GG-functions and their relations to $\mathcal{A}$-hypergeometric functions

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February 26, 2022

0 Introduction

In [1] I.M.Gelfand introduced the conception of hypergeometric functions, associated with the Grassmanian $G_{k,n}$ of $k$-dimensional subspaces in $\mathbb{C}^n$. The class of these functions includes many classical hypergeometric functions as particular cases. Series of consequent papers of I.M.Gelfand and his coauthors ([2]–[6]) are devoted to the theory of these functions. In particular, the paper [4] is devoted to hypergeometric functions associated with the Grassmanian $G_{3,6}$.

It should be mentioned that some well-known facts concerning classical hypergeometric functions were simply interpreted in these investigations. For example: 24 Kummer relations are known for the Gauss hypergeometric function $F(a,b,c;x)$, which was proved to be connected with the Grassmanian $G_{2,4}$. These relations arise from the natural action of the permutation group $S_4$ at $G_{2,4}$. The Appel function $F_1$ is connected with the Grassmanian $G_{2,5}$. The existence of two integral representations for $F_1$ by Euler integrals (one by single and one by double integrals) arises from the isomorphism of Grassmanians $G_{2,5}$ and $G_{3,5}$.

Further development of the theory of hypergeometric functions is associated with the paper of I.M.Gelfand, M.I.Graev and A.V.Zelevinsky [7]. In this paper general hypergeometric systems of equations were defined and their holonomy was proved. These general hypergeometric systems are called $\mathcal{A}$-hypergeometric systems or GGZ-systems.

Any $\mathcal{A}$-hypergeometric system is defined by a set $\mathcal{A} = \{\omega^1, \ldots, \omega^N\}$ of vectors of $\mathbb{Z}^n$ that linearly generate $\mathbb{C}^n$, and by a vector $\beta \in \mathbb{C}^n$. It consists of the

*The second author is supported by the Russian Foundation for Basic Research (grant 98–01–00798)
following equations for functions on $\mathbb{C}^N$:

\[
\sum_{j=1}^{N} \left( a_j \frac{\partial f}{\partial a_j} \right) \omega^j = f \cdot \beta \tag{0.1}
\]

\[
\prod_{\ell_j > 0} \left( \frac{\partial}{\partial a_j} \right)^{\ell_j} f = \prod_{\ell_j < 0} \left( \frac{\partial}{\partial a_j} \right)^{-\ell_j} f \tag{0.2}
\]

for any $(\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N$ such that $\sum_{j=1}^{N} \ell_j \omega^j = 0$. Analytic solutions of this system are called $\mathcal{A}$-hypergeometric functions.

Equation (0.1) is the condition of homogeneity of the function $f$ with respect to the following action of the complex torus $(\mathbb{C}^*)^n$ at $\mathbb{C}^N$:

\[t : (a_1, \ldots, a_n) \mapsto (t^{\omega^1} a_1, \ldots, t^{\omega^n} a_N),\]

where $t^{\omega^i} = t_1^{\omega^i_1} \cdots t_n^{\omega^i_n}$, $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$. By means of these conditions one can reduce the system (0.1), (0.2) to a system of $r = N - n$ equations for functions defined on the variety of orbits of the torus $(\mathbb{C}^*)^n$ in $\mathbb{C}^N$, i.e., for functions of $r$ arguments.

All classical hypergeometric functions and all hypergeometric functions associated with Grassmanians are $\mathcal{A}$-hypergeometric functions for appropriate sets $\mathcal{A}$.

For example, the Gauss hypergeometric function is $\mathcal{A}$-hypergeometric function associated with a set $\mathcal{A}$ of four vectors $\omega^1, \omega^2, \omega^3, \omega^4$ of 3-dimensional space, connected by the single linear relation $\omega^1 + \omega^2 - \omega^3 - \omega^4 = 0$.

In this paper we describe the results of the papers [8]–[10], where a new approach to the notion of general hypergeometric function was suggested.

Let us illustrate this approach at the example of Gauss hypergeometric function.

Consider the function

\[f(a, b, c; x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} x^n\]

(here $\Gamma$ is the Euler Gamma-function). This function differs from the Gauss function only by a constant multiplier. As a function of four arguments $a, b, c, x$, this function satisfies the following relations:

\[
\frac{d}{dx} f(a, b, c; x) = f(a+1, b+1, c+1; x)
\]

\[
a f(a, b, c; x) + x f(a+1, b+1, c+1; x) = f(a+1, b, c; x) \tag{0.3}
\]

\[
b f(a, b, c; x) + x f(a+1, b+1, c+1; x) = f(a, b+1, c; x)
\]
\[ cf(a, b, c; x) + xf(a + 1, b + 1, c + 1; x) = f(a, b, c - 1; x) \]

(the last three relations are called Gauss relations).

The Gauss differential equation for the function \( f \)

\[ x(1 - x) \frac{d^2 f}{dx^2} + \left( c - (a + b + 1)x \right) \frac{df}{dx} - abf = 0 \quad (0.4) \]

is a corollary of the equations (0.3). Therefore for definition and investigation of the function \( f \) one can start not with a second-order differential equation (0.4), but with the relations (0.3).

Such an approach can be applied to any \( A \)-hypergeometric system. The system of differential equations can be replaced by a system of linear relations between a function \( F \), its first-order partial derivatives, and its shifts with respect to parameters. Namely, instead of \( A \)-hypergeometric system (1.1), (1.2), we introduce a system of equations in the space of functions \( f(\beta, a) \) on \( \mathbb{C}^n \times \mathbb{C}^N \). This system consists of equations (1.1) and the following differential-difference equations:

\[ \frac{\partial f(\beta, a)}{\partial a_j} = f(\beta - \omega^j, a), \quad j = 1, \ldots, N \quad (0.5) \]

here the vectors \( \omega^j \) are not presumed to be integers.

System of equations (1.1), (1.3) is introduced and studied in [8]–[10]. There it is called GG-system associated with \( A = \{ \omega^1, \ldots, \omega^N \} \). Its analytical with respect to \( \beta \) and \( a \) solutions are called GG-functions associated with \( A \). Later, independently, and in a somewhat different form, these functions were introduced and studied in [12]. In [12] they are called quasi hypergeometric functions.

If vectors \( \omega^i \) are integer, then evidently (0.5) implies (1.2). In general case a GG-system cannot be reduced to \( A \)-hypergeometric systems, therefore solutions of GG-systems form a wider class of functions than \( A \)-hypergeometric functions. However, the structure of these solutions (their representations in the form of power series and in the form of integrals) is proved to be similar to the structure of solutions of \( A \)-hypergeometric functions.

One of advantages of the new approach is that “arguments” \( a \) and “parameters” \( \beta \) appear as arguments of same rights. This allows us, in particular, to regard solutions of GG-systems that are distributions with respect to \( \beta \) (see section 5).

In section 8 we give a general definition of GG-systems associated with an arbitrary complex Lie group. This definition includes the initial one as a particular case.
1 GG-systems and GG-functions

1.1 Definition of GG-system

Let $V$ be an $n$-dimensional complex vector space and let

$$A = \{\omega^1, \ldots, \omega^N\}, \quad N \geq n$$

be an arbitrary finite set of vectors from $V$ that linearly generates $V$.

**Definition 1** GG-system, associated with the set $A$, is the following system of equations in the space of functions $f(\beta, a)$ on $V \times \mathbb{C}^N$:

$$\frac{\partial f(\beta, a)}{\partial a_j} = f(\beta - \omega^j, a), \quad j = 1, \ldots, N \quad (1.1)$$

$$\sum_{j=1}^{N} a_j \frac{\partial f}{\partial a_j} \cdot \omega^j = f \cdot \beta \quad (1.2)$$

Solutions of this system in the class of analytical functions of $\beta$ and $a$ are called GG-functions associated with $A$.

**Notes.**

1. From this definition it follows, that GG-functions form a module over the ring of functions $u(\beta)$, satisfying the periodicity conditions:

$$u(\beta + \omega^j) = u(\beta), \quad j = 1, \ldots, N.$$

2. By virtue of (1.1), the equation (1.2) can be replaced by the following equation:

$$\sum_{j=1}^{N} a_j f(\beta - \omega^j, a) \cdot \omega^j = f \cdot \beta \quad (1.3)$$

1.2 An equivalent definition of GG-system

Let $L \subset \mathbb{C}^n$ be an arbitrary fixed linear subspace and let $L^\perp \subset (\mathbb{C}^N)'$ be the orthogonal complement to $L$, i.e., the set of all $\nu \in (\mathbb{C}^N)'$ such that $\langle \nu, \ell \rangle = 0$ for all $\ell \in L$.

**Definition 2** GG-system, associated with the linear subspace $L \subset \mathbb{C}^N$, is the following system of equations in the space of functions $F(\gamma, a), \gamma, a \in \mathbb{C}^N$:

$$\frac{\partial F(\gamma, a)}{\partial a_i} = F(\gamma - e_i, a), \quad i = 1, \ldots, N \quad (1.4)$$

where $e_1, \ldots, e_N$ is the standard basis of $\mathbb{C}^N$.

$$F(\gamma + \ell, a) = F(\gamma, a) \quad \text{for any } \ell \in L \quad (1.5)$$
\[ \sum_{i=1}^{N} \nu_i a_i F(\gamma - e_i, a) = \langle \nu, \gamma \rangle F(\gamma, a) \quad (1.6) \]

for any \( \nu \in L^\perp \).

Let us establish a connection between two definitions 1 and 2. To a set \( A = \{\omega^1, \ldots, \omega^N\} \) of vectors of \( V \) we correspond the following linear map:

\[ \pi : C^N \to V, \quad \pi(e_i) = \omega^i, \quad i = 1, \ldots, N \]

where \( e_1, \ldots, e_N \) is the standard basis of \( C^N \). Define \( L = \ker \pi \). Conversely, if a linear subspace \( L \subset C^N \) is given, then set \( V = C^N / L \) and define \( \omega^i, i = 1, \ldots, N \) as the image of \( e_i \) under the natural projection \( C^N \to V \).

It is evident that if \( f(\beta, a) \) is a function on \( V \times C^N \) and \( F(\gamma, a) \) is its pull-back to \( C^N \times C^N \), then the system (1.1),(1.2) for \( f \) is equivalent to the system (1.4)–(1.6) for \( F \).

1.3 Elementary and reducible cases

1° There exists a unique, up to a coefficient, GG-function, associated with \( L = C^N \), namely \( F(\gamma, a) = e^{a_1 + \ldots + a_N} \).

2° Any GG-function associated with the subspace \( L = 0 \) has the form

\[ F(\gamma, a) = u(\gamma) \prod_{j=1}^{N} \frac{a_j^{\gamma_j}}{\Gamma(\gamma_j + 1)} \]

where \( u(\gamma) \) is an arbitrary periodic function with the period 1 with respect to every \( \gamma_j \).

3° It can easily be checked that the GG-system on \( C^N \times C^N \) can be reduced to a GG-system on a space of lower dimension if one of the following conditions is satisfied:

1. The subspace \( L \subset C^N \) contains a nonzero coordinate subspace

2. The subspace \( L \) is contained in a proper coordinate subspace

3. \( L \) contains at least one vector of the form \( e_i - e_j, i \neq j \), where \( \{e_i\} \) is the standard basis of \( C^N \).

In the sequel we may exclude these cases out of consideration.

In terms of the definition 1 this amounts to the following: first, the cases \( n = 0 \) and \( n = N \) are excluded; second, vectors \( \omega^i \) are presumed to be nonzero, pairwise different, and such that every \( \omega^i \) can be represented as a linear combination of others.
1.4 Reduced GG-systems

We reduce a GG-system, associated with a set \( \mathcal{A} = \{\omega^1, \ldots, \omega^N\} \) of vectors of \( n \)-dimensional space \( V \), to a system of \( N \) equations for a function of \( r = N - n \) arguments.

Fix an arbitrary solution \( v(\beta, a) \) of the system (1.2) and an arbitrary basis \( \{l^i = (l^i_1, \ldots, l^i_n)\}_{i = 1, \ldots, r} \), \( r = N - n \) of the space \( L \).

Lemma 1 Any solution \( f(\beta, a) \) of the system (1.2) can be represented in the following form:

\[
f(\beta, a) = v(\beta, a)F(\beta, x)
\]

where \( x = (a^{l_1}, \ldots, a^{l_r}) \), \( a^{l_i} = a^{l_i_1}_{l_1} \cdots a^{l_i_n}_{l_n} \).

Lemma 2 Functions

\[
\frac{\partial v(\beta, a)/\partial a_i}{v(\beta - \omega^i, a)} \quad \text{and} \quad \frac{v(\beta, a)}{a_i v(\beta - \omega^i, a)} \quad i = 1, \ldots, N
\]

satisfy the equation (where \( \varphi \) is any of these functions):

\[
\sum_{i=1}^{N} \left( a_i \frac{\partial \varphi}{\partial a_i} \right) \omega^i = 0 \tag{1.8}
\]

Corollary These functions can be represented in the form:

\[
\frac{\partial v(\beta, a)/\partial a_i}{v(\beta - \omega^i, a)} = \varphi_i(\beta, x) \tag{1.9}
\]

\[
\frac{v(\beta, a)}{a_i v(\beta - \omega^i, a)} = \psi_i(\beta, x)
\]

where \( x = (a^{l_1}, \ldots, a^{l_r}) \).

Theorem 1 Any GG-function \( f(\beta, a) \) associated with \( \mathcal{A} = \{\omega^1, \ldots, \omega^N\} \) can be represented in the form (1.7), where \( v \) is an arbitrary fixed solution of the system (1.2) and \( F(\beta, x) \) is a solution of the following system

\[
F(\beta - \omega^i, x) = \varphi_i(\beta, x) \cdot F(\beta, x) + \sum_{j=1}^{r} l^i_j \psi_i(\beta, x)x_j \frac{\partial F}{\partial x_j}(\beta, x) \quad i = 1, \ldots, N ;
\]

functions \( \varphi_i \) and \( \psi_i \) are defined by equations (1.9).

Conversely, if \( F \) is a solution of the system (1.10), then the function \( f(\beta, a) \), defined by the equation (1.7), satisfies the GG-system (1.1), (1.2).

Definition 3 We call the system (1.10) reduced GG-system associated with \( \mathcal{A} \), and its solutions in the class of analytical functions of \( \beta \) and \( x \) reduced GG-functions associated with \( \mathcal{A} \).
Let us emphasize, that reduced GG-system depends not only on $A$, but also on the choice of the solution $v$ of the system (1.2) and of the basis $\{l^1, \ldots, l^r\}$ in $L$.

We shall indicate a function $v$ and a basis of $L$ such that the equation (1.11) has the simplest form.

**Definition 4** An $n$-element set $I \subset [1, N]$ is called a base if vectors $\omega^i \in V$, $i \in I$ are linearly independent (and hence form a basis of $V$).

Note that $I$ is a base if and only if $C^I \cap L = 0$, where $C^I \subset C^N$ is the coordinate subspace generated by the vectors $e_i$, $i \in I$.

With any base $I$ we associate the linear map

$\gamma : V \rightarrow C^N$

defined by the equalities

$\gamma(\omega^i) = e_i$, $i \in I$

Let us denote the $i$-th coordinate of $\gamma(\beta)$ by $\beta_i$ (so $\beta = \sum_{i \in I} \beta_i \omega^i$). It is evident that the function

$v(\beta, a) = a^{\gamma(\beta)} = \prod_{i \in I} a_{\beta_i}$ (1.11)

satisfies the equation (1.2). It is evident also that the vectors

$l^j = e_j - \gamma(\omega^j)$, $j \in J = [1, N] \setminus I$ (1.12)

belong to the subspace $L$ and are linearly independent. Therefore they form a basis of $L$.

**Definition 5** The system (1.10), where the function $v$ and the basis $\{l^j\}$ are defined by the equations (1.11) and (1.12), is called reduced GG-system associated with $A$ and the base $I$.

Solutions of this system are called reduced GG-functions.

**Proposition 1** The reduced GG-system associated with $A$ and $I$ has the following form:

$$\beta_i F(\beta, x) + \sum_{j \in J} l^j_i x_j \frac{\partial F(\beta, x)}{\partial x_j} = F(\beta - \omega^i, x) \quad \text{for } i \in I$$ (1.13)

where $l^j_i$ are coordinates of the vector $l^j \in C^N$, i.e., $l^j_i = -\gamma_i(\omega^j)$;

$$\frac{\partial F(\beta, x)}{\partial x_j} = F(\beta - \omega^j, x) \quad \text{for } j \in J$$ (1.14)
2 Description of solutions of GG-systems

2.1 Solutions of a reduced GG-system that are regular in a neighborhood of the point \( x = 0 \)

For an arbitrary base \( I \subset [1, N] \) we shall use the following notation:

\[
\Gamma_I(\beta) = \prod_{i \in I} \Gamma(\beta_i) ; \quad \beta \in V ,
\]

where \( \Gamma \) is the Euler Gamma-function and \( \beta_i \) are the coordinates of the vector \( \beta \) with respect to the basis \( \{\omega^i\}_{i \in I} \) of \( V \).

We shall write \( \mathbf{1} = (1, 1, \ldots, 1) \in V \).

**Theorem 2** Suppose \( F(\beta, x) \) is a solution of the reduced GG-system (1.13), (1.14), associated with a set \( A \) and a base \( I \). If \( F(\beta, x) \) is regular in a neighborhood of the point \( x = 0 \), then it has the following form:

\[
F(\beta, x) = \sum_m \frac{u\left(\beta - \sum_{j \in J} m_j \omega^j\right)}{m!} x^m , \quad (2.1)
\]

where \( m = (m_j)_{j \in J} \), \( m_j \in \mathbb{Z}_+ \), \( \frac{x^m}{m!} = \prod_{j \in J} \frac{x^{m_j}}{m_j!} \), and \( u(\beta) \) is a function on \( V \) that satisfies the periodicity condition:

\[
u(\beta - \omega^i) = u(\beta) \quad \text{for any } i \in I . \quad (2.2)
\]

Conversely, if \( F \) is a formal series of the form (2.1), where \( u(\beta) \) is an arbitrary function satisfying the periodicity condition (2.2), then \( F \) formally satisfies the equations of reduced GG-system (1.13), (1.14).

**Corollary.** Any solution of reduced GG-system (1.13), (1.14), regular in a neighborhood of \( x = 0 \), is determined uniquely by the initial term \( u(\beta)/\Gamma_I(\beta + 1) \) of its power series expansion in powers of \( x \).

**Definition 6** We denote formal power series (2.1), where

\[
u(\beta) = \exp(2\pi i \langle k, \beta \rangle) \quad \langle k, \beta \rangle = \sum_{i \in I} k_i \beta_i , \quad k \in \mathbb{Z}^I ,
\]

by \( F_{I,k} \). We call them reduced GG-series, associated with \( A \) and a base \( I \subset [1, N] \).
It follows from the definition that
\[ F_{I,0}(\beta, x) = \sum_m \frac{x^m}{\Gamma_I(\beta - \sum_{j\in J} m_j \omega^j + 1)} m! \]
\[ F_{I,k}(\beta, x) = \exp(2\pi i \langle k, \beta \rangle) F_{I,0} \left( \beta, e^{-2\pi i \langle k, \omega^j \rangle x} \right), \quad k \in \mathbb{Z}^I \]
where \( e^{-2\pi i \langle k, \omega^j \rangle x} \) is the vector with coordinates \( e^{-2\pi i \langle k, \omega^j \rangle x^j} \), \( j \in J \).

By \( l^j_i, i \in I \) denote coordinates of the vector \( \omega^j, j \in J \) with respect to the basis \( \{\omega^i\}_{i \in I} \).

**Proposition 2** If \( \Re \sum_{i \in I} l^j_i \geq -1 \) for all \( j \in J \),

\[ \text{then the series } F_{I,k} \text{ converge in a neighborhood of } x = 0 \text{ and thus define reduced GG-functions, which are regular in this neighborhood.} \]

**Remark** The series \( (2.1) \) can be represent also in the form
\[ F(\beta, x) = \sum_m u \left( \beta - \sum_{j \in J} m_j \omega^j \right) \frac{\Gamma_{I_1} \left( \beta - \sum_{j \in J} m_j \omega^j \right)}{\Gamma_{I_2} \left( \beta - \sum_{j \in J} m_j \omega^j + 1 \right)} \frac{x^m}{m!}, \]
where \( I = I_1 \sqcup I_2 \) is an arbitrary fixed partition of \( I \), and \( u \) satisfies the following conditions:
\[ u(\beta - \omega^i) = -u(\beta) \text{ if } i \in I_1, \quad u(\beta - \omega^i) = u(\beta) \text{ if } i \in I_2 \]

### 2.2 Solutions of GG-system (1.1), (1.2) that are regular in a neighborhood of coordinate subspaces of \( C^n \).

Let \( I \subset [1, N] \) be an arbitrary base and \( C^I \subset C^N \) be the coordinate subspace generated by vectors \( e_i, i \in I \). We say that \( a \in C^I \) is a generic point if \( a_i \neq 0 \) for all \( i \in I \).

**Theorem 3** Any solution of GG-system \( (1.1), (1.2) \) that is regular in a neighborhood of a generic point of \( C^I \), where \( I \subset [1, N] \) is an arbitrary base, has the following form:
\[ f(\beta, a) = \prod_{i \in I} a_i^{\beta_i} \sum_m \frac{u \left( \beta - \sum_{j \in J} m_j \omega^j \right)}{\Gamma_I \left( \beta - \sum_{j \in J} m_j \omega^j + 1 \right)} \prod_{j \in J} y^{m_j}_j, \quad (2.4) \]
where $u$ satisfies (2.2); $J = [1, N] \setminus I$, $y_j = a_j \prod_{i \in I} \beta_i$; $\beta_i$ and $\omega$ are coordinates of $\beta$ and $-\omega$, respectively, with respect to the basis $\{\omega\}_{i \in I}$.

Conversely, if $f$ is a formal series of the form (2.4), where $u(\beta)$ satisfies (2.2), then $f$ formally satisfies the equations of GG-system (1.1), (1.2).

**Definition 7** By $f_{I,k}(\beta,a)$, $k \in \mathbb{Z}_I$, we denote the formal series (2.4), where $u(\beta)$ = $\exp(2\pi i \langle k, \beta \rangle)$ for any fixed $\beta$.

It follows from the definition that GG-series and reduced GG-series are linked by the relation:

$$f_{I,k}(\beta,a) = a^{\beta} F_{I,k}(\beta,x)$$

where $a^{\beta} = \prod_{i \in I} a_i^{\beta_i}$ and $x \in \mathbb{C}$ is the vector with coordinates $x_j = a_j \prod_{i \in I} a_i^{\beta_i}$.

### 3 Relation to $\mathcal{A}$-hypergeometric functions

Consider the case when all vectors $\omega^1, \ldots, \omega^N$ that form the set $\mathcal{A}$ have integer coordinates with respect to some fixed basis of $V$. Then with the set $\mathcal{A}$ one can associate both GG-system (1.1), (1.2) and $\mathcal{A}$-hypergeometric system (0.1), (0.2).

We noted already that for any fixed $\beta$ a solution of the GG-system (1.1), (1.2) satisfies the $\mathcal{A}$-hypergeometric system (0.1), (0.2).

**Theorem 4** There exists a finite set $\{f_i(\beta,a)\}$ of GG-functions, associated with $\mathcal{A}$ and defined on $V \times U$, where $U \subset \mathbb{C}^N$ is some domain, such that

1. Any solution of GG-system (1.1), (1.2) on $V \times U$ can be uniquely represented in the form:

$$f(\beta,a) = \sum_i u_i(\beta) f_i(\beta,a)$$

where functions $u_i(\beta)$ satisfy the periodicity condition:

$$u_i(\beta + \omega^j) = u_i(\beta) \quad \text{for any} \quad j = 1, \ldots, N$$

2. For any fixed generic $\beta \in V$ the functions $f_i(\beta,a)$ form a basis in the space of solutions of $\mathcal{A}$-hypergeometric system (0.1), (0.2) on $U$.

Let us present a construction of such set of functions $f_i$.

Let $L \subset \mathbb{C}^N$ be the subspace of all vectors $(\ell_1, \ldots, \ell_N)$, satisfying the condition $\sum_{j=1}^N \ell_j \omega^j = 0$, and $\Lambda \subset (\mathbb{Z}^N)'$ be the lattice of integer vectors orthogonal to $L$. By $\Lambda' \subset (\mathbb{Z}^I)'$ denote the projection of $\Lambda$ to $(\mathbb{Z}^I)'$, where $\mathbb{Z}^I$ is the lattice
generated by the vectors \( e_i, i \in I \). In the sequel, we denote \( \mathcal{L}^I = (\mathbf{Z}^I)' / \Lambda^I \).

For any base \( I \) the set \( \mathcal{L}^I \) is finite.

Consider the GG-series \( f_{I,k}(\beta,a) \) from section 2.

Let \( \mathcal{J} \) be a subset of the set of all bases. By \( \mathcal{F}_J \) denote the set of GG-series \( f_{I,k} \), where \( I \) runs through bases from \( \mathcal{J} \), and \( k \) for every base \( I \) runs through one representative of every class \( \kappa \in \mathcal{L} = (\mathbf{Z}^I)' / \Lambda^I \).

**Theorem 5** There exist a subset \( \mathcal{J} \) of bases and a domain \( U \subset \mathbb{C}^N \) such that

1. all GG-series \( f_{I,k}(\beta,a) \in \mathcal{F}_J \) converge in \( U \)

2. the set \( \mathcal{F}_J \) of GG-functions satisfies the conditions of the theorem [4].

### 4 Integral representations of GG-functions

#### 4.1 Formal solutions

Consider a GG-system associated with a set \( \mathcal{A} = \{\omega^1, \ldots, \omega^n\} \) of vectors of a space \( V \) of dimension \( n \). Fix an arbitrary basis of \( V \) and for \( \beta \in V \) denote coordinates of \( \beta \) with respect to this basis by \( \beta_1, \ldots, \beta_n \). Also denote: \( T = V \setminus \Sigma \), where \( \Sigma \) is the union of all coordinate subspaces of \( V \).

**Proposition 3** Integrals of the following form formally satisfy the GG-system [1],[2],[3]:

\[
f(\beta,a) = \frac{1}{C} \exp \left( \sum_{j=1}^{N} a_j t^{\omega_j} \right) t^{-\beta} \prod_{i=1}^{n} \frac{dt_i}{t_i} ; \quad (4.1)
\]

where \( t^{\omega_j} = t_1^{\omega_j^1} \ldots t_n^{\omega_j^n} \), \( t^{-\beta} = t_1^{-\beta_1} \ldots t_n^{-\beta_n} \) and \( C \subset T \) is an arbitrary cycle of real dimension \( n \).

By the change of variables \( t_i = e^{s_i} \) the integral (4.1) can be represented in the following form:

\[
f(\beta,a) = \frac{1}{\Gamma} \exp \left( \sum_{j=1}^{N} a_j e^{(\omega_j,s)} - \langle \beta, s \rangle \right) ds_1 \ldots ds_n , \quad (4.2)
\]

where \( \Gamma \) is an \( n \)-dimensional cycle in the space \( V' \) dual to \( V \).

If the mentioned basis is the set \( \{\omega^i\}_{i \in I} \), where \( I \subset [1,N] \) is a base, then the integral (4.3) has the following form:

\[
f(\beta,a) = \frac{1}{C} \exp \left( \sum_{i \in I} a_i t^i + \sum_{j \notin I} a_j t^j \right) t^{-\beta} \prod_{i \in I} \frac{dt_i}{t_i} , \quad (4.3)
\]

where \( t^j = \prod_{i \in I} t_i^{\ell^j_i} \) and \( \ell^i_j \) are coordinates of \( \omega^j, j \notin I \) with respect to the chosen basis.
Proposition 4 For any base $I \subset [1, N]$ the following integrals formally satisfy the reduced GG-system (1.13), (1.14) associated with the base $I$:

$$F(\beta, x) = \int_C \exp \left( \sum_{i \in I} t_i + \sum_{j \notin I} x_j t^\ell_j \right) t^{-\beta} \prod_{i \in I} \frac{dt_i}{t_i}; \tag{4.4}$$

here $C \subset T$ is an arbitrary cycle of real dimension $n$.

Remark. If $\sum_{i \in I} x_j = 1$ for every $j \notin I$ then the following integrals formally satisfy also system (1.13), (1.14):

$$F(\beta, x) = \Gamma^{-1} (\beta I + 1) \int_C \left( 1 + \sum_{j \notin I} x_j e^{\omega_j / \omega_i} \right) \prod_{i \in I} x_j^{\beta_i - 1} ds, \tag{4.5}$$

where $\beta_I = \sum_{i \in I} \beta_i$ and $C$ is an arbitrary cycle of real dimension $n - 1$ in the hyperplane $\sum_{i \in I} s_i = 1$.

Similarly to (4.1), the integral (4.4) can be represented in the form:

$$F(\beta, x) = \int \exp \left( \sum_{i \in I} e^{\omega_i} + \sum_{j \notin I} x_j e^{\omega_j / \omega_i} - (\beta, s) \right) ds_1 \ldots ds_n \tag{4.6}$$

The problem is to find a family of cycles such that these integrals have nonempty domain of convergency. One may solve this problem only for the integrals (4.4).

4.2 Case $n = 1$.

In the case $n = 1$ the set $A$ is a set of nonegative numbers $\omega^1, \ldots, \omega^N \in \mathbb{C}$, and bases are one-element subsets of $[1, N]$. Integral (4.4) associated with an arbitrary base $I = \{i\}$ has the form:

$$F(\beta, x) = \int \exp \left( t + \sum_{j \neq i} x_j e^{\omega_j / \omega_i} \right) t^{-\beta - 1} dt \tag{4.7}$$

Proposition 5 Let $C \subset \mathbb{C}$ be a loop, going out of $-\infty$ by the real negative axis, around $0$, and again to $-\infty$ by the negative axis. If $\text{Re}(\omega_j / \omega_i) \leq 1$, then the integral (4.7) converge in a neighborhood of $x = 0$ and coincides with the function:

$$F_0(\beta, x) = 2\pi i \sum_m \left( \Gamma^{-1} \left( \beta - \sum_{j \neq i} \frac{\omega_j}{\omega_i} m_j + 1 \right) \prod_{j \neq i} \frac{x_j^m}{m_j!} \right)$$
4.3 Case \( n > 1 \).

Let us regard the integral (4.4) connected with an arbitrary base \( I \subset [1, N] \).

Proposition 6 If \( \sum_{i \in I} \text{Re} \ell_i \leq 1 \) for all \( j \in [1, N] \setminus I \) then there exists a cycle such that the integral (4.4) converges in a neighborhood of the point \( x = 0 \).

Let us present a description of the cycle \( C \). Let us regard the change of variables \( t = \rho u \), where \( \rho \in C \setminus \{0\} \) and \( u \) is a point of the plane \( U = \{ u \in T | \sum u_i = 1 \} \).

By means of this change the integral (4.4) can be reduced to the integral of the following form:

\[
F(\beta, x) = \int_C \exp \left( \rho + \sum_{j \notin I} x_j \rho^{s_j} \prod_{i \in I} u_i^{\ell_i} \right) \times (4.8) \\
\times \rho^{-\sum_{i \in I} \beta_i - 1} \prod_{i \in I} u_i^{-\beta_i - 1} \, du
\]

where \( r_i = \sum_{i \in I} \ell_i \) and \( du \) is a holomorphic volume form on \( U \). We shall describe the cycle \( C \) in coordinates \( \rho \) and \( u \).

To give a precise meaning to the integral (4.8) let us replace the multifunction on \( U \) under the integral by a function on a ramified covering \( \tilde{U} \cong C^{n-1} \) over \( U \). For this end let us introduce an infinite covering \( \tilde{T} \cong C^I \) over \( T \) such that the prototype of a point \( t \in T \) consists of points \( s \in C^I \) defined by the equality \( e^{st} = t_i, i \in I \). We define \( \tilde{U} \) as the prototype of \( U \) under this covering. We shall interprete the multiformal from integral (4.8) as single-valued form on \( \tilde{U} \).

Let us introduce a cycle \( \tilde{\Gamma} \subset \tilde{U} \), which is a multidimensional analog of double loop. The cycle \( \tilde{\Gamma} \) is associated with the simplex

\[
\Delta = \{ u \in U \cap R^I | u_i > 0 \},
\]

its construction see in [7], [11]. The desired cycle \( C \subset (C \setminus \{0\})^n \times \tilde{U} \) is equal to \( C_1 \times \tilde{\Gamma} \), where \( C_1 \subset C \setminus \{0\} \) is a loop around 0 along the negative axis (the same as \( C \) from the proposition 5).

4.4 Integral representation of GG-functions associated with real vectors.

With any base \( I \subset [1, N] \) we associate the following \( n \)-dimensional cycles \( \Gamma_k \subset C^I, k \in Z^I \):

\[
\Gamma_k = \{ s \in C^I | \text{Im} s_i = (2k_i + 1)\pi, i \in I \}
\]

Proposition 7 If all vectors \( \omega^1, \ldots, \omega^N \) are real (with respect to some basis in \( V \)), then the integral (4.7) over the cycle \( \Gamma_k \) converges in the domain

\[
\text{Re} \beta_i > 0, \quad i \in I,
\]
and thus is a reduced GG-function.

We denote this function by $F_k(\beta, x)$.

**Note.** The function $F_K$ can be extended by analiticity to all values of $\beta$.

## 5 Special cases

### 5.1 Case $N = n + 1$

In this case the reduced GG-functions are functions $F(\beta, x)$ on $\mathbb{C}^N \times \mathbb{C}$. The reduced GG-system has the following form:

\[
\beta_i F(\beta, x) + \ell_i x \frac{dF(\beta, x)}{dx} = F(\beta - e_i, x), \; i = 1, \ldots, n \tag{5.9}
\]

\[
\frac{dF(\beta, x)}{dx} = F(\beta + \ell, x) \tag{5.10}
\]

where $\ell = (\ell_1, \ldots, \ell_N)$ is an arbitrary fixed vector in $\mathbb{C}^N$. Every solution of (5.9), (5.10) that is regular in a neighborhood of $x = 0$ has the following form

\[
F(\beta, x) = \sum_{m=0}^{\infty} u(\beta + m\ell)c(m)\frac{x^m}{m!},
\]

where $c(m) = \left( \prod_{i=1}^{n} \Gamma(\beta_i + m\ell_i + 1) \right)^{-1}$, and $u$ is an arbitrary periodical function on $\mathbb{C}^n$ with period 1 with respect to every $\beta_i$.

**Examples.** If $n = 1$ then

\[
F(\beta, x) = \sum_{m=0}^{\infty} \frac{u(\beta + m\ell)}{\Gamma(\beta + m\ell + 1)} \frac{x^m}{m!}, \; \beta, \ell \in \mathbb{C}
\]

If $n = 2$ then

\[
F(\beta_1, \beta_2, x) = \sum_{m=0}^{\infty} \frac{u(\beta_1 + m\ell_1, \beta_2 + m\ell_2)}{\Gamma(\beta_1 + m\ell_1 + 1)\Gamma(\beta_2 + m\ell_2 + 1)} \frac{x^m}{m!}.
\]

The integral representation of the GG-functions has the form

\[
F(\beta, x) = \int_{\mathbb{C}} \exp(t_1 + \ldots + t_n + t_1^{-\ell_1} \ldots t_n^{-\ell_n}x) \prod_{i=1}^{n} t_i^{-\beta_i} dt_i
\]
5.2 Case $n = 1$.

In this case the reduced GG-functions are functions $F(\beta, x_1, \ldots, x_r)$ on $\mathbb{C} \times \mathbb{C}^r$. The reduced GG-system has the form:

\begin{align*}
\beta F(\beta, x) + \sum_{j=1}^{r} \ell^j x_j \frac{\partial F(\beta, x)}{\partial x_j} &= F(\beta - 1, x) \\
\frac{\partial F(\beta, x)}{\partial x_j} &= F(\beta + \ell^j, x)
\end{align*}

(5.11) (5.12)

where $\ell = (\ell^1, \ldots, \ell^r)$ is an arbitrary fixed vector in $\mathbb{C}^r$. Every solution of (5.11), (5.12) that is regular in a neighborhood of $x = 0$ has the following form

$$F(\beta, x) = \sum_{m} \left( \frac{u(\beta + \sum_{j=1}^{r} m_j \ell^j)}{\Gamma(\beta + \sum_{j=1}^{r} m_j \ell^j + 1)} \prod_{j=1}^{r} x_j^{m_j} \right),$$

where $u$ is an arbitrary periodic function on $\mathbb{C}$ with period 1.

The integral representation of the GG-functions has the form

$$F(\beta, x) = \int \exp(t + t^{-\ell^1} x_1 + \ldots + t^{-\ell^r} x_r) t^{-\beta - 1} dt$$

This integral converges for every $x$ in case when Re $\beta > 0$, Re $\ell^j > -1$, $j = 1, \ldots, r$ and $\Gamma$ is a loop going from $-\infty - i0$ to $-\infty + i0$ around the point 0.

6 GG-distributions

6.1 Definition of GG-distributions

Let us denote the space of compactly supported $C^\infty$-functions on $\mathbb{R}^n$ by $K_n$, and the space of Fourier transforms of functions from $K_n$ by $Z_n$, with natural topologies. According to Paley-Wiener theorem the space $Z_n$ consists of entire analytical functions $F(z_1, \ldots, z_n)$, satisfying the estimation:

$$|z_1^{q_1} \ldots z_n^{q_n} F(z_1, \ldots, z_n)| \leq C_q e^{a_1 |\text{Im} z_1| + \ldots + a_n |\text{Im} z_n|}$$

for any $q_1, \ldots, q_n = 0, 1, \ldots,$

We say that elements of dual spaces $K_n'$ and $Z_n'$, i.e., continuous linear functionals on $K_n$ and $Z_n$, are distributions on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. In particular, any continuous function $f(\xi)$ on $\mathbb{R}^n$ defines a continuous linear functional on $K_n$ by the following formula:

$$(f, F) = \int_{\mathbb{R}^n} f(\xi) F(\xi) d\xi$$
thus \( f \) can be regarded as a distribution on \( \mathbb{R}^n \).

Suppose \( f \) is a distribution on \( \mathbb{R}^n \). The Fourier transform of \( f \) is the linear functional \( \tilde{f} \) on \( Z_n \) such that for any function \( F \in K_n \) and its Fourier transform \( \tilde{F} \in Z_n \) the following equality holds:

\[
\left( \tilde{f}, \tilde{F}(z) \right) = (2\pi)^n \left( f, F(-\xi) \right).
\]

In particular, the Fourier transform of the function \( e^{i\langle a, \xi \rangle} \), where \( a = (a_1, \ldots, a_n) \), \( \langle a, \xi \rangle = \sum a_i \xi_i \), is the distribution \((2\pi)^n \delta(z - a) \in Z'_n\), defined by the equality

\[
\left( \delta(z - a), \tilde{F}(z) \right) = \tilde{F}(a)
\]

for any \( \tilde{F} \in Z'_n \).

We denote the space of analytical functions on \( \mathbb{C}^n \) with values in \( Z'_n \) by \( H = H_{n,r} \). We denote elements of \( H \) by \( \varphi(\beta, x), \beta \in \mathbb{C}^n, x \in \mathbb{C}^r \).

Let us regard the reduced GG-system (1.13),(1.14) associated with a set \( \mathcal{A} = \{\omega^1, \ldots, \omega^N\} \) of vectors of an \( n \)-dimensional linear space \( V \) and a base \( I \subset [1,N] \). Without loss of generality we may assume that \( I = [1,n] \) and denote \( \omega^i = e_i, i = 1, \ldots, n \) and \( \omega^{n+q} = \ell^q = \sum_{i=1}^n \ell^q_i e_i \) for \( q = 1, \ldots, r, r = N - n \). Then the equations of GG-system take the following form:

\[
\frac{\partial \varphi(\beta, x)}{\partial x_q} = \varphi(\beta - e_q, x), \quad q = 1, \ldots, r
\]

(6.1)

\[
\varphi(\beta - e_p, x) + \sum_{q=1}^r \ell^q_p \varphi(\beta - \ell^q, x) = \beta_p \varphi(\beta, x), \quad p = 1, \ldots, n
\]

(6.2)

where \( \{e_i\} \) is the standard basis of \( \mathbb{C}^n \). It is evident, that these equations have sense also for elements of the space \( H \).

**Definition 8** Elements of \( H \) that satisfy (6.1),(6.2) are called GG-distributions associated with \( \mathcal{A} \) and \( I \).

### 6.2 Description of GG-distributions

**Theorem 6** For any GG-system (6.1),(6.2) there exists a unique up to a constant multiplier GG-distribution \( f \). This distribution is the Fourier transform with respect to \( \xi = (\xi_1, \ldots, \xi_n) \) of the following function:

\[
F(\xi, x) = \exp \left( \sum_{p=1}^n e^{i \ell^p} + \sum_{q=1}^r x_q e^{i(\ell^q, \xi)} \right),
\]

where \( \ell^q, \xi = \ell^q_1 \xi_1 + \ldots + \ell^q_n \xi_n \).

Thus formally \( f \) is defined by the integral

\[
f(\beta, x) = \int_{\mathbb{R}^n} \exp \left( e^{i \ell^p} + \sum_{q=1}^r x_q e^{i(\ell^q, \xi)} - i \langle \beta, \xi \rangle \right) \, d\xi_1 \ldots d\xi_n
\]
6.3 Representation of the GG-distribution as series

**Theorem 7** The GG-distribution \( f(\beta, x) \) can be represented as the following power series with respect to \( x \):

\[
f(\beta, x) = \sum_{m \in \mathbb{Z}^n_+} \left( c_m(\beta) \prod_{q=1}^{k} \frac{x^{m_q}}{m_q!} \right),
\]

where

\[
c_m(\beta) = \sum_{r \in \mathbb{Z}^n_+} \left( \prod_{p=1}^{n} r_{p}^{-1} \right)^{-1} \delta \left( \beta_1 - \sum_{q=1}^{k} \ell_{q}^{1} m_q - r_1, \ldots, \beta_n - \sum_{q=1}^{k} \ell_{q}^{n} m_q - r_n \right),
\]

\( \delta(\beta_1, \ldots, \beta_n) \) is the delta-function on \( \mathbb{C}^n \).

In other words, for any function \( \varphi \in \mathbb{Z}^n_+ \) we have

\[
(f, \varphi) = \sum_{m \in \mathbb{Z}^n_+} (c_m, \varphi) \frac{x^m}{m!},
\]

where

\[
x^m = \prod_{q=1}^{k} \frac{x^{m_q}}{m_q!},
\]

\[
(c_m, \varphi) = \sum_{r \in \mathbb{Z}^n_+} (r!)^{-1} \varphi \left( \sum_{q=1}^{k} \ell_{q}^{1} m_q + r_1, \ldots, \sum_{q=1}^{k} \ell_{q}^{n} m_q + r_n \right).
\]

**Proposition 8** Series (6.3) converges for any \( \varphi \in \mathbb{Z}^n_+ \) and \( x \in \mathbb{C} \).

The distributions \( c_m \in \mathbb{Z}^n_+ \) are analytical functionals, i.e., \( (c_m, \varphi) \) can be represented in the form:

\[
(c_m, \varphi) = \int_{\Gamma_m} \frac{F_m(\beta) \varphi(\beta) d\beta}{\Gamma(\ell_p m - \beta_p)}
\]

where \( F_m \) is a function and \( \Gamma_m \subset \mathbb{C}^n \) is a surface of real dimension \( n \). In the case \( k = 1 \) we have:

\[
F_m(\beta) = \prod_{p=1}^{n} e^{\pi i (\ell_{p} m - \beta_p)} \Gamma(\ell_p m - \beta_p)
\]

and \( \Gamma_m = \gamma_{m_1}^1 \times \ldots \times \gamma_{m_n}^n \), where \( \gamma_{m_p}^p \subset \mathbb{C} \) is a contour that goes around poles of the function \( \Gamma(\ell_p m - \beta_p) \).
6.4 Connections with series of hypergeometric type

There exists a formal operation $\varphi \mapsto f$ that refers a GG-distribution $f$ to any GG-function $\varphi$ if $\varphi$ is defined by a series of hypergeometric type. This operation is replacing the multipliers $\Gamma(u)$ and $1/\Gamma(u+1)$ in the coefficients of the series by, respectively, the distributions $\gamma^+(u)$ and $\gamma^-(u)$, where

$$\gamma^+(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \delta(s + m), \quad \gamma^-(s) = \sum_{m=0}^{\infty} \frac{1}{m!} \delta(s - m);$$

$\delta(s)$ is the delta-function on $\mathbb{C}$.

The functions $\gamma^+(s)$ and $\gamma^-(s)$ satisfy the following functional relations:

$$\gamma^+(s + 1) = s\gamma^+(s), \quad \gamma^-(s - 1) = s\gamma^-(s),$$

i.e., the same relations as $\Gamma(s)$ and $1/\Gamma(s+1)$ respectively.

By theorem 6, the images under this operation of all hypergeometric-type series that satisfy system (1.1),(1.2) differ only by constant multipliers.

7 Resonance GG-systems

There exist GG-systems such that their solutions satisfy some additional differential equations under certain relations between parameters and certain regularity conditions. We call such GG-systems resonance.

7.1 Resonance sets

Let $A = \{\omega\}$ be a finite set of non-zero vectors linearly generating a space $V$. Let us introduce the following notation for an arbitrary vector $v \in V$:

$$A_v = \{\omega \in A \mid \omega + v \in A \cup \{0\}\}, \quad B_v = A \setminus A_v,$$

$L_v \subset V$ is the linear subspace generated by the vectors $\omega \in B_v$. In particular, $A_0 = A$, $B_0 = \emptyset$, $L_0 = 0$.

Definition 9 A vector $v \in V$ is called consistent with the set $A$ if $L_v$ is a proper subspace of $V$.

In particular, the vector $v = 0$ is consistent with any $A$. Obviously, if a vector $v \neq 0$ is consistent with $A$, then either $v = \omega' - \omega$ or $v = -\omega'$, where $\omega, \omega' \in A$. Thus vectors in general position are not consistent with $A$.

Example: $A = \{\omega^1, \ldots, \omega^{n+1}\}$, where $\omega^i$ are vectors from $\mathbb{C}^n$, and $\omega^1, \ldots, \omega^n$ are linearly independent. In this case all vectors $\omega^{n+1} - \omega^i$, $i = 1, \ldots, i$ as well as the vector $-\omega^{n+1}$ are consistent with $A$. 

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It is easily proved that for any consistent with \( A \) vector \( v \neq 0 \) we have:
\[
\text{codim } L_v = 1, \quad v \notin A, \quad v \notin L_v, \quad \text{and } A_v \cap L_v = \emptyset.
\]
This implies that any set \( A \) of \( N \) vectors of \( V \) such that a fixed nonzero vector \( v \) is consistent with \( A \), has the following form:
\[
A = \{ \omega^j - jv \mid j = 0, 1, \ldots, k_i - 1; i = 1, \ldots, r \} \cup \{-jv \mid j = 1, \ldots, k_0 - 1\},
\]
where \( \omega^1, \ldots, \omega^r, r < N \) are arbitrary pairwise different vectors such that the space, generated by them, has codimension 1 and does not contain \( v \), and \( k_0, \ldots, k_r \) are arbitrary natural numbers such that \( \sum_{i=1}^{r} k_i = N + 1 \) (if \( k_0 = 1 \), then the second set in (7.1) is assumed to be empty).

### 7.2 Resonance GG-systems

Let us regard a GG-system on \( V \times \mathbb{C}^N \), associated with a set \( A = \{ \omega \} \) of \( N \) nonzero vectors of \( V \), linearly generating \( V \). For convenience we shall enumerate coordinates in \( \mathbb{C}^N \) not by numbers, but by elements of \( A \), i.e., instead of \( a_i, \ i = 1, \ldots, N \) we shall write \( a_\omega, \ \omega \in A \). In this notation the GG-system has the following form:
\[
\frac{\partial f(\beta, a)}{\partial a_\omega} = f(\beta - \omega, a), \ \omega \in A \quad (7.2)
\]
\[
\sum_{\omega \in A} a_\omega \frac{\partial f}{\partial a_\omega} \cdot \omega = f(\beta, a) \cdot \beta \quad (7.3)
\]

**Definition 10** The GG-system (7.2), (7.3) is called resonance if there exists at least one nonzero consistent with \( A \) vector \( v \in V \).

**Definition 11** Suppose \( v \neq 0 \) is an arbitrary consistent with \( A \) vector. Let us associate with \( v \) the following hyperplane:
\[
\Pi_v = L_v + v
\]

**Theorem 8** Suppose GG-system (7.2), (7.3) is resonance, \( v \in V \) is a nonzero consistent with \( A \) vector, and \( f(\beta, a) \) is an arbitrary solution of the GG-system such that \( f \) is regular on the hyperplane \( \Pi_v \subset V \); then \( f \) satisfy the following additional relation:
\[
\sum_{\omega \in A_v} (\lambda, \omega) a_\omega \frac{\partial f(\beta, a)}{\partial a_{\omega + v}} = 0 \quad \text{for } \beta \in \Pi_v,
\]
where \( \lambda \neq 0 \) is the vector of the dual space \( V^* \), orthogonal to \( L_v \).
7.3 Example of resonance GG-system

Let $C^N = C^p \otimes C^n$ be the space of $p \times n$-matrices $a = \|a_{ij}\|$ ($p < n$), $A$ be the set of $N = pn$ vectors $\omega_{ij} = e_j + d_i \in C^n \oplus C^p$, where $e_1, \ldots, e_n$ and $d_1, \ldots, d_p$ are bases in $C^n$ and $C^p$ respectively, $V \subset C^n \oplus C^p$ be the linear space generated by the vectors $\omega_{ij}$.

The GG-system in the space of functions $f(\alpha, \beta; a), (\alpha, \beta) \in V, a \in C^N$, associated with the set $A$, has the following form:

$$\frac{\partial f(\alpha, \beta; a)}{\partial a_{ij}} = f(\alpha - e_j, \beta - d_i; a), \quad i = 1, \ldots, p, \quad j = 1, \ldots, n$$

Let

$$\sum_{i=1}^{p} a_{ij} \frac{\partial f}{\partial a_{ij}} = \alpha_j f, \quad j = 1, \ldots, n$$

(7.4)

$$\sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial a_{ij}} = \beta_i f, \quad i = 1, \ldots, p$$

Note that the condition $(\alpha, \beta) \in V$ is equivalent to the relation $\sum \alpha_j = \sum \beta_i$.

It is easily verified that all vectors $v_{ii'} = d_{i'} - d_i \in V$ are consistent with $A$.

We have: $A_{v_{ii'}} = \{\omega_{ij} \mid j = 1, \ldots, n\}$, and the hyperplane $\Pi_{v_{ii'}}$ is defined by the equation $\beta_i = -1$ and therefore does not depend on $i'$.

**Proposition 9** Any GG-function $f(\alpha, \beta; a)$ that is regular for $\beta_i = -1$, satisfy for $\beta_i = -1$ the following additional relations:

$$\sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial a_{ij}} = 0, \quad i' = 1, \ldots, p, \quad i \neq i'.$$

We denote restrictions of GG-functions $f(\alpha, \beta; a)$ to the plane in $V$, defined by the equations $\beta_i = -1, i = 1, \ldots, p$, by $\varphi(\alpha, a)$.

**Proposition 10** Functions $\varphi(\alpha, a)$ satisfy the following system of equations:

$$\sum_{i=1}^{p} a_{ij} \frac{\partial \varphi}{\partial a_{ij}} = \alpha_j \varphi, \quad j = 1, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} \frac{\partial \varphi}{\partial a_{ij}} = -\delta_{ii'} \varphi $$

(7.5)

$$\frac{\partial^2 \varphi}{\partial a_{ij} \partial a_{i'j'}} = \frac{\partial^2 \varphi}{\partial a_{ij} \partial a_{i'j}}$$

**Remark.** According to [1] the system (7.3) is the hypergeometric system of equations, associated with the Grassmanian $G_{p,n}$ of $p$-dimensional subspaces of $C^n$. We see that this system arises as resonance case of general GG-system (1.1), (1.2).
8 GG-functions associated with an arbitrary complex Lie group [14]

We give here a definition of GG-systems associated with an arbitrary complex Lie group $G$. This definition includes the definition of section 1 as a particular case. Earlier M.Kapranov [13] generalized the definition of $A$-hypergeometric system onto arbitrary reductive Lie groups.

Let $G$ be an arbitrary connected complex Lie group and let $v \mapsto vg, v \mapsto gv, \ g \in G$ be its actions by the right and left translations in the space of functions on $G$. Denote by $\mathcal{H}$ the space of analytic functions $v$ on $G$ such that the linear space generated by the right translations $vg, g \in G$, is finite-dimensional. Evidently $\mathcal{H}$ is an algebra with respect to multiplication of functions, and $\mathcal{H}$ is invariant under the right and left translations of $G$.

Let $H \subset \mathcal{H}$ and $V \subset \mathcal{H}$ be linear subspaces that are invariant under the right translations. Suppose that $H$ is finite dimensional and $V$ is closed under multiplication by elements of $H$.

**Definition 12** The following system of equations in the space of functions $f(v,h)$ on $V \times H$ is called a GG-system associated with $V$ and $H$:

$$f(vg,h) = f(v,h) \quad \text{for every } g \in G, \quad (8.6)$$

$$\frac{\partial f(v,h)}{\partial h^0} = f(h^0 v,h) \quad \text{for every } h^0 \in H \quad (8.7)$$

where $\frac{\partial f(v,h)}{\partial h^0} = \frac{d}{dt} f(v,h+th^0) \bigg|_{t=0}$. A solution $f(v,h)$ of equations (8.6),(8.7) linear with respect to $v$ and analytical with respect to $h$ is called a GG-function associated with $V$ and $H$.

**Remarks.** 1° If $h_1(g), \ldots, h_N(g)$ is a fixed basis in $H$ then the equations (8.7) have the following form in the coordinates $(x_1, \ldots, x_N)$ corresponding to this basis:

$$\frac{\partial f(v,x)}{\partial x_j} = f(h_j v, x), \ j = 1, \ldots, N \quad (8.8)$$

2° If $B$ is a basis in $V$, then GG-functions may be regarded as functions on $B \times H$.

The GG-system (1.1),(1.2) is a special case of the system (8.6),(8.7) when $G$ is an additive group $\mathbb{C}^n$, $H$ is the linear space with the basis $h_i(t) = e^{\langle \omega_i, t \rangle}$, $t \in \mathbb{C}^n$, and $V$ is linearly generated by all vectors $h_\beta(t) = e^{\langle \beta, t \rangle}$, $\beta \in (\mathbb{C}^*)^n$.

**Proposition 11** The following integrals formally satisfy the system (8.6),(8.7):

$$f(v, h) = \int \mathcal{C} e^{h(s)} v(g) \ dg, \quad (8.9)$$
where \( dg \) is a right-invariant holomorphic volume form and \( C \subset G \) is a cycle of real dimension equal to \( \dim C = \dim G \).

Let us consider a case when the orbits of \( G \) on the space \( H \) are algebraic submanifolds. Let \( P \subset \mathbb{C}[[\xi_1, \ldots, \xi_N]] \) be the subring of all polynomials \( p(\xi) = p(\xi_1, \ldots, \xi_N) \) such that \( p(h_1(g), \ldots, h_N(g)) \equiv 0 \), where \( h_1, \ldots, h_N \) is a fixed basis in \( H \).

Fix an arbitrary finite-dimensional subspace \( L \subset H \) and denote its dual by \( L^* \).

**Definition 13** The following system of equations in the space of functions \( F(h) \) on \( H \) with values in \( L^* \) is called the general GGZ-system associated with \( L \) and \( H \):

\[
\langle F(hg), v \rangle = \langle F(h), vy^{-1} \rangle \quad \text{for every } v \in L \text{ and } g \in G \quad (8.10)
\]

\[
p\left( \frac{\partial}{\partial x} \right) F = 0 \quad \text{for every } p \in P . \quad (8.11)
\]

Solutions of \((8.10),(8.11)\) in the class of analytic functions on \( H \) are called GGZ-functions associated with \( L \) and \( H \).

The \( A \)-hypergeometric system \((1.1),(1.2)\) is a special case of \((8.10),(8.11)\) when \( G \) is an additive group \( \mathbb{C}^n \), \( H \) is the linear space with the basis \( h_i(t) = e^{\langle \omega_i, t \rangle}, t \in \mathbb{C}^n, i = 1, \ldots, N \), and \( V \) is the one-dimensional space generated by \( e^{\langle \beta, t \rangle} \).

Let \( f(v, h) \) be a function on \( V \times H \) linear with respect to \( v \), and let \( L \subset V \). Define a function \( F(h) \) on \( H \) with values in \( L^* \) by the equality:

\[
\langle F(h), v \rangle = f(v, h) \quad \text{for any } v \in L
\]

**Proposition 12** If \( f(v, h) \) is a GG-function associated with \( V \) and \( H \), then \( F(h) \) is a GGZ-function associated with \( L \) and \( H \).

**Remark.** It is natural to define the q-analog of GG-system \((8.6),(8.7)\) as the system consisting of equations \((8.6)\) and of the equations obtained from equations \((8.8)\) by replacing the operators \( \partial/\partial x_j \) by corresponding q-differential operators. This system depends not only on the subspaces \( V \) and \( H \) but also on a choice of the basis \( h_1, \ldots, h_N \) in \( H \). Similarly one can define the q-analog of the general GGZ-system. The following functions formally satisfy the q-analog of the GG-system:

\[
f(v, h) = \int_C \prod_{j=1}^N \exp_q(x_j h_j(g)) v(g) dg ,
\]

where \( \exp_q \) is the q-exponential and \( dg \) and \( C \) are defined in the same way as in \((8.9)\).
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