An Example of $Z_N$–Graded Noncommutative Differential Calculus

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July 1999

Abstract

In this work, we consider the algebra $M_N(C)$ of $N \times N$ matrices as a cyclic quantum plane. We also analyze the coaction of the quantum group $\mathcal{F}$ and the action of its dual quantum algebra $\mathcal{H}$ on it. Then, we study the decomposition of $M_N(C)$ in terms of the quantum algebra representations. Finally, we develop the differential algebra of the cyclic group $Z_N$ with $d^N = 0$, and treat the particular case $N = 3$. 
1 Introduction

In the last decade, the concept of the noncommutative differential geometry, \cite{1} has been extensively developed. The most simple example of noncommutative differential geometry based on derivations is given by the Grassmannian of the matrix algebra $\mathcal{M}_N = M_N(C)$,\cite{2}. The matrix algebra $\mathcal{M}_N$ can also be considered as a cyclic quantum plane ($q^N = 1$) on which a coaction of quantum group $\mathcal{F}$ and an action of its dual $\mathcal{H}$ are naturally defined, and the associated Wess–Zumino differential complex is constructed, (\cite{3} and references therein) . Moreover, the notion of graded $q$–differential algebra with the condition $d^N = 0$, has been recently introduced,\cite{4}.

The main aim of this work is to study the noncommutative differential geometry of the cyclic group $\mathbb{Z}_N$, viewed as the subalgebra $M_{\text{diag}}^N$ of diagonal matrices of $\mathcal{M}_N$, as an example of $\mathbb{Z}_N$–graded noncommutative differential calculus.

This work is organized as follows: In section 2, we give a presentation of the space $M_N(C)$ as a cyclic Manin plane. In sections 3, we present the coaction and the action of the quantum group $\mathcal{F}$ and its dual $\mathcal{H}$ on $\mathcal{M}_N$ respectively and study the reduction of $\mathcal{M}_N$ under the representation of $\mathcal{H}$. In section 4, we construct the noncommutative differential complex of the cyclic group $\mathbb{Z}_N$ with a $\mathbb{Z}_N$–graded differential $d$, i.e. $d^N = 0$. Finally, in section 5 we treat in details the case $N = 3$. The section 6 is devoted to some conclusions and perspectives.

2 $\mathcal{M}_N \equiv M_N(C)$ as a cyclic quantum plane

The algebra of $N \times N$ matrices can be generated by two elements $x$ and $y$ obeying the relations:

$$xy = qyx$$
$$x^N = y^N = 1$$

where $q$ denotes a $N$–th root of unity:

$$q^N = 1 \ , \ q \neq 1 \ , \ \sum_{n=0}^{N-1} q^n = 0 \ , \ q^n = q^{n-N} \ , \ n \in \mathbb{Z}$$

and $1$ is the $N \times N$ unit matrix.

Explicitly $x$ and $y$ can be represented by the matrices:

$$x = \begin{pmatrix}
0 & 1 & \ldots & \ldots & \ldots \\
0 & 0 & 1 & \ldots & \ldots \\
\ldots & 0 & 1 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & 0 & 1 \\
1 & 0 & \ldots & \ldots & 0
\end{pmatrix} \quad , \quad y = \begin{pmatrix}
1 & \ldots & \ldots & \ldots & \ldots \\
\ldots & q & \ldots & \ldots & \ldots \\
\ldots & \ldots & q^2 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & q^{N-1}
\end{pmatrix}$$

We call the algebra generated by elements $x$ and $y$ satisfying the relations (\cite{1} and \cite{4}) the cyclic quantum plane $\mathcal{M}_N \equiv M_N(C)$. As a $N^2$–dimensional vector space, $\mathcal{M}_N$ is
spanned by the following basis:
\[ \{ \alpha^{rs} = x^r y^s; r, s = 0, 1, 2, \ldots, N - 1 \} \]
and is endowed by the following internal law:
\[ \alpha^{rs} \alpha^{mn} = f^{(rs)(mn)}_{(kl)} \alpha^{kl} \]
where \( x^r y^s = q^{sr} y^s x^r \) and:
\[ f^{(rs)(mn)}_{(kl)} = q^{-ms} \delta^{r+m}_k \delta^{s+n}_l. \]

The noncommutativity of the elements of \( \mathcal{M}_N \) is reflected by the following relation:
\[ \alpha^{rs} \alpha^{mn} = q^{(rn-ms)} \alpha^{mn} \alpha^{rs}. \]

We can also equip \( \mathcal{M}_N \) with a Lie structure by introducing the following commutation rule:
\[ [\alpha^{rs}, \alpha^{mn}] = C^{(rs)(mn)}_{(kl)} \alpha^{kl} \]
where the structure constants are given by:
\[ C^{(rs)(mn)}_{(kl)} = (q^{-ms} - q^{-nr}) \delta^{r+m}_k \delta^{s+n}_l. \]

Let us define a basis \( \{ e_{rs} \} \) of \( \text{Der}(\mathcal{M}_N) \), i.e. the Lie algebra of derivations (all are inner) of \( \mathcal{M}_N \) as follows:
\[ e_{rs} = \text{Ad}_{\alpha^{rs}} = [\alpha^{rs}, \cdot] \]
such that
\[ e_{rs} (\alpha^{mn}) = [\alpha^{rs}, \alpha^{mn}] = C^{(rs)(mn)}_{(kl)} \alpha^{kl} \]
and satisfying:
\[ [e_{rs}, e_{mn}] = C^{(rs)(mn)}_{(kl)} e_{kl}. \]

3 The quantum group \( \mathcal{F} \), its dual \( \mathcal{H} \) and reduction of \( \mathcal{M}_N \)

3.1 The quantum group \( \mathcal{F} \) and its coaction on \( \mathcal{M}_N \)

Let us construct the matrix quantum group generated by the quantum matrix: \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)
coacting on the coordinate doublet of the reduced quantum plane by the following left and right coactions:
\[ \left( \begin{array}{c} x' \\ y' \end{array} \right) = \delta_L \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a & b \\ c & d \end{array} \right) \otimes \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{array} \right) \]
\[(x'', y'') = \delta_R(x, y) = (x, y) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x \otimes a + y \otimes c \quad x \otimes b + y \otimes d).\]

Imposing that the quantities \(x', y'\) (and \(x'', y''\)) should satisfy the same relations as \(x\) and \(y\), one obtains the following defining relations of the quantum group \(\text{Fun}(GL_q(2))\)

\[
ab = qba \\
ac = qca \\
ad - da = (q - q^{-1})bc \\
bc = cb \\
bd = qdb \\
bd = qdc
\]

together with:

\[
a^N = d^N = 1 \quad c^N = b^N = 0.
\]

These latter represent an ideal \(I\), such that the resulting quantum group is the quotiented quantum group \(\text{Fun}(GL_q(2))/I\). The element \(\mathcal{D} = ad - qbc = da - q^{-1}bc\) is central and represents the \(q\)-determinant, and if we set it equal to 1, we get the quotiented \(\text{Fun}(SL_q(2))/I\).

The algebra defined by \(a, b, c, d\) and the above set of relations will be called \(\mathcal{F}\). Using the fact that \(a^N = 1\) and that:

\[
ad = 1 + qbc
\]

we obtain \(d = a^{N-1}(1 + qbc)\), so that \(d\) (or \(a\)) can be eliminated.

The algebra \(\mathcal{F}\) can therefore be linearly generated -as a vector space- by the elements \(a^\alpha b^\beta c^\gamma\) where \(\alpha, \beta, \gamma = 0, 1, 2, \ldots, N - 1\). We see that \(\mathcal{F}\) is a finite dimensional associative algebra, of dimension \(N^3\).

### 3.2 The quantum algebra \(\mathcal{H}\) and its action on \(\mathcal{M}_N\)

Using the interchange of multiplication and comultiplication by duality, we define the dual \(\mathcal{H}\) of \(\mathcal{F}\) as a quantum group of same dimension as \(\mathcal{F}\), generated by \(H^\alpha X_+^\beta X_-^\gamma; \alpha, \beta, \gamma \in \mathbb{Z}\), where \(X_+, X_-, H\) are defined by duality by means of the following pairing between generators:

\[
< H, a >= q \quad < H, b >= 0 \quad < H, c >= 0 \quad < H, d >= q^2 \\
< H^{-1}, a >= q^2 \quad < H^{-1}, b >= 0 \quad < H^{-1}, c >= 0 \quad < H^{-1}, d >= q \\
< X_+, a >= 0 \quad < X_+, b >= 1 \quad < X_+, c >= 0 \quad < X_+, d >= 0 \\
< X_-, a >= 0 \quad < X_-, b >= 0 \quad < X_-, c >= 1 \quad < X_-, d >= 0
\]

and the relations:

\[
H^N = 1 \\
X_+^N = X_-^N = 0
\]
\( \mathcal{H} \) acts on the reduced quantum plane \( \mathcal{M}_N \), since its dual \( \mathcal{F} \) coacts on it. There are again two possibilities, left or right, but we shall use the left action that is generally defined as follows. If we denote the right coaction of \( \mathcal{F} \) on \( \mathcal{M}_N \) as:

\[
\delta_R(z) = \sum_i z_i \otimes u_i
\]

then:

\[
X_L(z) = (Id \otimes < X_L, >) \circ \delta_R(z) = (Id \otimes < X_L, >)(\sum_i z_i \otimes u_i) = \sum_i < X_L, u_i > z_i,
\]

for \( z, z_i \in \mathcal{M}, X_L \in \mathcal{H}, u_i \in \mathcal{F} \).

It follows that the action of \( \mathcal{H} \) on \( \mathcal{M} \) is given by the following table:

|   | \( H \) | \( X_+ \) | \( X_- \) |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| \( x \) | \( qx \) | 0 | \( y \) |
| \( y \) | \( q^2 y \) | \( x \) | 0 |

For an arbitrary element of \( \mathcal{M} \), one finds the following expressions:

\[
H^L[x^r y^s] = q^{(r-s)} x^r y^s
\]

\[
X_+^L[x^r y^s] = q^r \left( \frac{1-q^{-2s}}{1-q^{-2}} \right) x^{r+1} y^{s-1}
\]

\[
X_-^L[x^r y^s] = q^s \left( \frac{1-q^{-2r}}{1-q^{-2}} \right) x^{r-1} y^{s+1}
\]

with \( r, s = 0, 1, 2, \ldots, N-1 \).

### 3.3 The reduction of the algebra \( \mathcal{M}_N \) into indecomposable representation of \( \mathcal{H} \)

The generator \( H \) always acts as an automorphism, for this reason, in order to study the invariant subspaces of \( \mathcal{M}_N \) under the left action of \( \mathcal{H} \), we have only to consider the action of \( X_+ \) and \( X_- \).

Forgetting numerical factors, the action of \( X_+ \) and \( X_- \) on a given element of \( \mathcal{M} \) can be written as follows:

\[
x^{r+1} y^{s-1} \Rightarrow x^r y^s \Leftrightarrow x^{r-1} y^{s+1}
\]

where \( X_- \) takes us from the left to the right and \( X_+ \) from the right to the left.

We verify that under the left action of \( \mathcal{H} \) the algebra of \( N \times N \) matrices can be decomposed
into a direct sum of $N$ subspaces of dimension $N$, according to :

$$
N_N = \{x^{N-1}y, x^{-1}y^2, x^{-2}y^3, \ldots, x^{N-3}y^2, x^{N-4}y^3, \ldots, xy^{N-2}, y^{N-1}\}
$$

$$
N_{N-1} = \{x^{N-2}y, x^{-2}y^2, x^{-3}y^3, \ldots, xy^{N-3}, y^{N-2}, x^{N-1}y^{N-1}\}
$$

$$
N_{N-2} = \{x^{N-3}y, x^{-3}y^2, x^{-4}y^3, \ldots, xy^{N-4}, y^{N-3}, x^{N-2}y^{N-2}, x^{N-1}y^{N-1}\}
$$

$$
N_{N-3} = \{x^{N-4}y, x^{-4}y^2, x^{-5}y^3, \ldots, xy^{N-5}, y^{N-4}, x^{N-3}y^{N-3}, x^{N-2}y^{N-2}, x^{N-1}y^{N-1}\}
$$

$$
N_{N-4} = \{x^{N-5}y, x^{-5}y^2, x^{-6}y^3, \ldots, xy^{N-6}, y^{N-5}, x^{N-4}y^{N-4}, x^{N-3}y^{N-3}, x^{N-2}y^{N-2},
$$

$$
N_{N-4}^{N-5}y^{N-1}\}
$$

.$$N_2 = \{x, y, x^{N-1}y^2, x^{-1}y^3, \ldots, x^3y^{N-2}, x^2y^{N-1}\}
$$

$$N_1 = \{1, x^{N-1}, y, x^{-2}y^2, x^{-3}y^3, \ldots, x^2y^{N-2}, xy^{N-1}\}
$$

such that :

$$\mathcal{M}_N = N_N \oplus N_{N-1} \oplus \ldots \oplus N_2 \oplus N_1$$

4 The $Z_N$–graded differential geometry of $Z_N$

First, let us recall that it is possible to construct a $Z_2$–graded noncommutative differential geometry of $\mathcal{M}_N$ based on derivations, by introducing a set of 1–forms $\theta^{kl}$ defined by the following duality relation, [2] :

$$\theta^{kl}(e_{mn}) = \delta^{kl} e_{mn} = \delta^k_m \delta^l_n.$$  

Then, using the $Z_2$–graded differential $d$ (and the wedge product), one easily describe the $Z_2$–graded noncommutative differential complex $(\Omega_{Der}(\mathcal{M}_N); d)$.  

Our main aim in this work is precisely to show that $\mathcal{M}_N$ itself, equiped with some well–defined differential $d$ satisfying $d^N = 0$, can be viewed as a $Z_N$–graded differential complex of the cyclic group $Z_N$.  

For this purpose, let us define a $Z_N$–grading on $\mathcal{M}_N$ such that :

$$|\alpha^{rs}| = \text{grading}(\alpha^{rs}) = r + s \mod(N).$$  

This means that a $Z_N$–grading equal to 1 is attributed to the fundamental objects 1, $x$ and $y$, and then the above decomposition of $\mathcal{M}_N$ is naturally equiped with the following $Z_N$–grading :

$$N_1 \rightarrow 0$$

$$N_2 \rightarrow 1$$

$$N_3 \rightarrow 2$$

.$$N_{N-2} \rightarrow N - 3$$

$$N_{N-1} \rightarrow N - 2$$

$$N_N \rightarrow N - 1$$
Consider the cyclic group of order \( N \), \( Z_N = \{1, y, y^2, y^3, \ldots, y^{N-1}\} \). Therefore, the algebra \( C^\infty(Z_N) \) of complex functions on \( Z_N \) can be realized as the algebra \( \mathcal{M}^{\text{diag}} \subset \mathcal{M}_N \) of diagonal complex \( N \times N \) matrices.

Starting from \( C^\infty(Z_N) \equiv \Omega^0(Z_N) = Z_N \), we can build the space of 1–forms \( \Omega^1(Z_N) \) by introducing a differential \( d_x : \Omega^0 \longrightarrow \Omega^1 \) associated to \( x \) and defined by :

\[
d_x(y^m) = [x, y^m] = (1 - q^{-m}) y^m.
\]

This means that the sub–space \( \Omega^1 = x\Omega^0 = \{x, xy, \ldots, xy^{N-1}\} \) constitutes the space of 1–forms.

This differential can be naturally extended to all other sub–spaces of \( \mathcal{M}_N \) such that :

\[
d_x : \Omega^k \longrightarrow \Omega^{k+1}
\]

\[
d_x(\alpha^{rs}) = x\alpha^{rs} - q^r \alpha^{rs} x = [x, \alpha^{rs}]_q = (1 - q^{-s}) \alpha^{(r+1),s}
\]

where the sub–space of \( k \)–forms is defined by :

\[
\Omega^k = x^k \Omega^0 = \{x^k, x^ky, \ldots, x^ky^{N-1}\}
\]

for \( k = 0, 1, \ldots, N - 1 \). It is easy to see that the degree of the differential forms is given by :

\[
\text{degree}(\alpha^{rs}) = r \mod(N)
\]

and that the wedge product between two arbitrary forms is nothing else than the usual matrix multiplication.

Then, the \( Z_N \)–graded differential complex \( (\Omega(Z_N), d) \), with \( d^N = 0 \), is completely built with :

\[
\Omega(Z_N) = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \ldots \oplus \Omega^{N-2} \oplus \Omega^{N-1} \approx M_N(C)
\]

Moreover, one can easily verify that the differential \( d \) satisfy a \( q \)-deformed Leibniz rule :

\[
d_x(\alpha^{rs}\alpha^{mn}) = (d_x(\alpha^{rs}))\alpha^{mn} + q^r \alpha^{rs} (d_x(\alpha^{mn}))
\]

and that effectively one has \( d^N = 0 \).

### 5 EXAMPLE : The cyclic group \( Z_3 \)

Let us now consider the case of \( Z_3 = \{1, y, y^2\} \), with :

\[
1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = y^3 , \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix} , \quad y^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q \end{pmatrix} .
\]

\[1 + q + q^2 = 0 \quad , \quad q = q^{-2} \quad , \quad q^2 = q^{-1} , q^3 = 1.\]
The algebra $C^\infty(Z_3)$ of complex functions on $Z_3$ is then identified with the sub–algebra $\mathcal{M}_3^{diag} \subset \mathcal{M}_3$ of diagonal complex $3 \times 3$ matrices, where $\mathcal{M}_3$ is generated by:

$$\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2\},$$

with

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. $$

If we attribute a $Z_3$–grading 1 to 1, x and y, then one has:

$$\{1, xy^2, x^2y\} \rightarrow 0$$

$$\{x, y, x^2y^2\} \rightarrow 1$$

$$\{x^2, y^2, xy\} \rightarrow 2$$

From the sub–space $\Omega^0 = Z_3$ of 0–forms, we build the two other subspaces of 1– and 2–forms respectively:

$$\Omega^1 = x\Omega^0 = \{x, xy, xy^2\}$$

$$\Omega^2 = x^2\Omega^0 = \{x^2, x^2y, x^2y^2\}.$$ 

by using the differential $d_x : \Omega^k \leftarrow \Omega^{k+1}$ defined by (4), i.e. :

$$d_x(1) = 0$$

$$d_x(y) = (1 - q^2)xy$$

$$d_x(y^2) = (1 - q)xy^2$$

$$d_x(x) = (1 - q)x^2$$

$$d_x(xy) = 0$$

$$d_x(xy^2) = (1 - q^2)x^2y^2$$

$$d_x(x^2) = (1 - q^2)1$$

$$d_x(x^2y) = (1 - q)y$$

$$d_x(x^2y^2) = 0$$

Then, the $Z_3$–graded differential algebra $\Omega(Z_3)$ is given by:

$$\Omega(Z_3) = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \approx \mathcal{M}_3(C)$$

with:

$$\Omega^k = x^kZ_3; \quad k = 0, 1, 2.$$ 

Finally, using the relations (5), we can easily verify that for arbitrary $\omega_p \in \Omega^p(Z_3)$ and $\omega_q \in \Omega^q(Z_3)$ one has :

$$d_x(\omega_p \omega_q) = (d_x\omega_p)\omega_q + q^p\omega_p(d_x\omega_q)$$

and

$$d^2_x(\omega_p) = [x, [x, [x, \omega_q]_q]]_q$$

$$= [x, [x, (x\omega - q^2\omega x)]_q]_q$$

$$= \ldots \ldots$$

$$= q^k(1 + q + q^2)[\ldots] + x^3\omega - \omega x^3$$

$$= 0.$$
6 Conclusion

In the last decade, noncommutative differential geometry became a very important research topic in Mathematical Physics. In this context, the role of the $C^*$–algebra of smooth complex functions on a ordinary manifold is played by an abstract associative not necessarily commutative $C^*$–algebra as analog of functions on noncommutative manifolds.

In order to define gauge theories on these noncommutative spaces, we need to define noncommutative differential calculus on them. In fact, several particle Physics models have been constructed on noncommutative spaces, for instance, on product spaces like $C^\infty(M) \otimes M_N(C), M_4 \times Z_N$, etc... , [2], [5].

In another hand, the matrix algebras $M_N$ are very often used in various fields of Physics. Moreover, its differential geometry is the most simple example of noncommutative differential geometry. In [3], the Wess–Zumino complex of $M_N$ is constructed. Nevertheless, following the Dubois-Violette’s approach, [2], we show how to construct the biggest sub–algebra of the noncommutative universal differential algebra of these matrix algebras, and present its decomposition into irreducible components by determining the eigenvalue equations of the associated Laplace–Beltrami operator, with a special interest to the case of $M_3(C)$, [6].

Actually, it seems very interesting to study the $Z_N$–graded differential geometry of some noncommutative spaces. We plan to treat this subject in a future paper in order to describe gauge theories on such spaces.

Acknowledgments

The authors would like to acknowledge Abdus Salam International Centre for Theoretical Physics where this work was realized under the Associateship scheme. They also would like to thank the Arab fund for financial support.
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