A NOTE ON ROUGH \( I \)-CONVERGENCE OF DOUBLE
SEQUENCES

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Abstract. In this paper we study some basic properties of rough \( I \)-convergent
double sequences in the line of Düntar [8]. We also study the set of all
rough \( I \)-limits of a double sequence and relation between boundedness and
rough \( I \)-convergence of a double sequence.

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\( I \)-limit.

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1. Introduction:

The concept of \( I \)-convergence of double sequences was introduced by Balcerzak et. al. [2]. The notion of \( I \)-convergence of a double sequence, which is
based on the structure of the ideal \( I \) of subsets of \( \mathbb{N} \times \mathbb{N} \), where \( \mathbb{N} \) is the set of
all natural numbers, is a natural generalization of the notion of convergence of
a double sequence in Pringsheim’s sense [17] as well as the notion of statistical
convergence of a double sequence [14].

A lot of work on \( I \)-convergence of double sequences can be found in ([3], [4],
[5], [7], etc.) and many others.

The concept of rough \( I \)-convergence of single sequences was introduced by Pal
et. al. [15] which is a generalization of the earlier concepts namely rough con-
vergence [16] and rough statistical convergence [1] of single sequences. Recently
rough statistical convergence of double sequences has been introduced by Mal-
lik and Maity [13] as a generalization of rough convergence of double sequences
[12] and investigated some basic properties of this type of convergence and also
studied relation between the set of statistical cluster points and the set of rough
limit points of a double sequence. Recently the notion of rough \( I \)-convergence
for double sequences has been introduced by Dündar [8]. In this paper we investigate some basic properties of rough $I$-convergence of double sequences in finite dimensional normed linear spaces which are not done earlier. We study the set of rough $I$-limits of a double sequence and also the relation between boundedness and rough $I$-convergence of a double sequence.

2. Basic Definitions and Notations

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

**Definition 2.1** (12). Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \| \cdot \|)$ and $r$ be a non-negative real number. $x$ is said to be $r$-convergent to $\xi \in X$, denoted by $x \overset{r}{\to} \xi$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $j, k \geq N_\epsilon$ we have $\|x_{jk} - \xi\| < r + \epsilon$.

In this case $\xi$ is called an $r$-limit of $x$.

It is clear that rough limit of $x$ is not necessarily unique (for $r > 0$). So we consider $r$-limit set of $x$ which is denoted by $LIM^r_x$ and is defined by $LIM^r_x = \{\xi \in X : x \overset{r}{\to} \xi\}$. $x$ is said to be $r$-convergent if $LIM^r_x \neq \emptyset$ and $r$ is called a rough convergence degree of $x$.

We recall that a subset $K$ of $\mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{m \to \infty \atop n \to \infty} \frac{K(n, m)}{n \cdot m},$$

where $K(n, m) = |\{(j, k) \in \mathbb{N} \times \mathbb{N} : j \leq n, k \leq m\}|$.

**Definition 2.2** (13). Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \| \cdot \|)$ and $r$ be a non negative real number. $x$ is said to be $r$- statistically convergent to $\xi$, denoted by $x \overset{r}{\text{st}-\to} \xi$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \epsilon\}$. In this case $\xi$ is called $r$-statistical limit of $x$.

Clearly for $r = 0$ from Definition 2.1 we get Pringsheim convergence of double sequences and from Definition 2.2 we get ordinary statistical convergence of double sequences.

**Definition 2.3.** A class $I$ of subsets of a nonempty set $X$ is said to be an ideal in $X$ provided

(i) $\emptyset \in I$.

(ii) $A, B \in I$ implies $A \cup B \in I$.

(iii) $A \in I, B \subset A$ implies $B \in I$.

$I$ is called a nontrivial ideal if $X \notin I$. 
Definition 2.4. A non empty class F of subsets of a nonempty set X is said to be a filter in X provided
(i) \( \phi \notin F \).
(ii) \( A, B \in F \) implies \( A \cap B \in F \).
(iii) \( A \in F, A \subseteq B \) implies \( B \in F \).
If I is a nontrivial ideal in X, \( X \neq \phi \), then the class
\[ F(I) = \{ M \subseteq X : M = X \setminus A \text{ for some } A \in I \} \]
is a filter on X, called the filter associated with I.

Definition 2.5 (4). A nontrivial ideal I in X is called admissible if \( \{ x \} \in I \) for each \( x \in X \).

Definition 2.6 (4). A nontrivial ideal I on \( \mathbb{N} \times \mathbb{N} \) is called strongly admissible if \( \{ i \} \times \mathbb{N} \) and \( \mathbb{N} \times \{ i \} \) belong to I for each \( i \in \mathbb{N} \).

Clearly every strongly admissible ideal is admissible. Throughout the paper we take I as a strongly admissible ideal in \( \mathbb{N} \times \mathbb{N} \).

Definition 2.7 (8). Let \( x = \{ x_{jk} \}_{j,k \in \mathbb{N}} \) be a double sequence in a normed linear space \((X, \| . \|)\) and \( r \) be a non negative real number. Then \( x \) is said to be rough ideal convergent or \( rI \)-convergent to \( \xi \), denoted by \( x \xrightarrow{rI} \xi \), if for any \( \varepsilon > 0 \) we have \( \{(j,k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| \geq r + \varepsilon \} \in I \). In this case \( \xi \) is called \( rI \)-limit of \( x \) and \( x \) is called rough \( I \)-convergent to \( \xi \) with \( r \) as roughness degree.

Throughout this paper \( x \) denotes the double sequence \( \{ x_{jk} \}_{j,k \in \mathbb{N}} \) in a normed linear space \((X, \| . \|)\) and \( r \) denotes a non negative real number.

For \( r = 0 \) we get the usual \( I \)-convergence of double sequences. But our main interest is on the case where \( r > 0 \). Because it may happen that a double sequence \( x = \{ x_{jk} \}_{j,k \in \mathbb{N}} \) is not \( I \)-convergent in usual sense but there exists a double sequence \( y = \{ y_{jk} \}_{j,k \in \mathbb{N}} \) which is \( I \)-convergent in usual sense and \( \| x_{jk} - y_{jk} \| \leq r \) for all \( (j,k) \in \mathbb{N} \times \mathbb{N} \) (or \( \{(j,k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - y_{jk} \| > r \} \in I \) for some \( r > 0 \)). Then \( x \) is \( rI \)-convergent.

From the definition it is clear that \( rI \)-limit of \( x \) is not necessarily unique (for \( r > 0 \)). So we consider \( rI \)-limit set of \( x \), which is denoted by \( I - LIM^r_x = \{ \xi \in X : x \xrightarrow{rI} \xi \} \). \( x \) is said to be \( rI \)-convergent if \( I - LIM^r_x \neq \emptyset \) and \( r \) is called a rough \( I \)-convergence degree of \( x \).

Definition 2.8. A double sequence \( x \) in X is said to be bounded if there exists a positive real number \( M \) such that \( \| x_{jk} \| < M \) for all \( (j,k) \in \mathbb{N} \times \mathbb{N} \).

Definition 2.9. A double sequence \( x \) in X is said to be \( I \)-bounded if there exists a positive real number \( M \) such that \( \{(j,k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} \| \geq M \} \in I \).

Definition 2.10 (5). A point \( \xi \in X \) is said to be an \( I \)-cluster point of a double sequence \( x = \{ x_{jk} \}_{j,k \in \mathbb{N}} \) if and only if for each \( \varepsilon > 0 \) the set \( \{(j,k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| < \varepsilon \} \notin I \). We denote the set of all \( I \)-cluster points of \( x \) by \( I(\Gamma_x) \).
Theorem 2.1. An $I$-bounded double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is $I$-convergent if and only if $I-\limsup x = I-\liminf x$.

Theorem 2.2. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a bounded double sequence of real numbers, then

(i) $I-\limsup x = \max I(\Gamma_x)$,

(ii) $I-\liminf x = \min I(\Gamma_x)$.

The above result is also true for $I$-bounded double sequences. So it can be stated as follows.

Theorem 2.3. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be an $I$-bounded double sequence of real numbers, then

(i) $I-\limsup x = \max I(\Gamma_x)$,

(ii) $I-\liminf x = \min I(\Gamma_x)$.

Theorem 2.4. For a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in a normed linear space $(X, \|\cdot\|)$ we have $\text{diam}(I-\text{LIM}_r x) \leq 2r$. In particular if $x \to I \xi$, then $I-\text{LIM}_r x = B_r(\xi) = \{y \in X : \|y - \xi\| \leq r\}$ and so $\text{diam}(I-\text{LIM}_r x) = 2r$.

Note 2.1. When $r=0$, then $\text{diam}(I-\text{LIM}_r x) = 0$. Therefore $I-\text{LIM}_r x$ is either $\emptyset$ or singleton. This implies the uniqueness of limit of $I$-convergent double sequence.

Theorem 2.5. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in $X$ and $c \in I(\Gamma_x)$. Then $\|\xi - c\| \leq r$ for all $\xi \in I-\text{LIM}_r x$ i.e. $I-\text{LIM}_r x \subseteq B_r(c)$.

We now consider an example of a double sequence which is rough $I$-convergent but not rough convergent.

Example 2.1. We consider the ideal $I_d = \{A \subset \mathbb{N} \times \mathbb{N} : d(A) = 0\}$. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in the normed linear space $(\mathbb{R}, \|\cdot\|)$ defined by

$$x_{jk} = \begin{cases} 2jk & \text{if } j \text{ and } k \text{ are squares}, \\ (-1)^{j+k} & \text{otherwise}. \end{cases}$$

Then

$$I_d-\text{LIM}_r x = \begin{cases} \emptyset & \text{if } r < 1 \\ [1-r, r] & \text{if } r \geq 1 \end{cases}$$

and $\text{LIM}_r x = \emptyset$ for all $r \geq 0$.

From the above example we see that $I-\text{LIM}_r x \neq \emptyset$ does not imply $\text{LIM}_r x \neq \emptyset$. But $\text{LIM}_r x \neq \emptyset$ always implies that $I-\text{LIM}_r x \neq \emptyset$. 
3. Main Results

We first establish a relation between boundedness and rough $I$-convergence of double sequences.

**Theorem 3.1.** If a double sequence $x = \{x_{jk}\}$ is bounded, then there exists $r \geq 0$ such that $I - LIM^r_x \neq \emptyset$.

*Proof.* The proof is similar to the proof of Theorem 2.4 [13], so is omitted. □

**Note 3.1.** Taking $I = \{A \in \mathbb{N} \times \mathbb{N} : d(A) = 0\}$, from Note 3.2 [13] we see that the converse of Theorem 3.1 is not true.

We now show that the converse of Theorem 3.1 is true if the double sequence $x$ is $I$-bounded.

**Theorem 3.2.** A double sequence $x$ is $I$-bounded if and only if there exists $r \geq 0$ such that $I - LIM^r_x \neq \emptyset$.

*Proof.* Let $x$ be an $I$-bounded double sequence. Then there exists a positive real number $M$ such that $A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk}\| \geq M\} \in I$. Let $r' = \sup\{\|x_{jk}\| : (j, k) \in \mathbb{N} \times \mathbb{N} \setminus A\}$. Then $0 \in I - LIM^{r'}_x$ and so $I - LIM^{r'}_x \neq \emptyset$.

Conversely, let $I - LIM^r_x \neq \emptyset$ for some $r \geq 0$. Let $\xi \in I - LIM^r_x$. Take $\varepsilon = 1$. Then $B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq 1 + r\} \in I$. Now $(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} \geq 1 + r + \|\xi\|\} \subseteq B$ and so $(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} \geq 1 + r + \|\xi\|\} \in I$. This shows that $x$ is $I$-bounded. □

Next we present an alternative proof of Theorem 2.4 [8] which gives a topological property of the $rI$-limit set of a double sequence.

**Theorem 3.3.** For all $r \geq 0$, the $rI$-limit set $I - LIM^r_x$, of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is closed.

*Proof.* Let $\xi$ be a limit point of $I - LIM^r_x$. Then for any $\varepsilon > 0$, $B^\varepsilon(\xi) \cap I - LIM^r_x \neq \emptyset$. Let $\alpha \in B^\varepsilon(\xi) \cap I - LIM^r_x$. Since $\alpha \in I - LIM^r_x$ so $A^\varepsilon(\alpha) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \alpha\| \geq r + \varepsilon\} \in I$. Let $B(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\}$. Now $(j, k) \notin A^\varepsilon(\alpha)$ implies $(j, k) \notin B(\varepsilon)$. Thus $(j, k) \in B(\varepsilon)$ implies $(j, k) \in A^\varepsilon(\alpha)$. This implies $B(\varepsilon) \subseteq A^\varepsilon(\alpha)$ and so $B(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\} \in I$. Therefore $\xi \in I - LIM^r_x$. Hence $I - LIM^r_x$ is a closed set in $X$. □

**Theorem 3.4.** Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in $X$. Then $x$ is $I$-convergent to $\xi$ if and only if $I - LIM^r_x = \overline{B^r(\xi)}$.

*Proof.* It directly follows from Theorem 2.4 that if $x$ is $I$-convergent to $\xi$, then $I - LIM^r_x = \overline{B^r(\xi)}$.

Conversely, let $I - LIM^r_x = \overline{B^r(\xi)}$. We have to show that $x$ is $I$-convergent to $\xi$, i.e. for all $a > 0$, $A(a) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq a\} \in I$. Now fixed
Let us choose \( r > 0 \) and \( \varepsilon > 0 \) such that \( r + \varepsilon < a \). For \( \xi \in I - LIM_\infty^x \), \( \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| \geq r + \varepsilon \} \in I \). Since \( \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| \geq a \} \subset \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| \geq r + \varepsilon \} \). So \( \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \| x_{jk} - \xi \| \geq a \} \in I \). Hence \( x \) is \( I \)-convergent to \( \xi \).

**Theorem 3.5.** Let \( (\mathbb{R}, \| . \|) \) be a strictly convex space and \( x = \{ x_{jk} \}_{j,k \in \mathbb{N}} \) be double sequence in \( \mathbb{R} \). For any \( r > 0 \), let \( y_1, y_2 \in I - LIM_\infty^x \) with \( \| y_1 - y_2 \| = 2r \). Then \( x \) is \( I \)-convergent to \( \frac{1}{2}(y_1 + y_2) \).

**Proof.** Let \( y_3 \) be an arbitrary \( I \)-cluster point of \( x \). Now since \( y_1, y_2 \in I - LIM_\infty^x \), so by Theorem 2.5 we have

\[
\| y_1 - y_3 \| \leq r \quad \text{and} \quad \| y_2 - y_3 \| \leq r.
\]

Then \( 2r = \| y_1 - y_2 \| \leq \| y_1 - y_3 \| + \| y_3 - y_2 \| \leq 2r \). Therefore \( \| y_1 - y_3 \| = \| y_2 - y_3 \| = r \). Now

\[
\frac{1}{2}(y_1 - y_2) = \frac{1}{2}((y_3 - y_1) + (y_2 - y_1)).
\]

Since \( \| y_1 - y_2 \| = 2r \), so \( \frac{1}{2} \| y_2 - y_1 \| = r \). Again since the space is strictly convex, so by (1) we get \( \frac{1}{2}(y_2 - y_1) = y_3 - y_1 = y_2 - y_3 \). Thus \( y_3 \) is the unique \( I \)-cluster point of the double sequence \( x \). Again by the given condition \( I - LIM_\infty^x \neq \emptyset \), so by Theorem 3.2 \( x \) is \( I \)-bounded. Since \( y_3 \) is the unique \( I \)-cluster point of the \( I \)-bounded double sequence \( x \), so by Theorem 2.1 and Theorem 2.3 \( x \) is \( I \)-convergent to \( y_3 = \frac{1}{2}(y_1 + y_2) \).

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