LENGTH CATEGORIES OF INFINITE HEIGHT

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Dedicated to Dave Benson on the occasion of his 60th birthday.

Abstract. For abelian length categories the borderline between finite and infinite representation type is discussed. Characterisations of finite representation type are extended to length categories of infinite height, and the minimal length categories of infinite height are described.

Contents

1. Introduction 1
2. Length categories 3
3. Grothendieck groups and almost split sequences 5
4. Effaceable functors and pure-injectives 9
5. Uniserial categories 12
6. Minimal length categories of infinite height 16

References 16

1. Introduction

An abelian category is a length category if it is essentially small and every object has a finite composition series [10]. The height of a length category is the supremum of the Loewy lengths of all objects.

The aim of this note is to explore the structure of length categories of infinite height. Length categories of finite height arise from artinian rings by taking the category of finite length modules. Also, length categories of infinite height are ubiquitous, and typical examples are the uniserial categories which are not of finite height. Recall that a length category is uniserial if every indecomposable object has a unique composition series [1]. For instance, the category of nilpotent finite dimensional representations of a cyclic quiver over any field is uniserial and of infinite height.

The paper is divided into three parts. First we extend known characterisations of finite representation type for module categories to more general length categories, including those of infinite height (Theorem 4.10). Then we show that uniserial categories satisfy these finiteness conditions (Corollary 5.4 and Theorem 5.10). In the final part, we describe the minimal length categories of infinite height, and it turns out that only uniserial categories occur (Theorem 6.1).

2. Length categories

In this section we collect some basic concepts that are relevant for the study of abelian length categories.

For a module $M$ over a ring $\Lambda$ let $\ell_\Lambda(M)$ denote its composition length.
Ext-finite categories. A length category \( C \) is (left) Ext-finite if for every pair of simple objects \( S \) and \( T \)
\[
\ell_{\text{End}_C(T)}(\text{Ext}_C^1(S, T)) < \infty.
\]
A length category \( C \) is equivalent to a module category (consisting of the finitely generated modules over a right artinian ring) if and only if the following holds [10]:

1. The category \( C \) has only finitely many simple objects.
2. The category \( C \) is Ext-finite.
3. The supremum of the Loewy lengths of the objects in \( C \) is finite.

Hom-finite categories. Let \( C \) be an essentially small additive category. Let us call \( C \) (left) Hom-finite if for all objects \( X, Y \) in \( C \) the \( \text{End}_C(Y) \)-module \( \text{Hom}_C(X, Y) \) has finite length. Clearly, this property implies that \( C \) is a Krull-Schmidt category, assuming that \( C \) is idempotent complete.

Lemma 2.1. Let \( C \) be a Krull-Schmidt category. Then \( C \) is Hom-finite provided that for all pairs of indecomposable objects \( X, Y \) the \( \text{End}_C(Y) \)-module \( \text{Hom}_C(X, Y) \) has finite length.

Proof. Choose decompositions \( X = \bigoplus X_i \) and \( Y = \bigoplus Y^j \), \( n_j > 0 \), such that the \( Y_j \) are indecomposable and pairwise non-isomorphic. Set \( Y' = \bigoplus Y_j \). Then
\[
\ell_{\text{End}_C(Y)}(\text{Hom}_C(X, Y)) = \sum \ell_{\text{End}_C(Y_j)}(\text{Hom}_C(X, Y_j))
\]
and
\[
\ell_{\text{End}_C(Y)}(\text{Hom}_C(X, Y)) = \ell_{\text{End}_C(Y')}\left(\text{Hom}_C(X, Y')\right)
= \sum \ell_{\text{End}_C(Y_j)}(\text{Hom}_C(X, Y_j))
\]
since
\[
\text{End}_C(Y')/\text{rad End}_C(Y') \cong \prod_j \text{End}_C(Y_j)/\text{rad End}_C(Y_j).
\]
Now the assertion follows. \( \square \)

Example 2.2. Let \( k \) be a commutative ring and \( C \) a \( k \)-linear category. If the \( k \)-module \( \text{Hom}_C(X, Y) \) has finite length for all \( X, Y \) in \( C \), then \( C \) and \( C^{\text{op}} \) are Hom-finite.

Finitely presented and effaceable functors. Let \( C \) be an abelian category. An additive functor \( F: C \to \text{Ab} \) is finitely presented if there is a presentation
\[
(2.1) \quad \text{Hom}_C(Y, -) \to \text{Hom}_C(X, -) \to F \to 0,
\]
and we call \( F \) effaceable if there is such a presentation such that the morphism \( X \to Y \) in \( C \) is a monomorphism. We denote by \( \text{Fp}(C, \text{Ab}) \) the category of finitely presented functors \( F: C \to \text{Ab} \) and by \( \text{Eff}(C, \text{Ab}) \) the full subcategory of effaceable functors. Note that \( \text{Fp}(C, \text{Ab}) \) is an abelian category, and \( \text{Eff}(C, \text{Ab}) \) is a Serre subcategory.

We recall the following duality. The assignment \( F \mapsto F^\vee \) given by
\[
F^\vee(X) = \text{Ext}_C^2(F, \text{Hom}_C(X, -))
\]
yields an equivalence
\[
(2.2) \quad \text{Eff}(C, \text{Ab})^{\text{op}} \cong \text{Eff}(C^{\text{op}}, \text{Ab}),
\]
where \( \text{Ext}_C^2(\cdot, \cdot) \) is computed in the abelian category \( \text{Fp}(C, \text{Ab}) \); see Theorem 3.4 in Chap. II of [5]. If \( 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0 \) is an exact sequence in \( C \) and \( F = \text{Coker} \text{Hom}_C(\alpha, -) \), then \( F^\vee \cong \text{Coker} \text{Hom}_C(-, \beta) \) and \( F^{\vee\vee} \cong F \).
The Yoneda functor
\[ \mathcal{C} \longrightarrow \text{Fp}(\mathcal{C}^{\text{op}}, \text{Ab}), \quad X \mapsto \text{Hom}_\mathcal{C}(-, X) \]
admits an exact left adjoint that sends \( \text{Hom}_\mathcal{C}(-, X) \) to \( X \); it annihilates the effaceable functors and induces an exact functor
\[ (2.3) \quad \frac{\text{Fp}(\mathcal{C}^{\text{op}}, \text{Ab})}{\text{Eff}(\mathcal{C}^{\text{op}}, \text{Ab})} \longrightarrow \mathcal{C} \]
which is an equivalence; see \([2, \text{p. 205}]\) and \([15, \text{III, Prop. 5}]\).

3. Grothendieck groups and almost split sequences

Let \( \mathcal{C} \) be an essentially small abelian category. The Grothendieck group \( K_0(\mathcal{C}) \) is the abelian group generated by the isomorphism classes \([C]\) of objects \( C \in \mathcal{C} \) subject to the relations \([C'] - [C] + [C'']\), one for each exact sequence \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) in \( \mathcal{C} \). Analogously, we write \( K_0(\mathcal{C}, 0) \) for the abelian group generated by the isomorphism classes \([C]\) of objects \( C \in \mathcal{C} \) subject to the relations \([C'] - [C] + [C'']\), one for each split exact sequence \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) in \( \mathcal{C} \). Thus there is a canonical epimorphism
\[ \pi: K_0(\mathcal{C}, 0) \longrightarrow K_0(\mathcal{C}) \]
Our aim is to find out when the kernel of \( \pi \) is generated by elements \([X] - [Y] + [Z]\) that are given by almost split sequences \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) in \( \mathcal{C} \).

Almost split sequences. Let \( \mathcal{C} \) be a Krull-Schmidt category. Recall from \([7]\) that a morphism \( \alpha: X \rightarrow Y \) in \( \mathcal{C} \) is left almost split if it is not a split mono and every morphism \( X \rightarrow Y' \) that is not a split mono factors through \( \alpha \). Dually, a morphism \( \beta: Y \rightarrow Z \) in \( \mathcal{C} \) is right almost split if it is not a split epi and every morphism \( Y' \rightarrow Z \) that is not a split epi factors through \( \beta \). An exact sequence \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) is almost split if \( \alpha \) is left almost split and \( \beta \) is right almost split.

We say that \( \mathcal{C} \) has almost split sequences if for every indecomposable object \( X \in \mathcal{C} \), there is an almost split sequence starting at \( X \) when \( X \) is non-injective, and there is an almost split sequence ending at \( X \) when \( X \) is non-projective.

Lemma 3.1. A morphism \( \alpha: X \rightarrow Y \) in \( \mathcal{C} \) is left almost split if and only if \( X \) is indecomposable and \( \alpha \) induces an exact sequence
\[ \text{Hom}_\mathcal{C}(Y, -) \longrightarrow \text{Hom}_\mathcal{C}(X, -) \longrightarrow F \longrightarrow 0 \]
in \( \text{Fp}(\mathcal{C}, \text{Ab}) \) such that \( F \) is a simple object.

Proof. See Proposition 2.4 in Chap. II of \([5]\). \(\square\)

Lemma 3.2. For an indecomposable object \( X \in \mathcal{C} \) are equivalent:
1. There is an almost split sequence \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) in \( \mathcal{C} \).
2. There is a simple object \( S \in \text{Eff}(\mathcal{C}, \text{Ab}) \) such that \( S(X) \neq 0 \).

Proof. (1) \( \Rightarrow \) (2): Use Lemma 3.1.
(2) \( \Rightarrow \) (1): The functor \( S \) admits a minimal projective presentation
\[ 0 \longrightarrow \text{Hom}_\mathcal{C}(Z, -) \longrightarrow \text{Hom}_\mathcal{C}(Y, -) \longrightarrow \text{Hom}_\mathcal{C}(X, -) \longrightarrow S \longrightarrow 0 \]
in \( \text{Fp}(\mathcal{C}, \text{Ab}) \) since \( \mathcal{C} \) is Krull-Schmidt. It follows from Proposition 4.4 in Chap. II of \([5]\) that the underlying sequence \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) is almost split in \( \mathcal{C} \). \(\square\)
Length and support. Let \( C \) be an essentially small additive category and suppose that \( C \) is Krull-Schmidt. Let \( \text{ind} C \) denote a representative set of the isoclasses of indecomposable objects. For an additive functor \( F : C \to \text{Ab} \) set
\[
supp(F) = \{X \in \text{ind} C \mid FX \neq 0\}
\]
and let \( \ell(F) \) denote the composition length of \( F \) in the category of additive functors \( C \to \text{Ab} \).

Lemma 3.3. For an additive functor \( F : C \to \text{Ab} \) we have
\[
\ell(F) = \sum_{X \in \text{ind} C} \ell_{\text{End}_C(X)}(FX).
\]

Proof. Let \( F : C \to \text{Ab} \) be a simple functor and \( FX \neq 0 \) for some \( X \in \text{ind} C \). Then we have \( supp(F) = \{X\} \) and \( \ell(F) = 1 = \ell_{\text{End}_C(X)}(FX) \). From this the assertion follows by induction on \( \ell(F) \). \( \square \)

Lemma 3.4. Let \( C \) be Hom-finite and \( F : C \to \text{Ab} \) a finitely generated functor. Then \( \ell(F) \) is finite if and only if \( supp(F) \) is finite.

Proof. Apply Lemma [3.3]. Clearly, \( supp(F) \) is finite when \( \ell(F) \) is finite. For the converse observe that \( \ell_{\text{End}_C(X)}(FX) \) is finite for all \( X \in C \) since \( C \) is Hom-finite and \( F \) is the quotient of a representable functor.

Remark 3.5. Let \( \text{Fp}(C, \text{Ab}) \) be abelian and \( F \in \text{Fp}(C, \text{Ab}) \). Then \( \ell(F) \) does not depend on the ambient category since every simple object in \( \text{Fp}(C, \text{Ab}) \) is simple in the category of all additive functors \( C \to \text{Ab} \).

Let \( C \) be Hom-finite and fix an object \( X \in C \). The assignment
\[
\chi_X : \text{Fp}(C, \text{Ab}) \to \mathbb{Z}, \quad F \mapsto \ell_{\text{End}_C(X)}(FX)
\]
induces a homomorphism \( K_0(\text{Fp}(C, \text{Ab})) \to \mathbb{Z} \).

Lemma 3.6. Let \( C \) be Hom-finite. Given functors \( F \) and \( (F_i)_{i \in I} \) in \( \text{Fp}(C, \text{Ab}) \),
\[
[F] \in ([F_i] \mid i \in I) \subseteq K_0(\text{Fp}(C, \text{Ab})) \quad \text{implies} \quad supp(F) \subseteq \bigcup_{i \in I} supp(F_i).
\]

Proof. Fix \( X \in \text{ind} C \). We have \( X \notin supp(F) \) if and only if \( \chi_X(F) = 0 \). Thus \( X \notin \bigcup_{i \in I} supp(F_i) \) implies \( \chi_X(F_i) = 0 \) for all \( i \in I \). If \( [F] \) is generated by the \( [F_i] \), then this implies \( \chi_X(F) = 0 \). Thus \( X \notin supp(F) \). \( \square \)

Relations for Grothendieck groups. Let \( C \) be an essentially small abelian category and consider the Yoneda functor
\[
C \to \text{Fp}(C, \text{Ab}), \quad X \mapsto h_X := \text{Hom}_C(X, -).
\]

Lemma 3.7. The Yoneda functor induces an isomorphism of abelian groups
\[
K_0(C, 0) \xrightarrow{\sim} K_0(\text{Fp}(C, \text{Ab})).
\]

Proof. The Yoneda functors identifies \( C \) with the full subcategory of projective objects in \( \text{Fp}(C, \text{Ab}) \). From this the assertion follows since every object in \( \text{Fp}(C, \text{Ab}) \) admits a finite projective resolution. \( \square \)

Given an almost split sequence \( 0 \to X \to Y \to Z \to 0 \) in \( C \), let \( S_X \) denote the corresponding simple functor in \( \text{Fp}(C, \text{Ab}) \) with \( supp(S_X) = \{X\} \); see Lemma [3.2].

Lemma 3.8. Let \( C \) be an abelian category. Then the following are equivalent:

1. The kernel of \( r : K_0(C, 0) \to K_0(C) \) is generated by elements \( [X] - [Y] + [Z] \) that are given by almost split sequences \( 0 \to X \to Y \to Z \to 0 \) in \( C \).
2. \( [F] \in ([S_X] \mid 0 \to X \to Y \to Z \to 0 \text{ almost split}) \) for all \( F \in \text{Eff}(C, \text{Ab}) \).
LENGTH CATEGORIES OF INFINITE HEIGHT

Proof. An exact sequence \( \eta: 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{C} \) gives rise to an exact sequence

\[
0 \to \text{Hom}_\mathcal{C}(Z, -) \to \text{Hom}_\mathcal{C}(Y, -) \to \text{Hom}_\mathcal{C}(X, -) \to F_\eta \to 0
\]

in \( \text{Fp}(\mathcal{C}, \text{Ab}) \) with \( [F_\eta] = [h_X] - [h_Y] + [h_Z] \). The assertion then follows by identifying \( [X] \) with \( [h_X] \) for all \( X \in \mathcal{C} \), keeping in mind Lemma 3.7. \( \square \)

Proposition 3.9. Let \( \mathcal{C} \) be an essentially small abelian Krull-Schmidt category. Consider the following conditions:

1. Every effeaceable finitely presented functor \( \mathcal{C} \to \text{Ab} \) has finite length.
2. The kernel of \( \pi: K_0(\mathcal{C}, 0) \to K_0(\mathcal{C}) \) is generated by elements \( [X] - [Y] + [Z] \) that are given by almost split sequences \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{C} \).

Then (1) implies (2) and the converse holds when \( \mathcal{C} \) is Hom-finite.

Proof. (1) \( \Rightarrow \) (2): Let \( S \) be a simple composition factor of an effaceable functor. Choosing a minimal projective presentation of \( S \) in \( \text{Fp}(\mathcal{C}, \text{Ab}) \) gives rise to an almost split sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{C} \) so that \( S = S_X \); see Lemma 3.2. Thus condition (2) in Lemma 3.8 holds.

(2) \( \Rightarrow \) (1): Fix \( F \in \text{Eff}(\mathcal{C}, \text{Ab}) \). Then Lemmas 3.6 and 3.8 imply that \( \text{supp}(F) \) is finite. From Lemma 3.4 it follows that \( F \) has finite length. \( \square \)

Remark 3.10. The property that every effaceable functor \( \mathcal{C} \to \text{Ab} \) has finite length is self-dual, thanks to the duality (2.2).

4. EFFACEABLE FUNCTORS AND PURE-INJECTIVES

The aim of this section is a characterisation of the length categories \( \mathcal{C} \) such that every effaceable finitely presented functor \( \mathcal{C} \to \text{Ab} \) has finite length. This involves the study of pure-injective objects, and we need to embed \( \mathcal{C} \) into a Grothendieck category.

Locally finitely presented categories. Let \( \mathcal{A} \) be a Grothendieck category. An object \( X \in \mathcal{A} \) is finitely presented if \( \text{Hom}_\mathcal{A}(X, -) \) preserves filtered colimits, and we denote by \( \text{fp} \mathcal{A} \) the full subcategory of finitely presented objects in \( \mathcal{A} \). The category \( \mathcal{A} \) is called locally finitely presented if \( \mathcal{A} \) has a generating set of finitely presented objects [10].

Let \( \mathcal{C} \) be an essentially small abelian category. We denote by \( \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab}) \) the category of left exact functors \( \mathcal{C}^{\text{op}} \to \text{Ab} \) and set \( \mathcal{A} = \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab}) \). Observe that \( \mathcal{A} \) is a locally finitely presented Grothendieck category. The category \( \mathcal{C} \) identifies with \( \text{fp} \mathcal{A} \) via the functor

\[
\mathcal{C} \to \mathcal{A}, \quad X \mapsto \text{Hom}_\mathcal{C}(-, X),
\]

and every object in \( \mathcal{A} \) is a filtered colimit of objects in \( \mathcal{C} \).

Locally noetherian categories. A Grothendieck category \( \mathcal{A} \) is called locally noetherian if \( \mathcal{A} \) has a generating set of noetherian objects. In that case finitely presented and noetherian objects in \( \mathcal{A} \) coincide.

A Grothendieck category \( \mathcal{A} \) is called locally finite if \( \mathcal{A} \) has a generating set of finite length objects. When \( \mathcal{A} \) is locally finite, then every noetherian object has finite length, since any object is the directed union of finite length subobjects. Thus finitely presented and finite length objects in \( \mathcal{A} \) coincide.
Purity. Let \( \mathcal{C} \) be an essentially small abelian category and \( \mathcal{A} = \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab}) \). We recall the following construction from [12, §3]. Set \( \hat{\mathcal{C}} = \text{Fp}(\mathcal{C}, \text{Ab})^{\text{op}} \) and \( \hat{\mathcal{A}} = \text{Lex}(\hat{\mathcal{C}}^{\text{op}}, \text{Ab}) \). Observe that \( \hat{\mathcal{C}} \) is abelian and identifies with \( \text{fp} \hat{\mathcal{A}} \). The functor
\[
\begin{array}{ccc}
C & \rightarrow & \hat{\mathcal{C}} \\
\downarrow & & \downarrow \\
A & \rightarrow & \hat{\mathcal{A}}
\end{array}
\]
is right exact and extends to a colimit preserving and fully faithful functor
\[
h: \mathcal{C} \rightarrow \hat{\mathcal{C}}, \quad X \mapsto \text{Hom}_C(X, -)
\]
that makes the following square commutative:
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{h} & \hat{\mathcal{C}} \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{h_*} & \hat{\mathcal{A}}
\end{array}
\]
Note that \( h_* \) is the left adjoint of \( h^*: \hat{\mathcal{A}} \rightarrow \mathcal{A} \) given by \( h^*(X) = X \circ h \).

There is a notion of purity for \( \mathcal{A} \). A sequence \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) in \( \mathcal{A} \) is pure-exact if \( \text{Hom}_\mathcal{A}(C, -) \) takes it to an exact sequence of abelian groups for all finitely presented \( C \in \mathcal{A} \). An object \( M \in \mathcal{A} \) is pure-injective if every pure monomorphism \( X \rightarrow Y \) induces a surjective map \( \text{Hom}_\mathcal{A}(Y, M) \rightarrow \text{Hom}_\mathcal{A}(X, M) \).

Lemma 4.1. (1) A sequence \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) in \( \mathcal{A} \) is pure-exact if and only if the induced sequence \( 0 \rightarrow \hat{X} \rightarrow \hat{Y} \rightarrow \hat{Z} \rightarrow 0 \) in \( \hat{\mathcal{A}} \) is exact.

(2) The functor \( X \mapsto \hat{X} \) identifies the pure-injective objects in \( \mathcal{A} \) with the injective objects in \( \hat{\mathcal{A}} \).

Proof. See Lemma 4 in §3.3 and Lemma 1 in §3.5 of [12].

The category \( \hat{\mathcal{A}} \) has enough injective objects. Thus every object in \( \mathcal{A} \) admits a pure monomorphism into a pure-injective object. We call such a morphism a pure-injective envelope if it becomes an injective envelope in \( \mathcal{A} \).

Example 4.2. Suppose that \( \mathcal{C} \) is Hom-finite. Then every finitely presented object in \( \mathcal{A} \) is pure-injective. This follows from Theorem 1 in §3.5 of [12].

Let us write \( \text{Ind} \mathcal{A} \) for a representative set of the indecomposable pure-injective objects in \( \mathcal{A} \), containing exactly one object for each isomorphism class. For a class \( \mathcal{X} \subseteq \mathcal{A} \) set
\[
\mathcal{X}^\perp = \{ M \in \text{Ind} \mathcal{A} \mid \text{Hom}_\mathcal{A}(X, M) = 0 \text{ for all } X \in \mathcal{X} \}.
\]

We recall the following detection result; see [18, Thm. 3.8] and [20, Thm. 4.2].

Proposition 4.3. Let \( \mathcal{X}, \mathcal{Y} \) be Serre subcategories of \( \hat{\mathcal{C}} \). Then
\[
\mathcal{X} \subseteq \mathcal{Y} \iff \mathcal{X}^\perp \supseteq \mathcal{Y}^\perp.
\]

Effaceable functors. We compute the orthogonal complement of the category of effaceable functors.

Lemma 4.4. We have \( \text{Eff}(\mathcal{C}, \text{Ab})^\perp = \{ M \in \text{Ind} \mathcal{A} \mid M \text{ is injective} \} \).

Proof. Fix \( F \in \text{Eff}(\mathcal{C}, \text{Ab}) \) with presentation (2.1) and \( M \in \text{Ind} \mathcal{A} \). Then we have \( \text{Hom}_\mathcal{A}(F, M) = 0 \) if and only if every morphism \( X \rightarrow M \) factors through \( X \rightarrow Y \). It follows that \( \text{Hom}_\mathcal{A}(F, M) = 0 \) when \( M \) is injective. If \( M \) is not injective, then there is a monomorphism \( \alpha: M \rightarrow N \) in \( \mathcal{A} \) that does not split. Moreover, \( \alpha \) is not a pure monomorphism since \( M \) is pure-injective. Thus we may assume that \( C = \text{Coker} \alpha \) is in \( \mathcal{C} \). Write \( N = \text{colim}_i N_i \) as a filtered colimit of objects in \( \mathcal{C} \). Then for some \( i \) the induced morphism \( \beta_i: N_i \rightarrow C \) is an epimorphism. Let \( \alpha_i: M_i \rightarrow N_i \) be the kernel of \( \beta_i \) and set \( F_i = \text{Coker} \text{Hom}_\mathcal{C}(\alpha_i, -) \). Then \( F_i \) is in \( \text{Eff}(\mathcal{C}, \text{Ab}) \) and \( \text{Hom}_\mathcal{A}(F_i, M) \neq 0 \) by construction.
Let \( \text{Fp}(\mathcal{C}, \text{Ab})_0 \) denote the full subcategory of finite length objects in \( \text{Fp}(\mathcal{C}, \text{Ab}) \).

**Lemma 4.5.** Let \( \mathcal{C} \) be a Krull-Schmidt category. An object in \( \text{Ind}\mathcal{A} \) belongs to \( (\text{Fp}(\mathcal{C}, \text{Ab})_0)^\perp \) if and only if it is not the pure-injective envelope of the source of a left almost split morphism in \( \mathcal{C} \).

**Proof.** An object \( M \in \text{Ind}\mathcal{A} \) belongs to \( (\text{Fp}(\mathcal{C}, \text{Ab})_0)^\perp \) if and only if \( \text{Hom}_\mathcal{A}(S, M) = 0 \) for every simple objects \( S \) in \( \mathcal{C} \). By Lemma 3.11, any simple object \( S \) in \( \mathcal{C} \) arises as the kernel of a morphism \( X \rightarrow Y \) that corresponds to a left almost split morphism \( X \rightarrow Y \) in \( \mathcal{C} \). Moreover, the morphism \( S \rightarrow X \) is an injective envelope in \( \mathcal{C} \) since \( \text{End}_\mathcal{C}(X) \) is local. It remains to observe that a morphism \( X \rightarrow M \) in \( \mathcal{A} \) is a pure-injective envelope if and only if \( X \rightarrow M \) is an injective envelope in \( \mathcal{A} \). \( \square \)

**Proposition 4.6.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck category and set \( \mathcal{C} = \text{fp}\mathcal{A} \). Suppose that \( \mathcal{C} \) is an abelian Krull-Schmidt category. Then the following conditions are equivalent:

1. Every effaceable finitely presented functor \( \mathcal{C} \rightarrow \text{Ab} \) has finite length.
2. Every indecomposable pure-injective object in \( \mathcal{A} \) is injective or the pure-injective envelope of the source of a left almost split morphism in \( \mathcal{C} \).

**Proof.** Effaceable functors and finite length functors form Serre subcategories in \( \text{Fp}(\mathcal{C}, \text{Ab}) \). Their orthogonal complements in \( \text{Ind}\mathcal{A} \) are described in Lemmas 4.4 and 4.5. It remains to apply Proposition 4.3. \( \square \)

**Fp-injective objects.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck category. An object \( X \in \mathcal{A} \) is called \( \text{fp}\)-injective if \( \text{Ext}^1_\mathcal{A}(C, X) = 0 \) for every finitely presented object \( C \in \mathcal{A} \).

Let \( \mathcal{C} \) be an essentially small abelian category and set \( \mathcal{A} = \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab}) \).

**Lemma 4.7.** A functor \( X \in \mathcal{A} \) is exact if and only if \( X \) is a \( \text{fp}\)-injective object.

**Proof.** Using the identification \( \mathcal{C} \overset{\sim}{\rightarrow} \text{fp}\mathcal{A} \), the functor \( X \) is exact iff for every exact sequence \( \eta: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( \text{fp}\mathcal{A} \) the induced sequence

\[
\text{Hom}_\mathcal{A}(\eta, X): 0 \rightarrow \text{Hom}_\mathcal{A}(C, X) \rightarrow \text{Hom}_\mathcal{A}(B, X) \rightarrow \text{Hom}_\mathcal{A}(A, X) \rightarrow 0
\]

is exact.

Now suppose \( \text{Ext}^1_\mathcal{A}(C, X) = 0 \). Clearly, this implies exactness of \( \text{Hom}_\mathcal{A}(\eta, X) \) for any exact \( \eta: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) in \( \text{fp}\mathcal{A} \). Conversely, let \( \mu: 0 \rightarrow X \rightarrow Y \rightarrow C \rightarrow 0 \) be exact in \( \mathcal{A} \) and write \( Y = \text{colim} \{ Y_j \} \); if \( Y_j \) is a filtered colimit of finitely presented objects. This yields an exact sequence \( \mu_j: 0 \rightarrow X_j \rightarrow Y_j \rightarrow C \rightarrow 0 \) in \( \text{fp}\mathcal{A} \) for some \( j \). Now exactness of \( \text{Hom}_\mathcal{A}(\mu_j, X) \) implies that \( \mu \) splits. \( \square \)

**Lemma 4.8.** Let \( X \in \mathcal{A} \). Then \( X \) is \( \text{fp}\)-injective in \( \mathcal{A} = \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab}) \).

**Proof.** We apply Lemma 4.7. Recall that \( \mathcal{C} = \text{Fp}(\mathcal{C}, \text{Ab})^{\text{op}} \). Thus \( X \) is exact for \( X = \text{Hom}_\mathcal{C}(-, C) \) with \( C \in \mathcal{C} \). Any object \( X \in \mathcal{A} \) is a filtered colimit of representable functors. Thus it remains to observe that a filtered colimit of exact functors is exact. \( \square \)

**Lemma 4.9.** Let \( \mathcal{A} \) be a locally noetherian Grothendieck category. Then every \( \text{fp}\)-injective object in \( \mathcal{A} \) is injective and decomposes into indecomposable objects.

**Proof.** When \( \mathcal{A} \) is locally noetherian, then finitely presented and noetherian objects in \( \mathcal{A} \) coincide and are therefore closed under quotients. Now apply Baer’s criterion to show that \( \text{fp}\)-injective implies injective. The decomposition into indecomposables follows from an application of Zorn’s lemma, using that \( \text{fp}\)-injective objects are closed under filtered colimits. \( \square \)
Finite type. We are now ready to extend some known characterisations of finite representation type for module categories to more general abelian categories, including the length categories of infinite height.

Recall that every essentially small abelian category $C$ with all objects in $C$ noetherian embeds into a locally noetherian Grothendieck category $A$ with $C \cong \text{fp} A$.

Theorem 4.10. Let $A$ be a locally noetherian Grothendieck category and set $C = \text{fp} A$. Suppose that $C$ is $\text{Hom}$-finite. Then the following are equivalent:

1. Every effaceable finitely presented functor $C \to \text{Ab}$ has finite length.
2. The kernel of $\pi: K_0(C,0) \to K_0(C)$ is generated by elements $[X] - [Y] + [Z]$ that are given by almost split sequences $0 \to X \to Y \to Z \to 0$ in $C$.
3. The category $C$ has almost split sequences, and every non-zero object in $A$ has an indecomposable direct summand that is finitely presented or injective.
4. The category $C$ has almost split sequences, and every indecomposable object in $A$ is finitely presented or injective.

Proof. We identify $\mathcal{C} \cong \text{fp} A$.

(1) $\equiv$ (2): See Proposition 3.9

(1) $\Rightarrow$ (3): Let $X \not= 0$ be an object in $A$. Suppose first that $\text{Hom}_A(S,X) \not= 0$ for a simple and effaceable $S \in \mathcal{C}$. Choose an injective envelope $\alpha: S \to C$ in $\mathcal{C}$. Then any non-zero morphism $\phi: S \to X$ factors through $\alpha$, since $X$ is fp-injective by Lemma 4.3. On the other hand, $\alpha$ factors through $\phi$ since $C$ is pure-injective by Example 3.2. Thus $C$ is isomorphic to a direct summand of $X$. Now suppose that $\text{Hom}_A(F,X) = 0$ for all effaceable $F \in \mathcal{C}$. Then $X: \text{Fp}(\text{C},\text{Ab}) \to \text{Ab}$ vanishes on $\text{Eff}((\text{C},\text{Ab}))$ and identifies with an exact functor $C^{\text{op}} \to \text{Ab}$ via the functor (2.3). Thus $X$ is injective and has an indecomposable summand since $A$ is locally noetherian; see Lemma 4.9.

Let $X \in \mathcal{C}$ be an indecomposable non-injective object. Then there is an epimorphism $\text{Hom}_C(X,-) \to F$ with $F \not= 0$ effaceable. The object $F$ has finite length and we may assume that $F$ is simple. It follows from Lemma 4.2 that there is an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$.

For every indecomposable non-projective $Z \in \mathcal{C}$, there is an almost split sequence $0 \to X \to Y \to Z \to 0$ in $A$, by Theorem 1.1 in [21]. The object $X$ is indecomposable, and therefore in $C$, by the first part of the proof.

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (1): Use Proposition 4.4.

Remark 4.11. Suppose the conditions in Theorem 4.10 hold. Then effaceable and finite length functors agree if and only if every injective object in $C$ is zero. This follows from the proof of Proposition 4.4.

Remark 4.12. Theorem 4.10 generalises various known characterisations of finite representation type for module categories.

For a ring $\Lambda$, let $\text{mod} \Lambda$ denote the category of finitely presented $\Lambda$-modules. Recall that $\Lambda$ has finite representation type if $\text{mod} \Lambda$ is a length category with only finitely many isomorphism classes of indecomposable objects.

(1) A ring $\Lambda$ has finite representation type if and only if every finitely presented functor $\text{mod} \Lambda \to \text{Ab}$ has finite length [3].

(2) An artin algebra $\Lambda$ has finite representation type if and only if for $C = \text{mod} \Lambda$ the kernel of $\pi: K_0(C,0) \to K_0(C)$ is generated by elements $[X] - [Y] + [Z]$ that are given by almost split sequences $0 \to X \to Y \to Z \to 0$ in $C$ [6, 11].

\textsuperscript{1}The proof of Theorem 4.10 is close to Butler’s original proof. Auslander’s proof is based on the use of a bilinear form on $K_0(C,0)$, following work of Benson and Parker on Green rings [9].
Using induction on $\ell$ we claim that $Y_{soc}(X)$ has finite length. Let $X$ be an indecomposable object and let $S$ be its socle series.

Proof. We need to show for all $U$ in $A$ that $\operatorname{End}_C(U)$ is isomorphic to $\operatorname{Hom}_C(U, Y)$ for some $Y$ in $A$. Let $X$ be an indecomposable object. If $\ell(X) = 0$, then $X$ is isomorphic to $\operatorname{soc}(X)$, and the claim is trivial. Otherwise, let $Y$ be the unique composition series of $X$. Then $\operatorname{soc}(X) \subseteq Y \subseteq \operatorname{soc}(X)$, and $\operatorname{soc}(X)$ is the smallest semisimple subobject of $X$. Thus $\ell(Y) = \ell(X) - 1$, and the claim follows by induction on $\ell(Y)$.

Uniserial categories. Recall that $C$ is uniserial if $C$ is a length category and each indecomposable object has a unique composition series.

Lemma 5.1. Let $C$ be an abelian length category. Then $C$ uniserial if and only if $\ell(X) = \ell(Y)$ for every indecomposable $X, Y \in C$.

Proof. Let $X, Y \in C$ be indecomposable. If $\ell(X) = \ell(Y)$, then the socle series of $X$ is the unique composition series of $X$.

Now assume $\ell(X) \neq \ell(Y)$. Then there exists some $n \geq 0$ such that $\operatorname{soc}^{n+1}(X)/\operatorname{soc}^n(X) = S_1 \oplus \ldots \oplus S_r$ with all $S_i$ simple and $r > 1$. Choose $n$ minimal and let $\operatorname{soc}^n(X) \subseteq U_i \subseteq X$ be given by $U_i/\operatorname{soc}^n(X) = S_i$. Then we have at least $r$ different composition series

$$0 = \operatorname{soc}^0(X) \subseteq \operatorname{soc}^1(X) \subseteq \ldots \subseteq \operatorname{soc}^n(X) \subseteq U_i \subseteq \ldots \subseteq \operatorname{soc}^{n+1}(X) \subseteq \ldots$$

of $X$. \hfill $\square$

Lemma 5.2. Let $C$ be a uniserial category. Then $C$ is Hom-finite.

Proof. We need to show for all $X, Y$ in $C$ that the $\operatorname{End}_C(Y)$-module $\operatorname{Hom}_C(X, Y)$ has finite length. It suffices to assume that $Y$ is indecomposable; see Lemma 2.2. We claim that

$$\ell_{\operatorname{End}_C(Y)}(\operatorname{Hom}_C(X, Y)) \leq \ell(X).$$

Using induction on $\ell(X)$ the claim reduces to the case that $X$ is simple. So let $S = \operatorname{soc}(Y)$ and write $E = E(Y)$ for its injective envelope. Note that $\operatorname{soc}(E) = Y$ for
n = ℓ(Y), by Lemma \[5.1\] Thus any endomorphism \( E \to E \) restricts to a morphism \( Y \to Y \). Write \( i: S \to Y \) for the inclusion. Then any morphism \( j: S \to Y \) induces an endomorphism \( f: E \to E \) such that \( f|_Y \circ i = j \). Thus the \( \text{End}_C(Y) \)-module \( \text{Hom}_C(S,Y) \) is cyclic, and it is annihilated by the radical of \( \text{End}_C(Y) \). Therefore \( \text{Hom}_C(S,Y) \) is simple. □

**Proposition 5.3.** Let \( C \) be a uniserial category. Then every non-zero object in \( \mathcal{A} \) has an indecomposable direct summand that belongs to \( C \) or is injective.

**Proof.** From Lemma \[5.1\] it follows that for every indecomposable injective object \( E \) in \( \mathcal{A} \) the subobjects form a linear chain

\[
0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots
\]

with \( E_n = \text{soc}^n(E) \) in \( C \) for all \( n \geq 0 \) and \( E = \bigcup_{n \geq 0} E_n \). Note that \( E = E_\ell(E) \) when \( \ell(E) < \infty \).

Fix \( X \neq 0 \) in \( \mathcal{A} \) and choose a simple subobject \( S \subseteq X \). Let \( U \subseteq X \) be a maximal subobject containing \( S \) such that \( S \subseteq U \) is essential; this exists by Zorn’s lemma. Then \( U \) is injective or belongs to \( C \). In the first case we are done. So assume \( U \in C \).

We claim that the inclusion \( U \to X \) is a pure monomorphism. To see this, choose a morphism \( C \to X/U \) with \( C \in C \). This yields the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
0 & \to & U & \to & V & \to & C & \to & 0 \\
0 & \to & U & \to & X & \to & X/U & \to & 0
\end{array}
\]

Write \( V = \bigoplus_i V_i \) as a direct sum of indecomposable objects. Then there exists an index \( i \) such that the composite \( S \to U \to V_i \to X \) is non-zero. Thus \( S \to V_i \) is essential and \( V_i \to X \) is a monomorphism. It follows from the maximality of \( U \) that \( U \to V_i \) is an isomorphism. Therefore the top row splits, and this yields the claim. It remains to observe that every object in \( C \) is pure-injective since \( C \) is Hom-finite; see Example \[4.2\] and Lemma \[5.2\] □

Let \( \text{Fp}(C, \text{Ab})_0 \) denote the full subcategory of finite length objects in \( \text{Fp}(C, \text{Ab}) \).

**Corollary 5.4.** Let \( C \) be a uniserial category. Then every effaceable finitely presented functor \( C \to \text{Ab} \) has finite length. Moreover effaceable and finite length functors agree if and only if all injective objects in \( C \) are zero. In that case we have an equivalence

\[
(5.1) \quad \frac{\text{Fp}(C^{\text{op}}, \text{Ab})}{\text{Fp}(C^{\text{op}}, \text{Ab})_0} \cong C.
\]

**Proof.** The first assertion follows from Proposition \[5.3\] and Theorem \[4.10\] keeping in mind that \( C \) is Hom-finite by Lemma \[5.2\]. Effaceable and finite length functors agree if and only if all injective objects in \( C \) are zero, by Remark \[4.11\]. Having this property, the equivalence is \( (2.3) \). □

**Remark 5.5.** An interesting instance of the equivalence \( (5.1) \) arises from the study of Greenberg modules; see \[13\] V, §4.1.8 and \[25\] §§4-5.

**Remark 5.6.** It would be interesting to have a more direct proof of Corollary \[5.4\] avoiding the embedding of \( C \) into a Grothendieck category.
Serre duality. Our next aim is a characterisation of uniserial categories that involves Serre duality. To this end recall the following characterisation in terms of Ext-quivers [1].

**Proposition 5.7.** A length category $C$ is uniserial if and only if it satisfies the following condition and its dual: For each simple object $S$ there exists, up to isomorphism, at most one simple object $T$ such that $\text{Ext}^1_C(S, T) \neq 0$, and in this case $\ell_{\text{End}_C(T)}(\text{Ext}^1_C(S, T)) = 1$.

We fix a field $k$ and write $D = \text{Hom}_k(-, k)$ for the usual duality. Let $C$ be a $k$-linear abelian category such that $\text{Hom}_C(X, Y)$ is finite dimensional for all $X, Y \in C$. Following [8], the category $C$ satisfies Serre duality if there exists an equivalence $\tau: C \to C$ with a functorial $k$-linear isomorphism

$$D \text{Ext}^1_C(X, Y) \to \text{Hom}_C(Y, \tau X)$$

for all $X, Y \in C$.

The functor $\tau$ is called Serre functor or Auslander-Reiten translation. Note that a Serre functor is $k$-linear and essentially unique provided it exists; this follows from Yoneda’s lemma.

The following result is well-known and describes the structure of a length category with Serre duality. Let us recall the shape of the relevant diagrams.

$$\tilde{A}_n: 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \longrightarrow n + 1$$

$$A_{\infty}^\infty: \cdots \circ \circ \circ \circ \circ \circ \circ \cdots$$

**Proposition 5.8.** Let $C$ be a $\text{Hom}$-finite $k$-linear length category and suppose $C$ admits a Serre functor $\tau$. Then $C$ is uniserial. The category $C$ admits a unique decomposition $C = \coprod_{i \in I} C_i$ into connected uniserial categories with Serre duality, where the index set equals the set of $\tau$-orbits of simple objects in $C$. The Ext-quiver of each $C_i$ is either of type $A_{\infty}^\infty$ (with linear orientation) or of type $\tilde{A}_n$ (with cyclic orientation).

**Proof.** We apply the criterion of Proposition 5.7 to show that $C$ is uniserial. To this end fix a simple object $S$. Then $\text{Ext}^1_C(S, T) \cong D \text{Hom}_C(T, \tau S) \neq 0$ for some simple object $T$ if and only if $T \cong \tau S$. Moreover, $\ell_{\text{End}_C(\tau S)}(\text{Ext}^1_C(S, \tau S)) = 1$. A quasi-inverse of $\tau$ provides a Serre functor for $C^{\text{op}}$. Thus the category $C$ is uniserial.

The structure of the Ext-quiver of $C$ follows from Proposition 5.7. The Serre functor acts on the set of vertices and the $\tau$-orbits provide the index set of the decomposition $C = \coprod_{i \in I} C_i$ into connected components. The Ext-quiver of $C_i$ is of type $A_{\infty}^\infty$ if the corresponding $\tau$-orbit is infinite. Otherwise, the Ext-quiver of $C_i$ is of type $\tilde{A}_n$ where $n + 1$ equals the cardinality of the $\tau$-orbit.

Auslander-Reiten duality. For an abelian category $A$ let $\text{St} A$ denote the stable category modulo injectives, which is obtained from $A$ by identifying two morphisms $\phi, \phi': X \to Y$ if

$$\text{Ext}^1_A(-, \phi) = \text{Ext}^1_A(-, \phi').$$

When $A$ has enough injective objects this means $\phi - \phi'$ factors through an injective object. We write $\text{Hom}_A(-, -)$ for the morphisms in $\text{St} A$.

Let us recall from [22 Corollary 2.13] the following version of Auslander-Reiten duality for Grothendieck categories, generalising the usual duality for modules over artin algebras [7].

**Proposition 5.9.** Let $A$ be a $k$-linear and locally finitely presented Grothendieck category. There exist a functor $\tau: \text{fp} A \to \text{St} A$ with a natural isomorphism

$$D \text{Ext}^1_A(X, -) \cong \text{Hom}_A(-, \tau X)$$

for all $X \in \text{fp} A$. 


Uniserial categories of infinite height. We are now ready to characterise uniserial categories of infinite height in terms of finitely presented functors.

**Theorem 5.10.** Let $\mathcal{C}$ be a $k$-linear length category such that $\text{Hom}_\mathcal{C}(X, Y)$ is finite dimensional for all $X, Y \in \mathcal{C}$. Then the following are equivalent:

1. A finitely presented functor $\mathcal{C} \to \text{Ab}$ is effaceable if and only if it has finite length.
2. The category $\mathcal{C}$ has Serre duality.
3. The category $\mathcal{C}$ is uniserial and all connected components have infinite height.

**Proof.** Set $\mathcal{A} = \text{Lex}(\mathcal{C}^{\text{op}}, \text{Ab})$ and identify $\mathcal{C} \xrightarrow{\sim} \text{fp} \mathcal{A}$.

1. $\Rightarrow$ (2): We claim that the functor $\tau : \mathcal{C} \to \text{St} \mathcal{A}$ in Proposition 5.9 yields a Serre functor for $\mathcal{C}$.

First observe that $\text{Hom}_\mathcal{A}(-, X) = \text{Hom}_\mathcal{A}(-, X)$ for every $X \in \mathcal{C}$. Condition (1) implies that every injective object in $\mathcal{C}$ is zero; see Remark 4.11. Now let $\phi : I \to X$ be a morphism in $\mathcal{A}$ with indecomposable injective $I$. Then $\text{Ker} \phi$ is indecomposable, and therefore injective or finitely presented, by Theorem 4.10. Thus $\phi = 0$.

The assumption on $\mathcal{C}$ in (1) is also satisfied by $\mathcal{C}^{\text{op}}$, thanks to the duality (2.2).

Thus $\text{Hom}_\mathcal{C}(X, -) = \text{Hom}_\mathcal{C}(X, -)$ for every $X \in \mathcal{C}$.

The functor $\tau : \mathcal{C} \to \text{St} \mathcal{A}$ in Proposition 5.9 lands in $\mathcal{C}$, because all indecomposable objects in $\text{St} \mathcal{A}$ belong to $\mathcal{C}$ by Theorem 4.10. In fact, the formula (5.2) yields an almost split sequence $0 \to \tau Z \to Y \to Z \to 0$ in $\mathcal{C}$ for every indecomposable object $Z$. Thus $\tau : \mathcal{C} \to \mathcal{C}$ is essentially surjective on objects, since the category $\mathcal{C}$ has almost split sequences by Theorem 4.10. The defining isomorphism for $\tau$ shows that $\tau$ is fully faithful, since $\text{Hom}_\mathcal{C}(-, -) = \text{Hom}_\mathcal{C}(X, -) = \text{Hom}_\mathcal{C}(X, -)$.

Indeed, the induced map $\tau_{X,Y} : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(\tau X, \tau Y)$ is a monomorphism. On the other hand, $\tau$ induces maps

$$\text{Hom}_\mathcal{C}(X, Y) \xrightarrow{\sim} \text{D}^2 \text{Hom}_\mathcal{C}(X, Y) \to \text{D Ext}^1_\mathcal{C}(Y, \tau X) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(\tau X, \tau Y)$$

where the middle one is the dual of the monomorphism

$$\text{Ext}^1_\mathcal{A}(-, \tau X) \to \text{D Hom}_\mathcal{A}(X, -)$$

from [22, Theorem 2.15]. Thus $\tau_{X,Y}$ is bijective.

2. $\Rightarrow$ (3): Use Proposition 5.8.

3. $\Rightarrow$ (1): Use Corollary 5.4.

### 6. Minimal length categories of infinite height

Throughout this section we fix an algebraically closed field $k$.

Our aim is an explicit description of the length categories of infinite height that are *minimal* in the sense that every proper closed subcategory has only finitely many isoclasses of indecomposable objects. Here, a full subcategory of an abelian category is *closed* if it is closed under subobjects and quotients.

**Theorem 6.1.** Let $\mathcal{C}$ be a $k$-linear length category with the following properties:

1. The category $\mathcal{C}$ has only finitely many isoclasses of simple objects.
2. The spaces $\text{Hom}_\mathcal{C}(X, Y)$ and $\text{Ext}^1_\mathcal{C}(X, Y)$ are finite dimensional for all objects $X, Y$ in $\mathcal{C}$.
3. The category $\mathcal{C}$ has infinite height.
4. Every proper closed subcategory of $\mathcal{C}$ has only finitely many isoclasses of indecomposable objects.
Then $\mathcal{C}$ is equivalent to the category of nilpotent finite dimensional representations of some cycle

$$Z_n: 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-1 \rightarrow n \quad (n \geq 1).$$

We do not prove the theorem as it stands. Instead we switch to an equivalent formulation involving representations of admissible algebras.

**Admissible algebras.** Let $A$ be a $k$-algebra and denote by $R$ its radical. We call $A$ admissible if it satisfies the following two conditions:

(a) The algebras $A/R$ and $R/R^2$ are finite dimensional.

(b) The algebra $A$ is separated and complete for the $R$-adic topology: $A \cong \lim_{\leftarrow i \geq 0} A/R^i$, canonically.

In the sequel, every admissible algebra $A$ will be considered together with its $R$-adic topology. Accordingly, an ideal $I$ of $A$ is open if $I$ contains some power $R^d$ iff $I$ is of finite codimension. In other words, our admissible algebras are exactly the profinite algebras satisfying (a). For more on profinite algebras, one may consult \[25\]. The closure of $I$ equals the intersection of all $I + R^d$.

Given an admissible algebra $A$, we are really interested in the category $\text{mod}_0 A$ of (left) $A$-modules of finite length. Condition (a) ensures that $\text{mod}_0 A$ has only finitely many isoclasses of simples and that the spaces $\text{Hom}_A(M, N)$ and $\text{Ext}_A^1(M, N)$ are finite dimensional for $M, N$ in $\text{mod}_0 A$. Condition (b) comes up naturally when one tries to recover $A$ from $\text{mod}_0 A$, up to Morita equivalence; see \[15\].

**Complete path algebras.** Let $Q$ be a finite quiver. The complete path algebra $k[Q]$ consists of the formal series $\sum_a a_u u$ where $u$ runs through the paths of $Q$ and $a_u \in k$. Here, the paths $e_i$ of length 0, corresponding to the vertices $i$, are included.

The multiplication is defined by

$$\left( \sum_{u} a_u u \right) \left( \sum_{v} b_v v \right) = \sum_{w} \left( \sum_{l_{uv}=w} a_u b_v \right) w.$$

For any integer $l \geq 0$, the elements $\sum_u a_u u$, where $u$ is restricted to the paths of length $\geq l$, form an ideal $k[Q]^{\geq l}$ of $k[Q]$; this ideal is precisely the $l$-th power of the radical of $k[Q]$. Consequently $k[Q]$ is admissible and, according to our convention, will always be considered together with its $k[Q]^{\geq 1}$-adic topology.

**Description of admissible algebras by quiver and relations.** Any admissible $k$-algebra $A$ that is basic, i.e., the quotient $A/R$ is a finite direct product of copies of $k$, can be presented as the complete path algebra of a finite quiver modulo a closed ideal: choose a decomposition $1_A = e_1 + \cdots + e_n$ of the unit element into pairwise orthogonal primitive idempotents; such a decomposition can be obtained by lifting the unique one for $A/R$. The quiver $Q_A$ then has vertices $1, \ldots, n$, and there are arrows $a_{ij}^{m}$: $i \rightarrow j$, $1 \leq m \leq n_{ji}$, where $n_{ji}$ is the dimension of $e_j(R/R^2)e_i$. Choose further elements $a_{ji}^{m} \in e_j Re_i$, $1 \leq m \leq n_{ji}$, whose classes form a basis of $e_j(R/R^2)e_i$. The choices uniquely determine a continuous homomorphism $k[Q_A] \rightarrow A$ that maps each $e_i$ to $e_i$ and each $a_{ji}^{m}$ to $a_{ji}^{m}$. This homomorphism is surjective; its kernel $I$ is contained in $k[Q_A]^{\geq 2}$ and necessarily closed. The presentation $A \leftarrow k[Q_A]/I$ allows one to interprete a module in $\text{mod}_0 A$ as a finite dimensional representation of $Q_A$ that satisfies the relations of some sufficiently small ideal $I + R^d$, depending on the module.
**Representation types.** Let $A$ be an admissible $k$-algebra.

The algebra $A$ is *representation-finite* if $\text{mod}_0 A$ has only finitely many isoclasses of indecomposables. In this case, $A$ is necessarily finite dimensional. Otherwise, there is some infinite dimensional indecomposable projective $P$, having a simple top, and $P$ admits indecomposable quotients of any finite length $\geq 1$.

The algebra $A$ is *mild* if any quotient $A/I$ by some closed ideal $I \neq 0$ is representation-finite.

**Remark 6.2.** In the definition of ‘mild’, the restriction to closed ideals is not necessary: if $I$ is an arbitrary ideal with closure $\bar{I}$, the algebras $A/I$ and $A/\bar{I}$ have the same finite dimensional modules. On the other hand, if $A/I$ is representation-finite, $I$ is even open.

**The main result.** Very special but frequently encountered examples of admissible algebras are the complete path algebras $k[Z_n]$ of the cyclic quivers

$$Z_n: \begin{array}{c} 1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} \cdots \xleftarrow{\alpha_{n-2}} n-1 \xleftarrow{\alpha_{n-1}} n \end{array} \ (n \geq 1).$$

They admit an alternative description: consider the discrete valuation $k$-algebra $\mathfrak{o} = k[\mathbb{T}]$ with maximal ideal $\mathfrak{m} = T k[\mathbb{T}]$ and fraction field $K$. Then $k[Z_n]$ is isomorphic to the following hereditary $\mathfrak{o}$-order in the simple $K$-algebra $M_n(K)$:

$$\begin{bmatrix} 0 & m & m & \cdot & m & m \\ 0 & 0 & m & \cdot & m & m \\ 0 & 0 & 0 & \cdot & m & m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & 0 & m \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix}$$

The Auslander-Reiten quiver of $\text{mod}_0 k[Z_n]$ is identified with $ZA_\infty/\tau^n$. If the orientation $\cdots \rightarrow 3 \rightarrow 2 \rightarrow 1$ of $A_\infty$ is used, then the vertex $(-i, l)$ corresponds to the uniserial indecomposable of length $l$ with top located at $i$.

It is well-known that $k[Z_n]$ is mild.

**Theorem 6.3.** If a $k$-algebra is admissible, basic, infinite dimensional, and mild, then it is isomorphic to $k[Z_n]$ for some $n \geq 1$.

Our result is close to well-known characterisations of hereditary orders; see [17]. The only novelty is that we can do without assuming a ‘large’ center or some ‘purity’ condition [14].

**Proof.** Let the $k$-algebra $A$ be admissible, basic, infinite dimensional, and mild. We denote by $R$ the radical of $A$ and fix a decomposition $1_A = e_1 + \cdots + e_n$ into pairwise orthogonal primitive idempotents.

**Step 1:** For any $i$, the algebra $e_i Ae_i$ is a quotient of $k[\mathbb{T}]$, and for any $i, j$, the space $e_j Ae_i$ is cyclic as a right module over $e_i Ae_i$ or as a left module over $e_j Ae_j$. Indeed, by well-known observations due to Jans [19] and Kupisch [23], a similar statement is true for each of the representation-finite algebras $A/R^i$. Our claim follows by passage to the limit.

**Step 2:** The $e_j Ae_j$-$e_i Ae_i$-bimodule $e_j Ae_i$ carries two natural filtrations. The first one is its radical filtration given by the subbimodules

$$\begin{aligned} (e_j Ae_i)^{2\mathbb{Z}} = \sum_{r+s=l} (e_j Re_j)^s A(e_i Re_i)^r. \end{aligned}$$

By step 1 it coincides with the radical filtration of $e_j Ae_i$ over $e_i Ae_i$ or over $e_j Ae_j$, it contains each non-zero subbimodule of $e_j Ae_i$ exactly once, and each quotient $(e_j Ae_i)^{2l}/(e_j Ae_i)^{2l+1}$ has dimension $\leq 1$. 

The second one is the filtration \((e_jR^me_i)_{m \geq 0}\) induced by the \(R\)-adic filtration on \(A\). Each term, being a submodule of \(e_jAe_i\), appears in the first filtration, and each quotient \(e_jR^me_i/e_jR^{m+1}e_i\) is semisimple over \(e_jAe_i\) and over \(e_jAe_j\).

Remembering that the second filtration is separated, we conclude that it is obtained from the first one by possibly introducing repetitions and that both filtrations define the same topology.

Step 3: Let \(e\) be an idempotent of \(A\) that is the sum of some of \(e_1, \ldots, e_n\). We claim that \(eAe\) is admissible: indeed, we just saw in step 2 that each space \(e_j(R/ReR)e_i\) (with \(e_i\) and \(e_j\) occurring in \(e\)) has dimension at most 1. Arguing as above, the topologies on \(e_jAe_i\) induced by the \(eRe\)-adic filtration on \(eAe\) and the \(R\)-adic filtration on \(A\) coincide. Therefore \(eAe \cong \lim_{l \geq 0} eAe/(eRe)^l\), canonically.

Of course, our claim would be wrong without the assumption of mildness of \(A\).

Step 4: Let \(e\) be as in step 3. We claim that \(eAe\) is mild. Indeed, let \(J\) be a non-zero ideal of \(eAe\) and \(I = AJA\) its extension to \(A\). Since \(A/I\) is in particular finite dimensional, the fully faithful left adjoint \(N \mapsto (A/I)e \otimes_{eAe/J} N\) of the restriction functor \(M \mapsto eM\) then maps \(\text{mod}_0 eAe/J\) into \(\text{mod}_0 A/I\). Since \(A/I\) is representation-finite, so is \(eAe/J\). Consequently, \(eAe\) is mild.

Step 5: For at least one \(i\), the algebra \(e_iAe_i\) is infinite dimensional, i.e., isomorphic to \(k[T]\). Otherwise all \(e_jAe_i\) and therefore \(A\) itself are finite dimensional by step 1.

Step 6: Consider the case \(n = 2\) (thus \(1_A = e_1 + e_2\)) and assume for definiteness that \(e_1Ae_1\) is isomorphic to \(k[\bar{T}]\) (step 5). Then \(e_1Ae_2Ae_1 \neq 0\); otherwise \(A/Ae_iA\) is still isomorphic to \(k[\bar{T}]\) and representation-infinite, contradicting the mildness of \(A\). Therefore \(e_1Ae_2Ae_1 = (e_1Re_1)^r\) for some uniquely determined integer \(r \geq 1\). We now transform the powers

\[
(e_1Re_1)^l
\]

into ideals of \(e_2Ae_2\):

\[
e_2A(e_1Re_1)^lAe_2
\]

and then back into ideals of \(e_1Ae_1\):

\[
e_1Ae_2A(e_1Re_1)^lAe_1 = (e_1Re_1)^r(e_1Re_1)^l(e_1Re_1)^r = (e_1Re_1)^{l+2r}.
\]

Since the sequence of ideals in (6.3) is strictly decreasing, so is the one in (6.2). In particular, also \(e_2Ae_2\) is isomorphic to \(k[\bar{T}]\) (step 5) and \(e_2Ae_1Ae_2 \neq 0\). Interchanging \(e_1\) and \(e_2\), we have \(e_2Ae_1Ae_2 = (e_2Re_2)^s\) for some uniquely determined integer \(s \geq 1\). We get

\[
(e_2Re_2)^{2s} = e_2Ae_1Ae_2Ae_1Ae_2 = e_2A(e_1Re_1)^rAe_2
\]

and

\[
s = \dim(e_2Re_2)^s/(e_2Re_2)^{2s} = \dim e_2Ae_1Ae_2/e_2A(e_1Re_1)^rAe_2 \geq r.
\]

By symmetry we even have \(r = s\). If \(r = s \geq 2\), the quiver of \(A\) is

\[
\begin{array}{c}
\circ \\
\longrightarrow \\
\circ \\
\end{array}
\]

and already \(A/R^2\) is representation-infinite: contradiction! Thus \(r = s = 1\) and \(A\) is isomorphic to \(k[Z_2]\).

Step 7: Since \(A\) clearly is connected, one immediately deduces from step 6: for any two distinct idempotents \(e_i\) and \(e_j\), \((e_i + e_j)A(e_i + e_j)\) is isomorphic to \(k[Z_2]\).

Step 7: Consider the case \(n = 3\) (thus \(1_A = e_1 + e_2 + e_3\)) and put \(R_i = e_iAe_i\), \(M_i = e_{i+1}Ae_i\), \(N_i = e_{i-1}Ae_i\). In this step the indices are taken modulo 3. By the previous step we have the relations

\[
R_{i+1}M_i = M_iR_i, \quad R_{i-1}N_i = N_iR_i, \quad N_{i+1}M_i = R_i, \quad M_{i-1}N_i = R_i.
\]
Now $M_{i+1} \neq 0$, since $N_{i+2}M_{i+1}M_i = R_{i+1}M_i \neq 0$, for instance, and therefore

$$M_{i+1}M_i = N_i R_i(i) = R_i(i)N_i,$$

for some uniquely determined integer $r(i) \geq 0$. Calculating in two ways:

$$M_{i+2}(M_{i+1}M_i) = M_{i+2}N_i R_i(i) = R_i(i+1),$$

$$(M_{i+2}M_{i+1})M_i = R_i(i+1)N_{i+1}M_i = R_i(i+1)+1,$$

we see that $r(i)$ is independent of $i$; denote this integer again by $r$. Similarly, we have

$$N_{i-1}N_i = M_i R_i = R_i +1 M_i$$

for all $i$ and some uniquely determined integer $s \geq 0$. Calculating again in two ways:

$$N_{i+2}(M_{i+1}M_i) = N_{i+2}N_i R_i = M_i R_i R_i = M_i R_i + s,$$

$$(N_{i+2}M_{i+1})M_i = R_{i+1}M_i = M_i R_i,$$

we see that $r + s = 1$ and hence that $(r, s)$ equals $(1, 0)$ or $(0, 1)$. This means that $A$ is isomorphic to $k[\mathbb{Z}_2]$.

Step 9: In the general case, taking into account steps 3 and 4 and applying steps 6 and 8, one immediately sees that the quiver of $A$ is a cycle. Since $A$ is infinite dimensional, there cannot be any relation.

\[\Box\]

References

[1] I. Kr. Amdal and F. Ringdal, Catégories unisérielles, C. R. Acad. Sci. Paris Sér. A-B \textbf{267} (1968), A85–A87 and A247–A249.

[2] M. Auslander, Coherent functors, in \textit{Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)}, 189–231, Springer, New York, 1966.

[3] M. Auslander, Representation theory of Artin algebras. II, Comm. Algebra \textbf{5} (1974), 269–310.

[4] M. Auslander, Large modules over Artin algebras, in \textit{Algebra, topology, and category theory (Oberwolfach, 1980)}, 1–17, Academic Press, New York, 1981.

[5] M. Auslander and I. Reiten, Representation theory of Artin algebras. III. Almost split sequences, Comm. Algebra \textbf{3} (1975), 239–294.

[6] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Izv. Akad. Nauk SSSR Ser. Mat. \textbf{53} (1989), no. 5, 1641–1674. Galois theory, rings, algebraic groups and their applications (Russian). Trudy Mat. Inst. Steklov. 183 (1990), 86–96, 225.

[7] F. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France \textbf{90} (1962), 323–448.

[8] P. Gabriel, Indecomposable representations. II, in \textit{Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)}, 81–104, Academic Press, London, 1973.

[9] W. H. Gustafson, On hereditary orders, Comm. Algebra \textbf{15} (1987), no. 1-2, 219–226.

[10] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. London Math. Soc. (3) \textbf{74} (1997), no. 3, 503–558.
19] J. P. Jans, On the indecomposable representations of algebras, Ann. of Math. (2) 66 (1957), 418–429.
20] H. Krause, The spectrum of a locally coherent category, J. Pure Appl. Algebra 114 (1997), no. 3, 259–271.
21] H. Krause, Morphisms determined by objects and flat covers, Forum Math. 28 (2016), no. 3, 425–435.
22] H. Krause, Auslander-Reiten duality for Grothendieck abelian categories, arXiv:1604.02813.
23] H. Kupisch, Symmetrische Algebren mit endlich vielen unzerlegbaren Darstellungen, I, J. Reine Angew. Math. 219 (1965), 1–25.
24] C. M. Ringel and H. Tachikawa, QF – 3 rings, J. Reine Angew. Math. 272 (1974).
25] C. Schoeller, Étude de la catégorie des algèbres de Hopf commutatives connexes sur un corps, Manuscripta Math. 3 (1970), 133–155.
26] J.-P. Serre, Gébres, Enseign. Math. (2) 39 (1993), no. 1-2, 33–85.

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