Energy Constraints and $F(T, T_G)$ Cosmology

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Abstract

The present paper is elaborated to discuss the energy condition bounds in a modified teleparallel gravity namely $F(T, T_G)$, involving torsion invariant $T$ and contribution from a term $T_G$, the teleparallel equivalent of the Gauss-Bonnet term. For this purpose, we consider flat FRW universe with matter contents as perfect fluid. We formulate the SEC, NEC, WEC and DEC in terms of some cosmic parameters including Hubble, deceleration, jerk and snap parameters. By taking two interesting models for $F(T, T_G)$ and some recent limits of these cosmic parameters, we explore the constraints on the free parameters present in both assumed models. We also discuss these constraints graphically in terms of cosmic time by taking power law cosmology into account.

Keywords: $F(T, T_G)$ theory; Raychaudhuri equation; Energy bounds.
PACS: 04.50.-h; 04.50.Kd; 98.80.Jk; 98.80.Cq.

1 Introduction

One of the most revolutionary investigation of the previous century is the accelerated expanding behavior of cosmos that motivates the scientists in a new
direction. This interesting fact is substantiated by numerous observational probes [1]-[7] etc. and leads to the existence of a new cryptic dominant ingredient in the cosmic matter distribution referred as dark energy (DE). In order to comprehend the nature of this new sort of energy, numerous attempts are made by incorporating some new terms in the Einstein-Hilbert Lagrangian density either in the matter sector or the gravitational sector of the action. As a result, these techniques offered a huge group of DE models including Chaplygin gas [8], cosmological constant [9], tachyon fields [10], quintessence [11], k-essence [12], modified theories like $f(R)$ gravity [13], Gauss-Bonnet gravity [14], $f(T)$ theory [15], $f(R,T)$ gravity [16] and scalar-tensor theories [17] that have numerous distinct and interesting cosmological applications.

Another interesting modification of Einstein’s relativity is obtained by introducing torsional formulation (torsion scalar $T$ which is obtained by contraction of torsion tensor) to explain the gravitational effects instead of curvature scalar [18]. In such a gravitational framework, Lagrangian density includes curvature less Weitzenb"ock connection as a replacement of torsion less Levi-Civita connection. This theory is referred as TEGR (teleparallel equivalent of general relativity) and has widespread applications in cosmology. Another comprehensive form of this theory has been proposed in literature [19] by replacing torsion scalar with a general function $f(T)$, known as $f(T)$ theory of gravity. Numerous significant cosmological aspects of this theory has been discussed in literature [19, 20]. Its another useful version is obtained by considering higher-torsion corrections just like the case of Gauss-Bonnet term [21] (arising from higher-curvature corrections), Lovelock combinations [22], Weyl combinations [23]. Based on this concept, Kofinas and Saridakis [24, 25] proposed a novel theory namely $F(T,G)$ gravity and then its generalized form $F(T,T,G)$ gravity and they also discussed its cosmological significance.

Energy condition bounds are used to explore the constraints on the free parameters arising from different DE models. These constraints has been discussed in various contexts like $f(R)$ gravity [26], $f(T)$ theory [27], $f(G)$ theory [28], $f(R)$ gravity with nonminimal interaction with matter [29], $f(R, L_m)$ gravity [30] and Brans-Dicke theory [31]. In this regard, Sharif and Saira [32] have discussed the energy conditions in the most general scalar-tensor gravity with dynamical equations involving second-order derivatives of scalar field for perfect fluid FRW geometry. Sharif and Zubair [33] have explored these constraints in a general theory $f(R,T)$ theory involving the trace of energy-momentum tensor. They also discussed the stability criteria
for such configuration using power law cosmology. They have also examined some models using energy inequalities for $f(R,T,R_{\mu \nu}T^{\mu \nu})$ gravity \[34\]. Recently, we have examined the energy bounds in a modified theory based on the non-minimal interaction of torsion scalar and perfect fluid matter using power law form of FRW cosmology \[35\]. We have derived the general inequalities involving different cosmic parameters and discussed them graphically.

In the present work, we are interested to discuss the energy constraints in $F(T,T_G)$ gravity using FRW universe model filled with perfect fluid matter. We derive these constraints in terms of cosmic parameters for two different proposed models of $F(T,T_G)$. The paper is designed in this layout. In the next section, we provide a general introduction of $F(T,T_G)$ theory and discuss the basic formulation of energy bounds. Section 3 is devoted to study these constraints for two different models of $F(T,T_G)$. Here we provide the graphical illustration of the obtained inequalities. Finally, we summarize the whole discussion.

2 Introduction to $F(T, T_G)$ Cosmology and General Formulation of Energy Bounds

In this section, we briefly explain some basic ingredients of TEGR and hence of $F(T, T_G)$. Here we also discuss the basic formulation of energy constraints. In tangent components, torsion and curvature tensor are defined as

\begin{align*}
T^a_{bc} & = \omega^a_{cb} - \omega^a_{bc} - C^a_{bc}, \\
R^a_{bcd} & = \omega^a_{bd,c} - \omega^a_{bc,d} + \omega^e_{bd}\omega^a_{ec} - \omega^e_{bc}\omega^a_{ed} - C^e_{cd}\omega^a_{be},
\end{align*}

where the source of parallel transportation, connection 1-form $\omega^a_{\mu}(x^\mu)$ in terms of vielbein field is given by $\omega^a_{\mu} = \omega^a_{\mu \nu}d\xi^\mu = \omega^a_{\nu}e^\nu$, while $C^e_{ab} = e^e_a e^b_\mu (e^\mu_{\nu, \mu} - e^\mu_{\nu, \mu})$ denote the structure coefficients arising from the vielbein commutation defined by

$$[e_a, e_b] = C^e_{ab} e_e.$$ 

Also, $g_{\mu \nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$, where $\eta_{ab}$ is the Minkowski metric. The contorsion tensor is defined in terms of torsion tensor as follows

$$K_{abc} = \frac{1}{2}(T_{cab} - T_{bca} - T_{abc}) = -K_{bac}. \tag{3}$$
In order to be consistent with the condition $R_{\text{bcd}} = 0$ (teleparallelism condition), we express the Weitzenböck connection as follows

$$\tilde{\omega}_\mu^\lambda = e_a^\lambda e_{\mu,\nu}^a,$$

while in terms of Levi-Civita connection, the Ricci scalar $R$ is given by

$$e\bar{R} = -eT + 2(eT^\mu_\nu)_\mu,$$

where

$$e = \det(e^\mu_a) = \sqrt{|g|}, \quad T = \frac{1}{4} T_{\mu\nu\lambda} + \frac{1}{2} T^{\mu\nu\lambda}T_{\lambda\mu\nu} - T^{\nu\mu}T^\lambda_{\nu\mu}.$$ 

Consequently, the Lagrangian density describing TEG in D-dimensions is given by

$$S_{\text{tel}} = -\frac{1}{2\kappa_D^2} \int_M d^Dx e T.$$  \hfill (4)$$

In a recent paper [24], teleparallel equivalent of Gauss-Bonnet theory has been proposed involving a new torsion scalar $T_G$, where, in Levi-Civita connection, the Gauss-Bonnet term is defined by

$$e\bar{G} = eT_G + \text{total diverg}$$ \hfill (5)$$

and the corresponding action takes the following form

$$S_{\text{tel}} = -\frac{1}{2\kappa_D^2} \int_M d^Dx e T_G.$$  \hfill (6)$$

Since both theories $f(T)$ and $f(T_G)$ arise independently, therefore a comprehensive theory involving both $T$ and $T_G$ as basic ingredient has been proposed by Kofinas and Saridakis defined by the following action

$$S_{\text{tel}} = -\frac{1}{2\kappa_D^2} \int_M d^Dx F(T, T_G).$$  \hfill (7)$$

In some certain limits of the function $F(T, T_G)$, other theories like GR, TEG, Einstein-Gauss-Bonnet theory etc. can be discussed. Energy constraints have many useful applications in GR as well as in modified gravity theories (discussion of various cosmological geometries). These
inequalities are firstly formulated in the context of GR for the derivation of some general results involving strong gravitational fields. In GR, four types of energy constraints are formulated using a well-known geometrical results referred as Raychaudhuri equation (explaining the dynamics of matter bits). These constraints are labeled as WEC, DEC, NEC and SEC. In a spacetime manifold, Raychaudhuri equation provides the temporal evolution of expansion scalar as a linear combination of Ricci tensor $R_{\mu \nu}$, shear tensor $\sigma_{\mu \nu}$ and rotation $\omega_{\mu \nu}$ given by

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu \nu} \sigma^{\mu \nu} + \omega_{\mu \nu} \omega^{\mu \nu} - R_{\mu \nu} u^\mu u^\nu,$$  

(8)

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu \nu} \sigma^{\mu \nu} + \omega_{\mu \nu} \omega^{\mu \nu} - R_{\mu \nu} k^\mu k^\nu,$$  

(9)

where $u^\mu$ and $k^\nu$ denote the tangent vectors to timelike and lightlike curves in the congruence. The attractive nature of gravity makes the congruence geodesic convergent (congruence gets closer to each other) and therefore leads to $\frac{d\theta}{d\tau} < 0$. Further, in some certain limits [36], the quadratic terms can be ignored and hence leads to inequalities

$$R_{\mu \nu} u^\mu u^\nu \geq 0, \quad R_{\mu \nu} k^\mu k^\nu \geq 0.$$  

These inequalities can be further formulated in terms of energy-momentum tensor and its trace by the inversion of the gravitational field equations as follows

$$(T_{\mu \nu} - \frac{T}{2} g_{\mu \nu}) u^\mu u^\nu \geq 0, \quad (T_{\mu \nu} - \frac{T}{2} g_{\mu \nu}) k^\mu k^\nu \geq 0.$$  

(10)

For the ideal case, i.e., perfect fluid matter, these inequalities provide the SEC, NEC, WEC, DEC as follows:

- **NEC**: $\rho + P \geq 0$,
- **SEC**: $\rho + P \geq 0, \quad \rho + 3P \geq 0$,
- **WEC**: $\rho \geq 0, \quad \rho + P \geq 0$,
- **DEC**: $\rho \geq 0, \quad \rho \pm P \geq 0$.  

(11)

In case of modified theories of gravity, these constraints can be defined by simply replacing the $\rho$ by $\rho_{eff}$ and $P$ by $P_{eff}$ with the assumption that the total matter contents behave like perfect fluid.
3 Energy Constraints in $F(T, T_G)$ Cosmology

Here we first formulate the gravitational field equations corresponding to the action (7) for FRW geometry with perfect fluid matter and then formulate the energy conditions for such a configuration. In the presence of matter sector, the action (7) takes the following form

$$S_{tel} = -\frac{1}{2\kappa^2_D} \int_M d^D x e F(T, T_G) + S_m.$$  \hfill (12)

We consider the flat FRW universe model with $a(t)$ as expansion radius given by

$$ds^2 = -dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).$$  \hfill (13)

The diagonal vierbein and the dual vierbein for this metric are

$$e^a_\mu = diag(1, a(t), a(t), a(t)),
\quad e^\mu_a = (1, a^{-1}(t), a^{-1}(t), a^{-1}(t)),$$

while the corresponding determinant is given by $e = a(t)^3$. The torsion scalar and Gauss-Bonnet equivalent term $T_G$ for this geometry in terms of Hubble parameter $H = \frac{\dot{a}}{a}$ are

$$T = 6H^2, \quad T_G = 24H^2(\dot{H} + H^2).$$ \hfill (14)

The gravitational field equations corresponding to the action (14) for this geometry are given by

$$F - 12H^2 F_T - T_G F_{T_G} + 24H^3 \dot{F}_{T_G} = 2\kappa^2 \rho_m,$$ \hfill (15)

$$F - 4(\dot{H} + 3H^2) F_T - 4H \dot{F}_T - T_G F_{T_G} + \frac{2}{3H} T_G \dot{F}_{T_G} + 8H^2 \ddot{F}_{T_G} = -2\kappa^2 P_m,$$ \hfill (16)

where $\rho_m$ and $P_m$ indicates the density and pressure of ordinary matter, respectively described by the ideal case of energy-momentum tensor given as follows

$$T_{\mu\nu} = (\rho_m + P_m) u_\mu u_\nu - P_m g_{\mu\nu}.$$ \hfill (17)
and $\kappa$ is the gravitational coupling constant. The gravitational field equations can also be written as

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_{DE}),$$ (18)

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_m + P_m + \rho_{DE} + P_{DE}),$$ (19)

where $\rho_{DE}$ and $P_{DE}$ are the density and pressure of dark energy, respectively given by

$$\rho_{DE} = \frac{1}{2\kappa^2}[6H^2 - F + 12H^2 F_T + T_G F_{TG} - 24H^3 \dot{F}_{TG}],$$ (20)

$$P_{DE} = \frac{1}{2\kappa^2}[-2(2\dot{H} + 3H^2) + F - 4(\dot{H} + 3H^2)F_T - 4H \dot{F}_T - T_G F_{TG} + \frac{2}{3H}T_G \dot{F}_{TG} + 8H^2 \ddot{F}_{TG}].$$ (21)

The above gravitational field equations can also be rewritten as follows

$$3H^2 = \kappa^2 \rho_{eff}, \quad 2\dot{H} = -\kappa^2 (\rho_{eff} + P_{eff}),$$ (22)

where $\rho_{eff} = \rho_m + \rho_{DE}$ and $P_{eff} = P_m + P_{DE}$. Furthermore, the derivatives are defined as

$$\dot{F}_T = F_T \dot{T} + F_{TT}T_G \dot{T}_G, \quad \dot{F}_{TG} = F_{TT} \dot{T}_G + F_{TGT} T_G \dot{T}_G,$$

$$\ddot{F}_{TG} = F_{TT} \ddot{T} + \frac{F_{TTT}}{2} T_G \ddot{T}_G + \frac{F_{TTG}}{2} T_G \ddot{T}_G + F_{TTG} \ddot{T}_G + F_{TGG} \dddot{T}_G + F_{TGTG} \dddot{T}_G,$$

with $F_{TT}, F_{TTG}, ...$ represent the second and higher-order differentiation with respect to $T$ and $T_G$. The energy bounds for any modified theories of gravity are defined in terms of effective density and pressure as follows:

**NEC**: $\rho_{eff} + P_{eff} \geq 0$,  
**SEC**: $\rho_{eff} + P_{eff} \geq 0$, $\rho_{eff} + 3P_{eff} \geq 0$,  
**WEC**: $\rho_{eff} \geq 0$, $\rho_{eff} + P_{eff} \geq 0$,  
**DEC**: $\rho_{eff} \geq 0$, $\rho_{eff} \pm P_{eff} \geq 0$. (25)
By inserting the corresponding values, the distinct inequalities in these energy constraints in terms of $T$, $T_G$ and $F$ can be written as

$$\rho_m + \frac{1}{2\kappa^2}[6H^2 - F + 12H^2F_T + T_G F_{TG} - 24H^3\dot{F}_{TG}] \geq 0, \quad (26)$$

$$\rho_m + P_m + \frac{1}{2\kappa^2}[-24H^3\ddot{F}_{TG} - 4\dot{H} - 4\dot{H} F_T - 4\dot{H} \dot{F}_T + \frac{2}{3\dot{H}}T_G \dddot{F}_{TG} + 8H^2\dddot{F}_{TG}] \geq 0, \quad (27)$$

$$\rho_m + P_m + \frac{1}{2\kappa^2}[-12H^2 - 2F - 2T_G F_{TG} - 24H^2F_T - 24H^3\dot{F}_{TG} - 12\dot{H} F_T - 12\dot{H} \dddot{F}_{TG} + \frac{2}{\dot{H}}T_G \dddot{F}_{TG} + 24H^2\dddot{F}_{TG}] \geq 0, \quad (28)$$

$$\rho_m - P_m + \frac{1}{2\kappa^2}[12H^2 - 2F + 24H^2F_T + 2T_G F_{TG} - 24H^3\dot{F}_{TG} + 4\dot{H} + 4\dot{H} F_T - \frac{2}{3\dot{H}}T_G \dddot{F}_{TG} - 8H^2\dddot{F}_{TG}] \geq 0. \quad (29)$$

We describe these conditions in terms of some cosmic parameters like deceleration parameter, jerk and snap parameters defined in terms of Hubble parameter by the following relations

$$q = -(1 + \frac{\dot{H}}{H^2}), \quad r = 2q^2 + q - \frac{\dot{q}}{\dot{H}}, \quad s = \frac{(r - 1)}{3(q - 1/2)}.$$

The first and higher order time rates of Hubble parameter can be expressed in terms of these parameters by the following relations

$$\ddot{H} = -H^2(1+q), \quad \dot{H} = H^3(j+3q+2), \quad \dddot{H} = H^4(s-4j-3q(q+4)-6). \quad (30)$$

The first and second-order time rates of torsion scalar $T$ and its Gauss-Bonnet equivalent term $T_G$ are given by

$$\dot{T} = 12H\dddot{H}, \quad \dot{T}_G = 24H^2(\dddot{H} + 2H\dddot{H}) + 48H\dddot{H}(\dot{H} + H^2), \quad (31)$$

$$\dddot{T} = 12(\dddot{H}^2 + H\dddot{H}), \quad \dddot{T}_G = 48\dddot{H}^3 + 144H\dddot{H}\dddot{H} + 288\dddot{H}^2H^2 + 24H^2\dddot{H} + 96H^3\dddot{H}. \quad (32)$$
In terms of cosmic parameters and recent value of Hubble parameter $H_0$, the terms $T$, $T_G$ and their derivatives can be expressed as follows

\begin{align*}
T &= 6H_0^2, \quad T_G = -24qH_0^1, \quad \dot{T} = 12H_0^3(1 + q), \quad (33) \\
\ddot{T} &= 12H_0^3(1 + q)^2 + 12H_0^1(1 + q), \quad (34) \\
\dot{T}_G &= 48H_0^5(1 + q)^2 - 96H_0^5(1 + q) + 24H_0^5(j + 3q + 2), \quad (35) \\
\ddot{T}_G &= \ -48H_0^6(1 + q)^3 - 144H_0^6(1 + q)(j + 3q + 2) + 288H_0^6(1 + q)^2 \\
&\quad + 24H_0^6(s - 4j - 3q(q + 4) - 6) + 96H_0^6(j + 3q + 2). \quad (36)
\end{align*}

In the following, we consider two particular models of $F(T, T_G)$ functions and discuss the corresponding constraints on the free parameters present in these models.

### 3.1 Model-1

Here, we consider the model

\[ F(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|}, \quad (37) \]

where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are free parameters to be constrained. For this model, the derivatives can be written as

\begin{align*}
\dot{F}_T &= \left[\{-\frac{1}{2} \beta_1 \beta_2 T(T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2}(|T_G|)^{-1/2}\} \dot{T} \right. \\
&\quad \left. + \frac{\alpha_2}{2}(|T_G|)^{-1/2} \dot{T}_G\right], \quad (38) \\
\dot{F}_{T_G} &= \left[\{-\frac{1}{2} \beta_1 \beta_2 T(T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2}(|T_G|)^{-1/2}\} \dot{T} \right. \\
&\quad \left. - \frac{\alpha_2}{4}(|T_G|)^{-3/2} \dot{T}_G\right], \quad (39) \\
\ddot{F}_{T_G} &= \left\{-\frac{3}{2} \beta_1 \beta_2 T(T^2 + \beta_2 T_G)^{-5/2} + \frac{\alpha_2}{2}(|T_G|)^{-1/2}\} \dot{T} \right. \\
&\quad \left. + \frac{3}{8} \beta_1 \beta_2 T(T^2 + \beta_2 T_G)^{-5/2} \dot{T}_G \right. \\
&\quad \left. + \frac{3}{8} \alpha_2 T \dot{T}_G + \{\frac{3}{8} \beta_1 \beta_2 T(T^2 + \beta_2 T_G)^{-5/2} \dot{T}_G \right. \\
&\quad \left. + \frac{3}{8} \alpha_2 T \dot{T}_G \right\} \ddot{T}_G. \quad (40)
\end{align*}
Inserting all these values in energy constraints (26)-(29), we get the following inequalities:

\[
\rho_{\text{eff}} + P_{\text{eff}} \geq 0 \Rightarrow \rho^m + P^m + \frac{1}{2\kappa^2} \left( \frac{2T_G}{3H} - 24H^3 \right) \left\{ \left( -\frac{1}{2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} \right)^2 + \frac{\alpha_2 T}{4} (|T_G|)^{-3/2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} \right\} \\
+ \frac{\alpha_2}{2} (|T_G|)^{-1/2} \dot{T} + \left( -\frac{\beta_1 \beta_2}{4} (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2 T}{4} (|T_G|)^{-3/2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} \right) \dot{T}_G \\
- 4\dot{H} - 4H \left\{ -1 + \beta_1 T (T^2 + \beta_2 T_G)^{-1/2} + 2\alpha_1 T + \alpha_2 (|T_G|)^{1/2} \right\} \\
- 4H \left\{ (\beta_1 (T^2 + \beta_2 T_G)^{-1/2}) - \beta_1 T^2 (T^2 + \beta_2 T_G)^{-3/2} + 2\alpha_1 \dot{T} \right\} \\
+ \left( -\frac{1}{2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2} (|T_G|)^{-1/2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} \right) \dot{T}_G \\
+ 8H^2 \left\{ \left( -\frac{1}{2} \beta_1 \beta_2 (T^2 + \beta_2 T_G)^{-3/2} + \frac{3}{2} \beta_1 \beta_2 T^2 (T^2 + \beta_2 T_G)^{-5/2} \right) \dot{T} \right\} \\
+ \left( 3 \beta_1 \beta_2 (T^2 + \beta_2 T_G)^{-3/2} + \frac{3\alpha_2 T}{8} (|T_G|)^{-5/2} \dot{T}_G \right) \right\} \geq 0, \quad (41) \\
\rho_{\text{eff}} \geq 0 \Rightarrow \rho^m + \frac{1}{2\kappa^2} [6H^2 - (-T + \beta_1 (T^2 + \beta_2 T_G)^{1/2}) \\
+ \alpha_1 T^2 + \alpha_2 (|T_G|)^{1/2}] + 12H^2 \left\{ -1 + \beta_1 T (T^2 + \beta_2 T_G)^{-1/2} + 2\alpha_1 T \right\} \\
+ \alpha_2 (|T_G|)^{1/2} + T_G \left( \frac{1}{2} \beta_1 \beta_2 (T^2 + \beta_2 T_G)^{-1/2} + \frac{\alpha_2 T}{2} (|T_G|)^{-1/2} \right) \\
- 24H^3 \left\{ \left( -\frac{1}{2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2} (|T_G|)^{-1/2} \dot{T} \right) \right\} \\
+ \left( -\frac{\beta_1 \beta_2}{4} (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2 T}{4} (|T_G|)^{-3/2} \dot{T}_G \right) \right\} \geq 0, \quad (42) \\
\rho_{\text{eff}} + 3P_{\text{eff}} \geq 0 \Rightarrow \rho^m + 3P^m + \frac{1}{2\kappa^2} [-12H^2 + 2(-T + \beta_1 (T^2 + \beta_2 T_G)^{1/2}) \\
+ \alpha_1 T^2 + \alpha_2 (|T_G|)^{1/2}] - 2T_G \left( \frac{1}{2} \beta_1 \beta_2 (T^2 + \beta_2 T_G)^{-1/2} \right) \\
+ \frac{\alpha_2 T}{2} (|T_G|)^{-1/2} \right\} - (24H^2 + 12H) \left\{ -1 + \beta_1 T (T^2 + \beta_2 T_G)^{-1/2} \\
+ 2\alpha_1 T + \alpha_2 (|T_G|)^{1/2} \right\} - 12H \left\{ (\beta_1 (T^2 + \beta_2 T_G)^{-1/2} - \beta_1 T^2 (T^2 + \beta_2 T_G)^{-3/2} + 2\alpha_1) \dot{T} \right\} \\
+ \left( -\frac{1}{2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2} (|T_G|)^{-1/2} \dot{T}_G \right) \right\} \\
+ \frac{2}{H} T_G - 24H^3 \left\{ \left( -\frac{1}{2} \beta_1 \beta_2 T (T^2 + \beta_2 T_G)^{-3/2} + \frac{\alpha_2}{2} (|T_G|)^{-1/2} \dot{T} \right) \right\} \right\} \geq 0,
Here, each of these inequalities depend on the values of \( \rho^m, P^m, \beta_1, \beta_2, \alpha_1, \alpha_2, T, T_G, \alpha_1, \alpha_2, \dot{T}, \dot{T}_G, H, H \) and \( \ddot{T}_G \). Firstly, we discuss these constraints using recent values of cosmic parameters like \( H_0, q_0, j_0 \) and \( s_0 \). For this purpose, we use the values proposed by Capozziello et al. \[37\]. These values are \( H_0 = 0.718, q_0 = -0.64, j_0 = 1.02 \) and \( s_0 = -0.39 \). We assume that the energy constraints are satisfied for ordinary matter quantities. Also, \( \kappa^2 = \frac{8\pi G}{c^4} \) is gravitational coupling constant and hence a positive quantity, therefore we only investigate the inequalities for DE source and find the possible constraints on the free parameters \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) by fixing any two of them. In terms of present day values, the above inequalities (41)-(44) result in the
following set of constraints given by

\[
\rho_{\text{eff}} = \{(-7.510\beta_1(-7.215 + \beta_2)(1.689 + \beta_2) + \sqrt{9.568 + 4.082\beta_2}(\alpha_2 \\
\times(59.004 + 25.175\beta_2) + (274.613 + 117.168\beta_2))/\{9.568 + 4.082\beta_2\}^2 \geq 0,
\]

\[
\rho_{\text{eff}} + p_{\text{eff}} = (-4.633\beta_1(-144.801 + \beta_2)(0.137 + \beta_2)(2.279 + \beta_2) + (229.584\alpha_1 \\
+53.403\alpha_2)(2.344 + \beta_2)^2 \sqrt{9.568 + 4.082\beta_2})/\{9.568 + 4.082\beta_2\}^2 \geq 0,
\]

\[
\rho_{\text{eff}} + 3p_{\text{eff}} = \{9.568 + 4.082\beta_2)^2/\{9.568 + 4.082\beta_2\}^2 \geq 0,
\]

\[
\rho_{\text{eff}} - p_{\text{eff}} = \{-70.101\beta_1(-0.889 + \beta_2)(2.293 + \beta_2)(10.790 + \beta_2) \\
+(1.480 \times 10^{-14} + 727.015\alpha_1 + 365.907\alpha_2)(2.344 + \beta_2)^2 \\
\times\sqrt{9.568 + 4.082\beta_2})/\{9.568 + 4.082\beta_2\}^2 \geq 0.
\]

We present the numerical evolution of above inequalities in terms of constants \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\). In right Figure 1, we show the variation of \(\rho_{\text{eff}} \geq 0\) versus parameters \(\alpha_1\) and \(\alpha_2\). In right panel, we set \(\beta_1 = 0.1\) and \(\beta_2 = 0.2\) and it is found that WEC can be satisfied for all positive values of \(\alpha_1\) and \(\alpha_2\). In left panel, we set the negative values of \(\beta_i (i = 1, 2)\), which requires \(\beta_2 \leq -2\). In this case, the constraint of WEC can be met if \(\alpha_i (i = 1, 2) > 0\). In Figure 2, we fix \(\alpha_i\) to determine the possible constraints on the parameters \(\beta_1\) and \(\beta_2\). In left panel, we explore the evolution for \(\alpha = 0.1\) and \(\alpha_2 = 0.2\), it can be seen that WEC is satisfied only if \(0 \leq \beta_2 \leq 7\). We also explore the variation of WEC for negative values of \(\alpha_i\) in right plot and find similar results.

The plots of NEC versus the parameters \(\alpha_i\) and \(\beta_i\) for model \(37\) are shown in Figure 3. In left plot, we fix \(\beta_1 = 0.1\) and \(\beta_2 = 0.2\) to show the variation for \(\alpha_i\), whereas in right plot, we show the variation of \(\beta_i\) for \(\alpha_1 = 0.1\) and \(\alpha_2 = 0.2\). It is interesting to mention here that one can get similar results for negative values of \(\alpha_i\) and \(\beta_i\). In case of SEC, we present the variations for free parameters in Figure 4 and discussed the difference in results. In left plot, we set \(\beta_1 = 0.1\) and \(\beta_2 = 0.2\), to determine the evolution of SEC versus \(\alpha_i\). It can be seen that SEC is satisfied if \(\alpha_i < 0\). We also explore the variation of SEC versus \(\beta_i\) which is found to be independent of signature of \(\alpha_i\). For the discussion of possible constraints arising due to DEC, we explore its evolution in Figure 5. In left plot, we show that DEC is satisfied only if \(\alpha_i > 0\) for all values of \(\beta_i\). For \(\alpha_1 = 0.1\) and \(\alpha_2 = 0.2\), DEC can be met if \(\beta_1 > 0\) and \(\beta_2 \geq -2\) as shown in right plot.
Figure 1: Evolution of WEC versus the parameters $\alpha_i (i = 1, 2)$ and $\beta_i (i = 1, 2)$. The left plot corresponds to parameters $\beta_1 = 0.1$ and $\beta_2 = 0.2$ and right plot corresponds to $\beta_1 = -10$ and $\beta_2 = -2$.

Figure 2: Evolution of WEC versus the parameters $\alpha_i (i = 1, 2)$ and $\beta_i (i = 1, 2)$. The left plot corresponds to parameters $\alpha_1 = 0.1$ and $\alpha_2 = 0.2$ and right plot corresponds to $\alpha_1 = -0.1$ and $\alpha_2 = -0.2$. 
Figure 3: Evolution of NEC versus the parameters $\alpha_i (i = 1, 2)$ and $\beta_i (i = 1, 2)$. The left plot corresponds to $\beta_1 = 0.1$ and $\beta_2 = 0.2$ and right plot corresponds to $\alpha_1 = 0.1$ and $\alpha_2 = 0.2$.

Figure 4: Evolution of SEC versus the parameters $\alpha_i (i = 1, 2)$ and $\beta_i (i = 1, 2)$. The left plot corresponds to $\beta_1 = 0.1$ and $\beta_2 = 0.2$ and right plot corresponds to $\alpha_1 = 0.1$ and $\alpha_2 = 0.2$. 
Figure 5: Evolution of DEC versus the parameters $\alpha_i (i = 1, 2)$ and $\beta_i (i = 1, 2)$. The left plot corresponds to $\beta_1 = 0.1$ and $\beta_2 = 0.2$ and right plot corresponds to $\alpha_1 = 0.1$ and $\alpha_2 = 0.2$.

The power law cosmology is described by $a(t) = a_0 t^m$, where $m$ is any arbitrary constant which further leads to $H = \frac{\dot{a}}{a}$. It is interesting to mention here that for $0 < m < 1$, the power law cosmology corresponds to decelerating universe, while for the values satisfying $m > 1$, this leads to accelerating cosmic model. The first, second and third-order time rates of Hubble parameter are

$$\dot{H} = -\frac{m}{t^2}, \quad \ddot{H} = \frac{2m}{t^3}, \quad \dddot{H} = -\frac{6m}{t^4}.$$  

We can also discuss the time rates of $T$ and $T_G$ in terms of cosmic time as follows

$$T = \frac{6m^2}{t^2}, \quad T_G = 24\frac{m^3(m-1)}{t^4}, \quad \dot{T} = -\frac{12m^2}{t^3}, \quad (45)$$

$$\dot{T}_G = -\frac{96m^3(m-1)}{t^5}, \quad \ddot{T} = \frac{36m^2}{t^4}, \quad \dddot{T}_G = -\frac{480m^3}{t^6} - \frac{144m^4}{t^6}. \quad (46)$$

One can find the energy constraints in power law cosmology for this theory by using the above defined relations. In Figures 6 and 7, we explore the evolution of constraints arising from weak, null, strong and dominant energy conditions in terms of $m$ and $t$ with $\alpha_1 = 10$, $\alpha_2 = 0.2$, $\beta_1 = 2$, $\beta_2 = 1$. In case of SEC, one need to set $\alpha_1 = 0.001$. We find that energy constraints can be satisfied for all values of $m$ and $t$. 

15
Figure 6: Evolution of WEC and NEC versus $m$ and $t$ with $\alpha_1 = 10$, $\alpha_2 = 0.2$, $\beta_1 = 2$, $\beta_2 = 1$.

Figure 7: Evolution of SEC and DEC versus $m$ and $t$ with $\alpha_1 = 10$, $\alpha_2 = 0.2$, $\beta_1 = 2$, $\beta_2 = 1$.
3.2 Model-2

In this section, we consider the following form of $F$ given by

$$F(T, T_G) = -T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2,$$

where $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_4$ are the free parameters to be constrained. For this model, the derivatives of $F$ can be written as

$$\begin{align*}
\dot{F}_T &= (2\beta_1 + 12\beta_3 T^2 + 4\beta_3 \beta_4 T_G) \dot{T} + 4\beta_3 \beta_4 T \dot{T}_G, \\
\dot{F}_{T_G} &= 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G, \\
\dot{F}_{T_G^2} &= 4\beta_3 \beta_4 T \ddot{T} + 4\beta_3 \beta_4 T \ddot{T}_G + 2\beta_3 \beta_4 T_G.
\end{align*}$$

For this model, the energy constraints take the following form

$$\begin{align*}
\rho_{\text{eff}} + P_{\text{eff}} &\geq 0 \Rightarrow \rho^m + P^m + \frac{1}{2\kappa^2} [ -24H^2(4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G) \\
&+ 4\beta_3 \beta_4 T_G \ddot{T} + 4\beta_3 \beta_4 T \ddot{T}_G ] \\
&+ 2T_G \{ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} + 8H^2 \{ 4\beta_3 \beta_4 T^2 \\
&+ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} \geq 0,
\end{align*}$$

(48)

$$\begin{align*}
\rho_{\text{eff}} &\geq 0 \Rightarrow \rho^m + \frac{1}{2\kappa^2} [ 6H^2 - (-T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2) \\
&+ 12H^2(-1 + 2\beta_1 T + 4\beta_3 T(T^2 + \beta_4 T_G)) + T_G(\beta_1 \beta_2 + 2\beta_3 \beta_4 (T^2 + \beta_4 T_G)) \\
&- 24H^3(4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G) \geq 0,
\end{align*}$$

(49)

$$\begin{align*}
\rho_{\text{eff}} + 3P_{\text{eff}} &\geq 0 \Rightarrow \rho^m + 3P^m + \frac{1}{2\kappa^2} [ -12H^2 + 2(-T + \beta_1(T^2 + \beta_2 T_G) \\
&+ \beta_3(T^2 + \beta_4 T)^2) - 2T_G(\beta_1 \beta_2 + 2\beta_3 \beta_4 (T^2 + \beta_4 T_G)) - (24H^2 + 12\dot{H}) \\
&\times (-1 + 2\beta_1 T + 4\beta_3 T(T^2 + \beta_4 T_G)) - (24H^3 - \frac{2T_G}{H}) \{ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} \\
&- 12H(\beta_1 \beta_2 + 4\beta_3 \beta_4 T \ddot{T} + 4\beta_3 \beta_4 T \ddot{T}_G) + 24H^2 \{ 4\beta_3 \beta_4 T^2 \\
&+ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} \geq 0,
\end{align*}$$

(50)

$$\begin{align*}
\rho_{\text{eff}} - P_{\text{eff}} &\geq 0 \Rightarrow \rho^m - P^m + \frac{1}{2\kappa^2} [ 12H^2 - 2(-T + \beta_1(T^2 + \beta_2 T_G) \\
&+ \beta_3(T^2 + \beta_4 T_G)) + (24H^2 + 4\dot{H})(-1 + 2\beta_1 T + 4\beta_3 T(T^2 + \beta_4 T_G)) \\
&+ 2T_G(\beta_1 \beta_2 + 2\beta_3 \beta_4 (T^2 + \beta_4 T_G)) + 4\dot{H} + 4H \{ (2\beta_1 \beta_2 + 4\beta_3 \beta_4 T_G) \ddot{T} \\
&+ 4\beta_3 \beta_4 T \ddot{T}_G \} - (24H^3 + \frac{2T_G}{3H}) \{ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} - 8H^2 \{ 4\beta_3 \beta_4 \ddot{T} \\
&+ 4\beta_3 \beta_4 T \ddot{T} + 2\beta_3 \beta_4 T_G \} \geq 0.
\end{align*}$$

(51)
Here each of these inequalities depend upon the values $\rho^m$, $P^m$, $H$, $k^2$, $T$, $T_G$, $\dot{H}$, $\dot{T}$, $\dot{T}_G$ and $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$ except the inequality given by Eq. (51) which does not depend upon the free parameter $\beta_2$. In terms of present day values of cosmic parameters $H$, $q$, $j$ and $s$, the above inequalities result in

$$
\rho_{\text{eff}} = 28.703\beta_1 + 640.765\beta_3 + 435.343\beta_3\beta_4 + 23.238\beta_3^2 \beta_4^2 \geq 0, \quad (52)
$$

$$
\rho_{\text{eff}} + p_{\text{eff}} = 13.777\beta_1 + 87.876\beta_3 + 393.686\beta_3\beta_4 + 8609.38\beta_3^2 \beta_4 - 10.685 \times \beta_3^2 \beta_4^2 \geq 0, \quad (53)
$$

$$
\rho_{\text{eff}} + 3p_{\text{eff}} = -2.227 - 16.074\beta_1 - 1017.9\beta_3 + 204.654\beta_3\beta_4 + 25828.1\beta_3^2 \geq 0, \quad (54)
$$

$$
\rho_{\text{eff}} - p_{\text{eff}} = 8.88178 \times 10^{-16} + 43.628\beta_1 + 1193.65\beta_3 + 582.718\beta_3\beta_4 - 8609.38\beta_3^2 \beta_4 + 57.1613\beta_3^2 \beta_4^2 \geq 0. \quad (55)
$$

Equations (52)-(55) show energy constraints in terms of $\beta_i$’s, obtained through the recent valued cosmic parameters. We show the evolution of these inequalities in Figures 8 and 9. In these plots, we fix $\beta_1 = 0.1$ and show variations against $\beta_3$ and $\beta_4$. Here WEC can be satisfied if $\beta_3 > 0$ and $\beta_4 > 0$, whereas in case of other constraints arising from NEC, SEC and DEC, one can set $\beta_3 \geq 0$. We also explore the energy constraints in terms of power law solution and set the parameters in a way to examine the evolution against $m$ and $t$. Figures 10 and 11 indicate that in power law cosmology, energy conditions for model (47) can be satisfied in terms of $m$ and $t$ for particular values of parameters $\beta_i$. 

Figure 8: Evolution of WEC and NEC versus $\beta_3$, $\beta_4$ with $\beta_1 = 0.1$. 

Here WEC can be satisfied if $\beta_3 > 0$ and $\beta_4 > 0$, whereas in case of other constraints arising from NEC, SEC and DEC, one can set $\beta_3 \geq 0$. We also explore the energy constraints in terms of power law solution and set the parameters in a way to examine the evolution against $m$ and $t$. Figures 10 and 11 indicate that in power law cosmology, energy conditions for model (47) can be satisfied in terms of $m$ and $t$ for particular values of parameters $\beta_i$. 

18
Figure 9: Evolution of SEC and DEC versus $\beta_3, \beta_4$ with $\beta_1 = 0.1$.

Figure 10: Evolution of WEC and NEC versus $m$ and $t$ with $\beta_1 = 0.001, \beta_3 = 0.2, \beta_4 = 0.03$.

Figure 11: Evolution of SEC with $\beta_1 = 0.001, \beta_3 = 0.2, \beta_4 = 0.03$ and DEC with $\beta_1 = 0.001, \beta_3 = -0.2, \beta_4 = -0.03$ versus $m$ and $t$. 

19
4 Summary and Discussion

In this paper, we have formulated the energy constraints in a general modified theory of gravity involving torsion scalar and a scalar equivalent to Gauss-Bonnet term. We have taken FRW universe model filled with perfect fluid matter. Firstly, we have defined these inequalities for general $F(T, T_G)$ by taking into account the effective energy density and its pressure. In order to be particular, we have considered two interesting models of $F(T, T_G)$ recently proposed in literature [24]. We have discussed the compatibility of the respective energy constraints for these models by fixing some of the free parameters. In order to examine these constraints, we have adopted two ways: introduction of some well-known cosmic parameters like Hubble, jerk, snap and deceleration parameters (we have used the recent limits of these parameters that are available in literature) and the power law cosmology.

Firstly, we explore the compatibility of energy conditions for a $F(T, T_G)$ model involving four free parameters namely $\beta_1, \beta_2, \alpha_1$ and $\alpha_2$ graphically. In plots, we have either fixed $\alpha_1, \alpha_2$ (by assuming their positive and negative values) and find the possible ranges of parameters $\beta_1, \beta_2$ or vice versa. It is seen that WEC can be satisfied for this model if we take $\alpha_1, \alpha_2 > 0$ for the fixed values fixed $\beta_i$ parameters within the range $0 < \beta_1, \beta_2 < 1$. It is interesting to mention here that if we consider some other positive large values of $\beta_i$, the positive ranges of $\alpha_i$ still remain valid. Furthermore, if we set negative values of $\beta_i$ then WEC can be satisfied only for $\alpha_1, \alpha_2 > 0$ if we impose the constraints $\beta_1 < 0$ and $\beta_2 \leq -2$. In the reverse case where we have fixed parameters $\alpha_1, \alpha_2$, either positive small values $0 < \alpha_i < 1$ or negative values satisfying $-1 < \alpha_i < 0$, WEC can be compatible with this model if $\beta_1, \beta_2$ satisfies the inequalities $0 \leq \beta_i \leq 7$.

Also, NEC constraints can be satisfied for this model if we take positive ranges of $\alpha_i$ parameters for the specified positive values of $\beta_i$ parameters. However, such a positive range of parameters $\alpha_i$ can be achieved so that NEC constraint remains valid, if one set parameters $\beta_1 = -10, \beta_2 = -2$. In a similar pattern, NEC constraint will be satisfied for small positive and negative values of $\alpha_i$ parameters with positive large values of $\beta_i$. In case of SEC, the inequalities will be satisfied only for $\alpha_i < 0$ when we fixed $0 < \beta_1, \beta_2 < 1$. Further, if we fix $\alpha_i$ and explore the possible ranges of $\beta_i$ so that SEC constraint remains compatible with this model, then it is observed that the inequality holds for $\beta_i > 0$ when $\alpha_i > 0$.

In case of power law cosmology, we have fixed all these four parameters
and found the possible variations of parameter $m$ and the cosmic time. It is seen that for positive fixed values of these parameters, the inequalities are satisfied $\forall \, m, \, t > 0$ except the case of SEC where one should fix $\alpha_1 = 0.001$ for the purpose.

For the second model involving four parameters $\beta_1, \beta_2, \beta_3$ and $\beta_4$, we have determined the numerical inequalities given by Eqns. (52)-(55) using the recent measures of cosmic parameters and discussed them graphically. It is interesting to mention here that all constraints arising from WEC, NEC, DEC and SEC are independent of the parameter $\beta_2$. We have found the possible variations of $\beta_3, \beta_4$ by fixing $\beta_1 = 0.1$ so that the inequalities are satisfied. It is seen that WEC remains valid if $\beta_3 > 0$ and $\beta_4 > 0$. However, for other constraints corresponding to NEC, SEC and DEC, one have to fix $\beta_3$ satisfying $\beta_3 \gtrless 0$. Further, we discuss the energy constraints using power law solution and found the possible variations of parameters $m$ and $t$ with all $\beta_i$ fixed. Plots indicate that in this case, energy constraints can be satisfied for $\forall \, m, \, t$ using particular values of $\beta_i$. It would be worthwhile to explore the possible ranges of the involved free parameters for other models of this gravity by making compatibility with the energy condition bounds.

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