EXOTIC LOW DENSITY FERMION STATES IN THE TWO MEASURES FIELD THEORY: NEUTRINO DARK ENERGY

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Abstract

There exist field theory models where the fermionic energy-momentum tensor contains a term proportional to $g_{\mu\nu}\bar{\psi}\psi$ which may contribute to the dark energy. We show that this new field theory effect can be achieved in the Two Measures Field Theory (TMT) in the cosmological context. TMT is an alternative gravity and matter field theory where the gravitational interaction of fermionic matter is reduced to that of General Relativity when the energy density of the fermion matter is much larger than the dark energy density. In this case also the 5-th force problem is solved automatically. In the opposite limit, where the magnitudes of fermionic energy density and scalar field dark energy density become comparable, nonrelativistic fermions can participate in the cosmological expansion in a very unusual manner. Some of the features of such Cosmo-Low-Energy-Physics (CLEP) states are studied in a toy model of the late time universe filled with homogeneous scalar field and uniformly distributed nonrelativistic neutrinos, and the following results are obtained: neutrino mass increases as $m \propto a^{3/2}$ ($a$ is the scale factor); the proportionality factor in the non-canonical contribution to the neutrino energy-momentum tensor (proportional to the metric tensor) approaches a constant as $a(t) \to \infty$
and therefore the non-canonical contribution to the neutrino energy density dominates over the canonical one \( \sim m/a^3 \sim a^{-3/2} \) at the late enough universe; hence the neutrino gas equation-of-state approaches \( w = -1 \), i.e. neutrinos in the CLEP regime behave as a sort of dark energy as \( a \to \infty \); the equation-of-state for the total (scalar field+neutrino) energy density and pressure also approaches \( w = -1 \) in the CLEP regime; besides the total energy density of such universe is less than it would be in the universe filled with the scalar field alone. An analytic solution is presented. A domain structure of the dark energy seems to be possible. We speculate that decays of the CLEP state neutrinos may be both an origin of cosmic rays and responsible for a late super-acceleration of the universe. In this sense the CLEP states exhibit simultaneously new physics at very low densities and for very high particle masses.

I. INTRODUCTION

In the light of the evidence that our universe is accelerating [1], one of the most fundamental questions facing modern physics is the nature of the dark energy [2]. In addition, the problem of the nature of the dark matter as well as the puzzling fact that at the present time the dark matter and the dark energy densities appear to be of the same order of magnitude, appear as a complete mystery. In the absence of a fundamental theory describing the nature of the dark energy and the dark matter, this observable ”cosmic coincidence” [3] makes it natural to assume that there should be a certain dynamical correlation or even a coupling between the dark energy and the dark matter. Having this in mind, in the context of quintessence scenarios [4] of an accelerating expansion for the present day universe a number of promising approaches have been developed in order to explain this observable fact, for example coupled quintessence [5], variable mass particles models [6] and different modifications of these and similar ideas [7].
Such modifications of the particle physics theory have a number of problems (see for example [8]). One of the most fundamental problems is the following: although there are justifications for coupling of the quintessence scalar $\phi$ to dark matter in the effective Lagrangian, it is not clear why similar coupling to the baryon matter is absent or essentially suppressed (such coupling would be the origin of a long range scalar force [9] because of the very small mass of $\phi$). This "fifth force" problem might be solved [10] if there would be a shift symmetry $\phi \rightarrow \phi + \text{const}$. But the quintessence potential itself does not possess this symmetry although at the present epoch it must be very flat [11].

Specific properties of neutrinos (which constitute a fraction of the dark matter) served as a basis for a number of models [12] concerning a possible coupling of neutrinos to the dark energy. Recently a new idea has been suggested [13] (see also [14]) that coupling of the neutrinos to a scalar allows to formulate the effective picture where the dark energy density depends on the neutrino mass (treated as a dynamical field). In this model the observable equation-of-state for the dark energy $w \approx -1$ is obtained if one assumes [13] that the neutrino mass $m_\nu$ depends on the density of the background nonrelativistic neutrinos $n_\nu$ in the following way: $m_\nu \propto n_\nu^w$. However, in the model [13] the energy density in neutrinos is small compared to the energy density in the total dark energy sector.

In this paper we show that in the framework of the Two Measures Field Theory (TMT) one can address the above goals while automatically solving some of the mentioned difficulties. For example, in spite of the presence of the scalar field $\phi$ potentials, the model allows a scale symmetry which involves the shift transformation $\phi \rightarrow \phi + \text{const}$. Some other results of our TMT model are the following:

1. The shift symmetry $\phi \rightarrow \phi + \text{const}$ gets spontaneous breakdown and as a consequence this results in the appearance of a dimensionful parameter.

2. Generically fermions in TMT are very much different from what one is used to in normal field theory. For example the fermion mass can depend upon the fermion density. Only if the local energy density of the fermion is much larger than the vacuum energy density, the fermion can have a constant mass. However this is exactly the case of atomic, nuclear and
particle physics, including accelerator physics and high density objects of astrophysics. This is why to such "high density" (in comparison with the vacuum energy density) phenomena we will refer as "normal particle physics conditions" and the appropriate fermion states in TMT we will call "regular fermions". For generic fermion states in TMT we will use the term "primordial fermions" in order to distinguish them from regular fermions.

3. One of the differences of the gravitational equations from those of the Einstein’s General Relativity (GR) consists in the appearance of a noncanonical term in the fermion energy-momentum tensor $\propto g_{\mu\nu}\bar{\psi}\psi$. There exists also an effective Yukawa coupling of the dilaton scalar field $\phi$ to primordial fermions that may be an origin of a long-range scalar force. However, the coupling of regular fermions to gravity approximates very closely that of GR with the correction being suppressed by factor of $\rho_{d.e.}/\rho_f$, where $\rho_{d.e.}$ and $\rho_f$ are the dark energy and local fermion energy densities respectively. The same is true also for the 5-th force suppression, i.e for the effective Yukawa coupling of regular fermions to the scalar field $\phi$.

4. In the opposite regime, where the local energy density of the primordial fermion is comparable with the vacuum energy density, TMT predicts the possibility of exotic states whose long range interactions are very much different from GR because of their noncanonical coupling to gravity and non-suppressed effective Yukawa coupling to scalar field $\phi$. In other words, for the fermion matter in such exotic states GR is not applicable.

As an example of such exotic states, in this paper we study solutions in a toy model of the late time universe filled with homogeneous scalar field $\phi$ and uniformly distributed nonrelativistic neutrinos. The following results are obtained as the local energy density of the neutrino becomes comparable with the vacuum energy density: neutrino mass increases as $m \sim a^{3/2}$ ($a$ is the scale factor); the non-canonical contribution to the neutrino energy-momentum tensor $\propto g_{\mu\nu}\bar{\psi}\psi$ dominates over the canonical one $\sim m/a^3$ and as a consequence the neutrino energy density scales as a sort of dark energy with equation-of-state approaching $w = -1$ as $a \to \infty$; the total (scalar field+neutrino) energy density scales as a sort of dark energy and approaches a constant as $a \to \infty$; besides the total energy density of such universe
is less than it would be in the universe free of fermionic matter at all. The latter means that in the framework of our toy model, regular nonrelativistic neutrino should undergo transition to an exotic state described above which we will denote Cosmo-Low-Energy-Physics (CLEP) state. Demonstration of this new field theory effect in the context of TMT is the purpose of this paper.

However before doing this in a systematic way in TMT, we show in Sec.2 that there is a possibility of obtaining a fermion contribution into the dark energy in the context of standard field theory. However, as we will explain, this attempt is not successful. In sections 3 and 4 we briefly review general ideas of TMT and formulate our scale and gauge invariant model. In Sec.5 we shortly discuss results for dark energy in the absence of fermions. Sec.6 contains a brief review of regular fermions in TMT and closely related to them question of reproducing GR. Sec.7 deals with the main subject of the paper: possible reorganization of the dark energy as a result of the appearance of the CLEP state neutrinos.

II. ATTEMPTS TO OBTAIN FERMIONIC DARK ENERGY IN THE CONTEXT OF STANDARD FIELD THEORY

In GR the free fermion part of the action has the form $S_{0}^{(f)} = \int d^{4}x \sqrt{-g} L_{f}$ where $L_{f}$ is the free fermion Lagrangian density in the Einstein-Cartan space-time. Then a potential cosmological-like term $\propto g_{\mu\nu} L_{f}$ naively appears but the Dirac equation forces $L_{f}$ to vanish.

Here we would like to give some feeling of the mechanism by which a term in the energy-momentum tensor proportional to $g_{\mu\nu} f(\tilde{\psi} \psi)$ (where $f(\tilde{\psi} \psi)$ is a function of $\tilde{\psi} \psi$) could be obtained in the framework of the standard (non TMT) field theories. Such a term may cause a negative fermion contribution to the pressure and therefore might play an important role for the total dark energy density. We give in this section two examples of models where on the classical level the fermion contributes a term in the energy-momentum tensor proportional to $g_{\mu\nu} f(\tilde{\psi} \psi)$. 
A. A model with coupling to topological density

If the action includes a topological density then the term proportional to $\gamma_{\mu\nu}\psi\bar{\psi}$ can appear in the energy-momentum tensor. As an example of a topological density we consider here $\Omega \equiv \epsilon^{\alpha\beta\mu\nu} Tr(F_{\alpha\beta}F_{\mu\nu})$ in a gauge theory.

For illustration let us consider a model where in addition to $L_f$, which includes now a minimal coupling to gauge fields, there is also the interaction of the fermion to the scalar field $\phi$ which in its turn couples to the topological density $\Omega$ (the latter is parity violating):

$$S_1 = \int \sqrt{-g}d^4x \left[ \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - \frac{1}{2}m_\phi^2\phi^2 + \lambda_f\phi\bar{\psi}\psi \right] + \int \lambda_{top}\phi\Omega d^4x.$$

(1)

By integrating out the $\phi$-field one can get (assuming that $\phi$ is slowly varying) the following fermion part of the effective action

$$S_{(ferm)}^{(eff)} = \int \sqrt{-g}d^4x \left[ L_f + \frac{\lambda_f^2}{2m_\phi^2}(\bar{\psi}\psi)^2 \right] + \int \frac{\lambda_f\lambda_{top}}{m_\phi^2}\Omega\bar{\psi}\psi d^4x.$$

(2)

Now the fermion equation yields

$$\sqrt{-g} \left[ L_f + \frac{\lambda_f^2}{2m_\phi^2}(\bar{\psi}\psi)^2 \right] + \frac{\lambda_f\lambda_{top}}{m_\phi^2}\Omega\bar{\psi}\psi = 0$$

(3)

Varying the effective action (2) with respect to $g_{\mu\nu}$ and using Eq.(3) we obtain the following fermion contribution to the energy-momentum tensor proportional to $g_{\mu\nu}$:

$$\Delta T_{(ferm)}^{\mu\nu} = g_{\mu\nu} \left[ \frac{\lambda_f\lambda_{top}}{m_\phi^2}\Omega\bar{\psi}\psi + \frac{\lambda_f^2}{2m_\phi^2}(\bar{\psi}\psi)^2 \right]$$

(4)

B. A model with fermion selfinteraction

Another model of interest is that of a fermion with a selfinteraction in curved space-time. The fermion part of the action we choose in the form

$$S = \int \sqrt{-g}d^4x \left[ L_f + \lambda(\bar{\psi}\psi)^l \right]$$

(5)

where $L_f$ is the free fermion Lagrangian density in the Einstein-Cartan space-time; $l$ is a dimensionless constant different from one; $\lambda$ is a coupling of the selfinteraction.
It follows from the fermion equation of motion that

\[ L_f = -l\lambda(\bar{\psi}\psi)^l \]  

(6)

Varying the action (5) with respect to \( g^{\mu\nu} \) and using Eq.(6) we obtain the following fermion contribution to the energy-momentum tensor proportional to \( g^{\mu\nu} \):

\[ \Delta T_{\mu\nu}^{(\text{ferm})} = (l - 1)\lambda(\bar{\psi}\psi)^l g_{\mu\nu} \]  

(7)

The terms proportional to \( g^{\mu\nu} \) in the above examples can be regarded as possible candidates for fermion contributions to the dark energy. However for this effect to be realized as a genuine dark energy in the present day universe one needs that the coefficient of the \( g^{\mu\nu} \) term to be approximately a space-time constant. But there are no reasons why these coefficients in Eqs.(4) or (7) should be constant even approximately. As we will see in TMT this goal can be achieved.

III. MAIN IDEAS OF THE TWO MEASURES THEORY

TMT is a generally coordinate invariant theory where the action has the form [15]-[23]

\[ S = \int L_1\Phi d^4x + \int L_2\sqrt{-g}d^4x \]  

(8)

including two Lagrangians \( L_1 \) and \( L_2 \) and two measures of integration: the usual one \( \sqrt{-g} \) and the new one \( \Phi \). The latter may be built by means either of four scalar fields \( \varphi_a \) \((a = 1, 2, 3, 4)\) or of a totally antisymmetric three index field \( A_{\alpha\beta\gamma} \).

A. Measure \( \Phi \) built of four scalar fields \( \varphi_a \)

The measure \( \Phi \) being a scalar density and a total derivative may be defined as follows\(^1\):

\(^1\)As it was shown in the recent paper [24], the incorporation of four scalar fields \( \varphi_a \) together with the scalar density \( \Phi \), Eq.(9), is a possible way to define local observables in the local quantum field theory approach to quantum gravity.
\[
\Phi = \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d.
\] (9)

To provide parity conservation one can choose for example one of \( \varphi_a \)'s to be a pseudoscalar. A shift of \( L_1 \) by a constant, \( L_1 \rightarrow L_1 + \text{const.} \), has no effect on the equations of motion. Similar shift of \( L_2 \) would lead to the change of the constant part of the Lagrangian coupled to the volume element \( \sqrt{-g} d^4 x \). In standard GR, this constant term is the cosmological constant. However in TMT the relation between the constant term of \( L_2 \) and the physical cosmological constant is very non trivial.

In addition to the above idea concerning the general structure of the action in TMT, there are only two basic assumptions:

1. The Lagrangian densities \( L_1 \) and \( L_2 \) may be functions of the matter fields, the metric, the connection (or spin-connection) but not of the "measure fields" \( \varphi_a \). In such a case, i.e. when the measure fields \( \varphi_a \) enter in the theory only via the measure \( \Phi \), the action (8) has the infinite dimensional symmetry \cite{17}: \( \varphi_a \rightarrow \varphi_a + f_a(L_1) \), where \( f_a(L_1) \) is an arbitrary function of \( L_1 \).

2. One should proceed in the first order formalism where all fields, including metric and connections (or vierbeins \( e^\mu_a \) and spin-connection \( \omega^{ab}_\mu \) in the presence of fermions) as well as the measure fields \( \varphi_a \) are independent dynamical variables. All the relations between them follow from equations of motion. The independence of the metric and the connection in the action means that we proceed in the first order formalism and the relation between connection and metric is not necessarily according to Riemannian geometry.

Varying the measure fields \( \varphi_a \), we get

\[
B^\mu_a \partial_\mu L_1 = 0
\] (10)

where

\[
B^\mu_a = \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d.
\] (11)

Since \( \text{Det}(B^\mu_a) = \frac{4\pi}{4!} \Phi^3 \) it follows that if \( \Phi \neq 0 \),
\[ L_1 = sM^4 = \text{const} \]  

where \( s = \pm 1 \) and \( M \) is a constant of integration with the dimension of mass.

**B. Measure \( \Phi \) built of a totally antisymmetric three index potential**

The measure \( \Phi \) may be constructed also [17] from a totally antisymmetric three index potential \( A_{\alpha\beta\gamma} \):

\[ \Phi = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu A_{\nu\alpha\beta}. \]  

(13)

To provide parity conservation we must here choose \( A_{\alpha\beta\gamma} \) to have negative parity. The shift symmetry \( L_1 \rightarrow L_1 + \text{const} \) as well as two basic assumptions similar to basic assumptions (1) and (2) of Sec.3.1 hold here too. However, the infinite dimensional symmetry \( \varphi_a \rightarrow \varphi_a + f_a(L_1) \) is replaced now by \( A_{\alpha\beta\gamma} \rightarrow A_{\alpha\beta\gamma} + \varepsilon_{\mu\alpha\beta\gamma} f^\mu(L_1) \) where \( f^\mu(L_1) \) are four arbitrary functions of \( L_1 \) and \( \varepsilon_{\mu\alpha\beta\gamma} \) is numerically the same as \( \varepsilon^{\mu\nu\alpha\beta} \).

Variation of \( A_{\alpha\beta\gamma} \) yields now

\[ \varepsilon^{\mu\nu\alpha\beta} \partial_\mu L_1 = 0, \]  

(14)

that implies Eq.(12) without the condition \( \Phi \neq 0 \) needed in the model with four scalar fields \( \varphi_a \).

**C. Generic features of TMT**

First of all one should notice the very important differences of TMT from scalar-tensor theories with nonminimal coupling:

1. In general, the Lagrangian density \( L_1 \) (coupled to the measure \( \Phi \)) may contain not only the scalar curvature term (or more general gravity term) but also all possible matter fields terms. This means that TMT modifies in general both the gravitational sector and the matter sector,
2. If the field $\Phi$ were the fundamental (non composite) one then the variation of $\Phi$ would result in the equation $L_1 = 0$ instead of (12), and therefore the dimensionful parameter $M$ would not appear.

Applying the first order formalism one can show (see for example Ref. [17]) that the resulting relation between metric and connection, as well as fermion and scalar fields equations contain the gradient of the ratio of the two measures

$$\zeta \equiv \frac{\Phi}{\sqrt{-g}}$$

which is a scalar field. It turns out that at least at the classical level, the measure fields $\varphi_a$ (in the model of Sec.3.1) or $A_{\alpha\beta\gamma}$ (in the model of Sec.3.2) affect the theory only through the scalar field $\zeta$.

The consistency condition of equations of motion has the form of algebraic constraint which determines $\zeta(x)$ as a function of matter fields. The surprising feature of the theory is that neither Newton constant nor curvature appear in this constraint which means that the geometrical scalar field $\zeta(x)$ is determined by the matter fields configuration locally and straightforward (that is without gravitational interaction).

By an appropriate change of the dynamical variables which includes a conformal transformation of the metric, one can formulate the theory in a Riemannian (or Riemann-Cartan) space-time. The corresponding conformal frame we call "the Einstein frame". The big advantage of TMT is that in the very wide class of models, the gravity and all matter fields equations of motion take canonical GR form in the Einstein frame. All the novelty of TMT in the Einstein frame as compared with the standard GR is revealed only in an unusual structure of the scalar fields effective potential, masses of fermions and their interactions with scalar fields as well as in the unusual structure of fermion contributions to the energy-momentum tensor: they appear to be $\zeta$ dependent. This is why the scalar field $\zeta(x)$ determined by the constraint, has a key role in these effects.
IV. SCALE AND GAUGE INVARIANT MODEL

To simplify the presentation of the main results we will study here a simplified gauge model which is Abelian, does not include the Higgs fields and quarks and chiral properties of fermions are ignored. In this simplified model the mass terms for Dirac spinors are included by hand from the very beginning into the Lagrangians $L_1$ and $L_2$. The matter content of our model includes the scalar field $\phi$, two so-called primordial fermion fields (the primordial neutrino $N$ and the primordial electron $E$) and electromagnetic field $A_\mu$. The latter is included in order to show the reasons why the gauge fields dynamics is canonical. Generalization to non-Abelian gauge models including also quarks, Higgs fields, the Higgs mechanism of mass generation and taking into account chiral properties of fermions is straightforward, see Ref. [23]. The presence of the scalar field $\phi$ allows to realize a spontaneously broken global scale invariance [18] which includes the shift transformation of $\phi$. We will see that in the cosmological context $\phi$ contributes to the present dark energy.

We allow in both $L_1$ and $L_2$ all the usual terms considered in standard field theory models in curved space-time. Keeping the general structure of Eq.(8) it is convenient to represent the action in the following form:

$$S = \int d^4x e^{\alpha\phi/M_p} \left[ (\Phi + b\sqrt{-g}) \left\{ -\frac{1}{\kappa} R(\omega, e) + \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu \right\} \right.$$  
$$- \int d^4x e^{2\alpha\phi/M_p} [\Phi V_1 + \sqrt{-g} V_2] - \int d^4x \sqrt{-g} \frac{1}{2} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$$  
$$+ \int d^4x e^{\alpha\phi/M_p} (\Phi + k\sqrt{-g}) \frac{i}{2} \sum_i \Psi_i \left( \gamma^\alpha e_\mu \nabla^{(i)}_\mu - \nabla^{(i)}_\mu \gamma^\alpha e_\mu \right) \Psi_i$$  
$$- \int d^4x e^{2\alpha\phi/M_p} \left[ (\Phi + h_E \sqrt{-g})\mu_E \bar{E} E + (\Phi + h_N \sqrt{-g})\mu_N \bar{N} N \right] \tag{16}$$

where $\Psi_i \ (i = N, E)$ is the general notation for the primordial fermion fields $N$ and $E$; $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$; $\mu_N$ and $\mu_E$ are the mass parameters; $\nabla^{(N)}_\mu = \tilde{\nabla}_\mu + \frac{1}{2} \omega^c_{\mu} \sigma_{cd} + i e A_\mu$; $R(\omega, V) = e^\mu e^{\nu} R_{\mu\nu\alpha\beta}(\omega)$ is the scalar curvature; $e_\mu^a$ and $\omega^a_{\mu\nu}$ are the vierbein and spin-connection; $g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$ and $R_{\mu\nu\alpha\beta}(\omega) = \partial_\mu \omega_{\nu\alpha\beta} + \omega_\mu^c \omega_{c\nu\alpha\beta} - (\mu \leftrightarrow \nu)$. $V_1$ and $V_2$ are constants with the dimensionality (mass)$^4$. When Higgs field is included into the model then $V_1$ and $V_2$ turn into functions of the Higgs field. As we will see later, in the Einstein
frame $V_1$, $V_2$ and $e^{\alpha \phi/M_p}$ enter in the effective potential of the scalar sector. Constants $b, k, h_N, h_E$ are non specified dimensionless real parameters and we will only assume that they have close orders of magnitude

$$b \sim k \sim h_N \sim h_E;$$  \hspace{1cm} (17)

$\alpha$ is a real parameter which we take to be positive.

The action (16) is invariant under the global scale transformations:

$$e^\alpha \rightarrow e^{\theta/2} e^\alpha, \quad \omega^\mu_{ab} \rightarrow \omega^\mu_{ab}, \quad \varphi_a \rightarrow \lambda_{ab} \varphi_b, \quad A_\alpha \rightarrow A_\alpha,$$

$$\phi \rightarrow \phi - \frac{M_p}{\alpha} \theta, \quad \Psi_i \rightarrow e^{-\theta/4} \Psi_i, \quad \overline{V}_i \rightarrow e^{-\theta/4} \overline{V}_i;$$

where $\theta = \text{const}$, $\lambda_{ab} = \text{const}$ and $\det(\lambda_{ab}) = e^{2\theta}$, \hspace{1cm} (18)

when the measure fields $\varphi_a$ are used for definition of the measure $\Phi$, as in Sec.3.1. If the definition (13) of Sec.3.2 is used then the scale transformation of the totally antisymmetric three index potential $A_{\alpha \beta \gamma}$ should be: $A_{\alpha \beta \gamma} \rightarrow e^{2\theta} A_{\alpha \beta \gamma}$.

The choice of the action (16) needs two observations:

1. We have chosen the kinetic term of $A_\mu$ in the conformal invariant form which is possible if it is coupled only to the measure $\sqrt{-g}$. Introducing the coupling of this term to the measure $\Phi$ would lead to the nonlinear field strength dependence in the $A_\mu$ equation of motion. One of the possible consequences of this may be non positivity of the energy density. Another consequence is a possibility of certain unorthodox effects, like space-time variations of the effective fine structure constant. This subject deserves a special study but it is out of the purposes of this paper where the model is studied setting $F_{\mu \nu} \equiv 0$ in the solutions.

2. With the aim to simplify the analysis of the results of the model (containing many free parameters) we have chosen the coefficient $b$ in front of $\sqrt{-g}$ in the first integral of (16) to be a common factor of the gravitational term $-\frac{1}{\kappa} R(\omega, e)$ and of the kinetic term for the field $\phi$. The research of more general case has been started in Ref. [22]. A more detailed
study of interesting consequences of this possibility will be presented in a separate paper [25].

We would like to stress that except for the modification of the general structure of the action based on the basic assumptions of TMT, we do not introduce into the action (16) any exotic terms and fields. On the other hand, in the framework of such an economic model, except for the above two notions, Eq.(16) describes the most general action of TMT satisfying the formulated above symmetries.

Variation of the measure fields ($\varphi_a$ in the model of Sec.3.1 or $A_{\alpha\beta\gamma}$ in the model of Sec.3.2) yields Eq.(12) where $L_1$ is now defined, according to Eq.(8), as the part of the integrand of the action (16) coupled to the measure $\Phi$. The appearance of the integration constant $sM^4$ in Eq.(12) spontaneously breaks the global scale invariance (18). In what follows we choose $s = +1$.

Except for the $A_\mu$ equation, all other equations of motion resulting from (16) in the first order formalism contain terms proportional to $\partial_\mu\zeta$ that makes the space-time non-Riemannian and equations of motion - non canonical. However, with the new set of variables ($\phi$ and $A_\mu$ remain unchanged)

$$\tilde{e}_{a\mu} = e^{\frac{4}{3}\phi/M_p}(\zeta + b)^{1/2}e_{a\mu}, \quad \tilde{g}_{\mu\nu} = e^{\frac{2\phi/M_p}{3}}(\zeta + b)g_{\mu\nu},$$

$$\Psi'_i = e^{-\frac{4}{3}\phi/M_p}(\zeta + b)^{1/2}\Psi_i, \quad i = N, E$$

which we call the Einstein frame, the spin-connections become those of the Einstein-Cartan space-time. Since $\tilde{e}_{a\mu}$, $\tilde{g}_{\mu\nu}$, $N'$ and $E'$ are invariant under the scale transformations (18), spontaneous breaking of the scale symmetry (by means of Eq.(12)) is reduced in the new variables to the spontaneous breaking of the shift symmetry

$$\phi \rightarrow \phi + \text{const.}$$

(20)

Notice that the Goldstone theorem generically is not applicable in this theory (see the second reference in Ref. [18])). The reason is the following. In fact, the scale symmetry (20) leads to a conserved dilatation current $j^\mu$. However, for example in the spatially flat FRW universe
the spatial components of the current $j^i$ behave as $j^i \propto M^4 x^i$ as $|x^i| \to \infty$. Due to this anomalous behavior at infinity, there is a flux of the current leaking to infinity, which causes the non conservation of the dilatation charge. The absence of the latter implies that one of the conditions necessary for the Goldstone theorem is missing. The non conservation of the dilatation charge is similar to the well known effect of instantons in QCD where singular behavior in the spatial infinity leads to the absence of the Goldstone boson associated to the $U(1)$ symmetry.

After the change of variables (19) to the Einstein frame and some simple algebra, the gravitational equations take the standard GR form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T_{\mu\nu}^{\text{eff}}$$

where $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$; the energy-momentum tensor $T_{\mu\nu}^{\text{eff}}$ is now

$$T_{\mu\nu}^{\text{eff}} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \tilde{g}_{\mu\nu} V_{\text{eff}}(\phi; \zeta) + T_{\mu\nu}^{(\text{em})} + T_{\mu\nu}^{(\text{ferm,can})} + T_{\mu\nu}^{(\text{ferm,noncan})},$$

$$V_{\text{eff}}(\phi; \zeta) = \frac{b (M^4 e^{-2\alpha\phi/M_p} + V_1) - V_2}{(\zeta + b)^2};$$

$T_{\mu\nu}^{(\text{em})}$ is the canonical energy momentum tensor of the electromagnetic field; $T_{\mu\nu}^{(\text{ferm,can})}$ is the canonical energy momentum tensor for (primordial) fermions $N'$ and $E'$ in curved space-time (including also interaction of $E'$ with $A_\mu$). $T_{\mu\nu}^{(\text{ferm,noncan})}$ is the noncanonical contribution of the fermions into the energy momentum tensor

$$T_{\mu\nu}^{(\text{ferm,noncan})} = -\tilde{g}_{\mu\nu} \Lambda_{\text{dyn}}^{(\text{ferm})}$$

where

$$\Lambda_{\text{dyn}}^{(\text{ferm})} \equiv Z_N(\zeta) m_N(\zeta) \overline{N'} N' + Z_E(\zeta) m_E(\zeta) \overline{E'} E'$$

and $Z_i(\zeta)$ and $m_i(\zeta) \ (i = N', E')$ are respectively
\[ Z_i(\zeta) \equiv \frac{(\zeta - \zeta^{(i)}_1)(\zeta - \zeta^{(i)}_2)}{2(\zeta + k)(\zeta + h_i)} \]

\[ m_i(\zeta) = \frac{\mu_i(\zeta + h_i)}{(\zeta + k)(\zeta + b)^{1/2}} \]

where

\[ \zeta^{(i)}_{1,2} = \frac{1}{2} \left[ k - 3h_i \pm \sqrt{(k - 3h_i)^2 + 8b(k - h_i) - 4kh_i} \right]. \]  \hspace{1cm} (27)

The noncanonical contribution \( T^{(\text{ferm,noncan})}_{\mu \nu} \) of the fermions into the energy momentum tensor has the transformation properties of a cosmological constant term but it is proportional to fermion densities \( \overline{\Psi}_i \Psi'_i \) \( (i = N', E') \). This is why we will refer to it as "dynamical fermionic \( \Lambda \) term". This fact is displayed explicitly in Eqs.(24),(25) by defining \( \Lambda^{(\text{ferm})}_{\text{dyn}} \).

\( T^{(\text{ferm,noncan})}_{\mu \nu} \) is somewhat similar to the contributions \( \Delta T_{\mu \nu} \) of the fermions into the energy momentum tensor obtained in the non-TMT models of Sec.2. But there are the following essential differences:

- The appearance of \( T^{(\text{ferm,noncan})}_{\mu \nu} \) in Eq.(22) is a direct result of the action (16) while the contribution \( \Delta T_{\mu \nu} \) in the first model of Sec.2 results from integrating out the slowly varying scalar field; in the second model of Sec.2 we introduced into the action the fermion self-interaction while our TMT action (16) has no such exotic terms.

- The contribution \( \Delta T_{\mu \nu} \) in the models of Sec.2 may be a source of departures from GR. On the contrary, as we will see in Sec.6, \( \Lambda^{(\text{ferm})}_{\text{dyn}} \) becomes negligible in gravitational experiments with regular matter.

- As we have noted in Sec.2, there are no reasons why the coefficient of \( g_{\mu \nu} \) in \( \Delta T_{\mu \nu} \) in the models of Sec.2 might be constant even approximately. But we will show in the context of a toy cosmological model in Sec.7 that a neutrino contribution into \( T^{(\text{ferm,noncan})}_{\mu \nu} \) may be a part of the cosmological constant term at the very late universe.

The ”dilaton” \( \phi \) field equation in the Einstein frame reads

\[ \Box \phi - \frac{\alpha}{M_p(\zeta + b)} \left[ M^4 e^{-2\alpha \phi/M_p} - (\zeta - b)V_1(\upsilon) + 2V_2(\upsilon)/\zeta + b \right] = -\frac{\alpha}{M_p} \Lambda^{(\text{ferm})}_{\text{dyn}}, \],

where \( \Box \phi = (\sqrt{-g})^{-1/2} \partial_{\mu}(\sqrt{-g}g^{\mu \nu} \partial_{\nu} \phi) \).
One can show that equations for the primordial fermions in terms of the variables (19) take the standard form of fermionic equations for $N'$ and $E'$ in the Einstein-Cartan space-time (see also Appendix A) where the standard electromagnetic interaction of $E'$ is present too. Note that the coupling of the fermions to the dilaton $\phi$ in the original action (which was only via exponents of $\phi$) disappear in the Einstein frame. All the novelty as compared with the standard field theory approach consists of the form of the $\zeta$ depending "masses" of the primordial fermions $N', E'$, second equation in Eq.(26). The fermion parts of the effective action in the Einstein frame is invariant under the global phase transformations of any of fermion fields exactly as it was in the original action. Therefore the conserved fermion 4-currents exist both in the original and in the Einstein frames. However in the Einstein frame these currents are not only covariantly conserved but also have the canonical form without any $\phi$ and $\zeta$ dependence.

The electromagnetic field equations are canonical due to our choice for the appropriate term in the action (16) to be conformally invariant.

The scalar field $\zeta$ (see its definition, Eq.(15)), is determined as a function of the scalar field $\phi$ and $\nabla_i \Psi_i'$ ($i = N', E'$) by the following constraint

$$\frac{1}{(\zeta + b)^2} \left\{ (b - \zeta) \left[ M^4 e^{-2\alpha \phi/M_p} + V_1(\nu) \right] - 2V_2(\nu) \right\} = \Lambda_{d_{dyn}}^{(ferm)}$$

The origin of the constraint is clear enough. There are two equations containing the scalar curvature: the first one is Eq.(12) and the second one follows from gravitational equations. The constraint is nothing but the consistency condition of these two equations.

One should point out an unexpected and very important fact, namely that the same function $\Lambda_{d_{dyn}}^{(ferm)}$, Eq.(25), emerges in three different places:

a) in the form of the noncanonical fermion contribution to the energy-momentum tensor, Eq.(24);

b) in the effective Yukawa coupling of the dilaton $\phi$ to fermions (see the right hand side of Eq.(28));

c) as the right hand side of the constraint.
Note that the original action (16) contains exponents of the scalar field $\phi$ and in particular the coupling of fermions with $\phi$ is realized in (16) only through the exponents of $\phi$. Nevertheless, except for the term $M^4 e^{-2\alpha \phi/M_p}$ originated by the scale symmetry breaking, the equations of motion in the Einstein frame do not contain explicitly the exponents of $\phi$.

However, the Yukawa-type coupling of the fermions to $\phi$ emerges in the Einstein frame, see the r.h.s. of Eq.(28).

It is interesting that non-explicit dependence on the exponent of $\phi$ in the equations of motion is actually present after solving the constraint (29) for $\zeta$. However this dependence is again in the form of $M^4 e^{-2\alpha \phi/M_p}$. Thus the exponential $\phi$-dependence in the equations of motion results only from the scale symmetry breaking. Recall that in the Einstein frame the scale symmetry transformations (18) are reduced to the shift symmetry $\phi \rightarrow \phi + \text{const.}$

Note finally that applying the constraint (29) to Eq.(28) one can reduce the latter to the form

$$\Box \phi - \frac{2\alpha \zeta^2}{\zeta + b^2} M_p^4 e^{-2\alpha \phi/M_p} = 0,$$

where $\zeta$ is a solution of the constraint (29). This result is true both in the presence of fermions and in their absence.

Generically, the constraint (29) determines $\zeta$ as a complicated function of $\phi$, $N'N'$ and $E'E'$. Substituting the appropriate solution for $\zeta$ into the equations of motion one can conclude that in general there is no sense, for example, to regard $V_{\text{eff}}(\phi, \nu; \zeta)$, Eq.(23), as the effective potential for the scalar field $\phi$ because it depends in a very nontrivial way on $N'N'$ and $E'E'$ as well. For the same reason, the fermion mass and $\Lambda_{\text{dyn}}^{(\text{term})}$ describe in general self-interactions of the primordial fermions depending also on the scalar field $\phi$. Therefore it is impossible, in general, to separate the terms of $T_{\mu\nu}$ describing the scalar field $\phi$ effective potential from the fermion contributions. Such mixing of the scalar field $\phi$ associated with dark energy, on the one hand, and fermionic matter, on the other hand, gives rise to a rather complicated system of equations when trying to apply the theory to general situations that could appear in astrophysics and cosmology. Notice that in such a case, the quantum theory
of fermion fields may be non perturbative: inserting solution for $\zeta$ into the effective fermion "mass", Eq.(26), it is easy to see that the "free" primordial fermion equation appears to be nonlinear in general. Considerable simplification of the situation occurs if for some reasons $\zeta$ appears to be constant or almost constant. Fortunately this is exactly what happens in many physically interesting situations.

V. DARK ENERGY IN THE ABSENCE OF MASSIVE FERMIONS

In the absence of massive fermions the constraint (29) determines $\zeta$ as the function of $\phi$ alone:

$$\zeta = \zeta_0(\phi) \equiv b - \frac{2V_2}{V_1 + M^4 e^{-2\alpha\phi/M_P}},$$

(31)

Note that the electromagnetic field does not enter in the constraint (29) and therefore the presence of the electromagnetic field does not affect the value of $\zeta_0$.

The effective potential of the scalar field $\phi$ results then from Eq.(23)

$$V_{\text{eff}}^{(0)}(\phi) \equiv V_{\text{eff}}(\phi; \zeta_0)|_{\phi}\psi\psi' = 0 = \frac{[V_1 + M^4 e^{-2\alpha\phi/M_P}]^2}{4[b(V_1 + M^4 e^{-2\alpha\phi/M_P}) - V_2]},$$

(32)

and the $\phi$-equation (28) is reduced to

$$\Box \phi + V_{\text{eff}}^{(0)}(\phi) = 0,$$

(33)

where prime sets derivative with respect to $\phi$.

The mechanism of the appearance of the effective potential (32) is very interesting and exhibits the main features of our TMT model\(^2\). In fact, the reasons of the transformation

\(^2\)The particular case of this model with $b = 0$ and $V_2 < 0$ was studied in Ref. [18]. The application of the TMT model with explicitly broken global scale symmetry to the quintessential inflation scenario was discussed in Ref. [19]. A particular case of the model (16) without explicit potentials, i.e. $V_1 = V_2 = 0$, has been studied in Ref. [22]; see also Appendix of the present paper.
of the prepotentials $V_1 e^{2\alpha \phi/M_p}$ and $V_2 e^{2\alpha \phi/M_p}$, coming in the original action (16), into the effective potential (32) are the following:

a) Transformation to the Einstein frame;

b) Spontaneous breakdown of the global scale symmetry which in the Einstein frame is reduced to the spontaneously broken shift symmetry (20);

c) The constraint which in the absence of fermions case takes the form (31).

It is easy to see that the gravitational equations (21) in the absence of fermions case become the standard Einstein equations of the model where the electromagnetic field and the minimally coupled scalar field $\phi$ with the potential $V_{\text{eff}}(\phi)$ are the sources of gravity.

The structure of the potential (32) allows to construct a model [18], [17] where zero vacuum energy is achieved without fine tuning when $V_1 + M^4 e^{-2\alpha \phi/M_p} = 0$. This and many other aspects of the dynamics of the scalar sector (including Higgs fields) will be studied in detail in a separate publication [25].

In what follows we will assume

$$V_1 > 0 \quad \text{and} \quad b > 0.$$  \hspace{1cm} (34)

Applying this model to the cosmology of the late time universe and assuming that the scalar field $\phi \rightarrow \infty$ as $t \rightarrow \infty$, we see that the evolution of the late time universe is governed by the dark energy density

$$\rho_{d,e}^{(0)} = \frac{1}{2} \dot{\phi}^2 + \Lambda^{(0)} + V_{q\text{-like}}^{(0)}(\phi).$$ \hspace{1cm} (35)

where $\Lambda^{(0)}$ is the cosmological constant

$$\Lambda^{(0)} = \frac{V_2^2}{4[bV_1 - V_2]}$$ \hspace{1cm} (36)

and $V_{q\text{-like}}^{(0)}(\phi)$ is the quintessence-like scalar field potential

$$V_{q\text{-like}}^{(0)}(\phi) = \frac{V_1 (bV_1 - 2V_2) + (bV_1 - V_2) M^4 e^{-2\alpha \phi/M_p}}{4[bV_1 - V_2][b(V_1 + M^4 e^{-2\alpha \phi/M_p}) - V_2]} \cdot M^4 e^{-2\alpha \phi/M_p}.$$ \hspace{1cm} (37)

The cosmological constant $\Lambda^{(0)}$ is the asymptotic value (as $t \rightarrow \infty$) of $\rho_{d,e}^{(0)}$ for the FRW universe in the model where massive fermions absent (the reason we emphasize this here
will be clear in Sec.7). \( \Lambda^{(0)} \) is positive provided \( bV_1 > V_2 \). The potential decreases to \( \Lambda^{(0)} \) monotonically if

\[
bV_1 > 2V_2
\]

that will be assumed in what follows.

There are two ways to provide the observable order of magnitude of the present day vacuum energy density by an appropriate choice of the parameters of the theory in the framework of the condition (38) but without supposition of an extreme smallness of \( V_1 \) and/or \( V_2 \):

1. If \( V_2 < 0 \) and \( bV_1 < |V_2| \) then \( \Lambda^{(0)} \approx \frac{V_2^2}{4|V_2|} \). In this case there is no need for \( V_1 \) and \( V_2 \) to be small: it is enough that the dimensionless quantity \( V_1/|V_2| \ll 1 \). This possibility is a kind of *seesaw* mechanism (see Refs. [18], [26]). For instance, if \( V_1 \) has the scale of electroweak symmetry breaking \( V_1 \sim (10^{3}\text{GeV})^4 \) and \( V_2 \) has the Planck scale \( |V_2| \sim (10^{18}\text{GeV})^4 \) then \( \Lambda^{(0)} \sim (10^{-3}\text{eV})^4 \).

2. If \( V_2 > 0 \) or alternatively \( V_2 < 0 \) and \( bV_1 > |V_2| \) then \( \Lambda^{(0)} \approx \frac{V_1}{4b} \). Hence the second possibility is to choose the *dimensionless* parameter \( b > 0 \) to be a huge number. In this case the order of magnitudes of \( V_1 \) and \( V_2 \) could be either as in the above case (i) or to be not too much different (or even of the same order). For example, if \( V_1 \sim (10^3\text{GeV})^4 \) then for getting \( \Lambda^{(0)} \sim (10^{-3}\text{eV})^4 \) one should assume that \( b \sim 10^{60} \). Note that \( b \) is the ratio of the coupling constants of the scalar curvature to the measures \( \sqrt{-g} \) and \( \Phi \) in the fundamental action of the theory (16).

**VI. GENERAL RELATIVITY AND REGULAR FERMIONS**

**A. Reproducing Einstein Equations and Regular Fermions**

In Sec.5 we have seen that in the absence of fermions case the gravitational equations (21) coincide with the Einstein equations. Analyzing Eqs.(21)-(27) in more general cases it is easy to see that Eqs.(21) and (22) are reduced to the Einstein equations in the appropriate
field theory model (i.e. when the scalar field, electromagnetic field and massive fermions are sources of gravity) if $\zeta$ is constant and

$$\Lambda_{dyn}^{\text{ferm}} = 0 \quad \text{or at least} \quad |T^{(\text{ferm,noncan})}_{\mu\nu}| \ll |T^{(\text{ferm,can})}_{\mu\nu}|. \quad (39)$$

According to Eqs.(24)-(27), in the case when a single massive fermion is a source of gravity, the condition (39) is realized if

$$Z_i(\zeta) \approx 0 \quad \Rightarrow \quad \zeta = \zeta_i^1 \quad \text{or} \quad \zeta = \zeta_i^2, \quad i = N', E', \quad (40)$$

where $\zeta_{i,2}^i$ are defined in Eqs.(27).

Recall that existence of a noncanonical contribution to the energy-momentum tensor (24), along with the $\zeta$ dependence of the fermion mass (26) discovered in Sec.4, displays the fact that generically primordial fermion is very much different from the regular one (see definitions in item (ii) of the Introduction section). To answer the question what are the characteristic features of the regular massive fermion we have to take into account the undisputed fact that the classical tests of GR deal only with regular fermionic matter. Hence we should identify the regular fermions with states of the primordial fermions satisfying the condition (40).

**B. Meaning of the Constraint and Regular Fermions**

We are going now to understand the meaning of the constraint (29). We start from the detailed analysis of two limiting cases

(1) in space-time regions without fermions;

(2) in space-time regions occupied by regular fermions;

and afterwards we will be able to formulate the meaning of the constraint in general case. Recall that the main goal of this paper realized in Sec.7 is to demonstrate a possibility of exotic states where fermion affects the dark energy, and this result is also based essentially on Eq.(29).
We will proceed keeping in mind that the parameters $V_1$, $V_2$ and/or $b$ are chosen appropriately, as in the discussion in items (i) or (ii) at the end of Sec.5 to provide a desirable order of magnitude of the cosmological constant $\Lambda^{(0)}$ in the absence of fermions case, Eq.(36).

It is convenient to divide the analysis into a few steps:

1. It follows from the condition (38) that $\zeta_0(\phi)$ (determined by Eq.(31)) has the same order of magnitude as the parameter $b$.

2. Recall that $V_{eff}^{(0)}(\phi)$ having the order of magnitude typical for the dark energy density (in the absence of fermions case) is obtained from $V_{eff}(\phi; \zeta)$, Eq.(23), as $\zeta = \zeta_0(\phi)$. Therefore with the help of the item (i) we conclude that each time when $\zeta$ has the order of magnitude close to that of the parameter $b$ (and if no special tuning is assumed) $V_{eff}(\phi; \zeta)$, Eq.(23), has the order of magnitude close to that of the dark energy density (in the absence of fermions case).

3. It is easy to see that each time when $\zeta$ has the order of magnitude close to that of the parameter $b$ (if no special tuning is assumed and in particular $\zeta \neq \zeta_0(\phi)$), the left hand side (l.h.s.) of the constraint (29) has the order of magnitude close to that of $V_{eff}(\phi; \zeta)$, Eq.(23), i.e. the l.h.s. of the constraint has the order of magnitude close to that of the dark energy density in the absence of fermions case.

4. Let us now turn to the right hand side (r.h.s.) of the constraint (29) in the presence of a single massive primordial fermion. It contains factor $m_i(\zeta)\overline{\Psi}_i\Psi'_i$, ($i = N', E'$) which have typical order of magnitude of the fermion canonical energy density $T_{00}^{(ferm, can)}$. If the primordial fermion is in a state of a regular fermion then according to the conclusion made at the end of the previous subsection, in the space-time region where the fermion is localized, the scalar $\zeta$ must be $\zeta = \zeta_1^i$ (or $\zeta = \zeta_2^i$). Therefore in the space-time region occupied by a single regular fermion, the r.h.s. of (29) is

$$\Lambda_{dyn}^{(ferm)}|_{regular} \equiv Z_i(\zeta_{1,2}^i)m_i(\zeta_{1,2}^i)\overline{\Psi}_i\Psi_i'_{regular} \quad (41)$$

Due to our assumption (17) it follows from the definitions (27) that both $\zeta_1^i$ and $\zeta_2^i$ have the order of magnitude close to that of $b$. Hence in the space-time region occupied by a single
regular fermion, the l.h.s. of (29) has the order of magnitude close to that of the dark energy density in the fermion vacuum. It is evident that in normal particle physics conditions, that is when the energy density of a single fermion \( \sim m_i(\zeta) \overline{\Psi}_i \Psi_i' \) is tens of orders of magnitude larger than the fermion vacuum energy density, the balance dictated by the constraint is satisfied in the present day universe just due to the condition (40).

5. In more general cases, i.e. when primordial fermion is in a state different from the regular one, the meaning of the constraint is similar: \textit{the balance between the scalar dark energy contribution to the l.h.s. of the constraint and the fermionic contribution to the r.h.s. of the constraint} is realized due to the factors \( Z_i(\zeta) \). In other words, \textit{the constraint describes the local balance between the fermion energy density and the scalar dark energy density} in the space-time region where the wave function of the primordial fermion is not equal to zero; \textit{by means of this balance the constraint determines the scalar} \( \zeta(x) \). Note also that due to this balance, \textit{the degree of localization of the fermion and values of} \( \zeta(x) \) \textit{may be strongly interconnected}. As we will see in Sec.7, this feature of fermions in TMT plays a \textit{key role in the mechanism providing a possibility for a primordial fermion to be either in the state of a regular fermion or in exotic states}.

One can suggest the following two alternative approaches to the question of how a primordial fermion can be realized as a regular one:

\textit{The first approach} discussed in Refs. [22], [23], is based on the idea of the "maximal economy". We start from one primordial fermion field for each type of fermions: one neutral primordial lepton field \( N \), one charged primordial lepton field \( E \) and similar for quarks. In other words we start from one generation of fermions (note that the gauge symmetry, for example \( SU(2) \times U(1) \), may be imposed in a usual way). Splitting of the primordial fermions into families occurs only in normal particle physics conditions, i.e. when the fermion energy density is huge in comparison with the vacuum energy density. One of the possibilities for this to be realized is the above mentioned condition \( Z_i(\zeta) \approx 0 \). The appropriate two constant solutions for \( \zeta \), i.e. \( \zeta = \zeta_{1,2} \), correspond to two different states of the primordial fermions with \textit{different constant masses} determined by the second equation in (26) where we have to
substitute $\zeta_{i,2}$ instead of $\zeta$. So, in the normal particle physics conditions, the scalar $\zeta$ plays the role of an additional degree of freedom determining different mass eigenstates of the primordial fermions which we want to identify with different fermion generations. Note that the classical tests of GR deal in fact with matter built of the fermions of the first generation (may be with a small touch of the second generation). This is why one can identify the states of the primordial fermions realized as $\zeta = \zeta_{1}^{(E)}$ (or $\zeta = \zeta_{2}^{(E)}$), it is detected as the regular electron $e$ (or muon $\mu$) and similar the primordial neutrino $N$ splits into the regular electron and muon neutrinos with masses respectively:

$$m_{e(\mu)} = \frac{\mu_{E}(\zeta_{1(2)}^{(E)} + h_{E})}{(\zeta_{1(2)}^{(E)} + k)(\zeta_{1(2)}^{(E)} + b)^{1/2}}; \quad m_{\nu_{e}(\nu_{\mu})} = \frac{\mu_{N}(\zeta_{1(2)}^{(N)} + h_{N})}{(\zeta_{1(2)}^{(N)} + k)(\zeta_{1(2)}^{(N)} + b)^{1/2}}$$

(42)

It turns out that there is only one more additional possibility to satisfy the constraint (29) when primordial fermion is in the normal particle physics conditions. This is the solution $\zeta^{i} = \zeta_{3}^{i} \approx -b$ which one can associate with the third generation of fermions (for details see Refs. [22], [23]). It is interesting that in contrast to the first two generations, the third generation defined by this way, may have gravitational interaction with unusual features since the condition (39) may not hold. The described splitting of the primordial fermions into three generations in the normal particle physics conditions is the family replication mechanism proposed in Refs. [22], [23].

The second approach is based on the idea that the three families of fermions of the standard model exist from the beginning in the original action, i.e. not to use the family replication mechanism for explanation of the observed three generations of fermions. In this case again, exactly as it was in the first approach, the primordial fermions turn into the regular fermions only in the normal particle physics conditions. Now however if we interpret the state of the primordial fermions with, for example, $\zeta^{(i)} = \zeta_{1}^{(i)}$ as the observable regular fermions, then some role should be assigned to the states with $\zeta^{(i)} = \zeta_{2}^{(i)}$ and $\zeta^{(i)} = \zeta_{3}^{(i)}$. By means of a choice of the parameters one can try for example to provide very large masses of the regular fermions with $\zeta^{(i)} = \zeta_{2}^{(i)}$ and $\zeta^{(i)} = \zeta_{3}^{(i)}$ that might explain the reason why
they are unobservable so far. However these questions are beyond of the goals of this paper and together with many other aspects of fermions in TMT will be studied in a separate publication.

C. Resolution of the 5-th Force Problem for Regular Fermions

Reproducing Einstein equations when the primordial fermions are in the states of the regular fermions is not enough in order to assert that GR is reproduced. The reason is that at the late universe, as $\phi \gg M_p$, the scalar field $\phi$ effective potential is very flat and therefore due to the Yukawa-type coupling of massive fermions to $\phi$, (the r.h.s. of Eq.(28)), the long range scalar force appears to be possible in general. The Yukawa coupling ”constant” is $\alpha \frac{m_i(Q)}{M_p} Z_i(\zeta)$. Applying our analysis of the meaning of the constraint in Sec.6.2, it is easy to see that for regular fermions with $\zeta^{(i)} = \zeta^{(i)}_{1,2}$ the factor $Z_i(\zeta)$ is of the order of the ratio of the vacuum energy density to the regular fermion energy density. Thus we conclude that the 5-th force is extremely suppressed for the fermionic matter observable in classical tests of GR. It is very important that this result is obtained automatically, without tuning of the parameters and it takes place for both approaches to realization of the regular fermions in TMT discussed in subsection 6.2.

VII. NONRELATIVISTIC NEUTRINOS AND DARK ENERGY

In Secs.5 and 6 we have studied two opposite limiting cases: one is realized if there are no fermions at all; the second one corresponds to the normal particle physics conditions. The latter means that for a single fermion $m_i(\zeta) |\bar{\Psi}_i\psi_i'$ is a huge magnitude in comparison with the vacuum energy density. It turns out that besides the normal particle physics situations, TMT predicts possibility of so far unknown states which can be realized, for example, in astrophysics and cosmology. Roughly speaking such exotic states may be created if the degree of localization of the fermion is very small.
A. The essence of the Cosmo-Particle Phenomena

1. Toy model I

To illustrate some of the properties of such states let us start from a simplest (but idealized) model describing the following self-consistent system: the spatially flat FRW universe filled with the homogeneous scalar field $\phi$ and a homogeneous primordial neutrino field $N'(t)$. The non-canonical contribution of the primordial neutrino $N'$ into the energy-momentum tensor reads

$$T_{\mu\nu}^{(N,\text{noncan})} = -\tilde{g}_{\mu\nu}\Lambda^{(N)}_{\text{dyn}}$$

(43)

where

$$\Lambda^{(N)}_{\text{dyn}} \equiv Z_N(\zeta)m_N(\zeta)\mathbf{N}^\dagger N'$$

(44)

$$Z_N(\zeta) = \frac{(\zeta - \zeta_1^{(N)})(\zeta - \zeta_2^{(N)})}{2(\zeta + k)(\zeta + h_N)}$$

$$m_N(\zeta) = \frac{\mu_N(\zeta + h_N)}{(\zeta + k)(\zeta + b)^{1/2}}$$

(45)

and $\zeta_{1,2}$ are defined by Eq.(27).

The neutrino has zero momenta and therefore one can rewrite $\mathbf{N}^\dagger N'$ in the form of density $\mathbf{N}^\dagger N' = u^\dagger u$ where $u$ is the large component of the Dirac spinor $N'$. The space components of the 4-current $\bar{e}_a^\mu N'\gamma^a N'$ equal zero. It follows from the 4-current conservation that $\mathbf{N}^\dagger N' = u^\dagger u = \frac{\text{const}}{a^3}$ where $a = a(t)$ is the scale factor. It is convenient to rewrite the constraint (29) (adjusted to the model under consideration) in the following form:

$$\frac{(b - \zeta)[M^4 e^{-2\alpha\phi/M_p} + V_1] - 2V_2}{(\zeta + b)^{3/2}} = \frac{(\zeta - \zeta_1^{(N)})(\zeta - \zeta_2^{(N)})}{(\zeta + k)^2} \mu_N \frac{\text{const}}{a^3}.$$  

(46)

A possible solution of the constraint for the expanding universe as the scale factor $a(t) \to \infty$ is identical to the one studied in Sec.5 where the fermion contribution is treated as negligible; in this case $\zeta$ is a function of $e^{-2\alpha\phi/M_p}$ alone and is independent of the scale factor.
There is however another solution where the decaying fermion contribution $u^\dagger u \sim \frac{\text{const}}{a^3}$ to the constraint is compensated by the appropriate behavior of the scalar field $\zeta$. Namely if expansion of the universe is accompanied by approaching $\zeta \to -k$ in such a way that $(\zeta + k)^2 \propto a^3$, then the r.h.s. of (46) approaches a constant. Note that the l.h.s. of (46) also approaches a constant if $\phi \to \infty$ as $a(t) \to \infty$ (recall we assume $\alpha > 0$). The described regime corresponds to a very unexpected state of the primordial neutrino with the following exotic features:

1. This state is different from the regular neutrino states since $-k \neq \zeta_{1,2}$ unless a fine tuning is made.

2. The effective mass of the neutrino in this state increases like $(\zeta + k)^{-1} \propto a^{3/2}$ and therefore $\rho_{(N,\text{canon})} = T_{00}^{(N,\text{canon})} = m_N(\zeta)u^\dagger u \propto a^{-3/2}$. At the same time the dynamical neutrino $\Lambda_{\text{dyn}}^{(N)}$ term approaches a constant: $\Lambda_{\text{dyn}}^{(N)} \propto (\zeta + k)^{-2}u^\dagger u \to \text{constant}$. This means that at the late time universe, the canonical neutrino energy density $\rho_{(N,\text{canon})}$ becomes negligible in comparison with the non-canonical neutrino energy density $\rho_{(N,\text{noncanon})} = T_{00}^{(N,\text{noncanon})} = -\Lambda_{\text{dyn}}^{(N)}$.

3. It follows immediately from Eq.(24) and item (ii) that such cold neutrino matter possesses a pressure $p_N$ and its equation of state in the late time universe approaches the form $p_N = -\rho_N = -\rho_{(N,\text{canon})}$ typical for a cosmological constant. Therefore the primordial neutrino in the described regime behaves as a sort of dark energy.

4. In the regime $\zeta \to -k$, the scalar field $\phi$ effective potential $V_{\text{eff}}(\phi; \zeta)$, Eq.(23), and the l.h.s. of the constraint (29) have the same order of magnitude while the r.h.s. of the constraint is $\Lambda_{\text{dyn}}^{(N)}$. Therefore in this toy model, contributions of the scalar field $\phi$ and the primordial neutrino into the dark energy density are of the same order of magnitude if the described above regime $\zeta \to -k$ is realized.

Thus TMT predicts a possibility of new type of states which we will refer as Cosmo-Low-Energy-Physics (CLEP) states.
2. Toy Model II: Uniformly distributed non-relativistic neutrinos

To a certain extent a more realistic model may be constructed by studying a possible effect of uniformly distributed non-relativistic (in the co-moving frame) neutrinos on cosmological scenarios in the context of TMT. In other words we are going to study solutions motivated by what was discussed above in the toy model I, but now in a model where the spatially flat FRW universe filled with the homogeneous scalar field $\phi$ and a cold gas of uniformly distributed non-relativistic neutrinos.

The study of the cosmology in the context of TMT becomes complicated in the presence of the fermionic matter because the averaging procedure of the matter and gravity equations of motion includes also the need to know the field $\zeta$ as a solution of the constraint. As it was discussed in the last paragraph of Sec.4, this generically results in the appearance of a nonlinear $N' N'$-dependence in all equations of motion. This makes the procedure of the cosmological averaging very unclear and perhaps practically impossible.

Significant simplifications of the described general situation we observed in Secs. 5 and 6 were related to the fact that in those particular cases the function $\zeta$ appears to be only $\phi$-dependent or constant (or approaching constant). The important thing we have learned in the toy model I is a possibility of the CLEP state where $\zeta$ is close to a constant $-k$. Recall that if the primordial neutrinos are in the regime $\zeta \approx -k$, their masses may be very large that justifies our choice of the gas of non-relativistic neutrinos. On the other hand, as we have seen in the toy model, the CLEP state can be realized if $\overline{N} N' \to 0$. A possible way to get up such a state might be spreading of the neutrino wave packet during its free motion lasting a very long (of the cosmological scale) time. However a considerable spreading of the wave packet is again possible if neutrino is non-relativistic.

One should note however that spreading is here much more complicated process than in the well known examples of quantum mechanics: decreasing of $\overline{N} N'$ (due to the spreading) is related to changing $\zeta$ which satisfies the nonlinear constraint equation. So we deal in this case with nonlinear quantum mechanics. The detailed study of the spreading of the wave
packet in TMT is a subject of considerable interest but in this paper we will concentrate our attention on the consequences of the assumption that states with $\zeta \approx -k$ are achievable for each of the particles of the gas of non-relativistic primordial neutrinos. The appropriate stage of the universe we will call the CLEP state because each of the neutrinos directly participate in the cosmological expansion: $\mathcal{N} N' \propto a^{-3}$ and $m_N \propto (\zeta + k)^{-1} \propto a^{3/2}$. We will see below that in the context of the cosmology of the late time universe such states provide lower energy of the universe than the states with $\zeta$ determined in Sec.5 where there are no fermions at all. In other words, independently of how the states with $\zeta \approx -k$ are realized they are energetically more preferable than states without massive fermionic matter at all.

In the CLEP state where $\zeta(x)$ approaches a constant ($\zeta \to -k$) the procedure of the cosmological averaging of the microscopic equations of motion becomes free of the mentioned difficulties. For example, taking into account that in the studied regime masses (45) of the non-relativistic primordial neutrinos are very large, one can ignore their kinetic energy. Then after averaging over spins of the neutrinos, the cosmological averaging of the microscopic non-canonical contribution to the energy-momentum tensor $T^{(N,\text{can})}_{\mu\nu}$ results in

$$<T^{(N,\text{can})}_{\mu\nu}>_{\text{cosm.av.}} = \delta^0_\mu \delta^0_\nu \frac{h_N - k}{(b - k)^{1/2}} \mu_N n (\zeta + k) a^3 + O\left(\frac{1}{a^3}\right),$$

(47)

where $n$ is a constant determined by the total number of the cold neutrinos entering the CLEP regime\(^3\), i.e. in the regime with $\zeta \approx -k$. Similarly, averaging of the $\Lambda^{(N)}_{\text{dyn}}$ term gives

$$<\Lambda^{(N)}_{\text{dyn}} >_{\text{cosm.av.}} = (b - k)^{1/2} (h_N - k) \mu_N n (\zeta + k) a^3 + O\left(\frac{1}{(\zeta + k) a^3}\right).$$

(48)

Hence the appropriate averaged expression of the microscopic non-canonical contribution to the energy-momentum tensor $T^{(N,\text{noncan})}_{\mu\nu}$, Eq.(43), is then

$$<T^{(N,\text{noncan})}_{\mu\nu}>_{\text{cosm.av.}} = -\tilde{g}_{\mu\nu} <\Lambda^{(N)}_{\text{dyn}} >_{\text{cosm.av.}}$$

(49)

\(^3\)We assume here that non-relativistic primordial neutrinos in the state with $\zeta \approx -k$ are stable and their total number is stabilized at the late time universe.
Taking into account the homogeneity of the scalar field $\phi$ and Eq.(48), the result of the cosmological averaging of the constraint (46) can be represented in the form

$$
\frac{(b + k)\left(M^4e^{-2\alpha\phi/M_p} + V_1\right) - 2V_2}{(b - k)^2} = (b - k)^{1/2}(h_N - k)\mu_N \frac{n}{(\zeta + k)^2a^3} + \mathcal{O}(\zeta + k)
$$

(Similar to the toy model of Sec.7.1.1, the constraint (50) allows a "CLEP solution" where the decay of the neutrino density with the expansion of the universe is accompanied by approaching $\zeta \to -k$ in such a way that $(\zeta + k)^{-2} \propto a^3$, and then the r.h.s. of (50) as well as $\Lambda^{(N)}_{\text{dyn}} > \text{cosm.av.}$ in Eq.(49) approach constants. At the same time, $<T^{(N,\text{can})}_{\mu\nu} > \text{cosm.av.}$, Eq.(47), approaches zero. Therefore in the CLEP regime, the neutrino energy-momentum tensor $<T^{(N)}_{\mu\nu} > \text{cosm.av.}$ is reduced to the approaching constant (as $a(t) \to \infty$) non-canonical part of the neutrino energy-momentum tensor

$$
<T^{(N)}_{\mu\nu} > \text{cosm.av.} = -\tilde{g}_{\mu\nu}(b - k)^{1/2}(h_N - k)\mu_N \frac{n}{(\zeta + k)^2a^3} + \mathcal{O}(\zeta + k)
$$

Now we want to obtain the homogeneous scalar field $\phi(t)$ contribution $T^{(\phi)}_{\mu\nu}$ into the energy-momentum tensor in the CLEP regime of our model, i.e. when due to the balance dictated by the constraint, uniformly distributed non-relativistic neutrinos demand that the averaged value of $\zeta$ approaches $-k$. Comparing the latter with the expression for $\zeta = \zeta_0$ in the case of the absence of fermion matter, Eq.(31), we conclude that the presence of the cold neutrino gas in the CLEP regime essentially changes $V_{\text{eff}}(\phi; \zeta)$, Eq.(23). In fact, instead of $V_{\text{eff}}^{(0)}(\phi)$ as $\zeta = \zeta_0$, Eq.(32), we obtain in the case $\zeta \to -k$

$$
V^{(\text{CLEP})}_{\text{eff}}(\phi) \equiv V_{\text{eff}}(\phi; \zeta)|_{\zeta \to -k} = \frac{b\left(M^4e^{-2\alpha\phi/M_p} + V_1\right) - 2V_2}{(b - k)^2} + \mathcal{O}(\zeta + k).
$$

We would like to stress that this reorganization of the scalar field $\phi$ dynamics in the presence of the cold neutrino gas in the CLEP regime is the direct result of a general feature of TMT following from the constraint (29): in the space-time region occupied by a fermion, the scalar field $\zeta$ provides a certain balance between the energy densities of the scalar field $\phi$ and fermion.
It is very interesting that in the CLEP regime the role of the constraint becomes still more important. So far we have discussed separately the neutrino contribution into the total energy-momentum tensor, Eq. (51), and the homogeneous scalar field $\phi$ contribution which can be written in a concise form

$$T_{\mu \nu}^{(\phi)} = \phi,_{\mu} \phi,_{\nu} - \frac{1}{2} \tilde{g}_{\mu \nu} \tilde{g}^{\alpha \beta} \phi,_{\alpha} \phi,_{\beta} + \tilde{g}_{\mu \nu} V_{\text{eff}}^{(\text{CLEP})}(\phi).$$

(53)

However due to the constraint (50), the separation of the neutrino and scalar field $\phi$ contributions into the total energy-momentum tensor written in the form

$$< T_{\mu \nu}^{(\text{tot})} >_{\text{cosm.av.}} = < T_{\mu \nu}^{(N)} >_{\text{cosm.av.}} + T_{\mu \nu}^{(\phi)}$$

(54)

loses clarity in the CLEP regime\(^4\). In fact, *using the constraint (50) one can represent "the neutrino contribution" $< T_{\mu \nu}^{(N)} >_{\text{cosm.av.}}$ into $< T_{\mu \nu}^{(\text{tot})} >_{\text{cosm.av.}}$ in terms of the scalar field $\phi$ alone* and thus the total energy and pressure in the CLEP state can be written in an equivalent form where they are only $\phi$-dependent:

$$\rho_{\text{tot}} \equiv < T_{00}^{(\text{tot})} >_{\text{cosm.av.}} = \frac{1}{2} \dot{\phi}^2 + U_{\text{eff}}^{(\text{tot})}(\phi).$$

(55)

$$p_{\text{tot}} \equiv < T_{ii}^{(\text{tot})} >_{\text{cosm.av.}} = \frac{1}{2} \dot{\phi}^2 - U_{\text{eff}}^{(\text{tot})}(\phi).$$

(56)

Here the effective potential $U_{\text{eff}}^{(\text{tot})}(\phi)$ is the sum

$$U_{\text{eff}}^{(\text{tot})}(\phi) \equiv \Lambda + V_{q-\text{like}}(\phi),$$

(57)

of the effective cosmological constant

$$\Lambda = \frac{V_2 - k V_1}{(b - k)^2}$$

(58)

and the potential

$$V_{q-\text{like}}(\phi) = -\frac{k}{(b - k)^2} M^4 e^{-2\alpha \phi/M_p} + {\mathcal{O}} (\zeta + k).$$

(59)

\(^4\)Recall that the constraint is just the consistency condition of the equations of motion.
$\Lambda$ and $V_{q-like}(\phi)$ are positive if

$$V_2 - kV_1 > 0 \quad \text{and} \quad k < 0$$

that we will assume in what follows. Recall also the conditions (34) and (38).

Thus, similar to the absence of fermions case discussed in Sec.5, the evolution of the late time universe filled with the homogeneous scalar field $\phi$ and the cold gas of uniformly distributed non-relativistic neutrinos in the state with $\zeta \approx -k$ proceeds as it would be in the standard field theory model (non-TMT) including both the cosmological constant and the scalar field $\phi$ with the exponential potential but without fermions. In other words the role of the gas of uniformly distributed non-relativistic neutrinos in the CLEP state consists only in the change of the dark energy in comparison with the absence of fermions case of Sec.5. Namely, the dark energy in the CLEP regime is also associated only with the homogeneous scalar field $\phi$ (as if there were no neutrinos) but now with the effective potential $U^{(tot)}_{eff}(\phi)$ instead of $V^{(0)}_{eff}(\phi)$, Eq.(32), obtained for the universe filled only with the homogeneous scalar field $\phi$.

The remarkable result consists in the fact that

$$V^{(0)}_{eff}(\phi) - U^{(tot)}_{eff}(\phi) = \frac{\left[(b + k) \left( V_1 + M^4 e^{-2\alpha\phi/M_p} \right) - 2V_2 \right]^2}{4(b-k)^2 \left[b \left( V_1 + M^4 e^{-2\alpha\phi/M_p} \right) - V_2 \right]} > 0,$$

where the conditions (34) and (38) have been used. This means that the universe with the gas of uniformly distributed non-relativistic neutrinos in the CLEP state is energetically more preferable than the one in the absence of fermions case.

**B. Cosmological Solutions for the Late Time Universe in the CLEP Regime**

Cosmological equations for a spatially flat FRW universe filled with the homogeneous scalar field $\phi$ and the cold gas of uniformly distributed non-relativistic neutrinos with wave function $N'(t)$ in the regime with $\zeta \approx -k$ are the following

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_p^2} \rho_{tot}$$

(62)
\[ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{2\alpha k}{(b-k)^2 M_p} M^4 e^{-2\alpha \phi/M_p} + \mathcal{O} \left( (\zeta + k)e^{-2\alpha \phi/M_p} \right) = 0, \]  

(63)

\[ i\gamma^0 \left( \partial_t + \frac{3\dot{a}}{2a} \right) N'(t) - m(\zeta) N'(t) = 0 \]  

(64)

Here \( \rho_{\text{tot}} \) is given by Eq.(55). The simplest way to obtain Eq.(63) is to use the scalar field \( \phi \) equation in the form given by Eq.(30). In Eq.(64) for the neutrino in the spatially flat FRW universe we have taken into account that the fermionic contribution to the spin-connection (A8) turns into zero in the rest frame of the fermion.

The solution of the neutrino equation is straightforward

\[ N'(t) = \frac{C}{a^{3/2}} \left( 1 \right) 000 \cdot e^{-i \int m(\zeta(t)) dt} \]  

(65)

where \( m(\zeta(t)) \) is defined by the second equation in (45) and \( C \) is a constant.

In the CLEP regime \( \zeta + k \propto a^{-3/2} \) and therefore the correction term \( \mathcal{O} \left( (\zeta + k)e^{-2\alpha \phi/M_p} \right) \) in (63) becomes negligible at the late time. The effective potential \( U_{\text{eff}}^{(\text{tot})}(\phi) \) monotonically decays to the effective cosmological constant \( \Lambda \), Eq.(58), as \( \phi \to \infty \). Hence for the class of solutions where \( \phi \to \infty \) and \( \dot{\phi} \to 0 \) as \( a(t) \to \infty \), the asymptotic behavior of the universe is governed by the effective cosmological constant \( \Lambda \).

One should note that the needed smallness of \( \Lambda \) cannot be here explained by means of a see-saw mechanism in spite of the fact that \( \Lambda \) is bounded from above(61) by \( \Lambda^{(0)} \) for which a see-saw mechanism was possible (see the end of Sec.5, item (i)). In fact, if we want to realize a similar situation in the case \( V_2 < 0 \) and \( -kV_1 < |V_2| \) then we obtain \( \Lambda < 0 \) which is of course unsatisfactory result.

However if \( V_2 > 0 \) or alternatively \( V_2 < 0 \) and \( -kV_1 > |V_2| \), the needed smallness of \( \Lambda \) is achieved again through a choice of huge value(s) of the dimensionless parameter(s) \( b \) and/or \( k \), similar to what was discussed in item (ii) at the end of Sec.5.

One more possibility is to get \( \Lambda \) very small by tuning the parameters such that \( V_2 - kV_1 \) is very small. If \( V_2 - kV_1 = 0 \) then \( \phi \) becomes the regular quintessence field with an exponential
potential.

For illustration of what kind of solutions one can expect, let us take the particular value for the parameter $\alpha$, namely $\alpha = \sqrt{3}/8$. Then for the late time universe in the CLEP regime, when $\zeta$ is close enough to $-k$, the cosmological equations allow the following analytic solution:

$$\phi(t) = \frac{M_p}{2\alpha} \varphi_0 + \frac{M_p}{2\alpha} \ln(M_p t), \quad a(t) \propto t^{1/3} e^{\lambda t}, \quad (66)$$

where

$$\lambda = \frac{1}{M_p} \sqrt{\frac{\Lambda}{3}}, \quad e^{-\varphi_0} = \frac{2(b - k)^2 M_p^2}{\sqrt{3}|k| M^4} \sqrt{\Lambda}. \quad (67)$$

and $\Lambda$ is determined by Eq.(58). The mass of the neutrino in such CLEP state increases exponentially in time and its $\phi$ dependence is double-exponential:

$$m_{N|\text{clep}} \sim (\zeta + k)^{-1} \sim a^{3/2}(t) \sim t^{1/2} e^{\frac{4}{3}\lambda t} \sim \exp \left[ \frac{3\lambda e^{-\varphi_0}}{2M_p} \exp \left( \frac{2\alpha M_p}{\phi} \right) \right]. \quad (68)$$

**VIII. CONCLUSIONS AND DISCUSSION**

**A. Main Results**

In this paper we have shown that the features of a fermion in TMT depend on its density and therefore states of the fermion (in this generalized sense) may be very much different from what is known about regular fermions. This is the reason why generically we should start in TMT from a so-called primordial fermion. Only in the normal particle physics conditions, i.e. when the fermion energy density has huge values in comparison with the vacuum energy density, the primordial fermion behaves as the regular fermion. Both the decoupling of fermion from the quintessence-like scalar field and restoration of Einstein equations occur also in this regime.

\footnote{For some details of the model with $V_1 = V_2 = 0$ see Appendix.}
It turns out that besides states of the regular fermions, the primordial fermion can be in exotic states which might be of interest in cosmology and astrophysics\textsuperscript{6}. In particular, for states of the primordial fermion with energy density close to the dark energy density, TMT predicts very interesting properties, like for example: the non-canonical (proportional to the metric tensor) term of the energy-momentum tensor becomes important; the mass of the fermion depends on its density, etc. We call such states Cosmo-Low-Energy-Physics (CLEP) states.

At this stage of research we have found out the solution in the CLEP regime for the model of the spatially flat FRW universe filled with the homogeneous scalar field $\phi$ and the cold gas of uniformly distributed non-relativistic neutrinos. The main features of the CLEP regime are the following:

- In the CLEP state, i.e. as the neutrino density approaches zero and $\zeta \rightarrow -k$, the canonical contribution of the uniformly distributed non-relativistic neutrinos to the energy-momentum tensor becomes negligible in comparison with their non-canonical (proportional to the metric tensor) contribution.

- Due to the constraint (29), the above neutrino contribution to the energy-momentum tensor can be expressed in term of the scalar field $\phi$ alone. Hence the gas of the uniformly distributed non-relativistic neutrinos in the CLEP state behaves as a genuine dark energy.

- The original scalar field $\phi$ effective potential is also $\zeta$ dependent and thus the CLEP state of the uniformly distributed non-relativistic neutrinos influences in a fundamental way the scalar field $\phi$ contribution to the dark energy.

- The late time universe in the CLEP regime is governed by both a cosmological constant and a decaying exponential potential.

\textsuperscript{6}Exotic fermion fields of different nature has been studied in Ref. [27].
• The mass of the neutrino in the CLEP state increases according to

\[ m_{N|\text{clep}} \propto (N' N')^{-1/2} \propto a^{3/2}(t) \]  

(69)

The latter feature allows to make a crude estimation for the present day mass of the CLEP neutrinos. It is clear that transition of cold neutrinos to the CLEP regime can occur after their decoupling, i.e. at the temperature \( \sim 0.1\,MeV \). Hence the ratio of the present scale factor \( a_0 \) to the scale factor at the neutrino decoupling epoch \( a_{\text{dec}} \) is \( a_0 / a_{\text{dec}} \sim 10^9 \).

Assume that the neutrino mass right before entering the CLEP regime is not less than the mass of the regular neutrino. Then using Eq.(69) one can estimate \( m_{N|\text{clep,present}} \sim 10^{14} m_\nu \) where \( m_\nu \) is of the typical order of a mass of a regular neutrino. For \( m_\nu \sim (10^{-2} - 1)eV \) it gives the following estimation for masses of the CLEP state neutrinos at the present day universe as a result of the cosmological expansion:

\[ m_{N|\text{clep}} \sim (1 - 10^2)TeV. \]  

(70)

One can obtain also a crude estimation of a typical size of the space region where the wave function of the present CLEP neutrino state is non-zero. Taking the mass parameter \( m_\nu \sim 10^{-2}eV \) right before entering the CLEP regime, we estimate that the linear size of the neutrino at this stage is larger than its Compton wave length \( \sim 10^{-3}cm \). Then the cosmological expansion results in the linear size of the support of the wave function of the present CLEP neutrino state to be larger than \( 10^6 cm \).

We did not study yet the transition of the regular neutrinos (which constitute at least a fraction of the dark matter) into the CLEP state (where neutrinos behave as a dark energy). We expect that a possible mechanism responsible for initiating this transition is spreading of the regular neutrino wave packet that has an effect the decreasing of the neutrino density; the latter is a necessary element of the CLEP regime. Due to the constraint (29), the description of this spreading requires the consideration of a non linear quantum mechanics which makes the complete solution of this problem complicated enough.
Nevertheless our statement is that the described CLEP regime is favorable from the energetic point of view. In fact, from the two possibilities existing as $\overline{N}N' \to 0$ (one is the normal one with the neutrino energy density decreasing to zero; another - the CLEP regime), the CLEP state provides the lowest total energy density.

The possibility of the CLEP state does not require tuning of the dimensionless parameters $b, k, h_N$ and values of $V_1$ and $V_2$ in the action (16) as well as of the value of the (positive) integration constant $M^4$ in Eq.(12). The effect is also not sensitive to the value of $n$ determined by the total number of cold neutrinos entering the CLEP regime. However the scale symmetry (18) (reduced in the Einstein frame to the shift symmetry (20)) and its spontaneous breakdown by means of Eq.(12) have a decisive role in the structure of the dynamically generated effective potential.

It is also very important that the gauge couplings of the neutrino that may appear in more realistic models, does not affect the possibility of the CLEP state, i.e. we are at this point free to use either the active neutrino with non-abelian gauge interactions or a sterile (non standard model) neutrino.

It is interesting to compare the CLEP regime solution of Sec.7 (in the context of TMT) with the results of Sec.2 where possibilities of fermion contributions to the dark energy have been demonstrated in the framework of regular (non TMT) toy models. In these non-TMT models, contributions to the energy-momentum tensor proportional to the metric tensor are really generated but there are no reasons for the proportionality factors to approach non zero constant values while this occurs in the CLEP regime.
Finally we would like to outline few possible further developments of the CLEP regime solutions.

- **Interactions of the CLEP State Neutrinos and Cosmic Rays.**

\( \bar{\nu}' \bar{\nu}' \) which is a probability density for non-relativistic neutrino in the co-moving frame, is anomalously small in the CLEP state: \((\bar{\nu}' \bar{\nu}')|_{\text{clep}} \propto a^{-3}(t)\). Therefore it is very hard to observe the primordial neutrinos in the CLEP states. Furthermore, an attempt to detect the fermion in the CLEP state should be accompanied by its localization, i.e developing a large \( \bar{\nu}' \bar{\nu}' \) that destroys the condition for the existence of the CLEP state.

Using our estimation for masses of the CLEP state neutrinos at the present day universe (70) and taking into account their standard weak interaction, one can speculate about possibility of decays of the CLEP state neutrinos. Smallness of the CLEP neutrino wave function leads to a very strong damping of the amplitude of the decay. However, propagator of the gauge boson exhibits a resonance enhancement of the amplitude of the decay of the the neutrino in the CLEP state \( N_{\text{clep}} \) when the energy \( E_1 \) of the charged lepton \( l^- \) produced in the vertex \( N_{\text{clep}} l^- W^+ \) (or \( N_{\text{clep}} \nu Z \)) satisfies \( E_1 \approx (m_N|_{\text{clep}}^2 - M_W^2)/(2m_N|_{\text{clep}}) \). This becomes possible when the mass of the CLEP state neutrinos \( m_N|_{\text{clep}} \) is larger than masses of the gauge bosons \( W^\pm \) and \( Z \).

In addition, the width of such weak decays of \( N_{\text{clep}} \) into three particles is enhanced because the phase space is proportional to \((m_N|_{\text{clep}})^5\). Estimations (70) for masses of the CLEP state neutrinos at the present day universe allow to expect that decays of the CLEP state neutrinos might be the origin of TeV cosmic rays. These ideas and preliminary estimations show that decays of the CLEP neutrino (both with masses below \( M_W, M_Z \), producing low energy cosmic rays, and with masses above \( M_W, M_Z \), producing TeV and even higher energy cosmic rays) deserve detailed study. In this
sense the CLEP states exhibit simultaneously new physics at very low densities and for very high particle masses.

• **Domain Structure of the Universe.** Existence of the CLEP states suggests the possibility of a domain structure of the dark energy. In fact, if for some reasons there are space-time regions empty of fermionic matter then the dark energy in such regions is governed by the scalar field with the potential (32). Let us call such regions as type I regions. As we have seen, the energy density of the type I regions is bigger than the energy density in the regions with CLEP state neutrinos (see Eq.(61)). Let us call the latter as type II regions. One can think of a typical situation when the bubble of the type II region is situated in the midst of the type I region. Then the difference of the pressures between the regions I and II causes the CLEP bubble to expand replacing the vacuum of the region I by the CLEP state. Due to this mechanical effect the size of the CLEP bubble should expand faster than the universe itself. Therefore the above estimations for the mass, Eq. (70), and size (see paragraph after Eq. (70)) of the present CLEP neutrino resulting from the cosmological expansion could be enhanced due to this mechanical effect.

• The difference between the dark energy states in regions I and II is related to the different values of the scalar $\zeta$ which is determined by the constraint (29). It is interesting that the electromagnetic field does not enter the constraint. Therefore, the dark energy bubbles including walls between them are transparent for radiation.

• **Super-Acceleration as a Possible Consequence of Decays of the CLEP State Neutrinos.** Let us consider a model of the present day universe containing both the regions I and the regions II and assume that the cosmologically averaged dark energy density $\rho_{d.e.}$ is a dominant fraction. The magnitude of $\rho_{d.e.}$ depends on the number of the regions II, i.e. on the number of the CLEP state neutrinos. Therefore the possibility of the discussed above decays of the CLEP state neutrinos may result in a growth of $\rho_{d.e.}$ that
exhibits itself via super-acceleration of the universe. It is interesting that these decays may become non negligible due to the mentioned above resonance enhancement of the amplitude only at a late enough epoch since the mass of the CLEP state neutrino should grow above masses of the gauge bosons. Hence one can expect a certain correlation between the cosmic rays physics on the one hand and the dark energy physics on the other hand.

- **Inhomogeneous CLEP States and Dark Matter.** The research presented in this paper is the first step in studying the CLEP states: here we have restricted ourselves in effects of the CLEP state in homogeneous cosmology. Possible inhomogeneous (local) effects of the CLEP states and their connection with dark matter will be a subject of future research.

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**APPENDIX A: CONNECTION IN THE ORIGINAL AND EINSTEIN FRAMES**

We present here what is the dependence of the spin connection $\omega_{\mu}^{ab}$ on $e^a_{\mu}$, $\zeta$, $\Psi$ and $\overline{\Psi}$. Varying the action (16) with respect to $\omega_{\mu}^{ab}$ and making use that

$$R(V, \omega) \equiv -\frac{1}{4\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} e^c_{\alpha} e^d_{\beta} R_{\mu\nu}^{ab}(\omega) \quad (A1)$$
we obtain
\begin{align}
\varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} & \left[ (\zeta + b_g) e^c_a D_\nu e^d_\beta + \frac{1}{2} \varepsilon^c_\alpha e^d_\beta \left( \xi_\nu + \frac{\alpha}{M_p} \phi_\nu \right) \right] \\
& + \frac{\kappa}{4} \sqrt{-g} (\zeta + k) \varepsilon^{\mu} \varepsilon_{abcd} \overline{\Psi} \gamma^5 \gamma^d \Psi = 0,
\end{align}
(A2)
where
\begin{align}
D_\nu e_{a\beta} & \equiv \partial_\nu e_{a\beta} + \omega^d_\nu e_{d\beta} \quad (A3)
\end{align}

The solution of Eq. (A2) is represented in the form
\begin{align}
\omega^{ab}_\mu = \omega^{ab}_\mu (e) + K^{ab}_\mu (e, \overline{\Psi}, \Psi) + K^{ab}_\mu (\zeta, \phi) 
\end{align}
(A4)
where
\begin{align}
\omega^{ab}_\mu (e) = e^a_\alpha e^b_\nu \{ \alpha \}_{\mu\nu} - e^b_\nu \partial_\mu e^a_\nu \quad (A5)
\end{align}
is the Riemannian part of the spin-connection,
\begin{align}
K^{ab}_\mu (e, \overline{\Psi}, \Psi) = \frac{\kappa}{8} \frac{\zeta + k}{\zeta + b_g} \eta_{cn} e_{d\mu} \varepsilon^{abcd} \overline{\Psi} \gamma^5 \gamma^n \Psi
\end{align}
(A6)
is the fermionic contribution that differs from the familiar one [28] by the factor $\frac{\zeta + k}{\zeta + b_g}$ and
\begin{align}
K^{ab}_\mu (\zeta, \phi) = \frac{1}{2 (\zeta + b_g)} \left( \zeta_\alpha + \frac{\alpha}{M_p} \phi_\alpha \right) (e^a_\mu e^{b\alpha} - e^b_\mu e^{a\alpha})
\end{align}
(A7)
is the non-Riemannian part of the spin-connection originated by specific features of TMT.

In the Einstein frame, i.e. in terms of variables defined by Eq.(19), the spin-connection read
\begin{align}
\omega^{ab}_\mu (\tilde{e}) + \frac{\kappa}{8} \eta_{cn} \tilde{e}_{d\mu} \varepsilon^{abcd} \overline{\Psi} \gamma^5 \gamma^n \Psi' \quad (A8)
\end{align}
which is exactly the spin-connection of the Einstein-Cartan space-time [28] with the vierbein $\tilde{e}_\mu^a$. 

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APPENDIX B: THE CLEP REGIME COSMOLOGICAL SOLUTIONS IN A MODEL WITHOUT EXPLICIT POTENTIALS

In Ref. [22] we have studied scalar field $\phi$ cosmology (without fermions) in a model without potentials in the action, i.e. $V_1 = V_2 = 0$ in (16). In the framework of the models with $V_1 = V_2 = 0$, in this Appendix we are going to discuss cosmological solutions in the CLEP regime.

In the model with $V_1 = V_2 = 0$, the constraint (29) is reduced to the following:

$$\frac{(b - \zeta)}{(\zeta + b)^2} M^4 e^{-2\alpha \phi/M} = \Lambda^{(\text{ferm})}_{\text{dyn}}$$

(A1)

A. The universe filled with the homogeneous scalar field $\phi$ alone.

Instead of the results of Sec.5 we obtain:

- the constraint (A1) yields $\zeta = b$;
- the cosmological constant $\Lambda^{(0)} = 0$
- the scalar field effective potential

$$V_{\text{eff}}^{(0)}(\phi)|_{V_1 = V_2 = 0} = \frac{M^4}{4b} e^{-2\alpha \phi/M}$$

(A2)

is generated only through the spontaneous breakdown of the shift symmetry (20) by means of Eq.(12).

B. The universe filled with the homogeneous scalar field $\phi$ and a cold gas of uniformly distributed non-relativistic neutrinos.

In the CLEP regime, i.e. as $\zeta \to -k$, Eqs.(47), (48) and (49) hold as well. However, instead of Eq.(50), the averaged constraint reads now

$$\frac{b + k}{(b - k)^2} M^4 e^{-2\alpha \phi/M} = (b - k)^{1/2} (h_N - k) \mu_N \frac{n}{(\zeta + k)^2 a^3} + O \left( (\zeta + k) e^{-2\alpha \phi/M} \right).$$

(B3)

Due to the constraint (??) the correction to the main term of $<T^{(N)}_{\mu\nu}>_{\text{cosm.av.}}$, Eq.(51), is now different:
\[ <T_{\mu\nu}^{(N)} >_{\text{cosm.av.}} = -\tilde{g}_{\mu\nu}(b - k)^{1/2}(h_N - k)\mu_N \frac{n}{(\zeta + k)^2a^3} + \mathcal{O} \left((\zeta + k)e^{-2\alpha\phi/M_p}\right). \]  

(B4)

Similar change of the correction term to \( V_{eff}^{(\text{CLEP})}(\phi) \), Eq.(52), takes place now:

\[
V_{eff}^{(\text{CLEP})}(\phi) = \frac{b \left(M^4 e^{-2\alpha\phi/M_p} + V_1\right) - V_2}{(b - k)^2} + \mathcal{O} \left((\zeta + k)e^{-2\alpha\phi/M_p}\right).
\]

(B5)

Keeping the definitions (53) and (54) and using the constraint \( ?? \) one can again represent the neutrino contribution into \( <T_{\mu\nu}^{(\text{tot})} >_{\text{cosm.av.}} \) in terms of the scalar field \( \phi \) alone and thus the total energy and pressure in the CLEP regime can be written in an equivalent form given by Eqs.(55) and (56) where now the cosmological constant equals zero and \( U_{eff}^{(\text{tot})}(\phi) \) reads

\[
U_{eff}^{(\text{tot})}(\phi) \equiv -\frac{k}{(b - k)^2} M^4 e^{-2\alpha\phi/M_p} + \mathcal{O} \left((\zeta + k)e^{-2\alpha\phi/M_p}\right).
\]

(B6)

Using Eqs.(B2) and (B6) we obtain the result similar to Eq.(61):

\[
V_{eff}^{(0)}(\phi) - U_{eff}^{(\text{tot})}(\phi) \equiv \frac{(b + k)^2}{4b(b - k)^2} M^4 e^{-2\alpha\phi/M_p} > 0.
\]

(B7)

Cosmological equations for a spatially flat FRW universe filled with the homogeneous scalar field \( \phi \) and the cold gas of uniformly distributed non-relativistic neutrinos in the state with \( \zeta \to -k \) coincide in the form with Eqs.(62) and (63) where \( \rho_{\text{tot}} \) is given by Eq.(55) with \( U_{eff}^{(\text{tot})}(\phi) \) given by Eq.(B6). Neglecting the corrections \( \propto (\zeta + k)e^{-2\alpha\phi/M_p} \) we deal with the quintessence field with an exponential potential. The appropriate solution is:

\[
\phi(t) = \frac{M_p}{2\alpha}\varphi_0 + \frac{M_p}{\alpha} \ln(M_p t); \quad a \propto t^{1/2\alpha},
\]

(B8)

where

\[
e^{-\varphi_0} = \frac{(b - k)^2}{4|k|\alpha^4}(3 - 2\alpha^2) \left(\frac{M_p}{M}\right)^4.
\]

(B9)

This solution describes an accelerating expansion if \( \alpha^2 < \frac{1}{2} \). It is easy to check that in this case the corrections \( \propto (\zeta + k)e^{-2\alpha\phi/M_p} \) we have neglected, approach zero faster than the main terms of the cosmological equations and the mass of the neutrino in such CLEP regime increases in time as

\[
m_N|_{\text{CLEP}} \sim (\zeta + k)^{-1} \sim t^{\frac{3}{2\alpha^2} - 1} > t^{1/2}.
\]

(B10)
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