Current Density Imaging through Acoustically Encoded Magnetometry: A Theoretical Exploration

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Abstract

The problem of determining a current density confined to a volume from measurements of the magnetic field it produces exterior to that volume is known to have non-unique solutions. To uniquely determine the current density, or the non-silent components of it, additional spatial encoding of the electric current is needed. In biological systems such as the brain and heart, which generate electric current associated with normal function, a reliable means of generating such additional encoding, on a spatial and temporal scale meaningful to the study of such systems, would be a boon for research. This paper explores a speculative method by which the required additional encoding might be accomplished, on the time scale associated with the propagation of sound across the volume of interest, by means of the application of a radially encoding pulsed acoustic spherical wave.

1 Introduction

The magnetic inverse problem consists of the estimation of the magnetic source, an electric current density, given the field it produces external to the source. However, as was shown long ago by Helmholtz [18], a current distribution inside a conductor cannot in general be determined uniquely from knowledge of the electromagnetic field exterior to the conductor. It is well known that some current configurations will be ”silent” in that they produce no electric or magnetic field on the surface of a sphere (the measurement shell) containing the current density. But this is not the only reason for the lack of invertibility. It is also known that even the non-silent currents can not be uniquely obtained through an inversion of the electromagnetic field data [2] [14]. To obtain a unique solution of the electromagnetic inverse problem

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additional spatial encoding beyond that obtained from electromagnetic measurements taken outside the brain or heart is needed.

The measurement of current density within a mechanically inaccessible volume is an important topic in biology. Magnetoencephalography (MEG), electroencephalography (EEG), magnetocardiography (MCG), and electrocardiography (ECG) have for decades been used as a means to study brain and heart function. In the associated literature the additional spatial encoding needed to obtain a unique inversion is supplied by means of mathematical or physical constraints for which there exists no independent justification or no reasonable estimates of error bounds. The result is a current density image of unknown veracity at best.

Instead of trying to supplement the electromagnetic field measurements with additional mathematical or physical constraints some researchers have attempted to use other means to encode the missing spatial information. Haider et al. [4] has presented a method by which measurement of the electric field external to a time-varying current density in combination with a spatially selective acoustic wave might be used to create a current density image. Witte et al. [11, 12], again working with measurements of the electric field, have investigated using the acousto-electric effect, wherein an applied acoustic wave produces a change in the resistivity of a medium, as a means of imaging current density. However, their work appears to neglect the effect of the acoustic wave on the primary current and does not make clear what components or properties of the current density can be uniquely determined.

The purpose of this paper is to present a theoretical exploration of a novel method which could potentially provide a unique inversion of the non-silent current density source from magnetic field data measured on a spherical shell in the presence of a spatially selective acoustic spherical wave. In particular an emphasis will be placed upon the imaging of current density within the brain (generated by neuronal activity) or a brain-like model system. This paper will not be concerned with how the acoustic wave might be generated and applied but various means for doing so, from standard ultrasound transducers to photoacoustic [20] interaction with biological tissue, exist. Although implementation of this method in the neuroscience setting would be challenging because of sensitivity and noise issues, and possibly because of issues with the proposed current density model, this work still has value in that it helps bring into focus some of the theoretical issues associated with current density imaging.

It has been shown [2, 14] that knowledge of the magnetic field on the surface of a spherical shell enclosing a compactly supported current density is specifically lacking (apart from the silence of some current) with respect the radial encoding of the current density. To obtain a radial encoding of the non-silent current density the method explored in this paper uses a spherical acoustic wave. In brief, as the acoustic wave propagates radially through the medium supporting the current density it perturbs the current density, and its magnetic field, in manner that depends upon the shape of the acoustic wave packet and its position (The speed of the wave is assumed to be known). When the magnetic field is sampled as the acoustic wave propagates through the medium sufficient radial encoding information may be obtained to yield a unique (in principal) inversion of the non-silent current density. The time scale associated with the acquisition of the data used to generate an image of the current density would be roughly equal to the time for the propagation of an acoustic wave across
the volume of interest plus some additional time to allow reflecting waves to dissipate. For a
system of brain-like spatial extent (speed of sound in brain is approximately 1500 m/s) this
would mean a time scale significantly less than a millisecond although signal averaging would
most certainly be needed to suppress noise.

This paper makes use of a highly idealized model of the brain which hopefully incorporates
enough realism to spark interest in further development of the proposed method. The
model consists of a set of assumptions about the geometry, mechanical properties and elec-
tromagnetic properties of the system. These assumptions include but are not limited to: (1)
The medium in which the current density is supported has uniform isotropic homogeneous
acoustic properties, (2) The mass density of the medium is uniform on the compact support
of the current density, (3) The conductivity of the medium is spherically symmetric, (4) The
attenuation and dispersion of the applied acoustic wave may be neglected, (5) The current
density does not vary in time and (6) The total charge density on the macroscopic scale is zero
at all times. Other assumptions will be introduced in a timely manner as the interaction of
the acoustic wave with the medium is explored. The assumptions associated with this simple
model are obvious shortcomings of this work but future efforts may be able to produce a more
realistic model without sacrificing the essential features of the proposed method.

The paper is organized as follows. In Section 2 a model is introduced which is hoped to
adequately describe the interaction of an acoustic spherical wave with a brain-like medium
that supports an electric current density source. In Section 3 an expression is derived (similar
to results given in [14, 1]) which makes clear that a limited set of radial moments associated
with the current density can be determined from the magnetic field data alone. So that the
reader may better appreciate the present state of MEG "imaging" it is also shown that most
of the data needed to uniquely determine a current density will, for spatial resolutions of
interest, be fixed by additional encoding or constraining data. In Section 4 it is shown how
magnetic field data obtained during application of an acoustic pulsed spherical wave could
be used to uniquely obtain the current density of this model. In Section 5 signal strength
and noise considerations are addressed. Finally Section 6 concludes with a brief discussion of
some of the obstacles associated with the proposed method. Topics for further exploration
are suggested throughout the paper.

2 Acoustic Wave in a Model Conductor

The current density within a biological-like medium of tensor conductivity $\sigma_{ij}(r)$ may be char-
acterized as the sum of a primary current $J_p(r)$ and a volume current $J_v(r) = \sum_{ij} \hat{x}_i \sigma_{ij}(r) E_j(r)$
(magnetization current density and polarization current density can usually be neglected)
where $E_j(r)$ are the Cartesian components of an electric field. This distinction between pri-
mary and secondary currents is associated with a choice of spatial scale. A conductivity is
always associated with a particular scale. In this case, where a distinction is made between
primary and volume current, the conductivity will be associated with a macroscopic scale or
volume $V_m$ that might contain on the order of thousands of neurons (The range of pyramidal
cell density in human cortex is $5,000 - 80,000$ mm$^{-3}$ and depends on cortical region [17]).
Since the conductivity is defined on this macroscopic scale then the Ohm’s Law current must be given by its product with an electric field of equal spatial scale. There will however be currents on the microscopic scale (perhaps a scale of a few neurons) that may be given by Ohm’s Law as well. However the Ohm’s Law expression on the microscopic scale will only be correct when it is expressed in terms of the electric field on the microscopic scale. The primary current can therefore be seen as a macroscopic quantity that is given by averaging over $V_m$ the product of the microscopic conductivity and microscopic electric field.

When the macroscopic conductivity is spherically symmetric, $\sigma_{ij}(r) = \sigma(r)\delta_{ij}$, it can be shown [5] that the radial component of the magnetic field exterior to a region supporting the current density is independent of the volume currents. We will assume that the conductivity is spherically symmetric prior to the application of the acoustic wave. Since the acoustic wave is assumed to be spherical then the spherical symmetry of the conductivity should be maintained at all times and the volume currents will not contribute to the radial component of the external magnetic field. Therefore we need only concern ourselves with the primary current in the remainder of this paper.

The source of the signals measured in magnetoencephalography is predominantly the primary current density in dendritic branches of neurons [5]. This current density is directed parallel to the long axis of the dendrite and is driven by an electric field established by energy consuming cellular processes which vary in time as ion channels open and close in response to neuronal activity. The connection between the current density and the microscopic scale electric field which drives it is via Ohm’s Law on the microscopic scale. For the sake of simplicity in this paper we will assume a static microscopic electric field, and therefore a steady current density, but generalization to a time varying field should be a relatively simple matter.

A model for the interaction of the acoustic field with the primary current will presently be described. To present this model we will first work with the microscopic current and then, as indicated above, average it over $V_m$ to obtain the macroscopic primary current. The microscopic current density $J(r)$ may be written as

$$J(r, t) = \rho^+(r, t)v^+(r, t) + \rho^-(r, t)v^-(r, t)$$

where $\rho^\pm$ and $v^\pm$ are the charge density and velocity respectively for the charge carriers. The velocity has a contribution from charge drift $v_d^\pm$ and bulk motion $v_b$ so that $v^\pm = v_d^\pm + v_b$ and

$$J(r, t) = \rho^+(r, t)v_d^+(r, t) + \rho^-(r, t)v_d^-(r, t) + [\rho^+(r, t) + \rho^-(r, t)]v_b.$$  

Using the relationship between drift velocity, mobility and electric field we may write

$$J(r, t) = [\rho^+(r, t)\mu^+(r, t) + \rho^-(r, t)\mu^-(r, t)]E(r, t) + [\rho^+(r, t) + \rho^-(r, t)]v_b.$$  

where $E(r, t)$ is the electric field on the microscopic scale and $\mu^\pm$ is the mobility of the charge carriers. For the sake of simplicity it has been assumed that there are only two ionic charge carriers but this restriction could easily be relaxed. We also assume that the mobility is isotropic and homogeneous. Relaxing this assumption (by assuming a tensor mobility) could be a topic for further exploration. However since the current is primarily directed along the
long axis of dendrites in the brain then this is not an unreasonable assumption with which to begin.

We will assume that prior to the application of the acoustic wave \( \mathbf{v}_b = 0 \) but during the application of the wave \( \mathbf{v}_b \neq 0 \). Also when the acoustic wave is applied the charge density, charge mobility and electric fields are perturbed such that:

\[
\rho^\pm = \rho_o^\pm + \Delta \rho^\pm \quad \mu^\pm = \mu_o^\pm + \Delta \mu^\pm \quad \mathbf{E} = \mathbf{E}_o + \Delta \mathbf{E} \tag{4}
\]

where the first term on the right of each of these equations is the quantity prior to application of acoustic wave and the second term is the small change due to the acoustic wave.

Substituting Equations (4) into Equation (3) and keeping all but the small second order terms:

\[
\mathbf{J} = \mathbf{J}_o + (\Delta \rho^+ \mu_o^+ + \Delta \rho^- \mu_o^-) \mathbf{E}_o + (\rho_o^+ \Delta \mu^+ + \rho_o^- \Delta \mu^-) \mathbf{E}_o + (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \Delta \mathbf{E} + \left[ \rho^+(\mathbf{r}, t) + \rho^-(\mathbf{r}, t) \right] \mathbf{v}_b \tag{5}
\]

where the primary current prior to the application of the acoustic wave is

\[
\mathbf{J}_o(\mathbf{r}, t) = (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \mathbf{E}_o. \tag{6}
\]

According to Jossinet et al. [9] for an adiabatic compression\[1\] of a fluid we can write

\[
\Delta \rho^\pm = \rho_o^\pm \beta_s \Delta p \tag{7}
\]

and

\[
\Delta \mu^\pm = \mu_o^\pm (H_p + m_T^\pm \Theta) \Delta p \tag{8}
\]

where \( \Delta p \) is the pressure change due to the acoustic wave and where the thermodynamic constants \( H_p, \beta_s, m_T^\pm \) and \( \Theta \) are defined in [9]. Therefore

\[
\mathbf{J} = \mathbf{J}_o + \beta_s (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \mathbf{E}_o \Delta p + H_p (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \mathbf{E}_o \Delta p + (m_T^+ \rho_o^+ \mu_o^+ + m_T^- \rho_o^- \mu_o^-) \mathbf{E}_o \Theta \Delta p + (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \Delta \mathbf{E} + \left[ \rho^+(\mathbf{r}, t) + \rho^-(\mathbf{r}, t) \right] \mathbf{v}_b \tag{9}
\]

or

\[
\mathbf{J} = \mathbf{J}_o + (\beta_s + H_p) \mathbf{J}_o \Delta p + (m_T^+ \rho_o^+ \mu_o^+ + m_T^- \rho_o^- \mu_o^-) \mathbf{E}_o \Theta \Delta p + (\rho_o^+ \mu_o^+ + \rho_o^- \mu_o^-) \Delta \mathbf{E} + \left[ \rho^+(\mathbf{r}, t) + \rho^-(\mathbf{r}, t) \right] \mathbf{v}_b \tag{10}
\]

In Equation (10) the term that involves the change \( \Delta \mathbf{E} \) in the microscopic electric field will now be investigated. The total microscopic charge density \( \rho_t = \rho^+ + \rho^- \) (which establishes the microscopic electric field driving the current along a dendrite) during application of the

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1The mechanical work done on a given volume of the fluid by the acoustic wave is converted into heat during a period of the wave. Since the time constant associated with thermal diffusion is much larger than the period of the acoustic wave then the wave propagates without any thermal exchange with the local environment - i.e., adiabatically.
acoustic wave will be \( \rho_t(r) = \rho_{to}(r) + \Delta \rho_t(r) \) where \( \rho_{to}(r) \) is the total charge density prior to application of the acoustic wave and \( \Delta \rho_t(r) \) is a small change in the total charge density due to the acoustic wave. Then

\[
\mathbf{E}_o(r) = \int \rho_{to}(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad \Delta \mathbf{E}(r) = \int \Delta \rho_t(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad (11)
\]

Since the total charge density is assumed to be zero on the macroscopic scale then to good approximation the electric field at any point \( r \) will be due to the microscopic distribution of charge arising locally within some small volume \( V_r \) (possibly smaller than macroscopic volume \( V_m \)) about \( r \) and Equation (11) becomes

\[
\mathbf{E}_o(r) = \int_{V_r} \rho_{to}(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad \Delta \mathbf{E}(r) = \int_{V_r} \Delta \rho_t(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad (12)
\]

Assume that the mass density of the acoustic medium is \( \rho(r,0) \) when at rest and at equilibrium with external forces. When an acoustic wave disturbs this initially resting medium a small perturbation, \( \Delta \rho(r,t) \), to the mass density occurs so that

\[
\rho(r,t) = \rho(r,0) + \Delta \rho(r,t). \quad (13)
\]

Since the drift velocity of the charge carriers can be safely assumed to be much less than the velocity of sound in the medium then the charge carriers will behave as though they are fixed relative to the medium on the time-scale of the acoustic wave propagation. In such a case the ratio of the mass density at time \( t \) to that at time zero will be equal to the ratio of the charge density \( \rho^\pm \) at time \( t \) to that at time zero or:

\[
\frac{\Delta \rho_t(r)}{\rho_{to}(r)} = \frac{\Delta \rho(r)}{\rho_o(r)} \quad (14)
\]

then for uniform initial mass density \( \rho_o(r) = \rho_o \)

\[
\Delta \mathbf{E}(r) = \frac{1}{\rho_o} \int_{V_r} \rho_{to}(r') \Delta \rho(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad (15)
\]

For a \( V_r \) on a scale less than the wavelength of the acoustic carrier frequency the mass density \( \Delta \rho(r') \) will vary little over \( V_r \) and to good approximation the integral becomes

\[
\Delta \mathbf{E}(r) = \frac{\Delta \rho(r)}{\rho_o} \int_{V_r} \rho_{to}(r') \frac{(r-r')}{|r-r'|^3} d^3r' \quad (16)
\]

\[
= \frac{\Delta \rho(r)}{\rho_o} \mathbf{E}_o(r) \quad (17)
\]

Combining Equations (10) and (17) using the relationship between \( \Delta \rho \) and \( \Delta p \) given in Equation (7) we write

\[
\mathbf{J} = \mathbf{J}_o + \beta_s (H_p) \mathbf{J}_o \Delta p + (m_T^+ \rho_o^+ \mu_o^+ + m_T^- \rho_o^- \mu_o^-) \mathbf{E}_o \Theta \Delta p + [\rho^+(r,t) + \rho^-(r,t)] \mathbf{v}_b \quad (18)
\]
To obtain the macroscopic current $\mathbf{J}$ we average the microscopic current over a volume $V$ on the macroscopic spatial scale of interest to obtain

$$\mathbf{J} = \mathbf{J}_o + (2\beta_s + H_p)\Delta p + \Theta(m_T^+\mu_o^+\rho_o^+\mathbf{E}_o\Delta p + m_T^-\mu_o^-\rho_o^-\mathbf{E}_o\Delta p)$$

(19)

where the horizontal line denotes an average over $V$. Since the bulk velocity $\mathbf{v}_b$ and acoustic pressure change $\Delta p$ vary little over $V$ (i.e. the acoustic wavelength is assumed to be much greater than the scale of $V$) we can write

$$\mathbf{J} = \mathbf{J}_o + (2\beta_s + H_p)\Delta p + \Theta(m_T^+\mu_o^+\rho_o^+\mathbf{E}_o + m_T^-\mu_o^-\rho_o^-\mathbf{E}_o)\Delta p$$

(20)

and since the macroscopic charge density is assumed to remain zero at all times then

$$\mathbf{J} = \mathbf{J}_o + (2\beta_s + H_p)\Delta p + \Theta(m_T^+\mu_o^+\rho_o^+\mathbf{E}_o + m_T^-\mu_o^-\rho_o^-\mathbf{E}_o)\Delta p.$$

(21)

Equation (21) may also be conveniently written as

$$\mathbf{J} = \mathbf{J}_o + (2\beta_s + H_p)\Delta p + \Theta(m_T^+\mu_o^+\rho_o^+\mathbf{E}_o + m_T^-\mu_o^-\rho_o^-\mathbf{E}_o)\Delta p$$

(22)

where $m_T = (m_T^+ + m_T^-)/2$, $\delta\mu^\pm = m_T^\pm - m_T$ and $\mathbf{J}_a^\pm$ is the macroscopic current density due to the signed charge carrier in the absence of the acoustic wave. The last term on the right of Equation (22) is due to the unequal mobilities of the charge carriers. If the only charge carriers were sodium and chloride ions then this term would be zero (see Table (I)). Regardless both contributions to the current density in the presence of the applied acoustic wave will be due to neuronal activity. This motivates defining an current density activity modulus $\mathbf{J}_a$ according to

$$\mathbf{J}_a = (2\beta_s + H_p)\Delta p + \Theta(m_T^+\mu_o^+\rho_o^+\mathbf{E}_o + m_T^-\mu_o^-\rho_o^-\mathbf{E}_o)$$

(23)

We then have

$$\mathbf{J} = \mathbf{J}_o + \mathbf{J}_a\Delta p.$$

(24)

We will assume that the medium is liquid-like and therefore only supports longitudinal acoustic waves. Such a spherical mass density wave propagating through an acoustic medium with velocity $v$ may be written as

$$\Delta p(r, t) = \frac{1}{r}\psi(r \pm vt).$$

(25)

Combining Equations (24) and (25) one obtains

$$\mathbf{J}(r, t) = \mathbf{J}_o(r) + \frac{1}{r}\psi(r \pm vt)\mathbf{J}_a(r)$$

(26)

\footnote{In human soft tissue the velocity of sound varies little with respect to tissue type and is usually assumed to be that of water, 1,482 m/s at 20° C, in medical diagnostic ultrasound applications.}
as the current density in the presence of the applied spherical acoustic wave. In the next section a relationship between the coefficients of a vector spherical harmonic expansion of the current density activity modulus $\mathbf{J}_a$ and the measured magnetic field will be derived. This relationship will be the imaging equation which when inverted yields an image of the non-silent part of the current density activity modulus.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$\beta_s$ & $4.56 \times 10^{-10}$ Pa$^{-1}$ \\
\hline
$H_p$ & $-1.73 \times 10^{-10}$ Pa$^{-1}$ \\
\hline
$\Theta$ & $1.4 \times 10^{-8}$ K Pa$^{-1}$ \\
\hline
$m_T$ (Na$^+$) & $2.48 \times 10^{-2}$ K$^{-1}$ \\
\hline
$m_T$ (K$^+$) & $2.17 \times 10^{-2}$ K$^{-1}$ \\
\hline
$m_T$ (Cl$^-$) & $2.48 \times 10^{-2}$ K$^{-1}$ \\
\hline
\end{tabular}
\caption{Thermodynamic constants [9].}
\label{tab:thermodynamic}
\end{table}

\section{The Magnetic Field Data}

In this section we develop a near-field (quasistatic) equation which relates the measured magnetic field to the current density activity modulus. Since Maxwell’s Laws are linear with respect to the current source then the first and second terms on the right side of Equation (26) will respectively result in temporally static and dynamic contributions to the measured magnetic field. The static contribution to the measured field can be removed by filtering or subtraction methods (see Section 4). We may therefore omit the contribution of the static current term to the measured field and consider only the effective current given by:

$$J_e(r, t) = \frac{1}{r} \psi(r \pm vt) J_a(r).$$

(27)

We will assume that measurements of the magnetic field are made on a spherical shell of radius $R$ enclosing the compactly supported current density. It is then natural to expand the magnetic field (and vector potential) in terms of vector spherical harmonics (VSH). The resulting expressions will allow a precise characterization of the quantities associated with the current density which can be determined uniquely. External to the current containing region we may write [8]:

$$A(r, t) = \frac{1}{c} \int \int \frac{J_e(r', t')}{|r' - r|} \delta \left( t' + \frac{|r - r'|}{c} - t \right) d^3r' dt'.$$

(28)

where $c$ is the speed of light. Using the current given by Equation (27) this becomes

$$A(r, t) = \frac{1}{c} \int \left[ \int \frac{1}{r'} \psi(r' \pm vt') \delta \left( t' + \frac{|r - r'|}{c} - t \right) dt' \right] \frac{J_a(r')}{|r' - r|} d^3r'.$$

(29)
and performing the temporal integration leads to
\[ A(r, t) = \frac{1}{c} \int \frac{1}{r'} \psi(r' \pm vt - \beta |r - r'|) \frac{\mathcal{J}_a(r')}{|r' - r|} \, d^3 r' \tag{30} \]

where \( \beta = v/c \). The speed of sound (water = 1482 m/s, animal tissue = 1540 m/s) is much less than the speed of light (3.0 x 10^8 m/s) so that \( \beta \ll 1 \) and since \( |r - r'| < R \) we can to excellent approximation write
\[ A(r, t) = \frac{1}{c} \int \frac{1}{r'} \psi(r' \pm vt) \frac{\mathcal{J}_a(r')}{|r' - r|} \, d^3 r' \tag{31} \]

which is the near-field (or quasistatic) approximation.

Using a vector spherical harmonic (VSH) expansion [16] of the infinite space Green’s function for the Laplace equation (see appendix A) and Equation (31) we may write the field outside of the sphere supporting the current density as
\[ A(r, t) = \frac{4\pi}{c} \sum_{ljm} \left( \int \frac{r'^{l-1}}{2l + 1} \psi(r' \pm vt) \mathcal{J}_a(r') \cdot Y^l_{jm}(\Omega') \, d^3 r' \right) \frac{Y^l_{jm}(\Omega)}{r^{l+1}}. \tag{32} \]

where \( j = 0, \ldots, \infty, l = j, j + 1, j - 1 \) (with the exception that \( l = 1 \) for \( j = 0 \)), and \( m = -j, -j + 1, \ldots, j - 1, j \). We will, as in the above equation, use the variable \( \Omega \) to denote the ordered pair of angular variables \((\theta, \phi)\).

If we write \( \mathcal{J}_a(r) \) in a VSH expansion (see Appendix A) as
\[ \mathcal{J}_a(r') = \sum_{l'j'm'} \alpha_{l'jm'}^j(r') Y_{l'm'}^{j'}(\Omega') \tag{33} \]
and substitute into Equation (32) then we arrive at
\[ A(r, t) = \frac{4\pi}{c} \sum_{ljm} \left( \int_0^R \frac{r'^{l+1}}{2l + 1} \psi(r' \pm vt) \alpha_{ljm}^j(r') \, dr' \right) \frac{Y^l_{jm}(\Omega)}{r^{l+1}}. \tag{34} \]

Taking the curl of Equation (34) (see Appendix A) and using the condition that \( \nabla \times \mathbf{B} = 0 \) outside the region of support for the current density then the magnetic field on the spherical measurement shell is:
\[ \mathbf{B}(\Omega, t) = -i \frac{4\pi}{c} \sum_{jm} \left( \frac{j}{2j + 1} \right)^{1/2} R^{-(j+2)} Y_{jm}^{j+1}(\Omega) \int_0^R r'^{j+1} \psi(r' \pm vt) \alpha_{jm}^j(r') \, dr'. \tag{35} \]

Therefore all \( Y_{jm}^{j+1} \) and \( Y_{jm}^{j-1} \) components of the current are silent in that they produce no field on the measurement shell. In addition any \( Y_{jm}^j \) component of the current for which the radial integral in Equation (35) is zero will also be silent.
It is well known that for regions where the current density vanishes a scalar magnetic field potential \( \Phi \) exists such that \( \mathbf{B} = -\nabla \Phi \) and which obeys the Laplace equation \( \nabla^2 \Phi = 0 \). Therefore, according to standard potential theory, a solution of the Laplace equation outside of the sphere can be obtained from knowledge of the normal derivative of the scalar potential on the surface enclosing the volume. Therefore all that is needed to fully determine \( \mathbf{B} \) outside the measurement sphere is the knowledge of \( B_r \) on the surface of the sphere. Using the definitions of the VSH given in appendix A one easily obtains \( B_r \):

\[
B_r(\Omega, t) = i \frac{4\pi}{c} \sum_{jm} \sqrt{\frac{j(j+1)}{2j+1}} R^{j+1} \int_0^R r^{j+1} \psi(r' \pm vt) \alpha_{jm}^j(r')dr' Y_{jm}(\Omega)
\]  

(36)

and we arrive at

\[
B_{jm}(t) = \int_0^R r^{j+1} \psi(r' \pm vt) \alpha_{jm}^j(r')dr'
\]  

(37)

where we have defined \( B_{jm}(t) = -i \frac{c}{4\pi} \frac{2j+1}{\sqrt{j(j+1)}} R^{j+2} B_{jm}(t) \) and \( B_{jm}(t) = \int B_r(\Omega, t) Y_{jm}^*(\Omega) d\Omega \).

This equation, the basic result of this section, connects the external magnetic field measurement in the presence of a spatially selective spherical acoustic wave to a single radial moment of each coefficient \( \alpha_{jm}^j(r) \).

Before moving on to the inversion of the magnetic field data it may be instructive to note that Equation (37) gives insight into the extent to which the inverse problem of MEG (\( \psi = 1 \)) is ill-posed. An \( N \)-point radial digital resolution will be defined by the the value of \( \alpha_{jm}^j(r) \), and hence the current density image, at \( N - 1 \) equidistant points \( r_n \) in the radial direction. For example, a 10-point radial resolution within a measurement shell of radius 10 cm would correspond to a 1.0 cm radial resolution. Let us assume that the current density is comprised of \( Y_{jm}^j \) components only (i.e., there are no silent parts of the current density) and that all \( \alpha_{jm}^j(r) \) are well approximated by a \( N^{th} \) order polynomial. To determine such a polynomial from its moments would require the knowledge of \( N + 1 \) moments. From Equation (37) we see that the field measurements give one moment only for each \( \alpha_{jm}^j(r) \). One can then state that for a unique \( N \)-point resolution inversion \( 1/(N+1) \) of the data is obtained from the field measurements and \( N/N + 1 \) of the data is obtained from the additional constraints. Then, practically speaking, to obtain a 1.0 cm radial digital resolution within a sphere of radius \( R = 10 \) cm only 10 percent of the data necessary for a unique inversion would come from the field measurement alone!

4 Inverting for the Non-Silent Current Density

In this section we investigate the use of Equation (37) in determining the coefficients \( \alpha_{jm}^j(r) \) and therefore the current density activity modulus. For the remainder of this paper the outgoing wave (i.e. argument is \( r - vt \)) will be assumed (An ingoing wave would be treated similarly). Since any physical acoustic wave must be causal then we must have \( \psi(r' - vt) = 0 \).
for $r' > vt$ so that Equation (37) becomes

$$B_{jm}(t) = \int_0^{vt} \psi(r' - vt)r'^{j+1}\alpha_{jm}^j(r')dr' \quad 0 < t < \frac{R}{v}. \quad (38)$$

Setting $r = vt$ (The acoustic wave speed $v$ is assumed to be known,) and defining the continuously measured data $D_{jm}(r) = B_{jm}(r/v)$ then Equation (38) can be written as

$$D_{jm}(r) = \int_0^r \psi(r' - r)g_{jm}(r')dr' \quad 0 < r < R \quad (39)$$

where $g_{jm}(r') = r'^{j+1}\alpha_{jm}^j(r')$.

Equation (39) is a Volterra integral equation of the first kind with convolution-type kernel. It relates the measured data $D_{jm}(r)$ to the unknown function $g_{jm}(r')$ on the interval $0 \leq r' \leq R$. Well established methods of solving for $g_{jm}$ can be found in the literature (see for example [7, 10, 13, 19]). The solutions are known to be unique if: (1) $D_{jm}(r')$ is continuously differentiable on $0 \leq r' \leq R$ and $D_{jm}(0) = 0$, (2) $\Psi(r' - r)$ and $\partial \Psi(r' - r)/\partial r$ are continuous on $0 \leq r' \leq r \leq R$, and (3) $\psi(0) \neq 0$.

Although the solution of Equation (39) is unique under expected conditions the solution is still ill-posed in the sense that it can be very sensitive to noise in the data $D_{jm}(r)$. Even though the underlying signal is expected to be differentiable on $0 \leq r' \leq R$ the addition of a noise component to the measured signal will create a $D_{jm}(r)$ that in general will not be differentiable. The violation of condition (1) is known to lead to an amplification of high frequency noise and the degree to which the condition is violated controls the amplification. Signal averaging and smoothing of the data can be used to diminish the effect of noise amplification but usually some type of regularization (Tikhonov regularization is common.) will be needed. Furthermore the usual consequence of regularization is that resolution decreases as regularization is increased.

To solve Equation (39) numerical methods are employed which take as input the data $D_{jm}(r_n)$ sampled at discrete points $r_n = n\Delta r = nv\Delta t$ where $\Delta t$ is the intersample interval. It can be shown [3] that solutions obtained through the discretization of the integral equation tend toward the exact solution as $\Delta t \to 0$. Therefore one might then think it advantageous to choose $\Delta t$ a small as possible. However a decrease in $\Delta t$ in the presence of data noise is expected to exacerbate the violation of condition (1) and increase sensitivity to noise. The above discussion highlights the difficulty in obtaining a theoretical bound on the radial resolution of the proposed method but doesn’t prevent establishing resolution by empirical means.

It may be advantageous to modulate the acoustic wave with a carrier frequency $k_o$ so that

$$\psi(r - vt) = \Omega(r - vt)e^{i2\pi(r-vt)k_o}. \quad (40)$$

This will have the effect of shifting the frequency spectrum of the signal and allow for the filtering of the DC component given by $\Psi$, as well as low frequency noise. The choice of the wave envelope $\Omega(r - vt)$ will undoubtedly have consequences with respect to the numerical inversion of Equation (39) but this will only briefly be addressed along with signal and noise considerations in the next section.
Signal, Noise and Resolution

In this section the signal and noise of the proposed method will be considered in rather broad terms. It is useful to estimate the signal strength of the proposed method (after filtering out the contribution of the DC term) compared to that of MEG. The ratio \( R_{\text{sig}} \) of the two signals would be given by

\[
R_{\text{sig}} = \frac{|J_a \Delta p|}{|J_o|}. \quad (41)
\]

From Table 1 note that the difference between \( m_T^+ \) and \( m_T^- \) for the most prevalent charge carriers in biological systems is about 10 percent. It is therefore justified to use the following approximation to the current density activity modulus to estimate \( R_{\text{sig}} \):

\[
J_a \approx (2\beta_s + H_p + \Theta m_T)J_o. \quad (42)
\]

(Note that Equation (42) would be exact if sodium and chloride ions are the only charge carriers.). Then

\[
R_{\text{sig}} \approx |(2\beta_s + H_p + \Theta m_T)\Delta p|. \quad (43)
\]

In diagnostic medical ultrasound imaging the overpressure \( \Delta p \) is typically in the range 0.5-5.0 MPa. Using the values given Table 1 an estimate of \( R_{\text{sig}} \approx 5.0 \times 10^{-3} \) is obtained for \( \Delta p = 5.0 \) MPa. Therefore the signal from any spherical shell of RMS thickness \( \sigma \) will be between 2 and 3 orders of magnitude smaller than that from the same shell in an MEG measurement. Since the signal strength of a neuromagnetic signal in MEG is typically in the range of 50-500 fT this is likely to present a challenge with respect to magnetometer sensitivities even with signal averaging. Whether larger overpressures could be used safely (avoiding cavitation), perhaps at higher carrier frequencies, to boost the signal strength without causing neuronal stimulation could be an area of experimental exploration. Additionally attenuation of the acoustic wave would be expected to increase with frequency presenting yet another challenge to obtaining a signal of sufficient strength. Note that in a model system built to demonstrate the proposed method perhaps \( R_{\text{sig}} \) could be made much larger than that estimated above.

Sources of magnetic and electronic noise in the proposed method may originate in the environment, the magnetometer, the magnetometer electronics and the body. For example environmental sources may include the acoustic wave source, communications systems (radio, television, and microwave transmitters), electric pumps, electric motors, large nearby moving objects (automobiles and elevators), and power distribution systems. Magnetometer related noise may be due to thermal noise in the magnetometer (for example, SQUID noise), noise in the supporting electronics and thermal Dewar noise. The electrical activity of neighboring organs such as muscle also generate a time varying magnetic field that may be considered to be noise. Even background brain alpha rhythms might be considered to be ”noise” if averaging (to increase signal-to-noise ratio) is used in an experiment which is time-locked to a stimulus. There are many means by which these sources of noise may be reduced and once
all means of reducing environmental noise are exhausted the ultimate remaining noise source will be that due to the thermally induced magnetic noise in the brain and body tissues. This noise has been estimated to be on the order of $0.1 \text{ fT}/\sqrt{\text{Hz}}$ [15].

If no carrier wave is used then the $\mathbf{J}_o$ term in Equation (24) might be removed by a subtraction method in which the signal is acquired in the presence and absence of the applied acoustic wave and the signals are subsequently subtracted to obtain the desired $\mathbf{J}_a$ term of Equation (24). If a carrier frequency is used then the $\mathbf{J}_o$ term of Equation (24) could be removed by high-pass filtering the temporal signal prior to deconvolution. Using a carrier frequency could potentially have another benefit in that the signal could be pushed into a frequency band significantly different from environmental noise sources and then be band-pass filtered to obtain a signal of reduced noise compared to that of the typical MEG experiment. However, the benefit of a carrier frequency may come at a cost. The assumption made in going from Equation (15) to Equation (16) involves the wave length of the acoustic wave and will be increasingly difficult to meet at smaller wavelengths associated with a carrier wave.

6 Discussion

We presently lack a relatively noninvasive tool to study intact brain activity on the spatial and temporal scales associated with most brain function. This paper explores one means by which a finer scale information of brain function might someday be accessible. The spirit of this work is to further the discussion of novel methods for relatively noninvasive studies of human brain activity. The author acknowledges that there would be many hurdles to achieving this goal by the proposed method, even for the simple geometry and assumptions considered here, and this paper does not present a comprehensive list of all such challenges. Instead this paper proposes a simple brain-like model for the interaction of an acoustic wave with a current carrying medium and shows how that interaction might potentially achieve the mapping of the non-silent component of the current density on the scales of interest to neuroscience.

In addition this work explores the interplay between signal, noise, safety, and inversion method that would ultimately place fundamental limitations on the proposed method. Such interplay is one known to all methods of imaging in the biological sciences. It is the hope of the author that this work and the work of those mentioned in this paper’s introduction will stimulate others to think about methods by which current density could be spatially encoded on time and spatial scales needed to fruitfully and safely probe brain function.

A Vector Spherical Harmonics

For convenience this appendix gives the definition and some properties of vector spherical harmonics [16]. The vector spherical harmonics may be defined and generated from the scalar
spherical harmonics according to:

\[
Y_{jm}^{j+1} = \sqrt{\frac{j+1}{2j+1}} \left( -e_r Y_{jm} + e_\theta \frac{1}{j+1} \frac{\partial Y_{jm}}{\partial \theta} + e_\phi \frac{im}{j+1} \frac{Y_{jm}}{\sin \theta} \right)
\]

\[
Y_{jm}^j = -e_\theta \frac{m}{\sqrt{j(j+1)}} \frac{Y_{jm}}{\sin \theta} - e_\phi \frac{i}{\sqrt{j(j+1)}} \frac{\partial Y_{jm}}{\partial \theta}
\]

\[
Y_{jm}^{j-1} = \sqrt{\frac{j}{2j+1}} \left( e_r Y_{jm} + e_\theta \frac{1}{j} \frac{\partial Y_{jm}}{\partial \theta} + e_\phi \frac{im}{j} \frac{Y_{jm}}{\sin \theta} \right).
\]

The vector spherical harmonics obey the orthogonality property

\[
\int_0^\pi \int_0^{2\pi} Y_{jm}^{*l'} Y_{jm}^l \sin \theta d\theta d\phi = \delta_{jj'} \delta_{ll'} \delta_{mm'}
\]  

(44)

and the following relations for the divergence operator:

\[
\nabla \cdot \left[ f(r)Y_{jm}^{j+1} \right] = -\sqrt{\frac{j+1}{2j+1}} \left( \frac{d}{dr} + \frac{j+2}{r} \right) f(r)Y_{jm}
\]  

(45)

\[
\nabla \cdot \left[ f(r)Y_{jm}^j \right] = 0
\]

\[
\nabla \cdot \left[ f(r)Y_{jm}^{j-1} \right] = \sqrt{\frac{j}{2j+1}} \left( \frac{d}{dr} - \frac{j-1}{r} \right) f(r)Y_{jm}
\]

and the curl operator:

\[
\nabla \times \left[ f(r)Y_{jm}^{j+1} \right] = i \sqrt{\frac{j}{2j+1}} \left( \frac{d}{dr} + \frac{j+2}{r} \right) f(r)Y_{jm}^j
\]  

(46)

\[
\nabla \times \left[ f(r)Y_{jm}^j \right] = i \sqrt{\frac{j}{2j+1}} \left( \frac{d}{dr} - \frac{j}{r} \right) f(r)Y_{jm}^{j+1}
\]

\[
+ i \sqrt{\frac{j+1}{2j+1}} \left( \frac{d}{dr} + \frac{j+1}{r} \right) f(r)Y_{jm}^{j-1}
\]

\[
\nabla \times \left[ f(r)Y_{jm}^{j-1} \right] = i \sqrt{\frac{j+1}{2j+1}} \left( \frac{d}{dr} - \frac{j-1}{r} \right) f(r)Y_{jm}^j.
\]

Since the vector spherical harmonics are complete on the space of functions for which \(\int |F(\Omega)|^2 d\Omega < \infty\) (i.e., square integrable functions of \(\Omega\)) then any function this space can be expanded as:

\[
F(r) = \sum_{ljm} \alpha_{ljm}^l(r) Y_{ljm}^l(\Omega).
\]
In particular the Green’s function for the infinite space Laplace equation can be written as

\[
\frac{\mathbf{J}(r')}{|r' - r|} = 4\pi \sum_{l} \sum_{m} \left( \frac{r'^l}{2l + 1} \mathbf{J}(r') \cdot \mathbf{Y}_{jm}(\Omega')d^3r' \right) \frac{\mathbf{Y}_{jm}(\Omega)}{r^{l+1}}
\]

(47)

where it has been assumed that \( r > r' \).

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