} \). Then we define:

(a) \( C/K \ast_C C/L \) to be the kernel of \( \phi \).

(b) \( C/K \circ_C C/L \) to be the image of \( \phi \).

Remark 4.32. The construction of \( C/K \ast_C C/L \) is dual to the construction which associates to two ideals \( I, J \) in a commutative ring \( R \) the quotient \((I \cap J)/(IJ)\). The constructions \( C/(K \oplus L) \) and \( C/K \circ_C C/L \) are dual to \( I \cap J \) and \( IJ \) respectively.

Theorem 4.33. Let \( f : A^* \rightarrow C^* \) be an \( n \)-connected map in \( c\mathcal{C}A^F \), \( n \geq 0 \). Suppose that \( L \) is an unstable coalgebra and \( g : K(L, 0) \rightarrow B^* \) a 0-connected map. Consider
the homotopy pullback diagram

\[
\begin{array}{ccc}
E^* & \longrightarrow & A^* \\
\downarrow & & \downarrow f \\
K(L, 0) & \underset{g}{\longrightarrow} & C^*
\end{array}
\]

We denote the bigraded coalgebras

\[A^* := \pi^*(A^*) \quad \text{and} \quad C^* := \pi^*(C^*)\]

in order to simplify the notation. Then:

(a) The object \(sk_{n+2}E^*\) is of type \(K_D(N, n + 1)\) where

(i) the unstable coalgebra \(D\) is isomorphic to \(A^0 \bigotimes C^0 L\).

(ii) the \(D\)-comodule \(N\) fits into a short exact sequence as follows,

\[0 \rightarrow (C/L \bigotimes C/A)^n \rightarrow N \rightarrow (L \bigotimes_c A)^{n+1} \rightarrow 0\]

If \(f\) induces a monomorphism in degree \(n + 1\), then \((L \bigotimes_c A)^{n+1} = 0\).

(b) If \(\pi^*(C^*)\) vanishes for \(s > n\), then the diagram

\[
\begin{array}{ccc}
sk_{n+2}E^* & \longrightarrow & A^* \\
\downarrow & & \downarrow f \\
K(L, 0) & \underset{g}{\longrightarrow} & C^*
\end{array}
\]

is a homotopy pushout.

A similar formula as in (a) is obtained by Massey/Peterson in [38, Theorem 4.1].

For the proof of this theorem, we will need the following technical lemmas.

**Lemma 4.34.** Let \(B \xrightarrow{f} C \xleftarrow{g} A\) be a diagram in \(\text{Coalg}_{\text{Vec}}\). Assume that

(a) \(B\) is concentrated in degree 0.

(b) \(f\) is an isomorphism in degrees \(< n\) and a monomorphism in degree \(n\).

Then \(\text{Cotor}^p_C(B, A)_q = 0\) for all pairs \((p, q) \neq (1, n)\) with \(p + q \leq n + 1\) and \(p > 0\).

**Proof.** Recall that

\[\text{Cotor}^p_C(B, A)_q = H_p(B \bigotimes_C I^*)_q\]

for any injective \(C\)-resolution \(A \xrightarrow{f} I^*\) of \(A \in \text{Comod}_C\). An injective \(C\)-resolution \(I^*\) of \(A\) can be given such that \(A \rightarrow I^0\) agrees with \(f\) in internal degrees \(q \leq n\).

Moreover, we can assume that the resolution has the property that \((I^p)_q = 0\) for \(0 \leq q < n\) and \(p > 0\). Then

\[(A \bigotimes_C I^p)_q = 0\]

for all \((p, q)\) with \(p > 0\) and \(0 \leq q < n\) and the assertion follows. \(\square\)

**Lemma 4.35.** Let \(K \xrightarrow{f} C \xleftarrow{g} L\) be inclusions of coalgebras (ungraded, graded or bigraded). Then there is an isomorphism

\[\text{Cotor}^1_C(K, L) \cong (C/K \bigotimes_C C/L)\]

**Proof.** Applying the functor \(- \bigotimes_C L\) to the short exact sequence of \(C\)-comodules:

\[0 \rightarrow K \rightarrow C \rightarrow C/K \rightarrow 0\]

we obtain an exact sequence

\[0 \rightarrow K \cap L \rightarrow L \rightarrow C/K \bigotimes_C L \rightarrow \text{Cotor}^1_C(K, L) \rightarrow 0.\]
On the other hand, applying the functor $\square_C(C/K)$ to the sequence,

$$0 \to L \to C \to C/L \to 0,$$

we get an exact sequence as follows,

$$0 \to L \square_C(C/K) \to C/K \to (C/L) \square_C(C/K) \to \cdots$$

Note then that $(C/K) *_C (C/K) := \ker((C/K \oplus L) \to (C/K) \otimes (C/L))$ is the image of the composition

$$L \square_C(C/K) \to C/K \to (C/K \oplus L)$$

This shows that $(C/K) *_C (C/L)$ fits in a short exact sequence

$$0 \to (L/K \cap L) \to L \square_C(C/K) \to (C/K) *_C (C/L) \to 0$$

and the claim follows.

**Remark 4.36.** Lemma 4.35 above is the (graded) dual of the well known formula

$$\text{Tor}_1^R(I, J) \cong (I \cap J)/(IJ),$$

where $I$ and $J$ are ideals in a commutative ring $R$.

**Proof of Theorem 4.33.** We use the spectral sequence (b) from Theorem 4.9 for the computation of $\pi^*(E^*)$ when $s \leq n + 1$. The identification of $\pi^n(E^*)$ is then immediate. As a consequence of (a bigraded variant of) Lemma 4.34, we have

$$\text{Cotor}^0_{C}(L, A)^q = 0,$$

as a graded vector space, for all $(p, q) \neq (1, n)$ with $p + q \leq n + 1$ and $p > 0$. This implies that $\pi^*(E^*)$ is trivial for all $0 < s < n + 1$. Since the cotensor product is left exact, there is a monomorphism of graded vector spaces,

$$(L \square_{C}A)^q \to (L \square_{C}C)^q$$

for all $q \leq n$. The map is also injective for $q = n + 1$ if $\pi^{n+1}(f)$ is injective. But $(L \square_{C}C)^q = 0$ for all $q > 0$, and consequently,

$$(L \square_{C}A)^q = 0$$

for all $0 < q \leq n$, because $L$ is concentrated in degree 0. By Lemma 4.35 we have an isomorphism

$$\text{Cotor}_{C}^1(L, A)^n \cong (C/L *_{C} C/\mathcal{C})^n$$

This graded vector space and $(L \square_{C}A)^{n+1}$ are the only potentially non-trivial objects of total degree $n + 1$ in the $E_2$-page. Since there is no place for non-trivial differentials in this degree, the proof of statement (a) is complete.

Claim (b) follows from the long exact sequence of cohomotopy groups and the 5-lemma. □

We list a few immediate consequences of Theorem 4.33. We will need a more general notion of primitivity.

**Definition 4.37.** Let $C \in (\mathbb{F}/\text{Coalg}_{\mathcal{V}_C})$ be a pointed graded coalgebra and $N$ a $C$-comodule. The sub-comodule $\text{Prim}_C(N)$ of the primitives under the $C$-coaction is the kernel of the map

$$N \xrightarrow{\Delta_N} C \otimes N \to (C/\mathbb{F}) \otimes N.$$

This is the subset of elements $n \in N$ such that $\Delta_N(n) = 1 \otimes n$. 
Corollary 4.38. Consider the special case of Theorem 4.33 where:
- \( L = \mathcal{F} \) (therefore \( E^* \) is the homotopy fiber of \( f \) at the chosen basepoint).
- \( \pi^{n+1}(f) \) is injective.

Let \( M = \mathcal{E}^0/\mathcal{A}^0 \) as a \( \mathcal{E}^0 \)-comodule. Then there is an isomorphism
\[
\pi^{n+1}(E^*) \cong \text{Prim}_{\mathcal{E}^0}(M).
\]
If \( M \) is a trivial \( \mathcal{E}^0 \)-comodule, then there is an isomorphism \( \pi^{n+1}(E^*) \cong M \).

Proof. It suffices to note that \((\mathcal{E}^0/\mathcal{F})*_{\mathcal{E}^0} M \cong \text{Prim}_{\mathcal{E}^0}(M)\). \( \square \)

Corollary 4.39. Let \( C \in \mathcal{C}\mathcal{A}, \ M \in \mathcal{V}C \) and consider the homotopy pullback in \( c\mathcal{C}\mathcal{A}^E \)
\[
\begin{array}{ccc}
E^* & \to & cC \\
\downarrow & & \downarrow \\
cC & \to & c(i_C(M))
\end{array}
\]
where \( C \to i_C(M) \) is the canonical inclusion. Then there is an isomorphism of \( C \)-comodules \( \pi^1(E^*) \cong M \).

Proof. Let \( D = i_C(M) \). It is enough to show that \( M*DM \cong M \). We have \( M*DM \cong \ker\{\phi: M \to M\square DM\} \). The map \( M \xrightarrow{\phi} M\square DM \subset M \otimes M \) can be factored as follows,
\[
\begin{array}{ccc}
M & \to & M \otimes M \\
\downarrow & & \downarrow \\
D & \xrightarrow{\Delta_D} & D \otimes D
\end{array}
\]
This shows that the map is trivial by definition of \( \Delta_D \) on \( M \). \( \square \)

Corollary 4.40. Let \( C^* \) be a pointed cosimplicial unstable coalgebra and \( n \geq 1 \). Then there is an isomorphism
\[
\pi^n(\Omega^nC^*) \cong \text{Prim}(\pi^0C^*).
\]
In particular, for a pointed unstable coalgebra \( C \), there is an isomorphism
\[
\pi^n(\Omega^nC) \cong \text{Prim}(C).
\]

Proof. Corollary 4.38 implies that \( \pi^1(\Omega C^*) \cong \text{Prim}(\pi^0(C^*)) \). Then the result follows by induction. \( \square \)

5. Cosimplicial spaces

In this section we study the resolution model structure on cosimplicial spaces obtained from the using the Eilenberg-MacLane space \( K(\mathbb{F}, n) \) as injective objects. The close connection to the resolution model structure \( c\mathcal{C}\mathcal{A}^E \) resulting from the representing property of the Eilenberg-MacLane spaces is used to import the homotopy theory developed in the previous section to cosimplicial spaces. We introduce a new piece of notation.

Notation 5.1. Let \( \text{sk}^n_c(-): \mathcal{S} \to \mathcal{S} \) denote the derived skeleton functor defined as the skeleton of a functorial Reedy cofibrant replacement.
5.1. Cosimplicial resolutions of spaces. Let $\mathcal{S}$ be the proper model category of simplicial sets and let $\mathbb{F}$ be any prime field. Consider $\mathcal{G} = \{K(\mathbb{F}, m) | m \geq 0\}$ as a set of injective objects for the category $\mathcal{S}$. Recall that $K(\mathbb{F}, m)$ denotes a fibrant model of an Eilenberg-MacLane space with only non-trivial homotopy group $\mathbb{F}$ in degree $m$ and that $E = H_\ast(\mathcal{G})$, where $E$ was defined in 4.1. We would like to consider the $\mathcal{G}$-resolution model structure on cosimplicial spaces in the sense of Section 3.

A map $f : X \to Y$ in $\text{Ho}(\mathcal{S})$ is $\mathcal{G}$-monic if and only if the induced map on mod-$p$ cohomology is surjective and if and only if the induced map $H_\ast (f) : H_\ast X \to H_\ast Y$ is injective. Recall that we call GEM any product of spaces of type $K(\mathbb{F}, m)$ for possibly various values of $m$. Every GEM is $\mathcal{G}$-injective.

Lemma 5.2. A fibrant $X \in \mathcal{S}$ is $\mathcal{G}$-injective if and only if it is a retract of a fibrant space weakly equivalent to a GEM. If a fibration $f : X \to Y$ is $\mathcal{G}$-injective, then the induced map $H_\ast (f) : H_\ast X \to H_\ast Y$ is $E$-injective.

Proof. $\mathcal{G}$-injectives are closed under retracts and weak equivalences, so the if part is immediate. There is a canonical $\mathcal{G}$-monic map (cf. [13, 4.5])

$$X \to \prod_{m \geq 0} \prod_{X \to K(\mathbb{F}, m)} K(\mathbb{F}, m)$$

which we can functorially factor as the composition of a $\mathcal{G}$-monic cofibration $i : X \to Z_X$ followed by a trivial fibration. In particular, $Z_X$ is again fibrant. If $X$ is $\mathcal{G}$-injective and fibrant, then the square

$$\begin{array}{ccc}
X & \to & X \\
\downarrow & & \downarrow \\
Z_X & \to & \ast
\end{array}$$

admits a lift, which means that $X$ is a retract of $Z_X$. This shows the only if part of the first statement. The second statement then follows from the fact that every $\mathcal{G}$-injective fibration $f : X \to Y$ is a retract of a fibration of the form (see [13, Lemma 3.10])

$$E \sim Y \times G \sim \mathbb{Z}_1 \sim Y$$

and the Künneth isomorphism. □

Thus, there are enough $\mathcal{G}$-injectives in the sense of Section 3. The following theorem is a direct application of Theorem 3.2.

Theorem 5.3. There is a proper simplicial model category $cS^G$ where the underlying category is the category of cosimplicial spaces, the weak equivalences are the $\mathcal{G}$-equivalences, the cofibrations are the $\mathcal{G}$-cofibrations and the fibrations are the $\mathcal{G}$-fibrations.

Proof. This is an application of Theorem 3.2. Right properness will be shown in Corollary 5.11  □

As a consequence of the representing property (2.14), we note that
• a map of cosimplicial spaces $f : X^\bullet \to Y^\bullet$ is a $G$-equivalence if and only if $H_\ast(f) : H_\ast(X^\bullet) \to H_\ast(Y^\bullet)$ is an $E$-equivalence of cosimplicial unstable coalgebras.

In particular, the functor $H_\ast$ descends to a functor

$$H_\ast : \text{Ho}(cS^G) \to \text{Ho}(cCA^E)$$

which also preserves finite products as a consequence of Theorem 2.11. More generally, Lemma 3.29 implies that

• a map $f : X^\bullet \to Y^\bullet$ is $n$-connected in $cS^G$ if and only if the induced map $H_\ast(f)$ is $n$-connected in $cCA^E$.

Also immediate is the following characterization of $G$-cofibrations:

• a map of cosimplicial spaces $f : X^\bullet \to Y^\bullet$ is a $G$-cofibration if and only if it is a Reedy cofibration and the map $H_\ast(f) : H_\ast(X^\bullet) \to H_\ast(Y^\bullet)$ is an $E$-cofibration.

These observations provide some evidence for the utility of this model category in our realization problem. First, the weak equivalences are designed so that they are actually detected in $cCA^E$ while at the same time a map of constant cosimplicial spaces $cX \to cY$ is a $G$-equivalence if and only if the underlying map of spaces $X \to Y$ is a homology equivalence. Second, the $G$-equivalences are detected on the $E_2$-page of the Tot-spectral sequence of a cosimplicial space, thus assuming convergence of the latter, we retain control of the properties of the resulting objects after totalization.

**Remark 5.4.** The monad $F : S \to S$ that sends an unpointed simplicial set $X$ to the free simplicial $F$-vector space $F[X]$ satisfies Assumption 3.20. This yields a fibrant replacement functor in $cS^G$ which commutes with filtered colimits. It is closely related to the Bousfield-Kan monad [10, Section 2.1] on pointed simplicial sets; in fact they become the same if one equips $X$ with a disjoint basepoint.

**Proposition 5.5.** The homology functor $H_\ast : cS^G \to cCA^E$ preserves the cofibers of 0-connected $G$-cofibrations $i : X^\bullet \to Y^\bullet$ (i.e. when the induced map $\pi^0 H_\ast(X^\bullet) \to \pi^0 H_\ast(Y^\bullet)$ is injective). In particular $H_\ast$ preserves the homotopy cofibers of 0-connected maps.

**Proof.** Let $i : X^\bullet \to Y^\bullet$ be a 0-connected $G$-cofibration and let $Z^\bullet$ be its cofiber. The induced map $H_\ast(i) : H_\ast(X^\bullet) \to H_\ast(Y^\bullet)$ is an $E$-cofibration and we denote its cofiber by $C^\ast$. We show that the canonical map $q : C^\ast \to H_\ast(Z^\bullet)$ is an isomorphism. Note that $\text{Hom}(C^\ast, H_\ast K(F, m))$ fits into a pullback square of simplicial groups

$$\begin{array}{ccc}
\text{Hom}(C^\ast, H_\ast K(F, m)) & \longrightarrow & \text{Hom}(H_\ast Y^\bullet, H_\ast K(F, m)) \\
\downarrow & & \downarrow H_\ast(i)^* \\
\text{Hom}(cF, H_\ast K(F, m)) & \longrightarrow & \text{Hom}(H_\ast X^\bullet, H_\ast K(F, m)).
\end{array}$$

Since the right hand vertical map is a fibration, the square is also a homotopy pullback. By (2.14), $H_\ast(i)^*$ can be identified with the fibration

$$i^* : [Y^\bullet, K(F, m)] \to [X^\bullet, K(F, m)].$$
It follows from the assumption that the map $i^*$ is surjective, that there are short exact sequences, for all $k, m \geq 0$,

$$0 \to \tilde{H}^m(Z^k) \to H^m(Y^k) \to H^m(X^k) \to 0.$$  

Applying (2.13) again, we conclude that the following is also a pullback square of simplicial groups

$$\begin{array}{ccc}
    \text{Hom}(H_*(Z^k), H_*(K(F, m))) & \longrightarrow & \text{Hom}(H_*(Y^k), H_*(K(F, m))) \\
    \downarrow & & \downarrow_{H_*(i)^*} \\
    \text{Hom}(c^n, H_*(K(F, m))) & \longrightarrow & \text{Hom}(H_*(X^k), H_*(K(F, m))).
\end{array}$$

So the map $q : C^* \to H_*(Z^k)$ induces natural isomorphisms

$$\text{Hom}(C^*, H_*(K(F, m))) \cong \text{Hom}(H_*(Z^k), H_*(K(F, m)))$$

which implies that $q$ an isomorphism by Proposition 2.15.

More generally, we have the following statement about preservation of homotopy pushouts.

**Proposition 5.6.** Let

$$\begin{array}{ccc}
    X^* & \longrightarrow & Y^* \\
    W^* & \longrightarrow & Z^*
\end{array}$$

be a homotopy $n$-cocartesian square in $cS^G$ where the map $i$ is 0-connected. Then the square

$$\begin{array}{ccc}
    H_*(X^*) & \longrightarrow & H_*(Y^*) \\
    \downarrow & & \downarrow \\
    H_*(W^*) & \longrightarrow & H_*(Z^*)
\end{array}$$

is homotopy $n$-cocartesian in $cCA^G$.

**Proof.** Assume that $i$ is a $G$-cofibration and consider the (homotopy) pushout

$$\begin{array}{ccc}
    X^* & \longrightarrow & Y^* \\
    W^* & \longrightarrow & P^*
\end{array}$$

and let $P^* \to Z^*$ be the canonical map which is $n$-connected by assumption. Then the induced map $H_*(P^*) \to H_*(Z^*)$ is $n$-connected. Let $C^*$ be the (homotopy) pushout of the maps

$$H_*(Y^*) \leftarrow H_*(X^*) \to H_*(W^*)$$

and $C^* \to H_*(P^*)$ the canonical map. Consider the diagram

$$\begin{array}{ccc}
    X^* & \longrightarrow & Y^* & \longrightarrow & Q_1^* \\
    \downarrow & & \downarrow & \longrightarrow & \downarrow \\
    W^* & \longrightarrow & P^* & \longrightarrow & Q_2^*
\end{array}$$
where the horizontal sequences are (homotopy) cofiber sequences and so the right hand side vertical map is a \( G \)-equivalence. The map \( i' \) is again 0-connected and hence by Proposition 5.5, the sequence of maps

\[
H_*(W^*) \to H_*(P^*) \to H_*(Q^*_2)
\]

is a (homotopy) cofiber sequence in \( c\mathcal{A}^E \). Then consider the following diagram

\[
\begin{array}{cccc}
H_*(W^*) & \xrightarrow{i'} & H_*(P^*) & \to H_*(Q^*_2) \\
\uparrow \text{id} & & \uparrow \sim & \\
H_*(W^*) & \to & C^* & \to H_*(Q^*_1)
\end{array}
\]

where the bottom sequence is also a (homotopy) cofiber sequence and the right vertical map is an \( E \)-equivalence. From this map of cofiber sequences follows that \( C^* \to H_*(P^*) \) is an \( E \)-equivalence. Since \( H_*(P^*) \to H_*(Z^*) \) is \( n \)-connected, it follows that \( C^* \to H_*(Z^*) \) is also \( n \)-connected, as required.

\[\square\]

5.2. Consequences of the Künneth theorem. In this subsection various important technical statements are proved, the most important one being the Homotopy Excision Theorem 5.12 for cosimplicial spaces.

**Proposition 5.7.** The homology functor \( H_* : c\mathcal{S}^G \to c\mathcal{A}^E \) preserves homotopy pullbacks.

This proposition will be useful for setting up the obstruction calculus. The following lemma shows that \( H_* \) preserves cotensors with a finite simplicial set. It generalizes Theorem 2.11.

**Lemma 5.8.** Let \( K \) be a finite simplicial set and \( X^* \) in \( c\mathcal{S} \). Then there is a natural isomorphism

\[H_*(\text{hom}(K, X^*)) \cong \text{hom}(K, H_*(X^*)�).\]

**Proof.** There is a natural isomorphism in each cosimplicial degree \( s \geq 0 \):

\[H_*(\text{hom}(K, X^*)) = H_*(\bigotimes K \text{ hom}(K, X^*)) = \bigotimes K H_*(X^*)� \text{ hom}(K, H_*(X^*))�.\]

These isomorphisms are clearly compatible with the cosimplicial structure maps.

\[\square\]

**Proposition 5.9.** The functor \( H_* : c\mathcal{S} \to c\mathcal{A} \) sends a \( G \)-cofree map on \( (G_s)_{s \geq 0} \) to an \( E \)-cofree map on \( (H_*G_s)_{s \geq 0} \) with the induced co-attaching maps. Moreover, \( H_* \) preserves pushbacks along a \( G \)-cofree map.

**Proof.** Let \( f : X^* \to Y^* \) in \( c\mathcal{S} \) be a \( G \)-cofree map on \( G \)-injectives \( (G_s)_{s \geq 0} \). By Lemma 5.17 there is a pullback diagram in \( c\mathcal{S} \) for all \( s \geq 0 \),

\[
\begin{array}{ccc}
\text{cosk}_s(f) & \to & \text{hom}(\Delta^s, cG_s) \\
\downarrow & & \downarrow \\
\text{cosk}_{s-1}(f) & \to & \text{hom}(\partial\Delta^s, cG_s)
\end{array}
\]

For every cosimplicial degree \( k \geq 0 \), we have isomorphisms

\[\text{hom}(\Delta^s, G_s)^k \cong \prod_{(\{k\} \Delta^s)^k} G_s \times \prod_{k \to s} G_s\]
and the right vertical map is given by the projection onto the first factor. Hence, for every cosimplicial degree \( k \geq 0 \), the left vertical map is isomorphic to the projection

\[
\cosk_{s-1}(f)^k \times \prod_{k \to s} G_s \to \cosk_{s-1}(f)^k.
\]

By Lemma 5.8, we conclude that the following square

\[
\begin{array}{ccc}
H_\ast(\cosk_s(f)) & \longrightarrow & \hom(\Delta^s, H_\ast G_s) \\
\downarrow & & \downarrow \\
H_\ast(\cosk_{s-1}(f)) & \longrightarrow & \hom(\tilde{\Delta}^s, H_\ast G_s)
\end{array}
\]

is a pullback diagram in \( cC\mathcal{A} \). Then \( H_\ast(\cosk_s(f)) \cong \cosk_s(H_\ast(f)) \) and the map \( H_\ast(f) : H_\ast(X^\bullet) \to H_\ast(Y^\bullet) \) is \( E \)-cofree as required. The second claim is similar. \( \square \)

**Corollary 5.10.** The homology functor \( H_\ast : cS^G \to cC\mathcal{A}^E \) sends quasi-\( G \)-cofree maps to \( E \)-cofree maps and \( G \)-fibrations to \( E \)-fibrations. Moreover, \( H_\ast \) sends pullbacks along a quasi-\( G \)-cofree Reedy fibration to pullbacks.

**Proof.** Let \( f : X^\bullet \to Y^\bullet \) be quasi-\( G \)-cofree. Then there are \( G \)-injective objects \((G_s)_{s \geq 0}\) and diagrams

\[
\begin{array}{ccc}
\cosk_s(f) & \overset{\sim}{\longrightarrow} & P^\bullet \\
\downarrow & & \downarrow \\
\cosk_{s-1}(f) & \longrightarrow & \hom(\tilde{\Delta}^s, cG_s)
\end{array}
\]

where the square is a pullback, and thus also Reedy homotopy pullback, and the map indicated by \( \sim \) is a Reedy equivalence. Applying \( H_\ast \) gives a diagram

\[
\begin{array}{ccc}
H_\ast(\cosk_s(f)) & \cong & H_\ast(P^\bullet) \\
\downarrow & & \downarrow \\
H_\ast(\cosk_{s-1}(f)) & \longrightarrow & \hom(\tilde{\Delta}^s, cH_\ast(G_s))
\end{array}
\]

where the square is a pullback by Proposition 5.9 and the indicated map is an isomorphism. It follows that \( H_\ast(f) \) is \( \mathcal{H} \)-cofree. If \( f : X^\bullet \to Y^\bullet \) is a \( G \)-fibration, then by Proposition 3.27 and Remark 5.4, \( f \) is a retract of a quasi-\( G \)-cofree map and the second claim follows. For the final claim, consider a pullback square

\[
\begin{array}{ccc}
E^\bullet & \longrightarrow & X^\bullet \\
\downarrow & f' \downarrow & \downarrow f \\
Y^\bullet & \longrightarrow & Z^\bullet
\end{array}
\]
where \( f \) is a quasi-\( \mathcal{G} \)-cofree Reedy fibration. Then there are \( \mathcal{G} \)-injective objects \((G_s)_{s \geq 0}\) and diagrams

\[
\begin{array}{ccc}
\text{cosk}_s(f') & \text{cosk}_s(f) \\
\downarrow & \downarrow \\
Q^* & P^* \\
\downarrow & \downarrow \\
\text{cosk}_{s-1}(f') & \text{cosk}_{s-1}(f) \\
\end{array}
\]

\[
\text{hom}(\Delta^s, cG_s) \\
\text{hom}(\Delta^s, cG_s)
\]

where the two front squares and the back square are pullbacks and the indicated maps are Reedy equivalences. The map \( \text{cosk}_s(f) \to \text{cosk}_{s-1}(f) \) is a Reedy fibration, which shows that the back square is a homotopy pullback and justifies why the top left map is a Reedy equivalence. Applying \( H_* \) yields the required result. \( \square \)

**Proof of Proposition 5.7.** By Proposition 3.21 any homotopy pullback in the \( \mathcal{G} \)-resolution model structure can be replaced up to \( \mathcal{G} \)-equivalence by a strict pullback diagram where the right hand vertical map is \( \mathcal{G} \)-cofree. Since \( H_* \) sends \( \mathcal{G} \)-equivalences to \( \mathcal{E} \)-equivalences, the result follows from Proposition 5.9. \( \square \)

**Corollary 5.11.** The model category \( cS^0 \) is right proper.

**Proof.** By Corollary 3.27 it suffices to show that the pullback of a \( \mathcal{G} \)-equivalence along a quasi-\( \mathcal{G} \)-cofree Reedy fibration is again a \( \mathcal{G} \)-equivalence. This follows from Corollary 5.10 and Corollary 4.13. \( \square \)

From Propositions 5.6 and 5.7 we also obtain the analogue of Theorem 4.17.

**Theorem 5.12 (Homotopy excision for cosimplicial spaces).** Let

\[
\begin{array}{ccc}
E^* & \rightarrow & X^* \\
\downarrow & & \downarrow f \\
Y^* & g & Z^* \\
\end{array}
\]

be a homotopy pullback square in \( cS^0 \) where \( f \) is \( m \)-connected and \( g \) is \( n \)-connected. Then the square is homotopy \( (m + n) \)-cocartesian.

**Proof.** Let \( W^* \) be the homotopy pushout of the diagram

\[
Y^* \leftarrow E^* \rightarrow X^* \\
Y^* \rightarrow Z^*
\]

and \( W^* \to Z^* \) the canonical map. We need to show that the induced map \( H_*(W^*) \to H_*(Z^*) \) is \( (m + n) \)-connected. We may assume that either \( m \) or \( n \) is \( \geq 0 \). By Proposition 5.7 the induced square

\[
\begin{array}{ccc}
H_*(E^*) & \rightarrow & H_*(X^*) \\
\downarrow & & \downarrow H_*(f) \\
H_*(Y^*) & \rightarrow & H_*(Z^*) \\
\end{array}
\]

is a homotopy pullback. The homotopy excision theorem for coalgebras shows that the square is also homotopy \( (m + n) \)-cocartesian. The homotopy excision theorem for coalgebras also shows that \( H_*(E^*) \to H_*(X^*) \) is \( n \)-connected and \( H_*(E^*) \to \)
$H_s(Y^*)$ is $m$-connected. Then, by Proposition $\ref{prop:freudenthal}$, $H_s(W)$ is the homotopy pushout of

$$H_s(Y^*) \leftarrow H_s(E^*) \rightarrow H_s(X).$$

and so the canonical map $H_s(W^*) \rightarrow H_s(Z^*)$ is $(m+n)$-connected. 

An important consequence of homotopy excision is the following version of the Freudenthal suspension theorem for the $G$-resolution model category of cosimplicial spaces. The super- and subscripts in the following statements indicate the respective

Corollary $\ref{cor:suspension}$. Let $X^*$ be a pointed cosimplicial space which is cosimplicially $n$-connected, i.e. \( \pi^s H_t(X^*) = 0 \) for $0 < s < n$ and $t > 0$ and \( \pi^0 H_0(X^*) = F \). Then the canonical map

$$\Sigma_c \Omega_f(X^*) \rightarrow X^*$$

is $2n$-connected.

In particular, we have

Corollary $\ref{cor:suspension2}$. Let $X^*$ be as above and assume in addition that $\pi^s H_s(X^*) = 0$ for $s > 2n - 1$. Then the canonical map

$$\Sigma_c \sk^c_{2n+1} \Omega_f(X^*) \rightarrow X^*$$

is a $G$-equivalence. In particular, $X^*$ admits a desuspension in $cS^G$.

5.3. Objects of type $L_C(M,n)$. In this subsection we define and construct objects of type $L_C(M,n)$ in $cS^G$. They are formal analogues of twisted Eilenberg-MacLane spaces in the setting of this resolution model category. These objects are closely related to the objects of type $K_C(M,n)$. This relation is a central fact that will allow the ‘algebraization’ of our obstruction theory.

Definition $\ref{def:objects}$. Let $C$ be an object of $\CA$. An object $X^*$ in $cS^G$ is said to be of type $L_C(M,0)$ if there are isomorphisms

$$\pi^s_c(X^*,G) \cong \begin{cases} \Hom_{\CA}(C,H_s G) & \text{if } s = 0, \\ 0 & \text{otherwise} \end{cases}$$

of $\mathcal{H}$-algebras defined in a general context in $\text{[B.11]}$ and in our setting identified in Theorem $\text{[C.12]}$. The data is equivalent to the structure of an unstable algebra.

Recall that $\sk^c_n(-)$ denotes the derived skeleton functor.

Definition $\ref{def:objects2}$. Let $C \in \CA$, $M \in V_C$ and $n \geq 1$. An object $X^*$ in $cS^G$ is said to be of type $L_C(M,n)$ if the following are satisfied:

(a) there are isomorphisms

$$\pi^s_c(X^*,G) \cong \begin{cases} \Hom_{\CA}(C,H_s G) & \text{if } s = 0, \\ \Hom_{\CA}(\iota_C(M),H_s G) & \text{if } s = n, \\ 0 & \text{otherwise} \end{cases}$$

where $H_s G$ is viewed as an object under $C$ with respect to the map $C \rightarrow F \rightarrow H_s G$. The first isomorphism is required to be an isomorphism of $\mathcal{H}$-algebras. The second one has to be an isomorphism of $\pi_0$-modules which are defined in $\text{[B.21]}$ and made explicit in Subsection $\text{[C.3]}$.

(b) there is a map $\eta : X^* \rightarrow X^*_0$ to an object of type $L(C,0)$ such that the composite map $\sk^c_n X^* \rightarrow X^* \rightarrow X^*_0$ is a $G$-equivalence.
A structured object of type $L_C(M,n)$ is a pair $(X^\bullet, \eta)$ where $X^\bullet$ is an object of type $L_C(M,n)$ and $\eta : X^\bullet \to X_0^\bullet$ is a map as in (b) above.

**Remark 5.17.** Let $X^\bullet \in cS^G$. Then the following statements are equivalent:

1. $X^\bullet$ is of type $L_C(M,n)$.
2. $X^\bullet$ is $(n+1)$-skeletal, i.e. $\text{sk}^c_{n+1}(X^\bullet) \cong X^\bullet$, and

   a. there are isomorphisms

   $$\pi^s H_\bullet(X^\bullet) \cong \begin{cases} C & \text{if } s = 0 \\ C[1] & \text{if } s = 2 \\ M & \text{if } s = n \\ M[1] & \text{if } s = n + 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (if $n \neq 2$, otherwise modify accordingly) where the first is an isomorphism of unstable coalgebras and the others are isomorphisms of $C$-comodules.

   b. as above.

The equivalence is an easy consequence of the spiral exact sequence once one recalls the following facts: In Subsection C.3 it is explained that an $H$-algebra is the same as an unstable algebra and that the $\pi_0$-module structure translates to the usual $\pi_0 H^\bullet(X^\bullet)$-action on $\pi_s H^\bullet(X^\bullet)$. Secondly, since we work over a field $\mathbb{F}$ the setting dualizes nicely to unstable coalgebras and comodules.

We will make use of the following key observation:

**Proposition 5.18.** Suppose $X^\bullet$ is a cosimplicial space such that

1. there are isomorphisms

   $$\pi^s H_\bullet(X^\bullet) \cong \begin{cases} C & \text{if } s = 0 \\ C[1] & \text{if } s = 2 \\ M & \text{if } s = n \\ 0 & \text{if } 0 < s < n, s \neq 2. \end{cases}$$

   (if $n \neq 2$, otherwise modify accordingly) where the first is an isomorphism of unstable coalgebras and the others are isomorphisms of $C$-comodules.

2. as above.

Then $\text{sk}^c_{n+1}(X^\bullet)$ is an object of type $L_C(M,n)$.

**Proof.** Switching to degreewise $\mathbb{F}$-vector space duals $(-)^\vee$ we obtain from (a) isomorphisms between the terms $\pi_s H^\bullet(X^\bullet)$ and the corresponding duals on the right side for $0 \leq s \leq n$. One plugs this into the spiral exact sequence with the knowledge that $\text{sk}^c_{n+1}$ kills all natural homotopy groups above dimension $n$. This forces an isomorphism

$$\pi_{n+2} H^\bullet(X^\bullet) \cong (\{\pi_n^a(X^\bullet, K(\mathbb{F}, n))\}_{n \geq 0})^\vee \cong M[1]$$

as $C$-comodules. Here the middle term is a $C$-comodule by Lemma [C.15]. By the same lemma the left side has the usual $C \cong \pi_0 H^\bullet(X^\bullet)$-comodule structure. So we have checked condition (2) from Remark 5.17.

Similarly to $K$-objects, it will also be convenient to allow objects of type $L_C(M,n)$ for $n = 0$. 

□
Definition 5.19. Let \( C \in \mathcal{C}A \) and \( M \in \mathcal{V}C \). An object of type \( L_C(M, 0) \) is simply an object of type \( L(\iota_C(M), 0) \). A structured object of type \( L_C(M, 0) \) is a pair \((X^\bullet, \eta)\) where \( X^\bullet \) is an object of type \( L_C(M, 0) \) and \( \eta : X^\bullet \to X_0^\bullet \) is a map to an object of type \( L(C, 0) \) which induces up to isomorphism the natural projection map \( \iota_C(M) \to C \) on \( \pi^0H_* \)-groups.

We now show how to construct objects of type \( L_C(M, n) \).

Construction 5.20. Let \( C \in \mathcal{C}A \). We first construct an object of type \( L(C, 0) \). The start of a cofree resolution of \( C \) gives a pullback square (in \( \mathcal{C} - \) but the maps are in \( \mathcal{C}A \))

\[
\begin{array}{ccc}
C & \longrightarrow & I^0 \\
\downarrow & & \downarrow \delta_0 \\
I^0 & \longrightarrow & I^1
\end{array}
\]

There exist GEMs \( G^0 \) and \( G^1 \) and maps \( \delta_0, \delta_1 : G^0 \to G^1 \) such that \( \delta_j = H_*(d_j^\prime) \).

Consider the homotopy pullback square in \( \mathcal{C}A^E \)

\[
\begin{array}{ccc}
X^\bullet & \longrightarrow & cG^0 \\
\downarrow & & \downarrow \\
cG^0 & \longrightarrow & cG^1
\end{array}
\]

By the homotopy excision theorem, this square is homotopy 1-cartesian in \( \mathcal{C} - \). This means that \( \pi^0H_*(X^\bullet) \) is isomorphic to \( C \) and the isomorphism is induced by a canonical map \( cC \to H_*(X^\bullet) \) in \( Ho(\mathcal{C}A^E) \). It follows that \( sk_1^C X^\bullet \) is of type \( L(C, 0) \).

Construction 5.21. Let \( C \in \mathcal{C}A \) and \( M \in \mathcal{V}C \). We give a construction of an object of type \( L_C(M, 1) \). Consider a homotopy pullback square as follows:

\[
\begin{array}{ccc}
E^\bullet & \longrightarrow & L(C, 0) \\
\downarrow & & \downarrow \\
L(C, 0) & \longrightarrow & L(\iota_C(M), 0)
\end{array}
\]

where the indicated \( L \)-objects are given by the construction above. We claim that \( sk_2H_*(E^\bullet) \) is an object of type \( K_C(M, 1) \). In fact, this requires a little bit more than what is immediately deducible from the homotopy excision because the latter only shows that \( \pi^1H_*(E^\bullet) \) injects into \( M \). However, there is a map of squares from the square below (cf. Remark 4.25):

\[
\begin{array}{ccc}
K_C(M, 1) & \longrightarrow & K(C, 0) \\
\downarrow & & \downarrow \\
K(C, 0) & \longrightarrow & K(\iota_C(M), 0)
\end{array}
\]
to the square induced by the homotopy pullback above

\[
\begin{array}{ccc}
\text{sk}_2 H_a(E^*) & \longrightarrow & H_a L(C,0) \\
\downarrow & & \downarrow \\
H_a L(C,0) & \longrightarrow & H_a L(\iota_C(M),0)
\end{array}
\]

which implies that \( M \) is also a retract of \( \pi_1 H_a(E^*) \), thus yielding a proof of the claim.

Now we claim that \( \text{sk}_2(E^*) \) is of type \( L_C(M,1) \). The spiral exact sequence shows the required isomorphism on \( \pi_0^s(-,-) \)-groups. Moreover, there is an epimorphism

\[
\pi_1^s(\text{sk}_2 E^*, G) \rightarrow \pi_1[\text{sk}_2 E^*, G] \cong \text{Hom}_{C/\mathcal{A}}(\iota_C(M), H_a G).
\]

The last isomorphism is a consequence of the spiral exact sequence and the fact that \( \text{sk}_2(E^*) \rightarrow E^* \) induces isomorphisms on the first two natural homotopy groups. Thus it suffices to show that this epimorphism is actually an isomorphism. This follows from the spiral exact sequence after we note that there is a commutative diagram

\[
\begin{array}{ccc}
\pi_0^s(L(C,0), \Omega G) & \longrightarrow & \pi_1^s(L(C,0), G) = 0 \\
\downarrow & & \downarrow \\
\pi_0^s(\text{sk}_2 E^*, \Omega G) & \longrightarrow & \pi_1^s(\text{sk}_2 E^*, G)
\end{array}
\]

which is induced by the map \( \text{sk}_2 E^* \rightarrow E^* \rightarrow L(C,0) \) and therefore the bottom connecting map is trivial.

**Construction 5.22.** Let \( n > 1, C \in \mathcal{C} \mathcal{A} \) and \( M \in V_C \). We give an inductive construction of an object of type \( L_C(M,n) \). Let \( X^* \) be of type \( L_C(M,n-1) \) and consider the homotopy pullback square

\[
\begin{array}{ccc}
E^* & \longrightarrow & \text{sk}_1^s(X^*) \\
\downarrow & & \downarrow \\
\text{sk}_1^s(X^*) & \longrightarrow & X^*
\end{array}
\]

where \( \text{sk}_1(X^*) \) is an object of type \( L(C,0) \). Then, by Theorem 5.12, it follows easily that \( \text{sk}_{n+1}^s(E^*) \) is an object of type \( L_C(M,n) \). We can always regard this as structured by declaring one of the maps to \( \text{sk}_1^s(X^*) \) to be the structure map.

**Remark 5.23.** Let \( (X^*, \eta) \) be a structured object of type \( L_C(M,n) \) and \( n > 0 \). Then the homotopy pushout of the maps

\[
X^*_0 \xrightarrow{\eta} X^* \xrightarrow{\eta} X^*_0
\]

is a structured object of type \( L_C(M,n-1) \).

It is a consequence of Proposition 5.28 below that structured objects of type \( L_C(M,n) \) are homotopically unique. To show this, we will objects of type \( L_C(M,n) \) with objects of type \( K_C(M,n) \). Although \( H_a(L_C(M,n)) \) is not an object of type
$K_C(M, n)$, we can extract such an object as follows. Consider the homotopy pullback square:

$$
\begin{array}{ccc}
D^* & \xrightarrow{j} & H_*(L_C(M, n)) \\
\downarrow & & \downarrow \\
\text{sk}_1 H_*(L(C, 0)) & \xrightarrow{} & H_*(L(C, 0))
\end{array}
$$

where the right hand side vertical map is the structure map and $\text{sk}_1 H_*(L(C, 0))$ is an object of type $K(C, 0)$. Then by the homotopy excision theorem (Theorem 4.17), we can conclude that $\text{sk}_{n+1}(D^*)$ is a structured object of type $K_C(M, n)$. For a fixed cofibrant choice of $L_C(M, n)$, let $K_C(M, n) := \text{sk}_{n+1}(D^*)$. Then there is a functor

$$
\phi(X^*): W^c_{\text{Hom}}(L_C(M, n), X^*) \rightarrow W^c_{\text{Hom}}(K_C(M, n), H_*X^*)
$$

which is defined on objects by sending

$$
L_C(M, n) \xrightarrow{f} U \xrightarrow{\sim} X^*
$$

to the object

$$
K_C(M, n) \rightarrow H_*(U) \xrightarrow{\sim} H_*(X^*)
$$

where the first map is canonically defined by $j$ and $f$. See Appendix [D] for the definition of $W^c_{\text{Hom}}(-, -)$ and its properties.

**Proposition 5.24.** The functor $\phi(X^*)$ induces a weak equivalence of classifying spaces.

**Proof.** We can always replace $X^*$ up to $G$-equivalence by a $G$-cofree object. So we assume that $X^*$ is $G$-cofree. Because both source and target of the map $\phi(X^*)$ preserve homotopy pullbacks, we can use the decomposition of Lemma [4.17] for cofree objects and induction to reduce the proof to the special case $X^* = \text{hom}_{\text{ext}}(\partial \Delta^*, cG)$ for a $G$-injective object $G$. Using Proposition [D.7] and Remark [D.8] we pass to the corresponding (derived) simplicial mapping spaces and proceed by direct inspection. We have

$$
\text{map}^{\text{der}}(L_C(M, n), (cG)^{\partial \Delta^*}) \simeq \text{map}^{\text{der}}(\partial \Delta^*, \text{map}^{\text{der}}(L_C(M, n), cG))
$$

and

$$
\text{map}^{\text{der}}(K_C(M, n), (cH_*G)^{\partial \Delta^*}) \simeq \text{map}^{\text{der}}(\partial \Delta^*, \text{map}^{\text{der}}(K_C(M, n), cH_*G))
$$

It is easy to see that the map $\phi((cG)^{\partial \Delta^*})$ is homotopic to $(\phi(cG))^{\partial \Delta^*}$. Therefore it suffices to consider only the case of $\phi(cG)$. Inspection shows that the maps on the two non-trivial homotopy groups of these mapping spaces, $\pi_0$ and $\pi_n$, are the duals of the maps on $\pi^0$ and $\pi^n$, respectively, induced by $K_C(M, n) \rightarrow D^* \rightarrow H_*(L_C(M, n))$, thus they are isomorphisms. \[\square\]

**Remark 5.25.** We record an obvious variation of the weak equivalence $\phi(X^*)$ for later purposes. Let

$$
W^c_{\pi_0}(L_C(M, n), X^*) \subset W^c_{\text{Hom}}(L_C(M, n), X^*)
$$

denote the subcategory defined by the maps $f: L_C(M, n) \rightarrow U \sim X^*$ which induce an isomorphism on $\pi^0 H_*$ and $\pi^n H_*$. Similarly let

$$
W^c_{\pi_0}(K_C(M, n), D^*) \subset W^c(K_C(M, n), D^*)
$$
denote the subcategory defined by maps \( u : K_C(M, n) \to V \sim D^\bullet \) which induce an isomorphism on \( \pi^0 \) and \( \pi^n \), that is, weak equivalences. Note that each of these is a (possibly empty) union of connected components of the corresponding categories. Therefore the functor \( \phi(X^\bullet) \) restricts to a weak equivalence between the classifying spaces of these subcategories.

**Remark 5.26.** Let \( Y^\bullet \) be a fibrant cosimplicial space and \( L(C, 0) \to Y^\bullet \) a map in \( cS^\emptyset \). Using standard homotopical algebra arguments, it can be shown that Proposition 5.24 extends to give weak equivalences between mapping spaces in the slice categories. That is, there are weak equivalences

\[
\text{map}^\text{der}_{L(C, 0)/cS^\emptyset}(L_C(M, n), Y^\bullet) \cong \text{map}^\text{der}_{cS^G/L_C}(K_C(M, n), H_*(Y^\bullet)).
\]

In particular, \( L_C(M, n) \) together with a choice of map \( L(C, 0) \to L_C(M, n) \) represent André-Quillen cohomology in cosimplicial spaces due to Theorem 4.28.

**Corollary 5.27.** Let \( Y^\bullet \) be a potential \( k \)-stage for \( D \) and \( n \leq k < \infty \) and fix a map \( \varphi : C \to D \). Then \( Y^\bullet \) can be viewed as an object under \( L(C, 0) \) via the map

\[
L(C, 0) \xrightarrow{\varphi} L(D, 0) = \text{sk}^0_1 Y^\bullet \to Y^\bullet
\]

and there is an identification natural in \( C \) and \( D \):

\[
\pi_*\text{map}^\text{der}_{L(C, 0)/cS^\emptyset}(L_C(M, n), Y^\bullet) \cong\begin{cases} 
AQ^n_\varphi(D, M), & \text{for } 0 \leq s \leq n \\
0, & \text{else}
\end{cases}
\]

After this short detour we are now ready to determine the homotopy type of the moduli space of structured objects of type \( L_C(M, n) \):

\[
\mathcal{M}(L_C(M, n) \to L(C, 0)).
\]

The definition of this space is completely analogous to the corresponding moduli space of \( K \)-objects, see subsection 4.3. It is the classifying space of the category \( \mathcal{W}(L_C(M, n) \to L(C, 0)) \) whose objects are structured objects of type \( L_C(M, n) \):

\[
X^\bullet \xrightarrow{\eta} X_0^\bullet
\]

and morphisms are square diagrams as follows

\[
\begin{array}{ccc}
X^\bullet & \xrightarrow{\eta} & X_0^\bullet \\
\downarrow & & \downarrow \\
Y^\bullet & \xrightarrow{\eta'} & Y_0^\bullet
\end{array}
\]

where the vertical arrows are weak equivalences in \( cS^\emptyset \).

**Proposition 5.28.** Let \( C \in \mathcal{C}A, M \in \mathcal{V}C \) and \( n \geq 0 \). Then:

(a) There is a weak equivalence \( \mathcal{M}(L(C, 0)) \cong B\text{Aut}(C) \).

(b) There is a weak equivalence \( \mathcal{M}(L_C(M, 0) \to L(C, 0)) \cong B\text{Aut}(M) \).

(c) There is a weak equivalence

\[
\mathcal{M}(L_C(M, n + 1) \to L(C, 0)) \cong \mathcal{M}(L_C(M, n) \to L(C, 0)).
\]
Proof. (a) By Proposition 4.26, it suffices to show that \( \mathcal{M}(L(C,0)) \simeq \mathcal{M}(K(C,0)) \). We apply Proposition 5.24 and compare directly the spaces of homotopy automorphisms of these two objects. We have a weak equivalence
\[
\text{map}^\text{der}(L(C,0), L(C,0)) \simeq \text{map}^\text{der}(K(C,0), H_*(L(C,0))).
\]
There is a functor
\[
F : \mathcal{W}_{\text{hom}}(K(C,0), H_*(L(C,0))) \rightarrow \mathcal{W}_{\text{hom}}(K(C,0), \text{sk}_2 H_*(L(C,0)))
\]
which takes a zigzag \( K(C,0) \rightarrow U \xrightarrow{\sim} H_*(L(C,0)) \) to the zigzag
\[
K(C,0) \xrightarrow{\sim} \text{sk}_2(K(C,0)) \rightarrow \text{sk}_2 U \xrightarrow{\sim} \text{sk}_2 H_*(L(C,0))
\]
where the last object is of type \( K(C,0) \). It is easy to see that this functor defines a homotopy inverse to the obvious map induced by \( \text{sk}_2 H_*(L(C,0)) \rightarrow H_*(L(C,0)) \). Then the result follows by passing to the appropriate components.

(b) Let \( \mathcal{MAP}_{\sim_\ast}(L_C(M,0), L(C,0)) \) and \( \mathcal{MAP}_{\sim}(K_C(M,0), K(C,0)) \) be the classifying spaces of the categories \( \mathcal{W}_{\sim_\ast}(L_C(M,0), L(C,0)) \) and \( \mathcal{W}_{\sim}(K_C(M,0), K(C,0)) \) respectively (see Remark 5.25). The results of Appendix D show that there is a homotopy fiber sequence
\[
\mathcal{MAP}_{\sim}(K_C(M,0), K(C,0)) \rightarrow \mathcal{M}(K_C(M,0) \rightarrow K(C,0))
\]
which, by (a), is a weak equivalence on base spaces and fibers. The result then follows from Proposition 4.26(b).

(c) Similarly to Proposition 4.26 using Theorem 5.12 (or Proposition 5.30 below).

As a consequence, we have the following

**Corollary 5.29.** Let \( C \in \mathcal{CA}, M \in \mathcal{VC} \) and \( n \geq 0 \). Then there is a weak equivalence
\[
\mathcal{M}(L_C(M, n) \rightarrow L(C,0)) \simeq B\text{Aut}_C(M).
\]
In particular, the moduli spaces of structured \( L \)-objects are connected.

5.4. **Postnikov decompositions.** Similarly to the case of unstable coalgebras, the homotopy excision theorem for cosimplicial spaces shows that the skeletal filtration of a cosimplicial space is principal, in the sense that the successive inclusions can be written as homotopy pushouts. This is a direct consequence of the following proposition.
Proposition 5.30. Let \( f : X^* \to Y^* \) be an \( n \)-connected map in \( cS^G \), \( n \geq 1 \), and \( C = \pi^0 H_\ast(Y^*) \). Let \( Z^* \) be the homotopy cofiber of \( f \) and \( M = \pi^n(H_\ast(Y^*)) \). Consider the homotopy pullback

\[
\begin{array}{ccc}
E^* & \longrightarrow & X^* \\
\downarrow & & \downarrow f \\
\sk^c_1(Y^*) & \longrightarrow & Y^*
\end{array}
\]

Then:

(a) \( M \) is naturally a \( C \)-comodule and the object \( \sk^c_{n+2}(E^*) \) together with the canonical map \( \sk^c_{n+2}(E^*) \to \sk^c_1(Y^*) \) define a structured object of type \( L_C(M, n+1) \).

(b) If \( Z^* \) is an object of type \( L_\bar{G}(M, n) \), then the diagram

\[
\begin{array}{ccc}
\sk^c_{n+2}(E^*) & \longrightarrow & X^* \\
\downarrow & & \downarrow f \\
\sk^c_1(Y^*) & \longrightarrow & Y^*
\end{array}
\]

is a homotopy pushout.

Proof. (a) This is a consequence of the homotopy excision theorems. The induced square

\[
\begin{array}{ccc}
H_\ast(E^*) & \longrightarrow & H_\ast(X^*) \\
\downarrow & & \downarrow f \\
H_\ast(\sk^c_1(Y^*)) & \longrightarrow & H_\ast(Y^*)
\end{array}
\]

is a homotopy pullback. Then Theorem 4.17(a) shows that \( H_\ast(E^*) \to H_\ast(\sk^c_1(Y^*)) \) is \( n \)-connected and \( \pi^{n+1}H_\ast(E^*) \) is isomorphic to \( M \). It follows that \( \sk^c_{n+2}(E^*) \) is an object of type \( L_C(M, n+1) \) (cf.

Remark 5.17). (b) follows easily from the long exact sequence of natural homotopy groups.

Similarly to unstable coalgebras, it follows that for every \( X^* \in cS^G \), the skeletal filtration

\[
\sk^c_1(X^*) \to \sk^c_2(X^*) \to \cdots \to \sk^c_n(X^*) \to \cdots \to X^*
\]

is defined by homotopy pushouts as follows

\[
\begin{array}{ccc}
L_C(M, n+1) & \longrightarrow & \sk^c_n(X^*) \\
\downarrow & & \downarrow \\
L(C, 0) & \longrightarrow & \sk^c_{n+1}(X^*)
\end{array}
\]

where \( C = \pi^0(H_\ast(X^*)) \) and \( M = \pi^n(H_\ast(X^*)) \). Moreover, the collection of maps \( w_n \) may be regarded as analogues of the Postnikov \( k \)-invariants in the context of cosimplicial spaces.

The next proposition reformulates and generalizes Proposition 5.30 in terms of moduli spaces. For \( X^* \in cS \) such that \( \sk^c_n(X^*) \simeq X^* \), for some \( n \geq 1 \), and \( M \) a coabelian \( \pi^0(H_\ast(X^*)) \)-comodule, let

\[
\mathcal{W}(X^* + (M, n))
\]
be the subcategory of cosimplicial spaces $Y^\bullet$ whose (derived) $n$-skeleton is $\mathcal{G}$-equivalent to $X^\bullet$ and $\pi^n(H_*Y^\bullet)$ is isomorphic to $M$ (as comodules). The morphisms are given by weak equivalences between such objects. The classifying space of this category, denoted by

$$\mathcal{M}(X^\bullet + (M,n)),$$

is the moduli space of $(n+1)$-skeletal extensions of $n$-skeletal object $X^\bullet$ by the comodule $M$.

The following is completely analogous to Proposition 4.30.

**Proposition 5.31.** Let $n \geq 1$ and suppose that $X^\bullet$ is an object of $cS^G$ such that $sk_n X^\bullet \simeq X^\bullet$. Let $M$ be an object of $V\pi^0(H_*X^\bullet))$. Then there is a natural weak equivalence

$$\mathcal{M}(X^\bullet + (M,n)) \simeq \mathcal{M}(L(C,0) \leftarrow L_C(M,n+1) \hookrightarrow X^\bullet).$$

**Proof.** Similarly to Proposition 4.30 using Proposition 5.30. □

The relationship between $L$- and $K$-objects can be further refined to show that ‘attachments’ of structured $L$-objects are determined ‘algebraically’ by the corresponding ‘attachments’ of structured $K$-objects. This can be expressed elegantly in terms of a homotopy pullback of the respective moduli spaces. See Appendix D for the definition and properties of these moduli spaces of maps.

**Theorem 5.32.** Let $X^\bullet \in cS^G$, $C \in \mathcal{C}A$ and $M \in VC$. Then there is a homotopy pullback square for every $n \geq 0$:

$$\begin{array}{ccc}
\mathcal{M}(L(C,0) \leftarrow L_C(M,n) \hookrightarrow X^\bullet) & \longrightarrow & \mathcal{M}(K(C,0) \leftarrow K_C(M,n) \hookrightarrow H_*(X^\bullet)) \\
\mathcal{M}(X^\bullet) & \longrightarrow & \mathcal{M}(H_*(X^\bullet)).
\end{array}$$

**Proof.** The top map is induced by a functor

$$\mathcal{W}(L(C,0) \leftarrow L_C(M,n) \hookrightarrow X^\bullet) \rightarrow \mathcal{W}(K(C,0) \leftarrow K_C(M,n) \hookrightarrow H_*(X^\bullet))$$

which is defined following the recipe of the definition of the functor $\phi(X^\bullet)$ and Proposition 5.24. Consider the following factorization of the diagram

$$\begin{array}{ccc}
\mathcal{M}(L(C,0) \leftarrow L_C(M,n) \hookrightarrow X^\bullet) & \longrightarrow & \mathcal{M}(K(C,0) \leftarrow K_C(M,n) \hookrightarrow H_*(X^\bullet)) \\
\mathcal{M}(L(C,0) \leftarrow L_C(M,n)) \times \mathcal{M}(X^\bullet) & \longrightarrow & \mathcal{M}(K(C,0) \leftarrow K_C(M,n)) \times \mathcal{M}(H_*(X^\bullet)) \\
\mathcal{M}(X^\bullet) & \longrightarrow & \mathcal{M}(H_*(X^\bullet)).
\end{array}$$

It suffices to show that both squares are homotopy pullbacks. By Proposition 5.24 and Theorem D.10 the induced map between the homotopy fibers of the top pair of vertical maps is a weak equivalence, thus the top square is a homotopy pullback. The bottom square is clearly a homotopy pullback by previous results on the moduli spaces of structured $K$- and $L$-objects. □
Remark 5.33. Let
\[ \mathcal{M}(L(C, 0) \leftarrow L_C(M, n) \Rightarrow X^*) \]
denote the moduli subspace of maps \( W \leftarrow U \rightarrow V \), where \( U \rightarrow W \) is a structured object of type \( L_C(M, n), V \simeq X^* \), and \( \phi(V)(f) \) is a weak equivalence. Similarly, let
\[ \mathcal{M}(K(C, 0) \leftarrow K_C(M, n) \Rightarrow C^*) \]
denote the moduli subspace of maps \( W \leftarrow U \rightarrow V \), where \( U \rightarrow W \) is a structured object of type \( K_C(M, n), V \simeq C^* \), and \( f \) is a weak equivalence. By Remark 5.25, it follows that the corresponding statement where we replace the arrows \( \rightarrow \) with arrows \( \Rightarrow \) in Theorem 5.32 is also true (with the same proof).

6. Moduli spaces of topological realizations

In this section we give a description of the moduli space of topological realizations \( \mathcal{M}_{\text{Top}}(C) \) of an unstable coalgebra \( C \). This moduli space is a space whose set of connected components is the set of non-equivalent realizations and the homotopy type of each component is that of the homotopy automorphisms of the corresponding realization. That is, \( \mathcal{M}_{\text{Top}}(C) \) is homotopy equivalent to
\[ \bigsqcup X \text{BAut}^h(X) \]
where the disjoint union is indexed over spaces \( X \) with \( H_\ast(X) \cong C \), one in each equivalence class. The actual definition of \( \mathcal{M}_{\text{Top}}(C) \) is given as a moduli space in the sense of Appendix D. This point of view, due to Dwyer and Kan, is essential in what follows. We emphasize that we work here with the Bousfield localization of spaces at \( H_\ast(-, \mathbb{F}) \)-equivalences, and accordingly equivalence classes of realizations are understood in this localized sense.

The description of the realization space \( \mathcal{M}_{\text{Top}}(C) \) breaks it into a tower of moduli spaces of approximate realizations. The precise meaning of this will be explained in Subsection 6.1. Moreover, this tower of moduli spaces is determined recursively by André-Quillen cohomology spaces of \( C \), i.e. spaces whose homotopy groups are André-Quillen cohomology groups of \( C \). Finally, in Subsection 6.3 we discuss how these results readily yield obstruction theories for the existence and uniqueness of realizations in terms of the André-Quillen cohomology of the unstable coalgebra.

The arguments of this section are heavily based on the homotopy excision theorems of the previous sections. This means that we are going to make frequent use of homotopy pullbacks and pushouts. Since these constructions will be required to be functorial for the arguments, we assume from the start fixed functorial models for such constructions, that can be found using standard methods of homotopical algebra, and omit the details pertaining to this or related issues.

6.1. Potential \( n \)-stages. Our description of the realization space of an unstable coalgebra will be given in terms of a sequence of moduli spaces of cosimplicial objects which in a certain sense approximate actual topological realizations regarded as constant cosimplicial objects. The meaning of this approximation is expressed in the model category \( cS^G \), and not in the model category of spaces.

If \( X \) is a realization of \( C \), i.e. \( H_\ast(X) \cong C \), then the associated cosimplicial object \( cX \in cS^G \) obviously has the property that \( H_\ast(cX) \) is an object of type \( K(C, 0) \). More generally, it will be shown in Theorem 6.13 that if \( X^* \) is a (fibrant)
cosimplicial space such that $H_\bullet(X^\bullet)$ is an object of type $K(C,0)$, then $\text{Tot}(X^\bullet)$ is a realization of $C$ assuming the convergence of the homology spectral sequence for $\text{Tot}$. This motivates the following notion of a cosimplicial space realizing (a cosimplicial resolution of) $C$.

**Definition 6.1.** A cosimplicial space $X^\bullet$ is called an $\infty$-stage for $C$ if $H_\bullet(X^\bullet)$ is an object of type $K_\bullet C,0$.

Before we state the following definition of an approximate $\infty$-stage, let us first discuss how one could attempt to construct one. The cosimplicial space $L_\bullet C,0$ is already at our disposal, satisfies $H_\bullet L_\bullet C,0 \in \mathcal{W}(K(C,0) + (C[1], 2))$.

A natural strategy then is to start an obstruction theory by ‘killing’ the non-trivial $C$-comodule in $\pi_2$ of $H_\bullet L_\bullet C,0$. If this is possible, then the spiral exact sequence will force a new non-trivial $C$-comodule in $\pi_3$ of a recognizable form. This motivates the definition.

**Definition 6.2.** A cosimplicial space $X^\bullet$ is called a potential $n$-stage for $C$, where $n \geq 0$, if $H_\bullet(X^\bullet) \in \mathcal{W}(eC + (C[n + 1], n + 2))$, i.e.

$$\pi^s H_\bullet(X^\bullet) \cong \begin{cases} C[n + 1] & \text{if } s = n + 2 \\ C & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

where the isomorphisms are between unstable coalgebras and $C$-comodules respectively.

Potential $n$-stages are introduced by Blanc [6, 5.6] under the name of Postnikov sections or approximations. The name here is inspired by the corresponding notion introduced in [7] for the realization problem of a $\Pi$-algebra.

There is a recognition principle of potential $n$-stages in terms of natural homotopy groups. It elucidates the sense in which a potential $n$-stage is an approximation to an actual topological realization up to homotopy level $n$ and it will also be useful in what follows. This is a key ingredient that uses in an essential way the extra structure of the spiral exact sequence obtained in Appendices [B, C].

**Proposition 6.3.** A cosimplicial space $X^\bullet$ is a potential $n$-stage for $C$ if and only if the following three conditions are satisfied:

(a) There are isomorphisms

$$\pi^0 H_\bullet(X^\bullet, G) \cong \text{Hom}_{\mathcal{C}_A}(C, H_\bullet(G))$$

of $\mathcal{H}$-algebras.

(b) $\pi^s(X^\bullet, G) = 0$ for $s > n$ and all $G \in \mathcal{G}$.

(c) $\pi_s[X^\bullet, G] = 0$ for $1 \leq s \leq n + 1$ and all $G \in \mathcal{G}$.

**Proof.** There is an equivalence of $\mathcal{H}$-algebras and unstable algebras by Theorem [C.12] with an inverse functor $u: \mathcal{H} - \text{Alg} \rightarrow \mathcal{U}_A$ exhibited in Corollary [C.13]. Lemma [C.15] identifies the relevant $\pi^0_0(X^\bullet, -)$-modules structures with $C^\omega \cong u(\pi^0_0(X^\bullet, -))$-module structures.

Now, by the spiral exact sequence in Theorem [3.13] and the cogenerating property of the Eilenberg-MacLane spaces $K(F, m) \in \mathcal{G}$ in Proposition [2.15], condition (a) is equivalent to $\pi^0 H_\bullet(X^\bullet) \cong C$, as unstable coalgebras. Similarly, condition (c)
is equivalent to $\pi^s H_\ast(X^\bullet) = 0$ for $1 \leq s \leq n + 1$. So both (a) and (c) are certainly true for potential $n$-stages. Further the spiral exact sequence yields isomorphisms of $\pi^s_0(X^\bullet, \ast)$-modules

$$\pi^s_\ast(X^\bullet, \ast) \cong \text{Hom}_{C,A}(C, H_\ast(\Omega^s(\ast)))$$

for all $s \leq n$. The spiral exact sequence continues with

$$\cdots \rightarrow \pi^s_{n+2}(X^\bullet, G) \rightarrow \pi^s_{n+2}[X^\bullet, G] \rightarrow \pi^s_{n+1}(X^\bullet, \Omega G) \rightarrow \pi^s_{n+1}(X^\bullet, G) \rightarrow 0$$

where the middle connecting map can be identified with the dual of a map of $C$-comodules from $C[n + 1]$ to itself. It follows that the connecting map must be an isomorphism and so we obtain inductively the vanishing of the higher natural homotopy groups. The converse is similar. 

From the last proposition, together with Proposition 5.30, it follows that if $X^\bullet$ is a potential $n$-stage for $C$, then $\text{sk}^C_m(X^\bullet)$ is a potential $m$-stage for $C$ for all $m \leq n$. Here $\text{sk}^C_n(\ast)$ denotes the derived $n$-skeleton, i.e. the $n$-skeleton of a functorial $G$-cofibrant replacement. Thus a potential $n$-stage should be thought of as a potential $n$-skeletal Postnikov truncation in $cS^G$ of an $\infty$-stage. We say that a potential $n$-stage $Y^\bullet$ extends or is over a potential $(n - 1)$-stage $X^\bullet$ if $\text{sk}^C_n(Y^\bullet) \cong X^\bullet$.

**Theorem 6.4.** Let $C \in C, A$ and $n \geq 1$. Suppose that

$$\alpha \colon X^\bullet_{n-1} \rightarrow X^\bullet$$

is a map in $cS^G$ where $X^\bullet_{n-1}$ is a potential $(n - 1)$-stage for $C$. Then $X^\bullet$ is a potential $n$-stage for $C$ over $X^\bullet_{n-1}$ if and only if there is a homotopy pushout square

$$
\begin{array}{ccc}
L_C(C[n], n + 1) & \xrightarrow{w_n} & X^\bullet_{n-1} \\
\downarrow & & \downarrow \alpha \\
L(C, 0) & \xrightarrow{} & X^\bullet
\end{array}
$$

where $w_n$ is a map such that the map $\phi(X^\bullet_{n-1})(w_n) : K_C(C[n], n + 1) \rightarrow H_\ast(X^\bullet_{n-1})$ defined in Proposition 5.23 is a weak equivalence.

**Proof.** Sufficiency is an easy consequence of the long exact sequence associated with this homotopy pushout in Proposition 6.12. The necessity is a direct consequence of Theorem 5.30. 

6.2. **The main results.** For an unstable coalgebra $C$, let $W_\ast(C)$ denote the subcategory of $cS^G$ whose objects are potential $n$-stages for $C$ and the morphisms are weak equivalences. The moduli space of potential $n$-stages $M_\ast(C)$ is defined to be the classifying space of this category.

The homotopy type of $M_0(C)$ has essentially already been determined, but we include the statement again for emphasis.

**Theorem 6.6.** There is a weak equivalence $M_0(C) \simeq B\text{Aut}(C)$. 

**Proof.** This is a reformulation of Proposition 5.23 since a potential 0-stage for $C$ is an object of type $L(C, 0)$. 

Note that there is a functor $\text{sk}^C_n : W_\ast(C) \rightarrow W_{n-1}(C)$ that sends a potential $n$-stage to its (derived) $n$-skeleton. Given a potential $(n - 1)$-stage $X^\bullet$, we denote by $W_\ast(C)_{X^\bullet}$ the subcategory of $W_\ast(C)$ over the component of $X^\bullet \in W_{n-1}(C)$. 

This, of course, may be empty. We denote its classifying space by $M_n^C$. This is the moduli space of potential $n$-stages over $X^\bullet$ in the sense of Proposition 6.4.

The following proposition is a refinement of Proposition 6.4 and is the first step to the description of the map $sk_n^C : M_n(C) \to M_{n-1}(C)$ in the theorem that follows.

**Proposition 6.7.** Let $X^\bullet$ be a potential $(n-1)$-stage. Then there is a homotopy pullback square

$$
\begin{array}{ccc}
\mathcal{M}_n(C)_X & \longrightarrow & \mathcal{M}(K(C,0) \leftarrow K_C(C[n], n+1) \triangleright H_\bullet(X^\bullet)) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{M}(X^\bullet) & \overset{H_\bullet}{\longrightarrow} & \mathcal{M}(H_\bullet(X^\bullet)).
\end{array}
$$

**Proof.** By Proposition 6.4 there are functors as follows:

$$
\begin{array}{ccc}
W_n(C)_X & \overset{\text{6.5}}{\longrightarrow} & W(L(C,0) \leftarrow L\mathcal{C}(C[n], n+1) \triangleright X^\bullet) \\
\downarrow \downarrow & & \downarrow \downarrow \\
W(X^\bullet) & \longrightarrow & W(X^\bullet)
\end{array}
$$

which make the diagram commute up to natural transformations. The top pair of functors induces an inverse pair of homotopy equivalences between the classifying spaces; the two composites are connected to the respective identity functors by zigzags of natural transformations. Then the result follows directly from Theorem 5.32 and Remark 5.33.

The following result and its corollary gives a description of the difference between $M_{n-1}(C)$ and $M_n(C)$ in terms of the André-Quillen cohomology of $C$.

**Theorem 6.8.** For every $n \geq 1$, there is a homotopy pullback square

$$
\begin{array}{ccc}
\mathcal{M}_n(C) & \longrightarrow & \mathcal{M}(K(C,0) \leftarrow K_C(C[n], n+2) \triangleright K(C,0)) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{M}_{n-1}(C) & \overset{H_\bullet}{\longrightarrow} & \mathcal{M}(K(C,0) \leftarrow K_C(C[n], n+2) \triangleright K(C,0)).
\end{array}
$$

where the map on the right is induced by the functor $(V \leftarrow U) \to (V \leftarrow U \to V)$.

**Proof.** It suffices to check this for each component of $M_{n-1}(C)$ by applying the last proposition. Let $X^\bullet$ be a potential $(n-1)$-stage for $C$. Then $H_\bullet(X^\bullet)$ is an object of $W(K(C,0) \leftarrow (C[n], n+1))$ and $\mathcal{M}(H_\bullet(X^\bullet))$ is one component of $\mathcal{M}(K(C,0) \leftarrow (C[n], n+1))$. By Proposition 4.30 this last moduli space is naturally weakly equivalent to

$$
\mathcal{M}(K(C,0) \leftarrow K_C(C[n], n+2) \triangleright K(C,0)),
$$
so this way we get the bottom map. Moreover, there is a homotopy commutative diagram

\[ \begin{array}{ccc}
\mathcal{M}(K(C, 0) \leftarrow K(C[C[n], n + 1) \rightleftharpoons H_\ast(X^\bullet)) & \rightarrow & \mathcal{M}(K(C[C[n], n + 2) \rightarrow K(C, 0)) \\
\mathcal{M}(H_\ast(X^\bullet)) & \rightarrow & \mathcal{M}(K(C, 0) \leftarrow K(C[C[n], n + 2) \rightarrow K(C, 0)) \\
\Delta & & \\
\end{array} \]

where:

(i) The top map is induced by the obvious forgetful functor and the recipe of Proposition 4.26(c).
(ii) The bottom map is induced by the recipe of Proposition 4.30 as explained above.
(iii) The map on the left is induced by the obvious forgetful functor.

The homotopy commutativity and homotopy pullback property of this square can be checked by considering separately the following two cases:

(a) \( X^\bullet \) can be extended to a potential \( n \)-stage. Then, by Proposition 6.7, \( H_\ast(X^\bullet) \) is weakly equivalent to \( K_C(C[n], n + 1) \) and hence the moduli space at the top left corner is non-empty. The weak equivalence indicated by \( \leftarrow \) determines a natural transformation between the two compositions in the square. Moreover, since the arrow \( \leftarrow \) indicates a weak equivalence, it follows easily that the top map in the square is actually a weak equivalence. Therefore the square is a homotopy pullback.

(b) \( X^\bullet \) cannot be extended to a potential \( n \)-stage. Then the space at the top left corner is empty and the commutativity of the diagram is a triviality. Moreover, the images of the bottom and right map lie in different connected components, therefore the square is a homotopy pullback in this case too.

Then the result follows from Proposition 6.7.

Theorem 6.8 can be used to identify the homotopy fibers of

\( \text{sk}_n^C: \mathcal{M}_n(C) \rightarrow \mathcal{M}_{n-1}(C) \)

in terms of the André-Quillen cohomology of \( C \). Let \( M \in \mathcal{V}_C, D \in C/CA \), and \( K_C(M, n) \) a pointed structured \( K \)-object. Following Theorem 4.28 we define the André-Quillen space of \( D \) with coefficients in \( M \) as follows

\[ AQ_C^C(D; M) := \text{map}_{C(C/CA)}^\text{der}(K_C(M, n), cD). \]

Note that this space has trivial homotopy groups in degrees greater than \( n \). Based on general results about moduli spaces (Theorem 11.10), there is a homotopy fiber sequence

\[ \text{map}_{C}^\text{der}(K_C(C[n], n + 2), cC) \rightarrow \mathcal{M}(K(C, 0) \leftarrow K_C(C[n], n + 2) \rightarrow K(C, 0)) \]

\[ \mathcal{M}(K(C, 0)) \times \mathcal{M}(K_C(C[n], n + 2) \rightarrow K(C, 0)) \]

where the homotopy fiber is the subspace of \( \text{map}_{C}^\text{der}(K_C(C[n], n + 2), cC) \) defined by the maps which induce a \( \pi^0 \)-isomorphism. Recall that we have weak equivalences:

\[ \mathcal{M}(K_C(C[n], n + 2) \rightarrow K(C, 0)) \cong B\text{Aut}_C(C[n]) \]
\[ \mathcal{M}(K(C,0)) \simeq B\text{Aut}^h(K(C,0)) \simeq B\text{Aut}(C). \]

In particular, both spaces are connected. It follows that the moduli space

\[ \mathcal{M}(K(C,0) \leftarrow K_C(C[n], n + 2) \rightarrow K(C,0)) \]

is the homotopy quotient of the mapping space

\[ \text{map}^\text{der}(K_C(C[n], n + 2), cC) = M \]

by the action of the homotopy automorphisms of \( K_C(C[n], n + 2) \) as a structured \( K \)-object and \( cC \).

The homotopy quotient of \( M \) under the action of \( \text{Aut}^h(cC) \simeq \text{Aut}(C) \) is homotopy equivalent to \( \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n]) \). This is obtained essentially by identifying sets of connected components of \( M \). There is an induced action of the homotopy automorphisms \( \text{Aut}^h(K_C(C[n], n + 2) \rightarrow cC) \simeq \text{Aut}_C(C[n]) \) on \( \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n]) \)

which has a homotopy fixed point at the basepoint. Indeed the associated action on \( M = \text{map}^\text{der}_\ast(K_C(C[n], n + 2), cC) \) has a fixed point at the basepoint - after modelling the spaces in terms of the external simplicial structure. Define

\[ \tilde{\mathcal{A}\mathcal{Q}_C^{n+2}}(C; C[n]) := \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n])//\text{Aut}^h(K_C(C[n], n + 2) \rightarrow cC) \]

\[ \simeq \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n])//\text{Aut}_C(C[n]) \]

to be the homotopy quotient. The inclusion of the basepoint in \( \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n]) \) yields a map

\[ B\text{Aut}_C(C[n]) \to \tilde{\mathcal{A}\mathcal{Q}_C^{n+2}}(C; C[n]) \]

which can be identified up to weak equivalence with the right vertical map \( \Delta \) of Theorem 6.11. The fiber of the last map is \( \Omega \mathcal{A}\mathcal{Q}_C^{n+2}(C; C[n]) \simeq \mathcal{A}\mathcal{Q}_C^{n+1}(C; C[n]) \), so we obtain the following as a corollary.

**Corollary 6.10.** Let \( X^\bullet \) be a potential \( n \)-stage for an unstable coalgebra \( C \). Then there is a homotopy pullback square

\[ \mathcal{A}\mathcal{Q}_C^{n+1}(C; C[n]) \rightarrow \mathcal{M}_n(C) \]

\[ \mathcal{M}_n(C) \leftarrow \mathcal{M}_{n-1}(C) \]

\[ \leftarrow \mathcal{M}_n(C) \]

\[ \mathcal{M}_n(C) \]

Our next goal is to compare the tower of moduli spaces \( \{\mathcal{M}_n(C)\} \) with the moduli space \( \mathcal{M}_X(C) \). The following theorem is an application of the results and methods of [17]; see Appendix [D] [18].

**Theorem 6.11.** Let \( C \) be an unstable coalgebra. Then there is a weak equivalence

\[ \mathcal{M}_X(C) \simeq \holim_n \mathcal{M}_n(C). \]

**Proof.** If \( \mathcal{W}_X(C) \) is empty, i.e. if there is no \( \infty \)-stage, then for every potential \( n \)-stage \( X^\bullet \) there is an \( m > n \) such that no potential \( m \)-stage \( Y^\bullet \) satisfies \( \text{sk}^n_{n+1}(Y^\bullet) \simeq X^\bullet \). It follows that the homotopy limit is also empty.

Let \( \mathbb{N} \) denote the poset of natural numbers. To bridge the gap from the setting of [17], we first replace \( \mathcal{W}_n(C) \), \( 0 \leq n \leq \infty \), with a new category of weak equivalences denoted by \( \mathcal{W}_n'(C) \). The objects are functors \( F : \mathbb{n} \to c\mathcal{S}^p \) (or \( F : \mathbb{N} \to c\mathcal{S}^p \) if \( n = \infty \)) such that:

(i) \( F(k) \) is a potential \( k \)-stage for \( C \),
On the other hand, it is easy to see that the homotopy colimit functor defines a functor in the other direction and that the two compositions are naturally weakly equivalent to the respective identity functors. Hence the classifying spaces of these categories are canonically defined up to homotopy equivalence for all identity functors. Therefore, the classifying spaces of these categories are canonically equivalent up to homotopy equivalence for all $n$.

The skeletal filtration of a potential $n$-stage $X^\bullet$:

\[ \cdots \rightarrow \text{sk}_k^c(X^\bullet) \rightarrow \text{sk}_{k+1}^c(X^\bullet) \rightarrow \cdots \]

defines a functor

\[ \{\text{sk}_k^c(-)\} : \mathcal{W}_n(C) \rightarrow \mathcal{W}_n'(C). \]

On the other hand, it is easy to see that the homotopy colimit functor defines a functor in the other direction

\[ \text{hocolim} : \mathcal{W}_n'(C) \rightarrow \mathcal{W}_n(C) \]

and that the two compositions are naturally weakly equivalent to the respective identity functors. Hence the classifying spaces of these categories are canonically identified up to homotopy equivalence for all $n$.

The connected components of $\mathcal{W}_n'(C)$ correspond to the equivalence classes of $\infty$-stages. Following [17], we say that two $\infty$-stages are conjugate if their $n$-skeleta are weakly equivalent for all $n$. By standard cofinality arguments, it suffices to prove the theorem for the components associated with a single conjugacy class of $\mathcal{W}_n'(C)$. Such a class is represented by the skeletal filtration $[\text{sk}_k^c X^\bullet]$ of a $\infty$-stage $X^\bullet$. Let $\text{Conj}(X^\bullet) \subseteq \mathcal{W}_n'(C)$ be the full subcategory of objects conjugate to $X^\bullet$, i.e., diagrams $F$ such that $F(n) = \text{sk}_{k+1}^c X^\bullet$. Let $\mathcal{W}_n'(C)_X \subseteq \mathcal{W}_n'(C)$ be the component containing $[\text{sk}_k^c(X^\bullet)]_{k \leq n}$. We emphasize that while $\mathcal{W}_n'(C)_X$ is connected, by definition, this will not be true for $\text{Conj}(X^\bullet)$ in general. Then an application of Theorem [D.12] shows the required weak equivalence

\[ \mathcal{M}_{\text{conj}}(X^\bullet) := B(\text{Conj}(X^\bullet)) \simeq \text{holim}_n \mathcal{W}_n'(C)_X \simeq \text{holim}_n \mathcal{M}(X(n)) \]

which is induced by the restriction maps.

Finally, we come to the comparison between $\infty$-stages and actual topological realizations of $C$. Let $\mathcal{W}_{\top}(C)$ denote the category of spaces whose objects are topological realizations of $C$, i.e., simplicial sets $X$ such that $H_*(X) \cong C$ in $\mathcal{C} \mathcal{A}$, and morphisms are $\mathbb{F}_p$-homology equivalences between simplicial sets. We define the classifying space of this category $\mathcal{M}_{\top}(C)$ to be the realization space of $C$. By Theorem [D.9] there is a weak equivalence

\[ \mathcal{M}_{\top}(C) \simeq \bigsqcup_X B\text{Aut}^h(X) \]

where the index set of the coproduct is the set of equivalence classes of realizations of $C$ with respect to $\mathbb{F}_p$-homology, with $X$ chosen in each class to be fibrant in the localized sense (i.e., $p$-complete), and $\text{Aut}^h(X)$ denotes the simplicial monoid of self-homotopy equivalences.

**Theorem 6.13.** Suppose that the homology spectral sequence of the cosimplicial $\infty$-stage for $C$ converges. Then the totalization functor induces a weak equivalence

\[ \mathcal{M}_{\mathcal{E}}(C) \simeq \mathcal{M}_{\top}(C). \]

**Proof.** Let $X^\bullet$ be an $\infty$-stage and assume that it is Reedy fibrant. The $E^2$-page of the homology spectral sequence of the cosimplicial space $X^\bullet$,

\[ \pi^* H_t(X^\bullet) \Rightarrow H_{t-s}(\text{Tot}(X^\bullet)), \]

is a weak equivalence. 

(ii) for every $k < m$, the map $F(k) \simeq \text{sk}_k^c F(k) \rightarrow \text{sk}_k^c F(m)$ is a weak equivalence.
is concentrated in the 0-line and so the spectral sequence collapses at this stage. The spectral sequence converges by assumption and therefore the edge homomorphism
\[(6.14)\]
\[H_*^\wedge (\text{Tot}(X^*)) \cong E_{0,*}^\infty \cong E_0^2 = \pi_0^*H_*^\wedge (X^*) \cong C\]
is an isomorphism. Since this isomorphism is induced by a map of spaces (see also [11, 2.4]):
\[\text{Tot}(X^*) \to X^0,\]
it follows that it is an isomorphism of unstable coalgebras. Hence \(\text{Tot}(X^*)\) is a topological realization of \(C\). This way we obtain a functor
\[\text{Tot}^f : W^\wedge_\infty (C) \to W^\wedge \text{Top}(C)\]
which sends an \(\infty\)-stage to the totalization of its (Reedy) fibrant replacement. On the other hand, there is a functor
\[c : W^\wedge \text{Top}(C) \to W^\wedge_\infty (C)\]
which sends a topological realization \(Y\) of \(C\) to the constant cosimplicial space \(eY\). The composite functors \(\text{Tot}^f \circ c\) and \(c \circ \text{Tot}^f\) are connected to the identity functors by natural weak equivalences
\[\text{Id} \to \text{Tot}^f \circ c\]
\[c \circ \text{Tot}^f \to (-)^f \leftarrow \text{Id}.\]
Therefore this pair of functors induces a pair of inverse homotopy equivalences between the associated moduli spaces and hence the result follows.

Bousfield’s convergence theorem of the homology spectral sequence (see [11, Theorem 3.4]) applies to the \(\infty\)-stage for \(C\) if \(C\) is a simply-connected unstable coalgebra, i.e. \(C_1 = 0\) and \(C_0 = \mathbb{F}\). Thus, we obtain the following

**Corollary 6.15.** Let \(C\) be a simply-connected unstable coalgebra. Then the totalization functor induces a weak equivalence \(\mathcal{M}_{\infty}(C) \simeq \mathcal{M}_{\text{Top}}(C)\).

**Remark 6.16.** Other convergence results for the homology spectral sequence were obtained by Shipley [53]. For example, the homology spectral sequence of an \(\infty\)-stage \(X^*\) converges if \(H_*^\wedge (X^n)\) and \(H_*^\wedge (\text{Tot}^f(X^*))\) have finite type and \(\text{Tot}^f(X^*)\) is \(p\)-good (see [53, Theorem 6.1]).

### 6.3. Obstruction theories.

In this subsection, we spell out in detail how the homotopical description of the realization space \(\mathcal{M}_{\text{Top}}(C)\) in terms of moduli spaces of potential \(n\)-stages gives rise to obstruction theories for the existence and uniqueness of realizations of \(C\). As is usually the case with obstruction theories, these are also described as branching processes: at every stage of the obstruction process, an obstruction to passing to the next stage can be defined, but this obstruction (and so also its vanishing) depends on the adventitious state of the process created so far by choices made at earlier stages.

Clearly, \(C\) is topologically realizable if and only if \(\mathcal{M}_{\text{Top}}(C)\) is non-empty. By Theorems 6.11 and 6.13 this is equivalent to the existence of an object in \(W^\wedge_\infty (C)\). Theorem 6.10 shows that a potential 0-stage always exists and that it is essentially unique. Suppose that this has been extended to a potential \((n-1)\)-stage \(X^*\). This defines a point \([X^*] \in \mathcal{M}_{n-1}(C)\). Theorem 6.8 and the discussion that followed gives a map
\[w : \mathcal{M}_{n-1}(C) \to \tilde{A}_\Omega_{\mathcal{C}}^{n+2}(C; C[n])\]
which associates to a potential \((n-1)\)-stage \(X^*\) the (class of) the André-Quillen cohomology class defining the Postnikov decomposition of 

\[ H_*(X^*) \in W(K(C,0) + (C[n], n+1)). \]

This is, in other words, the unique Postnikov \(k\)-invariant of \(H_*(X^*)\) in its invariant form. Theorem 6.12 shows that \(X^*\) is up to weak equivalence the \(n\)-skeleton of a potential \(n\)-stage if and only if \(w([X^*])\) lies in the component of the zero André-Quillen cohomology class. (We recall here that the component of \(A\Omega^{n+2}_C(C; C[n])\) corresponding to the zero André-Quillen cohomology class is fixed under the action of \(\text{Aut}_C(C[n])\).) We emphasize that the first such Postnikov invariant in the process, \(w([X^*])\), associated to a potential \(0\)-stage \(X^*\), is actually an invariant of \(C\) because \(\mathcal{M}_0(C)\) is connected. From the general comparison results of Baues/Blanc [3], this invariant seems to relate to an obstruction to endowing \(C\) (or its dual) with secondary operations.

Assuming that \(w([X^*])\) represents the trivial André-Quillen cohomology class and \(X^*\) can be extended to a potential \(n\)-stage, there will then be choices for an extension which can be read off Corollary 6.10. More specifically, there are homotopy fiber sequences as follows

\[
\begin{array}{cccc}
A\Omega^{n+1}_C(C; C[n]) & \longrightarrow & A\Omega^{n+1}_C(C; C[n]) \\
\downarrow & & \downarrow \\
\mathcal{M}_n(C)_X & \longrightarrow & \mathcal{M}(K_C(C[n], n+2) \to K(C,0)) \\
\downarrow & & \downarrow \\
\text{BAut}^h(X^*) & \longrightarrow & \text{BAut}^h(K_C(C[n], n+2))
\end{array}
\]

Then the set of non-weakly equivalent choices of potential \(n\)-stages over \(X^*\) is exactly the set of components of \(\mathcal{M}_n(C)_X\). There is an action of the group

\[ h\text{Aut}(K_C(C[n], n+2)) := \pi_1(\text{BAut}^h(K_C(C[n], n+2))) \]

on the components of the fiber

\[ A\Omega^{n+1}_C(C; C[n]) := \pi_0(A\Omega^{n+1}_C(C; C[n])). \]

By the long exact sequence of homotopy groups, the corresponding set of orbits is exactly the required set of possible extensions of \(X^*\), \(\pi_0(\mathcal{M}_n(C)_X)\). For every such choice of potential \(n\)-stage \(Y^*\) over \(X^*\), the same procedure leads to an obstruction \(w(Y^*)\) for finding a potential \((n+1)\)-stage over \(Y^*\). We emphasize that the Postnikov invariant of \(H_*(Y^*)\) is not determined by \(X^*\). According to Theorems 6.11 and 6.13 \(C\) admits a topological realization if and only if this procedure can be continued indefinitely to obtain an element in \(W'_n(C)\).

The problem of uniqueness of topological realizations of \(C\) can also be analysed by considering the tower of the moduli spaces. We have that \(C\) is uniquely realizable if and only if \(\mathcal{M}_{\text{Top}}(C)\) is connected. This means that at each stage of the obstruction theory where a choice of continuation presents itself there is exactly one choice that can be continued unhindered indefinitely. Assume that a topological realization \(X\) of \(C\) is given, which then defines basepoints in all moduli spaces. Then the number of different realizations is given by the Milnor exact sequence

\[ \pi_0(\mathcal{M}_n(C), [sk^c_{n+1}(\varepsilon X)]) \to \pi_0(\mathcal{M}_{\text{Top}}(C)) \to \lim \pi_0(\mathcal{M}_n(C)) \to \pi_1(\mathcal{M}_n(C), (n+1)C) \to \pi_2(\mathcal{M}_n(C), (n+2)C) \to \cdots \]
where
\[ \pi_1(\mathcal{M}_n(C), [\text{sk}^C_{n+1}(cX)]]) \cong \pi_0 \text{Aut}^h(\text{sk}^C_{n+1}(cX)). \]
Moreover, associated to this tower of pointed fibrations, there is a Bousfield-Kan homotopy spectral sequence which starts from André-Quillen cohomology and ends at the homotopy of the realization space.

We now turn to an obstruction theory for realizing maps between realized unstable coalgebras. That is, given a map \( \phi : H_*(X) \to H_*(Y) \), is there a realization \( f : X \to Y \), i.e. \( \phi = H_*(f) \), and if yes, what is the moduli space of all such realizations? This cannot be answered directly from the main theorems above, but an answer can be obtained by similar methods based on the established Postnikov decompositions of cosimplicial spaces.

For the question of existence, we proceed as follows. Under the assumption that \( C \) is simply-connected, this realization problem is equivalent to finding a realization \( f_x : cX \to cY \) in \( \text{Ho}(cS^\text{op}) \) such that \( H_*(f_x) = c(\phi) \). The existence of \( f_x \) is equivalent to the existence of a compatible sequence of maps in \( cS^\text{op} \):
\[
f_n : \text{sk}^C_n(cX) \to (cY)^f.
\]
Let \( C = H_*(X) \), \( D = H_*(Y) \) and a given map \( \phi : C \to D \). This map extends trivially to a map of cosimplicial objects
\[
\phi : K(C, 0) \to H_*(cY)
\]
which then gives by Proposition \([5.24]\) the starting point of the obstruction theory:
\[
f_1 : L(C, 0) \simeq \text{sk}^C_1(cX) \to (cY)^f.
\]
Assuming that \( f_n \) has been constructed, the obstruction to passing to the next step can be expressed in the following diagram:

\[
\begin{array}{ccc}
L(C, 0) & \xrightarrow{f_1} & (cY)^f \\
\downarrow f_0 & & \\
L_C(C[n], n+1) & \xrightarrow{\text{sk}^C_n(cX)} & (cY)^f \\
\downarrow & & \downarrow \\
L(C, n) & \xrightarrow{\text{sk}^C_{n+1}(cX)} & (cY)^f
\end{array}
\]

The dotted arrow exists if and only if the top composite factors up to homotopy through \( f_1 : L(C, 0) \to (cY)^f \). By Proposition \([5.24]\) this is equivalent to the existence of factorization in \( \text{Ho}(c\mathcal{A}^\mathcal{E}) \) of the associated map (which depends on \( f_n \))
\[
w(\phi; f_n) : K_C(C[n], n+1) \to cH_*(Y)
\]
through \( \phi : K(C, 0) \to cH_*(Y) \) (in \( \text{Ho}(c\mathcal{A}^\mathcal{E}) \)). Therefore \( w(\phi; f_n) \) determines an element in \( A Q^{k+1}_C(D; C[n]) \) whose vanishing is equivalent to the existence of an extension \( \text{sk}^C_{n+1}(cX) \to (cY)^f \).

As long as \( \phi \) can be realized, the problem of the uniqueness of realizations can be addressed similarly. In the diagram above, the choices of homotopies exhibiting the vanishing of the obstruction \( w(\phi; f_n) \), which correspond to elements of \( A Q^{k}_C(D; C[n]) \), lead to different extensions in general. The identification of these extensions depends on \( f_n \) and can be determined using standard obstruction-theoretic
methods. The realization is unique if and only if among all such intermediate extensions exactly one sequence of choices makes it to the end.

Appendix A. Cosimplicial objects

This section fixes some notation and recalls several basic constructions related to cosimplicial objects.

We denote by $\Delta$ the category of finite ordinals. Its objects are the natural numbers $n = \{0 < 1 < \ldots < n\}$ with $n \geq 0$. Its morphisms are the non-decreasing maps. Let $\mathcal{M}$ be a complete and cocomplete category. We denote by $c\mathcal{M}$ the category of cosimplicial objects in $\mathcal{M}$ which are the functors from $\Delta$ to $\mathcal{M}$. For an object $M$ in $\mathcal{M}$, let $c_M$ be the constant cosimplicial object at $M$.

**Definition A.1.** For a cosimplicial object $X^\bullet$ in $\mathcal{M}$ we define its $n$-th matching object by

$$M^n X^\bullet := \varprojlim_{[n] \to [k]} X_k^n \text{ in } \mathcal{M}$$

and its $n$-th latching object by

$$L^n X^\bullet := \varinjlim_{[k] \to [n]} X_k^n \text{ in } \mathcal{M}.$$  

On the category $c\mathcal{M}$ we have the Reedy model structure. A map $X^\bullet \to Y^\bullet$ is a

1. a Reedy equivalence if for all $n \geq 0$ the maps $X^n \to Y^n$ are weak equivalences in $\mathcal{M}$.
2. a Reedy cofibration if for all $n \geq 0$ the induced maps

$$X^n \cup_{L^n X^\bullet} L^n Y^\bullet \to Y^n$$

are cofibrations in $\mathcal{M}$.
3. a Reedy fibration if for all $n \geq 0$ the induced maps

$$X^n \to Y^n \times_{M^n Y^\bullet} M^n X^\bullet$$

are fibrations in $\mathcal{M}$.

**Definition A.2.** Assume that $\mathcal{M}$ is simplicially enriched, tensored and cotensored. Then the internal simplicial structure on $c\mathcal{M}$ is defined by

$$(X^\bullet \otimes^\text{int} K)^n = X^n \otimes K^n, \quad \text{hom}^\text{int}(K, X^\bullet)^n = \text{hom}(K, X^n)$$

and

$$\text{map}^\text{int}(X^\bullet, Y^\bullet)_n = \text{Hom}_{\mathcal{M}}(X^\bullet \otimes^\text{int} \Delta^n, Y^\bullet),$$

where $X^\bullet$ and $Y^\bullet$ are objects in $c\mathcal{M}$ and $K$ is a simplicial set.

**Proposition A.3.** The Reedy model structure on $c\mathcal{M}$ is compatible with the internal simplicial structure. The adjoint functors

$$(-)^0: c\mathcal{M}^\text{Reedy} \rightleftarrows \mathcal{M}: c$$

form a simplicial Quillen pair.

Assume now that $\mathcal{M}$ is a left proper model category, $\mathcal{G}$ a class of injective models and $c\mathcal{M}^\mathcal{G}$ the $\mathcal{G}$–resolution model category as defined in Subsection 3.1. The constant functor $c$ obviously maps weak equivalences to $\mathcal{G}$-equivalences, as defined in Subsection 3.1, but $(-)^0: c\mathcal{M}^\mathcal{G} \rightleftarrows \mathcal{M}: c$ is not a Quillen pair; $c$ does not preserve fibrations.
Lemma A.4. If $X \to Y$ is a $G$-injective fibration in $\mathcal{M}$, then $cX \to cY$ is a $G$-fibration. In particular, if $G$ is a $G$-injective object, then $cG$ is $G$-fibrant.

Proposition A.5. $\Delta^* \times - : \mathcal{M} \xrightarrow{\sim} c\mathcal{M} : \text{Tot}$ is a Quillen pair for both the Reedy and the $G$-resolution model structures.

Proof. This is proved in [13, Proposition 8.1].

For any $n \geq 0$ we denote by $\Delta_{\leq n}$ the full subcategory of $\Delta$ given by all the objects $\ell$ with $0 \leq \ell \leq n$. We denote the category of functors from $\Delta_{\leq n}$ to $\mathcal{M}$ by $c_n\mathcal{M}$. The inclusion functor $i_n : \Delta_{\leq n} \to \Delta$ induces a pullback functor $i_n^* : c\mathcal{M} \to c_n\mathcal{M}$ which possesses a right adjoint $\rho_n$ and a left adjoint $\lambda_n$. We define the $n$-th skeleton of a cosimplicial object $X^\bullet$ by

$$\sk_n X^\bullet := \lambda_n i_n^* X^\bullet$$

the $n$-th coskeleton of a cosimplicial object $X^\bullet$ by

$$\cosk_n X^\bullet := \rho_n i_n^* X^\bullet.$$ 

Because $i_n$ is full we have $(\sk_n X^\bullet)^s = X^s = (\cosk_n X^\bullet)^s$ for $0 \leq s \leq n$.

Lemma A.6. Let $X^\bullet$ be Reedy cofibrant. Then the maps $\sk_n X^\bullet \to \sk_{n+1} X^\bullet$ in $c\mathcal{M}$ are Reedy cofibrations for all $n \geq 0$. A dual statement holds for coskeleta.

Proof. See Berger/Moerdijk [3, Proposition 6.5].

Note that $\sk_n X^\bullet \to X^\bullet$ is usually not a $G$-cofibration as explained in Remark 3.7.

Proposition A.7. The adjoint pair $\sk_n : c\mathcal{M} \xrightarrow{\sim} c\mathcal{M} : \cosk_n$ is a Quillen adjunction for the Reedy model structure. The functor $\sk_n$ preserves trivial $G$-cofibrations (and dually, $\cosk_n$ preserves $G$-fibrations).

Proof. This is well-known for the Reedy model structure (see e.g. [3, Lemma 6.4]). The proof for the preservation of trivial $G$-cofibrations is based on the characterization of (trivial) $G$-cofibrations in [13, Proposition 3.13].

Let $f : X^\bullet \to Y^\bullet$ be a trivial $G$-cofibration. Then the cofibration

$$\sk_n(X^\bullet)^m \cup_{L^m(\sk_n(X^\bullet))} L^m(\sk_n(Y^\bullet)) \to \sk_n(Y^\bullet)^m$$

is $G$-monic if $m \leq n$ - since $f$ is a trivial $G$-cofibration - and if $m > n$, it is an isomorphism because

$$L^m(\sk_n(X^\bullet)) \xrightarrow{\cong} \sk_n(X^\bullet)^m$$

$$L^m(\sk_n(Y^\bullet)) \xrightarrow{\cong} \sk_n(Y^\bullet)^m$$

(see [3, Lemmas 6.2 and 6.3]). It follows that $\sk_n(f)$ is a trivial $G$-cofibration.

We review the construction of the external simplicial structure on $c\mathcal{M}$. This does not require the existence of an internal simplicial structure on $\mathcal{M}$. For any set $S$ and object $X$ of $\mathcal{M}$, we denote by

$$\coprod_S X$$

and

$$\prod_S X$$
the coproduct and product in $\mathcal{M}$ of $|S|$ copies of $X$. This construction is functorial in both variables $X$ and $S$. Thus, for objects $X^\bullet$ of $c\mathcal{M}$ and $L$ of $\mathcal{S}$, there is a functor $\Delta^{op} \times \Delta \to \mathcal{M}$ defined on objects by

$$([\ell],[m]) \mapsto \bigsqcup_{L_{\ell}} X^m.$$ 

Thus we can define the coend

$$X^\bullet \otimes_\Delta L := \int^\Delta \bigsqcup_{L_{\ell}} X^m \in \mathcal{M}$$

which is, by definition, the coequalizer given by

$$\bigsqcup_{[m] \to [\ell]} \bigsqcup_{L_{\ell}} X^m \rightrightarrows \bigsqcup_{\ell \geq 0} L_{\ell} X^\ell \to X^\bullet \otimes_\Delta L,$$

where the parallel arrows are defined in the obvious way using the maps induced by $[m] \to [\ell]$.

**Definition A.8.** Let $K$ be a simplicial set and $X^\bullet$ and $Y^\bullet$ objects of $c\mathcal{M}$. Then we define tensor, cotensor and mapping space objects as follows:

$$(X^\bullet \otimes^{ext} K)^n = X^\bullet \otimes_\Delta (K \times \Delta^n)$$

$$\text{hom}^{ext}(K,X^\bullet)^n = \prod_{K_n} X^n$$

$$\text{map}^{ext}(X^\bullet,Y^\bullet)_n = \text{Hom}_{c\mathcal{M}}(X^\bullet \otimes^{ext} \Delta^n, Y^\bullet)$$

This structure is called the *external simplicial structure* on $c\mathcal{M}$. We will usually drop the superscripts and often abbreviate

$$\text{hom}^{ext}(K,X^\bullet) \text{ by } (X^\bullet)^K.$$

**Definition A.9.** Let $K$ be a pointed simplicial set, $X^\bullet$ an object in $c\mathcal{M}$, and $Y^\bullet$ in $c\mathcal{M}_\ast$. We define

$$X^\bullet \otimes K = (X^\bullet \otimes^{ext} K)/(X^\bullet \otimes^{ext} \ast) \in c\mathcal{M}_\ast$$

obtaining as cofiber a canonical basepoint. We define

$$\overline{\text{hom}}(K,Y^\bullet)^n = \text{fib}[\text{hom}(K,Y^\bullet) \to \text{hom}(\ast,Y^\bullet) \cong Y^\bullet] \in c\mathcal{M}_\ast$$

as fiber taken at the basepoint of $Y^\bullet$ of the map induced by the basepoint of $K$. The map $K \to \ast$ together with the basepoint $\ast \to Y^\bullet \to \text{hom}(K,Y^\bullet)$, and hence for $\overline{\text{hom}}(K,Y^\bullet)$. We will often abbreviate

$$\overline{\text{hom}}(K,X^\bullet) \text{ by } K(X^\bullet).$$

By forgetting the basepoint of $K(Y^\bullet)$ one obtains an adjunction isomorphism

$$\text{Hom}_{c\mathcal{M}_\ast}(X^\bullet \otimes K,Y^\bullet) \cong \text{Hom}_{c\mathcal{M}}(X^\bullet,K(Y^\bullet)).$$

**Definition A.10.** Let $X^\bullet$ be an object in $c\mathcal{M}$ and $Y^\bullet$ an object in $c\mathcal{M}_\ast$. We denote the external mapping space $\text{map}(X^\bullet,Y^\bullet)$ in $c\mathcal{M}$ by

$$\text{map}(X^\bullet,Y^\bullet)$$

when we view it as a pointed simplicial set whose basepoint is given by the constant map $X^\bullet \to \ast \to Y^\bullet$. 
Remark A.11. For unpointed $X^\bullet$ and pointed $Y^\bullet$ there is a canonical isomorphism
\[ \text{map}(X^\bullet, Y^\bullet)_n \cong \text{Hom}_{cM^\ast}(X^\bullet \oplus \Delta^n, Y^\bullet) \]
of pointed sets which is natural in $n$, $X^\bullet$ and $Y^\bullet$.

Proposition A.12. Let $K$ be a simplicial set and $X^\bullet$ and $Y^\bullet$ objects of $cM$.

1. There are natural isomorphisms of simplicial sets
\[ \text{map}_{\Delta}(K, \text{map}(X^\bullet, Y^\bullet)) \cong \text{map}(X^\bullet \otimes K, Y^\bullet) \cong \text{map}(X^\bullet, (Y^\bullet)^K). \]

2. If $K$ and $Y^\bullet$ are pointed, but not $X^\bullet$, there are natural isomorphisms of pointed simplicial sets
\[ \text{map}_{\Delta^p}(K, \text{map}(X^\bullet, Y^\bullet)) \cong \text{map}_{cM^\ast}(X^\bullet \otimes K, Y^\bullet) \cong \text{map}(X^\bullet, K(Y^\bullet)). \]

Proof. Part (1) is proved in the simplicial case by Goerss/Jardine in [29, Theorem 2.5] and example [29, Example 2.8.(4)] illustrates the cosimplicial version. Part (2) is an easy consequence by direct inspection.

Although the external simplicial structure and the Reedy model structure are not compatible the next lemma states partial compatibility.

Lemma A.13. Let $f: X^\bullet \to Y^\bullet$ be a Reedy cofibration in $cM$ and $j: K \to L$ a cofibration in $S$. Then the map
\[ (X^\bullet \otimes^\text{ext} L) \cup_{(X^\bullet \otimes^\text{ext} K)} (Y^\bullet \otimes^\text{ext} K) \to Y^\bullet \otimes^\text{ext} L \]
is a Reedy cofibration which is a Reedy weak equivalence if $f$ is a Reedy weak equivalence. The various pointed analogues also hold.

Proof. For simplicial objects over $M$ this is proved in [29, VII.2.13].

Definition A.14. The collapsed boundary gives the $n$-sphere $S^n := \Delta^n/\partial \Delta^n$ a canonical basepoint. Let $X^\bullet$ be an object of $cM$. We define the $s$-th external suspension by
\[ \Sigma^s_{\text{ext}} X^\bullet = X^\bullet \oplus \Delta^s/\partial \Delta^s, \]
and, if $X^\bullet$ is pointed, the $s$-th external loop object by
\[ \Omega^s_{\text{ext}} X^\bullet = \text{hom}(\Delta^s/\partial \Delta^s, X^\bullet). \]
Again, we will usually drop the reference to the external structure. Note that $\Sigma^0 X^\bullet = X^\bullet \cup \ast$ supplies $X^\bullet$ with a disjoint basepoint.

The external and internal simplicial structures commute:

Lemma A.15. Given two simplicial sets $K$ and $L$ and a cosimplicial object $X^\bullet$, there are canonical natural isomorphisms
\[ \text{hom}^\text{int}(K, \text{hom}^\text{ext}(L, X^\bullet)) \cong \text{hom}^\text{ext}(L, \text{hom}^\text{int}(K, X^\bullet)), \]
where on the left the internal cotensor is applied degreewise. If $L$ and $X^\bullet$ are pointed there are canonical natural isomorphisms
\[ \text{hom}^\text{int}(K, \text{hom}(L, X^\bullet)) \cong \text{hom}(L, \text{hom}^\text{int}(K, X^\bullet)). \]

Proof. This is done by a direct inspection of the definitions.
Appendix B. The spiral exact sequence

A spiral exact sequence was first found in the context of Π-algebras and simplicial spaces by Dwyer/Kan/Stover [22] and goes hand in hand with resolution model structures. It was also used in the context of $E_8$-ring spectra in Goerss/Hopkins [26]. In contrast to all other instances of the spiral exact sequence we work here cosimplicially and unpointed. After the sequence is constructed in Subsection B.1 it is endowed with a module structure over its zeroth term in Subsection B.4. The associated spectral sequence is mentioned in Subsection B.5. All parts are written in the general language of universal algebra with input given only by Assumption 3.3.

B.1. Constructing the sequence. For the construction we follow the outline given in [26, 3.1.1]. Let $M$ and $G$ be as in Assumption 3.3.

Let $X'$ be a cosimplicial object in $M$. For any simplicial set $K$ we have the pair of adjoint functors $\text{ext}_{K}: cM \to \text{hom}_{\text{ext}}(p_{\leq 0}, c)$ that can be composed with the pair of adjoint functors $p_{\leq 0}: cM \to M: c$.

Definition B.1. This composition yields a functor $T: cM \to M; T X' = \text{ext}_{K} p_{\leq 0}(X')$.

For a fixed simplicial set $K$ we abbreviate $T K X' = \text{emb}_{K}(X')$.

Definition B.2. We have $T_{\Delta^n} X' \cong X'_{n}$ and $T_{\bar{\Delta}^n} X' \cong L^n X'$.

Definition B.3. The objects $T_{\Lambda^n} X' \cong L^n X'$ are the partial latching objects in [13, 3.12].

Definition B.4. If $K$ is a pointed simplicial set we define the functor $C_K: cM \to M_*$ by setting $C_K X' = (X' \otimes K)^0$. As for $T$ above, we actually obtain a functor $C: cM \times S_* \to M_*$. 

Remark B.5. The functor $C_K$ is left adjoint to the functor $K(c(-)) = \text{hom}(K, c(-)): M_* \to cM$, and there are natural pushout diagrams

$$
\begin{array}{ccc}
T_K X' & \longrightarrow & C_K X' \\
\downarrow & & \downarrow \\
T_L X' & \longrightarrow & C_L X' \\
\end{array}
$$

when $K \subset L$ is a map of pointed simplicial sets.

Remark B.6. Both adjoint pairs $(\otimes_{\text{ext}}, \text{hom}_{\text{ext}})$ and $((-)^0, c)$ are simplicially enriched but the simplicial structures on $cM$ changes between the external and the internal structure. In particular, the functors $T_K$ and $C_K$ are not enriched.

Lemma B.7. We have:
(1) For fixed (pointed) $K$ the functors
\[ T_K: c\mathcal{M} \to \mathcal{M} \quad \text{and} \quad C_K: c\mathcal{M} \to \mathcal{M}_* \]
are left Quillen if $c\mathcal{M}$ is equipped with the Reedy model structure.

(2) For fixed Reedy cofibrant $X^\bullet$ the functors
\[ T_{(-)}X^\bullet: \mathcal{S} \to \mathcal{M} \quad \text{and} \quad C_{(-)}X^\bullet: \mathcal{S}_* \to \mathcal{M}_* \]
maps cofibrations to $G$-monic cofibrations.

Proof. Follows from the more general Lemma A.13 or test the left lifting property through the adjoint pairs $\otimes^{\text{ext}}, \text{hom}^{\text{ext}}$ and $((-)^0, c)$. □

Definition B.8. We also need the partial matching objects of a simplicial group $G_*$. We set
\[ M^0_nG_* = \text{hom}(\Lambda^n_0, G_*), \]
where the cotensor is taken in the analogue of the external simplicial structure for simplicial groups.

The following statement is a special case of [13, Proposition 3.14]. Bousfield states the proposition in a section where $M$ is pointed but that property is irrelevant to the proof. It relies on [29, VII Lemma 1.24, Proposition 1.25].

Proposition B.9. Let $X^\bullet$ in $c\mathcal{M}$ be Reedy cofibrant and $G \in \mathcal{G}$ be fibrant. Then for all $n \geq 0$ there are canonical isomorphisms
\[ [L_n^0 X^\bullet, G] \cong M^0_n[X^\bullet, G]. \]

Definition B.10. It is convenient to shorten the notation
\[ \Delta^n_0 = \Delta^n/\Delta^n_0 \]
and define for a cosimplicial object $X^\bullet$ and every $n \geq -1$ the objects
\[ C^n X^\bullet = C_{\Delta^n_0} X^\bullet \quad \text{and} \quad Z^n X^\bullet = C_{\Delta^n/\partial \Delta^n} X^\bullet. \]
Here, $\Delta^{-1}$, $\Lambda^n_0$ and $\partial \Delta^n$ are to be interpreted as the empty set and
\[ Z^0 X^\bullet = C^0 X^\bullet = C_{\Delta^n_0} X^\bullet = X^\bullet \cup \ast \quad \text{and} \quad Z^{-1} X^\bullet = \ast. \]

For $n \geq 0$ the map $d^n_0: \Delta^{n-1} \to \Delta^n$ induces a cofibration
\[ d^n_0: \Delta^{n-1}/\partial \Delta^{n-1} \to \Delta^n \]
whose cofiber is isomorphic to $\Delta^n/\partial \Delta^n$. For Reedy cofibrant $X^\bullet$ it induces a cofiber sequence
\[ (B.11) \quad Z^{n-1} X^\bullet \xrightarrow{d^n_0} C^n X^\bullet \to Z^n X^\bullet \]
From this cofiber sequence we obtain a long exact sequence
\[ \ldots \to [Z^{n-1} X^\bullet, \Omega^{n+1} G] \xrightarrow{\delta} [Z^n X^\bullet, \Omega^n G] \xrightarrow{\gamma} [C^n X^\bullet, \Omega^n G] \xrightarrow{\delta} [Z^{n-1} X^\bullet, \Omega^n G] \to \ldots, \]
which can be spliced together to form an exact couple:
\[ (B.12) \]
\[ [Z^{n-1} X^\bullet, \Omega^n G] \xrightarrow{\beta} \]
\[ \xrightarrow{(1,-1)} \]
\[ \xleftarrow{\delta_{n,q}} [C^n X^\bullet, \Omega^n G] \xrightarrow{\gamma_{n,q}} [Z^n X^\bullet, \Omega^n G] \]
where the dotted boundary map $\beta_{n-1,q+1} : [Z^{n-1}X^\bullet, \Omega^{q+1}G] \to [Z^nX^\bullet, \Omega^qG]$ has bidegree $(1, -1)$. The brackets denote homotopy classes of maps in $\mathcal{M}_\ast$.

**Definition B.13.** We denote the differential of this exact couple by

$$d_{n,q} = \delta_{n,q} \circ \gamma_{n,q} : [C^nX^\bullet, \Omega^qG] \to [Z^{n-1}X^\bullet, \Omega^qG] \to [C^{n-1}X^\bullet, \Omega^qG].$$

Its first derived couple is called the *spiral exact sequence*. The associated spectral sequence will be called the *spiral spectral sequence*.

We proceed by describing the spiral exact sequence and its associated spectral sequence more explicitly.

**Definition B.14.** For a simplicial group $G^\bullet$ and $n \geq 1$ we let

$$N_nG^\bullet = \bigcap_{i=1}^n \ker [d_i : G_n \to G_{n-1}] \quad \text{and} \quad N_0G^\bullet = G_0$$
denote its normalization with differential $d_0 = d_0|_{N_nG^\bullet}$ and $d_0|_{G_0} = \ast$.

**Lemma B.15.** Let $G \in \mathcal{G}$ be fibrant and $X^\bullet$ be Reedy cofibrant.

1. For any $n \geq 0$ there is a natural isomorphism

$$[C^nX^\bullet, G]_{\mathcal{M}_\ast} \cong N_n[X^\bullet, G]_{\mathcal{M}}.$$

2. The differential $d$ of the exact couple (B.12) fits into the following commutative diagram:

$$
\begin{array}{ccc}
[C^nX^\bullet, G]_{\mathcal{M}_\ast} & \xrightarrow{d} & [C^{n-1}X^\bullet, G]_{\mathcal{M}_\ast} \\
\cong & & \cong \\
N_n[X^\bullet, G]_{\mathcal{M}} & \xrightarrow{d_0} & N_{n-1}[X^\bullet, G]_{\mathcal{M}}
\end{array}
$$

3. For any $n \geq 0$ there is a natural exact sequence

$$[C^{n+1}X^\bullet, G]_{\mathcal{M}_\ast} \xrightarrow{\delta_{n+1}} [Z^nX^\bullet, G]_{\mathcal{M}_\ast} \to \pi_n^h(X^\bullet, G) \to 0,$$

where the surjection can be identified via adjunction isomorphisms with the surjection

$$[X^\bullet, \Omega^n cG]_{c\mathcal{M}_{\text{Reedy}}} \to [X^\bullet, \Omega^n cG]_{c\mathcal{M} G}.$$

**Proof.** Let $X^\bullet$ be Reedy cofibrant, $G \in \mathcal{G}$ be fibrant and $n \geq 0$. The cofiber sequence

$$\Lambda_0^n \to \Delta^n \to \Delta_0^n$$

induces by Lemma B.17(1) a cofiber sequence

$$L_0^nX^\bullet \to X^n \to C^nX^\bullet$$
in $\mathcal{M}$. We obtain a long exact sequence

$$\ldots \to [C^nX^\bullet, G] \to [X^n, G] \to [L_0^nX^\bullet, G] \to [C^nX^\bullet, \Omega G] \to \ldots$$
of groups. The map $[X^n, G] \to [L_0^nX^\bullet, G]$ is surjective because it can be identified with the canonical map

$$[X^n, G] \to M_0^n[X^\bullet, G]$$
Clearly, the canonical map by Definition B.10 of the functor $Z$
Using Lemma A.15 there is an induced isomorphism $\pi$
ending with a surjection $X$
is well defined and surjective by our (co-)fibrancy conditions on $\text{Hom}$
their composite map is surjective. An element in its kernel has a representative in $X$
Here $\text{The spiral exact sequence is natural in } X^*$ and $G \in G$
and, in turn, the claimed isomorphism of part (1). To identify the differential in part (2) we use B.9 repeatedly and obtain the following commutative diagram:

\[
\begin{array}{ccc}
[C^n X^*, G] & \xrightarrow{d_0} & [Z^{n-1} X^*, G] \\
\downarrow \cong & & \downarrow \cong \\
\ker \{ \mathcal{A}_n \to [L^n_0 X^*, G] \} & \xrightarrow{d_0} & \ker \{ \mathcal{A}_{n-1} \to [L^{n-1} X^*, G] \} \\
\downarrow \cong & & \downarrow \cong \\
\ker \{ \mathcal{A}_n \to M^n_0 [X^*, G] \} & \xrightarrow{d_0} & \ker \{ \mathcal{A}_{n-1} \to M_{n-1} [X^*, G] \} \\
\downarrow \cong & & \downarrow \cong \\
N_n [X^*, G] & \xrightarrow{d_0} & N_{n-1} [X^*, G]
\end{array}
\]

Here $\mathcal{A}_k$ stands for $[X^k, G]$.

For part (3) we first recall the isomorphism $[X^*, \Omega^n cG]_{cM^\emptyset} \cong \pi^n_n (X^*, G)$.

By the Definition B.10 of the functor $Z^n$ there is an adjunction isomorphism
$\text{Hom}_{M^\emptyset} (Z^n X^*, G) \xrightarrow{\cong} \text{Hom}_{cM} (X^*, \Omega^n cG)$.

Using Lemma A.15 there is an induced isomorphism $[Z^n X^*, G]_{M^\emptyset} \xrightarrow{\cong} [X^*, \Omega^n cG]_{cM^{\text{Reedy}}}$.

Clearly, the canonical map $[X^*, \Omega^n cG]_{cM^{\text{Reedy}}} \to [X^*, \Omega^n cG]_{cM^\emptyset}$
is well defined and surjective by our (co-)fibrancy conditions on $X^*$ and $G$. Hence, their composite map is surjective. An element in its kernel has a representative in $\text{Hom}_{cM}(X^*, \Omega^n cG)$ that admits a dotted extension as follows:

\[
\begin{array}{ccc}
& & \Delta_0^{n+1} (cG) \\
& \downarrow & \\
X^* & \to & \Omega^n (cG)
\end{array}
\]

Therefore, any element in the kernel has a representative in $[C^{n+1} X^*, G]_{cM}$. $\square$

**Theorem B.16.** The spiral exact sequence is natural in $X^*$ and $G \in G$ and takes the form

\[
\ldots \to \pi_{p+1} [X^*, G] \xrightarrow{b} \pi_p [X^*, \Omega G] \xrightarrow{\delta} \pi_{p-1} [X^*, G] \xrightarrow{b} \pi_p [X^*, G] \to \ldots
\]

ending with a surjection $\pi^n_0 (X^*, G) \to \pi_1 [X^*, G]$ and an isomorphism $\pi^n_0 (X^*, G) \cong \pi_0 [X^*, G] =: \pi_0$. 
**Definition B.17.** In the spiral exact sequence we call the maps \( b \) satisfies a "Hurewicz theorem" 3.29. The map \( p \) of Lemma B.50.

**Example B.20.** algebras is an equivalence of categories. We let \( G \) products of objects in \( \mathcal{G} \) with \( \beta \).

**Corollary B.19.** The forgetful functor from \( \mathcal{H} \)-algebras in pointed sets to \( \mathcal{H} \)-algebras is an equivalence of categories.

**Example B.20.** Our motivating examples for \( \mathcal{H} \)-algebras are the following ones:

1. Every object \( X \) of \( \mathcal{M} \) yields an \( \mathcal{H} \)-algebra

   \[ F_X : \mathcal{G} \to \text{Set}, F_X(-) = [X, -]_{\mathcal{M}}. \]

   If \( \mathcal{M} = \mathcal{S} \) is the category of spaces and \( \mathcal{G} = \{ K_n = K(\mathbb{F}_p, n) | n \geq 0 \} \), this data is the same as considering \( H^*(X) \) as unstable algebra. This claim is proved in Appendix C.

2. Let \( X^* \) in \( c\mathcal{M} \) be Reedy cofibrant. For any \( s \geq 0 \) there are simplicial \( \mathcal{H} \)-algebras

   \[ G \mapsto \Omega^s[X^*, G]. \]

   Applying \( \pi_0 \) yields an \( \mathcal{H} \)-algebra structure on the homotopy groups

   \[ \pi_s[X^*, G]. \]

3. For any \( s \geq 0 \) the natural homotopy group

   \[ G \mapsto \pi_s^0(X^*, G) = [X^*, \Omega^s cG]_{c\mathcal{M}}^\circ \]

   is an \( \mathcal{H} \)-algebra.

**Proof.** Naturality in both \( X^* \) and \( G \in \mathcal{G} \) is clear. We exhibit now the terms of the derived couple of \([B.12] \). The first is given by

\[ E^2_{p,q} = \ker d_{p,q} / \text{im} d_{p+1,q} \cong \pi_p[X^*, \Omega^q G], \]

following from \([B.15] (2) \). By \([B.15] (3) \) this receives a map from

\[ \text{im} \{ \beta_{p,q} : [Z^p X^*, \Omega^q G] \to [Z^{p+1} X^*, \Omega^{q-1} G] \} \cong \text{coker} \delta_{p+1,q} \cong \pi^h_p(X^*, \Omega^q G) \]

and maps to

\[ \text{im} \beta_{p-2,q+1} \cong \text{coker} \delta_{p-1,q+1} \cong \pi^h_{p-2}(X^*, \Omega^{q+1} G), \]

with \( \beta_{p-2,q+1} : [Z^{p-2} X^*, \Omega^{q+1} G] \to [Z^{p-1} X^*, \Omega^q G] \). The surjection for \( p = 1 \) and isomorphism for \( p = 0 \) follow from the fact that \( C_0 X^* = Z^0 X^* = X^0 \sqcup * \).
B.3. Beck modules.

**Definition B.21.** Given an object $A$ in a pointed category $C$ with zero object $*$, an object in the slice category $C/A$ is an $A$-module if its structure map is a split epimorphism

$$p: E \xrightarrow{\sim} A$$

with a given section $s: A \to E$ such that $p$ is an abelian object in $C/A$. In the literature sometimes the term *Beck module* is used. A *morphism of $A$-modules* is a morphism of split epimorphisms restricting to the identity on $A$.

The objectwise fiber at the basepoint given by Corollary B.19,

$$M = \text{fib}[p: E \to A],$$

is an abelian object in $C$. Often we refer to an $A$-module only by $M$ and leave the split epimorphism, that is the action of $A$ on $M$, understood.

Our interest lies in modules over the following $H$-algebra.

**Definition B.22.** We fix a Reedy cofibrant $X^\bullet$ and let $\pi_0 = \pi_0^0(X^\bullet, G) \cong \pi_0[X^\bullet, G]$.

The content of the next Subsection [B.4] is to prove that the spiral exact sequence is a long exact sequence of $\pi_0$-modules. In the rest of this subsection we are going to describe the relevant $\pi_0$-module structures on its individual terms.

Let $Y^\bullet$ be a $G$-fibrant object in $cM_\ast$ and $K$ a pointed simplicial set. Consider the split $G$-fibration sequence from Definition A.9

$$K(Y^\bullet) \to (Y^\bullet)^K \xrightarrow{\epsilon_\ast} Y^\bullet,$$

where $\epsilon_\ast$ is given by evaluation at the basepoint and its section

$$s_{\text{ext}}: cG \to (cG)^K$$

is induced by $K \to \ast$. Just for the next example we consider the (singular complex of) the topological sphere $S^p$ and the topological disk $D^p$ or any other fibrant replacement of $\Delta^p/\partial \Delta^p$ and $\Delta^p_0$. Because they are now fibrant both $S^p$ and $D^p$ are coabelian cogroup objects in the homotopy category of spaces whose units can be chosen to supply a basepoint. Thus, for $K = S^p$ or $D^p$ and every $X^\bullet$ in $cM$ the split fibration sequence above yields an $[X^\bullet, Y^\bullet]_{cM_\ast}$-module structure on $[X^\bullet, K(Y^\bullet)]_{cM_\ast}$. In the $G$-model structure this replacement is harmless because the respective terms are $G$-equivalent. In contrast the Reedy homotopy type is changed and this intended in Lemma B.43. However, all our targets will be built from object in $G$ which are assumed to be abelian group objects in the homotopy category of $M$ whose unit lifts to a strict basepoint, see Assumption 3.3. This will allow us to be nonchalant about the models for $\Delta^p/\partial \Delta^p$ and $\Delta^p_0$, see Remark B.28.

**Example B.23.** For every $G \in G$ the split $G$-fibration sequence

$$0 \to \pi_0^0(X^\bullet, G) \to [X^\bullet, (cG)^{\Delta^p/\partial \Delta^p}]_{cM_\ast} \xrightarrow{\epsilon_\ast} \pi_0^0(X^\bullet, G) \to 0,$$

of $H$-algebras. This supplies the natural homotopy groups with a module structure over $\pi_0$. In fact, there is a $G$-equivalence of objects with section over $cG$ where in
the following diagram

\[
\begin{array}{ccc}
\Omega^p cG \times cG & \xrightarrow{(\gamma_{\text{ext}}, \epsilon_{\text{ext}})} & (cG)^{\Delta^p/\partial\Delta^p} \\
pr \downarrow \quad & & \downarrow \epsilon_g \\
cG & \equiv & cG
\end{array}
\]

both squares with maps to and from \(cG\) commute. This implies that the short exact sequence is isomorphic to the split exact sequence

\[
0 \rightarrow \pi_1^+(X^*, G) \rightarrow \pi_0^+(X^*, G) \times \pi_0^+(X^*, G) \xrightarrow{\sim} \pi_0^+(X^*, G) \rightarrow 0,
\]

of \(H\)-algebras.

**Definition B.25.** For a fibrant pointed simplicial set \(K\) we define

\[
\Omega^p K = \text{map}_S(\Delta^p/\partial\Delta^p, K) \equiv \text{map}_{S_*}((\Delta^p/\partial\Delta^p)^+, K).
\]

The functor \(\Omega_+ = \Omega^1\) is the free loop space functor commonly denoted by \(L\). Note the isomorphism

\[
\text{map}^\text{ext}(X^*, (cG)^{\Delta^p/\partial\Delta^p}) \cong \Omega_+ \text{map}(X^*, cG).
\]

**Example B.26.** Let \(X^*\) be Reedy cofibrant. The split cofiber sequence of pointed simplicial sets

\[
S^0 \xrightarrow{\sim} (\Delta^p/\partial\Delta^p)^+ \rightarrow \Delta^p/\partial\Delta^p,
\]

where the splitting collapses \(\Delta^p/\partial\Delta^p\) to the non-basepoint in \(S^0\), induces the split fibration sequence of simplicial abelian groups:

\[
\Omega^p[X^*, G] \rightarrow \Omega^p_+ [X^*, G] \xrightarrow{\sim} [X^*, G]
\]

Application of \(\pi_0\) gives \(\pi_0^+[X^*, G]\) a module structure over the \(H\)-algebra \(\pi_0\). As above, the induced short exact sequence is naturally isomorphic to the split short exact sequence

\[
\pi_0^+[X^*, G] \rightarrow \pi_0^+[X^*, G] \times \pi_0[X^*, G] \xrightarrow{\sim} \pi_0[X^*, G]
\]

of \(H\)-algebras.

**Remark B.28.** On \([X^*, (cG)^{\Delta^p/\partial\Delta^p}]_{\mathcal{M}^G}\) and \(\Omega_+[X^*, G]\) the relative group structures are the ones induced by the multiplication on \(G\) and not by the comultiplication of \(S^p\). An Eckmann-Hilton argument shows that on the level of \(G\)-homotopy classes both group structures coincide. This is our justification to drop any fibrant models for the sphere and the disk altogether and continue with \(\Delta^p/\partial\Delta^p\) and \(\Delta^0_+\).

**Definition B.29.** There is a split fibration sequence in \(\mathcal{M}_*\)

\[
\Omega_G \xrightarrow{\gamma_{\text{int}}} \Omega_+ G \xrightarrow{\epsilon_*} G,
\]

with section \(s_{\text{int}}: G \rightarrow \Omega_+ G\) induced by \(\Delta^1/\partial\Delta^1 \rightarrow *\), where

\[
\Omega_+ G := \text{hom}_{\mathcal{M}}^\text{int}(\Delta^1/\partial\Delta^1, G) \equiv \text{hom}_{\mathcal{M}_*}((\Delta^1/\partial\Delta^1)^+, G)
\]

is the “free loop space” on \(G\) and \(\epsilon_*\) is the evaluation at the basepoint. There is a weak equivalence

\[
\begin{array}{ccc}
\Omega G \times G & \xrightarrow{(\gamma_{\text{int}}, s_{\text{int}})} & \Omega_+ G \\
pr \downarrow \quad & & \downarrow \epsilon_g \\
G & \equiv & G
\end{array}
\]
such that both squares with maps to and from $G$ commute.

**Example B.30.** The $\mathcal{H}$-algebra $\pi^X_p(X^\bullet, \Omega G)$ is also a $\pi_0$-module and we describe explicitly how this structure is obtained. First one forms in $c\mathcal{M}_*$ a pullback square

$$
\begin{array}{ccc}
\text{pb}_1^p(G) & \xrightarrow{s'} & c(\Omega G)_{\Delta^p/\partial \Delta^p} \\
\downarrow & & \downarrow \\
\text{pb}_2^p(G) & \xrightarrow{s_{\text{ext}}} & c\Omega^* \Delta^p/\partial \Delta^p \\
\end{array}
$$

(B.31)

where $s_{\text{ext}}$ is the section coming from the basepoint of $\Delta^p/\partial \Delta^p$ and $e_*$ is the evaluation at the basepoint of $\Delta^1/\partial \Delta^1$. The vertical maps are $G$-fibrations with fiber $c(\Omega G)_{\Delta^p/\partial \Delta^p}$. Using the splitting $\omega_{\text{int}}$ from above one sees that there is a natural isomorphism

$$(\text{pb}_1^p(G) \to cG) \cong ((c\Omega G)_{\Delta^p/\partial \Delta^p} \times cG \xrightarrow{\text{pr}} cG)$$

of objects over $cG$. In the next diagram the maps $s_{\text{int}}$ and $e_* \circ s'$ are not even Reedy fibrations and so it is necessary to take a $G$-homotopy pullback.

$$
\begin{array}{ccc}
\text{pb}_2^p(G) & \xrightarrow{s_{\text{int}}} & \text{pb}_1^p(G) \\
\downarrow & & \downarrow \\
\text{pb}_2^p(G) & \xrightarrow{s_{\text{int}}} & c(\Omega G) \\
\end{array}
$$

(B.32)

Here $s_{\text{int}}$ is the section of $e$ and $e_*$ is the evaluation at the basepoint of $\Delta^p/\partial \Delta^p$. We obtain a split $G$-homotopy fiber sequence

$$\Omega^p c(\Omega G) \to \text{pb}_2^p(G) \xrightarrow{\sim} cG.$$  

The two step process can be accomplished in one single $G$-homotopy pullback:

$$
\begin{array}{ccc}
\text{pb}_2^p(G) & \xrightarrow{(s_{\text{ext}}, s_{\text{int}})} & c\Omega^* \Delta^p/\partial \Delta^p \\
\downarrow & & \downarrow \\
\text{pb}_2^p(G) & \xrightarrow{(e_* \circ s, e_*)} & cG \times c\Omega^* \Delta^p/\partial \Delta^p \times c\Omega G \\
\end{array}
$$

The left vertical map from Diagram (B.32) is given by $\text{pb}_2^p$ composed with projection onto the second factor. More explicitly, using the splitting $\omega_{\text{ext}}$ from there is a natural weak equivalence

$$(\text{pb}_2^p(G) \to cG) \simeq (\Omega^p c\Omega G \times cG \xrightarrow{\text{pr}} cG)$$

of objects over $cG$. One obtains a split short exact sequence

$$0 \to \pi^X_p(X^\bullet, \Omega G) \to [X^\bullet, \text{pb}_2^p(G)]_{c\mathcal{M}_*} \xrightarrow{\sim} \pi^X_p(X^\bullet, G) \to 0.$$  

This expresses the relevant $\pi_0$-module structure on $\pi^X_p(X^\bullet, \Omega G)$ over $\pi_0$ and again the middle term splits:

$$[X^\bullet, \text{pb}_2^p(G)]_{c\mathcal{M}_*} \cong \pi^X_p(X^\bullet, \Omega G) \times \pi_0.$$
The spiral exact sequence and $\pi_0$-modules. Now we describe an extra structure of the spiral exact sequence under Assumption 3.3.

The extra structure, being maps of modules over the $H$-algebra $\pi_0 = \pi_0^G(X^\bullet, G)$ is crucial in this approach to moduli spaces and the associated obstruction theory. The reader is invited to compare with Blanc/Dwyer/Goerss [7] and Goerss/Hopkins [28]. In our article it enters in Propositions 5.18 and 5.3.

We start by inspecting the Hurewicz maps in the spiral exact sequence. Observe that there is a natural isomorphism

\[ \text{Hom}_M(X^\bullet, G) \cong \text{map}^{\text{ext}}(X^\bullet, cG), \]

by example B.2. The canonical functor $M \to \text{Ho}(M)$ induces a map

\[ \text{map}^{\text{ext}}(X^\bullet, cG) \cong \text{Hom}_M(X^\bullet, G) \to [X^\bullet, G]. \]

**Proposition B.33.** The Hurewicz map $h$ is induced by $\chi$.

**Proof.** By our earlier construction of the spiral exact sequence, the Hurewicz map is induced by

\[ [C^{p+1}X^\bullet, G] \xrightarrow{\cong} N_{p+1}[X^\bullet, G] \]

\[ [Z^pX^\bullet, G] \xrightarrow{h'} Z_p[X^\bullet, G] \xrightarrow{\cong} N_p[X^\bullet, G] \xrightarrow{\cong} [X^\bullet, G] \]

where

\[ Z_p[X^\bullet, G] \cong \bigcap_{i=0}^p \ker \{(d')^*: [X^p, G] \to [X^{p-1}, G]\}. \]

Every element $\bar{x}$ in $\pi_p^G(X^\bullet, G)$ has a representative $x$ in

\[ x \in \text{Hom}_M(X^p, G) \to \text{Hom}_{M^s}(Z_p^G X^\bullet, G) \to [Z_p^G X^\bullet, G]_{M^s} \to \pi_p^G(X^\bullet, G), \]

where $Z^p X^\bullet \to X^p$ is a $G$-monic cofibration between cofibrant objects by Lemma B.37.2. The map $h'$, and therefore $h$, is induced by $\chi$.

**Corollary B.34.** The Hurewicz map is a $\pi_0$-module morphism.

**Proof.** The map $\chi$ directly yields a morphism of split $G$-homotopy fiber sequences:

\[ \Omega^p \text{map}(X^\bullet, cG) \xrightarrow{\Omega^p \chi = \chi^p} \Omega^p \text{map}(X^\bullet, cG) \xrightarrow{\text{map}(X^\bullet, cG)} \]

\[ \Omega^p[X^\bullet, G] \xrightarrow{\Omega^p \chi = \chi^p} \Omega^p[X^\bullet, G] \xrightarrow{\chi} [X^\bullet, G] \]

Applying $\pi_0$ supplies the $\pi_0$-module map.

To show that the shift maps

\[ s_{p,1}: \pi_p^G(X^\bullet, \Omega G) \to \pi_{p+1}^G(X^\bullet, G) \]

are compatible with these $\pi_0$-module structures we construct a zig-zag

\[ \Omega^p c(\Omega G) \xrightarrow{\cong} \Omega^p c(\Omega G) \to \Omega^{p+1} cG \]
of representing objects with a $G$-equivalence on the left. It induces the shift map after applying $[X^*, -]_{c\mathcal{M}}$. This zig-zag then is fitted into a morphisms of split homotopy fiber sequences over $cG$. This second way to construct the shift map is outlined in [22] in the “dual” setting of $\Pi$-algebras. The following discussion remains valid for the maps $s_{p, q}$ with $q \geq 1$, but we focus on the important case $s_{p, 1}$.

**Definition B.35.** For $G$ in $\mathcal{G}$ let

$$\text{Path}(G) = \text{hom}^\text{int}_{\mathcal{M}}(\Delta^1, G)$$

denote a path object for $G$ with fibration $(\text{ev}_0, \text{ev}_1): \text{Path}(G) \to G \times G$ which is a map of group objects in $\text{Ho} (\mathcal{M})$. As explained in Remark B.28 one would usually replace $\Delta^1$ with a fibrant coabelian cogroup object and obtain the group structure of $\text{Path}(G)$ from there. Up to homotopy this is not necessary. We write

$$PG = \text{fib}\{\text{ev}_0: \text{Path}(G) \to G\},$$

which is again a homotopy group object. It inherits a map $PG \to G$ induced by $\text{ev}_1$ with homotopy fiber $\Omega G$. We define an object $\bar{\Omega}^p c(\Omega G)$ in $c\mathcal{M}_{*}$ as pullback

$$\begin{array}{ccc}
\bar{\Omega}^p c(\Omega G) & \xrightarrow{\alpha} & \Delta_0^{p+1}(cPG) \\
\omega & \cong & \cong \\
\Omega^p c(\Omega G) & \to & \Omega^p cPG
\end{array}$$

where the lower map is induced by the canonical map $\Omega G \to PG$ and $PG$ is contractible, fibrant, and $G$-injective in $\mathcal{M}$. Hence, the vertical maps are trivial $G$-fibrations into $G$-fibrant targets.

**Lemma B.36.** There is a pullback square:

$$\begin{array}{ccc}
\bar{\Omega}^p c(\Omega G) & \xrightarrow{\sigma} & \Omega^{p+1}cG \\
\alpha & \downarrow & \downarrow \\
\Delta^{p+1}_0(cPG) & \to & \Delta^{p+1}_0(cG)
\end{array}$$

**Proof.** The commutative diagram in $c\mathcal{M}_{*}$ by adjointness:

$$\begin{array}{ccc}
\bar{\Omega}^p c(\Omega G) & \xrightarrow{\omega} & \Omega c(\Omega G) \\
\cong & \cong & \\
\bar{\Omega}^p c(\Omega G) & \xrightarrow{\sigma} & \Omega^{p+1}cG \\
\alpha & \downarrow & \downarrow \\
\Delta^{p+1}_0(cPG) & \xrightarrow{\cong} & \Delta^{p+1}_0(cG)
\end{array}$$

(B.37)

It has pullback squares on the back, left and right hand sides. Thus, the map $\sigma$ is induced as map between pullbacks and the front square is a pullback. □

**Remark B.38.** Some words on the previous diagrams are in order.
(1) The defining pullback square of \( \bar{\Omega}^p c(\Omega G) \) is obviously a homotopy pullback for the resolution model structure. The pullback square in Lemma B.36 is not. All the \( \simeq \)-labelled maps in Diagram (B.37) are \( \mathcal{G} \)-equivalences, but neither the back nor the front square are homotopy pullbacks. The map \( cPG \to cG \) is not a \( \mathcal{G} \)-fibration because the map \( PG \to G \) is not \( \mathcal{G} \)-injective. In fact, none of the vertical maps nor the maps from left to right are \( \mathcal{G} \)-fibrations.

(2) Giving a map \( X^* \to \bar{\Omega}^p c\Omega G \) is the same as prescribing the following data:
- a map \( f : Z^{p+1}X^* \to G \) in \( \mathcal{M}_* \) together with
- a nullhomotopy of the composite \( C^{p+1}X^* \to Z^{p+1}X^* \to G \) in \( \mathcal{M}_* \).

Definition B.39. We obtain a zig-zag in \( c\mathcal{M}_* \)

\[
\begin{align*}
\Omega^p c(\Omega G) & \xleftarrow{\simeq} \bar{\Omega}^p c(\Omega G) \xrightarrow{\sigma} \Omega^{p+1} c(G) \\
[X^*, \Omega^p c(\Omega G)]_{c\mathcal{M}^\text{Reedy}} & \xrightarrow{\simeq} [X^*, \Omega^{p+1} c(G)]_{c\mathcal{M}^\text{Reedy}}
\end{align*}
\]

that yields a map \( \beta_p : \Omega^p c(\Omega G) \to \Omega^{p+1} c(G) \) in \( \text{Ho}(c\mathcal{M}_*^\mathcal{G}) \).

Lemma B.40. The map \( \pi^2_p(X^*, \Omega G) \to \pi^2_{p+1}(X^*, G) \) induced by \( \beta_p \) is the shift map \( s_{p,1} \) of the spiral exact sequence.

Proof. The following diagram commutes:

\[
\begin{array}{ccc}
Z^pX^*, \Omega G & \xrightarrow{\beta_{p,1}} & Z^{p+1}X^*, G \\
\Downarrow \simeq & & \Downarrow \simeq \\
[X^*, \Omega^p c(\Omega G)]_{c\mathcal{M}^\text{Reedy}} & \xrightarrow{\beta_{p,1}} & [X^*, \Omega^{p+1} c(G)]_{c\mathcal{M}^\text{Reedy}}
\end{array}
\]

This surjects onto \( \mathcal{G} \)-homotopy classes. \( \square \)

Corollary B.41. The shift map \( s_{p,1} \) is a \( \pi_0 \)-module map.

Proof. Using the description of the \( \pi_0 \)-module structure of the natural homotopy groups in Examples B.23 and B.30 we are going to supply a zig-zag of split \( \mathcal{G} \)-homotopy fiber sequences

\[
\begin{align*}
\Omega^p c(\Omega G) & \xleftarrow{\omega} \bar{\Omega}^p c(\Omega G) \xrightarrow{\sigma} \Omega^{p+1} c(G) \\
\Omega^p c(\Omega G) \times cG & \xleftarrow{\simeq} \bar{\Omega}^p c(\Omega G) \times cG \xrightarrow{\simeq} \Omega^{p+1} cG \times cG
\end{align*}
\]

to which we then apply the functor \( [X^*, -]_{c\mathcal{M}^\mathcal{G}} \) proving the claim. But this is easy enough as indicated in the diagram. \( \square \)

We proceed by inspecting the remaining boundary maps in the spiral exact sequence. It is very useful to observe that most terms of the sequence are representable as functors of \( X^* \), even the bigraded homotopy groups \( \pi_p[X^*, G] \) for \( p \neq 1 \), as we are going to explain now. For \( p = 0 \) this is clear from the isomorphism \( \pi_0[X^*, G] \cong \pi^2_0(X^*, G) \).
**Definition B.42.** For $p \geq 2, q \geq 0$ we define $\Psi_p^q(G)$ in $cM_*$ by the following pullback square:

\[
\begin{array}{c}
\Psi_p^q(G) \\
\downarrow \tau \\
\Delta^p_0(c\Omega^qG)
\end{array} \right\} \rightarrow \begin{array}{c}
\Omega^{p-2}c(\Omega^{q+1}G) \\
\downarrow \sigma \\
\Omega^{p-1}c(\Omega^qG)
\end{array}
\]

Note that $\tau$ is a Reedy fibration because $\sigma$ is one by Lemma B.36. In particular, $\Psi^p_q(G)$ is Reedy fibrant. The vertical maps are $G$-fibrations with fiber $\Omega^p c\Omega^q G$. By looping the defining pullback square we get natural isomorphisms

\[
\Psi_p^q = \Psi_p^q(\Omega^q G)
\]

for $p \geq 2$ and $s, q \geq 0$. Similarly there are natural $G$-equivalences

\[
\Omega^* \Psi^p_q(G) \simeq \Psi^p_q(\Omega^q G).
\]

We concentrate on $\Psi_p^0(G) = \Psi_p^0(\Omega^0 G)$ and suppress the index $q$.

We point out that the next statement holds exactly for the model $\Delta^0_0$ of the disk and not for any fibrant replacements by cogroup objects.

**Lemma B.43.** For all $p \geq 2$ and every Reedy cofibrant $X^*$ maps in $cM$ of the form

\[
X^* \rightarrow \Psi^p(G)
\]

correspond bijectively to maps $C^p X^* \rightarrow G$ in $M_*$ together with a null homotopy of the composite

\[
C^{p-1} X^* \xrightarrow{d^0} C^p X^* \rightarrow G.
\]

Thus, there is a natural isomorphism

\[
[X^*, \Psi^p(G)]_{cM_{\text{Reedy}}} \cong \bigcap_{i=0}^{p} \ker \{(d^i)^* : [X^p, G] \rightarrow [X^{p-1}, G]\} \cong Z_p[X^*, G]_{\text{M}}
\]

of $H$-algebras.

**Proof.** Combining B.42 and B.30 yields a pullback square:

\[
\begin{array}{c}
\Psi^p(G) \\
\downarrow \tau \\
\Delta^p_0(cG)
\end{array} \rightarrow \begin{array}{c}
\Omega^{p-2}c(\Omega^q G) \\
\downarrow \sigma \\
\Omega^{p-1}c(\Omega^q G)
\end{array}
\]

(B.44)

Examining the square one deduces the characterization of maps into $\Psi^p(g)$. There is an adjunction isomorphism

\[
\text{Hom}_{cM_*}(C^p X^*, G) \cong \text{Hom}_{cM_*}(X^*, \Delta^p_0(cG)),
\]

and homotopies on the left correspond with homotopies in $\text{M}_*$ on the right by A.15 resulting in an isomorphism

\[
[C^p X^*, G]_{\text{M}} \cong [X^*, \Delta^p_0(cG)]_{cM_{\text{Reedy}}}
\]

This establishes the isomorphism as $H$-algebras. □
We move on to the $G$-homotopy level. The next proof follows the sketch from [22, Prop. 7.5]. Before we embark on the proof let us note that the pullback of a diagram

\[ A \rightarrow^f B \leftarrow^g C \]

in a model category is weakly equivalent to its homotopy limit if all object are fibrant and one of the maps $f$ or $g$ is a fibrations.

**Lemma B.45.** There are natural isomorphisms

\[ \pi_p[X^\bullet, G] \cong [X^\bullet, \Psi^p(G)]_{cM^G} \]

of $\pi_0$-modules for all $p \geq 2$.

**Proof.** There are natural surjections of $H$-algebras

\[ [X^\bullet, \Psi^p(G)]_{cM_{\text{Reedy}}} \twoheadrightarrow [X^\bullet, \Psi^p(G)]_{cM^G}, \]

and by Lemma [B.43]

\[ [X^\bullet, \Psi^p(G)]_{cM_{\text{Reedy}}} \cong Z_p[X^\bullet, G]_{\mathcal{M}} \twoheadrightarrow \pi_p[X^\bullet, G]. \]

It thus remains to show that these two maps of $H$-algebras have the same kernel.

We define a new object $P^1 \Psi^p_p(G)$ as a pullback of cosimplicial $H$-algebras:

\[ P^1 \Psi^p_p(G) \rightarrow \Delta^p_0(\text{cPath}(G)) \]

(B.46)

\[ \Psi^p_p(G) \times \Delta^p_{0+1}(cG) \twoheadrightarrow \Delta^p_0(cG) \times \Delta^p_0(cG) \]

The evaluation maps on the right are $G$-fibrations. Thus $\epsilon$ is a $G$-fibration, hence also the canonical map $\psi: P^p \Psi^p_p(G) \rightarrow \Psi^p_p(G)$, because it is the composition of $\epsilon$ with the projection to $\Psi^p_p(G)$.

We further claim that $P^p \Psi^p_p$ is $G$-contractible, i.e. $P^p \Psi^p_p(G) \cong \ast$. The splitting $G \rightarrow \text{Path}(G)$ induces an objectwise Reedy equivalence from the diagram

(B.47)

\[ \Psi^p_p(G) \times \Delta^p_{0+1}(cG) \rightarrow \Delta^p_0(cG)^2 \rightarrow \Delta^p_0(cG) \]

to Diagram (B.46). The respective homotopy limits become equivalent. The $G$-homotopy limit of (B.47) is weakly equivalent to the $G$-homotopy limit of the following diagram:

\[ \Psi^p_p(G) \twoheadrightarrow \Delta^p_0(cG) \overset{d^p}{\rightarrow} \Delta^p_0(cG). \]

Since $\tau$ is a $G$-fibration and all objects are $G$-fibrant, we can take the pullback as a model for the homotopy limit. By Diagram (B.44) this limit is isomorphic to the limit of

\[ \Delta^p_{0-1}(cPG) \rightarrow \Delta^p_{0-1}(cG) \leftarrow \Delta^p_{0+1}(cG) \]

where the right map is induced by

\[ \Delta^p_{0-1} \overset{d^p}{\rightarrow} \Delta^p_0 \overset{d^p}{\rightarrow} \Delta^p_{0+1}. \]

This composite map is constant. The limit reduces to the following product, where the canonical map

(B.48)

\[ \Delta^p_{0-1}(c\Omega G) \times \Delta^p_{0+1}(cG) \rightarrow P^p \Psi^p_p(G) \]

is a $G$-equivalence. Thus, $P^p \Psi^p_p(G)$ is $G$-contractible.
There is a map
\[ \phi : \Delta_0^{p+1}(cG) \rightarrow \Psi^P(G) \]
induced via the pullback diagram \([B.44]\) by the constant map
\[ \Delta_0^{p+1}(cG) \xrightarrow{\ast} \Delta_0^{p-1}(cPG) \]
and
\[ d^0 : \Delta_0^{p+1}(cG) \rightarrow \Delta_0^p(cG). \]
The respective maps to \(\Delta_0^{p-1}(cG)\) are equal since \(\Delta_0^{p-1} \rightarrow \Delta_0^{p+1}\) is constant. Inspecting the argument leading to \([B.48]\) yields the following commutative diagram
\[
\begin{array}{c}
\Delta_0^{p-1}(cG) \times \Delta_0^{p+1}(cG) \xrightarrow{\pi} P^P\Psi^P(G) \\
\downarrow \text{pr} \quad \quad \quad \downarrow \psi \\
\Delta_0^{p+1}(cG) \quad \quad \quad \quad \quad \quad \quad \quad \Psi^P(G)
\end{array}
\]
with the obvious projection on the left. This induces:
\[
\begin{array}{c}
[X^\bullet, P^P\Psi^P(G)]_{cM^\text{Reedy}} \xrightarrow{\psi_*} [X^\bullet, \Psi^P(G)]_{cM^\text{Reedy}} \\
\downarrow \text{pr}_* \quad \quad \quad \downarrow \\
[X^\bullet, \Delta_0^{p+1}(cG)]_{cM^\text{Reedy}} \xrightarrow{\phi_*} [X^\bullet, \Psi^P(G)]_{cM^\text{Reedy}}
\end{array}
\]
where \(\text{pr}_*\) is surjective and hence the horizontal cokernels are isomorphic. As already shown, \(P^P\Psi^P(G) \rightarrow \Psi^P(G)\) is a fibration from a contractible object in the resolution model structure. Thus the cokernel of \(\psi_*\) is \([X^\bullet, \Psi^P(G)]_{cM^\phi}\). The cokernel of \(\phi_*\) is identified via the following commutative diagram using Lemmata \([B.15]\) and \([B.48]\)
\[
\begin{array}{c}
[X^\bullet, \Delta_0^{p+1}(cG)]_{cM^\text{Reedy}} \xrightarrow{\phi_*} [X^\bullet, \Psi^P(G)]_{cM^\text{Reedy}} \\
\equiv \quad \quad \quad \equiv \\
[C^{p+1}X^\bullet, G]_{M^\phi} \equiv N_{p+1}[X^\bullet, G]_M \xrightarrow{d^0} \quad \quad Z_p[X^\bullet, G]_M
\end{array}
\]
The lower vertical cokernel is \(\pi_p[X^\bullet, G]\). We obtain the desired isomorphism
\[ \pi_p[X^\bullet, G] \cong [X^\bullet, \Psi^P(G)]_{cM^\phi} \]
as isomorphism of \(\mathcal{H}\)-algebras. \(\square\)

We can easily promote the isomorphism of \(\mathcal{H}\)-algebras to one of \(\pi_0\)-modules.

**Lemma B.49.** For all \(p \geq 2\) the isomorphisms from Lemma \([B.45]\)
\[ \pi_p[X^\bullet, G] \cong [X^\bullet, \Psi^P(G)]_{cM^\phi} \]
is an isomorphism of \(\pi_0\)-modules.

**Proof.** The dotted arrow in the next diagram
\[
\begin{array}{c}
0 \xrightarrow{\kappa} [X^\bullet, \Psi^P(G)]_{cM^\phi} \xrightarrow{\cong} [X^\bullet, \Psi^P(G)]_{cM^\phi} \times [X^\bullet, cG]_{cM^\phi} \xrightarrow{\cong} [X^\bullet, cG]_{cM^\phi} \xrightarrow{h_0} 0 \\
0 \xrightarrow{\kappa} \pi_p[X^\bullet, G] \xrightarrow{\cong} \pi_0\Omega^+_p[X^\bullet, G] \xrightarrow{\cong} \pi_0[X^\bullet, G] \xrightarrow{0}
\end{array}
\]
can be defined by the maps $\kappa$ and $h_0$ on the respective factors using the splitting $\Omega^p [X^*, G] \cong \Omega^p [X^*, G] \times [X^*, G]$ from Example B.26.

**Corollary B.50.** The spiral exact sequence for $s \geq 1$, more precisely the exact sequence

$$\cdots \to \pi_3^s (X^*, G) \to \pi_2 [X^*, G] \to \pi_0^s (X^*, \Omega^s G) \to \pi_1^s (X^*, G),$$

is obtained by applying the functor $[X^*, -]_{cM^0}$ to the homotopy fiber sequence

$$\Psi_3^s (G) \to \Omega^0 c\Omega G \to \pi_1 cG.$$

**Proof.** Consider the long exact sequence obtained from B.42 by applying $r_{X^*, \Omega}$ to the homotopy fiber sequence

$$\Psi_3^s (G) \to \Omega^0 c\Omega G \to \pi_1 cG.$$

Then use Lemma B.45. □

**Corollary B.51.** The boundary map $b$ in the spiral exact sequence is a $\pi_0$-module map.

**Proof.** There is an obvious morphism of split $G$-homotopy fiber sequences

$$\Psi^s (G) \to \Psi^s (G) \times cG \to cG,$$

where $\delta$ denotes the map from Definition B.42. □

It is done. Together, Corollaries B.34, B.41 and B.51 imply

**Theorem B.52.** The spiral exact sequence is a long exact sequence of $\pi_0$-modules.

**B.5. The spiral spectral sequence.** The spiral exact sequence can be derived further and yields the spiral spectral sequence.

**Corollary B.53.** The spiral spectral sequence takes the form

$$E^2_{p,q} = \pi_p [X^*, \Omega^q G] \quad \text{and} \quad d^p_2 : \pi_p [X^*, \Omega^q G] \to \pi_{p-2} [X^*, \Omega^{q+1} G]$$

The differential $d_2$ is a morphism of $H$-algebras. The terms $E^r_p$ form a $\pi_0$-modules and $d^p_r$ is morphism of those. If every $G$ admits deloopings $\Omega^q G$ for $q < 0$ in $G$ the spectral sequence converges strongly to $\text{colim}_k \pi_{p+k}^s (X^*, \Omega^{q-k} G)$.

**Proof.** The differential $d_2$ is an $H$-algebra map by Corollary B.50 and even more a $\pi_0$-module morphism by Theorem B.52. Convergence is proved by Boardman [9, Theorem 6.1(a)]. □

Lemma 2.11 from [13] constructs a natural isomorphism

$$\text{Tot} \text{hom}^\text{ext}_p (K, Y^*) \cong \text{hom}_{M^*} (K, \text{Tot} Y^*)$$

for all $K$ in $S$ and $Y^*$ in $cM$. This yields an isomorphism

$$\text{Tot} (K (Y^*)) \cong \text{hom}_{M^*} (K, \text{Tot} Y^*),$$

when both $K$ and $Y^*$ are pointed. Consequently,

$$\text{Tot} (\Omega^p_\text{ext} c\Omega^q G) \cong \Omega^{p+q} G$$
for all \( G \in \mathcal{G} \). Because \( \text{Tot} \) is a right Quillen functor for the \( \mathcal{G} \)-resolution model structure (see Proposition A.5) there is a map

\[
t_{p,q} : \pi^p_\bullet(X^\bullet, \Omega^q G) \cong [X^\bullet, \Omega^p \epsilon \Omega^q G]_{cM^q} \to [\text{Tot} X^\bullet, \Omega^{p+q} G],
\]

which is compatible with the shift maps, i.e., there is a commuting diagram

\[
\begin{array}{ccc}
\pi^p_\bullet(X^\bullet, \Omega^q G) & \xrightarrow{t_{p,q}} & \pi^q_\bullet(X^\bullet, \Omega^{p+q} G) \\
\text{[Tot} X^\bullet, \Omega^{p+q} G] & \xleftarrow{t_{p,q}} & \pi^{p+1}_\bullet(X^\bullet, \Omega^{p+q-1} G).
\end{array}
\]

Commutativity can be shown with the zig-zag from Definition B.39. One would like to relate the terms \( \text{colim}_k \pi^p_{p+k}(X^\bullet, \Omega^{p-k} G) \) and \([\text{Tot} X^\bullet, \Omega^{p+q} G]\) and ideally assert convergence to the latter. Such statements are hard to come by in this generality, but in the case of cosimplicial spaces and mod-p-GEMs there are well-known convergence theorems, compare Subsection C.5.

**Appendix C. \( \mathcal{H} \)-algebras and unstable algebras**

In Section 2 we gave a brief survey of the theory of unstable coalgebras. As explained there, these form the natural target category of singular homology with coefficients in \( \mathbb{F}_p \) or \( \mathbb{Q} \). Dually, unstable algebras form the natural target category for singular cohomology with coefficients in \( \mathbb{F}_p \). For precise definitions, we refer the reader to [51, 1.3 and 1.4]. Let \( \mathcal{UA} \) be the category of unstable algebras and \( \mathcal{U}^{\ell} \) the abelian category of unstable left modules. The rational case contains some surprisingly subtleties and will be discussed separately in Subsection C.4.

The main result of this appendix is Theorem C.12 which characterizes unstable algebras as product-preserving functors on the homotopy category of finite GEMs. Although this fact is certainly known and it has been mentioned in the literature (cf. [8, 2.1.1 and 5.1.5] and [4, Remark 0.4]), we have not been able to find a reference in the literature outlining the details of this characterization. This appendix is necessary for us to bring the results from Appendix B about the spiral exact sequence in a general context to bear on our realization problem in Subsection 6.1.

**C.1. The cohomology of Eilenberg-MacLane spaces.** Let \( \mathbb{F} = \mathbb{F}_p \) be a fixed prime field of positive characteristic. Let us abbreviate the notation for Eilenberg-MacLane spaces and write \( K_n = K(\mathbb{F}_p, n) \) for \( n \geq 0 \).

For \( n > 0 \), Cartan’s computation [14] of the \( \mathbb{F}_p \)-cohomology of \( K_n \) yields

(C.1) \[ H^*(K_n) \cong U(L(\mathbb{F}_p[n])), \]

where \( U : \mathcal{U}^{\ell} \to \mathcal{UA} \) is the free unstable algebra functor, \( L : \text{Vec} \to \mathcal{U}^{\ell} \) is the Steenrod-Epstein functor which is left adjoint to the forgetful functor, and \( \mathbb{F}_p[n] \) is the graded vector space consisting of a copy of \( \mathbb{F}_p \) in degree \( n \) and 0 otherwise.

The isomorphism holds also for \( n = 0 \). We have isomorphisms of algebras

\[ H^0(K_0) = H^0(K(\mathbb{F}_p, 0), \mathbb{F}_p) \cong \text{Hom}_{\text{Set}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[x]/(x^p - x), \]

and the right side is the free \( p \)-Boolean algebra on a single generator. We recall that an \( \mathbb{F}_p \)-algebra \( A \) is called \( p \)-Boolean if it satisfies \( x^p = x \) for all \( x \in A \). By definition, every unstable algebra over \( A_0 \) is \( p \)-Boolean in degree 0. Therefore we have natural isomorphisms, for all \( A \in \mathcal{UA} \),

\[ \text{Hom}_{\mathcal{UA}}(H^* K(\mathbb{F}_p, 0), A) \cong \text{Hom}_{\text{Alg}_p}(\mathbb{F}_p[x]/(x^p - x), A^0) \cong A^0. \]
and consequently, \( C.1 \) holds also for \( F = F_p \) and \( n = 0 \).

Altogether this says that the \( F_p \)-cohomology of \( K_n \) is the free unstable algebra on one generator in degree \( n \). Equivalently said, we have established the following

**Proposition C.2.** For all primes \( p \) and \( n \geq 0 \), there is an isomorphism

\[
\text{Hom}_{\mathcal{A}}(H^*(K(F_p, n)), A) \cong A^n
\]

which is natural in \( A \in \mathcal{U} \mathcal{A} \).

**C.2. Algebraic theories.** We review a few definitions and statements from the monograph [1] by Adámek/Rosický/Vitale.

**Definition C.3.** A small category \( D \) is sifted if finite products in \( \text{Set} \) commute with colimits over \( D \).

**Definition C.4.** An object \( A \) in a category \( \mathcal{A} \) is perfectly presentable if the Hom-functor \( \text{Hom}_{\mathcal{A}}(A, -): \mathcal{A} \to \text{Set} \) commutes with sifted colimits.

The following fact is proved as [1, Theorem 7.7] where a reference to a more general statement is given.

**Theorem C.5.** A functor between cocomplete categories preserves sifted colimits if and only if it preserves filtered colimits and reflexive coequalizers.

**Definition C.6.** An algebraic theory is a category \( \mathcal{T} \) with finite products. A \( \mathcal{T} \)-algebra is a set-valued functor from \( \mathcal{T} \) preserving finite products. A morphism of \( \mathcal{T} \)-algebras is a natural transformation of functors. The respective category will be denoted by \( \mathcal{T} \)-Alg.

If \( \mathcal{F} \) is a set of objects in a category \( \mathcal{A} \) we use the same symbol \( \mathcal{F} \) to denote the associated full subcategory of \( \mathcal{A} \). Its opposite category is written \( \mathcal{F}^{\text{op}} \) and \( \text{Set}^{\mathcal{F}^{\text{op}}} \) is the category of contravariant set-valued functors from \( \mathcal{F} \).

**Definition C.7.** A set \( \mathcal{F} \) of objects in a category \( \mathcal{A} \) is called a set of strong generators for \( \mathcal{A} \) if the functor

\[
h_{\mathcal{F}}: \mathcal{A} \to \text{Set}^{\mathcal{F}^{\text{op}}}, \quad A \mapsto \text{Hom}_{\mathcal{A}}(-, A)
\]

is faithful and reflects isomorphisms.

Now suppose that \( \mathcal{F} \) is closed under finite coproducts in \( \mathcal{A} \). Note that \( h_{\mathcal{F}}(A) = \text{Hom}_{\mathcal{A}}(-, A) \) sends coproducts in \( \mathcal{F} \) to products. Put differently, for each \( A \) the functors \( h_{\mathcal{F}}(A): \mathcal{F}^{\text{op}} \to \text{Set} \) preserves products and is, in particular, an \( \mathcal{F}^{\text{op}} \)-algebra. Thus, we actually obtain a functor

\[
h_{\mathcal{F}}: \mathcal{A} \to \mathcal{F}^{\text{op}} \dashv \text{Alg}.
\]

**Theorem C.9** (Adámek-Rosický-Vitale[1]). Suppose that the category \( \mathcal{A} \) is cocomplete and possesses a set \( \mathcal{F} \) of perfectly presentable strong generators which is closed under finite coproducts. Then \( \mathcal{A} \) is equivalent via the functor \( h_{\mathcal{F}} \) to \( \mathcal{F}^{\text{op}} \)-algebras.

**Proof.** See [1, Theorem 6.9].
C.3. **Unstable algebras are \( H \)-algebras.** We let \( \mathcal{G} \) be the set of Eilenberg-MacLane spaces \( K_n = K(F_p, n) \) for \( n \geq 0 \) and some fixed prime \( p \). We let \( \mathcal{H} \) be the full subcategory of the homotopy category of pointed spaces whose objects are the finite products of objects in \( \mathcal{G} \). These objects were called finite \( F_p \)-GEMs or shortly finite GEMs. The use of the letters \( \mathcal{G} \) and \( \mathcal{H} \) is consistent with Subsections B.3 and B.4. In particular, Definitions B.18 and C.6 can be merged; an \( \mathcal{H} \)-algebra is a product preserving functor from finite GEMs to sets. Let us consider the full subcategory spanned by the set of objects

\[
\mathcal{F} = \{ H^*(K) \mid K \in \mathcal{H} \} = H^*(\mathcal{H}) \subset \mathcal{U} \mathcal{A}
\]

Then the representing property of Eilenberg-MacLane spaces supplies an isomorphism of categories \( H^* : \mathcal{H}^{\text{op}} \overset{\cong}{\rightarrow} \mathcal{F} \). We note that \( \mathcal{F} \) is obtained from \( \mathcal{G} \) by completing with finite products, in the same way as \( \mathcal{H} \) is obtained from \( \mathcal{G} \).

**Lemma C.10.** Let \( F = F_p \) be a prime field of positive characteristic. Then the object \( H^*(K_n) \) is perfectly presentable in \( \mathcal{U} \mathcal{A} \) for every \( n \geq 0 \).

**Proof.** By Theorem C.5 it suffices to show that the representable functor

\[
\text{Hom}_{\mathcal{U} \mathcal{A}}(H^*(K_n), -) : \mathcal{U} \mathcal{A} \rightarrow \text{Set}
\]

preserves filtered colimits and reflexive coequalizers for all \( n \geq 0 \). We have seen in Proposition C.2 that for all \( F_p \) and \( n \geq 0 \), this representable functor is isomorphic to the functor \( D \mapsto D_n \). This clearly preserves filtered colimits. Let \( Q \) denote the colimit of the reflexive coequalizer diagram in \( \mathcal{U} \mathcal{A} \),

\[
\begin{array}{ccc}
D & \xrightarrow{\alpha} & E \\
\downarrow{\beta} & & \downarrow{\beta} \\
Q
\end{array}
\]

The remaining question is whether \( Q_n \) is a coequalizer of the induced diagram in \( \text{Set} \),

\[
\begin{array}{ccc}
D_n & \xrightarrow{\alpha_n} & E_n \\
\downarrow{\beta_n} & & \downarrow{\beta_n} \\
Q_n
\end{array}
\]

If \( Q' \) denotes the colimit of Diagram (C.11) in graded sets we will show that \( Q' \) is naturally an unstable algebra and has the necessary universal property in \( \mathcal{U} \mathcal{A} \). This forces an isomorphism \( Q \cong Q' \) in \( \mathcal{U} \mathcal{A} \) and finishes the argument.

For each \( d \in D \) we set \( \iota(d) = d - \sigma \alpha(d) \). Then \( \iota \) is an idempotent morphism of unstable left modules and there is a splitting

\[
D \cong E \oplus V
\]

in \( \mathcal{U}^\ell \) with

\[
E \cong \text{im} \sigma = \ker \iota \quad \text{and} \quad V = \text{im} \iota = \ker \alpha.
\]

Under the splitting, \( \alpha \) corresponds to the projection onto \( E \). There exists a map \( \beta' : V \rightarrow E \) of unstable left modules such that \( \beta \) can be written as \( \text{id}_E + \beta' \). Since \( \alpha \) is an algebra map, products of the form \( e v \) with \( e \in E \) and \( v \in V = \ker \alpha \) are in \( V \). Hence, the set quotient

\[
Q' \cong E/(e \sim e + \beta'(v))
\]
is the same as the quotient in $\mathcal{U}^f$. In fact, $Q'$ carries an algebra structure inherited from $E$ as one checks on representatives. Thus, $Q'$ is an object of $\mathcal{U}A$. The universal property can now be seen easily. □

Definition C.8 provides a functor $h_F : \mathcal{U}A \rightarrow \mathcal{F}^{\text{op}}$-Alg that we can compose with the isomorphism $\mathcal{H} \cong \mathcal{F}^{\text{op}}$ to obtain a functor

$$h : \mathcal{U}A \rightarrow \mathcal{H} - \text{Alg}.$$  

**Theorem C.12.** Let $\mathbb{F}$ be a prime field of positive characteristic. Then the functor $h$ is an equivalence of categories.

**Proof.** It is clear that the set $\{H^*(K_n) \mid n \geq 0\}$ is a strong generators for $\mathcal{U}A$. Lemma C.10 demonstrates that all its objects are perfectly presentable. In particular, $\mathcal{F}$ is a set of strong generators for $\mathcal{U}A$ which is closed under finite coproducts and whose elements are perfectly presentable. It follows from Theorem C.9 that $h$ is an equivalence from $\mathcal{U}A$ to $\mathcal{F}^{\text{op}}$-algebras. Finally, cohomology provides an isomorphism $\mathcal{F}^{\text{op}} \cong \mathcal{H}$. □

By Corollary B.19 every $\mathcal{H}$-algebra lifts uniquely to a product-preserving functor with values in pointed sets $\text{Set}_\ast$ which we called $\mathcal{H}$-algebras in pointed sets. In other words, the forgetful functor from pointed sets to sets induces an equivalence of categories from $\mathcal{H}$-algebras in pointed sets to $\mathcal{H}$-algebras and there is a zig-zag

$$\mathcal{U}A \xrightarrow{h} \mathcal{H} - \text{Alg} \leftarrow \mathcal{H} - \text{algebras in pointed sets}$$

of equivalences of categories.

It is useful to describe an inverse functor to $h$. Consider the inclusion functor

$$i : \mathcal{H}^{\text{op}} \cong \mathcal{F} \rightarrow \mathcal{U}A$$

and its left Kan extension

$$u = Li : \mathcal{H} - \text{Alg} \rightarrow \mathcal{U}A$$

along the Yoneda embedding $\mathcal{H}^{\text{op}} \rightarrow \mathcal{H} - \text{Alg}$. In the proof of Theorem C.9 (see [1, Theorem 6.9]) it is shown that $Li$ is an equivalence and an inverse to $h$. More explicitly, we claim that $u$ is naturally isomorphic to

$$F \in \mathcal{H} - \text{Alg} \mapsto \{F(K_n)\}_{n \geq 0},$$

where $\{F(K_n)\}_{n \geq 0}$ is a graded set together with all the operations induced by $\mathcal{H}$. To justify this consider the following commutative diagram of functors:

Here $f_1$ and $f_2$ are forgetful functors and their composite preserves and detects sifted colimits, see [1, Proposition 4.13]. The functor $f_2 = f_1 h$ also preserves and detects sifted colimits because $f_1$ does, see [1, Proposition 2.5]. Note that $\{F(K_n)\}_{n \geq 0} = f_2 f_1(F)$. By [1, Proposition 4.13] the Yoneda embedding is a cocompletion under sifted colimits. Hence, to identify the underlying graded set of $u(F)$ it suffices to examine only the representable functors $F$ coming from $\mathcal{H}$. By the Künneth
induce maps

\[ f_3f_1(y(K_m)) \cong \{H^n(K_m)\}_{n \geq 0} \cong f_3f_2(y(K_m)) \]

Thus we have proved

**Corollary C.13.** Let \( \mathbb{F} \) be a prime field of positive characteristic. The functor

\[ u: \mathcal{H}-\text{Alg} \to \mathcal{U}_A, \ u(F) = \{ F(K_n) \}_{n \geq 0} \]

is an inverse equivalence to \( h \).

**Example C.14.** Let \( X \) be a space. Via the evaluation functor in the previous corollary the \( \mathcal{H} \)-algebra \([X, G]\) becomes the same as the unstable algebra \( H^*(X) \). Similarly, for a cosimplicial space \( X^\bullet \) the \( \mathcal{H} \)-algebra \( \pi_s[X^\bullet, G] \) becomes an unstable algebra. This is a direct consequence of Theorem [C.12]

The \( \pi_0 \)-module structure is something familiar as well.

**Lemma C.15.** For any \( \mathcal{H} \)-algebra \( M \) which is a module over the \( \mathcal{H} \)-algebra \( A \) in the sense of Definition [B.21] the unstable algebra \( u(M) \) is a module over the algebra \( u(A) \) in the usual sense. In particular, for \( s > 0 \) the functor \( h \) identifies the \( \pi_0 \)-module \( \pi_sH^*(X^\bullet) \) with the \( \pi_0 \)-module \( \pi_s[X^\bullet, G] \).

**Proof.** For each map \( f: G_1 \to G_2 \) in \( \mathcal{H} \) one obtains an action map

\[ \phi_f: M(G_1) \times A(G_1) \to M(G_2) \]

satisfying the following relations given in [7] Section 4:

1. \( \phi_f(x, 0) = M(f)(x) \)
2. \( \phi_{g \cdot f}(x, a) = \phi_g(\phi_f(x, a), A(f)a) \),

This yields an \( u(A) \)-action on \( u(M) \) in the usual sense.

Now set \( A = [X^\bullet, -] \) and \( M = \Omega^p[X^\bullet, -] \). The splitting of the fibration sequence [B.27]

\[ \Omega^p[X^\bullet, G] \to \Omega^p_{+}[X^\bullet, G] \to [X^\bullet, G] \]

is induced by the constant map \( \Delta^p/\partial \Delta^p \to \ast \) and determines an isomorphism

\[ [X^\bullet, G] \times \Omega^p[X^\bullet, G] \cong \Omega^p_{+}[X^\bullet, G] \]

of functors to simplicial unstable left modules. This can be written explicitly as \( (a, \omega) \mapsto a + \omega \), where the based loop \( \omega \) is sent to the translated loop starting at \( a \). It can also be understood as the isomorphism between the total space and the product of fiber and base of a split principal fiber bundle.

Because of Theorem [C.12] and Corollary [C.13] we allow ourselves to identify the unstable algebra \( H^*(X^\bullet) \) and the \( \mathcal{H} \)-algebra \([X^\bullet, G]\). Let

\[ \mu: K_m \times K_n \to K_{m+n} \]

be the unique map sending \((u_m, u_n)\) to \( u_{m+n} \), where \( u_k \in H^k(K_k) \) is the fundamental class. Since \( \mathcal{H} \) is pointed there are inclusions \( K_m \to K_m \times K_n \leftarrow K_n \) that induce maps

\[ [X^\bullet, K_m] \times \Omega^p[X^\bullet, K_n] \to [X^\bullet, K_m \times K_n] \times \Omega^p[X^\bullet, K_m \times K_n] \]

which can be composed with the map induced by \( \mu \):

\[ [X^\bullet, K_m \times K_n] \times \Omega^p[X^\bullet, K_m \times K_n] \xrightarrow{\mu} [X^\bullet, K_{m+n}] \times \Omega^p[X^\bullet, K_{m+n}] \to \Omega^p[X^\bullet, K_{m+n}] \]
where the last map is the projection \( \phi_{id} \). Applying \( \pi \) yields the usual \( \pi H^*(X^*) \)-action on \( \pi H^*(X^*) \). \( \square \)

C.4. **Unstable algebras, rationally.** Since there are no non-trivial cohomology operations rationally, this case is much simpler and most of the above applies similarly, at least in positive degrees. Let us write \( K_n = K(\mathbb{Q}, n) \), \( n \geq 0 \), for the corresponding Eilenberg-MacLane space.

For \( n > 0 \), the computation of the \( \mathbb{Q} \)-cohomology of \( K_n \) yields

\[
\tag{C.16} H^*(K_n) \cong U(\mathbb{Q}[n]),
\]

where \( U(-) \) is the free graded commutative algebra functor. As a consequence, Proposition C.2 holds in this case as well.

However, there is a subtlety regarding the structure of a rational unstable algebra in degree 0. On the one hand, there is no obvious analogue of the \( p \)-Boolean property, and on the other hand, the rational cohomology of a space is set-like, i.e. isomorphic to the \( \mathbb{Q} \)-algebra \( \mathbb{Q}^X \) of functions \( X \to \mathbb{Q} \) for some set \( X \) which is also the dual algebra of the set-like \( \mathbb{Q} \)-coalgebra on \( X \) (cf. Definition 2.5).

The problem with simply defining rational unstable algebras to be non-negatively graded commutative \( \mathbb{Q} \)-algebras that are set-like in degree 0 is that the resulting theory is not algebraic in the sense of Theorem C.12. Again, we point out that this is highly desirable to enable a translation of the Appendix B to the main body of this article. Results of Heyneman/Radford [32, 3.7] show that for a reasonable set \( X \), the algebra \( \mathbb{Q}^X \) is (co)reflexive, which means that the canonical map to the finite (or restricted) dual of \( \mathbb{Q}^X \),

\[ \mathbb{Q}(X) \to \text{Hom}_{\text{Set}}(X, \mathbb{Q})^\circ, \]

is an isomorphism of \( \mathbb{Q} \)-coalgebras. We recall that

\[ (-)^\circ : \text{Alg}_{\mathbb{Q}}^{op} \to \text{Coalg}_{\mathbb{Q}} \]

is the right adjoint of the functor \( (-)^\vee : \text{Coalg}_{\mathbb{Q}} \to \text{Alg}_{\mathbb{Q}}^{op} \), taking a coalgebra to its linear dual algebra. See [55, Chapter IV] for a good account of the general properties of this functor. As a consequence, we have isomorphisms

\[ \text{Hom}_{\text{Alg}}(\mathbb{Q}^X, \mathbb{Q}^Y) \cong \text{Hom}_{\text{Coalg}}(\mathbb{Q}(Y), \mathbb{Q}(X)) \cong \text{Hom}_{\text{Set}}(Y, X) \]

showing that \( \text{Set}^{op} \) (for reasonable sets) is equivalent to set-like algebras and thus the degree 0 part of the proposed definition is not algebraic.

In relation to this we note several amusing facts. The isomorphisms above show that \( H^*(K_0) \) satisfies Proposition C.2 if \( A^0 \) there is set-like. However, the set-like property is not preserved under filtered colimits. It follows that \( H^0(K_0) \) is not perfectly presentable in the category of set-like \( \mathbb{Q} \)-algebras. Curiously though, mapping out of \( H^0(K_0) \) to set-like \( \mathbb{Q} \)-algebras preserves reflexive coequalizers.

Even in positive characteristic the relation between the \( p \)-Boolean and set-like properties is not absolutely tight; for example, the free \( p \)-Boolean \( \mathbb{F}_p \)-algebra on countably many generators is not set-like. There is, however, a duality theorem saying that a \( p \)-Boolean algebra \( A \) is isomorphic to the algebra of continuous \( \mathbb{F}_p \)-valued maps on \( \text{Spec}(A) \), see [34, Appendix].

Let \( \mathcal{G}_\mathbb{Q} \) denote the set of Eilenberg-MacLane spaces \( K_n = K(\mathbb{Q}, n) \) for \( n \geq 0 \), and let \( \mathcal{H}_\mathbb{Q} \) be the full subcategory of the homotopy category of pointed spaces whose objects are the finite products of objects in \( \mathcal{G}_\mathbb{Q} \). As before, \( \mathcal{H}_\mathbb{Q} \) is an algebraic
theory. Motivated especially by Theorem C.12 the following definition takes into account all these issues and is particularly suitable for our purposes.

**Definition C.17.** An unstable $\mathbb{Q}$-algebra is an $\mathcal{H}_\mathbb{Q}$-algebra.

Note that this definition is now formally the same as in the mod $p$ case.

**Example C.18.** A set-like $\mathbb{Q}$-algebra admits an obvious action from the algebra $H^0(K_0)$, the $\mathbb{Q}$-algebra of functions $\mathbb{Q} \to \mathbb{Q}$. In particular, the dual of an unstable $\mathbb{Q}$-coalgebra is an unstable $\mathbb{Q}$-algebra in the sense of Definition C.17.

**Example C.19.** Let $X$ be a space. The rational singular cohomology produces an unstable $\mathbb{Q}$-algebra as follows, $G_\mathbb{Q} \to \text{Set}$, $G \mapsto [X, G]$. Similarly, for a cosimplicial space $X^\bullet$, we have an unstable $\mathbb{Q}$-algebra $G \mapsto \pi_0[X^\bullet, G]$.

We note that

$$\pi_0(H^*(X^\bullet))^0 = \text{coeq}(H^0(X^1) \rightrightarrows H^0(X^0))$$

is actually again set-like.

**C.5. The cohomology spectral sequence.** Let us return to the spiral spectral sequence introduced in Subsection B.5 for a general resolution model category and specialize to $\mathcal{M}$ being the category of spaces and $G = \{K(\mathbb{F}, n) \mid n \geq 0\}$ where $\mathbb{F}$ is any prime field. The reindexed $E^2$-page for $G = K(\mathbb{F}, n)$ is identified as

$$E^2_{s,t} \cong \pi_s H^{n-t}(X^\bullet).$$

The exact couple (B.12) of $\mathcal{H}$-algebras

$$\ldots \to [Z^{m-1} X^\bullet, \Omega^{n+1} G] \to [Z^m X^\bullet, \Omega^n G] \to [C^m X^\bullet, \Omega^n G] \to [Z^{m-1} X^\bullet, \Omega^n G] \to \ldots$$

is obtained by passing from homology to cohomology in Rector’s construction of the homology spectral sequence [48]. Consequently, the spiral spectral sequence is, up to re-indexing, the linear dual of the homology spectral sequence of the cosimplicial space $X^\bullet$. In case the terms of the spectral sequence are all of finite type, the strong convergence results of Bousfield [11, Theorem 3.4] and Shipley [53, Theorem 6.1] for the homology spectral sequence imply analogous results for the convergence of the cohomology spectral sequence to $H^*(\text{Tot}^f(X^\bullet))$.

From results proved in Subsection B.4 and the identification of Lemma C.15 we obtain the following useful

**Proposition C.20.** The cohomology spectral sequence of a cosimplicial space $X^\bullet$ is a spectral sequence of $\pi_0 H^*(X^\bullet)$-modules.

**Appendix D. Moduli spaces in homotopy theory**

The approach to the definition of the moduli space of topological realizations of an unstable coalgebra that we take here follows and makes essential use of various results from a series of papers by Dwyer and Kan. In this appendix we recall the necessary background material and give a concise review of those results that are required in this paper. The interested reader should consult the original papers for a more complete account [17, 18, 19, 20] and [7].
D.1. **Simplicial localization.** Let \((\mathcal{C}, \mathcal{W})\) be a (small) category \(\mathcal{C}\) together with a subcategory of weak equivalences \(\mathcal{W}\) which contains the isomorphisms. In their seminal work, Dwyer and Kan introduced two constructions of a simplicially enriched category associated to \((\mathcal{C}, \mathcal{W})\). Each of the them is a refinement of the passage to the homotopy category \(\mathcal{C}[\mathcal{W}^{-1}]\), which is given by formally inverting all weak equivalences, and they uncover in a functorial way the rich homotopy theory hidden in the pair \((\mathcal{C}, \mathcal{W})\).

The simplicial localization \(L(\mathcal{C}, \mathcal{W})\) \cite{DwyerKan} is defined by a general free simplicial resolution

\[ L(\mathcal{C}, \mathcal{W})_n = F_n\mathcal{C}[(F_n\mathcal{W})^{-1}] \]

where \(n\) indicates the simplicial degree and \(F_n\) denotes the free category functor iterated \(n + 1\) times. As the simplicial set of objects is constant, \(L(\mathcal{C}, \mathcal{W})\) is a simplicially enriched category. We will follow the standard abbreviation and simply call such categories simplicial.

The following fact, saying that the simplicial localization preserves the homotopy type of the classifying space, will be useful.

**Proposition D.1** (Dwyer-Kan \cite{DwyerKan}). There are natural weak equivalences of spaces

\[ BC \simeq B(F_n\mathcal{C}) \simeq B(F_n\mathcal{C}[(F_n\mathcal{W})^{-1}]). \]

**Proof.** See \cite{DwyerKan} 4.3. \(\square\)

**Remark D.2.** As a side remark on the last proposition, we like to note that in the equivalent context of \(\infty\)-categories, one may regard the simplicial localization as given by taking a pushout along a disjoint union of copies of the inclusion of categories

\[ \bullet \longrightarrow \bullet \longrightarrow (\bullet \rightsquigarrow \bullet), \]

one for each morphism in \(\mathcal{W}\). Since this inclusion is a nerve equivalence (but not a Joyal equivalence), it follows that the homotopy type of the classifying space is invariant under such pushouts.

On the other hand, we also have the hammock localization \(L^H(\mathcal{C}, \mathcal{W})\) \cite{Hirschhorn} 3.1, \cite{Heller} 5.1 whose morphisms in simplicial degree \(n\) are given by “hammocks” of width \(n\):

\[
\begin{array}{c}
\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\
\end{array}
\]

This diagrams shows \(n\) composable morphisms between zigzag diagrams in \((\mathcal{C}, \mathcal{W})\) of the same shape, the same source and target, and whose components are weak
equivalences. This defines more explicitly a presentation of a simplicial category and is, for that reason, often more practical than the simplicial localization. The following proposition says that the two constructions lead to weakly equivalent simplicially enriched categories, i.e. there is zigzag of functors which induces homotopy equivalences of mapping spaces and equivalences of homotopy categories.

**Proposition D.3** (Dwyer-Kan [20]). There are natural weak equivalences of simplicial categories

\[ L^H(C, W) \xrightarrow{\sim} \text{diag}L^H(F_* C, F_* W) \xrightarrow{\sim} L(C, W). \]

*Proof.* See [20, 2.2]. □

D.2. Models for mapping spaces. If the pair \((C, W)\) is part of a simplicial model category structure on \(C\), the simplicial enrichment of \(L^H(C, W)\) agrees up to homotopy with the (derived) mapping spaces of the simplicial model category.

**Theorem D.4** (Dwyer-Kan [18]). Let \(C\) be a simplicial model category with weak equivalences \(W\) and \(X, Y \in C\) where \(X\) is cofibrant and \(Y\) is fibrant. Then there is a weak equivalence

\[ L^H(C, W)(X, Y) \simeq \text{map}_C(X, Y) \]

which is given by a natural zigzag of weak equivalences.

*Proof.* This is [18, 4.7] and is a special case of [18, 4.4]. The more general case applies also to non-simplicial model categories with \(\text{map}(-, -)\) replaced in this case by a functorial choice of derived mapping spaces defined in terms of (co)simplicial resolution of objects. □

**Remark D.5.** There are some obvious set-theoretical issues which need to be addressed here in order to extend the hammock localization to non-small categories such as model categories. We refer the reader to [18] for the details.

There is no direct way to compare the compositions in the simplicial category of cofibrant-fibrant objects \(C^\text{cf}_*\) and the hammock localization \(L^H(C^\text{cf}, W)\). However, there is a strengthening of Theorem D.3 (see [18, 4.8]) which says in addition that the weak equivalences of mapping spaces can be upgraded to a weak equivalence of simplicial categories. To achieve such a comparison, one considers the hammock localization applied degreewise to \(C^\text{cf}_*\). Then it is easy to see that that the obvious functors

\[ C^\text{cf}_* \xrightarrow{\sim} \text{diag}L^H(C^\text{cf}_*, W_\bullet) \xrightarrow{\sim} \text{diag}L^H(C_\bullet, W_\bullet) \xrightarrow{\sim} L^H(C, W) \]

are weak equivalences of simplicial categories.

In the presence of a homotopy calculus of fractions, the shapes of zigzag diagrams appearing in the definition of the hammock localization can be reduced to a single zigzag shape of arrow length 3. Although this reduction does not respect the composition in the category, which is given by concatenation and thus increases the length of the hammocks, it is useful in obtaining smaller models for the homotopy types of the mapping spaces.

Let \(\mathcal{M}AP(X, Y)\) denote the (geometric realization of the) subspace of the mapping space \(L^H(C, W)(X, Y)\), which is defined by 0-simplices of the form

\[ X \leftarrow \bullet \rightarrow \bullet \rightarrow Y. \]

More precisely, \(W_{\text{Hom}}(X, Y)\) will denote the category of such zigzag diagrams from \(X\) to \(Y\) and weak equivalences between them, and \(\mathcal{M}AP(X, Y)\) its classifying space.
**Proposition D.7** (Dwyer-Kan [20]). Let $\mathcal{C}$ be a model category with weak equivalences $\mathcal{W}$ and $X, Y \in \mathcal{C}$. Then there is a natural weak equivalence

$$\text{MAP}(X, Y) \simeq L^H(\mathcal{C}, \mathcal{W})(X, Y).$$

**Proof.** See [20, 6.2 and 8.4]. \hfill \Box

**Remark D.8.** If $X$ is cofibrant, then the smaller category $\mathcal{W}^\text{hom}(X, Y)$ whose objects are zigzags in which the first weak equivalence is the identity map has the same homotopy type (see [20] and [16]). The analogous statement when $Y$ is fibrant also holds.

**D.3. Moduli spaces.** We now turn to the definition of moduli spaces. Let $\mathcal{C}$ be a simplicial model category and $X \in \mathcal{C}$. Let $\mathcal{W}(X)$ denote the subcategory whose objects are the objects of $\mathcal{C}$ which are weakly equivalent to $X$ and the morphisms are weak equivalences between them. The classifying space $\mathcal{M}(X)$ of the category $\mathcal{W}(X)$ is called the moduli space for objects of type $X$. We remark that although $\mathcal{W}(X)$ is not a small category, it is nevertheless homotopically small which suffices for the purposes of extracting a well-defined homotopy type. We refer to [17] for more details related to this set-theoretical issue. Note that $\mathcal{M}(X)$ is always connected.

Combining the previous results, we obtain the following theorem.

**Theorem D.9** (Dwyer-Kan [17]). Let $\mathcal{C}$ be a simplicial model category and $X \in \mathcal{C}$ an object which is both cofibrant and fibrant. Then there is a weak equivalence

$$\mathcal{M}(X) \simeq \text{BAut}^h(X)$$

which is given by a natural zigzag of weak equivalences.

**Proof.** The combination of Propositions [D.1] and [D.3] gives a weak equivalence

$$\mathcal{M}(X) \simeq B(L^H(\mathcal{W}(X), \mathcal{W}(X))) \simeq B(L^H(\mathcal{W}(X), \mathcal{W}(X))(X, X)).$$

The map of simplicial monoids

$$L^H(\mathcal{W}(X), \mathcal{W}(X))(X, X) \longrightarrow L^H(\mathcal{C}, \mathcal{W})(X, X)$$

is the inclusion of those connected components whose 0-simplices define invertible maps in the homotopy category (see [18, 4.6]). Then from Theorem [D.4] and the weak equivalences of [D.6], we can conclude the required weak equivalence

$$\mathcal{M}(X) \simeq \text{BAut}^h(X)$$

by restriction to the appropriate components. \hfill \Box

The definition of moduli spaces clearly extends from objects to morphisms, or more general diagrams, by looking at the corresponding model category of diagrams. Following [7], for two objects $X$ and $Y$ in a model category $\mathcal{C}$, we write

$$\mathcal{M}(X \longrightarrow Y)$$

for the classifying space of a category $\mathcal{W}(X \longrightarrow Y)$ whose objects are maps

$$U \rightarrow V$$
where $U$ is weakly equivalent to $X$ and $V$ is weakly equivalent to $Y$, and whose morphisms are square diagrams as follows

\[
\begin{array}{c}
U 
\xrightarrow{\sim} 

\xrightarrow{\sim} 

\xrightarrow{\sim} 

\xrightarrow{\sim} 
\end{array}
\]

where the vertical maps are weak equivalences. More generally, for every property $\phi$ which applies to an object $(U \to V) \in \mathcal{W}(X \xrightarrow{\sim} Y)$ and whose satisfaction depends only on the respective component of $\mathcal{W}(X \xrightarrow{\sim} Y)$, we define

\[\mathcal{M}(X \xrightarrow{\phi} Y)\]

to be the classifying space of the corresponding subcategory. We can clearly combine and extend these definitions to define also moduli spaces of more complicated diagrams, e.g.,

\[\mathcal{M}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y)\]

There are various maps connecting these moduli spaces. Among them, we distinguish the following two types of maps.

- Maps which forget structure, e.g., the map $\mathcal{M}(X \xrightarrow{\sim} Y) \to \mathcal{M}(X) \times \mathcal{M}(Y)$ which is induced by the obvious forgetful functor.
- Maps which neutralize structure, e.g., the map $\mathcal{M}(X, Y) \to \mathcal{M}(X \xleftarrow{\sim} Y)$ which is induced by the functor $(X \sim U \to V \sim Y) \mapsto (U \to V)$.

The following theorem from [7] explains the relations between these types of moduli spaces in some important exemplary cases.

**Theorem D.10.** Let $\mathcal{C}$ be a model category and $X, Y, Z \in \mathcal{C}$. Then:

(a) The sequence

\[\mathcal{M}(X, Y) \to \mathcal{M}(X \xrightarrow{\sim} Y) \to \mathcal{M}(X) \times \mathcal{M}(Y)\]

is a homotopy fiber sequence.

(b) Suppose that $\mathcal{M}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y)$ is non-empty. Then there is a homotopy fiber sequence

\[\mathcal{M}(X, Y; \phi') \to \mathcal{M}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y) \to \mathcal{M}(Z \xleftarrow{\phi} X) \times \mathcal{M}(Y)\]

where $\mathcal{M}(X, Y; \phi') \subseteq \mathcal{M}(X, Y)$ denotes the subspace of maps which satisfy property $\phi'$.

**Proof.** (Sketch) Part (a) is [7] Theorem 2.9], but the proof seems to contain an error. A correction can be made following [16] where a similar error from [18] is corrected. Consider the category $\mathcal{W}^w(X \xrightarrow{\sim} Y)$ whose objects are the same as the objects of $\mathcal{W}(X \xrightarrow{\sim} Y)$ but morphisms are square diagrams as follows

\[
\begin{array}{c}
U 
\xrightarrow{\sim} 

\xrightarrow{\sim} 

\xrightarrow{\sim} 

\xrightarrow{\sim} 
\end{array}
\]

where both vertical maps are weak equivalences. Let $\mathcal{M}^w(X \xrightarrow{\sim} Y)$ denote the classifying space of this category. Although there is no natural functor in general
that connects the categories $\mathcal{W}(X \rightsquigarrow Y)$ and $\mathcal{W}_{\text{tw}}(X \rightsquigarrow Y)$, nevertheless there is a natural zigzag of weak equivalences

$$\mathcal{M}(X \rightsquigarrow Y) \simeq \mathcal{M}_{\text{tw}}(X \rightsquigarrow Y).$$

To see this, consider an auxiliary double category $\mathcal{W}_{\text{c}}(X \rightsquigarrow Y)$ defined as follows:

- the objects are the same as in $\mathcal{W}(X \rightsquigarrow Y)$,
- the horizontal morphisms are the morphisms of $\mathcal{W}(X \rightsquigarrow Y)$,
- the vertical morphisms are the morphisms of $\mathcal{W}_{\text{tw}}(X \rightsquigarrow Y)$,
- the squares are those diagrams that make everything commute.

The nerve of this double category is a bisimplicial set. We write $\mathcal{M}_{\text{c}}(X \rightsquigarrow Y)$ to denote the realization of its diagonal. By construction, there are natural maps $\mathcal{M}(X \rightsquigarrow Y) \to \mathcal{M}_{\text{c}}(X \rightsquigarrow Y) \leftarrow \mathcal{M}_{\text{tw}}(X \rightsquigarrow Y)$ and an application of [16, Proposition 3.9] -or its obvious analogue- shows that both maps are weak equivalences.

Similarly, we define $\mathcal{W}_{\text{Hom}}(X,Y)$ to be the analogous variant of the category $\mathcal{W}_{\text{Hom}}(X,Y)$ where one of the maps has the opposite variance. Again, by [16], we have a natural zigzag of weak equivalences:

$$B(\mathcal{W}_{\text{Hom}}^{\text{tw}}(X,Y)) \simeq \mathcal{M}_{\mathcal{A}}(X,Y).$$

Then it suffices to show that the sequence of functors

$$\mathcal{W}_{\text{Hom}}^{\text{tw}}(X,Y) \to \mathcal{W}_{\text{tw}}(X \rightsquigarrow Y) \to \mathcal{W}(X) \times \mathcal{W}(Y)^{\text{op}}$$

induces a homotopy fiber sequence of classifying spaces. This is an immediate application of Quillen’s Theorem B, as explained in [7, 2.9].

The proof of (b) is similar (cf. [7, 2.11]). Consider the category $\mathcal{W}_{\text{tw}}^{\text{tw}}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y)$ which has the same objects as $\mathcal{W}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y)$ but morphisms are commutative diagrams of the form:

$$
\begin{array}{c}
U \\
\sim
\end{array} \begin{array}{c}
V \\
\sim
\end{array} \begin{array}{c}
W \\
\sim
\end{array}
\begin{array}{c}
U' \\
\sim
\end{array} \begin{array}{c}
V \\
\sim
\end{array} \begin{array}{c}
W'
\end{array}
$$

Using the methods of [16], it can be shown that these two categories have homotopy equivalent classifying spaces. There is a forgetful functor

$$\mathcal{F} : \mathcal{W}_{\text{Hom}}^{\text{tw}}(Z \xleftarrow{\phi} X \xrightarrow{\phi'} Y) \to \mathcal{W}(Z \xleftarrow{\phi} X) \times \mathcal{W}(Y)^{\text{op}}.$$

Given $(U \leftarrow V, W) \in \mathcal{W}(Z \xleftarrow{\phi} X) \times \mathcal{W}(Y)^{\text{op}}$, the comma-category $\mathcal{F} \downarrow (U \leftarrow V, W)$ has as objects diagrams of the form:

$$
\begin{array}{c}
U' \\
\sim
\end{array} \begin{array}{c}
V' \\
\sim
\end{array} \begin{array}{c}
W' \\
\sim
\end{array}
\begin{array}{c}
U \\
\sim
\end{array} \begin{array}{c}
V \\
\sim
\end{array} \begin{array}{c}
W
\end{array}
$$

It is easy to see that there are functors which induce a pair of inverse equivalences $\mathcal{W}_{\text{Hom}}^{\text{tw}}(V, W) \simeq \mathcal{F} \downarrow (U \leftarrow V, W)$.

Then the required result is an application of Quillen’s Theorem B using (D.11) and Theorems [D.3] and [D.7].
D.4. A moduli space associated with a directed diagram. We discuss in
detail the case of a moduli space associated with a direct system of objects in a
model category. The diagrams that we actually have in mind for our applications
have the form of a dual Postnikov tower, but we will present the main result in
the general case of an arbitrary diagram. In the case of the model category of
simplicial sets, this result is a special case of the general theorem of [17], and in
fact the generalization to an arbitrary model category can also be treated similarly.

Let $\mathcal{M}$ be a simplicial model category. Let $\mathbf{N}$ denote the poset of natural numbers
and $\mathbf{n}$ the subposet $\{0 < 1 < \cdots < n\}$. Given a diagram $X : \mathbf{N} \to \mathcal{M}$, let $X_{\leq n}$
denote its restriction to $\mathbf{n}$. Two diagrams $X, Y : \mathbf{N} \to \mathcal{M}$ are called conjugate if
their restrictions $X_{\leq n}$ and $Y_{\leq n}$ are pointwise weakly equivalent. More generally,
given a small category $\mathcal{C}$ and diagrams $X, Y : \mathcal{C} \to \mathcal{M}$, these are called conjugate if for every
$J : \mathbf{n} \to \mathcal{C}$ the pullback diagrams $X \times J$ and $Y \times J$ are pointwise weakly equivalent.

Let $X : \mathbf{N} \to \mathcal{M}$ be a diagram. Consider the category Conj$_{X}$ of $\mathbf{N}$-diagrams
in $\mathcal{M}$ whose objects are the conjugates of $X$ and whose morphisms are pointwise
weak equivalences between conjugates. The moduli space of conjugates (called
classification complex in [17]) of $X$, denoted here by $\mathcal{M}_{\text{conj}}(X)$, is the classifying
space of the (homotopically small) category Conj$_{X}$. Similarly we define moduli
spaces $\mathcal{M}_{\text{conj}}(X)$ for more general diagrams $X : \mathcal{C} \to \mathcal{M}$.

We have canonical compatible restriction maps:

$$r_{n} : \mathcal{M}_{\text{conj}}(X) \to \mathcal{M}(X_{\leq n}) \simeq \text{BAut}^{h}(X_{\leq n}).$$

The last weak equivalence comes from Theorem D.9 applied to the (Reedy) model
category $\mathcal{M}^{\mathbf{n}}$.

Theorem D.12 (after Dwyer-Kan [17]). Let $\mathcal{M}$ be a simplicial model category and
$X : \mathbf{N} \to \mathcal{M}$ a diagram. Then there is a weak equivalence

$$\mathcal{M}_{\text{conj}}(X) \sim \text{holim}_{n}\mathcal{M}(X_{\leq n}).$$

Proof. Let $C = \bigsqcup_{n} \mathbf{n}$ be the coproduct of these posets. There is a functor $s : C \to C$
which is given by the inclusion functors $\mathbf{n} \to \mathbf{n} + 1$. The colimit of these inclusions
is $\mathbf{N}$ and there is a pushout

$$\begin{array}{ccc}
C \sqcup C^{\text{id} \times \text{id}} & \longrightarrow & C \\
| \downarrow \text{id} \times s | & & | \\
C & \longrightarrow & \mathbf{N}
\end{array}$$

Let $V = (v_{0} \to v_{2} \leftarrow v_{1})$ denote the zigzag poset with three elements. Consider
the pushout diagram

$$\begin{array}{ccc}
C \sqcup C & \longrightarrow & C \times V \\
| \downarrow f \downarrow g | & & | \\
C \times V & \longrightarrow & \mathbf{N}^{\mathbb{G}}
\end{array}$$

where $f$ (respectively $g$) sends the first copy of $\mathbf{n}$ to $\mathbf{n} \times \{v_{0}\}$ and the second copy
to $\mathbf{n} \times \{v_{1}\}$ (respectively $\mathbf{n} + 1 \times \{v_{1}\}$). There is an obvious transformation from
the latter pushout square to the first one. This induces a transformation from the
following pullback diagram
where $X_? \rightarrow X$ are given by pulling back the diagram $X$ to the respective categories and the maps in the square are likewise given by pullback functors, to the pullback diagram

(D.14) \[
\begin{array}{ccc}
\mathcal{M}_{\text{conj}}(X_{\mathbb{N}_\mathbb{C}}) & \longrightarrow & \mathcal{M}_{\text{conj}}(X_C) \\
\downarrow & & \downarrow \\
\mathcal{M}_{\text{conj}}(X_C) & \longrightarrow & \mathcal{M}_{\text{conj}}(X_{C \cup C})
\end{array}
\]

(To make sure that the squares are really pullbacks, we assume here that we work in a convenient category of topological spaces. Alternatively, it is also possible, and makes little difference, to work with simplicial sets throughout.)

We claim that the maps

\[ \mathcal{M}_{\text{conj}}(X) \rightarrow \mathcal{M}_{\text{conj}}(X_{\mathbb{N}_\mathbb{C}}) \]

\[ \mathcal{M}_{\text{conj}}(X_C) \rightarrow \mathcal{M}_{\text{conj}}(X_{C \times V}) \]

are weak equivalences. To see this note that the functors $C \times V \rightarrow C$ and $\mathbb{N}_\mathbb{C} \rightarrow \mathbb{N}$ admit sections $C \rightarrow C \times V$, $c \mapsto (c, v_1)$, and $\mathbb{N} \rightarrow \mathbb{N}_\mathbb{C}$, similarly. These give pairs of maps between the respective moduli spaces

\[ \mathcal{M}_{\text{conj}}(X_C) \hookrightarrow \mathcal{M}_{\text{conj}}(X_{C \times V}) \]

\[ \mathcal{M}_{\text{conj}}(X) \hookrightarrow \mathcal{M}_{\text{conj}}(X_{\mathbb{N}_\mathbb{C}}) \]

which are inverse weak equivalences because their composites are either equal to the identity or they can be connected to the identity by a zigzag of natural transformations. In fact, this zigzags is actually defined by natural transformations already available at the level of the indexing posets: it is essentially the zigzag connecting the identity functor on $V$ to the constant functor at $v_1 \in V$. Note that this uses that $X_{C \times V}$ is pulled back from $X : \mathbb{N} \rightarrow \mathcal{M}$ and so the values of $X_{C \times V}$ in the $V$-direction are weak equivalences.

Moreover, the square (D.14) is actually a homotopy pullback. The proof of this is similar to [17, Lemma 7.2]. The homotopy fibers of the vertical maps can be identified with homotopy equivalent spaces (the arguments are comparable to Theorem D.10). It follows that (D.13) is also a homotopy pullback. By [17, 8.3], we have a weak equivalence

\[ \mathcal{M}_{\text{conj}}(X_C) \rightarrow \prod_n \mathcal{M}(X_{\leq n}) \]

and so the homotopy pullback of (D.13) is weakly equivalent to $\text{holim}_n \mathcal{M}(X_{\leq n})$. This means that the map $\mathcal{M}_{\text{conj}}(X) \rightarrow \text{holim}_n \mathcal{M}(X_{\leq n})$, which is canonically induced by the restriction maps $r_n$, is a weak equivalence, as required. \[\square\]
Remark D.15. In our applications the diagram $X$ will correspond to a form of dual Postnikov tower, meaning in particular that each value $X(m)$ is determined by $X(n)$ for any $m > n$. More specifically, in such cases, one has an obvious weak equivalence $\mathcal{M}(X_{\leq n}) \simeq \mathcal{M}(X(n))$, which simplifies the statement of Theorem D.12.

Remark D.16. The methods of [17] apply more generally to arbitrary diagrams $X : C \to \mathcal{M}$. The only point of caution is related to making sure that certain categories of diagrams in $\mathcal{M}$ are again simplicial model categories so that the results of this section, most notably Theorem D.9, can be applied. If $\mathcal{M}$ is cofibrantly generated, then diagram categories over $\mathcal{M}$ can be equipped with such model structures, but as long as there is a model structure on the relevant diagram categories, cofibrant generation of the model structure on $\mathcal{M}$ is not needed.

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