On the direct decomposition of nilpotent expanded groups

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Nilpotent groups

Theorem (classical result from group theory)

Let \( G \) be a finite nilpotent group. Then \( G \) is isomorphic to a direct product of groups of prime power order.

Sketch of the proof

Let \( S_p \) be a \( p \)-Sylow subgroup of \( G \). Since \( G \) is nilpotent, \( N_G(H) > H \) for all \( H < G \). By Sylow, \( N_G(N_G(S_p)) \leq N_G(S_p) \), hence \( N_G(S_p) = G \), and thus \( S_p \trianglelefteq G \).
Theorem (Characterisations of nilpotent groups)

Let $G$ be a finite group, $k \in \mathbb{N}$. TFAE

1. $G$ is nilpotent of class $k$
   :\iff \text{the lower central series } \gamma_1(G) := G, \gamma_n(G) := [G, \gamma_{n-1}(G)]
   \text{satisfies } |\gamma_k(G)| > 1, |\gamma_{k+1}(G)| = 1;

2. $k$ is minimal in $\mathbb{N}$ with
   \[\exists p \in \mathbb{R}[x] : \deg(p) = k \text{ and } \forall n : |F_{V(G)}(n)| \leq 2^{p(n)};\]

3. the supremum of “the rank of commutator terms of $G$” is $k$
   (see \cite{Kearnes, 1999});

4. $|[\mathbf{G}, \mathbf{G}, \ldots, \mathbf{G}]_k| > 1 \text{ and } |[\mathbf{G}, \mathbf{G}, \ldots, \mathbf{G}]_{k+1}| = 1$ (see \cite{Mudrinski, 2009}).
Nilpotence for expanded groups

**Definition (Nilpotent expanded groups)**

Let $V = \langle V, +, -, 0, f_1, f_2, \ldots \rangle$ be an expanded group, $A, B \subseteq V$.

$$[A, B] := \{ p(a, b) \mid p \in \text{Pol}_2(V), \quad a \in A, b \in B, p(0, 0) = p(a, 0) = p(0, b) = 0 \}. $$

$V$ is *nilpotent of class $k$* if for $\gamma_1(V) := V, \gamma_n(V) := [V, \gamma_{n-1}(V)]$ we have $|\gamma_k(V)| > 1, |\gamma_{k+1}(V)| = 1$.

**Remarks on $[\bullet, \bullet]$**

- In expanded groups, we consider *ideals = 0-classes of congruences* instead of congruences.
- $[A, B]$ then corresponds to the *term-condition commutator* introduced in [Freese and McKenzie, 1987, McKenzie et al., 1987].
A nilpotent expansion of $\langle \mathbb{Z}_6, + \rangle$

Let $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ be defined by

| $x$ | $f(x)$ |
|-----|--------|
| 0   | 3      |
| 1   | 0      |
| 2   | 0      |
| 3   | 3      |
| 4   | 0      |
| 5   | 0      |

Then $V_6 := \langle \mathbb{Z}_6, +, -, 0, f \rangle$ is nilpotent of class 2, and its congruence lattice is a three element chain.
Facts on $V_6$

**Lemma**

$V_6$ is directly indecomposable, and $|F_{\mathcal{V}(V_6)}(n)| \geq 2^{2^n}$ for all $n \in \mathbb{N}$. 
Kearnes’s decomposition theorem

As a corollary of [Kearnes, 1999, Theorem 3.14] and [Hobby and McKenzie, 1988, Lemma 12.4], one obtains:

Theorem ([Kearnes, 1999])

Let $A$ be a finite Mal’cev algebra such that $\exists p \in \mathbb{R}[x]$ with

$$|F_\nu(A)(n)| \leq 2^{p(n)} \text{ for all } n \in \mathbb{N}.$$ 

Then $A$ is nilpotent and isomorphic to a direct product of algebras of prime power order.

Theorem ([Berman and Blok, 1987, Theorem 2])

Let $A$ be finite, in a congruence modular variety, of finite type, nilpotent, direct product of algebras of prime power order. Then

$$\exists p \in \mathbb{R}[x] : |F_\nu(A)(n)| = 2^{p(n)} \text{ for all } n \in \mathbb{N}.$$
Absorbing polynomials and supernilpotence

**Definition**

\[ V = \langle V, +, -, 0, f_1, f_2, \ldots \rangle \] expanded group, \( p \in \text{Pol}_n V \). \( p \) is absorbing \( \iff \forall x : 0 \in \{ x_1, \ldots, x_n \} \Rightarrow p(x_1, \ldots, x_n) = 0. \)

**Definition (supernilpotent)**

\( V \) expanded group, \( k \in \mathbb{N} \). \( V \) is supernilpotent of class \( k \) : \( \iff \)

1. there is a nonconstant absorbing \( p \in \text{Pol}_k(V) \), and
2. \( \forall n > k \) all \( n \)-ary absorbing polynomials are constant.
Lemma (Description of finite snp expanded groups)

Let $W$ be a finite expanded group, $k \in \mathbb{N}$. TFAE

1. $W$ is supernilpotent of class $k \in \mathbb{N}$;
2. $k$ is minimal in $\mathbb{N}$ with
   
   \[ \exists p \in \mathbb{R}[x] : \deg(p) = k \text{ and } \forall n : |F_{\mathcal{V}(W)}(n)| \leq 2^{p(n)}; \]

3. the supremum of “the rank of commutator terms of $W$” is $k$ (see [Kearnes, 1999]);

4. $|[\underbrace{W, W, \ldots, W}_k]| > 1$ and $|[\underbrace{W, W, \ldots, W}_{k+1}]| = 1$ (see [Mudrinski, 2009]).
Connections between nilpotent and supernilpotent

**Lemma (Groups)**

Let $G$ be group. Then $G$ is nilpotent of class $k \iff G$ is supernilpotent of class $k$.

**Remark**

$\Rightarrow$ requires commutator calculus; calculations done in [Aichinger and Ecker, 2006].

**Lemma (Expanded groups)**

A supernilpotent expanded group of class $k$ is nilpotent of class $\leq k$.

**Corollary of [Berman and Blok, 1987, Theorem 2]**

A finite nilpotent expanded group of finite type and prime power order is supernilpotent.
Connections between nilpotent and supernilpotent

**Theorem (EA, Mudrinski, 2011)**

Let $k \geq 1$, $m \geq 2$, $V = \langle V, +, -, 0, f_1, f_2, \ldots \rangle$ expanded group such that all $f_i$ are “multilinear” and of arity $\leq m$, and $V$ is nilpotent of class $k$. Then $V$ is supernilpotent of class $\leq m^{k-1}$.

**Remark (the bound can be attained)**

For all $k \geq 1$, $m \geq 2$, there is a finite nilpotent $V$ of class $k$ with all $f_i$ “multilinear” and of arity $\leq m$ such that $V$ is supernilpotent of class $m^{k-1}$. 
Definition (Characteristic of a prime section)

Let $V$ be an expanded group, and let $A \prec B \trianglelefteq V$, $[B, B] \leq A$. Then $\text{char}(A, B)$ is the exponent of $\langle B/A, + \rangle$.

Remark

$R := \langle P_0(V)/\text{Ann}(B/A), +, \circ \rangle$ is a ring with simple module $M := B/A$. Hence $\text{char}(A, B)$ is the characteristic of the division ring $\text{End}_R(B/A)$.

Characteristic is prime or zero

Let $V$ be an expanded group, and let $A \prec B \trianglelefteq V$, $[B, B] \leq A$. Then $\text{char}(A, B) \in \mathbb{P} \cup \{0\}$. 
Monochromatic expanded groups

Definition (A generalisation of “prime power order”)

Let $V$ be a solvable expanded group. $V$ is *monochromatic* if all prime sections in the ideal lattice have the same colour.

Theorem (EA, 2012)

Let $V$ be a supernilpotent expanded group whose ideal lattice is of finite height. Then $V$ is isomorphic to a direct product of finitely many monochromatic expanded groups.
Proof of this decomposition result

**Lemma**

Let $\mathbf{R}$ be a ring with unit, and let $\mathbf{M}$ be a unitary $\mathbf{R}$-module such that $\mathbf{M}$ has exactly three submodules; let $Q$ be the submodule different from 0 and $M$. Then the exponents of the groups $\langle M/Q, + \rangle$ and $\langle Q, + \rangle$ are equal.

**Lemma (cf. [Mayr, 2008, Lemma 3])**

Let $\mathbf{V}$ be a finite expanded group whose ideal lattice is a three element chain $\{0\} < Q < V$. We assume that the exponents of the groups $\langle Q, + \rangle$ and $\langle V/Q, + \rangle$ are different, and that $[V, V] = Q$ and $[V, Q] = 0$. Then $\mathbf{V}$ is not supernilpotent.
Main tool in the proof

The operation of the polynomial ring

\[
M := \{ p \in \operatorname{Pol}_1 V : p( V ) \subseteq Q, \\
p \text{ is constant on each } Q\text{-coset}\},
\]
\[
R := \mathbb{Z}[t], \ w \in V,
\]
\[
r \star_w m(x) := \sum_{i=0}^{\deg(r)} r_i \star m(x + i \star w) \quad \text{for } m \in M, x \in V.
\]

Use of this operation

- For all \( m \in \mathbb{N} \), there is \( w \in V, f \in M \) such that
  \[
  (t - 1)^m \star_w f \text{ is not constant}.
  \]
- From this, we will produce absorbing polynomials of arbitrary arity.
### Task

Produce absorbing nonconstant polynomial of arity $m$.

### Define a sequence

- Choose $f \in M$, $w \in W$ such that $(t - 1)^{m-1} \ast_w f$ is not constant.
- Define
  - $h^{(1)}(x_1) := f(x_1) - f(0)$.
  - $h^{(n)}(x_1, \ldots, x_n) :=$
    
    \begin{align*}
    h^{(n-1)}(x_1 + x_n, x_2, \ldots, x_{n-1}) & - h^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + \\
    h^{(n-1)}(0, x_2, \ldots, x_{n-1}) & - h^{(n-1)}(x_n, x_2, \ldots, x_{n-1}).
    \end{align*}

- Then $h^{(n)}(x_1, w, \ldots, w) =$
  
  $$((t - 1)^{n-1} \ast_w f)(x_1) - ((t - 1)^{n-1} \ast_w f)(0)$$
  for all $x_1 \in V$. 

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