A family of \((p, n)\)-gonal Riemann surfaces with several \((p, n)\)-gonal groups

SEBASTIÁN REYES-CAROCCA

Abstract. Let \(p \geq 3\) be a prime number and let \(n \geq 0\) be an integer such that \(p - 1\) divides \(n\). In this short note, we construct a family of \((p, n)\)-gonal Riemann surfaces of maximal genus \(2np + (p - 1)^2\) with more than one \((p, n)\)-gonal group.

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1. Introduction and statement of the result. Let \(S\) be a compact Riemann surface of genus \(g \geq 2\) and let \(\text{Aut}(S)\) denote its automorphism group. If \(p \geq 2\) is a prime number and \(n \geq 0\) is an integer, then \(S\) is called \((p, n)\)-gonal if there exists a group of automorphisms

\[ C_p \cong H \leq \text{Aut}(S) \]

such that the corresponding orbit space \(S/H\) has genus \(n\). The group \(H\) is called a \((p, n)\)-gonal group of \(S\).

Each compact Riemann surface with non-trivial automorphisms is \((p, n)\)-gonal for suitable values of \(p\) and \(n\). This simple fact shows that studying \((p, n)\)-gonal Riemann surfaces and their automorphisms is equivalent to studying the singular locus of the moduli space of compact Riemann surfaces.

\((p, n)\)-gonal Riemann surfaces and their automorphisms have been extensively considered over the last century as they generalize important and well-studied classes of Riemann surfaces, such as \((2, 0)\)-gonal or hyperelliptic, \((p, 0)\)-gonal or \(p\)-gonal, and \((2, n)\)-gonal or \(n\)-hyperelliptic Riemann surfaces, among others.

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Let $S$ be a $p$-gonal Riemann surface of genus $g \geq 2$. By the classical Castelnuovo-Severi inequality (see Accola’s book [1]), if
\[ g > (p - 1)^2, \]  
then the $p$-gonal group is unique in the automorphism group of $S$. A family of $p$-gonal Riemann surfaces of maximal genus $g = (p - 1)^2$ endowed with two $p$-gonal groups was constructed in [2], showing that the bound (1.1) is sharp. Furthermore, in the general case, following [3], if $S$ has two $p$-gonal groups, then they are conjugate in the automorphism group of $S$. An upper bound for the number of such groups was obtained in [4].

For $(p, n)$-gonality with $n \geq 1$, the Castelnuovo-Severi inequality ensures that if $S$ is a $(p, n)$-gonal Riemann surface of genus $g \geq 2$ and
\[ g > 2pn + (p - 1)^2, \]  
then the $(p, n)$-gonal group is unique in the automorphism group of $S$. In the general case, it was proved in [5] that if $S$ is a $(p, n)$-gonal Riemann surface of genus $g$ and $p > 2n + 1$, then all its $(p, n)$-gonal groups are conjugate in the automorphism group of $S$; an upper bound for the size of the corresponding conjugacy class was also determined in the same paper. Later, in [7], the uniqueness of the $(p, n)$-gonal group was proved to be true under the assumptions that the $(p, n)$-gonal group acts with fixed points and $p > 6n - 6$.

This short note is devoted to provide a family of $(p, n)$-gonal Riemann surfaces of maximal genus $g = 2pn + (p - 1)^2$ with two $(p, n)$-gonal groups. The existence of this family shows that the bound (1.2) is sharp for infinitely many pairs $(p, n)$.

**Theorem.** Let $p \geq 3$ be a prime number and let $n \geq 0$ be an integer such that $p - 1$ divides $n$. Set
\[ d = n/(p - 1) + 1. \]

Then there exists a complex $d$-dimensional family of $(p, n)$-gonal Riemann surfaces $S$ of genus
\[ g = 2np + (p - 1)^2 \]
with automorphism group of order $4p^2$ acting on $S$ with signature
\[ (0; 2, 2, 2, p, \ldots, d, p) \]
in such a way that each $S$ has more than one $(p, n)$-gonal group.

**Remark.** It is worth mentioning here the following observations which will follow from the proof of the theorem.

1. The result remains true if $p = 2$ and $n$ is odd.
2. If $n = 0$, our family agrees with the family constructed in [2].
2. Proof of the Theorem. Let $\Delta$ be a Fuchsian group of signature $(0; 2, 2, 2, p, \ldots, p)$ canonically presented as

$$\Delta = \langle \gamma_1, \ldots, \gamma_{d+3} : \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_4^p = \cdots = \gamma_{d+3}^p = \gamma_1 \cdots \gamma_{d+3} = 1 \rangle$$

and consider the group $G = D_p \times D_p$ (where $D_p$ denotes the dihedral group of order $2p$) presented in terms of generators $s_1, s_2, r_1, r_2$ and relations

$$s_1^2 = s_2^2 = r_1^2 = r_2^p = (s_1 r_1)^2 = (s_2 r_2)^2 = [s_1, r_2] = [s_1, s_2] = [r_1, r_2] = [r_1, s_2] = 1.$$

Existence of the family. By virtue of the classical Riemann existence theorem, the existence of the desired family follows after verifying that the Riemann-Hurwitz formula holds and after providing a surface-kernel epimorphism $\theta$ from $\Delta$ onto $G$.

Note that the equality

$$2(g - 1) = 4p^2(-2 + 3(1 - \frac{1}{2}) + d(1 - \frac{1}{p}))$$

shows that the Riemann-Hurwitz formula is satisfied for a branched $4p^2$-fold covering map from a compact Riemann surface of genus $g = 2np + (p - 1)^2$ onto the projective line, ramified over three values marked with 2 and $d$ values marked with $p$.

In addition, if $d$ is odd we can choose the surface-kernel epimorphism $\theta$ as

$$\theta(\gamma_1) = s_1, \quad \theta(\gamma_2) = s_2, \quad \theta(\gamma_3) = s_1 s_2 r_1 r_2, \quad \text{and} \quad \theta(\gamma_i) = \begin{cases} (r_1 r_2)^{-1} & \text{if } i \text{ is even}, \\ r_1 r_2 & \text{if } i \text{ is odd}, \end{cases}$$

and if $d$ is even, we can choose $\theta$ as

$$\theta(\gamma_1) = s_1, \quad \theta(\gamma_2) = s_2, \quad \theta(\gamma_3) = s_1 s_2 (r_1 r_2)^{-d/2}, \quad \text{and} \quad \theta(\gamma_i) = \begin{cases} r_1 & \text{if } i \text{ is even}, \\ r_2 & \text{if } i \text{ is odd}, \end{cases}$$

where $i \in \{4, \ldots, d + 3\}$.

The complex dimension of the family agrees with the complex dimension of the Teichmüller space associated to $\Delta$; namely, its dimension is $d$ (see, for example, [6]).

$(p, n)$-gonal groups. We denote the branched regular covering map given by the action of $G$ on $S$ by $\pi : S \to S/G$ and its branch values by $y_1, y_2, y_3, z_1, \ldots, z_d$, where each $y_k$ is marked with 2 and each $z_k$ is marked with $p$.

Assume that $d$ is odd. Consider the cyclic subgroups of order $p$

$$H_1 = \langle r_1 r_2 \rangle \quad \text{and} \quad H_2 = \langle r_1^{-1} r_2 \rangle$$

of $G$. We denote by $\pi_1$ and $\pi_2$ the branched regular covering maps given by the action of $H_1$ and $H_2$ on $S$ respectively. We observe that the fiber of $\pi$ over each $y_k$ does not contain any branch value of $\pi_1$ and $\pi_2$. In addition, for each $k$, the fiber of $\pi$ over $z_k$ has $4p$ elements; the isotropy group of $2p$ of them is isomorphic to $H_1$ and the remaining ones have isotropy group isomorphic to $H_2$. It follows that $\pi_1$ and $\pi_2$ ramify over $2pd$ values, each of them marked with $p$. Equivalently, the signature of the action of $H_j$ on $S$ is $(n_j; p; 2dp, p)$ where $n_j$ is the genus of $S/H_j$. We now consider the Riemann-Hurwitz formula to see that

$$2(g - 1) = p[2n_j - 2 + 2pd(1 - \frac{1}{p})]$$
and, after straightforward computations, one obtains that $n_j = n$ for $j = 1, 2$.

Assume that $d$ is even. Consider the cyclic subgroups of order $p$

$$H_1 = \langle r_1 \rangle \quad \text{and} \quad H_2 = \langle r_2 \rangle$$

of $G$ and let $\pi_1$ and $\pi_2$ be as before. As in the previous case, the fiber of $\pi$ over each $y_k$ does not contain any branch value of $\pi_1$ and $\pi_2$. For each $k$, the fiber of $\pi$ over $y_k$ has $4p$ elements; the isotropy group of them is isomorphic to $H_1$ if $k$ is odd and is isomorphic to $H_2$ if $k$ is even. It follows that $\pi_1$ and $\pi_2$ ramify over $2pd$ values, each of them marked with $p$. Equivalently, the signature of the action of $H_j$ on $S$ is $(n_j; p, 2dp, p)$ where $n_j$ is the genus of $S/H_j$. Similarly as previously done, the Riemann-Hurwitz formula ensures that $n_j = n$ for $j = 1, 2$.

In both cases, $H_1$ and $H_2$ are two $(p, n)$-gonal groups of $S$, as desired.

**Remark.** Note that if $d$ is odd, then the $(p, n)$-gonal groups are conjugate, but if $d$ even, then they are not.

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Sebastián Reyes-Carocca
Departamento de Matemática y Estadística
Universidad de La Frontera
Avenida Francisco Salazar
01145 Temuco
Chile
e-mail: sebastian.reyes@ufrontera.cl

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