ON THE GERSTEN–WITT COMPLEX OF AN AZUMAYA ALGEBRA WITH INVOLUTION

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Abstract. Let \((A, \sigma)\) be an Azumaya algebra with involution over a regular ring \(R\). We prove that the Gersten–Witt complex of \((A, \sigma)\) defined by Gille is isomorphic to the Gersten–Witt complex of \((A, \sigma)\) defined by Bayer-Fluckiger, Parimala and the author. Advantages of both constructions are used to show that the Gersten–Witt complex is exact when \(\dim R \leq 3\), \(\ind A \leq 2\) and \(\sigma\) is orthogonal or symplectic. This means that the Grothendieck–Serre conjecture holds for the group \(R\)-scheme of \(\sigma\)-unitary elements in \(A\) under the same hypotheses; \(R\) is not required to contain a field.

1. Introduction

Let \(R\) be a regular domain with fraction field \(K\) such that \(2 \in R^\times\). The Gersten conjecture for Witt groups, suggested by Pardon [27], predicts the existence of a cochain complex of Witt groups of the form

\[
0 \to W(R) \xrightarrow{d_{-1}} W(K) \xrightarrow{d_0} \bigoplus_{p \in \mathcal{R}(1)} W(k(p)) \xrightarrow{d_1} \bigoplus_{p \in \mathcal{R}(2)} W(k(p)) \xrightarrow{d_2} \cdots,
\]

called a Gersten–Witt complex for \(R\), which is exact when \(R\) is local. Here, \(\mathcal{R}(e)\) is the set of height-\(e\) primes in \(R\) and \(k(p)\) is the fraction field of \(R/p\). The map \(d_{-1}\) is base-change from \(R\) to \(K\), and the \(p\)-component of \(d_1\) is a second residue map \(W(K) \to W(k(p))\) in the sense of [32, p. 209].

Pardon [28] gave a construction of a Gersten–Witt complex for any regular \(R\), and proved its exactness if \(R\) is local of dimension 4 or less. Balmer and Walter [7] later gave another construction using triangulated hermitian categories, and also proved its exactness if \(R\) is local of dimension \(\leq 4\). Their approach revealed a spectral sequence underlying the Gersten–Witt complex, which led to further positive results about the exactness: It was established when \(R\) is semilocal and contains a field by Balmer, Gille, Panin and Walter [5] (see also [3]), when \(R\) is semilocal of dimension \(\leq 4\) by Balmer and Preeti [8], and when \(R\) is local and unramified of mixed characteristic by Jacobson [21]. Another construction, applying if \(R\) is essentially of finite type over a field, appears in Schmid [33].

Let \((A, \sigma)\) be an Azumaya algebra with involution over \(R\) (see [29]) and let \(\varepsilon \in \{\pm 1\}\). By introducing a modification to the Balmer–Walter construction, Gille [16], [18] defined a Gersten–Witt complex of Witt groups of \(\varepsilon\)-hermitian forms over \((A, \sigma)\), and proved its exactness when \(R\) is local and contains a field [19, Theorem 7.7]. Broadly speaking, the modification to [7] consisted of replacing the bounded derived category of finite projective \(R\)-modules and the duality \(\Hom(\_ , R)\) with a suitable subcategory of the bounded derived category of quasi-coherent sheaves over \(\text{Spec } R\) and the duality induced by a dualizing complex. When \((A, \sigma, \varepsilon) = (R, \text{id}_R, 1)\), Gille’s Gersten–Witt complex is isomorphic to the one defined by Balmer and Walter.

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Recently, Bayer-Fluckiger, Parimala and the author \cite{9} introduced another construction of a Gersten--Witt complex of \(\varepsilon\)-hermitian forms over \((A,\sigma)\), and proved it is exact when \(R\) is semilocal and \(\dim R \leq 2\) (\(R\) is not required to contain a field). In contrast with \cite{7,10,15}, this construction does not require the use of triangulated hermitian categories.

The main purpose of this work is to prove that the Gersten--Witt complexes defined by Gille and by Bayer-Fluckiger, Parimala and the author are in fact isomorphic. In the process, we streamline the Balmer–Walter construction of the Gersten–Witt complex, particularly the dévissage, and explain how it can be applied to Azumaya algebras with involution without using a dualizing complex as in \cite{10,18}. This is made possible thanks to a new method of transfer in hermitian categories, which is of independent interest; see Section 4.

The isomorphism between the constructions means that tools developed for individual constructions can be applied together. Specifically, it allows us to use the spectral sequence underlying Gille’s Gersten–Witt complex \cite{10,18} together with the 8-periodic exact sequence of \cite{15}, which is compatible with the Gersten–Witt complex of \cite{9}. By putting both of these together, we prove that the Gersten--Witt complex of \((A,\sigma)\) is exact when \(R\) is semilocal of dimension \(\leq 3\), the index of \(A\) is at most 2 and \(\sigma\) is orthogonal or symplectic; see Theorems 5.1.

This in turn establishes some open cases of the famous Grothendieck–Serre conjecture, which predicts that the map \(H^1_\text{ét}(R,G) \rightarrow H^1_\text{ét}(K,G)\) has trivial kernel for every regular local ring \(R\) and every reductive (connected) group scheme \(G \rightarrow \text{Spec} \, R\). See \cite[§5]{20} and \cite{12} for surveys of the many works discussing this conjecture. Our Theorem 5.1 and \cite[Proposition 8.7]{13} imply:

**Theorem 1.1.** Let \(R\) be a regular semilocal domain such that \(\dim R \leq 3\) and \(2 \in R^\times\), and let \(K\) be the fraction field of \(R\). Let \((A,\sigma)\) be an Azumaya \(R\)-algebra with involution such that \(\text{ind} \, A \leq 2\) and \(\sigma\) is orthogonal or symplectic. Let \(G\) denote the neutral component of the group \(R\)-scheme of elements \(a \in A\) satisfying \(a^\sigma a = 1\). Then \(H^1_\text{ét}(R,G) \rightarrow H^1_\text{ét}(K,G)\) is injective.

The paper is organized as follows: Section 2 recalls hermitian categories, Azumaya algebras with involution and some facts about Ext-groups. In Section 3 we recall the Gersten–Witt complex defined by Bayer-Fluckiger, Parimala and the author, and in Section 4 we recall the constructions of Balmer–Walter and Gille. The proof that the constructions yield isomorphic cochain complexes is given in Sections 5–7. In Section 5 we introduce a variant of transfer in hermitian categories, which is applied in Section 6 to define an isomorphism (called dévissage) between the respective terms of both Gersten–Witt complexes. This is shown to be an isomorphism of cochain complexes in Section 7. Finally, in Section 8 we apply the isomorphism to establish the exactness of the Gersten–Witt complex when \(\text{ind} \, A\) and \(\dim R\) are small.

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**Notation.** Throughout this paper, a ring means a commutative (unital) ring. Algebras are unital, but not necessarily commutative. We assume that 2 is invertible in all rings an algebras.

Unless otherwise indicated, \(R\) denotes a ring. Unadorned Hom-groups, Ext-groups and tensors are taken over \(R\). An \(R\)-ring means a commutative \(R\)-algebra. An \(R\)-algebra with involution is a pair \((A,\sigma)\) such that \(A\) is an \(R\)-algebra and \(\sigma : A \rightarrow A\) is an \(R\)-linear involution. An invertible \(R\)-module is a rank-1 projective \(R\)-module. Given \(p \in \text{Spec} \, R\), we write \(k(p)\) for the fraction ring of \(R/\mathfrak{p}\), and set \(M(\mathfrak{p}) = M \otimes k(p)\) for any \(R\)-module \(M\). If \(f : M \rightarrow N\) is an \(R\)-module...
homomorphism, let \(f(p)\) denote \(f \otimes \text{id}_{k(p)} : M(p) \to N(p)\). We also let \(\mu_2(R) = \{ r \in R : r^2 = 1 \}\).

Let \(A\) be an \(R\)-algebra. The center and the group of units of \(A\) are denoted \(\text{Cent}(A)\) and \(A^\times\), respectively. The category of all (resp. finite, finite projective, finite length) right \(A\)-modules is denoted \(\mathcal{M}(A)\) (resp. \(\text{Mod}_f(A), \mathcal{P}(A), \mathcal{M}_f(A)\)). Here, an \(A\)-module is said to be finite if it is finitely generated. The bounded derived category of \(\mathcal{P}(A)\) is denoted \(\mathcal{D}^b(A)\). Throughout, \(\text{Hom}\)-groups and Ext-groups taken over \(A\) are taken in the category of right \(A\)-modules. We write \(A_A\) to denote \(A\) when regarded as right module over itself.

2. Preliminaries

We recall some facts about hermitian categories, Azumaya algebras with involution and Ext-groups, setting notation along the way. For an extensive discussion, see \([23, \text{Chapter II}]\) and \([15, \S 1]\), for instance.

2A. Hermitian categories. As usual, a hermitian \(R\)-category is a triple \((\mathcal{C}, *, \omega)\) such that \(\mathcal{C}\) is an additive \(R\)-category, \(* : \mathcal{C} \to \mathcal{C}\) is a contravariant \(R\)-linear functor, and \(\omega : \text{id} \to **\) is a natural isomorphism satisfying \(\omega_P \circ \omega_P = \text{id}_{P^2}\) for all \(P \in \mathcal{C}\). We also say that \((*, \omega)\) is a hermitian structure on \(\mathcal{C}\).

Given \(\varepsilon \in \mu_2(R)\), an \(\varepsilon\)-hermitian space over \((\mathcal{C}, *, \omega)\) is a pair \((P, f)\), where \(P \in \mathcal{C}\) and \(f : P \to P^*\) is a morphism satisfying \(f = \varepsilon f^* \circ \omega_P\). We say that \((P, f)\) is unimodular if \(f\) is an isomorphism. The category of unimodular \(\varepsilon\)-hermitian spaces over \((\mathcal{C}, *, \omega)\) with isometries as morphisms is denoted \(\mathcal{H}^\varepsilon(\mathcal{C}, *, \omega)\).

Suppose in addition that \(\mathcal{C}\) is an exact category. We say that \((\mathcal{C}, *, \omega)\) is an exact hermitian category, or that \((*, \omega)\) is an exact hermitian structure on \(\mathcal{C}\), if \(* : \mathcal{C} \to \mathcal{C}\) is exact. In this case, we define the Witt group of \(\varepsilon\)-hermitian forms over \((\mathcal{C}, *, \omega)\), denoted \(W_\varepsilon(\mathcal{C}, *, \omega)\) or \(W_\varepsilon(\mathcal{C})\), as in \([1, \S 1]\). That is, \(W_\varepsilon(\mathcal{C})\) is the quotient of the Grothendieck group \(\mathcal{H}^\varepsilon(\mathcal{C}, *, \omega)\) (relative to orthogonal sum) by the subgroup generated by metabolic hermitian spaces. Here, a unimodular \(\varepsilon\)-hermitian space \((P, f)\) is called metabolic if there exists a short exact sequence \(0 \to P \to M \to L^* \to 0\) in \(\mathcal{C}\) such that \(L \to P\) is the kernel of the composition \(P \xrightarrow{f} P^* \to L^*\). The element of \(W_\varepsilon(\mathcal{C})\) represented by \((P, f) \in \mathcal{H}^\varepsilon(\mathcal{C})\) is the Witt class of \((P, f)\) and denoted \([P, f]\) or \([f]\).

Example 2.1. Let \((A, \sigma)\) be an \(R\)-algebra with involution and let \(M\) be an invertible \(R\)-module. Then \(\sigma\) and \(M\) induce an exact hermitian structure \((*, \omega) = (\text{id}_M, \omega_{*, M})\) on \(\mathcal{P}(A)\) as follows: The functor \(*\) sends an object \(P \in \mathcal{P}(A)\) to \(\text{Hom}_A(P, A \otimes M)\) endowed with the right \(A\)-module structure given by \((\alpha x) = a^\sigma(\phi x)\) \((\phi \in P^*, a \in A, x \in P)\), and a morphism \(\varphi\) to \(\text{Hom}_A(\varphi, A \otimes M)\). The isomorphism \(\omega_P : P \to P^{**}\) is given by \((\omega_P x) = (\phi x)^{\sigma \otimes \text{id}_M}\) \((x \in P, \phi \in P^*)\). The pair \((*, \omega)\) is an exact hermitian structure by \([3, \text{Example 1.2}]\).

Hermitian spaces over \((\mathcal{P}(A), *, \omega)\) and \((A \otimes M, \sigma \otimes \text{id}_M)\)-valued hermitian spaces over \((A, \sigma)\) in the sense of \([11, \S 1B]\) are essentially the same thing. Recall that the latter are pairs \((P, \hat{f})\) consisting of \(P \in \mathcal{P}(A)\) and a biadditive map \(\hat{f} : P \times P \to A \otimes M\) satisfying \(\hat{f}(xa, x')a'^\sigma = a^\sigma \hat{f}(x, x')a'\) and \(\hat{f}(x, x') = \varepsilon \hat{f}(x', x)a'^{\sigma \otimes \text{id}_M}\) for all \(x, x' \in P, a, a' \in A\). The map corresponding to \(f : P \to P^*\) is \(\hat{f} : P \times P \to A \otimes M\) given by \(\hat{f}(x, y) = (fx, y)\).

In the sequel, the category \(\mathcal{H}^\varepsilon(\mathcal{C}, *, \omega)\) and the Witt group \(W_\varepsilon(\mathcal{C}, *, \omega)\) will be denoted as \(\mathcal{H}^\varepsilon(\mathcal{C}, \sigma; M)\) and \(W_\varepsilon(\mathcal{C}, \sigma; M)\), respectively. We abbreviate this to \(\mathcal{H}^\varepsilon(\mathcal{C}, \sigma)\) and \(W_\varepsilon(\mathcal{C}, \sigma)\) when \(M = R\).

Let \((\mathcal{D}, \#, \eta)\) be another hermitian \(R\)-category and let \(\gamma \in \mu_2(R)\). Recall that a \(\gamma\)-hermitian functor from \((\mathcal{C}, *, \omega)\) to \((\mathcal{D}, \#, \eta)\) consists of a pair \((F, i)\), where
$F : \mathcal{C} \to \mathcal{D}$ is an additive $R$-functor and $i : F* \to \#F$ is a natural isomorphism satisfying $i_{F*} \circ F \omega_F = \gamma \cdot i_{\#F} \circ \eta_{F*}$ for all $P \in \mathcal{C}$. This induces a functor $F = \mathcal{H}(F, i) : \mathcal{H}_e(\mathcal{C}) \to \mathcal{H}_e(\mathcal{D})$ given by $F(P, f) = (FP, i_{P*} \circ Ff)$ on objects and acting as $F$ on morphisms. If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, then so is $\mathcal{H}(F, i)$, and $(F, i)$ is called a $\gamma$-hermitian equivalence. If $(\mathcal{C}, e_\mathcal{C})$ and $(\mathcal{D}, e_\mathcal{D})$ are exact hermitian categories and $F$ is exact, then we also have an induced group homomorphism $F = W(F, i) : W_e(\mathcal{C}) \to W_{e_\mathcal{D}}(\mathcal{D})$, which is an isomorphism when $F$ is an equivalence of exact categories.

2B. Azumaya algebras with involution. Following [15, §1.1], an $R$-algebra $A$ is called separable projective if it is separable as an $R$-algebra and projective as $R$-module, or equivalently, if $A$ is Azumaya over its center $\text{Cent}(A)$ and $\text{Cent}(A)$ is finite étale over $R$. An $R$-algebra with involution $(A, \sigma)$ is Azumaya if $A$ is separable projective over $R$, and the structure map $R \to \text{Cent}(A)^{(\sigma)} := \{a \in \text{Cent}(A) : a^{\sigma} = a\}$ is an isomorphism. See [15, §1] for further details.

We will work with separable projective $R$-algebras with involution throughout, rather than the more restricted class of Azumaya $R$-algebras with involution. This makes little difference in practice, because if $(A, \sigma)$ is a separable projective $R$-algebra with involution, then $(A, \sigma)$ is Azumaya over $\text{Cent}(A)^{(\sigma)}$ and $\text{Cent}(A)^{(\sigma)}$ is finite étale over $R$ [15, Example 1.20].

If $A$ is a separable projective $R$-algebra, then a right $A$-module $M$ is projective if and only if it is projective as an $R$-module [31, Proposition 2.14]. This fact, together with Schanuel’s lemma [21, Corollary 5.6], imply the following lemma.

Lemma 2.2. Let $A$ be a separable projective $R$-algebra and let $M \in \mathcal{M}(A)$. Then the projective dimension of $M$ as an $A$-module is at most the projective dimension of $M$ as an $R$-module. In particular, the right global dimension of $A$ is at most the global dimension of $R$.

Corollary 2.3. Let $A$ be a separable projective algebra over a regular ring $R$, and let $\mathcal{D}^b(\mathcal{M}_R(A))$ be the bounded derived category of $\mathcal{M}_R(A)$. Then the natural functor $\mathcal{D}^b(A) := \mathcal{D}^b(\mathcal{P}(A)) \to \mathcal{D}^b(\mathcal{M}_R(A))$ is an equivalence.

Proof. It is enough to show that all finite right $A$-modules have finite projective dimension (consult [34, Tags 0646, 0613]). Since $R$ is a regular, all finite $R$-modules have finite projective dimension [21, Theorem 5.94], and Lemma 2.2 completes the proof. \qed

2C. Homological Algebra. Let $A$ be a right noetherian $R$-algebra. Write $\mathcal{K}$ for the homotopy category of chain complexes in $\mathcal{M}_R(A)$, and $\mathcal{D} = \mathcal{D}^b(\mathcal{M}_R(A))$ for the derived category of $\mathcal{M}_R(A)$. We denote the shift-to-the-left-and-negate-the-differential functor $\mathcal{D} \to \mathcal{D}$ by $T$, that is, given a chain complex $P = (P_i, d_i)_{i \in \mathbb{Z}} = (\cdots \to P_2 \to P_1 \to P_0 \to d_0 \cdots)$, we have $TP = (P_{i-1}, -d_{i-1})_{i \in \mathbb{Z}}$.

As usual, given $M \in \mathcal{M}_R(A)$, a projective resolution of $M$ is a chain complex $P$ of objects in $\mathcal{P}(A)$ supported in non-negative degrees together with a map $\alpha = \alpha_P : P_0 \to M$ such that $\cdots \to P_1 \to P_0 \to M \to 0$ is exact. We use $\alpha$ to freely identify $\text{Ho}(P)$ with $M$.

Let $M, N \in \mathcal{M}_R(A)$ and $e \in \mathbb{N} \cup \{0\}$. Following [34, Tag 06XQ], we define $\mathcal{E}(M, N, e)$ as $\text{Hom}_\mathcal{D}(M, T^eN)$, where $M$ and $N$ are regarded as chain complexes concentrated in degree 0. Note that this definition does not require choosing a projective resolution of $M$, or an injective resolution of $N$. Rather, if $P$ is a projective resolution of $M$, then the map $\alpha_P : P_0 \to M$ defines a quasi-isomorphism $P \to M$ in $\mathcal{D}$, which gives rise to an isomorphism $\mathcal{E}(P, T^eN) \Rightarrow \mathcal{E}(M, N, e)$.
$\text{Hom}_C(P, T^e N)$ (see [31] Tag [05TG, 06XR]). It is straightforward to see that $\text{Hom}_C(P, T^e N)$ is $-e$-th homology group of the chain complex of $R$-modules

$\text{Hom}_A(P, N) := \{ \cdots \to \text{Hom}_A(P_{-1}, N) \to \text{Hom}_A(P_0, N) \to \text{Hom}_A(P_1, N) \to \cdots \}$

with $\text{Hom}_A(P_0, N)$ occurring in degree 0. Thus, we have an $R$-module isomorphism

$u : \text{Ext}^e_A(M, N) \to H_{-e}(\text{Hom}_A(P, N))$,

which we usually suppress. If $P'$ is another projective resolution of $M$, then there is a unique morphism $f : P \to P'$ in $D$ such that $H_0(f) = \phi$ and the induced morphism $H_{-e}(\text{Hom}_A(f, N)) : H_{-e}(\text{Hom}_A(P', N)) \to H_{-e}(\text{Hom}_A(P, N))$ coincides with $\text{Ext}^e_A(f, N) : \text{Ext}^e_A(M', N) \to \text{Ext}^e_A(M, N)$ upon identifying $\text{Ext}^e_A(M', N)$ and $\text{Ext}^e_A(M, N)$ with $H_{-e}(\text{Hom}_A(P', N))$ and $H_{-e}(\text{Hom}_A(P, N))$, respectively.

**Lemma 2.4.** Let $A$ and $B$ be a noetherian $R$-algebras. Let $D$ be the bounded derived category of $\mathcal{M}_l(A)$ and let $X, Y \in D$. If $B$ is flat over $R$, then the base-change map $\text{Hom}_D(X, Y) \otimes B \to \text{Hom}_{D^b(\mathcal{M}_l(A \otimes B))}(X \otimes B, Y \otimes B)$ is an isomorphism.

**Proof.** We abbreviate $X \otimes B$ to $X_B$, $A \otimes B$ to $A_B$, and so on. Suppose that $X$ is concentrated in degrees $n, \ldots, n + r$ and $Y$ is concentrated in degrees $m, \ldots, m + s$. We prove the lemma by induction on $r + s$. The case $r = s = 0$ is the famous fact that the natural map $\text{Ext}^e_A(M, N) \otimes B \to \text{Ext}^e_A(M_B, N_B)$ is an isomorphism when $B$ is flat over $R$, and $M$ and $N$ are right $M$-modules such that $N$ is finitely presented; see [31] Theorem 2.39, for instance. If $s > 0$, let $K = \ker(d_n : X_n \to X_{n+1})$ and let $X' = (0 \to X_{n-1} \to \text{im}(d_n) \to X_{n-2} \to \cdots)$. Then we have a distinguished triangle $T^n K \to X \to X' \to T^{n+1} K$ in $D$. Since $B$ is flat over $R$, $T^n K_B \to X_B \to X'_B \to T^{n+1} K_B$ is distinguished in $D^b(\mathcal{M}_l(A_B))$. These distinguished triangles give rise to a commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & \text{Hom}(T^{n+1} K, Y)_B & \to & \text{Hom}(X', Y)_B & \to & \text{Hom}(X, Y)_B & \to & \text{Hom}(T^n X, Y)_B & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \text{Hom}(T^{n+1} K_B, Y_B) & \to & \text{Hom}(X'_B, Y_B) & \to & \text{Hom}(X_B, Y_B) & \to & \text{Hom}(T^n X_B, Y_B) & \to & \cdots \\
\end{array}
\]

in which the top and bottom row are exact [33] Tag [0149]; the Hom-groups are taken in $D$ or $D^b(\mathcal{M}_l(A_B))$. Since $X'$ is concentrated in degrees $n + 1, \ldots, n + r$, the induction hypothesis and the Five Lemma imply that $(\ast)$ is an isomorphism. The case $s > 0$ is handled similarly. \qed

3. The Gersten–Witt Complex via Second-Residue Maps

Let $R$ be a regular ring, let $(A, \sigma)$ be a separable projective $R$-algebra with involution, and let $e \in \mu_2(R)$. We now recall the definition of the (augmented) Gersten–Witt complex of $e$-hermitian over $(A, \sigma)$ constructed in [9] using generalized...
second residue maps. This complex is denoted $GW^{A/R, \sigma, \varepsilon}$, or just $GW^{A, \sigma, \varepsilon}$, and takes the form

$$0 \to W_*(A, \sigma) \xrightarrow{d_{-1}} \bigoplus_{p \in R^{(0)}} \tilde{W}_*(A(p)) \xrightarrow{d_0} \bigoplus_{p \in R^{(1)}} \tilde{W}_*(A(p)) \xrightarrow{d_1} \bigoplus_{p \in R^{(2)}} \tilde{W}_*(A(p)) \to \cdots,$$

where $\tilde{W}_*(A(p))$ stands for $W_*(A(p), \sigma(p); \tilde{k}(p))$ (notation as in Example 2.1) and

$$\tilde{k}(p) := \text{Ext}^{\text{hgt} p}_{R_p}(k(p), R_p) \cong \text{Hom}_{R_p}(A_p/\mathfrak{p}^2, k(p))$$

(see [9] Proposition 1.9(iii)] regarding the isomorphism). Note that $\tilde{k}(p) \cong k(p)$ as $R$-modules, and thus $\tilde{W}_*(A(p)) \cong W_*(A(p), \sigma(p))$, but this isomorphism is not canonical unless $\text{hgt} p = 0$. We shall write $GW^{A, \sigma, \varepsilon}$ for the complex obtained from $GW^{A, \sigma, \varepsilon}$ by removing the $-1$-term.

The $p$-component of the differential $d_{-1}$ is given by localizing at $p$. For $e \geq 0$, the differential $d_e$ is defined to be $\sum_{p,q} \partial_{p,q}$, where the sum ranges over all $p \in R^{(e)}$ and $q \in R^{(e+1)}$ with $p \subseteq q$, and $\partial_{p,q}$ is defined as follows.

We base-change from $R$ to $R_q$ in order to assume that $R$ is a regular local ring of dimension $e + 1$ and $q$ is its maximal ideal. Set $S = R/p$, $m = q/p$, $\tilde{S} = \text{Ext}^{e}(S, R)$ and $\tilde{m}^{-1} = \text{Ext}^{e}(m, R)$. As explained in [9] §2.1, the localization-at-$p$ maps from $A \otimes \tilde{S}$ and $A \otimes \tilde{m}^{-1}$ to $A \otimes \tilde{k}(p)$ are injective, and when regarding $A \otimes \tilde{S}$ and $A \otimes \tilde{m}^{-1}$ as submodules of $A \otimes \tilde{k}(p)$, we have

$$(3.1) \quad A \otimes \tilde{m}^{-1} = \{x \in A \otimes \tilde{k}(p) \mid xm \subseteq A \otimes \tilde{S}\}.$$

Furthermore, we have $\text{Ext}^{e+1}_{R_q}(S, R) = 0$ and $\text{Ext}^{e}_{R_q}(k(q), R) = 0$, and therefore a short exact sequence $\tilde{S} \hookrightarrow \tilde{m}^{-1} \twoheadrightarrow k(q)$, induced by $m \hookrightarrow S \twoheadrightarrow k(q)$. Since $A$ is flat over $R$, the induced sequence

$$(3.2) \quad 0 \to A \otimes \tilde{S} \to A \otimes \tilde{m}^{-1} \to A \otimes \tilde{k}(q) \to 0$$

is also exact. The map indicated by $\tau$ will be denoted as $\tau_{p,q}$ or $\tau_{p,q,A}$ when there is a risk of confusion.

Let $(V, f) \in \mathcal{H}^e(A(p)) := \mathcal{H}^e(A(p), \sigma(p); \tilde{k}(p))$. An $A$-lattice in $V$ is a finitely generated $A \otimes S$-submodule $U$ of $V$ such that $U \cdot k(p) = V$. Given an $A$-lattice $U$ in $V$, write

$$U^f = \{x \in V : \tilde{f}(U, x) \subseteq A \otimes \tilde{S}\}.$$

Then $U^f$ is also an $A$-lattice in $V$ and $U^{f,f} = U$ [9 Lemma 2.1]. Moreover, this source also tells us that there exists an $A$-lattice $\tilde{U}$ in $V$ such that $U^f m \subseteq U \subseteq U^f$. Choosing an $A$-lattice $U$ in $V$ with $U^f m \subseteq U \subseteq U^f$, (3.1) allows us to define $\tilde{\partial}f : U^f/U \times U^{\text{f},f}/U \to A \otimes \tilde{k}(q)$ by

$$\tilde{\partial}f(x + U, y + U) = \tau_{p,q}(f(x, y)).$$

Write $(U^f/U, \partial f)$ for the corresponding hermitian space in $\mathcal{H}(A(q))$ and set

$$\partial_{p,q}[V, f] = [U^f/U, \partial f].$$

This is independent of the choice of $U$ [9 Lemma 2.3].

4. The Balmer–Walter Construction of The Gersten–Witt Complex

Let $R, A, \sigma, \varepsilon$ be as in Section 3. We proceed with recalling the Balmer–Walter construction of the Gersten–Witt complex of $R$, extending it to $\varepsilon$-hermitian forms over $(A, \sigma)$ in the process. We also recall Gille’s Gersten–Witt complex.

We refer the reader to [11] and [2] for all the necessary definitions concerning triangulated hermitian categories (also called triangulated categories with duality). A concise treatment is given in [11] §§1–2.
Let $(\ast, \omega)$ denote the hermitian structure induced by $\sigma$ on $\mathcal{P}(A)$ (Example 2.11 with $M = R$). As explained in [3, §2.6, §2.8], the bounded derived category of $\mathcal{P}(A)$, denoted $\mathcal{D}_b(A)$, inherits a triangulated hermitian structure, which we denote by $(\ast, 1, \omega)$. Specifically, if $P = (P_i, d_i)_{i \in \mathbb{Z}} = (\cdots \to P_i \xrightarrow{d_i} P_{i+1} \to \cdots)$, then $P^* = (P^*_i, d^*_i)_{i \in \mathbb{Z}} = (\cdots \to P^*_i \xrightarrow{d^*_i} P^*_{i+1} \to \cdots)$, and $\omega_p = (\omega_{P_i})_{i \in \mathbb{Z}}$. The corresponding $n$-th shifted hermitian structure (see [2, p. 131]) is $(D_n, \delta_n, \omega_n) := (T^n \circ \ast, (-1)^n, (-1)^{(n+1)/2} \omega)$, and we write $W^b_n(A, \sigma)$ or $W^b_n(A, \ast, 1, \omega)$ or $W^b_n(A, \omega)$ for the Witt group of $(\mathcal{D}_b(A), D_n, \delta_n, \omega_n)$.

For every $e \geq 0$, let $\mathcal{D}_e = \mathcal{D}_b(A)$ denote the full subcategory of $\mathcal{D}_b(A)$ consisting of chain complexes with homology $R$-supported in codimension $e$. Then we have a filtration,

$$
\mathcal{D}_b(A) = \mathcal{D}_0(A) \supset \mathcal{D}_1(A) \supset \mathcal{D}_2(A) \supset \cdots
$$

in which every term is a full subcategory of $\mathcal{D}_b(A)$ closed under shifts, mapping cones, isomorphisms, direct summands and $\ast$. (Note that $\mathcal{D}_b(A) = 0$ if $e > \dim R$.)

Balmer [1] Theorem 6.2] (see also [7, Theorem 7.1]) showed that the localization sequence $\mathcal{D}_{e+1} \to \mathcal{D}_e \to \mathcal{D}_e/\mathcal{D}_{e+1}$ gives rise to a long exact sequence of Witt groups,

$$
\cdots \to W^b_n(\mathcal{D}_{e+1}) \to W^b_n(\mathcal{D}_e) \to W^b_n(\mathcal{D}_e/\mathcal{D}_{e+1}) \xrightarrow{\delta_n} W^b_{n+1}(\mathcal{D}_{e+1}) \to \cdots,
$$

in which the left and middle maps are induced by the evident functors, and the right arrow is given by taking cones in the sense of [1, Definitions 5.16, 2.10]. Writing $\delta_{-1}$ for the natural map $W^b_n(D_0) \to W^b_n(D_0/D_1)$ and $\delta_{e}$ for the composition $W^b_n(\mathcal{D}_e/\mathcal{D}_{e+1}) \xrightarrow{\delta_n} W^b_{n+1}(\mathcal{D}_{e+1}) \to W^b_{n+1}(\mathcal{D}_{e+1}/\mathcal{D}_{e+2})$, we get a cochain complex

$$
0 \to W^b_n(D_0) \xrightarrow{d_{-1}} W^b_n(D_0/D_1) \xrightarrow{d_0} W^b_n(D_1/D_2) \xrightarrow{d_1} W^b_n(D_2/D_3) \xrightarrow{d_2} \cdots,
$$

which we denote by $\mathcal{B}^{A, R, \sigma, \varepsilon}_+\mathcal{D}_b$, or simply $\mathcal{B}^{A, \sigma, \varepsilon}_+$. This is the (augmented) Gersten–Witt complex of $(A, \sigma, \varepsilon)$ à la Balmer and Walter [7]. We write $\mathcal{B}^{A, \sigma, \varepsilon}_+$ for the complex obtained from $\mathcal{B}^{A, \sigma, \varepsilon}_+$ by removing the $-1$-term.

In the case $(A, \sigma, \varepsilon) = (R, \text{id}_R, 1)$, Balmer and Walter [7, §§6–7] defined a canonical isomorphism — called dévissage — between the respective terms of $\mathcal{B}^{A, \sigma, \varepsilon}_+$ and $\mathcal{GW}^{A, \sigma, \varepsilon}_+$. Bayer-Fluckiger, Parimala and the author [3, Proposition 2.7] showed that this isomorphism also respects the differentials. Here we shall generalize this to all $(A, \sigma, \varepsilon)$:

**Theorem 4.1.** There is a canonical isomorphism $s : \mathcal{B}^{A, \sigma, \varepsilon}_+ \to \mathcal{GW}^{A, \sigma, \varepsilon}_+$.

**Remark 4.2.** Gille’s Gersten–Witt complex [16, 18] is defined when $\dim R$ is finite, and is isomorphic to $\mathcal{B}^{A, \sigma, \varepsilon}_+$. This is explained in [16, §2.10, Example 4.4] and [18, pp. 349–350]. Briefly, let $\mathcal{D}_b^c(\mathcal{M}(A))$ denote the full subcategory of $\mathcal{D}_b(\mathcal{M}(A))$ consisting of chain complexes with finitely generated homology. Fix an injective resolution $I_* \in \mathcal{D}_b^c(\mathcal{M}(R))$ of $R$ and set $J_* = A \otimes_R I_*$. For every $F_* \in \mathcal{D}_b^c(\mathcal{M}(A))$, let $F^\#_*$ denote the Hom-chain complex $\text{Hom}_A(F_*, I_*)$, regarded as complex of right $A$-modules by twisting via $\sigma$. There is a natural isomorphism $\eta : F \to F^\#$ making $(\mathcal{D}_b^c(\mathcal{M}(A)), \#_!, \eta)$ into a triangulated hermitian category, see [18, §3.7]. Moreover, the isomorphism $A_A \cong J_*$ in $\mathcal{D}_b^c(\mathcal{M}(A))$ induces a natural isomorphism $i : P_*^\# \to P^\#_*$ for all $P_* \in \mathcal{D}_b^c(A) = \mathcal{D}_b^c(P(A))$ such that $(\text{id}, i) : (\mathcal{D}_b^c(A), \ast, 1, \omega) \to (\mathcal{D}_b^c(\mathcal{M}(A)), \#_!, \eta)$ is a 1-hermitian 1-exact functor (in the sense of [7, §4]). Gille’s Gersten Witt complex is defined exactly as $\mathcal{B}^{A, \sigma, \varepsilon}_+$ with the difference that $(\mathcal{D}_b^c(A), \ast, 1, \omega)$ is replaced with $(\mathcal{D}_b^c(\mathcal{M}(A)), \#_!, \eta)$. By Corollary 2.3, $(\text{id}, i)$ is an equivalence of triangulated hermitian categories. This equivalence respects the codimension filtration 4.4 on both $\mathcal{D}_b(A)$ and $\mathcal{D}_b^c(\mathcal{M}(A))$, so Gille’s Gersten–Witt complex is isomorphic to $\mathcal{B}^{A, \sigma, \varepsilon}_+$. 
Gille also showed that the respective terms of $\mathcal{B}_+^{A,\sigma}\epsilon$ and $\mathcal{GW}_+^{A,\sigma}\epsilon$ are isomorphic for all $A,\sigma,\epsilon$, but it is not clear that this isomorphism respects the differentials.

Theorem 4.1 affords a simple proof of the following proposition, which appears as Theorem 2.9 in [9].

**Proposition 4.3.** Let $R'$ be another regular ring and suppose that $A$ is equipped with an $R'$-algebra structure such that $(A,\sigma)$ is a separable projective $R'$-algebra with involution. Then $\mathcal{GW}_+^{A/R,\sigma}\epsilon \simeq \mathcal{GW}_+^{A/R',\sigma}\epsilon$.

**Proof.** Let $R_1 = \text{Cent}(A)\sigma$. Recall from [23] that $(A,\sigma)$ is Azumaya over $R_1$ and $R_1$ is finite étale over $R$, hence regular. Since the structure morphism $R' \to \text{Cent}(A)$ factors via $R_1$, it is enough to prove the proposition when $R' = R_1$. Now, going down and incomparability for prime ideals [24] Tags 00HS, 00GT imply that an $R'$-module is $R'$-supported in codimension $e$ if and only if it is $R$-supported in codimension $e$. Consequently, the filtration (14.1) of $D^b(A)$ is the same for $R$ and $R'$, so $\mathcal{B}_+^{A/R,\sigma}\epsilon \simeq \mathcal{B}_+^{A/R',\sigma}\epsilon$. Theorem 4.1 completes the proof.

The proof of Theorem 4.1 is given in the next three sections: Section 5 establishes a preliminary result about transfer in hermitian categories. This is used in the construction of $s = (s_n)_{n \geq -1}$, which is given in Section 6. Finally, Theorem 4.1 is proved in Section 7.

5. Transfer in Hermitian Categories

We introduce a variation of transfer into the endomorphism ring in hermitian categories, which will play a key role in the definition of the isomorphism $s$ of Theorem 4.1. To that end, we first recall hermitian forms valued in bimodules with involution in the sense of [13].

Let $A$ be an $R$-algebra (no involution on $A$ is assumed), and let $Z$ be an $(A^{op},A)$-progenerator, i.e., an $(A^{op},A)$-bimodule such that $Z_{\alpha}$ is finite projective, $A_{\alpha}$ is a summand of $Z_n$ for some $n \in \mathbb{N}$, and $A^{op} = \text{End}_A(Z)$; see [24] §18B for further details. Suppose further that $rz = zr$ for all $r \in R$, $z \in Z$, and let $\theta : Z \to Z$ be an $R$-automorphism (written exponentially) satisfying $(a^{op}zb)\theta = b^{op}za\theta$ and $z\theta = z$ for all $a, b \in A$, $z \in Z$. Finally, let $\varepsilon \in \mu_2(R)$.

Following [13] §2, define a $Z$-valued $\varepsilon\theta$-hermitian spaceootnote{In [13], the name is "(general) $\varepsilon\theta$-symmetric bilinear space".} over $A$ to be a pair $(P, \tilde{f})$ consisting of $P \in \mathcal{P}(A)$ and a biadditive map $\tilde{f} : P \times P \to Z$ satisfying $\tilde{f}(x\alpha, y\alpha') = a^{op}\tilde{f}(x, y)a'$ and $\tilde{f}(x, y) = \varepsilon \tilde{f}(x', y')$ for all $a, a' \in A$, $x, x' \in P$.

The pair $(Z, \theta)$ induces a hermitian structure $(\ast, \omega)$ on $\mathcal{P}(A)$ such that $\varepsilon$-hermitian forms over $\mathcal{P}(A)$ correspond to $Z$-valued $\varepsilon\theta$-hermitian forms: Given an object $P$ and a morphism $\phi$ in $\mathcal{P}(A)$, define $P^{\ast}$ to be $\text{Hom}_A(P,Z)$ endowed with the right $A$-module structure given by $(\phi a)x = a^{op}(\phi x)$ ($\phi \in P^{\ast}$, $a \in A$, $x \in P$), let $\phi^{\ast} = \text{Hom}_A(\phi, Z)$ and define $\omega_P : P \to P^{**}$ by $(\omega_P x)\phi = (\phi x)^{\theta}$ ($x \in P$, $\phi \in P^{\ast}$). Notice that $P^{\ast} \in \mathcal{P}(A)$ and $\omega_P$ is an isomorphism because $Z$ is an $(A^{op},A)$-progenerator, see [13] Lemmas 4.2, 4.4. If $(P, \tilde{f})$ is an $\varepsilon$-hermitian space over $(\mathcal{P}(A), \ast, \omega)$, then its corresponding $Z$-valued $\varepsilon\theta$-hermitian form over $A$ is $\tilde{f} : P \times P \to Z$ given by $\tilde{f}(x,y) = (f(x)y)$.  

**Example 5.1.** Let $(A,\sigma)$ be an $R$-algebra with involution and let $M$ be an invertible $R$-module. Take $Z = A \otimes M$, $\theta = \sigma \otimes \text{id}_M$, and make $Z$ into an $(A^{op},A)$-bimodule by setting $a^{op}zb := a^{op}zb$ ($a,b \in A$, $z \in Z$). Then $Z$-valued $\theta$-hermitian forms over $A$ are the same thing as $(A \otimes M, \sigma \otimes \text{id}_M)$-valued 1-hermitian forms over $(A,\sigma)$ in the sense of Example 2.1.
Let $(\mathcal{E}, \star, \omega)$ be any idempotent complete (see [11] §6) hermitian $R$-category and let $P_0$ be an object such that every object of $\mathcal{E}$ is a summand of $P_0^n$ for some $n \in \mathbb{N}$. By [23] Lemmas II.3.2.3, II.3.3.2 (for instance), the functor
\[ F = F_{P_0} := \text{Hom}_\mathcal{E}(P_0, -) \]
defines an equivalence from $\mathcal{E}$ to $\mathcal{P}(E)$, where $E := \text{End}_\mathcal{E}(P_0)$. Write $Z = F(P_0^n) = \text{Hom}_\mathcal{E}(P_0, P_0^n)$ and let $\theta : Z \to Z$ be given by $\varphi^\theta = \varphi^* \circ \omega_{P_0}$. We make $Z$ into an $(E^{\text{op}}, E)$-bimodule by setting $\alpha^{\text{op}} \cdot \varphi \cdot \beta = \alpha^* \circ \varphi \circ \beta$. One readily checks that $\varphi^{\text{op} \beta} = \varphi$ and $(\alpha^{\text{op}} \varphi \beta)^\theta = \beta^{\text{op}} \varphi^\theta \alpha$ for all $\alpha, \beta \in E$, $\varphi \in Z$. Moreover, $Z$ is an $(E^{\text{op}}, E)$-progenerator. Indeed, $F_* : E^{\text{op}} = \text{End}_E(P_0)^{\text{op}} \to \text{End}_E(Z)$ is an isomorphism, hence $\text{End}_E(Z) = E^{\text{op}}$, and since $P_0$ is isomorphic to a summand of $(P_0^n)^n$ for some $n$, the right $E$-module $E = FP_0$ is isomorphic to a summand of $Z^n = F(P_0^n)$ for the same $n$.

Let $(\tilde{s}, \tilde{\omega})$ denote the hermitian structure on $\mathcal{P}(E)$ induced by $Z$ and $\theta$. There is a natural isomorphism $i : F_* \to \tilde{s}F$ given by
\[ i_p : \text{Hom}_\mathcal{E}(P_0, P^*) \to \text{Hom}_E(\text{Hom}_\mathcal{E}(P_0, P), Z) \]
\[ \varphi \mapsto [\psi \mapsto \varphi^* \circ \omega_P \circ \psi] \]
(this is an isomorphism because $i_p = F \circ \text{Hom}(\omega_P, P_0^*) \circ \ast$). It is straightforward to check that
\[ (F, i) : (\mathcal{E}, \star, \omega) \to (\mathcal{P}(E), \tilde{s}, \tilde{\omega}) \]
is a 1-hermitian equivalence [see 23]. We call it the $P_0$-transfer. One readily checks that if $(P, f) \in H^e(\mathcal{E})$ and $F(P, f) = (FP, f_1)$, then the induced $Z$-valued $\varepsilon \theta$-hermitian form $\tilde{f}_1 : \text{Hom}_\mathcal{E}(P_0, P) \times \text{Hom}_\mathcal{E}(P_0, P) \to \text{Hom}_\mathcal{E}(P_0, P_0^n)$ is given by
\[ \tilde{f}_1(\varphi, \psi) = \varphi^* \circ f \circ \psi. \]

\textbf{Remark 5.2.} Ordinary transfer into the endomorphism ring, see [23] Proposition 2.4, requires one to specify a unimodular $\gamma$-hermitian form $h_0$ on the object $P_0$. Given such a form, we can define an $R$-involution $\sigma : E \to E$ by $\varphi^\sigma = h_0^{-1} \varphi^* h_0$ and identify $Z = F(P_0^* \varepsilon)$ with $E = FP_0$ via $Fh_0^{-1}$. Under this identification, $Z$-valued $\varepsilon \theta$-hermitian forms become $\gamma \varepsilon$-hermitian forms over $(E, \sigma)$, and we get a $\gamma$-hermitian equivalence $(\mathcal{E}, \ast, \omega) \to (\mathcal{P}(E), \ast, \sigma, R, \omega_\ast, R)$ (notation as in Example 2.1), which is precisely the transfer of [23] Proposition 2.4.

\section{Dévissage}

In this section, we construct the isomorphism $s = (s_e)_{e \geq -1}$ of Theorem 4.1. We use the same notation as in Section 4.

When $e = -1$, we define $s_{-1} : W_\varepsilon(A, \sigma) \to W_\varepsilon^0(\mathcal{D}^b(A), \ast, 1, \omega)$ to be the embedding-in-degree-0 homomorphism, which is an isomorphism by a theorem of Balmer [3 Theorem 4.3]. The case $e \geq 0$ will occupy the rest of this section and be concluded in Construction 6.7.

We begin by extending two results of Balmer and Walter [7 §§6–7] from $A = R$ to general $A$.

\textbf{Proposition 6.1.} There is an equivalence of triangulated hermitian $R$-categories,
\[ \text{loc} : \mathcal{D}^b(A)/\mathcal{D}^b_{e+1}(A) \xrightarrow{\sim} \prod_{p \in R^{e+1}} \mathcal{D}^b_p(A_p), \]
the $p$-component of which is given by localizing at $p$. The $p$-component of the implicit natural isomorphism $\text{loc} \ast \to \ast \text{loc}$ is the canonical isomorphism $(P^n)_p \to (P^n)_p^*$. 
Proof. It is routine to check that loc is a 1-exact 1-hermitian functor in the sense of [2] §4. It remains to check that loc is an equivalence of triangulated categories. Since the natural functor $\mathcal{D}^b(A) \to \mathcal{D}^b(\mathcal{M}_f(A))$ is an equivalence (Corollary 2.2), this can be shown as in the proof of [2, Proposition 1.7] (see also the proof of [1], Theorem 5.2).

Fix $p \in R(\mathcal{C})$ for the remainder of the discussion. Abusing the notation, we denote the triangulated hermitian structure that $\sigma$ induces on $\mathcal{D}^b(A_p)$ by $(\ast, 1, \omega)$ and write the corresponding shifted hermitian structures as $(D_p, \delta_p, \omega_n)_{n \in \mathbb{Z}}$. We further define

$$C(A_p)$$

to be the full subcategory of $\mathcal{D}^b_c(A_p)$ consisting of chain complexes $P$ supported in degrees $0, \ldots, e$ and satisfying $H_i(P) = 0$ for all $i \neq 0$. Then $D_e$ restricts to a duality on $C(A_p)$, because $\text{Ext}^i_A(M, A_p) = 0$ for any finite-length $A_p$-module $M$ and all $i \neq e$ [2, Proposition 1.7(i)]. Moreover, $C(A_p)$ is abelian. Indeed, $A_p$ has global dimension at most $e = \dim R_p$, and so $H_p$ defines an equivalence of $\mathcal{D}^b(A_p)$ to $\mathcal{M}_p(A_p)$, the category of finite-length $A_p$-modules. Thus, $(C(A_p), D_e, \omega_e)$ is an exact hermitian category, and we may consider its Witt group.

**Proposition 6.2.** There is an isomorphism

$$W_{\epsilon}(C(A_p)) \rightarrow W_{\epsilon}(\mathcal{D}^b_c(A_p), D_e, \delta_e, \omega_e) = W_{\epsilon}(\mathcal{D}^b_c(A_p))$$

defined by regarding a Witt class in $W_{\epsilon}(C(A_p))$ as a Witt class in $W_{\epsilon}(\mathcal{D}^b_c(A_p))$.

Proof. The proof is in the spirit of the proof of [2, Lemma 6.4].

Write $C := C(A_p)$. The hermitian structure $(D_e, \omega_e)$ on $C$ induces triangulated hermitian structures on $\mathcal{D}^b_c(C)$ and $\mathcal{K}^b_c(C)$ (the bounded homotopy category of $C$), both denoted $(\bar{D}, 1, \bar{\omega})$. We represent objects of $\mathcal{D}^b_c(C)$ and $\mathcal{K}^b_c(C)$ as bounded double chain complexes. Explicitly, the double complex $P_{\bullet, \bullet} = (P_{i,j}, h_{i,j}, v_{i,j})_{i,j \in \mathbb{Z}}$ (where $h_{i,j}$ are the horizontal differentials and $v_{i,j}$ are the vertical differentials) corresponds to $(\cdots \rightarrow P_{\bullet, \ast} \rightarrow P_{0, \bullet} \rightarrow P_{-1, \bullet} \rightarrow \cdots)$ in $\mathcal{D}^b_c(C)$ or $\mathcal{K}^b_c(C)$. Then $\bar{D}P_{\bullet, \bullet} = (P_{i,j}^\dagger, h_{i,j}^\dagger, v_{i,j}^\dagger, (1-1)^{i}v_{i-1,j}^\dagger)$ and $\bar{\omega}P_{\bullet, \bullet} = ((1-1)^{i}e_{i+1}^\dagger)_{i,j}$.

It is routine to check that the total complex construction

$$\text{Tot}: P_{\bullet, \bullet} \mapsto \left( \bigoplus_{i \in \mathbb{Z}} \bigoplus_{n \in \mathbb{Z}} \sum_i \left( h_{i,n-i} + (-1)^iv_{i,n-i} \right) \right)$$

defines a functor from $\mathcal{K}^b_c(C)$ to $\mathcal{D}^b_c(A_p)$. We claim that Tot extends via localization to $\mathcal{D}^b_c(C)$. To see this, let $H : \mathcal{D}^b_c(C) \to \mathcal{D}^b_c(\mathcal{M}_p(A_p))$ denote the exact functor induced by the equivalence $H_0 : C \to \mathcal{M}_p(A_p)$, namely, $HP_{\bullet, \bullet} = (H_0(P_{\bullet, \bullet}), H_0(h_{i,j}))_{i,j \in \mathbb{Z}}$. Consider the natural transformation $t_P : \text{Tot} P_{\bullet, \bullet} \to HP_{\bullet, \bullet}$ given as the composition $\text{Tot} P_{\bullet, \bullet} \mapsto P_{1,0} \mapsto P_{1,0}/\text{im} v_{1,1} \mapsto H_0(P_{\bullet, \bullet})$ in degree $i$. Since $H_0(P_{\bullet, \bullet}) = 0$ for $j > 0$, standard diagram chasing (e.g., as in the proof of [2, Tag 0EIR]) shows that $H_0(t_P) : H_0(\text{Tot} P_{\bullet, \bullet}) \to H_0(HP_{\bullet, \bullet})$ is an isomorphism. Thus, $t_P : \text{Tot} P_{\bullet, \bullet} \to HP_{\bullet, \bullet}$ is a natural quasi-isomorphism in $\mathcal{K}^b(\mathcal{M}_p(A_p))$. Now, if $f : P_{\bullet, \bullet} \to P'_{\bullet, \bullet}$ is a quasi-isomorphism in $\mathcal{K}^b(C)$, then $H_0(f) : H_0(HP_{\bullet, \bullet}) \to H_0(HP'_{\bullet, \bullet})$ is an isomorphism in $\mathcal{M}_p(A_p)$, and thus, so is $H_0(Tot f) : H_0(\text{Tot} P_{\bullet, \bullet}) \to H_0(\text{Tot} P'_{\bullet, \bullet})$. This means that Tot $f$ is a quasi-isomorphism, so Tot extends via localization to a functor $\mathcal{D}^b_c(C) \to \mathcal{D}^b_c(A_p)$.

Define $i : (\bar{D}, 1, \bar{\omega}) \to (\bar{D}, 1, \bar{\omega})$ is a 1-exact 1-hermitian functor in the sense of [2] §4.
We claim that \( \text{Tot} : D^b(C) \to D^b(A_p) \) is an equivalence of triangulated categories. To see this, fix an inverse functor \( F \) to \( H : D^b(C) \to D^b(M_\mathfrak{A}(A_p)) \), and let \( G : D^b(A_p) \to D^b(M_\mathfrak{A}(A_p)) \) be the functor induced by the inclusion \( \mathcal{P}(A_p) \to M_\mathfrak{A}(A_p) \). Then \( G \) is an equivalence by Corollary 23. Given \( M_\bullet \in D^b(M_\mathfrak{A}(A_p)) \), we observed above that there is a natural isomorphism \( \text{Tot} F M_\bullet = \text{Tot} P_\bullet \to H P_\bullet \cong M_\bullet \) in \( D^b(M_\mathfrak{A}(A_p)) \). Thus, \( G \text{Tot} F \) is equivalent to the natural functor \( D^b(M_\mathfrak{A}(A_p)) \to D^b(M_\mathfrak{A}(A_p)) \). The latter is an equivalence of triangulated categories by \([22, \S 1.15, \text{Lemma, Example (b)}]\), so \( \text{Tot} \) is an equivalence as well.

To conclude, \( (\text{Tot}, i) \) induces an isomorphism from \( W_0^p(D^b(C), D, \bar{1}, \bar{\omega}) \) to \( W_0^p(D^b(A_p), D_c, \delta_c, \omega_c) \). In addition, by \([2, \text{Theorem 4.3}]\), embedding-in-degree-0 induces an isomorphism \( W_\varepsilon(C, D_c, \omega_c) \to W_0^p(D^b(C), D, \bar{1}, \bar{\omega}) \). The composition of these two maps is the map considered in the proposition, so we are done.

Let \( C^0(A_p) \) denote the full subcategory of semisimple objects in \( C(A_p) \); it is closed under \( D_c \). Since the Jacobson radical of \( A_p \) is \( A_p \mathfrak{p} \) \([15, \text{Lemma 1.5}]\), and since \( A(\mathfrak{p}) \cong A_p/\mathfrak{p}A_p \) is semisimple artinian, the full subcategory of semisimple objects in \( M_\mathfrak{A}(A_p) \) is \( \mathcal{P}(A(\mathfrak{p})) \). As a result, the equivalence \( H_0 : C(A_p) \to M_\mathfrak{A}(A_p) \) restricts to an equivalence \( H_0 : C^0(A_p) \to \mathcal{P}(A(\mathfrak{p})) \).

**Proposition 6.3.** **The inclusion 1-hermitian functor \( C^0(A_p) \to C(A_p) \) induces an isomorphism \( W_\varepsilon(C^0(A_p), D_c, \omega_c) \to W_\varepsilon(C(A_p), D_c, \omega_c) \).**

**Proof.** This is a special case of a theorem of Quebbemann, Scharlau and Schulte \([29, \text{Corollary 6.9, Theorem 6.10}]\). We proceed with showing that \( W_\varepsilon(C^0(A_p), D_c, \omega_c) \) is canonically isomorphic to \( \tilde{W}(A(\mathfrak{p})) = W_\varepsilon(A(\mathfrak{p}), \sigma(\mathfrak{p}); \tilde{k}(\mathfrak{p})) \), which will finish the construction of \( s_c \). This was shown by Balmer and Walter \([7, \text{Theorem 6.1}]\) in the case \( A = R \), but their proof does not extend to our more general situation. Therefore we take a different approach, which is based on the transfer hermitian functor of Section 5. To that end, we introduce additional notation.

Let \( K \) denote a minimal resolution of the \( R_\mathfrak{p} \text{-module} \) \( k(\mathfrak{p}) \); recall from \([\mathcal{C}]\) that \( K = (K_1, d_i)_{i \in \mathbb{Z}} \) comes equipped with a morphism \( \alpha : K_0 \to k(\mathfrak{p}) \) which we use implicitly to identify \( H_0(K) \) with \( k(\mathfrak{p}) \). The actual choice of \( K \) will not matter in the end as long as \( K \) is supported in degrees \( 0, \ldots, c \). Write \( K_A = A \otimes K \) and note that \( H_0(K_A) = A(p) \) and \( K_A \in C^0(A_p) \) by the flatness of \( A \) over \( R \). We identify \( \text{End}_{D_\mathfrak{p}(A_\mathfrak{p})}(K_A) \) with \( A(p) \) via the isomorphism

\[
\text{End}_{C(A_p)}(K_A) \xrightarrow{\sim} \text{End}_{A}(A(p)) \xrightarrow{\sim} A(p).
\]

Now, as explained in Section 5, \( Z(K) := \text{Hom}_{D\mathfrak{p}(A_\mathfrak{p})}(K_A, D_c K_A) \) is naturally an \((A(p)^{op}, A(p))\)-bimodule. Explicitly, given \( \varphi, \tau \in A(p) \) which lift to \( a, b \in A \) and \( \varphi \in Z(K) \), we have \( \varphi \cdot \tau \tau = D_c(\ell_a \otimes \text{id}_K) \circ \varphi \circ (\ell_b \otimes \text{id}_K) \), where \( \ell_a : A_p \to A_p \) is left-multiplication by \( a \).

**Proposition 6.4.** **Let \( (*K, \omega(K)) \) denote the hermitian structure on \( \mathcal{P}(A(p)) \) induced by the \((A(p)^{op}, A(p))\)-bimodule \( Z(K) = \text{Hom}_{D\mathfrak{p}(A_\mathfrak{p})}(K_A, D_c K_A) \) and the map \( \theta(K) : Z(K) \to Z(K) \) given by \( \varphi^{(K)} = D_c \varphi \circ \omega_c K_A \). Then \( K_A \)-transfer (see Section 5) induces an exact 1-hermitian equivalence**

\[
(F, j) = (F^{(K)}, j^{(K)}): (C^0(A_p), D_c, \omega_c) \to (\mathcal{P}(A(p)), *(K), \omega(K)).
\]

**Proof.** This will follow from Section 5 once we show that every object in \( C^0(A_p) \) is a summand of \( K_A^n \) for some \( n \in \mathbb{N} \), and that \( F \) is exact. The former is a consequence of the equivalence \( H_0 : C^0(A_p) \to \mathcal{P}(A(p)) \) and the fact that every module in
\( \mathcal{P}(A(p)) \) is a summand of \( A(p)^n \) for some \( n \in \mathbb{N} \). The exactness of \( F \) is automatic because the categories \( \mathfrak{C}_0(A_p) \) and \( \mathcal{P}(A(p)) \) are abelian and semisimple.

Now, in order to show that \( W_z(\mathcal{C}_0(A_p), D_e, \omega_z) \cong \mathcal{W}_z(A(p)) \), it is enough to identify \( (Z(K), \theta(K)) \) with \( (A \otimes k(p), \sigma \otimes \text{id}_{k(p)}) \). In the end, it will turn out that the resulting isomorphism \( W_z(\mathcal{C}_0(A_p)) \to \mathcal{W}_z(A(p)) \) is independent of \( K \).

We begin by applying the construction of \( \mathcal{C}_0(A_p) \) with \( (A, \sigma) = (R, \text{id}_R) \) to form \( \mathcal{C}_0(R_p) \), which is equivalent to \( \mathcal{P}(k(p)) \) via \( H_0 \). The hermitian structure of \( \mathcal{P}(R_p) \) is denoted \((\vee, \zeta) \) and the induced shifted hermitian structures of \( \mathcal{D}^b(R_p) \) are denoted \((\Delta_n, \delta_n, \zeta_n)_{n \in \mathbb{Z}} \). Base-changing along the structure homomorphism \( R_p \to A_p \) induces a 1-hermitian functor \((G, t) : (\mathcal{P}(R_p), \vee, \zeta) \to (\mathcal{P}(A_p), \ast, \omega) \). Explicitly, for all \( P \in \mathcal{P}(R_p) \), we have \( GP = A \otimes P \) and \( t_P : A \otimes P \to (A \otimes P')^\ast \) is determined by \( (t_P(a \otimes \phi))(a' \otimes x) = a^\ast \cdot (\Delta x) \cdot a' \phi \in P^\ast, a, a' \in A, x \in P \). This, in turn, induces a 1-exact 1-hermitian functor \( \mathcal{D}^b(R_p) \to \mathcal{D}^b(A_p) \), which is also denoted \((G, t) \). For \( Q = (Q_i, d_i)_{i \in \mathbb{Z}} \in \mathcal{D}^b(R_p) \), we write \( GQ = A \otimes Q := (A \otimes Q_i, \text{id}_A \otimes d_i)_{i \in \mathbb{Z}} \) as \( QA \).

**Lemma 6.5.** For every \( P, Q \in \mathcal{D}^b(R_p) \), define a natural transformation

\[
\Phi_{P,Q} : A \otimes \text{Hom}_{\mathcal{D}^b(R_p)}(P, \Delta_e Q) \to \text{Hom}_{\mathcal{D}^b(A_p)}(PA, D_e QA)
\]

by \( \Phi_{P,Q}(a \otimes u) = t_Q \circ (\ell_a \otimes u) \), where \( \ell_a \) denotes left-multiplication by \( a \). Then:

(i) \( \Phi_{P,Q} \) is an isomorphism.

(ii) \( \Phi_{K,K} : A \otimes \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \to \text{Hom}_{\mathcal{D}^b(A_p)}(KA, D_e KA) = Z(K) \) is an isomorphism of \((A^\text{op}, A)\)-modules under which \( \sigma \otimes \text{id} \) corresponds to \( \theta(K) \) (see Proposition 6.4). Here, we regard \( A \otimes \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \) as an \((A^\text{op}, A)\)-bimodule by setting \( a^\text{op} \cdot x \cdot a' = a^\ast xa' \).

**Proof.** (i) The map \( \Phi_{P,Q} \) is a composition of the base-change map \( A \otimes H_0(P, \Delta_e Q) \to \text{Hom}(A \otimes P, A \otimes \Delta_e Q) \) and \( \text{Hom}(PA, t_Q) \). The first map is an isomorphism by Lemma 2.3 and the second is an isomorphism because \( t_Q \) is. (Recall that \( \mathcal{D}^b(R_p) \) and \( \mathcal{D}^b(M_f(R_p)) \) are equivalent because \( R_p \) is regular, and \( \mathcal{D}^b(A_p) \) is equivalent to \( \mathcal{D}^b(M_f(A_p)) \) by Corollary 2.3.)

(ii) The map \( \Phi_{K,K} \) is an isomorphism by (i), because \( K, \Delta_e K \in \mathcal{C}(R_p) \). It is a morphism of right \( A \)-modules because \( t_Q \circ (\ell_a \otimes u) = (t_Q \circ (\ell_a \otimes u)) \circ (\ell_b \otimes \text{id}) \) for all \( a, b \in A \). If we check that \( \sigma \otimes \text{id} \) corresponds to \( \theta(K) \) under \( \Phi_{K,K} \), then this will imply that \( \Phi_{K,K} \) is also a morphism of left \( A^\text{op} \)-modules, because \( a^\text{op} \cdot z = (z^a)^\text{op} \) for all \( z \in Z, a \in A \), and likewise for \( A \otimes \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \) and \( \sigma \otimes \text{id} \).

To see that \( \sigma \otimes \text{id} \) corresponds to \( \theta(K) \) under \( \Phi_{K,K} \), define \( \iota : \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \to \text{Hom}_{\mathcal{D}^b(A_p)}(KA, D_e KA) \) by \( \iota(u) = \Delta_e u \circ \zeta_e K \). It is routine to check that \( \varphi \mapsto D_e \varphi \circ \omega_e \zeta_e K \) corresponds to \( \sigma \otimes \iota \) under \( \Phi_{K,K} \), so it is enough to show that \( \iota \) is the identity. It is easy to check that \( \iota \) is \( R_p \)-linear and satisfies \( \iota \circ \iota = \text{id} \). Since \( \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \cong \text{Hom}(H_0(K), H_0(\Delta_e K)) = \text{Hom}(k(p), \text{Ext}_{R_p}^1(k(p), R_p)) \cong k(p) \), this means that \( \iota \in \{ \pm \text{id} \} \). To finish, it remains to exhibit a nonzero \( \varphi : K \to \Delta_e K \) such that \( \varphi = \iota(\varphi) \).

It is straightforward to check that \( \iota(\varphi) \) is a natural transformation from the contravariant functor \( K \mapsto \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \) to itself, so we may replace \( K \) with any chain complex quasi-isomorphic to it. In particular, we may assume that \( K \) is the Koszul complex \( K(E, s) \) associated to a regular sequence in \( R_p \) generating \( p_p \); see [3] Proposition 1.9 and the preceding comment, for instance. Fix an isomorphism \( \alpha : \Lambda^e E \to R_p \), and define \( \varphi : K \to \Delta_e K \) by \( \varphi(k) = (-1)^{e} \cdot \frac{1}{\alpha(1)} \alpha(k \wedge k') \) for all \( k \in \Lambda^e E, k' \in \Lambda^{e-1} E \). One readily checks that \( \varphi \in \text{Hom}_{\mathcal{D}^b(R_p)}(K, \Delta_e K) \), \( \varphi \neq 0 \) and \( \iota(\varphi) = \varphi \).
Now, by the discussion in (2.4) we have an isomorphism \( \tilde{k}(p) = \text{Ext}_{R_p}^1(k(p), R_p) \cong H_0(\Delta_c K) \). This gives rise to an isomorphism

\[
(6.1) \quad \text{Hom}_{D^b(A_p)}(K, \Delta_c K) \xrightarrow{H_0} \text{Hom}(\tilde{k}(p), H_0(\Delta_c K)) \xrightarrow{\phi \mapsto \phi(1)} H_0(\Delta_c K) \cong \tilde{k}(p),
\]

denoted \( \beta(K) \). By Lemma 6.4, this induces an \((A^{op}, A)\)-module isomorphism

\[
(6.2) \quad \Phi(K) := \Phi_{K,K} \circ (\text{id} \otimes \beta(K))^{-1} : A \otimes \tilde{k}(p) \to \text{Hom}_{D^b(A_p)}(K_A, D_c K_A) = Z(K)
\]

under which \( \sigma \otimes \text{id}_{\tilde{k}(p)} \) corresponds to \( \theta(K) \). The map \( (\Phi(K))^{-1} \) is the desired identification of \((Z(K), \theta(K)) \) with \((A \otimes \tilde{k}(p), \sigma \otimes \text{id}_{\tilde{k}(p)} \)\). Combining this with Proposition 6.3, we get an isomorphism

\[
(6.3) \quad F(K) : W_c(C(A_p), D_c, \omega_c) \to W_c(A(p), \sigma(p); \tilde{k}(p)) = W_c(A(p)).
\]

**Proposition 6.6.** With the above notation, \( F(K) \) is independent of the projective resolution \( K \) of \( k(p) \).

**Proof.** Let \( K' \) be another projective resolution of \( k(p) \) supported in degrees \( 0, \ldots, e \), and let \((P, \varphi) \in H^e(C(A_p)) \). We need to prove that \( F(K)(P, \varphi) = F(K')(P, \varphi) \).

By unfolding the definitions, we see that that \( \tilde{\phi} \) is the \( A \otimes \tilde{k}(p) \)-valued pairing on \( \text{Hom}_{D^b(A_p)}(K_A, P) \), where \( \tilde{\phi} \) is the \( A \otimes \tilde{k}(p) \)-valued pairing on \( \text{Hom}_{D^b(A_p)}(K_A, P) \) given by

\[
\tilde{\phi}(\varphi, \psi) = (\Phi(K))^{-1}(D_c \varphi \circ \epsilon f \circ \psi).
\]

Likewise, \( F(K')(P, \varphi) \) is represented by \( \text{Hom}_{D^b(A_p)}(K_A', P) \), \( \varphi' \) with \( \varphi' \) given by

\[
\tilde{\phi}'(\varphi, \psi) = (\Phi(K'))^{-1}(D_c \varphi \circ \epsilon f \circ \psi)
\]

for all \( \varphi, \psi \in \text{Hom}_{D^b(A_p)}(K_A', P) \).

By [11] Theorem 12.4 there exists an isomorphism \( \xi : K \to K' \) in \( D^b(R_p) \) such that \( H_0(\xi) = \text{id}_{k(p)} \). Then \( \xi_A : K_A \to K_A' \) is an isomorphism in \( D^b(A_p) \). We will show that \( \text{Hom}_{D^b(A_p)}(\xi_A, P) \) is an isometry from \( \varphi' \) to \( \varphi \), thus proving that \( F(K)(P, \varphi) = F(K')(P, \varphi) \). By the formulas for \( \varphi \) and \( \varphi' \), this amounts to showing that for all \( \varphi, \psi \in \text{Hom}_{D^b(A_p)}(K_A', P) \), we have

\[
(\Phi(K'))^{-1}(D_c \varphi \circ \epsilon f \circ \psi) = (\Phi(K))^{-1}(D_c \varphi \circ \epsilon f \circ (\psi \circ \xi_A)).
\]

Since the right hand side equals \( (\Phi(K'))^{-1}(D_c \varphi \circ (D_c \varphi \circ \epsilon f \circ \psi) \circ \xi_A) \), it is enough to check that \( (\Phi(K'))^{-1} = (\Phi(K))^{-1} \circ \text{Hom}_{D^b(A_p)}(\xi_A, D_c K_A) \), or equivalently, that

\[
(6.4) \quad \text{Hom}_{D^b(A_p)}(\xi_A, D_c K_A) \circ \Phi(K') = \Phi(K).
\]

By (6.2), in order to show (6.4), it is enough to show that the diagram

\[
A \otimes \tilde{k}(p) \xrightarrow{\text{id} \otimes (\beta(K))^{-1}} A \otimes \text{Hom}_{D^b(R_p)}(K, \Delta_c K) \xrightarrow{\Phi_{K,K}} \text{Hom}_{D^b(A_p)}(K_A, D_c K_A)
\]

commutes. The right square commutes by the naturality of \( \Phi_{\cdot, \cdot} \) in both inputs, so it is enough to prove that the triangle on the left commutes. This will follow if we show that \( \beta(K) \circ \text{Hom}_{D^b(R_p)}(\xi, \Delta_c K) = \beta(K') \). By the definition of \( \beta(K) \) in (6.1), this will follow once we show that the diagram

\[
\text{Hom}_{D^b(R_p)}(K, \Delta_c K) \xrightarrow{H_0} \text{Hom}(\tilde{k}(p), H_0(\Delta_c K)) \xrightarrow{\phi \mapsto \phi(1)} H_0(\Delta_c K) \xrightarrow{\psi^{-1}} \tilde{k}(p)
\]

commutes. The left square commutes by the naturality of \( \Phi_{\cdot, \cdot} \) in both inputs and outputs, so it is enough to prove that the right square commutes. This will follow if we show that \( \beta(K) \circ \text{Hom}_{D^b(R_p)}(\xi, \Delta_c K) = \beta(K') \). By the definition of \( \beta(K) \) in (6.1), this will follow once we show that the diagram

\[
\text{Hom}_{D^b(R_p)}(K', \Delta_c K') \xrightarrow{H_0} \text{Hom}(\tilde{k}(p), H_0(\Delta_c K')) \xrightarrow{\phi \mapsto \phi(1)} H_0(\Delta_c K') \xrightarrow{\psi^{-1}} \tilde{k}(p)
\]
commutes. Here, \( u \) is the isomorphism \( \text{Ext}^r_{R_p}(k(p), R_p) \to H_{-c}(\text{Hom}(K, R)) \), defined as in \( 2C \) and similarly for \( u' \). The left square commutes because the functor \( H_0 \) respects composition and \( H_0(\xi) = \text{id}_{k(p)} \). That the middle square commutes is a straightforward computation. Finally, we noted in \( 2C \) that the triangle on the right commutes. This completes the proof. \( \square \)

We conclude by defining \( s_e \) for \( e \geq 0 \).

**Construction 6.7.** For all \( e \in \mathbb{N} \cup \{0\} \), define

\[
 s'_e : B^A_{A, \sigma, \varepsilon} \cong W^c_\varepsilon(D^b(A)/D^b_{e+1}(A)) \cong \bigoplus_{p \in R^{(c)}} \tilde{W}_\varepsilon(A(p)) = GW^A_{A, \sigma, \varepsilon}
\]

as the composition of the isomorphisms

\[
\text{loc} : W^c_\varepsilon(D^b(A)/D^b_{e+1}(A)) \to \bigoplus_{p \in R^{(c)}} W^c_\varepsilon(D^b(A_p)),
\]

\[
\text{nat.map}^{-1} : \bigoplus_{p \in R^{(c)}} W^c_\varepsilon(D^b(A_p)) \to \bigoplus_{p \in R^{(c)}} W^c_\varepsilon(C^0(A_p)),
\]

\[
\bigoplus F(p) : \bigoplus_{p \in R^{(c)}} W^c_\varepsilon(C^0(A_p)) \to \bigoplus_{p \in R^{(c)}} \tilde{W}_\varepsilon(A(p)),
\]

constructed in Propositions \( 6.1 \) and \( 6.2 \) and \( 6.3 \), where we have written \( F(p) \) for \( F(K) \), as it is independent of \( K \). The second map is well-defined by Proposition \( 5.3 \).

Finally, set \( s_e = (-1)^e s'_e \).

### 7. Proof of Theorem 4.1

We use the notation of Sections \( 4 \) and \( 6 \).

Denote the \( e \)-th differentials of \( B^A_{A, \sigma, \varepsilon} \) and \( GW^A_{A, \sigma, \varepsilon} \) by \( d_e \) and \( d'_e \), respectively. We need to show that \( d'_e \circ s_e = s_{e+1} \circ d_e \). The case \( e = -1 \) is routine and is left to the reader. We assume that \( e \geq 0 \) henceforth. In this case, the theorem is equivalent to showing that for every \( p \in R^{(c)}, q \in R^{(c+1)} \) and \( (V, f) \in \mathcal{H}(A(p), \sigma(p); \xi(p)) \) (notation as in Example \( 2.1 \)), we have \( (s_{e+1} \circ d_e \circ s_e^{-1}[V, f])_q = \partial_{p, q}[V, f] \), or equivalently,

\[
(s'_{e+1} \circ d_e \circ s_e^{-1}[V, f])_q = -\partial_{p, q}[V, f],
\]

with the convention that \( \partial_{p, q} = 0 \) if \( p \not\in q \).

**Proof of (7.1) when \( p \not\in q \).** By Proposition \( 6.1 \), there is an isomorphism \( B^A_{A, \sigma, \varepsilon} \cong \bigoplus_{a \in R^{(c)}} W^c_\varepsilon(D^b(A_a)) \) for all \( e \geq 0 \). Under this isomorphisms, the map \( d_e : B^A_{A, \sigma, \varepsilon} \to B^A_{e+1, \sigma, \varepsilon} \) corresponds to a map \( d_e : \bigoplus_{a \in R^{(c)}} W^c_\varepsilon(D^b(A_a)) \to \bigoplus_{b \in R^{(c+1)}} W^c_\varepsilon(D^b_{e+1}(A_b)) \).

By Construction \( 6.7 \) it is enough to show that the composition

\[
W^c_\varepsilon(D^b_e(A_p)) \xrightarrow{i} \bigoplus_{a \in R^{(c)}} W^c_\varepsilon(D^b_{e+1}(A_a)) \xrightarrow{d_e} \bigoplus_{b \in R^{(c+1)}} W^c_\varepsilon(D^b_{e+1}(A_b)) \xrightarrow{p} W^c_\varepsilon(D^b_{e+1}(A_q))
\]

in which \( i \) and \( p \) are the evident embedding and projection, is 0.

We show this by comparing to the Gersten–Witt complex of \( (A_q, \sigma_q, \varepsilon) \). Indeed, localization-at-\( q \) defines a functor \( D^b(A) \to D^b(A_q) \) which respects the triangulated hermitian structures and codimension filtrations \( 4.1 \) on \( D^b(A) \) and \( D^b(A_q) \). Thus, it induces a morphism \( B^A_{A, \sigma, \varepsilon} \to B^A_{A_q, \sigma_q, \varepsilon} \). Applying Proposition \( 6.3 \) to \( A_q \), we see that \( B^A_{A_q, \sigma_q, \varepsilon} \cong \bigoplus_{a \in R_q^{(c)}} W^c_\varepsilon(D^b_{e+1}(A_q_a)) = \bigoplus_{a \in R_q^{(c)}, a \leq q} W^c_\varepsilon(D^b_{e+1}(A_q_a)), \) and likewise
$B_{c+1}^{A\alpha,\sigma,\varepsilon} \cong W_{c+1}^e(D_{c+1}^b(A))$. This gives rise to a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{a \in R(i)} W_{c}^e(D_{c}^b(A)) & \xrightarrow{\partial_c} & \bigoplus_{b \in R(i+1)} W_{c+1}^e(D_{c+1}^b(A)) \\
p' & & p \\
\bigoplus_{a \in R(i): a \in q} W_{c}^e(D_{c}^b(A)) & \xrightarrow{\partial_c'} & W_{c+1}^e(D_{c+1}^b(A))
\end{array}
$$

in which $p'$ and $p$ are induced by the morphism $B^{A,\sigma,\varepsilon} \to B_{c+1}^{A,\sigma,\varepsilon}$. It is straightforward to check that $p'$ and $p$ are just the evident projections (so $p$ agrees with $p$ from the previous paragraph). Since $p \not\subseteq q$, this implies that $p' \circ p = 0$, and thus $i \circ \partial_c \circ p = 0$.

We henceforth set our attention to the case $p \subseteq q$. It is easy to see that localizing at $q$ is harmless, so we further restrict to the case where $R$ is regular local and $q$ is its maximal ideal. We now evoke the notation used in the construction of $\partial_{p,q}$ in Section 3. In particular,

$$
S = R/p, \quad \tilde{S} = \text{Ext}_R^p(S, R),
$$

$$
m = q/p, \quad \tilde{m}^{-1} = \text{Ext}_R^q(m, R).
$$

We also let $A_S = A \otimes S$ and $\sigma_S = \sigma \otimes \text{id}_S$. Note that $S$ is a 1-dimensional domain, and thus Cohen-Macaulay as an $R$-module. By [9, Proposition 1.3(i)], the modules $\tilde{S}$ and $\tilde{m}^{-1}$ are $S$-torsion-free. Recall that localization-at-$p$ defines an embedding of both $\tilde{S}$ and $\tilde{m}^{-1}$ in $\tilde{k}(p) = \text{Ext}_R^p(k(p), R)$, and $\tilde{S} \subseteq \tilde{m}^{-1}$ as $R$-submodules of $\tilde{k}(p)$.

We introduce additional notation:

1. $\mathcal{D} = D^b(A)$. As in Section 3, the shifted hermitian structures of $\mathcal{D}$ induced by $\sigma$ are denoted $(D_n, \delta_n, \omega_n)_{n \in \mathbb{Z}}$.
2. $\mathcal{D}' = D^b(A_p)$. The shifted hermitian structures of $\mathcal{D}'$ induced by $\sigma_p$ are denoted $(D'_n, \delta_n, \omega'_n)_{n \in \mathbb{Z}}$.
3. The shifted hermitian structures of $D^b(R)$ induced by the involution $\text{id}_R$ are denoted $(\Delta_n, \delta_n, \zeta_n)_{n \in \mathbb{Z}}$ (as in Section 3).
4. $\mathcal{A}$ is the full subcategory of $\mathcal{D}$ consisting of chain complexes $P$ supported in non-negative degrees and satisfying $H_0(P) \in \mathcal{M}_e(A_S)$ and $H_i(P) = 0$ for all $i \neq 0$; this is an abelian category, because $H_0 : \mathcal{A} \to \mathcal{M}_e(A_S)$ is an equivalence (Lemma 2.2 [11, Theorem 12.4]).
5. $\mathcal{L}$ is the full subcategory of $\mathcal{M}_e(A_S)$ consisting of $A_S$-modules which are $S$-torsion-free. It inherits an exact structure from $\mathcal{M}_e(A_S)$.
6. $\mathcal{B}$ is the full subcategory of $\mathcal{A}$ consisting of complexes $P \in \mathcal{A}$ supported in degrees $0, \ldots, c$ and satisfying $H_0(P) \in \mathcal{L}$. It inherits an exact structure from the abelian category $\mathcal{A}$.
7. $(\ast, \tilde{\omega})$ is the exact hermitian structure on $\mathcal{L}$ obtained by applying Example 2.1 to $(A_S, \sigma_S)$ using the possibly non-invertible $S$-module $\tilde{S}$ and allowing modules in $\mathcal{L}$, rather than $\mathcal{P}(A_S)$. This is an exact hermitian structure by [9, Proposition 1.5, Corollary 1.8] ($\tilde{S}$ is a dualizing module for $S$ by [10, Theorem 3.3.7(b)]).
8. $\mathcal{L}$ is a minimal projective resolution of the $R$-module $S$.  


(n9) $K$ is a minimal projective resolution of the $R$-module $k(q)$.

(n10) $J$ is a minimal projective resolution of the $R$-module $m := q/p$.

(n11) $L_A := A \otimes L$, $K_A := A \otimes K$ and $J_A := A \otimes J$; they are objects of $A$ because since $A$ is flat over $R$.

(n12) The short exact sequence $m \rightarrow S \rightarrow k(q)$ gives rise to a distinguished triangle

$$J \rightarrow L \rightarrow K \rightarrow T J$$

in $D^b(R)$. Tensoring with $A$ gives rise to a distinguished triangle in $D$:

$$J_A \rightarrow L_A \rightarrow K_A \rightarrow T_A J_A.$$

(n13) $F := \text{Hom}(L_A, -) : A \rightarrow \mathcal{M}_I(A_S)$. The right $A_S$-module structure on Hom($L_A, P$) arises from the $R$-algebra isomorphism

$$\text{End}(L_A) \cong \text{End}_A(A_S) \cong A_S.$$

Explicitly, if $\pi \in A_S$ lifts to $a \in A$ and $f \in \text{Hom}(L_A, P)$, then $f \cdot \pi := f \circ (\ell_a \otimes \text{id}_L)$, where $\ell_a : A \rightarrow A$ is left-multiplication by $a$.

(n14) We endow $\text{Hom}(L_A, D_L A)$ with an $(A_S^{op}, A_S)$-bimodule structure as follows: if $\pi, \tau \in A_S$ lift to $a, b \in A$, then $\pi \cdot \tau := D_c(\ell_a \otimes \text{id}_L) \circ f \circ (\ell_b \otimes \text{id}_L)$ with $\ell_a$ as in (n13).

(n15) As in (6.1), there is an isomorphism $\beta : \text{Hom}_{D^b(R)}(L, \Delta_L) \rightarrow \text{Ext}_{\mathbb{N}}(S, R) = \tilde{S}$. We use $\Phi_{-,-}$ defined in Lemma (6.5(i)) and $\beta$ to construct an $R$-module isomorphism

$$\Phi(L) := \Phi_{L,L} \circ (\text{id} \otimes \beta)^{-1} : A \otimes \tilde{S} \rightarrow \text{Hom}(L_A, D_L A).$$

(n16) $K' = L_p \in D^b(R_p)$. It is a projective resolution of the $R_p$-module $k(p)$. We shall see below (Lemma 7.2) that $L$, and hence $K'$, is supported in degrees $0, \ldots, e$, so $K' \in C^e(A_q)$. Comparing (n15) with (6.2) shows that

$$(\Phi(L))_p = \Phi(K') : A \otimes k(p) \rightarrow \text{Hom}_{D^b(A_p)}(K'_{A}, D_c K'_{A}),$$

where $\Phi(K')$ is as in Section 6.1. In particular, we have an identification of $(A(p)^{op}, A(p))$-bimodules

$$\text{Hom}(L_A, D_c K_A)_p = \text{Hom}_{D^b}(K'_{A}, D_c K'_{A}).$$

We shall need several lemmas.

**Lemma 7.1.** There is a natural isomorphism

$$FP = \text{Hom}(L_A, P) \cong H_0(P)$$

for every $P \in A$. Moreover, there is a natural isomorphism $\text{Hom}(K_{A}, Q) \cong H_0(Q)$ for every $Q \in A$ with $H_0(Q) : q = 0$ such that the induced isomorphism

$$\text{Hom}(K_{A}, Q) \rightarrow H_0(Q) \rightarrow \text{Hom}(L_A, Q)$$

is $p^e$ (with $p$ as in (n12)).

**Proof.** Recall from [2C]{2} that the data of $L$ includes a map $\alpha_L : L_0 \rightarrow S$ which we use to identify $H_0(L)$ with $R$. Since $A$ is flat over $R$, this gives rise to an isomorphism $H_0(L_A) \rightarrow A_S$, denoted $(\alpha_L)_A$. Let $\psi_p : FP = \text{Hom}(L_A, P) \rightarrow H_0(P)$ denote the composition

$$(7.2) \quad \text{Hom}(L_A, P) \xrightarrow{H_0} \text{Hom}_A(A_S, H_0(P)) \xrightarrow{\psi_p \circ \phi(1)} H_0(P),$$

2Here, we are using the fact that forming Hom-sets and Ext-groups of finitely presented $A$-modules commutes with localization at $p$ up to a natural isomorphism, see [39, Theorems 2.38, 2.39]. Also, taking homology commutes with localization at $p$, because $R_p$ is flat over $R$. 

where in the first arrow, we used $\alpha_L$ to identify $H_0(L_A)$ with $A_S$. Explicitly, given $f : L_A \to P$, we have $\psi(f) = (H_0(f) \circ ((\alpha_L)_A)^{-1})(1_{A_S})$. The left arrow in (n22) is a bijection because $H_0 : A \to M_f(A_S)$ is an equivalence, and the right arrow is a bijection because $H_0(P) \cdot p = 0$ and $A_S = A/Ap$. Thus, $\psi_P$ is an isomorphism, and it is routine to check that it is natural in $P$.

The natural homomorphism $\text{Hom}(K_A, Q) \to H_0(Q)$ — call it $\psi_Q$ — is defined similarly, using the isomorphism $(\alpha_K)_A : H_0(K_A) \to A(q)$ induced by $\alpha_K : K_0 \to k(q)$. It is an isomorphism because $H_0(Q) \cdot q = 0$.

Checking the last assertion amounts to showing that $\psi_Q = \psi_Q \circ p^\circ$. Observe that in the construction of the first distinguished triangle in (n12), $p_0 : L \to K$, is taken to be compatible with $\alpha_L : L_0 \to S$ and $\alpha_K : K_0 \to k(q)$, and thus

$$(\alpha_K)_A \circ H_0(p) = \gamma \circ (\alpha_L)_A,$$

where $\gamma$ is the quotient map $A_S \to A(q)$. Now, for all $f : K_A \to P$, we have $\psi^Q(f) = (H_0(f) \circ ((\alpha_K)_A)^{-1})(1_{A(q)}) = (H_0(f) \circ H_0(p) \circ ((\alpha_L)_A)^{-1})(1_{A_S}) = \psi_P(p^f(f))$, as required.

**Lemma 7.2.** $(D_e, \omega_e)$ restricts to an exact hermitian structure on $B$.

**Proof.** By [9] Propositions 1.5, 1.7, for all $M \in \mathcal{L}$, we have $\text{Ext}^2_A(M, A) = 0$ if $i \neq e$, and $\text{Ext}^e_A(M, A)$ is $S$-torsion-free, so $(D_e, \omega_e)$ restricts to a hermitian structure on $B$. To see that $D_e$ is exact, consider a short exact sequence $P' \to P \to P''$ in $B$. Since $\text{Ext}^2_A(H_0(P'), A) = 0$ and $\text{Ext}^2_A(H_0(P''), A) = 0$, we have a short exact sequence $\text{Ext}^2_A(H_0(P''), A) \to \text{Ext}^e_A(H_0(P'), A) \to \text{Ext}^e_A(H_0(P''), A)$. This sequence is isomorphic to $H_0(D_eP) \to H_0(D_eP') \to H_0(D_eP'')$, so the latter is also short exact sequence in $\mathcal{M}_1(A_S)$. Since $H_0 : A \to \mathcal{M}_1(A_S)$ is an exact equivalence, it follows that $D_eP' \to D_eP \to D_eP''$ is a short exact sequence in $B$. $\square$

**Lemma 7.3.** Both $H_0$ and $F = \text{Hom}(L_A, -)$ restrict to an exact equivalence from $B$ to $\mathcal{L}$.

**Proof.** Since $H_0 \cong F$ as functors from $A$ to $\mathcal{M}(A_S)$ (Lemma [7]), it is enough to consider $H_0$. We already know that $H_0 : A \to \mathcal{M}_1(A_S)$ is an exact equivalence, and $H_0$ maps to $\mathcal{L}$ by the definition of $\mathcal{B}$, so it remains to check that $H_0 : B \to \mathcal{L}$ is essentially surjective. By [10], Proposition 1.5], every nonzero finite S-torsion-free S-module Cohen-Macaulay of dimension $1$. Thus, by the Auslander–Buchsbaum formula [10], Theorem 1.3.3 and Lemma [22], every right $A$-module in $\mathcal{L}$ has projective dimension at most $e$, and is therefore isomorphic to $H_0(P)$ for some $P \in \mathcal{B}$. $\square$

**Lemma 7.4.** $L$ and $J$ (see (n8) and (n10)) are supported in degrees $0, \ldots, e$, and $L_A, J_A \in \mathcal{L}$.

**Proof.** Applying Lemma 7.3 with $A = R$ shows that $S$ and $m$ are in the image of $H_0 : \mathcal{B} \to \mathcal{L}$ (when $A = R$). Since $L$ and $J$ are minimal $R$-projective resolutions of $S$ and $m$, respectively, the lemma follows. $\square$

**Lemma 7.5.** The isomorphism $\Phi^{(L)} : A \otimes \tilde{S} \to \text{Hom}(L_A, D_eL_A)$ of (n13) is an $(A_S^0, A_S)$-bimodule isomorphism. See [n14] for the module structure on the range; the module structure on $A \otimes \tilde{S}$ is determined by $a^p \cdot (x) = a^p x b$ if $a, b \in A, x \in A \otimes \tilde{S}$. Under this isomorphism, $a \otimes \Id_{\tilde{S}}$ corresponds to $\varphi \mapsto D_e \varphi \circ \omega_e L_A$.

**Proof.** Since $\tilde{S}$ is $S$-torsion-free and $A$ is finite projective over $R$, the $S$-module $A \otimes \tilde{S}$ is $S$-torsion-free. As $\Phi^{(L)}$ is an $R$-module isomorphism, Hom$(L_A, D_eL_A)$ is also $S$-torsion-free. It is therefore enough to prove the lemma after localizing at $p$. The required statements now follow from Lemma 6.3(ii) (applied to $K'$) and the second equation of (n16) $\square$
Lemma 7.6. For $P \in \mc{B}$, define a natural transformation $j_P : F\delta e P \to (FP)^\dagger$ by 

$$ (j_P \varphi)\psi = (\Phi^{(L)})^{-1}(D_e \varphi \circ \omega_{ε_P} \circ ψ) $$

for all $\varphi \in \text{Hom}(L_A, D_e P)$ and $ψ \in \text{Hom}(L_A, P)$. Then $(F, j) : (B, D_e, ω_e) \to (L, \delta, \tilde{ω})$ is an exact 1-hermitian equivalence.

Once suppressing $\Phi^{(L)}$, the hermitian functor $(F, j)$ is defined using the same formula as the $L_A$-transfer functor of Section 5, so we shall (abusively) refer to it as the $L_A$-transfer functor. Note, however, that it is not true in general that every object of $\mc{B}$ is a summand of a direct sum of copies of $L_A$.

Proof. By Lemma 7.3, $j$ satisfies $j$ remains to check that the Five Lemma, $j$ is natural and satisfies $j_{D, P} \circ Fω_{ε_P} = j_P^* \circ \tilde{ω}_{FP}$ for all $P \in \mc{B}$ is routine, thanks to Lemma 7.5. It remains to check that $j_P$ is an isomorphism for all $P \in B$.

We can find $n, m \in \mathbb{N}$ and an exact sequence $A^n_\mc{B} \to A^m_\mc{B} \to H_0(P) \to 0$ in $\mc{L}$. Since $H_0 : B \to L$ is an exact equivalence (Lemma 7.3), this gives rise to an exact sequence $L^e_n \to L^e_m \to P \to 0$ in $\mc{B}$, which in turn gives rise to exact sequences $0 \to FD_e P \to FD_e L^e_m \to FD_e L^e_n$ and $0 \to (FP)^\dagger \to (FL^e_m)^\dagger \to (FL^e_n)^\dagger$. The natural transformation $j$ determines a morphism between these sequences. It is routine to check that $j_L$ is an isomorphism, and hence so are $jL^e_n$ and $jL^e_m$. By the Five Lemma, $j_P$ is an isomorphism as well. □

Lemma 7.7. The isomorphisms $\Phi^{(L)}$ of (n15), the isomorphism $\Phi^{(K)}$ of (3.2), (with $K$ as in (n9)) and the exact sequence (7.3) fit into a commutative diagram of $(A^0\delta, A)$-bimodules

$$ (7.3) \quad \begin{array}{ccc}
A \otimes \tilde{S} & \xrightarrow{\Phi^{(L)}} & A \otimes \tilde{m}^{-1} \\
\downarrow \varphi^{(L)} & & \downarrow \varphi^{(j)} \\
\text{Hom}(L_A, D_e J_A) & \xrightarrow{(D_e \varphi) \circ j} & \text{Hom}(L_A, D_e J_A) \\
\downarrow \varphi^{(K)} & & \downarrow \varphi^{(K)} \\
\text{Hom}(K_A, D_e J_A) & & \text{Hom}(K_A, D_e J_A)
\end{array} $$

in which $\Phi^{(j)}$ is an isomorphism, and $t$ is the composition $t : \text{Hom}(L_A, D_e J) \xrightarrow{(D_e + 1)q} \text{Hom}(L_A, D_e J_A) \xrightarrow{(\rho e)^{-1}} \text{Hom}(K_A, D_e J_A)$ (the right arrow is invertible by Lemma 7.7).

Proof. Recall that $\tilde{S} = \text{Ext}^{e}_{\mc{B}}(S, R)$, $\tilde{m}^{-1} = \text{Ext}^{e}_{\mc{B}}(m, R)$ and $\tilde{k}(q) = \text{Ext}^{e+1}(k(q), R)$. As noted in (24), the exact sequence

$$ \tilde{S} \to \tilde{m}^{-1} \to \tilde{k}(q) $$

is isomorphic to

$$ H_0(\Delta_e L) \xrightarrow{H_0(\Delta_e q)} H_0(\Delta_e J) \xrightarrow{H_0(\Delta_e T^{-1}q)} H_0(\Delta_e T^{-1}K) = H_0(\Delta_{e+1} K) $$

(see (n12)). Applying Lemma 7.1 with $A = R$, we get a get a commutative diagram

$$ \begin{array}{ccc}
H_0(\Delta_e L) & \xrightarrow{H_0(\Delta_e q)} & H_0(\Delta_e J) & \xrightarrow{H_0(\Delta_e T^{-1}q)} & H_0(\Delta_{e+1} K) \\
\downarrow \text{Hom}(L, \Delta_e L) & \xrightarrow{(\Delta_e q)} & \text{Hom}(L, \Delta_e J) & \xrightarrow{(\Delta_e T^{-1}q)} & \text{Hom}(L, \Delta_{e+1} K) \\
\downarrow \text{Hom}(L, \Delta_e L) & \xrightarrow{(\Delta_e T^{-1}q)} & \text{Hom}(L, \Delta_{e+1} K) & \xrightarrow{\rho e} & \text{Hom}(L, \Delta_{e+1} K) \\
\downarrow \text{Hom}(L, \Delta_e L) & \xrightarrow{(\Delta_e T^{-1}q)} & \text{Hom}(L, \Delta_{e+1} K) & \xrightarrow{\rho e} & \text{Hom}(L, \Delta_{e+1} K)
\end{array} $$

in which the vertical and diagonal arrows are the natural isomorphisms provided by the lemma and the Hom-groups were taken in $D^b(R)$. 

By Lemma 6.8(i) (applied with $q$ in place of $p$ and $(P,Q) \in \{(L,L),(L,J),(L,T^{-1}K),(K,T^{-1}K)\}$), the tensor of the bottom row of the last diagram with $A$ is isomorphic via $\Phi$ to

$$\text{Hom}(L_A,D_cL_A) \xrightarrow{(D_c,i)} \text{Hom}(L_A,D_cJ_A) \xrightarrow{(D_c,i)^{-1}} \text{Hom}(L_A,D_cT^{-1}K_A) \xrightarrow{\epsilon^{p,p}} \text{Hom}(K_A,D_cT^{-1}K_A).$$

Putting everything together, we arrive at the following commutative diagram, in which the vertical maps are isomorphism.

$$\begin{array}{ccc}
A \otimes \tilde{S} & \xrightarrow{\text{Hom}(L_A,D_cL_A)} & A \otimes \tilde{m}^{-1} & \xrightarrow{\text{Hom}(L_A,D_cJ_A)} & A \otimes \tilde{k}(q) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}(L_A,D_cL_A) & \xrightarrow{(D_c,i)} & \text{Hom}(L_A,D_cJ_A) & \xrightarrow{(p^p)^{-1}(D_c,i)^{-1}} & \text{Hom}(K_A,D_cT^{-1}K_A)
\end{array}$$

Comparing the construction of the left-most, resp. right-most, vertical arrow with that of $\Phi^{(L)}$, resp. $\Phi^{(K)}$, shows that they coincide, so we have obtained the desired commutative diagram. 

Lemma 7.8. The functor $P \mapsto P_p : \mathcal{B} \to \mathcal{C}^0(A_p)$ (notation as in Section 2) is faithful.

Proof. Note first that any $P \in \mathcal{B}$ is supported in degrees $0, \ldots, e$ and satisfies $H_i(P) = 0$ for $i \neq 0$ and $H_0(P) \in \mathcal{M}_t(A_S)$. Since $R_p$ is flat over $R$, we have $H_i(P_p) = H_i(P)_p$ and thus $P_p \in \mathcal{C}^0(A_p)$.

There is a diagram of functors

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{P \mapsto P_p} & \mathcal{C}^0(A_p) \\
\downarrow H_0 & & \downarrow H_0 \\
\mathcal{L} & \xrightarrow{M \mapsto M_p} & \mathcal{M}_t(A(\mathfrak{p}))
\end{array}$$

which commutes up to natural isomorphism. By Lemma 7.3 and the discussion in Section 8 both vertical arrows are equivalences, and the bottom arrow is faithful because $\mathcal{L}$ consists of $S$-torsion-free $S$-modules. Thus, the top arrow is also a faithful functor. 

We are now ready to establish (4.1) in the case $\mathfrak{p} \subseteq \mathfrak{q}$. This will complete the proof of Theorem 4.1. Recall our assumption that $R$ is local and $\mathfrak{q}$ is its maximal ideal.

Proof (4.1) when $\mathfrak{p} \subseteq \mathfrak{q}$. Recall that we are given $(V,f) \in \mathcal{H}^{e}(A(\mathfrak{p}),\sigma(\mathfrak{p});\tilde{k}(\mathfrak{p}))$, and we wish to prove that

$$(s_{e+1}^{-1}d_e s_{e}^{-1}[V,f])_{\mathfrak{q}} = -\partial_{\mathfrak{p},\mathfrak{q}}[V,f].$$

The proof is divided into three steps. In the first step, we construct an object of $\mathcal{H}^{e}(\mathcal{D}_{\mathfrak{p}}(A)/\mathcal{D}_{\mathfrak{p}}(A),D_e,\omega_e)$ representing $s_{e}^{-1}[V,f]$. In the second step, we use that object to describe $\partial_{\mathfrak{p},\mathfrak{q}}[V,f]$, and in the third step we evaluate $s_{e+1}^{-1}d_e s_{e}^{-1}[V,f]$ and prove the desired equality.

As in Section 8 we abbreviate $\mathcal{D}_{\mathfrak{p}}(A) \to \mathcal{D}_{\mathfrak{c}}$. Recall that the shifted triangulated hermitian structures of $\mathcal{D}_{\mathfrak{c}} := \mathcal{D}_{\mathfrak{c}}^{b}(A_{\mathfrak{p}})$ are denoted $(D_{\mathfrak{e}},\delta_{\mathfrak{e}},\omega_{\mathfrak{e}})_{n \in \mathbb{Z}}$.

Step 1. Choose an $A$-lattice $U$ in $V$ such that $U^1 \cdot \mathfrak{m} \subseteq U \subseteq U^1$; see Section 8. We write $f_0$ for the map $U \to U^2 = \text{Hom}_{A_{\mathfrak{p}}}(U,A \otimes \tilde{S})$ given by $(fu)e = f(u,v)$. Then, up to natural identifications, $(f_0)_{\mathfrak{p}} = f$. 

Step 1.1. By Lemma 7.3, the $L_A$-transfer functor, $(F,j) : (\mathcal{B}, D_e, \omega_e) \to (\mathcal{L}, \tilde{\tau}, \tilde{\omega})$, is a 1-hermitian equivalence. We may therefore assume that $(U, f_0)$ is given as the $L_A$-transfer of a some (non-unimodular) $\epsilon$-hermitian space $(Y, g)$ over $(\mathcal{B}, D_e, \omega_e)$.

By the definition of $\mathcal{B}$, we have $(Y_p, g_p) \in \mathcal{H}^0(\mathcal{A}_p, D'_e, \omega'_e)$. We claim that $\tilde{F}(K') \sim (Y_p, g_p)$ (notation as in (6.3), (n16)) is isomorphic to $(V, f)$. Indeed, by Lemma 24 and (n16)

$$\text{Hom}_{\mathcal{D}'}(K'_A, Y_p) = \text{Hom}_{\mathcal{D}'}((L_A)_p, Y_p) \cong \text{Hom}(L_A, Y)_p = U_p = V.$$ 

Noting that $\Phi(K') = (\Phi(L))_p$ (up to natural identifications), it is routine to check that under the above isomorphism, $\tilde{F}(K') \sim (V, f)$.

Step 1.2. Let $Z$ be the cone of $g : Y \to D_e Y$ in $\mathcal{D}$. We claim that $Z \in \mathcal{C}^0(\mathcal{A}_q)$, i.e., $Z$ is supported in degrees $0, \ldots, e + 1$, $H_i(Z) = 0$ for $i \neq 0$, and $H_0(Z) \cong \text{coker}(U \to U^f) \in \mathcal{P}(A(q))$.

To see this, observe that we have a commutative diagram whose horizontal maps are isomorphisms:

\[
\begin{array}{c}
\text{H}_0(Y) \xrightarrow{\sim} FY \xrightarrow{Fg} Y \\
\text{H}_0(D_e Y) \xrightarrow{\sim} FD_e Y \xrightarrow{f_0} U_f
\end{array}
\]

Here, the horizontal maps of the left square are induced by the natural isomorphism $H_0 \to F$ of Lemma 7.3, the middle square commutes because $(U, f)$ is the $L_A$-transfer of $(Y, g)$ (Step 1.1, Lemma 7.3), and the third square commutes because $f_0 = f_{| U}$. By our choice of $U$, the map $U \to U^f$ is injective and its cokernel is an $A(q)$-module. Thus, the same applies to $H_0(Y) \to H_0(D_e Y)$. Now, by inspecting the hom long exact sequence associated to the exact triangle $Y \to D_e Y \to Z \to T Y$, we see that $H_0(Z) = 0$ for all $i \neq 0$ and $H_0(Z) \cong \text{coker}(U \to U^f) \in \mathcal{P}(A(q))$. Since $Y$ and $D_e Y$ are supported in degrees $0, \ldots, e$, the complex $Z$ is supported in degrees $0, \ldots, e + 1$, so $Z \in \mathcal{C}^0(\mathcal{A}_q)$.

Step 1.3. Next we claim that $(Y, g)$ represents a class in $W^c_e(D_e / D_{e+1})$ which maps onto $[Y_p, g_p]$ in $\bigoplus_{e \in R^c} \mathcal{W}_e^c (\mathcal{D}_p(\mathcal{A}_I))$ (see Proposition 6.1).

Since $(Y, g)$ clearly maps to $(Y_p, g_p)$ in $\mathcal{H}^c(D_p^c(\mathcal{A}_p), D'_e, \omega'_e)$, proving the claim amounts to showing that

(i) $g : Y \to D_e Y$ is an isomorphism in $D_e / D_{e+1}$, and

(ii) for all $t \in R^c - \{p\}$, the class $[Y_t, g_t]$ is trivial in $W^c_e(D_p^c(\mathcal{A}_I))$.

By the definition of $D_e / D_{e+1}$ (see 7.2, for instance), the morphism $g : Y \to D_e Y$ in $\mathcal{D}$ represents an isomorphism in $D_e / D_{e+1}$ if its cone lives in $D_{e+1}$. The latter follows from Step 1.2, so (i) holds.

To prove (ii), let $t \in R^c - \{p\}$. For all $i \in Z - \{0\}$, we have $H_i(Y_t) \cong H_i(Y)_t = 0$. Since $p - t \not\in \emptyset$ (otherwise $t \subseteq p$) and $H_0(Y) \cdot p \cong \emptyset$, we also have $H_0(Y_t) \cong H_0(Y)_t = 0$. It follows that $Y_t$ is isomorphic to the zero object in $D_p^c(\mathcal{A}_I)$, and thus $[Y_t, g_t] = 0$ in $W^c_e(D_p^c(\mathcal{A}_I))$.

Conclusion of Step 1. By putting Steps 1.1 and 1.3 together, we get $s'_t | Y, g | = [V, f_p]$ (see Construction 0.7).

Step 2. Since $U$ was chosen such that $U^f m \subseteq U \subseteq U^f$, we have $f(U^f, U^f) \subseteq A \otimes \tilde{m}^{-1}$ (see Section 3). Write $f_1 = f_{| U^f \times U^f} : U^f \times U^f \to A \otimes \tilde{m}^{-1}$. Then $\tilde{\delta} f : U^f / U \times U^f / U \to A \otimes \tilde{k}(q)$ is given by

$$\tilde{\delta} f(x + U, y + U) = T(f_1(x, y)) \quad (x, y \in U^f).$$
We now transform the presentation of the pairing \((\mathcal{U}^I/U, \partial f)\) into one defined by means of \((Y, g)\). We shall freely view \(U^+ = \text{Hom}_{A_S}(U, A \otimes \tilde{S})\) as an \(A\)-submodule of \(\text{Hom}_{A[p]}(V, A \otimes \tilde{k}(p))\) via the localization-at-\(p\) map.

Step 2.1. The diagram \((7.3)\) gives rise to an isomorphism \(f^{-1} \circ j_Y : FD_Y \to U^f\) (here, \(f^{-1}\) is restricted to a map from \(U^+\) to \(U^f\)). We claim that under this isomorphism, the form \(\hat{f}_2\) corresponds to the form \(\tilde{f}_2\) given by

\[
\tilde{f}_2(\varphi, \psi) = (\Phi_p^{(L)})^{-1}(D'_c \varphi_p \circ \varepsilon D'_c g_p^{-1} \circ \psi_p)
\]

for all \(\varphi, \psi \in FD_Y\). Indeed,

\[
\hat{f}_2(f^{-1}(j_Y \varphi), f^{-1}(j_Y \psi)) = (f(f^{-1}(j_Y \varphi)))(f^{-1}(j_Y \psi)) = (j_Y \varphi_p)(f^{-1}(j_Y \psi_p))
\]

\[
= (\Phi_p^{(L)})^{-1}(D'_c \varphi_p \circ \omega'_{s,Y_s} \circ (f^{-1}(j_Y \psi_p))) = (\Phi_p^{(L)})^{-1}(D'_c \varphi_p \circ \omega'_{s,Y_s} \circ (f^{-1}(j_Y \psi_p))) = \tilde{f}_2(\varphi, \psi).
\]

In the fourth equality we used the fact that \(f^{-1} \circ j_Y = (Fg)p^{-1}\), see \((7.4)\).

Step 2.2. Since \(U^f \mathfrak{m} \subseteq U\), every right \(A\)-module homomorphism \(A_S \to U^f\) restricts to a homomorphism \(A \otimes \mathfrak{m} \to U\). Recalling that \(H_0 : A \to \mathcal{M}_1(A_S)\) is an equivalence taking \(L_A\) to \(A_S\), \(J_A\) to \(A \otimes \mathfrak{m}\), and the morphism \(g : Y \to D_c Y\) to a morphism isomorphic to \(U \to U^f\) — see \((7.3)\) —, this means that for every \(\varphi \in \text{Hom}(L_A, D_c Y)\), there exists a unique \(\psi \in \text{Hom}(J_A, Y)\) such that \(g \circ \psi = \varphi \circ \iota\) (with \(\iota : J_A \to L_A\) defined in \((a12)\)). Write \(\text{res } \varphi := \psi\). Then

\[
(7.5) \quad g \circ \text{res } \varphi = \varphi \circ \iota
\]

for all \(\varphi \in FD_Y\).

Let \(\Phi^{(j)}\) be as in Lemma \((7.7)\). We claim that for all \(\varphi, \psi \in FD_Y\),

\[
\tilde{f}_2(\varphi, \psi) = (\Phi_p^{(j)})^{-1}(\varepsilon D_c(\text{res } \varphi) \circ \psi).
\]

Since \(U^f\) and \(A \otimes \mathfrak{m}^{-1}\) are \(S\)-torsion-free, it is enough to check this after localizing at \(\mathfrak{p}\). We check this using Lemma \((7.7)\).

\[
\hat{f}_2(\varphi, \psi) = (\Phi_p^{(L)})^{-1}(D'_c \varphi_p \circ \varepsilon D'_c g_p^{-1} \circ \psi_p)
\]

\[
= (\Phi_p^{(L)})^{-1}(D'_c \varphi_p \circ D'_c \varphi_p \circ \varepsilon D'_c g_p^{-1} \circ \psi_p)
\]

\[
= (\Phi_p^{(j)})^{-1}(D'_c g_p \circ (\text{res } \varphi_p) \circ \varepsilon D'_c g_p^{-1} \circ \psi_p)
\]

\[
= (\Phi_p^{(j)})^{-1}(D_c(\text{res } \varphi_p) \circ D'_c \varphi_p \circ \varepsilon D'_c g_p^{-1} \circ \psi_p).
\]

Step 2.3. Recall from Step 1.2 that \(Z\) denotes the cone of \(g : Y \to D_c Y\). In particular, we have a distinguished triangle

\[
(7.6) \quad Y \xrightarrow{\partial} D_c Y \xrightarrow{\varepsilon} Z \xrightarrow{\pi} TY.
\]

in \(D\). We observed in Step 1.2 that \(Z \in \mathcal{C}^0(A_k)\) and the sequence \(0 \to H_0(Y) \to H_0(D_c Y) \to H_0(Z) \to 0\) is exact. By Lemma \((7.4)\) this means that \(0 \to FY \to FD_c Y \to FZ \to 0\) is an exact sequence of right \(A\)-modules and \(p^0 : \text{Hom}(K_A, Z) \to \text{Hom}(L_A, Z) = FZ\) is an isomorphism. Thus, we have an exact sequence

\[
(7.7) \quad 0 \to FY \xrightarrow{Fg} FD_c Y \xrightarrow{(\varepsilon^0)^{-1} \circ F\varepsilon} \text{Hom}(K_A, Z) \to 0.
\]

The diagram \((7.3)\) specifies an isomorphism between the \(A\)-module homomorphism \(U \rightarrow U^f\) and \(FG : FY \rightarrow FD_c Y\), and so we get an isomorphism of \(A(p)\)-modules

\[
\text{Hom}(K_A, Z) \rightarrow U^f/U.
\]

Unfolding the definitions, this isomorphism is evaluated as follows. Given \(\alpha \in \text{Hom}(K_A, Z)\), the exactness of \((7.7)\) implies that there exists \(\varphi \in FD_c Y = \text{Hom}(L_A, D_c Y)\)
such that $\alpha = (p^o)^{-1}(Fu)\varphi$, or rather, $u \circ \varphi = \alpha \circ p$ (note that $Fu = u_0$). Since the isomorphism $FD_\varepsilon Y \to U^f$ is $f^{-1} \circ j_Y$, the element of $U^f/U$ corresponding to $\alpha$ is the image of $f^{-1}(j_Y\varphi)$ in $U^f/U$.

**Step 2.4.** The pairing $\partial \overline{\varepsilon} : U^f/U \times U^f/U \to A \otimes \bar{k}(q)$ induces an $A \otimes \bar{k}(q)$-valued pairing on $\text{Hom}(K_A, Z)$ via the isomorphism $\text{Hom}(K_A, Z) \to U^f/U$ of Step 2.3. Denote this pairing as $\partial \overline{f}_2$ and the corresponding $\varepsilon$-hermitian form by $\partial f_2$. We finish Step 2 by giving a way to evaluate

$$\partial f_2 : \text{Hom}(K_A, Z) \times \text{Hom}(K_A, Z) \to A \otimes \bar{k}(q).$$

Thus describing $\partial_{p,q}[V, f]$ in terms of $(Y, g)$ and the distinguished triangle \(7.6\).

Let $\alpha, \beta \in \text{Hom}(K_A, Z)$. We showed in Step 2.3 that there are $\varphi, \psi \in FD_\varepsilon Y$ such that

$$u \circ \varphi = \alpha \circ p,$$

$$u \circ \psi = \beta \circ p,$$

and the images of $\alpha$ and $\beta$ in $U^f/U$ are represented by $f^{-1}(j_Y\varphi)$ and $f^{-1}(j_Y\psi)$, respectively. Thus,

$$\partial f_2(\alpha, \beta) = \partial f(j^{-1}(j_Y\varphi), f^{-1}(j_Y\psi)) = T(f_1(f^{-1}(j_Y\varphi), f^{-1}(j_Y\psi))).$$

By Steps 2.1 and 2.2, the right hand side evaluates to $T((\Phi^{q,j})^{-1}(\varepsilon D_\varepsilon(\text{res} \varphi) \circ \psi))$, so by Lemma \[.\] we get

$$\partial f_2(\alpha, \beta) = T((\Phi^{q,j})^{-1}(\varepsilon D_\varepsilon(\text{res} \varphi) \circ \psi)),$$

$$= \varepsilon((\Phi^{q,j})^{-1} \circ (p^o)^{-1} \circ (D_{e+1}q)) \circ (\varepsilon D_\varepsilon(\text{res} \varphi) \circ \psi)$$

$$= \varepsilon((\Phi^{q,j})^{-1}((p^o)^{-1}(D_{e+1}q)) \circ \varepsilon D_\varepsilon(\text{res} \varphi) \circ \psi).$$

**Step 3.** We finally describe $s'_{e+1}d_e s^{-1}_{e-1}[V, f_p] = s'_{e+1}d_e[Y, g]$ and check that its negative agrees with $\partial_{p,q}[V, f_p]$, which is represented by $(\text{Hom}(K_A, Z), \partial f_2)$ defined Step 2.4.

**Step 3.1.** Recall the distinguished triangle \(7.6\). By [1] Theorem 2.6, we have a commutative diagram

$$\begin{CD}
Y @>g>> D_\varepsilon Y @>u>> Z @>v>> TY
\end{CD}$$

in which the top and bottom rows are distinguished triangles in $\mathcal{D}$ and $h$ is an isomorphism satisfying $h = \varepsilon \omega_{e+1} \circ D_{e+1}h$. We saw in Step 1.2 that $Z \in C^0(A_q) \subseteq D_{e+1}$. Thus, according to [1] Definitions 5.16, 2.10, $(Z, h)$ represents the image of $[Y, g]$ under the map $\partial e : W^e_\varepsilon(D_{e}/D_{e+1}) \to W^{e+1}_\varepsilon(D_{e+1})$ (see Section 3). Consequently, $(Z, h)$ represents $d_e[Y, g] \in W^{e+1}(D_{e+1}/D_{e+2})$. (In fact, $D_{e+2} = 0$ because we assume that $R$ is local with maximal ideal $q_1$.) Since $Z \in C^0(A_q)$, it follows that $(Z, h) = (Z_u, h_u)$ represents the image of $d_e[Y, g]$ in $W_\varepsilon(C^0(A_q), D_{e}, \omega_e)$, see Construction \[.\]. Consequently, with notation as in \[.\] (and $K$ as in \[.\]), we have

$$\hat{F}^{(K)}[Z, h] = s'_{e+1}d_e[y, g] = s'_{e+1}d_e s^{-1}_{e-1}[V, f_p].$$

Unfolding the definitions, $\hat{F}^{(K)}(Z, h)$ is the $\varepsilon$-hermitian space $(\text{Hom}(K_A, Z), h_1)$, where $h_1 : \text{Hom}(K_A, Z) \times \text{Hom}(K_A, Z) \to A \otimes \bar{k}(q)$ is given by

$$\hat{h}_1(\alpha, \beta) = (\Phi^{K})^{-1}(D_{e+1} \circ \varepsilon h \circ \beta)$$

for all $\alpha, \beta \in \text{Hom}(K_A, Z)$. 
Step 3.2. We now show that $h_1 = -\partial f_2$. By Steps 2.4 and 3.1, this would imply that $\delta_{e+1} \sigma_e^{-1}[V, f_p] = -\partial_{p, q}[V, f_p]$, completing the proof.

Let $\alpha, \beta \in \text{Hom}(K_A, Z)$, and let $\varphi, \psi \in \text{Hom}(L_A, Y)$ be elements satisfying (7.9).

By Steps 2.4 and 3.1, checking that $h_1(\alpha, \beta) = -\partial f_2(\alpha, \beta)$ amounts to showing that

$$\Phi(K)^{-1}(D_{e+1} \alpha \circ \varepsilon \circ h \circ \beta) = -\varepsilon(\Phi(K)^{-1}(p^e)^{-1}(D_{e+1} q \circ D_e (\text{res } \varphi) \circ \psi)),$$

or equivalently, that

$$(7.10) \quad D_{e+1} \alpha \circ \varepsilon \circ h \circ \beta \circ p = -D_{e+1} q \circ D_e (\text{res } \varphi) \circ \psi.$$

Consider the diagram

$$\begin{array}{cccccc}
J_A & \rightarrow & L_A & \rightarrow & K_A & \rightarrow \\
\downarrow \varphi \quad & & \downarrow \varepsilon \circ h \circ \beta \circ p & & \downarrow \text{res } \varphi \quad & \\
Y & \rightarrow & D_e (\text{res } \varphi) & \rightarrow & \text{res } \varphi & \rightarrow \\
\end{array}$$

in which the left square is (7.9), and the top and bottom rows are the distinguished triangles $[n(12)]$ and $[7,6]$. Since $D$ is a triangulated category, there exists $\alpha' \in \text{Hom}(K_A, Z)$ which makes the diagram commute. By the middle square and (7.8), we have $p^e \alpha' = u \circ \varphi = p^e \alpha$. Lemma (7.1) tells us that $p^e : \text{Hom}(K_A, Z) \rightarrow \text{Hom}(L_A, Z)$ is bijective, so $\alpha' = \alpha$. Plugging this into the right square of the diagram gives $T(\text{res } \varphi) \circ q = v \circ \alpha$. Applying $D_{e+1}$ to both sides, we get

$$D_{e+1} q \circ D_e (\text{res } \varphi) = D_{e+1} \alpha \circ D_{e+1} v.$$ Using this, the middle square of (7.9), and (7.8), we get

$$-D_{e+1} q \circ D_e (\text{res } \varphi) \circ \psi = -D_{e+1} \alpha \circ D_{e+1} v \circ \psi = D_{e+1} \alpha \circ h \circ u \circ \psi$$

$$= D_{e+1} \alpha \circ h \circ \beta \circ p,$$

so we proved (7.10). This completes the proof. \hfill \square

8. Azumaya Algebras of Index 2

Let $(A, \sigma)$ be an Azumaya algebra with involution over a regular ring $R$, and let $\varepsilon \in \mu_2(R)$. We finish by applying Theorem 1.1 to prove the exactness of $GW^A, \sigma, \varepsilon_+$ when $R$ is semilocal of dimension $\leq 3$, and $A = 2$ and $\sigma$ is orthogonal or symplectic. The proof will combine the exact octagon of (14), which was shown to be compatible with the differentials of $GW^A, \sigma, \varepsilon_+$ in [3] §6, together with the Gersten–Witt spectral sequence, introduced by Balmer and Walter [7] in the case $(A, \sigma, \varepsilon) = (R, \text{id}_R, 1)$, and by Gille [10], [11] in general. Theorem 4.4 is what allows us to use both of these tools at the same time.

Theorem 8.1. Assume $R$ is a regular semilocal domain of dimension $\leq 3$. If $\text{ind } A \leq 2$ and $\sigma$ is orthogonal or symplectic, then $GW^A, \sigma, \varepsilon_+$ is exact.

Proof. By [3] Theorem 8.7], it is enough to consider the following two cases:

1. $\text{deg } A = 1$;
2. $\text{deg } A = 2$, and there are $\lambda, \mu \in A^\times$ such that $\lambda^\sigma = -\lambda, \mu^\sigma = -\mu, \lambda \mu = -\mu \lambda$ and $\lambda^2 \in \text{Cent}(A)$.

Since $\sigma$ is orthogonal or symplectic, $\text{Cent}(A) = R$, and so case (1) is covered by [7] Corollary 10.4 and [3] p. 3. We proceed with case (2). Define $B, \tau_1, \tau_2$ as in [3] §6. Then $B$ is a quadratic étale $R$-subalgebra of $A$, $\tau_1$ is its standard $R$-involution and $\tau_2 = \text{id}_B$. Moreover, $A = B \otimes \mu B$ and $B = \text{Cent}(B) = R \otimes \lambda R$, so the set $\{1, \lambda, \mu, \mu \lambda\}$ is an $R$-basis of $A$. Since $\sigma$ maps $\lambda, \mu$, and $\mu \lambda$ to their negatives, $\sigma$ is necessarily symplectic (but $\varepsilon$ can be 1 or $-1$).
By [9, Theorem 6.2], the data of \( A, \sigma, \lambda, \mu \) gives rise to an 8-periodic exact sequence of Gersten-Witt complexes as defined in Section 3. Since \( G_{+}^{B, \tau_{2t-1}} = 0 \), it degenerates into a 7-term exact sequence:

\[
\begin{align*}
0 \to & \ G_{+}^{A, \sigma, 1} \to G_{+}^{B, \tau_{1}, -1} \to G_{+}^{A, \sigma, -1} \to G_{+}^{B, \tau_{2}, -1} \to \\
& G_{+}^{A, \sigma, -1} \to G_{+}^{B, \tau_{1}, -1} \to G_{+}^{A, \sigma, 1} \to 0.
\end{align*}
\]

We view this sequence as a double cochain complex. The columns \( G_{+}^{B, \tau_{1}, \pm 1} \) and \( G_{+}^{B, \tau_{2}, -1} \) are exact by [9, Theorem 9.4], and we have \( H^{i}(G_{+}^{A, \sigma, \pm 1}) = 0 \) by [9, Theorem 5.1], or [20, Theorem 8.4] (applies to \( GW_{chasing} \)) shows that the nullity of all cohomology groups in the following order: \( H^{-1} - H^{0} - H^{1} - \cdots \) of Theorem 4.1. Specifically, apply this lemma to assert the exactness of all cohomology groups in the following order: \( H^{-1}(G_{+}^{A, \sigma, 1}) = 0 \), \( H^{1}(G_{+}^{A, \sigma, 1}) = 0 \), \( H^{2}(G_{+}^{A, \sigma, 1}) = 0 \), \( H^{-1}(G_{+}^{A, \sigma, -1}) = 0 \), \( H^{1}(G_{+}^{A, \sigma, -1}) = 0 \), \( H^{2}(G_{+}^{A, \sigma, -1}) = 0 \).

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
H^{0}(G_{+}^{A, \sigma, -1}) & H^{1}(G_{+}^{A, \sigma, -1}) & H^{2}(G_{+}^{A, \sigma, -1}) & 0 & \cdots & \\
0 & 0 & 0 & 0 & \cdots & \\
H^{0}(G_{+}^{A, \sigma, 1}) & H^{1}(G_{+}^{A, \sigma, 1}) & H^{2}(G_{+}^{A, \sigma, 1}) & 0 & \cdots & \\
0 & 0 & 0 & 0 & \cdots & \\
H^{0}(G_{+}^{A, \sigma, -1}) & H^{1}(G_{+}^{A, \sigma, -1}) & H^{2}(G_{+}^{A, \sigma, -1}) & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

Figure 1. The \( E_{2} \) page of the Gersten–Witt spectral sequence of \((A, \sigma)\). The term \( H^{0}(G_{+}^{A, \sigma, 1}) \) is in the \((0, 0)\)-coordinate.

Remark 8.2. It is not straightforward to prove Theorem 8.1 directly using only the Gersten–Witt complex \( B_{+}^{A, \sigma, \varepsilon} \) of Balmer–Walter and Gille. Letting \( A, \sigma, \tau_{1}, \tau_{2} \) be as in the proof of Theorem 8.1, the functoriality of \( B_{+}^{A, \sigma, \varepsilon} \) with respect to hermitian functors does give rise to a 7-term sequence of cochain complexes analogous to (8.2):  

\[
\begin{align*}
0 \to & \ B_{+}^{A, \sigma, 1} \to B_{+}^{B, \tau_{1}, -1} \to B_{+}^{A, \sigma, -1} \to B_{+}^{B, \tau_{2}, -1} \to \\
& B_{+}^{A, \sigma, -1} \to B_{+}^{B, \tau_{1}, -1} \to B_{+}^{A, \sigma, 1} \to 0
\end{align*}
\]

but it is a priori not clear that (8.2) is exact in levels 1 and above, and the exactness at all levels is needed in the last paragraph of the proof. 

In more detail, let \( e \geq 0 \) and let us identify \( B_{+}^{A, \sigma, \varepsilon} \) with \( \bigoplus_{p \in \mathcal{R}(A)} W_{e}(A(p)) \cong \bigoplus_{p \in \mathcal{R}(A)} W_{e}(A(p), \sigma(p)) \) via the isomorphism constructed by Balmer–Walter [7] for \( A = R \), or Gille [10], [18] in general. One can show that the \( e \)-th level of (8.2)
decomposes as a direct sum of 7-term sequences \( \bigoplus_{p \in R(e)} S(p) \). It is expected that for each \( p \in R(e) \), the sequence \( S(p) \) is isomorphic to the exact sequence
\[
(8.3) \quad 0 \to W_1(A(p), \sigma(p)) \to W_1(B(p), \tau_1(p)) \to W_{-1}(A(\sigma), \sigma(p)) \to W_1(B(p), \tau_2(p)) \\
\to W_{-1}(A(p), \sigma(p)) \to W_{-1}(B(p), \tau_1(p)) \to W_1(A(p), \sigma(p)) \to 0
\]
considered in [15] Corollary 8.2 and earlier in Lewis [25], but when \( e \geq 1 \), it is not immediate that the maps in \( S(p) \) correspond to the maps appearing in (8.3), so we cannot assert that \( S(p) \) is exact without further work. (In fact, by crunching through Construction 6.7, it is possible to show that (S1) and (S2) are isomorphic and as a consequence assert the exactness of (S2) in levels 1 and above.)

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