T duality for boundary-non-critical strings

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Abstract
Recent work on the action of T duality on Dirichlet-branes is generalized to the case in which the open string satisfies boundary conditions that are neither Neumann nor Dirichlet. This is achieved by implementing T duality as a canonical transformation of the \( \sigma \)-model path integral. A class of boundary interactions that violate conformal symmetry is found to be T-dual of a correspondingly non-conformal class of boundary conditions. The analogy with some problems in boundary-non-critical quantum mechanics of interest for condensed matter is pointed out.

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A series of dramatic break-throughs has recently brought our understanding of critical open strings to levels unimaginable even just a few years ago [1]. Part of the motivation for the analysis reported in this Letter comes from contemplating the possibility that some of the technical tools that have contributed to the recent progress in the study of critical open strings might be generalized also to non-critical open strings, thereby providing an opportunity for new insight in this class of theories.

We are here particularly interested in open string theories in which the violations of conformal invariance originate from the physics of the boundary of the string. It is in fact plausible that these theories turn out to be more manageable than their bulk-non-critical counterparts, while still providing a reach laboratory for the exploration of exciting issues, such as the fate of the short-distance structure of the target-space geometry when world-sheet conformal invariance is violated [2, 3]. While we were writing this report we became aware of Ref. [4], whose findings appear to suggest that boundary-non-critical strings might also have a place in very conventional string-theory frameworks.

The expectation of the existence of “reasonably treatable” boundary-non-critical strings finds support in the established literature on boundary-conformal field theory [5, 6, 7], which has provided examples of boundary-non-critical deformations that preserve (most of) the simplicity (e.g. integrability) of the original conformal-invariant theory. Of course, a difficult task in the development of such “reasonably treatable” boundary-non-critical strings is the identification of non-conformal boundary deformations that be respectful (as much as possible) of the simplicity of critical open strings. In this letter we provide no answer to this most challenging aspect of the problem at hand; in fact, we limit ourself to the study of a specific type of boundary deformation: a linear-dilaton boundary background. While we find no reason to believe that this particular deformation should eventually turn out to be particularly simple and treatable, its analysis is intrinsically well motivated in light of the Liouville approach to target time in the context of string theory [2], or in the context of certain cosmological models based on strings [8].

The linear-dilaton boundary backgrounds considered here have the merit of providing us a concrete framework in which to start exploring the applicability to boundary-non-critical strings of T duality, which is one of the most important tools in the derivation of nonperturbative results of critical open strings. This is the central point of this Letter. We find that a conventional path-integral formalism [1] for the implementation of T duality transformations also applies to the boundary-non-critical case. The implications of T duality in this context are such that our linear-dilaton boundary backgrounds are T-dual of a correspondingly non-conformal class of boundary conditions. We report these observa-

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1We view the T duality as a canonical transformation in a σ-model path integral. This point of view was taken in the original literature on Kramers-Wannier duality in statistical physics, and recently, in the context of critical string theory, in ref. [9, 10].
tions as intuitively as possible via the analysis of two explicit cases, rather than providing general and formal derivations. The two cases considered are the flat 26-dimensional target space time and the 2-dimensional “cigar” target space time [11]. Additional intuition for our results can be gained by considering corresponding results in deformed boundary-conformal field theory and quantum mechanics. For example, among the recent results on two-anyon quantum mechanics one finds the observation [7, 12] that this scale-invariant system can be equivalently deformed either by a contact interaction $\delta^{(2)}(r)$ with running coupling $g(\mu)$ or by enforcing in the s-wave sector scale-dependent boundary conditions of the type

$$
[r|\nu|\psi(r) - w\rho^2|\nu|d(r|\nu|) \bigg|_{r=0} = 0 ,
$$

where $r$ is relative-position vector, $r \equiv |r|$, $\nu$ is the “statistical parameter” (a fundamental parameter of anyonic physics), while $\rho$ is a reference scale which together with the dimensionless parameter $w$ can be put in correspondence with the $g(\mu)$ characterizing the contact-interaction formulation of the problem.

### A. Flat 26-dimensional target space time.

Consider the case of a simple linear dilaton boundary background [8], in a non-critical string theory with central charge deficit $Q^2$

$$
S = \frac{1}{4\pi} \int d^2 \sigma \partial X^i \bar{\partial} X^i + \frac{1}{4\pi} \int d^2 \sigma \partial Y^j \bar{\partial} Y^j - \int_{\partial \Sigma} \hat{k} \eta^i X^i
$$

where $\eta$ is an n-dimensional constant number-valued vector, and (for later convenience) we have divided the 26 fields into $n$ fields of type $X$ (i.e. $i = 1 \ldots n$) and $26 - n$ fields of type $Y$ (i.e. $j = 1 \ldots 26 - n$). We have also used conventional notations $\hat{k}$ for the extrinsic curvature of the fiducial metric, $\Sigma$ for the world-sheet manifold, and $\partial \Sigma$ for the boundary of the world-sheet manifold. The fields $X$ and $Y$ are assumed to satisfy Neumann boundary conditions:

$$
\partial_\sigma \hat{X}^i = \partial_\sigma \hat{Y}^j = 0 \quad \text{on } \partial \Sigma
$$

where we use the notation $\partial_\sigma$ ($\partial_\tau$) for normal (tangent) derivatives on $\partial \Sigma$ and we also use the notation $\hat{\Phi}$ to emphasize that a field $\Phi$ is being evaluated on $\partial \Sigma$.

The path-integral formulation of this $\sigma$-model is:

$$
Z = \int DX \delta(\partial_\sigma \hat{X}^i) \delta(\partial_\sigma \hat{Y}^j) \exp \left[ - \int_{\Sigma} \frac{1}{4\pi} \left( \partial X^i \bar{\partial} X^i + \partial Y^j \bar{\partial} Y^j \right) + \int_{\partial \Sigma} \hat{k} \eta^i X^i \right] ,
$$

\footnote{Consistently with the objective of keeping our discussion as intuitive as possible we shall often implicitly or explicitly assume the fiducial geometry to be that of a disk, with constant extrinsic curvature (e.g. $\hat{k} = 2$ for a unit disk).}
where we indicated explicitly via a boundary delta-functional that the $X^i$ and the $Y^j$ are Neumann fields.

We wish to apply a functional T duality transformation on the fields $X^i$ in the path integral (4). We generalize recent work on T duality in the path-integral formalism (see e.g. Ref. [9]), which has been exclusively concerned with open strings satisfying conformal (Dirichlet or Neumann) boundary conditions, to the simple class of non-conformal boundary conditions that correspond to the boundary interaction described by our linear dilaton boundary background. We find that the same procedure of T-dualization is appropriate in the case of nonconformal boundary conditions. In particular, the T duality transformation has again (compare with Ref. [9]) as crucial element the introduction of a vectorial field variable corresponding to the partial derivative of the fields $X^i$ that are being dualized:

$$W^i_\alpha \equiv \partial_\alpha X^i.$$  

(5)

The fields $W$ are introduced in the path integral via the identity:

$$\int DW^i_\alpha \delta(W^i_\alpha - \partial_\alpha X) \delta(\epsilon_{\alpha\beta} \partial_\alpha W^j_\beta) = 1,$$

which takes of course into account the ‘Bianchi identity’:

$$\epsilon_{\alpha\beta} \partial_\alpha W^i_\beta = 0$$

(7)

The path integral (4) is therefore rewritten as:

$$Z = \int DX \, DY \, DW \, \delta(\partial_\sigma \hat{X}^i) \delta(\partial_\tau \hat{Y}^j) \delta(W^i_\alpha - \partial_\alpha X^i) \delta(\epsilon_{\alpha\beta} \partial_\alpha W^j_\beta) \delta(\hat{W}^i_\tau - \partial_\tau \hat{X}^i)$$

$$\int \delta(\hat{W}^i_\tau - \partial_\tau \hat{X}^i) \exp \left[ -\frac{1}{4\pi} \int_\Sigma (W^i_\alpha)^2 - \frac{1}{4\pi} \int_\Sigma \partial Y^j \partial Y^j + \int_{\partial\Sigma} k Q^i X^i \right]$$

$$= Z_Y \int DX^i \, DW^i_\alpha \, D\lambda^i_\alpha \, \delta(\partial_\sigma \hat{X}^i)$$

$$\exp \left[ -\frac{1}{4\pi} \int_\Sigma (W^i_\alpha)^2 - i \int_\Sigma \lambda^i_\alpha (\epsilon_{\alpha\beta} \partial_\alpha W^j_\beta) - i \int_\Sigma \lambda^i_\alpha (W^i_\alpha - \partial_\alpha X) \right]$$

$$\exp \left[ \int_{\partial\Sigma} k Q^i X^i - i \int_{\partial\Sigma} \lambda^i_\alpha \hat{W}^i_\sigma - i \int_{\partial\Sigma} \hat{\lambda}^i_\tau (\hat{W}^i_\tau - \partial_\tau \hat{X}^i) \right]$$

(8)

where (consistently with the notation already introduced for the normal and tangent derivatives) we denoted the normal (tangent) components of world-sheet vectors with a lower index $\sigma$ ($\tau$). (Summation is of course understood on all repeated indices apart from $\sigma$ and $\tau$ which are fixed labels for boundary fields.) We also introduced the short-hand notation

$$Z_Y = \int DY \, \delta(\partial_\sigma \hat{Y}^j) \exp \left[ -\frac{1}{4\pi} \int_\Sigma \partial Y^j \partial Y^j \right]$$

(9)
for the portion of the partition function that concerns the $Y^j$ fields, which are “spectators” of the T duality transformation being performed on the $X^i$ degrees of freedom; moreover we adopt the convention $\epsilon_{\sigma\tau} = 1$ and the following functional representation of a $\delta(\phi)$ constraint

$$\delta(\phi) = \int D\lambda e^{-i\int_M \phi \lambda}$$

(10)

with $M$ an appropriate manifold, indicating the range of definition of the arguments of the fields $\phi$. In our case $M = \Sigma$ or $\partial \Sigma$.

The Lagrange multipliers fields $\chi^i$ and $\lambda^i_{\alpha}$ play a highly non-trivial role in the T duality transformation; in particular, the fields $\chi^i$, which implement the Bianchi identity (7), turn out to be directly related to the fields that are T-dual to the fields $X^i$, just as expected from the analysis reported in Ref. [9].

It is convenient to rewrite (also using integration by parts) the partition function of (8) as

$$Z = Z_Y \int DX^i DW^i_{\alpha} D\chi^i D\lambda^i_{\alpha} \delta(\sigma \hat{X}^i)$$

$$\exp \left[ \int_{\Sigma} \left( \frac{i}{2\sqrt{\pi}} W^i_{\alpha} + \sqrt{\pi} \left( \epsilon_{\alpha\beta} \partial_\beta \chi^i - \lambda^i_{\alpha} \right) \right)^2 - \pi \int_{\Sigma} \left( \epsilon_{\alpha\beta} \partial_\beta \chi^i - \lambda^i_{\alpha} \right)^2 \right]$$

$$\exp \left[ i \int_{\Sigma} X^i \partial_\alpha \hat{X}^i - i \int_{\partial \Sigma} \hat{X}^i \left( \hat{\lambda}^i_{\sigma} + \partial_\tau \hat{X}^i + i k \eta^i \right) \right]$$

$$\exp \left[ -i \int_{\partial \Sigma} \hat{W}^i_{\tau} (\hat{\lambda}^i_{\tau} - \hat{\chi}^i) - i \int_{\partial \Sigma} \hat{W}^i_{\sigma} \hat{\lambda}^i_{\sigma} \right]$$

(11)

The functional integration over $X$ and $W$ can be done easily; one obtains (up to an irrelevant overall factor coming from the gaussian integration over $W$)

$$Z = Z_Y \int D\chi D\lambda \delta(\sigma \lambda_{\alpha}) \delta(\hat{\lambda}_{\sigma}) \delta(\hat{\chi} - \hat{\lambda}_{\tau}) \delta(\partial_\tau \hat{\lambda}_{\tau} + i k \eta^i + \hat{\lambda}_{\sigma})$$

$$\exp \left[ -\pi \int_{\Sigma} \left( \lambda^i_{\alpha} - \epsilon_{\alpha\beta} \partial_\beta \chi^i \right)^2 \right]$$

(12)

The fields $X^i_D$ that are T-dual to the fields $X^i$ are easily identified as the ones satisfying the relation

$$\frac{i}{2\pi} \epsilon_{\alpha\beta} \partial_\beta X^i_D \equiv \epsilon_{\alpha\beta} \partial_\beta \chi^i - \lambda^i_{\alpha}$$

(13)

whose consistency follows from the constraint $\partial_\alpha \lambda^i_{\alpha} = 0$ (see Eq. (12)).
Upon the change of variables $\chi^i \to X^i_D$, and disposing of the then trivial functional integration over the $\lambda$ fields, one can easily rewrite the partition function of (12) as (up to another irrelevant overall factor)

$$Z = Z_Y \int DX_D \delta(\partial_\tau \hat{X}^i_D + 2\pi Q \hat{k} \eta^i) \exp \left[ - \int_\Sigma (\partial_\alpha X^i_D)^2 \right]$$

(14)

where we also used the fact that (13), when combined to the constraint $\hat{\lambda}_i^\sigma = 0$ (see Eq. (12)), implies $\partial_\tau \hat{X}^i_D = -i2\pi \partial_\tau \hat{\chi}^i$.

The T duality transformation implemented via the path-integral manipulations that take from (4) to (14) evidently maps a Neumann open string with boundary interactions corresponding to the linear dilaton boundary background present in (4) into a free open string satisfying nonconformal boundary conditions (13)

$$\partial_\tau \hat{X}^i_D = -2\pi Q \hat{k} \eta^i .$$

(15)

This boundary condition reduces to the Dirichlet boundary condition in the limit $Q \to 0$. For every $Q \neq 0$ it encodes a “conformal anomaly” for the free T-dual theory that reflects the conformal anomaly of the corresponding boundary interactions of the original Neumann theory. As anticipated in the opening of the present Letter, this is reminiscent of certain dualities encountered in boundary conformal field theory (B) where one also maps a problem with nontrivial boundary interactions into a problem subject to nontrivial boundary conditions (B).

It is interesting to note at this stage that the above boundary condition might also have important implications for the quantization of the central charge. Consider for instance the boundary condition (13) in a cordino-disc planar geometry (annulus) with a flux passing through the middle of the two-dimensional surface. In this case, if we parametrize the external boundary $S^1$ of the disc by an angular variable $0 \leq \tau \leq 2\pi$, and assume for simplicity a linear dilaton along one direction only, say $\hat{X}^1$, assumed to be compact, then, the dual field $\hat{X}^1_D$ may have non-trivial winding number $n$ around $S^1$, $\hat{X}^1_D(2\pi) = \hat{X}^1_D(0) + 2\pi n R_D$, where $R_D$ is a compactification radius for the dual field. In that case, (13) could be integrated to yield

$$2\pi n R_D = -2\pi Q \int d\tau \hat{k} = -4\pi^2 Q$$

(16)

3It would be interesting to explore the relation of these findings with the recent observation (14) that fixed Dirichlet boundary conditions are not conformal invariant in the presence of a linear-dilaton bulk background (B).

4The representation, via canonical duality transformation, as a peculiar boundary condition, opens up the way of treating, in a simpler way, more complicated situations, like the Liouville boundary dynamics (B). There, the complications arise from the fact that the Liouville field on the boundary does not behave like a scalar field, in contrast to the linear dilaton case considered above. It would be interesting to derive the corrections to the boundary condition (15), that the Liouville field satisfies due to its non-trivial quantum dynamics (B), from the $\sigma$-model path integral method. This is left for future work.
taking into account the Euler characteristic of a disc to be one. The result is a quantization of the square-root $Q$ of the central charge deficit:

$$Q = -\frac{nR_D}{2\pi} \quad n \in \mathbb{Z} \quad (17)$$

This is consistent with the nature of the boundary non-conformal linear dilaton deformation which we introduced to begin with, (3), provided that the compactification radius of the field $i\hat{X}^1$ is $R_D^{-1}$, as required by T duality ($R = R_D^{-1}$), which is consistently reproduced in our approach here. Indeed, assuming the quantization condition (17) one finds that under a shift $\hat{X}^1(2\pi) = \hat{X}^1(0) + 2\pi n R_D^{-1}$ the action (3) changes by factors of $2i\pi m$ (with $m \in \mathbb{Z}$) and thus the partition function remains invariant.

It would be interesting to explore the relation of the result (17) with the analogous result that is found to hold in a certain type of (“bulk”) closed strings with compactified dimensions [16].

B. 2-dimensional “cigar” target space time.

The second example we have chosen in order to illustrate the doings of T duality in open-string theories with non-conformal boundary physics is the two-dimensional black hole with a cigar target-space metric [11]. For the closed-string version of this $\sigma$-model theory it is known that the duality transformations map the cigar metric into a funnel [18]. We shall rederive these results using the path-integral formulation of T duality; moreover, we shall generalize them to the open-string case and allow again our non-conformal linear-dilaton background on the boundary of the string.

From a physical viewpoint, the open-string analysis of this problem is motivated by the $D$-brane representation of the two-dimensional black hole. In fact, as argued in ref. [17], since the cigar metric has two asymptotically flat domains in target space time, one can get rid of one of them by orbifoldizing the solution to obtain a theory of unoriented open and closed strings in a black hole background, and a time-like orbifold singularity at the origin. All of the open string states of the model are confined to the orbifold singularity. As discussed in ref. [17] the model is dual, under target-space duality, to a conventional open string theory in the black hole geometry. This was the first example of a 1-brane (Dirichlet), in the context of two-dimensional strings.

Motivated by these considerations we shall discuss below the effect of our T duality canonical transformation, described above, on the path integral of a $\sigma$-model with a cigar-type metric and some open string excitations, represented by the existence of a boundary on the corresponding $\sigma$-model. Deviations from conformal invariance on the boundary may for example arise from recoil (or other matter back-reaction effects) of the Dirichlet
brane during scattering with some string matter.

Let us briefly first review what is known of the duality in the two-dimensional Euclidean stringy Black Hole. The standard $\sigma$-model action, obtained from gauging the $SL(2,\mathbb{R})/U(1)$ Wess-Zumino-Witten-Novikov (WZWN) conformal field theory action, is:

$$S_{WZW} = \frac{k}{4\pi} \int_{\Sigma} d^2\sigma \left[ \partial_{\alpha} \rho \partial_{\alpha} \rho + \tanh^2 \rho \partial_{\alpha} \theta \partial_{\alpha} \theta \right] - \frac{1}{8\pi} \int_{\Sigma} d^2\sigma \Phi(\rho) R^{(2)}$$

(18)

where $k$ is the level of the WZWN coset model $^{[11]}$, $\theta$ is a compact $U(1)$ coordinate (Euclidean time), and

$$\Phi(\rho) = \ln \cosh^2 \rho + \text{constant}.$$  

(19)

The above $\sigma$-model describes a two-dimensional string moving in a Euclidean black hole space time. The Hawking temperature of this black hole is given by the radius of the cigar at infinity:

$$\beta = T_{H}^{-1} = \sqrt{\frac{k}{2}}$$

(20)

Conformal invariance requires $k = 9/4$, since in that case the central charge $c = \frac{3k}{k-2} - 1$ becomes 26 $^{[11]}$.

It is convenient to absorb the constant $\frac{k}{2\pi}$ into a redefinition of $\rho \to \sqrt{2/k} \rho$. Then the action takes the form (which we shall use from now on):

$$S_{WZW} = \frac{1}{2\pi} \int_{\Sigma} d^2\sigma \left[ \partial_{\alpha} r \partial_{\alpha} r + \epsilon^2 \tanh^2 (\epsilon r) \partial_{\alpha} \theta \partial_{\alpha} \theta \right] - \frac{1}{8\pi} \int_{\Sigma} d^2\sigma \Phi(r) R^{(2)}$$

(21)

where $\epsilon = \sqrt{\frac{2}{k}}$ and

$$\Phi(r) = \ln \cosh^2 \epsilon r + \text{constant}.$$  

(22)

The dual transformation is given by:

$$G_{ab} \to G_{ab}^{-1}, \quad \Phi \to \Phi + \frac{\lambda}{2} \text{Indet} G$$

(23)

In this context, it should be noted that the duality transformation of ref. $^{[9]}$ has been applied earlier in ref. $^{[13]}$ in order to discuss the formal appearance, in a $\sigma$-model framework, of Dirichlet boundary conditions from the vacuum in a quantum foam picture of a generic $D$-brane system. A certain class of scale-dependent boundary conditions (boundary conditions that, like the ones considered in the present paper, are neither fixed Dirichlet, nor Neumann), has then been shown to result from the appearance of a Liouville mode to describe decoherence effects due to (boundary) recoil operators.
where $\chi$ is the Euler characteristic of the world-sheet manifold ($\chi = 2$ for sphere), and in the simplified case of a single compactified coordinate reduces to the standard $T$ duality $R \to 1/R$ [18]. Under the above transformation one obtains the dual $\sigma$-model action

$$S^{WZW}_{D} = \frac{1}{2\pi} \int_{\Sigma} d^2 \sigma [\partial_{\alpha} r \partial_{\alpha} r + \epsilon^2 \coth^2 (\epsilon r) \partial_{\alpha} \theta \partial_{\alpha} \theta] - \frac{1}{8\pi} \int_{\Sigma} d^2 \sigma \Phi(r, \theta) R^{(2)}$$

(24)

with

$$\Phi_D(r) = \ln [\epsilon^{-2} \sinh^2 \epsilon r] + \text{constant}$$

(25)

The Hawking temperature of this dual model is the inverse of (20):

$$\beta_D = \beta^{-1} = \sqrt{2/k}$$

(26)

This inversion of temperature is reminiscent (actually a simplified version) of the standard Kramers-Wannier duality in statistical physics (c.f. Ising model), to which the $T$ duality is related.

We shall show that these results can be easily rederived upon applying to the $\theta$ field the path-integral technique of dualization described above. This result will actually emerge as a corollary of the analysis of the theory (21) deformed by our nonconformal dilaton background

$$S^{WZW}_{def} = \frac{1}{2\pi} \int_{\Sigma} d^2 \sigma [\partial_{\alpha} r \partial_{\alpha} r + \epsilon^{-2} \tanh^2 (\epsilon r) \partial_{\alpha} \theta \partial_{\alpha} \theta] - \frac{1}{8\pi} \int_{\Sigma} d^2 \sigma \Phi(r) R^{(2)} + \int_{\partial\Sigma} \hat{k} Q \theta$$

(27)

Our analysis shall follow quite closely the one leading to (12), but take into account the fact that the cigar target space is curved. In particular, this implies that in the measure of integration of target-space-coordinate fields one should introduce the appropriate determinants $\sqrt{\det G}$ of the corresponding target space metric tensor, as required by target-space diffeomorphism invariance. The starting partition function is therefore

$$Z^{WZW}_{def} = \int D r D \theta \sqrt{\det G} \delta (\partial_{\alpha} r) \delta (\partial_{\sigma} \hat{\theta}) \exp \left[ -\frac{1}{2\pi} \int_{\Sigma} [\partial_{\alpha} r \partial_{\alpha} r + \epsilon^{-2} \tanh^2 (\epsilon r) \partial_{\alpha} \theta \partial_{\alpha} \theta] + \frac{1}{8\pi} \int_{\Sigma} \Phi(r) R^{(2)} + \int_{\partial\Sigma} \hat{k} Q \theta \right]$$

(28)

where again we indicated explicitly via functional Dirac $\delta$’s that the $r$ and $\theta$ are Neumann fields. In complete analogy with [8] we can rewrite the partition function $Z^{WZW}_{def}$ as

$$Z = \int D r D \theta D W \sqrt{\det G} \delta (\partial_{\alpha} r) \delta (\partial_{\sigma} \hat{\theta}) \delta (W_{\alpha} - \partial_{\alpha} \theta) \delta (\epsilon_{\alpha \beta} \partial_{\alpha} W_{\beta}) \delta (\hat{W}_{\alpha} - \partial_{\sigma} \hat{\theta}) \delta (\hat{W}_{\tau} - \partial_{\tau} \hat{\theta})$$

$$\exp \left[ -\frac{1}{2\pi} \int_{\Sigma} [\partial_{\alpha} r \partial_{\alpha} r + \epsilon^{-2} \tanh^2 (\epsilon r) (W_{\alpha})^2] + \frac{1}{8\pi} \int_{\Sigma} \Phi(r) R^{(2)} + \int_{\partial\Sigma} \hat{k} Q \theta \right]$$
\[ Z = \int Dr D\theta D\chi D\lambda \sqrt{\det G} \delta(\partial_\sigma \hat{r}) \delta(\partial_\sigma \hat{\theta}) \exp \left[ -\frac{1}{2\pi} \int [\partial_\alpha r \partial_\alpha r + \epsilon^{-2} \tanh^2(\epsilon r) (W_\alpha)^2 - \frac{\Phi(r)}{8\pi} J_{\Sigma} \right] (32) \]

This shows that the action of T duality on the boundary physics is completely analogous to the one discussed above for case of a flat (26-dimensional) target space; in fact, again the T duality maps a Neumann open string with boundary interactions corresponding to our linear dilaton background present in (28) into a free open string satisfying nonconformal boundary conditions

\[ \partial_\tau \hat{\theta}_D = -Q \hat{k} . \] (33)

Of course, the discussion following (15), on the quantization of the central charge deficit, applies to this case as well.

Concerning the bulk of the string it is explicit in (32) that the expected dual \( \tilde{G} \) of the original metric tensor \( G \) acts as target space metric tensor in the dual variables \( \{ r, \theta_D \} \). Eq.(34) also encodes the expected shift of the dilaton under T duality. Consistently with the fact that the dilaton is associated with purely quantum effects in the \( \sigma \)-model path.
integral [19], the duality transformation of the dilaton (23) comes from the measure of integration,
\[ \sqrt{\det G / \det \tilde{G}} = \exp \left( (1/2) \ln \left| \det G / \det \tilde{G} \right| \right) \] (34)

In fact, this term can be absorbed into a proper redefinition of the dilaton term. This can be seen most intuitively by observing that using world-sheet reparametrization invariance one can concentrate the world-sheet curvature on a single point on the world sheet \( \sigma_0 \):
\[ R^{(2)} \rightarrow 4\pi \chi \delta(\sigma - \sigma_0) \] (35)

where \( \chi \) is the Euler characteristic. Correspondingly, the dilaton term in (29), which so far remained unaffected by the change of variables, takes the form \( S_{\text{dil}} = \Phi(r(\sigma_0)) \). Observing that the dual metric is such that \( \det \tilde{G} = 1 / \det G \), and evaluating \( G \) at \( r(\sigma_0) \), one concludes that the last two terms of (32) encode a shift in the dilaton
\[ \Phi \rightarrow \Phi + \frac{\chi}{2} \ln \det G \] (36)

The point \( \sigma_0 \) on the world sheet is arbitrary due to reparametrization invariance which is assumed valid; hence the shift (36) should be considered as a generic shift of the dilaton field at any world-sheet point, just as required by the definition (23) of the duality transformation.

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