Localization in an interacting quasi-periodic fermionic chain

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We consider a many body fermionic system with an incommensurate external potential and a short range interaction in one dimension. We prove that, for certain densities and weak interactions, the zero temperature thermodynamical correlations are exponentially decaying for large distances, a property indicating persistence of localization in the interacting ground state. The analysis is based on Renormalization Group, and convergence of the renormalized expansion is achieved using fermionic cancellations and controlling the small divisor problem assuming a Diophantine condition for the frequency.

1. INTRODUCTION AND MAIN RESULTS

A. Introduction

The properties of a fermionic system, like the conduction electrons in a metal, are determined, when the interaction between particles is not taken into account, by the eigenfunctions of the single particle hamiltonian. In presence of an external periodic potential, the eigenfunctions are Bloch waves, and the zero temperature a.c. conductivity is vanishing (insulating behavior) or not (metallic behavior) depending if the Fermi level lies in correspondence of a gap in the single particle spectrum or not. A different way in which an external potential can produce an insulating behavior is known as Anderson localization [1]; in presence of certain potentials (like random ones, physically describing the presence of unavoidable impurities in the metal) the eigenfunctions of the single particle Hamiltonian can be exponentially localized and this produces an insulating behavior. Localization in the single particle Schrödinger equation with a random field has been indeed rigorously proved in various regimes of energy and disorder, starting from [2],[3]. Note that in one dimension typically any amount of disorder produces localization, while in three dimensions the disorder has to be sufficiently strong and a metal to insulator transition is expected varying the strength of the random field. Localization does not necessarily require disorder, as it has long been known [4] that also nonrandom systems with quasi-periodic potentials (or incommensurate in the lattice case) can present single particle localization. The one dimensional quasi-periodic Schrödinger equation has extended Bloch-Floquet eigenfunctions in the weak coupling regime [5],[6] and localized eigenfunctions in the strong coupling regime, see [7],[8],[9],[10], provided that some Diophantine condition is assumed. In the lattice case with a cosine potential, the weak or strong coupling regime are connected by a duality transformation [4], and in this case it can be proved [11] that the spectrum is a Cantor set. In any case, the case of 1D quasi-periodic potential resembles the 3D random case, as there is a transition between an extended and localized phase varying the strength of the potential.

A realistic description of metals must include the electron-electron interaction, so that the problem of the interplay between localization and interactions naturally arises [12]. In the physical literature the zero temperature thermodynamical properties of 1D interacting fermions with disorder has been analyzed in [13], [14], finding localized and delocalized regions; the quasi-periodic case have been studied in [15]. While such works concern the computation of the zero temperature thermodynamical quantities, in more recent times attention has been devoted also to the localization properties of excited states of interacting disordered many body systems, see [16],[17]. Evidence has been found that in several interacting systems with disorder all the eigenfunctions are localized for weak interactions, while stronger interactions can destroy localization, leading to a so-called many-body localization transition; similar properties has been found also in the quasi-periodic case [18], [19].

It should however remarked that not only the results about the excited states of the N-particle Hamiltonian but even the ground state properties (that is, the zero temperature thermodynamical quantities) are based on conjectures or approximations, and more quantitative results based on rigorous methods seem necessary. In particular, there is still no a rigorously established example of an interacting many body system in which localization, which was present in the non interacting single particle state, still persists in presence of interaction. The mathematical tools used for single particle localization in the disordered case can actually treat the case of a finite number of interacting disordered particles, see [20]. By using KAM or block Jacobi procedure, localization of most eigenstates (in the sense that the expectations of local observables are exponentially decaying) has been rigorously proved in [21] (see also [22]) in a many body interacting disordered fermionic chain , under a physically reasonable assumption that limits the amount of level attraction in the system. Evidence of localization for finite times in interacting disordered bosons has been found in [23].

There exist powerful methods, based on the version of Renormalization Group (RG) developed for constructive Quantum Field Theory, to compute the thermodynamical properties at zero temperature of interacting fermions. Such
techniques encounter at the moment some difficulty in the application to random fermions, but can be successfully applied in the case quasi-periodic or incommensurate potentials; this is not surprising as quasi-periodic potentials produce small divisors similar to the ones in the KAM Lindstedt series, whose convergence can be established by RG methods, see [24],[25]. We will therefore analyze the interplay of localization and interaction in the thermodynamical functions of interacting fermions with a quasi-periodic potential by RG methods. In particular, if Λ is a one dimensional lattice Λ = \{x ∈ \mathbb{Z},-[L/2] ≤ x ≤ [(L−1)/2]\}, we consider a system of fermions with Hamiltonian

\[ H_N = -\varepsilon \sum_{i=1}^{N} \partial_x^2 + u \sum_{i=1}^{N} \phi_x + \lambda \sum_{i,j=1}^{N} v(x_i - x_j) \]  

where \( \partial_x^2 f(x) = f(x + 1) + f(x - 1) - 2f(x) \), \( \phi_x = \bar{\phi}(\omega x) \) with \( \bar{\phi}(t) = \hat{\phi}(t + 1) \), \( \omega \) irrational and \( v(x - y) = \delta_{y,x+1} \) for definiteness. When \( \phi_x = \cos(\omega x 2\pi) \) and \( \lambda = 0 \) the above model is the interacting version of the Aubry-André model [4]. In absence of interactions between particles \( \lambda = 0 \) the eigenfunctions of \( H_N \) are the antisymmetric product of the single particle eigenfunctions of Schrödinger equation

\[ -\varepsilon \psi(x + 1) - \varepsilon \psi(x - 1) + u \phi_x \psi(x) = E \psi(x) \]  

which were extensively analyzed [5],[6],[7],[8],[9],[10]. In principle, the thermodynamical quantities could be obtained from such studies but, as a matter of fact, even in the \( \lambda = 0 \) case the only available results on the zero temperature properties of (1) were obtained by RG methods for functional integrals. Indeed in [26] the Grand canonical correlations with \( \lambda = 0 \) were written in terms of an expansion plagued by a small divisor problem, and convergence was proved in [26], for small \( \frac{\beta}{\varepsilon} \), suitable chemical potentials and assuming a Diophantine condition on the frequencies, that is \( |2\pi n\omega|_{2\pi} ≥ Cn^{-\frac{4}{3}} \) for any \( n \in \mathbb{Z}/\{0\} \) where \( ||.||_{2\pi} \) is the norm on the one dimensional torus with period \( 2\pi \). It was found a power law or an exponential decay of the zero temperature correlations at large distances depending if the chemical potential is inside a gap or not; that is metallic or a band insulator behavior. In the opposite limit when \( \varepsilon/\beta \) is large in the non interacting case \( \lambda = 0 \) it was proved in [27] that the correlations decay exponentially for suitable values of the chemical potential, in agreement with the localization properties of the single particle eigenfunction.

The only rigorous result for quasi-periodic interacting fermions is in [28], in which it was proved that for small \( \frac{\beta}{\varepsilon} \) and small \( \lambda \) there is still a power law decay of correlations for values of the chemical potential outside the gap, but the exponent is anomalous with a critical exponent signaling Luttinger liquid behavior. Therefore the metallic behavior, which was present in the non interacting case as consequence of the extended nature of the single particle eigenfunctions, persists also in presence of interaction (but one has a Luttinger liquid instead than a Fermi liquid). In addition if the chemical potential is inside a gap one has exponentially decay of correlation, and an anomalous exponent appears in the decay rate.

In the present paper we finally consider a system of interacting fermions with a large incommensurate potential, and we prove that, for suitable chemical potentials, the zero temperature thermodynamical correlations are exponentially decaying for large distances for weak interaction, a property indicating persistence of localization in the interacting ground state.

B. Thermodynamical quantities and solvable limits

We consider the Grand-canonical ensemble, in which one performs averages over the particle number. One introduces fermionic creation and annihilation operators \( a_x^+, a_x^- \) on the Fock space verifying \( \{a_x^-, a_y^-\} = \delta_{x,y} \delta_{x,y} \). The Fock space Hamiltonian corresponding to (1) can be written as

\[ H = -\varepsilon \sum_{x \in \Lambda} (a_{x+1}^+ a_x + a_x^+ a_{x+1}^-) + u \sum_{x \in \Lambda} \bar{\phi}(\omega x) a_x^+ a_x^- - \mu \sum_{x \in \Lambda} a_x^+ a_x^- + \lambda \sum_{x \in \Lambda} a_x^+ a_{x+1}^- a_{x+1}^- \]  

Using the Jordan-Wigner transformation the model can be mapped in the XXZ model with a coordinate dependent magnetic field \( h_x = \phi_x \).

Let us consider now the thermodynamical quantities in the grand-canonical ensemble. We consider the operators \( a_x^\pm = e^{x_0 H} a_x^\pm e^{-Hx_0} \), with

\[ x = (x,x_0), \quad 0 ≤ x_0 < \beta \]  

for some \( \beta > 0 \) (\( \beta^{-1} \) is the temperature); \( x_0 \) is the imaginary time and on it antiperiodic boundary conditions are imposed, that is, if \( a_x^\pm = a_{x,x_0,s}^\pm \), then \( a_{x,\beta}^\pm = -a_{x,0}^\pm \). The 2-point Schwinger function is defined as

\[ \frac{\text{Tr}[e^{-\beta H} T(a_x^+ a_y^-)]}{\text{Tr}[e^{-\beta H}]} = I(x_0 - y_0 > 0) \frac{\text{Tr}[e^{-\beta H} a_x^+ a_y^-]}{\text{Tr}[e^{-\beta H}]} - I(x_0 - y_0 ≤ 0) \frac{\text{Tr}[e^{-\beta H} a_x^+ a_y^-]}{\text{Tr}[e^{-\beta H}]} \]  

(5)
where $T$ is the time order product. The above quantity cannot be exactly computed, so that one has to rely on a perturbative expansion around some solvable limit. In particular the model is solvable in the free fermion limit ($\lambda = u = 0$), which is an extended phase and in the molecular limit, which is a localized phase; in order to investigate the interplay of localization and interaction we will perform an expansion around the molecular limit. Before doing that, let us discuss the main properties of the solvable limits.

In the free fermion limit, corresponding to $u = \lambda = 0$, the Hamiltonian can be written in diagonal form in momentum space. If we assume periodic boundary conditions and we set $a_x^\pm = \frac{1}{\beta} \sum_k e^{\pm ikx} \hat{a}_k^\pm$, with $k = \frac{2\pi}{L} n$ and $(\hat{a}_0^+, \hat{a}_0^-) = \delta_{x,x'} \delta_{k,k'}$ then ($\varepsilon = 1$ for definiteness)

$$H_0 = \sum_k (-\cos k + \mu) \hat{a}_k^+ \hat{a}_k^-$$  \hspace{1cm} (6)

The two points $\pm p_F$ are called Fermi points. The two point Schwinger function is equal to

$$G(x - y) = \frac{\text{Tr} \left[ e^{-\beta H_0} T(a_x^- a_y^+) \right]}{\text{Tr} \left[ e^{-\beta H_0} \right]} = \frac{1}{L} \sum_k e^{-ik(x-y)} \hat{G}(k, x_0 - y_0) =$$

$$= \frac{1}{L} \sum_k e^{-ik(x-y)} \left\{ \frac{e^{-(x_0 - y_0) \varepsilon(k)}}{1 + e^{\varepsilon(k)}} I(x_0 - y_0 > 0) - \frac{e^{-(\beta + x_0 - y_0) \varepsilon(k)}}{1 + e^{\varepsilon(k)}} I(x_0 - y_0 \leq 0) \right\}$$  \hspace{1cm} (7)

where $\varepsilon(k) = \mu - \cos k$ The function $\hat{G}(k, \tau)$ is defined only for $-\beta < \tau < \beta$, but we can extend it periodically over the whole real axis. The function $\hat{G}(k, \tau)$ is antiperiodic in $\tau$ of period $\beta$; hence its Fourier series is of the form

$$\hat{G}(k, \tau) = \frac{1}{\beta} \sum_{k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})} \hat{G}(k_0, k) e^{-ik_0 \tau}$$  \hspace{1cm} (8)

with

$$\hat{G}(k_0, k) = \int_0^\beta d\tau e^{ik \tau} \frac{e^{-\tau \varepsilon(k)}}{1 + e^{\varepsilon(k)}} = \frac{1}{\mp ik_0 + \varepsilon(k)}$$  \hspace{1cm} (9)

Note that the function $\hat{G}(k)$ is singular, in the limit $L \to \infty, \beta \to \infty$, at $k_0 = 0, k = \pm p_F$, with $\cos p_F = \mu$. $\pm p_F$ are the Fermi moments and close to them, that is for $k'$ small it behaves as

$$\hat{G}(k' \pm p_F, k_0) \sim \frac{1}{\mp ik_0 \pm \varepsilon(k')}$$  \hspace{1cm} (10)

Another solvable limit is the Molecular limit corresponding to $\lambda = \varepsilon = 0$. The Hamiltonian reduces to ($u = 1$ for definiteness)

$$H_0 = \sum_{x \in \Lambda} (\phi_x - \mu) a_x^+ a_x^-$$  \hspace{1cm} (11)

The 2-point function $g(x, y) = \langle T \{ a_x^- a_y^+ \} \rangle_{\beta, L}$ is equal to

$$g(x, y) = \delta_{x,y} \left\{ e^{-(x_0 - y_0)(\phi_x - \mu)} \frac{1}{1 + e^{-(\phi_x - \mu)}} I(x_0 - y_0 > 0) - \frac{e^{-(\beta + x_0 - y_0)(\phi_x - \mu)}}{1 + e^{-(\phi_x - \mu)}} I(x_0 - y_0 \leq 0) \right\} = \delta_{x,y} \hat{g}(x, x_0 - y_0)$$  \hspace{1cm} (12)

The function $\hat{g}(x, \tau)$ is defined only for $-\beta < \tau < \beta$, but we can extend it periodically over the whole real axis. This periodic extension is smooth in $\tau$ for $\tau \neq n\beta, n \in \mathbb{Z}$, but has a jump discontinuity at $\tau = n\beta$ equal to $(-1)^n$.

The function $g(x, y)$ is antiperiodic in $x_0 - y_0$ of period $\beta$; hence its Fourier series is of the form

$$g(x, y) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})} \hat{g}(x, k_0) e^{-ik_0 (x_0 - y_0)}$$  \hspace{1cm} (13)

with

$$\hat{g}(x, k_0) = \int_0^\beta d\tau e^{ik_0 \tau} \frac{e^{-\tau (\phi_x - \mu)}}{1 + e^{-\tau (\phi_x - \mu)}} = \frac{1}{\mp ik_0 + \phi_x - \mu}$$  \hspace{1cm} (14)
Let \( M \in \mathbb{N} \) and \( \chi(t) \) a smooth compact support function that is 1 for \( t \leq 1 \) and 0 for \( t \geq \gamma \), with \( \gamma > 1 \). Let \( D_{\beta} = D_{\beta} \cap \{ k_0 : \chi_0(\gamma^{-M}|k_0|) > 0 \} \), where \( D_{\beta} = \{ k_0 = \frac{2\pi}{\gamma} (n_0 + \frac{1}{2}), n_0 \in \mathbb{Z} \} \). If \( x_0 - y_0 \neq n\beta \), we can write

\[
g(x, y) = \lim_{M \to \infty} \frac{1}{\beta} \sum_{k_0 \in D_{\beta}} \chi(\gamma^{-M}|k_0|) \frac{e^{-ik_0(x_0 - y_0)}}{ik_0 + \phi_0 - \mu} \equiv \delta_{x, y} \frac{1}{\beta} \sum_{k_0 \in D_{\beta}} \frac{e^{-ik_0(x_0 - y_0)}}{ik_0 + \phi_0 - \mu} \equiv \lim_{M \to \infty} g(\leq M)(x, y)
\]

(15)

Because of the jump discontinuities, \( g(\leq M)(x, y) \) is not absolutely convergent but is pointwise convergent and the limit is given by \( g(x, y) \) at the continuity points, while at the discontinuities it is given by the mean of the right and left limits.

In particular, the above equality is not true for \( x_0 - y_0 = n\beta \), where the propagator is equal

\[
\tilde{g}(x, 0^-) \rightarrow_{\beta, L \to \infty} -I(\phi_x - \mu \leq 0)
\]

(16)

while the r.h.s. is equal to

\[
\frac{\tilde{g}(x, 0^-) + \tilde{g}(x, 0^+)}{2} \rightarrow_{\beta, L \to \infty} \left[ \frac{1}{2} - I(\phi_x - \mu \leq 0) \right]
\]

(17)

Assuming that \( \tilde{\phi}(x) \) is even, periodic \( \tilde{\phi}(x) = \tilde{\phi}(x + 1) \) and that there is only one \( \bar{x} \in (0, \frac{1}{\beta}) \) such that \( \mu = \tilde{\phi}(\bar{x}) \); setting \( x = x' \pm \bar{x} \) we have that, for small \( |\omega x'| \mod 1 \)

\[
\tilde{\phi}(x') + \rho \bar{x} - \mu = \rho \omega \bar{x} + r, \quad \rho = \pm
\]

(18)

therefore the 2-point function can be written as

\[
\tilde{g}(x' \pm \bar{x}, k_0) \sim \frac{1}{-ik_0 \pm r \omega \bar{x}}
\]

(19)

Note the similarity of (19) with (10); this analogy suggests to call \( \pm \bar{x} \) as Fermi coordinates, in analogy with the Fermi momenta \( \pm p_F \). In the special case of \( \phi_x = \cos(2\pi \omega x) \) (Almost-Mathieu operator), setting \( \varepsilon = u \)

\[
\tilde{G}(k, k_0) |_{k = 2\pi \omega x} = \tilde{g}(x, k_0)
\]

(20)

which is is a manifestation of the well known Aubry-duality.

C. Grassmann Integral representation

If \( B_{\beta, L} = \{ D_{\beta} \cup \Lambda \} \), we consider the Grassmann algebra generated by the Grassmannian variables \( \{ \psi_{x, k_0}^\pm \}_{x, k_0 \in B_{\beta, L}} \) and a Grassmann integration \( \int \[ \prod_{x, k_0 \in B_{\beta, L}} d\psi_{x, k_0}^- d\psi_{x, k_0}^+ \] \) defined as the linear operator on the Grassmann algebra such that, given a monomial \( Q(\psi^-, \psi^+) \) in the variables \( \psi_{x, k_0}^{\pm} \), its action on \( Q(\psi^-, \psi^+) \) is 0 except in the case \( Q(\psi^-, \psi^+) = \prod_{x, k_0 \in B_{\beta, L}} \psi_{x, k_0}^- \psi_{x, k_0}^+ \), up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommuting properties of the variables, by the condition

\[
\int \left[ \prod_{x, k_0 \in B_{\beta, L}} d\psi_{x, k_0}^+ d\psi_{x, k_0}^- \right] \prod_{x, k_0 \in B_{\beta, L}} \psi_{x, k_0}^- \psi_{x, k_0}^+ = 1
\]

(21)

We define also Grassmanian field as \( \psi_{x, k_0}^{\pm} = \frac{1}{\beta} \sum_{k_0 \in D_{\beta}} e^{\pm ik_0 x_0} \psi_{x, k_0}^{\pm} \) with \( x_0 = m_0 \frac{2\pi}{\gamma} \) and \( m_0 \in \{ 0, 1, ..., \gamma^M - 1 \} \). The "Gaussian Grassmann measure" is defined as

\[
P(d\psi) = \prod_{x, k_0 \in B_{\beta, L}} \beta d\psi_{x, k_0}^- d\psi_{x, k_0}^+ \tilde{g}(\leq M)(x, k_0) \exp\left\{ -\sum_{x, k_0} (\tilde{g}(\leq M)(x, k_0))^{-1} \psi_{x, k_0}^+ \psi_{x, k_0}^- \right\}
\]

(22)

We introduce the generating functional \( W_M(\phi) \) defined in terms of the following Grassmann integral (free boundary conditions in space are assumed)

\[
e^{W_M(\phi)} = \int P(d\psi)e^{-V^{(M)}(\psi) - B^{(M)}(\psi, \phi)}
\]

(23)
where $\psi^+_x$ and $\phi^+_x$ are Grassmann variables, $P(d\psi)$ has propagator
\[ g^{(\leq M)}(x, y) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0 \in \mathbb{D}_\beta} \chi(\gamma^{-M}[k_0]) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \phi_x - \phi_y} \] (24)
and $\int dx$ is a short form for $\int_{\mathbb{R}^d} \frac{1}{\beta} \sum_{x_0 \in \mathbb{Z}^d} \chi(\gamma^{-M}[k_0]) e^{-ik_0 x_0} dx_0$; moreover
\[ \mathcal{V}(M) = \lambda \int dx \psi^+_x \psi^+_x \psi^-_{x+e_1} \psi^-_{x+e_1} + \varepsilon \int dx [\psi^+_x \psi^-_{x+1} + \psi^+_x \psi^-_{x+1} + \psi^+_x \psi^-_{x+1} + \psi^+_x \psi^-_{x+1}] \]
\[ + \nu \int dx \psi^+_x \psi^+_x + \int dx \nu_C(x+1) \psi^+_x \psi^-_x + \int dx \nu_C(x) \psi^+_x \psi^-_{x+1} \psi^-_{x+1} \]
where
\[ \nu_C(x) = \frac{1}{2} \lambda \{ g(x,0^+) - \bar{g}(x,0^-) \} \] (25)
and $\bar{g}(x,0^-)$ was defined in (12). Finally
\[ B^{(M)}(\psi, \phi) = \int dx [\phi^+_x \psi^-_x + \psi^+_x \phi^-_x] \] (26)
Note that we expect that the chemical potential is modified by the interaction; in the analysis it is convenient to keep fixed the value of the Fermi coordinate in the free or interacting theory, therefore we write the chemical potential as $\phi_x + \nu$, where $\nu$ is a counterterm to be fixed so that the free and interacting Fermi coordinate are the same.

Let us define
\[ S^{M,\beta,L}_2(x, y) = \left. \frac{\partial^2}{\partial \phi^+_x \partial \phi^-_y} W_M(\phi) \right|_0 \] (27)
Note that $\lim_{M \to \infty} S^{M,\beta,L}_2$ can be written as a series in $\varepsilon, \lambda$ coinciding order by order with the series expansion for the Schwinger functions (5) with chemical potential $\mu = \phi_x + \nu$. Indeed each term of the series for (5) or $\lim_{M \to \infty} S^{M,\beta,L}_2$ can be expressed as a sum of integrals over propagators (respectively $g(x,y)$ (12) or $\lim_{M \to \infty} g^{(\leq M)}(x,y)$ (22)) which can be represented by Feynman graphs. The subset of graphs contributing to (5) and with no tadpoles coincides the the graphs contributing to $\lim_{M \to \infty} S^{M,\beta,L}_2$ and no vertices $\nu_C$. The integrands are different, as the propagators $g(x,y)$ (12) or $\lim_{M \to \infty} g^{(\leq M)}(x,y)$ (22)) are different at coinciding times. However the integrals are well defined and coincide, as the integrands of the graphs coincide except in a set of zero measure. Let us consider the remaining graphs. In the graphs with a tadpole in the expansion for $S^{M,\beta,L}_2$ there is a factor of the form
\[ g(x_1 - x) \nu_T(x+1) g(x_1 - x_2) \]
\[ \nu_T(x+1) = -\frac{\lambda}{2} [\bar{g}(x+1,0^+) + \bar{g}(x+1,0^-)] \] (28)
On the other hand, given a graph $G$ of this type, there is another graph $G$, which differs from it only because, in place of the term $\mathcal{V}(\phi)$ which produced the tadpole, there is a vertex $\nu_C(x+1)$. If we sum the values of $G$ and $\bar{G}$, we get a number which is equal to the value of $G$, with $-\lambda \bar{g}(x+1,1^-)$ replacing $\nu_T(x+1)$, so that the terms coincide with the analogous term in the expansion for (5). Therefore the perturbative expansion coincide. An analyticity argument, analogue to the one in Proposition 2.1 of [29], would allow to conclude the coincidence of (5) and $\lim_{M \to \infty} S^{M,\beta,L}_2$ beyond perturbation theory, once that the limit exists and certain analyticity properties are proved; this is quite standard and will be not repeated here for brevity, so we state our main results directly for the Grassmann integral.

D. Main result: localization in presence of interaction

We set $u = 1$ and we consider $\varepsilon, \lambda$ small. We define $T = \mathbb{R}/\mathbb{Z}$ the one dimensional torus, $||\theta||_1$ the norm, that is the absolute value of $\theta$ modulo 1 defined so that $0 \leq ||\theta||_1 \leq \frac{1}{2}$. Our main result is the following.

**Theorem 1.1** Let us consider $\phi_x = \tilde{\phi}(\omega x)$ an even function in $C^1(T)$, that is $\tilde{\phi}(x) = \tilde{\phi}(x+1)$ and $\phi_x = \phi_{-x}$, with $|\phi_x| \leq 1$ and $\phi(t)$ increasing for $0 < t < \frac{1}{2}$. We consider the 2-point function (27) with $\bar{x}$ half integer and so that $\partial_x \tilde{\phi}(\bar{x}) > 0$. Assume
\[ ||\omega x||_1 \geq C|x|^{-\gamma}, \text{ for any } 0 \neq x \in \mathbb{Z} \] (29)
For $\varepsilon$ small and $|\lambda| \leq \varepsilon^{2} + 2$ there exists a continuous function $\nu(\varepsilon, \lambda)$ such that, for any $N$, the limit
\[
\lim_{\beta \to \infty} \lim_{L \to \infty} \lim_{M \to \infty} S_2^{M, \beta, L}(x, y) = S_2(x, y)
\]
exists and verifies, for any $N \in \mathbb{N}$
\[
|S_2(x, y)| \leq C_N \frac{e^{-\kappa \log |\varepsilon|\pi n x |x-y|}}{1 + (|\sigma||x_0 - y_0|)^N}
\]
where $\sigma = O(\varepsilon^{2})$ and non vanishing and $\kappa, C_N$ positive constants.

The above theorem is proved by an expansion in $\lambda, \varepsilon$ around the molecular limit, considering the kinetic energy and and the many body interaction as perturbations and assuming the Fermi coordinate $x$ as equal to a half integer. If there is no interaction $\lambda = 0$, the exponential decay of the two point function is in agreement with the localization of the single particle eigenfunctions of the Schrödinger equation, see for instance lemma 4.3 of [9]. The above theorem says that the exponential decay persists in presence of interaction, for certain chemical potentials provided that the hopping is smaller than $O(\varepsilon^{3})$ for some positive $\gamma$, and the interaction is much smaller than the hopping. As the Grand-canonical averages reduces to the average over the ground state, such result indicates localization for the ground state eigenfunction of an interacting many body system.

A consequence of Theorem 1.1 combined with [28] is the existence of a quantum phase transition between an extended and a localized phase. Indeed it was proved in [28] in the small $\lambda, u$ case that even in presence of interaction the system has a metallic or a band insulating behavior; that is, for small $u$ and $\lambda (\varepsilon = 1)$ if $\mu = 1 - \cos pf$ then if $pf = mn\pi$ then $S_2(x, y)$ decays faster than any power (band insulator behavior) with rate $|\sigma| = O(1)$ where $\eta = \alpha \lambda + O(\lambda^2)$ with $\alpha > 0$ suitable constant; while if $|2pf + 2p\mu \pi n|_{2x} \geq C|n|^{-\gamma}$, for any $x \in \mathbb{Z}/\{0\}$ then it decays a a power law as $O((x - y)^{-1-\eta})$ (metallic behavior) with $\eta = b\lambda^{2} + O(\lambda^{3})$, $b$ a positive constant. Therefore, in presence of interaction increasing the amplitude of the quasi-periodic potential one moves from an extended to a localized phase.

### E. Sketch of the proof of Theorem 1.1 and contents

In order to prove Theorem 1.1 one has to face a small divisor problem resembling the one in KAM Linstedt series [24]; its origin lies in the fact that the expansion is in terms of sum of product of propagators $(-ik_0 + \phi_x - \phi_x)^{-1}$, and, due to the irrationality of $\omega$, propagators with very different $x$ can be very close. There are however essential differences with respect to KAM Linstedt series or in the non interacting $\lambda = 0$ case; in such cases the series can be represented in terms of tree diagrams, while in the present case the series are expressed in terms of diagrams with loops. The number of tree diagrams contributing to order $n$ in the perturbative expansion is $O(n!)$ and a $C_n^{\infty}$-bound on each diagram is sufficient for convergence; in presence of interaction, on the contrary, the number of diagrams $O(n!)^{\ell}$ and a similar bound on each diagram is not sufficient to achieve convergence. One has therefore to combine methods developed in the context of KAM with constructive Quantum Field theory techniques; in particular one has to use the fact that the fermionic expectations can be represented in terms of determinants.

We perform the analysis of the Grassmann integral (27) in an iterative way by using Renormalization Group methods. We start integrating the higher energy frequencies, see §2. Here there is not a small divisor problem but one has to show that the expansions are convergent using that the expansion can be written in term of Gram bounds. After the integration of the ultraviolet fields, we have to integrate the low energy modes (infrared scales) in which one has to face a small divisor problem, as discussed in §3. The theory is non-renormalizable according to power counting; the scaling dimension depends on the number of vertices in the subgraph, so that one has to improve the dimensions of all possible subgraphs with any number of external fields. In order to get such improvement, we have to exploit the incommensurability of the potential and take advantage from the diophantine condition on the frequency. One has to distinguish between two kind of terms in the effective potential, depending if the coordinates (measured from the Fermi coordinate) of the external fields are different (non-resonant terms) or equal (resonant terms). In the non resonant terms one uses the Diophantine condition to get good bounds, exploiting, roughly speaking, the idea that if the denominators associated to the external lines have similar small size but different coordinates, then the difference of coordinates is necessarily large (see lemma 3.2, 3.3 and 3.4 in §3.C). The result is somewhat similar to Bruno lemma as presented in [24], but new difficulties raise from the fact that the resonances have any number of external fields and not only two as in the non interacting case; in particular, one has to improve the bounds by a quantity proportional to the external lines for combinatorial reason, see §3.D. Regarding the resonances one uses that the local part of the terms with more than four external fields is vanishing. Moreover the resonances with two external fields produce a mass term implying an exponential decay in time; in particular the propagators associated to the two external fields have coordinate $x = x + n$ and $\phi_x = \phi_{x+n}$ either when $n = 0$ or $n = -2x$. The second case is responsible of the mass term while the first case produces the renormalization of the Fermi coordinates. Finally in §3.F we study also the flow of the running coupling constants and the two point functions, completing the theorem proof.
2. THE ULTRAVIOLET INTEGRATION

A. Ultraviolet and Infrared fields

We introduce a function $\chi_h(t, k_0) \in C^\infty(\mathbb{T} \times \mathbb{R})$, such that $\chi_h(t, k_0) = \chi_h(-t, -k_0)$ and $\chi_h(t, k_0) = 1$, if $\sqrt{k_0^2 + v^2|t|^2} \leq a \gamma^{-1}$ and $\chi_h(t, k_0) = 0$ if $\sqrt{k_0^2 + v^2|t|^2} \leq a \gamma^h$ with $a$ and $\gamma > 1$ suitable constants. We choose $a$ so that the supports of $\chi_0(\omega(x - \bar{x}), k_0)$ and $\chi_0(\omega(x + \bar{x}), k_0)$ are disjoint; note that the $C^\infty$ function on $\mathbb{T} \times \mathbb{R}$

$$\hat{\chi}^{u.v.}(\omega, k_0) = 1 - \chi_0(\omega(x - \bar{x}), k_0) - \chi_0(\omega(x + \bar{x}), k_0)$$

is equal to 0, if $\sqrt{k_0^2 + \phi_x^2} \leq b$, with $b$ a suitable constant. For reasons which will appear clear below, we choose $\gamma > 2\pi$. We can write then

$$g(x, y) = g^{(u.v.)(x, y)} + g^{(i.r.)(x, y)}$$

and

$$g^{(i.r.)(x, y)} = \sum_{\rho=\pm} g^{(\rho)(x, y)}$$

where

$$g^{(u.v.)(x, y)} = \frac{\delta_{x,y}}{\beta} \sum_{k_0 \in D_{\beta}} \chi(\gamma^{-M}|k_0|) \hat{\chi}^{u.v.}(\omega, k_0) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \phi_x - \phi_x}$$

$$g^{(\rho)(x, y)} = \frac{\delta_{x,y}}{\beta} \sum_{k_0 \in D_{\beta}} \chi_0(\omega(x - \rho \bar{x}), k_0) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \phi_x - \phi_x}$$

For definiteness, we start considering the generating function (23) with $\phi = 0$. The properties of Grassmann integrals imply that we can write

$$e^{W(0)} = \int P(d\psi)e^{-V(\psi)} = \int P(d\psi^{(i.r.)}) \int P(d\psi^{(u.v.)}) e^{-V(\psi^{(i.r.)}) + V^{(u.v.)})}$$

where $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ are gaussian Grassmann integrations with propagators respectively $g^{(u.v.)(x, y)}$ and $g^{(i.r.)(x, y)}$ and $\psi^{(u.v.)}$ and $\psi^{(i.r.)}$ are independent Grassmann variables. We can write

$$\int P(d\psi^{(u.v.)}) e^{-V(\psi^{(i.r.)}) + V^{(u.v.)})} = e^{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} E^{(v.n.)} e^{T^{(v.n.)}}} = e^{-\beta LE_0 - V^{(0)}(\psi^{(i,r.)})}$$

where $E^{T^{(v.n.)}}$ is the fermionic truncated expectation with respect to $P(d\psi^{(u.v.)})$; therefore

$$e^{W(0)} = e^{-\beta LE_0} \int P(d\psi^{(i.r.)}) e^{-V^{(0)}(\psi^{(i.r.)})}$$

where

$$\gamma^{(0)} = \sum_{n=2}^{\infty} \sum_{n_1} \int dx_{0,1} \ldots \sum_{n_2} \int dx_{0,2} \ldots \int dx_{0,n} W^{(h)}_{n}(x_1, \ldots, x_n) |\prod_{i=1}^{n} \psi^{(v.\varepsilon)}(x_i)^{(0)}|$$

Note that the kernel $W^{(h)}_{n}(x_1, \ldots, x_n)$ will contain in general Kronecker or Dirac deltas, and we define the $L_1$ norm as they would be positive functions.

**Lemma 2.1** The constant $E_0$ and the kernels $W^{(0)}_{n}$ are given by power series in $\lambda, \varepsilon, \nu$ convergent for $|\lambda|, |\varepsilon|, |\nu| \leq \varepsilon_0$, for $\varepsilon_0$ small enough and independent of $\beta, L, M$. They satisfy the following bounds:

$$|W^{(0)}_{n}|_{L_1} \leq \beta L C^n \varepsilon_0^k_n,$$

for some constant $C > 0$ and $k_n = \max\{1, n-1\}$. Moreover, $\lim_{M \to \infty} E_0$ and $\lim_{M \to \infty} W^{(0)}_{n}$ do exist and are reached uniformly, so that, in particular, the limiting functions are analytic in the same domain.
B. Proof of Lemma 3.1

We can write \( \chi(\gamma^{-M}|k_0|) = \sum_{j=-\infty}^{M-1} f_j(|k_0|) \) with, for \( j \leq M - 1, \) \( f_j(|k_0|) = \chi(\gamma^{-j}|k_0|) - \chi(\gamma^{-j+1}|k_0|) \) a smooth compact support function non vanishing for \( \gamma^{h-1} \leq |k_0| \leq \gamma^{h+1} \). Therefore

\[
g^{(u,v.)}(x, y) = \sum_{h=1}^{M} g^{(h)}(x, y), \tag{40}
\]

where

\[
g^{(h)}(x, y) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0} e^{ik_0(x_0-y_0)} - i k_0 + \phi_x - \phi_y \chi^{(u,v.)}(k_0, \omega x) f_h(|k_0|) = \delta_{x,y} g^{(h)}(x, x_0 - y_0) \tag{41}
\]

where we have used that \( \chi(\gamma^{-N}|k_0|) = \sum_{h=1}^{N} f_h(|k_0|) \), according to the definition after (15). By integration by parts, for any integer \( M \)

\[
|g^{(h)}(x, x_0 - y_0)| \leq \frac{C_M}{1 + |\gamma| |x_0 - y_0|} \tag{42}
\]

By using (40) we can write \( P(d\psi^{(u,v.)}) = \prod_{h=1}^{M} P(d\psi^{(h)}) \) and the corresponding decomposition of the field \( \psi^{(u,v.)}_{x,s} = \sum_{h=1}^{M} \psi^{(h)}_{x,s} \). Hence, we can integrate iteratively the fields \( \psi^{(h)} \), \( \psi^{(M-1)} \), ..., \( \psi^{(h)} \) with \( h \geq 1 \) and, if we define \( \psi^{(\leq 0)} = \psi^{(r,r)} \) and \( \psi^{(\leq h)} = \psi^{(r,r)} + \sum_{j=1}^{h} \psi^{(j)} \), if \( h \geq 0 \), we get:

\[
e^{W(0)} = e^{-L\beta E_h} \int P(d\psi^{(\leq h)}) e^{-V^{(h)}(\psi^{(\leq h)})} \tag{43}
\]

Let us consider first the effective potentials on scale \( h, V^{(h)}(\psi^{(\leq h)}) \). We want to show that they can be expressed as sums of terms, each one associated to an element of a family of labeled trees; we shall call this expansion the **tree expansion**.

The tree definition can be followed looking at Fig 1.

![Fig. 1: A tree \( \tau \in T_{h,n} \) with its scale labels.](image)

Let us consider the family of all trees which can be constructed by joining a point \( r \), the **root**, with an ordered set of \( \tilde{n} \geq 1 \) points, the **endpoints** of the **unlabeled tree**, so that \( r \) is not a branching point. \( \tilde{n} \) will be called the **order** of the unlabeled tree and the branching points will be called the **non trivial vertices**. The unlabeled trees are partially
orderd from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with $n$ end-points is bounded by $4^n$. We shall also consider the set $T_{h,n,M}$ of the labeled trees with $n$ endpoints (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h$ $\leq M$ with the root. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h,M+1]$, and we represent any tree $\tau \in T_{h,n,M}$ so that, if $v$ is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the scale of $v$, while the root $r$ is on the line with index $h$. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called trivial vertices. The set of the vertices will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex $v$ of a tree will be associated to its scale label $h_v$, defined, as above, as the label of the vertical line whom $v$ belongs to. Note that, if $v_1$ and $v_2$ are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

3) There is only one vertex immediately following the root, which will be denoted $v_0$; its scale is $h + 1$. If $v_0$ is an endpoint, the tree is called the trivial tree; this can happen only if $n + m = 1$.

4) Given a vertex $v$ of $\tau \in T_{h,n,M}$ that is not an endpoint, we can consider the subtrees of $\tau$ with root $v$, which correspond to the connected components of the restriction of $\tau$ to the vertices $w \geq v$; the number of endpoint of these subtrees will be called $n_v$. If a subtree with root $v$ contains only $v$ and one endpoint on scale $h_v + 1$, it will be called a trivial subtree.

5) Given an endpoint, the vertex $v$ preceding it is surely a non trivial vertex, if $n > 1$.

Our expansion is built by associating a value to any tree $\tau \in T_{h,n,M}$ in the following way.

First of all, given a normal endpoint $v \in \tau$ with $h_v = M + 1$, we associate to it one of the terms (note that to the $\varepsilon$ interaction two terms are associated) contributing to the potential $\Psi^{(M)}(\psi)$ while, if $h_v \leq M$, we associate to it one of the terms appearing in the following expression:

$$
-\nu(\psi^{(h_v)})_0 - \nu(\psi^{(h_v)})_e + \int d\lambda g^{(h_v,M)}(x + 1; 0)|\psi^+(\tau,\psi^{(h_v)})_x^-,\psi^-|<\tau,\psi^{(h_v)}_x|
$$



We associate to the label an index to specify which term is associated to the end-point. We introduce also a field label $f$ to distinguish the field variables appearing in the different terms associated to the endpoints; the set of field labels associated with the endpoint $v$ will be called $I_v$. Analogously, if $v$ is not an endpoint, we shall call $I_v$ the set of field labels associated with the endpoints following the vertex $v$; $x(f)$, $\varepsilon(f)$ will denote the space-time point, the $\varepsilon$ index of the Grassmann field variable with label $f$.

The previous definitions imply that, if $0 \leq h < M$, the following iterative equations are satisfied:

$$
-\Psi^{(h)}(\psi^{(\leq h)})_0 - \beta L_{h} = \sum_{n=1}^\infty \sum_{\tau \in T_{h,n,M}} \Psi^{(h)}(\tau,\psi^{(\leq h)})_0 + \beta L_{h},
$$

where, if $v_0$ is the first vertex of $\tau$ and $\tau_1, \ldots, \tau_s, s \geq 1$, are the subtrees with root in $v_0$,

$$
\Psi^{(h)}(\tau,\psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!} \sum_{\tau'_{h+1}} C_{h+1}[\Psi^{(h+1)}(\tau_1,\psi^{(\leq h+1)}); \ldots; \Psi^{(h+1)}(\tau_s,\psi^{(\leq h+1)})],
$$

where $\Psi^{(h+1)}(\tau_i,\psi^{(\leq h+1)})$ is equal to $\Psi^{(h+1)}(\tau_i,\psi^{(\leq h+1)})$ if the subtree $\tau_i$ contains more than one end-point, otherwise it is given by one of the terms contributing to the potentials in (25), if $h_v = M + 1$, or one of the addends in (44), if $h_v \leq M$, the choice depending on the label $a$.

Note that

$$
|\nu(\lambda g^{(h_v,M)}(x,0))| \leq C|\lambda|
$$

The above definitions imply, in particular, that, if $n > 1$ and $v$ is not an endpoint, then $N_v > 1$, with $N_v$ denoting the number of endpoints following $v$ on $\tau$; in fact the vertex preceding an end-point is necessarily non trivial, if $n > 1$.

Using its inductive definition, the right hand side of (45) can be further expanded, and in order to describe the resulting expansion we need some more definitions.
We associate with any vertex \( v \) of the tree a subset \( P_v \) of \( I_v \), the external fields of \( v \), and the set \( x_v \) of all space-time points associated with one of the end-points following \( v \). The subsets \( P_v \) must satisfy various constraints. First of all, \( |P_v| \geq 2 \), if \( v > v_0 \); moreover, if \( v \) is not an endpoint and \( v_1, \ldots, v_s \) are the \( s_v \geq 1 \) vertices immediately following it, then \( P_v \subseteq \cup_i P_{v_i} \); if \( v \) is an endpoint, \( P_v = I_v \). If \( v \) is not an endpoint, we shall denote by \( Q_v \) the intersection of \( P_v \) and \( P_{v_i} \); this definition implies that \( P_v = \cup_i Q_{v_i} \). The union \( I_v \) of the subsets \( P_v \setminus Q_v \), is, by definition, the set of the internal fields of \( v \), and is non empty if \( s_v > 1 \). Given \( \tau \in \mathcal{T}_{M,n,n} \), there are many possible choices of the subsets \( P_v, v \in \tau \), compatible with all the constraints. We shall denote \( P_\tau \), the family of all these choices and \( P \) the elements of \( \mathcal{P}_\tau \).

With these definitions, we can rewrite \( V^{(h)}(\tau, \psi^{(\leq h)}) \) in the r.h.s. of (45) as

\[
V^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{P \in \mathcal{P}_\tau} V^{(h)}(\tau, P) ,
\]

\[
\tilde{V}^{(h)}(\tau, P) = \int d\psi_{v_0} \psi^{(\leq h)}(P_{v_0}) K^{(h+1)}_{\tau, P}(x_{v_0}) ,
\]

(48)

where \( K^{(h+1)}_{\tau, P}(x_{v_0}) \) is defined inductively by the equation, valid for any \( v \in \tau \) which is not an endpoint,

\[
K^{(h+1)}_{\tau, P}(x_{v_0}) = \frac{1}{s_v} \prod_{i=1}^{s_v} [K^{(h+1)}_{v_i}(x_{v_i})] \mathcal{E}_h^{T}(\psi^{(h)}(P_{v_1} \setminus Q_{v_1}), \ldots, \psi^{(h)}(P_{v_{v_0}} \setminus Q_{v_{v_0}})) ,
\]

(49)

Moreover, if \( v_i \) is an endpoint, \( K^{(h+1)}_{v_{v_i}}(x_{v_0}) \) is equal to the kernel of one of the terms contributing to the potential in (25), if \( h_{v_i} = N + 1 \), or one of the four terms in (44), if \( h_{v_i} \leq N \); if \( v_i \) is not an endpoint, \( K^{(h+1)}_{v_{v_i}} = K^{(h+1)}_{\tau, P_{v_i}} \), where \( P_{v_i} = \{ P_w \mid w \in \tau_i \} \).

In order to get the final form of our expansion, we need a convenient representation for the truncated expectation in the r.h.s. of (49). Let us put \( s_v = s_{v_0}, \ P_t := P_{v_i} \setminus Q_{v_i} \); moreover we order in an arbitrary way the sets \( P_t^\pm := \{ f \in P_t, \varepsilon(f) = \pm \} \), we call \( f_{ij}^\pm \) their elements and we define \( x^{(i)} = \cup f \in P_t^+ x(f), y^{(i)} = \cup f \in P_t^- y(f), x_{ij} = x(f_{ij}^-), y_{ij} = x(f_{ij}^+). \) Note that \( \sum_{i=1}^s |P_t^-| = \sum_{i=1}^s |P_t^+| := k \), otherwise the truncated expectation vanishes. A couple \( l := (f_{ij}^-, f_{ij}^+) := (f_{i^-}, f_{i^+}) \) will be called a line joining the fields with labels \( f_{ij}^-, f_{ij}^+ \). Then, we use the Brydges-Battle-Federbush formula, if \( s > 1 \),

\[
\mathcal{E}_h^{T}(\tilde{\psi}^{(h)}(P_1), \ldots, \tilde{\psi}^{(h)}(P_s)) = \sum_{T \in T} \prod_{t \in T} \left[ g^{(h)}(x_t - y_t) \right] \int dP_T(t) \det G^{h,T}(t) ,
\]

(50)

where \( T \) is a set of lines forming an anchored tree graph between the clusters of points \( x^{(i)} \cup y^{(i)} \), that is \( T \) is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover \( t = \{ t_{ii'} \in [0, 1], 1 \leq i, i' \leq s \} \), \( dP_T(t) \) is a probability measure with support on a set of \( t \) such that \( t_{ii'} = u_{i'} \cdot u_i \) for some family of vectors \( u_i \in \mathbb{R}^s \) of unit norm.

\[
C_{ij, i'j'}^{h, T} = t_{ii'} \delta_{x_{ij}, y_{i'j'}} \left[ g^{(h)}(x_{ij} - x_0, ij - y_0, i'j') \right]_{\rho_{ij}, \rho_{i'j'}}^- ,
\]

(51)

with \( (f_{ij}^-, f_{ij}^+) \) not belonging to \( T \).

By inserting (50) in the r.h.s. of (49) we get

\[
V^{(h)}(\tau, P) = \sum_{T \in T} \int d\psi_{v_0} W_{\tau, P, T}(x_{v_0}) \prod_{f \in P_{v_0}} \psi^{(\leq h)}(\sigma(f))
\]

(52)

where

\[
W_{\tau, P, T}(x_{v_0}) = \prod_{v \not\in \text{e.p.}} \frac{1}{s_v} \int dP_{v_0}(t_v) \det G^{h_v, T_v}(t_v) \prod_{l \in T_v} \delta_{x_{0,l}, y_0} g^{(h_v)}(x_l; x_0, l - y_0, l)
\]

(53)

\( T \) is the set of the tree graphs on \( x_{v_0} \), obtained by putting together an anchored tree graph \( T_v \) for each non trivial vertex \( v \); \( v_1^*, \ldots, v_s^* \) are the endpoints of \( \tau \), \( f_{ij}^- \) and \( f_{ij}^+ \) are the labels of the two fields forming the line \( l \), “e.p.” is an abbreviation of “endpoint”. 
Note that we can eliminate the Kronecker deltas in the propagators in the spanning tree $T$, so that only a single sum over the coordinate remain and the coordinate of the external fields and of the fields in the determinants are assigned once that $x$, $T$ and $\tau$ are given, as the interaction is quasi local; we can then write

$$V^{(h)}(\tau, P) = \sum_{T \in T^*} \sum_{x} \int dx_{0,v_0} H_{\tau,P,T}(x, x_{0,v_0}) \prod_{f \in P_{v_0}} \psi_0^{(h)}(f)$$

(54)

where

$$H_{\tau,P,T}(x, x_{0,v_0}) = \prod_{v \not \in e.p.} \frac{1}{s_v!} \int dP_v(t_v) \det G^{h,v,T_v}(t_v) \prod_{l \in T_v} g^{(h,v)}(\hat{x}_l;x_0,l-y_0,l)$$

(55)

where there is a field $\bar{f}$ such that $\hat{x}(\bar{f}) = x$ and all the other coordinates $\hat{x}(f)$ are assigned once that $x$, $T$ and $\tau$ are given. We will call resonances the terms such that $\hat{x}(f) = x$ for ant $f \in P_{v_0}$.

In order to bound the above expression we introduce an Hilbert space $\mathcal{H} = \mathbb{R}^L \otimes \mathbb{R}^s \otimes L^2(\mathbb{R}^1)$ so that

$$G^{h,T}_{ij,i'j'} = \left( v_{x_{ij}} \otimes u_i \otimes A(x_{0,ij}-, x_{ij}), v_{y_{i'j'}} \otimes u_{i'} \otimes B(y_{0,i'j'}-, x_{ij}) \right)$$

(56)

where $v \in \mathbb{R}^L$ are unit vectors such that $(v_i, v_j) = \delta_{ij}$, $u \in \mathbb{R}^s$ are unit vectors $(u_i, u_i) = t_{ii'}$, and $A, B$ are vectors in the Hilbert space with scalar product

$$(A, B) = \int dz_0 A(x', x_0 - z_0) B^*(x', z_0 - y_0)$$

(57)

given by

$$A(x, x_0 - z_0) = \frac{1}{\beta} \sum_{k_0} e^{-ik_0(x_0-z_0)} \sqrt{\chi^{(u,v)}(k_0)(k_0^2 + (\phi_x - \mu)^2)}$$

$$B(x, y_0 - z_0) = \frac{1}{\beta} \sum_{k_0} e^{-ik_0(y_0-z_0)} \sqrt{\chi^{(u,v)}(k_0)(-ik_0 + \phi_x - \mu)}.$$

(58)

Moreover

$$\|A_h\|^2 = \int dz_0 |A_h(z)|^2 \leq C \gamma^{-3h}, \quad \|B_h\|^2 \leq C \gamma^{3h},$$

(59)

for a suitable constant $C$. 

---

**FIG. 2**: A tree $T_v$ connecting $S_v$ terms; inside the circles are the trees $T_{\bar{v}}$ with $v < \bar{v}$.
If $\varepsilon_0 = \max \{|\lambda|, |\nu|\}$, by using (49) and (50), we get the bound

\[
\frac{1}{\beta L} \sum_{\tau \in T} \sum_{\tau \in T} \sum_{x} \int d\tau_{0,0} |H_{\tau,\rho,T}(x,\tau_{0,0})| \leq \sum_{\tau \in T} \sum_{x} \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \max_{v \text{ not e.p.}} \left| \text{det} G^{h_v,T_0}(v) \right| \prod_{l \in T_0} \prod_{i \in T_0} \int d(x_{0,l} - y_{0,i}) \sup_x |g^{(h_v)}(x_l; x_{0,l} - y_{0,i})| \right]
\]

(60)

where, given the tree $\tau$, $T$ is the family of all tree graphs joining the space-time points associated to the endpoints, which are obtained by taking, for each non trivial vertex $v$, one of the anchored tree graph $T_v$ appearing in (50), and by adding the lines connecting the two vertices associated to non local endpoints. Note that the sum over the spatial coordinates is trivial thanks to the $\delta_{x,y}$ present in the propagators. Gram–Hadamard inequality, combined with (59), implies the dimensional bound:

\[
|\text{det} G^{h_v,T_0}(v)| \leq C^{s_v} \left( \prod_{l \in T_0} \left[ \prod_{x \text{ not trivial}} \gamma^{-h_v(x_l)} \right] \right)
\]

(61)

By the decay properties of $g^{(h)}(x)$ given by (42), it also follows that

\[
\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \prod_{l \in T_0} \int d(x_{0,l} - y_{0,i}) \sup_x |g^{(h_v)}(x_l; x_{0,l} - y_{0,i})| \leq C^{s_v} \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-h_v(x_l)}
\]

(62)

We can now perform the sum $\sum_{T \in T}$, which erases the $1/s_v!$ up to a $C^n$ factor. Then, by using the identity $\sum_{v \geq v_0} (s_v - 1) = n_v - 1$ and the bound $\sum_{v \geq v_0} [\sum_{l=1}^{s_v} |P_{v_l}| - |P_v| - 2(s_v - 1)] \leq 4n - 2(n - 1)$, we easily get the final bound

\[
\sum_{n=1}^{\infty} C^n \varepsilon_0^n \sum_{\tau \in T} \left[ \prod_{v \text{ not trivial}} \gamma^{-h_v(x_l)}(N_v - 1) \right] \leq C^n \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{-h_v(x_l)}
\]

(63)

where $v'$ is the non trivial vertex immediately preceding $v$ or $v_0$. This bound is suitable to control the expansion, if $n > 1$, since $N_v > 1$ for any non trivial vertex, as discussed below (47). If $n = 1$ the allowed trees have only one endpoint scale $h + 1$.

Note that $\sum_{T \in T}$ can be bounded by $\prod_{v \geq v_0} C^{s_v} \left( \prod_{l=1}^{s_v} |P_{v_l}| - |P_v| - 2(s_v - 1) \right) \leq C^n \prod_{v \geq v_0} 1$. In order to bound the sum over $\tau$, note that the number of unlabeled trees is $\leq 4^n$; moreover, as $N_v > 1$ and, if $v > v_0$, $2 \leq |P_v| \leq 4N_v - 2(N_v - 1)$, so that $N_v - 1 \geq |P_v|/6$.

\[
\left[ \prod_{v \text{ not trivial}} \gamma^{-h_v(x_l)}(N_v - 1) \right] \leq \left[ \prod_{v \text{ not trivial}} \gamma^{-\tilde{h}_v(x_l)} \right] \left[ \prod_{v \text{ not e.p.}} \gamma^{-\tilde{h}_v(x_l)} \right]
\]

(64)

The factor $\gamma^{-\tilde{h}_v(x_l)}$ can be used to bound the sum over the scale labels of the tree; moreover

\[
\sum_{P \in T} \gamma^{-\tilde{h}_v(x_l)} \leq C^n
\]

(65)

Since the constant $C$ is independent of $M, \beta, L$, the bounds above imply analyticity of the kernels in $\lambda$ and $\nu$, if $\varepsilon_0$ is small enough. It is an immediate consequence of the above bounds the proof of uniform convergence of the $M \to \infty$ limit; the proof of this is essentially identical to the one in [29] after (2.8) and it will not repeated here.

\section{The Infrared Integration and the Small Divisor Problem}

\subsection{Multiscale analysis}

The integration of the infrared (negative) scales has to be done in a different way, including the quadratic terms present in the effective potential producing a mass term. We describe the integration of the infrared scales by iteration; assume that we have integrated the fields $\psi^{(0)}, \ldots, \psi^{(h)}$ obtaining

\[
e^{-\beta L E_0} \int P(d\psi^{(0)}) e^{\nu^{(0)}}(\psi^{(0)}) = e^{-\beta L E_0} \int P(d\psi^{(h)}) e^{-\nu^{(h)}}(\psi^{(h)})
\]

(66)
where \( P(d\psi^{(\leq h)}) \) is the gaussian grassman measure with propagator, \( \rho = \pm \)
\[
g^{(\leq h)}_{\rho,\rho'}(x', y') = \delta_{x', y'} g^{(\leq h)}_{\rho,\rho'}(x', x_0 - y_0)
\] (67)

with
\[
g^{(\leq h)}_{\rho,\rho'}(x', x_0 - y_0) = \int dk_0 e^{ik_0(x_0-y_0)} \chi_h(\omega x', k_0) \left( -i\sigma_h + \nu \omega x' + r_{x'} \right)^{-1}
\] (68)

where \( \chi_h \) can be written as sum over trees (similar to the ones for \( \chi^{(0)} \)) and defined precisely below, and each tree with \( n \) end points contribute to \( \chi^{(h)} \) with a term of the form, after integrating the Koenecker deltas in the spanning tree as discussed before (54)
\[
\sum_{x'} \int dx_{0,1} \cdots \int dx_{0,n} H^{(h)}_{n,\rho_1,\ldots,\rho_n}(x'_1; x_{0,1}, \ldots, x_{0,n}) \prod_{i=1}^{n} \psi_{x',x_0,\rho_i}^{(\epsilon_i)(\leq h)}
\] (69)

where the coordinates of the external fields \( x'_i \) are assigned once that \( x \) and the labels of the tree are assigned. As in the previous section we call resonances the terms such that all the coordinates of the external points are equal
\[
x'_i = x'_1
\] (70)

We can split \( \chi^{(h)} \) in two parts
\[
\chi^{(h)} = \chi^{(h)}_{R} + \chi^{(h)}_{NR}
\] (71)

where in \( \chi^{(h)}_{R} \) are the resonant term (whose external fields verify (70)) while \( \chi^{(h)}_{NR} \) are the remaining terms.

We define a localization operation as a linear operation acting on \( \chi^{(h)} \) in the following way:

1. On the non resonant part of the effective potential is defined as
\[
\mathcal{L} \chi^{(h)}_{NR} = 0
\] (72)

2. On the resonant part of the effective potential its action consists in setting the time coordinate of the external fields equal
\[
\mathcal{L} \sum_{x'} \int dx_{0,1} \cdots dx_{0,n} H^{(h)}_{n,\rho_1,\ldots,\rho_n}(x'; x_{0,1}, \ldots, x_{0,n}) \prod_{i=1}^{n} \psi_{x',x_0,\rho_i}^{(\epsilon_i)(\leq h)}
\] (73)

We can write
\[
\mathcal{L} \chi^{(h)} = \gamma^h \nu_h F^{(h)}_v + F^{(h)}_z + s_h F^{(h)}_{\sigma} + F^{(h)}_{\lambda} + F^{(h)}_\zeta = s_h F^{(h)}_{\sigma} + \mathcal{L} \chi^{(h)}
\] (74)

where (note that \( H^{(h)}_n \) is translation invariant in the time direction)
\[
s_h = \frac{1}{\beta} \int dx_0 dy_0 H^{(h)}_{2,\rho - \rho}(0, x_0, y_0) \quad \zeta^h(x') = \frac{1}{\beta} \int dx_0 dy_0 \partial_x H^{(h)}_{2,\rho - \rho}(x', x_0, y_0)
\] (75)
\[
\nu_h = \frac{1}{\beta} \int dx_0 dy_0 H^{(h)}_{2,\rho}(x', x_0, y_0) \quad \zeta^h(x') = \frac{1}{\beta} \int dx_0 dy_0 \partial_x H^{(h)}_{2,\rho}(x', x_0, y_0)
\] (75)
\[
\lambda^h(x') = \frac{1}{\beta} \int dx_0 \cdots dx_{0,4} H^{(h)}_{4}(x'; x_{0,1}, x_{0,2}, x_{0,3}, x_{0,4})
\] (76)
and

\[
F^{(h)}_{\nu} = \sum_{\rho} \sum_{x'} \int dx_0 \bar{\psi}^{(\leq h)}_{x',\rho} \psi^{(\leq h)}_{x',\rho}
\]

Note that, for any integer \(N\)

\[
\begin{align*}
F^{(h)}_\zeta &= \sum_{\rho} \sum_{x'} \int dx_0 (\omega x') \bar{\zeta}_h(x') \psi^{(\leq h),+}_{x',\rho} \bar{\psi}^{(\leq h),-}_{x',-\rho} \\
F^{(h)}_\sigma &= \sum_{\rho} \sum_{x'} \int dx_0 \bar{\psi}^{(\leq h)}_{x',\rho} \psi^{(\leq h)}_{x',-\rho} \\
F^{(h)}_\lambda &= \sum_{x'} \int dx_0 \lambda_h(x') \bar{\psi}^{(\leq h),+}_{x',+} \psi^{(\leq h),-}_{x',-} \psi^{(\leq h)}_{x',-}
\end{align*}
\]

(77)

Note that the local terms with more than 6 fields are vanishing for \( n \geq 6 \) as there are at least two fields with the same \( \varepsilon, \rho \) and the same coordinate. Therefore the \( \mathcal{L} \) operation produces non-vanishing terms only on the terms with \( n = 2, 4 \). Note that \( \sigma_0 = O(\varepsilon^2) \).

We also define a renormalization operation as

\[
\mathcal{R} = 1 - \mathcal{L}
\]

(78)

so that we can rewrite (79) as

\[
\begin{align*}
\int P(d\psi^{(\leq h)}) e^{-\mathcal{L}V^{(h)}(\psi) - \mathcal{R}V^{(h)}(\psi)} &= \int P(d\psi^{(\leq h)}) e^{-s_h F^{(h)}_\sigma - \bar{\mathcal{L}}V^{(h)} - \mathcal{R}V^{(h)}} \\
\int \bar{P}(d\psi^{(\leq h)}) e^{-\mathcal{L}V^{(h)} - \mathcal{R}V^{(h)}}
\end{align*}
\]

with \( \bar{P}(d\psi^{(\leq h)}) \) with a propagator \( \bar{g}^{(\leq h)} \) coinciding with \( g^{(\leq h)} \) with \( \sigma_h \) replaced by \( \sigma_{h-1} \) with

\[
\sigma_{h-1} = \sigma_h + \chi_h s_h
\]

(80)

The effect of the \( \mathcal{R} \) operation is the following

\[
\mathcal{R} \sum_{x'} \int dx_{0,1} \ldots \int dx_{0,n} \mathcal{H}^{(h)}_{n,p_1,\ldots,p_n}(x'; x_{0,1}, \ldots, x_{0,n}) \prod_{i=1}^n \psi^{(\varepsilon_i,\leq h)}_{x',x_{0,i},p_i} = \sum_{x'} \int dx_{0,1} \ldots \int dx_{0,n} \mathcal{H}^{(h-1)}_{n,p_1,\ldots,p_n}(x'; x_{0,1}, \ldots, x_{0,n}) \prod_{i=1}^n \psi^{(\varepsilon_i,\leq h)}_{x',x_{0,i},p_i} - \prod_{i=1}^n \psi^{(\varepsilon_i,\leq h)}_{x',x_{0,i},p_i}
\]

(81)

We write then

\[
\int P(d\psi^{(\leq h-1)}) \int P(d\psi^{(h)}) e^{-\mathcal{L}V^{(h)} - \mathcal{R}V^{(h)}} = e^{-\beta \tilde{L} \tilde{E}_h} \int P(d\psi^{(\leq h-1)}) e^{-V^{(h-1)}(\psi^{(\leq h-1)})}
\]

(82)

where \( P(d\psi^{(\leq h-1)}) \) have propagator \( g^{(\leq h-1)} \) coinciding with (68) with \( h-1 \) replacing \( h \), and \( P(d\psi^{(h)}) \) has propagator \( g^{(h)} \) coinciding with with \( g^{(\leq h-1)} \) with \( \chi_{h-1} \) replaced by \( f_h = \chi_{h-1} - \chi_h \), where \( f_h \) a smooth has support in \( c_h^{-1} \leq |k_0^2 + v_0^2| \omega x' |^2 \leq c_{h+1}^{-1} \), for a suitable constant \( c \). Starting from the r.h.s. of (82), the procedure can be iterated. Note that, for any integer \( N \) and a suitable constant \( C_N \)

\[
|\bar{g}^{(h)}(x, x_0 - y_0)| \leq \frac{C_N}{1 + (\gamma \eta |x_0 - y_0|)^N}
\]

(83)

The above bound can be easily obtained integrating by parts.

**B. Tree expansion**

Again \( \mathcal{V}^{(h)} \) can be written as sum over trees, up to the following modifications to take into account the different multiscale integration procedure.

1. The scale index now is an integer taking values in \([h, 2] , h \) being the scale of the root.
2. With each vertex \( v \) of scale \( h_v = +1 \), which is not an endpoint, we associate one of the terms contributing to \( -V(0)(\psi(\leq h)) \), in the limit \( M = \infty \). With each endpoint \( v \) of scale \( h_v \leq 1 \) we associate one of local terms that contribute to \( \mathcal{L}V(h_v-1) \), and there is the constrain that \( h_v = h_{v'} + 1 \), if \( v' \) is the non trivial vertex immediately preceding it or \( v_0 \); to the end-points of scale \( h_v = 2 \) are associated one of the terms contributing to \( -V \) and there is not such a constrain.

3. With each trivial or non trivial vertex \( v > v_0 \), which is not an endpoint, we associate the \( \mathcal{R} = 1 - \mathcal{L} \) operator, acting on the corresponding kernel.

![FIG. 3: A tree \( \tau \in \mathcal{T}_{h,n} \) with its scale labels.]

A vertex \( v \) which is not an end-point such that the spatial coordinates \( x' \) in \( P_v \) are all equal is called resonant vertex, while if the coordinates are different is called non resonant vertex; the set of resonant vertices is denoted by \( \mathcal{H} \) and the set on non-resonant vertices is denoted by \( \mathcal{L} \). If \( v_1, \ldots, v_{s_v} \) are the \( s_v \geq 1 \) vertices following the vertex \( v \), we define

\[
S_v = S_v^L + S_v^H + S_v^2
\]

where \( S_v^L \) is the number of non resonant vertices following \( v \), \( S_v^H \) is the number of resonant vertices following \( v \), while \( S_v^2 \) is the number of trivial trees with root \( v \) associated to end-points.

If \( h \leq -1 \), the effective potential can be written in the following way:

\[
\mathcal{V}^{(h)}(\psi(\leq h)) + L\beta \bar{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi(\leq h))
\]

where, if \( v_0 \) is the first vertex of \( \tau \) and \( \tau_1, \ldots, \tau_s \ (s = s_{v_0}) \) are the subtrees of \( \tau \) with root \( v_0 \), \( \mathcal{V}^{(h)}(\tau, \psi(\leq h)) \) is defined inductively by the relation, if \( s > 1 \)

\[
\mathcal{V}^{(h)}(\tau, \psi(\leq h)) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_{h+1}^{T} \left[ \mathcal{V}^{(h+1)}(\tau_1, \psi(\leq h+1)) ; \cdots ; \mathcal{V}^{(h+1)}(\tau_s, \psi(\leq h+1)) \right]
\]

where \( \mathcal{V}^{(h+1)}(\tau_i, \psi(\leq h+1)) \):

1. it is equal to \( \mathcal{R} \mathcal{V}^{(h+1)}(\tau_i, \psi(\leq h+1)) \), with \( \mathcal{R} \) given by (81), if the subtree \( \tau_i \) is non trivial;

2. if \( \tau_i \) is trivial and \( h \leq -1 \), it is equal to one of the terms of \( \mathcal{L}V^{h+1} \) or, if \( h = 0 \), to one of the terms in the \( \mathcal{V} \).

By using (86) and the representation of the truncated expectations we get

\[
\mathcal{V}^{(h)} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{T \in \mathcal{T}} \int dx_{0,v_0} H_{\tau,P,T}(x, x_{0,v_0}) \prod_{f \in P_{v_0}} \psi^{(\leq h)}(f) \}
\]
where one of the spatial coordinates of the external fields $\prod_{f \in P_{v_0}} \psi^{(\leq h_0)}(f)$ is equal to $x'$ and the others are determined according to the following rule.

1. We define a tree $\bar{T}_v$, starting from $T_v$ and attaching to it the trees $T_{v_1}, \ldots, T_{v_{S_v}}$, associated to the vertices $v_1, \ldots, v_{S_v}$ following $v$ (graphically, we consider the tree $T_v$ in Fig. 2 and we replace the bubbles with the corresponding subtrees), and repeating this operation until the end-points are reached. The tree $\bar{T}_v$ is composed by a set of lines, representing propagators with scale $h_v \geq h_{\tilde{v}}$, connecting end-points $w$ of the tree $\tau$. Note that, contrary to $T_v$, the vertices of $\bar{T}_v$ are connected with at most four lines.

2. To each vertex $w$ of $\bar{T}_v$, is associated a coordinate $x_w$; if there are external fields $\psi^{(\leq h_v)}$ with coordinate $x_w$, we represent them as wiggly lines (see Fig. 4).

3. To each line $\ell$ of $\bar{T}_v$, we associate a label $a_{\ell} = 0, \pm 2x$.

4. To each vertex $w$ of $\bar{T}_v$, is associated a coordinate $x_w$, and to each line coming in or out $w$ is associated a factor $\delta_{i_w}$, where $i_w$ is a label identifying the lines connected to $w$. The vertices $w$ (which corresponds to the end-points of $\tau$) can be of type $\lambda, \nu, \lambda h, z h, z h$, and:

5. $\delta_{i_w} = 0$ if $w$ if it corresponds to a $\nu$ or $\nu h, z h$ end-point; $\delta_{i_w} = \pm 2x$ if $w$ if it correspond to a $\tilde{z} h$ end-point;

6. $\delta_{i_w} = \pm 1$ it corresponds to an $\varepsilon$ end-point; $\delta_{i_w} = (0, \pm 1)$ is a $\lambda$ end-point; $\delta_{i_w} = (0, \pm 2x)$ if is a $\lambda h$ end-point.

According to the above definitions, consider two vertices $w_1, w_2$ such that $x'_{w_1}$ and $x'_{w_2}$ are coordinates of the external fields, and let be $c_{w_1,w_2}$ the path (vertices and lines) in $\bar{T}_v$ connecting $w_1$ with $w_2$ (in the example in Fig. 4 the path is composed by $w_1, w_a, w_b, w_c, w_2$ and the corresponding lines ) as the path is a linear tree there is a natural orientation in the vertices, and we call $i_w$ the label of the line exiting from $w$ in $c_{w_1,w_2}$. Therefore the following relation holds

$$x'_{w_1} - x'_{w_2} = (\rho_{w_2} - \rho_{w_1}) x + \sum_{w \in c_{w_1,w_2}} \delta_{i_w} + \sum_{\ell \in c_{w_1,w_2}} a_{\ell} \tag{88}$$

The Diophantine condition implies a relation between the scale $h_v$ and the number of vertices between $w_2$ and $w_1$.

**Lemma 3.1** If $|c_{w_1,w_2}|$ is the number of vertices in the path $c_{w_1,w_2}$ with $x'_{w_1} \neq x'_{w_2}$ than, if $v'$ is the first vertex following $v$ in $\tau$

$$|c_{w_1,w_2}| \geq A \bar{x}^{-1} \gamma^{-h_v} \tag{89}$$

with a suitable constant $A$. 
Proof. Note that \( |\omega x''_w| \leq c_0^{-1} h_{w'}^{-1} \); there by using (88) and the Diophantine condition
\[
2c_0^{-1}\gamma h_{w'} \geq \|\omega x''_w\|_1 + \|\omega x''_w\|_1 \geq \|\omega (x''_{w_1} - x''_{w_2})\|_1 \tag{90}
\]
\[
= \|\omega (\rho_{w_2} - \rho_{w_1})\bar{x} + \sum_{w \in c_{w_1, w_2}} \delta_w + \sum_{\ell \in c_{w_1, w_2}} a_{\ell} \|_1 \geq
\]
\[
C_0|\rho_{w_2} - \rho_{w_1}|\bar{x} + \sum_{w \in c_{w_1, w_2}} \delta_w + \sum_{\ell \in c_{w_1, w_2}} a_{\ell} \geq C_0(4\bar{x}|c_{w_2, w_1})^{-\tau}
\]

The relation (88) is the analogue of the conservation of momentum rule in ordinary Feynman graphs. Lemma 3.1 says that there is a relation between the number of vertices and the scale of the external lines in the non resonant vertices; it is the analogue of Bruno lemma in KAM theory.

C. Power counting improvement and Diophantine condition

We define \( v_h = \tilde{v}_h \) where \( v_h \) are the running coupling constants. Therefore, each contribution from the tree \( \tau \in \mathcal{T}_{h,n} \) is proportional to a factor \( \varepsilon^n \).

**Lemma 3.2** If \( v \) is a vertex of \( \tau \) in \( \mathcal{T}_{h,n} \) which is not end-point, and \( v^*_i \) the end-points, and \( N_v = \sum_{i, v^*_i > v} 1 \) the number of end-points following \( v \) then
\[
\varepsilon^n \leq \varepsilon^{\frac{N_v}{2}} \prod_{v \text{ end. p.}} \varepsilon^{N_v 2^{h_{v'} - 1}} \tag{91}
\]

where \( v' \) is the vertex following \( v \) in \( \tau \).

**Proof** We can write
\[
\varepsilon^{\frac{1}{2}} = \prod_{h = -\infty}^{0} \varepsilon^{2^{h-2}} \tag{92}
\]

Given a tree \( \tau \in \mathcal{T}_{h,n} \), we consider an end-point \( v^* \) and the path in \( \tau \) from \( v^* \) to the root \( v_0 \); to each vertex \( v \) in such path with scale \( h_v \) we associate a factor \( \varepsilon^{2^{h_v-2}} \); repeating such operation for any end-point, the vertices \( v \) followed by \( N_v \) end-points are in \( N_v \) paths, therefore we can associate to them a factor \( \varepsilon^{N_v 2^{h_{v'} - 2}} \); finally we use that \( h_{v'} = h_v - 1 \).

It is an immediate consequence of Lemma 3.1 and Lemma 3.2 the following result, ensuring that we can extract from the \( \varepsilon^n \) factor a small factor to be associated to the non resonant vertices.

**Lemma 3.3** For any tree \( \tau \), if \( L \) is the set of non resonant vertices
\[
\varepsilon^{\frac{\tau}{2}} \leq \prod_{v \in L} \varepsilon^{A\bar{x}^{-1}\gamma^{-h_{v'}} 2^{h_{v'} - 1}} \tag{93}
\]

**Proof.** Note that if \( v \) is non resonant, there exists surely two external fields with coordinates \( x'_1, x'_2 \) such that \( x'_1 \neq x'_2 \); note that
\[
N_v \geq |c_{w_1, w_2}| \geq A\bar{x}^{-1}\gamma^{-h_{v'}} \tag{94}
\]
therefore
\[
\varepsilon^n \leq \prod_{v \in L} \varepsilon^{A\bar{x}^{-1}\gamma^{-h_{v'}} 2^{-h_{v'}+1}} \tag{95}
\]
D. Renormalized expansion

We want now to write the effective potential $\mathcal{V}^{(h)}$, see (86), in an equivalent way which is more suitable for the final bounds. Given a contribution to $\mathcal{V}^{h}$ corresponding to $\tau, T, P$, we consider the vertex $v^*$ of $\tau$ which are either non trivial or trivial but with some self-contraction of the external fields. We call vertices $v^*$ of depth $k$ the $v^*$ followed vertices of depth $< k$ and at least one with depth $k - 1$; the vertices $v^*$ followed only by end-points are of depth 1.

We start from the vertices $v^*$ of depth 1 and we distinguish three cases : 1) $v$ is non resonant or $|P_v| \leq 4$; 2) $v$ is resonant 3) $|P_v| \geq 6$. In case 2) the $\mathcal{R}$ operation acts on the external fields, as in (81); we can rewrite the difference of fields in the r.h.s. of (81) as sum of terms in which a field $\psi$ associated to $v$ to identify in the expansion the external fields of $\tau, T, P$.

The corresponding propagator can be written as

$$ D_{x_1,0,x_2,0}^{(\leq h_v)}(\leq h_v) = \psi_{x_1,0,x_2,0}^{(\leq h_v)} - \psi_{x_1,0,x_2,0}^{(\leq h_v)} $$

(96)

The corresponding propagator can be written as

$$ g^{(h_v)}(x_1,0 - z_0, x') - g^{(h_v)}(x_2,0 - z_0, x') = (x_0 - y_0) \int_0^1 dt\, \partial g^{(h_v)}(t(\tilde{x}_{12,0}(t) - z_0, x')) $$

(97)

where $\tilde{x}_{12,0}(t) = x_1,0 + t(x_2,0 - x_1,0)$ is an interpolated point between $x_{0,1}$ and $x_{0,2}$. We introduce then an extra label to identify in the expansion the external fields of $v$ to which is associated a $\psi$ or $D$ field.

Let us consider now case 3). If $|P_v| \geq 6$ we call $\bar{\rho}, \bar{\varepsilon}$ the labels of the external field whose number is maximal; the number of such external lines is $\geq |P_v|/4$. We consider a tree $\bar{T}_v$ and we associate to it another tree $\hat{T}_v$ eliminating from $\bar{T}_v$ all the trivial vertices not associated to any external line with label $\bar{\rho}, \bar{\varepsilon}$, and all the subtrees not containing any external line with label $\bar{\rho}, \bar{\varepsilon}$ (see Fig. 5 for an example), so that there is at least an external line associated to all end-points.

FIG. 5: A tree $\hat{T}_v$

The vertices $w$ of $\hat{T}$ are then only non trivial vertices or trivial vertices with external lines $\rho, \varepsilon$; all the end-points have associated an external line. In step 1 we consider end-points $w_a$ immediately followed by vertices $w_b$ with external lines (in the figure $w_4, w_{10}$). We can distinguish two cases. We call $x'_{w_a}$ and $x'_{w_b}$ the coordinate of the external fields associated to $w_a$ and $w_b$. If $x'_{w_a} \neq x'_{w_b}$ we consider the vertices in the path $c_{w_a,w_b}$ in $\hat{T}_v$, whose number $|c_{w_a,w_b}|$ is such that $M_{w_a,w_b} \geq A_{\psi}^{-\gamma} h_v/\tau$. If $x'_{w_a} = x'_{w_b}$ we can replace the $\psi$ field in $w_b$ with a $D$ field

$$ \psi_{\bar{\rho}, \bar{\varepsilon}}^{(\leq h_v - 1)} \equiv \psi_{\bar{\rho}, \bar{\varepsilon}}^{(\leq h_v - 1)} - \psi_{\bar{\rho}, \bar{\varepsilon}}^{(\leq h_v - 1)} $$

(98)

We consider now another tree obtained canceling the end-points $w_a$ and the resulting subtrees with no external lines and we proceed in a similar way, unless the tree has no end-points followed by vertices with external lines. In step 2 we consider in the resulting tree couple of endpoints followed by the same non trivial vertex (in the picture $w_1, w_2$); we call them $w_a, w_b$ and we proceed exactly as above distinguishing the two cases. We then cancel such end-points.
and the subtrees not containing external lines, so that the end-points are associated to external lines; we consider end-points followed by non trivial vertices with no external lines, and we proceed in a similar way. The resulting tree has again end-points with external lines followed by vertices with external lines (in the picture \(w_5\)), and we proceed as in step 1 before. We continue in this way so that at the end all except at most one vertex with external lines are considered. Note that by construction the different paths \(c_{w_a,w_b}\) do not overlap; for instance in Fig.5 the paths can be, if the corresponding coordinates are different, \(c_{w_{10},w_{11}}, c_{w_4,w_{13}}, c_{w_{11},w_2}, c_{w_5,w_4}, c_{w_5,w_7}, c_{w_7,w_{12}}, c_{w_3,w_{11}}\). This is an important point, as we will get a gain using the \(\varepsilon\) factors associated to each of the vertices in the path (therefore, if the path would be overlapping we would not get a gain proportional to the number of external fields). Similarly the difference of fields involve couples of points which are not overlapping.

We consider now the vertices \(v\) in \(\tau\) with depth increased by 1, and again we consider the corresponding tree \(\hat{T}_v\). The only differences with respect to the previous case is that some of the external fields can be \(D\) fields produced by the previous step. We want to avoid that difference of fields of order greater than 1 appear. Therefore if the vertex is a resonant vertex and one of the external line is a difference, then we consider the effect of the \(R\) operation as two separate terms. If it has more than 6 fields, than if one of the external field is a difference field then we do not perform the operation in the r.h.s. of (98). We proceed than in this way until the vertex with highest depth is reached. At the end of this procedure, the external field of each vertex can be \(\psi\) or \(D\) fields (no more than a single difference of fields can be produced). Now we write each propagator involving a \(D\) field as in the l.h.s. of (97). We start now from the vertex \(v\) with largest depth and we decompose the factors

\[
(x_0 - y_0) = \sum_r (x_{0,r} - x_{0,r-1})
\]

along the propagators of the spanning tree \(\hat{T}_v\). There are two possibilities: or such difference of coordinates correspond to the difference of coordinates of a propagator at scale \(h_v\), or to a propagator with some scale \(h_i\); in this second case we will consider separately all the the field differences in the vertices between \(v\) and \(v\).

In conclusion the final result can be written in the following way

\[
V^{(h)}(\tau, P) = \sum_{T \in T} \sum_{\alpha \in A_{r,T,p}} \sum_x \int dx_{0,v_0} H_{r,p,T,\alpha}(x, x_{0,v_0}) \prod_{f \in P_{v_0}} \partial^n f(x, x_{0,v_0}) \psi_{(x, x_{0,v_0})}(f) \]  

(100)

where \(A_{r,T,p}\) is set of indices allowing to distinguish the different terms produced by the \(R\) operation, by the decomposition of the zeros and by the improvements due to anticommutativity discussed above; moreover

\[
\sum_x \int dx_{0,v_0} H_{r,p,T}(x, x_{0,v_0}) = \]

(101)

\[
\varepsilon^n \left\{ \prod_{\text{not e.p.}} \frac{1}{v_{e.p.,\nu}^v} \left[ \prod_{v_{e.p.,\nu}} (\gamma^{h_v} x_{\nu}) \right] \left[ \prod_{v_{e.p.,\lambda}} (\gamma^{h_v} x_{\lambda}) \right] \prod_{v_{e.p.,\mu}} (\lambda_{h_v}) \prod_{v} dP_{T_v}(t_v) \det \bar{G}^{h_v,T_v}(t_v) \right\} \]

(102)

where \(\bar{x}\) can be interpolated points and \(\bar{G}^{h_v,T_v}\) is similar to \(G^{h_v,T_v}\) with some derivative applied on the \(\hat{\psi}\)

\[
G^{h_v,T_v}_{ij,i',j'} = t_{i,j} \delta_{x_i',y_{i'}} \partial_{0}^{h_v(j_i)} \partial_{0}^{h_v(j_i')} \left[ \bar{G}(h_v)(x_{0,ij} - y_{0,ij'}) \right] \rho_{i,j}^{h_v} \rho_{i',j'}^{h_v} \]

(103)

There is no need of a precise description of the various contributions to the sum over \(A_{r,T}\), but only need to know some very general properties, which follows from the previous construction.

1. There is a constant \(C\) such that, \(\forall T \in T_{r}, |A_T| \leq C^n\).

2. For any \(\alpha \in A_T\), the following inequality is satisfied

\[
\varepsilon^n \left[ \prod_{f \in I_{\alpha}} (\gamma^{h_v(f)} q_{0,\alpha}(f)) \right] \left[ \prod_{l \in T} (\gamma^{h_v(l)} h_{\alpha}(l)) \right] \leq \prod_{\psi \in H} \gamma^{h_v(\psi)} \prod_{\sigma \in G, |P_{\sigma}| \geq 6} \gamma^{(\partial_{\psi})_{h_v}} \]

(104)
In order to prove (103) we note that the propagator obtained contracting a $D$ field can be written as in the r.h.s. of (97) so that, for any $N$, $\alpha \geq 0$

$$|x_0 - y_0|^\alpha g^{(h_v)}(x_0 - y_0) \leq \frac{C_N}{1 + |\gamma_v|^N |x_0 - y_0|^N} \gamma^{-\alpha h_v}$$

and

$$|\partial^a g^{(h_v)}(x_0 - y_0)| \leq \frac{C_M}{1 + |\gamma_v|^N |x_0 - y_0|^N} \gamma^\alpha$$

Therefore, with respect to the bounds in which there are no $D$ fields, one has an extra factor in the bound

$$\left[ \prod_{f \in E_0} \gamma^{h_\alpha(f) c_{0,\alpha}(f)} \left[ \prod_{l \in T} \gamma^{b_\alpha(l)} c_{0,\alpha}(l) \right] \right] \cdot \left[ \prod_{v} \gamma^{\alpha_v (h_v - h_v)} \right] \leq \prod_{v} \gamma^{\alpha_v (h_v - h_v)}$$

where $\alpha_v$ is the number of $D$ fields external to $v$. By the construction discussed above, $\alpha_v \geq 1$ if $v$ is a resonant vertex $v \in H$ so that

$$\prod_{v} \gamma^{\alpha_v (h_v - h_v)} \leq \prod_{v \in H} \gamma^{h_v (h_v - h_v)} \prod_{v, |P_v| \geq 6} \gamma^{-(\alpha_v - 1)}$$

If $c_{w,w'}$ are the non overlapping paths joining two vertices $w, w'$ in $\hat{T}_v$ with external fields with $x_w \neq x_w'$ described above in the tree $\hat{T}_v$, we have

$$\varepsilon^{20} \leq \prod_{v \in \text{not.e.p.}} \varepsilon^{N_v 2h_v - 1} \leq \prod_{c_{w,w'}} \varepsilon^{c_{w,w'}[2h_{w'} - 1]}$$

as the total number of end-points in the set of all the paths $c_{w,w'}$ is at most equal to the number of vertices of $\hat{T}_v$ (which is also the number of end-points following $v$), as the paths are not overlapping. As $x_w \neq x_w'$ then $|c_{w,w'}| \geq A \varepsilon^{-1} \gamma^{-h_{v'}} / \tau$ and, if $\gamma^\tau / 2 = \gamma^\eta > 1$, for any vertex $v \in G$ and $\varepsilon$ small enough

$$\varepsilon^{c_{w,w'}[2h_{w'} - 1]} \leq \varepsilon^{B \gamma^{-h_{v'} / \tau} 2 h_{v'}} \leq \gamma^{-1}$$

In conclusion, for each vertex $v$ with $|P_v| \geq 6$ we have a factor $\gamma^{-1}$ (produced by the $D$ fields or by (109)) proportional to $|P_v|$ so that, for a suitable constant $C$

$$\varepsilon^{20} \prod_{v \in G, |P_v| \geq 6} \gamma^{-(\alpha_v - 1)} \leq C^n \prod_{v \in G, |P_v| \geq 6} \gamma^{-|P_v|}$$

Finally the fact that $|b_0| \leq 2$ and $|q_0| \leq 2$ follows from the discussion below (99).

### E. Bounds

In this section we get a bound for the kernels of the effective potential defined in (87).

**Lemma 3.4** If $v_h = (\tilde{\lambda}_h, \tilde{\nu}_h, \tilde{\omega}_h, \tilde{\zeta}_h) \equiv (\lambda_h, \alpha_h)$ then

$$\frac{1}{L^\beta} \sum_{\tau \in \tau_{h,n_\lambda,n_\alpha}} \sum_{P \in P_\tau} \sum_{T \in T_{\alpha,\tau}} \sum_{\alpha \in \mathcal{T}_{\alpha,\tau}} \sum_{x_0, v_0} \int dx_0, v_0 |H_{\tau, P, T, \alpha}(x, x_0, v_0)| \leq C^n |\log \varepsilon|^n \gamma^\alpha \varepsilon^{\frac{\tau}{h}} h^n (\gamma^{-h} \sup_k |\tilde{\lambda}_k|)^{n_\lambda} (\sup_k |\tilde{\alpha}_k|)^{n_\alpha}$$

where $C$ is a suitable constant and $n_\lambda, n_\alpha$ is the number of end-points of type $\lambda, \alpha$. 
Proof The matrix $\tilde{G}_{ij,ij'}^{h,T}$ can be written as

$$\tilde{G}_{ij,ij'}^{h,T} = \left( v_{x_{ij}} \otimes u_i \otimes A(x_{0,ij} -, x_{ij}) , v_{y'_{ij}}, \otimes u_{v'} \otimes B(y_{ij'}, -, x_{ij}) \right),$$

(122)

where $v \in \mathbb{R}^L$ are unit vectors such that $(v_i, v_j) = \delta_{ij}$, $u \in \mathbb{R}^s$ are unit vectors $(u_i, u_j) = t_{ij}$, and $A, B$ are vectors in the Hilbert space with scalar product

$$(A, B) = \int \, dz_0 A(x_0 - z_0, x') B^*(z_0 - y_0, x')$$

(113)

given by

$$A(x_0 - z_0, x') = \frac{1}{\beta} \sum_{k_0} e^{-i k_0 (x_0 - z_0)} \sqrt{f_h(k_0, x')} \mathbb{1} ,$$

(114)

$$B(y_0 - z_0, x') = \frac{1}{\beta} \sum_{k_0} e^{-i k_0 (y_0 - z_0)} \sqrt{f_h(k_0, y')} [A_h(k_0, x')]^{-1} ,$$

with $A_h$ defined in (67). Therefore

$$|\det \tilde{G}_{h, Tv}(t_v)| \leq \tilde{C}^n$$

(115)

We write the factor $\varepsilon^n$ in (102) as $\varepsilon^n \tilde{e} \tilde{\delta} \varepsilon^n \tilde{\gamma}$; we write $\varepsilon^n$ using Lemma 3.3 while the other factor is used in (103); therefore

$$\frac{1}{L \beta} \sum_g \int \, dx_{0,v_0} |H_{r,p,T,\alpha}(x, x_{0,v_0})| \leq \left[ \prod_{v \in L} \left[ \prod_{v \in H} \gamma^{-h_v} \gamma^0 \right] \prod_{v \in H} \gamma^{-h_v - h_v'} \right] \prod_{v \in H} \gamma^{-h_v - h_v'}$$

(116)

where $L$ is the set of non resonant clusters. We use that

$$\left[ \prod_{v \in H} \gamma^{-h_v - h_v} \right] = \left[ \prod_{v \in H} \gamma^{h_v - h_v'} \right]$$

(117)

where $v'$ is the first non trivial vertex following the non trivial vertex $v$ (note that all the vertices between $v$ and $v'$ are resonant). Note that

$$\left[ \prod_{v \in H} \gamma^{-h_v S_v} \right] \left[ \prod_{v \in H} \gamma^{h_v'} \right] \leq \gamma^h \left[ \prod_{v \in H} \gamma^{-h_v S_v} \right] \left[ \prod_{v \in H} \gamma^{h_v'} \right]$$

(118)

where we have used that no $R$ operation acts on $v_0$. Moreover by using (84)

$$\prod_{v \in H} \gamma^{-h_v S_v} = \left[ \prod_{v \in H} \gamma^{-h_v S_v^h} \right] \left[ \prod_{v \in H} \gamma^{-h_v S_v^L} \right]$$

(119)

Note that

$$\left[ \prod_{v \in H} \gamma^{-h_v S_v^h} \right] \left[ \prod_{v \in H} \gamma^{h_v'} \right] = 1$$

(120)

Moreover by (121)

$$\left[ \prod_{v \in H} \gamma^{-h_v S_v^L} \right] \left[ \prod_{v \in L} \varepsilon e^{A z^{-1} \gamma^{-h_v} 2^{h_v'}} \right] \leq \tilde{C}^n \prod_{v \in L} \gamma^{-h_v S_v^L} \gamma^{3 S_v^h} \leq \tilde{C}^n$$

(121)

following from the fact that, as $\gamma^+ / 2 \equiv \gamma^0 > 1$, for any $N$

$$e^{A z^{-1} \gamma^{-h_v} h} = e^{-|\log \varepsilon | A z^{-1} \gamma^{-h_v} h} \leq \varepsilon^{N \gamma h} \frac{N}{|\log \varepsilon | A z^{-1} \gamma^{-h} e^{-N}$$

(122)
as $e^{-\alpha x}x^N \leq \left[\frac{N}{\alpha}\right]^N e^N$. Therefore, by choosing $N = 3$ we get\
$\prod_{v \in E} e^{A_2 - r} - \gamma h - 2^{h'} \leq \bar{C} n^{1/2} \prod_{v \in \mathcal{S}_v} e^{3\bar{S}_v h_v}$ where $\bar{C} = \left[\frac{3}{log(2A_2)}\right]^{3e3}$ and we have used that the number of non-trivial vertices is smaller than the number of end-points $n$. Finally\
$$\prod_{v \in \mathcal{P}} \gamma^{-h_v} S_v^2 \prod_{v \in \mathcal{P}, \alpha} \gamma^h a_{\alpha v} \prod_{v \in \mathcal{P}, \gamma} |\lambda_{\alpha v}| \leq C n^{1/2} \prod_{v \in \mathcal{P}, \alpha} |\bar{\lambda}_{\alpha v}| \prod_{v \in \mathcal{P}, \gamma} |\gamma^{-h_v} \bar{\lambda}_{\alpha v}|$$ (123)\

Therefore we get the bound\
$$\frac{1}{L_{\beta}} \int dx_{v_0} |W_{\tau, p} T(x_{v_0})| \leq C^n e^{\frac{\pi}{2} \sum_{v \in \mathcal{P}} |\bar{\lambda}_{v}|} \prod_{v \in \mathcal{P}, \alpha} |\bar{\lambda}_{\alpha v}| \prod_{v \in \mathcal{P}, \gamma} |\gamma^{-h_v} \bar{\lambda}_{\alpha v}|$$ (124)\

The sum over $\mathcal{P}$ is done as in (65); the sum over $\gamma$ is over $C^n$ terms. The sum over the trees $\tau$ is done performing the sum of unlabelled trees and the sum over scales. The latter can be bounded by $|h|^m$, where $m$ is the number of non-trivial vertices, which is $\leq C^n$; indeed given the unlabeled tree, the scales of the trivial vertices and of the end-points are determined once that the scales of the non trivial vertices are determined; the former is bounded by $C^n$ so that (111) follows.\n
\section{The flow of the effective coupling}

It is an easy consequence of Lemma 3.5 the following result

\textbf{Lemma 3.5} If $\gamma h \geq \varepsilon^{2\bar{h}}$ then there exists an $\varepsilon_0$ and a choice $\nu$ such that for $\varepsilon \leq \varepsilon_0$ and $|\lambda| \leq \varepsilon^{2\bar{h} + 2}$ then there exists a suitable constant $C_1$ such that, for any $k \geq \bar{h}$\
$$|\bar{\lambda}_h| \leq |\bar{\lambda}| C_1 |\bar{\lambda}_h| \leq C_1$$ (125)\

\textit{Proof} We proceed by induction. The flow equation for $\nu_k$ is\
$$\nu_{k+1} = \gamma \nu_k + \gamma^{-k} \int dx_0 H_{2,pp}(0, x_0, 0)$$ (126)\
with $\nu_2 = \nu$. By iteration we get\
$$\nu_k = \gamma^{-k+1} (\bar{\nu} + \sum_{k' \geq k} \int dx_0 H_{2,pp}(0, x_0, 0))$$ (127)\
and by properly choosing $\bar{\nu}$ so that $\nu_h = 0$ we get\
$$\nu_k = -\gamma^{-k+1} \sum_{\bar{h} \leq k' \leq k} \int dx_0 H_{2,pp}(0, x_0, 0))$$ (128)\
and one can show by a fixed point argument, the existence of a bounded sequence of $\nu_k$ verifying (128) (the proof is identical to the one §A2.6 of [27]). Regarding the flow of $\bar{\xi}_h$ assume that (125) is true for $k \geq \bar{h}$. The flow equation for $\bar{\xi}_h$ is\
$$\bar{\xi}_h = \sum_{k \geq \bar{h}} \int dx_0 \partial_t H_{2}(k)$$ (129)\

Using lemma 3.4 and the fact that the derivative cancels a factor $\gamma^h$ we get for $\varepsilon$ small enough\
$$|\bar{\xi}_h| \leq \sum_{n=2}^{\infty} C^n C_1^n \varepsilon^{\frac{n}{2}} |h|^n \leq |h| C_2 (CC_1 |h| \varepsilon^{\frac{1}{4}}) \leq C_1$$ (130)\
where we use that $|h| \varepsilon^{\frac{1}{4}} \leq \varepsilon^{\frac{1}{4}}$ and $\gamma^{-k} |\bar{\lambda}| \leq \varepsilon$.\n
Similarly

\[
|\tilde{\lambda}_h| \leq |\lambda_0| + \sum_{n=2}^{\infty} \sum_{n_\lambda \geq 1}^{\infty} C^n C_1^\gamma k \varepsilon^\gamma |h|^n (\gamma^{-k}|\tilde{\lambda}_k|)^{n_\lambda} \leq \sum_{n=2}^{\infty} |h|^{n+1} \varepsilon^n C^n C_1^\gamma \sum_{n_\lambda = 1}^{\infty} |\lambda| (\gamma^{-h}|\tilde{\lambda}_k|)^{n_\lambda-1} \leq |\lambda| C_1
\] (131)

The above lemma says the the flow is bounded up to a scale \( \gamma^h \geq \varepsilon^{2\bar{x}} \). In order to integrate the smaller scales one has to use the mass term. Note indeed that if there exists two constants such that

\[
c_1 \varepsilon^{2\bar{x}} \leq \sigma_h \leq c_2 \varepsilon^{2\bar{x}}
\] (132)

then there exists a scale \( h^* \) with \( \gamma^{h^*} = O(\varepsilon^{2\bar{x}}) \) defined as the minimal \( h \) such that \( \gamma^h \geq \sigma_k \). For any integer \( M \) and a suitable constant \( C_M \)

\[
|g^{(\varepsilon^{h*})}(x_0 - y_0, x)| \leq \frac{C_M}{1 + (\gamma^{h*}|x_0 - y_0|)^M}
\] (133)

Indeed the denominator of the propagator is \( \geq c \gamma^{h*} \), so that the above bounds follow using integration by parts. The bound (133) says that the propagator corresponding to all the scales \( \leq h^* \) verifies the same bound of the single scale propagator; therefore we can bound the scale \( \leq h^* \) in a single step, and, by lemma 3.4 and 3.5 (for \( h > h^* = \log \varepsilon^{2\bar{x}} \)), convergence follows. It remains to prove (132) and that \( \tilde{\gamma}_h = O(\varepsilon^{2\bar{x}}) \). We can write (similar expressions hold for \( \varepsilon_h \))

\[
\sigma_h = \sum_{k \geq h} \int dx_0 H^{(k)}_{2,\rho,-\rho}(0, x_0, 0)
\] (134)

and

\[
H^{(h)}_{2} = H^{(a)(h)}_{2,\rho,-\rho} + H^{(b)(h)}_{2,\rho,-\rho}
\] (135)

where \( H^{(a)(h)}_{2,\rho,-\rho} \) is the sum over trees with \( n \leq 8\bar{x} \) and \( H^{(b)(h)}_{2,\rho,-\rho} \) is the sum over trees with \( n \geq 8\bar{x} + 1 \). The bound for \( H^{(b)(h)}_{2,\rho,-\rho} \) from lemma 3.4 is \( \leq C\varepsilon^{2\bar{x}+\frac{4}{\bar{x}}} \). Regarding \( H^{(a)(h)}_{2,\rho,-\rho} \) we again distinguish between trees with at least a \( \lambda, \lambda_h \) end-point and the rest; the former is bounded by \( C\gamma^{h}|\gamma^{-h}\lambda| \leq C\varepsilon^{2\bar{x}+1} \). Regarding the latter, it can be represented in terms of chain graphs with \( \varepsilon, \nu, \zeta \) end-points; if \( x'_{\ell} \) is the coordinate of any internal propagator with scale \( h \) and \( x' \) is the external coordinate, \( \neq x' \), \( a \) is a constant

\[
a \gamma^h \geq ||\omega x'||_1 + ||\omega x'_0||_1 \geq ||\omega x' - \omega x'_0|| \geq C_0 |x' - x'_0|^\gamma \geq C_0 |(8\bar{x})^2|^{-\gamma}
\] (136)

Moreover the graphs are \( O(\varepsilon^h) \) with \( k \geq 2\bar{x} \) (the sum of coordinates of internal vertices is \( 2\bar{x} \), and the difference of coordinate of the terms attached to \( \varepsilon \) vertices is \( \pm 1 \), and to \( \zeta \) is \( \pm 2\bar{x} \), but they are \( O(\varepsilon^{2\bar{x}}) \)). There is only one term contributing to lowest order; its value is \( \varepsilon^{2\bar{x}} a \) with

\[
a = \frac{\chi_{\geq h}}{\phi_{-\bar{x} + 1} - \phi_{\bar{x}}} \frac{\chi_{\geq h}}{\phi_{-\bar{x} + 2} - \phi_{\bar{x}}} \ldots \frac{\chi_{\geq h}}{\phi_{-1} - \phi_{\bar{x}}}
\] (137)

where \( \chi_{\geq h} \) is the cut-off function \( \chi_{u,v} + \sum_{\rho=\pm}^{0} \sum_{k=h}^{\infty} f^{(k)}(\omega(x - \rho \bar{x})) \). The terms proportional to \( \varepsilon^k \), with \( 2\bar{x} + 1 \leq k \leq 8\bar{x} \) have at most \( 8\bar{x} \) propagators bounded by (136); therefore

\[
\sigma_h = \varepsilon^{2\bar{x}} (a + O(\varepsilon^{2\bar{x}}) + O(\varepsilon^{2\bar{x}+\frac{4}{\bar{x}}}))
\] (138)

Therefore, for \( \varepsilon \leq O(\bar{x}!^{-a}) \) then (132) follows.

G. The 2-point function

We have finally to get a bound for the two-point function. First of all, we note that Lemma 3.4 and Lemma 3.5 immediately imply a bound for the kernel of the effective potential with two external lines, with coordinate \( x \) and \( y \).
Indeed in the trees $\tau \in \mathcal{T}_{h,n}$ with $n$ end-points contributing to $W_2^{(h)}$ there is necessarily a path $c_{w_1,w_2}$ in $\mathcal{T}_v$ connecting the points $w_1$, with $x_{w_1} = x$ and $w_2$ with $x_{w_2} = y$ such that by (88) $|x - y| \leq 8|c_{w_1,w_2}|$; moreover $|c_{w_1,w_2}| \leq n$ so that $n \geq \frac{1}{8}|x - y|$. Therefore no tree $\tau$ with $n < \frac{1}{8}|x - y|$ contribute to a kernel of the effective potential with external lines with coordinate $x$ and $y$; therefore by Lemma 3.4 and Lemma 3.5 we get, for $h \geq h^*$

$$
\frac{1}{\beta} \int dx_0 W_2^{(h)}(x, y) \leq \sum_{n \geq \frac{1}{8}|x - y|} C^n |h|^n \gamma \log \varepsilon \leq C \gamma^{h} \varepsilon^{|x - y|} \tag{139}
$$

with suitable $\alpha$ and $C$.

In order to bound the 2-point function we have to consider the multiscale integration with $\phi \neq 0$; we get

$$\phi_2(x, y) = \sum_{h = h^*}^{1} S_{2,h}(x, y) \tag{140}$$

and $S_{2,h}(x, y)$ are expressed in terms of a tree expansion similar to the one for $W_2^{(h)}$, where the only difference is that two external fields are replaced by propagators $g^{(k)}(x'; x_0 - z_0)$ and $g^{(i)}(y'; y_0 - z_0)$; therefore $S_{2,h}(x, y)$ (at $x, y$ fixed) verifies a bound similar to (139) with an extra external factor $C_N \frac{\gamma}{1 + \gamma N |x_0 - y_0|^N}$ for any $N$, that is

$$|S_{2,h}(x, y)| \leq \varepsilon^{|x - y|} \sum_{h = h^*}^{1} C_N \frac{\gamma}{1 + \gamma N |x_0 - y_0|^N} \tag{141}$$

In conclusion, by (138), for any $N$

$$|S_2(x, y)| \leq \sum_{h = h^*}^{1} \varepsilon^{|x - y|} \frac{C_N}{1 + \gamma N |x_0 - y_0|^N} \leq C_N \frac{e^{-\frac{1}{2} |x - y|}}{1 + |\sigma_h| |x_0 - y_0|^N} \tag{142}$$

so that (30) is proved.