Constraints and Generalized Gauge Transformations on Tree-Level Gluon and Graviton Amplitudes

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Abstract

Writing the fully color dressed and graviton amplitudes, respectively, as $A = \langle C|A \rangle = \langle C|M|N \rangle$ and $A_{gr} = \langle \tilde{N}|M|N \rangle$, where $|A\rangle$ is a set of Kleiss-Kuijf color ordered basis, $|N\rangle$, $|\tilde{N}\rangle$ and $|C\rangle$ are the similarly ordered numerators and color coefficients, we show that the propagator matrix $M$ has $(n - 3)(n - 3)!$ independent eigenvectors $|\lambda^0_j\rangle$ with zero eigenvalue, for $n$-particle processes. The resulting equations $\langle \lambda^0_j|A \rangle = 0$ are relations among the color ordered amplitudes. The freedom to shift $|N\rangle \rightarrow |N\rangle + \sum_j f_j|\lambda^0_j\rangle$ and similarly for $|\tilde{N}\rangle$, where $f_j$ are $(n - 3)(n - 3)!$ arbitrary functions, encodes generalized gauge transformations. They yield both BCJ amplitude and KLT relations, when such freedom is accounted for. Furthermore, $f_j$ can be promoted to the role of effective Lagrangian vertices in the field operator space.

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1 Introduction and Summary

In a series of papers, Bern and his collaborators [1], [2], [3], and others using string theory insights [4] or through the use of the BCFW recursion relations [5, 9], obtained some very interesting results regarding the number of independent amplitudes \((n - 3)!\) for \(n\) gluon scattering and the relationship between gravitational and non-abelian gauge amplitudes with the same external momenta. The derivations in [1] and arguments to arrive at these results rest heavily on duality between intrinsic dynamics and color kinematics and on what is termed generalized gauge transformations. Thus, one symbolically writes the color dressed \(n\)-gluon amplitude as [10]

\[
A^{(n)} = \sum_i \frac{c_i n_i}{(\prod s_j)_i},
\]

where \(c_i\) are color factors, \(n_i\) numerators made of kinematical invariants of the process, and \((\prod s_j)_i\) are appropriate products of inverse propagators. The conjecture, which has been subsequently proven, is that for channels with color factors satisfying

\[
c_i + c_j + c_k = 0,
\]

one can generate a set of numerators with only effective triple vertices such that the corresponding ones also satisfy

\[
n_i + n_j + n_k = 0.
\]

Furthermore, Bern, Carrasco and Johansson (BCJ) [1] argue that there are \((n - 3)(n - 3)!\) degrees of arbitrariness in specifying these numerators, which are made into generalized gauge transformations. Basing on these, they showed that the number of independent amplitudes for \(n\) gluon scattering is \((n - 3)!\) and that the corresponding graviton scattering amplitude is [1, 2, 3]

\[
A^{(n)}_{gr} = \sum_i \frac{n_i \tilde{n}_i}{(\prod s_j)_i},
\]

where \(\tilde{n}_i\) are another copy of numerators due to the same or a different gauge theory with

\[
\tilde{c}_i + \tilde{c}_j + \tilde{c}_k = 0,
\]

2The recursion relations found by Britto, Cachazo and Feng [6] among \(n\)-point color-ordered gauge theory tree-level amplitudes were proven in [5], based on certain complex shifts of pairs of external gluon momenta, and the on the behaviour of the tree level amplitude at large values of the complex shift parameter. In [7], the BCFW recursion relations were shown to originate in a set of identities, known as the largest time equation, which are obeyed in causal theories. In [8], the BCFW shifts were extended to triple shifts of external gluon momenta to address recursion relations at one loop level.
\[ \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0. \] (1.6)

These relations between gravity amplitudes and gauge theory amplitudes, which are inspired by the Kawai-Lewellen-Tye (KLT) relations [11] (for a more recent work see also [12]) and the BCJ relations have received a great deal of interest in the recent months: [13], [14], [15], [16], [17], [18], [19], and new identities (non-linear this time) were found among gauge theory amplitudes [20].

Since these results have profound consequences on how one may perform gauge amplitude calculations and more importantly how one may look at gravitational interaction, we would like to investigate the origin of the gauge freedoms. Somewhat surprisingly, we find that they come from the structure of the \((n-2)! \times (n-2)!\) propagator matrix, when we work with the Kleiss-Kuijf basis \(A^{(n)}_i [21]\),

\[ M \sim \left( \frac{1}{\prod s_j} \right), \] (1.7)

by which the color-ordered vector can be written as

\[ |A\rangle = M |N\rangle, \] (1.8)

where \(|N\rangle\) is a numerator vector with \(n_i\) as its entries

\[ |N\rangle = \begin{pmatrix} n_1 \\ n_2 \\ \cdots \\ n_{(n-2)!} \end{pmatrix}. \] (1.9)

We shall find that the matrix \(M\) has \((n-3)(n-3)!\) independent eigenvectors with zero eigenvalues

\[ M |\lambda_j^0\rangle = 0. \] (1.10)

As a consequence, one can add a linear combination of these null eigenvectors with the same number of arbitrary functions \(f_j\) as coefficients to the numerator vector without changing the value of the color ordered scattering amplitude vector

\[ |A\rangle = M(|N\rangle + \sum_j f_j |\lambda_j^0\rangle). \] (1.11)

\[ ^3 \text{In [16] the gauge freedom of Bern et al [1], [2] was interpreted as reparametrization invariance of the amplitude, such that it remains compatible with monodromy relations derived from string theory.} \]
One can of course interpret this as effectively making a transformation on the numerators
\[ |N\rangle \rightarrow |N'\rangle = |N\rangle + \sum_j f_j |\lambda^0_j\rangle. \] (1.12)

However, we should emphasize that we need not re-shuffle entries in the original numerator vector \( |N\rangle \) to accomplish the change. We just add a useful zero externally.

Another immediate outcome is that from (1.8) and (1.10), one obtains \((n - 3)(n - 3)!\) independent relations among the color ordered amplitudes
\[ \langle \lambda^0_j | A \rangle = \langle \lambda^0_j | M | N \rangle = 0, \] (1.13)
the precise form of which depends on our choice of basis for \( |A\rangle \) and \( |\lambda^0_j\rangle \).

To turn to gravity, let us be a bit more specific. We shall work in the Kleiss-Kuijf basis
\[ n_i = n(1, i_2, i_3, \ldots i_{n-1}, n). \] (1.14)
for the \( n\)-particle color ordered amplitudes, in which the labels denote the ordering of the external legs. There are obviously \((n - 2)!\) entries for the numerator vector \( |N\rangle_i = n_i \). We shall label the color factors in the same order and form a vector
\[ \langle C |_i \equiv c_i = c(1, i_2, i_3, \ldots i_{n-1}, n), \] (1.15)
in the same sequence as in \( |N\rangle \). It can be shown easily that (1.11) for the color dressed amplitude becomes
\[ A^{(n)} = \langle C | M | N \rangle, \] (1.16)
and (1.14) for the \( n\)-graviton amplitude
\[ A^{(n)}_{gr} = \langle \tilde{N} | M | N \rangle = \langle N | M | \tilde{N} \rangle, \] (1.17)
as \( M \) is symmetric. Let us denote
\[ \sum_j f_j |\lambda^0_j\rangle \equiv |\delta N\rangle. \] (1.18)

Clearly we have
\[ \delta A^{(n)} = \langle C | M | \delta N \rangle = 0, \] (1.19)
and
\[ \delta A^{(n)}_{gr} = \langle \delta \tilde{N} | M | N \rangle + \langle \tilde{N} | M | \delta N \rangle + \langle \delta \tilde{N} | M | \delta N \rangle = 0 \] (1.20)
term-wise because of (1.10). This is a statement of generalized gauge invariance.

3
In the recursive proof of squaring hypothesis of (1.4) for any $n$, Bern et al used BCFW complexification to relate the higher point numerator to the $\tilde{n}_i n_i$'s of lower points. They can differ by a set of gauge transformations at every $z$-pole, which must manage to cancel to give the whole amplitude a gauge independent construct. In our formulation, the existence of null eigenvectors of $M$ depends on overall energy momentum conservation and masslessness of all external legs, which are respected by BCFW. Thus $M$ and $|\delta N\rangle$ can be extended to yield (1.19) and (1.20). In fact, they are satisfied by the residues at every single $z$-pole of $M$, called channels, and moreover there is a unique $f_j$ at each channel to effect the necessary gauge transformation. In short, we find that our formulation is very natural for the study of the issues involved.

The plan of this article is as follows: in the next two sections (2 and 3) we shall specify the labeling $i$ of the numerators $n_i$ for four and five particle amplitudes in relation to the K-K basis. A set of Jacobi identities for them will summarize the eventual duality symmetry between $n_i$ and the associated color factors $c_i$. The propagator matrix $M$ will be given, from which a set of $(n-2)!$ null eigenvectors will be made explicit. The ranks of the null spaces for $n = 4, 5$ will be found by forming constraint matrices out of these null eigenvectors.

In Section 4, we shall expose the source of generalized gauge transformations, which is because of the existence of the null eigenvectors, again explicitly for $n = 4, 5$. We shall make use of specific choices of the gauge functions $f_j$ and null eigenvectors $|\lambda_j^0\rangle$ to show how relations between color ordered amplitudes and graviton amplitudes are reached, and how in general the structure of $f_j$ and $|\lambda_j^0\rangle$ connects the roles played by color Jacobi identities among $c_i$ for gauge amplitudes with those by $n_i$ for graviton amplitudes, to make gauge invariance possible for every channel of the scattering amplitudes.

Some concluding remarks are made in Section 5, where we also point out how natural it is to promote the gauge functions $f_j$ into effective Lagrangian vertices in field space to make manifest the semi-local nature of the necessary gauge transformations in recursive constructions of $n_i$.

We compile an Appendix to discuss the six particle scattering amplitudes.

## 2 Constraints on Four Gluon Amplitudes

In the Kleiss-Kuijf basis of $A(1234)$ and $A(1324)$, the amplitudes have simple poles in the $s_{12}, s_{14}$ and $s_{13}, s_{14}$ channels. Let us denote their corresponding numerators by $n_1 = n(12; 34), n_2 = n(23; 41)$ and $n_3 = n(13; 24), n_4 = n(32; 41)$, where the external gluons which share a vertex are denoted by pairs of indices not separated by semi-columns.
The cyclic order of the external gluons is indicated by the numerals 1 to 4 read clock-wise. Due to Jacobi identities we have
\[
n(ij; kl) = -n(ji; kl), \quad n(ij; kl) + n(ki; jl) + n(jk; il) = 0, \quad n(ij; kl) = n(lk; ji)
\]
which yield, in particular,
\[
n_4 + n_3 = n_1, \quad n_2 = -n_4.
\]
This leads to the following relationship between the Kleiss-Kuijf amplitudes and the basis of independent numerators, \(n_1, n_3\):
\[
\begin{pmatrix}
A(1234) \\
A(1324)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{s_{12}} + \frac{1}{s_{14}} & -\frac{1}{s_{14}} \\
-\frac{1}{s_{14}} & \frac{1}{s_{14}} + \frac{1}{s_{13}}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_3
\end{pmatrix},
\]
or, in shorthand notation,
\[
|A^{(4)}\rangle = M^{(4)}|N^{(4)}\rangle,
\]
where \(|A\rangle\) is the Kleiss-Kuijf amplitudes column matrix, \(M\) is the square matrix (and symmetric, in the chosen numerator basis) which encodes the pole structure of the amplitudes and \(|N\rangle\) is the numerator matrix.

The crucial observation made by Bern et al. in our language is that \(M\) has rank \((n - 3)!\) for a \((n - 2)!\) set of independent \(n\)-point gluon amplitudes. A related observation (see [4], [9]) is that the Kleiss-Kuijf amplitudes matrix \(A\) obeys a set of constraints which can be summarized by the following equation:
\[
\left( s_{i_2i_3} + s_{i_2i_4} + \cdots + s_{i_2i_{n-1}} + s_{i_2n} \right) A(1i_2i_3\ldots i_{n-1}n) \\
+ \left( s_{i_2i_3} + \cdots + s_{i_2i_{n-1}} + s_{i_2n} \right) A(1i_3i_2\ldots i_{n-1}n) \\
+ \cdots \\
+ \left( s_{i_2i_{n-1}} + s_{i_2n} \right) A(1i_3i_4\ldots i_{n-2}i_2i_{n-1}n) \\
+ s_{i_2n} A(1i_3i_4\ldots i_{n-1}i_2n) = 0.
\]
This means that there are eigenvectors \(\langle \lambda_j^0 \rangle\) of \(M\) with zero eigenvalue
\[
\langle \lambda_j^0 \rangle |M = 0,
\]
whose entries are \(s_{i_2i_3} + s_{i_2i_4} + \cdots + s_{i_2i_{n-1}} + s_{i_2n}\) at \((1i_2i_3\ldots i_{n-1}n)\), \ldots , \(s_{i_2n}\) at \((1i_3i_4\ldots i_{n-1}i_2n)\). There are \((n - 2)!\) such vectors, but only \((n - 3)(n - 3)!\) of which are
linearly independent. To see that we construct a constraint matrix $C$ by putting all these row null eigenvectors into a $(n-2)! \times (n-2)!$ matrix with the property
\[ C|A\rangle = CM|N\rangle = 0, \]  
which will be shown to have rank $(n-3)!$.

In the case of the four-point amplitude, it is easy to show that $M^{(4)}$ has rank 1. Equivalently, we have
\[ \langle \lambda_1^0 | = \langle s_{23} + s_{24} \rangle, \]
\[ \langle \lambda_2^0 | = \langle s_{34} s_{23} + s_{34} \rangle, \]
and the constraint matrix is
\[ C^{(4)} = \begin{pmatrix} s_{23} + s_{24} & s_{24} \\ s_{34} & s_{23} + s_{34} \end{pmatrix}. \]

The two constraints obeyed by the four-point Kleiss-Kuijf gluon amplitudes are not actually independent of each other. The rank of the constraint matrix $C^{(4)}$ is 1, which can be seen by noticing that, due to momentum conservation and the on-shellness of the external gluon momenta, the sum of the rows of the matrix $C^{(4)}$ is 0. This feature of the constraint matrix continues to hold for all $n$-point amplitudes.

### 3 Constraints on Five Gluon Amplitudes

The Kleiss-Kuijf basis of gluon amplitudes can be chosen to be: $A(12345)$, $A(14325)$, $A(13425)$, $A(12435)$, $A(14235)$, $A(13245)$. Each of these amplitudes has simple poles in the various kinematic invariants. We will denote the numerators associated with these poles as follows:

\begin{align*}
n_1 &= n(12;3;45), & n_2 &= n(23;4;51), & n_3 &= -n(12;5;34), & n_4 &= n(23;1;45), \\
n_5 &= n(51;2;34), & n_6 &= n(14;3;25), & n_7 &= n(32;5;14), & n_8 &= n(25;1;43), \\
n_9 &= n(13;4;25), & n_{10} &= -n(13;5;42), & n_{11} &= n(51;3;42), & n_{12} &= n(12;4;35), \\
n_{13} &= n(35;1;24), & n_{14} &= n(14;2;35), & n_{15} &= n(13;2;45). \end{align*}  

Our notation for the numerators associated with a certain tree level Feynman diagram is as follows: we specify the external gluons (with certain momenta and helicities) by

\[^4\text{For an early explicit calculation of five and six-gluon tree-level amplitudes see [22].}\]
numerals; those external gluons which are indicated by numerals not separated by semi-
columns share a common vertex; the remaining external gluon which joins some internal
lines in a vertex is indicated by a separate numeral; all external gluon lines are written in
a clock-wise order. Of course, due to Jacobi identities at the vertex on the internal lines,
the numerators obey straightforward identities associated with flipping the orientation of
the external line above or below the internal lines. For example:

\[ n(13; 5; 42) = -n(13; 42; 5). \]  (3.2)

Similarly, flipping the order of the external gluons with a common vertex brings a minus
sign, e.g.:

\[ n(12; 3; 45) = -n(21; 3; 45) = -n(12; 3; 54). \]  (3.3)

In addition we have relations between the fifteen numerators which also follow from the
use of Jacobi identities:

\[
\begin{align*}
n_1 + n_4 &= n_{15}, \\
n_1 - n_3 &= n_{12}, \\
n_6 - n_7 &= n_{14}, \\
n_2 + n_4 &= n_7, \\
n_9 - n_{15} &= n_{10}, \\
n_6 &= n_8 = n_9, \\
n_2 + n_5 &= n_{11}, \\
n_{11} - n_{10} &= n_{13}. 
\end{align*}
\]  (3.4)

In more generality, the numerators obey Jacobi-type identities

\[
\begin{align*}
n(ij; k; lm) &= -n(ji; k; lm), \\
n(ij; k; lm) + n(ki; j; lm) + n(jk; i; lm) &= 0, 
\end{align*}
\]  (3.5)

in addition to the time-reverse identity

\[ n(ml; k; ji) = -n(ij; k; lm). \]  (3.6)

Eliminating the other numerators in favor of \( n_1, n_6, n_9, n_{12}, n_{14}, n_{15} \), we can express the
Kleiss-Kuijf amplitudes in terms of the numerators

\[
\begin{pmatrix}
A(12345) \\
A(14325) \\
A(13425) \\
A(12435) \\
A(14235) \\
A(13245)
\end{pmatrix}
= M^{(5)}
\begin{pmatrix}
n_1 \\
n_6 \\
n_9 \\
n_{12} \\
n_{14} \\
n_{15}
\end{pmatrix}
\]  (3.7)
where $M^{(5)}$ is given by

\[ \begin{pmatrix}
\frac{1}{s_{12} s_{45}} + \frac{1}{s_{15} s_{44}} + \frac{1}{s_{21} s_{15}} & \frac{1}{s_{15} s_{34}} + \frac{1}{s_{21} s_{15}} & -\frac{1}{s_{15} s_{34}} - \frac{1}{s_{12} s_{34}} - \frac{1}{s_{23} s_{15}} & -\frac{1}{s_{23} s_{45}} - \frac{1}{s_{23} s_{15}} \\
\frac{1}{s_{15} s_{34}} + \frac{1}{s_{14} s_{23}} + \frac{1}{s_{15} s_{34}} & -\frac{1}{s_{15} s_{34}} - \frac{1}{s_{14} s_{23}} + \frac{1}{s_{15} s_{34}} & \frac{1}{s_{15} s_{24}} + \frac{1}{s_{15} s_{34}} & -\frac{1}{s_{15} s_{24}} - \frac{1}{s_{15} s_{24}} \\
-\frac{1}{s_{15} s_{34}} & -\frac{1}{s_{15} s_{34}} & \frac{1}{s_{15} s_{24}} + \frac{1}{s_{15} s_{24}} & -\frac{1}{s_{15} s_{24}} - \frac{1}{s_{15} s_{24}} \\
-\frac{1}{s_{15} s_{23}} & -\frac{1}{s_{14} s_{23}} - \frac{1}{s_{15} s_{23}} & -\frac{1}{s_{15} s_{24}} & \frac{1}{s_{15} s_{23}} + \frac{1}{s_{15} s_{24}} \\
-\frac{1}{s_{23} s_{45}} & -\frac{1}{s_{23} s_{15}} & -\frac{1}{s_{13} s_{24}} - \frac{1}{s_{15} s_{24}} & \frac{1}{s_{23} s_{45}} + \frac{1}{s_{23} s_{15}} \\
\end{pmatrix} \]

The symmetric matrix $M^{(5)}$ used to express the Kleiss-Kuijf five-point amplitudes in the basis of the independent numerators has rank 2 (and thus equal to $(n-3)!$ for $n$-point amplitudes).

This is also reflected by the existence of a constraint matrix such that $C^{(5)}|A^{(5)}\rangle = 0$ and $C^{(5)}M^{(5)} = 0$, where $C^{(5)}$ is obtained from specializing the constraint equations (2.5).
to the case of the chosen five-point Kleiss-Kuijf amplitude basis:

\[
\begin{pmatrix}
  s_{23} + s_{24} + s_{25} & 0 & s_{25} & 0 & 0 & s_{24} + s_{25} \\
  0 & s_{24} + s_{34} + s_{45} & s_{24} + s_{45} & 0 & 0 & s_{45} \\
  0 & s_{23} + s_{35} & s_{34} + s_{23} + s_{35} & 0 & s_{35} & 0 \\
  0 & s_{25} & 0 & s_{24} + s_{23} + s_{25} & s_{23} + s_{25} & 0 \\
  s_{45} & 0 & 0 & s_{34} + s_{45} & s_{24} + s_{34} + s_{45} & 0 \\
  s_{34} + s_{35} & 0 & 0 & s_{35} & 0 & s_{23} + s_{34} + s_{35}
\end{pmatrix}.
\]

The constraint matrix \( C^{(5)} \) has rank 4. In other words, not all constraint equations are independent, and so only four linear constraints can be used to express the six Kleiss-Kuijf five-point amplitudes in terms of each other, leaving us with two independent Kleiss-Kuijf amplitudes (that is, the number of independent color-ordered amplitudes is \((n-3)!\)). The null eigenvectors of matrix \( C^{(5)} \) (the elements of the kernel of \( C^{(5)} \)) can be constructed by hand:

\[
\begin{pmatrix}
  -s_{13}(s_{34} + s_{45})/s_{34}s_{12} - s_{45}s_{13}/s_{34}s_{25} - s_{45}(s_{23} + s_{35})/s_{34}s_{25} - s_{45}s_{13}/s_{34}s_{12} & 0 & 1 \\
  -s_{35}s_{14}/s_{34}s_{12} - s_{35}(s_{24} + s_{45})/s_{34}s_{25} - s_{35}s_{14}/s_{34}s_{25} - s_{14}(s_{34} + s_{35})/s_{34}s_{12} & 1 & 0
\end{pmatrix}.
\]

Of course, it is not a coincidence that the dimensionality of the null space of the matrix \( C^{(5)} \) and the rank of \( M^{(5)} \) are the same. The null eigenvectors of \( M^{(5)} \) can be found by solving \( M^{(5)}|\lambda^0\rangle = 0 \), which is identical with

\[
\langle \lambda^0 | M^{(5)} = 0,
\]

since \( M^{(5)} \) is symmetric. The existence of \( |\lambda^0\rangle \) implies that the amplitudes must obey a set of linear relations which follow from

\[
\langle \lambda^0 | A = 0.
\]

Fortunately, once one of the null eigenvectors is found, the others are obtained by permutations of indices. Let us assume that there is such a null vector of the form \( |\lambda^0_1\rangle = |a_1, a_2, a_3, 0, 0, 0\rangle \). Solving \( M^{(5)}|\lambda^0_1\rangle = 0 \) yields

\[
|\lambda^0_1\rangle = |s_{12}s_{45}, -s_{14}(s_{24} + s_{25}), s_{13}s_{24}, 0, 0, 0\rangle.
\]

The next null vector is of the form \( |\lambda^0_2\rangle = |b_1, b_2, 0, b_4, 0, 0\rangle \). We notice that \( a_3 = s_{13}s_{24} \) is the entry with indices 13425 in the Kleiss-Kuijf basis, and that \( b_4 \) corresponds to the
entry 12435=53421. Therefore this null vector can be obtained from $|\lambda_0^0\rangle$ by interchanging the indices 1 and 5. This gives

$$a_1 = s_{12}s_{45}, \quad 12345 \leftrightarrow 52341 = 14325 \quad b_2 = s_{25}s_{14}$$
$$a_2 = -s_{14}(s_{24} + s_{25}), \quad 14325 \leftrightarrow 54321 = 12345 \quad b_1 = -s_{45}(s_{24} + s_{12})$$
$$a_3 = s_{13}s_{24}, \quad 13425 \leftrightarrow 12435 \quad b_4 = s_{35}s_{24},$$

and so

$$|\lambda_2^0\rangle = | -s_{45}(s_{12} + s_{24}), s_{14}s_{25}, 0, s_{35}s_{24}, 0, 0\rangle.$$  \hspace{1cm} (3.14)

Similar arguments lead to the identification of the other two null eigenvectors:

$$|\lambda_3^0\rangle = |s_{12}s_{45}, -s_{25}(s_{14} + s_{24}), 0, 0, s_{35}s_{24}, 0\rangle$$ \hspace{1cm} (3.16)

$$|\lambda_4^0\rangle = | -s_{12}(s_{24} + s_{45}), s_{14}s_{25}, 0, 0, 0, s_{13}s_{24}\rangle.$$ \hspace{1cm} (3.17)

This set of null eigenvectors will be used in the next section.

## 4 Gauge Freedom

We have discussed the structure of the null eigenvectors of $M$ in the last section. A natural question may be what about the eigenvectors with non-zero eigenvalues. As one has

$$M = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|,$$ \hspace{1cm} (4.1)

one may suspect that these eigenvalues and vectors with $\lambda_i \neq 0$ may be essential in understanding the structure of the space of tree level amplitudes. For example, what linear combinations of the numerators should one be interested in to yield the amplitudes. However, this intuitive reasoning is not particularly rewarding, because both $\lambda_i$ and $|\lambda_i\rangle$ can have non-analytic dependence on kinematical invariants $s_i$. They must of course cancel out in any expression of physical significance, but their presence strongly indicates that we should seek a different path.

It turns out that the null eigenvectors are there to yield most of the information of this kind that we want to obtain. As we pointed out, we are free to add $\sum f_i |\lambda_i^0\rangle$ to $|N\rangle$. By adjusting these gauge functions $f_i$, we are able to isolate the combinations of $n_i$ which are relevant to the amplitudes and obtain relations amongst them. Furthermore, we can construct higher point $n_i$ recursively via BCFW continuation with lower point ones. The difference as it turns out is a gauge transformation. This opens up a new way to construct recursively an effective Lagrangian to study loops. What has to be done is of course to extend the on-shell gauge freedom to off-shell, which will be briefly touched on in the Concluding Remarks section.
4.1 Four Particle Amplitudes

To discuss gauge freedom, we begin by rewriting the color-dressed amplitude. We shall drop a common factor proportional to powers of the coupling constant. Thus,

\[ A^{(4)} = \frac{n(12; 34)c(1234)}{s_{12}} + \frac{n(13; 42)c(1342)}{s_{13}} + \frac{n(14; 23)c(1423)}{s_{14}}, \] (4.2)

where the color factors are

\[ c(1234) = \sum_g f_{a_1a_2g}f_{a_3a_4g}, \text{ etc.}, \] (4.3)

being the totally antisymmetric structure constants of the symmetry group. The antisymmetry and the Jacobi identity of these constants

\[ \sum_g \left( f_{a_1a_2g}f_{a_3a_4g} + f_{a_2a_3g}f_{a_1a_4g} + f_{a_3a_1g}f_{a_2a_4g} \right) = 0, \] (4.4)

are written into

\[ c(jikl) = -c(ijkl), \quad c(lkji) = c(ijkl), \quad c(ijkl) + c(jkil) + c(kijl) = 0, \] (4.5)

just as the \( n(ij; kl) \)'s. Simple algebra then gives

\[ A^{(4)} = \langle C^{(4)}|M^{(4)}|N^{(4)} \rangle, \] (4.6)

in which

\[ \langle C^{(4)} \rangle = \langle c(1234) c(1324) \rangle, \quad \langle N^{(4)} \rangle = \langle n(12; 34) n(13; 24) \rangle, \] (4.7)

and \( M^{(4)} \) is given in eq. (2.3).

The four graviton amplitude, according to the the squaring hypothesis, can be rearranged into

\[ A^{(4)}_{gr} = \frac{\tilde{n}(12; 34)n(12; 34)}{s_{12}} + \frac{\tilde{n}(13; 42)n(13; 42)}{s_{13}} + \frac{\tilde{n}(14; 23)n(14; 23)}{s_{14}} \]

\[ = \langle \tilde{N}^{(4)}|M^{(4)}|N^{(4)} \rangle, \] (4.8)

with

\[ \langle \tilde{N}^{(4)} \rangle = \langle \tilde{n}(12; 34) \tilde{n}(13; 24) \rangle. \] (4.9)

This rearrangement is again made possible by using the Jacobi identities of the \( n \)'s and \( \tilde{n} \)'s.
As already pointed out, $M^{(4)}$ has one null eigenvector. We may choose, for example
\[ \langle \lambda^0 | = \langle -s_{12} s_{13} |. \] (4.10)

Then, one can add
\[ |\delta N^{(4)} \rangle = f |\lambda^0 \rangle \] (4.11)
to the color-ordered amplitude equation
\[ |A^{(4)} = M^{(4)} |N^{(4)} \rangle \rightarrow |A^{(4)} \rangle = M^{(4)}(|N^{(4)} \rangle + |\delta N^{(4)} \rangle), \] (4.12)
where $f$ is an arbitrary function in general. However, depending on what issues we are interested in, we should choose $|\lambda^0 \rangle$ and $f$ accordingly. For example, if we choose
\[ f = -\frac{n(13; 24)}{s_{13}}, \] (4.13)
then
\[ \langle N^{(4)} | + \langle \delta N^{(4)} | = \langle n'_1 = n(12; 34) + \frac{s_{12}}{s_{13}} n(13; 24) 0 |, \] (4.14)
which gives
\[ A(1234) = -\frac{s_{13}}{s_{12}s_{14}} n'_1, \quad A(1324) = -\frac{1}{s_{14}} n'_1, \] (4.15)
or
\[ A(1234) = \frac{s_{13}}{s_{12}} A(1324), \] (4.16)
which implies that there is only one independent color-ordered amplitude.

Note that once we have fixed a particular set of numerators as basis, in our case $n(12; 34)$ and $n(13; 24)$, all others must be expressed in terms of them in making gauge transformations. Therefore, since
\[ n(13; 42) = -n(13; 24), \quad n(14; 23) = n(13; 24) - n(12; 34), \] (4.17)
we infer that the shifts of eq. (4.11)
\[ \delta n(12; 34) = -s_{12} f, \quad \delta n(13; 24) = s_{13} f, \] (4.18)
should give also
\[ \delta n(13; 42) = -s_{13} f, \quad \delta n(14; 23) = -s_{14} f. \] (4.19)

It is illuminating to check gauge invariance of the amplitudes by substituting eqs. (4.18-4.19) directly into the variation of eq. (4.2)
\[ \delta \mathbf{A}^{(4)} = -f (c(1234) + c(1342) + c(1423)) = 0, \] (4.20)
due to Jacobi identity of the last three indices. Similarly,

\[
\delta A^{(4)}_{\text{gr}} = -f(\tilde{n}(12; 34) + \tilde{n}(13; 42) + \tilde{n}(14; 23)) \\
- \tilde{f}(n(12; 34) + n(13; 42) + n(14; 23)) \\
+ f\tilde{f}(s_{12} + s_{13} + s_{14}),
\]  

(4.21)

upon making a similar gauge transformation on \(\tilde{n}\)'s. Each term in eq. (4.21) vanishes individually, because of the Jacobi identities of \(\tilde{n}\)'s, \(n\)'s and momentum conservation, respectively. They are made possible, because the kinematical invariants \(s_{12}, s_{13},\) and \(s_{14}\) so aptly generated by the null vector and then multiplied to \(f\) in eqs. (4.18-4.19) cancel out the propagators in eqs. (4.2) and (4.8). This is a general feature for higher point amplitudes also.

4.2 Five Particle Amplitudes

The color-dressed five gluon amplitude is

\[
A^{(5)} = \frac{c(12345)n(12; 3; 45)}{s_{12}s_{45}} + \frac{c(23451)n(23; 4; 51)}{s_{23}s_{51}} + \frac{c(34512)n(34; 5; 12)}{s_{34}s_{12}} \\
+ \frac{c(45123)n(45; 1; 23)}{s_{45}s_{23}} + \frac{c(51234)n(51; 2; 34)}{s_{51}s_{34}} + \frac{c(14325)n(14; 3; 25)}{s_{14}s_{25}} \\
+ \frac{c(32514)n(32; 5; 14)}{s_{32}s_{14}} + \frac{c(25143)n(25; 1; 43)}{s_{25}s_{43}} + \frac{c(13425)n(13; 4; 25)}{s_{13}s_{25}} \\
+ \frac{c(42513)n(42; 5; 13)}{s_{42}s_{13}} + \frac{c(51342)n(51; 3; 42)}{s_{51}s_{42}} + \frac{c(12435)n(12; 4; 35)}{s_{12}s_{35}} \\
+ \frac{c(35124)n(35; 1; 24)}{s_{35}s_{24}} + \frac{c(14235)n(14; 2; 35)}{s_{14}s_{35}} + \frac{c(13245)n(13; 2; 45)}{s_{13}s_{45}},
\]

(4.22)

where the color factors are

\[
c(ijklm) = \sum_{b,c} f_{a,b,c} f_{b,a,c} f_{c,a},
\]

(4.23)

which satisfy the same identities as \(n(ij; k; lm)\)

\[
c(jiklm) = -c(ijklm), \quad c(mlkji) = -c(ijklm), \\
c(ijklm) + c(jiklm) + c(kijlm) = 0.
\]

(4.24)

Using these, we can relate the color-ordered amplitudes of eq.(3.6) \(|A^{(5)}\rangle = M^{(5)}|N^{(5)}\rangle\) to color-dressed

\[
A^{(5)} = \langle C^{(5)}|A^{(5)}\rangle = \langle C^{(5)}|M^{(5)}|N^{(5)}\rangle,
\]

(4.25)
with
\[ \langle C^{(5)} | = \langle c(12345) \, c(14325) \, c(13425) \, c(12435) \, c(14235) \, c(13245) \rangle. \]  

(4.26)

Similarly, the five graviton amplitude is
\[ A_{gr}^{(5)} = \langle \tilde{N}^{(5)} | A^5 \rangle = \langle \tilde{N}^{(5)} | M^{(5)} | N^{(5)} \rangle, \]  

(4.27)

with the same ordering of indices for \( \tilde{N}^{(5)} \) as in \( N^{(5)} \), namely
\[ \langle \tilde{N}^{(5)} | = \langle \tilde{n}(12; 3; 45) \, \tilde{n}(14; 3; 25) \, \tilde{n}(13; 4; 25) \, \tilde{n}(12; 4; 35) \, \tilde{n}(14; 2; 35) \, \tilde{n}(13; 2; 45) \rangle. \]  

(4.28)

In a recursive proof of the squaring hypothesis to go from \( n = 4 \) to \( n - 5 \) graviton amplitudes, Bern et al explicitly obtained a gauge arbitrariness of the numerators. We would like to show that the form of the gauge transformations which warrants gauge invariance at each channel is predicated by the null eigenvectors of \( M^{(5)} \). For this purpose, it is convenient to take the four independent null eigenvectors as in eqs. (3.13, 3.15, 3.16 and 3.17).

The induced shifts on the numerators corresponding to
\[ |\delta N^5\rangle = \sum_{i=1}^{4} f_i |\lambda_i^0\rangle \]  

are
\[
\begin{align*}
\delta n(12; 3; 45) &= s_{12}s_{45}(f_1 - f_2 + f_3 - f_4) - s_{24}(s_{45}f_2 + s_{12}f_4), \\
\delta n(14; 3; 25) &= -s_{14}s_{25}(f_1 - f_2 + f_3 - f_4) - s_{24}(s_{14}f_1 + s_{25}f_3), \\
\delta n(13; 4; 25) &= s_{13}s_{24}f_1, \\
\delta n(12; 4; 35) &= s_{24}s_{35}f_2, \\
\delta n(14; 2; 35) &= s_{24}s_{35}f_3, \\
\delta n(13; 2; 45) &= s_{13}s_{24}f_4.
\end{align*}
\]  

(4.30)

Let us consider BCFW complexification of the amplitudes so that we can relate five-point graviton amplitude to three and four-point amplitudes where squaring hypothesis is known to be true. We shift the momenta \( p_1 \) and \( p_5 \) according to
\[ |\hat{1}\rangle = |1\rangle - z|5\rangle, \quad |\hat{5}\rangle = |5\rangle + z|1\rangle, \]  

(4.31)

which will give rise to poles in the complex \( z \)-plane due to the vanishing of \( \hat{s}_{12}, \hat{s}_{13}, \hat{s}_{14}, \hat{s}_{25}, \hat{s}_{35}, \) and \( \hat{s}_{45} \). We call these channels and examine the changes due to \( \delta n \) on the
color-dressed gluon amplitude and the graviton amplitude at each channel. For \( \hat{s}_{12} = 0 \), eq. (4.22) informs us that the change is

\[
\delta A^{(5)} = \frac{1}{s_{12}} \left[ c(12345) \delta \hat{n}(12; 3; 45) + \frac{c(3512) \delta \hat{n}(12; 5; 12)}{\hat{s}_{34}} + \frac{c(12435) \delta \hat{n}(12; 4; 35)}{\hat{s}_{35}} \right]
\]

(4.32)

According to eq. (4.30), we should have

\[
\begin{align*}
\delta \hat{n}(12; 3; 45) &= -s_{24} \hat{s}_{45} \hat{f}_2, \\
\delta \hat{n}(12; 4; 53) &= -\delta \hat{n}(12; 3; 45) = -s_{24} \hat{s}_{35} \hat{f}_2, \\
\delta \hat{n}(12; 5; 34) &= -\delta \hat{n}(12; 3; 45) - \delta \hat{n}(12; 4; 53) \\
&= s_{24}(\hat{s}_{45} + \hat{s}_{35}) \hat{f}_2 = -s_{24} \hat{s}_{34} \hat{f}_2.
\end{align*}
\]

(4.33)

Putting these into eq. (4.32), we have

\[
\delta A^{(5)} = -\frac{s_{24}}{s_{12}} [c(12345) + c(12534) + c(12453)] \hat{f}_2 = 0
\]

(4.34)

in view of a color Jacobi identity.

As for the five graviton amplitude, we replace the \( c(ijklm) \) in eq. (4.22) with \( \tilde{n}(ij; k; lm) \), whose variation is:

\[
\begin{align*}
\delta A^{(5)}_{gr} &= \frac{1}{s_{12}} \left[ \tilde{n}(12; 3; 45) \delta \tilde{n}(12; 3; 45) + \frac{\tilde{n}(12; 5; 34) \delta \tilde{n}(12; 5; 34)}{\hat{s}_{34}} + \frac{\tilde{n}(12; 4; 53) \delta \tilde{n}(12; 4; 53)}{\hat{s}_{35}} \right. \\
&+ \frac{\delta \tilde{n}(12; 3; 45) \tilde{n}(12; 3; 45)}{\hat{s}_{45}} + \frac{\delta \tilde{n}(12; 5; 34) \tilde{n}(12; 5; 34)}{\hat{s}_{34}} + \frac{\delta \tilde{n}(12; 4; 53) \tilde{n}(12; 4; 53)}{\hat{s}_{35}} \\
&+ \left. \frac{\delta \tilde{n}(12; 3; 45) \delta \tilde{n}(12; 3; 45)}{\hat{s}_{45}} + \frac{\delta \tilde{n}(12; 5; 34) \delta \tilde{n}(12; 5; 34)}{\hat{s}_{34}} + \frac{\delta \tilde{n}(12; 4; 53) \delta \tilde{n}(12; 4; 53)}{\hat{s}_{35}} \right]
\end{align*}
\]

(4.35)

whereby each term vanishes on account of Jacobi identities and momentum conservation, just as in the four particle case.
Let us briefly run through the other channels. For $\hat{s}_{13} = 0$, the relevant changes in numerators are

$$
\begin{align*}
\delta \hat{n}(13; 4; 25) &= \hat{s}_{13} s_{24} \hat{f}_1 = 0, \\
\delta \hat{n}(13; 2; 45) &= \hat{s}_{13} s_{24} \hat{f}_4 = 0, \\
\delta \hat{n}(42; 5; 13) &= -\delta \hat{n}(13; 2; 45) + \delta \hat{n}(13; 4; 25) = 0,
\end{align*}
$$

(4.36)

for $\hat{f}_1$ and $\hat{f}_4$ which are chosen not to blow up at $\hat{s}_{13} = 0$. In other words, a gauge transformation is not needed.

For $\hat{s}_{14} = 0$, eqs.(4.2), (4.9) and (4.30) instruct that we need the gauge shifts

$$
\begin{align*}
\delta \hat{n}(14; 3; 25) &= -s_{24} \hat{s}_{25} \hat{f}_3, \\
\delta \hat{n}(14; 2; 35) &= s_{24} \hat{s}_{35} \hat{f}_3, \\
\delta \hat{n}(32; 5; 14) &= -\delta \hat{n}(14; 2; 35) + \delta \hat{n}(14; 3; 25) \\
&= -s_{24} (\hat{s}_{35} + \hat{s}_{25}) \hat{f}_3 = s_{24} s_{23} \hat{f}_3.
\end{align*}
$$

(4.37)

The factors $\hat{s}_{25}$, $\hat{s}_{35}$ and $s_{23}$ are there to cancel out the matching propagators in $\delta A^{(5)}$ and $\delta A_{gr}^{(5)}$ so that we can use Jacobi identities and momentum conservation to obtain gauge invariance.

Along the same vein, for $\hat{s}_{25} = 0$, the gauge shifts are proportional to $\hat{f}_1$ with the matching factors $\hat{s}_{14}$, $\hat{s}_{13}$ and $s_{34}$ to facilitate gauge invariance in that channel. As for $\hat{s}_{35} = 0$, all the necessary shifts in $n_i$ are proportional to $\hat{s}_{35}$ and therefore we don’t have a gauge shift.

The last channel $\hat{s}_{45} = 0$ deserves a bit more exposition, because it was explicitly worked out by Bern et al when they constructed $n_i$ for five particles from those for three and four. Here

$$
\delta A^{(5)} = \frac{1}{s_{45}} \left[ \frac{c(12345) \delta \hat{n}(12; 3; 45)}{\hat{s}_{12}} + \frac{c(45123) \delta \hat{n}(45; 1; 23)}{s_{23}} + \frac{c(13245) \delta \hat{n}(13; 2; 45)}{\hat{s}_{13}} \right].
$$

(4.38)

In a familiar way by now, we find the shifts to be

$$
\begin{align*}
\delta \hat{n}(12; 3; 45) &= -s_{24} \hat{s}_{12} \hat{f}_4, \\
\delta \hat{n}(13; 2; 45) &= s_{24} \hat{s}_{13} \hat{f}_4, \\
\delta \hat{n}(45; 1; 23) &= -s_{24} s_{23} \hat{f}_4.
\end{align*}
$$

(4.39)

Thereupon

$$
\delta A^{(5)} = -\frac{s_{24}}{s_{45}} (c(12345) - c(45123) - c(13245)) \hat{f}_4,
$$

(4.40)

which is zero upon manipulating the color Jacobi identities a bit. Similarly, one has $\delta A_{gr}^{(5)} = 0$.  

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What we would like to note in this exercise is that the four different gauge functions \( f_i \) have their respective roles in different channels to activate non-trivial gauge transformations. What is more remarkable is that Bern et al found that the arbitrariness in constructing the five particle \( n \)’s from the three and four particle ones is a gauge transformation, which has the form of eq. (4.39). They gave a definite expression for \(- s_{24} \hat{f}_4\), but it depends on the \( n_i \)’s chosen by them.

There is a particular choice of \( f_i \) which will exhaust the gauge freedom for the shifted numerators. This is

\[
\begin{align*}
  f_1 &= -\frac{n(13; 4; 25)}{s_{13}s_{24}}, \quad f_2 = -\frac{n(12; 4; 35)}{s_{24}s_{35}}, \quad f_3 = -\frac{n(14; 2; 35)}{s_{24}s_{35}}, \quad f_4 = -\frac{n(13; 2; 45)}{s_{13}s_{24}},
\end{align*}
\]

which gives

\[
\langle N'^{(5)} | = \langle n'_1, \ n'_2, \ 0, \ 0, \ 0, \ 0 |,
\]

where

\[
\begin{align*}
  n'_1 &= n(12; 3; 45) - n(13; 425) \frac{s_{12}s_{45}}{s_{13}s_{24}} + n(12; 4; 35) \frac{s_{45}(s_{12} + s_{24})}{s_{24}s_{35}}, \\
  &\quad - n(14; 2; 35) \frac{s_{12}s_{45}}{s_{24}s_{35}} + n(13; 2; 45) \frac{s_{12}(s_{24} + s_{45})}{s_{13}s_{24}}, \\
  n'_2 &= n(14; 3; 25) + n(13; 425) \frac{s_{14}(s_{24} + s_{25})}{s_{13}s_{24}} - n(12; 4; 35) \frac{s_{14}s_{25}}{s_{24}s_{35}} \\
  &\quad + n(14; 2; 35) \frac{s_{25}(s_{14} + s_{24})}{s_{24}s_{35}} - n(13; 2; 45) \frac{s_{14}s_{25}}{s_{13}s_{24}}.
\end{align*}
\]

Clearly, the equation

\[
| A^{(5)} \rangle = M^{(5)} | N'^{(5)} \rangle \tag{4.44}
\]

tells us that we can solve for \( n'_1 \) and \( n'_2 \) in terms of any two of the six color-ordered amplitudes. For example, these may be

\[
\begin{align*}
  n'_1 &= s_{12}(s_{25}A(13425) - (s_{15} + s_{25})A(12435)), \\
  n'_2 &= s_{25}(- (s_{12} + s_{15})A(13425) + s_{12}A(12435)).
\end{align*}
\]

When we put these back into the right hand side of eq. (4.44), we obtain relations amongst the color ordered amplitudes, and into the equation

\[
\mathbf{A}^{(5)}_{gr} = \langle \tilde{N}'^5 | M^5 | N'^5 \rangle \tag{4.46}
\]

we have graviton amplitudes in terms of color-ordered gauge amplitudes, some KLT relations.
5 Concluding Remarks

We have identified in the context of S-matrix theory the origin of the constraints and the gauge freedom on gluon and graviton amplitudes. It is that the propagator matrix $M$ in $|A⟩ = M|N⟩$ has null eigenvectors $|λ_j^0⟩$. Their existence on the one hand reduces the number of independent entries (color-ordered amplitudes) in $|A⟩$ and on the other allows one to shift the numerator vector $|N⟩ \rightarrow |N⟩ + \sum_j f_j |λ_j^0⟩$, a set of gauge transformations. Because of the squaring hypothesis, this freedom transcends to the graviton amplitudes in the form of KLT relations.

What is even more compelling to investigate this subject is that in the recursive construction of scattering amplitudes, there is the necessity and the freedom to perform a gauge transformation to go from $n$ to $n+1$ particles. While globally the effects are nil on the fully color dressed gluon amplitudes and graviton amplitudes, which is meant by gauge invariance, it has profound implication on the semi-local characterization of interactions, particularly that aspect which has to do with the duality symmetry between color kinematics and numerator dynamics. We have in mind the promotion of the gauge functions $f_j$ into operator functionals, or effective Lagrangian. For example, for the five gluon scattering, eqs.(4.34), (4.40) and others inform us that the effective Lagrangian must yield for each channel $s_{ij}$

$$\delta^4(\sum_a p_a)\delta A^{(5)} = \delta^4(\sum_a p_a) \frac{1}{s_{ij}} (c(ijklm) + c(ijlmk) + c(ijmk)) f_{ijklm}$$

$$\sim \langle 0 | \int d^4x L_{eff}^{(5)}(x) | five \; gluon \; state \rangle,$$

where we have replaced $-s_{24}\hat{f}_j$ by $f_{ijklm}$ with a more elaborate set of indices to amplify the color and other attributes. From dimensional counting and the factorization of each channel into $2 \rightarrow 3$, it is not hard to convert the above up to a constant into the $L_{eff}^{(5)}(x)$ of Bern et al [2]. This approach takes on additional significance if through it one can unravel the meaning of the conventional highly non-local non-polynomial gravitational Lagrangians in perturbative studies. Hopefully a complementary principle may ensue. We shall return to these issues.

Note Added: As we prepared our paper for publication, we saw [23] which has overlap with our work.

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A Six Gluon Amplitudes

There are 24 independent color-ordered gluon amplitudes in the Kleiss-Kuijf basis, which we denote by $A(1i_2i_3i_4i_56)$ with $(i_2, i_3, i_4, i_5)$ equal to a permutation of indices $(2,3,4,5)$. Correspondingly, there are 24 independent numerators which we denote by $n(1j; k; l; m6)$, where as in sections 2 and 3, the external gluons which share a vertex are denoted by numerals which are not separated by a semi-column. The ordering of the gluons is clockwise. We use the shorthand notation

$$
n(12; 3; 4; 56) = n_1, \quad n(13; 2; 4; 56) = n_2, \quad n(13; 4; 2; 56) = n_3, \quad n(13; 4; 5; 26) = n_4 \\
n(12; 4; 3; 56) = n_5, \quad n(14; 2; 3; 56) = n_6, \quad n(14; 3; 2; 56) = n_7, \quad n(14; 3; 5; 26) = n_8 \\
n(12; 5; 4; 36) = n_9, \quad n(15; 2; 4; 36) = n_{10}, \quad n(15; 4; 2; 36) = n_{11}, \quad n(15; 4; 3; 26) = n_{12} \\
n(12; 3; 5; 46) = n_{13}, \quad n(13; 2; 5; 46) = n_{14}, \quad n(13; 5; 2; 46) = n_{15}, \quad n(13; 5; 4; 26) = n_{16} \\
n(12; 4; 5; 36) = n_{17}, \quad n(14; 2; 5; 36) = n_{18}, \quad n(14; 5; 2; 36) = n_{19}, \quad n(14; 5; 3; 26) = n_{20} \\
n(12; 5; 3; 46) = n_{21}, \quad n(15; 2; 3; 46) = n_{22}, \quad n(15; 3; 2; 46) = n_{23}, \quad n(15; 3; 4; 26) = n_{24}. \quad (A.1)
$$

Each of the 24 amplitudes can be written as a sum over 14-type Feynman diagrams. For example, for $A(123456)$ we have

$$
A(123456) = \frac{n_1}{s_{12}s_{13}s_{1234}} - \frac{n(12; 3; 6; 45)}{s_{12}s_{13}125} - \frac{n(12; 6; 3; 45)}{s_{12}s_{13}125} \\
+ \frac{n(12; 2; 3; 45)}{s_{16}s_{162}s_{1623}} + \frac{n(12; 6; 5; 34)}{s_{16}s_{162}s_{165}} + \frac{n(23; 4; 5; 61)}{s_{23}s_{234}s_{2345}} \\
- \frac{n(12; 34; 56)}{s_{34}s_{345}s_{3456}} - \frac{n(12; 3; 6; 45)}{s_{34}s_{345}s_{3456}} + \frac{n(12; 6; 3; 45)}{s_{34}s_{345}s_{3456}}, \quad (A.2)
$$

where the last two terms correspond to snow-flake Feynman diagrams, as their pole structure indicates. Using that the numerators obey Jacobi-type identities:

$$
n(ij; k; l; mn) = -n(ji; k; l; mn) = -n(ij; k; l; nm), \\
n(ij; k; l; mn) + n(ij; k; m; nl) + n(ij; k; n; lm) = 0, \\
n(ij; kl; mn) = n(ij; k; l; mn) - n(ij; l; k; mn), \quad (A.3)
$$

together with the time-reversed identity

$$
n(ij; k; l; mn) = n(nm; l; k; ji), \quad (A.4)
$$
and using that an external line joined with two internal lines at a vertex may be flipped under the internal lines at the price of an additional factor of \((-1)\), we can express the numerators in the color-ordered amplitude \(A(123456)\) in the chosen basis of independent numerators as follows:

\[
\begin{align*}
n(12; 3; 6; 45) &= -n_1 + n_{13} \\
n(12; 6; 3; 45) &= -n_1 - n_9 + n_{13} + n_{17} \\
n(61; 2; 3; 45) &= n_1 - n_4 + n_9 - n_{12} - n_{13} + n_{16} - n_{17} + n_{20} \\
n(12; 6; 5; 34) &= n_1 - n_5 + n_9 - n_{21} \\
n(23; 4; 5; 61) &= n_1 - n_2 - n_6 + n_7 + n_{11} - n_{12} - n_{22} + n_{23} \\
n(23; 4; 1; 56) &= -n_1 + n_2 + n_6 - n_7 \\
n(23; 1; 4; 56) &= -n_1 + n_2 \\
n(34; 2; 1; 56) &= n_1 - n_3 - n_5 + n_7 \\
n(34; 5; 2; 61) &= -n_1 + n_4 + n_5 - n_8 - n_9 + n_{12} + n_{21} - n_{24} \\
n(34; 2; 5; 61) &= -n_1 + n_3 + n_5 - n_7 - n_{10} + n_{12} + n_{22} - n_{24} \\
n(23; 1; 6; 45) &= n_1 - n_2 - n_{13} + n_{14} \\
n(12; 34; 56) &= n_1 - n_5 \\
n(61; 23; 45) &= n_1 - n_2 + n_{11} - n_{12} - n_{13} + n_{14} - n_{19} + n_{20}.
\end{align*}
\]  

(A.5)

Similar expressions may be found for the other elements of the Kleiss-Kuijf basis by permutation of indices, and the full propagator matrix \(M^{(6)}\) can be constructed in this way. It can be verified that the constraint matrix \(C^{(6)}\) has six null eigenvectors (while the expressions of these null eigenvectors may be found analytically, they are not particularly illuminating, and we do not give them here), and so, only 18 of the constraints are independent. The propagator matrix \(M^{(6)}\) has rank 6 (and thus equal to \((n - 3)!\)). We have separately computed the 18 null eigenvectors of the propagator matrix. Based on our previous discussion of the 4-point and 5-point amplitudes, these null eigenvectors can be chosen such that each 24-component vector has nontrivial entries among the 1,2,3,5,6,7 set in the chosen Kleiss-Kuijf basis \([A.1]\), plus one (and only one) more non-trivial component. For example,

\[
\begin{align*}
| & - (s_{13} + s_{23})(s_{36} + s_{34} + s_{45} + s_{46}) , s_{13}(s_{36} + s_{34} + s_{45} + s_{46}) , s_{13}(-s_{14} + s_{36}) , 0 , \\
& (s_{14} + s_{24})s_{35} , s_{14}s_{35} , (s_{23} + s_{35})s_{14} , 0 , s_{125}s_{36} , 0 , \ldots , 0) \\
| & s_{12} , s_{23} + s_{12} , s_{12} + s_{23} + s_{24} , -s_{26} , 0 , \ldots , 0 \rangle.
\end{align*}
\]  

(A.6)

are such null eigenvectors. For a concrete 6-point MHV amplitude, similar to the 5-point amplitude discussed by Bern et al. \([2]\), we can check that the shifts induced by the the
null eigenvectors are those needed to reconstruct the 6-point amplitude BCJ numerators (namely those in (A.1)) via BCFW shifts.

The fact that the null eigenvectors induce generalized gauge transformations at each channel is a consequence of \( \sum_j \hat{s}_\alpha \hat{M}_{ij}(z) \delta \hat{n}_j(z) \bigg|_{z = z_\alpha} = 0 \), for each channel \( \alpha \). This is true for any \( n \)-point amplitude. In principle, the arbitrary functions \( f_j \) which multiply the null eigenvector shifts can be appropriately chosen for each channel.

Given the pattern that the numerator matrix exhibits (sums over products of propagators of common \( n - 3 \)-tuples formed from adjacent gluon lines between two amplitudes in the Kleiss-Kuijf amplitude basis, with overall sign factors for certain entries: e.g. \( M^{(5)}_{12} = 1/(s_{15}s_{152}) + 1/(s_{15}s_{154}) \) for the numerator entry corresponding to \( 1' = A(12345) \) and \( 2' = A(14325) \) etc) one would hope that it is possible to write the structure of the null eigenvectors for a generic \( n \)-point numerator matrix. We plan to return to this question.

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