ORE EXTENSIONS OVER DUO RINGS

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Abstract

We show that there exist noncommutative Ore extensions in which every right ideal is two-sided. This answers a problem posed by Marks in [5]. We also provide an easy construction of one sided duo rings.

INTRODUCTION

Hirano, Hong, Kim and Park proved in [2] that an ordinary polynomial ring is one-sided duo only if it is commutative. Marks in [5], extended this result to Ore extensions, by showing that if a noncommutative Ore extension $R[x; \sigma, \delta]$ which is a duo ring on one side exists, then it has to be right duo, $\sigma$ must be not injective and $\delta \neq 0$ (see Theorem 1.2). He also obtained a series of necessary conditions for the Ore extension to be right duo (see Proposition 1.3).

The aim of this paper is to show that noncommutative Ore extensions which are right duo rings do exist and that the necessary conditions obtained by Marks are not sufficient for the Ore extension to be right duo.

In Section 2 we investigate corner extensions $R = A \oplus M$ of right duo rings $A$. In particular, we show in Theorem 2.4 that $R$ is right duo provided the $(A - A)$-bimodule $M$ is simple as a right $A$-module and faithful as a left $A$-module. This theorem together with Lemma 2.6 offer an easy way of constructing right duo rings which are not left duo. Rings obtained in this way will serve us as coefficient rings of Ore extensions which are right duo.

In Section 3 we determine all $\sigma$-derivations for a suitably chosen corner extension $R = A \oplus M$ and its endomorphism $\sigma$ (Theorem 3.3). This enable us to give, in Proposition 3.6 a classification of Ore extensions $R[x; \sigma, \delta]$ for $R$ and $\sigma$ as in Section 2.

*The research was supported by Polish KBN grant No. 1 P03A 032 27
Finally, Section 4 is devoted to description of Ore extensions from the previous section which are right duo rings.

The construction from the paper gives not only Ore extensions which are right duo. It provides also a ring $R$ with an endomorphism $\sigma$ such that, for any $n \in \mathbb{N}$, there exists a $\sigma$-derivation $\delta_n$ of $R$ such that every right ideal generated by a polynomial of degree smaller than $n$ is a two-sided ideal of $R[x;\sigma,\delta_n]$ but there exists a polynomial $f \in R[x;\sigma,\delta_n]$ of degree $n$ for which $fR[x;\sigma,\delta_n]$ is not a left ideal. All of those examples satisfy necessary conditions, obtained by Marks, for an Ore extension to be right duo.

1 PRELIMINARIES

All rings considered in this paper are associative with identity. Recall that a ring $R$ is called right (left) duo if every right (left) ideal of $R$ is a two-sided ideal.

For a subset $S$ of an $(R-R)$-bimodule $M$, $\text{l.ann}_R(S)$ will stand for the left annihilator of $S$ in $R$, i.e. $\text{l.ann}_R(S) = \{r \in R \mid rS = 0\}$. The right annihilator $\text{r.ann}_R(S)$ is defined similarly.

An Ore extension of a ring $R$ is denoted by $R[x;\sigma,\delta]$, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation, i.e. $\delta: R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in R$. Recall that elements of $R[x;\sigma,\delta]$ are polynomials in $x$ with coefficients written on the left. Multiplication in $R[x;\sigma,\delta]$ is given by the multiplication in $R$ and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$.

We say that a subset $S$ of $R$ is $(\sigma-\delta)$-stable, if $\sigma(S) \subseteq S$ and $\delta(S) \subseteq S$.

For $a \in R$, the map $d_a: R \to R$ defined by $d_a(r) = ar - \sigma(r)a$ is a $\sigma$-derivation. This $\sigma$-derivations is called the inner $\sigma$-derivation determined by the element $a$. A $\sigma$-derivation $\delta$ is called outer if it is not inner.

The following fact is well-known (see for example Lemma II.5.5[1]):

**Lemma 1.1.** Suppose that $\delta_1$ and $\delta_2$ are $\sigma$-derivations of a ring $R$. If $\delta_1 - \delta_2$ is an inner $\sigma$-derivation, then the Ore extensions $R[x;\sigma,\delta_1]$ and $R[x;\sigma,\delta_2]$ are $R$-isomorphic.

The next two results come from the paper [5] of Marks.

**Theorem 1.2.** Suppose that one of the following conditions holds.

1. The Ore extension $R[x;\sigma,\delta]$ is a left duo ring;
2. The Ore extension $R[x;\sigma,\delta]$ is a right duo ring and either $\sigma$ is injective or $\delta = 0$.

Then $R$ is commutative, $\sigma = \text{id}_R$ and $\delta = 0$, i.e. $R[x;\sigma,\delta] = R[x]$ is a commutative polynomial ring.
Proposition 1.3. Suppose that $R[x; \sigma, \delta]$ is a right duo ring which is noncommutative. Let $N = \bigcup_{i=1}^{\infty} \ker \sigma^i$. Then:

1. $\delta$ is an outer $\sigma$-derivation.
2. $R$ is a right duo ring.
3. Every ideal of $R$ is $(\sigma - \delta)$-stable.
4. For any $r \in R$, the sequence $\{\sigma^n(r)\}_{n \in \mathbb{N}}$ is eventually constant.
5. For any $r \in R$, the sequence $\{\sigma^n(\delta(r))\}_{n \in \mathbb{N}}$ is eventually zero.
6. $0 \neq N \subseteq J(R)$, where $J(R)$ denotes the Jacobson radical of $R$.
7. The factor ring $R[x; \sigma, \delta]/NR[x; \sigma, \delta]$ is isomorphic to the commutative polynomial ring $(R/N)[x]$.

All statements but (1) from the above proposition come from Lemma 7 and Theorem 11 of [5]. The statement (1) is a direct consequence of Lemma 1.1 and Theorem 1.2.

In [5], Marks also presented an example of a ring $R$ with an endomorphism $\sigma$ and $\sigma$-derivation $\delta$ which fulfills conditions (3) ÷ (7). In this example $\delta$ is an inner $\sigma$-derivation.

Let us observe that:

Proposition 1.4. Suppose that $R$, $\sigma$ and $\delta$ possess properties (2), (3) and (7) from Proposition 1.3. If $N = \bigcup_{i=1}^{\infty} \ker \sigma^i$ is a nil ideal of $R$, then every maximal one-sided ideal of $R[x; \sigma, \delta]$ is two-sided, i.e. $R[x; \sigma, \delta]$ is a quasi-duo ring.

Proof. Let $I$ be a nilpotent two-sided ideal of $R$. By assumption, $I$ is $(\sigma - \delta)$-stable, so $IR[x; \sigma, \delta]$ is also a nilpotent ideal of $R[x; \sigma, \delta]$. In particular, $IR[x; \sigma, \delta]$ is contained in the Jacobson radical $J$ of $R[x; \sigma, \delta]$.

Let $a \in N$. Since $R$ is a right duo ring and $a$ is a nilpotent element, $aR$ is a nilpotent two-sided ideal of $R$. Hence, by the above $NR[x; \sigma, \delta] \subseteq J$ follows. This implies that $NR[x; \sigma, \delta]$ is contained in any maximal one-sided ideal of $R[x; \sigma, \delta]$. Now, the thesis is an easy consequence of the fact that $R[x; \sigma, \delta]/(NR[x; \sigma, \delta]) \simeq (R/N)[x]$ is a commutative ring.

Let $R^\sigma = \{ r \in R \mid \sigma(r) = r \}$. Remark that statements (4) and (5) of Proposition 1.3 say that $R$, $\sigma$ and $\delta$ are of very special form. Namely, in the terminology of Lam (Cf. Definition 2.15[4]), $R^\sigma$ is a unital split corner ring of $R$, i.e. $R^\sigma$ is a unital subring of $R$, $R = R^\sigma \oplus N$ as abelian groups and $N$ is an ideal of $R$. The maps $\sigma$ and $\delta$ satisfy: for any $r \in N$, there exists $n \in \mathbb{N}$ such that $\sigma^n(r) = 0$ and $\delta(R) \subseteq N$. 
When seeking an example of a noncommutative Ore extension $R[x;\sigma,\delta]$ which is a right duo ring and the coefficient ring $R$ is one-sided noetherian, one can restrict his attention to more specific rings:

**Proposition 1.5.** Let $R$ be either left or right noetherian ring. Suppose that $R[x;\sigma,\delta]$ is a right duo ring which is noncommutative. Then there exists a noncommutative Ore extension $R'[x;\sigma',\delta']$, which is a right duo ring, such that:

1. $R' = A \oplus M$ where $A$ is a unital split corner subring of $R'$ with $M^2 = 0$ and $M \neq 0$.

2. $\sigma': R' \to R'$ is defined by $\sigma'(a + l) = a$, for any $a \in A$ and $l \in M$, and $\delta'(R') \subseteq M$.

**Proof.** We provide the proof in case $R$ is left noetherian. The case $R$ right noetherian can be done using the same arguments.

We know, by Proposition 1.3, that $R^{\sigma}$ is a split corner subring of $R = R^{\sigma} \oplus N$, where $0 \neq N = \bigcup_{i=1}^{\infty} \ker \sigma^i$ and $N \subseteq J(R)$.

Since $R$ is left noetherian, $N = \ker \sigma^m$ for some $m \in \mathbb{N}$, say $m$ is the smallest such number. If $m > 1$, then $\ker \sigma^{m-1}$ is a two-sided ideal of $R$ properly contained in $N$. Proposition 1.3(2) implies that $\ker \sigma^{m-1}$ is a $(\sigma - \delta)$-stable ideal. Therefore $\sigma$ and $\delta$ induce an endomorphism $\sigma''$ and a $\sigma''$-derivation $\delta''$ of $R/\ker \sigma^{m-1}$. The kernel of $\sigma''$ is equal to $N/\ker \sigma^{m-1} = N'' \neq 0$. Notice that $R[x;\sigma,\delta]/(\ker \sigma^{m-1}R[x;\sigma,\delta]) \simeq (R/\ker \sigma^{m-1})[x;\sigma'',\delta'']$ is a right duo ring as a homomorphic image of a right duo ring and it is not commutative, because $\ker \sigma'' \neq 0$. Therefore, eventually replacing $R$, $\sigma$ and $\delta$ by $R''$, $\sigma''$ and $\delta''$, respectively, we may assume that $m = 1$, i.e. $0 \neq N = \ker \sigma \subseteq J(R)$.

Let us observe that $N^2 \neq N$. Indeed, otherwise we would have $N = N^2 \subseteq J(R)N \subseteq N$, i.e. $J(R)N = N$. Notice that $N$ is finitely generated left $R$-module, as $R$ is left noetherian. Thus, Nakayama’s Lemma would imply $N = 0$.

Now, one can make a similar reduction, as in the first part of the proof, using $N^2 \neq N$ instead of $\ker \sigma^{m-1}$. This will result in a noncommutative Ore extension $R'[x;\sigma',\delta']$ such that it is a right duo ring, $R' = A \oplus M$, where $A = R^{\sigma'} = R^{\sigma}$, $M = \ker \sigma'$ and $M^2 = 0$. This proves (1) and the first property of (2). The fact that $\delta'(R') \subseteq M$ is a direct consequence of Proposition 1.3(5).

In Section 4 we will show that Ore extensions from the above proposition do exist.

Let us notice that it is easy to construct rings described in the statement (2) from Proposition 1.5. In fact, if $A$ is a unital split corner subring of $R = A \oplus M$ then $M$ is an $(A - A)$-bimodule. Conversely, when we have a ring $A$ and an $(A - A)$-bimodule $M$, then $A \oplus M$ is a ring with multiplication determined by
\[ M^2 = 0, \] multiplication in \( A \) and conditions \( am = a \cdot m, ma = m \cdot a \), for every \( a \in A \) and \( m \in M \), where \( \cdot \) denotes the bimodule action of \( A \) on \( M \). This ring can be viewed also as a subring of a triangular ring \( \begin{pmatrix} A & M \\ 0 & A \end{pmatrix} \) consisting of all matrices of the form \( \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \), where \( a \in A \) and \( m \in M \).

Henceforth, while writing \( R = A \oplus M \) we will always mean a ring constructed as above, i.e. \( A \) is a unital split corner subring of \( R = A \oplus M \) with \( M^2 = 0 \). We will always assume that the extension is nontrivial, i.e. \( M \neq 0 \).

\[ \sigma \] will denote the endomorphism of \( R = A \oplus M \) defined by \( \sigma|_A = \text{id}_A \) and \( \ker \sigma = M \). In the next two sections we will investigate when ring extensions of this kind are right duo and we will describe all \( \sigma \)-derivations of such rings.

## 2 CORNER EXTENSIONS WHICH ARE RIGHT DUO RINGS

Throughout this section \( A \) is a unital split corner subring of \( R = A \oplus M \), with \( M^2 = 0 \).

We begin this section with the following easy observation:

**Lemma 2.1.** Suppose \( R = A \oplus M \) is a right duo ring with \( M \neq 0 \). Then \( I = \text{l.ann}_A(M) \) is a two-sided ideal of \( R \) which is contained in \( \text{r.ann}_A(M) \). In particular, \( M \) has a natural structure of \( (A/I - A/I) \)-bimodule and \( R/I \cong A/I \oplus M \).

**Proof.** Let \( I = \text{l.ann}_A(M) \). Notice that \( IR = I(A \oplus M) = IA \subseteq I \). Thus \( I \) is a right ideal of \( R \) and, as \( R \) is right duo, \( I \) is a two-sided ideal. Then \( MI \subseteq M \cap I = 0 \), i.e. \( I \subseteq \text{r.ann}_A(M) \). Now it is standard to complete the proof of the lemma.

**Proposition 2.2.** Let \( A \) be a right duo ring and \( R = A \oplus M \), where \( M \) is an \( (A - A) \)-bimodule. Then:

1. Suppose that \( a \in A \) is such that \( aM = M \). Then, for any \( m \in M \), \( (a + m)R = aA + M \) is a two-sided ideal of \( R \).

2. Suppose that any right \( A \)-submodule of \( M \) is a \( (A - A) \)-subbimodule of \( M \) and \( aM = M \), for any \( 0 \neq a \in A \). Then \( R = A \oplus M \) is a right duo ring.

**Proof.** (1). Let \( a \in A \) be such that \( aM = M \). Then, for any \( m \in M \), \( (a + m)M = aM = M \). Therefore \( M \subseteq (a + m)R \). This yields that also \( aA \subseteq (a + m)R \) and we have \( aA + M \subseteq (a + m)R \subseteq aR + Mr \subseteq aR + M = aA + M \). This shows that \( (a + m)R = aA + M \). Since \( A \) is a right duo ring, \( aA \) is a two-sided ideal of \( A \). Now it is clear that \( (a + m)R \) is a two-sided ideal of \( R \).
(2). The assumptions imposed on $M$ imply that, for any $m \in M$, $mR$ is a two-sided ideal of $R$ while, by the statement (1), $(a + m)R$ is a two-sided ideal of $R$ for any $0 \neq a \in A$. This gives (2).

Remark 2.3. Let $R = A \oplus M$ be a ring satisfying the assumptions of the statement (2) from the above proposition. Then, using the proposition, one can easily give a description of the lattice of two-sided ideals of the right duo ring $R$ in terms of the lattice of two-sided ideals of $A$ and the lattice of $(A - A)$-subbimodules of $M$.

As an application of Proposition 2.2 we obtain the following:

Theorem 2.4. Let $A$ be a right duo ring and $R = A \oplus M$, where $M$ is an $(A - A)$-bimodule such that $M$ is faithful as a left $A$-module and simple as a right $A$-module. Then:

1. $R$ is a right duo ring.
Moreover:

2. $R$ is left duo iff $M$ is faithful as a right $A$-module (i.e. $A$ is a division ring) and simple as a left $A$-module.

Proof. (1). Let $0 \neq a \in A$. Since the module $AM$ is faithful, $aM$ is a nonzero submodule of the simple right $A$-module $MA$, so $aM = M$ follows. Now, it is easy to see that the thesis is a consequence of Proposition 2.2(2).

(2). Suppose that $R$ is left duo. Then, by the left hand version of Lemma 2.1, $I = r.ann_A(M) \subseteq l.ann_A(M) = 0$. Hence $I = 0$ and $M$ is faithful as a right $A$-module. Since $MA$ is also simple, $A$ is a right primitive, right duo ring. Thus $A$ is a division ring.

If $0 \neq N \subseteq AM$ is an $A$-submodule of $AM$ then $N$ is a two-sided ideal of $R$ since $R$ is left duo. Thus, in particular, $N$ is also a submodule of the simple $A$-module $MA$ and $N = M$ follows, showing that the left $A$-module $AM$ is simple.

On the other hand, if $M$ is simple and faithful both as a left and right $A$-module, then $A$ is a division ring, as $A$ is right duo, and $M$ is the only proper one-sided ideal of $R$. In particular, $R$ is left duo.

The examples of right duo rings given by the above theorem are, in some sense, minimal. Lemma 2.1 shows that a right duo ring $R = A \oplus M$ has a quotient of the form $A' \oplus M'$, where $A'M$ is a faithful $A'$-module. The following proposition offers another reduction.

Proposition 2.5. Suppose that $R = A \oplus M$ is a right duo ring and the $(A-A)$-bimodule $M$ has a maximal $(A - A)$-subbimodule. Then there exists an ideal $J$ of $R$ such that $R/J \simeq A' \oplus M'$, for some right duo ring $A'$, where $M'$ is an $(A' - A')$-bimodule which is simple as a right $A'$-module and faithful as a left $A'$-module.
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Proof. Let $N$ be a maximal $(A-A)$-subbimodule of $M$. Then $N$ is a two-sided ideal of $R$ and $R/N \cong A \oplus M'$, where $M'$ denotes the quotient $(A-A)$-bimodule $M/N$.

Notice that if $0 \neq W_A$ is a submodule of $M_A'$, then $W_A$ is a two-sided ideal of $R/N$ as it is a right ideal of a right duo ring $R/N$. In particular, $AW_A \subseteq W_A$, i.e. $0 \neq W_A$ is subbimodule of a simple bimodule $M'$. This means that $W = M'$ and shows that $M'$ is simple as right $A$-module. Now the proposition is a direct consequence of Lemma 2.1. Remark that the ideal $J$ from the proposition is equal to $\mathbb{l.ann}_A(A(M/N)) + N$.

In order to be able to make use of Theorem 2.4, we need the following:

Lemma 2.6. Let $A$ be a right duo ring. The following conditions are equivalent:

1. There exists an $(A,A)$-bimodule $M$ such that $M$ is faithful as left $A$-module and simple as right $A$-module.

2. There exist a right primitive ideal $P$ of $A$ and an injective homomorphism $\phi: A \to A/P$.

Proof. (1) $\Rightarrow$ (2). Let $P$ denote the annihilator of $M_A$. Then $P$ is right primitive ideal of $A$ and $A/P$ is a division ring as $A$ is a right duo ring. This means that, for any $0 \neq m \in M$, $r.\text{ann}_A(m) = P$.

Let us fix $0 \neq m \in M$ and consider $M$ as $(A-A/P)$-bimodule. Then $M = m(A/P)$ and for any $a \in A$, $am = m\phi_m(a)$ for a suitable element $\phi_m(a) \in A/P$. Notice that, because $r.\text{ann}_{A/P}(m) = 0$, the element $\phi_m(a)$ is uniquely determined by $a$. Thus we have a well-defined map $\phi = \phi_m: A \to A/P$. It is standard to check that $\phi$ is a ring homomorphism. If $\phi(a) = 0$, then $0 = m\phi(a)(A/P) = am(A/P) = aM$. Hence $a = 0$ follows, as the left $A$-module $AM$ is faithful. This shows that $\phi$ is injective.

(2) $\Rightarrow$ (1). Let $P$ be a right primitive ideal of $R$ and $\phi: A \to A/P$ an injective homomorphism. Then, as $A$ is right duo, $A/P$ is a division ring. Let $M$ be the one dimensional right vector space over $A/P$. Let us fix $0 \neq m \in M$ and define left $A$ module structure on $M$ by setting $a \cdot (mr) = m\phi(a)r$, for any $a \in A$ and $r \in A/P$. It is standard to check that this determines an $(A-A/P)$-bimodule structure on $M$. Notice that if $aM = 0$, then $m\phi(a) = 0$ and $a = 0$ follows, as $\phi$ is injective and $r.\text{ann}_{A/P}(m) = 0$. This induces the desired $(A-A)$-bimodule structure on $M$.

Remark 2.7. 1. In the proof of the implication (1) $\Rightarrow$ (2) from the above lemma, different choices of the element $0 \neq m \in M$ give different homomorphisms $\phi_m: A \to A/P$. In fact, one can check that $\phi_{mr} = r^{-1}\phi_m r$, for any $0 \neq r \in A/P$. 

2. The equivalence from the above lemma holds under a slightly weaker assumption that every maximal right ideal of $A$ is two-sided, i.e. $A$ is a right quasi-duo ring.

**Corollary 2.8.** Suppose that a right duo ring $A$ satisfies one of the equivalent conditions of the above lemma. Then:

1. $A$ is a right Ore domain.

2. If $A$ is an algebra over a field $K$ then $A$ is a division $K$-algebra, provided $A$ satisfies Nullstellensatz, i.e. for any simple $A$-module $N_A$, the division $K$-algebra $\text{End}_A(N_A)$ is algebraic over $K$ (for example when $A$ is a finitely generated commutative $K$-algebra or $\dim_K A < \#(K) - 1$, as cardinal numbers).

**Proof.** The statement (1) is clear as $A \simeq \phi(A)$ is a subring of the division ring $A/P$ and a domain which is a right duo ring is always a right Ore domain.

(2) Suppose the $K$-algebra $A$ satisfies Nullstellensatz and let $M$ denote the $(A - A)$-bimodule from Lemma 2.6. Then, since $M_A$ is simple, $\text{End}_A(M_A)$ is an algebraic division algebra over $K$.

Notice that $\psi: A \to \text{End}_A(M_A)$ defined by $\psi(a)(m) = am$, for $a \in A$ and $m \in M$, is a $K$-algebra homomorphism. Moreover $\psi$ is injective, since $A_M$ is a faithful as a left $A$-module. Therefore $A \simeq \psi(A)$ is a domain which is algebraic over $K$. This means that $A$ is a division algebra. $\square$

Of course, in general $A$ does not have to be a division ring if $A$ possesses an $(A - A)$-bimodule which is simple as a right $A$-module and faithful as left $A$-module.

**Example 2.9.** Let $A = K[x]$, where $K = F(X)$ denotes the field of rational functions over a field $F$ in the set $X = \{x_i \mid i = 0, 1, \ldots\}$ of indeterminates. Then the $F$-linear homomorphism $\phi: A \to K$ defined by $\phi(x) = x_1$ and $\phi(x_i) = x_{i+2}$, for $i = 0, 1, \ldots$, is injective. Thus, by Lemma 2.6 and Theorem 2.4, $R = A \oplus M$ is a right duo ring which is not left duo, where $M = K$ has the $(A - A)$-bimodule structure given by $a \cdot m = \phi(a)m$ and $m \cdot a = m\overline{a}$, for $m \in M$ and $a \in A$, where $\overline{a}$ denotes the canonical image of $a$ in $A/(x) = K$.

Theorem 2.4 together with Lemma 2.6 offers an easy way of constructing right duo rings which are not left duo.

Let us notice, that when $K = F(x)$ is the field of rational functions in indeterminate $x$ and $\phi: K \to K$ is an $F$-homomorphism given by $\phi(x) = x^2$, then the resulting right duo ring is exactly an old example of Asano (see Exercise 22.4A[3] and comments hereafter).
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Henceforth $A$ will stand for a commutative domain, $P$ for a maximal ideal of $A$, $K$ will denote the field $A/P$ and $\phi: A \to K$ a fixed injective homomorphism of rings. For any element $a \in A$, $\overline{a}$ will denote the canonical image of $a$ in $K = A/P$.

By Lemma 2.6 the right $K$ vector space $vK$ with the basis $\{v\}$ has a structure of $(A - A)$-bimodule given by $a \cdot vk = v\phi(a)k$ and $vk \cdot a = v\kappa a$, for any $a \in A$ and $k \in K$. $vK$ is faithful as a left $A$-module and simple as a right $A$-module. Thus, by Theorem 2.4 $R = A \oplus vK$ is a right duo ring.

From now on, $\sigma: R \to R$ stands for the endomorphism of $R$ given by $\sigma(a + v\ell) = a$, for any $a, \ell \in A$.

**Lemma 3.1.** Let $d_y$ denote the inner $\sigma$-derivation of $R$ determined by the element $y = c + v\ell \in A \oplus vK = R$, where $c, m \in A$. Then:

1. $d_y(a + v\ell) = y(a + v\ell)I = v\phi(c)\ell + v\ell(\overline{a} - \phi(a)) \in vK$, for any $a + v\ell \in R$. In particular, $d_y(v) = v\phi(c)$.

2. If $d_y(A) = 0$, then $d_y(a + v\ell) = v\phi(c)\ell$, for any $a + v\ell \in R$.

**Proof.** (1) By definition,

$$d_y(a + v\ell) = y(a + v\ell) - \sigma(a + v\ell)y = ca + v\ell a + v\phi(c)\ell - ac - v\phi(a)\ell = v\phi(c)\ell + v\ell(\overline{a} - \phi(a)).$$

Taking $a = 0$ and $\ell = 1$, we obtain $d_y(v) = v\phi(c)$.

(2) If $d_y(A) = 0$ then, by the statement (1), $v\ell(\overline{a} - \phi(a)) = 0$, for all $a \in A$, i.e. $d_y(a + v\ell) = v\phi(c)\ell$, for $a, \ell \in A$. □

For $\overline{c} \in K$, define $\delta_\omega: R \to R$ by setting $\delta_\omega(a + v\ell) = v\overline{c}\ell$, for any $a, \ell \in A$. Keeping the above notion, we have:

**Lemma 3.2.** For any $\overline{c} \in K$, $\delta_\omega$ is a $\sigma$-derivation of $R = A \oplus vK$. Moreover $\delta_\omega$ is an inner $\sigma$-derivation iff $\overline{c} \in \phi(A)$.

**Proof.** Let $r_1 = a + v\ell, r_2 = b + v\ell \in R = A \oplus vK$, where $a, b, \ell, m \in A$. By making direct computations we have:

$$\delta_\omega(r_1r_2) = \delta_\omega(ab + v(\phi(a)\ell + \ell b)) = v\overline{c}(\phi(a)\ell + \ell b) \quad \text{and}$$

$$\sigma(r_1)\delta_\omega(r_2) + \delta_\omega(r_1)r_2 = a \cdot v\overline{c}\ell + v\overline{c}\ell(b + v\ell) = v\overline{c}\phi(a)\ell + v\overline{c}\ell b.$$

This shows that $\delta_\omega$ is a $\sigma$-derivation.

Notice that $\delta_\omega(A) = 0$ and, by Lemma 3.1(2), every inner $\sigma$-derivation of $R$ such that $\delta(A) = 0$ is of the form $\delta(a + v\ell) = v\phi(c)\ell$ for suitable $c \in A$. Hence $\delta_\omega$ is inner iff there is $c \in A$ such that $\overline{c} = \phi(c)$. This completes the proof of the lemma. □
Recall that Lemma 3.1 describes all inner \( \sigma \)-derivations of \( R \). Therefore, the following theorem gives a description of all \( \sigma \)-derivations of \( R \).

**Theorem 3.3.** Let \( \delta \) be a nonzero \( \sigma \)-derivation of \( R = A \oplus vK \). Then:

1. There exists \( \varpi \in K \) such that \( \delta(v) = v\varpi \).

2. If \( \delta(vK) = 0 \), then one of the following conditions holds:
   
   \( (a) \) \( \delta \) is an inner \( \sigma \)-derivation of \( R \)
   
   \( (b) \) \( \phi = \text{id}_K \), i.e. \( R = K \oplus vK \) is a commutative ring, \( \delta \) is an outer \( \sigma \)-derivation and there exists a derivation \( d \) of the field \( K \) such that \( \delta(a + vb) = vd(a) \), for any \( a, b \in K \).

3. Let \( \varpi \in K \) be such that \( \delta(v) = v\varpi \). Then \( (\delta - \delta_\omega)(vK) = 0 \), i.e. \( \delta - \delta_\omega \) is a \( \sigma \)-derivation satisfying the assumption of the statement (2).

**Proof.** (1). Let \( \delta(v) = a + v\varpi \), where \( a, \omega \in A \). Since \( \sigma(v) = 0 \), we have:

\[
0 = \delta(0) = \delta(v^2) = \sigma(v)\delta(v) + \delta(v)v = \delta(v)v = v\phi(a).
\]

Hence \( a = 0 \) follows, as \( \phi \) is injective. This gives (1).

(2). Let \( \delta \) be a \( \sigma \)-derivation of \( R = A \oplus vK \) such that \( \delta(vK) = 0 \). First we claim that \( \delta(A) \subseteq vK \). To this end, let \( a \in A \) and \( \delta(a) = c + v\sigma \in R \), for some \( c, \sigma \in A \). Then \( 0 = \delta(v\phi(a)) = \delta((a \cdot v)) = \sigma(a) \cdot \delta(v) + \delta(a) \cdot v = (c + v\sigma)\phi) \). Hence, as \( \phi \) is injective, \( c = 0 \). This proves the claim.

Since \( \delta \neq 0 \) and \( \delta(vK) = 0 \), we may pick an element \( a_0 \in A \) such that \( \delta(a_0) \neq 0 \). By the first part of the proof \( \delta(a_0) = v\varpi \), for some \( \varpi \in A \).

Let \( b \in A \). Then \( \delta(b) \in vK \), so \( a_0 \cdot \delta(b) = \delta(b)\phi(a_0) \) and \( \delta(b) \cdot a_0 = \delta(b)\varpi \). Computing \( \delta(a_0 b) = \delta(ba_0) \) we obtain \( v\varpi + a_0 \cdot \delta(b) = \delta(b) \cdot a_0 + v\phi(b) \). Using this, one can see that

\[
\delta(a_0 b) = \delta(b)\varpi + v\varpi - \phi(a_0) \quad \text{for any} \quad b \in A.
\]

Let \( c \in A \) be such that \( \varpi = a_0 - \phi(a_0) \in K \).

If \( \varpi = 0 \) then, using the equation (3.1) and the fact that \( \varpi \neq 0 \), we get \( b - \phi(b) = 0 \), for any \( b \in A \). Recall that \( \phi: A \to A/P = K \) is injective, and \( \bar{b} \) is a natural image of \( b \in A \) in \( K \). Therefore, \( P \) has to be equal to 0, i.e. \( A = K \) is a field and \( \phi = \text{id}_K \). Then \( R = K \oplus vK \) is a commutative ring (isomorphic to \( K[x]/(x^2) \)). By the first part of the proof of (2), \( \delta(K) \subseteq vK \). Therefore, for any \( a \in K \), \( \delta(a) = vd(a) \) for a suitable element \( d(a) \in K \). It is clear that the element \( d(a) \) is uniquely determined by \( a \). Moreover, as also \( \delta(vK) = 0 \), \( \delta(a + vb) = d(a) \), for any \( a, b \in K \). Now, it is standard to check that the map \( \delta: R \to R \), defined by the above formula, is a \( \sigma \)-derivation of \( R \) iff \( d: K \to K \) is a derivation of the field \( K \). Moreover, by Lemma 3.1(1), such nonzero \( \sigma \)-derivation is always outer, i.e. the statement (b) holds.
Suppose $0 \neq \overline{c} \in K$. Because $K$ is a field, there exists $c' \in A$ such that $\overline{cc'} = 1$. Then, again making use of the equation (3.1), we obtain $\delta(b) = vs\overline{c}(\overline{b} - \phi(b))$ for any $b \in B$. Now, the fact that $\delta(vK) = 0$ and Lemma 3.1(1) yield that $\delta = d_y$ is the inner $\sigma$-derivation determined by the element $y = vs\overline{c} \in R$, i.e. the statement (a) holds. This completes the proof of (2).

(3). By (1), there is $\omega \in K$ such that $\delta(v) = v\omega$. Notice that $(\delta - \delta_\omega)(v) = 0$ and $\sigma(v) = 0$. Therefore

$$(\delta - \delta_\omega)(vl) = (\delta - \delta_\omega)(v \cdot l) = \sigma(v) \cdot (\delta - \delta_\omega)(l) + (\delta - \delta_\omega)(v) \cdot l = 0,$$

for any $l \in A$. This means that $(\delta - \delta_\omega)(vK) = (\delta - \delta_\omega)(v \cdot A) = 0$ and the statement (3) follows.

The above theorem together with Lemma 3.2 give us immediately the following:

**Corollary 3.4.** Suppose that $R = A \oplus vK$ is noncommutative. For a $\sigma$-derivation $\delta$ of $R$, the following conditions are equivalent:

1. $\delta$ is an outer $\sigma$-derivation of $R$.

2. There exist $\overline{w} \in K$ and $y \in R$, such that:
   
   (i) $\overline{w} \notin \phi(A)$;
   
   (ii) $\delta = \delta_\omega + d_y$.

If $R$ is commutative (i.e. $\phi = \text{id}_A$) then, By Lemma 3.2, $\delta_\omega$ is an inner derivation of $R$, for any $\overline{w} \in K$. Thus, by the above theorem, we also get:

**Corollary 3.5.** Suppose $R = A \oplus vK$ is commutative. Then $R = K \oplus vK$ and for a $\sigma$-derivation $\delta$ of $R$, the following conditions are equivalent:

1. $\delta$ is an outer $\sigma$-derivation of $R$.

2. There exist a nonzero derivation $d$ of $K$, such that $\delta(a + vb) = vd(a)$, for any $a, b \in K$.

As a direct application of the above Corollaries and Lemma 1.1 we obtain the following classification of Ore extensions over our ring $R = A \oplus vK$:

**Proposition 3.6.** Let $\delta$ be a $\sigma$-derivation of $R = A \oplus vK$. Then:

1. Suppose that $R$ is noncommutative. Then the Ore extension $R[x; \sigma, \delta]$ is $R$-isomorphic either to $R[x; \sigma]$ or to $R[x; \sigma, \delta_\omega]$, for some $\overline{w} \in K \setminus \phi(A)$, where $\delta_\omega(a + vl) = v\overline{w}l$, for any $a, l \in A$.

2. Suppose that $R$ is commutative. Then $R = K \oplus vK$ and the Ore extension $R[x; \sigma, \delta]$ is $R$-isomorphic either to $R[x; \sigma]$ or to $R[x; \sigma, \hat{\delta}]$, where $\hat{\delta}(a + vb) = vd(a)$, for any $a, b \in K$, and $d$ denotes a nonzero derivation of $K$.
4 ORE EXTENSIONS WHICH ARE RIGHT DUO RINGS

We will continue to use the notation from the previous section. Recall that $A$ is a commutative domain, $K = A/P$, where $P$ is a fixed maximal ideal of $A$. $R = A \oplus vK$ is a split corner extension of $A$ with $(vK)^2 = 0$. The left and right actions of $A$ on the right $K$-linear vector space $vK$ are given by $a \cdot v = v\phi(a)$ and $v \cdot a = \overline{a}$, for $a \in A$.

Recall that $\sigma : R \to R$ denotes the endomorphism of $R$ given by $\sigma(a + vl) = a$ for any $a, l \in A$. For a fixed element $\overline{x} \in K$, $\delta_\omega$ stands for the $\sigma$-derivation of $R$ defined in Lemma 4.2, i.e. $\delta_\omega(a + vl) = v\overline{a}l$, for any $a, l \in K$.

**Lemma 4.1.** For any $a \in A$ and $f \in R[x; \sigma, \delta_\omega]$, we have:

1. $xa = ax$ and $xv = v\overline{x}$;
2. $vfv = 0$

In particular, $vR[x; \sigma, \delta_\omega]$ is a two-sided ideal of $R[x; \sigma, \delta_\omega]$ with $(vR[x; \sigma, \delta_\omega])^2 = 0$.

**Proof.** The easy proof is left to the reader. $\square$

For $\overline{x} \in K$, let $\phi_\omega$ denote the extension of $\phi$ to a homomorphism $\phi_\omega : A[x] \to K \subseteq K[x]$ given by $\phi_\omega(x) = \overline{x}$.

We also extend $\Phi : A \to K$ to a homomorphism $\Phi : A[x] \to K[x]$, by setting $\Phi(x) = x$.

Let $vK[x]$ denote the free right $K[x]$-module generated by the element $v$. Then $vK[x]$ has a structure of an $(A[x] - A[x])$-bimodule given by $f \cdot v = v\phi_\omega(f)$ and $v \cdot f = v\overline{f}$, for $f \in A[x]$. Thus we can consider the ring $T_\omega = A[x] \oplus vK[x]$. With the help of Lemma 4.1 one can easily check that:

**Lemma 4.2.** 1. If $f \in A[x] \subseteq R[x; \sigma, \delta_\omega]$, then $fv = v\phi_\omega(f)$
2. The map $\Phi : R[x; \sigma, \delta_\omega] \to T_\omega$ defined by $\Phi((a + vl)x^k) = ax^k \oplus v\overline{l}x^k$, for any $a, l \in A$, is an isomorphism of rings.
3. If $N$ is an $A[x]$-submodule of the right $A[x]$-module $vK[x] \subseteq T_\omega$, then $N$ is a $(A[x] - A[x])$-subbimodule of $vK[x]$.

**Definition 4.3.** For any polynomial $f = \sum_{k=0}^{n}(a_k + vl_k)x^k \in R[x; \sigma, \delta_\omega]$ we set:

1. $f_A = \sum_{k=0}^{n}a_k x^k \in A[x] \subseteq R[x; \sigma, \delta_\omega]$ and $f_v = f - f_A$.
2. $D_f = \sum_{k=0}^{n} \phi(a_k)\overline{\omega}^k \in K$, that is $D_f = \phi_\omega(f_A)$. 
Proof. Let \( \omega \in K \) be transcendental over the subfield generated by \( \phi(A) \subseteq K \), then \( D_f = 0 \) iff \( a_k = 0 \) for all \( 0 \leq k \leq n \), i.e. \( f = f_o \in vR[x; \sigma, \delta_\omega] \).

Combining Lemmas 4.1 and 4.2(1) we get the following:

**Remark 4.4.** Let \( f \in R[x; \sigma, \delta_\omega] \). Then: \( fv = f_Av = vD_f \). In particular, \( fv = 0 \) iff \( D_f = 0 \).

Now we are in position to prove the following:

**Proposition 4.5.** For a polynomial \( f \in R[x; \sigma, \delta_\omega] \), the following conditions are equivalent:

1. \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \);

2. One of the following conditions holds:

   (a) \( D_f \neq 0 \);
   
   (b) \( f_A = 0 \), i.e. \( f \in vR[x; \sigma, \delta_\omega] \);
   
   (c) \( vf = 0 \) and \( f = f_A \).

**Proof.** Let \( f = \sum_{k=0}^{n}(a_k + v\omega^k)x^k \in R[x; \sigma, \delta_\omega] \), with \( a_n + v\omega^n \neq 0 \).

(2) \( \Rightarrow \) (1). By Lemma 4.2, the ring \( R[x; \sigma, \delta_\omega] \) is isomorphic to \( T_\omega = A[x] \oplus vK[x] \). Let \( f_A = \sum_{k=0}^{n}a_kx^k \in A[x] \). Notice that \( D_f = D_{f_A} = \phi_\omega(f_A) \).

Suppose that \( D_f \neq 0 \). Then, by Lemma 4.2, \( f_AvK[x] = v\phi_\omega(f_A)K[x] = vK[x] \). Therefore, by Proposition 2.2(1), \( f_AT_\omega = fT_\omega \) is a two-sided ideal of \( T_\omega \). This yields that \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \), provided \( D_f \neq 0 \).

Suppose \( f_A = 0 \), i.e. all coefficients of \( f \) are in \( vK \). In this case, Lemma 4.2(3) implies that \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \).

Finally, suppose that the statement (c) of (2) holds. Let \( g \in R[x; \sigma, \delta_\omega] \). Since \( vf = 0 \) and \( f = f_A \), we obtain \( gf = gAf = gAf_A = fAgA = fgA \). This shows that \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \).

(1) \( \Rightarrow \) (2). Let \( fR[x; \sigma, \delta_\omega] \) be a two-sided ideal of \( R[x; \sigma, \delta_\omega] \), where \( f = \sum_{k=0}^{n}(a_k + v\omega^k)x^k \in R[x; \sigma, \delta_\omega] \) with \( a_n + v\omega^n \neq 0 \). Suppose that \( D_f = 0 \). We shall prove that either (b) or (c) of the statement (2) holds.

By Remark 4.4, for any \( g \in R[x; \sigma, \delta_\omega] \) we have \( fg = f_gA \). We claim that \( vf = 0 \). Since \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal, \( vf \in fR[x; \sigma, \delta_\omega] \). Thus, for some \( g \in R[x; \sigma, \delta_\omega] \) we have:

\[
(4.1) \quad vf = vA = fg = f_A.
\]

Then \( vf = (f + v)xgA = fAgA + v_xgA \) implies \( 0 = fAgA \) in the domain \( A[x] \), whence \( f_A = 0 \) or \( g_A = 0 \). In either case, Equation (4.1) implies \( vf = 0 \).

It now suffices to show that either \( f_o = 0 \) or \( f_A = 0 \). To this end, assume that \( f_A \neq 0 \). Since \( vf = 0 \), we have \( f_A \in P[x] \subseteq A[x] \) (where \( A/P = K \)). Thus,
in particular, \( P \neq 0 \). Choose a nonzero element \( p \in P \). Since \( pf \in fR[x; \sigma, \delta_\omega] \), for some \( h \in R[x; \sigma, \delta_\omega] \) we have:

\[(4.II) \quad pf = f_{AP} + f_v \phi(p) = fh = f_A h_A + f_v h_A.\]

Hence \( f_{AP} = f_A h_A \) in the domain \( A[x] \), so \( h_A = p \in P \). Therefore \( f_v h_A = f_v p = f_v \overline{p} = 0 \) and Equation \((4.II)\) implies \( f_v \phi(p) = 0 \). Since \( p \neq 0 \) and \( \phi \) is injective, \( f_v = 0 \) follows. This completes the proof of the proposition. \( \square \)

Recall that for \( f = \sum_{k=0}^{n} (a_k + v_l k)x^k \in R[x; \sigma, \delta_\omega] \), \( D_f = \sum_{k=0}^{n} \phi(a_k)\overline{x}^k \). Notice that if either \( \overline{x} \) is transcendental over the subfield \( \overline{\phi(A)} \) of \( K \) generated by \( \phi(A) \) or \( \overline{x} \) is algebraic over \( \overline{\phi(A)} \) of degree greater than \( \deg f \), then \( D_f = 0 \) iff \( f \in vR[x; \sigma, \delta_\omega] \). Hence, by the above proposition, \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \).

On the other hand, if \( \overline{x} \) is algebraic, say of degree \( n \), then there exists a polynomial \( g = \sum_{k=0}^{n} a_k x^k \in A[x] \subseteq R[x; \sigma, \delta_\omega] \) of degree \( n \), such that \( D_g = 0 \). Then, by the above proposition, the right ideal \( (g + v)R[x; \sigma, \delta_\omega] \) is not two-sided. Therefore we obtain:

**Corollary 4.6.** Let \( \overline{x} \in K \) and \( \overline{\phi(A)} \) denote the subfield of \( K \) generated by \( \phi(A) \). Then:

1. If \( \overline{x} \) is transcendental over \( \overline{\phi(A)} \) then \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \), for any \( f \in R[x; \sigma, \delta_\omega] \), i.e. \( R[x; \sigma, \delta_\omega] \) is a right duo ring.

2. If \( \overline{x} \) is algebraic of degree \( n + 1 \) over \( \overline{\phi(A)} \), for some \( n \geq 0 \), then:
   
   (a) for every polynomial \( f \in R[x; \sigma, \delta_\omega] \) of degree \( \deg(f) \leq n \), \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \);
   
   (b) there exists a polynomial \( f \in R[x; \sigma, \delta_\omega] \) of degree \( n + 1 \) such that \( fR[x; \sigma, \delta_\omega] \) is not a two-sided ideal of \( R[x; \sigma, \delta_\omega] \).

When \( P = 0 \), i.e. \( R = K \oplus vK \), then \( vf \neq 0 \), for any polynomial \( f \in R[x; \sigma, \delta_\omega] \) with \( f_A \neq 0 \). Thus, in this case, Proposition 4.5 boils down to:

**Corollary 4.7.** Suppose \( R = K \oplus vK \) and \( f \in R[x; \sigma, \delta_\omega] \). Then \( fR[x; \sigma, \delta_\omega] \) is a two-sided ideal of \( R[x; \sigma, \delta_\omega] \) iff either \( D_f \neq 0 \) or \( f \in vR[x; \sigma, \delta_\omega] \).

Now we are in position to prove the following:

**Theorem 4.8.** Let \( A \) be a commutative domain with a maximal ideal \( P \), \( \phi : A \to A/P = K \) an injective homomorphism and \( R = A \oplus vK \) the associated unital split corner extension of \( A \). Then:

1. \( R[x; \sigma, \delta] \) is a quasi-duo ring.
2. The following statements are equivalent:

(a) \( R[x; \sigma, \delta] \) is a right duo ring;

(b) There exists \( \overline{\omega} \in K \), such that \( \overline{\omega} \) is transcendental over the subfield of \( K \) generated by \( \phi(A) \) and \( R[x; \sigma, \delta] \) is \( R \)-isomorphic to \( R[x; \sigma, \delta_{\omega}] \).

Proof. (1). By virtue of Proposition 3.6 it is enough to prove the statement in case the Ore extension \( R[x; \sigma, \delta] \) is as described in the proposition. Then, in any case, \( I = vK \) is a nilpotent \((\sigma - \delta)\)-stable ideal of \( R \) and \( R[x; \sigma, \delta]/IR[x; \sigma, \delta] \cong A[x] \) is commutative. Now, one can complete the proof of (1) using similar arguments as in the proof of Proposition 1.4.

(2). The implication \((b) \Rightarrow (a)\) is given by Corollary 1.6(1).

\((a) \Rightarrow (b)\). By Theorem 1.2(2), the Ore extension \( R[x; \sigma] \) is never right duo. Thus, in view of Proposition 3.6 and Corollary 4.6(2), it is enough to show that if \( R \) is commutative and \( \delta \) is of the form described in Proposition 3.6(2), then \( R[x; \sigma, \delta] \) is not right duo. To this end, suppose that \( R = K \oplus vK \) is commutative, \( d \) is a derivation of \( K \) and \( \delta(a + vb) = vd(a) \), for any \( a, b \in K \). We claim that \( vx \not\in xR[x; \sigma, \delta] \), i.e. \( xR[x; \sigma, \delta] \) is not a left ideal. Suppose that \( vx = xg \), for some polynomial \( g \in R[x; \sigma, \delta] \). Since \( \sigma(v) = \delta(v) = 0 \), \( xv = 0 \) and we may assume that \( g = \sum_{i=0}^{n} a_i x^i \), where \( 0 \neq a_n, a_{n-1}, \ldots, a_0 \in K \). Then \( \deg(vx) = \deg(xg) = n + 1 \) and \( g = a \in K \) follows. Hence \( vx = ax + vd(a) \), which is impossible. Thus \( vx \not\in xR[x; \sigma, \delta] \) and \( R[x; \sigma, \delta] \) is not right duo, provided \( R \) is commutative. \( \Box \)

Remark 4.9. One can easily check, with the help of Theorem 3.3(2)(b), that if \( R \) and \( \delta \neq 0 \) are as in the proof of the implication \((a) \Rightarrow (b)\) in the above theorem, then \( R[x; \sigma, \delta] \) satisfies all necessary conditions from Proposition 1.3 for an Ore extension to be right duo.

Example 4.10. Let \( K = L(x_i \mid i = 0, 1, \ldots), A = K[x], \phi \) and \( R = A \oplus M \) be as in Example 2.9. Then \( \phi(A) \subseteq L(x_i \mid i = 1, 2, \ldots) = L \). Thus \( x_0 \in K \) is transcendental over \( L \) and Theorem 4.8(2) implies that \( R[x; \sigma, \delta_{x_0}] \) is a right duo ring.

It is easy to construct an example of a field \( K \) with an endomorphism \( \phi \) such that all possibilities from Corollary 4.6 occur.

Example 4.11. Let \( R = K \oplus vK \), where \( K = F(X) \) is a field of rational functions over a field \( F \) in the set \( X = \{x_i \mid i = 0, 1, 2, \ldots\} \) of indeterminates and \( \phi: K \rightarrow K \) is an \( F \)-linear homomorphism defined by setting \( \phi(x_i) = (x_{i+1})^{i+1} \) for \( x_i \in X \).

It is easy to see that \( x_0 \) is transcendental over \( \phi(K) \) as \( \phi(K) \subseteq F(X \setminus \{x_0\}) \), while \( x_i \) is algebraic over \( \phi(K) \) of degree \( i \) for all \( x_i \in X \setminus \{x_0\} \).

Thus, by Corollaries 4.6 4.7 and Lemma 3.2 respectively, we have:
1. $R[x;\sigma,\delta_x]$ is a right duo ring.

2. If $k \geq 1$, then $(x^k-(x_k)^k)R[x;\sigma,\delta_x]$ is not a two-sided ideal of $R[x;\sigma,\delta_x]$ but $fR[x;\sigma,\delta_x]$ is a two-sided ideal, for all polynomials $f \in R[x;\sigma,\delta_x]$ with $\deg(f)<k$.

3. If $k \geq 2$, then $\delta_x$ is an outer $\sigma$-derivation of $R$.

One can check that, for any $k \geq 2$, the Ore extension $T=R[x;\sigma,\delta_x]$ from the above example satisfies all necessary conditions from Proposition 1.3 for the ring $T$ to be right duo. This means that conditions obtained by Marks in [5] for an Ore extension to be right duo are not sufficient.

We close the paper formulating the following problems:

**Problem 1.** Let $B$ be a ring. Find necessary and sufficient conditions, in terms of properties of $B$, $\tau$ and $\delta$, for an Ore extension $B[x;\tau,\delta]$ to be a right duo ring.

**Problem 2.** Let $B$ be a split corner subring of $T = B \oplus M$ with $M^2 = 0$. Find necessary and sufficient conditions, in terms of properties of $B$ and the $(B-B)$-bimodule $M$, for the ring $T$ to be right duo.

Theorems 2.4 and 4.8 provide examples of right duo rings of the form $B\oplus M$ as above. Those are of the form described in Proposition 2.2.

If $B$ is a commutative noetherian ring then, as the following proposition shows, $B[x;\tau,\delta]$ is never a one-sided duo ring, except the case $B[x;\tau,\delta] = B[x]$. Nevertheless, by Remark 4.9 there exist such noncommutative Ore extensions which satisfy all necessary conditions from Proposition 1.3. By Proposition 1.4 these Ore extensions are quasi-duo rings.

**Proposition 4.12.** Let $B$ be a commutative noetherian ring. If the Ore extension $B[x;\tau,\delta]$ is a right (left) duo ring, then $\tau = \text{id}_B$ and $\delta = 0$, i.e. $B[x;\tau,\delta] = B[x]$ is a commutative polynomial ring.

**Proof.** If $B[x;\tau,\delta]$ is left duo, then the thesis is a consequence of Theorem 1.2.

Suppose that $B[x;\tau,\delta]$ is a right duo ring which is noncommutative. Then, by Proposition 1.5, there exists a noncommutative Ore extension $R[x;\sigma,\delta]$ which is right duo, such that $R = A \oplus M$ is a split corner extension of a ring $A$ with $M^2 = 0$, where $M = \ker \sigma \neq 0$ and $\sigma|_A = \text{id}_A$. Since $R$ is a factor ring of $B$, $R$ is commutative and noetherian.

By Proposition 2.5 there is an ideal $J$ of $R$ such that $R/J \simeq A' \oplus M'$, where $M'$ is an $(A' - A')$-bimodule which is simple as a right $A'$-module and faithful as a left $A'$-module. Proposition 1.3 guarantees that $J$ is a $(\sigma - \delta)$-stable, so $R[x;\sigma,\delta]/(JR[x;\sigma,\delta]) \simeq (R/J)[x;\sigma,\delta]$, where $\sigma$ and $\delta$ denote also the maps induced on $R/J$, i.e. replacing $R$ by $R'$, we may assume that the commutative ring $R = A \oplus M$, where $M$ is simple as a right $A$-module and
faithful as a left $A$-module. Thus $R = A \oplus M$ is a ring considered in Sections 3 and 4. Now, since $R$ is commutative, Theorem 4.8(2) yields that $R[x; \sigma, \delta]$ is not right duo. This contradicts our assumption and completes the proof of the proposition.

\begin{flushright} \Box \end{flushright}

\section*{References}

[1] Brown, K.A.; Goodearl, K.R.; Lectures on algebraic quantum groups, Birkhauser Verlag, 2002.

[2] Hirano, Y.; Hong, C.-H.; Kim, J.-Y; Park, J.K.; On strongly bounded rings and duo rings, Comm. Algebra, 23(6) (1995), 2199-2214.

[3] Lam, T.Y.; Exercises in Classical Ring Theory, Problem Books in Mathematics, Springer, 2003.

[4] Lam, T.Y.; Corner Ring Theory: A Generalization of Peirce Decompositions I, Proc. International Conference in Algebras, Modules, and Rings, Marcel Dekker, New York (to appear).

[5] Marks, G.; Duo Rings and Ore extensions, J. Algebra 280(2) (2004) 463-471

\textbf{Acknowledgments.} I thank Greg Marks for careful reading the manuscript and for simplifying the arguments used in the proof of Proposition 4.5.