The role of density in the energy conservation for the isentropic compressible Euler equations

Yanqing Wang∗ Yulin Ye† and Huan Yu ‡

Abstract

In this paper, we study Onsager’s conjecture on the energy conservation for the isentropic compressible Euler equations via establishing the energy conservation criterion involving the density $\rho \in L^k(0, T; L^l(T^d))$. The motivation is to analysis the role of the integrability of density of the weak solutions keeping energy in this system, since almost all known corresponding results require $\rho \in L^\infty(0, T; L^\infty(T^d))$. Our results imply that the lower integrability of the density $\rho$ means that more integrability of the velocity $v$ are necessary in energy conservation and the inverse is also true. The proof relies on the Constantin-E-Titi type and Lions type commutators on mollifying kernel.

MSC(2000): 35Q30, 35Q35, 76D03, 76D05

Keywords: compressible Euler equations; Onsager’s conjecture; energy conservation; vacuum

1 Introduction

In 1949, Onsager [30] conjectured that every weak solution of the incompressible homogeneous Euler equations with Hölder continuity exponent $\alpha > 1/3$ must conserve energy and there exists a weak solution with energy dissipation to the Euler equations when $\alpha < 1/3$. Let $v$ represent the fluid velocity field and $\pi$ stand for the scalar pressure. The classical incompressible Euler equations describing inviscid fluids read

\[
\begin{cases}
v_t + \text{div} (v \otimes v) + \nabla \pi = 0, \\
\text{div} v = 0, \\
v(x, 0) = v_0.
\end{cases}
\] (1.1)

The positive part of Onsager’s conjecture was proved by Constantin-E-Titi in [10], where they showed that energy is conserved for a weak solution $v$ in the Besov

∗College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, P. R. China Email: wangyanqing20056@gmail.com
†Corresponding author. School of Mathematics and Statistics, Henan University, Kaifeng, 475004, P. R. China. Email: ylye@vip.henu.edu.cn
‡School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, P. R. China Email: yuhuandreamer@163.com
space $L^3(0,T;B_{3,\infty}^\alpha(T^3))$ with $\alpha > 1/3$. For $\alpha = 1/3$, it is shown that every weak solution $v$ of the Euler equations on $[0,T]$ conserves energy provided $v \in L^{3}(0,T;B_{3,c(N)}^{1/3})$ by Cheskidov-Constantin-Friedlander-Shvydkoy in \cite{6}, where $B_{3,c(N)}^{1/3} = \{(u \in B_{3,\infty}^{1/3}, \lim_{q \to \infty} \int_0^T 2^q \|\Delta_q u\|_{L^3}^3 dt = 0)\}$. After a series of papers by De Lellis and Szekelyhidi \cite{12,14,16}, Isett \cite{22} successfully solved the second part of Onsager’s conjecture in three-dimensional space. At this stage, the progress of Onsager’s conjecture of the 3D incompressible Euler equations \begin{equation} \tag{1.1} \end{equation} is satisfactory.

Note that the density $\rho$ of the flow in Euler equations \begin{equation} \tag{1.1} \end{equation} is a constant. A natural extension of Onsager’s conjecture is to consider critical regularity of conservation of energy in the density-dependent (nonhomogeneous) Euler system and the compressible Euler equations. Important progress involving generalized Onsager’s conjecture in both equations has been made (see \cite{1,4,11,19,24,29} and references therein). Before we state these results, we recall the nonhomogeneous Euler system \begin{equation} \tag{1.2} \end{equation}
and the compressible Euler equations \begin{equation} \tag{1.3} \end{equation}
In general, one complements equations \begin{equation} \tag{1.2} \end{equation} or \begin{equation} \tag{1.3} \end{equation} with initial data
\begin{equation} \tag{1.4} \end{equation}
where we define $v_0 = 0$ on the sets $\{x \in \Omega : \rho_0 = 0\}$. In the present paper, we consider the periodic case, which means $\Omega = T^d$ with dimension $d \geq 2$.

In particular, for the density-dependent Euler equations \begin{equation} \tag{1.2} \end{equation}, by means of Constantin-E-Titi type commutators on mollifying kernel, Feireisl-Gwiazda-Gwiazda-Wiedemann \cite{19} first established the following energy conservation criterion of weak solutions: if the weak solutions satisfy
\begin{equation} \tag{1.5} \end{equation}
with $2\alpha + \beta > 1$ and $0 \leq \alpha, \beta \leq 1$, then the energy is locally conserved. Introducing the Lions type commutators on mollifying kernel, Chen and Yu \cite{4} derived the sufficient conditions for weak solutions of the nonhomogeneous Euler equations keeping energy as follow: If there holds
\begin{equation} \tag{1.5} \end{equation}
Then the energy is conserved.
For the isentropic compressible Euler equations (1.3), Feireisl-Gwiazda-Gwiazda-Wiedemann \[19\] show that

\[
0 \leq \rho \leq \bar{\rho} \text{ a.e. in } (0, T) \times \mathbb{T}^d, \pi \in C^2[\rho, \bar{\rho}]
\]
\[
\rho, \rho v \in B^\beta_{3, \infty}(0, T) \times \mathbb{T}^d, v \in B^{\alpha}_{3, \infty}(0, T) \times \mathbb{T}^d,
\]
and \(0 \leq \alpha, \beta \leq 1\) such that \(\beta > \max\{1 - 2\alpha, \frac{1}{2}(1 - \alpha)\}\) guarantee that the energy of weak solutions is locally conserved. Very recently, the hypothesis \(\pi \in C^2\) was improved to \(\pi \in C^{1, \gamma - 1}\) with \(1 \leq \gamma < 2\) by Akramov-Debiec-Skipper-Wiedemann in \[1\]. Moreover, under the density \(\rho \in L^\infty([0, T] \times \mathbb{T}^d)\), sufficient conditions for weak solutions of system \(1.2\) and \(1.3\) to conserve the energy balance in isentropic compressible Euler equations \(1.3\).

From the results mentioned above, almost sufficient conditions involving non-constant density for persistence of energy requires that \(\rho \in L^\infty(0, T; L^\infty(\mathbb{T}^d)).\) This can be derived from \(\text{div } v = 0\) in the nonhomogeneous Euler system. However, for compressible case, the natural regularity of the density is only \(\rho \in L^\infty(0, T; L^l(\mathbb{T}^d))\). A natural question is that under the hypothesis \(\rho \in L^\infty(0, T; L^l(\mathbb{T}^d))\) with \(l < \infty\) rather than \(\rho \in L^\infty(0, T; L^\infty(\mathbb{T}^d))\), what is the minimum regularity to guarantee the energy balance in isentropic compressible Euler equations \(1.3\).

We state our first result for isentropic compressible Euler equations (1.3) away from vacuum as follows.

**Theorem 1.1.** Let the pair \((\rho, v)\) be a weak solution of isentropic compressible Euler equations in the sense of definition \([2, 4]\). Suppose that the density \(\rho\) and the velocity \(v\) satisfy

\[
0 < c \leq \rho \in L^{\max\{\frac{p}{p - 3}, \frac{\rho^{(\gamma - 1)}}{2}\}}(0, T; L^{\max\{\frac{q}{q - 3}, \frac{q(\gamma - 1)}{2}\}}(\mathbb{T}^d)),
\]
\[
\nabla \rho \in L^{\frac{p}{p - 3}}(0, T; L^{\frac{\rho}{\rho - 3}}(\mathbb{T}^d)), \partial_t \in L^{\frac{p}{p - 3}}(0, T; L^{\frac{\rho}{\rho - 3}}(\mathbb{T}^d)),
\]
\[
v \in L^p(0, T; B^{\alpha}_{q, \infty}(\mathbb{T}^d)), \alpha > 1/3, p, q > 3.
\]

Then the energy is locally conserved, that is,

\[
\int_{\mathbb{T}^d} \partial_t \left( \frac{1}{2} \rho |v|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx = 0,
\]

in the sense of distributions on \((0, T)\).

**Remark 1.1.** By small modification in the proof of Theorem \(1.1\), our result also holds for \(p = 3, q = 3\). Then taking \(p = q = 3\) in \((1.6)\), one derives sufficient conditions for local energy balance of weak solutions

\[
0 < c \leq \rho \in L^\infty(0, T; L^\infty(\mathbb{T}^d)), \nabla \rho \in L^\infty(0, T; L^\infty(\mathbb{T}^d)), \partial_t \in L^\infty(0, T; L^\infty(\mathbb{T}^d)),
\]
\[
v \in L^3(0, T; B^{\alpha}_{3, \infty}(\mathbb{T}^d)), \alpha > 1/3,
\]

which is an analogue of classical Constantin-E-Titi theorem for the Onsager conjecture of the 3D compressible Euler equations with the density away from vacuum.

**Remark 1.2.** This theorem implies that lower integrability of the density \(\rho\) means that more integrability of the velocity \(v\) are necessary in energy conservation of the isentropic compressible Euler equations and the inverse is also true.
Remark 1.3. It seems that this is the first energy conservation criterion for the compressible Euler equations, which allows the density to require its upper bound.

Remark 1.4. All results here can be viewed as addressing what is the minimum regularity posed on both the density and the velocity to guarantee the energy balance in isentropic compressible Euler equations (1.3).

Corollary 1.2. The energy conservation (1.7) is valid provided one of the following two conditions is satisfied

\[(1) \quad \varrho \in L^{\frac{p}{\gamma - 3}}(0, T; L^{\gamma\
- 3}(\mathbb{T}^d)), \varrho_t \in L^{\frac{p}{\gamma - 3}}(0, T; L^{\frac{2}{\gamma - 3}}(\mathbb{T}^d)), \nabla \varrho \in L^{\frac{p}{\gamma - 3}}(0, T; L^{\frac{2}{\gamma - 3}}(\mathbb{T}^d)),\]
\[v \in L^p(0, T; B^{\alpha}_{q, \infty}(\mathbb{T}^d)), \alpha > 1/3, 3 \leq p, q \leq \frac{3\gamma - 1}{\gamma - 1}. \quad \text{(1.8)}\]

\[(2) \quad \varrho \in L^{\frac{3\gamma - 1}{2}}(0, T; L^{\frac{3\gamma - 1}{2}}(\mathbb{T}^d)), \varrho_t \in L^{\frac{3\gamma - 1}{2}}(0, T; L^{\frac{3\gamma - 1}{2}}(\mathbb{T}^d)), \nabla \varrho \in L^{\frac{3\gamma - 1}{2}}(0, T; L^{\frac{3\gamma - 1}{2}}(\mathbb{T}^d)),\]
\[v \in L^{\frac{3\gamma - 1}{2}}(0, T; B^{\alpha}_{q, \infty}(\mathbb{T}^d)), \alpha > 1/3. \quad \text{(1.9)}\]

Next, we extend the energy conservation up to the initial time. In this case, the more condition posed on density are needed. The precise result is the following.

Theorem 1.3. Let the pair \((\varrho, v)\) be a weak solution in the sense of definition 2.1. Assume that there holds

\[0 < c \leq \varrho \in L^k(0, T; L^l(\mathbb{T}^d)), \]
\[k \geq \max\{\frac{p}{p - 3}, \frac{p(\gamma - 1)}{2}, \frac{(\gamma - 1)(d + q)p}{2q - d(p - 3)}\}, l \geq \max\{\frac{q}{q - 3}, \frac{q(\gamma - 1)}{2}\}, \]
\[\nabla \sqrt{\varrho} \in L^{2\gamma(\frac{2q - 2q}{\gamma - 3}) - \frac{2q}{\gamma - 3}}(0, T; L^{\frac{2q}{\gamma - 3}}(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^{\frac{2q}{\gamma - 3} - \frac{2q}{\gamma - 3}}(0, T; L^{\frac{2q}{\gamma - 3}}(\mathbb{T}^d)),\]
\[v \in L^p(0, T; B^{\alpha}_{q, \infty}(\mathbb{T}^d)), \alpha > 1/3, p, q > 3, q > \frac{d(p - 3)}{2}, \text{ and } v_0 \in L^{\max\{\frac{2\alpha}{3\gamma - 1}, \frac{2\alpha}{\gamma - 1}\}}(\mathbb{T}^d). \quad \text{(1.10)}\]

Then the energy is globally conserved, namely, for any \(t \in [0, T]\),

\[E(t) = E(0), \quad \text{(1.11)}\]

where \(E(t) = \int_{\mathbb{T}^d} \left(\frac{1}{2} \varrho |v|^2 + \frac{\kappa}{\gamma} \varrho^{\frac{\gamma}{p}}\right) dx. \)

Remark 1.5. It should be noted that this result also holds for \(p = 3, q = 3\) just by small modification in the proof of Theorem 1.3.

Corollary 1.4. Let the weak solution \((\varrho, v)\) of compressible Euler equations meets one of the following condition

\[(1) \quad 0 < c \leq \varrho \in L^\infty(0, T; L^{\frac{q}{\gamma - 3}}), \partial_t \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{q}{\gamma - 3}}(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{q}{\gamma - 3}}(\mathbb{T}^d)),\]
\[v \in L^3(0, T; B^{\alpha}_{q, \infty}(\mathbb{T}^d)), \alpha > 1/3, 3 \leq q \leq \frac{3\gamma - 1}{\gamma - 1} \text{ and } v_0 \in L^2(\mathbb{T}^d). \quad \text{(1.12)}\]
(2) $0 < c \leq q \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)),$  

$$v \in L^3(0, T; B^{\frac{\alpha}{1-\gamma}}_{\frac{3q-1}{2}}(\mathbb{T}^d)), \alpha > 1/3, \text{ and } v_0 \in L^2(\mathbb{T}^d).$$

(1.13)

Then the relation (1.11) is valid.

**Remark 1.6.** Here the minimum integrability of the density in space direction is $\frac{3q-1}{2}$ for energy conservation. It is an open problem to derive energy conservation criterion under the condition $q \in L^\infty(0, T; L^1(\mathbb{T}^d)).$

Finally, we consider the energy conservation of the weak solutions of the isentropic compressible Euler equations allowing vacuum.

**Theorem 1.5.** Let the pair $(\varrho, v)$ be a solution of in the sense of definition 2.1. Suppose that there holds

$$0 \leq \varrho \in L^k(0, T; L^1(\mathbb{T}^d)),$$

$$k \geq \max\{\frac{p}{p-3}, \frac{p(\gamma - 1)}{2}, \frac{(\gamma - 1)(n + q)p}{2q - n(p - 3)}\}, l \geq \max\{\frac{q}{q - 3}, \frac{q(\gamma - 1)}{2}\},$$

$$\nabla \sqrt{\varrho} \in L^{\frac{2kp}{p-3-k}}(0, T; L^\infty(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^{\frac{2kp}{p-3-k}}(0, T; L^\infty(\mathbb{T}^d)),$$

$$v \in B^{\frac{\alpha}{1-\gamma}}_{p, \infty}(0, T; B^{\frac{\alpha}{1-\gamma}}_{q, \infty}(\mathbb{T}^d)), \beta \geq \alpha > 1/3, p, q > 3, q > \frac{d(p - 3)}{2}, \text{ and } v_0 \in L^{\max\{\frac{2k}{p-3}, \frac{q}{q-3}\}}(\mathbb{T}^d).$$

Then the energy is globally conserved, namely, for any $t \in [0, T]$,

$$E(t) = E(0),$$

(1.14)

where $E(t) = \int_{\mathbb{T}^d} \left(\frac{1}{2} |v|^2 + \kappa \frac{\varrho^\gamma}{\gamma - 1}\right) dx.$

**Remark 1.7.** This result also holds for $p = 3, q = 3$ by small modification in the proof of Theorem 1.5.

**Corollary 1.6.** Let the weak solution $(\varrho, v)$ of compressible Euler equations meets one of the following conditions

(1) $0 \leq \varrho \in L^\infty(0, T; L^\infty(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^\infty(0, T; L^\infty(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^\infty(0, T; L^\infty(\mathbb{T}^d)),$  

$$v \in B^{\frac{\beta}{1-\gamma}}_{3, \infty}(0, T; B^{\frac{\alpha}{1-\gamma}}_{3, \infty}(\mathbb{T}^d)), \beta \geq \alpha > 1/3, \text{ and } v_0 \in L^2(\mathbb{T}^d).$$

(2) $0 \leq \varrho \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)),$  

$$v \in B^{\frac{\beta}{3}}_{3, \infty}(0, T; B^{\frac{\alpha}{3}}_{q, \infty}(\mathbb{T}^d)), \beta \geq \alpha > 1/3, 3 < q < \frac{3\gamma - 1}{\gamma - 1} \text{ and } v_0 \in L^2(\mathbb{T}^d).$$

(3) $0 \leq \varrho \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)), \partial_t \sqrt{\varrho} \in L^\infty(0, T; L^{\frac{3q-1}{2}}(\mathbb{T}^d)),$  

$$v \in B^{\frac{\beta}{3}}_{3, \infty}(0, T; B^{\frac{\alpha}{3}}_{q, \infty}(\mathbb{T}^d)), \beta \geq \alpha > 1/3, \text{ and } v_0 \in L^2(\mathbb{T}^d).$$

Then the relation (1.15) is valid.

**Remark 1.8.** In the spirit of above theorems, we will consider the energy equality of weak solutions in the isentropic compressible Navier-Stokes equations without upper bound of the density in a forthcoming paper [33].
We would like to mention that infinitely many weak solutions of the isentropic compressible Euler equations have been constructed in [15, 17]. The uniqueness of dissipative solutions with regularity assumption to the isentropic Euler equations can be found in [20].

The rest of this paper is divided into three sections. In Section 2, we present the auxiliary lemmas including the Constantin-E-Titi type and Lions type commutators on mollifying kernel. Section 3 is devoted to the Onsager conjecture on the energy conservation for the isentropic compressible Euler equations.

2 Notations and some auxiliary lemmas

First, we introduce some notations used in this paper. For \( p \in [1, \infty] \), the notation \( L^p(0, T; X) \) stands for the set of measurable functions on the interval \((0, T)\) with values in \(X\) and \(\|f(t, \cdot)\|_X\) belonging to \(L^p(0, T)\). The classical Sobolev space \(W^{k,p}(\Omega)\) is equipped with the norm \(\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|=0}^k \|D^\alpha f\|_{L^p(\Omega)}\). For \(1 \leq q \leq \infty\) and \(0 < \alpha < 1\), the homogeneous Besov space \(\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)\) is the space of functions \(f\) on the \(d\)-dimensional torus \(\mathbb{T}^d = [0, 1]^d\) for which the semi-norm

\[
\|f\|_{\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)} = \left\| |y|^{-\alpha} \left\| f(x - y) - f(x) \right\|_{L^q(\mathbb{T}^d)} \right\|_{L^\infty(\mathbb{R}^d)} < \infty,
\]

and the nonhomogeneous Besov space \(B^\alpha_{q,\infty}(\mathbb{T}^d)\) is the set of functions \(f \in L^q(\mathbb{T}^d)\) for which the norm

\[
\|f\|_{B^\alpha_{q,\infty}(\mathbb{T}^d)} = \|f\|_{L^q(\mathbb{T}^d)} + \|f\|_{\dot{B}^\alpha_{q,\infty}(\mathbb{T}^d)} < \infty.
\]

Likewise, we define \(B^\alpha_{q,\infty}(0, T)\) via replacing \(\mathbb{T}^d\) by \((0, T) \times \mathbb{T}^d\) in the above. A similar definition of Besov norms on the whole space \(\mathbb{R}^d\) can be referred to in [3].

Let \(\eta : \mathbb{R}^d \to \mathbb{R}\) be a standard mollifier.i.e. \(\eta(x) = C_0 e^{-\frac{1}{|x|^2}}\) for \(|x| < 1\) and \(\eta(x) = 0\) for \(|x| \geq 1\), where \(C_0\) is a constant such that \(\int_{\mathbb{R}^d} \eta(x) dx = 1\). For \(\epsilon > 0\), we define the rescaled mollifier \(\eta_\epsilon(x) = \frac{1}{\epsilon^d} \eta(\frac{x}{\epsilon})\). For any function \(f \in L^1_{loc}(\Omega)\), its mollified version is defined as

\[
f^\epsilon(x) = (f \ast \eta_\epsilon)(x) = \int_{\mathbb{R}^d} f(x - y)\eta_\epsilon(y) dy, \quad x \in \Omega_\epsilon,
\]

where \(\Omega_\epsilon = \{ x \in \Omega : d(x, \partial \Omega) > \epsilon \}\). To deal with the vacuum vase, we need the space-time mollifier. Abusing notation slightly, we also denote \(\eta\) be non-negative smooth function supported in the space-time ball of radius 1 and its integral equals to 1. We define the rescaled space-time mollifier \(\eta_\epsilon(t, x) = \frac{1}{\epsilon^d} \eta(\frac{t}{\epsilon}, \frac{x}{\epsilon})\)

\[
f^\epsilon(t, x) = \int_0^T \int_{\Omega} f(s, y)\eta_\epsilon(t - s, x - y) dy ds.
\]

Moreover, for simplicity, we denote by

\[
\int_0^T \int_{\mathbb{T}^d} f(t, x) dx dt = \int_0^T \int f \quad \text{and} \quad \|f\|_{L^p(0, T; X)} = \|f\|_{L^p(X)}.
\]

**Definition 2.1.** A pair \((\rho, v)\) is called a weak solution to (1.3) with initial data (1.4) if \((\rho, v)\) satisfies
(i) equation (1.3) holds in $D'(0, T; \Omega)$ and
\[ P(\varrho), g|v|^2 \in L^\infty(0, T; L^1(\Omega)), \]
(2.2)

(ii) the density $\varrho$ is a renormalized solution of (1.3) in the sense of [12].

(iii) the energy inequality holds
\[ E(t) \leq E(0), \]
where $E(t) = \int_\Omega \left( \frac{1}{2} |\varrho|^2 + \kappa \frac{\varrho^2}{T-\tau} \right) dx.$

The following lemma is the Lions type commutators on space-time mollifying kernel, which was stated in [4, 23] and whose proof can be found in [32]. We refer the reader to (ii) the energy inequality holds
\[ E(t) \leq E(0), \]
for the original version.

Lemma 2.1. ([4, 23, 22]) Let $1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.\ Let $\partial$ be a partial derivative in space or time, in addition, let $\partial \varphi, \nabla f \in L^{p_1}(0, T; L^{q_1}(\mathbb{T}^d))$, $g \in L^{p_2}(0, T; L^{q_2}(\mathbb{T}^d))$. Then, there holds
\[ \left\lVert \partial (fg^\varepsilon) - \partial (fg^\varepsilon) \right\rVert_{L^p(0, T; L^q(\mathbb{T}^d))} \]
\[ \leq C \left( \left\lVert \partial \varphi \right\rVert_{L^{p_1}(0, T; L^{q_1}(\mathbb{T}^d))} + \left\lVert \nabla f \right\rVert_{L^{p_1}(0, T; L^{q_1}(\mathbb{T}^d))} \right) \left\lVert g \right\rVert_{L^{p_2}(0, T; L^{q_2}(\mathbb{T}^d))}, \]
(2.4)
for some constant $C > 0$ independent of $\varepsilon, f$ and $g$. Moreover,
\[ \partial (fg^\varepsilon) - \partial (fg^\varepsilon) \to 0 \quad \text{in } L^p(0, T; L^q(\mathbb{T}^d)), \]
as $\varepsilon \to 0$ if $p_2, q_2 < \infty$.

Lemma 2.2. Assume that $0 < \alpha_i \leq \beta_i, \ i = 1, 2$ and $1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, for any $f \in B^{\beta_1}_{p_1, \infty}(0, T); B^{\alpha_1}_{q_1, \infty}(\mathbb{T}^d)$, $g \in B^{\beta_2}_{p_2, \infty}(0, T); B^{\alpha_2}_{q_2, \infty}(\mathbb{T}^d)$ and $\varepsilon > 0$, there holds
\[ \left\lVert (fg)^\varepsilon - f^\varepsilon g^\varepsilon \right\rVert_{L^p(0, T; L^p(\mathbb{T}^d))} \leq C \varepsilon^{\alpha_1 + \alpha_2} \left\lVert f \right\rVert_{B^{\beta_1}_{p_1, \infty}(0, T); B^{\alpha_1}_{q_1, \infty}(\mathbb{T}^d)} \left\lVert g \right\rVert_{B^{\beta_2}_{p_2, \infty}(0, T); B^{\alpha_2}_{q_2, \infty}(\mathbb{T}^d)}, \]
(2.5)
where $f^\varepsilon$ and $g^\varepsilon$ are defined in (2.1).

Remark 2.1. If one modifies the functions $f$ and $g$ just in space direction, the inequality (2.5) reduces to
\[ \left\lVert (fg)^\varepsilon - f^\varepsilon g^\varepsilon \right\rVert_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C \varepsilon^{\alpha_1 + \alpha_2} \left\lVert f \right\rVert_{L^{p_1}(0, T; B^{\alpha_1}_{q_1, \infty}(\mathbb{T}^d))} \left\lVert g \right\rVert_{L^{p_2}(0, T; B^{\alpha_2}_{q_2, \infty}(\mathbb{T}^d))}, \]
(2.6)
Proof. We recall the following identity observed by Constantin-E-Titi in [10]
\[
\begin{align*}
(fg)^\varepsilon(t, x) - f^\varepsilon g^\varepsilon(t, x) \\
= & \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta_\varepsilon(\tau, y) \left[ f(t - \tau, x - y) - f(t, x) \right] \left[ g(t - \tau, x - y) - g(t, x) \right] d\tau dy \\
- & (f - f^\varepsilon)(g - g^\varepsilon)(t, x) \\
= & G(t, x) - (f - f^\varepsilon)(g - g^\varepsilon)(t, x).
\end{align*}
\]
First, the Hölder’s inequality and Minkowski inequality yield that

\[
\|G\|_{L^p((0,T);L^q(T^d))} \leq \int_{-\varepsilon}^{\varepsilon} \int_{|y| \leq \varepsilon} \eta_\varepsilon(\tau,y) \left[ f(t-\tau,x-y) - f(t-\tau,x) + f(t-\tau,x) - f(t,x) \right]
\]

\[
\left[ g(t-\tau,x-y) - g(t-\tau,x) + g(t-\tau,x) - g(t,x) \right] \, dy\,d\tau
\]

\[
\leq \int_{-\varepsilon}^{\varepsilon} \int_{|y| \leq \varepsilon} \eta_\varepsilon(\tau,y) \left[ \|f(t-\tau,x-y) - f(t-\tau,x)\|_{L^p_1(L^q_1)} + \|f(t-\tau,x) - f(t,x)\|_{L^p_1(L^q_1)} \right]
\]

\[
\left[ \|g(t-\tau,x-y) - g(t-\tau,x)\|_{L^p_2(L^{q_2})} + \|g(t-\tau,x) - g(t,x)\|_{L^p_2(L^{q_2})} \right] \, dy\,d\tau
\]

\[
\leq C_{\varepsilon^{\alpha_1+\alpha_2}} \|f\|_{B^\alpha_{p_1,\infty}((0,T);B^\alpha_{q_1,\infty}(T^d))} \|g\|_{B^{\beta_2}_{p_2,\infty}((0,T);B^{\beta_2}_{q_2,\infty}(T^d))},
\]

and

\[
\|f - f^\varepsilon\|_{L^p(0,T;L^q(T^d))} \leq C \varepsilon^{\alpha+\alpha_2} \|f\|_{B^\alpha_{p_1,\infty}((0,T);B^\alpha_{q_1,\infty}(T^d))} \|g\|_{B^{\beta_2}_{p_2,\infty}((0,T);B^{\beta_2}_{q_2,\infty}(T^d))},
\]

where we have used the following facts that for any \( u \in B^\beta_{p,\infty}((0,T);B^\alpha_{q,\infty}(T^d)) \) with \( \beta \geq \alpha > 0, 1 \leq p, q \leq \infty \) and a.e. \( y \in \mathbb{R}^d \),

\[
\|u(\cdot-y) - u(\cdot)\|_{L^p(0,T;L^q(T^d))} \leq C |y|^{\alpha} \|u\|_{B^\alpha_{p,\infty}((0,T);B^\alpha_{q,\infty}(T^d))},
\]

\[
\|u^\varepsilon - u\|_{L^p(0,T;L^q(T^d))} \leq C \varepsilon^{\alpha} \|u\|_{B^\beta_{p,\infty}((0,T);B^\alpha_{q,\infty}(T^d))},
\]

\[
\|\nabla u^\varepsilon\|_{L^p(0,T;L^q(T^d))} \leq C \varepsilon^{\alpha-1} \|u\|_{B^\beta_{p,\infty}((0,T);B^\alpha_{q,\infty}(T^d))},
\]

which can be deduced from periodicity of the function \( u \) in essentially the same manner as derivation of [3, Lemma 2.1] and [19]. The proof of this lemma is completed. \( \square \)

**Lemma 2.3.** Let \( p, q, p_1, q_1, p_2, q_2 \in [1, +\infty) \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume \( f \in L^{p_1}(0,T;L^q(T^d)) \) and \( g \in L^{p_2}(0,T;L^{q_2}(T^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(T^d))} \to 0, \quad \text{as } \varepsilon \to 0.
\]

**Proof.** By the triangle inequality, one have

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(T^d))} \leq C \left( \|(fg)^\varepsilon - (fg)\|_{L^p(0,T;L^q(T^d))} + \|f - f^\varepsilon\|_{L^p(0,T;L^q(T^d))} + \|g - g^\varepsilon\|_{L^p(0,T;L^q(T^d))} \right)
\]

\[
\leq C \left( \|(fg)^\varepsilon - (fg)\|_{L^p(0,T;L^q(T^d))} + \|f - f^\varepsilon\|_{L^{p_1}(0,T;L^{q_1}(T^d))} \|g\|_{L^{p_2}(0,T;L^{q_2}(T^d))} \right)
\]

\[
+ \|f^\varepsilon\|_{L^{p_1}(0,T;L^{q_1}(T^d))} \|g - g^\varepsilon\|_{L^{p_2}(0,T;L^{q_2}(T^d))},
\]

then, together with the properties of the standard mollification, we can obtain (2.10). \( \square \)
Lemma 2.4 (31). Let $X \hookrightarrow B \hookrightarrow Y$ be three Banach spaces with compact imbedding $X \hookrightarrow Y$. Further, let there exist $0 < \theta < 1$ and $M > 0$ such that
\[ \|v\|_B \leq M \|v\|_X^{1-\theta} \|v\|_Y^\theta \quad \text{for all } v \in X \cap Y. \] (2.11)

Denote for $T > 0$,
\[ W(0,T) := W^{s_0,r_0}((0,T),X) \cap W^{s_1,r_1}((0,T),Y) \] (2.12)
with
\[ s_0, s_1 \in \mathbb{R}; \quad r_0, r_1 \in [1,\infty], \]
\[ s_\theta := (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s^* := s_\theta - \frac{1}{r_\theta}. \] (2.13)

Assume that $s_\theta > 0$ and $F$ is a bounded set in $W(0,T)$. Then, we have

If $s_* \leq 0$, then $F$ is relatively compact in $L^p((0,T),B)$ for all $1 \leq p < p^* := -\frac{1}{s_*}$.

If $s_* > 0$, then $F$ is relatively compact in $C((0,T),B)$.

3 Onsager conjecture on the energy conservation for the isentropic compressible Euler equations

We will use the framework of [4] to prove main theorems in this section. First, we reformulate the equations after mollifier the equations. Second, we invoke the Constantin-E-Titi type and Lions type commutators on mollifying kernel to pass the limits. Third, it is to get the energy conservation up to the initial time.

3.1 Non-vacuum case

Proof of Theorem 1.1. For the non-vacuum case, it is sufficient to mollify $v$ in space direction. For any smooth function $\phi(t)$ which is compact support in $(0, +\infty)$, multiplying the momentum equation in (1.3) by $(\phi(t)v^\varepsilon)^\varepsilon$, then integrating over $(0, T) \times \Omega$, we arrive at
\[ \int_0^T \int \phi(t)v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon + \text{div} (\rho v \otimes v)^\varepsilon + \kappa \nabla \pi (\rho)^\varepsilon \right] = 0. \] (3.1)

To apply the commutators on mollifying kernel to pass the limit of $\varepsilon$, we first need to rewrite equations (3.1). By some straightforward computation, we see that
\[ \int_0^T \int \phi(t)v^\varepsilon \partial_t (\rho v)^\varepsilon = \int_0^T \int \phi(t)v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t)v^\varepsilon \partial_t (\rho v^\varepsilon) = \int_0^T \int \phi(t)v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t)\rho \partial_t |v^\varepsilon|^2 \] (3.2)
\[ + \int_0^T \int \phi(t)\rho |v^\varepsilon|^2. \]

Using integration by parts many times and the equation (1.3), one deduces that
\[ \int_0^T \int \phi(t)v^\varepsilon \text{div} (\rho v \otimes v)^\varepsilon \]
Thanks to integration by parts once again, we infer that

\[
\int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon - (\varrho v) \otimes v^\varepsilon\right) + \int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon\right)
\]
\[
= - \int_0^T \int \phi(t) \nabla v^\varepsilon \left((\varrho v \otimes v)^\varepsilon - \varrho((v \otimes v)^\varepsilon) + \varrho((v \otimes v)^\varepsilon) - (\varrho v) \otimes v^\varepsilon\right)
+ \int_0^T \int \phi(t) \left(\text{div}\left((\varrho v)^\varepsilon\right) + \frac{1}{2} \varrho v \nabla |v^\varepsilon|^2\right)
\]
\[
= \int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon - \varrho((v \otimes v)^\varepsilon)\right) - \int_0^T \int \phi(t) \nabla v^\varepsilon \varrho((v \otimes v)^\varepsilon - v \otimes v^\varepsilon)
+ \frac{1}{2} \int_0^T \int \phi(t) \text{div}\left((\varrho v)^\varepsilon\right)^2
\]
\[
= \int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon - \varrho(v \otimes v)^\varepsilon\right) - \int_0^T \int \phi(t) \nabla v^\varepsilon \varrho(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon
- \frac{1}{2} \int_0^T \int \phi(t) \partial_t \varrho|v^\varepsilon|^2. \quad (3.3)
\]

Inserting (3.4) into (3.3), we arrive at

\[
\int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon\right)
\]
\[
= \int_0^T \int \phi(t) v^\varepsilon \text{div}\left((\varrho v \otimes v)^\varepsilon - \varrho(v \otimes v)^\varepsilon\right) - \int_0^T \int \phi(t) \nabla v^\varepsilon \varrho((v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon)
+ \frac{1}{2} \int_0^T \int \phi(t) |v^\varepsilon|^2 |\varrho^\varepsilon - \varrho_t| + \frac{1}{2} \int_0^T \int \phi(t) |v^\varepsilon|^2 (\varrho_t^\varepsilon - \varrho_t). \quad (3.5)
\]

We also reformulate the pressure term as

\[
\kappa \int_0^T \int \phi(t) v^\varepsilon \nabla (\varrho^\gamma)^\varepsilon = \kappa \int_0^T \int \phi(t) [v^\varepsilon \nabla (\varrho^\gamma)^\varepsilon - v \nabla (\varrho^\gamma)] + \kappa \int_0^T \int \phi(t) v \nabla (\varrho^\gamma). \quad (3.6)
\]
With the help of the integration by parts and the mass equation in (1.3) again, we know that
\[
\kappa \int_0^T \int \phi(t)v \cdot \nabla (\phi^\gamma) = - \kappa \int_0^T \int \phi(t) \phi^\gamma - \phi v \text{div } v \\
= \kappa \int_0^T \int \phi(t) \phi^\gamma - \phi (v + \nabla \phi) \\
= \kappa \frac{1}{\gamma} \int_0^T \int \phi(t) \partial_t \phi^\gamma + \kappa \frac{1}{\gamma} \int_0^T \int \phi(t)v \cdot \nabla \phi^\gamma,
\]
which leads to that
\[
\kappa \int_0^T \int \phi(t)v \cdot \nabla (\phi^\gamma) = \frac{1}{\gamma - 1} \int_0^T \int \phi(t)\partial_t \phi^\gamma. \quad (3.7)
\]
Plugging (3.7) into (3.6), we conclude that
\[
\kappa \int_0^T \int \phi(t)v^\varepsilon \nabla (\phi^\gamma)^\varepsilon = \kappa \int_0^T \int \phi(t)[v^\varepsilon \nabla (\phi^\gamma)^\varepsilon - v \nabla (\phi^\gamma)] + \kappa \frac{1}{\gamma - 1} \int_0^T \int \phi(t)\partial_t \phi^\gamma. \quad (3.8)
\]
Substituting (3.2), (3.4) and (3.8) into (3.11), we observe that
\[
- \int_0^T \int \phi(t)\partial_t \left( \frac{|v^\varepsilon|^2}{2} + \kappa \frac{1}{\gamma - 1} \phi^\gamma \right) \\
= - \int_0^T \int \phi(t)v^\varepsilon \left[ \partial_t (\phi v)^\varepsilon - \partial_t (\phi v^\varepsilon) \right] \\
+ \int_0^T \int \phi(t)v^\varepsilon \text{div} [(\phi v \otimes v)^\varepsilon - \phi (v \otimes v)^\varepsilon] - \int_0^T \int \phi(t) \nabla v^\varepsilon \phi [v \otimes v]^\varepsilon - v^\varepsilon \otimes v^\varepsilon] \\
+ \frac{1}{2} \int_0^T \int \phi(t)|v^\varepsilon|^2 \text{div} [\phi v^\varepsilon - (\phi v)^\varepsilon] - \frac{1}{2} \int_0^T \int \phi(t)|v^\varepsilon|^2 (\phi^\gamma - \phi_t) \\
- \int_0^T \int \phi(t)\kappa|v^\varepsilon \nabla (\phi^\gamma)^\varepsilon - v \nabla (\phi^\gamma)|. \quad (3.9)
\]
At this stage, we can apply the commutators on mollifying kernel to pass the limit in the right hand-side of (3.11). In light of the Hölder’s inequality and Lions type commutators on mollifying kernel Lemma (2.1), we get
\[
\int_s^t \int \phi(t)v^\varepsilon \left[ \partial_t (\phi v)^\varepsilon - \partial_t (\phi v^\varepsilon) \right] \leq C|v^\varepsilon|_{L^p(L^q)} ||\partial_t (\phi v)^\varepsilon - \partial_t (\phi v^\varepsilon)||_{L^p(L^{q+\tau})} \\
\leq C||v||_{L^p(L^q)}^2 ||\partial_t||_{L^{p-\tau}(L^q)} + ||\nabla \phi||_{L^{p-\tau}(L^{q+\tau})}; \quad (3.10)
\]
\[
\left| \int_0^T \int \phi(t)v^\varepsilon \text{div} [(\phi v \otimes v)^\varepsilon - \phi (v \otimes v)^\varepsilon] \right| \\
\leq C||\text{div} [(\phi v \otimes v)^\varepsilon - \phi (v \otimes v)^\varepsilon]||_{L^p(L^q)} ||v^\varepsilon||_{L^p(L^q)} \\
\leq C||\text{div} [(\phi v \otimes v)^\varepsilon - \phi (v \otimes v)^\varepsilon]||_{L^p(L^q)} ||v^\varepsilon||_{L^p(L^q)} \\
\leq C||v^\varepsilon||_{L^p(L^q)}^2 ||\partial_t||_{L^{p-\tau}(L^{q+\tau})} + ||\nabla \phi||_{L^{p-\tau}(L^{q+\tau})}; \quad (3.11)
\]
and
\[\int_0^T \int |v^\varepsilon|^2 \text{div}[qv^\varepsilon - (qv)^\varepsilon] \leq C\|v^\varepsilon\|^2 \|\text{div}[qv^\varepsilon - (qv)^\varepsilon]\|_{L^p(L^{\frac{n}{p}})} \leq C\|v^\varepsilon\|_{L^p(L^{\frac{n}{p}})} + \|\nabla \varrho\|_{L^p(L^{\frac{n}{p}})}\]  \hfill (3.12)

In addition, we derive from Lemma 2.1 that, as \(\varepsilon \to 0\),
\[\int_0^T \int v^\varepsilon \left[ \partial_t (qv^\varepsilon) - \partial_t (qv)^\varepsilon \right] \to 0,\]
\[\int_0^T \int |v^\varepsilon|^2 \text{div}[qv^\varepsilon - (qv)^\varepsilon] \to 0,\]
\[\int_0^T \int |v^\varepsilon|^2 \text{div}[qv^\varepsilon - (qv)^\varepsilon] \to 0.\]

The classical Constantin-E-Titi type commutators on mollifying kernel (2.6) and the H\"older’s inequality allow us to obtain
\[\int_0^T \int \phi(t) \nabla v^\varepsilon \varrho[(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon] \leq C\|\varrho\|_{L^p(L^{\frac{n}{p}})} \|(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon\|_{L^p(L^{\frac{n}{p}})} \|\nabla v^\varepsilon\|_{L^p(L^q)} \leq C\varepsilon^{3\alpha - 1} \|\varrho\|_{L^p(L^{\frac{n}{p}})} \|v^\varepsilon\|^3_{L^p(B^\alpha_{q,\infty})},\]  \hfill (3.13)

which in turn means
\[\int_0^T \int \phi(t) \nabla v^\varepsilon \varrho[(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon] \to 0.\]  \hfill (3.14)

The H\"older’s inequality guarantees that
\[\int_0^T \int \phi(t)|v^\varepsilon|^2 (\varrho^\varepsilon - \varrho_t) \leq C\||v^\varepsilon|^2\|_{L^p(L^{\frac{n}{p}})} \|\varrho^\varepsilon - \varrho_t\|_{L^p(L^{\frac{n}{p}})} \leq C\varepsilon^{3\alpha - 1} \|\varrho\|_{L^p(L^{\frac{n}{p}})} \|v^\varepsilon\|^3_{L^p(B^\alpha_{q,\infty})}.\]  \hfill (3.15)

As a consequence, the standard properties of the mollification help us to see that, as \(\varepsilon \to 0\),
\[\int_0^T \int \phi(t)|v^\varepsilon|^2 (\varrho^\varepsilon - \varrho_t) \to 0.\]

It follows from the H\"older’s inequality that
\[\|\nabla (\varrho^\gamma)\|_{L^p(L^{\frac{n}{p}})} = \gamma\|\nabla \varrho(\varrho^{\gamma - 1})\|_{L^p(L^{\frac{n}{p}})} \leq C\|\nabla \varrho\|_{L^p(L^{\frac{n}{p}})} \|\varrho\|_{L^{p(\frac{n}{2})}} \|\varrho^{\gamma - 1}\|_{L^{p(\frac{n}{2})}} \|\varrho(\varrho^{\gamma - 1})\|_{L^{p(\frac{n}{2})}} \leq C\varepsilon^{3\alpha - 1} \|\varrho\|_{L^p(L^{\frac{n}{p}})} \|v^\varepsilon\|^3_{L^p(B^\alpha_{q,\infty})},\]  \hfill (3.16)

which in turn implies that
\[\int_0^T \int \phi(t)[v^\varepsilon \nabla (\varrho^\gamma)^\varepsilon - v \nabla (\varrho^\gamma)] \to 0.\]  \hfill (3.17)

Combining the above estimates (3.10)-(3.14) and (3.19), we get
\[\int_0^T \int \phi(t) \partial_t \left( \frac{1}{2} \varrho |v|^2 + \kappa \frac{\varrho^{\gamma}}{\gamma - 1} \right) dx = 0.\]  \hfill (3.18)
Proof of Theorem 1.3. A slight modify the above proof allows us to obtain the global energy conservation up to the initial time. Indeed, repeating the derivation above, we infer that

$$- \int_0^T \int \phi(t) \left( \frac{|v|^2}{2} + \kappa \frac{1}{\gamma - 1} \phi \right) dt = - \int_0^T \int \phi(t) v^\epsilon \left[ \partial_t (gv)^\epsilon - \partial_t (gv^\epsilon) \right]$$

$$+ \int_0^T \int \phi(t) v^\epsilon \text{div} [(gv \otimes v)^\epsilon - g(v \otimes v)^\epsilon] - \int_0^T \int \phi(t) \nabla v^\epsilon g [(v \otimes v)^\epsilon - v^\epsilon \otimes v^\epsilon]$$

$$+ \frac{1}{2} \int_0^T \int \phi(t) |v^\epsilon|^2 \text{div} [(gv^\epsilon)^\epsilon - (gv^\epsilon)^\epsilon] - \frac{1}{2} \int_0^T \int \phi(t) |v^\epsilon|^2 (g_t - \delta_t)$$

$$- \int_0^T \int \phi(t) \kappa [v^\epsilon \nabla (g^\epsilon)^\epsilon - \nabla (g^\epsilon)].$$  \hspace{1cm} (3.19)

Note that the triangle inequality and Hölder’s inequality guarantee that

$$\| \partial_t \phi \|_{L^{\frac{2q}{q-2}}(L^{\frac{2q}{q-3}})} \leq C \| \sqrt{\phi} \partial_t \sqrt{\phi} \|_{L^{\frac{2q}{q-3}}(L^{\frac{2q}{q-3}})} \leq C \| \sqrt{\phi} \|_{L^{2k}(L^{2l})} \| \partial_t \sqrt{\phi} \|_{L^{\frac{2kq}{2kq-3l-p}}(L^{\frac{2kq}{2kq-3l-p}})},$$  \hspace{1cm} (3.20)

and

$$\| \nabla \phi \|_{L^{\frac{2q}{q-2}}(L^{\frac{2q}{q-3}})} \leq C \| \sqrt{\phi} \nabla \sqrt{\phi} \|_{L^{\frac{2q}{q-3}}(L^{\frac{2q}{q-3}})} \leq C \| \sqrt{\phi} \|_{L^{2k}(L^{2l})} \| \nabla \sqrt{\phi} \|_{L^{\frac{2kq}{2kq-3l-p}}(L^{\frac{2kq}{2kq-3l-p}})},$$  \hspace{1cm} (3.21)

Substituting (3.20) and (3.21) into (3.10) - (3.12) and (3.16), following the path of (3.18), we conclude that

$$- \int_0^T \int \phi(t) \left( \frac{|v|^2}{2} + \kappa \frac{1}{\gamma - 1} \phi \right) dt = 0.$$

The next objective is to get the energy equality up to the initial time $t = 0$ by the similar method in [34], for the convenience of the reader and the integrity of the paper, we give the details.

First we prove the continuity of $\sqrt{\phi} v(t)$ in the strong topology as $t \to 0^+$. To do this, we define the function $f$ on $[0, T]$ as

$$f(t) = \int_{\mathbb{T}^d} (gv)(t, x) \cdot \varphi(x) dx, \text{ for any } \varphi(x) \in \mathcal{D}(\mathbb{T}^d),$$

which is a continuous function with respect to $t \in [0, T]$. Moreover, since

$$g \in L^{\infty}(0, T; L^\gamma(\mathbb{T}^d)) \text{ and } \sqrt{\phi} v \in L^\infty(0, T; L^2(\mathbb{T}^d)),$$

we can obtain $gv \in L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d))$.

From the momentum equation, we have

$$\frac{d}{dt} \int_{\mathbb{T}^n} (gv)(t, x) \cdot \varphi(x) dx = \int_{\mathbb{T}^n} gv \otimes v : \nabla \varphi(x) - \pi \text{div} \varphi(x) dx,$$

which is bounded for any function $\varphi \in \mathcal{D}(\mathbb{T}^d)$. Then it follows from the Corollary 2.1 in [18] that

$$gv \in C([0, T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\mathbb{T}^d)).$$  \hspace{1cm} (3.22)
On the other hand, since
\[ \nabla \varphi^\gamma \in L^{k,p-1} \left( L^{\frac{1}{q-p+1}} \right), \]
\[ \partial_t \varphi^\gamma \in L^{k,p-1} \left( L^{\frac{1}{q-p+1}} \right) \hookrightarrow L^{k,p-1} \left( L^{\frac{1}{q-p+1}} \right), \]
and
\[ \nabla \sqrt{\varphi} \in L^{2k,p-2} \left( L^{\frac{2q-p}{q-p}} \right), \partial_t \sqrt{\varphi} \in L^{2k,p-2} \left( L^{\frac{2q-p}{q-p}} \right) \hookrightarrow L^{2k,p-2} \left( L^{\frac{2q-p}{q-p}} \right). \]

Hence, using the Aubin-Lions Lemma 2.4, we can obtain

\[ \varphi^\gamma \in C([0,T]; L^{\frac{1}{q-p+1}}(\mathbb{T}^d)) \text{ and } \sqrt{\varphi} \in C([0,T]; L^{\frac{2q}{q-p}}(\mathbb{T}^d)), \]
for \( k \geq \frac{(\gamma-1)(d+p)}{2q-d+3} \), \( p > 3 \) and \( q > \max\{3, \frac{d(p-3)}{2}\} \).

Meanwhile, using the natural energy (3.24), (3.25) and (3.26), we have

\[
0 \leq \lim_{t \to 0} \int |\sqrt{\varphi v} - \sqrt{\varphi_0}v_0|^2 dx \\
= 2\lim_{t \to 0} \left( \int \frac{1}{2}|\varphi|^2 + \frac{1}{\gamma - 1} \varphi^\gamma \right) dx - \int \left( \frac{1}{2} \varphi_0 |v_0|^2 + \frac{1}{\gamma - 1} \varphi_0^\gamma \right) dx \\
+ 2\lim_{t \to 0} \left( \int \sqrt{\varphi_0}v_0 \left( \sqrt{\varphi_0}v_0 - \sqrt{\varphi}v \right) dx + \frac{1}{\gamma - 1} \int (\varphi_0^\gamma - \varphi^\gamma) dx \right) \\
\leq 2\lim_{t \to 0} \int \sqrt{\varphi_0}v_0 \left( \sqrt{\varphi_0}v_0 - \sqrt{\varphi}v \right) dx \\
= 2\lim_{t \to 0} \int v_0 (\varphi_0 v_0 - v_0) dx + 2\lim_{t \to 0} \int v_0 \sqrt{\varphi} (\sqrt{\varphi} - \sqrt{\varphi_0}) dx = 0,
\]
from which it follows

\[ \sqrt{\varphi}v(t) \to \sqrt{\varphi}v(0) \text{ strongly in } L^2(\mathbb{T}^d) \text{ as } t \to 0^+. \] (3.25)

Similarly, one has the right temporal continuity of \( \sqrt{\varphi}v \) in \( L^2(\mathbb{T}^d) \), hence, for any \( t_0 \geq 0 \), we infer that

\[ \sqrt{\varphi}v(t) \to \sqrt{\varphi}v(t_0) \text{ strongly in } L^2(\mathbb{T}^d) \text{ as } t \to t_0^+. \] (3.26)

Before we go any further, it should be noted that (3.18) remains valid for function \( \phi(t) \) belonging to \( W^{1,\infty} \) rather than \( C^1 \), then for any \( t_0 > 0 \), we redefine the test function \( \phi(t) \) as \( \phi_{\tau} \) for some positive \( \tau \) and \( \alpha \) such that \( \tau + \alpha < t_0 \), that is

\[
\phi_{\tau}(t) = \begin{cases} 
0, & 0 \leq t \leq \tau, \\
\frac{t}{\tau}, & \tau \leq t \leq \tau + \alpha, \\
1, & \tau + \alpha \leq t \leq t_0, \\
\frac{t_0 - t}{\alpha}, & t_0 \leq t \leq t_0 + \alpha, \\
0, & t_0 + \alpha \leq t.
\end{cases}
\] (3.27)

Then substituting this test function into (3.18), we arrive at

\[ -\int_\tau^{\tau+\alpha} \int_0^\alpha \left( \frac{1}{2} \varphi v^2 + \frac{1}{\gamma - 1} \varphi^\gamma \right) dx + 1 \int_{t_0}^{t_0+\alpha} \int_0^\alpha \left( \frac{1}{2} \varphi v^2 + \frac{1}{\gamma - 1} \varphi^\gamma \right) dx = 0. \] (3.28)
Taking $\alpha \to 0$ and the Lebesgue point Theorem, we deduce that

$$-\int \left(\frac{1}{2} \varrho v^2 + \kappa \frac{1}{\gamma - 1} \varrho^\gamma \right)(\tau) \, dx + \int \left(\frac{1}{2} \varrho v^2 + \kappa \frac{1}{\gamma - 1} \varrho^\gamma \right)(t_0) \, dx = 0. \quad (3.29)$$

Finally, letting $\tau \to 0$, using (3.22) and (3.25), we can obtain

$$\int \left(\frac{1}{2} \varrho v^2 + \kappa \frac{1}{\gamma - 1} \varrho^\gamma \right)(t_0) \, dx = \int \left(\frac{1}{2} \varrho_0 v_0^2 + \kappa \frac{1}{\gamma - 1} \varrho_0^\gamma \right) \, dx. \quad (3.30)$$

Then we complete the proof of Theorem 1.3. \hfill \Box

### 3.2 Vacuum case

**Proof of Theorem 1.5.** For the vacuum case, we need to mollify $v$ in both the time and space directions. With the proof of of Theorem 1.3 in hand, we just need to replace (3.13) by the following estimate

$$\int_0^T \int \phi(t) \nabla v^\varepsilon \cdot [(v \otimes v)^\varepsilon - v^\varepsilon \otimes v^\varepsilon] \, dx \leq C \varepsilon^{3(1-\alpha)} \|\varrho\|_{L^p \left(\frac{2}{L^{\frac{q}{2}}} \right)} \|\nabla v^\varepsilon\|_{L^p \left(\frac{2}{L^{\frac{q}{2}}} \right)} \|v^\varepsilon\|_{B^0_{p,\infty}(B^\alpha_{q,\infty})}^3 \|v^\varepsilon\|_{B^0_{p,\infty}(B^\beta_{q,\infty})}. \quad (3.31)$$

where the Constantin-E-Titi type commutators (2.5) in Lemma 2.2 and the Hölder’s inequality are used. The rest proof is the same as the one in the last theorem. \hfill \Box

### Acknowledgement

Wang was partially supported by the National Natural Science Foundation of China under grant (No. 11971446, No. 12071113 and No. 11601492). Ye was partially supported by the National Natural Science Foundation of China under grant (No.11701145) and China Postdoctoral Science Foundation (No. 2020M672196). Yu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11901040), Beijing Natural Science Foundation (BNSF) (No. 1204030) and Beijing Municipal Education Commission (KM202011232020).

### References

[1] I. Akramov, T. Debiec, J. W. D. Skipper and E. Wiedemann, Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. Anal. PDE. 13 (2020), 789–811

[2] C. Bardos, P. Gwiazda, A. Świerczewska-Gwiazda, E. S. Titi and E. Wiedemann, Onsager’s conjecture in bounded domains for the conservation of entropy and other companion laws. Proc. R. Soc. A, 475 (2019), 18 pp.
[3] D. Chae, On the conserved quantities for the weak solutions of the Euler equations and the quasi-geostrophic equations, Commun. Math. Phys. 266 (2006), 197–210.

[4] R. M. Chen and C. Yu, Onsager’s energy conservation for inhomogeneous Euler equations, J. Math. Pures Appl. 131 (2019), 1–16.

[5] R. M. Chen, A. F. Vasseur and C. Yu, Global ill-posedness for a dense set of initial data to the isentropic system of gas dynamics. arXiv:2103.04905, 2021.

[6] A. Cheskidov and P. Constantin, S. Friedlander and R. Shvydkoy, Energy conservation and Onsager’s conjecture for the Euler equations. Nonlinearity, 21 (2008), 1233–52.

[7] A. Cheskidov and X. Luo, Energy equality for the Navier-Stokes equations in weak-time Onsager spaces. Nonlinearity, 33 (2020), 1388–1403.

[8] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. Comm. Pure. Appl. Math., 58(2015), 1157–1190.

[9] E. Chiodaroli, O. Kreml, V. Mácha, and S. Schwarzacher. Non-uniqueness of admissible weak solutions to the compressible Euler equations with smooth initial data. Trans. Amer. Math. Soc., 374 (2021), 2269–2295.

[10] P. Constantin, E. Weinan and E.S. Titi, Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. Commun. Math. Phys. 165 (1994), 207–209.

[11] T. Drivas and G. Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. Commun. Math. Phys., 359 (2018), 733–763.

[12] R. J. DiPerna and P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math., 98 (1989), 511–547.

[13] C. De Lellis and L J. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal., 195 (2010), 225–260.

[14] C. De Lellis, L J. Székelyhidi. The Euler equations as a differential inclusion, Ann. Math. 170 (2009), 1417–1436.

[15] C. De Lellis, L J. Székelyhidi. Dissipative continuous Euler flows. Invent Math. 193(2013 ), 377–407.

[16] C. De Lellis and L J. Székelyhidi. Dissipative Euler flows and onsager’s conjecture. J Eur Math Soc. 16 (2014), 1467–1505.

[17] T. Luo, C. Xie and Z. Xin. Non-uniqueness of admissible weak solutions to compressible Euler systems with source terms. Adv. Math., 291(2016) 542–583.

[18] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, 2004.

[19] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda and E. Wiedemann, Regularity and energy conservation for the compressible euler equations. Arch Ration Mech Anal. 223 (2017), 1375–1395.

[20] E. Feireisl, S. S. Ghoshal and A. Jana, On uniqueness of dissipative solutions to the isentropic Euler system., Commun. Partial Differ. Equ., 44 (2019), 1285–1298.
[21] P. Gwiazda, M. Michálek and A.Świerczewska-Gwiazda, A note on weak solutions of conservation laws and energy/entropy conservation. Arch Ration Mech Anal., 229 (2018), 1223–1238.

[22] P. Isett, A proof of Onsager’s conjecture. Ann. of Math. 188 (2018), 871–963.

[23] I. Lacroix-Violet and A. Vasseur, Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit, J. Math. Pures Appl. 114 (2018), 191–210.

[24] T. M. Leslie and R. Shvydkoy, The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations J. Differential Equations. 261 (2016), 3719–3733.

[25] J. L. Lions, Sur la régularité et l’unicité des solutions turbulentes des équations de Navier Stokes. Rend. Semin. Mat. Univ. Padova, 30 (1960), 16–23.

[26] P. L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford University Press, New York, 1998.

[27] P. L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2. Compressible Models, Oxford University Press, New York, 1998.

[28] Q. Nguyen, P. Nguyen and B. Tang, Onsager’s conjecture on the energy conservation for solutions of Euler equations in bounded domains. J. Nonlinear Sci. 29 (2019), 207–213.

[29] Q. Nguyen, P. Nguyen and B. Tang, Energy conservation for inhomogeneous incompressible and compressible Euler equations. J. Differential Equations, 269 (2020), 7171–7210.

[30] L. Onsager, Statistical hydrodynamics, Nuovo Cim. (Suppl.) 6 (1949), 279–287.

[31] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1987), 65–96.

[32] Y. Ye, Y. Wang and W. Wei, Energy equality in the isentropic compressible Navier-Stokes equations allowing vacuum. [arXiv:2108.09425]

[33] Y. Ye, Y. Wang and H. Yu, Energy equality for the isentropic compressible Navier-Stokes equations without upper bound of the density. Preprint. 2021.

[34] C. Yu. Energy conservation for the weak solutions of the compressible Navier-Stokes equations. Arch. Ration. Mech. Anal. 225 (2017), 1073–1087.