A family of 3D $H^2$-nonconforming tetrahedral finite elements for the biharmonic equation

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Abstract  In this article, a family of $H^2$-nonconforming finite elements on tetrahedral grids is constructed for solving the biharmonic equation in 3D. In the family, the $P_\ell$ polynomial space is enriched by some high order polynomials for all $\ell \geq 3$ and the corresponding finite element solution converges at the order $\ell - 1$ in $H^2$ norm. Moreover, the result is improved for two low order cases by using $P_6$ and $P_7$ polynomials to enrich $P_4$ and $P_5$ polynomial spaces, respectively. The error estimate is proved. The numerical results are provided to confirm the theoretical findings.

Keywords  $H^2$-nonconforming element, finite element method, biharmonic problem, tetrahedral grid

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1 Introduction

We consider the biharmonic equation:

$$\Delta^2 u = f \quad \text{in} \, \Omega,$$

$$u = \partial_n u = 0 \quad \text{on} \, \partial \Omega,$$

where $\Omega$ is a bounded 3D polyhedral domain, $\partial_n u = \nabla u^T \mathbf{n}$, and $\mathbf{n}$ is the unit outer normal vector to $\partial \Omega$.

The weak formulation of (1.1) reads: Find $u \in H^2_0(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H^2_0(\Omega).$$

Here, $H^2_0(\Omega) = \{v \in H^2(\Omega) \mid v = \partial_n v = 0 \text{ on } \partial \Omega\}$ and $H^2(\Omega)$ is the standard Sobolev space $[1]$. The bilinear form in (1.2) is defined by

$$a(u, v) = \int_\Omega D^2 u : D^2 v \, dx, \quad (f, v) = \int_\Omega f v \, dx,$$
where $D^2 u = \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \right)_{i,j} \quad (1 \leq i, j \leq 3)$ is a $3 \times 3$ tensor.

There are many numerical methods for the biharmonic equation (1.2) such as the finite element method. In a finite element method, a finite dimensional space $V_h$ of piecewise polynomials is constructed to approximate $H^2_0(\Omega)$ functions. If the finite element space is a subspace of $H^2_0(\Omega)$, it is called a conforming finite element. Otherwise, it is called a nonconforming element. In a conforming finite element method, the subspace $V_h$ must be globally $C^1$. One advantage of a conforming element is that the error of a numerical solution only depends on the approximation power of the finite element space. But a globally $C^1$ differentiable element requires a high degree of polynomials. On 2D triangular meshes, the lowest order conforming element is the Argyris $P_3$ element [2, 5]. Such an element can be reduced a little to the Bell element [5,17,18] with 18 degrees of freedom by restricting a $P_4$ polynomial to a $P_3$ polynomial for normal derivatives on three edges of a triangle. On 3D tetrahedral meshes, a family of conforming elements of polynomials of degree 9 and the above was constructed by Zhang [22, 24]. On rectangular meshes in 2D and 3D, the problem is relatively simple. The classic Bogner-Fox-Schmit (BFS) $C^1$-$Q_3$ element [5,18] can be easily extended to any higher degree, higher space dimension and higher smoothness [9,23]. A minimal polynomial degree $C^m$-$Q_k$ conforming element on $n$-dimensional rectangular grids was also proposed by Hu and Zhang [9,12].

However, the strong continuity requirement and the high degrees of freedom with higher order derivatives of conforming elements are not computationally desirable. There have been many nonconforming elements developed. The space of a nonconforming element, with fewer degrees of freedom on each element, is not a subspace of $C^1$ functions, and even not a subspace of $C^0$ functions. The minimal degree nonconforming element for the biharmonic equation in 2D is the Morley element, with six degrees of freedom on each triangle, which was extended to any dimension in [20]. Like the Morley element, both the Veubake elements [6] and the new class of Zienkiewicz-type (NZT) element [19] are convergent with the order $O(h)$ in an energy norm. In a higher order nonconforming finite element method, the $P_1$ polynomial is usually enriched with higher order polynomials. A second order method on 2D triangular meshes was proposed by Gao et al. [7] with two $P_3$ polynomials added to the $P_3$ polynomial space. In [21], the $P_3$ polynomial space was enriched by six $P_6$ polynomials, and in [4], four $P_5$ polynomials, four $P_7$ polynomials and four $P_8$ polynomials to achieve a second order nonconforming element in 3D. A class of 12 degrees of freedom triangular plate bending elements with quadratic rate of convergence was constructed in [15], where the shape function spaces are the $P_3$ polynomial space enriched by two $P_5$ polynomials. A family of 3D elements were constructed in [8], by using $P_{1+5}$ polynomials to enrich the $P_0$ polynomial space.

Recently, a new estimate technology was proposed in [14] by Hu and Zhang, generalizing the ideas of [10,13]. The error estimate is based on two continuity hypotheses on the gradient jump and the function value jump across $(d - 1)$-dimensional internal sides. The theory was applied to construct a second order nonconforming element on triangular grids, enriching the $P_3$ polynomial space by two $P_4$ polynomials on each triangle which is equivalent to the element constructed by Shi et al. [16] where a class of 12 degrees of freedom triangular elements of second order was proposed by using the double set parameter method. Similarly, based on this theory, Hu and Zhang [11] constructed a second order element in 3D with the $P_3$ polynomial space enriched by eight $P_4$ polynomials. In addition, this results in the lowest polynomial degree element of second order approximation in 3D so far. Compared with other 3D $H^2$-nonconforming elements, that element does not require vertex continuity. The aim of this work is to extend the $P_3$ nonconforming element to a family of $P_\ell$ nonconforming elements for all $\ell$. For large $\ell$, it is shown that the minimum polynomial degree of enriched polynomials is $\ell + 4$, and such a family of $H^2$-nonconforming finite elements is desired on tetrahedral meshes. It is noted that the polynomial degree of the elements in [8] is one degree higher than that of the elements in this work. But for small $\ell$, $\ell + 4$ is not the minimum polynomial degree for the $\ell - 1$ order of convergence. For example, when $\ell = 3$, $\ell + 1$ is the minimum degree as shown in [14]. In this work, for the $\ell = 4$ and $\ell = 5$ cases, the $P_4$ and $P_5$ polynomial spaces are enriched by $P_{6}$, $P_7$ polynomials, respectively. These are the lowest degree of enriched polynomials that can be found for these two cases so far.

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1) Hu J, Zhang S. Constructions of nonconforming finite elements for fourth order elliptic problems and applications.
The rest of this article is organized as follows. In Section 2, we introduce two hypotheses and present an energy-norm error estimate based on the two hypotheses. In Section 3, we construct a family of $H^2$-nonconforming elements for all polynomial degrees on tetrahedral grids. In Section 4, a lower order polynomials is used to replace $P_7$ and $P_8$ polynomials for the third and fourth order element methods and all explicit basis functions for the two elements are given in this section and the appendix. Finally, we present some numerical results to confirm the theoretical results.

2 Hypotheses and abstract theory

Let $T_h = \{T\}$ be a regular tetrahedral grid on $\Omega$ (see [3]), and $h$ be the mesh size of $T_h$. Let $F$ and $e$ be a two-dimensional face triangle and a one-dimensional edge of the element $T$, respectively. Let $P_\ell(G)$ represent the space of polynomials of degree less than or equal to $\ell$ over $G$. Let $F_h$ be the set of all two-dimensional face triangles of $T_h$. Let $\omega_F$ be the union of two elements sharing the two-dimensional face triangle $F$. Given an integer $\ell > 0$, let $V_{h,\ell}$ be the nonconforming element space of $H^2_0(\Omega)$ on the mesh $T_h$, defined in (3.12) below. Let $\Pi_\ell,G$ be the $L^2$ projection operator onto $P_\ell(G)$.

The finite element problem, discretizing the biharmonic equation (1.1) is: Find $u_h \in V_{h,\ell}$ such that

$$\langle D^2 u_h, D^2 v_h \rangle_h = \langle f, v_h \rangle, \quad \forall v_h \in V_{h,\ell},$$

(2.1)

where the discrete inner product is defined as

$$\langle \cdot, \cdot \rangle_h^2 = \sum_{T \in T_h} \langle \cdot, \cdot \rangle_T^2.$$

The existence and uniqueness of the solutions in (2.1) follow from the norm $| \cdot |_{2,h} = \langle D^2 \cdot, D^2 \cdot \rangle_h^{1/2}$ (to be proved) on $V_{h,\ell}$. By the second Strang’s lemma [18], we have

$$|u - u_h|_{2,h} \leq C \inf_{v_h \in V_{h,\ell}} |u - v_h|_{2,h} + \sup_{0 \neq \omega_h \in V_{h,\ell}} \frac{|\langle f, w_h \rangle - \langle D^2 u, D^2 w_h \rangle_h|}{|w_h|_{2,h}}.$$

To bound the error in the second term, i.e., the consistency error, the following two hypotheses were proposed in [14,16].

**Hypothesis 2.1.** For all internal face triangles $F$ of $T_h$, assume

$$\int_F [\nabla_h v_h] \cdot q dS = 0, \quad \forall q \in (P_{\ell-2}(F))^3 \text{ and } \forall v_h \in V_{h,\ell},$$

(2.2)

where $[ \cdot ]$ is the jump across $F$, $\nabla_h$ and $D^2_h$ are the discrete counterpart of $\nabla$ and $D^2$, respectively, defined element-wisely. For all domain boundary face triangles $F$ of $T_h$, assume

$$\int_F \nabla_h v_h \cdot q dS = 0, \quad \forall q \in (P_{\ell-2}(F))^3 \text{ and } \forall v_h \in V_{h,\ell}.$$

(2.3)

**Hypothesis 2.2.** For all internal face triangles $F$ of $T_h$, assume

$$\int_F [v_h] q dS = 0, \quad \forall q \in P_{\ell-3}(F) \text{ and } \forall v_h \in V_{h,\ell}.$$  

(2.4)

For all domain boundary face triangles $F$ of $T_h$, assume

$$\int_F v_h q dS = 0, \quad \forall q \in P_{\ell-3}(F) \text{ and } \forall v_h \in V_{h,\ell}.$$

(2.5)
Theorem 2.3 (See [14, Theorem 2.1]). Assume $V_{h,\ell}$ satisfies Hypotheses 2.1 and 2.2 with $\ell \geq 3$, and that the seminorm
\[
\|D^2_h \|_0 = (D^2, D^2)_h^{1/2}
\]
defines a norm over the nonconforming finite element space $V_{h,\ell}$. Let $u_h$ and $u$ be the solution of (2.1) and (1.1), respectively. Then
\[
\|D^2_h (u - u_h)\|_0 \leq C \inf_{s_h \in V_{h,\ell}} \|D^2_h (u - s_h)\|_0 + C \left( \sum_{F \in \mathcal{F}_h} \|(I - \Pi_{f,\omega_F})D^2 u\|_{0,\omega_F}^2 \right)^{1/2} + \left( \sum_{T \in \mathcal{T}_h} h^4 \|(I - \Pi_{f-1,T})f\|_{0,T}^2 \right)^{1/2}.
\]

3 A family of $H^2$-nonconforming finite elements

In this section, we construct a family of $H^2$-nonconforming finite element spaces for the biharmonic problem in 3D. Based on Hypotheses 2.1 and 2.2, the dual basis of the finite element space in 3D, i.e., the degrees of freedom of the finite element, consists of

\[
\begin{align*}
E^{(\ell)}(v) &= \frac{1}{|e|} \int_{e} vP_{\ell-2}(e) \, ds \quad \text{on all edges}, \\
F^{(\ell)}(v) &= \frac{1}{|F|} \int_{F} vP_{\ell-3}(F) \, dS \quad \text{on all face triangles}, \\
T^{(\ell)}(v) &= \frac{1}{|T|} \int_{T} vP_{\ell-4}(T) \, dx \quad \text{on all tetrahedrons}, \\
N^{(\ell)}(v) &= \frac{1}{|F|} \int_{F} \partial_{n} vP_{\ell-3}(F) \, dS \quad \text{on all face triangles}.
\end{align*}
\]

When $\ell = 2$, (3.2) and (3.3) drop and this element is the 3D Morley element. When $\ell = 3$, (3.3) drops. Therefore, in this article, we only consider the cases $\ell \geq 3$.

What is the dimension of the dual basis (3.1)--(3.4)? How many high order polynomials are needed to enrich each polynomial space $P_{\ell}(T)$ so that the enriched space can fulfill (3.1)--(3.4)? What is the minimum degree of enriched polynomials for the base polynomial space $P_{\ell}(T)$? These questions would lead to a proper definition of the enriched space $P^+_{\ell}(T)$ below. Next, we define the enriched polynomial space $P^+_{\ell}(T)$.

Lemma 3.1. For $\ell \geq 27$, the number of local degrees of freedom of (3.1)--(3.4) is bigger than the dimension of $P_{\ell+3}(T)$.

Proof. The number of degrees of freedom of (3.1) to (3.4) is
\[
\frac{1}{6} (\ell^3 + 18\ell^2 - \ell - 18),
\]
which is bigger than the dimension
\[
\frac{1}{6} (\ell^3 + 15\ell^2 + 74\ell + 120)
\]
of $P_{\ell+3}$ when $\ell \geq 27$. $\blacksquare$

Therefore, the minimum polynomial degree of enriched polynomials is $\ell + 4$ for the $P_{\ell}$ polynomial space when $\ell \geq 27$. The difference between the number of degrees of freedom of (3.1)--(3.4) and the dimension of $P_{\ell}(T)$ is
\[
\frac{1}{6} (\ell^3 + 18\ell^2 - \ell - 18) - \frac{1}{6} (\ell^3 + 6\ell^2 + 11\ell + 6) = (2\ell^2 - 2\ell) - 4 = 4(\ell - 1)(\ell - 0)/2 - 4 = 4 \dim P_{\ell-2}(F) - 4.
Thus, the $P_{1\ell}$ polynomial space has to be enriched by $\dim P_{\ell-2}(F) - 1$ high order polynomials on each face triangle. We aim to construct $4(\dim P_{\ell-2}(F) - 1)$ high order polynomials. The next lemma shows some $P_{\ell+4}$ functions which vanish on degrees of freedom of (3.1), (3.2) and (3.4). Moreover, their derivatives will also vanish on three face triangles of $T$.

**Lemma 3.2.** Let $F_m$ be a face triangle of $T$. A function

$$v \in b^2_{F_m} P_{\ell-2}(T) = \{ \lambda_i^2 \lambda_j^2 \lambda_k^2 q \mid q \in P_{\ell-2}(T) \},$$

where the face triangle $F_m$, the $m$-th face, is formed by three vertices $i$, $j$ and $k$, and $\lambda_i$ is a linear function valued 1 at vertex $i$ and 0 at the rest vertices, is unisolvent by the following degrees of freedom:

1. $\int_{F_m} v P_{\ell-3}(F_m) dS$,
2. $\int_T v P_{\ell-4}(T) d\mathbf{x}$,
3. $\int_{F_m} \partial_{n_m} v P_{\ell-2}(F_m) dS$.

When $\ell = 3$, (ii) of (3.5) drops.

**Proof.** Without loss of generality, we prove it on $F_4$. Let

$$\phi = b_{F_4}^2 q \in b_{F_4}^2 P_{\ell-2}(T)$$

and suppose it vanishes on all degrees of freedom of (3.5)(i)–(3.5)(iii). The number of these degrees of freedom is

$$\dim P_{\ell-4}(T) + \dim P_{\ell-3}(F) + \dim P_{\ell-2}(F) = \dim P_{\ell-2}(T).$$

Therefore, we only need to prove $\phi \equiv 0$. At first we want to prove if $\phi$ vanishes on all degrees of freedom then $\phi$ satisfies

$$\int_{F_4} (r \cdot \nabla\phi) q_{\ell-2} dS = 0, \quad \forall q_{\ell-2} \in P_{\ell-2}(F_4)$$

(3.6)

for all $r \in \mathbb{R}^3$. Let $t$ be the tangential vector, the projection of $r$ on the face $F_4$, $t = r - (r \cdot n_4) n_4$. Here, $n_4$ is the unit outer normal to the face $F_4$. It follows from (3.5)(iii) that

$$\int_{F_4} (r \cdot \nabla\phi) q_{\ell-2} dS = \int_{F_4} ((t + (r \cdot n_4) n_4) \cdot \nabla\phi) q_{\ell-2} dS$$

$$= \int_{F_4} (t \cdot \nabla\phi) q_{\ell-2} dS + \int_{F_4} ((r \cdot n_4) n_4 \cdot \nabla\phi) q_{\ell-2} dS$$

$$= \int_{F_4} (t \cdot \nabla\phi) q_{\ell-2} dS.$$

Noting $\phi = b_{F_4}^2 q$ that $(\phi q_{\ell-2}) |_{\partial F_4} = 0$, we have by (3.5)(i),

$$\int_{F_4} (t \cdot \nabla\phi) q_{\ell-2} dS = \int_{F_4} \partial_t (\phi q_{\ell-2}) dS - \int_{F_4} \phi (t \cdot \nabla q_{\ell-2}) dS$$

$$= -\int_{F_4} \phi (t \cdot \nabla q_{\ell-2}) dS = 0,$$

since $t \cdot \nabla q_{\ell-2} \in P_{\ell-3}(F_4)$. Therefore, we have proved (3.6). Next, let $r$ in (3.6) be

$$r = c(\nabla\lambda_2 \times \nabla\lambda_3), \quad c = 1/(\nabla\lambda_2 \times \nabla\lambda_3) \cdot \nabla\lambda_1,$$

where the box product is nonzero, a scaled volume of the tetrahedron $T$. Then

$$\nabla\phi \cdot r = (2b_{F_4} q \nabla b_{F_4} + b_{F_4}^2 \nabla q) \cdot r.$$
By (3.5)(ii),

\[ q = \frac{1}{2} F_4 q \lambda_2 \lambda_3 \nabla \lambda_1 + \lambda_1 \lambda_3 \nabla \lambda_2 + \lambda_1 \lambda_2 \nabla \lambda_3 \cdot \mathbf{r} + b_{F_4}^2 \nabla q \cdot \mathbf{r} \]

Therefore, for

\[ d \]

where

\[ t \]

functions of (3.9) on the four faces:

Repeating the above arguments with

\[ q_{\ell-2} = \lambda_1 q_{\ell-3}/2 \]

for an arbitrary \( q_{\ell-3} \in P_{\ell-3}(F_4) \), and consequently the first term is zero due to (3.5)(i). Since \( \frac{\lambda_1 \lambda_3 \lambda_2}{2} > 0 \) on \( F_4 \) except at \( \partial F_4 \), the \( P_{\ell-3}(F_4) \) polynomial vanishes,

\[ (\nabla q \cdot \mathbf{r}) |_{F_4} = 0. \] (3.7)

Repeating the above arguments with

\[ \mathbf{r} = c (\nabla \lambda_3 \times \nabla \lambda_1) \quad \text{and} \quad \mathbf{r} = c (\nabla \lambda_1 \times \nabla \lambda_2), \]

we obtain (3.7) in the other two linearly independent directions and consequently

\[ \nabla q |_{F_4} = 0, \] (3.8)

which implies

\[ \partial_{n_4} q |_{F_4} = 0 \quad \text{and} \quad \partial q |_{F_4} = 0 \]

for any tangential vector \( t \) on the face \( F_4 \). Thus, \( q |_{F_4} \) is a constant. By (3.5)(i) this constant is 0. Therefore, for \( \ell = 3 \) we have \( q = 0 \). If \( \ell \geq 4 \), we obtain

\[ q = \lambda_2^2 q_0 \]

for some \( q_0 \in P_{\ell-4}(T) \).

By (3.5)(ii), \( q_0 = 0 \). Thus \( q = 0 \). The proof is completed. \( \square \)

Note that the number of degrees of freedom of (3.5)(iii) is \( \dim P_{\ell-2}(F) \). Since we only need

\[ \dim P_{\ell-2}(F) - 1 \]

high order polynomials on each face, one basis function associated with (3.5)(iii) should be removed. The nodal basis of the space \( b_{F_4}^2 P_{\ell-2}(T) \), dual to the degrees of freedom of (3.5)(i)–(3.5)(iii), is

\[
\begin{align*}
\phi_{1,F_4}^{(1)}, \phi_{2,F_4}^{(1)}, \ldots, & \phi_{(\ell-2)(\ell-1)/2,F_4}^{(1)}; \\
\phi_{1,F_4}^{(2)}, \phi_{2,F_4}^{(2)}, \ldots, & \phi_{(\ell-3)(\ell-2)/(\ell-1)/6,F_4}^{(2)}; \\
\phi_{1,m}, & \phi_{2,m}, \ldots, \phi_{d_0,m}, \phi_{d_0+1,m};
\end{align*}
\] (3.9)

where \( d_0 = (\ell - 1)(\ell - 0)/2 - 1 \) and the last basis function satisfies

\[ \frac{1}{|F_m|} \int_{F_m} \partial_{n,m} \phi_{d_0+1,m} dS = 1. \] (3.10)

Except the last basis function \( \phi_{d_0+1,m} \), we enrich the \( P_{\ell} \) polynomial space by the third group basis functions of (3.9) on the four faces:

\[ P_{\ell}^+(T) = P_{\ell}(T) + \text{span}_{1 \leq m \leq d_0} \{ \phi_{1,m}, \ldots, \phi_{d_0,m} \}. \] (3.11)
Note that each $\phi_{i,m} (= b^2_{F_m} q_{\ell-2})$ above vanishes on six edges of $T$ and it is a nodal basis function of $P^+_\ell(T)$ associated with the fourth group degrees of freedom of (3.4).

Then the family of nonconforming finite element spaces is defined via the local space $P^+_\ell(T)$. We have

$$V_{h,\ell} = \left\{ v \in L^2(\Omega) \mid v|_T \in P^+_\ell(T), \int_F v p_{\ell-3} ds \text{ is continuous at internal edges of } T, \int_F v p_{\ell-3} dS \text{ and } \int_F \partial_n v p_{\ell-3} dS \text{ are continuous on internal face triangles of } T, \int_F v p_{\ell-3} ds = \int_F v p_{\ell-3} dS = 0, \text{ at boundary edges and on face triangles of } T \right\},$$

where $P^+_\ell(T)$ is defined in (3.11) and $p_\ell$ denotes a general $P_\ell$ polynomial. We are going to show that the shape function space $P^+_\ell(T)$ is unsolved by degrees of freedom of (3.1)–(3.4).

**Theorem 3.3.** The shape function space $P^+_\ell(T)$ is unsolved by degrees of freedom of (3.1)–(3.4).

**Proof.** First, (3.1)–(3.3) and the four vertex valuations $\{ v(x_m) \}$ form a dual basis for $P_\ell(T)$. This can be verified as follows: If all the degrees of freedom of $v \in P_\ell(T)$ vanish, then (1) $v$ vanishes on each edge because it vanishes at two end points of the edge and its moments of the order $\leq \ell - 2$ on the edge also vanishes; (2) $v$ vanishes on each face triangle because it vanishes on the 3 edge of the face triangle and its moments of the order $\leq \ell - 3$ on the face triangle vanishes too; (3) $v$ vanishes on the tetrahedron $T$ because it vanishes on the four face triangles of $T$ and its moments of the order $\ell - 4$ on $T$ also vanishes.

The corresponding basis functions of $P_\ell(T)$, dual to (3.1)–(3.3) and the following four degrees of freedom of

$$\frac{1}{|F_m|} \int_{F_m} \partial_{n_m} v dS, \quad m = 1, \ldots, 4$$

are

$$\phi_{1, P_1}, \ldots, \phi_{d_1, P_1}, \phi_{d_1+1, P_1}, \phi_{d_1+2, P_1}, \phi_{d_1+3, P_1}, \phi_{d_1+4, P_1},$$

where $d_1 = \dim P_\ell(T) - 4$. Note that the last four basis functions vanish for (3.1)–(3.3) and satisfy the following orthogonal property with respect to the degrees of freedom of (3.13), i.e.,

$$\int_{F_m} \partial_{n_m} \phi_{d_1+i, P_\ell} dS = \begin{cases} 0, & \text{if } i \neq m, \\ \text{non-zero,} & \text{if } i = m, \end{cases} \quad i, m = 1, 2, 3, 4.$$

In fact, we have the following expression for such a function, independent of $\lambda_1, \lambda_2$ and $\lambda_3$:

$$\phi_{d_1+4, P_\ell} = \lambda_4 (a_1 \lambda_1 + a_2 \lambda_1 + \cdots + a_{\ell} \lambda_1^{\ell-1}),$$

where $a_1 \neq 0, a_1 + a_2 + \cdots + a_{\ell} = 1$. This can be verified by applying degrees of freedom of (3.1)–(3.3) to it. We have

$$\frac{1}{|F_4|} \int_{F_4} \partial_{n_4} \phi_{d_1+4, P_\ell} dS = -|\nabla \lambda_4| |a_1| |F_4| \neq 0,$$

where $|F_4|$ is the area of the $F_4$. But on any other face triangle, say $F_1$, we have

$$\int_{F_1} \partial_{n_1} \phi_{d_1+4, P_\ell} dS = \int_{F_1} n_1 \cdot \nabla \phi_{d_1+4, P_\ell} dS$$

$$= \int_{F_1} \left( \frac{\n_2 \times \n_4}{\n_2 \times \n_4} \cdot \n_1 - \frac{c_1 t_1}{\n_2 \times \n_4} \cdot \n_1 \right) \cdot \nabla \phi_{d_1+4, P_\ell} dS$$

$$= \int_{F_1} \frac{\n_2 \times \n_4}{\n_2 \times \n_4} \cdot \n_1 \cdot \nabla \phi_{d_1+4, P_\ell} dS.$$
where $t_1$ is a tangent vector on $F_1$ such that
\[
\mathbf{n}_2 \times \mathbf{n}_4 = c_0 \mathbf{n}_3 + c_1 t_1 = [(\mathbf{n}_2 \times \mathbf{n}_4) \cdot \mathbf{n}_1] \mathbf{n}_1 + c_1 t_1,
\]
\[
(\mathbf{n}_2 \times \mathbf{n}_4) \cdot \mathbf{n}_1 = 6|T| \neq 0,
\]
\[
\int_{F_1} t_1 \cdot \nabla \phi_{d_i+4,p} dS = \int_{\partial F_1} (t_1 \cdot t_{\partial F_1}) \phi_{d_i+4,p} dS = 0
\]
by (3.1), and $\mathbf{n}_4$ is parallel to $\nabla \phi_{d_i+4,p}$ everywhere by (3.15).

We now show that these functions from (3.14) and (3.9) form a (not dual to the degrees of freedom of (3.1)-(3.4)) basis of $P_\ell^+(T)$. We only need to show they are linearly independent. Assume that
\[
u = \sum_{i=1}^{d_i} a_i \phi_{i,p} + \sum_{i=1}^{4} b_i \phi_{d_i+1,p} + \sum_{m=1}^{d_0} \sum_{i=1}^{d_0} c_{m,i} \phi_{i,m} = 0
\]
for the parameters $a_i, b_i$ and $c_{m,i}$. Sequentially,
apply degrees of freedom of (3.1)-(3.3) to $\nu \Rightarrow a_i = 0$,
apply four degrees of freedom of (3.13) to $\nu \Rightarrow b_i = 0$,
apply the rest degrees of freedom of (3.4) to $\nu \Rightarrow c_{m,i} = 0$.

This completes the proof. \hfill $\blacksquare$

Finally, we present a convergence theorem of the family of finite elements.

**Theorem 3.4.** The equation (2.1) has a unique solution $u_h \in V_{h,\ell}$. Moreover, the $H^2$ semi-norm error estimate with $u \in H^{\ell+1}(\Omega) \cap H^0_0(\Omega)$ is given by
\[
\|D^2(u - u_h)\|_0 \lesssim C \ell^{-1}|u|_{\ell+1}.
\]

**Proof.** Let $u_h$ be a solution to (2.1) with $f = 0$ there. On each tetrahedron, $D^2 u_h = 0$ which implies that $\nabla_h u_h$ is a piecewise constant vector there. The jump condition (2.1) indicates that $\nabla_h u_h$ is a global constant vector. By the normal derivative boundary condition, we have $\nabla_n u_h = 0$. Thus, $u_h$ is a piecewise constant on each tetrahedron. Combining the jump condition (2.2), we know that $u_h$ is a global constant. Then, the function value boundary condition implies $u_h = 0$. Thus, the square linear system of (2.1) has a unique solution.

By (3.1)-(3.4), Theorem 2.3 and the standard interpolation theory, the theorem is proved. \hfill $\blacksquare$

### 4 The lower order situation

According to Lemma 3.1, $P_{\ell+4}$ polynomials are needed for enrichment when $\ell \geq 27$.

However, when $\ell$ is small, the enriched $P_{\ell}$ polynomial space, defined in (3.9), can be improved by using lower degree polynomials compared with $P_{\ell+4}$ polynomials. In this section, we mainly focus on finding the optimal degree of such an enriched $P_{\ell}$ function space for $\ell = 4, 5$.

For convenience, we use $N_{m,i}^{(\ell)}$, $m = 1, \ldots, 4$ to represent the $l$-th normal derivative degree of freedom on the $m$-th face, i.e.,
\[
N_{m,i}^{(\ell)}(v) = \frac{1}{|F_m|} \int_{F_m} \partial_n(v) \lambda_{i1}^l \lambda_{i2}^l \lambda_{i3}^l ds, \quad l = 1, \ldots, \dim(P_{\ell-2}(F_m)),
\]
where
\[
l_1 + l_2 + l_3 = \ell - 2
\]
Lastly, the rest eight $P^E_{m,l}$ represents that $\lambda^1\lambda^2\lambda^3_k$ is the $l$-th basis function on $P_{\ell-2}(F_m)$. Here, we suppose that the face $F_m$ consists of the vertices $i, j$ and $k$. Similarly, we can define $E^{(l)}_{m,i}$, $F^{(l)}_{m,i}$ and $T^{(l)}_i$ as

$$\begin{align*}
E^{(l)}_{m,i}(v) &= \frac{1}{|e_m|} \int_{e_m} \partial_n(v)\lambda^1_i \lambda^2_j ds, \quad m = 1, \ldots, 6, \quad l = 1, \ldots, \dim(P_{\ell-2}(e_m)), \quad l_1 + l_2 = \ell - 2, \\
F^{(l)}_{m,i}(v) &= \frac{1}{|F_m|} \int_{F_m} \partial_n(v)\lambda^1_i \lambda^2_j \lambda^3_k dS, \quad m = 1, \ldots, 4, \quad l = 1, \ldots, \dim(P_{\ell-3}(F_m)), \quad l_1 + l_2 + l_3 = \ell - 3, \\
T^{(l)}_i(v) &= \frac{1}{|T|} \int_{T} \partial_n(v)\lambda^1_i \lambda^2_j \lambda^3_k d\mathbf{x}, \quad l = 1, \ldots, \dim(P_{\ell-4}(T)), \quad l_1 + l_2 + l_3 + l_4 = \ell - 4.
\end{align*}$$

4.1 Enriched $P_4(T)$ element in 3D

When $\ell = 4$, the degrees of freedom of (3.1)–(3.4) can be equivalently rewritten as follows, respectively.

We have

$$\begin{align*}
E^{(4)}_{m,i}(v) &= \frac{1}{|e_m|} \int_{e_m} v\lambda^1_i \lambda^2_j ds, \quad m = 1, \ldots, 6; \\
F^{(4)}_{m,i}(v) &= \frac{1}{|F_m|} \int_{F_m} v\lambda^1_i \lambda^2_j \lambda^3_k dS, \quad m = 1, \ldots, 4,
\end{align*}$$

where $(l_1, l_2, l_3)$ is $(2, 0, 0), (0, 2, 0), (0, 0, 2), (0, 1, 1), (1, 0, 1), (1, 1, 0)$ for $l = 1, 2, 3, 4, 5, 6$, respectively.

The number of degrees of freedom of (4.1)–(4.4) is 55 which is less than $\dim P_3(T) = 56$. However, there are no enough linearly independent polynomials in $P_3(T)$ with respect to (4.1)–(4.4). In fact, there are four nonzero functions in $P_3(T)$ which vanish for all the degrees of freedom of (4.1)–(4.4). To see it, define

$$b_i = \lambda^2_i \left(\lambda^3_3 - \frac{15}{8} \lambda^3_5 + \frac{15}{14} \lambda_1 - \frac{5}{28}\right), \quad i = 1, 2, 3, 4.$$ 

Note that $b_i$ vanishes for all the degrees of freedom of (4.1)–(4.4). Hence, the lowest polynomial degree for the enrichment is six.

Next, we present 24 functions associated with the degrees of freedom of (4.4). The 24 basis functions consist of four $P_3$ functions which are defined in (3.15) and twenty $P_6$ functions which vanish for the degrees of freedom of (4.1)–(4.3).

Firstly, letting $\ell = 4$ in (3.15) we get four functions as

$$\begin{align*}
\tilde{\phi}_{m,0}^{(N,4)} &= \frac{1}{4} (35\lambda^4_m - 60\lambda^3_m + 30\lambda^2_m - 4\lambda_m), \quad m = 1, \ldots, 4.
\end{align*}$$

Next, we define the following twelve $P_6$ functions:

$$\begin{align*}
\tilde{\phi}_{m,t}^{(N,4)} &= \frac{1}{2} \lambda^2_i \lambda_m (2\lambda^3_m - 56\lambda^2_m + 21\lambda_m - 2), \quad 1 \leq m \neq t \leq 4.
\end{align*}$$

Lastly, the rest eight $P_6$ functions are defined in (4.8) as follows:

$$\begin{align*}
\tilde{\phi}_{m,4+t}^{(N,4)} &= \frac{1}{2} b_r (2\lambda^3_m - 56\lambda^2_m + 21\lambda_m - 2), \quad 1 \leq m \neq t \leq 4.
\end{align*}$$
Recall that $b_{F_t}$ is the cubic face bubble function with respect to the face $F_t$. In fact, there are twelve functions in (4.8) but we only need eight of them. Note that the 28 functions in (4.6)–(4.8) vanish for the degrees of freedom of (4.1)–(4.3) and satisfy

$$r_n^m(N)\frac{\delta m,n 2l_1 l_2 l_3!}{(l_1 + l_2 + l_3 + 2)!}, \quad 1 \leq n, m \leq 4, \quad l = 1, \ldots, 6,$$

(4.9)

where $r_n$ is the distance between vertex $n$ and the face triangle $F_n$;

$$r_n^m(N)\frac{\delta m,n 2l_1 l_2 l_3!}{(l_1 + l_2 + l_3 + 2)!},$$

(4.10)

where $1 \leq m, n \leq 4, 1 \leq t \neq m \leq 4, l = 1, \ldots, 6$ and $i, j$ and $k$ are the index of three vertices on $F_m$;

$$r_n^m(N)\frac{\delta m,n 2l_1 l_2 l_3!}{(l_1 + l_2 + l_3 + 2)!},$$

(4.11)

where $1 \leq m, n \leq 4, 1 \leq t \neq m \leq 4, l = 1, \ldots, 6$. Thus, we define the following shape function space:

$$\bar{P}_4^+(T) = P_4(T) + \bar{B}_4(T),$$

(4.12)

where

$$\bar{B}_4(T) = \text{span}\{-\phi_{m,1}, \phi_{1,7}, \phi_{1,8}, \phi_{2,5}, \phi_{3,5}, \phi_{3,6}, \phi_{4,6}, \phi_{4,7}\}$$

(4.13)

with $1 \leq m \neq t \leq 4$. Note that the multi index $(m, t)$ of $\phi_{m,4+t}$ are chosen as

$$(1, 3), (1, 4), (2, 4), (2, 1), (3, 1), (3, 2), (4, 2), (4, 3).$$

In fact, such a selection is not unique but symmetric. The 20 functions of $\bar{B}_4(T)$ are linearly independent, which will be shown in Theorem 4.1 below. Then the global finite element space is defined by

$$V_{h,4} = \left\{ v \in L^2(\Omega) \left| v_T \in \bar{P}_4^+(T), \begin{array}{l}
\int_e v p_{2d} ds \text{ is continuous at internal edges of } T_h, \\
\int_F v p_1 dS \text{ and } \int_F \partial_n v p_2 dS \text{ are continuous on internal face triangles of } T_h, \\
\int_e v p_{2d} = \int_F v p_1 dS = \int_F \partial_n v p_2 dS = 0,
\end{array} \right. \right\}$$

at boundary edges and on boundary face triangles of $T_h$.

Next, we construct the rest 31 $P_4$ functions which do not vanish for normal derivative moments of (4.4).

Firstly, we define the following 18 $P_4$ function as

$$\begin{align*}
\tilde{\phi}_{m,1}^{(E,4)} &= 60b_{e_m}(7\lambda_i^2 - 6\lambda_i + 1), \\
\tilde{\phi}_{m,2}^{(E,4)} &= 60b_{e_m}(21\lambda_i\lambda_j - 6\lambda_i - 6\lambda_j + 2), \\
\tilde{\phi}_{m,3}^{(E,4)} &= 60b_{e_m}(7\lambda_j^2 - 6\lambda_j + 1),
\end{align*}$$

(4.14)

where $\lambda_i$ and $\lambda_j$ ($i < j$) are two barycentric coordinates of the edge $e_m$, and $b_{e_m} = \lambda_i\lambda_j, m = 1, \ldots, 6$.

We also need the following twelve $P_4$ functions:

$$\begin{align*}
\tilde{\phi}_{m,1}^{(F,4)} &= 180b_{F_m}(7\lambda_i - 2), \\
\tilde{\phi}_{m,2}^{(F,4)} &= 180b_{F_m}(7\lambda_j - 2), \\
\tilde{\phi}_{m,3}^{(F,4)} &= 180b_{F_m}(7\lambda_k - 2),
\end{align*}$$

(4.15)
where $\lambda_i, \lambda_j$ and $\lambda_k$ are three barycentric coordinates of triangle face $F_m$, and $b_{F_m} = \lambda_i \lambda_j \lambda_k$, $m = 1, \ldots, 4$. The last one $P_4$ function that we need is

$$\tilde{\phi}_1^{(T,4)} = 840\lambda_1\lambda_2\lambda_3\lambda_4. \quad (4.16)$$

The 31 $P_4$ functions, defined in (4.14)–(4.16), satisfy

$$\mathbb{E}_{n,l}(\tilde{\phi}_m^{(E,4)}) = \delta_{n,m}\delta_{l,t}, \quad \mathbb{F}_{r,l}(\tilde{\phi}_m^{(E,4)}) = 0, \quad \mathcal{T}_1(\tilde{\phi}_m^{(E,4)}) = 0, \quad (4.17)$$

where $1 \leq l, t \leq 3$, $1 \leq r \leq 4$, $1 \leq m, n \leq 6$;

$$\mathbb{E}_{n,l}(\tilde{\phi}_1^{(F,4)}) = 0, \quad \mathbb{F}_{r,l}(\tilde{\phi}_1^{(F,4)}) = \delta_{r,m}\delta_{l,t}, \quad \mathcal{T}_1(\tilde{\phi}_1^{(F,4)}) = 0, \quad (4.18)$$

where $1 \leq l, t \leq 3$, $1 \leq m, r \leq 4$, $1 \leq n \leq 6$;

$$\mathbb{E}_{n,l}(\tilde{\phi}_1^{(T,4)}) = 0, \quad \mathbb{F}_{r,l}(\tilde{\phi}_1^{(T,4)}) = 0, \quad \mathcal{T}_1(\tilde{\phi}_1^{(T,4)}) = 1, \quad (4.19)$$

where $1 \leq l \leq 3$, $1 \leq r \leq 4$, $1 \leq n \leq 6$.

**Theorem 4.1.** $\tilde{P}_4^+(T)$ is unisolvent for the degrees of freedom of (4.1)–(4.4).

**Proof.** Assume that

$$u_h = \sum_{m=1}^{6} \sum_{l=1}^{3} a_{m,l} \tilde{\phi}_m^{(E,4)} + \sum_{m=1}^{4} \sum_{l=1}^{4} b_{m,l} \tilde{\phi}_m^{(F,4)} + c \tilde{\phi}_1^{(T,4)} + \sum_{m=1}^{4} \sum_{t \in S_m} d_{m,t} \tilde{\phi}_m^{(N,4)} = 0$$

for the parameters $a_{m,l}, b_{m,l}, c$ and $d_{m,t}$. Here,

$$S_m = \{ t \in \mathbb{Z} \mid 0 \leq t \leq 4, t \neq m \} \cup \{ j + 4, k + 4 \},$$

where $j$ is the index of the second vertex and $k$ is the index of the third on $F_m$. We only need to show if $u_h$ vanishes for the degrees of freedom of (4.1)–(4.4) then $u_h$ vanishes. From (4.17)–(4.19), applying degrees of freedom of (4.1)–(4.3) to $u_h \Rightarrow a_{m,l} = 0$, $b_{m,t} = 0$, $c = 0$.

Next, we show that the remaining 24 coefficients $d_{m,t}, t \in S_m (m = 1, \ldots, 4)$ are also zeros. An application of the functions $r_m \delta_{m,t}(\cdot)$ $(m = 1, \ldots, 4, l = 1, \ldots, 6)$ to $u_h$, yields the matrix diag$\{A, A, A, A\}$ after exchanging rows with

$$A = \begin{pmatrix}
1/6 & 1/15 & 1/90 & 1/60 & 1/60 \\
1/6 & 1/90 & 1/15 & 1/90 & 1/60 \\
1/6 & 1/90 & 1/15 & 1/60 & 1/180 \\
1/12 & 1/60 & 1/60 & 1/180 & 1/180 \\
1/12 & 1/60 & 1/60 & 1/180 & 1/90 \\
1/12 & 1/60 & 1/60 & 1/60 & 1/90
\end{pmatrix}.$$

The fact $\det(A) \neq 0$ completes the proof. □

**Theorem 4.2.** The finite element solution $u_h \in V_{h,A}$ in (2.1) satisfies

$$|u - u_h|_{2,h} \leq Ch^3 |u|_5,$$

where $u \in H^5(\Omega) \cap H_0^2(\Omega)$ and $C$ is independent of $h$. 
4.2 Enriched $P_5(T)$ in 3D

The degrees of freedom for this case are

\[
E_{m,l}^{(5)}(v) = \frac{1}{| \epsilon_m |} \int_{\epsilon_m} v \lambda_i^1 \lambda_j^2 ds, \quad m = 1, \ldots, 6, \tag{4.20}
\]

where $(l_1, l_2)$ is $(3, 0), (0, 3), (2, 1), (1, 2)$ for $l = 1, \ldots, 4$, respectively;

\[
E_{m,l}^{(5)}(v) = \frac{1}{| F_m |} \int_{F_m} v \lambda_i^1 \lambda_j^2 \lambda_k^3 dS, \quad m = 1, \ldots, 4; \tag{4.21}
\]

here, $(l_1, l_2, l_3)$ is

\[
(2, 0, 0), (0, 2, 0), (0, 0, 2), (0, 1, 1), (1, 0, 1), (1, 1, 0)
\]

for $l = 1, \ldots, 6$, respectively;

\[
T_l^{(5)}(v) = \frac{1}{| T |} \int_T v \lambda_i d\mathbf{x}, \quad l = 1, \ldots, 4, \tag{4.22}
\]

\[
N_{m,l}^{(5)}(v) = \frac{1}{| F_m |} \int_{F_m} \partial_n v \lambda_i^1 \lambda_j^2 \lambda_k^3 dS, \quad m = 1, \ldots, 4, \tag{4.23}
\]

where the multi index $(l_1, l_2, l_3)$ is

\[
(3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (2, 0, 1), (2, 1, 1), (1, 2, 0), (1, 0, 2), (1, 1, 2), (1, 1, 1)
\]

for $l = 1, \ldots, 10$, respectively. We need $P_7$ polynomials because the number of degrees of freedom of (4.20)–(4.23) is 92, which is bigger than $\dim P_6(T) = 84$. Next, we present 40 functions associated with degrees of freedom of (4.23), which vanish for degrees of freedom of (4.20)–(4.22). Firstly, letting $\ell = 5$ in (3.15) we get four functions

\[
\tilde{\phi}_{m,0}^{(N,5)} = -\frac{1}{5} \lambda_m (126 \lambda_m^4 - 280 \lambda_m^3 + 210 \lambda_m^2 - 60 \lambda_m + 5), \quad m = 1, \ldots, 4. \tag{4.25}
\]

Secondly, we define the 24 $P_7$ functions as follows:

\[
\tilde{\phi}_{m, l}^{(N,5)} = -\lambda_i^1 \lambda_m (66 \lambda_m^4 - 120 \lambda_m^3 + 72 \lambda_m^2 - 16 \lambda_m + 1), \quad 1 \leq m \neq l \leq 4, \tag{4.26}
\]

\[
\tilde{\phi}_{m, 4+l}^{(N,5)} = -\frac{1}{4} \lambda_j^2 \lambda_m (99 \lambda_m^4 - 165 \lambda_m^3 + \lambda_m - 135 \lambda_m^2 + 180 \lambda_m^1 + 54 \lambda_m^2 \\
- 54 \lambda_i \lambda_m - 6 \lambda_m + 4 \lambda_i), \quad 1 \leq m \neq l \leq 4. \tag{4.27}
\]

Define the following twelve $P_7$ functions:

\[
\tilde{\phi}_{m, 8+t}^{(N,5)} = -b_{5t} (66 \lambda_m^4 - 120 \lambda_m^3 + 72 \lambda_m^2 - 16 \lambda_m + 1), \quad 1 \leq m \neq t \leq 4. \tag{4.28}
\]

The eight functions which we need, can be gotten by choosing the multi index $(m, t)$ as

\[
(1, 3), (1, 4), (2, 4), (2, 1), (3, 1), (3, 2), (4, 2), (4, 3).
\]

In fact, such a selection is not unique but symmetric. In addition, the last four functions that we need are

\[
\tilde{\phi}_{m,13}^{(N,5)} = \frac{1}{4} \lambda_k b_{5t} (165 \lambda_m^3 - 180 \lambda_m^2 + 54 \lambda_m - 4), \quad m = 1, \ldots, 4, \tag{4.29}
\]

where $i$ and $k$ are the index of the first and the third vertices on $F_m$, respectively.

Thus, we get all 40 functions associated with degrees of freedom of (4.23). Similarly, define the shape function space as

\[
\tilde{\mathcal{P}}_7^+ = P_7(T) + \tilde{B}_5(T) \tag{4.30}
\]
with $\tilde{B}_5(T) = \text{span}\left\{ \tilde{\phi}_{m,5}, \tilde{\phi}_{m,4}, \tilde{\phi}_{m,3}, \tilde{\phi}_{m,2}, \tilde{\phi}_{m,1}, \tilde{\phi}_{m,6}, \tilde{\phi}_{m,5}, \tilde{\phi}_{m,4}, \tilde{\phi}_{m,3}, \tilde{\phi}_{m,2}, \tilde{\phi}_{m,1} \right\}$, $1 \leq m \neq t \leq 4$. The 36 functions of $B_5(T)$ are linearly independent, which will be shown in Theorem 4.3 below. Then the global finite element space is defined by

$$V_{h,5} = \left\{ v \in L^2(\Omega) \mid v|_T \in \mathcal{P}_5^+(T), \right. $$

$$\int_e v p_3 ds \text{ is continuous at internal edges of } \mathcal{T}_h, $$

$$\int_F v p_4 dS \text{ and } \int_F \partial_n v p_3 dS \text{ are continuous on internal face triangles of } \mathcal{T}_h, $$

$$\int_e v p_3 ds = \int_F v p_2 dS = \int_F \partial_n v p_3 dS = 0, $$

at boundary edges and on boundary face triangles of $\mathcal{T}_h \}.$

Next, we define 52 $P_5$ functions which do not vanish for degrees of freedom of (4.23). First of all, the following 24 $P_5$ functions with an edge bubble $b_{e_m} = \lambda_i \lambda_j$ are needed:

$$\tilde{\phi}_{m,1}^{(E,5)} = 60 b_{e_m} (42 \lambda_i^3 - 56 \lambda_i^2 + 21 \lambda_i - 2),$$

$$\tilde{\phi}_{m,2}^{(E,5)} = 60 b_{e_m} (42 \lambda_j^3 - 56 \lambda_j^2 + 21 \lambda_j - 2),$$

$$\tilde{\phi}_{m,3}^{(E,5)} = 60 b_{e_m} (252 \lambda_i^2 \lambda_j - 168 \lambda_i \lambda_j - 56 \lambda_i^2 + 42 \lambda_i + 21 \lambda_j - 6),$$

$$\tilde{\phi}_{m,4}^{(E,5)} = 60 b_{e_m} (252 \lambda_j^2 \lambda_i - 168 \lambda_i \lambda_j - 56 \lambda_j^2 + 42 \lambda_j + 21 \lambda_i - 6),$$

where $\lambda_i$ and $\lambda_j$ ($i < j$) are two barycentric coordinates of the edge $e_m$, $m = 1, \ldots, 6$. Next, we need the following twenty-four $P_5$ basis functions with a face bubble function $b_{f_m} = \lambda_i \lambda_j \lambda_k$:

$$\tilde{\phi}_{m,1}^{(F,5)} = 1260 b_{f_m} f_1(\lambda_i), \quad \tilde{\phi}_{m,2}^{(F,5)} = 1260 b_{f_m} f_1(\lambda_j),$$

$$\tilde{\phi}_{m,3}^{(F,5)} = 1260 b_{f_m} f_1(\lambda_k), \quad \tilde{\phi}_{m,4}^{(F,5)} = 2520 b_{f_m} f_2(\lambda_i, \lambda_j),$$

$$\tilde{\phi}_{m,5}^{(F,5)} = 2520 b_{f_m} f_2(\lambda_i, \lambda_k), \quad \tilde{\phi}_{m,6}^{(F,5)} = 2520 b_{f_m} f_2(\lambda_j, \lambda_k),$$

where $\lambda_i, \lambda_j$ and $\lambda_k$ are three barycentric coordinates of the face $F_m$, and

$$f_1(\lambda_i) = 12 \lambda_i^2 - 8 \lambda_i + 1, \quad f_2(\lambda_i, \lambda_j) = 18 \lambda_i \lambda_j - 4 \lambda_i - 4 \lambda_j + 1, \quad m = 1, \ldots, 4.$$ 

The last four $P_5$ functions that we need are

$$\tilde{\phi}_{m}^{(T,5)} = 3360 \lambda_1 \lambda_2 \lambda_3 \lambda_4 (9 \lambda_m - 2), \quad m = 1, \ldots, 4.$$ 

In addition, the 52 $P_5$ functions satisfy

$$E_{n,t}^{(5)} (\tilde{\phi}_{m,t}^{(E,5)}) = \delta_{m,n} \delta_{l,t}, \quad F_{l,n}^{(5)} (\tilde{\phi}_{m,t}^{(E,5)}) = 0, \quad T_l^{(5)} (\tilde{\phi}_{m,t}^{(E,5)}) = 0,$$

where $1 \leq m, n \leq 6, 1 \leq l, t \leq 4$;

$$E_{n,t}^{(5)} (\tilde{\phi}_{m,t}^{(F,5)}) = 0, \quad F_{l,n}^{(5)} (\tilde{\phi}_{m,t}^{(F,5)}) = \delta_{m,t} \delta_{n,l}, \quad T_l^{(5)} (\tilde{\phi}_{m,t}^{(F,5)}) = 0,$$

where $1 \leq n, t \leq 6, 1 \leq m, l \leq 4$;

$$E_{n,l}^{(5)} (\tilde{\phi}_{m}^{(T,5)}) = 0, \quad F_{l,n}^{(5)} (\tilde{\phi}_{m}^{(T,5)}) = 0, \quad T_l^{(5)} (\tilde{\phi}_{m}^{(T,5)}) = \delta_{l,t},$$

where $1 \leq n \leq 6, 1 \leq l, t \leq 4$.

**Theorem 4.3.** $\mathcal{P}_5^+(T)$ is unsolved for degrees of freedom of (4.20)–(4.23).
Proof. Let \( u_h \in \mathcal{P}_5^+(T) \) vanish for all 92 degrees of freedom of (4.20)–(4.23). Then we only need to show \( u_h = 0 \). Assume that

\[
\begin{align*}
u_h &= \sum_{m=1}^{6} \sum_{t=1}^{4} a_{m,t} \phi_{m,t}^{(E,5)} + \sum_{m=1}^{6} \sum_{t=1}^{4} b_{m,t} \phi_{m,t}^{(F,5)} + \sum_{m=1}^{4} c_m \phi_{m}^{(T,5)} + \sum_{m=1}^{4} \sum_{l \in S_m} d_{m,t} \phi_{m,l}^{(N,5)} = 0
\end{align*}
\]

for the parameters \( a_{m,t}, b_{m,t}, c_m \) and \( d_{m,t} \). Here,

\[
S_m = \{ t \in \mathbb{Z} \mid 0 \leq t \leq 8, t \neq m, t \neq m + 4 \} \cup \{ j + 8, k + 8, 13 \},
\]

where \( j \) and \( k \) are the index of the second and the third vertices on \( F_m \), respectively. Sequentially, applying degrees of freedom of (4.20)–(4.22) to \( u_h \) we have

\[
a_{m,t} = 0, \quad b_{m,t} = 0, \quad c_m = 0.
\]

Then we apply the functional \( r_m N_m^{(3)}(\cdot) \) \( (m = 1, \ldots, 4, l = 1, \ldots, 10) \) to \( u_h \), to get the following matrix for the linear system (4.31) after exchanging rows:

\[
\begin{pmatrix}
a_n & a_n & 0 & c_1 e_1 & 0 & c_2 e_2 & 0 & 0 & 0 & 0 \\
0 & 0 & a_n & 0 & c_1 e_1 & 0 & c_2 e_2 & 0 & 0 & 0 \\
0 & c_2 e_2 & 0 & 0 & a_n & 0 & c_1 e_1 & 0 & 0 & 0 \\
0 & c_1 e_1 & 0 & c_2 e_2 & 0 & 0 & a_n & 0 & 0 & 0
\end{pmatrix},
\]

where \( c_1 = -c_2 = \frac{1}{1680} \), and \( e_t \) is the \( t \)-th canonical basis vector of \( \mathbb{R}^{10} \). It holds that

\[
[A, a] = \begin{pmatrix}
1/10 & 1/21 & 1/210 & 1/210 & 1/28 & 1/560 & 1/560 & 1/105 & 1/105 & 1/168 \\
1/10 & 1/210 & 1/21 & 1/210 & 1/560 & 1/28 & 1/560 & 1/420 & 1/105 & 1/420 \\
1/30 & 1/210 & 1/210 & 1/21 & 1/560 & 1/28 & 1/105 & 1/420 & 1/168 & 1/420 \\
1/30 & 1/105 & 1/210 & 1/630 & 1/168 & 1/240 & 1/1680 & 1/420 & 1/210 & 1/420 \\
1/30 & 1/105 & 1/630 & 1/210 & 1/630 & 1/168 & 1/420 & 1/210 & 1/1680 & 1/840 \\
1/30 & 1/630 & 1/105 & 1/210 & 1/630 & 1/168 & 1/420 & 1/630 & 1/420 & 1/1680 \\
1/30 & 1/210 & 1/105 & 1/630 & 1/210 & 1/630 & 1/420 & 1/1680 & 1/420 & 1/210 \\
1/30 & 1/630 & 1/210 & 1/105 & 1/630 & 1/210 & 1/420 & 1/630 & 1/1680 & 1/2520 \\
1/60 & 1/420 & 1/420 & 1/420 & 1/840 & 1/840 & 1/840 & 1/630 & 1/630 & 1/1680
\end{pmatrix}.
\]

One can check the matrix in (4.32) is invertible which completes the proof.

\[
\square
\]

**Theorem 4.4.** The finite element solution \( u_h \in \mathcal{V}_{h,5} \) in (2.1) satisfies

\[
|u - u_h|_{2,h} \leq Ch^4 |u|_{6},
\]

where \( u \in H^6(\Omega) \cap H_0^2(\Omega) \) and \( C \) is independent of \( h \).

5 Numerical tests

Let the domain of the boundary value problem (1.1) be the unit cubic \( \Omega = (0, 1)^3 \). The exact solution is

\[
u(x, y, z) = 2^{10} (x - x^2)^2 (y - y^2)^2 (z - z^2)^2.
\]

(5.1)

We choose a family of uniform grids, shown in Figure 1 for all tests.
Figure 1  The first three grids on the unit cube domain $\Omega$

| Table 1 | The error and the order of convergence, by the $P_3 + 8P_7$ finite element (4.12) (with $\ell = 3$) |
| Grid | $\| u - u_h \|_0$ | $h^n$ | $\| u - u_h \|_1$ | $h^n$ | $\| u - u_h \|_2$ | $h^n$ |
|-------|----------------|--------|----------------|--------|----------------|--------|
| 1     | 0.0506877      | 0.0    | 0.3979739      | 0.0    | 3.8919205      | 0.0    |
| 2     | 0.0367926      | 0.5    | 0.2479380      | 0.7    | 3.7328475      | 0.1    |
| 3     | 0.0053230      | 2.8    | 0.0424815      | 2.5    | 1.3958787      | 1.4    |
| 4     | 0.0005108      | 3.4    | 0.0049778      | 3.1    | 0.3761410      | 1.9    |
| 5     | 0.0000398      | 3.7    | 0.0005033      | 3.3    | 0.0923416      | 2.0    |

| Table 2 | The error and the order of convergence, by the $P_4 + 20P_8$ finite element (3.11) (with $\ell = 4$) |
| Grid | $\| u - u_h \|_0$ | $h^n$ | $\| u - u_h \|_1$ | $h^n$ | $\| u - u_h \|_2$ | $h^n$ |
|-------|----------------|--------|----------------|--------|----------------|--------|
| 1     | 0.0510863      | 0.0    | 0.3390932      | 0.0    | 4.3623945      | 0.0    |
| 2     | 0.0105988      | 2.3    | 0.0802464      | 2.1    | 1.7878478      | 1.3    |
| 3     | 0.0042500      | 4.6    | 0.0057934      | 3.8    | 0.3314596      | 2.4    |
| 4     | 0.0000079      | 5.8    | 0.0003274      | 4.1    | 0.0476001      | 2.8    |

| Table 3 | The error and the order of convergence, by the $P_4 + 20P_6$ finite element (4.12) |
| Grid | $\| u - u_h \|_0$ | $h^n$ | $\| u - u_h \|_1$ | $h^n$ | $\| u - u_h \|_2$ | $h^n$ |
|-------|----------------|--------|----------------|--------|----------------|--------|
| 1     | 0.0249382      | 0.0    | 0.3040140      | 0.0    | 5.9311094      | 0.0    |
| 2     | 0.0041104      | 2.6    | 0.0645440      | 2.2    | 2.0850503      | 1.5    |
| 3     | 0.0001847      | 4.5    | 0.0061410      | 3.4    | 0.3909556      | 2.4    |
| 4     | 0.0000052      | 5.1    | 0.0004154      | 3.9    | 0.0531467      | 2.9    |

We first solve the biharmonic problem (1.1) with the exact solution (5.1) by the $P_3$ finite element method (3.11) (with $\ell = 3$), i.e., the $P_3 + 8P_7$ element ($P_3$ polynomials plus eight $P_7$ polynomials on each tetrahedron). We can see, from Table 1, that the numerical solution converges at orders 2, 3 and 4 in $H^2$-norm, $H^1$-norm and $L^2$-norm, respectively.

We next solve the biharmonic problem (1.1) with the exact solution (5.1) by the $P_4 + 20P_8$ finite element method (3.11) (with $\ell = 4$), i.e., the full $P_4$ polynomials plus twenty $P_8$ polynomials on each tetrahedron. In Table 2, we list the orders of convergence of the numerical solutions, which are 3, 4 and 5 in $H^2$-norm, $H^1$-norm and $L^2$-norm, respectively.

Lastly, we apply the low-order $V_{h,4}$ (4.12) method to solving the 3D biharmonic equation. Here, the full $P_4$ polynomial space is enriched by twenty $P_6$ polynomials, on each tetrahedron. We call it the $P_4 + 20P_6$ element. Due to a better condition number, this method is more stable than the above $P_4 + 20P_8$ element. From Table 3, we can see the $P_4 + 20P_6$ element converges also at orders 3, 4 and 5 in $H^2$-norm, $H^1$-norm and $L^2$-norm, respectively.

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Appendix A

We will present all basis functions of $P_4 + 20P_6$ in this appendix. Basis functions of $P_5 + 36P_6$ can be found in [11]. Let three vertices index $(i, j, k)$ on $F_m$ be $(2, 3, 4), (3, 4, 1), (4, 1, 2), (1, 2, 3)$ for $m = 1, \ldots, 4$, respectively. Let $r_m(l), 1 \leq m \neq l \leq 4$ be the local index of vertex $l$ on $m$-th face, for example, $r_3(1) = 2, r_4(1) = 3$.

Appendix A.1  Basis functions of the $P_4 + 20P_6$ element

The shape function space of the $P_4 + 20P_6$ element can be equivalently rewritten as

$$\tilde{P}_4^\pm(T) = P_4(T) + \tilde{B}_4,$$

$$\tilde{B}_4 = \text{span}\{\lambda_2^3(3\lambda_m - 4), b_{F_1}\lambda_1^2(3\lambda_1 - 4), b_{F_2}\lambda_1^2(3\lambda_1 - 4), b_{F_3}\lambda_2^2(3\lambda_2 - 4), b_{F_4}\lambda_2^2(3\lambda_2 - 4), b_{F_5}\lambda_3^2(3\lambda_3 - 4)\}.$$
\[ b_F \lambda_3^2(3\lambda_3 - 4), b_F \lambda_3^2(3\lambda_4 - 4), b_F \lambda_4^2(3\lambda_4 - 4), b_F \lambda_4^2(3\lambda_4 - 4), 1 \leq m \neq t \leq 4. \]

Firstly, we show the following 24 basis functions dual to the degrees of freedom of (4.4):

\[ \phi^{(N,4)}_{m,1} = 6r_m (\phi^{(N,4)}_{m,0} + 5\phi^{(N,4)}_{m,i} - 10\phi^{(N,4)}_{m,j} + 10\phi^{(N,4)}_{m,k}), \]
\[ \phi^{(N,4)}_{m,2} = 6r_m (-4\phi^{(N,4)}_{m,0} + 5\phi^{(N,4)}_{m,i} + 10\phi^{(N,4)}_{m,j} + 5\phi^{(N,4)}_{m,k}), \]
\[ \phi^{(N,4)}_{m,3} = 6r_m (-4\phi^{(N,4)}_{m,0} - 5\phi^{(N,4)}_{m,i} + 10\phi^{(N,4)}_{m,j} + 5\phi^{(N,4)}_{m,k}), \]
\[ \phi^{(N,4)}_{m,4} = 12r_m (11\phi^{(N,4)}_{m,0} - 10\phi^{(N,4)}_{m,j} - 15\phi^{(N,4)}_{m,k} + 75\phi^{(N,4)}_{m,k+4}), \]
\[ \phi^{(N,4)}_{m,5} = 6r_m (-3\phi^{(N,4)}_{m,0} - 5\phi^{(N,4)}_{m,i} - 5\phi^{(N,4)}_{m,j} + 5\phi^{(N,4)}_{m,k} + 50\phi^{(N,4)}_{m,k+4}), \]
\[ \phi^{(N,4)}_{m,6} = 6r_m (-3\phi^{(N,4)}_{m,0} - 5\phi^{(N,4)}_{m,i} - 5\phi^{(N,4)}_{m,j} + 5\phi^{(N,4)}_{m,k} + 50\phi^{(N,4)}_{m,k+4}), \]

where \( m = 1, \ldots, 4 \). These 24 functions vanish for degrees of freedom of (4.1)–(4.3) and satisfy

\[ N^{(4)}_{m,l}(\phi^{(N,4)}_{m,i}) = \delta_{m,n} \delta_{l,t}, \quad 1 \leq m, n \leq 4, \quad 1 \leq l, t \leq 6. \]

Next, we present the following basis function dual to the degree of freedom of (4.3):

\[ \phi^{(T,4)}_{1} = \phi^{(T,4)}_{1,4} + \sum_{m=1}^{4} \sum_{l=1}^{3} \frac{2}{r_m} \phi^{(N,4)}_{m,1} + \sum_{m=1}^{4} \sum_{l=4}^{6} \frac{4}{r_m} \phi^{(N,4)}_{m,1}. \quad \text{(A.1)} \]

Here, \( \phi^{(T,4)}_{1} \) vanishes for degrees of freedom of (4.1), (4.2) and (4.4) and satisfies

\[ \Phi^{(4)}_{1,4}(\phi^{(T,4)}_{1}) = 1. \]

Then, 12 basis functions dual to degrees of freedom of (4.2) are shown as follows:

\[ \phi^{(F,4)}_{m,1} = \phi^{(F,4)}_{m,1} - \frac{1}{r_i} (6\phi^{(N,4)}_{i,1} + 6\phi^{(N,4)}_{i,2} + 2\phi^{(N,4)}_{i,3} + 2\phi^{(N,4)}_{i,4} + 4\phi^{(N,4)}_{i,6}) \]
\[ + \frac{1}{r_j} (6\phi^{(N,4)}_{j,3} + 2\phi^{(N,4)}_{j,4} + 2\phi^{(N,4)}_{j,5} + \frac{1}{r_k} (6\phi^{(N,4)}_{k,2} + 2\phi^{(N,4)}_{k,4} + \phi^{(N,4)}_{k,6}) \]
\[ - (n_m)^T \left( \frac{2n_i}{r_i} + \frac{6n_m}{r_m} \right) \phi^{(N,4)}_{m,1} + \left( \frac{n_k}{r_k} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,5} + \left( \frac{n_j}{r_j} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,6}, \]
\[ \phi^{(F,4)}_{m,2} = \phi^{(F,4)}_{m,2} - \frac{1}{r_j} (6\phi^{(N,4)}_{j,1} + 2\phi^{(N,4)}_{j,2} + 6\phi^{(N,4)}_{j,3} + 2\phi^{(N,4)}_{j,4} + 4\phi^{(N,4)}_{j,5} + 2\phi^{(N,4)}_{j,6}) \]
\[ + \frac{1}{r_k} (6\phi^{(N,4)}_{k,3} + 2\phi^{(N,4)}_{k,4} + \phi^{(N,4)}_{k,6}) + \frac{1}{r_i} (6\phi^{(N,4)}_{i,1} + \phi^{(N,4)}_{i,5} + 2\phi^{(N,4)}_{i,6}) \]
\[ - (n_m)^T \left( \frac{2n_i}{r_i} + \frac{6n_m}{r_m} \right) \phi^{(N,4)}_{m,2} + \left( \frac{n_k}{r_k} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,5} + \left( \frac{n_j}{r_j} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,6}, \]
\[ \phi^{(F,4)}_{m,3} = \phi^{(F,4)}_{m,3} - \frac{1}{r_k} (2\phi^{(N,4)}_{k,1} + 6\phi^{(N,4)}_{k,2} + 6\phi^{(N,4)}_{k,3} + 4\phi^{(N,4)}_{k,5} + 2\phi^{(N,4)}_{k,6}) \]
\[ + \frac{1}{r_i} (6\phi^{(N,4)}_{i,2} + \phi^{(N,4)}_{i,4} + 2\phi^{(N,4)}_{i,6}) + \frac{1}{r_j} (6\phi^{(N,4)}_{j,1} + 2\phi^{(N,4)}_{j,5} + \phi^{(N,4)}_{j,6}) \]
\[ - (n_m)^T \left( \frac{2n_k}{r_k} + \frac{6n_m}{r_m} \right) \phi^{(N,4)}_{m,3} + \left( \frac{n_i}{r_i} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,5} + \left( \frac{n_j}{r_j} + \frac{2n_m}{r_m} \right) \phi^{(N,4)}_{m,6}, \]

where \( m = 1, \ldots, 4 \). These 12 basis functions satisfy

\[ \Phi^{(4)}_{m,l}(\phi^{(F,4)}_{m,i}) = \delta_{m,n} \delta_{l,t}, \quad 1 \leq m, n \leq 4, \quad 1 \leq l, t \leq 3, \]

and vanish for the degrees of freedom of (4.1), (4.3) and (4.4). Finally, we give the last 18 basis functions which are dual to the degrees of freedom of (4.2), i.e.,

\[ \phi^{(E,4)}_{m,1} = \phi^{(E,4)}_{m,1} + \frac{1}{r_i} (6\phi^{(N,4)}_{i,1} + 2\phi^{(N,4)}_{i,3} + 2\phi^{(N,4)}_{i,4} + 2\phi^{(N,4)}_{i,5} + \phi^{(N,4)}_{i,7} + 2\phi^{(N,4)}_{i,8}) \]
\[ + \frac{1}{r_j} (6\phi^{(N,4)}_{j,3} + 2\phi^{(N,4)}_{j,4} + \phi^{(N,4)}_{j,6}) + \frac{1}{r_k} (6\phi^{(N,4)}_{k,2} + 2\phi^{(N,4)}_{k,5} + \phi^{(N,4)}_{k,6}). \]


\[ +2\phi^{(N,4)}_{i,\tau(i)+3} + \frac{2}{r_j} \phi^{N,4}_{j,\tau(j)} + \left( \frac{2n_i}{r_i} + \frac{2n_j}{r_j} \right)^T \left( \phi^{(N,4)}_{i,\tau(i)} n_i + \phi^{N,4}_{j,\tau(j)} n_j \right), \]

\[ \phi^{(E,4)}_{m,2} = \phi^{(E,4)}_{m,2} + \frac{1}{r_i} \left( 12 \phi^{(N,4)}_{i,\tau(i)} + 2\phi^{(N,4)}_{i,\tau(i)+3} + 2\phi^{(N,4)}_{j,\tau(j)+3} \right) + \left( 12 \phi^{(N,4)}_{j,\tau(j)} + 2\phi^{(N,4)}_{j,\tau(j)+3} \right) \]

\[ + \left( \frac{2n_i}{r_i} + \frac{2n_j}{r_j} \right)^T \left( \phi^{(N,4)}_{i,\tau(i)+3} n_i + \phi^{(N,4)}_{j,\tau(j)+3} n_j \right), \]

\[ \phi^{(E,4)}_{m,3} = \phi^{(E,4)}_{m,3} + \frac{1}{r_j} \left( 6 \phi^{(N,4)}_{j,\tau(j)} + 2\phi^{(N,4)}_{j,\tau(j)+3} + 2\phi^{(N,4)}_{j,\tau(j)+3} + 2\phi^{(N,4)}_{j,\tau(j)+3} \right) \]

\[ + \left( \frac{2n_i}{r_i} + \frac{2n_j}{r_j} \right)^T \left( \phi^{(N,4)}_{i,\tau(i)+3} n_i + \phi^{(N,4)}_{j,\tau(j)+3} n_j \right), \]

where \( i \) and \( j \) (\( i < j \)) form the \( m \)-th edge of the tetrahedron, \( m = 1, \ldots, 6 \). ̃\( i \) and ̃\( j \) (̃\( i < ̃j \)) are the other two index of vertices on the tetrahedron (i.e., if \((i, j) = (2, 4)\) then \((̃i, ̃j) = (1, 3)\)). They vanish for degrees of freedom of (4.2)–(4.4) and satisfy

\[ \mathbb{E}^{(4)}_{m,l}(\phi^{(E,4)}_{n,t}) = \delta_{m,n} \delta_{l,t}, \quad 1 \leq m, n \leq 6, \quad 1 \leq l, t \leq 3. \]