On \( MT \)-Convexity

Mevliüt TUNC and Hüseyin YILDIRIM

Abstract. In this paper, one new classes of convex functions which is called \( MT \)-convex functions are given. We also establish some Hadamard-type inequalities.

1. Introduction

The following definition is well known in the literature: A function \( f : I \to \mathbb{R} \), \( \emptyset \neq I \subseteq \mathbb{R} \), is said to be convex on \( I \) if inequality

\[
    f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \). Geometrically, this means that if \( P, Q \) and \( R \) are three distinct points on the graph of \( f \) with \( Q \) between \( P \) and \( R \), then \( Q \) is on or below chord \( PR \).

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and \( a, b \in I \) with \( a < b \). The following double inequality:

\[
    f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known in the literature as Hadamard’s inequality (or H-H inequality) for convex function. Keep in mind that some of the classical inequalities for means can come from (1.2) for convenient particular selections of the function \( f \). If \( f \) is concave, this double inequality hold in the inversed way.

In [3], Pachpatte established two Hadamard-type inequalities for product of convex functions.

Theorem 1. Let \( f, g : [a, b] \subseteq \mathbb{R} \to [0, \infty) \) be convex functions on \( [a, b] \), \( a < b \). Then

\[
    \frac{1}{b - a} \int_a^b f(x)g(x) \, dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)
\]

and

\[
    2f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)g(x) \, dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b)
\]

2000 Mathematics Subject Classification. 26D15.
Key words and phrases. convexity, AM-GM inequality.
where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

We recall the well-known AM-GM inequality for $n$ positive real numbers which can be stated as follows.

If $p_1, ..., p_n$ are positive numbers which sum to 1 and $f$ is a real continuous function that is concave up, then

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right)$$

A special case is

$$\sqrt[n]{x_1x_2...x_n} \leq \frac{x_1 + x_2 + ... + x_n}{n}$$

with equality iff $x_1 = x_2 = ... = x_n$.

We recall the well-known AM-GM inequality for two positive real numbers which can be stated as follows.

If $x, y \in \mathbb{R}^+$, then

$$\sqrt{xy} \leq \frac{x + y}{2}$$

with equality if and only if $x = y$.

This inequality has many simple proofs. For example, a proof based on the concavity of the logarithmic function is presented in various sources, and the original reference is Jensen’s paper [1]. A proof based on induction, given by Cauchy in 1821, is presented in many sources, as for example in [2], pp. 1–2.

In the following section our main results are given. We establish new a class of convex functions and then we obtain new Hadamard type inequalities for the new class of convex function.

**Definition 1.** [See [4]] Two functions $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are said to be similarly ordered, shortly $f$ s.o. $g$, if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for every $x, y \in X$.

**2. MT–Convexity and Related Results**

**Remark 1.** If we take $x = t$ and $y = 1 - t$ in (1.4), we have

$$1 \leq \frac{1}{2\sqrt{t(1-t)}}$$

for all $t \in (0, 1)$.

**Definition 2.** A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of $MT(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the inequality;

$$f(t x + (1 - t) y) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{\sqrt{1 - t}}{2\sqrt{t}} f(y).$$

**Remark 2.** In (2.2), if we take $t = 1/2$, inequality (2.2) reduce to Jensen convex.
THEOREM 2. Let \( f \in MT (I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1 [a, b] \). Then

\[
(2.3) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx
\]

and

\[
(2.4) \quad \frac{2}{b - a} \int_a^b \tau(x) \, f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

where \( \tau(x) = \sqrt{\frac{(b-x)(x-a)}{b-a}} \), \( x \in [a, b] \).

PROOF. Since \( f \in MT (I) \), we have, for all, \( x, y \in I \) (with \( t = \frac{1}{2} \) in (2.2)) that

\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}
\]

i.e. with \( x = ta + (1-t)b \), \( y = (1-t)a + tb \),

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{2} (f(ta + (1-t)b) + f((1-t)a + tb)).
\]

By integrating, we get

\[
(2.5) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{2} \left[ \int_0^1 f(ta + (1-t)b) \, dt + \int_0^1 f((1-t)a + tb) \, dt \right],
\]

Since

\[
\int_0^1 f(ta + (1-t)b) \, dt = \int_0^1 f((1-t)a + tb) \, dt = \frac{1}{b - a} \int_a^b f(u) \, du,
\]

we get the inequality (2.3) from (2.5).

For the proof of (2.4), we first note that if \( f \in MT (I) \), then for all \( a, b \in I \) and \( t \in [0, 1] \), it yields

\[
2 \sqrt{t(1-t)} f(ta + (1-t)b) \leq tf(a) + (1-t) f(b)
\]

and

\[
2 \sqrt{t(1-t)} f((1-t)a + tb) \leq (1-t) f(a) + tf(b).
\]

By adding these inequalities and integrating on \( t \) over \([0, 1] \), we obtain

\[
(2.6) \quad \int_0^1 \sqrt{t(1-t)} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] dt \leq \frac{f(a) + f(b)}{2}
\]

Therefore,

\[
(2.7) \quad \int_0^1 \sqrt{t(1-t)} f(ta + (1-t)b) \, dt = \int_0^1 \sqrt{t(1-t)} f((1-t)a + tb) \, dt
\]

\[
= \frac{1}{b - a} \int_a^b \frac{\sqrt{(b-x)(x-a)}}{(b-a)^2} f(x) \, dx.
\]

We get (2.4) by combining (2.6) with (2.7) and the proof is completed.

The constant 1 in (2.3) is the best possible because this inequality obviously reduces to an equality for the function \( f(x) = 1 \) for all \( a \leq x \leq b \). Additionally,
Moreover, this function is to be in the class $MT(I)$, because
\[
\frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{(1-t)}}{2\sqrt{t}} f(y) \geq \frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{(1-t)}}{2\sqrt{t}} = g(t)
\]
\[
\geq \min g(t) = g\left(\frac{1}{2}\right) \quad 0 < t < 1
\]
\[
= 1 \geq f(tx + (1-t)y)
\]
for all $x, y \in [a, b]$ and $t \in [0, 1]$. Thus, the proof is completed. \qed

\textbf{Remark 3.} In (2.3), if we take $x = \frac{a+b}{2}$, inequality (2.3) reduce to Jensen’s inequality.

\textbf{Theorem 3.} Let $f \in MT(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then
\[
\frac{\pi}{2} f\left(\frac{a+b}{2}\right) \leq f(a) + f(b).
\]

\textbf{Proof.} Since $f \in MT(I)$, we have
\[
f\left(\frac{a+b}{2}\right) \leq f\left(\frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}\right)
\]
\[
\leq \frac{1}{2} \left( f(ta + (1-t)b) + f((1-t)a + tb) \right)
\]
\[
\leq \frac{1}{2} \left( \frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{(1-t)}}{2\sqrt{t}} \right) (f(a) + f(b))
\]
Moreover
\[
4\sqrt{t}(1-t)f\left(\frac{a+b}{2}\right) \leq f(a) + f(b)
\]
By integrating, we get
\[
4f\left(\frac{a+b}{2}\right) \int_0^1 \sqrt{t}(1-t)dt \leq f(a) + f(b)
\]
\[
\frac{\pi}{2} f\left(\frac{a+b}{2}\right) \leq f(a) + f(b).
\]
The proof is completed. \qed

\textbf{Theorem 4.} Let $f, g \in MT(I)$, $a, b \in I$ with $a < b$ and $fg \in L_1[a,b]$. Then we have the inequality
\[
(2.8) \quad \frac{1}{b-a} \int_a^b \mu(x) f(x) g(x) \, dx
\]
\[
\leq \frac{1}{12} [f(a)g(a) + f(b)g(b)] + \frac{1}{24} [f(a)g(b) + f(b)g(a)]
\]
where $\mu(x) = \frac{(b-x)(x-a)}{(b-a)^2}$, $x \in [a,b]$. 

ON MT–CONVEXITY

Proof. Since \( f, g \in MT(I) \), we have
\[
\begin{align*}
    f(t a + (1 − t) b) & \leq \frac{\sqrt{t}}{2 \sqrt{1 − t}} f(a) + \frac{\sqrt{1 − t}}{2 \sqrt{t}} f(b) \\
    g(t a + (1 − t) b) & \leq \frac{\sqrt{t}}{2 \sqrt{1 − t}} g(a) + \frac{\sqrt{1 − t}}{2 \sqrt{t}} g(b)
\end{align*}
\]
Since \( f \) and \( g \) are nonnegative, we write that
\[
\begin{align*}
    & f(t a + (1 − t) b) g(t a + (1 − t) b) \\
    \leq & \frac{t}{4 (1 − t)} f(a) g(a) + \frac{1}{4} (f(a) g(b) + f(b) g(a)) + \frac{1 − t}{4t} f(b) g(b) \\
    = & \frac{t^2}{4t (1 − t)} f(a) g(a) + \frac{t (1 − t)}{4t (1 − t)} (f(a) g(b) + f(b) g(a)) + \frac{(1 − t)^2}{4t (1 − t)} f(b) g(b)
\end{align*}
\]
Consequently,
\[
\begin{align*}
    t (1 − t) f(t a + (1 − t) b) g(t a + (1 − t) b) \\
    \leq & \frac{1}{4} \left\{ t^2 f(a) g(a) + t (1 − t) [f(a) g(b) + f(b) g(a)] + (1 − t)^2 f(b) g(b) \right\}
\end{align*}
\]
By integrating, we get
\[
\begin{align*}
    & \frac{1}{b − a} \int_a^b \frac{(b − x)(x − a)}{(b − a)^2} f(x) g(x) dx \\
    \leq & \frac{1}{12} [f(a) g(a) + f(b) g(b)] + \frac{1}{24} [f(a) g(b) + f(b) g(a)].
\end{align*}
\]
The proof is completed. \( \square \)

Remark 4. If we choose \( x = \frac{a + b}{2} \) in the inequality (2.5), we obtain the special state of the inequality (1.3).

Theorem 5. Let \( f, g \) be similarly ordered, nonnegative and MT–convex functions on \( I, a, b \in I \) with \( a < b \) and \( f g \in L_1[a, b] \). Then we have the inequality
\[
\frac{1}{b − a} \int_a^b \mu(x) f(x) g(x) dx \leq \frac{f(a) g(a) + f(b) g(b)}{8}
\]
where \( \mu(x) = \frac{(b−x)(x−a)}{(b−a)^2} \), \( x \in [a, b] \).

Proof. Since \( f, g \in MT(I) \), we have
\[
\begin{align*}
    f(t a + (1 − t) b) & \leq \frac{\sqrt{t}}{2 \sqrt{1 − t}} f(a) + \frac{\sqrt{1 − t}}{2 \sqrt{t}} f(b) \\
    g(t a + (1 − t) b) & \leq \frac{\sqrt{t}}{2 \sqrt{1 − t}} g(a) + \frac{\sqrt{1 − t}}{2 \sqrt{t}} g(b)
\end{align*}
\]
Since \( f \) and \( g \) are nonnegative and similarly ordered, we write that
\[
f(ta + (1 - t)b)g(ta + (1 - t)b) \leq \frac{t}{4(1 - t)}f(a)g(a) + \frac{1}{4}(f(a)g(b) + f(b)g(a)) + \frac{1 - t}{4t}f(b)g(b)
\]
\[
= \frac{t^2}{4t(1 - t)}f(a)g(a) + \frac{t(1 - t)}{4t(1 - t)}(f(a)g(b) + f(b)g(a)) + \frac{(1 - t)^2}{4t(1 - t)}f(b)g(b)
\]
\[
\leq \frac{t^2}{4t(1 - t)}f(a)g(a) + \frac{t(1 - t)}{4t(1 - t)}(f(a)g(a) + f(b)g(b)) + \frac{(1 - t)^2}{4t(1 - t)}f(b)g(b)
\]
Consequently,
\[
4t(1 - t)f(ta + (1 - t)b)g(ta + (1 - t)b) \leq tf(a)g(a) + (1 - t)f(b)g(b)
\]
By integrating, we get
\[
\frac{4}{b - a} \int_a^b \mu(x)f(x)g(x)dx \leq \frac{1}{2}[f(a)g(a) + f(b)g(b)].
\]
The proof is completed. \( \square \)

References

[1] J.L.W.V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, Acta Mathematica 30 (1906), 175–193.
[2] B. Bollobás, Linear Analysis, an introductory course (Cambridge Univ. Press 1990).
[3] B.G. Pachpatte, On some inequalities for convex functions, RGMIA Res. Rep. Coll., 6 (E), 2003.
[4] H.J. Skala, On the characterization of certain similarly ordered super-additive functionals, Proceedings of the American Mathematical Society, 126 (5) (1998), 1349-1353.