Dual representation of expectile-based expected shortfall and its properties

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Abstract  An expectile can be considered a generalization of a quantile. While expected shortfall is a quantile-based risk measure, we study its counterpart—the expectile-based expected shortfall—where expectile takes the place of a quantile. We provide its dual representation in terms of a Bochner integral. Among other properties, we show that it is bounded from below in terms of the convex combination of expected shortfalls, and also from above by the smallest law invariant, coherent, and comonotonic risk measures, for which we give the explicit formulation of the corresponding distortion function. As a benchmark to the industry standard expected shortfall, we further provide its comparative asymptotic behavior in terms of extreme value distributions. Based on these results, we finally explicitly compute the expectile-based expected shortfall for selected classes of distributions.

Keywords  Expectile, Expected shortfall, Tail conditional expectation, Dual representation, Coherent risk measure

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1. Introduction

Many risk measures proposed for the quantification of financial risks are given either directly or indirectly in terms of the quantile of the distribution of the loss profile. This includes the value at risk \( q_\alpha(L) := \inf \{m : P[L \leq m] \geq \alpha \} \) and the derivation of it is similar to the tail conditional expectation and the expected shortfall, respectively,

\[
TCE_\alpha(L) := E [L|L > q_\alpha(L)] \quad \text{and} \quad ES_\alpha(L) := \frac{1}{1 - \alpha} \int_\alpha^1 q_u(L)du.
\]

While the expected shortfall is coherent in the sense of Artzner et al. [2], the value at risk and tail conditional expectation are not sub-additive and hence not coherent, see [14, 29]. For diversification purposes, the expected shortfall is therefore preferred to these two quantile-based risk measures. However, the expected shortfall is not elicitable. It has been recently discussed that elicitability is
a useful property from a backtesting viewpoint, see Chen [7], Emmer et al. [13], Gneiting [15], Ziegel [32]. In terms of elicitability and coherency, the expectile $e_\alpha$ which was first introduced by Newey and Powell [25] and defined as the unique solution of
\[
\alpha E \left[ (L - e_\alpha(L))^+ \right] = (1 - \alpha) E \left[ (L - e_\alpha(L))^- \right]
\]
is the only alternative coherent law invariant risk measure which is elicitable as shown by Weber [31], Ziegel [32], and Bellini and Bignozzi [3].

As discussed by Bellini et al. [5], the expectile can be seen as a generalization of the quantile. We therefore revisit the former quantile-based risk measure by considering the expectile instead of quantile, namely,

\[
\text{tce}_\alpha(L) := E [L | L > e_\alpha(L)] \quad \text{and} \quad \text{es}_\alpha(L) := \frac{1}{1 - \alpha} \int_0^1 e_u(L)du,
\]

hereafter referred to as expectile-based tail conditional expectation and expectile-based expected shortfall, respectively.

The notion of expectile-based tail conditional expectation $\text{tce}_\alpha$ and expected shortfall $\text{es}_\alpha$ is relatively new. Expectile-based tail conditional expectation was first introduced in Taylor [30] for the estimation of the expected shortfall from the expectile for a loss profile with continuous distribution. Though it is positive homogeneous and cash invariant, however, it is not monotone and sub-additive in general, see Daouia et al. [9]. For this reason, Daouia et al. [9] criticized the estimation $\text{ES}_\alpha$ from $\text{tce}_\alpha$ and proposed $\text{es}_\alpha$ which is coherent. They further showed that for the Fréchet type of extreme value distribution, $\text{tce}_\alpha$ and $\text{es}_\alpha$ are asymptotically equivalent.

In this paper, we systematically study the properties of these expectile-based risk measures under the light of recent results obtained in Tadese and Drapeau [27]. Since the expectile-based expected shortfall $\text{es}_\alpha$ is a coherent risk measure, we provide its dual set in terms of a Bochner integral:

\[
\mathcal{Q}_{\text{es}} = \left\{ Y \in \mathcal{Q}: Y = \frac{1}{1 - \alpha} \int_0^1 Y(u)du \quad \text{for some} \quad Y \in \mathcal{Y} \quad \text{with} \quad \int_0^1 \|Y(u)\|_\infty du < \infty \right\},
\]

where $\mathcal{Q}$ is the set of probability densities in $L^\infty$, and $\mathcal{Y}$ is the set of strongly measurable functions $Y : (\alpha, 1) \to L^\infty$ such that $Y(u)$ is in the dual set of $e_u$ for almost every $u$. We further bound $\text{es}_\alpha$ from below in terms of convex combination of expected shortfalls, that is,

\[
\sup_{0 < \beta < 1} \{ (1 - \gamma_\beta) \text{ES}_\beta(L) + \gamma_\beta E[L] \} \leq \text{es}_\alpha(L),
\]

where $\gamma_\beta$ has an explicit expression given by Relation (4.2). Though $\text{es}_\alpha$ is not comonotonic, in the sense of Delbaen [11] we provide the smallest comonotonic risk measure dominating $\text{es}_\alpha$, that is,

\[
\text{es}_\alpha(L) \leq R_\varphi(L) := \int_0^1 \varphi'(t)q_{1-t}(L)dt,
\]

where the concave distortion function $\varphi$ is explicitly given by Relation (4.1). To compare the value of $\text{es}_\alpha$ with respect to the industry standard expected shortfall and value at risk, we provide their asymptotic relative behavior for each extreme value distribution type—Fréchet, Weibull, and Gumbel. Finally, based on the present result we provide an explicit expression for $\text{es}_\alpha$ for several classical distributions.

The rest of the paper is organized as follows. In section 2, we introduce some basic notations, and definitions of the quantile- and expectile-based risk measures. In section 3, we address the dual
representation of expectile-based expected shortfall. In section 4, we provide properties, bounds, and asymptotic results for the expectile-based expected shortfall. Section 5 illustrates those results with examples for some loss profiles with a known distribution.

2. Basic definitions and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Throughout, $L^1$ and $L^\infty$ denote the set of integrable and essentially bounded random variables identified in the $P$-almost sure sense, respectively. For each $L$ in $L^1$, $F_L$ represents its cumulative distribution. We also denote by $Q$ the set of densities in $L^\infty$ for probability measures that are absolutely continuous with respect to $P$, that is,

$$Q = \{Y \in L^\infty : Y \geq 0 \text{ and } E[Y] = 1\}.$$ 

We say $R: L^1 \to (-\infty, \infty]$ is a coherent risk measure, if $R$ is

- **Monotone**: $R(L_1) \leq R(L_2)$ whenever $L_1 \leq L_2$;
- **Cash-invariant**: $R(L - m) = R(L) - m$ for each $m$ in $\mathbb{R}$;
- **Sub-additive**: $R(L_1 + L_2) \leq R(L_1) + R(L_2)$;
- **Positive homogeneous**: $R(\lambda L) = \lambda R(L)$ for every $\lambda \geq 0$.

A coherent risk measure is further called Fatou continuous, if $R(L) \leq \liminf_n R(L_n)$ whenever $(L_n)$ is a sequence dominated in $L^1$ and converges to $L$ $P$-almost surely.

For $L$ in $L^1$, we consider the quantile-based functions

- **Value at Risk**: for $0 < \alpha < 1$,
  $$q_\alpha(L) = \inf\{m : F_L(m) \geq \alpha\}.$$ 
- **Tail Conditional Expectation**: for $0 < \alpha < 1$,
  $$TCE_\alpha(L) = E[L|L > q_\alpha(L)].$$
- **Expected Shortfall**: for $0 < \alpha < 1$,
  $$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 q_\alpha(L)du.$$ 

It is well known that the value at risk is not sub-additive—not even convex—and therefore not a coherent risk measure, see [14, 29]. While the expected shortfall is coherent and coincides with the tail conditional expectation for loss profiles with continuous distributions, in general, $ES_\alpha \geq TCE_\alpha$ and $TCE_\alpha$ may not be sub-additive, see [1, 2, 14].

For confidence level $\alpha$ in $[1/2, 1)$, the expectile $e_\alpha$ of $L$ in $L^1$ is defined as the unique solution of

$$\alpha E[(L - e_\alpha(L))^+] = (1 - \alpha)E[(L - e_\alpha(L))^-]. \quad (2.1)$$

Since we mainly concentrate on expectiles, throughout this paper we assume that $1/2 \leq \alpha < 1$.

It turns out that the expectile is a law invariant, finite-valued, and the only elicitable and coherent risk measure, see [3, 12, 31, 32]. From [5], its dual representation is given by

$$e_\alpha(L) = \max_{Y \in Q_\alpha} E[LY],$$

where

$$Q_\alpha := \left\{Y \in Q : \gamma \leq Y \leq \frac{\alpha \gamma}{1-\alpha} \text{ for some } \gamma \in \left[\frac{1-\alpha}{\alpha}, 1\right]\right\}. \quad (2.2)$$
Remark 2.1 Following [27, Proposition 3.2], it can be taken that
\[ \gamma = \frac{1 - \alpha}{\alpha + (1 - 2\alpha)\beta^*}, \]
for some \( \beta^* \) in \([P[L < e_\alpha], P[L \leq e_\alpha]]\) as a candidate for Relation (2.2). In this particular case, \( \gamma \) depends on \( \alpha \).

Since expectile can be seen as a generalization of quantile, if \( q_\alpha \) is replaced by \( e_\alpha \) in the definition of \( TCE_\alpha \) and \( ES_\alpha \), we then get the expectile-based functions on \( L^1 \) defined as
- **Expectile-based Tail Conditional Expectation**: for \( 1/2 \leq \alpha < 1 \),
  \[ tce_\alpha(L) = E[L_jL > e_\alpha(L)]. \]
- **Expectile-based Expected Shortfall**: for \( 1/2 \leq \alpha < 1 \),
  \[ es_\alpha(L) = \frac{1}{1 - \alpha} \int_\alpha^1 e_u(L)du. \]

If \( L \) has a continuous distribution, it holds that \( tce_\alpha(L) = ES_{\beta^*}(L) \), where \( \beta^* = P[L \leq e_\alpha(L)] \). It is also known that \( tce_\alpha \) is not monotone and sub-additive in general and, hence, not a coherent risk measure, see [9]. However, \( es_\alpha \) is coherent and has the following properties.

**Proposition 2.2** The expectile-based expected shortfall is law invariant, \((-\infty, \infty]\)-valued, coherent, and Fatou continuous.

**Proof** It is known that the map \( u \mapsto e_\alpha(L) \) is continuous on \([1/2, 1]\), see [5, 25]. It implies that \( e_\alpha(L) \) is measurable. Since \( E[L] \leq e_u(L) \) for each \( u \in [\alpha, 1] \), it holds that the integration is well defined and the range of \( es_\alpha \) is a subset of \((-\infty, \infty]\). The law invariance and coherent properties of \( es_\alpha \) directly follow from the expectile. Let \((L_n)\) be a sequence dominated in \( L^1 \) and converging to \( L \) almost surely. Since a finite-valued coherent risk measure is Fatou continuous, it holds that \( e_\alpha \) is also Fatou continuous, see [19]. The Fatou continuity of \( e_\alpha \) together with Fatou’s Lemma—since \( e_u \) is bounded from below by the expectation—yields
\[
es_\alpha(L) = \frac{1}{1 - \alpha} \int_\alpha^1 e_u(L)du \leq \frac{1}{1 - \alpha} \int_\alpha^1 \left( \liminf_n e_u(L_n) \right) du \leq \liminf_n \frac{1}{1 - \alpha} \int_\alpha^1 e_u(L_n)du = \liminf_n es_\alpha(L_n).
\]
This ends the proof of the proposition. \( \square \)

3. Dual representation of expectile-based expected shortfall

The expectile-based expected shortfall is law invariant, coherent, and Fatou continuous. Hence, it admits a representation of the form
\[ es_\alpha(L) = \sup_{Y \in Q es} E[LY], \]
for some \( Q es \subseteq Q \) which is called the dual set of \( es_\alpha \), see [6, 8, 19]. This section is dedicated to describe the set \( Q es \). Throughout this section, we consider the measurable space \((I, I, \mu)\), where \( I = [\alpha, 1] \), \( I \) is the Borel sigma-algebra of \( I \), and \( \mu \) is the Lebesgue measure on \( I \). We denote by \( L_s^0(L^\infty) \) the space of all step functions on \( I \) with values in \( L^\infty \) identified \( \mu \)-almost everywhere, that is,
where \( Y_n \) is a sequence in \( L^0(L^\infty) \) such that \( \|Y_n(u) - Y(u)\|_\infty \to 0 \) \( \mu \)-almost everywhere. Finally, \( L^0(I) \) denotes the space of all real-valued random variables on \( I \) identified \( \mu \)-almost everywhere. Throughout this section, all equalities and inequalities in \( L^0(I) \) are identified in the \( \mu \)-almost everywhere sense. Clearly, \( u \mapsto \|Y\|_\infty(u) \) and \( u \mapsto \langle L, Y(u) \rangle := E[LY(u)] \) with \( L \in L^1 \) and \( Y \in L^0(L^\infty) \) are in \( L^0(I) \). It also holds that \( L \mapsto e(L) \) is a function from \( L^1 \) to \( L^0(I) \).

Proposition 3.1. The expectile-based expected shortfall admits the representation

\[
es_{\alpha}(L) = \sup_{Y \in \mathcal{Y}} \frac{1}{1 - \alpha} \int_I E[LY]d\mu, \tag{3.1}
\]

where

\[
\mathcal{Y} = \left\{ Y \in L^0(L^\infty) : E[LY(u)] = 1 \text{ and } \gamma(u) \leq Y(u) \leq \frac{u\gamma(u)}{1 - u} \right\}
\]

for some \( \gamma \in L^0(I) \) such that \( \frac{1 - u}{u} \leq \gamma(u) \leq 1 \).

Furthermore, \( Y^* \) in \( \mathcal{Y} \) is optimal if and only if \( Y^*(u) \) is the optimal density of \( e_u \) for \( \mu \)-almost all \( u \) in \( I \).

Proof. As a result of Relation (2.2), it follows that \( Y(u) \in Q_u \) for \( \mu \)-almost all \( u \) in \( I \). This implies that \( e(L) \geq E[LY(\cdot)] \) for all \( Y \) in \( \mathcal{Y} \) and therefore

\[
es_{\alpha}(L) \geq \sup_{Y \in \mathcal{Y}} \frac{1}{1 - \alpha} \int_I E[LY]d\mu. \tag{3.2}
\]

For each \( n \) in \( \mathbb{N} \), we consider the partition \( \Pi^n \) of \([0, 1]\) given by

\[
\Pi^n := \left\{ t^n_k = \frac{\alpha(n-k) + k}{n} : k = 0, \cdots, n \right\}.
\]

Let

\[
\mathcal{Y}^n := \left\{ Y(u) = \sum_{k=0}^{n-1} Y^n_{t_k}(u) 1_{(t_k^n, t_{k+1}^n]}(u) : Y^n_{t_k} \in Q_{t_k^n} \text{ for all } k = 0, \cdots, n-1 \right\}.
\]

For \( Y \) in \( \mathcal{Y}^n \), it holds that \( Y \in L^0(L^\infty) \) and \( E[Y(u)] = 1 \). Since \( Y^n_{t_k} \in Q_{t_k^n} \), there exist \( \gamma^n_{t_k} \in \mathbb{R} \) such that

\[
\gamma^n_{t_k} \leq Y^n_{t_k} \leq \frac{t_k^n \gamma^n_{t_k}}{1 - t_k^n}.
\]

\footnote{When the measure \( \mu \) is finite, most literature defines strong measurability in terms of finite-valued functions. In this particular situation, this definition of measurability is equivalent to the current definition of measurability, see Hille and Phillips [16], for instance.}
for all \( k = 0, \ldots, n - 1 \). Define
\[
\gamma(u) := \sum_{k=0}^{n-1} \gamma(t^n_k, t^n_{k+1})(u).
\]

It follows that \( \gamma \) in \( L^0(I), (1-u)/u \leq \gamma(u) \leq 1 \) and \( \gamma(u) \leq Y(u) \leq u\gamma(u)/(1-u). \) Hence, \( Y^n \subseteq Y \) and Relation (3.2) further implies that
\[
es_{\alpha}(L) \geq \sup_n \left\{ \sup_{Y \in Y^n} \frac{1}{1-\alpha} \sum_{k=0}^{n-1} E[LY_{t^n_k}](t^n_{k+1} - t^n_k) \right\} = \sup_n \frac{1}{1-\alpha} \int_I e^n_u(L)d\mu, \quad (3.3)
\]
where \( e^n_u(L) := \sum_{k=0}^{n-1} e_{t^n_k}(L)(t^n_{k+1} - t^n_k) \) which is a sequence in \( L^0(I) \) such that \( e^n_u(L) \) converges to \( e(L) \) \( \mu \)-almost everywhere. The monotone convergence theorem together with Relation (3.3) yields Relation (3.1).

Let \( Y^* \) in \( Y \) be given. By the definition of \( Y \), we always have \( e_u(L) - E[LY^*(u)] \geq 0 \). If \( Y^* \) is optimal, then \( \int_I (e(L) - E[LY^*])d\mu = 0 \) and, hence, \( e(L) = E[LY] \). The converse statement is clear thus ending the proof. \( \square \)

**Remark 3.2** Following the proof of Proposition 3.1 and Remark 2.1, it can be chosen that
\[
\gamma^n(u) = \sum_{k=0}^{n-1} \frac{1 - t^n_k}{t^n_k + (1 - 2t^n_k)\beta(t^n_k)} \gamma(t^n_k, t^n_{k+1})(u),
\]
for some \( \beta(t^n_k) \in [P[L < e_{t^n_k}], P[L \leq e_{t^n_k}]] \) to approximate the random variable \( \gamma \) given in the definition of \( Y \).

Finally, to provide the dual representations of \( es_{\alpha} \), we need the Bochner integral. The step function \( Y = \sum_{n=1}^{\infty} Y_n 1_{I_n} \) in \( L^0_s(L^\infty) \) is said to be Bochner integrable with respect to the measure \( \mu \), provided that \( \int_Y ||Y||_\infty d\mu = \sum_{n=1}^{\infty} ||Y_n||_\infty \mu(I_n) < \infty \). In this case, the Bochner integral of \( Y \) is denoted by \( \int_I Yd\mu \) and given by
\[
\int_I Yd\mu := \sum_{n=1}^{\infty} Y_n \mu(I_n).
\]

A function \( Y \) in \( L^0(L^\infty) \) is also said to be Bochner integrable, if there exists a Bochner integrable sequence \( (Y_n) \) in \( L^0_s(L^\infty) \) such that \( ||Y_n(u) - Y(u)||_{\infty} \to 0 \) \( \mu \)-almost everywhere and \( \int_I ||Y_n(u) - Y(u)||_{\infty} \mu(du) \to 0 \). In this case, the Bochner integral of \( Y \) with respect to \( \mu \) is given by
\[
\int_I Yd\mu := \lim_{n \to \infty} \int_Y Y_n d\mu.
\]
It is well known that \( Y \) in \( L^0(L^\infty) \) is Bochner integrable if and only if \( \int_I ||Y(u)||_{\infty} \mu(du) \) is finite. For a Bochner integrable function \( Y \) in \( L^0(L^\infty) \), it also holds that
\[
E \left[ L \int_I Yd\mu \right] = \int_I E[LY]d\mu, \quad \text{for all } L \in L^1,
\]
see [16], for instance. With this at hand, the dual representation reads as follows.

**Theorem 3.3** The expectile-based expected shortfall \( es_{\alpha} \) admits the dual representation
\[
es_{\alpha}(L) = \sup_{Y \in \mathcal{Q}_{es}} E[LY],
\]
where \( \mathcal{Q}_{es} \) is the \( \sigma(L^\infty, L^1) \)-closure of the non-empty and convex set.
\[ \mathcal{Q}_{es} = \left\{ Y \in \mathcal{Q} : Y = \frac{1}{1 - \alpha} \int_{I} Y \, d\mu \text{ for some } Y \in \mathcal{Y} \text{ with } \int_{I} ||Y||_{\infty} \, d\mu < \infty \right\}. \]

Furthermore, \( Y^* \in \mathcal{Q}_{es} \) is optimal for \( es_{\alpha} \) if and only if \( Y^* \) in \( \mathcal{Y} \) for which \( Y^* = \frac{1}{1 - \alpha} \int_{I} Y^* \, d\mu \) is optimal for Relation (3.1).

**Remark 3.4** Note that, even if \( \mathcal{Q}_{es} \) is convex, it is not clear that it is closed. Indeed, it depends on the regularity of the set of strongly measurable functions \( \mathcal{Y} \). Furthermore, a maximizer is not always to be expected since \( es_{\alpha} \) is not necessarily finite.

**Proof** Let \( Y \) be in \( \mathcal{Q}_{es} \), that is, \( Y = \frac{1}{1 - \alpha} \int_{I} Y \, d\mu \) for some Bochner integrable function \( Y \) in \( \mathcal{Y} \). Relation (3.4) and Proposition 3.1 yield

\[ E[LY] = E \left[ \frac{1}{1 - \alpha} L \int_{I} Y \, d\mu \right] = \frac{1}{1 - \alpha} \int_{I} E[LY] \, d\mu \leq es_{\alpha}(L). \]

It follows that

\[ \sup_{Y \in \mathcal{Q}_{es}} E[LY] \leq es_{\alpha}(L). \quad (3.5) \]

For each \( Y \) in \( \mathcal{Y}^n \), it holds that

\[ \int_{I} ||Y||_{\infty} \, d\mu = \sum_{k=0}^{n-1} ||Y_{k}||_{\infty}(t_{k+1} - t_{k}) = \frac{1 - \alpha}{n} \sum_{k=0}^{n-1} ||Y_{k}||_{\infty} < \infty, \]

implying that every element of \( \mathcal{Y}^n \) is Bochner integrable. Hence, Relation (5) and (6) yield

\[ es_{\alpha}(L) = \sup_{n} \left\{ \sup_{Y \in \mathcal{Q}_{es}^n} E[LY] \right\} \leq \sup_{Y \in \mathcal{Q}_{es}} E[LY], \quad (3.6) \]

where

\[ \mathcal{Q}_{es}^n = \left\{ Y \in \mathcal{Q} : Y = \frac{1}{1 - \alpha} \int_{I} Y \, d\mu \text{ for some } Y \in \mathcal{Y}^n \right\}. \]

The last inequality follows from the fact that \( \mathcal{Q}_{es}^n \subseteq \mathcal{Q}_{es} \) for all \( n \) in \( \mathbb{N} \). Relation (3.5) and (3.6) yield

\[ es_{\alpha}(L) = \sup_{Y \in \mathcal{Q}_{es}} E[LY]. \]

Clearly, \( \mathcal{Q}_{es} \) is non-empty and a convex subset of \( \mathcal{Q} \). Hence, taking the \( \sigma(L^\infty, L^1) \)-closure \( \mathcal{Q}_{es} \) does not affect the supremum. The last assertion directly follows from Proposition 3.1.

**4. Properties of expectile-based expected shortfall**

**4.1 Comonotonicity**

A coherent risk measure \( R : L^1 \to (-\infty, \infty] \) is said to be comonotonic, if \( R(L_1 + L_2) = R(L_1) + R(L_2) \) for each comonotone pairs\(^1\) of loss profiles \( L_1 \) and \( L_2 \). It is well known that the quantile-based expected shortfall is comonotonic, while the expectile is not, see [13, 14, 26], for instance. It is therefore not astonishing that the expectile-based expected shortfall is not comonotonic as shown in the following example.

\(^1\)We say \( L_1 \) and \( L_2 \) in \( L^1 \) are comonotone, if \( (L_1(\omega) - L_1(\omega'))(L_2(\omega) - L_2(\omega')) \geq 0 \) for all \( (\omega, \omega') \in \Omega \times \Omega \).
Example 4.1 Let \( \varphi \) be the concave distortion function given by

\[
\varphi(t) = \begin{cases} 
0 & \text{if } t = 0, \\
1 - \frac{1 - \alpha}{2} & \text{if } t = 1/2, \\
-\frac{t}{1-2t} \left[ 1 - \frac{1 - t}{(1-\alpha)(1-2t)} \ln \left( 2\alpha - 1 + \frac{1-\alpha}{t} \right) \right] & \text{if } t \neq 1/2 \text{ and } 0.
\end{cases}
\] (4.1)

Then, by Example 5.1, \( \varphi \) corresponds to the distortion probability measure \( C_\varphi(A) = \text{es}_\alpha(1_A) \). For \( L \sim \text{Unif}[0,1] \), following Example 5.2, we get \( \text{es}_{0.66}(L) \approx 0.706 \). The Choquet integral of \( L \) with respect to \( C_\varphi \) is \( R_\varphi(L) = \int_0^1 \varphi(P[L > x])dx = \int_0^1 \varphi(1-x)dx \approx 0.758 \). Hence, \( R_\varphi(L) \neq \text{es}_{0.66}(L) \) and by [14, Corollary 4.95] \( \text{es}_\alpha \) is not comonotonic.

In the spirit of [11, Theorem 6], the following proposition provides the smallest comonotonic risk measure that dominates \( \text{es}_\alpha \) uniformly on \( L^1 \).

Proposition 4.2 Let \( \varphi \) be the distortion function given by Relation (4.1), it holds that

\[
\text{es}_\alpha(L) \leq R_\varphi(L) := \int_0^1 \varphi'_+(t)q_{1-t}(L)dt,
\]

where \( \varphi'_+ \) is the right-hand derivative of \( \varphi \). Moreover, \( R_\varphi \) is the smallest law invariant, coherent, and comonotonic risk measure dominating \( \text{es}_\alpha \) uniformly for each \( L \) in \( L^1 \). In particular, \( \text{es}_\alpha(1_A) = R_\varphi(1_A) = \varphi(P[A]) \) for each \( A \in \mathcal{F} \).

Proof Following [11, Theorem 6], for each \( u \) in \([1/2,1]\), \( e_u \) is dominated uniformly for each \( L \) in \( L^1 \) by the smallest law invariant, coherent, and comonotonic risk measure as

\[
e_u(L) \leq \int_0^1 \frac{u(1-u)}{((2u-1)t + 1-u)^2} q_{1-t}(L)dt.
\]

Using Fubini’s theorem yields

\[
\text{es}_\alpha(L) \leq \frac{1}{1-\alpha} \int_0^1 \left( \int_0^1 \frac{u(1-u)}{((2u-1)t + 1-u)^2} q_{1-t}(L)dt \right) du
\]

\[
= \int_0^1 \left( \frac{1}{1-\alpha} \int_0^{1-\alpha} \frac{u(1-u)}{((1-2u)t + u)^2} du \right) q_{1-t}(L)dt
\]

\[
= \int_0^1 \varphi'_+(t)q_{1-t}(L)dt,
\]

where

\[
\varphi'_+(t) = \begin{cases} 
\frac{-1}{(1-2t)^2} \left[ 1 + \frac{(1-2t)(1-\alpha)}{t + (1-2t)(1-\alpha)} - \frac{1}{(1-\alpha)(1-2t)} \ln \left( 2\alpha - 1 + \frac{1-\alpha}{t} \right) \right] & \text{for } t \neq 1/2, \\
2(1-\alpha) \left( 1 - \frac{2}{3}(1-\alpha) \right) & \text{for } t = 1/2.
\end{cases}
\]

Clearly, \( R_\varphi \) is the smallest law invariant, coherent and comonotonic risk measure that dominates \( \text{es}_\alpha \) uniformly. From Example 5.1, we also have that \( \text{es}_\alpha(1_A) = \varphi(P[A]) \). \( \square \)

4.2 Quantile-based versus expectile-based expected shortfall

For a given confidence level \( \alpha \) in \([1/2,1]\), the expectile is less conservative than expectile-based expected shortfall, that is, \( e_\alpha \leq \text{es}_\alpha \), see [9], for instance. However, as compared to the quantile-
based expected shortfall, it holds that $es_\alpha \leq ES_\alpha$ or $es_\alpha > ES_\alpha$ depending on the considered loss profile, see Figure 1.

Using the lower bounds of the expectile given in [27, Proposition 3.1], we provide a family of lower bounds for $es_\alpha$ in terms of the convex combination of expected shortfalls as follows.

**Proposition 4.3** For each $\beta$ in $(0, 1)$, it holds that

$$(1 - \gamma_\beta)ES_\beta(L) + \gamma_\beta E[L] \leq es_\alpha(L),$$

where

$$\gamma_\beta := \begin{cases} 
\frac{1 - \alpha}{2\beta - 1} - \frac{1 - \beta}{(1 - \alpha)(2\beta - 1)^2} \ln \left( 2\alpha - 1 + \frac{1 - \alpha}{1 - \beta} \right) & \text{for } \beta = 1/2, \\
\frac{1 - \alpha}{2\beta - 1} - \frac{1 - \beta}{(1 - \alpha)(2\beta - 1)^2} \ln \left( 2\alpha - 1 + \frac{1 - \alpha}{1 - \beta} \right) & \text{for } \beta \neq 1/2.
\end{cases}$$

(4.2)

Furthermore, the risk measure $R_\alpha$, defined as

$$R_\alpha(L) := \sup_{0 < \beta < 1} \{(1 - \gamma_\beta)ES_\beta(L) + \gamma_\beta E[L]\}, \quad L \in L^1$$

is law invariant and coherent such that $R_\alpha(L) \leq es_\alpha(L)$ uniformly for $L$ in $L^1$ and $R_\alpha(1_A) = es_\alpha(1_A)$ for every $A \in \mathcal{F}$.

**Proof** Let $u$ in $[\alpha, 1)$ be given. From [27, Proposition 3.1], for each $\beta$ in $(0, 1)$ we get

$$\left( 1 - \frac{1 - u}{(1 - 2u)\beta + u} \right) ES_\beta(L) + \frac{1 - u}{(1 - 2u)\beta + u} E[L] \leq e_u(L).$$

Integrating both sides of the above inequality with respect to $u$ gives the first result of the proposition.

The law invariant and coherent property of $R_\alpha$ directly follows from the properties of $ES_\beta$. Let
Let $A$ be in $\mathcal{F}$ such that $P[A] = 1 - p$ with $0 < p < 1$. A simple computation using $\beta = p$ yields $(1 - \gamma_p)ES_p + \gamma_p E[1_A] = es_\alpha(1_A)$. Hence, $R_\alpha(1_A) = es_\alpha(1_A)$.

\begin{remark}
Note that the inequality $R_\alpha \leq es_\alpha$ can be strict as shown in Example 5.6.
\end{remark}

As shown in Figure 2 below, $\gamma_\beta$ is an increasing and convex function on $(0, 1)$ as a function of $\beta$.

The expectile can be uniformly dominated by the convex combination of expected shortfalls, see [27], for instance. However, this is not the case for the expectile-based expected shortfall as shown by the following proposition.

\begin{proposition}
The expectile-based expected shortfall $es_\alpha$ cannot be dominated uniformly for $L$ in $L^1$ by coherent risk measures of the form
\[ (1 - \lambda)ES_\beta(L) + \lambda ES_\delta(L), \]
for some $0 \leq \beta < 1$, $0 \leq \lambda \leq 1$, and $0 \leq \delta < 1$.
\end{proposition}

\begin{proof}
We know that the concave distortion function that corresponds to
\[ (1 - \lambda)ES_\beta(L) + \lambda ES_\delta(L) \]
is given by
\[ \varphi_{\lambda, \beta, \delta}(t) := (1 - \lambda) \left( \frac{t}{1 - \beta} \land 1 \right) + \lambda \left( \frac{t}{1 - \delta} \land 1 \right) \quad \text{for } 0 \leq t \leq 1. \]

Furthermore, for the distortion function $\varphi$ given by Relation (4.1) its tangent at $(0, 0)$ and $(1, 1)$ is $x = 0$ and $y = \gamma_0 x + 1 - \gamma_0$, respectively. Suppose there exists a risk measure of the form $(1 - \lambda)ES_\beta + \hat{\lambda} ES_\delta$ dominating $es_\alpha$ for some $0 \leq \hat{\beta}, \hat{\delta} < 1$ and $0 \leq \hat{\lambda} \leq 1$. It follows that $\varphi(t) \leq \varphi_{\lambda, \hat{\beta}, \hat{\delta}}(t)$ for each $t$ in $[0, 1]$. Without loss of generality, we take the smallest one from these bounds, that is, $\varphi_{\lambda, \hat{\beta}, \hat{\delta}}$ is tangent to the graph of $\varphi$ at $(0, 0)$ and $(1, 1)$. This contradicts the fact that the $y$-axis is tangent to $\varphi$ at $(0, 0)$. Hence, our supposition is false and there is no such upper bound, see Figure 3.
4.3 Asymptotic behavior of expectile-based expected shortfall

For a given confidence level $\alpha$, the quantile-based tail conditional expectation and expected shortfall coincide, that is, $TCE_\alpha = ES_\alpha$, provided that the distribution of $L$ is continuous, see [14, 29]. However, this is not true in general for $tce_\alpha$ and $es_\alpha$, with neither dominating the other depending on the considered loss profile, see Figure 1. As discussed in section 4.2, it also holds that $ES_\alpha > es_\alpha$ or $ES_\alpha \leq es_\alpha$, depending on the considered loss profile. When $F_L$ is attracted by value at risk and expectile, [28] gives an asymptotic behavior of expected shortfall in terms of value at risk, and [27] also studies the asymptotic behavior of $es_\alpha$ in terms of $tce_\alpha$ and $ES_\alpha$ for Fréchet-type distributions. Following these results, we are interested in showing the asymptotic behavior of $es_\alpha$ with respect to $tce_\alpha$ and $ES_\alpha$ for loss profiles attracted \(^1\) by extreme value distribution $H$. It is well known that $H$ can only be either Weibull ($\Psi_\eta$), Gumbel ($\Lambda$), or Fréchet ($\Phi_\eta$) with parameter $\eta > 0$, where $\Psi_\eta(x) = \exp(-(\eta x)^\eta)$ for $x < 0$, $\Lambda(x) = \exp(-e^{-x})$ for $x \in \mathbb{R}$, and $\Phi_\eta(x) = \exp(-x^{-\eta})$ for $x > 0$, see [10, 24], for instance.

The following proposition states the asymptotic relationship between $es_\alpha$, $tce_\alpha$, and $ES_\alpha$ based on each class of extreme value distribution.

**Proposition 4.6** Let $\hat{x} := \sup \{m \in \mathbb{R}: F_L(m) < 1 \}$. If $0 < \hat{x} \leq \infty$, as the confidence level $\alpha$ goes to 1, it holds that

(i) (Fréchet) If $F_L$ is in $MDA(\Phi_\eta)$ with $\eta > 1$,

\[
es_\alpha(L) \sim tce_\alpha(L) \sim (\eta - 1)^{-\frac{1}{\eta}} ES_\alpha(L) \quad \text{and} \quad \frac{tce_\alpha(L)}{e_\alpha(L)} \sim \frac{ES_\alpha(L)}{q_\alpha(L)}.
\]

---

\(^1\)We say $F_L$ is attracted by an extreme value distribution function $H$ and denoted by $MDA(H)$ if there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ for each $n$ in $\mathbb{N}$ such that $\lim_{n \to \infty} F_L^n(c_n x + d_n) = H(x)$. 

---
(ii) (Weibull) If $F_L$ is in $MDA(\Psi_\eta)$ with $\eta > 0$, 
\[
\frac{\hat{x} - ES_\alpha(L)}{\hat{x} - es_\alpha(L)} = o(1) \quad \text{and} \quad \frac{\hat{x} - tce_\alpha(L)}{\hat{x} - e_\alpha(L)} \sim \frac{\hat{x} - ES_\alpha(L)}{\hat{x} - q_\alpha(L)}.
\]

(iii) (Gumbel) If $F_L$ is in $MDA(\Lambda)$, 
\[
tce_\alpha(L) \sim \frac{ES_\alpha(L)}{q_\alpha(L)} \quad \text{and} \quad \frac{\hat{x} - tce_\alpha(L)}{\hat{x} - e_\alpha(L)} \sim \frac{\hat{x} - ES_\alpha(L)}{\hat{x} - q_\alpha(L)}
\]
for the case $\hat{x} = \infty$ and $\hat{x} < \infty$, respectively.

If further $F_L(x) = 1 - \exp(-x^\tau r(x))$ for some slowly varying function $r$ and constant $\tau > 0$ such that 
\[
\lim_{x \to \infty} \left( \frac{r(cx)}{r(x)} - 1 \right) \ln r(x) = 0 \quad (4.3)
\]
for some constant $c > 0$, it holds that 
\[
es_\alpha(L) \sim tce_\alpha(L) \sim ES_\alpha(L).
\]

**Proof** The Fréchet case directly follows from [9, Proposition 3] and [27, Proposition 4.1]. For the Weibull case, from [23, Proposition 3.3], we get $\hat{x} - q_\alpha(L) = o(\hat{x} - e_\alpha(L))$ as $\alpha$ goes to 1. It follows that 
\[
\hat{x} - ES_\alpha(L) = \frac{1}{1 - \alpha} \int_\alpha^1 (\hat{x} - q_\alpha(L))du = \frac{1}{1 - \alpha} \int_\alpha^1 (\hat{x} - e_\alpha(L))g(u)du,
\]
for some function $g$ such that $g(u)$ goes to 0 as $u$ goes to 1. Hence, 
\[
\hat{x} - ES_\alpha(L) \sim \frac{1}{1 - \alpha} \int_\alpha^1 (\hat{x} - e_\alpha(L))g(\alpha)du = g(\alpha)(\hat{x} - e_\alpha(L)).
\]
That is, $\hat{x} - ES_\alpha(L) = o(\hat{x} - e_\alpha(L))$. Since $tce_\alpha(L) = ES_{\beta^*}(L)$, it follows that 
\[
tce_\alpha(L) = q_{\beta^*}(L) + \frac{1}{1 - \beta^*} E[(L - q_{\beta^*}(L))^+].
\]
Rearranging gives 
\[
tce_\alpha(L) \sim \frac{1}{e_\alpha(L)} + \frac{1}{(1 - \beta^*)e_\alpha(L)} E[(L - e_\alpha(L))^+]. \quad (4.4)
\]
Relation (4.4) can be rewritten as 
\[
\frac{\hat{x} - tce_\alpha(L)}{\hat{x} - e_\alpha(L)} = 1 - \frac{1}{(1 - \beta^*)(\hat{x} - e_\alpha(L))} E[(L - e_\alpha(L))^+]. \quad (4.5)
\]
Following [22, Lemma 3.2 and Remark 3.3], as $x$ goes to $\hat{x}$ we get 
\[
\frac{1}{(\hat{x} - x)(1 - F_L(x))} E[(L - x)^+] \sim \frac{1}{\eta + 1}.
\]
Since as $\alpha$ goes to 1, we have $e_\alpha$ goes to $\hat{x}$, it follows that 
\[
\frac{1}{(\hat{x} - e_\alpha(L))(1 - \beta^*)} E[(L - e_\alpha(L))^+] \sim \frac{1}{\eta + 1}.
\]
Hence, Relation (4.5) yields

\[\text{A measurable function } r : \mathbb{R} \to \mathbb{R} \text{ is said to be slowly varying if } \lim_{x \to \infty} \frac{r(x)}{x^\tau} = 1 \text{ for each } x \in \mathbb{R}.\]
\[
\frac{\hat{x} - tce_\alpha(L)}{\hat{x} - e_\alpha(L)} \sim 1 - \frac{1}{\eta + 1} = \frac{\eta}{\eta + 1} \sim \frac{\hat{x} - ES_\alpha(L)}{\hat{x} - q_\alpha(L)}.
\]

The relation for \((\hat{x} - ES_\alpha(L))/(\hat{x} - q_\alpha(L))\) is due to [22, Theorem 3.4].

As for the Gumbel case, from Mao et al. [23, Relation (3.20)], we have

\[
E[(L - e_\alpha(L))^+] = \begin{cases} 
  e_\alpha(L)\alpha(1 - \beta^*), & \hat{x} = \infty, \\
  (\hat{x} - e_\alpha(L))\alpha(1 - \beta^*), & \hat{x} < \infty.
\end{cases}
\tag{4.6}
\]

The Relations (4.4), (4.5), and (4.6) together with [22, Theorem 3.4] yield

\[
\frac{tce_\alpha(L)}{e_\alpha(L)} \sim 1 \sim \frac{ES_\alpha(L)}{q_\alpha(L)} \quad \text{and} \quad \frac{\hat{x} - tce_\alpha(L)}{\hat{x} - e_\alpha(L)} \sim 1 \sim \frac{\hat{x} - ES_\alpha(L)}{\hat{x} - q_\alpha(L)}
\]

for cases \(\hat{x} = \infty\) and \(\hat{x} < \infty\), respectively. The last asymptotic result holds as a result of \(e_\alpha(L) \sim q_\alpha(L)\) under the given conditions, see [4, Proposition 2.4]. This ends the proof of the proposition.

According to Proposition 4.6, for the Fréchet and Gumbel (with condition (4.3)) cases, as the confidence level \(\alpha\) goes to 1, the expectile-based tail conditional expectation and expectile-based shortfall are equivalent. Here, both \(tce_\alpha\) and \(es_\alpha\) can be interpreted as the expectation of the loss under the event that the loss exceeds \(e_\alpha\). Figure 4 provides a graphical illustration for the ratio of \(ES_\alpha/es_\alpha\) for Pareto, standard Student \(t\), beta, and standard normal distributions. Note that the Pareto distribution with parameter \(a\) is attracted by Fréchet type \(MDA(\Phi_a)\), the standard Student \(t\) distribution with \(\nu\) degree of freedom is also attracted by Fréchet type \(MDA(\Phi_\nu)\). The beta distribution is attracted by Weibull type \(MDA(\Psi_1)\) and the standard normal distribution is attracted by Gumbel type \(MDA(\Lambda)\).

---

Figure 4  Graph of \(ES_\alpha/es_\alpha\) (for Pareto, standard Student \(t\), and standard normal distributions) and \((\hat{x} - ES_\alpha)/(\hat{x} - es_\alpha)\) (for the beta distribution).
5. Examples

In this section, we provide explicit or semi-explicit computations of expectile-based tail conditional expectation and expected shortfall for Bernoulli, uniform, beta, exponential, standard normal, Konker, Pareto, and Student $t$ distributions. We also illustrate some of the results of section 4.1.

Example 5.1 (Bernoulli). Let $L \sim \text{Bern}(p)$ for some $p$ in $(0, 1)$. From [11, 24], we get

$$e_\alpha(L) = \frac{\alpha p}{(2\alpha - 1)p + 1 - \alpha}.$$  

After integration

$$e_{s\alpha} = \begin{cases} \frac{1 - 1 - \alpha}{p} & \text{if } p = 1/2, \\ \frac{1 - \alpha}{1 - 2p} \left[ 1 - \frac{1 - p}{(1 - \alpha)(1 - 2p)} \ln \left( \frac{2\alpha - 1 + \frac{1 - \alpha}{p} \right) \right] & \text{if } p \neq 1/2. \end{cases}$$

Example 5.2 (Uniform). Let $F_L(x) = x$ for $x$ in $[0, 1]$. From [27], we have

$$q_\alpha(L) = \alpha, \quad \text{and} \quad e_\alpha(L) = \frac{\sqrt{\alpha(1 - \alpha)} - \alpha}{1 - 2\alpha} = \beta^* = P[L \leq e_\alpha(L)].$$

We also have $ES_\alpha(L) = 1 - \frac{1 - \alpha}{2}$ and $tce_\alpha(L) = 1 - \frac{1 - \beta^*}{2}$. After integration,

$$e_{s\alpha}(L) = \frac{1}{2} - \frac{1}{4(1 - \alpha)} \ln \left( \frac{1 + 2\sqrt{\alpha(1 - \alpha)}}{1 - 2\sqrt{\alpha(1 - \alpha)}} \right) + \frac{\sqrt{\alpha(1 - \alpha)}}{2(1 - \alpha)}.$$  

Following [17], $F_L$ is attracted by Weibull type $MDA(\Psi_1)$. It also holds that $\hat{x} = 1$ and a simple computation shows

$$\frac{1 - ES_\alpha(L)}{1 - e_{s\alpha}(L)} = o(1) \quad \text{and} \quad \frac{1 - tce_\alpha(L)}{1 - e_\alpha(L)} = \frac{1}{2} = \frac{1 - ES_\alpha(L)}{1 - q_\alpha(L)}.$$  

Example 5.3 (Beta). For $a > 0$, let $F_L(x) = x^a$ with $x$ in $[0, 1]$. From [27], we have

$$q_\alpha(L) = \alpha^{\frac{1}{a}} \quad \text{and} \quad ES_\alpha(L) = \frac{a \left( 1 - \alpha^{\frac{1}{a} + 1} \right)}{(1 - \alpha)(a + 1)}.$$  

From [27], we also get $e_\alpha(L) = (\beta^*)^{1/a}$ and $tce_\alpha(L) = ES_{\beta^*}(L)$, where $\beta^*$ solves

$$\alpha(a + 1) + (1 - 2\alpha)\beta^* = a\alpha(\beta^*)^{\frac{a}{a}}.$$  

According to [18] and [24], $e_\alpha(L) = q_\alpha(\tilde{L})$, where $\tilde{L}$ is a random variable with probability density function given by

$$g(x) = \frac{F_L(x) E[L] - E[L_1|L \leq x]}{(2xF_L(x) - E[L_1|L \leq x]) + E[L] - x}^2, \quad 0 < F_L(x) < 1.$$  

In the case of the beta distribution, it follows that

$$g(x) = \frac{a}{a + 1} \frac{x^a(1 - x)}{(2x^{a+1} - (a + 1)x + a)^2}, \quad 0 < x < 1,$$  

where $a > 0$.
which yields
\[
es_{\alpha}(L) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} q_{\alpha}(\tilde{L}) d\tilde{u} = E \left[ \tilde{L} | \tilde{L} > q_{\alpha}(\tilde{L}) \right]
= \frac{a}{(1 - \alpha)(a + 1)} \int_{(\beta^*)^\frac{1}{\alpha}}^{1} \frac{x^{a+1}(1 - x)}{(2x^{a+1} - (a + 1)x + a)^2} dx.
\]

It is known that \( F_{\tilde{L}} \) belongs to the Weibull type \( MDA(\Psi_1) \), see \([22, 23]\), for instance. We also have \( \tilde{x} = 1 \) and as a result of Proposition 4.6, it holds that
\[
\frac{1 - ES_{\alpha}(L)}{1 - es_{\alpha}(L)} = o(1) \quad \text{and} \quad \frac{1 - tce_{\alpha}(L)}{1 - e_{\alpha}(L)} \sim \frac{1 - ES_{\alpha}(L)}{1 - q_{\alpha}(L)}.
\]

Example 5.4 (Exponential). Let \( F_{\tilde{L}}(x) = 1 - \exp(-x) \) for \( x \geq 0 \). Then, \( ES_{\alpha}(L) = 1 - \ln(1 - \alpha) \) and \( e_{\alpha}(L) = 1 + W \left( (2\alpha - 1)/(e(1 - \alpha)) \right) \), where \( W \) is the Lambert function, see \([27]\). We also have
\[
g(x) = \frac{x e^{-x}}{(1 - x - 2e^{-x})^2}, \quad x \geq 0.
\]
Hence,
\[
es_{\alpha}(L) = \frac{1}{1 - \alpha} \int_{1 + W \left( \frac{2\alpha - 1}{e(1 - \alpha)} \right)}^{\infty} \frac{x^2 e^{-x}}{(1 - x - 2e^{-x})^2} dx.
\]
From \([4]\), we get \( e_{\alpha}(L) \sim q_{\alpha}(L) \). This implies that \( es_{\alpha} \sim ES_{\alpha}(L) \sim tce_{\alpha}(L) \).

Example 5.5 (Standard normal). Let \( L \) be a random variable with a standard normal distribution with cumulative probability distribution and density function \( \Phi \) and \( \phi \), respectively. From \([24]\), we get
\[
ES_{\alpha}(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha}.
\]
Following \([27]\), we also get
\[
e_{\alpha}(L) = \Phi^{-1}(\beta^*),
\]
where \( \beta^* \) solves the equation
\[
\Phi^{-1}(\beta^* - \left( 1 - \frac{1 - \alpha}{\alpha + (1 - 2\alpha)\beta^*} \right) \frac{\phi(\Phi^{-1}(\beta^*))}{1 - \beta^*} = 0.
\]
We also have
\[
g(x) = \frac{\phi(x)}{(x(1 - 2\Phi(x)) - 2\phi(x))^2}, \quad x \in \mathbb{R}.
\]
Hence,
\[
es_{\alpha}(L) = \frac{1}{1 - \alpha} \int_{\Phi^{-1}(\beta^*)}^{\infty} \frac{x\phi(x)}{(x(1 - 2\Phi(x)) - 2\phi(x))^2} dx.
\]
It is well known that a random variable with a normal distribution is attracted by the Gumbel type \( MDA(\Lambda) \). From \([4]\), we get \( e_{\alpha}(L) \sim q_{\alpha}(L) \). This implies that \( es_{\alpha} \sim ES_{\alpha}(L) \sim tce_{\alpha}(L) \).

\(^1\) \( W \) is a function such that \( xe^{\nu} = y \) if and only if \( x = W(y) \).
Example 5.6 (Konker). Let $F_L(x) = \frac{4 + x^2 + x \cdot \sqrt{x^2 + 4}}{2(x^2 + 4)}$ for $x$ in $\mathbb{R}$. From [20], we get

$$e_\alpha(L) = q_\alpha(L) = \frac{2\alpha - 1}{\sqrt{\alpha(1 - \alpha)}}.$$ 

It implies that

$$es_\alpha(L) = ES_\alpha(L) = 2\sqrt{\frac{\alpha}{1 - \alpha}}.$$ 

In this case, $F_L$ is attracted by the Fréchet type $MDA(\Phi_2)$. Clearly, it holds that

$$es_\alpha(L) = ES_\alpha(L) = tce_\alpha(L)$$

and

$$\frac{tce_\alpha(L)}{e_\alpha(L)} = \frac{ES_\alpha(L)}{q_\alpha(L)}.$$ 

In the case of the Konker distribution, the risk measure $R_\alpha$ given in Proposition 4.3 may be strictly less than $es_\alpha$. For instance, we get

$$R_{0.5}(L) = \sup_{0 < \beta < 1} \left\{ 2 \left( 1 - \frac{1}{2\beta - 1} + \frac{2(1 - \beta)}{(2\beta - 1)^2} \ln \left( \frac{1}{2(1 - \beta)} \right) \right) \sqrt{\beta} \right\}$$

$$\approx 1.53 < es_{0.5}(L) = 2.$$ 

Example 5.7 (Pareto). For $a > 1$ and $x \geq 0$, let $F_L(x) = 1 - (1/(x + 1))^a$. For $a = 2$, 

$$q_\alpha(L) = \frac{1}{\sqrt{1 - \alpha}} - 1, \quad ES_\alpha(L) = \frac{2}{\sqrt{1 - \alpha}} - 1, \quad \text{and} \quad e_\alpha(L) = \sqrt{\frac{\alpha}{1 - \alpha}}.$$ 

From [27], we also have $\beta^* = 1 - (1 - \alpha)/(1 + 2\sqrt{\alpha(1 - \alpha)})$ and we get

$$tce_\alpha(L) = 2\sqrt{\frac{1}{1 - \alpha} + 2e_\alpha(L) - 1} \quad \text{and} \quad es_\alpha(L) = e_\alpha(L) + \frac{\arcsin(\sqrt{1 - \alpha})}{1 - \alpha}.$$ 

It is known that $F_L$ is attracted by the Fréchet type $MDA(\Phi_2)$, see [21, 23], for instance. A simple computation shows that

$$tce_\alpha(L) \sim es_\alpha(L) \sim ES_\alpha(L) \quad \text{and} \quad \frac{ES_\alpha(L)}{q_\alpha(L)} \sim 2 \sim \frac{tce_\alpha(L)}{e_\alpha(L)}.$$ 

Example 5.8 (Student $t$). Let $L$ be a random variable with a standard Student $t$ distribution with cumulative probability distribution and density function $T_\nu$ and $t_\nu,$ respectively, with $\nu$ degree of freedom. From [24], we get

$$ES_\alpha(L) = \frac{t_\nu(T_\nu^{-1}(\alpha))}{1 - \alpha} \left( \frac{\nu + (T_\nu^{-1}(\alpha))^2}{\nu - 1} \right).$$ 

Following [27], we also get

$$e_\alpha(L) = T_\nu^{-1}(\beta^*),$$

where $\beta^*$ solves the equation

$$T_\nu^{-1}(\beta^*) - \left( 1 - \frac{1 - \alpha}{\alpha + (1 - 2\alpha)\beta^*} \right) \frac{t_\nu(T_\nu^{-1}(\beta^*))}{1 - \beta^*} \left( \frac{\nu + (T_\nu^{-1}(\beta^*))^2}{\nu - 1} \right) = 0.$$
A simple computation yields
\[ g(x) = \frac{(\nu - 1)(\nu + x^2)t_\nu(x)}{(2(xT_\nu(x)(\nu - 1) + (\nu + x^2)t_\nu(x)) - (\nu - 1)x)^2}, \quad x \in \mathbb{R}. \]

Hence,
\[ es_\alpha(L) = \frac{1}{1 - \alpha} \int_{\frac{1}{1+\alpha}}^{\infty} \frac{(\nu - 1)x(\nu + x^2)t_\nu(x)}{(2(xT_\nu(x)(\nu - 1) + (\nu + x^2)t_\nu(x)) - (\nu - 1)x)^2} dx. \]

*It is well known that a random variable with a Student t distribution with \( \nu \) degree of freedom is attracted by the Fréchet type MDA(\( \Phi_\nu \)).* It follows that
\[ es_\alpha(L) \sim tce_\alpha(L) \sim (\nu - 1)^{-\frac{1}{2}} ES_\alpha(L). \]

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