The Ricci flow of asymptotically hyperbolic mass and applications

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Abstract

We consider the evolution of the asymptotically hyperbolic mass under the curvature-normalized Ricci flow of asymptotically hyperbolic, conformally compactifiable manifolds. In contrast to asymptotically flat manifolds, for which ADM mass is constant during Ricci flow, we show that the mass of an asymptotically hyperbolic manifold of dimension $n \geq 3$ decays smoothly to zero exponentially in the flow time. From this, we obtain a no-breathers theorem and a Ricci flow based, modified proof of the scalar curvature rigidity of zero-mass asymptotically hyperbolic manifolds. We argue that the nonconstant time evolution of the asymptotically hyperbolic mass is natural in light of a conjecture of Horowitz and Myers, and is a test of that conjecture. Finally, we use a simple parabolic scaling argument to produce a heuristic “derivation” of the constancy of ADM mass under asymptotically flat Ricci flow, starting from our decay formula for the asymptotically hyperbolic mass under the curvature-normalized flow.

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1 Introduction

The positive energy theorem, also called the positive mass theorem, asserts that if a complete asymptotically flat manifold has nonnegative scalar curvature, it has nonnegative Arnowitt-Deser-Misner (ADM) mass. One of the most interesting aspects of this theorem is the variety of methods by which it has been proved, each method rich in mathematical content or physical insight and often in both. There are, for example, the Schoen-Yau minimal surface proof [28], Witten’s spinorial method [31], the spacetime fastest curves approach [24], Geroch’s inverse mean curvature flow [12, 19, 7], and even a method based on 3-manifold geometrization [27]. (Most methods of proof also require a restriction on the dimension \( n \). The Witten method has no dimension restriction, but requires that the manifold be spin. The work of [20] has no dimensional or topological restriction.)

A second statement, usually referred to as the rigidity, says that if in addition the mass is zero, then the metric is the flat metric on \( \mathbb{R}^n \). What is usually shown is that zero mass implies Ricci flatness. While this suffices if \( n = 3 \), for \( n > 3 \) one can invoke the splitting theorem to show that the unique Ricci flat and asymptotically flat complete metric is the flat Euclidean metric on \( \mathbb{R}^n \). Both the positivity theorem and the rigidity statement have counterparts for asymptotically anti-de Sitter spacetimes [1] and for asymptotically hyperbolic Riemannian manifolds [22, 4, 30, 9, 3].

Recently, Haslhofer [16] has shown that rigidity can be proved on asymptotically flat manifolds by using the properties of Ricci flow. His argument depends on two important properties of Ricci flow. These are that, during the flow of an asymptotically flat metric, (i) the ADM mass remains constant [11, 23] and (ii) the scalar curvature increases monotonically. Both these facts are easy to show, once it is known that the flow of an asymptotically flat metric remains asymptotically flat. Then the Ricci flow beginning at a zero mass, zero scalar curvature metric that is not Ricci flat, and thus not a fixed point of the flow, will produce a zero-mass metric with scalar curvature everywhere nonnegative and somewhere positive. Once this is done a conformal transformation can be found to return the scalar curvature to zero, strictly lowering the mass and therefore rendering it negative, while still maintaining asymptotic flatness. However, this would violate the positive energy theorem. Therefore, the initial Ricci curvature had to be zero.

Very recently, Bahuaud [6] has shown that the Ricci flow of conformally compactifiable metrics always exists for some time interval during which the evolving metric remains conformally compactifiable. Qing, Shi, and Wu [26] have studied the long-time existence and convergence of conformally compactifiable, asymptotically hyperbolic metrics. The flow in this case is so-called normalized flow, given by

\[
\frac{\partial g_{ij}}{\partial t} = -2 \left( R_{ij} + (n-1) g_{ij} \right) = -2 E_{ij} .
\]

Under various appropriate asymptotic conditions, noncompact manifolds admit a definition of mass. Intrinsic geometric flows can sometimes preserve these asymptotic conditions and, when they do, a natural problem is to determine the evolution of mass during the flow; see for example [11, 23, 8, 21].

Though the ADM mass in the asymptotically flat case is constant under Ricci flow [11, 23], there are good reasons to think that mass in the asymptotically hyperbolic
case will not always remain constant during the flow (1.1). One such reason is that, if it were to remain always constant, then a version of the rigidity argument above could be used to falsify an otherwise quite plausible form of positive energy conjecture of Horowitz and Myers [18], a conjecture which has consequences for physics, which we now briefly describe.

For \( \dim M \geq 3 \), consider the family of metrics on \( M \) given by

\[
ds^2 = \frac{dr^2}{r^2 (1 - \frac{1}{r^n})} + r^2 \left[ (1 - \frac{1}{r^n}) d\xi^2 + \sum_{i=3}^n d\theta_i^2 \right],
\]

(1.2)

with \( r \in [1, \infty) \), \( \xi \in [0, 4\pi/n] \), and \( \theta_i \in [0, a_i] \) where \( 0 < a_3 \leq \cdots \leq a_n \). Then (1.2) yields a family (parametrized by the \( a_i \)) of smooth metrics on \( \mathbb{R}^2 \times T^{n-2} \), where \( T^{n-2} \) is an \((n - 2)\)-torus. (A parameter, sometimes denoted \( r_0 \) or \( M \), often appears in descriptions of the metric (1.2) [18], but has no significance and can be removed by rescaling the coordinates. Another parameter, \( \ell \), the radius of curvature at infinity, also sometimes appears but can be removed by homothetic rescaling.) These metrics are asymptotically (locally) hyperbolic and have scalar curvature \( R = -n(n - 1) \). We can obtain an Einstein metric from (1.2) by adding an extra dimension, say with coordinate \( \tau \), and adding either \(-r^2 d\tau^2\) with \( \tau \in \mathbb{R} \) to (1.2) to obtain a Lorentzian metric which is sometimes called the AdS soliton, or by adding \( r^2 d\tau^2 \) with \( \tau \in S^1 \) to (1.2) to obtain a Riemannian metric that has been called a toric black hole [2]. To avoid confusion with Ricci (and other) solitons and to distinguish (1.2) from the AdS soliton Einstein metric from which (1.2) is induced on a slice, we will refer to the metric (1.2) by the term Horowitz-Myers geon. (In physics, geon loosely connotes a nonsingular, stable, localized concentration of curvature sometimes, but not necessarily, associated with nontrivial topology. The metric (1.2) is not a Ricci soliton.)

A strong formulation of the Horowitz-Myers conjecture [18] is that each member of this family minimizes the hyperbolic mass amongst all metrics that asymptote to it at large \( r \) and have scalar curvature \( R = -n(n - 1) \). Interestingly, the mass of (1.2) is negative: it is \(-\frac{4\pi}{n} \prod_{i=3}^n a_i \) (or \(-\frac{1}{4\pi G} \prod_{i=3}^n a_i \) in the physics normalization achieved by multiplying by \( \frac{1}{16\pi G} \) where \( G \) is Newton’s constant). Hence, the Horowitz-Myers conjecture plays the role of a “positive” mass conjecture for this class of asymptotic structures, if “positive” is interpreted to mean \( \geq -\frac{1}{4\pi G} \prod_{i=3}^n a_i \). The conjecture arises from the hope, as yet unrealized, that the AdS/CFT correspondence in physics could be extended to field theories and geometries that do not have supersymmetry. Accepted “low-energy” physical theories (e.g., QCD, the theory of the strong nuclear force) are not supersymmetric, and if AdS/CFT could be extended to them, it would provide a new tool to study these theories nonperturbatively (though QCD with massive quarks is also not conformal). Horowitz and Myers argue that the veracity of their conjecture would follow if a nonsupersymmetric AdS/CFT correspondence were to exist. Hence, the conjecture provides a nice test of these ideas.

The Horowitz-Myers conjecture can be used to predict that the mass strictly increases, at least initially, along the solution of (1.1) that develops from initial data (1.2). To see how, note that the metric (1.2) is not Einstein, so it is not a fixed point of (1.1). The scalar curvature of (1.2) is \(-n(n - 1)\) everywhere, and it is easy to show that under the flow it will remain \( \geq -n(n - 1) \) and will become \( > -n(n - 1) \) somewhere. We
can stop the flow at some time \( t > 0 \) and return the scalar curvature to \(-n(n-1)\) by a conformal transformation found by solving the Yamabe equation, which can always be done in this case \[5\]. This conformal transformation lowers the mass. If the mass had remained constant (or had decreased) under the flow, then the combination of flow followed by a conformal transformation would produce a metric of mass less than the soliton’s mass, violating the conjecture of Horowitz and Myers.

We will in fact confirm this prediction, by showing:

**Theorem 1.1.** Let \( g(t), \ t \in [0,T), \) be an asymptotically hyperbolic solution of (1.1) developing from a metric \( g(0) = g_0 \) of mass \( m_0 \) on a manifold \( M \) with \( \dim M \geq 2 \). Then the mass \( m(t) \) of \( g(t) \) obeys

\[
m(t) = m_0 e^{-(n-2)t}.
\]  

(1.3)

For \( g_0 \) a Horowitz-Myers geon, then \( m_0 = -\frac{4\pi}{n} \prod_{i=3}^{n} a_i < 0 \) and \( n \geq 3 \), so from (1.3) we see that \( m(t) \) is strictly monotonic increasing, as predicted from the Horowitz-Myers conjecture. This also helps to explain why the soliton metrics have negative, rather than zero, mass, for if the mass were zero initially, it would remain so under the flow. Then a conformal transformation could be found to perturb the metric to one of negative mass, preserving \( R + n(n+1) \geq 0 \). A complete, zero mass, \( R + n(n+1) = 0 \) metric that is not Ricci flat cannot minimize mass.

Though formula (1.3) may stand in contrast to the constancy of ADM mass in asymptotically flat Ricci flow, these two situations are actually connected by a heuristic argument based on parabolic scaling. We outline this argument in Section 6 and use it to provide a heuristic derivation, starting from (1.3), of the constancy of ADM mass during asymptotically flat Ricci flow.

Various results, including some already known and some generalizations thereof, follow from Theorem 1.1. One that follows immediately is

**Corollary 1.2.** Let \( g(t), \ t \in [0,T), \) be an asymptotically hyperbolic solution of (1.1) developing from a metric \( g(0) = g_0 \) of mass \( m_0 \) on a manifold \( M \) with \( \dim M \geq 3 \). Then there exist times \( 0 \leq t_1 < t_2 < T \) such that \( m(t_1) = m(t_2) \) iff \( m(t) = m_0 = 0 \) for all \( t \in [0,T) \).

So-called breather solutions of (1.1) are solutions that are periodic up to a diffeomorphism. That is, a flow is a breather if there are times \( t_1 \neq t_2 \) during the flow and a diffeomorphism \( \varphi \) such that \( g(t_2) = \varphi^* g(t_1) \). A breather of the form \( g(t) = \varphi_t^* g(0) \) for all \( t \) is called a soliton (solitons of (1.1) are generally referred to as expanding Ricci solitons). Einstein metrics may be regarded as solitons of (1.1) for which \( \varphi_t \) is independent of \( t \).

**Corollary 1.3 (No Massive Breathers/Solitons).** Let \( g(t) \) be as in Corollary 1.2. Let \( 0 \leq t_1 < t_2 < T \) be such that \( g(t_2) = \varphi_{t_2,t_1}^* g(t_1) \), where \( \varphi_{t_2,t_1} - \text{id} \in \mathcal{O}(x^\tau) \), \( \tau > \frac{1}{2} \dim M \), and \( x \) is a defining function (see section 2). Then \( m(t) = m_0 = 0 \) for all \( t \in [0,T) \).
Proof. If the diffeomorphism \( \varphi_{t_2, t_1} \) obeys \( \varphi_{t_2, t_1} - \text{id} \in O(x^0) \), then \( m(\varphi_{t_2, t_1}^* g(t_1)) = m(g(t_1)) \) by Theorem 3.4 of \([17]\) (or Theorem 2.3 of \([10]\); see also Theorem 2.3 of \([9]\)). For a breather, we have \( m(\varphi_{t_2, t_1}^* g(t_1)) = m(g(t_2)) \) and thus \( m(g(t_2)) = m(g(t_1)) \) for some \( t_1 \neq t_2 \). Then \( m_0 = 0 = m(t) \) by Corollary 1.2. \( \square \)

It follows that asymptotically hyperbolic Einstein manifolds cannot have nonzero mass, as was first proved in section 5 of \([4]\).

Our results do not preclude Einstein metrics, solitons and, more generally, breathers of undefined mass.

Asymptotically hyperbolic manifolds can be assigned a boundary-at-infinity \( \partial_\infty M \); see Definition 2.1 below. For example, the Horowitz-Myers geon has \( \partial_\infty M \simeq T^{n-1} \), while standard hyperbolic space has \( \partial_\infty M \simeq S^{n-1} \). For \( \partial_\infty M \simeq S^{n-1} \), we have \([22, 4, 30, 9, 4]\).

Proposition 1.4 (Rigidity). Let \( 3 \leq n = \dim M \leq 6 \). Let \( M \) admit a class \( \mathcal{G} \) of metrics whose elements are asymptotically hyperbolic with \( \partial_\infty M \simeq S^{n-1} \) and \( E[g] := R + n(n-1) \geq 0 \). If \( M \) is not spin, then further restrict \( \mathcal{G} \) to those metrics whose mass aspect function (see Definition 2.3) is of semi-definite sign. If \( m[g] = 0 \) for some \( g \in \mathcal{G} \) then \( (M, g) \) is isometric to standard hyperbolic space.

In fact, as with the asymptotically flat case \([16]\), the rigidity theorem can be shown to be a consequence of the behaviour of mass under the flow; that is, it is a corollary of Theorem 1.1. This is shown in section 5, using a modified version of the proof given in \([3]\). We replace the variation of the metric used therein by one based on the flow \([11]\).

Finally, from this we obtain

Corollary 1.5. Let \( 3 \leq n = \dim M \leq 6 \), and let \( g(t_2) = \varphi_{t_2, t_1}^* g(t_1) \in \mathcal{G} \), with \( \mathcal{G} \) as in Proposition 1.4, and \( g \) and \( \varphi_{t_2, t_1} \) as in Corollary 1.3. Then \( g \) is isometric to standard hyperbolic space.

As a special case, there are no nontrivial steady solitons of the flow \([11]\) with \( R \geq -n(n+1) \) and boundary-at-infinity \( \partial_\infty M \simeq S^{n-1} \). This has already been shown in \([14]\) by a different method, in the case of \( \partial_\infty M \) connected but of otherwise arbitrary topology. This result is applicable to a technique to find Einstein metrics using numerical Ricci flow, developed principally by Wiseman (see \([32]\) and citations therein) and collaborators.

This paper is organized as follows. In section 2, we describe asymptotically hyperbolic manifolds and mass formulae. In section 3, we consider the normalized Ricci flow of an asymptotically hyperbolic manifold. In section 4, we prove Theorem 1.1. In section 5, we prove Proposition 1.4. In section 6, we provide a heuristic argument that, starting from \([13]\), one can predict that the mass of an asymptotically flat metric will remain constant during Ricci flow. An appendix discusses the equivalence of our version of Wang’s mass formula and the Chruściel-Herzlich mass formula. A second appendix outlines an alternative derivation, where we compute the behaviour of the mass under Ricci-DeTurck flow and then pass to the Ricci flow result by a pullback.

Our index convention for the curvature tensor is such that \( \hat{R}_j^i x^k y^j z^k := (\nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z)^i \). We maintain the positions of indices on the Riemann tensor when...
raising or lowering, so for example $R_{ijkl} := g_{im} R^{m}_{jkl}$. We define the Laplacian $\Delta T$ on a tensor $T$ with components $T^{k\ldots l}_{i\ldots j}$ in a coordinate basis so that the components of $\Delta T$ are $\Delta T^{k\ldots l}_{i\ldots j} = g^{pq} \nabla_p \left( \nabla_q T^{k\ldots l}_{i\ldots j} \right)$.

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2 Asymptotically hyperbolic manifolds

There are two approaches to the question of what it means for a manifold to be asymptotic to a hyperbolic manifold; see for example [17]. One is based on the Penrose conformal approach first employed in general relativity, while the other uses charts and fall-off conditions. The two approaches are commensurate, in that conformal compactification implies that curvature and its derivatives obey some mild decay conditions.

Definition 2.1. A complete manifold $(M, g)$ is called (smoothly) conformally compactifiable if there is a manifold-with-boundary $(\tilde{M}, \tilde{g})$, a function $\rho \in C^\infty(\tilde{M})$ obeying $\rho(p) = 0 \iff p \in \partial\tilde{M}$ and $d\rho|_p \neq 0$ whenever $p \in \partial\tilde{M}$, and a smooth diffeomorphism $\psi$ from the interior of $\bar{M}$ onto $M$ such that $\rho^2 \psi^* g = \tilde{g}$ is a smooth metric on $\tilde{M}$. The function $\rho$ is called the defining function for the boundary-at-infinity $\partial\tilde{M} =: \partial\infty M$.

Definition 2.2. If $(M, g)$ is conformally compactifiable with defining function $\rho$ such that

- the metric on $\partial\infty M$ induced by $\tilde{g}$ has constant sectional curvature $k$, where either $k = 0$ or $k = 1$ (or $k = -1$, but we will not treat that case in the sequel), and
- $\rho$ obeys
  \[ |d\rho|^2 := \tilde{g}^{ij} \partial_i \rho \partial_j \rho = 1 + O(\rho) \]  

then we call $(M, g)$ asymptotically hyperbolic.

Our definition is only local in that $\partial\infty M$ can have the topology of a spherical space form (for $k = 1$) or a flat manifold (for $k = 0$). A standard calculation shows that (2.1) implies that

$$|E[g]| \equiv |Ric[g] + (n - 1)g| = O(\rho)$$  \hspace{1cm} (2.2)$$

whenever $\tilde{g} \in C^2(\tilde{M})$ (here $|T_{ij}|^2 := g^{ik} g^{jl} T_{ij} T_{kl}$).

We will consider a much more limited class of metrics, admitting a definition of mass. We first consider the $k = 1$ case, with $\partial\infty M \cong S^{n-1}$ with the canonical (round) metric $\tilde{g} := g(S^{n-1}, \text{can})$ (one can also consider spherical space forms). This case was treated by Wang [30]. Specifically, Wang treats metrics which have an expansion

$$g = \text{csch}^2(x) \left[ dx^2 + \tilde{g} + x^n \kappa/n + O(x^{n+1}) \right] ,$$  \hspace{1cm} (2.3)$$
where $x := \text{arcsinh} (\rho)$, $\kappa$ is a tensor on $\partial_\infty M$ (sometimes called the mass aspect tensor).

Then Wang defines the mass to be 

$$m \equiv m[g] := \int_{S^{n-1}} \tilde{g}^{AB} \kappa_{AB} d\tilde{\mu},$$

with $d\tilde{\mu}$ the volume element defined by $\tilde{g}$. This definition agrees with a definition given by Chruściel and Herzlich [9] (see Appendix A). However, the Chruściel-Herzlich definition has the virtue of being formulated in a general way that includes the cases where $\partial_\infty M$ is a spherical space form or a closed flat ($k = 0$) or hyperbolic ($k = -1$) manifold. We will need to treat the $k = 0$ case, so we will extend the Wang definition to include it.

**Definition 2.3.** Consider the set of asymptotically hyperbolic metrics for which, in a collar neighbourhood of $\partial_\infty M$ coordinatized by $(x^i) = (x, y^A)$, the metric can be written as 

$$g = \frac{1}{\rho_{(k)}^2 (x)} \left[ dx^2 + g_{(k)} + x^n \kappa/n + O(x^{n+1}) \right],$$

$$\rho_{(k)} (x) = \begin{cases} \sinh (x), & \text{if } k = 1, \\ x, & \text{if } k = 0. \end{cases}$$

Here $g_{(k)}$ denotes a metric on $\partial_\infty M$ of constant sectional curvature $k = 0$ or $k = 1$, $\kappa = \kappa_{AB} dy^A dy^B$ is a symmetric tensor on level sets of $x$, and $O(x^{n+1})$ denotes a symmetric tensor on $\partial_\infty M$ whose components in the $\left\{ \frac{\partial}{\partial y^A} \right\}$ basis are each bounded above in magnitude by $C x^{n+1}$ for some constant $C$ as $x \to 0$. We define the mass of such a metric to be 

$$m \equiv m[g] := \int_{\partial_\infty M} g_{(k)}^{AB} \kappa_{AB} d\mu_{(k)},$$

where $d\mu_{(k)}$ is the $g_{(k)}$ metric volume element on $\partial_\infty M$, and $g_{(k)}^{AB} \kappa_{AB} : \partial_\infty M \to \mathbb{R}$ is called the mass aspect function.

**3 Term-by-term flow**

Bahuaud [6] proves the short time existence of a solution to the Ricci-DeTurck flow of asymptotically hyperbolic, conformally compactifiable metrics $g(t)$ having the form 

$$g(t) = \frac{dx^2 + \hat{h}(x, y^A) + v(t, x, y^A)}{x^2},$$

where $y^A$ are coordinates on $\partial_\infty M$ and $\hat{h}$ is a fixed, time independent metric on level sets of $x$, so that $\frac{dx^2 + \hat{h}}{x^2} =: h$ is the initial asymptotically hyperbolic metric and $v(0, x, y^A) = 0$. This means that $x$ is a special defining function for $h$, and $x$ is fixed in time during Bahuaud’s Ricci-DeTurck flow. The DeTurck diffeomorphism $\varphi_t$ is generated by a vector field $X = X^k \frac{\partial}{\partial x^k} = \frac{\partial \varphi_t}{\partial t}$ which vanishes on the conformal boundary (see [6].
section 2, or appendix B for the explicit expression for $X$). This means that the components of $X$ in the coordinate basis of (3.1) are $O(x^2)$. Then the pullback metric is

$$
\varphi_t^* g(t) = \frac{\varphi_t^* (dx^2 + \dot{h}) + \varphi_t^* v}{(x \circ \varphi_t)^2}
\quad = \frac{dx^2 + \dot{h} + v(t,x,y^A) + (\varphi_t^* - \text{id})(dx^2 + \dot{h} + v)}{(x \circ \varphi_t)^2}
\quad = \frac{dx^2 + \dot{h} + \bar{v}(t,x,y^A)}{(x \circ \varphi_t)^2}.
$$

(3.2)

where $\bar{v} := v + (\varphi_t^* - \text{id})(dx^2 + \dot{h} + v)$. Since $X$ vanishes at $x = 0$, then $X^k = \frac{d}{dt} (x^k \circ \varphi_t) \in O(x^2)$ in our $(x^k) = (x,y^A)$ coordinate basis. Integrating this over $t \in [0,T]$ with $\varphi_0 = \text{id}$ yields

$$
x^k \circ \varphi_t = x^k + O(x^2),
$$

(3.3)

where $z \in O(x^2)$ means that $|z| < cT x^2$ for $t \in [0,T]$, where $cT \in \mathbb{R}_+$ can depend on $T$. Since $x^1 = x$, this yields $x \circ \varphi_t = x + O(x^2)$. Moreover, differentiating (3.3), then $\frac{\partial}{\partial x^j} (x^k \circ \varphi_t) = \delta^k_j + O(x)$, from which it follows that $[\varphi_t^* - \text{id}]_j^k \in O(x)$. Hence, since in addition $v \in O(x)$, then $\bar{v} \in O(x)$.

We may therefore write the pullback metric which evolves under (1.1) as

$$
g(t) = \frac{1}{\rho^2_k(x)} \tilde{g}(t) = \frac{1}{\rho^2_k(x)} \left[ dx^2 + g(k) + x^m \kappa(t)/m + O(x^{m+1}) \right],
$$

$$
\rho_k(x) = \begin{cases} 
\sinh(x), & \text{if } k = 1, \\
x, & \text{if } k = 0,
\end{cases}
$$

(3.4)

$$
\text{Ric}[g_k] = (n-2) kg_k,
$$

where we take $1 \leq m \leq n = \dim M$ and where $\kappa(t) := \kappa_{ij}(t) dx^i dx^j$. This is more general than (2.5) as we do not assume that $m = n$ (indeed, we start our iteration below with $m = 1$) and we allow that that $\kappa_{11}$ and $\kappa_{1A}$ can be nonzero ($\rho$ is not assumed to be a special defining function at arbitrary $t$). Note that $\kappa_{ij}$ is a function of the coordinates $y^A$ on level sets of $x$. The metric induced from $\tilde{g}$ on $x = \text{const}$ hypersurfaces is

$$
\tilde{g}_{AB} := g(k)_{AB} + x^m \kappa_{AB}/m + O(x^{m+1}),
$$

(3.5)

and we also define

$$
\tilde{K}_{AB} := \frac{1}{2} \frac{\partial}{\partial x} \tilde{g}_{AB} = \frac{1}{2} x^{m-1} \kappa_{AB} + O(x^m).
$$

(3.6)

This is nearly the extrinsic curvature (but not quite, since $\frac{\partial}{\partial x}$ need not be a unit vector). Although the components of $\tilde{g}_{AB}$ are simply $\tilde{g} \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right)$, we find it somewhat useful to maintain a notational distinction between $\tilde{g}_{AB}$ and $\tilde{g}_{AB}$.
We wish to prove that if the initial metric has this form with \( m = n \), then so does the flowing metric at any time \( t > 0 \) along the flow. To do so, we must first compute the right-hand side of (1.1) for a metric of the above form.

We employ several standard expressions. The first is the radial matrix Riccati equation:

\[
\frac{\partial}{\partial x} K_{AB} - K_{AC} K^{C}_{B} = -\tilde{R}^{1}_{A1B} + O(x^{2m-2}) ,
\]

(3.7)

where \( \tilde{R}^{1}_{A1B} = \langle dx, \widetilde{\text{Riem}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \frac{\partial}{\partial y} \rangle \) (the “1” indicates the \( x \)-direction), and the tilde denotes the curvature tensor of \( \tilde{g} = g/\rho^{2}(k) \). The \( O(x^{2m-2}) \) correction term occurs only because \( K_{AB} \) approximates the extrinsic curvature of \( x = \text{const} \) surfaces only to this order. The next ingredients are the equations of Gauss, Codazzi, and Mainardi.

If we write \( \tilde{K} := \tilde{K}_{AA} \) and if \( \hat{R}_{AB} \) denotes the Ricci curvature of the connection \( \hat{\nabla} \) of the induced metric \( \hat{g} \), these yield

\[
\tilde{R}_{11} = \hat{R} - \hat{R}^{2} - \tilde{K}_{AB} \tilde{K}^{AB} ,
\]

(3.8)

\[
\tilde{R}_{1A} = \tilde{R}_{A1} = \tilde{\nabla}^{B} \tilde{K}_{AB} - \tilde{\nabla} A \tilde{K} ,
\]

(3.9)

\[
\tilde{R}_{AB} = \hat{R}_{AB} + \hat{R}^{1}_{A1B} + \hat{K}_{AC} \hat{K}^{C}_{B} - \tilde{K}_{AB} \tilde{K} .
\]

(3.10)

We also need that Taylor’s theorem gives

\[
\hat{R}_{AB} = R_{AB}[g(k)] + O(x^{m}) = (n - 2)kg(k)_{AB} + O(x^{m}) .
\]

(3.11)

The trace of (3.7), together with (3.6), allows us to estimate \( \tilde{R}_{11} \). We can estimate \( \tilde{R}_{1A} \) immediately from equation (3.9). To estimate \( \tilde{R}_{AB} \) we combine (3.7) and (3.10) and use (3.6). We obtain

\[
\tilde{R}_{11} = -\frac{1}{2} (m - 1)x^{m-2} g^{AB}_{(k)} \kappa_{AB} + O(x^{m-1}) ,
\]

(3.12)

\[
\tilde{R}_{1A} = \tilde{R}_{A1} = O(x^{m-1}) ,
\]

(3.13)

\[
\tilde{R}_{AB} = (n - 2)kg_{(k)}AB - \frac{1}{2} (m - 1)x^{m-2} \kappa_{AB} + O(x^{m-1}) ,
\]

(3.14)

where, when \( m = 1 \), these expressions are valid if we interpret \( (m - 1)x^{m-2} \) as being identically zero, including at \( x = 0 \).

Another standard expression relates the Ricci curvature \( R_{ij} \) of \( g \) to the Ricci curvature \( \tilde{R}_{ij} \) of \( \tilde{g} \). It is

\[
R_{ij} = \tilde{R}_{ij} + \frac{1}{\rho^{2}(k)} \left[ (n - 2)\tilde{\nabla}_{i} \tilde{\nabla}_{j} \rho^{2}(k) + \tilde{g}_{ij} \tilde{\Delta} \rho^{2}(k) \right] - (n - 1)\tilde{g}_{ij} \frac{\tilde{\nabla}^{2} \rho^{2}(k)}{\rho^{4}(k)} ,
\]

(3.15)

where \( \tilde{\Delta} := \tilde{g}^{ij} \tilde{\nabla}_{i} \tilde{\nabla}_{j} \) and \( \tilde{\nabla}^{2} \rho^{2}(k) := \tilde{g}^{ij} \tilde{\nabla}_{i} \rho^{2}(k) \tilde{\nabla}_{j} \rho^{2}(k) \). After straightforward computa-
tion using expressions (3.12–3.15), we obtain

\[
R_{11} = -\frac{(n-1)}{\rho_{(k)}} - \frac{1}{2}(n-1)x^{m-2}\kappa_{11} - \frac{1}{2}(m-2)x^{m-2}g^{CD}_{(k)}\kappa_{CD} \\
+ O(x^{m-1}) ,
\]

(3.16)

\[
R_{1A} = R_{A1} = -\frac{(n-1)}{m}x^{m-2}\kappa_{1A} + O(x^{m-1}) ,
\]

(3.17)

\[
R_{AB} = -\frac{(n-1)}{\rho_{(k)}} \left( kg_{(k)AB} + \frac{1}{m}x^{m}\kappa_{AB} \right) \\
+ \frac{1}{2} \left[ g^{CD}_{(k)}\kappa_{CD} + \left( \frac{2n-2}{m} - 1 \right)\kappa_{11} \right] x^{m-2}g_{(k)AB} \\
+ \frac{1}{2}(n-m-1)x^{m-2}\kappa_{AB} + O(x^{m-1}) .
\]

(3.18)

Using \( E_{ij} := R_{ij} + (n-1)g_{ij} \), then we obtain

\[
E_{11} = -\frac{1}{2}(m-2) \left[ \frac{(n-1)}{m}\kappa_{11} + g^{CD}_{(k)}\kappa_{CD} \right] x^{m-2} + O(x^{m-1}) ,
\]

(3.19)

\[
E_{1A} = E_{A1} = O(x^{m-1}) ,
\]

(3.20)

\[
E_{AB} = \frac{1}{2} \left( \frac{2n-2}{m} - 1 \right)\kappa_{11} + g^{CD}_{(k)}\kappa_{CD} \right] x^{m-2}g_{(k)AB} \\
+ \frac{1}{2}(n-m-1)x^{m-2}\kappa_{AB} + O(x^{m-1}) .
\]

(3.21)

**Proposition 3.1.** Let \( g(t) \) be a solution of (1.1) of the form (3.4) on some interval \( t \in [0,T) \). If the expansion (3.4) for the initial metric \( g(0) \) begins at order \( m = n \), then so does the expansion for \( g(t) \).

**Proof.** From (3.4), \( \frac{\partial g_{ij}}{\partial t} = \frac{1}{m}x^{m-2}\frac{\partial \kappa_{ij}}{\partial t} + O(x^{m-1}) \). Then (1.1) yields

\[
\frac{\partial \kappa_{ij}}{\partial t} = -\frac{2m}{x^{m-2}}E_{ij} + O(x) = A_{ij}^{kl}\kappa_{kl} + O(x) ,
\]

(3.22)

where \( A \) is a matrix with components given (using (3.19–3.21)) by

\[
A_{1111} = (m-2)(n-1) ,
\]

(3.23)

\[
A_{11}^{CD} = m(m-2)g^{CD}_{(k)} ,
\]

(3.24)

\[
A_{AB}^{11} = -m \left( \frac{2n-2}{m} - 1 \right) g_{(k)AB} ,
\]

(3.25)

\[
A_{AB}^{CD} = -mg_{(k)AB}g^{CD}_{(k)} - m(n-m-1)\delta_{A}^{C}\delta_{B}^{D} .
\]

(3.26)

Taking the limit \( x \to 0 \), we obtain a linear system for \( \kappa_{ij}(t) \). If \( m < n \), then \( \kappa_{ij}(0) = 0 \) by assumption. But then, by uniqueness, \( \kappa_{ij}(t) = 0 \). \qed
4 Ricci flow and mass

We now wish to consider the Ricci flow developing from an initial metric of the form (2.5). This means that \( m = n \) and that \( \kappa_{11}(0) = \kappa_{A1}(0) = 0 \). Proposition 3.1 then guarantees that \( m = n \) throughout the flow, but it does not guarantee that \( \kappa_{11}(t) = 0 \), nor that \( \kappa_{A1}(t) = 0 \), at any \( t > 0 \). Since we wish to compute the mass of the flowing metric, we would have to transform this metric back to the form (2.5) (vanishing \( \kappa_{ij} \)) whenever we wish to compute the mass. Alternatively, we can transform the mass formula to a form valid for arbitrary \( \kappa_{ij} \).

To this end, consider a metric of the form

\[
g = \frac{1}{\rho^2_{(k)}(x')} \left[ dx'^2 + g_{(k)} + \frac{x'^n}{n} \kappa_{ij} dx^i dx^j + O(x'^{n+1}) \right]
\]

(4.1)

generalizing the form (2.5) to allow for possibly nonzero \( \kappa_{11} \) and \( \kappa_{1A} \). The coordinate transformation

\[
y^A = z^A + \frac{x'^{n+1}}{n(n+1)} g_{AB}^{(k)}(0) \kappa_{1B} .
\]

(4.2)

brings (4.1) to the form

\[
g = \frac{1}{\rho^2_{(k)}(x')} \left[ dx'^2 + g_{(k)} + \frac{x'^n}{n} \left( \kappa_{11} dx'^2 + 2 \kappa_{1A} dx^A dz^B + \kappa_{AB} dz^A dz^B \right) + O(x'^{n+1}) \right],
\]

(4.3)

The further transformation

\[
x = x' \left( 1 + \frac{1}{2n^2 \kappa_{11}} x'^n \right)
\]

(4.4)

brings the metric to the form

\[
g = \frac{1}{\rho^2_{(k)}(x')} \left[ dx^2 + g_{(k)} + \frac{x^n}{n} \left( \kappa_{AB} + \frac{1}{n} \kappa_{11} g_{(k)AB} \right) dy^A dy^B + O(x^{n+1}) \right].
\]

(4.5)

Once we have this form, Lemma 3.10 of [3] shows that further transformations eliminate the higher order terms in \( g_{11} \) and \( g_{1A} \), returning \( g \) to the form (2.5) but with \( \kappa_{AB} \) replaced by \( \kappa_{AB} + \frac{1}{n} \kappa_{11} g_{(k)AB} \). Thus we have proved the following:

**Lemma 4.1.** The mass of a metric of the form (4.1) is

\[
m \equiv m[g] := \int_{\partial_M} \left( g_{AB}^{(k)} \kappa_{AB} + \frac{(n-1)}{n} \kappa_{11} \right) d\mu_k.
\]

(4.6)

We will give the integrand in the above formula a name.
Definition 4.2. The mass aspect function \( \sigma : \partial_\infty M \to \mathbb{R} \) is
\[
\sigma := g^{AB}(k) \kappa_{AB} + \frac{(n-1)}{n} \kappa_{11} .
\]
(4.7)

Proposition 4.3. Let \( g(t) \) be as in proposition 3.1, with the expansion (3.4) for \( g(0) \) beginning at order \( m = n \). Then the mass aspect of \( g(t) \) evolves as
\[
\sigma(t) = \sigma_0 e^{-(n-2)t} ,
\]
(4.8)
where \( \sigma_0 = \sigma(0) \).

Proof. By proposition 3.1, the metric takes the form (4.1) at each \( t \in [0,T) \). The flow of \( \kappa_{ij}(t) \) is then obtained by setting \( m = n \) in (3.22–3.26). This gives
\[
\frac{\partial \kappa_{11}}{\partial t} = (n-2) \left[ (n-1) \kappa_{11} + ng^{CD}(k) \kappa_{CD} \right] ,
\]
\[
\frac{\partial \kappa_{AB}}{\partial t} = -(n-2) \kappa_{11} g^{(k)AB} + n \left[ \kappa_{AB} - g^{(k)AB} g^{CD}(k) \kappa_{CD} \right] .
\]
(4.9) (4.10)
Forming the appropriate linear combination, we thus obtain
\[
\frac{\partial \sigma}{\partial t} = -(n-2) \sigma ,
\]
(4.11)
from which the proposition follows.

Theorem 1.1 now immediately follows from Proposition 4.3.

Proof of Theorem 1.1. Integrate (4.8) over \( \partial_\infty M \).

In passing, we also obtain a theorem known from [4] for \( k = 1 \) (with \( \partial_\infty M \)) and generalize it to the \( k = 0 \) case.

Lemma 4.4. If an asymptotically hyperbolic metric is Einstein, it cannot have nonzero mass.

Proof. Setting \( m = n \) in (3.19) yields \( E_{11} = -\frac{1}{2}(n-2)\sigma \). The Einstein condition is \( E_{ij} = 0 \), so in particular \( E_{11} = 0 \). Thus, the mass aspect vanishes.

When \( \partial_\infty M \cong S^n \), a partial converse of this result is proved in the next section.

5 Rigidity

In this section, we give a variant of a standard proof [3] of Proposition 1.4 based on the flow (1.1) and following ideas in [16]. The variant method relies on Bahuaud’s short-time existence [6] theorem to construct a metric variation. By way of contrast, [3] directly posits a suitable variation (the first Newton approximation to a solution of (1.1)). In return for this added complexity at one point in the argument, our method
gains in simplicity at another by avoiding the necessity in \([3]\) (section 3.2.3 thereof) of replacing a solution of the Yamabe equation by a solution of its linearization.

To begin, we say that the mass aspect of an asymptotically hyperbolic metric \(g\) is of semi-definite sign if \(\sigma : \partial_\infty \to \mathbb{R}\) is either a nonnegative function or a nonpositive function; i.e., if \(\sigma(p)\sigma(q) \geq 0 \forall p, q \in \partial_\infty M\). We recall from Proposition 1.4 the class \(\mathcal{G}\) of metrics whose elements are asymptotically hyperbolic with \(\partial_\infty M \cong S^{n-1}\) and with well-defined mass, obeying \(E[g] := R + n(n-1) \geq 0\), \(3 \leq n = \text{dim} M \leq 6\), and further restricted if \(M\) is not spin to contain only metrics whose mass aspect is of semi-definite sign. We take \(3 \leq n = \text{dim} M \leq 6\).

**Theorem 5.1** (Positive Energy Theorem \([30, 9, 3]\)). If \(g \in \mathcal{G}\) then \(m[g] \geq 0\).

**Proof.** For the spin case, see \([30, 9]\). For the semi-definite mass aspect case, see \([3]\). □

**Proof of Proposition 1.4.** By way of contradiction, assume that \(g \in \mathcal{G}\) and \(E_{ij}|_p \neq 0\) at some \(p \in M\). Without loss of generality, we can take \(E|_p > 0\), for if it is not, then we can take \(g\) to be initial data for a flow \(g(t)\) solving (1.1). By \([6]\) and Proposition 3.1, \(g(t)\) will exist at least on a non-empty interval \([0, T]\) and will remain asymptotically hyperbolic with well-defined mass. Then a standard derivation shows that under the flow (1.1), \(E := R + n(n-1)\) evolves according to

\[
\frac{\partial E}{\partial t} = \Delta E + 2E_{ij}E^{ij} - 2(n-1)E.
\]

(5.1)

By the maximum principle and since we assume that \(E(0) \geq 0\), then \(E(t) \geq 0 \forall t \in [0, T]\). Furthermore, inspection of (5.1) shows that if \(E(0) = 0\) and \(E_{ij}(0)|_p \neq 0\), then \(\frac{\partial E}{\partial t}(0)|_p > 0\), and so \(E(t)|_p > 0\) for \(t \in (0, T)\) for some \(0 < T < T_1\). Furthermore, by Theorem 1.1, \(m(t) = 0\).

Hence, we now have \(E \geq 0\) on \(M\) and a \(p \in M\) such that \(E|_p > 0\), and we have \(m = 0\). When \(k = 1\), Proposition 3.13 of \([3]\) now shows that there is a conformal transformation of \(g\) which produces a new metric \(\hat{g} = w^{\frac{4}{n-2}}g\), \(w \to 1\) on approach to \(\partial_\infty M\), such that \(\hat{g}\) is strongly asymptotically hyperbolic, the scalar curvature of \(\hat{g}\) obeys \(\bar{E} := \bar{R} + n(n-1) = 0\), and the mass aspect of \(\hat{g}\) is pointwise strictly lower than that of \(g\): \(\hat{\sigma} < \sigma\). The proof of this proposition relies on the existence of a suitable solution of the Yamabe prescribed scalar curvature equation on \((M, g)\), which was proved in \([5]\). Of course, it then follows that the mass of \(\hat{g}\) is strictly lower than that of \(g\): \(\hat{m} := m(\hat{g}) < m = m(g) = 0\). But if \(g \in \mathcal{G}\), then \(\hat{g} \in \mathcal{G}\), since \(\bar{E} = 0\) and since the flow and conformal transformation preserve strong asymptotic hyperbolicity. Furthermore, if \((M, g)\) is spin then so is \((M, \hat{g})\) since possession of a spin structure is a topological property, and if \(\sigma\) is of semi-definite sign, then \(\hat{m} = 0 \Rightarrow \hat{\sigma} = 0\), and so \(\hat{\sigma} < \sigma\) implies that \(\hat{\sigma} < 0\). Either way, this contradicts the positive mass theorem \([30, 9, 3]\). Hence \(E_{ij} \equiv 0\) pointwise on \(M\). Then, using the Einstein manifold rigidity result of \([25]\) valid for \(3 \leq n = \text{dim} M \leq 6\), \((M, g)\), we see that \((M, g)\) is isometric to standard hyperbolic space. □
6 Parabolic scaling and ADM mass

The Ricci flow of mass has arisen in the physics of string theory and the renormalization group. In that context, [15] exhibits an explicit example of a 2-dimensional Ricci soliton whose ADM mass (in 2-dimensions, this is the deficit angle of the cone to which the metric asymptotes) remains constant during Ricci flow. The flow has a $t \to \infty$ limit, which is flat space. Thus the mass remains constant during the flow and equal to the mass of the initial metric, but the mass of the limit manifold is zero. This behaviour persists in much more general circumstances. In [11, 23] it was shown that ADM mass is constant under asymptotically flat Ricci flow in all dimensions $n \geq 3$, and in [23] it was shown that rotationally symmetric, asymptotically flat Ricci flow approaches flat space in the $t \to \infty$ limit if no minimal surface is present in the initial data.

A short heuristic argument which we now outline shows that the behaviour of mass in the asymptotically flat flow can be “derived”, based on reasoning that follows from (1.3). When the asymptotic sectional curvature is $1/\ell^2$ and is no longer normalized to 1, parabolic scaling leads us to replace $t \mapsto t/\ell^2$ in (1.3):

$$m(t) := m_0 e^{-(n-2)t/\ell^2}.$$  \hspace{1cm} (6.1)

This is the formula for the flow of mass under the evolution

$$\frac{\partial g_{ij}}{\partial t} = -2 \left( R_{ij} + \frac{(n-1)}{\ell^2} g_{ij} \right),$$  \hspace{1cm} (6.2)

which is obtained from (1.1) by the replacements $t \mapsto t/\ell^2$ and $g \mapsto g/\ell^2$. If we now consider the limit $\ell \to \infty$ at any fixed $t$, we see that (6.1) becomes simply $m(t) = m_0 = \text{const}$, (6.2) becomes the usual Ricci flow, and since the asymptotic radius of curvature $\ell$ goes to infinity the solution $g$ becomes asymptotically flat. However, if instead we take $t \to \infty$ first, before sending $\ell \to \infty$, the mass would still of course approach 0 in this limit. While this discussion is quite far from rigourous, we see that it reproduces the known behaviour of the mass of an asymptotically flat metric under Ricci flow.

Finally, we note that [15] conjectured that an initially positive mass will decrease under a string theory process called tachyon condensation, which can be modelled by Ricci flow. Asymptotically flat spacetimes can have two notions of mass, called ADM and Bondi [29]. The conjecture referred directly to the Bondi mass, but the Ricci flow occurs on a Riemannian manifold where there is only ADM mass, whose flow behaviour is at best a very coarse manifestation of the conjectured behaviour. When one passes to asymptotically anti-de Sitter spacetimes, the Lorentzian analogue of asymptotically hyperbolic manifolds, the distinction between these two notions of mass disappears. Perhaps this is why the smooth mass evolution (1.3) under (1.1) seems to reflect so well the expectation arising from physics of tachyon condensation.

A Equivalence of Wang and Chruściel-Herzlich masses

In his paper [30], Wang defines the mass (2.6) for $k = 1$ and $\partial_\infty M \simeq S^{n-1}$. This mass is equivalent to the Chruściel-Herzlich mass [9], which is formulated in a chart-independent manner. Let $b$ and $g$ be any two Riemannian metrics on a manifold $M^n$. 

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Let $D$ be the Levi-Civita connection of $b$ and let $\nabla$ be the Levi-Civita connection of $g$. The Chruściel-Herzlich mass of $(M^n, g)$ with respect to the reference metric $b$ is

$$m_{\text{CH}}(g) = \lim_{R \to \infty} \int_{N_R} \mathbb{U}(V \circ \phi^{-1}) dS_i,$$

(A.1)

$$\mathbb{U}^i = \sqrt{g} \left\{ V \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) D_j g_{kl} + \left( g^{jk} \nabla^i V - g^{ik} \nabla^j V \right) e_{jk} \right\},$$

(A.2)

$$e_{jk} := g_{jk} - b_{jk},$$

(A.3)

where $V : M_{\text{ext}} \to \mathbb{R}$ is a smooth $O(r)$ function on a collar neighbourhood $M_{\text{ext}}$ of $\partial_\infty M$, $\phi^{-1} : M_{\text{ext}} \to [R, \infty) \times N$ is a smooth diffeomorphism, and $dS_i := n_i dA_R$ is the induced volume (i.e., area) element on $N_R$.

In (2.6) we extended Wang’s definition to the case of $k = 0$, with $\partial_\infty M$ a flat torus. The Chruściel-Herzlich mass [9] formulation already includes this case. Here we show that our $k = 0$ extension agrees with the chart-independent $k = 0$ Chruściel-Herzlich formulation, as required for the proof of Corollary 1.3. We will do this by evaluating the Chruściel-Herzlich mass in the coordinate gauge in which the metric takes the form (2.5) (so that $\kappa_{1i} = \kappa_{i1} = 0$) and checking that the Chruściel-Herzlich expression for the mass reduces to (2.6) with $k = 0$. The $k = 1$ case is similar and straightforward, and has been reported in [9].

In the $k = 0$ case, for which $N$ is an $(n - 1)$-torus, we have (up to a scale for $V$, which we fix to be 1)

$$V(r) = r,$$

(A.4)

$$b = \frac{dr^2}{r^2} + r^2 g(0) = \frac{dr^2}{r^2} + r^2 \sum_{A=2}^n d\theta_A^2,$$

(A.5)

$$e_{jk} = g_{jk} - b_{jk} = \frac{1}{n_{r^{n-1}}} \kappa_{jk} + O(1/r^{n-1}),$$

(A.6)

where in the last equation we used the form (2.5) for $g$ with $k = 0$ and $\rho(0)(r) = r = 1/x$, and in the sequel we will use $A, B \in \{2, \ldots, n\}$ to denote coordinates on hypersurfaces $r = \text{const.}$

It is now a matter of plugging (2.5, A.4–A.6) into (A.2). We write

$$\mathbb{U}^i = \sqrt{g} r \left\{ A^i + B^i \right\},$$

(A.7)

$$A^i = \left( g^{jk} g^{il} - g^{ij} g^{kl} \right) D_j g_{kl},$$

(A.8)

$$B^i = \left( \frac{\nabla^i r}{r} g_{jk} - \frac{\nabla^j r}{r} g_{ik} \right) e_{jk}$$

$$= \left( g^{il} g^{jk} - g^{ij} g^{kl} \right) \frac{\nabla^i r}{r} e_{jk},$$

(A.9)

and find that

$$A^1 = \frac{(n - 1)}{n_{r^{n-1}}} g_{AB}^{AB} \kappa_{AB} + O(1/r^n),$$

(A.10)

$$B^1 = \frac{1}{n_{r^{n-1}}} g_{AB}^{AB} \kappa_{AB} + O(1/r^n),$$

(A.11)
and $A^A = O(1/r^n)$, $B^A = O(1/r^n)$. Substituting (A.10) and (A.11) into (A.7) yields

$$U_i = \sqrt{g} \left\{ \frac{1}{r^{n-2}} g^{AB}_0 \kappa_{AB} \delta_i^1 + O(1/r^{n-1}) \right\} .$$  \hspace{1cm} (A.12)

Plugging this into (A.1) yields

$$m_{CH}(g) = \lim_{R \to \infty} \int_{N_R} U_i dS_i = \int_N g^{AB}_0 \kappa_{AB} dV(g_0) ,$$  \hspace{1cm} (A.13)

which is the mass formula (2.6) for the $k = 0$ case. As discussed in the Introduction, for the metric (1.2) this formula yields a mass of $-\frac{4\pi}{n} \prod_{i=3}^n a_i$.

**B  Ricci-DeTurck flow**

Bahuaud’s normalized Ricci-DeTurck flow \cite{6} is

$$\frac{\partial g}{\partial t} = -2 E + L_X g \equiv -2 (\text{Ric} + (n - 1)g) + L_X g ,$$  \hspace{1cm} (B.1)

where $L_X g$ is the Lie derivative of $g$ along the DeTurck vector field $X$. Pulling back along the diffeomorphism generated by $X$ yields the normalized Ricci flow (1.1). One can apply the iteration of section 3 directly to (B.1), rather than to (1.1) as is done in the main text. We outline the main steps. For comparison purposes, we fix $k = 1$ so that $\partial M$ carries a metric $g^{(1)} = g(S^{n-1}, \text{can})$ with sectional curvature 1, and adopt Bahuaud’s notation

$$g = h + v ,$$  \hspace{1cm} (B.2)

where $g(0) = h$ so that $v(0) = 0$, $\frac{\partial h}{\partial t} = 0$, and the components of $v$ in an orthonormal basis of $h$ are $O(x)$. Since $h$ is now the initial metric, we write (using $\sinh \rho = \frac{4x}{4 - x^2}$)

$$h := x^{-2} \tilde{h} := x^{-2} \left[ dx^2 + (1 - (x/2)^2)^2 g(S^{n-1}, \text{can}) + \frac{1}{n} x^n \kappa_{AB} \right] ,$$  \hspace{1cm} (B.3)

where $x$ is a special defining function for the asymptotically hyperbolic metric $h$. Note that $\kappa$ here corresponds to the initial metric “perturbation” and is constant along the flow, unlike in the main text. We write

$$v(t) = \frac{1}{m} w_{ij}(t) x^{m-2} ,$$  \hspace{1cm} (B.4)

where $w(t, x) = w_0(t) + O(x)$. We then consider the cases $m \leq n$, using that $w(0) = 0$ when $m \leq n$.

Let $\nabla$ be the Levi-Cività connection of $g$ with connection coefficients $\Gamma^k_{ij}$ and let $\tilde{\nabla}$ be the Levi-Cività connection of $h$ with connection coefficients $\tilde{\Gamma}^k_{ij}$. The DeTurck vector field $X = X^k \frac{\partial}{\partial x^k}$ has components

$$X^k = g^{ij} \left( \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij} \right) = h^{kl} h_{ij} \left( \tilde{\nabla}_i v_{jl} - \frac{1}{2} \tilde{\nabla}_l v_{ij} \right) + O(v^2, v \cdot \partial v) .$$  \hspace{1cm} (B.5)
A calculation then yields
\[
X_1 = -\frac{(n-2)}{m} w_{11} x^{m-1} + \left( \frac{m-2}{2m} \right) \left( w_{11} - g^{AB}_{(1)} w_{AB} \right) x^{m-1} + O(x^m), \quad (B.6)
\]
\[
X_A = \left( \frac{m-n}{m} \right) w_{1A} x^{m-1} + O(x^m). \quad (B.7)
\]

We then compute that
\[
(\mathcal{L}Xg)_{11} = -2(n-2) w_{11} x^{m-2} + (m-2) \left( w_{11} - g^{AB}_{(1)} w_{AB} \right) x^{m-2} + O(x^{m-1}), \quad (B.8)
\]
\[
(\mathcal{L}Xg)_{1A} = (\mathcal{L}Xg)_{1A} = \frac{1}{m} (m-n)(m+1) w_{1A} x^{m-2} + O(x^{m-1}), \quad (B.9)
\]
\[
(\mathcal{L}Xg)_{AB} = \left[ \frac{2(n-2)}{m} w_{11} x^{m-2} - \frac{(m-2)}{m} \left( w_{11} - g^{CD}_{(1)} w_{CD} \right) x^{m-2} \right] g_{(1)AB} x^{m-2} + O(x^{m-1}). \quad (B.10)
\]

For \( m < n \), the evolution equations for the metric perturbation under DeTurck flow are obtained by replacing \( \kappa \) by \( w \) in equations (3.19–3.21) and adding to these the appropriate Lie derivative component from equations (B.8–B.10) to form \( \xi_{ij} := E_{ij} - \frac{1}{2} \mathcal{L}Xg_{ij} \). Then (3.22) is replaced by
\[
\frac{\partial w_{ij}}{\partial t} = -\frac{2m}{x^{m-2}} \xi_{ij} + O(x) =: A_{ijkl} w_{kl} + O(x). \quad (B.11)
\]

Then (B.11) is a linear system with zero initial data (cf equation (3.22)), and hence \( w_{ij}(t) = 0 \) for \( m < n \), so Proposition 3.1 holds for Bahuaud’s Ricci-DeTurck flow.

For \( m = n \), we must replace \( \kappa \) in (3.19–3.21) by \( \kappa + w \) (but not in (B.8–B.10)) and proceed as above, adding (B.8–B.10) to \( E \). In particular, we are interested in the evolution of the combination \( \frac{(n-1)}{n} w_{11} + g^{AB}_{(1)} w_{AB} \), as this is the time-evolving part of the mass aspect. But, setting \( m = n \) in (B.8) and (B.10), we obtain
\[
\frac{(n-1)}{n} (\mathcal{L}Xg)_{11} + g^{AB}_{(1)} (\mathcal{L}Xg)_{AB} = O(x^{n-1}) \quad (B.12)
\]

Thus, the Lie derivative term in (B.1) does not contribute to the evolution of the mass aspect. Then it is easy to see that Proposition 4.3 applies also to Bahuaud’s Ricci-DeTurck flow, and then so does Theorem 1.1.

Finally, now we may put \( m = n \) in (B.6) and (B.7) and raise the index, obtaining \( X^k \in O(x^{n+1}) \). Thus \( \phi^k_t = \text{id} + O(x^n) \), so the mass is invariant under the pullback of \( g \) by \( \phi^k_t \) (see, for example, Theorem 3.4 of [17]). This reproduces the Ricci flow result in the main text.

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