Resistive axisymmetric equilibria with arbitrary flow

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An analysis of axisymmetric equilibria with arbitrary incompressible flow and finite resistivity is presented. It is shown that with large aspect ratio approximation or vanishing poloidal current, a uniform conductivity profile is consistent with equilibrium flows. Also a comment made on coexistence of both toroidal and poloidal flows in an axisymmetric field-reversed configuration.

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Calculating the equilibrium is one of the fundamental problems of magnetically confined plasmas. Most studies are directed toward finding an ideal (i.e. infinitely conducting) magnetohydrodynamic (MHD) equilibria in an axisymmetric plasma. The earliest calculations are those of Grad and Shafranov, leading to the famous Grad-Shafranov equation \( \nabla \times \mathbf{B} = 0 \). The ideal and static Grad-Shafranov equation is a differential equation in the magnetic flux function \( \psi \) with two arbitrary surface quantities as the pressure \( p(\psi) \) and the poloidal current \( I(\psi) \).

Consequently, there have been attempts to include various effects e.g. mass flow into the equilibrium equations. An equivalent of Grad-Shafranov equation in an axisymmetric ideal plasma with arbitrary flow has been given by Hameiri. In some recent works, Steinhauer et al. deals with a generalization of Grad-Shafranov equilibria in a multi-fluid with flow and Throumoulopoulos and Tasso consider a helically symmetric equilibria with flow. The situation with flow becomes more realistic when one realizes the existence of equilibrium flows both in toroidal and poloidal directions in tokamaks following momentum deposition through heating by neutral beam injection. With equilibrium flows, the resultant governing differential equation does not remain always elliptic. The investigation of a general MHD equilibrium becomes much more complicated when one tries to include the effects of other important factors, say of viscous stress tensor. Recently Ren et al. have studied the deformation of magnetic island by including the effect of sheared flow and viscosity into an ideal two-dimensional MHD equilibrium configuration. However, there is an element of inconsistency whether an ideal equilibrium is realistic. Heuristically, one ignores the resistivity in the Ohm’s law while calculating the equilibrium, but then a resistive stability analysis based on a stationary equilibrium remains questionable as long as the field diffusion is not taken into account. Montgomery et al. have investigated the problem on non-ideal static axisymmetric equilibria. There have also been attempts to calculate resistive axisymmetric equilibrium with only toroidal flow. It has been further argued that tokamak equilibrium flow is either purely toroidal or the poloidal component is small and quickly damped by magnetic pumping. So there is a natural tendency to exclude the poloidal flow while calculating an equilibrium. But when one considers finite conductivity with purely equilibrium toroidal flow, the conductivity (hence the resistivity) becomes a function of space. In general, the resistivity is not a flux function irrespective of equilibrium flow. In this report, we ask the very pertinent question, whether the situation changes in presence of poloidal flow. As we show that a uniform resistivity profile is consistent in presence of poloidal flow, whereas it has been shown that a scalar pressure equilibrium can not have uniform resistivity. Further we show that in a field-reversed (FRC) axisymmetric configuration with no toroidal magnetic field, both toroidal and poloidal equilibrium flows can coexist with finite resistivity, which is not found to be the case with ideal equilibrium.

We consider the equilibrium resistive MHD equations with plasma flow. The equations are

\[
\nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)
\]

\[
\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = 0, \nabla \times \mathbf{B} = 0, \quad (2)
\]

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = j/\sigma, \quad (3)
\]

\[
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p, \quad (4)
\]

where the symbols have their usual meanings. We use a right handed cylindrical system \((r, \theta, z)\) with \(z\) as the axis of symmetry, \(\theta\) as the toroidal angle, and \(r\) along the major radius of an axisymmetric device. We assume the plasma flow to be arbitrary (toroidal and poloidal) and axisymmetry is assumed i.e. \(\partial / \partial \theta = 0\). The plasma resistivity \(\eta = \sigma^{-1}\) is assumed to be an unspecified function of \(r\) and \(z\). We have further assumed here that the equilibrium is maintained in a steady-state through resistive diffusion. The magnetic induction equation allows us to write the magnetic field as

\[
\mathbf{B} = \frac{1}{r} \nabla \psi \times \hat{\mathbf{e}}_\theta + \frac{I}{r} \hat{\mathbf{e}}_\theta, \quad (5)
\]

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where $\psi$ is the magnetic flux function which is the azimuthal component of the vector potential $A$ and $I$ is the current function. Similarly, following the continuity equation, Eq. (4), we can express the plasma equilibrium velocity as

$$\mathbf{v} = \frac{1}{\rho r} \nabla \varphi \times \hat{e}_\theta + \omega r \hat{e}_\theta,$$

where $\varphi$ is the velocity stream function and $\omega = v_\theta / r$ is the toroidal angular velocity. We also assume that the flow is incompressible i.e. $\nabla \cdot \mathbf{v} = 0$.

Because the flow is now in both toroidal and poloidal direction, the poloidal component of current, $j_p$, need not vanish. In general the current can be expressed as

$$j = -\frac{1}{r} \Delta^* \psi \hat{e}_\theta + \frac{1}{r} \nabla I \times \hat{e}_\theta,$$

where $\Delta^*$ is the elliptic operator defined by $\Delta^* \psi = r^2 \nabla \cdot \left( \frac{1}{r} \nabla \psi \right)$. Taking curl of the Ohm’s law, Eq. (5), we have

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \nabla \times (j / \sigma)$$

with

$$\mathbf{v} \times \mathbf{B} = \frac{1}{\rho r^2} \nabla \varphi \times \nabla \psi - \frac{1}{\rho r^4} \nabla \varphi + \omega \nabla \psi.$$

The $\hat{e}_\theta$ component of Eq. (5) can be now written as

$$\hat{e}_\theta \cdot \nabla \omega \times \nabla \psi - \hat{e}_\theta \cdot \nabla \left( \frac{I}{\rho r^2} \right) \times \nabla \psi$$

$$= \frac{1}{\sigma r} \left( \frac{1}{\sigma} \nabla \rho \times \nabla \psi \right) + \frac{2}{r} \frac{\partial I}{\partial r} - \nabla^2 I.$$

We now invoke the large aspect ratio expansion and assume that toroidal magnetic field, to the first approximation, can be written $B_\theta \approx B_\theta r_0 / r$. Here, $B_\theta$ is the value of the toroidal magnetic field at center of the cylindrical cross section of the torus at distance $r_0$ from the axis of symmetry. To this effect we have the current function $I \approx B_\theta r_0 = \text{const.}$ Under these assumption, the above equation can be written as

$$\hat{e}_\theta \cdot \nabla \omega \times \nabla \psi = -\frac{I}{\rho r^2} \hat{e}_\theta \cdot \nabla \rho \times \nabla \varphi$$

$$- \frac{2I}{\rho r^3} \hat{e}_\theta \cdot \nabla r \times \nabla \varphi.$$  

The first term on the right hand side of the above equation vanishes by virtue of the incompressibility condition and the continuity equation. We neglect the second term because of its $1/r^3$ dependence and find that the toroidal angular velocity $v_\theta / r = \omega(\psi)$ becomes a surface quantity. This also further means that $j = j_\theta \hat{e}_\theta$ with $j_\theta = 0$. We, however, note that the condition $\omega = \omega(\psi)$ is identically satisfied in a field-reversed configuration (FRC) where $I = 0$ and large aspect ratio approximation is not required. It can be noted here that without any approximation, $\omega$ becomes a flux function when one considers ideal equilibrium or resistive equilibrium with only toroidal flow.

Now, we consider the momentum equation Eq. (6) and its $\hat{e}_\theta$ component. With the above approximations, we can write Eq. (6) as

$$j_\theta \hat{e}_\theta \times \mathbf{B}_p = \nabla P + \rho \nabla \left[ \frac{1}{2 \rho r^2} \left( \nabla \varphi \right)^2 \right]$$

$$- \nabla \cdot \left( \frac{1}{\rho r^2} \nabla \varphi \right) \nabla \varphi - \omega' \left( \nabla \varphi \times \nabla \psi \right)$$

$$- \frac{2 \omega}{r} \frac{\partial \varphi}{\partial z} \hat{e}_\theta - \omega^2 r \nabla r.$$

In the above equation $\mathbf{B}_p$ is the poloidal component of the magnetic field and (’) denotes derivative with respect to $\psi$. Taking the $\hat{e}_\theta$ component of the above equation, we have,

$$\hat{e}_\theta \cdot \nabla \varphi \times \nabla (\omega r^2) = 0,$$

which means that $\varphi \equiv \varphi(\omega r^2)$. We take the simplest situation of $\varphi \propto \omega r^2$ which yields another surface quantity, $\varphi / r^2 = \zeta(\psi)$. However, it is important to note that, $\zeta(\psi)$ is not an arbitrary function in the sense that it is proportional to the toroidal velocity $\omega(\psi)$ i.e. the toroidal and poloidal flows are no longer independent. Physically, one can understand this by noting that finite resistivity allows plasma motion across the the flux surfaces. Equivalently, toroidal flow, in a resistive axisymmetric plasma, is always associated with poloidal flow.

Because of equilibrium flow, however, plasma pressure $p$ is no longer a flux function now. Taking the $\mathbf{B}_p$ component of the momentum equation (Eq. (13)), we have

$$\mathbf{B}_p \cdot \left[ \frac{\nabla P}{\rho} + \nabla \left( \frac{1}{2 \rho r^2} \left( \nabla \varphi \right)^2 - \frac{1}{2} \omega^2 r^2 \right) \right]$$

$$= \frac{1}{\rho} \nabla \cdot \left( \frac{1}{\rho r^2} \nabla \zeta(r^2) \right) \mathbf{B}_p \cdot \nabla (\zeta^2).$$

Depending upon the equation of state, now, several options are possible. However, we note that density, in general, is not a flux function in presence of arbitrary plasma flow. This can be easily seen from the equation of continuity Eq. (8), after applying the incompressibility condition,

$$\hat{e}_\theta \cdot \nabla \varphi \times \nabla \rho = 0.$$  

It can be seen from the above expression that $\rho$ is not a surface quantity. We note here that axisymmetric equilibrium with incompressible equilibrium flows are generally associated with constant density magnetic surfaces.

One is also free to choose density as a flux function in case of resistive axisymmetric equilibrium with only incompressible toroidal flow.  

2
Taking the \( \mathbf{e}_\theta \) component of Ohm’s law Eq.(3) along with Eq.(1), we have an expression for plasma conductivity,

\[
\sigma \left( E_0 r_0 + \frac{2}{\rho} r \zeta B_r \right) + \Delta^* \psi = 0,
\]

where \( E_0 \) is the longitudinal externally applied electric field at major radius \( r = r_0 \). We immediately see from the above expression that conductivity, in general, is a space dependent quantity.

In what follows, we shall consider two cases with (i) uniform and constant density and (ii) a nonuniform density. In the second case, we consider isentropic magnetic surfaces. We now assume that plasma density is uniform with two arbitrary flux functions \( \chi \) (\( \psi \)).

Bernoulli’s equation, \( \frac{\partial}{\partial r} \rho \frac{\partial}{\partial r} \int \frac{1}{2} \frac{1}{r^2} \Delta^* (\zeta r^2) B_p \cdot \nabla (\zeta r^2) \]

\[ = \frac{1}{r^2} \Delta^* (\zeta r^2) B_p \cdot \nabla (\zeta r^2). \]  \[ (17) \]

Integration of the above equation yields the equivalent Bernoulli’s equation,

\[ p + \frac{(\nabla \varphi)^2}{2 r^2} - \frac{\omega^2}{2} = \int \frac{1}{B_p} \frac{1}{r^2} \Delta^* (\zeta r^2) B_p \cdot \nabla (\zeta r^2) \]

\[ + \chi(\psi), \]  \[ (18) \]

where the integration is along a magnetic field line and \( \chi(\psi) \) is an arbitrary surface quantity. The solubility condition further requires that

\[ \oint \frac{1}{B_p} \frac{1}{r^2} \Delta^* (\zeta r^2) B_p \cdot \nabla (\zeta r^2) = 0. \]  \[ (19) \]

We further assume that part of the pressure gradient that varies within a magnetic flux tube has no \( \nabla \psi \) component \( \| \), \( r \).

\[ \nabla \psi \cdot \nabla \oint \frac{1}{B_p} \frac{1}{r^2} \Delta^* (\zeta r^2) B_p \cdot \nabla (\zeta r^2) = 0. \]  \[ (20) \]

Together with Eq.(18) and the above assumption, the \( \nabla \psi \) component of the momentum equation yields the equivalent Grad-Shafranov equation,

\[ \Delta^* \psi + r^2 \left( \chi' + \omega' \omega r \right) + \frac{\Delta^* (\zeta^2 r^2) \psi \cdot \nabla (\zeta^2 r^2)}{|\nabla \psi|^2} = 0, \]  \[ (21) \]

with two arbitrary flux functions \( \chi(\psi) \) and \( \omega(\psi) \). The primes refer derivative with respect to \( \psi \).

We now consider the second case where we consider a nonuniform density. With incompressible flow, magnetic surfaces with constant entropy is quite a reasonable approximation in ideal MHD. However, considering long resistive diffusion time, the right hand side of Eq.(16) can be neglected and we can continue to proceed with isentropic magnetic surfaces \( \therefore \). The equation of state can now be written as, \( p = S \rho^\gamma \), where, \( S(\psi) \) is the entropy which is a flux function and \( \gamma \) is the ratio of specific heats. We now write \( B_p \cdot \nabla p/\rho \) as \( B_p \cdot \nabla [\gamma S \rho^{\gamma-1}/(\gamma - 1)] \), so that equivalent Bernoulli’s equation can be written as,

\[ \Theta(\psi) + \int \frac{dl}{B_p} \frac{1}{\rho} \nabla \cdot \left( \frac{1}{\rho^\gamma r^2} \nabla (\zeta^2 r^2) \right) B_p \cdot \nabla (\zeta^2 r^2) \]

\[ = \frac{\gamma - 1}{\gamma - 1} S \rho^{\gamma-1} + \frac{1}{2 \rho^\gamma r^2} \nabla (\varphi^2) - \frac{1}{2} \omega^2 r^2, \]  \[ (22) \]

where \( \Theta(\psi) \) is arbitrary. As we have assumed previously, it requires a solubility condition and the equivalent to the assumption \( (20) \). We can then continue to write the equivalent Grad-Shafranov equation by taking the \( \nabla \psi \) component of the momentum equation and applying the Bernoulli’s law Eq.(22),

\[ \Delta^* \psi + r^2 \left( \Theta' + \omega' \omega r - S' \rho^{\gamma-1} \right) \]

\[ = \frac{\rho |\nabla \psi|^2}{r^2} \nabla \cdot \left[ \frac{1}{\rho r^2} \nabla (\zeta^2 r^2) \right] \nabla \psi \cdot \nabla \psi. \]  \[ (23) \]

In the above equation we have four arbitrary surface quantities i.e. \( \Theta(\psi), \omega(\psi) \), and \( S(\psi) \) and the primes denote derivative with respect to \( \psi \).

We have derived the differential equations, equivalent to the Grad-Shafranov equation, for resistive axisymmetric plasma with arbitrary equilibrium flows. These equilibrium equations Eqs.(21), (23) have to be solved subject to conductivity constraint Eq.(16). Further, in a field-reversed configuration (FRC) with no toroidal magnetic field, it can be seen from Eq.(13) that both poloidal and toroidal flow can coexist.

We now show that a uniform conductivity profile is consistent with resistive axisymmetric equilibria with arbitrary flow. A simple examination of Eq.(4), though reveals that uniform conductivity may be possible with scalar pressure equilibrium in presence of flow, it however provides no easier way of proving it. We note that the usual procedure for solving Eqs.(21) and (23) requires specifying \textit{a priori} dependence of the respective arbitrary functions on \( \psi \). However, in the presence of finite resistivity, the resistivity constraint Eq.(14) can be used to solve for \( \psi \), which can be uniquely determined if the right hand side of Eq.(16) is specified \( \therefore \). It should be noted here with caution whether the resultant solution for \( \psi \) corresponds to realistic profiles for other physical quantities such as pressure, density, velocity etc. However, our sole aim, here, is to demonstrate the existence of a solution consistent with uniform resistivity in presence of flows.

From Eq.(13) we know that \( \rho \equiv \rho(\varphi) \), and assume that \( \rho \propto \varphi \). We now assume that conductivity is uniform in space so that the resulting Eq.(16) can be written as,

\[ \Delta^* \psi + \frac{\alpha}{r^2} \frac{\partial \psi}{\partial z} = \beta, \]  \[ (24) \]

In the above equation we have four arbitrary surface quantities i.e. \( \Theta(\psi), \omega(\psi) \), and \( S(\psi) \) and the primes denote derivative with respect to \( \psi \).

We have derived the differential equations, equivalent to the Grad-Shafranov equation, for resistive axisymmetric plasma with arbitrary equilibrium flows. These equilibrium equations Eqs.(21), (23) have to be solved subject to conductivity constraint Eq.(16). Further, in a field-reversed configuration (FRC) with no toroidal magnetic field, it can be seen from Eq.(13) that both poloidal and toroidal flow can coexist.

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where α and β are arbitrary constants. It is worthwhile mentioning at this point that Eq. (24) can not be used in case of very small resistivity. In the limit of vanishing resistivity (large β in the above equation), the solution of Eq. (24) contains short scale spatial dependence (boundary layers), not present in case of ideal equilibrium and may lead to unphysical results.

Note that the above equation is a elliptic equation and can be treated as boundary value problem. Following Zheng et al. [22], we assume a solution of the form

$$
ψ_h(r, z) = \sum_{n=0,1,2,...} f_n(r)z^n
$$

for the homogeneous part of Eq. (24). We however retain the odd terms in the summation to take care of the asymmetric-term in Eq. (24). For simplicity we assume that $f_n(r) = 0$ for $n \geq 3$, which, however, can be extended up to any number of terms if required, about which we shall make a comment later. Substituting Eq. (25) in the equivalent homogeneous equation for Eq. (24) we can solve for the functions $f_{0,1,2}(r)$. The homogeneous solution of Eq. (24) is then given by,

$$
ψ_h(r, z) = a_1r^2\{4r^2 + 16z^2 - α^2[4(ln r - 4 ln r + 2 - r^2] + 8αz(r^2 - 2 ln r)\} + a_2r^2(2 ln r - 1) - α^2 ln r(ln r + 1) + 4αz ln r + 4z^2

+ a_3r^2[α(2 ln r - 1) + 4z] + a_4r^2,
$$

where $a_i$s are arbitrary constants to be determined from the boundary conditions. A particular solution of Eq. (24) is $ψ_p = βr^2(2 ln r - 1)/4$. So the complete solution of Eq. (24) is

$$
ψ = ψ_h + ψ_p,
$$

which can be verified by direct substitution. For a conducting circular boundary of a toroidal axisymmetric device, the constant flux ($ψ$) contours are shown in Fig.1 (a) which shows a scaler pressure equilibrium. The solution for $σ ∝ r^2$ is shown in Fig.1 (b). Note that $σ ∝ r^2$ is the only possible solution for resistive axisymmetric equilibrium without flow [22].

In principle the expansion in Eq. (24) should be retained with a large number of terms which will result a equally large number of arbitrary constants for the solution in $ψ$. These constants can then be used to shape any arbitrarily shaped plasma boundary.

In passing, we would like to note that resistive field diffusion ($\partial B/\partial t \neq 0$) is intrinsically involved with non-stationary equilibria ($v \neq 0$). However, a series of ideal quasi-stationary equilibrium states can be built up with $\partial B/\partial t = 0$ in which, the effect of finite resistivity is only to slowly evolve the equilibrium in a diffusive time scale [22].

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1. H. Grad and H. Rubin, in Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958, edited by United Nations (United Nations Publications, Geneva, 1958), Vol. 31, pp. 190.
2. V. D. Shafranov, Sov. Phys. JETP 6, 545 (1958).
3. R. Lüst and A. Schlüter, Z. Naturforsch 12a, 850 (1957).
4. E. Hameiri, Phys. Fluids 26, 230 (1983).
5. L. C. Steinhauser, Phys. Plasmas 6, 2734 (1999).
6. G. N. Throumoulopoulos and H. Tasso, “Ideal magnetohydrodynamic equilibria with helical symmetry and incompressible flows”, e-print physics/9907004 (to be published in J. Plasma Phys.).
7. S. Suckewer, H. P. Eubank, R. J. Goldston et al., Phys. Rev. Lett. 43, 207 (1979).
8. K. Brau, M. Bitter, R. J. Goldston et al., Nucl. Fusion 23, 1643 (1983).
9. S. D. Scott, M. Bitter, H. Hsuan et al., in Proceedings of the 14st European Conference on Controlled Fusion and Plasma Physics, Madrid, 1987 (European Physical Society, Geneva 1987), Vol. 11D, pp. 65.
10. G. N. Throumoulopoulos, J. Plasma Phys. 59, 303 (1998).
11. S. Semenzato, R. Gruber, and H. P. Zehrfeld, Comp. Phys. Rep. 1, 389 (1984).
C. Ren, M. S. Chu, and J. D. Callen, Phys. Plasmas 6, 1203 (1999).

D. Montgomery, J. W. Bates, and H. R. Lewis, J. Plasma Phys., (1997).

D. Dobrott, S. C. Prager, and J. B. Taylor, Phys. of Fluids 20, 1850 (1977).

A. B. Hassam and R. M. Kulsrud, Phys. Fluids 21, 2271 (1987).

J. W. Bates and H. R. Lewis, Phys. Plasmas 3, 2395 (1996).

R. A. Clemente and R. L. Viana, Plasma Phys. Control. Fusion 41, 567 (1999).

K. Avinash, S. N. Bhattacharyya, and B. J. Green, Plasma Phys. Control. Fusion 34, 465 (1992).

G. N. Throumoulopoulos and G. Pantis, Plasma Phys. Control. Fusion 38, 1817 (1996).

G. N. Throumoulopoulos and H. Tasso, Phys. Plasmas 4, 1492 (1997).

A. I. Morozov and L. S. Solovév, in Reviews of Plasma Physics, edited by M. A. Leontovich (Consultants Bureau, New York, 1980), Vol. 8, pp. 1.

S. B. Zheng, A. J. Wootton, and R. Solano, Phys. Plasmas 3, 1176 (1996).

H. Grad and J. Hogan, Phys. Rev. Lett. 24, 1337 (1979).