ON THE STRUCTURE OF ABELIAN HOPF ALGEBRAS

TILMAN BAUER

Abstract. We study the structure of the category of graded, connected, commutative and cocommutative Hopf algebras over a perfect field $k$ of characteristic $p$. Under a pure-injectivity condition, which in particular is always fulfilled for Hopf algebras of finite type, every $p$-torsion object in this category is uniquely a direct sum of explicitly given indecomposables. This classification is essentially equivalent to the classification of nonnegatively graded $k[s,t]/(st)$-modules, where $|s| = 1$ and $|t| = -1$. It gives rise to a similar classification of not necessarily $p$-torsion objects that are either free as commutative algebras or cofree as cocommutative coalgebras.

1. Introduction

Building on work of Hopf [Hop41] and Borel [Bor54], Milnor and Moore’s classical paper [MM65] described the structure of graded, connected, commutative Hopf algebras over a field $k$ as algebras and a classification as Hopf algebras in the case where they are primitively generated. In the simplest case, when the characteristic of $k$ is 0 and the Hopf algebras in question are abelian (i.e., commutative and cocommutative), this is the end of the story. In that case, the primitives and indecomposables coincide and thus $H$ is primitively generated. However, a classification in positive characteristic seemed less approachable. Some years later, Schoeller [Sch70] observed that the category of abelian Hopf algebras over a perfect field $k$ of characteristic $p$ is abelian and thus, by the Freyd–Mitchell embedding theorem, isomorphic to the category of graded modules over a ring, which she described. This is Dieudonné theory in a graded setting.

Although Dieudonné theory has been very successfully applied in various topological and algebraic contexts, it seems that the original question of classification has fallen into a bit of neglect. This paper aims to fix this to a certain extent.

Let $k$ be a perfect field of characteristic $p$. Let $H$ be the category of nonnegatively graded, connected (i.e. $H_0 = k$), abelian Hopf algebras over $k$. We will refer to objects of $H$ simply as “Hopf algebras.”

Our main classification result concerns Hopf algebras $H$ that are $p$-torsion in the sense that the multiplication-by-$p$ map $[p]$ in the abelian group $\text{End}(H)$ is trivial, and that satisfy the following condition, which uses the Frobenius ($p$th power) map $F: H_i \to H_{pi}$ and its dual, the Verschiebung $V: H_{pi} \to H_i$:

\begin{align*}
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\end{align*}
Definition. A $p$-torsion Hopf algebra $H$ over $k$ is pure-injective if the following condition holds: For each sequence $(n_i)_{i \geq 0}$ of nonnegative numbers and each primitive element $x \in P(H)$, if a solution $\{x_i\}$ exists for every finite subset of the set of equations

$$x_0 = x; \quad V(x_i) = F^{n_i}x_{i-1}$$

then a solution $\{x_i\}$ for the whole system exists.

Note that any Hopf algebra of finite type is pure-injective, as is any injective Hopf algebra. This is not the usual categorical definition of pure-injectivity, but it is equivalent to it, as we will show.

Theorem 1.1. Every pure-injective $p$-torsion Hopf algebra over a perfect field of characteristic $p$ decomposes uniquely into a tensor product of indecomposable Hopf algebras. A pure-injective $p$-torsion Hopf algebra is indecomposable if and only if it is isomorphic to a Hopf algebra $H(r, m, I)$ described below, indexed by a tuple $(r, m, I)$ with $r \in \mathbb{N}$, $m \in \mathbb{N}_0 \cup \{\infty\}$, $I \subset \{1, \ldots, m\}$ (resp. $I \subset \mathbb{N}$ if $m = \infty$).

For Hopf algebras of finite type, this theorem was proven by Touzé [Tou21, Section 9] in the context of “exponential functors,” using results from [CB18] on representations of string algebras.

Similarly to the classification of finitely generated modules over PIDs, the uniqueness statement is not (falsely) claiming that the decomposition is natural. It is instead to be understood as a Krull–Schmidt property, i.e. that the unordered sequence of tuples $(r, m, I)$ occurring in the decomposition into indecomposables is the same for every such decomposition.

Explicitly, the Hopf algebra $H(r, m, I)$ is given as an algebra by

$$H(r, m, I) = k[x_0, x_1, \ldots, x_m]/\left(x_i^p - \begin{cases} x_i & i \in I \\ 0 & i \notin I \end{cases}, \quad |x_i| = rp^i,\right.$$ 

and it has a unique coalgebra structure that makes it indecomposable as a Hopf algebra. Note that $H(r, m, I)$ has dimension 1 in degrees $0, r, 2r, \ldots, (p^m+1 - 1)r$ (resp. in degrees $r, 2r, \ldots$ if $m = \infty$) and is trivial in all other dimensions.

Non-pure-injective $p$-torsion Hopf algebras can be more complicated; in particular, there are indecomposables that are not of finite type (Example 3.8).

Similarly, a satisfying classification does not exist for general (non-$p$-torsion) Hopf algebras, even under a pure-injectivity condition. There are indecomposable Hopf algebras of arbitrarily large dimension in any given degree (Example 6.4). The situation greatly improves if $H$ is free as a (commutative) algebra or cofree as a (cocommutative) coalgebra:

Theorem 1.2. Given any tuple $(r, m, I)$, there exist

- a unique Hopf algebra $H_f(r, m, I)$ which is free as an algebra, and
- a unique Hopf algebra $H_c(r, m, I)$ which is cofree as a coalgebra

such that $H_f(r, m, I)/[p] \cong H(r, m, I) \cong H_c(r, m, I)/[p]$. Any Hopf algebra which is free as an algebra or cofree as a coalgebra and whose mod-$[p]$ reduction is pure-injective, decomposes uniquely into a tensor product of Hopf algebras isomorphic to $H_f(r, m, I)$ (resp. $H_c(r, m, I)$).

The condition of being free as an algebra is strictly weaker than being free over a coalgebra, or being a projective object in Hopf algebras.
In the general case, we can at least classify all Hopf algebras \( H \) with a given indecomposable reduction modulo \( [p] \). We define:

**Definition.** A basic Hopf graph is a (finite or infinite) connected chain of arrows with end points in \( \mathbb{N}_0 \times \mathbb{N}_0 \) with the following properties:

1. Every arrow goes either from \((i, j)\) to \((i+1, j)\), from \((i+1, j)\) to \((i, j)\), from \((i, j+1)\) to \((i+1, j)\), or from \((i+1, j+1)\) to \((i, j)\).
2. There is an arrow with end point \((0, 0)\).
3. Every arrow shares each of its end point with exactly one other arrow, unless the end point lies on the \(x\)-axis (i.e. is of the form \((i, 0)\)), in which case it may share its end point with no other arrow.

Denote by \( \mathcal{HG} \) the set of basic Hopf graphs.

Thus a basic Hopf graph looks something like this:

To any basic Hopf graph \( \Gamma \) and natural number \( r \), we can associate a triple \((r, m_\Gamma, I_\Gamma)\) as in Theorem 1.1, where \( m_\Gamma \) is the horizontal length of \( \Gamma \) and \( i \in I_\Gamma \) iff there is a right or right-down pointing arrow from \( i-1 \) to \( i \) in \( \Gamma \).

**Theorem 1.3.** For every basic Hopf graph \( \Gamma \) and \( r \geq 1 \), there exists a unique Hopf algebra \( H = H(r, \Gamma) \) such that \( H/[p] \cong H(r, m_\Gamma, I_\Gamma) \) and with the properties:

1. \( V: H_{rp} \to H_{rp-1} \) is injective iff \( \Gamma \) contains an arrow from \((i, j)\) to \((i-1, j)\) or an arrow from \((i-1, j)\) to \((i, j-1)\);
2. \( F: H_{rp-1} \to H_{rp} \) is injective iff \( \Gamma \) contains an arrow from \((i-1, j)\) to \((i, j)\) or an arrow from \((i, j)\) to \((i-1, j-1)\).

Any Hopf algebra whose mod-[\(p\)] reduction is isomorphic to some \( H(r, m, I) \) is itself isomorphic to \( H(r, \Gamma) \) for exactly one pair \((r, \Gamma)\).

Once translated into statements about Dieudonné modules, none of the methods and proofs in this paper use particularly sophisticated techniques.

An early attempt at classifying Hopf algebras was made in [Wra67], but that paper contains some key errors that invalidate its main results, such as that the category of Hopf algebras does not have projectives or injectives (it does), that the homological dimension is 1 (it is 2 [Sch70]), and a classification of indecomposable objects. One such error is the assumption that two Hopf algebras which are isomorphic both as algebras and as coalgebras, must be isomorphic as Hopf algebras [Wra67, 4.9]. A counterexample to this is Kuhn’s “cautionary example” [Kuh20, Example 1.13].

**Remark 1.4.** If \( H \) is a graded-commutative Hopf algebra and \( \text{char}(k) > 2 \) then there is a natural splitting \( H = H^{\text{even}} \otimes H^{\text{odd}} \) [Bon96, Proposition A.4] where \( H^{\text{even}} \) is concentrated in even degrees and \( H^{\text{odd}} \) is the exterior algebra on the odd-dimensional primitive elements of \( H \). Thus odd Hopf algebras are simply classified by the underlying graded \( k \)-vector space of primitive elements. We therefore lose
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2. $p$-typical splittings and the Dieudonné equivalence

Let $k$ be a perfect field of characteristic $p$. We recall the classical Dieudonné equivalence in the graded context [Sch70]. Denote by $W(k)$ the ring of $p$-typical Witt vectors of $k$ and by $\text{frob}: W(k) \to W(k)$ its Frobenius, i.e. the unique lift of the $p$th power map on $k$ to a linear map on $W(k)$. Denote by $\text{DMod}$ the category of Dieudonné modules, i.e. positively graded $W(k)$-modules together with additive operators $F: M_n \to M_{pn}$ and $V: M_{pn} \to M_n$, for each $n$, satisfying

- $FV = VF = p$
- $F(\lambda x) = \text{frob}(\lambda) F(x)$ for $\lambda \in W(k)$, $x \in M$;
- $V(\text{frob}(\lambda)x) = \lambda V(x)$ for $\lambda \in W(k)$, $x \in M$.

By convention, $V(x) = 0$ unless $p \mid |x|$.

**Theorem 2.1** ([Sch70]). There is an equivalence of abelian categories

$$D: \mathcal{H} \to \text{DMod}.$$  

Let us call a module $M \in \text{DMod}$ $p$-typical of type $j$, where $j$ is an natural number coprime to $p$, if $M_i = 0$ unless $i = jp^a$ for some $a \geq 0$. Denote the full subcategory of $p$-typical Dieudonné modules of type $j$ by $\text{DMod}^{(j)}$. Similarly, a Hopf algebra $H$ is called $p$-typical of type $j$ if $H_i = 0$ unless $i = jp^k$ for some $k$. Denote the full subcategory of $p$-typical Hopf algebras of type $j$ by $\mathcal{H}^{(j)}$.

**Lemma 2.2** ([Sch70, §2],[Rav75]). A Hopf algebra $H$ is $p$-typical of type $j$ iff $D(H)$ is a $p$-typical Dieudonné module of type $j$. There are splittings of abelian categories

$$\text{DMod} \simeq \prod_{(j,p)=1} \text{DMod}^{(j)} \quad \text{and} \quad \mathcal{H} \simeq \prod_{(j,p)=1} \mathcal{H}^{(j)}$$

compatible with the Dieudonné functor $D$. Finally, for any $j$ coprime to $p$, there is an equivalence of categories

$$\Lambda^{(j)}: \text{DMod}^{(j)} \cong \text{Mod}_R,$$

where $R = W(k)[s,t]/(st-p)$ is a $\mathbb{Z}$-graded commutative ring with $|s| = 1$, $|t| = -1$, and $\text{Mod}_R$ denotes the category of nonnegatively graded modules over it.

**Proof.** The preservation of $p$-typicality under the Dieudonné functor follows from its construction. The $p$-typical splitting on the module level is obvious because $F$ and $V$ preserve the type $j$. The corresponding splitting of Hopf algebras follows.

For the last statement, construct a functor $\Lambda^{(j)}: \text{DMod}^{(j)} \to \text{Mod}_R$ by

$$\Lambda^{(j)}(M)_n = M_{jp^n} \quad \text{with } W(k)\text{-module structure given by } \lambda x = \lambda^{p^n} x$$

and define $s: \Lambda^{(j)}(M)_n \to \Lambda^{(j)}(M)_{n+1}$ by $s(x) = F(x)$ and similarly $t(x) = V(x)$. Since $k$ is perfect, $\text{frob}$ is bijective on $W(k)$, and $\Lambda^{(j)}$ is an equivalence of categories. \qed
Using this, the classification of Hopf algebras is thus reduced to the classification of modules in \( \text{Mod}_R \). We let

\[
(2.3) \quad D^{(j)} : \mathcal{H}^{(j)} \xrightarrow{\mathcal{L}} \text{DMod}^{(j)} \xrightarrow{\mathcal{A}^{(j)}} \text{Mod}_R
\]

denote the composite equivalence between \( p \)-typical Hopf algebras of type \( j \) and modules over \( \mathcal{R} \).

Similarly, the classification of \( p \)-torsion Hopf algebras is reduced to the category of modules in \( \text{Mod}_R \), where \( R = \mathcal{R}/(p) = k[s, t]/(st) \).

3. The classification of graded \( k[s, t]/(st) \)-modules

In this section, we will consider the category of \( \mathbb{Z} \)-graded \( R \)-modules and restrict to nonnegatively graded \( R \)-modules only when necessary.

3.1. Duality of \( R \)-modules. Given an \( R \)-module \( M \), we denote by \( M^{\text{op}} \) the (co-variant) dual given by

\[
(M^{\text{op}})_m = M_{-m}
\]

This becomes an \( R \)-module by interchanging the \( s \)- and \( t \)-action, and descends to a duality of \( R \)-modules (recall that \( R = \mathcal{R}/(p) \)).

Remark 3.1. Note that this duality is not meaningful on the level of Hopf algebras since it does not send nonnegatively graded modules to nonnegatively graded modules. The duality that corresponds to Cartier duality (linear duals of Hopf algebras) is Pontryagin (or Matlis) duality \( M^* = \text{Hom}_{W(k)}(M, CW(k)) \), which does not satisfy \( M^{**} = M \) unless \( M \) is of finite length. We will use the simple-minded covariant dual to save work in the sections that follow.

3.2. Chains in \( R \)-modules. The fundamental concept in our study of \( R \)-modules is that of a chain, which is a sequence of elements connected by either \( s \)- or \( t \)-multiplication.

Definition. A chain \( C \) in \( M \in \text{Mod}_R \) is a sequence of elements \( a_i \in M_i \) in consecutive degrees \( i \) such that either \( t(a_i) = a_{i-1} \) or \( s(a_i) = a_{i+1} \). We allow finite chains \( (i \in \{m, \ldots, n\}) \) as well as infinite chains in either or both directions.

We write \( \text{Ch}_{m,n} \) for the set of chains from degree \( m \) to degree \( n \), allowing \( n \) to be \( \infty \) and \( m \) to be \( -\infty \).

We write \( [m, n] \) for the range \( \{m, \ldots, n\} \) when \( -\infty < m \leq n < \infty \), \( [m, \infty] = \{m, m+1, \ldots\} \), \( [-\infty, n] = \{\ldots, n-1, n\} \) and \( [-\infty, \infty] = \mathbb{Z} \).

A signature is a tuple \( \sigma = (m, n, I) \) where \( -\infty \leq m \leq n \leq \infty \) and \( I \subseteq [m, n-1] \).

We say that a chain \( C = (a_i)_{i \in \omega} \in \text{Ch}_{m', n'} \), for \( m' \leq m \) and \( n' \geq n \), adheres to the signature \( \sigma = (m, n, I) \) if:

1. \( a_i = 0 \) unless \( m \leq i \leq n \);
2. \( s(a_i) = a_{i+1} \) if \( i \in I \);
3. \( t(a_{i+1}) = a_i \) if \( i \notin I \).

We write \( \text{Ch}_\sigma \) for the \( k \)-vector space of chains adhering to the signature \( \sigma \).

Note that \( \text{Ch}_{m,n} \) is not an abelian group since sums of chains of distinct signatures usually are not chains. A chain has a unique signature if and only if all of its elements are nonzero.
Definition. We say that a chain \( C \) is strongly degenerate if it contains a zero element, and degenerate if it is a \( k \)-linear combination of strongly degenerate chains (of the same size, and adhering to a common signature).

Example 3.2. Consider the following module \( M \):

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\quad d
\quad b
\quad c
\quad a
\]

In this and similar pictures, the grading is given by the height (the degree of \( a \) is 0, the degree of \( b \) and \( c \) is 1, and the degree of \( d \) is 2). All letters are generators of a copy of \( k \), downward arrows denote a nontrivial \( t \) (thus \( t(b) = t(c) = a \)), and upward arrows denote a nontrivial \( s \) (thus \( s(b) = d \)).

In \( M \), the chain \((a, b, d)\) is degenerate (but not strongly degenerate), as it is the sum of the degenerate chains \((0, b - c, d)\) and \((a, c, 0)\). The chains \((a, b)\), \((a, c)\), and \((b, d)\) are all nondegenerate.

Definition. We say that a chain \((a_m, \ldots, a_n)\) is lower closed if \( m = -\infty \) or \( t(a_m) = 0 \), upper closed if \( n = \infty \) or \( s(a_n) = 0 \), and closed if it is both.

Note that every chain \( C \) has a unique upper closure \( C_u \), lower closure \( C_l \), and closure \( C \), given as the minimal (upper/lower) closed chain containing \( C \). Example 3.2 shows that the closure of a nondegenerate chain may well become degenerate.

We introduce some more notation:

Definition.

- \( \text{Ch}^+_m = \bigcup_{m \leq n \leq \infty} \text{Ch}_{m,n} \) is the set of upward chains;
- \( \text{Ch}^-_m = \bigcup_{-\infty \leq m \leq n} \text{Ch}_{m,n} \) is the set of downward chains.
- \( \text{Ch}^+_a \) and \( \text{Ch}^-_a \) are the sets of upward and downward chains, respectively, with extremal element \( a \in M \).
- \( \text{Ch}^{\pm}_a \) is the set \( \text{Ch}^+_a \cap \text{Ch}^-_a \), the set of upward/downward chains adhering to the signature \( \sigma \) with extremal element \( a \).
- For a chain \( C \in \text{Ch}_{m,n} \) and \( m \leq m' \leq n' \leq n \), we denote by \( C|^{n'}_{m'} \in \text{Ch}_{m',n'} \) the subchain in degrees \( m' \) through \( n' \); we use the same notation for signatures, \( a|^{n'}_{m'} \).

Finally, we consider modules generated by chains:

Definition. A submodule \( M < N \) is called a chain module if it is freely generated by the elements of a chain \( C \) that is not strongly degenerate. Note that this implies that \( C \) is closed. For a signature \( \sigma = (m, n, I) \), the chain module associated to \( \sigma \) is the unique module \( M(\sigma) \) such that \( M(\sigma)_i = k(x_i) \) for \( m \leq i \leq n \) and \( M(\sigma)_i = 0 \) otherwise, and such that \( (x_m, \ldots, x_n) \) is a chain of signature \( \sigma \).

3.2.1. Degenerate chains. Under \( R \)-linear maps, degenerate chains map to degenerate chains and thus preimages of nondegenerate chains are nondegenerate. Subchains of nondegenerate chains are nondegenerate, but concatenations of nondegenerate chains can become degenerate (cf. Ex. 3.2).
Lemma 3.3. A finite chain \( C \in \text{Ch}_\sigma \) is degenerate iff \( C = C^u + C^l \) for some \( C^u \in \text{Ch}_\sigma^+ (0) \) and some \( C^l \in \text{Ch}_\sigma^- (0) \).

**Proof.** It is clear that \( C \) is degenerate if it can be written as such a sum, so let us consider the converse. We use induction on \( n-m \). The cases \( n-m = 0 \) and \( n-m = 1 \) are clear. Let \( C = C_1 + \cdots + C_k \) be a decomposition into strongly degenerate chains. Without loss of generality, suppose that \( C_1, \ldots, C_l \) have vanishing bottom element and \( C_{l+1}, \ldots, C_k \) have nonvanishing bottom element.

The subchains \( C'_i = C_i|_{m+1}^n \) for \( i = l+1, \ldots, k \) exhibit the corresponding subchain \( C' = (C_{l+1} + \cdots + C_k)|_{m+1}^n \) as degenerate. Thus \( C' = (C'^u)^u + (C'^l)^l \) by inductive assumption. By setting \( C^u \) to be the sum of the downward extension of \( (C')^u \) with 0 and \( C_1 + \cdots + C_l \) and \( C^l \) the downward extension of \( (C')^l \) with the bottom element of \( C \), we obtain the desired decomposition. \( \square \)

**Lemma 3.4.** An (infinite) chain \( C \) is nondegenerate if and only if all finite subchains are nondegenerate.

**Proof.** Suppose \( C \) is degenerate, with \( C = C_1 + \cdots + C_k \) a decomposition into strongly degenerate summands. Suppose \( C_i = (a_{ij}) \) has \( a_{ij} = 0 \). Then the finite subchain of \( C \) contained within degrees \( \min(j_i) \) and \( \max(j_i) \) is degenerate.

For the converse direction, suppose that \( C \) is a chain and the finite subchain \( C'_m \) from degree \( m \) to degree \( n \) is degenerate. Let \( C'_m = C^u + C^l \) be a decomposition as in Lemma 3.3. Let \( C_1 \subset C^u \cup C^|_{n+1}^\infty \) and \( C_2 = C^l \cup C^|_{-\infty}^{m-1} \) be degenerate and \( C = C_1 + C_2 \), so \( C \) is degenerate. \( \square \)

3.2.2. A total order on signatures.

**Definition.** We denote the set of signatures with fixed \( m \) and \( n \) by \( \text{Sig}^{m,n} \), the sets of signatures with bottom degree \( m \) by \( \text{Sig}^+(m) \), and the sets of signatures with top degree \( n \) by \( \text{Sig}^-(n) \), in the same way as for chains.

We define a total order on \( \text{Sig}^+(m) \) as a variant of the lexicographic order. Let \( \sigma = (m,n,I) \) and \( \sigma' = (m,n',I') \) be distinct signatures. Suppose that \( k \geq m \) is the maximal number such that

\[ k \leq n, \quad k \leq n', \quad \text{and} \quad I \cap [m,k] = I' \cap [m,k]. \]

Then \( \sigma' < \sigma \) iff the following hold:

- \( k \notin I \) and
- \( k = n' \) or \( k \in I' \).

Otherwise, \( \sigma' > \sigma \).

Similarly, we define a total order on \( \text{Sig}^-(n) \) by duality: if \( \sigma = (m,n,I) \) is a signature then the dual signature is defined as

\[ \sigma^{op} = (-n,-m,\{ -i \mid i \notin I, m \leq i \leq n \}) \]

If \( \sigma \in \text{Sig}^-(n) \) then \( \sigma^{op} \in \text{Sig}^+(n) \), and we define \( \sigma < \sigma' \) iff \( \sigma^{op} > (\sigma')^{op} \).

Explicitly: \( (m',n,I') < (m,n,I) \) iff there exists a \( k \) such that \( I \cap \{ k, \ldots, n \} = I' \cap \{ k, \ldots, n \} \) and

- \( k-1 \notin I \) and
• \( k = m' \) or \( k - 1 \in I' \).

The duality of these orders is designed to be compatible with the covariant duality of modules: \( C \) is a chain in \( M \) adhering to \( \sigma \) if and only if the dual chain \( C^{\text{op}} \) in \( M^{\text{op}} \) adheres to \( \sigma^{\text{op}} \).

We can depict a signature \( \sigma \in \text{Sig}^+(m) \) graphically by a chain of \( s \) and \( t \), depending on whether \( i \in S \) or not. In particular, for any signatures \( x, y \) and \( z \), the ascending order satisfies

\[(xsy) < (x) < (xtz),\]

and the descending order satisfies

\[(ysx) < (x) < (ztx).\]

Note that the ascending order on \( \text{Sig}^+(m) \) is neither a well-ordering nor a co-well-ordering:

\[() < (t) < (tt) < \cdots \quad \text{and} \quad () > (s) > (ss) > \cdots.\]

The same is true for the descending order unless one restricts to signatures of nonnegatively graded modules, in which case \( \text{Sig}^-(n) \) is finite.

The order on signatures is compatible with whether an element is the bottom (or top) element of a chain of that signature:

**Lemma 3.5.** Let \( a \in M_m, m \leq n \leq \infty \), and \( \sigma > \tau \in \text{Sig}(m,n) \) be two signatures such that \( \text{Ch}_\sigma^+(a) \) is nonempty. Then \( \text{Ch}_\tau^+(a) \) is also nonempty.

Similarly, for \( b \in M_n \), if \( \text{Ch}_\tau^-(b) \) is nonempty then so is \( \text{Ch}_\sigma^-(b) \).

**Proof.** Let \( \sigma = (m,n,I) \) and \( \tau = (m',n',J) \). Then there exists \( k \geq m \) such that \( \sigma^k|_m = \tau^k|_m \) and \( k \in J - I \). Thus \( C = (a_i)_{m \leq i \leq n} \in \text{Ch}_\sigma^+(a) \) then \( t(a_{k+1}) = a_k \) and thus \( s(a_k) = 0 \). Hence the chain \( (a_m, \ldots, a_k, 0, \ldots, 0) \) adheres to \( \tau \).

The statement about downward chains follows from duality. \( \square \)

**Definition.** We say that a chain \( C \in \text{Ch}_{m,n} \) is **normalized** if there exists an \( m' \in [m, n] \) such that \( C|_{m'} \) is nondegenerate and \( C|_{m'+1} = 0 \).

In particular, nondegenerate chains are automatically normalized.

**Lemma 3.6.** Let \( a \in M_m - \{0\} \) and \( C \in \text{Ch}_\sigma^+(a) \). Then there is a normalized chain \( C' \in \text{Ch}_\sigma^+(a) \) and the signature of the nondegenerate part of \( C' \) is greater or equal to \( \sigma \), and equal if and only if \( C \) is nondegenerate. If \( C \) is degenerate, then \( C' \) can be chosen upper closed.

**Proof.** We use induction in the length of the chain \( C \). If \( C \) has length 1 (the minimal possible length since \( a \neq 0 \)) then \( C \) is nondegenerate and hence normalized.

Let \( \sigma = (m, n, I) \). If \( C \) is degenerate, let \( C = C^n + C^l \) as in Lemma 3.3. Then \( C^l \in \text{Ch}_\sigma^+(a) \). Let \( C' = C^l|_{m'_{\cdot}} \) be the longest subchain that is not strongly degenerate. It has signature \( \sigma' = \sigma|_{m'_{\cdot}} \). We necessarily must have \( j \in I \). Thus \( \sigma' > \sigma \). Even if \( C' \) is not yet normalized, there is a normalized chain \( C'' \in \text{Ch}_{\sigma'}(a) \) of signature at least \( \sigma' \), which can be extended with zeroes to a normalized chain \( C'' \in \text{Ch}_{\sigma'}(a) \).

The following proposition says that a chain is nondegenerate if its signature is extremal:
Proposition 3.7. Let $\sigma = (m, n, I)$ be a signature, $a \in M_m$, $b \in M_n$, and $C \in \text{Ch}_\sigma$ a chain between $a$ and $b$. Then the following are equivalent:

1. $C$ is degenerate;
2. There is an upper closed chain $C' \in \text{Ch}_{m,n} \cap \text{Ch}^+(a)$ adhering to a bigger signature than $\sigma$;
3. There is a chain $C' \in \text{Ch}_{m,n} \cap \text{Ch}^+(a)$ adhering to a bigger signature than $\sigma$;
4. There is a lower closed chain $C' \in \text{Ch}_{m,n} \cap \text{Ch}^-(b)$ adhering to a smaller signature than $\sigma$;
5. There is a chain $C' \in \text{Ch}_{m,n} \cap \text{Ch}^-(b)$ adhering to a smaller signature than $\sigma$.

Proof. A chain $C$ in $M$ is degenerate iff its dual chain $C^{\text{op}}$ in $M^{\text{op}}$ is, so it suffices to prove (1) $\iff$ (2) $\iff$ (3).

The implication (1) $\Rightarrow$ (2) follows immediately from Lemma 3.6. Clearly, (2) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (1), let $C' \in \text{Ch}_{m',n}^+(a) \cap \text{Ch}_{m,n}$, $\sigma' = (m, n', I') > \sigma = (m, n, I)$ and $n' \leq n$. Denote the elements of $C$ by $a = a_m, a_{m+1}, \ldots, a_n = b$ and the elements of $C'$ by $a = a'_m, a'_{m+1}, \ldots, a'_n = b'$.

According to the definition of the order, there exists a $k$ with $I \cap [m, k] = I' \cap [m, k]$ such that $k \in I - I'$. (The case $k = n$ is excluded because by assumption, $k \leq n' \leq n$ and thus in that case, $\sigma = \sigma'$.)

This means that $s(a_k) = a_{k+1}$, and either $k = n' < n$ or $k < n'$ and $t(a'_{k+1}) = a'_k$.

In the former case, $C'$ must be strongly degenerate and also adheres to $\sigma$. However, $C - C'$ has bottom element 0 and thus is also strongly degenerate. Thus $C = C' + (C - C')$ is degenerate.

In the latter case (i.e., $n' > k$ and $t(a'_{k+1}) = a'_k$), we must have that $s(a'_k) = 0$ and hence there is a degenerate chain $C'' = (a'_m, \ldots, a'_k, 0, \ldots, 0)$ adhering to the signature $\sigma$. Moreover, $C - C''$ is a degenerate chain as well, thus $C$ is degenerate.

3.2.3. Height and order. It follows from Prop. 3.7 that for every nonzero $a \in M_m$ and $n \geq m$, there exists at most one signature $\sigma(a, n)$ such that $\text{Ch}_{\sigma(a, n)}^+(a)$ contains a nondegenerate chain. Moreover, since subchains of nondegenerate chains are nondegenerate, $\sigma(a, n+1) |_{m}^{n} = \sigma(a, n)$, which means that the sequence $\{\sigma(a, n)\}$ of all elements $\sigma(a, n)$ that exist, converges in the order topology to a unique signature.

Definition. The height $\text{ht}(a)$ of a nonzero element $A \in M$ is the signature

$$\lim_{n \to \infty} \sigma(a, n),$$

where the limit is taken over the (finite or infinite) set of $n$ such that $\sigma(a, n)$ exists.

Similarly, the order $\text{o}(a)$ of a nonzero element $A \in M$ is defined as dual of the height of its dual:

$$\text{o}(a) = (\text{ht}(a^{\text{op}}))^{\text{op}},$$

where $a^{\text{op}} \in M^{\text{op}}$.

An element $a$ of height $\sigma$ does not necessarily admit a nondegenerate chain in $\text{Ch}_{\sigma}^+(a)$; if it does, we call it a limit element. (Of course, the issue only arises for elements of infinite height.) Similarly, if $a$ admits a nondegenerate chain in $\text{Ch}_{\sigma}^-(a)$, we call it a colimit element.
An $R$-module in which all elements are limit and colimit elements is called complete.

The difference between elements of infinite heights and limit elements is analogous to the question of the vanishing of the derived inverse limit $\lim^1(\cdots \mathbb{Z} / A \mathbb{Z} / A)$ of the multiplication-by-$p$ map on an abelian $p$-group $A$. An element $a \in A$ can be $p^n$-divisible for all $n$ without admitting a sequence $(a_i)$ with $a_0 = a$ and $pa_{i+1} = a_i$. The following example illustrates this idea for $R$-modules:

**Example 3.8.** For any two $R$-modules $M$, $N$ with $M_0 = k(x)$, $N_0 = k(y)$, $sx = 0 = sy$, we define the bouquet $M \vee N = (M \oplus N)/(x - y)$.

Let $\sigma = (0, \infty, I)$ be any signature with $0 \notin I$ and such that $I$ has infinite complement. Note that $M(\sigma|_0)$ is a submodule of $M(\sigma)$ iff $n \notin I$. Define an $R$-module as the infinite bouquet $NP I(\sigma|_0) = \bigvee_{n \notin I} M(\sigma|_n)$. The module $NP I(0, \infty, \emptyset)$ is depicted below:

```
      x33
      /    \
  x11    x22
      \    /  \
       x0  x21 x32
            \   \
            x31 \\
```

We also define the module $PI(\sigma)$ as $PI(\sigma) = NP I(\sigma) \vee M(\sigma)$.

Denote the standard basis elements in the $M(\sigma)$ summand by $x_{\infty,i}$.

In both $NP I(\sigma)$ and $PI(\sigma)$, $\text{ht}(x_0) = \sigma$, but only in $PI(\sigma)$ is $x_0$ a limit element. By replacing the basis elements $x_{n,i}$ by $x_{n,i} - x_{\infty,i}$ in $PI(\sigma)$, we see that $PI(\sigma) \cong M(\sigma) \oplus \bigoplus_{n \notin I, n \geq 1} M(\sigma|_n)$ is isomorphic to a direct sum of chain modules.

It will follow from Thm. 3.19 that $NP I(\sigma)$ cannot be a direct sum of chain modules. We will continue the study of this example in Ex. 6.3.

**Lemma 3.9.** The following are equivalent for an element $a \in M_m$:

1. $a$ is a limit element.
2. For every $\sigma \in \text{Sig}_{m, \infty}$ such that $\text{Ch}_{\sigma|_m}(a)$ is nonempty for all $n$, $\text{Ch}_\sigma(a)$ is nonempty.

**Proof.** The direction (2) $\Rightarrow$ (1) is immediate by choosing $\sigma = \text{ht}(a)$. For the converse, let $C_n \in \text{Ch}_{\sigma|_m}(a)$. If $C_n$ is degenerate for some $n \geq m$ then by Lemma 3.3, there exists another chain $C'_n \in \text{Ch}^+|_{\sigma|_m}(a) \cap \text{Ch}^-(0)$. This chain can be extended upwards by zeroes to produce a chain in $\text{Ch}^+(a)$. 

If, on the other hand, all $C_n$ are nondegenerate, then the existence of a (nondegenerate) element in $Ch_n(a)$ follows from the definition of limit elements. □

Lemma 3.10.

(1) If $ht(a)$ is finite then there exists an upper closed nondegenerate chain of signature $ht(a)$ in $Ch^+(a)$.

(2) If $ht(a) = (m, n, I)$, $n \leq \infty$, then for every $m \leq n'$ with $n' \notin I$, there exists an upper closed nondegenerate chain of signature $ht(a) | m$ in $Ch^+(a)$.

(3) If $\sigma(a)$ is finite then there exists a lower closed nondegenerate chain of signature $\sigma(a)$ in $Ch^-(a)$.

(4) If $C \in Ch^+(a)$ is nondegenerate and upper closed then $\sigma(C) \leq ht(a)$.

(5) If $C \in Ch^-(a)$ is nondegenerate and lower closed then $\sigma(C) \geq o(a)$.

Proof. (4): Suppose that $\sigma = \sigma(C) = (m, n, I)$ and $ht(a) = (m', n', I') > \sigma$. Then we must have $n' > n$, $\sigma = ht(a) | m$, and $n \in I'$. Let $C'$ be a nondegenerate chain of signature $\sigma' = ht(a) | m + 1$. Since $C$ is upper closed, $C \in Ch(\sigma')$ and is degenerate as such because its top element is zero. Similarly, $C' - C$ is degenerate because its bottom element is zero. Thus the sum, $C''$, is degenerate, a contradiction.

(1): Let $C \in Ch_n(a)$ be a nondegenerate chain of signature $\sigma = (m, n, I) = ht(a)$ and suppose that $C$ is not upper closed. Let $C'$ be the unique extension of $C$ to $Ch_{n+1}(a)$. This chain $C'$ must be degenerate, because otherwise $ht(a)$ would not have maximal length. $C'$ adheres (uniquely) to the signature $\sigma' = (m, n + 1, I \cup \{n\}) < \sigma$. By Lemma 3.6, there exists a normalized, upper closed chain $C'' \in Ch_{\sigma}(a)$. Let $D \in Sig_{m \leq n}$ denote its nondegenerate part, which is also upper closed. By Lemma 3.6 again, $\sigma(D) > \sigma'$. Since in $Sig_{m \leq n+1}$, $\sigma$ is a direct successor to $\sigma'$, $\sigma(D) \geq \sigma$. But by (4), we also have $\sigma(D) \leq \sigma$, thus $D$ is a nondegenerate, upper closed chain in $Ch_{ht(a)}(a)$.

For (2), let $I' = [m, \infty] - I$. For every $n \in I'$, let $C_n \in Ch_{\sigma | m+1}(a)$ be a nondegenerate chain. By assumption, $C_n | m$ is upper closed and, as a subchain of a nondegenerate chain, nondegenerate.

Assertions (5) and (3) are dual to (4) and (1), respectively. □

Lemma 3.11. Let $f : N \to M$ be a morphism of $R$-modules and $a \in N$. If $C \in Ch^+_{\sigma}(a)$ is a nondegenerate, upper closed chain then $\sigma \leq ht_M(f(a))$.

Proof. If $f(C)$ is still nondegenerate then the claim follows from Lemma 3.10(4).

If, on the other hand, $f(C)$ is degenerate in $M$ then by Prop. 3.9 there is a chain $D$ in $Ch^+(f(a))$ adhering to a bigger signature $\tau$, where $\tau$ and $\sigma$ have the same length. Hence, by Lemma 3.6, there also exists a normalized chain $C' \in Ch^+(f(a))$ with upper closed, nondegenerate part of signature $\tau' > \tau > \sigma$. Since $\tau' \leq ht(f(a))$ by Lemma 3.10, the claim follows. □

Lemma 3.12.

(1) For $f : N \to M$ and $a \in N$, $ht_N(a) \leq ht_M(f(a))$ and $o_N(a) \geq o_M(f(a))$.

(2) Let $a \in N_m$, $b \in M_m$. Then

$$ht_{N \oplus M}(a, b) = \min(ht_N(a), ht_M(b))$$

and

$$o_{N \oplus M}(a, b) = \max(o_N(a), o_M(b))$$
(3) If \( a, b \in M_m \) with \( a, b \) and \( a + b \) nonzero, then
\[
\text{ht}(a + b) \geq \min(\text{ht}(a), \text{ht}(b)).
\]
and
\[
o(a + b) \leq \max(o(a), o(b))
\]

Proof:

(1) If \( \text{ht}(a) \) is finite then the claim follows from Lemma 3.11 and Lemma 3.10(1).
If \( \text{ht}(a) = \sigma = (m, \infty, I) \) is infinite then we consider two cases. If \([m, \infty] - I\)
is finite, with maximal element \( n \), then there exists a chain \( C \in \text{Ch}_n^+(a) \) and Lemma 3.11 yields the result.

In the remaining case, there exists a sequence of upper closed, finite, nondegenerate chains \( C_i \) in \( \text{Ch}_n^+(a) \) with \( \lim_{i \to \infty} \sigma_i = \text{ht}(a) \). Again by Lemma 3.11, \( \sigma_i \leq \text{ht}(f(a)) \) for all \( i \) and thus \( \text{ht}(a) \leq \text{ht}(f(a)) \).

The claim about orders is dual.

(2) The projection \( N \oplus M \to N \) shows that \( \text{ht}_{N \oplus M}((a, b)) \leq \text{ht}_N(a) \), and similarly \( \text{ht}_N(a) \leq \text{ht}_{N \oplus M}(a, b) \leq \min(\text{ht}_N(a), \text{ht}_M(b)) \). To show equality, suppose that \( \sigma = \text{ht}_N(a) \leq \text{ht}_M(b) = \tau \).
If \( a \) is a limit element, there exists a nondegenerate, upper closed chain \( C \in \text{Ch}_n^+(a) \).
By Lemma 3.5, there also exists a (degenerate unless \( \sigma = \tau \)) upper closed chain \( D \in \text{Ch}_n^+(b) \).
Then the chain \( (C, D) \in \text{Ch}_n^+(a, b) \) is upper closed and nondegenerate since its projection to \( N \) is nondegenerate. By Lemma 3.10, \( \text{ht}_{M \oplus N}((a, b)) \geq \sigma \).
In case \( a \) is not a limit element, choose a sequence \( C_i \) of upper closed, nondegenerate chains with signatures converging to \( \sigma \) and proceed as before.

The claim about orders is dual.

(3) This follows from the previous part, using the addition homomorphism \( M \oplus M \to M \). \( \square \)

### 3.3. Injective \( R \)-modules.

From now on, we will concentrate on nonnegatively graded \( R \)-modules. The injective objects are very easily classified:

**Lemma 3.13.** A connected \( R \)-module \( M \) is injective if and only if the following three conditions are met:

(i) \( \ker s = \text{im} t \) in all degrees;
(ii) \( \ker t = \text{im} s \) in all positive degrees; and
(iii) If \( x \in \text{im}(s)_{n+1} \) and \( y \in \text{im}(t)_{n-1} \) then there exists \( z \in M_n \) such that \( x = s(z) \) and \( y = t(z) \).

Proof. The three conditions boil down to the lifting problems for the monomorphisms which are the inclusions of the three ideals \( (s) \), \( (t) \), and \( (s, t) \) into \( R \), respectively. More precisely, since we are dealing with nonnegatively graded modules, they correspond to the inclusions
\[
(\Sigma^n I)_{\geq 0} \hookrightarrow (\Sigma^n R)_{\geq 0}
\]
for the three ideals \( I = (s), (t), \) and \( (s, t) \), respectively, and any \( n \geq 0 \). This shows that injective \( R \)-modules satisfy the three conditions. Conversely, since the mentioned three ideals are precisely the graded prime ideals of \( R \), the refined Baer criterion shows that the three conditions are sufficient for the injectivity of \( M \). \( \square \)

**Corollary 3.14.** A chain module \( M(\sigma) \) is injective if and only if \( \sigma = (0, \infty, [0, n]) \) for some \( n \leq \infty \).
Proof. If $M$ was finite with top degree $n$ then $s(M_n) = 0$ and hence $M_n \subseteq \text{im}(t)$, a contradiction. So $M$ has to be infinite.

If $M$ had bottom degree $n > 0$ then $t(M_n) = 0$ and hence $M_n \subseteq \text{im}(s)$, again a contradiction. So $M_0 \cong k$.

If there was some $n \in I$ with $n - 1 \not\in I$ then we have a class $x_n \in M_n$, unique up scaling, with $s(x_n) = x_{n+1} \neq 0$ and $t(x_n) = x_{n-1} \neq 0$. This means that the lifting problem of type (iii) with $x = x_{n+1}$ and $y = 0$ does not have a solution. \hfill \square

Corollary 3.15. Any injective $R$-module is a direct sum of injective chain modules.

Proof. First note that a nontrivial injective module $M$ must have nontrivial $M_0$. Thus let $M$ be injective and $a \in M_0 - \{0\}$. By injectivity, either $s^n(a) \neq 0$ for all $n$ or there is an $n \geq 0$ such that $s^n(a) \neq 0$ and there are classes $a_{n+1}, a_{n+2}, \ldots$ such that $t(a_{i+1}) = a_i$ and $a_{n+1} = s^n(a)$. Thus $M$ contains an injective chain module, which by injectivity must be a direct summand. The claim follows by transfinite induction since arbitrary direct sums of of injectives over $R$ are injective. \hfill \square

3.4. Purity. For any ring $R$, a submodule $P$ of an $R$-module $M$ is called pure if its inclusion map stays injective after tensoring with any other $R$-module. Direct summands are examples of pure submodules, and in fact every pure submodule is a filtered colimit of direct summands [AR94, 2.30]. Thus the notion of purity is really only interesting when studying infinitely generated modules, and in fact turns out to be a central notion in that case. An equivalent characterization of purity is: $P < M$ is pure iff each linear equation $Ax = \beta$ with $\beta \in P^m$, $A \in R^{m \times n}$, that is solvable in $M$, is also solvable in $P$ (cf. [Lam99, Theorem 4.89]).

A module $I$ is called pure-injective if $\text{Hom}(M, I) \to \text{Hom}(P, I)$ is surjective for every pure morphism $P \to M$, i.e. if maps into $I$ extend along pure monomorphisms. The module $I$ is algebraically compact if every system of (infinitely many) $R$-linear equations
\begin{equation}
\left\{ \sum_{j \in J} r_{ij} x_j = b_i \right\}_{i \in I}
\end{equation}
in possibly infinitely many variables $x_j$ has a solution under the assumption that every finite subset of equations does. Of course, every single equation in (3.16) has only finitely many terms, i.e. for each $i$, $r_{ij} = 0$ for all but finitely many $j$, but the required solution $x_j$ does not need to satisfy that all but finitely many $x_j$ are zero. Again, it is a standard result [War69] that pure-injective modules are algebraically compact and vice versa.

In this subsection, we will show that linear systems of equations, as used in characterizing pure morphism and pure-injective modules, can be replaced by the simpler notion of chains in our setting. We continue to only consider connected $R$-modules.

Remark 3.17. Note that the set of chains $C \in \text{Ch}_\sigma^+(a)$ of signature $\sigma = (m, n, I)$ (where $m \leq n \leq \infty$) is equal to the solution set of the system of linear equations in the variables $(a_i)_{i \in [m, n]}$:
\begin{align*}
am = a; & \quad \left\{ \begin{array}{l}
am - ta_{m+1} = 0; \quad m \notin I \\
sam - a_{m+1} = 0 \quad m \in I 
\end{array} \right.
\end{align*}
**Definition.** A submodule $P$ of an $R$-module $M$ is **pre-pure** if it is pure as a $k[s]$- and as a $k[t]$-module.

Concretely, this means that if $t^b x = \beta$ has a solution $x \in M$ (with $b \geq 0$, $\beta \in P$) then it also has a solution $x \in P$, and similarly for the equation $s^a x = \beta$.

**Lemma 3.18.** The following are equivalent for a submodule $P < M$:

1. $P$ is pre-pure and for each nonnegative integers $n$, $a$, $b$ and $\beta \in P$, the system of equations
   \[
   \begin{align*}
   s^a x &= 0 \\
   t^b x &= \beta
   \end{align*}
   \]
   is solvable in $M$ if and only if it is solvable in $P$.

2. $P$ is pure.

3. $P$ is pre-pure and every nondegenerate chain in $P$ is also nondegenerate in $M$.

4. For every $a \in P$, $\text{ht}_P(a) = \text{ht}_M(a)$ and $\alpha_P(a) = \alpha_M(a)$.

**Proof.** (1) $\Rightarrow$ (2): We first consider the special case of a system of equations of the form
   \[
   \begin{align*}
   s^a x &= \beta_1 \\
   t^b x &= \beta_2
   \end{align*}
   \]
   for $a, b > 0$. Suppose that there exists a solution $z \in M$. Then by pre-purity, there exists $y \in P$ such that $s^a y = \beta_1$. Now consider the system of equations
   \[
   \begin{align*}
   s^a x &= 0 \\
   t^b x &= \beta_2 - t^b y.
   \end{align*}
   \]
   Since $z - y$ solves this system in $M$, there must be a solution $w$ in $P$ as well. But then $y + w$ solves the original system in $P$.

   Now let $Ax = \beta$ be an arbitrary linear equation in $M$ with $\beta \in P^m$, $A \in R^{m \times n}$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be the first column of $A$. We consider two cases:

   - All coefficients are multiples of some coefficient $\alpha_i$ for some $i$, without loss of generality $i = 1$. This happens if either $\alpha_1 \in k^x$, all $\alpha_i \in (s)$, or all $\alpha_i \in (t)$. Then by a Gauss elimination step, $Ax = \beta$ is equivalent to a system of the form
     \[
     \begin{pmatrix}
     \alpha_1 & l \\
     0 & B
     \end{pmatrix} x = \beta.
     \]
     By induction, $Bx_{>2} = \beta_{>2}$ has a solution in $P$, and the first equation
     \[
     \alpha_1 x_1 = \beta_1 - l \cdot x_{>2}
     \]
     is solvable with $x_1 \in P$ by hypothesis because $\alpha_1 = \lambda s^a$ or $\alpha_1 = \lambda t^b$ for some $\lambda \in k^x$. (It could also happen that $\alpha_1 = 0$, but in that case the above equation would not be solvable in $M$ unless $\beta_1 - l \cdot x_{>2} = 0$.)

   - There are two indices $i$ and $j$ such that $\alpha_i = \lambda s^a$ and $\alpha_j = \mu t^b$, $a, b > 0$, $\lambda, \mu \in k^x$, and such that every $\alpha_q$ is a multiple of either $\alpha_i$ or $\alpha_j$. We assume without loss of generality that $(i,j) = (1,2)$. Then by a Gauss elimination step, $Ax = \beta$ is equivalent to a system of the form
     \[
     \begin{pmatrix}
     s^a & l_1 \\
     t^b & l_2 \\
     0 & B
     \end{pmatrix} x = \beta.
     \]
Once again, by induction, $Bx_{\geq 2} = \beta_{\geq 3}$ has a solution in $P$, and the remaining equations are

$$\begin{align*}
s^a x_1 &= \beta_1 - l_1 \cdot x_{\geq 2} \\
t^b x_1 &= \beta_2 - l_2 \cdot x_{\geq 2}
\end{align*}$$

By the special case discussed before, this system has a solution $x_1 \in P$.

(2) $\Rightarrow$ (3): Suppose that $P$ is a pure submodule, so in particular it is pre-pure. To show that every nondegenerate chain $C$ in $P$ stays nondegenerate in $M$, consider first the case that the inclusion $i: P \to M$ has a retraction $p: M \to P$, i.e. $P$ is a direct summand. Then $C = p(i(C))$, i.e. $i(C)$ is the preimage of a nondegenerate chain and hence nondegenerate.

Now let $M_i$ be a filtered diagram of modules with colimit $M$ and $\iota_i: P \to M_i$ compatible split inclusions (the splittings $p_i: M_i \to P$ do not have to commute with the maps in the diagram.) Suppose $C = C_1 + \cdots + C_n$ becomes degenerate in $M$, with each $C_j$ strongly degenerate. By Lemma 3.4, we may without loss of generality assume that $C$ is finite. Then $C_1, \ldots, C_n$ lie in a common module $M_i$ and are strongly degenerate there, thus $C$ is degenerate in $M_i$, a contradiction to the first case.

(3) $\Rightarrow$ (4): Let $P$ be pre-pure, $a \in P$ and

$$\sigma = (m, n, I) = \text{ht}_P(a) \neq \text{ht}_M(a) = (m, n', I') = \tau.$$ 

By Lemma 3.12, we must have $\sigma < \tau$. By the definition of the order relation, there exists $k$ such that $I' \cap [m, k] = I' \cap [m, k], k \notin I'$, and either $k = n$ or $k \in I$. The former case is precluded by pre-purity. The latter case would imply by Prop. 3.7 that a nondegenerate chain of signature $\sigma$ in $P$ would become degenerate in $M$, contrary to the assumption. The argument about the order is dual.

(4) $\Rightarrow$ (1): Suppose that $P < M$ is a submodule with the property that $\text{ht}_P(a) = \text{ht}_M(a)$ and $\text{o}_P(a) = \text{o}_M(a)$ for all $a \in P$.

We consider the three cases of systems of equations:

1. $t^b x = \beta$ for some $b \geq 1, \beta \in P_m$ has a solution $x \in M$. If the equation had no solution in $P$ then $\text{ht}_P(\beta)|_{m+b} < (m, n, \emptyset) = \text{ht}_M(\beta)|_{m+b}$ and thus $\text{ht}_P(\beta) < \text{ht}_M(\beta)$, a contradiction to the height condition.

2. $s^a x = \beta$ for some $a \geq 1, \beta \in P_m$ has a solution $x \in M$. By the dual argument, a solution in $P$ must exist due to the order condition.

3. $\begin{align*}
s^a x &= 0 \\
t^b x &= \beta
\end{align*}$ for some $a, b \geq 0, \beta \in P_m - \{0\}$ has a solution $x \in M$. We may without loss of generality assume that $s^{a-1}(x) \neq 0$ for all such solutions $x$ since we could otherwise decrease $a$.

Thus there exists a upper closed chain $C$ in $\text{Ch}^+(\beta)$ in $M$ adhering to the signature $\sigma = (m, m+a+b, [m+a, m+b-1])$, which by Lemma 3.6 we may assume to be nondegenerate. By Lemma 3.10, $\sigma \leq \text{ht}_M(\beta) = \text{ht}_P(\beta)$. If $\sigma = \text{ht}_P(\beta)$ then we are done, since a chain of signature $\sigma$ in $P$ gives a solution in $P$ to the linear system. If $\sigma < \text{ht}_P(\beta)$ then the definition of the order implies that either there exists a solution $x \in P$ with $s^{a'} = 0$ for $a' < a$ (contradicting minimality of $a$) or there exists a longer nondegenerate chain $C' \in \text{Ch}^+(\beta)$ with larger signature in $P$, which means that $C'|_{m+a+b}$ still solves the system.

□
3.4.1. Pure-injectivity. To study pure-injectivity in $R$-modules, we will have to consider finite and infinite systems of linear equations over $R$. Fortunately, in our case, this condition can be reduced to statements about finite and infinite chains.

**Theorem 3.19.** Let $N$ be an $R$-module. Then $N$ is pure-injective if and only if every element of $M$ is a limit element.

**Proof.** Only the "if" direction needs proof. Let $A \subseteq B$ be a pure submodule and $f: A \to N$ a homomorphism. We need to show that $f$ extends to $B$ if every element of $N$ is a limit element. The argument is somewhat inspired by [EM02, Proof of Thm. 5.1.2].

Let us call an extension $\tilde{g}: \tilde{B} \to N$ of $f$, where $A \leq \tilde{B} \leq B$, a partial homomorphism if for every system of linear equations

$$M y = b \quad (M \in R^{n \times m}, b \in \tilde{B}^n)$$

that is solvable in $B$, the system

$$M y = \tilde{g}(b)$$

is solvable in $N$.

By an argument completely analogous to the proof of (1) $\Rightarrow$ (2) in Lemma 3.18, $(\tilde{B}, \tilde{g})$ is a partial homomorphism if and only if

- If $uy = b$ is solvable in $B$ for some $b \in \tilde{B}$ and $u \in \{ s^a, t^b \}$ then $uy = \tilde{g}(b)$ is solvable in $N$; and
- If $\{ s^a y = 0; t^b y = b \}$ is solvable in $B$ for some $b \in \tilde{B}$, then $\{ s^a y = 0; t^b y = \tilde{g}(b) \}$ is solvable in $N$.

The set of partial homomorphism is ordered in the obvious way, nonempty since $(A, f)$ is in it, and satisfies the ascending chain condition, so Zorn’s Lemma gives a maximal partial homomorphism $\tilde{B} \to N$.

If $B = \tilde{B}$ then we have found an extension, so suppose that $c \in B - \tilde{B}$ is an element in the complement. Choose $c$ of minimal degree, so that $t(c) \in \tilde{B}$.

We consider the following cases:

1. $c$ has finite height $\sigma = (m, n, I)$ and there exists a nondegenerate chain in $\text{Ch}_\tau(c)$ all of whose elements lie in $B - \tilde{B}$.
2. There exists an $n$ and a nondegenerate chain $C \in \text{Ch}_\tau(c) \cap \text{Ch}_{m,n}$ with top element in $\tilde{B}_n$.
3. $c$ has infinite height and there exist, for every $n \geq m$, a nondegenerate chain $C_n \in \text{Ch}_\tau(c) \cap \text{Ch}_{m,n}$ all of whose elements lie in $B - \tilde{B}$.

**Cases (1) and (2):**

In case (1), let $C = (c_{m-1}, \ldots, c_n)$ be a chain with $c_{m-1} = t(c), c_m = c$ and $C_{i_n}$ is nondegenerate. Then $C$ is a solution to a finite system of linear equations with right hand side in $\tilde{B}$. By the partial homomorphism condition for $\tilde{B}$, there is a chain $D = (d_{m-1}, \ldots, d_n) \in \text{Ch}_\tau(g(t(c)))$ in $N$.

Similarly, in case (2), a shortest nondegenerate chain $C = (c_{m-1}, \ldots, c_n)$ with $c_{m-1} = t(c), c_m = c$, and $c_n = b$ is a solution to a finite system of linear equations with right hand side in $B$. By the partial homomorphism condition for $\tilde{B}$, there is a chain $D = (d_{m-1}, \ldots, d_n)$ from $d_{m-1} = g(c_{m-1}) = g(t(c))$ to $g(c_n) = g(b)$ in $N$.

In either of the cases, let $B' = \tilde{B} + \langle C \rangle$, the sum of $\tilde{B}$ and the chain module spanned by $C$. 

Define $h: B' \to N$ as the extension of $\tilde{g}$ defined by $h(c_i) = d_i$. Note that this is well-defined by construction.

To see that $(B', h)$ is a partial homomorphism, it suffices to consider equations of the form $uy = c_i$ for $u \in \{s^a, t^b\}$ and pairs of equations of the form $\{s^n y = 0; t^n y = c_i\}$ for $m \leq i < n$.

If $s^n y = c_i$ is solvable for some $y \in B$, $i \in [m, n]$, and $a \geq 1$ then firstly, $i - a \geq m$ by the minimality of $m$. Moreover, $s^n c_{i-a} = c_i$ because $C$ is nondegenerate. Thus $s^n h(c_{i-a}) = h(c_i)$ provides the desired solution in $N$.

If $t^n y = c_i$ is solvable for some $y \in B$, $i \in [m, n]$, and $b \geq 1$ then it is also solvable for some $y \in B'$ by construction and the nondegeneracy of $C$. Thus again, $t^n h(y) = h(c_i)$. In fact, we can choose $y = c_{i+b}$, and we must have that $i + b \leq n$.

Finally, suppose that $\{s^n y = 0; t^n y = c_i\}$ is solvable for some $y \in B$, $i \in [m, n]$, and $a, b \geq 1$. Let $c' = s^a c_{i+b}$. If $c' = 0$ then we are done by the previous case, so assume the opposite. Then there are two possibilities. Either $i + a + b < n$, in which case $c' = c_{i+a+b}$. Since another solution $y$ exists and defines a chain adhering to a larger signature, the chain $C$ must be degenerate, a contradiction. In the other case, $c' \in B$. Consider $z = c_{i+b} - y$. We have that $t^n z = 0$ and $s^n z = c'$. Thus there exists a solution $w \in N$ to $t^n w = 0, s^n w = \tilde{g}(b')$. Then $y = w - h(c_{i+b})$ solves $\{s^n y = 0; t^n y = h(c_i)\}$.

Now consider case (3). For every $n \geq m$, $n \notin I$, let $C_n = (c_{a,m}, \ldots, c_{n,n})$ be a nondegenerate upper closed chain in $Ch^+_{\sigma|_m}(c)$. If the complement of $I$ is finite (implying that $c$ is a limit element), let $C_\infty \in Ch^+_{c}(c)$ be a nondegenerate chain.

Now let

$$B' = \tilde{B} + \sum_i (C_i),$$

where $i$ runs through those indices $m \leq i \leq \infty$ for which $C_i$ is defined. If $t(c) = 0$ then we can extend $\tilde{g}: \tilde{B} \to N$ to $B'$ by zero on all $C_i$ summands, and we are done. Thus suppose $t(c) \neq 0 \in \tilde{B}$.
Then the augmented chains $C_n \cup \{ t(c) \}$ can be lifted to chains $D_n \in \Ch_{m,n}(X) \cap \Ch^+(\tilde{g}(t(c)))$ by case 1, which however may be degenerate. Nonetheless, by Lemma 3.9 there exists a chain $D = (d_{m-1}, d_m, \ldots) \in \Ch^+(\tilde{g}(t(c)))$ adhering to the signature $\sigma$ extended down to degree $m - 1$ (if $m > 0$). Now define $h: B' \to N$ by $\tilde{g}$ on $\tilde{B}$ and

$$h(c_{n,i}) = d_i.$$  

Again, by construction, $h$ is an extension of $\tilde{g}$ to $B'$. 

Now, repeating the argument to show that $(B', h)$ is a partial homomorphism, consider the basic linear equations one at a time. 

Suppose $s^a y = \gamma$ has a solution in $B$, where $\gamma = \sum_{n=i}^{\infty} \lambda_n c_{n,i}$ and all but finitely many $\lambda_n$ are zero. Then $i - a \geq m$ as before and

$$s^a \left( \sum_{n=i}^{\infty} \lambda_n c_{n,i-a} \right) = \gamma.$$  

Applying $h$ to both sides gives a solution in $N$.

If $t^b y = \gamma$, with notations as in the previous case and $\sum_n \lambda_n = 1$, we have that $t^{-a} \gamma = c$ and $h(\gamma) = d_i$. Since the chain $(c = t^{b+i-a} y, \ldots, ty, y)$ is nondegenerate, the height of $c$ must be at least as large, and there must exist an $n'$ such that $t^n c_{n',i+b} = c$. Since $h(c_{n,i}) = d_i$ for any $n$, we have a solution $t^h(c_{n',i+b}) = d_i = h(\gamma)$.

As for the last case, suppose $\{ s^a y = 0; t^b y = \gamma \}$ is solvable for some $y \in B$ and $\gamma$ as above. Let $y' = c_{n',i+b}$ be the solution to $t^h y = \gamma$ constructed above and $c' = s^a y'$. As before, we may assume $c' \neq 0$ because otherwise we have found a solution $h(y')$. Again, we study two possibilities. Either $i + a + b < n$, in which case $c' = c_{n',i+a+b}$. Since another solution $y$ exists and defines a chain adhering to a larger signature, the chain $C$ must be degenerate, a contradiction. In the other case, $c' \in \tilde{B}$. Consider $z = y' - y$. We have that $t^h z = 0$ and $s^a z = c'$. Thus there exists a solution $w \in N$ to $t^h w = 0$, $s^a w = \tilde{g}(b')$. Then $y = w - h(c_{i+b})$ solves $\{ s^a y = 0; t^b y = h(\gamma) \}$.

\[ \square \]

**Corollary 3.21.** Arbitrary direct sums of pure-injective $R$-modules are pure-injective, and any chain module is pure-injective.

\[ \square \]

**Proof.** Let $M_i$ be a collection of pure-injective $R$-modules and $a \in M = \bigoplus_i M_i$. Then $a$ is contained in a finite subsum $M'$, and its height in $M'$ is the same as its height in $M$ by Lemma 3.18 because inclusions of direct summands are pure. Since finite sums (finite products) of pure-injective modules are pure-injective, $a$ is a limit element in $M'$ and thus also in $M$.

That chain modules are pure-injective follows immediately from Theorem 3.19. \[ \square \]

**Theorem 3.22.** The pure-injective $R$-modules exactly the direct sums of chain modules.

**Proof.** We first show:

**Claim:** Every nontrivial pure-injective $R$-module has a chain module as a direct summand.

Indeed, let $c \in M$ be a nonzero element of minimal degree. Let $C$ be a nondegenerate chain in $\Ch_{ht(c)}^+(c)$, which exists by pure-injectivity, and let $N < M$ be the chain module generated by it. Then the inclusion $N < M$ is pure by Lemma 3.18(4).
and $N$ itself is pure-injective, so the inclusion $N \to M$ has a retraction, proving the claim.

Now consider the set $S$ of all collections $C = (C_i) \subseteq \text{Ch}_0^+$ of nondegenerate chains in $M$ with bottom element in degree 0 such that

- Every $C_i$ spans a pure chain submodule $M_i$ of $M$;
- The map $N_C = \bigoplus M_i \to M$ is injective.

The set $S$ is ordered by inclusion and nonempty by the claim. For every $C \in S$, $N_C < M$ is pure. Since directed colimits of pure submodules are pure, and directed colimits of direct sums by maps that are inclusions of summands are again direct sums, the ascending chain condition in Zorn’s lemma is satisfied, so let $C$ be a maximal element and $N = N_C$ the associated module. Then $N \to M$ is split because $N$ is pure-injective by the corollary. Let $Q$ be a complement of $N$ in $M$. Then $Q_0 = 0$ because otherwise, by the claim, we could choose a new pure and pure-injective submodule of $Q$, contradicting maximality. Repeating this process for the downward shift $Q(-1)$ inductively, we obtain an isomorphism of $M$ with a direct sum of chain modules. □

**Corollary 3.23.** An $R$-module $M$ is pure-injective if and only if every element in $\ker(t)$ is a limit element.

**Proof.** We show by induction that for each $k \geq 1$, $\ker(t^k)$ consists of limit elements, the case $k = 1$ given by assumption. This suffices since $M$ is nonnegatively graded and hence, for every $a \in M$, $t^k a = 0$ for $k > |a|$.

Suppose $a \in M_m - \{0\}$ with $t^k a = 0$ and $b = t^k(a) \neq 0$. Let $\sigma = \text{ht}(a)$. If $\sigma$ is finite then there is nothing to show, so assume $\sigma = (m, \infty, I)$ is infinite. For $n \geq m$, let $C_n$ be a nondegenerate chain in $\text{Ch}_{\sigma|m}^+(a)$ Let $\sigma' = (m-k, \infty, I)$ and $C'_n \in \text{Ch}_{\sigma'|m}^+(b)$ the unique extension of $C_n$. ($C'_n$ may be degenerate.)

Since $t(b) = 0$, $b$ is a limit element, and there exists a nondegenerate chain $D' \in \text{Ch}_{\tau|m}^+(b)$, where $\tau = \text{ht}(b)$. Since $\tau \geq \sigma'$, $\tau$ has to have length at least $k$. Thus let $d = D_m'$ be the element in degree $m$. Since $\tau \geq \sigma'$, $\tau|_m^\infty \geq \sigma$ and by Lemma 3.5, there exists a chain $D \in \text{Ch}_{\tau|m}^+(d)$.

Since $C_n - D|^n_m \in \text{Ch}_{\sigma|m}^+(a-d)$ for all $n$ and $t^k(a-d) = b-b = 0$, by induction and Lemma 3.9, there exists a chain $K \in \text{Ch}_{\rho}(a-d)$. Thus $K + D \in \text{Ch}_{\sigma|m}^+(a)$. □

We next address the question of uniqueness of the decomposition of Thm. 3.22. This is essentially the Krull-Schmidt-Azumaya theorem in a graded context. We use it in the form proved by Gabriel:

**Theorem 3.24** ([Gab62, §I.1, Théorème 1]). Let $\mathcal{C}$ be a Grothendieck category. If $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ are two families of indecomposable objects in $\mathcal{C}$ with local endomorphism rings such that $\bigoplus_{i \in I} M_i \cong \bigoplus_{j \in J} N_j$ then there exists a bijection $h: I \to J$ such that $M_i \cong N_{h(i)}$. □

**Corollary 3.25.** Let $M \cong \bigoplus_{j=1}^m M(\sigma_j) \xrightarrow{\phi} \bigoplus_{i=1}^{m'} M(\sigma'_i)$. Then the sets $\{\sigma_j\}_j$ and $\{\sigma'_i\}_i$ are equal.

**Proof.** A Grothendieck category is an AB5 category with a generator, and the category of nonnegatively graded $R$-modules satisfies these conditions. We also have that $\text{End}(M(\sigma)) \cong k$ is indeed local. Thus Theorem 3.24 applies to give the result. □
4. The classification of abelian \( p \)-torsion Hopf algebras

With the results of the previous section in hand, all ingredients are in place to prove the first main theorem.

Continue to let \( k \) be a perfect field of characteristic \( p \). As an object of an abelian category, any Hopf algebra \( H \) has a multiplication-by-\( p \) map, classically denoted by \([p] : H \to H\). This map is given by the \( p \)-fold comultiplication followed by the \( p \)-fold multiplication:

\[
[p] : H \xrightarrow{\Delta^{p-1}} H^{\otimes p} \xrightarrow{\mu^{p-1}} H.
\]

In this section, we classify pure-injective \( p \)-torsion Hopf algebras. From an algebro-geometric point of view, \( p \)-torsion Hopf algebras represent \( \mathbb{F}_p \)-module schemes rather than abelian group schemes. Since the Dieudonné equivalence is an equivalence of abelian categories, the full subcategory of \( p \)-torsion Hopf algebras is equivalent to the category of \( p \)-torsion Dieudonné modules. By Lemma 2.2, the category of \( p \)-typical \( p \)-torsion Hopf algebras (of any type) is thus equivalent to the category of nonnegatively graded modules over \( R = R/(p) \cong k[s, t]/(st) \).

**Lemma 4.1.** A \( p \)-torsion Hopf algebra \( H \) is pure-injective (in the sense of the definition given in the introduction) iff the Dieudonné modules \( D^{(j)}(H) \) are pure-injective (in the sense of Subsection 3.4) for all \( j > 0 \).

**Proof.** Since the \( F \) and \( V \) operators preserve the \( p \)-typical splitting, it is clear that \( H \) is pure-injective iff all of its \( p \)-typical factors are pure-injective, thus assume that \( H \) is pure-injective of type \( j \). Under the Dieudonné equivalence \( D^{(j)} : \mathcal{H} \to \text{Mod}_R \), we have that

\[
D^{(j)}(F) = s \quad \text{and} \quad D^{(j)}(V) = t.
\]

Moreover, \( P(H) \cong \ker(s) : D^{(j)}(H) \to D^{(j)}(H) \), and hence every \( x \in P(H) \) corresponds to an element \( D^{(j)}(x) \in \ker(s) \). (This is a slight abuse of notation, since there is no natural transformation \( H \to D^{(j)}(H) \).)

Thus the system of equations

\[
x_0 = x; \quad V(x_i) = F^{m_i}x_{i-1}
\]

from the definition of pure-injectivity of Hopf algebras translates to a system of equations

\[
y_0 = D^{(j)}(x); \quad t(y_i) = s^{m_i}y_{i-1}
\]

The result follows from Lemma 3.9 and Cor. 3.23. \( \square \)

**Proof of Theorem 1.1.** Let \( r = jp^{r'} \) with \( p \nmid j \).

Define \( H(r, m, I) = D^{(j)}\Sigma' M(m, I) \), using the \( p \)-typical Dieudonné equivalence (2.3). By Lemma 2.2, every Hopf algebra splits naturally and uniquely into a tensor product of \( p \)-typical parts of various types, so it suffices to show the theorem for \( p \)-typical Hopf algebras of type \( j \). The equivalence of pure-injectivity as used in Thm. 1.1 and pure-injectivity of Dieudonné modules is Lemma 4.1, the existence of the splitting follows from Corollary 3.22, and the uniqueness from Corollary 3.25.

It suffices to show that the characterization of \( H(r, m, I) \) in the introduction is correct. Let \( H \) be any Hopf algebra isomorphic as algebras to

\[
k[x_0, x_1, \ldots, x_m]/\left(x_i^{p-1} - \begin{cases} x_i; & i \in I \\ 0; & i \notin I \end{cases}, \quad |x_i| = rp^j.\right.
\]
Then $H$ is $j$-typical, and $M = D^{(j)}(H) \cong \langle x_0, x_1, \ldots, x_m \rangle$ with $s(x_{i-1}) = x_i$ iff $i \in I$. In order for $H$, and hence $M$, to be indecomposable, it is necessary that $tx_i = x_{i-1}$ if $i \notin I$, because otherwise, $M$ would split as $\langle x_0, \ldots, x_{i-1} \rangle \oplus \langle x_i, \ldots, x_m \rangle$. Thus $M \cong \Sigma^r M(m, I)$.

**Example 4.2.** Primitively generated Hopf algebras are Hopf algebras for which the canonical map $PH \to H \to QH$ from primitives to indecomposables is surjective. For the associated $R$-modules $M$, this translates to the map ker$(t) \to$ coker$(s)$ being surjective. Thus for each $m \in M_i$, there exists an $m' \in M_{i-1}$ such that $t(m + sm') = 0$. By induction, $tm' = 0$ and hence $tm = 0$. This shows that primitively generated Hopf algebras have trivial Verschiebung, and in particular are $p$-torsion and pure-injective. Hence they split into copies of $H(r, m, I)$. The Hopf algebra $H(r, m, I)$ is primitively generated exactly if $I = [1, m]$, recovering the abelian case of the classification in [MM65, Theorem 7.16] and generalizing it to non-finite-type Hopf algebras. This is not new – by [Web85], any connected $F_p(s)$-module splits into a sum of cyclic modules.

5. **Hopf Algebras that are Free as Algebras or Cofree as Coalgebras**

A classification of general abelian Hopf algebras seems too much to ask for, even in the finite-type case; we will give some hopefully illuminating examples of the complexity of the problem in the next section. However, we can classify the Hopf algebras that reduce to the Hopf algebras $H(r, m, I)$ of Section 4 and use this to classify all Hopf algebras that are either free as algebras or cofree as coalgebras and whose mod-$p$ reduction is pure-injective.

**Remark 5.1.** The forgetful functor from the category $\mathcal{H}$ to connected, graded, commutative algebras has a right adjoint, the cofree Hopf algebra on an algebra. Similarly, the forgetful functor from $\mathcal{H}$ to connected, graded, cocommutative coalgebras has a left adjoint, the free Hopf algebra on a coalgebra. A Hopf algebra that is free on a coalgebra is also free as an algebra, but not vice versa. Similarly, a Hopf algebra that is cofree on an algebra is also cofree as a coalgebra, but not vice versa.

**Definition.** A Hopf algebra $H$ is called basic if its mod-$p$ reduction $H/[p]$ is indecomposable pure-injective and $H_1 \neq 0$.

By Theorem 1.1, $H$ is thus basic iff $H/[p] \cong H(0, m, I)$ for some $0 \leq m \leq \infty$ and $I \subseteq [1, m]$.

We will use the graphical model of “basic Hopf graphs” from the introduction for indexing basic Hopf algebras. Projection to the $x$-axis associates to each basic Hopf graph $\Gamma$ a pair $(m_\Gamma, I_\Gamma)$, where $m$ is the maximal $x$-coordinate of all arrow end point (possibly $\infty$) and $i \in I$ iff there is an arrow $(i - 1, j)$ to $(i, j')$ for some $j, j'$. Moreover, each Hopf graph $\Gamma$ comes with a vector $v_\Gamma$ of length $m_\Gamma + 1$ consisting of the $y$-coordinates of the vertices of $\Gamma$.

Conversely, a pair $(m, I)$ and a vector $v$ of length $m + 1$ gives rise to a unique basic Hopf graph if

1. $v_0 = 0$
2. if $i \in I$ then $v_i \in \{v_{i-1} - 1, v_{i-1}\}$
3. if $i \notin I$ then $v_i \in \{v_{i-1}, v_{i-1} + 1\}$
4. if $m < \infty$ then $v_m = 0$. 

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Fig. 5.2 and 5.3 show the two examples of basic Hopf graphs from the introduction together with data \( m, I, v \) explained above.

![Figure 5.2](image1.png) \( m_\Gamma = 4, I_\Gamma = \{2, 4\}, v_\Gamma = (0, 1, 1, 1, 0) \)

![Figure 5.3](image2.png) \( m_\Gamma = \infty, I_\Gamma = \{1\}, v_\Gamma = (0, 0, 1, 2, 3, \ldots) \)

**Definition.** Let \( \Gamma \) be a basic Hopf graph. Define an \( \mathcal{R} \)-module \( M(\Gamma) \) as follows:

\[
M(\Gamma)_i = \begin{cases} 
W(k)/p^{(m_i)}\langle x_i \rangle; & i \leq m_{\Gamma}; \\
0; & i > m_{\Gamma}; 
\end{cases}
\]

together with \( s \)- and \( t \)-multiplications given, for \( 1 \leq i \leq m \), by

\[
sx_{i-1} = \begin{cases} 
x_i; & i \in I_\Gamma; \\
px_i; & i \notin I_\Gamma; 
\end{cases}
\]

\[
tx_i = \begin{cases} 
px_{i-1}; & i \in I_\Gamma; \\
x_{i-1}; & i \notin I_\Gamma. 
\end{cases}
\]

We call an \( \mathcal{R} \)-module \( M \) basic if its associated Hopf algebra \( (D(j))^{-1}(M) \) is basic, or equivalently, by Theorem 1.1, if \( M/(p) \cong M(m, I) \) for some \( (m, I) \).

**Theorem 5.5.** An \( \mathcal{R} \)-module \( M \) is basic with \( M/(p) \cong M(m, I) \) if and only if \( M \cong M(\Gamma) \) for some basic Hopf graph \( \Gamma \) with \( (m_\Gamma, I_\Gamma) = (m, I) \). Moreover, \( M \) is uniquely determined up to isomorphism by the properties

1. t: \( M_i \rightarrow M_{i-1} \) is injective iff \( \Gamma \) contains an arrow from \((i, j)\) to \((i - 1, j)\) or an arrow from \((i, j)\) to \((i, j - 1)\), and
2. s: \( M_{i-1} \rightarrow M_i \) is injective iff \( \Gamma \) contains an arrow from \((i - 1, j)\) to \((i, j)\) or an arrow from \((i, j)\) to \((i - 1, j - 1)\).

**Proof.** It is evident from the definition that \( M(\Gamma)/(p) \cong M(m_\Gamma, I_\Gamma) \), so the “only if” direction is clear.

Let \( M \) be a basic \( \mathcal{R} \)-module with \( M/(p) \cong M(m, I) \). Then \( M_i = W_{v_i+1}(k) \) for some positive integers \( v_i \) for \( i \leq m \) and \( M_i = 0 \) for \( i > m \). Note that the condition \( p = st = ts \) implies that \( v_i \in \{v_{i-1} - 1, v_i, v_{i+1}\} \) for all \( i \geq 0 \).

We inductively construct lifts \( \tilde{x}_i \in M_i \) of the defining generators \( x_i \in M(m, I)_i \) and show that the \( v_i \) satisfy the conditions for being \( v_I \) of a basic Hopf graph \( \Gamma \) with \( (m, I) = (m_\Gamma, I_\Gamma) \). Since \( tM_0 = 0 \) for dimensional reasons, \( pM_0 = 0 \) as well, so \( M_0 = k \) and \( v_0 = 0 \). Let \( \tilde{x}_0 \) be the unique lift of \( x_0 \).

Suppose now that \( \tilde{x}_0, \ldots, \tilde{x}_{i-1} \) are constructed satisfying (5.4). If \( i \in I \), define \( \tilde{x}_i = s(\tilde{x}_{i-1}) \). Then \( \tilde{x}_i \) is a lift of \( x_i \) and hence a generator, and we have that
For any Dieudonné functor of type 1 from (2.3). By construction, 

Conversely, if \( i \notin I \), but \( i \leq m \), \( t: M_i/(p) \rightarrow M_{i-1}/(p) \) is bijective, hence \( t: M_i \rightarrow M_{i-1} \) is surjective. Let \( \tilde{x}_i \) be any preimage of \( \tilde{x}_{i-1} \) under \( t \). Then \( \tilde{x}_i \) is a lift of \( x_i \) and hence a generator. Again, \( v_i = v_{i-1} - 1 \) is not possible because of the surjectivity of \( t: M_i \rightarrow M_{i-1} \). \( \square \)

**Proof of Theorem 1.3.** Given \( \Gamma \), define \( H(\Gamma) = (D^{(1)}(M(\Gamma))) \), using the \( p \)-typical Dieudonné functor of type 1 from (2.3). By construction, \( H(\Gamma)/[p] \cong H(0, m_\Gamma, I_\Gamma) \). For any \( p \)-typical Hopf algebra \( H \) of type 1 with \( M = D^{(1)}(H) \), we have that

\[
D^{(1)}(F: H_p \rightarrow H_p^i) = s: M_i \rightarrow M_{i+1}
\]

and

\[
D^{(1)}(V: H_{p^{i+1}} \rightarrow H_{p^i}) = t: M_{i+1} \rightarrow M_i
\]

The claimed classification of basic Hopf algebras thus follows directly from Theorem 5.5. Moreover, let \( H \) be any Hopf algebra whose mod-\( p \) reduction is indecomposable. Let \( r \) be the smallest positive integer such that \( H_r \neq 0 \). Then by Theorem 1.1, \( H \) is concentrated in degrees \( rp^i \). Denote by \( H' \) the regraded Hopf algebra with \( H'_j = H_{r^j} \). Then \( M' = D^{(1)}(H') \cong M(\Gamma) \) is basic, and \( H \cong H(r, \Gamma) \). \( \square \)

It seems that most Hopf algebras one encounters “in nature” are tensor products of basic ones. We will not attempt to support this claim with many examples, but one important one is the following:

**Example 5.6.** Consider the Hopf algebra \( H = H^*(BU; k) \cong k[c_1, c_2, \ldots] \), where \( BU \) is the classifying space of the infinite unitary group and \( c_i \) are the universal Chern classes in degree \( 2i \). By dimensional considerations, \( H \) splits into a tensor product of one \( p \)-typical part of type \( j \) for each \( p \nmid j \). The type-1 part \( \Lambda_p \) is the Hopf algebra representing the \( p \)-typical Witt vector functor. It is basic: \( \Lambda_p \cong H(\Gamma) \) for

\[
\Gamma:
\]

**Lemma 5.7.** A Hopf algebra \( H \) is free as a commutative algebra iff the Frobenius \( F \) on \( DH \) is injective. It is cofree as a cocommutative coalgebra iff the Verschiebung \( V \) on \( DH \) is surjective.

**Proof.** The \( p \)th power map of \( H \) is a frob-twisted \( k \)-linear Hopf algebra morphism \( F: H \rightarrow H \) inducing the maps \( F: DH \rightarrow DH \) of Dieudonné modules. If \( H \) is free as a commutative algebra, \( F \) will therefore be injective. Conversely, suppose \( H \) is not free as a commutative algebra. Every Hopf algebra over a field is a directed colimit of Hopf algebras that are finitely generated as algebras. If \( \{x_i\} \) is a minimal set of algebra generators for \( H \), then there exists a polynomial \( q \) in the \( x_i \) which is zero in \( H \). Since this polynomial only involves finitely many of the \( x_i \), it is contained in a finitely generated Hopf algebra. Thus it suffices to show that \( F \) is noninjective in the case where \( H \) is a finite type. By Borel’s theorem on
the classification of algebra structures on connected, graded Hopf algebras [MM65, Theorem 7.11], a Hopf algebra of finite type that is not free as an algebra has a noninjective Frobenius. The argument for the Verschiebung is dual. □

This gives a simple characterization of when a basic Hopf graph corresponds to a free resp. cofree Hopf algebra:

**Corollary 5.8.** The Hopf algebra $H(\Gamma)$ is

- free as an algebra if $\Gamma$ is infinite and consists of horizontal right arrows and diagonal left arrows only;
- cofree as a coalgebra if $\Gamma$ is infinite and consists of horizontal left arrows and diagonal left arrows only.

Thus the only basic Hopf algebra that is both free as an algebra and cofree as a coalgebra is $\Lambda_p$ of Example 5.6.

**Theorem 5.9.** Let $H$ be a Hopf algebra whose mod-$[p]$ reduction is pure-injective. If $H$ is free as an algebra or cofree as a coalgebra then $H$ is a tensor product of basic Hopf algebras.

**Proof.** Without loss of generality, suppose that $H$ is $p$-typical and $M$ its associated $R$-module. Choose a decomposition

$$\bigoplus_{i=1}^n \gamma_i : \bigoplus_{i=1}^n \Sigma^r \text{M}(m_i, I_i) \xrightarrow{\gamma_i} \text{M}/(p)$$

as in Theorem 1.1, where $n \in \mathbb{N} \cup \{\infty\}$. By Theorem 1.3, there are basic Hopf graphs $\Gamma_i$ and maps

$$\gamma_i : \Sigma^r \text{M}(\Gamma_i) \rightarrow \text{M}$$

such that $\bigoplus_{i=1}^n \gamma_i$ is a surjective map and $\gamma_i = \gamma_i/(p)$.

We denote the defining generator of $\Sigma^r \text{M}(\Gamma_i)$ in degree $j$ by $x_{i,j}$ and identify it with its image in $\text{M}$ under $\gamma_i$. To see that $\bigoplus_{i=1}^n \gamma_i$ is a injection, we need to show that

$$\bigoplus_{i=1}^n \alpha_i x_{i,j} = 0 \implies \alpha_i x_{i,j} = 0$$

for all $\alpha_i \in k$.

Here we implicitly assume that only finitely many $\alpha_i$ are nontrivial if $n = \infty$.

We show this by induction in the degree $j$. For $j = 0$, $M(\Gamma_i)_0 \cong M(m_i, I_i)_0$ is $p$-torsion, and by definition, the nonzero $x_{i,0}$ are linearly independent. Assume that (5.10) holds for $j - 1$. Since modulo $p$, the $x_{i,j}$ are linearly independent, we have that $p \mid \alpha_i$ for all $i$. But $p = st$, and since $s$ is injective by assumption, we have that

$$\sum_{i=1}^n \alpha_i x_{i,j} = 0 \Rightarrow \alpha_i x_{i,j} = 0$$

for all $\alpha_i \in k$.

We have that $tx_{i,j} \in \{x_{i,j-1}, px_{i,j-1}\}$, so by induction, $\frac{\alpha_i}{p}tx_{i,j} = 0$ and hence $s(\frac{\alpha_i}{p}tx_{i,j}) = \alpha_i x_{i,j} = 0$ for all $i$.

This completes the proof that $\bigoplus_{i=1}^n \gamma_i$ is an isomorphism.

The result for Hopf algebras which are cofree as coalgebras follows from dualization. □
Proof of Theorem 1.2. Given Theorem 5.9, the only thing that remains is to classify those basic Hopf algebras of mod-$p$ type $(m, I)$ which are free as an algebras (the other case being dual). This corresponds to $s$ being injective. The characterization of basic $R$-modules with injective $s$ from Theorem 5.5 shows that there is exactly one basic Hopf graph $\Gamma$ with this property for any given $(m, I)$. \hfill $\Box$

We will now compare the property of $H$ being free as an algebra to two stronger conditions. The first condition is being a projective object. The second condition is, morally, to be free over a graded, connected coalgebra. However, this notion is unnecessarily restrictive and is incompatible with the property of being $p$-typical – a $p$-typical Hopf algebra can never be free in this sense. Instead, we consider the property of being free over a $p$-polar graded coalgebra [Bau22, Bau24].

Definition. A graded $p$-polar $k$-coalgebra is a graded vector space $C$ together with a $k$-linear map

$$\Delta: C_{pn} \to \left(C_n \otimes \cdots \otimes C_n\right)^{\Sigma_p}$$

which, with this structure, is a retract of a graded $k$-coalgebra.

The dual definition of this notion was given in [Bau24], along with the dual version of the following theorem:

Theorem 5.11. The functor which associates to a graded $k$-coalgebra the free graded Hopf algebra over it factors naturally through the category of $p$-polar coalgebras.

In other words, there exists a free Hopf algebra functor on $p$-polar coalgebras, and the usual free Hopf algebra functor on coalgebras really only depends on the underlying $p$-polar coalgebra structure. This free functor sends $p$-typical $p$-polar coalgebras (in the obvious meaning) to $p$-typical Hopf algebras of the same type.

Proposition 5.12. Let $H = H(r, \Gamma)$ be an irreducible Hopf algebra. Then

1. $H$ is free on a graded, connected $p$-polar coalgebra iff $\Gamma$ has $m = \infty$, $I_H = \{i \mid i \geq n\}$, $(v_H)_i = \min(i, n)$ for $0 \leq n \leq \infty$ (Figs. 5.13 and 5.14)

   ![Figure 5.13](image1)

   $m = \infty$, $I = [4, \infty]$, $v_H = (0, 1, 2, 3, 3, \ldots)$

   ![Figure 5.14](image2)

   $m = \infty$, $I = \emptyset$, $v_H = \{0, 1, 2, 3\}$

2. $H$ is projective if, in addition, $p \nmid r$. 


Proof. (1) In [Kuh20], Kuhn showed that for every finitely generated \( \mathcal{R}/(s) \)-module \( M \), there exists a Hopf algebra \( H(M) \) with indecomposables \( M \), and in [Bau24], the author showed that \( H(M) \) is free on a \( p \)-polar coalgebra and (noncanonically) unique with this property. Moreover, the construction \( H \) sends direct sums to tensor products. Indecomposable \( \mathcal{R}/(s) \)-modules are suspensions of modules of the form \( k[t]/(t^{n+1}) \), with corresponding Hopf algebra isomorphic to \( H(\Gamma_n) \) with \( \Gamma_n \) as displayed in Fig. 5.13 for \( n = 4 \), or of the form \( k[t, t^{-1}]/k[t] \), with corresponding Hopf algebra \( H(\Gamma_{\infty}) \) as displayed in Fig. 5.14.

(2) This is well-known, e.g. by [Sch70, Théorème 3.2]. \( \square \)

6. Some Examples

In this section, we collect a few examples that show that our results are sharp in many respects.

Example 6.1. This example is to show that the grading is crucial and that, in particular, the connectedness assumption on \( H \) cannot be dropped.

Suppose \( H \) is an ungraded abelian Hopf algebra over a perfect field \( k \). By [Fon77], \( H \) naturally splits into \( H^u \otimes H^m \), where \( H^u \) denotes the unipotent part (also known as conilpotent) and \( H^m \) is the part of multiplicative type. The latter is classified by an abelian group \( A \) with a continuous action of the absolute Galois group \( \Gamma \) of \( k \); if the \( \Gamma \)-action is trivial then the associated Hopf algebra is just the group ring \( k[A] \). So a classification of \( H^m \), under some finiteness assumptions, is feasible. Note that connected, graded Hopf algebras are automatically unipotent for degree reasons.

Dieudonné theory works as expected for ungraded Hopf algebras (at least when \( H^m \) represents a functor taking values in abelian \( p \)-groups), and unipotent Hopf algebras correspond to Dieudonné modules on which \( V \) acts nilpotently. Let \( W \) be an \( n \)-dimensional \( \mathbb{F}_p \)-vector space, \( \phi: W \to W \) an endomorphism, and construct a Dieudonné module \( M_\phi = V \oplus V \) with \( V(x, y) = (0, x) \) and \( F(x, y) = (0, \phi(x)) \). Then it is easy to see that \( M_\phi \cong M_\psi \) if and only if \( \phi \) and \( \psi \) are conjugate, and \( M_\phi \) is decomposable iff \( W \) has a \( \phi \)-invariant direct sum decomposition. So already this finite-dimensional example shows that the moduli of indecomposable modules is rather large and, in an imprecise meaning, positive-dimensional (it grows with the size of \( k \)).

Example 6.2. Products of pure-injective modules are pure-injective, but pure-injective \( \mathcal{R} \)-modules decompose as a sum of chain modules by Thm. 3.22. This leads, for instance, to a direct sum decomposition into uncountable many chain modules of the module

\[
M = \prod_{i=0}^{\infty} \mathcal{R}/(t, s^i)(x_i),
\]

where the generators \( x_i \) have degree \( i \).

Example 6.3. We revisit Example 3.8 with the non-pure-injective submodule \( NPI(\sigma) \) of the pure-injective module \( PII(\sigma) \). The inclusion \( NPI(\sigma) \to PII(\sigma) \) is pure by Lemma 3.18. Indeed, if \( f: NPI(\sigma) \to M \) is any morphism into a pure-injective module \( M \) then \( f \) can be extended to \( NPI(\sigma) \) by sending the infinite chain of \( x_{\infty,*} \) to any chain in \( \text{Ch}_*^+(f(x_0)) \), which must exist because \( f(x_0) \) is a
limit element. Thus \( NP I(\sigma) \) is the pure-injective hull of \( NPI(\sigma) \). The quotient, \( NP I(\sigma)/NPI(\sigma) \), is the pure-injective chain module \( Ch(\sigma|\infty) \), which again is pure-injective.

I do not know of an example of an \( R \)-module of pure-injective dimension greater than 1.

**Example 6.4.** In this example, we will construct arbitrarily large indecomposable \( R \)-modules of finite type (and hence Hopf algebras), showing in particular that non-\( p \)-torsion Hopf algebras do not decompose into tensor products of basic Hopf algebras.

Let \( \Gamma_i \) be the basic Hopf graph

![Basic Hopf Graph](image)

with \( m = i + 4 \).

One checks readily that there are no nontrivial maps \( M(\Gamma_i) \rightarrow M(\Gamma_j) \) unless \( i = j \).

There is a unique nontrivial extension \( 0 \rightarrow \Sigma k \rightarrow M_i \rightarrow M(\Gamma_i) \rightarrow 0 \) for each \( i \), resulting in a basic Hopf algebra with graph

![Basic Hopf Algebra Graph](image)

Thus \( \text{Ext}^1(M(\Gamma_i), \Sigma k) \cong k(\alpha_i) \). For any \( N > 0 \), the class

\[
(\alpha_1, \ldots, \alpha_N) \in \text{Ext}^1 \left( \bigoplus_{i=1}^N M(\Gamma_i), \Sigma k \right) \cong \bigoplus_{i=1}^N \text{Ext}^1(M(\Gamma_i), \Sigma k)
\]

thus defines an \( R \)-module \( M \). I claim that this module is indecomposable. We use the following result, which is a special case of [HW06, Theorem 2.3]:

**Lemma 6.5.** Let \( Q \) be a finitely generated \( R \)-module of positive depth, \( T \) an indecomposable finitely generated \( R \)-module of finite length, and \( \alpha \in \text{Ext}^1(Q, T) \). Suppose that for each \( f \in \text{End}(Q) - \{0\} \), \( f^* \alpha \neq 0 \). Then \( \alpha \) represents an indecomposable module.

In our case, \( Q = \bigoplus_{i=1}^N M(\Gamma_i), T = \Sigma k, \) and \( \alpha = (\alpha_1, \ldots, \alpha_k) \). We have that \( \text{End}(Q) \cong k^N \) because there are no nontrivial maps \( M(\Gamma_i) \rightarrow M(\Gamma_j) \) for \( i \neq j \). For \( f = (f_1, \ldots, f_N) \in \text{End}(Q) \), we have that \( f^* \alpha = (f_1 \alpha_1, \ldots, f_N \alpha_N) \) and since all \( \alpha_i \) are nontrivial, \( f^* \alpha \neq 0 \).

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