Sharpest estimate of electric field from a conductive rod and application

Xiaoping Fang\(^1\) | Youjun Deng\(^2\) | Hongyu Liu\(^3\)

\(^1\) School of Mathematics and Statistics, Key Laboratory of Hunan Province for Statistical Learning and Intelligent Computation, Hunan University of Technology and Business, Changsha, China
\(^2\) School of Mathematics and Statistics, Central South University, Changsha, Hunan, China
\(^3\) Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong SAR, China

Correspondence
Youjun Deng, School of Mathematics and Statistics, Central South University, Changsha, Hunan, China.
Email: youjundeng@csu.edu.cn

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Abstract
We are concerned with the quantitative study of the electric field perturbation due to the presence of an inhomogeneous conductive rod embedded in a homogenous conductivity. We sharply quantify the dependence of the perturbed electric field on the geometry of the conductive rod. In particular, we accurately characterize the localization of the gradient field (i.e., the electric current) near the boundary of the rod where the curvature is sufficiently large. We develop layer-potential techniques in deriving the quantitative estimates and the major difficulty comes from the anisotropic geometry of the rod. The result complements and sharpens several existing studies in the literature. It also generates an interesting application in EIT (electrical impedance tomography) in determining the conductive rod by a single measurement, which is also known as the Calderón’s inverse inclusion problem in the literature.

KEYWORDS
asymptotic analysis, conductivity equation, electrical impedance tomography, Neumann–Poincaré operator, rod inclusion, single measurement
1 | INTRODUCTION

1.1 | Mathematical setup

Initially focusing on the mathematics, but not the physics, we consider the following elliptic PDE system in $\mathbb{R}^2$: 

$$\begin{aligned}
\begin{cases}
\nabla \cdot (\sigma(\mathbf{x}) \nabla u(\mathbf{x})) = 0, & \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \\
u(\mathbf{x}) - H(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}), & |\mathbf{x}| \to \infty,
\end{cases}
\end{aligned}$$

(1)

where $u(\mathbf{x}) \in H^1_{\text{loc}}(\mathbb{R}^2)$ is a potential field, and $\sigma(\mathbf{x})$ is of the following form:

$$\sigma := (\sigma_0 - 1)\chi(D) + 1, \quad \sigma_0 \in \mathbb{R}^+, \text{ and } \sigma_0 \neq 1,$$

(2)

with $D$ being a bounded domain with a connected complement $\mathbb{R}^2 \setminus \overline{D}$. $H(\mathbf{x})$ in (1) is a (nontrivial) harmonic function in $\mathbb{R}^2$, which stands for a background potential. We are mainly concerned with the quantitative properties of the solution $u(\mathbf{x})$ in (1) and in particular, its dependence on the geometry of $D$. To that end, we next introduce the rod-geometry of $D$ for our subsequent study. Let $\Gamma_0$ be a straight line of length $L \in \mathbb{R}^+$ with $\Gamma_0 = (x_1, 0), x_1 \in (-L/2, L/2)$. Let $\mathbf{n} := (0, 1)$ and define the two points $P := (-L/2, 0)$ and $Q := (L/2, 0)$. Then, the rod $D$ is introduced as $D = \overline{D^a} \cup \overline{D^f} \cup \overline{D^b}$, where $D^f$ is defined by

$$D^f := \{\mathbf{x}; \mathbf{x} = \Gamma_0 \pm \mathbf{t} \mathbf{n}, \mathbf{t} \in (-\delta, \delta)\}, \quad \delta \in \mathbb{R}^+.$$

(3)

The two end-caps $D^a$ and $D^b$ are two half disks with radius $\delta$ and centering at $P$ and $Q$, respectively. More precisely,

$$D^a = \{\mathbf{x}; |\mathbf{x} - P| < \delta, x_1 < -L/2\}, \quad D^b = \{\mathbf{x}; |\mathbf{x} - Q| < \delta, x_1 > L/2\}.$$

It can be verified that $D$ is of class $C^{1,\alpha}$ for a certain $\alpha \in \mathbb{R}^+$. In what follows, we define $S^c := \partial D^c = \partial(D^a \cup D^b)$, and $S^f := \partial D^f$. Specially, $S^f = \Gamma_1 \cup \Gamma_2$, where $\Gamma_j, j = 1, 2$ are defined by

$$\Gamma_1 = \{\mathbf{x}; \mathbf{x} = \Gamma_0 - \delta \mathbf{n}\}, \quad \Gamma_2 = \{\mathbf{x}; \mathbf{x} = \Gamma_0 + \delta \mathbf{n}\}.$$

(4)

Moreover, we always use $\mathbf{z}_x$ and $\mathbf{z}_y$ to signify the projections of $\mathbf{x} \in S^f$ and $\mathbf{y} \in S^f$ on $\Gamma_0$, respectively.

The elliptic PDE system (1) describes the perturbation of an electric field $H(\mathbf{x})$ due to the presence of a conductive body $D$. $u(\mathbf{x})$ signifies the electric potential, and $\sigma(\mathbf{x})$ signifies the conductivity of the space. The homogeneous background space possesses a conductivity being normalized to be 1, whereas the conductive rod $D$ possesses an inhomogeneous conductivity being $\sigma_0$. The perturbed electric potential is $u - H$ and the gradient field $\nabla(u - H)$ is the corresponding electric current.

1.2 | Background discussion and literature review

The conductivity equation (1) is a fundamental problem in many existing studies. It is the master equation for the electrical impedance tomography (EIT), which is an important medical imaging
modality and it can also find important applications in materials science; see, e.g., Refs. 1–5 and the references cited therein. There are rich results in the literature devoted to the quantitative properties of the solution to (1) and its geometric relationship to the conductive inclusion $D$. In this work, we derive an accurate characterization of the perturbed electric field $u - H$ and its dependence on the geometry of $D$. There are mainly two motivations for our study as described in what follows.

First, in Refs. 6–9, the authors studied the electric field perturbation from thin/slender structures, which are originated from the study of imaging crack defects in EIT.6,7,10 This is closely related to the current study. Indeed, the geometric setup in the aforementioned works are more general than the one considered in the present article. However, with the specific rod-geometry, we can derive an accurate characterization of the perturbed electric field and its geometric dependence on $D$. In fact, in our asymptotic formula of the electric field $u - H$ (with respect to $\delta \ll 1$), the leading-order term is exact and can be used to fully decode $D$. This is in a sharp contrast to the existing studies mentioned above, which inevitably involve some qualitative estimates due to the more general geometries. The rod-geometry, though special, also possesses several local features that are worth our investigation, which is the second motivation of our study as described below.

Clearly, the studies mentioned above are mainly concerned with extracting the global geometry of $\partial D$ from the perturbed electric field $u - H$. On the other hand, there are studies on the relationships between the local geometry of $\partial D$ and the perturbed field $u - H$. In fact, there are classical results concerning the singularities of the solution near a boundary corner point.11 Roughly speaking, if $\partial D$ possesses a corner, then the solution $u$ locally around the corner point can be decomposed into the sum of a regular part and a singular part. Such a qualitative property has been used to establish novel uniqueness and stability results for the Calderón inverse inclusion problem in EIT by a single partial boundary measurement.12–14 The Calderón inverse inclusion problem is a longstanding problem in the literature and we present more background discussion in Section 4.

In Ref. 15, the corner singularities of a conductive inclusion have been characterized in terms of the generalized polarization tensors associated with the electric potential $u$, and the results are directly applied to EIT. Recently, in Ref. 16, the authors consider the case that $\partial D$ is smooth, but possesses high-curvature points. In two dimensions, a high-curvature point means that the extrinsic curvature of the boundary curve $\partial D$ at that point is sufficiently large. It is shown in Ref. 16 that the quantitative property of $\nabla u$ around the high-curvature point enables one to recover the local part of $\partial D$ around that high-curvature point. However, the sharp curvature dependence of $\nabla u$ in Ref. 16 is established through numerically refining the upper-bound estimate in Ref. 17.

As mentioned before, the rod-geometry possesses a few interesting local features that consolidate the numerical study in Ref. 16. First, it is geometrically anisotropic where the two dimensions are of different scales. In fact, the curvature at the two end-caps (i.e., $S^c = \partial D^a \cup \partial D^b$) is $\delta^{-1} \geq 1$, whereas the curvature at the facade part (i.e., $S^f$) is 0. Hence, the rod-geometry, though special, provides rich insights on the curvature dependence of the electric field with respect to the shape of the conductive inclusion. In fact, we see that the perturbed electric energy is localized at the two end-caps of the rod. Similar to Ref. 16, the result enables us to rigorously justify that one can uniquely determine the conductive rod by a single measurement in EIT. It is emphasized that in three dimensions or in the case that the rod is curved, the situation would become much more complicated. Hence, we mainly consider the case with a straight rod in the two dimensions. Nevertheless, even in such a case, the mathematical analysis is technically involved and highly non-trivial. We develop layer potential techniques to tackle the problem. It turns out that the so-called Neumann-Poincaré (NP) operator plays a critical role in our analysis. We would like to mention that the NP operator and its spectral properties have received considerable attention recently in
the literature due to its important applications in several intriguing fields of mathematical physics, including plasmon resonances and invisibility cloaking.\textsuperscript{18–20} Finally, we would also like to mention in passing that more general rod-geometries were also considered in the literature in different contexts of physical importance.\textsuperscript{30–32}

The rest of the paper is organized as follows. In Section 2, we derive several auxiliary results and in Section 3, we present the main results on the quantitative analysis of the solution $u$ to (1) with respect to the geometry of the inclusion $D$. Finally, we consider in Section 4 the application of the quantitative result derived in Section 3 to Calderón’s inverse inclusion problem.

2  \qquad \textbf{SOME AUXILIARY RESULTS}

In this section, we establish several auxiliary results for our subsequent use. We first present some preliminary knowledge on the layer potential operators for solving the conductivity problem (1), and we also refer to Ref. 2, 33 for more related results and discussions.

2.1  \quad \textbf{Layer potentials}

Let $G$ be the radiating fundamental solution to the Laplacian $\Delta$ in $\mathbb{R}^2$, which is given by

$$G(x) = \frac{1}{2\pi} \ln |x|, \quad x \neq 0. \quad (5)$$

For any bounded Lipschitz domain $B \subset \mathbb{R}^2$, with connected complement, we denote by $S_B : L^2(\partial B) \to H^1(\mathbb{R}^2 \setminus \partial B)$ the single-layer potential operator given by

$$S_B[\phi](x) := \int_{\partial B} G(x - y)\phi(y) \, ds_y, \quad (6)$$

and $K_B^w : H^{-1/2}(\partial B) \to H^{-1/2}(\partial B)$ the NP operator:

$$K_B^w[\phi](x) := \text{p.v.} \frac{1}{2\pi} \int_{\partial B} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) \, ds_y, \quad (7)$$

where p.v. stands for the Cauchy principle value. In (7) and also in what follows, unless otherwise specified, $\nu$ signifies the exterior unit normal vector to the boundary of the domain concerned. It is known that the single-layer potential operator $S_B$ is continuous across $\partial B$ and satisfies the following trace formula:

$$\frac{\partial}{\partial \nu} S_B[\phi]_{\pm} = \left( \pm \frac{1}{2} I + K_B^w \right)[\phi] \quad \text{on} \quad \partial B, \quad (8)$$

where $\frac{\partial}{\partial \nu}$ stands for the normal derivative and the subscripts $\pm$ indicate the limits from outside and inside of a given inclusion $B$, respectively.

By using the layer-potential techniques, one can readily find the integral solution to (1) by

$$u = H + S_D[\varphi], \quad (9)$$
where the density function \( \varphi \in H^{-1/2}(\partial D) \) is determined by

\[
\varphi = (\lambda I - \mathcal{K}_D^a)^{-1} \left[ \frac{\partial H}{\partial v} \right]_{|\partial D}.
\]  

Here, the constant \( \lambda \) is defined by

\[
\lambda := \frac{\sigma_0 + 1}{2(\sigma_0 - 1)}.
\]

### 2.2 Asymptotic expansion of the NP operator

In what follows, we always suppose that \( \delta \ll 1 \). We present some asymptotic expansions of the NP operator with respect to \( \delta \). Recalling that \( \partial D = S^a \cup S^f \cup S^b \), we decompose the NP operator into several parts accordingly. To that end, we introduce the following boundary integral operator:

\[
\mathcal{K}_{S,S'}[\varphi](\mathbf{x}) := \chi(S') \frac{1}{2\pi} \int_S \frac{\langle x - y, v_x \rangle}{|x - y|^2} \varphi(y) ds_y, \quad \text{for} \quad S \cap S' = \emptyset. \tag{11}
\]

It is obvious that \( \mathcal{K}_{S,S'} \) is a bounded operator from \( L^2(S) \) to \( L^2(S') \). For the case \( S = S^c \), we mean \( S^c = S^a \cup S^b \). In what follows, we define \( S_1^a \) and \( S_1^b \) by

\[
S_1^a = \{ \mathbf{x}; |\mathbf{x} - P| = 1, x_1 < -L/2 \}, \quad S_1^b = \{ \mathbf{x}; |\mathbf{x} - Q| = 1, x_1 > L/2 \}. \tag{12}
\]

For the subsequent use, we also introduce the following regions:

\[
\Omega_\delta(P) := \{ \mathbf{x}; |P - z_\mathbf{x}| = O(\delta), \mathbf{x} \in S^f \}, \tag{13}
\]

\[
\Omega_\delta(Q) := \{ \mathbf{x}; |Q - z_\mathbf{x}| = O(\delta), \mathbf{x} \in S^f \}. \tag{14}
\]

Define \( \bar{\varphi}(\mathbf{x}) := \varphi(\mathbf{x}) \), where \( \mathbf{x} \in S^a, S^b \) and \( \bar{x} \in S_1^a, S_1^b \).

We can prove the following result on the asymptotic expansion of the NP operator.

**Lemma 1.** The NP operator \( \mathcal{K}_D^a \) admits the following asymptotic expansion:

\[
\mathcal{K}_D^a[\varphi](\mathbf{x}) = \mathcal{K}_0[\varphi](\mathbf{x}) + \delta \mathcal{K}_1[\varphi](\mathbf{x}) + O(\delta^2), \tag{15}
\]

where \( \mathcal{K}_0 \) is defined by

\[
\mathcal{K}_0[\varphi](\mathbf{x}) = \chi(S^a) \left( \mathcal{K}_{S^f,S^a}[\varphi](\mathbf{x}) + \frac{1}{4\pi} \int_{S_1^a} \bar{\varphi}(\mathbf{y}) ds_y \right) + \chi(S^b) \left( \mathcal{K}_{S^f,S^b}[\varphi](\mathbf{x}) + \frac{1}{4\pi} \int_{S_1^b} \bar{\varphi} \right) \tag{16}
\]

\[
+ A_{1,2,1}[\varphi] + A_{1,2}[\varphi] + \chi(\Omega_\delta(P)) \mathcal{K}_{S_1^a,S^f}[\varphi](\mathbf{x}) + \chi(\Omega_\delta(Q)) \mathcal{K}_{S_1^b,S^f}[\varphi](\mathbf{x}),
\]
and

\[ K_1[\phi] = \chi(S^b) \frac{\langle x - P, \nu_x \rangle}{2\pi |x - P|^2} \int_{S_1^a} \phi + \chi(S^a) \frac{\langle x - Q, \nu_x \rangle}{2\pi |x - Q|^2} \int_{S_1^b} \phi \]

\[ + \chi(S^f \setminus \tau_\delta(P)) \left( \frac{\delta}{|x - P|^2} \int_{S_1^a} (1 - \langle \tilde{y} - P, \nu_x \rangle) \hat{\phi}(\tilde{y}) d\tilde{y} + o \left( \frac{\delta}{|x - P|^2} \right) \right) \]

\[ + \chi(S^f \setminus \tau_\delta(Q)) \left( \frac{\delta}{|x - Q|^2} \int_{S_1^b} (1 - \langle \tilde{y} - Q, \nu_x \rangle) \hat{\phi}(\tilde{y}) d\tilde{y} + o \left( \frac{\delta}{|x - Q|^2} \right) \right). \tag{17} \]

Here, the operators \( A_{\Gamma_1, \Gamma_2} \) and \( A_{\Gamma_2, \Gamma_1} \) are defined by

\[ A_{\Gamma_1, \Gamma_2}[\phi](x) = \frac{1}{\pi} \chi(\Gamma_2) \int_{\Gamma_1} \frac{\delta}{|x - y|^2} \phi(y) dy, \]

\[ A_{\Gamma_2, \Gamma_1}[\phi](x) = \frac{1}{\pi} \chi(\Gamma_1) \int_{\Gamma_2} \frac{\delta}{|x - y|^2} \phi(y) dy. \tag{18} \]

**Proof.** First, one has the following separation:

\[ K_D^a[\phi](x) = \frac{1}{2\pi} \int_{S^f} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \phi(y) dy + \frac{1}{2\pi} \int_{S_a \cup S^b} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \phi(y) dy \]

\[ = \colon A_1[\phi](x) + A_2[\phi](x). \tag{19} \]

Note that for \( x, y \in \Gamma_j, j = 2, 3 \), one can easily obtain that \( \langle x - y, \nu_x \rangle = 0 \). Thus, one has

\[ A_1[\phi](x) = \frac{1}{2\pi} \int_{S_f} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \phi(y) dy \]

\[ = \chi(S^a \cup S^b) \frac{1}{2\pi} \int_{S^f} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \phi(y) dy \]

\[ + \chi(\Gamma_1) \frac{1}{2\pi} \int_{\Gamma_2} \frac{\langle (x - 2\delta \nu_x, y) + 2\delta \nu_x, \nu_x \rangle}{|x - y|^2} \phi(y) dy \]

\[ + \chi(\Gamma_2) \frac{1}{2\pi} \int_{\Gamma_1} \frac{\langle (x - 2\delta \nu_x, y) + 2\delta \nu_x, \nu_x \rangle}{|x - y|^2} \phi(y) dy \]

\[ = K_{S^f, S^c}[\phi](x) + \delta \chi(\Gamma_1) \frac{1}{\pi} \int_{\Gamma_2} \frac{1}{|x - y|^2} \phi(y) dy + \delta \chi(\Gamma_2) \frac{1}{\pi} \int_{\Gamma_1} \frac{1}{|x - y|^2} \phi(y) dy. \tag{20} \]
On the other hand,

\[
A_2[\phi](x) = \frac{1}{2\pi} \int_{S^a \cup S^b} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy
\]

\[
= \chi(S^a) \frac{1}{2\pi} \int_{S^a} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy + \chi(S^b) \frac{1}{2\pi} \int_{S^b} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy
\]

\[
+ \chi(S^f) \frac{1}{2\pi} \int_{S^a \cup S^b} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy
\]

\[
= \chi(S^a) \frac{1}{2\pi} \int_{S^a} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy + \chi(S^b) \frac{1}{2\pi} \int_{S^b} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy
\]

\[
+ \chi(S^f) \frac{1}{2\pi} \int_{S^a \cup S^b} \frac{(x - y, \nu_x)}{|x - y|^2} \phi(y) dy
\]

\[
= \chi(S^a) \frac{1}{4\pi} \int_{S^a} \frac{\phi(y)}{|x - y|^2} dy + \chi(S^b) \frac{1}{4\pi} \int_{S^b} \frac{\phi(y)}{|x - y|^2} dy + K_{S^b, S^a}[\phi] + K_{S^a, S^b}[\phi] + K_{S^f, S^f}[\phi].
\]

For \( y \in S^a \) and \( x \in S^b \), by using Taylor’s expansion, one has

\[
|x - y| = |x - P - (y - P)| = |x - P| - \delta(y - P) = |x - P| + \delta(x - P, y - P) + O(\delta^2).
\]

Thus, one has

\[
K_{S^a, S^b}[\phi](x) = \delta \frac{(x - P, \nu_x)}{2\pi|x - P|^2} \int_{S^a} \frac{\phi(y)}{|x - y|^2} dy + O(\delta^2).
\]

Similarly, one can obtain

\[
K_{S^b, S^a}[\phi](x) = \delta \frac{(x - Q, \nu_x)}{2\pi|x - Q|^2} \int_{S^b} \frac{\phi(y)}{|x - y|^2} dy + O(\delta^2).
\]

For \( x \in S^f \), \( y \in S^a \), by direct computations, one can obtain

\[
K_{S^a, S^f}[\phi](x) = \delta^2 \int_{S^a} \frac{1 - (\tilde{y} - P, \nu_x)}{|x - P|^2 - 2\delta(x - P, \tilde{y} - P) + \delta^2} \frac{1}{|x - y|^2} \phi(y) dy.
\]

We decompose \( S^f = (S^f \setminus t_\delta(P)) \cup t_\delta(P) \), then one has

\[
K_{S^a, S^f}[\phi](x) = \frac{\delta^2}{|x - P|^2} \int_{S^a \setminus \{x - P\}} (1 - (\tilde{y} - P, \nu_x)) \frac{1}{|x - P|^2 - 2\delta(x - P, y - P) + \delta^2} \phi(y) dy + O\left(\frac{\delta^2}{|x - P|^2}\right), \quad x \in S^f \setminus t_\delta(P).
\]

Similarly, one can derive the asymptotic expansion for \( K_{S^b, S^f} \). By substituting (23)–(25) back into (21) and combining (20) one finally achieves (15), which completes the proof.

The proof is complete. ■

**Lemma 2.** The operators \( \mathcal{A}_{\Gamma_j, \Gamma_k, \{j, k\} = \{1, 2\}, \{2, 1\}} \) defined in (18) are bounded operators from \( L^2(\Gamma_j) \) to \( L^2(\Gamma_k) \). Furthermore, the operators \( \chi(t_\delta(P))K_{S^a, S^f} \) and \( \chi(t_\delta(Q))K_{S^b, S^f} \) are bounded operators from \( L^2(S^a) \) to \( L^2(S^f) \), and from \( L^2(S^b) \) to \( L^2(S^f) \), respectively.
Proof. We only prove that \( A_{\Gamma_2, \Gamma_1} \) is a bounded operator \( L^2(\Gamma_2) \to L^2(\Gamma_1) \). First, for \( \phi_1 \in L^2(\Gamma_1) \) and \( \phi_2 \in L^2(\Gamma_2) \), one has

\[
\langle A_{\Gamma_2, \Gamma_1} \phi_2, \phi_1 \rangle_{L^2(\Gamma_1)} = \frac{1}{2\pi} \left| \int_{\Gamma_1} \int_{\Gamma_2} \frac{\delta}{|x - y|^2} \phi_2(y) \phi_1(x) ds_y ds_x \right|
\]

\[
\leq \frac{1}{4\pi} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\delta}{|x - y|^2} \phi_2^2(y) ds_y ds_x + \frac{1}{4\pi} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\delta}{|x - y|^2} \phi_1^2(x) ds_y ds_x
\]

\[
= \frac{1}{4\pi} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{\delta}{|x_1 - y_1|^2 + 4\delta^2} dx_1 \phi_2^2(y) dy_1 + \frac{1}{4\pi} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \frac{\delta}{|x_1 - y_1|^2 + 4\delta^2} dy_1 \phi_1^2(x) dx_1
\]

\[
= \frac{1}{8\pi} \int_{-L/2}^{L/2} \left( \arctan \frac{L/2 - y_1}{2\delta} - \arctan \frac{-L/2 - y_1}{2\delta} \right) \phi_2^2(y) dy_1
\]

\[
+ \frac{1}{8\pi} \int_{-L/2}^{L/2} \left( \arctan \frac{L/2 - x_1}{2\delta} - \arctan \frac{-L/2 - x_1}{2\delta} \right) \phi_1^2(x) dx_1
\]

\[
\leq C(\|\phi_1\|_{L^2(\Gamma_1)}^2 + \|\phi_2\|_{L^2(\Gamma_2)}^2),
\]

where the constant \( C \) is independent of \( \delta \). By following a similar argument as in Ref. 2 (p. 18), one can show that \( A_{\Gamma_2, \Gamma_1} \) is a bounded operator \( L^2(\Gamma_2) \to L^2(\Gamma_1) \).

Lemma 3. Suppose \( x \in S^c \), then for any function \( \phi \in L^2(S^f) \), which satisfies

\[
\phi(y) = -\phi(y + 2\delta n), \quad y \in \Gamma_1,
\]

there holds

\[
\mathcal{K}_{S^f/\{I_2(P) \cup I_2(Q)\},S^c}[\phi](x) = o(1).
\]

Proof. Note that \( S^f = \Gamma_1 \cup \Gamma_2 \). Straightforward computations show that

\[
\mathcal{K}_{S^f/\{I_2(P) \cup I_2(Q)\},S^c}[\phi](x) = \frac{1}{2\pi} \int_{S\setminus\{I_2(P) \cup I_2(Q)\}} \frac{\langle x - y, v_x \rangle}{|x - y|^2} \phi(y) ds_y
\]

\[
= \frac{1}{2\pi} \int_{S\setminus\{I_2(P) \cup I_2(Q)\}} \frac{\langle x - z_y - \delta v_y, v_x \rangle}{|x - z_y - \delta v_y|^2} \phi(y) ds_y
\]

\[
= \frac{1}{2\pi} \int_{S\setminus\{I_2(P) \cup I_2(Q)\}} \frac{\langle x - z_y, v_x \rangle}{|x - z_y|^2} \phi(y) ds_y + o(1)
\]

\[
= \frac{1}{2\pi} \int_{\Gamma\setminus\{I_2(P) \cup I_2(Q)\}} \frac{\langle x - z_y, v_x \rangle}{|x - z_y|^2} \phi(y) ds_y
\]

\[
+ \frac{1}{2\pi} \int_{\Gamma\setminus\{I_2(P) \cup I_2(Q)\}} \frac{\langle x - z_y, v_x \rangle}{|x - z_y|^2} \phi(y + 2\delta n) ds_y + o(1) = o(1),
\]

which completes the proof.
3 | QUANTITATIVE ANALYSIS OF THE ELECTRIC FIELD

In this section, we present the quantitative analysis of the solution to the conductivity equation (1) as well as its geometric relationship to the inclusion $D$.

3.1 | Several auxiliary lemmas

Recall that $u$ is represented by (9). We first derive some asymptotic properties of the density function $\varphi$ in (10). Let $z_x \in \Gamma_0$ be defined by

$$z_x = \begin{cases} x + \delta n, & x \in \Gamma_1, \\ x - \delta n, & x \in \Gamma_2. \end{cases}$$

(28)

One has the following asymptotic expansion for $H$ around $\Gamma_0$:

$$H(x) = H(z_x) + \nabla H(z_x) \cdot (x - z_x) + \mathcal{O}(|x - z_x|^2) = H(z_x) + \delta \nabla H(z_x) \cdot (x - z_x) + \mathcal{O}(\delta^2),$$

(29)

for $x \in S^f$ and $\bar{x} \in S_1^f$. Similarly, one has

$$H(x) = H(P) + \nabla H(P) \cdot (x - P) + \mathcal{O}(|x - P|^2) = H(P) + \delta \nabla H(P) \cdot \nu_x + \mathcal{O}(\delta^2),$$

(30)

for $x \in S^a$ and $\bar{x} \in S_1^a$. Moreover,

$$H(x) = H(Q) + \nabla H(Q) \cdot (x - Q) + \mathcal{O}(|x - Q|^2) = H(Q) + \delta \nabla H(Q) \cdot \nu_x + \mathcal{O}(\delta^2),$$

(31)

for $x \in S^b$ and $\bar{x} \in S_1^b$.

We now can show the following asymptotic result.

Lemma 4. Suppose $\varphi$ is defined in (10), then one has

$$\varphi(x) = \begin{cases} (\lambda I - A_\delta)^{-1}[(\partial_x \cdot H(\cdot,0))(x_1) + \delta (\lambda I - A_\delta)^{-1}[\partial_{x_2}^2 H(\cdot,0)](x_1)] \\ + \chi(t_5(P) \cup t_5(Q))\mathcal{O}(\delta^{2(1-\epsilon)}) + \mathcal{O}(\delta^2), & x \in \Gamma_0 \setminus (t_5(P) \cup t_5(Q)), \\ (\lambda I - \mathcal{K}_1^a)^{-1}[\nabla H(P) \cdot \nu] + o(1), & x \in S^a \cup t_5(P), \\ (\lambda I - \mathcal{K}_2^a)^{-1}[\nabla H(Q) \cdot \nu] + o(1), & x \in S^b \cup t_5(Q), \end{cases}$$

(32)

where $0 < \epsilon < 1$ and the operator $A_\delta$ is defined by

$$A_\delta[\psi](x_1) := \frac{1}{2\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2} \psi(y_1)dy_1, \quad \psi \in L^2(-L/2,L/2).$$

(33)

The operators $\mathcal{K}_1^a$ and $\mathcal{K}_2^a$ are defined by

$$\mathcal{K}_1^a[\varphi_1](x) := \int_{S^a \cup t_5(P)} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \varphi(y)ds_y + \chi(t_5(P))A_{S^a \cap t_5(P)}[\varphi_1](x),$$

$$\mathcal{K}_2^a[\varphi_2](x) := \int_{t_5(Q) \cup S^b} \frac{\langle x - y, \nu_x \rangle}{|x - y|^2} \varphi(y)ds_y + \chi(t_5(Q))A_{S^b \cap t_5(Q)}[\varphi_2](x),$$

(34)
respectively.

**Proof.** Because

\[
(\lambda I - \mathcal{K}_D^n)[\varphi] = \left[ \frac{\partial H}{\partial \nu} \right]_{\partial D}.
\]

By combining (15) and (29) one can readily verify that

\[
(\lambda I - \mathcal{K}_3^n)[\varphi](x) = \nabla H(z_x) \cdot \nu_x + o(1), \quad x \in S^f.
\]

By using (16), one thus has

\[
\begin{align*}
\lambda \varphi(x) &- \frac{1}{\pi} \int_{\Gamma_2} \frac{\delta}{|x - y|^2} \varphi(y) dy = -\nabla H(z_x) \cdot \nu + o(1), \quad x \in \Gamma_1 \setminus (t_5(P) \cup t_5(Q)), \\
\lambda \varphi(x) &- \frac{1}{\pi} \int_{\Gamma_1} \frac{\delta}{|x - y|^2} \varphi(y) dy = \nabla H(z_x) \cdot \nu + o(1), \quad x \in \Gamma_2 \setminus (t_5(P) \cup t_5(Q)).
\end{align*}
\] (35)

By direct computations, one can show

\[
\begin{align*}
\lambda \varphi(x_1, -\delta) &- \frac{1}{\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2} \varphi(y, \delta) dy_1 \\
&= -\partial_{x_2} H(x_1, 0) + o(1), \quad |x_1| \leq L/2 - O(\delta), \\
\lambda \varphi(x_1, \delta) &- \frac{1}{\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2} \varphi(y_1, -\delta) dy_1 \\
&= \partial_{x_2} H(x_1, 0) + o(1), \quad |x_1| \leq L/2 - O(\delta).
\end{align*}
\] (36)

Thus, one can derive that \(\varphi(x_1, -\delta) = -\varphi(x_1, \delta) + o(1)\), for \(|x_1| \leq L/2 - O(\delta)\). Furthermore, for \(x \in S_1^a\), by making use of (16), (30), and Lemma 6, one has

\[
(\lambda I - \mathcal{K}_1^n)[\varphi](x) = \nabla H(P) \cdot \nu_x + o(1), \quad \text{in} \quad S^a \cup t_5(P).
\] (37)

In a similar manner, one can show that

\[
(\lambda I - \mathcal{K}_2^n)[\varphi](x) = \nabla H(Q) \cdot \nu_x + o(1), \quad \text{in} \quad S^b \cup t_5(Q),
\] (38)

and so the last equation in (32) follows.

Next, by combining (15), (16), and (17) again for \(x \in \Gamma_j \setminus (t_5(P) \cup t_5(Q))\), \(j = 1, 2\), and using the second and third equations in (32), one has

\[
\begin{align*}
\lambda \varphi(x_1, (-1)^j\delta) &- \frac{1}{2\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2} \varphi(y_1, (-1)^{j+1}\delta) dy_1 \\
&= (-1)^j \partial_{x_2} H(x_1, 0) + \delta \partial_{x_2} H(x_1, 0) + \chi(t_5(P) \cup t_5(Q)) O(\delta^{2(1-\varepsilon)}) + O(\delta^2), \quad 0 < \varepsilon < 1,
\end{align*}
\] (39)
which verifies the first equation in (32) and completes the proof.

Before presenting our main result, we need to further analyze the operator $A_\delta$ defined in (33)

**Lemma 5.** Suppose $A_\delta$ is defined in (33), then it holds that

$$A_\delta[y^n_1](x_1) = \frac{1}{2}x_1^n + o(1), \quad x \in \Gamma_j \setminus (t_\delta(P) \cup t_\delta(Q)), \quad n \geq 0. \quad (40)$$

**Proof.** We use deduction to prove the assertion. Because $x \in \Gamma_j \setminus (t_\delta(P) \cup t_\delta(Q))$, one has

$$|L/2 - x_1| = \mathcal{O}(\delta^\varepsilon), \quad \text{and} \quad |L/2 + x_1| = \mathcal{O}(\delta^\varepsilon), \quad 0 \leq \varepsilon < 1.$$ 

Then, for $n = 0$, it is straightforward to verify that

$$A_\delta[1](x_1) = \frac{1}{\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2}dy_1 = \frac{1}{2\pi} \left( \arctan \frac{L/2 - x_1}{2\delta} - \arctan \frac{-L/2 - x_1}{2\delta} \right) = \frac{1}{2} + o(1). \quad (41)$$

Next, we suppose that (40) holds for $n \leq N$. Then, by using change of variables, one can derive that

$$A_\delta[y^{N+1}_1](x_1) = \frac{1}{\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2}y_1^Ny_1dy_1$$

$$= \frac{1}{2\pi} \int_{-L/2-x_1}^{L/2-x_1} \frac{1}{1 + t^2}y_1^N(2\delta t + x_1)dt$$

$$= \frac{1}{\pi} \delta \int_{-L/2-x_1}^{L/2-x_1} \frac{1}{1 + t^2}y_1^N t dt + x_1 \frac{1}{2}x_1^N + o(1)$$

$$= \frac{1}{\pi} \delta \mathcal{O}(\ln(1 + \delta^{2(\varepsilon^{-1})})) + \frac{1}{2}x_1^{N+1} + o(1) = \frac{1}{2}x_1^{N+1} + o(1),$$

which completes the proof.

The following lemma is also of critical importance

**Lemma 6.** There holds the following that

$$\int_{S^p \cup t_\delta(P)} (\lambda I - K^p_1)^{-1}[\nabla H(P) \cdot \nu] = -2\delta \left( \lambda - \frac{1}{2} \right)^{-1} \partial x_1 H(P) + o(\delta), \quad (43)$$

$$\int_{S^b \cup t_\delta(Q)} (\lambda I - K^b_2)^{-1}[\nabla H(Q) \cdot \nu] = 2\delta \left( \lambda - \frac{1}{2} \right)^{-1} \partial x_1 H(Q) + o(\delta).$$
Proof. For any $f \in L^2(\partial D)$, we consider the following boundary integral equation

$$(\lambda I - K^w_D)[\phi] = f.$$  \hfill (44)

By using the decomposition (16) (see also (37) and (38)), one has

$$
\chi(S^a \cup t_\delta(P))(\lambda I - K^w_1 + o(1))[\phi] + \chi(S^b \cup t_\delta(Q))(\lambda I - K^w_2 + o(1))[\phi] \\
+ A_{\Gamma_2,\Gamma_1}[\phi] + A_{\Gamma_1,\Gamma_2}[\phi] + \chi(t_\delta(P) \cup t_\delta(Q))\mathcal{O}(\delta^{3(1-\epsilon)}) + \mathcal{O}(\delta^2) = f, \quad 0 < \epsilon < 1.
$$ \hfill (45)

Note that $\partial D$ is of $C^{1,\alpha}$. By taking the boundary integral of both sides of (44) on $\partial D$ and making use of (45), one then has

$$
\left(\lambda - \frac{1}{2}\right) \int_{\partial D} \phi = \int_{S^a \cup t_\delta(P)} (\lambda I - K^w_1 + o(1))[\phi] + \int_{S^b \cup t_\delta(Q)} (\lambda I - K^w_2 + o(1))[\phi] \\
+ \int_{\Gamma_1} A_{\Gamma_2,\Gamma_1}[\phi] + \int_{\Gamma_2} A_{\Gamma_1,\Gamma_2}[\phi] + o(\delta) = \int_{\partial D} f.
$$ \hfill (46)

By assuming $f = \chi(S^a \cup t_\delta(P))\nabla H(P) \cdot \nu$ and plugging into (46), one thus has

$$
\left(\lambda - \frac{1}{2}\right) \int_{S^a \cup t_\delta(P)} (\lambda I - K^w_1 + o(1)^{-1}[\nabla H(P) \cdot \nu] = \int_{S^a \cup t_\delta(P)} \nabla H(P) \cdot \nu = -2\delta \partial_{x_1} H(P),
$$ \hfill (47)

which verifies the first equation in (43). Similarly, by assuming $f = \chi(S^b \cup t_\delta(Q))\nabla H(q) \cdot \nu$, one can prove the second equation in (43). The proof is complete. \hfill \blacksquare

3.2 Sharp asymptotic approximation of the solution $u$

With Lemmas 4, 5, and 6, we can now establish one of the main results of this paper as follows.

**Theorem 1.** Let $u$ be the solution to (1) and (2), with $D$ of the rod-shape described in Section 1.1. Then, for $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{D}$, it holds that

$$
u(\mathbf{x}) = H(\mathbf{x}) + \delta \frac{1}{2\pi} \left(\lambda - \frac{1}{2}\right)^{-1} \int_{-L/2}^{L/2} \ln \frac{(x_1 - y_1)^2 + x_2^2}{(x_1 + L/2)^2 + x_2^2} \partial_{y_2} H(y_1,0) dy_1 \\
+ \delta \frac{1}{\pi} \left(\lambda - \frac{1}{2}\right)^{-1} \int_{-L/2}^{L/2} \frac{x_2}{(x_1 - y_1)^2 + x_2^2} \partial_{y_2} H(y_1,0) dy_1 \\
+ \delta \frac{1}{2\pi} \left(\lambda - \frac{1}{2}\right)^{-1} \ln \frac{(x_1 - L/2)^2 + x_2^2}{(x_1 + L/2)^2 + x_2^2} \partial_{x_1} H(L/2,0) + o(\delta).
$$ \hfill (48)
Proof. By using (9) and Taylor’s expansion along with $\Gamma_0$, one has

$$
\begin{align*}
\mathbf{u}(\mathbf{x}) = & H(\mathbf{x}) + \int_{S^f \setminus (\partial_{\delta}(P) \cup \partial_{\delta}(Q))} G(\mathbf{x} - \mathbf{z}_y) \varphi(\mathbf{y}) \, ds_y \\
+ & \delta \int_{S^f \setminus (\partial_{\delta}(P) \cup \partial_{\delta}(Q))} \nabla_y G(\mathbf{x} - \mathbf{z}_y) \cdot \nu_y \varphi(\mathbf{y}) \, ds_y \\
+ & \int_{S^a \cup \partial_{\delta}(P)} G(\mathbf{x} - \mathbf{z}_y) \varphi(\mathbf{y}) \, ds_y + \int_{S^b \cup \partial_{\delta}(Q)} G(\mathbf{x} - \mathbf{z}_y) \bar{\varphi}(\bar{\mathbf{y}}) \, ds_{\bar{\mathbf{y}}} + O(\delta^2).
\end{align*}
$$

(49)

First, by using (39), one can derive that

$$
\int_{S^f} G(\mathbf{x} - \mathbf{z}_y) \varphi(\mathbf{y}) \, ds_y = 2\delta \int_{\Gamma_1 \setminus (\partial_{\delta}(P) \cup \partial_{\delta}(Q))} G(\mathbf{x} - \mathbf{z}_y)(\lambda I - A_\delta)^{-1} [\partial_{\mathbf{x}_2}^2 H(\cdot, 0)](y_1) \, dy_1 + o(\delta)
$$

(50)

$$
\frac{1}{2\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \int_{-L/2}^{L/2} \ln((x_1 - y_1)^2 + x_2^2) \partial_{y_2}^2 H(y_1, 0) \, dy_1 + o(1).
$$

(51)

Similarly, one has

$$
\int_{S^f} \nabla_y G(\mathbf{x} - \mathbf{z}_y) \cdot \nu_y \varphi(\mathbf{y}) \, ds_y = \frac{1}{\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \int_{-L/2}^{L/2} \frac{x_2}{(x_1 - y_1)^2 + x_2^2} \partial_{y_2}^2 H(y_1, 0) \, dy_1 + o(1).
$$

(51)

By using Lemma 6, one then obtains that

$$
\int_{S^b \cup \partial_{\delta}(Q)} G(\mathbf{x} - \mathbf{z}_y) \varphi(\mathbf{y}) \, ds_y = \int_{S^b \cup \partial_{\delta}(Q)} G(\mathbf{x} - \mathbf{Q}) \varphi(\mathbf{y}) \, ds_y + o(\delta)
$$

$$
\frac{1}{\pi} \ln |\mathbf{x} - \mathbf{Q}| \left( \lambda - \frac{1}{2} \right)^{-1} \partial_{\mathbf{x}_1} H(\mathbf{Q}) + o(\delta)
$$

(52)

$$
\frac{1}{2\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \ln((x_1 - L/2)^2 + x_2^2) \partial_{\mathbf{x}_1} H(L/2, 0) + o(\delta).
$$

(52)

Noting that

$$
\int_{\partial D} \varphi(\mathbf{y}) \, ds_y = \int_{\partial D} (\lambda I - \mathcal{K}_D^n)^{-1} \left[ \frac{\partial H}{\partial \mathbf{y}} \right](\mathbf{y}) \, ds_y = 0,
$$
and by combining (50), one can readily show that

\[
\int_{S^1 \cup \partial(P)} G(x - z_{j}) \varphi(y) \, ds_{y} = \int_{S^1 \cup \partial(P)} G(x - P) \varphi(y) \, ds_{y} + o(\delta)
\]

\[
= -\frac{1}{2\pi} \ln |x - P| \left( \int_{S^1 \cup \partial(Q)} \varphi(y) \, ds_{y} \right) + 2 \int_{\Gamma_{1}} \left( \lambda I - A_{1} \right)^{-1} [\partial_{y_{2}}^{2} H(\cdot, 0)](y_{1}) \, dy_{1} \right) + o(\delta)
\]

(53)

\[
= -\delta \frac{1}{2\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \ln((x_{1} + L/2)^{2} + x_{2}^{2}) \partial x_{1} H(L/2, 0)
\]

\[
- \delta \frac{1}{2\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \ln((x_{1} + L/2)^{2} + x_{2}^{2}) \int_{-L/2}^{L/2} \partial_{y_{2}}^{2} H(y_{1}, 0) \, dy_{1} + o(\delta).
\]

Finally, by substituting (50)–(53) into (49) one has (48), which completes the proof. ■

We finally can derive the sharp asymptotic expansion of the electric field as follows.

**Theorem 2.** Suppose \( H(x) = a \cdot x \), where \( a = (a_{1}, a_{2}) \in \mathbb{R}^{2} \). Then for \( x \in \mathbb{R}^{2} \setminus D \), the electric field \( u \) satisfies

\[
u(x) = a \cdot x + \delta \frac{1}{\pi} \left( \lambda - \frac{1}{2} \right)^{-1} a_{2} \left( \arctan \left( \frac{L/2 - x_{1}}{x_{2}} \right) + \arctan \left( \frac{L/2 + x_{1}}{x_{2}} \right) \right)
\]

\[
+ \delta \frac{1}{2\pi} \left( \lambda - \frac{1}{2} \right)^{-1} a_{1} \ln \left( \frac{x_{1} - L/2 + x_{2}^{2}}{x_{1} + L/2 + x_{2}^{2}} \right) + o(\delta).
\]

(54)

Furthermore, the perturbed gradient field admits the following asymptotic expansion:

\[
\nabla u(x) = a + \delta \frac{1}{\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \left( f_{2}(x) a_{1} - f_{1}(x) a_{2} \right) + o(\delta).
\]

(55)

where the functions \( f_{j}, j = 1, 2 \) are defined by

\[
f_{1}(x) := \frac{x_{2}}{(x_{1} - L/2)^{2} + x_{2}^{2}} - \frac{x_{2}}{(x_{1} + L/2)^{2} + x_{2}^{2}}
\]

\[
f_{2}(x) := \frac{x_{1} - L/2}{(x_{1} - L/2)^{2} + x_{2}^{2}} - \frac{x_{1} + L/2}{(x_{1} + L/2)^{2} + x_{2}^{2}}.
\]

(56)

**Proof.** The proof is given by using (48) together with direct computations. ■
3.3 Quantitative analysis and numerical illustrations

Define the following vector field:

\[
\mathbf{E}^s := \delta \frac{1}{\pi} \left( \lambda - \frac{1}{2} \right)^{-1} \left( f_2(x)a_1 - f_1(x)a_2 \right) \left( f_1(x)a_1 + f_2(x)a_2 \right).
\]  (57)

According to (55), \( \mathbf{E}^s \) is the leading order term of the perturbed gradient field. It is noted that the distribution of \( |\mathbf{E}^s| \) is independent of the uniform gradient potential \( a \). In fact, one has

\[
|\mathbf{E}^s|^2 = \delta^2 \frac{1}{\pi^2} \left( \lambda - \frac{1}{2} \right)^{-2} (a_1^2 + a_2^2)(f_1(x)^2 + f_2(x)^2).
\]  (58)

Moreover, further computations show that

\[
f_1(x)^2 + f_2(x)^2 = \left( \frac{1}{|x-Q|} - \frac{1}{|x-P|} \right)^2 + \frac{2}{|x-P||x-Q|} \left( 1 - \frac{\langle x-P, x-Q \rangle}{|x-P||x-Q|} \right).
\]  (59)

One can thus derive that \( |\mathbf{E}^s| \) is maximized near the two caps (high curvature parts) of the inclusion \( D \). In fact, near the caps one has

\[
|x-P| = \delta + o(\delta), \quad \text{or} \quad |x-Q| = \delta + o(\delta).
\]

By (59) one then has

\[
f_1(x)^2 + f_2(x)^2 = \delta^{-2}(1 + o(1)),
\]  (60)

while near the centering parts of the rod,

\[
f_1(x)^2 + f_2(x)^2 = O(1).
\]

To better illustrate the result, we next present some numerical solutions with different background fields. The parameters of the rod-shape inclusion are selected as follows:

\[
\sigma_0 = 2, \quad L = 10, \quad \delta = 5 \times \tan(\pi/36) \approx 0.4374.
\]  (61)

We choose three different uniform background fields, i.e., \( \mathbf{a} = (1, 0), (0, 1), (1, 1) \), respectively, and plot the absolute values of the leading-order perturbed fields as well as the corresponding gradient fields, which are scaled for better display. It is clearly shown from Figure 1 to Figure 3 that the gradient fields behave much stronger near the high curvature parts of the inclusion.

4 APPLICATION TO CALDERÓN INVERSE INCLUSION PROBLEM

In this section, we consider the application of the quantitative results derived in the previous section to the Calderón inverse inclusion problem. To that end, we let \( D \) denote a generic rod.
inclusion that is obtained through rigid motions performed on special case described in Section 1.1. We write $D(L, \delta, z_0, \sigma_0)$ to signify its dependence on the length $L$, width $\delta$, position $z_0$ (which is the geometric center of $D$), as well as the conductivity parameter $\sigma_0$. Consider the conductivity system (1) associated with a generic inclusion described above. The inverse inclusion problem is concerned with recovering the shape of the inclusion, namely, $\partial D$, independent of its content $\sigma_0$, by measuring the perturbed electric field $(u - H)$ away from the inclusion. This is one of the central problems in EIT, which forms the fundamental basis for the electric prospecting. The case with a single measurement, namely, the use of a single probing field $H$, is a longstanding problem in the literature. The existing results for the single-measurement case are mainly concerned with specific shapes including discs/balls and polygons/polyhedrons\textsuperscript{14,15,34–38} as well as the other general shapes but with a priori conditions; see Refs. 39–44. As discussed earlier, in Ref. 16, the local recovery of the highly curved part of $\partial D$ was also considered. Next, using the asymptotic result quantitative result in Theorem 2, we show that one can uniquely determine a conductive inclusion up to an error level $\delta \ll 1$.

**Theorem 3.** Let $D_j = D_j(L_j, \delta_j, z_0^{(j)}, \sigma_0^{(j)})$, $j = 1, 2$, be two conductive rods such that $L_j \sim 1, \delta_j \sim \delta \ll 1$ and $\sigma_0^{(j)} \sim 1$ for $j = 1, 2$. Let $u_j$ be the corresponding solution to (1) associated with $D_j$ and a
FIGURE 3  \( \mathbf{a} = (1, 1) \). Left: perturbed field \( |u - \mathbf{a} \cdot \mathbf{x}| \) (scaled) Right: perturbed gradient field \( |\nabla u - \mathbf{a}| \) (scaled)

given nontrivial \( H(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} \). Suppose that

\[
  u_1 = u_2 \quad \text{on} \; \partial \Sigma, \tag{62}
\]

where \( \Sigma \) is a bounded simply connected Lipschitz domain enclosing \( D_j \). Then it cannot hold that

\[
  \text{dist}(D_1, D_2) \gg \delta. \tag{63}
\]

Proof. First, by (62), we know that \( u_1 = u_2 \) in \( \mathbb{R}^2 \setminus \Sigma \) and hence by unique continuation, we also know that \( u_1 = u_2 \) in \( \mathbb{R}^2 \setminus (D_1 \cup D_2) \). Next, because the Laplacian is invariant under rigid motions, we note that the quantitative result in Theorem 2 still holds for \( D_j \). By contradiction, we assume that (63) holds. It is easily seen that there must be one cap point, say \( \Theta_0 \in \partial D_1 \), which lies away from \( D_2 \) and \( \text{dist}(\Theta_0, D_2) \gg \delta \). Hence, one has \( u_1(\Theta_0) = u_2(\Theta_0) \). Now, we arrive at a contradiction by noting that using Theorem 2, one has \( u_1(\Theta_0) \sim 1 \), whereas \( u_2(\Theta_0) \sim \delta \ll 1 \).

The proof is complete. \( \blacksquare \)

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DATA AVAILABILITY STATEMENT

This is a mathematical paper containing all the necessary theoretical proofs and numerical illustrations. There are no data to be reported concerning this work.

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