On Some Integrals Over the Product of Three Legendre Functions

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Abstract

The definite integrals \( \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_v(x)^3 \, dx \), \( \int_{-1}^{1} x(1-x^2)^{(v-1)/2} P_v(x) P_{v+1}(x)^2 \, dx \), and \( \int_{-1}^{1} x(1-x^2)^{(v-1)/2} P_v(x)^2 P_{v+1}(x) \, dx \) are evaluated in closed form, where \( P_v \) is the Legendre function of degree \( v \), and \( \text{Re} \, v > -1 \). Special cases of these formulae are related to certain integrals over elliptic integrals that have arithmetic interest.

Keywords: Legendre functions, multiple elliptic integrals, elliptic integrals as hypergeometric functions, asymptotic expansions, multiscale methods, finite Hilbert transforms

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1 Introduction

In this brief note, we present analytic proofs of the following integral formulae for \( \text{Re} \, v > -1 \):

\[
\int_{-1}^{1} \frac{(P_v(x))^3}{(1-x^2)^{(v-1)/2}} \, dx = [3 - 2 \cos(v \pi)] \int_{-1}^{1} \frac{(P_v(x))^2 P_v(-x)}{(1-x^2)^{(v-1)/2}} \, dx = \frac{3 - 2 \cos(v \pi)}{\pi} \left( \frac{\cos \frac{v \pi}{2}}{2^v} \right)^3 \left( \frac{\Gamma(1+v/2)}{\Gamma(1 + v/2)} \right)^4 ;
\]

\[
\int_{-1}^{1} \frac{x(P_{v+1}(x))^3}{(1-x^2)^{(v-1)/2}} \, dx = [2 \cos(v \pi) - 3] \int_{-1}^{1} \frac{x(P_{v+1}(x))^2 P_{v+1}(-x)}{(1-x^2)^{(v-1)/2}} \, dx = \frac{3 - 2 \cos(v \pi)}{8 \pi} \left( \frac{\cos \frac{v \pi}{2}}{2^v} \right)^3 \left( \frac{\Gamma(1+v/2)}{\Gamma(1 + v/2)} \right)^4 .
\]

As the special cases of Legendre functions \( P_{-1/2}, P_{-1/3}, P_{-2/3}, P_{-1/4}, P_{-3/4} \) and \( P_{-1/6}, P_{-5/6} \) are related to the complete elliptic integrals of the first kind \( K(k) = \int_0^\pi (1 - k^2 \sin^2 \theta)^{-1/2} \, d\theta \) [Ref. [1], Chap. 33], the evaluation in Eq. (1.1) allows us to compute certain challenging integrals over elliptic integrals.

While a recent manuscript by M. Rogers, J. G. Wan and I. J. Zucker [2] was in preparation, one of the authors (J. G. Wan) sent me a draft that contained a proof for the evaluation

\[
\int_{0}^{1} \frac{[K(\sqrt{1-k^2})]^3}{\sqrt{k(1-k^2)^3/4}} \, dk = \frac{3(\Gamma(1/4))^8}{32\sqrt{2\pi}^2},
\]

and asked me if there is an analytic verification for another identity they discovered via numerical experiments:

\[
\int_{0}^{1} \frac{[K(\sqrt{1-k^2})]^2 K(k)}{\sqrt{k(1-k^2)^3/4}} \, dk \geq \frac{\Gamma(1/4)^8}{32\sqrt{2\pi}^2}.
\]

On the same day of correspondence (Feb. 21, 2013), I wrote back my deduction of Eq. (1.4) from Eq. (1.3) along with a generalization to Legendre functions of arbitrary degrees (i.e. the first equality in Eq. (1.1)). My proof is reproduced below as Proposition 2.1 verbatim, in the form as was communicated to the authors of [2]. Later on, I realized that one can evaluate the integral in Eq. (1.4) without the prior knowledge of Eq. (1.3) drawing on the \( v = -1/2 \) scenario of the second equality in Eq. (1.1). The computation of the integral \( \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_v(x)^2 P_{v+1}(x) \, dx, \text{Re} \, v > -1 \) is elaborated in Proposition 3.1, where the connections between Eqs. (1.1) and (1.2) are also revealed.

The proofs in this note build on some spherical harmonic techniques, Tricomi transform identities, and Hansen-Heine type scaling analysis developed in [3], which are independent of the lattice sum methods in [2]. In the current version of [2], the authors have announced the availability of a proof for Eq. (1.4) based on modular forms, which will appear elsewhere [4]. Their arithmetic proof will draw on [3] and will generalize Eq. (1.4) along another direction. Notwithstanding sharp differences in our methods and motivations, I wish to express my sincere gratitude to the authors of [2] for their inspirational work and friendly communications.

1

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2 An Application of Tricomi Pairing

Proposition 2.1 We have the following evaluations:

\[
\frac{(1/2)^{3}}{32\sqrt{2}\pi^{2}} = \frac{1}{3} \int_{0}^{1} \frac{[K(\sqrt{1-k^{2}})^{2}]^{2}K(k)}{\sqrt{1-k^{2}}^{3/4}}dk = \frac{1}{3} \int_{0}^{1} \frac{[K(\sqrt{1-k^{2}})^{2}]^{2}K(k)}{\sqrt{1-k^{2}}^{3/4}}dk = \frac{1}{3} \int_{0}^{1} \frac{[K(\sqrt{1-k^{2}})^{2}]^{2}K(k)}{\sqrt{1-k^{2}}^{3/4}}dk
\]

which is a special case of

\[
\int_{-1}^{1} \frac{(P_{\nu}(x))^{3}}{[1-(1-x^{2})^{1/2}]}dx = [3-2\cos(n\pi)] \int_{-1}^{1} \frac{(P_{\nu}(x))^{2}P_{\nu}(-x)}{(1-x^{2})^{1/2}}dx, \quad \text{Re}\nu > -1,
\]

where \( P_{\nu}(1-2x) = \frac{2}{\pi}F_{1}\left[\frac{-\nu+1}{2},1|x\right] \) stands for the Legendre function of the first kind of degree \( \nu \). (One has \( P_{-1/2}(x) = \frac{\pi}{2}\sqrt{1-x^{2}} \) for \( -1 < x < 1 \).)

Proof We shall prove Eq. (2.2) for \( -1 < \nu < 0 \). The rest of our claims will then follow from analytic continuation and the special case where \( \nu = -1/2 \).

We first recall the following Cauchy principal values involving fractional degree Legendre functions [Ref. [3], Propositions 4.2 and 4.4]:

\[
\mathcal{P} \int_{-1}^{1} \frac{(1 + \zeta)^{n}P_{\nu}(-\zeta)}{\pi(x-\zeta)^{2/3}} d\zeta = \frac{(1+x)^{n-1}Q_{\nu}(x)}{\pi(1-x)^{a}}, \quad -1 < x < 1, a, n \in \mathbb{Z}_{\geq 0}, \text{Re}(\nu - n) > -1.
\]

Here,

\[
Q_{\nu}(x) := \frac{\pi}{2\sin(\nu\pi)}[\cos(\nu\pi)P_{\nu}(x) - P_{\nu}(-x)], \quad \nu \in \mathbb{C} \setminus \mathbb{Z}; \quad Q_{\nu}(x) := \lim_{\nu \to n} Q_{\nu}(x), \quad n \in \mathbb{Z}_{\geq 0}
\]

defines the Legendre functions of the second kind for \( -1 < x < 1 \). We will also need a familiar integral formula [Ref. [6], Eq. 11.336]:

\[
\mathcal{P} \int_{-1}^{1} \frac{(1 + \zeta)^{n-1}}{\pi(x-\zeta)^{2}} d\zeta = \frac{(1+x)^{n-1}Q_{\nu}(x)}{\pi(1-x)^{a}} \cot(\pi n), \quad -1 < x < 1, 0 < a < 1.
\]

We note that the Tricomi transform \( \mathcal{F} : \mathcal{L}^{p}(-1,1) \rightarrow \mathcal{L}^{p}(-1,1) \) for \( 1 < p < +\infty \), defined by

\[
(\mathcal{F} f)(x) := \mathcal{P} \int_{-1}^{1} f(\zeta) d\zeta = \int_{-1}^{1} f(\zeta) d\zeta, \quad \text{a.e. } x \in (-1,1)
\]

satisfies a Parseval-type identity [Ref. [6], Eq. 11.237]:

\[
\int_{-1}^{1} f(x)g(x)dx + \int_{-1}^{1} g(x)f(x)dx = 0,
\]

and the Hardy-Poincaré-Bertrand formula [Ref. [6], Eq. 11.52]:

\[
\mathcal{F}[f(\mathcal{F} g)] = (\mathcal{F} f)(\mathcal{F} g) - fg,
\]

for any inputs \( f \in \mathcal{L}^{p}(-1,1), p > 1; g \in \mathcal{L}^{q}(-1,1), q > 1 \) and \( \frac{1}{p} + \frac{1}{q} < 1 \).

For \( -1 < \nu < 0 \), we set \( n = 0 \) in Eq. (2.4), \( a = (1-\nu)/2 \) in Eq. (2.6) and apply Eq. (2.8) to \( f(x) = (1+x)^{\nu}P_{\nu}(x) \), \( g(x) = (1+x)^{1-\nu}/(1-x)^{1-\nu} \), which results in the following identity:

\[
\frac{\int_{-1}^{1} P_{\nu}(\zeta)\cot(\zeta)\frac{(1+\nu)}{2}Q_{\nu}(\zeta) d\zeta}{(1-\zeta^{2})^{1/2}} = \frac{2}{\pi}Q_{\nu}(x)\cot(\zeta)\frac{(1+\nu)/2}{(1-x^{2})^{1-\nu}2}, \quad -1 < x < 1.
\]
We may pair up Eqs. (2.3) and (2.9) in an application of the Parseval-type identity (Eq. (2.7)):

\[
\int_{-1}^{1} P_r(x) \cot \left( \frac{1-v}{2} \pi \right) + \frac{2}{\pi} Q_r(x) \left[ P_r(x) \right]^2 - \left( P_r(-x) \right)^2 \left( 1 - x^2 \right)^{(1-v)/2} \sin(v \pi) \, dx = -2 \int_{-1}^{1} \frac{2}{\pi} Q_r(x) \cot \left( \frac{1-v}{2} \pi \right) - P_r(x) \, P_r(-x) \, dx. \tag{2.10}
\]

As we spell out \( Q_r(x) \) using Eq. (2.5), we may reduce Eq. (2.10) into a vanishing identity

\[
\int_{-1}^{1} \frac{P_r(x) + P_r(-x)}{(1 - x^2)^{(1-v)/2}} \left[ \left( P_r(x) \right)^2 - 2[2 - \cos(v \pi)] P_r(x) P_r(-x) + [P_r(-x)]^2 \right] \, dx = 0.
\]

After we exploit the invariance of the factor \((1 - x^2)^{(1-v)/2}\) under the reflection \(x \rightarrow -x\), we may deduce Eq. (2.2) from the equation above. \( \blacksquare \)

3 Some Integrals Over the Product of Three Legendre Functions

**Proposition 3.1** (a) For \( \Re v > -1 \), we have the integral identities:

\[
\int_{-1}^{1} \frac{[P_r(x)]^2 P_r(-x)}{(1 - x^2)^{(1-v)/2}} \, dx = -\frac{8}{3} \int_{-1}^{1} \frac{x[P_{v+1}(x)]^2 P_{v+1}(-x)}{(1 - x^2)^{(1-v)/2}} \, dx, \tag{3.1}
\]

\[
\int_{-1}^{1} \frac{[P_r(x)]^2}{(1 - x^2)^{(1-v)/2}} \, dx = \frac{8}{3} \int_{-1}^{1} \frac{x[P_{v+1}(x)]^3}{(1 - x^2)^{(1-v)/2}} \, dx, \tag{3.2}
\]

and the recursion relation:

\[
\int_{-1}^{1} \frac{[P_r(x)]^2 P_r(-x)}{(1 - x^2)^{(1-v)/2}} \, dx = -\frac{64(v+2)^4}{(v+1)^4} \int_{-1}^{1} \frac{[P_{v+2}(x)]^2 P_{v+2}(-x)}{(1 - x^2)^{(1+v)/2}} \, dx. \tag{3.3}
\]

(b) We have the following evaluation:

\[
\int_{-1}^{1} \frac{[P_r(x)]^2 P_r(-x)}{(1 - x^2)^{(1-v)/2}} \, dx = \frac{1}{\pi} \left( \frac{\cos \frac{v \pi}{2}}{2^v} \right)^3 \left[ \frac{\Gamma \left( \frac{1+v}{2} \right)}{\Gamma \left( \frac{1+v}{2} \right)} \right]^4, \quad \Re v > -1. \tag{3.4}
\]

**Proof** (a) Using the standard recursion relations for Legendre functions, one can directly verify that

\[
\frac{d}{dx} [(1 - x^2)^{-(v+1)/2} P_{v+1}(x)] = (v+1)(1 - x^2)^{-(v+3)/2} P_v(x),
\]

\[
\frac{d}{dx} [(1 - x^2)^{v+1} P_v(x) P_v(-x)] = (v+1)(1 - x^2)v[P_v(x) P_{v+1}(-x) - P_v(-x) P_{v+1}(x)],
\]

so an integration by parts leads to the following identity:

\[
\Phi^r (v) := \int_{-1}^{1} (1 - x^2)^{(v-1)/2} [P_r(x)]^2 P_r(-x) \, dx = -\int_{-1}^{1} (1 - x^2)^{(v-1)/2} P_{v+1}(x)[P_r(x) P_{v+1}(-x) - P_v(-x) P_{v+1}(x)] \, dx
\]

\[
= -\int_{-1}^{1} (1 - x^2)^{(v-1)/2} P_v(x) [P_{v+1}(x) P_{v+1}(-x) - [P_{v+1}(-x)]^2] \, dx, \quad \Re v > -1. \tag{3.5}
\]

Integrating over \((1 - x^2)^{-(v+3)/2} P_v(x)\) again, and exploiting the relation

\[
\frac{d}{dx} (1 - x^2)^{v+1} P_{v+1}(x) P_{v+1}(-x) = \frac{d}{dx} [(1 - x^2)^{v+1} P_{v+1}(-x)]^2
\]

\[
= (1 - x^2)^v [2x[P_v(x)]^2 + 2x P_{v+1}(x) P_{v+1}(-x) - 2(v+2) P_{v+2}(x) P_{v+2}(-x)]
\]

\[
- (v+2) P_{v+2}(x) P_{v+1}(-x) + (v+2) P_{v+1}(x) P_{v+2}(-x),
\]

we may deduce

\[
\Phi^r (v) = \frac{4}{v+1} \int_{-1}^{1} (1 - x^2)^{(v-1)/2} x[P_{v+1}(x)]^2 P_{v+1}(-x) \, dx
\]

\[
- \frac{2(v+2)}{v+1} \int_{-1}^{1} (1 - x^2)^{(v-1)/2} P_v(x)[P_{v+1}(x) P_{v+1}(-x) - [P_{v+1}(-x)]^2] \, dx, \quad \Re v > -1 \tag{3.6}
\]
from Eq. 3.5 and a few reflection transformations \( x \mapsto -x \). Now, with the recursion relation \((2v+3)xP_{v+1}(x) = (v+2)P_{v+2}(x)+(v+1)P_v(x)\), we may take a linear combination of Eqs. 3.5 and 3.6 that eliminates \( P_v \) and \( P_{v+2} \):

\[
\Phi_L(v) = -\frac{8}{3} \int_{-1}^{1} (1-x^2)^{(v-1)/2} x[P_{v+1}(x)]^2P_{v+1}(-x) \, dx. \tag{3.7}
\]

Clearly, Eq. 3.7 entails Eq. 3.1. The proof of Eq. 3.2 is essentially similar, if not simpler:

\[
[3-2\cos(v\pi)]\Phi_L(v) := \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_v(x) \, dx = 2 \int_{-1}^{1} (1-x^2)^{(v-1)/2} xP_v(x)[P_{v+1}(x)]^2 \, dx \tag{3.5}
\]

\[
= \frac{8(v+2)}{2v+5} \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_{v+2}(x)[P_{v+1}(x)]^2 \, dx \tag{3.6}
\]

\[
= \frac{8}{3} \int_{-1}^{1} (1-x^2)^{(v-1)/2} x[P_{v+1}(x)]^3 \, dx. \tag{3.7}
\]

From Eq. 3.6, we may use integration by parts to compute

\[
[3-2\cos(v\pi)]\Phi_L(v) = \frac{4}{2v+5} \int_{-1}^{1} (1-x^2)^{(3v+5)/2} P_{v+1}(x) \frac{d}{dx}[(1-x^2)^{-v-2}P_{v+2}(x)]^2 \, dx
\]

while the same method also brings us

\[
2(v+2) \int_{-1}^{1} (1-x^2)^{(v-1)/2} x[P_{v+2}(x)]^2 P_{v+1}(x) \, dx = \int_{-1}^{1} (1-x^2)^{(3v+5)/2} xP_{v+2}(x) \frac{d}{dx}[(1-x^2)^{-v-2}P_{v+2}(x)]^2 \, dx
\]

\[
= -2(v+2) \int_{-1}^{1} x[P_{v+2}(x)]^2 P_{v+1}(x) \, dx - \int_{-1}^{1} (1-x^2)^{(v-1)/2} [1-4(v+2)x^2][P_{v+2}(x)]^3 \, dx. \tag{3.9}
\]

Therefore, a combination of Eqs. 3.3 and 3.9 results in an identity valid for \( \text{Re} v > -1 \):

\[
[3-2\cos(v\pi)]\Phi_L(v) = \frac{4(111v^2+38v+33)}{3(2v+3)(2v+5)} \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_{v+2}(x)^3 \, dx - \frac{16(2v+3)}{3(2v+5)}[3-2\cos(v\pi)]\Phi_L(v+2). \tag{3.10}
\]

Meanwhile, using the Legendre differential equation for \( P_v(x) \) and integration by parts, one can establish the following vanishing identity:

\[
0 = \int_{-1}^{1} P_v(x) \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} [(1-x^2)^{(v+1)/2} P_{v+2}(x)] \right] \, dx + \nu(v+1) \int_{-1}^{1} (1-x^2)^{(v+1)/2} P_v(x)[P_{v+2}(x)]^2 \, dx
\]

\[
+ \frac{2(2v+3)}{v+1} \int_{-1}^{1} (1-x^2) \frac{dP_v(x)}{dx} \frac{d}{dx} [(1-x^2)^{(v+1)/2} P_{v+2}(x)] \, dx - 2v(2v+3) \int_{-1}^{1} (1-x^2)^{(v+1)/2} P_v(x)[P_{v+2}(x)]^2 \, dx
\]

\[
= 2(v+2) \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_{v+1}(x)^2 [(v+2)P_{v+1}(x) - (2v+3)P_{v+2}(x)] \, dx
\]

\[
+ \frac{1}{v+1} \int_{-1}^{1} (1-x^2)^{(v-1)/2} [P_{v+1}(x)]^2 [(2v+3)^2 - 3v^2] P_{v+1}(x) + (v+2)(5v^2 + 16v + 13)P_{v+2}(x) \, dx. \tag{3.11}
\]

Here, in the last step of Eq. 3.11, we have just spelt out the integrands literally, relying on the basic recursion relation \((2\mu+1)xP_{\nu+1}(x) = (\mu+1)P_{\nu+1}(x)+\mu P_{\nu-1}(x)\) wherever necessary. With the aid of Eqs. 3.5, 3.6, and 3.9, we may recast Eq. 3.11 into the following identity for \( \text{Re} v > -1 \):

\[
[3-2\cos(v\pi)]\Phi_L(v) = \frac{2(23v^3 + 111v^2 + 177v + 93)}{3(2v+3)(2v^2 + 8v + 7)} \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_{v+2}(x)^3 \, dx
\]

\[
+ \frac{8(2v+3)(3-v^2)}{3(v+1)(2v^2 + 8v + 7)}[3-2\cos(v\pi)]\Phi_L(v+2). \tag{3.12}
\]

One can now eliminate the expression \( \int_{-1}^{1} (1-x^2)^{(v-1)/2} P_{v+2}(x)^3 \, dx \) from Eqs. 3.10 and 3.12 so as to verify the recursion relation

\[
\Phi_L(v) = -\frac{64(v+2)^4}{(v+1)^4} \Phi_L(v+2), \quad \text{Re} v > -1,
\]

as stated in Eq. 3.3.
(b) We denote the left- and right-hand sides of Eq. \(3.4\) by \(\Phi_L(\nu)\) and \(\Phi_R(\nu)\), respectively. We shall show that the functions \(\Phi_L(\nu)\) and \(\Phi_R(\nu)\) share enough common characteristics to warrant the truthfulness of the identity in Eq. \(3.4\).

Firstly, we point out that both sides of Eq. \(3.4\) agree on non-negative even integers \(\Phi_L(2m) = \Phi_R(2m), m \in \mathbb{Z}_{\geq 0}\), and \(\Phi_L(\nu) = O((\nu - n)^3)\) for each positive odd integer \(n = 2m + 1, m \in \mathbb{Z}_{\geq 0}\), so that \(\Phi_L(\nu)\) or \(\Phi_R(\nu)\) is bounded as \(\nu\) approaches any positive odd integer. Here, one may verify \(\Phi_L(0) = \Phi_R(0) = \pi\) through direct computation, and the recursion relation for \(\Phi_L(\nu)\) (Eq. \(3.3\)) consequently brings us \(\Phi_L(2m) = \Phi_R(2m), m \in \mathbb{Z}_{\geq 0}\). To show that \(\Phi_L(\nu) = O((\nu - n)^3)\) for \(n = 2m + 1, m \in \mathbb{Z}_{\geq 0}\), we compute the first- and second-order derivatives of \(\Phi_L(\nu)\) at \(\nu = n\). Bearing in mind that \(P_n(-x) = -P_n(x)\) when \(n\) is a positive odd integer, we may differentiate Eq. \(2.2\) in \(\nu\) and simplify the equation above into

\[
3 \int_{-1}^{1} \frac{[P_n(x)]^2}{1 - x^2} \frac{\partial P_n(x)}{\partial \nu} \, dx = \partial \nu \bigg|_{\nu = n} \left[ (3 - 2\cos(n\pi)) \int_{-1}^{1} \frac{[P_n(x)]^2 P_n(-x)}{1 - x^2} \, dx \right] \]

and using the Leibniz rule of differentiation, we obtain

\[
\partial^2 \nu \bigg|_{\nu = n} \int_{-1}^{1} \frac{[P_n(x)]^3}{1 - x^2} \, dx = 6 \int_{-1}^{1} \frac{P_n(x)}{1 - x^2} \left( \frac{\partial P_n(x)}{\partial \nu} \right)^2 \, dx + 3 \int_{-1}^{1} \frac{[P_n(x)]^2}{1 - x^2} \left( \frac{\partial^2 P_n(x)}{\partial \nu^2} \right) \, dx
\]

appealing again to the symmetry \(P_n(x) = -P_n(-x)\), we see that the last two lines in Eq. \(3.14\) add up to

\[
-10 \int_{-1}^{1} \frac{P_n(x)}{1 - x^2} \left( \frac{\partial P_n(x)}{\partial \nu} \right)^2 \, dx = -5 \int_{-1}^{1} \frac{[P_n(x)]^3}{1 - x^2} \, dx
\]

thus we may reach the result \(\Phi_L(\nu) = O((\nu - n)^3)\) where \(n = 2m + 1, m \in \mathbb{Z}_{\geq 0}\). In view of the triple zero at \(\nu = 1\), we may use the recursion relation for \(\Phi_L(\nu) = 64(\nu + 2)^4\Phi_L(\nu + 2)/(\nu + 1)^4\) (Eq. \(3.3\)) to analytically continue \(\Phi_L(\nu)\) as a meromorphic function for \(\nu \in \mathbb{C} \setminus \{-1, -3, -5, \ldots\}\), so that all the negative odd integers are simple poles. Based on the facts gleaned so far, we know that the ratio \(\Phi_L(\nu)/\Phi_R(\nu)\) is analytic in the whole complex \(\nu\)-plane, and \(\Phi_L(2m)/\Phi_R(2m) = 1\) for \(m \in \mathbb{Z}\).

Secondly, we show that the entire function

\[
f(\nu) := \frac{1}{\sin \frac{\pi}{2\nu}} \frac{\Phi_L(\nu)}{\Phi_R(\nu) - 1}, \quad \nu \in \mathbb{C}
\]

has at most \(O(\mathbb{C} \nu^{3/2})\) growth rate when \(\nu\) tends to infinity in a certain manner. More precisely, we will be concerned with \(\nu\) residing on a square contour \(C_N\) with vertices \(2N - \frac{1}{2} - 2iN, 2N - \frac{1}{2} + 2iN, -2N - \frac{1}{2} + 2iN,\) and \(-2N - \frac{1}{2} - 2iN\), where \(N\) is a positive integer. For \(\eta \in \mathbb{R}\), the conical function

\[
P_{-\frac{1}{2} + i\eta}(\cos \theta) = 2F_1 \left( \frac{1}{2} - i\eta, \frac{1}{2} + i\eta, \frac{1}{2}; \sin^2 \frac{\theta}{2} \right) = 1 + \sum_{k=1}^{\infty} \prod_{n=1}^{k} \left( \frac{4n^2 + 2(n - 1) - 1}{4n^2} \right) \frac{\sin 2k \theta}{2^k} \quad 0 \leq \theta < \pi
\]

is strictly positive and increasing in \(\theta\). This allows us to deduce the following a priori bound estimate for \(\Re \nu = -1/2:\)

\[
|\Phi_L(\nu)| \leq \int_{-1}^{1} \frac{[P_n(x)]^2 P_n(-x)}{1 - x^2} \, dx \leq \int_{-1}^{1} \frac{[P_n(x)]^2 P_n(-x)}{(1 - x^2)^{1/2}} \, dx \leq \int_{0}^{\pi/2} \frac{[P_n(\cos \theta)]^2 + P_n(-\cos \theta)P_n(\cos \theta)P_n(-\cos \theta)}{\sqrt{\sin \theta}} \, d\theta
\]

\[
\leq [\Phi_L(0)]^2 \int_{0}^{\pi/2} \frac{P_n(\cos \theta)}{\sqrt{\sin \theta}} \, d\theta + \int_{0}^{\pi/2} \frac{[P_n(-\cos \theta)]^2 P_n(\cos \theta)}{\sqrt{20/\pi}} \left( \frac{\pi \sin \theta}{2\theta} \right)^{3/2} \, d\theta
\]

\[
= \left[ \frac{\Gamma(\frac{5}{8})}{4\pi^2} \right]^2 \left( \frac{\sqrt{2} + 2\cos \frac{(4\nu + 1)\pi}{4}}{\Gamma(1 + \frac{3}{2})} \right)^3 \frac{\pi^2}{4} \int_{0}^{\pi/2} [P_n(-\cos \theta)]^2 P_n(\cos \theta) \left( \frac{\sin \theta}{\theta} \right)^{3/2} \, d\theta.
\]
where we have computed the integral \( \int_0^\pi P_\nu(\cos \theta)/\sqrt{\sin \theta} \, d\theta \) using a result of T. M. MacRobert [7], and quoted the standard evaluation of \( P_\nu(0) \). Next, we recall the asymptotic behavior of conical functions [Ref. 3, pp. 473-474] for \( \eta \to +\infty, \theta \in [0, \pi/2] \):

\[
P_{\frac{1}{2} + i\eta}(\cos \theta) \sim \sqrt{\frac{\theta}{\sin \theta}} I_0(\eta \theta), \quad P_{\frac{1}{2} + i\eta}(-\cos \theta) \sim \frac{2 \cosh(\eta \pi)}{\pi} \sqrt{\frac{\theta}{\sin \theta}} K_0(\eta \theta),
\]

(3.17)

where \( I_0 \) and \( K_0 \) are modified Bessel functions of zero order. This allows us to evaluate the limit

\[
\lim_{\eta \to +\infty} \eta^{3/2} \left| \Phi_R \left( -\frac{1}{2} + i\eta \right) \sin \left( \frac{2(\eta - 1)i\eta}{4} \right) \right|^2 = \lim_{\eta \to +\infty} \eta^{3/2} \left| \Phi_R \left( -\frac{1}{2} + i\eta \right) \sin \left( \frac{2(\eta - 1)i\eta}{4} \right) \right|^2
\]

\[
= \lim_{\eta \to +\infty} \eta^{3/2} \left| \cos^2(\eta \pi) \right| \frac{\int \eta^{n/2} \left| I_0(\eta \theta) (K_0(\eta \theta)) \right|^2 d\theta}{\sqrt{\pi}} = \pi \frac{\int \eta^{n/2} (K_0(\eta \theta))^2 d\theta}{\sqrt{\pi}} < +\infty.
\]

(3.18)

Combining the results in Eqs. 3.16 and 3.18 we see that \( f(\nu) = O(\text{Im} \nu)^{3/2} \) is true for \( \text{Re} \nu = -\frac{1}{2} \). As \( \Phi_L(\nu) \) and \( \Phi_R(\nu) \) share the same recursion relation, we have \( f(\nu) = -f(\nu + 2) \). Thus, the estimate \( f(\nu) = O(\text{Im} \nu)^{3/2} \) holds for the two vertical sides of the square contour \( C_N \), where \( \text{Re} \nu = -2N - \frac{1}{2} \) and \( \text{Re} \nu = 2N - \frac{1}{2} \). To estimate the growth bound on the two horizontal sides of the square contour \( C_N \), we begin with the scenarios where \( \text{Im} \nu = 2N > 10 \) and \( 1 \leq \text{Re} \nu \leq 3 \). After a slight modification of Eq. 3.16 we may put down

\[
|\Phi_L(\nu)| \leq \int_0^{\pi/2} |P_\nu(\cos \theta)^2 P_\nu(-\cos \theta)(\sin \theta)^\text{Re} \nu d\theta + \int_0^{\pi/2} |P_\nu(-\cos \theta)^2 |P_\nu(\cos \theta)(\frac{\pi \sin \theta}{2\theta})^{3/2} \theta^{\text{Re} \nu} d\theta.
\]

(3.19)

As the asymptotic formulae in Eq. 3.17 remain valid when \( i\eta \) is replaced with \( (2\nu + 1)/2 \), we may employ a scaling argument as in Eq. 3.18 to deduce

\[
\frac{1}{\sin^2(\nu \pi)} \int_0^{\pi/2} |P_\nu(-\cos \theta)^2 |P_\nu(\cos \theta)(\frac{\pi \sin \theta}{2\theta})^{3/2} \theta^{\text{Re} \nu} d\theta = O\left( \frac{1}{\text{Im} \nu^{\nu + 1}} \right), \quad \text{as Im} \nu \to +\infty, 1 \leq \text{Re} \nu \leq 3.
\]

(3.20)

Likewise, we have \( \csc(\nu \pi) \int_0^{\pi/2} |P_\nu(-\cos \theta)(\sin \theta)^\text{Re} \nu d\theta = O(|\text{Im} \nu|^{-\nu - 1}) \). By the mean value theorem for integration, there exists an acute angle \( \alpha_x, \epsilon (0, \pi/2) \) such that

\[
P_\nu(\cos \alpha_x) = \sqrt{\frac{\alpha_x}{\sin \alpha_x}} O\left( \left| I_0 \left( \frac{(2\nu + 1)\alpha_x}{2i} \right) \right| \right), \quad I_0 \left( \frac{(2\nu + 1)\alpha_x}{2i} \right) = \frac{1}{\pi} \int_0^{\pi} e^{\frac{2\nu + 1}{2\pi} \sin \theta} \cos \theta d\theta \leq e^{\pi |\text{Im} \nu|/2}
\]

and

\[
\int_0^{\pi/2} |P_\nu(\cos \theta)^2 |P_\nu(-\cos \theta)(\sin \theta)^\text{Re} \nu d\theta = |P_\nu(\cos \alpha_x)|^2 \int_0^{\pi/2} |P_\nu(-\cos \theta)(\sin \theta)^\text{Re} \nu d\theta = O\left( \frac{\sin(\nu \pi) e^{\pi |\text{Im} \nu|}}{\text{Im} \nu^{\nu + 1}} \right).
\]

(3.21)

From Eqs. 3.19, 3.21 one can confirm the bound estimate \( f(\nu) = O(|\text{Im} \nu|^{3/2}) \) for \( \text{Im} \nu = 2N > 10 \) and \( 1 \leq \text{Re} \nu \leq 3 \). By complex conjunction and recursion, one can extend this result to all the points on the the two horizontal sides of the square contour \( C_N \). This completes the task stated at the beginning of the paragraph.

Lastly, by an application of Cauchy’s integral formula, we know that the second derivative for the function \( f(\nu), \nu \in \mathbb{C} \) vanishes everywhere:

\[
f''(\nu) = \frac{1}{\pi i} \lim_{N \to -\infty} \int_{C_N} \frac{f(z)}{(z - \nu)^3} \, dz = 0, \quad \forall \nu \in \mathbb{C},
\]

so \( f(\nu) \) must be an affine function \( f(\nu) = a \nu + b \) for two constants \( a, b \in \mathbb{C} \). However, as we have the recursion \( f(\nu) = -f(\nu + 2) \), such an affine function must be identically zero. This eventually verifies Eq. 3.4.

**Remark**

During the course of deriving recursion relations for \( \Phi_L(\nu) \), we have obtained various integrals that are equal to \( \Phi_L(\nu) \) times an elementary function of \( \nu \). A sophisticated by-product of the proof above is the following:

\[
\frac{1}{\pi} \int_{-1}^{1} \left( \frac{1}{1 - x^2} \right)^{(1 - \nu^2)} \frac{1}{1 - \nu^2} \, dx = \left( \frac{3 - 2 \cos(\nu \pi)}{2} \right)^3 \frac{\Gamma(\frac{1 + \nu}{2})^4}{\left( \Gamma(\frac{1 + \nu}{2}) \right)^4}, \quad \text{Re} \nu > -1.
\]

(3.22)

Here, the first line in Eq. 3.22 can be proved in a similar vein as Eq. 2.2 except that one chooses \( n = 2 \) in the application of Eq. 2.4. The second line in Eq. 3.22 is a result of Eqs. 3.4 and 3.10.
Remark It so happens that Eq. (3.4) can be rewritten as

\[
\int_{-1}^{1} \frac{(P_{v}(x)x)^{2}P_{v}(-x)}{(1-x^{2})(1-v^{2})} \, dx = \left[ \frac{P_{v}(0)}{2^{v}} \right]^{3} \int_{-1}^{1} \frac{d x}{(1-x^{2})(1-v^{2})}. \tag{3.4}
\]

At the moment, we are not aware of a heuristic interpretation for Eq. (3.4) that is simpler than the foregoing multi-step proof of Eq. (3.4).

Combining the results from Propositions 2.1 and 3.1, we have verified Eq. (1.1) in its entirety. The special case where \( v = -1/2 \) corresponds to Eqs. (1.3) and (1.4). In the next corollary, we apply Ramanujan’s theory of elliptic functions on alternative bases [Ref. 1, Chap. 33] to Legendre functions of fractional degrees \( P_{-1/3} = P_{-2/3}, P_{-1/4} = P_{-3/4}, P_{-1/6} = P_{-5/6} \), so as to deduce closed-form evaluations of certain integrals over elliptic integrals. The ratios of gamma functions will be simplified so that only \( \Gamma(1/3), \Gamma(1/4) \) and \( \Gamma(1/5) \) are retained in the final presentation of the fractions [Ref. 9, §54].

Corollary 3.2 We have the following identities

\[
\frac{3^{7/2}}{2^{19/3}} \frac{[\Gamma(1/3)]^{12}}{\pi^{7}} = \int_{-1}^{1} \frac{(P_{-1/3}(x))^{3}}{(1-x^{2})^{3/2}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{3 \sqrt{2}(1+p+p^{2})^{3}}{(1+2p)^{11/6}(1-p^{2})^{1/2}p(2+p)^{1/3}} \left[ K \left( \sqrt{p(2+p)} \right) \right]^{3} \, dp;
\]

\[
\frac{3^{7/2}}{2^{22/3}} \frac{[\Gamma(1/3)]^{12}}{\pi^{7}} = \int_{-1}^{1} \frac{(P_{-1/3}(x))^2P_{-1/3}(-x)}{(1-x^{2})^{3/2}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{\sqrt{3}(1+p+p^{2})^{3}}{(1+2p)^{11/6}(1-p^{2})^{1/2}p^{2}(2+p)^{1/3}} \left[ K \left( \sqrt{p(2+p)} \right) \right]^{3} \, dp; \tag{3.23}
\]

\[
\frac{3^{7/2}}{2^{13/3}} \frac{[\Gamma(1/4)]^{12}}{\pi^{7}} = \int_{-1}^{1} \frac{(P_{-1/3}(x))^{3}}{(1-x^{2})^{3/6}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{\sqrt{3}(1+p+p^{2})^{4}}{(1+2p)^{11/6}(1-p^{2})^{1/2}p^{2}(2+p)^{2/3}} \left[ K \left( \sqrt{p(2+p)} \right) \right]^{3} \, dp; \tag{3.24}
\]

\[
\frac{3^{4}}{2^{19/3}} \frac{[\Gamma(1/4)]^{12}}{\pi^{7}} = \int_{-1}^{1} \frac{(P_{-1/3}(x))^{2}P_{-1/3}(-x)}{(1-x^{2})^{3/6}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{\sqrt{3}(1+p+p^{2})^{4}}{(1+2p)^{11/6}(1-p^{2})^{1/2}p^{2}(2+p)^{2/3}} \left[ K \left( \sqrt{p(2+p)} \right) \right]^{3} \, dp; \tag{3.25}
\]

\[
\frac{\sqrt{1+\sqrt{2}(1+2\sqrt{2})}}{2^{5/2}(2+\sqrt{2})^{4}} \frac{[\Gamma(1/5)]^{8}}{\pi^{3} [\Gamma(1/4)]^{4}} = \int_{-1}^{1} \frac{(P_{-1/4}(x))^{3}}{(1-x^{2})^{5/8}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{(2-t)(K(\sqrt{t}))^{3}}{(1-t)^{1/4}t^{1/4}} \, dt = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{(2-t)(K(\sqrt{t}))^{2}K(\sqrt{1-t})}{(1-t)^{1/4}t^{1/4}} \, dt; \tag{3.27}
\]

\[
\frac{(1+\sqrt{2})^{2}}{2^{5/2}(2+\sqrt{2})^{4}} \frac{[\Gamma(1/5)]^{8}}{\pi^{3} [\Gamma(1/4)]^{4}} = \int_{-1}^{1} \frac{(P_{-1/4}(x))^{2}P_{-1/4}(-x)}{(1-x^{2})^{5/8}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{(2-t)(K(\sqrt{t}))^{2}K(\sqrt{1-t})}{(1-t)^{1/4}t^{1/4}} \, dt; \tag{3.28}
\]

\[
\frac{(2-\sqrt{2})^{3/2}(3+\sqrt{2})}{2^{15/4}} \frac{[\Gamma(1/5)]^{8}}{\pi^{3} [\Gamma(1/4)]^{4}} = \int_{-1}^{1} \frac{(P_{-1/4}(x))^{3}}{(1-x^{2})^{7/8}} \, dx = \frac{8}{\pi^{3}} \int_{0}^{1} \frac{(2-t)^{2}(K(\sqrt{t}))^{3}}{(1-t)^{3/4}t^{3/4}} \, dt \tag{3.29};
\]
\[
\frac{(2 - \sqrt{3})^{3/2}}{2^{15/4}} \frac{[\Gamma(\frac{1}{3})]^8}{\pi^3[\Gamma(\frac{1}{4})]^{11/4}} = \int_{-1}^{1} \frac{(P_{-1/4}(x))^2P_{-1/4}(-x)}{(1-x^2)^{3/4}} \, dx = \frac{8}{\pi^3} \int_{0}^{1} \frac{1}{(2-t)^2[2E(\sqrt{t}) - K(\sqrt{t})]^3} \, dt
\]

\[
= \frac{8}{\pi^3} \int_{0}^{1} \frac{(1-t)^2[2E(\sqrt{t}) - K(\sqrt{t})]^3}{2(1-t)^{3/2}t^{3/4}} \, dt;
\]

\[
3^{3/2}(2 - \sqrt{3}) \frac{[\Gamma(\frac{1}{4})]^8}{\pi^5} = \int_{-1}^{1} \frac{(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{8}{\pi^3} \int_{0}^{1} \frac{1}{2^{5/6}[t(1-t)]^{1/6}} \, dt;
\]

\[
3 \frac{[\Gamma(\frac{1}{4})]^8}{2^{4}(1 + \sqrt{3}) \pi^5} = \int_{-1}^{1} \frac{(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{8}{\pi^3} \int_{0}^{1} \frac{1}{2^{3/4}(1-t + t^2)[2E(\sqrt{t}) - K(\sqrt{t})]^3} \, dt;
\]

\[
3^{3/2}(1 + \sqrt{3}) \frac{[\Gamma(\frac{1}{4})]^8}{2^{5} \pi^5} = \int_{-1}^{1} \frac{(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{8}{\pi^3} \int_{0}^{1} \frac{1}{2^{1/6}[t(1-t)]^{5/6}} \, dt,
\]

which are special cases of Eq. 1.7.

**Remark** Thanks to the contiguous relations of Legendre functions, and the differentiation formulae for the Legendre function \(P_\nu\), we may reckon that \(\nu = 1/2\) in Eq. 1.1 and \(\nu = -1/2\) in Eq. 1.2 would bring us

\[
\frac{384\pi^3}{[\Gamma(\frac{1}{4})]^8} = \int_{-1}^{1} \frac{(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{8}{\pi^3} \int_{0}^{1} \frac{\sqrt{2}2E(\sqrt{t}) - K(\sqrt{t})^3}{[t(1-t)]^{1/4}} \, dt
\]

\[
= 3 \int_{-1}^{1} \frac{x(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{24}{\pi^3} \int_{0}^{1} \frac{\sqrt{2}2E(\sqrt{t}) - K(\sqrt{t})^3}{[t(1-t)]^{1/4}} \, dt;
\]

\[
\frac{9[\Gamma(\frac{1}{4})]^8}{32\pi^5} = \int_{-1}^{1} \frac{x(P_{1/2}(x))^3}{(1-x^2)^{3/4}} \, dx = \frac{24}{\pi^3} \int_{0}^{1} \frac{(1-2t)[2E(\sqrt{t}) - K(\sqrt{t})]^3}{\sqrt{2}[t(1-t)]^{3/4}} \, dt,
\]

as one may reckon that \(P_{1/2}(1-2t) = \frac{4}{3}[2E(\sqrt{t}) - K(\sqrt{t})].\)

**Remark** It might be worth noting that the special values of gamma functions appearing in Corollary 3.2 have also arisen from certain lattice sums. It would be thus interesting to see the deduction of these integral formulae from modular forms and special values of \(L\)-functions.

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