Research Article

A Posteriori Error Analysis for a New Fully Mixed Isotropic Discretization of the Stationary Stokes-Darcy Coupled Problem

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In this paper, we develop an a posteriori error analysis for the stationary Stokes-Darcy coupled problem approximated by conforming the finite element method on isotropic meshes in \( \mathbb{R}^d \), \( d \in \{2, 3\} \). The approach utilizes a new robust stabilized fully mixed discretization. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution plus the stabilization terms. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient.

1. Introduction

There are many serious problems currently facing the world in which the coupling between groundwater and surface water is important. These include questions such as predicting how pollution discharges into streams, lakes, and rivers making its way into the water supply. This coupling is also important in technological applications involving filtration. We refer to the nice overview [1] and the references therein for its physical background, modeling, and standard numerical methods. One important issue in the modeling of the coupled Darcy-Stokes flow is the treatment of the interface condition, where the Stokes fluid meets the porous medium. In this paper, we only consider the so-called Beavers-Joseph-Saffman condition, which was experimentally derived by Beavers and Joseph in [2], modified by Saffman in [3], and later mathematically justified in [4–7].

There are three popular formulations of the coupled Darcy-Stokes flow, namely, the primal formulation, the mixed formulation in the Darcy region, or the fully mixed formulation, see, for example, the works [8–16] for some mathematical analysis. Two different mixed formulations have been studied by Layton et al. in [15]. The first one enforces the weak continuity of the normal component of the velocity field on the interface, by employing a Lagrange multiplier, while the second one imposes the strong continuity in the functional space. We can call these two mixed formulations the weakly coupled formulation and the strongly coupled formulation, respectively. The weakly coupled formulation gives more freedom in the choice of the discretization in the Stokes side and the Darcy side singly. The authors in [9–11, 15, 17–19] employ the weakly coupled formulation. The works on the strongly coupled formulation have been based on the development of a unified discretization, namely, the Stokes side and the Darcy side are discretized using the same finite element. This approach simplifies the numerical implementation, only if the unified discretization is not significantly more complicated than the commonly used discretizations for the Darcy and Stokes problems. The authors in [8, 20] have proposed a conforming, unified finite element for the strongly coupled mixed formulation. Superconvergence analysis of the finite element methods for the Stokes-Darcy system was studied by Chen et al. in [21]. Other less restrictive discretizations as the nonconforming unified approach [12, 19] or the discontinuous Galerkin (DG) approach have been analyzed in [13, 14, 22]. Due to its
discontinuous nature, some (DG) discretizations for the coupled Darcy-Stokes problem may break the strong coupling in the discrete level [13, 14], as they impose the normal continuity across the interface via interior penalties.

The adaptive techniques have become indispensable tools and unavoidable in the field of study behavior of the error committed during solving partial differential equations (PDE). A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equidistribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [23], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

A posteriori error estimations have been widely studied for both the mixed formulations of the Darcy flow [24–26] and the Stokes flow [27–36]. However, only a few works exist for the coupled Darcy-Stokes problem, see for instance [18, 37–40]. The works in [18, 38] concern the strongly coupled mixed formulation where a H(div) conforming and nonconforming finite element methods have been employed. The papers [37, 39] concern the weakly coupled mixed formulation while [39] uses the primal formulation on the Darcy side. The authors in [40] employ a fully mixed formulation where Raviart-Thomas elements have been used to approximate the velocity in both the Stokes domain and the Darcy domain, and constant piecewise for approximating the pressure.

In [16], a stabilized finite element method for the stationary-mixed Stokes-Darcy problem has been proposed for the fully mixed formulation. The authors have used the well-known MINI elements (P1b − P1) to approximate the velocity and pressure in the conduit for the Stokes equation. To capture the fully mixed technique in the porous medium region linear Lagrangian elements, P1 has been used for hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements have been used for Darcy velocity. An a priori error analysis is performed with some numerical tests confirming the convergence rates. However, to our best knowledge, they did not talk about the adaptive method for the fully mixed discretization proposed in [16]. In this case, our main objective is to perform an a posteriori error analysis by constructing reliable and efficient indicator errors. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. We prove that our indicator errors are efficiency and reliability and then are optimal. The difference between our paper and the reference [40] is that our discretization uses MINI elements (P1b − P1) to approximate the velocity and pressure in the conduit for Stokes equations; P1 −Lagrange elements to approximate hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements have been used for Darcy velocity. As a result, an additional term is included in the error estimator that measures the stability of the method. In order to treat appropriately this stability term, we further need a special Helmholtz decomposition ([18], Theorem 3), a regularity result ([18], Theorem 4), and an estimate of the stability error ([18], Theorem 5).

The paper is organized as follows. Some preliminaries and notation are given in Section 2. The efficiency result is derived using the technique of bubble function introduced by Verfürth [41] and used in a similar context by Carstensen [25, 42]. In Section 3, the a posteriori error estimates are derived. We offer our conclusion and the further works in Section 4.

2. Preliminaries and Notations

2.1. Model Problem. We consider the model of a flow in a bounded domain $\Omega \subset \mathbb{R}^d (d = 2$ or $3$), consisting of a porous medium domain $\Omega_p$, where the flow is a Darcy flow, and an open region $\Omega_f$, where the flow is governed by the Stokes equations. The two regions are separated by an interface $\Gamma = \partial \Omega_p \cap \partial \Omega_f$. Let $\Gamma_l = \partial \Omega_l \backslash \Gamma$, $l = f, p$. Each interface and boundary is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). We denote by $n_l$ (resp., $n_p$) the unit outward normal vector along $\partial \Omega_l$ (resp., $\partial \Omega_p$). Note that on the interface $\Gamma$, we have $n_f = -n_p$. Figure 1 shows a sketch of the problem domain, its boundaries, and some other notations.

The fluid velocity and pressure $u_f(x)$ and $p(x)$ are governed by the Stokes equations in $\Omega_f$:

\[
\begin{aligned}
\begin{cases}
-2\nu \nabla \cdot \text{D}(u_f) + \nabla p &= \mathbf{f}_f \text{ in } \Omega_f, \\
\nabla \cdot u_f &= 0 \text{ in } \Omega_f,
\end{cases}
\end{aligned}
\]

where $\mathbf{T} = -p \mathbb{I} + 2\nu \text{D}(u_f)$ denotes the stress tensor, and

$\text{D}(u_f) = (1/2)(\nabla u_f + (\nabla u_f)^T)$ represents the deformation tensor. The porous media flow is governed by the following Darcy equations on $\Omega_p$, through the fluid velocity $u_p(x)$ and the piezometric head $\phi(x)$:

\[
\begin{aligned}
\begin{cases}
\mathbf{u}_p &= -K \nabla \phi \text{ in } \Omega_p, \\
\nabla \cdot u_p &= \mathbf{f}_p \text{ in } \Omega_p,
\end{cases}
\end{aligned}
\]

We impose impermeable boundary conditions, $u_p \cdot n_p = 0$ on $\Gamma_p$, on the exterior boundary of the porous media region, and no-slip conditions, $u_f = 0$ on $\Gamma_f$, in the Stokes region. Both selections of boundary conditions can be modified. On $\Gamma$, the interface coupling conditions are conservation of mass, balance of forces, and a tangential condition on the fluid region’s velocity on the interface. The correct tangential condition is not completely understood (possibly due to matching a pointwise velocity in the fluid region with an averaged or homogenized velocity in the porous region). In this paper, we take the Beavers-Joseph-Saffman (−Jones), see [2–7], interfacial coupling:

\[
\mathbf{u}_f \cdot \mathbf{n}_f + u_p \cdot n_p = 0 \text{ on } \Gamma, \tag{3}
\]

\[
-\mathbf{n}_f \cdot \mathbb{T} \cdot \mathbf{n}_f = p - 2\nu \mathbf{n}_f \cdot \text{D}(u_f) \cdot \mathbf{n}_f = \rho g \phi \text{ on } \Gamma, \tag{4}
\]
2.2. Notations and the Weak Formulation.

In this part, we first introduce some Sobolev spaces [43] and norms. If \( W \) is a bounded domain in \( \mathbb{R}^d \) and \( m \) is a nonnegative integer, the Sobolev space \( H^m(W) = W^{m,2}(W) \) is defined in the usual way with the usual norm \( \| \cdot \|_{m,W} \) and seminorm \( | \cdot |_{m,W} \). In particular, \( H^0(W) = L^2(W) \) and we write \( \| \cdot \|_W \) for \( \| \cdot \|_{0,W}. \)

Similarly, we denote by \( (\cdot, \cdot)_W \) the \( L^2(W)[L^2(W)]^N \) or \([L^2(W)]^{d \times d}\) inner product. For shortness, if \( W \) is equal to \( \Omega \), we will drop the index \( \Omega \), while for any \( m \geq 0 \), \( \| \cdot \|_{m,\Omega} = \| \cdot \|_{m,\Omega}, \quad | \cdot |_{m,\Omega} = | \cdot |_{m,\Omega}, \) and \( (\cdot, \cdot)_\Omega \) for \( l = f, s. \) The space \( H^m(\Omega) \) denotes the closure of \( C^\infty(\Omega) \) in \( H^m(\Omega) \).

Let \( [H^m(\Omega)]^d \) be the space of vector-valued functions \( \mathbf{v} = (v_1, \ldots, v_d) \) with components \( v_j \in H^m(\Omega) \). The norm and the seminorm on \([H^m(\Omega)]^d \) are given by

\[
\| \mathbf{v} \|_{m,\Omega} = \left( \sum_{i=1}^d |v_i|^2 \right)^{1/2},
\]

\[
| \mathbf{v} |_{m,\Omega} = \left( \sum_{i=1}^d |v_i|^2 \right)^{1/2}.
\]

For a connected open subset of the boundary \( E \subset \partial \Omega \cup \partial \Omega_f \), we write \( \langle \cdot, \cdot \rangle_E \) for the \( L^2(E) \) inner product (or duality pairing), that is, for scalar-valued functions \( \lambda, \sigma \), one defines:

\[
\langle \lambda, \sigma \rangle_E = \int_E \lambda \sigma ds.
\]

By setting the space

\[
H_{div} = H(\text{div}; \Omega_f) = \left\{ \mathbf{v}_p \in [L^2(\Omega_f)]^d : \nabla \cdot \mathbf{v}_p \in L^2(\Omega_f) \right\},
\]

we introduce the following spaces:

\[
\mathbf{X}_f := \left\{ \mathbf{v}_f \in [H^1(\Omega_f)]^d : \mathbf{v}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f \right\},
\]

\[
Q_f = L^2(\Omega_f),
\]

\[
\mathbf{X}_p := \left\{ \mathbf{v}_p \in H(\text{div}; \Omega_f) : \mathbf{v}_p \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_p \right\},
\]

\[
Q_p = L^2(\Omega_p).
\]

For the spaces \( \mathbf{X}_f \) and \( \mathbf{X}_p \), we define the following norms:

\[
\| \mathbf{v}_f \|_{1} = \sqrt{\| \nabla \mathbf{v}_f \|^2_{L^2(\Omega_f)} + \| \mathbf{v}_f \|^2_{L^2(\Omega_f)}} \quad \text{with} \quad \| \mathbf{v}_f \|_{1,\Omega_f} = \| \nabla \mathbf{v}_f \|_{L^2(\Omega_f)}, \quad \forall \mathbf{v}_f \in \mathbf{X}_f,
\]

\[
\| \mathbf{v}_p \|_{div} = \sqrt{\| \nabla \cdot \mathbf{v}_p \|^2_{L^2(\Omega_f)} + \| \mathbf{v}_p \|^2_{L^2(\Omega_f)}}, \quad \forall \mathbf{v}_p \in \mathbf{X}_p.
\]

The variational formulation of the steady-state Stokes-Darcy problem (1)–(5) reads as find \( (\mathbf{u}_f, p ; \mathbf{u}_p, \psi) \in (\mathbf{X}_f, Q_f ; \mathbf{X}_p, Q_p) \) satisfying

\[
a_f(\mathbf{u}_f, \mathbf{v}_f) - b_f(\mathbf{v}_f, p) + c_f(\mathbf{v}_f, \psi) = (\mathbf{f}_f, \mathbf{v}_f), \quad \forall \mathbf{v}_f \in \mathbf{X}_f,
\]

\[
b_f(\mathbf{u}_f, q) = 0, \quad \forall q \in Q_f,
\]

\[
a_p(\mathbf{u}_p, \mathbf{v}_p) - b_p(\mathbf{v}_p, \psi) - c_f(\mathbf{v}_p, \psi) = 0, \quad \forall \mathbf{v}_p \in \mathbf{X}_p,
\]

\[
b_p(\mathbf{u}_p, \psi) = \rho g \left( \mathbf{f}_p, \psi \right)_{\Omega_p}, \quad \psi \in Q_p,
\]

where the bilinear forms are defined as

\[
a_f(\mathbf{u}_f, \mathbf{v}_f) = 2\nu (D(\mathbf{u}_f), D(\mathbf{v}_f))_{\Omega_f}
\]

\[
+ \sum_{j=1}^{d-1} \frac{\alpha}{\sqrt{\tau_j \cdot K_{\tau_j}}} < \mathbf{u}_j, \mathbf{v}_j >_{\tau_j, \Omega_f},
\]

\[
a_p(\mathbf{u}_p, \mathbf{v}_p) = \rho g (K^{-1} \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p},
\]

\[
b_f(\mathbf{v}_f, p) = (p, \nabla \cdot \mathbf{v}_f)_{\Omega_f}.
\]
\[ b_p(v, \phi) = \rho g(\phi, \nabla \cdot v_p) \Omega, \]

\[ c_r(v_f, \phi) = \rho g < \phi, v_f \cdot n_f >_T. \] (16)

After introducing, for \( U = (u_f, p, u_p, \phi) \in X_f \times Q_p \times X_p \times Q_p \times H \) and \( V = (v_f, q, v_p, \psi) \in X_f \times Q_p \times X_p \times Q_p \),

\[ \mathcal{L}(U, V) = a_f(u_f, v_f) - b_f(v_f, p) + b_p(u_p, v_p) - b_p(v_p, \phi) + c_f(v_f) + c_p(u_f, v_p), \]

\[ \mathcal{F}(V) = (f, v_f) \Omega + \rho g (f_p, \psi), \] (17)

the weak formulation (12)–(15) can be equivalently written as follows: find \( U \in H \) satisfying

\[ \mathcal{L}(U, V) = \mathcal{F}(V), \quad \forall V \in H. \] (18)

It is easy to verify that this variational formulation is well-posedness [16].

To end this section, we recall the following Poincaré, Korn’s, and the trace inequalities, which will be used in the later analysis; there exist constant \( C_p, C_K \), and \( C_o \), only depending on \( \omega_f \) such that for all \( v_f \in X_f \),

\[ \| v_f \| \leq C_p \| v_f \|_1, \]

\[ |v_f| \leq C_K \| \nabla (v_f) \|_{\Omega}, \] (19)

\[ \| v_f \|_{L^2(\Omega)} \leq C_o \| v_f \|_{H^1(\Omega)}. \]

Besides, there exists a constant \( \tilde{C}_o \), that only depends on \( \Omega, \) such that for all \( \psi \in Q_p \),

\[ \| \psi \|_{L^2(\Omega)} \leq \tilde{C}_o \| \psi \|_{L^2(\Omega)}^{1/2} \| \psi \|_{H^1(\Omega)}^{1/2}. \] (20)

2.3. Fully Mixed Isotropic Discretization. First, we consider the family of triangulations \( T_h \) of \( \Omega \), consisting of \( T_h^f \) and \( T_h^p \), which are regular triangulations of \( \Omega \) and \( \Omega_p \), respectively, where \( h > 0 \) is a positive parameter. We also assume that on the interface \( \Gamma \) the two meshes of \( T_h^f \) and \( T_h^p \), which form the regular triangulation \( T_h = T_h^f \cup T_h^p \), coincide.

The domain of the uniformly regular triangulation \( \Omega_j \) \( \cup \Omega_p \) is such that \( \tilde{\Omega} = \{ K : K \in T_h \} \) and \( h = \max_{K \in T_h} h_K \). There exist positive constants \( c_1 \) and \( c_2 \) satisfying \( c_1 h \leq h_K \leq c_2 p_K \). To approximate the diameter \( h_K \) of the triangle (or tetrahedral) \( K, p_K \) is the diameter of the greatest ball included in \( K \). Based on the subdivisions \( T_h^f \) and \( T_h^p \), we can define finite element spaces \( X_{h_f} \subset X_f, Q_{h_f} \subset Q_f, X_{h_p} \subset X_p, \) and \( Q_{h_p} \subset Q_p \). We consider the well-known MINI elements \( P1b-P1 \) (16–P1) to approximate the velocity and the pressure in the conduits for Stokes equations [44]. To capture the fully mixed technique in the porous medium region linear Lagrangian elements, \( P1 \) are used for hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements are used for Darcy velocity [45].

In the fluid region, we select for the Stokes problem the finite element spaces \( (X_{h_f}, Q_{h_f}) \) that satisfy the velocity-pressure inf-sup condition: there exists a constant \( C_f > 0 \), independent of \( h \), such that,

\[ \inf_{0 \neq \phi \in Q_{h_f}} \sup_{0 \neq \psi \in X_{h_f} \setminus \{0\}} \frac{b_p(v, \phi)}{\| v \|_{\Omega} \| \phi \|_{\Omega}} \geq C_f, \] (21)

In the porous region, we use the finite element spaces \( (X_{h_p}, Q_{h_p}) \) that also satisfy a standard inf-sup condition: there exist a constant \( C_p > 0 \) such that for all \( \psi \in Q_{h_p} \),

\[ \inf_{0 \neq \phi \in Q_{h_p}} \sup_{0 \neq \psi \in X_{h_p} \setminus \{0\}} \frac{b_p(v, \phi)}{\| v \|_{div} \| \phi \|_{\Omega}} \geq C_p, \] (22)

Then, the finite element discretization of (18) is to find \( U_h \in H_h = X_{h_f} \times Q_{h_f} \times X_{h_p} \times Q_{h_p} \) such that

\[ \mathcal{L}(U_h, V_h) + J_R(U_h, V_h) = \mathcal{F}(V_h), \quad \forall V_h \in H_h. \] (23)

This is the natural discretization of the weak formulation (18) except that the stabilized term \( J_R(U_h, V_h) \) is added. This bilinear form \( J_R(\ldots) \) is defined by

\[ J_R(U_h, V_h) = \frac{\delta}{h} \left( (u_h^0 - u_p^h) \cdot n_f, (v_h^0 - v_p^h) \cdot n_f \right)_\Gamma, \quad 0 < h < 1. \] (24)

We are now able to define the norm on \( H_h \):

\[ \| V \|_{H_h} = \sqrt{\| v_h^0 \|_{L^2(\Omega)}^2 + \| q_h^0 \|_{L^2(\Omega)}^2 + \| v_h^0 \|_{L^2(\Omega)}^2 + \| q_h^0 \|_{L^2(\Omega)}^2}, \] (25)

We have the following results (see [16], Theorem 2 and Theorem 3):

**Theorem 1.** There exists a unique solution \( U_h \in H_h \) to problem (23) and if the solution \( U \in H \) of the continuous problem (18) is smooth enough, then we have

\[ \| U - U_h \|_h \leq C(U)h. \] (26)

Below, in order to avoid excessive use of constants, the abbreviation \( x \leq y \) stand for \( x \leq cy \), with \( c \) a positive constant independent of \( x, y \), and \( \Omega \).

**Remark 2.** (Galerkin orthogonality relation). Let \( U = (u_f, p, u_p, \phi) \in H \) be the exact solution and \( U_h = (u_{h_f}, p_h, u_{h_p}, \phi_h) \in H_h \) be the finite element solution. Then, for any \( V_h = (v_{h_f}, \ldots) \in H_h \),
\( p_h, \mathbf{v}_h, \psi_h \in H_h \), and using technical regularity result Theorem 4 below, we can subtract (18) to (23) to obtain the Galerkin orthogonality relation:

\[
\mathcal{L}_h(U - U_h, V_h) = \mathcal{L}_h(U, V_h) - \mathcal{L}_h(U_h, V_h) \\
= \mathcal{L}(U, V_h) - \mathcal{L}(U_h, V_h) \\
= \mathcal{F}(V_h) - \mathcal{F}(V_h) = 0. 
\] (27)

Thus, we have the relation:

\[
2\nu (D(\mathbf{e}_j), D(\mathbf{v}_j))_{\Omega_j} + \sum_{j=1}^{d-1} \frac{\alpha}{\sqrt{\tau_j}} \| \mathbf{e}_j \cdot \mathbf{r}_j, \mathbf{v}_j \|_{\Gamma_j} \\
- (\mathbf{e}_p, \nabla \cdot \mathbf{v}_h)_{\Omega_j} + (q_h, \nabla \cdot \mathbf{e}_j)_{\Omega_j} + \rho g \left( [K^{-1} \mathbf{e}_p, \mathbf{v}_h]_{\Omega_p} - [\lambda \phi, \psi]_{\Gamma} \right) = 0. 
\] (28)

where here and below, the errors in the velocity and in the pressure of Stokes equations and errors in the hydraulic and Darcy velocity equations are, respectively, defined by

\[
\mathbf{e}_j := \mathbf{u}_j - \mathbf{u}_{jh}, \\
\mathbf{e}_p := \mathbf{p} - p_h, \\
\lambda \phi := \phi - \phi_h. 
\] (29)

3. A Posteriori Error Analysis

3.1. Some Technical Results. Our a posteriori analysis requires some analytical results that are recalled. We define the space

\[
\mathcal{H} = \left\{ \mathbf{v} \in H(\text{div}, \Omega) : \mathbf{v}|_{\Omega_j} \in X_j \right\} 
\] (30)

with the norm

\[
\| \mathbf{v} \|_{\mathcal{H}} := \sqrt{\| \mathbf{v} \|_{1,\Omega_j}^2 + \| \mathbf{v}_p \|_{\Omega_p}^2 + \| \nabla \mathbf{v}_p \|_{\Omega_p}^2}. 
\] (31)

The first one concerns a sort of Helmholtz decomposition of elements of \( \mathcal{H} \). Recall first that if \( d = 3 \),

\[
H_0(\text{curl}, \Omega) = \left\{ \psi \in L^2(\Omega)^3 : \text{curl} \psi \in L^2(\Omega)^3 \text{ and } \psi \times n = 0 \text{ on } \partial \Omega \right\}. 
\] (32)

**Theorem 3.** (see Ref. [18], page 708). Any \( \mathbf{v} \in \mathcal{H} \) admits the Helmholtz-type decomposition

\[
\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, 
\] (33)

where \( \mathbf{v}_0, \mathbf{v}_1 \in \mathcal{H}_j \) but satisfying \( \mathbf{v}_0 \in H^1(\Omega)^d \),

\[
\mathbf{v}_1 = \begin{cases} 
0 & \text{in } \Omega_f, \\
\text{curl} \beta_p & \text{in } \Omega_p, 
\end{cases} 
\] (34)

where \( \beta_p \in H^1(\Omega) \) if \( d = 2 \), while \( \beta_p \in H^1(\Omega_p)^3 \cap H_0(\text{curl}, \Omega_p) \) if \( d = 3 \), with the estimate

\[
\| \mathbf{v}_0 \|_{1,\Omega} + \| \beta_p \|_{1,\Omega_p} \lesssim \| \mathbf{v} \|_{\mathcal{H}}. 
\] (35)

The second result that we need is a regularity result for the solution \( U = (u_j, p, u_p, \phi) \in H \) of (18) is the following theorem:

**Theorem 4.** (see [18], page 710). Let \( U \in H \) be the unique solution of (18). If \( \mathbf{K} \in \mathcal{C}^{1,1}(\Omega) \), then there exists \( \varepsilon > 0 \) such that

\[
u_{\Omega_p} \in [H^{1/2+\varepsilon}(\Omega_p)]^d. 
\] (36)

Let us finish this section by an estimation of the stability error (see [18], Theorem 5):

**Theorem 5.** For any \( U_h = (u_{jh}, p_h, u_{ph}, \phi_h) \in H_h \), we have

\[
\inf_{W_h \in H \cap \mathcal{H}} \| U_h - W_h \|_h \leq I_r(U_h, U_h). 
\] (37)

3.2. Error Estimator. In order to solve the Stokes-Darcy coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the lower and upper error bounds are derived in Section 3.3.

3.2.1. Error Equations. The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier and that involve the data at hand. Thus, we define the error equations: let \( U = (u_j, p, u_p, \phi) \in H \) be the exact solution and \( U_h = (u_{jh}, p_h, u_{ph}, \phi_h) \in H_h \) be the finite element solution. Then, for any \( \mathbf{V}_h = (v_{jh}, q_h, v_{ph}, \psi_h) \in H_h \) and \( \mathbf{V} = (v_j, p, v_p, \phi) \in H \), using the Helmholtz decomposition (Theorem 3), we have

\[
\mathcal{L}_h(U - U_h, \mathbf{V}) = \mathcal{L}_h(U - U_h, \mathbf{V} - \mathbf{V}_h) \\
= \sum_{K \in \mathcal{T}_h} \left( \mathbf{R}_k(U_h), \mathbf{V} - \mathbf{V}_h \right)_K + \sum_{K \in \mathcal{T}_h} \left( \mathbf{R}_k(U_h), \mathbf{V} - \mathbf{V}_h \right)_K, 
\] (38)
where

\[
\begin{aligned}
\left( R_K^e(U_h), V - V_h \right)_K &= (f_j + 2\nu V : D(u_{j,h}) - Vp_h, v_f - v_{j,h})_K \\
&= \left( q_h \cdot V \cdot u_{j,h} \right)_K - \sum_{E \in \mathcal{E}_h(0 \cup K)} \sum_{j=1}^{d-1} H_j E \left( \begin{array}{c}
\frac{2}{\nu K} D(u_{j,h}) \cdot \tau_j + \frac{\alpha}{\sqrt{\nu K}} \frac{u_{j,h} \cdot \tau_j}{\tau_j} (v_f - v_{j,h}) \cdot \tau_j \\
\end{array} \right)_E \\
+ \sum_{E \in \mathcal{E}_h(0 \cup K)} (p_h - 2\nu V : D(u_{j,h}) \cdot n_f - \rho \phi_{j,h}(v_f - v_{j,h}) \cdot n_f)_E.
\end{aligned}
\]  

(40)

\[
\begin{aligned}
\left( R_K^e(U_h), V - V_h \right)_K &= \left( \text{curl} (\rho g K^{-1} u_{ph} + \nabla \phi_h), \beta_p - \beta_{ph} \right)_K \\
&= \left( \rho g \left( f_p - \nabla \cdot u_{ph} \right), \psi - \psi_h \right)_K \\
&- \sum_{E \in \mathcal{E}_h(0 \cup K)} (\rho g K^{-1} u_{ph} + \nabla \phi_h) \times n_E, \psi - \psi_h)_E \\
+ \sum_{E \in \mathcal{E}_h(0 \cup K)} \left( [\rho g \phi_h n_E]_E \right)_K \beta_p - \beta_{ph})_E \\
- \sum_{E \in \mathcal{E}_h(0 \cup K)} (\rho g \phi_h n_E)_E, [v - \psi_h])_E.
\end{aligned}
\]  

3.2.2. Residual Error Estimators

Definition 6 (a posteriori error indicators). The residual error estimator is locally defined by

\[
\Theta_K = \left[ \Theta_{K,j} + \Theta_{K,p}^2 \right]^{1/2} \text{ for each } K \in \mathcal{T}_h,
\]  

(42)

where

\[
\begin{aligned}
\Theta_{K,j}^2 &= \frac{1}{h_K^2} \left\| f_{j,h} + 2\nu V : D(u_{j,h}) - Vp_h \right\|_K^2 + \left\| V : u_{j,h} \right\|_K^2 + \sum_{E \in \mathcal{E}_h(0 \cup K)} \left( \frac{1}{h_E} \left\| 2Vn_f : D(u_{j,h}) \cdot \tau_j + \frac{\alpha}{\sqrt{\nu K}} \frac{u_{j,h} \cdot \tau_j}{\tau_j} (v_f - v_{j,h}) \cdot \tau_j \right\|_E \right) \\
&\quad + \sum_{E \in \mathcal{E}_h(0 \cup K)} \left( \frac{1}{h_E} p_h - 2\nu V : D(u_{j,h}) \cdot n_f - \rho \phi_{j,h}^2 \right)_E.
\end{aligned}
\]  

(43)

\[
\begin{aligned}
\Theta_{K,p}^2 &= \frac{1}{h_K^2} \left\| \text{curl} (\rho g K^{-1} u_{ph} + \nabla \phi_h) \right\|_K^2 + \left\| \rho g \left( f_p - \nabla \cdot u_{ph} \right) \right\|_K^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(0 \cup K \cup \partial_\gamma \mathcal{T}_p)} \left( \frac{1}{h_E} \left\| [\rho g (K^{-1} u_{ph} + \nabla \phi_h)]_E \cdot n_E \right\|_E \right)^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(0 \cup K \cup \partial_\gamma \mathcal{T}_p)} \left( \frac{1}{h_E} \left\| \rho g \phi_n p_E \right\|_E \right)^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(0 \cup K \cup \partial_\gamma \mathcal{T}_p)} \left( \frac{1}{h_E} \left\| \phi_{j,h} (u_{j,h} - u_{ph}) \cdot n_f \right\|_E \right)^2.
\end{aligned}
\]  

(44)

The global residual error estimator is given by

\[
\Theta := \left[ \sum_{K \in \mathcal{T}_h} \Theta_K^2 \right]^{1/2}.
\]  

(45)

Furthermore, denote the local and global approximation terms by

\[
\zeta_{K} = \left\{ \begin{array}{ll}
h_K \left\| f_j - f_{j,h} \right\|_K^2, & \forall K \in \mathcal{T}_h, \\
\rho g \left\| f_p - f_{p,h} \right\|_K, & \forall K \in \mathcal{T}_h.
\end{array} \right.
\]  

(46)

where the global function $f_{j,h} : \Omega \rightarrow \mathbb{R}^d$ is defined by

\[
f_{j,h} = f_K = \frac{1}{|K|} \int_{K} f_j(x) dx, \quad \forall K \in \mathcal{T}_h.
\]  

(47)

while in $\Omega_T$, we take $f_{h,K} = f_K$ for all $K \in \mathcal{T}_h$, as the unique element of $\mathbb{P}^1(K)$ such that

\[
\int_{K} f_K(x) q(x) dx = \int_{K} f(x) q(x) dx, \quad \forall q \in \mathbb{P}^1(K).
\]  

(48)

Remark 7. The residual character of each term on the right-hand side of (43) and (44) is quite clear since if $(u_{j,h}, p_h, \phi_h) \in H_h$ would be the exact solution of (18), then they would vanish.

3.2.3. Analytical Tools

(1) Inverse Inequalities. In order to derive the lower error bounds, we proceed similarly as in [25, 42] (see also [46]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$, and $E \in \mathcal{E}(K)$, we let $b_K$ and $b_K$ be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [41]). In particular, $b_K$ satisfies $b_K \in \mathbb{P}^1(K)$, supp $(b_K) \subseteq K$, $b_K = 0$ on $\partial K$, and $0 \leq b_K \leq 1$ on $K$. Similarly, $b_K \in \mathbb{P}^1(K)$, supp $(b_K) \subseteq \omega_E = \{K' \in \mathcal{T}_h : E \in \mathcal{E}(K')\}$, $b_K = 0$ on $\partial K$, and $0 \leq b_K \leq 1$ in $\omega_E$.

We also recall from [47] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \rightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)|E = p$, $\forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of $L$, that is, the componentwise application of $L$, is denoted by $L$. Additional properties of $b_K$, $b_E$, and $L$ are collected in the following lemma (see [47]).

Lemma 8. Given $k \in \mathbb{N}^*$, there exist positive constants depending only on $k$ and shape-regularity of the triangulations (minimum angle condition), such that for each simplex $K$ and
E ∈ \mathcal{F}(K), there hold

\begin{align}
\|q\|_K &\leq \|q_{h_K}^{1/2}\|_K \leq \|q\|_{K^*}, \quad \forall q \in \mathbb{P}^4(K), \quad (49) \\
|q_{h_K}^{1/2}|_{L^1(K)} &\leq h_K^{1/2} |q|_{L^1(K)}, \quad \forall q \in \mathbb{P}^4(K), \quad (50) \\
\|p\|_K &\leq \|p_{h_K}^{1/2}\|_K \leq \|p\|_{E^*}, \quad \forall p \in \mathbb{P}^4(E), \quad (51) \\
\|L(p)\|_K + h_K |L(p)|_{L^1(K)} &\leq h_K^{1/2} \|p\|_{E^*}, \quad \forall p \in \mathbb{P}^4(E). \quad (52)
\end{align}

(2) Continuous Trace Inequality.

**Lemma 9** (continuous trace inequality). There exists a positive constant \( \beta_1 > 0 \) depending only on \( \sigma_0 \) such that

\[ \|v\|^2_{\mathcal{T}} \leq \beta_1 \|v\|_{K}\|v\|_{L^1(K)}, \quad \forall K \in \mathcal{T}_h, \forall v \in \left[H^1(K)\right]^d. \quad (53) \]

(3) Clément Interpolation Operator. In order to derive the upper error bounds, we introduce the Clément interpolation operator \( I_{Cl}^h : H^1_0(\Omega) \rightarrow \mathcal{P}^d(\mathcal{T}_h) \) that approximates optimally nonsmooth functions by continuous piecewise linear functions:

\[ \mathcal{P}^d(\mathcal{T}_h) = \{ v \in C^0(\Omega) : \forall K \in \mathcal{T}_h, \forall v \in [H^1(K)]^d \}. \quad (54) \]

In addition, we will make use of a vector-valued version of \( I_{Cl}^h \), that is, \( I_{Cl}^h : [H^1_0(\Omega)]^d \rightarrow \mathcal{P}^d(\mathcal{T}_h)^d \), which is defined componentwise by \( I_{Cl}^h \). The following lemma establishes the local approximation properties of \( I_{Cl}^h \) (and hence of \( I_{Cl}^h \)), for a proof see [48], Section 3.

**Lemma 10.** There exist constants \( C_1, C_2 > 0 \), independent of \( h \), such that for all \( v \in H^1_0(\Omega) \), we obtain

\[ \mathcal{L}_h(U - U_h, V) = \mathcal{L}_h(U - U_h, V - V_h) = \sum_{K \in \mathcal{T}_h} \left( R^e_K(U_h, V - V_h) \right)_K + \sum_{K \in \mathcal{T}_h} \left( R^f_K(U_h, V - V_h) \right)_K, \quad (57) \]

where

\[ (R^e_K(U_h, V - V_h))_K = (f_h + 2v \cdot (x - \phi_0) - (x - \phi_0) \cdot (x - \phi_0))_K \]

and

\[ \left( R^f_K(U_h, V - V_h) \right)_K = \left( \int_{\partial \Omega} (p_h - 2v_{\partial \Omega} \cdot (x - \phi_0)) \cdot (x - \phi_0) \cdot (x - \phi_0) \right)_K, \quad (58) \]

The inf-sup condition of \( \mathcal{L}_h \) leads to

\[ \|U - U_h\|_h \leq C \sup_{V \in \mathcal{V}_h} \frac{\|\mathcal{L}_h(U - U_h, V)\|_h}{\|V\|_h}. \quad (60) \]

Now, using the error equation (57)–(59), Cauchy-Schwarz inequality, and the Clément operator of Lemma 10, we deduce the estimate (56). The proof is complete.

3.3.2. Efficiency Result. To prove local efficiency for \( w \in \Omega \), let us denote by

\[ \|\psi_v\|_w^2 = \sum_{K \in \mathcal{C}(\Omega)} |\psi_v|^2_{1,K} + \sum_{K \in \mathcal{C}(\Omega)} \left( \|\nabla \psi_v\|_{1,K}^2 + \|\text{div} \psi_v\|_{1,K}^2 \right) \]

and

\[ \|\psi_v \times \psi_v\|_{1,w}^2 = \sum_{K \in \mathcal{C}(\Omega)} J_K(\psi_v, \psi_v), \quad (61) \]

Proof. We take \( \psi_v = 0 = q_h \) in error equation (38)–(41) and we obtain

\[ \mathcal{L}_h(U - U_h, V) = \mathcal{L}_h(U - U_h, V - V_h) = \sum_{K \in \mathcal{T}_h} \left( R^e_K(U_h, V - V_h) \right)_K + \sum_{K \in \mathcal{T}_h} \left( R^f_K(U_h, V - V_h) \right)_K, \quad (57) \]

where

\[ \left( R^e_K(U_h, V - V_h) \right)_K = (f_h + 2v \cdot (x - \phi_0) - (x - \phi_0) \cdot (x - \phi_0))_K \]

and

\[ \left( R^f_K(U_h, V - V_h) \right)_K = \left( \int_{\partial \Omega} (p_h - 2v_{\partial \Omega} \cdot (x - \phi_0)) \cdot (x - \phi_0) \cdot (x - \phi_0) \right)_K, \quad (58) \]

The inf-sup condition of \( \mathcal{L}_h \) leads to

\[ \|U - U_h\|_h \leq C \sup_{V \in \mathcal{V}_h} \frac{\|\mathcal{L}_h(U - U_h, V)\|_h}{\|V\|_h}. \quad (60) \]

Now, using the error equation (57)–(59), Cauchy-Schwarz inequality, and the Clément operator of Lemma 10, we deduce the estimate (56). The proof is complete.
where
\[
J_K(v_f, v_j) := \sum_{k \in \mathcal{K}(\Omega) \cap \mathcal{E}(K)} \delta h_k^{-1} \left\| [v_j]_k \right\|_E^2;
\]
(62)

\[
\| (\epsilon, \lambda) \|_W := \| \epsilon \|_W + \| \lambda \|_W.
\]

The main result of this subsection can be stated as follows.

**Theorem 12** (efficiency of \( \Theta \)). Under the assumptions of Theorem 4, the following lower error bound holds:

\[
\Theta_K \leq \left\| (e_f, e_p) \right\|_{h_{\tilde{w}_K}} + \left\| (e_p, \lambda_p) \right\|_{\tilde{w}_K} + \sum_{K \subset \tilde{w}_K} \xi_K',
\]
where \( \tilde{w}_K \) is a finite union of neighboring elements of \( K \).

**Proof.** We begin by bounding each term of the residuals separately.

(i) **Element residual in \( \Omega_f \):** to estimate \( h_{K_p}^2 \) \( \| f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h \|_K^2 \), we choose in error equation \( (57) - (59) \) for each \( K \in \mathcal{K}_h \), \( V = (v_1, 0, v_2, 0) \) and \( \nu = (0, 0, 0, 0) \) with \( v_0 = 0 \) on \( \partial \Omega_p \),

\[
v_{f/K} = \begin{cases} f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h & \text{on } \Omega_f \setminus K, \\ 0 & \text{on } \partial \Omega_p \setminus K. \end{cases}
\]

(64)

for obtained, \( \mathcal{L}_h(U - U_h, V) = \| [f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h] \|_{K_p}^2 + (f_j - f_{h/K}) \| \nabla p_h \|_{K}^2. \) Noted that \( \| f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h \|_{K}^2 \) \( b_{K_p}^2 \) \( \| f_j - f_{h/K} \| \nabla p_h \|_{K}^2. \) Because \( \mathcal{L}_h(U - U_h, V) = (\mathcal{D}(e_f, v_f))_K \) in this case, then we have \( (\mathcal{D}(e_f, v_f))_K = \| [f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h] \|_{K_p}^2 + (f_j - f_{h/K}) \| \nabla p_h \|_{K}^2. \). The first inverse inequality (49) and the Cauchy-Schwarz inequality lead to

\[
\| f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h \|_{K_p}^2 - \| (f_j + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h) \|_{K_p}^2 \leq |(f_j - f_{h/K}) \| \nabla p_h \|_{K}^2.
\]

(65)

Using inverse inequality (50), we deduce the estimate:

\[
h_{K_p} \| f_{h/K} + 2vV \cdot \nabla (u_{f/K}) - \nabla p_h \|_{K} \leq \left\| (e_f, e_p) \right\|_{h_{\tilde{w}_K}} + \xi_K'.
\]

(66)

(ii) **Divergence element residual in \( \Omega_f \) (estimation of \( \| \nabla u_{f/K} \|_{K}^2 \):** for each \( K \in \mathcal{K}_h \), we have,

\[
\| \nabla u_{f/K} \|_K^2 \leq \| u_{f/K} - u_{f/K} \|_{1,K}.
\]

(67)

(iii) **Element residual in \( \Omega_p \):** we have for each \( K \in \mathcal{K}_h \),

\[
\| \rho g (f_{p/K} - \nabla \cdot u_{f/K}) \|_K = \| \rho g (f_{p/K} - \nabla \cdot u_{f/K}) - \rho g (f_{p/K} - \nabla \cdot u_{f/K}) \|_K
\]

\[
= \| \rho g \text{ div}_h(u_p - u_{p/K}) - \rho g (f_{p/K} - f_{p/K}) \|_K \leq \left\| (e_f, e_p) \right\|_{h_{\tilde{w}_K}} + \xi_K.
\]

(68)

(iv) **Curl element residual in \( \Omega_p \):** for \( K \in \mathcal{K}_h \), we set \( C_K = \text{curl} \left( \rho g K^{-1} u_{p/K} + \nabla \phi_{f/K} \right) \) and \( W_K = C_K b_{K_p} \). Hence, we notice that \( W_K \) belongs to \( \mathbf{H} \) and is divergence free; therefore, by Equations (57)–(59), we obtain with \( V = \mathbf{0} \) and \( \beta_p = W_K \), \( \mathcal{L}_h(U - U_h, W_K) = (R_K^2(U_h), W_K)_K = \| \text{curl} \left( \rho g K^{-1} u_{p/K} + \nabla \phi_{f/K} \right) \|_{K}. \)

The first inverse inequality (49) and the Cauchy-Schwarz inequality lead to

\[
\| \text{curl} \left( \rho g K^{-1} u_{p/K} + \nabla \phi_{f/K} \right) \|_{K} \leq \| \epsilon_f, e_p \|_{h_{\tilde{w}_K}} + \| (e_p, \lambda_p) \|_{\tilde{w}_K}.
\]

(69)

Again, the inverse inequality (49) allows to get

\[
\| \text{curl} \left( \rho g K^{-1} u_{p/K} + \nabla \phi_{f/K} \right) \|_{K} \leq \| (e_f, e_p) \|_{h_{\tilde{w}_K}} + \| (e_p, \lambda_p) \|_{\tilde{w}_K}.
\]

(70)

(v) **Interface elements on \( \Gamma \):** we fix an edge \( E \) included in \( \Gamma \) and for a constant \( r_{E} \) fixed later on a unit vector \( N \), we consider

\[
W_E = r_{E} b_{E} N,
\]

(71)

that clearly belongs to \( \mathbf{H} \). Hence, by residual equation (57)–(59), we obtain with \( V_h = \mathbf{0} \),

\[
\mathcal{L}_h(U - U_h, W_E) = \left( \mathbf{R}_K^2(U_h), W_E \right)_K + \left( \mathbf{R}_K^2(U_h), W_E \right)_K.
\]

(72)
included in $\tilde{\Omega}_f$ (resp., $\tilde{\Omega}_p$) having $E$ as edge/face, and

\[
\left( R^e_f (U_h), W^e_f \right)_{K_f} = (f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W^e_f)_{K_f} - (q \nabla \cdot u_{j,h})_{K_f}
\]

\[
- \sum_{E \in E_h (\partial K, \partial \Omega_p)} \sum_{i=1}^d \left( \frac{\alpha}{\sqrt{\tau_j} \cdot (K - \tau_j)} u_{j,h} \cdot \tau_j, W^e_f \cdot \tau_j \right)_E
\]

\[
+ \sum_{E \in E_h (\partial K, \partial \Omega_p)} \left( p_h - 2uV \cdot D(u_{j,h}) \cdot \nabla \phi_h + (p_0 - \rho g \phi_h) E \cdot n \right)_E,
\]

\[
(\mathcal{R}^e (U_h), W^e)_{K_p} = \left( \begin{array}{c}
\nabla (\rho g K^{-1} u_{ph} + \nabla \phi_h), \beta_p \nabla \cdot \nabla \phi_h \nabla \cdot \nabla \cdot \nabla \phi_h
\end{array} \right)_{K_p}
\]

\[
+ \left( \rho g \left( f_p - \nabla \cdot u_{ph} \right), \nabla \cdot \nabla \phi_h \nabla \cdot \nabla \cdot \nabla \phi_h \right)
\]

\[- \sum_{E \in E_h (\partial K, \partial \Omega_p)} \left( \rho g K^{-1} u_{ph} + \nabla \phi_h \right) \times n_E, \psi)_E
\]

\[
+ \sum_{E \in E_h (\partial K, \partial \Omega_p)} \left( \left( \rho g \phi_h n_E \right) E \cdot \beta_p \right)_E
\]

\[- \sum_{E \in E_h (\partial K, \partial \Omega_p)} \left( \rho g \phi_h n_E \right) E \cdot \left( W^e_f \right)_E.
\]

(73)

Taken $W_E = 0$ in $K_p$, $q = 0$ in $\Omega$ and for each $j = 1, \cdots$, $d - 1$, $r_E = 2uV \cdot D(u_{j,h}) \cdot \tau_j + (\alpha/\sqrt{\tau_j} \cdot (K - \tau_j)) u_{j,h} \cdot \tau_j$ with $N = \tau_j$. We have

\[
\mathcal{A}_h (U - U_h, W_E) = (f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E)_{K_f}
\]

\[- \left( 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - (f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E)_{K_f}
\]

\[- \left( 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

\[- \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

\[- \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

(74)

Hence,

\[
\left\| \mathcal{A}_h (U - U_h, W_E) \right\|^2 = \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

\[- \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

\[- \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

\[- \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f} - \left( f_j + 2uV \cdot D(u_{j,h}) - \nabla p_h, W_E \right)_{K_f}
\]

Inverse inequalities (50) and (51) and Cauchy-Schwarz inequality lead to

\[
h^1_E \left\| 2uV \cdot D(u_{j,h}) \cdot \tau_j + \frac{\alpha}{\sqrt{\tau_j} \cdot (K - \tau_j)} u_{j,h} \cdot \tau_j \right\| \leq \left\| \left( \epsilon_j, \epsilon_j \right) \right\|_{h, w_E}
\]

\[
+ \left\| \left( \epsilon_p, \lambda \phi \right) \right\|_{w_E} + \sum_{K \in w_E} \xi_K.
\]

(76)

with $w_E = K_f \cup K_p$. 

(77)

(78)

(79)

(80)

(81)

(82)
Thus,
\[
\left\| (K^{-1}u_{\phi} + \nabla \phi_h) \times n_p \right\|_E \leq \left\| (e_f, e_p) \right\|_{h,K} + \left\| (e_p, \lambda) \right\|_{K}.
\]  
(83)

The estimates (66), (67), (70), (76), (77), (81), and (83) provide the desired local lower error bound.

4. Summary

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Stokes-Darcy system. A residual type a posteriori error estimator is provided that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods.

5. Nomenclature

(i) \( \Omega \subset \mathbb{R}^d, d \in \{2, 3\} \) bounded domain
(ii) \( \Omega_p \) : the porous medium domain
(iii) \( \Omega_f \) : the fluid region
(iv) \( \Gamma = \partial \Omega_f \cap \partial \Omega_p \)
(v) \( \Omega_f = \Omega \setminus (\{ \Omega_p \cup \Gamma \} \setminus \Gamma_1 ) \)
(vi) \( \Gamma_1 = \partial \Omega_f \setminus \Gamma \setminus \partial \Omega_p \)
(vii) \( n_j \) (resp., \( n_p \)) the unit outward normal vector along \( \partial \Omega_f \) (resp., \( \partial \Omega_p \))
(viii) \( u \) : the fluid velocity
(ix) \( p \) : the fluid pressure
(x) In 2D, the curl of a scalar function \( w \) is given as usual by
\[
\text{curl } w = (\partial w/\partial x_2 - \partial w/\partial x_1)^T
\]  
(84)

(xi) In 3D, the curl of a vector function \( w = (w_1, w_2, w_3) \)
is given as usual by \( \text{curl } w := \nabla \times w \), namely,
\[
\text{curl } w = (\partial w_3/\partial x_2 - \partial w_2/\partial x_3, \partial w_1/\partial x_3 - \partial w_3/\partial x_1, \partial w_2/\partial x_1 - \partial w_1/\partial x_2)
\]  
(85)

(xii) \( P^k \): the space of polynomials of total degree not larger than \( k \)
(xiii) \( \mathcal{T}_h \): triangulation of \( \Omega \)

(xiv) \( \mathcal{T}_h \): the corresponding induced triangulation of \( \Omega_h \), \( l \in \{ f, p \} \)
(xv) For any \( K \in \mathcal{T}_h \), \( h_K \) is the diameter of \( K \) and \( \rho_K = 2r_K \) is the diameter of the largest ball inscribed into \( K \)
(xvi) \( h = \max_{K \in \mathcal{T}_h} h_K \) and \( \sigma_h = \max_{K \in \mathcal{T}_h} (h_K/\rho_K) \)
(xvii) \( \mathcal{E}_h \): the set of all the edges or faces of the triangulation
(xviii) \( \mathcal{E}(K) \): the set of all the edges (\( N = 2 \)) or faces (\( N = 3 \)) of a element \( K \)
(xix) \( \mathcal{N}_h = \bigcup_{K \in \mathcal{T}_h} \mathcal{N}(K) \)
(xx) \( \mathcal{N}(K) \): the set of all the vertices of a element \( K \)
(xxi) \( \mathcal{N}_h = \bigcup_{K \in \mathcal{T}_h} \mathcal{N}(K) \)
(xxii) For \( \mathcal{A} \subset \Omega \), \( \mathcal{E}_h(\mathcal{A}) = \{ E \in \mathcal{E}_h : E \subset \mathcal{A} \} \)
(xxiii) For \( E \in \mathcal{E}_h \), we associate a unit vector \( n_E \) such that \( n_E \) is orthogonal to \( E \) and equals to the unit exterior normal vector to \( \partial \Omega \)
(xxiv) For \( E \in \mathcal{E}_h \), \( |\phi|_E \) is the jump across \( E \) in the direction of \( n_E \)
(xxv) In order to avoid excessive use of constants, the abbreviations \( x \leq y \) and \( x \sim y \) stand for \( x \leq cy \) and \( c_1x \leq y \leq c_2x \), respectively, with positive constants independent of \( x, y \), or \( F \)

Data Availability

There are no data underlying the findings in this paper to be shared.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Discacciati and A. Quarteroni, "Navier-Stokes/Darcy coupling: modeling, analysis, and numerical approximation," *Revista Matemática Complutense*, vol. 22, pp. 315–426, 2009.
[2] G. Beavers and D. Joseph, "Boundary conditions at a naturally permeable wall," *Journal of Fluid Mechanics*, vol. 30, no. 1, pp. 197–207, 1967.
[3] P. Saffman, "On the boundary condition at the interface of a porous medium," *Studies in Applied Mathematics*, vol. 1, pp. 93–104, 1971.
[4] W. Jäger and A. Mikelić, "On the boundary conditions of the contact interface between a porous medium and a free fluid," *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, vol. 23, 1996.
[5] W. Jäger and A. Mikelić, "On the interface boundary condition of beavers, joseph and saffman," *SIAM Journal on Applied Mathematics*, vol. 60, pp. 1111–1127, 2000.
[40] G. Gatica, R. Oyarzúa, and F.-J. Sayas, “A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem,” Computer Methods in Applied Mechanics and Engineering, vol. 200, no. 21-22, pp. 1877–1891, 2011.

[41] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, Chichester, UK, 1996.

[42] C. Carstensen and G. Dolzmann, “A posteriori error estimates for mixed FEM in elasticity,” Numerische Mathematik, vol. 81, no. 2, pp. 187–209, 1998.

[43] R. Adams, J. Fournier, and S. Spaces, Pure and Applied Mathematics, vol. 140, Elsevier, Amsterdam, 2nd edition, 2003.

[44] D. N. Arnold, F. Brezzi, and M. Fortin, “A stable finite element for the Stokes equations,” Calcolo, vol. 21, no. 4, pp. 337–344, 1984.

[45] F. Brezzi, J. J. Douglas, and L. D. Marini, “Two families of mixed finite elements for second order elliptic problems,” Numerische Mathematik, vol. 47, no. 2, pp. 217–235, 1985.

[46] G. Gatica, “A note on the efficiency of residual-based a-posteriori error estimators for some mixed finite element methods,” Electronic Transactions on Numerical Analysis, vol. 17, pp. 218–233, 2004.

[47] R. Verfürth, “A posteriori error estimation and adaptive mesh-refinement techniques,” Journal of Computational and Applied Mathematics, vol. 50, no. 1-3, pp. 67–83, 1994.

[48] P. Clément, “Approximation by finite element functions using local regularisation,” RAIRO Modélisation Mathématique et Analyse Numérique, vol. 9, pp. 77–84, 1975.