On Affine Hilbert Functions of Unions of Layers in Finite Grids

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Abstract

The affine Hilbert function is a classical algebraic object that has been central, among other
tools, to the development of the polynomial method in combinatorics. Owing to its concrete
connections with Gröbner basis theory, as well as its applicability in several areas like computa-
tional complexity, combinatorial geometry, and coding theory, an important line of enquiry is to
understand the affine Hilbert function of structured sets of points in the affine space.

In this work, we determine the affine Hilbert function (over the reals) of arbitrary unions
of layers of points in a uniform grid (a finite grid with the component sets having equispaced
points), where each layer of points is determined by a fixed sum of components for all the points.
This extends a result of Bernasconi and Egidi (Inf. Comput. 1999) from the Boolean cube
setting to the uniform grid setting.

Our proofs follow a similar outline as that of Bernasconi and Egidi. However, there are
two bottlenecks that arise in the uniform grid setting. We resolve these by using (i) a classical
fact that a symmetric Jordan basis exists for the function space on a uniform grid, and (ii) an
extension to multisets of an algebraic interpretation by Friedl and Rónyai (Discrete Math. 2003)
of the notion of order shattering.

The affine Hilbert function is, in fact, a stronger notion than the finite-degree Zariski closure,
which is yet another important tool in the polynomial method toolkit. We conclude by giving an
alternate proof of a combinatorial characterization of a variant of finite-degree Zariski closures,
for unions of layers in uniform grids, obtained in an earlier work of the author (arXiv, 2021).

Notations. \( \mathbb{R} \) denotes the set of all real numbers, \( \mathbb{Z} \) denotes the set of all integers, \( \mathbb{N} \) denotes
the set of all nonnegative integers, and \( \mathbb{Z}^+ \) denotes the set of all positive integers.

1 Introduction

We will work over the field \( \mathbb{R} \). For any two integers \( a \leq b \), by abuse of notation, we will denote
the interval of all integers between \( a \) and \( b \) by \([a, b]\). Further, the integer interval \([1, n]\) will also be
denoted by \([n]\). By a uniform grid, we mean a finite grid of the form
\( [0, k_1 - 1] \times \cdots \times [0, k_n - 1] \),
for some \( k_1, \ldots, k_n \in \mathbb{Z}^+ \). Consider a uniform grid
\( G = [0, k_1 - 1] \times \cdots \times [0, k_n - 1] \). For any \( x = (x_1, \ldots, x_n) \in G \),
define the weight of \( x \) as \( \text{wt}(x) = \sum_{i=1}^n x_i \). We define a subset \( A \subseteq G \) to be
weight-determined if
\[
    x \in A, \ y \in G, \ \text{wt}(y) = \text{wt}(x) \implies y \in A.
\]

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Let $N = \sum_{i=1}^{n} (k_i - 1)$. It follows that there is a one-to-one correspondence between weight-determined sets of $G$ and subsets of $[0, N]$ – a subset $E \subseteq [0, N]$ corresponds to a symmetric subset $\overline{E} \subseteq G$, defined as the set of all elements $x \in G$ satisfying $\text{wt}(x) \in E$. When $E = \{j\}$, a singleton set, we will denote $\overline{E}$ by $\overline{j}$. By a layer, we mean a weight-determined set of the form $\overline{j}$. Thus, an arbitrary weight-determined set is a union of layers. We will freely use the one-to-one correspondence and identify the weight-determined set $E$ with $\overline{E}$ without mention, whenever convenient. This identification will be clear from the context. In addition, for $E \subseteq [0, N]$, we will denote $|E|$ by $[G]_E$. It is then immediate that $[G]_j = [G]_{N-j}$, for all $j \in [0, N]$.

It is a classical result by de Bruijn, Tengbergen, and Kruyswijk \cite{dBvETK51} that $G$ is Sperner, assuming the canonical partial order on $G$: for $a, b \in G$, we have $a \leq b$ if and only if $a_i \leq b_i$ for all $i \in [n]$. Hence, $G$ is unimodal, that is,

$$\left[ \begin{array}{c} G \\ 0 \end{array} \right] \leq \cdots \leq \left[ \begin{array}{c} G \\ \lfloor N/2 \rfloor \end{array} \right] \geq \cdots \geq \left[ \begin{array}{c} G \\ N \end{array} \right].$$

We will stick with the above notations whenever we consider uniform grids. Further, we will assume throughout that $k_i \geq 2$, for all $i \in [n]$.

The affine Hilbert function. For any subset $A \subseteq \mathbb{R}^n$, let $V(A)$ denote the vector space of all functions $A \rightarrow \mathbb{R}$. For $d \in \mathbb{N}$, let $V_d(A)$ denote the subspace of all functions that admit a polynomial representation with degree at most $d$. The affine Hilbert function of $A$ is defined by $H_d(A) = \dim V_d(A)$, $d \in \mathbb{N}$. (See, for instance, Cox, Little, and O’Shea \cite[Chapter 9, Section 3]{CLO15} for an introduction.)

This is a well-studied object in the literature, and the following is a list of some important works.

- Bernasconi and Egidi \cite{BE99} characterized the affine Hilbert functions of symmetric sets of the Boolean cube (sets that are invariant under permutations of coordinates), and used these to study the computational complexity of symmetric Boolean functions.
- Nie and Wang \cite{NW15} used affine Hilbert functions (over fields of prime characteristic) to give bounds on the sizes of finite-degree Zariski closures of subsets of the affine space over a finite field. These bounds yielded solutions to some problems in combinatorial geometry.
- Affine Hilbert functions also implicitly appear in the theory of Reed-Muller codes, and more generally, polar codes. See, for instance, Abbe, Shpilka, and Ye \cite{ASY21} for some recent development.

1.1 Motivation

Consider the Boolean cube $\{0, 1\}^n$. For any $x \in \{0, 1\}^n$, the Hamming weight is defined to be

$$|x| := \text{wt}(x) = \sum_{i \in [n]} x_i.$$  

Let $S_n$ denote the permutation group on $[n]$. We say a subset $A \subseteq \{0, 1\}^n$ is symmetric if

$$(x_1, \ldots, x_n) \in A, \sigma \in S_n \implies (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in A.$$ 

It is easy to check that symmetric sets in $\{0, 1\}^n$ are exactly the weight-determined sets in $\{0, 1\}^n$.  


Remark 1.1. In general, over uniform grids, weight-determined sets and symmetric sets (sets invariant under permutations of coordinates) are different. For instance, take a uniform grid \( G = [0, k_1 - 1] \times \cdots \times [0, k_n - 1] \), where \( n \geq 2 \), \( k_i \geq 3 \) for all \( i \in [n] \), and \( k_i \neq k_j \) for some \( i, j \in [n], i \neq j \). Then trivially, \( G \) is a weight-determined set that is not symmetric. Further, the Boolean cube \( \{0, 1\}^n \subseteq G \) is a symmetric set that is not weight-determined.

The following results serve as primary motivation for our work.

(a) **Affine Hilbert functions of symmetric sets of the Boolean cube over \( \mathbb{R} \)**

Let \( N \in \mathbb{Z}^+ \) and \( d \in [0, N] \). For any \( E \subseteq [0, N] \), consider the enumerations

\[
[0, d] \setminus E = \{t_1 < \cdots < t_1\} \quad \text{and} \quad E \setminus [0, d] = \{w_1 < \cdots < w_r\}.
\]

This enumeration was considered by Bernasconi and Egidi [BE99] in the context of affine Hilbert functions, and so we define the pair of sequences \( \{\{t_1 < \cdots < t_1\}, \{w_1 < \cdots < w_r\}\} \) to be the \((N, d)\)-BE enumeration of \( E \).

Bernasconi and Egidi [BE99] characterized the affine Hilbert functions of all symmetric sets of the Boolean cube.

**Theorem 1.2 ([BE99]).** Consider the Boolean cube \( \{0, 1\}^n \). For any \( d \in [0, n] \) and \( E \subseteq [0, n] \), if \( \{\{t_1 < \cdots < t_1\}, \{w_1 < \cdots < w_r\}\} \) is the \((n, d)\)-BE enumeration of \( E \),

\[
H_d(E) = \sum_{w \in E \cap [0, d]} \binom{n}{w} \sum_{j=1}^{\min(d,r)} \min\left\{ \binom{n}{t_j}, \binom{n}{w_j} \right\}.
\]

Further, they use Theorem 1.2 to generalize a result of Smolensky [Smo93] on non-approximability of some Boolean functions by low-degree polynomials.

(b) **Affine Hilbert functions of special symmetric sets of the Boolean cube in positive characteristic**

Fix a prime \( p \) and let \( q \) be a power of \( p \). For \( a \in [0, n] \) and \( s \in [q - 1] \), define

\[
E_{n,a,s,q} = \{j \in [0, n] : j \equiv t \pmod{q}, \text{ for some } t \in [a, a + s - 1]\}.
\]

Felszeghy, Hegedűs, and Rónyai [FHR09] prove the following.

**Theorem 1.3 ([FHR09]).** Fix the field \( \mathbb{F}_p \), where \( p \) is a prime, and let \( q \) be a power of \( p \). Let \( a \in [0, n] \) and \( s \in [q - 1] \) such that \((n - s - q)/2 < a \leq (n - s + q)/2\), and let \( m = \min\{a + s - 1, n - a\}\).

- If \( d \in [0, m] \), then

\[
H_d(E_{n,a,s,q}) = \sum_{i=0}^{\lfloor d/q \rfloor} \sum_{k=0}^{i} \binom{n}{d - iq - k}.
\]

- If \( d \in [m + 1, n] \), then

\[
H_d(E_{n,a,s,q}) = \sum_{i=-\lfloor r/q \rfloor}^{\lfloor (n+s-m-1)/q \rfloor} \sum_{k=0}^{i} \binom{n}{m + iq - k} - \sum_{i=1}^{\lfloor (n+s-1)/q \rfloor} \sum_{k=0}^{s-1} \binom{n}{d + iq - k}.
\]
Further, they use Theorem 1.3 to obtain an upper bound on the size of set systems satisfying some restricted intersection conditions, thereby generalizing and proving a conjecture by Babai and Frankl (online manuscript, latest version [BF20]).

Motivated by the above results, we concern ourselves with the following question.

**Question 1.4.** Fix a field $\mathbb{F}$. Let $G$ be a uniform grid (if $\mathbb{F}$ has zero characteristic), or the Boolean cube $\{0,1\}^n$ (if $\mathbb{F}$ has positive characteristic). For every $E \subseteq [0,N]$ ($N = n$ in the Boolean cube setting), characterize (combinatorially) the affine Hilbert function $H_d(E)$, $d \in [0,N]$.

### 1.2 Our result

Our main theorem is an extension of Theorem 1.2 to weight-determined sets in uniform grids, over $\mathbb{R}$. Recall that for a uniform grid $G$, and for any $j \in [0,N]$, $[G_j] = |j|$.

**Theorem 1.5** (Affine Hilbert functions of general weight-determined sets).

Let $G$ be a uniform grid. For any $d \in [0,N]$ and $E \subseteq [0,N]$, if $(\{t_{\ell} < \cdots < t_1\}, \{w_1 < \cdots < w_r\})$ is the $(N,d)$-BE enumeration of $E$, then

$$H_d(E) = \sum_{w \in E \cap [0,d]} \left[ \begin{array}{c} G \\ w \end{array} \right] + \sum_{j=1}^{\min\{\ell,r\}} \min \left\{ \left[ \begin{array}{c} G \\ t_j \end{array} \right], \left[ \begin{array}{c} G \\ w_j \end{array} \right] \right\}.$$

Thus, Theorem 1.5 extends Theorem 1.2 to uniform grids.

### 1.3 Uniform grid vis à vis Boolean cube: Bottlenecks in extending Theorem 1.2, and their resolutions

It is interesting to note that the proof of Theorem 1.2 does not readily extend to give Theorem 1.5. More specifically, there are two main arguments which do not extend to the setting of uniform grids. We list these bottlenecks here, and outline how we solve them.

Let us first fix a few more notations. Let $G$ be a uniform grid. Consider the polynomial ring $\mathbb{R}[X]$ over the indeterminates $(X_1, \ldots, X_n)$. For any $\alpha \in \mathbb{N}^n$, we define the monomial $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Now, for any $i \in [n]$ and $a \in \mathbb{N}$, define $Y_i^{(a)} = X_i(X_i - 1) \cdots (X_i - a + 1)$. Further, for any $\alpha \in \mathbb{N}^n$, define $Y^{(\alpha)} = Y_1^{(\alpha_1)} \cdots Y_n^{(\alpha_n)}$. Finally, for any $D, E \subseteq [0,N]$, let $\text{Ev}_{D,E} \in M_{D \times E}(\mathbb{R})$ be the evaluation matrix defined by

$$\text{Ev}_{D,E}(\alpha, \beta) = Y^{(\alpha)}(\beta) = \alpha! \binom{\beta}{\alpha}, \text{ for all } \alpha \in D, \beta \in E.$$  

It follows by definition that $H_d(E) = \text{rank}(\text{Ev}_{[0,d],E})$, for all $E \subseteq [0,N]$ and $d \in [0,N]$.

(a) Consider the statement of Theorem 1.5 for a single layer. In particular, since $G$ is unimodal, it claims the following.

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1Note that we have $\alpha! := \prod_{i \in [n]} \alpha_i!$ and $\binom{\beta}{\alpha} := \prod_{i \in [n]} \binom{\beta_i}{\alpha_i}$, for any $\alpha, \beta \in \mathbb{N}^n$. 

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Claim. For any \(d \in [0, N]\), if \(j \in [d, N - d]\), then \(H_d(j) = \binom{d}{j}\).

In the Boolean cube setting, this claim follows easily by Wilson’s rank formula [Wil90, Theorem 1], in characteristic zero. The proof of Wilson’s rank formula is via induction on the number of indeterminates \(n\). It is quite straightforward to check that such an inductive argument would fail in the case of larger uniform grids. This is the first bottleneck.

We remedy this issue by dropping the inductive argument altogether. Instead, we make use of a classical fact about ranked posets. To get to it, let us consider some definitions. Let (\(P, \leq\)) be a ranked poset with rank function \(\rho\), and let \(h = \max_{a \in P} \rho(a)\). For every \(t \in [0, h]\), let \(P_t = \{a \in P : \rho(a) = t\}\). For any \(A \subseteq P\), let \(V(A)\) denote the vector space of all functions \(A \to \mathbb{F}\). It is then clear that \(V(P) = V(P_0) \oplus \cdots \oplus V(P_h)\). For any \(a \in P\), by abuse of notation, we denote by \(a\), the function on \(P\) that takes the value 1 at \(a\), and the value 0 everywhere else. Define the up operator on \(P\) to be the linear map \(U : V(P) \to V(P)\) given by

\[
U(a) = \sum_{b \in P, a \leq b} b, \quad \text{for all } a \in P.
\]

For the uniform grid \(G\), it is easy to see that the above mentioned claim will follow quickly if we can show that the restriction of the up operator \(U : V(j) \to V(j + 1)\) is injective, for all \(j \in [0, \lfloor N/2 \rfloor - 1]\). Incidentally, this is an immediate corollary of a classical result that a symmetric Jordan basis exists for \(V(G)\).\(^2\) See Canfield [RC80], Proctor, Saks, and Sturtevant [PSS80], and Proctor [Pro82] for proofs of existence, as well as Srinivasan [Sri11] for a constructive proof. (The preliminaries on symmetric Jordan basis are also covered in these references.) The formal details are presented in Section 3.

(b) Consider the following technical lemma about ranks of evaluation matrices. We need this lemma in an inductive argument in the proof of Theorem 1.5, in Section 4, and we will prove this lemma in Section 6.

**Lemma 1.6.** Let \(G\) be a uniform grid. For \(d \in [0, \lfloor N/2 \rfloor]\) and any \(E \subseteq [N - d + 1, N]\),

\[
\text{rank}(\text{Ev}_{d,E}) = \text{rank}(\text{Ev}_{d,\text{min}E}).
\]

In the Boolean cube setting, this lemma again follows by a simple induction on the number of indeterminates \(n\). And yet again, such an induction fails for larger uniform grids. This is the second bottleneck.

The notion of order shattering provides an alternative approach, which remedies this issue. In the Boolean cube setting, Friedl and Rónyai [FR03, Corollary 18] used order shattering and generalized Wilson’s rank formula to determine the rank of \(\text{Ev}_{D,w}\), for any \(D \subseteq [0, \lfloor n/2 \rfloor]\) with \(w \in [\max D, n - \max D]\). This proves the above lemma by a dualization, via the identity: \((\text{Ev}_{D,E})^t = \text{Ev}_{n - E, n - D}\), for all \(D, E \subseteq [0, n]\).\(^3\) Such an identity is clearly not true over larger uniform grids. Nevertheless, we can extend the notion of order shattering and its properties (in an elementary manner) to uniform grids, and give a proof of the lemma, without appealing to any dualization. The formal details are presented in Section 6.

\(^2\)Note that here, we consider \(G\) as a ranked poset with rank function \(wt\).

\(^3\)Here, for any \(F \subseteq [0, n]\), we have \(n - F := \{n - a : a \in F\}\).
1.4 An application: the finite-degree Z*-closure of weight-determined sets

Let \( G \) be a uniform grid. For any \( d \in [0, N] \) and \( A \subseteq G \), the degree-\( d \) Zariski (Z-) closure of \( A \), denoted by \( \text{Z-cl}_{G,d}(A) \), is defined to be the common zero set, in \( G \), of all polynomials with degree at most \( d \), that vanish at each point in \( A \). Formally introduced by Nie and Wang [NW15] (for any set of points in the affine space) with the aim to further the polynomial method in combinatorial geometry, it has appeared implicitly much earlier in the literature (see, for instance, Wei [Wei91], Heijnen and Pellikaan [HP98], Keevash and Sudakov [KS05], and Ben-Eliezer, Hod, and Lovett [BEHL12]).

In an earlier work of the author [Ven21], the finite-degree Z-closures were considered in the context of polynomial covering problems for weight-determined sets in a uniform grid. Since these polynomial covering problems impose vanishing conditions on polynomial functions on weight-determined sets, it is a natural question to determine the finite-degree Z-closures of weight-determined sets. As noted in [Ven21], the finite-degree Z-closure of a weight-determined set need not be weight-determined, and so a variant of the Z-closures is more suitable – the finite-degree Z*-closure – defined exclusively for weight-determined sets.

Let \( G \) be a uniform grid. For any \( d \in [0, N] \) and \( E \subseteq [0, N] \), the degree-\( d \) Z*-closure of \( E \), denoted by \( \text{Z*-cl}_{G,d}(E) \), is defined to be the maximal weight-determined set contained in \( \text{Z-cl}_{G,d}(E) \). The finite-degree Z*-closure is clearly a weight-determined set. So we can use our identification of weight-determined sets with subsets of \([0, N]\), while describing these closures. Further, in the Boolean cube setting, by definition, the finite-degree Z-closures and Z*-closures coincide for symmetric sets.

For any \( d \in [0, N] \), consider the set operator \( L_{N,d} : 2^{[0,N]} \to 2^{[0,N]} \), defined in [Ven21] as follows. For any \( E = \{t_1 < \cdots < t_s\} \subseteq [0, N] \), let

\[
L_{N,d}(E) = \begin{cases} 
E & \text{if } s \leq d \\
[0, t_{s-d}] \cup E \cup [t_{d+1}, N] & \text{if } s \geq d + 1
\end{cases}
\]

For \( k \in \mathbb{Z}^+ \), recursively define \( L_{N,d}^{k+1} = L_{N,d} \circ L_{N,d}^k \). Further, for any \( E \subseteq [0, N] \), define \( \overline{T}_{N,d}(E) = \bigcup_{k \geq 1} L_{N,d}^k(E) \).

The combinatorial characterization of finite-degree Z*-closures in [Ven21] is for a subclass of uniform grids, called strictly unimodal uniform (SU\(^2\)) grids. A uniform grid \( G \) is said to be SU\(^2\) if

\[
\begin{bmatrix} G \\ 0 \\ \vdots \end{bmatrix} < \begin{bmatrix} G \\ \lfloor N/2 \rfloor \\ \vdots \end{bmatrix} = \begin{bmatrix} G \\ \lceil N/2 \rceil \\ \vdots \end{bmatrix} > \begin{bmatrix} G \\ N \end{bmatrix}.
\]

**Theorem 1.7 ([Ven21, Theorem 1.7]).** Let \( G \) be an SU\(^2\) grid. For any \( d \in [0, N] \) and \( E \subseteq [0, N] \),

\[
\text{Z*-cl}_{G,d}(E) = \overline{T}_{N,d}(E).
\]

The following fact connects the notions of finite-degree Z*-closures and affine Hilbert functions, for weight-determined sets. This shows that the affine Hilbert function is a stronger notion. The proof of the fact is obvious from, for instance, Nie and Wang [NW15] and the definitions.

**Fact 1.8.** Let \( G \) be a uniform grid. For any \( d \in [0, N] \) and \( E \subseteq [0, N] \), we have \( H_d(E) = H_d(\text{Z*-cl}_{G,d}(E)) \).

Using Theorem 1.5 and Fact 1.8, we will give an alternate proof of Theorem 1.7.
1.5 Related work

Prior to our work, there have been several attempts to characterize affine Hilbert functions, as well as related notions, for special subclasses of symmetric sets in the Boolean cube, over both fields of positive and zero characteristic. Besides the finite-degree Z-closures, two other important notions concerned with essentially the same circle of ideas are the Gröbner basis, and the set of standard monomials of the vanishing ideal of the set of points. For quick introductions, refer Cox, Little, and O’Shea [CLO15], and Nie and Wang [NW15].

A selection of the prior work on Gröbner bases, standard monomials, affine Hilbert functions, and finite-degree Z-closures for symmetric sets of the Boolean cube, in addition to the ones mentioned so far, are as follows.

- Hegedűs and Rónyai [HR03] characterized the reduced Gröbner basis for a single layer, with respect to all lexicographic orders (over all fields), and further generalized this characterization to linear Sperner families (over characteristic zero) in [HR18].
- Felszeghy, Ráth, and Rónyai [FRR06] studied a lex game to give a combinatorial criterion for a squarefree monomial to be a standard monomial of a symmetric set (over all fields).
- Srinivasan and the author [SV21] obtained a characterization of finite-degree Z-closures of a subclass of symmetric sets, in positive characteristic.

A digression: why consider only uniform grids? A typical grid that is nonuniform would be of the form \( H = \{ t_0^{(1)} < \cdots < t_k^{(1)} \} \times \cdots \times \{ t_0^{(n)} < \cdots < t_k^{(n)} \} \), where the differences \( t_{i+1}^{(j)} - t_i^{(j)} \), \( i \in [0, k_j - 2] \) are not equal, for some \( j \in [n] \). In this way, a natural way to define the weight of an element \( x \in H \) is \( \text{wt}(x) := \sum_{j \in [n]} s_j \), where \( x_j = t_j^{(j)} \), for all \( j \in [n] \).

Consider an example: let \( G = \{0, 1, 2\}^2 \) and \( H = \{0, 1, 3\}^2 \). For both grids, the dimensions are the same, and \( N = 4 \). Consequently, \( \binom{G}{j} = \binom{H}{j} \), for all \( j \in [0, 4] \). However, the grid \( G \) is uniform, but the grid \( H \) is nonuniform. It is easy to see that \( H_d(\{a, 2\}) > H_d(\{2\}) \), for all \( a \in [0, 4], a \neq 2 \) over \( G \), but \( H_d(\{a, 2\}) = H_d(\{2\}) \), for all \( a \in [0, 4], a \neq 2 \) over \( H \). In view of Theorem 1.5, this shows that the affine Hilbert functions over \( H \) do not depend only on the weights of the points in the set, but also on the coordinates of the points, which is undesirable. So the ‘uniform’ condition ensures that the setting is nice enough for the results to be combinatorially neat.

Organization of the paper. In Section 2, we will look at a few preliminaries required for the rest of the discussion. In Section 3, we will determine the affine Hilbert function of a single layer in a uniform grid. This includes the resolution of the first bottleneck, discussed in Subsection 1.3. In Section 4, we will determine the affine Hilbert functions of all weight-determined sets in a uniform grid. Here, we will assume Lemma 1.6, which captures the second bottleneck, discussed in Subsection 1.3. In Section 5, we will discuss the application of our characterization of affine Hilbert functions to determine the finite-degree Z*-closures of weight-determined sets. Finally, in Section 6, we will prove Lemma 1.6, thereby resolving the second bottleneck.

\[ \text{In fact, } H \text{ is a ranked poset with wt as the rank function.} \]
2 Preliminaries

In this short section, let us gather some preliminaries that we require, some of which we have already mentioned in the Introduction (Section 1).

Posets and grids. Let \((P, \leq)\) be a finite poset. For any \(a \in P\), we denote \(\Delta(a) = \{b \in P : b \leq a\}\), and \(\nabla(a) = \{b \in P : b \geq a\}\). A subset \(D \subseteq P\) is a downset if
\[
a \in D, \ b \in P, \ b \leq a \implies b \in D.
\]

We are specifically interested in the uniform grid \(G = [0, k_1 - 1] \times \cdots \times [0, k_n - 1]\), which is a poset with respect to the natural order: for \(a, b \in G\), we have \(a \leq b\) if and only if \(a_i \leq b_i\), for all \(i \in [n]\). Further, let \(K = \max_{i\in[n]} k_i\), and for any \(a \in G\), define \(\text{lex-wt}(a) = \sum_{i\in[n]} a_i K^{n-i}\). The following is an easy observation.

Observation 2.1. The function \(\text{lex-wt}\) defines a total order \(\preceq\) on \(G\), given by \(a \preceq b\) if and only if \(\text{lex-wt}(a) \leq \text{lex-wt}(b)\). In fact, \(\preceq\) is the lexicographic order on \(G\) induced by the order \(1 > \cdots > n\) on the set of coordinates.

Polynomial representations. We denote the polynomial ring over \(\mathbb{R}\) in the indeterminates \(X = (X_1, \ldots, X_n)\) by \(\mathbb{R}[X]\). For any \(P(X) \in \mathbb{R}[X]\) and \(\alpha \in \mathbb{N}^n\), let \(\text{coeff}(\alpha, P)\) denote the coefficient of \(X^\alpha\) in \(P(X)\). Recall that for a uniform grid \(G\), \(V(G)\) denotes the vector space of all functions \(G \to \mathbb{R}\). A fundamental result that we will require is Alon’s Combinatorial Nullstellensatz [Alo99].

Theorem 2.2 ([Alo99]). Let \(G\) be a uniform grid. The set of monomials \(\{X^\alpha : \alpha \in G\}\) is a basis of the vector space of functions \(V(G)\).

An easy corollary that follows by elementary linear algebra is the following. We state it without proof.

Corollary 2.3. Let \(G\) be a uniform grid. Any set of polynomials \(\{f_\alpha(X) : \alpha \in G\}\) satisfying
\[
\text{coeff}(\alpha, f_\alpha) \neq 0, \quad \text{and} \quad f_\alpha(X) \in \text{span}\{X^\beta : \beta \in \Delta(\alpha)\}, \quad \text{for all } \alpha \in G,
\]
is a basis for \(V(G + a)\), for all \(a \in \mathbb{N}^n\).

For any polynomial \(P(X) \in \mathbb{R}[X]\), we denote \(Z(P) = \{a \in G : P(a) = 0\}\). Recall that for any \(i \in [n]\) and \(a \in \mathbb{N}\), define \(Y_i^{(\alpha)} = X_i^a \cdot \cdots \cdot X_i^{a+1}\). Further, for any \(\alpha \in \mathbb{N}^n\), define \(Y^{(\alpha)} = Y_1^{(\alpha_1)} \cdots Y_n^{(\alpha_n)}\). So we have, \(Z(Y^{\alpha}) = G \setminus \nabla(\alpha)\), for all \(\alpha \in G\).

Finite-degree \(Z\)-closures and \(Z^*\)-closures. Let \(G\) be a uniform grid. For any \(d \in [0, N]\) and \(A \subseteq G\), the degree-\(d\) Zariski (\(Z\)-) closure of \(A\) is defined as \(Z_{-}\text{-cl}_{G,d}(A) = Z(f(X) \in \mathbb{R}[X] : f(a) = 0, \text{ for all } a \in A; \deg(f) \leq d}\). Further, for a weight-determined set \(E, E \subseteq [0, N]\), the degree-\(d\) \(Z^*\)-closure of \(E\) is defined to be the maximal weight-determined subset of \(Z_{-}\text{-cl}_{G,d}(E)\). So \(Z_{+}\text{-cl}_{G,d}(E)\) is a weight-determined set. We will, therefore, use our identification of weight-determined sets in \(G\) with subsets of \([0, N]\), and identify \(Z_{-}\text{-cl}_{G,d}(E)\) with \(Z_{+}\text{-cl}_{G,d}(E) \subseteq [0, N]\).

---

\[^5\]Strictly speaking, this is the weak version of the Combinatorial Nullstellensatz [Alo99, Theorem 2]. The strong version [Alo99, Theorem 1] describes a reduced Gröbner basis of the vanishing ideal of \(G\).

\[^6\]We have the obvious definition \(G + a = \{b + a : b \in G\}\).
Gröbner basis theory. For quick introductions to the relevant notions from Gröbner basis theory, refer Cox, Little, and O’Shea [CLO15] – Chapter 2 (for Gröbner basis), Chapter 5 (for standard monomials\(^7\)), and Chapter 9 (for affine Hilbert function) – and Nie and Wang [NW15] (for finite-degree Z-closures).

Let \( G \) be a uniform grid. For any \( S \subseteq G \), let \( \text{SM}_\leq(S) \) denote the set of \textit{standard monomials} of the vanishing ideal of \( S \), with respect to a monomial order \( \leq \). Further, for a polynomial \( P(\mathbb{X}) \in \mathbb{F}[\mathbb{X}] \), let \( \text{LM}_\leq(P) \) denote the \textit{leading monomial} of \( P \) (with respect to \( \leq \)). The following are basic facts about the inter-relationships between Gröbner bases, standard monomials, affine Hilbert functions, and finite-degree Z-closures.

\textbf{Fact 2.4.} (a) For a monomial order \( \leq \) and \( S \subseteq G \), if \( \mathcal{G}_\leq(S) \) is a Gröbner basis of the vanishing ideal of \( S \) with respect to \( \leq \), then

(i) \( \text{SM}_\leq(S) = \{ \mathbb{X}^\alpha : \mathbb{X}^\alpha \text{ does not divide } \text{LM}_\leq(P) \text{, for any } P(\mathbb{X}) \in \mathcal{G}_\leq(S) \} \).

(ii) for any \( x \in G \) and \( d \in [0, N] \),

\[ x \in \text{Z-cl}_{G,d}(S) \iff P(x) = 0 \text{, for all } P(\mathbb{X}) \in \mathcal{G}_\leq(S), \deg(P) \leq d. \]

(b) For any \( S \subseteq G \), \( x \in G \), and \( d \in [0, N] \),

\[ x \in \text{Z-cl}_{G,d}(S) \iff H_d(S \cup \{x\}) = H_d(S). \]

Note that Fact 1.8, in fact, follows from Fact 2.4 (b).

Let \( G \) be a uniform grid. Let \( \preceq \) denote the lexicographic order on \( G \) induced by the order \( 1 > 2 > \cdots > n \) on the coordinates. For any \( A \subseteq G \), let \( \text{SM}_\leq(A) \) denote the set of standard monomials of \( \mathcal{I}(A) \), the vanishing ideal of \( A \). The set \( \text{SM}_\leq(A) \) is a downset of monomials. The following is the well-known Footprint bound.

\textbf{Proposition 2.5 (Footprint bound [CLO15, Chapter 1, Proposition 4]).} For any \( A \subseteq G \), the set of monomials \( \text{SM}_\leq(A) \) is a basis of the vector space of functions \( V(A) \simeq \mathbb{R}[\mathbb{X}] / \mathcal{I}(A) \).

\textbf{Some more definitions.} Let \( G \) be a uniform grid. For any \( D, E \subseteq [0, N] \), let \( \text{Ev}_{D,E} \in M_{D \times E}(\mathbb{R}) \) be the evaluation matrix defined by

\[ \text{Ev}_{D,E}(\alpha, \beta) = \mathbb{Y}^{(\alpha)}(\beta) = \alpha! \binom{\beta}{\alpha}, \text{ for all } \alpha \in D, \beta \in E. \]

Further, for any \( D \subseteq [0, N] \), define

\[ \text{diag}_D = \text{diag}(\alpha! : \alpha \in D) \in M_{D \times D}(\mathbb{R}). \]

We denote the zero vector in \( \mathbb{N}^n \) by \( 0^n \).

\(^7\)The terminology ‘standard monomials’, however, is not used in Cox, Little, and O’Shea [CLO15].

\(^8\)A linear order \( \leq \) on the set of all monomials in \( n \) indeterminates \( X_1, \ldots, X_n \) is a \textit{monomial order} if \( (i) \ 1 \leq u \) for every monomial \( u \), and \( (ii) \) for monomials \( u, v \) with \( u \leq v \), we have \( uw \leq vw \) for every monomial \( w \).

\(^9\)Note that we have \( \alpha! := \prod_{i \in [n]} \alpha_i! \) and \( \binom{\beta}{\alpha} := \prod_{i \in [n]} \binom{\beta_i}{\alpha_i} \), for any \( \alpha, \beta \in \mathbb{N}^n \).
3 Affine Hilbert function of a single layer

Recall that in a uniform grid $G$, a layer is a weight-determined set of the form $j; j \in [0, N]$. In this section, we will determine the affine Hilbert functions of single layers. Our main result is the following.

**Theorem 3.1** (Affine Hilbert function of a single layer). Let $G$ be a uniform grid and $d, w \in [0, N]$. Then $H_d(w) = \min \{ [G]_d, [G]_w \}$.

As mentioned in Subsection 1.3, the arguments in the Boolean cube setting do not extend to the uniform grid setting, and we will appeal to a classical result that a symmetric Jordan basis exists for $V(G)$. Several proofs of existence are known (Canfield [RC80], Proctor, Saks, and Sturtevant [PSS80], Proctor [Pro82]), and a constructive proof was given by Srinivasan [Sri11]. An immediate consequence is the injectivity of the up operator, as mentioned in Subsection 1.3. The following is the formal statement, in the form that we require it.

**Proposition 3.2** ([RC80, PSS80, Pro82, Sri11]). Let $G$ be a uniform grid and $d \in [0, N - 1]$. Consider the matrix $U_{d,d+1} \in M_{d \times d+1}(\mathbb{R})$ defined by

$$
U_{d,d+1}(\alpha, \beta) = \begin{cases} 1, & \alpha \leq \beta \\ 0, & \text{otherwise} \end{cases}
$$

Then $U_{d,d+1}$ has full rank, that is, $\text{rank}(U_{d,d+1}) = \min \{ [G]_d, [G]_{d+1} \}$.

We will also need another lemma. For $i \in [n]$, let $e_i \in \{0, 1\}^n$ be defined by $(e_i)_j = 1$ if and only if $j = i$.

**Lemma 3.3.** Let $G$ be a uniform grid. For any $d, w \in [0, N], d < w$, we have as functions in $V(w)$,

$$
Y(\alpha) = \frac{1}{w - d} \sum_{\beta \geq \alpha, \beta \in [d+1]} Y(\beta), \text{ for all } \alpha \in [d].
$$

**Proof.** Let $\alpha \in [d]$ and $I_\alpha = \{i \in [n] : \beta_i < k_i - 1\}$. We then have, as functions in $V(w)$,

$$
Y(\alpha) \left( \sum_{i \in I_\alpha} X_i - \left( w - \sum_{i \notin I_\alpha} (k_i - 1) \right) \right) = 0.
$$

This gives

$$
\left( w - \sum_{i \notin I_\alpha} (k_i - 1) \right) Y(\alpha) = Y(\alpha) \left( \sum_{i \in I_\alpha} X_i \right)
= \sum_{i \in I_\alpha} (Y(\alpha)(X_i - \alpha_i) + \alpha_i Y(\alpha))
= \sum_{i \in I_\alpha} Y(\alpha+e_i) + Y(\alpha) \left( \sum_{i \in I_\alpha} \alpha_i \right).
$$

Now $w - \sum_{i \notin I_\alpha} (k_i - 1) - \sum_{i \in I_\alpha} \alpha_i = w - \text{wt}(\alpha) = w - d > 0$. So we get

$$
Y(\alpha) = \frac{1}{w - d} \sum_{i \in I_\alpha} Y(\alpha+e_i) = \frac{1}{w - d} \sum_{\beta \geq \alpha, \beta \in [d+1]} Y(\beta).
$$

\[\square\]
Repeatedly applying Lemma 3.3 immediately gives us an important corollary.

**Corollary 3.4.** Let $G$ be a uniform grid. For any $d, w \in [0, N]$, $d < w$,

$$V_d(w) \subseteq \text{span}\{Y^{(\alpha)} : \alpha \in \mathbb{J}\}.$$ 

For every $w \in [0, N]$, we clearly have $Ev_{w,w} = \text{diag}_w$, and so $\text{rank}(Ev_{w,w}) = \left[ \begin{array}{c} G \\ w \end{array} \right]$. Further, Proposition 3.2 and Lemma 3.3 together immediately give us the following.

**Corollary 3.5.** Let $G$ be a uniform grid. For any $d, w \in [0, N]$, $d < w$, we have

$$Ev_{d,w} = \frac{1}{(w-d)!}U_{d,d+1} \cdots U_{w-1,w}\text{diag}_w.$$ 

In particular, if $w \leq \lfloor N/2 \rfloor$, $Ev_{d,w}$ has full rank, that is, $\text{rank}(Ev_{d,w}) = \left[ \begin{array}{c} G \\ d \end{array} \right]$. Also in particular, for $d < d' < w$, we have

$$Ev_{d,w} = \frac{1}{(w-d)!} \cdots \frac{1}{(w-d' + 1)!} Ev_{d',w} = \frac{1}{(d' - d)!} (w-d' - d)^{w-d'} Ev_{d',w}.$$ 

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Since $H_d(w) = \dim V_d(w)$, the evaluation matrix of concern to us is $Ev_{[0,d],w} = 
\begin{bmatrix} Ev_{0,w} \\
Ev_{1,w} \\
\vdots \\
Ev_{d,w} 
\end{bmatrix}$. If $d \geq w$, then we have $Ev_{[0,d],w} = 
\begin{bmatrix} Ev_{0,w} \\
\vdots \\
Ev_{d,w} \\
\vdots \\
Ev_{d,w} 
\end{bmatrix}$, and $Ev_{w,w} = \text{diag}_w$. So $H_d(w) = 
\begin{bmatrix} Ev_{d,w} \\
\vdots \\
Ev_{w,w} \\
\vdots \\
Ev_{0,w} 
\end{bmatrix}$

$\text{rank}(Ev_{[0,d],w}) = \text{rank}(Ev_{w,w}) = \left[ \begin{array}{c} G \\ w \end{array} \right].$

Now suppose $d < w$. By Corollary 3.4, we conclude that $H_d(w) = \text{rank}(Ev_{d,w})$. If $w \leq \lfloor N/2 \rfloor$, then by Corollary 3.5, we have $H_d(w) = \min \{ \left[ \begin{array}{c} G \\ d \end{array} \right], \left[ \begin{array}{c} G \\ w \end{array} \right] \}$. Now suppose $w > \lfloor N/2 \rfloor$. Then $N - w \leq \lfloor N/2 \rfloor$. Note that for any $P(X) \in \mathbb{R}[X]$, we have $P(X_1, \ldots, X_n) \in \text{span}\{Y^{(\alpha)} : \alpha \in \left[ \begin{array}{c} G \\ d \end{array} \right] \}$ if and only if $P(k_1 - 1 - X_1, \ldots, k_n - 1 - X_n) \in \text{span}\{Y^{(\alpha)} : \alpha \in \left[ \begin{array}{c} G \\ d \end{array} \right] \}$. This means $V_d(w)$ and $V_d(N - w)$ are isomorphic vector spaces, and so $H_d(w) = H_d(N - w)$. The proof is then complete by previous arguments, by noting that $\left[ \begin{array}{c} G \\ w \end{array} \right] = \left[ \begin{array}{c} G \\ N-w \end{array} \right].$ 

### 4 Affine Hilbert functions of general weight-determined sets

In this section, we will determine the affine Hilbert functions of general weight-determined sets in a uniform grid. We proceed by first determining the affine Hilbert functions of a subclass of weight-determined sets. Our main theorem will reduce to the case of this subclass.

#### 4.1 A special case

Let begin by describing the subclass of weight-determined sets we are interested in; we call these *interval-compatible* weight-determined sets. Let $I \subseteq [0, N]$ be an interval. We say a set $E = \{w_t : t \in I\} \subseteq [0, N]$ of distinct integers is $I$-compatible if

- $E$ is a uniform grid.
- Every interval-compatible weight-determined set $E$ is $I$-compatible.
Lemma 4.2.

Proof. We first note that it is enough to prove the following claim. We will need to generalize a bit and consider a \( \tilde{\alpha} \)-compatible. Then

\[
Y = \sum_{t \in I} \min \left\{ \left[ \frac{G}{t} \right], \left[ \frac{G}{w_t} \right] \right\}.
\]

Theorem 4.1 (Affine Hilbert functions of interval-compatible weight-determined sets).

Let \( G \) be a uniform grid. Let \( I = [c, d] \subseteq [0, N] \) be an interval and \( E = \{ w_t : t \in I \} \subseteq [0, N] \) be \( I \)-compatible. Then

\[
\text{We need two lemmas to proceed. The first lemma, stated below, generalizes Corollary 3.4. We will need to generalize a bit and consider a shifted analogue of the polynomial \( Y^{(a)} \). For } i \in [n] \text{ and any } a, \beta \in \mathbb{N}, \text{ define } Y_t^{(a, \beta)} = \prod_{i \in [0,b-1]} (X_i - (a + t)). \text{ Further, for any } \alpha, \beta \in \mathbb{N}^n, \text{ define } Y^{(a, \alpha, \beta)} = \prod_{i \in [n]} Y_i^{(\alpha_i, \beta_i)}; \text{ therefore, we have } Y^{(a, \alpha, \beta)} = Y^{(a)} Y^{(\alpha, \beta)}.
\]

Lemma 4.2. Let \( G \) be a uniform grid. If \( d \in [0, N] \) and \( E \subseteq [0, N] \) such that \( d < \min E \), then

\[
V_d(E) \subseteq \text{span} \{ Y^{(\beta)} : \beta \in [d + 1, d + |E|] \}.
\]

Proof. We first note that it is enough to prove the following claim.

Claim. \( Y^{(a)} \in \text{span} \{ Y^{(\beta)} : \beta \in [d + 1, d + |E|] \} \), for every \( a \in \mathbb{N} \).

Indeed if this claim is true, then for any \( a \in [0, d - 1] \), we will inductively get

\[
Y^{(a)} \in \text{span} \{ Y^{(\beta)} : \beta \in [\text{wt}(a) + 1, \text{wt}(a) + |E|] \}
\]

\[
\subseteq \text{span} \{ Y^{(\beta)} : \beta \in [\text{wt}(a) + 2, \text{wt}(a) + 1 + |E|] \} \quad \text{applying the claim to all } Y^{(\beta)} : \beta \in \text{[wt}(a) + 1]
\]

\[
\subseteq \cdots
\]

\[
\subseteq \text{span} \{ Y^{(\beta)} : \beta \in [d + 1, d + |E|] \}.
\]

Now fix a \( a \in \mathbb{N} \). Define \( G_a = [a_1, k_1 - 1] \times \cdots \times [a_n, k_n - 1] \). Consider the polynomial \( P(\bar{X}) = \prod_{w \in E} \left( \sum_{i \in [n]} X_i - w \right) \). Clearly \( P \neq 0 \) as a function in \( V(G_a) \). By Corollary 2.3, in \( V(G_a) \), \( P \) admits a unique representation \( \tilde{P}(\bar{X}) \in \text{span} \{ Y^{(a, \beta)} : a + \beta \in G_a \} \). Since \( \deg(P) \leq |E| \), in fact, we get \( \tilde{P}(\bar{X}) \in \text{span} \{ Y^{(a, \alpha, \beta)} : a + \beta \in G_a, \text{wt}(\beta) \leq |E| \} \). Let \( \tilde{P}(\bar{X}) = \sum_{a + \beta \in G_a, \text{wt}(\beta) \leq |E|} c_{\beta} Y^{(a, \beta)} \). So the constant term in this representation is \( c_0 = \tilde{P}(\alpha) = P(\alpha) = \prod_{w \in E} (d - w) \neq 0 \), since \( d < \min E \).

Now note that in \( G \), \( Z(\bar{Y}^{(a)}) = G \setminus G_a \) and so as functions in \( V(E) \), we have \( \bar{Y}^{(a)} = 0 \), which gives

\[
Y^{(a)} \left( \sum_{\alpha + \beta \in G_a, \text{wt}(\beta) \leq |E|} c_{\beta} Y^{(a, \beta)} \right) = 0, \quad \text{that is, } \bar{Y}^{(a)} \left( \prod_{w \in E} (d - w) + \sum_{\alpha + \beta \in G_a, \beta \neq 0^a, \text{wt}(\beta) \leq |E|} c_{\beta} Y^{(a, \beta)} \right) = 0.
\]

12
This implies
\[
\left( \prod_{w \in E} (d - w) \right)^{Y(\alpha)} = - \sum_{\alpha + \beta \in G_{\alpha}} c_{\beta} Y(\alpha) Y(\alpha) Y(\alpha) \\
\Rightarrow Y(\alpha) = \left( \prod_{w \in E} (d - w) \right) \sum_{\alpha + \beta \in G_{\alpha}} c_{\beta} Y(\alpha + \beta).
\]

Thus \( Y(\alpha) \in \text{span}\{ Y(\gamma) : \gamma \in [d+1, d+|E|] \} \), since \( \text{wt}(\alpha + \beta) \in [d+1, d+|E|] \) for \( \beta \neq 0^n \). 

The second lemma is Lemma 1.6, which remedies the second bottleneck mentioned in Subsection 1.3. As mentioned there, this lemma will be proved using the notion of order shattering, extended to multisets. We defer the discussion on order shattering, and the proof of the lemma to Section 6. Let us recall the statement of the lemma.

**Lemma 1.6.** Let \( G \) be a uniform grid. For \( d \in [0, \lfloor N/2 \rfloor] \) and any \( E \subseteq [N - d + 1, N] \),
\[
\text{rank}(Ev_{d,E}) = \text{rank}(Ev_{d,min,E}).
\]

The proof of Theorem 4.1 is captured by the following proposition.

**Proposition 4.4.** Let \( G \) be a uniform grid. Let \( I := [c, d] \subseteq [0, N] \) be an interval and \( E := \{ w_t : t \in I \} \subseteq [0, N] \) be \( I \)-compatible. Then
\[
\text{rank}(Ev_{I,E}) = \sum_{t \in I} \min \left\{ \left[ \frac{G}{t} \right], \left[ \frac{G}{w_t} \right] \right\}.
\]

**Proof.** Note that for every \( t \in I \), by Theorem 3.1, we have \( \text{rank}(Ev_{t,w_t}) = \min \{ \left[ \frac{G}{t} \right], \left[ \frac{G}{w_t} \right] \} \). So we need to prove that \( \text{rank}(Ev_{I,E}) = \sum_{t \in I} \text{rank}(Ev_{t,w_t}) \). We prove this by induction on \( |I| \). If \( |I| = 1 \), then the claim is trivial.

Now suppose \( I = [c, d] \subseteq [0, N] \), \( c < d \) and assume the claim is true for any interval \( J \subseteq [0, N], |J| < |I| \). Consider
\[
 Ev_{[c,d],E} = \begin{bmatrix}
 Ev_{d,w_d} & Ev_{d,E \setminus \{w_d\}} \\
 Ev_{d-1,w_d} & \vdots \\
 Ev_{c+1,w_d} & Ev_{[c,d-1],E \setminus \{w_d\}} \\
 Ev_{c,w_d} & 
\end{bmatrix}.
\]

Note that by definition of \( E \), we have \( t < w_d \), for all \( t \in [c, d - 1] \). So by Corollary 3.5, we have
\[
 Ev_{t,w_d} = \frac{1}{w_d - t} U_{t,t+1} Ev_{t+1,w_d}, \quad \text{for all } t \in [c, d - 1].
\]
Then we get

\[
\begin{bmatrix}
I_{d \times d} & O & \cdots & O & O \\
\left(\frac{-1}{w_d - (d-1)}\right)U_{d-1,d} & I_{d-1 \times d-1} & \cdots & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & \cdots & I_{c+1 \times c+1} & O \\
O & O & \cdots & \left(\frac{-1}{w_d - c}\right)U_{c,c+1} & I_{c,c} \\
\end{bmatrix}
\begin{bmatrix}
\operatorname{Ev}_{d,w_d} \\
\operatorname{Ev}_{d,E \{t_d\}} \\
\operatorname{Ev}_{d-1,w_d} \\
\vdots \\
\operatorname{Ev}_{c+1,w_d} \\
\operatorname{Ev}_{c,w_d} \\
\end{bmatrix}
= 
\begin{bmatrix}
\operatorname{Ev}_{d,w_d} \\
\operatorname{Ev}_{d,E \{w_d\}} \\
O \\
\vdots \\
O \\
O \\
\end{bmatrix}
\begin{bmatrix}
F \\
F_{d-1,w_d} \\
\vdots \\
F_{c,w_d} \\
\end{bmatrix}
\]

Since we have premultiplied \( \operatorname{Ev}_{[c,d],E} \) with an invertible matrix in the above step, using Theorem 3.1 and Lemma 1.6, we get

\[
\operatorname{rank}(\operatorname{Ev}_{[c,d],E}) = \operatorname{rank}(\operatorname{Ev}_{d,w_d}) + \operatorname{rank}(F) = \min \left\{ \left[ \begin{array}{c} G \\ d \end{array} \right] + \left[ \begin{array}{c} G \\ w_d \end{array} \right], \left[ \begin{array}{c} G \\ c \end{array} \right] + \left[ \begin{array}{c} G \\ c \end{array} \right] \right\} + \operatorname{rank}(F).
\]

Let us now determine \( F \). Considering \( F \) as a block matrix with the obvious block division

\[
F = \begin{bmatrix}
F_{d-1,w_d} & \cdots & F_{d-1,w_c} \\
\vdots & \ddots & \vdots \\
F_{c,w_d} & \cdots & F_{c,w_c} \\
\end{bmatrix},
\]

we observe that for all \( s, t \in [c, d - 1] \),

\[
F_{s,w_t} = \left(\frac{-1}{w_d - s}\right)U_{s,s+1}\operatorname{Ev}_{s+1,w_t} + \operatorname{Ev}_{s,w_t}
\]

\[
= \begin{cases} 
\operatorname{Ev}_{s,w_t}, & w_t \leq s, \\
\left(\frac{w_t - s}{w_d - s} + 1\right)\operatorname{Ev}_{s,w_t}, & w_t > s 
\end{cases}
\]

\[
= \begin{cases} 
O, & w_t < s \\
I_{w_t \times w_t}, & w_t = s \\
\left(\frac{w_d - w_t}{w_d - s}\right)\operatorname{Ev}_{s,w_t}, & w_t > s 
\end{cases}
\]

\[
= \left(\frac{w_d - w_t}{w_d - s}\right)\operatorname{Ev}_{s,w_t}.
\]

Thus we have

\[
F = \text{diag}\left(\left(\frac{1}{w_d - s}\right)I_{s+1 \times s}: s \in [c, d - 1]\right) \cdot \operatorname{Ev}_{[c,d-1], E \{w_d\}} \cdot \text{diag}(w_d - w_t)I_{w_t \times w_t} : w_t \in E \setminus \{w_d\}).
\]
By the induction hypothesis, this implies

\[
\text{rank}(F) = \text{rank}(\text{Ev}_{[c,d-1],E \setminus \{w_d\}}) = \sum_{t \in [c,d-1]} \min \left\{ \left[ G_t \right], \left[ G_{w_t} \right] \right\}.
\]

This completes the proof. \(\square\)

Theorem 4.1 is now immediate.

**Proof of Theorem 4.1.** Since \(I = [c, d] = [(c-1)+1, (c-1)+|E|]\), by Lemma 4.2, we have \(V_{c-1}(E) \subseteq \text{span}\{Y^\alpha : \alpha \in \mathcal{I}\}\). Thus \(V_d(E) \subseteq \text{span}\{Y^{(w)} : w \in \mathcal{W}\}\). Therefore, by Proposition 4.4,

\[
H_d(E) = \text{rank}(\text{Ev}_{I,E}) = \sum_{t \in I} \min \left\{ \left[ G_t \right], \left[ G_{w_t} \right] \right\}.
\]

### 4.2 The general case

We will now determine the affine Hilbert functions of all weight-determined sets. For convenience, let us restate our main theorem. Recall that for \(d \in [0, N]\) and \(E \subseteq [0, N]\), the \((N, d)\)-BE enumeration of \(E\) is the pair of sequences \((\{t_\ell < \cdots < t_1\}, \{w_1 < \cdots < w_r\})\) such that

\[
[0, d] \setminus E = \{t_\ell < \cdots < t_1\} \quad \text{and} \quad E \setminus [0, d] = \{w_1 < \cdots < w_r\}.
\]

**Theorem 1.5 (Affine Hilbert functions of general weight-determined sets).**

Let \(G\) be a uniform grid. For any \(d \in [0, N]\) and \(E \subseteq [0, N]\), if \((\{t_\ell < \cdots < t_1\}, \{w_1 < \cdots < w_r\})\) is the \((N, d)\)-BE enumeration of \(E\), then

\[
H_d(E) = \sum_{w \in E \cap [0, d]} \left[ G_w \right] + \sum_{j=1}^{\min\{\ell, r\}} \min \left\{ \left[ G_{t_j} \right], \left[ G_{w_j} \right] \right\}.
\]

**Proof.** Let \(m = \min\{\ell, r\}\). Then we can write

\[
(E \cap [0, d]) \cup \{t_m < \cdots < t_1\} = E' \cup I,
\]

where \(I = [c, d]\) is an interval for some \(c \in [0, d+1]\), \((I = \emptyset\) when \(c = d+1\).\) and max \(E'+1 < \min I.\)

For each \(t \in I\),

- if \(t \in E \cap [0, d]\), then set \(v_t = t\).
- if \(t = t_j\), for some \(j \in [m]\), then set \(v_t = w_{t_j}\).

Let \(E'' = \{v_t : t \in I\}\). Then clearly \(E = E' \cup E''\) and further, \(E''\) is \(I\)-compatible. So we need to show that

\[
H_d(E) = \sum_{w \in E'} \left[ G_w \right] + \sum_{t \in I} \min \left\{ \left[ G_v \right], \left[ G_t \right] \right\}.
\]

Let us first prove the following claim.

**Claim.** \(V_d(E) \subseteq \text{span}\{Y^{(w)} : w \in E' \cup I\}\).

\(^{10}\)We take \(\max(\emptyset) = -\infty\) and \(\min(\emptyset) = \infty\), as convention.
Proof of Claim. Since $I = [c, d]$, it is enough to show that $V_{c-1}(E) \subseteq \{\mathbb{Y}(a) : a \in E' \cup I\}$. Let us prove this by induction on $|E'|$. If $|E'| = 0$, then $E = E''$, which is $I$-compatible, and so the claim is true by Lemma 4.2, since $I = [c, d] = [(c - 1) + 1, (c - 1) + |E|]$. Now suppose $|E'| \geq 1$ and let $a = \min E'$. Let $f \in V_{c-1}(E)$. Since $\max E' + 1 < \min I$, we have $a < c - 1$. Then by Corollary 3.4, we have $V_{c-1}(a) = V_a(a) = V(a)$. Let $P(\mathbb{X}) \in \text{span}\{\mathbb{Y}(a) : a \in a\}$ be a polynomial representation of $f|\alpha \in V_{c-1}(a) = V(a)$. Since $f \in V_{c-1}(E)$, by induction hypothesis, let $Q(\mathbb{X}) \in \text{span}\{\mathbb{Y}(a) : a \in (E' \setminus \{a\}) \cup I\}$ be a polynomial representation of $f - P \in V_{c-1}(E \setminus \{a\})$. Note that $Q|\alpha = 0$. So $P(\mathbb{X}) + Q(\mathbb{X}) \in \text{span}\{\mathbb{Y}(a) : a \in E' \cup I\}$ is a polynomial representation of $f \in V_{c-1}(E)$. This completes the proof of the claim.

By the above claim, we conclude that $H_d(E) = \text{rank}(Ev_{E' \cup I, E}) = \text{rank}(Ev_{E' \cup I, E \cup E''})$. Now note that for any $j \in E'$, as a function in $V(j)$, we have $\mathbb{Y}(\gamma)(\beta) = \gamma! \delta_{\beta, \gamma}$, for all $\beta, \gamma \leq j$. So we get $Ev_{E', E'} = I_{E'} + N$, for some upper triangular matrix $N \in M_{E' \times E'}(\mathbb{R})$ with 0-s on the diagonal. Also clearly, $Ev_{I, E'} = O$ since $\max E' + 1 < \min I$. Thus

$$Ev_{E' \cup I, E' \cup E''} = \begin{bmatrix} Ev_{I, E'} & Ev_{I, E'} \\ Ev_{E', E''} & Ev_{E', E'} \end{bmatrix} = \begin{bmatrix} Ev_{I, E'} & O \\ Ev_{E', E''} & I_{E'} + N \end{bmatrix}.$$ 

This gives

$$H_d(E) = \text{rank}(I_{E'} + N) + \text{rank}(Ev_{I, E''}) = \sum_{w \in E'} G_{w} + \text{rank}(Ev_{I, E''}).$$

Since $E''$ is $I$-compatible, the proof is complete by Proposition 4.4.

5 Application: Determining finite-degree $Z^*$-closures using affine Hilbert functions

In this section, we will use Theorem 1.5 to give an alternative proof of the combinatorial characterization of finite-degree $Z^*$-closures of weight-determined sets in a uniform grid (Theorem 1.7), as given in an earlier work of the author [Ven21]. This characterization was further used to obtain bounds on some versions of polynomial covering problems for weight-determined sets.

The following are some important properties of finite-degree $Z^*$-closures of weight-determined sets.

**Proposition 5.1 ([Ven21, Proposition 2.2]).** Let $G$ be a uniform grid and $d \in [0, N]$.

(a) $Z^*\text{-cl}_{G,d}(E)$ is weight-determined, for all $E \subseteq [0, N]$.

(b) $Z^*\text{-cl}_{G,d}$ is a closure operator.

(c) $Z^*\text{-cl}_{G,d+1}(E) \subseteq Z^*\text{-cl}_{G,d}(E)$, for all $E \subseteq [0, N]$.

We note that in [Ven21], Theorem 1.7 follows from two lemmas. We will simply state these lemmas and give alternative proofs for these, using Theorem 1.5. Note that we will apply Theorem 1.5, in this setting, via Fact 1.8.
Let us make a definition for convenience, which would help us keep track of the summands in the expression of the affine Hilbert function as given in Theorem 1.5. For any $d, \in [0, N], E \subseteq [0, N], |E| \geq d + 1$, define a sequence
\[
H_d[E] = ((u_1, v_1), \ldots, (u_{d+1}, v_{d+1})),
\]
where

- $E = \{u_1 < \cdots < u_{d+1} < \cdots < u_{|E|}\}$ and \{\{v_1, \ldots, v_{d+1}\} = [0, d]$.  
- for any $j \in [d + 1]$, if $u_j \leq d$ then $v_j = u_j$.
- if $E \setminus [0, d] = \{u_s < \cdots < u_{d+1} < \cdots < u_{|E|}\}$, then $v_{d+1} < \cdots < v_s$.

A sequence satisfying the above properties is clearly unique, and so $H_d[E]$ is well-defined. The following important observation is immediate from Theorem 1.5, and a glance at the definition of BE enumeration.

**Observation 5.2.** For any $d, \in [0, N]$ and $E \subseteq [0, N], |E| \geq d + 1$,
\[
\text{if } H_d[E] = ((u_1, v_1), \ldots, (u_{d+1}, v_{d+1})),
\]
then
\[
H_d(E) = \sum_{j \in [d+1]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\}.
\]

The first lemma holds over any uniform grid, and identifies a collection of layers which are certain to lie in the finite-degree $Z^*$-closures of weight-determined sets.

**Lemma 5.3 (Closure Builder Lemma [Ven21, Lemma 3.3])** Let $G$ be a uniform grid. If $d, \in [0, N]$ and $E \subseteq [0, N]$ such that $|E| \geq d + 1$, then
\[
[0, \min E] \cup [\max E, N] \subseteq Z^*\text{-cl}_{G, d}(E).
\]

**Proof using Theorem 1.5.** It is enough to show that $[0, \min E] \subseteq Z^*\text{-cl}_{G, d}(E)$; the other containment can be argued similarly. Let $a, \in [0, N], a < \min E$. Suppose $a \leq d$. Then we have
\[
H_d[E] = ((u_1, v_1), (u_2, v_2), \ldots, (u_{d+1-(a+1)}), (u_{d+1-(a+1)}, a), \ldots, (u_{d+1}, 0)),
\]
and
\[
H_d[E \cup \{a\}] = ((a, a), (u_1, v_1), \ldots, (u_{d+1-(a+1)}), (u_{d+1-(a+1)}, a), \ldots, (u_{d+1}, 0)).
\]
Further we have $j - 1 \leq u_j \leq N - (d + 1) - j$, for every $j \in [d + 1]$. So by Observation 5.2, we get
\[
H_d(E) = \sum_{j \in [d+1-(a+1)]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\} + \sum_{j \in [0, a]} \min \left\{ \begin{bmatrix} G \\ u_{d+1-j} \end{bmatrix}, \begin{bmatrix} G \\ j \end{bmatrix} \right\}
\]
\[
= \sum_{j \in [d+1-(a+1)]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\} + \sum_{j \in [0, a]} \begin{bmatrix} G \\ j \end{bmatrix},
\]
and
\[
H_d(E \cup \{a\}) = \begin{bmatrix} G \\ a \end{bmatrix} + \sum_{j \in [d+1-(a+1)]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\} + \sum_{j \in [a]} \min \left\{ \begin{bmatrix} G \\ u_{d+1-j} \end{bmatrix}, \begin{bmatrix} G \\ j - 1 \end{bmatrix} \right\}
\]
\[
= \frac{G}{a} + \sum_{j \in [d+1-(a+1)]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\} + \sum_{j \in [a]} \begin{bmatrix} G \\ j - 1 \end{bmatrix}
\]
\[
= \sum_{j \in [d+1-(a+1)]} \min \left\{ \begin{bmatrix} G \\ u_j \end{bmatrix}, \begin{bmatrix} G \\ v_j \end{bmatrix} \right\} + \sum_{j \in [0, a]} \begin{bmatrix} G \\ j \end{bmatrix}.
\]
So by Fact 1.8, we get \( a \in Z^*-\text{cl}_{G,d}(E) \).

Now suppose \( a > d \). This implies \( \min E > d \). Then we have

\[
H_d(E) = ((u_1, d), (u_2, d - 1), \ldots, (u_d, 0)),
\]
and \( H_d(E \cup \{a\}) = ((a, d), (u_1, d - 1), \ldots, (u_d, 0)) \).

So by Observation 5.2, we get

\[
H_d(E) = \sum_{j \in [0,d]} \binom{G}{j} = H_d(E \cup \{a\}).
\]

So by Fact 1.8, we get \( a \in Z^*-\text{cl}_{G,d}(E) \). This completes the proof. ☐

The second lemma characterizes the finite-degree \( Z^* \)-closure of \( T_{N,i} \), \( i \in [0,N] \) in an \( SU^2 \) grid.

**Lemma 5.4 ([Ven21, Lemma 3.5]).** Let \( G \) be an \( SU^2 \) grid. For every \( i \in [0,N] \),

\[
Z^*-\text{cl}_{G,d}(T_{N,i}) = \begin{cases} T_{N,i}, & i \leq d \\ [0,N], & i > d \end{cases}
\]

**Proof using Theorem 1.5.** Note that the result is trivial if \( i > \lfloor N/2 \rfloor \). So assume \( i \leq \lfloor N/2 \rfloor \). If \( d < i \), then \([0,d] \subseteq T_{N,i}\) and we have

\[
H_d[T_{N,i}] = ((0,0), \ldots, (d,d)) = H_d[T_{N,i} \cup \{a\}], \quad \text{for all } a \in [i, N - i].
\]

So by Fact 1.8 and Observation 5.2, \([i, N - i] \subseteq Z^*-\text{cl}_{G,d}(T_{N,i})\), that is, \( Z^*-\text{cl}_{G,d}(T_{N,i}) = [0,N] \).

Now suppose \( i \leq d \). Then by Proposition 5.1 (c), we have \( T_{N,i} \subseteq Z^*-\text{cl}_{G,d}(T_{N,i}) \subseteq Z^*-\text{cl}_{G,i}(T_{N,i}) \).

So it is enough to prove \( Z^*-\text{cl}_{G,i}(T_{N,i}) = T_{N,i} \). For any \( a \in [i, N - i] \), we have

\[
H_i[T_{N,i}] = ((0,0), \ldots, (i - 1, i - 1), (N - i + 1, i)),
\]
and \( H_i[T_{N,i} \cup \{a\}] = ((0,0), \ldots, (i - 1, i - 1), (a, i)) \).

Therefore, by Observation 5.2,

\[
H_i(T_{N,i}) = \sum_{j \in [0,i-2]} \binom{G}{j} + 2 \binom{G}{i - 1},
\]
and \( H_i(T_{N,i} \cup \{a\}) = \sum_{j \in [0,i-1]} \binom{G}{j} = H_i(T_{N,i}) + \left( \binom{G}{i} - \binom{G}{i - 1} \right) \)

since \( G \) is \( SU^2 \). So by Fact 1.8, we get \( Z^*-\text{cl}_{G,i}(T_{N,i}) = T_{N,i} \). ☐
6 Order shattering in uniform grids, an algebraic lemma, and the proof of Lemma 1.6

In this section, we consider the notion of order shattering. Introduced by Aldred and Anstee [AA95], the algebraic interpretation of order shattering, in the setting of the Boolean cube, was given by Anstee, Rónyai, and Sali [ARS02], and Freidl and Rónyai [FR03]. This extends to uniform grids, and can be proven using the ideas in Bollobas, Leader, and Rónyai [BLR89], as well as in [AA95, ARS02, FR03].

Let us first look at the main result required to prove Lemma 1.6, and then move on to consider the theory of order shattering to the extent necessary.

6.1 An algebraic lemma, and the proof of Lemma 1.6

For convenience, let us restate Lemma 1.6.

\textbf{Lemma 1.6.} Let $G$ be a uniform grid. For $d \in [0, \lfloor N/2 \rfloor]$ and any $E \subseteq [N - d + 1, N]$, 
\[ \text{rank}(E_{v^d, E}) = \text{rank}(E_{v^d, \min E}). \]

We require the following algebraic lemma.

\textbf{Lemma 6.2.} Let $G$ be a uniform grid, and let $\preceq$ be the lexicographic order on $G$ induced by the order $1 > \cdots > n$ on the coordinates. For any $i, j \in [0, \lfloor N/2 \rfloor]$, if $i \leq j$, then $SM_{\preceq}(i) \subseteq SM_{\preceq}(j)$.

We will prove Lemma 6.2 in Subsection 6.2; for now, let us assume Lemma 6.2, and prove Lemma 1.6.

\textbf{Proof of Lemma 1.6.} Note that for any $E \subseteq F \subseteq [N - d + 1, N]$ with $\min E = \min F$, it is clear that 
\[ \text{rank}(E_{v^d, \min E}) \leq \text{rank}(E_{v^d, E}) \leq \text{rank}(E_{v^d, F}). \]

So it is enough to prove $\text{rank}(E_{v^d, \{w,N\}}) = \text{rank}(E_{v^d, w})$, for all $w \in [N - d + 1, N]$. We will prove this by induction on $w \in [N - d + 1, N]$. The assertion is trivial if $w = N$. So now consider any $w \in [N - d + 1, N]$, $w < N$. We have
\[ E_{d,[w,N]} = \begin{bmatrix} E_{v^d,w} & E_{d,w+1} & E_{v^d,[w+2,N]} \end{bmatrix}. \]

By induction hypothesis, we have $\text{rank}(E_{d,[w+1,N]}) = \text{rank}(E_{d,w+1})$, and so there exists an invertible matrix $C \in M_{[w+1,N] \times [w+1,N]}(\mathbb{R})$ such that $\text{rank}(E') = \text{rank}(E_{[w,N]})$, where
\[ E' := E_{d,[w,N]} \cdot \begin{bmatrix} I_{w \times w} & O \\ O & C \end{bmatrix} = \begin{bmatrix} E_{v^d,w} & E_{d,w+1} & O \end{bmatrix}. \]

Therefore, by Theorem 3.1, it is enough to show that $\text{rank} \left( \begin{bmatrix} E_{v^d,w} & E_{d,w+1} \end{bmatrix} \right) = \left[ \begin{smallmatrix} C_{w} \\ w \end{smallmatrix} \right]$. Without loss of generality, let $B \subseteq d$ such that the rows of $E_{B,w+1}$ form a basis of the rowspace of $E_{d,w+1}$. So there exists an invertible matrix $R \in M_{[w \times w]}$ such that $\text{rank}(E'') = \text{rank} \left( \begin{bmatrix} E_{v^d,w} & E_{d,w+1} \end{bmatrix} \right)$, where
\[ E'' := R \cdot \begin{bmatrix} E_{v^d,w} & E_{d,w+1} \end{bmatrix} = \begin{bmatrix} E_{v^B,w} & E_{v^B,w+1} \\ F & O \end{bmatrix}. \]
Further, by Theorem 3.1, we get \( |B| = \text{rank}(\text{Ev}_{B,w+1}) = \begin{bmatrix} G \end{bmatrix}_{w+1} \), and
\[
\text{rank}(E'') = \text{rank} \left( \begin{bmatrix} \text{Ev}_{B,w} & \text{Ev}_{B,w+1} \end{bmatrix} \right) + \text{rank}(F) = \begin{bmatrix} G \end{bmatrix}_{w+1} + \text{rank}(F).
\]

Now once again, we note that for any \( P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}] \), we have \( P(X_1, \ldots, X_n) \in \text{span}\{\mathbb{Y}^{(\alpha)} : \text{wt}(\alpha) \leq d\} \) if and only if \( P(k_1 - 1 - X_1, \ldots, k_n - 1 - X_n) \in \text{span}\{\mathbb{Y}^{(\alpha)} : \text{wt}(\alpha) \leq d\} \). This means for any \( i \in [0, \lceil N/2 \rceil] \), the vector spaces \( V(i) \) and \( V(N-i) \) are isomorphic, and further, \( \text{SM}_{\leq}(i) = \text{SM}_{\leq}(N-i) \).

So by Lemma 6.2, we have \( \text{SM}_{\leq}(w+1) \subseteq \text{SM}_{\leq}(w) \). This gives
\[
\{\mathbb{Y}^{(\alpha)} : \alpha \in B\} \subseteq \text{span}(\text{SM}_{\leq}(w+1)) \subseteq \text{span}(\text{SM}_{\leq}(w)),
\]
and so \( \{\mathbb{Y}^{(\alpha)} : \alpha \in B\} \) is linearly independent in \( V(w) \), that is, \( \text{rank}(\text{Ev}_{B,w}) = \begin{bmatrix} G \end{bmatrix}_{w+1} \).

Finally, we have \( \begin{bmatrix} \text{Ev}_{B,w} \\ F \end{bmatrix} = R \cdot \text{Ev}_{d,w} \), and so, by Theorem 3.1,
\[
\text{rank}(F) = \text{rank}(\text{Ev}_{d,w}) - \text{rank}(\text{Ev}_{B,w}) = \begin{bmatrix} G \\ w \end{bmatrix} - \begin{bmatrix} G \\ w+1 \end{bmatrix}.
\]
This implies \( \text{rank}(E'') = \begin{bmatrix} G \\ w \end{bmatrix} \), which completes the proof. \( \square \)

### 6.2 Order shattering and the proof of Lemma 6.2

In this subsection, we will move on to mention two results pertaining to order shattering, in the setting of uniform grids. These results are known in the Boolean cube setting, and the proofs of the extensions to uniform grids is straightforward; we will, therefore, omit these proofs. Lemma 6.2 will then be obvious from these two results.

Let us also consider slightly different notations, for convenience. We consider the uniform grid \( G \) as a collection of multisets given by
\[
G = \{a : [n] \to \mathbb{N} : a(i) \in [0, k_i - 1], \text{ for all } i \in [n]\}.
\]
Note that for any \( a \in G \), we have \( \text{wt}(a) = \sum_{i \in [n]} a(i) \). We will take advantage of the following abuse of notations.

- For any \( i \in [n] \) and \( a \in G \), we denote ‘\( i \in a \)’ if \( a(i) \geq 1 \).
- For any \( a, b \in G \), we denote ‘\( a \leq b \)’ if \( a(i) \leq b(i) \), for all \( i \in [n] \).
- For any \( a, b \in G \), we define \( a \cap b \in G \) by \( (a \cap b)(i) = \min\{a(i), b(i)\} \), for all \( i \in [n] \).
- We denote the zero function on \([n]\) (that is, the empty multiset) by \( \emptyset \).

Let \( \preceq \) be the lexicographic order on \( G \) induced by the order 1 > · · · > n on the coordinates. For any \( b \in G \), let \( \tau(b) = \max\{i \in [n] : b(i) \geq 1\} \), and define \( b^* \in G \) by
\[
b^*(i) = \begin{cases} 0, & \text{if } i \leq \tau(b), \\ k_i - 1, & \text{if } i > \tau(b). \end{cases}
\]
Further, for any \( b \in G \), \( b \neq \emptyset \), let \( b^- \) be the predecessor of \( b \) with respect to \( \preceq \).

We recursively define that \( A \subseteq G \) **order shatters** \( b \in G \) (with respect to \( \preceq \)) if the following hold:
(a) if \( \emptyset = \emptyset \), then \( |A| \geq 1 \),

(b) if \( \emptyset \neq \emptyset \), then there exist \( A', A'' \subseteq A \) such that

(i) \( a' = b' \) for all \( a' \in A' \).

(ii) \( a'' < b'' \) for all \( a'' \in A'' \).

(iii) \( a' \cap b' = a'' \cap b'' \) for all \( a', a'' \in A'' \).

(iv) \( |A'| + |A''| \geq |\Delta b| \).

(v) \( A' \) and \( A'' \) both order shatter \( b \).

Further, we denote \( \text{ord-str}_{\leq}(A) = \{ b \in G : A \text{ order shatters } b \} \).

Let us now collect the two results that we require. The first result is an extension of a result by [ARS02], from the Boolean cube setting to uniform grids. The proof is a straightforward extension to uniform grids; therefore, we omit it.

**Proposition 6.3.** Let \( G \) be a uniform grid. Then for any \( A \subseteq G \), \( \text{ord-str}_{\leq}(A) = \text{SM}_{\leq}(A) \).

The second result is an extension of a result in [FR03], from the Boolean cube setting to uniform grids. The proof is a straightforward extension to uniform grids; therefore, we omit it.

**Proposition 6.4.** Let \( G \) be a uniform grid. For any \( i, j \in [0, \lfloor N/2 \rfloor] \), if \( i \leq j \), then

\[
\text{ord-str}_{\leq}(i) = \text{ord-str}_{\leq}(j) \cap [0, i].
\]

Lemma 6.2 then follows immediately from Proposition 6.3 and Proposition 6.4.

**Proof of Lemma 6.2.** Let \( i, j \in [0, \lfloor N/2 \rfloor] \), \( i \leq j \). By Proposition 6.4, we have \( \text{ord-str}_{\leq}(i) = \text{ord-str}_{\leq}(j) \cap [0, i] \), which implies \( \text{ord-str}_{\leq}(i) \subseteq \text{ord-str}_{\leq}(j) \). The proof is then complete by Proposition 6.3. \( \square \)

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**References**

[AA95] Robert E.L. Aldred and Richard P. Anstee. On the density of sets of divisors. *Discrete Mathematics*, 137(1-3):345–349, 1995.  
https://doi.org/10.1016/0012-365X(93)E0114-J. 19
[Alo99] Noga Alon. Combinatorial Nullstellensatz. *Combinatorics, Probability and Computing*, 8(1-2):7–29, 1999. https://doi.org/10.1017/S0963548398003411.

[ARS02] Richard P. Anstee, Lajos Rónyai, and Attila Sali. Shattering news. *Graphs and Combinatorics*, 18(1):59–73, 2002. https://doi.org/10.1007/s003730200003.

[ASY21] Emmanuel Abbe, Amir Shpilka, and Min Ye. Reed–Muller Codes: Theory and Algorithms. *IEEE Transactions on Information Theory*, 67(6):3251–3277, 2021. https://doi.org/10.1109/TIT.2020.3004749.

[BE99] Anna Bernasconi and Lavinia Egidi. Hilbert Function and Complexity Lower Bounds for Symmetric Boolean Functions. *Inf. Comput.*, 153(1):1–25, August 1999. https://doi.org/10.1006/inco.1999.2798.

[BEHL12] Ido Ben-Eliezer, Rani Hod, and Shachar Lovett. Random low-degree polynomials are hard to approximate. *Computational Complexity*, 21(1):63–81, 2012. https://doi.org/10.1007/s00037-011-0020-6.

[BF20] László Babai and Péter Frankl. Linear algebra methods in combinatorics, Version 2.1. *Department of Computer Science, The University of Chicago*, page 150, 2020. https://people.cs.uchicago.edu/~laci/CLASS/HANDOUTS-COMB/BaFrNew.pdf.

[BLR89] Béla Bollobás, Imre Leader, and Andrew J. Radcliffe. Reverse Kleitman inequalities. *Proceedings of the London Mathematical Society*, 3(1):153–168, 1989. https://doi.org/10.1112/plms/s3-58.1.153.

[CLO15] David A Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Springer, 2015. https://doi.org/10.1112/plms/s3-58.1.153.

[dBvETK51] N.G. de Bruijn, C. van Ebbenhorst Tengbergen, and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wiskunde (2)*, 23:191–193, 1951. https://research.tue.nl/files/4373475/597494.pdf.

[FHR09] Bálint Felszeghy, Gábor Hegedűs, and Lajos Rónyai. Algebraic Properties of Modulo $q$ Complete $f$-Wide Families. *Combinatorics, Probability and Computing*, 18(3):309–333, 2009. https://doi.org/10.1017/S0963548308009619.

[FR03] Katalin Friedl and Lajos Rónyai. Order shattering and Wilson’s theorem. *Discrete Mathematics*, 270(1):127–136, 2003. https://doi.org/10.1016/S0012-365X(02)00869-5.

[FRR06] Bálint Felszeghy, Balázs Ráth, and Lajos Rónyai. The lex game and some applications. *Journal of Symbolic Computation*, 41(6):663–681, 2006. https://doi.org/10.1016/j.jsc.2005.11.003.

[HP98] P. Heijnen and R. Pellikaan. Generalized Hamming weights of $q$-ary Reed-Muller codes. *IEEE Transactions on Information Theory*, 44(1):181–196, 1998. https://doi.org/10.1109/18.651015.

[HR03] Gábor Hegedűs and Lajos Rónyai. Gröbner bases for complete uniform families. *Journal of Algebraic Combinatorics*, 17(2):171–180, 2003. https://doi.org/10.1023/A:1022934815185.
