A coupled Hartree system with Hardy-Littlewood-Sobolev critical exponent: existence and multiplicity of high energy positive solutions

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Abstract

This paper deals with a coupled Hartree system with Hardy-Littlewood-Sobolev critical exponent

\[
\begin{align*}
-\Delta u + (V_1(x) + \lambda_1)u &= \mu_1(\frac{|x|^{-4} * u^2}{u})u + \beta(\frac{|x|^{-4} * v^2}{v})u, \quad x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2)v &= \mu_2(\frac{|x|^{-4} * v^2}{v})v + \beta(\frac{|x|^{-4} * u^2}{u})v, \quad x \in \mathbb{R}^N,
\end{align*}
\]

where \( N \geq 5, \lambda_1, \lambda_2 \geq 0 \) with \( \lambda_1 + \lambda_2 \neq 0 \), \( V_1(x), V_2(x) \in L^\infty(\mathbb{R}^N) \) are nonnegative functions and \( \mu_1, \mu_2, \beta \) are positive constants. Such system arises from mathematical models in Bose-Einstein condensates theory and nonlinear optics. By variational methods combined with degree theory, we prove some results about the existence and multiplicity of high energy positive solutions under the hypothesis \( \beta > \max\{\mu_1, \mu_2\} \).

Keywords: Hartree system; Hardy-Littlewood-Sobolev critical exponent; Lack of compactness; Positive solution.

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1 Introduction and main results

In this paper, we look for standing waves of the two-component nonlinear Hartree system

\[
\begin{align*}
\frac{i}{\partial_t} \Phi_1 &= -\Delta \Phi_1 + V_1(x) \Phi_1 - \mu_1(K(x) * |\Phi_1|^2)\Phi_1 - \beta(K(x) * |\Phi_2|^2)\Phi_1, \\
\frac{i}{\partial_t} \Phi_2 &= -\Delta \Phi_2 + V_2(x) \Phi_2 - \mu_2(K(x) * |\Phi_2|^2)\Phi_2 - \beta(K(x) * |\Phi_1|^2)\Phi_2, \\
(x, t) &\in \mathbb{R}^N \times \mathbb{R}^+, \quad \Phi_j = \Phi_j(x, t) \in \mathbb{C}, \quad \Phi_j(x, t) \to 0, \text{ as } |x| \to +\infty, \quad j = 1, 2,
\end{align*}
\]

where \( i \) is the imaginary unit, \( \Phi_j (j = 1, 2) \) is the corresponding condensate amplitudes, \( K(x) \) is a nonnegative function which possesses information about the self-interaction between the particles, \( \mu_j \) is the interspecies

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scattering length and coupling constant $\beta$ is the intraspecies scattering length: $\mu_j > 0$ corresponds to the attractive and $\mu_j < 0$ to the repulsive self-interactions; similarly, the coupling constant $\beta > 0$ corresponds to the attraction and $\beta < 0$ to the repulsion between the two components in the system, for more details we refer to [19, 48]. The system (1.1) can also be found in the studies of nonlinear optics [5]. Physically, the solution $\Phi_j$ in system (1.1) denotes the $j$-th component of the beam in Kerr-like photorefractive media. The positive constant $\mu_j$ indicates the self-focusing strength in the component of the beam, and the coupling constant $\beta$ measures the interaction between the first and the second component of the beam.

In order to find standing waves of the form

$$(\Phi_1(x,t),\Phi_2(x,t)) = (e^{i\lambda_1 t}u(x),e^{i\lambda_2 t}v(x)),$$

then we are led to study the solutions of the following problem

$$
\begin{align*}
-\Delta u + (V_1(x) + \lambda_1)u &= \mu_1(K(x) * u^2)u + \beta(K(x) * v^2)u, & x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2)v &= \mu_2(K(x) * v^2)v + \beta(K(x) * u^2)v, & x \in \mathbb{R}^N.
\end{align*}
$$

(1.3)

For any $\beta \neq 0$, the system (1.3) possesses a trivial solution $(0,0)$ and a pair of semi-trivial solutions with one component being zero, which have the form $(u,0)$ or $(0,v)$. Usually, we look for solutions of system (1.3) which are different from the preceding ones. A solution $(u,v)$ such that $u \neq 0, v \neq 0$, is non-trivial and $u > 0, v > 0$, respectively positive solution. A solution is called a ground state solution if its energy is minimal among the energy of all the non-trivial solutions of system (1.3).

Let $K(x)$ be a Dirac-delta function, that is $K(x) = \delta(x)$, then system (1.3) transforms into the following Schrödinger system

$$
\begin{align*}
-\Delta u + (V_1(x) + \lambda_1)u &= \mu_1 u^3 + \beta u^2 v, & x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2)v &= \mu_2 v^3 + \beta u^2 v, & x \in \mathbb{R}^N.
\end{align*}
$$

(1.4)

Due to the important application in physics, the system (1.4) in low dimensions ($N = 1, 2, 3$) has been widely investigated. Not only the existence but also the qualitative properties of solutions for system (1.4) have been established by researchers via variational methods, Lyapunov-Schmidt reduction method or bifurcation method, see for example [3, 4, 7, 8, 13, 14, 17, 30, 31, 32, 35, 39, 40, 45, 46, 51] and the references therein. Note that, if $N \leq 3$, the nonlinearity and coupling terms in system (1.4) are of subcritical growth with respect to Sobolev critical exponent, the cubic nonlinearities and coupled terms are all of critical growth for $N = 4$ and super-critical growth for $N \geq 5$. Here we only introduce some results closely related to our paper. Chen and Zou [13, 14] considered the following Brézis-Nirenberg type problem

$$
\begin{align*}
-\Delta u + \lambda_1 u &= \mu_1 u^{2^* - 1} + \beta u^{2_{\lambda_2} - 1}v^{2_{\lambda_2}}, & x \in \Omega \\
-\Delta v + \lambda_2 v &= \mu_2 v^{2^* - 1} + \beta u^{2_{\lambda_1}}v^{2_{\lambda_1} - 1}, & x \in \Omega
\end{align*}
$$

(1.5)

$$
u, v \geq 0, x \in \Omega, \quad u = v = 0, x \in \partial \Omega
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$. They established the existence, uniqueness and limit behaviour of positive ground state solutions in some ranges of $\lambda_1, \lambda_2, \beta$. It turned out that results in the higher dimensions are quite different from those in $N = 4$. Recently, some existence and multiplicity results for a Coron type system (1.5) in a bounded domain with one or multiple small holes were obtained in [31, 43]. In contrast, there are
very few results for system (1.4) on the whole space $\mathbb{R}^4$. Wu and Zou [54] proved the existence of positive ground state solutions for system (1.4) with steep potential wells. Moreover, they studied the phenomenon of phase separation of ground state solutions as $\beta \to -\infty$. The positive solutions of system (1.4) are also considered at higher energy levels than the ground state energy level, see [10, 29] for example. By considering the functional constrained on a subset of the Nehari manifold consisting of functions invariant with respect to a subgroup of $O(N+1)$, Clapp and Pistoia [10] proved that system (1.4) has infinitely many fully nontrivial solutions, which are not conformally equivalent. Let $\lambda_1 = \lambda_2 = 0$ and $N = 4$ in system (1.4), Liu and Liu [29] proved the existence of positive solutions once that $\|V_1\|_{L^2} + \|V_2\|_{L^2}$ is small enough and $\beta > \max\{\mu_1, \mu_2\}$. It extends the celebrated work for semilinear Schrödinger equation by Benci and Cerami [6] to Schrödinger system (1.4). Suppose that $\lambda_1 = \lambda_2 > 0$ in system (1.4), the existence result in [29] was extended by Guo et al. [25]. More novelty, by establishing a new type of global compactness result, they overcame the loss of compactness of Palais-Smale sequence, and succeeded in proving the multiplicity of positive solutions via variational methods.

In the following, we are concerned about system (1.3) with a Riesz potential response function $K(x),$ more precisely, $K(x) = |x|^{-\alpha}, \alpha \in (0, N)$ and

$$\begin{cases}
-\Delta u + (V_1(x) + \lambda_1)u = \mu_1(|x|^{-\alpha} * u^2)u + \beta(|x|^{-\alpha} * v^2)u, & x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2)v = \mu_2(|x|^{-\alpha} * v^2)v + \beta(|x|^{-\alpha} * u^2)v, & x \in \mathbb{R}^N.
\end{cases}$$

(1.6)

Clearly, semi-trivial solutions of system (1.6) correspond to solutions of Choquard-Pekar type equation

$$-\Delta u + (V_j(x) + \lambda_j)u = \mu_j(|x|^{-\alpha} * u^2)u, \quad x \in \mathbb{R}^N, \quad j = 1, 2.$$  

(1.7)

Such a equation appeared as early as 1954, in a work by Pekar describing the quantum mechanics of apolaron at rest, see [42]. We also refer to [26, 36] for more physical backgrounds. In the last few decades, the equation (1.7) and its general form have been extensively investigated, not only on subcritical case [2, 20, 28, 33, 34, 37, 38], but also critical case [1, 13, 21, 22, 23].

Compared with a single equation, there are very few results available on the coupled Hartree system (1.6). The first attempt is due to Yang, Wei and Ding [54], they studied a singular perturbed problem related to system (1.6) and proved the existence of ground state solutions for $\beta$ large enough. Recently, Wang and Shi [49] proved the existence and non-existence of ground state solutions to system (1.6) with $\alpha = 1, N = 3$. The existence of normalized solutions to system (1.6) was proved by Wang and Yang [50] under suitable ranges for the parameters $\mu_1, \mu_2, \beta$. All the papers mentioned above deal with the subcritical case. For the critical case $\alpha = 4$ (the nonlocal terms in system (1.6) are critical with respect to Hardy-Littlewood-Sobolev inequality, see Lemma 2.1 below), Gao et al. [20] proved that system (1.6) exists a positive solution with its functional energy lying in $(c_\infty, \min\{S_{2\mu_1}^{\frac{2}{4\mu_1}}, S_{2\mu_2}^{\frac{2}{4\mu_2}}, 2c_\infty\})$, if $\lambda_1 = \lambda_2 = 0$ and $V_1, V_2$ satisfy

$(C_1)$ $V_1(x), V_2(x) \geq 0, \ \forall x \in \mathbb{R}^N$;

$(C_2)$ $V_1(x), V_2(x) \in L^\frac{2}{\alpha}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$;

$(C_3)$ $V_1(x), V_2(x)$ satisfy

$$0 < \frac{\beta - \mu_2}{2\beta - \mu_1 - \mu_2} C(N, 4)^{-\frac{1}{2}} \|V_1\|_{L^\infty} + \frac{\beta - \mu_1}{2\beta - \mu_1 - \mu_2} C(N, 4)^{-\frac{1}{2}} \|V_2\|_{L^\infty}$$

$$< \min\left\{ \frac{\beta^2 - \mu_1\mu_2}{\mu_1(2\beta - \mu_1 - \mu_2)}, \sqrt{\frac{\beta^2 - \mu_1\mu_2}{\mu_2(2\beta - \mu_1 - \mu_2)}}, \sqrt{2} \right\} S_{HL} - S_{HL},$$
where \( c_\infty \) denotes by the ground state energy level and

\[
S_{HL} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} (|x|^{-4} \ast u^2) u^2 dx)^{\frac{1}{2}}}
\]

Note that \( V_j \in L^{\frac{2}{N}}(\mathbb{R}^N) \) and \( \lambda_j = 0, j = 1, 2 \), imply that the only limit problem for \([1.6]\) is

\[
\begin{aligned}
-\Delta u &= \mu_1 (|x|^{-\alpha} \ast u^2) u + \beta (|x|^{-\alpha} \ast v^2) u, \quad x \in \mathbb{R}^N, \\
-\Delta v &= \mu_2 (|x|^{-\alpha} \ast v^2) v + \beta (|x|^{-\alpha} \ast u^2) v, \quad x \in \mathbb{R}^N, \\
u, v &\in D^{1,2}(\mathbb{R}^N),
\end{aligned}
\]

and this fact was a key ingredient in the approach of \([20]\).

On the other hand, \( \lambda_j > 0 \) is very significant by its physical meaning, as one can readily seen in the ansatz \([1.2]\). In this paper, we are ready to investigate the existence and multiplicity of positive solutions for system \([1.6]\) in such cases. Let us consider the following coupled Hartree system with Hardy-Littlewood-Sobolev critical exponent

\[
\begin{aligned}
-\Delta u + (V_1(x) + \lambda_1) u &= \mu_1 (|x|^{-4} \ast u^2) u + \beta (|x|^{-4} \ast v^2) u, \quad x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2) v &= \mu_2 (|x|^{-4} \ast v^2) v + \beta (|x|^{-4} \ast u^2) v, \quad x \in \mathbb{R}^N.
\end{aligned}
\]

(1.8)

Here and always in the sequel we assume that \( N \geq 5 \), constants \( \lambda_j \geq 0, \mu_j > 0, \beta \in \mathbb{R} \setminus \{0\}, j = 1, 2 \) and potentials \( V_j(x) \) satisfy

(A1) \( V_1(x), V_2(x) \geq 0, \forall x \in \mathbb{R}^N, \quad V_1(x) + V_2(x) \neq 0; \)

(A2) \( V_1(x), V_2(x) \in L^{\frac{2}{N}}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N), q > \frac{N}{2}; \)

(A3) \( V_1(x), V_2(x) \) verify

\[
\frac{\beta - \mu_1}{2\beta - \mu_1 - \mu_2} \| V_1 \|_{L^{\frac{2}{N}}} + \frac{\beta - \mu_2}{2\beta - \mu_1 - \mu_2} \| V_2 \|_{L^{\frac{2}{N}}}
\]

\[
< \min \left\{ \sqrt{\frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}}, \sqrt{\frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)}} \right\} S - S,
\]

where \( S \) is the best constant in the Sobolev embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \).

We remark that, if \( V_j(x) + \lambda_j \geq 0, \) but \( V_j(x) + \lambda_j \neq 0 \) for \( j = 1, 2 \), then it is easy to prove that system \([1.8]\) does not have any ground state solutions (see Lemma \([2.7]\) below for computation details). Therefore we must look for solutions at higher energy levels than ground state energy level. The main results of our paper are stated as follows.

**Theorem 1.1.** Let \( \beta > \max\{\mu_1, \mu_2\} \), \( \lambda_1, \lambda_2 \geq 0 \) and \( \lambda := \max\{\lambda_1, \lambda_2\} > 0. \)

(i) Suppose that \( V_1(x), V_2(x) \) satisfy (A1)-(A2), then there exists \( \lambda^* > 0 \), such that if \( \lambda \in (0, \lambda^*) \), system \([1.8]\) exists at least a positive solution.

(ii) Suppose that \( V_1(x), V_2(x) \) satisfy (A1)-(A3), then there also exists \( \lambda^{**} > 0 \) with \( \lambda^{**} < \lambda^* \) such that if \( \lambda \in (0, \lambda^{**}) \), system \([1.8]\) has at least two distinct positive solutions.

**Remark 1.1.** In the proof of Theorem \([1.1]\) the requirement \( V_1(x), V_2(x) \in L^q_{loc}(\mathbb{R}^N), q > \frac{N}{2} \) is used only in showing that a non-trivial non-negative solution of system \([1.8]\) is indeed a positive solution of system \([1.8]\) by the Harnack inequality in \([23]\).
In the case $\lambda_1 = \lambda_2 = 0$, Gao et al. [20] obtained a high energy positive solution of system (1.8) by using Linking theorem combined with a global compactness result. Compared with this, the problem we considered is much more difficult due to the existence of potential terms $\lambda_1 u$ and $\lambda_2 v$. The major difficulties and challenges we faced are: first, the main aim in our paper is to investigate the multiplicity of positive solutions of system (1.8). However, the classical Linking theorem can only prove the existence of Palais-Smale sequences, but fail to obtain the multiplicity of Palais-Smale sequences. In the paper [12], Cerami and Passaseo developed a new variational trick and proved the multiplicity of solutions for scalar equation with Neumann boundary in half space $\mathbb{R}^N_+$ under similar hypotheses as we stated. Also, the similar arguments are used to study the other equations, see [1, 11, 24] for example. Obviously, the methods developed for the scalar equation are not directly applicable to system (1.8), several difficulties arise because of the nonlocal interaction terms with critical exponent. Second, dealing with such kind of critical elliptic system, we have to overcome the loss of compactness and distinguish the solutions from the semi-trivial ones. To the best of our current knowledge, the only available compactness result associated with system (1.8) is given by Gao et al. [20]. Unfortunately, the global compactness result in [20] can not work well on our problem, since the work space we chosen is totally different from the setting $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ in [20]. The loss of compactness makes the study of system (1.8) very complicated and substantially different, some new ideas must be introduced. In order to overcome the lack of compactness, we adapt some ideas from [12] together with new techniques of analysis for Hartree system, and give a complete description for the Palais-Smale sequences of the corresponding energy functional in the product spaces $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. This is the first attempt in this direction, and our result in this aspect is new and original. To the end, adopting this description, we succeed in proving the existence and multiplicity of positive solutions via degree theory, which can be seen as an improvement of the existence results by Gao et. al [20].

The paper is organized as follows. In section 2, we introduce some preliminary results and obtain a nonexistence result. In section 3, we establish a global compactness result and investigate the behavior of Palais-Smale sequences of $I$. In section 4, we make the proof of some technical lemmas that will be used in Section 5. At end of this paper, we will prove our main results. Our notations are standard. We will use $C, C_i, i \in \mathbb{N}$, to denote different positive constants from line to line.

## 2 Preliminary results and nonexistence result

Let $X$ and $Y$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, we define the standard norm on the product space $X \times Y$ as following

$$\|(x, y)\|_{X \times Y}^2 = \|x\|_X^2 + \|y\|_Y^2.$$ 

Particularly, if $X$ and $Y$ are two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, then $X \times Y$ is also a Hilbert space with the inner product

$$\langle (x, y), (\varphi, \phi) \rangle_{X \times Y} = \langle x, \varphi \rangle_X + \langle y, \phi \rangle_Y.$$ 

Let the Sobolev space $D^{1,2}(\mathbb{R}^N)$ be endowed with the norm and inner product respectively

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \langle u, v \rangle_{D^{1,2}} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx.$$
Then we define the product space $H_0 := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ with norm
\[
\|(u,v)\|_{H_0}^2 := \|(u,v)\|_{D^{1,2}\times D^{1,2}}^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,dx.
\]
Recalling that the standard norm of $H^1(\mathbb{R}^N)$ be endowed with the norm
\[
\|u\|_{H^1}^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx.
\]
Obviously, if $\lambda_1, \lambda_2 > 0$, then $\| \cdot \|_{\lambda_j}$, $j = 1, 2$, are equivalent to $\| \cdot \|_{H^1}$, where
\[
\|u\|^2_{\lambda_j} := \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_j u^2)\,dx.
\]
In this paper, the work space we chosen depends on the sign of $\lambda_j$ ($j = 1, 2$). Here and in the sequel we always require $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda := \max\{\lambda_1, \lambda_2\} > 0$. Set
\[
\|(u,v)\|_H^2 := \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2,
\]
where $H$ is defined by
\[
H = \begin{cases} 
H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), & \text{if } \lambda_1 > 0, \lambda_2 > 0, \\
H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), & \text{if } \lambda_1 > 0, \lambda_2 = 0, \\
D^{1,2}(\mathbb{R}^N) \times H^1(\mathbb{R}^N), & \text{if } \lambda_1 = 0, \lambda_2 > 0.
\end{cases}
\]
Let $u^+ := \max\{0, u\}$ and $u^- := \max\{0, -u\}$, $u = u^+ - u^-$ and similarly for $v$, $v = v^+ - v^-$. In order to obtain the positive solutions of system (1.8), we are ready to consider the modified system
\[
\begin{cases} 
-\Delta u + (V_1(x) + \lambda_1)u = \mu_1(|x|^{-4} * |u^+|^2)u^+ + \beta(|x|^{-4} * |v^+|^2)u^+, & x \in \mathbb{R}^N, \\
-\Delta v + (V_2(x) + \lambda_2)v = \mu_2(|x|^{-4} * |v^+|^2)v^+ + \beta(|x|^{-4} * |u^+|^2)v^+, & x \in \mathbb{R}^N.
\end{cases}
\tag{2.1}
\]
If $(u, v)$ is a solution of system (2.1), multiplying the first equation by $u^-$ and the second equation by $v^-$ in system (2.1) and integrating on $\mathbb{R}^N$, then we get
\[
\int_{\mathbb{R}^N} (|\nabla (u^-)|^2 + |\nabla (v^-)|^2)\,dx + \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)(u^-)^2\,dx + \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)(v^-)^2\,dx = 0,
\]
which implies that $u \geq 0$, $v \geq 0$. By assumptions (A1)-(A2) and the strong maximum principle, we can prove $u > 0$, $v > 0$. Thus, $(u, v)$ is a positive solution of system (1.8). In the sequel, we focus our attention on the modified system (2.1) and shall to prove the existence and multiplicity of non-trivial solutions to system (2.1) by variational methods. Next, we first introduce the well-known Hardy-Littlewood-Sobolev inequality, which will be frequently used, see [27].

**Lemma 2.1.** *(Hardy-Littlewood-Sobolev inequality)* Suppose $\alpha \in (0, N)$, and $p, r > 1$ with $\frac{1}{p} + \frac{1}{r} + \frac{\alpha}{N} = 2$. Let $f \in L^p(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(p, r, \alpha, N)$, independent of $f$ and $g$, such that
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\alpha}\,dxdy \right| \leq C(p, r, \alpha, N)\|f\|_{L^p} \cdot \|g\|_{L^r},
\tag{2.2}
\]
where $\| \cdot \|_{L^p} = \left( \int_{\mathbb{R}^N} |u|^p\,dx \right)^{\frac{1}{p}}$. If $p = r = \frac{2N}{2N-\alpha}$, then
\[
C(p, r, \alpha, N) = C(N, \alpha) = \pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-\alpha}{2}\right)} \right\}^{-\frac{\alpha}{N}}.
\]
In this case, the equality in (2.2) is achieved if and only if \( f \equiv (\text{const.}) g \) and
\[
g(x) = A(\gamma^2 + |x - \bar{a}|^2)^{-\frac{(2N - n)}{2}}
\]
for some \( A \in \mathbb{C}, \bar{a} \in \mathbb{R}^N \) and \( 0 \neq \gamma \in \mathbb{R} \).

To study system (2.1) by variational methods, we define the energy functional \( I(u, v) : H \to \mathbb{R} \) by
\[
I(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + |\nabla v|^2 + (V_1(x) + \lambda_1)u^2 + (V_2(x) + \lambda_2)v^2 \right) dx
- \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mu_1 |u^+(x)|^2 |u^+(y)|^2 + 2\beta |u^+(x)|^2 |v^+(y)|^2 + 2\gamma |u^+(x)||v^+(y)|^2 dx dy.
\]

Since \( V_1(x), V_2(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \), then in view of Hölder inequality and Hardy-Littlewood-Sobolev inequality, \( I \) is well defined in \( H \) and belongs to \( C^1(H, \mathbb{R}) \). In what follows, we define the Nehari manifold
\[
\mathcal{N} := \{(u, v) \in H \mid (u, v) \neq (0, 0), \langle I'(u, v), (u, v) \rangle = 0 \},
\]
where
\[
\langle I'(u, v), (u, v) \rangle = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + |\nabla v|^2 + (V_1(x) + \lambda_1)u^2 + (V_2(x) + \lambda_2)v^2 \right) dx
- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mu_1 |u^+(x)|^2 |u^+(y)|^2 + 2\beta |u^+(x)|^2 |v^+(y)|^2 + 2\gamma |u^+(x)||v^+(y)|^2 dx dy.
\]

Moreover, we set
\[
c = \inf_{(u, v) \in \mathcal{N}} I(u, v).
\]

We remark that the existence of non-trivial solutions to system (2.1) is closely related to the corresponding problem
\[
\begin{cases}
-\Delta u = \mu_1(|x|^{-4} * |u^+|^2)u^+ + \beta(|x|^{-4} * |v^+|^2)u^+, & x \in \mathbb{R}^N, \\
-\Delta v = \mu_2(|x|^{-4} * |v^+|^2)v^+ + \beta(|x|^{-4} * |u^+|^2)v^+, & x \in \mathbb{R}^N.
\end{cases}
\tag{2.3}
\]

Throughout the paper we denote by \( I_\infty : H_0 \to \mathbb{R} \) the energy functional whose critical points are solutions of system (2.3), that is
\[
I_\infty(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx
- \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mu_1 |u^+(x)|^2 |u^+(y)|^2 + 2\beta |u^+(x)|^2 |v^+(y)|^2 + 2\gamma |u^+(x)||v^+(y)|^2 dx dy.
\]

Also the analogues of \( \mathcal{N} \) and \( c \) can be denoted by \( \mathcal{N}_\infty \) and \( c_\infty \) respectively, more precisely,
\[
\mathcal{N}_\infty := \{(u, v) \in H_0 \mid (u, v) \neq (0, 0), \langle I_\infty'(u, v), (u, v) \rangle = 0 \}
\]
and
\[
c_\infty = \inf_{(u, v) \in \mathcal{N}_\infty} I_\infty(u, v).
\]

To avoid semi-trivial solutions in searching for non-trivial solutions of system (2.1), we first investigate the single elliptic equation
\[
-\Delta u + (V_j(x) + \lambda_j)u = \mu_j(|x|^{-4} * |u^+|^2)u^+, \quad j = 1, 2,
\]

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of which the associated energy functional $J_j : H^1(\mathbb{R}^N) \to \mathbb{R}$ (or $D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$, if $\lambda_j = 0$) by

$$J_j(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (V_j(x) + \lambda_j) u^2 dx - \frac{\mu_j}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u^+(x)|^2 |u^+(y)|^2}{|x-y|^4} dxdy.$$ 

Consider the infimum

$$m_j = \inf_{u \in \mathcal{M}_j} J_j(u),$$

defined on the Nehari manifold

$$\mathcal{M}_j = \{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, \langle J'_j(u), u \rangle = 0 \}, \text{ if } \lambda_j > 0,$$

or

$$\mathcal{M}_j = \{ u \in D^{1,2}(\mathbb{R}^N) \mid u \neq 0, \langle J'_j(u), u \rangle = 0 \}, \text{ if } \lambda_j = 0.$$

If $V_j = \lambda_j = 0$, we then denote the analogues of $J_j$, $m_j$, $\mathcal{M}_j$ by $J_j^{\infty}$, $m_j^{\infty}$, $\mathcal{M}_j^{\infty}$ respectively.

It is well-known that infimum $m_j^{\infty}$ is attained by

$$W_{\delta,z,j}(x) = \mu_j^{-\frac{1}{2}} U_{\delta,z}(x),$$

where

$$U_{\delta,z}(x) = C_N \left( \frac{\delta}{\delta^2 + |x-z|^2} \right)^{\frac{N-2}{2}}, \quad \delta > 0, \quad z \in \mathbb{R}^N \quad (2.4)$$

is the unique positive solution of

$$-\Delta u = (|x|^{-4} * u^2)u, \quad (2.5)$$

and constant $C_N$ only depends on dimension $N$, see [18, 23]. Moreover, by calculation we have

$$\int_{\mathbb{R}^N} |\nabla U_{\delta,z}(x)|^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_{\delta,z}(x)|^2 |U_{\delta,z}(y)|^2}{|x-y|^4} dxdy = S^2_{HL} = C(N,4)^{-1} S^2.$$ 

and

$$m_j^{\infty} = \frac{S^2_{HL}}{4 \mu_j}.$$ 

**Lemma 2.2.** Let $N \geq 5$ and nonnegative function $V_j(x) \in L^{\frac{N}{4}}(\mathbb{R}^N)$, then

$$m_j = m_j^{\infty} = \frac{S^2_{HL}}{4 \mu_j}.$$ 

**Proof:** For arbitrary $u \in \mathcal{M}_j$, since $V_j(x)$ and $\lambda_j$ are both nonnegative, then

$$J_j(u) \geq m_j^{\infty},$$

which implies that $m_j \geq m_j^{\infty}$.

To show that the opposite inequality holds, we then consider the sequence

$$\Phi_n(x) = \mu_j^{-\frac{1}{2}} U_{\delta_n,0}(x),$$

where $\delta_n \to 0^+$ as $n \to \infty$. On one hand, since $N \geq 5$, then

$$\lambda_j \int_{\mathbb{R}^N} |\Phi_n|^2 dx = \frac{\lambda_j \delta_n^2}{\mu_j} \int_{\mathbb{R}^N} |U_{1,0}(x)|^2 dx = o_n(1),$$

and
Corollary 2.3. Let \((u, v) \in N\) be a critical point of \(I\) constrained on \(N\). Assume that

\[
I(u, v) < \min \left\{ \frac{S_{HL}}{4\mu_1}, \frac{S_{HL}}{4\mu_2} \right\},
\]

then \(u \neq 0\) and \(v \neq 0\).
Lemma 2.4. For arbitrary \((u, v) \in H \setminus \{(0, 0)\}\), let \((\tau_{(u,v)} u, \tau_{(u,v)} v)\), \((t_{(u,v)} u, t_{(u,v)} v)\) be the projections of \((u, v)\) on \(N_\infty\) and \(N\) respectively. Then

\[
\tau_{(u,v)} \leq t_{(u,v)}.
\]

Proof: Since \(V_j(x)\) and \(\lambda_j\) are both nonnegative, then by direct calculations we get

\[
\tau_{(u,v)}^2 = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx}{\int_{\mathbb{R}^N} \mu_1 u^2 dx + \int_{\mathbb{R}^N} \mu_2 v^2 dx} \geq \frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2},
\]

which implies that \(\tau_{(u,v)} \leq t_{(u,v)}\).

In what follows, we always require \(\beta > \max\{\mu_1, \mu_2\}\). Let

\[
k_1 = \frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}, \quad k_2 = \frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2},
\]

The next two lemmas are essentially proved in [20] and we omit them.

Lemma 2.5. ([20, Lemma 2.3]) If \(\beta > \max\{\mu_1, \mu_2\}\), then

\[
c_\infty = \frac{1}{4} (k_1 + k_2) S_{HL}^2
\]

and any ground state solutions of system (2.3) must be of the form

\[(u, v) = (\sqrt{k_1} U_{\delta, z}, \sqrt{k_2} U_{\delta, z})\]

for some \(\delta \in \mathbb{R}\) and \(z \in \mathbb{R}^N\).

Lemma 2.6. ([20, Corollary 3.5]) Let \(\beta > \max\{\mu_1, \mu_2\}\). If \((u, v) \in H_0\) is a non-trivial classical positive solution of system (2.3), then we have

\[(u, v) = (\sqrt{k_1} U_{\delta, z}, \sqrt{k_2} U_{\delta, z})\]

for some \(\delta > 0\) and \(z \in \mathbb{R}^N\). Moreover, each non-trivial classical positive solution \((u, v) \in H_0\) of system (2.3) is a ground state solution.

In [27], Gao et al. proved the uniqueness result above by using of the method of moving spheres in integral form. We would like to point out that the uniqueness result plays an important role in proving the compactness of Palais-Smale sequence in a suitable energy interval.

Lemma 2.7. Suppose that \(N \geq 5\), \(\beta > \max\{\mu_1, \mu_2\}\), \(\lambda_1, \lambda_2 \geq 0\) and \(\lambda := \max\{\lambda_1, \lambda_2\} > 0\). If \(V_1(x), V_2(x)\) satisfy assumptions (A1)-(A2), then \(c_\infty = c\) and \(c\) is not attained.

Proof: Let \(\delta_n \to 0^+\) as \(n \to \infty\), we define \(\Phi_n(x) = U_{\delta_n, 0}(x)\). Following by the arguments in the proof of Lemma 2.2, we then get

\[
\int_{\mathbb{R}^N} (V_1(x) + \lambda_1) |\Phi_n(x)|^2 dx + \int_{\mathbb{R}^N} (V_2(x) + \lambda_2) |\Phi_n(x)|^2 dx = o_n(1).
\]
Let $t_n > 0$ be such that
\[
t_n^2 = \frac{(k_1 + k_2)\int_{\mathbb{R}^N} |\nabla \Phi_n|^2 dx + k_1 \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|\Phi_n|^2 dx + k_2 \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|\Phi_n|^2 dx}{(k_1 + k_2)\int_{\mathbb{R}^N} \frac{|\Phi_n|^2}{|x-y|^4} dx dy},
\]
then $(t_n \sqrt{k_1} \Phi_n, t_n \sqrt{k_2} \Phi_n) \in \mathcal{N}$ and $t_n \to 1$ as $n \to \infty$, in which we use \cite{2,3} and
\[
\mu_1 k_1^2 + \mu_2 k_2^2 + 2\beta k_1 k_2 = k_1 + k_2.
\]
Moreover,
\[
c \leq I(t_n \sqrt{k_1} \Phi_n, t_n \sqrt{k_2} \Phi_n) = \frac{t_n^2 (k_1 + k_2)}{4} \int_{\mathbb{R}^N} |\nabla \Phi_n|^2 dx + \frac{t_n^2 k_1}{4} \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|\Phi_n|^2 dx + \frac{t_n^2 k_2}{4} \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|\Phi_n|^2 dx
\]
\[
= \frac{1}{4}(k_1 + k_2) \int_{\mathbb{R}^N} |\nabla \Phi_n|^2 dx + o_n(1)
\]
\[
= \frac{1}{4}(k_1 + k_2) S_{HL}^2 + o_n(1).
\]
Taking the limit $n \to \infty$ in the equality above, then
\[
c \leq \frac{1}{4}(k_1 + k_2) S_{HL}^2 = c_\infty.
\]
Let $(u, v) \in \mathcal{N}$ be arbitrarily chosen and set $\tau_{(u,v)} > 0$ be such that $(\tau_{(u,v)} u, \tau_{(u,v)} v) \in \mathcal{N}_\infty$. Recalling that $\lambda_1$ and $V_\beta(x)$ are both nonnegative, then by Lemma \cite{2,3} we get $\tau_{(u,v)} \leq 1$. Furthermore, by computation we get
\[
c_\infty \leq I_\infty(\tau_{(u,v)} u, \tau_{(u,v)} v)
\]
\[
= \frac{1}{4}(\tau_{(u,v)}) \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx \right)
\]
\[
\leq \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx \right) + \frac{1}{4} \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)u^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)v^2 dx
\]
\[
= I(u, v),
\]
which leads to $c_\infty \leq c$. Therefore, $c = c_\infty$.

In the end of the proof, we shall to prove that $c$ is not attained. We argue by contradiction and assume that $c$ is attained by $(u_0, v_0) \in \mathcal{N}$. It follows by Lemma \cite{2,3} that there exists $\tau_0 \in (0, 1]$ such that $(\tau_0 u_0, \tau_0 v_0) \in \mathcal{N}_\infty$. Thus, by calculation we have
\[
c_\infty \leq I_\infty(\tau_0 u_0, \tau_0 v_0)
\]
\[
= \frac{\tau_0^2}{4} \left( \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx \right)
\]
\[
\leq \frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx \right) + \frac{1}{4} \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|u_0|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|v_0|^2 dx
\]
\[
= c = c_\infty,
\]
which implies that
\[
\tau_0 = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|u_0|^2 dx + \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|v_0|^2 dx = 0. \quad (2.9)
\]
Moreover, $c_\infty$ is attained by $(u_0, v_0)$. Since $\beta > \max\{\mu_1, \mu_2\}$, then by Lemma 2.4 we get

$$c_\infty = \frac{1}{4}(k_1 + k_2)S_{HL}^2 < \min\left\{\frac{1}{4\mu_1}S_{HL}^2, \frac{1}{4\mu_2}S_{HL}^2\right\} = \min\{m_1, m_2\},$$

which implies that $u_0 \neq 0$ and $v_0 \neq 0$. By the strong maximum principle, $u_0 > 0$ and $v_0 > 0$ for all $x \in \mathbb{R}^N$. Thus, there is a contradiction with (2.9) because of the behavior of Palais-Smale sequences of $L$. Therefore, based on the arguments above, we prove that $c$ is not attained.

\[ \blacksquare \]

### 3 Some compactness results

In this paper, the problem we studied is affected by lack of compactness due to the unboundedness of $\mathbb{R}^N$ and to the critical exponent. In this section, we introduce a global compactness result and investigate the behavior of Palais-Smale sequences of $I$, see Proposition 3.3 below. We would like to point out that, a version of that compactness result for single equations and coupled systems can be found in [6, 25, 31, 47] by different approach.

We recall that a sequence $\{(u_n, v_n)\} \subset H$ is called a Palais-Smale sequence for $I$, if

$$\sup_n I(u_n, v_n) < +\infty, \quad I'(u_n, v_n) \to 0 \quad \text{in} \quad H^{-1},$$

where $H^{-1}$ is the dual space of $H$.

Before stating the global compactness result, we first introduce a Brézis-Lieb type lemma for nonlocal terms, which is proved in [20], so we omit it here.

**Lemma 3.1.** ([20, Lemma 4.1]) Let $N \geq 5$ and $\{(u_n, v_n)\}$ to be a bounded sequence in $L^{\frac{2N}{N-4}}(\mathbb{R}^N) \times L^{\frac{2N}{N-4}}(\mathbb{R}^N)$ such that $(u_n, v_n) \rightharpoonup (u, v)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then

$$\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (|x|^{-4} * |u_n|^2)|u_n^+|^2 dx - \int_{\mathbb{R}^N} (|x|^{-4} * |(u_n - u)|^2)|u_n - u|^2 dx \right) = \int_{\mathbb{R}^N} (|x|^{-4} * |u^+|^2)|u^+|^2 dx$$

and

$$\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (|x|^{-4} * |v_n|^2)|v_n^+|^2 dx - \int_{\mathbb{R}^N} (|x|^{-4} * |(v_n - v)|^2)|v_n - v|^2 dx \right) = \int_{\mathbb{R}^N} (|x|^{-4} * |v^+|^2)|v^+|^2 dx,$$

**Lemma 3.2.** Suppose that $\lambda_1, \lambda_2 \geq 0$ with $\lambda := \max\{\lambda_1, \lambda_2\} > 0$ and suppose that $(u, v)$ is the solution of

$$\begin{cases}
-\Delta u + \lambda_1 u = \mu_1(|x|^{-4} * u^2)u + \beta(|x|^{-4} * v^2)u, & x \in \mathbb{R}^N, \\
-\Delta v + \lambda_2 v = \mu_2(|x|^{-4} * v^2)v + \beta(|x|^{-4} * u^2)v, & x \in \mathbb{R}^N.
\end{cases}
$$

(3.1)

(a) If $\lambda_1 > 0, \lambda_2 > 0$, then $(u, v) = (0, 0)$.

(b) If $\lambda_1 = 0, \lambda_2 > 0$, then $v = 0$ and $u$ solves

$$-\Delta u = \mu_1(|x|^{-4} * u^2)u. \quad (3.2)$$
(c) If $\lambda_1 > 0$, $\lambda_2 = 0$, then $u = 0$ and $v$ solves

$$-\Delta v = \mu_2(|x|^{-4} * v^2)v.$$  

**Proof:** In what follows, we prove the lemma by Pohozaev identity. More precisely, our proof follows by a classical strategy of testing the equation against $x \cdot \nabla u$, which is made rigorous by multiplying by cut-off functions, see [52, Appendix B] and [38, Proposition 3.5] for details.

Since $\lambda_1, \lambda_2 \geq 0$ with $\lambda := \max\{\lambda_1, \lambda_2\} > 0$, then the following cases can happen: $\lambda_1, \lambda_2 > 0$, or $\lambda_1 = 0, \lambda_2 > 0$, or $\lambda_1 > 0, \lambda_2 = 0$.

Let $(u, v)$ be a solution of system (3.1). Then by the standard elliptic regularity theory, $(u, v) \in W^{2, p}_{loc}(\mathbb{R}^N) \times W^{2, p}_{loc}(\mathbb{R}^N)$, $p \geq 1$, see [38, Theorem 2]. Let $\phi(x) \in C^1_0(\mathbb{R}^N)$ be a cut-off function such that $\phi(x) = 1$ on $B_1(0)$. Multiplying the first equation by $u_\rho = \phi(\rho x)(x \cdot \nabla u)$, the second equation by $v_\rho = \phi(\rho x)(x \cdot \nabla v)$ and integrating by parts, then the following properties hold:

\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla u_\rho dx + \lambda_1 \int_{\mathbb{R}^N} uu_\rho dx = \mu_1 \int_{\mathbb{R}^N} (|x|^{-4} * u^2)u_\rho dx + \beta \int_{\mathbb{R}^N} (|x|^{-4} * v^2)u_\rho dx \tag{3.3}
\]

and

\[
\int_{\mathbb{R}^N} \nabla v \cdot \nabla v_\rho dx + \lambda_2 \int_{\mathbb{R}^N} vv_\rho dx = \mu_2 \int_{\mathbb{R}^N} (|x|^{-4} * v^2)v_\rho dx + \beta \int_{\mathbb{R}^N} (|x|^{-4} * u^2)v_\rho dx. \tag{3.4}
\]

We compute for every $\rho > 0$,

\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla u_\rho dx = \int_{\mathbb{R}^N} \phi(\rho x) \left( |\nabla u|^2 + x \cdot \nabla \left( \frac{|\nabla u|^2}{2} \right) (x) \right) dx + \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi(\rho x))(\rho x \cdot \nabla u) dx \\
= \int_{\mathbb{R}^N} \left( 2 - N \right) \phi(\rho x) - \rho x \cdot \nabla \phi(\rho x) \right) \frac{|\nabla u(x)|^2}{2} dx + \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi(\rho x))(\rho x \cdot \nabla u) dx.
\]

By Lebesgue’s dominated convergence theorem, we have

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^N} \nabla u \cdot \nabla u_\rho dx = \frac{2 - N}{2} \int_{\mathbb{R}^N} |\nabla u|^2. \tag{3.5}
\]

Next, by calculation we have

\[
\int_{\mathbb{R}^N} uu_\rho dx = \int_{\mathbb{R}^N} u(x) \phi(\rho x)x \cdot \nabla u(x) dx \\
= \int_{\mathbb{R}^N} \phi(\rho x)x \cdot \nabla \left( \frac{u^2}{2} \right)(x) dx \\
= - \int_{\mathbb{R}^N} \left( N \phi(\rho x) + \rho x \cdot \nabla \phi(\rho x) \right) \frac{|u(x)|^2}{2} dx.
\]

By Lebesgue’s dominated convergence theorem again, it holds that

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^N} uu_\rho dx = - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 dx. \tag{3.6}
\]

By a similar argument as above, we can also use the Lebesgue’s dominated convergence theorem to conclude
that

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u u_\rho \, dx = \lim_{\rho \to 0} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) \phi(\rho x) x \cdot \nabla \left( \frac{u^2}{2} \right)(x) \, dx
\]

\[
= - \lim_{\rho \to 0} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) \frac{u^2}{2} [N \phi(\rho x) + \rho x \cdot \nabla \phi(\rho x)] \, dx
\]

\[
+ \lim_{\rho \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2x(x-y)|u(x)|^2|u(y)|^2 \phi(\rho x)}{|x-y|^6} \, dx \, dy
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2x^2 - 2xy)|u(x)|^2|u(y)|^2}{|x-y|^6} \, dx \, dy
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x^2 - 2xy + y^2)|u(x)|^2|u(y)|^2}{|x-y|^6} \, dx \, dy
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u^2 \, dx. 
\]

Arguing as above, we can also prove that

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^N} (|x|^{-4} * v^2) u u_\rho \, dx
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * v^2) u^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2x(x-y)|u(x)|^2|v(y)|^2}{|x-y|^6} \, dx \, dy. 
\]

Therefore, by (3.5)–(3.8), we prove that (3.3) equals to

\[
- \frac{N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N\lambda_1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u^2 \, dx - \frac{N}{2} \beta \int_{\mathbb{R}^N} (|x|^{-4} * u^2) u^2 \, dx
\]

\[
+ \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2x^2 - 2xy)|u(x)|^2|v(y)|^2}{|x-y|^6} \, dx \, dy. 
\]

Repeating the argument as above, we can also prove that (3.3) equals to

\[
- \frac{N}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \frac{N\lambda_2}{2} \int_{\mathbb{R}^N} |v|^2 \, dx
\]

\[
= - \frac{N}{2} \int_{\mathbb{R}^N} (|x|^{-4} * v^2) v^2 \, dx - \frac{N}{2} \beta \int_{\mathbb{R}^N} (|x|^{-4} * u^2) v^2 \, dx
\]

\[
+ \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2x^2 - 2xy)|v(x)|^2|u(y)|^2}{|x-y|^6} \, dx \, dy. 
\]

Note that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\beta(2x^2 - 2xy)|u(x)|^2|v(y)|^2}{|x-y|^6} \, dx \, dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\beta(2x^2 - 2xy)|v(x)|^2|u(y)|^2}{|x-y|^6} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\beta(2x^2 - 2xy)|u(x)|^2|v(y)|^2}{|x-y|^6} \, dx \, dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\beta(2x^2 - 2xy)|v(x)|^2|u(y)|^2}{|x-y|^6} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\beta(2x^2 - 4xy + 2y^2)|u(x)|^2|v(y)|^2}{|x-y|^6} \, dx \, dy
\]

\[
= 2\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2|v(y)|^2}{|x-y|^4} \, dx \, dy,
\]
by (3.9)-(3.10) we get
\[
- \frac{N-2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{N}{2} \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)dx
\]
\[
= - \frac{N-2}{2} \left[ \mu_1 \int_{\mathbb{R}^N} (|x|^{-4} \ast u^2)u^2 dx + \mu_2 \int_{\mathbb{R}^N} (|x|^{-4} \ast v^2)v^2 dx \right] - (N-2)\beta \int_{\mathbb{R}^N} (|x|^{-4} \ast v^2)u^2 dx.
\]
(3.11)

On the other hand, since \((u,v)\) is a pair solution of system (3.11), then
\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx + \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)dx
\]
\[
= \mu_1 \int_{\mathbb{R}^N} (|x|^{-4} \ast u^2)u^2 dx + \mu_2 \int_{\mathbb{R}^N} (|x|^{-4} \ast v^2)v^2 dx + 2\beta \int_{\mathbb{R}^N} (|x|^{-4} \ast v^2)u^2 dx.
\]
(3.12)

Thus, it follows by (3.11) and (3.12) that
\[
\int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)dx = 0.
\]
(3.13)

If \(\lambda_1 > 0\) and \(\lambda_2 > 0\), then by (3.13), we have \((u,v) = (0,0)\), thus \((a)\) occurs. If \(\lambda_1 = 0\) and \(\lambda_2 > 0\), then by (3.13), again, we get \(v=0\), moreover \(u\) solves
\[-\Delta u = \mu_1 (|x|^{-4} \ast u^2)u.\]

Thus, \((b)\) is right. Finally, repeating the arguments above, we can also prove that \((c)\) is true. 

**Theorem 3.3.** Suppose that \(\lambda_1, \lambda_2 \geq 0\) with \(\lambda := \max\{\lambda_1, \lambda_2\} > 0\), let \((A_1)-(A_2)\) hold and \(\{(u_n,v_n)\} \subset H\) be a Palais-Smale sequence of functional \(I\) at level \(d\).

(a) If \(\lambda_1 > 0, \lambda_2 > 0\), then there exist a solution \((u^0,v^0)\) of system (2.1), \(\ell\) sequences of positive numbers \(\{\sigma_n^k\} (1 \leq k \leq \ell)\) and \(\ell\) sequences of points \(\{y_n^k\} (1 \leq k \leq \ell)\) in \(\mathbb{R}^N\), such that

\[
\|(u_n,v_n)\|^2_H = \|(u^0,v^0)\|^2_H + \sum_{k=1}^\ell \left\| \left( \sigma_n^k \left( \frac{u^k \sigma_n^k}{\sigma_n^k} \right) \right) \right\|^2_{H_0} + o_n(1)
\]
and
\[
I(u_n,v_n) = I(u^0,v^0) + \sum_{k=1}^\ell I_\infty(u^k,v^k) + o_n(1),
\]

where \(\sigma_n^k \to 0\) as \(n \to \infty\) and \((u^k,v^k) \neq (0,0)\) solves system (2.9).

(b) If \(\lambda_1 = 0, \lambda_2 > 0\), then there exist a solution \((u^0,v^0)\) of system (2.1), \(\ell_1\) sequence of points \(\{z_n^k\} (1 \leq k \leq \ell_1)\), \(\ell_2\) sequences of positive numbers \(\{\sigma_n^k\} (1 \leq k \leq \ell_2)\) and \(\ell_2\) sequences of points \(\{y_n^k\} (1 \leq k \leq \ell_2)\) in \(\mathbb{R}^N\), such that

\[
\|(u_n,v_n)\|^2_H = \|(u^0,v^0)\|^2_H + \sum_{k=1}^{\ell_1} \|w^k(x-z_n^k,0)\|_{H_0}^2
\]
\[
+ \sum_{k=1}^{\ell_2} \left\| \left( \sigma_n^k \left( \frac{u^k \sigma_n^k}{\sigma_n^k} \right) \right) \right\|^2_{H_0} + o_n(1)
\]
and
\[
I(u_n,v_n) = I(u^0,v^0) + \sum_{k=1}^{\ell_1} I_\infty(u^k,0) + \sum_{k=1}^{\ell_2} I_\infty(u^k,v^k) + o_n(1),
\]
where \(|z_n^k| \to \infty, \sigma_n^k \to 0\) as \(n \to \infty\), \((u^k, v^k) \neq (0, 0)\) solves system (2.3) and \((w^k, 0)\) is a semi-trivial solution of system (2.3).

(c) If \(\lambda_1 > 0, \lambda_2 = 0\), then there exist a solution \((v^0, v^0)\) of system (2.1), \(\ell_1\) sequence of points \(\{z_n^k\} (1 \leq k \leq \ell_1)\), \(\ell_2\) sequences of positive numbers \(\{\sigma_n^k\} (1 \leq k \leq \ell_2)\) and \(\ell_2\) sequences of points \(\{y_n^k\} (1 \leq k \leq \ell_2)\) in \(\mathbb{R}^N\) such that

\[
\| (u_n, v_n) \|_H^2 = \| (u^0, v^0) \|_H^2 + \sum_{k=1}^{\ell_1} \| (0, w^k(x - z_n^k)) \|_{H_0}^2 + \sum_{k=1}^{\ell_2} \| (\frac{\sigma_n^k}{\lambda_1^2} u^k(x - y_n^k), (\frac{\sigma_n^k}{\lambda_2^2} v^k(x - y_n^k)) \|_{H_0}^2 + o_n(1),
\]

and

\[
I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell_1} I_\infty(0, w^k) + \sum_{k=1}^{\ell_2} I_\infty(u^k, v^k) + o_n(1),
\]

where \(|z_n^k| \to \infty, \sigma_n^k \to 0\) as \(n \to \infty\), \((u^k, v^k) \neq (0, 0)\) solves system (2.3) and \((0, w^k)\) is a semi-trivial solution of system (2.3).

**Proof:** (a) Let us first consider the case \(\lambda_1, \lambda_2 > 0\), and now \(H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). Since \(\{(u_n, v_n)\} \subset H\) is a sequence of Palais-Smale sequence for the functional \(I\), then by computation we prove that \(\{(u_n, v_n)\}\) is bounded in \(H\). Without loss of generality, we may suppose that \((u_n, v_n) \rightharpoonup (u^0, v^0)\) in \(H\), \((u_n, v_n) \to (u^0, v^0)\) a.e. in \(\mathbb{R}^N \times \mathbb{R}^N\) and \((u_n, v_n) \to (u^0, v^0)\) in \(L^2_{\text{loc}}(\mathbb{R}^N) \times L^2_{\text{loc}}(\mathbb{R}^N)\), where \((u^0, v^0)\) is a pair of solution to system (2.1).

Let \((u^1_n, v^1_n) = (u_n, v_n) - (u^0, v^0)\), then

\[
(u^1_n, v^1_n) \to (0, 0) \text{ in } H, \quad (u^1_n, v^1_n) \to (0, 0) \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \text{ and in } L^2_{\text{loc}}(\mathbb{R}^N) \times L^2_{\text{loc}}(\mathbb{R}^N). \tag{3.14}
\]

It follows by Lemma 3.3 and the classical Brézis-Lieb lemma [52] that

\[
I_{\lambda, \infty}(u^1_n, v^1_n) = I(u_n, v_n) - I(u^0, v^0) + o_n(1)
\]

and

\[
I'_{\lambda, \infty}(u^1_n, v^1_n) = I'(u_n, v_n) - I'(u^0, v^0) + o_n(1) = o_n(1),
\]

where

\[
I_{\lambda, \infty}(u, v) := I_\infty(u, v) + \frac{\lambda_1}{2} \int_{\mathbb{R}^N} u^2 dx + \frac{\lambda_2}{2} \int_{\mathbb{R}^N} v^2 dx.
\]

Therefore, \(\{(u^1_n, v^1_n)\}\) is a Palais-Smale sequence for \(I_{\lambda, \infty}\).

If \((u^1_n, v^1_n) \to (0, 0)\) in \(H\), then we have done. If \((u^1_n, v^1_n) \to (0, 0)\), but \((u^1_n, v^1_n) \nrightarrow (0, 0)\) in \(H\). Then there exists a positive constant \(\tilde{c} > 0\) such that

\[
\| (u^1_n, v^1_n) \|_H^2 \geq \tilde{c} > 0,
\]

that is

\[
\int_{\mathbb{R}^N} (|\nabla u^1_n|^2 + \lambda_1 |u^1_n|^2) dx + \int_{\mathbb{R}^N} (|\nabla v^1_n|^2 + \lambda_2 |v^1_n|^2) dx \geq \tilde{c}.
\]
As we known that \{ (u_n^1, v_n^1) \} is a Palais-Smale sequence for \( I_{\lambda, \infty} \), then by Hardy-Littlewood-Sobolev inequality and Young inequality,

\[
\hat{c} \leq \int_{\mathbb{R}^N} (|\nabla u_n^1|^2 + \lambda_1 |u_n^1|^2) \, dx + \int_{\mathbb{R}^N} (|\nabla v_n^1|^2 + \lambda_2 |v_n^1|^2) \, dx
\]

\[
= \mu_1 \int_{\mathbb{R}^N} (|x|^{-4} * (|u_n^1|^2))(|u_n^1|^2) \, dx + \mu_2 \int_{\mathbb{R}^N} (|x|^{-4} * (|v_n^1|^2))(v_n^1)^2 \, dx
\]

\[
+ 2\beta \int_{\mathbb{R}^N} (|x|^{-4} * (|u_n^1|^2))(u_n^1)^2 \, dx + o_n(1)
\]

\[
\leq C \left[ \left( \int_{\mathbb{R}^N} (|u_n^1|^2)^2 \, dx \right)^{\frac{4}{n-4}} + \left( \int_{\mathbb{R}^N} (v_n^1)^2 \, dx \right)^{\frac{4}{n-4}} \right] + o_n(1),
\]

which implies that there exists a positive constant \( \hat{c} > 0 \) such that

\[
\int_{\mathbb{R}^N} (|u_n^1|^2)^2 \, dx + \int_{\mathbb{R}^N} (v_n^1)^2 \, dx \geq \hat{c}.
\]

(3.15)

We assume without loss of generality that \( \|(v_n^1)^+\|_{L^{2^*}} \geq \frac{1}{2} \hat{c} \) and set

\[
t_n = \frac{\|(u_n^1)^+\|_{L^{2^*}}}{\|(v_n^1)^+\|_{L^{2^*}}} \geq 0,
\]

then by the Sobolev inequality and Hardy-Littlewood-Sobolev inequality, we get

\[
\mathcal{S}(t_n + 1)(\|(v_n^1)^+\|_{L^{2^*}}^2)
\]

\[
= \mathcal{S}(\|(u_n^1)^+\|_{L^{2^*}}^2 + \|(v_n^1)^+\|_{L^{2^*}}^2)
\]

\[
\leq \|(u_n^1, v_n^1)\|_{H^N}^2
\]

\[
= \mu_1 \int_{\mathbb{R}^N} (|x|^{-4} * (|u_n^1|^2))(u_n^1)^2 \, dx + \mu_2 \int_{\mathbb{R}^N} (|x|^{-4} * (|v_n^1|^2))(v_n^1)^2 \, dx
\]

\[
+ 2\beta \int_{\mathbb{R}^N} (|x|^{-4} * (|u_n^1|^2))(u_n^1)^2 \, dx + o_n(1)
\]

\[
\leq C(N, 4)(\mu_1(\|(u_n^1)^+\|_{L^{2^*}}^2 + \| v_n^1 \|_{L^{2^*}}^2) + 2\beta \| (u_n^1)^+ \|_{L^{2^*}}^2 \| (v_n^1)^+ \|_{L^{2^*}}^2) + o_n(1)
\]

\[
= C(N, 4)(\mu_1 t_n^2 + 2\beta t_n + \mu_2)\| v_n^1 \|_{L^{2^*}}^2 + o_n(1),
\]

which leads to

\[
\| v_n^1 \|_{L^{2^*}}^2 \geq \frac{\mathcal{S}(t_n + 1) - o_n(1)}{C(N, 4)(\mu_1 t_n^2 + 2\beta t_n + \mu_2)}.
\]

A direct computation shows that

\[
\inf_{t \geq 0} \frac{(1 + t)^2}{\mu_1 t^2 + 2\beta t + \mu_2} = k_1 + k_2
\]

(see [29] Lemma 2.3), then

\[
\int_{\mathbb{R}^N} (|\nabla u_n^1|^2 + |\nabla v_n^1|^2) \, dx \geq \mathcal{S}(\|(u_n^1)^+\|_{L^{2^*}}^2 + \|(v_n^1)^+\|_{L^{2^*}}^2)
\]

\[
= \mathcal{S}(t_n + 1)(\|(v_n^1)^+\|_{L^{2^*}}^2)
\]

\[
\geq \frac{\mathcal{S}^2(t_n + 1)^2}{C(N, 4)(\mu_1 t_n^2 + 2\beta t_n + \mu_2)} - o_n(1)
\]

\[
\geq (k_1 + k_2)\mathcal{S}^2_H - o_n(1) = 4c_\infty - o_n(1).
\]

(3.16)
We claim that 
\[ d^1_n := \max_{i \in \mathbb{N}} \left( \int_{P_i} \left( |(u_n^1)^+| + |(v_n^1)^+| \right) dx \right)^{\frac{p}{p^*}} \geq C_0 > 0, \]  
(3.17)
where \( P_i \) are hypercubes with disjoint interior and unitary sides with \( \mathbb{R}^N = \sum_{i \in \mathbb{N}} P_i, i \in \mathbb{N} \). Indeed, by calculation we have

\[
0 < \hat{c} \leq \int_{\mathbb{R}^N} \left( |(u_n^1)^+|^2 + |(v_n^1)^+|^2 \right) dx = \sum_{i=1}^{\infty} \int_{Q_i} \left( |(u_n^1)^+|^2 + |(v_n^1)^+|^2 \right) dx \\
\leq (d^1_n)^{2^*-2} \sum_{i=1}^{\infty} \left[ \int_{P_i} \left( |(u_n^1)^+|^2 + |(v_n^1)^+|^2 \right) dx \right]^{\frac{p}{p^*}} \\
\leq (d^1_n)^{2^*-2} \sum_{i=1}^{\infty} \left[ \left( \int_{P_i} |(u_n^1)|^2 \right)^{\frac{p}{p^*}} + \left( \int_{P_i} |(v_n^1)|^2 \right)^{\frac{p}{p^*}} \right] \\
\leq C_1 (d^1_n)^{2^*-2} \sum_{i=1}^{\infty} \left[ \int_{P_i} (|\nabla u_n|^2 + |\lambda_1| u_n^2) dx \right. \\

\left. + \int_{P_i} (|\nabla v_n|^2 + \lambda_2 |v_n|^2) dx \right] \\
= C_1 (d^1_n)^{2^*-2} \left[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\lambda_1| u_n^2) dx \right. \\

\left. + \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \lambda_2 |v_n|^2) dx \right] \\
\leq C_2 (d^1_n)^{2^*-2},
\]

where \( C_2 \) is a positive constant independent of \( i \) and the last inequality is due to the boundedness of \( (u_n^1, v_n^1) \) in \( H \). Thus, the claim holds true. We can also observe that

\[
\max_{i \in \mathbb{N}} \left( \int_{P_i} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \right) \geq \max_{i \in \mathbb{N}} \left( \int_{P_i} \left( |(u_n^1)^+|^2 + |(v_n^1)^+|^2 \right) dx \right)^{\frac{p}{p^*}} S \geq SC_0^2.
\]

Define a concentration function of \( (u_n^1, v_n^1) \) by

\[
Q_n(r) = \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} (|\nabla u_n|^2 + |\nabla v_n|^2) dx
\]

Since \( Q_n(0) = 0 \), \( Q_n(\infty) \geq 4c_\infty - a_n(1) \) and \( Q_n(r) \) is continuous in \( r \), then there exists \( y_n^1 \in \mathbb{R}^N, z_n > 0 \) such that

\[
Q_n(z_n) = \int_{B_{z_n}(y_n^1)} (|\nabla u_n|^2 + |\nabla v_n|^2) dx = \delta < \min \left\{ \frac{2c_\infty}{\lambda_2}, \frac{SC_0^2}{2} \right\}
\]

(3.18)

where \( L \) is the least number of balls with radius 1 covering a ball of radius 2, and \( \delta > 0 \) is independent of \( n \).

We see that \( \sigma_n \) is bounded, otherwise, for large \( n \),

\[
Q_n(\sigma_n) \geq \max_{i \in \mathbb{N}} \left( \int_{P_i} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \right) \geq SC_0^2 > \delta,
\]

which leads to a contradiction.

We set

\[
(\hat{u}_n^1, \hat{v}_n^1) := \sigma_n \left( (u_n^1, v_n^1 (\sigma_n x + y_n^1), v_n^1 (\sigma_n x + y_n^1)) \right),
\]

(3.19)

then

\[
\int_{\mathbb{R}^N} (|\nabla \hat{u}_n^1|^2 + |\nabla \hat{v}_n^1|^2) dx = \int_{B_{r(0)}} (|\nabla u_n|^2 + |\nabla v_n|^2) dx < \infty
\]

and

\[
\int_{B_{r(0)}} (|\nabla \hat{u}_n^1|^2 + |\nabla \hat{v}_n^1|^2) dx = \delta.
\]

(3.20)
Hence, there exists \((u^1, v^1) \in H_0\) such that \((\hat{u}^1_n, \hat{v}^1_n) \rightharpoonup (u^1, v^1)\) in \(H_0\) and \((\hat{u}^1_n, \hat{v}^1_n) \rightarrow (u^1, v^1)\) a.e. on \(\mathbb{R}^N \times \mathbb{R}^N\).

We claim that \((u^1, v^1) \neq (0, 0)\) is a solution of system \((2.3)\). Indeed, arguing as in \cite{17} (see also \cite{41} Lemma 3.6), \cite{25} Theorem 3.1, we can find \(\rho \in [1, 2]\) such that \(\hat{u}^1_n - u^1 \rightarrow 0\) and \(\hat{v}^1_n - v^1 \rightarrow 0\) in \(H^{1/2,2}(\partial B_\rho(0))\).

Then, the solutions \(\phi_{1,n}, \phi_{2,n}\) of

\[
\begin{cases}
-\Delta \phi = 0, \quad x \in B_3(0) \setminus B_\rho(0), \\
\phi|_{\partial B_\rho(0)} = \hat{u}^1_n - u^1, \quad \phi|_{\partial B_1(0)} = 0, 
\end{cases}
\]

and

\[
\begin{cases}
-\Delta \phi = 0, \quad x \in B_3(0) \setminus B_\rho(0), \\
\phi|_{\partial B_\rho(0)} = \hat{v}^1_n - v^1, \quad \phi|_{\partial B_1(0)} = 0, 
\end{cases}
\]

respectively, satisfying

\[
\phi_{1,n} \rightarrow 0, \quad \phi_{2,n} \rightarrow 0 \quad \text{in} \quad H^1(B_3(0) \setminus B_\rho(0)). \tag{3.21}
\]

Let

\[
\varphi_{1,n} = \begin{cases}
\hat{u}^1_n - u^1, \quad x \in B_\rho(0), \\
\phi_{1,n}, \quad x \in B_3(0) \setminus B_\rho(0), \\
0, \quad x \in \mathbb{R}^N \setminus B_3(0),
\end{cases}
\]

and

\[
\varphi_{2,n} = \begin{cases}
\hat{v}^1_n - v^1, \quad x \in B_\rho(0), \\
\phi_{2,n}, \quad x \in B_3(0) \setminus B_\rho(0), \\
0, \quad x \in \mathbb{R}^N \setminus B_3(0),
\end{cases}
\]

then

\[
\|\varphi_{j,n}\|_{L^2(\mathbb{R}^N)} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty, \quad j = 1, 2.
\]

Let

\[
\hat{\varphi}_{j,n} = \sigma_n \frac{N-2}{2} \varphi_{j,n}(\frac{x}{\sigma_n}), \quad j = 1, 2.
\]

By \((3.21)-(3.24)\), we get

\[
\|\hat{\varphi}_{1,n}\|_{L^2(\mathbb{R}^N)}^2 = \|\varphi_{1,n}\|_{L^2(\mathbb{R}^N)}^2 + \lambda_1 \sigma_n^2 \|\varphi_{1,n}\|_{L^2(\mathbb{R}^N)}^2 = \|\hat{u}^1_n - u^1\|^2_{L^2(\mathbb{R}^N(0))} + o_n(1). \tag{3.24}
\]

Similarly,

\[
\|\hat{\varphi}_{2,n}\|_{L^2(\mathbb{R}^N)}^2 = \|\varphi_{2,n}\|_{L^2(\mathbb{R}^N)}^2 + \lambda_2 \sigma_n^2 \|\varphi_{2,n}\|_{L^2(\mathbb{R}^N)}^2 = \|\hat{v}^1_n - v^1\|^2_{L^2(\mathbb{R}^N(0))} + o_n(1). \tag{3.25}
\]

Note that \(\{(u^1_n, v^1_n)\}\) is a sequence of Palais-Smale sequence for \(I_{\lambda,\infty}\), then

\[
\langle I_{\infty}'(\hat{u}^1_n, \hat{v}^1_n), (\varphi_{1,n}, \varphi_{2,n}) \rangle = \langle I_{\lambda,\infty}'(u^1_n, v^1_n), (\hat{\varphi}_{1,n}, \hat{\varphi}_{2,n}) \rangle + o_n(1) = o_n(1).
\]
Furthermore, we get
\[
o_{n}(1) = \langle \mathcal{I}_\infty' \left( \hat{u}^1_n, \hat{u}^1_n \right), (\varphi_{1,n}, \varphi_{2,n}) \rangle \\
= \int_{B_\rho(0)} \nabla \hat{u}^1_n \nabla (\hat{u}^1_n - u^1) \, dx + \int_{B_\rho(0)} \nabla \hat{v}^1_n \nabla (\hat{v}^1_n - v^1) \, dx \\
- \mu_1 \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 (\hat{u}^1_n)^+(y)(\hat{u}^1_n - u^1)(y) \, dx \, dy}{|x-y|^4} \\
- \mu_2 \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n)^+(x)|^2 (\hat{v}^1_n)^+(y)(\hat{v}^1_n - u^1)(y) \, dx \, dy}{|x-y|^4} \\
- \beta \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{(\hat{u}^1_n)^+(x)(\hat{u}^1_n - u^1)(x)(\hat{v}^1_n)^+(y) \, dx}{|x-y|^4} \\
- \beta \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{(\hat{v}^1_n)^+(y)(\hat{v}^1_n - v^1)(y) \, dx}{|x-y|^4} \, dy + o_n(1)
\]  
(3.26)

Since \((\hat{u}^1_n, \hat{v}^1_n)\) is bounded in \(H_0\), then \((\hat{u}^1_n)\) is bounded in \(L^{\frac{2N}{N-2}}\) and \(\hat{u}^1_n \to u^1\) a.e. in \(\mathbb{R}^N\), we have \(|(\hat{u}^1_n)^+|^2 \to |(u^1)^+|^2\) in \(L^{\frac{2N}{N-2}}(\mathbb{R}^N)\). By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \(L^{\frac{2N}{N-2}}\) to \(L^{\frac{2N}{N-2}'}(\mathbb{R}^N)\) and then
\[
\int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 \, dx}{|x-y|^4} \to \int_{\mathbb{R}^N} \frac{|(u^1)^+|^2 \, dx}{|x-y|^4} \quad \text{in} \quad L^{\frac{2N}{N-2}'}(\mathbb{R}^N).
\]
Combining this with \((\hat{u}^1_n)^+ \to (u^1)^+\) in \(L^{\frac{2N}{N-2}}\), we then get
\[
(\hat{u}^1_n)^+(y) \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 \, dx}{|x-y|^4} \to (u^1)^+(y) \int_{\mathbb{R}^N} \frac{|(u^1)^+(x)|^2 \, dx}{|x-y|^4} \quad \text{in} \quad L^{\frac{2N}{N-2}'}(\mathbb{R}^N),
\]
which implies that
\[
\lim_{n \to \infty} \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 (\hat{u}^1_n)^+(y)(\hat{u}^1_n - u^1)(y) \, dx \, dy}{|x-y|^4} = \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(u^1)^+(x)|^2 (u^1)^+(y)(u^1)(y) \, dx \, dy}{|x-y|^4}
\]
\[
= \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(u^1)^+(x)|^2 |(u^1)^+(y)|^2 \, dx \, dy + o_n(1)}{|x-y|^4}.
\]
Therefore,
\[
\int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 (\hat{u}^1_n)^+(y)(\hat{u}^1_n - u^1)(y) \, dx \, dy}{|x-y|^4}
\]
\[
= \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n)^+(x)|^2 |(\hat{u}^1_n)^+(y)|^2 \, dx \, dy - \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(u^1)^+(x)|^2 |(u^1)^+(y)|^2 \, dx \, dy + o_n(1)}{|x-y|^4}
\]
\[
= \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n - u^1)^+(x)|^2 |(\hat{u}^1_n - u^1)^+(y)|^2 \, dx \, dy + o_n(1)}{|x-y|^4}
\]

Similarly, we can also prove the following properties hold:
\[
\int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n)^+(x)|^2 (\hat{v}^1_n)^+(y)(\hat{v}^1_n - v^1)(y) \, dx \, dy = \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n - v^1)^+(x)|^2 |(\hat{v}^1_n - v^1)^+(y)|^2 \, dx \, dy + o_n(1)},
\]
\[
\int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n)^+(x)|^2 (\hat{v}^1_n)^+(y)(\hat{u}^1_n - u^1)(y) \, dx \, dy = \int_{B_\rho(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n - v^1)^+(x)|^2 |(\hat{v}^1_n - u^1)^+(y)|^2 \, dx \, dy + o_n(1)},
\]
By (3.21) and the scale invariance again, we have

\[
\int_{B_r(0)} \int_{\mathbb{R}^N} \frac{(\hat{u}^1_n(x))^2(\hat{v}^1_n(y))^2}{|x-y|^4} dx dy = \int_{B_r(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n-u^1)(x)|^2|\hat{v}^1_n-v^1(y)|^2}{|x-y|^4} dx dy + o_n(1).
\]

Inserting the equalities above into (3.23), we then get

\[
o_n(1) = \int_{B_r(0)} \nabla \hat{u}^1_n \nabla (\hat{u}^1_n-u^1) dx + \int_{B_r(0)} \nabla \hat{v}^1_n \nabla (\hat{v}^1_n-v^1) dx
- \mu_1 \int_{B_r(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n-u^1)^+(x)|^2}{|x-y|^4} dxdy
- \mu_2 \int_{B_r(0)} \int_{\mathbb{R}^N} \frac{|(\hat{v}^1_n-v^1)^+(x)|^2}{|x-y|^4} dxdy
- \beta \int_{B_r(0)} \int_{\mathbb{R}^N} \frac{|(\hat{u}^1_n-u^1)^+(x)|^2}{|x-y|^4} dxdy + o_n(1).
\]

By (3.24) and the scale invariance again, we have

\[
o_n(1) = \int_{\mathbb{R}^N} |\nabla \varphi_{1,n}|^2 dx + \int_{\mathbb{R}^N} |\nabla \varphi_{2,n}|^2 dx - \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{1,n}^+(x)^2}{|x-y|^4} dxdy
- \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{2,n}^+(y)^2}{|x-y|^4} dxdy - 2\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{1,n}^+(x)^2}{|x-y|^4} dxdy
= \int_{\mathbb{R}^N} |\nabla \varphi_{1,n}|^2 dx + \int_{\mathbb{R}^N} |\nabla \varphi_{2,n}|^2 dx - \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{1,n}^+(x)^2}{|x-y|^4} dxdy
- \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{2,n}^+(y)^2}{|x-y|^4} dxdy - 2\beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_{1,n}^+(x)^2}{|x-y|^4} dxdy.
\]

If \((\hat{\varphi}^1_{1,n}, \hat{\varphi}^2_{2,n}) \neq (0, 0)\), we define \(t_n > 0\) by

\[
t^2_n = \frac{\| (\hat{\varphi}^1_{1,n}, \hat{\varphi}^2_{2,n}) \|^2_{H_0}}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mu_1 |\varphi_{1,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2 + \mu_2 |\varphi_{2,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2 + 2\beta |\varphi_{1,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2}{|x-y|^4} dxdy}.
\]

Then \((t_n \hat{\varphi}_{1,n}, t_n \hat{\varphi}_{2,n}) \in N_{\infty}\) and moreover,

\[
e_{\infty} \leq I_{\infty}(t_n \hat{\varphi}_{1,n}, t_n \hat{\varphi}_{2,n})
= \frac{1}{4} t^2_n \int_{\mathbb{R}^N} \left( |\nabla \varphi_{1,n}|^2 + |\nabla \varphi_{2,n}|^2 \right) dx
\leq \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\| (\varphi_{1,n}, \varphi_{2,n}) \|^2_{H_0}}{\mu_1 |\varphi_{1,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2 + \mu_2 |\varphi_{2,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2 + 2\beta |\varphi_{1,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2}{|x-y|^4} dxdy,
\]

which implies that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mu_1 |\hat{\varphi}_{1,n}^+(x)|^2|\varphi_{1,n}^+(y)|^2 + \mu_2 |\hat{\varphi}_{2,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2 + 2\beta |\hat{\varphi}_{1,n}^+(x)|^2|\varphi_{2,n}^+(y)|^2}{|x-y|^4} dxdy
\leq \frac{1}{4 e_{\infty}} \left( \| (\nabla \varphi_{1,n})^2 + |\nabla \varphi_{2,n}|^2 \right) dx^2.
\]
From (3.24) and (3.25), we then get

\[ u \]

Thus we assert that (\( \tilde{v}_1, \tilde{v}_2 \)) in (3.18). By (3.14), (3.19) and (3.21),

\[ \lim_{n \to \infty} \int_{B_r(0)} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx = 0. \]

Then by (3.27), (3.28), we get

\[ o_n(1) \geq \left( 1 - \frac{1}{4c_\infty} \int_{B_r(0)} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx \right) \cdot \int_{\mathbb{R}^N} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) dx. \]

Note that from (3.20),

\[ \int_{B_r(0)} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx \leq L \int_{B_r(0)} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx < 2c_\infty. \]

Thus for \( n \to \infty \),

\[ \int_{\mathbb{R}^N} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) dx \to 0. \]

From (3.21) and (3.23), we then get

\[ \int_{B_r(0)} |\nabla (\tilde{u} - u)|^2 dx \to 0, \quad \int_{B_r(0)} |\nabla (\tilde{v} - v)|^2 dx \to 0. \]

Thus we assert that (\( u^1, v^1 \)) \( \neq (0, 0) \) due to (3.24).

Furthermore, we point out that \( \sigma_n \to 0 \) as \( n \to \infty \). Indeed, if not, we may assume that \( \sigma_n \to \sigma^* > 0 \) as \( n \to \infty \). Let

\( (\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + y_n), v_n(x + y_n)), \)

where \( y_n \) is defined in (3.18). By (3.14), (3.19) and (\( u^1, v^1 \)) \( \neq (0, 0) \), we get \( |y_n| \to \infty \) as \( n \to \infty \). Then, \( \{ (\tilde{u}_n, \tilde{v}_n) \} \) is a Palais-Smale sequence of \( I_{\lambda, \infty} \) and also bounded in \( H \). We may assume that \( (\tilde{u}_n(x), \tilde{v}_n(x)) \to (\tilde{u}, \tilde{v}) \) in \( H \). Moreover, we can find that \( (\tilde{u}, \tilde{v}) \) solves the system (3.1). Then by Lemma 3.2, we assert that \( (\tilde{u}, \tilde{v}) = (0, 0) \), which implies that \( (\tilde{u}_n, \tilde{v}_n) \to (0, 0) \) in \( H \). Thus, \( \tilde{u}_n \to 0, \tilde{v}_n \to 0 \) in \( L_{x}^{2}(\mathbb{R}^N) \). On the other hand, using again \( (\tilde{u}_n, \tilde{v}_n) \to (u^1, v^1) \neq (0, 0) \) in \( H_0 \), there exists a \( r_0 > 0 \) such that

\[ \int_{B_{r_0}(0)} (|u^1|^2 + |v^1|^2) dx > 0. \]

Moreover,

\[ \lim_{n \to \infty} \int_{B_{r_0}(0)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2) dx = \lim_{n \to \infty} \int_{B_{r_0}(0)} \sigma_n^{-2} (|\tilde{u}_n(\sigma_n^{-1} x)|^2 + |\tilde{v}_n(\sigma_n^{-1} x)|^2) dx \]

\[ = \lim_{n \to \infty} \int_{B_{r_0}(\sigma_n)} \sigma_n^2 (|\tilde{u}_n(y)|^2 + |\tilde{v}_n(y)|^2) dy \]

\[ = \lim_{n \to \infty} \int_{B_{r_0}(0)} (\sigma^*)^2 (|u^1(y)|^2 + |v^1(y)|^2) dy > 0. \]
This contradicts with the fact that \( \tilde{u}_n^1 \to 0, \tilde{v}_n^1 \to 0 \) in \( L^2_{loc}(\mathbb{R}^N) \).

In what follows, we want to prove that \( (u^1, v^1) \) is a nonzero solution of system (2.3). For arbitrary \( \varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N) \), then

\[
(I_{I_n}(\tilde{u}_n^1, \tilde{v}_n^1), (\varphi_1, \varphi_2)) = (I_{I_n}(\tilde{u}_n^1, \tilde{v}_n^1), (\tilde{\varphi}_1, \tilde{\varphi}_2)) + o_n(1) = o_n(1),
\]

where \( \tilde{\varphi}_j = \sigma_n^{-\frac{N-2}{2}} \varphi_j(\frac{x}{\sigma_n}), j = 1, 2 \). Then, \( (u^1, v^1) \) solves the system (2.3).

Note that the bubble \( \sigma_n^{-\frac{N-2}{2}} \varphi(x) \) does not belong to \( H \). Technically, we need to modify it so that we can eliminate this bubble from the original sequence \( (u^1_n, v^1_n) \) to obtain a new Palais-Smale sequence of \( I_{I, \infty} \) at a lower level. We define

\[
(u^2_n(x), v^2_n(x)) := (u^1_n(x), v^1_n(x)) - \sigma_n^{-\frac{N-2}{2}} \varphi(x) \frac{u^1(x) - y_n^1}{\sigma_n^2} \varphi(x), \varphi(x) \frac{v^1(x) - y_n^1}{\sigma_n^2} \varphi(x),
\]

where \( \varphi(x) \) is a cut-off function satisfying \( \varphi(x) = 1 \) if \( |x| \leq (0, 1) \), \( \varphi(x) = 0 \) if \( |x| > 2 \).

In the following, we want to prove that \( \{(u^2_n(x), v^2_n(x))\} \) is a sequence of Palais-Smale sequence for \( I_{I, \infty} \) in \( H \) and \( (u^2_n(x), v^2_n(x)) \to (0, 0) \) in \( H \). In fact, in order to prove

\[
(u^2_n(x), v^2_n(x)) \to (0, 0) \quad \text{in} \quad H,
\]

it is sufficient to prove that

\[
\sigma_n^{-\frac{N-2}{2}} \left( \varphi \left( \frac{x - y_n^1}{\sigma_n^2} \right) u^1 \left( \frac{x - y_n^1}{\sigma_n^2} \right), \varphi \left( \frac{x - y_n^1}{\sigma_n^2} \right) v^1 \left( \frac{x - y_n^1}{\sigma_n^2} \right) \right) \to (0, 0) \quad \text{in} \quad H. \tag{3.29}
\]

Recalling that \( \varphi \) is a cut-off functions, then by calculation we have

\[
\| \left( \varphi \left( \frac{x - y_n^1}{\sigma_n^2} \right) \frac{u^1(x)}{\sigma_n^2}, \varphi \left( \frac{x - y_n^1}{\sigma_n^2} \right) \frac{v^1(x)}{\sigma_n^2} \right) \|_{L^2 \times L^2}^2 = \sigma_n^2 \int_{\mathbb{R}^N} |\varphi(\sigma_n^2 x)|^2 |u^1(x)|^2 \, dx + \sigma_n^2 \int_{\mathbb{R}^N} |\varphi(\sigma_n^2 x)|^2 |v^1(x)|^2 \, dx
\]

\[
\leq \sigma_n^2 \int_{|x| \leq 2\sigma_n^2} |u^1(x)|^2 \, dx + \sigma_n^2 \int_{|x| \leq 2\sigma_n^2} |v^1(x)|^2 \, dx
\]

\[
\leq C \sigma_n \left( \int_{|x| \leq 2\sigma_n^2} |u^1|^2 \, dx \right)^{\frac{2}{p}} + C \sigma_n \left( \int_{|x| \leq 2\sigma_n^2} |v^1|^2 \, dx \right)^{\frac{2}{p}} \to 0, \quad \text{as} \quad n \to \infty, \tag{3.30}
\]

where \( \| (u, v) \|_{L^p \times L^p}^p = \int_{\mathbb{R}^N} |u|^p \, dx + \int_{\mathbb{R}^N} |v|^p \, dx \). On the other hand, direct calculation shows that

\[
\| \varphi \left( \frac{x - y_n^1}{\sigma_n^2} \right) \frac{u^1(x)}{\sigma_n^2} - \sqrt{u^1} \left( \frac{x - y_n^1}{\sigma_n^2} \right) \|_{L^{2,1}}^2 \to 0,
\]

\[
\| \varphi \left( \frac{\sigma_n^2 x}{2} \right) u^1 - u^1 \|_{L^{2,1}}^2 \to 0. \tag{3.31}
\]
as \( \sigma_n \to 0 \). Since \( \sigma_n^{\frac{N-2}{2}} u^1 \left( \frac{x-y_1}{\sigma_n} \right) \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \), then \( \varphi \left( \frac{x-y_1}{\sigma_n} \right) \sigma_n^{\frac{N-2}{2}} u^1 \left( \frac{x-y_1}{\sigma_n} \right) \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \). Similar to (3.31), we can also prove that \( \varphi \left( \frac{x-y_1}{\sigma_n} \right) \sigma_n^{\frac{N-2}{2}} v^1 \left( \frac{x-y_1}{\sigma_n} \right) \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \). Therefore,

\[
\sigma_n^{\frac{N-2}{2}} \left( \varphi \left( \frac{x-y_1}{\sigma_n} \right) u^1 \left( \frac{x-y_1}{\sigma_n} \right), \varphi \left( \frac{x-y_1}{\sigma_n} \right) v^1 \left( \frac{x-y_1}{\sigma_n} \right) \right) \to (0, 0) \text{ in } H_0.
\]

Thus, by (3.31) and (3.32), we then prove that (3.29) is true. Notice that \( (u_n^1(x), v_n^1(x)) \to (0, 0) \) in \( H \), then by (3.29),

\[
(u_n^2(x), v_n^2(x)) \to (0, 0) \text{ in } H.
\]

Based on the properties (3.30) and (3.31), together with Lemma 3.1, we have

\[
I_{\lambda, \infty}(u_n^2, v_n^2) = I_{\lambda, \infty}(u_n^1, v_n^1) - I_{\infty}(u^1, v^1) + o_n(1)
\]

and

\[
\left\| (u_n, v_n) \right\|^2_H = \left\| (u^0, v^0) \right\|^2_H + \left\| (u_n^1, v_n^1) \right\|^2_H + o_n(1)
\]

\[
\left\| (u^0, v^0) \right\|^2_H + \left\| (u_n^2, v_n^2) \right\|^2_H + \left\| \sigma_n^{\frac{N-2}{2}} u^1 \left( \frac{x-y_1}{\sigma_n} \right), \sigma_n^{\frac{N-2}{2}} v^1 \left( \frac{x-y_1}{\sigma_n} \right) \right\|^2_{H_0} + o_n(1).
\]

Taking test function \((\tilde{\varphi}, \tilde{\psi}) \in H \) with \( \left\| (\tilde{\varphi}, \tilde{\psi}) \right\|_H \leq 1 \), then by Lemma 3.1 again, we have

\[
\langle I_{\lambda, \infty}^\prime(u_n^2, v_n^2), (\tilde{\varphi}, \tilde{\psi}) \rangle
\]

\[
= \langle I_{\lambda, \infty}(u_n^1, v_n^1), (\tilde{\varphi}, \tilde{\psi}) \rangle
\]

\[
- \langle I_{\lambda, \infty}^\prime(u_n^1, v_n^1), (\varphi - \frac{N-2}{2} u^1) \left( \frac{x-y_1}{\sigma_n} \right), \left( \frac{x-y_1}{\sigma_n} \right) \left( \frac{x-y_1}{\sigma_n} \right) \rangle \psi \left( \frac{x-y_1}{\sigma_n} \right) + o_n(1)
\]

\[
= \langle I_{\lambda, \infty}(u_n^1, v_n^1), (\tilde{\varphi}, \tilde{\psi}) \rangle - \langle I_{\lambda, \infty}^\prime(u_n^1, v_n^1), (\frac{N-2}{2} u^1) \left( \frac{x-y_1}{\sigma_n} \right), \left( \frac{x-y_1}{\sigma_n} \right) \left( \frac{x-y_1}{\sigma_n} \right) \rangle \psi + o_n(1)
\]

\[
= \langle I_{\lambda, \infty}(u_n^1, v_n^1), (\tilde{\varphi}, \tilde{\psi}) \rangle + o_n(1).
\]

Note that \((u_n^1, v_n^1)\) is a sequence of Palais-Smale sequence for \( I_{\lambda, \infty} \), then

\[
\left\| I_{\lambda, \infty}^\prime(u_n^2, v_n^2) \right\|_{H^{-1}} = \sup_{(\tilde{\varphi}, \tilde{\psi}) \in H, \left\| (\tilde{\varphi}, \tilde{\psi}) \right\|_H \leq 1} \langle I_{\lambda, \infty}^\prime(u_n^2, v_n^2), (\tilde{\varphi}, \tilde{\psi}) \rangle
\]

\[
= \left\| I_{\lambda, \infty}^\prime(u_n^1, v_n^1) \right\|_{H^{-1}} + o_n(1) \to 0, \text{ as } n \to \infty.
\]

Thus \((u_n^2, v_n^2)\) is a sequence of Palais-Smale sequence for \( I_{\lambda, \infty} \). Moreover, by (3.33) we get

\[
I(u_n, v_n) = I_{\lambda, \infty}(u_n^1, v_n^1) + I(u^0, v^0) + o_n(1) = I_{\lambda, \infty}(u_n^2, v_n^2) + I_{\infty}(u^1, v^1) + I(u^0, v^0) + o_n(1).
\]

If \((u_n^2, v_n^2) \to (0, 0) \) in \( H \), then we have done. Otherwise, we can iterate the procedure. Taking into account that at every step \( \ell \), we then prove that

\[
\left\| (u_n, v_n) \right\|_H^2 = \left\| (u^0, v^0) \right\|_H^2 + \left\| (u_{n+1}^{\ell+1}, v_{n+1}^{\ell+1}) \right\|_H^2
\]

\[
+ \sum_{k=1}^{\ell} \left\| \left( \sigma_n^{k-2} u_k^1 \left( \frac{x-y_k^1}{\sigma_n^k} \right), \sigma_n^{k-2} v^1 \left( \frac{x-y_k^1}{\sigma_n^k} \right) \right) \right\|_H^2 + o_n(1)
\]

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and

\[ I(u_n, v_n) = I(u_0, v_0) + \sum_{k=1}^\ell I_\infty(u^k, v^k) + I_{\lambda, \infty}(u^{\ell+1}_n, v^{\ell+1}_n) + o_n(1). \]

Since \( I_\infty(u^k, v^k) \geq \frac{1}{4}(k_1 + k_2)S^2_{H,L} \), then

\[ I(u_n, v_n) \geq I(u_0, v_0) + \frac{\ell}{4}(k_1 + k_2)S^2_{H,L} + o_n(1). \]

Hence, the iteration must terminate at a finite index \( \ell \) such that \( \|(u^{\ell+1}_n, v^{\ell+1}_n)\|_H \to 0 \). So

\[ \|(u_n, v_n)\|_H^2 = \|(u_0, v_0)\|_H^2 + \sum_{k=1}^\ell \left\| \left( (\sigma_n^k)^{-\frac{N-2}{2}} u^k \frac{x - y^k_n}{\sigma_n^k}, (\sigma_n^k)^{-\frac{N-2}{2}} v^k \frac{x - y^k_n}{\sigma_n^k} \right) \right\|_{H_0}^2 + o_n(1) \]

and

\[ I(u_n, v_n) = I(u_0, v_0) + \sum_{k=1}^\ell I_\infty(u^k, v^k) + o_n(1). \]

(b) Let us now consider the case \( \lambda_1 = 0, \lambda_2 > 0 \), in this case \( H := D^{1,2}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). We remark that, to obtain the proof of case (b), one need to adjust some arguments displayed in the proof of case (a).

Repeat the argument in the previous case, we have \((u_n, v_n) \to (u_0, v_0)\) in \( H \), where \((u_0, v_0)\) is a pair of solution to system (2.1). Let \((u^1_n, v^1_n) = (u_n, v_n) - (u_0, v_0)\), then \((u^1_n, v^1_n)\) is a sequence of Palais-Smale sequence of \( I_{\lambda, \infty} \). If \((u^1_n, v^1_n) \to (0, 0)\) in \( H \), then we have done. If \((u^1_n, v^1_n) \to (0, 0)\), but \((u^1_n, v^1_n) \not\to (0, 0)\) in \( H \). Setting

\[ (\tilde{u}^1_n(x), \tilde{v}^1_n(x)) := (u^1_n(x + y^1_n(x), v^1_n(x + y^1_n(x))), \]

then we prove that \((\tilde{u}^1_n, \tilde{v}^1_n) \to (\tilde{u}^1, \tilde{v}^1)\) in \( H \), where \((\tilde{u}^1, \tilde{v}^1)\) solves system (3.1). By Lemma 3.2(b), \((\tilde{u}^1, \tilde{v}^1) = (w^1, 0)\), where \( w^1 \) solves scalar equation (3.2).

If \( w^1 = 0 \), then by the same argument as in proof of case (a), we can prove \((u^1, v^1) \neq (0, 0)\). Next, we need to modify \((u^2_n(x), v^2_n(x))\) as follows:

\[ (u^2_n(x), v^2_n(x)) := (u^1_n(x), v^1_n(x)) - \sigma_n^{\frac{N-2}{2}} \left( u^1 \frac{x - y^1_n(x)}{\sigma_n}, \varphi(x - y^1_n(x)) v^1 \frac{x - y^1_n(x)}{\sigma_n} \right). \]

It is easy to check that \((u^2_n(x), v^2_n(x)) \to (0, 0)\) in \( D^{1,2}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \).

If \( w^1 \neq 0 \), then \((\tilde{u}^1_n, \tilde{v}^1_n) \to (w^1, 0) \neq (0, 0)\) in \( D^{1,2}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). We directly modify

\[ (u^2_n(x), v^2_n(x)) := (u^1_n(x), v^1_n(x)) - (w^1(x - y^1_n(x), 0). \]

We claim that \( y^1_n \to \infty \) as \( n \to \infty \). If not, we then suppose that \( \{y^1_n\} \) is bounded. Since \( u^1_n \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \), we have \( u^1_n \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Hence, \( u^1_n(x + y^1_n) \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), namely, \( \tilde{u}^1_n \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). On the other hand, as \( \tilde{u}^1_n \to w^1 \) in \( D^{1,2}(\mathbb{R}^N) \), we have \( \tilde{u}^1_n \to w^1 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). This is impossible due to \( w^1 \neq 0 \). Since \( |y^1_n| \to \infty \), then \( w^1(x - y^1_n) \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \). So \((u^2_n(x), v^2_n(x)) \to (0, 0)\) in \( D^{1,2}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \), and we set \( z^1_n = y^1_n \) in this case.

Based on above discussion, the rest proof is standard. We iterate the procedure as in proof in case (a) and prove that

\[ \|(u_n, v_n)\|_H^2 = \|(u_0, v_0)\|_H^2 + \|(u_0^{\ell+1}, v_0^{\ell+1})\|_H^2 + \sum_{k=1}^{\ell_1} \left( \|w^k(x - z^k_n, 0)\|_{H_0}\right)^2 + \sum_{k=1}^{\ell_2} \left( \|\sigma_n^{-\frac{N-2}{2}} u^k \frac{x - y^k_n}{\sigma_n^k}, \sigma_n^{-\frac{N-2}{2}} v^k \frac{x - y^k_n}{\sigma_n^k}\right) \|_{H_0}^2 + o_n(1) \]
and

\[ I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell_1} I_\infty(u^k, 0) + \sum_{k=1}^{\ell_2} I_\infty(u^k, v^k) + I_{\lambda, \infty}(u^{\ell+1}_n, v^{\ell+1}_n) + o_n(1). \]

Recalling that \( I_\infty(u^k, v^k) \geq \frac{1}{2}(k_1 + k_2)S^2_{HL}, \) then the iteration must terminate at a finite index \( \ell \) such that \( \|u^{\ell+1}_n, v^{\ell+1}_n\|_H \to 0, \) and then \( \ell = \ell_1 + \ell_2. \) Therefore,

\[
\| (u_n, v_n) \|_H^2 = \| (u^0, v^0) \|_H^2 + \sum_{k=1}^{\ell_1} \| (u^k(x - z_n^k), 0) \|_{H_0}^2 \]

\[ + \sum_{k=1}^{\ell_2} \| (\sigma_n^k - \frac{\|u\|}{\|v\|}u^k, \sigma_n^k - \frac{\|u\|}{\|v\|}v^k) \|_{H_0}^2 + o_n(1) \]

and

\[ I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell_1} I_\infty(u^k, 0) + \sum_{k=1}^{\ell_2} I_\infty(u^k, v^k) + o_n(1), \]

(c) The proof of case (c) is similar to the proof of case (b), in the sequel we only omit it for keeping our paper a suitable length.

**Corollary 3.4.** If \( \{(u_n, v_n)\} \subset \mathcal{N} \) is a Palais-Smale sequence for the constrained functional \( I|_\mathcal{N} \) at level \( d \in (c_\infty, \min\{\frac{S^2_{HL}}{4\mu_1}, \frac{S^2_{HL}}{4\mu_2}, 2c_\infty\}), \) then \( \{(u_n, v_n)\} \) is relatively compact.

**Proof:** Suppose \( \{(u_n, v_n)\} \) is a sequence of Palais-Smale sequence for \( I|_\mathcal{N} \) at level \( d \), that is, \( I(u_n, v_n) \to d \) and \( I'(u_n, v_n) \to 0 \) as \( n \to +\infty. \) Since \( \mathcal{N} \) is a natural constraint, then \( I'(u_n, v_n) \to 0 \) as \( n \to +\infty. \)

It is easy to prove that \( \{(u_n, v_n)\} \) is bounded in \( H \). Without loss of generality, we may assume that \( (u_n, v_n) \to (u^0, v^0) \) in \( H \) up to a subsequence. In the following, we will divide the proof into three cases depending on the signs of \( \lambda_1 \) and \( \lambda_2 \).

**Case 1:** If \( \lambda_1, \lambda_2 > 0 \). It follows by Theorem 3.3(a) that there exist a solution \( (u^0, v^0) \) of system \( \text{(2.1)} \), \( \ell \) sequences \( \{(u^k, v^k)\} \) of solutions of system \( \text{(2.3)}, \) \( 1 \leq k \leq \ell, \) such that

\[ I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell} I_\infty(u^k, v^k) + o_n(1). \]

If \( \ell = 0 \), then we have done. Otherwise, we may suppose \( \ell > 0. \) Since \( d < 2c_\infty, \) then \( \ell = 1, \) that is

\[ d = I(u^0, v^0) + I_\infty(u^1, v^1). \]

If \( (u^0, v^0) \neq (0, 0), \) then from Lemma 2.4, we get

\[ d = I(u^0, v^0) + I_\infty(u^1, v^1) > c + c_\infty = 2c_\infty, \]

which contradicts to the assumption \( d \in (c_\infty, \min\{\frac{S^2_{HL}}{4\mu_1}, \frac{S^2_{HL}}{4\mu_2}, 2c_\infty\}). \)

If \( (u^0, v^0) = (0, 0). \) Since \( u^1 \geq 0 \) and \( v^1 \geq 0, \) by Lemma 2.6 and the uniqueness of positive solutions for the scalar equation

\[ -\Delta u = \mu_j(|x|^{-4} \ast |u|^2)u, \quad j = 1, 2, \quad x \in \mathbb{R}^N, \]
we desert that \((u^1, v^1)\) must be, up to translation and dilation, one of the following three solutions 

\[
(\sqrt{k_1}U_{1,0}, \sqrt{k_2}U_{1,0}), \quad \left(\frac{1}{\sqrt{\mu_1}}U_{1,0}, 0\right), \quad (0, \frac{1}{\sqrt{\mu_2}}U_{1,0}),
\]

where \(k_1 = \frac{\beta_1}{\beta_1 - \mu_1}, k_2 = \frac{\beta_2}{\beta_2 - \mu_2}\). Therefore,

\[
either d = c_\infty, \text{ or } d = \frac{1}{4\mu_1} \|U_{1,0}\|_{D^{1,2}}^2 = \frac{S_{\mu_1}^2}{4\mu_1}, \text{ or } d = \frac{1}{4\mu_2} \|U_{1,0}\|_{D^{1,2}}^2 = \frac{S_{\mu_2}^2}{4\mu_2},
\]

which also contradicts to assumption \(d \in (c_\infty, \min\{\frac{S_{\mu_1}^2}{4\mu_1}, \frac{S_{\mu_2}^2}{4\mu_2}, 2c_\infty\}\). Therefore \((u_n, v_n) \to (u^0, v^0)\) in \(H\).

**Case 2:** If \(\lambda_1 = 0, \lambda_2 > 0\). Then from Theorem 3.3(b) we assert that there exist \(\ell_2\) sequences \(\{(u^k, v^k)\}\) of solutions of system (2.3a), \(1 \leq k \leq \ell_2\), \(\ell_1\) sequences \(\{(u^k, 0)\}\) of semi-trivial solution of system (2.3a), \(1 \leq k \leq \ell_1\), such that

\[
I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell_1} I_\infty(u^k, 0) + \sum_{k=1}^{\ell_2} I_\infty(u^k, v^k) + o_n(1).
\]

First, we are going to show \(\ell_1 = 0\). We suppose by contradiction that \(\ell_1 > 0\), then we have

\[
I(u_n, v_n) \geq I(u^0, v^0) + \sum_{k=1}^{\ell_1} I_\infty(u^k, 0) + o_n(1) \geq I_\infty(u^1, 0) + o_n(1),
\]

Which implies that \(d \geq \frac{S_{\mu_2}^2}{4\mu_1}\) and there is a contradiction with \(d \in (c_\infty, \min\{\frac{S_{\mu_1}^2}{4\mu_1}, \frac{S_{\mu_2}^2}{4\mu_2}, 2c_\infty\}\). Hence, we get

\[
I(u_n, v_n) = I(u^0, v^0) + \sum_{k=1}^{\ell_2} I_\infty(u^k, v^k) + o_n(1).
\]

By the same arguments in the proof of case 1, we can also get \(\ell_2 = 0\). So \((u_n, v_n) \to (u^0, v^0)\) in \(H\).

**Case 3:** If \(\lambda_1 > 0, \lambda_2 = 0\). In this case, we can prove \((u_n, v_n) \to (u^0, v^0)\) in \(H\) by repeating the arguments in the proof of case 2, so we omit it here.

\[\blacksquare\]

### 4 Some basic estimates

For \((u, v) \in H_0 \setminus \{(0, 0)\}\), we introduce a barycenter function

\[
\xi(u, v) = \int_{\mathbb{R}^N} \frac{x}{1+|x|} (\mu_1(u^+)^2 + 2\beta(u^+) \frac{\partial}{\partial x} (v^+) \frac{\partial}{\partial x} + \mu_2(v^+)^2) \, dx
\]

and set

\[
\gamma(u, v) = \int_{\mathbb{R}^N} \frac{x}{1+|x|} \cdot \xi(u, v) \cdot \left(\mu_1(u^+)^2 + 2\beta(u^+) \frac{\partial}{\partial x} (v^+) \frac{\partial}{\partial x} + \mu_2(v^+)^2\right) \, dx
\]

to estimate the concentration of \((u, v)\) around its barycenter. It is not difficult to find that the maps \(\xi\) and \(\gamma\) are continuous with respect to the \(H_0\) norm and for \(t > 0\) and \((u, v) \in H_0\) such that \((u^+, v^+) \neq (0, 0)\),

\[
\xi(tu, tv) = \xi(u, v), \quad \gamma(tu, tv) = \gamma(u, v).
\]
Lemma 4.1. Suppose that $\lambda_1, \lambda_2 \geq 0$, $\lambda = \max\{\lambda_1, \lambda_2\} > 0$ and $\beta > \max\{\mu_1, \mu_2\}$. Then
\[
c^* = \inf\{I(u, v) \mid (u, v) \in \mathcal{N}, \xi(u, v) = 0, \gamma(u, v) = \frac{1}{2}\} > c_\infty. \tag{4.2}
\]

**Proof:** From Lemma 2.7 we have $c^* \geq c = c_\infty$. We argue by contradiction and assume that $c^* = c_\infty$. Then by Ekeland’s variational principle, there exists a sequence of $\{(u_n, v_n)\} \subset \mathcal{N}$ such that, as $n \to \infty$,
\[
\begin{cases}
\xi(u_n, v_n) \to 0, & \gamma(u_n, v_n) \to \frac{1}{2}, \\
I(u_n, v_n) \to c_\infty, & (I_{\mathcal{N}})'(u_n, v_n) \to 0.
\end{cases}
\tag{4.3}
\]

As Nehari manifold $\mathcal{N}$ is natural constraint, then $\{(u_n, v_n)\}$ is a sequence of Palais-Smale sequence of $I$ at $c_\infty$. By Lemma 2.7 and Theorem 3.3, we deduce that there exist sequences of $\delta_n > 0$ and $y_n \in \mathbb{R}^N$, and a nonzero solution $(u, v)$ of system (2.3) satisfying $I_\infty(u, v) = c_\infty$ such that
\[
\begin{aligned}
&\left(\delta_n^{\frac{N-2}{2}} u_n(\delta_n x + y_n), \delta_n^{\frac{N-2}{2}} v_n(\delta_n x + y_n)\right) \to (u, v) \text{ in } H_0.
\end{aligned}
\]

Then $(u, v)$ must be a positive solution of system (4.3). Otherwise, if $u \neq 0, v = 0$ or $u = 0, v \neq 0$, then $u > 0$ or $v > 0$ for $x \in \mathbb{R}^N$. By the uniqueness of positive solutions of

\[
-\Delta u = \mu_j(|x|^{-4*}|u|^2)u, \quad j = 1, 2, \quad x \in \mathbb{R}^N,
\]

then $u = \frac{1}{\sqrt{\mu_1}} U_{\delta_0}y_n$ or $v = \frac{1}{\sqrt{\mu_2}} U_{\delta_0}y_n$ for some $\delta > 0$ and $y \in \mathbb{R}^N$. Furthermore, we deduce that

\[
c_\infty = I_\infty(u, 0) = \frac{1}{4\mu_1} S_{HL}^2, \quad \text{or} \quad c_\infty = I_\infty(0, v) = \frac{1}{4\mu_2} S_{HL}^2,
\]

which is absurd due to

\[
c_\infty = \frac{k_1 + k_2}{4} S_{HL}^2, \quad \frac{1}{\mu_1} > k_1 + k_2, \quad \frac{1}{\mu_2} > k_1 + k_2.
\]

Thus we have
\[
(u_n, v_n) = (\sqrt{k_1} U_{\delta_n}y_n, \sqrt{k_2} U_{\delta_n}y_n) + (\varphi_n, \psi_n), \tag{4.4}
\]

where $(\varphi_n, \psi_n) \to (0, 0)$ in $H_0$ as $n \to \infty$.

Next, we would like to show that
\[
\begin{cases}
\lim_{n \to \infty} \delta_n = \delta_0 > 0, & (a) \\
\lim_{n \to \infty} y_n = y_0 \in \mathbb{R}^N, & (b).
\end{cases}
\tag{4.5}
\]

In order to prove (4.5) (a), we start to show that $\{\delta_n\}$ is bounded. Otherwise, then there exists a subsequence $\{\delta_{n_j}\}$ of $\{\delta_n\}$, such that $\lim_{n \to \infty} \delta_n = \infty$. Then for each $r > 0$, we have

\[
\lim_{n \to \infty} \int_{B_r(0)} |U_{\delta_n}y_n(x)|^2 dx = 0.
\]

From (4.3) we know that $\lim_{n \to \infty} \xi(u_n, v_n) = 0$, then
\[
\frac{1}{2} = \lim_{n \to \infty} \gamma(u_n, v_n)
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|x|}{1 + |x|} (\mu_1 k_1^2 \frac{d^2}{dx^2} + 2\beta k_1^2 k_2^2 + \mu_2 k_2^2) |U_{\delta_n}y_n|^2 dx
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|x|}{1 + |x|} |U_{\delta_n}y_n|^2 dx + o_n(1)
\]

\[
\geq \frac{r}{1 + r},
\]

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which is impossible provided that \( r > 1 \). Hence, \( \{\delta_n\} \) must be bounded. Up to a subsequence, we may suppose that \( \delta_n \to \delta_0 \geq 0 \) as \( n \to \infty \). If \( \delta_0 = 0 \), then for each \( r > 0 \), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_r(y_n)} |U_{\delta_n,y_n}(x)|^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_r(0)} |U_{\delta_n,0}(x)|^2 \, dx = 0.
\]
By the assumption \( \lim_{n \to \infty} \xi(u_n,v_n) = 0 \) again, then we get for each \( r > 0 \),
\[
\frac{|y_n|}{1 + |y_n|} = \left| \frac{y_n}{1 + |y_n|} - \xi(u_n,v_n) \right| + o_n(1) \\
= \frac{\int_{\mathbb{R}^N} \left| \frac{x}{1 + |x|} - \frac{y_n}{1 + |y_n|} \right| (\mu_1 k_1^2 + 2\beta k_1^2 k_2^2 + \mu_2 k_2^2) |U_{\delta_n,y_n}|^2 \, dx}{\int_{\mathbb{R}^N} |U_{\delta_n,y_n}|^2 \, dx} + o_n(1) \\
= \frac{\int_{B_r(y_n)} \left| \frac{x}{1 + |x|} - \frac{y_n}{1 + |y_n|} \right| |U_{\delta_n,y_n}|^2 \, dx}{\int_{B_r(y_n)} |U_{\delta_n,y_n}|^2 \, dx} + o_n(1) \\
\leq 2r + o_n(1).
\]
Thus, we assert that \( |y_n| \to 0 \) as \( n \to \infty \). Repeating the calculation above, one can easily find that
\[
0 \leq \gamma(u_n,v_n) \leq \frac{\int_{\mathbb{R}^N} \left| \frac{x}{1 + |x|} - \frac{y_n}{1 + |y_n|} \right| |U_{\delta_n,y_n}|^2 \, dx}{\int_{\mathbb{R}^N} |U_{\delta_n,y_n}|^2 \, dx} + o_n(1) \leq 2r + o_n(1),
\]
from which it follows that
\[
\lim_{n \to \infty} \gamma(u_n, v_n) = 0.
\]
Obviously, there is a contradiction with (4.3). Thus, (4.5) (a) holds true.

Now we shall to show that \( \{y_n\} \subset \mathbb{R}^N \) and is bounded. We argue by contradiction and suppose that \( \lim_{n \to \infty} |y_n| = \infty \). Then for each \( \varepsilon > 0 \), we can find \( r_0 = r_0(\varepsilon) > 0 \) such that for all \( n \in \mathbb{N} \),
\[
\int_{\mathbb{R}^N \setminus B_{r_0}(y_n)} |U_{\delta_0,y_n}|^2 \, dx = \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |U_{\delta_0,0}|^2 \, dx < \varepsilon. \tag{4.6}
\]
For such a fixed \( r_0 \), there exists \( N^* \in \mathbb{N} \) such that for each \( n \geq N^* \) and \( x \in B_{r_0}(y_n) \),
\[
\left| \frac{x}{1 + |x|} - \frac{y_n}{1 + |y_n|} \right| < \varepsilon. \tag{4.7}
\]
By (4.6) and (4.7), for \( n \) large enough, we prove that
\[
\left| \xi(u_n, v_n) - \frac{y_n}{1 + |y_n|} \right| \leq \frac{\int_{\mathbb{R}^N} \left| \frac{x}{1 + |x|} - \frac{y_n}{1 + |y_n|} \right| |U_{\delta_0,y_n}|^2 \, dx}{\int_{\mathbb{R}^N} |U_{\delta_0,y_n}|^2 \, dx} \\
\leq \frac{\varepsilon \int_{B_{r_0}(y_n)} |U_{\delta_0,y_n}|^2 \, dx + o_n(1)}{\int_{\mathbb{R}^N} |U_{\delta_0,y_n}|^2 \, dx} + \frac{2 \int_{\mathbb{R}^N \setminus B_{r_0}(y_n)} |U_{\delta_0,y_n}|^2 \, dx}{\int_{\mathbb{R}^N} |U_{\delta_0,y_n}|^2 \, dx} \\
\leq C\varepsilon,
\]
where positive constant \( C \) is independent of \( y_n \) and \( r_0 \). Based on the inequality above together with the assumption \( \lim_{n \to \infty} |y_n| = \infty \), we get
\[
\lim_{n \to \infty} |\xi(u_n, v_n)| = 1,
\]
which contradicts with (4.3). Therefore, we have proved that \( \{ y_n \} \) is bounded in \( \mathbb{R}^N \) and then (4.5)(b) holds true.

Recalling that \( V_1(x), V_2(x) \) are nonnegative and \( \lambda_1, \lambda_2 \geq 0 \) with \( \lambda = \max \{ \lambda_1, \lambda_2 \} > 0 \), by (4.4) and (4.5) we get

\[
\begin{align*}
4c_\infty &= \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)dx + \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|u_n|^2dx + \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|v_n|^2dx \right] \\
&\geq (k_1 + k_2) \int_{\mathbb{R}^N} \nabla |U_{\delta_0,y_0}|^2dx + \int_{\mathbb{R}^N} (k_1V_1(x) + k_2V_2(x))|U_{\delta_0,y_0}|^2dx \\
&\quad + \int_{B_{\delta_0}(y_0)} (k_1\lambda_1 + k_2\lambda_2)|U_{\delta_0,y_0}|^2dx \\
&> (k_1 + k_2) S_{HL}^2 = 4c_\infty,
\end{align*}
\]

which is impossible and thus the proof is completed. \( \blacksquare \)

**Lemma 4.2.** Suppose that \( \lambda_1, \lambda_2 \geq 0, \lambda = \max \{ \lambda_1, \lambda_2 \} > 0 \) and \( \beta > \max \{ \mu_1, \mu_2 \} \). Then

\[
c^{**} = \inf \{ I(u,v) \mid (u,v) \in \mathcal{N}, \xi(u,v) = 0, \gamma(u,v) \geq \frac{1}{2} \} > c_\infty. \tag{4.8}
\]

**Proof:** By an analogous argument in the proof of Lemma 4.1, we succeed in proving (4.8). Here we only give a brief proof for the reader’s convenience. Obviously, \( c^{**} \geq c_\infty \). If \( c^{**} = c_\infty \), then it follows by the Ekeland’s principle that there exists a sequence \( \{ (u_n, v_n) \} \subset \mathcal{N} \) such that, as \( n \to \infty \),

\[
\xi(u_n, v_n) \to 0, \quad \gamma(u_n, v_n) \geq \frac{1}{2} \tag{4.9}
\]

and

\[
I(u_n, v_n) \to c_\infty, \quad (I|_{\mathcal{N}})'(u_n, v_n) \to 0. \tag{4.10}
\]

Moreover, \( \{ (u_n, v_n) \} \) is a sequence of Palais-Smale sequence of \( I \) at level \( c_\infty \). By the same computations made in Lemma 4.1, we then have

\[
(u_n, v_n) = (\sqrt{k_1}U_{\delta_n,y_n}, \sqrt{k_2}U_{\delta_n,y_n}) + (\tilde{\varphi}_n, \tilde{\psi}_n)
\]

where \( \delta_n > 0, y_n \in \mathbb{R}^N \) and \( (\tilde{\varphi}_n, \tilde{\psi}_n) \to 0 \) in \( H_0 \).

We claim that \( \{ \delta_n \} \) is bounded. Otherwise, \( \delta_n \to \infty \). Then

\[
c_\infty = \lim_{n \to \infty} I(u_n) - \frac{1}{4} \langle (I'(u_n, v_n)), (u_n, v_n) \rangle \\
= \lim_{n \to \infty} \frac{1}{4} \left[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)dx + \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)|u_n|^2dx \\
+ \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)|v_n|^2dx \right] \\
\geq \lim_{n \to \infty} \frac{1}{4} \left[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)dx + \lambda_1 \int_{B_{\delta_n}(y_n)} |u_n|^2dx + \lambda_2 \int_{B_{\delta_n}(y_n)} |v_n|^2dx \right] \\
\geq \frac{k_1 + k_2}{4} S_{HL}^2 + \lim_{n \to \infty} \frac{1}{4} \left( \lambda_1 \int_{B_{\delta_n}(y_n)} |u_n|^2dx + \lambda_2 \int_{B_{\delta_n}(y_n)} |v_n|^2dx \right) \\
\geq \frac{k_1 + k_2}{4} S_{HL}^2 + \frac{\lambda_1 k_1}{4} \delta_n^2 \int_{B_{\delta_n}(y_n)} |U_{1,0}|^2dx + \frac{\lambda_2 k_2}{4} \delta_n^2 \int_{B_{\delta_n}(y_n)} |U_{1,0}|^2dx + o(\delta_n^2) \to \infty,
\]

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which leads to a contradiction. Hence, \(\{\delta_n\}\) is bounded, and up to a subsequence, we may assume that
\[
\lim_{n \to \infty} \delta_n = \delta^*. \]
Working again as the proof of Lemma 4.1 we can also prove that \(\delta^* > 0\) and bounded sequence \(\{y_n\}\) satisfies \(y_n \to y^*\). Thus,
\[
c_\infty = \lim_{n \to \infty} \frac{1}{4} \left[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)dx + \int_{\mathbb{R}^N} (V_1(x) + \lambda_1)u_n^2 dx + \int_{\mathbb{R}^N} (V_2(x) + \lambda_2)v_n^2 dx \right]
\geq \frac{(k_1 + k_2)}{4} S_{HL}^2 + \frac{\lambda_1 k_1}{4} \int_{B_{\gamma s}(y')} |U_{\beta, yr}|^2 dx + \frac{\lambda_2 k_2}{4} \int_{B_{\gamma s}(y')} |U_{\beta, yr}|^2 dx
> \frac{k_1 + k_2}{4} S_{HL}^2 = c_\infty.
\]
Obviously, this is impossible. Therefore, \(c^{**} = c_\infty\) cannot occur and then we complete the whole proof. 

Note that \(k_1 = \frac{\beta - \mu_2}{\beta - \mu_1} \cdot \frac{1}{\beta - \mu_1} \mu_2 \), \(k_2 = \frac{\beta - \mu_1}{\beta - \mu_2} \cdot \frac{1}{\beta - \mu_2} \mu_1 \) and \(\beta > \max\{\mu_1, \mu_2\} \). From assumption \((A_3)\), we have
\[
0 < \frac{\beta - \mu_2}{2\beta - \mu_1} C(N, 4)^{-\frac{1}{2}} \|V_1\|_{L^{2N}} + \frac{\beta - \mu_1}{2\beta - \mu_2} C(N, 4)^{-\frac{1}{2}} \|V_2\|_{L^{2N}}
< \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} S_{HL} - S_{HL}.
\]
Let us consider
\[
f(t) = 2^{-\frac{1-a}{2}} \min \left\{ \frac{\beta - \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta - \mu_1}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} S_{HL} - S_{HL}.
\]
It is not difficult to prove that \(f(t)\) is continuous and increasing in \([0, 1]\). Moreover, \(f(0) \leq 0\) and \(f(1) > 0\).

Let us choose a constant \(a \in (0, 1)\) such that \(f(a) > 0\), and
\[
0 < \frac{\beta - \mu_2}{2\beta - \mu_1} C(N, 4)^{-\frac{1}{2}} \|V_1\|_{L^{2N}} + \frac{\beta - \mu_1}{2\beta - \mu_2} C(N, 4)^{-\frac{1}{2}} \|V_2\|_{L^{2N}}
= 2^{-\frac{1-a}{2}} \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} S_{HL} - S_{HL}.
\tag{4.11}
\]
Next, we fix a constant \(\tilde{c}\) such that
\[
c_\infty < \tilde{c} < \min \left\{ \frac{c^* + c_\infty}{2}, 2^{1-a} c_\infty \right\}.
\tag{4.12}
\]
We remark that, as \(f(a) > 0\), then
\[
2^{-\frac{1-a}{2}} \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} S_{HL} - S_{HL} > 0
\Rightarrow 2^{-\frac{1-a}{2}} \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} > 1
\Rightarrow \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} > 2^{1-a}
\Rightarrow \min \left\{ \frac{\beta^2 - \mu_1 \mu_2}{\mu_1 (2\beta - \mu_1 - \mu_2)}, \frac{\beta^2 - \mu_1 \mu_2}{\mu_2 (2\beta - \mu_1 - \mu_2)} \right\} > 2^{1-a}
\Rightarrow \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\} \geq 2^{1-a} c_\infty.
\]
In the following, $\vartheta(x)$ is a function that belongs to $C_0^\infty(B_1(0))$ and satisfies the properties below:

\[
\begin{align*}
(\text{i}) \quad & \vartheta(x) \in C_0^\infty(B_1(0)), \quad \vartheta(x) \geq 0, \forall x \in B_1(0); \\
(\text{ii}) \quad & \vartheta(x) \text{ is symmetric and } |x_1| \leq |x_2| \Rightarrow \vartheta(x_1) > \vartheta(x_2); \\
(\text{iii}) \quad & \int_{\mathbb{R}^N} |\nabla \vartheta|^2 dx = \int_{\mathbb{R}^N} (|x|^{-\frac{N+2}{2}} * |\vartheta|^2)|x|^2 dx > S_{HL}^2; \\
(\text{iv}) \quad & (\sqrt{k_1} \vartheta, \sqrt{k_2} \vartheta) \in \mathcal{N}_\infty, \quad I_\infty(\sqrt{k_1} \vartheta, \sqrt{k_2} \vartheta) = \Sigma \in (c_\infty, \bar{c}).
\end{align*}
\]  

Moreover, for each $\delta > 0$ and $y \in \mathbb{R}^N$, we set

\[
\vartheta_{\delta,y} = \begin{cases} 
\delta^{-\frac{N+2}{2}} \vartheta\left(\frac{x-y}{\delta}\right), & x \in B_\delta(y); \\
0, & x \notin B_\delta(y).
\end{cases}
\]

We remark that by the definition of $\vartheta_{\delta,y}$ and by variable change, it follows that for each $\delta > 0$ and $y \in \mathbb{R}^N$,

\[
\|\vartheta\|_{L^{2^*}} = \|\vartheta\|_{L^{2^*}(B_1(0))} = \|\vartheta_{\delta,y}\|_{L^{2^*}(\mathbb{R}^N)} = \|\vartheta_{\delta,y}\|_{L^{2^*}}.
\]  

**Lemma 4.3.** Suppose that $(A_1)$-$(A_2)$ hold, then

(a) \( \lim_{\delta \to 0} \sup \int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx : y \in \mathbb{R}^N \) = 0, \( j = 1, 2; \)

(b) \( \lim_{\delta \to \infty} \sup \int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx : y \in \mathbb{R}^N \) = 0, \( j = 1, 2; \)

(c) \( \lim_{\rho \to \infty} \sup \int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx : |y| = r, y \in \mathbb{R}^N, \delta > 0 \) = 0, \( j = 1, 2. \)

**Proof:** Let $y \in \mathbb{R}^N$ be chosen arbitrarily, then for each $\delta > 0$, by Hölder inequality we have

\[
\int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx = \int_{B_\delta(y)} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx \\
\leq \|V_j\|_{L^{\frac{N}{N-2}}(B_\delta(y))} \|\vartheta\|_{L^{2^*}(B_1(0))}^2 \leq C \|V_j\|_{L^{\frac{N}{N-2}}(B_\delta(y))},
\]

where $C$ is a positive constant independent of $\delta$. Then,

\[
\sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}(x)|^2 dx \leq C \sup_{y \in \mathbb{R}^N} \|V_j\|_{L^{\frac{N}{N-2}}(B_\delta(y))}.
\]  

Since

\[
\lim_{\delta \to 0} \|V_j\|_{L^{\frac{N}{N-2}}(B_\delta(y))} = 0 \text{ uniformly in } y \in \mathbb{R}^N,
\]

then Lemma 4.3(a) follows from (1.15).

To prove (b), we fix arbitrarily $y \in \mathbb{R}^N$ and note for each $\rho > 0$ and $\delta > 0$,

\[
\int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta,y}|^2 dx = \int_{B_\rho(0)} V_j(x)|\vartheta_{\delta,y}|^2 dx + \int_{\mathbb{R}^N \setminus B_\rho(0)} V_j(x)|\vartheta_{\delta,y}|^2 dx \\
\leq \|V_j\|_{L^{\frac{N}{N-2}}(B_\rho(0))} \|\vartheta_{\delta,y}\|_{L^{2^*}(B_\rho(0))}^2 + \|V_j\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N \setminus B_\rho(0))} \|\vartheta_{\delta,y}\|_{L^{2^*}(\mathbb{R}^N \setminus B_\rho(0))}^2 \\
\leq \|V_j\|_{L^{\frac{N}{N-2}}(B_\rho(0))} \sup_{y \in \mathbb{R}^N} \|\vartheta_{\delta,y}\|_{L^{2^*}(B_\rho(0))}^2 + \bar{C} \|V_j\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N \setminus B_\rho(0))},
\]

in which constant $\bar{C}$ is independent of $\delta$ and $\rho$. By the fact that

\[
\lim_{\delta \to \infty} \|\vartheta_{\delta,y}\|_{L^{2^*}(B_\rho(0))} = 0, \text{ uniformly in } y \in \mathbb{R}^N,
\]
Lemma 4.4. Let \( c \) which contradicts with \( (4.17) \). So Lemma 4.3 \((4.18)\) and Lemma 4.3 \((4.19)\), we conclude that \( \lim_{n \to \infty} \delta_n = \tilde{\delta} > 0 \). By \( (4.16) \) and the assumption that \( V_j(x) \in L^\infty(\mathbb{R}^N) \), we have

\[
\lim_{n \to \infty} \|V_j\|_{L^\infty(B_{\delta_n}(y_n))} = 0.
\]

Furthermore, by Hölder inequality we get

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V_j(x)|\vartheta_{\delta_n,y_n}|^2 \, dx \leq \lim_{n \to \infty} \left[ \|V_j\|_{L^\infty(B_{\delta_n}(y_n))} \cdot \left\|\vartheta_{\delta_n,y_n}\right\|_{L^{2^*}(B_{\delta_n}(y_n))}^2 \right] = 0,
\]

which contradicts with \( (4.17) \). So Lemma 4.3 \((c)\) holds true.

Lemma 4.4. Let \( \langle x|y \rangle_{\mathbb{R}^N} \) be the inner product of vector \( x,y \in \mathbb{R}^N \). For fixed \( r > 0 \), the following relations hold:

\( a \) \( \lim_{\delta \to 0} \{ \gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) : y \in \mathbb{R}^N \} = 0; \)

\( b \) \( \lim_{\delta \to \infty} \{ \gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) : |y| \leq r \} = 1. \)

\( c \) \( \langle \xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) | y \rangle_{\mathbb{R}^N} > 0, \forall y \in \mathbb{R}^N \setminus \{0\}, \forall \delta > 0. \)

Proof: \( a \) For any \( \delta > 0 \) and \( y \in \mathbb{R}^N \), we get

\[
\left| \frac{y}{1+|y|} - \xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) \right| \leq \frac{\int_{\mathbb{R}^N} \left| \frac{y}{1+|y|} - \frac{x}{1+|x|} \right| |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} \leq 2\delta,
\]

where in the last inequality we use the property that \( \left| \frac{y}{1+|y|} - \frac{x}{1+|x|} \right| < 2\delta \) for any \( x \in B_{\delta}(y) \). Then

\[
0 \leq \gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})
= \frac{\int_{\mathbb{R}^N} \left| \frac{y}{1+|y|} - \xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) \right| |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx}
\leq \frac{\int_{\mathbb{R}^N} \left| \frac{y}{1+|y|} - \frac{x}{1+|x|} \right| |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} + \frac{1}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} \leq 4\delta,
\]

which implies that \( \lim_{\delta \to 0} \gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) = 0 \) uniformly in \( y \in \mathbb{R}^N \).

\( b \) Let us first show that

\[
\lim_{\delta \to \infty} \xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) = 0,
\]

(4.18)
uniformly for \( y \in B_r(0) \). In fact, since \( \vartheta_{\delta,0} \) is radially symmetric, then

\[
\int_{\mathbb{R}^N} \frac{x}{1 + |x|} \frac{\vartheta_{\delta,0}}{|\vartheta_{\delta,0}|^2} \, dx = 0.
\]

So, by calculations we get

\[
|\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})| = \left| \int_{\mathbb{R}^N} \frac{x}{1 + |x|} (\mu_1 k_1^{\frac{2}{k_1}} + 2\beta k_1^{\frac{2}{k_1}} k_2 + \mu_2 k_2^{\frac{2}{k_2}}) |\vartheta_{\delta,y}|^2 \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^N} \frac{x}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^N} \frac{x}{1 + |x|} (|\vartheta_{\delta,y}|^2 - |\vartheta_{\delta,0}|^2) \, dx \right|
\]

\[
\leq \frac{1}{\mathbb{R}^N |\vartheta_{\delta,y}|^2} \int_{\mathbb{R}^N} |\vartheta_{\delta,y}^2 - \vartheta_{\delta,0}^2| \, dx \to 0,
\]

as \( \delta \to \infty \), uniformly for \( y \in B_r(0) \). So, (4.18) holds true.

Next, for any \( \varepsilon > 0 \), we fix a constant \( \rho = \rho(\varepsilon) > 0 \) such that \( \frac{1}{1 + \rho} < \frac{\varepsilon}{3} \). Then for such a fixed constant \( \rho > 0 \), it is easy to prove that

\[
\lim_{\delta \to \infty} \int_{B_r(0)} |\vartheta_{\delta,y}|^2 \, dx = 0
\]

uniformly in \( y \in B_r(0) \). Thus, there exists a constant \( \delta_0 > 0 \) such that for each \( \delta > \delta_0 \),

\[
\frac{1}{\|\vartheta\|^2} \int_{B_r(0)} |\vartheta_{\delta,y}|^2 \, dx < \frac{\varepsilon}{3}
\]

uniformly in \( y \in B_r(0) \). For fixed constant \( \delta_0 \), considering (4.18), we can also deduce that for each \( \delta \in (\delta_0, \infty) \) and \( y \in B_r(0) \), there holds

\[
|\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})| < \frac{\varepsilon}{3}.
\]

By the definition of function \( \gamma \), we have

\[
\gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) \leq 1 + |\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})| \leq 1 + \frac{\varepsilon}{3}.
\]

On the other hand, for the same assumptions on \( \delta \) and \( y \), that is \( \delta \in (\delta_0, \infty) \) and \( y \in B_r(0) \), we get

\[
\gamma(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) = \frac{\int_{\mathbb{R}^N} \left| \frac{x}{1 + |x|} - \xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}}) \right| |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx}
\]

\[
\geq \frac{\int_{\mathbb{R}^N} \frac{|x|}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} - |\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})|
\]

\[
\geq \frac{\int_{\mathbb{R}^N \setminus B_r(0)} \frac{|x|}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} - |\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})|
\]

\[
\geq \rho \frac{1}{1 + \rho} - \frac{\int_{B_r(0)} |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} |\vartheta_{\delta,y}|^2 \, dx} - |\xi(\sqrt{k_1\vartheta_{\delta,y}}, \sqrt{k_2\vartheta_{\delta,y}})|
\]

\[
\geq 1 - \frac{1}{1 + \rho} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \geq 1 - \varepsilon.
\]
Finally, based on the arguments above, we can easily get

\[ \lim_{\delta \to \infty} \gamma(\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}}) = 1 \]

uniformly for \( y \in B_r(0) \), and the proof of (b) is completed.

(c) For any \( x \in \mathbb{R}^n \) with \( \langle x \mid y \rangle_{\mathbb{R}^n} > 0 \), there holds \( -x - y > |x - y| \), which together with \( [4.13] (ii) \) implies that \( \vartheta_{\delta,y}(x) \geq \vartheta_{\delta,y}(-x) \) for any \( x \in \mathbb{R}^n \) with \( \langle x \mid y \rangle_{\mathbb{R}^n} > 0 \). Moreover, meas \( \{ x \in \mathbb{R}^n : \langle x \mid y \rangle_{\mathbb{R}^n} > 0, \vartheta_{\delta,y}(x) > \vartheta_{\delta,y}(-x) \} > 0 \). Then by computation we get for \( \delta > 0 \) and any \( y \in \mathbb{R}^n \setminus \{0\} \),

\[
\int_{\mathbb{R}^n} \frac{\langle x \mid y \rangle_{\mathbb{R}^n}}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx = \int_{\{ x \in \mathbb{R}^n : \langle x \mid y \rangle_{\mathbb{R}^n} > 0 \}} \frac{\langle x \mid y \rangle_{\mathbb{R}^n}}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx + \int_{\{ x \in \mathbb{R}^n : \langle x \mid y \rangle_{\mathbb{R}^n} < 0 \}} \frac{\langle x \mid y \rangle_{\mathbb{R}^n}}{1 + |x|} |\vartheta_{\delta,y}|^2 \, dx
\]

which implies that

\[
\left( \xi(\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}}) \mid y \right)_{\mathbb{R}^n} = \frac{\int_{\mathbb{R}^n} \langle x \mid y \rangle_{\mathbb{R}^n} \left( \mu_1 k_1^2 + 2\beta k_1 k_2 + 2\beta k_2^2 \right) |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^n} (\mu_1 k_1^2 + 2\beta k_1 k_2 + 2\beta k_2^2) |\vartheta_{\delta,y}|^2 \, dx} > 0
\]

for any \( y \in \mathbb{R}^n \setminus \{0\} \) and \( \delta > 0 \).

\[ \blacktriangleleft \]

5 Proof of main results

We first introduce the notation \( I_0 \) to define the functional \( I \) with \( \lambda_1 = \lambda_2 = 0 \), that is

\[
I_0(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} \left( |\nabla u|^2 + |\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2 \right) \, dx - \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\mu_1 |u^+(x)|^2 |u^+(y)|^2 + 2\beta |u^+(x)|^2 |v^+(y)|^2 + \mu_2 |v^+(x)|^2 |v^+(y)|^2}{|x - y|^4} \, dx \, dy.
\]

According to the definition above, we set Nehari manifold

\[
N_0 := \{ (u, v) \in H_0 \setminus \{(0, 0)\} : (I_0'(u, v), (u, v)) = 0 \}.
\]

For each \( \vartheta_{\delta,y} \), we set

\[
\vartheta_{\delta,y} := (\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}})
\]

and the projections

\[
\hat{\vartheta}_{\delta,y} := t_{\delta,y} \vartheta_{\delta,y} \in N, \quad \hat{\vartheta}_{\delta,y} := t_{\delta,y} \vartheta_{\delta,y} \in N_0, \quad t_{\delta,y}, t_{\delta,y,0} > 0. \tag{5.1}
\]

Let us define

\[
\xi \circ \vartheta_{\delta,y} := \xi(\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}}), \quad \gamma \circ \vartheta_{\delta,y} := \gamma(\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}}),
\]

and in analogous way \( \xi \circ \hat{\vartheta}_{\delta,y}, \xi \circ \hat{\vartheta}_{\delta,y}, \gamma \circ \hat{\vartheta}_{\delta,y}, \gamma \circ \hat{\vartheta}_{\delta,y} \).

Observe that, by \( [4.1] \),

\[
\xi \circ \vartheta_{\delta,y} = \xi \circ \hat{\vartheta}_{\delta,y} = \xi \circ \hat{\vartheta}_{\delta,y}, \quad \gamma \circ \vartheta_{\delta,y} = \gamma \circ \hat{\vartheta}_{\delta,y} = \gamma \circ \hat{\vartheta}_{\delta,y} \tag{5.2}
\]

[35]
Lemma 5.1. Let $t_{\delta,y,0}$ be as in (5.1), then the following relations hold:

(a) $\lim_{\delta \to 0} \sup \{|t_{\delta,y,0} - 1| : y \in \mathbb{R}^N\} = 0$;
(b) $\lim_{\delta \to \infty} \sup \{|t_{\delta,y,0} - 1| : y \in \mathbb{R}^N\} = 0$;
(c) $\lim_{\delta \to \infty} \sup \{|t_{\delta,y,0} - 1| : y \in \mathbb{R}^N, |y| = r, \delta > 0\} = 0$.

Proof: By (4.13) and the definition of $\vartheta_{\delta,y}$, we get $(\sqrt{k_1}\vartheta_{\delta,y}, \sqrt{k_2}\vartheta_{\delta,y}) \in \mathcal{N}_\infty$. Then by calculation we have

\[
1 = \frac{\int_{\mathbb{R}^N} (|\nabla(\sqrt{k_1}\vartheta_{\delta,y})|^2 + |\nabla(\sqrt{k_2}\vartheta_{\delta,y})|^2) \, dx}{\int_{\mathbb{R}^N} (\mu_1 k_1^2 + \mu_2 k_2^2 + 2\beta k_1 k_2)(|x|^{-4} * |\vartheta_{\delta,y}|^2)|\vartheta_{\delta,y}|^2 \, dx} - \frac{\int_{\mathbb{R}^N} (k_1 V_1(x) + k_2 V_2(x)) |\vartheta_{\delta,y}|^2 \, dx}{\int_{\mathbb{R}^N} (\mu_1 k_1^2 + \mu_2 k_2^2 + 2\beta k_1 k_2)(|x|^{-4} * |\vartheta_{\delta,y}|^2)|\vartheta_{\delta,y}|^2 \, dx}.
\]

Obviously, the properties (a)-(c) hold true by Lemma 4.3.

Lemma 5.2. If (A_1) and (A_2) hold, then there exist constants $\bar{r} > 0$ and $0 < \delta_1 < \frac{1}{2} < \delta_2$ such that

\[
\gamma \circ \tilde{\vartheta}_{\delta_1,y} < \frac{1}{2}, \quad \forall y \in \mathbb{R}^N, \tag{5.3}
\]

and

\[
\gamma \circ \tilde{\vartheta}_{\delta_2,y} > \frac{1}{2}, \quad \forall y \in \mathbb{R}^N, \quad |y| < \bar{r} \tag{5.4}
\]

where $\bar{c}$ is defined in (4.12) and

\[\mathcal{H} := \{ (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N : \delta \in [\delta_1, \delta_2], |y| < \bar{r} \}. \tag{5.5}\]

Proof: For every $\delta > 0$ and $y \in \mathbb{R}^N$, by calculation we have

\[
I_0 \circ \tilde{\vartheta}_{\delta,y} = I_0(t_{\delta,y,0} \sqrt{k_1}\vartheta_{\delta,y}, t_{\delta,y,0} \sqrt{k_2}\vartheta_{\delta,y})
\]

\[
= (k_1 + k_2) t_{\delta,y,0}^2 \int_{\mathbb{R}^N} |\nabla \tilde{\vartheta}_{\delta,y}|^2 \, dx + t_{\delta,y,0}^2 \int_{\mathbb{R}^N} (k_1 V_1(x) + k_2 V_2(x)) |\vartheta_{\delta,y}|^2 \, dx
\]

\[
- t_{\delta,y,0}^2 \int_{\mathbb{R}^N} (\mu_1 k_1^2 + 2\beta k_1 k_2 + \mu_2 k_2^2)(|x|^{-4} * |\vartheta_{\delta,y}|^2)|\vartheta_{\delta,y}|^2 \, dx
\]

\[
= (k_1 + k_2) t_{\delta,y,0}^2 \int_{\mathbb{R}^N} |\nabla \tilde{\vartheta}_{\delta,y}|^2 \, dx + t_{\delta,y,0}^2 \int_{\mathbb{R}^N} (k_1 V_1(x) + k_2 V_2(x)) |\vartheta_{\delta,y}|^2 \, dx
\]

\[
- (k_1 + k_2) t_{\delta,y,0}^2 \int_{\mathbb{R}^N} (|x|^{-4} * |\vartheta_{\delta,y}|^2)|\vartheta_{\delta,y}|^2 \, dx.
\]

By (5.2), Lemma 4.3(a), Lemma 4.3(a), Lemma 5.1(a) and (4.13), we prove that there exists $\delta_1 \in (0, \frac{1}{2})$ such that $\gamma \circ \tilde{\vartheta}_{\delta_1,y} < \frac{1}{2}$ and $I_0 \circ \tilde{\vartheta}_{\delta_1,y} < \bar{c}$ for every $y \in \mathbb{R}^N$.

Moreover, by Lemma 4.3(c), Lemma 5.1(c) and (1.13) again, we can choose $\bar{r} > 0$ such that, if $|y| = \bar{r}$, then $I_0 \circ \tilde{\vartheta}_{\bar{r},y} < \bar{c}$ is satisfied for every $\delta > 0$.

In the end, once $\bar{r}$ is fixed, then the existence of $\delta_2 > \frac{1}{2}$ such that $\gamma \circ \tilde{\vartheta}_{\delta_2,y} > \frac{1}{2}$ and $I_0 \circ \tilde{\vartheta}_{\delta_2,y} < \bar{c}$ for all $y \in \mathbb{R}^N$ with $|y| \leq \bar{r}$, follows from (5.2), Lemma 4.3(b), Lemma 4.4(b), Lemma 5.1(b) and (4.13).
Lemma 5.3. Let $\delta_1, \delta_2, \bar{r}$ and $\mathcal{H}$ be defined as Lemma 5.2. Then there exist $(\tilde{\delta}, \tilde{y}) \in \partial \mathcal{H}$ and $(\delta, y) \in \mathcal{H}$ such that
\[
\xi(\sqrt{k_1 \delta_{\tilde{\delta}, \tilde{y}}}, \sqrt{k_2 \delta_{\tilde{\delta}, \tilde{y}}}) = 0, \quad \gamma(\sqrt{k_1 \delta_{\tilde{\delta}, \tilde{y}}}, \sqrt{k_2 \delta_{\tilde{\delta}, \tilde{y}}}) > \frac{1}{2}
\] (5.6)
and
\[
\xi(\sqrt{k_1 \delta_{\tilde{\delta}, \tilde{y}}}, \sqrt{k_2 \delta_{\tilde{\delta}, \tilde{y}}}) = 0, \quad \gamma(\sqrt{k_1 \delta_{\tilde{\delta}, \tilde{y}}}, \sqrt{k_2 \delta_{\tilde{\delta}, \tilde{y}}}) = \frac{1}{2}
\] (5.7)

Proof: By the symmetric property of $\partial_{\delta_2,0}$, we have $\xi(\sqrt{k_1 \partial_{\delta_2,0}}, \sqrt{k_2 \partial_{\delta_2,0}}) = 0$. Moreover, by Lemma 5.2 together with (5.2), we get $\gamma(\sqrt{k_1 \partial_{\delta_2,0}}, \sqrt{k_2 \partial_{\delta_2,0}}) > \frac{1}{2}$. Hence, (5.6) is trivial if we choose $(\tilde{\delta}, \tilde{y}) = (\delta_2, 0)$.

For any $(\delta, y) \in \mathcal{H}$, we set
\[
\Theta(\delta, y) = (\gamma(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}), \xi(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}))
\]
and denote the homotopy map $T : [0, 1] \times \partial \mathcal{H} \to \mathbb{R} \times \mathbb{R}^N$ as following
\[
T(\delta, y, s) = (1 - s)(\delta, y) + s\Theta(\delta, y).
\] (5.8)

To prove (5.7), it is enough to prove that
\[
deg(\Theta, \mathcal{H}, (\frac{1}{2}, 0)) = 1.
\] (5.9)

We remark that $deg(Id, \mathcal{H}, (\frac{1}{2}, 0)) = 1$, if we can prove that for each $(\delta, y) \in \partial \mathcal{H}$ and $s \in [0, 1]$, $T(\delta, y, s) \neq (\frac{1}{2}, 0)$, then (5.9) follows easily from the topological degree homotopy invariance.

Next, we are going to verify that for any $(\delta, y) \in \partial \mathcal{H}$ and $s \in [0, 1]$,
\[
((1 - s)\delta + s\gamma(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}), (1 - s)y + s\xi(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}})) \neq (\frac{1}{2}, 0).
\] (5.10)

Let $\partial \mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ with
\[
\mathcal{H}_1 = \{(\delta, y) \in \partial \mathcal{H} : |y| \leq \bar{r}, \delta = \delta_1\},
\mathcal{H}_2 = \{(\delta, y) \in \partial \mathcal{H} : |y| \leq \bar{r}, \delta = \delta_2\},
\mathcal{H}_3 = \{(\delta, y) \in \partial \mathcal{H} : |y| = \bar{r}, \delta \in [\delta_1, \delta_2]\}.
\]

If $(\delta, y) \in \mathcal{H}_1$, then by (5.2) - (5.3) we get
\[
(1 - s)\delta_1 + s\gamma(\sqrt{k_1 \partial_{\delta_1,y}}, \sqrt{k_2 \partial_{\delta_1,y}}) < \frac{1}{2}(1 - s) + \frac{s}{2} < \frac{1}{2}.
\]
If $(\delta, y) \in \mathcal{H}_2$, then by (5.2) and (5.3),
\[
(1 - s)\delta_2 + s\gamma(\sqrt{k_1 \partial_{\delta_2,y}}, \sqrt{k_2 \partial_{\delta_2,y}}) > \frac{1}{2}(1 - s) + \frac{s}{2} > \frac{1}{2}.
\]
If $(\delta, z) \in \mathcal{H}_3$, then from Lemma 4.1(c) we observe that
\[
((1 - s)y + s\xi(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}) | y)_{\mathbb{R}^N} = (1 - s)|y|^2 + s(\xi(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}) | y)_{\mathbb{R}^N} > 0,
\]
which implies that
\[
(1 - s)y + s\xi(\sqrt{k_1 \partial_{\delta,y}}, \sqrt{k_2 \partial_{\delta,y}}) \neq 0.
\]
Lemma 5.4. Let $\delta_1, \delta_2, r$ and $\mathcal{H}$ be defined as Lemma 4. Assume that $(A_3)$ holds, then

$$
\mathcal{K} = \sup\{I_0 \circ \hat{\vartheta}_{\delta,y} : (\delta, y) \in \mathcal{H}\} < \min\left\{ \frac{S_{HH}}{4\mu_1}, \frac{S_{HL}}{4\mu_2}, 2c_\infty \right\}. \tag{5.11}
$$

**Proof:** For each $(\delta, y) \in \mathcal{H}$, by the definition of $\hat{\vartheta}_{\delta,y}$ and (4.13) we get

$$
I_0 \circ \hat{\vartheta}_{\delta,y} = \frac{t_{\delta,y,0}}{4}(k_1 + k_2) \int_{\mathbb{R}^N} \left| \nabla \vartheta_{\delta,y} \right|^2 dx + \frac{t_{\delta,y,0}^2}{4} \int_{\mathbb{R}^N} (k_1V_1(x) + k_2V_2(x)) \left| \vartheta_{\delta,y} \right|^2 dx
$$

$$
\leq \frac{t_{\delta,y,0}}{4}(k_1 + k_2) \left| \vartheta_{\delta,y} \right|^2_{D^1,2} + \frac{t_{\delta,y,0}^2}{4} \left( k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}} \right) \left| \vartheta_{\delta,y} \right|_{L^{\infty}}^2
$$

$$
\leq \frac{t_{\delta,y,0}}{4}(k_1 + k_2) \left| \vartheta_{\delta,y} \right|^2_{D^1,2} + \frac{t_{\delta,y,0}^2}{4} \left( k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}} \right) \left| \vartheta_{\delta,y} \right|_{L^{\infty}}^2
$$

$$
= t_{\delta,y,0}^2 \left( 1 + \frac{k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}}}{(k_1 + k_2)S} \right) \Sigma. \tag{5.12}
$$

Notice that $(t_{\delta,y,0}^2 \sqrt{k_1 \vartheta_{\delta,y}}, t_{\delta,y,0}^2 \sqrt{k_2 \vartheta_{\delta,y}}) \in N_0$, then

$$
t_{\delta,y,0}^2 = \frac{(k_1 + k_2) \int_{\mathbb{R}^N} \left| \nabla \vartheta_{\delta,y} \right|^2 dx + \int_{\mathbb{R}^N} (k_1V_1(x) + k_2V_2(x)) \left| \vartheta_{\delta,y} \right|^2 dx}{(k_1 + k_2) \int_{\mathbb{R}^N} (|x|^{-4} + \left| \vartheta_{\delta,y} \right|^2) \left| \vartheta_{\delta,y} \right|^2 dx}.
$$

Since $(\sqrt{k_1 \vartheta_{\delta,y}}, \sqrt{k_2 \vartheta_{\delta,y}}) \in N_\infty$, then

$$
t_{\delta,y,0}^2 = \frac{(k_1 + k_2) \int_{\mathbb{R}^N} \left| \nabla \vartheta_{\delta,y} \right|^2 dx + \int_{\mathbb{R}^N} (k_1V_1(x) + k_2V_2(x)) \left| \vartheta_{\delta,y} \right|^2 dx}{(k_1 + k_2) \int_{\mathbb{R}^N} (|x|^{-4} + \left| \vartheta_{\delta,y} \right|^2) \left| \vartheta_{\delta,y} \right|^2 dx}
$$

$$
\leq 1 + \frac{k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}} + \left| \vartheta_{\delta,y} \right|_{L^{\infty}}^2}{(k_1 + k_2)S} \Sigma \tag{5.13}
$$

Let us insert (5.13) into (5.12), then by (4.11) and (4.13) we get

$$
I_0 \circ \hat{\vartheta}_{\delta,y} \leq \left( 1 + \frac{k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}}}{(k_1 + k_2)S} \right)^2 \Sigma
$$

$$
< \left( 1 + \frac{k_1 \left| V_1 \right|_{L^{\frac{N}{2}}} + k_2 \left| V_2 \right|_{L^{\frac{N}{2}}}}{(k_1 + k_2)S} \right)^2 \bar{c}
$$

$$
= \left( 1 + \frac{\beta - \mu_2}{(2\beta - \mu_1 - \mu_2)S} \left| V_1 \right|_{L^{\frac{N}{2}}} + \frac{\beta - \mu_1}{(2\beta - \mu_1 - \mu_2)S} \left| V_2 \right|_{L^{\frac{N}{2}}} \right)^2 \bar{c}
$$

$$
\leq \min\left\{ \frac{S_{HL}}{4\mu_1}, \frac{S_{HL}}{4\mu_2}, 2c_\infty \right\}.
$$
Lemma 5.5. Let $\delta_1, \delta_2, r, \mathcal{H}$ as in Lemma 5.2, then there exists a number $\lambda^* > 0$ such that for each $\lambda := \max\{\lambda_1, \lambda_2\} \in (0, \lambda^*)$, the following relations hold:

\[
\gamma \circ \tilde{\delta}_{\delta, y} < \frac{1}{2}, \quad \forall y \in \mathbb{R}^N, \tag{5.14}
\]

\[
\gamma \circ \tilde{\delta}_{\delta, y} > \frac{1}{2}, \quad \forall y \in \mathbb{R}^N, \quad |y| < r \tag{5.15}
\]

and

\[
\tilde{K} := \sup\{I \circ \tilde{\delta}_{\delta, y} : (\delta, y) \in \partial \mathcal{H} \} < \bar{c}. \tag{5.16}
\]

Furthermore, if (A3) holds, then $\lambda^*$ can be found such that for each $\lambda := \max\{\lambda_1, \lambda_2\} \in (0, \lambda^*)$, in addition to (5.14)-(5.16),

\[
\tilde{s} := \sup\{I \circ \tilde{\delta}_{\delta, y} : (\delta, y) \in \mathcal{H} \} < \min \left\{ \frac{\mathcal{S}^2_{HL}}{4\mu_1}, \frac{\mathcal{S}^2_{HL}}{4\mu_2}, 2c_\infty \right\} \tag{5.17}
\]

is also satisfied.

**Proof:** From (5.2), we know that

\[
\gamma \circ \tilde{\delta}_{\delta, y} = \gamma \circ \tilde{\delta}_{\delta, y}, \quad \forall (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N.
\]

Then (5.14) and (5.16) can be seen as a direct consequence of Lemma 5.2. Recalling that $(\sqrt{k_1} \partial_{\delta, y}, \sqrt{k_2} \partial_{\delta, y}) \in \mathcal{N}_\infty$ and $\tilde{\delta}_{\delta, y} := (t_{\delta, y} \sqrt{k_1} \partial_{\delta, y}, t_{\delta, y} \sqrt{k_2} \partial_{\delta, y}) \in \mathcal{N}$, then by computation we get

\[
1 = \frac{(k_1 + k_2) \int_{\mathbb{R}^N} |\nabla \partial_{\delta, y}|^2 \, dx}{(\mu_1 k_1^2 + 2\beta k_1 k_2 + \mu_2 k_2^2) \int_{\mathbb{R}^N} (|x|^{-4} * |\partial_{\delta, y}|^2) |\partial_{\delta, y}|^2 \, dx} = t_{\delta, y}^2 - \frac{k_1 \int_{\mathbb{R}^N} (V_1(x) + \lambda_1) |\partial_{\delta, y}|^2 \, dx + k_2 \int_{\mathbb{R}^N} (V_2(x) + \lambda_2) |\partial_{\delta, y}|^2 \, dx}{(k_1 + k_2) \int_{\mathbb{R}^N} (|x|^{-4} * |\partial_{\delta, y}|^2) |\partial_{\delta, y}|^2 \, dx}
\]

and

\[
\int_{\mathbb{R}^N} \lambda_j |\partial_{\delta, y}|^2 \, dx = \lambda_j \delta^2 \int_{B_1(0)} |\partial_{y, y, 0}|^2 \, dx, \tag{5.18}
\]

which implies that

\[
\lim_{\lambda \to 0} \sup_{(\delta, y) \in \mathcal{H}} |t_{\delta, y} - t_{\delta, y, 0}| = 0, \tag{5.19}
\]

where $\lambda := \max\{\lambda_1, \lambda_2\}$. Thus, if $\lambda$ is suitably small, then (5.16) and (5.17) follows straightly by (5.25), (5.11), (5.18) and (5.19).

With the help of the previous estimates, we can prove Theorem 1.1 by deformation arguments that we can use the global compactness result obtained in Section 3. Before the proof, we first introduce a notation for the sublevel sets:

\[
I^c := \{(u, v) \in \mathcal{N} : I(u, v) \leq c \}.
\]

**Proof of Theorem 1.1** In this part, we always assume that $\lambda \in (0, \lambda^*)$, where $\lambda^*$ is stated in Lemma 5.5.

To prove Theorem 1.1 we first prove that a critical level exists in $(c_\infty, \bar{c})$ under assumptions (A1) and (A2). If in additional (A3) holds and $\lambda \in (0, \lambda^{**})$, then another critical level exists in $(\bar{c}, \min\{\frac{\mathcal{S}^2_{HL}}{4\mu_1}, \frac{\mathcal{S}^2_{HL}}{4\mu_2}, 2c_\infty\})$.

**Step 1.** Let us first assume that (A1) and (A2) hold. By using (4.8), (4.11), (5.6), (5.16) and (4.12), we have

\[
c_\infty < c^{**} \leq I \circ \tilde{\delta}_{\delta, y} \leq \tilde{K} < \bar{c} < c^*.
\]

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We claim that the functional $I|_{\mathcal{N}}$ has a critical level in $(c_\infty, \bar{c})$. We argue by contradiction and suppose that it is incorrect. From Corollary 4.14, we know that $I$ constrained on $\mathcal{N}$ satisfies the Palais-Smale condition in the energy interval $(c_\infty, \bar{c})$. Then in virtue of the deformation lemma (Lemma 2.3), we can prove that there exist a constant $\sigma_1 > 0$ such that $c^{**} - \sigma_1 > c_\infty$, $\bar{K} + \sigma_1 < \bar{c}$ and a continuous function

$$\eta : [0, 1] \times I^{-} \to I^{+}$$

such that

$$\eta(0, (u, v)) = (u, v), \quad \forall (u, v) \in I^{+};$$
$$\eta(s, (u, v)) = (u, v), \quad \forall (u, v) \in I^{c^{**}-\sigma_1}, \, \forall s \in [0, 1];$$
$$I \circ \eta(s, (u, v)) \leq I(u, v), \quad \forall (u, v) \in I^{+}, \, \forall s \in [0, 1];$$
$$\eta(1, I^{+}) \subset I^{c^{**}-\sigma_1}. \tag{5.21}$$

From (5.16) and (5.22), we observe that

$$(\delta, y) \in \partial \mathcal{H} \implies I \circ \tilde{\delta}, y \leq \tilde{K} \implies I \circ \eta(1, \tilde{\delta}, y) \leq c^{**} - \sigma_1. \tag{5.23}$$

For $s \in [0, 1]$ and $\delta, y \in \mathcal{H}$, we define

$$\Gamma(\delta, y, s) = \begin{cases} T(\delta, y, 2s), & s \in [0, \frac{1}{2}] ; \\ (\gamma \circ \eta(2s - 1, \tilde{\delta}, y), \xi \circ \eta(2s - 1, \tilde{\delta}, y)), & s \in [\frac{1}{2}, 1], \end{cases}$$

where $T$ is defined as (5.8). As shown in Lemma 5.3, we have already proved

$$\forall s \in [0, \frac{1}{2}], \quad \forall (\delta, y) \in \partial \mathcal{H}, \quad \Gamma(\delta, y, s) \neq \left( \frac{1}{2}, 0 \right). \tag{5.24}$$

Moreover, by (5.14), (5.16) and (5.21), we deduce that

$$I \circ \eta(2s - 1, \tilde{\delta}, y) \leq I \circ \tilde{\delta}, y \leq \tilde{K} < \bar{c} < c^{*}, \quad \forall s \in [\frac{1}{2}, 1], \, (\delta, y) \in \partial \mathcal{H},$$

which implies that

$$\forall s \in [\frac{1}{2}, 1], \quad \forall (\delta, y) \in \partial \mathcal{H}, \quad \Gamma(\delta, y, s) \neq \left( \frac{1}{2}, 0 \right). \tag{5.25}$$

Furthermore, by (5.21), (5.24) and the continuity of $\Gamma$, we prove that there exists $(\tilde{\delta}, \tilde{y}) \in \partial \mathcal{H}$ such that

$$\xi \circ \eta(1, \tilde{\delta}, \tilde{y}) = 0, \quad \gamma \circ \eta(1, \tilde{\delta}, \tilde{y}) \geq \frac{1}{2}. \tag{5.26}$$

In view of the definition of $c^{**}$ and (5.26), we then get

$$I \circ \eta(1, \tilde{\delta}, \tilde{y}) \geq c^{**},$$

which contradicts with (5.23). Therefore, the functional $I|_{\mathcal{N}}$ has at least a critical point $(u_\ell, v_\ell) \in \mathcal{N}$ such that $I(u_\ell, v_\ell) \in (c_\infty, \bar{c})$. And $(u_\ell, v_\ell)$ is also a critical point of $I$, since $\mathcal{N}$ is a natural constraint. Moreover, $u_\ell \geq 0$ and $v_\ell \geq 0$. Note that $I(u_\ell, v_\ell) < \bar{c} < \min\left\{ \frac{S_{m1}}{4\ell_1}, \frac{S_{m2}}{4\ell_2} \right\}$, then by Corollary 2.3, we prove that $u_\ell \neq 0$ and $v_\ell \neq 0$. Since $V_j(x) \in L^q_{loc}(\mathbb{R}^N)$ with $q > \frac{N}{4}$, then $u_\ell, v_\ell$ satisfy

$$-\Delta u_\ell = a(x)u_\ell, \quad -\Delta v_\ell = b(x)v_\ell$$

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where
\[ a(x) = -(V(x) + \lambda_1) + \mu_1(|x|^{-4} \ast u^2_T) + \beta(|x|^{-4} \ast u^2_T) \in L^\infty_{\text{loc}}(\mathbb{R}^N), \]
\[ b(x) = -(V(x) + \lambda_2) + \mu_2(|x|^{-4} \ast u^2_T) + \beta(|x|^{-4} \ast u^2_T) \in L^\infty_{\text{loc}}(\mathbb{R}^N). \]

The Brézis-Kato theorem (see [9]) implies that \( u_T, v_T \in L^q_{\text{loc}}(\mathbb{R}^N) \) for all \( 1 \leq q_1 < \infty \). Thus, \( u_T, v_T \in C^{0,\alpha} \) by classical regularity results. Hence, by the Harnack inequality (see [44]), we conclude concluding the first part proof of Theorem 1.1.

Step 2. In the sequel, we assume that \( (A_3) \) holds true also. From [4.12], [4.11], [5.14] and [5.17], we get
\[ \bar{c} < c^* \leq I \circ (\bar{\gamma}_{\delta,y}) \leq \bar{s} < \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\}. \]

We are ready to show that \( I|_{\mathcal{K}} \) has a critical level in the interval \( (\bar{c}, \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\}) \). Arguing as above, we suppose by contradiction that there does not exist any critical levels in \( (\bar{c}, \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\}) \).

By Corollary 3.4 again, we know that \( I|_{\mathcal{K}} \) satisfies the Palais-Smale condition in \( (\bar{c}, \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\}) \).

By the deformation lemma again, then we prove that there exist a positive number \( \sigma_2 \) such that
\[ c^* - \sigma_2 > \bar{c}, \quad \bar{s} + \sigma_2 < \min \left\{ \frac{S_{HL}^2}{4\mu_1}, \frac{S_{HL}^2}{4\mu_2}, 2c_\infty \right\} \]
and a continuous function
\[ \check{\eta}: [0, 1] \times I^{\bar{s} + \sigma_2} \to I^{\bar{s} + \sigma_2} \]
such that
\[ \check{\eta}(0, (u, v)) = (u, v) \quad \forall (u, v) \in I^{\bar{s} + \sigma_2}; \]
\[ \check{\eta}(s, (u, v)) = (u, v) \quad \forall (u, v) \in I^{c^* - \sigma_2}, \forall s \in [0, 1]; \]
\[ I \circ \check{\eta}(s, (u, v)) \leq I(u, v) \quad \forall (u, v) \in I^{\bar{s} + \sigma_2}, \forall s \in [0, 1]; \]
\[ \check{\eta}(1, I^{\bar{s} + \sigma_2}) \subset I^{c^* - \sigma_2}. \]

With the help of (5.17) and the properties listed above, we can easily find that
\[ \forall (\delta, y) \in \mathcal{K} \Rightarrow I \circ \check{\gamma}_{\delta,y} \leq \bar{s} \Rightarrow I \circ \check{\eta}(1, \check{\gamma}_{\delta,y}) \leq c^* - \sigma_2. \]

Let \( s \in [0, 1] \) and \( (\delta, y) \in \mathcal{K} \), we define the map
\[ \check{\Gamma}(\delta, y, s) = \begin{cases} T(\delta, y, 2s), & s \in [0, \frac{1}{2}]; \\ \left( \gamma \circ \check{\eta}(2s - 1, \check{\gamma}_{\delta,y}), \xi \circ \check{\eta}(2s - 1, \check{\gamma}_{\delta,y}) \right), & s \in \left[ \frac{1}{2}, 1 \right], \end{cases} \]
where \( T \) is defined in [5.8]. As proved in Lemma 5.3, we have
\[ \forall s \in [0, \frac{1}{2}], \forall (\delta, y) \in \partial \mathcal{K}, \check{\Gamma}(\delta, y, s) \neq \left( \frac{1}{2}, 0 \right). \]

On the other hand, since
\[ \forall (\delta, y) \in \partial \mathcal{K} \Rightarrow I \circ \check{\gamma}_{\delta,y} \leq \check{K} < \bar{c} < c^* - \sigma_2, \]
then
\[ \check{\Gamma}(\delta, y, s) = \check{\Gamma}(\delta, y, \frac{1}{2}) = T(\delta, y, 1), \quad \forall s \in \left[ \frac{1}{2}, 1 \right], \forall (\delta, y) \in \partial \mathcal{K}. \]
So
\[\hat{\Gamma}(\delta, y, s) \neq \left( \frac{1}{2}, 0 \right), \quad \forall s \in \left[ \frac{1}{2}, 1 \right], \quad \forall (\delta, y) \in \partial\mathcal{H}.\]

By the homotopy invariance of topological degree together with (5.1), we prove that there exist a pair of \((\hat{\sigma}, \hat{y}) \in \mathcal{H}\) such that
\[\xi \circ \hat{\eta}(1, \tilde{\psi}_{\delta, \hat{y}}) = 0, \quad \gamma \circ \hat{\eta}(1, \tilde{\psi}_{\delta, \hat{y}}) = \frac{1}{2}\]
and
\[I \circ \hat{\eta}(1, \tilde{\psi}_{\delta, \hat{y}}) \geq c^*,\]
which contradicts with (5.27). Then the constrained functional \(I|_\mathcal{X}\) has at least a critical point \((u_h, v_h)\) with \(I(u_h, v_h) \in (\bar{c}, \min\{\frac{c^2}{4\mu_1}, \frac{c^2}{4\mu_2}, 2c_\infty\})\). Moreover, \((u_h, v_h)\) is a critical point of functional \(I\) and \((u_h, v_h)\) is definite in sign. Arguing by the same methods in Step 1, we can also prove that \((u_h, v_h)\) is a positive solution of system (2.1), and also of system (1.8). Finally, we complete the whole proof of Theorem 1.1.

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