H(div)-CONFORMING FINITE ELEMENT TENSORS WITH CONSTRAINTS

LONG CHEN AND XUEHAI HUANG

ABSTRACT. A unified construction of H(div)-conforming finite element tensors, including vector element, symmetric matrix element, traceless matrix element, and, in general, tensors with linear constraints, is developed in this work. It is based on the geometric decomposition of Lagrange elements into bubble functions on each sub-simplex. Each tensor at a sub-simplex is decomposed into tangential and normal components. The tangential component forms the bubble function space, while the normal component characterizes the trace. Some degrees of freedom can be redistributed to \((n - 1)\)-dimensional faces. The developed finite element spaces are H(div)-conforming and satisfy the discrete inf-sup condition. Intrinsic bases of the constraint tensor space are also established.

1. INTRODUCTION

Hilbert complexes play a fundamental role in the theoretical analysis and the design of stable numerical methods for partial differential equations [5, 7, 2, 15]. Recently, in [8], Arnold and Hu have developed a systematic approach to derive new Hilbert complexes from well-understood differential complexes, such as the de Rham complex. In space \(\mathbb{R}^n\), for \(0 \leq k \leq n\), let \(\text{Alt}^{k,n-1} := \text{Alt}^k \otimes \text{Alt}^{n-1}\) be the tensor product of alternating multilinear functional spaces, \(H^s\) be the standard Sobolev space with real index \(s\), and \(\kappa_{k}\) be the Koszul operator for the de Rham complex. Below, we rotate the right end column of the Bernstein-Gelfand-Gelfand (BGG) diagram in [8] and switch the ordering of the index in [8] to match the row action of the operator \(\text{div}\).

\[
\begin{array}{ccccccccc}
H^q_{n-1} \otimes \text{Alt}^{n,n} & \xrightarrow{\kappa_{n-1}} & H^q_{n-1} \otimes \text{Alt}^{n,n-1} & \xrightarrow{\text{div}} & H^q_{n} \otimes \text{Alt}^{0,n-1} \\
& & & & & & & \downarrow \text{div} & \\
\downarrow \text{div} & & & & & & & & \\
\end{array}
\]

The algebraic operator \(s^{k,n-1} : \text{Alt}^{k,n-1} \rightarrow \text{Alt}^{k-1,n}\), for \(1 \leq k \leq n\) along the \(\searrow\) direction, is defined as

\[
s^{k,n-1} \omega (w_1, \ldots, w_{k-1}) (v_1, \ldots, v_n) := \sum_{i=1}^{n} (-1)^{i-1} \omega (v_i, w_1, \ldots, w_{k-1}) (v_1, \ldots, \widehat{v_i}, \ldots, v_n)
\]

\[
\forall v_1, \ldots, v_n, w_1, \ldots, w_{k-1} \in \mathbb{R}^n.
\]

The tensor space \(\mathcal{X}\) is defined as

\[
\mathcal{X} := \ker(s^{k,n-1}) \cap \text{Alt}^{k,n-1}, \quad 1 \leq k \leq n - 1.
\]
A tensor $\omega \in \text{Alt}^{k,n-1}$ can be represented by a matrix $A = (a_{\sigma,i}) \in \mathbb{R}^{(\mathbb{S}^n) \times n}$, which is called a proxy matrix, and then $X$ is a subset of matrices satisfying certain linear constraints. For simplicity, we will use matrix $A$ to represent an element in $X$. Given a domain $\Omega \subset \mathbb{R}^n$, define $H(\text{div}, \Omega; X)$ space as

$$H(\text{div}, \Omega; X) = \{ A \in L^2(\Omega; X) : \text{div} A \in L^2(\Omega; \mathbb{R}^n) \},$$

where the divergence operator is applied to each row of $A$.

Notable examples are $H(\text{div}; \mathbb{S})$ with the symmetric matrix $\mathbb{S}$, which plays an important role in the discretization of the elasticity equation in the mixed form [34, 4, 9], and $H(\text{div}; \mathcal{T})$ with the traceless matrix $\mathcal{T}$, which is used in the mixed form of the linearized Einstein-Bianchi system [38, 17, 29].

Given a simplicial mesh $T_h$ of $\Omega$, we shall develop a systematic construction of $H(\text{div})$-conforming finite element subspaces of $H(\text{div}, \Omega; X)$ space. Take a proxy matrix $A$ for an elementwise smooth tensor $\omega \in L^2(\Omega; X).$ $H(\text{div})$-conformity means $A n_F$ is continuous across each $(n-1)$-dimensional face $F$ of $T_h$. Namely, each row of $A$ is an $H(\text{div})$-conforming vector function.

It is the constraint $s^{k,n-1}(A) = 0$ that makes the finite element construction difficult, as the constraint and the normal continuity should be satisfied simultaneously. For example, the symmetry of the tensors makes it a challenging problem to construct $H(\text{div}; \mathbb{S})$-conforming finite elements. Arnold and Winther [9] constructed such an element in two dimensions, and later it was extended to higher dimensions in [31, 3, 1]. Hu and Zhang [30, 28] constructed $H(\text{div}; \mathbb{S})$-conforming finite elements with full polynomial spaces with matched order of approximation.

The approach we shall use is the geometric decomposition, which leads to explicit bases for finite elements. The geometric decomposition is an important tool for finite element analysis. For example, it is used in [26] to construct a local and bounded co-chain projection to the discrete de Rham complexes. The finite element system in [22] also originates from the geometric decomposition. The geometric decomposition of standard finite element de Rham complexes is well-studied in [6, 5, 27], and in [21] for nodal finite element de Rham complexes. Recently geometric decomposition has been extended to smooth finite elements and smooth finite element de Rham and Stokes complexes [16, 19].

We shall integrate the geometric decomposition of the Lagrange element with tangential-normal ($t$-$n$) decompositions on subsimplices to produce geometric decompositions of $H(\text{div})$-conforming finite element vectors and tensors, exhibiting normal continuity across all $(n-1)$-dimensional faces. In a similar way, hierarchical geometric decomposition of $H(\text{div})$-conforming finite elements in two and three dimensions is discussed in [24, 25, 12]. A geometric decomposition of $H(\text{div})$-conforming finite element vectors with a different tangential-normal decomposition in three dimensions is also shown in [21]. While these prior studies offer similar insights, our methodology introduces a novel level of generality. A significant aspect of our contribution is the expansion of geometric decomposition techniques to effectively manage tensors subjected to specific constraints.

To satisfy the constraint while still keeping normal continuity, the crucial step is to get a $t$-$n$ decomposition of $X$ with respect to the second component in $\text{Alt}^{k,n-1}$, i.e.,

$$X = \mathcal{F}^f(X) \oplus \mathcal{N}^f(X),$$

where $\mathcal{F}^f(X) = (\text{Alt}^k \otimes \mathcal{F}) \cap \ker(s^{k,n-1})$ and $\mathcal{N}^f(X) = \pi_X(\text{Alt}^k \otimes \mathcal{N}^f)$ with an oblique (non-orthogonal) projection operator $\pi_X : \text{Alt}^k \otimes \mathcal{N}^f \to X$ so that one constraint is used only once either in $\mathcal{F}^f(X)$ or $\mathcal{N}^f(X)$.

This induces the geometric decomposition

$$P_r(T; X) = \bigoplus_{\tau=0}^n \bigoplus_{f \in \Delta_T} b_f P_{r-(\tau+1)}(f) \otimes \left[ \mathcal{F}^f(X) \oplus \mathcal{N}^f(X) \right].$$

The tangential component will contribute to the polynomial bubble space

$$B_r(\text{div}; T; X) := \bigoplus_{\tau=1}^n \bigoplus_{f \in \Delta_T} \left[ b_f P_{r-(\tau+1)}(f) \otimes \mathcal{F}^f(X) \right],$$

and the normal component $b_f P_{r-(\tau+1)}(f) \otimes \mathcal{N}^f(X)$ to the trace.
As a direct result of decomposition (1), the following DoFs
\[(2a) \quad \omega(v_i), \quad i = 0, \ldots, n, \omega \in \mathcal{X},\]
\[(2b) \quad (\omega, \eta)_f, \quad \eta \in P_{r-(\ell+1)}(f) \otimes \mathcal{N}_\ell(\mathcal{X}), \quad f \in \Delta\ell(T_h), \quad \ell = 1, \ldots, n-1,\]
\[(2c) \quad (\omega, \eta)_T, \quad \eta \in B_r(\text{div}; T; \mathcal{X}), \quad T \in T_h,\]
will determine a space $V_h \subset \mathcal{H}(\text{div}; \Omega; \mathcal{X})$. Here we single out the vertex DoFs to emphasize the finite element function is continuous on vertices.

Discrete inf-sup condition will be established with requirement $r \geq n + 1$ and with modification of DoFs for $r \geq k + 1$ for $1 \leq k \leq n - 2$. Variants can be constructed by further tuning DoFs (2), which will recover the existing $\mathcal{H}(\text{div}; \mathcal{S})$ elements [18, 28, 30] and $\mathcal{H}(\text{div}; \mathbb{T})$ elements [17, 29].

The geometric decomposition approach in this paper is not easy to extend to the case $\mathcal{X} = \mathcal{S} \cap \mathbb{T}$, which requires special care and super-smoothness of DoFs; see the recent work [32].

The rest of this paper is organized as follows. Section 2 covers foundational concepts, including simplex, barycentric coordinates, Bernstein polynomials, and a geometric decomposition of Lagrange elements. Sections 3 and 4 explore the geometric decompositions of vector face elements and matrix face elements, respectively. Section 5 focuses on the constraint tensor space $\mathcal{X}$ and its bases. The geometric decomposition of $\mathcal{H}(\text{div})$-conforming tensors is developed in Section 6. As the language of differential form is abstract, in the first few sections we present the results using vector and matrix language and then move to the differential forms in Sections 5 and 6.

2. Notation and Background

We summarize the most important notation and integer indices in the beginning:

- $\mathbb{R}^n : n$ is the dimension of the ambient Euclidean space and $n \geq 2$;
- $P_r : r$ is the degree of the polynomial and $r \geq 0$;
- $\Lambda^k : k$ is the order of the differential form and $0 \leq k \leq n$;
- $\Delta\ell(T) : \ell$ is the dimension of a sub-simplex $f \in \Delta\ell(T)$ and $0 \leq \ell \leq n$.

2.1. Simplex and sub-simplices. Let $T \subset \mathbb{R}^n$ be an $n$-dimensional simplex with vertices $v_0, v_1, \ldots, v_n$ in general position. Following [6], we let $\Delta(T)$ denote all the subsimplices of $T$, while $\Delta\ell(T)$ denotes the set of subsimplices of dimension $\ell$, for $0 \leq \ell \leq n$.

For a sub-simplex $f \in \Delta\ell(T)$, we will overload the notation $f$ for both the geometric simplex and the algebraic set of indices. Namely on one hand $f = \{f(0), \ldots, f(\ell)\} \subseteq \{0, 1, \ldots, n\}$, and on the other hand

$$f = \text{Convex}(v_{f(0)}, \ldots, v_{f(\ell)}) \in \Delta\ell(T)$$

is the $\ell$-dimensional simplex spanned by the vertices $v_{f(0)}, \ldots, v_{f(\ell)}$. If $f \in \Delta\ell(T)$, for $\ell = 0, \ldots, n-1$, then $f^* \in \Delta_{n-\ell-1}(T)$ denotes the sub-simplex of $T$ opposite to $f$. Algebraically treating $f$ as a subset of $\{0, 1, \ldots, n\}$, $f^* \subseteq \{0, 1, \ldots, n\}$ so that $f \cup f^* = \{0, 1, \ldots, n\}$, i.e., $f^*$ is the complement of set $f$. Geometrically,

$$f^* = \text{Convex}(v_{f^*(1)}, \ldots, v_{f^*(n-\ell)}) \in \Delta_{n-\ell-1}(T)$$

is the $(n-\ell-1)$-dimensional simplex spanned by vertices not contained in $f$. We refer to [19, Fig. 2] for an illustration of $f$ and $f^*$.

Denote by $F_i$ the $(n-1)$-dimensional face opposite to vertex $v_i$, i.e., $F_i = \{i\}^*$. Here capital $F$ is reserved for an $(n-1)$-dimensional face of $T$. For lower dimensional sub-simplices, we sometimes use more conventional notation. For example, the vertex will be denoted by $v_i$ and the edge formed by $v_i$ and $v_j$ will be denoted by $e_{ij}$. 


2.2. \(t\)-n bases. For an \(\ell\)-dimensional sub-simplex \(f \in \Delta_i(T)\), choose \(\ell\) linearly independent tangential vectors \(\{t_1^f, \ldots, t_\ell^f\}\) of \(f\) and \(n-\ell\) linearly independent normal vectors \(\{n_1^f, \ldots, n_{n-\ell}^f\}\) of \(f\). The set of \(n\) vectors \(\{t_1^f, \ldots, t_\ell^f, n_1^f, \ldots, n_{n-\ell}^f\}\) forms a basis of \(\mathbb{R}^n\). Notice that for \(\ell = 0\), i.e., at vertices, there are no tangential vectors, and for \(\ell = n\), there are no normal vectors. Define the tangent plane and normal plane of \(f\) as

\[\mathcal{T}^f := \text{span}\{t_i^f, i = 1, \ldots, \ell\}, \quad \mathcal{N}^f := \text{span}\{n_i^f, i = 1, \ldots, n-\ell\}.\]

All vectors are normalized but \(\{t_i^f\}\) or \(\{n_i^f\}\) may not form an orthonormal basis.

Inside the subspace \(\mathcal{T}^f\), we can find a basis \(\{\hat{t}_1^f, \ldots, \hat{t}_\ell^f\}\) dual to \(\{t_1^f, \ldots, t_\ell^f\}\), i.e., \(\langle \hat{t}_i, t_j \rangle = \delta_{i,j}\) for \(i, j = 1, \ldots, \ell\). Similarly we have a basis \(\{\hat{n}_1^f, \ldots, \hat{n}_{n-\ell}^f\}\) of \(\mathcal{N}^f\) and \(\langle \hat{n}_i, n_j \rangle = \delta_{i,j}\) for \(i, j = 1, \ldots, n-\ell\). As \(\mathcal{T}^f \perp \mathcal{N}^f\), the basis \(\{\hat{t}_1^f, \ldots, \hat{t}_\ell^f, \hat{n}_1^f, \ldots, \hat{n}_{n-\ell}^f\}\) is also dual to \(\{t_1^f, \ldots, t_\ell^f, n_1^f, \ldots, n_{n-\ell}^f\}\).

Given a sub-simplex \(f \in \Delta_i(T)\), we now present two bases for its normal plane \(\mathcal{N}^f\) constructed in [13].

Recall that we label \(F_i\) as the \((n-1)\)-dimensional face opposite to the \(i\)-th vertex. Then \(f \subseteq F_i\) for \(i \in f^*\). One basis is composed by unit normal vectors of all such \((n-1)\)-dimensional faces:

\[\{n_{F_i}, i \in f^*\},\]

and will be called the face normal basis.

We now give its dual basis in \(\mathcal{N}^f\). For \(f \in \Delta_i(T), \ell = 0, 1, \ldots, n-1\) and \(i \in f^*\), let \(f \cup \{i\}\) denotes the \((\ell + 1)\)-dimensional face in \(\Delta_{i+1}(T)\) with vertices \(\{i, f(0), \ldots, f(\ell)\}\). Let \(n_{f \cup \{i\}}^f\) be a unit normal vector of \(f\) but tangential to \(f \cup \{i\}\). The basis

\[\{n_{f \cup \{i\}}^f, i \in f^*\}\]

will be called the tangential normal basis.

**Lemma 2.1.** For \(f \in \Delta_i(T), \ell = 0, 1, \ldots, n-1\) and \(i \in f^*\), the rescaled tangential normal basis \(\{n_{f \cup \{i\}}^f / \langle n_{f \cup \{i\}}^f, n_{F_i} \rangle, i \in f^*\}\) of \(\mathcal{N}^f\) is dual to the face normal basis \(\{n_{F_i}, i \in f^*\}\).

**Proof.** Clearly \(n_{f \cup \{i\}}^f, n_{F_i} \in \mathcal{N}^f\) for \(i \in f^*\). It suffices to prove

\[n_{f \cup \{i\}}^f \cdot n_{F_j} = 0 \quad \text{for} \quad i, j \in f^*, i \neq j,\]

which follows from the fact \(f \cup \{i\} \subseteq F_j\) and \(n_{f \cup \{i\}}^f \in \mathcal{T}^{f \cup \{i\}}\). \(\square\)

**Example 2.2.** An important example is \(f \in \Delta_0(T)\), i.e., \(f\) is a vertex. Without loss of generality, let \(f = \{0\}\). Then \(n_{\{0\}}^f\) is a unit normal vector of edge \(\{0, i\}\): \(t_{0i}\) or \(t_{00}\) depending on the orientation. Its dual basis is \(\{n_{F_i} / \langle n_{F_i}, t_{0i} \rangle, i = 1, \ldots, n\}\). See Fig. 1 (a).

**Example 2.3.** Let \(f = \{0, 1, 2, 3\}\) be an edge of a tetrahedron. Then we have two bases of the normal plane \(\mathcal{N}^f\):

\[\{n_{F_1}, n_{F_2}\}\]

and \(\{n_{\{0,1\}}^f, n_{\{0,2\}}^f\}\). They are dual to each other with an appropriate rescaling. See Fig. 1 (b).

2.3. Barycentric coordinates and Bernstein polynomials. For a domain \(\Omega \subseteq \mathbb{R}^n\) and integer \(r \geq 0\), \(\mathbb{P}_r(\Omega)\) denotes the space of real valued polynomials defined on \(\Omega\) of degree less than or equal to \(r\). For simplicity, we let \(\mathbb{P}_r = \mathbb{P}_r(\mathbb{R}^n)\). Hence, if \(n\)-dimensional domain \(\Omega\) has nonempty interior, then \(\dim \mathbb{P}_r(\Omega) = \dim \mathbb{P}_r = \binom{r+n}{n}\). When \(\Omega = \{v\}\) is a point, \(\mathbb{P}_r(v) = \mathbb{R}\) for all \(r \geq 0\). And we set \(\mathbb{P}_r(\Omega) = \{0\}\) when \(r < 0\). Let \(\mathbb{H}_r(\Omega)\) denote the space of real valued polynomials defined on \(\Omega\) of degree \(r\).

For \(n\)-dimensional simplex \(T\), we denote by \(\lambda_0, \lambda_1, \ldots, \lambda_n\) the barycentric coordinate functions with respect to \(T\). That is \(\lambda_i \in \mathbb{P}_1(T)\) and \(\lambda_i(v_j) = \delta_{i,j}\), \(0 \leq i, j \leq n\), where \(\delta_{i,j}\) is the Kronecker delta function. The functions \(\{\lambda_i, i = 0, 1, \ldots, n\}\) form a basis for \(\mathbb{P}_1(T)\). \(\sum_{i=0}^n \lambda_i(x) = 1\), and \(0 \leq \lambda_i(x) \leq 1, i = 0, 1, \ldots, n\), for \(x \in T\). The
sub-simplices of $T$ correspond to the zero sets of the barycentric coordinates. Indeed \( f = \{ x \in T \mid \lambda_i(x) = 0, i \in f^* \} \) for \( f \in \Delta_f(T) \).

We will use the multi-index notation \( \alpha \in \mathbb{N}^n \), meaning \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with integer \( \alpha_i \geq 0 \). We define \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( |\alpha| := \sum_{i=1}^{n} \alpha_i \). We will also use the set \( \mathbb{N}^{0:n} \) of multi-indices \( \alpha = (\alpha_0, \ldots, \alpha_n) \), with \( \lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} \) for \( \alpha \in \mathbb{N}^{0:n} \).

We introduce the simplicial lattice \([19, 16]\), which is also known as the principal lattice \([37]\). A simplicial lattice of degree \( r \) and dimension \( n \) is a multi-index set of \( n + 1 \) components and with fixed length \( r \), i.e.,
\[
T^n_r = \{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{0:n} \mid \alpha_0 + \alpha_1 + \ldots + \alpha_n = r \} .
\]

An element \( \alpha \in T^n_r \) is called a node of the lattice. The Bernstein representation of polynomial of degree \( r \) on a simplex \( T \) is
\[
\mathbb{P}_r(T) := \text{span}\{\lambda^\alpha = \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}, \alpha \in T^n_r \} .
\]

In the Bernstein form, for an \( f \in \Delta_f(T) \),
\[
\mathbb{P}_r(f) = \text{span}\{\lambda^\alpha = \lambda_{f(0)}^{\alpha_0} \lambda_{f(1)}^{\alpha_1} \cdots \lambda_{f(\ell)}^{\alpha_\ell}, \alpha \in T^n_r \} .
\]

Through the natural extension defined by the barycentric coordinate, \( \mathbb{P}_r(f) \subseteq \mathbb{P}_r(T) \). The bubble polynomial of \( f \) is a polynomial of degree \( \ell + 1 \):
\[
b_f := \lambda_f = \lambda_{f(0)} \lambda_{f(1)} \cdots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f) .
\]

We have the following property of the bubble polynomial \( b_f \).

**Lemma 2.4.** Let \( f, e \in \Delta(T) \). If \( f \not\subseteq e \), then \( b_f |_e = 0 \).

**Proof.** As \( f = (f \cap e^*) \cup (f \cap e) \) and \( f \not\subseteq e \), we conclude \( f \cap e^* \neq \emptyset \). So \( b_f \) contains \( \lambda_i \) for some \( i \in e^* \) and consequently \( b_f |_e = 0 \). \( \square \)

In particular, \( b_f \) vanishes at all sub-simplices other than \( f \) with dimensions \( \leq \dim f \), and higher dimensional sub-simplices not containing \( f \).
2.4. Geometric decomposition of Lagrange elements. We begin with the geometric decomposition of the Lagrange element. The following geometric decomposition of Lagrange element is given in [6] without proofs. As it is the foundation of other geometric decompositions in later sections, we present it using our notation and provide a detailed proof. We refer to [6, Fig. 2.1] for an illustration of this geometric decomposition. For the 0-dimensional face, i.e., a vertex \( v \), we understand that \( \int_v u \, ds = u(v) \) for \( 1 \in \mathbb{P}_r(v) = \mathbb{R} \).

**Theorem 2.5** (Geometric decomposition of Lagrange element, (2.6) in [6]). For the polynomial space \( \mathbb{P}_r(T) \) with \( r \geq 1 \) on an \( n \)-dimensional simplex \( T \), we have the following decomposition

\[
\mathbb{P}_r(T) = \bigoplus_{f=0}^n \bigoplus_{f \in \Delta(f)} b_f \mathbb{P}_{r-(\ell+1)}(f).
\]

And the function \( u \in \mathbb{P}_r(T) \) is uniquely determined by DoFs

\[
\int_f u \, p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_f(T), \ell = 0, \ldots, n.
\]

**Proof.** We first prove the decomposition (3). Each component \( b_f \mathbb{P}_{r-(\ell+1)}(f) \subset \mathbb{P}_r(T) \) and the sum is direct due to the property of \( b_f \), cf. Lemma 2.4. Then count the dimensions and use the combinatorial identity

\[
\sum_{\ell=0}^n \binom{n+1}{\ell+1} \binom{r-1}{r-\ell-1} = \binom{n+r}{r},
\]

which can be proved by looking at the coefficient of \( x^r \) in \((1 + x)^{n+1}(1 + x)^{r-1} = (1 + x)^{n+r}\).

To prove the unisolvency, we choose a basis \( \{\phi_i\} \) of \( \mathbb{P}_r(T) \) by the decomposition (3) and denote DoFs (4) as \( \{N_i\} \). By construction, the dimension of \( \{\phi_i\} \) matches the number of DoFs \( \{N_i\} \). The DoF-Basis matrix \( (N_i(\phi_j)) \) is thus square and block lower triangular in the sense that for \( \phi_f \in b_f \mathbb{P}_{r-(\ell+1)}(f) \),

\[
\int_{\Delta_f} \phi_f p \, ds = 0, \quad e \in \Delta_m(T) \text{ with } m \leq \ell \text{ and } e \neq f, p \in \mathbb{P}_{r-dim e-1}(e)
\]
due to the property of \( b_f \) established in Lemma 2.4. Each diagonal block matrix is the Gram matrix

\[
\int_f pq b_f \, dx_f, \quad p, q \in \mathbb{P}_{r-(\ell+1)}(f),
\]
in the measure \( b_f \, dx_f \) and thus symmetric and positive definite. In particular, it is invertible. So the unisolvency follows from the invertibility of this lower triangular matrix; see below for an illustration.

\[
N_f \setminus \phi_f \begin{array}{cccccc}
0 & 1 & \ldots & n-1 & n \\
\hline
0 & \square & 0 & \ldots & 0 & 0 \\
1 & \square & \square & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n-1 & \square & \square & \ldots & \square & 0 \\
n & \square & \square & \ldots & \square & \square \\
\end{array}
\]

\( \square \)

**Remark 2.6.** It is important to note that \( \mathbb{P}_{r-(\ell+1)}(f) = \{0\} \) when \( r < \ell + 1 \). As a result, the last non-zero term in (3) corresponds to \( \ell \leq \min\{r-1, n\} \). This implies that the degree of the polynomial dictates the dimension of the sub-simplex in decomposition (3). For instance, with quadratic polynomials, the summation includes only edge bubbles and excludes face bubbles and higher dimensions. Despite this, the full summation notation \( \bigoplus_{\ell=0}^n \) is retained.
for simplicity, with the implicit understanding that the range of non-zero sub-spaces will automatically truncate the limits.

Let \( \{ T_h \} \) be a family of partitions of \( \Omega \) into nonoverlapping simplices with \( h_T := \text{diam}(T) \) and \( h := \max_{T \in T_h} h_T \). The mesh \( T_h \) is conforming in the sense that the intersection of any two simplices is either empty or a common lower sub-simplex. Let \( \Delta_\ell(T_h) \) be the set of all \( \ell \)-dimensional sub-simplices of the partition \( T_h \) for \( \ell = 0, 1, \ldots, n \). The Lagrange finite element space

\[
S^r_h := \{ v \in C(\Omega) : v|_{T} \in \mathbb{P}_r(T), \forall T \in T_h, \text{ DoFs (4) are single-valued} \},
\]

has the geometric decomposition

\[
S^r_h = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_\ell(T_h)} b_f \mathbb{P}_{r-(\ell+1)}(f).
\]

Here we extend the polynomial \( b_f \mathbb{P}_{r-(\ell+1)}(f) \) to each element \( T \) containing \( f \) by the Bernstein form in the barycentric coordinate and thus it is a piecewise polynomial function and continuous in \( \Omega \). Consequently \( S^r_h \subset H^1(\Omega) \) and the dimension of \( S^r_h \) is

\[
\dim S^r_h = \sum_{\ell=0}^{n} |\Delta_\ell(T_h)| \binom{r-1}{\ell},
\]

where \( |\Delta_\ell(T_h)| \) is the cardinality, i.e., the number of \( \ell \)-dimensional simplices in \( T_h \).

The geometric decomposition of the vector Lagrange elements is a straightforward generalization:

\[
\mathbb{P}_r(T; \mathbb{R}^n) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_\ell(T)} [b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{R}^n].
\]

In (6), a fixed orthonormal basis of \( \mathbb{R}^n \) is implicitly assumed in which the vector is expanded. It is usually the Cartesian coordinate describing the domain \( \Omega \).

3. Geometric Decompositions of Vector Face Elements

In this section we consider \( H(\text{div}) \)-conforming vector finite elements with local shape function space \( \mathbb{P}_r(T; \mathbb{R}^n) \). Define \( H(\text{div}, \Omega) := \{ v \in L^2(\Omega; \mathbb{R}^n) : \text{div} v \in L^2(\Omega) \} \). For a subdomain \( K \subseteq \Omega \), the trace operator for the \( \text{div} \) operator is

\[
\text{tr}_K^{\text{div}} v = v \cdot n|_{\partial K} \quad \text{for} \quad v \in C(K),
\]

where \( n \) denotes the outwards unit normal vector of \( \partial K \). Given a triangulation \( T_h \) and a piecewise smooth function \( u \), it is well known that \( u \in H(\text{div}, \Omega) \) if and only if \( n_F \cdot u \) is continuous across all faces \( F \in \Delta_{n-1}(T_h) \), which can be ensured by having DoFs on faces. An \( H(\text{div}) \)-conforming finite element is thus also called a face element.

3.1. Examples in three dimensions. We first use three-dimensional examples to illustrate the main idea. Recall that the geometric decomposition of the vector Lagrange elements in three dimensions reads

\[
\mathbb{P}_r(T; \mathbb{R}^3) = \bigoplus_{\ell=0}^{3} \bigoplus_{f \in \Delta_\ell(T)} [b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{R}^3].
\]

A fixed orthonormal basis \( \{ e_i \}_{i=1}^{3} \) of \( \mathbb{R}^3 \) is used in (7). See Fig. 2 (a).

An \( H(\text{div}) \) function is a vector proxy of an \( (n-1) \)-form; see Section 5. As a differential form, it is an intrinsic quantity independent of the choice of coordinates/frames. Based on this observation, we shall choose different frames at different sub-simplex \( f \in \Delta_\ell(T) \).

For \( f \in \Delta_\ell(T) \) with \( \ell = 0, 1, 2, 3 \), the tangent plane and normal plane of \( f \) are

\[
\mathcal{T}^f = \text{span}\{ t^f_i, i = 1, \ldots, \ell \}, \quad \mathcal{N}^f = \text{span}\{ n^f_i, i = 1, \ldots, 3 - \ell \}.
\]

Then \( \mathbb{R}^3 \) admits a tangential-normal \((t-n)\) decomposition \( \mathbb{R}^3 = \mathcal{T}^f \oplus \mathcal{N}^f \). Coupled with the bubble polynomials, we obtain a \( t-n \) decomposition of \( \mathbb{P}_r(T; \mathbb{R}^3) \) as

\[
\mathbb{P}_r(T; \mathbb{R}^3) = \bigoplus_{\ell=0}^{3} \bigoplus_{f \in \Delta_\ell(T)} [B_{r} \mathcal{T}^f \oplus B_{r} \mathcal{N}^f],
\]

where \( B_r \mathcal{T}^f \) and \( B_r \mathcal{N}^f \) are the second-order bubble polynomials in \( \mathcal{T}^f \) and \( \mathcal{N}^f \), respectively.
where
\[ \mathbb{B}_r \mathcal{T}^f = b_j \mathbb{P}_{r-(t+1)}(f) \otimes \mathcal{T}^f, \quad \mathbb{B}_r \mathcal{N}^f = b_j \mathbb{P}_{r-(t+1)}(f) \otimes \mathcal{N}^f. \]

Notice that for a vertex \( v \in \Delta_0(T) \), \( \mathbb{B}_r \mathcal{T}^v = \{0\} \), and \( \mathbb{B}_r \mathcal{N}^v = \lambda_v \mathbb{R}^3 \) as \( b_v = \lambda_v \mathbb{P}_{r-1}(v) = \mathbb{R} \), and \( \mathcal{N}^v = \mathbb{R}^3 \).

Define the polynomial bubble space \( \mathbb{B}_r(\mathbf{div}, T) := \ker(\mathbf{tr}^\text{div}) \cap \mathbb{P}_r(T; \mathbb{R}^3) \), where \( \mathbf{tr}^\text{div} \mathbf{u} = \mathbf{u} \cdot \mathbf{n}_{\partial T} \). The tangential component will form the \( \mathbf{div} \) bubble space: for \( r \geq 2 \), it holds that
\[ \mathbb{B}_r(\mathbf{div}, T) = \bigoplus_{t=1}^3 \bigoplus_{f \in \Delta_t(T)} \mathbb{B}_r \mathcal{T}^f. \]

Verification of \( \mathbb{B}_r \mathcal{T}^f \subseteq \mathbb{B}_r(\mathbf{div}, T) \) is straightforward. For face \( F \) not containing \( f \), \( b_f|_F = 0 \). For face \( F \) containing \( f \), \( \mathbf{tr}^\text{div} \mathbf{u} = \mathbf{u} \cdot \mathbf{n}_F = 0 \) as \( t_i^f \cdot \mathbf{n}_F = 0 \).

(a) Fixed Cartesian basis \( \{e_i\}_{i=1}^3 \) is used for the vector Lagrange element. (b) In Nédélec/BDM element, face normal basis \( \{\mathbf{n}_F, f \subseteq F\} \) is used for \( \mathcal{N}^f \). (c) Facewise redistribution of the normal DoFs for the quadratic face element.

**Figure 2.** Classical face elements can be obtained by a special \( t-n \) decomposition and face-wise redistribution of normal components.

The normal component will contribute to the trace. We can derive the second family of Nédélec element \([36, 11]\), which generalizes the Brezzi-Douglas-Marini (BDM) element \([10]\) in two dimensions, from a special \( t-n \) basis. For face \( F \in \Delta_2(T) \), choose two linearly independent tangential vectors \( t^F_1, t^F_2 \) of \( F \) and one normal vector \( \mathbf{n}_F \) of \( F \). For edge \( e \in \Delta_1(T) \), choose a tangential vector \( t^e \) of \( e \) and the face normal basis vectors \( \{\mathbf{n}_{F_1}, \mathbf{n}_{F_2}, \mathbf{n}_{F_3}\} \), where \( F_1, F_2, F_3 \) are two faces containing \( e \). For vertex \( v \in \Delta_0(T) \), choose \( \{\mathbf{n}_{F_1}, \mathbf{n}_{F_2}, \mathbf{n}_{F_3}\} \), where \( F_1, F_2, F_3 \) are three faces containing vertex \( v \). For tetrahedron \( T \), we keep the canonical basis \( \{e_i\}_{i=1}^3 \), which is considered as tangential vectors of \( T \). See Fig. 2 (b).

We then group normal components \( \{\mathbb{B}_r \mathcal{N}^f, f \in \Delta(T)\} \) face by face. On each face \( F \), again by the geometric decomposition of Lagrange element, the normal components at different sub-simplices \( f \in \Delta(F) \) will determine the scalar function \( \mathbf{u} \cdot \mathbf{n}_F \). In Fig. 2 (c), we show such face-wise redistribution for a quadratic face element.

**Example 3.1** (Nédélec element/BDM element). Taking \( \mathcal{N}^f := \text{span} \{\mathbf{n}_{F_i}, f \subseteq F_i\} \), the following DoFs
\[ (v \cdot \mathbf{n}_{F_i})|_{F_i}(v), \quad F_i \in \Delta_2(T), v \in \Delta_0(F_i), \]
\[ \int_e (v \cdot \mathbf{n}_{F_i}) p \, ds, \quad F_i \in \Delta_2(T), e \in \Delta_1(F_i), p \in \mathbb{P}_{r-2}(e), \]
\[ \int_{F_i} (v \cdot \mathbf{n}_{F_i}) p \, ds, \quad F_i \in \Delta_2(T), p \in \mathbb{P}_{r-3}(F_i), \]
\[ \int_T v \cdot p \, dx, \quad p \in \mathbb{B}_r(\mathbf{div}, T) \]
define the Nédélec element/BDM element. DoFs (8a)-(8c) are all located on face $F_i$, and uniquely determine $(v \cdot n_{F_i}) |_{F_i} \in \mathbb{P}_r(F_i)$. Hence DoFs (8a)-(8c) can be combined to one DoF

$$\int_{F_i} (v \cdot n_{F_i}) \, p \, ds, \quad F_i \in \Delta_2(T), \, p \in \mathbb{P}_r(F_i).$$

The interior DoF (8d) can be further replaced by $p \in (\mathbb{P}_{r-2}(T; \mathbb{R}^n) \oplus \mathbb{H}_{r-2}(T; \mathbb{K}) \mathbf{x})$ with $\mathbb{K}$ being the skew-symmetric matrix space; see [18]. Therefore DoFs (8) induce the Nédélec/BDM element; see Fig. 3 (a).

Different $H(\text{div})$-conforming finite elements can be obtained by different $t$-$n$ basis.

**Example 3.2 (Stenberg element).** Taking $\{n_i^v\}_{i=1}^3 = \{e_i\}_{i=1}^3$ and $\{n_i^e\}_{i=1}^2 = \{n_{F_i}\}_{i=1}^2$ as two face $F_i$ sharing $e$, the following DoFs

$$v(v), \quad v \in \Delta_0(T),$$

$$\int_{e_i} (v \cdot n_{F_i}) \, p \, ds, \quad F_i \in \Delta_2(T), \, e \in \Delta_1(F_i), \, p \in \mathbb{P}_{r-2}(e),$$

$$\int_{F} v \cdot n_{F} \, p \, ds, \quad F \in \Delta_2(T), \, p \in \mathbb{P}_{r-3}(F),$$

$$\int_{T} v \cdot p \, dx, \quad p \in \mathbb{B}_r(\text{div}, T)$$

induce the Stenberg element [40], which is continuous at vertices; see Fig. 3 (b).

---

**Figure 3.** Different $H(\text{div})$-conforming finite elements can be obtained by different $t$-$n$ decompositions.

**Example 3.3 (Christiansen-Hu-Hu element).** Taking $\{n_i^v\}_{i=1}^3 = \{e_i\}_{i=1}^3$ and $\{n_i^e\}_{i=1}^2$ depending only on $e$, the following DoFs

$$v(v), \quad v \in \Delta_0(T),$$

$$\int_{e_i} (v \cdot n_{F_i}^e) \, p \, ds, \quad e \in \Delta_1(T), \, p \in \mathbb{P}_{r-2}(e), \, i = 1, 2,$$

$$\int_{F} (v \cdot n_{F}) \, p \, ds, \quad F \in \Delta_2(T), \, p \in \mathbb{P}_{r-3}(F),$$

$$\int_{T} v \cdot p \, dx, \quad p \in \mathbb{B}_r(\text{div}, T)$$
define the Christiansen-Hu-Hu (CHH) face element [21, Section 3.5], which has extra continuity at vertices and on the normal planes of edges; see Fig. 3 (c).

3.2. Face elements in arbitrary dimension. We now present and prove the result in arbitrary \( n \geq 2 \) dimensions. Again the main idea is using different and in general non-orthonormal \( t \)-\( n \) decomposition adapted to different sub-simplices.

3.2.1. Bubble spaces. For an \( \ell \)-dimensional sub-simplex \( f \in \Delta_\ell(T) \), choose \( \ell \) linearly independent tangential vectors \( \{ t_1^f, \ldots, t_\ell^f \} \) of \( f \) and \( n - \ell \) linearly independent normal vectors \( \{ n_1^f, \ldots, n_{n-\ell}^f \} \) of \( f \). The set of \( n \) vectors \( \{ t_1^f, \ldots, t_\ell^f, n_1^f, \ldots, n_{n-\ell}^f \} \) forms a basis of \( \mathbb{R}^n \). Notice that for \( \ell = 0 \), i.e., at vertices, there are no tangential vectors and for \( \ell = n \), there are no normal vectors. Define the tangent plane and normal plane of \( f \) as

\[
\mathcal{T}^f := \text{span}\{ t_i^f, i = 1, \ldots, \ell \}, \quad \mathcal{N}^f := \text{span}\{ n_i^f, i = 1, \ldots, n - \ell \}.
\]

All vectors are normalized but \( \{ t_i^f \} \) or \( \{ n_i^f \} \) may not be an orthonormal basis. Coupled with the bubble polynomial, for \( r \geq 1 \), define

\[
\mathbb{B}_r \mathcal{T}^f = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{T}^f, \quad \mathbb{B}_r \mathcal{N}^f = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{N}^f.
\]

Lemma 3.4. The shape function space \( \mathbb{P}_r(T; \mathbb{R}^n) \) has a geometric decomposition

\[
(9) \quad \mathbb{P}_r(T; \mathbb{R}^n) = \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} \left[ \mathbb{B}_r \mathcal{T}^f \oplus \mathbb{B}_r \mathcal{N}^f \right].
\]

A function \( u \in \mathbb{P}_r(T; \mathbb{R}^n) \) is uniquely determined by the DoFs: for all \( f \in \Delta_\ell(T), \ell = 0, 1, \ldots, n, \)

\[
(10a) \quad \int_f (u \cdot t_i^f) \, p \, ds, \quad i = 1, \ldots, \ell, \quad p \in \mathbb{P}_{r-(\ell+1)}(f),
\]

\[
(10b) \quad \int_f (u \cdot n_j^f) \, p \, ds, \quad j = 1, \ldots, n - \ell, \quad p \in \mathbb{P}_{r-(\ell+1)}(f).
\]

Proof. Since \( \{ t_1^f, \ldots, t_\ell^f, n_1^f, \ldots, n_{n-\ell}^f \} \) forms a basis of \( \mathbb{R}^n \), DoFs (10a)-(10b) are equivalent to

\[
\int_f u \cdot p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{R}^n).
\]

Then the unisolvence follows from Theorem 2.5 for the Lagrange element.

Next we use \( \mathbb{B}_r \mathcal{T}^f \) or \( \mathbb{B}_r \mathcal{N}^f \) to characterize the kernel or range of the trace operator, respectively. Define the polynomial bubble space

\[
\mathbb{B}_r(\text{div} \, T) := \ker(\text{tr}^{\text{div}}) \cap \mathbb{P}_r(T; \mathbb{R}^n).
\]

It is obvious that \( \mathbb{B}_0(\text{div} \, T) = \{ 0 \} \) and \( \mathbb{B}_1(\text{div} \, T) = \{ 0 \} \).

Lemma 3.5. For \( r \geq 2 \), it holds that

\[
(11) \quad \mathbb{B}_r(\text{div} \, T) = \bigoplus_{\ell=1}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r \mathcal{T}^f,
\]

and

\[
(12) \quad \text{tr}^{\text{div}} : \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r \mathcal{N}^f \to \text{tr}^{\text{div}} \mathbb{P}_r(T; \mathbb{R}^n)
\]

is a bijection. Consequently

\[
\dim \mathbb{B}_r(\text{div} \, T) = \sum_{\ell=1}^{n} \binom{n+1}{\ell+1} \binom{r-1}{\ell} \binom{\ell}{1}.
\]
Proof. Verification of
\[ \bigoplus_{e=1}^{n} \bigoplus_{f \in \Delta_{e}(T)} \mathbb{B}_{r} \mathcal{F}^{f} \subseteq \mathbb{B}_{r}(\text{div}, T) \]
is straightforward. For face \( F \) not containing \( f \), \( b_{f}|_{F} = 0 \). For face \( F \) containing \( f \), \( \text{tr}^{\text{div}} u = u \cdot n_{F} = 0 \) as \( t_{f}^{F} \cdot n_{F} = 0 \) for \( u \in \mathbb{B}_{r} \mathcal{F}^{f} \).

Then apply the trace operator to the decomposition (9) and use \( \text{tr}^{\text{div}}(\mathbb{B}_{r} \mathcal{F}^{f}) = 0 \) to obtain
\[ \text{tr}^{\text{div}} \left( \bigoplus_{e=0}^{n-1} \bigoplus_{f \in \Delta_{e}(T)} \mathbb{B}_{r} \mathcal{N}^{f} \right) = \text{tr}^{\text{div}} \mathbb{P}_{r}(T; \mathbb{R}^{n}). \]
So the map \( \text{tr}^{\text{div}} \) in (12) is onto.

We then prove it is also injective. For an \( f \in \Delta_{\ell}(T) \), recall that \( \{ n_{F_{i}} \mid i \in f^{*} \} \) is the face normal basis of \( \mathcal{N}^{f} \) and \( \{ \hat{n}_{F_{i}} \mid i \in f^{*} \} \) is its dual basis of \( \mathcal{N}^{f} \). We expand \( u \in \bigoplus_{e=0}^{n-1} \bigoplus_{f \in \Delta_{e}(T)} \mathbb{B}_{r} \mathcal{N}^{f} \) in the dual basis, i.e.,
\[ u = \sum_{e=0}^{n-1} \sum_{f \in \Delta_{e}(T)} \sum_{i \in f} b_{f} p_{f}^{i} \hat{n}_{F_{i}} \text{ with } p_{f}^{i} \in \mathbb{P}_{r-\ell+1}(f). \]
We will prove if \( \text{tr}^{\text{div}} u = 0 \), then \( u = 0 \).

To do so, we consider the operator \( N_{f}^{\ell}(u) = (u \cdot n_{F_{i}})|_{f} \). Condition \( \text{tr}^{\text{div}} u = 0 \) implies \( N_{f}^{\ell}(u) = 0 \) for all \( f \in \Delta(T) \) and \( i \in f^{*} \). By the choice of the basis of \( \mathcal{N}^{f} \) and the property of bubble polynomial \( b_{f} \), the corresponding \( N \)-matrix is block lower triangular with diagonal matrices in the diagonal block; see (5) for an illustration. Therefore \( N_{f}^{\ell}(u) = 0 \) implies \( u = 0 \). More specifically, we have \( N_{f}^{\ell}(u) = p_{f}^{i} = 0 \) for \( v \in \Delta_{0}(T) \) and \( i \in f^{*} \). After that, we apply \( N_{f}^{\ell}(u) \) to conclude \( p_{f}^{i} = 0 \) for \( f \in \Delta_{\ell}(T) \). Using this forward substitution argument for the lower triangular matrix, we conclude all coefficient polynomials \( p_{f}^{i} = 0 \).

Once we have proved the map \( \text{tr} \) in (12) is bijective, we conclude (11) from the decomposition (9).

With this characterization, tangential DoFs in (10a) can be merged as one \( \mathbb{B}_{r}(\text{div}, T) \).

3.2.2. Brezzi-Douglas-Marini element. Given an \( f \in \Delta_{\ell}(T) \), we choose \( \{ n_{F}, f \subseteq F \in \Delta_{n-1}(T) \} \) as the basis for its normal plane \( \mathcal{N}^{f} \).

Lemma 3.6 (Local BDM element). The shape function space \( \mathbb{P}_{r}(T; \mathbb{R}^{n}) \) is uniquely determined by the DoFs
\[ \int_{F} v \cdot n_{F} p \, ds, \quad p \in \mathbb{P}_{r}(F), F \in \Delta_{n-1}(T), \]
\[ \int_{T} v \cdot p \, dx, \quad p \in \mathbb{B}_{r}(\text{div}, T). \]

Proof. By the geometric decomposition of \( \mathbb{P}_{r}(F) \) element, (13a) can be decomposed into
\[ \int_{F} (v \cdot n_{F})|_{F} p \, ds, \quad F \in \Delta_{n-1}(T), f \in \Delta_{\ell}(F), \]
\[ p \in \mathbb{P}_{r-\ell+1}(f), \ell = 0, 1, \ldots, n-1. \]

We switch the ordering of \( f \) and \( F \) to
\[ \int_{F} (v \cdot n_{F})|_{F} p \, ds, \quad f \in \Delta_{\ell}(T), F \in \Delta_{n-1}(T), f \subseteq F, \]
\[ p \in \mathbb{P}_{r-\ell+1}(f), \ell = 0, 1, \ldots, n-1, \]
which is exactly (10b) according to our choice of normal basis.

Therefore given a \( v \in \mathbb{P}_{r}(T; \mathbb{R}^{n}) \), if (13a) vanishes, then \( \text{tr} v = 0 \) and consequently \( v \in \mathbb{B}_{r}(\text{div}, T) \). Finally the vanishing DoF (13b) implies \( v = 0 \).
One benefit of using the decomposed version (14) instead of the merged one (13a) is that the well documented Lagrange basis functions can be used in the implementation of the face element; see [13]. We call the change from (14) to (13a) the facewise redistribution of normal DoFs. Namely by using the face normal basis, we can redistribute the DoFs on \( f \) to each \((n-1)\)-dimensional face \( F \).

To glue local finite elements to form an \( H(\text{div}) \)-conforming finite element, we need to enforce continuity of \( \mathbf{v} \cdot \mathbf{n}_F \) by choosing \( \mathbf{n}_F \) depending only on \( F \) not element \( T \).

**Lemma 3.7** (BDM space). For each \( F \in \Delta_{n-1}(\mathcal{T}_h) \), choose a normal vector \( \mathbf{n}_F \). For the shape function space \( \mathbb{P}_r(T; \mathbb{R}^n) \), the following DoFs

\[
\begin{align*}
(15a) & \quad \int_F \mathbf{v} \cdot \mathbf{n}_F \mathbf{p} \, ds, \quad \mathbf{p} \in \mathbb{P}_r(F), F \in \Delta_{n-1}(\mathcal{T}_h), \\
(15b) & \quad \int_T \mathbf{v} \cdot \mathbf{p} \, dx, \quad \mathbf{p} \in \mathbb{B}_r(\text{div}, T), T \in \mathcal{T}_h,
\end{align*}
\]

define an \( H(\text{div}) \)-conforming space

\[
V_{\text{BDM}} = \{ \mathbf{v}_h \in H(\text{div}, \Omega) : \mathbf{v}_h|_F \in \mathbb{P}_r(T; \mathbb{R}^n), \forall T \in \mathcal{T}_h \}.
\]

**Proof.** On each element \( T \), DoFs (15) will determine a function in \( \mathbb{P}_r(T; \mathbb{R}^n) \) by Lemma 3.6. DoF (15a) will determine the trace \( \mathbf{v} \cdot \mathbf{n}_F \) on \( F \) independent of the element containing \( F \), and thus the function is \( H(\text{div}, \Omega) \)-conforming. \( \Box \)

We have the geometric decomposition of the global BDM element space

\[
V_{\text{BDM}} = \bigoplus_{F \in \Delta_{n-1}(\mathcal{T}_h)} \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_k(F)} \mathbb{B}_r, \mathcal{N}^f(F, \Omega) \oplus \bigoplus_{T \in \mathcal{T}_h} \mathbb{B}_r(\text{div}, T),
\]

and

\[
\dim V_{\text{BDM}} = |\Delta_{n-1}(\mathcal{T}_h)| \binom{n+r-1}{r} + |\Delta_n(\mathcal{T}_h)| \sum_{\ell=1}^{n} \binom{n+1}{\ell+1} \binom{r-1}{\ell}.
\]

Here

\[
\mathbb{B}_r, \mathcal{N}^f(F, \Omega) := \{ \mathbf{v}_h \in H(\text{div}, \Omega) : \mathbf{v}_h|_T \in \mathbb{B}_r, \mathcal{N}^f \text{ for } T \in \mathcal{T}_h, F \subseteq T, \mathbf{v}_h|_{T'} = 0 \text{ for } T' \in \mathcal{T}_h, F \not\subseteq T' \}.
\]

3.2.3. **Stenberg’s type element.** We can construct an \( H(\text{div}) \)-conforming element with more continuity on the normal plane of lower dimensional sub-simplices by choosing a global basis of \( \mathcal{N}^f \).

In the \( t-n \) decomposition, if a basis vector \( \mathbf{t}_f^i \) or \( \mathbf{n}_f^i \) depends only on \( f \) not on element \( T \) containing \( f \), we call it **global** and otherwise \( \mathbf{t}_f^i(T) \) or \( \mathbf{n}_f^i(T) \) is local and the corresponding DoFs are different for different \( T \) containing \( f \). For a global basis vector, the corresponding DoF (10a) or (10b) only depends on \( f \) and thus imposes continuity in that direction. In the extreme case, if all \( t-n \) basis vectors are global, we obtain the Lagrange element.

The following is a generalization of Stenberg element by imposing more continuity on the normal plane for sub-simplices with dimension from 0 to \( m \) for some \(-1 \leq m \leq n-2\). When \( m = 0 \), it is the original Stenberg’s element [40], i.e., only continuous at vertices. When \( m = n-2 \), it is Christiansen-Hu-Hu element constructed in [21]. When \( m = -1 \), no DoF (16a) exists and thus reduces to the BDM element.
Lemma 3.8 (Stenberg type element). Let $-1 \leq m \leq n - 2$. For each $f \in \Delta_{\ell}(\mathcal{T}_h)$ with $\ell \leq m$, we choose $n - \ell$ normal vectors $\{n_1^f, \ldots, n_{n-\ell}^f\}$. Then the DoFs
\begin{equation}
\int_f v \cdot n_i^f \, p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f), \ f \in \Delta_{\ell}(\mathcal{T}_h),
\end{equation}
\begin{equation}
\int_f (v \cdot n_F) \big|_F \, p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f), F \in \Delta_{n-1}(\mathcal{T}_h), \ f \in \Delta_{\ell}(F),
\end{equation}
\begin{equation}
\int_T v \cdot p \, dx, \quad p \in \mathbb{B}_r(\mathrm{div}, T), \ T \in \mathcal{T}_h,
\end{equation}
will determine a space $V^r_{\text{Stenberg}} \subset H(\mathrm{div}, \Omega)$.

Proof. For $T \in \mathcal{T}_h$ and $f \in \Delta_{\ell}(T)$, both $\{n_1^f, \ldots, n_{n-\ell}^f\}$ and $\{n_F, F \in \Delta_{n-1}(T), f \in F\}$ are bases of the normal plane $\mathcal{N}^f$. Then the number of DoFs (16) restricted to $T$ equals to the number of DoFs (13). DoF (16a) determines DoF (15a) for $\ell = 0, \ldots, m$. Thus we conclude the result from Lemma 3.7.

We have the geometric decomposition of the global Stenberg element space
\[
V^r_{\text{Stenberg}} = \bigoplus_{\ell=0}^m \bigoplus_{f \in \Delta_{\ell}(\mathcal{T}_h)} \mathbb{B}_r(\mathcal{N}^f(\Omega)) \bigoplus \bigoplus_{F \in \Delta_{n-1}(\mathcal{T}_h)} \bigoplus_{\ell=m+1}^{n-1} \bigoplus_{f \in \Delta_{\ell}(F)} \mathbb{B}_r(\mathcal{N}^f(F, \Omega))
\]
and
\[
dim V^r_{\text{Stenberg}} = \sum_{\ell=0}^m |\Delta_{\ell}(\mathcal{T}_h)|(n-\ell) \binom{r-1}{\ell} + |\Delta_{n-1}(\mathcal{T}_h)| \sum_{\ell=m+1}^{n-1} \binom{n}{\ell+1} \binom{r-1}{\ell} + |\Delta_{n}(\mathcal{T}_h)| \sum_{\ell=1}^n \binom{n+1}{\ell+1} \binom{r-1}{\ell} \binom{\ell}{1}.
\]

Here
\[
\mathbb{B}_r(\mathcal{N}^f(\Omega)) := \{v_h \in H(\mathrm{div}, \Omega) : v_h|_T \in \mathbb{B}_r(\mathcal{N}^f) \text{ for } T \in \mathcal{T}_h, f \subseteq T, \ v_h|_{T'} = 0 \text{ for } T' \in \mathcal{T}_h, f \nsubseteq T'\}.
\]
Clearly $V^r_{\text{Stenberg}} \subseteq \mathbb{B}_{\text{D}}$, and $\dim V^r_{\text{Stenberg}} < \dim \mathbb{B}_{\text{D}}$ for $0 \leq m \leq n - 2$.

We introduce an $n$-dimensional smoothness vector $r = (r_0, r_1, \ldots, r_{n-1})^T \in \mathbb{R}^n$ to characterize the smoothness of the finite element space at sub-simplices of dimension $\ell \in (0, 1, \ldots, n-1)$. For the space $V^r_{\text{Stenberg}}$ defined by DoFs (16), the smoothness vector is given by
\[
r = (0, -\frac{1}{n-1}, \ldots, -\frac{m}{n-1}, -1, \ldots, -1)^T.
\]
For an $\ell$-dimensional face $f \in \Delta_{\ell}(\mathcal{T}_h)$, a smoothness parameter of $-\frac{\ell}{n-1}$ means the vector is discontinuous only on the $\ell$-dimensional tangential space and continuous on the normal plane $\mathcal{N}^f$. The value $-1$ indicates that the DoF is redistributed to faces, and the vector is discontinuous on the tangential space of $(n-1)$-dimensional faces. The same smoothness vector will determine the same global finite element space, although DoFs may be slightly different; see the modification of DoFs in Corollary 3.11.
Here are the smoothness vectors for the examples in Section 3.1: Nédélec element/BDM element: \( r = -1 \) with \( m = -1 \), Stenberg element: \( r = (0, -1, -1)^T \) with \( m = 0 \), and CHH element: \( r = (0, -1/2, -1)^T \) with \( m = 1 \). These vectors give the smoothness properties of the respective finite element spaces at vertices, edges, and faces.

### 3.3. Discrete inf-sup condition.

In the unisolvence of vector Lagrange elements, cf. the proof of Lemma 3.4, any basis of \( \mathbb{R}^n \) at \( f \) is allowed. The \( t \)-\( n \) basis is used for two purposes: 1. the \( H(\text{div}) \)-conformity; 2. the discrete \( \text{div} \) stability.

If the \( H(\text{div}) \)-conformity is the only concern, we can simply choose the Lagrange element. Another and more important consideration is from the \( \text{div} \) stability. At the continuous level, we have the \( \text{div} \) stability: \( \text{div}: H(\text{div}, \Omega) \to L^2(\Omega) \) is surjective, which has a continuous right inverse. A regular right inverse in \( H^1(\Omega) \) also exists [23].

By the Euler’s formula for homogenous degree polynomials \( \mathbb{H}_{r-1}(T) \), i.e., \( \text{div}(xq) = (r - 1 + n)q \) for any \( q \in \mathbb{H}_{r-1}(T) \), clearly we have \( \text{div} \mathbb{P}_r(T; \mathbb{R}^n) = \mathbb{P}_{r-1}(T) \). Hence the discrete \( \text{div} \) stability in one element always holds. We discuss the global version. Let

\[
V_h := \{ v_h \in H(\text{div}, \Omega) : v_h|_T \in \mathbb{P}_r(T; \mathbb{R}^n) \text{ for each } T \in \mathcal{T}_h \},
\]

\[
Q_h := \{ q_h \in L^2(\Omega) : q_h|_T \in \mathbb{P}_{r-1}(T) \text{ for each } T \in \mathcal{T}_h \}.
\]

The discrete \( \text{div} \) stability refers to \( \text{div} V_h = Q_h \) and the operator \( \text{div} \) has a continuous right inverse.

We will use the \( L^2 \)-inner product \((\cdot, \cdot)_T\) and define the orthogonal complement of a subspace \( V \subset L^2(T) \) as \( V^\perp \), i.e.,

\[
V^\perp = \{ u \in L^2(T) : (u, v)_T = 0 \quad \forall \ v \in V \}.
\]

We first give the following characterization of the range of the \( \text{div} \) operator on the bubble polynomial space.

**Lemma 3.9** (Lemma 3.6 in [18]). It holds

\[
\text{div} \mathbb{B}_r(\text{div}, T) = \mathbb{P}_{r-1}(T) \cap \mathbb{R}^\perp, \quad r \geq 0.
\]

**Proof.** When \( r = 0, 1 \), both sides are zero. Therefore we focus on \( r \geq 2 \). The inclusion \( \text{div}(\mathbb{B}_r(\text{div}, T)) \subseteq (\mathbb{P}_{r-1}(T) \cap \mathbb{R}^\perp) \) is proved through integration by parts

\[
(\text{div} \ v, p)_T = -(\ v, \text{grad} p)_T = 0, \quad p \in \ker(\text{grad}) = \mathbb{R}.
\]

If \( \text{div}(\mathbb{B}_r(\text{div}, T)) \neq \mathbb{P}_{r-1}(T) \cap \mathbb{R}^\perp \), there exists a \( p \in \mathbb{P}_{r-1}(T) \cap \mathbb{R}^\perp \) and \( p \perp \text{div}(\mathbb{B}_r(\text{div}, T)) \), which is equivalent to \( \text{grad} p \perp \mathbb{B}_r(\text{div}, T) \). Expand the vector \( \text{grad} p \) in the basis \( \{ n_i, i = 1, \ldots, n \} \) as \( \text{grad} p = \sum_{i=1}^n q_i n_i \) with \( q_i \in \mathbb{P}_{r-2}(T) \). Then set \( v_p = \sum_{i=1}^n |\nabla \lambda_i| q_i \lambda_0 \lambda_i t_{i,0} \in \mathbb{B}_r(\text{div}, T) \), where \( t_{i,0} := v_0 - v_i \). We have

\[
(\text{grad} p, v_p)_T = \sum_{i=1}^n \int_T q_i^2 \lambda_0 \lambda_i \, dx = 0,
\]

which implies \( q_i = 0 \) for \( i = 1, \ldots, n \), i.e., \( \text{grad} p = 0 \) and \( p = 0 \) as \( p \in \mathbb{P}_{r-1}(T) \cap \mathbb{R}^\perp \). \( \Box \)

Next we verify the discrete \( \text{div} \) stability.

**Proposition 3.10.** Let \( r \geq 1 \) and \( V_{\text{BDM}} \) be the BDM space defined in Lemma 3.7 and \( Q_h \) defined by (17). It holds the discrete inf-sup condition

\[
\| q_h \|_0 \lesssim \sup_{v_h \in V_{\text{BDM}}} \frac{(\text{div} v_h, q_h)}{\| v_h \|_0 + \| \text{div} v_h \|_0} \quad \forall \ q_h \in Q_h.
\]
Clearly we have
\begin{equation}
\|v\|_1 \lesssim \|q_h\|_0.
\end{equation}

Let \(\tilde{v}_h \in V_{BDM}\) satisfy
\[
\int_F \tilde{v}_h \cdot n_F \, ds = \int_F v \cdot n_F \, ds, \quad F \in \Delta_{n-1}(T_h),
\]
and the rest DoFs vanish. By the scaling argument,
\begin{equation}
\|\tilde{v}_h\|_0 + \|\tilde{v}_h\|_0 \lesssim \|v\|_1 \lesssim \|q_h\|_0.
\end{equation}

Clearly we have \(\text{div}(\tilde{v}_h - v)|_T \in \mathbb{P}_{r-1}(T) \cap \mathbb{R}^d\) for each \(T \in T_h\). By Lemma 3.9, there exists \(b_h \in L^2(\Omega; \mathbb{R}^n)\) such that \(b_h|_T \in B_r(\text{div}, T)\) for each \(T \in T_h\), and
\begin{equation}
\text{div} b_h = \text{div}(v - \tilde{v}_h), \quad \|b_h\|_{0,T} \lesssim h_T \|\text{div}(\tilde{v}_h - v)\|_{0,T}.
\end{equation}

Take \(v_h = \tilde{v}_h + b_h \in V_{BDM}\). By (19) and (21), it holds
\begin{equation}
\text{div} v_h = \text{div} \tilde{v}_h + \text{div} b_h = \text{div} v = q_h.
\end{equation}

It follows from (20) and (21) that
\[
\|v_h\|_0 + \|\text{div} v_h\|_0 = \|v_h\|_0 + \|q_h\|_0 \leq \|\tilde{v}_h\|_0 + \|b_h\|_0 + \|q_h\|_0 \lesssim \|\tilde{v}_h\|_0 + \|\text{div} \tilde{v}_h\|_0 + \|q_h\|_0.
\]

Combining (22)-(23) yields (18). \(\square\)

For the Stenberg-type element, the continuity at normal planes introduces some constraint and makes the discrete inf-sup condition harder. As all bubble functions are treated locally, \(\text{div} B_r(\text{div}, T) = \mathbb{P}_{r-1}(T) \cap \mathbb{R}^d\) still holds. We only need to show \(\cup_{T \in T_h} \mathbb{P}_0(T)\) is in the range of \(\text{div} V_h\), which requires the face average \(\int_F v \cdot n_F \, ds\).

We propose the following modification to have a better discrete inf-sup condition. Denote by
\[
B_{r,m+1}(F) := \left[ \oplus_{\ell=m+1}^{n-1} \oplus_{f \in \Delta_\ell} \mathbb{P}_{r-\ell}(f) \right] \mathbb{P}_0(F), \quad -1 \leq m \leq n - 2.
\]

**Corollary 3.11.** With the same setting in Lemma 3.8 and further assume \(r \geq m + 2\) for \(-1 \leq m \leq n - 2\). Replacing DoF (16b) by
\begin{equation}
\int_F v \cdot n_F \, q \, ds, \quad F \in \Delta_{n-1}(T_h), \quad q \in (B_{r,m+1}(F) \cap \mathbb{P}_0(F)) \oplus \mathbb{P}_0(F),
\end{equation}
will define the same finite element space.

**Proof.** When \(-1 \leq m \leq n - 2\) and \(r \geq m + 2\), \(\dim B_{r,m+1}(F) \geq |\Delta_{m+1}(F)| \geq \dim \mathbb{P}_0(F)\). Therefore the number of DoFs remains the same as DoF (16b).

Vanishing DoFs (16a) will imply \(v \cdot n_F \big|_{F} \in B_{r,m+1}(F)\), which can be determined by (24). So the unisolvence follows.

As the change will not affect the continuity, it will define the same finite element space. \(\square\)

Now \(\int_F v \cdot n_F \, ds\) is in DoF (24). The proof of the following result is identical to that of Proposition 3.10.

**Proposition 3.12.** Let \(-1 \leq m \leq n - 2\) and \(r \geq m + 2\). Let \(V_{h,m}^r\) be the \(H(\text{div})\)-conforming finite element defined in Lemma 3.8. The following discrete inf-sup condition holds with a constant independent of \(h\)
\begin{equation}
\|q_h\|_0 \lesssim \sup_{v_h \in V_{h,m}^r} \frac{\langle \text{div} v_h, q_h \rangle}{\|v_h\|_0 + \|\text{div} v_h\|_0} \quad \forall q_h \in Q_h.
\end{equation}
4. Geometric Decompositions of Matrix Face Elements

In this section, we generalize the geometric decomposition of $H(\text{div})$-conforming vector finite elements to two $H(\text{div})$-conforming matrix finite elements: the traceless matrix $T$ and the symmetric matrix $S$.

4.1. Traceless matrix elements. We consider the $H(\text{div}, \Omega; T)$-conforming finite element spaces, where $T \in \mathbb{R}^{n \times n}$ is the set of square matrices with vanishing trace, i.e., the sum of the diagonal is zero.

We start from the tensor product of the Lagrange element and $T$:

$$\mathbb{P}_r(T; T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} \left[ b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes T \right].$$

That is each component of the matrix function is a Lagrange element of degree $r$ and thus is continuous. To be $H(\text{div})$-conforming, however, normal continuity is sufficient.

To impose the normal continuity of a traceless matrix function, the key is a $t$-n decomposition at each sub-simplex. Here the $t$-n decomposition is with respect to the second component in the tensor product form.

Given a sub-simplex $f \in \Delta_\ell(T)$, choose a $t$-n basis $\{t^f_i, n^f_j\}_{i=1}^{\ell} \times_{j=1}^{n-\ell}$ and decompose $\mathbb{R}^n = \mathcal{T}^f \oplus T^f$. All basis vectors are normalized but may not be mutually orthogonal. By the tensor product the $n \times n$ matrix space $\mathbb{M}$ has the following decomposition

$$\mathbb{M} = (\mathbb{R}^n \otimes \mathcal{T}^f) \oplus (\mathbb{R}^n \otimes T^f).$$

For a matrix $A \in \mathbb{M}$, $\text{tr}^\text{div}(A) = An_f$ and thus $\text{tr}^\text{div}(\mathbb{R}^n \otimes \mathcal{T}^f) = 0$. The normal component $\mathbb{R}^n \otimes T^f$ will contribute to the normal trace.

We then modify (26) for $\mathbb{M}$ to impose the traceless constraint while not changing $\text{tr}^\text{div}$. When computing the trace of a matrix, we use $\text{trace}(u \otimes v) = v \cdot u$. We pick up the element $t^f_i \otimes t^f_i \in \mathbb{R}^n \otimes \mathcal{T}^f$ and use it to modify the basis in $(\mathbb{R}^n \otimes \mathcal{T}^f) \oplus (\mathbb{R}^n \otimes T^f)$ to get the following $t$-n decomposition on $f \in \Delta_\ell(T)$ for $\ell \geq 1$:

$$\mathcal{T}^f(T) := \text{span}\{n^f_j \otimes t^f_i, 1 \leq i \leq \ell, 1 \leq j \leq n - \ell\}$$

$$\oplus \text{span}\{t^f_i \otimes t^f_j - (t^f_i \cdot t^f_j)t^f_i \otimes t^f_i, 1 \leq i, j \leq \ell\},$$

$$\mathcal{T}^f(T) := \text{span}\{t^f_i \otimes n^f_j, 1 \leq i \leq \ell, 1 \leq j \leq n - \ell\}$$

$$\oplus \text{span}\{n^f_i \otimes n^f_j - (n^f_i \cdot n^f_j)t^f_i \otimes t^f_i, 1 \leq i, j \leq n - \ell\}.$$
For Lemma 4.2. Follow the same proof of Lemma 3.5, we have the characterization of the bubble space.

Proof. Since \( T = \mathcal{T}^f(T) \oplus \mathcal{M}^f(T) \), DoFs (27a)-(27b) are equivalent to

\[
\int_f \mathbf{A} : \mathbf{q} \, ds, \quad \mathbf{q} \in \mathbb{P}_{r-(\ell+1)}(f) \otimes T, \ f \in \Delta_\ell(T), \ell = 0, \ldots, n.
\]

Then the unisolvence follows from the unisolvence of the Lagrange element. \( \square \)

Define the bubble polynomial space

\[
\mathbb{B}_r(\text{div}, T; T) := \mathbb{P}_r(T; T) \cap \ker(\text{tr}^\text{div}).
\]

Follow the same proof of Lemma 3.5, we have the characterization of the bubble space.

**Lemma 4.2.** For \( r \geq 2 \), it holds that

\[
\mathbb{B}_r(\text{div}, T; T) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r(\mathcal{M}^f(T)),
\]

and

\[
\text{tr}^\text{div} : \bigoplus_{\ell=0}^{r-1} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r(\mathcal{M}^f(T)) \to \text{tr}^\text{div} \mathbb{P}_r(T; T)
\]

is a bijection.

Similar to the generalized Stenberg element, we can redistribute some normal DoFs onto the \((n-1)\)-dimensional faces to obtain the following \( H(\text{div}; \Omega; T) \) element.

**Theorem 4.3** \((H(\text{div}; \Omega))\)-conforming finite elements. Let \( 0 \leq m \leq n-2 \). For each \( f \in \Delta_\ell(T_h) \) with \( \ell \leq m \), we choose \( n-\ell \) normal vectors \( \{n^f_1, \ldots, n^f_{n-\ell}\} \). For each \( F \in \Delta_{n-1}(T_h) \), choose a normal vector \( n_F \). Then the DoFs

\[
(28a) \quad \mathbf{A} \in \mathcal{L}^2(\Omega; T), \ \mathbf{v} \in \Delta_0(T_h),
\]

\[
(28b) \quad \int_f (\mathbf{A} n^f_i) \cdot \mathbf{q} \, ds, \quad f \in \Delta_\ell(T_h), \mathbf{q} \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{R}^n),
\]

\[
\quad \quad \quad \quad \quad \quad i = 1, \ldots, n-\ell, \ \ell = 1, \ldots, m,
\]

\[
(28c) \quad \int_f (\mathbf{A} n^f_F) \cdot \mathbf{q} \, ds, \quad F \in \Delta_{n-1}(T_h), \ f \in \Delta_\ell(F),
\]

\[
\quad \quad \quad \quad \quad \quad \mathbf{q} \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{R}^n), \ \ell = m+1, \ldots, n-1,
\]

\[
(28d) \quad \int_T \mathbf{A} : \mathbf{q} \, dx, \quad T \in T_h, \ \mathbf{q} \in \mathbb{B}_r(\text{div}, T; T),
\]

will determine a space \( V^r(\Omega) \subset H(\text{div}; \Omega; T) \), where

\[
V^r(\Omega) := \{ \mathbf{A} \in L^2(\Omega; T) : \mathbf{A}|_T \in \mathbb{B}_r(T; T) \ \forall \ T \in T_h, \ \text{DoFs} \ (28a)-(28b) \ \text{are single-valued across} \ f \in \Delta_\ell(T_h) \ \text{for} \ \ell = 0, \ldots, m, \ \text{DoF} \ (28c) \ \text{is single-valued across} \ F \in \Delta_{n-1}(T_h) \}.
\]

Proof. For \( T \in T_h \) and \( f \in \Delta_\ell(T) \), both \( \{n^f_1, \ldots, n^f_{n-\ell}\} \) and \( \{n_F, F \in \Delta_{n-1}(T), f \subset F\} \) are basis of the normal plane \( \mathcal{M}^f \). Then DoFs (28) restricted to \( T \) are equivalent to DoFs (27), thus uniquely determine \( \mathbb{P}_r(T; T) \).

For \( F \in \Delta_{n-1}(T_h) \), DoFs (28a)-(28b) restricted to \( F \) will determine

\[
\int_f (\mathbf{A} n^f_F) \cdot \mathbf{q} \, ds, \quad f \in \Delta_\ell(F), \ \mathbf{q} \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{R}^n), \ \ell = 0, \ldots, m.
\]

Then by the unisolvence of Lagrange element in Theorem 2.5, \( (\mathbf{A} n^f_F) \cdot \mathbf{q} \) is determined by DoFs (28a)-(28c) restricted to \( F \). Therefore \( V^r(\Omega) \subset H(\text{div}; \Omega; T) \). \( \square \)

To have a better discrete \( \text{div} \) stability, we modify the face DoFs.
Corollary 4.4. With the same setting in Theorem 4.3 and further assume \( r \geq m + 2 \) for \( 0 \leq m \leq n - 3 \), and \( r \geq n + 1 \) for \( m = n - 2 \). Replacing DoF (28c) by

\[
\int_F (A_{\mathbf{F}}) \cdot \mathbf{q} \, ds, \quad F \in \Delta_{n-1}(T_h), \quad \mathbf{q} \in \left[ (B_{r,m+1}(F) \cap \mathbb{P}_1^+(F)) \oplus \mathbb{P}_1(F) \right] \otimes \mathbb{R}^n,
\]

will define the same finite element space.

Proof. When \( m = n - 2 \), we require \( r \geq n + 1 \) so that \( r - (m + 1) \geq 1 \) and \( \dim B_{r,m+1}(F) \geq \dim \mathbb{P}_1(F) \). When \( 0 \leq m \leq n - 3 \) and \( r \geq m + 2 \), \( \dim B_{r,m+1}(F) \geq |\Delta_{m+1}(F)| \geq \dim \mathbb{P}_1(F) \). Therefore the number of DoFs remains the same as DoF (28c).

Vanishing DoFs (28a)-(28b) will imply \( A_{\mathbf{F}}|_{F} \in B_{r,m+1}(F) \otimes \mathbb{R}^n \), which can be determined by (29). So the unisolvence follows. As the change will not affect the continuity, it will define the same finite element space.

DoF (29) is more friendly for verifying the discrete inf-sup condition and (28c) is better for the uni-solvence and implementation.

When \( m = 0 \), it is the generalization of Stenberg element for vector functions to traceless tensor functions. Almost all DoFs are redistributed to face \( F \) except at the vertex, where the traceless constraint is imposed. The case \( n = 3 \), \( r \geq 2 \) is the \( H(\text{div}; T) \) element constructed in [17].

Remark 4.5. Comparing with the vector face element, cf. Lemma 3.8. \( m \) starts from \( 0 \) not \(-1 \). Namely the \( H(\text{div}; T) \)-element should be continuous at vertices. We argue that the continuity at vertices is also necessary. Take a vertex in \( \Delta_0(T) \), for example \( v_0 \). Then \( A_{\mathbf{F}_i}|_{v_0} \) is determined by the vector \( A_{\mathbf{F}_i}|_{F_i} \in \mathbb{R}^n \) for \( i = 1, \ldots, n \). If it is continuous on each face but not on vertices, the number of elements in \( A_{\mathbf{F}_i}(v_0) \) is \( n \) for each face. Running \( i \) from 1 to \( n \), \( A(v_0) \) is determined by \( n^2 \) conditions, which is more than \( \dim T = n^2 - 1 \).

4.2. Discrete div stability for traceless tensors. We first show the div stability of the bubble space \( B_r(\text{div}; T; \mathbb{T}) \). Denote by \( t_{i,j} \) the edge vector from \( v_i \) to \( v_j \). By computing the constant directional derivative \( t_{i,j} \cdot \nabla \lambda_\ell \) by values on the two vertices, we have

\[
t_{i,j} \cdot \nabla \lambda_\ell = \delta_{j \ell} - \delta_{i \ell} = \begin{cases} 1, & \text{if } \ell = j, \\ -1, & \text{if } \ell = i, \\ 0, & \text{if } \ell \neq i, j. \end{cases}
\]

Lemma 4.6. The set of traceless tensors \( \{\nabla \lambda_i \otimes t_{i+1,j}\} \) is a basis of the traceless tensor space \( \mathbb{T} \). Its dual basis is \( \{t_{i,j} \otimes \nabla \lambda_\ell \} \). All indices are modulo \( n \).

Proof. It suffices to prove

\[
(t_{\ell,k} \otimes \nabla \lambda_\ell) : (\nabla \lambda_i \otimes t_{i+1,j}) = \delta_{ik} \delta_{j \ell}
\]

for \( 0 \leq i, k \leq n, j \in \{0, \ldots, n\} \setminus \{i, i+1\} \), and \( \ell \in \{0, \ldots, n\} \setminus \{k, k+1\} \).

When \( i = k \), by \( \ell \neq i, i + 1 \), it follows

\[
(t_{\ell,k} \otimes \nabla \lambda_\ell) : (\nabla \lambda_i \otimes t_{i+1,j}) = \nabla \lambda_\ell \cdot t_{i+1,j} = \delta_{j \ell}.
\]

When \( i = \ell \), by \( i \neq j \), it follows

\[
(t_{\ell,k} \otimes \nabla \lambda_\ell) : (\nabla \lambda_i \otimes t_{i+1,j}) = 0.
\]

When \( i \neq k, \ell \), clearly \( (t_{\ell,k} \otimes \nabla \lambda_\ell) : (\nabla \lambda_i \otimes t_{i+1,j}) = 0 \). \hfill \Box

Let \( RT = \{ax + b : a \in \mathbb{R}, b \in \mathbb{R}^n\} \). For a matrix \( A \), define \( \text{dev} \ A = A - \frac{1}{n} \text{trace}(A) I \) as the projection of \( A \) to \( \mathbb{T} \). It is straightforward to verify \( \ker(\text{dev grad}) = RT \). Again let \( RT^\perp \) be the \( L^2 \)-orthogonal complement in \( L^2(T; \mathbb{R}^n) \).
Lemma 4.7. For each \( T \in T_h \), it holds
\[
\text{div} \, B_r(\text{div}, T; T) = P_{r-1}(T; \mathbb{R}^n) \cap RT^\perp.
\]

**Proof.** It follows from the integration by parts that
\[
\text{div} \, B_r(\text{div}, T; T) \subseteq (P_{r-1}(T; \mathbb{R}^n) \cap RT^\perp).
\]

We claim the equality holds. If \( \text{div} \, B_r(\text{div}, T; T) \subset (P_{r-1}(T; \mathbb{R}^n) \cap RT^\perp) \), then there exists \( \mathbf{u} \in P_{r-1}(T; \mathbb{R}^n) \cap RT^\perp \) satisfying the orthogonality condition \( (\mathbf{u}, \text{div} \, \mathbf{A})_T = 0 \) for any \( \mathbf{A} \in B_r(\text{div}, T; T) \). Equivalently
\[
(\text{dev} \, \mathbf{u}, \mathbf{A})_T = 0 \quad \forall \mathbf{A} \in B_r(\text{div}, T; T).
\]

By expressing \( \text{grad} \, \mathbf{u} = \sum_{i=0}^n \sum_{j \in \{0, \ldots, n\} \setminus \{i, i+1\}} q_{ij} (t_{j,i} \otimes \nabla \lambda_j + \frac{1}{n} I) \) with \( q_{ij} \in P_{r-2}(T) \), we choose
\[
\mathbf{A} = \sum_{i=0}^n \sum_{j \in \{0, \ldots, n\} \setminus \{i, i+1\}} \lambda_{i+1,j} q_{ij} \nabla \lambda_i \otimes t_{i+1,j} \in B_r(\text{div}, T; T).
\]

Then we have
\[
\sum_{i=0}^n \sum_{j \in \{0, \ldots, n\} \setminus \{i, i+1\}} (\lambda_{i+1,j} q_{ij}, q_{ij})_T = 0.
\]

Therefore \( q_{ij} = 0 \) for all \( i \) and \( j \). Thus \( \mathbf{u} = 0 \). \( \square \)

We mention that characterization (30) in three dimensions is firstly proved in [29].

**Proposition 4.8** (Discrete inf-sup condition for \( H(\text{div}; T) \)-conforming finite elements). Let \( 0 \leq m \leq n - 2 \). Let \( r \geq m + 2 \) for \( 0 \leq m \leq n - 3 \), and \( r \geq n + 1 \) for \( m = n - 2 \). Let \( V^r(T) \) be defined in Theorem 4.3. It holds the discrete inf-sup condition
\[
\|\mathbf{v}_h\|_0 \lesssim \sup_{\mathbf{A}_h \in V^r(T)} \frac{(\text{div} \, \mathbf{A}_h, \mathbf{v}_h)}{\|\mathbf{A}_h\|_0 + \|\text{div} \, \mathbf{A}_h\|_0} \quad \forall \mathbf{v}_h \in Q_h,
\]

where \( Q_h := \{ \mathbf{v}_h \in L^2(\Omega; \mathbb{R}^n) : \mathbf{v}_h|_T \in P_{r-1}(T; \mathbb{R}^n) \text{ for each } T \in \mathcal{T}_h \} \).

**Proof.** First there exists \( \mathbf{A} \in H^1(\Omega, \mathcal{T}) \) such that
\[
\text{div} \, \mathbf{A} = \mathbf{v}_h, \quad \|\mathbf{A}\|_1 \lesssim \|\mathbf{v}_h\|_0.
\]

Thanks to DoFs (28) and (29), we let \( \widetilde{\mathbf{A}}_h \in V^r(\mathcal{T}) \) satisfy
\[
\int_F (\widetilde{\mathbf{A}}_h \mathbf{n}_F) \cdot \mathbf{q} \, ds = \int_F (\mathbf{A}_0 \mathbf{n}_F) \cdot \mathbf{q} \, ds, \quad \mathbf{q} \in P_1(F; \mathbb{R}^n), F \in \Delta_{n-1}(\mathcal{T}_h),
\]

and other DoFs vanish. By the scaling argument,
\[
\|\widetilde{\mathbf{A}}_h\|_0 + \|\text{div} \, \widetilde{\mathbf{A}}_h\|_0 \lesssim \|\mathbf{A}\|_1 \lesssim \|\mathbf{v}_h\|_0.
\]

Then through integration by parts we have \( \text{div} (\mathbf{A}_h - \mathbf{A})|_T \in P_{r-1}(T; \mathbb{R}^n) \cap RT^\perp \) for each \( T \in \mathcal{T}_h \). By Lemma 4.7, there exists \( \mathbf{b}_h \in L^2(\Omega; \mathcal{T}) \) such that \( \mathbf{b}_h|_T \in B_r(\text{div}, T; T) \) for each \( T \in \mathcal{T}_h \), and
\[
\text{div} \, \mathbf{b}_h = \text{div} (\mathbf{A} - \mathbf{A}_h), \quad \|\mathbf{b}_h\|_{0,T} \lesssim h_T \|\text{div} (\mathbf{A}_h - \mathbf{A})\|_{0,T}.
\]

Take \( \mathbf{A}_h = \mathbf{A}_h + \mathbf{b}_h \in V^r(T) \). By (32) and (34), it holds
\[
\text{div} \, \mathbf{A}_h = \text{div} \, \widetilde{\mathbf{A}}_h + \text{div} \, \mathbf{b}_h = \text{div} \, \mathbf{A} = \mathbf{v}_h.
\]
It follows from (33) and (34) that
\[
\| A_h \|_0 + \| \text{div} \ A_h \|_0 = \| A_h \|_0 + \| v_h \|_0 \leq \| \bar{A}_h \|_0 + \| b_h \|_0 + \| v_h \|_0
\]
(36)
\[
\lesssim \| \bar{A}_h \|_0 + \| b_h \|_0 \| \text{div} \ A_h \|_0 + \| v_h \|_0 \lesssim \| v_h \|_0.
\]
Combining (35)-(36) yields (31).

4.3. Symmetric matrix elements. We start from the tensor product of the Lagrange element and the symmetric matrix $S$:
\[
P_r(T; S) = \bigoplus_{i=0}^{n} \bigoplus_{f \in \Delta(T)} [b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes S].
\]
The construction process is similar to the traceless case in which $t$-$n$ decompositions of $S$ at sub-simplices are the key. Additional complication arises as the $n(n-1)/2$ symmetry constraints are more complicated than only 1 traceless constraint.

Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$. Then $M = \{\sum_{i,j=1}^{n} a_{ij} v_i \otimes v_j, a_{ij} \in \mathbb{R}\}$ is the space of $n \times n$-matrices. An element $A \in M$ can be identified with the coefficient matrix $(a_{ij})$ and will be still denoted by $A$. For a better explanation, we illustrate an $n \times n$-matrix by $n \times n$ blocks; see Fig. 4 for the case $n = 3$. Block $(i,j)$ corresponds to the basis function $v_i \otimes v_j$. We will identify $n(n+1)/2$ blocks and modify corresponding basis function to form a basis of $S$ with consideration of the normal continuity.

![Diagram](image)

\textbf{Figure 4.} Several $t$-$n$ decompositions for $S$ in $\mathbb{R}^3$. Blocks with the same symbol (circle, square, or diamond) are in the same constraint sequence and the white block is used as the pair index. Color of the block represents: Green: free rows and free blocks; Blue: all free indices not in free rows; Red: bubble functions. Blue or green blocks are free indices in $N_f(S)$. All white blocks are pair indices and the corresponding coefficients are determined by the free variables through the constraints.

Given an $f \in \Delta(T)$, choose a $t$-$n$ basis $\{t_f, n_f\}_{i=1, \ldots, \ell}$ and decompose $\mathbb{R}^n = \mathcal{F}^f \oplus N_f$. All basis vectors are normalized. We have the matrix decomposition
\[
M = (\mathbb{R}^n \otimes \mathcal{F}^f) \oplus (\mathbb{R}^n \otimes N_f).
\]
We modify each component to impose the symmetric constraint. For the tangential component, we simply take
\[
\mathcal{F}^f(S) := (\mathbb{R}^n \otimes \mathcal{F}^f) \cap S.
\]
Using the tensor product with the scalar bubble polynomial, we can construct
\[
\mathbb{B}_r \mathcal{F}^f(S) := b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{F}^f(S) \subset \ker(\text{tr}^{\text{div}}) \cap \mathbb{P}_r(T; S).
\]
We provide a specific example for \( n = 3, \ell = 2 \), i.e., a \( t-n \) decomposition on a face \( F \) of a tetrahedron. Choose a \( t-n \) basis \( \{ t^F_1, t^F_2, n_F \} \). An element in \( \mathbb{R}^n \otimes \mathcal{T}^F \) has the matrix representation \(
abla \)\( \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \)\), where \(*\) represents a generic value. The symmetric constraint implies an element in \( \mathcal{T}^F(S) \) has the form \(
abla \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). The value \( a_{21} \) is left blank since due to the symmetric constraint it is equal to \( a_{12} \). We call \( (1, 2) \) a free index while \( (2, 1) \) is the pair index of \( (1, 2) \).

We then move to the normal component. A naive definition of \( \mathcal{N}^f(S) \) would be \( (\mathbb{R}^n \otimes \mathcal{N}^f) \cap S \). Unfortunately in general for three subspaces of a vector space:

\[
(A \cap C) \oplus (B \cap C) \subseteq (A \oplus B) \cap C.
\]

And the equality may not hold. We continue the example with \( n = 3, \ell = 2 \). An element in \( \mathbb{R}^n \otimes \mathcal{N}^f \) has the matrix form \( \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \) w.r.t the basis \( \{ t^F_1, t^F_2, n_F \} \). An element in \( (\mathbb{R}^n \otimes \mathcal{N}^f) \cap S \) has the form \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \). Then

\[
\dim((\mathbb{R}^n \otimes \mathcal{T}^F) \cap S) + \dim((\mathbb{R}^n \otimes \mathcal{N}^f) \cap S) = 3 + 1 = 4 < 6 = \dim S.
\]

The discrepancy is due to the fact that the constraints \( a_{13} = a_{31} \) and \( a_{23} = a_{32} \) have been used duplicately in both \( (\mathbb{R}^n \otimes \mathcal{T}^F) \cap S \) and \( (\mathbb{R}^n \otimes \mathcal{N}^f) \cap S \). We need to make sure one constraint is used only once either in \( \mathcal{T}^f(S) \) or \( \mathcal{N}^f(S) \).

To this end, for each constraint, we set the index \( (j, i) \) with \( i < j \) as the pair index and call \( (i, j) \) a free index. The pair index value \( a_{ij} \) is determined by the free variable \( a_{ji} \) through the constraint \( a_{ij} = a_{ji} \).

We introduce the concept: normal constraints. We call the constraint \( a_{ij} = a_{ji}, i \neq j \), a normal constraint if both \( (i, j) \) and \( (j, i) \) are in the normal component, i.e., \( i, j \geq \ell + 1 \) w.r.t the \( t-n \) basis \( \{ t^F_i, n^F_j \}_{i=1, ..., \ell} \). For example, in Fig. 4 (b), for an edge \( e \in \Delta_1(T) \), there are \( (n - 1)(n - 2)/2 \) normal constraints. As the constraint involves two entries, no normal constraints for \( F \in \Delta_{n-1}(T) \); see Fig. 4 (c). The normal constraints will be imposed inside the normal components.

For non-normal constraint, in the pair index \( (j, i) \), \( i \) is inside the tangential component. We can use it to change the basis without affecting the normal trace. As an example, consider the circle block in Fig. 4 (b)-(c), the basis function is changing from \( t \otimes n \) to \( \text{sym}(t \otimes n) = (n \otimes t + t \otimes n)/2 \) and \( 2 \text{sym}(t \otimes n)n_F = (t \otimes n)n_F \). By doing this way, we ensure each constraint is used only once and the normal trace remains unchanged.

In summary, for \( f \in \Delta_\ell(T), \ell = 0, \ldots, n \), define

\[
\mathcal{T}^f(S) := \text{span}\{ \text{sym}(t^F_i \otimes t^F_j), 1 \leq i \leq j \leq \ell \};
\]

\[
\mathcal{N}^f(S) := \text{span}\{ \text{sym}(t^F_i \otimes n^F_j), 1 \leq i \leq \ell, 1 \leq j \leq n - \ell \}
\]

\[
\oplus \text{span}\{ \text{sym}(n^F_i \otimes n^F_j), 1 \leq i \leq j \leq n - \ell \}.
\]

By counting the dimensions, it is easy to show the direct decomposition

\[
S = \mathcal{T}^f(S) \oplus \mathcal{N}^f(S).
\]

Then tensor product with the scalar bubble function to obtain

\[
\mathbb{B}_r \mathcal{T}^f(S) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{T}^f(S), \quad \mathbb{B}_r \mathcal{N}^f(S) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{N}^f(S).
\]

With the \( t-n \) decomposition, (37) can be rewritten as

\[
\mathbb{P}_r(T; S) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} [\mathbb{B}_r \mathcal{T}^f(S) \oplus \mathbb{B}_r \mathcal{N}^f(S)].
\]
Define the polynomial bubble space
\[ \mathbb{B}_r(\text{div}, T; S) := \ker(\text{tr}^\text{div}) \cap \mathbb{P}_r(T; S). \]
Again the tangential component \( \mathbb{B}_r(\mathcal{T}^f(S)) \) will contribute to the bubble space. Namely for \( r \geq 2 \), it holds that
\[ \mathbb{B}_r(\text{div}, T; S) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r(\mathcal{T}^f(S)). \]
Characterization of \( \mathbb{B}_r(\text{div}, T; S) \) in (38) is new and different with the one given in [28, 30].

Unlike the traceless case, not all normal DoFs can be redistributed to faces since the normal constraint should be imposed on \( \mathcal{N}^f \otimes \mathcal{N}^f \) with a global normal basis \( \{ n^f_i \} \), which can be thought of as the super-smoothness induced by the constraint. For example, for symmetric matrix \( A \), the restriction of \( A \) to the normal plane of \( f \), which is a symmetric matrix of smaller size \( (n - \ell) \times (n - \ell) \), should be continuous due to DoF (39b). The tangential-normal component can be redistributed to face \( F \). Therefore, in DoFs (39), (39c) is posed globally and (39d) is facewisely.

**Theorem 4.9** \( (H(\text{div}; S)-\text{conforming finite elements}) \). Let \( 0 \leq m \leq n-2 \). For each \( f \in \Delta_\ell(T_h) \), we choose a global \( t-n \) basis \( \{ t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell} \} \). Then the DoFs
(39a) \[ A(v), \quad v \in \Delta_0(T_h), A \in \mathbb{S}, \]
(39b) \[ \int_f ((n^f_i)^\top A n^f_j) q \, ds, \quad f \in \Delta_\ell(T_h), q \in \mathbb{P}_{r-(\ell+1)}(f), \]
\[ 1 \leq i \leq j \leq n - \ell, \quad \ell = 1, \ldots, n-1, \]
(39c) \[ \int_f ((t^f_i)^\top A n^f_j) q \, ds, \quad f \in \Delta_\ell(T_h), q \in \mathbb{P}_{r-(\ell+1)}(f), \]
\[ i = 1, \ldots, \ell, \quad j = 1, \ldots, n - \ell, \quad \ell = 1, \ldots, m, \]
(39d) \[ \int_f ((t^f_i)^\top A n_f) |_F \ q \, ds, \quad F \in \Delta_{n-1}(T_h), \quad f \in \Delta_\ell(F), q \in \mathbb{P}_{r-(\ell+1)}(f), \]
\[ i = 1, \ldots, \ell, \quad \ell = m + 1, \ldots, n-1, \]
(39e) \[ \int_T A : q \, dx, \quad T \in T_h, \quad q \in \mathbb{B}_r(\text{div}, T; S), \]
will determine a space \( V^r(S) \subset H(\text{div}, \Omega; S) \), where
\[ V^r(S) := \{ A \in L^2(\Omega; S) : A |_T \in \mathbb{P}_r(T; S) \quad \forall T \in T_h, \}
DoFs (39a)-(39b) are single-valued across \( f \in \Delta_\ell(T_h) \) for \( \ell = 0, \ldots, n-1, \)
DoF (39c) is single-valued across \( f \in \Delta_\ell(T_h) \) for \( \ell = 1, \ldots, m, \)
DoF (39d) is single-valued across \( F \in \Delta_{n-1}(T_h) \).

**Proof.** For \( T \in T_h \) and \( f \in \Delta_\ell(T) \), both \( \{ n^f_1, \ldots, n^f_{n-\ell} \} \) and \( \{ n_F, F \in \Delta_{n-1}(T_h), f \subset F \} \) are basis of the normal plane \( \mathcal{N}^f \). DoFs (39a)-(39d) restricted to \( T \) will determine normal component \( \mathbb{B}_r(\mathcal{T}^f(S)) \) and (39e) for the tangential component \( \mathbb{B}_r(\mathcal{T}^f(S)) \), thus uniquely determine \( \mathbb{P}_r(T; S) \).

For \( F \in \Delta_{n-1}(T_h) \), DoFs (39a)-(39d) restricted to \( F \) will determine
\[ \int_f (A n_f) |_F \cdot q \, ds, \quad f \in \Delta_\ell(F), q \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{R}^n), \quad \ell = 0, \ldots, n-1, \]
and thanks to the unisolvence of Lagrange element in Theorem 2.5 will determine \((\mathbf{A}_h)\big|_F\). Therefore \(V^r(S) \subset H(\text{div}; \Omega; S)\).

When \(m = n - 2\), it is the Hu-Zhang element [28, 30]. When \(m = 0\), DoF (39d) can be further merged to one and lead to the modification in [18, Lemma 4.5]

\[
\int_F (\Pi_F \mathbf{A}_h \nabla) \cdot \mathbf{q} \; ds, \quad F \in \Delta_{n-1}(T_h), \mathbf{q} \in \text{ND}_{r-2}(F),
\]

where \(\text{ND}_{r-2}(F) := \{ \mathbf{q} \in \mathbb{P}_{r-1}(F; \mathbb{R}^{n-1}) : \mathbf{q} \cdot \mathbf{x} \in \mathbb{P}_{r-1}(F) \} \) and \(\Pi_F\) is the projection of a vector to the plane \(\mathcal{T}^F\).

4.4. Discrete \(\text{div}\) stability for symmetric tensors. For each \(T \in T_h\), the range of the \(\text{div}\) operator on the bubble space of symmetric tensors [28, 30] is

\[
\text{div} B_r(\text{div}; T; \mathbb{S}) = \mathbb{P}_{r-1}(T; \mathbb{R}^n) \cap \text{RM}^\perp,
\]

where \(\text{RM} = \{ \mathbf{N} \mathbf{x} + \mathbf{b} : \mathbf{N} \in \mathbb{K}, \mathbf{b} \in \mathbb{R}^n \}\) and \(\text{RM}^\perp\) is the \(L^2\)-orthogonal complement in \(L^2(T; \mathbb{R}^n)\). It can be proved similar to Lemma 4.7 and an abstract version will be proved in Lemma 6.9.

Applying the same argument as in Proposition 4.8, we derive the \(\text{div}\) stability of space \(V^r(S)\) from (41).

**Proposition 4.10.** Let \(r \geq n + 1\) and \(V^r(S)\) be defined in Theorem 4.9. It holds the discrete inf-sup condition

\[
\| \mathbf{v}_h \|_0 \lesssim \sup_{\mathbf{A}_h \in V^r(S)} \frac{(\text{div} \mathbf{A}_h, \mathbf{v}_h)}{\| \mathbf{A}_h \|_0 + \| \text{div} \mathbf{A}_h \|_0} \quad \forall \mathbf{v}_h \in Q_h.
\]

Notice that due to the extra normal continuity, we cannot modify the face DoFs to relax the degree requirement \(r \geq n + 1\) to \(r \geq 2\) as we have done for the traceless element in Proposition 4.8. Thanks to DoF (40), the tangential-normal component contains \(\mathbb{P}_1(F)\) for \(r \geq 2\). Lower order \(H(\text{div}; S)\)-conforming finite elements are designed by enriching the symmetric quadratic polynomial space with only \((n + 1)\)-order normal-normal face bubbles in [33], which have only \(n(n + 1)^2\) DoFs for the reduced one.

5. Constraint Tensor Spaces

In this section we shall introduce the constraint tensor space \(\mathbb{X}\) as a kernel space, and discover bases of \(\mathbb{X}\). We first recall some background on differential forms, then give concrete formulae on the algebraic operator \(s^{k,n-1}\) and define the constraint tensor space \(\mathbb{X}\) as the kernel of \(s^{k,n-1}\). Lastly we present two bases of \(\mathbb{X}\).

5.1. Background on differential forms.

5.1.1. Increasing sequence. We mainly follow the notation set in [6] but with some simplification. For non-negative integers \(a, b, l, m\), with \(0 \leq b - a \leq m - l\), define the set of increasing sequences as

\[
\Sigma(a : b, l : m) := \{ \sigma : \{a, \ldots, b\} \to \{l, \ldots, m\} \mid \sigma(a) < \sigma(a + 1) < \cdots < \sigma(b) \}.
\]

We will overload the notation \(\sigma\) as its range, i.e., for \(\sigma \in \Sigma(a : b, l : m)\), we use the same notation \(\sigma\) to refer to the set \(\{\sigma(i) \mid i = a, \ldots, b\}\). The set \(\Sigma(0 : k, 0 : n)\) will be mainly used for the description of sub-simplices, and \(\Sigma(1 : k, 1 : n)\) for \(k\) differential forms in \(\mathbb{R}^n\). For \(\sigma \in \Sigma(0 : k, 0 : n)\), \(f_{\sigma} \in \Delta_k(T)\) is the sub-simplex formed by vertices with index \(\{\sigma(0), \ldots, \sigma(k)\}\). On the other hand, for \(f \in \Delta_k(T)\), the index of its vertices can be sorted in ascending order to get an increasing sequence \(\sigma_f\).

For \(\sigma \in \Sigma(0 : k, 0 : n)\), denote by \(\sigma^c \in \Sigma(1 : n - k, 0 : n)\) the complementary map characterized by

\[
\sigma \cup \sigma^c = \{0, 1, \ldots, n\}.
\]

For \(\sigma \in \Sigma(1 : k, 1 : n)\), its complementary map \(\sigma^c \in \Sigma(1 : n - k, 1 : n)\) satisfies

\[
\sigma \cup \sigma^c = \{1, \ldots, n\}.
\]
For the unique element in \( \Sigma(a : b, a : b) \), we simply write it as \([a : b]\).

We follow [35] to introduce notation on the addition and subtraction of increasing sequences. Let \( \sigma \in \Sigma(a : b, l : m) \). If \( q \in [l : m] \setminus \sigma \), then we write \( \sigma + q = q + \sigma \) for the unique element of \( \Sigma(a : b + 1, l : m) \) with image \( \sigma \cup \{q\} \).

In that case, we also write \( \epsilon(q, \sigma) \) for the signum of the permutation that orders the sequence \([q, \sigma(a), \ldots, \sigma(b)]\) in the ascending order. For \( q \in \sigma \), \( \sigma - q \) is the unique element in \( \Sigma(a : b - 1, l : m) \) s.t. \( (\sigma - q) + q = \sigma \).

5.1.2. Differential forms. We consider an \( n \)-dimensional domain \( \Omega \subset \mathbb{R}^n \). Usually we choose a Cartesian coordinate and describe a point \( x = (x_1, \ldots, x_n) \in \Omega \) in this coordinate. We also use \( \mathbb{R}^n \) to denote the \( n \)-dimensional linear vector space, which can be identified with the space of points by identifying a point \( x \) with the vector \( x = \partial x \). We use \( \partial x \), as the unit vector from the origin \( o \) to point \( (0,1, \ldots, 1,0) \), which is considered as an element in the tangent space \( T_x\Omega \). Its dual basis of \( (\mathbb{R}^n)^* \) is denoted by \( \{dx_i\}_{i=1}^n \), i.e., \( dx_i(\partial x_j) = \delta_{ij} \). We use the standard inner product of vectors to make \( \mathbb{R}^n \) a Hilbert space, which introduces an inner product on \( (\mathbb{R}^n)^* \): \( \langle dx_i, dx_j \rangle = \delta_{ij}, i, j = 1, \ldots, n \).

We shall reserve notation \( \{dx_i\}^n_{i=1} \) for the orthonormal basis induced by the ambient orthonormal coordinate of \( \mathbb{R}^n \).

A generic basis will be denoted by \( \{dy_i\}^n_{i=1} \), which may not be orthonormal. We can find another basis \( \{dy_i\}^n_{i=1} \) dual to \( \{dx_i\}^n_{i=1} \) in the sense that \( \langle dy_i, dy_j \rangle = \delta_{ij} \). Indeed let \( M = (\langle dy_i, dy_j \rangle)^n_{i,j=1} \). Then \( (dy_1, \ldots, dy_n)^T = M^{-1}(dy_1, \ldots, dy_n)^T \). When \( \{dy_i\}^n_{i=1} \) is orthonormal, \( dy_i = dy_i \) for \( i = 1, \ldots, n \) as \( M \) is identity.

For a vector space \( V \), we define the space of exterior \( k \)-forms as the alternating multilinear functional space on \( V^k := \bigwedge^k V \) and denote it by \( \bigwedge^k(V) \) or simply \( \bigwedge^k \) if \( V \) is clear in the context. By definition, \( \bigwedge^k \subset (V^k)^* \).

The best way to study a \( k \)-form is through the action on \( k \) vectors in \( V \).

Let \( \omega \in \bigwedge^p \) and \( \eta \in \bigwedge^q \), we define the wedge product \( \omega \wedge \eta \in \bigwedge^{p+q} \) as:

\[
(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_{\sigma} \text{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),
\]

where the sum is over all permutations \( \sigma \) of \( \{1, \ldots, p+q\} \), for which \( \sigma(1) < \sigma(2) < \cdots < \sigma(p) < \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q) \), and \( \text{sign}(\sigma) \) is the signature of the permutation \( \sigma \). We have the determinant formula on the wedge product. For \( \omega_i \in V^*, v_i \in V, i = 1, \ldots, p, \)

\[
(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, \ldots, v_p) = \det (\omega_i(v_j))_{i,j=1,\ldots,p}.
\]

For a smooth manifold \( \Omega \), a \( k \)-th order differential form is a section of the tangent bundle \( \bigcup_{x \in \Omega} \bigwedge^k(T_x\Omega) \), where \( T_x\Omega \) is the tangent space at \( x \). The linear space formed by all \( k \)-th differential forms is denoted by \( \bigwedge^k(\Omega) \), or simply \( \Lambda^k \). As \( \Omega \) is a domain in \( \mathbb{R}^n \), given any point \( x \) in the interior of \( \Omega \), the tangent space \( T_x\Omega \) is isomorphism to \( T_o\Omega \) by shifting the origin to \( x \). That is we can use one basis \( \{dy_i\} \) for all \( \bigwedge^1(T_x\Omega) \), \( x \in \Omega \).

For \( \sigma \in \Sigma(1 : k, 1 : n) \), we extend the multi-index notation to write \( dy_\sigma \in \bigwedge^k \):

\[
dy_\sigma := dy_{\sigma(1)} \wedge \cdots \wedge dy_{\sigma(k)}.
\]

An element \( \omega \in \Lambda^k(\Omega) \) thus has a representation

\[
(42) \quad \omega = \sum_{\sigma \in \Sigma(1 : k, 1 : n)} a_\sigma(x) \, dy_\sigma, \quad x \in \Omega.
\]

For a manifold, the basis \( \{dy_\sigma\}^n_{\sigma=1} \) is defined on a local chart where in (42), as \( \Omega \) is flat, a global coordinate is used.

Using (42), we define the exterior derivative \( d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega) \) as: for \( \omega = \sum_{\sigma \in \Sigma(1 : k, 1 : n)} a_\sigma(x) \, dy_\sigma \), define \( d\omega \in \Lambda^{k+1}(\Omega) \) by

\[
d\omega = \sum_{\sigma \in \Sigma(1 : k, 1 : n)} \sum_{i \not\in \sigma} \partial_i a_\sigma \, dy_i \wedge dy_\sigma = \sum_{\sigma \in \Sigma(1 : k, 1 : n)} \sum_{i \not\in \sigma} (\partial_i a_\sigma) \, \epsilon(i, \sigma) \, dy_{i+\sigma}.
\]

It can be verified that this definition of \( d\omega \) is independent of the choice of bases.
The Hodge star for the ambient orthonormal basis \( \{ dx_i \}_{i=1}^n \) is defined as
\[
\star dx_i = (-1)^{i-1} dx^i, \quad \star dx^i = (-1)^{n-i} dx_i,
\]
which satisfy
\[
dx_i \wedge \star dx_i = dx, \quad dx^i \wedge \star dx^i = dx,
\]
with the volume \( dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \). By definition, \( \star \omega = (-1)^{n-1} \omega \) for \( \omega \in \text{Alt}^1 \) or \( \text{Alt}^{n-1} \).

We extend the definition to a generic coordinate and define
\[
\star dy_i := (-1)^{i-1} dy^i \quad \text{satisfying} \quad dy_i \wedge \star dy_i = \delta_{i,j} dy.
\]

**5.1.3. Inner product.** An intrinsic definition of an inner product on \( \text{Alt}^k \) is
\[
\langle \omega, \eta \rangle := \sum_{\sigma \in \Sigma(1:k,1:n)} \omega(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)})\eta(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)}), \]
where \( (e_1, \ldots, e_n) \) is any orthonormal basis of \( \mathbb{R}^n \). Then by definition
\[
\omega \wedge \star \eta = \langle \omega, \eta \rangle dx, \quad \omega, \eta \in \text{Alt}^k.
\]

Recall that \( \{ dx_i \}_{i=1}^n \) is an orthonormal basis of \( \text{Alt}^1 \), i.e., \( \langle dx_i, dx_j \rangle = \delta_{i,j}, i,j = 1, \ldots, n \). It is naturally extended to an orthonormal basis \( \{ dx_\sigma, \sigma \in \Sigma(1:k,1:n) \} \) of \( \text{Alt}^k \), i.e.
\[
\langle dx_\sigma, dx_{\sigma'} \rangle = \delta_{\sigma,\eta'}, \quad \sigma, \eta \in \Sigma(1:k,1:n).
\]

The duality of \( \{ dy_i \}_{i=1}^n \) and \( \{ dy_i \}_{i=1}^n \) are also extended to the k-forms
\[
\langle dy_\sigma, dy_{\sigma'} \rangle = \delta_{\sigma,\eta'}, \quad \sigma, \eta \in \Sigma(1:k,1:n).
\]

But \( \{ dy_i \}_{i=1}^n \) may not be orthogonal.

For \( \omega, \eta \in \Lambda^k(\Omega) \), a further integral over the domain is included, i.e.,
\[
\langle \omega, \eta \rangle_\Omega = \int_\Omega \langle \omega, \eta \rangle dx, \quad \omega, \eta \in \Lambda^k(\Omega).
\]

For a sub-manifold \( f \) of \( \Omega \), the volume form \( dx \) will induce the one for \( f \) and denoted by \( dx_f \). Define
\[
\langle \omega, \eta \rangle_f = \int_f \langle \omega, \eta \rangle dx_f.
\]

For \( \omega, \eta \in \Lambda^k(\Omega) \) with expression
\[
\omega = \sum_{\sigma \in \Sigma(1:k,1:n)} a_\sigma(x) dx_\sigma, \quad \text{and} \quad \eta = \sum_{\sigma \in \Sigma(1:k,1:n)} b_\sigma(x) dx_\sigma,
\]
it is easy to prove that
\[
\langle \omega, \eta \rangle_\Omega = \sum_{\sigma \in \Sigma(1:k,1:n)} \int_\Omega a_\sigma(x)b_\sigma(x) dx,
\]
and (44) is invariant when changing to another orthonormal basis. For non-orthonormal basis, transformation will enter the formulae of the inner product.

Denote by \( \mathbb{P}_r \Lambda^k(\Omega) \) the space with polynomial coefficients, and \( L^2 \Lambda^k(\Omega) \) is the space with square-integrable coefficient functions. The space \( H\Lambda^k(\Omega) := \{ \omega \in L^2 \Lambda^k(\Omega) : d\omega \in L^2 \Lambda^{k+1}(\Omega) \} \). When \( k = n - 1 \), \( H\Lambda^{n-1}(\Omega) \) is isomorphism to \( H(\text{div}, \Omega) := \{ \upsilon \in L^2(\Omega; \mathbb{R}^n) : \text{div} \upsilon \in L^2(\Omega) \} \).
5.1.4. Proxy vectors of differential forms. Representation (42) enables us to identify a differential form with a vector function:

\[ \omega \leftrightarrow (a_\sigma)_{\sigma \in \Sigma(1:k;1:n)} \]

and \((a_\sigma)\) is called a vector proxy of \(\omega\). Be aware that, by definition, the differential form is coordinate independent while a vector proxy depends on the coordinate.

For a 1-form \(\omega = \sum_{i=1}^n u_i \, dx_i \in \Lambda^1\), define

\[ \text{Prox}_1(\omega) = u = (u_1, u_2, \ldots, u_n)^T. \]

For a vector \(t = (t_1, t_2, \ldots, t_n)^T\) representing the tangent vector \(\sum_{i=1}^n t_i \partial x_i\), the action is

\[ \omega(t) = \sum_{i,j=1}^n u_i t_j \langle dx_i, \partial x_j \rangle = u \cdot t. \]

For \(\omega \in \Lambda^{n-1}\), we can write \(\omega = \sum_{i=1}^n u_i \star dx_i\), which induces an isomorphism

\[ \text{Prox}_{n-1} : \omega \rightarrow u = (u_1, u_2, \ldots, u_n)^T. \]

By definition, we have

\[ \text{Prox}_1(\star \omega) = (-1)^{n-1} \text{Prox}_{n-1}(\omega), \quad \omega \in \Lambda^{n-1}, \]

\[ \text{Prox}_{n-1}(\star \omega) = \text{Prox}_1(\omega), \quad \omega \in \Lambda^1. \]

Notice that the proxy vectors are defined using an orthonormal basis. Using the proxy vectors, we can change the wedge product to the inner product of vectors

\[ \langle \omega, \eta \rangle = (\text{Prox}_1 \omega, \text{Prox}_{n-1}(\star \eta)), \quad \omega, \eta \in \Lambda^1, \]

\[ \langle \omega, \eta \rangle = (\text{Prox}_k \omega, \text{Prox}_k \eta), \quad \omega \in \Lambda^k, \eta \in \Lambda^k, k = 1 \text{ or } n-1, \]

\[ \omega \wedge \eta = (\text{Prox}_1 \omega, \text{Prox}_{n-1} \eta) dx, \quad \omega \in \Lambda^1, \eta \in \Lambda^{n-1}. \]

For \(\omega \in \Lambda^{n-1}\), the representation of \(d\omega\) using the proxy vector is \(\text{div } u\), i.e.,

\[ d\omega = (\text{div } u) \, dx_i, \quad \text{with } u = \text{Prox}_{n-1} \omega. \]

The Prox operator is a bijection. More precisely, given a vector \(u = (u_1, \ldots, u_n)^T\) represented in the ambient coordinate, let \(\omega = \sum_{i=1}^n u_i \, dx_i \in \Lambda^1\) and \(\star \omega = \sum_{i=1}^n u_i \star dx_i \in \Lambda^{n-1}\). Then \(\text{Prox}_1(\omega) = \text{Prox}_{n-1}(\star \omega) = u\).

To resemble the notation of differential forms, we introduce notation

\[ d\omega := \text{Prox}^{-1}_1(u) = \sum_{i=1}^n u_i \, dx_i, \quad \star d\omega := \text{Prox}^{-1}_{n-1}(u) = \sum_{i=1}^n u_i \star dx_i. \]

Here in \(d\omega\) \(d\) is understood as a dual operator mapping a tangent vector \(u\) to a co-tangent vector \(d\omega \in \Lambda^1\), and the symbol \(d\) is not associated to any differentiation. A textbook notation of \(d\omega\) is \(\text{div } u\) and \(\star d\omega\) is \(\text{curl } u\).

Denote the proxy vector of \(\{d\hat{y}_i\}_{i=1}^n\) by \(\{v_i\}_{i=1}^n\) and \(\{d\hat{y}_i\}_{i=1}^n\) by \(\{\hat{v}_i\}_{i=1}^n\). Then \(\{\hat{v}_i\}_{i=1}^n\) is dual to \(\{v_i\}_{i=1}^n\) in the sense that \((\hat{v}_i, v_j) = \delta_{i,j}\). Treat \(v_i\) as a column vector and form the matrix \(V = (v_1, \ldots, v_n)\) and \(\hat{V} = (\hat{v}_1, \ldots, \hat{v}_n)\). The gram matrix is \(M = V^T V\). Then we have the relation \(\hat{V} = VM^{-1}\).

From

\[ (v_i, \text{Prox}_{n-1}(\star d\hat{y}_j)) \, dx = d\hat{y}_i \wedge \star d\hat{y}_j = \delta_{i,j} \, dy = \delta_{i,j} \det(V) \, dx, \]

\[ (\hat{v}_i, \text{Prox}_{n-1}(\star d\hat{y}_j)) \, dx = d\hat{y}_i \wedge \star d\hat{y}_j = \delta_{i,j} \, dy = \delta_{i,j} \det(\hat{V}) \, dx, \]
we get

\[ \text{Prox}_{n-1}(s y_i) = \det(V) \text{Prox}_1(dy_i) = \det(V) \hat{v}_i, \]

\[ \text{Prox}_{n-1}(s y_i) = \det(\hat{V}) \text{Prox}_1(dy_i) = \det(\hat{V}) v_i, \]

where the scaling \( \det(V) \) or \( \det(\hat{V}) \) is due to the non-orthogonality. In view of proxy vectors, * is like a kind of dual operator mapping a vector to its dual vector.

5.1.5. Trace operator. For \( F \in \Delta_{n-1}(T) \), let the trace operator \( \text{tr}^{\text{div}}_F : \Lambda^{n-1}(T) \to \Lambda^{n-1}(F) \) be the pullback of the inclusion \( F \hookrightarrow T \). That is for any tangent vectors \( v_1, \ldots, v_{n-1} \) of \( F \), we also treat them as tangent vectors of \( T \) and define

\[ \text{tr}^{\text{div}}_F \omega(v_1, \ldots, v_{n-1}) := \omega(v_1, \ldots, v_{n-1}), \quad \omega \in \Lambda^{n-1}(T). \]

We denote by \( \text{tr}^{\text{div}} : \Lambda^{n-1}(T) \to \cup_{F \in \partial T} \Lambda^{n-1}(F) \) as \( \text{tr}^{\text{div}} \omega|_F = \text{tr}^{\text{div}}_F \omega \).

Let \( n_F \) be the normal vector of \( F \) so that the orientation of \( F \), which is given by the volume \( dx_F \in \Lambda^{n-1}(F) \), and \( n_F \) form a consistent orientation of the ambient orthonormal basis. Then

\[ \text{tr}^{\text{div}}_F \omega = u \cdot n_F \, dx_F \quad \text{with} \quad u = \text{Prox}_{n-1} \omega. \]

On the other hand, for any \( p \in L^2(F) \), we have

\[ \langle \omega, p \ast dn_F \rangle_F = \int_F u \cdot n_F p \, dx_F = \int_F p \, \text{tr}^{\text{div}}_F \omega. \]

Based on (45), we can discuss the trace operator in the more familiar vector function setting. The trace operator for space \( H(\text{div}, T) \)

\[ \text{tr}^{\text{div}} : H(\text{div}, T) \to H^{-1/2}(\partial T) \]

is a continuous extension of \( \text{tr}^{\text{div}} u = u \cdot n|_{\partial T} \) defined on smooth functions.

5.1.6. Differential forms in the barycentric coordinates. As \( \sum_{i=0}^n \lambda_i = 1 \), \( \sum_{i=0}^n d\lambda_i = 0 \) and \( \{d\lambda_0, \ldots, d\lambda_n\} \) is not a basis of \( \text{Alt}^k \). A vertex as the origin, without loss of generality, say \( v_0 \), then \( \{d\lambda_1, \ldots, d\lambda_n\} \) forms a basis of \( \text{Alt}^k \). In general, through the index \( \sigma \), there is one-to-one correspondence between \( \Delta_{k-1}(F_0) \) and \( \text{Alt}^k(T) \).

Namely for \( \sigma \in \Sigma(1 : k : 1 : n) \), \( f_\sigma \) is a \((k-1)\)-dimensional simplex in \( \Delta_{k-1}(F_0) \) with vertices \( \{\sigma(1), \ldots, \sigma(k)\} \) and

\[ d\lambda_\sigma = d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k)} \in \text{Alt}^k. \]

We can also write as \( d\lambda_f \) assuming the index of the vertices of \( f \) is sorted in the ascending order.

The 1-form \( d\lambda_i \) has a vector representation \( \nabla \lambda_i \), which is a scaled normal vector \( n_F \) of face \( F_i \). For a simplex \( f \in \Delta_k(T) \), \( \{\nabla \lambda_i, i \in f^*\} \) are \( n-\ell \) normal vectors of \( f \) and can span the normal plane of \( f \). The vector representations of \( \{\nabla \lambda_i, i \in f^*\} \) are scaling of \( n-\ell \) normal vectors \( t_{\sigma(0)0}(i) \) of \( f_{\sigma} \) and can span the tangent plane of \( f_{\sigma} \). This is illustrated in Fig. 5.

5.2. An algebraic operator. Define \( \text{Alt}^{k,i}(\mathbb{R}^n) = \text{Alt}^k(\mathbb{R}^n) \otimes \text{Alt}^i(\mathbb{R}^n) \) for \( k, i = 0, 1, \ldots, n \). In particular \( \text{Alt}^{k,n-1}(\mathbb{R}^n) = \text{Alt}^k(\mathbb{R}^n) \otimes \text{Alt}^{n-1}(\mathbb{R}^n) \cong \text{Alt}^{k,n-1}(\mathbb{R}^n) \otimes \mathbb{R}^n \). To simplify the notation, \( \text{Alt}^{k,i}(\mathbb{R}^n) \) is abbreviated as \( \text{Alt}^{k,i} \). In [8], the algebraic operator \( G^{k,n-1} : \text{Alt}^{k,n-1} \to \text{Alt}^{k-1,n} \) is defined as

\[ G^{k,n-1}(w_1, \ldots, w_{k-1})(v_1, \ldots, v_n) := \sum_{i=1}^n (-1)^{i-1} \omega(v_1, w_1, \ldots, w_{k-1})(v_1, \ldots, \hat{v}_i, \ldots, v_n) \]

\[ \forall v_1, \ldots, v_n, w_1, \ldots, w_{k-1} \in \mathbb{R}^n. \]
Recall that we have reserved \( \{ dx_i \} \) for a fixed orthonormal basis of \( \text{Alt}^1(\mathbb{R}^n) \). We are going to derive more concrete forms of operator \( s^{k,n-1} \) in a generic basis \( \{ dy_i \} \), which may not be orthonormal. We expand \( \omega \in \text{Alt}^{k,n-1} \) in this basis as

\[
\omega = \sum_{i=1}^{n} \sum_{\sigma \in \Sigma(1:k,1:n)} a_{\sigma,i} dy_\sigma \otimes dy_i.
\]

An element in \( \text{Alt}^{k,n-1} \) can be identified as a matrix \( A = (a_{\sigma,i}) \) of size \( \binom{n}{k} \times n \) indexed by \((\sigma,i)\) for \( i = 1, 2, \ldots, n \) and \( \sigma \in \Sigma(1:k,1:n) \).

**Lemma 5.1.** For \( \omega = \sum_{i=1}^{n} \sum_{\sigma \in \Sigma(1:k,1:n)} a_{\sigma,i} dy_\sigma \otimes dy_i \), we have

\[
s^{k,n-1} \omega = \sum_{\tau \in \Sigma(1:k-1,1:n)} \left( \sum_{i \in \tau^c} \epsilon(i,\tau) a_{i+\tau,i} \right) dy_\tau \otimes dy_i.
\]

**Proof.** Let \( \{ \partial y_i \} \) be the basis of the tangent space dual to \( \{ dy_i \} \). That is \( dy_i(\partial y_j) = \delta_{ij} \) for \( 1 \leq i, j \leq n \). Given \( \sigma \in \Sigma(1:k,1:n) \), we use the notation \( \partial y_\sigma \) to denote \( k \) vectors \( \partial y_{\sigma(1)}, \ldots, \partial y_{\sigma(k)} \). Then \( dy_\sigma(\partial y_\sigma') = \delta_{\sigma,\sigma'} \) for \( \sigma, \sigma' \in \Sigma(1:k,1:n) \). To get the coefficient of \( s^{k,n-1} \omega \) for the component \( dy_\tau \otimes dy, \tau \in \Sigma(1:k-1,1:n) \), we check the action

\[
s^{k,n-1} \omega \left( \partial y_{\tau(1)}, \ldots, \partial y_{\tau(k-1)} \right) (dy_1, \ldots, dy_n)
= \sum_{i=1}^{n} (-1)^{i-1} \omega \left( \partial y_{i+\tau}, \partial y_{\tau(1)}, \ldots, \partial y_{\tau(k-1)} \right) (\partial y_1, \ldots, \partial y_i, \partial y_{i+\tau}, \ldots, dy_n)
= \sum_{i \in \tau^c} \omega \left( \epsilon(i,\tau) \partial y_{i+\tau} \right) \left( (-1)^{i-1} \partial y_i \right)
= \sum_{i \in \tau^c} \epsilon(i,\tau) a_{i+\tau,i}.
\]

If \( i \in \tau \), then vectors \( \partial y_i, \partial y_{\tau(1)}, \ldots, \partial y_{\tau(k-1)} \) are linearly dependent and thus the term vanishes. So only \( i \in \tau^c \) are left in the summation. \( \square \)

For a given \( \tau \in \Sigma(1:k-1,1:n) \), we call the sequence of index \( \{ (i_m + \tau, i_m) \}, i_m \in \tau^c, m = 1, 2, \ldots, n-k+1 \), the **constraint sequence** of \( \tau \), which can be also written as \( \{ (\sigma_m, i_m), m = 1, 2, \ldots, n-k+1, \sigma_m = i_m + \tau \} \). The length of the constraint sequence is \( |\tau^c| = n-k+1 \). Without loss of generality, we can sort as \( i_1 < i_2 < \ldots < i_{n-k+1} \). The first one \( (i_1 + \tau, i_1) \) will be called the pair index of the constraint sequence.
We provide some visualization of the constraint sequence. The tensor product \( dy_\sigma \otimes * dy_i \) can be visualized as follows: for each sub-simplex \( f_\sigma \), we attach a vector of length \( n \) to \( \tau \). The sub-index \( i \) in \( * dy_i \) corresponds to the \( i \)-th component of this vector. See 5-edge stars in Fig. 6. We can associate the \((k-1)\)-form \( dy_\tau \) with the sub-simplex \( f_\tau \) of dimension \( k-2 \), then \( \{ f_{i+\tau} \}_{i \in \tau} \) corresponds to all \((k-1)\)-dimensional sub-simplices of \( F_0 \) (excluding \( f_{0+\tau} \) as index 0 is not used in differential forms) using \( \tau \) as a boundary face. See Fig. 6.

If we identify entries of the matrix proxy as nodes of a graph, a constraint sequence will define a path of nodes. See Fig. 7. Indices in different constraint sequences are different. Namely for \( \tau \neq \tau' \), \((i+\tau, i) \neq (j+\tau', j)\) as either \( i \neq j \) or \( i + \tau \neq j + \tau' \). On the graph, different constraint sequences will correspond to disjoint paths.

Since only the value \( a_{\sigma,i} \) on the constraint sequence will contribute to the image \( s_{k,n-1} \), we conclude that

\[
\sum_{i=1}^{n-1} (dy_\sigma \otimes * dy_i) = 0 \iff i \in \sigma^c.
\]

For each row, i.e., for a fixed \( \sigma \), there are \( k \) entries \((\sigma, i) \), \( i \in \sigma \) on \( k \) different constraint sequences and the rest \( n-k \) entries are not in any constraint sequence.

We can identify \( \text{Alt}^{k-1,n} \) as a vector in \( \mathbb{R}^{\text{dim } \text{Alt}^{k-1}} \). With the matrix and vector representations, the \( s_{k,n-1} \) operator induces an operator from matrix \( A \) to a vector in \( \mathbb{R}^{\text{dim } \text{Alt}^{k-1}} \) and will be still denoted by \( s_{k,n-1} \). We collect the coefficients of a constraint sequence and denote by \( a_{\tau} = (a_{\tau, m + \nu, i_m})_{m=1,2,...,n-k+1} \). Let \( e_\tau = (\epsilon(i_m, \tau))_{m=1,2,...,n-k+1} \) be the corresponding sign vector. We can write the operator as

\[
(s_{k,n-1}A)_\tau = a_{\tau} \cdot e_\tau, \quad \tau \in \Sigma(1: k-1, 1:n).
\]

That is the action is along each constraint sequence.

**Lemma 5.2.** For \( k = 1, \ldots, n-1 \), the operator \( s_{k,n-1} : \text{Alt}^{k,n-1} \rightarrow \text{Alt}^{k,n-1} \) is onto. And \( s_{n,n-1} : \text{Alt}^{n,n-1} \rightarrow \text{Alt}^{n-1,n} \) is a bijection and its proxy \( s_{n,n-1} \) is the transpose operator.

**Proof.** First consider \( k = 1, \ldots, n-1 \). By the linearity, it suffices to prove for \( \tau = [1, \ldots, k-1] \), there exists \( \omega = \sum_{i=1}^{n-1} \sum_{\sigma \in \Sigma(1: k, 1:n)} a_{\sigma,i} dy_\sigma \otimes * dy_i \) such that \( s_{k,n-1} \omega = dy_\tau \otimes dy \).

For a given \( \tau \), we just pick up one \( \sigma = i + \tau \) from its constraint sequence and set the coefficient be \( \epsilon(i, \tau) \). More precisely, take \( a_{[1,\ldots,k],k} = \epsilon(k, [1 : k - 1]) = (-1)^{k-1} \), and \( a_{\sigma,i} = 0 \) for the rest. Then

\[
s_{k,n-1} \omega = \sum_{\tau \in \Sigma(1: k-1, 1:n)} \left( \sum_{\sigma \in \tau} \epsilon(i, \tau) a_{i+\tau,i} \right) dy_\tau \otimes dy = dy_\tau \otimes dy.
\]
Next consider $k = n$. For $\omega = \sum_{i=1}^{n} a_i \, d y_i \otimes \ast d y_i$, we have

$$s^{n,n-1} \omega = \sum_{\tau \in \Sigma(1:n-1;1:n)} \left( \sum_{i \in \tau^c} \epsilon(i, \tau) a_i \right) d y_\tau \otimes d y = \sum_{i=1}^{n} a_i \ast d y_i \otimes d y.$$  

Namely $s^{n,n-1}$ maps the row vector $(a_1, \ldots, a_n)$ to a column vector $(a_1, \ldots, a_n)^\top$.

If we identify the row and column vector by the transpose, we can also say $s^{n,n-1}$ is the identity operator.  

5.3. **Constraint tensor spaces.** Now we are ready to introduce the tensor space 

$$X := \ker(s^{k,n-1}) \cap \text{Alt}^{k,n-1} = \{ \omega \in \text{Alt}^{k,n-1} \mid s^{k,n-1} \omega = 0 \}, \quad 1 \leq k \leq n - 1.$$ 

When $k = n$, as $s^{n,n-1}$ is bijection, $X = \{0\}$ is trivial. So throughout the rest of the paper, we will consider the non-trivial case $1 \leq k \leq n - 1$. For a given basis $\{d y_i\}$, it will be more convenient to work on the matrix representation

$$X := \left\{ \omega = \sum_{i=1}^{n} \sum_{\sigma \in \Sigma(1:k,1:n)} a_{\sigma,i} \, d y_\sigma \otimes \ast d y_i \mid A = (a_{\sigma,i}) \in \mathbb{R}^{\binom{n}{k} \times n} : \right.$$ 

$$\sum_{i \in \tau^c} \epsilon(i, \tau) a_{i+\tau,i} = 0, \quad \forall \tau \in \Sigma(1:k-1,1:n) \right\}.$$ 

As $s^{k,n-1}$ is surjective,

$$\dim X = \dim \text{Alt}^{k,n-1} - \dim \text{Alt}^{k-1,n} = n \binom{n}{k} - \left( \binom{n}{k-1} \right) = (n-k) \binom{n+1}{k}.$$ 

**Figure 7.** Constraint sequences for $n = 4$, $k = 2$ with sign $\epsilon(i, \tau)$. A constraint sequence will define a path of nodes. Different constraint sequences will correspond to disjoint paths. The white circle denotes the pair index of each constraint sequence which is a non-free index. Other circles in yellow are free indices. For each row, there are $k$ entries $(\sigma, i)$, $i \in \sigma$ on $k$ different constraint sequences and the rest $n - k$ entries are not in any constraint sequence.
For the orthonormal basis \{d_{x_i}\}, we introduce

\[ H(\text{div}, \Omega; X) := \left\{ A = (a_{\sigma,i}) \in L^2(\Omega, X) : \text{div} A \in L^2(\Omega, \mathbb{R}^{|i|}) \right\} \]

with \text{div} A := (\sum_{i=1}^{n} \partial_{x_i} a_{\sigma,i}), i.e., the divergence operator is applied row-wise. Its differential form version is

\[ H(d_{n-1}, \Omega; X) := \left\{ \omega \in L^2(\Lambda^{k,n-1}(\Omega)) : s_{k,n-1}\omega = 0, d_{n-1}\omega \in L^2(\Lambda^{k,n}(\Omega)) \right\}, \]

where the exterior derivative \(d_{n-1}\) is applied to the component \(\Lambda^{n-1}\) in \(\Lambda^{k,n-1}\). In view of the matrix proxy \(A\) in the orthonormal basis \{d_{x_i}\}, the trace on face \(F\)

\[ \text{tr}^F_{\text{div}} A = AN_F \]

is a column vector of length \(\binom{n}{k}\), and should be continuous on the \((n-1)\)-dimensional faces across simplices.

**Example 5.3.** Consider \(k = n - 1\). For \(\omega = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \ast dy_i \otimes dy_j\), we have

\[ s_{n-1,n-1}\omega = \sum_{\tau \in \Sigma(1,n-2,1,n)} \left( \sum_{\iota \in \tau} (-1)^{(|1:n|\setminus|i+r|)-1} \epsilon(i, \tau)a_{[1:n] \setminus [i+r],i} \right) dy_{\iota} \otimes dy_\tau \]

\[ = \sum_{1 \leq i < j \leq n} (-1)^{i+j} (a_{j,i} - a_{i,j}) dy_{(i,j)} \otimes dy_{(j,i)} \]

where \(dy_{(i,j)} := dy_1 \wedge \cdots \wedge dy_i \wedge \cdots \wedge dy_j \wedge \cdots \wedge dy_n\). In terms of the matrix proxy, it holds

\[ s_{n-1,n-1}(A) = 2\text{vskw}(A) \quad \text{with} \quad A = (a_{i,j})_{n \times n} \in \mathbb{R}^{n \times n}, \]

where operator \text{vskw} : \mathbb{R}^{n \times n} \to \mathbb{R}^{(n-1)/2} is defined by

\[ (\text{vskw}(A))_\sigma = \frac{1}{2}(-1)^{i+j} (a_{j,i} - a_{i,j}) \quad \text{with} \quad \sigma = [i,j] \in \Sigma(1 : 2, 1 : n). \]

Thus \(X = \mathbb{S}\) consists of all symmetric matrices.

**Example 5.4.** Consider \(k = 1\). For \(\omega = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \ast dy_j \otimes dy_i\), we have

\[ s_{1,n-1}\omega = \sum_{i=1}^{n} a_{i,i} dy_i \]

In terms of the matrix proxy,

\[ s_{1,n-1}(A) = \text{trace}(A). \]

Thus \(X = T\) is the traceless matrix space.

5.4. **Bases of the constraint tensor space.** Recall that

\[ \text{Alt}^{k,n-1} = \text{span}\{ dy_{\sigma} \otimes dy_i, \sigma \in \Sigma(1 : k, 1 : n), i = 1, \ldots, n \}, \quad 1 \leq k \leq n. \]

We shall modify the basis function \(dy_{\sigma} \otimes dy_i\) to get a basis of \(X = \text{Alt}^{k,n-1} \cap \ker(s_{k,n-1})\).

Recall that we consider the non-trivial case: \(1 \leq k \leq n - 1\) so that the length of the constraint sequence \(n - k + 1\) is greater than or equal to 2. We shall define an oblique (non-orthogonal) projection operator \(\pi_X\) applied to \(dy_{\sigma} \otimes dy_i\) for index pair \((\sigma, i)\).

When \(i \in \sigma^c\), \(s_{k,n-1}(dy_{\sigma} \otimes dy_i) = 0\), and thus we keep it unchanged, i.e.,

\[ \pi_X(dy_{\sigma} \otimes dy_i) = dy_{\sigma} \otimes dy_i, \quad i \in \sigma^c. \]
For each constraint sequence \((\sigma_m, i_m)_{m=1,\ldots,n-k+1}\), set \((\sigma_i, i_1)\) as the pair index and modify basis functions to, for \(m = 1, \ldots, n-k+1\),
\[
\pi_X(dy_{\sigma_m} \otimes * dy_{i_m}) := dy_{\sigma_m} \otimes * dy_{i_m} - \epsilon(i_m, \tau) \epsilon(i_1, \tau) dy_{\sigma_{i_1}} \otimes * dy_{i_1}.
\]
In terms of the coefficient vector, \(\pi_X\) will map the vector \(a_\tau = (0, \ldots, 1, \ldots, 0)\) to the vector
\[
\tilde{a}_\tau = (-\epsilon(i_m, \tau) \epsilon(i_1, \tau), 0, \ldots, 1, \ldots, 0),
\]
so that the constraint \(\tilde{a}_\tau \cdot \epsilon_\tau = 0\) is satisfied. By the linear combination, we get the mapping \(\pi_X : \Omega^{k,n-1} \rightarrow X\).

An index \((\sigma, i)\) will be called a free index if \(\pi_X(dy_{\sigma} \otimes * dy_i) \neq 0\). By definition, only \(\pi_X(dy_{\sigma_{i_1}} \otimes * dy_{i_1}) = 0\), i.e., only the pair index of each constraint sequence is not free. Therefore the number of basis functions is reduced by one for each constraint sequence. In total, we remove \(\binom{n}{k-1}\) basis functions of \(\Omega^{k,n-1}\) and obtain a basis of \(X\):
\[
X = \text{span}\{\pi_X(dy_{\sigma} \otimes * dy_i), (\sigma, i) \text{ is free}\}.
\]
In Section 6, we will use \(\pi_X\) to define a \(t\)-\(n\) decomposition of \(X\) and construct finite element subspaces of \(H(\text{div}, \Omega; X)\).

Next we will present intrinsic bases of \(X\) using the barycentric coordinate.

**Lemma 5.5.** For any \(\sigma \in \Sigma(0 : n-k, 0 : n)\) and \(i = 1, \ldots, n-k\), it holds
\[
d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*} \in X,
\]
where
\[
d\lambda_{[\sigma(0), \sigma(i)]^*} = d\lambda_{[0:n]-\sigma(0)-\sigma(i)} = d\lambda_{0} \wedge \cdots \wedge d\lambda_{\sigma(0)} \wedge \cdots \wedge d\lambda_{\sigma(i)} \wedge \cdots \wedge d\lambda_{n}.
\]

**Proof.** We treat \(\sigma(0)\) as the origin. Let \(y_i = \lambda_{\sigma(i)}\) for \(i = 1, \ldots, n-k\), and \(y_{n-k+i} = \lambda_{\sigma^*(i)}\) for \(i = 1, \ldots, k\). Then
\[
d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*} = (-1)^{i-1}(dy_{n-k+i} \wedge \cdots \wedge dy_{n}) \otimes * dy_i \in X\text{ for }i = 1, \ldots, n-k\text{ as }i \leq [n-k+1, \ldots, n]^c. \]

The vector proxy of \(d\lambda_{[\sigma(0), \sigma(i)]^*}\) is a scaling of the edge vector \(e_{\sigma(0)\sigma(i)}\), which is on the tangent plane of \(f_{\sigma} \in \Delta_{n-k}(T)\). The \(k\)-form \(d\lambda_{\sigma^*}\) is the volume of the normal plane of \(f_{\sigma}\), i.e., \(\mathcal{N}_{\sigma^*}\) of dimension \(k\). Their tensor product for all \(f \in \Delta_{n-k}(T)\) forms a basis of \(X\).

The inner product of \(k\)-form \((\omega, \eta)\) can be extended to \(\Omega^{k,n-1}\) by the tensor product. Define \(P_X\) as the orthogonal projection from \(\Omega^{k,n-1}\) to \(X\) w.r.t. the inner product \(\langle \cdot, \cdot \rangle\).

**Lemma 5.6.** The set
\[
\{d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*}\}_{i=1,\ldots,n-k}
\]
in \(X\) is dual to the set
\[
\{P_X(\ast d\lambda_{\sigma^*} \otimes d\lambda_{\sigma(i)}\})\}_{\sigma \in \Sigma(0 : n-k, 0 : n)}
\]
in the sense that: for any \(\sigma, \tau \in \Sigma(0 : n-k, 0 : n)\), and \(i, j = 1, \ldots, n-k\),
\[
\langle d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*}, P_X(\ast d\lambda_{\tau^*} \otimes d\lambda_{\tau(j)}\}) \rangle = \delta_{\sigma,\tau} \delta_i \delta_j.
\]

**Proof.** Since \(d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*} \in X\), by definition of \(P_X\) and the inner product \(\langle \cdot, \cdot \rangle\), it suffices to prove that: for any \(\sigma, \tau \in \Sigma(0 : n-k, 0 : n)\), and \(i, j = 1, \ldots, n-k\),
\[
\langle d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*}, d\lambda_{\tau^*} \otimes d\lambda_{\tau(j)} \rangle \neq 0
\]
if and only if
\[
\sigma = \tau \text{ and } i = j.
\]

By definition,
\[
\langle d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0), \sigma(i)]^*}, d\lambda_{\tau^*} \otimes d\lambda_{\tau(j)} \rangle = (d\lambda_{\sigma^*} \wedge d\lambda_{\tau^*}) \otimes (d\lambda_{[\sigma(0), \sigma(i)]^*} \wedge d\lambda_{\tau(j)}).
Then \( (d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*}) \wedge (d\lambda_{\tau-\tau(j)} \otimes d\lambda_{\tau(j)}) \neq 0 \) is equivalent to

\[
\tau(j) \in \{\sigma(0),\sigma(i)\} \quad \text{and} \quad \tau - \tau(j) \subset \sigma.
\]

This indicates \( \tau \subseteq \sigma \). We finish the proof by the fact \( \tau \) and \( \sigma \) have the same length. \( \square \)

We are in the position to present intrinsic bases of \( \mathbb{K} \) using the barycentric coordinates.

**Theorem 5.7 (Intrinsic bases of \( \mathbb{K} \)).** The set

\[
\{ d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*} \}_{\sigma \in \Sigma(0:n-k,0:n)}
\]

is a basis of \( \mathbb{K} \). Its dual basis is

\[
(47) \quad \{ P_{\mathbb{K}}(\star \cdot d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*}) \}_{\sigma \in \Sigma(0:n-k,0:n)}.
\]

**Proof.** The number of the set \( \{ d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*} \}_{\sigma \in \Sigma(0:n-k,0:n)} \) is \( (n-k) \binom{n+1}{k} \), which equals to \( \dim \mathbb{K} \), cf. (46). Hence it suffices to prove that they are linearly independent. Assume there exist \( c_{\sigma,i} \in \mathbb{R} \) for each \( \sigma \in \Sigma(0:n-k,0:n) \) and \( i = 1, \ldots, n-k \) such that

\[
\sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} c_{\sigma,i} d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*} = 0.
\]

Then apply the wedge product with \( d\lambda_{\tau-\tau(j)} \otimes d\lambda_{\tau(j)} \) for \( \tau \in \Sigma(0:n-k,0:n) \) and \( 0 \leq j \leq n-k \), due to Lemma 5.6, we obtain \( c_{\tau,j} = 0 \). As \( (\tau,j) \) runs over the whole set \( \Sigma(0:n-k,0:n) \times \{1, \ldots, n-k\} \), we conclude all \( c_{\tau,j} \) vanishes. \( \square \)

**Example 5.8.** When \( k = n-1 \), \( f_\sigma \) is an edge and the vector proxy of \( d\lambda_{\sigma^*} \) is a scaling of the tangent vector \( t_{\sigma(0)\sigma(1)} \) of \( f_\sigma \). A basis of \( \mathbb{K} \) is thus given by \( \{ d\lambda_{\sigma^*} \otimes d\lambda_{[\sigma(0),\sigma(i)]^*} \}_{\sigma \in \Sigma(0:1,0:n)} \) and the dual basis is \( \{ \text{sym}(\star \cdot d\lambda_{\sigma(0)} \otimes \star \cdot d\lambda_{[\sigma(0),\sigma(1)]}) \}_{\sigma \in \Sigma(0:1,0:n)} \). Equivalently, in terms of the vector proxy, a basis of \( \mathbb{S} \) is \( \{ t^e \otimes t^e \} \) for \( e \in \Delta_i(T) \) and the dual basis is \( \{ \text{sym}(n_e \otimes n_e) \} \) for \( e = \text{Convex}(v_i,v_j) \in \Delta_i(T) \), which are crucial in designing the \( H(\text{div};\mathbb{S}) \) element [28, 30, 14] and useful in the Regge calculus [20].

**Example 5.9.** When \( k = 1 \), \( f_\sigma \) is an \( (n-1) \)-dimensional face \( F \) and the vector proxy of \( d\lambda_{\sigma^*} \) is \( n_F \). In the matrix proxy, a basis of \( \mathbb{T} \) is \( \{ n_F \otimes t^F_i \}_{i=1}^{F \in \Delta_{n-1}(T)} \), which is discovered in [29] and presented in Lemma 4.6.

### 5.5. Formulae on the projections.

We will present an explicit formula for \( P_{\mathbb{K}} \). Recall that the basis \( \{ d\hat{y}_i \}_{i=1}^n \) is dual to \( \{ d\hat{y}_i \}_{i=1}^n \) in the sense that \( \langle d\hat{y}_i, d\hat{y}_j \rangle = \delta_{i,j} \) for \( i, j = 1, \ldots, n \). The duality also holds for corresponding bases of \( \text{Alt}^k \); see (43).

**Lemma 5.10.** It holds

\[
(48) \quad \mathbb{K}^+ = \text{span}\{ \hat{e}_{\tau}^A, \; \tau \in \Sigma(1:k-1,1:n) \},
\]

where

\[
(49) \quad \hat{e}_{\tau}^A := \sum_{i \in \tau^+} \epsilon(i, \tau) d\hat{y}_{i+\tau} \otimes \star d\hat{y}_i.
\]

For \( \omega \in \text{Alt}^{k,n-1} \) expanded in the basis \( \omega = \sum_{\sigma,i} a_{\sigma,i} d\hat{y}_\sigma \otimes \star d\hat{y}_i \), it holds

\[
(50) \quad P_{\mathbb{K}} \omega = \sum_{\tau \in \Sigma(1:k-1,1:n)} \frac{a_{\tau} \cdot e_{\tau}}{n-k+1} \hat{e}_{\tau}^A,
\]
where \( P_{X^\perp} := I - P_X \). Consequently for \( \sigma \in \Sigma(1 : k, 1 : n) \),

\[
(P_X \omega)_{\sigma, i} = \begin{cases} a_{\sigma, i} - \frac{A_{\tau} \cdot e_i}{n - k + 1} \varepsilon(i, \tau), & i \in \sigma^c, \\ a_{\sigma, i}, & i \in \sigma, \tau = \sigma - i. \end{cases}
\]

**Proof.** Let \( \eta = \sum_{i=1}^{n} \sum_{\sigma \in \Sigma(1 : 1 : n)} a_{\sigma, i} dy_\sigma \otimes * dy_i \in X \). Then

\[
(\hat{e}_r^A, \eta) = \sum_{i \in \tau^c} \varepsilon(i, \tau) a_i + \tau, i = 0, \quad \tau \in \Sigma(1 : k - 1, 1 : n).
\]

That is \( \hat{e}_r^A \perp X \). As any two constraint sequence vectors are disjointed, \( \{ \hat{e}_r^A, \tau \in \Sigma(1 : k - 1, 1 : n) \} \) is linear independent and (48) follows from the dimensions match.

Let \( \varepsilon_r^A = \sum_{i \in \tau^c} \varepsilon(i, \tau) dy_i \otimes * dy_i \), which also forms a basis of \( X^\perp \). Formula (50) holds by testing with \( \varepsilon_r^A \)

\[
\langle \omega, \varepsilon_r^A \rangle = A_{\tau} \cdot e_r, \quad \langle \hat{e}_r^A, \varepsilon_r^A \rangle = n - k + 1.
\]

Combining (50) and \( P_X = I - P_{X^\perp} \) gives (51). \( \square \)

The constraint tensor spaces \( X \) and \( X^\perp \) are defined intrinsically using properties of differential forms, which is independent of choices of the basis. In the proof above, we use different bases \( \{ dy_i \} \) or \( \{ dy_i \} \) for the ease of computing the projection.

## 6. Geometric Decomposition of \( H(\text{div}) \)-conforming Tensors with Constraints

In this section, we generalize the geometric decomposition of the \( H(\text{div}) \)-conforming vector finite element to the \( H(\text{div}) \)-conforming tensor finite element. We decompose \( P_t(T; X) \) into a direct sum of the tangential bubble subspace and a normal subspace. Then we present DoFs and show the \( H(\text{div}) \)-conformity and the discrete inf-sup condition.

### 6.1. Decomposition of the constraint tensor space

We start from the tensor product of the Lagrange element with \( X \):

\[
P_t(T; X) = \bigoplus_{t=0}^{n} \bigoplus_{f \in \Delta_t(T)} \left[ b_f P_{t-(t+1)}(f) \otimes X \right].
\]

For an \( \ell \)-dimensional face \( f \in \Delta_\ell(T) \), there is a matrix function \( A^f \in \mathbb{R}^{(\ell) \times n} \) satisfying the constraint \( s^{k,n-1}(A^f) = 0 \). The vector \( H(\text{div}) \) element is \( k = 0 \) for which the matrix \( A \) is degenerated to a vector of length \( n \) and no constraint is imposed. For \( 1 \leq k \leq n - 1 \), it is the constraint \( s^{k,n-1}(A^f) = 0 \) that makes the finite element construction difficult as the constraint and the normal continuity should be satisfied simultaneously.

As before, for a face \( f \in \Delta_\ell(T) \), we choose a \( t \)-basis \( \{ t_1^f, \ldots, t_\ell^f, n_1^{f \perp}, \ldots, n_\ell^{f \perp} \} \), where the set of \( \ell \) tangential vectors \( \{ t_1^f, \ldots, t_\ell^f \} \) is a basis of the tangent plane \( \mathcal{T}_f \) of \( f \) and the set of \( n - \ell \) normal vectors \( \{ n_1^{f \perp}, \ldots, n_{\ell^{f \perp}} \} \) forms a basis of the normal plane \( \mathcal{N}_f \) of \( f \). All basis vectors are normalized but may not be orthogonal. We write \( i \in \mathcal{I}_f \) and \( i \in \mathcal{N}_f \) to emphasize the range of the index.

Inside the subspace \( \mathcal{T}_f \), we can find a basis \( \{ \hat{t}_1^f, \ldots, \hat{t}_\ell^f \} \) dual to \( \{ t_1^f, \ldots, t_\ell^f \} \), i.e., \( \hat{t}_i^f \in \mathcal{T}_f \) and \( \langle t_j^f, \hat{t}_i^f \rangle = \delta_{i,j} \) for \( i, j \in \mathcal{I}_f \). Similarly we have a basis \( \{ \hat{n}_1^{f \perp}, \ldots, \hat{n}_{\ell^{f \perp}}^{f \perp} \} \) of \( \mathcal{N}_f \) and \( \langle \hat{n}_i^{f \perp}, n_j^{f \perp} \rangle = \delta_{i,j} \) for \( i, j \in \mathcal{N}_f \). As \( \mathcal{T}_f \perp \mathcal{N}_f \), the basis \( \{ \hat{t}_1^f, \ldots, \hat{t}_\ell^f, \hat{n}_1^{f \perp}, \ldots, \hat{n}_{\ell^{f \perp}}^{f \perp} \} \) is also dual to \( \{ t_1^f, \ldots, t_\ell^f, n_1^{f \perp}, \ldots, n_{\ell^{f \perp}} \} \). Let \( V = (t_1^f, \ldots, t_\ell^f, n_1^{f \perp}, \ldots, n_{\ell^{f \perp}}) \) and \( \hat{V} = (\hat{t}_1^f, \ldots, \hat{t}_\ell^f, \hat{n}_1^{f \perp}, \ldots, \hat{n}_{\ell^{f \perp}}) \).
We say the basis \( \{ dy^f_i \} \) is the basis of \( \text{Alt}^1 \) corresponding to a \( t-n \) basis if

\[
\text{Prox}_1(dy^f_i) = \begin{cases} 
    t_i^f & \text{for } i \in \mathcal{T}_f, \\
    n_{i-t}^f & \text{for } i \in \mathcal{N}_f.
\end{cases}
\]

Then its dual \( \{ \hat{dy}^f_i \} \) has the vector proxy

\[
\text{Prox}_1(\hat{dy}^f_i) = \begin{cases} 
    \hat{t}_i^f & \text{for } i \in \mathcal{T}_f, \\
    \hat{n}_{i-t}^f & \text{for } i \in \mathcal{N}_f.
\end{cases}
\]

We extend the domain of Prox and \( \text{Prox}^{-1} \) to subspaces. For example, \( \text{Prox}^{-1}_{n-1} \mathcal{T}_f = \text{span} \{ \text{Prox}^{-1}_{n-1}(\hat{t}_i^f), i = 1, \ldots, \ell \} = \text{span}(\text{Prox}^{-1}_{n-1}(\hat{t}_i^f), i = 1, \ldots, \ell) \). Then

\[
\text{Alt}^{k,n-1} = \text{span}\{ d\hat{y}^f_\sigma \otimes * dy^f_i, \sigma \in \Sigma(1 : k, 1 : n), i = 1, \ldots, n \} = (\text{Alt}^k \otimes \text{Prox}^{-1}_{n-1} \mathcal{T}_f) \oplus (\text{Alt}^k \otimes \text{Prox}^{-1}_{n-1} \mathcal{N}_f).
\]

We introduce the concept normal constraints. A constraint with the constraint sequence \( \{(\sigma_{i_m}, i_m)_{m=1,\ldots,n-k+1}\} \) is called a normal constraint if all \( i_m \in \mathcal{N}_f \). The normal constraints will be imposed inside the normal component. Recall that we sort the constraint sequence s.t. \( i_1 < i_2 < \ldots < i_{n-k+1} \) and set \( (\sigma_{i_1}, i_1) \) as the pair index. So for a non-normal constraint, the pair index \( (\sigma_{i_1}, i_1), i_1 \in \mathcal{T}_f \) is in the tangential component for \( \dim f \geq 1 \). Also recall that non-pair indices are free indices.

Define, for \( f \in \Delta_\ell(T) \) with \( 0 \leq \ell \leq n \),

\[
\mathcal{T}_f(X) := \text{span}\{ \pi_X(dy^f_\sigma \otimes * dy^f_i), (\sigma, i) \text{ is free, } i \in \mathcal{T}_f \},
\]

\[
\mathcal{N}_f(X) := \text{span}\{ \pi_X(dy^f_\sigma \otimes * dy^f_i), (\sigma, i) \text{ is free, } i \in \mathcal{N}_f \}.
\]

For \( \ell = 0 \), i.e., at vertex \( v \in \Delta_0(T) \), we understand \( \mathcal{T}_0(X) = \{0\} \) as no tangent plane and \( \mathcal{N}^0(X) = X \) as \( \mathcal{N} = \mathbb{R}^n \).

**Lemma 6.1.** Given a \( t-n \) basis of a face \( f \in \Delta_\ell(T) \), we have the following decomposition

\[
X = \mathcal{T}_f(X) \oplus \mathcal{N}_f(X).
\]

Their dimensions are

\[
\dim \mathcal{T}_f(X) = \ell \left( \begin{array}{c} n \\ k \end{array} \right) + \left( \begin{array}{c} n - \ell \\ n - k + 1 \end{array} \right) - \left( \begin{array}{c} n \\ k - 1 \end{array} \right),
\]

\[
\dim \mathcal{N}_f(X) = (n - \ell) \left( \begin{array}{c} n \\ k \end{array} \right) - \left( \begin{array}{c} n - \ell \\ n - k + 1 \end{array} \right).
\]

**Proof.** By construction, the sum is direct. It suffices to count the dimension. The number of constraints is \( \dim \text{Alt}^{k-1} \).

By the proof of the surjectivity of \( s^{k,n-1} \), all constraints are linearly independent.

Therefore

\[
\dim X = \dim \text{Alt}^{n-1} \times \dim \text{Alt}^k - \dim \text{Alt}^{k-1} = n \dim \text{Alt}^k - \dim \text{Alt}^{k-1}.
\]

For each normal constraint, it will remove one index in \( \mathcal{N}_f \). So

\[
\dim \mathcal{N}_f(X) = (n - \ell) \times \dim \text{Alt}^k - \# \text{ normal constraints}.
\]

If a constraint is non-normal, then the pair index is in the tangential component. So

\[
\dim \mathcal{T}_f(X) = \ell \times \dim \text{Alt}^k - \# \text{ non-normal constraints}.
\]
Sum these two and use the fact
\[ \text{# normal constraints} + \text{# non-normal constraints} = \text{# all constraints} = \dim \text{Alt}^{k-1} \]
to conclude \( \dim X = \dim \mathcal{E}^f(X) + \dim \mathcal{N}^f(X) \).

The number of the normal constraints is \( \binom{n-\ell}{n-k+1} \) (among \( n-\ell \) indices of the normal plane, choose \( n-k+1 \) to form the constraint sequence \( \{i_m, m = 1, \ldots, n-k+1\} \)) and thus the number of the non-normal constraints is \( \binom{n}{k-1} - \binom{n-\ell}{n-k+1} \).

\[ \square \]

6.2. **Geometric decomposition of polynomial constraint tensors.** Define the bubble polynomial space
\[ \mathbb{B}_r(div, T; X) := \mathbb{P}_r(T; X) \cap \ker(\text{tr}^\text{div}). \]
There is no bubble polynomial for lower degree \( r = 0, 1 \).

**Lemma 6.2.** We have \( \mathbb{P}_0(div, T; X) = \mathbb{P}_1(div, T; X) = 0 \).

**Proof.** Take \( \omega \in \mathbb{B}_r(div, T; X) \) with \( r = 0, 1 \). Since \( \{ \mathbf{n}_{F_0}, \ldots, \mathbf{n}_{F_{\ell-1}}, \mathbf{n}_{F_{\ell+1}}, \ldots, \mathbf{n}_{F_n}\} \) form a basis of \( \mathbb{R}^n \), and \( (\text{tr}^\text{div}, \omega)(\nu_j) = 0 \) for \( 0 \leq i \neq j \leq n \), we get \( \omega(\nu_j) = 0 \). Thus, \( \omega = 0 \). \[ \square \]

The tangential component contributes to the bubble. The normal component will contribute to the normal trace. Coupled with the bubble polynomials, we define, for \( f \in \Delta(\alpha) \) with \( 0 \leq \ell \leq n \),
\[ \mathbb{B}_r \mathcal{E}^f(X) := b_f \mathbb{P}_r-\ell+1(f) \otimes \mathcal{E}^f(X), \quad \mathbb{B}_r \mathcal{N}^f(X) := b_f \mathbb{P}_r-\ell+1(f) \otimes \mathcal{N}^f(X). \]

**Theorem 6.3** (Characterization of div bubble tensors). For \( r \geq 2 \), it holds that
\[ \mathbb{B}_r(div, T; X) = \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta(\alpha)} \mathbb{B}_r \mathcal{E}^f(X), \]
and
\[ \text{tr}^\text{div} : \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta(\alpha)} \mathbb{B}_r \mathcal{N}^f(X) \rightarrow \text{tr}^\text{div} \mathbb{P}_r(T; X) \]
is a bijection. Consequently
\[ \dim \mathbb{B}_r(div, T; X) = \sum_{\ell=1}^{n} \binom{n+1}{\ell+1} \binom{r-1}{\ell} \binom{n}{k} + \binom{n-\ell}{n-k+1} - \binom{n}{k-1} \]

**Proof.** Notice that the trace operator is applied to the second component in the tensor product \( dy^\ell_F \otimes *dy^\ell_F \) and \( \text{tr}^\text{div}(s * dy^\ell_F) = \det(V) \mathbf{n}_F \cdot \hat{t}^\ell_F \mathbf{d}x_F = 0 \) if \( i \in \mathcal{E}^f \) and \( f \in \Delta(F) \). The modification in \( \pi(X)(dy^\ell_F \otimes *dy^\ell_F) \) will use the pair index in \( \mathcal{E}^f \) and thus remains the normal trace free. For \( f \notin \Delta(F) \), the bubble function \( b_f |_F = 0 \).

So we have verified \( \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta(\alpha)} \mathbb{B}_r \mathcal{E}^f(X) \subseteq \mathbb{B}_r(div, T; X) \).

The rest is the same as Lemma 3.5. \[ \square \]

We propose the following finite element for the constraint tensor \( X \).

**Lemma 6.4.** For each \( f \in \Delta_\ell \), choose a \( t-n \) basis \( \{ t^\ell_1, \ldots, t^\ell_k, n^\ell_1, \ldots, n^\ell_{n-\ell} \} \) for \( \ell = 1, \ldots, n-1 \). Let \( \{ dy_i \} \) be the corresponding basis of \( \text{Alt}^1 \), and \( \{ \tilde{dy}_i \} \) be its dual basis. The shape function space \( \mathbb{P}_r(T; X) \) is uniquely determined by the DoFs
\begin{align*}
(52a) \quad \omega(\nu_i), \quad i = 0, \ldots, n, \omega \in X, \\
(52b) \quad (\omega, \eta)_f, \quad \eta \in \mathbb{P}_r-\ell+1(f) \otimes \{ dy^\ell_i \otimes *\tilde{dy}^\ell_i \ | \ (\sigma, i) \ is \ free, i \in \mathcal{N}^f \}, \\
(52c) \quad (\omega, \eta)_T, \quad \eta \in \mathbb{B}_r(div, T; X).
\end{align*}
Proof. Recall the duality
\[ \langle dy^l_{\sigma} \otimes * dy^l_{i}, d\tilde{y}^l_{\sigma} \otimes \tilde{d}y^l_{i} \rangle = \delta_{\sigma,\eta} \delta_{i,j} \]
for all \(1 \leq i, j \leq n\), and \(\sigma, \eta \in \Sigma(1 : k, 1 : n)\). In \(\pi_X(dy^l_{\sigma} \otimes * dy^l_{i}) = dy^l_{\sigma} \otimes \tilde{d}y^l_{i} - \epsilon(i, \tau) \epsilon(i_1, \tau) dy^l_{i_1} \otimes dy^l_{i_1} \), the pair index \((\sigma_1, i_1)\) is not free and thus the duality still holds for all free indices
\[ \langle \pi_X(dy^l_{\sigma} \otimes * dy^l_{i}), d\tilde{y}^l_{\sigma} \otimes \tilde{d}y^l_{i} \rangle = \delta_{\sigma,\eta} \delta_{i,j}, \quad (\sigma, i) \text{ and } (\eta, j) \text{ are free.} \]

Now assume (52a)-(52b) vanishes. For \(\omega = \sum_{\text{free}(\sigma, i)} c_{(\sigma, i)} b_{f} \pi_X(dy^l_{\sigma} \otimes * dy^l_{i})\), the DoF \((\omega, c_{(\sigma, i)} d\tilde{y}^l_{\sigma} \otimes \tilde{d}y^l_{i})\) for \(i \in \mathcal{N}^f\) will define a function \(P_{\omega}(\cdot)\). By Lemma 6.4, on each simplex, DoFs (52) will define a function \(V^h\) will determine a space \(H(\text{div})\) and the formula
\[ \text{by choosing a global } V^h \in H(\text{div}, \Omega; \mathbb{X}) \text{ we can write out an explicit basis function, } \]
\[ \text{Proof.} \]

DoF (52b) is in the spirit of the Petrov-Galerkin method, where the test function \(d\tilde{y}^l_{\sigma} \otimes \tilde{d}y^l_{i}\) is different from the trial function \(\pi_X(dy^l_{\sigma} \otimes \tilde{d}y^l_{i})\). This change is important as \(\text{Prox}(dy^l_{\sigma} \otimes \tilde{d}y^l_{i}) = c\text{Prox}(d\tilde{y}^l_{\sigma} \otimes \tilde{d}y^l_{i})\) for \(i \in \mathcal{N}^f\) will contain the normal component only, which will determine the normal trace to ensure the \(H(\text{div})\)-conformity.

In view of the vector proxy, usually we can choose an orthonormal basis for \(\mathcal{T}^f\) so that \(\hat{t}^l_i = \tilde{t}^l_i\) for \(i \in \mathcal{T}^f\). We use the normal vector \(n^f_i\) to define DoFs while use its dual basis \(\tilde{n}^f_i\) to expand the shape function.

Similar to [13], we can write out an explicit basis function
\[ \phi_\alpha \pi_X(dy^l_{\sigma} \otimes * dy^l_{i}), \quad \alpha \in \mathbb{T}^f(\hat{f}), (\sigma, i) \text{ is free, } f \in \Delta\ell(T), \ell = 0, \ldots, n, \]
where \(\phi_\alpha\) is the nodal basis of Lagrange element at lattice point \(\alpha\), and \(\mathbb{T}^f(\hat{f})\) is the set of lattice points whose geometric embedding is in the interior of \(f\).

6.3. \(H(\text{div})\)-conforming finite element spaces. We shall glue local finite element spaces to form an \(H(\text{div})\)-conforming subspace of \(H(\text{div}, \Omega; \mathbb{X})\) by choosing a global \(t\)-\(n\) basis \(\{ t^f_1, \ldots, t^f_{\ell}, n^f_1, \ldots, n^f_{n-\ell} \}\), i.e., depending only on \(f\) not the element containing \(f\).

**Theorem 6.5** \((H(\text{div})\)-conforming finite element with global \(t\)-\(n\) bases). For each \(f \in \Delta\ell(T^h), \ell = 0, \ldots, n-1\), choose a \(t\)-\(n\) basis \(\{ t^f_1, \ldots, t^f_{\ell}, n^f_1, \ldots, n^f_{n-\ell} \}\) depending only on \(f\). Let \(\{ dy^l_1 \} \) be the corresponding basis of \(\text{Alt}^1\), and \(\{ d\tilde{y}^l_1 \} \) be its dual basis. Then the following DoFs
\[
\begin{align*}
(53a) & \quad \omega(v_i), i = 0, \ldots, n, \omega \in \mathbb{X}, \\
(53b) & \quad (\omega, \eta)_f, \quad \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{ d\tilde{y}^l_1 \otimes * d\tilde{y}^l_1 \mid (\sigma, i) \text{ is free}, i \in \mathcal{N}^f\}, \\
(53c) & \quad (\omega, \eta)_T, \quad \eta \in \mathbb{P}_{r}(\text{div}, T; \mathbb{X}), T \in \mathcal{T}_h,
\end{align*}
\]
will determine a space \(V^h \subset H(\text{div}, \Omega; \mathbb{X})\).

**Proof.** By Lemma 6.4, on each simplex, DoFs (53) will define a function \(\omega \in \mathbb{P}_{r}(T; \mathbb{X})\). We only need to verify the trace is uniquely determined by (53a)-(53b).

For a face \(F \in \Delta_{n-1}(T^h)\), we have a normal vector \(n_F\) depending only on \(F\) and the formula
\[ \langle \tau F^\text{div}, \omega, d\tilde{y}^l_\sigma \rangle = (\omega, d\tilde{y}^l_\sigma \otimes * d^n_F) \]
for \(F \in \Delta(F), n_F \in \mathcal{N}^f\), we can expand \(n_F = \sum_{i=1}^{n-\ell} c_i^f n_i^f\) and \(* d^n_F = \sum_{i=1}^{n-\ell} c_i^f * d^n_i^f = \sum_{i \in \mathcal{N}^f} c_i^f * d\tilde{y}_i^f\)
when \((\sigma, i)\) is free and \(i \in \mathcal{N}^f\). \((\omega, d\tilde{y}_\sigma \otimes * d\tilde{y}_i^f)\) is given by the DoFs (53a)-(53b).
Then consider the case \((\sigma, i)\) is not free and \(i \in \mathcal{N}_f\). Namely \((\sigma, i) = (i_1 + \tau, i_1)\) is a pair index for a constraint sequence. As \(i_1 = i \in \mathcal{N}_f\), this constraint is a normal constraint, i.e., all \(i_m \in \mathcal{N}_f\) for \(m = 1, \ldots, n - k + 1\). We can express \(\epsilon(i_1, \tau) d\hat{y}_f^+ \otimes * d\hat{y}_i = \epsilon^A - \sum_{m=2}^{n-k+1} \epsilon(i_m, \tau) d\hat{y}_{i_{m-1}^+}^+ \otimes * d\hat{y}_{i_m}^+\), where \(\epsilon^A \in \mathbb{R}^1\) is defined in (49). As \(\omega \in \mathcal{X}\) and \(\epsilon^A \in \mathbb{R}^1\), \(\langle \omega, \epsilon^A \rangle = 0\) and consequently

\[
(\omega, \epsilon(i_1, \tau) d\hat{y}_f^+ \otimes * d\hat{y}_i) = - \sum_{m=2}^{n-k+1} \epsilon(i_m, \tau)(\omega, d\hat{y}_{i_{m-1}^+}^+ \otimes * d\hat{y}_{i_m}^+). \]

Notice that the index \((i_m + \tau, i_m)\) is free and \(i_m \in \mathcal{N}_f\) for \(m \geq 2\). So \((\omega, d\hat{y}_f^+ \otimes * d\hat{y}_i)\) can be also determined by DoFs (53a)-(53b) even \((\sigma, i)\) is not free.

It follows that

\[
(\text{tr}^\text{div}_f \omega, \eta)_f, \quad f \in \Delta_\ell(F), \eta \in \mathbb{P}_{\ell-1}(f) \otimes \text{Alt}^k, \ell = 0, \ldots, n - 1.
\]

can be determined by DoFs (53a)-(53b). As \(\text{tr}^\text{div}_f \omega \in \mathbb{P}_{\ell}(F) \otimes \text{Alt}^k\), by the uni-solvence of the vector Lagrange element, we conclude \(\text{tr}^\text{div}_f \omega\) is uniquely determined by DoFs (53a)-(53b). \(\square\)

DoF (53a) implies the continuity at vertices. We argue that the continuity at vertices is also necessary. Take a vertex in \(\Delta_0(T)\), for example \(v_0\). Then \((A n_{F_i})(v_0)\) is determined by the row vector \((A n_{F_i})|_{F_i} \in \Lambda^k\) for \(i = 1, \ldots, n\), where \(F_i \in \Delta_{n-1}(T)\) is the face opposite to \(v_i\). If it is continuous on each face not on vertices, the number of elements in \((A n_{F})(v_0)\) is \(\dim \text{Alt}^k\) for each face. Running \(i\) from 1 to \(n\), \(A(v_0)\) is determined by \(n \dim \text{Alt}^k\) conditions, which is more than \(\dim \mathcal{X}\). In other words, the constraint makes the tensor product of vector DoFs fails and introduce additional smoothness.

6.4. **Facewise redistribution.** DoFs (53a)-(53b) implies stronger continuity on the normal plane. We shall further redistribute some DoFs to faces \(F \in \Delta_{n-1}(T)\).

To do so, we introduce the concept of the **free block** and the **normal constraint block**. For a fixed \(\sigma\), the row vector \((\sigma, i), i = \ell + 1: n\) is called a free row if no index is in a normal constraint. All free rows will form a sub-matrix called the free block. The rest is called the normal constraint block, which contains all normal constraints. Do not confuse the free row with the free index. All indices of a free row are free. But a row with all free indices may not be a free row. A free index can be associated to a normal constraint sequence. See the second and third rows in Fig. 8.

For indices in the free block, the pair index is in the tangential component, which will not change the normal trace. For a normal constraint, the pair index is still in the normal constraint block and changing a basis of \(\mathcal{N}_f\) may destroy the \(H(\text{div})\)-conformity. Therefore a global basis \(\{n_f^\ell\} \) of \(\mathcal{N}_f\) is chosen to impose the normal constraints for the normal constraint block. For example, DoFs (39b) for \(n_f^\ell \sigma n_j\) in the \(H(\text{div}, \mathbb{S})\) element are normal constraints.

For a free row, we have a vector in \(\mathbb{R}^n\) consists of \([a_{\sigma,1}, \ldots, a_{\sigma,\ell}, a_{\sigma,\ell+1}, \ldots, a_{\sigma,n}]^T\). The first \(\ell\) components are in the tangential component and their values are determined locally as the element-wise div bubble polynomials. The part \([a_{\sigma,\ell+1}, \ldots, a_{\sigma,n}]^T\) is in the normal component and the corresponding DoFs can be redistributed facewise by using the face normal basis \(\{n_F, f \subseteq F, F \in \Delta_{n-1}(T)\}\) of \(\mathcal{N}_f\). In short, a free row is just like a vector \(H(\text{div})\) element.

**Theorem 6.6** \((H(\text{div})\)-conforming finite element with face redistribution). For each \(f \in \Delta_\ell(T_h), \ell = 0, \ldots, n - 1\), choose a global t-n basis \(\{t_i^f, \ldots, t_i^f, n_i^f, \ldots, n_i^{f-\ell}\}\) depending only on \(f\). Let \(\{d\hat{y}_i\}\) be the corresponding basis of
Alt\(^1\), and \(\{\hat{d}y_i\}\) be its dual basis. The following DoFs

\[(54a)\quad \omega(u_i), \; i = 0, \ldots, n, \; \omega \in \mathbb{X},\]

\[(54b)\quad (\omega, \eta)_f, \; \eta \in P_{r-(\ell+1)}(f) \otimes \{\hat{d}y_{\sigma}^f \otimes \hat{d}y_{i}^f \mid (\sigma, i) \text{ is free and in the normal constraint block}\},
\]

\[\quad f \in \Delta_\ell(T_h), \; \ell = 1, \ldots, n-1,\]

\[(54c)\quad (\text{tr}^\text{div}_F \omega, \eta)_f, \; \eta \in P_{r-(\ell+1)}(f) \otimes \{\hat{d}y_{\sigma}^f \mid \sigma \text{ is in the free block}\},
\]

\[\quad F \in \Delta_{n-1}(T_h), \; f \in \Delta_\ell(F), \; \ell = 1, \ldots, n-1,\]

\[(54d) \quad (\omega, \eta)_T, \; \eta \in B_r(\text{div}, T; \mathbb{X}), \; T \in T_h,\]

will determine a space \(V_h \subset H(\text{div}, \Omega; \mathbb{X}).\)

**Proof.** Local unisolvence on an element. We write DoF (54c) as

\[(55)\quad (\text{tr}^\text{div}_F \omega, \hat{d}y_{\tau}^f)_f = (\omega, \hat{d}y_{\tau}^f \otimes \hat{d}y_{\sigma}^f), \; f \subseteq F,\]

where \(\{n_T^f, f \subseteq F, F \in \Delta_{n-1}(T)\}\) is the face normal basis of \(\mathcal{N}^f\) in element \(T\). As \(\{n_T^f, i \in f^*\}\) and \(\{n_j^f, j = 1, \ldots, n-\ell\}\) are different bases of the same space \(\mathcal{N}^f\), DoF (55) will also determine

\[(56)\quad (\omega, \hat{d}y_{\tau}^f \otimes \hat{d}y_{\sigma}^f)_f, \; i = \ell + 1, \ldots, n.\]

Together with (54b), we obtain DoFs (53b). The number of DoFs remains the same as in a free row all indices are free. Then we conclude the unisolvence from Theorem 6.5.

Global conformity across elements. The continuity

\[ (\text{tr}^\text{div}_F \omega, \eta)_f, \; \eta \in P_{r-(\ell+1)}(f) \otimes \{\hat{d}y_{\sigma}^f \mid \sigma \text{ is in the normal constraint block}\} \]
is implied by DoFs (54a)-(54b) as the global \( t_n \) basis \( \{ t_1^n, \ldots, t_f^n, n_1^n, \ldots, n_{n-\ell}^n \} \) depending only on \( f \). Together with DoF (54c), we conclude \( \text{tr}^{\text{div}} \omega \) is continuous. \( \square \)

**Remark 6.7.** It seems that we can also try to redistribute the rows in the normal constraint block as DoF (56) can be derived from DoF (55). The problem comes from the fact the basis \( \{ n_1^f, f \subseteq F, F \in \Delta_{n-1}(T) \} \) is element-dependent. Expand a global basis vector \( n_i^f \) with index \( i > \ell \) and corresponding DoFs can be redistributed to faces. For the DoFs situated on a sub-simplex \( F \), we choose global normal vectors \( n_i^F \) which share a global DoF labeling, and local tangential vectors \( t_i^F \) which have different labeling in different elements.

We count the size of the normal constraint block. The normal constraint block disappears when \( \ell \geq k \). For the construction of the normal constraint block, there exist some \( i > \ell \) such that \( \tau = \sigma - i \in \Sigma(1 : k - 1, 1 : n) \) satisfies \( \tau^c \subseteq [\ell + 1 : n] \), which is equivalent to \( \tau^c \subseteq [\ell + 1 : n] \). Hence the number of rows in the constraint block is \( \binom{n - \ell}{n - k} \); among all \( n - \ell \) indices of the normal plane, choose \( n - k \) indices to form \( \sigma^c \). When \( \ell = 0 \), \( \binom{n}{n - k} = \binom{k}{k} \), i.e., all rows belong to the constraint block. Consequently DoFs at vertices cannot be redistributed facewisely. We thus give another justification of the continuity at vertices.

### 6.5. Discrete inf-sup condition.

For a smooth tensor \( u = (u_\sigma) \) with index \( \sigma \in \Sigma(1 : k, 1 : n) \), let grad \( u \) be a tensor with size \( \binom{n}{k} \times n \) given by

\[
(\text{grad } u)_{\sigma,i} := \partial_{x_i} u_\sigma.
\]

**Lemma 6.8.** It holds that

\[
\ker(P_{\chi} \text{grad}) = P_0(T; \mathbb{R}^{\binom{n}{k}}) + \mathbb{X}^\perp x.
\]

**Proof.** Noting that grad \( (P_0(T; \mathbb{R}^{\binom{n}{k}}) + \mathbb{X}^\perp x) = \mathbb{X}^\perp \), hence

\[
P_0(T; \mathbb{R}^{\binom{n}{k}}) + \mathbb{X}^\perp x \subseteq \ker(P_{\chi} \text{grad}).
\]

By (31) in [8], \( \ker(P_{\chi} \text{grad}) \subseteq P_1(T; \mathbb{R}^{\binom{n}{k}}) \). Take \( c + Ax \in \ker(P_{\chi} \text{grad}) \). By grad \( (c + Ax) = A \), we have \( P_\chi A = 0 \), i.e. \( A \in \mathbb{X}^\perp \). Therefore \( \ker(P_{\chi} \text{grad}) \subseteq P_0(T; \mathbb{R}^{\binom{n}{k}}) + \mathbb{X}^\perp x \). \( \square \)

We introduce notation \( RX := \ker(P_{\chi} \text{grad}) \). Examples are \( RX = RM \) the rigid motion for \( X = S \), and \( RX = RT \) for \( X = T \), where \( RT := P_0(T; \mathbb{R}^n) + xP_0(T) \) is the lowest Raviart-Thomas element. In general, \( RX \) is the Whitney form \( P_0 \Lambda^k + \kappa_k P_0 \Lambda^{k+1} \), which is another characterization of \( \ker(P_{\chi} \text{grad}) \).

Operator \( P_{\chi} \text{grad} \) is the proxy of \( P_{\chi} \text{d} := (-1)^{n-1} \ast P_{\chi} \text{d} : \Lambda^{k,0} \rightarrow \Lambda^{k,1} \). Indeed \( \int_T \omega \wedge P_{\chi} \text{d} \eta = (-1)^{n-1} \int_T \omega \wedge \ast \ast \text{d} \eta = \int_T \omega \wedge \text{d} \eta \), then the integration by parts holds

\[
\int_T \omega \wedge \eta = (-1)^n \int_T \omega \wedge P_{\chi} \text{d} \eta + \int_{\partial T} \text{tr}^{\text{div}} \omega \wedge \eta
\]
for any $\omega \in H(\text{div}; T; X)$ and $\eta \in \Lambda^{k,0}$. In the matrix and vector proxy, we have
\[
\int_T (\text{div} \mathbf{A}) \cdot \mathbf{u} \, dx = -\int_T \mathbf{A} : \nabla \mathbf{u} \, dx + \int_{\partial T} (\mathbf{A} n) \cdot \mathbf{u} \, dS.
\]

We consider the finite elements defined in Theorem 6.6. Define the global finite element space
\[
V_h := \{ \omega_h \in L^2(\Omega; X) : \omega_h |_{T} \in \mathbb{P}_r(T; X) \quad \forall \, T \in \mathcal{T}_h, \}
\]
the DoFs (54a)-(54b) is single-valued across $f \in \Delta_\ell(\mathcal{T}_h)$ for $\ell = 0, \ldots, n - 1$, and the DoF (54c) is single-valued across $F \in \Delta_{n-1}(\mathcal{T}_h)$,
\[
Q_h := \{ q_h \in L^2(\Omega; \Lambda^k) : q_h |_{T} \in \mathbb{P}_{r-1}(T; \Lambda^k) \quad \forall \, T \in \mathcal{T}_h \}.
\]

Thanks to Theorem 6.6, $V_h \subset H(\text{div}; \Omega; X)$. We are going to verify the discrete inf-sup condition $\text{div} \, V_h = Q_h$ if $r \geq n + 1$. The following characterization of the range of the $\text{div}$ operator on the bubble polynomial space is an abstract version of results (30) and (41) established in [28, 30, 29].

**Lemma 6.9.** For each $T \in \mathcal{T}_h$, it holds
\[
\text{div} \mathcal{B}_r(\text{div}; T; X) = \mathbb{P}_{r-1}(T; \Lambda^k) \cap RX^\perp.
\]

**Proof.** When $r = 0, 1$, (59) is obviously true as both sides are zero. We thus consider $r \geq 2$.

Apply the integration by parts (57) to get
\[
\text{div} \mathcal{B}_r(\text{div}; T; X) \subseteq (\mathbb{P}_{r-1}(T; \Lambda^k) \cap RX^\perp).
\]

Next we focus on the proof of the other side
\[
(\mathbb{P}_{r-1}(T; \Lambda^k) \cap RX^\perp) \subseteq \text{div} \mathcal{B}_r(\text{div}; T; X).
\]

For simplicity, write $d\lambda_{\sigma^\ast} \otimes d\lambda_{[\sigma(0), \sigma(i)]^\ast}$ as $\phi_{[\sigma,i]}$ for each $\sigma \in \Sigma(0 : n-k, 0 : n)$ and $i = 1, \ldots, n-k$. By Theorem 5.7, $\{ \phi_{[\sigma,i]} \}_{i=1,\ldots,n-k}$ is a basis of $X$, whose dual basis (appropriate rescaling of (47)) is denoted by $\{ \psi_{\tau,j} \}_{\tau \in \Sigma(0:n-k,0:n)}$, that is $\psi_{\tau,j} \in X$, $\langle \phi_{[\sigma,i]}, \psi_{\tau,j} \rangle = 1$ for $\sigma = \tau$ and $i = j$, otherwise it vanishes.

Consider the edge $e = e_{\sigma(0), \sigma(i)}$. The vector proxy of $d\lambda_{[\sigma(0), \sigma(i)]}$ is proportional to $t^e$. Coupled with the edge bubble function $b_e = \lambda_{\sigma(0)} \lambda_{\sigma(i)}$, the vector function $b_e t^e$ satisfies
\[
\mathbf{n}_F \cdot b_e t^e |_{F} = 0, \quad F \in \Delta_{n-1}(T),
\]
as if the edge $e \not\subseteq F$, then $b_e |_{F} = 0$; otherwise $\mathbf{n}_F \cdot t^e = 0$. Therefore $\lambda_{[\sigma(0), \sigma(i)]} \phi_{[\sigma,i]} \in \mathcal{B}_2(\text{div}; T; X)$.

If $(\mathbb{P}_{r-1}(T; \Lambda^k) \cap RX^\perp) \not\subseteq \text{div} \mathcal{B}_r(\text{div}; T; X)$, then there exists $u \in \mathbb{P}_{r-1}(T; \Lambda^k) \cap RX^\perp$ satisfying $(u, \text{div} \omega)_T = 0$ for any $\omega \in \mathcal{B}_r(\text{div}; T; X)$. Equivalently
\[
(P_{\text{grad}} u, \omega)_T = 0 \quad \forall \, \omega \in \mathcal{B}_r(\text{div}; T; X).
\]

By expressing $P_{\text{grad}} u = \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} q^{\sigma,i} \psi_{[\sigma,i]}$ with $q^{\sigma,i} \in \mathbb{P}_{r-2}(T)$, we choose
\[
\omega = \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} \lambda_{[\sigma(0), \sigma(i)]} q^{\sigma,i} \phi_{[\sigma,i]} \in \mathcal{B}_r(\text{div}; T; X).
\]

Then we have
\[
\sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} (\lambda_{[\sigma(0), \sigma(i)]} q^{\sigma,i}, q^{\sigma,i})_T = 0.
\]

Therefore $q^{\sigma,i} = 0$ for all $i$ and $\sigma$ and consequently $u = 0$. \qed
Employing the same argument as the proof of (25), the discrete inf-sup condition follows from (59).

**Lemma 6.10.** Let \( r \geq n + 1 \). Let \( V_h \subset H(div; \Omega; \mathbb{X}) \) be the finite element space defined in Theorem 6.5 and \( Q_h \) be the space defined in (58). It holds

\[
\text{div} V_h = Q_h.
\]

**Proof.** Take \( q_h \in Q_h \). By (37) in [8], \( \text{div} H^1(\Omega; \mathbb{X}) = L^2(\Omega; \mathbb{X}) \). Then there exists \( \omega \in H^1(\Omega; \mathbb{X}) \) satisfying \( \text{div} \omega = q_h \). Let \( \omega_1 \) be the Scott-Zhang interpolation [39] of \( \omega \) in the tensor \( r \)th order Lagrange element space \( \mathbb{S}_r^k \otimes \mathbb{X} \) such that

\[
(\text{tr}^\text{div}_F (\omega - \omega_1), \eta)_F = 0, \quad \eta \in \mathbb{P}_{r-n}(F) \otimes \text{Alt}^k, F \in \Delta_{n-1}(T_h).
\]

Since \( r \geq n + 1 \), \( RX \subset \mathbb{P}_1(T; \text{Alt}^k) \subseteq \mathbb{P}_{r-n}(T; \text{Alt}^k) \), the last equation together with the integration by parts implies \( \text{div}(\omega - \omega_1)|_T \in \mathbb{P}_{r-1}(T; \text{Alt}^k) \cap \text{RX}^k \). By (59), there exists \( \omega_2 \in V_h \) such that \( \omega_2|_T \in \mathbb{B}_r(\text{div}, T; \mathbb{X}) \) for each \( T \in T_h \) and \( \text{div} \omega_2 = \text{div}(\omega - \omega_1) \). Finally take \( \omega_h = \omega_1 + \omega_2 \in V_h \) to get \( \text{div} \omega_h = q_h \). \( \square \)

For \( \ell \geq k \), all DoFs can be redistributed facewisely as no normal constraint block exists. We can further modify the DoFs to get the discrete inf-sup condition with degree \( r \geq k + 1 \) relaxing the requirement \( r \geq n + 1 \) for \( k = 1, \ldots, n - 2 \).

**Theorem 6.11** (\( H(div) \)-conforming finite element with a better inf-sup condition). Let \( 1 \leq k \leq n - 2 \) and \( r \geq k + 1 \). For each \( f \in \Delta_{\ell}(T_h) \), \( \ell = 0, \ldots, n - 1 \), choose a global t-n basis \( \{ t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell} \} \) depending only on \( f \). Let \( \{ \text{d}y_i \} \) be the corresponding basis of \( \text{Alt}^1 \), and \( \{ \text{d}y_i^* \} \) its dual basis. The DoFs

\[
\begin{align*}
(60a) & \quad \omega(v_i), i = 0, \ldots, n, \omega \in \mathbb{X}, \\
(60b) & \quad (\omega, \eta)_f, \quad \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{ \text{d}y^*_i \otimes \ast \text{d}y^*_i \mid (\sigma, i) \text{ is free and in the constraint block} \}, \\
(60c) & \quad (\text{tr}^\text{div}_F \omega, \eta)_f, \quad \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{ \text{d}y^*_i \mid \sigma \text{ is in the free block} \}, \\
(60d) & \quad (\text{tr}^\text{div}_F \omega, \eta)_F, \quad \eta \in \mathbb{P}_1(F) \oplus (\mathbb{B}_{r,k}(F) \cap \mathbb{P}_1(F)) \otimes \text{Alt}^k, F \in \Delta_{n-1}(T_h), \\
(60e) & \quad (\omega, \eta)_T, \quad \eta \in \mathbb{B}_r(\text{div}, T; \mathbb{X}), T \in T_h,
\end{align*}
\]

will determine a space \( V_h \subset H(div; \Omega; \mathbb{X}) \).

**Proof.** The condition \( k \leq n - 2 \) is to ensure

\[
\dim \mathbb{B}_{k+1,k}(F) = |\Delta_k(F)| = \binom{n - 1}{k + 1} \geq n = \dim \mathbb{P}_1(F) \quad \text{if } k \leq n - 2.
\]

So that we can modify the face DoF to (60d). Vanishing DoFs (60a)-(60c) will imply \( \text{tr}^\text{div}_F \omega \in \mathbb{B}_{r,k}(F) \otimes \text{Alt}^k \), which can be decomposed into (60d). \( \square \)

As \( \eta \in \mathbb{P}_1(F) \otimes \text{Alt}^k \) is included in DoF (60d), we acquire the following discrete inf-sup condition by applying the same argument as the proof of Lemma 6.10.

**Corollary 6.12.** Let \( 1 \leq k \leq n - 2 \) and \( r \geq k + 1 \). Let \( V_h \subset H(div; \Omega; \mathbb{X}) \) be the finite element space defined in Theorem 6.11 and \( Q_h \) be the space defined in (58). It holds

\[
\text{div} V_h = Q_h.
\]
## References

1. S. Adams and B. Cockburn. A mixed finite element method for elasticity in three dimensions. *J. Sci. Comput.*, 25(3):515–521, 2005.
2. D. N. Arnold. *Finite Element Exterior Calculus*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2018.
3. D. N. Arnold, G. Awanou, and R. Winther. Finite elements for symmetric tensors in three dimensions. *Math. Comp.*, 77(263):1229–1251, 2008.
4. D. N. Arnold, J. Douglas, Jr., and C. P. Gupta. A family of higher order mixed finite element methods for plane elasticity. *Numer. Math.*, 45(1):1–22, 1984.
5. D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.
6. D. N. Arnold, R. S. Falk, and R. Winther. Geometric decompositions and local bases for spaces of finite element differential forms. *Comput. Methods Appl. Mech. Engrg.*, 198(21-26):1660–1672, 2009.
7. D. N. Arnold and K. Hu. Complexes from complexes. *Found. Comput. Math.*, 21(6):1739–1774, 2021.
8. D. N. Arnold and R. Winther. Mixed finite elements for elasticity. *Numer. Math.*, 92:401–419, 2002.
9. F. Brezzi, J. Douglas, and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numer. Math.*, 47(2):217–235, 1985.
10. F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin. Mixed finite elements for second order elliptic problems in three variables. *Numer. Math.*, 51(2):237–250, 1987.
11. D. A. Castro, P. R. B. Devloo, A. M. Farias, S. M. Gomes, D. de Siqueira, and O. Durán. Three dimensional hierarchical mixed finite element approximations with enhanced primal variable accuracy. *Comput. Methods Appl. Mech. Engrg.*, 306:479–502, 2016.
12. C. Chen, L. Chen, X. Huang, and H. Wei. Geometric decomposition and efficient implementation of high order face and edge elements. *Commun. Comput. Phys.*, 35(4):1045–1072, 2024.
13. L. Chen, J. Hu, and X. Huang. Fast auxiliary space preconditioners for linear elasticity in mixed form. *Math. Comp.*, 87(312):1601–1633, 2018.
14. L. Chen and X. Huang. Decoupling of mixed methods based on generalized Helmholtz decompositions. *SIAM J. Numer. Anal.*, 56(5):2796–2825, 2018.
15. L. Chen and X. Huang. Geometric decompositions of the simplex lattice and smooth finite elements in arbitrary dimension. *arXiv preprint arXiv:2111.10712*, 2021.
16. L. Chen and X. Huang. Discrete Hessian complexes in three dimensions. In *The virtual element method and its applications*, volume 31 of *SEMA SIMAI Springer Ser.*, pages 93–135. Springer, Cham, 2022.
17. L. Chen and X. Huang. Finite elements for div- and div-div-conforming symmetric tensors in arbitrary dimension. *SIAM J. Numer. Anal.*, 51(2):1932–214, 2013.
18. L. Chen and X. Huang. Finite element de Rham and Stokes complexes in three dimensions. *Math. Comp.*, 93(345):55–110, 2024.
19. S. H. Christiansen. On the linearization of Regge calculus. *Numer. Math.*, 119(4):613–640, 2011.
20. S. H. Christiansen, J. Hu, and K. Hu. Nodal finite element de Rham complexes. *Numer. Math.*, 139(2):411–446, 2018.
21. S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-preserving discretization. *Acta Numer.*, 20:1–119, 2011.
22. M. Costabel and A. McIntosh. On Bogovskiǐ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.*, 265(2):297–320, 2010.
23. D. De Siqueira, P. R. B. Devloo, and S. M. Gomes. Hierarchical high order finite element approximation spaces for $H(div)$ and $H(curl)$. In G. Kreiss, P. Lötstedt, A. Målqvist, and M. Neytcheva, editors, *Numerical Mathematics and Advanced Applications 2009*, pages 269–276, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.
24. D. De Siqueira, P. R. B. Devloo, and S. M. Gomes. A new procedure for the construction of hierarchical high order $H(div)$ and $H(curl)$ finite element spaces. *J. Comput. Appl. Math.*, 240:204–214, 2013.
25. R. S. Falk and R. Winther. Local bounded cochain projections. *Math. Comp.*, 83(290):2631–2656, 2014.
26. J. Gopalakrishnan, L. E. García-Castro, and L. F. Demkowicz. Nédélec spaces in affine coordinates. *Comput. Math. Appl.*, 49(7-8):1285–1294, 2005.
27. J. Hu. Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the higher order case. *J. Comput. Math.*, 33(3):283–296, 2015.
28. J. Hu and Y. Liang. Conforming discrete Gradgrad-complexes in three dimensions. *Math. Comp.*, 90(330):1637–1662, 2021.
29. J. Hu and S. Zhang. A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids. *Sci. China Math.*, 58(2):297–307, 2015.
30. J. Hu and S. Zhang. Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the lower order case. *Math. Models Methods Appl. Sci.*, 26(9):1649–1669, 2016.
31. K. Hu, T. Lin, and B. Shi. Finite elements for symmetric and traceless tensors in three dimensions. *arXiv preprint arXiv:2311.16077*, 2023.
[33] X. Huang, C. Zhang, Y. Zhou, and Y. Zhu. New low-order mixed finite element methods for linear elasticity. *Adv. Comput. Math.*, 50(2):Paper No. 17, 31, 2024. 23

[34] C. Johnson and B. Mercier. Some equilibrium finite element methods for two-dimensional elasticity problems. *Numer. Math.*, 30(1):103–116, 1978. 2

[35] M. W. Licht. On basis constructions in finite element exterior calculus. *Adv. Comput. Math.*, 48(2):Paper No. 14, 36, 2022. 24

[36] J. C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 50:57–81, 1986. 8

[37] R. A. Nicolaides. On a class of finite elements generated by Lagrange interpolation. II. *SIAM J. Numer. Anal.*, 10:182–189, 1973. 5

[38] V. Quenneville-Belair. *A New Approach to Finite Element Simulations of General Relativity*. ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–University of Minnesota. 2

[39] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990. 42

[40] R. Stenberg. A nonstandard mixed finite element family. *Numer. Math.*, 115(1):131–139, 2010. 9, 12

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697, USA

Email address: chenlong@math.uci.edu

CORRESPONDING AUTHOR. SCHOOL OF MATHEMATICS, SHANGHAI UNIVERSITY OF FINANCE AND ECONOMICS, SHANGHAI 200433, CHINA

Email address: huang.xuehai@sufe.edu.cn