The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface.

Kieran G. O’Grady
October 1 1995

Abstract. We prove that the weight-two Hodge structure of moduli spaces of torsion-free sheaves on a K3 surface is as described by Mukai (the rank is arbitrary but we assume the first Chern class is primitive). We prove the moduli space is an irreducible symplectic variety (by Mukai’s work it was known to be symplectic). By work of Beauville, this implies that its $H^2$ has a canonical integral non-degenerate quadratic form; Mukai’s recepee for $H^2$ includes a description of Beauville’s quadratic form. As an application we compute higher-rank Donaldson polynomials of K3 surfaces.

Recently Jun Li [Li1, Li2] determined the stable rational Hodge structure on the $n$-th cohomology, for $n \leq 2$, of moduli spaces of rank-two torsion-free sheaves on an arbitrary projective surface (stable means: for large enough dimension of the moduli space). It seems worthwhile to study in greater detail the integral Hodge structure of these moduli spaces: in this paper we consider the case of a K3 surface. Mukai [M1, M2, M3] studied extensively moduli of sheaves on a K3: in [M2] he determined the Hodge structure of two-dimensional moduli spaces, and in [M3, Th. (5.15)] there is a beautiful description of the weight-two Hodge structure when the rank is at most two, and the dimension of the moduli space is greater than two. In this paper we prove that Mukai’s description of $H^2$ is valid in arbitrary rank. (But notice that we assume the first Chern class is primitive.)

Statement of the results.

Let $S$ be a projective K3 surface. The Mukai lattice [M2, § 2] consists of $H^*(S; \mathbb{Z})$ endowed with the symmetric bilinear form

$$
\langle \alpha, \beta \rangle := -\int_S \alpha^* \wedge \beta,
$$

where for $\alpha = \alpha^0 + \alpha^1 + \alpha^2 \in H^*(S)$, with $\alpha^i \in H^{2i}(S)$, we let $\alpha^* := \alpha^0 - \alpha^1 + \alpha^2$. Setting

$$
F^0H^*(S) := H^*(S), \quad F^1H^*(S) := H^0(S) \oplus F^1H^2(S) \oplus H^4(S), \quad F^2H^*(S) := F^2H^2(S),
$$

$(H^*(S; \mathbb{Z}), F^*)$ becomes a weight-two Hodge structure, with the additional datum of Mukai’s quadratic form defined above. If $F$ is a sheaf on $S$, following Mukai we set

$$
v(F) := ch(F)(1 + \omega) = \text{rk}(F) + c_1(F) + (\chi(F) - r) \omega,
$$

where $\omega \in H^4(S; \mathbb{Z})$ is the fundamental class. Notice that, since the intersection form of $S$ is even, $v(F) \in H^*(S; \mathbb{Z})$, and of course $v(F) \in F^1H^*(S)$. An element of $F^1 \cap H^*(S; \mathbb{Z})$ will be called a Mukai vector. Let $H$ be an ample divisor on $S$, and $v$ be a Mukai vector: we let $M_v(S, H)$ be the moduli space of Gieseker-Maruyama $H$-semistable torsion-free sheaves $F$ on $S$ such that $v(F) = v$. (We abbreviate to $M_v(H)$ whenever the surface $S$ is fixed.) Thus $M_v(H)$ is a projective scheme. Now assume $H$ is $v$-stabilizing, i.e. that all sheaves parametrized by $M_v(H)$ are $H$-slope-stable (see Proposition (II.1)). In this case the moduli space is smooth [M1, Th. (0.3)], of dimension equal to

$$
d(v) := 2 + \langle v, v \rangle.
$$
Furthermore there exists a quasi-tautological family of sheaves [M2, Th. (A.5)], i.e. a sheaf $\mathcal{F}$ on $S \times \mathcal{M}_v(H)$, flat over $\mathcal{M}_v(H)$, such that if $[F] \in \mathcal{M}_v(H)$ represents the isomorphism class of $F$, then

$$\mathcal{F}|_{S \times [F]} \cong F^{\oplus \sigma}$$

for some $\sigma > 0$. We can assume $\sigma$ is independent of $[F]$, and we will denote it by $\sigma(\mathcal{F})$. Letting $\pi: S \times \mathcal{M}_v \to S$ and $\rho: S \times \mathcal{M}_v \to \mathcal{M}_v$ be the projections, we set

$$\theta_{\mathcal{F}}(\alpha) := \frac{1}{\sigma(\mathcal{F})} \rho_* [\text{ch}(\mathcal{F})^*(1 + \pi^*\omega)\pi^*\alpha]_3,$$

where $\alpha \in H^*(S; \mathbb{C})$, and $[\cdot]_3$ is the component belonging to $H^6(S \times \mathcal{M}_v(H))$. If $\mathcal{E}$ is another quasi-tautological sheaf then there exist vector bundles $\xi, \eta$ on $\mathcal{M}_v(H)$ such that $\mathcal{E} \otimes \rho^*\xi \cong \mathcal{F} \otimes \rho^*\eta$ [M2, Th. (A.5)]. From this it follows easily that the restriction of $\theta_{\mathcal{F}}$ to $v^\perp \subset H^*(S; \mathbb{C})$ is independent of the choice of a quasi-tautological family: thus we get a canonical Mukai map [M3, (5.14)]:

$$\theta_v: v^\perp \to H^2(\mathcal{M}_v(H); \mathbb{C}).$$

We will prove the following

(2) **Main Theorem.** Keeping notation as above, assume that $v^1$, the component of $v$ belonging to $H^2(S; \mathbb{Z})$, is primitive. Let $H$ be a $v$-stabilizing ample divisor on $S$. Assume that the expected dimension of $\mathcal{M}_v(H)$ is greater than 2, i.e. that $(v, v) > 0$. Then:

1. The moduli space $\mathcal{M}_v(H)$ is an irreducible symplectic variety, i.e. simply connected with a symplectic form spanning $H^{2,0}$, deformation equivalent to a symplectic projective birational model of $T^{[n]}$, where $T$ is a projective $K3$ surface, $n := d(v)/2$, and $T^{[n]}$ is the Hilbert scheme parametrizing length-$n$ subschemes of $T$. In particular, by Beauville [B, Th. 5], $H^2(\mathcal{M}_v(H); \mathbb{Z})$ has a canonical integral quadratic form.

2. The map $\theta_v$ is an isomorphism of integral Hodge structures, and an isometry if $v^\perp$ is provided with the restriction of Mukai’s form, and $H^2(\mathcal{M}_v(H))$ is provided Beauville’s quadratic form.

**Comments.**

- Mukai [M2] proved that if the dimension of $\mathcal{M}_v(H)$ is two, i.e. $(v, v) = 0$, then the map $\theta_v$ induces an isomorphism of Hodge structures between $v^\perp/Cv$ and $H^2(\mathcal{M}_v(H); \mathbb{C})$ (the only hypothesis is that semistability is equivalent to stability). Our proof of the main theorem can be easily adapted to this case (with the hypothesis that $v^1$ is primitive). However in a certain sense Mukai’s proof is preferable: it is conceptual and it “explains” the definition of $\theta_v$ and of the Mukai lattice. A conceptual proof of Theorem (2) would be very interesting: it might indicate how to extend the result to arbitrary surfaces.

- We have some restrictions on the choice of $v$: it is natural to expect that the theorem holds whenever semistability coincides with stability.

- The key observation in our proof of the main theorem is the following: if $S$ is an elliptic $K3$ and $v^1$ has degree one on the elliptic fibers, then the moduli space is birational to $S^{[n]}$. It is interesting to notice that one can use the main theorem to show that in general the moduli space is not birational to any $T^{[n]}$ [M3, p.167].

**Plan of the paper.** First we prove the main theorem when $S$ is an elliptic $K3$ with a section, $v^1$ has degree one on the fibers of the elliptic fibration, and the ample divisor $H$ is suitable in the sense of Friedman [F1]. In this case the moduli space is birational to $S^{[n]}$; one possible route to this result would be to proceed as in [F2, Th. (3.14)], i.e. to construct every stable vector-bundle as an elementary modification of a fixed rigid bundle. Since we also want to prove Item (2) of the
main theorem, we will proceed differently. We will prove that \( M_v(H) \) is birational to a moduli space of rank one less; iterating we get down to rank one, and since the moduli space of rank-one torsion-free sheaves on \( S \) (with \( c_2 = n \)) is isomorphic to \( S^{[n]} \), we conclude that \( M_v(H) \) is birational to \( S^{[n]} \). This procedure allows us to verify Item (2) by induction on the rank, the case of rank one being trivial; all of this is done in Section I. In Section II we prove the theorem in general: the idea is to deform \( S \) to an elliptic \( K3 \) (this part is similar to an argument in [GoHu]). In Section III we give an application of the main theorem: we compute higher-rank Donaldson polynomials. The last section is devoted to the proof of some technical results on polarizations.

Acknowledgments. The present work owes a lot to the papers of Mukai [M1, M2, M3].

I. The case of an elliptic \( K3 \) surface.

In order to state the main result of this section we need some preliminaries on the choice of a polarization. For a surface \( S \), let \( A(S) \subset H^{1,1}(S; \mathbb{R}) \) be the ample cone, i.e. the real convex cone spanned by Chern classes of ample divisors.

**Definition.** Let \( k > 0 \). A \( k \)-wall of \( A(S) \) consists of the intersection \( L^\perp \cap A(S) \), where \( L \) is a divisor on \( S \) such that \(-k \leq L^2 < 0\).

An open \( k \)-chamber \( C \subset A(S) \) is a connected component of the complement of the union of all \( k \)-walls. An ample divisor on \( S \) is \( k \)-generic if it does not belong to any \( k \)-wall.

Let \( S \) be a \( K3 \), and \( v \in H^*(S; \mathbb{Z}) \) be a Mukai vector. We will often consider \(|v|\)-walls, where

\[
|v| := \frac{(\langle v, v \rangle)^2}{4} + \frac{(\langle v, v \rangle)^4}{2}.
\]

(I.0.2)

Now we specialize to the case of an elliptic \( K3 \), i.e. a \( K3 \) surface \( S \) together with a linear pencil \( |C| \), where \( C \subset S \) is an elliptic curve. (Thus \(|C|\) defines a morphism to \( \mathbb{P}^1 \) with generic fiber an elliptic curve.)

**Definition.** Let \((S, |C|)\) be an elliptic \( K3 \), and \( k \) be a positive integer. An ample divisor \( H \) on \( S \) is \( k \)-suitable if it is \( k \)-generic and if \( C \) is in the closure of the unique open chamber containing \( H \).

Notice that \( H \) is \( k \)-generic (suitable) if and only if any ample divisor in \( \mathbb{Q}_+H \) is \( k \)-generic (suitable); hence it makes sense to consider \( k \)-suitable polarizations. One can show in general that \( k \)-suitable polarizations exist for any \( k \). We will only need the following special case.

**Lemma.** Let \( S \) be an elliptic \( K3 \) surface with a section \( \Sigma \) of the elliptic fibration, and such that \( Pic(S) = \mathbb{Z}[\Sigma] \oplus \mathbb{Z}[C] \), where \( C \) is an elliptic fiber. Let \( H \sim (a\Sigma + bC) \) be an ample divisor such that

\[
\frac{b}{a} \geq k + 1.
\]

Then \( H \) is \( k \)-suitable.

**Proof.** Proving that \( H \) is \( k \)-suitable is equivalent to proving that \( sign(L \cdot H) = sign(L \cdot C) \) for all divisors \( L \in Pic(S) \) satisfying (I.0.1). (Notice that \((L \cdot C) \neq 0\) for any such \( L \).) Clearly it suffices to test only the \( L \) such that \((L \cdot C) > 0\). Let \( L \sim (x\Sigma + yC) \). Then \((L \cdot C) > 0\) is equivalent to \( x > 0 \), and (I.0.1) reads

\[
0 < 2x(x - y) \leq k.
\]

(*)
Using our hypothesis and the positivity of $a, x$ we get

$$L \cdot H = (b - 2a)x + ay = a \cdot \left[ \left( \frac{b}{a} - 1 \right) x - (x - y) \right] \geq a \cdot (kx - (x - y)) .$$

Since $(\ast)$ holds, and since $x, (x - y)$ are positive integers, we have $kx \geq 2(x - y)$. By the above inequality we conclude that $(L \cdot H) > 0$. q.e.d.

A divisor $D$ (or the first Chern class of a divisor) on an elliptic $K3$ is a numerical section if $D \cdot C = 1$, where $C$ is an elliptic fiber. This section is devoted to proving the following result.

(I.0.4) **Theorem.** Let $S$ be an elliptic $K3$. Assume that the elliptic fibration has a section, and that $\rho(S) = 2$. Let $v \in H^*(S; \mathbb{Z})$ be a Mukai vector such that $v^1$ is a numerical section and $d(v) > 2$. Let $H$ be a $|v|$-suitable ample divisor on $S$. Then $\mathcal{M}_v(H)$ is an irreducible symplectic variety birational to $S[n]$, where $n := d(v)/2$. Furthermore the map

$$\theta_v: H^*(S; \mathcal{C}) \to H^2(\mathcal{M}_v(H); \mathcal{C})$$

is an isomorphism of integral Hodge structures, and an isometry.

I.1. Preliminaries.

Let $V$, $T$ be schemes. A family of sheaves on $V$ parametrized by $T$ consists of a sheaf $\mathcal{F}$ on $V \times T$, flat over $T$. For $t \in T$ we let $\mathcal{F}_t := \mathcal{F}|_{V \times \{t\}}$.

(I.1.1). We recall some well-known facts about stable bundles on elliptic curves [A] which will be useful in the course of the section. Let $C$ be an elliptic curve. Then up to isomorphism there exists a unique stable bundle $V_r^L$ on $C$ of given rank $r$ and determinant $L$. If $L$ has degree one the picture is particularly simple. By Hirzebruch-Riemann-Roch, Serre duality and stability we have $h^0(V_r^L) = 1$. A non-zero section of $V_r^L$ gives rise to a non-split exact sequence

$$0 \to \mathcal{O}_C \to V_r^L \to V_{r-1}^L \to 0 .$$

Conversely, since $\dim \text{Ext}^1(V_{r-1}^L, \mathcal{O}_C) = 1$, the bundle $V_r^L$ can be constructed as the unique non-split extension above.

Cayley-Bacharach. We collect some well-known criteria for local-freeness of extensions [GrHa, 729-731] involving the Cayley-Bacharach property. Let $S$ be a surface. Let $L, M$ be line-bundles on $S$. Let $X, Y$ be 0-dimensional reduced subschemes of $S$, and $I_X, I_Y$ be their ideal sheaves. We consider extensions

$$0 \to I_X \otimes L \to F \to I_Y \otimes M \to 0 . \quad (\text{I.1.2})$$

(I.1.3). Let $P \in Y$. If there exists a section of $L^{-1} \otimes M \otimes K_S$ vanishing at all points of $(Y - P)$ and non-zero at $P$, then (I.1.2) is split in a neighborhood of $P$, and hence $F$ is singular at $P$.

Now assume that the image of the evaluation map

$$e_Y: H^0(L^{-1} \otimes M \otimes K_S) \to H^0(L^{-1} \otimes M \otimes K_S|_Y)$$

has codimension one. If (I.1.2) is non-split at one point of $Y$, the following converse of (I.1.3) holds.
(I.1.4) Suppose $\text{Im}(e_Y)$ has codimension one, and assume that Extension (I.1.2) is non-split in the neighborhood of one point (at least) of $Y$. Let $P \in Y$. If all the sections of $L^{-1} \otimes M \otimes K_S$ vanishing at $(Y - P)$ vanish also at $P$, then $F$ is locally-free at $P$.

(I.1.5) Suppose that $h^1(I_X \otimes M^{-1} \otimes L) = 0$. Let hypotheses be as in (I.1.4), except that instead of assuming (I.1.2) is non-split in the neighborhood of one point of $Y$, we only assume (I.1.2) is globally non-split. Then the conclusion of (I.1.4) hold.

Proof of (I.1.3)-(I.1.4)-(I.1.5). To prove the three statements consider the exact sequence

$$H^1(Hom(I_Y \otimes M, I_X \otimes L)) \to \text{Ext}^1(I_Y \otimes M, I_X \otimes L) \xrightarrow{\partial} H^0(\text{Ext}^1(I_Y \otimes M, I_X \otimes L)) \xrightarrow{\iota} H^2(Hom(I_Y \otimes M, I_X \otimes L)).$$

Identifying the dual of the last term with $H^0(M \otimes L^{-1} \otimes K_S)$ via Serre duality, and the dual of $H^0(\text{Ext}^1(I_Y \otimes M, I_X \otimes L))$ with $H^0(M \otimes L^{-1} \otimes K_S)_{|Y}$ via Grothendieck duality, the transpose of $\iota$ gets identified with $e_Y$. From this (I.1.3) follows at once. To prove (I.1.4) let $e_{(Y - P)}: H^0(L^{-1} \otimes M \otimes K_S) \to H^0(L^{-1} \otimes M \otimes K_S)_{|(Y - P)}$ be evaluation. The obvious map $\text{Im}(e_Y) \to \text{Im}(e_{(Y - P)})$ is an injection by hypothesis, and since $\text{Im}(e_Y)$ has codimension one we conclude that $e_{(Y - P)}$ is surjective. Let $\tau \in H^0(\text{Ext}^1(I_Y \otimes M, I_X \otimes L))$ be the image under $g$ of the extension class corresponding to (I.1.2); since $e_Y$ is the transpose of $f$, $\tau$ is annihilated by $\text{Im}(e_Y)$. If the extension is split at $P$, we actually have $\tau \in H^0(\text{Ext}^1(I_{(Y - P)} \otimes M, I_X \otimes L))$, and since $e_{(Y - P)}$ is surjective we conclude that $\tau = 0$, contradicting the assumption that (I.1.2) is non-split at one point of $Y$. Finally (I.1.5) follows from (I.1.4) because if $h^1(I_X \otimes M^{-1} \otimes L)$ vanishes then $g$ is an injection. \textbf{q.e.d.}

The following proposition states a remarkable property of suitable polarizations: the validity of this result is the reason for introducing the notion of suitability. The proof will be given in Section IV.

(I.1.6) Propostion. Let $S$ be an elliptic K3, with $|C|$ the elliptic fibration. Let $v \in H^*(S; \mathbb{Z})$ be a Mukai vector such that $\langle v^1, C \rangle$ and $v^0$ are coprime, and let $H$ be a $|v|$-suitable ample divisor on $S$.

1. If a torsion-free sheaf $F$ with $v(F) = v$ is $H$-slope-semistable then it is $H$-slope-stable. In particular $H$ is $v$-stabilizing.
2. If $[F] \in \mathcal{M}_v(H)$ the restriction of $F$ to the generic $C_t \in |C|$ is stable.
3. Conversely if $F$ is a torsion-free sheaf on $S$ with $v(F) = v$, such that the restriction of $F$ to the generic $C_t \in |C|$ is stable, then $F$ is $H$-slope-stable.

I.2. Outline of the section.

Let $S$ be a K3 surface.

(I.2.1) Definition. Two Mukai vectors $v, w \in H^*(S; \mathbb{Z})$ are equivalent ($v \sim w$) if there exists a line bundle $\xi$ such that $w = ch(\xi) \cdot v$.

Thus if $F$ is a sheaf on $S$ then $v(F \otimes \xi) \sim v(F)$. Assume $v \sim w$: since multiplication by $ch(\xi)$ is an isometry of the Mukai lattice we have $|v| = |w|$ (see (I.0.2)), hence open $|v|$-chambers coincide with open $|w|$-chambers. From now on we assume $S$ is elliptic with elliptic pencil $|C|$: if $v \sim w$ a polarization is $|v|$-suitable if and only if it is $|w|$-suitable. If $H$ is $|v|$-suitable the map

$$\mathcal{M}_v(H) \ni [F] \mapsto [F \otimes \xi] \in \mathcal{M}_w(H)$$

5
is an isomorphism: in fact if \([F] \in \mathcal{M}_v(H)\) then \(F\) is slope-stable by Proposition (I.1.6), and slope-stability is preserved by tensorization. Furthermore, since multiplication by \(ch(\xi)\) is an isometry of the Mukai lattice, Theorem (I.0.4) holds for \(\mathcal{M}_v(H)\) if and only if it holds for \(\mathcal{M}_w(H)\). Thus we are allowed to replace \(v\) by any equivalent vector; we will use this freedom to normalize the Mukai vector of Theorem (I.0.4) as follows. Let \([F] \in \mathcal{M}_v(H); \) since \(v^1\) is a numerical section

\[ \chi(F \otimes [kC]) = \chi(F) + k. \]

Set

\[ \chi(v) := \chi(F), \text{ where } [F] \in \mathcal{M}_v(H), \]

and let \(w := v \cdot ch \left( \left(1 - \chi(v) \right) C \right) \). Then \(w^1\) is again a numerical section, and furthermore \(\chi(w) = 1\). Therefore it suffices to prove Theorem (I.0.4) under the additional hypothesis that \(\chi(v) = 1\): in this case we say \(v\) is \textit{normalized}.

The proof of Theorem (I.0.4) goes roughly as follows. We can assume \(v\) is normalized; if \([F] \in \mathcal{M}_v(H)\) then by stability \(h^2(F)\) vanishes and hence \(h^0(F) \geq \chi(v) = 1\). Suppose that \(h^0(F) = 1\), then we can have a canonical sequence

\[ 0 \to O_S \to F \to E \to 0. \]

Assume also that \(E\) is torsion-free (this will be the case if \(F\) is locally-free): since \(H\) is \([v]\)-suitable, it will follow that \(E\) is \(H\)-slope-stable. Let \(\mathcal{M}_w(H)\) be the moduli space to which the isomorphism class of \(E\) belongs: by mapping \([F]\) to \([E]\) we get a rational map \(\varphi\) from \(\mathcal{M}_v(H)\) to \(\mathcal{M}_w(H)\). Since \(\chi(E) = -1\), we see that \(h^1(E') \geq 1\) for all \([E'] \in \mathcal{M}_w(H)\). It turns out that \(h^1(E') = 1\) for the generic \([E'] \in \mathcal{M}_w(H)\). Since \(h^1(E') = \dim \text{Ext}^1(E', O_S)\) this means that \(\varphi\) has degree one, and thus \(\mathcal{M}_v(H)\) is birational to \(\mathcal{M}_w(H)\). Normalizing \(w\) and repeating this argument one gets down to rank one, i.e. \(S^{[n]}\). This is the method by which we will define a birational map between \(\mathcal{M}_v(H)\) and \(S^{[n]}\); the details are in the next subsection. In order to prove that \(\theta_v\) is an isomorphism of Hodge structures we will construct a subset \(U\) of \(\mathcal{M}_v(H)\), and a tautological family \(\mathcal{F}\) of sheaves on \(S\) parametrized by \(U\). Since the complement of \(U\) has codimension two, one has that \(H^2(U) \cong H^2(\mathcal{M}_v(H))\), and that \(\theta_v\) is determined by \(\theta_{\mathcal{F}}\). This is the longest part of the proof, the difficulty being that we must perform semistable reduction along certain divisors; it takes up Subsections I.4-I.5. Finally we will be able to verify that \(\theta_v\) is an isometry by carrying out a purely numerical computation: this is the content of the last subsection.

\textbf{I.3. The moduli space is birational to} \(S^{[n]}\).

For the rest of this section we assume \(S, v, H\) are as in Theorem (I.0.4), except that we do not assume \(d(v) > 2\), only \(d(v) \geq 0\). Furthermore we suppose that \(v\) is normalized. We will systematically omit \(H\) from our notation. We let \([C]\) be the elliptic pencil, and \(\Sigma\) be its section. Set

\[ r := v^0 = \text{rank of } F, \text{ for } [F] \in \mathcal{M}_v \quad n = d(v)/2 = \text{half the dimension of } \mathcal{M}_v. \]

Then

\[ v = r + \Sigma + (n - r^2 + r)C + (1 - r)\omega. \tag{I.3.1} \]

In particular \(v\) is determined by \(r, n\); we will denote \(\mathcal{M}_v\) by \(\mathcal{M}_r^{2n}\). The main result of this subsection is the following.

\textbf{(I.3.2) Proposition.} If non-empty the moduli space \(\mathcal{M}_r^{2n}\) is an irreducible symplectic variety birational to \(S^{[n]}\).

In the next subsection we will show that in fact \(\mathcal{M}_r^{2n}\) is always non-empty. Before proving Proposition (I.3.2) we need to discuss certain Brill-Noether loci. Let \(W^d_v \subset \mathcal{M}_v\) be the subset parametrizing locally-free sheaves \(F\) such that \(h^0(F) = (d + 1)\) (we assume \(d \geq 0\).
(I.3.3) Proposition. Keep notation and assumptions as above. Then \( W^d_v \) is of pure dimension, and
\[
\text{cod}(W^d_v, M_v) = \max\{d + 1 - \chi(v), 0\}.
\]
We will prove Proposition (I.3.3) at the end of this subsection.

Proof of Proposition (I.3.2). It suffices to prove that \( M_{2n}^2 \) is birational to \( S^{[n]} \); in fact since \( M_{2n}^2 \) is symplectic \([M1]\), and since \( S^{[n]} \) is symplectic irreducible, it will follow that also \( M_{2n}^2 \) is symplectically irreducible (i.e. simply connected with a symplectic form spanning \( H^{2,0} \)). The proof is by induction on \( r \). When \( r = 1 \) we have \( M_{2n}^2 \cong S^{[n]} \), hence there is nothing to prove. So let’s suppose \( r > 1 \). First we notice that the (open) subset of \( M_{2n}^2 \) parametrizing locally-free sheaves is dense: as is well-known this follows from the fact that for all \( [F] \in M_{2n}^2 \) we have \( h^2(\text{ad} F^{**}) = 0 \).

By Proposition (I.3.3) the open subset \( W_r^0 := W_r^0 \) is dense in \( M_{2n}^2 \). If \([F] \in W_r^0\) then \( F \) fits into a unique exact sequence
\[
0 \to \mathcal{O}_S \to F \to Q \to 0.
\]
The quotient \( Q \) is torsion-free by the following.

(I.3.4) Claim. Keep notation as above. Let \([F] \in M_v\), with \( F \) locally-free, and suppose that \( h^0(F) = k + 1 > 0 \). Then \( F \) fits into a unique exact sequence
\[
0 \to \mathcal{O}_S(kC) \xrightarrow{\alpha} F \to Q \to 0, \tag{I.3.5}
\]
where \( Q \) is torsion-free.

Proof of the claim. Let \( f : S \to \mathbb{P}^1 \) be the elliptic fibration. Since \( F \) is torsion-free \( f_* F \) is also torsion-free, hence locally-free. By (I.1.1) \( f_* F \) is of rank one, and thus it is a line-bundle; since \( h^0(f_* F) = h^0(F) \) we must have \( f_* F = \mathcal{O}_{\mathbb{P}^1}(k) \). The natural map \( f^* f_* F \to F \) gives rise to (I.3.5), which is clearly unique. To prove that \( Q \) is torsion-free we must show that the divisorial part of the zero-locus of \( \alpha \), call it \( \text{div}(\alpha) \), is empty. Let \( C_t \) be a generic elliptic fiber; by (I.1.6) the restriction of \( F \) to \( C_t \) is stable, and hence \( \text{div}(\alpha) \cap C_t = \emptyset \). Since all the elliptic fibers are irreducible (\( \rho(S) = 2 \)), we conclude that \( \text{div}(\alpha) \) is a union of elliptic fibers. But if \( \alpha \) vanishes on an elliptic fiber then we get a (non-zero) map \( \mathcal{O}_S((k+1)C) \to F \), contradicting the assumption \( h^0(F) = k + 1 \). We conclude that \( \text{div}(\alpha) = \emptyset \), and thus \( Q \) is torsion-free.

We go back to the proof of Proposition (I.3.2). Let
\[
w := v(Q) = v(F) - 1 - \omega.
\]
By (I.1.6) and by (I.1.1) the restriction of \( Q \) to a generic elliptic fiber is stable, and hence by (I.1.6) we conclude that \( Q \) itself is stable. Thus \([Q] \in M_w\). Since \( w^1 = v^1(F) \), it is a numerical section and hence \( M_w\) is one of the moduli spaces we are considering. (But we do not normalize \( w \) for the moment.) From \( h^0(F) = 1 \) we get \( h^0(Q) = 0 \). By Serre duality \( H^2(Q) \cong \text{Hom}(Q, \mathcal{O}_S)^* \), and by stability of \( Q \) the last group is zero. Thus \( \chi(Q) = -h^1(Q) \). Since \( \chi(Q) = \chi(F) - \chi(\mathcal{O}_S) = -1 \) we conclude that \( h^1(Q) = 1 \). Hence \([Q] \in A_w\), where \( A_w \subset M_w \) is the open subset
\[
A_w := \{[Q'] \in M_w | h^1(Q') = 1\} = \{[Q'] \in M_w | h^0(Q') = 0\}.
\]
To sum up: we have defined a map
\[
\varphi : W^0_r \to A_w.
\]
As is easily checked this map is a morphism. Let \([Q'] \in \text{Im} \varphi\); since by Serre duality \( H^1(E) \cong \text{Ext}^1(E, \mathcal{O}_S)^* \), the morphism \( \varphi \) is injective. We claim that \( \text{Im} \varphi \) is an open non-empty subset of
\( A_w \). Let \([F] \in W^0_v\) (\([F]\) exists because by hypothesis \( \mathcal{M}_{2n}^n \) is non-empty) and set \([Q] = \varphi([F])\). By definition of \( A_w \), we have \( h^1(Q') = 1 \) for all \([Q'] \in A_w\), and since \( \dim \text{Ext}^1(Q', \mathcal{O}_S) \cong H^1(Q')^* \) there exists a unique non-trivial extension, call it \( F' \), of \( Q' \) by \( \mathcal{O}_S \). By openness of stability the sheaf \( F' \) is stable for \([Q']\) varying in an open (non-empty) subset of \( A_w \); this proves that \( \text{Im} \varphi \) is open in \( \mathcal{M}_w \). It suffices to show that

\[
\dim\{[Q'] \in \mathcal{M}_w \mid h^0(Q') > 0\} < \dim \mathcal{M}_w.
\]

If \( w^0 = (r - 1) > 1 \) the locus parametrizing locally-free sheaves is dense in \( \mathcal{M}_w \), and hence the inequality follows from Proposition (I.3.3). If \( w^0 = 1 \) one easily checks the inequality by hand. Since \( \varphi \) is injective, and since its image is dense in \( \mathcal{M}_w \), it defines a birational map between \( \mathcal{M}_{2n}^n \) and \( \mathcal{M}_w \). Normalizing \( w \) one gets \( \mathcal{M}_w \cong \mathcal{M}_{2n-1}^n \). By the inductive hypothesis \( \mathcal{M}_{2n-1}^n \) is birational to \( S^{[n]} \), and hence so is \( \mathcal{M}_{2n}^n \).

**Proof of Proposition (I.3.3).** First suppose \( \chi(v) \geq (d + 1) \). Let \([F] \in \mathcal{M}_v \). By Serre duality and stability \( H^2(F) = \text{Hom}(F, \mathcal{O}_S)^* = 0 \), and hence \( h^0(F') \geq \chi(v) \). Thus \( W_v^d \) is an open (eventually empty) subset of \( \mathcal{M}_v \), and hence the proposition holds in this case (the empty set has any codimension). From now on we assume that \( \chi(v) \leq d \). One can describe \( W_v^d \) as a determinantal variety: this is possible by standard methods because \( H^2(F) \) vanishes for \([F] \in \mathcal{M}_v \).

The dimension formula for determinantal varieties gives that

\[
\text{cod}(W_v^d, \mathcal{M}_v) \leq \max\{(d + 1)(d + 1 - \chi(v)), 0\}.
\]

(1.3.6)

We first deal with a special case.

**(I.3.7) Lemma.** Keeping notation as above, suppose that \( d = 0 \) and \( \chi(v) \leq 0 \). Then \( W_v^0 \) is smooth of pure dimension, and

\[
\text{cod}(W_v^0, \mathcal{M}_v) = 1 - \chi(v).
\]

**Proof.** Let \([F] \in W_v^0 \). The non-zero section of \( F \) gives an exact sequence

\[
0 \to \mathcal{O}_S \to F \to Q \to 0.
\]

(I.3.8)

By Claim (I.3.4) \( Q \) is torsion-free. There is an exact sequence [O, Prop. (1.17)]

\[
0 \to T[F]W_v \to \text{Ext}^1(F, F) \xrightarrow{\beta} \text{Ext}^1(\mathcal{O}_S, Q).
\]

To compute \( \text{rk} \beta \) consider the exact sequence

\[
\text{Ext}^1(F, F) \xrightarrow{\gamma} \text{Ext}^1(F, Q) \to \text{Ext}^2(F, \mathcal{O}_S) \to \text{Ext}^2(F, F) \to \text{Ext}^2(F, Q).
\]

(*)

By Serre duality \( \text{Ext}^2(F, Q) \cong \text{Hom}(Q, F)^* \). This last group is zero: in fact \( Q \) is stable because its restriction to a generic elliptic fiber is stable, hence \( \text{Hom}(Q, Q) \cong \mathbb{C} \), and since (I.3.8) is not split we conclude that \( \text{Hom}(Q, F) = 0 \). Furthermore

\[
\text{Ext}^2(F, F) \cong \mathbb{C} \quad \text{Ext}^2(F, \mathcal{O}_S) \cong \text{Hom}(\mathcal{O}_S, F)^* \cong \mathbb{C},
\]

and hence the map \( \gamma \) in (*) is surjective. From the exact sequence

\[
\text{Ext}^1(F, Q) \xrightarrow{\gamma} \text{Ext}^1(\mathcal{O}_S, Q) \to \text{Ext}^2(Q, Q) \to \text{Ext}^2(F, Q) = 0.
\]
we get $\dim \text{Im} \eta = h^1(Q) - 1$. Since $\beta = \eta \circ \gamma$ we conclude that 
\[ \text{cod}(T[F]W^0_v, T[F]M_v) = h^1(Q) - 1. \]

Since $h^0(Q) = h^2(Q) = 0$, we have $h^1(Q) = -\chi(Q)$. From (I.3.8) we get $\chi(Q) = \chi(F) - 2$, and hence 
\[ \text{cod}(T[F]W^0_v, T[F]M_v) = 1 - \chi(F) = 1 - \chi(v). \]

The lemma follows at once from this equality together with Inequality (I.3.6). \[ \text{q.e.d.} \]

Now we prove Proposition (I.3.3) in general. Let $[F] \in M_v$ with $F$ locally-free. By Claim (I.3.4) we have $h^0(F(-dC)) = 1$ if and only if $[F] \in W^d_v$. Hence, if $w := v \cdot ch([-dC])$, tensorization by $[-dC]$ defines an isomorphism between $W^d_v$ and $W^0_w$, and also of course between $M_v$ and $M_w$. Thus 
\[ \text{cod}(W^d_v, M_v) = \text{cod}(W^0_w, M_w) = 1 - \chi(w) = 1 - \chi(v) + d, \]

where the second equality holds by Lemma (I.3.7). \[ \text{q.e.d.} \]

**I.4. A large open subset of $M^{2n}_r$, for $r \leq 2$.**

Given $r \geq 1$ and $n \geq 0$ we will construct an open non-empty subset $U^{2n}_r \subset M^{2n}_r$ and a tautological family of sheaves on $S$ parametrized by $U^{2n}_r$. Together with Proposition (I.3.2) this will establish that $M^{2n}_{r}$ is an irreducible symplectic variety birational to $S^{[n]}$. Furthermore since the complement of $U^{2n}_r$ has codimension at least two the map $\theta_v: H^*(S) \to H^2(M^{2n}_r)$ will be completely determined by a tautological family on $S \times U^{2n}_r$. The construction of $U^{2n}_r$ and the relative tautological family is by induction on $r$: the idea is to imitate the picture for stable vector bundles on an elliptic curve (see (I.1.1)). First we will deal with the cases $r = 1, 2$. For simplicity’s sake we fix $n$ and we often omit it from our notation: $M^{2n}_r$ will be denoted by $M_r$, etc. The moduli space $M_1$ will be tacitly identified with $S^{[n]}$. Let $f: S \to \mathbb{P}^1$ be the elliptic fibration.

**I.4.1 Definition.** $U_1 \subset S^{[n]}$ is the set consisting of $[Z]$ such that:

1. $\# Z_{\text{red}} \geq (n - 1)$, and $Z \cap \{\text{critical points of } f\} = \emptyset$,
2. $h^0(I_Z \otimes [(n - 2)C]) = 0$, and if $h^0(I_Z \otimes [(n - 1)C]) > 0$ then $Z$ is reduced,
3. if $Z \cap \Sigma \neq \emptyset$ then $Z$ is reduced, $Z \cap \Sigma$ consists of a single point, and $h^0(I_Z \otimes [(n - 1)C]) = 0$.

**I.4.2 Remark.** The complement of $U_1$ in $S^{[n]}$ has codimension two.

Our choice of a tautological family on $S \times U_1$ (recall that we identify $M_1$ with $S^{[n]}$) is 
\[ \mathcal{F}^1 := I_Z \otimes \pi^*[\Sigma + nC], \]

where $Z \subset S \times U_1$ is the tautological subscheme. Notice that if $x \in U_1$ then $v(\mathcal{F}^1_x)$ is the normalized Mukai vector $v$ with $v^0 = 1, \langle v^1, C \rangle = 1, d(v) = 2n$, i.e. the vector $v$ of (I.3.1) with $r = 1$. Thus $\mathcal{F}^1$ is indeed a tautological family of sheaves on $S \times U_1$. The reason for restricting to the subset $U_1$ of the whole rank-one moduli space $S^{[n]}$ will become apparent when we deal with higher-rank moduli spaces. Now let’s move to the case of rank two. We will construct a family of stable rank-two sheaves parametrized by $U_1$, this family will define a classifying morphism $U_1 \to M_2$, and $U_2$ will be defined as the image of this morphism. The first step is to construct a family of extensions on $S$ parametrized by $U_1$. If $[Z] \in U_1$ then by Items (2)-(3) of (I.4.1) together with Serre duality

\[ \dim \text{Ext}^1(I_Z \otimes [\Sigma + (n - 2)C], \mathcal{O}_S) = h^1(I_Z \otimes [\Sigma + (n - 2)C]) = 1, \]
\[ \text{Ext}^i(I_Z \otimes [\Sigma + (n - 2)C], \mathcal{O}_S) = 0 \text{ for } i = 0, 2. \] (I.4.3)
Hence if $\rho: S \times U_1 \to U_1$ is projection,

$$\xi_2 := \text{Ext}^1_{\rho}(F^1 \otimes \pi^*[-2C], \mathcal{O}_{S \times U_1})$$

is a line-bundle on $U_1$. The exact sequence

$$0 = H^1(\text{Ext}^0_{\rho}(-, -)) \to \text{Ext}^1(F^1 \otimes \pi^*[-2C] \otimes \rho^*\xi_2, \mathcal{O}_{S \times U_1}) \to$$

$$\to H^0(\text{Ext}^1(\xi_2^{-1} \otimes F^1 \otimes \pi^*[-2C], \mathcal{O}_{S \times U_1})) \to H^2(\text{Ext}^0_{\rho}(-, -)) = 0$$

shows that there is a unique non-trivial tautological extension

$$0 \to \mathcal{O}_{S \times U_1} \to \mathcal{E}_2 \to F^1 \otimes \pi^*[-2C] \otimes \rho^*\xi_2 \to 0. \tag{I.4.4}$$

Since $\mathcal{O}_{S \times U_1}$ and $F^1$ are families of torsion-free sheaves parametrized by $U_1$, so is $\mathcal{E}_2$. If $[Z] \in U_1$ then $\mathcal{E}_2[Z]$ is the unique non-split extension

$$0 \to \mathcal{O}_S \to \mathcal{E}_2[Z] \to I_Z \otimes [\Sigma + (n - 2)C] \to 0. \tag{I.4.5}$$

A computation shows that

(I.4.6) Keeping notation as above, $\nu(\mathcal{E}_2[Z])$ is normalized and

$$\nu^0 = 2, \quad \langle \nu^1, C \rangle = 1, \quad 2 + \langle \nu(\mathcal{E}_2[Z]), \nu(\mathcal{E}_2[Z]) \rangle = 2n.$$  

Thus $\nu(\mathcal{E}_2[Z])$ is the vector $\nu$ of (I.3.1) with $r = 2$.

If $\mathcal{E}_2[Z]$ is stable then by the above computation the isomorphism class of $\mathcal{E}_2[Z]$ is represented by a point of $\mathcal{M}_2$. Before analyzing stability we prove the following proposition.

(I.4.7) Proposition. Let $Q$ be a torsion-free stable sheaf on $S$, with $\langle c_1(Q), C \rangle = 1$. Suppose that for all $t \in \mathbb{P}^1$ we have

$$\text{Hom}(Q|_{C_t}, \mathcal{O}_{C_t}) = 0. \tag{I.4.8}$$

Let

$$0 \to [kC] \to F \to Q \to 0 \tag{I.4.9}$$

be a non-split extension.

1. The sheaf $F$ is slope-stable.
2. Letting

$$s := \dim \text{Ext}^1(Q, [kC]) - 1,$$

there are exactly $s$ elliptic fibers $C_u$ (counted with appropriate multiplicities) such that the restriction of (I.4.9) to $C_u$ splits.

Proof. By (I.4.8) the sheaf $\text{Hom}_f(Q, [kC])$ is zero. From the exact sequence

$$0 \to H^1(\text{Hom}_f(Q, [kC])) \to \text{Ext}^1(Q, [kC]) \to H^0\left(\text{Ext}^1_f(Q, [kC])\right) \to H^2(\text{Hom}_f(Q, [kC]))$$

we conclude that

$$\text{Ext}^1(Q, [kC]) \cong H^0\left(\text{Ext}^1_f(Q, [kC])\right). \tag{I.4.10}$$

Since $Q$ is torsion-free it is $f$-flat, hence the Euler characteristic $\chi(Q|_{C_t}, [kC]|_{C_t})$ is independent of the elliptic fiber $C_t$, and thus (I.4.8) implies that $\dim \text{ Ext}^1(Q|_{C_t}, [kC]|_{C_t})$ is constant. Now let $C_t$ be a generic elliptic fiber. Since $Q$ is stable its restriction to $C_t$ is stable by (I.1.6). By (I.1.1) we get $\dim \text{ Ext}^1(Q|_{C_t}, [kC]|_{C_t}) = 1$, and hence $\lambda := \text{ Ext}^1_f(Q, [kC])$ is a line-bundle. By Equality (I.4.10) the extension class of (I.4.9) corresponds to a non-zero section of this line-bundle, and the elliptic fibers $C_u$ for which the restriction of (I.4.9) to $C_u$ is trivial are in one-to-one correspondence with the zeroes of this section. This implies Item (1), by (I.1.1)-(I.1.6). It also implies Item (2) if we notice that

$$\dim \text{ Ext}^1(Q, [kC]) = h^0(\lambda) = \deg \lambda + 1. \tag{q.e.d.}$$
Consider the Ext-sequence associated to (I.4.13):

\[ \varepsilon_{[Z]} \]

where equivalent to \( Z \) since the complement of \( D \) divisor we let \( E \) elliptic fiber is unstable, and by Proposition (I.1.6) we conclude that \( \varepsilon_{[Z]} \) is stable. 

Since for \( [Z] \in U_1 \) we have \( \ell(Z) = n \), the above corollary shows that \( \varepsilon_{[Z]} \) is stable for the generic \( [Z] \). We let \( L \) be the divisor on \( U_1 \) consisting of points \( [Z] \) such that

\[ h^0(I_Z \otimes [(n - 1)C]) > 0, \]

i.e. such that \( Z \) contains two points lying on the same elliptic fiber. If \( D \subset S \) is an effective reduced divisor we let \( D_1 \) be the reduced divisor on \( S^{[n]} \) defined by

\[ D_1 := \{ [Z] \in S^{[n]} | Z \cap D \neq \emptyset \}. \]

Since the complement of \( U_1 \) in \( S^{[n]} \) has codimension two (I.4.2) we can identify the group of divisors on \( U_1 \) with the group of divisors in \( S^{[n]} \); we will use the same symbol for corresponding divisors on \( U_1 \) and \( S^{[n]} \). The corollary above states that \( \varepsilon_{[Z]} \) is stable if \( [Z] \notin L \cup \Sigma_1 \). We will show that if \( [Z] \in L \cup \Sigma_1 \) then \( \varepsilon_{[Z]} \) is indeed unstable.

**Semistable reduction along \( L \).** Notice that if \( n < 2 \) then \( L = \emptyset \), hence throughout this subsubsection we assume that \( n \geq 2 \). Let \( [Z] \in L \); since \( [Z] \in L \), and since by Item (2) of (I.4.1) the scheme \( Z \) is reduced, we can write uniquely \( Z = Z_0 \cup W \), where \( Z_0 \) consists of two points belonging to the same elliptic fiber \( C_0 \). To simplify notation we set \( E := \varepsilon_{[Z]} \). We recall that \( E \) is the unique non-trivial extension:

\[ 0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow I_Z \otimes [\Sigma + (n - 2)C] \rightarrow 0. \quad (I.4.12) \]

The sheaf \( E \) is unstable for the following reason. From

\[ \dim \text{Hom}(I_Z \otimes [\Sigma + (n - 2)C]|_{C_t}, \mathcal{O}_{C_t}) = \begin{cases} 0 & \text{if } t \neq 0, \\
1 & \text{if } t = 0, \end{cases} \]

it follows that \( \text{Ext}_1^1(I_Z \otimes [\Sigma + (n - 2)C], \mathcal{O}_S) \cong \mathcal{O}_{\text{pt}}(-1) \oplus C_0 \), where \( C_0 \) is the skyscraper sheaf at 0. Now notice that, setting \( k = 0 \) and \( Q := I_Z \otimes [\Sigma + (n - 2)C] \), Equality (I.4.10) holds, because for its validity it is sufficient that (I.4.8) is satisfied for the generic elliptic fiber. Hence we conclude that the restriction of the non-split extension (I.4.12) to an elliptic fiber \( C_t \) is split for \( t \neq 0 \), and non-split for \( t = 0 \). (Strange as it may sound.) Hence the restriction of \( E \) to the generic elliptic fiber is unstable, and by Proposition (I.1.6) \( E \) itself must be unstable. It is also clear that a destabilizing sequence must have some relation with the fiber \( C_0 \). Let’s exhibit a destabilizing sequence. Consider the natural exact sequence

\[ 0 \rightarrow I_W \otimes [\Sigma + (n - 3)C] \rightarrow I_Z \otimes [\Sigma + (n - 2)C] \rightarrow \iota_* \mathcal{O}_{C_0}(-P) \rightarrow 0, \quad (I.4.13) \]

where \( \iota: C_0 \hookrightarrow S \) is the inclusion, and \( P \in C_0 \) is the point such that \( (\Sigma \cap C_0) + P \) is linearly equivalent to \( Z_0 \) (as divisors on \( C_0 \)). We claim that \( \sigma \) lifts to a map \( \tilde{\sigma}: I_W \otimes [\Sigma + (n - 3)C] \rightarrow E \). Consider the Ext-sequence associated to (I.4.13):

\[ 0 = \text{Hom}(I_W \otimes [\Sigma + (n - 3)C], \mathcal{O}_S) \rightarrow \text{Ext}^1(\iota_* \mathcal{O}_{C_0}(-P), \mathcal{O}_S) \rightarrow \text{Ext}^1(I_Z \otimes [\Sigma + (n - 2)C], \mathcal{O}_S) \rightarrow \text{Ext}^1(I_W \otimes [\Sigma + (n - 3)C], \mathcal{O}_S). \]
The obstruction to lifting $\alpha$ is given by $\gamma(e)$, where $e$ is the extension class of (I.4.12). A computation shows that the first Ext$^1$-group appearing above is one-dimensional. Since the second Ext$^1$-group is also one-dimensional we see that $\gamma = 0$, and hence $\sigma$ lifts to a map $\tilde{\sigma}: I_W \otimes [\Sigma + (n - 3)C] \to E$. The lift is unique because $h^0([\Sigma - (n - 3)C]) = 0$. We claim that the quotient $E/\text{Im} \tilde{\sigma}$ is torsion-free. First notice that since $\sigma$ is an isomorphism outside $C_0$, the quotient is certainly torsion-free outside $C_0$. By (I.1.5) $E$ is locally-free along $C_0$, and clearly also $I_W \otimes [\Sigma + (n - 3)C]$ is locally-free along $C_0$. Thus it suffices to show that $\tilde{\sigma}$ does not vanish at the generic point of $C_0$. Assume the contrary: we would get a non-zero map $I_W \otimes [\Sigma + (n - 2)C] \to E$, which is absurd. Thus $E/\text{Im} \tilde{\sigma}$ is torsion-free; since its rank is one it is isomorphic to $I_Y \otimes M$, where $Y$ is some zero-dimensional subscheme of $S$, and $M$ is a line-bundle. Computing Chern classes one gets that $M = [C]$. Restricting $\tilde{\sigma}$ to $C_0$ one gets that $Y = P$. Hence we have an exact sequence

$$0 \to I_W \otimes [\Sigma + (n - 3)C] \xrightarrow{\tilde{\sigma}} E \to I_P \otimes [C] \to 0.$$  

(rem I.4.14)

This is the Harder-Narasimhan filtration of $E$; since we do not need this statement we omit its (easy) proof. By uniqueness of (I.4.14) we can globalize the construction to all of $E^2|_{S \times L}$: letting $P \subset S \times L$ and $W \subset S \times L$ be the subschemes swept out by $P$ and $W$ as $[Z]$ varies in $L$, there exist line bundles $\eta_1$, $\eta_2$ on $L$ such that we have an exact sequence

$$0 \to I_W \otimes \pi^*[\Sigma + (n - 3)C] \otimes \rho^*\eta_1 \to E^2|_{S \times L} \to I_P \otimes \pi^*[C] \otimes \rho^*\eta_2 \to 0.$$  

Letting $\iota^L: S \times L \hookrightarrow S \times U_1$ be the inclusion, we define $G^2$ to be the elementary modification of $E^2$ associated to the above exact sequence, i.e. $G^2$ is the sheaf on $S \times U_1$ fitting into the exact sequence

$$0 \to G^2 \to E^2 \xrightarrow{\iota_*^L (I_P \otimes \pi^*[C] \otimes \rho^*\eta_2)} 0.$$  

(I.4.15)

The sheaf $G^2$ is $U_1$-flat [F2, Lemma (A.3)], i.e. we can view it as a family of sheaves on $S$ parametrized by $U_1$.

(I.4.16) Proposition. The sheaf $G^2|_Z$ is torsion-free for all $[Z] \in U_1$, and stable if $[Z] \in (U_1 - \Sigma_1)$.

If $[Z] \in (U_1 - L)$ then $G^2|_Z \cong E^2|_Z$, hence the sheaf $G^2|_Z$ is torsion-free for $[Z] \in (U_1 - L)$, and, by Corollary (I.4.11), stable for $[Z] \in (U_1 - L - \Sigma_1)$. We are left with showing that $G^2|_Z$ is torsion-free and stable for $[Z] \in L$. Assume $[Z] \in L$; for simplicity’s sake set $G := G^2|_Z$. The sheaf $G$ is naturally an extension: in fact restricting (I.15) to $S \times \{Z\}$ we obtain the exact sequence

$$0 \to I_P \otimes [C] \to G \to I_W \otimes [\Sigma + (n - 3)C] \to 0.$$  

(I.4.17)

Thus $G$ is torsion-free. Stability will be proved in various steps. First we consider the case when $n > 2$.

(I.4.18) Claim. Keeping notation as above, assume $n > 2$, i.e. $W \neq \emptyset$. There exists at least one point of $W$ at which $G$ is locally-free.

Let’s show that the above claim implies $G$ is stable. Consider $G^{**}$: from (I.17) we obtain the exact sequence

$$0 \to [C] \to G^{**} \to I_{W_0} \otimes [\Sigma + (n - 3)C] \to 0,$$  

(I.4.19)

where $W_0 \subset W$ is the subset of points at which $G$ is locally-free (recall that by Item (2) of (I.4.1) the scheme $W$ is reduced). By (I.18) we know $W_0 \neq \emptyset$, and hence the extension class of (I.19) is non-trivial. Since $W_0 \cap \Sigma = \emptyset$ and $W_0$ intersects every elliptic fiber in at most one point, we can
apply Proposition (I.4.7) to the extension (I.4.19) and we conclude that $G^{**}$ is slope stable. Thus also $G$ is slope-stable.

**Proof of Claim (I.4.18).** As usual let $E := \mathcal{E}^2_{[Z]}$. The Kodaira-Spencer map

$$T_{[Z]} S^{[n]} \to \text{Ext}^1(E, E)$$

is an isomorphism by the following criterion.

(I.4.20). Let $Q$ be a stable torsion-free sheaf on $S$, such that $c_1(Q) \cdot C = 1$. Assume that

$$h^0(Q) = 0 \quad h^1(Q) = \dim \text{Ext}^1(Q, \mathcal{O}_S) = 1.$$  

Let $F$ be the unique non-trivial extension

$$0 \to \mathcal{O}_S \to F \to Q \to 0.$$  

Then $F$ is simple, and furthermore the Kodaira-Spencer map $\text{Ext}^1(Q, Q) \to \text{Ext}^1(F, F)$ corresponding to varying $Q$ and deforming (uniquely by (†)) the extension (⁎) is an isomorphism.

**Proof.** The proof consists in diagram chasing. We leave the details to the reader.  

The following is the key ingredient in the proof of Claim (I.4.18).

(I.4.21). There exists at least one point $Q \in W$ such that the restriction

$$\text{Ext}^1(E, E) \xrightarrow{\phi_Q} \text{Ext}^1(E_Q, E_Q),$$

is surjective. (Here $E_Q$ is the localization of $E$ at $Q$.)

**Proof.** The local-to-global exact sequence for $\text{Ext}$ gives an exact sequence

$$0 \to H^1(\text{Hom}(E, E)) \to \text{Ext}^1(E, E) \xrightarrow{\Phi} \bigoplus_{Q \in W} \text{Ext}^1(E_Q, E_Q).$$  

Let’s prove that

$$h^1(\text{Hom}(E, E)) = 3.$$  

Considering the natural exact sequence

$$0 \to \text{Hom}(E, E) \to \text{Hom}(E^{**}, E^{**}) \to \bigoplus_{Q \in W} C_Q \to 0$$

we get

$$h^1(\text{Hom}(E, E)) = \dim \text{Hom}(E, E) - h^0(E^{*} \otimes E^{**}) + (n - 2) + h^1(E^{*} \otimes E^{**}).$$  

Considering the destabilizing sequence for $E^{**}$, one proves that $h^0(E^{*} \otimes E^{**}) = (n - 2)$. Applying Hirzebruch-Rieman-Roch and Serre duality to $E^{*} \otimes E^{**}$ we get $h^1(E^{*} \otimes E^{**}) = 2$. Since by (I.4.20) we have $\dim \text{Hom}(E, E) = 1$, Equation (†) gives (**). Now let’s prove (I.4.21), arguing by contradiction. For all $Q \in W$ we have $E_Q \cong \mathcal{O} \oplus I_Q$, where $I_Q$ is the maximal ideal at $Q$, and hence $\dim \text{Ext}^1(E_Q, E_Q) = 3$. Assuming that for all $Q \in W$ the restriction map $\phi_Q$ is not surjective we
conclude from (*) and (**) that \( \dim \text{Ext}^1(E, E) < 2n \). This is absurd because \( \dim \text{Ext}^1(E, E) = 2n \) (by Hirzebruch-Riemann-Roch and simplicity of \( E \)). \( \text{q.e.d.} \)

Now we are ready to prove Claim (I.4.18). Let \( Q \in W \) be a point such that (I.4.21) holds at \( Q \). By (I.4.20) and (I.4.21) the map

\[
T_{[Z]} S^n \rightarrow \text{Ext}^1(E_Q, E_Q),
\]

obtained by composing Kodaira-Spencer with restriction to \( \text{Ext}^1(E_Q, E_Q) \), is surjective, i.e. the map from a neighborhood of \([Z]\) to the versal deformation space of \( E_Q \cong I_Q \oplus \mathcal{O} \) induced by \( \mathcal{E}^2 \) is a submersion. This means that there are local coordinates \( x, y \) on \( S \), centered at \( Q \), and local coordinates \( u, v, t \) on \( S^n \), centered at \([Z]\), such that \( \mathcal{E}^2 \) is locally generated by \( \alpha, \beta, \gamma \) with the single relation

\[
(x - u)\alpha + (y - v)\beta + t\gamma = 0.
\]

By (I.1.3)-(I.1.5) the sheaf \( \mathcal{E}^2_{[Z']} \), for \([Z']\) near \([Z]\), is singular exactly when \([Z'] \in L \), and then it is singular at each point of \( W \). Hence a local equation for \( L \) is given by \( \{ t = 0 \} \). Changing local generators of \( \mathcal{E}^2 \) if necessary we can assume that \( \alpha, \beta \) generate, for \( t = 0 \), the destabilizing subsheaf \( I_W \otimes [\Sigma + (n - 3)] \). Then the elementary modification \( G^2 \) (see (I.1.15)) is locally generated by \( \alpha, \beta \) and \( t\gamma \). By the above relation these elements generate a locally-free (rank-two) sheaf. \( \text{q.e.d.} \)

This finishes the proof of Proposition (I.4.16) when \( n > 2 \). Now we examine the case \( n = 2 \). The destabilizing sequence (I.4.14) becomes

\[
0 \rightarrow [\Sigma - C] \rightarrow E \rightarrow I_P \otimes [C] \rightarrow 0, \quad \text{(I.4.22)}
\]

and \( E \) is locally-free. Exact sequence (I.4.17) becomes

\[
0 \rightarrow I_P \otimes [C] \rightarrow G \rightarrow [\Sigma - C] \rightarrow 0. \quad \text{(I.4.23)}
\]

Let \( C_t \) be a generic elliptic fiber: we will prove \( G|_{C_t} \) is stable. This will establish that \( G \) is stable by Proposition (I.1.6). The vector bundle \( G|_{C_t} \) is stable if and only if the restriction of (I.4.23) to \( C_t \) is non-split. To determine the relevant extension class we proceed as follows. Consider \( \mathcal{E}^2_{[Z']}|_{C_t} \) for \([Z'] \in U_t \): by (I.4.22) this is unstable for \([Z'] \in L \), while by (I.1.6) it is stable (for \( C_t \) generic) if \([Z'] \in (U_t - L - \Sigma_1) \). Furthermore, restricting Exact sequence (I.4.15) to \( C_t \times U_t \), we see that \( G^2|_{C_t \times U_t} \) is obtained from \( \mathcal{E}^2|_{C_t \times U_t} \) by applying the first step of semistable reduction to the bundles parametrized by \( L \). Hence by [O, (1.11)] the extension class of the restriction of (I.4.23) to \( C_t \) is given by the following recipe. Let

\[
\alpha : T_{[Z]} S^2 \rightarrow H^1(\mathcal{O}_{C_t}(-\Sigma))
\]

be the composition of Kodaira-Spencer and the natural map \( H^1(\text{ad}E) \rightarrow H^1(\mathcal{O}_{C_t}(-\Sigma)) \) induced by restriction of Exact sequence (I.4.22). If \( x \in T_{[Z]} S^2 \) is a tangent vector, the class \( \alpha(x) \) represents the obstruction to lifting to first order, in the direction \( x \), the restriction of (I.4.22) to \( C_t \). Thus \( \alpha \) vanishes on \( T_{[Z]} L \), and hence it induces a map

\[
\overline{\alpha} : T_{[Z]} S^2 / T_{[Z]} L \rightarrow H^1(\mathcal{O}_{C_t}(-\Sigma)).
\]

The extension class of the restriction of (I.4.23) to \( C_t \) is given by a generator of \( \text{Im} \overline{\alpha} \). By (I.4.20) the Kodaira-Spencer map is surjective, and thus it suffices to show that

\[
H^1(\text{ad}E) \rightarrow H^1(\mathcal{O}_{C_t}(-\Sigma)) \quad \text{(I.4.24)}
\]
is non-zero. Consider

\[ H^1(\text{ad}E) \to H^1(\text{ad}E|_{C_i}) \xrightarrow{\beta} H^2(\text{ad}E \otimes [-C]) . \]

By Serre duality the transpose of \( \beta \) is identified with the restriction map

\[ t^\beta : H^0(\text{ad}E \otimes [C]) \to H^0(\text{ad}E|_{C_i}) . \]

Let’s examine \( H^0(\text{ad}E \otimes [C]) \). Using (I.4.22) one verifies that any \( \varphi \in \text{Hom}(E, E\otimes[C]) \) decomposes as \( \varphi = \varphi^0 + \varphi^1 \), where \( \varphi^0 \in H^0([C]) \otimes \text{Id}_E \), and \( \varphi^1 \) is a constant multiple of the composition

\[ E \to I_P \otimes [C] \to [C] \to E \otimes [C], \]

where the last map is given by the non-zero section of \( E \). Thus \( H^0(\text{ad}E \otimes [C]) \) is one-dimensional; let \( \varphi \) be a generator. Then

\[ \varphi|_{C_i} = \begin{bmatrix} \eta_{11} & \eta_{12} \\ 0 & -\eta_{11} \end{bmatrix}, \]

where \( \eta_{11} \in H^0(\mathcal{O}_{C_i}) \), and \( \eta_{12} \in H^0(\mathcal{O}_{C_i}(\Sigma)) \) (recall that \( E|_{C_i} \cong \mathcal{O}_{C_i}(\Sigma) \oplus \mathcal{O}_{C_i} \)), and if \( C_i \) is generic then \( \eta_{11} \neq 0 \). Now let \( \tau \in H^1(\text{ad}E|_{C_i}) \). Then

\[ \tau = \begin{bmatrix} \tau_{11} & 0 \\ \tau_{21} & -\tau_{11} \end{bmatrix}, \]

where \( \tau_{11} \in H^1(\mathcal{O}_{C_i}) \), and \( \tau_{21} \in H^1(\mathcal{O}_{C_i}(-\Sigma)) \). Since \( \text{Im}^t \beta \) is generated by \( \varphi|_{C_i} \), Serre duality tells us that \( \tau \in \text{Ker} \beta \) if and only if

\[ 0 = \text{Tr} \begin{bmatrix} \tau_{11} & 0 \\ \tau_{21} & -\tau_{11} \end{bmatrix} \cdot \begin{bmatrix} \eta_{11} & \eta_{12} \\ 0 & -\eta_{11} \end{bmatrix} = 2\tau_{11}\eta_{11} + \tau_{21}\eta_{12} . \]

Since \( \eta_{11} \neq 0 \) we conclude that the generator \( \tau \) of \( \text{Ker} \beta \) has \( \tau_{21} \neq 0 \). Let \( \tilde{\tau} \in H^1(\text{ad}E) \) be a lift of \( \tau \). The image of \( \tilde{\tau} \) under the map (I.4.24) is equal to \( \tau_{21} \), which is non-zero. Hence \( \text{Im} \tilde{\tau} \) is non-zero, and \( G|_{C_i} \) is stable. This proves Proposition (I.4.16) in the case \( n = 2 \), and finishes the proof of the proposition.

**Semistable reduction along \( \Sigma_1 \).** Notice that \( \Sigma_1 \) is empty if \( n = 0 \), hence we assume throughout this subsection that \( n \geq 1 \). Let \( [Z] \in \Sigma_1 \). By Item (3) of Definition (I.4.1) the intersection \( Z \cap \Sigma \) consists of a single reduced point \( P \), and \( Z = P \cup W \), where \( W \) is reduced of length \( (n - 1) \). We let \( C_0 \) be the elliptic fiber through \( P \). To simplify notation we set \( G := G^2_{[Z]} \). Since \( G \) is the same as \( \mathcal{E}^2_{[Z]} \), it is presented as the unique non-trivial extension

\[ 0 \to \mathcal{O}_S \to G \to I_Z \otimes [\Sigma + (n - 2)C] \to 0. \]  

(I.4.25)

The sheaf \( G^2_{[Z]} \) is unstable; one argues exactly as when we explained why \( \mathcal{E}^2_{[Z]} \) is unstable for \( [Z] \in L \). We will not go through the same argument again. Instead we directly exhibit a destabilizing sequence for \( G \). Let

\[ 0 \to I_W \otimes [\Sigma + (n - 3)C] \xrightarrow{\tau} I_Z \otimes [\Sigma + (n - 2)C] \to \iota_*\mathcal{O}_{C_0} \to 0, \]  

(I.4.26)
be the natural exact sequence, where \( \iota: \mathcal{C}_0 \hookrightarrow S \) is the inclusion. We claim that \( \sigma \) lifts to a map \( \tilde{\sigma}: \mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}] \to \mathcal{G} \). Consider the Ext-sequence associated to (I.4.26):

\[
0 = \text{Hom}(\mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}], \mathcal{O}_S) \to \text{Ext}^1(\iota_* \mathcal{O}_{\mathcal{C}_0}, \mathcal{O}_S) \to \\
\to \text{Ext}^1(\mathcal{I}_Z \otimes [\Sigma + (n - 2)\mathcal{C}], \mathcal{O}_S) \xrightarrow{\iota} \text{Ext}^1(\mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}], \mathcal{O}_S) .
\]

The obstruction to lifting \( \alpha \) is given by \( \gamma(e) \), where \( e \) is the extension class of (I.4.25). Since the first Ext\(^1\)-group in the above exact sequence has dimension zero, and since also the second Ext\(^1\)-group is one-dimensional, we see that \( \gamma = 0 \), and hence \( \sigma \) lifts to a map \( \tilde{\sigma}: \mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}] \to \mathcal{G} \). The lift is unique because \( h^0([-\Sigma - (n - 3)\mathcal{C}]) = 0 \). The quotient \( \mathcal{G}/\text{Im}\tilde{\sigma} \) is torsion-free: the argument is the same as the one given to show that \( E/\text{Im}\tilde{\sigma} \) is torsion-free when we performed semistable reduction along \( L \), we omit to repeat it. Since \( E/\text{Im}\tilde{\sigma} \) is torsion-free of rank one, it is isomorphic to \( \mathcal{I}_Y \otimes M \), where \( Y \) is some zero-dimensional subscheme of \( S \) and \( M \) is a line-bundle. Computing Chern classes one gets that \( M = [\mathcal{C}] \) and \( Y \) is empty. Hence we get

\[
0 \to \mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}] \to \mathcal{G} \to [\mathcal{C}] \to 0 . 
\]

This sequence gives the Harder-Narasimhan filtration of \( \mathcal{G} \). By unicity the above exact sequence globalizes: thus there are line-bundles \( \theta_1, \theta_2 \) on \( \Sigma_1 \) such that we have

\[
0 \to \mathcal{I}_W \otimes \pi^*[\Sigma + (n - 3)\mathcal{C}] \otimes \rho^* \theta_1 \to \mathcal{G}^2|_{S \times \Sigma_1} \to \pi^*[\mathcal{C}] \otimes \rho^* \theta_2 \to 0 ,
\]

where \( \mathcal{W} \) is the scheme swept out by \( W \) as \( [Z] \) varies in \( \Sigma_1 \). Let \( \mathcal{F}^2 \) be the sheaf on \( S \times \mathcal{U}_1 \) defined by

\[
0 \to \mathcal{F}^2 \to \mathcal{G}^2 \to \iota_{\Sigma_1}^*(\pi^*[\mathcal{C}] \otimes \rho^* \theta_2) \to 0 ,
\]

where \( \iota_{\Sigma_1}: \Sigma_1 \hookrightarrow \mathcal{U}_1 \) is the inclusion. The sheaf \( \mathcal{F}^2 \) is \( \mathcal{U}_1 \)-flat ([F2, Lemma (A.3)]), hence we can view it as a family of sheaves on \( S \) parametrized by \( \mathcal{U}_1 \).

**Proposition (I.4.29)** Keep notation as above. If \( [Z] \in \mathcal{U}_1 \) then \( v \left( \mathcal{F}^2_{[Z]} \right) \) is the normalized Mukai vector \( v \) with \( v^0 = 2, (v^1, \mathcal{C}) = 1 \), and \( d(v) = 2n \), i.e. it is equal to the vector of (I.3.1) with \( r = 2 \). The sheaf \( \mathcal{F}^2 \) is a family of stable torsion-free sheaves on \( S \), parametrized by \( \mathcal{U}_1 \).

**Proof.** Since \( \mathcal{F}^2 \) is \( \mathcal{U}_1 \)-flat, and since \( \mathcal{U}_1 \) is connected, the vector \( v \left( \mathcal{F}^2_{[Z]} \right) \) is independent of \( [Z] \). If \( [Z] \in (\mathcal{U}_1 - L - \Sigma_1) \) then \( \mathcal{F}^2_{[Z]} = \mathcal{E}^2_{[Z]} \), thus the first statement follows from (I.4.6). For \( [Z] \in (\mathcal{U}_1 - \Sigma_1) \) we have \( \mathcal{F}^2_{[Z]} = \mathcal{G}^2_{[Z]} \), hence the only thing left to prove is that \( \mathcal{F}^2_{[Z]} \) is stable and locally-free for \( [Z] \in \Sigma_1 \). Restricting (I.4.28) to \( S \times [Z] \), for \( [Z] \in \Sigma_1 \), one gets

\[
0 \to [\mathcal{C}] \to \mathcal{F}^2_{[Z]} \to \mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}] \to 0 . 
\]

For each \( t \in \mathbb{P}^1 \) one has \( \text{Hom}(\mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}]|_{C_t}, [\mathcal{C}]|_{C_t}) = 0 \), because of Item (3) of (I.4.1). Thus by Proposition (I.4.7) the sheaf \( \mathcal{F}^2_{[Z]} \) is stable if the extension (I.4.30) is non-split. The extension class of (I.4.30) is obtained as follows. Let

\[
\alpha: T_{[Z]} \mathcal{S}^{[n]} \to \text{Ext}^1(\mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}], [\mathcal{C}])
\]

be the composition of Kodaira-Spencer for \( \mathcal{G}^2 \) and of the map

\[
\beta: \text{Ext}^1(\mathcal{G}^2_{[Z]}, \mathcal{G}^2_{[Z]}) \to \text{Ext}^1(\mathcal{I}_W \otimes [\Sigma + (n - 3)\mathcal{C}], [\mathcal{C}]).
\]
associated to \((\text{I}.4.27)\). For \(x \in T[Z]S^{[n]}\) the class \(\alpha(x)\) is the obstruction to lifting \((\text{I}.4.27)\) to first order in the direction of \(x\). Thus \(\alpha\) vanishes on \(T[Z]\Sigma_1\), and it induces a map
\[
\overline{\alpha}T[Z]S^{[n]}/T[Z]\Sigma_1 \to \text{Ext}^1(I_W \otimes [\Sigma + (n - 3)C], [C]).
\]
The extension class of \(\ast\) is equal to the generator of \(\text{Im} \overline{\alpha} [O, (\text{I}.11)]\). By \((\text{I}.4.20)\) the Kodaira-Spencer map is surjective, hence it suffices to check that the map \(\beta\) is non-zero. One checks that the lemma below applies to the exact sequence \((\text{I}.4.27)\), and hence \(\beta\) does not vanish. This proves Extension \((\text{I}.4.30)\) is non-split, and hence \(\mathcal{F}^2_{[Z]}\) is stable by Proposition \((\text{I}.4.7)\).

**Lemma.** Let \(A, B\) be stable torsion-free sheaves on \(S\), with
\[
\dim \text{Ext}^1(A, B) \geq 2 \quad \text{Hom}(A, B) = \text{Hom}(B, A) = 0.
\]
Suppose we have a non-trivial extension
\[
0 \to A \to F \to B \to 0.
\]
Then the map \(\Phi: \text{Ext}^1(F, F) \to \text{Ext}^1(A, B)\) is non-zero.

**Proof.** The proof consists of diagram chasing. First one proves that \(F\) is simple, and thus by Serre duality \(\text{Ext}^2(F, F) \cong \mathbb{C}\). Then one gets surjectivity of \(\text{Ext}^1(F, F) \to \text{Ext}^1(F, B)\). Finally one considers
\[
\text{Ext}^1(F, B) \xrightarrow{\Phi} \text{Ext}^1(A, B) \to \text{Ext}^2(B, B) \to \text{Ext}^2(F, B) \cong \text{Hom}(B, F)^* = 0.
\]
Since \(B\) is stable it is simple, and thus \(\dim \text{Ext}^2(B, B) = 1\). Since \(\dim \text{Ext}^1(A, B) \geq 2\) the map \(\Psi\) is non-zero, and hence \(\Phi\) does not vanish.

Since \(\mathcal{F}^2\) is a family of stable sheaves on \(S\) parametrized by \(\mathcal{U}_1\), with normalized Mukai vector \(v\), it defines a classifying morphism \(\varphi: \mathcal{U}_1 \to \mathcal{M}_2\).

**Proposition-Definition.** Keep notation as above. Let \(\mathcal{U}_2 := \varphi(\mathcal{U}_1)\). Then \(\mathcal{U}_2\) is an open subset of \(\mathcal{M}_2\), and its complement has codimension two. The map \(\varphi: \mathcal{U}_1 \to \mathcal{U}_2\) is an isomorphism, and hence we can view \(\mathcal{F}^2\) as a tautological family of sheaves on \(S\) parametrized by \(\mathcal{U}_2\).

**Proof.** On the subset \((\mathcal{U}_1 - L - \Sigma_1)\) the map \(\varphi\) is injective. In fact if \([F] \in \varphi(\mathcal{U}_1 - L - \Sigma_1)\) then \(h^0(F) = 1\) by Items (2)-(3) of \((\text{I}.4.1)\), and the inverse \(\varphi^{-1}\) on \(\varphi(\mathcal{U}_1 - L - \Sigma_1)\) is obtained by associating to \([F]\) the (zero-dimensional) zero-locus of a non-zero section of \(F\). Thus \(\varphi\) has degree one onto its image. Let \(\theta\) be a symplectic form on \(\mathcal{M}_2\) [M1]. Since the complement of \(\mathcal{U}_1\) in \(S^{[n]}\) has codimension two, \(\varphi^*\tau\) extends to a two-form on all of \(S^{[n]}\). This two-form is non-zero because \(\varphi\) has degree one, and hence it is a symplectic form. This implies that \(\varphi\) is an embedding: since \(\text{dim} \mathcal{U}_1 = \text{dim} \mathcal{M}_2\) and since \(\mathcal{M}_2, \mathcal{U}_1\) are smooth we conclude that \(\varphi\) is an isomorphism onto \(\mathcal{U}_2\), and that \(\mathcal{U}_2\) is open in \(\mathcal{M}_2\). Now let’s prove that the complement of \(\mathcal{U}_2\) has codimension at least two. By Proposition \((\text{I}.3.2)\) we know \(\mathcal{M}_2\) is irreducible, hence \(\varphi\) defines a birational map
\[
\varphi: S^{[n]} \cdots \to \mathcal{M}_2.
\]
Assume that \((\mathcal{M}_2 - \mathcal{U}_2)\) contains a divisor \(D\). The map \(\varphi^{-1}\) is a morphism on some (non-empty) open subset \(D^0 \subset D\) because \(S^{[n]}\), \(\mathcal{M}_2\) are both irreducible symplectic. Applying Zariski’s main theorem to a resolution of indeterminacies of \(\varphi\) we see that \(\varphi^{-1}(D^0) \subset (S^{[n]} - \mathcal{U}_1)\). Since \((S^{[n]} - \mathcal{U}_1)\) has codimension two, we conclude that \(\varphi^{-1}\) contracts \(D^0\). This is absurd because the pull-back by \(\varphi^{-1}\) of a symplectic form on \(S^{[n]}\) is a symplectic form on \(\mathcal{M}_2\). Thus \((\mathcal{M}_2 - \mathcal{U}_2)\) has codimension at least two in \(\mathcal{M}_2\).

**q.e.d.**

**I.5. A large open subset of \(\mathcal{M}_r\), for \(r \geq 3\).**

We retain notations and conventions of the previous subsection. For each \(r \geq 3\) we will construct a subscheme \(\mathcal{U}_r\) of \(\mathcal{M}_r\) and a tautological family \(\mathcal{F}^r\) of sheaves on \(S\) parametrized by \(\mathcal{U}_r\). In order to construct \(\mathcal{F}^r\) we need the following inductive hypotheses:
Let $r \geq 2$. If $x \in U_r$ then $h^0(F^r_x \otimes [-2C]) = 0$.

Let $r \geq 2$, and let $y \in \Sigma_1 \subset U_r$ (this makes sense by (I.5.1)). Then $F^r_y$ is locally-free. There exists exactly one elliptic fiber $C_u$ such that $\text{Hom}(F^r_y|_{C_u}, O_{C_u}) \neq 0$. Furthermore $F^r_y|_{C_u} \cong O_{C_u} \oplus V$, where $\text{Hom}(V, O_{C_u}) = 0$.

Let $r \geq 2$. If $x \in (U_r - \Sigma_1)$ then $\text{Hom}(F^r_x|_{C_t}, O_{C_t}) = 0$ for all elliptic fibers $C_t$.

Let $r \geq 2$ be given, and assume (I.5.2)-(I.5.4) hold for this $r$: we will construct $F^{r+1}$. Let $x \in U_r$. Since the Mukai vector corresponding to $M_r$ is normalized we have $\chi(F^r_x) = 1$. Let $C_s$, $C_t$ be two elliptic fibers; the exact sequence

$$0 \to F^r_x \otimes [-2C] \to F^r_x \to F^r_x|_{C_s \cup C_t} \to 0$$

shows that $\chi(F^r_x \otimes [-2C]) = -1$. By (I.5.2) $h^0(F^r_x \otimes [-2C]) = 0$, and by Serre duality together with stability of $F^r_x$,

$$H^2(F^r_x \otimes [-2C]) \cong \text{Hom}(F^r_x \otimes [-2C], O_S)^* = 0. \quad (I.5.5)$$

We conclude that $H^1(F^r_x \otimes [-2C])$ is one-dimensional. By Serre duality $\text{Ext}^1(F^r_x \otimes [-2C], O_S)$ is one-dimensional. Thus if $\rho: S \times U_r \to U_r$ is projection,

$$\xi_{r+1} := \text{Ext}^1(F^r_x \otimes [-2C], O_{S \times U_r})$$

is a line-bundle on $U_r$. Let $E^{r+1}$ be the tautological extension

$$0 \to O_{S \times U_r} \to E^{r+1} \to F^r \otimes \pi^*[-2C] \otimes \rho^*\xi_{r+1} \to 0.$$ 

Since $F^r$ is $U_r$-flat, so is $E^{r+1}$. If $x \in U_r$ then $E^{r+1}_x$ is the unique non-trivial extension

$$0 \to O_S \to E^{r+1}_x \to F^r_x \otimes [-2C] \to 0. \quad (I.5.6)$$

Lemma. Let $r \geq 2$ be given, and assume (I.5.4) holds for this $r$. Then $E^{r+1}$ is a family of torsion-free sheaves on $S$ parametrized by $U_r$. If $x \in U_r$, $v(E^{r+1}_x)$ is the normalized Mukai vector $v$ such that

$$v^0 = (r + 1) \quad (v^1, C) = 1 \quad d(v) = 2n, \quad (I.5.8)$$

i.e. the vector corresponding to $M_r$. For $x \in (U_r - \Sigma_1)$ the sheaf $E^{r+1}_x$ is stable, and furthermore for all elliptic fibers $C_t$ one has $\text{Hom}(E^{r+1}_x|_{C_t}, O_{C_t}) = 0$.

Proof. $U_r$-flatness follows immediately from $U_r$-flatness of $F^r$. Since $F^r_x$ is torsion-free, so is $E^{r+1}_x$. To show that $v(E^{r+1}_x)$ is normalized consider the long exact cohomology sequence associated to (I.5.6). Equation (I.5.8) is proved by a simple computation. Stability follows from Item (1) of Proposition (I.4.7). By Item (2) of Proposition (I.4.7) the restriction of (I.5.6) to any elliptic fiber is non-split, and hence the result follows from Remark (I.5.9) below.

Remark. Let $C_t$ be an elliptic fiber, and $0 \to O_{C_t} \to V \to W \to 0$ be a non-trivial extension. If $\text{Hom}(W, O_{C_t}) = 0$ then $\text{Hom}(V, O_{C_t}) = 0$.

Let’s show that for $y \in \Sigma_1$ the sheaf $E^{r+1}_y$ is not stable. By (I.5.3) there exists a unique elliptic fiber $C_u$ such that $\text{Hom}(F^r_y \otimes [-2C]|_{C_u}, O_{C_u}) \neq 0$, this group is one-dimensional, and if $\beta$ is a generator, the corresponding map is surjective. We define $H^r_y$ to be the sheaf on $S$ fitting into the exact sequence

$$0 \to H^r_y \to F^r_y \otimes [-2C] \to \iota^u_u O_{C_u} \to 0, \quad (I.5.10)$$

where $\iota^u: C_u \hookrightarrow S$ is the inclusion. At this point we add to our list of inductive hypotheses one last item.
Let $r \geq 2$ be given. Let $y \in \Sigma_1$. Then $\mathcal{H}_y^r$ is locally-free, and $\text{Hom}(\mathcal{H}_y^r|_C, \mathcal{O}_C) = 0$ for all elliptic fibers $C_t$.

We claim that $\alpha$ lifts to $\mathcal{E}_y^{r+1}$; this will give a destabilizing subsheaf of $\mathcal{E}_y^{r+1}$. Consider the exact sequence

$$
\text{Hom}(\mathcal{H}_y^r, \mathcal{O}_S) \to \text{Ext}^1(\mathcal{H}_y^r|_C, \mathcal{O}_S) \to \text{Ext}^1(\mathcal{F}_y^r \otimes [-2C], \mathcal{O}_S) \xrightarrow{\gamma} \text{Ext}^1(\mathcal{H}_y^r, \mathcal{O}_S).
$$

Letting $e$ be the extension-class of (I.5.6), $\gamma(e)$ is the obstruction to lifting $\alpha$ to $\mathcal{E}_y^{r+1}$. Since $\mathcal{H}_y^r|_C = \mathcal{F}_y^r|_C$ for $t \neq u$, the restriction of $\mathcal{H}_y^r|_C$ to the generic elliptic fiber is stable, and hence the first group of the above sequence is zero. A straightforward computation shows that the first $\text{Ext}^1$-group is one-dimensional; since we know that also the second $\text{Ext}^1$-group has dimension one, $\gamma$ must vanish. This proves that $\alpha$ lifts to a map $\tilde{\alpha}: \mathcal{H}_y^r \to \mathcal{E}_y^{r+1}$.

Claim. Let $r \geq 2$ be given. Assume that (I.5.2), (I.5.3) and (I.5.11) hold for $r$. Then

$$
\mathcal{E}_y^{r+1}/\text{Im}\tilde{\alpha} \cong [C].
$$

Proof. The quotient $\mathcal{E}_y^{r+1}/\text{Im}\tilde{\alpha}$ is clearly of rank one. Let’s start by showing that it is torsion-free. By (I.5.11) the sheaf $\mathcal{H}_y^r$ is locally-free. By (I.5.3) $\mathcal{F}_y^r \otimes [-2C]$ is locally-free, hence by Exact sequence (I.5.6) also $\mathcal{E}_y^{r+1}$ is locally-free. Thus, since outside of $C_u$ the map $\tilde{\alpha}$ is an injection, $\mathcal{E}_y^{r+1}/\text{Im}\tilde{\alpha}$ has torsion if and only if it drops rank along all of $C_u$. Let’s assume that this is indeed the case: since $\text{Ker}(\tilde{\alpha}|_{C_u}) \subset \text{Ker}(\alpha|_{C_u})$, and since the second kernel is a line-bundle, we conclude that the two kernels coincide. We claim this implies that $\tilde{\alpha}$ extends to a map $\mathcal{F}_y^r \otimes [-2C] \to \mathcal{E}_y^{r+1}$.

In fact consider the subsheaf $\text{Sat}(\text{Im}\tilde{\alpha}) \subset \mathcal{E}_y^{r+1}$ associated to the presheaf $\text{Sat}(\text{Im}\tilde{\alpha}) := \{\sigma \in (\mathcal{E}_y^{r+1})_U | \varphi \sigma \in \text{Im}\tilde{\alpha} \text{ for some } 0 \neq \varphi \in \mathcal{O}_U\}$.

$\text{Sat}(\text{Im}\tilde{\alpha})$ is locally-free, and since $\text{Ker}(\tilde{\alpha}|_{C_u})$ is of rank one, it fits into an exact sequence

$$
0 \to \mathcal{H}_y^r \xrightarrow{\tilde{\alpha}} \text{Sat}(\text{Im}\tilde{\alpha}) \to \mathcal{O}_C \xrightarrow{\iota_u} 0.
$$

Since $\text{Ker}(\tilde{\alpha}|_{C_u}) = \text{Ker}(\alpha|_{C_u})$, we have $\text{Sat}(\text{Im}\tilde{\alpha}) \cong \mathcal{F}_y^r \otimes [-2C]$. Thus $\tilde{\alpha}$ extends to a map $\mathcal{F}_y^r \otimes [-2C] \to \mathcal{E}_y^{r+1}$. This is impossible because (I.5.6) does not split. Hence $\mathcal{E}_y^{r+1}/\text{Im}\tilde{\alpha}$ is torsion-free, and therefore isomorphic to $\mathcal{I}_Y \otimes \lambda$, for a line-bundle $\lambda$ and a zero-dimensional subscheme $Y$ of $S$. Computing Chern classes one gets that $\lambda \cong [C]$ and that $Y = \emptyset$.

Thus we get a unique exact sequence:

$$
0 \to \mathcal{H}_y^r \to \mathcal{E}_y^{r+1} \to [C] \to 0.
$$

Since $c_1(\mathcal{H}_y^r, C) = 1$, the above exact sequence shows that the restriction of $\mathcal{E}_y^{r+1}$ to the generic fiber is unstable; by (I.1.6) we conclude that $\mathcal{E}_y^{r+1}$ is unstable. By unicity of the above construction, we can globalize it to all of $S \times \Sigma_1$. Hence there exists a line-bundle $\eta_{r+1}$ on $S \times \Sigma_1$ and a surjection $\mathcal{E}_y^{r+1}|_{S \times \Sigma_1} \to \eta_{r+1}$, which restrict on $S \times \{y\}$ to the above destabilizing quotient for $\mathcal{E}_y^{r+1}$. We define $\mathcal{F}_x^{r+1}$ as the elementary modification of $\mathcal{E}_y^{r+1}$ fitting into the exact sequence

$$
0 \to \mathcal{F}_x^{r+1} \to \mathcal{E}_y^{r+1} \xrightarrow{\iota_x} \eta_{r+1} \to 0.
$$

Lemma. Let $r$ be given, and assume (I.5.2)-(I.5.11) hold. Then $\mathcal{F}_x^{r+1}$ is a family of torsion-free stable sheaves on $S$. If $x \in U$, then $v(\mathcal{F}_x^{r+1})$ is equal to the normalized Mukai vector $v$. 

19
satisfying (I.5.8). If $y \in \Sigma_1$ there exists exactly one elliptic fiber $C_u$ such that $\Hom(F_{y}^{r+1}|_{C_u}, \mathcal{O}_{C_u})$ is non-zero. Furthermore $F_{y}^{r+1}|_{C_u} \cong \mathcal{O}_{C_u} \oplus V$, where $\Hom(V, \mathcal{O}_{C_u}) = 0$.

**Proof.** Since $E^{r+1}$ is $U_r$-flat, the sheaf $F^{r+1}$ is flat by [F2, Lemma (A.3)].

If $x \in U_r - \Sigma_1$ then $F_{x}^{r+1} = E_{x}^{r+1}$. \hfill (I.5.15)

If $y \in \Sigma_1$ restriction of (I.13) to $S \times \{y\}$ gives

$$0 \to [C] \to F_{y}^{r+1} \to H_{y}^{r} \to 0.$$ \hfill (I.5.16)

In particular $F_{y}^{r+1}$ is torsion-free, hence $F^{r+1}$ is a family of torsion-free sheaves on $S$ parametrized by $U_r$. From this it follows that there is a two-step locally-free resolution of $F^{r+1}$, and hence the Chern character of $F_{y}^{r+1}$ is a locally constant function of $x \in U_r$. Thus $v(F_{y}^{r+1})$ is a locally constant function of $x \in U_r$. Since, by (I.5.1), $U_r$ is irreducible, this function is constant; hence Equality (I.5.15) and Lemma (I.5.7) imply that $v(F_{y}^{r+1})$ is as stated. Let’s prove stability. For $x \in (U_r - \Sigma_1)$ this follows from (I.15) together with Lemma (I.5.7). Let $y \in \Sigma_1$. Proceeding exactly as in the proof of Proposition (I.4.29) one shows that Extension (I.5.16) is non-trivial, and hence stability of $F_{y}^{r+1}$ follows from (I.4.7) together with (I.5.11). What is left to prove are the last two statements. By (I.11) we can apply Proposition (I.4.7) to Extension (I.5.16): since $\dim \Ext^1(H_{y}^{r}, [C]) = 2$ we get that there exists a unique elliptic fiber $C_u$ such that the restriction of (I.16) to $C_u$ is trivial. By Remark (I.5.9) together with (I.11) we conclude that $\Hom(F_{y}^{r+1}|_{C_u}, \mathcal{O}_{C_u}) \neq 0$ if and only if $t = u$. That $F_{y}^{r+1}|_{C_u}$ splits as stated follows immediately from (I.5.11). \hfill q.e.d.

Now we can define $U_{r+1}$. By Lemma (I.14) the sheaf $F^{r+1}$ defines a classifying morphism $\varphi : U_r \to M_{r+1}$.

(I.5.17) **Proposition-Definition.** Keep notation as above. Let $U_{r+1} := \varphi(U_r)$. Then $U_{r+1}$ is an open subset of $M_{r+1}$, and its complement has codimension two. The morphism $\varphi : U_r \to U_{r+1}$ is an isomorphism onto its image. Hence we can view $F^{r+1}$ as a tautological family of sheaves on $S$ parametrized $U_{r+1}$.

**Proof.** The proof goes exactly as the proof of Proposition (I.4.31). \hfill q.e.d.

What is left to do is to prove inductively that hypotheses (I.5.1)-(I.5.11) hold for all $r \geq 2$. Item (I.5.1) is obvious.

**Proof of (I.5.2).** To verify (I.5.2) for $r = 2$ one considers separately the three cases $x \in (U_2 - L - \Sigma_1)$, $x \in L$, and $x \in \Sigma_1$. Set $x = [Z]$. In the first case $F_{[Z]}^{2} = E_{[Z]}^{2}$; tensoring Exact sequence (I.4.5) by $[-2C]$, and using Definition (I.4.1), one gets the desired result. In the second case $F_{[Z]}^{2} = G_{[Z]}^{2}$, and one gets the result arguing similarly, with Sequence (I.4.17) replacing (I.4.5). In the third case one proceeds in the same way, with (I.4.30) replacing (I.4.5). Now let’s prove the inductive step. If $x \in (U_{r+1} - \Sigma_1)$ then $F_{x}^{r+1} = E_{x}^{r+1}$ (of course we identify $U_{r+1}$ with $U_r$); tensoring (I.5.6) by $\mathcal{O}_S(-2C)$ and applying the inductive hypothesis one gets the result. Let $y \in \Sigma_1$. Tensoring (I.5.16) by $\mathcal{O}_S(-2C)$ we see that it suffices to show that $h^0(H_{y}^{r+1} \otimes [-2C]) = 0$. By (I.5.10) $H_{y}^{r+1}$ is a subsheaf of $F_{y}^{r+1} \otimes [-2C]$ hence the result follows from the inductive hypothesis.

**Proof of (I.5.3).** First we verify that (I.5.3) holds for $r = 2$. We start by showing that $F_{y}^{2}$ is locally-free for all $y \in \Sigma_1$. The proof is by contradiction. Assume $F_{[Z]}^{2}$ is singular for some $[Z] \in \Sigma_1$. Let’s prove that this implies $F_{[Z]}^{2}$ is singular for all $[Z] \in \Sigma_1$. Since $\Sigma_1 \cap L = \emptyset$, there exists $[Z] \in (U_2 - L)$ such that $F_{[Z]}^{2}$ is singular. The locus of $[Z] \in (U_2 - L)$ such that $F_{[Z]}^{2}$
is singular has codimension at most one, because $F^2$ has rank two. If $[Z] \in \langle U_2 - L - \Sigma_1 \rangle$ then $F^2_{[Z]} = E^2_{[Z]}$; since $E^2_{[Z]}$ fits into the non-trivial extension (I.4.5) it follows from (I.1.5) together with Definition (I.4.1) that $F^2_{[Z]}$ is locally-free. Since $\Sigma_1$ is irreducible we conclude that $F^2_{[Z]}$ is singular for all $[Z] \in \Sigma_1$. Let $[Z] \in \Sigma_1$, and set $Z = P \cap W$, where $P := Z \cap \Sigma$. Since $F^2_{[Z]}$ fits into exact sequence (I.4.30) its singular points belong to $W$. On the other hand we claim that if $n > 2$ (we will treat the case $n = 2$ separately) there exists at least one point of $W$ at which $F^2_{[Z]}$ is locally-free. In fact consider the exact sequence

$$H^1 (\text{Hom}(I_W \otimes \Sigma + (n - 3)C), [C]) \to \text{Ext}^1 (I_W \otimes \Sigma + (n - 3)C), [C]) \xrightarrow{\tau} H^0 ((\text{Ext}^1 (I_W \otimes \Sigma + (n - 3)C), [C])) \to H^2 (\text{Hom}(I_W \otimes \Sigma + (n - 3)C), [C]) \ . \ (I.5.18)$$

Since $n > 2$ the first term is zero, and the claim follows immediately. Now fix $P_0 \in \Sigma$, and consider

$$\Omega_0 := \{ [W] \in (S - P_0)^{[n-1]} | [P_0 \cup W] \in \Sigma_1 \} .$$

Each $[W] \in \Omega_0$ is the disjoint union of the subset consisting of the points at which $F^2_{[W \cup P_0]}$ is locally-free and of its complement. Since for each $[W] \in \Omega_0$ these subsets are both non-empty, we conclude that the incidence locus

$$\bar{\Omega}_0 := \{ ([W], Q) \in \Omega_0 \times S | Q \in W \}$$

has two components. This is absurd, and hence $F^2_{[Z]}$ is locally-free for all $[Z] \in \Sigma_1$, if $n > 2$. Now let’s suppose that $n = 2$. Assume $F^2_{[Z]}$ is singular. Consider Exact sequence (I.5.18): since $W$ consists of one point, and since

$$\dim H^0 ((\text{Ext}^1 (I_W \otimes \Sigma + (n - 3)C), [C])) = 1 ,$$

the image under $\tau$ of the extension class of (I.4.30) must be zero, i.e. the extension class lives in the first cohomology group of (I.5.18). Since this group is one-dimensional, we conclude that the isomorphism class of $F^2_{[Z]}$ only depends on $W$, and not on $P$. This is absurd because $U_2$ is a subset of the moduli space $M_2$. We have finished proving that $F^2_{[Z]}$ is locally-free for all $[Z] \in \Sigma_1$. To prove the second and third statements of (I.5.3) one applies Item (2) of Proposition (I.4.7) to the non-split exact sequence (I.4.30): this gives that there exists a unique elliptic fiber $C_u$ such that the restriction of (I.4.30) to $C_u$ splits. By Remark (I.5.9) and Definition (I.4.1) one concludes that $\text{Hom}(F^2_{[Z]}|_{C_u}, O_{C_u}) \neq 0$ if and only if $t = u$, and that the splitting at $t = u$ is as claimed. It remains to verify the inductive step: this was proved in Lemma (I.5.14).

**Proof of (I.5.4).** Let’s prove (I.5.4) for $r = 2$. First consider $x \in (U_2 - \Sigma_1 - L)$. Set $x = [Z]$. Then $F^2_{[Z]} = E^2_{[Z]}$. By Item (2) of Proposition (I.4.7) the restriction of (I.4.5) to any elliptic fiber is non-split, and hence Statement (I.5.4) follows from Remark (I.5.9). We are left with proving that (I.5.4) holds for $x \in L$ (and $r = 2$).

**Claim.** Assume there exists $x \in L$ and an elliptic fiber $C_u$ such that $\text{Hom}(F^2_{[Z]}|_{C_u}, O_{C_u}) \neq 0$. Then $F^2_{[Z]}|_{C_u}$ is singular.

**Proof of the claim.** Since $x \in L$, we have $F^2_x = G^2_x$. To simplify notation set $G := G^2_x$. Then we have

$$0 \to I_P \otimes [C] \to G \to I_W \otimes [\Sigma + (n - 3)C] \to 0 ,$$

21
with notation as in (I.4.17). By (I.4.18) the sheaf $G$ is locally-free at one at least of the points of $W$. By (I.1.4) and Definition (I.4.1) the sheaf $G$ is locally-free at all points of $W$, and hence we get

$$0 \to [C] \to G^{**} \to I_W \otimes [\Sigma + (n - 3)C] \to 0,$$

(†)

Since $\text{Ext}^1(I_W \otimes [\Sigma + (n - 3)C], [C])$ is one-dimensional, the restriction of (†) to any elliptic fiber $C_t$ is non-split by Item (2) of Proposition (I.4.7). Thus by Remark (I.5.9) $\text{Hom}(G^{**}|_{C_t}, \mathcal{O}_{C_t}) = 0$ for all elliptic fibers $C_t$. Since $G|_{(S-P)} \cong G^{**}|_{(S-P)}$ we conclude that $P \in C_u$, i.e. $G|_{C_u}$ is singular. q.e.d.

Let’s go back to the proof that (I.5.4) holds for $r = 2$ and $x \in L$. Assume that (I.5.4) does not hold for some $x \in L$; we will arrive at a contradiction. Let $G$, $C_u$ be as above. Let $\text{Def}^0(G|_{C_u})$ be the subspace of the deformation space $\text{Def}(G|_{C_u})$ parametrizing deformations “fixing the determinant”. The restriction map $r_u: \text{Def}(G) \to \text{Def}^0(G|_{C_u})$ is surjective. In fact it suffices to check that $h^2(\text{ad}F^{**} \otimes [-C]) = 0$; by Serre duality this is equivalent to $h^0(\text{ad}F^{**} \otimes [C]) = 0$, and this follows from stability of $F^{**}|_{C_t}$ for a generic elliptic fiber $C_t$. Letting $V \subset \text{Def}^0(F|_{C_u})$ be the subspace parametrizing deformations $A$ such that $\text{Hom}(A, \mathcal{O}_{C_u}) \neq 0$, one verifies easily that the generic point of $V$ parametrizes a locally-free sheaf. Since the restriction map $r_u$ is surjective we conclude that there exists $x' \in U_2$ arbitrarily near to $x$ such that $\text{Hom}(F^2_{x'}|_{C_u}, \mathcal{O}_{C_u}) \neq 0$ and $F^2_{x'}|_{C_u}$ is locally-free. By the previous claim $x' \notin L$. Since $\Sigma \cap L = \emptyset$ we can assume $x' \in (U_2 - L - \Sigma_1)$. This is absurd: by Item (I.5.3), which we have just proved, $\text{Hom}(F^2_{x'}|_{C_u}, \mathcal{O}_{C_u}) \neq 0$ for such an $x'$. Now let’s prove the inductive step. Since $x \in (U_{r+1} - \Sigma_1)$ we have $F^{r+1}_x = E^{r+1}_x$ (we are identifying $U_{r+1}$ with $U_r$). By Remark (I.5.9) and Inductive hypothesis (I.5.4) it suffices to show that the restriction of (I.5.6) to any elliptic fiber is non-split. This follows at once from Item (2) of Proposition (I.4.7).

**Proof of (I.5.11).** By (I.5.3), which we have just proved, $F^r_y$ is locally-free for $r \geq 2$. Exact sequence (I.5.10) proves that also $H^r_y$ is locally-free for $r \geq 2$. Now let’s prove that

$$\text{Hom}(H^r_y|_{C_t}, \mathcal{O}_{C_t}) = 0$$

for all elliptic fibers $C_t$. (I.5.19)

For this it is convenient to notice that $H^r_y$ has been defined also when $r = 1$. In fact Exact sequence (I.5.10) in the case $r = 1$ becomes Exact sequence (I.4.26). Thus

$$H^1_y = I_W \otimes [\Sigma + (n - 3)C].$$

Exact sequences (I.5.12)-(I.5.16) for $r = 1$ reduce to (I.4.27) and (I.4.30) respectively. We will prove inductively that (I.5.19) holds for all $r \geq 1$. It holds for $r = 1$ because by Definition (I.4.1) $W$ intersects every elliptic fiber in at most one reduced point, not belonging to $\Sigma$. Now assume (I.5.19) holds for $r \geq 1$. Replacing $r$ by $r + 1$ in (I.5.10), and tensorizing the exact sequence by $[2C]$, we get

$$0 \to H^{r+1}_x \otimes [2C] \to F^{r+1}_x \to \iota^u_x \mathcal{O}_{C_u} \to 0.$$

Thus local sections of $H^{r+1}_x \otimes [2C]$ consist of local sections of $F^{r+1}_x$ whose projection to $[C]|_{C_u} \cong \mathcal{O}_{C_u}$ is zero, where the projection is defined by the splitting of (I.5.16) along $C_u$. Let $\sigma$ be a non-zero section of $[C]$ which vanishes on $C_u$: by (I.5.16) this gives a section of $F^{r+1}_x$, which we still denote by $\sigma$. Since $\sigma$ vanishes on $C_u$, it belongs to $H^0(\mathcal{H}^{r+1}_x \otimes [2C])$; notice that as a section of this locally-free sheaf it is nowhere zero along $C_u$. Considering (I.5.16) we see that the resulting exact sequence is:

$$0 \to \mathcal{O}_S \to H^{r+1}_x[2C] \to H^r_x \to 0. \quad (I.5.20)$$
The restriction of the above extension to a generic elliptic fiber coincides with the restriction of \((I.5.16)\), and hence it is non-trivial. By the inductive hypothesis \(\text{Hom}(H^r_s|_{C_i}, \mathcal{O}_{C_i}) = 0\) for all elliptic fibers. Thus we can apply Item (2) of Proposition \((I.4.7)\) to Extension \((I.5.20)\). A computation gives \(\dim \text{Ext}^1(H^r_s, \mathcal{O}_S) = 1\), and hence by \((I.4.7)\) the restriction of the above extension to any elliptic fiber is non-trivial. By \((I.5.9)\) and the inductive hypothesis we conclude that \((I.5.11)\) holds with \(r\) replaced by \((r + 1)\).

This ends the proof that the tautological families \(F^r\) are defined for all \(r\).

**I.6. Proof of Theorem \((I.0.4)\).**

As explained in Subsection \((I.2)\) we can assume \(v\) is normalized, that is \(\mathcal{M}_v(H) = \mathcal{M}_v^{2n}\) for \(r = v^0\) and \(n = d(v)/2\). By Subsections \((I.4-I.5)\) we know \(\mathcal{M}_v^{2n}\) is non-empty, thus by Proposition \((I.3.2)\) it is birational to \(S[n]\). We are left with proving that \(\theta_v\) is an isomorphism of integral Hodge structures, and that it preserves the quadratic forms. As in the previous subsections we think \(n\) is fixed, and we omit it from our notation whenever this shouldn’t cause confusion. We let

\[v_r := r + c_1(\Sigma + (n - r^2 + r)C) + (1 - r)\omega\]

be the Mukai vector corresponding to \(\mathcal{M}_v^{2n}\), and we set \(\theta_r := \theta_{v_r}\). A lattice \((\Lambda, q)\) consists of a free \(\mathbb{Z}\)-module \(\Lambda\) provided with an integral quadratic form \(q\). A homomorphism between lattices \((\Lambda_1, q_1), (\Lambda_2, q_2)\) consists of a homomorphism of modules \(f: \Lambda_1 \to \Lambda_2\) such that \(q_1(\alpha) = q_2(f(\alpha))\) for all \(\alpha \in \Lambda_1\).

**Proposition \((I.6.1)\).** Keep notation as above and assume \(n \geq 1\). The map \(\theta_r\) is integral, i.e.

\[\theta_r \left( u_r^\perp \cap H^*(S; \mathbb{Z}) \right) \subset H^2(\mathcal{M}_r; \mathbb{Z})\]

This map is a homomorphism of lattices, where \(u_r^\perp \cap H^*(S; \mathbb{Z})\) is provided with Mukai’s quadratic form, given by \((1)\), and \(H^2(\mathcal{M}_r; \mathbb{Z})\) is provided with Beauville’s canonical quadratic form \(B_{\mathcal{M}_r}\) \([B, \text{Th. (5)}]\).

**Proof of Proposition \((I.0.4)\) assuming \((I.6.1)\).** Our first step is to identify the quadratic form \(B_{\mathcal{M}_v^{2n}}\). For this we need the lemma below. This result is well-known to experts \([M3, \text{Prop. (5.8)}]\); we give a proof for the reader’s convenience.

**Proposition \((I.6.2)\).** Let \(X, Y\) be birational irreducible symplectic projective varieties. Let \(f: X \to Y\) be a birational map. There exist open subsets \(i: U \in X, j: V \in Y\), whose complements have codimension at least two, such that \(f\) restricted to \(U\) is regular and it defines an isomorphism \(f|_U: U \cong V\). Let \(f^*: H^2(X; \mathbb{Z}) \cong H^2(Y; \mathbb{Z})\) be the isomorphism obtained as the composition

\[H^2(X; \mathbb{Z}) \xrightarrow{i^*} H^2(U; \mathbb{Z}) \xrightarrow{j^*} H^2(V; \mathbb{Z}) \xrightarrow{\sim} H^2(Y; \mathbb{Z})\]

which is independent of \(U, V\). Then for all \(\alpha \in H^2(X)\)

\[B_X(\alpha) = B_Y(f^*\alpha)\]

\[\text{(I.6.3)}\]

**Proof.** Let \(X \leftarrow Z \to Y\) be a resolution of indeterminacies of \(f\), with \(Z\) smooth. Let’s show that the exceptional divisors of \(g\) are the same as the exceptional divisors of \(h\). If \(\tau_X \in H^0(\Omega^2_X)\) then there exists \(\tau_Y \in H^0(\Omega^2_Y)\) such that

\[g^*\tau_X = h^*\tau_Y\]

\[\text{(*)}\]
Now assume $\tau_X \neq 0$. Let $\dim X = \dim Y = 2n$. If $E$ is an exceptional divisor of $g$ then $\wedge^n (g^* \tau_X)$ vanishes on $E$, hence by (\*) does $\wedge^n (h^* \tau_Y)$. Since $\tau_Y$ is non-degenerate this implies that $E$ is an exceptional divisor of $h$. Thus every exceptional divisor of $g$ is an exceptional divisor of $h$. Reversing the rôles of $X$ and $Y$ we conclude that the exceptional divisors of $g$ and $h$ are the same. Clearly the first statement holds if we set $U := g(\cup_i E_i)$, $V := h(\cup_i E_i)$, where $\{E_i\}$ is the collection of all exceptional divisors. Equivalently the isomorphism $f^\sharp$ is obtained from the decomposition

$$g^* H^2(X; \mathbb{Z}) \oplus \oplus \mathbb{Z} c_1(E_i) = H^2(Z; \mathbb{Z}) = h^* H^2(Y; \mathbb{Z}) \oplus \oplus \mathbb{Z} c_1(E_i).$$

To prove the second statement we recall [B, p. 772] that Beauville’s quadratic form $B_X$ is the unique non-zero integral primitive positive multiple of the quadratic form $q_X$ defined by

$$q_X(\alpha) = \frac{n}{2} \int_X (\tau_X \tau_X)^n - 1 \alpha^2 + (1 - n) \int_X \tau_X^n \tau_X \alpha \cdot \int_X \tau_X^n \tau_X^n \alpha.$$

Now express $q_X(\alpha)$ and $q_Y(\alpha)$ as integrals over $Z$ of the appropriate forms pulled-back by $g$ and $h$ respectively, and use the relation

$$g^* \alpha = h^* f^\sharp \alpha + \sum_i n_i E_i,$$

valid for some integers $n_i$. Then, as is easily checked, in order to prove (I.6.3) it suffices to show that

$$(g^* \tau_X g^* \tau_X)^n - 1 |_{E_i} = 0 = (h^* \tau_Y h^* \tau_Y)^n - 1 |_{E_i}$$

(\dag)

for any exceptional divisor $E_i$. Since $E_i$ is contracted by $g$ and by $h$, and since the fibers of $g|_{E_i}$ are distinct from the fibers of $h|_{E_i}$, at every point of $E_i$ we have

$$\dim \text{span of Ker}(g|_{E_i}) \text{ and Ker}(h|_{E_i}) \geq 2.$$

The above span is clearly contained in $\text{Ker}(g^*|_{E_i}) = \text{Ker}(h^* \tau_Y|_{E_i})$, and hence this kernel is at least two-dimensional. This implies (\dag). q.e.d.

The proposition above allows us to identify the Beauville form of $\mathcal{M}^{2n}$ with that of $S^{[n]}$. The latter is described as follows [B, p. 777-778]. There is an inclusion $\sigma: H^2(S; \mathbb{Z}) \hookrightarrow H^2(S^{[n]}; \mathbb{Z})$ obtained composing the natural "symmetrization map" $H^2(S; \mathbb{Z}) \hookrightarrow H^2(S^{[n]}; \mathbb{Z})$, where $S^{[n]}$ is the $n$-fold symmetric product of $S$, with the pull-back map $\epsilon^*: H^2(S^{[n]}) \rightarrow H^2(S^{[n]})$, where $\epsilon: S^{[n]} \rightarrow S^{[n]}$ is the morphism mapping a subscheme to the 0-cycle associated to it. One has

$$H^2(S^{[n]}; \mathbb{Z}) = \sigma(H^2(S; \mathbb{Z})) \oplus \mathbb{T} Z,$$

(I.6.4)

where $T \in H^2(S^{[n]}; \mathbb{Z})$ is the (unique) class such that $2T$ is cohomologous to the divisor parametrizing non-reduced subschemes.

*(I.6.5) Description of $B_{S^{[n]}}$. The direct sum (I.6.4) is orthogonal for $B_{S^{[n]}}$. The restriction of $B_{S^{[n]}}$ to $\sigma(H^2(S; \mathbb{Z}))$ is equal to the intersection form on $H^2(S; \mathbb{Z})$, and $B_{S^{[n]}(T)} = -2(n - 1)$.*

Let’s go back to the proof of Proposition (I.0.4). Since the map $\theta_r$ clearly preserves type, all we must show is that the homomorphism of lattices is in fact an isomorphism. Since the Mukai form $\langle \cdot, \cdot \rangle$ has discriminant one, the discriminant of its restriction to $\nu_r^\perp$ is equal to $\langle \nu_r, \nu_r \rangle = 2(n - 1)$. In particular $\theta_r$ is injective because $n > 1$. Since $\text{rk} \nu_r^\perp = 23 = \text{rk} H^2(S^{[n]}) = \text{rk} H^2(M_r^{2n})$ the image $\theta_r(H^*(S; \mathbb{Z}))$ has finite index, say $s$, in $H^2(M_r^{2n}; \mathbb{Z})$. By Proposition (I.6.1) the map $\theta_r$ is a
homomorphism of lattices, and hence the discriminant of $B_{M^2}$ is equal to $s^2(2n - 2)$. By (I.6.2) together with (I.6.5) the discriminant of $B_{M^2}$ is equal to $2(n - 1)$, thus $s = 1$, i.e. $\theta_r$ is an isomorphism.

**Proof of Proposition (I.6.1).** Let $U_r \subset M_r$ be the open subset constructed in Subsections I.4-I.5. By (I.4.2)-(I.4.31)-(I.5.17) the complement of $U_r$ in $M_r$ has codimension two, and hence $H^2(M_r) \cong H^2(U_r)$. Thus $\theta_r$ is determined by the map $\theta_{F^r}: H^r(S) \to H^2(U_r)$ given by

$$\theta_{F^r}(\alpha) := \rho_* [ch(F^r)^*(1 + \omega)\alpha]_3,$$

where $\rho: S \times U_r \to U_r$ is projection, and we have denoted by the same symbol classes in $H^r(S)$ and their pull-back to $H^r(S \times U_r)$. By (I.4.31) and (I.5.1) there is an isomorphism $\varphi: U_r \cong U_1$, and $U_1$ is an open subset of $S^{[n]}$ whose complement has codimension two. Thus by Proposition (I.6.2) $\varphi$ induces an isomorphism of the lattices $(H^2(M_r; \mathbb{Z}, B_{M_r}))$ and $(H^2(S^{[n]}; \mathbb{Z}, B_{S^{[n]}}))$. We will always tacitly identify these lattices.

The proof of Proposition (I.6.1) will be by induction on $r$. By the construction of $F^{r+1}$ we have two exact sequences:

$$0 \to F^{r+1} \to E^{r+1} \to \iota_* Q_{r+1} \to 0,$$

$$0 \to O_S \times U_r \to E^{r+1} \to F^r \otimes [-2C] \otimes \rho^* \xi_{r+1} \to 0.$$  \hspace{1cm} (I.6.6) \hspace{1cm} (I.6.7)

Here $\iota$ is the inclusion of $S \times (L \cup \Sigma_1)$ if $r = 1$, and of $S \times \Sigma_1$ if $r \geq 2$, while $Q_{r+1}$ is a rank-one sheaf. From the two exact sequences above one gets

$$\theta_{F^{r+1}}(\alpha) = \theta_{F^r}(\alpha \cdot e^{2C}) + \langle v_r, \alpha \cdot e^{2C} \rangle c_1(\iota_{r+1}) - \rho_* [ch(\iota_* Q_{r+1})^*(1 + \omega)\alpha]_3.$$  \hspace{1cm} (I.6.8)

Since $\xi_{r+1} = Ext^1(\rho^*(F^r \otimes [-2C], O_{S \times U_r})$, using (I.4.3)-(I.5.2)-(I.5.5) we get $c_1(\iota_{r+1}) = c_1(\rho(\iota_{F^r} \otimes [-2C]))$. Applying Grothendieck-Riemann-Roch one gets

$$c_1(\iota_{r+1}) = c_1(e^{2C}) + \rho_* [c_1(F^r)\omega]_3.$$  \hspace{1cm} (I.6.9)

We need some notation. First we recall that if $\alpha \in H^r(S)$ then $\alpha^1$ is the component of $\alpha$ belonging to $H^{2n}(S)$, and that $\omega \in H^4(S; \mathbb{Z})$ is the fundamental class. For $\alpha \in H^r(S)$ we set $\alpha^1 := \sigma(\alpha^1) \in H^2(S^{[n]})$. If $\alpha^1$ is integral and $A \subset S$ is a real surface representing it (i.e. representing its Poincaré dual), then $\alpha^1$ is represented by the subset of $S^{[n]}$ parametrizing schemes intersecting $A$. By abuse of notation we will often denote by the same symbol a divisor and its first Chern class.

**Rank one.** Since $F^1 = I_Z \otimes [\Sigma + nC]$, we have

$$\theta_{F^1}(\alpha) = \rho_* [ch(I^1_Z) e^{-\Sigma - nC}(1 + \omega)\alpha]_3.$$  \hspace{1cm} (I.6.10)

First we compute $ch(I^1_Z)$. Let $j: Z \hookrightarrow S \times U_1$ be the inclusion, and let

$$\Gamma := \{(p, [Z]) \in S \times U_1 | p \text{ is a non-reduced point of } Z \}.$$  \hspace{1cm} (I.6.11)

Then

$$ch(j_* O_Z) = [Z] - \frac{1}{2} [\Gamma] + \text{(higher order)},$$

$$ch(I^1_Z) = 1 - ch(j_* O_Z) = 1 - [Z] + \frac{1}{2} [\Gamma] + \text{(higher order)},$$

25
where the first equality follows from Grothendieck-Riemann-Roch. Substituting in the expression for \( \theta_{\mathcal{F}_1} \) one gets
\[
\theta_{\mathcal{F}_1}(\alpha) = -\rho_* \{ [\mathcal{Z}] \cdot (\alpha^1 - \alpha^0(\Sigma + nC)) \} - \frac{1}{2} \alpha^0 \rho_* [\Gamma].
\]
Since for \( \beta \in H^2(S) \) we have \( \rho_* \{ [\mathcal{Z}] \cdot \beta \} = \beta_1 \), and since \( \rho_* [\Gamma] = 2T \), we get
\[
\theta_{\mathcal{F}_1}(\alpha) = -\alpha_1 + \alpha^0(\Sigma_1 + nC_1 - T).
\]  \hfill (I.6.10)

In particular \( \theta_{\mathcal{F}_1} \) is integral, and hence also \( \theta_1 \). Using Proposition (I.6.2) together with (I.6.5) we conclude that
\[
B_{M_1}(\theta_{\mathcal{F}_1}(\alpha), \theta_{\mathcal{F}_1}(\alpha)) = \alpha^1 \cdot \alpha^1 - 2 \alpha^0 \left( \alpha^1 \cdot (\Sigma + nC) \right).
\]
Since \( \alpha, v_1 = 0 \) is equivalent to \( \alpha^2 = \alpha^1 \cdot (\Sigma + nC) \), the map \( \theta_1 \) is a homomorphism of lattices. This finishes the proof of Proposition (I.6.1) when the rank is one.

**Rank two.** Applying Equation (I.6.9) one gets
\[
c_1(\xi_2) = -(n - 2)C_1 - \Sigma_1 + T.
\]

Now let’s compute \( \rho_* [ch(\iota_* Q_2)^*(1 + \omega)\alpha]_3 \). Let
\[
\iota_{\Sigma_1}: S \times \Sigma_1 \hookrightarrow S \times U_2, \quad \iota^L: S \times L \hookrightarrow S \times U_2
\]
be the inclusions, and set \( Q_{\Sigma_1} := Q_2|_{S \times \Sigma_1}, Q^L := Q_2|_{S \times L} \). Then
\[
Q_{\Sigma_1} \cong \pi^*[C] \otimes \rho^* \lambda_1, \quad Q^L \cong \pi^*[C] \otimes I_\Omega \otimes \rho^* \lambda_2,
\]
where \( \lambda_1, \lambda_2 \) are line bundles on \( U_2 \), and
\[
\Omega := \{ (P, x) \mid x \in L \text{ and } P \text{ is the singular point of } Q_2|_{S \times \{x\}} \}.
\]

Applying Grothendieck-Riemann-Roch to \( \iota_{\Sigma_1}, \iota^L \) one gets
\[
\begin{align*}
ch(\iota^*_{\Sigma_1} Q_{\Sigma_1}) &\equiv \rho^*[\Sigma_1] + \pi^*[C] \cdot \rho^*[\Sigma_1] \pmod{H^3(U_2)), \quad (I.6.11) \\
ch(\iota^*_{\Sigma_1} Q^L) &\equiv \rho^*[L] + \pi^*[C] \cdot \rho^*[L] - [\iota^*_{\Omega}] \pmod{H^3(U_2)), \quad (I.6.12)
\end{align*}
\]

We claim that the following linear equivalence among divisors on \( S^{[n]} \) holds:
\[
L \sim (n - 1)C_1 - T. \quad (I.6.13)
\]

**Proof of (I.6.13).** Since \( S^{[n]} \) is simply connected it suffices to prove that the divisors are cohomologous, and for this it suffices to verify that they intersect any 2-homology class in the same number of points. If \( \beta \) is a two-cycle on \( S \), and \( W \subset S \) is a set of \( (n - 1) \) points disjoint from \( \beta \), let
\[
\beta_W := \{ [Z] \in S^{[n]} \mid Z = P \cup W, \text{ where } P \in \beta \}.
\]
Then \( \beta_W \) is a two-cycle on \( S^{[n]} \), and the annihilator of \( c_1(T) \) is spanned by the \( \beta_W \). Clearly we have
\[
\langle c_1(L), \beta_W \rangle = (n - 1)\langle c_1(C), \beta \rangle = (n - 1)\langle c_1(C_1) - c_1(T), \beta_W \rangle.
\]
Now choose a point \( P \in \Sigma \) and a subset \( W \subset S \) of \((n - 2)\) points disjoint from \( \Sigma \). Let \( \tilde{\Sigma} \subset S^{[n]} \) be the curve given by
\[
\tilde{\Sigma} := \{ [Z_Q] \in S^{[n]} | Z_Q = Q \cup P \cup W \text{ where } Q \in \Sigma \},
\]
where the scheme structure of \( Z_P \) at \( P \) consists of the double point contained in \( \Sigma \). Since
\[
\langle c_1(L), \tilde{\Sigma} \rangle = n - 2, \quad \langle c_1(C_1), \tilde{\Sigma} \rangle = 1, \quad \langle c_1(T), \tilde{\Sigma} \rangle = 1,
\]
the intersections of the two sides of Equation (I.6.13) with \( \tilde{\Sigma} \) are equal. Since \( H_2(S^{[n]}) \) is spanned by \( \tilde{\Sigma} \) and the classes \( \beta_W \), we conclude that (I.6.13) holds. \( \text{q.e.d.} \)

Using (I.6.13) together with Equations (I.6.11)-(I.6.12) one obtains that
\[
\rho_* [ch(t_*Q_2)^*(1 + \omega)\alpha]_3 = (n - 1) \left( \int_S \alpha^1 \cdot C - \alpha^2 \right) C_1 - \left( \int_S \alpha^0 \omega - \alpha^1 \cdot C + \alpha^2 \right) \Sigma_1 - \left( \int_S \alpha^1 \cdot C - \alpha^2 \right) T.
\]
At this point we have all the elements needed to apply Formula (I.6.8). The result is
\[
\theta_{F_2}(\alpha) = -\alpha_1^1 - \left( \int_S (n - 2)\alpha^0 \omega + (n - 2)\alpha^1 \cdot \Sigma + (n^2 - 3n + 3)\alpha^1 \cdot C - (2n - 3)\alpha^2 \right) C_1 - \left( \int_S \alpha^0 \omega + \alpha^1 \cdot \Sigma + (n - 1)\alpha^1 \cdot C - 2\alpha^2 \right) \Sigma_1 + \left( \int_S \alpha^1 \cdot C \right) T.
\]
In particular \( \theta_{F_2} \) is integral. Since \( \langle \alpha, v_2 \rangle = 0 \) is equivalent to \( \alpha^0 = (2\alpha^2 - \alpha^1 \cdot \Sigma - (n - 2)\alpha^1 \cdot C) \), one gets that
\[
\theta_2(\alpha) = -\alpha_1^1 - \left( \int_S (n - 1)\alpha^1 \cdot C - \alpha^2 \right) C_1 - \left( \int_S \alpha^1 \cdot \Sigma + (n - 1)\alpha^1 \cdot C - 2\alpha^2 \right) \Sigma_1 + \left( \int_S \alpha^1 \cdot C \right) T.
\]
A tedious but straightforward computation shows that \( \theta_2 \) is a homomorphism of lattices.

**Rank at least three.** Let \( \Omega \subset H^*(S; \mathbf{Z}) \) be the \( \mathbf{Z} \)-span of \( \{1, C, \Sigma, \omega\} \). Since the restriction of the Mukai form to \( \Omega \) is unimodular, \( H^*(S; \mathbf{Z}) \) is the orthogonal direct sum of \( \Omega \) and of \( \Omega^\perp \). Notice that \( \Omega^\perp \subset v_r^\perp \), and thus
\[
v_r^\perp = (v_r^\perp \cap \Omega^\perp) \oplus_{\perp} \Omega^\perp.
\]
Let \( r \geq 1 \). By Formula (I.6.8) the restrictions of \( \theta_{F_{r+1}} \) and \( \theta_{F_r} \) to \( \Omega^\perp \) are equal, and hence by induction we have
\[
\theta_r |_{\Omega^\perp} = \theta_1 |_{\Omega^\perp}
\]
for all \( r \geq 1 \). In order to examine the restriction of \( \theta_r \) to \( v_r^\perp \cap \Omega \), it is convenient to introduce the class \( \beta_r := v_r - 2(n - 1)C \); notice that \( \beta_r \in v_r^\perp \cap \Omega \). A computation gives
\[
\langle \beta_r, \beta_r \rangle = -2(n - 1).
\]
Let \( \Lambda_r := \{ v_r, \beta_r \}^\perp \cap \Omega \). As is easily checked we have an orthogonal direct sum decomposition
\[
v_r^\perp \cap \Omega = \mathbf{Z}\beta_r \oplus_{\perp} \Lambda_r.
\]
27
A computation gives
\[ \Lambda_r := \{ x + yC + z\omega \mid (r-1)x + y - rz = 0 \}. \]  
(\text{I.6.18})

(\text{I.6.19}) \textbf{Lemma.} Keep notation as above. If \( r \geq 3 \) then
\[ \theta_{\mathcal{F}_r} (x + yC + z\omega) = x(C_1 - (r-2)\Sigma_1) - y\Sigma_1 + z(r\Sigma_1 - C_1), \quad \theta_{\mathcal{F}_r}(\beta_r) = T. \]

We will prove the lemma at the end of the section. First we finish the proof of Proposition (\text{I.6.1}) assuming the lemma. By Equality (\text{I.6.17}) and Lemma (\text{I.6.19}) the restriction of \( \theta_r \) to \( v_r^+ \cap \Omega \) is integral. By (\text{I.6.15}) the restriction of \( \theta_r \) to \( \Omega^\perp \) is also integral. By (\text{I.6.14}) we conclude that \( \theta_r \) is integral. Now let’s prove that \( \theta_r \) is a homomorphism of lattices. Since \( \theta_1 \) is a lattice homomorphism, and since \( \theta_1(\Omega^\perp) \) is the orthogonal complement of \( \{C_1, \Sigma_1, T\} \), it follows that \( \theta_r|_{\Omega^\perp} \) is a lattice homomorphism (use (\text{I.6.15})) and that \( \theta_r(\Omega^\perp) \) is perpendicular to \( \{C_1, \Sigma_1, T\} \). By Lemma (\text{I.6.19}) the image \( \theta_r(v_r^+ \cap \Omega) \) is contained in the \( \mathbb{Z} \)-span of \( \{C_1, \Sigma_1, T\} \), hence it is perpendicular to \( \theta_r(\Omega^\perp) \).

It remains to verify that the restriction of \( \theta_r \) to \( v_r^+ \cap \Omega \) is a lattice homomorphism. If we restrict \( \theta_r \) to \( \mathbb{Z}\beta_r \) we get a lattice homomorphism by Formula (\text{I.6.16}). Lemma (\text{I.6.19}) shows that \( \theta_r(\beta_r) \) is perpendicular to \( \theta_r(\Lambda_r) \). Hence from (\text{I.6.17}) we see that we are reduced to proving that the restriction of \( \theta_r \) to \( \Lambda_r \) is a lattice homomorphism. This consists of a straightforward computation (use (\text{I.6.18})).

\textbf{Proof of Lemma (I.6.19).} It follows from Exact sequences (\text{I.6.6})-(\text{I.6.7}) that for \( r \geq 2 \) one has
\[ \rho_* \left[ c_1(\mathcal{F}^{r+1})\omega \right]_3 = \rho_* \left[ c_1(\mathcal{F}^r)\omega \right]_3 + r c_1(\xi_{r+1}) - \Sigma_1. \]

Equation (\text{I.6.8}) gives the following formulae, for \( r \geq 2 \):
\begin{align*}
\theta_{\mathcal{F}_{r+1}}(1) &= \theta_{\mathcal{F}_r}(e^2C) + (r+1)c_1(\xi_{r+1}) + \Sigma_1, \\
\theta_{\mathcal{F}_{r+1}}(C) &= \theta_{\mathcal{F}_r}(C) + c_1(\xi_{r+1}), \\
\theta_{\mathcal{F}_{r+1}}(\Sigma) &= \theta_{\mathcal{F}_r}(\Sigma + 2\omega) - (r^2 + r + 2 - n)c_1(\xi_{r+1}) - \Sigma_1, \\
\theta_{\mathcal{F}_{r+1}}(\omega) &= \theta_{\mathcal{F}_r}(\omega) - r c_1(\xi_{r+1}) + \Sigma_1. 
\end{align*}
(\text{I.6.20})

Using (\text{I.6.6})-(\text{I.6.7}) and (\text{I.6.13}) one gets
\[ \rho_* \left[ c_1(\mathcal{F}^2)\omega \right]_3 = -(2n - 3)C_1 - 2\Sigma_1 + 2T. \]

Formula (\text{I.6.9}) together with the equality above and the computations for rank two give
\[ c_1(\xi_3) = (n - 1)C_1 - T. \]

Plugging this value in Formulae (\text{I.6.20}), and using the computations for rank two, one obtains Lemma (\text{I.6.19}) for \( r = 3 \). In order to prove Lemma (\text{I.6.19}) for \( r \geq 4 \) we add to the lemma the following statement.

(\text{I.6.21}). Keep notation as above. If \( r \geq 4 \) then
\[ \rho_* \left[ c_1(\mathcal{F}^{r-1})\omega \right]_3 = -r\Sigma_1 + C_1, \\
c_1(\xi_r) = 0. \]

The proof of Lemma (\text{I.6.19}) for \( r \geq 4 \) is by induction. First one verifies that (\text{I.6.21}) and Lemma (\text{I.6.19}) hold for \( r = 4 \) by a computation, and then the inductive step is an easy consequence of Formulae (\text{I.6.20}).

\textbf{II. Proof of the main theorem.}

We will need the following technical result; its proof is deferred to the Section IV.
(II.1) Proposition. Let $S$ be a $K3$ surface, and $v \in H^*(S; \mathbb{Z})$ be a Mukai vector. Suppose that $v^0$ and the order of divisibility of $v^1$ are coprime.

1. If an ample divisor on $S$ is $|v|$-generic then it is also $v$-stabilizing.
2. Let $H$ be a $v$-stabilizing ample divisor on $S$ (for example this is the case if $H$ is $|v|$-generic, by Item (1)). Let $L$ be a $|v|$-generic ample divisor on $S$, and let $C$ be the unique open $|v|$-chamber containing $L$. If $H \in C$ then $H$-slope-stability is the same as $L$-slope-stability, and hence $\mathcal{M}_v(H) = \mathcal{M}_v(L)$.

By the above proposition we can assume, in proving the Main Theorem, that $H$ is $v$-generic. We will prove Theorem (2) by first deforming $S$ to an elliptic surface, and then invoking Theorem (1.0.4). Let $K_{2d}$ be the moduli space of polarized $K3$ surfaces of degree $2d$, i.e. couples $(S, H)$, where $S$ is a $K3$ and $H$ is a primitive ample divisor on $S$ with $H^2 = 2d$. Let $[S, H] \in K_{2d}$, and assume $v \in H^*(S; \mathbb{Z})$ is a Mukai vector such that

$$v^1 = \pm c_1(H). \quad \text{(II.2)}$$

Given any $[S_x, H_x] \in K_{2d}$ the Mukai vector $v$ makes sense in $H^*(S_x; \mathbb{Z})$, because of (II.2), and hence we can consider the moduli space $\mathcal{M}_v(S_x, H_x)$. Let

$$\mathcal{K}_{2d}(v) := \{[S_x, H_x] \in K_{2d} | H_x \text{ is } v\text{-generic}\}.$$

This is an open subset of $K_{2d}$; in fact its complement is the union of a finite set of components of the Noether-Lefschetz locus.

(II.3) Proposition. Keep notation as above, and assume that (II.2) is satisfied. Suppose there exists $[T, L] \in K_{2d}(v)$ such that Theorem (2) holds for the moduli space $\mathcal{M}_v(T, L)$. Given any other $[S, H] \in K_{2d}(v)$ Theorem (2) holds for the moduli space $\mathcal{M}_v(S, H)$.

Proof. Let $\mathcal{H}_{2d}(v)$ be the parameter space (open subset of a Hilbert scheme) for $K3$ surfaces $[S, H] \in K_{2d}$ embedded in projective space by a high multiple of $H$. Let $S_{2d} \to \mathcal{H}_{2d}(v)$ be the tautological family of (embedded) surfaces. By [Ma] there exists a relative moduli space $\pi: \mathcal{M}_v(2d) \to \mathcal{H}_{2d}(v)$: the fiber of $\pi$ over a surface $S_x$ embedded by a multiple of $H_x$ is isomorphic to $\mathcal{M}_v(S_x, H_x)$. By [M1, Th. (1.17)] $\pi$ is a submersion at every point (here we use the fact that on $K_{2d}(v)$ semistability implies stability). Since $\mathcal{H}_{2d}(v)$ is irreducible (by irreducibility of $K_{2d}(v)$) and $\pi$ is proper [Ma], the map $\pi$ is surjective. Hence, by irreducibility of $\mathcal{H}_{2d}(v)$, the moduli space $\mathcal{M}_v(S, H)$ is a deformation of $\mathcal{M}_v(T, L)$, and therefore Item (1) of Theorem (2) holds for $\mathcal{M}_v(S, H)$. To prove Item (2) notice that one can construct a relative quasi-tautological family of sheaves on $S_{2d} \times_{\mathcal{H}_{2d}(v)} \mathcal{M}_v(2d)$ (see the proof of [M2, Th. (A.5)]). Hence the map $\theta_v; v^1 \cap H^*(S_x) \to H^2(\mathcal{M}_v(S_x, H_x))$ is a locally constant function of $x \in \mathcal{H}_{2d}(v)$. Since $\mathcal{H}_{2d}(v)$ is irreducible we conclude that Item (2) of Theorem (2) holds for $\mathcal{M}_v(S, H)$.

First we prove Theorem (2) under particularly favourable hypotheses.

(II.4) Proposition. Let hypotheses be as in the statement of Theorem (2). In addition suppose that $v^1 = c_1(H)$, where $H$ is a (primitive) ample divisor on $S$, and that

$$H^2 \geq 2|v|, \quad H^2 \geq 4. \quad \text{(II.5)}$$

Then Theorem (2) holds for the moduli space $\mathcal{M}_v(S, H)$.

Proof. Let $T$ be an elliptic $K3$ with $\text{Pic}(S) = \mathbb{Z}[\Sigma] \oplus \mathbb{Z}[C]$, where $\Sigma$ is a section of the elliptic fibration, and $C$ is an elliptic fiber. Let $d := H^2/2$, and set $L := (\Sigma + (d + 1)C)$. By (II.5) we have
\(d \geq 2\), hence \(L\) is an ample divisor on \(T\). Since \(L\) is primitive, and since \(L^2 = 2d = H^2\), both \([S, H]\) and \([T, L]\) belong to the same moduli space \(K_{2d}\). By (II.5) together with Lemma (I.0.3) the polarization \(L\) is \(|v|\)-suitable, in particular \(|v|\)-generic. The polarization \(H\) is \(|v|\)-generic by hypotheses, hence \([T, L], [S, H] \in K_{2d}(v)\). Since \(L\) is \(|v|\)-suitable, and since \(v^1 (= c_1(L))\) is a numerical section, Theorem (I.0.4) tells us that the Main Theorem holds for \(\mathcal{M}_v(T, L)\). By Proposition (II.3) we conclude that the Main Theorem holds also for \(\mathcal{M}_v(S, H)\). \(\text{q.e.d.}\)

Next we concentrate on the case when \(\rho(S)\) (the rank of \(\text{Pic}(S)\)) is at least two, with no further hypotheses.

(II.6) Lemma. Let \(S\) be a a projective K3 surface with \(\rho(S) \geq 2\). Let \(v \in H^*(S; Z)\) be a Mukai vector with \(v^1\) primitive, and \(C\) be an open \(v\)-chamber. There exists a Mukai vector \(w\) equivalent to \(v\) such that \(w^1 = c_1(H)\), where \(H\) is a primitive ample divisor belonging to \(C\). We can choose \(w\) so that \(H^2\) is arbitrarily large.

Proof. Let’s show that there exists an ample divisor \(H \in C\) such that

a. \(H\) is primitive, and
b. \(c_1(H) \equiv v^1 (\mod v^0 H^{1,1}_Z)\).

There exists a \(Z\)-basis of \(H^{1,1}_Z(S)\) consisting of elements of \(C\); we identify \(H^{1,1}_Z(S)\) with \(Z^\rho\) via such a basis. There exist positive integers \(m, a_i\) (for \(i = 1, \ldots, \rho\)), with \(gcd\{a_1, \ldots, a_\rho\} = 1\), such that

\[ v^1 \equiv (ma_1, \ldots, ma_\rho) (\mod v^0 H^{1,1}_Z). \]

Since \(v^1\) is primitive, we have \(gcd\{m, v^0\} = 1\). Let \(H\) be the divisor (class) such that

\[ c_1(H) = (ma_1 + v^0 n_1, \ldots, ma_\rho - 1 + v^0 n_{\rho - 1}, ma_\rho), \]

where, for all \(i = 1, \ldots, \rho - 1,\)

- \(n_i > 0,\)
- \(gcd\{m, n_i\} = 1,\) and
- if \(p\) is a prime dividing \(a_\rho\) but not \(m\), then \(p\) divides each \(n_i\).

Since all the coordinates of \(c_1(H)\) are strictly positive, \(H\) belongs to \(C\), in particular it is ample. By construction \(H\) satisfies Items (a) and (b). If the \(n_i\) are arbitrarily large then \(H^2\) is arbitrarily large. Now let \(\xi\) be the line bundle on \(S\) such that

\[ c_1(H) = v^1 + c_1^0(\xi), \quad \text{i.e.} \quad c_1(\xi) = (n_1, \ldots, n_{\rho - 1}, 0). \]

If \(w := ch(\xi)v\) then \(w^1 = c_1(H)\), hence \(w\) satisfies the conclusions of the lemma. \(\text{q.e.d.}\)

(II.7) Proposition. Let hypotheses be as in the statement of Theorem (2). In addition suppose that \(\rho(S) \geq 2\). Then Theorem (2) holds for \(\mathcal{M}_v(H)\).

Proof. Let \(C\) be an open \(|v|\)-chamber such that \(H \in \overline{C}\). By Lemma (II.6) there exists a Mukai vector \(w\) equivalent to \(v\) such that \(w^1 = c_1(L)\), where \(L\) is a primitive ample divisor such that \(L \in C\) and

\[ L^2 \geq 2|v|, \quad L^2 \geq 4. \]

Since \(w\) is equivalent to \(v\) we have \(\mathcal{M}_w(L) = \mathcal{M}_v(L)\), and by Proposition (II.1) \(\mathcal{M}_v(L) = \mathcal{M}_v(H)\). Hence it is equivalent to prove that Theorem (2) holds for \(\mathcal{M}_w(L)\). By the above inequality we can apply Proposition (II.4), hence the Main Theorem does indeed hold for \(\mathcal{M}_w(L)\). \(\text{q.e.d.}\)
We are left with proving that the Main Theorem holds when \( \rho(S) = 1 \). Assume \( S \) is such a surface. Let \( H \) be the ample generator of \( \text{Pic}(S) \), and set \( d := H^2/2 \). Since \( v^1 \) is primitive we have \( v^1 = \pm c_1(H) \), hence we can apply Proposition (II.3). By surjectivity of the period map for polarized \( K3 \) surfaces there exists \( [T, L] \in K_{2d}(v) \) with \( \rho(T) \geq 2 \). By Proposition (II.7) the Main Theorem holds for \( \mathcal{M}_v(T, L) \), and thus Proposition (II.3) shows that Theorem (2) holds for \( \mathcal{M}_v(S, H) \). This finishes the proof of the Main Theorem.

### III. Higher-rank Donaldson polynomials.

Let \((S, H)\) be a polarized \( K3 \) surface, and \( v \in H^*(S; \mathbb{Z}) \) be a Mukai vector; we assume that \( H \) is \( v \)-stabilizing. If the rank (i.e. \( v^0 \)) is two then one can associate to \( \mathcal{M}_v \) a certain Donaldson polynomial function on \( H_2(S) \). The (algebro-geometric) definition extends to any rank as follows. Let \( \mathcal{F} \) be a quasi-tautological family of sheaves on \( S \times \mathcal{M}_v \), and let \( \rho: S \times \mathcal{M}_v \to \mathcal{M}_v \) be projection. We define \( \mu_{\mathcal{F}}: H_2(S) \to H^2(\mathcal{M}_v) \) by setting

\[
\mu_{\mathcal{F}}(\alpha) := \rho_* \left[ \frac{-1}{\sigma(\mathcal{F})} \chi_2(\mathcal{F}) + \frac{1}{2\sigma(\mathcal{F})^2 \cdot \text{rk}(\mathcal{F})} \chi_1^2(\mathcal{F}) \right] \cdot \alpha.
\]

If \( \mathcal{G} \) is another quasi-tautological family of sheaves, then there exist vector bundles \( \xi, \eta \) on \( \mathcal{M}_v \) such that \( \mathcal{F} \otimes \rho_* \xi \cong \mathcal{G} \otimes \rho_* \eta \) \([2, \text{Th. (A.5)}]\); it follows that \( \mu_{\mathcal{F}} = \mu_{\mathcal{G}} \), and hence we can set

\[
\mu_v := \mu_{\mathcal{F}} \quad \text{where} \quad \mathcal{F} \text{ is any quasi-tautological family.}
\]

Now let \( q_v: H_2(S; \mathbb{Q}) \to \mathbb{Q} \) be the polynomial given by

\[
q_v(\alpha) := \int_{\mathcal{M}_v} \mu_v(\alpha)^{d(v)},
\]

where \( d(v) := 2 + \langle v, v \rangle \) is the (complex) dimension of \( \mathcal{M}_v \). If \( v^0 = 2 \) then \( q_v \) equals the corresponding Donaldson polynomial by a theorem of Morgan \([\text{Mo}]\). Donaldson’s polynomials of \( K3 \) surfaces have been computed long ago; we will show that the formula for rank two holds also in higher rank.

**Proposition (III.1)**. Keeping notation as above, assume that \( v^1 \) is primitive. Set \( d(v) = 2n \), and let \( Q: H_2(S; \mathbb{Q}) \to \mathbb{Q} \) be the intersection form. Then

\[
q_v = \frac{2n!}{n!2^n} Q^n.
\]

The above result is a straightforward corollary of our Main Theorem (2) if \( \dim \mathcal{M}_v > 2 \), and of results of Mukai if \( \dim \mathcal{M}_v \leq 2 \), together with a theorem of Fujiki \([\text{Fu}, \text{Th. (4.7)}]\). We will provide a simple proof of Fujiki’s result avoiding hyperkähler structures.

**Theorem (Fujiki [\text{Fu}, \text{Th. (4.7)}])**. Let \( X \) be an irreducible Kähler symplectic compact manifold of (complex) dimension \( 2n \). There exists \( \lambda_X \in \mathbb{Q} \) (call it the Fujiki constant of \( X \)) such that

\[
\int_X \beta^{2n} = \lambda_X \cdot B_X(\beta)^n \tag{III.2}
\]

for all \( \beta \in H^2(X) \), where \( B_X: H^2(X) \to \mathbb{C} \) is the quadratic form associated to \( X \) by Beauville \([\text{B}, \text{Th. (5)}]\). Fujiki’s constant is independent of the birational (symplectic) model of \( X \).

**Proof**. Let \( p_X: H^2(X) \to \mathbb{C} \) be the degree-\( 2n \) polynomial defined by the left-hand side of (III.2). Consider the versal deformation space \( f: \mathcal{X} \to D \) of \( X \), and for \( s \in D \) let \( X_s := f^{-1}(s) \). Let \( u_s: H^2(X_s) \cong H^2(X) \) be the isomorphism given by the Gauss-Manin connection. Then

\[
p_X(u_s^* \beta) = p_X(\beta). \tag{*}
\]
If $\varphi_s \in H^{2,0}(X_s)$ then $p_X(\varphi_s) = 0$ by type consideration, and thus by (*) we have

$$p_X(u_s\varphi_s) = 0.$$ 

By Beauville [B, Th. (5)] the collection

$$\{(u_s\varphi_s) | s \in D, \ 0 \neq \varphi_s \in H^{2,0}(X_s)\}$$

fills out an open (in the analytic topology) subset of the smooth quadric where $B_X$ vanishes. Hence we deduce that $p_X = B_X \cdot p'_X$ for some degree-$2(n-1)$ polynomial $p'_X$. Now consider

$$p_X(u_s\varphi_s, \ldots, u_s\varphi_s, u_s\varphi_s) = n^{-1}B_X(u_s\varphi_s, u_s\varphi_s) \cdot p'_X(u_s\varphi_s).$$

Since the left-hand side vanishes by type consideration (unless $n = 1$, in which case we are done), and since $B_X(u_s\varphi_s, u_s\varphi_s) > 0$ by [B, Th. 5], we conclude that $p'_X(u_s\varphi_s) = 0$ for all $\varphi_s$, and hence $B_X(p'_X)$. Thus $p_X = B_X p''_X$ for some degree-$2(n-2)$ polynomial $p''_X$. Evaluating

$$p_X(u_s\varphi_s, \ldots, u_s\varphi_s, u_s\varphi_s, \ldots, u_s\varphi_s) \quad i \leq (n-1),$$

and arguing similarly we conclude that $p_X = \lambda_X B^n_X$ for some constant $\lambda_X$. That $\lambda_X \in \mathbb{Q}$ follows from integrality of $B_X$ [B] and of $p_X$. Finally let's show that if $f: X \to Y$ is birational then $\lambda_X = \lambda_Y$. Since $f, f^{-1}$ are isomorphisms in codimension one they induce an isomorphism $f^*: H^2(X) \cong H^2(Y)$, and one has $B_X(\beta) = B_Y(f^*\beta)$ for all $\beta \in H^2(X)$. We claim that if $\varphi_X \in H^{2,0}(X)$ then

$$\lambda_X B_X(\varphi_X + \overline{\varphi}_X)^n = \int_X (\varphi_X + \overline{\varphi}_X)^2^n = \int_Y (f^*\varphi_X + f^*\overline{\varphi}_X)^2^n = \lambda_Y B_Y(f^*\varphi_X + f^*\overline{\varphi}_X)^n.$$ 

In fact let $X \supset U \cong V \subset Y$ be the open subsets identified by $f$. By Hartog's Theorem $\varphi_X|U = \varphi_X|V$ extends to a holomorphic two-form on all of $Y$, which necessarily represents $f^*\varphi_X$. Since the integrals appearing above can be computed by integrating over $U$ and $V$ respectively, we conclude that they are equal. Since $B_X(\varphi_X + \overline{\varphi}_X) \neq 0$ when $\varphi_X \neq 0$ we conclude that $\lambda_X = \lambda_Y$. \textbf{q.e.d.}

**Claim.** Let notation and hypotheses be as in Proposition (III.1). The Fujiki constant of $M_v$ is equal to $(2n)!/(n!^2)^n$.

**Proof.** If $\dim M_v = 2$ then by a theorem of Mukai [M2] the moduli space is a $K3$ surface, and hence the claim holds. If $\dim M_v > 2$ then by Theorem (2) the moduli space is deformation equivalent to a symplectic birational model of $T^{[n]}$, call it $X$, where $T$ is another projective $K3$, and $2n = d(v)$. In fact the proof of Theorem (2) shows that $M_v$ can be deformed to $X$ through symplectic projective varieties. The Fujiki constant is clearly invariant under deformations through symplectic Kähler varieties, hence $\lambda_{M_v} = \lambda_X$. By Fujiki's Theorem $\lambda_X = \lambda_{T^{[n]}}$. A simple calculation shows that

$$\lambda_{T^{[n]}} = \frac{(2n)!}{n!2^n}. $$

This finishes the proof of the claim. \textbf{q.e.d.}

**Proof of Proposition (III.1).** Assume first that $\dim M_v > 2$. Let $\iota_v: H^2(S) \to H^*(S)$ be the map defined by

$$\iota_v(\alpha) := \alpha + \left(\frac{1}{v_0} \int_S v^1 \wedge \alpha\right) \omega,$$

32
where \( \omega \in H^4(S; \mathbb{Z}) \) is the fundamental class. The image of \( \iota_v \) is contained in \( v^\perp \), hence it makes sense to compose \( \iota_v \) and \( \theta_v \). A straightforward computation gives

\[
\mu_v = -\theta_v \circ \iota_v.
\]

If \( Q \) is the intersection form, then \( \iota_v^*(\alpha, \alpha) = Q(\alpha) \). Since \( \theta_v \) is an isometry between \( (v^\perp, \langle, \rangle) \) and \( (H^2(M_v), B_{M_v}) \) (by the Main Theorem) we conclude that

\[
B_{M_v}(\mu_v(\alpha)) = Q(\alpha).
\]

The proposition follows at once from this equality together with Fujiki’s Theorem and Claim (III.3). Now assume that \( \dim M_v = 2 \): the same argument as above applies thanks to Mukai’s description of \( (H^2(M_v), B_{M_v}) \) as \( v^\perp/C_v \) [M2]. Finally when \( \dim M_v = 0 \), the proposition is equivalent to the statement that \( M_v \) consists of a single reduced point. We proved in Section I that moduli spaces on an elliptic surface, with \( v^1 \) a numerical section, are non-empty. This implies that the moduli space is a reduced point by [M2, Cor. (3.6)].

**IV. Appendix.**

In this section we will prove Propositions (II.1)-(I.1.6). We let \( S \) be a projective smooth irreducible surface. For a torsion-free sheaf \( F \) on \( S \) we set

\[
\Delta_F := c_2(F) - \frac{r_F - 1}{2r_F} c_1(F)^2,
\]

where \( r_F \) is the rank of \( F \). We will be exclusively interested in slope-(semi)stability, and hence we will systematically omit the prefix ”slope”.

**Lemma.** Let \( H \) be a polarization of \( S \). Let \( F \) be a strictly \( H \)-semistable torsion-free sheaf on \( S \), and let

\[
0 \to A \to F \xrightarrow{\pi} B \to 0
\]

be destabilizing, i.e. \( [r_{FC_1}(A) - r_A c_1(F)] \cdot H = 0 \). Then

\[
-\frac{r_F^3}{2} \Delta_F \leq [r_{FC_1}(A) - r_A c_1(F)]^2 \leq 0.
\]

**IV.1**

Furthermore the right inequality is an equality only if \( [r_{FC_1}(A) - r_A c_1(F)] = 0 \).

**Proof.** Replacing \( A \) by \( \pi^{-1}(\text{Tor}(B)) \), where \( \text{Tor}(B) \) is the torsion subsheaf of \( B \), we can assume \( B \) is torsion-free. First suppose \( c_1(F) = 0 \). Since \( A \) is slope destabilizing \( c_1(A) \cdot H = 0 \), and hence by Hodge index \( c_1(A)^2 \leq 0 \), with equality only if \( c_1(A) = 0 \). We are left with proving the other inequality. Since \( c_1(B) = -c_1(A) \) we have

\[
c_2(F) = c_2(A) + c_2(B) - c_1(A)^2.
\]

Both \( A \) and \( B \) are semistable, and thus by Bogomolov’ inequality

\[
c_2(A) \geq \frac{r_A - 1}{2r_A} c_1(A)^2 \quad c_2(B) \geq \frac{r_B - 1}{2r_B} c_1(A)^2.
\]

Substituting into the previous equality one gets the left inequality in (IV.1). If \( c_1(F) \neq 0 \), we formally set \( F_0 := F \otimes (\det F)^{-1/r_F} \). Then \( c_1(F_0) = 0 \), and applying the previous argument to \( F_0 \) one gets (IV.1).

\( \text{q.e.d.} \)
Proof of Item (1) of Proposition (II.1). If $S$ is a $K3$, then

$$
\frac{r^3}{2} \Delta_F = |v(F)|,
$$

and hence, using the notation of the previous lemma, we have

$$
-|v(F)| \leq [r_{FC1}(A) - r_{AC1}(F)]^2 \leq 0.
$$

By the hypothesis on the divisibility of $c_1(F)$ we see that $[r_{FC1}(A) - r_{AC1}(F)] \neq 0$, and thus $[r_{FC1}(A) - r_{AC1}(F)]^\perp \cap A(S)$ is a $v(F)$-wall. Since $[r_{FC1}(A) - r_{AC1}(F)] \cdot H = 0$, $H$ belongs to this wall.

(IV.2) Remark. In fact we have proved something slightly stronger, namely that if $F$ is an $H$-semistable torsion-free sheaf on $S$ with $v(F) = v$, then $F$ is $H$-stable.

Proof of Item (2) Proposition (II.1). Item (2) follows at once from the following.

(IV.3) Lemma. Let $S$ be a $K3$ surface, and let $H_0$, $H_1$ be ample divisors on $S$. Assume $F$ is a torsion-free sheaf on $S$ such that the order of divisibility of $c_1(F)$ is coprime to $\text{rk}(F)$. If $F$ is $H_0$-slope-stable and $H_1$-slope-unstable, then there exists a $|v(F)|$-wall $\xi^\perp$ such that $\xi \cdot H_0$, $\xi \cdot H_1$ are of opposite signs (and non-zero).

Proof. Since $|v(F^{**})| \leq |v(F)|$ we can replace $F$ by $F^{**}$, and assume that $F$ is locally-free. One can extend in the obvious way the notion of (semi)stability to arbitrary elements of $A(S)$. Letting $\Lambda := R_+c_1(H_0) \oplus R_+c_1(H_1)$, we set

$$
\Lambda^s := \{H \in \Lambda | F \text{ is } H\text{-stable}\}
$$

$$
\Lambda^u := \{H \in \Lambda | F \text{ is } H\text{-unstable}\}.
$$

Let $H \in \Lambda^s$, and let $E$ be an $H$-desemistabilizing subsheaf of $F$; then, by continuity, $E$ is $H'$-desemistabilizing for all $H'$ near $H$, and hence $\Lambda^u$ is an open subset of $\Lambda$. Let’s show that also $\Lambda^s$ is open in $\Lambda$. Set $H_t := (1-t)H_0 + tH_1$ for $t \in [0,1]$. Since $\Lambda^s$ is a (positive) cone it suffices to show that the set of $t$ such that $H_t \in \Lambda^s$ is open. Assume $H_T \in \Lambda^s$. Let $I \subset [0,1]$ be the subset of points $t$ such that

$$
[r_{FC1}(E) - r_{EC1}(F)] \cdot H_t = 0
$$

for some $H_1$-destabilizing subsheaf $E \subset F$ with torsion-free quotient. The set of $H_1$-destabilizing subsheaves of $F$ with torsion-free quotient is a bounded set (Grothendieck), and hence $I$ is a finite set. Define $t_0$ by setting $t_0 := \min I$. Since $\Lambda^s$ is a convex cone, we have $t > 7$ for all $t \in I$, and hence $t_0 > 7$. We claim that $F$ is $H_{t_0}$-stable for all $t < t_0$. In fact let $E \subset F$ be a subsheaf with torsion-free quotient. If $E$ is not $H_{t_0}$-destabilizing then, since $F$ is $H_0$-stable, $E$ is not $H_{t_0}$-destabilizing for all $t < 1$, in particular for $t < t_0$. If $E$ is $H_{t_0}$-desemistabilizing then, since

$$
\varphi_E(t) := [r_{FC1}(E) - r_{EC1}(F)] \cdot H_t
$$

is a linear function of $t$, and since $\varphi_E(1) > 0$, $\varphi_E(0) < 0$, it will be negative for all $t < t_E$, where $t_E \in [0,1]$ is the unique solution of $\varphi_E(t) = 0$. Clearly $t_E$ is a rational number and hence $t_E \in I$. Thus $t_0 \leq t_E$, and we conclude that $E$ is not desemistabilizing for all $t < t_0$. This finishes the proof that $\Lambda^s$ is open. Now we can prove Lemma (IV.3). The nonempty subsets $\Lambda^s, \Lambda^u \subset \Lambda$ are open and disjoint; since $\Lambda$ is connected there must exist $H \in \Lambda$ not belonging to either of these.
subsets, i.e. $F$ is strictly $H$-semistable. Replacing $H$ by an appropriate multiple of itself, we can assume $H$ is an integral class. By Item (1) of Proposition (II.1) we conclude that $H$ belongs to a $|v(F)|$-wall; this wall separates $H_0$ from $H_1$.

**Proof of Proposition (I.1.6).** By definition a $|v|$-suitable polarization is also $|v|$-generic, and hence Item (1) follows from Proposition (II.1). In proving Items (2)-(3) we can assume $F$ is locally-free: in fact if it is singular replace it by $F^{**}$, and observe that stability for $F$ and $F^{**}$ is the same, and that $|v(F^{**})| < |v(F)|$. We will prove Item (2) by contradiction. Assume that the restriction $F|_{C_t}$ to the generic elliptic fiber $C_t$ is not stable. Since $\langle v^1(F), C \rangle$ and $v^0(F)$ are coprime the restriction $F|_{C_t}$ is unstable. Thus there exists a proper subsheaf $A \subset F$ whose restriction to $C_t$ is a desemistabilizing subsheaf, for the generic elliptic fiber $C_t$. Hence $[r_Fc_1(A) - r_Ac_1(F)] \cdot C > 0$. If $N$ is sufficiently large then $[r_Fc_1(A) - r_Ac_1(F)] \cdot (NC + H) > 0$, and hence $F$ is $(NC + H)$-unstable. Since $H$ is $v$-stabilizing, $F$ is $H$-stable, and hence by Lemma (IV.3) the polarizations $H$ and $(NC + H)$ are separated by a $|v|$-wall, contradicting the hypothesis that $H$ is $|v|$-suitable. Thus $F|_{C_t}$ is stable for the generic elliptic fiber $C_t$. Now let’s prove Item (3). First let’s show that $F$ is $(NC + H)$-stable, for $N$ sufficiently large. In fact let $D \in |H|$ be a smooth curve (if no such $D$ exists we replace $H$ by a high multiple of itself), and set

$$d := \max_{0 \neq E \subset F} \{ r_F \deg_D E - r_E \deg_D F \},$$

where $\deg_D E = c_1(E|_D)$, $\deg_D F = c_1(F|_D)$. Let $C_1, \ldots, C_N \in |C|$ be generic elliptic fibers, and $A \subset F$ be a subsheaf with $0 < r_A < r_F$. Then

$$[r_Fc_1(A) - r_Ac_1(F)] \cdot (NC + H) = r_F \deg_D A - r_A \deg_D F + \sum_{i=1}^N r_F \deg_{C_i} A - r_A \deg_{C_i} F.$$

By stability of $F|_{C_t}$, the right-hand side is bounded above by $(d - N)$, which is negative because $N \gg 0$. Since $A \subset F$ is arbitrary we conclude that $F$ is $(NC + H)$-stable. Proceeding as in the proof of Item (2) we conclude that $F$ is also $H$-stable. This finishes the proof of Proposition (I.1.6).

**References.**

[A] M. F. Atiyah. *Vector bundles over an elliptic curve*, Proc. London Math. Soc. 7 (1957), 414-452.

[B] A. Beauville. *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Differential Geom. 18 (1983), 755-782.

[Fu] A. Fujiki. *On the de Rham cohomology group of compact Kähler symplectic manifolds*, Algebraic Geometry Sendai 1985 (1987), Adv. Studies in Pure Math. no. 10, Kinokuniya Tokio and North-Holland Amsterdam, 105-165.

[F1] R. Friedman. *Rank two vector bundles over regular elliptic surfaces*, Invent. math. 96 (1989), 283-332.

[F2] R. Friedman. *Vector bundles and SO(3)-invariants for elliptic surfaces III: the case of odd fiber degree*, preprint (1994).

[Li1] J. Li. *The first two Betti numbers of the moduli spaces of vector bundles on surfaces*, preprint (1995).

[Li2] J. Li. *Picard groups of the moduli spaces of vector bundles over algebraic surfaces*, preprint (1995).

[GoHu] L. Göttsche, D. Huybrechts. *Hodge numbers of moduli spaces of stable bundles on K3 surfaces*, preprint, MPI/94-80.
[GrHa] P. Griffiths, J. Harris. *Principles of algebraic geometry*, John Wiles & sons, 1978.

[Ma] M. Maruyama. *Moduli of stable sheaves II*, J. Math. Kyoto Univ. 18 (1978), 557-614.

[Mo] J. Morgan. *Comparison of the Donaldson polynomial invariants with their algebro-geometric analogues*, Topology 32 (1993), 449-489.

[M1] S. Mukai. *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. math. 77 (1984), 101-116.

[M2] S. Mukai. *On the moduli space of bundles on K3 surfaces I*, M. F. Atiyah and others, Vector bundles on algebraic varieties, Bombay colloquium 1984, Tata Institute for fundamental research studies in mathematics no. 11 (1987), 341-413.

[M3] S. Mukai. *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*, Sugaku Expositions, vol. 1, n. 2 (1988), 139-174.

[MR] V. B. Mehta and A. Ramanathan. *Restriction of stable sheaves and representations of the fundamental group*, Invent. Math. 77 (1984), 163-172.

[O] K. O’Grady. *Relations among Donaldson polynomials of certain algebraic surfaces*, to appear on Forum Mathematicum.

Università di Salerno, Facoltà di Scienze, Baronissi (Sa) - Italia

TEGRADY@MAT.UNITROMA1.IT