Bayesian Proportional Resource Allocation Games
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Abstract—We consider a proportional allocation mechanism that gives to each user an amount of a resource proportional to the user’s bid. We study a corresponding Bayesian game in which each user has incomplete information on the state or type of the other users. We prove the existence of a Bayes-Nash equilibrium. Furthermore, under some mild assumptions, we establish asymptotic efficiency: we show that the per user social welfare achieved at any Bayes-Nash equilibrium is asymptotically equal to the maximum possible.

I. INTRODUCTION

Communication networks typically serve large populations of heterogeneous users with different preferences and utilities. Serving these diverse users in a socially efficient manner requires a resource allocation mechanism through which the users can express their preferences. Of course, for any given mechanism, one needs to take into account the possibility that a user can act strategically. Whether the mechanism delivers good performance in the face of such strategic behavior is a question that is addressed within a game-theoretic framework, by studying the efficiency properties (in terms of the resulting social welfare) of the corresponding Nash equilibria.

In this paper, we offer a Bayesian analysis of the proportional allocation mechanism introduced by Kelly [5] in the context of communication networks. In this mechanism, each user \( i \) submits a bid \( w_i \) (the actual payment to be made by a user). Then, the network manager allocates to each user a fraction of a divisible resource (e.g., bandwidth) in proportion to the user’s bid. This resource allocation mechanism can also be interpreted in terms of a market clearing process: the network manager sets a clearing price \( p \), which is proportional to the sum of the bids, and allocates an amount \( w_i/p \) of the resource to user \( i \).

When users are price takers (i.e., take the clearing price as a given, and do not take into account the effect of their own bid on the price), this mechanism is socially efficient: it results in allocations that maximize aggregate utility [5]. If the users act strategically (i.e., if they do take into account the impact of their own bids) the aggregate utility at a resulting allocation (at a Nash equilibrium) can be as low as 75% of the optimal [3]. Furthermore, under certain assumptions, this mechanism has the best possible efficiency loss guarantee among a broader class of proportional allocation mechanisms [6], [4].

On the other hand, when the number of users is large, one can hope that the resulting Nash equilibrium is close to a fully efficient (“competitive”) equilibrium. This is indeed the case, in the limit of a large number of users, under the mild assumption that a socially optimal allocation gives a vanishing fraction of the total resource to any single user (see Corollary 2.8 in p. 56 of [2]). This is also the case (under certain assumptions), with high probability, when each of a large number of users possesses a utility function drawn randomly and independently from a certain class of utility functions [8].

Note that the setting of [8] is in some sense Bayesian, in that the users’ utilities are drawn from a given distribution. On the other hand, the equilibrium concept being studied in [8] is the same (non-Bayesian) Nash equilibrium considered in earlier works. Such a focus on Nash equilibria of randomly drawn but deterministic games requires either (i) an implicit assumption that every user knows the utility function (type) of every other user, or, (ii) that there are repeated interactions and a learning process through which every user can obtain enough information on the behavior of the other users. Option (i) is clearly unappealing. Option (ii) implies that in reality users are faced with a repeated game, for which the relevance of the Nash equilibrium of the one-shot game can be questioned. Furthermore, the second option is clearly inapplicable in a volatile environment where users come and go quickly, in a time scale that is fast compared to the time it takes for a learning process to converge.

Because of the above mentioned drawbacks of focusing on the Nash equilibria of deterministic games, we are led to consider a Bayesian game: utility functions (types) are drawn at random (as in [8]) from a known probability distribution. Each player then submits a bid based only on its own type and knowledge of the distribution from which types are drawn. The relevant solution concept
in this setting is that of a Bayes-Nash equilibrium. Its advantage is that it can be computed locally by each user, without knowing the types of the other users. Thus, the Bayesian formulation that we consider can capture a one-shot game (no repeated interactions), without placing unrealistic assumptions on the available information.

The rest of the paper is organized as follows. In Section 2, we introduce the model to be studied. In Section 3, we will address the existence of a Bayes-Nash equilibrium. In Section 4, we argue that the social welfare achieved at a Bayes-Nash equilibrium is approximately the same as the optimal social welfare, as the number of users increases to infinity.

II. The Model

In this section, we define the model, introduce our notation, and specify the resource allocation mechanism of interest.

A. Formal description of the model

Our model consists of the following elements:

(a) Users: A set \( \mathcal{N} = \{1, 2, \ldots, N\} \) of \( N \geq 2 \) users. We typically use the dummy variable \( n \) to index the users.

(b) States: A common (for all users) finite set of states, \( \mathcal{S} = \{1, 2, \ldots, \mathcal{S}\} \). We use \( s_n \) to denote the state of user \( n \). A vector \( s = (s_1, s_2, \ldots, s_N) \in \mathcal{S}^N \) is called a state vector.

(c) Resource: An amount \( C > 0 \) of a divisible resource (referred to as bandwidth), to be allocated to the users. A nonnegative vector \( b = (b_1, \ldots, b_N) \) that satisfies \( \sum_{n=1}^{N} b_n \leq C \) is a feasible allocation.

(d) Utility functions: We are given a collection of concave utility functions \( U_s : [0, \infty) \rightarrow [0, \infty) \), \( s \in \mathcal{S} \). Here, \( U_s \) is the utility function of a user who happens to be at state \( s \). We define the utility function vector by \( \mathbf{U} = (U_1, \ldots, U_s) \).

(e) State randomness: The state \( s_n \) of each user \( n \) is a realization of an \( \mathcal{S} \)-valued random variable \( S_n \). We define the random state vector \( \mathbf{S} \) by \( \mathbf{S} = (S_1, \ldots, S_N) \). Let \( p_{n,s} \) be the probability that the state of user \( n \) is \( s \), i.e., \( p_{n,s} = \mathbb{P}(S_n = s) \). The probability vector of user \( n \) is \( \mathbf{p}_n = (p_{n,1}, p_{n,2}, \ldots, p_{n,s}) \), and lies in the \( \mathcal{S} \)-dimensional unit simplex, which we denote by \( \mathcal{P} \). We define the (overall) probability vector \( \mathbf{p}^N = (\mathbf{p}_1, \ldots, \mathbf{p}_N) \), which is an element of \( \mathcal{P}^N \).

We will use \( \mathcal{M} = (\mathcal{N}, \mathcal{S}, C, \mathbf{U}, \mathbf{p}^N) \) to denote a particular fully specified model. We will say that a model is symmetric if the distribution of the random vector \( (S_1, \ldots, S_N) \) is invariant under permutations of its elements; in particular, the probability vector \( \mathbf{p}_n \) is the same for all \( n \), and each user has the same probability of being at a particular state. Furthermore, all users at the same state form the same beliefs (i.e., conditional distribution) on the part of the state vector that they do not observe.

In general (that is, without the symmetry assumption), we allow some users to have zero probability of being at certain states. On the other hand, without loss of generality, we assume that for each state \( s \in \mathcal{S} \), there exists some user \( n \) for which \( p_{n,s} > 0 \). For any user \( n \), we use \( s_{-n} = (s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_N) \) and \( \mathbf{S}_{-n} = (S_1, \ldots, S_{n-1}, S_{n+1}, \ldots, S_N) \) to indicate the states of the other users.

B. Interpretation

The user states that we have introduced correspond to what are commonly referred to as "types" in game theory. Note that the type of a user determines completely its preferences. Let us pause to mention some possible interpretations of these states in a communication network context.

(a) Consider a wireless communication network with frequency division multiplexing. Here \( C \) represents the total bandwidth available and \( b_n \) is the bandwidth allocated to user \( n \). The state \( s_n \) of user \( n \) represents the quality of its channel, and determines the throughput that the user can obtain using the given bandwidth. Accordingly, the throughput and utility of user \( n \) is a function of both \( b_n \) and \( s_n \).

(b) Consider a wireline link with capacity \( C \). We interpret \( b_n \) as the rate allocated to user \( n \), and interpret \( s_n \) as the "type" of the user: users of different types make different uses of their allocated rate (e.g., voice versus video, or elastic versus time-critical communications), and accordingly derive different utilities from any given allocated rate.

C. Resource allocation mechanisms

We will compare the social welfare (the total expected utility derived by all the users) achieved by two mechanisms. The first is a centralized resource allocation mechanism where the agents divide the available resources so that the aggregate utility of all users is maximized. The second is the proportional allocation mechanisms in Kelly [5]: each user submits a bid, interpreted as a payment, and bandwidth is allocated in proportion to the bids. We will assume quasilinear payoffs, i.e., that the payoff of a user is equal to the difference between the utility derived and the user’s bid.
On the other hand, when calculating the social welfare, we will only consider the resulting total expected utility; payments are viewed simply as transfers that do not affect social welfare. This way, we can focus on the efficiency of the resulting allocations, and ask the question of whether Kelly’s mechanism results in an efficient utilization of the available resources. We now proceed to define the two mechanisms formally.

**(PO) Ex Post Optimal Allocation.** An allocation based on knowledge of the entire state vector allocates a certain bandwidth \( b_{n,s} \) to agent \( n \) when the state vector is \( s = (s_1, s_2, \ldots, s_N) \). An optimal allocation chooses, for each state vector \( s \), a vector \( (b_{1,s}, \ldots, b_{N,s}) \) to maximize the social welfare

\[
\sum_{n=1}^{N} \sum_{s \in S^n} U_{s_n}(b_{n,s}) \mathbb{P}(S = s).
\]

It is not hard to see that this is equivalent to maximizing \( \sum_{n=1}^{N} U_{s_n}(b_{n,s}) \) separately for each \( s \). We let \( W_{PO}(M) \) be the optimal social welfare associated with a model \( M \).

**(BG) Bayesian Game.** Here, each user knows its own state, but not the state of the other users. User \( n \), if found at state \( s \), submits a bid \( w_{n,s} \geq 0 \). Without loss of generality, we assume that if \( p_{n,s} = 0 \), then \( w_{n,s} = 0 \). We use the notation \( w_s \) to denote the strategy vector of user \( n \), i.e., \( w_s = (w_{n,1}, \ldots, w_{n,s}) \). \( w \) to denote the strategy profile \( w = (w_1, \ldots, w_N) \), and \( w_{-n} \) to denote the strategy profile of all users other than \( n \), so that \( \hat{w} = (w_n, w_{-n}) \). The bandwidth allocated to user \( n \) when the state vector is \( s = (s_1, \ldots, s_N) \), is

\[
b_{n,s}(w) = \begin{cases} 0, & \text{if } w_{n,s} = 0 \\ \sum_{i=1}^{N} w_{i,s}, & \text{otherwise.} \end{cases}
\]

At a state \( s \) with \( \mathbb{P}(S_n = s) > 0 \), user \( n \) is faced with an expected payoff of the form

\[
Q_{n,s}(w_{n,s}, w_{-n}) = \sum_{s_{-n} \in S^{N-1}} U_{s_n}(b_{n,s}(w)) \mathbb{P}(S_{-n} = s_{-n} \mid S_n = s).
\]

For a given model \( M \) and strategy profile \( w \), the expected payoff of user \( n \) is

\[
Q_n(w) = \sum_{s \in S} p_{n,s} Q_{n,s}(w_{n,s}, w_{-n}).
\]

The (expected) payoff functions \( Q_n(w) \) define an \( N \)-person game, which falls within the class of Bayesian games. We are interested in the pure Nash equilibria of this game, which we will be referring to as Bayes-Nash equilibria (or BNE, for short). Note that a strategy profile \( w \) is a BNE if and only if there is no profitable deviation from \( w_{n,s} \) whenever \( p_{n,s} > 0 \), or

\[
p_{n,s} Q_{n,s}(w_{n,s}, w_{-n}) \leq p_{n,s} Q_{n,s}(w_{n,s}, w_{-n}), \quad \forall n \in N, \quad \forall s \in S.
\]

The social welfare achieved at a strategy profile \( w \) is

\[
\sum_{n=1}^{N} \sum_{s \in S^n} U_{s_n}(b_{n,s}(w)) \mathbb{P}(S = s).
\]

If the set of BNEs is not empty, we are interested in the social welfare achieved at the “worst” BNE (the infimum of the social welfare over all BNEs), which we denote by \( W_{BG}(M) \).

### III. Bayesian Game: Existence of Equilibria

In this section, we address (under mild assumptions) the existence of a BNE, and the existence of a symmetric BNE for symmetric games.

Recall from Section II that a model is symmetric if the distribution of the state vector is invariant under permutations. When the model is symmetric, we also say that the corresponding Bayesian game is symmetric. A strategy profile \( w \) is called symmetric if there exist some \( w_s, s \in S \), such that

\[
w_{n,s} = \hat{w}_s, \quad \forall n \in N, \quad \forall s \in S.
\]

A symmetric BNE is a symmetric strategy profile which is a BNE.

The main result of this section follows.

**Theorem 3.1:** Consider the Bayesian game (BG) defined in Section II. Suppose that for each state \( s \in S \), the utility function \( U_s \) is concave, strictly increasing, and continuous on \([0, \infty)\). Then, a BNE is guaranteed to exist. Furthermore, if the game is symmetric, there exists a symmetric BNE.

Theorem 3.1 can be viewed as an extension to a Bayesian setting of the non-Bayesian existence result of Hajek and Gopalakrishnan [1]. In that work, existence was proved by showing that the Nash equilibrium conditions are equivalent to the optimality conditions for a related convex optimization problem. Assuming that the utility functions of all users are concave, strictly increasing, and continuous, then the convex optimization problem has a unique optimal solution, which is also a unique Nash equilibrium of the game (see [3] for details). Furthermore, the game is symmetric if and only if all users have the same utility function. In that case, the optimal solution of the convex optimization
problem (and therefore, the Nash equilibrium as well) is symmetric. However, for the Bayesian games studied in this paper, a reformulation in terms of a convex optimization problem is not apparent (and is unlikely to be possible), and the proof is more complicated. The main technical difficulty to be overcome is that the payoff is a discontinuous function of the strategy profile. We provide here a high-level summary of the proof.

We first consider a “bounded game” in which we fix some \( B > 0 \) and restrict the bids \( w_{n,s} \) to lie in the compact set \([0, B]\). We then construct a perturbed game in which we fix some \( \varepsilon > 0 \) and let the bandwidth allocated to a user be \( Cw_{n,s}/(\varepsilon + \sum_i w_{i,s}) \); this is equivalent to introducing an additional virtual user who always submits a bid of \( \varepsilon > 0 \). Using Rosen’s existence theorem [7], we obtain that a BNE \( w^\varepsilon \) exists for this perturbed bounded game. Taking the limit as \( \varepsilon \to 0 \), we consider a limit point of \( w^\varepsilon \), and show that it is a BNE of the unperturbed bounded game. Finally we show that a BNE of the unperturbed bounded game is also a BNE of the original Bayesian game, as long as \( B \) is chosen large enough.

For the case of symmetric games, the proof uses a standard argument to show that \( w^\varepsilon \) can be taken to be symmetric, which by taking the limit as \( \varepsilon \to 0 \) leads to a symmetric BNE of the original game.

We note that Theorem 3.1 is not true under the weaker assumption that the utility functions are only non-decreasing, even for the deterministic case (single state). While the perturbed game always has a Nash equilibrium, the original game need not have one.

IV. Bayesian games with a large number of users

In this section, we consider an asymptotic analysis, as the number \( N \) of users grows, and the capacity also grows proportionally; we will assume, for concreteness, that \( C = N \). On the other hand, we will fix the utility functions \( U_s \) associated to the different states, independently of \( N \). (This corresponds, for example, to a system that serves a few predetermined types of users, as the user population grows.) We will make a number of assumptions that guarantee that no single user can have a sizable impact on the equilibrium unit price of the divisible resource. Thus, the users effectively become price takers, and we obtain a situation similar to the competitive equilibria in the economics literature. It is then natural to expect that in the limit, a BNE is efficient when compared to an ex post optimal allocation.

On the technical side, a key step is provided by the fact that, at a BNE, the bids of different users that happen to be at the same state must be very close (cf. Theorem 4.1); loosely speaking, we can say that a BNE must be close to symmetric, and this is true even without assuming that the original game is symmetric.

The proofs are somewhat long and tedious, and are omitted due to space limitations.

A. Assumptions, preliminaries, and approximate symmetry of BNEs

In this subsection, we introduce certain assumptions under which we can establish an upper and a lower bound (independent of \( N \)) on the users’ bids, and show that a BNE must be approximately symmetric.

Assumption 4.1: We have \( U_s(b) \leq D \), for all \( b \geq 0 \) and \( s \in S \), where \( D \) is some constant (independent of \( N \)).

Lemma 4.1: Under Assumption 4.1, if \( w \) is an associated BNE, and if \( p_{n,s} > 0 \), then \( w_{n,s} \leq D \).

Assumption 4.2: For each \( s \in S \), the right derivative of \( U_s \) at zero, denoted \( U'_s(0) \), is infinite.

Lemma 4.2: Suppose that Assumptions 4.1 and 4.2 hold, and that the utility functions \( U_s \) are strictly increasing. Then, there exists a position constant \( \pi \), completely determined by the utility functions, such that for any BNE \( w \), any \( n \), and any \( s \) that satisfies \( p_{n,s} > 0 \), we have \( w_{n,s} \geq \pi \).

By Lemmas 4.1 and 4.2, the bid of any individual user is too small relative to the total to make a difference. Thus, the situation faced by two users that happen to be at the same state is essentially the same. Accordingly, their bids (at a BNE) will be approximately the same. In Theorem 4.1 and in subsequent results in this section, we make an additional assumption that the states of different users are independent. This assumption is mostly for convenience and can be relaxed to various forms of weak dependence. We also make the assumption that the utility functions are strictly concave. It is unclear whether this assumption can be relaxed.

Theorem 4.1: Suppose that the states of different users are independent. Suppose, furthermore, that all the utility functions \( U_s \) are strictly increasing, strictly concave, continuously differentiable on \((0, \infty)\), and satisfy Assumptions 4.1 and 4.2. Fix some \( \varepsilon > 0 \). Then, there exists an integer \( N_0 \) such that if the number \( N \) of users is at least \( N_0 \), and if \( w \) is a BNE, then

\[
|w_{n,s} - w_{m,s}| \leq \varepsilon,
\]

for any \( s \in S \) and any pair of users \( n \) and \( m \) for which \( p_{n,s} > 0 \) and \( p_{m,s} > 0 \).

B. Asymptotic efficiency

We are now ready to proceed to our asymptotic efficiency results. Any discussion of asymptotics needs
to refer to a sequence of games, with an increasing number of users; this corresponds to a sequence of models/games, which we will denote by $\mathcal{M}^N$. In the particular asymptotics that we consider, we increase the number of users (and proportionally scale the capacity); we require the user states to be independent, but we allow the distribution of each user’s state to be different and fairly arbitrary; and, crucially, we fix the space of possible states and the utility functions associated with each state. When necessary, we will use a superscript $N$ to indicate quantities associated with the $N$-user model $\mathcal{M}^N$. For example, $p_{n,s}^N$ stands for the probability that $S_n = s$ in the $N$-user model.

We will also make one additional assumption, which guarantees that (with high probability) there are enough users present at each state. This assumption introduces a minimal amount of statistical regularity, which we exploit in our proof. We use the notation $p_s^N = \frac{1}{N} \sum_{n=1}^{N} p_{n,s}^N$.

**Assumption 4.3:** There exists some $N_0$ and a positive absolute constant $h$ such that
\[ p_s^N \geq h, \quad \forall s \in S, \quad \forall N \geq N_0. \]

Our result compares $W_{BG}(\mathcal{M}^N)/N$, the per user welfare at a BNE, with $W_{PO}(\mathcal{M}^N)/N$, the ex post optimal (centralized) welfare; see Section II for the precise definitions. It turns out that they are both equal, asymptotically, to $W_{CE}(\mathcal{M}^N)/N$, defined as the optimal value of the objective function in the following certainty equivalent optimization problem:

\[
\text{maximize} \quad \sum_{s=1}^{S} p_s^N U_s(b_s^N) \\
\text{subject to} \quad \sum_{s=1}^{S} p_s^N b_s^N \leq 1, \quad b_s^N \geq 0.
\]

Notice that this is the ex post optimization problem obtained if it were known that the number of users at state $N$ was exactly equal to its expected number, $p_s^N \cdot N$.

**Theorem 4.2:** Consider a sequence of models/games $\mathcal{M}^N$ in which:
(a) $C = N$, for all $N$;
(b) the set $S$ of states is the same for all $N$;
(c) the state probabilities $p_{n,s}$ satisfy Assumption 4.3;
(d) the utility functions $U_s : [0, \infty) \to \mathbb{R}$ associated with each state $s$ do not change with $N$, are strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfy Assumptions 4.1-4.2.

Then,
\[
\lim_{N \to \infty} \frac{W_{BG}(\mathcal{M}^N)}{N} - \frac{W_{CE}(\mathcal{M}^N)}{N} = 0.
\]

As an immediate corollary of Theorem 4.2, we obtain that the asymptotic efficiency loss at a BNE of the Bayesian game, compared to an ex post optimal allocation, is zero. This result incorporates two effects: first, there is no efficiency loss due to the incomplete information pattern imposed by decentralization; second, there is no efficiency loss due to the strategic behavior of the users.

**Corollary 4.1:** Under the assumptions of Theorem 4.2,
\[
\lim_{N \to \infty} \frac{W_{BG}(\mathcal{M}^N)}{N} = 1.
\]

The basic idea behind our results is that due to independence and the large number of users, statistical fluctuations are washed out. Thus, each user is faced with an approximately deterministic situation. Furthermore, because of the boundedness of the utility functions, no user has any reason to place a large bid, and therefore no single user can have a significant effect on the overall allocation. Therefore, the Bayesian game becomes approximately the same as a model with perfect information and a large number of small users, a setting for which efficiency is naturally expected.

Let us now discuss the scope of the asymptotic efficiency result.

(a) For the deterministic setting considered in [3] an efficiency loss of 25% is possible. However, the examples that demonstrate such an efficiency loss violate several of our assumptions: they involve utility functions that are linear (hence violate our Assumptions 4.1 and 4.2) and also change with $N$; furthermore they involve a single user with “large market power” and $N - 1$ “small” users. Such a situation violates our Assumption 4.3, which requires that each user type be a positive fraction of the user population.

(b) The results in [8] show that when the utility functions are drawn from some particular distributions, asymptotic efficiency fails to obtain. The examples in [8] involve unbounded utility functions and also an infinite set of possible utility functions, which is quite different from our assumptions. It is an interesting question to study the efficiency loss of Bayes-Nash equilibria when the random choice of the utility functions is as in [8].
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