Stochastic Algorithms for Advanced Risk Budgeting

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Abstract

Modern portfolio theory has provided for decades the main framework for optimizing portfolios. Because of its sensitivity to small changes in input parameters, especially expected returns, the mean-variance framework proposed by Markowitz (1952) has however been challenged by new construction methods that are purely based on risk. Among risk-based methods, the most popular ones are Minimum Variance, Maximum Diversification, and Risk Budgeting (especially Equal Risk Contribution) portfolios. Despite some drawbacks, Risk Budgeting is particularly attracting because of its versatility: based on Euler’s homogeneous function theorem, it can indeed be used with a wide range of risk measures. This paper presents sound mathematical results regarding the existence and the uniqueness of Risk Budgeting portfolios for a very wide spectrum of risk measures and shows that, for many of them, computing the weights of Risk Budgeting portfolios only requires a standard stochastic algorithm.

Keywords: portfolio optimization, risk budgeting, risk measures, volatility, expected shortfall, stochastic algorithm, stochastic gradient descent.

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1 Introduction

Seventy years ago, Markowitz [28] transformed the financial problem of asset allocation into a simple (quadratic) mathematical optimization problem. His modern portfolio theory has opened the quest for quantitative tools to better invest in financial markets and construct optimized investment portfolios. Markowitz’ model was soon generalized by Tobin [41] who introduced a risk-free asset into the framework and proved what is now called a mutual fund separation theorem. Then, [26], [29], [38] and [42] independently developed the capital asset pricing model (CAPM). Undoubtedly, Markowitz’ modern portfolio theory, Tobin’s mutual fund separation theorem and the CAPM have shaped the asset management industry: diversification is the cornerstone of asset allocation; volatility is the most popular measure of portfolio risk; so-called “α” and “β” are ubiquitous concepts to describe strategies, not to talk about the recurrent debates between passive and active investing.

Although mathematically sound, Markowitz’ idea of finding the asset allocation which maximizes the portfolio expected return given a variance constraint – the so-called mean-variance optimization – raises practical issues. In particular, choosing (or estimating) the vector of expected returns that one inputs in the mean-variance framework is a difficult and very sensitive task as small differences in expected returns may yield significant differences in the final portfolio (see [7]).

A large set of portfolio construction methods have been proposed to mitigate these issues such as the famous Black-Litterman model [8] that uses the CAPM as a baseline for the expected returns. Some of them simply do not rely on any expected return. The simplest example is that of equally-weighted portfolios (see for instance [15]) which is input-agnostic and nevertheless (or, maybe, for that reason) considered an interesting benchmark beyond traditional capitalization-weighted ones. Most quantitative portfolio construction methods that are independent of expected returns are in fact risk-based methods: they focus on risk mitigation and ignore (at least mathematically) return maximization. Most of them rely only on the covariance matrix of asset returns. The portfolio construction method based on finding the weights that minimize the ex ante variance of the portfolio return is a typical example: it corresponds to the least risky portfolio on Markowitz’ efficient frontier – the so-called Minimum Variance portfolio. The Most-Diversified portfolio approach is based on maximizing the diversification ratio introduced in [12]. Most-Diversified portfolios are appealing and their properties are analyzed in [13]. A third popular risk-based approach is Risk Budgeting, which is at the core of this paper, and which, broadly speaking, consists in allocating risk rather than capital in line with given budgets. Risk Budgeting has been studied and largely advocated by Roncalli and his coauthors in an interesting series of papers (e.g. [9], [10], [25], [27], [37]) and in the reference book [36]. One of the most adopted Risk Budgeting methods is Equal Risk Contribution (ERC) which corresponds to the case where the risk budgets are chosen as equals. The popularity of ERC is due to the fact that it corresponds to the “least-concentrated” portfolio in terms of risk and it is relatively insensitive to small errors in the covariance matrix estimation, compared to the Most-Diversified and Minimum-Variance portfolios (see [14]).

As mentioned above, Risk Budgeting focuses on the contribution of each asset to the portfolio risk, rather than on the portfolio risk itself. Therefore, it relies on a mathematical framework to handle the decomposition of the total portfolio risk into asset-wise risk contributions. Such a

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1The case of covariance matrices also raises issues. There is an important literature on covariance / correlation matrix cleansing (see [29] for shrinkage methods, [24] for eigenvalue clipping, and [11] for the recent rotationally invariant estimators).
breakdown of risk is performed thanks to Euler’s homogeneous function theorem because of its axiomatic justification (see [36] and [40]). In particular, Risk Budgeting is versatile and relies solely on the positive homogeneity and the differentiability of the chosen risk measure.

Various frameworks have been proposed to build and study Risk Budgeting portfolios. The initial framework proposed in [27] studies ERC portfolios when volatility is the risk measure. The authors prove that whenever the covariance matrix of asset returns is positive-definite, an ERC portfolio exists and is unique. For that purpose, they regard the equations defining a Risk Budgeting portfolio as the first order condition (up to rescaling) of a constrained minimization problem. Extensions to other risk measures are present in various papers: [37] reintroduces expected returns and deals with risk measures that are a linear combination of expected return and volatility and [9] deals with Risk Budgeting when the risk measure is Expected Shortfall (see also [25]).

In terms of computational methods, several approaches have been proposed. The reference methods rely (up to a rescaling factor) on the numerical approximation of the solution of an unconstrained convex minimization problem, i.e. the optimization of the Lagrangian associated with the usual constrained minimization problem of the above literature. Many gradient descent algorithms have been proposed. Beyond simple methods, a Nesterov acceleration technique is applied in [39] and [26] preferred a cyclic coordinate descent one. In most cases, the reference risk measure is the volatility of portfolio returns, or at least a risk measure that may be easily computed. For instance, when asset returns are distributed according to a Gaussian mixture, [25] discusses the case of Risk Budgeting with Expected Shortfall as a way to control the “skewness risk” of portfolios. When the risk measure and / or the distribution of asset returns are more complex (see for instance [18]), the above methods can be challenged by simulation-based numerical methods. This was the initial motivation of this paper.

In fact, our goal in this paper is to provide a unique framework that applies to a wide spectrum of risk measures. For that purpose, we first revisit some existence and uniqueness results about optimal portfolios. Then, for a large range of risk measures including deviation measures (like volatility) and spectral risk measures (like Expected Shortfall), we show that building Risk Budgeting portfolios boils down to using a standard stochastic gradient descent (SGD) approach.

In Section 2, we introduce some notations, define formally the Risk Budgeting problem and formally prove the existence and the uniqueness of a Risk Budgeting portfolio. In that section, we borrow a lot from the existing literature. In particular, our proof is based on the introduction of a convex minimization problem whose first order condition is (up to rescaling) the condition that defines Risk Budgeting portfolios. In Section 3, we start by summarizing the current use of Expected Shortfall in the Risk Budgeting literature and extend the semi-analytic formula of [25] to Student-t mixture distributions. We then show that Expected Shortfall does not necessarily need to be computed in the definition of the convex minimization problem because Expected Shortfall has itself a variational characterization in the form of a minimum: the famous Rockafellar-Uryasev formula (see [34], for instance). In particular, for almost any distribution of asset returns, a stochastic gradient descent enables to compute the Risk Budgeting portfolio when Expected Shortfall is the risk measure. Section 4 generalizes the approach and shows that the set of risk measures for which building a Risk Budgeting portfolio boils down to using a standard stochastic gradient descent is very large. Numerical examples are provided and discussed in Section 5.

2See also [10] for an extension beyond ERC portfolios to general Risk Budgeting portfolios.
2 Risk Budgeting: definition, existence and uniqueness

2.1 Notations and statement of the problem

Let us first define a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). This probability space will be used throughout the paper and we assume that all the random variables involved in the paper are defined on it.

Hereafter, so-called risk measures are some functions mapping a random variable (regarded as a loss) to a real number assessing the risk of this random variable. To be compatible with our analysis of the Risk Budgeting problem, we require that they satisfy two classical properties, as defined below:

**Definition 1.** A function \(\rho : L^0(\Omega, \mathbb{R}) \rightarrow \mathbb{R}\) is said to be a RB-compatible risk measure if it satisfies the following assumptions:

\[
\begin{align*}
(\text{PH}) & \quad \forall Z \in L^0(\Omega, \mathbb{R}), \forall \lambda \geq 0, \quad \rho(\lambda Z) = \lambda \rho(Z) \quad \text{(positive homogeneity)}, \\
(\text{SA}) & \quad \forall Z_1, Z_2 \in L^0(\Omega, \mathbb{R}), \quad \rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2) \quad \text{(sub-additivity)}.
\end{align*}
\]

As a consequence, a RB-compatible risk measure is convex.

**Remark 1.** Coherent risk measures (see [3]) are RB-compatible risk measures but the converse is not true. In particular, volatility is a RB-compatible risk measure but not a coherent risk measure (see the discussion in Section 4).

In order to define Risk Budgeting portfolios, let us consider a financial universe with \(d\) assets. Their returns are stacked in a random vector \(X\) with values in \(\mathbb{R}^d\), i.e. \(X \in L^0(\Omega, \mathbb{R}^d)\). Portfolios are identified with weight vectors that belong to the simplex \(\Delta_d = \{\theta \in \mathbb{R}^d_+ | \theta_1 + \ldots + \theta_d = 1\}\).

To any \(d\)-dimensional random vector \(X\) and any RB-compatible risk measure \(\rho\), we can associate a function \(R_{\rho,X}\) (hereafter denoted by \(R\) when there is no ambiguity) defined by

\[R : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \quad y \mapsto -\rho(-y'X).\]

If the portfolio weights are \(\theta \in \Delta_d\), then \(\theta'X\) is the return of the portfolio and \(R(\theta)\) is the risk associated with this portfolio.

In what follows, we shall always assume that \(R\) is continuous on \(\mathbb{R}^d_+\) and continuously differentiable on \((\mathbb{R}^d_+)^d\). Moreover, we assume that \(R(\theta) > 0\) for any \(\theta \in \Delta_d\), i.e., the risk of a portfolio is always positive.

We can now recall the usual definition of the Risk Budgeting problem associated with the function \(R\).

**Definition 2.** Let \(\Delta^0_d = \{\theta \in (\mathbb{R}^d_+)^d | \theta_1 + \ldots + \theta_d = 1\}\) and let \(b \in \Delta^0_d\.

We say that \(\theta \in \Delta^0_d\) solves RB\(_b\) if and only if

\[\theta_i \partial_i R(\theta) = b_i R(\theta),\]

for every \(i \in \{1, \ldots, d\}\).
2.2 Theoretical results on Risk Budgeting

Given the above definition of Risk Budgeting portfolios, two questions naturally arise:

1. the existence of a vector of weights $\theta$ that solves RB$_b$ for a given vector of budgets $b \in \Delta_d^>$;

2. the uniqueness of such a vector of weights $\theta$.

The following theorem solves the first point.

**Theorem 1.** Set $b \in \Delta_d^>$. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuously differentiable convex and increasing function. Let the function $\Gamma_g : (\mathbb{R}^+_n)^d \rightarrow \mathbb{R}$ be defined by

$$\Gamma_g : y \mapsto g(\mathcal{R}(y)) - \sum_{i=1}^d b_i \log y_i.$$

There exists a unique minimizer $y^*$ of the function $\Gamma_g$ and $\theta^* := \frac{y^*}{\sum_{i=1}^d y_i} \in \Delta_d^>$ solves RB$_b$.

**Proof.** Since $\mathcal{R}(\theta) > 0$ for every $\theta \in \Delta_d^>$ by assumption, (PH) implies $\mathcal{R}(y) > 0$ for every $y \in (\mathbb{R}^+_n)^d$. Therefore, $\Gamma_g(y) = g(\mathcal{R}(y)) - \sum_{i=1}^d b_i \log y_i$ is well defined for all $y \in (\mathbb{R}^+_n)^d$.

Let us then notice that $\Gamma_g$ is strictly convex since $g$ is convex and increasing, $\mathcal{R}$ is convex, and $y \in (\mathbb{R}^+_n)^d \rightarrow -\sum_{i=1}^d b_i \log y_i$ is strictly convex.

To prove the existence of a minimizer to the function $\Gamma_g$, for any $\theta \in \Delta_d^>$, let us introduce the function $\gamma_{g,\theta} : \mathbb{R}^+_n \rightarrow \mathbb{R}$ defined by

$$\gamma_{g,\theta} : \lambda \mapsto \Gamma_g(\lambda \theta) = g(\lambda \mathcal{R}(\theta)) - \sum_{i=1}^d b_i \log \theta_i - \log \lambda.$$

We first notice that $\lim_{\lambda \rightarrow +\infty} \gamma_{g,\theta}(\lambda) = \lim_{\lambda \rightarrow +\infty} \gamma_{g,\theta}(\lambda) = +\infty$ [1]. By continuity, there exists $\lambda^*(\theta)$ such that $\gamma_{g,\theta}(\lambda) \geq \gamma_{g,\theta}(\lambda^*(\theta))$ for every $\lambda > 0$. Let us show by contradiction that $\theta \mapsto \lambda^*(\theta)$ is bounded.

For that purpose, assume the existence of a sequence $(\theta_n)_n$ with values in $\Delta_d^>$ such that $\lambda_n := \lambda^*(\theta_n) \rightarrow +\infty$. We can then extract a subsequence $(\theta_{\varphi(n)})_n$ that converges towards $\theta \in \Delta_d$ and such that $\lambda_{\varphi(n)} \rightarrow +\infty$. For all $n$, $\lambda_n$ satisfies the first order condition $\gamma'_{g,\theta_{\varphi(n)}}(\lambda_{\varphi(n)}) = 0$, i.e.

$$\mathcal{R}(\theta_{\varphi(n)})(g'(\lambda_{\varphi(n)} \mathcal{R}(\theta_{\varphi(n)}))) = \frac{1}{\lambda_{\varphi(n)}} = 0.$$

Therefore, if $x_n := \lambda_{\varphi(n)} \mathcal{R}(\theta_{\varphi(n)})$, then we have $x_n g'(x_n) = 1$ for all $n$. However, because $\lim_{n \rightarrow +\infty} \lambda_{\varphi(n)} = +\infty$ and $\lim_{n \rightarrow +\infty} \mathcal{R}(\theta_{\varphi(n)}) = \mathcal{R}(\theta) > 0$, we have $\lim_{n \rightarrow +\infty} x_n = +\infty$. This contradicts $x_n g'(x_n) = 1$ for all $n$, because $g$ is a convex and increasing function. Therefore, we have proved that $\theta \mapsto \lambda^*(\theta)$ is bounded: there exists a constant $M$ such that $\lambda^*(\theta) \leq M$.

For every $y \in (\mathbb{R}^+_n)^d$, we have

$$\Gamma_g(y) = \gamma_{g,y} / \sum_i y_i \left( \sum_i y_i \right) \geq \gamma_{g,y} / \sum_i y_i \left( \lambda^* \left( \frac{y}{\sum_i y_i} \right) \right) = \Gamma_g \left( \frac{y}{\sum_i y_i}, \lambda^* \left( \frac{y}{\sum_i y_i} \right) \right).$$

[1]For the latter point, note that the convexity of $g$ implies $g(\lambda \mathcal{R}(\theta)) \geq g(c) + g'(c)(\lambda \mathcal{R}(\theta) - c)$ for any $c$ such that $g'(c) > 0$ (which exists since $g$ is strictly increasing).
Now, as the nonempty setting \(D\), the uniqueness of the minimizer is a consequence of the strict convexity of \(\Gamma\) as defined in Theorem 1, and let \(y\) be the unique minimizer of \(\Gamma\).

For any \(y \in C_M\), if there exists \(j \in \{1, \ldots, d\}\) such that \(y_j < \varepsilon\), then, by definition of \(\varepsilon\),

\[
\Gamma_y(y) = g(R(y)) - \sum_{i=1}^{d} b_i \log y_i \geq g(0) - \sum_{i=1}^{d} b_i \log y_i \\
\geq g(0) - \sum_{i \neq j} b_i \log y_i - b_j \log \varepsilon \geq g(0) - \log M \sum_{i \neq j} b_i - b_j \log \varepsilon \\
\geq g(0) - \log M(1 - b_j) - b_j \log \varepsilon \geq \Gamma_{\bar{y}}(\bar{y}).
\]

After having proved the existence of a solution to the Risk Budgeting problem, let us deal with uniqueness.

The uniqueness of the minimizer is a consequence of the strict convexity of \(\Gamma\).

Now, as \(y^*\) is an interior minimum of \(\Gamma_y\), we have

\[
\forall i \in \{1, \ldots, d\}, \quad g'(R(y^*))\partial_i R(y^*) - \frac{b_i}{y_i^*} = 0,
\]

or, equivalently,

\[
\forall i \in \{1, \ldots, d\}, \quad y_i^* g'(R(y^*))\partial_i R(y^*) = b_i.
\]

Summing over \(i \in \{1, \ldots, d\}\), Euler’s homogeneous function theorem gives \(R(y^*) g'(R(y^*)) = 1\).

Therefore, we get

\[
\forall i \in \{1, \ldots, d\}, \quad y_i^* \partial_i R(y^*) = b_i R(y^*).
\]

Setting \(\theta^* = y^* / \sum_{i=1}^{d} y_i^*\) and using (PH), we see that \(\theta\) solves RB.

After having proved the existence of a solution to the Risk Budgeting problem, let us deal with uniqueness.

**Theorem 2.** For some \(b \in \Delta^d_+\), let \(\theta \in \Delta^d_+\) be a solution of RB. Let \(g : \mathbb{R}_+ \to \mathbb{R}\) be a continuously differentiable convex and increasing function. Consider the map \(\Gamma_g\) as defined in Theorem 1, and let \(y^*\) be the unique minimizer of \(\Gamma_g\).

Then, we have

\[
\theta = \frac{y^*}{\sum_{i=1}^{d} y_i^*}.
\]

\(^4\)It contains \(\bar{y}\) by definition of \(\varepsilon\).
Proof. The function $h : \lambda \in \mathbb{R}_+ \mapsto \mathcal{R}(\lambda \theta)g'(\mathcal{R}(\lambda \theta))$ is continuous because it is still the case for $\mathcal{R}$ and $g'$. Since $h(0) = 0$ and $\lim_{\lambda \to +\infty} h(\lambda) = +\infty$, there exists $\lambda \in \mathbb{R}_+$ such that $h(\lambda) = 1$.

Defining $y := \vec{\lambda} \theta$, we obtain, for all $i \in \{1, \ldots, d\}$,

$$y_i \partial_i \mathcal{R}(y) g'(\mathcal{R}(y)) = \vec{\lambda} \theta_i \partial_i \mathcal{R}(\vec{\lambda} \theta) g'(\mathcal{R}(\vec{\lambda} \theta)) = \frac{\vec{\lambda} \theta_i \partial_i \mathcal{R}(\vec{\lambda} \theta)}{\mathcal{R}(\vec{\lambda} \theta)} = \frac{\theta_i \partial_i \mathcal{R}(\theta)}{\mathcal{R}(\theta)} = g(b_i),$$

because $\theta$ is a solution of $\text{RB}_b$ and $\mathcal{R}$ (resp. $\partial_i \mathcal{R}$) is positively homogeneous of degree 1 (resp. 0). In other words, this yields

$$\partial_i \mathcal{R}(y) g'(\mathcal{R}(y)) = \frac{b_i}{y_i} = 0, \forall i \in \{1, \ldots, d\},$$

and $y$ is a critical point of the convex function $\Gamma_y$. We conclude that $y^* = y = \vec{\lambda} \theta$. Since $\theta \in \Delta_d^{>0}$, we must have $\theta = y^* / \sum_{i=1}^d y_i^*$.

The above theorems prove the existence of a unique solution to the Risk Budgeting problem for any vector of positive budgets and provide a variational characterization (up to rescaling) of the vector of weights. The link between the Risk Budgeting problem and the unconstrained convex minimization problem we defined in Theorem 1 was already noticed by several authors, notably [10, 20, 36, 39], when $g$ is the identity map (a sketch of the proof of our theorems 1 and 2 has for instance been proposed in [39, Section 2.2.2.2]). Nonetheless, to the best of our knowledge, no complete mathematical proof of the existence and uniqueness of Risk Budgeting portfolios has been stated in the literature until now. Moreover, the possibility of choosing functions $g$ beyond the identity function will prove to be useful for developing our general framework in the next sections.

Remark 2. The above theorems are also useful to shed light on the advantage of Risk Budgeting for building portfolios. Indeed, specifying budgets rather than weights allows somehow to reduce risk as it has been shown in [39]. Indeed, for any $b \in \Delta_d^{>0}$, if $\theta \in \Delta_d^{>0}$ is a solution of $\text{RB}_b$, then $\mathcal{R}(\theta) \leq \mathcal{R}(b)$.

3 Risk Budgeting with Expected Shortfall

3.1 The importance of Expected Shortfall

In the first papers advocating the Risk Budgeting approach, the chosen risk measure was volatility. Volatility indeed constitutes a reasonable choice of risk measure, especially when the probability distributions of asset returns do not exhibit asymmetry and/or heavy tails.

denoting by $\Sigma$ the covariance matrix of asset returns, the volatility of the portfolio defined by the vector of weights $y$ is obviously $\mathcal{R}(y) := \sqrt{y^T \Sigma y}$. The latter quantity is easily computed.

$^{5}$If $g^*$ is the minimizer of $y \mapsto \mathcal{R}(y) - \sum_{i=1}^d b_i \log y_i$ on $(\mathbb{R}_+^d)$ and $\lambda := \sum_{i=1}^d y_i^*$, then we have seen that $\theta = g^*/\lambda$ and

$$\mathcal{R}(\theta) = \frac{1}{\lambda} \mathcal{R}(g^*) \leq \frac{1}{\lambda} \left( \mathcal{R}(\lambda b) - \sum_{i=1}^d b_i \log(\lambda b_i) + \sum_{i=1}^d b_i \log y_i^* \right) \leq \mathcal{R}(b) - \frac{1}{\lambda} \sum_{i=1}^d b_i \log \left( \frac{b_i}{y_i^*} \right) \leq \mathcal{R}(b),$$

because of the non-negativeness of relative entropy.
and the Risk Budgeting problem can be efficiently solved using a gradient descent procedure, to minimize over \((\mathbb{R}^+)^d\) the function

\[
\Gamma_{x \mapsto x^2} : y \mapsto (\mathcal{R}(y))^2 - \sum_{i=1}^{d} b_i \log y_i = y'\Sigma y - \sum_{i=1}^{d} b_i \log y_i.
\]

Nonetheless, it is well-known that asset and portfolio returns exhibit skewed and heavy-tailed distributions. And numerous studies show that excess returns reward investors for carrying the risk of sudden and significant losses [24, 30]. Therefore, to more efficiently deal with such distributional features in portfolio management, other risk measures have been proposed in the literature. In particular, Expected Shortfall is an alternative risk measure considered in the recent Risk Budgeting literature. For instance, [25] proposed to use Expected Shortfall to construct Risk Budgeting portfolios because the latter risk measure allows to focus on the left tail of P&L distributions only, contrary to volatility.

When the chosen risk measure is Expected Shortfall, there is no simple formula for \(\mathcal{R}(y)\) in general. There exist nevertheless some cases in which Expected Shortfall is easily computed. For instance, when asset returns are distributed according to a mixture of two Gaussian distributions as in [25], there exist semi-analytic expressions for Expected Shortfalls. Then, the above gradient procedure works fine because it can rely on semi-analytic formulas.

In this section, we first show that some semi-analytic expressions for Expected Shortfall are available whenever the underlying asset returns are distributed according to a Student-t mixture. Therefore, we extend the scope of the probability distributions for which the above gradient descent procedure should be the reference method for solving Risk Budgeting problems. Note that we must choose a level \(\alpha\) so that the Expected Shortfall (at that level) of all long-only portfolios is positive. Then, we propose a general method that does not rely on any parametric assumption for the joint law of the asset returns, but it requires to use a stochastic gradient descent rather than a simple gradient descent.

### 3.2 A parametric model with semi-analytic expressions

There are many possible choices for modeling the joint distribution of asset returns. Because asset returns often exhibit skewed and heavy-tailed distributions, Student-t mixtures are natural candidates. Student-t distributions indeed generate heavy tails, and mixing them allows to model skewness when some of them are non centered. Now, we show that Expected Shortfall can be computed in a semi-analytic manner when the vector of asset returns \(X\) is a mixture of \(N\) multivariate Student-t distributions. This means \(X\) can be written

\[
X = 1\{C = 1\}X_1 + \ldots + 1\{C = N\}X_N,
\]

where:

- for all \(i \in \{1, \ldots, N\}\), \(X_i\) follows a \(d\)-dimensional Student-t distribution \(t(\mu_i, \Lambda_i, \nu_i)\) with location parameter \(\mu_i\), positive-definite scale matrix \(\Lambda_i\), and \(\nu_i > 1\) degrees of freedom, i.e. \(X_i\) has a density

\[
f(x|\mu_i, \Lambda_i, \nu_i) := \frac{\Gamma((\nu_i + d)/2)}{\Gamma(\nu_i/2)\nu_i^{d/2}\pi^{d/2}\det(\Lambda_i)^{1/2}} \left(1 + \frac{1}{\nu_i}(x - \mu_i)'\Lambda_i^{-1}(x - \mu_i)\right)^{-(\nu_i + d)/2}.
\]

\(6\)Expected Shortfall is a coherent risk measure (see [2]) and it is therefore a RB-compatible risk measure.

\(7\)Unlike volatility, Expected Shortfall depends on expected returns, which can be seen as a disadvantage. Nonetheless, it is always possible to translate random variables by their expectations to capture skewness risk independently of expected returns (see Section 4).
• $C$ is a discrete random variable with values in \( \{1, \ldots, N\} \) and \( P(C = i) = p_i \) for all \( i \in \{1, \ldots, N\} \), \( \sum_{i=1}^{N} p_i = 1 \);

• $C, X_1, \ldots, X_N$ are mutually independent.

For $y \in \mathbb{R}^d$, the probability density function and the cumulative distribution function of the loss $-y'X$ are respectively

\[
f_{-y'X}(z) = \sum_{i=1}^{N} \frac{p_i}{\sqrt{y'\Lambda_i y}} f_{t_{\nu_i}} \left( \frac{z + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right),
\]

and

\[
F_{-y'X}(z) = \sum_{i=1}^{N} \frac{p_i}{\sqrt{y'\Lambda_i y}} F_{t_{\nu_i}} \left( \frac{z + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right),
\]

where $f_{t_{\nu_i}}$ and $F_{t_{\nu_i}}$ denote respectively the density function and the cumulative distribution function of a standard Student-t distribution with \( \nu_i \) degrees of freedom.

Note that the latter distributions are continuous. Recall that the Value-at-Risk at level \( \alpha \in (0, 1) \) of a continuous real-valued random variable \( Z \) with positive density (regarded as a loss), denoted by \( \text{VaR}_\alpha(Z) \), is defined by \( P(Z \geq \text{VaR}_\alpha(Z)) = 1 - \alpha \) or equivalently \( P(Z \leq \text{VaR}_\alpha(Z)) = \alpha \). Moreover, the Expected Shortfall at level \( \alpha \in (0, 1) \) of a continuous real-valued random variable \( Z \) with positive density (regarded as a loss), denoted by \( \text{ES}_\alpha(Z) \), is defined by \( \text{ES}_\alpha(Z) = \mathbb{E}[Z | Z \geq \text{VaR}_\alpha(Z)] \).

Therefore, the Value-at-Risk at level \( \alpha \in (0, 1) \) associated with the loss \(-y'X\) - hereafter denoted by \( \text{VaR}_\alpha(-y'X) \) - is characterized by the relationship

\[
\sum_{i=1}^{N} p_i F_{t_{\nu_i}} \left( \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right) = \alpha.
\]

The associated Expected Shortfall at level \( \alpha \), which is the expected loss given that the loss exceeds \( \text{VaR}_\alpha(-y'X) \), is then

\[
\text{ES}_\alpha(-y'X) = \frac{1}{1 - \alpha} \sum_{i=1}^{N} \frac{p_i}{\sqrt{y'\Lambda_i y}} \int_{\text{VaR}_\alpha(-y'X)}^{+\infty} z f_{t_{\nu_i}} \left( \frac{z + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right) dz
\]

\[
= \frac{1}{1 - \alpha} \sum_{i=1}^{N} p_i \left\{ \int_{\text{VaR}_\alpha(-y'X)}^{+\infty} \left( u \sqrt{y'\Lambda_i y} - y'\mu_i \right) f_{t_{\nu_i}}(u) \, du \right\}
\]

\[
= \frac{1}{1 - \alpha} \sum_{i=1}^{N} p_i \left\{ \sqrt{y'\Lambda_i y} \left( \int_{\text{VaR}_\alpha(-y'X)}^{+\infty} \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \, du \right) - y'\mu_i \left( \int_{-\infty}^{\text{VaR}_\alpha(-y'X)} \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \, du \right) \right\}
\]

\[
= \frac{1}{1 - \alpha} \sum_{i=1}^{N} p_i \left\{ \sqrt{y'\Lambda_i y} \left( \nu_i + \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right)^2 \right\} - y'\mu_i \left( \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right)
\]

\[
- y'\mu_i F_{t_{\nu_i}} \left( \frac{\text{VaR}_\alpha(-y'X) + y'\mu_i}{\sqrt{y'\Lambda_i y}} \right)
\]
where we used the identity $\int_{t}^{+\infty} u f_{t_{u}}(u)du = (\nu + t^2)f_{t_{u}}(t)/(\nu - 1)$, obtained by a direct calculation.

Given that $\text{VaR}_\alpha(-y’X)$ can easily be computed with a root-solving algorithm, the above expression can be regarded as semi-analytic. In particular, following Theorem 1 a simple gradient descent procedure can be used to compute Risk Budgeting portfolios in this framework.

### 3.3 Towards a stochastic optimization problem

Although some parametric models as in Section 3.2 could be used, they can fall short of being a good representation of asset returns. For that reason, more complex models might be preferred despite the lack of a semi-analytic expression for Expected Shortfall. For example, in order to capture joint tails, [18] proposed some Pareto distributions for the left tail of individual asset returns and a Vine copula to model the dependence structure. For such a setting, in order to use a gradient descent procedure, one needs to estimate the Expected Shortfall term at each step of the optimization algorithm. This typically requires a computer-intensive approach as Monte Carlo simulations are usually necessary to estimate Value-at-Risk and then Expected Shortfall.

An alternative route consists in estimating Expected Shortfall and computing the solution of the Risk Budgeting problem at the same time. This route is based on the variational characterization of Expected Shortfall known as the Rockafellar-Uryasev formula (see [34]):

$$ES_\alpha(Z) = \inf_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{1 - \alpha} \mathbb{E}[ (Z - \zeta_+) ] \right\},$$

for any continuous real-valued random variable $Z \in L^1(\mathbb{R})$ with positive density (regarded as a loss). Moreover, the infimum in the above formula is in fact a minimum and the minimizer is $\text{VaR}_\alpha(Z)$.

Using Rockafellar-Uryasev formula, the function $\Gamma_{id}$ in Theorem 1 is written

$$\Gamma_{id} : y \mapsto \min_{\zeta \in \mathbb{R}} \mathbb{E} \left[ \zeta + \frac{1}{1 - \alpha} (-y’X - \zeta)_+ \right] - \sum_{i=1}^{d} b_i \log y_i.$$

Therefore, solving RB$_b$ boils down to solving the stochastic optimization problem

$$\min_{(y, \zeta) \in (\mathbb{R}^d)^2 \times \mathbb{R}} \mathbb{E} \left[ \zeta + \frac{1}{1 - \alpha} (-y’X - \zeta)_+ \right] - \sum_{i=1}^{d} b_i \log y_i. \quad (1)$$

More precisely, once the solution $(y^*, \zeta^*)$ of the stochastic optimization problem has been found, typically using a stochastic gradient descent, we simply need to normalize $y^*$ and $\theta := y^*/\sum_{i=1}^{d} y_i^*$ solves RB$_b$.

By using the above variational characterization, we have seen that solving the Risk Budgeting problem with Expected Shortfall is reduced to solving a stochastic optimization problem. It must be noted that this approach can be applied to a large range of asset return distributions. We only require that such distributions have a finite mean and a positive density w.r.t. the Lebesgue measure. This approach is therefore, somehow, universal in terms of asset return distributions.

Is this approach specific to Expected Shortfall? We tackle this question in the next section and show that similar ideas can be used for a large set of risk measures.
4 Extension to other RB-compatible risk measures

4.1 Introduction and preliminary remarks

Consider a RB-compatible risk measure $\rho$. The ideas outlined in Section 3.3 formally apply when there exists a continuously differentiable, convex and increasing function $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$g(\mathbb{R}(y)) = g(\rho(-y'X)) = \min_{\zeta \in \mathcal{Z}} \mathbb{E}[H(\zeta, -y'X)]$$

for some set $\mathcal{Z}$ and some function $H$. Indeed, in such a case, the RB$_b$ problem boils down to the stochastic optimization problem

$$\min_{y \in (\mathbb{R}_+^*)^d, \zeta \in \mathcal{Z}} \mathbb{E}[H(\zeta, -y'X) - \sum_{i=1}^d b_i \log y_i].$$

Like in Section 3.3 if $(y^*, \zeta^*)$ is a solution of the above stochastic optimization problem, then $\theta := \frac{y^*}{\sum_{i} y_i}$ solves RB$_b$.

It is noteworthy that the theorems of Section 3 involve a function $g$. In the case of Expected Shortfall, we used the identity function, i.e. $g(x) = x$, and that case deserves several remarks.

Risk measures that can be characterized as minimizers have significantly attracted the attention of the academic literature (see [19] and the concept of elicitability). In spite of their attractiveness for optimization problems, risk measures characterized by minima have been less studied. An important paper dealing with risk measures in the form of minima is [31] in which the authors introduce a general framework involving the famous quadrangles made of error, deviation, regret and risk. In their setting, risk measures are minima and even minima of expected values in the case of expectation quadrangles. In their interesting paper [16], Embrechts and his coauthors introduced the notions of Bayes pair and Bayes risk measure which are related to our problem when $g = \text{Id}$: a pair of risk measures $(\eta, \rho)$ is called a Bayes pair if there exists a measurable function $G : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\eta(Z) = \arg\min_{\zeta \in \mathbb{R}} \mathbb{E}[G(\zeta, Z)] \quad \text{and} \quad \rho(Z) = \min_{\zeta \in \mathbb{R}} \mathbb{E}[G(\zeta, Z)].$$

In addition, if $\eta$ satisfies that $\eta(Z + c) = \eta(Z) + c$ for any scalar $c$, then $\rho$ is called a Bayes risk measure. In particular, they show that convex combinations of Expected Shortfall and expectation (called ES/E mixtures) constitute the unique class of Bayes risk measures that are also coherent risk measures.

In fact, if $\rho$ is a RB-compatible risk measure such that $\mathcal{R}_{\rho, X}(y) = \min_{\zeta \in \mathcal{Z}} \mathbb{E}[H(\zeta, -y'X)]$ for some map $H$, then, for $\beta \in \mathbb{R}_+$ and $\delta \in \mathbb{R}$, $\tilde{\rho}$ defined by $\tilde{\rho}(Z) := \beta \rho(Z) + \delta E[Z]$ is a RB-compatible risk measure too, and $\mathcal{R}_{\tilde{\rho}, X}(y) = \min_{\zeta \in \mathcal{Z}} \mathbb{E}[\beta H(\zeta, -y'X) - \delta y'X]$. In particular, our ideas apply to linear combinations of Expected Shortfall and expectation terms (with positive coefficient for the Expected Shortfall terms). Linear combinations of Expected Shortfall and expectation are in fact quite common in the literature. As we discussed, they appear in the recent literature on Bayes pairs. They also appear for instance in the context of risk measures derived from the notion of optimized certainty equivalent (a utility function-based decision theoretic criterion that

\footnote{Of course, we must choose a risk measure that is positive for all long-only portfolios, otherwise our Risk Budgeting problem does not make sense.}
was first introduced by [5]: for a large class of utility functions $u$, [6] introduced sub-additive risk measures of the form
\[ \rho : Z \mapsto \inf_{\zeta \in \mathbb{R}} \left\{ \zeta - \mathbb{E}[u(-Z + \zeta)] \right\}. \]
In particular, if $u(\xi) = a\xi_+ - b\xi_-$ with $0 \leq a < 1 < b$, we get
\[ \rho(Z) = \inf_{\zeta \in \mathbb{R}} \left\{ \zeta - \mathbb{E}[a(-Z + \zeta)_+ - b(-Z + \zeta)_-] \right\} = a\mathbb{E}[Z] + (1-a)\mathbb{E}_{\frac{-1}{b-a}}(Z), \]
which is indeed a linear combination (with positive coefficients) of an expectation and an Expected Shortfall.

Another linear combination of Expected Shortfall and expectation appears when one wants to factor out expectation from Expected Shortfall (think of the initial motivations behind risk-based methods – see Section 1). In that case, we obtain the positive risk measure given by
\[ \rho(Z) = \mathbb{E}_{\alpha}(Z) - \mathbb{E}[Z] = \min_{\zeta \in \mathbb{R}} \mathbb{E} \left[ \zeta - Z + \frac{1}{1-\alpha}(Z - \zeta)_+ \right] = \min_{\zeta \in \mathbb{R}} \mathbb{E} \left[ \frac{\alpha}{1-\alpha}(Z - \zeta)_+ + (Z - \zeta)_- \right]. \]
The above discussion was bound to the specific case where $g = \text{Id}$. Freedom in the choice of $g$ is however important. As an example, if $g(x) = x^2$, $Z = \mathbb{R}$ and $H(\zeta, Z) = (Z - \zeta)^2$, we indeed have, in the case where volatility is the risk measure, that
\[ g(\mathcal{R}(y)) = \min_{\zeta \in \mathbb{Z}} \mathbb{E}[H(\zeta, -y'X)]. \]
In other words, the Risk Budgeting problem with volatility as the risk measure can be solved using our new approach.\(^9\)

The natural question is now to evaluate the range of RB-compatible risk measures that can be seen as minima of some expected criteria. As we will see, many RB-compatible risk measures can be written, for well chosen sets $Z$ and functions $H$, as $\min_{\zeta \in \mathbb{Z}} \mathbb{E}[H(\zeta, -y'X)]$. Classical generalizations of Expected Shortfall include spectral risk measures: they have such a representation. Moreover, classical extensions of volatility include a large class of deviation measures based on $L^p$ norms, that even includes recent risk measures like variantiles (see [43]): we will see they have such a representation too.

### 4.2 Spectral risk measures

Spectral risk measures are well-adopted risk measures that allow to amplify (or reduce) the impact of greater losses through a distortion function, as defined in [1]. Formally, they are defined as
\[ \rho_h(Z) := \int_0^1 \text{VaR}_s(Z) h(s) \, ds, \]
for some functions $h : [0, 1] \to \mathbb{R}_+$ that are non-decreasing, right-continuous and satisfy $\int_0^1 h(s) \, ds = 1$ with $h(0) = 0$.

Expected Shortfall is a particular spectral risk measure since, for any continuous and $L^1$ random variable $Z$ with positive density, we have
\[ \text{ES}_\alpha(Z) = \frac{1}{1-\alpha} \int_0^1 \text{VaR}_s(Z) \, ds = \int_0^1 \text{VaR}_s(Z) \frac{1}{1-\alpha} 1\{s \in (\alpha, 1)\} \, ds. \]
\(^9\)Solving the problem by gradient descent is of course not the best numerical approach in this case.
In other words, Expected Shortfall corresponds to setting \( h(s) = \frac{1}{1-s} \mathbf{1} \{ s \in (\alpha,1) \} \): it takes into account Value-at-Risks above a fixed threshold uniformly and fully dismisses the others. Spectral risk measures extend Expected Shortfall in that they can assign a variety of weights to the different loss quantiles.

For a continuous and \( L^1 \) random variable \( Z \) with positive density, note that

\[
\rho_h(Z) = \int_0^1 \text{VaR}_s(Z) h(s) \, ds = -\int_0^1 \frac{d}{ds} \left( (1-s) \text{ES}_s(Z) \right) h(s) \, ds
\]

\[
= \int_0^1 (1-s) \text{ES}_s(Z) \, dh(s) - \left[ h(s)(1-s) \text{ES}_s(Z) \right]_0^1.
\]

where we use Riemann–Stieltjes integral. As \( \lim_{s \to 0^+} h(s)(1-s) \text{ES}_s(Z) = h(0) \mathbb{E}[Z] = 0 \) and \( \lim_{s \to 1^-} h(s)(1-s) \text{ES}_s(Z) = h(1) \lim_{s \to 1^-} \mathbb{E}[Z \mathbb{1} \{ Z \geq \text{VaR}_s(Z) \}] = 0 \) because \( Z \in L^1 \), the bracket term vanishes and we get

\[
\rho_h(Z) = \int_0^1 (1-s) \min_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{1-s} \mathbb{E}[(Z - \zeta)_+] \right\} \, dh(s)
\]

\[
= \min_{\zeta(s) \in Z} \int_0^1 (1-s) \left\{ \zeta(s) + \frac{1}{1-s} \mathbb{E}[(Z - \zeta(s))_+] \right\} \, dh(s)
\]

\[
= \min_{\zeta(s) \in Z} \mathbb{E} \left[ \int_0^1 \left\{ (1-s) \zeta(s) + (Z - \zeta(s))_+ \right\} \, dh(s) \right],
\]

where \( Z \) is the set of measurable functions on \( (0,1) \).

Therefore, solving Risk Budgeting problems for spectral risk measures boils down to solving stochastic optimization problems, but in infinite dimension. In practice, we need to discretize the integrals to get a finite-dimensional stochastic optimization problem. These approximate solutions correspond to risk measures that are linear combinations (with positive coefficients) of a finite number of Expected Shortfall terms for different risk levels.

As in the case of Expected Shortfall, it is possible to factor out expected returns from spectral risk measures. Noticing that \( \int_0^1 (1-s) \, dh(s) = 1 \), we obtain the positive risk measure

\[
\rho_h(Z) - \mathbb{E}[Z] = \min_{\zeta(s) \in Z} \mathbb{E} \left[ \int_0^1 \left\{ (1-s)(\zeta(s) - Z) + (Z - \zeta(s))_+ \right\} \, dh(s) \right]
\]

\[
= \min_{\zeta(s) \in Z} \mathbb{E} \left[ \int_0^1 \left\{ s(Z - \zeta(s))_+ + (1-s)(Z - \zeta(s))_- \right\} \, dh(s) \right].
\]

4.3 Deviation measures

Deviation measures are ubiquitous in statistics and applied mathematics. See [32, 33, 31] for a discussion on deviation measures in risk management particularly. Standard deviation is surely the most popular member of this class, and it is commonly used in finance. Indeed, being the square root of a quadratic form, standard deviation is appealing for introducing the risk dimension in portfolio optimization problems. It is however symmetrical because the volatility of \( X \) is the same as the volatility of \( -X \), while there is a great benefit in considering gains and losses asymmetrically in finance. In what follows, we propose a large class of RB-compatible risk measures that contains both symmetrical and asymmetrical deviation measures and for which our ideas of stochastic optimization (see Section 3.3) apply.
**Proposition 3.** Let \( a, b \in \mathbb{R}^*_+ \) and let the function \( \psi_{a,b} : \mathbb{R} \to \mathbb{R}^+ \) be defined by
\[
\psi_{a,b}(z) = az_+ + bz_-.
\]
Set \( p \in [1, +\infty) \). Let \( F \) be a finite-dimensional subspace of \( L^p(\Omega, \mathbb{R}) \) and \( \rho : L^p(\Omega, \mathbb{R}) \to \mathbb{R} \) be defined by
\[
\rho(Z) = \inf_{f \in F} \mathbb{E}[\psi_{a,b}(Z-f)^p]^{\frac{1}{p}}.
\]
Then, \( \rho \) is a RB-compatible risk measure and the infimum in the definition of \( \rho \) is in fact a minimum.

**Proof.** For \( \lambda > 0 \), we have
\[
\rho(\lambda Z) = \inf_{f \in F} \mathbb{E}
\]
\[
\left[\psi_{a,b}((Z-f)^p)\right]^{\frac{1}{p}} = \inf_{f \in F} \mathbb{E}
\]
\[
\left[\lambda^p \psi_{a,b}\left(Z - \frac{f}{\lambda}\right)^p\right]^{\frac{1}{p}}
\]
\[
= \lambda \inf_{f \in F} \mathbb{E}
\]
\[
\left[\psi_{a,b}\left(Z - \frac{f}{\lambda}\right)^p\right]^{\frac{1}{p}} = \lambda \inf_{f \in F} \mathbb{E}[\psi_{a,b}(Z-f)^p]^{\frac{1}{p}} = \lambda \rho(Z).
\]
Since \( \rho(0) = \inf_{f \in F} \mathbb{E}[\psi_{a,b}(-f)^p]^{\frac{1}{p}} = 0 \), \( \rho \) is positively homogeneous.

Coming to sub-additivity, it is clear that \( \psi_{a,b} \) is sub-additive. Thus, for all \( Z_1, Z_2 \in L^p(\Omega, \mathbb{R}) \), we have
\[
\rho(Z_1 + Z_2) = \inf_{f \in F} \mathbb{E}[\psi_{a,b}(Z_1 + Z_2 - f)^p]^{\frac{1}{p}} = \inf_{f_1, f_2 \in F} \mathbb{E}[\psi_{a,b}(Z_1 + Z_2 - f_1 - f_2)^p]^{\frac{1}{p}}
\]
\[
\leq \inf_{f_1, f_2 \in F} \mathbb{E}[\psi_{a,b}(Z_1 - f_1 + Z_2 - f_2)]^{\frac{1}{p}}
\]
\[
\leq \inf_{f_1, f_2 \in F} \mathbb{E}[\psi_{a,b}(Z_1 - f_1)]^{\frac{1}{p}} + \mathbb{E}[\psi_{a,b}(Z_2 - f_2)]^{\frac{1}{p}}
\]
\[
\leq \rho(Z_1) + \rho(Z_2),
\]
where we used the triangular inequality for the \( L^p \) norm.

Let us now consider a sequence \((f_n)_n\) of maps in \( F \) such that \( \mathbb{E}[\psi_{a,b}(Z-f_n)^p]^{\frac{1}{p}} \leq \rho(Z) + \frac{1}{n+1} \). We have
\[
\mathbb{E}[f_n^p]^{\frac{1}{p}} \leq \mathbb{E}[Z^p]^{\frac{1}{p}} + \mathbb{E}[|Z-f_n|^p]^{\frac{1}{p}} \leq \mathbb{E}[Z^p]^{\frac{1}{p}} + \frac{1}{\min(a,b)} \mathbb{E}[\psi_{a,b}(Z-f_n)^p]^{\frac{1}{p}}
\]
\[
\leq \mathbb{E}[Z^p]^{\frac{1}{p}} + \frac{1}{\min(a,b)} (\rho(Z) + 1)
\]
and therefore \((f_n)_n\) is bounded in \( F \). Up to a subsequence, it converges therefore, for the \( L^p \) norm, towards a random variable \( f \) in \( F \) and we have
\[
\mathbb{E}[\psi_{a,b}(Z-f)^p]^{\frac{1}{p}} \leq \mathbb{E}
\]
\[
\left[\psi_{a,b}(Z-f_n) + \max(a,b)|f_n-f|^p\right]^{\frac{1}{p}}
\]
\[
\leq \mathbb{E}[\psi_{a,b}(Z-f_n)^p]^{\frac{1}{p}} + \max(a,b)\mathbb{E}[|f_n-f|^p]^{\frac{1}{p}}
\]
\[
\leq \rho(Z) + \frac{1}{n+1} + \max(a,b)\mathbb{E}[|f_n-f|^p]^{\frac{1}{p}}.
\]
Sending \( n \) to \(+\infty\), we see that \( f \) is a minimizer and that the infimum in the definition of \( \rho \) is indeed a minimum. \( \square \)
Since \( \rho(Z + c) = \rho(Z) \) for all \( Z \in L^p(\Omega, \mathbb{R}) \) and any constant \( c \), the latter risk measures \( \rho \) are called deviation measures – see [35]. They are compatible with our framework for \( g(x) = x^p \).

Interestingly, the above family contains some familiar risk measures for particular choices of \( F \), \( a \), \( b \) and \( p \):

(a) when \( a = b \), we get symmetrical measures as
- standard deviation when \( F = \text{span}(1) \), \( a = b = 1 \) and \( p = 2 \);
- median absolute deviation (MAD), i.e. \( \mathbb{E}[|Z - \text{median}(Z)|] \), when \( F = \text{span}(1) \), \( a = b = 1 \) and \( p = 1 \) (the minimum is reached for the median of the portfolio losses).

(b) When \( a \neq b \), we retrieve some asymmetrical measures, for instance
- Expected Shortfall at level \( \alpha \) minus expectation, i.e. \( \text{ES}_\alpha(Z) - \mathbb{E}[Z] \), for \( \alpha \in (0, 1) \), when \( F = \text{span}(1) \), \( a = \alpha/(1 - \alpha) \), \( b = 1 \) and \( p = 1 \) (the minimum is reached for \( \text{VaR}_\alpha(Z) \));
- the square root of the variate at level \( \alpha \) (see [43]) when \( F = \text{span}(1) \), \( a = \sqrt{\alpha} \), \( b = \sqrt{1 - \alpha} \) and \( p = 2 \) (the minimum is reached for the expectile at level \( \alpha \)).

The above examples show the relevance of this family of risk measures, which are particular cases of deviation measures. When the goal is to focus on heavy tail risks, it should be relevant to impose \( p > 2 \). Extensions beyond the space \( F \) of constant random variables can also be considered to focus on residual risk (when \( F \) is spanned by factors, in the same spirit as [33]).

## 5 Numerical results

### 5.1 Convergence of the stochastic gradient descent algorithm

In this section, we want to illustrate our results and ideas in a simple case. We consider a set of 4 assets and construct an ERC portfolio \(^{10}\) for Expected Shortfall at a confidence level \( \alpha = 95\% \). Our goal is to compare the performances of a SGD algorithm with that of a standard procedure \(^{11}\).

For what follows, we assume the joint distribution of our asset returns is given by a mixture of two multivariate Student-t distributions. More precisely, we assume that \( X \) has the density w.r.t. the Lebesgue measure

\[
f_X(x) := p f(x|\mu_1, \Lambda_1, \nu_1) + (1 - p) f(x|\mu_2, \Lambda_2, \nu_2).
\]

To work with a realistic model, it has been calibrated using daily returns of Apple Inc. (AAPL), JPMorgan Chase & Co. (JPM), Pfizer Inc. (PFE) and Exxon Mobil Corporation (XOM) over the period August 2008–April 2022. Our model parameters are then estimated using the expectation-maximization algorithm and rounded for the purpose of illustration. Fixing the degrees of freedom at \( \nu_1 = 4.0 \) and \( \nu_2 = 2.5 \) a priori, we obtained that the weight is \( p = 0.7 \), the location vectors are \( \mu_1 = (0.001, 0.001, 0.001, 0.003)' \) and \( \mu_2 = (-0.001, -0.002, -0.001, -0.002)' \), and the scale matrices are

\[
\Lambda_1 = \begin{bmatrix} 0.00010 & 0.00005 & 0.00002 & 0.00003 \\ 0.00005 & 0.00010 & 0.00002 & 0.00002 \\ 0.00002 & 0.00002 & 0.00010 & 0.00002 \\ 0.00003 & 0.00002 & 0.00002 & 0.00010 \end{bmatrix} \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} 0.00040 & 0.00010 & 0.00010 & 0.00020 \\ 0.00010 & 0.00010 & 0.00008 & 0.00009 \\ 0.00010 & 0.00008 & 0.00010 & 0.00007 \\ 0.00020 & 0.00009 & 0.00007 & 0.00020 \end{bmatrix}.
\]

\(^{10}\)This corresponds to setting risk budgets to \( b = (0.25, 0.25, 0.25, 0.25) \).

\(^{11}\)Although we call our method a SGD method, we use a mini-batch implementation because of its computational advantage compared to vanilla stochastic gradient descent where one considers a single data point at each iteration.
Since Expected Shortfall has a semi-analytic form in our multivariate Student-t mixture model (Section 3.2), we can easily compute the Risk Budgeting portfolio using the L-BFGS-B algorithm. The resulting portfolio $\theta$ is given in Table 1. We confirm that $\theta$ solves $RB_b$ by noticing that the risk contributions $\theta_i \partial_i R(\theta)$ are the same for all assets. This portfolio constitutes a reliable reference to evaluate the convergence and the accuracy of alternative (stochastic) methods. It is referred to as the reference portfolio in what follows.

| Asset | $\theta_i$ | $\theta_i \partial_i R(\theta)$ |
|-------|------------|-------------------------------|
| 1     | 0.17958    | 0.00806                       |
| 2     | 0.28127    | 0.00806                       |
| 3     | 0.30483    | 0.00806                       |
| 4     | 0.23432    | 0.00806                       |

Table 1: Reference portfolio weights and risk contributions.

Let us now come to the use of SGD methods to compute Risk Budgeting portfolios. In order to solve the stochastic optimization problem presented in Section 3.3 using SGD, we require sample points for asset returns. In what follows, we draw $10^6$ sample points with the multivariate Student-t mixture distribution of the random variable $X$. We use a mini-batch size of 128 and 10 epochs. Figure 1 illustrates the dynamics of the $(y, \zeta)$ pair in (1) and that of the associated $\theta_{SGD}$ through the iterations.

The final estimator $y^{SGD}_*$ is computed using a standard Polyak-Ruppert averaging that relies on the last 20% of all iterations. We always use this variance reduction technique for the SGD method throughout the paper. The resulting portfolio $\theta^{SGD}_*$ and its deviation from the reference portfolio are given in Table 2. We clearly see that $\theta^{SGD}_*$ is very close to the reference portfolio ($\|\theta - \theta^{SGD}_*\|_1 = 0.00076$).

We stop the algorithm when the infinity norm of a projected gradient vector is less than $10^{-6}$.

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12We stop the algorithm when the infinity norm of a projected gradient vector is less than $10^{-6}$. 

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| Asset | $\theta_i^{SGD}$ | $|\theta_i - \theta_i^{SGD}|$ |
|-------|-----------------|------------------|
| 1     | 0.17954         | 0.00005          |
| 2     | 0.28165         | 0.00038          |
| 3     | 0.30449         | 0.00034          |
| 4     | 0.23432         | 0.00001          |

Table 2: Risk Budgeting portfolio weights obtained by using the SGD method with a Polyak-Ruppert averaging.

5.2 Speed and accuracy of various methods for different portfolio sizes

Semi-analytic formulas for Expected Shortfall are only available in very specific cases. In general, the approximated computation of an Expected Shortfall without using the Rockafellar-Uryasev formula requires a sample of returns $X = \{x_1, \ldots, x_n\}$. Such returns are observed historically or may be simulated based on a model. Then, invoke the usual empirical estimator

$$\hat{\mathbb{E}}S_\alpha(y, X) = \frac{\sum_{i=1}^n (-y'x_i) \mathbb{1}\{-y'x_i \geq \hat{q}_\alpha\}}{\sum_{i=1}^n \mathbb{1}\{-y'x_i \geq \hat{q}_\alpha\}},$$

where $\hat{q}_\alpha$ is the empirical Value-at-Risk of level $\alpha$, i.e. it is the empirical $\alpha$-quantile associated with the set of portfolio losses $\{-y'x_1, \ldots, -y'x_n\}$.

In this case, the computation of the Risk Budgeting portfolio corresponds to solving the optimization problem given by

$$\min_{y \in (\mathbb{R}^*_+)^d} \left\{ \hat{\mathbb{E}}S_\alpha(y, X) - \sum_{i=1}^d b_i \log y_i \right\}, \tag{2}$$

and normalizing the minimizer.

To solve Problem (2), one can use a standard gradient descent approach where, at each iteration, the gradient is approximated using finite differences, i.e. $\partial_y \hat{\mathbb{E}}S_\alpha(-y'X)$ is approximated by

$$\frac{\hat{\mathbb{E}}S_\alpha(y + he_i, X) - \hat{\mathbb{E}}S_\alpha(y, X)}{h},$$

where $e_i$ denotes the $i^{th}$ vector of the canonical basis of $\mathbb{R}^d$ and $h$ is a small tuning parameter. Hereafter, we set $h = 10^{-4}$.

Two methods can be derived from the above remarks to compute Risk Budgeting portfolios for Expected Shortfall. The first one corresponds to using the same sample of asset returns throughout the optimization process to estimate these risk measures. It is the only option (besides the SGD method) when working with historical samples in a model-free setting. This method will be called the one-sample benchmark gradient descent (OSBGD) approach. Such ideas can also be used in a model-based setting where one can run the algorithm on a simulated sample. This alternative way of working requires a model for $X$ and is based on simulating a new sample of asset returns at each iteration to estimate the new Expected Shortfall. We call the latter method the multi-sample benchmark gradient descent (MSBGD) approach. The

\textsuperscript{13}There exist several ways of computing empirical quantiles. We shall use the default method of Python NumPy package which corresponds to the method 7 in \cite{21}.
advantage of this second method is that it allows to manage smaller sample sizes and thus faster calculations at each iteration, while allowing for comparable – or in fact higher – accuracy in the long run.

Our implementation of the OSBGD method is a gradient descent procedure which is based on the Barzilai-Borwein methodology: at each step, the gradient is calculated using finite differences over a fixed sample and the step size is determined by a cheap approximation of the Hessian (see [4]). The stopping rule is based on the difference between two consecutive values of the objective function in Problem (2) and the algorithm is stopped if that difference drops below $10^{-6}$. The MSBGD method is very similar to the above procedure, but the gradient is calculated using a new sample drawn from the chosen model of asset returns at each iteration. The algorithm is stopped after a fixed number of iterations rather than using a stopping rule because of the stochasticity of gradient approximations. The final estimator is computed by averaging the last iterations.

This section provides results about accuracy and speed of the SGD, OSBGD and MSBGD methods in model-free and model-based settings. Our accuracy measure is the Manhattan / $\| \cdot \|_1$ distance between the reference portfolio and the portfolio obtained by the optimization method.

We are interested in building ERC portfolios of $d \in \{10, 20, 50, 100, 200, 350\}$ assets for Expected Shortfall at the confidence level $\alpha = 95\%$. In practice, as we are never in possession of neither the true distribution of asset returns nor the reference portfolio, we cannot measure the accuracy of the portfolio computed by the presented methods. To conduct our empirical analysis, we adopt a simulation-based approach. We define a data generating process (DGP) and assume that it reflects the true distribution of asset returns. We call it $DGP_{true}$. We then draw $n = 3500$ data points from $DGP_{true}$ to generate a synthetic historical sample $X_{hist}$. The use of $X_{hist}$ to compute Risk Budgeting portfolios is twofold: we can follow the model-free approach, where we can run the SGD and OSBGD methods using $X_{hist}$, or, alternatively, the model-based approach, where we can fit a model on $X_{hist}$ and proceed with simulated samples $X_{sim}$ drawn from the estimated model. The latter choice allows to use all three methods (SGD, OSBGD and MSBGD). A detailed description of all these procedures is given in Appendix A.

Start with the model-free approach and compute Risk Budgeting portfolios using the SGD and OSBGD methods using $X_{hist}$. For the SGD method, we use a mini-batch size of 128 and stop it after 100 epochs. Table 3 documents the accuracy and computation time of both methods for different portfolio sizes.

| $d$ | Accuracy | Time |
|-----|----------|------|
|     | SGD      | OSBGD| SGD | OSBGD |
| 10  | 5.46 (1.63) | 5.47 (1.65) | 1.08 (0.01) | 0.05 (0.02) |
| 20  | 6.63 (1.82) | 6.63 (1.78) | 1.18 (0.01) | 0.14 (0.09) |
| 50  | 7.26 (1.64) | 7.28 (1.68) | 1.32 (0.01) | 0.34 (0.12) |
| 100 | 7.67 (1.06) | 7.69 (1.05) | 1.52 (0.01) | 0.76 (0.30) |
| 200 | 7.60 (1.42) | 7.60 (1.41) | 1.91 (0.02) | 1.32 (0.43) |
| 350 | 7.83 (1.73) | 7.73 (1.62) | 2.52 (0.01) | 2.53 (1.09) |

The sample size $n$ is chosen so as to represent the typical size of historical samples in the equity world where we often deal with a maximum of 10–15 years of daily return data.
Table 3: Accuracy of the Risk Budgeting portfolios obtained by the SGD and OSBGD methods for different numbers of assets under historical samples and computation time of algorithms (in seconds). The accuracy measure corresponds to $100\|\theta - \theta^{\text{method}}\|_1$. Figures correspond to means and standard deviations (in parentheses) computed by repeating the process $m = 50$ times with $X_{\text{hist}}$ drawn from $m$ different DGP$_{\text{true}}$ for each $d$.

Table 3 shows that one can get reasonably close to the true Risk Budgeting portfolio up to a certain level with a limited amount of historical data. We observe that both methods yield very similar results in terms of accuracy. It is intuitive that both methods produce similar results after an almost complete process of the information contained in the same inputs. In terms of computation time, the OSBGD method is efficient especially when constructing portfolios with a small number of assets. The advantage of OSBGD over SGD however disappears as the portfolio dimension $d$ grows.

The alternative to the model-free approach is to follow the model-based approach. It uses $X_{\text{hist}}$ to evaluate a model that is believed to reflect the true behavior of asset returns. Then, such a model allows us to draw large simulated samples without being restricted by the size of the historical sample. The primary risk of this approach is the mis-specification of the true distribution of asset returns. We therefore want to consider three cases. The first one corresponds to a situation where the estimated model perfectly matches the true distribution of asset returns – DGP$_{\text{true}}$. This case is not realistic but worth to analyze. Of course, we expect to obtain results very close to the reference portfolios. In the second case, we correctly specify the family of the true distribution of $X$: we assume that DGP$_{\text{true}}$ is really a mixture of two multivariate Student-t distributions. Naturally, in our case, well-specifying the parametric family of the $X$ distribution corresponds to the case where we fit a mixture of two multivariate Student-t distributions to $X_{\text{hist}}$. This yields an estimated model DGP$_{\text{SM}}$ [15]. For the third method, we mis-specify the family of the $X$ distribution. In our case, we fit a mixture of two multivariate Gaussian distributions to $X_{\text{hist}}$ and obtain a “wrong” model DGP$_{\text{GM}}$. Obviously, any other parametric family can be assumed. The mis-specified case is here a Gaussian mixture so as not to excessively deviate from DGP$_{\text{true}}$.

Table 4 shows the results of the model-based approach. For the SGD and OSBGD methods, we run the algorithm using a (fixed) simulated sample $X_{\text{sim}}$ of size $10^6$. We use a mini-batch size of 128 and stop it after 4 epochs for the SGD method. For the MSBGD method, the size of the sample $X_{\text{sim}}$ – that is repeatedly drawn over again at each iteration – is chosen to be $10^5$ and we stop the algorithm after 60 iterations. The final estimator is computed by averaging last 5 iterations.

In Table 4 we observe that the model-based approach can yield more accurate results than the model-free approach whose results are shown in Table 3. It is of course true in the case of a perfect specification of the distribution of asset returns. Interestingly, this is still true in the case of a correct specification of the family of the $X$ distribution. However, we see that it generally does not hold any longer in the case of a mis-specified model. We can conclude that model-based methods can challenge model-free methods if we are confident about our choice of the family of the true distribution of asset returns. Another important result concerns the computation time of methods. The MSBGD method is faster than the OSBGD method since the gradient can be computed much faster when $X_{\text{sim}}$ is of size $10^5$ compared to the case of OSBGD where it is of size $10^6$.

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[15] We use the expectation-maximization algorithm with fixed degrees of freedom: $\nu_1 = 4.0$ and $\nu_2 = 2.5$.

[16] No substantial improvement is observed after 4 epochs and 60 iterations for the SGD and MSBGD methods respectively. The sample size used in the MSBGD method is chosen to be $10^5$ because, using smaller sample sizes like $10^3$ and $10^4$, we cannot get accurate portfolios due to very poor quality of gradient approximation. On the other hand, a larger sample size like $10^6$ excessively slows down the process for a negligible improvement in terms of accuracy.
10⁶. However, its performance in terms of speed is also far from being close to the performance of the SGD method especially for high dimensional portfolios. The results show that SGD is the fastest among the three methods to obtain accurate portfolios independently of the number of assets if we adopt the model-based approach.

|                  |                  |                  |                  |                  |
|------------------|------------------|------------------|------------------|------------------|
|                  | d    | SGD (Std) | OSBGD (Std) | MSBGD (Std) | SGD (Std) | OSBGD (Std) | MSBGD (Std) |
| Well-specified   | 10   | 0.37 (0.10)| 0.35 (0.10)| 0.34 (0.08)| 3.42 (0.12)| 12.82 (2.43)| 9.64 (0.35)  |
| (Parameters)     | 20   | 0.42 (0.09)| 0.41 (0.12)| 0.37 (0.12)| 3.75 (0.06)| 26.00 (4.64)| 16.49 (0.89) |
|                  | 50   | 0.52 (0.12)| 0.49 (0.12)| 0.38 (0.10)| 4.46 (0.09)| 66.10 (13.01)| 32.77 (0.52) |
|                  | 100  | 0.51 (0.04)| 0.49 (0.05)| 0.40 (0.12)| 5.58 (0.06)| 141.21 (23.62)| 62.27 (1.19) |
|                  | 200  | 0.53 (0.08)| 0.53 (0.10)| 0.40 (0.05)| 9.05 (0.62)| 275.14 (52.35)| 123.96 (1.35) |
|                  | 350  | 0.62 (0.09)| 0.54 (0.09)| 0.41 (0.08)| 17.51 (0.76)| 522.43 (41.14)| 221.69 (1.04) |
| Well-specified   | 10   | 2.91 (1.30)| 2.91 (1.31)| 3.00 (1.36)| 3.69 (0.30)| 11.97 (2.43)| 9.18 (0.51)  |
| (Family)         | 20   | 3.22 (2.03)| 3.20 (2.02)| 3.23 (1.87)| 3.81 (0.12)| 24.97 (5.54)| 16.79 (0.59) |
|                  | 50   | 3.91 (1.63)| 3.93 (1.64)| 3.93 (1.74)| 4.41 (0.05)| 67.41 (11.98)| 32.00 (1.32) |
|                  | 100  | 3.25 (1.36)| 3.24 (1.36)| 3.23 (1.37)| 5.52 (0.04)| 130.35 (14.46)| 60.73 (1.34) |
|                  | 200  | 3.70 (1.62)| 3.65 (1.64)| 3.61 (1.57)| 8.77 (0.90)| 278.52 (44.32)| 118.22 (1.19) |
|                  | 350  | 4.25 (0.81)| 4.03 (0.75)| 3.97 (0.75)| 16.41 (0.52)| 588.99 (53.37)| 213.48 (1.02) |
| Mis-specified    | 10   | 8.38 (3.01)| 8.29 (3.01)| 8.45 (2.99)| 3.63 (0.29)| 14.66 (3.08)| 9.08 (0.28)  |
|                  | 20   | 8.01 (2.55)| 8.01 (2.54)| 8.01 (2.53)| 3.89 (0.17)| 29.98 (6.14)| 16.89 (0.68) |
|                  | 50   | 8.76 (2.75)| 8.77 (2.79)| 8.70 (2.78)| 4.44 (0.04)| 107.07 (80.33)| 32.53 (2.14) |
|                  | 100  | 8.55 (5.06)| 8.54 (5.07)| 8.49 (5.01)| 5.53 (0.10)| 156.53 (58.07)| 60.57 (1.48) |
|                  | 200  | 9.23 (3.94)| 9.22 (3.94)| 9.11 (3.74)| 8.85 (0.68)| 337.73 (94.27)| 118.50 (2.16) |
|                  | 350  | 10.45 (3.20)| 10.00 (2.99)| 10.01 (3.04)| 16.72 (0.65)| 637.16 (153.91)| 213.95 (1.08) |

Table 4: Accuracy of the Risk Budgeting portfolios obtained by the three methods for different numbers of assets for samples drawn from DGP_{True}, DGP_{SM} and DGP_{GM} and computation time of algorithms (in seconds). The accuracy measure corresponds to \(100\|\theta - \hat{\theta}_{\text{method}}\|_1\). Figures correspond to means and standard deviations (in parentheses) computed by repeating the process \(m = 50\) times with samples drawn from \(m\) different DGP_{True}, DGP_{SM} and DGP_{GM} for each \(d\).

5.3 Risk Budgeting for the allocation of negatively skewed assets

Choosing the right risk measure that correctly reflects the true risk of an investment is an important task in portfolio construction. In this section, we build Risk Budgeting portfolios using the SGD method and for the risk measures mentioned in Section 4. Our aim is to observe the impact of the choice of a risk measure on the Risk Budgeting portfolio when dealing with negatively skewed assets that exhibit jump risk. Indeed, negative skewness of asset returns distributions is generally associated with large negative jumps in asset prices. A reasonable modeling approach (see [9] for a similar model) is to assume a two-component mixture model. Each component represents a different state of the market, typically a “normal” state and a “stressed” state where the probability of a downward jump substantially increases.

Here, consider three assets and a mixture of two multivariate Gaussian distributions: the joint density of their returns \(X\) is given by

\[
f_X(x) = p\phi(x|\mu_1, \Sigma_1) + (1-p)\phi(x|\mu_2, \Sigma_2),
\]

where \(\phi(\mu, \Sigma)\) denotes the probability density function of a multivariate Gaussian distribution \(\mathcal{N}(\mu, \Sigma)\) and \(p\) is the probability of being in the “normal” market state. We consider the following parameters which are chosen carefully to be realistic. The vector of expected returns in the
normal market state is $\mu_1 = (0.02, 0.06, 0.10)'$. In the stressed market state, the expected returns of some assets dramatically decrease because $\mu_2 = (-0.15, -0.30, 0.10)'$. The covariance matrices in the two different states are

$$
\Sigma_1 = \begin{bmatrix}
0.0064 & 0.0080 & 0.0048 \\
0.0080 & 0.0400 & 0.0240 \\
0.0048 & 0.0240 & 0.0090
\end{bmatrix} \quad \text{and} \quad \Sigma_2 = \begin{bmatrix}
0.0289 & 0.0230 & 0.0048 \\
0.0230 & 0.0800 & 0.0240 \\
0.0048 & 0.0240 & 0.1000
\end{bmatrix}.
$$

We construct ERC portfolios of these three assets, choosing several risk measures: volatility $\sigma(Z)$, median absolute deviation $MAD(Z) = \min_{\zeta \in \mathbb{R}} E[|Z - \zeta|]$, Expected Shortfall $ES_{\alpha}$, spectral risk measure $S_h$ where the distortion function $h$ is a power function and quantile $\nu_{\alpha}(Z) = \min_{\zeta \in \mathbb{R}} E[\alpha(Z - \zeta)^2 + (1 - \alpha)(Z - \zeta)^2]$. We also examine the impact of including or excluding expected loss when relevant.

To be more explicit, $S_h$ is based on the distortion function $h(s) = s^{1/c-1/c}$ for some $c \in (0, 1]$. As $c$ gets close zero, $h(s)$ attributes more weight to larger $s$ values, making the risk measure more sensitive to extreme losses. In this section, we consider the case $c = 0.05$. This function $h$ might reflect the risk profile of an investor better than a step function – as in the case of Expected Shortfall – because it assigns increasing weights to larger losses in a smooth way.

Table 5 shows the estimated RB portfolios for two different values of $p \in \{0.8, 1\}$.

| risk measure | Asset 1 | Asset 2 | Asset 3 |
|--------------|---------|---------|---------|
| $p = 1$      | Volatility 0.60916 0.22200 0.16884  
MAD 0.60951 0.22185 0.16864  
$ES_{0.95} - E$ 0.60972 0.22190 0.16838  
$S_h - E$ 0.60969 0.22200 0.16831  
MAD + $E$ 0.58872 0.22046 0.19082  
$ES_{0.95}$ 0.60342 0.22168 0.17490  
$S_h$ 0.60252 0.22169 0.17579  
$\nu_{0.99}$ 0.59850 0.22138 0.18012 |
| $p = 0.8$    | Volatility 0.52700 0.22882 0.24418  
MAD 0.54790 0.22644 0.22566  
$ES_{0.95} - E$ 0.46458 0.22612 0.30929  
$S_h - E$ 0.47528 0.22727 0.29745  
MAD + $E$ 0.45476 0.20345 0.34180  
$ES_{0.95}$ 0.44055 0.21511 0.34434  
$S_h$ 0.44515 0.21510 0.33975  
$\nu_{0.99}$ 0.45719 0.21327 0.32954 |

Table 5: Risk Budgeting portfolios for different risk measures under the assumption of normal ($p = 1$) and negatively skewed (Gaussian mixture) asset returns ($p = 0.8$).

When $p = 1$, asset returns follow a multivariate Gaussian distribution and do not exhibit skewness. We obtain very similar Risk Budgeting portfolios for all the risk measures insensitive to expected loss, i.e. volatility, MAD, $ES_{0.95} - \ E$ and $S_h - \ E$, because these risk measures are proportional to one another in the Gaussian case. When expected loss comes into play, it slightly impacts the allocation. Overall, there is no apparent advantage to using risk measures over volatility in the absence of negative skewness.
When we introduce skewness by setting \( p = 0.8 \), we obtain significantly different Risk Budgeting portfolios. Volatility and MAD seem to capture part of the higher risk induced by the likelihood of observing a stressed market. However, symmetrical deviation measures are not ideal to deal with skewed asset returns since they do not account for the direction of the asymmetry. Adding expected loss to MAD, i.e. using the risk measure MAD + E, considerably impacts the allocation and tilts the weights in accordance with expected returns (and hence skewness). When we look at Expected Shortfall at \( \alpha = 0.95 \), we observe a larger impact of skewness compared to the two previous symmetrical deviation measures. Factoring out expected loss from Expected Shortfall \( \text{ES}_{0.95} - E \) does not significantly impact the portfolio allocation because such an expected loss is very small relative to the large losses that Expected Shortfall captures. Using spectral risk measures \( S_h \) and \( S_h - E \) yield portfolios which are similar to those obtained with Expected Shortfall. Similarly, the use of the extreme variantile \( \psi_{0.99} \) allows to capture skewness risk.

**Conclusion**

In this paper, we provide an analysis of the Risk Budgeting problem. First, we provide mathematical results that prove the existence of a unique solution to the Risk Budgeting problem. Then, in light of the rising interest for constructing Risk Budgeting portfolios for Expected Shortfall instead of volatility, we show that such a task can be performed using gradient descent tools when a mixture of multivariate Student-t distributions is assumed for asset returns. More generally, in model-based or model-free settings, this is still the case using stochastic gradient descent and by exploiting a variational characterization of Expected Shortfall. Beyond Expected Shortfall, we show that the Risk Budgeting problem actually boils down to a stochastic optimization problem for a wide range of popular risk measures. We provide numerical results that validate our theoretical findings and discuss the computational advantage associated with the stochastic optimization viewpoint introduced in this paper.

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**Data Availability Statement**

The data that support the findings of this study are available from the corresponding author upon reasonable request.
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A Description of the different procedures to evaluate the performances of SGD, OSBGD and MSBGD by simulation

• We define DGP\textsubscript{true} that can generate $d$-dimensional return vectors. Since it is not convenient and realistic to manually define the parameters of DGP\textsubscript{true}, as the number of parameters rapidly increases with respect to $d$, we use the following systematic approach: for a given $d$, we randomly select $d$ stocks from the S&P500 Index components (in April 2022) and fit a mixture of two multivariate Student-t distributions using the expectation-maximization algorithm on daily return data (August 2008–April 2022) with fixed degrees of freedom ($\nu_1 = 4.0$ and $\nu_2 = 2.5$). The estimated values of $p$, $\mu_1$, $\mu_2$, $\Sigma_1$ and $\Sigma_2$ are then used as the parameters of DGP\textsubscript{true}.

• We draw the sample $X_{\text{hist}}$ of size $n = 3500$ from DGP\textsubscript{true}.

• From $X_{\text{hist}}$, we propose two main approaches to compute the Risk Budgeting portfolio:
  
  – Model-free approach, where we rely on
    
    * SGD to solve Problem (1), using $X_{\text{hist}}$;
    * OSBGD to solve Problem (2), using $X_{\text{hist}}$ to compute the gradient at each iteration.
  
  – Model-based approach, where we first estimate a model using $X_{\text{hist}}$ to define DGP\textsubscript{est}. Then, invoke
    
    * SGD to solve Problem (1), using a sample $X_{\text{sim}}$ of size $10^6$ drawn from DGP\textsubscript{est};
    * OSBGD to solve Problem (2), using always the same sample $X_{\text{sim}}$ of size $10^6$ initially drawn from DGP\textsubscript{est} to compute the gradient at each iteration;
    * MSBGD to solve Problem (2), using a new sample $X_{\text{sim}}$ of size $10^5$ drawn from DGP\textsubscript{est} repeatedly to compute the gradient at each iteration.

In the model-based approach, estimating DGP\textsubscript{est} using $X_{\text{hist}}$ is the key step. Ideally, the ultimate goal is to get a model as close as possible to DGP\textsubscript{true}. In this paper, we take consider three different situations:

* we perfectly estimate the model, i.e. DGP\textsubscript{est} is formally equivalent to DGP\textsubscript{true};
* we correctly specify the family of the $X$ distribution which means that we fit a mixture of two multivariate Student-t distributions applying the expectation-maximization algorithm on $X_{\text{hist}}$ with fixed degrees of freedom ($\nu_1 = 4.0$ and $\nu_2 = 2.5$); the estimated models is denoted DGP\textsubscript{SM};
* we do not correctly specify the family of the $X$ distribution and fit a mixture of two multivariate Gaussian distributions applying the expectation-maximization algorithm on $X_{\text{hist}}$; the estimated model is denoted DGP\textsubscript{GM}.