Spectrograms of one-dimensional isotropic turbulence

V A Frost
Ishlinsky Institute for Problems in Mechanics, RAS, Pr. Vernadskogo, 101-1,
Moscow, 119526, Russia
frost@ipmnet.ru

Abstract. An equation for describing the evolution of the spectrogram velocity amplitudes of individual wave components placed on a finite field, the interaction of the components in which satisfies the triad rule: \( k = k' \pm k'' \), is proposed. The details of the energy transport along the spectrum, the decay and power supply modes are considered. In the absence of dissipation and pumping the total intensity of the system that is analogy of energy is unchanged. The results of a numerical calculation of the transition to white noise (in the absence of dissipation), the decay regimes at various Reynolds numbers, and the transition to the equilibrium state for different pumping by the large-scale components, are obtained. The results obtained correspond to known experimental data obtained for real turbulent flows. In the one-dimensional case, the correlation function and the third moment that determines the structure change in the field structure upon decay is restored.

1. Introduction
The article is devoted to development of the method for receiving realization of the isotropic turbulent velocity field which can be used for testing various hypotheses of closing of the statistical moment equations. Usage of concrete realization which represent the main characteristic of whole statistical ensemble allows to avoid the difficulties of closing of the moment equations used for the description of turbulent fields.

The isotropic turbulent velocity field is the simplest example suitable for testing of various methods of creation of the closure methods of description of the turbulent phenomena. For the turbulent field there is a large number of experimental data on decay of turbulence in the uniform mean flow behind turbulence-producing grids that can be used for the analysis of methods of closing. Assuming that a realization of the isotropic turbulent field is observed, it is possible to use these experimental data for testing of the hypotheses used for closing. Analysis of three various hypotheses: K.Hasselmann [1], Yu.M. Lytkin and G.G. Chernykh [2] and N.I. Akatnov and E. N. Bystrova [3,4] , has shown [5] that it isn't possible to make the accurate choice of the most exact approach, most likely, because of incomplete compliance of experimental data for conditions of isotropic turbulence. Differences from the regularities corresponding to an isotropy at experiments with turbulence-produced grids of irregular shape [6-8] are clearly expressed.

Usage for our purposes a direct numerical integration of the Navier-Stokes equations is extremely expensive, and the necessary restriction of flow field significantly reduces area in which isotropy conditions are satisfied.

It would seem that to get the field corresponding to the isotropic turbulence, it is possible by setting a certain spectrum and having played the amplitudes and phases of large, but finite, number of
combinations of three-dimensional harmonics that satisfy the continuity condition [9]. But the distribution of third moments that are an object of hypotheses of closing received in this case because of an extremity of a set of harmonics and because of the possible requirements connected to limitation of area of a velocity field turn out very different from experimentally observable. When using the Navier-Stokes equation for calculation of time evolution of a field it is necessary to consider boundary conditions, which leads that the flow conditions is different from the needed conditions of isotropy.

The equations for various average values of parameters of the field such as average velocity, energy of turbulence, average dissipation, etc. (see for example [10]) and more difficult LES and RANS methods [11] that are applied only for flows in the conditions of far from an isotropy can't solve our problem.

The next way of receive and study of properties of the isotropic turbulent field of is offered.

The difficulties associated with the flow boundary, can be overcome by a transition to the spectral representation (to a wave space), and associated with no closure of the equations for averages, using the specific implementations of fields and corresponding selection of the initial conditions. The use of a concrete implementation leads to the fact that the role of the spectrum of the process now takes the spectrogram.

For the one-dimensional field the system of the equations describing evolution on time of a spectrogram is constructed in which interaction of separate harmonicas happens similar to the interactions described by Navier-Stokes's equation. The calculations illustrating development of the processes having the analogs in real turbulent streams are carried out.

The following designations and terminology are used:

i) common: \( \nu \) – the dissipative factor (inverse Reynolds number);

ii) in the physical (configuration) space: \( u = u(x,t) \) – turbulent velocity fluctuations; \( U_a(x) \) – an average velocity field; \( A_i(x) \) – an external source of fluctuations; \( R(r) = \langle u(x)u(x+r) \rangle \) – of a field correlation function; \( T(r) = \langle u(x)u(x)u(x+r) \rangle \) – the moment of the third order of a field \( u \);

iii) in the wave space: \( i \) – number of a wave; \( k_i \) – the wave number that represents the wavelength corresponding to \( i \); \( A_i \) – coefficient in representation of a field \( u = \sum A_i \sin(k_i x) \); \( B_i \) – the coefficient in the representation of the average field; \( a(k) \) – density of the amplitudes distribution; \( S = \sum A(k) \) – total intensity of a field; \( k_0 \) – shift of the initial distribution of amplitudes on \( k \) axis; \( \sigma \) – the parameter determining width of the initial distribution of amplitudes; \( S_0 \) – initial total intensity of a field; \( N_o \) – normalizing multiplier.

iv) terminology: instead of the term "spectrogram" is often used the term "spectrum"; for the coefficients \( A_i \) and \( B_i \) the term "amplitude" is used: for total intensity, sometimes for the purpose of abbreviation uses the term "energy".

As an example for creation of the equation for the spectrogram, the Burgers equation [12, 13] for some one-dimensional function \( u(x,t) \) is considered with added the analog of some average field \( U(x,t) \) what interaction with \( u(x,t) \) can give to increase energy of field \( u(x,t) \).

\[
\frac{\partial}{\partial t} u(x,t) = u(x,t) \frac{\partial}{\partial x} u(x,t) + \frac{\partial}{\partial x} (u(x,t) U(x)) + \nu \frac{\partial^2}{\partial x^2} u(x,t)
\] (1)

The different forms of the equation, that similar to the equation (1), were considering with different conditions of the energy supplying to the flow [14–16]. From all results given in [12–16] it is necessary to mark following:

i) the existence of the exact solution of the equation (1) on finite and infinite intervals;
ii) the existence on a finite interval of \( x \) the critical Reynolds number \( Re_m = 2\pi m/B_{2m} \) depending on the wave number \( m \) and the corresponding amplitude \( B_{2m} \) of an external field. If the Reynolds number exceeds its critical value the increase of amplitude \( A_m \) take place;

ii) the numerical calculation of development of fluctuation showed [13] that, having arisen, fluctuations grow during the initial stage of process then begin to decrease and eventually is disappearing;

iii) the requirements of conservation of energy can be fulfilled only using the corresponding boundary conditions which formulation is difficult for an infinite interval \( x \), and for finite one it is is connected with restriction of the domain where the flow is isotropic.

2. Statement of the problem

On the interval of the positive integers \( i = 1..., N \) the set of the wave numbers \( k_i = ik_m \), where \( k_m \) is the scale determining the range of wave numbers, is define

The field \( u(x,t) \) is considered in the form of decomposition on a set of orthogonal and normalized functions

\[
u(x,t) = \sum_{j=1}^{N} A_j \phi_j(x)
\]

(2)

For a limited set of wave numbers \( \phi_j(x) = \sin(kx) \) can be chosen.

The nonlinear term in the equation (1) describes the reorganization of the field after substitution in the equation of expression (2) becomes a sum of terms of type \( \sin(k_i x) \cos(k_j x) \). Using a trigonometric ratio:

\[2\sin(k')\cos(k^*) = \sin(k' + k^*) + \sin(k' - k^*)\]

allows to write down an equation analog (1) as the system of the equations for \( i = 1., N \):

\[
\frac{dA_i}{dt} = \frac{1}{2} \sum_{j=1}^{N} k_j (A_{i+j} + B_{i+j}) A_j + \frac{1}{2} \sum_{j=1}^{N} k_j (A_{i-j} + B_{i-j}) A_j - \sum_{j=1}^{N} k_j A_k A_j + \frac{dA_{i-l}}{dt} - \nu k_i^2 A_i
\]

(3)

Here \( k_i \) – a wave number; \( A_k = A_k(t) \) – the appropriate amplitude; \( B_k \) – coefficients in representation of a field of \( U \); \( \nu \) – inverse to the Reynolds number constructed on initial intensity of a field, a characteristic wave number, and the real coefficient \( \nu \) defining dissipation; \( dA_{n,j}/dt \) – intensity of an external source.

The third term of the right part is entered into the equation (3) for the saving value \( \sum_{j=1}^{N} A_j \) in the case of \( \nu = 0 \) and the absence of the fields \( A_j \) and \( U \)

The first two items in the right part describe the arising or an increase in the intensity of the respective amplitudes, the third item describes the reduction of the existed nonzero amplitude \( A_i \) due to influence on the other amplitudes, and, finally, the last element of the right side describes the decrease in amplitude due to dissipation.

The result is the system (3) describes the change of the set of harmonic amplitudes of the implementation of the field \( u(x,t) \). It is assumed that the properties of the solutions of this equation represent the properties of the whole set of possible implementations.

Considering the system (3) in the case when only one of \( A_i \neq 0 \), assuming that all other equal to zero, we get:
\[
\frac{dA_i}{dt} = k_i A_i \left( U_{2i} - \nu k_i \right)
\]

From where follows that the condition of stability will be: \(Re_1 = \frac{U_{2i}}{\nu k_i} = 1\).

Such form of critical number means that large-scale mode are the most sensitive, and the wave number of the arising fluctuation is two times less the wave number of the harmonics which are the cause of their occurrence.

Consideration of the system on a bounded set of wave numbers demands that when carrying out calculations only such transitions upon which all wave numbers \(k\) and \(k + k'\) and \(k - k'\) also belong to the \((k_i, k_N)\) interval are considered, only in this case there is no energy removal from the considered region.

Use of a bounded part of the positive integers provides the energy conservation. In the absence of \(U(x, t) = 0\) and dissipation \(\nu = 0\) the set (3) will be:

\[
\frac{dA_i}{dt} = \frac{1}{2} \sum_{j=1}^{N} k_j A_{i+j} A_j + \frac{1}{2} \sum_{j=1}^{N} k_j A_{i-j} A_j - \sum_{j=1}^{N} k_j A_k A_j
\]

(4)

Very similar equation can be received for a one-dimensional spectrum of turbulence when using the hypothesis for third moment of K. Hasselmann [1] for the Corssin equation [17]. Produced in this work the transition from continuous functions to the functions described on a finite set of elements that is applicable also to equations similar to equation [17]. This transition, from continuous functions to functions on the finite sets of elements previously was used by us to calculate of the probability density function [18, 19].

The calculation is performed as a solution of the Cauchy problem for initial conditions for amplitudes and determination of the distributions of \(B(k_i)\) and/or \(A_n\) and carried out by a method equivalent to the method of Runge-Kutta second order by time.

For calculation of a field in configuration space it is necessary to identify value \(A(k)\) as \(a(k)dk\) where \(a(k)\) is density in the interval of \(k \in (k_i, k_i + dk)\). Use of a ratio (2) due to a multiplicity of wave numbers leads to appearance of sharp bursts which can be avoided, selecting in a random way value \(k\) from values of an interval \(k \in (k_i, k_i + dk)\), and selecting also in a random way the phase shift \(d\phi \in (0, 2\pi)\) corresponding to this wave number.

3. White noise

In the absence of dissipation and energy supply the total intensity doesn't change. In this case there must be a stationary solution \(A_i = \text{invar}(i)\) of the set (4) corresponding to the process called by "white noise", that is the process the correlation of which \(\langle u(x)u(x + r) \rangle\) is delta–function \(\delta(r)\).

Substitution \(A_i = \text{invar}(i)\) in the equation (4) shows that \(A_i = \text{invar}(i)\) is the solution. Since this is limited diapason of wave numbers is considered in which there are no higher wave numbers, the correlation \(\langle u(x)u'(x + r) \rangle \neq \delta(r)\). On a finite interval of \(k_i\) this solution leads to a correlation differing from delta–function and defined by the length of the interval used.

For obtaining a time evolution of amplitudes \(A_i\) the system (4) with initial conditions of \(A_i = N_a\exp\left(-\left(k_i - k_d\right)^2/(2\sigma)^2\right)\) where \(k_d\) defines shift of distribution towards lagers \(k\), and \(\sigma\) is width of the initial distribution, and value \(N_a\) is the initial total intensity of the field, is used. The parameter values are: \(N_a = 1, \sigma = 10, k_d = 20\).

In figure 1 development of process of transition to finite distribution of amplitude \(A_i\) is pointed. Thick curves correspond to initial and finite distributions. The formation of the harmonicas with large
wave numbers and the peaks at small $k$ are noticeable, these peaks illustrate the emergence of the beats at low–wave numbers. Over time, these peaks arose, first rising and then smoothed.

![Image](image_url)

**Figure 1.** Time evolution of a spectrum in the course of tending to uniform distribution. The thick curves correspond to initial distribution and the distribution at the end of calculation.

4. The decay

The considered process corresponds to the conditions observed at a turbulent flow in the channel downstream behind the turbulence–produced grids, which formed the initial state of the turbulent field. In this case the initial state of the flow is modeled by an initial form of the spectrogram. Calculation of decay process is performed according to the equation (3) with $A_n$ and $B_n$ equal to zero, with the initial condition $A_i = N_a \exp \left(-k_i^2/(2\sigma)\right)$, where $\sigma$ determines a width of initial distribution, and $N_a$ is determined by condition $\sum A_i = S_0$ where $S_0$ defines initial intensity of the field. The parameter values are: $S_0 = 1$, $\sigma = 10$.

The microscale $\lambda$ is the parameter which being changing in the decay process and its changing characterizes the evolution of the field structure. In the configuration space its value is defined with correlation function $R(r) = \langle u(x)u(x + r) \rangle$:

$$\lambda^2 = -R(0) \left. \frac{\partial^2 R}{\partial r^2} \right|_{r=0}$$

Analog of this relation in wave space (in our case of a finite set of wave numbers) will be:

$$\lambda^2 = \sum_{i=1}^{N} A_i \left/ \sum_{i=1}^{N} k_i^2 A_i \right.$$
The time evolution of the squared microscale $\lambda^2$, the inverse intensity $1/S$ and Reynolds number $Re = \lambda S/\nu$ are shown in figure 2. The dashed lines distinguish the intervals at which changes occur under the linear law. The temporal evolution of the total intensity $S$ is shown in figure 3.

![Figure 2. Decay process.](image)

![Figure 3. Time evolution of the total intensity $S(t)$.](image)

The experimental results of [20] (see figure 2 from [20]), are in good agreement with the results obtained in the present work. The coincidence of the curves $\lambda^2$ from [20] and our data is just amazing. In both cases there is a long–linearity of change of $\lambda^2$, and a small drop in the curves at the initial stage of the decay. Deviations from linearity at the final stage in figure 2 are observed when the value of $\lambda^2$ exceeds the observed experimental values.

Coincidence of results for the curves $1/S$ and $U^2/\langle u(x)^2 \rangle$ [20] is slightly worse. The linearity of $1/S$ begins a bit later than for the analogical dependence $U^2/\langle u(x)^2 \rangle$ and in addition at large times there is deviation from linearity. Whereas in the experimental dependences nonlinearity of the curve $U^2/\langle u(x)^2 \rangle$ at large distances from the grid are absent.

It should be noted that the regularities connected with the Reynolds number differ for real case and the considering case. The assumption of constancy of the Reynolds number under experimental conditions leads to the following equation for microscale $\lambda^2$:

$$\frac{d\lambda^2}{dt} = 10\nu,$$

what is confirmed by experimental data [20]. At the same time the similar result for settlement conditions forms out on condition of constancy of the combination $\lambda^2 S$ rather than the Reynolds number constancy. This leads to:

$$\frac{d\lambda^2}{dt} = \nu,$$

that corresponds to the monotonic decreasing the Reynolds number during the decay.

The time evolution of the Reynolds numbers $Re_\lambda$ calculated from the results of several studies [20–23] are given in figure 4. The most part of curves can’t be treated as a constancy of Reynolds number in decay process. Some results are, especially, received in [22] is very close to that presented.
in figure 2. The reason of wide range (5÷75) limiting values of Reynolds number isn't clear. For the unknown reason for large times the various values of Reynolds numbers are formed.

![Figure 4](image_url)

**Figure 4.** The time evolution of the Reynolds numbers: 1 – behind grids, made of parallel spaced cylindrical rods [21], 2 – standard grid [21], 3 – [21], 4 – [6], 5 – [22], 6 – the results of the calculation of the Navier–Stokes equations [23].

In the following section the influence of dissipation on decay process is studying.

4.1. **Influence of the dissipation intensity** \( \nu \) **on the microscale** \( \lambda \).

Results of calculation of time evolution of microscale \( \lambda \) for various values of the dissipation intensity \( \nu \) are given in figure 5. The rate of \( \lambda^2 \) increasing is proportional to the value \( \nu \). At small values \( \nu \) the distinction from linear growth is noticeable. Results for the dependence \( \lambda^2 \) on \( \nu t \) are given in figure 6. In these coordinates the slope of the curves are identical, and the shift along ordinate axis is explained by differences in initial part of curves, that is illustrated figure 7 in where the respective initial stages of dependences from time are given.

![Figure 5](image_url)

**Figure 5.** Time evolution of \( \lambda^2 \).

![Figure 6](image_url)

**Figure 6.** As figure 6 vs \( \nu t \).
4.2. Spectrograms
Results of calculation of the change of values $A_k(t)$ in the decay process are given in the following figures. In figure 8 for one value $\nu$ the normalized spectra $A_k/A_0$ are given. On an initial stage the extension towards large wave numbers is forming, then amplitudes of large wave numbers are decreasing and the normalized spectrum $A_k/A_0$ ceases to change.
In figure 9 for a number of values $\nu$ changing in the range $0.005$ till $0.04$ the limited forms normalized spectra $A_k/A_0$ are given.

![Figure 7. Initial stage of decay.](image)

![Figure 8. Change of the spectrum in the decay process, thick lines correspond to initial distribution and distribution at time of the end of calculation.](image)

![Figure 9. The normalized spectra at time of the end of calculation for various values $\nu$.](image)

The transformation

$$k_i^n = k_i^0 + \left(k_i^0 - k_i^0\right)\left(\frac{\nu}{\nu_0}\right)^{-0.55}$$
allows to reduce all curves to a single one, in our case (figure 10) to dashed curve corresponding to the greatest $\nu$.

In figure 10 the dependences of normalized spectra at time of the end of calculation are pointed as $\log(A_i/A_j)$. Straight dashed lines select intervals of large wave numbers on which the viscosity plays a defining role. At large wave numbers all curves are well approximated by straight lines and the expression (5) leads to the following type of the spectrum at large wave numbers:

$$A_i \sim \exp(-\nu^{0.55}k)$$

![Figure 10](image)

**Figure 10.** Too that figure 8 in semi logarithmic coordinates.

5. **The influence of the external field and external forces**

The stationary regimes can be received only at special impact to amplitude $A_i$ which can be executed by several ways. The first method is as follow: at each step of calculation, produced an increase of certain amplitudes, which compensates for the change in the spectrum under the influence of dissipation. The quantity $B_i$ characterizes distribution over wave numbers the intensity of such influence (a rate of pumping). This is model of the fan operation which continuously adds the energy into the flow.

The second method: the growth of quantity $A_i$ proceeds due to the action of a certain stationary external field which spectral representation $B_i$ is known. This way models the velocity perturbation growth due to an existence of the mean velocity gradient as it has in real flows.

5.1. **Fan mode**

The calculation was performed for zero initial distribution of values $A_i = 0$, which corresponds to the activation of the fan at the initial absence of flow. The distribution was set uniformly distributed on the 15 lower wave numbers and was normalized to set the intensity of the impact.

In figure 11 the regime of going to a steady state is displayed. Dependences of the quantities $\lambda^2$ and total intensity $S$ are given.

The results can be explained by the existence of an initial time interval when the jet of the fan raised in the initially quiet medium consists at first from of the large enough fractions which size is specified by fan construction, and hence at first the turbulent viscosity is small and energy of system
sharply increases. Amount of input in flow of energy far exceeds the energy which disappears under the action of turbulent viscosity. However, the disintegration of the jet leads to the emergence of small–scale formations, the growth of the turbulent viscosity occurs and the excess initially stored energy dissipated. The stationary condition in which all inputted energy equal to dissipated one is forming.

In figure 12 the initial and stationary spectra for the regime of modeling the fan are presented.

5.2. Influence of the external field
In this case the initial distribution \( A_i = 0.01 \) of was given. Distribution of the amplitudes of the external field \( B_i \) was specified, also as in the previous calculation, uniformly distributed on 15 lowest wave numbers, and the intensity of external field action was normalized.

In figure 13 the regime of transition to stationary state under the external field action is presented. In this case the display scale of the parameter \( \lambda^2 \) was changed with the aim to illustrate a small non–monotonicity of curves.

In figure 14 the initial and formed under the influence of the external field spectra are given.
In both analyzed cases (the fan mode and external field influence) the results practically coincide, except for a small bend down the curve $u$ at the case when the influence of external field is considered.

In this case such a behavior of the curve $u$ as above is explained due to an existence of initial period on which amplitudes of the lowest wave numbers firstly grow. It is also confirmed by the features of microscale change. The studied examples are typical illustrations of the initial conditions influence on the character of solution. The initial fields being specified in more wide range of the wave numbers would significantly increase an initial turbulent viscosity.

The considered examples are typical illustrations of the influence of initial conditions on the nature of phenomenon.

6. **Configuration space. Third–order moments**
The use for restoring a field in physical space of the relation $u(x,t) = \sum A_i(t) \sin(k_i x)$ due to the wave numbers multiplicity that is caused the accepted form of wave numbers interval leads to emergence of sharp velocity changes at separate points $x$. It is possible to avoid this if to recall that we deal with approximation of the continuous distribution of amplitudes $A(k,t)$ and, therefore, the values of $A_i$ represent a contribution of the interval designated by $k_i$ into the integral $\int a(k)dk$. This means $A_i = a(k_i)dk$. To avoid these difficulties, when calculating the field $u(x)$, one have to execute a random choice of the wave number $k_i$ from the interval $(k_i, k_{i+1})$. It is also possible to choose random phase shift from the range $(0,2\pi)$.

In figure 15 the example of realization of the field $u(x,t)$ is presented. In the next figure 16 the correlation function $R(r) = \langle u(x)u(x+r) \rangle$ and the statistical moment of the field $u(x,t)$ third–order $T(r) = \langle u(x)u(x)u(x+r) \rangle$, which in form nearly coincides with the derivative of correlation function are presented.

![Figure 15. Example of realization of the field $u(x,t)$.](image1)

![Figure 16. Correlation function $R(r)=\langle u(x)u(x+r) \rangle$, third–order moment $T(r)=\langle u(x)u(x)u(x+r) \rangle$ and the derivative $dR(r)/dr$.](image2)

7. **Conclusion**
This research of the properties of isotropic turbulent phenomena is based on four main assumptions.

1. Using the equations for a certain implementation of random phenomenon to which, undoubtedly, liquid whirls belong allows to avoid the difficulties arising by transition from
the dynamic equations (for example Navier–Stokes equations) to the equations for averages when nonlinearity of the dynamic equations results in their no closure. It is supposed that the received description of separate implementation rather precisely corresponds to properties of a whole ensemble.

2. The transition to wave space, which is natural in case of studying the isotropic fields, enables one to avoid the difficulties related to existence of the boundary conditions being necessary in configuration space that violates, in principle, requirements of isotropy.

3. The select of the finite set of wave numbers located in a finite segment of the positive integers allows one to receive the discrete description of the process of the calculating in which the interaction of different components of the wave field does not lead to extension of the set of wave numbers.

4. And, perhaps, the main thing. The interaction of the wave-field amplitudes related to nonlinearity of dynamic equations is using the triad rule \( k = k' + k'' \).

Numerical solving the Cauchy problem allows to receive results for the regime of the dissipative decay of the field, the regimes of energy input directly or as a result of interaction with external field that, in many details, are in agreement with known results for real turbulent flows.

Especially it should be noted that the considered approach allows one to account for the influence of initial field structure which can have significant effect on the results received.

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