Labyrinthine dissipative patterns

I Bordeu

Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile.
E-mail: ibordeu@imperial.ac.uk

Abstract. A wide range of physical systems exhibit labyrinthine patterns as transient or steady states. One of the routes by which labyrinths emerge is through the destabilization of a localized structure through a curvature instability. In this work, we will study the emergence of labyrinths through this mechanism in different physical contexts, including nonlinear cavity optics, vegetation dynamics and chemical reactions. Furthermore, it is shown that the resulting labyrinths can be classified depending on their connectivities.

1. Introduction

Systems which are kept out of thermodynamic equilibrium can exhibit multistability of states. These states can be homogeneous or inhomogeneous. The latter are sustained by the injection and dissipation of energy matter and/or momentum, and correspond to a mechanism by which the systems self-organize [1, 2], minimizing—at least locally—some physical quantity, in variational systems this quantity is their energy, sometimes called nonequilibrium or Lyapunov functional [3]. It is in this framework that dissipative structures emerge [1]. This structures can be localized, this is, they are finite spatial domains (with a non diverging energy), or extended, corresponding to periodic or non periodic spatial structures which may possess a characteristic wavelength that defines them [2, 4, 5, 6].

It has been observed that a wide range of dissipative systems exhibit a state (or multiple states) of equilibrium in which labyrinth structures are formed, such as, experimental observations in magnetic fluids [7], cholesteric liquid crystals [8, 9], Langmuir monolayers [10], vegetation patterns [11], nonlinear cavity optics [12], among others. Such structures are characterized for having a well defined wavelength, but a small correlation length, this is, the pattern has no privileged direction, generating in this way an elaborated spatial structure [13]. This definition is purely statistical, thus, based on the average spatial distribution of the structure. In this work we study the route by which a localized spot destabilizes, and by an elliptical deformation it elongates, this elongation generates a stripe structure which suffers from a transversal instability propagating to all available space in the form of a labyrinthine structure, this curvature mechanism has been reported in [14]. Based on this result, we will test the universality of the mechanism presented, by showing its existence in a variety of physical systems. Namely, through a general competition-redistribution type of model and through the Gray-Scott reaction-diffusion model it will be shown that the transition from localized to
Curvature instability mechanism. Initially circular localized structure (left) and time evolved extended labyrinthine pattern final state (right) for Eq. (1). Parameters $\eta = -0.065$, $\epsilon = 2.45$, $\nu = 2.0$. Pseudo-spectral simulation (periodic boundary conditions), grid of $1024 \times 1024$ points and spacing $dx = 0.5$.

extended structure is also observed in the context of vegetation dynamics and chemical reactions, respectively. The different type of labyrinths observed will lead us to a simple classification of the labyrinth structures based on their connectivity.

2. From a localized to an extended labyrinthine pattern

2.1. Modified Swift-Hohenberg equation

The Swift-Hohenberg equation [15] is a real nonlinear partial differential equation (PDE), deduced in the context of fluid dynamic to describe the amplitude of the pattern formed by Rayleigh-Bénard convection cells. By means of a generalized version of the Swift-Hohenberg equation it has been recently described how a localized structure loses stability, elongating, suffering a transversal instability to then generate an extended labyrinthine pattern [14]. The modified Swift-Hohenberg equation considered describes the evolution of a real scalar field $u(x, y, t)$, through an isotropic, symmetric under reflection, nonlinear PDE, which includes an constant term which breaks the field reflection symmetry. The modified Swift-Hohenberg equation reads

$$\frac{\partial u(x, y, t)}{\partial t} = \eta + \epsilon u(x, y, t) - u(x, y, t)^3 - \nu \nabla^2 u(x, y, t) - \nabla^4 u(x, y, t),$$

(1)

This equation was deduced originally as an approximation to the Maxwell-Bloch equations, in the context of nonlinear cavity optics to describe the nascent optical bistability with transversal effect [16, 17]. This equation predicted the existence and stability of circular [17] and rodlike [18] localized structures, and it also exhibits extended patterns, such as, labyrinths.

The field $u(x, y, t)$ accounts for the amplitude of the electric field, and $x$ and $y$ are spatial coordinates and $t$ is time. The bifurcation parameter $\epsilon$ measures the input field amplitude. The $\eta$ parameter breaks the field reflection symmetry $u \rightarrow -u$. The parameter $\nu$ stands for the diffusion coefficient, positive values of $\nu$ lead to the formation of spatial patterns. The 2D Laplacian operator $\nabla^2 = \partial^2_{xx} + \partial^2_{yy}$ and the 2D bilaplacian operator $\nabla^4$ act on the transverse plane $(x, y)$, and they stand for the transport mechanisms or spatial coupling.

The modified Swift-Hohenberg equation (1) can be derived from a Lyapunov functional, i.e.

$$\frac{\partial u}{\partial t} = -\frac{1}{2} \frac{\delta F[u, \nabla u, \nabla^2 u]}{\delta u},$$

(2)

where $F$ is the Lyapunov functional or energy [19] and reads

$$F \equiv -\iint_{\mathbb{R}^2} dxdy \left[ 2\eta u + \epsilon u^2 - \frac{u^4}{2} + \nu (\nabla u)^2 - (\nabla^2 u)^2 \right].$$

(3)
It is clear that the \( \eta \) term accounts for the asymmetry between the homogeneous states. By using the solutions of Eq. (1), this functional satisfies

\[
\frac{dF}{dt} = -\int_{\mathbb{R}^2} dx dy (\partial_t u)^2 \leq 0.
\] (4)

This property guarantees that a system described by Eq. (1) evolves towards a state of minimal energy. The modified Swift-Hohenberg equation exhibits, for \( \epsilon > 0 \), coexistence between homogeneous and pattern states [20, 21], this allows the existence of stable localized structures.

This localized structures are azimuthally symmetric and occupy only a portion of the available space (see Fig. 1 left panel).

![Figure 1](image1.png)

**Figure 2.** Curvature instability mechanism for the generation of a labyrinth from a localized structure as initial condition for the modified Swift-Hohenberg equation (1). Simulation made for parameters \( \eta = -0.065, \epsilon = 2.45, \nu = 2.0 \). Pseudo-spectral method was used with periodic boundary conditions, grid of 1024 × 1024 points and spacing \( dx = 0.5 \).

### 2.2. Curvature instability mechanism

As described in reference [14], Eq. (1) supports a curvature instability over localized spots. This instability deforms the spot into an elongated elliptical shape, producing a rod-like structure. This rod structure suffers from a transversal instability, undulations appear in its core. The non-saturating dynamics that follow take the system into an disordered extended structure that covers all the available space. Such structure shall be called a labyrinthine pattern or simply labyrinth, which is characterized for not having a global order or regularity. Due to the variational origin of Eq. (1) and given that the temporal dynamic moves towards a minimum of the Lyapunov functional (4), we know that the labyrinthine structure will reach a stationary state asymptotically. In this case, the labyrinth is composed by a single interconnected structure or invaginated structure [14] (which protrudes from the more stable homogeneous state). Defects such as, dislocations, disinclination, and phase fronts contribute to the spatial disorder of the labyrinth. This behavior occurs far from any pattern forming instability and exists in a regime of bistability between homogeneous steady states.

### 3. Vegetation dynamics and chemical reaction models

The curvature instability mechanism for the formation of labyrinthine patterns from initially localized structures requires of few ingredients as the phenomenon is observed even in the simplest of the pattern forming equations as is Eq. (1).
3.1. Competition-redistribution model

In this section, a minimal but general model for vegetation pattern forming systems will be introduced. In this model the curvature instability mechanism also emerges in a regime of bistability of homogeneous steady states far from the pattern forming instability.

The emergence of spatial organization in extended landscapes has been an intriguing phenomenon for both botanists and nonlinear scientists community for decades and are still a matter of great discussion [22, 24, 25]. One of the transversal agreements among scientists is that competition for soil resources such as water and nutrients leads to self-organization phenomenon. Depending on the topography of the terrain, the climate conditions and the vegetation species, the process of self-organization leads to the formation of a wide variety of patterns, such as, tiger bush [22, 23], or fairy circles [24, 25].

![Temporal evolution](image)

**Figure 3.** Curvature instability mechanism for the generation of a labyrinth from a localized patch as initial condition for the interaction-redistribution model (5). Simulation made for parameters $\eta = 0.085$, $\kappa = 22.6$, $\Delta = 0.005$, $\Gamma = 0.5$ and $\alpha = 0.125$. Finite differences method was used with periodic boundary conditions, grid of 200×200 points and spacing $dx = 0.5$.

Taking into account relation that exists between the structure of individual plants and the competition-facilitation interactions in a vegetation community is that the generic interaction-redistribution models emerge [22, 26]. These models have been successful in the prediction of formation of localized structures in arid and semi-arid environments, where the resource scarcity induces high competition between plants through their root networks. In this context, a localized structure emerges as a patch of vegetation in an unpopulated homogeneous state. By considering a logistic nonlinearity with nonlocal coupling between plants Tlidi et al. [27] where able via a weak gradient approximation, to deduce a mean-field nonlinear partial differential equation for the temporal evolution of the phytomass density $\rho(x, y, t)$:

$$\partial_t \rho = -\rho \left( \eta - \kappa \rho + \rho^2 \right) + (\Delta - \Gamma \rho) \nabla^2 \rho - \alpha \rho \nabla^4 \rho,$$

where $(x, y)$, and $t$ are the spatial coordinates and time, respectively. This equation supposes an homogeneous and isotropic environment, and assumes that the interaction between plants does not change though their size. The parameters: $\eta$ accounts for the decrease-to-growth rate ratio; $\kappa$ is the facilitation-to-competition susceptibility ratio; $\Delta$ is proportional to the square root of the facilitation-to-competition range ratio, this three parameters are positive-defined. The parameters $\Gamma$ and $\alpha$ are the nonlinear diffusion coefficients. The existence of a nonlinear hyperdiffusion is fundamental for Eq. (5) to satisfy the physical constrain for the phytomass density $\rho(x, y, t) \geq 0$. However, the presence of the nonlinear diffusion terms $u \nabla^2 u$ and $u \nabla^4 u$ render Eq. (5) nonvariational. Thus, Eq. 5 cannot be written as the variation of a Lyapunov functional.

Nonvariational systems can exhibit permanent complex spatiotemporal dynamics, such as chaos, intermittency, turbulence, among others.
The real order parameter equation (5) constitutes the simplest model of spatial dynamics in which competitive interactions between individuals occur locally.

The homogeneous solutions of Eq. (5) are (i) $\rho_s^0 = 0$, which correspond to a territory without vegetation, a bare state, (ii) and two non-zero states given by $\rho_s^\pm = (\kappa \pm \sqrt{\kappa^2 - 4\eta})/2$, a vegetated state. When $\kappa \leq 0$, only the homogeneous steady state $\rho_s^+$, is physically consistent, for $\eta < 0$. It decreases monotonously with $\mu$ and vanishes at $\eta = 0$. When $\kappa > 0$, the viable homogeneous solution extends up to the limit point $\rho_L = \kappa/2$ and $\eta_L = \kappa^2/4$. In the range $0 < \eta < \eta_L$, the biomass density exhibits a bistable behavior: the stable homogeneous branches of solutions $\rho_s^0$ and $\rho_s^+$ coexist with the intermediate unstable branch $\rho_s^-$. The former solution is always unstable.

The upper homogeneous state $\rho_s^+$ undergoes a Turing instability characterized by an intrinsic wavelength

$$\Lambda_m = 2\pi \sqrt{2\alpha/\sqrt{\Gamma/\alpha - \Delta/\rho_m}}$$

(6)

The threshold $\eta_m$, associated with this instability is solution of the following cubic equation:

$$(2\Gamma \rho_m - \Delta)^2 = 4\alpha \rho_m^2 (2\rho_m - k)$$

(7)

We now focus on parameter regime where the uniform plant distribution exhibit bistability ($\kappa > 0$) as we know, from the previous section that bistability is a necessary ingredient for the curvature mechanism to exist. For parameters $\eta_m < \eta < \eta_L$ the homogeneous state becomes unstable with respect to the Turing instability. Thus, infinitesimal fluctuation around the uniform plant distribution $\rho_s^+$ will spontaneously evolution towards stationary, spatially periodic distributions of the biomass density which will invade the whole system. However, when a stable localized vegetation patch is considered (these exist for $\eta > \eta_L$) as an initial condition (high aridity), see Fig. 4, and the aridity is then varied in such quantity that the system reaches the coexistence of homogeneous states, then the localized structure suffers and elliptical deformation caused by the curvature instability and elongates into a rod structure (see Fig. 3 at $t_2$) in a process identical as previously observed for Eq. (1). Afterwards, the rod structure suffers from a transversal instability (see Fig. 3 at $t_3$), in this case the whole structure

Figure 4. Bifurcation diagram for $\kappa = 0.6$, $\Delta = 0.005$, $\Gamma = 0.5$ and $\alpha = 0.125$, of the homogeneous plant population states (blue), dashed lines indicate unstable regime. Stable localized structures are observed for values of $\eta > \eta_C$. When the aridity $\eta$ is decreased below $\eta_L$ the curvature instability mechanism is observed.
is compromised and not only the central portion as it was observed in the Swift-Hohenberg case (Fig. 1 for $t_3$). As time further increases, the labyrinthine structure invades all the system.

Here, we have described the emergence of the curvature instability mechanism over a localized structure which by turn generates the emergence of a labyrinthine structure. This takes place when entering with an initially stable localized structure to the bistability parameter region.

3.2. Gray-Scott model

Pushing further the idea of universality of the curvature instability mechanism for the formation of labyrinths, we now enter the domain of chemical reactions. It is well-known that the reaction and diffusion of the components of a chemical reaction can produce an immense variety of spatial structures, such as lamellar structures and self-replicating spots [28, 29], localized structures [30, 31] and Turing structure [28, 32, 33], and exhibit complex spatiotemporal dynamics, such as oscillatory dynamics, breathing localized structure and spatiotemporal chaos [34, 35].

In a series of seminal works [36, 37, 38] P. Gray and S.K. Scott introduced a variant of the autocatalytic model for glycolisis first proposed by Sel’kov [34]. This prototype autocatalytic reaction (i.e. the product of the reaction is also the catalyst) is described by two irreversible processes

\[
A + 2B \rightarrow 3B, \\
B \rightarrow C, \tag{8}
\]

where $A$, $B$ and $C$ are chemical species, $C$ is a non-reactive product (inert) and the system is kept out-of-equilibrium by a permanent injection of reactive $A$, called feeding. In their papers, Gray and Scott show that this simple model presents multistability, hysteresis cycles, patterns, oscillatory dynamics and other exotic patterns.

![Figure 5. Bifurcation diagram the concentration $V$ for $k = 0.061$, homogeneous states (blue), dashed lines indicate unstable regime. The $\sqrt{F}$-curve shows the position of the saddle-node bifurcation. For $F_c < F < F_h$ stable localized structures are observed (numerically). When changing the feeding parameter for a localized structure to values $F_r < F < F_c$, the curvature instability mechanism is observed. Reconnection zone indicates the zone where a radial instability is observed and structures can reconnect (see. Fig. 7). Here $F_h = 0.097$, $F_c = 0.085$ and $F_r = 0.078.$](image-url)
Through a rescaling and including the diffusion phenomena in the reactions, a nonlinear partial differential equation can be deduced for the temporal evolution of the concentrations $U$ and $V$ of $A$ and $B$, respectively [28]. This set of equations read

$$\begin{align}
\frac{\partial U}{\partial t} &= D_u \nabla^2 U - UV^2 + F(1 - U), \\
\frac{\partial V}{\partial t} &= D_v \nabla^2 V + UV^2 - (F + k)V,
\end{align}$$

(9)

which are an example of reaction-diffusion equations, as they model the local evolution $U$ and $V$, and the diffusion process of the chemical species. In Eq. (9) $D_u$ and $D_v$ correspond to the diffusion coefficients associated to $U$ and $V$, respectively. $F$ represents the feeding rate of the reactive $A$ and dimensionless constant $k$ (killing rate) accounts for the rate of the second reaction. Unlike the interaction-redistribution model presented for vegetation dynamics, Eq. (5), the Gray-Scott model Eq. (9) does not have a variational structure, which is the reason behind its rich phenomenology which includes, from oscillatory dynamics, self-replication [28] and chaos [35] and others mentioned before. The existence and stability of patterns and localized spots (or single pulses) as homoclinic solutions of Eq. (9) has been well studied both in one and two dimensional cases [30, 39, 40, 41, 42]. This system supports a trivial homogeneous steady state $(U^0, V^0) = (1, 0)$ which is always stable. When the discriminator $d = 1 - 4(F + k)^2 > 0$ two additional homogeneous steady states $(U^\pm, V^\pm)$ appear, the first

$$(U^+, V^+) = \left( \frac{1}{2}(1 + \sqrt{d}), \frac{F}{2(F + k)}(1 - \sqrt{d}) \right)$$

(10)

is always unstable. The second state

$$(U^-, V^-) = \left( \frac{1}{2}(1 - \sqrt{d}), \frac{F}{2(F + k)}(1 + \sqrt{d}) \right)$$

(11)

is stable when $-V^2 + k < 0$ and $(F + k)(V^2 - F) > 0$. For $F > 1/4$ the system only exhibits the trivial steady state. For $F < 1/4$ and $k < 1/16$ the system exhibits three homogeneous steady states (one unstable). The bifurcation diagram for the homogeneous states is shown in Fig. (5). For a detailed study on the bifurcations and instabilities of Eq. 9 see Ref. [33] and references therein.

![Figure 6. Curvature instability mechanism for the generation of a labyrinth from a localized concentration peak as initial condition for the Grey-Scott model, Eqs. 9. Simulation made for parameters $F = 0.080$, $k = 0.061$. Finite differences method was used with periodic boundary conditions, grid of $1024 \times 1024$ points and spacing $dx = 1$.](image)

As indicated in Fig. 5, localized structures are observed in $F_c < F < F_h$, is $F$ is increased further the system falls to the homogeneous state $V^0 = 0$. When the parameter $F$ is reduced
below \( F_c \), perturbations become unstable and the structure suffers a curvature instability characterized by the elongation of the structure (see Fig. 6). As observed for the modified Swift-Hohenberg Eq. (1) and for the interaction-redistribution models, Eq. (5), the concentration field suffers then a transversal instability characterized by the wiggling of the central section of the structure. Non saturating evolution leads to the formation of a complex labyrinthine structure as seen in Fig. 6.

When the feeding parameter is decreased further \((F < F_c)\), an initially circular localized structure suffers from a radial instability as seen in Fig. 7 at \( t_2 \), there two spots where given as initial conditions. As the radius of the structures grow, they suffer from a curvature instability. Non saturation causes the structure to propagate. Differently from what is observed from every labyrinthine structure showed previously, here the tips merge, this causes reconnection of the structures. In the transition from \( t_4 \) to \( t_5 \) one can realize that the two initially distinct structures reconnect into one single labyrinth.

![Figure 7](image-url) Alternative mechanism for the generation of a labyrinth from a localized concentration peak as initial condition for the Grey-Scott model 9. Simulation made for parameters \( F = 0.070 \), \( k = 0.061 \). In this parameter zone occurs reconnection between structures. Finite differences method was used with periodic boundary conditions, grid of 1024\( \times \)1024 points and spacing \( dx = 1 \).

4. Labyrinth connectivities
The curvature instability of a localized structure was observed in the three systems presented previously. In the three cases (modified Swift-Hohenberg, interaction-redistribution and Grey-Scott models) an initially localized structure suffers from the curvature instability mechanism to finally fall into an extended labyrinthine structure. However each of these structures are different (see Figs. 2.t_5, 3.t_5, 6.t_5 and 7.t_6). Starting from a single localized structure the Swift-Hohenberg model Eq. (1), generates a single fully connected structure, this is, starting in the higher field on can reach every other higher value field without leaving it. The same happens in the fist labyrinth from the Gray-Scott model (cf. Fig. 6) only that in this case the structure shows no dislocations as the structure remains in a single line labyrinth. In the case of the interaction-redistribution model Eq. (3), starting from a single localized structure, the final labyrinthine state is disintegrated into a high number of small structures, thus, this labyrinth is highly disjoint. On contrary, when considering two initial structures in Fig. 7, the structures where able to reconnect increasing the connectivity from the initial state.
A classification of the labyrinth structures originated from localized initial states can be done based on the connectivity difference between the final and the initial states. If $\Delta = C_f - C_i$, where $C_i$ and $C_f$ are the initial and final number of disjoint structure, then, if $\Delta > 0$ we will say labyrinth is dissociative, if $\Delta = 0$ it is neutral and if $\Delta < 0$ the labyrinth is associative. We can conjecture after the previous observations that the type of labyrinth will depend on the capacity of a structure of preserving its integrity, here, surface tension, which keeps the structure together will play a fundamental role. In this sense, the type of labyrinths a system exhibits will give information of the surface properties of the system.

5. Conclusions
In this work the curvature destabilization of a localized structure had been show as a mechanism for the generation of extended labyrinthine patterns. Furthermore, the mechanism has been shown to exist in a wide range of physical systems including vegetation dynamics through a general interaction-redistribution model, and chemical reactions through the Gray-Scott reaction diffusion equations. It has been shown that depending not only in the context in which labyrinths emerge but also on the parameters considered for simulation/experimentation, different labyrinths emerge and even more, they can be classified as associative, neutral or dissociative, based on the difference between the initial and final connectivity of their structures. Future work will address quantitatively the effect of surface tension in the type of labyrinth formed.

Acknowledgments
I want to thank Professor Marcel G. Clerc for his support and valuable comments during the development of this work. I also thank Professor Mustapha Tlidi for his support. During the development of this work I was supported by CONICYT, Becas de Magister Nacional, number 22130947.

References
[1] Glansdorff P and Prigogine I (1971) Thermodynamic Theory of Structures, Stability and Fluctuations (Wiley, New York).
[2] Nicolis G and Prigogine I (1977) Self-Organization in Non Equilibrium Systems (J. Wiley & Sons, New York).
[3] Graham R and Tel T (1984) Phys. Rev. Lett. 52 1.
[4] Pismen L M (2006) Patterns and Interfaces in Dissipative Dynamics (Springer, Berlin).
[5] Cross M C and Hohenberg P C (1993) Rev. Mod. Phys. 65 851.
[6] Descalzi O, Clerc M G, Residori S, and Assanto G (2011) Localized States in Physics: Solitons and Patterns (Springer, New York).
[7] Dickstein A J, Erramilli S, Goldstein R E, Jackson D P, and Langer S A (1993) Science 261.
[8] Ribiere P and Oswald P (1990) Journal de Physique 51 16.
[9] Oswald P, Baudry J, and Pirkl S (2000) Physics Reports 337 1.
[10] Heinig P, Helseth L E, and Fischer Th M (2004) New. J. Phys. 6 189.
[11] von Hardenberg J, Meron E, Shachak M, and Zarmi Y (2001) Phys. Rev. Lett. 87 19.
[12] Taranenko V B, Staliunas K, and Weiss C O (1998) Phys. Rev. Lett. 81 11.
[13] Le Berre M, Ressayre E, Tallet A, Pomeau Y, and Di Menza L (2002) Phys. Rev. E. 66 026203.
[14] Bordeu I, Clerc M G, Lefever R, and Tlidi M (2015) Communications in Nonlinear Science and Numerical Simulation 29 482.
[15] Swift J and Hohenberg P C (1977) Phys. Rev. A 15 319.
[16] Tlidi M, Georgiou M, and Mandel P (1993) Phys. Rev. A 48 4605.
[17] Tlidi M, Mandel P, and Lefever R (1994) Phys. Rev. Lett. 73 640.
[18] Bordeu I and Clerc M G (2015) Phys. Rev. E 92 042915.
[19] Vladimirov A G, Lefever R, and Tlidi M (2011) Phys. Rev. A 84 043848.
[20] Couillet P, Riera C, and Tresser C (2000) Prog. Theor. Phys. Supp. 139.
[21] del Campo F, Haudin F, Rojas R G, Bortolozzo U, Clerc M G, and Residori S (2012) Phys. Rev. E 86 036201.
[22] Lefever R and Lejeune O (1997) Bulletin of Mathematical Biology 59 263.
[23] Lejeune O and Tlidi M (1999) Journal of Vegetation Science **10** 201.
[24] Juergens N (2013) Science **339**.
[25] Fernandez-Oto C, Tlidi M, Escaff D, and Clerc M G (2014) Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **372**.
[26] Martínez-García R, Calabrese J M, Hernandez-Garcia E, and Lopez C (2013) Geophysical Research Letters **40** 6143.
[27] Lejeune O, Tlidi M, Couteron P (2002) Phys. Rev. E **66** 010901.
[28] Pearson J E (1993) Science **261** 189.
[29] Lee K J and Swinney H L (1995) Phys. Rev. E **51** 1899.
[30] Koga S, Kuramoto Y (1980) Progress of Theoretical Physics **63** 106.
[31] Vanag V K, Epstein I R (2007) Chaos: An Interdisciplinary Journal of Nonlinear Science **17** 037110.
[32] Lee K J, McCormick W D, Ouyang Q, and Swinney H L (1993) Science **261** 192.
[33] Mazin W, Rasmussen K E, Moskilde E, Borckmans P, and Dewel G (1996) Mathematics and Computers in Simulation **40** 371.
[34] Sel’Kov E E (1968) European Journal of Biochemistry **4** 79.
[35] Nishiura Y and Ueyama D (2001) Physica D: Nonlinear Phenomena **150** 137.
[36] Gray P and Scott S K (1983) Chemical Engineering Science **38** 29.
[37] Gray P and Scott S K (1984) Chemical Engineering Science **39** 1087.
[38] Gray P and Scott S K (1985) The Journal of Physical Chemistry **89** 22.
[39] Doelman A, Gardner R A, and Kaper T J (1998) Physica D: Nonlinear Phenomena **122** 1.
[40] Hale J K, Peletier L A, and Troy W C (2000) SIAM Journal on Applied Mathematics **61** 102.
[41] Wei J (2001) Physica D: Nonlinear Phenomena **148** 20.
[42] Muratov C B and Osipov V V (2002) SIAM Journal on Applied Mathematics **62** 1463.