DIRECTIONAL ASYMPTOTIC CONES OF GROUPS EQUIPPED WITH BI-INARIANT METRICS

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Abstract. Given a bi-invariant metric on a group, we construct a version of an asymptotic cone without using ultrafilters. The new construction, called the directional asymptotic cone, is a contractible topological group equipped with a complete bi-invariant metric and admits a canonical Lipschitz homomorphism to the standard asymptotic cone. Moreover, the directional asymptotic cone of a countable group is separable.

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1. Introduction

1.1. Historical context. In 1984 van den Dries and Wilkie [9] published an alternative proof of Gromov’s celebrated theorem on polynomial growth. Their proof uses methods of non-standard analysis and provides the first construction that uses ultrafilters of what we know now as the asymptotic cone of a group equipped with the word metric. The construction generalises to all metric spaces and was popularised by Gromov in [6] where he provided computations for a substantial
family of examples and posed many inspiring questions. Asymptotic cones became a well established tool of metric geometry and geometric group theory [3]. It is an unpleasant feature, however, that the asymptotic cone of a space depends on the choice of a non-principal ultrafilter \( \omega \) and a scaling sequence \( d \), and it is often an unapproachable space.

While asymptotic cones are defined for arbitrary metric spaces, in this paper we are interested in groups \( G \) equipped with conjugation invariant norms or, equivalently, bi-invariant metrics, see Paragraph 2.1 below. In this case, the asymptotic cone \( \text{Cone}_\omega(G, d) \) is a group equipped with a bi-invariant metric, rendering it a topological group. This paper grew out of the interest in these groups.

1.2. Asymptotic cones. Intuitively, the asymptotic cone of a group \( G \) (or any metric space) is thought of as “its image from far away”. For example, the asymptotic cone of a bounded group (or space) is a point, that of \( \mathbb{Z} \) is \( \mathbb{R} \) and that of \( \mathbb{Z}^n \) is \( \mathbb{R}^n \) (both with the \( L^1 \)-metric).

The definition of the asymptotic cone presented below is not the most general one and we refer the reader to the excellent exposition by Drutu and Kapovich [3] for details.

Let \( G \) be equipped with a conjugation invariant norm \( \| \cdot \| \). Choose a non-principal ultrafilter \( \omega \) on \( \mathbb{N} \) and a scaling sequence \( d = (d_n) \), i.e a sequence \( d_n > 0 \) with \( \lim_n d_n = \infty \). Consider the set \( B(G, d) \) of all sequences \( (g_n) \subseteq G \) such that \( \left( \frac{1}{d_n} \| g_n \| \right) \) is bounded. Recall that a non-principal ultrafilter allows us to compute \( \omega \)-limits of any bounded sequence in \( \mathbb{R} \). The ultrafilter “picks up” a convergent subsequence in a way that makes the usual calculus of limits work. Define an equivalence relation on \( B(G, d) \) where \( (g_n) \sim (h_n) \) if \( \omega \)-lim \( \frac{1}{d_n} \| g_n^{-1} h_n \| = 0 \). Then \( \text{Cone}_\omega(G, d) \) is the set of equivalence classes denoted \( \{g_n\} \). The norm on \( \text{Cone}_\omega(G, d) \) is defined by \( \|\{g_n\}\| = \omega\text{-lim}_n \frac{\|g_n\|}{d_n} \) and the group structure is \( \{g_n\} \cdot \{h_n\} = \{g_n h_n\} \). It is well defined due to the conjugation-invariance of the norm - the fact first observed by Calegari and Zhuang [2].

1.3. Directional asymptotic cones. Let us dwell on the isometry \( \text{Cone}_\omega(\mathbb{Z}, d) \cong \mathbb{R} \). It sends \( \{z_n\} \in \text{Cone}_\omega(\mathbb{Z}, d) \) to \( \omega\text{-lim}_n \frac{1}{d_n} z_n \in \mathbb{R} \). Now, look at some \( r > 0 \) in \( \mathbb{R} \) and set \( z_n := (-1)^n [rd_n] \). Clearly \( z_n \in B(\mathbb{Z}, d) \), however the sequence \( \frac{z_n}{d_n} \) is not convergent. Its accumulation points are \( \pm r \) and \( \omega\text{-lim}_n \frac{z_n}{d_n} = \pm r \) depending on the choice of the ultrafilter \( \omega \). On the other hand, the sequence \( z'_n := [rd_n] \) will have \( \omega\text{-lim} \frac{z'_n}{d_n} = r \) for any non-principal ultrafilter. The difference between the two sequences is clear. The first, \( \{z_n\} \), lacks “direction”, a deficiency
that is circumvented by the ultrafilter $\omega$. The second, $(z'_n)$, has a well defined “direction” in $\mathbb{Z}$. In the case of $\mathbb{Z}$, the sequences in $B(\mathbb{Z}, d)$ that “lack direction” appear to be superfluous. In some precise sense (in this case, via the inclusion $\mathbb{Z} \subseteq \mathbb{R}$) the idea of “direction” is encapsulated by the fact that $\frac{z_n}{d_n}$ is Cauchy while $\frac{z'_n}{d_n}$ is not.

This example highlights the excessive amount of sequences used in the definition of the classical asymptotic cones. This excess is conveniently dealt with by the ultrafilter, however, this convenience comes with a price, which is misleadingly invisible in the example of $\text{Cone}_\omega(\mathbb{Z}, d)$. First, $\text{Cone}_\omega(G, d)$ may depend on the choice of the ultrafilter $\omega$ as proven by Drutu and Sapir [4] who found an example of a finitely generated group with infinitely many non-homeomorphic asymptotic cones. See also [3, page 189] for a more detailed context. We don’t know examples of groups with bi-invariant metrics whose asymptotic cones depend on the choice of an ultrafilter.

Due to the “bloated” amount of sequences, ultrafilter cones tend to be unmanageably large. For example, the cone of a separable space may become non-separable. For example, the asymptotic cone of a Gromov hyperbolic space is an $\mathbb{R}$-tree [6, Example 2.B (b)] hence it is non-separable in general. Since regular trees embed isometrically into the Cayley graph of the bi-invariant metric on the free group $F_2$, due to [1, Theorem 1.D], the asymptotic cone of $F_2$ with the bi-invariant word norm is non-separable. The complexity of asymptotic cones is exhibited by their fundamental groups which are very often uncountable [6, Example 2.B (c')].

This paper grew out of an attempt to avoid this excess. Our main construction is a version of the asymptotic cone for groups $G$ equipped with bi-invariant metrics which we call the directional asymptotic cone. We emphasise that unlike the asymptotic cone which is defined for any metric space, our construction is specific to groups with bi-invariant metric. With the example above in mind, our first task is to establish a notion of “directions” in the group $G$. Inspired by Cayley graphs, every $g \in G$ is a “direction”. Directions should be thought of as “non commutative vectors”, so we would want to be able to “add” and “rescale” them. Thus, we define the home for directions to be the group $F_\mathbb{R}(G) = *_G \mathbb{R}$, the free product of $\mathbb{R}$, one copy for each $g \in G$. We suitably equip this group with a bi-invariant metric and take its completion (as a metric space), i.e the space of all Cauchy sequences. For any unbounded subset $T$ of the interval $(0, \infty)$ which we call a scaling set (Definition 5.1), we define the directional asymptotic cone $\hat{C}(G, T)$ as a suitable quotient. The elements of $\hat{C}(G, T)$ can be thought of as Cauchy sequences
in \( G \) as its metric is rescaled by factors taken from \( T \) ("moving further and further away from the group"). This construction is carried out in Sections 5 and 6. We will write \( \hat{C}(G) \) for the maximal scaling set \( T = (0, \infty) \).

Our construction depends on the scaling set \( T \subseteq (0, \infty) \), however, there are natural epimorphisms \( \hat{C}(G, T) \to \hat{C}(G, T') \) whenever \( T' \subseteq T \). We say that \( G \) is independent of scaling, see Section 9, if these maps are isometries. In this case \( \hat{C}(G) \) has particularly nice properties.

**Relation to the ultrafilter cone.** For any non-principal ultrafilter \( \omega \) and scaling sequence \( d \) there is a natural homomorphism
\[
\rho_{\omega, d} : \hat{C}(G) \to \text{Cone}_\omega(G, d)
\]
which is Lipschitz with constant 1 (Proposition 10.2). We show in Theorem 10.3 that it is injective if and only if \( G \) is independent of scaling.

In fact, under some favourable conditions \( \rho_{\omega, d} : \hat{C}(G) \to \text{Cone}_\omega(G, d) \) is an isometric embedding. For example, Theorems 9.9 and 10.3 show that this is the case for any group \( G \) with the property that the commutator subgroup of any finitely generated subgroup of \( G \) is bounded in \( G \). For example, we show in Propositions 11.2 and 11.3 that

- \( \hat{C}(\mathbb{Z}^n) \) is isometric to \( \mathbb{R}^n \) with the \( L^1 \)-norm if \( \mathbb{Z}^n \) is equipped with this norm. Moreover, \( \rho_{\omega, d} : \hat{C}(\mathbb{Z}^n) \xrightarrow{\approx} \text{Cone}_\omega(\mathbb{Z}^n, d) \) is an isometry for any ultrafilter \( \omega \) and any scaling sequence \( d \). See Proposition 11.2.

- Any nilpotent group \( G \) is independent of scaling, \( \hat{C}(G) \) is abelian, and \( \hat{C}(G) \to \text{Cone}_\omega(G, d) \) is an isometric embedding for any ultrafilter \( \omega \) and any scaling sequence \( d \). See Proposition 11.3.

1.4. **Algebraic and metric properties.**

**Functoriality.** \( \hat{C}(\mathcal{G}) \) is a functor from the category of groups with bi-invariant metrics and Lipschitz homomorphisms to the category of complete groups with bi-invariant metrics and Lipschitz homomorphisms (Propositions 6.11 and 6.4).

**Metric properties.** The directional asymptotic cone shares metric properties with the ultrafilter cone, for example

- \( \hat{C}(G, T) \) is a complete metric space (Proposition 6.4).
- \( \hat{C}(G, T) \) is a length space (Corollary 7.4).
Unlike the ultrafilter cones which tend to be wild metric spaces with very complicated fundamental groups, the directional asymptotic cone is quite tame. For example

- If $G$ is countable then $\hat{C}(G)$ is separable i.e it is a Polish group (Proposition 7.2).
- $\hat{C}(G, T)$ is a contractible and $\text{End}(G)$ with the compact-open topology is also contractible for a large family of scaling sets which includes $T = [0, \infty)$. See Theorem 7.10.

The contractibility of $\hat{C}(G)$ should be contrasted with the ultrafilter cones which, in general, can have a very non-trivial topology. For example, the asymptotic cone of $G = \bigoplus_n \mathbb{Z}/n\mathbb{Z}$ equipped with the word metric associated with generators of the form $(0, \ldots, 0, 1, 0, \ldots)$ is not simply connected, due to Karlhofer [8, Proposition 7.2]. On the other hand, we don’t know a single example of a group equipped with the bi-invariant word metric associated with finite normal generating set such that its asymptotic cone is not contractible.

Products. By Theorem 8.3 for any scaling set $T$

$$\hat{C}(G \times H, T) \cong \hat{C}(G, T) \times \hat{C}(H, T).$$

Extensions. Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an extension with $\pi$ Lipschitz. If $N$ is bounded in $G$ and if there exists a Lipschitz set theoretic section $H \to G$ then by Theorem 8.4 $\pi$ induces a bi-Lipschitz equivalence

$$\hat{C}(G, T) \xrightarrow{\hat{\pi}(\cdot)} \hat{C}(H, T).$$

In particular it is an isomorphism of groups.

If $N \trianglelefteq G$ it is possible to equip $H = G/N$ with a natural quotient norm (see Definition 2.5). By Corollary 8.5 if $N$ is bounded in $G$ then $\hat{C}(G, T) \to \hat{C}(H, T)$ is an isometry. For example,

- If $G$ is a finitely generated nilpotent then its commutator subgroup $[G, G]$ is bounded with respect to the bi-invariant (word) metric by [1, Theorem 5.H]. Since the group is finitely generated this metric Lipschitz dominates any bi-invariant metric, hence the commutator subgroup is bounded. Therefore $\hat{C}(G, T) \to \hat{C}(G_{ab})$ is an isometry. In particular $\hat{C}(G)$ is abelian by Proposition 8.1.

- The same is true for Thompson’s group $F$ or, more generally, for Higman-Thompson groups, which follows from [5, Theorem 1.1] and the subsequent discussion. The above mentioned theorem states that the commutator subgroup of a Higman-Thompson
group $G$ is six-uniformly simple. This means that the commutator subgroup is a simple group and given any nontrivial element $1 \neq g \in [G, G]$ every other element is a product of at most six conjugates of $g^{\pm 1}$. In particular, the bi-invariant metric associated with the conjugacy classes of $g^{\pm 1}$ has diameter at most six.

1.5. Examples. The results above enable us to compute the directional asymptotic cones of some specific groups. Of course, one issue is to equip groups with conjugation invariant norms. If $A \subseteq G$ is a (normally) generating set, there is an associated standard conjugation invariant word norm $\| \|_A$ where $\|g\|_A$ is the smallest integer $n$ such that $g$ can be expressed as a product of conjugates of elements of $A$ and their inverses. See Definition 2.12.

Abelian and nilpotent groups. As we mentioned above, if $G$ is nilpotent then by Proposition 11.3 it is independent of scaling, $\hat{\mathcal{C}}(G)$ is abelian, and $\rho_{\omega,d} : \hat{\mathcal{C}}(G) \to \text{Cone}_\omega(G, d)$ are isometric embeddings.

If $G$ is a finitely generated nilpotent group and is equipped with the standard conjugation invariant word metric then, in fact, $\rho_{\omega,d} : \hat{\mathcal{C}}(G) \to \text{Cone}_\omega(G, d)$ is an isometry of groups which are bi-Lipschitz equivalent to $(\mathbb{R}^n, \| \|_1)$, and hence to $\mathbb{R}^n$ with the Euclidean norm, where $n = \dim_{\mathbb{R}} G_{ab} \otimes \mathbb{R}$. This is the content of Example 11.5.

A special case is $(\mathbb{Z}^n, \| \|_1)$ in which case $\hat{\mathcal{C}}(\mathbb{Z}^n) \cong (\mathbb{R}^n, \| \|_1)$ by Proposition 11.2.

Solvable groups. If $G$ is a finitely generated solvable group with finitely generated nilpotent commutator subgroup and equipped with the standard conjugation invariant word norm. Then $\hat{\mathcal{C}}(G)$ is bi-Lipschitz equivalent to $(\mathbb{R}^n, \| \|_1)$, where $n$ is the rank of the abelianisation of $G$. This follows from Proposition 11.4 since $[G, G]$ is bounded in $G$ by [1] Theorem 5.K and the argument used for nilpotent group in the section about extensions above. Notice that $(\mathbb{R}^n, \| \|_1)$ is bi-Lipschitz equivalent to $\mathbb{R}^n$ with the Euclidean norm.

Commutator subgroup. The directional cone of the commutator subgroup $[G, G]$ of a group $G$ equipped with the commutator length is abelian. See Example 11.7. The same is true for any ultrafilter asymptotic cone which is easy to verify (cf. [2] where the authors discuss more general verbal groups).
Free groups. Let $F_n$ be the free group on $n$ generators equipped with the bi-invariant word metric associated with the standard generating set. Let $\Theta \subseteq F_n$ be a subset consisting of representatives of conjugacy classes of elements that are not proper powers. Let $\tau: \Theta \to \mathbb{R}$ denote the stable length of the word metric (see Section 6.8) and let $F_{\mathbb{R}}(\Theta; \tau)$ denote the real free group on $\Theta$ equipped with the bi-invariant word metric associated with $\tau$ (see Section 4). In Section 12 we show that $F_n$ is independent of scaling and we construct an injective Lipschitz homomorphism

$$F_{\mathbb{R}}(\Theta; \tau) \to \hat{\mathcal{C}}(F_n)$$

with dense image. We conjecture that it is an isometry. We also show that the directional cone of a free group is independent of scaling and hence the natural homomorphisms $\rho_{\omega, d}: \hat{\mathcal{C}}(F_n) \to \text{Cone}_\omega(F_n, d)$ are injective (Theorem 12.1).

1.6. Relation to the construction of Calegari and Zhuang. Let $F$ denote a free group let and $W \subseteq F$ which we refer to as a set of words. Let $G$ be a group. The set of $W$-verbal elements (or simply verbal elements) in $G$ is

$$X_W = \bigcup_{\varphi \in \text{Hom}(F, G)} \varphi(W).$$

The $W$-verbal subgroup of $G$ is $G_W = \langle X_W \rangle$. We equip $G_W$ with the standard word norm $\| \cdot \|_{X_W}$ with respect to the set of generators $X_W$.

In [2] Calegari and Zhuang study verbal subgroups making use of the asymptotic cones of $G_W$ (with the verbal word norm). They fix a non-principal ultrafilter $\omega$ and the scaling sequence $d = (d_n)$ where $d_n = n$, and denote $\hat{A}_W(G) = \text{Cone}_\omega(G_W, d)$, or simply $\hat{A}_W$. Clearly, $\hat{A}_W$ depends on the choice of the ultrafilter (and a scaling sequence $d$). Then in [2, Section 3.4, Definition 3.8] they define a subgroup $A_W \leq \hat{A}_W$ which they call the real cone of $G$. This group consists of all elements of $\text{Cone}_\omega(G_W, d)$ of the form $\{g_1^{[t_1n]} \cdots g_k^{[t_kn]}\}_n$, where $g_i \in G_W$ and $t_i \in \mathbb{R}$ and $\lfloor \cdot \rfloor$ is the floor function. It is not clear (to us) to what extent the group $A_W$ depends on the choice of the ultrafilter.

The construction of the real cone is tightly related to our construction of the directional asymptotic cone of $G_W$ as follows. Let $F_{\mathbb{R}}(G_W)$ denote the free amalgamated product $\ast_{g \in G_W} \mathbb{R}$, see Section 4. A key feature of the directional asymptotic cone described in Section 6.3 is that it is equipped with a canonical homomorphism $\eta: F_{\mathbb{R}}(G_W) \to \hat{\mathcal{C}}(G_W)$ whose image is dense. By composing it with the homomorphism $\rho: \hat{\mathcal{C}}(G_W) \to \text{Cone}_\omega(G_W, d)$ (see Proposition 10.2) we obtain a
homomorphism $\rho' : F_\mathbb{R}(G_W) \to \hat{A}_W$. By easy inspection of $\eta$ and $\rho$, the real cone $A_W$ is precisely the image of $\rho'$. Since $\rho$ is Lipschitz and $\hat{C}(G_W)$ is a complete metric space, the closure of $A_W$ in $\hat{A}_W$ is equal to the image of the directional cone $\hat{C}(G_W)$ in $\text{Cone}_\omega(G_W, d)$.

Some of the fundamental properties of the real cone proven in [2] have their analogues for directional cones, and in fact, they follow from them. For example, in [2, Lemma 3.9 and 3.10] it is shown that $A_W$ has an action of $\mathbb{R}$ and consequently that it is a contractible space. Similarly, in Theorem 7.10 we show $\hat{C}(H)$ is contractible. We also show that it admits an action of the monoid $[0, \infty)$ and with only a bit more effort one can show this action extends to an action of $\mathbb{R}$.

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### 2. Conjugation invariant norms

2.1. Let $G$ be a group. A **norm** on $G$ is a function $\nu : G \to [0, \infty)$ that satisfies the following conditions:

1. $\nu(g) = 0 \iff g = 1$ (non-degeneracy or point separation);
2. $\nu(g) = \nu(g^{-1})$ (symmetry);
3. $\nu(gh) \leq \nu(g) + \nu(h)$ (triangle inequality).

By relaxing the condition of separation of points we obtain a **pseudo-norm** or a **degenerate norm** on $G$. A (pseudo) norm is called **conjugation-invariant** if in addition for every $g, h \in G$,

$$\nu(gh^{-1}) = \nu(g).$$

Conjugation invariant (pseudo) norms are in one-to-one correspondence with bi-invariant (pseudo) metrics on $G$ via $d_\nu(g, h) = \nu(g^{-1}h)$. In this case $G$ becomes a topological group (but this is not necessarily the case if the norm is not conjugation invariant).

2.2. Let $f : (G, \mu) \to (H, \nu)$ be a homomorphism of groups equipped with pseudonorms, whence pseudometrics. Then $f$ is a Lipschitz function if and only if there exists $C \geq 0$ such that $\nu(f(g)) \leq C \mu(g)$ for all $g \in G$.

2.3. A pseudometric $\bar{d}$ on $\bar{X}$ induces an equivalence relation $\bar{x} \sim \bar{y} \iff \bar{d}(\bar{x}, \bar{y}) = 0$ and a metric space $(X, d)$ on the set of equivalence classes where $d([\bar{x}], [\bar{y}]) = \bar{d}(\bar{x}, \bar{y})$.

Let $\nu$ be a conjugation-invariant pseudonorm on $G$ and let

$$G_0 = \{g \in G : \nu(g) = 0\}.$$
Definition 2.5. Let $\| \|$ be a conjugation-invariant pseudonorm on $G$. Then for any $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ in $G$

$$\| (g_1 \cdots g_n)^{-1}(h_1 \cdots h_n) \| \leq \sum_{i=1}^{n} \| g_i^{-1}h_i \|.$$ 

Proof. Induction on $n$. Set $g = g_1 \cdots g_n$ and $g' = g_1 \cdots g_{n-1}$ and $h = h_1 \cdots h_n$ and $h' = h_1 \cdots h_{n-1}$. Then $\| g^{-1}h \| = \| g_n^{-1}g'^{-1}h'h_n \| \leq \| g_n^{-1}h_n \| + \| g_n^{-1}g'^{-1}h'h_n \| = \| g_n^{-1}h_n \| + \| g'^{-1}h' \| \leq \sum_{i=1}^{n} \| g_i^{-1}h_i \|$. □

Lemma 2.4. Let $\| \|$ be a conjugation-invariant pseudonorm on $G$. Then for any $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ in $G$

Then for any $g \in G$ and $h \in G/G$, the pseudometric $\| \|$ on $G/G$ is well defined. The symmetry of $\| \|$ follows from the symmetry of $\| \|$ and since $\pi^{-1}(h^{-1}) = (\pi^{-1}(h))^{-1}$. Consider $h_1, h_2 \in H$. Given $\epsilon > 0$, choose some $g \in \pi^{-1}(h_1)$ such that $\| g \| < \| h_1 \| + \epsilon$. Since $\pi^{-1}(h_1h_2) = g_1 \cdot \pi^{-1}(h_2)$, we get that

$$\| h_1h_2 \| = \inf \{ \| g \| : g \in \pi^{-1}(h_1) \}.$$ 

Proof. Surjectivity of $\pi$ implies that $\| \|$ is well defined. The symmetry $\| h \| = \| h^{-1} \|$ follows from the symmetry of $\| \|$ and since $\pi^{-1}(h^{-1}) = (\pi^{-1}(h))^{-1}$. Consider $h_1, h_2 \in H$. Given $\epsilon > 0$, choose some $g \in \pi^{-1}(h_1)$ such that $\| g \| < \| h_1 \| + \epsilon$. Since $\pi^{-1}(h_1h_2) = g_1 \cdot \pi^{-1}(h_2)$, we get that

$$\| h_1h_2 \| = \inf \{ \| g \| : g \in \pi^{-1}(h_1) \}.$$ 

Then $\| \|$ is conjugation invariant if $\| \|$ is, in which case it is a metric quotient, hence Lipschitz with constant 1.

Lemma 2.6. Let $\pi : G \to H$ be a surjective homomorphism and let $\| \|$ be a pseudonorm on $G$. For any $h \in H$ set

$$\| h \| = \inf \{ \| g \| : g \in \pi^{-1}(h) \}.$$ 

Then $\| \|$ is a pseudonorm on $H$. It is conjugation invariant if $\| \|$ is, in which case it is a metric quotient, hence Lipschitz with constant 1.

Proof. Surjectivity of $\pi$ implies that $\| \|$ is well defined. The symmetry $\| h \| = \| h^{-1} \|$ follows from the symmetry of $\| \|$ and since $\pi^{-1}(h^{-1}) = (\pi^{-1}(h))^{-1}$. Consider $h_1, h_2 \in H$. Given $\epsilon > 0$, choose some $g \in \pi^{-1}(h_1)$ such that $\| g \| < \| h_1 \| + \epsilon$. Since $\pi^{-1}(h_1h_2) = g_1 \cdot \pi^{-1}(h_2)$, we get that

$$\| h_1h_2 \| = \inf \{ \| g \| : g \in \pi^{-1}(h_1) \}.$$ 

Then $\| \|$ is conjugation invariant if $\| \|$ is, in which case it is a metric quotient, hence Lipschitz with constant 1.
**Remark.** Lemma 2.6 is valid for groups with pseudonorms but fails for general pseudometric spaces.

**Proposition 2.7.** Let $\pi: G \to H$ be a metric quotient. If $G$ is a complete metric group then so is $H$.

**Proof.** Let $h_1, h_2, \ldots$ be a Cauchy sequence in $H$. Since $(h_n)$ is convergent iff it contains a convergent subsequence, by passage to a subsequence if necessary we may assume that $\|h_k^{-1}h_n\|_H < \frac{1}{2^k}$ for every $k \leq n$. For every $k \geq 1$ denote $\delta_k = h_k^{-1}h_{k+1}$. Thus, $\|\delta_k\|_H < \frac{1}{2^k}$.

Lift $h_1$ arbitrarily to some $g_1 \in G$. By construction of $\| \|_H$, every $\delta_k$ lifts to some $\Delta_k \in G$ such that $\|\Delta_k\| < \frac{1}{2^k}$. For every $k \geq 1$ set $g_k = g_1 \cdot \Delta_1 \cdots \Delta_{k-1}$. Since $\pi$ is a homomorphism

$$\pi(g_k) = \pi(g_1)\pi(\Delta_1) \cdots \pi(\Delta_{k-1}) = h_1 \delta_1 \cdots \delta_{k-1} = h_k$$

for all $k$. Consider some $m \geq 1$. For any $k \geq m$

$$\|g_m^{-1}g_k\|_G = \|\Delta_m \cdots \Delta_{k-1}\|_G < \sum_{i=m}^{k-1} \frac{1}{2^i} < \frac{1}{2^{m-1}}$$

which is arbitrarily small for sufficiently large $m$. It follows that $(g_n)$ is a Cauchy sequence in $G$, and by hypothesis it converges to some $g \in G$. Since $\pi$ is Lipschitz, the sequence $h_n = \pi(g_n)$ converges to $\pi(g)$.

**Proposition 2.8.** Suppose that $\pi_G: G \to G'$ and $\pi_H: H \to H'$ are metric quotients. Suppose that there are homomorphisms $\varphi$ and $\varphi'$ rendering the following diagram commutative

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
\pi_G \downarrow & & \downarrow \pi_H \\
G' & \xrightarrow{\varphi'} & H'
\end{array}
\]

If $\varphi$ is Lipschitz then $\varphi'$ is Lipschitz with the same constant.

**Proof.** Let $C > 0$ be the Lipschitz constant of $\varphi$. Given $g' \in G'$, for any $\epsilon > 0$ choose $g \in G$ such that $\pi_G(g) = g'$ and $\|g\| < \|g'\| + \frac{\epsilon}{C}$. Since $\pi_H$ is Lipschitz with constant 1,

$$\|\varphi'(g')\| = \|\varphi'(\pi_G(g))\| = \|\pi_H(\varphi(g))\| \leq \|\varphi(g)\| \leq C\|g\| < C\|g'\| + \epsilon.$$

This completes the proof since $\epsilon$ is arbitrary.

The next result is left to the reader.

**Proposition 2.9.** Let $f: G \to G'$ and $g: G' \to G''$ be homomorphisms between groups equipped with conjugation invariant pseudonorms. If $f$ and $g$ are metric quotients then so is $g \circ f$. If $f$ and $g \circ f$ are metric quotients then so is $g$. 
2.10. By a **length function** on a set $A$ we mean a function 
$$\mu : A \rightarrow [0, \infty).$$

Let $G$ be a group and $A \subseteq G$ a normal generating set. That is, $G$ is the smallest normal subgroup containing $A$. Let $\mu$ be a length function on $A$. The conjugation invariant **word norm associated to** $\mu$ is defined by

$$\|g\|_\mu = \inf \left\{ \sum_{i=1}^{n} \mu(a_i) : g = (g_1a_1^{\pm 1}g_1^{-1}) \cdots (g_na_n^{\pm 1}g_n^{-1}), \ a_i \in A, g_i \in G \right\}.$$  

It is straightforward to check that $g \mapsto \|g\|_\mu$ is a conjugation invariant pseudonorm. The details are left to the reader.

Notice that $\|a\|_\mu \leq \mu(a)$ and that in general a strict inequality may hold. Also, even if $\mu(a) \neq 0$ for all $a \in A$, the resulting $\| \|_\mu$ may be degenerate. For example, $A = \{ \frac{1}{n} \}$ (normally) generates $\mathbb{Q}$ and $\mu(\frac{1}{n}) = \frac{1}{n^2}$ yields a vanishing pseudonorm on $\mathbb{Q}$ since $\|\frac{1}{n}\|_\mu = \lim_{k \to \infty} k\mu(\frac{1}{nk}) = 0$.

**Proposition 2.11.** The norm $\| \|_\mu$ on $G$ defined above has the property that for any homomorphism $f : G \rightarrow H$ where $H$ is equipped with a conjugation invariant pseudonorm, if $\|f(a)\|_H \leq C \cdot \mu(a)$ for all $a \in A$ then $f$ is Lipschitz with constant $C$.

**Proof.** Choose $g \in G$. For any presentation $g = u_1 \cdots u_n$, where each $u_i$ is conjugate of some $a_i^{\pm 1}$ we get that

$$\|f(g)\|_H \leq \sum_i \|f(u_i)\|_H = \sum_i \|f(a_i^{\pm 1})\|_H = \sum_i \|f(a_i)\|_H \leq C \sum_i \mu(a_i).$$

This shows that $\|f(g)\|_H \leq C \|g\|_\mu$. $\square$

We illustrate the above with the next important example and the subsequent two propositions.

**Definition 2.12.** Suppose that $G$ is normally generated by $A$. The associated **standard conjugation invariant word norm** $\| \|_A$ is the word norm in Section 2.10 with respect to the constant length function $1 : A \rightarrow [0, \infty)$. Thus, $\|g\|_A$ is the minimum integer $n \geq 0$ such that $g$ can be written as a product of conjugates of elements of $A$ and their inverses. It is easy to check that this is a norm rather than a pseudo norm.
Example 2.13. The standard basis $E = \{e_1, \ldots, e_n\}$ normally generates $\mathbb{Z}^n$ and the standard word norm $\|\|_E$ coincides with the $L^1$-norm $\|(x_1, \ldots, x_n)\|_1 = \sum |x_i|.$

Proposition 2.14. Suppose $G$ is normally generated by finite subsets $A,A'$. Then the standard conjugation invariant word norms $\|\|_A$ and $\|\|_{A'}$ are bi-Lipschitz equivalent.

Proof. Since the roles of $A$ and $A'$ are interchangeable, it suffices to show that $\text{id}: (G,\|\|_A) \to (G,\|\|_{A'})$ is Lipschitz. The norms take integer values only so we may set $C = \max\{\|a\|_{A'} : a \in A\}$. Then $\|\text{id}(a)\|_{A'} \leq C = C \cdot \|a\|_A$. The result follows from Proposition 2.11. □

Proposition 2.15. Let $\pi: G \to H$ be a group epimorphism and $A \subseteq G$ a normally generating set. Set $B = \pi(A)$ and consider the standard conjugation invariant norm $\|\|_A$ and $\|\|_B$ on $G$ and $H$ respectively. Then $\pi: (G,\|\|_A) \to (H,\|\|_B)$ is a metric quotient map.

Proof. Let $\|\|_H$ denote the quotient pseudonorm on $H$ defined in Lemma 2.6. Choose some $h \in H$. We will show that $\|h\|_H = \|h\|_B$ and by this complete the proof. By the description of $\|\|_A$ and $\|\|_B$ in Definition 2.12 and since $B = \pi(A)$ it follows that $\|h\|_B \leq \|g\|_A$ for any $g \in \pi^{-1}(h)$, hence $\|h\|_B \leq \|h\|_H$. Conversely, by the same description it is clear that if $\|h\|_B = n$ then there exists $g \in \pi^{-1}(h)$ such that $\|g\|_A \leq n$ and therefore $\|h\|_H \leq n = \|h\|_B$. □

2.16. Let $(X,d)$ be a pseudometric space. Let $C(X)$ denote the set of all Cauchy sequences $x = (x_n)$ in $X$. There is a pseudometric on $C(X)$ defined by $\delta(x,y) = \lim_n d(x_n,y_n)$. The associated metric space denoted $(\hat{X},\hat{d})$ is the completion of $X$. There is an embedding $X \to C(X)$ which sends any $x \in X$ to the constant sequence. Its image is dense in $C(X)$. If $X$ is a metric space (rather than pseudometric) then the resulting $X \to \hat{X}$ is an isometric embedding with dense image.

Remark 2.17. By construction $\hat{X}$ is always a bona fide metric space even if we started with a pseudo metric space $X$.

3. Word norms on the free group

Assumption. Throughout this section we assume that a set $A$ is equipped with a length function $\mu$.

Definition 3.1. Denote the free group generated by a set $A$ by $F(A) = *_A \mathbb{Z}.$
Clearly $A$ is a normal generating set of $F(A)$. We will write

$$F(A; \mu)$$

to denote the group $F(A)$ equipped with the conjugation invariant pseudonorm $\| \|_\mu$ described in Section 2.10 which we henceforth denote by $\| \|_{F(A; \mu)}$. When $\mu$ is understood from the context we will write $F(A)$ for $F(A; \mu)$ and $\| \|_{F(A)}$ for $\| \|_{F(A; \mu)}$.

As in [1], there is a description of the pseudonorm $\| \|_{F(A)}$ as a cancellation pseudonorm which we now outline.

3.2. Let $A$ be a set. Denote

$$W(A) = \{\text{all words in the alphabet } A\}.$$ 

It is a monoid under concatenation of words. In fact, it is the free monoid generated by $A$.

Let $w \in W(A)$ be a word. A sequence in $w = a_1 \cdots a_n$ is a word of the form $u = a_{i_1} \cdots a_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. That is, it is a “subword” of $w$. We denote by $w - u$ the word obtained by the removal of the symbols in $u$ from $w$. An interval in $w$ is a sequence of consecutive letters i.e a sequence of the form $a_k a_{k+1} \cdots a_{k+\ell-1}$ where $\ell \geq 0$ and $k \geq$ and $k + \ell \leq n$. A word of the form $a^\ell = a \cdots a$ is called a syllable.

If $\mu: A 	o [0, \infty)$ is a length function it extends to a unique homomorphism of monoids

$$\mu: W(A) \to [0, \infty), \quad \mu(a_1 \cdots a_k) = \sum_{i=1}^{k} \mu(a_i).$$

We call $\mu(w)$ the total length or the $\mu$-length of $w \in W(A^{\pm 1})$.

3.3. Let $A^{-1}$ denote a set containing the symbols $a^{-1}$ for every $a \in A$. We will write $A^{\pm 1} = A \sqcup A^{-1}$.

The free group $F(A)$ is the set of equivalence classes in $W(A^{\pm 1})$ under the equivalence relation generated by the relation $aa^{-1} \sim \epsilon \sim a^{-1}a$ where $\epsilon$ denotes the empty word and $a \in A$. Each element of $F(A)$ is represented by a unique reduced word $w$, a word which contains no interval of the form $aa^{-1}$ or $a^{-1}a$.

Let $w \in W(A^{\pm 1})$. A sequence $u$ in $w$ is called a cancellation sequence if $w - u$ represents the trivial element in $F(A)$. Notice that every word has at least one cancellation sequence (the word itself).

Given a length function $\mu$ on $A$, extend it to $A^{\pm 1}$ by setting $\mu(a^{-1}) := \mu(a)$. In turn, this extends to a monoid homomorphism $\mu: W(A^{\pm 1}) \to$
Thus,
\[
\mu(a_1^{\pm 1} \cdots a_k^{\pm 1}) = \sum_{i=1}^{k} \mu(a_i).
\]

The norm \( \| \cdot \|_{F(A;\mu)} \) on \( F(A) \) has the following useful description.

**Proposition 3.4.** Consider \( w \in F(A) \) represented by \( w \in W(A^{\pm 1}) \).

Then
\[
\| w \|_{F(A;\mu)} = \min \{ \mu(u) : u \text{ is a cancellation sequence in } w \}.
\]

In particular, the right hand side is independent of the choice of \( w \) and \( \| w \|_{F(A;\mu)} \) is equal to the total length of a \( \mu \)-minimal cancellation sequence \( u \) in \( w \).

We omit the proof as it is similar to the proof in [I, Proposition 2.E] in the case that \( \mu \) is the constant function \( \mu = 1 \).

**Corollary 3.5.** For every \( w \in F(A) \) there exist \( u_1, \ldots, u_n \in F(A) \) such that \( w = u_1 \cdots u_n \) and each \( u_i \) is conjugate to \( a_i^{\pm 1} \) for some \( a_i \in A \) and \( \| w \|_{F(A)} = \sum_{i=1}^{n} \mu(a_i) \).

**Proof.** If \( u = a_1^{\pm 1} \cdots a_n^{\pm 1} \) is a \( \mu \)-minimal cancellation sequence in a word \( w \) representing \( w \) then \( w \) is a product of conjugates of \( a_i^{\pm 1} \). \( \square \)

**Corollary 3.6.** If \( w = a^d \) for some \( a \in A \) then \( \| w \|_{F(A)} = d \cdot \mu(a) \).

**Proof.** The only cancellation sequence in \( w = a^d \) is \( w \) itself. \( \square \)

**Definition 3.7.** Let \( \mu \) be a length function on \( A \). Set
\[
A^\# = \{ a \in A : \mu(a) > 0 \}.
\]

**3.8.** Let \( w \in W(A) \) be a word. Denote by \( w^\# \) the word obtained by deleting from \( w \) all the symbols with zero length, whence retaining only the symbols in \( A^\# \). Thus, we obtain a function \( W(A) \rightarrow W(A^\#) \).

If \( u \) is a sequence in \( w \) let \( u^\# \) denote the sequence in \( w \) obtained by adding to \( u \) all the symbols in \( w \) of length zero.

One easily verifies that
\[
\mu(w) = \mu(w^\#)
\]
\[
\mu(u) = \mu(u^\#)
\]
\[
(w - u)^\# = w^\# - u^\# = w - u^\#.
\]

The function of \( W(A^{\pm 1}) \rightarrow W(A^{\pm 1^\#}) \) defined by \( w \mapsto w^\# \) gives rise to a homomorphism \( F(A) \xrightarrow{w \mapsto w^\#} F(A^\#) \).

The proof of the following facts is straightforward and left to the reader.
Proposition 3.9. Let \( w \in F(A) \). Then

(i) \( \|w\|_{F(A)} = \|w\|_{F(A^\#)} \).

(ii) The pseudonorm on \( F(A^\#) \) as a subgroup of \( F(A) \) coincides with the norm on \( F(A^\#) \) associated with the length function \( \mu|_{A^\#} \).

(iii) Suppose that \( w \in F(A^\#) \). If \( \|w\|_{F(A)} = 0 \) then \( w = 1 \). \( \square \)

Corollary 3.10. If \( \mu > 0 \) then the pseudonorm \( \|\|_\mu \) on \( F(A) \) is a norm.

Proof. In this case \( A^\# = A \). \( \square \)

We will now discuss the naturality of the construction \( F(A; \mu) \). Any function \( f: A \to B \) gives rise to a natural homomorphism \( f_*: F(A) \to F(B) \).

Definition 3.11. Let \( \mu_A \) and \( \mu_B \) be length functions on sets \( A \) and \( B \). A function \( f: A \to B \) is called proto-Lipschitz if there exists \( C \geq 0 \) such that \( \mu_B(f(a)) \leq C\mu_A(a) \) for all \( a \in A \).

Recall that any function \( f: A \to H \) where \( H \) is a group extends uniquely to a homomorphism \( f_*: F(A) \to H \). The next result is immediate from Proposition 2.11.

Proposition 3.12. If \( f: A \to H \) is proto-Lipschitz with constant \( C \), namely \( \|f(a)\|_H \leq C\mu(a) \) for all \( a \in A \), then \( f_*: F(A) \to H \) is Lipschitz with constant \( C \). \( \square \)

Definition 3.13. Let \( A \) be a set and \( d \) an integer. Let

\[ \psi_d: F(A) \to F(A) \]

be the unique homomorphism which extends the function \( A \to F(A) \) given by \( a \mapsto a^d \).

The homomorphisms \( \psi_d \) have the following naturality property whose proof is immediate.

Proposition 3.14. Let \( f: F(A) \to F(B) \) be a homomorphism with the property that for every \( a \in A \) there exists \( b \in B \) and an integer \( k \) such that \( f(a) = b^k \). Then for any integer \( d \) the following square commutes

\[
\begin{array}{ccc}
F(A) & \xrightarrow{\psi_d} & F(A) \\
\downarrow f & & \downarrow f \\
F(B) & \xrightarrow{\psi_d} & F(B)
\end{array}
\]

\( \square \)
The following result is central to this paper.

**Lemma 3.15.** For any $g \in F(A)$

$$\|\psi_d(g)i\_\mu = d \cdot \|g\|_\mu.$$

**Proof.** For any $a \in A$,

$$\|\psi_d(a)i\| = \|a^d\| \leq d \cdot \|a\| \leq d \cdot \mu(a).$$

By Proposition 2.11, $\psi$ is Lipschitz with constant $d$, i.e. for any $g \in F(A)$

$$\|\psi_d(g)i\|_\mu \leq d\|g\|_\mu.$$ 

For any $g \in F(A)$ let $\ell(g)$ denote the length of the reduced word $g$ representing $g$. We will use induction on $n = \ell(g)$ to prove that

$$\|\psi_d(g)i\|_\mu \geq d \cdot \|g\|_\mu.$$ 

The base case $n = 0$ is a triviality since $g = 1$. Suppose $\ell(g) = n$ and write $g = a_1 \cdots a_n$ in a reduced form where $a_i \in A \cup A^{-1}$. So $g$ is a word representing $g$. Then

$$\psi_d(g) = a_1^d \cdots a_n^d.$$ 

Crucially, the right hand side is a reduced word which we denote by $\psi_d(g)$. Choose a $\mu$-minimal cancellation sequence $b$ in $\psi_d(g)$. In order for the reduction process of $\psi_d(g) - b$ to the empty word to start, it is clear that $b$ must contain at least one syllable $a_i^d$. So we write $b = b_1a_i^db_2$.

Set $g'_1 = a_1 \cdots a_{i-1}$ and $g'_2 = a_{i+1} \cdots a_n$. Then $b_1 \subseteq \psi_d(g'_1)$ and $b_2 \subseteq \psi_d(g'_2)$. Notice that $g'$ is not necessarily reduced, but clearly $\ell(g') < \ell(g) = n$ where $g' \in F(A)$ is the element represented by $g'$. Also notice that since the norm is conjugation-invariant,

$$\|g\|_\mu = \|g'_1a_i g'_2\|_\mu \leq \|a_i\|_\mu + \|g'_1 g'_2\|_\mu = \mu(a_i) + \|g'\|_\mu.$$ 

Since $\psi_d(g) = \psi_d(g'_1)a_i^d\psi_d(g'_2)$ and since $b$ contains the entire syllable $a_i^d$, the word $\psi_d(g'_1)\psi_d(g'_2) - b_1b_2$ is equal to $\psi_d(g) - b$. Hence, $b_1b_2$ is a cancellation sequence in $\psi_d(g')$. By the induction hypothesis on $g'$ we get the following chain of inequalities

$$\|\psi_d(g)i\|_\mu = d \cdot \mu(a_i) + \mu(b_1) + \mu(b_2)$$

$$= d \cdot \mu(a_i) + \mu(b_1b_2)$$

$$\geq d \cdot \mu(a_i) + \|\psi_d(g')\|_\mu$$

$$\geq d \cdot \mu(a_i) + d \cdot \|g'\|_\mu$$

$$= d(\mu(a_i) + \|g'\|_\mu)$$

$$\geq d \cdot \|g\|_\mu.$$
This completes the proof. □

4. The groups $F_{\mathbb{R}}(A)$

**Running assumption:** In this section all sets $A, B, \ldots$ are assumed to be equipped with length functions $\mu_A, \mu_B, \ldots$ (see Section 2.10).

**Definition 4.1.** Let $A$ be a set. Denote

$$F_{\mathbb{R}}(A) \overset{\text{def}}{=} *_{A}\mathbb{R},$$

the free product of a family of copies of the group $\mathbb{R}$ indexed by $A$. More generally, if $R$ is any unital ring, write $F_{R}(A)$ for the free amalgamated product $*_{A}R$.

**Example 1.** $F(\mathbb{A}) = *_{\mathbb{A}}\mathbb{Z}$ and $F_{\mathbb{Q}}(\mathbb{A}) = *_{\mathbb{A}}\mathbb{Q}$ are naturally subgroups of $F_{\mathbb{R}}(\mathbb{A})$.

**Summary of this section:** The purpose of this section is to define a conjugation invariant norm $\| \|_\mu$ on $F_{\mathbb{R}}(A)$ which extends $\| \|_\mu$ on $F(A)$, see Definition 4.9. We give two alternative descriptions of this norm. One as a limit of the norm of $F(\mathbb{A}; \mu)$, see Theorem 4.18. The other as a cancellation norm in Theorem 4.26. Crucially, we define homomorphisms $\psi_t : F_{\mathbb{R}}(A) \to F_{\mathbb{R}}(A)$ which extend the homomorphisms $\psi_d$ in Definition 3.13 and the properties in Lemma 3.15.

**Definition 4.2.** Let $A(\mathbb{R})$ denote the set whose elements are the symbols $a(r)$ for all $a \in A$ and all $r \in \mathbb{R}$.

4.3. It is clear that $F_{\mathbb{R}}(A)$ has the following presentation. Generators are the elements of $A(\mathbb{R})$ subject to the relations

\[(2) \quad a(0) = 1 \quad \text{and} \quad a(r) \cdot a(s) = a(r + s).
\]

These relations clearly imply that for any $k \in \mathbb{Z}$

$$a(r)^k = a(kr)$$

and in particular $a(r)^{-1} = a(-r)$. Every element of $F_{\mathbb{R}}(A)$ is represented by a unique (possibly empty) word

$$a_1(r_1) \cdots a_n(r_n)$$

in which $r_i \neq 0$ for all $1 \leq i \leq n$ and $a_i \neq a_{i+1}$ for all $1 \leq i \leq n - 1$. Such words are called **reduced**.

The inclusions $\mathbb{Z} \subseteq \mathbb{R}$ and $\mathbb{Q} \subseteq \mathbb{R}$ give natural inclusions

$$F(\mathbb{A}) \leq F_{\mathbb{R}}(\mathbb{A})$$

$$F_{\mathbb{Q}}(\mathbb{A}) \leq F_{\mathbb{R}}(\mathbb{A}).$$

These are the subgroups generated by the symbols $a(r)$ where $r \in \mathbb{Z}$ or $r \in \mathbb{Q}$ respectively.
Definition 4.4. For any $t \in \mathbb{R}$ define a homomorphism
$$\psi_t : F_\mathbb{R}(A) \to F_\mathbb{R}(A)$$
which on generators is defined by $\psi_t(a(r)) = a(tr)$.

One checks that $\psi_t$ respects the relations (2) and therefore it is well defined, and that it is a homomorphism. Thus,

$$\psi_t(a_1(r_1) \cdots a_n(r_n)) = a_1(tr_1) \cdots a_n(tr_n).$$

Proposition 4.5. The following holds for $\psi_t : F_\mathbb{R}(A) \to F_\mathbb{R}(A)$.

(i) $\psi_0$ is the trivial homomorphism.
(ii) $\psi_1$ is the identity homomorphism.
(iii) $\psi_t \circ \psi_s = \psi_{st}$.

In particular $\psi_t$ is an isomorphism for all $t \neq 0$.

Proof. Immediate from (3). \qed

Any function $f : A \to B$ gives rise to a natural homomorphism
$$f_* : F_\mathbb{R}(A) \to F_\mathbb{R}(B)$$
which on generators in $A(\mathbb{R})$ has the effect $f_*(a(r)) = f(a)(r) \in B(\mathbb{R})$. Indeed, this assignment respects the relations (2) so $f_*$ is a well defined homomorphism.

Proposition 4.6. For any $f : A \to B$ the following square commutes for any $t \in \mathbb{R}$.

$$\begin{array}{c}
F_\mathbb{R}(A) \xrightarrow{f_*} F_\mathbb{R}(B) \\
\downarrow \psi_t \\
F_\mathbb{R}(A) \xrightarrow{f_*} F_\mathbb{R}(B)
\end{array}$$

Proof. It suffices to check on generators $a(r) \in A(\mathbb{R})$. Indeed,
$$f_*(\psi_t(a(r))) = f_*(a(tr)) = f(a)(tr) = \psi_t(f(a)(r)) = \psi_t(f_*(a(r))).$$ \qed

Definition 4.7. Given a length function $\mu$ on $A$ we extend it to a length function $\mu_\mathbb{R}$ on $A(\mathbb{R})$ by
$$\mu_\mathbb{R}(a(r)) = |r| \cdot \mu(a).$$

4.8. The natural map $A(\mathbb{R}) \to F_\mathbb{R}(A)$ is not injective. However the only obstruction is that the preimage of $1 \in F_\mathbb{R}(A)$ is the set $\{a(0)\}_{a \in A}$. Throughout, we will abuse notation and write $A(\mathbb{R})$ for its image in $F_\mathbb{R}(A)$. 

Since $\mu_\mathbb{R}(a(0)) = 0$ for all $a \in A$, the function $\mu_\mathbb{R}$ factors uniquely through the image of $A(\mathbb{R})$ in $F_\mathbb{R}(A)$, and we will abuse notation and write $\mu_\mathbb{R}$ for both functions.

**Definition 4.9.** Let $\mu$ be a length function on $A$. Let $F_\mathbb{R}(A; \mu)$ denote the group $F_\mathbb{R}(A)$ equipped with the conjugation invariant pseudonorm $\| \|_{\mu_\mathbb{R}}$ in Section 2.10 obtained from the length function $\mu_\mathbb{R}$ on $A(\mathbb{R}) \subseteq F_\mathbb{R}(A)$.

The basic naturality property of this construction is:

**Proposition 4.10.** Let $f : A \to B$ be a proto-Lipschitz function with constant $C$ (Definition 3.11). Then $f_* : F_\mathbb{R}(A) \to F_\mathbb{R}(B)$ is Lipschitz with constant $C$.

**Proof.** For any $a(r) \in A(\mathbb{R})$,

$$\|f_*(a(r))\|_{F_\mathbb{R}(B)} = \|f(a(r))\|_{F_\mathbb{R}(B)} \leq \mu_B, \mathbb{R}(f(a)(r)) = |r| \cdot \mu_B(f(a)) \leq C|r|\mu_A(a) = C\mu_\mathbb{R}(a(r)).$$

The result follows from Proposition 2.11.

**Lemma 4.11.** $F_Q(A)$ is dense in $F_\mathbb{R}(A)$.

**Proof.** Given $\alpha = a_1(r_1) \cdots a_n(r_n)$ in $F_\mathbb{R}(A)$ consider elements $\beta \in F_Q(A)$ of the form $\beta = a_1(q_1) \cdots a_n(q_n)$ where $q_i \in \mathbb{Q}$. By Lemma 2.4 $\|\alpha^{-1}\beta\|_{F_\mathbb{R}(A)} \leq \sum_{i=1}^n \mu(a_i) \cdot |r_i - q_i|$ which is arbitrarily small for a suitable choice of $q_i$.

The next lemma is fundamental to this paper. Compare with Lemma 3.15.

**Lemma 4.12.**

(a) For any $w \in F_\mathbb{R}(A)$ and any $t \in \mathbb{R}$

$$\|\psi_t(w)\|_{F_\mathbb{R}(A)} = |t| \cdot \|w\|_{F_\mathbb{R}(A)}.$$

(b) For any $w \in F_\mathbb{R}(A)$ there exists $C_w \geq 0$ such that for any $s, t \in \mathbb{R}$

$$\|\psi_s(w)^{-1}\psi_t(w)\|_{F_\mathbb{R}(A)} \leq C_w |s - t|.$$

**Proof.**

(a) For any $a(r) \in A(\mathbb{R})$,

$$\|\psi_t(a(r))\|_{F_\mathbb{R}(A)} = \|a(tr)\|_{F_\mathbb{R}(A)} \leq |tr|\mu(a) = |t|\mu_\mathbb{R}(a(r)).$$

Proposition 2.11 shows that $\psi_t$ is Lipschitz with constant $|t|$. 
If $t = 0$ then $\psi_t$ is trivial so $\|\psi_t(w)\| = t\|w\|$ trivially. So we assume $t \neq 0$. By Proposition 4.5 $\psi_{1/t} \circ \psi_t$ is the identity. Since $\psi_{1/t}$ is Lipschitz with constant $\frac{1}{|t|}$, for any $w \in F_R(A)$

$$\|w\| = \|\psi_{1/t}(\psi_t(w))\| \leq \frac{1}{|t|}\|\psi_t(w)\| \leq \frac{1}{|t|}|t| \cdot \|w\| = \|w\|.$$  

Therefore the inequalities are equalities and in particular $\|\psi_t(w)\| = |t| \cdot \|w\|.$

(b) Let $w = a_1(r_1) \cdots a_k(r_k)$ be a word representing $w$. Set $C_w = \sum_i |r_i|\mu(a_i)$. Notice that $\psi_t(w)$ is represented by $\prod_i a_i(tr_i)$ and $\psi_s(w)$ by $\prod_i a_i(sr_i)$. Lemma 2.4 implies

$$\|\psi_t(w)^{-1}\psi_s(w)\| \leq \sum_i \|a_i(tr_i)^{-1}a_i(sr_i)\| = \sum_i \|a_i(sr_i - tr_i)\| \leq \sum_i |sr_i - tr_i|\mu(a_i) = C_w|s - t|.$$  

Our next goal is to give alternative, more explicit, descriptions of the norm of $F_R(A)$.

**Definition 4.13.** For any $x \in \mathbb{R}$ set

$$\|x\| = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \quad \text{(the floor of } x) \\ \lceil x \rceil & \text{if } x \leq 0 \quad \text{(the ceiling of } x) \end{cases}.$$  

We leave it to the reader to verify that for all $x, y \in \mathbb{R}$ and $p \geq 0$

$$\|-x\| = -\|x\|,$$

$$\|\|x\| - \|y\| - \|x - y\|\| \leq 1,$$

$$\|\|px\| - p\|x\|\| \leq p.$$  

Concatenation makes the set of words in the alphabet $A(\mathbb{R})$ with concatenation of words as the monoidal operation, we write $ww'$ or $w \cdot w'$ for the concatenation. It is also equipped with an involution $w \mapsto w^{-1}$ where $(a_1(r_1) \cdots a_n(r_n))^{-1} = a_n(-r) \cdots a_1(-r_1)$, thus $w^{-1}$ represents the inverse of the element in $F_R(A)$ that $w$ represents.

**Definition 4.14.** Let $w = a_1(r_1) \cdots a_k(r_k)$ be a word in the alphabet $A(\mathbb{R})$. For any $t \in \mathbb{R}$ let $w\{t\}$ be the following element of $F(A)$

$$w\{t\} = a_1^{\lfloor tr_1 \rfloor} \cdots a_k^{\lfloor tr_k \rfloor}.$$  

Given $t \in \mathbb{R}$, the assignment $w \mapsto w\{t\}$ gives a function from the monoid of words in the alphabet $A(\mathbb{R})$ to the free group $F(A)$. We obtain the following equalities, the second follows since $\|tr\| = -\|tr\|$.

$$ww'\{t\} = w\{t\}w'\{t\}$$

Proposition 4.15. Let $w$ be a word in the alphabet $A(\mathbb{R})$ representing $w \in F_\mathbb{R}(A)$. Then

$$\lim_{t \to \infty} \psi_{1/t}(w\{t\}) = w$$

in the metric space $F_\mathbb{R}(A)$.

Proof. Say, $w = a_1(r_1) \cdots a_n(r_n)$. By Definition 4.14 and (3),

$$\psi_{1/t}(w\{t\}) = a_1(\|tr_1\|/t) \cdots a_n(\|tr_n\|/t).$$

Lemma 2.4 then implies

$$\|w^{-1} \cdot \psi_{1/t}(w\{t\})\|_{F_\mathbb{R}(A)} \leq \sum_{i=1}^n \|a_i(\|tr_i\|/t - r_i)\|_{F_\mathbb{R}(A)} \leq \sum_{i=1}^n \|\|tr_i\|/t - r_i\|_{F_\mathbb{R}(A)} \leq \mu(a_i).$$

This tends to 0 as $t \to \infty$. □

Definition 4.16. Let $T \subset (0, \infty)$ be an unbounded subset of $\mathbb{R}$. Let $(X, d)$ be a pseudometric space. A function $f : T \to X$ is called Cauchy if for any $\epsilon > 0$, if $t, t' \in T$ are sufficiently large then $|f(t) - f(t')| < \epsilon$.

It is clear that in this case, if $(X, d)$ is complete then $\lim_{t \in T} f(t)$ exists.

Lemma 4.17. Let $W$ denote the set of words in the alphabet $A(\mathbb{R})$.

(a) For any $w \in W$

$$f_w(t) = \frac{1}{t} \|w\{t\}\|_{F(A)}, \quad (t > 0)$$

is a Cauchy function relative to $T = (0, \infty)$.

(b) If $w, w' \in W$ represent the same element in $F_\mathbb{R}(A)$ then

$$\lim_{t \to \infty} \frac{1}{t} \|w\{t\}\|_{F(A)} = \lim_{t \to \infty} \frac{1}{t} \|w'\{t\}\|_{F(A)}.$$

Proof. [a] Throughout we will write $\| \|$ for the norm in $F(A; \mu)$. Fix a word $w = a_1(r_1) \cdots a_k(r_k)$. For any $t > 0$, $\|w\{t\}\| = \|a_1^{\|tr_1\|} \cdots a_k^{\|tr_k\|}\|$, hence

$$\|w\{t\}\| \leq \sum_{i=1}^k \mu(a_i) \|tr_i\| \leq t \sum_{i=1}^k \mu(a_i)|r_i|.$$
For any \( t \geq 1 \) set \( g_w(t) = \frac{1}{|t|} \| w(t) \| \). Then
\[
|g_w(t) - f_w(t)| = \left( \frac{1}{|t|} - \frac{1}{t} \right) \| w(t) \| \leq \left( \frac{1}{|t|} - 1 \right) \sum_i \mu(a_i)|r_i| \xrightarrow{t \to \infty} 0.
\]

Therefore \( f_w \) and \( g_w \) are equivalent and it suffices to show that \( g_w \) is Cauchy. Suppose that \( s, t > 1 \). Lemmas 3.15 and 2.4 imply
\[
\left| \frac{1}{|t|} \| w(t) \| - \frac{1}{|s|} \| w(s) \| \right| = \frac{1}{|t|} |s| \cdot \| w(t) \| - |t| \cdot \| w(s) \| |
\leq \frac{1}{|t|} \left| \| \psi_{|s|}(w(t)) \| - \| \psi_{|t|}(w(s)) \| \right|
\leq \frac{1}{|t|} \left| \| \psi_{|t|}(w(s))^{-1} \psi_{|s|}(w(t)) \| \right|
\leq \frac{1}{|t|} \sum_{i=1}^n \| a_i^{[s] \cdot \| tr_i \| - |t| \cdot \| sr_i \|} \|
= \sum_{i=1}^n \mu(a_i) \cdot \left| \| tr_i \| - \frac{|sr_i|}{|s|} \right| \xrightarrow{s,t \to \infty} 0.
\]

It suffices to show that \( \frac{1}{t} \| w(t)^{-1} w'(t) \| \xrightarrow{t \to \infty} 0 \). By the presentation (2) of \( F_\mathbb{R}(A) \) it suffices to prove this in the two cases: (i) \( w = w_1 w_2 \) and \( w' = w_1 \cdot a(0) \cdot w_2 \), and (ii) \( w = w_1 \cdot a(r + s) \cdot w_2 \) and \( w' = w_1 \cdot a(r) a(s) \cdot w_2 \).

Case (i) is trivial since \( a(0)\{t\} = 1 \in F(A) \) for any \( t \), so \( w\{t\} = w\{t\} \). Case (ii) follows from (5) and the conjugation invariance of \( \| \|_{F(A)} \),
\[
\frac{1}{t} \| w(t)^{-1} w'(t) \| =
\leq \frac{1}{t} \| (w_1 \{t\} \cdot a^{[tr + ts]} \cdot w_2 \{t\})^{-1} \cdot (w_1 \{t\} \cdot a^{[tr]} a^{[ts]} \cdot w_2 \{t\}) \|
= \frac{1}{t} \| a^{[tr + ts - tr + ts]} \|
= \frac{1}{t} \mu(a) \| tr \| + \| ts \| - \| tr + ts \| \xrightarrow{t \to \infty} 0.
\]

\[ \square \]

**Theorem 4.18.** For any \( w \in F_\mathbb{R}(A) \) and any word \( w \) in the alphabet \( A(\mathbb{R}) \) representing it,
\[
\| w \|_{F_\mathbb{R}(A;\mu)} = \lim_t \frac{1}{t} \| w\{t\} \|_{F(A;\mu)}.
\]

**Proof.** We omit \( \mu \) from the notation. We also write \( \| \| \) for the norm in \( F(A) \). Consider an arbitrary presentation \( w = u_1 \cdots u_n \) for some \( u_i \in
$F_\mathbb{R}(A)$ conjugate to $a_i(r_i)$ in $A(\mathbb{R})$. Choose words $u_i$ in the alphabet $A(\mathbb{R})$ representing $u_i$ of the form $v_i a_i(r_i) v_i^{-1}$ for some words $v_i$. Then $u_i \{t\}$ is conjugate to $a_i^{[tr_i]}$ in $F(A)$. Set $w' = u_1 \cdots u_n$. Then $w'$ represents $w$ and

$$\frac{1}{t} \|w' \{t\}\| = \frac{1}{t} \| \prod_{i=1}^{n} u_i \{t\} \| \leq \frac{1}{t} \sum_{i=1}^{n} \|a_i^{[tr_i]}\| = \sum_{i=1}^{n} \|a_i^{[tr_i]}\| \mu(a_i)$$

Taking the limit we get

$$\lim_{t \to 0} \frac{1}{t} \|w \{t\}\|_{F(A)} = \lim_{t \to 0} \frac{1}{t} \|w' \{t\}\|_{F(A)} \leq \sum_i \mu_R(a_i(r_i)).$$

Since the presentation $w = u_1 \cdots u_n$ was arbitrary,

$$\lim_{t \to 0} \frac{1}{t} \|w \{t\}\|_{F(A)} \leq \|w\|_{F_R(A)}.$$

We prove the reverse inequality. Write $w = a_1(r_1) \cdots a_n(r_n)$. Consider an arbitrary $\epsilon > 0$. Observe that for all sufficiently large $t$,

$$\sum_{i=1}^{n} |r_i - \frac{[tr_i]}{t}| \mu(a_i) < \epsilon.$$

By Corollary 3.5, $w \{t\} = u_1 \cdots u_k$ where each $u_i$ is conjugate to $b_i^{\pm 1}$ for some $b_i \in A$ and $\|w \{t\}\| = \sum_{i=1}^{k} \mu(b_i)$. By definition, $w \{t\} = a_1^{[tr_1]} \cdots a_n^{[tr_n]}$ so $\psi_{1/t}(w \{t\}) = a_1 (\frac{[tr_1]}{t}) \cdots a_n (\frac{[tr_n]}{t})$. By Lemma 2.4 if $t \gg 0$

$$\|w^{-1} \cdot \psi_{1/t}(w \{t\})\|_{F_R(A)} \leq \sum_{i=1}^{n} \|a_i (\frac{[tr_i]}{t}) - r_i\|_{F_R(A)}$$

$$\leq \sum_{i=1}^{n} \|\frac{[tr_i]}{t} - r_i\| \mu(a_i) < \epsilon.$$

Since $\psi_{1/t}(w \{t\}) = \prod_{i=1}^{k} \psi_{1/t}(u_i)$ and $\psi_{1/t}(u_i)$ is conjugate to $b_i (\frac{1}{t})$,

$$\|\psi_{1/t}(w \{t\})\|_{F_R(A)} \leq \sum_{i=1}^{k} \|b_i (\frac{1}{t})\|_{F_R(A)} \leq \sum_{i=1}^{k} \frac{1}{t} \mu(b_i) = \frac{1}{t} \|w \{t\}\|_{F(A)}.$$

Gathering all this we get

$$\|w\|_{F_R(A)} < \|\psi_{1/t}(w \{t\})\|_{F_R(A)} + \epsilon \leq \frac{1}{t} \|w \{t\}\|_{F(A)} + \epsilon.$$

Taking the limit we get

$$\|w\|_{F_R(A)} \leq \lim_{t \to 0} \frac{1}{t} \|w \{t\}\|_{F(A)} + \epsilon.$$
Corollary 4.19. The pseudonorm on $F(A)$ as a subgroup of $F_{\mathbb{R}}(A)$ coincides with the pseudonorm $\| \|_{F(A)}$ on $F(A)$ (Definition 3.11).

Proof. Consider $w \in F(A)$. Then $w = a_1^{d_1} \cdots a_n^{d_n}$, and as an element in $F_{\mathbb{R}}(A)$ it is represented by the word $w = a_1(d_1) \cdots a_n(d_n)$ where $d_i \in \mathbb{Z}$. For any $k \in \mathbb{Z}$ observe that $w\{k\} = a_1^{kd_1} \cdots a_n^{kd_n} = \psi_k(w)$. It follows from Lemma 3.15 and Theorem 4.18 that

$$\|w\|_{F_{\mathbb{R}}(A)} = \lim_{t \to \infty} \frac{1}{t} \|w\{t\}\|_{F(A)} = \lim_{k \to \infty} \frac{1}{k} \|w\{k\}\|_{F(A)} = \lim_{k \to \infty} \frac{1}{k} \|\psi_k(w)\|_{F(A)} = \|w\|_{F(A)}.$$
4.23. Recall that \( A^\# = \{ a \in A : \mu(a) > 0 \} \). Given a word \( w \) in the alphabet \( A(\mathbb{R}) \) we form a word \( w^\# \) in the alphabet \( A^\#(\mathbb{R}) \) by deleting from \( w \) all the symbols \( a(r) \) with \( \mu(a) = 0 \). The assignment \( w \mapsto w^\# \) respects the relations (2) so it induces a homomorphism \( F_{\mathbb{R}}(A) \xrightarrow{w \mapsto w^\#} F(A^\#) \).

If \( u \) is a sequence in \( w \), let \( u^\# \) denote the sequence in \( w \) obtained by adjoining to \( u \) all the symbols \( a(r) \) in \( w \) with \( \mu(a) = 0 \). One easily verifies that the equations (1) hold.

**Lemma 4.24.** Let \( w \in F_{\mathbb{R}}(A) \) be represented by a word \( w \) in the alphabet \( A^\#(\mathbb{R}) \subseteq A(\mathbb{R}) \). If \( \|w\|_{F_{\mathbb{R}}(A)} = 0 \) then \( w = 1 \).

*Proof.* The reduced form of \( w \) is another word in the alphabet \( A^\#(\mathbb{R}) \) representing \( w \), therefore we may assume \( w \) is reduced. Then \( w = a_1(r_1) \cdots a_n(r_n) \) for some \( r_i \neq 0 \), and \( a_i \neq a_{i+1} \) and \( n \geq 0 \).

Suppose that \( n \geq 1 \). For any \( 1 \leq i \leq n \), it is clear that \( \left\| r_i t \right\|_t > \frac{|r_i|}{2} \) for all \( t \gg 0 \). Therefore,

\[
\varepsilon := \min\left\{ \frac{|r_i|}{2} : 1 \leq i \leq n \right\} < \min\left\{ \left| \frac{|r_i|}{t} \right| : 1 \leq i \leq n \right\} \quad (t \gg 0).
\]

Clearly, \( \|r_i t\| \neq 0 \) for all \( i \) if \( t \gg 0 \). Therefore

\[
w\{t\} = a_1^{[r_1]} \cdots a_n^{[r_n]}
\]

is a non trivial reduced word in \( F(A) \). If \( u \subseteq w\{t\} \) is a cancellation sequence then \( u \) must contain at least one syllable in \( w\{t\} \) so

\[
\mu(u) \geq \min_i \left| \left| r_i t \right| \right| : \mu(a_i) \geq t\varepsilon \quad (t \gg 0).
\]

Then \( \|w\{t\}\|_{F(A)} \geq t\varepsilon \) by Proposition 4.18. It follows from Theorem 4.18 that \( \|w\|_{F_{\mathbb{R}}(A)} \geq \varepsilon > 0 \) which contradicts the hypothesis. Therefore \( n = 0 \), whence \( w = 1 \). \( \square \)

The next observation is left to the reader.

**Proposition 4.25.** Let \( g_1, \ldots, g_n \) and \( h_0, \ldots, h_n \) be elements in a group \( \Gamma \) and set \( \gamma = h_0 g_1 h_1 g_2 \cdots g_n h_n \). Then \( \gamma = h_0 \cdots h_n \cdot \tilde{g} \) where \( \tilde{g} \) is a product of conjugates of \( g_1, \ldots, g_n \). \( \square \)

**Theorem 4.26.** Let \( w \) be a word in the alphabet \( A(\mathbb{R}) \) representing \( w \in F_{\mathbb{R}}(A) \). Then

\[
\|w\|_{F_{\mathbb{R}}(A)} = \min\{ \mu(u) : u \text{ is a cancellation sequence in } w \}
\]

We call a cancellation sequence in \( w \) which attains the minimum \( \mu \)-minimal.
Proof. Write \( w = a_1(r_1) \cdots a_n(r_n) \). If \( u = a_1(s_1) \cdots a_n(s_n) \) is a cancellation sequence then by Proposition 4.25 \( w \) is a product of conjugates of \( a_i(s_i) \) and therefore

\[
\|w\|_{F_k(A)} \leq \sum_i \|a_i(s_i)\|_{F_k(A)} \leq \sum_i \mu_R(a_i(s_i)) = \sum_i \|s_i\|_\mu(a_i).
\]

Since this holds for any cancellation sequence

\[
\|w\|_{F_k(A)} \leq \inf \{ \mu(u) : u \text{ is a cancellation sequence in } w \}.
\]

It remains to find a cancellation sequence \( u \) such that \( \mu(u) = \|w\|_{F_k(A)} \).

For every \( t > 0 \) we note that

\[
w\{t\} = a_1^{r_1t} \cdots a_n^{r_nt} \in F(A).
\]

Choose a \( \mu \)-minimal cancellation sequence \( v(t) \) in \( w\{t\} \). By applying the homomorphism \( F(A) \xrightarrow{w \mapsto w^\#} F(A^\#) \) from Section 3.3 it follows from (1) that \( u(t) := v(t)^\# \) is also a cancellation sequence in \( w\{t\} \) and since \( \mu(u(t)) = \mu(v(t)) \), it is also \( \mu \)-minimal. We write \( u(t) = a_1^{m_1(t)} \cdots a_n^{m_n(t)} \).

By definition \( m_i(t) \in [0, \|r_it\|] \) and \( \prod_i a_i^{\|r_it\|-m_i(t)} \) is the trivial element in \( F(A) \) and \( \|w\{t\}\|_{F(A)} = \sum_i \|m_i(t)\|_\mu(a_i) \). Furthermore, \( m_i(t) = \|r_it\| \) whenever \( \mu(a_i) = 0 \).

Now, \( \frac{m_i(t)}{t} \in [0, \|r_it\|/t] \subseteq [0, r_i) \). By the compactness of \( \prod_{i=1}^n [0, r_i) \) there exists a sequence \( t_1, t_2, \ldots \) such that \( \lim_k t_k = \infty \) and such that the limits \( s_i := \lim_k \frac{m_i(t_k)}{t_k} \) exist. Consider the word in the alphabet \( A(\mathbb{R}) \)

\[
u = a_1(s_1) \cdots a_n(s_n).
\]

This is a sequence in \( w \) since \( s_i \in [0, r_i) \). Let \( w' \in F_\mathbb{R}(A) \) be the element represented by

\[
w - u = a_1(r_1 - s_1) \cdots a_n(r_n - s_n).
\]

Observe that if \( \mu(a_i) = 0 \) then \( r_i = s_i \) so \( w - u \) and \( (w - u)^\# \) represent the same element in \( F_\mathbb{R}(A) \). By Proposition 4.25 for any \( k \geq 1 \) we get

\[
w\{t\} = \prod_i a_i^{\|t_i(r_i-s_i)\|} = \prod_i a_i^{\|t_k r_i\|-m_i(t_k)} \cdot a_i^{\|t_k(r_i-s_i)\|+m_i(t_k)-\|t_k r_i\|}
\]
is a product of conjugates of $a_i^{{\|t_k(r_i-s_i)\|+m_i(t_k)-\|tkr_i\|}}$. By Theorem 4.18
\[
\|w\|_{F_k(A)} = \lim_k \frac{1}{t_k} \|w\{t_k\}\|_{F(A)}
\leq \lim_k \frac{1}{t_k} \sum_i \|a_i^{{\|t_k(r_i-s_i)\|+m_i(t_k)-\|tkr_i\|}}\|_{F(A)}
= \lim_k \sum_i \left|\frac{\|t_k(r_i-s_i)\|+m_i(t_k)-\|tkr_i\|}{t_k}\right| \mu(a_i) = 0.
\]
Lemma 4.24 applied to $(w-u)^\#$ implies that $w$ is the trivial element, so $u$ is a cancellation sequence in $w$. By Theorem 4.18
\[
\mu(u) = \sum_i |s_i| \mu(a_i) = \sum_i \lim_k \frac{m_i(t_k)}{t_k} \mu(a_i) = \lim_k \frac{1}{t_k} \|w\{t_k\}\|_{F(A)} = \|w\|_{F_k(A)}.
\]
This completes the proof. \qed

Corollary 4.27. Let $\mu^\#$ denote the restriction of $\mu$ to $A^\#$, see Section 4.26. Then the pseudonorm of $F_k(A^\#)$ as a subgroup of $F_k(R;\mu)$ coincides with the pseudonorm induced by $\mu^\#$. Moreover, $\|w^{-1}w^\#\|_{F_k(A)} = 0$ for every $w \in F_k(A)$ and the pseudonorm on $F_k(A^\#)$ is a norm.

Proof. Given $w \in F_k(A^\#)$, then as an element in $F_k(R;\mu)$ we can choose a representing word in the alphabet $A(R)$ consisting only of symbols in $A^\#(R)$. Theorem 4.26 implies that $\|w\|_{F_k(A^\#)} = \|w\|_{F_k(A)}$. The pseudonorm on $F_k(A^\#)$ is a norm by Lemma 4.24. By Proposition 4.25 any $w \in F_k(A)$ is the product of $w^\#$ with an element $v$ which is a product of conjugates of elements in $A \setminus A^\#$. Then $\|w^{-1}w^\#\|_{F_k(A)} = \|v\|_{F_k(A)} = 0$. \qed

Proposition 4.28. $F_k(A)$ is a geodesic pseudometric space.

Proof. We write $\|\|$ for the norm of $F_k(A)$. Since the metric on $F_k(A)$ is invariant with respect to translations, it suffices to find for every $w \in F_k(A)$ a geodesic to the identity element. Write $w = a_1(r_1)\cdots a_n(r_n)$. Let $u = a_1(s_1)\cdots a_n(s_n)$ be a $\mu$-minimal cancellation sequence, see Theorem 4.26. Define $\alpha: [0, 1] \to F_k(A)$ by
\[
\alpha(t) = a_1(r_1 - s_1 t)\cdots a_n(r_n - s_n t).
\]
It is clear that $\alpha(0) = w$ and $\alpha(1) = 1$. Consider some $0 \leq t' < t \leq 1$. Then $\alpha(t) = \prod_i a_i(r_i - ts_i)$ and $\alpha(t') = \prod_i a_i(r_i - ts'_i)$. By Lemma 2.4 and the relations (2)
\[
\|\alpha(t)^{-1}\alpha(t')\| \leq \sum_i \|a_i(t's_i - ts_i)\| \leq |t-t'| \sum_i \mu(a_i)|s_i| = |t-t'| \cdot \|w\|.
\]
In particular, given $0 \leq t_1 \leq t_2 \leq 1$ we get
\[
\|w\| = \|\alpha(0)\alpha(1)^{-1}\|
\leq \|\alpha(0)\alpha(t_1)^{-1}\| + \|\alpha(t_1)\alpha(t_2)^{-1}\| + \|\alpha(t_2)\alpha(1)^{-1}\|
\leq t_1\|w\| + (t_2 - t_1)\|w\| + (1 - t_2)\|w\|
= \|w\|.
\]
Hence, equality holds everywhere and, in particular,
\[
\|\alpha(t_1)\alpha(t_2)^{-1}\| = |t_2 - t_1| \cdot \|w\|.
\]
It follows that $\alpha$ is a geodesic. \hfill \Box

5. The group of asymptotic directions

**Running assumption:** As in the previous section all sets $A, B, \ldots$ are assumed to be equipped with length functions $\mu_A, \mu_B, \ldots$ (see Section 2.10). Thus, $F(A)$ and $F_{\mathbb{R}}(A)$ are equipped with the associated conjugation-invariant word pseudonorms from Sections 3 and 4.

**Definition 5.1.** A **scaling set** is an unbounded subset of the interval $(0, \infty)$.

**Notation.** Throughout $T, T'$ etc. will denote scaling sets.

Recall from Definition 4.16 that a function $f : T \to X$ where $(X, d)$ is a pseudometric space is called Cauchy if
\[
\sup\{d(f(s_1), f(s_2)) : s_1, s_2 \in T \cap [t, \infty)\} \to_{t \to \infty} 0.
\]

**Definition 5.2.** Let $T \subseteq (0, \infty)$ be unbounded. A function
\[
w : T \to F(A; \mu)
\]
is called an **asymptotic direction** if the function $f_w : T \to F_{\mathbb{R}}(A; \mu)$ defined by
\[
f_w(t) = \psi_{1/t}(w(t))
\]
is Cauchy. Let
\[
\mathcal{D}(A, T; \mu) \subseteq \prod_{t \in T} F(A)
\]
denote the group of all asymptotic directions in $F(A; \mu)$ with respect to $T$. When $\mu$ is understood from the context we write $\mathcal{D}(A, T)$. When $T = (0, \infty)$ we write $\mathcal{D}(A; \mu)$ or $\mathcal{D}(A)$. 
To justify the terminology we need to show that $\mathcal{D}(A)$ is a subgroup of $\prod_{t \in T} F(A)$. This follows easily from the triangle inequality, the symmetry of the norm and the fact that $\psi_t$ are homomorphisms. The details are left to the reader.

If $x_1, x_2, \ldots$ is a Cauchy sequence in a pseudo metric space $(X, d)$ and $y \in X$ is a basepoint, then $d(x_n, y)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, this justifies the following definition.

**Definition 5.3.** For any $w \in \mathcal{D}(A, T)$ define a conjugation invariant pseudonorm on $\mathcal{D}(A, T)$ by

$$\|w\|_{\mathcal{D}(A, T)} = \lim_{t \in T} \|\psi_{1/t}(w(t))\|_{F(A)}.$$ 

We write $\|\cdot\|_{\mathcal{D}(A)}$ when $T$ (and $\mu$) is understood from the context.

That $\|\cdot\|_{\mathcal{D}(A)}$ is a conjugation-invariant pseudonorm follows easily from the fact that $\|\cdot\|_{F(A)}$ has this property and that $\psi_t$ is a homomorphism. It follows from Lemma 4.12 and Corollary 4.19 that

$$\|w\|_{\mathcal{D}(A)} = \lim_{t \in T} \frac{1}{t} \|w(t)\|_{F(A)}.$$ 

With the notation of Section 2.3 we make the following definition.

**Definition 5.4.** The group of asymptotic directions on $A$ is

$$\hat{\mathcal{D}}(A, T; \mu) = \mathcal{D}(A, T; \mu)/\mathcal{D}(A, T; \mu)_0.$$ 

We will omit $T$ and $\mu$ from the notation whenever they are understood from the context. Let $\|\cdot\|_{\hat{\mathcal{D}}(A)}$ be the induced conjugation-invariant norm. We will denote by $[w]$ the image of $w \in \mathcal{D}(A)$ in $\hat{\mathcal{D}}(A)$.

**Proposition 5.5.** The group $\mathcal{D}(A, T)$ is a complete pseudo metric space and hence, $\hat{\mathcal{D}}(A, T)$ is a complete metric spaces.

**Proof.** Let $w_1, w_2, \ldots$ be a Cauchy sequence in $\mathcal{D}(A)$. We will find a pseudo limit $v$ in $\mathcal{D}(A)$. Since a Cauchy sequence converges if and only if it contains a convergent subsequence, we may assume by passage to a subsequence that

$$\|w_n^{-1} w_{n+1}\|_{\mathcal{D}(A)} < \frac{1}{2^{n+1}}$$

for all $n \geq 1$. It follows from the definition of $\|\cdot\|_{\mathcal{D}(A)}$ that there exist $r_1, r_2, \ldots$ such that for any $n \geq 1$ the following hold.

(a) For any $s, t \in T$ such that $s, t > r_n$

$$\|\psi_{1/s}(w(s))^{-1} \cdot \psi_{1/t}(w_n(t))\|_{F(A)} < \frac{1}{2^n}.$$ 

(This is because $w_n \in \mathcal{D}(A)$).
(b) For any \( t \in T \) such that \( t > r_n \)
\[
\| \psi_{1/t}(w_{n+1}(t) \cdot w_n(t)^{-1}) \|_{F_k(A)} < \frac{1}{2^{n+1}}.
\]
(This is because \( \|w_{n+1}w_n^{-1}\|_{\mathcal{D}(A)} < \frac{1}{2^{n+1}} \).)

By increasing the \( r_n \)'s we may arrange that \( 0 < r_1 < r_2 < \ldots \) and that \( r_n \to \infty \). For convenience we set \( r_0 = 0 \) and \( w_0 = 1 \), the constant function \( w_0: T \to F(A) \). Observe that for any \( j \geq i \geq 1 \) and any \( t \in T \cap (r_{j-1}, \infty) \)
\[
\|\psi_{1/t}(w_i(t))^{-1}w_j(t)\|_{F_k(A)} \leq \sum_{k=i}^{j-1} \|\psi_{1/t}(w_k(t))^{-1}w_{k+1}(t)\|_{F_k(A)} < \sum_{k=i}^{j-1} \frac{1}{2^{k+1}} < \frac{1}{2^i}.
\]

Define \( v: T \to F(A) \) as follows. For any \( t \in T \) there exists a unique \( n \geq 0 \) such that \( r_n < t \leq r_{n+1} \). Set
\[
v(t) = w_n(t).
\]
We claim that \( v \in \mathcal{D}(A) \). To see this, given \( \epsilon > 0 \) choose \( n \geq 1 \) such that \( \frac{1}{2^n} < \epsilon \). Consider some \( s, t > r_n \) and suppose that \( s \geq t \). Then \( r_i < t \leq r_{i+1} \) and \( r_j < s \leq r_{j+1} \) for some \( j \geq i \geq n \). Since \( n \geq 1 \),
\[
\|\psi_{1/t}(v(t))^{-1}\psi_{1/s}(v(s))\|_{F_k(A)} = \|\psi_{1/t}(w_i(t))^{-1}\psi_{1/s}(w_j(s))\|_{F_k(A)}
\leq \|\psi_{1/t}(w_i(t))^{-1}\psi_{1/s}(w_i(s))\|_{F_k(A)} + \|\psi_{1/s}(w_i(s))^{-1}w_j(s)\|_{F_k(A)}
< \frac{1}{2^i} + \frac{1}{2^n} < \frac{1}{2^n} \leq \frac{1}{2^n} < \epsilon.
\]

It remains to show that \( w_n \to v \) in \( \mathcal{D}(A) \). Given \( \epsilon > 0 \) choose \( m \geq 1 \) such that \( \frac{1}{2^m} < \epsilon \). Suppose that \( n \geq m \). If \( t > r_n \) then \( r_i < t \leq r_{i+1} \) for some \( i \geq n \) and since \( n \geq m \geq 1 \)
\[
\|\psi_{1/t}(w_n(t)^{-1}v(t))\|_{F_k(A)} = \|\psi_{1/t}(w_n(t)^{-1}w_i(t))\|_{F_k(A)} < \frac{1}{2^i} \leq \frac{1}{2^m}.
\]
Taking the limit \( t \to \infty \) we get for all \( n \geq m \)
\[
\|w_n^{-1}v\|_{\mathcal{D}(A)} \leq \frac{1}{2^m} < \epsilon.
\]
We deduce that \( w_n \to v \). \( \square \)

Recall that elements of \( \mathcal{D}(A) \) are functions \( w: T \to F(A) \) such that \( f_w(t) = \psi_{1/t}(w(t)) \) is a Cauchy function into \( F_k(A) \), hence is convergent in its completion. We will now define \( \hat{\Psi}: \hat{\mathcal{D}}(A) \to \hat{F}_k(A) \) and \( \hat{v}: F_k(A) \to \hat{\mathcal{D}}(A) \) and show that they are well defined in Proposition 5.8 below.
Definition 5.6. Define a homomorphism $\Psi : \mathcal{D}(A,T) \to \hat{F}_R(A)$ by
$$\Psi(w) = \lim_{t \to T} \psi_{1/t}(w(t)).$$

Define a homomorphism $\hat{\Psi} : \hat{\mathcal{D}}(A) \to \hat{F}_R(A)$ by
$$\hat{\Psi}([w]) = [\Psi(w)].$$

Definition 5.7. Define a homomorphism $\iota : F_R(A) \to \hat{\mathcal{D}}(A,T)$ as follows. Given $w \in F_R(A)$ choose a word $w$ in the alphabet $A(\mathbb{R})$ which represents $w$. There results a function $w\{\cdot\} : T \to F(A)$ defined by $t \mapsto w\{t\}$, see Definition 4.14. Set
$$\iota(w) = [w\{\cdot\}].$$

We say that a homomorphism $\varphi : G \to H$ between groups equipped with pseudonorms is norm preserving if $\|\varphi(g)\| = \|g\|$ for all $g \in G$.

Proposition 5.8. The function $\hat{\Psi}$ and $\iota$ are well defined homomorphisms. Moreover, $\hat{\Psi}$ is an isometry $\hat{\mathcal{D}}(A,T) \cong \hat{F}_R(A)$ and $\hat{\Psi} \circ \iota$ is the canonical homomorphism $F_R(A) \to \hat{F}_R(A)$. In particular, $\iota$ is norm preserving and its image is dense in $\hat{\mathcal{D}}(A)$.

Proof. We will omit $T$ from the notation. It is clear that $\Psi$ is well defined and it is a homomorphism because the norm on $\mathcal{D}(A)$ is conjugation invariant so it makes $\mathcal{D}(A)$ a topological group. It is norm preserving because for any $w \in D(A)$
$$\|\Psi(w)\|_{\hat{F}_R(A)} = \|\lim_{t \to \infty} \psi_{1/t}(w(t))\|_{\hat{F}_R(A)} = \lim_{t \to \infty} \|\psi_{1/t}(w(t))\|_{\hat{F}_R(A)} = \lim_{t \to \infty} \|\psi_{1/t}(w(t))\|_{F_R(A)} = \|w\|_{\mathcal{D}(A)}.$$ 

In particular $\text{Ker}(\Psi) = \mathcal{D}(A)_0$. So $\Psi$ factors through $\hat{\Psi}$ which is therefore a norm preserving monomorphism, i.e an isometric embedding. Since $\hat{\mathcal{D}}(A)$ is complete by Proposition 5.5, the image of $\hat{\Psi}$ is closed in $\hat{F}_R(A)$.

We now prove that $\iota$ is well defined. Suppose that $w$ represents $w \in F_R(A)$. It follows from Proposition 4.15 that $\psi_{1/t}(w\{t\})$ is a Cauchy function in $F_R(A)$, hence by definition $w\{\cdot\} \in \mathcal{D}(A)$. By the same proposition, if $w'$ is another word representing $w$ then
$$\|w\{\cdot\}^{-1}w'\{\cdot\}\|_{\mathcal{D}(A)} = \lim_{t} \|\psi_{1/t}(w\{t\})^{-1} \cdot w'\{t\}\|_{F_R(A)} = \lim_{t} \|\psi_{1/t}(w\{t\})^{-1} \cdot \psi_{1/t}(w'\{t\})\|_{F_R(A)} = \|w^{-1}w\|_{F_R(A)} = 0.$$ 

So $\iota$ is independent of choices, hence it is well defined. It is a homomorphism because $(ww')\{t\}) = w\{t\} \cdot w'\{t\}$ by (5).
By Proposition 4.15 for any \( w \in F_{\mathbb{R}}(A) \) and a representing word \( w \),
\[
\hat{\Psi}(\iota(w)) = \hat{\Psi}([w]) = \lim_{t} \psi_{1/t}(w\{t\}) = w.
\]
(The limit is in \( F_{\mathbb{R}}(A) \)). Therefore \( \Psi \circ \iota \) is the canonical homomorphism \( F_{\mathbb{R}}(A) \to \hat{F}_{\mathbb{R}}(A) \). In particular, the image of \( \hat{\Psi} \) contains a dense subset of \( F_{\mathbb{R}}(A) \) and since it is closed, \( \hat{\Psi} \) is surjective, hence an isometry. Since \( F_{\mathbb{R}}(A) \to \hat{F}_{\mathbb{R}}(A) \) is norm preserving, so is \( \iota \).

Next, we show that \( \mathcal{D}(-,T) \) and \( \hat{\mathcal{D}}(-,T) \) are functorial with respect to proto-Lipschitz functions (Definition 3.11) and with respect to inclusions \( T' \subseteq T \) of scaling sets. Recall that any function \( f : A \to B \) induces a homomorphism \( f_{*} : F(A) \to F(B) \).

**Proposition 5.9.** The assignments \( A \mapsto \mathcal{D}(A,T) \) and \( A \mapsto \hat{\mathcal{D}}(A,T) \) are functors from the category of sets equipped with length functions and with proto-Lipschitz functions as morphisms to the category of groups with conjugation invariant (pseudo) norms.

In more detail, a proto-Lipschitz \( f : A \to B \) with constant \( C \) induces homomorphisms \( \mathcal{D}(f) : \mathcal{D}(A,T) \to \mathcal{D}(B,T) \) and \( \hat{\mathcal{D}}(f) : \hat{\mathcal{D}}(A,T) \to \hat{\mathcal{D}}(B,T) \) defined by \( \mathcal{D}(f)(w) = f_{*} \circ w \) and \( \hat{\mathcal{D}}(f)([w]) = [f_{*} \circ w] \), both are Lipschitz with constant \( C \). Also, \( \mathcal{D}(g \circ f) = \mathcal{D}(g) \circ \mathcal{D}(f) \) and \( \mathcal{D}(\text{id}) = \text{id} \), and similarly \( \hat{\mathcal{D}}(g \circ f) = \hat{\mathcal{D}}(g) \circ \hat{\mathcal{D}}(f) \) and \( \hat{\mathcal{D}}(\text{id}) = \text{id} \).

**Proof.** Let \( f : A \to B \) be proto Lipschitz with constant \( C \). By Proposition 2.11 both \( f_{*} : F(A) \to F(B) \) and \( f_{R}(f) : F_{\mathbb{R}}(A) \to F_{\mathbb{R}}(B) \) are Lipschitz homomorphisms with constant \( C \). Consider some \( w \in \mathcal{D}(A) \) and set \( v = f_{*} \circ w \). Then \( v : T \to F(B) \). In particular \( \mathcal{D}(f_{*}) \) carries \( \mathcal{D}(A)_{0} \) into \( \mathcal{D}(B)_{0} \) so \( \mathcal{D}(f) \) induces a homomorphism \( \hat{\mathcal{D}}(f) : \hat{\mathcal{D}}(A) \to \hat{\mathcal{D}}(B) \) which is Lipschitz with constant \( C \) and \( \hat{\mathcal{D}}(f)([w]) = [f_{*} \circ w] \). The functorial properties of \( A \mapsto \mathcal{D}(A,T) \)
follow from those of \( A \mapsto F(A) \), i.e \((g \circ f)_* = g_* \circ f_*\) and \( \text{id}_* = \text{id} \). Hence the functorial properties of \( A \mapsto \hat{D}(A, T) \).

\[ \hat{D}(-, T') \rightarrow \hat{D}(-, T), \quad \hat{\text{res}}_{T'}^T : \hat{D}(-, T') \rightarrow \hat{D}(-, T), \quad \hat{\text{res}}_{T'}^T ([w]) = [w|_T] \]

The second of which is a natural isometry.

**Proof.** Clearly, if \( u : T' \rightarrow F_{\mathbb{R}}(A) \) is Cauchy then so is \( u|_T \). Therefore, \( \text{res}_{T'}^T : D(A, T') \rightarrow D(A, T) \) defined by \( w \mapsto w|_T \) is a well defined homomorphism. It is norm preserving because clearly

\[
\| w|_T \|_{D(A, T)} = \lim_{t \in T} \| \psi_{1/t}(w(t)) \|_{F_{\mathbb{R}}(A)} = \lim_{t \in T'} \| \psi_{1/t}(w(t)) \|_{F_{\mathbb{R}}(A)} = \| w \|_{D(A, T')}.
\]

It is a natural transformation of functors because by inspection the following square commutes for any proto-Lipschitz \( f : A \rightarrow B \).

\[
\begin{array}{ccc}
D(A, T') & \xrightarrow{\text{D}(f, T')} & D(B, T') \\
\downarrow \text{res}_{T'}^T & & \downarrow \text{res}_{T'}^T \\
D(A, T) & \xrightarrow{\text{D}(f, T)} & D(B, T).
\end{array}
\]

Since it is norm preserving there results a natural transformation of functors \( \hat{\text{res}}_{T'}^T : \hat{D}(-, T') \rightarrow \hat{D}(-, T) \) defined by \( \hat{\text{res}}_{T'}^T ([w]) = [w|_T] \). By inspection the following triangle is commutative

\[
\begin{array}{ccc}
\hat{D}(A, T') & \xrightarrow{\hat{\text{res}}_{T'}^T} & \hat{D}(A, T) \\
\downarrow \hat{\Psi} & & \downarrow \hat{\Psi} \\
F_{\mathbb{R}}(A) & & F_{\mathbb{R}}(A)
\end{array}
\]

Since the arrows \( \hat{\Psi} \) are isometries by Proposition 5.8, \( \hat{\text{res}}_{T'}^T \) is a natural isometry.

\[ \square \]

Our next goal is to prove that \( \hat{D}(A, T) \) are contractible spaces.

Let \( f : A \times B \rightarrow Y \) be a function (of sets). Given \( a \in A \) we write \( f_a : B \rightarrow Y \) for the restriction of \( f \) to \( \{ a \} \times B \cong B \). Similarly, \( f_b \) is the restriction to \( A \times \{ b \} \cong A \). The proof of the following lemma is left to the reader.
Lemma 5.11. Suppose that $A \subseteq X$ is a dense subspace of a metric space and $Y$ is a complete metric space. Let $T$ be a metric space and endow $T \times X$ with the metric $\max\{d_T, d_X\}$. Suppose that $f : T \times A \to Y$ is a function with the following properties.

(i) For any $t \in T$ the function $f_t : A \to Y$ is Lipschitz with constant $C_t$.

(ii) If $J \subseteq T$ is bounded then $\{C_t : t \in J\}$ is a bounded subset of $\mathbb{R}$.

(iii) For any $a \in A$ the resulting function $f_a : T \to Y$ is continuous. Then $f$ extends uniquely to a continuous function $\tilde{f} : T \times X \to Y$. □

Proposition 5.12. There is a continuous function $h : \mathbb{R} \times \hat{D}(A, T) \to D(A, T)$ such that the following diagram commutes for all $t \in \mathbb{R}$.

$$
\begin{array}{ccc}
F_\mathbb{R}(A) & \xrightarrow{\psi_t} & F_\mathbb{R}(A) \\
\downarrow & & \downarrow \\
\hat{D}(A, T) & \xrightarrow{h_t} & \hat{D}(A, T),
\end{array}
$$

Moreover, each $h_t$ is a homomorphism and

(i) $\|h_t(x)\|_{\hat{D}(A)} = |t| \cdot \|x\|_{\hat{D}(A)}$

(ii) $h_s \circ h_t = h_{st}$ and $h_1 = \text{id}$ and $h_0$ is the trivial homomorphism.

Proof. By composing with the isometry $\hat{\Psi}$ we may replace $\hat{D}(A)$ with $\overline{F_\mathbb{R}(A)}$ and the map $\iota$ with the canonical homomorphism $i : F_\mathbb{R}(A) \to F_\mathbb{R}(A)$. Since the latter is norm preserving, and since $\psi_t$ carries $F_\mathbb{R}(A)_0$ into itself by Lemma 4.12(a), we may further replace $i$ with the isometric embedding $i : F_\mathbb{R}(A)/F_\mathbb{R}(A)_0 \to \overline{F_\mathbb{R}(A)}$ and write $F_\mathbb{R}(A^\#)$ for $F_\mathbb{R}(A)/F_\mathbb{R}(A)_0$. Define $f : \mathbb{R} \times F_\mathbb{R}(A^\#) \to \overline{F_\mathbb{R}(A)}$ by $f(t, w) = i(\psi_t(w))$. Then $f_t$ is Lipschitz with constant $t$ for any $t \in \mathbb{R}$ and $f_w$ is Lipschitz with constant $C_w$ for any $w \in F_\mathbb{R}(A^\#)$ by Lemma 4.12. The conditions of Lemma 5.11 hold and we obtain a continuous $h : \mathbb{R} \times \overline{F_\mathbb{R}(A)} \to \overline{F_\mathbb{R}(A)}$ which restricts to $f$. The commutativity of the square follows. (i) and (ii) follow from the density of $F_\mathbb{R}(A^\#)$ in $\overline{F_\mathbb{R}(A)}$ and from Lemma 4.12(a) and Proposition 4.5. □

Corollary 5.13. $\hat{D}(A, T)$ is contractible via a homotopy $h_t$ defined on the unit interval $[0, 1]$ such that each $h_t$ is a homomorphism. Hence, $\text{End}(\hat{D}(A, T))$ equipped with the compact-open topology is contractible.

Recall that a (pseudo) metric space $X$ is called a length space if for any $x, y \in X$ and any $\epsilon > 0$ there exists a path $\gamma : I \to X$ from $x$ to $y$ such that $\ell(\gamma) < d(x, y) + \epsilon$, where $\ell(\gamma)$ is the length of $\gamma$. 
Lemma 5.14. Let $G$ be a group equipped with a conjugation invariant norm and $H$ a dense subgroup. Suppose that $H$ is a length space and that $G$ is complete. Then $G$ is a length space.

Proof. Since the metric on $G$ is invariant it suffices to show that for any $g \in G$ and any $\epsilon > 0$ there exists a path $\gamma$ from $1$ to $g$ such that $\ell(\gamma) < \|g\| + \epsilon$. Choose a sequence $h_n \in H$ such that $h_n \to g$. We may assume that $\|g^{-1}h_n\| < \frac{1}{2^n-1}$ for all $n \geq 1$, hence $\|h_n^{-1}h_{n+1}\| < \frac{1}{2^n-1}$. For every $n \geq 2$ set $\Delta_n = h_{n-1}^{-1}h_n \in H$. Then $\|\Delta_n\| < \frac{1}{2^n}$ for all $n \geq 2$. Since $H$ is a length space, choose paths $\delta_n : I \to H$ from $1$ to $\delta_n$ such that $\ell(\delta_n) < \frac{1}{2^n-1}$. Observe that for any $t \in I$

$$\|\delta_n(t)\| \leq \ell(\delta_n[0,t]) \leq \ell(\delta_n) < \frac{1}{2^n-1}.$$ 

Given $\epsilon > 0$ choose $n \geq 2$ such that $\frac{\epsilon}{2} < \frac{1}{2^n-2}$. Choose a path $\beta : I \to H$ from $1$ to $h_n$ such that $\ell(\beta) < \|h_n\| + \frac{\epsilon}{2}$. Observe that for every $k \geq n$

$$h_k = h_n\Delta_{n+1} \cdots \Delta_k.$$ 

Define paths $\alpha_k : I \to H$ by $\alpha_k = \beta \cdot \delta_{n+1} \cdots \delta_k$, i.e

$$\alpha_k(t) = \beta(t) \cdot \delta_{n+1}(t) \cdots \delta_k(t), \quad (t \in I).$$

It is clear that $\alpha_k$ are continuous because the norm on $G$ makes it a topological group. Since $G$ is complete, $\alpha_k$ converges uniformly to some $\alpha : I \to G$ which is therefore also continuous. Clearly $\alpha(0) = 1$ and

$$\alpha(1) = \lim_{k \geq n} \alpha_k(1) = \lim_{k \geq n} h_n \cdot \Delta_{n+1} \cdots \Delta_k = \lim_{k \geq n} h_k = g.$$ 

So $\alpha$ is a path from $1$ to $G$. By Lemma 2.4 for any $t, t' \in I$

$$\|\alpha_k(t)^{-1}\alpha(t')\| \leq \|\beta(t)^{-1}\beta(t')\| + \sum_{i=n+1}^{k} \|\delta_i(t)^{-1}\delta_i(t')\|.$$ 

It follows that $\ell(\alpha_k) \leq \ell(\beta) + \sum_{i=n+1}^{k} \ell(\delta_i)$. Since the convergence is uniform,

$$\ell(\alpha) = \lim_{k \geq n} \ell(\alpha_k) \leq \ell(\beta) + \frac{1}{2^n-1} < \|h_n\| + \frac{\epsilon}{2} + \frac{1}{2^n-1} < \|g\| + \epsilon.$$ 

This completes the proof. \hfill \qed

Corollary 5.15. $\hat{\mathcal{D}}(A,T)$ is a length space.

Proof. By Proposition we may replace $\hat{\mathcal{D}}(A,T)$ with $\hat{\mathcal{F}}_{\mathbb{R}}(A)$. The natural completion homomorphism $\iota : \mathcal{F}_{\mathbb{R}}(A) \to \hat{\mathcal{F}}_{\mathbb{R}}(A)$ is norm preserving, so its image is a geodesic space by Proposition 4.28. Apply Lemma 5.14. \hfill \qed
6. The directional asymptotic cone

Running assumption: Throughout this section \( T, T', \ldots \) denote scaling sets (Definition 5.1).

All groups \( G \) are assumed to be equipped with a conjugation invariant norm. In this case \( G \) is equipped with a canonical length function

\[
\mu(g) = \| g \|.
\]

Whenever the length function on \( G \) is not specified it is meant that it is \( \mu \). For example, \( F(G) \) and \( D(G, T) \) stand for \( F(G; \mu) \) and \( D(G, T; \mu) \) etc.

6.1. We denote the generators of \( F(G) \) by \( \bar{g} \) where \( g \in G \). Similarly, the generators of \( F(\mathbb{R}) \) are denoted \( \bar{g}(r) \) for \( g \in G \) and \( r \in \mathbb{R} \).

Notice that \( g^{-1} \neq \bar{g}^{-1} \) in \( F(G) \). The first element is a generator, the other is the inverse of a different generator. Similarly, \( \bar{g}(-r) = \bar{g}(r)^{-1} \neq g^{-1}(r) \) in \( F(\mathbb{R}) \).

The assignment \( \bar{g} \mapsto g \) gives rise to a canonical homomorphism

\[
\pi : F(G) \to G.
\]

It is Lipschitz with constant 1 by Proposition 2.11 (recall that the length function is \( \mu(g) = \| g \| \)).

Given a scaling set \( T \), there results a homomorphism

\[
\pi_* : \prod_{t \in T} F(G) \to \prod_{t \in T} G, \quad \pi'_*(w) = \pi \circ w.
\]

That is, given a function \( w : T \to F(G) \) then \( \pi'_*(w)(t) = \pi(w(t)) \) for all \( t \in T \).

Recall from Definition 5.2 that \( D(G, T) \) is a subgroup of \( \prod_{t \in T} F(G) \) equipped with the norm in Definition 5.3. Set

\[
\mathcal{C}(G, T) \overset{\text{def}}{=} \pi'_*(D(G, T)).
\]

Thus, \( \mathcal{C}(G, T) \) is the image of \( D(G, T) \) under \( \pi_* : \prod_T F(G) \to \prod_T G \). As in the previous sections we will suppress \( T \) from the notation whenever it doesn’t cause confusion.

By slight abuse of notation \( \pi_* : D(G, T) \to \mathcal{C}(G, T) \) will also denote the restriction of \( \pi_* \) to \( D(G, T) \). Equip \( \mathcal{C}(G, T) \) with the quotient conjugation-invariant pseudonorm in Lemma 2.6 i.e for any \( g : T \to G \) in \( \mathcal{C}(G, T) \)

\[
\| g \|_{\mathcal{C}(G, T)} = \inf \left\{ \| w \|_{D(G, T)} : w \in \pi_*^{-1}(g) \right\}.
\]
Definition 6.2. The directional asymptotic cone of $G$ is the metrification of $C(G, T)$, see Section 2.3. That is, 
\[ \hat{C}(G, T) = C(G, T)/C(G, T)_0 \]
equipped with the induced conjugation-invariant norm. We will write $[g]$ for the image of $g \in C(G, T)$ in $\hat{C}(G, T)$.

6.3. It is clear from the definitions that $\pi_*: D(G, T) \to C(G, T)$ carries $D(G, T)_0$ into $C(G, T)_0$. There results an epimorphism of normed groups 
\[ \hat{\pi}: \hat{D}(G, T) \to \hat{C}(G, T). \]
By the definition of $\| \cdot \|_{\hat{C}(G)}$ and since the quotients $D(G, T) \to \hat{D}(G, T)$ and $C(G, T) \to \hat{C}(G, T)$ are norm preserving, one easily checks that 
\[ \| [g] \|_{\hat{C}(G)} = \inf \left\{ \| [w] \|_{\hat{D}(G)} : [w] \in \hat{\pi}^{-1}([g]) \right\}. \]
Thus, $\| \cdot \|_{\hat{C}(G)}$ is the quotient norm in Lemma 2.6 induced from $\| \cdot \|_{\hat{D}(G)}$ and $\hat{\pi}$ is a metric quotient. It follows from Proposition 5.8 that the image of the composition 
\[ F_{\mathbb{R}}(G) \to \hat{D}(G, T) \xrightarrow{\hat{\pi}} \hat{C}(G, T) \]
is dense.

Proposition 6.4. $\hat{C}(G, T)$ is a complete normed group.
Proof. Immediate from Section 6.3 and Propositions 2.7 and 5.8. \qed

The following two fundamental lemmas will be used repeatedly.

Lemma 6.5. For any $g \in C(G, T)$ 
\[ \limsup_{t \in T} \frac{1}{t} \| g(t) \| \leq \| g \|_{C(G)}. \]
Proof. Choose an arbitrary $\epsilon > 0$. By definition of $\| \cdot \|_{C(G)}$ there exists $w \in D(G)$ such that $\pi_*(w) = g$ and $\| w \|_{D(G)} < \| g \|_{C(G)} + \epsilon$. Since $\pi: F(G) \to G$ is Lipschitz with constant 1 we get $\| g(t) \| \leq \| w(t) \|_{F(G)}$ for all $t \in T$. It follows from (0) 
\[ \limsup_{t \in T} \frac{1}{t} \| g(t) \| \leq \limsup_{t \in T} \frac{1}{t} \| w(t) \|_{F(G)} = \| w \|_{D(G)} < \| g \|_{C(G)} + \epsilon. \]
Since $\epsilon > 0$ was arbitrary, the result follows. \qed

Lemma 6.6. For any function $g: T \to G$, 
\[ \limsup_{t \in T} \frac{1}{t} \| g(t) \| = 0 \]
if and only if $g \in C(G, T)$ and $\| g \|_{C(G)} = 0$. 

Lemma 4.12 and Corollary 4.19 imply that for any s, t > 0
\[\|\psi_{1/t}(w(t))^{-1}\psi_{1/s}(w(s))\|_{F_\text{b}(G)} \leq \|\psi_{1/t}(g(t))\|_{F_\text{b}(G)} + \|\psi_{1/s}(g(s))\|_{F_\text{b}(G)}\]
\[= \frac{1}{t}\|g(t)\|_{\text{F}(G)} + \frac{1}{s}\|g(s)\|_{\text{F}(G)} = \frac{1}{t}\|g(t)\| + \frac{1}{s}\|g(s)\|\].

This is arbitrarily small if s, t ≫ 0, so w(t) ∈ D(G). Clearly g = π∗ o w, so g ∈ C(G). Moreover, by (6)
\[\|g\|_{C(G)} \leq \|w\|_{D(G)} = \lim_{t} \frac{1}{t}\|w(t)\|_{\text{F}(G)} = \lim_{t} \frac{1}{t}\|g(t)\| = 0.

This prove the “only if” part of the lemma. The opposite implication follows from Lemma 6.5. □

Recall the homomorphisms \(\hat{\pi}: \hat{D}(G, T) \to \hat{C}(G, T)\) and \(\iota: F_\text{R}(G) \to \hat{D}(G, T)\) from Section 6.3 and Definition 5.7.

Lemma 6.7. Consider some \(g \in G\) and \(r \in \mathbb{R}\).

(i) For any integer \(p \neq 0\),
\[\hat{\pi} \circ \iota(\overline{g(r)}) = \hat{\pi} \circ \iota(\overline{g^p(r)})\).

(ii) For any \(h \in G\)
\[\hat{\pi} \circ \iota(hgh^{-1}(r)) = \hat{\pi} \circ \iota(g(r)).\]

Proof. ([i] By definition \(\hat{\pi}(\iota(\overline{g(r)}))\) is represented by the function \(\gamma(t) = g^{\|r\|}\) and \(\hat{\pi}(\iota(\overline{g^p(r)}))\) is represented by \(\gamma'(t) = g^{p\|r/p\|}\). They represent the same element in \(\hat{C}(G)\) by Lemma 6.6 since \(\|\|r\| - p\|\frac{r}{p}\|\) < |p| so
\[\lim_{t \in T} \frac{1}{t}\|\gamma(t)^{-1}\gamma'(t)\| = \lim_{t \in T} \frac{1}{t}\|g^{p\|r/p\| - \|r\|}\| \leq \lim_{t \in T} \|g\| - \frac{\|\|r\| - p\|\frac{r}{p}\|\|}{t} = 0.\]

([ii] By definition \(\hat{\pi} \circ \iota(\overline{g(r)})\) is represented by \(\gamma(t) = g^{\|r\|}\) and \(\hat{\pi} \circ \iota(hgh^{-1}(r))\) is represented by \(\gamma'(t) = (hgh^{-1})\|r\| = hg\|r\|h^{-1}\). Since
\[\lim_{t \in T} \frac{1}{t}\|\gamma(t)^{-1}\gamma'(t)\| = \lim_{t \in T} \frac{1}{t}\|g^{-\|r\|}hg\|r\|h^{-1}\|\]
\[\leq \lim_{t \in T} \frac{1}{t}\|g^{-\|r\|}hg\|r\|\| + \lim_{t \in T} \frac{1}{t}\|h^{-1}\|
\[= \lim_{t \in T} \frac{1}{t}(\|h\| + \|h^{-1}\|) = 0,\]
they represent the same element in \(\hat{C}(G)\) by Lemma 6.6. □
6.8. Let $G$ be a group equipped with conjugation invariant norm. By Fekete’s Lemma

$$\tau(g) \overset{\text{def}}{=} \inf_n \frac{\|g^n\|}{n} = \lim_n \frac{\|g^n\|}{n}$$

which is called the **stable length** or **translation length** of $g \in G$. Observe that $\tau(g^p) = |p| \cdot \tau(g)$ for any integer $p$ and that $\tau(gh^{-1}) = \tau(g)$ for any $h \in G$.

**Corollary 6.9.** Consider a generator $\alpha = \overline{f}(r) \in \hat{F}(G)$. Then

$$\|\hat{\pi}(\nu(\alpha))\|_{\hat{C}(G,T)} = |r| \cdot \tau(g).$$

**Proof.** Assume $r \neq 0$. First, $[h] = \hat{\pi}(\nu(\alpha))$ is represented by the function $h: T \to G$ given by $h(t) = g^{[rt]}$. It follows from Lemma 6.5 that $\limsup_{t \in T} \frac{1}{|t|} \|h(t)\| \leq \|[h]\|_{\hat{C}(G,T)}$. Since $T$ is unbounded and since $\|g^{-n}\| = \|g^n\|$, we get from 6.8 above

$$\limsup_{t \in T} \frac{1}{|t|} \|g^{[rt]}\| = |r| \cdot \limsup_{n \to \infty} \|g^n\| = |r| \cdot \tau(g).$$

Thus $\|[h]\|_{\hat{C}(G,T)} \geq \tau(g)$. By Lemma 6.7(1), for any $k > 0$ we get $[h] = [\hat{\pi}((\nu(g^k)^{(r)})])$ in $\hat{C}(G,T)$. Since $\hat{\pi}$ is a metric quotient, see paragraph 6.3 and by Proposition 5.8

$$\|[h]\|_{\hat{C}(G,T)} \leq \|\nu(g^k)^{(r)}\|_{\hat{D}(G,T)}$$

$$= \|\nu(g^k)^{(r)}\|_{\hat{F}(G)}$$

$$= |r| \cdot \frac{1}{k} \|g^k\| \xrightarrow{k \to \infty} |r| \cdot \tau(g).$$

We deduce that $\|\hat{h}\|_{\hat{C}(G,T)} = \tau(g)$. \(\square\)

We will use Proposition 5.8 to identify $\hat{D}(G,T)$ with $\hat{F}(\hat{G};\mu)$ and abusively write $\hat{\pi}: \hat{F}(\hat{G};\mu) \to \hat{C}(G,T)$ for the metric quotient homomorphism in paragraph 6.3.

**Proposition 6.10.** Consider $\Theta \subseteq G$ with the property that for any $g \in G$ there exists $\theta \in \Theta$ and $p \in \mathbb{Z}$ such that $g$ is conjugate to $\theta^p$. Let $\text{id}_{\mu}^\tau: \hat{F}(G;\mu) \to \hat{F}(G;\tau)$ denote the identity homomorphism of the underlying groups.
Then there exists a homomorphism \( \hat{\xi} : \hat{F}_R(G; \tau) \to \hat{C}(G; T, \mu) \) which renders the following diagram commutative.

\[
\begin{array}{ccc}
F_R(G; \mu) & \xrightarrow{\iota} & F_R(G; \mu) \\
\downarrow{\text{id}_\mu} & & \downarrow{\text{id}_\mu} \\
F_R(G; \tau) & \xrightarrow{\iota} & F_R(G; \tau) \\
\downarrow{\text{incl}} & & \downarrow{\text{incl}} \\
F_R(\Theta; \tau) & \xrightarrow{\iota} & F_R(\Theta; \tau)
\end{array}
\]

Moreover, \( \xi_\Theta \) is a metric quotient homomorphism.

Proof. Set \( \xi = \hat{\pi} \circ \iota \circ (\text{id}_\mu)^{-1} \). By Corollary 6.9 and Proposition 2.11, \( \xi \) is Lipschitz with constant 1. By the universal property of the completion and Proposition 6.4 there exists a unique homomorphism \( \hat{\xi} \) as in the diagram, Lipschitz with constant 1, and rendering the upper triangle commutative. Set \( \hat{\xi}_\Theta = \hat{\xi} \circ \text{incl} \). The diagram is commutative. It remains to show that \( \hat{\xi}_\Theta \) is a metric quotient homomorphism.

Define a homomorphism \( \sigma : F_R(G; \tau) \to F_R(\Theta; \tau) \) as follows. For a generator \( \bar{g}(r) \), choose \( \theta \in \Theta \) and \( p \in \mathbb{Z} \) such that \( g \) is conjugate (via some element \( h \in G \)) to \( \theta^p \) and set \( \sigma(\bar{g}(r)) = \bar{\theta}(pr) \). If \( g \in \Theta \) then we choose \( \theta = g \) and \( p = 1 \). The relations (2) in paragraph 4.3 are satisfied by this assignment and it therefore extents to a unique homomorphism \( \sigma : F_R(G; \tau) \to F_R(\Theta; \tau) \). It is Lipschitz with constant 1 by Proposition 2.11 because for a generator \( \bar{g}(r) \)

\[
\| \sigma(\bar{g}(r)) \|_{F_R(\Theta; \tau)} = \| \bar{\theta}(pr) \| = |pr| \cdot \tau(\theta) = |r| \cdot \tau(\theta^p) = |r| \cdot \tau(h \theta^p h^{-1}) = |r| \cdot \tau(g) = \| \bar{g}(r) \|_{F(G; \tau)}.
\]

Then \( \sigma \) extends to a homomorphism \( \bar{\sigma} : \hat{F}_R(G; \tau) \to \hat{F}_R(\Theta; \tau) \), Lipschitz with constant 1. It is clear that \( \sigma \circ \text{incl} \) is the identity on \( \hat{F}_R(\Theta; \tau) \), hence \( \bar{\sigma} \) is a left inverse for \( \hat{\text{incl}} \). Furthermore, we claim that \( \hat{\xi}_\Theta \circ \bar{\sigma} = \hat{\xi} \).

Indeed using the density of \( F_R(G; \tau) \) in its completion, it suffices to check, with the aid of Lemma 6.7, that on generators \( \bar{g}(r) \in F_R(G; \tau) \)

\[
(\hat{\xi}_\Theta \circ \bar{\sigma})(\iota(\bar{g}(r))) = (\hat{\xi}_\Theta \circ \iota)(\sigma(\bar{g}(r))) = (\hat{\xi}_\Theta \circ \iota)(\bar{\theta}(pr)) = \hat{\xi} \circ \iota(\bar{g}(r)) = \hat{\xi}(\iota(\bar{g}(r))).
\]
Consider some \([g] \in \hat{\mathcal{C}}(G; \mu)\). Since \(\hat{\pi}\) is a metric quotient, given \(\epsilon > 0\) there exists \(\hat{\alpha} \in F_{\mathbb{R}}(G; \mu)\) such that \(\hat{\pi}(\hat{\alpha}) = [g]\) and \(\|\hat{\alpha}\|_{F_{\mathbb{R}}(G; \mu)} < \|\hat{g}\| + \epsilon\). Set \(\hat{\alpha}_\Theta = \hat{\sigma}(\hat{id}_\mu(\hat{\alpha}))\). Then \(\hat{\xi}_\Theta(\hat{\alpha}_\Theta) = \hat{\xi}(\hat{id}_\mu(\hat{\alpha})) = [g]\). Since \(\hat{\sigma}\) and \(\hat{id}_\mu\) are Lipschitz with constant 1
\[
\|\hat{\alpha}_\Theta\|_{F_{\mathbb{R}}(G; \tau)} \leq \|\hat{\alpha}\|_{F_{\mathbb{R}}(G; \mu)} < \|\hat{g}\|_{\hat{\mathcal{C}}(G)} + \epsilon.
\]

Since \(\hat{\xi}_\Theta\) is Lipschitz with constant 1 it now follows that it is a metric quotient homomorphism. \(\Box\)

We will now address the naturality of the construction \(\hat{\mathcal{C}}(G, T)\). By inspection, any homomorphism \(\varphi: G \rightarrow H\) fits to the commutative diagram
\[
\begin{array}{ccc}
F(G) & \xrightarrow{\varphi_\ast} & F(H) \\
\pi \downarrow & & \downarrow \pi \\
G & \xrightarrow{\varphi} & H.
\end{array}
\]

**Proposition 6.11** (Naturality). The assignments \(G \mapsto \mathcal{C}(G, T)\) and \(G \mapsto \hat{\mathcal{C}}(G, T)\) give rise to functors on the category of groups with conjugation-invariant (pseudo) norm and Lipschitz homomorphisms between them. Moreover, in relation to the naturality of \(\mathcal{D}(-, T)\) and \(\hat{\mathcal{D}}(-, T)\) in Proposition 5.9, the homomorphisms \(\pi: \mathcal{D}(-, T) \rightarrow \mathcal{C}(-, T)\) and \(\hat{\pi}: \hat{\mathcal{D}}(-, T) \rightarrow \hat{\mathcal{C}}(-, T)\) are natural transformations of functors.

In more detail, any Lipschitz homomorphism \(\varphi: G \rightarrow H\) between groups equipped with conjugation-invariant pseudonorm gives rise to Lipschitz homomorphisms
\[
\mathcal{C}(\varphi): \mathcal{C}(G, T) \rightarrow \mathcal{C}(H, T) \quad \text{defined by } \mathcal{C}(\varphi)(g) = \varphi \circ g
\]
\[
\hat{\mathcal{C}}(\varphi): \hat{\mathcal{C}}(G, T) \rightarrow \hat{\mathcal{C}}(H, T) \quad \text{defined by } \hat{\mathcal{C}}(\varphi)([g]) = [\varphi \circ g]
\]

They render the following squares commutative
\[
\begin{array}{ccc}
\mathcal{D}(G, T) & \xrightarrow{\mathcal{D}(\varphi)} & \mathcal{D}(H, T) \\
\pi_\ast \downarrow & & \downarrow \pi_\ast \\
\mathcal{C}(G, T) & \xrightarrow{\mathcal{C}(\varphi)} & \mathcal{C}(H, T)
\end{array}
\quad
\begin{array}{ccc}
\hat{\mathcal{D}}(G, T) & \xrightarrow{\hat{\mathcal{D}}(\varphi)} & \hat{\mathcal{D}}(H, T) \\
\hat{\pi}_\ast \downarrow & & \downarrow \hat{\pi}_\ast \\
\hat{\mathcal{C}}(G, T) & \xrightarrow{\hat{\mathcal{C}}(\varphi)} & \hat{\mathcal{C}}(H, T)
\end{array}
\]

We abusively denote both \(\mathcal{C}(\varphi)\) and \(\hat{\mathcal{C}}(\varphi)\) by \(\varphi_\ast\). Then \(\text{id}_\ast = \text{id}\) and \((\psi \circ \varphi)_\ast = \psi_\ast \circ \varphi_\ast\).

**Proof.** We will omit \(T\) from the notation. Let \(K(G)\) denote the kernel of \(\pi: \mathcal{D}(G) \rightarrow \mathcal{C}(G)\). Similarly, \(\hat{K}(G)\) denotes the kernel of \(\hat{\pi}: \hat{\mathcal{D}}(G) \rightarrow \hat{\mathcal{C}}(G, T)\).
$\hat{\mathcal{C}}(G)$. By definition, if $w \in K(G)$ then $\pi \circ w$ is the constant function $1: T \to G$.

Suppose that $\varphi: G \to H$ is Lipschitz with constant $C$. By Proposition 5.9 there are Lipschitz homomorphisms $\mathcal{D}(\varphi): \mathcal{D}(G) \xrightarrow{w \mapsto \varphi \circ w} \mathcal{D}(H)$ and $\hat{\mathcal{D}}(\varphi): \hat{\mathcal{D}}(G) \xrightarrow{\hat{\varphi} \circ \hat{w}} \hat{\mathcal{D}}(H)$. By the commutativity of $(\ref{eq:commutative})$, for any $w \in K(G)$

$$\pi_*(\mathcal{D}(\varphi)(w)) = \pi \circ \mathcal{D}(\varphi)(w) = \pi \circ \varphi_* \circ w = \varphi \circ \pi \circ w = 1.$$ 

So $\mathcal{D}(\varphi)(w) \in K(H)$ and therefore $\mathcal{D}(\varphi)(K(G)) \subseteq K(H)$.

If $[w] \in \hat{K}(G)$ then $\pi \circ w$ represents the trivial element in $\hat{\mathcal{C}}(G)$ so by Lemma 6.6, $\lim_{t \in T} \|\pi(w(t))\| = 0$. By the commutativity of $(\ref{eq:commutative})$

$$\hat{\pi}(\hat{\mathcal{D}}(\varphi)([w])) = \hat{\pi}([\varphi_* \circ w]) = [\pi \circ \varphi_* \circ w] = [\varphi \circ \pi \circ w].$$

Since $\varphi$ is Lipschitz, $\lim_{t \in T} \|\varphi \circ \pi \circ w\| = 0$. By Lemma 6.6 $\hat{\mathcal{D}}(\varphi)([w])$ is the trivial element in $\hat{\mathcal{C}}(H)$. It follows that $\hat{\mathcal{D}}(\varphi)(\hat{K}(G)) \subseteq \hat{K}(H)$.

As a result there exist unique homomorphisms $\mathcal{C}(\varphi): \mathcal{C}(G) \to \mathcal{C}(H)$ and $\hat{\mathcal{C}}(\varphi): \hat{\mathcal{C}}(G) \to \hat{\mathcal{C}}(H)$ rendering the squares in the statement of the proposition commutative. By Proposition 2.8 these are Lipschitz homomorphisms with constant $C$. The functorial properties follows from the surjectivity of $\pi_*$ and $\hat{\pi}$ which imply that $\mathcal{C}(\varphi)$ and $\hat{\mathcal{C}}(\varphi)$ filling in the above diagrams commutative are unique.

We now address the naturality of the construction $\hat{\mathcal{C}}(G, T)$ with respect to $T$.

**Proposition 6.12.** An inclusion $T \subseteq T'$ of scaling sets gives rise to natural transformations of functors

$$\text{res}_{T'}^T: \mathcal{C}(-, T') \to \mathcal{C}(-, T), \quad \text{res}_{T'}^T(g) = g|_T$$

$$\hat{\text{res}}_{T'}^T: \hat{\mathcal{C}}(-, T') \to \hat{\mathcal{C}}(-, T), \quad \hat{\text{res}}_{T'}^T([g]) = [g|_T]$$

Moreover, $\hat{\text{res}}_{T'}^T: \hat{\mathcal{C}}(G, T') \to \hat{\mathcal{C}}(G, T)$ is a metric quotient.

**Proof.** Fix $G$. For any scaling set $S$ and let $K(S)$ denote the kernel of $\pi_*: \mathcal{D}(G, S) \to \mathcal{C}(G, S)$.

First, we show that there are homomorphisms $\text{res}_{T'}^T: \mathcal{C}(G, T') \to \mathcal{C}(G, T)$ and $\hat{\text{res}}_{T'}^T: \hat{\mathcal{C}}(G, T') \to \hat{\mathcal{C}}(G, T)$ rendering the following squares
commutative where the arrows in the first rows are defined in Proposition [5.10]

\[ \mathcal{D}(G, T') \xrightarrow{\text{res}_{T'}^T} \mathcal{D}(G, T) \quad \mathcal{C}(G, T') \xrightarrow{\text{res}_{T'}^T} \mathcal{C}(G, T) \]

\[ \hat{\pi} \downarrow \quad \hat{\pi} \downarrow \]

\[ \mathcal{D}(G, T) \xrightarrow{\hat{\pi}} \hat{\mathcal{D}}(G, T) \quad \mathcal{C}(G, T) \xrightarrow{\hat{\pi}} \hat{\mathcal{C}}(G, T) \]

We consider the first diagram. Choose some \( w \in K(T') \). Then \( \pi_*(w) = \pi \circ w \) is the constant function \( 1 : T' \to F(G) \). Therefore, \( \pi_*(\text{res}_{T'}^T(w)) = \pi \circ w|_T \) is the constant function \( 1 : T \to F(G) \), so \( \text{res}_{T'}^T \) in the first row carries \( K(T') \) into \( K(T) \). Since \( \pi_* \) is surjective, the dotted arrow \( \text{res}_{T'}^T \) rendering the square commutative exists and is unique. Its uniqueness and the fact that \( \text{res}_{T'}^T \) in the first row is a natural transformation, shows that the same holds for \( \text{res}_{T'}^T \) at the bottom. By inspection, \( \text{res}_{T'}^T(g) = g|_T \).

We now consider the second square. Choose some \([w] \in \hat{K}(T')\). Notice that

\[ \hat{\pi}(\hat{\text{res}}_{T'}^T([w])) = \hat{\pi}([w]|_T) = [\pi \circ w|_T]. \]

Since \( \hat{\pi}([w]) = [\pi \circ w] \) is the trivial element in \( \hat{\mathcal{C}}(G, T') \), Lemma [6.6] implies

\[ 0 = \lim_{t \in T} \frac{1}{t} \| \pi \circ w(t) \| = \lim_{t \in T} \frac{1}{t} \| \pi \circ w|_T(t) \|. \]

It follows from the same lemma that \( \hat{\pi}(\hat{\text{res}}_{T'}^T([w])) \) is the trivial element in \( \hat{\mathcal{C}}(G, T) \). Therefore \( \text{res}_{T'}^T \) at the first arrow carries \( \hat{K}(T') \) into \( \hat{K}(T) \). Since \( \hat{\pi} \) is surjective the dotted arrow \( \hat{\text{res}}_{T'}^T \) rendering the square commutative exists and is unique. The same argument above shows that It is a natural transformation of functors and that it is given by the formula \( \hat{\text{res}}_{T'}^T([g]) = [g]|_T \). Proposition [5.10] shows that \( \hat{\text{res}}_{T'}^T \) in the first row is an isometry. Since \( \hat{\pi} \) is a metric quotient (see Section [6.3]), Proposition [2.9] shows that the dotted arrow \( \hat{\text{res}}_{T'}^T \) is a metric quotient too. \( \square \)

**Proposition 6.13.** The homomorphism \( \hat{\text{res}}_{T'}^T : \hat{\mathcal{C}}(G, T') \to \hat{\mathcal{C}}(G, T) \) is an isometry if and only if it is injective.

**Proof.** \( \hat{\mathcal{C}}(G, T) \) and \( \hat{\mathcal{C}}(G, T') \) are metric groups (not just pseudometric). \( \square \)
7. Metric and topological properties of the directional asymptotic cone

Running assumptions: All group $G, H, \ldots$ are equipped with a conjugation invariant norm. Throughout $T, T', \ldots$ denote scaling sets.

7.1. By Proposition 6.4, $\hat{\mathcal{C}}(G, T)$ is a complete metric space.

Proposition 7.2. If $G$ is countable then $\hat{\mathcal{C}}(G, T)$ is a separable.

Proof. Clearly $\mathcal{F}_Q(G)$ is countable, so $\hat{\mathcal{D}}(G, T)$ is separable by Lemma 4.11 and Proposition 5.8. Since $\hat{\pi}$ is a metric quotient, $\hat{\mathcal{C}}(G, T)$ is separable. □

Lemma 7.3. Suppose that $G$ is a group equipped with a conjugation invariant pseudonorm and that $\pi : G \to H$ is a metric quotient. If $G$ is a length space then so is $H$.

Proof. The norms are invariant to translation so it suffices to prove that for any $h \in H$ and $\epsilon > 0$ there exists path $\gamma : I \to H$ from 1 to $h$ such that $\ell(\gamma) < \|h\| + \epsilon$. Choose $g \in G$ such that $\|g\| < \|h\| + \frac{\epsilon}{2}$ and a path $\beta : I \to G$ from 1 to $g$ with $\ell(\beta) < \|g\| + \frac{\epsilon}{2}$. Set $\gamma = \pi \circ \beta$. Since $\pi$ is Lipschitz with constant 1 we get $\ell(\gamma) \leq \ell(\beta) < \|h\| + \epsilon$. □

Corollary 7.4. $\hat{\mathcal{C}}(G, T)$ is a length space.

Proof. By Section 6.3, $\hat{\pi} : \hat{\mathcal{D}}(G, T) \to \hat{\mathcal{C}}(G, T)$ is a metric quotient. Apply Corollary 5.15 and Lemma 7.3. □

Our next goal is to prove that under some conditions on the scaling set, the directional asymptotic cone is a contractible group.

Definition 7.5. A scaling set $T$ is called ample if for every $s > 0$ there exists a function $\sigma : T \to T$ such that $\lim_{t \in T} \sigma(t) = s$.

Example 7.6. (i) The maximal scaling set $T = (0, \infty)$ is ample. In this case $\sigma(t) = st$.

(ii) $T = \{1, 2, 3, \ldots\}$ is ample. In this case $\sigma(t) = \lfloor st \rfloor$.

(iii) Similarly, if $T$ is a non-constant arithmetic sequence of positive integers, then it is ample.

(iv) $T = \{2^n : n = 0, 1, 2, \ldots\}$ is not ample.

The following lemma is an exercise in Calculus and is left to the reader.

Lemma 7.7. Let $T$ be a scaling set, $f : T \to \mathbb{R}$ a (bounded) function and $\sigma : T \to T$ a function such that $\lim_{t \in T} \sigma(t) = \infty$. Then

$$\limsup_{t \in T} f(\sigma(t)) \leq \limsup_{t \in T} f(t).$$
Recall the homomorphism \( \iota: F_B(G) \to \hat{D}(G,T) \) from Definition 5.7 and Proposition 5.8.

**Lemma 7.8.** Suppose that \( T \) is an ample scaling set. For \( \alpha \in F_B(G) \) and \( s > 0 \) set \( [g] = \hat{\pi}(\iota(\alpha)) \) and \( [h] = \hat{\pi}(\iota(\psi_s(\alpha))) \). Then
\[
\limsup_{t \in T} \frac{1}{t} \|h(t)\|_G \leq s \cdot \|[g]\|_{\hat{C}(G,T)}.
\]

**Proof.** Suppose that \( \alpha = \overline{\alpha}(r_1) \cdots \overline{\alpha}(r_k) \). By the definitions of \( \psi_s \), of \( \iota \) and of \( \hat{\pi} \), for any \( t \in T \)
\[
g(t) = g_1^{\|r_1\|} \cdots g_n^{\|r_n\|} \quad \text{and} \quad h(t) = g_1^{\|r_1\|} \cdots g_n^{\|r_n\|}.
\]
Let \( \sigma: T \to T \) be a function such that \( \lim_{t \in T} \frac{\sigma(t)}{t} = s \). Since \( s > 0 \) it is clear that \( \lim_{t \in T} \sigma(t) = \infty \). Let \( k: T \to G \) be the function \( k(t) = g_1^{\|r_1\|} \cdots g_n^{\|r_n\|} \). By Lemma 2.4, for any \( t \in T \)
\[
\frac{1}{t} \|h(t)^{-1}k(t)\| \leq \sum_{i=1}^n \|g_i\| \cdot \left\| \frac{\|r_i\|}{t} - \frac{\|r_i\|}{t} \right\| \xrightarrow{t \to \infty} 0.
\]
Use Lemmas 7.7 and 6.5 to estimate:
\[
\limsup_{t \in T} \frac{1}{t} \|h(t)\| = \limsup_{t \in T} \frac{1}{t} \|k(t)\|
\]
\[
= \limsup_{t \in T} \frac{\sigma(t)}{t} \cdot \|g_1^{\|r_1\|} \cdots g_n^{\|r_n\|}\|
\]
\[
\leq s \cdot \|g_1^{\|r_1\|} \cdots g_n^{\|r_n\|}\|
\]
\[
\leq s \cdot \|[g]\|_{\hat{C}(G,T)}.
\]
\[\square\]

Recall the continuous map \( h: [0, \infty) \times \hat{D}(G,T) \to \hat{D}(G,T) \).

**Lemma 7.9.** Suppose that \( T \) is ample. Let \( K \) denote the kernel of \( \hat{\pi}: \hat{D}(G,T) \to \hat{C}(G,T) \). Then \( h_s(K) \leq K \) for any \( s \geq 0 \).

**Proof.** We suppress \( T \) from the notation. When \( s = 0 \) there is nothing to prove since \( h_s \) is the trivial homomorphism. So we assume \( s > 0 \).

Consider some \([u] \in K\) represented by \( u \in D(G)\). By Proposition 5.8 the image of \( \iota \) is dense in \( \hat{D}(G) \). Choose a sequence \( \alpha_k \in F_B(G) \) such that \( \lim_k \iota(\alpha_k) = [u] \). Since \( \hat{\pi} \) is continuous,
\[
\lim_k \hat{\pi}(\iota(\alpha_k)) = \hat{\pi}([u]) = 1.
\]
(The limit is in \( \hat{C}(G) \)). Denote \([w] = h_s([u])\) for some \( w \in D(G)\). Our goal is to show that \( \hat{\pi}([w]) \) is the trivial element in \( \hat{C}(G) \).
Denote \([v_k] = \iota(\psi_s(\alpha_k))\) for some \(v_k \in \mathcal{D}(G)\). Then by Proposition 5.12
\[
[w] = h_s([u]) = h_s(\lim_k \iota(\alpha_k)) = \lim_k \iota(\psi_s(\alpha_k)) = \lim[v_k].
\]
(the limit is in \(\hat{\mathcal{D}}(G)\)). Since \(\hat{\pi}\) is continuous,
\[
\lim_k \hat{\pi}([v_k]) = \hat{\pi}([w]).
\]
(The limit is in \(\hat{\mathcal{C}}(G)\)). By definition \(\hat{\pi}([w]) = [\pi \circ w]\). By the triangle inequality and the subadditivity of \(\limsup\), for any \(k \geq 1\)
\[
\limsup_{t \in T} \frac{1}{t}||\pi(w(t))|| \leq \limsup_{t \in T} \frac{1}{t}||\pi(v_k(t)^{-1}w(t))|| + \limsup_{t \in T} \frac{1}{t}||\pi(v_k(t))||.
\]
By definition \(\hat{\pi}([v_k]^{-1}[w])\) is represented by the function \(\pi(v_k(t)^{-1}w(t))\) so by Lemma 6.5 the first term on the right hand side is estimated by
\[
\limsup_{t \in T} \frac{1}{t}||\pi(v_k(t)^{-1}w(t))|| \leq ||\hat{\pi}([v_k]^{-1}[w])||_{\hat{\mathcal{C}}(G)} \xrightarrow{k \to \infty} 0.
\]
For the second term, notice that \(\pi \circ v_k\) represents \(\hat{\pi}([v_k]) = \hat{\pi}(\iota(\psi_s(\alpha_k)))\) so (Lemma 7.3) yields
\[
\limsup_{t \in T} \frac{1}{t}||\pi(v_k(t))|| \leq s \cdot ||\hat{\pi}(\iota(\alpha_k))||_{\hat{\mathcal{C}}(G)} \xrightarrow{k \to \infty} 0.
\]
Hence \(\limsup_{t \in T} \frac{1}{t}||\pi(w(t))|| = 0\) and Lemma 6.6 shows that \(\hat{\pi}([w]) = 1\).

**Theorem 7.10.** Suppose that \(T\) is ample. There exists a continuous function \(h: [0, \infty) \times \hat{\mathcal{C}}(G, T) \to \hat{\mathcal{C}}(G, T)\) rendering the following diagram commutative
\[
\begin{array}{ccc}
[0, \infty) \times \hat{\mathcal{D}}(G, T) & \xrightarrow{h} & \hat{\mathcal{D}}(G, T) \\
\uparrow_{[0, \infty) \times \hat{\pi}} & & \downarrow_{\hat{\pi}} \\
[0, \infty) \times \hat{\mathcal{C}}(G, T) & \xrightarrow{\hat{h}} & \hat{\mathcal{C}}(G, T).
\end{array}
\]
It has the property that \(\hat{h}_t: \hat{\mathcal{C}}(G, T) \to \hat{\mathcal{C}}(G, T)\) are homomorphisms such that \(\|\hat{h}_t(x)\|_{\hat{\mathcal{C}}(G)} = t \cdot \|x\|_{\hat{\mathcal{C}}(G)}\) and \(\hat{h}_0 = \text{id}\) and \(\hat{h}_1 = \hat{id}\) is the trivial homomorphism and \(\hat{h}_t \circ \hat{h}_s = \hat{h}_{ts}\). In particular, \(\hat{\mathcal{C}}(G, T)\) is contractible and \(\text{End}(\hat{\mathcal{C}}(G, T))\) (suitably topologized) is contractible.

**Proof.** We suppress \(T\) from the notation. By Lemma 7.9 for every \(s \geq 0\) there is a homomorphism \(\hat{h}_s: \hat{\mathcal{C}}(G) \to \hat{\mathcal{C}}(G)\) such that the following
We obtain a function \( \hat{h} \) as in the statement of the theorem which renders the square in its statement commutative. By Proposition 5.12
\[
\|h_t(x)\|_{\hat{D}(G)} = t \cdot \|x\|_{\hat{D}(G)}
\]
for all \( x \in \hat{D}(G) \). Since \( \hat{\pi} \) is a metric quotient, it follows that \( \|\hat{h}_t(x)\|_{\hat{C}(G)} \leq t \cdot \|x\|_{\hat{C}(G)} \) for all \( x \in \hat{C}(G) \), namely \( \hat{h}_t \) is Lipschitz with constant \( t \). This also shows that \( \hat{h}_s : [0, \infty) \rightarrow [0, \infty) \) is Lipschitz for any \( x \in \hat{C}(G) \), hence continuous. Apply Lemma 5.11 with \( A = X = Y = \hat{C}(G) \) and \( f = \hat{h} \) to deduce that \( \hat{h} \) is continuous.

Since \( \hat{\pi} \) is surjective, \( \hat{h}_1 \) rendering the diagram commutative is unique and therefore \( \hat{h}_1 = \text{id} \) and \( \hat{h}_0 \) is trivial and \( \hat{h}_t \circ \hat{h}_s = \hat{h}_{ts} \), because \( h_t \) has these properties. In particular \( \hat{h} \) gives a homotopy from \( \hat{h}_1 = \text{id}_{\hat{C}(G)} \) and the trivial homomorphism \( \hat{h}_0 \), so \( \hat{C}(G) \) is contractible. Since \( \hat{h}_t \) are homomorphism, this contraction is a homotopy in the category of groups, so \( \text{End}(\hat{C}(G)) \) with the compact-open topology is contractible as well. 

\[\square\]

8. Algebraic properties of the directional asymptotic cone

**Running assumptions:** All group \( G, H, \ldots \) are equipped with a conjugation invariant norm. Throughout \( T, T', \ldots \) denote scaling sets.

**Proposition 8.1.** If \( G \) is abelian then \( \hat{C}(G,T) \) is abelian. More generally, if \( G \) is nilpotent of class \( n \) then so is \( \hat{C}(G,T) \).

**Proof.** The product in \( C(G,T) \) is induced by the point-wise product of functions \( f : T \rightarrow G \) and \( \hat{C}(G,T) \) is a quotient. \[\square\]

Recall that \( g \in G \) is called **distorted** if its stable length vanishes i.e \( \tau(g) = 0 \), see paragraph 6.8

**Proposition 8.2.** If \( G \) is bounded then \( \hat{C}(G,T) = 1 \). More generally, if every element in \( G \) is distorted then \( \hat{C}(G,T) = 1 \).

**Proof.** Recall that \( \hat{\pi} \circ \iota : F_\mathbb{R}(G) \rightarrow \hat{C}(G,T) \) is Lipschitz with dense image, see Proposition 5.8 and paragraph 6.3 It remains to show its image it trivial, which is the case if the image of every generator
\[ \alpha = \overline{g}(r) \in F_r(G) \text{ is trivial.} \]

Now \([h] := \hat{p}(\iota(\alpha))\) is represented by \(h: T \to G\) where \(h(t) = g^{\|rt\|}\). Then

\[
\limsup_{t \in T} \frac{1}{t} \|h(t)\| = \limsup_{t \in T} \frac{\|rt\|}{t} \cdot \frac{1}{\|rt\|} \|g^{\|rt\|}\| = |r| \cdot \tau(g) = 0
\]

because \(T\) is unbounded. Lemma 6.6 implies that \([h] = 1\).

Let \(G\) and \(H\) be (pseudo) normed groups. The associated \(L^1\) (pseudo) norm on \(G \times H\) is defined by \(\|g \times h\|_{G \times H} := \|g\|_G + \|h\|_H\). Clearly if \(\|\cdot\|_G\) and \(\|\cdot\|_H\) are conjugation-invariant then so is \(\|\cdot\|_{G \times H}\).

**Theorem 8.3.** Equip \(G \times H\) with the associated \(L^1\) norm. Then for any scaling set \(T\) there is a natural isometry of groups

\[
\hat{C}(G \times H, T) \xrightarrow{\ [g \times h] \mapsto [g] \times [h]} \hat{C}(G, T) \times \hat{C}(H, T)
\]

where the codomain is equipped with the associated \(L^1\) norm.

**Proof.** Let \(\lambda^G: G \times H \to G\) and \(\lambda^H: G \times H \to H\) denote the canonical projections. Let \(i^G: G \to G \times H\) and \(i^H: H \to G \times H\) be the canonical inclusions. They are clearly Lipschitz with constant 1. By Proposition 6.11 they induce homomorphisms \(\lambda^G_* := \hat{\chi}(\lambda^G)\) and \(\lambda^H_* := \hat{\chi}(\lambda^H)\) and \(i^G_* := \hat{\chi}(i^G)\) and \(i^H_* := \hat{\chi}(i^H)\), all are Lipschitz with constant 1. By the description of \(i^G_*\) and \(i^H_*\) in By Proposition 6.11 and since \(G\) and \(H\) commute in \(G \times H\), it follows that their images in \(\hat{\chi}(G \times H)\) commute. There results a homomorphism

\[
i: \hat{\chi}(G) \times \hat{\chi}(H) \to \hat{\chi}(G \times H), \quad i([g], [h]) = i^G_*([g]) \cdot i^H_*([h]).
\]

Then \(i\) is Lipschitz with constant 1 because the metric on the domain is the \(L^1\) metric so

\[
\|i([g], [h])\|_{\hat{\chi}(G \times H)} = \|i^G_*([g]) \cdot i^H_*([h])\| \leq \|i^G_*([g])\| + \|i^H_*([h])\|
\]

\[
\leq \|[g]\| + \|[h]\| = \|([g], [h])\|_{\hat{\chi}(G) \times \hat{\chi}(H)}.
\]

Use \(\lambda^G_*\) and \(\lambda^H_*\) to define a homomorphism

\[
\Lambda = : \hat{\chi}(G \times H) \xrightarrow{(\lambda^G_*, \lambda^H_*)} \hat{\chi}(G) \times \hat{\chi}(H).
\]

By the description of the homomorphisms in Proposition 6.11 it is clear that \(\Lambda\) and \(i\) are inverses of each other because for any \([g] \in \hat{\chi}(G)\) and \([h] \in \hat{\chi}(H)\) represented by \(g: T \to G\) and \(h: T \to H\),

\[
\Lambda(i([g(t)], [h(t)])) = \Lambda([g(t) \times 1 \cdot [1 \times h(t)])) = ([g], [h])
\]

This show that \(\Lambda \circ i\) is the identity. Similarly, since any function \(f: T \to G \times H\) is a pair of functions \(g(t)\) and \(h(t)\), it follows that \(i \circ \Lambda\) is the identity.
The homomorphisms $\lambda^G, \lambda^H$ give rise to a homomorphism
$$\lambda: F_\mathbb{R}(G \times H) \to F_\mathbb{R}(G) \times F_\mathbb{R}(H)$$
which on generators has the effect $\lambda(g \times h(r)) = (g(r), h(r))$. One easily checks that this assignment respects the relations (2) in Paragraph 4.3, hence $\lambda$ is well defined. It is Lipschitz with constant 1 by Proposition 2.11 because for a generators we get
$$\|\lambda(g \times h(r))\| = (\|g(r)\| + \|h(r)\|) = |r| \cdot (\|g\| + \|h\|) = |r| \cdot \|g \times h(r)\|.$$ 
By checking generators of $F_\mathbb{R}(G \times H)$ and using that $G, H$ commute in $G \times H$, it follows that the following square commutes
$$\begin{array}{ccc}
F_\mathbb{R}(G \times H) & \xrightarrow{\lambda} & F_\mathbb{R}(G) \times F_\mathbb{R}(H) \\
\hat{\pi} \circ \iota & & (\hat{\pi} \circ \iota) \\
\hat{\mathcal{C}}(G \times H) & \xrightarrow{\Lambda} & \hat{\mathcal{C}}(G) \times \hat{\mathcal{C}}(H).
\end{array}$$
Upon completion we get a homomorphism $\hat{\lambda}: F_\mathbb{R}(G \times H) \to F_\mathbb{R}(G) \times F_\mathbb{R}(H)$, Lipschitz with constant 1. By Proposition 5.8 and Paragraph 6.3 we get the commutative diagram
$$\begin{array}{ccc}
F_\mathbb{R}(G \times H) & \xrightarrow{\hat{\lambda}} & F_\mathbb{R}(G) \times F_\mathbb{R}(H) \\
\hat{\pi} & & (\hat{\pi}, \hat{\pi}) \\
\hat{\mathcal{C}}(G \times H) & \xrightarrow{\Lambda} & \hat{\mathcal{C}}(G) \times \hat{\mathcal{C}}(H).
\end{array}$$
where the vertical arrows are metric quotient homomorphisms. Then $\Lambda$ is Lipschitz with constant 1 by Proposition 2.8. Since its inverse $i$ is also Lipschitz with constant 1 both of them are isometries. \(\square\)

**Theorem 8.4.** Let $f: G \to H$ be a surjective homomorphism, Lipschitz with constant $C$. Assume further that

(a) $\text{Ker}(f)$ is a bounded subgroup of $G$.

(b) $f$ admits a set-theoretic section $s: H \to G$ (not necessarily a homomorphism) such that there exists $C'$ such that $\|s(h)\|_G \leq C'\|h\|$ for all $h \in H$.

Then $\hat{\mathcal{C}}(f, T): \hat{\mathcal{C}}(G, T) \to \hat{\mathcal{C}}(H, T)$ is an isomorphism, Lipschitz with constant $C$, whose inverse is the homomorphism
$$s_!: \hat{\mathcal{C}}(H, T) \xrightarrow{[h] \mapsto [sch]} \hat{\mathcal{C}}(G, T)$$
which is Lipschitz with constant $C'$. 
Our next goal is to prove that there is a homomorphism $s$ commutative. Choose some $\left[ \cdot \right]$ in the statement of the theorem that renders the square $C$.

\textbf{Proof.}\ We will suppress $T$ from the notation and write $K$ for $\ker(f)$.

First, if $k: T \to G$ is a function with image in $K$ then by Lemma \ref{lem:homomorphism} $k \in C(G)$ and $\left[ k \right] = 1 \in \hat{C}(G)$ because

$$\limsup_{t \in T} \frac{1}{t} \|k(t)\| \leq \limsup_{t \in T} \frac{\text{diam}_C(K)}{t} = 0.$$  

Therefore, if $g: T \to G$ is in $C(G)$ then $g \cdot k \in C(G)$ and $\left[ g \right] = [g \cdot k]$ in $\hat{C}(G)$.

By Proposition \ref{prop:homomorphism} $f$ and $s$ induce homomorphisms $f_*: F(G) \to F(H)$ and $s_*: F(H) \to F(G)$, Lipschitz with constants $C$ and $C'$. Recall that $\pi^G: F(G) \to G$ and $\pi^H: F(H) \to H$ are the canonical homomorphisms. Since $f \circ s = \text{Id}$ it follows that $f_* \circ s_* = \text{Id}$, so \eqref{eq:homomorphism} implies that for any $w \in F(H)$

$$f(\pi^G(s_*(w))) = \pi^H(f_*(s_*(w))) = \pi^H(w) = f(s(\pi^H(w))).$$

We deduce that for every $w \in F(H)$

$$\pi^G(s_*(w)) \text{ and } s(\pi^H(w)) \text{ differ by an element of } K.$$  

By Proposition \ref{prop:homomorphism} $f$ induces $\hat{s}(f): \hat{C}(G) \to \hat{C}(H)$, Lipschitz with constant $C$ given by $\hat{s}(f)([g]) = [f \circ g]$. A-priori a similar $\hat{s}(s)$ does not exist since $s$ is not a homomorphism. By Proposition \ref{prop:homomorphism} $s$ induces $\hat{D}(s): \hat{D}(H) \to \hat{D}(G)$, Lipschitz with constant $C'$, given by $\hat{D}(s)([w]) = [s_* \circ w]$. We obtain the following solid diagram

$$\begin{array}{ccc}
\hat{D}(H) & \xrightarrow{\hat{D}(s)} & \hat{D}(G) \\
\downarrow_{\hat{s}^H} & & \downarrow_{\hat{s}^G} \\
\hat{C}(H) & \xrightarrow{s_1} & \hat{C}(G).
\end{array}$$

Our next goal is to prove that there is a homomorphism $s_1$ of the form given in the statement of the theorem that renders the square commutative. Choose some $[h] \in \hat{C}(H)$ represented by $h: T \to H$ in $\hat{C}(H)$. Choose some $w \in \hat{D}(H)$ which lifts $h$, namely $w: T \to F(H)$ is in $\hat{D}(H)$ and $h = \pi^H_*(w) = \pi^H \circ w$. Set

$$s_1([h]) \overset{\text{def}}{=} \hat{s}^G(\hat{D}(s)([w])) = \hat{s}^G([s_* \circ w]) = [\pi^G \circ s_* \circ w].$$

We have observed above that $\pi^G \circ s_* \circ w$ and $s \circ \pi^H \circ w$ differ by some $k: T \to G$ with values in $K$ and therefore they represent the same element in $\hat{C}(G)$, i.e

$$s_1([h]) = [s \circ \pi^H \circ w] = [s \circ h].$$
In particular $s_1([h])$ is independent of the choice of $w$. In order to prove that $s_1$ is independent of the choice of representative $h: T 	o H$, observe that $s(h_1) s(h_2)$ and $s(h_1 h_2)$ differ by an element of $K$ for any $h_1, h_2 \in H$.

If $h': T \to H$ and $h: T \to H$ represent the same element in $\hat{C}(H)$ then $h' = h \cdot \delta$ for some $\delta: T \to H$ which represents the trivial element in $\hat{C}(H)$. By Lemma 6.6 this is equivalent to $\lim sup_{t \in T} \frac{\| \delta(t) \|}{t} = 0$. Then for any $t \in T$

$$s(h'(t)) = s(h(t) \cdot \delta(t)) = s(h(t)) \cdot s(\delta(t)) \cdot k(t)$$

for some $k: T \to K \subseteq G$. We have seen that $[k] = 1$. Since $s$ is proto-Lipschitz, $\lim sup_{t \in T} \frac{\| s(\delta(t)) \|}{t} \leq \lim sup_{t \in T} C' \frac{\| \delta(t) \|}{t} = 0$, hence $[s \circ \delta] = 1$. We deduce that $[s \circ h'] = [s \circ h]$, so $s_1$ is independent of representatives, and is therefore well defined. It is a homomorphism because given $[h_1], [h_2] \in \hat{C}(H)$ observe that for any $t \in T$

$$s(h_1(t) \cdot h_2(t)) = s(h_1(t)) \cdot s(h_2(t)) \cdot k(t)$$

for some $k: T \to K \subseteq G$, so $s_1([h_1 \cdot h_2]) = [s \circ (h_1 \cdot h_2)] = [(s \circ h_1) \cdot (s \circ h_2) \cdot k] = s_1([h_1]) \cdot s_1([h_2])$.

Clearly, by its definition $s_1$ renders the square above commutative. Since $\hat{\pi}^H$ and $\hat{\pi}^G$ are metric quotient maps and $\hat{D}(s)$ is Lipschitz with constant $C'$, Proposition 2.8 shows that $s_1$ is Lipschitz with constant $C'$.

Since $f \circ s = \text{Id}$, for any $[h] \in \hat{C}(H)$

$$\hat{C}(f)(s_1([h])) = [f \circ s \circ h] = [h].$$

Hence, $\hat{C}(f) \circ s_1 = \text{Id}$ and in particular $s_1$ injective. Given $[g] \in \hat{C}(G)$ set $[h] = \hat{C}(f)([g]) = [f \circ g]$. Since $s$ is a section, $s \circ f \circ g$ differs from $g$ by some $k: T \to K \subseteq G$ and in particular $[s \circ f \circ g] = [g]$. Then $s_1([h]) = [s \circ f \circ g] = [g]$.

This shows that $s_1$ is surjective. Hence it is an isomorphism and the same is true for $\hat{C}(f)$. \hfill \Box

**Corollary 8.5.** Let $\pi: G \to H$ be a metric quotient with bounded kernel. Then $\hat{C}(\pi): \hat{C}(G, T) \to \hat{C}(H, T)$ is an isometry.

**Proof.** Since $\pi$ is a metric quotient (Definition 2.5) it is Lipschitz with constant 1 and, given $\epsilon > 0$ one can choose a section $s: H \to G$ such that $\| h \|_H \leq \| s(h) \|_G \leq \| h \|_H (1 + \epsilon)$. By Theorem 8.4 $\hat{C}(\pi): \hat{C}(G, T) \to \hat{C}(H, T)$ is bijective, Lipschitz with constant 1 and its inverse $s_1$ is
Lipschitz with constant \((1 + \epsilon)\). Since \(\epsilon\) was arbitrary, \(s_i\) is Lipschitz with constant 1, hence \(\hat{C}(\pi)\) is an isometry.

The next result deals with the directional asymptotic cones of subgroups of \(G\) which normally generated by words in \(G\) of a fixed pattern. Examples include the commutator subgroup of \(G\), or the subgroups \(\Gamma^n G\) of the upper central series.

Let \(w \in F(X)\) be thought of as a word in the alphabet \(X\). Given an assignment of elements of \(G\) to the letters in \(X\), i.e. a function \(\alpha : X \to G\), let \(w(\alpha) \in G\) denote the evaluation of the word \(w\) at the assignment \(\alpha\). Let \(w(G)\) denote the subset of \(G\) of all such evaluations. Notice that if \(N \leq G\) then \(w(N) \leq N\) so \(w(G/N) = w(G)N/N\).

**Proposition 8.6.** Consider some \(w \in F(X)\). Let \(H \leq G\) be the normal subgroup generated by \(w(G)\). Equip \(w(G)\) with a bounded length function \(\mu\) and equip \(H\) with the associated conjugation invariant pseudonorm in Section 2.10. Then \(w(\hat{C}(H,T)) = 1\) for any scaling set \(T\).

**Proof.** We suppress \(T\) from the notation. Say \(\mu\) is bounded by \(C\).

Consider a function \(\alpha : X \to \hat{C}(H)\). Then \(\alpha(x) = [h(x)]\) for some \(h(x) : T \to H\) in \(C(H)\). Thus \(\alpha \in (H^T)^X\) and it corresponds to \(\alpha' \in (H^X)^T\) where \(\alpha'(t)(x) = \alpha(x)(t)\). By inspection \(w(\alpha) \in \hat{C}(H)\) is represented by the function \(k : T \to H\) defined by \(k(t) = w(\alpha'(t))\). Since \(w(\alpha'(t))\) is a generator of \(H\) and since \(\mu\) is bounded, it follows that \(\|k(t)\| \leq \mu(w(\alpha'(t))) \leq C\). Therefore \(\limsup_{t \in T} \frac{\|k(t)\|}{\|k(t)\|} = 0\) and by Lemma 6.6 \(w(\alpha) = [k] = 1\). Since \(\alpha\) was arbitrary, \(w(\hat{C}(H)) = 1\).

9. **Independence of scaling**

**Running assumptions:** All group \(G, H, \ldots\) are equipped with a conjugation invariant norm. Throughout \(T, T', \ldots\) denote scaling sets.

**Definition 9.1.** \(G\) is called **independent of scaling** if for any inclusion \(T \subseteq T'\) of scaling sets, \(\tilde{\text{res}}^{T'}_T : \hat{C}(G, T') \to \hat{C}(G, T)\) is an isometry.

**Notation.** For the remainder of the section, when we write \(\hat{C}(G)\) we mean \(\hat{C}(G, (0, \infty))\), i.e the full scaling set.

It is clear that if \(T \subseteq T' \subseteq T''\) are scaling sets then \(\tilde{\text{res}}^{T''}_T \circ \tilde{\text{res}}^{T'}_T = \tilde{\text{res}}^{T''}_T\). Applying this to the maximal scaling set \(T'' = (0, \infty)\) we get:

**Proposition 9.2.** A group \(G\) is independent of scaling if and only if the homomorphisms \(\tilde{\text{res}}_T : \hat{C}(G) \to \hat{C}(G, T)\) are isometries for all scaling sets \(T\).
9.3. Let \([g] \in \hat{C}(G, T)\). If \(\lim_{t \to T} \frac{\|g(t)\|}{t}\) exists then it is independent of the representative because by Lemma 6.6, \(g, h: T \to G\) represent the same element in \(\hat{C}(G, T)\) if and only if \(\lim_{t \to T} \frac{\|g(t) - h(t)\|}{t} = 0\).

Lemma 9.4. Let \(G\) be a group. If \(\lim_{t} \frac{\|g(t)\|}{t}\) exist for any \([g] \in \hat{C}(G)\) then \(G\) is independent of scaling.

Proof. By Propositions 9.2 and 6.13 we need to show that \(\widehat{\text{Res}}: \hat{C}(G) \to \hat{C}(G, T)\) are injective. If \([g] \in \hat{C}(G)\) is in the kernel, then by Lemma 6.6 \(\lim_{t \to T} \frac{\|g(t)\|}{t} = 0\), and since \(T \subseteq (0, \infty)\) is unbounded \(\lim_{t \to T} \frac{\|g(t)\|}{t} = 0\), so \([g] = 1\).

Recall that \(F_\mathbb{Q}(A)\) denotes the subgroup of \(F_\mathbb{R}(A)\) generated by the symbols \(a(q)\) where \(a \in A\) and \(q \in \mathbb{Q}\). Once again, we will consider the homomorphisms \(\hat{\pi}: \hat{D}(G, T) \to \hat{C}(G, T)\) and \(\iota: F_\mathbb{R}(G) \to \hat{D}(G, T)\) from Section 6.3 and Definition 5.7.

Lemma 9.5. Suppose that \(\lim_{t \to T} \frac{\|h(t)\|}{t}\) exists for any \([h] \in \hat{C}(G)\) in the image of \(F_\mathbb{Q}(G)\) under \(\hat{\pi} \circ \iota\). Then \(\lim_{t \to T} \frac{\|g(t)\|}{t}\) exists for all \([g] \in \hat{C}(G)\). If, in addition, for every \([h]\) of the form above

\[
\lim_{t \to T} \frac{\|h(t)\|}{t} = \|[h]\|_{\hat{C}(G)},
\]

then for every \([g] \in \hat{C}(G)\)

\[
\|[g]\|_{\hat{C}(G)} = \lim_{t \to T} \frac{\|g(t)\|}{t}.
\]

Proof. We need to show that \(\frac{\|g(t)\|}{t}\) is Cauchy. Choose \(\epsilon > 0\). Since \(\iota(F_\mathbb{R}(G))\) is dense in \(\hat{C}(G)\) by Proposition 5.8 and \(F_\mathbb{Q}(G)\) is dense in \(F_\mathbb{R}(G)\) by Lemma 4.11 and \(\hat{\pi}\) is Lipschitz, we can choose \(\alpha \in F_\mathbb{Q}(G)\) such that \([h] = \hat{\pi}(\iota(\alpha))\) is at distance \(< \frac{\epsilon}{2}\) from \([g]\). That is, \(\|[g]\|^{-1} \cdot [h]\|_{\hat{C}(G)} < \frac{\epsilon}{2}\). By Lemma 6.5

\[
\limsup_{t \to T} \frac{\|g(t)\| - \|h(t)\|}{t} \leq \limsup_{t \to T} \frac{\|g(t) - h(t)\|}{t} \leq \|[g]^{-1} [h]\|_{\hat{C}(G)} < \frac{\epsilon}{2}.
\]

By hypothesis \(L = \lim_{t \to T} \frac{\|h(t)\|}{t}\) exists, so \(\frac{\|g(t)\|}{t} - L < \frac{\epsilon}{2}\) for all \(t \gg 0\). In particular \(\frac{\|g(t)\|}{t}\) is Cauchy, as needed.

Assume that \(\lim_{t \to T} \frac{\|h(t)\|}{t} = \|[h]\|_{\hat{C}(G)}\) for all \([h] \in \hat{C}(G)\) in the image of \(F_\mathbb{Q}(G)\). Given \([g] \in \hat{C}(G)\) choose a sequence \([h_n] \in \hat{C}(G)\) such that \(\|[g]^{-1} [h_n]\|_{\hat{C}(G)} < \frac{1}{n}\). It follows from the display above (and since we already know that \(\frac{\|g(t)\|}{t}\) is Cauchy) that

\[
\lim_{t \to T} \frac{\|g(t)\|}{t} - \|[h_n]\|_{\hat{C}(G)} < \frac{1}{n}.
\]
Therefore \( \lim_{t \to 0} \frac{\|g(t)\|}{t} = \lim_n \| [h_n]\|_{\hat{C}(G)} = \| [g]\|_{\hat{C}(G)} \) for any \( [g] \in \hat{C}(G) \), as claimed.

Lemma 9.7 below gives a criterion for the first condition in Lemma 9.5 and Theorem 9.9 gives a condition for the second.

**Lemma 9.6.** Any element of \( \hat{C}(G) \) in the image of \( F_\mathbb{Q}(G) \) under \( \hat{\pi} \circ \iota \) is the image of an element \( \alpha \in F_\mathbb{Q}(G) \) of the form \( \overline{g_1} (\frac{1}{m}) \cdots \overline{g_k}(\frac{1}{m}) \) for some \( g_1, \ldots, g_k \in G \) and some integer \( m > 0 \).

**Proof.** Suppose \( [h] \in \hat{C}(G) \) has the form \( \hat{\pi}(\iota(\alpha')) \) where \( \alpha' \in F_\mathbb{Q}(G) \). Then \( \alpha' = \overline{g_1}(\frac{1}{m}) \cdots \overline{g_k}(\frac{1}{m}) \) for some integers \( p_i \) and \( m > 0 \). Set \( \alpha = \overline{g_1^p}(\frac{1}{m}) \cdots \overline{g_k^p}(\frac{1}{m}) \). It follows from Lemma 9.7 that
\[
[h] = \hat{\pi}(\iota(\alpha')) = \prod_{i=1}^n \hat{\pi}(\iota(\overline{g_i}(\frac{1}{m}))) = \prod_{i=1}^n \hat{\pi}(\iota(g_i^p(\frac{1}{m}))) = \hat{\pi}(\iota(\alpha)).
\]
\[\Box\]

**Lemma 9.7.** Suppose that for any \( g_1, \ldots, g_k \in G \) the limit
\[
\lim_{n \to \infty} \frac{1}{n} \| g_1^n \cdots g_k^n \|
\]
exists. Then \( \lim_{t \to 0} \frac{\|h(t)\|}{t} \) exists for any \( [h] \in \hat{C}(G) \) in the image of \( \hat{\pi} \circ \iota : F_\mathbb{Q}(G) \to \hat{C}(G) \).

**Proof.** By Lemma 9.6 \( [h] = \hat{\pi}(\iota(\alpha)) \) where \( \alpha = \overline{g_1}(\frac{1}{m}) \cdots \overline{g_k}(\frac{1}{m}) \), thus \( h(t) = \overline{g_1^t}(\frac{1}{m}) \cdots \overline{g_k^t}(\frac{1}{m}) \). Since \( 0 \leq t - m\|\frac{1}{m}\| < m \), i.e. \( t - m\|\frac{1}{m}\| \) is bounded as a function of \( t \),
\[
\lim_{t \to \infty} \frac{\|h(t)\|}{t} = \lim_{t \to \infty} \frac{1}{m\|\frac{1}{m}\|} \| g_1^\frac{t}{m} \cdots g_k^\frac{t}{m} \| = \lim_{n \to \infty} \frac{1}{m\|} \| g_1^n \cdots g_k^n \|
\]
and the limit exists by the hypothesis. \[\Box\]

**Definition 9.8.** A group \( G \) has locally bounded commutators if the commutator subgroup of any finitely generated subgroup \( H \leq G \) is bounded, i.e \( [H, H] \) is a bounded subgroup of \( G \).

**Theorem 9.9.** Suppose that \( G \) has locally bounded commutators. Then
(a) \( G \) is independent of scaling.
(b) \( \| [g]\|_{\hat{C}(G)} = \lim_{t \to 0} \frac{\|g(t)\|}{t} \) for any \( [g] \in \hat{C}(G) \).
(c) \( \hat{C}(G) \) is abelian.

**Proof.** Consider some \( [h] \) in the image of \( \hat{\pi} \circ \iota : F_\mathbb{Q}(G) \to \hat{C}(G) \). By Lemma 9.6 \( [h] = \hat{\pi}(\iota(\alpha)) \) for some \( \alpha \in F_\mathbb{Q}(G) \) of the form \( \alpha = \overline{g_1}(\frac{1}{m}) \cdots \overline{g_k}(\frac{1}{m}) \). Set \( \alpha' = \overline{g_1} \cdots \overline{g_k}(\frac{1}{m}) \in F_\mathbb{Q}(G) \) and \( [h'] = \hat{\pi}(\iota(\alpha')) \).
Let $H$ be the subgroup of $G$ generated by $g_1, \ldots, g_k$. Clearly, for any $t > 0$ there exists $\gamma(t) \in [H,H]$ such that
\[ h(t) = g_1^{[t/m]} \cdots g_k^{[t/m]} = (g_1 \cdots g_k)^{[t/m]} \cdot \gamma(t) = h'(t) \cdot \gamma(t). \]

By Lemma 6.6 $\gamma$ represents the trivial element in $\hat{\mathcal{C}}(G)$ since by assumption $[H,H]$ is bounded in $G$. Therefore $[h] = [h']$ in $\hat{\mathcal{C}}(G)$ and by Corollary 6.9
\[
\lim_{t \to 0} \frac{1}{t} \|h(t)\| = \lim_{t \to 0} \frac{1}{t} \|h'(t)\| = \lim_{t \to 0} \frac{1}{t} \|(g_1 \cdots g_k)^{[t/m]}\|
\]
\[ = \lim_{n \to \infty} \frac{1}{n} \|(g_1 \cdots g_k)^n\| = \tau(g_1 \cdots g_k) = \|[h']\|_{\hat{\mathcal{C}}(G)} = \|[h]\|_{\hat{\mathcal{C}}(G)}. \]

The conditions of Lemma 9.5 are fulfilled by $G$, hence (b) holds, and by Lemma 9.4 $G$ is independent of scaling.

It remains to prove point (c). First, suppose that $[u], [v] \in \hat{\mathcal{C}}(G)$ are in the image of $F_R(G)$ under $\hat{\pi} \circ \iota$, see Section 6.3. Then $u(t) = h_1^{[r_1 t]} \cdots h_k^{[r_k t]}$ and $v(t) = g_1^{[s_1 t]} \cdots g_k^{[s_k t]}$ for some $h_i, g_i \in G$ and $r_i, s_i \in \mathbb{R}$. Let $K$ be the subgroup of $G$ generated by $h_i, g_i$ and let $z$ denote the commutator $[u,v]$. Then $z(t) \in [K,K]$ for any $t > 0$. Since by the assumption $[K,K]$ is bounded in $G$, Lemma 6.6 shows that $[z] = 1$ in $\hat{\mathcal{C}}(G)$. Thus, $[u] \cdot [v] = [v] \cdot [u]$ for any $[u], [v] \in \hat{\mathcal{C}}(G)$ in the image of $F_R(G)$.

Let $x, y \in \hat{\mathcal{C}}(G)$ be arbitrary. We will show that $xyx^{-1}y^{-1} = 1$. Choose some $\epsilon > 0$. By Section 6.3 there are $x', y' \in \hat{\mathcal{C}}(G)$ in the image of $F_R(G)$ such that $\|x^{-1}x'\|_{\hat{\mathcal{C}}(G)} < \epsilon$ and $\|y^{-1}y'\|_{\hat{\mathcal{C}}(G)} < \epsilon$. We leave it to the reader to check that $(xyx^{-1}y^{-1}) \cdot (x'y'x'^{-1}y'^{-1})$ is a product of four conjugates of $(x^{-1}x')^\pm 1$ and $(y^{-1}y')^\pm 1$. But we have seen above that $x'y'x'^{-1}y'^{-1} = 1$. Hence, $\|xyx^{-1}y^{-1}\|_{\hat{\mathcal{C}}(G)} < 4\epsilon$. Since $\epsilon > 0$ was arbitrary $xyx^{-1}y^{-1} = 1$.

10. Relation to the ultralimit asymptotic cone

Running assumptions: All group $G, H, \ldots$ are equipped with a conjugation invariant norm. Throughout $T, T', \ldots$ denote scaling sets.

10.1. Let $T$ be a scaling set. A $T$-valued scaling sequence is a function $d: \mathbb{N} \to T$ such that $\lim_{n \to \infty} d(n) = \infty$. We call $d$ a scaling sequence if $T = (0, \infty)$. We will write $d_n$ for $d(n)$ when this is convenient, which is the usual practice.

Throughout this section $\omega$ denotes a non principal ultrafilter on $\mathbb{N}$. The elements of $\text{Cone}_\omega(G,d)$ are denoted by $\{g_n\}$ where $\{g_n\}$ is a sequence in $G$ such that $\limsup_n \frac{1}{d_n} \|g_n\|_G < \infty$. 


Proposition 10.2. Let $T$ be a scaling set. For any $T$-valued scaling sequence and any non principal ultrafilter $\omega$ there is a well defined canonical homomorphism
\[
\rho_{\omega,d} : \hat{\mathcal{C}}(G,T) \xrightarrow{[g] \mapsto [g \circ d]} \text{Cone}_\omega(G,d).
\]
It is Lipschitz with constant 1.

Proof. Since $d_n \in T$ and $\lim_{n \to \infty} d_n = \infty$, Lemma 6.6 implies
\[
\limsup_n \frac{1}{d_n} \|g(d_n)\| \leq \limsup_{t \in T} \frac{1}{t} \|g(t)\| \leq \|[g]\|_{\hat{\mathcal{C}}(G,T)} < \infty.
\]
Thus, if $g \in \mathcal{C}(G,T)$ then $g \circ d : \mathbb{N} \to G$ represents an element in $\text{Cone}_\omega(G,d)$. To see that $\rho$ is independent of representatives, suppose that $g, g' \in \mathcal{C}(G,T)$ represent the same element in $\hat{\mathcal{C}}(G,T)$. Then $[g^{-1}g']$ is the trivial element in $\hat{\mathcal{C}}(G,T)$ so Lemma 6.6 implies
\[
\lim_n \frac{1}{d_n} \|g(d_n)^{-1}g'(d_n)\| = \lim_{t \in T} \frac{1}{t} \|g(t)^{-1}g'(t)\| = 0.
\]
Therefore $\{g \circ d\} = \{g' \circ d\}$ in $\text{Cone}_\omega(G,d)$. It follows that $\rho$ is well defined. It is a homomorphism because the group structures on $\hat{\mathcal{C}}(G,T)$ and $\text{Cone}_\omega(G,d)$ are induced from that of $G$. \qed

Recall Definition 9.1 of independence of scaling. Also recall that $\hat{\mathcal{C}}(G)$ denotes $\hat{\mathcal{C}}(G,(0,\infty))$ with the maximal scaling set $(0,\infty)$.

Theorem 10.3. The following are equivalent for a group $G$.

(a) $G$ is independent of scaling.

(b) The homomorphism $\rho_{\omega,d} : \hat{\mathcal{C}}(G) \to \text{Cone}_\omega(G,d)$ is injective for any non-principal ultrafilter $\omega$ and any scaling sequence $d$.

If, in addition, $\|[g]\|_{\hat{\mathcal{C}}(G)} = \lim_{n \to \infty} \frac{1}{n} \|g(t)\|$ for every $[g] \in \hat{\mathcal{C}}(G)$, then $\rho_{\omega,d}$ is an isometric embedding.

Proof. (a) $\implies$ (b) Fix some $\omega$ and $d$ and consider some $[g] \in \hat{\mathcal{C}}(G)$ in the kernel of $\rho = \rho_{\omega,d}$. Then $\omega \cdot \lim_n \frac{1}{d_n} \|g(d_n)\| = 0$ and therefore there is a subsequence $\{n_k\}_{k \geq 1}$ such that $\lim_{k \to \infty} \frac{1}{d_{n_k}} \|g(d_{n_k})\| = 0$. By further passage to a subsequence we may assume that $\{d_{n_k}\}_k$ is increasing. Set $T = \{d(n_k) : k \geq 1\}$. This is a scaling set and we set $[h] = \text{res}_T([g])$. Then $h = g{\mid}_T$ and since $d_{n_k}$ is increasing
\[
\limsup_{t \in T} \frac{1}{t} \|h(t)\| = \limsup_k \frac{1}{d(n_k)} \|g(d(n_k))\| = 0.
\]
By Lemma 6.6 $[h] = 1$, but by hypothesis $\text{res}_T$ is injective, so $[g] = 1$. 

Consider a scaling set \( T \). Choose a scaling sequence \( d_1 < d_2 < \ldots \) in \( T \) (this can be done since \( T \) is unbounded). By inspection the injective homomorphism \( \rho_{\omega,d} : \hat{\mathcal{C}}(G) \to \text{Cone}_\omega(G, d) \) factors as

\[
\hat{\mathcal{C}}(G) \xrightarrow{\overline{\rho_{\omega,d}}} \hat{\mathcal{C}}(G, T) \xrightarrow{\rho_{\omega,d}} \text{Cone}_\omega(G, d).
\]

Therefore \( \overline{\rho_{\omega,d}} \) is injective and by Proposition 6.13 it is an isometry.

Suppose that \( \lim_{t>0} \frac{1}{t} \| g(t) \| = \| [g] \|_{\hat{\mathcal{C}}(G)} \) where \( [g] \in \hat{\mathcal{C}}(G) \). Clearly

\[
\| \rho([g]) \| = \| \{ g \circ d \} \| = \omega \cdot \lim_n \frac{1}{d_n} \| g(d_n) \| \text{ and the } \omega \text{-limit is an honest limit by the assumption. Then } \| \rho([g]) \| = \| [g] \|_{\hat{\mathcal{C}}(G)} \text{ and } \rho \text{ is an isometry.}
\]

\[\square\]

11. Examples: Abelian and nilpotent groups and verbal norms

We begin with the simplest examples of abelian groups.

**Proposition 11.1.** Let \( \mathbb{Z} \) be equipped with the standard metric. Then

(a) \( \hat{\mathcal{C}}(\mathbb{Z}) \) is isometric to \( \mathbb{R} \) with the usual metric.

(b) \( \mathbb{Z} \) is independent of scaling (Definition 7.1) thus \( \hat{\mathcal{C}}(\mathbb{Z}, T) \) is isometric to \( \mathbb{R} \) for any scaling set \( T \).

(c) \( \rho : \hat{\mathcal{C}}(\mathbb{Z}) \to \text{Cone}_\omega(\mathbb{Z}, d) \) is an isometry for any \( \omega \) and \( d \).

**Proof.** We choose the scaling set \( T = (0, \infty) \). By Proposition 5.8 and Paragraph 6.3 the homomorphism \( \hat{\pi} \circ \iota : F_\mathbb{R}(\mathbb{Z}) \to \hat{\mathcal{C}}(\mathbb{Z}) \) is Lipschitz with constant 1 and has dense image. Therefore it suffices to show that the image of \( \hat{\pi} \circ \iota \) is isometric to \( \mathbb{R} \).

With the notation in Paragraph 6.1 let \( L \leq F_\mathbb{R}(\mathbb{Z}) \) be the subgroup \( L = \{ \overline{1}(r) : r \in \mathbb{R} \} \). We claim that

\[\hat{\pi} \circ \iota(F_\mathbb{R}(\mathbb{Z})) = \hat{\pi} \circ \iota(L).\]

Since \( \hat{\pi} \circ \iota \) is a homomorphism it suffices to show that the image of any generator \( \alpha = \overline{1}(r) \in F_\mathbb{R}(\mathbb{Z}) \) is in the image of \( L \). Set \( [h] = \hat{\pi}(\iota(\alpha)) \). It is represented by the function \( h : T \to \mathbb{Z} \) given by \( h(t) = n \cdot \lfloor rt \rfloor \). Consider \( \alpha' = \overline{1}(nr) \in L \) and set \( [h'] = \hat{\pi}(\iota(\alpha')) \) represented by \( h'(t) = \lfloor nrt \rfloor \). Now,

\[\limsup_{t>0} \frac{1}{t} \| nrt \| - n \| rt \| \leq \limsup_{t>0} \frac{1}{t} |n| = 0.\]

By Lemma 6.6 \( [h^{-1}h']_{\hat{\mathcal{C}}(\mathbb{Z})} = 0 \) so \( [h] = [h'] \) as required.

By the definition of the metric on \( F_\mathbb{R}(\mathbb{Z}) \) it is clear that \( L \) is isometric to \( \mathbb{R} \) via \( r \mapsto \overline{1}(r) \). It is left to show that \( \hat{\pi} \circ \iota : L \to \hat{\mathcal{C}}(\mathbb{Z}) \) is an isometric embedding. Indeed, the stable length of any \( n \in \mathbb{Z} \) is \( |n| \) so by Lemma

\[\| \hat{\pi}(\iota(\overline{1}(r))) \|_{\hat{\mathcal{C}}(\mathbb{Z})} = |r| \cdot \tau(1) = |r| = \| \overline{1}(r) \|_L.\]
Theorems 9.9 and 10.3 show that $Z$ is independent of scaling and that $\rho_{\omega,d}: \mathcal{C}(Z) \to \text{Cone}_\omega(Z,d)$ is an isometric embedding. It is, in fact, an isometry since it is well known that $\text{Cone}_\omega(Z,d) \cong \mathbb{R}$.

**Proposition 11.2.** Equip $Z^n$ with the standard $L^1$-norm $\|v\| = \sum_i |v_i|$.

(a) $\mathcal{C}(Z^n, T)$ is isometric to $\mathbb{R}^n$ with the standard $L^1$-norm.

(b) $Z^n$ is independent of scaling and $\rho_{\omega,d}: \mathcal{C}(Z^n) \to \text{Cone}_\omega(Z^n,d)$ is an isometry.

**Proof.** This follows from Proposition 11.1 and Theorem 8.3. The second assertion follows from Theorems 9.9 and 10.3 and the well known fact that $\text{Cone}_\omega(Z^n,d) \cong \mathbb{R}^n$.

We move on to study nilpotent groups.

**Proposition 11.3.** Suppose that $G$ is nilpotent. Then

(a) $G$ is independent of scaling,

(b) $\mathcal{C}(G)$ is abelian and

(c) $\rho_{\omega,d}: \mathcal{C}(G) \to \text{Cone}_\omega(G,d)$ is an isometric embedding for any ultrafilter $\omega$ and scaling sequence $d$.

**Proof.** It is shown in [1, Theorem 5.H] that the commutator subgroup of any finitely generated nilpotent group is bounded. Since subgroups of nilpotent groups are nilpotent, it follows that $G$ has locally bounded commutators, Definition 9.8. Theorems 9.9 and 10.3 show that $G$ is independent of scaling, $\mathcal{C}(G)$ is abelian, and that $\rho_{\omega,d}$ are isometric embeddings.

**Proposition 11.4.** Let $G$ be normally generated by $A$ and equipped with the standard conjugation invariant word norm $\|\|_A$, Definition 2.12. Suppose that $[G,G]$ is bounded and that the image of $A$ in $G_{ab}$ is finite. Then

(a) $G$ is independent of scaling,

(b) $\mathcal{C}(G) \to \text{Cone}_\omega(G,d)$ is an isometry, and

(c) both groups are bi-Lipschitz equivalent to $(\mathbb{R}^n, \|\|_1)$ where $n = \dim G_{ab} \otimes \mathbb{R}$.

**Proof.** Let $\pi: G \to G_{ab}$ be the quotient with kernel $[G,G]$. Set $B = \pi(A)$. By assumption $B$ is finite, and it generates $G_{ab}$. By Proposition 2.15 $\pi: (G, \|\|_A) \to (G_{ab}, \|\|_B)$ is a metric quotient. Since $[G,G]$ is bounded Corollary 8.5 implies that $\mathcal{C}(\pi)$ is an isometry $\mathcal{C}(G,T) \to \mathcal{C}(G_{ab},T)$ for any scaling set $T$. But any abelian group is independent of scaling by Theorem 9.9, hence $G$ is independent of scaling. This proves point (a).
Let $T$ denote the torsion subgroup of $G_{ab}$ and set $H = G_{ab}/T$. Equip $H$ with the quotient norm, Lemma 2.6, and let $\tau: G_{ab} \to H$ be the quotient map. By Proposition 2.15, the norm on $H$ is the standard word norm $\| \cdot \|_{\tau(B)}$. Clearly $H \cong \mathbb{Z}^n$ for some $n$ and let $\lambda: H \to \mathbb{Z}^n$ be an isomorphism. It follows from Proposition 2.14 that $\lambda$ is a bi-Lipschitz equivalence. The naturality of the maps $\rho_{\omega,d}$, Section 10, implies the commutativity of the following diagram:

$$
\begin{array}{cccccc}
\hat{C}(G) & \longrightarrow & \hat{C}(G_{ab}) & \longrightarrow & \hat{C}(H) & \longrightarrow & \hat{C}(\mathbb{Z}^n) \\
\rho \downarrow & & \rho \downarrow & & \rho \downarrow & & \rho \\
\text{Cone}_\omega(G,d) & \longrightarrow & \text{Cone}_\omega(G_{ab},d) & \longrightarrow & \text{Cone}_\omega(H,d) & \longrightarrow & \text{Cone}_\omega(\mathbb{Z}^n)
\end{array}
$$

We have seen above that $\hat{C}(\pi)$ is an isometry. The boundedness of $[G,G]$ also easily implies that $\pi_*$ is an isometry. Since $T$ is finite it is bounded in $G_{ab}$ and the same argument shows that $\hat{C}(\tau)$ and $\tau_*$ are isometries. Since $\lambda$ is a bi-Lipschitz equivalence $\hat{C}(\lambda)$ is a bi-Lipschitz equivalence by Proposition 6.11, and $\lambda_*$ is a bi-Lipschitz equivalence by easy inspection of the definition of ultrafilter cones. By Proposition 11.2 the leftmost vertical arrow $\rho$ is an isometry between groups that are isometric to $({\mathbb{R}}^n, \| \cdot \|_1)$. Therefore $\rho$ at the first column is a bi-Lipschitz equivalence of groups which are bi-Lipschitz equivalent to $({\mathbb{R}}^n, \| \cdot \|_1)$. Since it is also an isometric embedding, it is an isometry and this proves (b) and (c). \hfill \square

**Example 11.5.** Let $G$ be a finitely generated nilpotent group equipped with the standard conjugation invariant word norm. Then $G$ is independent of scaling, $\rho_{\omega,d}: \hat{C}(G) \to \text{Cone}_\omega(G,d)$ is an isometry for any ultrafilter $\omega$ and any scaling sequence $d$, both groups are bi-Lipschitz equivalent to $({\mathbb{R}}^n, \| \cdot \|_1)$. Indeed, we can apply Proposition 11.4 since $[G,G]$ is bounded by [11, Theorem 5.H].

**Example 11.6.** Let $G$ be a group, all of whose elements have finite order. Then $\hat{C}(G) = 1$ by Proposition 8.2. For example $S_\infty$, the group of finitely supported permutations of $\{1,2,\ldots\}$ is a torsion group and therefore $\hat{C}(S_\infty) = 1$. In contrast, Karlhofer [8] proved that the ultrafilter cone $\text{Cone}_\omega(S_\infty)$ with respect to any non-principal ultrafilter is a nontrivial contractible simple group in which every element is a commutator.

**Example 11.7.** Let $G$ be a group and equip $H = [G,G]$ with the commutator length norm. Then $\hat{C}(H)$ is abelian. More generally, let $H = \Gamma_n(G)$ be the $n^{th}$ group in the lower central series of $G$. That
is, \( H \) is the subgroup of \( G \) generated by all commutators of length \( n \). Equip \( H \) with the conjugation-invariant word norm associated with this set of generators. Then \( \hat{C}(H) \) is nilpotent of class \( n \). Indeed, let \([x_1, x_2, \ldots, x_n]\) denote iterated commutator. Apply Proposition 8.6 with the word \( w = [x_1, x_2, \ldots, x_n] \in F(x_1, \ldots, x_n) \).

12. Example: The free group

**Assumptions:** In this we study the free group \( G = F(A) \) equipped with the standard conjugation-invariant word norm, i.e the word norm associated to the constant length function \( \mu(a) = 1 \).

**Theorem 12.1.** \( F(A) \) is independent of scaling. The homomorphisms \( \rho_{\omega,d} : \hat{C}(F(A)) \rightarrow \text{Cone}_{\omega}(F(A), d) \) are injective.

**Proof.** The hypothesis of Lemma 9.7 is satisfied for \( F(A) \) by [7, Theorem 1.1, Example 1.2]. Then \( F(A) \) is independent of scaling by Lemmas 9.5 and 9.4. The rest follows from Theorem 10.3. \( \square \)

We do not know whether \( \rho_{\omega,d} \) are isometries. In the remainder of this section we will give some details about the structure of \( \hat{C}(F(A)) \). Once again, our results are incomplete.

**Definition 12.2.** Let \( \Gamma \) be a group. An element \( g \in \Gamma \) is called **pure** if \( g = h^n \) for some \( h \in G \) and some \( n \in \mathbb{Z} \) implies \( h = g^{\pm 1} \) and \( n = \pm 1 \). Let \( \text{Pure}(\Gamma) \) denote the set of pure elements in \( \Gamma \).

**Example.** The identity element \( e \in \Gamma \) is never a pure element. The only pure elements in \( \mathbb{Z} \) are \( \pm 1 \). Divisible groups contain no pure elements.

12.3. \( \text{Pure}(\Gamma) \) is closed under conjugation and inverses. The relation \( g \sim g' \) if \( g' \) is conjugate to \( g^{\pm 1} \) is an equivalence relation on \( \text{Pure}(\Gamma) \).

For the remainder of this section we fix

\[ \Theta \subseteq \text{Pure}(F(A)), \]

a set of cyclically reduced representatives for the equivalence classes in \( \text{Pure}(F(A)) \). Notice that \( \Theta \) does not contain the identity element.

Recall the homomorphisms \( \hat{\pi} \) and \( \iota \) from paragraph 6.3 and Definition 5.7. Recall that \( F(\Theta; \tau) \) denotes \( F(\Theta) \) equipped with the conjugation invariant norm defined in paragraph 2.10 with respect to the stable length function \( \tau : \Theta \rightarrow \mathbb{R} \).

**Theorem 12.4.** Set \( G = F(A) \). The restriction of \( \hat{\pi} \circ \iota : F_{\mathbb{R}}(G) \rightarrow \hat{C}(G) \) to \( F_{\mathbb{R}}(\Theta) \) is an injective homomorphism

\[ \xi_{\Theta} : F_{\mathbb{R}}(\Theta; \tau) \rightarrow \hat{C}(G) \]
with dense image. It is Lipschitz with constant 1 and \( \| \xi(\alpha) \| = \| \alpha \| \) for all generators \( \alpha = \overline{\theta}(r) \in F_\mathbb{R}(\Theta) \). Upon completion,

\[
\hat{\xi}_\Theta : \hat{F}(\Theta; \overline{\theta}) \to \hat{C}(G)
\]

is a metric quotient homomorphism (Definition 2.5).

**Conjecture.** The map \( \hat{\xi} \) is an isometry.

**Notation.** The remainder of this section is devoted to the proof of Theorem 12.4. Throughout we denote

\[ G = F(A) \]

Since \( F(A) \) is independent of scaling we will work with the scaling set \( T = (0, \infty) \). Throughout \( \mu : G \to \mathbb{R} \) denotes the norm function and \( \tau \) the stable length.

**12.5.** Let us recall several elementary facts about the free group \( F(A) \).

Let \( w \) be a word in the alphabet \( A \sqcup A^{-1} \). We will write \( |w| \) for the **length** of the word \( w \), i.e., the number of letters in the alphabet \( A \sqcup A^{-1} \) appearing in it. We call \( w \) **reduced** if it contains no subword of the form \( aa^{-1} \) or \( a^{-1}a \) which we call an **elementary relation**. The elements of \( F(A) \) are equivalence classes of words and each is represented by a unique reduced word. Any word \( w \) may be reduced by successively eliminating elementary relations; The resulting reduced word is independent of the order elementary relations are eliminated.

At this stage it is worthwhile to highlight some elementary facts related to this process of reduction. Let us fix terminology and say that a **sequence** in a word \( w \) is just a collection of symbols (letters) in \( w \), while a **subword** or an **interval** is a sequence of consecutive symbols in \( w \). Clearly, the reduced form of \( w \) is obtained by removing a collection of symbols from \( w \), which form a disjoint union of intervals in \( w \). In the sequel we will use repeatedly the following observation. Consider a sequence \( u \) in \( w \). Let \( w - u \) denote the word obtained by removing the symbols in \( u \) from \( w \). Then the reduced form of \( w - u \) is obtained from \( w \) by the removal of \( i \) disjoint intervals whose union contains the symbols in \( u \). Clearly, \( i \leq |u| \).

Suppose that \( w, u \) are reduced words. Their concatenation \( wu \) may or may not be reduced. It is not reduced only if an elementary relation appears at the juncture. In this case the reduction sequence of \( wu \) is unique and it is obtained in a unique way by successively removing an elementary relation at the juncture. Thus, \( w = w'v \) and \( u = v^{-1}u' \) for some suffix \( v \) of \( w \) and \( w'u' \) is the reduced form of \( wu \).

Suppose that \( w = w_1 \cdots w_n \) and \( u = u_1 \cdots u_m \) are **reduced** words expressed as the concatenation of non-empty (reduced) words. We say
that \( w_i \) and \( u_j \) collide if in the process of reduction of \( wu \) “they meet each other” in the sense that there are letters in \( w_i \) that are cancelled with those of \( u_j \) in the unique (!) process of reduction of \( wu \).

More precisely, \( w_i \) and \( u_j \) collide if

\[
\begin{align*}
\max \left\{ \sum_{k=1}^{i-1} |w_k|, \sum_{k=1}^{j-1} |u_k| \right\} < \min \left\{ |v|, \sum_{k=1}^{i} |w_k|, \sum_{k=1}^{j} |u_k| \right\}.
\end{align*}
\]

Figure 1 illustrates the situation. The “collision” is at an interval of length

\[
\ell = \min \left\{ |v|, \sum_{k=1}^{i} |w_k|, \sum_{k=1}^{j} |u_k| \right\} - \max \left\{ \sum_{k=1}^{i-1} |w_k|, \sum_{k=1}^{j-1} |u_k| \right\}.
\]

Let \( g \) be a cyclically reduced word in the alphabet \( A \sqcup A^{-1} \). Notice that the word \( g^n = g \cdots g \) formed by the \( n \)-fold concatenation of \( g \) is (cyclically) reduced. We say that a word \( v \) is a \( g \)-packet if it is a sub-word (i.e an interval) in the word \( g^n = g \cdots g \) for some \( n \neq 0 \) (including negative \( n \)). It is clear that any interval in a \( g \)-packet is itself a \( g \)-packet. It is also clear that a \( g \)-packet of length \( k \cdot |g| \) is conjugate to \( g^{\pm k} \) in \( F(A) \). The following is therefore clear

**Proposition 12.6.** Let \( w = w_n \cdots w_1 \) and \( u = u_1 \cdots u_k \) be reduced words. Suppose that in the process of reduction of \( wu \) the subwords \( w_i \) and \( u_j \) collide at interval of length \( \ell \). If \( w_i \) is a \( g \)-packet and \( u_j \) is an \( h \)-packet, and if \( \ell = \text{lcm}(|g|, |h|) \) then \( g^{\pm k} \) is conjugate to \( h^{\pm m} \) where \( k = \ell/|g| \) and \( m = \ell/|h| \).

Recall \( \Theta \subseteq F(A) \) from paragraph 12.3

**Proposition 12.7.** For any \( 1 \neq g \in G = F(A) \) there exists some \( \theta \in \Theta \) such that \( g \) is conjugate to \( \theta^n \) for some \( n \neq 0 \).

**Proof.** Use induction on \( |g| \). By replacing \( g \) with a conjugate, we may assume that \( g \) is cyclically reduced. If \( g \) is pure, we are done. Otherwise
$g = h^k$ for some $h \in G$ and $k \in \mathbb{Z}$ and $k \geq 2$. Then $h$ is cyclically reduced and $|g| = k \cdot |h|$. Apply the induction hypothesis to $h$. \hfill \square

Lemma 12.8. Let $\theta_1, \theta_2 \in G$ be cyclically reduced and pure elements. Suppose that $\theta_1^{n_1} = \theta_2^{n_2}$ for some $n_1, n_2 \neq 0$. Then $\theta_1 = \theta_2^{\pm 1}$.

Proof. By replacing $\theta_1$ with $\theta_1^{-1}$ if $n_1 < 0$ and replacing $\theta_2$ with $\theta_2^{-1}$ if $n_2 < 0$ we may assume that $n_1, n_2 > 0$.

Clearly $\theta_1^{n_1}$ and $\theta_2^{n_2}$ are cyclically reduced. Set $\ell_1 = |\theta_1|$ and $\ell_2 = |\theta_2|$. The result is obvious if $\ell_1 = \ell_2$ by comparing the reduced words $\theta_1^{n_1} = \theta_2^{n_2}$. So we assume that $\ell_1 < \ell_2$ and write $m = \gcd(\ell_1, \ell_2)$. Set $p_1 = \frac{\ell_1}{m}$ and $p_2 = \frac{\ell_2}{m}$. Then $\theta_1 = \eta_0 \cdots \eta_{p_1-1}$ with $|\eta_i| = m$. Similarly $\theta_2 = \zeta_0 \cdots \zeta_{p_2-1}$ with $|\zeta_i| = m$. Clearly, $\eta_0 = \zeta_0$ since these are the first $m$ symbols in $\theta_1^{n_1} = \theta_2^{n_2}$. By the Chinese remainder theorem for any $0 \leq r < p_2$ there exists $0 \leq q < p_2$ such that $qp_1 = r \mod p_2$. That means that the $q$-th occurrence of $\eta_0$ in $\theta_1^{n_1}$ is equal to $\zeta_r$ in some occurrence of $\theta_2$ in $\theta_2^{n_2}$ (notice that $p_1|n_2$). Therefore $\zeta_i = \eta_0$ for all $i$. By a similar argument $\eta_i = \zeta_0$ for all $i$. So $\theta_1 = \eta_0^{p_1}$ and $\theta_2 = \eta_0^{p_2}$. The result follows since $\theta_1$ and $\theta_2$ are pure. \hfill \square

Lemma 12.9. Consider $\theta_1, \theta_2 \in \Theta$ and suppose that $\ell \geq \text{lcm}(|\theta_1|, |\theta_2|)$. Suppose that there exists a reduced word $v$ such that $|v| \geq \ell$ and such that $v$ is a $\theta_1$-packet and $v^{-1}$ is a $\theta_2$-packet. Then $\theta_1 = \theta_2$.

Proof. By taking a suffix of $v$ we may assume that $\ell = \text{lcm}(|\theta_1|, |\theta_2|)$. Indeed, such a suffix remains reduced and a $\theta_i$-packet ($i = 1, 2$). Set $n_1 = \frac{\ell}{|\theta_1|}$ and $n_2 = \frac{\ell}{|\theta_2|}$. Then $v$ is a cyclic permutation of $\theta_1^{n_1}$ and $v^{-1}$ is a cyclic permutation of $\theta_2^{n_2}$. Hence $\theta_1^{\pm n_1}$ is a cyclic permutation of $\theta_2^{\pm n_2}$. Then $\theta_1^{\pm n_1} = \theta_3^{\pm n_2}$ for some $\theta_3$ which is a cyclic permutation of $\theta_2$. Lemma 12.8 implies that $\theta_1 = \theta_3^{\pm 1}$, so $\theta_1$ is conjugate to $\theta_2^{\pm 1}$. That is, $\theta_1$ and $\theta_2$ belong to the same equivalence class in $\mathcal{P}(G)$, so $\theta_1 = \theta_2$ by the definition of $\Theta$. \hfill \square

It is well known that $G = F(A)$ has no distorted elements, namely $\tau(g) > 0$ for all $1 \neq g \in G$. To see this, observe that for every nontrivial $g \in F(A)$ there exists a homogeneous quasimorphism $\psi: F(A) \to \mathbb{R}$ such that $\psi(g) > 0$ [11, Section 4.E]. It follows that $\tau(g) > 0$.

Lemma 12.10. Consider cyclically reduced $g \in G$. Then the removal of $m$ symbols from $g^n$ results in a word whose reduced form is obtained from $g^n$ by the removal of at most $m$ intervals of total length at most $\frac{|g|(|1+|g|)}{\tau(g)}m$.

In particular it is a product of at most $m + 1$ $g$-packets of total length at least $n|g| - \frac{|g|(|1+|g|)}{\tau(g)}m$. 
Therefore, it is clear that for every union of the sequences $u$ by following the reduction process that yields $w$ of $g$ into non-empty intervals where $P$. Let us now consider a sequence $u$ of $n$ symbols in $G$. Thus, the removal of a single interval $J$ containing $u$. This completes the proof.

**Lemma 12.11.** Fix some numbers $\ell > 0$ and $N > 0$. Suppose that $P_1 = \{I_1, \ldots, I_{m_1}\}$ and $P_2 = \{J_1, \ldots, J_{m_2}\}$ are partitions of $\{1, \ldots, N\}$ into non-empty intervals where $N \geq \ell(m_1+m_2)$. Then there are $I_i \in P_1$ and $J_j \in P_2$ which intersect at an interval of length at least $\ell$.

**Proof.** We will assume that the intervals in $P_1$ (resp. $P_2$) are ordered in an increasing manner, i.e $1 \in I_1$ and $I_1I_2\cdots I_k$ is an interval for every $1 \leq k \leq m_1$. Use (strong) induction on $N$. We may assume without
loss of generality that $|I_{m_1}| \leq |J_{m_2}|$ namely that $I_{m_1} \subseteq J_{m_2}$. Choose $1 \leq k \leq m_1$ maximal with the property that

$$I_{m_1-k+1} \cdots I_{m_1} \subseteq J_{m_2}.$$ 

Set $K = I_{m_1-k+1} \cdots I_{m_1}$ for short.

If the length of one of $I_{m_1-k+1}, \ldots, I_{m_1}$ is at least $\ell$ then we are done. So we assume that the length of all these $k$ intervals is $< \ell$.

Suppose that $|J_{m_2} \setminus K| \geq \ell$. Then $K$ is a proper interval in $\{1, \ldots, N\}$ and in particular $k \leq m_1 - 1$. The maximality of $k$ implies that $I_{m_1-k} \supsetneq J_{m_2} \setminus K$ and in particular $J_{m_2}$ and $I_{m_1-k}$ intersect at an interval of length $\geq \ell$, and we are done again.

So we assume that $|J_{m_2} \setminus K| < \ell$. This implies that $|J_{m_2}| < (k+1)\ell$. Since $m_1 \leq k$ and $m_2 \geq 1$,

$$|J_{m_2}| < (k+1)\ell \leq (m_1 + m_2)\ell \leq N.$$ 

So $J_{m_2}$ is a proper interval in $\{1, \ldots, N\}$, and hence so is $K$. This implies that $m_2 \geq 2$, that $k \leq m_1 - 1$. The maximality of $k$ implies that $I_{m_1-k} \setminus J_{m_2} \neq \emptyset$. Set $N' = N - |J_{m_2}|$. Clearly, $N' < < N$. Set

$$P'_1 = \{I_1, \ldots, I_{m_1-k-1}, I_{m_1-k} \setminus J_{m_2}\}$$

$$P'_2 = \{J_1, \ldots, J_{m_2-1}\}.$$ 

Both are partitions of $\{1, \ldots, N'\}$ consisting of $m'_1 = m_1 - k$ and $m'_2 = m_2 - 1$ intervals. Also

$$N' = N - |J_{m_2}| > N - (k+1)\ell \geq (m_1 + m_2)\ell - (k+1)\ell = (m'_1 + m'_2)\ell.$$ 

By the induction hypothesis there are $I' \in P'_1$ and $J' \in P'_2$ which intersect at an interval of length $\geq \ell$. Hence one of the intervals $I_1, \ldots, I_{m_1-k}$ and $J_1, \ldots, J_{m_2-1}$ intersect at an interval of length $\geq \ell$. This completes the induction step. \hfill \Box

**Lemma 12.12.** Consider $\theta_1, \ldots, \theta_p \in \Theta$ where $p \geq 1$. Assume that $\theta_i \neq \theta_{i+1}$ for all $1 \leq i < p$. Set $\ell = \text{lcm}(|\theta_1|, \ldots, |\theta_p|)$, and for every $1 \leq i \leq p$ set

$$\kappa_i = \frac{|\theta_i|}{9\ell + \frac{1+2|\theta_i|}{\tau(\theta_i)}|\theta_i|}.$$ 

Let $n_1, \ldots, n_p$ be integers such that $|n_i| > \frac{6\ell}{|\theta_i|}$ for all $1 \leq i \leq p$. Then

$$\|\theta_1^{n_1} \cdots \theta_p^{n_p}\|_G \geq \min_{1 \leq i \leq p} |n_i| \cdot \kappa_i.$$ 

**Proof.** Let $g$ be the (not necessarily reduced) word $\theta_1^{n_1} \cdots \theta_p^{n_p}$. Let $u$ be a sequence in $g$ of length

$$m \leq \min_{1 \leq i \leq p} |n_i| \cdot \kappa_i.$$
We will show that \( g - u \) is not the trivial element and by this complete the proof since the norm on \( G \) coincides with the cancellation norm.

Partition the sequence \( u \) to subsequences \( u_i \subseteq \theta_i^{n_i} \) of length \( m_i \) each \((1 \leq i \leq p)\). Thus, \( m = \sum_i m_i \). By Lemma 12.10 the removal of \( u_i \) from \( \theta_i^{n_i} \) yields, after reduction, a reduced word \( w_i \) which is a product of at most \( m_i + 1 \) \( \theta_i \)-packets of total length

\[
|w_i| \geq |n_i| \cdot |\theta_i| - m_i \cdot |\theta_i| \cdot \frac{1+|\theta_i|}{\tau(\theta_i)} \\
\geq |n_i| \cdot |\theta_i| - m \cdot |\theta_i| \cdot \frac{1+|\theta_i|}{\tau(\theta_i)} \quad \text{(since } m_i \leq m \text{)} \\
\geq m_i |\theta_i| - m \cdot |\theta_i| \cdot \frac{1+|\theta_i|}{\tau(\theta_i)} \quad \text{(since } m \leq |n_i| \text{)} \\
= 9 \ell m \quad \text{(by definition of } \kappa_i). \]

Then \( g - u \) is equivalent to \( w_1 \cdots w_p \). For any \( 1 \leq i < p \) we will now examine the reduction of \( w_i w_{i+1} \) at the juncture. This reduction boils down to a word \( v_i \) such that \( w_i = w_i' v_i \) and \( w_{i+1} = v_i^{-1} w_{i+1}' \) and \( w_i w_{i+1}' \) is reduced. We claim that for every \( 1 \leq i < p \)

\[
|v_i| \leq \min \left\{ \frac{|w_i|}{3}, \frac{|w_{i+1}|}{3} \right\} + \ell. \tag{8} \]

Assume false, i.e. \( |v_i| > \min \left\{ \frac{|w_i|}{3}, \frac{|w_{i+1}|}{3} \right\} + \ell \). Then \( |v_i| > (3m + 1) \ell \).

If \( m = 0 \) then \( m_i = 0 \) for all \( i \) and then \( u \) is empty so \( w_i = \theta_i^{n_i} \) for all \( i \). Then \( v_i \) is a \( \theta_i \)-packet and \( v_i^{-1} \) is a \( \theta_{i+1} \)-packet of length \( > (3m + 1) \ell \geq \ell \geq \text{lcm}(|\theta_i|, |\theta_{i+1}|) \). Lemma 12.9 implies that \( \theta_i = \theta_{i+1} \) which is a contradiction. This proves (8) if \( m = 0 \). So we assume that \( m \geq 1 \).

Given \( 1 \leq i < p \), recall that \( w_i \) is a product of at most \( m_i + 1 \) \( \theta_i \)-packets and similarly \( w_{i+1} \) is a product of at most \( m_{i+1} + 1 \) \( \theta_{i+1} \)-packets. Therefore \( v_i \) is a product of at most \( m_i + 1 \) \( \theta_i \) packets and \( v_i^{-1} \) is a product of at most \( m_{i+1} + 1 \) \( \theta_{i+1} \)-packets. If we set \( N = |v_i| \), then the partition of \( w_i \) into the \( \theta_i \)-packets gives a partition \( P_1 \) of \( \{1, \ldots, N\} \) (the letters in \( v_i \)) into at most \( m_i + 1 \) intervals. Similarly, the partition of \( w_{i+1} \) into the \( \theta_{i+1} \)-packets gives a partition \( P_2 \) of \( \{1, \ldots, N\} \) into at most \( m_{i+1} + 1 \) intervals. Then

\[
N = |v_i| > (3m + 1) \ell \geq (m + 2) \ell = ((m_i + 1) + (m_{i+1} + 1)) \ell. \]

Lemma 12.11 implies that in the process of reduction of \( w_i w_{i+1} \) there is a collision of a \( \theta_i \)-packet in \( w_i \) with a \( \theta_{i+1} \)-packet in \( w_{i+1} \) at an interval \( v_i' \) in \( v_i \) of length at least \( \ell \geq \text{lcm}(|\theta_i|, |\theta_{i+1}|) \). Lemma 12.9 implies that \( \theta_i = \theta_{i+1} \) which is a contradiction. This proves (8) in the case \( m \geq 1 \).

Let us set \( v_0 \) and \( v_p \) to be the empty words, which we regard as a prefix of \( w_1 \) and a suffix of \( w_p \). Clearly (8) applies to then as well,
where we agree that \( w_0 \) and \( w_{p+1} \) are empty. Then given \( 1 \leq i \leq p \)
\[
|w_i| - |v_{i-1}| - |v_i| \geq |w_i| - 2|\frac{|w_i|}{3} - 2\ell = |\frac{|w_i|}{3} - 2\ell.
\]
If \( m = 0 \) then in particular \( m_i = 0 \), in which case \( w_i = \theta_i^{n_i} \), so
\[
|\frac{|w_i|}{3} - 2\ell = |\frac{|n_i|}{3} - 2\ell > \frac{6\ell}{3} - 2\ell = 0.
\]
If \( m > 0 \) then
\[
|\frac{|w_i|}{3} - 2\ell \geq \frac{9\ell m}{3} - 2\ell \geq \ell > 0.
\]
This proves that \( v_{i-1}^{-1} \) and \( v_i \) are disjoint intervals in \( w_i \) for all \( 1 \leq i \leq p \), so \( w_i = v_{i-1}^{-1} \tilde{w}_i v_i \) for non empty subword \( \tilde{w}_i \). It follows that the reductions at the junctures of \( w_1 \cdots w_p \) do not affect each other so the reduced form of \( w_1 \cdots w_p \) is \( \tilde{w}_1 \cdots \tilde{w}_p \) which is non-trivial since \( \tilde{w}_i \) are not empty and reduced, and there may be no reductions at the junctures. This completes the proof. \( \square \)

**Proof of Theorem 12.4.** Let \( \alpha = \overline{\theta}_1(r_1) \cdots \overline{\theta}_p(r_p) \) be an non-trivial element of \( F_\mathbb{R}(\Theta) \) represented in reduced form, i.e \( \theta_1, \ldots, \theta_p \in \Theta \) where \( p > 0 \), and \( r_i \neq 0 \) and \( \theta_i \neq \theta_{i+1} \). We will show that \( \| \hat{\pi}(\iota(\alpha)) \|_{\hat{C}(G)} > 0 \).

Set \( [g] = \hat{\pi}(\iota(\alpha)) \). By definition it is represented by the function
\[
g(t) = \theta_1^{[r_1 t]} \cdots \theta_p^{[r_p t]}, \quad (g: (0, \infty) \to G).
\]
Set \( r = \min\{ |r_1|, \ldots, |r_p| \} \). Clearly \( r > 0 \). Set \( \ell = \text{lcm}(|\theta_1|, \ldots, |\theta_p|) \). Clearly, \( |[r_1 t]|, |\theta_1| > 6\ell \) for all sufficiently large \( t \). Let \( \kappa_i \) be defined as in Lemma 12.12. Clearly \( \kappa_i > 0 \) for all \( i \), so \( \kappa := \min\{ \kappa_1, \ldots, \kappa_p \} > 0 \).

It follows by that lemma that
\[
\limsup_{t \to \infty} \frac{1}{t} \| g(t) \| \geq \limsup_{t \to \infty} \min_{1 \leq i \leq p} \frac{|[r_i t]|}{t} \cdot \kappa_i \geq \limsup_{t \to \infty} \frac{|[r_i t]|}{t} \kappa = |r| \kappa > 0.
\]
Now apply Lemma 6.5 to deduce that \([g]\) is not the trivial element in \( \hat{C}(G) \). Therefore \( \text{Ker}(\hat{\pi} \circ \iota |_{F_\mathbb{R}(\Theta)}) \) is trivial.

Set \( \xi_\Theta = \hat{\pi} \circ \iota |_{F_\mathbb{R}(\Theta)} \). By Proposition 12.7 \( \Theta \subseteq G \) satisfies the conditions of Proposition 6.10. In its notation \( \xi_\Theta \) is the composition \( \hat{\pi} \circ \iota \circ \text{id}_\mu^* \circ \text{incl} \) and it follows (since \( \hat{C}(G) \) is complete) that \( \xi_\Theta \) is Lipschitz with constant 1, has dense image and \( \hat{\xi}_\Theta = \hat{\xi}_\Theta \) is a metric quotient homomorphism. If \( \alpha = \overline{\theta}(r) \in F_\mathbb{R}(\Theta) \) is a generator then \( \| \xi_\Theta(\alpha) \| = \| \hat{\pi}(\iota(\alpha)) \| = |r| \cdot \tau(\theta) = \| \overline{\theta}(r) \| \) by Lemma 6.7 and Corollary 6.9. \( \square \)

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