Exponential elliptic boundary value problems on a solid torus in the critical of supercritical case

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Abstract: In this paper we investigate the behavior and the existence of positive and non-radially symmetric solutions to nonlinear exponential elliptic model problems defined on a solid torus $\mathbb{T}$ of $\mathbb{R}^3$, when data are invariant under the group $G = O(2) \times I \subset O(3)$. The model problems of interest are stated below:

$$(P_1) \quad \Delta v + \gamma = f(x)e^v, \quad v > 0 \quad \text{on} \quad T, \quad v|_{\partial T} = 0.$$ 

and

$$(P_2) \quad \Delta v + a + fe^v = 0, \quad v > 0 \quad \text{on} \quad T,$$

$$\frac{\partial v}{\partial n} + b + ge^v = 0 \quad \text{on} \quad \partial T.$$ 

We prove that exist solutions which are $G$–invariant and these exhibit no radial symmetries. In order to solve the above problems we need to find the best constants in the Sobolev inequalities in the exceptional case.

Key words: Exponential problems · Solid torus · Sobolev inequalities · Critical of supercritical exponent

Mathematics Subject Classification (2000) 35J66 · 46E35 · 35B33
1 Introduction

In recent years, significant progress has been made on the analysis of a number of important features of nonlinear partial differential equations of elliptic and parabolic type. The study of these equations has received considerable attention, because of their special mathematical interest and because of practical applications of the torus in scientific research today. For example in Astronomy, investigators study the torus which is a significant topological feature surrounding many stars and black holes [26]. In Physics the torus is being explored at the National Spherical Torus Experiment (NSTX) at Princeton Plasma Physics Laboratory to test the fusion physics principles for the spherical torus concept at the MA level [36]. In Biology some investigators interested in circular DNA molecules detected in a large number of viruses, bacteria, and higher organisms. In this topologically very interesting type of molecule, superhelical turns are formed as the Watson-Crick double helix winds in a torus formation [25].

Let the solid torus be represented by the equation

\[ T = \left\{ (x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - l)^2 + z^2 \leq r^2, \ l > r > 0 \right\}, \]

and the subgroup \( G = O(2) \times I \) of \( O(3) \). Note that the solid torus \( T \subset \mathbb{R}^3 \) is invariant under the group \( G \).

We consider the following nonlinear exponential elliptic boundary problems

\((P_1)\) \quad \Delta u + \gamma = f(x)e^u, \ v > 0 \quad \text{on} \ T, \quad v|_{\partial T} = 0

and

\((P_2)\) \quad \Delta u + a + fe^u = 0, \ v > 0 \quad \text{on} \ T, \quad \frac{\partial u}{\partial n} + b + ge^u = 0 \quad \text{on} \ \partial T,

where \( \Delta u = -\nabla^i \nabla_i u \) is the Laplacian of \( u \), \( \frac{\partial}{\partial n} \) is the outer unit normal derivative, \( f, g \) are two smooth \( G \)-invariant functions and \( \gamma, a, b \in \mathbb{R} \).

Clearly, a radially symmetric solution is a \( G \)-invariant solution, for any subgroup \( G \) of \( O(n) \). The converse problem is considered in this paper, that is we prove that there exist positive solutions which are \( G \)-invariant and non-radially symmetrical if \( G = O(2) \times I \).

Problems \((P_1)\) and \((P_2)\) own their origin to the "Nirenberg Problem" posed in 1969–70 in the following way:
Given a (positive) smooth function $f$ on $(\mathbb{S}^2, g_0)$ (close to the constant function, if we want), is it the scalar curvature of a metric $g$ conformal to $g_0$? ($g_0$ is the standard metric whose sectional curvature is 1) (see [3]).

Recall that, if we write $g$ in the form $g = e^u g_0$, the problem is equivalent to solving the equation:

$$\Delta u + 2 = fe^u.$$

Nirenberg Problem has been studied extensively and is completely solved (see [2], [8], [45], [37], [19]). Further, we refer the reader to [14, 15], [13], [11], [12], [39], [40], [41], [9], [10], [33], [38], [7], [30], [1], [16], [43], in which the authors study this problem or its generalization.

Best constants in Sobolev inequalities are fundamental in the study of non-linear PDEs on manifolds, because of their strong connection with the existence and the multiplicity of the solutions of the corresponding problems (see for example [46], [5], [47], [35], [34], [29], [6, 7], [22, 23, 24] and the references therein). It is also well-known, that Sobolev embeddings can be improved in the presence of symmetries in the sense that we obtain continuous embeddings in higher $L^p$ spaces, that it, allow us to solve equations with higher critical exponents (see for example [42], [28], [20], [21], [35], [27], [6, 7], [30, 31], [22, 23, 24] and the references therein). Especially, in our case we solve problems with the highest supercritical exponent (critical of supercritical).

Let:

$$C^\infty_{0,G} = \{ v \in C^\infty_0(T) : v \circ \tau = v, \forall \tau \in G \},$$

$$C^\infty_G = \{ v \in C^\infty(T) : v \circ \tau = v, \forall \tau \in G \}$$

and

$$L^p_G = \{ v \in L^p(T) : v \circ \tau = v, \forall \tau \in G \}$$

that is, the spaces of all $G-$invariant functions under the action of the group $G = O(2) \times I$.

We define the Sobolev space $H^p_{1,G}(T)$, $p \geq 1$ as the completion of $C^\infty_G(T)$ with respect to the norm

$$\| v \|_{H^p_{1,G}} = \| \nabla v \|_p + \| v \|_p$$

and $\dot{H}^p_{1,G}(T)$ as the closure of $C^\infty_{0,G}(T)$ in $H^p_{1,G}(T)$. 

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In [22] we proved that for any \( p \in [1, 2) \) real, the embedding \( H^p_{1,G}(T) \hookrightarrow L^q_{G}(T) \) is compact for \( 1 \leq q < \frac{2p}{2-p} \), while the embedding \( H^p_{1,G}(T) \hookrightarrow L^{r/p}_{G}(T) \) is only continuous. Also, in [23] we proved that for any \( p \in [1, 2) \) real, the embedding \( H^p_{1,G}(T) \hookrightarrow L^q_{G}(\partial T) \) is compact for \( 1 \leq q < \frac{p}{2} \), while the trace embedding \( H^p_{1,G}(T) \hookrightarrow L^r_{G}(\partial T) \) is only continuous. Additionally, we observe that if \( \frac{3}{2} < p < 2 \) then \( q = \frac{2p}{2-p} > 6 = \frac{3}{3-2} \) and \( \tilde{q} > \frac{p}{2-p} > 4 = \frac{2(3-1)}{3-2} \), that is the exponents \( q \) and \( \tilde{q} \) are supercritical.

In this paper, we study the exceptional case when \( p = n-k = 3-1 = 2 \). In this case \( H^2_{1,G}(T) \not\subset L^\infty_{G}(T) \), however, when \( v \in H^2_{1,G}(T) \) we have \( e^v \in L^1_{G}(T) \), \( e^v \in L^1_{G}(\partial T) \) and the exponent \( p = 2 \) is the critical of supercritical.

This paper is organized as follows:

In Section 2, we recall some definitions and we present the two lemmas on which are based the proofs of the theorems concerning the best constants. Proofs of these lemmas are in Section 6. Section 3 is devoted in the presentation of results of the paper. In Section 4, we determine the best constants \( \mu \) and \( \tilde{\mu} \) of the inequalities:

\[
\int_T e^v dV \leq C \exp \left[ \mu \| \nabla v \|^2_2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV \right]
\]

and

\[
\int_{\partial T} e^v dS \leq C \exp \left[ \tilde{\mu} \| \nabla v \|^2_2 + \frac{1}{4\pi^2 r l} \int_{\partial T} v dS \right],
\]

In section 5, we use the above two inequalities, in order to solve the nonlinear exponential elliptic problems \((P_1)\) and \((P_2)\). Concerning problem \((P_1)\), we prove the existence of solutions of the associated variational problem. We study problem \((P_2)\) in the same way as the \((P_1)\), except its last part (case 4 of Theorem 3.4), which is based upon the method of upper solutions and lower solutions.

2 Notations and Preliminary Results

For completeness we cite some background material and results from [23]. Let \( \mathcal{A} = \{(\Omega_i, \xi_i) : i = 1, 2\} \) be an atlas on \( T \) defined by

\[
\Omega_1 = \{(x, y, z) \in T : (x, y, z) \not\in H^\infty_{XZ}\},
\]

\[
\Omega_2 = \{(x, y, z) \in T : (x, y, z) \not\in H^\infty_{XZ}\},
\]

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where
\[ H^+_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y = 0\} \]
\[ H^-_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x < 0, y = 0\} \]
and \( \xi_i : \Omega_i \to I_i \times D, i = 1, 2, \) with \( I_1 = (0, 2\pi), \ I_2 = (-\pi, \pi), \)
\[ D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 < 1\}, \ \partial D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 = 1\}, \]
\[ \xi_i(x, y, z) = (\omega_i, t, s), \ i = 1, 2 \] with \( \cos \omega_i = \frac{x}{\sqrt{x^2+y^2}}, \ \sin \omega_i = \frac{y}{\sqrt{x^2+y^2}}, \) where
\[ \omega_1 = \begin{cases} \arctan \frac{y}{x}, & x \neq 0 \\ \pi/2, & x = 0, y > 0 \\ 3\pi/2, & x = 0, y < 0 \end{cases} \]
\[ \omega_2 = \begin{cases} \arctan \frac{y}{x}, & x \neq 0 \\ \pi/2, & x = 0, y > 0 \\ -\pi/2, & x = 0, y < 0 \end{cases} \]
and
\[ t = \frac{\sqrt{x^2+y^2}-l}{r}, \ s = \frac{z}{r}, \ 0 \leq t, s \leq 1. \]
The Euclidean metric \( g \) on \((\Omega, \xi) \in \mathcal{A}\) can be expressed as
\[ (\sqrt{g} \circ \xi^{-1})(\omega, t, s) = r^2(l + rt). \]
For any \( G \)–invariant \( \nu \) we define the functions \( \phi(t, s) = (\nu \circ \xi^{-1})(\omega, t, s). \)
Then we have:
\[ \int_T e^{\nu}dV = 2\pi r^2 \int_D e^{\phi(t,s)}(l + rt) \ dt \ ds \]  \hspace{1cm} (1)
\[ ||\nabla \nu||^2_{L^2(T)} = 2\pi \int_D |\nabla \phi(t,s)|^2(l + rt) \ dt \ ds \]  \hspace{1cm} (2)
and
\[ \int_{\partial T} e^{\nu}dS = 2\pi r \int_{\partial D} e^{\phi(t,0)}(l + rt) \ dt, \]  \hspace{1cm} (3)
where by \( \phi \) we denote the extension of \( \phi \) on \( \partial D. \)
Consider a finite covering \((T_j)_{j=1,...,N}\), where
\[ T_j = \{(x, y, z) \in \mathcal{T} : (\sqrt{x^2+y^2}-l_j)^2 + (z-z_j)^2 < \delta_j^2\} \]
is a tubular neighborhood (an open small solid torus) of the orbit \( O_{P_j} \) of \( P_j \) under the action of the group \( G. \) \( P_j(x_j, y_j, z_j) \in \mathcal{T} \) and \( l_j = \sqrt{x_j^2+y_j^2} \) is the horizontal distance of the orbit \( O_{P_j} \) from the axis \( z'z \) and \( \delta_j = l_j \varepsilon_j \) for any \( \varepsilon_j > 0. \)
Then the following lemmas hold:
Lemma 2.1 1. For all $\varepsilon > 0$, there exists a constant $C_\varepsilon$, such that for all $v \in \tilde{H}^2_{1,G}(T_j)$ the following holds:

\[
\int_{T_j} e^v dV \leq C_\varepsilon \exp \left[ (1 + c\varepsilon) \frac{1}{32\pi^2 l_j} \| \nabla v \|_2^2 \right],
\]

where $c > 0$.

2. For all $\varepsilon > 0$, there exist constants $C_\varepsilon$ and $D_\varepsilon$, such that for all $v \in \tilde{H}^2_{1,G}$ the following holds:

\[
\int_T e^v dV \leq C_\varepsilon \exp \left[ \left( \frac{1}{32\pi^2 (l-r)} + \varepsilon \right) \| \nabla v \|_2^2 + D_\varepsilon \| v \|_2^2 \right].
\]

In addition the constant $\frac{1}{32\pi^2 (l-r)}$ is the best constant for the above inequality.

Lemma 2.2 Let $T$ be the solid torus, $2\pi^2 r^2 l$ be the volume of $T$ and $4\pi^2 rl$ be the volume of $\partial T$, then for all $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that:

1. For all functions $v \in \mathcal{H}_G$ the following inequality holds

\[
\int_T e^v dV \leq C_\varepsilon \exp \left[ (\mu + \varepsilon) \| \nabla v \|_2^2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV \right]
\]

(4)

2. For all functions $v \in \mathcal{H}_G$ the following inequality holds

\[
\int_{\partial T} e^v dS \leq C_\varepsilon \exp \left[ (\mu + \varepsilon) \| \nabla v \|_2^2 + \frac{1}{4\pi^2 rl} \int_{\partial T} v dS \right],
\]

(5)

where, for the first inequality, $\mu = \frac{1}{32\pi^2 (l-r)}$ if $\mathcal{H}_G = \tilde{H}^2_{1,G}$ and $\mu = \frac{1}{16\pi^2 (l-r)}$ if $\mathcal{H}_G = H^2_{1,G}$.

For the second inequality $\mu > \frac{1}{8\pi^2 (l-r)}$ for all $v \in H^2_{1,G}$.

The constant $\mu$ is the best constant for the above inequalities.

3 Statement of Results

3.1 Best constants on the solid torus

We have the following theorem:
Theorem 3.1 Let $T$ be the solid torus, $2\pi^2r^2l$ be the volume of $T$ and $4\pi r l$ be the volume of $\partial T$, then there exists a constant $C$ such that:

1. For all functions $v \in \mathcal{H}_G$ the following inequality holds

\[
\int_T e^v dV \leq C \exp \left[ \mu \|\nabla v\|_2^2 + \frac{1}{2\pi^2r^2l} \int_T v dV \right] \tag{6}
\]

2. For all functions $v \in \mathcal{H}_G$ the following inequality holds

\[
\int_{\partial T} e^v dS \leq C \exp \left[ \mu \|\nabla v\|_2^2 + \frac{1}{4\pi^2 r l} \int_{\partial T} v dS \right], \tag{7}
\]

where, for the first inequality, $\mu = \frac{1}{32\pi^2(l-r)}$ if $\mathcal{H}_G = \overset{\circ}{H}^2_{1,G}$ and $\mu = \frac{1}{16\pi^2(l-r)}$ if $\mathcal{H}_G = H^2_{1,G}$.

For the second inequality $\mu > \frac{1}{8\pi^2(l-r)}$ for all $v \in H^2_{1,G}$.

The constant $\mu$ is the best constant for the above inequalities.

Remark 3.1 In [32] Faget proved that for a compact 3–dimensional manifold without boundary the first best constant for inequality (6) is $\mu_3 = \frac{2}{81\pi}$ and the map: $H^1 \ni v \rightarrow e^v \in L^1$ is compact. Clearly, the best constant $\mu_3$ depends only on the dimension 3 of the manifold. For the solid torus, we prove that the first best constant for the same inequality (6) is $\mu = \frac{1}{32\pi^2(l-r)}$ and the map: $\overset{\circ}{H}^2_{1,G} \ni v \rightarrow e^v \in L^1_G$ is compact. In this case, the best constant $\mu$ depends on the geometry of the solid torus.

Corollary 3.1 For all $v \in \overset{\circ}{H}^2_{1,G}$ such that $\|\nabla v\|_2^2 \leq 2\pi(l+r)$ and for all $\alpha \leq 4\pi$ the following holds:

\[
\int_T e^{\alpha v^2} dV \leq C \cdot 2\pi^2 r^2 l \tag{8}
\]

where the constant $C$ is independent of $v \in \overset{\circ}{H}^2_{1,G}$. The constant $\alpha \leq 4\pi$ is the best, in the sense that, if $\alpha > 4\pi$ the integral in the inequality is finite but it can be made arbitrary large by an appropriate choice of $v$.

Remark 3.2 Corollary 3.1 is a special case of the result of Moser [44].
3.2 Resolutions of the Problems

For the problem

(P₁) \[ \Delta v + \gamma = f(x)e^v, \ v > 0 \quad \text{on} \ T, \ v|_{\partial T} = 0. \]

we have the theorem:

**Theorem 3.2** Consider a solid torus \( \overline{T} \) and the function \( f \) continuous and \( G \)-invariant.
Then the problem \( (P_1) \) accepts a solution that belongs to \( C_0^\infty \), if one of the following holds:

(a) \( \sup_T f < 0 \) if \( \gamma < 0 \).
(b) \( \int_T f dV < 0 \) and \( \sup_T f > 0 \) if \( \gamma = 0 \).
(c) \( \sup_T f > 0 \) if \( 0 < \gamma < \frac{8(l-r)}{lr^2} \).

For the problem

(P₂) \[ \Delta v + a + f e^v = 0, \ v > 0 \quad \text{on} \ T, \]
\[ \frac{\partial v}{\partial n} + b + g e^v = 0 \quad \text{on} \ \partial T, \]

we have the next theorem:

**Theorem 3.3** Consider a solid torus \( \overline{T} \) and the smooth functions \( f, g \) \( G \)-invariant and not both identical 0. If \( a, b \in \mathbb{R} \) and \( R = 2\pi^2 r^2 a + 4\pi^2 r b \), the problem \( (P_2) \) accepts a solution that belongs to \( C_0^\infty \) in each one of the following cases:

1. If \( a = b = 0 \) the necessary and sufficient condition is \( f \) and \( g \) not both \( \geq 0 \) and that \( \int_T f dV + \int_{\partial T} g dS > 0 \).
2. If \( a \geq 0 \) and \( b \geq 0 \), \( f, g \) not both \( \geq 0 \) everywhere and \( 0 < R < 4\pi^2(l-r) \). Particularly, if \( g = 0 \) then we can substitute the last condition with \( 0 < R < 8\pi^2(l-r) \).
3. If \( R > 0 \) (respectively \( R < 0 \)) it is necessary that \( f, g \) not both \( \geq 0 \) everywhere (respectively \( \leq 0 \)). Then there exists a solution of the problem in each one of the following cases:

(a) \( a < 0, b > 0, f < 0, g \leq 0 \) and \( b < \frac{l-r}{lr} \) if \( g \neq 0 \) or \( b < \frac{2(l-r)}{lr} \) if \( g = 0 \).
(b) \( a > 0, b < 0, f \leq 0, g < 0 \) and \( a < \frac{2(l-r)}{tr^2} \).

(c) \( a > 0, b < 0, f \geq 0, g > 0 \) and \( a < \frac{2(l-r)}{tr^2} \) if \( g \not\equiv 0 \) or \( a < \frac{4(l-r)}{tr^2} \) if \( g \equiv 0 \).

(d) \( a < 0, b > 0, f > 0, g \geq 0 \) and \( b < l-r \).

4. If \( a \leq 0, b \leq 0 \), not both = 0, it is necessary \( \int_T f dV + \int_{\partial T} g dS > 0 \). Then there exists a non empty subset \( S_{f, g} \) of \( \mathbb{R}^2 = \{(a, b) \neq (0, 0) : a \leq 0, b \leq 0\} \) with the property that if \((c, d) \in S_{f, g} \) then \((c', d') \in S_{f, g} \) for any \( c' \geq c, d' \geq d \) and such that the problem (P_2) has a solution if and only if \((a, b) \in S_{f, g} \).

\( S_{f, g} = \mathbb{R}^2 \) if and only if the functions \( f, g \) are \( \not\equiv 0 \) and \( \geq 0 \). For all \((a, b) \in \mathbb{R}^2 \) there exist functions \( f \) and \( g \) such that \( \int_T f dV + \int_{\partial T} g dS > 0 \) and \((a, b) \not\in S_{f, g} \).

4 Proofs of the Theorem concerning the best constants

Proof of Theorem 3.1 1. We give a proof by contradiction based on Lemma 2.2.

Assume that for any \( C_\alpha \), there exist \( v_\alpha \in \mathcal{H}_{1,G}^2 \) with \( \int_T v_\alpha dV = 0 \) such that

\[
\int_T e^{v_\alpha} dV > C_\alpha \exp \left( \mu \int_T |\nabla v_\alpha|^2 dV \right).
\]  \hspace{1cm} (9)

Set \( \phi_\alpha(t, s) = (v_\alpha \circ \xi^{-1})(\omega, t, s) \). By (9) because of (1) and (2) we obtain sequentially

\[
2\pi r^2 \int_D e^{\phi_\alpha}(l+rt)dt ds > C_\alpha \exp \left( 2\pi \mu \int_D |\nabla \phi_\alpha|^2 (l+rt) dt ds \right),
\]

\[
2\pi r^2 (l+r) \int_D e^{\phi_\alpha} dt ds > C_\alpha \exp \left( 2\pi \mu (l-r) \int_D |\nabla \phi_\alpha|^2 dt ds \right),
\]

and since \( \mu = \frac{1}{32\pi^2(l-r)} \) (see part 1 of Lemma 2.2) we have

\[
\int_D e^{\phi_\alpha} dt ds > \frac{C_\alpha}{2\pi r^2(l+r)} \exp \left( \frac{1}{16\pi} \int_D |\nabla \phi_\alpha|^2 dt ds \right).
\]
The last inequality means that for any $c_\alpha$, there exists $\phi_\alpha \in \dot{H}^{2}_{1}(D)$ with $\int_{D} \phi_\alpha dt ds = 0$, such that

$$\int_{D} e^{\phi_\alpha} dt ds > c_\alpha \exp \left( \frac{1}{16\pi} \int_{D} |\nabla \phi_\alpha|^2 dt ds \right),$$

which is a contradiction, (see Theorem 1 in [17]).

2. The proof of this part is similar to the proof of the first one. Let us sketch it. Assume that for any $\tilde{C}_\alpha$, there exist $\upsilon_\alpha \in H^{2}_{1,G}$ with $\int_{\partial T} \upsilon_\alpha dS = 0$ such that

$$\int_{\partial T} e^{\upsilon_\alpha} dS > \tilde{C}_\alpha \exp \left( \mu \int_{T} |\nabla \upsilon_\alpha|^2 dV \right)$$

(10)

and define the function $\phi_\alpha$ as in the first part.

By (10) because of (2) and (3) we take the inequality

$$\int_{\partial D} e^{\phi_\alpha} dt > \tilde{c}_\alpha \exp \left( \mu \int_{D} |\nabla \phi_\alpha|^2 dt ds \right),$$

where $\tilde{c}_\alpha = \frac{\tilde{c}_\alpha}{2\pi r(l+r)}$.

The last inequality is false (see Theorem 3 in [18]) and the theorem is proved.

□

Proof of Corollary 3.1. Given $\varepsilon > 0$, let $(T_j)_{j=1,...,N}$ be a finite covering of $\bar{T}$, where

$$T_j = \{ Q \in \mathbb{R}^3 : d(Q, O_{P_j}) < \delta_j, \quad \delta_j = l_j \varepsilon_j \quad \text{and} \quad \varepsilon_j \leq \varepsilon \}. $$

For $\bar{T}$ we build a $G$–invariant partition of unity $(h_j)_{j=1,...,N}$ relative to the $T_j$’s. If we denote $\Phi = \upsilon \circ \xi_j^{-1}$, $\Phi \in \dot{H}^{2}_{1}(D)$, for all $\upsilon \in \dot{H}^{2}_{1,G}$, following the
same argument as in the Lemma 2.1 we obtain

\[ \int_T e^{\alpha \nu^2} dV = \int_T \left( \sum_{j=1}^N h_j \right) e^{\alpha \nu^2} dV \]

\[ = \sum_{j=1}^N \int_{T_j} h_j e^{\alpha \nu^2} dV \]

\[ = \sum_{j=1}^N \int_{I \times D} (h_j \circ \xi_j^{-1}) e^{\alpha \nu^2} \circ \xi_j^{-1} (\sqrt{g} \circ \xi_j^{-1}) d\omega dt ds \]

\[ = 2\pi \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \nu^2} \circ \xi_j^{-1} \delta_j^2 (l_j + \delta_j t) dt ds \]

\[ = \frac{1}{\pi} \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} 2\pi^2 \delta_j^2 l_j (1 + \varepsilon_j t) dt ds \]

\[ \leq (1 + \varepsilon) \frac{1}{\pi} \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} Vol(T_j) dt ds \leq \]

\[ \leq (1 + \varepsilon) \frac{1}{\pi} Vol(T) \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} dt ds \]

\[ = (1 + \varepsilon) 2\pi r^2 l \int_D \sum_{j=1}^N (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} dt ds \]

\[ = (1 + \varepsilon) 2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds \]

or

\[ \int_T e^{\alpha \nu^2} dV \leq (1 + \varepsilon) 2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds. \quad (11) \]

Because of \(|\nabla \nu|^2 \leq 2\pi (l + r)\) and (2) we obtain \(|\nabla \Phi|^2 \leq 1\) and according to Theorem 2.47 of [3] for all \(\Phi \in \tilde{H}^1(D)\) with \(|\nabla \Phi|^2 \leq 1\) and for any \(\alpha \leq 4\pi\) the following inequality holds

\[ \int_D e^{\alpha \Phi^2} dt ds \leq C\pi, \quad (12) \]
where the constant $C$ is the same for all open and bounded subsets of $\mathbb{R}^2$.

Thus, from inequalities (11) and (12) we obtain

$$\int_T e^{\alpha v^2} dV \leq (1 + \varepsilon) C^2 \pi r^2 l. \quad (13)$$

Suppose now that inequality (13) does not hold for $\varepsilon = 0$. That is, there exists $v \in H^2_{1,G}$ with $\|\nabla v\|_2^2 \leq 2\pi (l + r)$ and $\theta > 0$ such that the following inequality holds

$$\int_T e^{\alpha v^2} dV \geq (1 + \theta) C^2 \pi r^2 l \quad (14)$$

By (14), and because of (11) we obtain

$$(1 + \varepsilon) 2\pi r^2 l \int_D e^{\alpha \Phi^2} dt \geq (1 + \theta) C^2 \pi r^2 l. \quad (15)$$

Since (15) holds for any $\varepsilon > 0$ we can choose $\varepsilon$ such that $\varepsilon < \theta$ and (15) yields

$$2\pi r^2 l \int_D e^{\alpha \Phi^2} dt \geq \frac{1 + \theta}{1 + \varepsilon} C^2 \pi r^2 l$$

or

$$\int_D e^{\alpha \Phi^2} dt > C\pi. \quad (16)$$

But according to Theorem 2.47 of [3] for all $\Phi \in \hat{H}^2_1(D)$ the following

$$\int_D e^{\alpha \Phi^2} dt \leq C\pi$$

holds. Thus (16) is false and the corollary is proved. \hfill \Box

5 Proofs of the Theorems concerning the problems

Proof of Theorem 3.2 We see that if $f$ is a constant the problem can be solved immediately. If $f = 0$ and $\gamma = 0$, solutions are all the constants. If $\gamma f > 0$ the constant $ln(\gamma/f)$ is the solution.

Consider the functional

$$I(v) = \int_T |\nabla v|^2 dV + 2\gamma \int_T vdV,$$
the set
\[ A = \left\{ \upsilon \in H_{1,G}^2 : \int_T f e^\upsilon dV = \gamma \text{Vol}(T) \right\} \]
and denote
\[ \nu = \inf_{\upsilon \in A} I(\upsilon). \]
If \( \gamma > 0 \), in order \( A \neq \emptyset \), it is necessary \( f \) to be somewhere positive, if \( \gamma < 0 \) it’s necessary \( f \) to be somewhere negative, and if \( \gamma = 0 \) it is necessary \( f \) to change sign. In the following we accept that \( f \) satisfies the above necessary condition and it is not a constant.

(a) \( \gamma < 0 \) and \( f \) negative everywhere.

Combining Jensen’s inequality:
\[ \frac{1}{\text{Vol}(T)} \int_T \upsilon dV \leq \ln \left( \frac{1}{\text{Vol}(T)} \int_T e^\upsilon dV \right) \]
along with the following inequality:
\[ \frac{1}{\text{Vol}(T)} \int_T e^\upsilon dV \leq \frac{1}{\text{Vol}(T) \sup f} \int_T f(x)e^\upsilon dV = \frac{\gamma}{\sup f} \]
we obtain
\[ \int_T \upsilon dV \leq \text{Vol}(T) \ln \left( \frac{\gamma}{\sup f} \right) \]
and thus
\[ I(\nu) \geq 2\gamma \text{Vol}(T) \ln \left( \frac{\gamma}{\sup f} \right). \]
From the last inequality we conclude that \( \nu \) is finite.

Let \( \{\upsilon_i\} \in A \) be a minimizing sequence of \( I \), that is \( I(\upsilon_i) \rightarrow \nu \). If we take \( I(\upsilon_i) \leq 1 + \nu \) we obtain
\[ \int_T |\nabla \upsilon_i|^2 dV + 2\gamma \int_T \upsilon_i dV \leq 1 + \nu \]
thus
\[ 1 + \nu - 2\gamma \int_T \upsilon_i dV \geq \int_T |\nabla \upsilon_i|^2 dV \geq 0 \]
and
\[ \int_T \upsilon_i dV \leq \frac{1 + \nu}{2\gamma}. \]

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By (17) and (18) we obtain $|\int_T v_i dV| < C$, where $C$ is a constant. In addition, we have

$$\int_T |\nabla v_i|^2 dV \leq 1 + \nu - 2\gamma \int_T v_i dV \leq 1 + \nu - 2\gamma C.$$  

Thus $\{v_i\}$ is bounded in $H^2_{1,G}(T)$ and there exists a subsequence of $v_i$, denoted again by $v_i$ and a function $\bar{v}$ such that:

(a) $\{v_i\} \rightharpoonup \bar{v}$ on $H^2_{1,G}(T)$, (by Banach’s Theorem),
(b) $\{v_i\} \rightharpoonup \bar{v}$ on $L^2_G(T)$, (by Kondrakov’s Theorem),
(c) $\{v_i\} \rightarrow \bar{v}$ a.e., (by Proposition 3.43 of [3]) and
(d) $\{e^{vi}\} \rightarrow e^{\bar{v}}$ on $L^1_G(T)$, (by Theorem 3.1).

From (c) arises that $\bar{v}$ is $G$–invariant and so $\bar{v} \in A$, thus $I(\bar{v}) \geq \nu$. From (d) we conclude that

$$\|\bar{v}\|_{H^2_1} \leq \lim_{i \to \infty} \inf \|v_i\|_{H^2_1} = \nu$$

and by definition of $\nu$ we obtain $I(\bar{v}) = \nu$.

Using the variation method we can prove that $\bar{v}$ is a week solution of the corresponding Euler equation and, by the regularization Theorem of [48] and Theorem 3.54 of [3], we conclude that $\bar{v} \in C^\infty_G$.

(b) $\gamma = 0$ and $f$ changes sign.

In this case we need the extra condition $\int_T f(x) dV < 0$, because if we multiply the equation of the problem by $e^{-v}$ and integrate we obtain

$$\int_T f(x) dV = \gamma \int_T e^{-v} dV - \int_T e^{-v} |\nabla v|^2 dV,$$

the second part of this equality is negative.

Since $\gamma = 0$,

$$I(v) = \int_T |\nabla v|^2 dV$$

and considering $\int_T v dV = 0$, if we define the set

$$\tilde{A} = \left\{ v \in H^2_{1,G} : \int_T v dV = 0, \int_T f e^v dV = 0 \right\}$$

we will have

$$\nu = \inf_{v \in \tilde{A}} I(v) \geq 0.$$
In the following we work in the same way as in \((a)\).
Thus, there exists a minimizing subsequence of \(v_i\), denoted again by \(v_i\) that’s converge on a function \(\bar{v} \in \tilde{A}\).

If \(\kappa\) and \(\lambda\) are the Lagrange multipliers, the Euler equation is

\[
\Delta \bar{v} + \kappa = \lambda f(x)e^\bar{v}.
\]

Intergrading by parts, because of \(\int_T f e^\bar{v}dV = 0\), we obtain \(\kappa = 0\) and for the function \(\bar{v}\) holds

\[
\Delta \bar{v} = \lambda f(x)e^\bar{v}.
\] (19)

By equation (19) we obtain that \(\bar{v}\) is not constant, because of \(\int_T f(x)dV < 0\), and so \(\lambda \neq 0\). In addition, multiplying the same equation by \(e^{-\bar{v}}\) and integrating by parts we obtain \(\lambda \int_T f(x)e^\bar{v}dV < 0\) and then \(\lambda > 0\).

Finally, is easy to check that the solution of the equation is \(\bar{v} - \ln \lambda\).

\(c\) \(\gamma > 0\) and \(f\) somewhere positive.
Consider the same variation problem as in case \((a)\) and suppose that \(f\) is somewhere positive, which is the necessary condition to be \(A \neq \emptyset\), since \(\sup_T f > 0\). We have

\[
\gamma \text{Vol} (V) = \int_T f e^\nu dV \leq \sup f \int_T e^\nu dV.
\] (20)

In addition by Theorem 3.1 we have

\[
\int_T e^\nu dV \leq C \exp \left\{ (\mu + \epsilon) \int_T |\nabla \nu|^2 dV + \frac{1}{\text{Vol} (V)} \int_T \nu dV \right\}.
\] (21)

From (20) and (21) we obtain

\[
\gamma \text{Vol} (V) \leq C \sup f \exp \left\{ (\mu + \epsilon) \int_T |\nabla \nu|^2 dV + \frac{1}{\text{Vol} (T)} \int_T \nu dV \right\},
\]

\[
\frac{\gamma \text{Vol} (V)}{C \sup f} \leq \exp \left\{ (\mu + \epsilon) \int_T |\nabla \nu|^2 dV + \frac{1}{\text{Vol} (T)} \int_T \nu dV \right\},
\]

\[
\ln \left( \frac{\gamma \text{Vol} (T)}{C \sup f} \right) \leq (\mu + \epsilon) \int_T |\nabla \nu|^2 dV + \frac{1}{\text{Vol} (T)} \int_T \nu dV,
\]

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\[ 2\gamma \text{Vol}(T) \ln \left( \frac{\gamma \text{Vol}(T)}{C_{\sup f}} \right) \leq 2\gamma \text{Vol}(T) (\mu + \varepsilon) \int_T |\nabla v|^2 \, dV + 2\gamma \int_T v \, dV, \]

\[ 2\gamma \text{Vol}(T) \ln \left( \frac{\gamma \text{Vol}(T)}{C_{\sup f}} \right) \leq 2\gamma \text{Vol}(T) (\mu + \varepsilon) \int_T |\nabla v|^2 \, dV + I(v) - \int_T |\nabla v|^2 \, dV, \]

\[ I(v) \geq 2\gamma \text{Vol}(T) \ln \left( \frac{\gamma \text{Vol}(T)}{C_{\sup f}} \right) + [1 - 2\gamma \text{Vol}(T)(\mu + \varepsilon)] \int_T |\nabla v|^2 \, dV \]

or

\[ I(v) \geq [1 - 2\gamma \text{Vol}(T)(\mu + \varepsilon)] \int_T |\nabla v|^2 \, dV + C', \tag{22} \]

where \( \mu = \frac{1}{32\pi^2(l-r)} \) and \( C' = 2\gamma \text{Vol}(T) \ln \left( \frac{\gamma \text{Vol}(T)}{C_{\sup f}} \right) \).

So, for \( \gamma < \frac{8(l-r)}{l+r} \), we have that \( I(v) \) is bounded below.

Thus if \( v_i \in A \) is a minimizing sequence of \( I \), by equation (22) we obtain that \( \| \nabla v_i \|_2^2 \leq C_1 \), and by equations (20) and (21) that \( \int_T v_i \, dV \geq C_2 \), where \( C_1 \) and \( C_2 \) are constants. Since \( \nu = \inf_{v \in A} I(v) \) and \( \lim_{i \to \infty} I(v_i) = \nu \) we may assume that \( I(v_i) < \nu + 1 \) and so \( \int_T v_i \, dV \leq C_3 \), where \( C_3 \) is a constant. Thus \( \{v_i\} \) is bounded in \( H^2_{1,G}(T) \) and then the rest of the proof is the same as in case (a).

**Proof of Theorem 3.3** Following [19], let \( v \in C^\infty_G(T) \) be a solution of (P_2). We observe that integration by parts yields

\[ \int_T (\Delta v + a + f e^v) \, dV = 0, \]

\[ -\int_{\partial T} \frac{\partial v}{\partial n} \, dS + \int_T (a + f e^v) \, dV = 0, \]

\[ \int_{\partial T} (b + ge^v) \, dS + \int_T (a + f e^v) \, dV = 0, \]

\[ a \int_T dV + b \int_{\partial T} dS + \int_T f e^v \, dV + \int_{\partial T} g e^v \, dS = 0, \]

\[ a \text{Vol}(T) + b \text{Vol}(\partial T) + \int_T f e^v \, dV + \int_{\partial T} g e^v \, dS = 0 \]
namely

\[ K(v) = aVol(T) + bVol(\partial T) + \int_T f e^v dV + \int_{\partial T} ge^v dS = 0. \]  

(23)

Multiplying by \( e^{-v} \) and integrating by parts also implies

\[ \int_T (e^{-v} \Delta v + ae^{-v} + f) dV = 0 \]

\[ -\int_{\partial T} e^{-v} \frac{\partial v}{\partial n} dS + \int_T (ae^{-v} + f) dV - \int_T e^{-v} |\nabla v|^2 dV = 0, \]

\[ \int_{\partial T} (e^{-v} b + g) dS + \int_T (ae^{-v} + f) dV - \int_T e^{-v} |\nabla v|^2 dV = 0 \]

namely

\[ a \int_T e^{-v} dV + b \int_{\partial T} e^{-v} dS + \int_T f dV + \int_{\partial T} g dS - \int_T e^{-v} |\nabla v|^2 dV = 0. \]  

(24)

Moreover, if \( v \in H^2_{1,G}(T) \), according to [3], [49] and [18] and because of theorem 3.1, for any \( q \geq 1 \), \( v \in L^q_G(T) \), \( v \in L^q_G(\partial T) \) and \( e^v \in L^q_G(T) \).

Set

\[ I(v) = \frac{1}{2} \int_T |\nabla v|^2 dV + a \int_T v dV + b \int_{\partial T} v dS \]

and

\[ A = \{ v \in H^2_{1,G} : K(v) = 0 \}. \]

Our aim is the minimization of \( I(v) \) on \( A \).

1. Case \( a = b = 0 \), \( \int_T f dV + \int_{\partial T} g dS > 0 \) and \( f \) and \( g \) not both \( \geq 0 \).

Since \( f \) and \( g \) are not both identically 0, the solutions of equation \( (P_2) \) are not constant functions. Hence if \( v \) is a solution we have

\[ \int_T e^{-v} |\nabla v|^2 dV > 0 \]  

(25)

and then by (24) and (25) yield

\[ \int_T f dV + \int_{\partial T} g dS > 0. \]
Since \( a = b = 0 \) and \( K(v) = \int_T f e^v dV + \int_{\partial T} g e^v dS \) in order \( A = \{ v \in H^2_1(T) : K(v) = 0 \} \neq \emptyset \) it’s necessary \( f \) and \( g \) not to be both \( \geq 0 \).

Inversely, if \( f \) and \( g \) are not both \( \geq 0 \), we will prove that \( A \neq \emptyset \).

Because of \( \int_T f dV + \int_{\partial T} g dS > 0 \),
we have
\[
\{ f(T) \cup g(\partial T) \} \cap (0, +\infty) \neq \emptyset \quad \text{and} \quad \{ f(T) \cup g(\partial T) \} \cap (-\infty, 0) \neq \emptyset.
\]

Define a \( C^\infty \) function \( \eta : [0, +\infty) \to [0, 1] \) such that \( \eta = 1 \) in \([0, 1/2]\), \( \eta = 0 \) in \([1, +\infty)\) and examine the following two cases:

(i) \( f \) changes sign on \( T \).

There are two tori \( T_1 \) and \( T_2 \) contained in \( T \) such that \( f > 0 \) on \( T_1 \) and \( f < 0 \) on \( T_2 \). Let the points \( P_i, i = 1, 2 \) belong to the central orbits \( O_{P_i}, i = 1, 2 \) of \( T_i, i = 1, 2 \), respectively and let
\[
T_1 = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_1})^2 + (z - z_{P_1})^2 < \delta^2 \right\}
\]
and
\[
T_2 = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_2})^2 + (z - z_{P_2})^2 < \delta^2 \right\},
\]
where \( l_{P_i} = \sqrt{x_{P_i}^2 + y_{P_i}^2}, i = 1, 2 \) the horizontal distance of the orbit \( O_{P_i}, i = 1, 2 \) from the axis \( z'z \).

Set
\[
\alpha = \int_{T\setminus(T_1 \cup T_2)} f dV + \int_{\partial T} g dS,
\]
and suppose that \( \alpha \geq 0 \). Then
\[
\alpha_0 = \alpha + \int_{T_1} f dV > 0.
\]

Consider the continuous function
\[
\sigma(t) = \int_{T_2} f(P) \exp \left[ t \eta \left( \frac{d(P, O_{P_2})}{\delta} \right) \right] dV, \ t \in \mathbb{R},
\]
where \( d \) is the Euclidean distance in \( \mathbb{R}^3 \).

Since \( \lim_{t \to +\infty} \sigma(t) = -\infty \) and \( \lim_{t \to -\infty} \sigma(t) = 0 \), there exists \( t_0 \in \mathbb{R} \) such that
σ(t_0) = -α_0.

Hence if we define the function \( v \in C_0^\infty(T) \) as

\[
v(P) = \begin{cases} 
  t_0 \eta(d(P, O)/\delta), & P \in T_2 \\
  0, & P \notin T_2
\end{cases}
\]

by definition of \( \sigma \) we obtain

\[
\sigma(t_0) = \int_{T_2} f(P)e^{v(P)}dV
\]

and then

\[
\int_{T} f(P)e^{v(P)}dV = -\alpha_0
\]

From the last equality we have

\[
\int_{T_2} f e^v dV + \int_{T_1} f dV = 0,
\]

\[
\int_{T_2} f e^v dV + \int_{T_2 \setminus (T_1 \cup T_2)} f dV + \int_{\partial T} g dS + \int_{T_1} f dV = 0,
\]

\[
\int_{T_2} f e^v dV + \int_{T_2 \setminus T_1} f dV - \int_{T_1} f dV + \int_{\partial T} g dS + \int_{T_1} f dV = 0,
\]

\[
\int_{T_2} f e^v dV + \int_{T_2 \setminus T_1} f dV + \int_{\partial T} g dS = 0
\]

and from this by definition of \( v \) we obtain

\[
\int_{T_2} f e^v dV + \int_{T_2 \setminus T_1} f e^v dV + \int_{\partial T} g e^v dS = 0,
\]

\[
\int_{T} f e^v dV + \int_{\partial T} g e^v dS = 0.
\]

This means that \( v \in A \) and hence \( A \neq \emptyset \).

(ii) \( f \) does not change sign on \( T \).

If \( f \equiv 0 \) and \( g \) changes sign, following arguments of the previous case, we construct a function \( v \in C_0^\infty(\overline{T}) \) such that \( \int_{\partial T} g e^v dS = 0 \), hence \( K(v) = 0 \)
and $A \neq \emptyset$.

If $f \not\equiv 0$, let us suppose that $f \geq 0$ and $K(u) = 0$. Then there exist $P_1 \in T$ and $P_2 \in \partial T$ such that $f(P_1) > 0$ and $g(P_2) < 0$.

Consider the tori

$$T_1 = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2 - l_{P_1}})^2 + (z - z_{P_1})^2 < \delta^2 \right\}$$

and

$$T_2 = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2 - l_{P_2}})^2 + (z - z_{P_2})^2 < \delta^2 \right\},$$

where $\delta$ is small enough, such that $T_1 \cap T_2 = \emptyset$, $f > 0$ a.e. in $T_1$ and $g < 0$ a.e. in $T_2 \cap \partial T$.

Set

$$\beta = \int_{T \setminus T_1} f \, dV + \int_{\partial T \setminus T_2} g \, dS + \int_{\partial T \cap T_2} g(P) \exp \left[ t\eta \left( \frac{2d(P, O_{P_2})}{\delta} \right) \right] \, dS$$

and choose $t$ large enough such that $\beta < 0$.

Denote

$$T_{2(\delta/2)} = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2 - l_{P_2}})^2 + (z - z_{P_2})^2 < \left( \frac{\delta}{2} \right)^2 \right\}$$

and define a function $\vartheta \in C^\infty(T)$, $0 \leq \vartheta(P) \leq 1$ such that $\vartheta = 1$ in a neighborhood of $\partial T \cap T_{2(\delta/2)}$, $\vartheta = 0$ out of $T_2$ and its support $K$ to have small enough measure such that the following holds

$$\gamma = \int_K f(P) \exp \left[ t\vartheta(P)\eta \left( \frac{2d(P, O_{P_2})}{\delta} \right) \right] \, dV - \int_K f(P) \, dV < -\beta.$$

Consider now the continuous function

$$\hat{\sigma}(t) = \int_{T_1} f(P) \exp \left[ t\eta \left( \frac{d(P, O_{P_1})}{\delta} \right) \right] \, dV, \ t \in \mathbb{R}.$$
Define now the function \( \nu \in \mathcal{C}^\infty(T) \) by

\[
\nu(P) = \begin{cases}
t' \eta \left( d(P, O_{P_1}) / \delta \right), & P \in T_1 \\
t \vartheta(P) \eta \left( 2d(P, O_{P_2}) / \delta \right), & P \in T_2 \\
0, & P \notin T_1 \cup T_2
\end{cases}
\]

We have

\[
\beta = \int_{T \setminus T_1} f dV + \int_{\partial T \setminus T_2} g dS + \int_{\partial T \cap T_2} g e^\nu dS
\]

\[
= \int_{T \setminus T_1} f dV + \int_{\partial T \setminus T_2} g e^\nu dS + \int_{\partial T \cap T_2} g e^\nu dS
\]

\[
= \int_{T \setminus T_1} f dV + \int_{\partial T} g e^\nu dS \quad (27)
\]

and

\[
\gamma = \int_{T_2} f(P) \exp \left[ t \vartheta(P) \eta \left( \frac{2d(P, O_{P_2})}{\delta} \right) \right] dV
\]

\[
- \int_{T_2 \setminus K} f(P) \exp \left[ t \vartheta(P) \eta \left( \frac{2d(P, O_{P_2})}{\delta} \right) \right] dV - \int_{K} f(P) dV
\]

\[
= \int_{T_2} f e^\nu dV - \int_{T_2 \setminus K} f dV - \int_{K} f dV. \quad (28)
\]

By (26), (27) and (28) we now obtain

\[
\int_{T \setminus T_1} f dV - \int_{T_2 \setminus K} f dV - \int_{K} f dV + \int_{T_1} f e^\nu dV + \int_{T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS = 0,
\]

\[
\int_{T \setminus T_1} f dV - \int_{T_2} f dV + \int_{T_1} f e^\nu dV + \int_{T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS = 0,
\]

\[
\int_{T \setminus (T_1 \cup T_2)} f dV + \int_{T_1 \cup T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS = 0,
\]

\[
\int_{T \setminus (T_1 \cup T_2)} f e^\nu dV + \int_{T_1 \cup T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS = 0,
\]
\[ \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS = 0. \]

Hence \( v \in A \) and \( A \neq \emptyset \).

We observe that if \( K(v) = 0 \) then \( K(v + c) = 0 \) for any constat \( c \). So we can suppose that \( \int_T vdV = 0 \) for any \( v \in A \).

Set

\[ \mu = \inf_{v \in A} \left\{ \int_T |\nabla v|^2 dV : \int_T vdV = 0 \right\} \geq 0. \]

Let \( \{v_i\} \) be a minimizing sequence. Since \( \sup_i(\|\nabla v_i\|^2) < +\infty \), this is bounded in \( H^2_{1,G}(T) \). Thus there exists a subsequence \( \{v_{i_j}\} \) and a function \( v \in H^2_{1,G}(T) \) such that:

(a) \( \{v_{i_j}\} \rightharpoonup v \) on \( H^2_{1,G}(T) \), (by Banach’s Theorem),
(b) \( \{v_{i_j}\} \to v \) on \( L^q_G(T), q \geq 1 \), (by Kondrakov’s Theorem),
(c) \( \{v_{i_j}\} \to v \) a.e., (by Proposition 3.43 of [3]),
(d) \( \{e^{v_{i_j}}\} \to e^v \) (by Theorem 3.11) and
(e) \( \{v_{i_j}\} \to v \) a.e., on \( \partial T \) and \( \{e^{v_{i_j}}\} \to e^v \) on \( L^q_G(\partial T) \),

where by \( v_i \) and \( v \) we denote the trace of \( v_{i_j} \) and \( v \) on \( \partial T \), respectively (by Theorem 4 of [15]).

The latter implies

\[ \lim_{i \to \infty} \int_T v_{i_j} dV = \int_T vdV \]

and

\[ \lim_{i \to \infty} \left( \int_T f e^{v_{i_j}} dV + \int_{\partial T} g e^{v_{i_j}} dS \right) = \int_T f e^v dV + \int_{\partial T} g e^v dS = 0. \]

From the last two equalities along with (c) arises that \( v \in A \) and \( \int_T vdV = 0 \), hence, by definition of \( \mu \), \( \|\nabla v\|^2_2 \geq \mu \).

From (b) and using Theorem 3.17 of [3] we obtain

\[ \|\nabla v\|^2_2 \leq \lim_{i \to \infty} \inf \|\nabla v_{i_j}\|^2_2 = \mu. \]

Thus, by definition of \( \mu \), \( \|\nabla v\|^2_2 = \mu \) and the \( \inf \|\nabla v_{i_j}\|^2_2 \) is attained, where \( v_i \in A \) and \( \int_T v_i dV = 0 \).

If \( \kappa \) and \( \lambda \) are the Lagrange multipliers, the Euler equation is

\[ \int_T \nabla^i v \nabla_i hdV + \kappa \left( \int_T f e^v hdV + \int_{\partial T} g e^v hdS \right) + \lambda \int_T hdV = 0, \quad (29) \]
for all $h \in H_1^2(T)$.

Since $K(\nu) = 0$, for $h = 1$ arises $\lambda = 0$, and for $h = \nu$, $\kappa \neq 0$. (If $\kappa = 0$, $\|\nabla \nu\|_2 = 0$ and since $\int_T \nu dV = 0$, $\nu = 0$ a.e. thus $K(\nu) > 0$, which is false). According to Theorem 1 of [18] the solution $\nu \in H_1^2$ of (29) is $C^\infty$ and if satisfies:

$$
\begin{align*}
\Delta \nu + \kappa f e^\nu &= 0 \quad \text{in } T \\
\frac{\partial \nu}{\partial n} + \kappa g e^\nu &= 0 \quad \text{on } \partial T
\end{align*}
$$

(30)

Setting $h = e^{-\nu}$ in (29) we find

$$
\kappa = \left( \int_T f dV + \int_{\partial T} g dS \right)^{-1} \int_T |\nabla \nu|^2 e^{-\nu} dV > 0
$$

(31)

and then $\nu - \ln \kappa$ is a solution of $(P_2)$.

2. Case $a \geq 0$, $b \geq 0$, not both $\equiv 0$ and

$$
f^{-1}\left((-\infty, 0)\right) \neq \emptyset \quad \text{or} \quad g^{-1}\left((-\infty, 0)\right) \neq \emptyset.
$$

In this case we have

$$
R = aVol(T) + bVol(\partial T) > 0
$$

and by (23)

$$
\int_T f e^\nu dV + \int_{\partial T} g e^\nu dS < 0.
$$

Then, if $f, g$ are not both $\geq 0$, $A \neq \emptyset$.

By Theorem 3.1 arises that, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$
\int_T e^\nu dV \leq C_\varepsilon \exp \left(1 + \varepsilon\right) \frac{1}{16\pi^2 (l - r)} \|\nabla \nu\|^2_2 + \frac{1}{Vol(T)} \int_T \nu dV
$$

(32)

and

$$
\int_{\partial T} e^\nu dS \leq C_\varepsilon \exp \left(1 + \varepsilon\right) \frac{1}{8\pi^2 (l - r)} \|\nabla \nu\|^2_2 + \frac{1}{Vol(\partial T)} \int_{\partial T} \nu dS, \quad (33)
$$

for all $\nu \in H_1^2$.

From the definitions of $K(\nu)$ and $R$ and by (23) we obtain

$$
R = \left| \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS \right| \leq \left( \max_T |f| \right) \int_T e^\nu dV + \left( \max_{\partial T} |g| \right) \int_{\partial T} e^\nu dS
$$
and using (32), (33) we obtain

\[
R \leq \left( \max_T |f| \right) C_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{16\pi^2 (l - r)} \| \nabla v \|^2_2 + \frac{1}{Vol(T)} \int_T v dV \right] \\
+ \left( \max_{\partial T} |g| \right) C_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \| \nabla v \|^2_2 + \frac{1}{Vol(\partial T)} \int_{\partial T} v dS \right]
\]

\[
\leq \left( \max_T |f| \right) C_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \| \nabla v \|^2_2 + \frac{1}{Vol(T)} \int_T v dV \right] \\
+ \left( \max_{\partial T} |g| \right) C_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \| \nabla v \|^2_2 + \frac{1}{Vol(\partial T)} \int_{\partial T} v dS \right].
\]

The last inequality gives

\[
\inf_{v \in A} \left\{ \int_T v dV + (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} Vol(T) \| \nabla v \|^2_2 \right\} = c_T(\varepsilon) > -\infty \quad (34)
\]

and

\[
\inf_{v \in A} \left\{ \int_{\partial T} v dS + (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} Vol(\partial T) \| \nabla v \|^2_2 \right\} = c_{\partial T}(\varepsilon) > -\infty. \quad (35)
\]

By (34), (35) we obtain

\[
I(v) = \frac{1}{2} \int_T |\nabla v|^2 dV + a \int_T v dV + b \int_{\partial T} v dS
\]

\[
\geq \left[ \frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} (a Vol(T) + b Vol(\partial T)) \right] \| \nabla v \|^2_2 + ac_T(\varepsilon) + bc_{\partial T}(\varepsilon)
\]

\[
= \left[ \frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} R \right] \| \nabla v \|^2_2 + ac_T(\varepsilon) + bc_{\partial T}(\varepsilon). \quad (36)
\]

If we assume \( R < 4\pi^2 (l - r) \) and if we choose \( \varepsilon > 0 \) such that \( c = \frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} R > 0 \), by (36) we conclude that \( \mu = \inf_{v \in A} I(v) > -\infty \).

Let \( \{v_i\}_{i \in \mathbb{N}}, v_i \in A \) be a minimizing sequence of \( I(v) \) such that

\[
\mu \leq I(v_i) \leq \mu + 1 \quad (37)
\]

for any \( i \in \mathbb{N} \) (36) and (37) yield

\[
0 \leq \| \nabla v_i \|^2_2 \leq \frac{I(v_i) - ac_T(\varepsilon) - bc_{\partial T}(\varepsilon)}{c} \leq \frac{\mu + 1 - ac_T(\varepsilon) - bc_{\partial T}(\varepsilon)}{c} < +\infty.
\]
By (34), (35) and (36) we also obtain
\[ \int_T \nu_i \, dV \geq c_T(\varepsilon) - (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \text{Vol}(T) (\mu + 1) = C_T \] (38)

and
\[ \int_{\partial T} \nu_i \, dS \geq c_{\partial T}(\varepsilon) - (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \text{Vol}(\partial T) (\mu + 1) = C_{\partial T}. \] (39)

By the definition of \( I(\nu) \) and because of (37) yields
\[ a \int_T \nu_i \, dV + b \int_{\partial T} \nu_i \, dS \leq I(\nu_i) \leq \mu + 1. \]

The last relation, because of (38), (39) gives us
\[ \int_T \nu_i \, dV \leq \frac{\mu + 1}{a} - C_{\partial T} \quad \text{if} \quad a \neq 0 \] (40)

and
\[ \int_{\partial T} \nu_i \, dS \leq \frac{\mu + 1}{b} - C_T \quad \text{if} \quad b \neq 0. \] (41)

By (37), (38), (40) and (41) we have
\[ \left| \int_T \nu_i \, dV \right| \leq C_1 \] (42)

and
\[ \left| \int_{\partial T} \nu_i \, dS \right| \leq C_2. \] (43)

Since the inequality
\[ \int_T \phi^2 \, dV \leq C \left( \int_T |\nabla \phi|^2 \, dV + \frac{1}{\text{Vol}(T)} \int_T \phi^2 \, dV \right) \] (44)

holds for any \( \phi \in H^2_{1,G}(T) \), taking into account that (37) and (38) also hold, it follows that \( \{v_i\}_{i \in \mathbb{N}}, v_i \in A \) is bounded in \( L^2_{G}(T) \). Moreover, since (38) holds we conclude that \( \sup_{i \in \mathbb{N}} \left( \|v_i\|_{H^2_{1,G}} \right) < \infty \). Hence, as in the previous
Recall that, if \( \nu \) is the Lagrange multiplier, the Euler equation is

\[
\int_T \nabla^i \nu \nabla_i h dV + a \int_T h dV + b \int_{\partial T} h dS = \nu \left( \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS \right),
\]

for all \( h \in H^2_1(T) \).

For \( h = 1 \) since \( K(\nu) = 0 \) we find

\[
\nu = \left( a \text{Vol}(T) + b \text{Vol}(\partial T) \right)^{-1} \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS - 1 = 1.
\]

Using the same arguments as in case 1, we prove that \( \nu \in C^\infty_G(T) \) and that is a solution of (P_2).

If \( g \equiv 0 \) we have

\[
R = \left| \int_T f e^\nu dV \right| \leq \left( \max_T |f| \right) \int_T e^\nu dV
\]

\[
\leq \left( \max_T |f| \right) C_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{16\pi^2 (l - r)} \| \nabla \nu \|_2^2 + \frac{1}{\text{Vol}(T)} \int_T \nu dV \right].
\]

Hence, if \( R < 8\pi^2(l - r) \), following the same process as above we prove that (P_2) has a solution.

3. Suppose that \( R > 0 \) and \( a, b \) not both \( \geq 0 \) (the case \( R < 0 \) and \( a, b \) not both \( \leq 0 \) can be treated in the same way).

By (23) it is necessary that \( f, g \) are not both \( \geq 0 \) everywhere. Then \( A \neq \emptyset \).

(a) \( a < 0, b > 0, f < 0, g \leq 0 \) and \( b \text{Vol}(\partial T) < 4\pi^2(l - r) \) if \( g \not\equiv 0 \) or \( b \text{Vol}(\partial T) < 8\pi^2(l - r) \) if \( g \equiv 0 \).

Since \( f \in C^\infty_G(T) \) is negative everywhere and \( T \) is compact, there exists \( \delta > 0 \) such that \( |f| \geq \delta > 0 \).

If \( \nu \in A \) we have

\[
|R| = \left| \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS \right| = \int_T |f| e^\nu dV + \int_{\partial T} |g| e^\nu dS
\]

and by elementary inequality \( e^x \geq 1 + x, x \in \mathbb{R} \) we obtain

\[
|R| \geq \int_T |f| e^\nu dV \geq \delta \int_T e^\nu dV \geq \delta \int_T (1 + \nu) dV = \delta \text{Vol}(T) + \delta \int_T \nu dV.
\]
Since $a < 0$ we finally obtain
\[ a \int_T v dV \geq a \left( \frac{|R|}{\delta} - Vol(T) \right). \tag{47} \]

By (33) implies that for any $\varepsilon > 0$ there exists a constant $\tilde{C}_\varepsilon$ such that
\[ |R| \leq \tilde{C}_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{8\pi^2 (l - r)} \| \nabla v \|_2^2 + \frac{1}{Vol(\partial T)} \int_{\partial T} v dS \right]. \tag{48} \]

By (48) we obtain
\[ b \int_{\partial T} v dS \geq b Vol(\partial T) \ln \frac{|R|}{\tilde{C}_\varepsilon} - (1 + \varepsilon) \frac{b Vol(\partial T)}{8\pi^2 (l - r)} \| \nabla v \|_2^2. \tag{49} \]

By the definition of $I(v)$ and (47), (49) we obtain
\[ I(v) \geq \left[ \frac{1}{2} - (1 + \varepsilon) \frac{b Vol(\partial T)}{8\pi^2 (l - r)} \right] \| \nabla v \|_2^2 + b Vol(\partial T) \ln \frac{|R|}{\tilde{C}_\varepsilon} + a \left( \frac{|R|}{\delta} - Vol(T) \right). \tag{50} \]

If $b Vol(\partial T) < 4\pi^2 (l - r)$, that is $b < \frac{l - r}{l}$ and $\varepsilon$ is chosen small enough, (50) implies that $I(v)$ is bounded below for all $v \in A$ and we can prove the existence of a solution of $(P_2)$ as in the previous cases.

If $g \equiv 0$, it suffices to assume that $b < \frac{2(l-r)}{lr}$ and then by (48) we obtain
\[ |R| = \int_T |f| e^u dV \leq \tilde{C}_\varepsilon \exp \left[ (1 + \varepsilon) \frac{1}{16\pi^2 (l - r)} \| \nabla v \|_2^2 + \frac{1}{Vol(\partial T)} \int_{\partial T} v dS \right] \]
and we continue as above.

**Case (b)** $a > 0$, $b < 0$, $f \leq 0$, $g < 0$ and $a Vol(T) < 4\pi^2(l - r)$.

We work as in the previous case and, supposing that $a < \frac{2(l-r)}{lr^2}$ we conclude the existence of a solution of $(P_2)$.

Cases (c) and (d) are similar to (b).

**4.** Case $a \leq 0$, $b \leq 0$, not both = 0.

By (24) it is necessary to assume that $\int_T f dV + \int_{\partial T} g dS > 0$ and by (23) arises that $f$, $g$ are not both $\leq 0$ a.e.

The proof of this case is based upon the method of upper solutions and lower
solutions and is the same as the one in Theorem 2, case (iv) of [19].

Let us sketch the proof: It suffices to find functions $\upsilon_-, \upsilon_+ \in C_0^\infty(T)$ such that $\upsilon_+ \geq \upsilon_-$ which satisfy the equations

$$
\begin{aligned}
\Delta \upsilon_+ + a + fe^{\upsilon_+} &\geq 0 \quad \text{in} \quad T \\
\frac{\partial \upsilon_+}{\partial n} + b + ge^{\upsilon_+} &\geq 0 \quad \text{on} \quad \partial T
\end{aligned}
$$

(51)

and

$$
\begin{aligned}
\Delta \upsilon_- + a + fe^{\upsilon_-} &\leq 0 \quad \text{in} \quad T \\
\frac{\partial \upsilon_-}{\partial n} + b + ge^{\upsilon_-} &\leq 0 \quad \text{on} \quad \partial T
\end{aligned}
$$

(52)

respectively.

We denote by $P_{(a,b)}$ the nonlinear problem $(P_2)$ and solve this case in four steps. More precisely, we prove that:

1. For any $u \in C^0_G(T)$, $P_{(a,b)}$ accepts a lower solution $\upsilon_-$ such that $\upsilon_- \leq u$.
2. For any $u \in C^0_G(T)$, $P_{(a,b)}$ accepts an upper solution $\upsilon_+$ such that $\upsilon_+ \geq u$.
3. Choosing $f$, $g$ appropriately, the set $S_{f,g}$ can be contained in $\mathbb{R}_2^- = \{(a, b) \neq (0, 0) : a \leq 0, b \leq 0\}$ strictly.
4. If $f$, $g$ are $\not\equiv 0$ and nonnegative everywhere then $S_{f,g} = \mathbb{R}_2^-$.

\[\square\]

6 Proofs of the Lemmas

Proof of Lemma 2.1 1. Let $\varepsilon_0 > 0$ and $(T_j)_{j=1,\ldots,N}$ be a finite covering of $\bar{T}$, where

$$
T_j = \{Q \in \mathbb{R}^3 : d(Q, O_{P_j}) < \delta_j, \quad \delta_j = l_j \varepsilon_j \quad \text{and} \quad \varepsilon_j \leq \varepsilon_0\}
$$

Then for any $u \in C_{0,G}^\infty(T_j)$ by (11) we obtain

$$
\int_{T_j} e^u dV = \int_{I \times D} e^{\varepsilon_j^{-1}} (\sqrt{g} \circ \xi_j^{-1}) d\omega dtds
$$

$$
= 2\pi l_j \delta_j^2 \int_D e^\phi (1 + \frac{\delta_j}{l_j}t) dtds
$$

$$
\leq 2\pi l_j \delta_j^2 (1 + \varepsilon_0) \int_D e^\phi dtds.
$$
From this and by Theorem 1 of [17] we have

\[
\int_{T_j} e^\nu dV \leq 2\pi l_j \delta_j^2 C (1 + \varepsilon_0) \exp \left( \mu_2 \int_D |\nabla \phi|^2 dtds \right),
\]

where \( \mu_2 = \frac{1}{16\pi} \) is the best constant of Sobolev inequality

\[
\int_D e^f dV \leq C \exp \left[ \mu_2 \|\nabla f\|_2^2 \right],
\]

with \( f \in \dot{H}^2(D) \).

Moreover from [2] we obtain

\[
\int_{T_j} |\nabla \psi|^2 dV = 2\pi \int_D |\nabla \phi|^2 (l_j + \delta_j t) dtds \\
\geq 2\pi l_j (1 - \varepsilon_0) \int_D |\nabla \phi|^2 dtds,
\]

thus

\[
\int_D |\nabla \phi|^2 dtds \leq \frac{1}{2\pi l_j} \frac{1}{1 - \varepsilon_0} \int_{T_j} |\nabla \psi|^2 dV.
\]

Finally, we have

\[
\int_{T_j} e^\psi dV \leq C_{\varepsilon_0} \exp \left[ (1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \int_{T_j} |\nabla \psi|^2 dtds \right],
\]

where \( C_{\varepsilon_0} = 2\pi l_j \delta_j^2 C (1 + \varepsilon_0) \) and \( \frac{1}{1 - \varepsilon_0} = 1 + c\varepsilon_0, c > 0 \).

2. Let us choose \( \delta > 0 \) such that the torus \( \bar{T} \) is covered by \( N \) open subsets

\[
T_{j, \delta/2} = \{Q \in T : d(Q, O_{P_j}) < \delta/2\}
\]

We consider the decreasing real valued \( C^\infty \) function \( \Psi(r) \), which equals 1 for \( 0 \leq r \leq \delta/2 \) and 0 for \( r \geq \delta \) and we note \( \Psi_j(Q) = \Psi(d(Q, O_{P_j})) \).

The \( \Psi_j \)'s defined on \( T_j = \{Q \in T : d(Q, O_{P_j}) < \delta\} \) are \( G \)-invariant, but they are not a partition of unity.
Let \( v \in C_{0,G}^\infty(T) \). Then \((v\Psi_j) \in C_{0,G}^\infty(T_j)\) and from the first part of this lemma we obtain

\[
\int_{T_j} e^{v\Psi_j} dV \leq C \exp \left( (1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \| \nabla (v\Psi_j) \|^2 \right)
\]

(Because of the following relation)

\[
\| \nabla (v\Psi_j) \|^2 \leq \| \Psi_j \nabla v \|^2 + 2 \| \Psi_j \nabla v \| \| v \nabla \Psi_j \| + \| v \nabla \Psi_j \|^2.
\]

and since for all \( \varepsilon_0 > 0 \) a constant \( D_{\varepsilon_0} \) exists such that

\[
\| \Psi_j \nabla v \| \| v \nabla \Psi_j \| \leq \varepsilon_0 \| \Psi_j \nabla v \|^2 + D_{\varepsilon_0} \| v \nabla \Psi_j \|^2,
\]

we obtain

\[
\| \nabla (v\Psi_j) \|^2 \leq (1 + 2\varepsilon_0) \| \nabla v \|^2 + \tilde{D} \| v \|^2,
\]

where \( \tilde{D} = (2D_{\varepsilon_0} + 1) (\sup_{T} |\nabla \Psi_j|^2) \).

From (53) because of the last inequality we have

\[
\int_{T_j} e^{v\Psi_j} dV \leq C \exp \left( (1 + c\varepsilon_0) (1 + 2\varepsilon_0) \frac{\mu_2}{2\pi l_j} \| \nabla v \|^2 + \tilde{D} \| v \|^2 \right),
\]

(54)

where \( D = (1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \tilde{D} \).

Since \( \inf l_j = l - r \) given \( \varepsilon > 0 \) we can choose \( \varepsilon_0 \) small enough such that from (54) we obtain

\[
\int_T e^v dV \leq \sum_{i=1}^N \int_{T_j,\delta/2} e^v dV
\]

\[
\leq \sum_{i=1}^N \int_{T_j,\delta} e^{v\Psi_j} dV
\]

\[
\leq C \exp \left( \left( \frac{\mu_2}{2\pi (l - r)} + \varepsilon \right) \| \nabla v \|^2 + D \| v \|^2 \right),
\]

and so we have the desired inequality.

Now we need to prove that the constant \( \frac{\mu_2}{2\pi (l - r)} \) is the best constant \( \mu \) such that the inequality

\[
\int_T e^v dV \leq C \exp \left( (\mu + \varepsilon) \| \nabla v \|^2 + D \| v \|^2 \right)
\]

is satisfied.
holds for all $v \in H_{1,G}^2$.

For that purpose, for all $\varepsilon$, we need to find a sequence $(v_{\alpha}) \in H_{1,G}^2$, such that for all $\Delta, E \in \mathbb{R}$ the following holds:

$$
\lim_{\alpha \to 0} \frac{\|\nabla v_{\alpha}\|_2^2 + \Delta \|v_{\alpha}\|_2^2 + E}{\ln \int_T e^{v_{\alpha}} dV} \leq \frac{2\pi(l - r)}{\mu^2} + \varepsilon. \quad (55)
$$

Let us consider the orbit $O_{inf}$ of minimum length $2\pi(l - r)$. For any $\varepsilon_0 > 0$, let

$$
T_{j_0} = \{Q \in \mathbb{R}^3 : d(Q, O_{inf}) < \delta, \quad \delta = \varepsilon_0(l - r)\},
$$

where $d(Q, O_{inf})$ denotes the distance from $Q$ to the orbit $O_{inf}$.

It is easy to prove that

$$
d(Q, O_P) = \delta d_D(\xi_{j_0}(Q), O) = \delta \sqrt{t^2 + s^2}, \quad (56)
$$

where $d_D$ denotes the distance in the disc $D$ centered on $O$.

For all $\alpha > 0$ define the functions $(v_{\alpha})$ by

$$
v_{\alpha}(Q) = \begin{cases}
-2\ln(\alpha + d^2(Q, O_{inf})) + 2\ln(\alpha + \delta^2), & \text{if } Q \in T \cap T_{j_0}
\end{cases},

if \ Q \in T \setminus T_{j_0}.
$$

Since $v_{\alpha}$ depends only on the distance to $O_{inf}$, $v_{\alpha} \in H_{1,G}^2(T_{j_0})$.

Setting $\phi_{\alpha} = v_{\alpha} \circ \xi_{j_0}^{-1}$ we obtain

$$
\int_T e^{v_{\alpha}} dV = \int_{I \times D} e^{v_{\alpha} \circ \xi_{j_0}^{-1}}(\sqrt{g} \circ \xi_{j_0}^{-1}) d\omega dt ds

= \int_{I \times D} e^{\phi_{\alpha}} \delta^2 ((l - r) + \delta t) d\omega dt ds

= 2\pi (l - r) \delta^2 \int_D e^{\phi_{\alpha}} \left(1 + \frac{\delta}{l - r} t\right) dt ds

\geq 2\pi (l - r) \delta^2 (1 - \varepsilon_0) \int_D e^{\phi_{\alpha}} dt ds. \quad (57)
$$

Hence, by definition of $v_{\alpha}$ and because of (56) for all $\xi_{j_0}(Q) = (t, s) \in D$ we obtain

$$
\phi_{\alpha}(\xi_j(Q)) = \ln \left(\frac{\alpha + \delta^2}{\alpha + \delta^2 (t^2 + s^2)}\right)^2
$$
thus
\[ \int_D e^{\phi_\alpha} dt ds = \int_D \left( \frac{\alpha + \delta^2}{\alpha + \delta^2 (t^2 + s^2)} \right)^2 dt ds. \]

Changing variables in the latter equality we obtain
\[
\int_D e^{\phi_\alpha} dt ds = \int_0^{2\pi} \int_0^1 \frac{(\alpha + \delta^2)^2 r}{(\alpha + \delta^2 r^2)^2} dr d\theta
= \frac{(\alpha + \delta^2)^2 \pi}{\delta^2} \int_0^1 \frac{\alpha + \delta^2 r^2}{\alpha + \delta^2 r^2} dr
= \frac{(\alpha + \delta^2)^2 \pi}{\alpha}. \quad (58)
\]

By (57) and (58) we have
\[
\int_T e^{\nu_\alpha} dV \geq (1 - \varepsilon_0) 2\pi (l - r) \delta^2 \frac{(\alpha + \delta^2) \pi}{\alpha}
\geq (1 - \varepsilon_0) 2\pi^2 (l - r) \delta^4 \frac{1}{\alpha}
= C_{\varepsilon_0} \frac{1}{\alpha},
\]
and then
\[
\ln \int_T e^{\nu_\alpha} dV \geq \ln C_{\varepsilon_0} + \ln \frac{1}{\alpha}, \quad (59)
\]
where \( C_{\varepsilon_0} = (1 - \varepsilon_0) 2\pi^2 (l - r) \delta^4. \)

Moreover, because of (2) we have
\[
\|\nabla u_\alpha\|_2^2 = 2\pi \int_D |\nabla \phi_\alpha(t, s)|^2 ((l - r) + \delta t) dt ds
\leq (1 + \varepsilon_0) 2\pi (l - r) \int_D |\nabla \phi_\alpha(t, s)|^2 dt ds. \quad (60)
\]

Since
\[
|\nabla \phi_\alpha(t, s)|^2 = |\nabla [-2 \ln (\alpha + \delta^2 (t^2 + s^2)) + 2 \ln (\alpha + \delta^2)]|^2
= |-2\nabla \ln (\alpha + \delta^2 (t^2 + s^2))|^2
= 4 \left| \left( \frac{2\delta^2 t}{\alpha + \delta^2 (t^2 + s^2)}, \frac{2\delta^2 s}{\alpha + \delta^2 (t^2 + s^2)} \right) \right|^2
= \frac{16\delta^4 (t^2 + s^2)}{[\alpha + \delta^2 (t^2 + s^2)]^2},
\]
\]
we have

\[
\int_D |\nabla \phi_\alpha(t, s)|^2 dt ds = \int_D \frac{16\delta^4 (t^2 + s^2)}{[\alpha + \delta^2 (t^2 + s^2)]^2} dt ds = 2\pi \int_0^1 \frac{16\delta^4 r^2}{(\alpha + \delta^2 r^2)^2} dr.
\]

Changing variables we obtain

\[
\int_D |\nabla \phi_\alpha(t, s)|^2 dt ds = 16\pi \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau
\]

\[
= \frac{1}{\mu^2} \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau. \quad (61)
\]

We further define the function

\[
h(\alpha) = \ln \frac{1}{\alpha} - \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau, \alpha > 0
\]

and changing the variable we obtain

\[
h(\alpha) = \ln \frac{1}{\alpha} - \int_0^{\delta^2} \frac{\tau}{\alpha^2 (1 + \frac{\tau}{\alpha})^2} d\tau
\]

\[
= \int_{\delta^2}^{\delta^2/\alpha} \frac{1}{u} du - \int_0^{\delta^2/\alpha} \frac{u}{(1 + u)^2} du
\]

\[
= \int_{\delta^2}^{\delta^2/\alpha} \left( \frac{1}{u} - \frac{u}{(1 + u)^2} \right) du - \int_0^{\delta^2} \frac{u}{(1 + u)^2} du
\]

\[
= \int_{\delta^2}^{\delta^2/\alpha} \left[ 1 - (1 + u^{-1})^{-2} \right] u^{-1} du - \int_0^{\delta^2} \frac{u}{(1 + u)^2} du
\]

and because of

\[
\left[ 1 - (1 + u^{-1})^{-2} \right] u^{-1} = \left[ 1 - \left( \frac{1 + 1}{u} \right)^{-2} \right] \frac{1}{u} = \left[ 1 - \left( \frac{u}{u + 1} \right)^2 \right] \frac{1}{u}
\]

\[
= \frac{1}{u + 1} \cdot \frac{2u + 1}{u + 1} \cdot \frac{1}{u} < \frac{2}{u + 1} \cdot \frac{1}{u} < \frac{2}{u^2}
\]

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we finally obtain
\[
\lim_{\alpha \to 0} h(\alpha) = \int_{\delta^2}^{\infty} \left[ 1 - (1 + u^{-1})^{-2} \right] u^{-1} du - \int_{0}^{\delta^2} \frac{u}{(1 + u)^2} du
\]
\[
\leq \int_{\delta^2}^{\infty} 2u^{-2} du - \int_{0}^{\delta^2} \frac{u}{(1 + u)^2} du = C_0.
\]
Thus, for any \(\alpha > 0\) close to 0 the following holds:
\[
\int_{\delta^2}^{\infty} \frac{\tau}{(\alpha + \tau)^2} d\tau = \ln \frac{1}{\alpha} + C_1. \tag{62}
\]
From (60), (61) and (62) we obtain
\[
\|\nabla \nu_\alpha\|_2^2 \leq \frac{(1 + \varepsilon_0) 2\pi (l - r)}{\mu^2} \ln \frac{1}{\alpha} + C. \tag{63}
\]
On the other hand we have
\[
\|\nu_\alpha\|_2^2 = 2\pi \delta^2 \int_{D} |\phi_\alpha|^2 ((l - r) + \delta t) dtds
\]
\[
\leq (1 + \varepsilon_0) 2\pi (l - r) \delta^2 \int_{D} |\phi_\alpha|^2 dtds
\]
\[
= C_0 \int_{D} \left| 2 \ln \left( \frac{\alpha + \delta^2}{\alpha + \delta^2 (t^2 + s^2)} \right) \right|^2 dtds
\]
\[
= 4C_0 \int_{D} \left( \ln (\alpha + \delta^2) - \ln (\alpha + \delta^2 (t^2 + s^2)) \right)^2 dtds
\]
\[
\leq 8\pi C_0 \int_{D} \left[ 2 \ln^2 (\alpha + \delta^2) + 2 \ln^2 (\alpha + \delta^2 (t^2 + s^2)) \right] dtds
\]
\[
= 8\pi C_0 \int_{0}^{1} 2 \ln^2 (\alpha + \delta^2) r dr + \int_{0}^{1} 2 \ln^2 (\alpha + \delta^2 r^2) r dr
\]
\[
= 8\pi C_0 \ln^2 (\alpha + \delta^2) \int_{0}^{1} 2r dr + \frac{8\pi C_0}{\delta^2} \int_{0}^{1} \ln^2 (\alpha + \delta^2 r^2) 2\delta^2 r dr
\]
\[
= 8\pi C_0 \ln^2 (\alpha + \delta^2) + \frac{8\pi C_0}{\delta^2} \int_{0}^{1} \ln^2 (\alpha + \delta^2 r^2) (\alpha + \delta^2 r^2) \prime dr
\]
\[
= C_1 + C_2 \int_{\alpha}^{\alpha + \delta^2} \ln^2 \zeta d\zeta
\]
\[
C_1 + C_2 \left[ \zeta \left( \ln^2 \zeta - 2 \ln \zeta + 2 \right) \right]^{\alpha + \delta^2} \\
= C_1 + C_2 \left[ \left( \alpha + \delta^2 \right) \left( \ln^2 \left( \alpha + \delta^2 \right) - 2 \ln \left( \alpha + \delta^2 \right) + 2 \right) \right] \\
- C_2 \alpha \left( \ln^2 \alpha - 2 \ln \alpha + 2 \right),
\]

and since \( \lim_{\alpha \to 0^+} (\alpha \ln \alpha) = \lim_{\alpha \to 0^+} (\alpha \ln^2 \alpha) = 0 \) we have
\[
\| \nu_\alpha \|_2^2 \leq C_1 + C_2 C_3 = C.
\]

Finally, from (59), (63) and (64) for any \( \Delta, E \in \mathbb{R} \) the following holds:
\[
\frac{\| \nabla \nu_\alpha \|_2^2 + \Delta \| \nu_\alpha \|_2^2 + E}{\ln \int_T e^{\nu_\alpha} dV} \leq \frac{(1 + \varepsilon_0) \frac{2\pi (l-r)}{\mu_2}}{\ln \frac{1}{\alpha} + \ln C_{\varepsilon_0}},
\]
thus,
\[
\lim_{\alpha \to 0} \frac{\| \nabla \nu_\alpha \|_2^2 + \Delta \| \nu_\alpha \|_2^2 + E}{\ln \int_T e^{\nu_\alpha} dV} \leq (1 + \varepsilon_0) \frac{2\pi (l-r)}{\mu_2}.
\]

For any \( \varepsilon > 0 \) consider \( \varepsilon_0 > 0 \) such that \( (1 + \varepsilon_0) \frac{2\pi (l-r)}{\mu_2} \leq \frac{2\pi (l-r)}{\mu_2} + \varepsilon \) and so from (65) we obtain our result. \( \square \)

**Proof of Lemma 2.2** Following arguments similar to those in [4] and [30] we prove the first and second part of the lemma, respectively.

1. Our aim here is to find a constant \( C_\varepsilon \), such that for any \( \varepsilon > 0 \) and for all functions \( \nu \in \mathcal{H}_G \), with \( \int_T \nu dV = 0 \) the following inequality holds
\[
\int_T e^\nu dV \leq C_\varepsilon \exp \left[ \left( \mu + \varepsilon \right) \| \nabla \nu \|_2^2 \right],
\]
where \( \mu = \frac{1}{16\pi L} \) if \( \mathcal{H}_G = \overset{\circ}{H}^2_{1,G} \) and \( \mu = \frac{1}{8\pi L} \) if \( \mathcal{H}_G = H^2_{1,G} \).

(i) Let \( \nu \in C_{0,\overset{\circ}{G}}(T) \) with \( \int_T \nu dV = 0 \) and \( \bar{\nu} = sup(\nu, 0) \).

Then \( \bar{\nu} \in \overset{\circ}{H}^2_{1,G}(T) \) and
\[
\int_{\mathbb{R}^3} \bar{\nu} dx = \frac{1}{2} \int_{\mathbb{R}^3} \nu dx, \quad \int_{\mathbb{R}^3} |\nabla \bar{\nu}| dx \leq \int_{\mathbb{R}^3} |\nabla \nu| dx.
\]
For any \( t \in \mathbb{R} \), denote by \( m_t(\nu) \) the measure of the set
\[
\Omega_t(\nu) = \{ x \in T : \nu(x) \geq t \}.
\]

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Given $v \in C_{0,G}^\infty(T)$, $m_t(v)$ is a decreasing function of $t$, not necessarily continuous. Let $m > 0$ depending on $\varepsilon$. Then, for a given $v \in C_{0,G}^\infty(T)$ two different cases can occur: whether there exists $s \geq 0$ such that $m_s(v) \geq m$ or not.

(a) Suppose there exists $s \geq 0$ such that $m_s(v) \geq m$.

If we denote

$$S = \sup\{s \in \mathbb{R} : m_s(v) \geq m\},$$

we will have $S \geq 0$, $m_{S+1}(v) < m$ and $m_{S/2}(v) \geq m$.

According to Lemma 2.1 we have the following

$$\int_T e^v dV = e^{S+1} \int_T e^{v-(S+1)} dV \leq e^{S+1} \int_T e^{\widetilde{\nabla} v} dV$$

$$\leq e^{S+1} C_{\varepsilon/2} \exp \left[ \left( \mu + \frac{\varepsilon}{2} \right) \|\nabla v\|_2 + D_{\varepsilon/2} \left\| v - (S + 1) \right\|_2 \right].$$

(66)

Since $\int_T v dV = 0$, and $\|\tilde{v}\|_1 = \frac{1}{2} \|v\|_1$ by Poincaré inequality there exists a constant $C_1$ such that

$$\|\tilde{v}\|_1 = \frac{1}{2} \|v\|_1 \leq C_1 \|\nabla v\|_2,$$

(67)

and since $S + 1 > 0$ we obtain

$$\left\| v - (S + 1) \right\|_1 \leq \|\tilde{v}\|_1 \leq C_1 \|\nabla v\|_2.$$

(68)

From the last two inequalities, by Hölder’s inequality and the Sobolev continuous and compact embedding of $H_{1,G}^2$ in $L_G^p(T)$, we obtain

$$\left\| v - (S + 1) \right\|_2^2 \leq m_{S+1}^{1/2}(v) \left\| v - (S + 1) \right\|_4^2$$

$$< m^{1/2} \left\| v - (S + 1) \right\|_4^2$$

$$\leq m^{1/2} C_2 \|\nabla v\|_2^2,$$

(69)

where $C_2$ is a constant independent of $v$ and $\mu$.

By the definition of $\Omega_t(v)$ we have that

$$\Omega_0(v) = \{x \in T : v(x) \geq 0\}$$

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and

\[ \Omega_{S/2}(v) = \{ x \in T : v(x) \geq S/2 \}. \]

Thus

\[ \| \tilde{v} \|_1 = \int_{\Omega_{S/2}(v)} v dV \geq \int_{\Omega_{S/2}(v)} \frac{S}{2} dV = \frac{S}{2} m_{S/2}(v). \] (70)

From (67) and (70), since \( m_{S/2}(v) \geq m \), we obtain

\[ S \leq \frac{2}{m_{S/2}(v)} \| \tilde{v} \|_1 \leq \frac{2}{m} C_1 \| \nabla v \|_2. \] (71)

The elementary inequality

\[ x < Sx^2 + \frac{1}{S}, \quad x \in \mathbb{R}, S > 0, \]

with \( x = \| \nabla v \|_2 \) yields

\[ \| \nabla v \|_2 < S \| \nabla v \|_2^2 + \frac{1}{S}, \quad S > 0. \] (72)

From (71), and because of (72), we obtain

\[ S \leq \frac{2C_1}{m} \left( S \| \nabla v \|_2^2 + \frac{1}{S} \right), \]

and with \( \frac{2C_1}{m} = \frac{m}{S} \) we obtain

\[ S \leq m \| \nabla v \|_2^2 + 4C_1^2 m^{-3} = m \| \nabla v \|_2^2 + C_3 m^{-3}. \] (73)

Thus, from (66), and because of (69) and (73), we obtain

\[ \int_T e^v dV \leq C_\varepsilon \exp \left[ \left( \frac{\mu_2}{L} + \frac{\varepsilon}{2} + D_{\varepsilon/2} C_2 m^{1/2} + m \right) \| \nabla v \|_2^2 \right] \] (74)

where \( C_\varepsilon = C_{\varepsilon/2} \exp (C_3 m^{-3} + 1) \).

(b) Suppose now that \( m_s(v) < m \) for any \( s \geq 0 \).

By Lemma 2.1 we have the following

\[ \int_T e^v dV \leq \int_T e^{\tilde{v}} dV \leq C_{\varepsilon/2} \exp \left[ \left( \mu + \frac{\varepsilon}{2} \right) \| \nabla \tilde{v} \|_2^2 + D_{\varepsilon/2} \| \tilde{v} \|_2^2 \right] \]
or
\[
\int_T e^\nu dV \leq C_{\varepsilon/2} \exp \left[ \left( \mu + \frac{\varepsilon}{2} \right) \|\nabla \nu\|_2^2 + D_{\varepsilon/2} \|\nu\|_2^2 \right].
\]
(75)

In this case, \( m_0(\nu) \leq m \) and so
\[
\|\tilde{v}\|_2^2 \leq m_0^{1/2}(\nu) \|\tilde{v}\|_4^2 < m^{1/2} \|\tilde{v}\|_4^2 \leq m^{1/2} C_2 \|\nabla \nu\|_2^2.
\]
(76)

From (75), and because of (76) we obtain
\[
\int_T e^\nu dV \leq C_{\varepsilon/2} \exp \left[ \left( \mu + \frac{\varepsilon}{2} + D_{\varepsilon/2} C_2 m^{1/2} + m \right) \|\nabla \nu\|_2^2 \right].
\]
(77)

In both cases we have to choose \( m > 0 \) such that
\[
D_{\varepsilon/2} C_2 m^{1/2} + m < \frac{\varepsilon}{2}
\]
and
\[
C_\varepsilon = C_{\varepsilon/2} \exp \left( \frac{2C_1}{m^2} + 1 \right)
\]
so, for all \( \nu \in H_{1,G}^0 \) with \( \int_T \nu dV = 0 \) the following inequality holds
\[
\int_T e^\nu dV \leq C_\varepsilon \exp \left[ (\mu + \varepsilon) \|\nabla \nu\|_2^2 \right],
\]
(78)

where \( \mu = \frac{1}{16\pi L} \).

(ii) Let now \( \nu \in H_{1,G}^2 \). Following the same steps as in the first part of Lemma 2.1 by Theorem 3 of [15], for all \( \nu \in C_G^\infty(T) \), we obtain
\[
\int_{T_j} e^\nu dV \leq C \exp \left[ (1 + c\varepsilon) \frac{1}{16\pi^2 l_j} \int_{T_j} \|\nabla \nu\|_2^2 dV \right].
\]
(79)

Consequently, since \( C_G^\infty(T) \) is dense in \( H_{1,G}^2 \) and (79) holds for any \( j = 1, 2, \ldots, N \), by the second part of Lemma 2.1, we conclude that, for all \( \varepsilon > 0 \), there are constants \( C_\varepsilon \) and \( D_\varepsilon \) such that for all \( \nu \in H_{1,G}^2 \) the following holds
\[
\int_T e^\nu dV \leq C_\varepsilon \exp \left[ (\mu + \varepsilon) \|\nabla \nu\|_2^2 + D_\varepsilon \|\nu\|_2^2 \right],
\]

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where \( \mu = \frac{1}{8\pi L} \) is the best constant for this inequality.

Following the same steps as in the first part of this lemma we derive that for any \( \varepsilon > 0 \) and for all functions \( v \in H^{1}_{1,G} \), with \( \int_{T} v dV = 0 \) the following inequality holds

\[
\int_{T} e^{v} dV \leq C_{\varepsilon} \exp \left[ (\mu + \varepsilon) \| \nabla v \|_{2}^{2} \right],
\]

(80)

where \( \mu = \frac{1}{8\pi L} \).

By parts (i) and (ii) of the lemma we conclude that, for all \( \varepsilon > 0 \), there exists constant \( C_{\varepsilon} \) such that for all \( v \in \tilde{H}_{1,G}^{2} \) or \( v \in H^{2}_{1,G} \) with \( \int_{T} v dV = 0 \), inequalities (78) and (80) hold respectively.

Finally, we observe that if \( \tilde{v} = v - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \) we have

\[
\int_{T} \tilde{v} dV = \int_{T} \left( v - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \right) dV = \int_{T} v dV - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \int_{T} dV
= \int_{T} v dV - \frac{1}{Vol(T)} Vol(T) \int_{T} v dV = 0,
\]

and so, rewriting (78) and (80) with \( \tilde{v} = v - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \) we obtain:

\[
\int_{T} e^{v - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV} dV \leq C_{\varepsilon} \exp \left[ (\mu + \varepsilon) \| \nabla \left( v - \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \right) \|_{2}^{2} \right],
\]

\[
e^{-\frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV} \int_{T} e^{v} dV \leq C_{\varepsilon} \exp \left[ (\mu + \varepsilon) \| \nabla v \|_{2}^{2} \right]
\]

or

\[
\int_{T} e^{v} dV \leq C_{\varepsilon} \exp \left[ (\mu + \varepsilon) \| \nabla v \|_{2}^{2} + \frac{1}{2\pi^{2}r^{2}l} \int_{T} v dV \right],
\]

and the first part of the lemma is proved.

2. Let \( v \in C_{G}^{\infty}(T) \), with \( \int_{\partial T} v dS = 0 \), \( \phi = v \circ \xi^{-1} \) and \( n \) the outward unit normal.
By Stoke’s theorem we have

\[
\int_{\partial T} e^\nu dS = 2\pi r^2 \int_{\partial D} e^\phi (l + rt) d\sigma_D \\
\leq 2\pi r^2 (l + r) \int_{\partial D} e^\phi d\sigma_D \\
= 2\pi r^2 (l + r) \int_D \text{div}(e^\phi n) dt ds \\
= 2\pi r^2 (l + r) \int_D [\text{div}(n + \phi)] e^\phi dt ds \\
\leq 2\pi r^2 (l + r) \left[ C_0 \int_D e^\phi dt ds + \int_D |\nabla \phi| e^\phi dt ds \right], \tag{81}
\]

where \( C_0 = \sup_D |\text{div} n| \).

By (81), and because of Theorem 4 of [17] arises

\[
\int_{\partial T} e^\nu dS \leq 2\pi r^2 (l + r) \left[ C_0 \tilde{C} \exp (\mu \|\nabla \phi\|_2^2) + \int_D |\nabla \phi| e^\phi dt ds \right]. \tag{82}
\]

By Hölder’s inequality and by Theorem 3 of [18] we obtain

\[
\int_D |\nabla \phi| e^\phi dt ds \leq \|\nabla \phi\|_2 \left( \int_D e^{2\phi} dt ds \right)^{1/2} \leq \tilde{C} \|\nabla \phi\|_2 \exp (2\tilde{\mu} \|\nabla \phi\|_2^2), \tag{83}
\]

where \( \tilde{\mu} \) a is constant greatest than 1/8\pi.

From the elementary inequality \( t \leq C_1 \exp (\varepsilon_0 t^2) \), \( t \geq 0, \varepsilon_0 > 0 \) and \( C_1 \) a constant with arbitrary \( \varepsilon_0 > 0 \) and \( t = \|\nabla \phi\|_2 \) we obtain

\[
\|\nabla \phi\|_2 \leq C_1 \exp (\varepsilon_0 \|\nabla \phi\|_2^2). \tag{84}
\]

Combining inequalities (82), (83) and (84) we obtain

\[
\int_{\partial T} e^\nu dS \leq 2\pi r^2 (l + r) \left[ C_0 \tilde{C} \exp (\tilde{\mu} \|\nabla \phi\|_2^2) + \tilde{C} \|\nabla \phi\|_2 \exp (2\tilde{\mu} \|\nabla \phi\|_2^2) \right] \\
\leq 2\pi r^2 (l + r) \\
\times \left[ C_0 \tilde{C} \exp (\tilde{\mu} \|\nabla \phi\|_2^2) + \tilde{C} C_1 \exp (\varepsilon_0 \|\nabla \phi\|_2^2) \exp (2\tilde{\mu} \|\nabla \phi\|_2^2) \right] \\
\leq 2\pi r^2 (l + r) \tilde{C} (C_0 + C_1) \exp [(2\tilde{\mu} + \varepsilon_0) \|\nabla \phi\|_2^2].
\]
Since
\[ \| \nabla \phi \|^2 = \int_D |\nabla \phi|^2 \, dt \, ds \leq \frac{1}{L} \int_T |\nabla \upsilon|^2 \, dV, \]
the latter inequality becomes
\[ \int_{\partial T} e^\upsilon \, dS \leq 2\pi r^2 (l + r) \bar{C} (C_0 + C_1) \exp \left( \frac{2\bar{\mu} + \varepsilon_0}{L} \int_T |\nabla \upsilon|^2 \, dV \right) \]
\[ \leq C \exp \left( \frac{2\bar{\mu} + \varepsilon_0}{L} \int_T |\nabla \upsilon|^2 \, dV \right). \]
Given \( \varepsilon > 0 \), we can choose \( \varepsilon_0 > 0 \) such that
\[ \frac{2\bar{\mu} + \varepsilon_0}{L} < \frac{2\bar{\mu}}{L} + \varepsilon = \mu + \varepsilon, \]
and the last inequality yields
\[ \int_{\partial T} e^\upsilon \, dS \leq C \exp \left( (\mu + \varepsilon) \int_T |\nabla \upsilon|^2 \, dV \right). \]
(85)
Rewriting (85) with \( \bar{\upsilon} = \upsilon - \frac{1}{4\pi r^2 l} \int_{\partial T} \upsilon \, dS \) yields the second inequality of the lemma. \( \square \)

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