Strong versus Weak Coupling Confinement 
in $\mathcal{N} = 2$ Supersymmetric QCD

A. Marshakov$^{a,b}$ and A. Yung$^{c,d}$

$^a$Theory Department, P. N. Lebedev Physics Institute and Institute for Theoretical and Experimental Physics, Moscow, Russia
$^b$Yukawa Institute for Theoretical Physics, Kyoto, Japan
$^c$Petersburg Nuclear Physics Institute, Gatchina, Russia
$^d$William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, USA

Abstract

We consider $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $SU(N_c) = SU(N+1)$ and $N_f$ number of quark matter multiplets, being perturbed by a small mass term for the adjoint matter, so that its Coulomb branch shrinks to a number of isolated vacua. We discuss the vacuum where $r = N$ quarks develop VEV’s for $N_f \geq 2N - 2N_c - 2$ (in particular, we focus on the $N_f = 2N$ and $N_f = 2N+1$ cases).

In the equal quark mass limit at large masses this vacuum stays at weak coupling, the low-energy theory has $U(N)$ gauge symmetry and one observes the non-Abelian confinement of monopoles. As we reduce the average quark mass and enter the strong coupling regime the quark condensate transforms into the condensate of dyons. We show that the low energy description in the strongly-coupled domain for the original theory is given by $U(N)$ dual gauge theory of $N_f \geq 2N$ light non-Abelian dyons, where the condensed dyons still cause the confinement of monopoles, and not of the quarks, as can be thought by naive duality.
1 Introduction

In this paper we consider $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $SU(N_c) = SU(N + 1)$ and $N_f$ fundamental quark matter multiplets. This $\mathcal{N} = 2$ theory is perturbed by a small mass term $\mu \text{Tr}\Phi^2$ for adjoint matter, so that the Coulomb branch shrinks to a number of isolated vacua, whose study was initiated in [1]. We continue the program of studying the features of confinement in $\mathcal{N} = 2$ supersymmetric QCD in the regime, when one can trust the semiclassical string solutions. Mostly interesting part of this study concerns the case, when on the intermediate scales $\sqrt{\mu m}$ (for $\mu \ll m$, the average quark mass) the non-Abelian $SU(N)$ symmetry is restored and one may think of a non-Abelian confinement. In [1] we have found the confinement of monopoles with the non-Abelian spectrum at weak coupling, and a natural next question is what happens to this picture, if one manages to move it into the strong coupling regime of the original theory. For that purpose we need to use the details of the Seiberg-Witten exact solution [2, 3].

In order to achieve this aim, we start with the vacuum, where maximum number of quarks condense in general position. If $N_f \geq 2N$ (in particular, we focus on the cases $N_f = 2N = 2N_c - 2$ and $N_f = 2N + 1 = 2N_c - 1$) this vacuum (where $N$ quarks have nonvanishing condensates) stays at weak coupling, and (in the equal quark mass limit) at low energies the theory in the vicinity of such vacuum has a non-Abelian description in terms of the non-asymptotically free $SU(N) \times U(1) \simeq U(N)$ gauge theory [4, 5, 1] with $N_f$ light quark flavors. Condensation of quarks ensures the monopole confinement: such theory supports the non-Abelian strings [6, 7, 8, 9] (see also reviews [12, 10, 11, 13]), which confine monopoles. We address the following question: what happen to the confinement when we change parameters of the theory and go to the strong coupling regime.

We shall see, that as we reduce the average quark mass and enter the strong coupling domain in the "mass-plane", the original quarks change their quantum numbers and transform into dyons. We show that the low energy effective theory is given by the $U(N)$ dual gauge theory with $N_f \geq 2N$ light non-Abelian dyons. The dual theory at strong coupling also supports the non-Abelian strings, much as the original one. However, we find that these strings in the dyonic condensate still confine monopoles! Thus, in contrast to naive expectations, one still get a confinement of monopoles (not quarks!) at the dual strong-coupling regime of $\mathcal{N} = 2$ supersymmetric QCD.

This picture can be compared with studied recently in [14, 15] for $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $U(N)$ and $N \leq N_f < 2N$ flavors. The parameter interpolating between weak and strong coupling regimes is the coefficient of the Fayet-Iliopoulos (FI) term $\xi$, which plays the same role as $\sqrt{\mu m}$ in our setup. At large values of $\xi$ the theory with $N \leq N_f$ is in weak-coupled phase, while at small $\xi$ this theory goes into the strong coupling regime. It has been shown in [14, 15], that the theory upon reducing $\xi$ exhibits a crossover transition, and below crossover it is described in terms of dual $U(N_f - N)$ theory of $N_f$ dyons, similar to the Seiberg duality [16, 17] in $\mathcal{N} = 1$ supersymmetric QCD, where emergence of the dual gauge group $U(N_f - N)$ was first recognized, see also [4].

Although the transition in the full theory is smooth the low energy effective descriptions in two regimes differ drastically: the gauge groups and spectra of light states are different, and the
perturbative and non-perturbative states (mesons formed by monopole-dyon pairs connected by confining strings) interchange upon passing from one regime to another. Moreover, it is shown in [14, 15] that in the strong coupling region at small $\xi$, the states confined by non-Abelian strings are still monopoles, which is very close to the conclusions of present paper. However, below we shall see, that the low energy effective theories for $N_f \geq 2N = 2N_c - 2$ are essentially different from the $N_f < 2N$ case. We show that in our case with large number of flavors, the low energy description is still given by the $U(N)$ dual gauge theory with $N_f \geq 2N$ light non-Abelian dyons, and there is no crossover, in contrast to the case $N_f < 2N$, when the dual gauge group becomes $U(N_f - N)$.

The paper is organized as follows. In sect. 2 we present our theory and in sect. 3 study, what happens with the quark $r = 2$ vacuum in the weak coupling regime - at large $m$ - semiclassically. Mostly in this paper we consider the simplest but mostly illustrative case of $N_c = 3$ or $N = 2$, so that the gauge group in ultraviolet is $SU(3)$. In sect. 4 we use the Seiberg-Witten curves and differentials to study our theory at strong coupling at small $m$. In sect. 5 we present the action of the dual theory, describing effectively the low energy limit of our theory at small $m$ and, finally, in sect. 6 we calculate the fluxes of strings leading confinement of monopoles in dual theory. Sect. 7 contains our conclusions, and some details are collected in Appendices.

2 Classical theory

2.1 Supersymmetric QCD

The particular theory we are going to consider in this paper is $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $SU(N_c) = SU(3)$ and $N_f = 4, 5$ flavors of fundamental matter hypermultiplets (quarks). Generically we choose different values for the quark multiplets masses, $m_A \neq m_B$, $A, B = 1, \ldots, N_f$. However, our final goal is to consider the case when at least some of quark masses coincide $\Delta m_{AB} \equiv m_A - m_B \to 0$. In this limit the global symmetry $SU(N)_{C + F} = SU(2)_{C + F}$ emerges, which is responsible in particular for presence of the nonabelian strings and the nonabelian features of confinement.

The bosonic part of the action of our model reads

$$S = \int d^4x \left[ \frac{1}{4g^2} \text{Tr}(F_{\mu\nu})^2 + \frac{1}{g^2} \text{Tr}(D_\mu \Phi D_\mu \Phi^\dagger) + \right.$$

$$+ |\nabla_\mu Q_A|^2 + \left|\nabla_\mu \tilde{Q}^A\right|^2 + V(Q, \tilde{Q}, \Phi) \left. \right]$$

(2.1)

Here $D_\mu$ is the covariant derivative in the adjoint representation of $SU(N_c)$, while

$$\nabla_\mu = \partial_\mu - iA_\mu = \partial_\mu - iA_\mu^a T^a$$

(2.2)

is covariant derivative in the fundamental representation. Normally we suppress the color matrix indices, they can be restored, for example, in the $SU(3)$ case using the Gell-Mann matrices $T^a = \frac{1}{2} \delta^{ab} T^a$ and normalized as $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. The adjoint scalar $\Phi = \sqrt{2} a^a T^a$ is a superpartner of the
gauge field in the vector supermultiplet. The scalar components of quark hypermultiplets $Q^A_k$ and $\tilde{Q}^A_k$ are $N_c \times N_f = 3 \times N_f$ matrices with color ($k = 1, \ldots, N_c$) and flavor ($A = 1, \ldots, N_f$) indices.

The potential $V(Q, \tilde{Q}, \Phi)$ in the Lagrangian (2.1) is a sum of the FI $D$- (the first line) and $F$- (the second line) terms,

$$V(Q, \tilde{Q}, \Phi) = g^2 \left( \bar{Q}^A Q_A - \bar{\tilde{Q}}^A \tilde{Q}_A + \frac{1}{g^2} f^{abc} a^b a^c \right)^2 + g^2 \left| \sqrt{2} \tilde{Q}^A Q_A + \mu a^A \right|^2 + \sum_{A=1}^{N_f} \left( \left| (\Phi + m_A) Q^A \right|^2 + \left| \tilde{Q}_A (\Phi + m_A) \right|^2 \right)$$

(2.3)

where the sum over repeated flavor indices $A$ is implied.

The potential (2.3) implies, that the original $SU(3)$ theory is perturbed by adding a small mass term for the adjoint matter, via the superpotential $W = \mu \text{Tr} \Phi^2$. Generally speaking, this superpotential breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$, and the Coulomb branch shrinks to a number of isolated $\mathcal{N} = 1$ vacua $[4, 5]$. In the limit of $\mu \to 0$ these vacua correspond to special singular points on the Coulomb branch, when a pair of monopoles/dyons or quarks become massless. $N_c = 3$ of them (often referred to as the Seiberg-Witten vacua) are always at strong coupling; these vacua also exist in $\mathcal{N} = 1$ pure $SU(N_c)$ gauge theory.

There are also vacua of a different type, to be referred to as quark vacua, which may or may not be at weak coupling, dependently on the values of the quark masses $m_A$, for $m_A \gg \Lambda_{SU(3)}$ quark vacua with $\langle Q^A \rangle \neq 0$ are in the weak coupling regime. These vacua are characterized by an integer $r$, counting the number of condensed flavors, and the number of gauge non-equivalent ones equals to $(N_c - r) C_r^{N_f} [4, 5, 1]$. For the $SU(3)$ gauge theory one has therefore $2N_f$ vacua with $r = 1$ and $N_f(N_f - 1)/2$ with $r = 2$.

Below we concentrate mostly on physics in $r = 2$ vacua. In particular, for these vacua we have the phenomenon of restoration of global $SU(2)_{C+F}$ symmetry if two masses of the condensed quarks coincides, when the $\mathbb{Z}_2$-strings develop orientational zero modes and therefore are called non-Abelian $[6, 7, 8, 9]$.

### 2.2 Quark vacua

Consider $r = 2$ vacuum with nonvanishing VEV’s $\langle Q^A \rangle \neq 0$ of, say, $A = 1, 2$ flavors (12-vacuum). At large non-degenerate values of masses $m_A$, $A = 1, 2$, this vacuum is in weak coupling and semiclassical analysis is applicable. The adjoint scalar develop the following VEV’s (see [1] for more details):

$$\Phi = - \begin{pmatrix} m_1 \\ m_2 \\ -m_1 - m_2 \end{pmatrix}$$

(2.4)
Figure 1: Roots and fundamental weights for the $SU(3)$ gauge group in its Cartan plane. The roots are canonically normalized, as $\alpha^2 = 2$. The simple roots are $\alpha_{12} = \alpha_1 - \alpha_2$ and $\alpha_2$, while $\alpha_1 = \alpha_{12} + \alpha_2$ is the “highest” root. The notations are chosen for the roots to be orthogonal $\alpha_i \cdot \mu_j = \delta_{ij}$, $i, j = 1, 2$ to the weights $\mu_1$ and $\mu_2$ of the fundamental representations $3$ (the weights of the dual fundamental representation $\bar{3}$ are depicted with dashed lines).

The same can be rewritten in $T^a$-components as

$$\sqrt{2} \langle a^3 \rangle = \langle \alpha_{12} \cdot a \rangle = -\Delta m, \quad \langle a^8 \rangle = -\sqrt{\frac{3}{2}} \langle \mu_{12} \cdot a \rangle = -\sqrt{6}m$$

(2.5)

where we have introduced

$$m = \frac{1}{2}(m_1 + m_2), \quad \Delta m = m_1 - m_2$$

(2.6)

By gauge rotations the VEV’s of two quark flavors can be chosen as

$$Q^k_A = \delta^k_A \sqrt{\mu M_A}, \quad A = 1, 2, \quad k = 1, 2$$

$$M_1 = 2m_1 + m_2, \quad M_2 = m_1 + 2m_2$$

(2.7)

In the limit $\mu \to 0$ we move towards Coulomb branch where two quarks become massless. The conditions of their masslessness follow from (2.3), (2.5). They read

$$a_A + m_A = 0, \quad A = 1, 2$$

(2.8)

where

$$a_A = \mu_A \cdot a = \begin{cases} \frac{a^8}{\sqrt{6}} + \frac{a^3}{\sqrt{2}} & A = 1 \\ \frac{a^8}{\sqrt{6}} - \frac{a^3}{\sqrt{2}} & A = 2 \end{cases}$$

(2.9)

In general the mass of a BPS state is given by Seiberg-Witten formula [2, 3]

$$m(q_e, q_m) = |q_e \cdot a + q_m \cdot a_D + B_A m_A|$$

(2.10)
where $a$ and $a_D$ are two vectors in the Cartan plane of $SU(3)$ gauge group with Cartesian components $(a^3, a^8)$, $(a_D^3, a_D^8)$ respectively, while electric and magnetic charges of a BPS state are given by the vectors $q_e$ and $q_m$ from the (dual) Cartan plane, see fig. 1. The flavor charge $B = \{B_A\}$ is a $N_f$-vector determined by the transformation properties of a state with respect to the global flavor group $U(N_f)$ broken down to $U(1)^{N_f}$ in the case of generic masses $m_A$ (for example, the flavor charge of the first quark is $B_1 = (1,0,\ldots)$, the second quark is $B_2 = (0,1,0,\ldots)$, etc). From (2.7) it follows that massless quarks in the 12-vacuum have electric charges given by two weight vectors $\mu \equiv \sqrt{2} (a^3)$, which does not break $N_f = 2$ supersymmetry [18, 19]. Of course, keeping higher order terms in $\mu\text{Tr}\Phi^2$ would inevitably explicitly break $N_f = 2$ SUSY.

From (2.10), (2.5) one also gets

$$m_{\mu_{1}}^{A=2} = |\mu_1 \cdot a + m_2| = |\Delta m| = |\mu_2 \cdot a + m_1| = m_{\mu_{2}}^{A=1}$$ (2.11)

for the W-boson of the $SU(2)$, generated by the root vector $\alpha_{12} = \alpha_1 - \alpha_2$.

In the special case, when $\Delta m_{12} = m_1 - m_2 = 0$, the masses (2.11) and (2.12) vanish, and this is a sign of (partial) restoration of the non-Abelian symmetry. Indeed, the $SU(3)$ gauge group is broken now to $U(2) \simeq SU(2) \times U(1)$ at high scale $m = m_1 = m_2$. In the effective low-energy $U(2)$ theory at small adjoint mass $\mu$ there is a crucial simplification. Since the chiral superfield $A = a + \sqrt{2} \Delta \theta + F_\theta \theta^2$, the $N = 2$ superpartner of the $U(1)$ gauge field (embedded in the $T^8$-direction into the $SU(3)$ gauge group), it not charged under the gauge group, the superpotential $W = \mu \text{Tr} \Phi^2$ can be truncated into the linear superpotential $W_A \sim \xi A$, with

$$\xi = 6\mu m$$ (2.13)

which does not break $N_f = 2$ supersymmetry [18, 19]. Of course, keeping higher order terms in $\mu\text{Tr} \Phi^2$ would inevitably explicitly break $N_f = 2$ SUSY.

The bosonic part of the low energy effective action of the $U(2)$ theory reads

$$S = \int d^4x \left[ \frac{1}{4g_2^2} (F_{\mu
u}^a)^2 + \frac{1}{4g_1^2} (F_{\mu
u}^8)^2 + \frac{1}{g_2^2} |D_{\mu} a^a|^2 + \frac{1}{g_1^2} |\partial_{\mu} a^8|^2 
+ |\nabla_{\mu} Q_A|^2 + \left| \nabla_{\mu} \bar{Q}^A \right|^2 + V(Q_A, \bar{Q}^A, a^a, a^8) \right]$$ (2.14)

Here $\alpha = 1, 2, 3$, and $D_{\mu}$ is the covariant derivative in the adjoint representation of $SU(2)$, while

$$\nabla_\mu = \partial_\mu - \frac{i}{2\sqrt{3}} A_\mu - i A_\mu^a \frac{\tau^a}{2}$$ (2.15)

with the Pauli $\tau^a$-matrices, normalized as $\text{Tr}(\tau^a \tau^b) = 2\delta_{ab}$. The coupling constants $g_1$ and $g_2$ correspond to the $U(1)$ and $SU(2)$ sectors respectively, these couplings are equal at the scale $m$ of breaking the $SU(3)$ symmetry down to $SU(2) \times U(1)$, but generally split below this scale due to nontrivial renormalization group flow in the $SU(2)$ sector (except for the conformal case with $N_f = 4$). The $U(1)$ charges $\pm 1/2\sqrt{3}$ of the fundamental matter fields are fixed by
normalization of the $T^8$-generator in the original $SU(3)$ gauge. Note that in the above action color indices of quark fields $Q^k_A$ run over the restricted $SU(2)$ subset $k = 1, 2$.

The scalar potential $V(Q_A, \tilde{Q}^A, a^\alpha, a)$ in the action (2.14) has the form,

$$
V(Q_A, \tilde{Q}^A, a^\alpha, a) = \frac{g_2^2}{2} \left( \tilde{Q}^A \frac{\tau^\alpha}{2} Q_A - Q^A \frac{\tau^\alpha}{2} \tilde{Q}_A + \frac{1}{g_2^2} \varepsilon^{\alpha\beta\gamma} \bar{a}^\beta a^\gamma \right)^2 + \\
+ \frac{g_4^2}{24} \left( \tilde{Q}^A Q_A - \tilde{Q}_A Q^A \right)^2 + 2g_2^2 \left| \tilde{Q}^A \frac{\tau^\alpha}{2} Q_A \right|^2 + \frac{g_2^2}{6} \left| Q^A Q_A - \xi \right|^2 + \\
+ \sum_{A=1}^{N_f} \left( \left( \frac{a_4^8}{\sqrt{6}} + \frac{\tau^\alpha a^\alpha}{\sqrt{2}} + m_A \right) Q_A \right)^2 + \left( \left( \frac{a_4^8}{\sqrt{6}} + \frac{\tau^\alpha a^\alpha}{\sqrt{2}} + m_A \right) \tilde{Q}_A \right)^2
$$

(2.16)

The adjoint fields in 12-vacuum now develop the VEV’s

$$
\langle a^3 \rangle = 0, \quad \langle a^8 \rangle = -\sqrt{6} m
$$

(2.17)

being just particular case of (2.5) at $m = m_1 = m_2$. The quark’s condensates in this case acquire the color-flavor locked form

$$
\langle Q^k_A \rangle = \langle \tilde{Q}^k_A \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \end{pmatrix}_{N_f}
$$

(2.18)

where we restrict the color and flavor indices to $k = 1, 2$ and $A = 1, \ldots, N_f$ and present the quark fields as $2 \times N_f$ matrices in the tensor product of color and flavor spaces.

These $r = 2$ vacua exist in a theory with two flavors already, other flavors do not play any role in the classical theory. Note however, that if the number of flavors is not large enough (for example, $N_f = 2$) the non-Abelian nature of low energy theory on the Coulomb branch is ruined at $\Delta m = 0$ by the strong coupling effects, and the low energy theory becomes Abelian (for example, it is well known that $W$-bosons decay due to the presence of curves of marginal stability and the gauge sector contains only two photons). Hence, we need to consider a theory with $N_f = 4$ or $N_f = 5$ in order to ensure that $SU(2) \times U(1)$ low energy theory is conformal or infrared free with $\beta_{SU(2)} \leq 0$, and does not run into strong coupling\(^1\). In this case classically unbroken on the Coulomb branch (at $\mu = 0$) gauge symmetry $SU(2) \times U(1)$ remains unbroken on the quantum level [4].

The color-flavor locked form of the quark VEV’s in (2.18) and the absence of VEV of the adjoint scalar $a^\alpha$ condensate (2.17) result in the fact that, while both gauge and flavor $SU(2)$ subgroups are broken by the quark condensation, the diagonal $SU(2)_{C+F}$ survives as a global symmetry. It is seen explicitly, since all states in the low-energy theory combine into multiplets of $SU(2)_{C+F}$. In particular, on Coulomb branch at $\mu = 0$ the “wrong flavor” quarks (2.11) become massless at $\Delta m = 0$ and are unified with the always massless in $r = 2$ vacuum “right flavor” quarks (2.8) to form $2 \otimes 2 = 3 + 1$ of $SU(2)_{C+F}$, while two massless photons $A^8_\mu$ and $A^3_\mu$ are combined with $W^\pm$-bosons (with vanishing masses (2.12) at $\Delta m \to 0$) to form a singlet ($A^8_\mu$)

\(^1\)An alternative way is to consider the theory away from the Coulomb branch at $\xi \gg \Lambda_{\cal N} = 2$, see e.g. [8, 12].
and a triplet \((A^a_i)\) of \(SU(2)_{C+F}\), i.e. the spectrum of the theory becomes really non-Abelian in this limit. Away from the Coulomb branch (at non-zero \(\mu\)) the presence of this symmetry leads to emergence of the orientational zero modes of the \(\mathbb{Z}_2\)-strings in the model, and the strings become non-Abelian \([6, 7, 9, 8]\), see also the reviews \([10, 11, 12, 13]\). In what follows we consider the theory with large number of flavors \((N_f = 4, 5)\), and study whether the theory at \(r = 2\) vacuum preserves its non-Abelian nature at \(\Delta m = 0\) as we reduce \(m\) and go into strong coupling regime through the monodromies which turn quarks into dyons.

In the case of coinciding masses the Higgs branches of the moduli space can be also developed. The dimension of these Higgs branches can be easily counted, see e.g. \([4, 1]\). For example, from the root \((2.17)\) on the Coulomb branch one gets the Higgs branch of dimension

\[
\dim \mathcal{H} = 4N(N_f - N)
\]

\[\bigg|_{N=2} = 8(N_f - 2)
\]

which simply counts the difference between (real) number of scalar quarks and the number of F-term, D-term and gauge constraints. The global unbroken group in this case is \(SU(2)_{C+F} \times SU(N_f - 2) \times U(1)\).

In the case of large number of quarks \(N_f \geq N_c\) the baryonic branch can also appear, where the VEV’s of the baryonic operators

\[
B \sim \epsilon^{A_1 \ldots A_{N_c}} \epsilon^{k_1 \ldots k_{N_c}} Q^{k_1}_{A_1} \ldots Q^{k_{N_c}}_{A_{N_c}}\]

and

\[
\tilde{B} \sim \epsilon^{A_1 \ldots A_{N_c}} \tilde{Q}^{k_1}_{A_1} \ldots \tilde{Q}^{k_{N_c}}_{A_{N_c}}
\]

do not vanish, in addition to the meson fields \(M^B_A \sim \tilde{Q}^B_k Q^k_A\). It happens, for example in the \(SU(3)\) case (see \((2.4)\)), if the flavor masses satisfy the constraint

\[
m_1 + m_2 + m_A = 0, \quad A \neq 1, 2
\]

and the third components \(Q^3_A\) and \(\tilde{Q}^A_3\) condense \([4, 1]\). This phenomenon can happen, say, in the theory with \(N_f = 5\) flavors, with the pairwise coincident masses \(m_1 = m_3\) and \(m_2 = m_4\) but different \(m_5\). However, we shall see in what follows, that the baryonic branch constraint \((2.21)\) does not intersect with the strong-coupled domain in the mass plane and, therefore, does not influence our conclusions about the effective theory at strong coupling.

### 2.3 Non-Abelian semilocal strings

Now we will briefly review some aspects of the non-Abelian strings in our theory \((2.1)\). The non-Abelian strings in \(\mathcal{N} = 2\) QCD with \(N_f = N\) arise, since the Abelian \(\mathbb{Z}_N\)-vortex solutions break the \(SU(N)_{C+F}\) global group, and therefore acquire the orientational zero modes, associated with rotations of their color flux inside the non-Abelian group \(SU(N)\). The global group on the \(\mathbb{Z}_N\)-string solution is broken down to \(SU(N-1) \times U(1)\), and as a result, the moduli space of the non-Abelian string is described by the coset space

\[
\frac{SU(N)}{SU(N-1) \times U(1)} \sim \mathbb{C}P^{N-1}
\]

and the low-energy effective world sheet theory contains the \(\mathcal{N} = 2\) SUSY two dimensional \(\mathbb{C}P^{N-1}\) sigma-model \([6, 7, 8, 9]\), see also \([20]\) about the non-supersymmetric case.
If one adds the “extra” quark flavors with degenerate masses, the strings emerging in the theory with \( N_f > N \) become semilocal, (see [21] for a comprehensive survey of the Abelian semilocal strings). It means, that the transverse size of such string is no longer fixed, but becomes an additional modulus of a string solution. In particular, the string solutions on the Higgs branches (typical in the multiflavor theories) are commonly not fixed-radius, but rather semilocal strings.

The non-Abelian semilocal strings in \( \mathcal{N} = 2 \) QCD with \( N_f > N \) (studied in [6, 9, 22, 23]) have both orientational and size moduli. The effective two-dimensional theory which describes the internal dynamics of the non-Abelian semilocal string is \( \mathcal{N} = (2, 2) \) “toric” sigma model, which includes fields associated with both orientational and size zero modes of the string. Its bosonic action in the gauge formulation (which assumes taking the limit \( e^2 \to \infty \)) has the form

\[
S = \int d^2 x \left( |\nabla_i n^P|^2 + |\tilde{\nabla}_j \rho^K|^2 + \frac{1}{4 e^2} (F_{ij})^2 + \frac{1}{e^2} |\partial_i \sigma|^2 + \frac{e^2}{2} (|n^P|^2 - |\rho^K|^2 - 2\beta)^2 + \right. \\
\left. + \sum_{P=1}^{N_f} \left( \sqrt{2}\sigma + m_P \right)^2 |n^P|^2 + \sum_{K=N+1}^{N_f} \left( \sqrt{2}\sigma + m_K \right)^2 |\rho^K|^2 \right) \tag{2.23}
\]

The world-sheet fields \( n^P, P = 1, ..., N, \) and \( \rho^K, K = N + 1, ..., N_f \) correspond to the orientational and size moduli correspondingly, and have charges +1 and −1 with respect to the auxiliary two-dimensional \( U(1) \) gauge group, so that the covariant derivatives, are \( \nabla_i = \partial_i - iA_i \) and \( \tilde{\nabla}_j = \partial_j + iA_j \) respectively, where \( i, j = 1, 2 \). Small mass differences \( |m_A - m_B| \) lift orientational and size zero modes generating a shallow potential on the moduli space. The \( D \)-term condition

\[
|n^P|^2 - |\rho^K|^2 = 2\beta, \tag{2.24}
\]

is implemented in the limit \( e^2 \to \infty \). Moreover, in this limit the two-dimensional gauge field \( A_i \) and its \( \mathcal{N} = 2 \) bosonic superpartner \( \sigma \) become auxiliary and can be integrated out. The two-dimensional FI D-term \( \beta \) is related to the four-dimensional coupling as

\[
\beta = \frac{2\pi}{g_2^2} \tag{2.25}
\]

We should mention here, that already the semilocality of strings destroys the confinement of monopoles [24, 22]. The reason is that the transverse size of the string can grow indefinitely. When it becomes comparable with the distance between the sources of magnetic flux (monopoles), the linear confining potential between these sources is replaced by the Coulomb-like potential, which does not confine. In order to preserve the confinement of monopoles by strings in our theory we should lift the size zero modes keeping mass differences \( |m_P - m_K| \), \( P = 1, ..., N, K = N + 1, ..., N_f \) small, but nonvanishing.

The detailed discussion of the non-Abelian strings can be found in [10, 11, 12, 13]. Here we would like just to point out, that they arise in effective low-energy theories with actions like (2.14) basically independent of the nature of the fields. We shall use this fact below, when discussing the effective gauge theory of light dyons in the strong coupled domain for the original theory (2.14). Much in the same way as for (2.14), the similar action of the effective theory has non-Abelian string solutions, and their fluxes are determined by asymptotic of the gauge fields at spatial infinity. The main question to be addressed below is what are exactly the states, being confined by non-Abelian semilocal strings in the dual low-energy theory.
3 Semiclassical analysis of the mass plane

Our main interest in this paper is to study the basic features of the non-Abelian confinement in Seiberg-Witten theory. It has been proposed long ago [1], that one way to do this is to start with the $SU(N_c)$, $N_c \geq 3$ (at least $SU(3)$) gauge theory in the ultraviolet, in order to be able to restore partially the non-Abelian gauge group at the intermediate mass scale $\sqrt{\mu m} \ll m$ in a weakly-coupled effective theory around some $\mathcal{N} = 1$ vacuum. In order to get non-Abelian confinement one should require sufficiently large number of flavors: it is restricted from the top by common $N_f \leq 2N_c$, or $\beta_{UV} = 2N_c - N_f > 0$ to have well-defined asymptotically free theory in UV, but the effective theory should have $\beta_{eff} = \beta_{SU(2)} \leq 0$ in order to stay at weak coupling [4] and to be able to trust semiclassical string solutions, which ensure confinement. In particular, for the $SU(3)$ supersymmetric QCD, broken down to the only possible non-Abelian group $SU(2)$ in this case, one can consider theory with $N_f = 4$ or $N_f = 5$ flavors. For further simplicity, we shall take pairwise coinciding masses of flavor multiplets, say $m_1 = m_3$ and $m_2 = m_4$, and therefore our “phase diagram” would depend on only two parameters. The extra flavor mass $m_5$ (in the $N_f = 5$ theory; in $N_f = 4$ theory it is absent) almost does not enter the game in semiclassical regime: it is large and basically only renormalizes the scale $\Lambda^2 \equiv \Lambda^2_{N_f=4} \rightarrow m_5 \Lambda |_{N_f=5}$. The difference of the exact $N_f = 5$ picture from that of $N_f = 4$ we discuss below in sect. 4.4.

Hence, one can start with large $m_{1,2} \gg \Lambda$ and stay at quark $r = 2$ vacuum, where the matrix of complex scalar from $\mathcal{N} = 2$ vector multiplet is taken in the form, say, (2.4) in the 12-vacuum with the condensed two first flavors. Then $Q_A^k \sim \delta_A^k \neq 0$, for $k,A = 1,2$, solve the equations $(\Phi^k_l + m_A \delta^k_l)Q_A^l = 0$, for the critical values of the superpotential, i.e. the first and second massless quarks have the color charges $\mu_1$ and $\mu_2$ from the triangle of fundamental representation $3$ ($u$ and $d$ quarks in the terminology of [1]) for the scalar matrix $\Phi$ is taken in the gauge (2.4) (we forget for a second about extra flavors with $A = 3,4$, and return to their influence to our conclusions later). The quark condensate breaks gauge group completely on the scale $\langle Q_A \rangle \sim \sqrt{\mu m A}$, but at the intermediate energy scales $\sqrt{\mu m} < E < m$ (when $m_1 \sim m_2 \sim m$) we get generically an effective $U(1) \times U(1)$ theory, but at $\Delta m = m_1 - m_2 \rightarrow 0$ this gauge group enlarges to $SU(2) \times U(1) \simeq U(2)$. The “phase diagram” in the $(m_1,m_2)$-plane can be depicted as on fig. 2, certainly this is only a “real section” of the two-dimensional space of complex masses $(m_1,m_2) \in \mathbb{C}^2$.

Three lines, depicted at fig. 2, correspond to coinciding eigenvalues of the matrix (2.4). The dashed line when $m_1 = m_2$ is the line of $SU(2) \times U(1)$ theory with $N_f = 4$ light fundamental multiplets, charged w.r.t. $SU(2)$ group. The effective action of this theory is given by eq. (2.14). It has therefore $\beta_{eff} = \beta_{SU(2)} = 4 - N_f = 0$, and stays at weak coupling, if we fix non-running coupling to be small $\frac{1}{g^2} \sim \log \frac{\Lambda}{\Lambda} \gg 1$ at the scale, when $SU(3)$ breaks down to $SU(2) \times U(1)$. The corresponding $SU(2)$ subgroup corresponds to the $\alpha_{12}$-direction on fig. 1, and we have doublet of light quarks ($\mu_1$ and $\mu_2$) in the fundamental representation of this $SU(2)$.

Two solid lines

\[ 2m_1 + m_2 = 0, \quad m_1 + 2m_2 = 0 \quad (3.1) \]

are different: here, in each case, one deals with restoration of a $SU(2)$ subgroup, now either in $\alpha_1$ or in $\alpha_2$ direction (fig. 1). Each ($\alpha_1$- or $\alpha_2$-) $SU(2)$ subgroup interacts with only two.
charged light flavors, with the charges given either by \( \mu_1 \) or by \( \mu_2 \), due to \( \mu_i \cdot \alpha_j = \delta_{ij} \).

Hence, the corresponding effective beta-function \( \beta_{\text{eff}} = \beta_{SU(2)} = 4 - N_f = 2 > 0 \), and the effective theory falls into strong-coupling regime by the Seiberg-Witten mechanism. Thus, our semiclassical considerations are not reliable in the vicinity of (3.1), and we will have to extract more information from the exact solutions to study the vicinity of these lines: the “fat lines” with the width of the order of \( \Lambda \).

Remember now, that the picture at fig. 2 is in fact real slice of the full complex picture, and each straight line has real codimension 2. It means, that one can go around each solid line, naively separating two “weakly coupled” sectors, moving to complex domain and keeping \( |m_A| \gg \Lambda, A = 1, 2 \), taking into account the weak-coupling monodromies. Restricting ourselves to sit in the \( r = 2 \) vacuum (where first two flavors condense) in one of the sectors, we shall preserve this condition in all weakly coupled domains, once it had been chosen (12-vacuum for our choice). The consistency of this picture is better seen in different from (2.4) gauge, when the (real) eigenvalues of \( \Phi \) are ordered (see Appendix A), but we shall use mostly the gauge (2.4) below, as more adequate for our purposes.

As we already noticed, at the vicinity of solid lines from fig. 2 the theory around \( r = 2 \) vacuum falls into the strong coupling regime. Differently one can say, that on these lines \( r = 2 \) vacuum collides with on of the \( r = 1 \) vacua, since matrix (2.4) acquires simultaneously the form

\[
\Phi\big|_{2m_1+m_2=0} = -\begin{pmatrix} -\frac{m_2}{2} & m_2 \\ m_2 & -\frac{m_2}{2} \end{pmatrix}, \quad \Phi\big|_{m_1+2m_2=0} = -\begin{pmatrix} m_1 & -\frac{m_1}{2} \\ -\frac{m_1}{2} & -m_1 \end{pmatrix}
\]

(3.2)

The theory at \( r = 1 \) vacuum has, in addition to the light quark, a light monopole or dyon, with the electric and magnetic charges \( (n_e, n_m) = (0, 1) \) and \( (n_e, n_m) = (1, 1) \) in units of the orthogonal root to the charge of quark (see detailed discussion in [1]). The semiclassically
Figure 3: Fine structure of the (real slice of) the picture at fig. 2 in \((m_1, m_2)\)-plane, when the equations of border lines (3.1) are replaced by \(m_1 + 2m_2 = \pm 2\Lambda\) and \(2m_1 + m_2 = \pm 2\Lambda\). Light regions correspond to weak coupling regime with condensates of two light quarks. In light-grey regions one deals with condensed (mutually orthogonal) light quarks and dyons, while in the dark region - rhombus around the origin, we get a doublet of light dyons.

unbroken \(SU(2)\)-subgroups in (3.2) are “broken back” to \(U(1)\) on the scale \(\Lambda\), say

\[
\Phi|_{r=1} \sim \begin{pmatrix} -\frac{m_2}{2} \pm \Lambda \\ m_2 \\ -\frac{m_2}{2} \mp \Lambda \end{pmatrix},
\]

(3.3)
i.e. the \(SU(2) \subset SU(3)\) subgroup exists only for \(\Lambda \to 0\), while the \(\Lambda\)-splitting in (3.2) is governed at strong coupling by the original Seiberg-Witten mechanism [2]. Therefore, the exact behavior around the solid lines at fig. 2 is describe by the effective strongly-coupled \(SU(2)\) gauge theory with \(N_f = 2\) light fundamental multiplets, see Appendix B and sect. 4.3.

This transparent example obeys, however, an important internal problem. When the UV gauge group \(SU(3)\) breaks at the scale \(m\) into \(SU(2) \times U(1)\), the \(SU(2)\) theory with \(N_f = 4\) flavors is conformal, and its coupling does not run, being fixed by

\[
e^{-\frac{8\pi^2}{g^2}} = \left(\frac{\Lambda}{m}\right)^{2N_c-N_f} = \left(\frac{\Lambda}{m}\right)^2
\]

(3.4)

If average mass \(m\) goes below \(\Lambda = \Lambda_{SU(3)}\), the effective low-energy conformal \(SU(2)\) theory we consider is in the strong coupling regime, so that the naive semiclassical analysis is not applicable. The coupling of quantum \(SU(2)\) theory with \(N_f = 4\) is being moreover renormalized by the instanton effects (see e.g. [25, 26, 27]), and therefore goes beyond our control at \(m \leq \Lambda\).
In order to avoid this one has to consider instead the theory with \( N_f = 5 \), which becomes the IR free in the \( SU(2) \) sector. In this theory we shall always take \( m_A \sim m \), and \( |\delta m_{AB}| \ll |m| \), for any \( A, B = 1, \ldots, N_f = 5 \). At energies \( E > |m| \) this is an asymptotically free \( SU(3) \) \( N = 2 \) supersymmetric gauge theory with \( \beta_3 = 6 - N_f = 1 \), i.e. \( 1/g_3^2 \sim (\log E/\Lambda) \), while at \( E < |m| \) this is a zero-charge \( SU(2) \) theory with \( \beta_2 = 4 - N_f = -1 \), or \( 1/g_2^2 \sim -\log(\E/\tilde{\Lambda}) \). At \( E = m \) these two lines intersect, what gives

\[
\tilde{\Lambda} = \max \left( \frac{m^2}{\Lambda}, \Lambda \right),
\]

where we have also taken into account that at \( |m| < \Lambda \) the scale of breaking of the \( SU(3) \) gauge symmetry down to \( U(2) \) is determined by \( \Lambda \).

Below we consider the case, when \( m_1 = m_3, \) and \( m_2 = m_4, \) while \( m_5 \) can vary in different ranges. More precisely our low energy \( U(2) \) theory with five flavors is not asymptotically free and stays at weak coupling if

\[
|m_A - m_B| \ll \tilde{\Lambda}, \quad A, B = 1, 2, 5
\]

i.e. all mass differences are essentially small.

## 4 Exact solution of \( N_c = 3 \) and \( N_f = 4, 5 \) theories

### 4.1 Seiberg-Witten theory for supersymmetric QCD

Generic curve for \( \mathcal{N} = 2 \) supersymmetric QCD with \( N_c \) colors and \( N_f \) flavors can be written in the form \([28, 29, 30, 31]\)

\[
y^2 = P(x)^2 - 4Q(x)
\]

(4.1)

where

\[
P(x) = \prod_{i=1}^{N_c} (x - \phi_i), \quad \sum_{i=1}^{N_c} \phi_i = -\Lambda \delta_{N_f, 2N_c - 1}
\]

(4.2)

\[
Q(x) = \Lambda^{2N_c - N_f} \prod_{A=1}^{N_f} (x + m_A)
\]

with two polynomials of powers \( N_c \) and \( N_f \) respectively. Semiclassically the roots \( \{\phi_i\}, \ i = 1, \ldots, N_c \) coincide with the eigenvalues of the matrix \( \Phi \) of the condensate of the complex scalar from the vector multiplet of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory, but being computed exactly they (or, better, their symmetric functions) are got corrected in (dependent upon \( \Lambda \) and \( m_A \) way) due to the instanton effects. The curve (4.1) can be also re-written as \([32]\)

\[
w + \frac{Q(x)}{w} = P(x)
\]

(4.3)

or

\[
W + \frac{1}{W} = \frac{P(x)}{\sqrt{Q(x)}}
\]

(4.4)
with \( y = w - \frac{Q(x)}{w} = \left(W - \frac{1}{W}\right)\sqrt{Q(x)} \).

The curves (4.1), (4.3) or (4.4) are endowed with a generating differential

\[
dS \sim x \frac{dw}{w} = x \frac{dW}{W} + \frac{1}{2} x \frac{dQ}{Q} = x \frac{dP}{P} - x \frac{P}{2y} \frac{dQ}{Q} + \frac{1}{2} x \frac{dQ}{Q} \tag{4.5}
\]

where it is chosen to have the residues

\[
\text{res}_{P_-^A} dS = m_A \cdot P \Big|_{x=-m_A} - \frac{m_A}{2} = -m_A \tag{4.6}
\]

at the points \( P_A \) with \( x(P_+^A) = -m_A \) at one of the sheets of (4.1). The variation of (4.5) at constant \( W \) gives rise to

\[
\delta dS \sim \frac{dx}{y} \left( \delta P(x) - \frac{1}{2} P \frac{\delta Q(x)}{Q(x)} \right) \tag{4.7}
\]

In the case of \( SU(3) \) gauge group it is convenient to introduce explicitly

\[
P(x) = (x - \phi_1)(x - \phi_2)(x - \phi_3) = x^3 - ux - v \tag{4.8}
\]

with, for \( N_f < 5 \) and \( \phi_3 = -\phi_1 - \phi_2 \),

\[
\begin{align*}
u &= \phi_1^2 + \phi_2^2 + \phi_1 \phi_2 \\
v &= -\phi_1 \phi_2 (\phi_1 + \phi_2)
\end{align*} \tag{4.9}
\]

so that for the Seiberg-Witten periods one gets

\[
a_i = \frac{1}{2\pi i} \oint_{A_i} dS, \quad a_i^P = \frac{1}{2\pi i} \oint_{B_i} dS \tag{4.10}
\]

### 4.2 Conformal \( N_f = 4 \) theory

First we consider the \( N_f = 4 \) case with the pairwise coinciding masses, when the Seiberg-Witten curve becomes

\[
y^2 = (x - \phi_1)^2(x - \phi_2)^2(x + \phi_1 + \phi_2)^2 - 4\Lambda^2(x + m_1)^2(x + m_2)^2, \tag{4.11}
\]

To describe the 12-vacuum exactly in terms of (4.11) it is necessary to ensure that this curve has two double roots, determined by the quark masses in semiclassical limit. This is easily obtained - for the pairwise coinciding masses - by putting exactly any two roots of the polynomial (4.8) to coincide with the masses \( m_{1,2} \) with the opposite sign. For (2.4), this is

\[
\phi_1 = -m_1, \quad \phi_2 = -m_2 \tag{4.12}
\]

so that the curve (4.11) turns into

\[
y^2 = (x + m_1)^2(x + m_2)^2 (\left(x - m_1 - m_2\right)^2 - 4\Lambda^2) =
\]

\[
= (x + m_1)^2(x + m_2)^2 (\left(x - M\right)^2 - 4\Lambda^2) \equiv (x + m_1)^2(x + m_2)^2 Y^2 \tag{4.13}
\]

\[
M = m_1 + m_2
\]
where

\[ Y^2 = (x - M)^2 - 4\Lambda^2 \]  

(4.14)

coincides with the formal "pure \( U(1) \)" \( \mathcal{N} = 2 \) SUSY gauge theory curve [33, 34], with the only VEV given here by \( M = m_1 + m_2 \).

Two contours \( A_1 \) and \( A_2 \) shrink for the curve (4.13), (see fig. 4), and the associated periods \( a_{1,2} = \frac{1}{2\pi i} \oint_{A_{1,2}} dS \) (and therefore \( a_3 \) and \( a_8 \), see (2.9)) reduce to the residue integrals (see Appendix B). Here we choose canonical basis of the \( (A, B) \)-cycles on (4.11) as follows: the \( A_k \)-cycle surrounds the cut, which shrinks to \( x = \phi_k \simeq -m_k \), both for \( k = 1, 2 \), while the dual \( B_k \)-cycles obey \( A_i \circ B_j = \delta_{ij} \), see fig. 4. The values of the residues of the differential (4.5) on the curve (4.13) correspond to exact vanishing of the effective masses of the quarks \( A = 1, 2 \) in the 12-vacuum (at \( \mu = 0 \), see (2.9))

\[
\begin{align*}
  a_1 + m_1 &= \frac{a_3}{\sqrt{2}} + \frac{a_8}{\sqrt{6}} + m_1 = 0 \\
  a_2 + m_2 &= -\frac{a_3}{\sqrt{2}} + \frac{a_8}{\sqrt{6}} + m_2 = 0 
\end{align*}
\]  

(4.15)

In (4.15) we have used that the charges of these two quarks in the gauge (2.4) are given by two weights of the algebra (see fig. 1)

\[
Q_1^1 : \quad n_e = \frac{\mu_1}{\sqrt{2}}, \quad n_m = 0 \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( \frac{1}{2}, 0; \frac{1}{2\sqrt{3}}, 0 \right) 
\]  

\[
(4.16)
\]

and

\[
Q_2^1 : \quad n_e = \frac{\mu_2}{\sqrt{2}}, \quad n_m = 0 \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( -\frac{1}{2}, 0; \frac{1}{2\sqrt{3}}, 0 \right) 
\]  

\[
(4.17)
\]

of the \( SU(3) \) gauge group, broken down generally to \( U(1) \times U(1) \) by nonvanishing \( \Delta m_{AB} \neq 0 \). Here \( n_e^3, n_m^3 \) and \( n_e^8, n_m^8 \) are electric and magnetic charges of a state with respect to two Cartan generators of \( SU(3) \) gauge group, \( T^3 \) and \( T^8 \) respectively.
The main outcome of this solution is absence of any corrections of the order of $\Lambda$ to the first two $\phi$’s in (4.12), which means that in the equal mass limit these two $\phi$’s become exactly equal. This is a signal of restoration of the non-Abelian $U(2) \simeq SU(2) \times U(1)$ gauge group at the root of the Higgs branch (at $\mu = 0$). We have expected this in sect. 3 in semiclassical at large masses $m$. Now we see, that this phenomenon occurs for arbitrary $m$, in particular, if we reduce $m$ and go all the way to the strong coupling region at $m \ll \Lambda$. As was already mentioned, the physical reason for the emergence of the non-Abelian gauge group is that the dual low-energy effective theory with the dual gauge group $U(2) \simeq SU(2) \times U(1)$ at $m \ll \Lambda$ is not asymptotically free in the equal mass limit and stays at weak coupling. Therefore, the classical analysis showing that the non-Abelian gauge group is restored at the root of the Higgs branch remains intact in quantum theory.

The curve (4.13) has two double roots at

$$e_1 = e_2 = -m_1, \quad e_3 = e_4 = -m_2. \quad (4.18)$$

while the remaining two roots of the ”$U(1)$-curve” are at

$$e_5 = m_1 + m_2 - 2\Lambda, \quad e_6 = m_1 + m_2 + 2\Lambda, \quad (4.19)$$

In the monopole singularity the other roots coincide, e.g. $e_2 = e_5$, see fig. 4, and the $B_1$-contour shrinks producing a regular period. Indeed (cf. with (B.11)),

$$a_1^D = a_3^D + \sqrt{3}a_8^D = \frac{1}{2\pi i} \oint_{B_1} dS =$$

$$= -\frac{i}{\pi} \left( \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2} + \left( m_1 + \frac{m_2}{2} \right) \log \frac{2m_1 + m_2 - \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2}}{2m_1 + m_2 + \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2}} \right) \quad (4.20)$$

Again, we fix its real part to ensure that (4.20) vanishes at $2m_1 + m_2 = 2\Lambda$, corresponding to the masslessness of the monopole with the charges

$$n_e = 0, \quad n_m = \frac{\alpha_1}{\sqrt{2}} \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( 0, \frac{1}{2}; 0, \frac{\sqrt{3}}{2} \right). \quad (4.21)$$

which is one of three $SU(3)$ elementary monopoles, whose charges are determined by the roots of the $su(3)$ algebra, see fig. 1. Exchanging in (4.20) $m_1 \leftrightarrow m_2$, one gets at $m_1 + 2m_2 = 2\Lambda$ vanishing of the mass

$$a_2^D = -a_3^D + \sqrt{3}a_8^D = \frac{1}{2\pi i} \oint_{B_2} dS =$$

$$= -\frac{i}{\pi} \left( \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2} + \left( m_1 + \frac{m_2}{2} \right) \log \frac{m_1 + 2m_2 - \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2}}{m_1 + 2m_2 + \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2}} \right) \quad (4.22)$$

---

2This is in perfect agreement with [4], where non-Abelian dual gauge groups $U(r)$ were identified at the roots of non-baryonic Higgs branches in the $SU(N)$ gauge theory with $N_f$ massless quarks, $r < 2N_f$.

3Strictly speaking, this is true only for the theory with $N_f = 5$, which we consider in sect.4.4.
of the monopole with the charge

\[ n_e = 0, \ n_m = \frac{\alpha_2}{\sqrt{2}} \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( 0, \ -\frac{1}{2}; 0, \ -\frac{\sqrt{3}}{2} \right). \]

Clearly, the imaginary part \( \text{Im}(a_1^D) = 0 \) vanishes also at \( 2m_1 + m_2 = -2\Lambda \), and similarly \( \text{Im}(a_2^D) = 0 \) if \( m_1 + 2m_2 = -2\Lambda \). However, the real parts of expression (4.20) at \( 2m_1 + m_2 = -2\Lambda \) equals to \( 2m_1 + m_2 = -\alpha_1 \cdot a \), or to the mass of the W-boson with the charge \( \alpha_1 \). Hence, at \( 2m_1 + m_2 = -2\Lambda \) one gets the massless dyon with the charge

\[ n_e = \frac{\alpha_1}{\sqrt{2}}, \ n_m = \frac{\alpha_1}{\sqrt{2}} \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( \frac{1}{2}, \ -\frac{1}{2}; \frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} \right). \]

and similarly, the massless dyon with the charge

\[ n_e = \frac{\alpha_2}{\sqrt{2}}, \ n_m = \frac{\alpha_2}{\sqrt{2}} \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = \left( -\frac{1}{2}, \ -\frac{1}{2}; \frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} \right). \]

at \( m_1 + 2m_2 = -2\Lambda \).

### 4.3 Permutation of the branch points

Vanishing of the monopole masses \( a_1^D \) by (4.21) at \( 2m_1 + m_2 = 2\Lambda \) and, similarly, of \( a_2^D \) at \( m_1 + 2m_2 = 2\Lambda \) leads to appearance of massless monopoles in 12-vacuum with massless quarks (4.15), what means that we arrive at the Argyres-Douglas (AD) points/lines [35, 36]. The AD point corresponds to particular value of the mass parameters (or a curve in the space of all mass parameters, as in our theory), where the mutually nonlocal states become massless simultaneously. From (4.18) and (4.19) one finds, that in our \( N_c = 3, N_f = 4 \) theory there are four AD lines where the \( r = 2 \) 12-vacuum collides with the magnetic singularities. These are at

\[ 2m_1 + m_2 = 2\Lambda, \quad e_1 = e_2 = e_5 = -m_1 \]
\[ 2m_1 + m_2 = -2\Lambda, \quad e_1 = e_2 = e_6 = -m_1 \]

and at

\[ 2m_2 + m_1 = 2\Lambda, \quad e_3 = e_4 = e_5 = -m_2 \]
\[ 2m_2 + m_1 = -2\Lambda, \quad e_3 = e_4 = e_6 = -m_2 \]

producing the quantum regularization (depicted at fig. 3) of two classical lines (3.1) from fig. 2 where the semiclassical approximation is no longer valid.

As we reduce the masses \( m \) and pass from weak coupling into the strong coupling domain, crossing (4.26) and/or (4.27), along the Coulomb branch at \( \mu = 0 \), the quantum numbers of massless quarks \( Q_1 \) and \( Q_2 \) change due to nontrivial monodromies in the space of masses\(^4\). The complex \( m \)-plane has cuts, crossing these periods \( a \) and \( a^D \) change linearly, accordingly

\(^4\)Note that \( U(2) \) theory with \( N_f = 4, 5 \) does not have monodromies, associated with \( \Delta m_{AB} \) since this theory is not asymptotically free, to be compared with studied in [15, 14].
changing the quantum numbers of corresponding states (there monodromies were studied in [37] in the SU(2) gauge theory through a monodromy matrix approach). Let us demonstrate now, that as we pass through the AD lines (4.26), (4.27) the root pairings on the curve (4.11) change and light quarks transform into the light dyons.

Take real \( m_1 \) and \( m_2 \), and consider the first AD line in (4.26). On this line the 12-vacuum with two massless quarks (4.16) and (4.17) collides with \( r = 1 \) vacuum where quark \( Q_2^2 \) with the charges (4.17) and the monopole (4.21) are massless. The roots \( e_3 \) and \( e_4 \) are far away and, therefore, the charges of the \( Q_2^2 \) quark (4.17) do not change, and we focus on the colliding roots \( e_1 \), \( e_2 \) and \( e_5 \), \( e_6 \) (see (4.26)) as we decreasing \( m \). Hence, this situation is described by an effective SU(2) curve with four roots and two cuts, and it is described in detail in Appendix C.

In order to see this, let us slightly split the degenerate branch points of (4.13) by shifting \( \phi_1 \) from its solution (4.12), parameterizing the shift as

\[
x = -m_1 + z, \quad \phi_1 = -m_1 + \delta \phi, \quad 2m_1 + m_2 = 2\Lambda + \epsilon, \tag{4.28}
\]

and relabel the roots (4.18), (4.18) after the shift \( e_i \rightarrow z_i = e_i + m_1 \), \( i = 1, \ldots, 6 \). The curve (4.11) now acquires the form

\[
y^2 = (m_1 - m_2)^2 \left[ (z - \delta \phi)^2 (z - \epsilon - 2\Lambda + \delta \phi)^2 - 4\Lambda^2 z^2 \right] \sim \left[ z^2 - (4\Lambda + \epsilon) z + 2\Lambda \delta \phi \right] \left[ z^2 - z \epsilon + 2\Lambda \delta \phi \right] \tag{4.29}
\]

where we have omitted the overall factor and inessential terms one can neglect in this approximation. The polynomial in the r.h.s. of (4.29) obviously factorizes into the product of two quadratic polynomials, whose roots are the ends of the cut, surrounding the \( A_1 \) and \( A_3 \) cycle (see fig. 4), while the degenerate \( A_2 \)-cut is separated from (4.29) for a while, i.e. the situation is indeed reduced to the curve of effective SU(2) theory with two flavors, see details in Appendix C. One can study therefore with the help of (4.29) what happens literally, when the ends of the first and third cuts touch each other.

Comparing the last relation from (4.28) with (4.26) one finds that there are two critical regimes for parameter \( \epsilon \) in (4.29), the first is at \( \epsilon \sim 0 \) or the vicinity of upper AD-line (the first equation in (4.26)), while in the vicinity of lower AD-line (the second equation in (4.26)) one should take \( \epsilon = -4\Lambda + \tilde{\epsilon} \), with \( \tilde{\epsilon} \sim 0 \).

Consider, first, \( \epsilon \sim 0 \) (and let us be interested in domain, where \( z \ll \Lambda \)). Equation (4.29) turns into

\[
y^2 \sim \left( z^2 - 4\Lambda z + \delta \right) \left( z^2 - \epsilon z + \delta \right) \tag{4.30}
\]

with \( \delta = 2\Lambda \delta \phi \). It has ”large root” \( z_6 \approx 4\Lambda \) of the first quadratic polynomial, which is far away from its second root \( z = z_1 \approx \frac{1}{2}\delta \phi \approx 0 \), they correspond to \( e_6 \) and \( e_1 \) respectively in (4.26). Two roots of the second quadratic polynomial at

\[
z_{5,2} = \frac{\epsilon}{2} \pm \sqrt{\frac{\epsilon^2}{4} - \delta} \tag{4.31}
\]

At \( \epsilon > 0 \) we have \( z_1 \approx z_2 \) ( for the minus sign in (4.31)), while \( z_5 \approx \epsilon \). The contour \( A_1 \) goes therefore around a ”small” cut, and the corresponding period integral is associated with the
mass $a_1 + m_1$ of light quark $Q^1_1$, while the roots $z_6 \approx 4\Lambda$ and $z_5 \approx \epsilon$ are the ends of the cut, surrounded by $A_3$-cycle.

At $\epsilon \to 0$ the $B_1$-cycle also degenerates, which means that monopole with the charge (4.21) also becomes massless. Then, when $\epsilon < 0$ becomes negative, the picture changes drastically, now two roots at $z = z_1 \approx 0$ and $z = z_5$ become close to each other, while $z = z_2$ is negative of the order of $|\epsilon|$. After such transition we get a small $A_1 + B_1$-cycle, with the mass of light state, corresponding to the period integral along the $A_1 + B_1$-contour, see Fig. 4 (see also details in Appendix C).

This means, that the massless quark $Q^1_1$ transforms into the massless dyon $D^1_1$ with the quantum numbers

$$D^1_1: \quad n_e = \frac{\mu_1}{\sqrt{2}}, \quad n_m = \frac{\alpha_1}{\sqrt{2}} \quad \text{or} \quad (n^3_e, n^3_m; n^8_e, n^8_m) = \left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

or $n^1 = \frac{1}{\sqrt{2}}(\mu_1 \oplus \alpha_1)$, while the charge (4.17) of the quark $Q^2_2$ does not change. Hence, the quantum numbers of the massless quark $Q^1_1$ in the $r = 2$ vacuum, after the collision with the monopole singularity, become shifted by the monopole magnetic charge.

Analogously, one can analyze the first AD line in (4.27) where the $r = 2$ vacuum collides with another $r = 1$ vacuum, containing massless quark $Q^1_1$ and the monopole with the charge (4.23). Now $Q^1_1$ does not change its charge, while the quark $Q^2_2$ acquires additionally the charge of the monopole (4.23). As a result, below both AD lines, i.e. inside the rhombus in Fig. 3, we end up with two massless dyons: the dyon $D^1_1$ with the charge $n^1 = \frac{1}{\sqrt{2}}(\mu_1 \oplus \alpha_1)$ (4.32), and

$$D^2_2: \quad n_e = \frac{\mu_2}{\sqrt{2}}, \quad n_m = \frac{\alpha_2}{\sqrt{2}} \quad \text{or} \quad (n^3_e, n^3_m; n^8_e, n^8_m) = \left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

or $n^2 = \frac{1}{\sqrt{2}}(\mu_2 \oplus \alpha_2)$. The quark masslessness conditions (4.15) at small $m$, inside the rhombus, are replaced by the dyon masslessness conditions, namely,

$$a_1 + a^D_1 + m_1 = \mu_1 \cdot a + \alpha_1 \cdot a^D + m_1 = \frac{a_3}{\sqrt{2}} + \frac{a^D_3}{\sqrt{2}} + \frac{a_8}{\sqrt{6}} + \frac{\sqrt{3}}{2} a^D + m_1 = 0$$

$$a_2 + a^D_2 + m_2 = \mu_2 \cdot a + \alpha_2 \cdot a^D + m_2 = -\frac{a_3}{\sqrt{2}} - \frac{a^D_3}{\sqrt{2}} + \frac{a_8}{\sqrt{6}} + \frac{\sqrt{3}}{2} a^D + m_2 = 0$$

Let us point out here the crucial fact, that both electric and magnetic charges of our massless dyons (4.32), (4.33) are $\pm \frac{1}{2}$ with respect to the $\tau^3$-generator of the dual $U(2)$ gauge group, i.e. they can belong to the fundamental representation of this group; moreover, all dyons $D^l_1$ ($l = 1, 2$) form the color doublets. This is another confirmation of the conclusion we already made above, that the non-Abelian $SU(2)$ factor of the dual gauge group gets restored in the equal mass limit.

Going further, and crossing the lower AD lines at Fig. 3, corresponding to the lower equations in (4.26), (4.27), one comes back to the $12$-vacuum we had in weak coupling, where the electric $A$-cycles are small. The dyons $D^k_1$, $k, A = 1, 2$ with the charges (4.32) and (4.33) change
Figure 5: Weak-coupling parabolas in $N_f = 5$ theory. Here the curves (4.40) and (4.42) are plotted at $m_5 = 5\Lambda$, and the straight line $\Sigma = m_1 + m_2 + m_5 = 0$ is the baryonic branch condition, see (4.39).

their quantum numbers on these lines due to the presence of massless dyons with the charges $\alpha_1 \oplus \alpha_1$ and $\alpha_2 \oplus \alpha_2$ (on (4.26) and (4.27) correspondingly), whose both electric and magnetic constituents are given entirely in terms of the root vectors, and turn back into the light quarks. The detailed analysis can be found in Appendices B,C, and we skip it in the main text.

4.4 $N_f = 5$ IR free theory

Let us turn now to the peculiarities of the above analysis in the $N_f = 5$ case. If $m_1 = m_3$ and $m_2 = m_4$ the $N_f = 5$ curve (4.1), (4.2) acquires the form

$$y^2 = (x - \phi_1)^2(x - \phi_2)^2(x + \phi_1 + \phi_2 + \Lambda)^2 - 4\Lambda(x + m_1)^2(x + m_2)^2(x + m_5)$$ (4.35)

and if we are exactly in the $r = 2$ 12-vacuum, similarly to the $N_f = 4$ case it degenerates to

$$y^2 = (x + m_1)^2(x + m_2)^2Y^2$$ (4.36)

where now

$$Y^2 = p^2 - 4\Lambda(x + m_5)$$
$$p = x - M + \Lambda = x - m_1 - m_2 + \Lambda$$ (4.37)

is the curve of the "formal $U(1)$" theory with one flavor.

Compare to the $N_f = 4$ case, the roots (4.18) for the curve (4.36) remain intact, while instead of (4.19), one gets from (4.37)

$$e_5 = M + \Lambda - 2\sqrt{\Lambda\Sigma}, \quad e_6 = M + \Lambda + 2\sqrt{\Lambda\Sigma}$$ (4.38)
with $M = m_1 + m_2$ and
\[ \Sigma = M + m_5 = m_1 + m_2 + m_5 \] (4.39)
is the discriminant of (4.37), whose vanishing means that we come to the origin of the baryonic branch.

It is easy to see now, that instead of the AD straight lines (4.26) and (4.27) of the $N_f = 4$ theory, colliding of the roots $e_5 = e_1 = e_2 = -m_1$ and $e_6 = e_1 = e_2 = -m_1$ result into two branches of parabola
\[ Y_1^2 = (2m_1 + m_2)^2 - \Lambda(2m_5 + m_2) + \Lambda^2 = 0 \] (4.40)
in the mass $(m_1, m_2)$-plane, see fig. 5. At $m_5 \gg m_{1,2}$, after renormalization $\Lambda m_5 \rightarrow \Lambda_4^2$ one comes back to the straight lines from (4.26), but generally one find from (4.40) that the size of the strong coupled domain around the classical line $2m_1 + m_2 = 0$ from fig. 2 is rather\\n\[ \sqrt{\Lambda \Delta m} = \sqrt{\Lambda(m_5 - m_1)} \] in $N_f = 5$ theory (if we fix $\Delta m$), which can be seen, rewriting (4.40) as
\[ \left( m_1 + m_2 - \frac{\Lambda}{2} \right)^2 = \Lambda(m_5 - m_1) \] (4.41)
Like in $N_f = 4$ case on one branch of this parabola the $r = 2$ vacuum collides with the $r = 1$ vacuum, where quark $Q_2^2$ and monopole with charges (4.21) are massless, while on the other branch it collides with another $r = 1$ vacuum with the massless quark $Q_2^2$ and dyon (4.24).

If instead we collide $e_5 = e_3 = e_4 = -m_2$ and $e_6 = e_3 = e_4 = -m_2$, one gets the second parabola
\[ Y_2^2 = (m_1 + 2m_2)^2 - \Lambda(2m_5 + m_1) + \Lambda^2 = 0 \] (4.42)
or
\[ \left( m_2 + m_1 - \frac{\Lambda}{2} \right)^2 = \Lambda(m_5 - m_2) \] (4.43)
with the same properties. On two branches of this parabola our \( r = 2 \) vacuum collides with \( r = 1 \) vacua with massless quark \( Q_1^1 \) and the monopole (4.23) or dyon (4.25) respectively. The intersection of two parabolas is depicted at fig. 5, 6 and we see how the rhombus of \( N_f = 4 \) theory is deformed in the \( N_f = 5 \) case. These results can be also confirmed by direct calculation of Seiberg-Witten periods

\[
a_k^D = \frac{1}{2\pi i} \oint_{B_k} dS = -\frac{i}{\pi} \left( Y_k + \left( M + \frac{m_k + m_5}{2} \right) \log \frac{M + m_k + \Lambda - Y_k + \frac{m_k - m_5}{2} \log \frac{a + b_+ m_k - b_- Y_k}{a + b_+ m_k + b_- Y_k}}{a + b_+ m_k - b_- Y_k} \right), \quad k = 1, 2
\]

(4.44)

with \( Y_{1,2} \) being defined in (4.40), (4.42). Formulas (4.44) show that the conclusions about \( N_f = 4 \) case remain almost intact in the case of five flavors. At \( Y_k = 0, k = 1, 2 \) (i.e. for each parabola (4.40), (4.42) in the mass plane) the imaginary part of the corresponding period \( \text{Im}(a_k^D) \) vanishes, while the real part \( \text{Re}(a_k^D) \) jumps when passing from the positive to negative branch of the corresponding \( k \)-th parabola.

Permutation of roots upon crossing these AD-lines can be analyzed much in the same way as in \( N_f = 4 \) case. Say, on the first branch of the parabola (4.41) the Seiberg-Witten curve (4.35) gives the equation

\[
z(z^2 - \epsilon z + \delta) = 0
\]

(4.45)

which is the same as eq. (4.30) (taken at the first AD line) with \( \delta = 2\sqrt{\Lambda(m_5 - m_1)} \delta \phi \), while \( \epsilon \) is defined via relation

\[
2m_1 + m_2 - \Lambda = 2\sqrt{\Lambda(m_5 - m_1)} + \epsilon.
\]

(4.46)

In fact, the scheme we considered in previous section does not at all depend upon the details of the model, and is completely described by the effective \( SU(2) \) curve with two cuts, which is considered in Appendix C. Therefore the result of the root permutation is the same: inside the "deformed rhombus" massless quarks \( Q_1^1 \) and \( Q_2^2 \) transform into massless dyons \( D_1^1 \) and \( D_2^2 \) with charges (4.32) and (4.33) respectively.

Hence, we come now already close to the main conclusion of our paper. Decreasing the values of the fundamental masses, we turn the original theory into the strong coupling regime. Due to monodromies in the space of masses, the original light quarks in our model change their quantum numbers and turn into the light dyons with the charges (4.32) and (4.33). The exact analysis of the Seiberg-Witten theory shows that this happens, in particular, on the AD lines, surrounding the strongly coupled domain in the mass space. The exact shape of the strongly-coupled domain is slightly different in the \( N_f = 4 \) and \( N_f = 5 \) theories, but that does not influence the main conclusions. The \( N_f = 4 \) theory is conformal, and therefore one cannot really ensure for it the weakly coupled regime for the dual theory of light dyons. However, in the \( N_f = 5 \) case in the regime (3.6), inside the domain surrounded by parabolas on fig. 6 (we shall still refer to it as to rhombus), the effective dual theory of light dyons is at weak coupling, and the semiclassical analysis of the properties of confinement [1] is directly applicable. In order to do this, we are going to write the effective action, and study the charges of the string solutions.
5 Low-energy effective action at strong coupling

In this section we construct the effective low-energy theory at strong coupling, i.e. inside the rhombus on fig. 3, or in the "rhombus" surrounded by two parabolas at fig. 6. We call it dual to the original theory with the action (2.14). In this region we keep \(|m_A - m_B|\) small (see (3.6)) and \(m \ll \Lambda\).

As was shown above, the massless quarks \(Q_A^1\) and \(Q_A^2\) are transformed into the massless dyons \(D_A^1\) and \(D_A^2\) with the charges (4.32) and (4.33); the latter form a fundamental representation of the gauge group \(SU(2)\). According to the charges (4.32), (4.33), the third component of the \(SU(2)\) dual gauge field has to be the following linear combination

\[
B_\mu^3 = \frac{1}{\sqrt{2}} (A_\mu^3 + A_\mu^{3D})
\]

of the gauge fields. If the dual non-Abelian gauge group is restored, the components \(B_{\mu}^{1,2}\) of the gauge field become massless at \(m_1 = m_2\). Let us check, whether this is indeed the case.

The electric and magnetic charges of the dual \(W\)-bosons \(B_\mu^\pm\) coincide with the charges of the operators \(\hat{D}_3^A A_\mu^A\) and \(\hat{D}_3^A D_\mu^A\). From (4.32), (4.33) we get therefore for these charges \((\mu_1 - \mu_2) \oplus (\alpha_1 - \alpha_2) = \alpha_{12} \oplus \alpha_{12}\), see fig. 1, or, in physical normalization, more explicitly

\[
B_{\mu}^\pm : \quad n_e = n_m = \pm \frac{\alpha_{12}}{\sqrt{2}}, \quad \text{or} \quad (n_e^3, n_m^3; n_e^8, n_m^8) = (\pm 1, \pm 1; 0, 0)
\]

These charges determine the mass of these states by the Seiberg–Witten mass formula [2]. One gets from (4.34)

\[
m_{B_{\mu}^\pm} = \sqrt{2} |a_3 + a_3^D| = |\Delta m_{12}|
\]

At \(\Delta m_{12} \to 0\), the dual \(W\)-boson mass (5.3) vanishes, as was expected, and therefore the fields (5.1) and (5.2) can be unified into the adjoint multiplet \(B_\mu^\alpha\), \((\alpha = 1, 2, 3)\), of the non-Abelian \(SU(2)\) factor of the gauge group of the dual theory.

The light dyons \(D_A^k\) \((k = 1, 2)\) are also charged with respect to the \(U(1)\) gauge group, associated with the \(T^8\)-generator of the underlying \(SU(3)\) gauge group broken in the dual theory down to \(SU(2) \times U(1) \simeq U(2)\). According to the dyon charges (4.32), (4.33) the dual photon field is given by the following linear combination

\[
B_\mu^8 = \frac{1}{\sqrt{10}} (A_\mu^8 + 3A_\mu^{8D})
\]

of the dual gauge fields. It turns out, that the dyons \(D_A^k\) and the gauge fields \(B_\mu^\alpha\) \((\alpha = 1, 2, 3)\), \(B_\mu^8\), together with their superpartners, are the only light states to be included in the dual low-energy effective theory inside the rhombus of strong-coupled regime. All other states are either heavy (with masses of the order of \(\hat{\Lambda}\) the scale (3.5) of the dual theory) or decay on the curves of marginal stability [2, 3, 37, 38, 15, 14].
It means, that the bosonic part of the effective low-energy action of the dual theory inside the rhombus can be written in the form

$$S_{\text{dual}} = \int d^4x \left[ \frac{1}{4g_2^2} (F_{\mu\nu}^\alpha)^2 + \frac{1}{4\tilde{g}_1^2} (F_{\mu\nu}^8)^2 + \frac{1}{\tilde{g}_2^2} |D_\mu b^\alpha|^2 \right]$$

$$+ \frac{1}{\tilde{g}_2^2} |\partial_\mu b^8|^2 + |\nabla_\mu D^A|^2 + \left| \nabla_\mu \tilde{D}_\alpha \right|^2 + V(D, \tilde{D}, b^\alpha, b^8) \right)$$

where $b^8$ and $b^\alpha$, so that

$$b^8 = \frac{1}{\sqrt{10}} (a^8 + 3a_D^8), \quad b^3 = \frac{1}{\sqrt{2}} (a^3 + a_D^3) \quad \text{for} \quad \alpha = 3$$

are the scalar $\mathcal{N} = 2$ superpartners of gauge fields $B_\mu^8$ and $B_\mu^\alpha$, while $F_{\mu\nu}^8$ and $F_{\mu\nu}^\alpha$ are their field strengths, and the gauge couplings $\tilde{g}_1$ and $\tilde{g}_2$ correspond to the $U(1)$ and $SU(2)$ subgroups respectively.

The covariant derivatives are defined in accordance with the charges of the dyons, namely

$$\nabla_\mu = \partial_\mu - i \left( \sqrt{2} B_\mu^\alpha \frac{\tau^\alpha}{2} + \sqrt{\frac{5}{6}} B_\mu^8 \right)$$

The scalar potential $V(D, \tilde{D}, b^\alpha, b^8)$ in the action (5.5) is

$$V(D, \tilde{D}, b^\alpha, b^8) = \frac{\tilde{g}_2^2}{4} \left( \tilde{D}^A \tau^\alpha D_A - \tilde{D}^A \tau^\alpha \tilde{D}_A \right) + \frac{5}{12} \tilde{g}_1^2 \left( |D_A|^2 - |\tilde{D}_A|^2 \right) +$$

$$+ \tilde{g}_2^2 \left| \tilde{D}^A \tau^\alpha D_A + \frac{\partial \mathcal{W}}{\partial b^\alpha} \right|^2 + \tilde{g}_1 \left| \sqrt{\frac{5}{3}} \tilde{D}^A D_A + \frac{\partial \mathcal{W}}{\partial b^8} \right|^2$$

$$+ \sum_{A=1}^{N_f} \left[ \left| \left( \sqrt{\frac{5}{3}} b^8 + \tau^\alpha b^\alpha + m_A \right) D_A \right|^2 + \left| \left( \sqrt{\frac{5}{3}} b^8 + \tau^\alpha b^\alpha + m_A \right) \tilde{D}_A \right|^2 \right]$$

Now let us turn directly to the desired limit of equal quark masses, $\Delta m_{AB} = 0$. The vacuum of the theory (5.5) is located, due to (4.34), (5.6), at the following values of the scalar condensates

$$\langle b^8 \rangle = -\sqrt{\frac{3}{5}} m_1 + m_2 = -\sqrt{\frac{3}{5}} m, \quad \langle b^\alpha \rangle = 0$$

while the VEV’s of the dyon fields are determined by the FI F-term coming from the derivatives of $\mathcal{N} = 1$ deformation superpotential $\mathcal{W}$ in (5.8). For these derivatives one can write from general principles

$$\frac{\partial \mathcal{W}}{\partial \bar{b}^8} = \tilde{\mu} \lambda + \ldots, \quad \frac{\partial \mathcal{W}}{\partial \bar{b}^\alpha} = c \tilde{\mu} b^\alpha + \ldots$$

where $\tilde{\mu} = \text{const} \mu$ and dots stand for higher powers of fields $b^8$ and $b^\alpha$ we ignore at small $\tilde{\mu}$, while the (inessential for our purposes) constant $c$ can be in principle determined from the exact solution. This gives

$$\langle D_A^k \rangle = \langle \tilde{D}_A^\alpha \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
where we restrict ourselves to the case $N_f = 5$, while
\[ \tilde{\xi} = \sqrt{\frac{3}{5}} \mu \Lambda \]  (5.12)

One can also calculate the dimension of the Higgs branch which emerges in the equal mass limit. It results in
\[ \dim \mathcal{H}\big|_{\text{rhombus}} = 4NN_f - 2N^2 - N^2 - N^2 = 4N(N_f - N)\big|_{N=2} = 8(N_f - 2) \]  (5.13)

where we have to take into account the $4NN_f$ real dyon degrees of freedom and subtract $2N^2 F$-term conditions, $N^2$ $D$-term conditions and, finally, $N^2$ phases eaten by the Higgs mechanism, where $N = 2$ for the $U(2)$ dual gauge group. The dimension of the Higgs branch of the dual theory (5.13) coincides, as expected, with the dimension of the Higgs branch (2.19) in the original low energy theory (2.14).

From (5.9) and (5.11) we see again that both gauge $U(2)$ and flavor $SU(N_f)$ groups, are broken in the vacuum, but the color-flavor locked form of (5.11) guarantees that the diagonal global $SU(2)_{C+F}$ survives. More exactly, the unbroken global group of the dual theory is $SU(2)_{C+F} \times SU(N_f - 2)_F \times U(1)$ and coincides again with the global group of the original theory (for generic quark masses is broken down to $U(1)^{N_f-1}$). Much in the same way as in the original theory (2.14), the presence of the global $SU(2)_{C+F}$ group leads to formation of the non-Abelian strings in the dual theory (5.5).

### 6 Confined monopoles

Since quarks are in the Higgs phase in the weak coupling regime of the original theory at large $m$ in $r = 2$ vacuum, the monopoles are confined. Two of three $SU(3)$ elementary monopoles with the charges $\mathbf{a}_{1,2}$, or $(0, \pm \frac{1}{2}, 0, \frac{\sqrt{3}}{2})$, see (4.21) and (4.23), are attached to the ends of the elementary strings while the third one with the charge $\mathbf{a}_{12} = \mathbf{a}_1 - \mathbf{a}_2$ or $(0, 1; 0, 0)$ becomes a string junction of two elementary strings [1]. Inside the rhombus at small $m$ the dual theory (5.5) is in the weak coupling regime and we can use it to study confinement at strong coupling. In this domain the light dyons condense instead quarks, therefore, here we deal with oblique confinement [39].

In this section we determine the elementary string fluxes of the strings in the dual theory (5.5) inside the rhombus and show that still the elementary monopole fluxes can be absorbed by strings. Hence, it is the monopoles being still confined, much in the same way as in the original $U(2)$ theory (2.14) at weak coupling.

Consider, first, the elementary string $S_1$ arising due to the winding of dyon $D^1_1$. At $r \to \infty$ (in the transverse plane with polar co-ordinates $(r, \theta)$ to the direction of the string) one has
\[ D^1_1|_{r, \to \infty} \sim \sqrt{\xi} e^{i\theta}, \quad D^2_2|_{r, \to \infty} \sim \sqrt{\xi} \]  (6.1)

see (5.11). Taking into account the dyon charges (4.32), (4.33) we derive the behavior of the
gauge potentials at infinity,
\[ n^1 \cdot (A_i \oplus A_i^D) = \frac{1}{\sqrt{2}} (\mu_1 \cdot A_i + \alpha_1 \cdot A_i^D) = \frac{A_3^i}{2} + \frac{A_{3D}^i}{2} + \frac{\sqrt{3}}{2} A^8_i \sim \partial_i \theta \]
\[ n^2 \cdot (A_i \oplus A_i^D) = \frac{1}{\sqrt{2}} (\mu_2 \cdot A_i + \alpha_2 \cdot A_i^D) = -\frac{A_3^i}{2} - \frac{A_{3D}^i}{2} + \frac{\sqrt{3}}{2} A^8_i \sim 0 \]
(6.2)
which, in turn, implies both
\[ A^3_i + A_{3D}^i \sim \partial_i \theta \]
\[ \frac{A^8_i}{\sqrt{3}} + \sqrt{3} A^{8D}_i \sim \partial_i \theta \]
(6.3)
The combinations orthogonal to those of (6.3) are required to vanish at infinity:
\[ A^3_i - A_{3D}^i \sim 0 \]
\[ A^{8D}_i - 3 A^8_i \sim 0 \]
As a result one gets for each component
\[ A^3_i \sim \frac{1}{2} \partial_i \theta, \quad A_{3D}^i \sim \frac{1}{2} \partial_i \theta, \]
\[ A^8_i \sim \frac{3\sqrt{3}}{10} \partial_i \theta, \quad A^{8D}_i \sim \frac{3\sqrt{3}}{10} \partial_i \theta \]
(6.4)
The string charges are defined in terms of the fluxes
\[ \oint dx_i (A_{3D}^i, A^i_3; A^{8D}^i, A^8_i) = 4\pi (-n^3_e, n^3_m; -n^8_e, n^8_m) \]
(6.5)
This definition ensures that the string has the same charge as a trial dyon which can be attached to the string endpoint (not necessarily being present in the spectrum of the theory). In particular, according to this definition, the charge of the $S_1$-string in dual theory with the fluxes (6.4) is
\[ n_{S_1} = \left( -\frac{1}{4}, -\frac{1}{4}, -\frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \right) \]
(6.6)
In a similar way one determines the charges of another $\mathbb{Z}_2$-elementary string, arising due to winding at spatial infinity of the second dyon $D^2_2$:
\[ n_{S_2} = \left( \frac{1}{4}, -\frac{1}{4}, -\frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \right) \]
(6.7)
Now we can check that, each of three $SU(3)$ monopoles can be indeed confined by these two strings. Say, for the monopoles with the charges $0 \oplus \alpha_{1,2}$ or $(0, \pm \frac{1}{2}; 0, \frac{\sqrt{3}}{2})$ one has
\[ \frac{1}{\sqrt{2}} (0 \oplus \alpha_1) = (0, \frac{1}{2}; 0, \frac{\sqrt{3}}{2}) = n_{S_1} + \frac{7}{10} n^1 + \frac{2}{10} n^2, \]
\[ \frac{1}{\sqrt{2}} (0 \oplus \alpha_2) = (0, -\frac{1}{2}; 0, \frac{\sqrt{3}}{2}) = n_{S_2} + \frac{2}{10} n^1 + \frac{7}{10} n^2 \]
(6.8)
where $n^1$ and $n^2$ are the charges of the $D^1_1$ and $D^2_2$ dyons (4.32), (4.33). Formula (6.8) shows, that only a part of the monopole flux is confined to the string, while the remaining part is
screened by dyon condensate. Finally, for the third $SU(3)$ $\alpha_{12}$-monopole one gets from (6.8), that

$$\frac{1}{\sqrt{2}}(0 \oplus \alpha_{12}) = (0, 1; 0, 0) = n_{S_1} - n_{S_2} + \frac{1}{2}(n^1 - n^2)$$

(6.9)

and we find that it is also confined, being a junction of two elementary strings $S_1$ and $S_2$.

We see that although the quark charges change as we pass from the weak to strong coupling domains, and they become dyons, this does not happen to the monopoles: the monopole states do not change their charges, and they are confined by elementary strings in both domains at large and small $m$. However, inside the rhombus in dual theory there is a peculiarity: the monopole flux is only partially carried by attached string, and the remaining part of it is screened by the dyon condensate.

Hence, our result provides an explicit counterexample to the commonly accepted belief, that if monopoles are confined in the original theory, then quarks should be confined in the dual theory. We have demonstrated above however, that it is monopoles monopoles rather than quarks are confined in the strong coupling domain at small $m$. Similar results are obtained in [14] for $\mathcal{N} = 2$ supersymmetric $\tilde{U}(N)$ QCD with number of flavors $N_f < 2N$.

7 Conclusions

In this paper we have answered to the long-standing question, what happens at strong coupling to the confinement of monopoles in supersymmetric QCD, caused by the flux tubes in the effective theory around $r = 2$ quark vacua [1]. In order to do this, the picture from the original theory at large values of quark masses has been moved into the domain of small masses, or strongly coupled original theory. Fortunately, using the exact Seiberg-Witten solution to the $\mathcal{N} = 2$ supersymmetric gauge theories, one can go beyond the semiclassical approximation and study at least, what happens to the quantum numbers of the light fields.

We have considered in this paper the supersymmetric QCD with large number of flavors (masses some of them were taken coinciding for simplicity) and demonstrated, that when going towards the domain of small masses the light quarks acquire magnetic charges. This can be seen by careful study of permutation of the branch points and the period integrals on (almost) singular curves in the vicinity of colliding vacua. As a result, in the domain of small masses one deals with the effective theory of light dyons.

When masses of the original condensed quarks coincide, the non-Abelian gauge symmetry is partially restored. The exact form of the Seiberg-Witten curves shows, that the same phenomenon occurs in the dual effective theory of dyons, i.e. at equal values of masses it becomes non-Abelian. As a consequence, the string or flux tube solutions in the effective theory acquire the non-Abelian structure, quite in the same way, as in the original theory, and moreover, the spectrum of the confined objects is again non-Abelian.

The charges of confined objects are determined by string fluxes and charges of the condensates. The analysis of the last section shows, that despite the condensate in the dual theory is formed by dyons and its charge contains the magnetic component, the charges of the confined objects, i.e. those parts of string’s fluxes, not being screened by the condensate, are still
magnetic, and not electric. The generic reason for this is distinction between the weight-quark charges, and the root-monopole ones. Coming to the strongly coupled domain, a quark can acquire the monopole charge, but cannot get rid of the original electric weight-vector. Hence, the quarks cannot be confined by strings, formed due to the condensation of dyons, which possess the quark electric charges in addition to the magnetic ones: the states, confined by these strings, are still monopoles.

Of course, there are $r = 0$ vacua in our theory where the monopoles (or dyons with root electric charges) condense, and electric flux tubes are formed. This triggers confinement of quarks according to Seiberg-Witten scenario \[2\]. However, this provides a model with only Abelian confinement. As we have shown in this paper, if being interested in the non-Abelian confinement, which occurs due to the formation of non-Abelian strings, we ultimately end up with the confinement of monopoles. This conclusion is in accordance with similar results, obtained in \[14\] for the $\mathcal{N} = 2$ supersymmetric $U(N)$ theory with $N_f < 2N$.

Although $\mathcal{N} = 2$ supersymmetric QCD does not looks like the real world QCD, the supersymmetric gauge theories can be considered as certain "theoretical laboratory" for studying the properties of realistic confinement. One can be really amused, how the strongly coupled region can be described in terms of the dual theory, and the properties of all objects we are interested in are under the full control. Hence, studying the different phases of supersymmetric QCD provides an important experience - what can in principle happen in gauge theory at strong coupling, and is therefore very useful not only in the context of mathematical physics, but enriching our intuition before attacking the problems of the real world.

Acknowledgments

One of us (AY) is grateful to A. Gorsky and M. Shifman for useful discussions.

The work of AM was supported by the Russian Federal Nuclear Energy Agency, by Grant of Support for Scientific Schools LSS-1615.2008.2, by the RFBR grants 08-01-00667, 09-02-90493-Ukr, 09-02-93105-CNRSL, 09-01-92440-CE, by Kyoto University, and by the Dynasty Foundation. The work of AY was supported by FTPI, University of Minnesota, by the RFBR Grant 09-02-00457a and by Grant of Support for Scientific Schools LSS-11242003.2. AM thanks IHES at Bures-sur-Yvette, where a part of this work had been done, and the Yukawa Institute for Theoretical Physics, where the work has been completed.

Appendix

A Classical picture in different gauges

Here we present for completeness the explicit pictures for the condensate charges in two different gauges. In addition to the gauge (2.4) we use throughout the main text, we relate it to the ”global” or ”real” gauge, where all eigenvalues of $\Phi = \text{diag}(\phi_1, \phi_2, \phi_3)$ are chosen to be real and ordered, say $\phi_1 < \phi_2 < \phi_3$. We always take the 12-vacuum, with condensed two first flavors
Figure 7: Classical picture for the 12-vacuum in the mass $(m_1, m_2)$-plane $(M \equiv m_1 + m_2)$, for the gauge, when the eigenvalues of $\Phi$ are real and ordered $\phi_1 < \phi_2 < \phi_3$. The diagonal part of the $\Phi$-matrix at each corner, and the direction of each $A = 1, 2$ condensed flavor are presented explicitly. The long root vector at each line shows also the color-direction of each restored SU(2)-subgroup in this gauge.

with the bare masses $m_A$, $A = 1, 2$.

The directions of the condensed flavors in color space are depicted in the global gauge directly at each sector, as well as the form of the matrix $\Phi$, which is different from (2.4) just by particular permutation of the eigenvalues. On the straight lines one gets a restoration of SU(2)-subgroup, again in each case its direction in the color space is given by a root, presented explicitly at fig. 7.

The dashed line differs from the solid ones by the fact that both quarks on each side of it are charged w.r.t. the restored SU(2), and passing through this line just exchanges the colors of two condensed flavors, which can be seen in semiclassical regime. In contrast to that, on each solid line, only one of the quark flavors is charged w.r.t. restored SU(2) group, and it flips it direction in the color space, while the other one is remained intact. This process goes in regime beyond the semiclassical approximation, and is governed by the Seiberg-Witten mechanism [2, 3]. In the first case, on the dashed lines, flipping of the quark’s charge is caused by the massless $(1, 0)$ W-boson in the restored SU(2), while on the dashed lines it is caused by the massless $(1, 1) = [2, 1]$ dyon.

The picture from fig. 7 can be gauge transformed to fix the matrix $\Phi$ being always of the form (2.4), in the whole $(m_1, m_2)$-plane. The result is presented on fig. 8, and we see that all vectors in the color space are just rotated, consistently for the weights of quarks and roots of the restored SU(2) subgroups. For the sake of simplicity (e.g. all the restored SU(2)-subgroups have the same color direction on different branches of the same solid line) we use the gauge (2.4) and that of fig. 8 in the main text, though the consistency between the different sectors of
Figure 8: Classical picture for the 12-vacuum in the mass $(m_1, m_2)$-plane for the gauge, when matrix $\Phi$ is always of the form (2.4). The direction of each $A = 1, 2$ condensed flavor then remains the same in each sector, while the color-direction of each restored $SU(2)$-subgroup are rotated correspondingly.

the weakly-coupled theory and the charges of Seiberg-Witten dyons is better seen in the global gauge, i.e. at fig. 7.

B Period integrals on degenerate curves

In the basic example of $N_c = 2$, $N_f = 2$ theory with the coinciding masses $m_1 = m_2 = m$ one gets for (4.1)

$$y^2 = (x^2 - u)^2 - 4\Lambda^2(x + m_1)(x + m_2) = (x^2 - u)^2 - 4\Lambda^2(x + m)^2$$  \hspace{1cm} (B.1)

and the generating differential (4.5) turns for the pairwise coinciding masses into

$$dS \sim \frac{xdP}{y} - \frac{x}{2y} \frac{dQ}{Q} + \frac{1}{2} x \frac{dP}{y} - x \frac{dq}{q} + x \frac{dq}{q}$$

$$q(x) = \prod_{B=1}^{N_f/2} (x + m_B)$$  \hspace{1cm} (B.2)

where we have chosen $m_{B+N_f/2} = m_B$, $B = 1, \ldots, N_f/2$ for even number of flavors $N_f$. The curve (B.1) just corresponds to $P(x) = x^2 - u$ and $q(x) = x + m$.

When exactly at quark vacuum $u = u_Q = m^2$, the curve (B.1) degenerates further to

$$y^2 = (x + m)^2 ((x - m)^2 - 4\Lambda^2) \equiv (x + m)^2 Y^2$$

$$Y^2 = (x - m)^2 - 4\Lambda^2$$  \hspace{1cm} (B.3)
and the Seiberg-Witten differential (4.5) turns into
\[ dS = \frac{xdx}{Y} + \frac{xdx}{x+m} \]  
(4.4)

Due to (4.6) and to the fact, that on degenerate curve the position of the mass pole at \( x = -m \) (for both flavors) coincides with the degenerate cut, the differential (4.4) is normalized by\(^5\)
\[
\frac{1}{2\pi i} \oint_{x=-m} dS_+ = \frac{1}{2\pi i} \oint_{A^+} dS_+ = a = -m
\]  
(4.5)

\[
\frac{1}{2\pi i} \oint_{x=-m} dS_- = \text{res}_{x=-m} dS_- - \frac{1}{2\pi i} \oint_{A^-} dS_- = -2m + a = -m
\]

obviously true for (4.4). Since the curve (B.3) is rational, the differential (B.4) can be easily integrated, giving rise to
\[
S = Y + m \log(x - m + Y) + x - m \log(x + m)
\]  
(4.6)

In order to compute the desired \( B \)-period (the monopole mass), one has to take the difference \( S_+|_{x=-m} - S_-|_{x=-m} \) of the values of (4.6) on two different sheets of the Riemann surface (B.3).

This is not possible to do by direct substitution of \( x = -m \) into (4.6) due to the logarithmic singularity, i.e. the curve (B.3) is "too degenerate". Let us then regularize it and denote the distance between the position of the pole and the nearest end of the shrinking cut by \( \epsilon^\pm \), dependently on the sheet \( Y = Y_\pm \) of (B.3). The values of these \( \epsilon^\pm = \epsilon^\pm(m, \Lambda) \) can be determined as follows (see e.g. [34]): the differential
\[
d\phi = \frac{dw}{w} \quad \text{(B.3)} \quad \frac{dx}{Y} + \frac{dx}{x+m}
\]  
(4.7)

should have constant periods [40], moreover, its \( B \)-periods on (B.3) can be just chosen vanishing. Integrating (4.7) up to
\[
\phi = \log(x - m + Y) + \log(x + m)
\]  
(4.8)

and putting \( \phi_+|_{x=-m} - \phi_-|_{x=-m} \equiv \phi_+|_{x=-m+\epsilon^+} - \phi_-|_{x=-m+\epsilon^-} = 0 \), one gets (at \( \epsilon^\pm \to 0 \))
\[
\log \frac{\epsilon^+}{\epsilon^-} = \log \frac{m + \sqrt{m^2 - \Lambda^2}}{m - \sqrt{m^2 - \Lambda^2}}
\]  
(4.9)

Therefore
\[
S_+|_{x=-m} - S_-|_{x=-m} = -m \log \frac{\epsilon^+}{\epsilon^-} + 2 \frac{Y|_{x=-m}}{x=-m} + m \log \frac{-2m + Y|_{x=-m}}{-2m - Y|_{x=-m}} =
\]
\[= 4\sqrt{m^2 - \Lambda^2} + 2m \log \frac{m - \sqrt{m^2 - \Lambda^2}}{m + \sqrt{m^2 - \Lambda^2}} \]  
(4.10)

\(^5\)Here again the "homological" normalization of the charges and periods is used, so that the quark’s quantum numbers are \((n_e, 0) = (\frac{1}{2}, 0) = [1, 0]. \) To get the “physical normalization” \( a \), one should just renormalize in (4.5) (and in (B.12) below) \( a = n_e \cdot \hat{a} = \frac{1}{2}a. \)
Figure 9: Permutation of the branch points at \( t \sim 0 \). The \( A \)-cycle "catches" the \( B \)-cycle and turns into \( A + B \), which means that massless \([1, 0]\) quark emits the massless anti-monopole and turns into the massless dyon \( \tilde{a} + m + a^D \to 0 \).

Hence, evaluating \( B \)-period on degenerated curve (B.3) gives rise to explicit formula (cf. the result with [41, 42])

\[
\frac{1}{2\pi i} \left( S_+|_{x=-m} - S_-|_{x=-m} \right) = -\frac{i}{\pi} \left( 2\sqrt{m^2 - \Lambda^2} + m \log \frac{m - \sqrt{m^2 - \Lambda^2}}{m + \sqrt{m^2 - \Lambda^2}} \right)
\]

(C.11)

showing, that \( \text{Im}(a_D)|_{m=\pm\Lambda} = 0 \). However, one should also consider carefully the real part of (B.11), taking into account the logarithmic cut. We fix it to vanish at \( m = \Lambda \), then

\[
\text{Re}(a_D)|_{m=\Lambda} = 0
\]

\[
\text{Re}(a_D)|_{m=-\Lambda} = 2m = -2a = -a
\]

(C.12)

It means, that when quark singularity \( u_Q \) collides with \( u_M \) we have massless monopole with \( |a_D| = 0 \), while when \( u_Q \) collides with \( u_D \), one gets the vanishing mass of the \([1, 1] = [2, 1]\) dyon, \( |a_D + 2a| = |a_D + a| = 0 \). Quite in a similar way, using the degenerate curves (4.13), (4.36), (4.37) and corresponding limit of the Seiberg-Witten differential (4.5) one computes the periods (4.20), (4.22) and (4.44).

C Permutation of the branching points on SU(2) curve

Let us now turn to the issue of permutation of the branching points of the "basic" Seiberg-Witten curve (B.1) in the vicinity of its degenerate form (B.3). Obviously one can rewrite (B.1) as

\[
y^2 = [x^2 - u - 2\Lambda(x + m)][x^2 - u + 2\Lambda(x + m)] = \]

(C.13)

\[
= [z^2 - 2(m + \Lambda)z + \delta][z^2 - 2(m - \Lambda)z + \delta]
\]
where the co-ordinate \( x = -m + z \) is just shifted, and the distance \( \delta \) is introduced

\[
u = m^2 - \delta, \quad \delta > 0
\]  

(C.14)

so that for \( \delta \to 0 \) one gets back to (B.3). Introducing also

\[
t = m - \Lambda
\]  

(C.15)

the curve (C.13) acquires the form

\[
y^2 = \left[ z^2 - 2(2\Lambda + t)z + \delta \right] \left[ z^2 - 2t z + \delta \right]
\]  

\[\approx \left| \tilde{t} \right| \ll \Lambda \left[ z^2 - 2\Lambda z + \delta \right] \left[ z^2 - 2t z + \delta \right]
\]  

(C.16)

convenient to study around the monopole singularity, where the r.h.s. reflects what happens, when we approach the region \( m \sim \Lambda \) and \( t \ll \Lambda \). The roots of (C.16) are at

\[
z_0 \simeq \frac{\delta}{4\Lambda} \approx 0, \quad z_{\Lambda} \simeq 4\Lambda \to \infty
\]  

and

\[
z_\pm = t \pm \sqrt{\tilde{t}^2 - \delta}
\]  

(C.17)

If \( \tilde{t}^2 \gg \delta \) and \( t \) is real positive, the obvious ordering is

\[
(z_0 \simeq \frac{\delta}{4\Lambda} \approx 0) < (z_- \simeq \frac{\delta}{2\tilde{t}}) < (z_+ \simeq 2\tilde{t}) < (z_{\Lambda} \simeq 4\Lambda \approx \infty)
\]  

(C.18)

so that the natural choice for the \( A \)-cycle is around \( z_0 \) and \( z_- \), while for the dual \( B \)-cycle from \( z_- \) to \( z_+ \), see fig. 9. At positive \( t \) the \( A \)-cycle, drawn as a contour around \( z_0 \) and \( z_- \) at fig. 9, is “small” i.e. almost shrinks. However at negative \( t \), the roots \( z_+ \) and \( z_0 \) become close to each other, and the shrinking contour is now \( A + B \), cf. with sect. 4.3. Thus, the quark picks up the monopole charge and transforms into the massless dyon with the mass \( |a + a_D + m| = |\frac{1}{2}a + a_D + m| = 0 \).

To study another interesting domain one needs to introduce

\[
\tilde{t} = m + \Lambda
\]  

(C.19)

instead of (C.15) and consider the curve (C.13) for \( \tilde{t} \ll \Lambda \), i.e. as

\[
y^2 = \left[ z^2 - 2\tilde{t} z + \delta \right] \left[ z^2 - 2(\tilde{t} - 4\Lambda) z + \delta \right] \simeq \left| \tilde{t} \right| \ll \Lambda \left[ z^2 - 2\tilde{t} z + \delta \right] \left[ z^2 - 2\tilde{t} z + \delta \right]
\]  

(C.20)

which is basically just a result of replacement \( t \to 4\Lambda - \tilde{t} \). Instead of (C.17) one gets the roots

\[
z_+ \mapsto \tilde{z}_0 \simeq -\frac{\delta}{4\Lambda} \approx 0, \quad z_- \mapsto \tilde{z}_{\Lambda} \approx -4\Lambda
\]  

(C.21)

\[
z_0 \simeq \frac{\delta}{4\Lambda} \approx 0 \mapsto \tilde{z}_-, \quad z_{\Lambda} \approx 4\Lambda \mapsto \tilde{z}_+
\]  

where

\[
\tilde{z}_\pm = \tilde{t} \pm \sqrt{\tilde{t}^2 - \delta}
\]  

(C.22)
Figure 10: Exchange of roots in the regime $\tilde{t} \sim 0$. The "small" cycle $A+B$ (the upper picture), encircling $\tilde{z}_0$ and $\tilde{z}_-$, is combined with the degenerate $2A+B$ at $\tilde{z}_+ = \tilde{z}_-$ (taken with the opposite sign), and they form again the “quark cycle”.

The "small" $A+B$ cycle is now surrounding the points $z_+ \mapsto \tilde{z}_0$ and $z_0 \mapsto \tilde{z}_-$, and at $\tilde{t} \gg \sqrt{\delta}$ we have the natural ordering

$$
(\tilde{z}_A \simeq -4\Lambda \approx -\infty) < (\tilde{z}_0 \simeq -\frac{\delta}{4\Lambda} \approx 0) < (\tilde{z}_- \simeq \frac{\delta}{2\tilde{t}}) < (\tilde{z}_+ \simeq 2\tilde{t})
$$

As we reduce $\tilde{t}$ the dyon cycle $2A+B$ becomes also degenerate. At negative $\tilde{t}$ the roots $\tilde{z}_0$ and $\tilde{z}_+$ become close to each other, and the contour $(A+B) - (2A+B) = -A$ almost shrinks, see fig. 10. This gives again a quark cycle, corresponding to the massless quark.

References

[1] A. Marshakov and A. Yung, Nucl. Phys. B647 (2002) 3 [arXiv:hep-th/0202172].
[2] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, [arXiv:hep-th/9407087].
[3] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484, [arXiv:hep-th/9408099].
[4] P. Argyres, M. Plesser and N. Seiberg, Nucl. Phys. B471 (1996) 159, [arXiv:hep-th/9603042].
[5] G. Carlino, K. Konishi and H. Murayama, Nucl. Phys. B590 (2000) 137, [arXiv:hep-th/0005076].
[6] A. Hanany and D. Tong, JHEP 0307, 037 (2003) [arXiv:hep-th/0306150].
[7] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B673, 187 (2003) [arXiv:hep-th/0307287].

[8] M. Shifman and A. Yung, Phys. Rev. D70, 045004 (2004) [arXiv:hep-th/0403149].

[9] A. Hanany and D. Tong, JHEP 0404, 066 (2004) [arXiv:hep-th/0403158].

[10] D. Tong, TASI Lectures on Solitons, arXiv:hep-th/0509216.

[11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A 39, R315 (2006) [arXiv:hep-th/0602170].

[12] M. Shifman and A. Yung, Rev. Mod. Phys. 79 1139 (2007) [arXiv:hep-th/0703267]; Supersymmetric Solitons, Cambridge University Press, 2009.

[13] D. Tong Quantum Vortex Strings: A Review, arXiv:0809.5060 [hep-th].

[14] M. Shifman and A. Yung, Phys. Rev. D79, 125012 (2009) [arXiv:0904.1035 [hep-th]].

[15] M. Shifman and A. Yung, Phys. Rev. D 79, 105006 (2009) arXiv:0901.4144 [hep-th].

[16] N. Seiberg, Nucl. Phys. B435, 129 (1995) [arXiv:hep-th/9411149].

[17] K. A. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. 45BC, 1 (1996) [arXiv:hep-th/9509066].

[18] A. Hanany, M. J. Strassler and A. Zaffaroni, Nucl. Phys. B513, 87 (1998) [hep-th/9707244].

[19] A. Vainshtein and A. Yung, Nucl. Phys. B614, 3 (2001) [arXiv:hep-th/0012250].

[20] V. Markov, A. Marshakov and A. Yung, Nucl. Phys. B709 (2005) 267 [arXiv:hep-th/0408235].

[21] A. Achucarro and T. Vachaspati, Phys. Rept. 327, 347 (2000) [arXiv:hep-ph/9904229].

[22] M. Shifman and A. Yung, Phys. Rev. D73, 125012 (2006) [arXiv:hep-th/0603134].

[23] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Phys. Rev. D76, 105002 (2007) [arXiv:0704.2218 [hep-th]].

[24] K. Evlampiev and A. Yung, Nucl. Phys. B662 (2003) 120 [hep-th/0303047].

[25] N. Dorey, T. Hollowood, V. Khoze and M. Mattis, Phys. Rept. 371 (2002) 231-459, arXiv:hep-th/0206063

[26] T. Grimm, A. Klemm, M. Marino and M. Weiss, JHEP 0708 (2007) 058 [arXiv:hep-th/0702187].

[27] A. Marshakov, A. Mironov and A. Morozov, JHEP 0911 (2009) 048 [arXiv:0909.3338 [hep-th]].
[28] A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 283 [arXiv:hep-th/9505075].

[29] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74, 3931 (1995) [arXiv:hep-th/9411057].

[30] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B344, 169 (1995) [arXiv:hep-th/9411048].

[31] P. Argyres, M. Plesser, and A. Shapere, Phys. Rev. Lett. 75, 1699 (1995) [arXiv:hep-th/9505100].

[32] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, Phys.Lett., B380 (1996) 75-80, [arXiv:hep-th/9603140].

[33] A. Losev, A. Marshakov and N. Nekrasov, in Ian Kogan memorial volume From fields to strings: circumnavigating theoretical physics, 581-621; [arXiv:hep-th/0302191].

[34] A. Marshakov and N. Nekrasov, JHEP 0701 (2007) 104, hep-th/0612019; A. Marshakov, Theor.Math.Phys. 154 (2008) 362 [arXiv:0706.2857[hep-th]].

[35] P. C. Argyres and M. R. Douglas, Nucl. Phys. B448, 93 (1995) [arXiv:hep-th/9505062].

[36] P. Argyres, M. Plesser, N. Seiberg, and E. Witten, Nucl. Phys. B461, 71 (1996) [arXiv:hep-th/9511154].

[37] A. Bilal and F. Ferrari, Nucl. Phys. B516, 175 (1998) [arXiv:hep-th/9706145].

[38] M. Shifman, A. Vainshtein and R. Zwicky, J. Phys. A 39, 13005 (2006) hep-th/0602004.

[39] G. ’t Hooft, in 1981 Cargé`se Summer School Lecture Notes on Fundamental Interactions, NATO Adv. Study Inst. Series B: Phys., Vol. 85, ed. M. Lévy et al. (Plenum Press, New York, 1982) [reprinted in G. ’t Hooft, Under the Spell of the Gauge Principle (World Scientific, Singapore, 1994), page 514]; Nucl. Phys. B190, 455 (1981); see also J. Cardy and E. Rabinovici, Nucl. Phys. B205, 1 (1982); J. Cardy, Nucl. Phys. B205, 17 (1982).

[40] I. Krichever, Commun. Pure. Appl. Math. 47 (1992) 437 [arXiv: hep-th/9205110].

[41] N. Dorey, JHEP 9811 (1998) 005 [arXiv:hep-th/9806056]; N. Dorey, T. Hollowood and D. Tong, JHEP 9905 (1999) 006 [arXiv:hep-th/9902134].

[42] A. Hanany and K. Hori, Nucl. Phys. B513, 119 (1998) [arXiv:hep-th/9707192].