On the Cardinality of Possible Worlds in Discrete Spacetime Structures

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Abstract

We propose a first-order theory of discrete causality based on a variation of causal sets where the light cone of an event is represented by finitely branching trees. We argue through basic topological properties of Cantor space that under certain assumptions about the spacetime structure and causation, given any event \( x \), if all worldlines extending the event \( x \) are ‘eventually deterministic’, then the number of alternate universes with respect to \( x \) is exactly \( \aleph_0 \). We also observe that if there are countably many alternate universes with respect to \( x \), then at least one of these universes must be necessarily ‘decidable’ in the sense that there is an algorithm which determines whether or not any given event belongs to the given worldline. We finally point out that there can be only countably many universes which have an ending.

Keywords Multiverse, discrete spacetime, determinism, causal sets, closed timelike curves, effectively closed sets, Cantor space, computable trees.

The many-worlds interpretation (MWI) of quantum mechanics has been a topic of interest since the inception of the idea in Hugh Wheeler’s doctoral thesis and a following publication [12]. His work was carried under the supervision of John Archibald Wheeler, hence it is usually referred to as the Everett-Wheeler interpretation of quantum mechanics. The original article has more than 1400 citations to the date, and it has been depicted in the pop culture with many cultural references to the idea of parallel universes. Furthermore, with the inception of quantum computing as a field of importance, and through David Deutsch’s, one of the early leading figures in quantum computing, open adoption of this interpretation in his book [7], the topic has been revitalized. Although the exact ratio of physicists considering this as their main choice of interpretation of quantum mechanics is unknown and debated [23], it is considered as an alternative to the generally accepted Copenhagen interpretation (CI).

There are different CI and MWI formulations, but to crudely summarize the main difference between these two interpretations one can focus on the problem

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of measurement. In the CI, the wavefunction depicting the quantum state of a physical object ‘collapses’ at the moment of measurement into one of the possible states. The probability distribution before the point of measurement becomes irrelevant, and the physical reality only consists of the outcome of this measurement. Conversely, in the MWI, the point of measurement does not cause an irreversible effect, all the probabilities before continue to exist in orthogonal universes or worlds. Although this interpretation solves the measurement problem of quantum mechanics, it opens up many lanes of inquiry.

One intriguing question concerns exactly how many worlds \([14, 17]\) there are in this interpretation. If we assume that all universes have to come to an end, then the number of possible worlds is obviously finite. On the other hand if we do not have any restriction on the lifespan of universes, then clearly there may be infinitely many possible worlds. However, as Cantor famously pointed out nearly 150 years ago \([5]\), there are different sizes of infinities. So it becomes a natural question to ask exactly how many possible worlds there are if there exist infinitely many of them. In this study we try to devise a rigorous account of exactly how many worlds there might be in different scenarios in the MWI, under reasonable assumptions about spacetime, causality, and events.

It is a major and longstanding debate in the philosophy of physics whether spacetime fabric admits a discrete or continuous structure. There are various physical theories like general relativity that support the continuous view based on Lorentzian manifolds, as well as theories that rely on a discrete space conception such as quantum field gravity. This dichotomy also determines how we perceive events and the way they occur in spacetime. In this work we take the discrete route and represent events as discrete indivisible spacetime entities. This is also the route taken in what is known today as causal set theory, introduced in \([3]\) and studied by many researchers, including works in \([9, 4, 19, 22, 25, 26, 10]\). Most of the works in causal set theory, such as \([24, 27, 15]\) introduce the discreteness property while preserving Lorentz invariance. We consider a rather ‘digitalized’ version of causal sets so as to discretize the light cone where events occur in least possible time ticks. Our assumptions also show some similarities with the branching space-time, which originally goes back to Belnap \([1]\), who proposed a branching space-time structure as a ‘simple blend of relativity and indeterminism’ \(\text{[1]}\). The project was criticized in \([11]\) where the authors suggested to prune some branches from branching space-time. Müller \([20]\) then introduced a differential-geometrical version of branching space-times as a generalized non-Hausdorff manifold.

Consequently with our version of causal sets, under appropriate assumptions on the properties of the universe, one can describe light cones as finitely branching tree structures and define worldlines as infinite paths on such trees or equivalently as members of \(2^N\) Cantor space.\(\text{[2]}\) Using some facts about effectively closed subsets

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1 See Belnap, et al. (2021) \([2]\) for the revised version of branching space-time.

2 In fact we will use the space \(n^N\) for \(n \in \mathbb{N}\), which we shall refer to it with the same name.
of Cantor space, we observe that for any event \( x \), if all worldlines extending \( x \) are ‘eventually deterministic’, in the sense that there exists an event above which there is no more split, then the number of alternate universes with respect to \( x \) is exactly \( \aleph_0 \). Furthermore, if there are countably many alternate universes with respect to \( x \), then at least one of them must be decidable in the sense that there is an effective uniform method, i.e. an algorithm, which determines whether or not any given event belongs to the given worldline.

The paper is organised as follows. In Section 1, we start by giving some basic assumptions about the universe, the concept of causality, and events. In Section 2 we translate these physical concepts into mathematical domain by viewing light cones of events as finitely branching trees. Consider the following statements:

(i) All worldlines extending a given event \( x \) are ‘eventually deterministic’.\(^3\)

(ii) The number of alternate universes with respect to \( x \) is exactly \( \aleph_0 \).

(iii) There exists a ‘decidable’ universe.

We show that (i) is a sufficient condition for (ii). We also show that (ii) implies (iii), and thus (i) implies (iii).

In other words, we argue that given any event \( x \), if all worldlines extending the event \( x \) are ‘eventually deterministic’, then the number of alternate universes with respect to \( x \) is exactly \( \aleph_0 \). We then observe that if there are countably many alternate universes with respect to \( x \), then at least one of these universes must be necessarily ‘decidable’ in the sense that there is an algorithm which determines whether or not any given event belongs to the given worldline.

1 Basic Assumptions

This section lays out the assumptions about the structure and the property of the physical universe and of the concept of causality in order to prepare us to work in the mathematical domain. The reason one must articulate these hypotheses is that it is the only way to move the subject matter into the mathematical domain so as to develop some logico-mathematical results about them. Although the assumptions given in this paper may be disputable, the results we show will be rigorous and clear. One can discuss whether any or all of these hypotheses are not representative of the physical reality that we are living in, however we tried to keep these assumptions on the most fundamental level and in accordance with up-to-date cosmological understanding to the best of our abilities.

We begin by giving the following assumptions about the universe and then move onto giving some axioms concerning causality and events. In the final part we give our definition of worldline and how it can be represented in tree structures.

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\(^3\)We will define what we mean by eventually deterministic in Section 2.3.
The order between these items is irrelevant. However we shall list them starting from the simplest assumption and then get more specific.

I. Assumptions about the universe.

(a) The universe is expanding indefinitely.

(b) (i) There are finitely many atoms in the universe at any fixed moment. (ii) Every atom consists of finitely many indivisible parts.

(c) By (a) and (b), it follows that the observable universe is finite.

(d) Physical space is not dense. That is, for any two distinct points \( x \) and \( y \), there are at most finitely many points in-between \( x \) and \( y \) in the Euclidean sense. That is,

\[
|\{ z : \text{bet}(z, x, y) \}| < \aleph_0,
\]

where \( \text{bet}(z, x, y) \) denotes that the point \( z \) is between \( x \) and \( y \).

(e) No-signalling principle is valid (i.e. events are bounded by the speed of light in their future light cones).

II. Assumptions about causality and events

(a) Let \( x \) and \( y \) be two events. We let \( x \leq y \) mean “event \( x \) is a cause of event \( y \)” (or “event \( y \) is the effect of \( x \)”). We let \( x < y \) mean \( x \leq y \) and \( x \neq y \).

Any given event \( x \) defines a double light cone consisting of lower and upper parts, where the lower cone is defined as the collection of all possible past events that end up with \( x \). That is, the lower cone of \( x \) is

\[
\{ y : y \leq x \}.
\]

The upper cone represents all possible future events that stems from \( x \), which is defined as the set

\[
\{ y : x \leq y \}.
\]

(b) Causality is a transitive. That is, if \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

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4Whether the universe has a beginning or not will not be relevant to our analysis and will not affect the results for reasons that will become clear later. However, since we will be concerned with spacetime light cones of events in its general form, we may assume without loss of generality that the past and future cones are both unbounded and so we will make our analysis under the assumption that the future light cone of any event is potentially unbounded. Nevertheless we will be primarily concerned with the future light cone of events.

5In fact, whether space is discrete or continuous is not a settled question in physics as both are consistent with different physical theories. See Forrest for a discussion.

6By event we mean an instantaneous situation or action that is associated with a point in spacetime. Events are primitive objects of the domain of discourse in the theory of causal sets.
(c) Causality is not dense, i.e., if \( x \leq y \) then there do not exist infinitely many events occurring between the event \( x \) and the event \( y \).\(^7\)

(d) We say that two events \( x \) and \( y \) are incomparable (or spacelike separated) if neither \( x \leq y \) nor \( y \leq x \). Two incomparable events \( x \) and \( y \) cannot be seen as a single event. Similarly, no event can be decomposed into smaller incomparable sub-events. In other words, all events are ‘atomic’ (see Section 2.2 for details).

(e) For every event \( x \), since there are finitely many atoms—by I(b) and I(d)—there can only be finitely many events \( y_1, \ldots, y_n \) such that \( y_i \) is the immediate successor (or immediate effect) of \( x \), meaning that \( x < a < y_i \) for no event \( a \) (See Section 2.2 for details).

(f) A closed timelike curve is a sequence of events that happen between two distinct events \( x \) and \( y \) such that \( x \leq y \) and \( y \leq x \). There exist closed timelike curves iff the anti-symmetric property does not hold for causality. In other words, there are closed timelike curves iff it is not the case that \( x = y \) whenever \( x \leq y \) and \( y \leq x \).\(^8\)

(g) It follows from I(b), II(c), and II(e) that given any event \( x \), the collection of all future events that stem from \( x \) forms a subset of \( 2^{<N} \) (see Section 2.2 for details).

(h) The collection of all events which end up with \( x \) gives the lower cone of \( x \) and it can be similarly represented by a subset of \( 2^{<N} \) (see Section 2.2 for details).

III. Assumptions on worldlines

(a) Let \( x \) be an event. A worldline of \( x \) is sequence of events

\[ \ldots, p_2(x), p_1(x), x, f_1(x), f_2(x), \ldots \]

where each \( p_i(x) \) denotes the past events such that \( p_i(x) \leq x \) and that \( p_{i+1}(x) \leq p_i(x) \) for every \( i \), whereas \( f_i(x) \) denotes the future events satisfying that \( x \leq f_i(x) \) and \( f_i(x) \leq f_{i+1}(x) \) for every \( i \).

Note that there may exist many worldlines in the possible world interpretation, i.e., if the universe is non-deterministic, whereas there can only be one worldline associated with a given event in a deterministic picture. We do not endorse a particular position here, but we will later see what follows if we assume each position.

We shall now discuss about the consequences of these basic assumptions and also argue what worldlines actually correspond to in our presentation.

\(^7\)This is also known as the locally finiteness condition in causal sets.

\(^8\)In most theories of causal sets, the anti-symmetric property is implicitly assumed. We follow the same tradition.
2 Worldlines as real numbers

We first give some basic notions and facts about effectively closed subsets of Cantor space and computable trees. These notions can be found in more detail in [8], and [6]. We then elaborate more on the basic assumptions given in Section 1 to associate these notions with the physical concepts. Finally we give some results and corollaries that will apply to causal sets.

2.1 Computable subsets of Cantor space

When we say a computable set we mean a set $A \subseteq \mathbb{N}$ for which there is an algorithm such that for any given natural number $n$, the algorithm can decide whether or not $n \in A$. Of course any countable set is in one-to-one correspondence with the set of natural numbers under an appropriate pairing function. The definition then generalizes to any countable set under a suitable pairing function. Let us start by giving the notation for strings and trees.

Strings. A string is a finite sequence of symbols. We denote finite strings with lowercase Greek letters like $\sigma, \tau, \eta, \rho, \pi, \upsilon$. We let $\sigma^* \tau$ denote the concatenation of $\sigma$ followed by $\tau$. We let $\sigma \subseteq \tau$ denote that $\sigma$ is an initial segment of $\tau$. For example, $1001 \subseteq 10011$. We say a string $\sigma$ is incompatible with $\tau$ if neither $\sigma \subseteq \tau$ nor $\tau \subseteq \sigma$. Otherwise we say that $\sigma$ is compatible with $\tau$. Similarly, we say that $\sigma$ is an extension of $\tau$ if $\tau \subseteq \sigma$.

Trees and $\Pi_1^0$ classes. A set $T$ of strings is upward closed if $\sigma \in T$ and $\sigma \subseteq \tau$ then $\tau \in T$. A tree is an upward closed set of strings. For $n \in \mathbb{N}$, an $n$-ary tree is a tree with at most $n$ many branchings for each node. We say that a set $A$ lies on $T$ if there exist infinitely many $\sigma$ in $T$ such that $\sigma \subseteq A$. A set $A$ is a path on $T$ if $A$ lies on $T$. So if $A$ is a path on $T$, then every initial segment of $A$ is in $T$. We denote the set of infinite paths of $T$ by $[T]$. We say that a string $\sigma \in T$ is infinitely extendible if there exists some $A \supseteq \sigma$ such that $A \in [T]$. A tree $T$ is perfect if every infinitely extendible string in $T$ has at least two incompatible extensions in $T$. If $\sigma, \tau \in T$ and $\sigma \subseteq \tau$ and there does not exist $\sigma'$ with $\sigma \subseteq \sigma' \subseteq \tau$ then we say that $\tau$ is an immediate successor of $\sigma$ in $T$ and $\sigma$ is the immediate predecessor of $\tau$ in $T$.

Definition 1. We say that a tree $T$ is computable if for any string $\sigma$, there is an algorithm which decides whether or not $\sigma \in T$.

Definition 2. (i) Let $T \subseteq n^\mathbb{N}$ be a tree. The set of infinite paths through $T$ is defined as

$$[T] = \{ A : \forall n(A \upharpoonright n \in T) \}.$$  

\footnotetext{We will use the terms computable and decidable interchangably.}
where $A \upharpoonright n$ denotes the initial segment of the path $A$ up to length $n$.

(ii) A class $A \subseteq n^\mathbb{N}$ is $\Pi^0_1$ (or effectively closed) if there exists a computable relation $\varphi(n, X)$ such that $X \in A$ if and only if $\forall n \varphi(n, X)$, where $n$ ranges over $\mathbb{N}$ and $X$ ranges over $n^\mathbb{N}$.

Now from the definitions, it is easy to see the following.

**Theorem 1.** Let $A \subseteq n^\mathbb{N}$ be a class for some natural number $n$. The following are known to be equivalent.

(i) $A = [T]$ for some computable tree $T$.

(ii) $A$ is effectively closed.

Since we are working in Cantor space, we shall mention the compactness property of it. Although compactness can be provided by various forms, the most well known form is given by König’s lemma [16].

**Lemma 1** (König’s Lemma). If $T$ is a finitely branching infinite tree, then $T$ has an infinite path.

We shall now give some topological properties about trees.

**Definition 3.** Let $n \in \mathbb{N}$ and let $T \subseteq n^{<\mathbb{N}}$ be a tree.

(i) For any given $\sigma \in T$, we let $T_\sigma$ be the subtree of nodes compatible with $\sigma$ and be defined as

$$T_\sigma = \{ \tau \in T : \sigma \text{ is compatible with } \tau \}.$$  

(ii) A path $A \in [T]$ is said to be isolated if there exists a string $\sigma \in T$ such that $[T_\sigma] = \{A\}$, in which case we say that $\sigma$ isolates $A$. Otherwise $A$ is called a limit point.

**Definition 4.** We say that $\sigma \in T$ is infinitely extendible in $T$ if there exists some $A \supseteq \sigma$ such that $A \in [T]$.

Note that when $\sigma$ isolates $A$, there are no incompatible infinite extensions of $\sigma$ in $T$.

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[i6] Effectively closed means computably closed.
2.2 Translation between mathematical and physical contexts

Next we elaborate on some of the assumptions given in Section 1 and then argue that given any event $x$, the future light cone of $x$ can be seen as an effectively closed subset of Cantor space, i.e., a $\Pi^0_1$ class. So the worldlines form members of such classes. Hence one can treat strings in a computable tree as spacetime events in light cones, and treat worldlines as infinite paths on the tree through which the effectively closed set is represented.

Elaboration of II(d):

Given an event $x$, a split of at least two immediate effects of $x$ defines incomparable spacetime events (see Figure 1). In the MWI, thus, a split is usually considered as a way of defining alternate universes.

Hence the assumption given in II(d) ensures the existence of alternate universes relative to any event. One may ask what would happen if there were mergings in the light cone of an event. It may be possible that two incomparable events may later merge into a single event and continue along the same worldline (see Figure 2). Even if separate universes merge into a single universe at some point in the future—in which case the structure of causal sets could possibly be represented by a directed acyclic graph—in our case one can just use trees to represent light cones. Furthermore, splitting and merging events will not have any effect on the cardinality of worldlines. The reason is the following. The number of paths from the beginning of the split to the merge always is finite. Let us take Figure 2 as an illustration. A split occurs at event $b$ and a merging occurs at event $l$. Now there can only be finitely many events that are in the past of $l$ and the future of $b$. That is, there are finitely many $z$ such that $b \leq z$ and $z \leq l$. We shall call the collection of such $z$’s a split-merge region. The number of paths that start with $b$ and end up with $l$ is clearly finite. This then has no effect on the number of worldlines in the future of $b$ if the future of $b$ contains infinitely many paths.

Elaboration of II(e): We think of event as an indivisible entity in our discrete spacetime conception. By I(c), since the observable universe is finite and there are
finitely many entities in the universe, there can only be finitely many immediate effects of an event. This is due to that there are finitely many positions a particle can be in at each next time click, where the number of positions is bounded by the size of the observable universe and the speed of light.\textsuperscript{11} This means that there are finitely many possible effects that emerge from every event. This also guarantees that the future light cone of an event (and similarly the past light cone, as these two light cones are symmetric with each other) can be represented by a finitely branching tree. The elements of the tree consists of spacetime events partially ordered by causation. So our first observation is the following: Given any event \(x\), the future (and the past) light cone of \(x\) can be thought of as a finitely branching tree of strings. Since the universe is expanding indefinitely, the tree is potentially infinite in the limit.

\textbf{Elaboration of II(f):} In causal sets, usually the anti-symmetric property is assumed outright, and this is also what we adopt as well. However, one may ask what follows if one chooses not to add the anti-symmetric property as an axiom. The anti-symmetric property, it seems, ensures that the light cone of an event does not contain any cycles. In fact this corresponds to the concept of closed timelike curves in the casual structure. If \(x \neq y\) whenever \(x \leq y\) and \(y \leq x\), then this means that the spacetime region that lays in-between the events \(x\) and \(y\) is a closed timelike curve. So the anti-symmetric property fails iff there exists a closed timelike curve. We leave it as an open question that how many possible universes there would be if one decided to reject the anti-symmetric property. Note that the

\textsuperscript{11}This discussion can possibly be extended to show that under Lorentz transformation, there exist a suitable pairing function that can translate the order of events/branches between different reference frames while maintaining the overall causal structure of forward and backward branching trees, however, we did not delve further into this line of inquiry.
representation of light cones changes in case one assumes the existence of closed timelike curves.

**Elaboration of II(g):** It follows from I(b), II(c), II(e) and from the previous paragraph that given any event \(x\), the collection of all future events from \(x\) forms a subset of \(n^{<N}\). II(c) ensures that if \(y\) is an immediate successor of \(x\), then there is no spacetime event \(z\) satisfying that \(x < z < y\). So given the future light cone of \(x\) as a tree, all nodes in the tree are separated due to this condition. So II(c) makes the concept of *immediate effect* available for use. And so whenever \(y\) is an immediate effect of \(x\), this can be simply shown by a directed branch from \(x\) to \(y\), where \(x\) is put just below \(y\) in the tree. Combining this with II(e) and I(d), the future light cone of an event \(x\) forms a collection of events which are the immediate effects of \(x\), the immediate effects of effects of \(x\), and so on. This defines a subset of \(n^{<N}\) for some natural \(n\) since there are only finitely many immediate effects of \(x\).

**Elaboration of II(h):** Since we work with cardinal arithmetic and we want to calculate the number of universes relative to a given event \(x\), we may imagine the past light cone as the symmetry of future light cone. But we need to look at two possibilities where they might differ from each other.

Possibility 1: The universe has no beginning. Assuming that the past of \(x\) is eternal, future and past light cones of \(x\) can be taken as two separate tree structures where one is just an inverted version of the other. So the same structure holds for both past and future light cones of an event \(x\). That is, they are both infinite subsets of \(n^{<N}\).

Possibility 2: The universe has a beginning. In this case, the number of events that end up with \(x\) is finite.

Assuming that the universe is indefinitely expanding, the number of future events of \(x\) is greater than or equal to the number events in the past of \(x\). So this means that, in the worst case, we can just work with the future light cone of \(x\) to calculate the cardinality of universes relative to \(x\). The dominant factor without loss of generality can thus be assumed to come from the number of worldlines that extend \(x\).

**Elaboration of III(a):** The definition given in III(a) guarantees that a worldline of a given event \(x\) is a sequence of successive events in the past and the future of \(x\). A future worldline of \(x\) is just a path that lies on the future light cone of \(x\). Since we argued in the elaboration of II(h) that it is sufficient to work with the future light cone, it is worth noting that the future light cone of \(x\) is simply a subset of \(n^{<N}\) as argued in the elaboration of II(g). But since the future light cone is represented by a finitely branching tree \(T\) of strings, where \(x\) is the root of the tree and every successor of \(x\) is an effect of \(x\), the infinite branches that lie on \(T\) give us the future worldlines of \(x\), that is, possible universes that emerge from \(x\).
We have not quite argued about how to form a tree of strings from events. Let \( x \) be a spacetime event. The future light cone of \( x \) can be formed as a tree \( T \subseteq n^{<\mathbb{N}} \) for some natural number \( n \) inductively as follows: We let \( x \) be the empty string, hence the root of the tree. Let \( \sigma \in T \) represents an event \( y \). Every immediate effect \( z_i \) of \( y \) is defined by \( \sigma * i \) in \( T \).

Now one may also ask if the tree defined by the effects of an event \( x \) constitutes a computable tree. This is in fact trivial since we really take the full tree \( n^{<\mathbb{N}} \) to begin with, where \( n \) is bounded by the space that a particle can travel from \( x \) with a speed of light. So the tree \( n^{<\mathbb{N}} \) consists of all immediate effects of \( x \), all immediate effects of immediate effects of \( x \), and so on. The future light cone of \( x \) is then always computable and it basically corresponds to the full \( n \)-ary tree. By \( \text{I}(e) \), the immediate effects of any given event is bounded by the region of space through which light can travel in a least possible time tick. It is clear that the number of infinite paths on a full \( n \)-ary tree is uncountable since it is perfect.

Given a spacetime event \( x \), it should now be clear that the future light cone of \( x \) is a full \( n \)-ary tree. The class of all infinite paths (worldlines) on this tree gives us an effectively closed sets. So now properties about members of effectively closed sets can be translated into a property of worldlines extending the event \( x \). Each future worldline is an infinite path on the tree of effects of \( x \). So every infinite path can be thought of as a possible universe starting from a given spacetime event \( x \). Of course \( x \) can always be defined as the first cause (such as the Big Bang event) if we assume that the universe has a beginning.

### 2.3 Alternate universes in causal sets

The axioms of causal sets given in Section 1 describes a discrete spacetime structure on which a partial order is defined by causality. Given any event \( x \), the future light cone of \( x \), which is represented by a full \( n \)-ary tree \( T \), defines all alternate universes that originate from \( x \). These universes are defined by the collection of all infinite paths on \( T \), and this set is necessarily an uncountable set since it is perfect. But we ask under what conditions there could be less universes. Clearly if all alternative universes have an end, then there must be only finitely many universes since there will always be a point in each universe after which no event occurs. A more interesting worldline is that which is isolated by an event. An isolated path on \( T \) describes a worldline which eventually stabilizes and stops to split further. We observe that this corresponds to the following notion.

**Definition 5.** Let \( x \) be an event. A worldline \( p \) of \( x \) is called *eventually deterministic* if \( p \) is an isolated path on the future light cone of \( x \).

Now we will make use of the following results.

**Theorem 2.**

(i) Any isolated member of an effectively closed set is computable.

(ii) An effectively closed set is called *special* if it does not contain a computable member. Every special effectively closed set has \( 2^{\aleph_0} \) many members. By
contrapositive, if an effectively closed set is countable, then it contains a computable member.

(iii) Any countable closed subset of Cantor space contains an isolated member.

**Proof.** (i) Let $T$ be a computable tree and let $A$ be a path on $T$. Assume that $A \in [T]$ isolated. Then there is a string $\sigma \subseteq A$ such that no path on $T$ except $A$ extends $\sigma$. But then by K"onig’s Lemma, for any $n > |\sigma|$, there exists a unique $\tau \supseteq \sigma$ such that the subtree of $T$ above $\tau$ is infinite. To compute $A \upharpoonright n$ for $n > |\sigma|$, we find $m \geq n$ such that exactly one $\tau \supseteq \sigma$ of length $n$ has an extension of length $m$ in $T$. Then, $A \upharpoonright n = \tau$.

(ii) By (i), since there is no computable member in $[T]$, every member of $T$ splits. Therefore the number of elements of $[T]$ is $2^{\aleph_0}$.

(iii) Let $[T]$ be a closed set for a tree $T$. If $[T]$ has no isolated points, then $T$ is perfect and thus $[T]$ is uncountable.

One consequence of this theorem concerns classes with finitely many members.

**Corollary 1.** Let $\mathcal{P}$ be a finite $\Pi^0_1$ class. Then every member of $\mathcal{P}$ is computable.

The rest is just translating the results into the physical domain. We can argue that given any event $x$, if all worldlines extending $x$ are eventually deterministic, then there are countably many alternate universes with respect to $x$. Consider the future light cone of $x$, which is described by a tree $T$. This is simply due to the fact that if every member of $[T]$ is eventually deterministic—and so all isolated—then $[T]$ defines a discrete set. Thus $[T]$ is necessarily countable.

It also follows from Theorem 2 that if there are countably many universes extending a given event $x$, then at least one of these universes is decidable.

Another corollary is that there can only be countably many universes which have an ending. To see this, let $x$ be an event. Consider the future light cone of $x$. But there cannot be uncountably many finite paths extending $x$ since each would have to be an isolated path.

### 3 Discussion and Conclusion

In this work we tried to explore the relationship between the many-worlds interpretation of quantum mechanics and the cardinality of the possible worlds in different circumstances in the parallel universes under the given interpretation. Initially, we started by defining what an event is with respect to the given interpretation of spacetime, and showed how worldlines could be constructed based on this. We then associated worldlines with the infinite paths of computable trees, and the collection of paths of these trees with effectively closed subsets of Cantor space, (in fact $n^N$). Using some known topological properties of the Cantor space and computability, under the assumptions we made with respect to the nature
of spacetime (i.e., that being discrete), we derived some results through finitely branching trees containing all possible worldlines as infinite paths, irrespective of whether there is an initial event (like Big Bang) or an eternal universe model. Our observation revealed the following results; if all worldlines extending \( x \) are ‘eventually deterministic’, in the sense that there exists an event above which there is no more split, then the number of alternate universes with respect to \( x \) is exactly \( \aleph_0 \). Furthermore, if there are countably many alternate universes with respect to \( x \), then at least one of them must be decidable in the sense that there is an effective uniform method, i.e. an algorithm, which determines whether or not any given event belongs to the given worldline. We also pointed out that the number of universes which have an end can only be countable, and so the number of universes with no end is uncountable.

The summary and discussion above can also be understood in the following manner. In the MWI of quantum mechanics, there exist all the possible universes with all possible configurations. Universes with no beginning, universes with singular beginnings, universes with no end, or multiple ends, eternal universes with no ends, universes with no beginning but that will end, and so on. In our analysis, we identified a relationship that states, under certain discrete spacetime assumptions, given an event \( x \), if all worldlines extending that event lead to ‘eventually deterministic’ futures, then at least one of them must be decidable. The inverse of this conditional statement may not hold. Hence, our work actually identifies that for a certain type of a universe—where events lead to ‘eventually deterministic’ futures—there is a decidable worldline and that there are exactly \( \aleph_0 \) many eventually deterministic universes. We believe that, the property of decidability is an important distinction for an interpretation of quantum mechanics, that is, models that are unable to produce decidable worldlines should not be considered as physically possible.

We consider this work as a minor step toward the bigger goal of creating rigorous mathematical tools to apply on the problem of distinguishing between different interpretations of quantum mechanics. As the field of quantum technologies progress rapidly, the need to address the still unresolved questions from the early days of quantum mechanics should be identified as a worthy endeavour, and we want to strongly encourage the readers of this work to contemplate on these issues. Formulation of clear experiments to rule out certain interpretations can only come from delving deep into the assumptions and their implications, and it is a long and arduous journey that awaits the community of researchers both in foundations of quantum mechanics and philosophy of physics.

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