A CENTRAL LIMIT THEOREM FOR STAR-GENERATORS OF $S_\infty$, Which RELATES TO TRACELESS CCR-GUE MATRICES

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Abstract. We prove a limit theorem concerning the sequence of star-generators of $S_\infty$, where the expectation functional is provided by a character of $S_\infty$ with weights $(w_1, \ldots, w_d, 0, 0, \ldots)$ in the Thoma classification. The limit law turns out to be the law of a “traceless CCR-GUE” matrix, an analogue of the traceless GUE where the off-diagonal entries $g_{i,j}$ satisfy the commutation relation $g_{i,j}g_{j,i} = g_{j,i}g_{i,j} + (w_j - w_i)$. The special case $w_1 = \cdots = w_d = 1/d$ yields the law of a bona fide traceless GUE matrix, and we retrieve a result of Köstler and Nica from 2021, which in turn extended a result of Biane from 1995.

1. Introduction

The sequence of star-generators $(1, 2), (1, 3), \ldots, (1, n), \ldots$ of the infinite symmetric group $S_\infty$ has received a good amount of attention in recent years. This was due to nice combinatorial facts found about factorizations of arbitrary permutations into products of star-generators, and also because of the direct connection that star-generators have with the important family of Jucys-Murphy elements of $\mathbb{C}[S_\infty]$. See for instance the presentation of results shown in the introduction to [7].

An interesting point of view, going back to a paper of Biane [2], is to treat the star-generators as a sequence of selfadjoint random variables in the $*$-probability space $(\mathbb{C}[S_\infty], \varphi)$, where $\varphi$ is the canonical trace on the group algebra $\mathbb{C}[S_\infty]$. In particular, one of the results in [2] is a limit theorem with CLT flavour which holds in this framework, and finds the semicircle law of Wigner to be the limit law.

The canonical trace $\varphi$ on $\mathbb{C}[S_\infty]$ is the $d \to \infty$ limit of a significant sequence of trace-states $\varphi_d : \mathbb{C}[S_\infty] \to \mathbb{C}$, $d \in \mathbb{N}$, given by the so-called “block characters” of $S_\infty$ (see e.g. the presentation in [8]). In the recent paper [11] it was observed that the CLT theorem for the star-generators of $S_\infty$ still holds in the $*$-probability space $(\mathbb{C}[S_\infty], \varphi_d)$ for a finite value of $d \in \mathbb{N}$. Moreover, the limit law for the CLT can in this case be identified as the law of a known $d \times d$ random Hermitian matrix, the “traceless GUE matrix”.

Moving one step further, we note that the block character which defines $\varphi_d$ is a special case of extremal character of $S_\infty$. In the classification of such extremal characters that was given by Thoma [14], this block character is encoded by the sequence

$$\left(1/d, \ldots, 1/d, 0, \ldots, 0, \ldots\right)$$

with $d$ occurrences of $1/d$ at the beginning.

(The Thoma classification actually encodes a character by using two sequences of numbers in $[0, \infty)$; but in this paper we are only dealing with situations where the second Thoma sequence has all its terms equal to 0.)

The present paper continues the study of the CLT theorem for star-generators, in the setting where instead of the sequence (1.1), we consider a sequence

$$\left\{ (w_1, \ldots, w_d, 0, \ldots, 0, \ldots), \text{ with } d \geq 2 \text{ and with } w_1 \geq w_2 \geq \cdots \geq w_d > 0 \text{ such that } \sum_{i=1}^{d} w_d = 1. \right\}$$

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As explained below, the CLT result still holds in connection to a tuple of weights as in (1.2), and continues to belong to the general category of “exchangeable CLT” theorems. This is stated as Theorem 2.5 of the paper, which we prove by reduction to a basic exchangeable CLT result of Bożejko and Speicher [6]. Moreover, the limit law \( \mu_w \) arising in Theorem 2.5 still has a neat realization as the law of an appropriate version of \( d \times d \) traceless GUE matrix. But the extension from the framework of (1.1) to the one of (1.2) is non-trivial in the following two respects.

(a) When looking at the law of large numbers that typically precedes a CLT, one finds the case \( w_1 = w_2 = \cdots = w_d = 1/d \) to be the only one where the centering of the star-generators is done in the usual way, by subtracting a scalar multiple of the unit of \( \mathbb{C}[S_\infty] \). For any other tuple \( (w_1, \ldots, w_d) \), the centering has to be done by subtracting a (non-scalar) operator \( A_0 \) in the GNS representation of the character; this naturally moves the framework of the CLT to the \( \ast \)-probability \( (\mathcal{M}, \text{tr}) \), where \( \mathcal{M} \) is the von Neumann algebra generated by the said GNS representation, and \( \text{tr} \) is the natural trace that \( \mathcal{M} \) comes equipped with. We explain how the centering goes in Section 2.2, and then discuss this in more detail in Section 3 of the paper.

(b) The CLT limit law \( \mu_w \) corresponding to a general tuple \( w = (w_1, \ldots, w_d) \) of weights as in (1.2) still is realized as the law of a special \( d \times d \) matrix \( M \); but the entries of \( M \) are now living in the non-commutative world of canonical commutation relations (CCR). More precisely: for \( 1 \leq i, j \leq d \) with \( w_i \neq w_j \), the \( (i, j) \)-entry \( a_{i,j} \) of \( M \) and its adjoint \( a_{j,i} = a_{i,j}^* \) are set to satisfy the relation \( a_{i,j}a_{j,i} = a_{j,i}a_{i,j} + (w_j - w_i) \). This forces \( a_{i,j} \) to no longer be a random variable in the usual sense. But nevertheless, the description of \( a_{i,j} \) clearly fits within a notion of “CCR-complex-Gaussian random variable”, and the resulting matrix \( M \) is a natural CCR-analogue for a traceless GUE matrix. These points are explained in Section 2.3 below, where we also state the main result of the paper, Theorem 2.9.

Organisation of the paper. Besides the present Introduction, the paper has 7 sections.

- Section 2 gives a general presentation of the framework and results of the paper. The CLT result is stated as Theorem 2.5 in Section 2.2, then in Section 2.3 we introduce the notion of traceless CCR-GUE matrix and we give the statement of Theorem 2.9. Section 4 gives a glimpse of the underlying combinatorics which connects Theorems 2.5 and 2.9, the proofs of both these theorems require fiddling with pair-partitions \( \pi \) of sets \( \{1, \ldots, k\} \), only that this fiddling is done in two different ways, which associate two different permutations, “\( \tau_\pi \) vs. \( \sigma_\pi \)”, to the same pair-partition \( \pi \). Proposition 2.11 of Section 2.4 explains how the cycle structure of \( \tau_\pi \) can be read from \( \sigma_\pi \) – this is a key-point towards the proof of Theorem 2.9.

- In Section 3 we set the framework used throughout the paper, and we review the law of large numbers which introduces the operator \( A_0 \) used for centering in our CLT result.

- In Section 4 we review the setting for an exchangeable CLT, and we prove Theorem 2.5.

- Sections 5-7 are devoted to studying the moments of the limit law \( \mu_w \) which arises in Theorem 2.5. More precisely: in Section 5 we clarify (cf. Proposition 5.6) how the said moments are expressed in terms of the permutations \( \tau_\pi \). In Section 6 we present in more detail the connection \( \tau_\pi \leftrightarrow \sigma_\pi \) advertised in the presentation of results from Section 2.4. This is then used in Section 7 in order to obtain a formula where the moments of \( \mu_w \) are written directly in terms of permutations \( \sigma_\pi \) (cf. Proposition 7.6).

- Finally, in Section 8 we discuss in more detail the notion of traceless CCR-GUE matrix. We obtain (cf. Proposition 8.4) a Wick-style formula for the joint moments of the entries
2. Presentation of results

2.1. Description of framework, the law of large numbers.

Notation 2.1. (Some general notation.) As is customary, we denote by $S_\infty$ the group of all finite permutations $\sigma$ of $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ (thus $\sigma : \mathbb{N} \to \mathbb{N}$ is bijective, and there exists $n_0 \in \mathbb{N}$ such that $\sigma(n) = n$ for $n > n_0$). We will write permutations $\sigma \in S_\infty$ in cycle notation, where $\sigma$ is expressed as a product of disjoint cycles, and every $n \in \mathbb{N}$ not specifically included in a cycle is assumed to be a fixed point of $\sigma$. Important instances of such writing are provided by the sequence of star-transpositions $(\gamma_n)_{n=1}^\infty$ defined as

$$\gamma_1 = (1, 2), \gamma_2 = (1, 3), \ldots, \gamma_n = (1, n+1), \ldots,$$

and by the forward cycles $(\eta_n)_{n=1}^\infty$, which are

$$\eta_1 = (1) (= \text{unit of } S_\infty), \eta_2 = (1, 2), \eta_3 = (1, 2, 3), \ldots, \eta_n = (1, 2, \ldots, n), \ldots$$

Notation 2.2. (The character $\chi$.) Throughout the whole paper we fix a $d \geq 2$ and a tuple of weights,

$$w = (w_1, \ldots, w_d),$$

with $w_1 \geq w_2 \geq \cdots \geq w_d > 0$ and $w_1 + \cdots + w_d = 1$. We consider the power sums

$$p_n := w_1^n + w_2^n + \cdots + w_d^n, \quad n \in \mathbb{N},$$

thus getting a sequence of numbers $1 = p_1 > p_2 > \cdots > p_n > \cdots > 0$. With this in hand, we then define $\chi : S_\infty \to \mathbb{R}$ by putting

$$\chi(\sigma) := \prod_{\substack{V \text{ orbit of } \sigma, \ \ |V| \geq 2}} p_{|V|}.$$ (Concrete example: $\sigma = (1, 3, 2)(5, 6) \in S_\infty$ has $\chi(\sigma) = p_2 \cdot p_3$. In the case when $\sigma = \eta_1 =$ unit of $S_\infty$, the empty product on the right hand side of (2.5) is taken to be equal to 1.)

It is immediate that $\chi$ is a class-function, that is, the value $\chi(\sigma)$ only depends on the conjugacy class of $\sigma$ in $S_\infty$. It is, moreover, not hard to prove that $\chi$ is a function of positive type, hence it is what one calls a character of the group $S_\infty$. Some more advanced considerations (see e.g. [1, Section 4.2]) show that $\chi$ is actually an extremal character of $S_\infty$, appearing in the Thoma classification of such characters in the way indicated in (1.2) above.

Notation 2.3. (GNS.) We will use the notation $U : S_\infty \to B(H)$ for the GNS representation of $\chi$. Thus $H$ is a Hilbert space over $\mathbb{C}$, endowed with a map

$$S_\infty \to H : \sigma \mapsto \hat{\sigma},$$

such that

1. the linear span of the image of the map (2.6) is dense in $H$;
2. for every $\sigma, \tau \in S_\infty$ we have $\langle \hat{\sigma}, \hat{\tau} \rangle = \chi(\sigma \tau^{-1})$.

(Since $\chi$ is a class-function, the $\sigma \tau^{-1}$ in (2) can be replaced by any of $\tau \sigma^{-1}, \sigma^{-1} \tau$, or $\tau^{-1} \sigma$.) Then, for every $\sigma \in S_\infty$, the operator $U(\sigma) \in B(H)$ is defined via the requirement that

$$[U(\sigma)](\tau) = \hat{\sigma} \tau, \quad \forall \tau \in S_\infty.$$
It is easily verified that the definition of \( U(\sigma) \) makes sense and that the map \( \sigma \mapsto U(\sigma) \) is a representation of \( S_\infty \) by unitary operators on \( H \). Note that for a star-transposition \( \gamma_n \), the operator \( U(\gamma_n) \) is a symmetry, so in particular it is self-adjoint.

**Remark 2.4. (Law of large numbers.)** In our setting, this amounts to the fact that the averages of \( U(\gamma_n) \)'s are convergent in the strong operator topology. We denote
\[
\text{SOT} - \lim_{n \to \infty} \frac{1}{n} \left( U(\gamma_1) + \cdots + U(\gamma_n) \right) =: A_0 \in B(H).
\]
The operator \( A_0 \) was studied in detail in [9], and played a crucial role in the considerations of non-commutative dynamics made in that paper. For the sake of keeping the presentation self-contained, we include in Section 3 below the relatively easy proofs of the facts about \( A_0 \) needed here, in particular the convergence in (2.8) and the relevant fact that
\[
\text{Spectrum}(A_0) = \{ t \in (0, 1) \mid \exists 1 \leq i \leq d \text{ such that } w_i = t \}.
\]
From (2.9) it follows that \( A_0 \) is an invertible selfadjoint operator – but note that the case \( w_1 = \cdots = w_d = 1/d \) is the only one when \( A_0 \) comes out as a scalar multiple of the identity operator on \( H \).

### 2.2. The theorem of CLT type, and the limit law \( \mu_w \).

The next step to be taken, once the limit \( A_0 \) from (2.8) was identified, is to subtract \( A_0 \) out of the \( U(\gamma_n) \)'s and seek a limit in law for the rescaled averages
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (U(\gamma_i) - A_0).
\]
The law of the operators (2.10) is to be considered with respect to the vector-state defined by the vector \( \hat{\eta}_1 \in H \), where recall that (coming from (2.2)) we use the notation “\( \eta_1 \)” for the unit of \( S_\infty \). Equivalently, the law of these operators is to be considered in the \( W^\ast \)-probability space \( (M, \text{tr}) \), where \( \text{tr} \) is the natural trace-state of the von Neumann algebra \( M \subseteq B(H) \) generated by the \( U(\gamma_n) \)'s – cf. discussion in Section 3.1 below.

It turns out that the following holds.

**Theorem 2.5.** For every \( n \in \mathbb{N} \), let \( \mu_n \) denote the law of the rescaled average (2.10), in the sense described above. Then, when \( n \to \infty \), the probability measures \( \mu_n \) have a \( \text{weak}^\ast \)-limit \( \mu_w \), which depends on the tuple of weights \( w = (w_1, \ldots, w_d) \) fixed in Notation 2.2.

Theorem 2.5 falls under the general umbrella of “exchangeable CLT” results. Indeed, it is easily seen that upon putting
\[
\hat{U}_n := U(\gamma_n) - A_0 \in M, \quad n \in \mathbb{N},
\]
we get a sequence \( (\hat{U}_n)_{n=1}^\infty \) of elements of \( M \) which are centred and exchangeable with respect to the trace \( \text{tr} \). In Section 4 of the paper we provide an elementary combinatorial proof that the operators \( \hat{U}_n \) also satisfy the “vanishing singleton property” used as hypothesis in a basic exchangeable CLT result of Bożejko and Speicher, Theorem 0 in [6]. This makes Theorem 2.5 come out as a corollary of the said theorem from [6]. An alternative, more high-powered approach is to obtain Theorem 2.5 by reducing it to [10, Theorem 9.4],
a result which applies to sequences of non-commutative random variables displaying weaker probabilistic symmetries than what we have here.

We emphasize the fact that in order for the CLT mechanism to kick in, it is important that the centering of the $U(\gamma_n)$’s goes with $\tilde{U}_n := U(\gamma_n) - A_0$. This differs from the usual centering procedure, which is done by subtracting a scalar multiple of the unit, and is commonly denoted by putting a circle on the top of the element which is being centered. In the case at hand we would write $\tilde{U}_n := U(\gamma_n) - \lambda 1_M$, with $\lambda = \text{tr}(U(\gamma_n)) = p_2$. But unless we are in the special case with $w_1 = \cdots = w_d = 1/d$, using the operators $\tilde{U}_n$ does not lead to a theorem of CLT type.

2.3. Realization of $\mu_w$ as the law of a traceless CCR-GUE matrix.

The GUE model is one of the most referenced models of random Hermitian matrix – see e.g. [11, Chapter 3]. For the purposes of this paper, it is convenient to consider a GUE matrix $G = \{g_{i,j}\}_{i,j=1}^d$ with entries rescaled such that the expected normalized trace of $G^2$ is $E(\text{tr}_d(G^2)) = 1$. The so-called “traceless GUE” is less referenced in the literature, but is for instance mentioned in Biane’s paper [2] cited above. It is the random matrix $M$ obtained from the above $G$ by projecting the random vector $(g_{1,1}, \ldots, g_{d,d}) \in \mathbb{R}^d$ onto the hyperplane of equation $t_1 + \cdots + t_d = 0$. Thus

$$M := G - \frac{g_{1,1} + \cdots + g_{d,d}}{d} I_d,$$

where $I_d$ is the identity $d \times d$ matrix. It follows, in particular, that the diagonal entries of $M$ are linearly dependent. They form a Gaussian family of centred random variables with covariance matrix $C = [c_{i,j}]_{i,j=1}^d$, where:

$$c_{i,j} = (d-1)/d^2 \text{ for } 1 \leq i \leq d \text{ and } c_{i,j} = c_{j,i} = -1/d^2 \text{ for } 1 \leq i < j \leq d.$$

The result obtained in [11, Theorem 1.1] can be phrased like this: if the tuple of weights $w = (w_1, \ldots, w_d)$ from Notation 2.2 happens to have $w_1 = \cdots = w_d = 1/d$, then the limit law $\mu_w$ from Theorem 2.5 is precisely the law (a.k.a. the average empirical eigenvalue distribution) of the traceless GUE matrix $M$ from Equation (2.12). The main point of the present paper is to indicate a nice phenomenon which appears when some of the weights $w_1, \ldots, w_d$ are allowed to be different from each other:

$$\begin{align*}
\text{The limit law } \mu_w \text{ still is the law of a matrix } M \text{ like in (2.12).} \\
\text{Only that now an off-diagonal entry } g_{i,j} \text{ and its adjoint } g_{j,i} = g_{i,j}^* \text{ are set to satisfy the relation } g_{i,j}g_{j,i} = g_{j,i}g_{i,j} + (w_j - w_i).
\end{align*}$$

Of course, if $i, j \in \{1, \ldots, d\}$ are such that $w_i \neq w_j$, then the commutation relation stated in (2.14) forces $g_{i,j}$ to no longer be a random variable in the usual sense. In order to substantiate the statement made in (2.14), we thus introduce the following notion.

**Definition 2.6.** (CCR-analogue of a complex Gaussian random variable.) Let $(\mathcal{A}, \varphi)$ be a $\ast$-probability space and let $\omega_{(1,\ast)}, \omega_{(\ast,1)}$ be two parameters in $(0, \infty)$. We will say that an element $a \in \mathcal{A}$ is a centred CCR-complex-Gaussian element with parameters $\omega_{(1,\ast)}$ and $\omega_{(\ast,1)}$ when we have the commutation relation

$$a^*a = aa^* + (\omega_{(\ast,1)} - \omega_{(1,\ast)})1_A,$$
and we have the expectation formula

\[ \varphi(a^p (a^*)^q) = \begin{cases} 0, & \text{if } p \neq q; \\ p! \omega_{(1,\ast)}^p, & \text{if } p = q. \end{cases} \quad \text{for } p, q \in \mathbb{N} \cup \{0\}. \tag{2.16} \]

**Remark 2.7.** The term “Gaussian” used in Definition 2.6 is justified by looking at the special case when \( \omega_{(1,\ast)} = \omega_{(\ast,1)} =: \omega \). In such a case, Equation (2.15) says that \( a \) commutes with \( a^* \) (hence can be treated like a complex random variable from classical probability), while (2.16) becomes the formula giving the joint moments of \( f \) and \( \overline{f} \) for a centred complex Gaussian variable \( f \) of variance \( \omega \).

We note that the commutation in (2.15) together with the prescription from (2.16) determine all the joint moments of \( a \) and \( a^* \). In particular one gets, symmetrically to the second branch of (2.16), that \( \varphi((a^*)^p a^q) = p! \omega_{(\ast,1)}^p \) for every \( p \in \mathbb{N} \). This is a special case of a “Wick-style” formula for computing joint moments of \( a \) and \( a^* \) which is given in Proposition S.3 in the body of the paper.

We next introduce a CCR-analogue for the notion of traceless GUE matrix. Concerning the diagonal elements \( a_{1,1}, \ldots, a_{d,d} \) appearing in the next definition, observe that the covariance matrix \( C \) given in (2.17) below is a generalization of (2.13) from the description of a traceless GUE. The fact that \( a_{1,1}, \ldots, a_{d,d} \) form a Gaussian family can be understood in the sense that their joint moments are given by a rather standard Wick formula (cf. review in Definition 8.2.1 below).

**Definition 2.8.** (CCR analogue for a traceless GUE matrix.) Let \( (\mathcal{A}, \varphi) \) be a \(*\)-probability space. Suppose we have a family of unital \(*\)-subalgebras of \( \mathcal{A} \), denoted as

\[ \{ \mathcal{A}_0 \} \cup \{ \mathcal{A}_{i,j} \mid 1 \leq i < j \leq d \}, \]

which are commuting independent (a standard notion – cf. review in Definition 8.2.1 below). Suppose moreover that we have some elements of \( \mathcal{A} \), as follows.

(i) For every \( 1 \leq i < j \leq d \), we have a centred CCR-complex-Gaussian element \( a_{i,j} \in \mathcal{A}_{i,j} \) with parameters \( w_j \) and \( w_i \) (meaning that in Definition 2.6 we take \( \omega_{(1,\ast)} = w_j \) and \( \omega_{(\ast,1)} = w_i \)). We put \( a_{j,i} := a_{i,j}^* \in \mathcal{A}_{i,j} \), thus getting the relation \( a_{j,i} a_{i,j} = a_{i,j} a_{j,i} + (w_i - w_j) 1_{\mathcal{A}} \).

(ii) \( \mathcal{A}_0 \) is commutative and we have selfadjoint elements \( a_{1,1}, \ldots, a_{d,d} \in \mathcal{A}_0 \) which form a centred Gaussian family with covariance matrix \( C = [c_{i,j}]_{i,j=1}^d \), where:

\[ c_{i,i} = w_i - w_i^2 \text{ for } 1 \leq i \leq d, \quad \text{and} \quad c_{i,j} = c_{j,i} = -w_i w_j \text{ for } 1 \leq i < j \leq d. \tag{2.17} \]

Then the selfadjoint matrix \( M = [a_{i,j}]_{1 \leq i,j \leq d} \in M_d(\mathcal{A}) \) is said to be a traceless CCR-GUE matrix with parameters \( w_1, \ldots, w_d \).

The main result of the present paper is then stated as follows.

**Theorem 2.9.** Let \( (\mathcal{A}, \varphi) \) be a \(*\)-probability space, and let \( M = [a_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A}) \) be a traceless CCR-GUE matrix with parameters \( w_1, \ldots, w_d \), as in the preceding definition. Consider the linear functional \( \varphi_\omega : M_d(\mathcal{A}) \to \mathbb{C} \) defined by

\[ \varphi_\omega(X) = \sum_{i=1}^d w_i \varphi(x_{i,i}), \quad \text{for } X = [x_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A}). \tag{2.18} \]
Then the law of $M$ in the $*$-probability space $(M_d(A),\varphi_w)$ is equal to the limit law $\mu_w$ from the theorem of CLT type discussed in Section 2.2.

Remark 2.10. In the special case when $w_1 = \cdots = w_d = 1/d$, the traceless CCR-GUE matrix $M$ from Definition 2.8 becomes a usual traceless GUE matrix, as defined in Equation 2.12 at the beginning of this subsection. Let us simply denote by $\mu_d$ the limit law $\mu_w$ which appears in this special case, and let us use the notation $\nu_d$ for the law (a.k.a. “average empirical eigenvalue distribution”) of the actual GUE matrix $G$. It turns out that $\mu_d$ and $\nu_d$ are related by a nice convolution formula:

$\nu_d = \mu_d \ast \mathcal{N}(0,1/d^2),$  

(2.19)

where $\mathcal{N}(0,1/d^2)$ is the centred normal distribution of variance $1/d^2$. In the paper [11], the formula (2.19) was used in order to write explicitly the Laplace transform of the limit law $\mu_d$ (which is easily done, based on the fact that the Laplace transform of $\mu_d$ is well-known – see e.g. the Section 3.3 of the monograph [1]).

The paper [11] is exclusively devoted to the case when $w_1 = \cdots = w_d = 1/d$. It is perhaps amusing to note that the authors of that paper were not aware of the interpretation of $\mu_d$ as distribution of the traceless GUE, and derived Equation (2.19) directly from the interpretation of $\mu_d$ as limit law in the CLT for star-generators. Once we know that $\mu_d$ is the distribution of the traceless GUE matrix $M$, it becomes of course much easier to obtain (2.19) via a calculation based on the relation $G = M + \frac{\sqrt{d}}{d}I_d$, which is a re-writing of the definition of $M$ in (2.12).

2.4. Underlying combinatorics: two permutations associated to a $\pi \in \mathcal{P}_2(k)$.

In this subsection we point out an interesting combinatorial phenomenon which connects the Theorems 2.5 and 2.9 stated above. The limit law $\mu_w$ turns out to be symmetric with finite moments of all orders, and the proofs of both Theorems 2.5 and 2.9 rely on explicit formulas for the even moments of $\mu_w$. In both cases, these explicit formulas express the moment of order $k$ of $\mu_w$ as a summation over the set $\mathcal{P}_2(k)$ of pair-partitions of $\{1, \ldots, k\}$; and in both cases the general term of the summation, indexed by a $\pi \in \mathcal{P}_2(k)$, involves a certain permutation in $S_\infty$ that is associated to $\pi$. But then things go as follows.

- The moment formulas coming from Theorem 2.5 use a permutation “$\tau_{\pi}$” associated to $\pi$, where $\tau_{\pi}$ is defined as a certain product of star-transpositions.

- The moment formulas needed in Theorem 2.9 are Wick-style formulas, and use a permutation “$\sigma_{\pi}$” associated to $\pi$, where $\sigma_{\pi}$ is obtained by simply viewing every pair of $\pi$ as a transposition.

At first sight, the permutations $\tau_{\pi}$ and $\sigma_{\pi}$ do not appear to be directly related to each other. For a concrete example, say for instance that $k = 8$ and we are dealing with

$\pi = \{\{3, 8\}, \{4, 7\}, \{1, 6\}, \{2, 5\}\} \in \mathcal{P}_2(8).$

In order to be consistent with the notation used in the body of the paper, we listed the pairs $V_1, \ldots, V_4$ of $\pi$ in decreasing order of their maximal elements: $V_1 = \{3, 8\}, \ldots, V_4 = \{2, 5\}.$

The permutation $\sigma_{\pi}$ simply is the product of the 4 disjoint transpositions $(3, 8), (4, 7), (1, 6)$ and $(2, 5)$. The permutation $\tau_{\pi}$ is found via a more elaborate procedure, described as follows: on a picture of $\pi$ as shown in Figure 1 below, we draw $\gamma_1$’s on top of the two elements of
from Equation (2.4), the character \( \chi \) and the GNS representation \( U \) as this is the information needed in order to compute \( \chi \)

For a general \( \pi \in \mathcal{P}_2(k) \), the product of star-transpositions which defines \( \tau_\pi \) has 2 occurrences of each of \( \gamma_1, \ldots, \gamma_{k/2} \), thus cannot move any number \( n > (k + 2)/2 \), and belongs to the subgroup \( S_{(k+2)/2} \) of \( S_\infty \). We are, in fact, only interested in the sizes of orbits of \( \tau_\pi \), as this is the information needed in order to compute \( \chi(\tau_\pi) \); it is not obvious, though, how these sizes of orbits can be expressed in terms of the “other” permutation \( \sigma_\pi \in S_k \subseteq S_\infty \).

The nice fact we want to signal here is that the desired orbit sizes can actually be read from the permutation \( \eta_{k+1} \cdot \sigma_\pi \in S_{k+1} \), where \( \eta_{k+1} = (1, 2, \ldots, k, k + 1) \). More precisely, the following holds.

**Proposition 2.11.** Let \( \pi = \{V_1, \ldots, V_{k/2}\} \in \mathcal{P}_2(k) \) and let us consider the permutations \( \tau_\pi \in S_{(k+2)/2} \) and \( \sigma_\pi \in S_k \), as described above. Consider moreover the decomposition

\[
\{1, \ldots, k+1\} = R_1 \cup \cdots \cup R_p
\]

of \( \{1, \ldots, k+1\} \) into orbits of \( \eta_{k+1} \cdot \sigma_\pi \). Then the decomposition of \( \{1, \ldots, (k+2)/2\} \) into orbits of \( \tau_\pi \) has \( p \) orbits, of cardinalities

\[
|R_1 \cap B|, \ldots, |R_p \cap B| \quad \text{(in some order)},
\]

where \( B := \{\max(V_i) \mid 1 \leq i \leq k/2\} \cup \{k+1\} \).

In the concrete example shown in Figure 1: one has

\[
\eta_9 \cdot \sigma_\pi = (1, 2, \ldots, 9) \cdot (3, 8)(4, 7)(1, 6)(2, 5) = (1, 7, 5, 3, 9)(2, 6)(4, 8).
\]

Intersecting the orbits of \( \eta_9 \cdot \sigma_\pi \) with \( B = \{5, 6, 7, 8, 9\} \) splits \( B \) into a \( 3 + 1 + 1 \) partition, indeed of the same type as the partition of \( \{1, \ldots, 5\} \) into orbits of \( \tau_\pi \).

**Proposition 2.11** (or rather, a slight generalization shown in Section 6 below, which allows \( \pi \) to have singleton blocks) is the combinatorial link connecting Theorems 2.5 and 2.9.

3. Background: the framework of \((\mathcal{M}, \text{tr})\) and the law of large numbers

In this section we clarify what is the non-commutative probability space we will work with, and we discuss the law of large numbers which is a prerequisite for a theorem of CLT kind. We will use the framework considered in Section 2.1: the tuple of weights \( w = (w_1, \ldots, w_d) \) that is fixed for the whole paper, the symmetric power sums \( 1 = p_1 > p_2 > \cdots > p_n > \cdots \) from Equation (2.4), the character \( \chi : S_\infty \to \mathbb{R} \) defined by using the \( p_n \)'s in Equation (2.5), and the GNS representation \( U : S_\infty \to B(\mathcal{H}) \) of the character \( \chi \), which was introduced in Notation 2.3.
3.1. The $W^*$-probability space $(\mathcal{M}, \text{tr})$.

Remark 3.1. 1° Recall that we use the notation $\hat{\sigma}$ for the vector in $\mathcal{H}$ corresponding to a permutation $\sigma \in S_\infty$. From the prescription of inner products stated in Notation 2.3 it is clear that $\|\hat{\sigma}\| = 1$ for every $\sigma \in S_\infty$, while for any $\sigma \neq \tau$ in $S_\infty$, the common value of $\langle \hat{\sigma}, \hat{\tau} \rangle$ and $\langle \hat{\tau}, \hat{\sigma} \rangle$ falls in the interval $(0, 1)$. This also has the immediate consequence that the map $S_\infty \ni \sigma \mapsto \hat{\sigma} \in \mathcal{H}$ is injective, since

$$\|\hat{\sigma} - \hat{\tau}\|^2 = 2(1 - \chi(\sigma\tau^{-1})) > 0, \; \forall \sigma \neq \tau \in S_\infty.$$ 

2° In general, we cannot count on the set $\{\hat{\sigma} \mid \sigma \in S_\infty\} \subseteq \mathcal{H}$ to be linearly independent. But we can at least be sure that the sequence $(\hat{\gamma}_n)_{n=1}^\infty$ is linearly independent. This is because, for every $n \geq 1$, the Gram matrix associated to the vectors $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$ has entries

$$\langle \hat{\gamma}_i, \hat{\gamma}_j \rangle = \chi(\gamma_i\gamma_j) = \begin{cases} 1, & \text{if } i = j, \\ p_3, & \text{if } i \neq j; \end{cases}$$

this Gram matrix is then found to be invertible, since $0 < p_3 < 1$.

Notation and Remark 3.2. 1° It is immediate that the linear span of the operators $U(\sigma)$ is a unital *-subalgebra of $B(\mathcal{H})$ and, consequently, that the closure

$$\mathcal{M} := \overline{\text{span}}_{\text{WOT}}\{U(\sigma) : \sigma \in S_\infty\} \subseteq B(\mathcal{H})$$  

is a von Neumann algebra of operators on $\mathcal{H}$.

2° Consider the special vector $\hat{\eta}_1 \in \mathcal{H}$ (where recall that “$\eta_1$” stands for the unit of $S_\infty$). Then $\hat{\eta}_1$ is a trace-vector for $\mathcal{M}$; that is, the vector-state

$$\text{tr} : \mathcal{M} \to \mathbb{C}, \; \text{tr}(T) := \langle T(\hat{\eta}_1), \hat{\eta}_1 \rangle$$  

for $T \in \mathcal{M}$ is a trace. The trace property of tr is an immediate consequence of the fact that $\chi$ is a class-function: one has $\text{tr}(U(\sigma)U(\tau)) = \chi(\sigma\tau) = \chi(\tau\sigma) = \text{tr}(U(\tau)U(\sigma))$ for every $\sigma, \tau \in S_\infty$, which then implies that $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in \mathcal{M}$.

3° Standard arguments using the representation $U : S_\infty \to B(\mathcal{H})$ defined by multiplication on the right $(U(\sigma)) \hat{\tau} = \tau\hat{\sigma}^{-1}$ for $\sigma, \tau \in S_\infty$ yield the fact that the linear map

$$\mathcal{M} \ni T \mapsto T(\hat{\eta}_1) \in \mathcal{H}$$  

is injective. Hence putting $\|T\|_2 := \|T(\hat{\eta}_1)\|$ for $T \in \mathcal{M}$ defines a norm on $\mathcal{M}$. A known property of this norm (see e.g. [13, Section II.2]) is that it metrizes the SOT-topology on the unit ball $\mathcal{B} := \{T \in \mathcal{M} \mid \|T\| \leq 1\}$. So, in particular: if $T$ and $(T_n)_{n=1}^\infty$ are in $\mathcal{B}$, then verifying that $\lim_{n \to \infty} \|T_n - T\|_2 = 0$ will be sufficient to ensure that $(T_n)_{n=1}^\infty$ is SOT-convergent to $T$.

4° As a consequence of the injectivity of the map [63], the trace-state tr is faithful, i.e. has the property that $\text{tr}(T^*T) > 0$ for every $T \neq 0$ in $\mathcal{M}$.

In summary: we see that tr is a WOT-continuous faithful trace-state of the von Neumann algebra $\mathcal{M}$. The couple $(\mathcal{M}, \text{tr})$ is what one refers to as a tracial $W^*$-probability space; this $(\mathcal{M}, \text{tr})$ is the framework we will use throughout the paper.
3.2. The operator $A_0 \in \mathcal{M}$ and the law of large numbers for $U(\gamma_n)$’s.

We now start looking at the sequence of operators $U(\gamma_n)$ in $\mathcal{M}$. Observe that the $U(\gamma_n)$’s are linearly independent; this is clear from the fact that, upon applying these operators to the vector $\hat{\gamma}_1$, we get the linearly independent sequence of vectors $(\gamma_n)_{n=1}^{\infty}$ of $\mathcal{H}$. It turns out that the sequence $(U(\gamma_n))_{n=1}^{\infty}$ is WOT-convergent. The limit is a special operator $A_0 \in \mathcal{M}$ used in [9], which played a crucial role in the considerations of non-commutative dynamics made in that paper. For the sake of keeping the presentation self-contained, we include below the relatively easy proofs of the facts we need, in particular (cf. Proposition 3.4) of the occurrence of $A_0$ in the law of large numbers we are interested in.

Proposition 3.3. The sequence $(U(\gamma_n))_{n=1}^{\infty}$ has a WOT-limit $A_0 \in \mathcal{M}$, where $A_0 = A_0^*$, and $\|A_0\| \leq 1$. The operator $A_0$ can be described via its action on vectors, as follows:

\begin{equation}
\text{For every } \sigma, \tau \in S_\infty, \text{ one has } \langle A_0(\hat{\sigma}), \hat{\tau} \rangle = \frac{p_{|\hat{\tau}|}}{p_{|\tau|}} \langle \hat{\sigma}, \hat{\tau} \rangle,
\end{equation}

where $V$ is the orbit of $\sigma \tau^{-1}$ which contains the number 1.

Proof. We start by verifying a formula related to (3.4), stated as follows: let $\sigma, \tau \in S_k$ (i.e. such that $\sigma(n) = \tau(n) = n$ for all $n > k$), and let $V$ be the orbit of $\sigma \tau^{-1}$ which contains the number 1. Then

\begin{equation}
\langle [U(\gamma_n)](\hat{\sigma}), \hat{\tau} \rangle = \frac{p_{1+|V|}}{p_{|V|}} \langle \hat{\sigma}, \hat{\tau} \rangle, \quad \forall n \geq k.
\end{equation}

Indeed, the equality stated in (3.5) amounts to

$$\chi(\gamma_n \sigma \tau^{-1}) = \frac{p_{1+|V|}}{p_{|V|}} \chi(\sigma \tau^{-1}), \quad \forall n \geq k.$$  

This follows directly from the formula defining $\chi$ in (2.5), combined with the following observation about orbit structure: when $n \geq k$, the permutation $\gamma_n \sigma \tau^{-1}$ has the same orbits as $\sigma \tau^{-1}$, with the only difference that the number $n + 1$ is now inserted into the orbit $V$ of $\sigma \tau^{-1}$, right before the occurrence of 1 in that orbit. (Denoting $(\tau \sigma^{-1})(1) := h \in \{1, \ldots, k\}$; one has that $\sigma \tau^{-1}$ sends $h$ directly to 1, while $\gamma_n \sigma \tau^{-1}$ sends $h \mapsto n + 1 \mapsto 1$.)

From (3.5) it easily follows, by approximating arbitrary $\xi_1, \xi_2 \in \mathcal{H}$ with linear combinations of vectors $\hat{\sigma}, \hat{\tau}$, that the limit

\begin{equation}
\beta(\xi_1, \xi_2) := \lim_{n \to \infty} \langle [U(\gamma_n)](\xi_1), \xi_2 \rangle
\end{equation}

exists for every $\xi_1, \xi_2 \in \mathcal{H}$. Equation (3.6) defines a sesquilinear functional $\beta$ on $\mathcal{H}$, with the property that $|\beta(\xi_1, \xi_2)| \leq \|\xi_1\| \cdot \|\xi_2\|$ for all $\xi_1, \xi_2 \in \mathcal{H}$, and a standard argument infers from here the existence of an $A_0 \in B(\mathcal{H})$ such that $\beta(\xi_1, \xi_2) = (A_0(\xi_1), \xi_2)$ for all $\xi_1, \xi_2 \in \mathcal{H}$. In combination with (3.6), the latter formula says that the sequence $(U(\gamma_n))_{n=1}^{\infty}$ converges WOT to $A_0$. We have that $A_0 \in \mathcal{M}$ with $A_0^* = A_0$ and $\|A_0\| \leq 1$, because the $U(\gamma_n)$’s have these properties. Finally, making $n \to \infty$ in Equation (3.5) gives the formula (3.4). \qed

Proposition 3.4. (Law of large numbers.) Let $A_0 \in \mathcal{M}$ be as in Proposition 3.3. Then

\begin{equation}
\text{SOT - } \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} U(\gamma_i) = A_0.
\end{equation}

Proof. We will prove that

\begin{equation}
\left\| \frac{1}{n} \sum_{i=1}^{n} U(\gamma_i) - A_0 \right\|_2^2 = \frac{1 - p_3}{n}, \quad \forall n \geq 1.
\end{equation}
is obtained by using the following procedure:

Let \( \{ x_i \} \) be a tuple obtained out of \( \{ \gamma_i \} \) by replacing its infinite values by some “new, distinct finite values”. That is, \( \{ \gamma_i \} \) gives \( \{ x_i \} \). We thus focus on the induction step: we fix an \( \gamma_3 \) such that \( \gamma_3 \rightarrow \infty \) and at the third equality sign we made use of (3.10). Upon putting all these things together and plugging them into (3.9), we find that the quantity appearing in (3.9) is indeed equal to \( (1 - p_3)/n \), as claimed in (3.8).

The calculation of \( \| A_0(\tilde{\eta}) \|^2 \) (equivalently, of \( \text{tr}(A_0^2) \)) shown above gives a special case of a formula for moments with respect to the trace, which we record in the next proposition.

**Proposition 3.5.** Consider a tuple \( \mathbf{j} : \{1, \ldots, k\} \rightarrow \mathbb{N} \cup \{\infty\} \) and the operators \( T_1, \ldots, T_k \in \mathcal{M} \) defined by

\[
(3.11) \quad T_h = \left\{ \begin{array}{ll}
U(\gamma_j(h)), & \text{if } j(h) \in \mathbb{N}, \\
A_0, & \text{if } j(h) = \infty
\end{array} \right\}, \quad 1 \leq h \leq k.
\]

Let \( \mathbf{j} : \{1, \ldots, k\} \rightarrow \mathbb{N} \) be a tuple obtained out of \( \mathbf{j} \) by replacing its infinite values by some “new, distinct finite values”. That is, \( \mathbf{j} \) is obtained by using the following procedure:

\[
(3.12) \quad \text{let } Q := \{ q \in \mathbb{N} \mid \exists 1 \leq h \leq k \text{ such that } j(h) = q \};
\]

\[
\text{chooses an injective function } f : \mathbf{j}^{-1}(\infty) \rightarrow \mathbb{N} \setminus Q;
\]

\[
\text{define } \mathbf{j} : \{1, \ldots, k\} \rightarrow \mathbb{N} \text{ by putting } j(h) = \left\{ \begin{array}{ll}
j(h), & \text{if } j(h) \in \mathbb{N}, \\
f(h), & \text{if } j(h) = \infty
\end{array} \right\}.
\]

Then one has that

\[
(3.13) \quad \text{tr}(T_1 \cdots T_k) = \text{tr}\left( U(\gamma_j(1)) \cdots U(\gamma_j(k)) \right) = \chi(\gamma_{j(1)} \cdots \gamma_{j(k)}).
\]

**Proof.** We proceed by induction on the cardinality \( \ell = j^{-1}(\infty) \). The base case of the induction is \( \ell = 0 \); in this case (3.13) holds trivially, since the procedure described in (3.12) gives \( \mathbf{j} = \mathbf{i} \). We thus focus on the induction step: we fix an \( \ell \geq 1 \), we assume the statement of the proposition holds for \( \ell - 1 \), and we will prove that it holds for \( \ell \) as well.

So let \( \mathbf{j} : \{1, \ldots, k\} \rightarrow \mathbb{N} \cup \{\infty\} \) with \( |j^{-1}(\infty)| = \ell \), and let \( T_1, \ldots, T_k \in \mathcal{M} \) be as in (3.11). We consider a tuple \( \mathbf{j} \) defined as in (3.12), and we aim to prove that (3.13) holds.
We choose an \( h_o \in i^{-1}(\infty) \subseteq \{1, \ldots, k\} \) and an \( n_o \in \mathbb{N} \) such that \( n_o > \max(Q) \) and \( n_o > \max\{f(h) \mid h \in i^{-1}(\infty)\} \), where \( f \) is the function used in (3.12) in order to construct the tuple \( j \). For every \( n \geq n_o \) we consider the tuples \( j^{(n)} : \{1, \ldots, k\} \to \mathbb{N} \cup \{\infty\} \) and \( j^{(n)} : \{1, \ldots, k\} \to \mathbb{N} \) defined by

\[
(3.14) \quad j^{(n)}(h) := \begin{cases} j(h), & \text{if } h \neq h_o, \\ n, & \text{if } h = h_o \end{cases}, \quad \text{and} \quad j^{(n)}(h) := \begin{cases} j(h), & \text{if } h \neq h_o, \\ n, & \text{if } h = h_o \end{cases}.
\]

For such \( n \) we also consider the operators \( T^{(n)}_1, \ldots, T^{(n)}_k \in M \) defined by the same recipe as in Equation (3.11), but by starting from the tuple \( j^{(n)} \) instead of \( j \).

For every \( n \geq n_o \), the pre-image \( (i^{(n)})^{-1}(\infty) \) has cardinality \( \ell - 1 \), hence the induction hypothesis applies to \( j^{(n)} \). It is immediate that, when used on \( j^{(n)} \), the procedure described in (3.12) can be arranged to lead to the tuple \( j^{(n)} \). We thus obtain that

\[
(3.15) \quad \text{tr}(T^{(n)}_1 \cdots T^{(n)}_k) = \chi(j^{(n)}_1) \cdots j^{(n)}_k), \quad n \geq n_o.
\]

We leave it as an exercise to the reader to check that, for every \( n \geq n_o \), due to how \( j^{(n)} \) is defined in (3.14) (and due to the specifics of how \( j \) is obtained), the permutation \( \gamma^{(n)}_1 \cdots j^{(n)}_k \) has the same orbit structure as \( \gamma^{(n)}_1 \cdots j^{(n)}_k \). The character \( \chi \) thus takes the same value on these two permutations. We come to the conclusion that the right-hand side of (3.15) is actually independent of \( n \), and constantly equal to \( \chi(\gamma^{(n)}_1 \cdots j^{(n)}_k) \).

Now to the left-hand side of (3.15): we observe that the operators indicated there can also be described as

\[
(3.16) \quad T^{(n)}_h = \begin{cases} T_h, & \text{if } h \neq h_o, \\ U(\gamma_n), & \text{if } h = h_o. \end{cases}
\]

Thus \( T^{(n)}_h \) differs from \( T_h \) only on position \( h = h_o \), where we have

\[
\text{WOT} - \lim_{n \to \infty} T^{(n)}_h = \text{WOT} - \lim_{n \to \infty} U(\gamma_n) = A_0 = T_{h_o}.
\]

This further implies that \( \text{WOT} - \lim_{n \to \infty} T^{(n)}_1 \cdots T^{(n)}_k = T_1 \cdots T_k \), and upon applying the WOT-continuous functional \( \text{tr} \) to the latter limit we find that

\[
\lim_{n \to \infty} \text{tr}(T^{(n)}_1 \cdots T^{(n)}_k) = \text{tr}(T_1 \cdots T_k).
\]

We conclude that making \( n \to \infty \) in (3.15) leads to the Equation (3.13) we were after. \( \square \)

**Corollary 3.6.**

1° For every \( k \in \mathbb{N} \), one has that \( \text{tr}(A^k_0) = p_{k+1} \).

2° The scalar spectral measure of \( A_0 \) with respect to \( \text{tr} \) is equal to \( \sum_{i=1}^d w_i \delta_{w_i} \) (convex combination of Dirac measures).

3° The spectrum of \( A_0 \) is equal to \( \{t \in (0, 1) \mid \exists 1 \leq i \leq d \text{ such that } w_i = t\} \).

**Proof.** 1° This is the special case of Proposition 3.5 when the tuple \( j \) under consideration has \( j(h) = \infty \) for all \( 1 \leq h \leq k \). Indeed, in this special case we can pick the tuple \( j : \{1, \ldots, k\} \to \mathbb{N} \) to be defined by \( j(h) = h, 1 \leq h \leq k \), and Equation (3.13) simply tells us that \( \text{tr}(A^k_0) = \chi(\gamma_1 \cdots \gamma_k) \). Since \( \gamma_1 \cdots \gamma_k = (k+1, k, \ldots, 2, 1) \), a cycle of length \( k+1 \), it follows that \( \text{tr}(A^k_0) = p_{k+1} \), as claimed.
2° Let $\mu$ be the scalar spectral measure of $A_0$ with respect to $\text{tr}$, and let $\mu' := \sum_{i=1}^d w_i \delta_{w_i}$. For every $k \geq 1$ we have
\[
\int_{\mathbb{R}} t^k d\mu'(t) = \sum_{i=1}^d w_i \cdot (w_i)^k = \sum_{i=1}^d w_i^{k+1} = p_{k+1} = \text{tr}(A_0^k) = \int_{\mathbb{R}} t^k d\mu(t),
\]
where at the last two equality signs we used the result of 1° above and the definition of a scalar spectral measure. Thus $\mu$ and $\mu'$ are two compactly supported probability measures which have the same moments, and this implies that $\mu = \mu'$.

3° Since $\text{tr}$ is faithful, the spectrum of $A_0$ is known (see e.g. [12, Proposition 3.15]) to be equal to the support $\text{supp}(\mu)$, where $\mu$ is the scalar spectral measure from the proof of part 2°. But from 2° it is clear that $\text{supp}(\mu) = \{ t \in (0,1) \mid \exists 1 \leq i \leq d \text{ such that } w_i = t \}$. 

4. THE THEOREM OF CLT TYPE FOR THE SEQUENCE OF OPERATORS $U(\gamma_n) - A_0$

Now that we identified in Proposition 3.4 the law of large numbers for the sequence of represented star-generators $U(\gamma_n)$, we go to the next step usually taken towards a theorem of CLT type: we subtract the limit $A_0$ out of the $U(\gamma_n)$’s, and we seek to find a limit in law for the rescaled averages
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^n (U(\gamma_i) - A_0).
\]
This is covered by [10, Theorem 9.4], which applies to sequences with certain weaker probabilistic symmetries. For the reader’s convenience, we will give a self-contained presentation, tailored to the specific instance of CLT that we want to address.

4.1. Review of the CLT for exchangeable sequences.

Notation 4.1. (Review of some basic notation about set-partitions.) Let $k$ be in $\mathbb{N}$.

1° $\mathcal{P}(k)$ will denote the set of all partitions of $\{1, \ldots, k\}$. A partition $\pi \in \mathcal{P}(k)$ is thus of the form $\pi = \{V_1, \ldots, V_\ell\}$ where $V_1, \ldots, V_\ell$ (the blocks of $\pi$) are non-empty pairwise disjoint sets with $V_1 \cup \cdots \cup V_\ell = \{1, \ldots, k\}$.

2° On $\mathcal{P}(k)$ we consider the partial order given by reverse refinement: for $\pi, \rho \in \mathcal{P}(k)$ we write “$\pi \leq \rho$” to mean that every block of $\pi$ is contained in some block of $\rho$.

3° We denote $\mathcal{P}_2(k) := \{ \pi \in \mathcal{P}(k) \mid \text{ every block } V \in \pi \text{ has } |V| = 2 \}$ (where $\mathcal{P}_2(k) = \emptyset$ if $k$ is odd). The elements of $\mathcal{P}_2(k)$ are called pair-partitions.

4° We will treat a tuple $i \in \mathbb{N}^k$ as a function $i : \{1, \ldots, k\} \rightarrow \mathbb{N}$. One defines the kernel of such a tuple as the partition $\ker(\pi) \in \mathcal{P}(k)$ defined as follows: two numbers $p, q \in \{1, \ldots, k\}$ belong to the same block of $\ker(\pi)$ if and only if $i(p) = i(q)$.

Notation and Remark 4.2. (Exchangeable sequences.) Let $(A, \varphi)$ be a $*$-probability space and let $(a_n)_{n=1}^\infty$ be a sequence of selfadjoint elements of $A$. Quantities of the form
\[
\varphi(a_{\underline{i}(1)} \cdots a_{\underline{i}(k)}), \text{ with } k \in \mathbb{N} \text{ and } i : \{1, \ldots, k\} \rightarrow \mathbb{N},
\]
go under the name of joint moments of \((a_n)_{n=1}^\infty\). We say that \((a_n)_{n=1}^\infty\) is exchangeable to mean that its joint moments are invariant under the natural action of \(S_\infty\), that is:

\[
\varphi(a_{j(1)} \cdots a_{j(k)}) = \varphi(a_{\sigma j(1)} \cdots a_{\sigma j(k)})
\]

for every \(k \in \mathbb{N}\) and \(\overline{i}, \overline{j} : \{1, \ldots, k\} \to \mathbb{N}\) for which \(\exists \sigma \in S_\infty\) such that \(\overline{j} = \sigma \circ \overline{i}\).

It is easily seen that for two tuples \(\overline{i}, \overline{j} : \{1, \ldots, k\} \to \mathbb{N}\), the existence of a \(\sigma \in S_\infty\) such that \(\overline{j} = \sigma \circ \overline{i}\) is equivalent to the fact that \(\ker(\overline{i}) = \ker(\overline{j})\). Thus the condition (4.1) could be equivalently written as

\[
\varphi(a_{j(1)} \cdots a_{j(k)}) = \varphi(a_{i(1)} \cdots a_{i(k)})
\]

for every \(k \in \mathbb{N}\) and \(\overline{i}, \overline{j} \in \mathbb{N}^k\) such that \(\ker(\overline{i}) = \ker(\overline{j})\).

**Definition 4.3.** Let \((\mathcal{A}, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be an exchangeable sequence of selfadjoint elements of \(\mathcal{A}\).

1° The function on partitions associated to \((a_n)_{n=1}^\infty\) is \(t : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C}\) defined as follows: for every \(k \in \mathbb{N}\) and \(\pi \in \mathcal{P}(k)\) we put

\[
\begin{cases}
t(\pi) := \varphi(a_{\overline{i}(1)} \cdots a_{\overline{i}(k)}) & \text{where } \overline{i} \in \mathbb{N}^k \\
\text{any } k\text{-tuple such that } \ker(\overline{i}) = \pi.
\end{cases}
\]

This formula is unambiguous due to (4.2) above.

2° The sequence \((a_n)_{n=1}^\infty\) is said to have the singleton vanishing property when the function \(t\) defined in (4.3) satisfies:

\[
\begin{cases}
t(\pi) = 0 & \text{whenever } \pi \in \sqcup_{k=1}^\infty \mathcal{P}(k) \\
\text{has at least one block } V \text{ with } |V| = 1.
\end{cases}
\]

Note that the singleton vanishing property guarantees centering: in the special case when \(\pi\) is the unique partition in \(\mathcal{P}(1)\), (4.4) says that \(\varphi(a_n) = 0\) for every \(n \in \mathbb{N}\).

**Theorem 4.4.** (CLT for an exchangeable sequence, following [6].)

Let \((\mathcal{A}, \varphi)\) be a \(*\)-probability space and let \((a_n)_{n=1}^\infty\) be a sequence of selfadjoint elements of \(\mathcal{A}\) which is exchangeable and has the singleton vanishing property. Let \(t : \sqcup_{k=1}^\infty \mathcal{P}(k) \to \mathbb{C}\) be the function on partitions associated to \((a_n)_{n=1}^\infty\) in Definition 4.3. Consider the linear functional \(\mu : \mathbb{C}[X] \to \mathbb{C}\) defined via the prescription that \(\mu(1) = 1\) and that

\[
\mu(X^k) = \sum_{\rho \in \mathcal{P}_2(k)} t(\rho), \quad \forall k \in \mathbb{N},
\]

with the empty sum on the right-hand side of (4.5) being read as 0 for \(k\) odd. Then:

1° \(\mu\) is positive (that is, \(\mu(P \cdot \overline{P}) \geq 0\) for every \(P \in \mathbb{C}[X]\)).

2° For every \(n \in \mathbb{N}\) put

\[
s_n := \frac{1}{\sqrt{n}} (a_1 + \cdots + a_n) \in \mathcal{A}.
\]

Then \((s_n)_{n=1}^\infty\) converges in moments to \(\mu\), that is, one has \(\lim_{n \to \infty} \varphi(s_n^k) = \mu(X^k)\) for every \(k \in \mathbb{N}\). \(\square\)
4.2. Back to our framework.

Notation and Remark 4.5. We now return to the framework of the $W^*$-probability space $(\mathcal{M}, \text{tr})$ from Section 3.1. We use the same notation as there, and we put:

$$U_n := U(\gamma_n) \quad \text{and} \quad \hat{U}_n := U_n - A_0, \quad n \in \mathbb{N}. \quad (4.7)$$

Our plan is to prove that $(\hat{U}_n)_{n=1}^{\infty}$ is exchangeable and has the singleton vanishing property, and then invoke the general Theorem 4.4 reviewed above.

As explained in Section 2.2, the notation “\( \hat{U}_n \)” goes in the same spirit as the commonly used notation \( U_n := U_n - \text{tr}(U_n)1_{\mathcal{M}} = U_n - p_21_{\mathcal{M}} \). The element \( \hat{U}_n \) has in particular the centering property: \( \text{tr}(U_n) = \text{tr}(U_n) - \text{tr}(A_0) = p_2 - p_2 = 0 \). However, unless we are in the special case that our weights are \( w_1 = \cdots = w_d = 1/d \), one has \( \hat{U}_n \neq \hat{U}_n \). This distinction is important, as the development shown below (more precisely, the verification of the singleton vanishing property) would not work in connection to the elements \( \hat{U}_n \).

In order to verify the desired properties of \( (\hat{U}_n)_{n=1}^{\infty} \), it is convenient that we first introduce another bit of terminology concerning set-partitions.

Notation and Remark 4.6. Let \( k \) be a positive integer.

1° It is customary that for \( \pi, \rho \in P(k) \) one denotes

$$\pi \wedge \rho := \{ V \cap W \mid V \in \pi, W \in \rho, V \cap W \neq \emptyset \}. \quad (4.8)$$

It is immediate that \( \pi \wedge \rho \) (called the meet of \( \pi \) and \( \rho \)) belongs to \( P(k) \) and is the largest common lower bound of \( \pi \) and \( \rho \) with respect to the partial order by reverse refinement.

2° For every subset \( S \subseteq \{1, \ldots, k\} \) we will denote by \( \pi_S \), the partition in \( P(k) \) which has a block equal to \( \{1, \ldots, k\} \setminus S \) and has a singleton block at \( h \) for every \( h \in S \).

3° We will be interested in meets of the form \( \pi \wedge \pi_S \) for \( \pi \in P(k) \) and \( S \subseteq \{1, \ldots, k\} \). We note that the blocks of \( \pi \wedge \pi_S \) are described as follows:

- every \( h \in S \) becomes a singleton block in \( \pi \wedge \pi_S \);
- for every \( V \in \pi \) which is not contained in \( S \), we have that \( V \setminus S \) is a block of \( \pi \wedge \pi_S \).

Proposition 4.7. 1° The sequence \( (U_n)_{n=1}^{\infty} \) is exchangeable in \((\mathcal{M}, \text{tr})\).

2° The sequence \( (\hat{U}_n)_{n=1}^{\infty} \) is exchangeable in \((\mathcal{M}, \text{tr})\).

3° Let \( u \) be the function on partitions associated to the \( U_n \)'s and let \( t \) be the function on partitions associated to the \( \hat{U}_n \)'s. Then for every \( k \in \mathbb{N} \) and \( \pi \in P(k) \) we have

$$t(\pi) = \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|S|} u(\pi \wedge \pi_S). \quad (4.8)$$

Proof. 1° Let \( \mathbf{i}, \mathbf{j} : \{1, \ldots, k\} \rightarrow \mathbb{N} \) be tuples such that \( \ker(\mathbf{i}) = \ker(\mathbf{j}) \). It is easily seen that there exists \( \sigma \in S_{\infty} \) such that \( \gamma_{\mathbf{j}(h)} = \sigma^{-1} \gamma_{\mathbf{i}(h)} \sigma \) for every \( 1 \leq h \leq k \). Upon applying the representation \( U \) we therefore find that

$$U_{\mathbf{i}(1)} \cdots U_{\mathbf{i}(k)} = U(\sigma)^{-1} \cdot (U_{\mathbf{j}(1)} \cdots U_{\mathbf{j}(k)}) \cdot U(\sigma),$$

which in turn implies that

$$\text{tr}(U_{\mathbf{i}(1)} \cdots U_{\mathbf{i}(k)}) = \text{tr}(U_{\mathbf{j}(1)} \cdots U_{\mathbf{j}(k)}).$$
Since the right-hand side of (4.9) only depends on $\ker(\tilde{\pi})$, this equation will imply that $(\hat{U}_n)_{n=1}^\infty$ is exchangeable, and will also provide a formula connecting the functions $t$ and $u$, which is precisely what is stated in (4.8).

Let us then fix a tuple $\vec{i} : \{1, \ldots, k\} \to \mathbb{N}$, for which we verify that (4.9) holds. We have

$$
\text{tr}(\hat{U}_{\vec{i}(1)} \cdots \hat{U}_{\vec{i}(k)}) = \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|S|} u(\pi \land \pi_S), \quad \text{where } \pi = \ker(\vec{i}) \in \mathcal{P}(k).
$$

The required formula (4.9) follows when we substitute (4.11) into (4.10).

**Proposition 4.8.** The sequence $(\hat{U}_n)_{n=1}^\infty$ has the singleton vanishing property.

**Proof.** Fix a partition $\pi \in \mathcal{P}(k)$ which is known to have a singleton block at $p$, for some $1 \leq p \leq k$. Our goal for the proof is to verify that $t(\pi) = 0$, where, same as in the preceding proposition, $t$ denotes the function on partitions associated to $(\hat{U}_n)_{n=1}^\infty$.

Consider the self-map $\Phi$ of the power set of $\{1, \ldots, k\}$ which is defined by putting

$$
\Phi(S) = \begin{cases} 
S \setminus \{p\}, & \text{if } p \in S, \\
S \cup \{p\}, & \text{if } p \notin S
\end{cases}, \quad \text{for } S \subseteq \{1, \ldots, k\}.
$$

It is clear that $\Phi$ is a bijection from $2^{\{1, \ldots, k\}}$ onto itself. Another (elementary, but important) detail concerning $\Phi$ is that, due to our hypothesis that $\{p\}$ is a block of $\pi$, one has:

$$
\pi \land \pi_S = \pi \land \pi_{\Phi(S)}, \quad \forall S \subseteq \{1, \ldots, k\}.
$$

We can then start from the formula for $t(\pi)$ provided by Proposition 4.7 and write:

$$
t(\pi) = \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|S|} u(\pi \land \pi_S)
$$

$$
= \frac{1}{2} \left( \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|S|} u(\pi \land \pi_S) + \sum_{S \subseteq \{1, \ldots, k\}} (-1)^{|\Phi(S)|} u(\pi \land \pi_{\Phi(S)}) \right)
$$

$$
= \frac{1}{2} \left( \sum_{S \subseteq \{1, \ldots, k\}} ((-1)^{|S|} + (-1)^{|\Phi(S)|}) u(\pi \land \pi_S) \right),
$$

for $1 \leq p \leq k$. This completes the proof.
Remark 4.10. Quite obviously, the law of the element \( s \) have now obtained the proof of Theorem 2.5, where the linear functional \( \mu \)'s from (4.15) converge to \( \mu \) in moments.

Finally, it is clear that for every \( S \subseteq \{1, \ldots, k\} \) we have \( |\Phi(S)| = |S| \pm 1 \), and hence \((-1)^{|S|} + (-1)^{|\Phi(S)|} = 0 \). This shows that all the terms of the sum indicated in (4.14) are equal to 0, and leads to the required conclusion that \( \mathbf{t}(\pi) = 0 \).

Since the sequence \( (\hat{U}_n)_{n=1}^\infty \) was now found to satisfy the hypotheses of Theorem 4.4 we can therefore invoke this theorem and draw the following conclusion.

**Corollary 4.9.** (CLT for the sequence of elements \( \hat{U}_n \)) Consider the elements

\[
(4.15) \quad s_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{U}_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n (U(\gamma_i) - A_0) \in \mathcal{M}, \quad n \geq 1.
\]

On the other hand, let \( \mathbf{t} \) be the function on partitions associated to \( (\hat{U}_n)_{n=1}^\infty \) and consider the linear functional \( \mu : \mathbb{C}[X] \to \mathbb{C} \) defined via the prescription that \( \mu(1) = 1 \) and that

\[
(4.16) \quad \mu(X^k) = \sum_{\rho \in \mathcal{P}_2(k)} \mathbf{t}(\rho), \quad \forall k \in \mathbb{N},
\]

with the empty sum on the right-hand side of (4.16) being read as 0 for \( k \) odd. Then \( \mu \) is a positive functional and the \( s_n \)'s from (4.15) converge to \( \mu \) in moments.

**Remark 4.10.** Quite obviously, the law of the element \( s_n \) indicated in Equation (4.15) is the same as the law \( \mu_n \) mentioned in Theorem 2.5. We would therefore like to claim that we have now obtained the proof of Theorem 2.5 where the linear functional \( \mu \) indicated in the preceding corollary is an algebraic incarnation of the law “\( \mu_w \)” from Theorem 2.5. There is however a difference between the weak*-convergence needed for the conclusion of Theorem 2.5 and the convergence in moments which is provided by Corollary 4.9. The well-known argument to be invoked in such a situation (see e.g. [3, Theorem 30.2]) is that the limit law \( \mu \) is uniquely determined by its moments. This in turn is known to follow (cf. [3, Theorem 30.1]) if we can provide some bounds on the moments of \( \mu \) which ensure that the series \( \sum_{k=0}^\infty (\mu(X^k)/k!)z^k \) has a positive radius of convergence.

In order to get the desired bounds on the moments \( \mu(X^k) \), let us look back at the functions on partitions \( \mathbf{u} \) and \( \mathbf{t} \) that were introduced in Proposition 1.7.3. We first observe that \( |\mathbf{u}(\pi)| \leq 1 \) for every \( \pi \in \bigcup_{k=1}^\infty \mathcal{P}(k) \), as is clear from the fact that \( \mathbf{u}(\pi) \) is computed as the value of \( \chi \) on a suitable product of star transpositions. By plugging this observation into the formula (4.8) of Proposition 1.7.3 we therefore find that

\[
|\mathbf{t}(\pi)| \leq \sum_{S \subseteq \{1, \ldots, k\}} |\mathbf{u}(\pi \wedge \pi_S)| \leq 2^k, \quad \forall k \in \mathbb{N} \text{ and } \pi \in \mathcal{P}(k).
\]

The latter inequality can in turn be plugged into Equation (4.16) of Corollary 4.9 to conclude that for every \( k \in \mathbb{N} \) we have

\[
(4.17) \quad |\mu(X^k)| \leq 2^k \cdot |\mathcal{P}(k)| = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 2^k \cdot (k-1)!!, & \text{if } k \text{ is even}. \end{cases}
\]

The bound obtained in (4.17) is clearly sufficient for our purposes. Indeed, for \( k \) even we have \((2^k \cdot (k-1)!!)/k! = 2^{k/2}/(k/2)!!\), which shows that the series \( \sum_{k=0}^\infty (\mu(X^k)/k!)z^k \) has an
infinite radius of convergence. Thus Corollary 4.9 together with this bound on moments, implies Theorem 2.5.

Remark 4.11. We mention that the idea of associating a function on pair-partitions to an extremal character of $S_\infty$ has been considered a while ago in [5]. See pages 215-216 (in particular the figure on page 216) of [5], where it is explained how that function on pair-partitions is computed. Calculations made on various special examples don’t seem to suggest, however, a connection between the construction from [5] and the functions on partitions $t$ and $u$ that appeared in Proposition 4.7 of the present paper.

5. Moments of the limit law $\mu_w$, in terms of permutations $\tau_\pi$

In the present section we do some further processing of the formula (4.16) for an even moment of the limit law $\mu_w$. Our goal, reached in Proposition 5.6, is to describe such a moment in a way which only refers to how the character $\chi: S_\infty \to \mathbb{C}$ evaluates products of star-transpositions "$\tau_\pi$" of the kind discussed in Section 1.4 of the Introduction. It turns out to be convenient to consider such permutations $\tau_\pi$ not only when $\pi$ is a pair-partition (as mentioned in Section 1.4) but also in the case when $\pi$ has some singleton blocks.

Notation 5.1. Let $k$ be a positive integer. We denote by $\mathcal{P}_{\leq 2}(k)$ the set of all partitions $\pi \in \mathcal{P}(k)$ such that every block $V \in \pi$ has cardinality 1 or 2 – that is, $V$ is either a “singleton-block” or a “pair”. For $\pi \in \mathcal{P}_{\leq 2}(k)$, the number of singleton-blocks in $\pi$ will be denoted as $|\pi|_1$, and the number of pairs in $\pi$ will be denoted as $|\pi|_2$. We thus have $|\pi|_1 + |\pi|_2 = |\pi|$, the total number of blocks of $\pi$, while $|\pi|_1 + 2|\pi|_2 = k$, the total number of points partitioned by $\pi$.

Note that, as a consequence of the latter equation, $|\pi|_1$ is sure to be of same parity as $k$.

Definition 5.2. (The permutation $\tau_\pi$.) Let $k \in \mathbb{N}$ and let $\pi \in \mathcal{P}_{\leq 2}(k)$. We construct a permutation $\tau_\pi \in S_\infty$, defined as a product of star-transpositions $\gamma_m$, as follows.

Step 1. Denoting $|\pi| =: \ell$, we list the blocks $V_1, \ldots, V_\ell$ of $\pi$ in decreasing order of their maximal elements, and for $1 \leq i \leq \ell$ we put

$$a_i := \min(V_i), \quad b_i := \max(V_i).$$

The convention on the order in which we consider $V_i$’s reflects on the $b_i$’s, and gives:

$$k = b_1 > b_2 > \cdots > b_\ell.$$

Step 2. Let $\underline{r}: \{1, \ldots, k\} \to \{1, \ldots, \ell\}$ be the tuple which takes the value $i$ on the block $V_i$, for every $1 \leq i \leq \ell$. In other words, $\underline{r}$ is described by the formula

$$\underline{r}(a_i) = \underline{r}(b_i) = i, \quad \forall 1 \leq i \leq \ell.$$ 

Step 3. We put

$$\tau_\pi := \gamma_{\underline{r}(1)} \cdots \gamma_{\underline{r}(k)} \in S_{\ell+1} \subseteq S_\infty.$$ 

The relation $\tau_\pi \in S_{\ell+1}$ holds because (5.4) only uses $\gamma_1, \gamma_2, \ldots, \gamma_\ell$, with each of them appearing either once or two times in the product indicated there; as a consequence, $\tau_\pi$ cannot move any number $n > \ell + 1$. 

Example 5.3. For a concrete example, suppose that \( k = 7 \) and that we are dealing with 
\[ \pi = \{ \{1, 6\}, \{2, 5\}, \{3\}, \{4, 7\} \} \in \mathcal{P}_{\leq 2}(7). \]

In Step 1 of the preceding definition we thus have \( \ell = 4 \), and the order in which we consider the blocks of \( \pi \) is 
\[ V_1 = \{4, 7\}, V_2 = \{1, 6\}, V_3 = \{2, 5\}, V_4 = \{3\}. \]

For a pictorial rendering of Steps 2 and 3, Figure 2 shows a drawing of \( \pi \) where \( \gamma_1 \)'s are placed on top of \( V_1 \), \( \gamma_2 \)'s are placed on top of \( V_2 \), and so on. The resulting product of star-transpositions is, for this example: 
\[ \tau_\pi = \gamma_2 \gamma_3 \gamma_4 \gamma_1 \gamma_3 \gamma_2 \gamma_1 = (1, 5, 4, 2) \in S_5. \]

Remark 5.4. Let \( \pi \) be a partition in \( \mathcal{P}_{\leq 2}(k) \). The specifics of how we constructed the permutation \( \tau_\pi \) will be useful later on in the paper; but if we only want to know how the character \( \chi \) applies to \( \tau_\pi \), then let us note that one simply has
\[ \chi(\tau_\pi) = u(\pi), \]
where \( u \) is the function on partitions associated to the sequence \((U_n)_{n=1}^\infty \) (as considered in Proposition 4.7). In other words, one can write
\[ \chi(\tau_\pi) = \text{tr}(U_{\vec{i}(1)} \cdots U_{\vec{i}(k)}) = \chi(\gamma_{\vec{i}(1)} \cdots \gamma_{\vec{i}(k)}) \]
where \( \vec{i} : \{1, \ldots, k\} \to \mathbb{N} \) is any tuple such that \( \ker(\vec{i}) = \pi \). This is because the permutation \( \gamma_{\vec{i}(1)} \cdots \gamma_{\vec{i}(k)} \) appearing in (5.6) belongs to the same conjugacy class of \( S_\infty \) as \( \tau_\pi \), and thus the character \( \chi \) must take the same value on them.

Now let us fix an even \( k \in \mathbb{N} \) and let us look again at the formula (4.16) obtained in Section 3 for the moment of order \( k \) of our limit law of interest. In order to move further, let us focus on one of the terms \( t(\rho) \) appearing on the right-hand side of (4.16). We have at our disposal a formula, established in Proposition 4.7.3, which expresses \( t(\rho) \) as a summation over subsets \( S \subseteq \{1, \ldots, k\} \). Our next point is that the latter summation actually has a lot of cancellations, and can be simplified in the way indicated by the next lemma.

Lemma 5.5. Let \( t \) be the function on partitions associated to the sequence \((U_n)_{n=1}^\infty \) (same as in Section 4.2). Let \( k \) be an even positive integer, and let \( \rho \) be in \( \mathcal{P}_{2}(k) \). Then
\[ t(\rho) = \sum_{\pi \in \mathcal{P}_{\leq 2}(k)} (-1)^{|\pi|_1/2} \chi(\tau_\pi), \]
where the non-negative even integer \( |\pi|_1 \) is picked from Notation 5.1.

Proof. We start from the formula for \( t(\rho) \) provided by Equation (4.8) of Proposition 4.7 and in that formula we group terms according to what is \( \rho \wedge \pi_S \). Noting that \( \rho \wedge \pi_S \leq \rho \)
for all $S \subseteq \{1, \ldots, k\}$ (which implies in particular that $\rho \land \pi_S \in \mathcal{P}_{\leq 2}(k)$), we find that

$$t(\rho) = \sum_{\pi \in \mathcal{P}_{\leq 2}(k), \pi \leq \rho} \left( \sum_{S \subseteq \{1, \ldots, k\} \text{ with } \rho \land \pi_S = \pi} (-1)^{|S|} \right) u(\pi).$$

The observation made in (5.5) of Remark 5.4 allows us to re-write this as

(5.8) $$t(\rho) = \sum_{\pi \in \mathcal{P}_{\leq 2}(k), \pi \leq \rho} \left( \sum_{S \subseteq \{1, \ldots, k\} \text{ with } \rho \land \pi_S = \pi} (-1)^{|S|} \right) \chi(\tau_\pi).$$

The cancellations we need are in the inside sum on the right-hand side of (5.8). More precisely, we will prove that for every $\pi \in \mathcal{P}_{\leq 2}(k)$ such that $\pi \leq \rho$ one has:

(5.9) $$\sum_{S \subseteq \{1, \ldots, k\} \text{ with } \rho \land \pi_S = \pi} (-1)^{|S|} = (-1)^{|\pi_1|/2}.$$

It is clear that plugging (5.9) into (5.8) will lead to the formula (5.7) claimed by the lemma.

We are thus left to fix a partition $\pi \in \mathcal{P}_{\leq 2}(k)$ such that $\pi \leq \rho$, for which we verify that (5.9) holds. Let us write explicitly $\rho = \{V_1, \ldots, V_p, W_1, \ldots, W_q\}$, where the pairs $V_1, \ldots, V_p$ are also appearing in $\pi$, while each of $W_1, \ldots, W_q$ is broken into two singleton-blocks of $\pi$ (which, in particular, forces the relation $|\pi_1| = 2q$). We leave it as an exercise to the reader to check the elementary fact that for a subset $S \subseteq \{1, \ldots, k\}$ one has:

(5.10) $$\left( \rho \land \pi_S = \pi \right) \Leftrightarrow \left( S \cap V_i = \emptyset \text{ for every } 1 \leq i \leq p, \text{ and } S \cap W_j \neq \emptyset \text{ for every } 1 \leq j \leq q \right).$$

By following the description indicated on the right-hand side of (5.10), it is immediate that the sets $S \subseteq \{1, \ldots, k\}$ such that $\rho \land \pi_S = \pi$ are enumerated, without repetitions, by taking the following steps:

(5.11)

| Step 1. Pick an integer $q'$ such that $0 \leq q' \leq q$. |
| Step 2. Select $q'$ of the pairs $W_1, \ldots, W_q$. Say we selected the pairs $\{W_j \mid j \in Q'\}$, where $Q' \subseteq \{1, \ldots, q\}$ has $|Q'| = q'$. |
| Step 3. Choose one element of each of the pairs selected in Step 2. That is, for every $j \in Q'$ choose a $c_j \in W_j$. |
| Step 4. Put $S := \{c_j \mid j \in Q'\} \cup \left( \bigcup_{j \in Q''} W_j \right)$, where $Q'' = \{1, \ldots, q\} \setminus Q'$. |

Note that every $S$ produced by (5.11) has a cardinality $|S|$ of the same parity as $q'$.

So then, the sum on the left-hand side of (5.9) is found to be equal to:

(5.12) $$\sum_{q' = 0}^{q} \binom{q}{q'} \cdot 2^{q'} \cdot (-1)^{q'},$$

where: the binomial coefficient corresponds to the choice of $Q'$ in Step 2; the factor $2^{q'}$ corresponds to the choices of $c_j$’s made in Step 3; and the factor $(-1)^{q'}$ corresponds to the $(-1)^{|S|}$ from the summation on the left-hand side of (5.9).

But (5.12) is, however, the binomial expansion for $(1 + 2 \cdot (-1))^q$, and is thus equal to $(-1)^q$. Since $q = |\pi_1|/2$, we have indeed arrived to the right-hand side of (5.9).

The new form of Equation (4.16) that was announced at the beginning of the section is then stated as follows.
Proposition 5.6. For an even \( k \in \mathbb{N} \), the limit law \( \mu_w \) from Theorem 2.5 has
\[
\int_{\mathbb{R}} t^k \, d\mu_w(t) = \sum_{\pi \in \mathcal{P}_{\leq 2}(k)} (-1)^{\left|\pi\right|/2} \cdot (\left|\pi\right| - 1)!! \cdot \chi(\tau_\pi),
\]
where the non-negative even integer \( \left|\pi\right| \) is picked from Notation 5.7.

Proof. As discussed in Remark 4.10, the moment of order \( k \) of \( \mu_w \) can be viewed as \( \mu(X^k) \), where \( \mu : \mathbb{C}[X] \to \mathbb{C} \) is the linear functional appearing in Corollary 4.9. Moreover, Equation (4.16) of Corollary 4.9 expresses \( \mu(X^k) \) as a summation over \( \rho \in \mathcal{P}_2(k) \). In that summation we replace every term by using Lemma 5.5 and then we interchange the order of the summations over \( \rho \) and \( \pi \). We get
\[
\sum_{\pi \in \mathcal{P}_{\leq 2}(k)} \left( \sum_{\rho \in \mathcal{P}_2(k) \atop \rho \succeq \pi} (-1)^{\left|\pi\right|/2} \chi(\tau_\pi) \right).
\]
This is exactly the quantity indicated on the right-hand side of (5.13), since for every \( \pi \in \mathcal{P}_{\leq 2}(k) \) there are \( (\left|\pi\right| - 1)!! \) pairings \( \rho \in \mathcal{P}_2(k) \) such that \( \rho \succeq \pi \). \( \Box \)

6. Combinatorics: \( \sigma_\pi, \tau_\pi \) and a Relation between Them

Notation and Remark 6.1. (The permutation \( \sigma_\pi \).) Consider a partition \( \pi \in \mathcal{P}_{\leq 2}(k) \), which we write explicitly in the same way as in Step 1 of Definition 5.2: \( \pi = \{V_1, \ldots, V_\ell\} \), with \( \min(V_i) =: a_i \) and \( \max(V_i) =: b_i \) for all \( 1 \leq i \leq \ell \), and where \( k = b_1 > \cdots > b_\ell \). In Definition 5.2 we introduced a permutation \( \tau_\pi \) associated to \( \pi \), and this was important for discussing the moments of the limit law \( \mu_w \) of our Theorem 2.5. But frequently encountered in the literature there is however another permutation associated to \( \pi \), which we will denote as \( \sigma_\pi \), and is simply defined as
\[
\sigma_\pi := \prod_{1 \leq i \leq \ell \atop a_i \neq b_i} (a_i, b_i) \in S_k \subseteq S_\infty.
\]
At first sight, the permutations \( \sigma_\pi \) and \( \tau_\pi \) do not appear to be related to each other. But it turns out that there exists a non-trivial connection, at the level of orbit structures, between \( \tau_\pi \) and the product \( \eta_{k+1} \cdot \sigma_\pi \), where \( \eta_{k+1} \in S_{k+1} \) is the forward cycle from Notation 2.1. We will first explain how this goes on a concrete example (cf. Example 6.3), then the general case is addressed in Proposition 6.6 at the end of the section.

Notation 6.2. Notation as above. In addition to \( a_1, \ldots, a_\ell \) and \( b_1, \ldots, b_\ell \) we put
\[
a_0 = b_0 := k + 1,
\]
and we denote
\[
B_\pi := \{b_\ell, \ldots, b_1, b_0\} \subseteq \{1, \ldots, k + 1\}.
\]

Example 6.3. Let us look again at Example 5.3 of the preceding section, where we had \( k = 7 \) and \( \pi = \{\{1, 6\}, \{2, 5\}, \{3\}, \{4, 7\}\} \in \mathcal{P}_{\leq 2}(7) \), with blocks listed in the order
\[
V_1 = \{4, 7\}, \ V_2 = \{1, 6\}, \ V_3 = \{2, 5\}, \ V_4 = \{3\}.
\]
In this example, the set \( B_\pi \) from (6.3) comes out as \( B_\pi = \{3, 5, 6, 7, 8\} \).
We have \( \sigma_\pi = (1, 6)(2, 5)(4, 7) \in S_7 \). In order to connect to \( \tau_\pi \), it turns out to be relevant
to examine how the set \( B_\pi \) is partitioned by its intersections with orbits of \( \eta_{k+1} \cdot \sigma_\pi \). In the
example at hand we have

\[
\eta_8 \cdot \sigma_\pi = (1, 2, \ldots, 8) \cdot (1, 6)(2, 5)(4, 7) = (1, 7, 5, 3, 4, 8)(2, 6) \in S_8,
\]

and intersecting \( B_\pi \) with the orbits of \( \eta_8 \cdot \sigma_\pi \) thus gives

\[
B_\pi = \{3, 5, 7, 8\} \cup \{6\},
\]
a partition of type \( 4 + 1 \). Now, we saw in Example 5.3 that \( \tau_\pi = (1, 5, 4, 2)(3) \in S_5 \),
which also has an orbit structure of type \( 4 + 1 \). Proposition 6.6 below claims this is not a
coincidence. In order to arrive to that general statement, we will use a lemma, preceded by
another bit of notation.

**Notation 6.4 (Induced permutation).** Let \( \tau \in S_\infty \) and let \( A \subseteq \mathbb{N} \) be finite and non-empty.
We will use the notation \( \tau|_A \) for the permutation in \( S_\infty \) which is obtained by only retaining
how \( \tau \) acts on \( A \).

If \( A \) happens to be invariant with respect to \( \tau \), then \( \tau|_A \) is simply obtained by keeping
the cycles of \( \tau \) that are contained in \( A \), while every \( n \in \mathbb{N} \setminus A \) becomes a fixed point.

In general, \( \tau|_A \) maps every \( a \in A \) to the first element of the orbit \( \tau(a), \tau^2(a), \ldots \) which
is found to be again in \( A \) (while every \( n \in \mathbb{N} \setminus A \) becomes a fixed point). This is very nicely
followed in cycle notation: in order to obtain \( \tau|_A \) we start from the cycle notation for \( \tau \),
and remove the elements of \( \mathbb{N} \setminus A \) that might appear in it.

[A concrete example: if \( \tau = (1, 3, 2)(5, 6) \) and \( A = \{1, 2, 5, 7\} \), then \( \tau|_A = (1, 3, 2)(5, 6) = (1, 2)(5) = (1, 2) \].]

**Lemma 6.5.** Let \( \pi \in \mathcal{P}_{\leq 2}(k) \) and consider the permutations \( \sigma_\pi, \tau_\pi \) introduced above (in
Notation 6.4 and in Definition 5.2, respectively). Let us form the induced permutation

\[
(\sigma_\pi \cdot \eta_{k+1}^{-1})|_{B_\pi} \in S_{k+1}, \tag{6.4}
\]

where \( \eta_{k+1} \in S_{k+1} \) is the forward cycle and \( B_\pi = \{b_\ell, \ldots, b_1, b_0\} \subseteq \{1, \ldots, k+1\} \) is as in
Notation 6.2. The permutation (6.4) relates to \( \tau_\pi \) in the following way:

\[
\left\{ \begin{array}{l}
\text{if } i, j \in \{0, \ldots, \ell\} \text{ are such that } (\sigma_\pi \cdot \eta_{k+1}^{-1})|_{B_\pi}(b_i) = b_j, \\
\text{then it follows that } \tau_\pi(i + 1) = j + 1.
\end{array} \right. \tag{6.5}
\]

**Proof.** The case when in (6.5) we have \( i = 0 \) is slightly different from the others. But we can
afford to not discuss this case, due to the following observation: if we assume that all cases
with \( i \neq 0 \) have been verified, then the case \( i = 0 \) automatically follows. Indeed, the permutation
\( (\sigma_\pi \cdot \eta_{k+1}^{-1})|_{B_\pi} \) gives in particular a bijection \( f \) from the set \( B_\pi = \{b_0, b_1, \ldots, b_\ell\} \) onto
itself. Denote \( f(b_0) = b_{j_0}, f(b_1) = b_{j_1}, \ldots, f(b_\ell) = b_{j_\ell} \), (where \( \{j_0, \ldots, j_\ell\} \) is a permutation
of \( \{0, 1, \ldots, \ell\} \) ) and suppose it was verified that
\[
\pi_{\pi}(1 + 1) = j_1 + 1, \; \pi_{\pi}(2 + 1) = j_2 + 1, \ldots, \pi_{\pi}(\ell + 1) = j_\ell + 1.
\]

By equating the set-differences
\[
\{1, \ldots, \ell + 1\} \setminus \{\tau_{\pi}(2), \ldots, \tau_{\pi}(\ell + 1)\} = \{1, \ldots, \ell + 1\} \setminus \{j_1 + 1, \ldots, j_\ell + 1\}
\]
we then find that \( \tau_{\pi}(1) = j_0 + 1 \), which is precisely the missing case \( i = 0 \) of (6.5).
For the remaining part of the proof we fix \( i, j \in \{0, \ldots, \ell \} \), with \( i \neq 0 \), and where 
\[
(\sigma_\pi \cdot \eta_{k+1}^{-1})_{\mid B_\pi} (b_i) = b_j.
\]
The latter equation entails the existence of a \( q \geq 1 \) such that
\[
(\sigma_\pi \cdot \eta_{k+1}^{-1})^q(b_i) = b_j, \quad \text{while } (\sigma_\pi \cdot \eta_{k+1}^{-1})^p(b_i) \notin B_\pi \text{ for every } 1 \leq p < q.
\]
Based on (6.6), we have to prove that \( \tau_\pi(i + 1) = j + 1 \).

Subcase \( q = 1 \). It is instructive to work out in detail the case when in (6.6) we have \( q = 1 \), i.e. we simply have \( (\sigma_\pi \cdot \eta_{k+1}^{-1})(b_i) = b_j \). By applying \( \sigma_\pi \) to both sides of this equation, and upon recalling that \( \sigma_\pi \) swaps \( a_j \) and \( b_j \) for every \( 0 \leq j \leq \ell \), we find that
\[
(\sigma_\pi \cdot \eta_{k+1})^{-1}(b_i) = a_j.
\]
Let us assume for a moment that in (6.7) we have \( b_i > 1 \), and the equation thus says that \( a_j = b_i - 1 \). On the other hand, let us look at the permutation \( \tau_\pi \) and let us follow how it acts on the number \( i + 1 \). Recall from Definition 5.2 (cf. Equation (5.3)) that \( \tau_\pi \) is defined as a product of star-transpositions, which we write here by putting into evidence the factors on the consecutive positions \( a_j \) and \( b_i \):
\[
(6.8) \quad \tau_\pi = \left( \prod_{1 \leq h < a_j} \gamma_{\{h\}} \right) \cdot \gamma_{\ell(a_j)} \gamma_{\{h\}} \cdot \left( \prod_{b_i < h \leq k} \gamma_{\{h\}} \right).
\]
Concerning the product (6.8), we make the following observations.

(i) The part \( \prod_{b_i < h \leq k} \gamma_{\{h\}} \) does not move \( i + 1 \) at all, since it has no factors “\( \gamma_i \)”.

(ii) The part \( \gamma_{\ell(a_j)} \gamma_{\{h\}} \) comes to \( \gamma_j \gamma_i \), and sends \( i + 1 \) to \( j + 1 \): \( \gamma_j(\gamma_i(i + 1)) = \gamma_j(1) = j + 1 \).

So from here on we have to watch for what happens to \( j + 1 \). But:

(iii) The part \( \prod_{1 \leq h < a_j} \gamma_{\{h\}} \) of (6.8) does not move \( j + 1 \), since it has no factors “\( \gamma_j \)”.

So overall, the conclusion is that \( \tau_\pi(i + 1) = j + 1 \), as we wanted to prove.

In order to complete the discussion of the case \( q = 1 \), we must also examine the subcase when in (6.6) we have \( b_i = 1 \). This happens precisely when the partition \( \pi \) has a singleton-block at \( \{1\} \), which appears last in the numbering of blocks of \( \pi \): \( \nu = \{1\} \), with \( a_k = b_i = 1 \). Thus it must be that \( i = \ell \), while the equation \( a_j = \eta_{k+1}^{-1}(1) = k + 1 \) implies \( j = 0 \). Our desired conclusion about the action of \( \tau_\pi \) thus comes to \( \tau_\pi(\ell + 1) = 1 \). This is indeed true, and is verified by writing \( \tau_\pi = \gamma_{\{1\}} \cdot \left( \prod_{2 \leq h \leq k} \gamma_{\{h\}} \right) \), where \( \prod_{2 \leq h \leq k} \gamma_{\{h\}} \) does not move \( \ell + 1 \), while \( \gamma_{\{1\}} \) is \( \gamma_\ell \), sending \( \ell + 1 \) to \( 1 \).

Subcase \( q > 1 \). In what follows we assume that \( q \) of (6.6) is \( \geq 2 \).

It is clear that the orbit of \( b_i \) under the action of \( \sigma_\pi \cdot \eta_{k+1}^{-1} \) is contained in \( \{1, \ldots, k + 1\} \). Hence the numbers \( (\sigma_\pi \cdot \eta_{k+1}^{-1})^p(b_i) \) with \( 1 \leq p < q \) are in \( \{1, \ldots, k + 1\} \setminus B_\pi \), which forces every such \( (\sigma_\pi \cdot \eta_{k+1}^{-1})^p(b_i) \) to be of the form \( a_i(p) = \min(V_i(p)) \) for a pair \( V_i(p) = \{a_i(p), b_i(p)\} \) of \( \pi \). We thus have:
\[
(6.9) \quad \begin{cases}
(\sigma_\pi \cdot \eta_{k+1}^{-1})(b_i) = a_i(1), & (\sigma_\pi \cdot \eta_{k+1}^{-1})(a_i(1)) = a_i(2), \\
& \ldots, (\sigma_\pi \cdot \eta_{k+1}^{-1})(a_i(\ell-2)) = a_i(\ell-1), & (\sigma_\pi \cdot \eta_{k+1}^{-1})(a_i(\ell-2)) = b_j.
\end{cases}
\]

Similarly to how we did in (6.7) above, we apply \( \sigma_\pi \) to both sides of each of the equalities listed in (6.9), and we arrive to:
\[
(6.10) \quad \eta_{k+1}^{-1}(b_i) = b_i(1), \quad \eta_{k+1}^{-1}(a_i(1)) = b_i(2), \ldots, \eta_{k+1}^{-1}(a_i(q-2)) = b_i(q-1), \quad \eta_{k+1}^{-1}(a_i(q-1)) = a_j.
\]
With the possible exception of the last one listed, the instances of \( \eta_{k+1}^{-1} \) that appear in (6.10) are genuine subtractions by 1. This is because all of \( b_i(1), \ldots, b_i(q-1) \) are elements of some
blocks of $\pi$, and are thus numbers from $\{1, \ldots, k\}$, which cannot be equal to $k + 1$. We may therefore re-write (6.8) in the form
\begin{equation}
(6.11) \quad b_i - 1 = b_i(1), \quad a_i(1) - 1 = b_i(2), \ldots, \quad a_i(q-2) - 1 = b_i(q-1), \quad \eta_{k+1}^{-1}(a_i(q-1)) = a_j,
\end{equation}
where for the last occurrence of $\eta_{k+1}^{-1}$ we have to also consider the possibility that $i(q-1) = \ell$ (with $a_\ell = 1$) and $j = 0$ (with $a_0 = k + 1$).

We now move to examine how the permutation $\tau_\pi$ acts on the number $i + 1$. Assume first that we are in the situation with $a_i(q-1) > 1$, when the last equality in (6.11) just says that $a_j = a_i(q-1) - 1$. We break the product defining $\tau_\pi$ in a way which follows the same idea as we used in (6.8), but where we now separate multiple pieces of the product:
\begin{equation}
\tau_\pi = \left( \prod_{1 \leq h < a_j} \gamma_{\pi(h)} \right) \cdot \gamma_{\pi(a_j)} \gamma_{\pi(a_i(q-1))} \cdot \left( \prod_{a_i(q-1) < h < b_i(q-1)} \gamma_{\pi(h)} \right) \cdot \gamma_{\pi(b_i(q-1))} \gamma_{\pi(a_i(q-2))} \cdot \ldots
\end{equation}
\begin{equation}
(6.12) \quad \ldots \cdot \gamma_{\pi(b_i(2))} \gamma_{\pi(a_i(1))} \cdot \left( \prod_{a_i(1) < h < b_i(1)} \gamma_{\pi(h)} \right) \cdot \gamma_{\pi(b_i(1))} \gamma_{\pi(b_i)} \cdot \left( \prod_{b_i < h \leq k} \gamma_{\pi(h)} \right).
\end{equation}
The reader should have no difficulty to check that, in (6.12): $\gamma_{\pi(b_i(1))} \gamma_{\pi(b_i)}$ sends $i + 1$ to $i(1) + 1$, then $\gamma_{\pi(b_i(2))} \gamma_{\pi(a_i(1))}$ sends $i(1) + 1$ to $i(2) + 1$, and so on until we reach $\gamma_{\pi(a_j)} \gamma_{\pi(a_i(q-1))}$ sending $i(q - 1) + 1$ to $j + 1$, while all the products $\prod_{h} \gamma_{\pi(h)}$ do not participate in the action. Hence overall we get that $\tau_\pi(i + 1) = j + 1$, as we wanted to prove. An illustration of how things look in this case (with $q \geq 2$ and $j \neq 0$) appears in Figure 3.

![Figure 3](image)
Figure 3. An illustration of the case $q = 3$, $j \neq 0$ in the proof of Lemma 6.5
The picture shows $\gamma_i$ on top of position $b_i$ and $\gamma_j$ on top of position $a_j$, also $\gamma_i(1)$ on top of positions $b_i(1)$, $a_i(1)$ and $\gamma_i(2)$ on top of positions $b_i(2)$, $a_i(2)$.

Note that $a_j = a_i(q-1) - 1$, $b_i(2) = a_i(q-1) - 1$, $b_i(1) = b_i - 1$.

In order to finalize the discussion of the case when $q > 1$, we must also examine the leftover subcase with $i(q - 1) = \ell$ and $j = 0$, when the last equality stated in (6.11) amounts to $\eta_{k+1}^{-1}(1) = k + 1$. The way to break into pieces the product which defines $\tau_\pi$ now becomes:
\begin{equation}
\tau_\pi = \gamma_{\pi(1)} \left( \prod_{1 < h < b_i(1)} \gamma_{\pi(h)} \right) \cdot \gamma_{\pi(b_i(1))} \gamma_{\pi(a_i(q-2))} \cdot \ldots
\end{equation}
\begin{equation}
(6.13) \quad \ldots \cdot \gamma_{\pi(b_i(2))} \gamma_{\pi(a_i(1))} \cdot \left( \prod_{a_i(1) < h < b_i(1)} \gamma_{\pi(h)} \right) \cdot \gamma_{\pi(b_i(1))} \gamma_{\pi(b_i)} \cdot \left( \prod_{b_i < h \leq k} \gamma_{\pi(h)} \right).
\end{equation}
The steps of the action of $\tau_\pi$ on $i + 1$ are then the following: $\gamma_{\pi(b_i(1))} \gamma_{\pi(b_i)}$ sends $i + 1$ to $i(1) + 1$, then $\gamma_{\pi(b_i(2))} \gamma_{\pi(a_i(1))}$ sends $i(1) + 1$ to $i(2) + 1$, and so on until we arrive to $\gamma_{\pi(b_i(q-2))} \gamma_{\pi(a_i(q-2))}$ sending $i(q - 2) + 1$ to $i(q - 1) + 1 = \ell + 1$, and at the very end $\gamma_{\pi(1)}$ sends $\ell + 1$ to 1. (Same as was the case in (6.12), the “$\prod_{h} \gamma_{\pi(h)}$” pieces listed in (6.13) do
not participate in the action.) The value of \( j \) used in this case is \( j = 0 \); thus the conclusion coming out of (6.13), that \( \tau_\pi(i + 1) = 1 \), completes the proof of this lemma. \( \square \)

**Proposition 6.6.** Consider the same framework and notation as in the preceding lemma. Let \( R_1, \ldots, R_p \) be the orbits of \( \eta_{k+1} \cdot \sigma_\pi \) which intersect \( B_\pi \). Then: \( \tau_\pi \) has exactly \( p \) orbits that are contained in \( \{1, \ldots, \ell + 1\} \), and the sizes of these orbits are (upon suitable re-ordering) equal to \( |R_1 \cap B_\pi|, \ldots, |R_p \cap B_\pi| \).

**Proof.** We first note that, since \( \eta_{k+1} \cdot \sigma_\pi \) is the inverse of \( \sigma_\pi \cdot \eta_{k+1}^{-1} \), we may view the sets \( R_1, \ldots, R_p \) as the orbits of \( \sigma_\pi \cdot \eta_{k+1}^{-1} \) which intersect \( B_\pi \). Due to this fact and to how an induced permutation is defined (cf. Notation 6.4), it is immediate that \( \sigma_\pi \cdot \eta_{k+1}^{-1} \) can be viewed as the orbits of the bijection \( \varphi : B_\pi \to B_\pi \) defined by

\[
\varphi(b) = \left( (\sigma_\pi \cdot \eta_{k+1}^{-1})_{|B_\pi}\right)(b), \forall b \in B_\pi.
\]

On the other hand, the preceding lemma gives the relation

\[
f \circ \varphi = \psi \circ f \quad \text{(equality of maps from } B_\pi \text{ to } \{1, \ldots, \ell + 1\}),
\]

where \( \varphi \) is as in (6.14), while \( \psi : \{1, \ldots, \ell + 1\} \to \{1, \ldots, \ell + 1\} \) is defined by \( \psi(i) = \tau_\pi(i) \) for \( 1 \leq i \leq \ell + 1 \), and \( f : B_\pi \to \{1, \ldots, \ell + 1\} \) simply maps \( b_i \) to \( i + 1 \) for every \( 0 \leq i \leq \ell \). When combined with the observation from the preceding paragraph, (6.15) implies that the orbits of the bijection \( \psi \) are \( f(R_1 \cap B_\pi), \ldots, f(R_p \cap B_\pi) \). But the orbits of \( \psi \) are precisely the orbits of \( \tau_\pi \) which are contained in \( \{1, \ldots, \ell + 1\} \). Thus the sizes of the latter orbits are \( |R_1 \cap B_\pi|, \ldots, |R_p \cap B_\pi| \), as claimed. \( \square \)

The preceding proposition was about the permutation \( \eta_{k+1} \cdot \sigma_\pi \in S_{k+1} \), which has some (possibly) non-trivial orbits contained in \( \{1, \ldots, k + 1\} \) and then fixes every \( n > k + 1 \). It is useful to complement the statement of Proposition 6.6 with the following observation.

**Proposition 6.7.** The list of orbits \( R_1, \ldots, R_p \) mentioned in Proposition 6.6 consists precisely of all the orbits of \( \eta_{k+1} \cdot \sigma_\pi \) that are contained in \( \{1, \ldots, k + 1\} \).

**Proof.** For every \( 1 \leq j \leq p \), we clearly have that

\[
R_j \cap B_\pi \neq \emptyset \implies R_j \cap \{1, \ldots, k + 1\} \neq \emptyset \implies R_j \subseteq \{1, \ldots, k + 1\}.
\]

It remains to show, conversely, that if \( R \) is an orbit of \( \eta_{k+1} \cdot \sigma_\pi \) such that \( R \subseteq \{1, \ldots, k + 1\} \), then \( R \cap B_\pi \neq \emptyset \) (which then implies that \( R \) is counted among \( R_1, \ldots, R_p \)). We prove \( R \cap B_\pi \neq \emptyset \) by displaying a specific element of \( R \) which is sure to be in \( B_\pi \), as follows:

\[
(\eta_{k+1} \cdot \sigma_\pi)^{-1}(m) \in B_\pi.
\]

For the proof of (6.16), we first dispose of the case when \( m = 1 \), where we compute that

\[
(\eta_{k+1} \cdot \sigma_\pi)^{-1}(1) = \sigma_\pi(\eta_{k+1}^{-1}(1)) = \sigma_\pi(k + 1) = k + 1 = b_0 \in B_\pi.
\]

Suppose now that \( R \) has \( m := \min(R) > 1 \). We claim that the number \( m - 1 \) cannot be of the form \( \max(V_i) \) for one of the blocks \( V_1, \ldots, V_{\ell'} \) of \( \pi \). Indeed, if that would be the case, then the number \( m' := (\eta_{k+1} \cdot \sigma_\pi)^{-1}(m) \in R \) would come out as

\[
m' = \sigma_\pi(\eta_{k+1}^{-1}(m)) = \sigma_\pi(m - 1) = \sigma_\pi(\max(V_i)) = \min(V_i);
\]

hence \( m' = \min(V_i) \leq \max(V_i) = m - 1 < m \), in contradiction with the assumption that \( m \) is the minimum element of \( R \). We conclude that \( m - 1 \) has to be of the form \( \min(V_i) \) for an \( 1 \leq i \leq \ell \), when a calculation similar to the one above yields the desired conclusion:
The right-hand side of (7.1) are obtained, without repetitions, by starting with an arbitrary \( \tau \) since \( Q \) crossing pair-partition.

It follows that the right-hand side of (7.1) is equal to \( \chi(\tau) \).

Example 6.8. It is instructive to see what happens when \( \pi \) from Proposition 6.6 is a non-crossing pair-partition. This means that \( k \) is even, \( \pi \) has \( \ell \) pairs \( V_1 = \{a_1, b_1\}, \ldots, V_\ell = \{a_\ell, b_\ell\} \) with \( \ell = k/2 \) and \( a_1 < b_1, \ldots, a_\ell < b_\ell \), and it is not possible to find some \( 1 \leq i, j \leq \ell \) such that \( a_i < a_j < b_i < b_j \). It is easy to see that in this case \( \tau \) is the identity permutation – thus when looking at the orbits of \( \tau \) that are contained in \( \{1, \ldots, \ell + 1\} \), we will find \( \ell + 1 \) orbits, each of them of cardinality 1. Proposition 6.6 then gives us a non-trivial statement about the orbits of \( \eta_{k+1} \cdot \sigma_\pi \) that are contained in \( \{1, \ldots, k + 1\} \): there are \( \ell + 1 \) such orbits \( R_1, \ldots, R_{\ell+1} \), and \( |R_j \cap B_\pi| = 1 \) for every \( 1 \leq j \leq \ell + 1 \).

7. Moments of the limit law \( \mu_{w_\pi} \) in terms of the permutations \( \sigma_\pi \)

We now resume the discussion about the even moments of the limit law \( \mu_{w_\pi} \), from where it was left, in Equation (5.13) at the end of Section 5. That equation uses the values of the character \( \chi \) on permutations \( \tau \) with \( \pi \in \mathcal{P}_{\leq 2}(k) \). In this section we show how such a value \( \chi(\tau) \) can be described by using the “other” permutation \( \sigma_\pi \) associated to \( \pi \), and then we record (cf. Proposition 7.6) what this has to say concerning the moments of \( \mu_{w_\pi} \).

Proposition 7.1. Let \( \pi \in \mathcal{P}_{\leq 2}(k) \) and let \( \sigma_\pi, \tau_\pi \in S_{\infty} \) be the permutations associated to \( \pi \) that were discussed in the preceding sections. One has

\[
\chi(\tau_\pi) = \sum_{i : \{1, \ldots, k+1\} \to \{1, \ldots, d\}, \text{constant under action of } \eta_{k+1} \cdot \sigma_\pi} w_i(b_0) w_i(b_1) \cdots w_i(b_\ell),
\]

where \( k + 1 = b_0 > b_1 > \cdots > b_\ell \geq 1 \) and the cycle \( \eta_{k+1} \in S_{k+1} \) are as in Proposition 6.6 while the weights \( w_1 \geq w_2 \geq \cdots \geq w_d > 0 \) are as in Notation 2.3.

Proof. Same as in Propositions 6.6 and 6.7 let \( R_1, \ldots, R_p \) be the orbits of \( \eta_{k+1} \cdot \sigma_\pi \) which are contained in \( \{1, \ldots, k + 1\} \). All the tuples \( j : \{1, \ldots, k + 1\} \to \{1, \ldots, d\} \) considered on the right-hand side of (7.1) are obtained, without repetitions, by starting with an arbitrary tuple \( j : \{1, \ldots, p\} \to \{1, \ldots, d\} \) and by putting

\[
i(h) = j(1) \text{ for } h \in R_1, \ldots, j(h) = j(p) \text{ for } h \in R_p.
\]

It follows that the right-hand side of (7.1) is equal to

\[
\sum_{j : \{1, \ldots, p\} \to \{1, \ldots, d\}} w_{|R_1 \cap B_\pi|} \cdots w_{|R_p \cap B_\pi|},
\]

where \( B_\pi := \{b_\ell, \ldots, b_1, b_0\} \subseteq \{1, \ldots, k + 1\} \). Moreover, the latter sum clearly factors as

\[
\left( \sum_{j=1}^d w_j^{|R_1 \cap B_\pi|} \right) \cdots \left( \sum_{j=1}^d w_j^{|R_p \cap B_\pi|} \right).
\]

Now, Proposition 6.6 assures us that the the orbits of \( \tau_\pi \) that are contained in \( \{1, \ldots, \ell + 1\} \) can be listed as \( Q_1, \ldots, Q_p \), in such a way that we have \( |Q_i| = |R_i \cap B_\pi| \) for every \( 1 \leq i \leq p \). Since \( \tau_\pi \) fixes every \( m > \ell + 1 \), the definition of \( \chi \) gives

\[
\chi(\tau_\pi) = p_{|Q_1|} \cdots p_{|Q_p|} = \left( \sum_{j=1}^d w_j^{|Q_1|} \right) \cdots \left( \sum_{j=1}^d w_j^{|Q_p|} \right).
\]
Upon comparing this with the product indicated in (7.2), we see that \( \chi(\tau_\pi) \) is indeed equal to the right-hand side of (7.1), as claimed. \( \square \)

We next note that the formula (7.1) obtained above can be re-written in a way which doesn’t explicitly refer to \( k + 1 \) – this goes by replacing the occurrence of the cycle \( \eta_k = (1, \ldots, k) \in S_k \subseteq S_\infty \).

**Corollary 7.2.** In the same framework and notation as in Proposition 7.1, one has

\[
\chi(\tau_\pi) = \sum_{\tilde{\iota} : \{1, \ldots, k\} \to \{1, \ldots, d\}, \text{constant under action of } \eta_k \cdot \sigma_\pi} w_{\tilde{\iota}(1)} \cdot (w_{\tilde{\iota}(b_1)} \cdots w_{\tilde{\iota}(b_\ell)}).
\]

**Proof.** Let us denote

\[ \mathcal{I} := \{ \tilde{\iota} : \{1, \ldots, k\} \to \{1, \ldots, d\} | \tilde{\iota} \text{ is constant under the action of } \eta_k \cdot \sigma_\pi \}, \]

\[ \mathcal{J} := \{ j : \{1, \ldots, k\} \to \{1, \ldots, d\} | j \text{ is constant under the action of } \eta_k \cdot \sigma_\pi \}. \]

Moreover, for every \( \tilde{\iota} : \{1, \ldots, k\} \to \{1, \ldots, d\} \) we will use the notation “\( r(\tilde{\iota}) \)” for the restriction of \( \tilde{\iota} \) to \( \{1, \ldots, k\} \).

For the partition \( \pi \in \mathcal{P}_{\leq 2}(k) \) we are working with, recall that we denote \( a_1 = \min(V_1) \), where \( V_1 \) is the block of \( \pi \) which contains the number \( k \). That is: either \( V_1 \) is a singleton block at \( k \) and \( a_1 = b_1 = k \), or \( V_1 \) is a pair \( \{a_1, b_1\} \) with \( a_1 < b_1 = k \). Direct inspection shows that the permutation \( \eta_k \circ \sigma_\pi \) sends \( a_1 \mapsto k + 1 \mapsto 1 \), while \( \eta_k \circ \sigma_\pi \) sends directly \( a_1 \mapsto 1 \). (This includes the possibility that \( a_1 = 1 \), when \( \eta_k \circ \sigma_\pi \) swaps \( 1 \) with \( k + 1 \), while \( \eta_k \circ \sigma_\pi \) has \( 1 \) as a fixed point.) As a consequence, every \( \tilde{\iota} \in \mathcal{I} \) has \( \tilde{\iota}(a_1) = \tilde{\iota}(k + 1) = \tilde{\iota}(1) \), and every \( j \in \mathcal{J} \) has \( j(a_1) = j(1) \). A further consequence of this observation is that the following statements (a) and (b) are holding:

(a) If \( \tilde{\iota} \in \mathcal{I} \), then \( r(\tilde{\iota}) \in \mathcal{J} \). It thus makes sense to consider the map \( r : \mathcal{I} \to \mathcal{J} \).

(b) The restriction map \( r : \mathcal{I} \to \mathcal{J} \) defined in (a) is bijective.

The verifications needed in order to obtain (a) and (b) are straightforward. For instance: for checking that the map \( r \) is surjective we start with a tuple \( \tilde{\iota} \in \mathcal{J} \) and check that

\[ \tilde{\iota}(h) := \begin{cases} \tilde{\iota}(h), & \text{if } 1 \leq h \leq k \\ \tilde{\iota}(1), & \text{if } h = k + 1 \end{cases} \]

for \( 1 \leq h \leq k + 1 \). Thus the change of variable provided by \( r \) converts the summation from the right-hand side of (7.1) into the summation on the right-hand side of (7.3), as required. \( \square \)
Remark 7.3. The next corollary records what comes out when the expression for \( \chi(\tau_\pi) \) that was just obtained is replaced back in the formula (5.13) for an even moment of the limit law \( \mu_n \). When stating the corollary, we will make two adjustments in the notation.

1° The fact that \( \hat{\eta} \) is constant under the action of \( \eta_k \cdot \sigma_\pi \) can be written for short as

\[
(7.4) \quad \hat{\eta} \circ (\eta_k \cdot \sigma_\pi) = \hat{\eta},
\]

with the slight abuse of notation that \( \eta_k \cdot \sigma_\pi \in S_k \subseteq S_\infty \) is now treated as a function from \( \{1, \ldots, k\} \) to itself (while the tuple \( \hat{\eta} \) is viewed as a function from \( \{1, \ldots, k\} \) to \( \{1, \ldots, d\} \)). We note, moreover, that upon composing with \( \sigma_\pi \) on the right in (7.4) and upon taking into account that \( \sigma_\pi^2 \) is the identity permutation, we can equivalently re-write (7.4) as

\[
(7.5) \quad \hat{\eta} \circ \eta_k = \hat{\eta} \circ \sigma_\pi.
\]

2° In order to better keep in mind that the numbers \( b_1, \ldots, b_k \) depend on \( \pi \), we will re-write the product \( w_{\hat{\eta}^{|1|}}(b_1) \cdots w_{\hat{\eta}^{|k|}}(b_k) \) that appeared in (7.3) in the form \( \prod_{V \in \pi} w_{\hat{\eta}^{|\max(V)|}}(V) \).

Corollary 7.4. For an even \( k \in \mathbb{N} \), the moment \( \int_\mathbb{R} t^k d\mu_n(t) \) is equal to

\[
(7.6) \quad \sum_{\hat{\eta} : \{1, \ldots, k\} \to \{1, \ldots, d\}} \left( \sum_{\pi \in P_{\leq 2}(k), \ \text{with} \ \hat{\eta}_{|\pi|} = \hat{\eta}_{\sigma_\pi}} (-1)^{|\pi|/2} \cdot (|\pi| - 1)!! \cdot w_{\hat{\eta}^{|1|}}(1) \cdot \prod_{V \in \pi} w_{\hat{\eta}^{|\max(V)|}}(V) \right).
\]

Proof. Upon plugging the formula (7.3) into (5.13), we find \( \int_\mathbb{R} t^k d\mu(t) \) written as a double sum indexed by

\[
\left\{ (\pi, \hat{\eta}) \mid \pi \in P_{\leq 2}(k) \text{ and } \hat{\eta} : \{1, \ldots, k\} \to \{1, \ldots, d\}, \text{ such that } \hat{\eta} \circ \eta_k = \hat{\eta} \circ \sigma_\pi \right\}.
\]

In (7.6) we simply record this double sum, in the form of an iterated sum. \( \square \)

For a further bit of processing of the moment formula obtained in (7.6), we observe that the double factorial appearing in that formula can be made to disappear, if we do the following trick: merge in pairs the singleton blocks of the partition \( \pi \), and use a colouring of the pairs of the resulting pair-partition. We thus proceed as follows.

Notation and Remark 7.5. (Bicoloured pair-partitions.) 1° We will use the name bi-coloured pair-partition for a pair-partition \( \rho \) of \( \{1, \ldots, k\} \) where every pair of \( \rho \) was painted in either blue or red. The set of all bi-coloured pair-partitions of \( \{1, \ldots, k\} \) will be denoted by \( P_2^{(b-r)}(k) \) (with the usual convention that \( P_2^{(b-r)}(k) = \emptyset \) for \( k \) odd).

2° For \( k \) even, we will use a “red-pair-breaking” map \( P_2^{(b-r)}(k) \ni \rho \mapsto \pi \in P_{\leq 2}(k) \) described as follows: \( \pi \) is obtained by taking every red block of \( \rho \) and breaking it into two singleton blocks. It is useful to note that, for every \( \pi \in P_{\leq 2}(k) \), the pre-image of \( \pi \) under this red-pair-breaking map has cardinality equal to \( (|\pi| - 1)!! \). The double factorial appears because we are counting the number of ways of grouping the \( |\pi| \) singleton blocks of \( \pi \) into \( |\pi|/2 \) red pairs of a bicoloured pair-partition \( \rho \).

3° For \( k \) even and \( \rho \in P_2^{(b-r)}(k) \) we will denote \( \sigma_\rho^{(\text{blue})} := \prod_{\{a,b\} \text{ pair of } \rho} \text{ blue} \ (a,b) \in S_k \).

The formula obtained in Corollary 7.4 is then further processed as follows.
Proposition 7.6. For every $k \in \mathbb{N}$, the moment $\int_{\mathbb{R}} \mu_{\text{w}}(t)^k \, dt$ is equal to

$$\sum_{i\in\{1,\ldots,k\} \to \{1,\ldots,d\}} w_i(1) \left( \sum_{\rho \in \mathcal{P}_{2}^{(b-r)}(k), \text{red, } p < q} \prod_{\{p,q\} \in \rho} w_{i}(q) \cdot \prod_{\{p,q\} \in \rho, \text{blue, } p < q} (-w_{i}(p)w_{i}(q)) \right).$$

Proof. For $k$ odd, both quantities involved are equal to 0. For $k$ even, the expression in (7.7) is obtained from the one in (7.6) when we use the surjection $\mathcal{P}_{2}^{(b-r)}(k) \to \mathcal{P}_{\leq 2}(k)$ indicated in Remark 7.5.2. \qed

8. Wick-style formulas and the CCR-GUE model for the law $\mu_{\text{w}}$

In this section we examine the CCR-GUE matrix $M$ that was introduced in Definition 2.8 and we give the proof of the main result of the paper, Theorem 2.9. This theorem claims that the law of $M$ (considered in the natural $*$-probability space where $M$ lives) coincides with the limit law $\mu_{\text{w}}$ obtained in Theorem 2.5.

The next remark gives an outline of how the section is organised.

Remark 8.1. Recall from Definition 2.8 that the entries $\{a_{i,j} \mid 1 \leq i,j \leq d\}$ of $M$ are for the most part commuting, as they are drawn from a family of commuting independent subalgebras

$$\mathcal{A}_0 \cup \{\mathcal{A}_{i,j} \mid 1 \leq i < j \leq d\}$$

of $\mathcal{A}$. The notion of “commuting independent subalgebras” is reviewed in Definition 8.2.1 below. Note that this notion does not require an individual subalgebra $\mathcal{A}_{i,j}$ to be commutative – and in fact $\mathcal{A}_{i,j}$ will surely not be commutative if the weights $w_i, w_j$ are distinct.

Concerning the algebra $\mathcal{A}_0 \subseteq \mathcal{A}$, where we pick the diagonal entries $a_{1,1}, \ldots, a_{d,d}$ of $M$; this one can be assumed to be commutative, and the formula used for computing joint moments of $a_{1,1}, \ldots, a_{d,d}$ is the usual Wick formula for real Gaussian random variables. We review this Wick formula in Definition 8.2.2.

On the other hand: for $1 \leq i < j \leq d$ we will derive a CCR-analogue of the Wick-formula for a complex Gaussian random variable, which allows us to compute the $*$-moments of $a_{i,j} \in \mathcal{A}_{i,j}$. The precise statement of how this goes is given in Proposition 8.3.

Together with the commuting independence of the subalgebras from (8.1), the two Wick formulas (usual version and CCR-version) will then give us a Wick-style summation formula for a general joint moment of the full family of entries $a_{i,j}$ of $M$. This formula is derived in Proposition 8.3.

Finally, the proof of Theorem 2.9 is easily obtained by comparing the moments of the law of $M \in M_d(\mathcal{A})$ (as they come out when we use the Wick-style formula of Proposition 8.3 against the description of moments of $\mu_{\text{w}}$ that was obtained in Proposition 7.6).

Definition 8.2. (Review of some basic notions needed below.)

1. Commuting independent subalgebras. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $(\mathcal{A}_{\lambda})_{\lambda \in \Lambda}$ be a family of unital $*$-subalgebras of $\mathcal{A}$. We say that the $\mathcal{A}_{\lambda}$’s are commuting independent to mean that the following two conditions (i) + (ii) are fulfilled:

(i) For every $\lambda_1 \neq \lambda_2$ in $\Lambda$ and every $x_1 \in \mathcal{A}_{\lambda_1}, x_2 \in \mathcal{A}_{\lambda_2}$, one has that $x_1x_2 = x_2x_1$.

(ii) Whenever $\lambda_1, \ldots, \lambda_k \in \Lambda$ are distinct ($\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq k$), one has

$$\varphi(x_1x_2 \cdots x_k) = \varphi(x_1)\varphi(x_2) \cdots \varphi(x_k), \quad \forall x_1 \in \mathcal{A}_{\lambda_1}, \ldots, x_k \in \mathcal{A}_{\lambda_k}. $$
2° Gaussian family of selfadjoints, via the Wick formula. Let \((A, \varphi)\) be a \(*\)-probability space where the algebra \(A\) is commutative. Suppose we are given a family \(x_1, \ldots, x_d\) of selfadjoint elements of \(A\), and a symmetric matrix \(C = [c_{ij}]_{i,j=1}^d \in M_d(\mathbb{R})\). We say that \(x_1, \ldots, x_d\) form a centred Gaussian family with covariance matrix \(C\) to mean that the following formula for computation of joint moments is holding: for every \(k \geq 1\) and every tuple \(\iota : \{1, \ldots, k\} \to \{1, \ldots, d\}\), one has

\[
\varphi(x_{\iota(1)} \cdots x_{\iota(k)}) = \sum_{\rho \in P_2(k)} \prod_{\{p,q\} \in \rho} c_{\iota(p)\iota(q)},
\]

with the usual convention that for \(k\) odd the right-hand side of (8.2) is to be read as “0”.

**Proposition 8.3.** (Wick’s lemma for a CCR-complex-Gaussian element.)

Let \((A, \varphi)\) be a \(*\)-probability space and suppose that \(a \in A\) is a centred CCR-complex-Gaussian element with parameters \(\omega_{(1, \ast)}, \omega_{(\ast, 1)} \in (0, \infty)\), in the sense of Definition 2.6. This means that \(a\) and \(a^*\) satisfy the commutation relation (2.15) and that the expectations \(\varphi(a^p(a^q)^*)\) are evaluated according to the formula (2.16) from that definition. We also put, for convenience, \(\omega_{(1, 1)} = \omega_{(\ast, \ast)} = 0\).

Then, for every \(k \in \mathbb{N}\) and \((\varepsilon(1), \ldots, \varepsilon(k)) \in \{1, \ast\}^k\), one has

\[
\varphi(a^{\varepsilon(1)} \cdots a^{\varepsilon(k)}) = \sum_{\rho \in P_2(k)} \prod_{\{p,q\} \in \rho, \; p < q} \omega(\varepsilon(p), \varepsilon(q)),
\]

with the usual convention that for \(k\) odd the right-hand side of (8.3) is to be read as “0”.

**Proof.** We will proceed by induction on the number of “\((\ast, 1)\)-inversions” \(\text{inv}(\varepsilon)\) in a tuple \(\varepsilon \in \sqcup_{k=1}^\infty \{1, \ast\}^k\), where for \(k \geq 1\) and \(\varepsilon = (\varepsilon(1), \ldots, \varepsilon(k)) \in \{1, \ast\}^k\) we denote

\[
\text{inv}(\varepsilon) := \left| \left\{ (p, q) \mid 1 \leq p < q \leq k, \; \varepsilon(p) = \ast, \; \varepsilon(q) = 1 \right\} \right|.
\]

The base-case of the induction is the one when \(\text{inv}(\varepsilon) = 0\). This refers to the situation when there is no instance of a \(*\) preceding a 1 among the components of \(\varepsilon\), hence when \(\varepsilon\) is of the form \((1, \ldots, 1, \ast, \ldots, \ast)\). In this case, the left-hand side of (8.3) is of the form \(\varphi(a^u(a^v)^*)\) for some \(u, v \geq 0\) with \(u + v = k\), and is therefore explicitly described by Equation (2.16). It is an easy counting exercise, left to the reader, to check that in this special situation the right-hand side of Equation (8.3) also matches the formula indicated in Equation (2.16).

For the rest of the proof we work on the induction step of the argument. We thus fix an \(\ell \geq 1\), we assume that (8.3) holds for all the tuples \(\varepsilon \in \sqcup_{k=1}^\infty \{1, \ast\}^k\) which have \(\text{inv}(\varepsilon) \leq \ell - 1\), and we prove that it also holds for tuples \(\varepsilon\) which have \(\text{inv}(\varepsilon) = \ell\). To this end, we consider a tuple \(\varepsilon_\rho = (\varepsilon_\rho(1), \ldots, \varepsilon_\rho(k))\) with \(\text{inv}(\varepsilon_\rho) = \ell\). This particular \(\varepsilon_\rho\) is also fixed for the rest of the proof, and our goal is to verify that (8.3) holds for it. We will run the verification by assuming that the length \(k\) of \(\varepsilon_\rho\) is such that \(k \geq 3\) (if \(k \leq 2\), then (8.3) follows from immediate calculations of moments of order 1 and 2).

Since \(\text{inv}(\varepsilon_\rho) = \ell > 0\), there has to exist a \(j \in \{1, \ldots, k - 1\}\) such that \(\varepsilon_\rho(j) = \ast\) and \(\varepsilon_\rho(j + 1) = 1\). That is, the monomial

\[
a^{\varepsilon_\rho} := a^{\varepsilon_\rho(1)} \cdots a^{\varepsilon_\rho(k)}
\]

has an \(a^*\) on position \(j\) which is immediately followed by an \(a\) on position \(j + 1\). Let \(\varepsilon\) be the tuple in \(\{1, \ast\}^k\) which is obtained by swapping the positions \(j\) and \(j + 1\) of \(\varepsilon_\rho\):

\[
\varepsilon(j) = 1, \; \varepsilon(j + 1) = \ast, \; \text{and} \; \varepsilon(p) = \varepsilon_\rho(p) \text{ for all } p \in \{1, \ldots, k\} \setminus \{j, j + 1\}.
\]
We notice that:

\[
\begin{aligned}
a^{\varepsilon_o} &= (a^{\varepsilon(1)} \ldots a^{\varepsilon(j-1)})(a^*a)(a^{\varepsilon(j+2)} \ldots a^{\varepsilon(k)}) \\
&= (a^{\varepsilon(1)} \ldots a^{\varepsilon(j-1)})(aa^* + (\omega_{(s,1)} - \omega_{(1,s)})1_\mathcal{A})(a^{\varepsilon(j+2)} \ldots a^{\varepsilon(k)}) \\
&= a^* + (\omega_{(s,1)} - \omega_{(1,s)})a^{\varepsilon'}, \\
\end{aligned}
\]

where \(\varepsilon' := (\varepsilon_o(1), \ldots, \varepsilon_o(j-1), \varepsilon_o(j+2), \ldots, \varepsilon_o(k))\) is obtained by removing the components \(j\) and \(j+1\) out of \(\varepsilon_o\), and where \(a^{\varepsilon}, a^{\varepsilon'}\) are defined in the same way as \(a^{\varepsilon_o}\) was defined in (8.4). We hence conclude that:

(8.5) \[\varphi(a^{\varepsilon_o}) = \varphi(a^{\varepsilon}) + (\omega_{(s,1)} - \omega_{(1,s)})\varphi(a^{\varepsilon'}).\]

Now, it is immediately verified that \(\text{inv}(\varepsilon) = \text{inv}(\varepsilon_o) - 1 = \ell - 1\), and also that \(\text{inv}(\varepsilon') < \text{inv}(\varepsilon_o) = \ell\); hence the induction hypothesis applies to both \(\varepsilon\) and \(\varepsilon'\), and assures us that we have \(\varphi(a^{\varepsilon}) = \Sigma(\varepsilon)\) and \(\varphi(a^{\varepsilon'}) = \Sigma(\varepsilon')\), where we denote:

\[
\Sigma(\varepsilon) := \sum_{\rho \in \mathcal{P}_2(k)} \prod_{\{p,q\} \in \rho} \omega(\varepsilon(p),\varepsilon(q)), \quad \Sigma(\varepsilon') := \sum_{\rho' \in \mathcal{P}_2(k-2)} \prod_{\{p,q\} \in \rho'} \omega(\varepsilon'(p),\varepsilon'(q)).
\]

Recall that what we need to prove is the equality \(\varphi(a^{\varepsilon_o}) = \Sigma(\varepsilon_o)\), with

\[
\Sigma(\varepsilon_o) := \sum_{\rho \in \mathcal{P}_2(k)} \prod_{\{p,q\} \in \rho} \omega(\varepsilon_o(p),\varepsilon_o(q)).
\]

In view of (8.5), the desired equality \(\varphi(a^{\varepsilon_o}) = \Sigma(\varepsilon_o)\) will follow if we can verify that

(8.6) \[\Sigma(\varepsilon_o) = \Sigma(\varepsilon) + (\omega_{(s,1)} - \omega_{(1,s)})\Sigma(\varepsilon').\]

So we are left to establish that (8.6) holds. To that end, let us also consider the sum

(8.7) \[\tilde{\Sigma} = \sum_{\rho \in \mathcal{P}_2(k), \{j,j+1\} \not\in \rho} \prod_{\{p,q\} \in \rho} \omega(\varepsilon_o(p),\varepsilon_o(q)).\]

We make the observation that:

(8.8) \[\Sigma(\varepsilon_o) - \tilde{\Sigma} = \omega_{(s,1)} \cdot \Sigma(\varepsilon') \quad \text{and} \quad \Sigma(\varepsilon) - \tilde{\Sigma} = \omega_{(1,s)} \cdot \Sigma(\varepsilon').\]

Indeed, for the first Equation (8.8) we write the difference \(\Sigma(\varepsilon_o) - \tilde{\Sigma}\) as

\[
\sum_{\rho \in \mathcal{P}_2(k), \{j,j+1\} \not\in \rho} \prod_{\{p,q\} \in \rho, \; \text{with} \; p < q} \omega(\varepsilon_o(p),\varepsilon_o(q)) = \omega(\varepsilon_o(j),\varepsilon_o(j+1)) \cdot \sum_{\rho' \in \mathcal{P}_2(k-2)} \prod_{\{p,q\} \in \rho', \; \text{with} \; p < q} \omega(\varepsilon'(p),\varepsilon'(q))
\]

where at the first equality sign we let the running pair-partition \(\rho'\) be obtained out of \(\rho\) by removing its pair \(\{j,j+1\}\) and then redenoting the elements of \(\{1, \ldots, k\} \setminus \{j,j+1\}\) as \(1, \ldots, k-2\). The second Equation (8.8) is obtained in a similar manner, where now we view \(\varepsilon'\) as the restriction of \(\varepsilon\), and the factor \(\omega(\varepsilon(j),\varepsilon(j+1))\) that has to be treated separately is equal to \(\omega_{(1,s)}\).

Finally, subtracting the second Equation (8.8) out of the first leads precisely to the formula (8.6) that we had been left to prove. \(\square\)

Now we put together the Wick formulas described above, to get a formula for the joint moments of the entries of the CCR-GUE matrix.
Proposition 8.4. Let $M = [a_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A})$ be the CCR-GUE matrix introduced in Definition 2.8. Then for every $k \in \mathbb{N}$ and $\iota, \underline{j} : \{1, \ldots, k\} \to \{1, \ldots, d\}$ we have
\begin{equation}
\varphi(a_{\iota(1),\underline{j}(1)} \cdots a_{\iota(k),\underline{j}(k)})
= \sum_{\rho \in \mathcal{P}_2^{(b-r)}(k)} \left( \prod_{\{p,q\} \in \rho, \text{ blue}, \ p < q} \delta_{\iota(p),\underline{j}(q)} \delta_{\iota(q),\underline{j}(p)} w_{\iota(p)} \delta_{\iota(q),\underline{j}(q)} \cdot \prod_{\{p,q\} \in \rho, \text{ red}, \ p < q} \delta_{\iota(p),\underline{j}(p)} \delta_{\iota(q),\underline{j}(q)} \cdot (-w_{\iota(p)} w_{\iota(q)}) \right),
\end{equation}
where $\mathcal{P}_2^{(b-r)}(k)$ is the set of bicoloured pair-partitions from Notation 7.5.

Proof. Throughout the whole proof we fix a $k \in \mathbb{N}$ and two tuples $\iota, \underline{j} : \{1, \ldots, k\} \to \{1, \ldots, d\}$, for which we will verify that (8.9) holds. It will come in handy to use the following notation:
\begin{equation}
\begin{cases}
\quad \text{Let } P_o := \{1 \leq p \leq k \mid \iota(p) = \underline{j}(p)\}.
\quad \text{For every } 1 \leq u < v \leq d, \text{ let } \quad P_{u,v} := \{1 \leq p \leq k \mid (\iota(p),\underline{j}(p)) = (u,v) \text{ or } (\iota(p),\underline{j}(p)) = (v,u)\}.
\end{cases}
\end{equation}
It is immediate that the sets introduced in (8.10) are pairwise disjoint (with the possibility that some of them are empty), and their union is equal to $\{1, \ldots, k\}$. We will run the calculations below by making the assumption that all these sets have even cardinality. We leave it as an exercise to the reader to verify that if this was not the case (i.e. either $|P_o|$ or one of the $|P_{u,v}|$’s was odd), then calculations similar to those shown below would quickly end in the conclusion that both sides of (8.9) are equal to 0. Also: we will write the remaining part of this proof by assuming we have $P_o \neq \emptyset$. For the case when $P_o = \emptyset$, one simply has to ignore all the considerations pertaining to $P_o$ (e.g. the discussion about the quantities $C_o$ and $C_o'$) which appear below.

On the left-hand side of (8.9), the hypothesis (i) of commuting independence entails a factorization
\begin{equation}
\varphi(a_{\iota(1),\underline{j}(1)} \cdots a_{\iota(k),\underline{j}(k)}) = C_o \cdot \prod_{\substack{1 \leq u < v \leq d \with P_{u,v} \neq \emptyset}} C_{u,v},
\end{equation}
where the quantities $C_o$ and $C_{u,v}$ are given by
\begin{equation}
C_o = \varphi\left( \prod_{p \in P_o} a_{\iota(p),\underline{j}(p)} \right) \quad \text{and} \quad C_{u,v} = \varphi\left( \prod_{p \in P_{u,v}} a_{\iota(p),\underline{j}(p)} \right).
\end{equation}
We note that $\prod_{p \in P_o} a_{\iota(p),\underline{j}(p)}$ is a commuting product and that $C_o$ can be evaluated by using the Wick formula reviewed in Definition 5.2. A product $\prod_{p \in P_{u,v}} a_{\iota(p),\underline{j}(p)}$ is non-commuting, hence it is important to mention that its factors are written in the increasing order of the elements of $P_{u,v}$; each of these factors is either an $a_{u,v}$ or an $a_{u,v}'$, and $C_{u,v}$ can thus be evaluated by using the CCR-Wick formula from Proposition 5.3.

We now turn to the right-hand side of (8.9), which we re-write more concisely by making the following ad-hoc definition: we say that $\rho \in \mathcal{P}_2^{(b-r)}(k)$ is compatible with $\iota$ and $\underline{j}$ when $\{j\} + \{jj\}$ below are holding:
\begin{equation}
\begin{cases}
\quad (j) \quad \text{For every blue pair } \{p, q\} \text{ of } \rho \text{ one has } \iota(p) = \underline{j}(q) \text{ and } \underline{j}(q) = \iota(p); \quad
\quad (jj) \quad \text{For every red pair } \{p, q\} \text{ of } \rho \text{ one has } \iota(p) = \iota(q) \text{ and } \underline{j}(p) = \underline{j}(q).
\end{cases}
\end{equation}
The conditions (j) + (jj) from (8.13) are precisely addressing the Kronecker deltas on the right-hand side of (8.9), and the expression written there takes the form:

\[
\sum_{\rho \in \mathcal{P}_2^{(b-r)}(k) \text{ compatible with } i,j} \left( \prod_{\{p,q\} \in \rho, \text{ blue, } p < q} w_{i}(q) \cdot \prod_{\{p,q\} \in \rho, \text{ red, } p < q} (-w_{i}(p)w_{j}(q)) \right).
\]

We next observe (via direct comparison between (8.10) and (8.13)) that if \( \rho \in \mathcal{P}_2^{(b-r)}(k) \) is compatible with \( i \) and \( j \), then every pair \( \{p,q\} \) of \( \rho \) is contained either in \( P_o \) or in a \( P_{u,v} \), with the colour of \( \{p,q\} \) being governed by the following rules:

\[
\begin{cases}
\text{If } \{p,q\} \subseteq P_{u,v} \text{ for some } 1 \leq u < v \leq d, \text{ then } \{p,q\} \text{ has colour blue.} \\
\text{If } \{p,q\} \subseteq P_o \text{ and } i(p) \neq i(q), \text{ then } \{p,q\} \text{ has colour red.} \\
\text{If } \{p,q\} \subseteq P_o \text{ and } i(p) = i(q), \text{ then } \{p,q\} \text{ may be either blue or red.}
\end{cases}
\]

Thus every \( \rho \in \mathcal{P}_2^{(b-r)}(k) \) compatible with \( i \) and \( j \) gets to be identified to a family

\[
\{\rho^{(o)}\} \cup \{\rho^{(u,v)} \mid 1 \leq u < v \leq d \text{ with } P_{u,v} \neq \emptyset\}
\]

where \( \rho^{(o)} \) is a bicoloured pair-partition of \( P_o \), every \( \rho^{(u,v)} \) is a pair-partition of \( P_{u,v} \), and each of these partitions must satisfy some compatibility conditions with \( i \) and \( j \) (inherited from the compatibility conditions satisfied by \( \rho \)). It is important to note that the compatibility conditions which have to be satisfied by \( \rho^{(o)} \) and the ones that have to be satisfied by the various \( \rho^{(u,v)} \)'s do not interfere with each other. So what we get here is a bijection

\[
\rho \in \mathcal{P}_2^{(b-r)}(k) \text{ compatible with } i,j \Longleftrightarrow \mathcal{P}^{(o)} \times \prod_{1 \leq u < v \leq d} \mathcal{P}^{(u,v)},
\]

where \( \mathcal{P}^{(o)} \) is a certain set of bicoloured pair-partitions of \( P_o \), and every \( \mathcal{P}^{(u,v)} \) is a certain set of pair-partitions of \( P_{u,v} \).

Now, the bijection observed in (8.17) can be used as a change of variable in the indexing set of the summation (8.14), and this leads to a factorization:

\[
\sum_{\rho \in \mathcal{P}_2^{(b-r)}(k) \text{ compatible with } i,j} \left( \prod_{\{p,q\} \in \rho, \text{ blue, } p < q} w_{i}(q) \cdot \prod_{\{p,q\} \in \rho, \text{ red, } p < q} (-w_{i}(p)w_{j}(q)) \right) = C'_o \cdot \prod_{1 \leq u < v \leq d} C'_{u,v},
\]

where \( C'_o \) is expressed as a summation over \( \mathcal{P}^{(o)} \) and every \( C'_{u,v} \) is expressed as a summation over \( \mathcal{P}^{(u,v)} \).

The last thing which needs to be observed is that the summation \( \sum_{\mathcal{P}^{(o)}} \) which gives \( C'_o \) is nothing but the Wick expansion for the expectation \( \varphi(\prod_{p \in P_o} a_{i(p)\overline{j(p)}}) \) that had appeared in (8.12); consequently, we have \( C'_o = C_o \). Likewise, for \( 1 \leq u < v \leq d \) with \( P_{u,v} \neq \emptyset \), the summation \( \sum_{\mathcal{P}^{(u,v)}} \) which gives \( C'_{u,v} \) is the CCR-Wick expansion for the expectation \( \varphi(\prod_{p \in P_{u,v}} a_{i(p)\overline{j(p)}}) \) in (8.12); consequently, we have \( C'_{u,v} = C_{u,v} \). We conclude that the two sides of Equation (8.9) are indeed equal to each other, since they were expressed as \( C_o \cdot \prod_{u,v} C_{u,v} \) and \( C'_o \cdot \prod_{u,v} C'_{u,v} \), respectively. \( \blacksquare \)

\footnote{Trying to give a detailed description of what are \( \mathcal{P}^{(o)} \) and \( \mathcal{P}^{(u,v)} \) would make the notation become quite heavy. We believe it is less painful and nevertheless quite convincing to see how the bijection (8.17) and the factorization (8.18) work in a relevant concrete example, such as the one presented immediately following to the present proof.}
Example 8.5. In order to have a better grasp of what the proof of Proposition 8.4 is doing, it is useful to run the steps of the proof in the setting of a concrete example. Assume for instance that \( k = 10 \) and we are looking at

\[
\varphi(a_{1,1} a_{1,2} a_{1,1} a_{3,3} a_{2,1} a_{3,1} a_{1,2} a_{1,3} a_{2,2} a_{2,1}).
\]

The tuples \( i, j \in \{1, \ldots, 10\}^2 \) used in this example are thus

\[
(8.20) \quad i = (1, 1, 1, 3, 2, 3, 1, 1, 2, 2) \quad \text{and} \quad j = (1, 2, 1, 3, 1, 1, 2, 3, 2, 1).
\]

The set of positions in \( \{1, \ldots, 10\} \) where we see elements \( a_{i,i} \) is \( P_o = \{1, 3, 4, 9\} \). We observe that we also have \( P_{1,2} = \{2, 5, 7, 10\} \) (positions where we see an \( a_{1,2} \) or an \( a_{2,1} \)), and \( P_{1,3} = \{6, 8\} \) (positions where we see an \( a_{1,3} \) or an \( a_{3,1} \)). The other sets \( P_{u,v} \) defined in (8.10) of the proof of Proposition 8.4 (e.g. \( P_{2,3} \), or \( P_{1,v} \) with \( 4 < v < d \)) are empty and will simply not appear in the further discussion of the example.

Now let us start to follow the steps of the proof of Proposition 8.4. Commuting independence allows us to re-arrange the product of \( a_{i,j} \)'s from (8.19) as

\[
(a_{1,1} a_{1,1} a_{3,3} a_{2,2}) \times (a_{1,2} a_{2,1} a_{1,2} a_{2,1}) \times (a_{3,1} a_{1,3})
\]

and to factor its expectation as \( C_o \cdot C_{1,2} \cdot C_{1,3} \) where

\[
C_o = \varphi(a_{1,1} a_{1,1} a_{3,3} a_{2,2}), \quad C_{1,2} = \varphi(a_{1,2} a_{2,1} a_{1,2} a_{2,1}), \quad C_{1,3} = \varphi(a_{3,1} a_{1,3}).
\]

We note that \( C_o \) can be evaluated by using the Wick formula reviewed in Definition 8.2:

\[
C_o = c_{1,1}c_{3,2} + c_{1,3}c_{1,2} + c_{1,2}c_{1,3}
\]

\[
= (w_1 - w_1^2)(-w_2 w_3) + (-w_1 w_3)(-w_1 w_2) + (-w_1 w_2)(-w_1 w_3)
\]

\[
= 3w_1^2 - w_1 w_2 w_3,
\]

where the entries \( c_{i,j} \) of the covariance matrix \( C \) are as stated in Proposition 8.4. For \( C_{1,2} \) and \( C_{1,3} \) we use the CCR-Wick formula discussed in Proposition 8.3 which gives

\[
C_{1,2} = w_2^2 + w_1 w_2 \quad \text{and} \quad C_{1,3} = w_1.
\]

The two terms in the formula for \( C_{1,2} \) correspond to the pairings \( \{2, 5\}, \{7, 10\} \) and \( \{2, 10\}, \{5, 7\} \) in \( \mathcal{P}_2(P_{1,2}) \), while the formula for \( C_{1,3} \) uses the unique pairing in \( \mathcal{P}_2(P_{1,3}) \).

Let us now check what pairings \( \rho \in \mathcal{P}_2^{(b-r)}(10) \) are compatible with the tuples \( i \) and \( j \) indicated in (8.20). It is clear that the \( a_{3,1} \) on position 6 must belong to a blue pair of \( \rho \), and this pair can only be \( \{6, 8\} \). Likewise, the occurrences of \( a_{1,2} \) on positions 2 and 7 have to be paired (in some order) to the occurrences of \( a_{2,1} \) on positions 5 and 10, which will make for two more blue pairs of \( \rho \). Finally, the \( a_{i,i} \)'s on positions 1, 3, 4, 9 remain to be paired among themselves; this will make for some red pairs of \( \rho \), with the exception of the fact that if we pair together the two occurrences of \( a_{1,1} \), that pair could be either red or blue. This discussion leads precisely to the bijection indicated in (8.17) of Proposition 8.4. On the right-hand side of (8.17) we have, for this example, a Cartesian product \( \mathcal{P}^{(o)} \times \mathcal{P}^{(1,2)} \times \mathcal{P}^{(1,3)} \) where:

- \( \mathcal{P}^{(o)} \) consists of the pair-partitions \( \{\{1, 3\}, \{4, 9\}\} \) of \( P_o \).
- \( \mathcal{P}^{(1,2)} \) consists of the pair-partitions \( \{\{2, 5\}, \{7, 10\}\} \) of \( P_{1,2} \).
- \( \mathcal{P}^{(1,3)} \) consists of the pair-partition \( \{\{6, 8\}\} \) of \( P_{1,3} \).
We leave it to the reader to use the concrete descriptions of $\mathcal{P}^{(a)}$, $\mathcal{P}^{(1,2)}$ and $\mathcal{P}^{(1,3)}$ in order to write down explicitly what we get in the factorization \((8.18)\). This factorization will produce some constants $C'_0, C'_{1,2}, C'_{1,3}$, which are found (another exercise for the reader) to coincide precisely with the $C_0, C_{1,2}$ and $C_{1,3}$ that were computed above.

### 8.6. Proof of Theorem 2.9

Since we saw in Remark 4.10 that the law $\mu_\omega$ is uniquely determined by its moments, it will suffice to pick a $k \in \mathbb{N}$ and verify that the moment of order $k$ of the law of $M \in M_d(A)$ is equal to the moment of order $k$ of $\mu_\omega$. The said moment of the law of $M$ is

$$
\varphi_\omega(M^k) = \sum_{i=1}^{d} w_i \varphi((i, i)\text{-entry of } M^k)
= \sum_{i=1}^{d} w_i \varphi\left( \sum_{i_2, \ldots, i_k = 1} a_{i_2, a_{i_3} \cdot \ldots \cdot a_{i_{k-1}, i_k} a_{i_k, i}} \right)
\tag{8.21}
$$

Now pick a tuple $i : \{1, \ldots, k\} \to \{1, \ldots, d\}$ and observe that if we put $j := i \circ \eta_k$ (that is, we put $j(1) = i(2), \ldots, j(k - 1) = i(k), j(k) = i(1)$), then the Wick formula \((8.9)\) used for these $i$ and $j$ gives:

$$
\varphi(a_{i(1), i(2)}, \ldots, a_{j(k-1), j(k)} a_{j(k), j(1)})
= \sum_{\rho \in \mathcal{P}_2^{(b-\text{red})}(k)} \left( \prod_{\{p, q\} \in \rho, \text{blue, } p < q} \delta_{i(p), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (q)} \delta_{\bar{i}(q), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (p)} w_i(q) \cdot \prod_{\{p, q\} \in \rho, \text{red, } p < q} \delta_{\bar{j}(p), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (p)} \delta_{\bar{j}(q), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (q)} \cdot (-w_{\bar{j}(p)} w_{\bar{j}(q)}) \right).
\tag{8.22}
$$

But for any $\rho \in \mathcal{P}_2^{(b-\text{red})}(k)$, the definition of the permutation $\sigma_{\rho}^{\text{blue}}$ can be used to evaluate the product of Kronecker $\delta$‘s on the right-hand side of \((8.22)\), as follows:

$$
= \prod_{\{p, q\} \in \rho, \text{blue, } p < q} \delta_{i(p), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (q)} \delta_{\bar{i}(q), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (p)} \cdot \prod_{\{p, q\} \in \rho, \text{red, } p < q} \delta_{\bar{i}(p), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (p)} \delta_{\bar{i}(q), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (q)}
$$

$$
= \prod_{h=1}^{k} \delta_{(\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (h), (\mathcal{O}_{\text{red}} c_{\mathcal{O}_{\text{red}}}) (h)} = \begin{cases} 1, & \text{if } i \circ \sigma_{\rho}^{\text{blue}} = i \circ \eta_k, \\ 0, & \text{otherwise.} \end{cases}
$$

The conclusion of the preceding paragraph is that, for every $i : \{1, \ldots, k\} \to \{1, \ldots, d\}$, Equation \((8.22)\) evaluates the expectation $\varphi(a_{i(1), i(2)}, \ldots, a_{i(k-1), i(k)} a_{i(k), i(1)})$ to

$$
\sum_{\rho \in \mathcal{P}_2^{(b-\text{red})}(k)} \left( \prod_{\{p, q\} \in \rho, \text{blue, } p < q} w_i(q) \cdot \prod_{\{p, q\} \in \rho, \text{red, } p < q} (-w_{\bar{i}(p)} w_{\bar{i}(q)}) \right).
\tag{8.23}
$$

When substituting this into \((8.21)\), we find that $\varphi_\omega(M^k)$ is precisely given by the formula we had found in Proposition 7.6 for the moment of order $k$ of $\mu_\omega$. \qed
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