HAMILTONIAN 2-FORMS IN KÄHLER GEOMETRY,
II GLOBAL CLASSIFICATION

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Abstract. We present a classification of compact Kähler manifolds admitting a hamiltonian 2-form (which were classified locally in part I of this work). This involves two components of independent interest.

The first is the notion of a rigid hamiltonian torus action. This natural condition, for torus actions on a Kähler manifold, was introduced locally in part I, but such actions turn out to be remarkably well behaved globally, leading to a fairly explicit classification: up to a blow-up, compact Kähler manifolds with a rigid hamiltonian torus action are bundles of toric Kähler manifolds.

The second idea is a special case of toric geometry, which we call orthotoric. We prove that orthotoric Kähler manifolds are diffeomorphic to complex projective space, but we extend our analysis to orthotoric orbifolds, where the geometry is much richer. We thus obtain new examples of Kähler–Einstein 4-orbifolds.

Combining these two themes, we prove that compact Kähler manifolds with hamiltonian 2-forms are covered by blow-downs of projective bundles over Kähler products, and we describe explicitly how the Kähler metrics with a hamiltonian 2-form are parameterized. We explain how this provides a context for constructing new examples of extremal Kähler metrics—in particular a subclass of such metrics which we call weakly Bochner-flat.

We also provide a self-contained treatment of the theory of compact toric Kähler manifolds, since we need it and find the existing literature incomplete.

This paper is concerned with the construction of explicit Kähler metrics on compact manifolds, and has several interrelated motivations. The first is the notion of a hamiltonian 2-form, introduced in part I of this series [1].

Definition 1. Let φ be any (real) $J$-invariant 2-form on the Kähler manifold $(M, g, J, \omega)$ of dimension $2m$. We say φ is hamiltonian if

\[ \nabla_X \phi = \frac{1}{2} (d \text{tr} \phi \wedge JX - d^c \text{tr} \phi \wedge X) \]

for any vector field $X$, where $\text{tr} \phi = \langle \phi, \omega \rangle$ is the trace with respect to $\omega$. When $M$ is a Riemann surface ($m = 1$), this equation is vacuous and we require instead that $\text{tr} \phi$ is a Killing potential, i.e., a hamiltonian for a Killing vector field $J \text{grad}_g \text{tr} \phi$.

A second motivation is the notion of a weakly Bochner-flat (WBF) Kähler metric, by which we mean a Kähler metric whose Bochner tensor (which is part of the curvature tensor) is co-closed. By the differential Bianchi identity, this is equivalent (for $m \geq 2$) to the condition that $\rho + \frac{\text{Scal}}{2(m+1)} \omega$ is a hamiltonian 2-form, where $\rho$ is the Ricci form. WBF Kähler metrics are extremal in the sense of Calabi, i.e., the symplectic gradient of the scalar curvature is a Killing vector field, and provide

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a class of extremal Kähler metrics which include the Bochner-flat Kähler metrics studied by Bryant [10] and products of Kähler–Einstein metrics. The geometry of WBF Kähler metrics is tightly constrained, because the more specific the normalized Ricci form is, the closer the metric is to being Kähler–Einstein, while the more generic it is, the stronger the consequences of the hamiltonian property.

A hamiltonian 2-form $\phi$ induces an isometric hamiltonian $\ell$-torus action on $M$ for some $0 \leq \ell \leq m$, which we call the order of $\phi$. This says nothing for $\ell = 0$, but for $\ell = m$, it means that $M$ is toric. Toric Kähler manifolds are well understood, and a third motivation for our work is to extend this understanding to certain torus actions with $0 < \ell < m$. We introduce the notion of a rigid hamiltonian $\ell$-torus action and prove that a compact Kähler manifold with such an action has a blow-up which is biholomorphic to a bundle of toric Kähler $2\ell$-manifolds.

We shall be particularly interested in the projective bundles of the form $M = P(\mathcal{L}_0 \otimes \mathbb{C}^{d_0+1} \oplus \cdots \oplus \mathcal{L}_\ell \otimes \mathbb{C}^{d_\ell+1}) \to S$, where $\mathcal{L}_0, \ldots, \mathcal{L}_\ell$ are line bundles over a compact Kähler manifold $S$ and the $\ell$-torus action is induced by scalar multiplication on the vector bundles $\mathcal{L}_j \otimes \mathbb{C}^{d_j+1}$, with $d_j \geq 0$. The blow-up of $M$ along the submanifolds determined by setting the $j$th fibrewise homogeneous coordinate (in $\mathcal{L}_j \otimes \mathbb{C}^{d_j+1}$) to zero, for $j = 0, \ldots, \ell$, is a bundle of toric Kähler 2$\ell$-manifolds: the projective bundle $P(\tilde{\mathcal{L}}_0 \oplus \cdots \oplus \tilde{\mathcal{L}}_\ell) \to \mathbb{C}P^{d_0} \times \cdots \times \mathbb{C}P^{d_\ell} \times S$, where $\tilde{\mathcal{L}}_j = \mathcal{O}(0, \ldots, 0, -1, 0, \ldots) \otimes \mathcal{L}_j$ (with $\mathcal{O}(-1)$ over the $j$th factor $\mathbb{C}P^{d_j}$).

When $\ell = 1$, projective line bundles have been well-used, since the seminal work of Calabi [11], in the construction of explicit examples of extremal Kähler metrics. The idea to consider blow-downs was introduced by Koiso and Sakane [25, 26], who constructed Kähler–Einstein metrics in this way. Our fourth motivation is to provide a general framework for constructing extremal Kähler metrics on projective bundles and their blow-downs, and in doing so we obtain new examples.

The toric Kähler $2m$-manifolds arising from hamiltonian 2-forms of order $m$ are of a special class, which we call orthotoric. Compact orthotoric Kähler manifolds are necessarily biholomorphic to complex projective space, but there are many more examples on orbifolds. Our final motivation is to study Kähler metrics on toric orbifolds, especially orthotoric orbifolds, and to obtain new examples.

The main goal of this paper is to show that a compact Kähler manifold with a hamiltonian 2-form of order $\ell$ is necessarily biholomorphic to a projective bundle $M$ of the form described above, and conversely to show precisely how to construct Kähler metrics with hamiltonian 2-forms of order $\ell$ on such bundles.

We hope however, that with the various motivations discussed above, the Reader who does not share our enthusiasm for hamiltonian 2-forms will find something of interest in this paper. Hamiltonian 2-forms rather provide a device that unifies and underlies the above themes. The journey to our main result, and its consequences, yield a number of results of independent interest.

- We obtain necessary and sufficient first order boundary conditions for the compactification of compatible Kähler metrics on toric symplectic orbifolds, clarifying work of Abreu [12], whose proofs we do not understand (see Remark 3 and § 1.4).
- We introduce and study rigid hamiltonian torus actions, and orthotoric Kähler manifolds and orbifolds.
- We construct new explicit Kähler–Einstein metrics on 4-orbifolds.
- We unify and extend constructions of Kähler metrics on projective bundles, obtaining new weakly Bochner-flat and extremal Kähler metrics on projective line bundles and on the projective plane bundle $P(\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathbb{C}^2) \to \mathbb{C}P^1$. 
We have attempted to make this paper as independent as possible from the first part [4]. However, we shall make essential use of the local classification of Kähler manifolds with a hamiltonian 2-form of order \( \ell \), so we recall the result here. The Reader who is not interested in hamiltonian 2-forms per se, could take this local classification result as a (rather complicated) definition of the class of Kähler metrics that we wish to classify globally.

We define the momentum polynomial of a hamiltonian 2-form \( \phi \) to be

\[
(2) \quad p(t) := (-1)^m \text{pf}(\phi - t\omega) = t^m - (\text{tr} \phi) t^{m-1} + \cdots + (-1)^m \text{pf} \phi
\]

where the pfaffian is defined by \( \phi \wedge \cdots \wedge \phi = (\text{pf} \phi) \omega \wedge \cdots \wedge \omega \).

**Theorem 1.** [4] Let \((M, g, J, \omega)\) be a connected Kähler \(2m\)-manifold with a hamiltonian 2-form \(\phi\). Then:

(i) the functions \(p(t)\) on \(M\) (for each \(t \in \mathbb{R}\)) are Poisson-commuting hamiltonians for Killing vector fields \(K(t) := J \text{grad}_g p(t)\);

(ii) there is a monic polynomial \(p_c(t)\) with constant coefficients such that \(p(t) = p_c(t)p_{nc}(t)\) and, if \(p_{nc}(t) = \sum_{r=0}^\ell (-1)^r \sigma_r t^{\ell-r}\) (with \(0 \leq \ell \leq m\)), then the Killing vector fields \(K_r := J \text{grad}_g \sigma_r\) \((r = 1, \ldots, \ell)\) are linearly independent on a connected dense open subset \(M^0\) of \(M\). The integer \(\ell\) is called the order of \(\phi\).

On the open subset \(M^0\), the roots \(\xi_1, \ldots, \xi_\ell\) of \(p_{nc}(t)\) are smooth, functionally independent and everywhere pairwise distinct, and they extend continuously to \(M\). Denote by \(\eta_a, a = 1, \ldots, N\) \((N \leq m - \ell)\) the different constant roots of \(p_c(t)\) and by \(d_\eta\) their multiplicities. Then there are (positive or negative definite) Kähler metrics \((g_\eta, \omega_\eta)\) of real dimension \(2d_\eta\), functions \(F_1, \ldots, F_\ell\) of one variable, and 1-forms \(\theta_1, \ldots, \theta_\ell\) with \(\theta_r(K_s) = \delta_{rs}\) such that the Kähler structure on \(M^0\) is of the form

\[
(3) \quad g = \sum_{a=1}^N p_{nc}(\eta_a)g_\eta + \sum_{j=1}^\ell \frac{p'(\xi_j)}{F_j(\xi_j)} d\xi_j^2 + \sum_{r=1}^\ell \frac{F_j(\xi_j)}{p'(\xi_j)} \left( \sum_{r=1}^\ell \sigma_{r-1}(\hat{\xi}_j) \theta_r \right)^2,
\]

\[
\omega = \sum_{a=1}^N p_{nc}(\eta_a)\omega_\eta + \sum_{r=1}^\ell d\sigma_r \wedge \theta_r, \quad d\theta_r = \sum_{a=1}^N (-1)^r \eta_a^{\ell-r} \omega_\eta,
\]

and the hamiltonian 2-form \(\phi\) is given by

\[
(4) \quad \phi = \sum_{a=1}^N \eta_a p_{nc}(\eta_a)\omega_\eta + \sum_{r=1}^\ell (\sigma_r d\sigma_1 - d\sigma_{r+1}) \wedge \theta_r
\]

with \(\sigma_{\ell+1} = 0\). (Here \(\sigma_{\ell-1}(\hat{\xi}_j)\) denote the elementary symmetric functions of the roots with \(\xi_j\) omitted. We remark also that \(p'(\xi_j) = p_c(\xi_j) \prod_{k \neq j} (\xi_j - \xi_k)\).)

We shall obtain our global description of compact Kähler manifolds admitting a hamiltonian 2-form of order \(\ell\) by exploiting three aspects of the local geometry revealed by Theorem [4]:

(i) The components \(g(K_r, K_s)\) of the metric are constant on fibres of the momentum map \((\sigma_1, \ldots, \sigma_\ell)\): \(M \to \mathbb{R}^\ell\). (This holds on all of \(M\) by continuity.)

(ii) The Kähler quotient metrics \(\sum_{a=1}^N p_{nc}(\eta_a)g_\eta\) are simultaneously diagonalizable (with respect to \(\sum_{a=1}^N g_\eta\)) with constant eigenvalues for each fixed \((\sigma_1, \ldots, \sigma_\ell)\).

(iii) The roots \(\xi_1, \ldots, \xi_\ell\) of \(p_{nc}\) have orthogonal gradients.

In [4], these properties were interpreted by saying that \((M, g, J, \omega)\) is given locally by a rigid hamiltonian \(\ell\)-torus action with semisimple Kähler quotient and orthotoric fibres. We shall see that this is not far from being true globally.
If $M$ is compact, the closure of the group of hamiltonian isometries of $M$ generated by $K_1, \ldots, K_\ell$ is a torus $\mathbb{T}$ (with $\ell \leq \dim \mathbb{T} \leq m$). When $\ell = m$, $K_1, \ldots, K_m$ generate a torus action, and $M$ is a toric Kähler manifold. In the first section we review the necessary background of toric Kähler geometry and introduce a suitable invariant language. Then, in section 2, we pursue a similar theory for $\ell < m$ when property (i) holds. In particular we prove that $\dim \mathbb{T} = \ell$ so there is a global rigid $\ell$-torus action. We provide a generalized Calabi construction for such actions which classifies them up to covering when the Kähler quotient is semisimple, i.e., when property (ii) is also satisfied. In section 3, we study toric Kähler manifolds (and orbifolds) satisfying property (iii) in general, and here we exhibit new explicit Kähler–Einstein metrics on compact 4-orbifolds. In section 4, we obtain a complete description of compact Kähler manifolds with hamiltonian 2-forms, which we use to construct new examples of compact weakly Bochner-flat and extremal Kähler manifolds. In subsequent work we shall construct many more examples and classify weakly Bochner-flat Kähler metrics in dimension 6.

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**1. Hamiltonian actions and toric geometry**

We begin by reviewing hamiltonian torus actions, paying particular attention to the theory of toric Kähler manifolds. Toric Kähler geometry can be studied either from the complex or symplectic viewpoint, and we adopt, primarily, the latter. Furthermore, with a view to applications, we do not restrict attention to manifolds, but also consider orbifolds: this is a natural context in toric symplectic geometry [21][29]. We refer to [6][19][29] for general information about torus actions.
on symplectic manifolds, and to [12, 14, 17, 18] for further information about toric Kähler manifolds and orbifolds.

Our treatment has some novel features: in particular we obtain first order boundary conditions for the compactification of compatible Kähler metrics on toric symplectic manifolds, and to [1, 12, 14, 17, 18] for further information about toric Kähler manifolds and orbifolds.

1.1. Hamiltonian torus actions. Let $\mathbb{T}$ be an $\ell$-dimensional torus, with Lie algebra $\mathfrak{t}$, acting effectively on a symplectic $2m$-manifolds $(M, \omega)$, and for $\xi \in \mathfrak{t}$ denote by $X_{\xi}$ the corresponding vector field on $M$. Then we say that the action is hamiltonian if there is a $\mathbb{T}$-invariant smooth map $\mu: M \to \mathfrak{t}^*$, called a momentum map for the action, such that $i_{X_{\xi}} \omega = -(d\mu, \xi)$ for any $\xi \in \mathfrak{t}$.

Remark 1. Note that our actions are hamiltonian in the strong sense that $\mu$ is $\mathbb{T}$-invariant (if $\mu$ has a critical point—as it does in the compact case—this is automatic). Since $\mathbb{T}$ is abelian, this implies that $\omega(X_{\xi}, X_{\eta}) = 0$ for any $\xi, \eta \in \mathfrak{t}$. We also remark that the action determines and is determined by $\mu$ up to a constant.

We shall normally be interested in the case that $(M, \omega, \mu)$ has a compatible almost Kähler structure, i.e., a $\mathbb{T}$-invariant metric $g$ and almost complex structure $J$ with $\omega(X, Y) = g(JX, Y)$. Such compatible metrics always exist.

We shall make significant use of the symplectic slice theorem for $\mathbb{T}$-orbits in $M$, which we now recall. Let $\mathbb{T} \cdot x$ be such an orbit for $x \in M$. Since $\mathbb{T} \cdot x$ is isotropic with respect to $\omega$, the isotropy representation of $T_x M$ on $T_x M$ induces a $2(m-k)$-dimensional symplectic representation on $V_x := T_x(\mathbb{T} \cdot x)^0/T_x(\mathbb{T} \cdot x)$, where $T_x(\mathbb{T} \cdot x)^0$ denotes the annihilator with respect to $\omega_x$ of $T_x(\mathbb{T} \cdot x)$ in $T_x M$. This is called the symplectic isotropy representation.

Using the metric $g_x$, $T_x M$ is an orthogonal direct sum of the subspaces

$T_x(\mathbb{T} \cdot x) \cong \mathfrak{t}/t_x, \quad JT_x(\mathbb{T} \cdot x) \cong (t/t_x)^* \cong \mathfrak{t}^*_x, \quad V_x$

where $\mathfrak{t}^*_x$ the annihilator of $t_x$ in $t^*$ (identified with $JT_x(\mathbb{T} \cdot x)$ using $\omega_x$).

Lemma 1. Let $(M, g, J, \omega)$ be an almost Kähler manifold with an isometric hamiltonian $\mathbb{T}$-action. Fix $x \in M$ and a splitting $\chi: \mathfrak{t} \to t_x$ of the inclusion.

Then the action of $T_x$ on the symplectic isotropy representation $V_x$ is effective, and there is a symplectic form $\omega_0$ on the normal bundle $N = \mathbb{T} \times \mathbb{T}_x (\mathfrak{t}^0_x \oplus V_x) \to \mathbb{T} \cdot x$ and a symplectomorphism $f$ from a neighbourhood of the zero section $0_N$ in $N$ to a neighbourhood of $\mathbb{T} \cdot x$ in $M$ such that:

- the obvious $\mathbb{T}$-action on $N$ by left multiplication is hamiltonian with momentum map $\mu_0([\alpha, v]) = \alpha + \mu V(v) \circ \chi$, where $\alpha$ is an element of the fibre belonging to $\mathfrak{t}^0_x \subset \mathfrak{t}$, and $\mu V: V_x \to \mathfrak{t}^*_x$ is the momentum map of the symplectic isotropy representation;
- $f$ is $\mathbb{T}$-equivariant, is equal to the bundle projection along $0_N$, and its fibre derivative along $0_N$ is the natural identification of the vertical bundle with $N$.

Proof. The normal exponential map provides a $\mathbb{T}$-equivariant diffeomorphism from a neighbourhood of $0_N \cong \mathbb{T} \cdot x$ of the normal bundle $N = \mathbb{T} \times \mathbb{T}_x (\mathfrak{t}^0_x \oplus V_x) \to \mathbb{T} \cdot x$ (with the natural $\mathbb{T}$ action induced by left multiplication on $\mathbb{T}$) to a neighbourhood of $\mathbb{T} \cdot x$ in $M$; then $\mathbb{T}$ acts effectively on $N$ while $\mathbb{T}_x$ acts trivially on $\mathfrak{t}^0_x$, so $\mathbb{T}_x$ acts effectively on $V_x$.

The chosen projection $\chi: \mathfrak{t} \to t_x$ identifies the normal bundle $N$ with the symplectic quotient of $T^* \mathbb{T} \times V_x$, by the diagonal action of $\mathbb{T}_x$ (since $T^* \mathbb{T} \cong \mathbb{T} \times t^*$). The induced symplectic form is $\mathbb{T}$-invariant with the given momentum map.
The pullback of $\omega$ by the normal exponential map gives another symplectic form $\omega_1$ on a neighbourhood of $0_N$ in $N$, agreeing with $\omega_0$ along $0_N$ ($\omega_1$ and $\omega_0$ both equal $\omega_x$ at $T_{(x,0)}N \cong T_xM$). By the equivariant relative Darboux theorem, there is a $\mathbb{T}$-equivariant diffeomorphism $h$ of $N$ fixing $0_N$, with $dh = Id$ there, and such that $h^*\omega_1 = \omega_0$ on a neighbourhood $U$ of $0_N$ in $N$. Then $f = \exp ch$ is the equivariant symplectomorphism we seek. \hfill \Box

This result easily generalizes to orbifolds—see [29] Lemma 3.5 and Remark 3.7.

1.2. Toric manifolds and orbifolds. A connected $2m$-dimensional symplectic manifold or orbifold $(M, \omega)$ is said to be toric if it is equipped with an effective hamiltonian action of an $m$-torus $\mathbb{T}$ with momentum map $\mu: M \to \mathfrak{t}^*$. Compact toric symplectic manifolds were classified by Delzant [14], and this classification was extended to orbifolds by Lerman–Tolman [29]. Essentially, they are classified by the image of the momentum map $\mu$, which is a compact convex polytope in $\mathfrak{t}^*$, but this statement requires some interpretation, particularly in the orbifold case.

Definition 2. Let $t$ be an $m$-dimensional real vector space. Then a rational Delzant polytope $(\Delta, \Lambda, u_1, \ldots, u_n)$ in $\mathfrak{t}^*$ is a compact convex polytope $\Delta \subset \mathfrak{t}^*$ equipped with normals belonging to a lattice $\Lambda$ in $t$

$$(5) \quad u_j \in \Lambda \subset \mathfrak{t}$$

$(j = 1, \ldots, n, \ n > m)$ such that

$$(6) \quad \Delta = \{ x \in \mathfrak{t}^* : L_j(x) \geq 0, \ j = 1, \ldots, n \}$$

with

$$L_j(x) = \langle u_j, x \rangle + \lambda_j$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and such that for any vertex $x \in \Delta$, the $u_j$ with $L_j(x) = 0$ form a basis for $t$. If the normals form a basis for $\Lambda$ at each vertex, then $\Delta$ is said to be integral, or simply a Delzant polytope.

The term rational refers to the fact that the normals span an $m$-dimensional vector space over $\mathbb{Q}$. A rational Delzant polytope is obviously $m$-valent, i.e., $m$ codimension one faces and $m$ edges meet at each vertex: by (6) the codimension one faces $F_1, \ldots, F_n$ are given by $F_j = \Delta \cap \{ x \in \mathfrak{t}^* : L_j(x) = 0 \}$, so that $u_j$ is an inward normal vector to $F_j$. In the integral case, the $u_j$ are necessarily primitive, and so are uniquely determined by $(\Delta, \Lambda)$. In general, the primitive inward normals are $u_j/m_j$ for some positive integer labelling $m_j$ of the codimension one faces $F_j$, so rational Delzant polytopes are also called labelled polytopes [29]. However, it turns out to be more convenient to encode the labelling in the normals. Note that $\lambda_1, \ldots, \lambda_n$ are uniquely determined by $(\Delta, \Lambda, u_1, \ldots, u_n)$.

The rational Delzant theorem [14, 29] states that compact toric symplectic orbifolds are classified (up to equivariant symplectomorphism) by rational Delzant polytopes (with manifolds corresponding to integral Delzant polytopes). Given such a polytope, $(M, \omega)$ is obtained as a symplectic quotient of $\mathbb{C}^n$ by an $(n - m)$-dimensional subgroup $G$ of the standard $n$-torus $(S^1)^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$: precisely, $G$ is the kernel of the map $(S^1)^n \to \mathbb{T} = \mathbb{T}/2\pi\Lambda$ induced by the map $(x_1, \ldots, x_n) \mapsto \sum_{j=1}^n x_j u_j$ from $\mathbb{R}^n$ to $t$, and the momentum level for the symplectic quotient is the image in $\mathfrak{g}^*$ of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n*}$ under the transpose of the natural inclusion of the Lie algebra $\mathfrak{g}$ in $\mathbb{R}^n$.

Conversely, a toric symplectic orbifold gives rise to a rational Delzant polytope as the image $\Delta$ of its momentum map $\mu$, where $\Lambda$ is the lattice of circle subgroups, and the positive integer labelling $m_j$ of the codimension one faces $F_j$ is determined.
Remark 2. Toric symplectic manifolds and orbifolds are simply connected (as topological spaces—the inverse image of the union of the faces meeting a given vertex is contractible, and the complement has codimension two). However one can consider orbifold coverings and quotients: a compact convex polytope with chosen normals (giving a basis for $\mathfrak{t}$ at each vertex) is a rational Delzant polytope with respect to any lattice satisfying (4). In particular, if $\Lambda$ is a (finite index) sublattice of $\Lambda'$, then the torus $T' = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ is the quotient of $T = \mathbb{R}^2/\mathbb{Z}$ by a finite abelian group $\Gamma \cong \mathbb{Z}/\Lambda$. The corresponding toric symplectic orbifolds $M$ and $M'$ (under the tori $T$ and $T'$) are related by a regular orbifold covering: $M' = M/\Gamma$.

Clearly there is a ‘smallest’ lattice $\Lambda$ satisfying (4), namely the lattice generated by the normals $u_1, \ldots, u_n$. This is a sublattice of any other lattice $\Lambda'$ with $u_j \in \Lambda'$, so any toric symplectic orbifold $M'$, corresponding to such a $\Lambda'$, is a quotient of the toric symplectic orbifold $M$ (corresponding to $\Lambda$) by a finite abelian group $\Gamma$.

In fact $M$ is the universal orbifold cover of $M'$ in the sense of [35]. One may also characterize $\Lambda$ as the unique lattice containing $u_1, \ldots, u_n$ for which $G$ is connected, i.e., $M$ is a symplectic quotient of $\mathbb{C}^n$ by a $(n - m)$-subtorus of $(S^1)^n$.

1.3. Compatible Kähler metrics: local theory. We turn now to the study of compatible Kähler metrics on toric symplectic orbifolds. On the union $M^0 := \mu^{-1}(\Delta^0)$ of the generic orbits, such metrics have an explicit description due to Guillemin [17]. Orthogonal to the orbits is a rank $m$ distribution spanned by commuting holomorphic vector fields $JX_\xi$ for $\xi \in \mathfrak{t}$. Hence there is a function $t : M^0 \to \mathbb{R}^2/\mathbb{Z}$, defined up to an additive constant, such that $dt(JX_\xi) = 0$ and $dt(X_\xi) = \xi$ for $\xi \in \mathfrak{t}$. The components of $t$ are ‘angular variables’, complementary to the components of the momentum map $\mu : M^0 \to \mathfrak{t}^*$, and the symplectic form in these coordinates is simply

$$\omega = \langle d\mu \wedge dt \rangle,$$

where the angle brackets denote contraction of $t$ and $t^*$.

These coordinates identify each tangent space with $t \oplus t^*$, so any $\mathbb{T}$-invariant $\omega$-compatible almost Kähler metric is given by

$$g = \langle d\mu, G, d\mu \rangle + \langle dt, H, dt \rangle,$$

where $G$ is a positive definite $S^2$-valued function on $\Delta^0$, $H$ is its inverse in $S^2 t^*$—observe that $G$ and $H$ define mutually inverse linear maps $t^* \to t$ and $t \to t^*$ at each point—and $\langle \cdot, \cdot \rangle$ denotes the pointwise contraction $t^* \times S^2 t \times t^* \to \mathbb{R}$ or the dual contraction. The corresponding almost complex structure is defined by

$$J dt = -\langle G, d\mu \rangle$$

from which it follows that $J$ is integrable if and only if $G$ is the hessian of a function on $\Delta^0$ [17].

Remark 3. The description of $\mathbb{T}$-invariant $\omega$-compatible Kähler metrics on $M^0$ shows that they are parameterized by functions on $\Delta^0$ with positive definite hessian. There is a subtle point here, however, which is often overlooked in the literature, namely that the angular coordinates $t$ depend on the (lagrangian) orthogonal distribution to the $\mathbb{T}$-orbits in $M^0$, and there is no reason for two metrics to have the same orthogonal distribution. This is not a problem on $M^0$, since the obvious map sending one set of angular coordinates to another is an equivariant symplectomorphism, but this symplectomorphism may not extend to $M$. 

by the fact that the local uniformizing group of every point in $\mu^{-1}(F_0)$ is $\mathbb{Z}/m_j \mathbb{Z}$. (Here, and elsewhere, for any face $F$, we denote by $F^0$ its interior.)
The Delzant construction realizes \((M, \omega)\) as a symplectic quotient of \(\mathbb{C}^n\), so there is an obvious choice of a ‘canonical’ compatible Kähler metric \(g_0\), namely the one induced by the flat metric on \(\mathbb{C}^n\). An explicit formula for this Kähler metric in symplectic coordinates was obtained by Guillemin \cite{Guillemin}, and extended to the orbifold case by Abreu \cite{Abreu}. On \(M^0\), the canonical metric is given by \((\mathbf{S})\) with \(G\) equal to
\[
\frac{1}{2} \text{Hess}\left(\sum_{j=1}^{n} L_j(\mu) \log |L_j(\mu)|\right) = \frac{1}{2} \sum_{j=1}^{n} \frac{u_j \otimes u_j}{L_j(\mu)}.
\]
Hence the induced metric on \(\Delta^0\) is \(\frac{1}{2} \sum_{j=1}^{n} d(L_j)^2 / L_j\). (See also \cite{Abreu}.)

1.4. **Compatible Kähler metrics: compactification.** On any compact toric symplectic manifold or orbifold, the canonical metric \(g_0\) is globally defined on \(M\)—by construction. The study of other globally defined Kähler metrics is greatly facilitated by the following elementary lemma (see also \cite{Calderbank} and Remark \(\text{(ii)}\) below).

**Lemma 2.** Let \((M, \omega)\) be a toric symplectic 2m-manifold or orbifold with momentum map \(\mu: M \rightarrow \Delta \subset \mathbb{T}^*\), and suppose that \((g_0, J_0)\), \((g, J)\) are compatible almost Kähler metrics on \(M^0 = \mu^{-1}(\Delta^0)\) of the form \((\mathbf{S})-\mathbf{(3)}\), given by \(G_0, G\) and the same angular coordinates, and such that \((g_0, J_0)\) extends to an almost Kähler metric on \(M\). Then \((g, J)\) extends to an almost Kähler metric on \(M\) provided that
\[
\begin{align*}
\text{(11)} & \quad G - G_0 \quad \text{is smooth on } \Delta, \\
\text{(12)} & \quad G_0 HG_0 - G_0 \quad \text{is smooth on } \Delta.
\end{align*}
\]

**Remark 4.** (i) We use here the fact that any \(\mathbb{T}\)-invariant smooth function on \(M\) is the pullback by \(\mu\) of a smooth function on \(\Delta\) (this follows from the symplectic slice theorem and \cite{Guillemin}; see \cite{PD}).

(ii) For generators \(X_\xi, X_\eta\) of the \(\mathbb{T}\)-action, \(g_0(X_\xi, X_\eta)\) is a \(\mathbb{T}\)-invariant smooth function on \(M\), hence the pullback of a smooth function on \(\Delta\). Thus \(H_0\) is a smooth \(S^2 t^*\)-valued function on \(\Delta\) (degenerating on \(\partial \Delta\)). Condition \(\text{(11)}\) thus implies that \(H_0 G\) is smooth on \(\Delta\). We claim that in the presence of \(\text{(11)}, \text{(12)}\) is equivalent to \(H_0 G\) being nondegenerate on \(\Delta\). Indeed, if \(H_0 G\) is nondegenerate, its inverse \(H G_0\) is smooth on \(\Delta\); now composing \(G - G_0\) on the right by this we obtain \(\text{(12)}\). Conversely, multiplying by \(H_0\) we deduce from \(\text{(12)}\) that \(H G_0\) is smooth on \(\Delta\), so \(H_0 G\) is nondegenerate.

**Proof of Lemma 2.** The key point is that it suffices to show \(g\) is smooth on \(M\): it will then be nondegenerate because it is compatible with \(\omega\) (equivalently if \(J\) extends smoothly to \(M\), it is an almost complex structure on \(M\) by continuity). For the smoothness of \(g\), we simply compute the difference
\[
g - g_0 = \langle d\mu, G - G_0, d\mu \rangle + \langle dt, H - H_0, dt \rangle
\]
\[
= \langle d\mu, G - G_0, d\mu \rangle + \langle J_0 d\mu, G_0 HG_0 - G_0, J_0 d\mu \rangle.
\]
Now \(\mu, g_0\) and \(J_0\) are smooth on \(M\), hence so is \(g\) by \(\text{(11)-(12)}\). \(\square\)

According to Abreu \cite{Abreu}, when \(g_0\) is the canonical metric on \((M, \omega)\), these conditions are not only sufficient but necessary for the compactification of \(g\). However, in our view there are some shortcomings in his (rather sketchy) proof. In particular he does not address the issue of the dependence of the angular coordinates on the metric. The following observation only partially resolves this difficulty.

**Lemma 3.** Let \((M, \omega)\) be a compact toric symplectic manifold with two compatible almost Kähler metrics which induce the same \(S^2 t^*\)-valued function \(G\) on the interior...
of the Delzant polytope. Then there is an equivariant symplectomorphism of $M$ sending one metric to the other.

Proof. By Remark 3, such a symplectomorphism exists on $M^0$. It extends uniquely to $M$, since $M^0$ is dense and $(M,g)$ is a complete. The extension is a distance isometry by continuity, and is therefore smooth by a standard argument. 

Note that this lemma makes essential use of the completeness of $(M,g)$. It can, however, be extended to compact orbifolds, for instance by lifting the distance isometry to compatible uniformizing charts.

On the other hand we learn nothing about the dependence of the angular coordinates on metrics which induce different $S^2t$-valued functions on the interior of the Delzant polytope. Thus the above lemma does not suffice to clarify Abreu’s proof.

The issue of the compactification of toric Kähler metrics is an important one. We shall therefore establish precise necessary and sufficient compactification conditions by a self-contained argument. Our proof also has the merit of being elementary and, modulo the above lemma, local, in contrast to [1, 2], where the existence of a global biholomorphism is used. Indeed, compactification is about boundary conditions, so it is a local question. We shall present these boundary conditions in a form more closely analogous to the well-known conditions in complex dimension one. As a warm-up for the rest of the subsection we first recall this case.

Let $(M,\omega)$ be a compact toric symplectic 2-orbifold. This must be an orbifold 2-sphere (i.e., equivariantly homeomorphic to $\mathbb{C}P^1$ with the standard circle action, but the two fixed points may be orbifold singularities), equipped with a rotation invariant area form. On $M^0$, which is diffeomorphic to $\mathbb{C}^\times$, a compatible Kähler metric takes the form

\begin{equation}
 g = \frac{d\mu^2}{\Theta(\mu)} + \Theta(\mu)dt^2,
\end{equation}

where $\omega = d\mu \wedge dt$. The rational Delzant polytope is an interval $[\alpha, \beta] \in \mathfrak{t}^*$ with normals $u_\alpha, u_\beta \in \mathfrak{t}$. If we identify a generator of the lattice $\Lambda$ in $\mathfrak{t}$ with $1 \in \mathbb{R}$ (chosen so that $u_\alpha$ is positive), then $t: M^0 \to t/2\pi\Lambda$ becomes a coordinate of period $2\pi$, and the orbifold singularities have cone angles $2\pi/m_\alpha, 2\pi/m_\beta$ where $m_\alpha = u_\alpha, m_\beta = -u_\beta \in \mathbb{Z}^+$.

Since $\Theta(\mu)$ is the norm squared of the Killing vector field, $\Theta$ is smooth on $[\alpha, \beta]$, positive on the interior, and zero at the endpoints. On the other hand, $\mu$ is a Morse function (i.e., the two critical points are nondegenerate—this follows easily using a symplectic slice) and $dd^c\mu = \Theta'(\mu)\omega$, so that $\Theta'(\alpha)$ and $\Theta'(\beta)$ are nonzero.

Now let $\hat{U} \subset \mathbb{R}^2$ be an orbifold chart covering an $S^1$-invariant neighbourhood $U = \hat{U}/\mathbb{Z}_{m_\alpha}$ of $\mu^{-1}(\alpha)$, where $\mathbb{Z}_{m_\alpha}$ acts in the standard way on $\mathbb{R}^2$ and the covering map $\pi$ sends 0 to $\mu^{-1}(\alpha)$. The $S^1$-action on $U$ lifts to one on $\hat{U}$, fixing 0 and commuting with $\mathbb{Z}_{m_\alpha}$. Now $\hat{t} = t \circ \pi/m_\alpha$ is a coordinate of period $2\pi$ on $\hat{U} \setminus \{0\}$ while $\hat{\mu} = m_\alpha(\mu \circ \pi)$ is the momentum map of the $S^1$ action on $\hat{U}$, with respect to $\hat{\omega} = d\hat{\mu} \wedge d\hat{t} = \pi^*\omega$. The pull back of $g$ to $\hat{U} \setminus \{0\}$ is

\begin{equation}
 \hat{g} = \frac{d\hat{\mu}^2}{m_\alpha^2 \Theta(\hat{\mu}/m_\alpha)} + m_\alpha^2 \Theta(\hat{\mu}/m_\alpha)dt^2.
\end{equation}

If this metric compactifies smoothly at 0 we must have $m_\alpha \Theta'(\alpha) = 2$ (see [22]). With an analogous argument at $\mu^{-1}(\beta)$, we deduce that $u_\alpha \Theta'(\alpha) = 2 = u_\beta \Theta'(\beta)$.

To show that these conditions are sufficient for the smooth extension of $g$ (in the orbifold sense) to $M$, we put $r^2/2 = \mu - \alpha$ and let $t$ have period $2\pi/m_\alpha$. Since $\Theta(\alpha) = 0, g$ differs from a multiple of $g_0 = dr^2 + \frac{1}{4}\Theta'(\alpha)^2 r^2 dt^2$ by a smooth bilinear
form on \( M \), vanishing at \( \mu = \alpha \). Clearly the condition \( \Theta'(\alpha) = 2/m_\alpha \) provides a smooth (orbifold) extension of \( g_0 \) to \( \mu^{-1}(\alpha) \) by considering \((r,t/m_\alpha)\) to be the polar coordinates in a uniformising chart. The other endpoint is analogous.

To summarize, \( g \) given by (13) is globally defined on a toric orbifold whose rational Delzant polytope is \([\alpha, \beta] \subset t^*\), with normals \( u_\alpha, u_\beta \in t \), if and only if \( \Theta \) smooth on \([\alpha, \beta] \), with
\[
\Theta(\alpha) = 0 = \Theta(\beta), \quad \Theta'(\alpha) u_\alpha = 2 = \Theta'(\beta) u_\beta
\]
and \( \Theta \) positive on \((\alpha, \beta)\). The derivative conditions make invariant sense, since \( \Theta \) takes values in \((t^*)^2\), so its derivative takes values in \( t^* \). Also note that the conditions are manifestly independent of the choice of lattice (as they should be).

In order to generalize this criterion to the case \( m > 1 \), we introduce some notation. For any face \( F \subset \Delta \), we denote by \( t_F \subset t \) the vector subspace spanned by the inward normals \( u_j \in t \) to all codimension one faces of \( \Delta \), containing \( F \); thus the codimension of \( t_F \) equals the dimension of \( F \). Furthermore, the annihilator \( t_F^\perp \) of \( t_F \) in \( t^* \) is naturally identified with \((t/t_F)^*\).

**Proposition 1.** Let \((M, \omega)\) be a compact toric symplectic \(2m\)-manifold or orbifold with momentum map \( \mu: M \to \Delta \subset t^*\) and \( H \) be a positive definite \( S^2 t^* \)-valued function on \( \Delta^0 \). Then \( H \) comes from a \( \mathbb{T} \)-invariant, \( \omega \)-compatible almost Kähler metric \( g \) via (13) if and only if it satisfies the following conditions:

- [smoothness] \( H \) is the restriction to \( \Delta^0 \) of a smooth \( S^2 t^* \)-valued function on \( \Delta \);
- [boundary values] for any point \( y \) on the codimension one face \( F_j \subset \Delta \) with inward normal \( u_j \), we have
\[
H_y(u_j, \cdot) = 0 \quad \text{and} \quad (dH)_y(u_j, u_j) = 2u_j,
\]
where the differential \( dH \) is viewed as a smooth \( S^2 t^* \otimes t \)-valued function on \( \Delta \);
- [positivity] for any point \( y \) in interior of a face \( F \subset \Delta \), \( H_y(\cdot, \cdot) \) is positive definite when viewed as a smooth function with values in \( S^2(t/t_F)^* \).

**Proof.** We first prove the necessity of these conditions. Let \((M, \omega, \mu)\) be a compact toric symplectic orbifold with polytope \( \Delta \), and \((g, J)\) a compatible Kähler metric. For any \( x \in M \) and \( \xi, \eta \in t \), we put \( H_{\mu(x)}(\xi, \eta) = g_x(X_\xi, X_\eta) \). Clearly \( H \) is an \( S^2 t^* \)-valued function on \( \Delta \) and the smoothness and positivity properties follow immediately from the definition.

It remains to establish the boundary values (15) for \( y = \mu(x) \) in a codimension one face \( F_j \). The vanishing of \( H_y(u_j, \cdot) = 0 \) is immediate from the definition (the Killing vector field corresponding to \( u_j \) vanishes on \( \mu^{-1}(F_j) \)). This implies in particular that \( dH_y(u_j, u_j) \) is proportional to \( u_j \). To obtain the correct constant, we use a symplectic slice, as in Lemma 11 to pullback the metric \( g \) to the normal bundle \( N \) of the orbit \( \mathbb{T} \cdot x \) for a point \( x \in M \) with \( \mu(x) = y \), and restrict to the symplectic isotropy representation \( V_x \). By construction, the Killing vector field corresponding to \( u_j \) induces the generator \( X \) of the standard circle action on \( V_x \), and the metric induced by \( g \) agrees to first order at \( 0 \) with the constant metric \( g_0 \) given by \( g_x \). It is now straightforward to check that the constant is 2 (indeed, \((V_x, g_0, \omega_0)\) is a toric Kähler 2-orbifold, so we have already computed this above).

Now we explain why the given conditions are sufficient to conclude \( H \) that comes from a smooth compatible metric on \((M, \omega)\).

We know that the function \( H_0 = G_0^{-1} \), with \( G_0 \) defined by (10), does correspond to a globally defined invariant Kähler metric on \((M, \omega)\) (and so it satisfies the given
conditions, as one can easily check directly). By virtue of Lemma \ref{lem:11}, it is enough to show that for any \( H = G^{-1} \) satisfying the given conditions, the sufficient conditions \((11)\)–\((12)\) are satisfied. As explained in Remark \ref{rem:4} we have to check that both \( HG_0 \) and \( G - G_0 \) are smoothly extendable about each point \( y_0 \in \partial \Delta \). We shall establish this by a straightforward argument using Taylor’s Theorem.

Suppose that \( y_0 \) belongs to the interior of a \( k \)-dimensional face \( F \) of \( \Delta \). Let us choose a vertex of \( F \). Since \( \Delta \) is a rational Delzant polytope, the affine functions \( L_i(y) = \langle u_i, y \rangle + \lambda_i \) which vanish at this vertex form a coordinate system on \( \Delta \). By reordering the inward normals \( u_1, \ldots, u_m \), we can suppose that these coordinate functions are \( L_1(y), \ldots, L_m(y) \) (so \( u_1, \ldots, u_m \) form a basis for \( t_F \)) and that \( L_1(y), \ldots, L_{m-k}(y) \) vanish on \( F \) (so \( u_1, \ldots, u_{m-k} \) span \( t_F \)). We set \( y_i = L_i(y) - L_i(y_0) \) for \( i = 1, \ldots, m \). These functions also form a coordinate system on \( \Delta \), with \( y_0 \) corresponding to the origin, and \( y_1, \ldots, y_{m-k} \) vanish on \( F \).

We now let \( H_{ij}(y) = H_y(u_i, u_j) \) and let \( (G_{ij}(y)) \) be the inverse matrix to \( (H_{ij}(y)) \) (which is the matrix of \( G \) with respect to the dual basis). Similarly we define inverse matrices \( (H_{ij}^0(y)) \) and \( (G_{ij}^0(y)) \). The conditions (i)–(iii) imply:

- \( H_{ij}(y) \) are smooth functions on \( \Delta \);
- on any codimension one face \( F_i \) containing \( F \) (with inward normal \( u_i \), \( i = 1, \ldots, m - k \)), we have
  \begin{equation}
  H_{ij}(y) = H_{ji}(y) = 0 \quad \text{for all } j = 1, \ldots, m \quad \text{and } \partial H_{ii}/\partial y_i = 2.
  \end{equation}
- the matrix \( (H_{ij}(y))_{i,j=m-k+1}^m \) is positive definite on the interior of \( F \);

We conclude from (16) that for \( i = 1, \ldots, m - k \), \( H_{ij}(y) = H_{ji}(y) = 0(y_i) \) (for all \( j = 1, \ldots, m \)) and \( H_{ii}(y) = 2y_i(1 + O(y_i)) \), where \( O(y_i) \) denotes the product of \( y_i \) with a smooth function of \( y \).

Putting these conditions together, we then have:

\[
H_{ij}(y) = 2y_i \delta_{ij} + y_i y_j F_{ij}(y) \quad \text{for } i, j = 1, \ldots, m - k
\]
\[
H_{ij}(y) = y_i F_{ij}(y) \quad \text{for } i = 1, \ldots, m - k \quad \text{and } j = m - k + 1, \ldots, m,
\]

where \( F_{ij} \) are smooth functions. (Recall also that \( H_{ij} = H_{ji} \).

It follows that \( \det(\lambda(y)) = 2^{m-k} y_1 y_2 \cdots y_{m-k} P(y) \) where the function \( P(y) = \det(\lambda(y))_{i,j=m-k+1}^m + O(y_1) + O(y_2) + \cdots O(y_{m-k}) \) is positive at the origin. Since the same holds for \( H_{ij}^0(y) \) it follows that \( \det(\lambda(y)) \det(\lambda_{ij}^0(y)) \) can be extended to the origin as a smooth and positive function.

On the other hand \( G_{pq}(y) \) is the determinant of a cofactor matrix of \( (H_{ij}(y)) \) divided by \( \det(\lambda(y)) \). This will be smooth if the determinant of the cofactor is \( O(y_i) \) for each \( i = 1, \ldots, m - k \). We see that this is true unless \( 1 \leq p = q \leq m - k \), in which case we obtain \( G_{pq}(y) = (1 + O(y_p))/2y_p \). The same holds for \( G_{pq}^0(y) \).

We deduce that \( G - G_0 \) is smooth at \( y_0 \), and hence \( H_0G \) is smooth at \( y_0 \). Since it is nondegenerate there, its inverse \( HG_0 \) is also smooth. \( \square \)

Remark 5. By continuity, it suffices that the boundary conditions (15) hold on the interior of the codimension one faces. However they and their tangential derivatives imply that for a point \( y \) on any face \( F \subset \Delta \), we have

\begin{equation}
H_y(u_j, \cdot) = 0 \quad \text{and} \quad (dH)_y(u_j, u_k) = 2\delta_{jk} u_j
\end{equation}

for any inward normals \( u_j, u_k \) in \( t_F \).

The proof also shows that our first order conditions are equivalent to (11)–(12) with \( G_0 \) given by (10), thus establishing the validity of (2)–see Remark (4)(ii).
1.5. **Toric complex manifolds and orbifolds.** We now turn briefly to the complex point of view on toric Kähler manifolds and orbifolds. Given a rational Delzant polytope $(\Delta, \Lambda, u_1, \ldots, u_n)$, we obtain a complex subgroup $G^c$ of $(\mathbb{C}^\times)^n$ as the complexification of $G$. The relation between complex quotients and symplectic quotients then shows that the canonical complex structure on the toric symplectic orbifold $(M, \omega)$ constructed from $\Delta$ is equivariantly biholomorphic to the quotient by $G^c$ of a dense open subset $\mathbb{C}^n_s$ of $\mathbb{C}^n$ given by

$$C^c_n = \bigcup_f C^c_{\mathfrak{p}}, \quad C^c_{\mathfrak{p}} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_j = 0 \text{ iff } L_j(x) = 0 \text{ for } x \in F^0 \}.$$  

Thus $C^c_n$ is $\mathbb{C}^n$ with the coordinate subspaces removed that do not correspond to faces of $\Delta$. Observe that the complex quotient only depends on the inward normals (which determine $G^c$) and the combinatorics of the faces (which determine $C^c_n$), i.e., by specifying which sets of codimension one faces have nonempty intersection. These data can be encoded in a family of convex simplicial cones called a *fan*.

Furthermore, *any* $\mathbb{T}$-invariant $\omega$-compatible complex structure on $M$ is equivariantly biholomorphic to the standard one (see [29] for the result in the general orbifold case). Of course this biholomorphism does not preserve $\omega$ in general. Thus two toric Kähler manifolds (or orbifolds) are equivariantly biholomorphic if and only if they have the same fan.

1.6. **Restricted toric manifolds.** Toric Kähler manifolds can be used to provide examples of Kähler manifolds with non-toric isometric hamiltonian torus actions simply by restricting the action to a subtorus. These torus actions can be surprisingly complicated in general. However, the subtori generated by a subset of the normals to the Delzant polytope have much simpler actions.

**Example.** 1. We can illustrate this in the simplest nontrivial case of $S^1$ actions on $\mathbb{C} P^2$, which is toric under the action of $\mathbb{C} P^2$, which is toric under the action of $\mathbb{T} \cong S^1 \times S^1$ given by $(\lambda_1, \lambda_2) : [z_0, z_1, z_2] \mapsto [\lambda_1 z_0, \lambda_1 z_1, \lambda_2 z_2]$. The ‘tame’ $S^1$ subgroups generated by the normals are given by $\lambda_1 = 0, \lambda_2 = 0$ or $\lambda_1 = \lambda_2$. The momentum map of the $S^1$ action is then the projection of the momentum map of $\mathbb{T}$ along the corresponding face of the Delzant polytope $\Delta$ (which is a simplex). The momentum map of ‘wild’ $S^1$ subgroups, such as $\lambda_1 = \lambda_2^2, \lambda_2 = \lambda_3^3$, are given by more general projections. We wish to draw attention to two distinctions between these two types of $S^1$ action.

(i) For tame actions, the momentum map of the $S^1$ action has no critical values on the interior of the momentum interval, whereas for wild actions it does.

(ii) For tame actions, the orbits of the complexified action (of $\mathbb{C}^\times$) have smooth closures, whereas for wild actions, they do not—for instance they are singular cubics for the case $\lambda_1 = \lambda_2^2, \lambda_2 = \lambda_3^3$.

The blow up of $\mathbb{C} P^2$ at a point is the first Hirzebruch surface $F_1 = P(O \oplus O(1)) \to \mathbb{C} P^1$. If this point is one of the three fixed points of the $\mathbb{T}$-action corresponding to a vertex of $\Delta$, then the standard fibrewise $S^1$ action on $F_1$ descends to the tame $S^1$ action on $\mathbb{C} P^2$ corresponding to the opposite edge. Thus a tame $S^1$ action realises $\mathbb{C} P^2$ as the blowdown of a toric bundle (of projective lines) over $\mathbb{C} P^1$.

We generalize this by considering torus actions on blowdowns of toric bundles (with fibre any toric Kähler manifold) over a product of complex projective spaces.

Let $\mathcal{V}$ be a toric Kähler $2\ell$-manifold, under a torus $\mathbb{T}$, with Delzant polytope $\Delta$. By the Delzant construction $\mathcal{V}$ is $(\mathbb{T}$-equivariantly symplectomorphic to) a symplectic quotient of $\mathbb{C}^n$ by an $n - \ell$ dimensional subgroup $G$ of the standard $n$-torus $\mathbb{T}^n$ (with $\mathbb{T} = \mathbb{T}^n/G$). From [15] $\mathcal{V}$ is also $(\mathbb{T}^c$-equivariantly biholomorphic to) the holomorphic quotient $\mathbb{C}^n_s/G^c$, where $\mathbb{C}^n_s$ is the set of stable points in $\mathbb{C}^n$. 
Given integers $d_1, \ldots, d_n \geq 0$, there are now two constructions we can make.

(i) Let $\mathbb{C}^D = \mathbb{C}^{d_1+1} \times \mathbb{C}^{d_2+1} \times \cdots \times \mathbb{C}^{d_n+1}$. Then we have a block diagonal action of $\mathbb{T}^n$ on $\mathbb{C}^D$ as a subtorus of the standard torus $\mathbb{T}^N$: the $i$th circle in $\mathbb{T}^n$ acts by scalar multiplication on $\mathbb{C}^{d_i+1}$ and trivially on the other factors. Since $G$ is a subtorus of $\mathbb{T}^n$, we can form the symplectic quotient of $\mathbb{C}^D$ by $G$ and this will be diffeomorphic to the stable quotient by $G^c$. Let us denote the corresponding manifold by $M$.

The standard Kähler structure on $\mathbb{C}^D$ can be written in block diagonal momentum coordinates $(x_1, \ldots, x_n)$ of $\mathbb{T}^n$ as

\[
\tilde{g}_0 = \sum_{j=1}^n \left( x_j g_j + \frac{dx_j^2}{2x_j} + 2x_j \theta_j^2 \right)
\]

\[
\tilde{\omega} = \sum_{j=1}^n (x_j \omega_j + dx_j \wedge \theta_j), \quad d\theta_j = \omega_j
\]

where $x_j = r_j^2/2$, for the radial coordinate $r_j$ on $\mathbb{C}^{d_j+1}$, and $g_j$ is the Fubini–Study metric on $\mathbb{C}^{d_j}$, normalized so that $\theta_j$ is the connection 1-form of the Hopf fibration and $\frac{1}{2}g_j + \theta_j^2$ is the round metric on the unit sphere $S^{2d_j+1}$: we obtain the flat metric in spherical polar coordinates on each $\mathbb{C}^{d_j+1}$ factor by substituting $x_j = r_j^2/2$.

This induces a Kähler structure on $M$ by writing the momentum coordinates $x_j = L_j(\mu)$ of $\mathbb{T}^n$ in terms of the momentum map $\mu$ of $\mathbb{T}$, where $L_1, \ldots, L_n$ are the affine functions defining $\Delta$. The resulting Kähler metric, in the notation of (1.3) is

\[
g'_0 = \sum_{j=1}^n L_j(\mu)g_j + \langle d\mu, G_0, d\mu \rangle + \langle \theta, H_0, \theta \rangle,
\]

\[
\omega' = \sum_{j=1}^n L_j(\mu)\omega_j + \langle d\mu \wedge \theta \rangle, \quad d\theta = \sum_{j=1}^n \omega_j \otimes u_j
\]
with \( \mathbf{G}_0 \) given by (19) and \( \mathbf{H}_0 \) is inverse to \( \mathbf{G}_0 \). This reduces to the canonical toric Kähler structure on \( \mathcal{V} \) when \( d_j = 0 \) for all \( j \).

Our aim is to show that there is a compatible Kähler structure on \( M \) generalizing the given toric Kähler structure of \( \mathcal{V} \), which is determined by an arbitrary matrix \( \mathbf{H} \) satisfying the necessary and sufficient compactification conditions of Proposition 1. To do this, and to understand better the holomorphic geometry of \( M \), we consider another construction.

(ii) Let \( \tilde{M} = \bigoplus_{j=1}^n \mathcal{O}(-1) \to \prod_{j=1}^n \mathbb{C}P^{d_j} \), where \( \mathcal{O}(-1) = \mathcal{O}(0, \ldots, 0, -1, 0, \ldots) \) is the line bundle which is \( \mathcal{O}(-1) \) over \( \mathbb{C}P^{d_j} \) and trivial over the other factors. Let \( \tilde{M}^0 = \prod_{j=1}^n \mathcal{O}(-1)^{c_j} \) be the associated holomorphic principal \( (\mathbb{C}^\times)^n \)-bundle (given by removing the zero section from each line bundle). Now \( \tilde{M} = \tilde{M}_0 \times_{(\mathbb{C}^\times)^n} \mathbb{C}^n \) admits a fibre-preserving holomorphic action of \( (\mathbb{C}^\times)^n \).

Since \( G \) is a subtorus of \( \mathbb{T}^n \) we can form the holomorphic stable quotient of \( \tilde{M} \) by \( G^c \) to obtain a complex manifold \( \hat{M} \). We see immediately that \( \hat{M} = M^0 \times_{\mathbb{T}^n} \mathcal{V} \) where \( M^0 = \tilde{M}^0 / G^c \). Thus \( \hat{M} \) is a bundle of toric complex manifolds.

It is easy to see how constructions (i) and (ii) are related, since \( \mathcal{O}(-1) \to \mathbb{C}P^{d_j} \) is the blow-up of \( \mathbb{C}^{d_j+1} \) at the origin, so that \( \tilde{M} \) is (equivariantly biholomorphic to) the blow-up of \( \mathbb{C}^D = \bigcup_j \mathbb{C}^{d_j+1} \) along the union over \( j \) of the coordinate subspaces with zero in the \( j \)th factor. The stable quotients of \( \tilde{M} \) and \( \mathbb{C}^N \) that we consider are related by this blow-up (by construction), and so \( \hat{M} \) is (equivariantly biholomorphic to) a blow-up of \( M \).

The Kähler structure (19) on \( M \) therefore pulls back to give a Kähler structure on \( \hat{M} \), except that the metric and symplectic form degenerate on the exceptional divisor. Again by construction, this induces the canonical toric Kähler structure of \( \mathcal{V} \) on each fibre of \( \hat{M} \).

Let \( \mathbf{G} = \mathbf{H}^{-1} \) be the matrices inducing the given toric Kähler structure on \( \mathcal{V} \). Then we obtain a new Kähler structure on \( \hat{M} \), degenerating on the exceptional divisor and inducing the given toric Kähler structure on each fibre:

\[
g' = \sum_{j=1}^n L_j(\mu) g_j + \langle d\mu, \mathbf{G}, d\mu \rangle + \langle \theta, \mathbf{H}, \theta \rangle,
\]

(20)

\[
\omega' = \sum_{j=1}^n L_j(\mu) \omega_j + \langle d\mu \wedge \theta \rangle, \quad d\theta = \sum_{j=1}^n \omega_j \otimes u_j.
\]

There is no reason a priori why this should descend to \( M \) (in particular, the complex structure is different). Nevertheless, it does, because of the strong control over the boundary behaviour of \( \mathbf{H} \) given by Proposition 1.

**Proposition 2.** The degenerate Kähler structure (20) on \( \hat{M} \) descends to give a (nondegenerate) Kähler structure on \( M \).

**Proof.** We know that (19) is globally defined smooth Kähler structure on \( M \). We shall show that (20) defines a compatible Kähler metric on the same symplectic manifold (with the same angular coordinates). For this, it suffices to show that the difference \( g' - g_0' \) is smooth on \( M \). However, since the compatible Kähler metrics defined on \( \mathcal{V} \) by \( \mathbf{H} \) and \( \mathbf{H}_0 \) are smooth, Proposition 1 and Remark 5 show that \( \mathbf{G} - \mathbf{G}_0 \) and \( \mathbf{G}_0 \mathbf{H} \mathbf{G}_0 - \mathbf{G}_0 \) are smooth functions on the Delzant polytope \( \Delta \) of \( \mathcal{V} \). Now the momentum map \( \mu \) on \( (M, \omega') \) is smooth, with image \( \Delta \). It therefore follows, as in the proof of Lemma 2, that \( g' - g_0' \) is smooth. \( \square \)
2. Rigid Hamiltonian Torus Actions

In this section, we introduce the notion of a rigid Hamiltonian torus action. Toric Kähler manifolds automatically carry such an action: our goal is to extend some of the rigid properties of toric Kähler manifolds to rigid torus actions in general, and to classify them. In the first three subsections we study respectively the differential topology, symplectic geometry and biholomorphic type of compact (smooth) Kähler manifolds with such an action, then we combine these threads to describe the Kähler geometry. In the final subsection, we specialize to the case that the torus action is ‘semisimple’ and give a generalized Calabi construction of all compact Kähler manifolds with a semisimple rigid torus action.

2.1. Stratification of the Momentum Polytope. Before defining the torus actions we will consider, we establish a couple of basic facts. We shall make essential use the convexity theorem of Atiyah and Guillemin–Sternberg [5, 19].

Lemma 4. Let $T$ be a torus in the group of Hamiltonian isometries of a compact connected Kähler manifold $(M,g,J,\omega)$, which is the closure of the group generated by $\ell$ Hamiltonian Killing vector fields $K_r = J\nabla g \sigma_r$ ($r = 1, \ldots, \ell$) that are independent on a dense open set. Suppose that $g(K_r,K_s)$ depends only on $(\sigma_1, \ldots, \sigma_\ell)$ for $r,s = 1, \ldots, \ell$. Then:

(i) the torus $T$ has dimension $\ell$;

(ii) the image of the momentum map $\mu: M \to t^*$ of $T$ is a compact convex polytope such that $\mu$ is regular (i.e., submersive) as a map to the interior of any of its faces.

Proof. By the Atiyah–Guillemin–Sternberg convexity theorem [5, 19], the image of $\mu$ is a compact convex polytope $\Delta$ in $t^*$, the convex hull of the finite image $I$ of the fixed point set of $T$. The momentum coordinates $\sigma = (\sigma_1, \ldots, \sigma_\ell)$ are related to $\mu$ by the natural inclusion

$$\mathbb{R}^\ell \cong \text{span}(K_1, \ldots, K_\ell) \subseteq t,$$

which in turn gives rise to a linear projection $\pi: t^* \to R^\ell$ such that $\sigma = \pi \circ \mu$.

Let us first consider the image of $\Delta$ by $\pi$. We claim that $\pi$ is injective on $I$. Indeed, since $K_1, \ldots, K_\ell$ generate $T$, the fixed point set is precisely the set of common zeros of $K_1, \ldots, K_\ell$, and since $g(K_r,K_s)$ depends only on $\sigma$, the preimage of an element of $\pi(\Delta)$, containing an element of $I$, consists entirely of elements of $I$. Now $I$ is finite and the preimages of $\pi$ are convex, so each such preimage has just one point.

Second, we note that the set of regular values of $\sigma$ is connected. Indeed, the critical point set of $\sigma$ in $M$ has codimension at least two—it is the set where the holomorphic $\ell$-vector $K_1^{1,0} \wedge \cdots \wedge K_\ell^{1,0}$ vanishes—so the set of regular points $U$ is connected. Now as $g(K_r,K_s)$ depends only on $\sigma$, the inverse image of a critical value consists entirely of elements of $I$. Now $I$ is finite and the preimages of $\pi$ are convex, so each such preimage has just one point.

Third, consider the orbits of the commuting vector fields $JK_1, \ldots, J K_\ell$—this is the gradient flow of $\sigma$, and so the orbit of any regular point consists entirely of regular points and its boundary points are all critical. Now regular points map to regular values and critical points map to critical values, so by the connectivity of the regular values, all regular orbits have the same image—and the closure is the image of $\sigma$ since regular values are dense.

These facts implies the conclusions of the lemma as follows.

(i) Suppose $x$ is a regular point of $\sigma$ and $\mu(x)$ belongs to a closed face $F$ of $\Delta$. Then the $T^c$ orbit of $x$ also maps to $F$, where $T^c$ is the complexification of $T$. Since
the orbit under $JK_1, \ldots, JK_\ell$ is contained in the $T^c$ orbit, $\pi$ maps $F$ onto $\text{Im} \, \sigma$. Now $\pi$ is bijective on vertices, so $F = \Delta$. In other words the inverse image (under $\pi$) of a regular value of $\sigma$ meets no proper face of $\Delta$: this clearly implies $\pi$ is bijective, hence $\mu = \sigma$ and $\dim \mathbb{T} = \ell$.

(ii) We have seen that the image of the closure $C$ of any regular $T^c$ orbit is the whole of $\Delta$. Atiyah [5] shows that the inverse image in $C$ of any open face $F^0$ is a single $T^c$-orbit and $\mu$ is a submersion from this orbit to $F^0$. Since this is true for all regular orbits, and the union of the regular orbits is dense, the claim follows. □

**Definition 3.** Let $(M, g, J, \omega)$ be a connected Kähler $2m$-manifold with an effective isometric hamiltonian action of an $\ell$-torus $T$ with momentum map $\mu: M \to \mathfrak{t}^*$. We say the action is rigid iff for all $x \in M$, $R^*_x g$ depends only on $\mu(x)$, where $R_x: T \to T \cdot x \subset M$ is the orbit map.

In other words, for any two generators $X_\xi, X_\eta$ of the action $(\xi, \eta \in \mathfrak{t})$, $g(X_\xi, X_\eta)$ is constant on the levels of the momentum map $\mu$. We remark that the inverse image of a critical value of $\mu$ can be approximated (to first order on a dense open subset) by inverse images of nearby regular values. Hence it suffices to know that the generators have constant inner products on the generic level sets of $\mu$. Thus part (i) of Lemma 4 implies that on a compact manifold, a *local* rigid torus action (as in [4]) is necessarily a global one. In particular, on a compact Kähler manifold with a hamiltonian 2-form of order $\ell$, the associated Killing vector fields $K_1, \ldots, K_\ell$ generate a rigid $\ell$-torus action. Another example is any toric Kähler manifold.

Part (ii) of Lemma 4 has further consequences for compact Kähler manifolds with a rigid torus action.

**Proposition 3.** Suppose $(M, g, J, \omega)$ is a compact connected Kähler manifold of dimension $2m$, with a rigid hamiltonian $\ell$-torus action with momentum map $\mu$ whose image is a compact convex polytope $\Delta$.

(i) If $F$ is a $k$-dimensional closed face $(0 \leq k \leq \ell)$ of $\Delta$, then $M_F := \mu^{-1}(F)$ is a compact totally geodesic Kähler submanifold of $M$ of dimension $2(m_F + k)$ $(0 \leq m_F \leq m - \ell)$ with a rigid hamiltonian action of a $k$-torus $T/T_F$, where $T_F$ is the intersection of the isotropy subgroups of points in $M_F$.

(ii) If $F^0$ is the interior of $F$, then $M^0_F := \mu^{-1}(F^0) \cong F^0 \times P_F$ where $P_F$ is a compact manifold of dimension $2m_F + k$ with a locally free action of $T/T_F$. Moreover, the levels of $\mu$ are compact connected submanifolds of $M$.

**Proof.** (i) Let $T_F$ be the intersection of the isotropy subgroups of points in $M_F$. Then the connected component of the identity in $T_F$ is an $(\ell - k)$-dimensional subtorus of $T$, and $M_F$ is a connected component of its fixed point set. Since $T_F$ acts on $M$ effectively by hamiltonian isometries, $M_F$ is a compact totally geodesic Kähler submanifold of $M$, of dimension at most $2m - 2(\ell - k)$. By definition, $M_F$ carries an effective hamiltonian action of $T/T_F$ (which is connected, hence a $k$-torus), so it has dimension at least $2k$. The momentum map is essentially $\mu$, viewed as a map from $M_F$ to the affine span of $F$, so the action is rigid.

(ii) By Lemma 4, the critical values of $\mu$, regarded in the above way, are precisely the boundary points of $F$, and $\mu$ is regular as a map from $M^0_F$ to $F^0$. The gradient flow of $\mu$ commutes with $T$ and hence provides an equivariant trivialization of $M^0_F$. Thus $M^0_F$ is diffeomorphic to $F^0 \times P_F$ and the action of $T/T_F$ is given by an effective locally free action on $P_F$, with trivial action on $F^0$. The levels of $\mu$ are smooth since any point in the image of $\mu$ is in some open face; they are connected by [5]. □
This shows that ‘wild’ $S^1$ actions on $\mathbb{CP}^2$ (as a symplectic manifold) of Example 1 cannot be rigid with respect to any compatible Kähler metric. One can easily check that the ‘tame’ actions are rigid with respect to the Fubini–Study metric.

2.2. The symplectic isotropy representations. We now wish to obtain precise information about the symplectic isotropy representations of the torus action. If $\mu(x)$ belongs to an open $k$-dimensional face $F^0$, then the Lie algebra $t_x$ of the isotropy group $T_x \geq T_F$ of $x$ is the vector subspace of elements of $t$, annihilated by the elements of the vector subspace of $t^*$ parallel to $F$; indeed this is clearly the image of $d\mu_x$, and $t_x$ is the kernel of the transpose of $d\mu_x$.

Since the orbit $T \cdot x$ is $k$-dimensional, the symplectic isotropy representation $V_x = T_x(T \cdot x)^0/T_x(T \cdot x)$ of $T_x$ (and its Lie algebra $t_x$) has dimension $m - k$. Hence it is an orthogonal direct sum of $m - k$ complex 1-dimensional representations with (not necessarily distinct) characters $T_x \rightarrow S^1$. Differentiating this action gives the weights $\alpha_1, \ldots, \alpha_{m-k}$ of the action of $t_x$, which are integral elements of $t_x^*$. Since the $t_x$ action is effective, the weights $\alpha_1, \ldots, \alpha_{m-k}$ span $t_x^*$, and we order them so that $\alpha_1, \ldots, \alpha_{\ell-k}$ form a basis for $t_x^*$.

**Lemma 5.** Suppose $\mu(x)$ belongs to an open $k$-dimensional face $F^0$ of $\Delta$ and let $V_x$ be the symplectic isotropy representation of $T_x$ at $x \in M$.

(i) The induced $t_x$ action has exactly $\ell - k$ distinct nonzero weights.

(ii) $T_x$ is connected.

**Proof.** (i) We choose a projection $\chi: t \rightarrow t_x$ and introduce a symplectic slice as in Lemma 1. Thus there is a $T$-equivariant symplectomorphism from a neighborhood $U$ of the zero section $0_N$ in the normal bundle $N \rightarrow T \cdot x$ to a neighborhood of $T \cdot x$ in $(M, \omega)$, where the normal bundle $N = T \times T_x (t_x^0 \oplus V_x) \rightarrow T \cdot x$ is realised as a symplectic quotient of $T^*T \times \Sigma^*$ by the diagonal action of $T_x$. The symplectomorphism identifies $0_N$ with $T \cdot x$ and its differential along the zero section is essentially the identity map. Let us denote the pullback of $(g, J, \omega)$ by $(g_0, J_0, \omega_0)$. We then have that $g_0$ agrees with $g_x$ at $x$.

We now bring in the rigidity condition that the induced metric on $T$ depends only on $\mu$. This implies that for any vector fields $X_\xi, X_\eta$ ($\xi, \eta \in t$) induced by the action of $T_x$ on $(U, g_0, J_0, \omega_0, g_0(X_\xi, X_\eta)$, as a function on $U$, depends only on the momentum map $\mu_0$ of $N$, $\mu_0(\alpha, v) = \alpha + \mu v$ with $\mu v = \frac{1}{2} \sum_{i=1}^{m-k} |z_i|^2 \alpha_i$, where $z_1, \ldots, z_{m-k}$ are the standard complex coordinates on the weight spaces in $V_x$. It follows from [32] that $(\text{being smooth on } U)$ $g_0(X_\xi, X_\eta)$ is a smooth function of $\mu_0$. In particular, for $\alpha = 0 \in t_x^0$, $g_0(X_\xi, X_\eta)$ is a smooth function of $\mu$.

Thus on $V_x \cap U$, $d(g_0(X_\xi, X_\eta))$ is a pointwise linear combination of the components of

\[
(22) \quad d(g_0(X_\xi, X_\eta)) = \sum_{i=1}^{m-k} (z_i \bar{z}_i + \bar{z}_i z_i) \alpha_i(\xi) \alpha_i(\eta).
\]
If we differentiate $d(g_0(X_\xi, X_\eta)) = \langle d\mu_v, B(\xi, \eta) \rangle$ with respect to $\pi_t$, using (21) and (22), and evaluate at the origin of $V_\xi$, the error terms and derivative of $B$ go away. Equating coefficients of $dz_1, \ldots, dz_{m-k}$ therefore gives

$$2\alpha_i(\xi)\alpha_i(\eta) = \alpha_i(B_0(\xi, \eta))$$

for all $i$, i.e., $B_0^0\alpha_i = 2\alpha_i \otimes \alpha_i$. (We remark that this generalizes the conditions (17) in the toric case.) Now $\alpha_1, \ldots, \alpha_{\ell-k}$ is a basis for $t^*_F$, so we may write $\alpha_{\ell-k+1}, \ldots , \alpha_{m-k}$ as $\alpha_i = \sum_{j=1}^{\ell-k} \lambda_{ij}\alpha_j$. We then deduce from $B_0^0\alpha_i = 2\alpha_i \otimes \alpha_i$ that

$$\lambda_{ij}\lambda_{ik} = \delta_{jk}\lambda_{ij}.$$  

Thus for each $i$, $\lambda_{ij}$ is nonzero for at most one $j$, and then equal to one, i.e., for any $i = \ell - k + 1, \ldots, m - k$, the weight $\alpha_i$ is either zero, or it is one of $\alpha_1, \ldots, \alpha_{\ell-k}$.

(ii) We prove that all isotropy groups of the $T$-action are connected. Since the gradient flow of $\mu$ commutes with $\mathbb{T}$, it suffices to prove this near a fixed point $y$ of the $T$-action, where the symplectic slice gives a $T$-equivariant symplectomorphism with a neighbourhood of the origin in a symplectic vector space $V_y$. Now since the $T$-action on $V_y$ is effective, with $\ell$-distinct nonzero weights, these form a basis for the dual lattice. This ensures the isotropy groups of points in $V_y$ are connected. □

Part (i) of Lemma 5 is the key to the theory of rigid hamiltonian torus actions. In particular it allows us to refine Proposition 3.

**Proposition 4.** Suppose $(M, g, J, \omega)$ is a compact connected Kähler manifold with a rigid hamiltonian $\ell$-torus action, as in Proposition 3.

(i) If $F^0$ is an open $k$-dimensional face, then the isotropy group of all points in $M^0_F$ is an $(\ell - k)$-torus $T_F$, and the isotropy representations are all equivalent, with the distinct nonzero weights in $t^*_F$ forming a basis for the lattice dual to the lattice of circle subgroups of $t_F$.

(ii) The image $\Delta$ of $\mu$ is a Delzant polytope.

(iii) $P_F$ is a principal $k$-torus bundle (under $\mathbb{T}/T_F$) over a compact manifold $S_F$ of dimension $2m_F$, with a family of Kähler structures parameterized by $F^0$.

**Proof.** (i) This is immediate from Lemma 5: the distinct nonzero weights form a basis for $t_F$, the Lie algebra of the (connected) isotropy group of any point in $M^0_F$.

(ii) Applying this to a fixed point, observe that the directions of the distinct nonzero weights are the edges meeting the corresponding vertex of $\Delta$. There are $\ell$ of these and the dual basis gives a basis for the lattice of circle subgroups of $T$ consisting of normals to the faces meeting the vertex.

(iii) By Proposition 3, $P_F$ has a locally free action of $\mathbb{T}/T_F$, and by Lemma 5 the isotropy groups are connected, so the action is free. Hence $P_F$ is a principal $\mathbb{T}/T_F$ bundle over a compact manifold $S_F$. Choosing a point $v$ in $F^0$ identifies $S_F$ with the Kähler quotient of $M_F$ at momentum level $v$. □

### 2.3. The complexified torus action

We now turn to the structure of the orbits of the complexified torus action. If the $T$ action is generated by vector fields $K_1, \ldots, K_\ell$, then the complexified action of $\mathbb{T}^c$ is generated by the (real) holomorphic vector fields $K_1, \ldots, K_\ell, JK_1, \ldots, JK_\ell$. These are linearly independent on a dense open set (since the $T$ action is hamiltonian) and generate a foliation of $M$ by complex orbits, whose generic leaf is $2\ell$-dimensional. As we have already remarked in (2.1) $JK_1, \ldots, JK_\ell$ generate the gradient flow of $\mu$, and therefore the momentum image of a $2k$-dimensional leaf is a $k$-dimensional open face $F^0$ of $\Delta$; the isotropy
group of any point in this leaf is the complexification $T^c_F$ of $T_F$ and the closure (in $M$) of the leaf maps onto the closed face $F$.

To understand the complex orbits further, we reinterpret $V_x$ as the fibre of the normal bundle to $T^c\cdot x$ at $x$, carrying the complex isotropy representation, and we linearize the $T^c\cdot x$ action using a holomorphic slice rather than a symplectic one.

In general, let $G$ be a compact Lie group of hamiltonian isometries of a Kähler manifold $M$, and let $G^c$ be the complexification, which acts holomorphically on $M$. Then the holomorphic slice theorem [20, 31] states that if $G^c\cdot x$ is the orbit through $x \in M$ with isotropy representation $(G_{\mathfrak{x}}, V_x)$, then there is a $G^c$-equivariant biholomorphism from a neighbourhood of $G^c\cdot x$ in $M$ to a neighbourhood of the zero section in $G^c \times_{G_{\mathfrak{x}}} V_x \to G^c\cdot x$.

Remark 6. For many purposes, it suffices to know that a neighbourhood of $x$ is locally $G^c$-equivariantly biholomorphic to a neighbourhood of the zero section in $G^c \times_{G_{\mathfrak{x}}} V_x$. This is quite easy to establish. Indeed, let $\psi : U \to M$ be a holomorphic chart with $\psi(0) = x$ and $d\psi_0 = Id$, where $U$ is an open neighbourhood of the origin in $T_x M$. We can assume $U$ and $\psi$ are $G_x$-equivariant by averaging, since $G_x$ is compact. Now by acting with $G$, we obtain a $G$-equivariant biholomorphism $\psi$ from a neighbourhood $\tilde{U}$ of $G\cdot x$ in $M$ to a neighbourhood of the zero section in $G \times_{G_{\mathfrak{x}}} \tilde{V}_x \to G\cdot x$. Here $\tilde{V}_x$ is the orthogonal complement of $T_x(G\cdot x)$: note $\tilde{V}_x = V_x \oplus W_x$ where $V_x$ is the orthogonal complement of $T_x(G^c\cdot x)$, and $W_x = JT_x(G\cdot x)$.

Now since $\psi$ is holomorphic and $G$-equivariant, it is (locally) $G^c$-equivariant. This is only a local result, because the domain $\tilde{U}$ is a priori only $G$-invariant, not $G^c$-invariant. The hard part of the holomorphic slice theorem is to show such a ‘local’ slice can be analytically continued to a $G^c$-invariant neighbourhood of $G^c\cdot x$.

Lemma 6. Suppose $\mu(x)$ belongs to an open $k$-dimensional face $F^0$ of $\Delta$ and let $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_{\ell-k}$ be the complexifications of the circle subgroups of the isotropy subgroup $T_F$ dual to the basis of distinct nonzero weights in the symplectic isotropy representation of $T_F$.

Then $T^c_F = \mathbb{T}_1^c \times \cdots \times \mathbb{T}_{\ell-k}^c$ and there is a $T^c\cdot x$ equivariant biholomorphism from a neighbourhood $U$ of $T^c\cdot x$ in $M$ to a neighbourhood $W$ of the zero section in $T^c \times_{T^c_F} (V_0 \oplus V_1 \oplus \cdots \oplus V_{\ell-k}) \to T^c / T^c_F$ where $V_0$ is the trivial representation (possibly zero), while for $i = 1, \ldots, \ell - k, V_i$ is a nonzero vector space carrying the standard action of $\mathbb{T}_i^c \cong \mathbb{C}^\times$ by scalar multiplication, with $\mathbb{T}_j^c$ acting trivially for $j \neq i$. Under this biholomorphism:

(i) the $p$-dimensional faces $F^i$ meeting $F$ correspond bijectively to $(p-k)$-element subsets $\mathcal{J}_p \subseteq \{1, \ldots, \ell - k\}$ in such a way that

$\bullet \ M_{F^p} \cap U$ is the intersection of $W$ with those elements whose $V_j$ component vanishes for $j \in \{1, \ldots, \ell - k\} \setminus \mathcal{J}_p$;

(ii) if $Y$ is a $p$-dimensional complex orbit with $x \in \overline{Y} \subseteq M_{F^p}$, $\dim F^i = p$ then there are one dimensional subspaces of $V_j$ for $j \in \mathcal{J}_p$ such that

$$Y \cap U \cong T^c \times_{T^c_F} \bigoplus_{j \in \mathcal{J}_p} V_j$$

under the obvious inclusion into $T^c \times_{T^c_F} (V_0 \oplus \cdots \oplus V_{\ell-k})$.

Proof. By the holomorphic slice theorem there is a $V_x$-equivariant biholomorphism from a neighbourhood of $T^c\cdot x$ to neighbourhood of the zero section in $T^c \times_{T^c_F} V_x$, where $V_x$ is normal to $T^c\cdot x$ at $x$. Equivalently, $V_x$ is the symplectic isotropy representation of $T_F$, now equipped with the natural complexified action of $T^c_F$. By Lemma 5 the distinct nonzero weights of the $t_F$ action on $V_x$ are dual to a
basis for the lattice of circle subgroups of $\mathbb{T}_F$, and we take the $V_i$’s to be the weight spaces (with $V_0$ the zero weight space). This gives what we want.

(i) It is clear that the faces $F'$ containing $F$ correspond to subsets $\mathcal{J}_{F'}$ of $\{1, \ldots, \ell - k\}$ with $\mathbb{T}_j$ acting nontrivially on $M_{F'}$ for $j \in \mathcal{J}_{F'}$. The biholomorphism identifies $M_{F'} \cap U$ with those elements of $W$ whose isotropy group is contained in $\mathbb{T}_j$. Since the latter is the product of the $\mathbb{T}_j$ for $j \in \{1, \ldots, \ell - k\} \setminus \mathcal{J}_{F'}$, the result follows.

(ii) Under the biholomorphism, the complex orbits $Y$ near $\mathbb{T}^c \cdot x$ are all of the form $\mathbb{T}^c \times \mathbb{T}_j \cdot (v_0 + U_j \times \cdots \times U_{\ell-k})$, where $v_0 \in V_0$ and either $U_j = L_j := L_j \setminus \{0\}$, where $L_j$ is a one-dimensional subspace of $V_j$, or $U_j = \{0\} \subset V_j$.

If $Y$ is a $p$-dimensional orbit in $M_{F'}$, then these two cases occur accordingly as $j \in \mathcal{J}_{F'}$ or not. Clearly $x \in \overline{Y}$ if and only if $v_0 = 0$, and then the biholomorphism identifies $\overline{Y} \cap U$ with $\bigoplus_{j \in \mathcal{J}_{F'}} L_j$ as stated. \hfill \Box

Lemma\[1\] gives a lot of information about the equivariant holomorphic geometry of $M$. For instance, applying it at a fixed point gives a $\mathbb{T}^c$-equivariant chart from a neighbourhood of the fixed point to $U_0 + V_1 \oplus \cdots \oplus V_\ell$, where $V_1, \ldots, V_\ell$ are the nontrivial weight spaces associated to the corresponding vertex $v$ of $\Delta$, and $U_0$ is a neighbourhood of the origin in the trivial weight space $V_0$. In the toric case, $V_0 = 0$ and $\dim V_j = 1$ for all $j$, and we obtain the linear charts underlying the toric complex manifold. In the general case, such charts provide a finite atlas, since there are finitely many vertices $v$ and they have compact preimages $S_v = \mu^{-1}(v)$.

**Proposition 5.** Suppose $(M, g, J, \omega)$ is a compact connected Kähler manifold with a rigid hamiltonian $\ell$-torus action, as in Proposition\[2\]

(i) The closure of a $2k$-dimensional complex orbit in $M$ is a toric Kähler submanifold of $M$ whose Delzant polytope is a $k$-dimensional face $F$ of $\Delta$.

(ii) For any $k$-dimensional face $F$ of $\Delta$, $M_F^0 = F^0 \times P_F$ is a holomorphic principal $\mathbb{T}^c/\mathbb{T}_F$-bundle over a complex manifold $S_F$.

(iii) The blow-up of $M_F$ along the inverse images of the codimension one faces of $F$ is equivariantly biholomorphic to the total space of $M_F^0 \times \mathbb{T}^c/\mathbb{T}_F \cdot \mathcal{V}_F \to S_F$ for some smooth toric complex manifold $\mathcal{V}_F$.

(iv) If $F$ is a $k$-dimensional face, with the $(k-1)$-dimensional face $F'$ in its boundary, then $S_F$ is a holomorphic $\mathbb{C}P^d$-bundle over $S_{F'}$ with $d = m_F - m_{F'} \geq 0$.

Furthermore if $Q_F$ denotes the fibrewise Hopf fibration over the $\mathbb{C}P^d$-bundle $S_F \to S_{F'}$, then $P_F \to S_F$ is the pullback of $P_{F'} \to S_{F'}$ along the $S^{2d+1}$-bundle map $Q_F \to S_{F'}$ composed with the $S^1$-bundle map $Q_F \to S_F$.

**Proof.** (i) For all $x \in M$, any complex orbit has a smooth closure along $\mathbb{T}^c \cdot x$ by Lemma\[3\]. Hence the closures of the complex orbits are smoothly embedded, and become toric Kähler manifolds under the induced metric. We have already remarked that $\mu$ maps any such orbit closure to a face $F$ of $\Delta$, and clearly $\mu$, viewed as a map to the affine span of $F$ (with a choice of origin), is a momentum map for the induced toric action.

(ii) For convenience, we prove this result for $F = \Delta$: the general result follows by replacing $M$ with $M_F$ and $\mathbb{T}^c$ by $\mathbb{T}^c/\mathbb{T}_F$.

Since $\mathbb{T}^c$ acts freely on $M^0$ it defines a holomorphic fibration over $S_\Delta$. To verify that the fibration is locally trivial, observe that a neighbourhood of a $\mathbb{T}^c$ orbit in $M^0$ is equivariantly biholomorphic to a neighbourhood of the zero section in $\mathbb{T}^c \times V_0 \to \mathbb{T}^c$. The latter, being $\mathbb{T}^c$-invariant, is of the form $\mathbb{T}^c \times U_0$, and the projection to $U_0$ gives the required local trivialization. Since $\mathbb{T}^c$ acts simply transitively on the fibers, $M^0$ is a principal $\mathbb{T}^c$-bundle over $S_\Delta$. 

(iii) We again prove the result when the face is the whole polytope $\Delta$.

We first consider the blow-up $\hat{M}$ of $M$ along all $M_F$ with $F$ codimension one in $\Delta$. (Of course the blow-up is trivial if $M_F$ already has complex codimension one in $M$). Thus, $\hat{M}$ is the complex manifold obtained from $M$ by replacing each $M_F$ by its projectivized normal bundle $\hat{M}_F$; these become divisors (i.e. of complex codimension one) in $\hat{M}$, and the $T^c$ action lifts naturally to $\hat{M}$. Lemma 9 shows that the generic $T^c$ orbits for the lifted action have disjoint smooth closures in $\hat{M}$, and this gives a holomorphic fibration of $M$ whose fibres are all toric Kähler manifolds with Delzant polytope $\Delta$. In particular (forgetting the symplectic structure) they are all isomorphic toric complex manifolds \[18, 29\].

Let $\mathcal{V}_\Delta$ be a toric complex manifold in this isomorphism class, and choose a basepoint on the generic orbit $\mathcal{V}_\Delta^0$ to identify it with $T^c$. Then there is an equivariant biholomorphism $M^0 \times_{T^c} \mathcal{V}_\Delta^0 \to M^0 = \hat{M}^0$ (here $\hat{M}^0$ stands for the subset of points of $\hat{M}$ with generic $T^c$ orbits; it is the same as $M^0$ because the blow-up is the identity on the complement of the exceptional divisor). Since $\mathcal{V}_\Delta$ has the same isotropy representations as the fibres of $\hat{M}$, this extends to an equivariant biholomorphism $M^0 \times_{T^c} \mathcal{V}_\Delta \to \hat{M}$ (indeed the holomorphic slices of Lemma 6 provide the extension).

(iv) Consider, as in (iii), the blow-up $\hat{M}$ of $M$ along its codimension one faces. This is equivariantly biholomorphic to $M^0 \times_{T^c} \mathcal{V}_\Delta$ and for any face $F$, the inverse image $\hat{M}_F$ of $F$ in $\hat{M}$ is $M^0 \times_{T^c} \mathcal{V}_F$ (where only $T^c/T^c_F$ acts effectively on $\mathcal{V}_F$, which is the inverse image of $F$ in $\mathcal{V}_\Delta$).

Now $\mathcal{V}_0^0_F$ is equivariantly biholomorphic to a $T^c_F/T^c_F$ bundle over $\mathcal{V}_0^F$, namely the punctured normal bundle of $\mathcal{V}_0^F$ in $\mathcal{V}_F$, so it follows that the same is true for $\hat{M}_F^0$; it is equivariantly biholomorphic to the punctured normal bundle of $\hat{M}_F^0$, in $\hat{M}_F$. Passing to the blow-down, we deduce that $M_0^F$ is equivariantly biholomorphic to the punctured normal bundle of $M_0^F$, in $M_F$, which is a $T^c/T^c_F$-equivariant bundle with $T^c_F/T^c_F$ acting by scalar multiplication on the fibres.

The quotient by $T^c/T^c_F$ identifies $S_F$ biholomorphically with a bundle over $S_F$. To describe this bundle, we first divide the punctured normal bundle of $M_0^F$, by $T^c_F/T^c_F$ to obtain the projectivized normal bundle as a $T^c/T^c_F$-equivariant $CP^d$ bundle over $M_0^F$, with trivial action on the fibres. Now the quotient by $T^c/T^c_F$, shows that $S_F \to S_F$ is a holomorphic $CP^d$-bundle.

The unit normal bundle of $M_0^F$, is the sphere bundle induced by the Hopf fibration over the projectivized normal bundle and the result follows.

This shows that ‘wild’ $S^1$ actions on $CP^2$ (as a complex manifold) discussed in Example 1 cannot be rigid with respect to any compatible Kähler metric. On the other hand, we noted there that the complex orbits of ‘tame’ $S^1$ actions do indeed have smooth closures.

2.4. Kähler geometry of rigid hamiltonian torus actions. Given a Kähler $2m$-manifold $M$ with a rigid hamiltonian action of an $\ell$-torus $T$, we have obtained a description of the equivariant biholomorphism type of $M$, stratified by the inverse images of the faces of the momentum polytope $\Delta$: $M^0$ is a principal $T^c$-bundle over a complex manifold $S_\Delta$ of dimension $2m_\Delta$, with $m_\Delta = m - \ell$, and there is a toric complex manifold $\mathcal{V}_\Delta$ such that the blow up of $M$ along the codimension one faces of $\Delta$ is biholomorphic to $M^0 \times_{T^c} \mathcal{V}_\Delta \to S_\Delta$: mutatis mutandis, the inverse image $M_F = \mu^{-1}(F)$ of a face of $\Delta$ has the same structure; further if $F_1, \ldots, F_n$ denote the codimension one faces of $\Delta$, then $S_{F_j}$ has dimension $2m_{F_j} \leq 2m_\Delta$ and $S_\Delta$ is a $CP^{d_j}$-bundle over $S_{F_j}$ with $d_j = m_\Delta - m_{F_j}$, and we say a blow-down occurs over $F_j$.
if \( d_j > 0 \). (We remark that if \( F' \) is a codimension one face of \( F \), it must be \( F \cap F' \) for some codimension one face \( F_j \) of \( \Delta \). We then have \( m_F - m_{F'} = d_j = m_\Delta - m_{F_j} \).

It remains to describe the Kähler structure of \( M \) in terms of this equivariant biholomorphism type. To do that we first recall some equivalent formulations of the rigidity condition established (locally) in [4].

Suppose, generally, that \( M \) is a Kähler manifold endowed with an isometric Hamiltonian action of an \( \ell \)-torus \( \mathbb{T} \) with momentum map \( \mu \). For a contractible open subset \( U \) of the regular values of \( \mu \), the gradient flow of \( \mu \) identifies \( \mu^{-1}(U) \) with \( \mu^{-1}(v) \times U \) for any \( v \) in \( U \), and hence \( \mu^{-1}(U)/\mathbb{T} \cong S \times U \) for a complex manifold \( S \), with a family \( \omega_h \) of compatible symplectic forms on the fibres of \( S \times U \to U \). We can therefore define the derivative \( \partial_\mu \omega_h \) with respect to \( \mu \), and this will be a 2-form on \( S \) with values in \( t \). Now \( \mu^{-1}(U) \) is a principal \( \mathbb{T} \)-bundle with connection over \( S \times U \), so it has a curvature form \( \Omega \), which is also a closed 2-form with values in \( t \). If \( \partial_\mu \omega_h = \Omega \) on \( S \times U \) we say that the rigid Duistermaat–Heckman property holds (so-called because it holds in cohomology by work of Duistermaat and Heckman).

We then have the following global version of [4] Proposition 8.

**Lemma 7.** For an isometric Hamiltonian \( \mathbb{T} \)-action the following are equivalent.

(i) The action is rigid.

(ii) The \( \mathbb{T}^c \)-orbits are totally geodesic.

(iii) The orthogonal distribution to the \( \mathbb{T}^c \)-orbits is \( \mathbb{T}^c \)-invariant.

(iv) The rigid Duistermaat–Heckman property holds.

**Proof.** This is essentially the same as [4] Proposition 8]. Let \( X \) denote a vector field which is orthogonal to a \( \mathbb{T}^c \)-orbit. The rigidity condition is equivalent to the statement that \( \partial_\mu (g(K_r, K_s)) = -2g(\nabla_{K_r} K_s, X) \) vanishes along the given orbit for all such vector fields \( X \). Since \( J \) is parallel and \( K_s \) is holomorphic this is equivalent to the fact that the \( \mathbb{T}^c \)-orbit is parallel. It is easy to compute that this condition is equivalent to the fact that \( \mathcal{L}_{K_r} X \) and \( \mathcal{L}_{JK_s} X \) are orthogonal to the given \( \mathbb{T}^c \) orbit for all \( X, r = 1, \ldots, \ell \), i.e., the orthogonal distribution is \( \mathbb{T}^c \)-invariant.

(iv) is equivalent to the local rigidity of the action on \( M^0 \) by the Pedersen–Poon construction (see [30] [4]); this implies rigidity on \( M \) by continuity. \( \square \)

We next show that a compact Kähler manifold with a rigid Hamiltonian action of a torus gives rise in a natural way to the following data.

**Definition 4.** Let \( \mathcal{V} \) be a compact toric Kähler manifold under an \( \ell \)-torus \( \mathbb{T} \) with Delzant polytope \( \Delta \). Then rigid Hamiltonian data for \( \mathcal{V} \) consists of a quadruple \( (\mathcal{V}_F, S_F, P_F, \omega_F) \) for each face \( F \) of \( \Delta \), where:

(i) \( \mathcal{V}_F \) is the inverse image of \( F \) in \( \mathcal{V} \), which is a compact toric Kähler manifold under \( \mathbb{T} \times \mathbb{T}_F \), where \( \mathbb{T}_F \) is the isotropy subgroup of \( \mathbb{T} \) associated to \( F \);

(ii) \( S_F \) is a compact complex manifold which is a holomorphic projective space bundle over \( S_F \), for any codimension one face \( F' \) of \( F \);

(iii) \( \pi: P_F \to S_F \) is a principal \( \mathbb{T}/\mathbb{T}_F \)-bundle with connection \( \theta_F: TP_F \to \mathfrak{t}/\mathfrak{t}_F \), whose curvature \( \Omega_F \in C^\infty(S_F, \Lambda^{1,1}S_F \otimes \mathfrak{t}/\mathfrak{t}_F) \) pulls back to the fibres of \( S_F \to S_F' \) to give the Fubini–Study metric in \( 2\pi c_1(\mathcal{O}(1)) \) tensored with \( \omega_F' \) normal to the codimension one face \( F' \);

(iv) \( \omega_F \) is a section of (the pullback of) \( \Lambda^{1,1}S_F \) over \( S_F \times F \), which

- is positive on \( S_F \times F^0 \),
- satisfies \( d_\mu \omega_F = \Omega_F \) on \( S_F \times \{v\} \) for all \( v \in F^0 \),
- and whose restriction to \( S_F \times F' \), for any codimension one face \( F' \) of \( F \), is the pullback of \( \omega_{F'} \) along the map \( S_F \times F' \to S'_F \times F' \).
Proposition 6. Let $M$ be a compact connected Kähler $2m$-manifold with a rigid Hamiltonian action of an $r$-torus $\mathbb{T}$ and momentum map $\mu: M \to \Delta$. Then there are rigid Hamiltonian data $(\mathcal{V}_F, S_F, P_F, \omega_F)$ (for the faces $F$ of $\Delta$) associated to a toric Kähler manifold $\mathcal{V}$ with Delzant polytope $\Delta$ such that:

- the pullback of the Kähler metric on $M_F = \mu^{-1}(F)$ to the fibres of the blow-up $\tilde{M}_F \cong P_F \times_{\mathbb{T}} V_F$ (see Proposition 4) is induced by the Kähler metric on $V_F$;
- $S_F$ is the Kähler quotient of $M_F$ by $\mathbb{T}/T_F$ and the Kähler quotient metric at momentum level $v \in F^0$ is induced by $\omega_F$ on $S_F \times \{v\}$;
- the orthogonal distribution to the generic fiber $T/F$ pulls back to the blow-up $\tilde{M}_F$, biholomorphically onto the complex orbit closures in $\mathcal{V}$, and the smooth dependence of the Kähler form on $\mu$ of $\omega$ induces a Kähler structure on each complex orbit closure, which depends only on the blow-up $\tilde{M}_F$ and $d\mu$.

In particular, on $M^0 \cong P_\Delta \times_{\mathbb{T}} V_\Delta$, the Kähler structure is given by

$$g = h_0 + \langle \mu, h \rangle + \langle d\mu, G \rangle + \langle \theta, G^{-1}, \theta \rangle, \quad \omega = \Omega_0 + d\langle \mu, \theta \rangle = \Omega_0 + \langle \mu, \Omega \rangle + \langle d\mu \wedge \theta \rangle,$$

where $\omega_\Delta = \Omega_0 + \langle \mu, \Omega \rangle$, $h_0 + \langle \mu, h \rangle$ is the corresponding family of hermitian metrics, $\theta = \theta_\Delta$, $\Omega = \Omega_\Delta$, the toric Kähler metric on $V_\Delta^0$ is given by $\Omega$, for $G: \Delta^0 \to S^2 t$, and (as before) angled brackets denote pointwise contractions.

Proof. It suffices to prove the result for the whole polytope $\Delta$. We know by Propositions 4 and 5 that the blow-up $\tilde{M}$ is equivariantly biholomorphic to $M^0 \times_{\mathbb{T}} V_\Delta \cong P_\Delta \times_{\mathbb{T}} V_\Delta$ for a toric complex manifold $V_\Delta$, and the fibres of $P_\Delta \times_{\mathbb{T}} V_\Delta \to S_\Delta$ map biholomorphically onto the complex orbit closures in $M$. The Kähler metric of $M$ induces a Kähler structure on each complex orbit closure, which depends only on the momentum map $\mu$. Since $\mu$ is $\mathbb{T}$-invariant, there is a toric Kähler structure on $V_\Delta$, with Delzant polytope $\Delta$, such that the fibres of $P_\Delta \times_{\mathbb{T}} V_\Delta$, with the metric induced from $V_\Delta$, map isometrically onto the complex orbit closures in $M$.

The Kähler metric on $M^0$ induces a principal $\mathbb{T}$-connection on $M^0 \to B_\Delta = S_\Delta \times \Delta^0$ (the orthogonal distribution to the fibres), and by Lemma 7 this is the pullback of a principal $\mathbb{T}$-connection $\theta$ on $\pi: P_\Delta \to S_\Delta$. The lemma also shows that the family $\omega_\Delta$ of Kähler forms induced on $S_\Delta$ depends affinely on $\mu \in \Delta^0$ and $\pi^*d\mu \omega_\Delta = d\theta$, so the linear part is the curvature $\Omega$ of the connection $\theta$ for all $\mu \in \Delta^0$; $\omega_\Delta$ is therefore smoothly defined for all $\mu$.

The Kähler form on $M$ pulls back to the blow-up $\tilde{M}$ to give a 2-form which degenerates on the exceptional divisor. Using the description of this divisor given in Proposition 5 and the smooth dependence of the Kähler form on $\mu$, it follows that the Kähler form $\omega_\Delta$ approaches the pullback of $\omega_F$ along $S_\Delta \to S_F$ as $\mu \sim F^0 \subset F$, for a codimension one face $F$ of $\Delta$. We then deduce that the pullback of $d\mu \omega_\Delta$ to a fibre of $S_\Delta \to S_F$ takes values, for $\mu \in F^0$, in the annihilator $t_F$ of $T_{\mu} F$, i.e., is of the form $\Omega \otimes u_F$, where $u_F$ is the primitive inward normal to $F$, and $\Omega$ is a $(1,1)$-form on $S_F$. Since the normal bundle to the divisor $\tilde{M}_F$ in $\tilde{M}$ must have degree $-1$ on each fibre of $S_\Delta \to S_F$ and $\Omega$ is the curvature of a connection on this degree $-1$ line bundle, we must have $[-\Omega/2\pi] \in c_1(\mathcal{O}(-1))$.

To show that $\Omega$ is the Futaki–Sata metric in its Kähler class, we take $v \in F^0$, the interior of a codimension one face of $\Delta$, and construct a symplectic slice, as in Lemma 1 to a point $x$ in $\mu^{-1}(v)$ projecting to the given fibre of $S_\Delta \to S_F$. Thus a neighbourhood of $\mathbb{T} \cdot x$ in $M$ is equivariantly symplectomorphic to a neighbourhood $U$ of the zero section $0_N \cong \mathbb{T} \cdot x$ of the normal bundle $N = \mathbb{T} \times_{\mathbb{T}} (T_{\mu} F \oplus V_x) \to \mathbb{T} \cdot x$, with the obvious $\mathbb{T}$-action, and canonical symplectic form $\omega_0$. Pulling back the Kähler structure of $M$, and restricting to the fibre $V_x$ at $x$, gives a Kähler metric on a neighbourhood of the origin in $V_x$ with a rigid Hamiltonian circle action of $\mathbb{T}_F$. 
and constant symplectic form. Observe that the Kähler quotient \( P(V_x) \) of \( V_x \setminus \{0\} \) by \( \mathbb{T}_F \) is a fibre of \( S_\Delta \to S_F \).

Let \( z = r^2/2 \) be half the distance squared to the origin in \( V_x \)—which is the momentum map of the \( \mathbb{T}_F \) action contracted with \( u_F \in \mathfrak{t}_F \). Then the Kähler structure on \( V_x \) may be written

\[
g = zh + \frac{dz^2}{H(z)} + H(z)\theta^2, \quad \omega = z\Omega + dz \wedge \theta,
\]

for some function \( H(z) \), where \( d\theta = \Omega \) and \( \Omega \) is as before, and \( (h, \Omega) \) is independent of \( z \) (the Kähler quotient depends affinely on \( z \) and degenerates at \( z = 0 \)). The vector field dual to \( \theta \) generates the \( S^1 \) action, and this preserves \( z \), so it is tangent to the level surfaces of \( z \) (which are spheres), and generates a (topological) Hopf fibration of them. Now \( z \) is a function of the geodesic distance to \( z = 0 \) (the geodesic distance is obtained by integrating \( 1/\sqrt{H(z)} \)). For smooth compactification at \( z = 0 \), the metric on geodesic spheres must have constant curvature when \( z \to 0 \). Hence \( (h, \Omega) \) must tend to the Fubini–Study metric, so that \( \theta \) tends to the standard connection as \( z \to 0 \). Since \( (h, \Omega) \) is independent of \( z \), it is the Fubini–Study metric.

The explicit form of the metric on \( M^0 \) easily follows from Lemma 4 and Proposition 6. Note that a similar formula can be established on \( M^0_F = \mu^{-1}(F^0) \) for any face \( F \), but an origin needs to be chosen in \( F \) so that \( \mu |_{M_F} \) can be considered to take values in \( (t/t_F)^* = t^0_F \). □

Remark 7. In the absense of blow-downs, \( \Omega_0 + \langle \mu, \Omega \rangle \) is positive for all \( \mu \) in \( \Delta \), and for all \( F, S_F = S_\Delta, P_F = P_\Delta/\mathbb{T}_F \), with the induced Kähler metrics and connections; then the data of this proposition clearly do define (uniquely) a Kähler metric on \( M \) with a rigid hamiltonian action of \( T \). However, the existence of the connection \( \theta \) implies integrality conditions on the curvature form \( \Omega \), and the compactification of the toric Kähler metric on \( \mathcal{V}_\Delta \) implies boundary conditions on \( G \).

When there are blow-downs, it is difficult to describe the data needed to construct the Kähler metric on \( M \), because of the family of fibrations \( S_F \to S_F \); the Kähler quotient metrics are related by pullback, and the fibrations and pullbacks must commute with. Rather than attempt this in full generality, we restrict attention to a special case, which is all we shall need for the application to hamiltonian 2-forms.

2.5. Semisimple actions and the generalized Calabi construction.

Definition 5. A hamiltonian torus action is semisimple if for any regular value \( v \) of the momentum map \( \mu \), the derivative with respect to \( \mu \) of the family \( \omega_h \) of Kähler forms on the complex quotient \( S \) is parallel and diagonalizable with respect to \( \omega_h \) at \( \mu = v \). (Observe that \( S \) is well defined, as a complex orbifold at least, for \( \mu \) in the connected component \( U_v \) of \( v \) in the regular values, since the gradient flow of \( \mu \) is transitive on \( U_v \).)

Integrating this condition, we deduce that on any connected component of the regular values of \( \mu \), the corresponding Kähler quotient metrics \( \omega_h \) are simultaneously diagonal with the same Levi-Civita connections. Thus for a semisimple rigid hamiltonian torus action, there is a symplectic \((1, 1)\)-form \( \Omega_S \) on \( S_\Delta \) such that the family of Kähler forms induced by \( \mu \in \Delta^0 \) are parallel and simultaneously diagonalizable with respect to \( \Omega_S \).

Definition 6. By generalized Calabi data of dimension \( m \), rank \( \ell \), we mean:

(i) a \( 2(m - \ell) \)-dimensional product \( S \) of \( N \geq 0 \) Kähler manifolds \( (S_a, \pm g_a, \pm \omega_a) \) of dimension \( 2m_a > 0 \) (if \( \ell = m \), \( N = 0 \));
(ii) a compact toric $2\ell$-dimensional Kähler manifold $\mathcal{V}$ with Delzant polytope $\Delta \subset t^*$ and momentum map $\mu_\mathcal{V} : \mathcal{V} \to \Delta$;

(iii) a principal $\mathbb{T}$-bundle $P \to S$, with a principal connection of curvature $\Omega \in C^\infty(S, \Lambda^1, S \otimes \mathfrak{t})$, where $\mathbb{T}$ is the $\ell$-torus acting on $\mathcal{V}$;

(iv) a $(1,1)$-form $\Omega_0$ on $S$ such that $\Omega_0 + \langle v, \Omega \rangle$ is positive for $v \in \Delta^0$;

(v) constants $c_{a0} \in \mathbb{R}$ and $c_a \in \mathfrak{t}$ such that $\Omega_0 = \sum_{a=1}^N c_{a0} \omega_a$ and $\Omega = \sum_{a=1}^N c_a \omega_a$;

(vi) a subset $\mathcal{C} \subset \{1, \ldots, N\}$ such that for $a \notin \mathcal{C}$, $\{v \in \Delta : c_{a0} + \langle v, c_a \rangle = 0\}$ is empty, while for $a \in \mathcal{C}$ are distinct codimension one faces of $\Delta$ with (primitive) inward normals $u_a \in \mathfrak{t}$, and $S_a = \mathbb{C}P^{d_a}$ with $d_a > 0$, $\pm g_a$ is a Fubini–Study metric and $c_a \otimes \omega_a/2\pi \in u_a \otimes c_1(\mathcal{O}(-1))$.

Given these data we define the manifold $\hat{M} = P \times_\mathbb{T} \mathcal{V} = M^0 \times_{\mathbb{T}^c} \mathcal{V} \to S$, where $M^0 = P \times_\mathbb{T} \mu_\mathcal{V}^{-1}(\Delta^0)$. Since the curvature 2-form of $P$ has type $(1,1)$, $M^0$ becomes a holomorphic principal $\mathbb{T}^c$-bundle with connection and $\hat{M}$ is a complex manifold. The toric Kähler structure on $\mathcal{V}$ endows $\hat{M}$ with a fibrewise metric and ‘momentum map’ $\mu : \hat{M} \to \Delta^c$; indeed, being $\mathbb{T}$ invariant, the momentum map $\mu_\mathcal{V}$ of $\mathcal{V}$ can be defined on $\hat{M} = P \times_\mathbb{T} \mathcal{V}$.

According to (vi), the set $\mathcal{C}$ corresponds bijectively to a subset $\mathcal{B}$ of the codimension one faces of $\Delta$, and for $F \in \mathcal{B}$ corresponding to $a \in \mathcal{C}$, the connection on $\hat{M}_F := \hat{\mu}^{-1}(F)$ is flat over each fibre of $S \to \prod_{b \neq a} S_b$. This gives a $\mathbb{C}P^{d_a}$ fibration of $\hat{M}_F$ such that the normal bundle to $\hat{M}_F$ in $\hat{M}$ is a line bundle which has degree $-1$ on each $\mathbb{C}P^{d_a}$ fibre. Since a tubular neighbourhood of $\hat{M}_F$ in $\hat{M}$ is diffeomorphic to a neighbourhood of the zero section in the normal bundle, it follows that the topological space $M$, obtained by contracting $\hat{M}$ along the $\mathbb{C}P^{d_a}$ fibration of each such $\hat{M}_F$, is a smooth manifold and $M^0$ is an open dense submanifold.

If the Kähler structure given by (24) (which pulls back to the fibrewise metric on the fibres of $M \to S$) extends smoothly to $M$, then we say that this Kähler manifold $(M, g, J, \omega)$ is given by the generalized Calabi construction (with blow-downs).

We shall see that the contraction $\hat{M} \to M$ realises $\hat{M}$ (with the complex structure described above) as a blow-up of $M$. We therefore refer to this contraction as a blow-down. Our main result shows that all generalized Calabi data give rise to a generalized Calabi construction, and that this classifies compact Kähler manifolds with a semisimple rigid Hamiltonian torus actions up to a covering.

**Theorem 2.** Let $M$ be a compact connected Kähler $2m$-manifold with a semisimple rigid Hamiltonian action of an $\ell$-torus $\mathbb{T}$ and momentum map $\mu : M \to \Delta \subset t^*$. Then some cover of $M$ is given by the generalized Calabi construction.

Conversely, for any generalized Calabi data (i)–(vi), the generalized Calabi construction produces a smooth Kähler manifold with a semisimple rigid Hamiltonian action of an $\ell$-torus.

**Proof.** We construct the generalized Calabi data from Proposition 6 by imposing the condition that the action is semisimple. As remarked in [4 §3.3], the condition that $\Omega_0$ and the components of $\Omega$ are simultaneously diagonalizable and parallel (with respect to some Kähler metric $\Omega_S$) implies that the (distinct) eigendistributions $\mathcal{H}_a$ ($a = 1, \ldots, N$) are parallel. By the deRham decomposition theorem, some cover of $S_\Delta$ (for instance the universal cover), is a Kähler product $(S, \Omega_S) = \prod_{a=1}^N (S_a, \omega_a)$ (note that $S$ may not be compact). The generalized Calabi data (i)–(v) are then obtained from Proposition 6 by setting $\mathcal{V} = V_\Delta$, pulling back $P_\Delta$, $\theta_\Delta$, $\Omega_0$ and $\Omega$ to give a principal bundle $P$ with connection over $S$, and defining the constants $c_{a0}$ and $c_a$ by (v).
Let $B$ be the set of codimension one faces $F$ of $\Delta$ such that a blow-down occurs (i.e., $M_F$ is not a divisor); then $\hat{M}$ is the blow-up of $M$ along $M_F$ with $F \in B$. The pullback of the metric to $\hat{M}$ degenerates on the fibres of a $\mathbb{C}P^d$-bundle $S_\Delta \to S_F$ for some $d > 0$. Now $\mathbb{C}P^d$ is simply connected, so this is covered by a $\mathbb{C}P^d$-bundle with total space $S$, whose base is a cover of $S_F$. Hence there must be at least one $a$ such that $c_{a0} + \langle v, c_a \rangle = 0$ for $v \in F^0$; since $\mathbb{C}P^d$ does not admit a Kähler product metric, this $a$ is unique, and $S_F$ is covered by $\prod_{b \neq a} S_b$, while $S_a = \mathbb{C}P^{da}$ with $d_a = d$. On the other hand $c_{a0} + \langle v, c_a \rangle$ is an affine function of $v$, so it can vanish on at most one codimension one face of the Delzant polytope $\Delta$. Thus $B$ corresponds bijectively to a subset $C \subset \{1, \ldots, N\}$. Now note that for any face $F$, with $v \in F^0$, the metric induced on $S_F$ is nondegenerate, so $c_{a0} + \langle v, c_a \rangle$ does not vanish on $\Delta$ for $a \notin C$. This establishes (vi).

The pullback of $\hat{M}$ to $S$ is a cover of $\hat{M}$, and by construction, this descends to $M$. Hence, up to a cover, $M$ is obtained from the generalized Calabi construction.

Conversely, given the data of Definition 6, we will prove that there exists a smooth compact Kähler manifold $(M, g, J, \omega)$ with a semisimple rigid hamiltonian action of the $\ell$-torus $\mathbb{T}$ given by the generalized Calabi construction. The main difficulty is to deal with the blow-downs.

Let us suppose there are $k \geq 0$ blow-downs: then, after reordering, we may assume $C = \{1, \ldots, k\}$ and that $S = \mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_k} \times S''$ for some Kähler product $S''$. The conditions (iii) and (v) of Definition 6 imply that $\Omega'' := \sum_{a=k+1}^N c_a \omega_a$ is the curvature of a principal $T$-bundle $P'' \to S''$. We are going to let $M$ be of the form $P'' \times T M'$, where $M'$ is a $2(\ell + d_1 + \cdots + d_k)$ dimensional Kähler manifold with a rigid semisimple isometric hamiltonian action of $T$, obtained from the generalized Calabi construction with respect to the following data:

(i) $S' = \mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_k}$;
(ii) $(V, \omega, \mu, \Delta)$, with the given compatible toric Kähler metric;
(iii) a principal $T$-bundle $P'$ with curvature form $\Omega' = \sum_{a=1}^k c_a \omega_a$;
(iv) $\Omega_0' = \sum_{a=1}^k c_{a0} \omega_a$;
(v) the given constants $c_{a0}$ and $c_a$ for $a = 1, \ldots, k$;
(vi) $C = \{1, \ldots, k\}$.

Since the data for $M$ are generalized Calabi data, so are these data for $M'$. If $M'$ can be constructed with these data, it follows from Proposition 6 and Remark 7 that $M$ is equipped with a Kähler metric and a semisimple rigid action of $T$; using the first part of the theorem we also see that $M$ is given by the generalized Calabi construction associated to the initial data.

Thus it remains only to establish the generalized Calabi construction for $M'$. However, such an $M'$ is obtained as a restricted toric Kähler manifold, the construction of which we discussed already in §1.6. □

Just as toric complex manifolds may be described in terms of linear charts, i.e., in terms of a family of vector spaces, each with a decomposition into one dimensional subspaces, glued together by Laurent monomials, so bundles of toric complex manifolds (arising in the generalized Calabi construction without blow-downs) may be described (by the holomorphic slice theorem) in terms of families of vector bundles, each a direct sum of line bundles, glued together in a similar way. The simplest case is the case of projective bundles $P(L_0 \oplus L_1 \oplus \cdots \oplus L_\ell) \to S$, which are obtained by gluing together the vector bundles $\bigoplus_{k \neq j} L_k$ for $j = 0, \ldots, \ell$. This is the only case we shall need in the sequel.
3. Orthotoric Geometry

We now return to our primary aim: the classification of compact Kähler manifolds endowed with a hamiltonian 2-form. In this section we treat the case when the order of the hamiltonian 2-form is maximal, and therefore the corresponding Kähler manifolds are toric. Motivated by the orthogonality of the gradients of the roots of the momentum polynomial, see Theorem [1] we define orthotoric Kähler manifolds and orbifolds, and classify the compact ones.

3.1. The polytope of an orthotoric orbifold.

Definition 7. An orthotoric Kähler manifold (or orbifold) $M$ is a toric Kähler $2m$-manifold (or orbifold) with a momentum map $\sigma = (\sigma_1, \ldots, \sigma_m)$ and (rational) Delzant polytope $\Delta = \sigma(M)$, such that on the dense open set $M^0 = \sigma^{-1}(\Delta^0)$ of regular points of $\sigma$, the roots $\xi_1, \ldots, \xi_m$ of the momentum polynomial $\sum_{r=0}^m (-1)^r \sigma_r t^{m-r}$ ($\sigma_0 = 1$) are smoothly defined, pairwise distinct and functionally independent, and the Kähler metric has the explicit form

$$g = \sum_{j=1}^m \frac{\Delta_j}{\Theta_j(\xi_j)} d\xi_j^2 + \sum_{j=1}^m \frac{\Theta_j(\xi_j)}{\Delta_j} \left( \sum_{r=1}^m \sigma_{r-1}(\hat{\xi}_j) \, dt_r \right)^2$$

$$\omega = \sum_{j=1}^m d\xi_j \wedge \left( \sum_{r=1}^m \sigma_{r-1}(\hat{\xi}_j) \, dt_r \right) = \sum_{r=1}^m d\sigma_r \wedge dt_r,$$

for functions $\Theta_1, \ldots, \Theta_m$ of one variable. Here $\Delta_j = \prod_{k \neq j} (\xi_j - \xi_k)$.

Clearly the gradients of $\xi_1, \ldots, \xi_m$ are orthogonal with respect to $g$. Conversely, it was shown in [1, §3.4] that this property characterizes orthotoric Kähler manifolds (and the result applies equally to orbifolds).

Note that the basis $K_1, \ldots, K_m$ of the Lie algebra of the torus identifies it with $\mathbb{R}^m$, and we view the invariant 1-forms $dt_1, \ldots, dt_m$ as the dual basis of $\mathbb{R}^m$.

Proposition 7. Let $M$ be a compact orthotoric Kähler $2m$-manifold or orbifold with momentum map $\sigma = (\sigma_1, \ldots, \sigma_m)$ and rational Delzant polytope $\Delta$.

(i) $\Delta$ is the (one to one) image under the elementary symmetric functions of a domain of the form

$$D = \{(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m : \alpha_j \leq \xi_j \leq \beta_j\}$$

where

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{m-1} \leq \alpha_m < \beta_m.$$

Thus, setting $\sigma_0 = 1$, $\Delta = \{(\sigma_1, \ldots, \sigma_m) : (-1)^{m-j} \sum_{r=0}^m (-1)^r \sigma_r \alpha_j^{m-r} \leq 0$ and $(-1)^{m-j} \sum_{r=0}^m (-1)^r \sigma_r \beta_j^{m-r} \geq 0$ for $j = 1, \ldots, m\}$. This is a simplex if and only if $\alpha_{j+1} = \beta_j$ for $j = 1, \ldots, m-1$.

(ii) If $M$ is nonsingular (i.e., a manifold), then $\Delta$ is a simplex.

Proof. (i) $\sigma_1, \ldots, \sigma_m$ are the elementary symmetric functions of the roots $\xi_1, \ldots, \xi_m$ of the momentum polynomial, and we want to find the domain $D$ in the $\xi_j$ coordinates corresponding to $\Delta$. We first remark that this domain must be bounded. Also, the functions $\Theta_j(\xi_j)$ must be nonzero on the interior $D^0$ of $D$ in order that the metric be finite and nondegenerate.
Now consider in particular the metric on the torus given by

\[ g(K_r, K_s) = \sum_{j=1}^{m} \frac{\Theta_j(\xi_j) \sigma_{r-1}(\xi_j) \sigma_{s-1}(\xi_j)}{\Delta_j}. \]

The determinant of this matrix is (up to a sign) \( \prod_{j=1}^{m} \Theta_j(\xi_j) \). As we approach a special orbit of the \( m \)-torus action, i.e., as \( \sigma \) approaches the boundary of \( \Delta \), this must tend to zero, i.e., at least one of the functions \( \Theta_j \) of one variable must tend to zero. Since these functions are nonvanishing on \( D^0 \), it follows that \( D^0 \) is a domain of the form \( \prod_{j=1}^{m} (\alpha_j, \beta_j) \), where \( \Theta_j \) is nonvanishing on the interval \((\alpha_j, \beta_j)\) and tends to zero at the endpoints. Now \( \xi_1, \ldots, \xi_m \) must be pairwise distinct on \( D^0 \), so we may assume (after reordering) that \( \xi_1 < \cdots < \xi_m \) on \( D^0 \). Hence

\[ \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots < \beta_{m-1} \leq \alpha_m < \beta_m. \]

Noting that the elementary symmetric functions are affine in each variable, we readily check that this domain does indeed map bijectively to a convex polytope. Indeed any \((\xi_1, \ldots, \xi_m)\) in \( D \) satisfy

\[ (-1)^{m-j} \prod_{k=1}^{m} (\alpha_j - \xi_k) \leq 0, \quad (-1)^{m-j} \prod_{k=1}^{m} (\beta_j - \xi_k) \geq 0 \]

for all \( j = 1, \ldots, m \); equality is attained in one of these expressions on any face, and in any of these expressions on some face. Expanding in terms of the elementary symmetric functions of \((\xi_1, \ldots, \xi_m)\) gives the explicit description of \( \Delta \).

A compact convex polytope in \( \mathbb{R}^{m*} \) is a simplex if and only if it has \( m + 1 \) vertices. The vertices of \( D \) are the points where \( \xi_j \in \{\alpha_j, \beta_j\} \) for all \( j = 1, \ldots, m \). Now observe that a vertex of \( D \) maps to a vertex of \( \Delta \) if and only if it does not lie on one of the diagonals \( \xi_j = \xi_k \) for \( j \neq k \).

(ii) We shall show that \( \alpha_{j+1} = \beta_j \) for \( j = 1, \ldots, m - 1 \). Suppose for contradiction that this does not hold for some \( j \in \{1, \ldots, m - 1\} \) and consider the four vertices

\[
\begin{align*}
(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_m), & \quad (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j, \beta_{j+1}, \alpha_{j+2}, \ldots, \alpha_m), \\
(\alpha_1, \ldots, \alpha_{j-1}, \beta_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_m), & \quad (\alpha_1, \ldots, \alpha_{j-1}, \beta_j, \beta_{j+1}, \alpha_{j+2}, \ldots, \alpha_m).
\end{align*}
\]

Since \( \alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1} \) these four points map to four distinct vertices spanning a two dimensional face of \( \Delta \) \((\xi_k = \alpha_k \) defines a hyperplane\). Now any face of a Delzant polytope is Delzant (as one easily checks) and the Delzant property is invariant under affine transformation. Hence we may as well map this 2-dimensional face into \( \mathbb{R}^2 \) by sending \((\sigma_1, \ldots, \sigma_m)\) to \((\sigma_1 - a_1, \sigma_2 - a_1 \sigma_1 + a_1^2 - a_2)\), where \( a_1 \) and \( a_2 \) are the first two elementary symmetric functions of \( \{\alpha_k : k \neq j, j+1\} \); in terms of \( \xi_j, \xi_{j+1} \) (fixing \( \xi_k = \alpha_k \) for \( k \neq j, j+1 \)), this formula gives \((\xi_j + \xi_{j+1}, \xi_j \xi_{j+1})\), and so our face gets mapped to the quadrilateral with vertices

\[
(\alpha_j + \alpha_{j+1}, \alpha_j \alpha_{j+1}), \quad (\alpha_j + \beta_{j+1}, \alpha_j \beta_{j+1}), \quad (\beta_j + \alpha_{j+1}, \beta_j \alpha_{j+1}), \quad (\beta_j + \beta_{j+1}, \beta_j \beta_{j+1})
\]

and normals (up to scale)

\[
(\alpha_j, -1), \quad (\beta_j, -1), \quad (\alpha_{j+1}, -1), \quad (\beta_{j+1}, -1).
\]

Again, \( \alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1} \), so these four normals point in distinct directions, and so cannot be scaled to form a basis for the same lattice at each vertex. Our quadrilateral is therefore not Delzant, hence neither is \( \Delta \), a contradiction. \( \square \)

For the rest of this subsection we suppose \( \Delta \) is a simplex: the above proposition shows that this is necessarily true if \( M \) is nonsingular.
By the Delzant construction, any symplectic orbifold $M$ whose rational Delzant polytope is a simplex is a symplectic quotient of $\mathbb{C}^{m+1}$ by a one dimensional subgroup $G$ of $(S^1)^{m+1}$. From the relation between complex and symplectic quotients, cf. [18], it follows that $M$ is a quotient of a weighted projective space $\mathbb{C}P_{a_0, \ldots, a_m}^m$ — here $a_0, \ldots, a_m \in \mathbb{Z}^+$ have highest common factor 1 and $\mathbb{C}P_{a_0, \ldots, a_m}^m$ is the quotient of $\mathbb{C}^{m+1} \setminus \{0\}$ by the holomorphic action

$$(z_0, \ldots, z_m) \mapsto (\zeta^{a_0} z_0, \ldots, \zeta^{a_m} z_m) \quad \text{for} \quad \zeta \in \mathbb{C}^\times;$$

note that $\mathbb{C}P_{1, \ldots, 1}^m$ is the usual (nonsingular) $\mathbb{C}P^m$.

We want to describe $\Delta$ more explicitly as a rational Delzant simplex. We put $\beta_0 = \alpha_1$, so $\Delta$ is the image under the elementary symmetric functions of the domain

$$D = \{(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m : \beta_{j-1} \leq \xi_j \leq \beta_j\}$$

where

$$\beta_0 < \beta_1 < \cdots < \beta_{m-1} < \beta_m.$$  

**Proposition 8.** Let $M$ be a compact orthotoric Kähler 2m-orbifold whose Delzant polytope $\Delta$ is the image of (28) under the elementary symmetric functions.

(i) $\Delta = \{\sigma : \{v_j, \sigma\} + \kappa_j \geq 0\}$, where $\kappa_j = \beta_j^m / \prod_{k \neq j} (\beta_j - \beta_k)$ and

$$v_j = \left( -\beta_j^{m-1} / \prod_{k \neq j} (\beta_j - \beta_k), \ldots, (-1)^r \beta_j^{m-r} / \prod_{k \neq j} (\beta_j - \beta_k), \ldots, (-1)^m / \prod_{k \neq j} (\beta_j - \beta_k) \right).$$

The codimension one faces of $\Delta$ are $F_0, \ldots, F_m$, where

- $F_0$ is the image of the boundary component $\xi_1 = \beta_0$ of $D$,
- $F_m$ is the image of the boundary component $\xi_m = \beta_m$ of $D$, and
- $F_j$, for $j = 1, \ldots, m - 1$, is the union of the images of the boundary components $\xi_j = \beta_j$ and $\xi_{j+1} = \beta_j$ of $D$.

(ii) The normals are of the form $u_j = 2n_j v_j / c$, where $c > 0$ and $n_j \in \mathbb{Z}^+$ ($j = 0, \ldots, m$) have highest common factor 1; then $M$ is equivariantly biholomorphic to an orbifold quotient of $\mathbb{C}P_{a_0, \ldots, a_m}^m$, where $n_j = \prod_{k \neq j} a_k$.

(iii) $M$ is nonsingular if and only if it is biholomorphic to $\mathbb{C}P^m$ if and only if $n_j = 1$ (for all $j$) and the lattice of circle subgroups is generated by $u_0, \ldots, u_m$. The dual lattice in $\mathbb{R}^{m*}$ is then generated by

$$\theta_{p, q} = \sum_{r=0}^{m} \frac{1}{2} c (\sigma^2_r(\beta_q) - \sigma^2_r(\beta_p)) dt_r$$

where $\sigma^2_r(\beta_p)$ denotes the $r$th elementary symmetric function of the $m$ variables $\{\beta_j : j = 0, \ldots, m, j \neq p\}$.

**Proof.** (i) When $\Delta$ is a simplex, the inequalities in (27) may be written

$$\prod_{k=1}^{m} (\beta_j - \xi_k) \prod_{k \neq j} (\beta_j - \beta_k) > 0$$

for all $j = 0, \ldots, m$, which immediately gives the stated form of $\Delta$. (Note that the apparent codimension two face $\xi_j = \beta_j = \xi_{j+1}$ is ‘straightened out’ by the elementary symmetric functions; this is why $\Delta$ has only $m + 1$ faces, not $2m$.)

(ii) From the form of the simplex $\Delta$ it is immediate that the normals $u_0, \ldots, u_m$ are positive multiples of $v_0, \ldots, v_m$. They belong to a common lattice if and only if the linear dependence relation among them can be written $\sum_{j=0}^{m} u_j / n_j = 0$, where $n_0, \ldots, n_m$ are nonzero rational numbers. We now observe that the $v_j$'s already satisfy $\sum_{j=0}^{m} v_j = 0$ by the Vandermonde identity (cf. [10, Appendix B]). Hence we must have $u_j = C n_j v_j$ for some nonzero constant $C$ and without loss of generality.
we can take $C$ and $n_j$’s to be positive and suppose $n_0, \ldots, n_m$ are integers with highest common factor 1. We then put $C = 2/c$.

We have already seen that any toric Kähler orbifold with polytope a simplex is equivariantly biholomorphic to an orbifold quotient of a weighted projective space. It remains to show that the integers $n_j$ are related to the weights $a_k$ by $n_j = \prod_{k \neq j} a_k$. For this, we note that any weighted projective space has an orbifold quotient whose simplex is standard with respect to the lattice $\Lambda$, i.e., the primitive normals sum to zero. The primitive normals are $u_j/m_j$ and Abreu shows that the labels (in this case) are given by $m_j = \prod_{k \neq j} a_k$. Since $\sum_{j=0}^m v_j = 0$, and the $m_j$ have highest common factor 1, we have $m_j = n_j$.

(iii) The only (orbifold quotient of a) weighted projective space which is nonsingular is $\mathbb{C}P^m$. Clearly $M$ is equivariantly biholomorphic to $\mathbb{C}P^m$ if and only if the $n_j$ all equal 1 and the lattice $\Lambda$ of circle subgroups is the minimal one. In terms of the vector fields $K_1, \ldots, K_m$, it follows that vector fields generating $\Lambda$ are

$$X_j = \frac{2}{c} \sum_{r=1}^{m} \frac{(-1)^r \beta_j^{m-r} K_r}{\prod_{k \neq j} (\beta_j - \beta_k)},$$

with $\sum_{j=0}^m X_j = 0$. To see that \((\ref{eqn:brackets})\) generate the dual lattice, we note that

$$\theta_{p,q} = \sum_{r=1}^{m} \frac{1}{2c} \sigma_{r-1}(\hat{\beta}_p, \hat{\beta}_q)(\beta_p - \beta_q)dt_r$$

for $0 \leq p < q \leq m$, where $\sigma_{r-1}(\hat{\beta}_p, \hat{\beta}_q)$ is the $(r-1)$st elementary symmetric function of the $m-1$ variables $\{\beta_j : j = 0, \ldots, m, j \neq p, q\}$. We compute

$$\theta_{p,q}(X_j) = \sum_{r=1}^{m} \frac{(-1)^r \sigma_{r-1}(\hat{\beta}_p, \hat{\beta}_q)(\beta_p - \beta_q)\beta_j^{m-r}}{\prod_{k \neq j} (\beta_j - \beta_k)}$$

$$= \frac{\prod_{k \neq j} (\beta_j - \beta_k)}{\prod_{k \neq j} (\beta_j - \beta_k)} (\beta_q - \beta_p) = \delta_{jq} - \delta_{jp}$$

and the result follows.

The constant $c$ determines the scale of $M$: the symplectic volume is proportional to $1/c$. The other constants $\beta_0, \ldots, \beta_m$ are related to the fact that the Killing vector fields $K_1, \ldots, K_m$ do not necessarily form an integral basis.

We remark that all simplices are equivalent under affine transformation, and so for any $\beta_0 < \cdots < \beta_m$, any rational Delzant simplex is equivalent to the simplex of this proposition for some lattice $\Lambda$ in $\mathbb{R}^m$ and some normals $u_j = 2n_jv_j/c \in \Lambda$.

3.2. Compactification of orthotoric Kähler metrics. We next establish necessary and sufficient conditions for the compactification of the orthotoric Kähler metric \((\ref{eqn:compactification})\) on a compact $2m$-orbifold $M$. We obtain these conditions by specializing those of Proposition \([\ref{prop:toric}]\) to the orthotoric case.

**Proposition 9.** Let $M$ be a compact symplectic $2m$-orbifold such that the rational Delzant polytope $\Delta \subset \mathbb{R}^{m*}$ is the image of $\prod_{j=1}^{m}[\alpha_j, \beta_j]$ under the elementary symmetric functions, where

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots < \beta_{m-1} \leq \alpha_m < \beta_m.$$
Let $L^\alpha_j(\sigma) = \langle u^\alpha_j, \sigma \rangle + \lambda^\alpha_j$ and $L^\beta_j(\sigma) = \langle u^\beta_j, \sigma \rangle + \lambda^\beta_j$ where

\[
\lambda^\alpha_j = -c_j^\alpha a_j^m, \quad u^\alpha_j = c_j^\alpha (\alpha_j^{m-1}, \ldots (-1)^{r-1} \alpha_j^{m-r}, \ldots (-1)^{m-1}), \\
\lambda^\beta_j = -c_j^\beta b_j^m, \quad u^\beta_j = c_j^\beta (\beta_j^{m-1}, \ldots (-1)^{r-1} \beta_j^{m-r}, \ldots (-1)^{m-1}),
\]

and the constants $c_j^\alpha, c_j^\beta \in \mathbb{R}$ are such that the normals of $\Delta$ are the distinct elements among $u_j^\alpha, u_j^\beta$, i.e., $\Delta = \{ \sigma \in \mathbb{R}^{m*} : L^\alpha_j(\sigma) \geq 0$ and $L^\beta_j(\sigma) \geq 0$ for $j = 1, \ldots, m \}$, but if $\alpha_{j+1} = \beta_j$, we have $c_{j+1}^\alpha = c_j^\beta$ as the normals $u_j^\alpha, u_{j+1}^\beta$ are then not distinct.

Then the Kähler metric (25), defined for $\xi_j \in (\alpha_j, \beta_j)$, extends to an orthotoric Kähler metric on $M$ if and only if for $j = 1, \ldots, m$, $\Theta_j$ is the restriction to $(\alpha_j, \beta_j)$ of a smooth function $\Theta$ on $\bigcup_{j=1}^m [\alpha_j, \beta_j]$ satisfying (for $j = 1, \ldots, m$):

\[
\Theta(\alpha_j) = 0 = \Theta(\beta_j),
\]

(32)

\[
\Theta'(\alpha_j)c_j^\alpha = 2 = \Theta'(\beta_j)c_j^\beta;
\]

(33)

\[
(-1)^{m-j}\Theta > 0 \quad \text{on} \quad (\alpha_j, \beta_j).
\]

Proof. By (25), $H$ is given by

\[
H_{rs} = \sum_{j=1}^m \frac{\Theta_j(\xi_j) \sigma_{r-1}(\xi_j) \sigma_{s-1}(\xi_j)}{\Delta_j}.
\]

This is a smooth and symmetric function of $\xi_1, \ldots, \xi_m$, so by Glaeser [15], it is a smooth function of $\sigma_1, \ldots, \sigma_m$. The positivity condition is clear, so it remains to consider the boundary conditions (32). We must show these are equivalent to (15).

The form of the normals shows that $H(u_i^\alpha, \cdot)$ is given by

\[
\sum_{r=1}^n H_{rs}(u_i^\alpha)_r = \sum_{j=1}^m \frac{c_j^\alpha \Theta_j(\xi_j) (-1)^{r-1} \sigma_{r-1}(\xi_j) \alpha_i^{m-r} \sigma_{s-1}(\xi_j)}{\Delta_j}
\]

\[
= \sum_{j=1}^m \frac{c_j^\alpha \Theta_j(\xi_j) \sigma_{s-1}(\xi_j) \prod_{k \neq j} (\alpha_i - \xi_k)}{\Delta_j}.
\]

On the codimension one face $\xi_i = \alpha_i$, this reduces to $c_j^\alpha \Theta_j(\alpha_i) \sigma_{s-1}(\xi_i)$, which vanishes for all $s$ if and only if $\Theta_j(\alpha_i) = 0$. For the derivative conditions we differentiate

\[
H(u_i^\alpha, u_i^\alpha) = \sum_{j=1}^m \frac{(c_j^\alpha)^2 \Theta_j(\xi_j) \prod_{k \neq j} (\alpha_i - \xi_k)^2}{\Delta_j}
\]

and evaluate along $\xi_i = \alpha_i$ to obtain

\[
(c_j^\alpha)^2 \Theta_j'(\alpha_i) \prod_{k \neq i} (\alpha_i - \xi_k) \left. d\xi_i = c_j^\alpha \Theta_j'(\alpha_i) d \prod_{k=1}^m (c_j^\alpha(\alpha_i - \xi_k) \right|_{\xi_i = \alpha_i}.
\]

This equals $2u_i^\alpha$ if and only if $c_j^\alpha \Theta_j'(\alpha_i) = 2$. The boundary conditions at the $\beta$ endpoints are analogous.

Note that (32) could be taken as the definition of the constants $c_j^\alpha$ and $c_j^\beta$. However, these are then required to satisfy positivity and integrality conditions, since $(-1)^{m-j}c_j^\alpha$ and $(-1)^{m-j+1}c_j^\beta$ must be positive for the normals to be inward pointing, while $u_j^\alpha$ and $u_j^\beta$ must belong to a common lattice in $\mathbb{R}^m$.

We summarize our results for the case that the rational Delzant polytope is a simplex. The following is immediate from Propositions 7, 8, and 9.
Theorem 3. Let $M$ be a compact orthotoric $2m$-manifold or orbifold with moment map $\mathbf{\sigma}$, whose rational Delzant polytope is a simplex $\Delta$ with normals $u_0, u_1, \ldots, u_m \in \mathbb{R}^m$, where the Kähler metric is given by \([25]\) on $M^0 = \mathbf{\sigma}^{-1}(\Delta^0)$.

(i) $M$ is equivariantly biholomorphic to a toric orbifold quotient of $\mathbb{C}P^m_{\alpha_0, \ldots, \alpha_m}$ and, with $n_j = \prod_{k \neq j} a_k$, there are constants $\beta_0 < \beta_1 < \cdots < \beta_m$, $c > 0$, and a smooth function $\Theta$ on $[\beta_0, \beta_m]$, such that for $j = 0, \ldots, m$:

\[
\frac{\beta^{m-1}_j}{\prod_{k \neq j} (\beta_j - \beta_k)} \cdots \frac{-1)^r \beta^{m-r}_j}{\prod_{k \neq j} (\beta_j - \beta_k)} \cdots \frac{(-1)^m}{\prod_{k \neq j} (\beta_j - \beta_k)}
\]

\[\Theta_j = \Theta \quad \text{on} \quad [\beta_{j-1}, \beta_j];\]

\[(-1)^{m-j} \Theta > 0 \quad \text{on} \quad (\beta_{j-1}, \beta_j);\]

\[\Theta(\beta_j) = 0, \quad \Theta'(\beta_j) = -\frac{c}{n_j} \prod_{k \neq j} (\beta_j - \beta_k).\]

(ii) Conversely, given constants $\beta_0 < \beta_1 < \cdots < \beta_m$, $c > 0$ and a smooth function $\Theta$ on $[\beta_0, \beta_m]$ satisfying \([30]\)–\([37]\), the Kähler metric given by \([25]\) and \([35]\) defines an orthotoric structure on $\mathbb{C}P^m_{\alpha_0, \ldots, \alpha_m}$ and its toric orbifold quotients, such that the rational Delzant polytope is the image of $[\beta_0, \beta_1] \times [\beta_1, \beta_2] \times \cdots \times [\beta_m, \beta_1]$ under the elementary symmetric functions, with normals given by \([34]\).

(iii) Any (nonsingular) compact orthotoric Kähler $2m$-manifold $M$ arises in this way (with $n_j = 1$ for $j = 0, \ldots, m$) and is equivariantly biholomorphic to $\mathbb{C}P^m$.

3.3. Examples on weighted projective spaces.

3.3.1. The Fubini–Study metric. We recall from \([1]\) §5.4 that an orthotoric Kähler metric has constant holomorphic sectional curvature $c$ if and only if $\Theta_j = \Theta_0$ for all $j = 1, \ldots, m$, where $\Theta_0$ is a polynomial of degree $m + 1$ with distinct roots and leading coefficient $-c$. We then have $\Theta_0(t) = -c \prod_{j=0}^{m} (t - \beta_j)$ with $\beta_0 < \cdots < \beta_m$, which clearly satisfies \([30]\)–\([37]\). Thus we see directly that this orthotoric metric is defined on $\mathbb{C}P^m$, in accordance with \([1]\) §2.4, where it was shown more generally that the Fubini–Study metric on $\mathbb{C}P^m$ admits hamiltonian 2-forms of arbitrary order $\leq m$, in one to one correspondence with Killing potentials.

This form of the Fubini–Study metric is familiar for $m = 1$, when \([25]\) yields

\[g = \frac{d\xi^2}{c(\xi - \beta_0)(\beta_1 - \xi)} + \frac{c(\xi - \beta_0)(\beta_1 - \xi)dt^2}.\]

Setting $2\xi = (\beta_1 - \beta_0)z + \beta_0 + \beta_1$, $t = 2\psi/c(\beta_1 - \beta_0)$ and rescaling $g$ by $c$, we get

\[g_{FS} = \frac{dz^2}{1 - z^2} + (1 - z^2)dy^2.\]

In arbitrary dimension $m$, the Fubini–Study metric is the ‘canonical’ metric associated to its simplex, hence is given here by \([8]\) with

\[G = \frac{1}{2} \text{Hess} \left( \sum_{j=0}^{m} L_j(\mathbf{\sigma}) \log |L_j(\mathbf{\sigma})| \right)\]

where

\[L_j(\mathbf{\sigma}) = \langle u_j, \mathbf{\sigma} \rangle + \lambda_j = \frac{2 \prod_{k=1}^{m} (\beta_j - \xi_k)}{c \prod_{k \neq j} (\beta_j - \beta_k)}.\]
It follows that
\[
\sum_{r,s=1}^{m} G_{rs} d\sigma_r d\sigma_s = \frac{1}{2} \sum_{j=0}^{m} L_j \left( \frac{dL_j}{L_j} \right)^2 = \frac{1}{c} \sum_{j=0}^{m} \prod_{k\neq j}^{m} (\beta_j - \beta_k) \left( \sum_{k=1}^{m} \frac{d\xi_k}{\xi_k - \beta_j} \right)^2
\]
which immediately yields the orthotoric description (25), since the inner sum over \( j \) is \( \Delta_q \delta_{pq} \) by the Lagrange interpolation formula.

3.3.2. Bochner-flat metrics. More generally, any extremal orthotoric metric \( \Theta \) for which \( \Theta_j = \Theta \) is necessarily Bochner-flat \( \cite{4} \ §5.4 \); in this case \( \Theta \) must be a polynomial of degree less or equal to \( m + 2 \), and the boundary conditions imply that \( \Theta \) has \( m + 1 \) or \( m + 2 \) distinct roots. The former case gives the Fubini–Study metric and its orbifold quotients, while the latter recovers the Bochner-flat examples of \( \cite{10} \), which are defined on \( \mathbb{CP}^m \) for distinct weights \( a_0, \ldots, a_m \). Indeed, for any positive integers \( a_0 > \cdots > a_m \) we take the metric (25) with
\[
\Theta_j(t) = \Theta(t) = -(t - \beta) \Theta_0(t) = c(t - \beta) \prod_{j=0}^{m} (t - \beta_j),
\]
where \( c > 0 \) is a homothety factor for the metric and we deduce from (37) that the real numbers \( \beta_0 < \cdots < \beta_m < \beta \) and \( c > 0 \) satisfy
\[
(38) \quad \beta_j = \beta - \frac{a_j}{\prod_{k=0}^{m} a_k}.
\]
This metric is Bochner-flat (see \( \cite{4} \) Proposition 16) and compactifies on \( \mathbb{CP}^m \) (see Theorem 9). As shown by Bryant \( \cite{10} \), there are actually Bochner-flat metrics on \( \mathbb{CP}^m \) for any choice of weights. There is an easy way to see this using the relation between Bochner-flat metrics and flat CR structures found by Webster \( \cite{37} \). Indeed \( \mathbb{CP}^m \) is a quotient of \( S^{2m+1} \) by a weighted \( S^1 \)-action by CR automorphisms of the flat CR structure, and the Sasakian structure induced by the associated Reeb field gives rise to a Bochner-flat Kähler metric on the quotient.

The Bochner-flat metrics on weighted projective spaces are all toric (see \( \cite{2} \) for the general form in momentum coordinates). However, when the weights are not distinct, they are not orthotoric (apart from the Fubini–Study metric): from our point of view, the Bochner-flat metric is endowed with a natural hamiltonian 2-form which is (an affine deformation of) the normalized Ricci form \( \cite{4} \) and it has order \( m \) if and only if the weights \( a_j \) are distinct.

Remark 8. Note that the orthotoric Bochner-flat Kähler metric on a weighted projective space is unique (up to isomorphism and scale): \( \beta_0, \ldots, \beta_m \) are determined as above (the choice of \( \beta \) can be absorbed in the coordinate freedom). In fact a stronger uniqueness result is true: the Bochner-flat metric is the unique extremal Kähler metric (up to isomorphism and scale) on any weighted projective space. To see this, recall that the second deRham cohomology group of \( \mathbb{CP}^{m}_a \) is one dimensional, so there is only one Kähler class up to scale (this follows, for instance, by the Smith–Gysin sequence for the space of orbits, \( \mathbb{CP}^{m}_a \), of the weighted \( S^1 \)-action on the \( (2m + 1) \)-sphere); therefore the uniqueness result of Guan \( \cite{16} \) (which readily generalizes to orbifolds) applies to the Kähler class of \( \mathbb{CP}^{m}_a \).

The uniqueness implies that any toric \( 2m \)-orbifold of constant scalar curvature, whose rational Delzant polytope is a simplex, is an orbifold quotient of \( \mathbb{CP}^m \). Note that the Futaki invariant of \( \mathbb{CP}^m \) vanishes if and only if \( a_0 = a_1 = \cdots = a_m \).
3.4. Kähler–Einstein orthotoric surfaces. In this subsection we present new examples of Kähler–Einstein metrics on compact orbifolds. As we have seen in the previous subsection, we have to work beyond the context of weighted projective spaces, so we consider polytopes with more than \( m + 1 \) codimension one faces. We restrict attention to complex orbifold surfaces \( (m = 2) \) in order to make the construction completely explicit. In this case, a polytope with more than \( m + 1 = 3 \) faces necessarily has \( 2m = 4 \) faces and we are in the ‘generic’ case where the roots \( \xi_1, \xi_2 \) are everywhere distinct on \( \Delta \).

According to [4 §5.3], an orthotoric Kähler metric on a 4-orbifold is Kähler–Einstein if and only if \( \Theta_j(t) = -P_j(t)/C, \; j = 1, 2 \) for some positive constant \( C \), and some \( \pm \)-monic polynomials \( P_j \) of degree 3, such that \( P_1(t) - P_2(t) = c \) where \( c \) is a constant. The Bochner tensor vanishes precisely when \( c = 0 \), and the metric is then the Fubini–Study metric. We therefore assume that \( c \neq 0 \) in order to obtain new examples. Also, for compactness, the scalar curvature must be positive (otherwise the Ricci tensor would be nonpositive, contradicting the existence of Killing vector fields with zeros), which implies that the polynomials \( P_j \) are monic.

It remains to solve the compactification conditions. For simplicity, we shall take the lattice \( \Lambda \subset \mathbb{R}^2 \) to be \( \mathbb{Z}^2 \) or a sublattice. The conditions of Proposition 9 can then be satisfied by supposing that \( P_j \) has integer roots (including the endpoints \( \alpha_j \) and \( \beta_j \)) and \( C \) is chosen so that \( 2/\Theta'(\alpha_j) = c_j^\alpha = -2C/P_j'(\alpha_j) \) and \( 2/\Theta'(\beta_j) = c_j^\beta = -2C/P_j'(\beta_j) \) are all integers for \( j = 1, 2 \).

The condition \( (32) \) implies that \( P_1 \) and \( P_2 \) have three distinct roots, \( p_1 < q_1 < r_1 \) and \( p_2 > q_2 > r_2 \), respectively. The condition \( P_1 - P_2 = c \) reads

\[
\begin{align*}
p_1 + q_1 + r_1 &= p_2 + q_2 + r_2, \\
p_1^2 + q_1^2 + r_1^2 &= p_2^2 + q_2^2 + r_2^2.
\end{align*}
\]

Positivity and \( (32) \) give \( \alpha_1 = p_1, \; \beta_1 = q_1, \; \alpha_2 = q_2, \; \beta_2 = p_2 \) and hence (without loss) \( q_1 < q_2 \). Taking the roots to be all integral, we note that, up to an affine deformation of the hamiltonian 2-form and orbifold coverings/quotients, we can also assume that \( \gcd(p_1, q_1, r_1, p_2, q_2, r_2) = 1 \) and

\[ p_1 + q_1 + r_1 = 0 = p_2 + q_2 + r_2. \]

A class of solutions to this problem is obtained by taking any coprime positive integers \( (p, q) \) with \( p > q \) and putting

\[ p_1 = -p, \; q_1 = -q, \; r_1 = p + q, \; p_2 = p, \; q_2 = q, \; r_2 = -p - q. \]

With these assumptions, we have

\[
\begin{align*}
\Theta_1(\xi) &= -\frac{(\xi + p)(\xi + q)(\xi - p - q)}{C}, \\
\Theta_2(\xi) &= -\frac{(\xi - p)(\xi - q)(\xi + p + q)}{C}.
\end{align*}
\]

The corresponding Delzant polytope \( \Delta \) is the quadrilateral with vertices

\[ (0, -p^2), \; (0, -q^2), \; (p - q, -pq), \; (q - p, -pq) \]

and one-dimensional faces \( F_j^{\alpha}, \; F_j^{\beta}, \; j = 1, 2 \) determined by the lines

\[ \{ \sigma : \ell_j^{\alpha}(\sigma) = 0 \}, \; \{ \sigma : \ell_j^{\beta}(\sigma) = 0 \}, \]
orbifold covers \( \hat{\Delta} \) singular by the normals of \( \Delta \), see Remark 2) give rise only to Sasaki–Einstein orbifolds.

M groups of \( M \) of \( M \) are abelian \([29]\) so that if \( S \) \( S \) is nonsingular even though \( S \) is singular \([28]\) \( S \) is nonsingular if and only if all local uniformizing groups of \( M \) \( M \) become a compact, Einstein, non-Kähler hermitian complex orbifold surface (see \([28]\) for a classification in the smooth case).

(ii) A similar construction yields a countable family of compact orbifold complex surfaces supporting orthotoric weakly Bochner-flat metrics which are neither Bochner-flat nor Kähler–Einstein (see \([31]\) for a classification in the smooth case).

(iii) According to \([3]\), the primitive part of the hamiltonian 2-form \( \phi \) associated to \( g_{p,q} \) defines an integrable almost-complex structure \( I \) on \( M(p,q) \), which is compatible with \( g_{p,q} \) but induces the opposite orientation to the one of \( M(p,q) \). With respect to this structure, \((M(p,q), g_{p,q}, I)\) become a compact, Einstein, non-Kähler hermitian complex orbifold surface (see \([28]\) for a classification in the smooth case).

We summarize our construction as follows.

**Theorem 4.** There is a family of nonequivalent compact Kähler–Einstein orthotoric orbifold surfaces \((M(p,q), g_{p,q})\), depending on coprime positive integers \( q < p \).

**Remark 9.** (i) According to the results of \([3]\), the primitive part of the hamiltonian 2-form \( \phi \) associated to \( g_{p,q} \) defines an integrable almost-complex structure \( I \) on \( M(p,q) \), which is compatible with \( g_{p,q} \) but induces the opposite orientation to the one of \( M(p,q) \). With respect to this structure, \((M(p,q), g_{p,q}, I)\) become a compact, Einstein, non-Kähler hermitian complex orbifold surface (see \([28]\) for a classification in the smooth case).

(ii) A similar construction yields a countable family of compact orbifold complex surfaces supporting orthotoric weakly Bochner-flat metrics which are neither Bochner-flat nor Kähler–Einstein (see \([3]\) for a classification in the smooth case).

(iii) According to \([3]\), any Kähler–Einstein orbifold \((M, J, \omega)\) of complex dimension \( m \) gives rise to a Sasaki–Einstein structure on the total space \( S \) of a principal \( S^1 \) bundle over \( M \) (which is suitably associated to the canonical bundle of \( M \)). In general, \( S \) is an \((2m + 1)\)-dimensional orbifold rather than a manifold, but it may happen that \( S \) is nonsingular even though \( M \) is singular \([3] \): in fact, \( S \) is nonsingular if and only if all local uniformizing groups of \( M \) inject into the structure group \( S^1 \) (see \([3]\) Theorem 2.3)). In the case of toric Kähler orbifolds all local uniformizing groups are abelian \([29]\) so that if \( S \) is nonsingular, then all local uniformizing groups of \( M \) must be cyclic. Using this observation one can show that the universal orbifold covers \( \tilde{M}(p,q) \) of \( M(p,q) \) (the one which corresponds to the lattice generated by the normals of \( \Delta \), see Remark \([3]\) give rise only to singular 5-dimensional Sasaki–Einstein orbifolds.
4. Compact Kähler manifolds with Hamiltonian 2-forms

We now combine the work of the previous three sections to classify, up to a covering, compact Kähler manifolds with a Hamiltonian 2-form. In the case of Kähler surfaces we can refine the classification. We end by giving some examples of extremal and weakly Bochner-flat Kähler metrics with Hamiltonian 2-forms.

4.1. General classification. There are two parts to a general classification of compact Kähler manifolds with a Hamiltonian 2-form. First, we must classify the possible equivariant biholomorphism types of manifolds which can admit a Hamiltonian 2-form. Second, we describe the compatible Kähler structures on such manifolds which do admit Hamiltonian 2-forms.

The equivariant biholomorphism type is described in parts (ii)–(iv) of the following theorem: we show that, up to a blow-up and a covering, a compact Kähler manifold with a Hamiltonian 2-form. In part (v) of the theorem, we show that the relevant Kähler structures are given by a special case of the generalized Calabi construction with $\mathcal{V} = \mathbb{CP}^\ell$. Conversely we show that this construction produces compact Kähler manifolds with a Hamiltonian 2-form.

Theorem 5. Let $(M, g, J, \omega)$ be a compact connected Kähler $2m$-manifold with a Hamiltonian 2-form $\phi$ of order $\ell \geq 0$, with nonconstant roots $\xi_1, \ldots, \xi_\ell$ and (distinct) constant roots $\eta_1, \ldots, \eta_N$, $N \geq 0$.

(i) The elementary symmetric functions $(\sigma_1, \ldots, \sigma_\ell)$ of $(\xi_1, \ldots, \xi_\ell)$ are the components of the momentum map $\sigma: M \to \mathbb{R}^{\ell\ell}$ of an $\ell$-torus $T^\ell$ ≤ Isom$(M, g)$. The image $\Delta$ of $\sigma$ is a Delzant simplex in $\mathbb{R}^{\ell\ell}$, whose interior is the image under the elementary symmetric functions of a domain $D = \prod_{j=1}^\ell (\beta_{j-1}, \beta_j)$ with $\beta_0 < \beta_1 < \cdots < \beta_\ell$.

(ii) Let $S_\Delta$ be the stable quotient of $M$ by the complex torus $T^\ell$ and let $\hat{M}$ be the blow-up of $M$ along the inverse image of the codimension one faces $F_0, F_1, \ldots, F_\ell$ of $\Delta$. Then there are holomorphic line bundles $L_0, L_1, \ldots, L_\ell$ over $S_\Delta$ (uniquely determined up to overall tensor product with a holomorphic line bundle) such that $\hat{M}$ is $T^\ell$-equivariantly biholomorphic to $P(L_0 \oplus L_1 \oplus \cdots \oplus L_\ell) \to S_\Delta$.

(iii) $S_\Delta$ is covered by a product $S$ of $N$ Hodge Kähler manifolds $(S_a, \pm \omega_a)$ of dimension $2d_a$, indexed by the constant roots $\eta_a$ ($S$ is a point if $N = 0$). There are constants $c, C_1, \ldots, C_N$ such that for $j = 0, \ldots, \ell$, $a = 1, \ldots, N$

\begin{equation}
\frac{1}{2}c \left( \prod_{k \neq j} (\eta_a - \beta_k) \right) (C_a(\eta_a - \beta_j) + 1) [\omega_a / 2\pi]
\end{equation}

is an integral cohomology class on $S_a$, and the pullback of $L_j$ to $S$ is a tensor product $\bigotimes_{a=1}^N \pi_a^* L_{j,a}$, where $\pi_a$ is the projection of $S$ to $S_a$ and $L_{j,a} \to S_a$ is a holomorphic line bundle with first Chern class given by [11].

(iv) The subset $B$ of those $j \in \{0, \ldots, \ell\}$ for which the blow up over the face $F_j$ is nontrivial corresponds bijectively to a subset $C$ of $\{1, \ldots, N\}$ such that for $j \in B$ corresponding to $a \in C$, $\eta_a = \beta_j$, $S_a = \mathbb{CP}^{d_a}$, $\pm \omega_a$ is the Fubini–Study metric on $S_a$ of constant holomorphic sectional curvature $\pm c \prod_{k \neq j} (\beta_j - \beta_k)$, and (without loss) $L_{j,a} = \mathcal{O}(-1)$ and $L_{k,a} = \mathcal{O}$ for $k \neq j$.

For a $\not\in C$ either $\eta_a < \beta_0$ or $\eta_a > \beta_\ell$. 
(v) The Kähler metric on \( M \) and its pullback to \( \hat{M} \) are determined by the explicit metric \([4] \) on \( M^0 \), where:

- the pullback to \( S = \prod_{a=1}^{N} S_a \) of the Kähler quotient metric on \( S_\Delta \) induced by 
  \[ \sigma(\xi_1, \ldots, \xi_\ell) \in \Delta^0 \] 
  is the Kähler product metric

\[
\sum_{a=1}^{N} \left( \prod_{j=1}^{\ell} (\eta_a - \xi_j) \right) d_a;
\]

- \( \theta_1, \ldots, \theta_\ell \) are the components of a connection on \( \hat{M} \to S_\Delta \) associated to a principal \( \mathbb{T} \)-connection;

- for \( j = 1, \ldots, \ell \), \( F_j(t) = p_c(t) \Theta(t) \), where \( p_c(t) = \prod_{a=1}^{N} (t - \eta_a) d_a \),

\[
(-1)^{m-j} \Theta > 0 \quad \text{on} \quad (\beta_j, \beta_j),
\]

\[
\Theta(\beta_j) = 0, \quad \Theta'(\beta_j) = -c \prod_{k \neq j} (\beta_j - \beta_k),
\]

and the metric on the \( \mathbb{CP}^\ell \)-fibres of \( \hat{M} \to S_\Delta \) is the orthotoric Kähler metric \([23] \) with \( \Theta_j(t) = \Theta(t) \).

Conversely, suppose \( S \) is a product of Hodge Kähler manifolds \((S_a, \pm g_a, \pm \omega_a)\) and constants \( \beta_0, \ldots, \beta_\ell, \eta_1, \ldots, \eta_N, c_0, c_1, \ldots, c_N \) satisfying the conditions in (i)–(iv) above and such that \([12] \) is positive for \( \sigma(\xi_1, \ldots, \xi_\ell) \in \Delta^0 \).

Then there is a complex manifold \( M \) obtained by a blow-down of a projective bundle \( \hat{M} = P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S \) which gives rise to these data. Further, for any smooth function \( \Theta \) on \([\beta_0, \beta_\ell] \) satisfying \([13] – [14] \), a Kähler metric of the form \([11] \), with \( F_j(t) = p_c(t) \Theta(t) \), is globally defined on \( M \) and admits a hamiltonian 2-form of order \( \ell \).

Proof. Consider the explicit form \([3] \) of the metric on the open subset \( M^0 \) of \( M \). By Lemma \([4] \) the map \( \sigma = (\sigma_1, \ldots, \sigma_\ell) : M \to \mathbb{R}^\ell \) generates a rigid hamiltonian torus action, and the Kähler quotient metric (i.e., \([12] \)) is clearly semisimple, so that by Theorem \([2] \) there is a cover of \( M \) which is given by the generalized Calabi construction. We shall therefore suppose \( \Gamma \) is trivial in the following.

The Kähler metrics \( \pm \omega_a \) are determined by \([12] \) and the constants \( c_{a0} \) and \( c_a = (c_{a1}, \ldots, c_{a\ell}) \) appearing in the generalized Calabi data are

\[
c_{a0} = \eta_a^\ell, \quad c_{ar} = (-1)^r \eta_a^{\ell-r}, \quad r = 1, \ldots, \ell.
\]

Since the roots \( \xi_1, \ldots, \xi_\ell \) of the momentum polynomial \( p_{nc}(t) \) are smooth, functionally independent and pairwise distinct on \( M^0 \), with orthogonal gradients, the toric Kähler manifold \( \mathcal{V} \) appearing in the generalized Calabi data is orthotoric.

(i) \( \sigma \) is a momentum map by definition, and by Proposition \([7] \) its image \( \Delta \) is a simplex as stated. In particular, the codimension one faces \( F_0, F_1, \ldots, F_\ell \) correspond to the boundary points \( \beta_0 < \beta_1 < \cdots < \beta_\ell \) of \( D \), and \( \mathcal{V} \) is biholomorphic to \( \mathbb{CP}^\ell \).

(ii) \( M^0 \) is a holomorphic principal \( \mathbb{T}^c \)-bundle over \( S \) and the blow up of \( \hat{M} \) along the inverse image of the codimension one faces is equivariantly biholomorphic to a projective bundle \( M^0 \times_{\mathbb{T}^c} \mathbb{CP}^\ell \) with a global fibre preserving \( \mathbb{T}^c \) action. This action identifies \( \hat{M} \) with \( P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \) for holomorphic line bundles \( \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_\ell \) over \( S \) (uniquely determined as stated) in such a way that the \( 2(\ell-1) \)-dimensional orbits of \( \mathbb{T}^c \) in each fibre are orbits of elements of \( P(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_\ell) \) with one homogeneous
coordinate vanishing. We label the line bundles so that the codimension one face $F_j$ corresponds to the orbit of $\mathbb{T}^c$ with $L_j$ component vanishing.

(iii) We only need to construct the line bundles $L_{j,a}$ and establish the formulas \((41)\) for their first Chern classes. The explicit form \((3)\) of the metric on the principal bundle $M^0$ shows that the connection $1$-forms $\theta_r$, with $\theta_r(K_s) = \delta_{rs}$, satisfy

\[
d\theta_r = \sum_{a=1}^{N} (-1)^r \eta_a^{\ell - r} \omega_a,
\]

where $\pm \omega_a$ are the Kähler forms of the globally defined metrics $\pm g_a$ on $S_a$. Note that the $\theta_r$ are not necessarily integral. The integral principal connection forms are those which evaluate to integers on the Euler fields $X_0, \ldots, X_\ell$, which, according to Proposition \((8)\) are given by \((30)\):

\[
\theta_{p,q} = \sum_{r=0}^{\ell} \frac{1}{2} c(\sigma^p_r(\hat{\beta}_q) - \sigma^q_r(\hat{\beta}_p)) \theta_r,
\]

More specifically, this is the connection form of the line bundle $L_p^{-1} \otimes L_q$ (up to a sign convention). The curvature form of $L_p^{-1} \otimes L_q$ is therefore

\[
d\theta_{p,q} = \sum_{a=1}^{N} \sum_{r=0}^{\ell} \frac{1}{2} c(-1)^r (\sigma^p_r(\hat{\beta}_q) - \sigma^q_r(\hat{\beta}_p)) \eta_a^{\ell - r} \omega_a
\]

\[
= \sum_{a=1}^{N} \frac{1}{2} c \left( \prod_{k \neq q} (\eta_a - \beta_k) - \prod_{k \neq p} (\eta_a - \beta_k) \right) \omega_a.
\]

It follows that for each $a = 1, \ldots, N$, the corresponding $2$-form in this sum is integral in the sense that the cohomology class

\[
\frac{1}{2} c \left( \prod_{k \neq q} (\eta_a - \beta_k) - \prod_{k \neq p} (\eta_a - \beta_k) \right) [\omega_a/2\pi]
\]

is in the image of $H^2(S_a, \mathbb{Z})$ in $H^2(S_a, \mathbb{R})$. If $\eta_a = \beta_j$ for some $j$, we deduce (by taking $p = j$, $q \neq j$) that $\frac{1}{2} c (\prod_{k \neq j} (\eta_a - \beta_k)) \omega_a$ is integral. Otherwise, this will differ from an integral class by a constant. Hence there are constants $C_1, C_2, \ldots, C_N$ such that for each $j = 0, \ldots, \ell$ and $a = 1, \ldots, N$, the $2$-form

\[
\frac{1}{2} c \left( \prod_{k \neq j} (\eta_a - \beta_k) \right) (C_a(\eta_a - \beta_j) + 1) \omega_a
\]

is also integral. Now the Lefschetz Theorem for $(1,1)$-classes implies that there are holomorphic line bundles $L_{j,a}$ with connection over $S_a$ whose curvature forms are given by \((19)\): the first Chern classes are then as stated in \((11)\). It follows that $L_j$ is the tensor product of $\bigotimes_{a=1}^{N} \pi_a^* L_{j,a}$ by a flat line bundle $F_j$. Since any flat line bundle on $S$ is a tensor product of flat line bundles pulled back from the factors $S_a$, we may use the freedom in the choice of $L_{j,a}$ to make $F_j$ trivial.

Finally, note that for each $a$, the Chern classes $c_1(L_{j,a})$ cannot vanish for all $j$. It follows that the manifold $S_a$ is Hodge, i.e., admits a Kähler metric whose Kähler class is integral in cohomology.

(iv) For any $\sigma$ in $\Delta^0$, the Kähler quotient metric \((12)\) is global on $S$, so that $p_{uc}(\eta_a) = \prod_{j=1}^{\ell} (\xi_j - \eta_a)$ does not vanish on $\Delta^0$. Hence no $\eta_a$ can belong to any of the open intervals $(\beta_{j-1}, \beta_j)$. Clearly when $\eta_a = \beta_j$ for some $j = 0, \ldots, \ell$, $p_{uc}(\eta_a)$ vanishes on the codimension one face $F_j$ of $\Delta$ and this is precisely the condition that
a blow-down occurs (over the factor $S_a$). The rest is immediate from the definition of generalized Calabi data apart from the normalization of the Fubini–Study metric on $S_a$. For this we note that the formula (41) gives

$$c_1(L_{j,a}) = \frac{1}{2}c \left( \prod_{k \neq j}(\beta_j - \beta_k) \right) [\omega_a/2\pi]$$

(and $c_1(L_{k,a}) = 0$ for $k \neq j$). Since $L_{j,a}$ has to be $O(-1)$, we must have

$$[\rho_a/2\pi] = (d_a + 1)c_1(L_{j,a}) = \frac{1}{2}c(d_a + 1) \left( \prod_{k \neq j}(\beta_j - \beta_k) \right) [\omega_a/2\pi],$$

where $\rho_a = \frac{Scal_a}{2d_a} \omega_a$ is the Ricci form of the Fubini–Study metric $\pm g_a$. The holomorphic sectional curvature $\pm \frac{1}{d_a(d_a + 1)} Scal_a$ is therefore as stated. (Note that this can be always achieved by rescaling $\omega_a$.)

(v) This is immediate from the explicit form of the metric, the generalized Calabi construction, and the necessity of the conditions of Theorem 3 for the compactification of orthotropic Kähler metrics on $\mathbb{C}P^\ell$.

For the converse, observe first that the integrality conditions ensure (by the Lefschetz Theorem for $(1,1)$-classes) that there are holomorphic line bundles $L_{j,a}$ over $S_a$ with first Chern classes given by (41), equipped with compatible connections whose curvatures are given by (49), and we define $L_j = \prod_{a=1}^N \pi_a^* L_{j,a}$.

Let $\tilde{M} = P(L_0 \oplus L_1 \oplus \cdots \oplus L_\ell) \cong M^0 \times _{\mathbb{T}^\ell} \mathbb{C}P^\ell$, where $\mathbb{T}^\ell$ acts by scalar multiplication on each line bundle $L_j$ modulo overall scalar multiplication on the direct sum (which acts trivially on the projective bundle), $M^0$ is the union of the open $\mathbb{T}^\ell$ orbits in each fibre of $\tilde{M} \to S$, and $\mathbb{C}P^\ell$ is toric under $\mathbb{T}^\ell$.

By (15) (see also Theorem 2), $\tilde{M}$ has a blow-down $M$, which collapses a family of divisors (which are closures of complex codimension one $\mathbb{T}^\ell$-orbits) corresponding to $\eta_a \in \mathcal{C}$ along the $\mathbb{C}P^{d_a}$ fibrations induced by the connection on $\tilde{M}$.

Because of the sufficiency of the conditions of Theorem 3 for the compactification of orthotropic Kähler metrics on $\mathbb{C}P^\ell$, we have generalized Calabi data for the construction of a Kähler metric on $M$ using Theorem 2, where $\mathcal{V} = \mathbb{C}P^\ell$ equipped with this orthotropic structure, the connection has curvature (15), and the constants are given by (49). On $M^0$ the Kähler structure is given by (41).

The hamiltonian 2-form $\phi = \sum_{r=1}^\ell (\sigma_r d\tau_1 - d\sigma_{r+1}) \wedge dt_r$ (defined on $M^0$) also extends on $M$. Indeed, it follows from [1] §2.2 that the 2-jet of $\phi$ is a parallel section (over $M^0$) of a vector bundle with linear connection globally defined on $M$. Since $M \setminus M^0$ has codimension at least two in $M$, $\phi$ extends to the whole of $M$. \qed

Remark 10. It follows from the proof of Theorem 2 that $M$ is a bundle of restricted toric Kähler manifolds over $\prod_{a \in \mathcal{C}} S_a$. The typical fibre $X$ is a toric Kähler manifold of dimension $2k$, $k = \ell + \sum_{a \in \mathcal{C}} d_a$, obtained as a blow-down of a $\mathbb{C}P^\ell$ bundle over a product of $\# \mathcal{C} < \ell + 1$ projective spaces as in §1.8 and admits a hamiltonian 2-form of order $\ell$. However, by [1] §§2.4, §5.4, the Fubini–Study metric on $\mathbb{C}P^k$ admits a hamiltonian 2-form of order $\ell$ with any number of distinct constant roots between 0 and $\ell + 1$, with all factors in the Kähler quotient being blown down over some face of the Delzant polytope of $\mathbb{C}P^\ell$. It follows that $X$ is biholomorphic to $\mathbb{C}P^k$, and $M$ is a bundle of projective spaces over $\prod_{a \in \mathcal{C}} S_a$. However, the metric on the fibres need not be orthotropic unless $k = \ell$. In fact it is not hard to see directly that the blow-down of $P(L_0 \oplus O \oplus L_1 \oplus O \oplus \cdots \oplus L_\ell \oplus O(-1) \oplus \cdots \oplus L_\ell \oplus O) \to S' \times \mathbb{C}P^d$ is biholomorphic to $P(L_0 \oplus L_1 \oplus \cdots L_\ell \oplus \mathbb{C}^{d+1} \oplus \cdots \oplus L_\ell) \to S'$. 
Kähler surfaces with Hamiltonian 2-forms. In this subsection we specialize to the case that $(M, g, J, \omega)$ is a smooth compact Kähler surface with a nontrivial Hamiltonian 2-form; here nontrivial means that $\phi$ is not a constant multiple of $\omega$. We obtain a complete classification, overcoming the issue of coverings raised in the previous subsection. We first recall that if $\phi$ is a nontrivial Hamiltonian 2-form, then for any real numbers $a, b$ ($a \neq 0$), the affine deformation $a\phi + b\omega$ is again a nontrivial Hamiltonian 2-form (of the same order as $\phi$).

**Proposition 10.** Let $(M, g, J)$ be a connected Kähler surface not of constant holomorphic sectional curvature. Then $(M, g, J)$ admits at most one (up to an affine deformation) nontrivial Hamiltonian 2-form, even locally.

**Proof.** According to [3] Lemmas 2 and 6, the primitive part $\phi_0$ of a nontrivial Hamiltonian 2-form $\phi$ defines (on the open dense subset $U$ where $\phi_0 \neq 0$) a conformally Kähler hermitian structure $(g, I)$, such that $I$ and $J$ induce opposite orientations on $U$; then the antiselfdual tensor $W^-$ of $g$, with respect to orientation induced by $J$, has degenerate spectrum on $U$, hence on $M$; moreover, $\phi_0$ is an eigenform of $W^-$, whose eigenvalue, at each point where $W^-$ does not vanish, is the unique simple eigenvalue of $W^-$. We also know that $\phi$ commutes with the Ricci form $\rho$—see [4], §2.2—so on an open subset where $\rho_0 \neq 0$, $\phi_0$ is proportional to $\rho_0$.

It follows that on any open subset where $W^- \neq 0$ or $\rho_0 \neq 0$, the primitive parts of two nontrivial Hamiltonian 2-forms, $\phi$ and $\phi'$, are related by $\phi_0 = f\phi'_0$ for a smooth function $f$; since the primitive part of any Hamiltonian 2-form satisfies $d(\phi_0/|\phi_0|^2) = 0$ (see [3] Lemma 2), $f$ must be a constant, i.e., $\phi_0 = a\phi'_0$. By unique continuation [4], §2.2, this equality holds everywhere on $M$ and so $\phi - a\phi'$ is a Hamiltonian 2-form with vanishing primitive part, hence a multiple of $\omega$. Thus $\phi$ and $\phi'$ are affinely equivalent unless $W^-$ and $\rho_0$ are identically zero. 

**Remark 11.** The above result is optimal: according to [4], §2.3, each of the manifolds $\C P^2$, $\C^2$ and $\C H^2$ endowed with its canonical Kähler structure admits a 9-dimensional family of nontrivial Hamiltonian 2-forms.

**Theorem 6.** Let $(M, J)$ be a compact complex surface which supports a Kähler metric $g$ with a nontrivial Hamiltonian 2-form $\phi$. Then the following cases occur.

(i) $\phi$ is of order zero; then $(M, J)$ is biholomorphic to a compact locally symmetric Kähler surface of reducible type.

(ii) $\phi$ is of order one; then $(M, J)$ is biholomorphic to either $\C P^2$ or to a ruled surface of the form $P(O \oplus L) \to S$ where $S$ is a compact complex curve and $L$ is a holomorphic line bundle over $S$ of positive degree.

(iii) $\phi$ is of order two; then $(M, J)$ is biholomorphic to $\C P^2$.

Each complex surface listed in (i)–(iii) above admits (infinitely many) Kähler metrics with nontrivial Hamiltonian 2-forms of the corresponding order.

**Proof.** (i) If the order of $\phi$ is zero, i.e., if $\phi$ is parallel, then, by the deRham decomposition theorem, the universal cover $(\tilde{M}, \tilde{g})$ of $(M, g)$ is a Kähler product $(\tilde{U}_1 \times \tilde{U}_2, g_1 \times g_2)$ where each $\tilde{U}_i$ biholomorphic to $\C P^1$, $\C H^1$ or $\C$, equipped with a Kähler metric $g_i$. Taking the conjugate complex structure on one of the factors defines a Kähler structure $(\tilde{g}, I)$ on $\tilde{M}$, with the opposite orientation to $(g, J)$. By a result of Kotschick [27], $(M, J)$ is either a geometric complex surface [35] or is a minimal ruled surface.

If $(M, J)$ is geometric complex surface, the fundamental group acts biholomorphically and isometrically with respect to the product of constant curvature metrics on $\tilde{U}_i$, i.e., $(\tilde{M}, \tilde{g})$ carries a reducible locally symmetric Kähler structure.
If \((M, J)\) is a minimal ruled surface, then it is biholomorphic to the total space of the projectivization \(P(E)\) of a rank 2 holomorphic vector bundle \(E\) over a compact complex curve \(S\) (see for instance [7]) and so, without loss, \(U_1\) is the universal cover of \(S\) and \(U_2 = \mathbb{CP}^1\): the Kähler product metric \(\tilde{g} = g_1 \times g_2\) must be compatible with the holomorphic splitting. If \(U_1 = \mathbb{CP}^1\) as well, then \(M = \mathbb{CP}^1 \times \mathbb{CP}^1\) so it admits a product symmetric structure. Suppose \(U_1 = \mathbb{C}\) or \(\mathbb{CH}^1\); by Liouville’s Theorem, any holomorphic isometry of \((\tilde{M}, \tilde{g})\) has the form \(\Psi(z, w) = (\psi_1(z), \psi_2(z, w))\), where \(\psi_1\) is a holomorphic isometry of \((U_1, g_1)\) and (for any fixed \(z\)) \(w \mapsto \psi_2(z, w)\) is a holomorphic isometry of \((\mathbb{CP}^1, g_2)\). Since \(\psi_1\) is a holomorphic automorphism of \(U_1\), it preserves a constant curvature metric on \(U_1\); similarly, since \(\text{Isom}(g_2)\) is a compact subgroup of \(PSL(2, \mathbb{C})\), it lies in a conjugate of \(PSU(2)\) and hence preserves a constant curvature metric on \(\mathbb{CP}^1\). Thus the fundamental group preserves the product of constant curvature metrics on \(U_1\) and \(\mathbb{CP}^1\), so \((M, J)\) is again a geometric complex surface supporting a reducible locally symmetric Kähler structure.

(ii) Suppose now that \(\phi\) has order 1. By Theorem 5, after blowing up \(M\) at most once, we get a compact complex surface \(\hat{M}\) which is a holomorphic \(\mathbb{CP}^1\)-bundle over a compact complex curve \(S\), i.e., \(M\) is a ruled complex surface [7]. If \(S \cong \mathbb{CP}^1\), then \(M\) is either \(\mathbb{CP}^1 \times \mathbb{CP}^1\) or a Hirzebruch surface \(F_k = P(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{CP}^1\).

Of these surfaces, only \(F_1\) is not minimal: it is the blow-up \(\mathbb{CP}^2\) at one point. We conclude that \(M\) is either \(\mathbb{CP}^2\) or can be written as \(P(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{CP}^1\), \(k \in \mathbb{Z}\). If \(S\) has genus \(g(S) \geq 1\), by using again Theorem 5, we have \(M = \hat{M}\) and therefore \(M\) is a (minimal) ruled surface \(P(E)\) over a compact complex curve \(S\), with the induced \(\mathbb{C}^\times\)-action tangent to the projective fibers. Clearly in the latter case \(E\) must be split, and so without loss, \(E = \mathcal{O} \oplus \mathcal{L}\).

As a final point, we have to show that we can assume \(\deg \mathcal{L} > 0\) (or \(k > 0\) in the case of \(F_k\)). But this is an immediate consequence of Theorem 5: the formula [11] specializes to give (see also [49]) \(c_1(\mathcal{L}) = \frac{1}{4}c(\beta_1 - \beta_0)[\omega_S/2\pi]\), where \(\beta_0 < \beta_1\) and \(c \neq 0\), while \(\pm \omega_S\) is the Kähler structure induced on stable quotient \(S\). Thus \(\deg \mathcal{L} \neq 0\) and since \(P(E) \cong \mathcal{L} \otimes \mathcal{L}^*\), we can assume that \(\deg \mathcal{L} > 0\).

(iii) This is an immediate consequence of Proposition 4.

It follows from Theorem 5 that each complex surface listed in Theorem 6 does admit infinitely many (non-isometric) Kähler metrics with nontrivial hamiltonian 2-forms of the corresponding order.

\(\Box\)

Remark 12. The complex surfaces in Theorem 6 also admit extremal Kähler metrics with nontrivial hamiltonian 2-forms, see [11, 33].

4.3. Examples: extremal and weakly Bochner-flat Kähler metrics. We turn now to the construction of particular types of Kähler metrics with hamiltonian 2-forms. From this point of view, the notion of a hamiltonian 2-form is simply a device which provides constructions of interesting Kähler manifolds, and this uses very little of the theory that we have developed: the converse part of Theorem 4 which essentially amounts to the sufficiency of the conditions for the compactification of a toric Kähler metric and for the construction of Kähler metrics on blow-downs. In fact, we shall mainly restrict attention here to metrics on projective line bundles (with no blow-downs) where these issues are trivial.

We recall from [4] how Bochner-flat, weakly Bochner-flat and extremal Kähler metrics with hamiltonian 2-forms arise. A Kähler manifold \(M\) is Bochner-flat if the Bochner tensor (a component of the Kähler curvature) vanishes, weakly Bochner-flat (WBF) if the Bochner tensor is co-closed, and extremal if the scalar curvature is a Killing potential (i.e., its symplectic gradient is a Killing vector field).
By the differential Bianchi identity, a Kähler metric is WBF if and only if its normalized Ricci form $\tilde{\rho} = \rho + \frac{\text{Scal}}{2(m+1)}\omega$ is a hamiltonian 2-form. It follows that a WBF Kähler manifold is extremal. Any Kähler–Einstein manifold is WBF, since $\tilde{\rho}$ a constant multiple of $\omega$; however, the hamiltonian 2-form in this case is trivial. To deal with this, and the case of extremal Kähler metrics, we shall suppose that there is a nontrivial hamiltonian 2-form $\phi$ on $M$ such that $\tilde{\rho} = a\phi + b\omega$ in the case of WBF Kähler metrics, and such that the scalar curvature $\text{Scal} = a\text{tr}_{\omega} \phi + b$ in the case of extremal Kähler metrics (for constants $a,b$).

Suppose that we have a Kähler manifold $(M, g, J, \omega)$ with a hamiltonian 2-form $\phi$ of order $\ell$ where the Kähler quotient is a product of $N$ Kähler manifolds $S_a$ of dimension $2d_a$, corresponding to the constant roots $\eta_a$ of $\phi$. The Kähler metric then has the explicit form $\tilde{\omega}$ and there is the following local classification result [4].

(i) $g$ is extremal, with $\text{Scal}$ as above, if and only if
- for all $j$, $F_j''(t) = \tilde{p}_c(t)q(t)$, where $\tilde{p}_c(t) = \prod_{a=1}^{N} (t - \eta_a)^{d_a-1}$ and $q$ is a polynomial of degree $\ell + N$ independent of $j$;
- for all $a$, $g_a$ has constant scalar curvature $q(\eta_a)/\prod_{b \neq a} (\eta_a - \eta_b)$.

$g$ then has constant scalar curvature if and only if $q$ has degree $\ell + N - 1$.

(ii) $g$ is weakly Bochner-flat, with $\tilde{\rho}$ as above, if and only if
- for all $j$, $F_j'(t) = \tilde{p}_c(t)q(t)$, where $\tilde{p}_c(t) = \prod_{a=1}^{N} (t - \eta_a)^{d_a}$ and $q$ is a polynomial of degree $\ell + 1$ independent of $j$;
- for all $a$, $S_a$ is Kähler–Einstein with scalar curvature $d_aq(\eta_a)$.

$g$ is then Kähler–Einstein if and only if $q$ has degree $\ell$.

(iii) [10] $g$ is Bochner-flat, with $\tilde{\rho}$ as above, if and only if
- for all $j$, $F_j(t) = \tilde{p}_c(t)q(t)$ where $\tilde{p}_c(t) = \prod_{a=1}^{N} (t - \eta_a)^{d_a+1}$ and $q$ is a polynomial of degree $\ell + 2 - N$ independent of $j$;
- for all $a$, $S_a$ has constant holomorphic sectional curvature and scalar curvature $d_a(d_a + 1)q(\eta_a)\prod_{b \neq a} (\eta_a - \eta_b)$.

$g$ has constant holomorphic sectional curvature if and only if $q$ has degree $\ell + 1 - N$.

We want to combine this local classification with the global construction of Theorem 5. To do this, we have to satisfy the boundary conditions of [14], and the integrality conditions for the first Chern classes $c_1(\mathcal{L}_{j,a})$ given by [11].

Remark 13. In practice, we need enough freedom in the choice of $F(t)$ and the constants both to satisfy these boundary conditions and to prescribe the first Chern classes freely (up to some open conditions), since otherwise we face potentially nontrivial diophantine problems on our data. Let us analyse the implications of this in the case that there are no blow-downs, i.e., $M = P(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_\ell) \rightarrow S$ where $S$ is the product of compact Hodge complex manifolds $S_1, \ldots, S_N$. Thus we have $N(\ell + 1)$ integrality conditions, together with $2(\ell + 1)$ boundary conditions for the function $F(t) = p_c(t)\Theta(t)$, giving $(N + 2)(\ell + 1)$ constraints on $F(t)$ and the constants $\beta_0, \ldots, \beta_\ell, \eta_1, \ldots, \eta_N$ and $c, C_1, \ldots, C_N$. Three of these constants, say $c, \beta_0, \beta_\ell$ are useless for satisfying the constraints, since there is a homothety freedom $g \mapsto kg$ in the Kähler metric and an affine freedom $\xi_j \mapsto a\xi_j + b$ in the orthotoric coordinates. We therefore have $2N + \ell - 1$ effective constants. This leaves $N(\ell + 1) + 2(\ell + 1) - 2N - \ell + 1 = (N + 1)(\ell - 1) + 4$ constraints on $F(t)$. Subtracting this from the number of coefficients defining $F(t)$ gives the expected dimension of the moduli space of solutions, which we require to be nonnegative.

(i) In the extreme case, $F(t)$ is determined by $\ell + 3 + N$ constants, giving $N(2 - \ell)$ dimensional moduli and forcing $\ell \leq 2$. 

(ii) In the WBF case, $F(t)$ is determined by $\ell + 3$ constants, giving $N(1 - \ell)$ dimensional moduli and forcing $\ell \leq 1$

(iii) In the Bochner-flat case, $F(t)$ is determined by $\ell + 3 - N$ constants, giving $-N\ell$ dimensional moduli and forcing $\ell = 0$, $N \leq 2$. (This parameter count agrees with the classification of Bryant: the only compact Bochner-flat Kähler manifolds are products of at most two constant holomorphic sectional curvature manifolds.)

We concentrate here on WBF Kähler metrics on projective line bundles, by assuming the existence of a hamiltonian 2-form $\phi$ of order 1. In this case, once we fix the base manifolds $S_a$ and the line bundles $L_{j,a}$, the moduli are zero dimensional.

4.3.1. The general setting. In order to render our discussion as self-contained as possible, we first recall our notations. Let $(S_a, \pm g_a, \pm \omega_a)$, $a = 1, \ldots N$, be compact connected Kähler manifolds of real dimension $2d_a$, associated to the distinct constant roots $\eta_a$ of the hamiltonian 2-form. A Kähler metric with a hamiltonian 2-form of order 1 is defined on a projective line bundle over $S = S_1 \times \cdots \times S_N$, using a metric of the form

$$g = \sum_{a=1}^{N} (z - \eta_a)g_a + \frac{\prod_{a=1}^{N} (z - \eta_a)^{d_a}}{F(z)} dz^2 + \frac{F(z)}{\prod_{a=1}^{N} (z - \eta_a)^{d_a}} \theta^2,$$

$$\omega = \sum_{a=1}^{N} (z - \eta_a)\omega_a + dz \wedge \theta,$$

$$d\theta = \sum_{a=1}^{N} \omega_a,$$

where we normalize the momentum interval for $z$ to $[-1,1]$ and require $|\eta_a| > 1$. Note that each Kähler metric $g_a$ can be positive or negative definite, depending on the sign of $\eta_a$, and for convenience it is taken here with the opposite sign to the one used in equation (3)---observe that $p_{\text{nc}}(\eta_a) = \eta_a - z$ rather than $z - \eta_a$). It is convenient to set $\eta_a = -1/x_a$; now the sign of $g_a$ is the sign of $x_a$.

The projective line bundle is $M = P(O \oplus L) \cong P(O \oplus L^{-1})$, where up to a sign convention, $\theta$ is a connection form on the principal $S^1$-bundle associated to $L$ with curvature $d\theta$. By Theorem 5, $g$ compactifies on $M$ when $F(z)$ satisfies the following boundary conditions (for the fibrewise compactification on $CP^1$):

$$F(\pm 1) = 0, \quad F'(\pm 1) = \mp 2p_c(\pm 1).$$

For the existence of $L$, we require that $\omega_a$ is integral, i.e., $[\omega_a/2\pi]$ is in the image of $H^2(S_a, \mathbb{Z})$ in $H^2(S_a, \mathbb{R})$, and we write $L = \bigotimes_a L_a$, where $L_a$ is (the pullback to $M$ of) a line bundle on $S_a$ with $c_1(L_a) = [\omega_a/2\pi]$.

In order to obtain WBF Kähler metrics, the $S_a$ must Kähler–Einstein, i.e., with Ricci form $\rho_a = s_a\omega_a$. Since $[\rho_a/2\pi]$ is an integral class, the first Chern class of the anti-canonical bundle, $s_a = p_a/\rho_a$ for integers $p_a, \rho_a$. If $s_a \neq 0$, we take $p_a$ maximal so that the anti-canonical bundle has a $p_a$th root (i.e., $[\rho_a/2\pi p_a]$ is a primitive class); then $L_a$ is $K^{-s_a/p_a}$ twisted by a flat line bundle.

Remark 14. If $S_a$ is a Riemann surface $\Sigma_g$ of genus $g$, then $p_a = 2|g - 1|$, while if $S_a = CP^{d_a}$, then $p_a = d_a + 1$ so that $K^{-1/p_a} = O(1)$. More generally, if the scalar curvature of $S_a$ is positive, then $p_a \leq d_a + 1$ by Kobayashi–Ochiai [21].

The remaining conditions to obtain a WBF metric (as in (4.3 ii) above) are

$$F'(z) = p_c(z)(b_1 z^2 + b_0 z + b_1),$$

where $p_c(z) = \prod_{a=1}^{N} (z - \eta_a)^{d_a}$, and

$$2s_a = b_1\eta_a^2 + b_0\eta_a + b_1.$$
Using the boundary conditions \((52)\) and the equation \((53)\) for \(F'\), we deduce that \(b_0 = -2\) and \(b_1 = -b_{-1}\). So, re-naming \(b_{-1}\) to \(B\), equation \((53)\) becomes
\[
F'(z) = p_c(z) (B(z^2 - 1) - 2z)
\]
and \((54)\) gives
\[
B(1 - x_a^2) = 2x_a (s_a - 1).
\]
g is Kähler–Einstein if and only if \(B = 0\), which holds if and only if \(s_a = 1/x_a\) for all \(a\). (This implies in particular that the base factors have positive scalar curvature.)

On the other hand, given the above, then \((52)\) is satisfied if and only if we set
\[
F(z) = \int_{-1}^{1} p_c(t) (B(t^2 - 1) - 2t) dt
\]
and
\[
\int_{-1}^{1} p_c(t) (B(t^2 - 1) - 2t) dt = 0.
\]
Since \(F'(z)\) only changes sign once on the interval \((-1, 1)\), \(F(z)\) as defined above will not have any zeroes between \(z = -1\) and \(z = 1\). Therefore, as the sign of \(F(z)\) equals the sign of \(p_c(z)\) between \(-1\) and \(1\), the metric \(g\) will be positive definite.

So, in conclusion, the problem of constructing a WBF Kähler metric on \(M\) (for given Kähler–Einstein manifolds \(S_a\) with \(s_a = p_a/q_a\)) reduces to finding solutions \(B, x_1, \ldots, x_N\) to \((56)\) and \((57)\). However, \(p_c(t)(1 - t^2)\) has constant sign on \((-1, 1)\), so \(B\) is uniquely determined by \((57)\): substituting for \(B\) from \((55)\) (for each \(a\)) it suffices to show that there exist distinct \((x_1, \ldots, x_N)\) with \(0 < |x_a| < 1\) such that
\[
h_a(x_1, \ldots, x_N) := \int_{-1}^{1} \tilde{p}_c(t) H_a(t) dt
\]
vanishes for \(a = 1, \ldots, N\), where \(\tilde{p}_c(t) = \prod_{b=1}^{N} (x_b t + 1)^{d_b}\) and
\[
H_a(t) = x_a (x_a s_a - 1)(1 - t^2) + t(1 - x_a^2) = x_a s_a (1 - t^2) + (t - x_a)(x_a t + 1)
\]

**Remark 15.** If \(s_b \neq s_a, x_b\) cannot equal \(x_a\), again since \(p_c(t)(1 - t^2)\) has constant sign on \((-1, 1)\). Hence if \(x_a = x_b\), then \(s_a = s_b\) and \(S_a \times S_b\) is Kähler–Einstein. Thus we do not actually need to check that \(x_1, \ldots, x_N\) are distinct: if \(x_a = x_b\), we still get a WBF Kähler metric, but the hamiltonian 2-form has fewer distinct constant roots.

**4.3.2. WBF Kähler metrics over Kähler–Einstein manifolds.** We consider the simplest case \(N = 1\), when \(S = S_1\) is a Kähler–Einstein manifold. Replacing the momentum coordinate \(z\) by \(-z\) if necessary (and dropping the 1 subscripts) we may suppose that we have to find \(0 < x < 1\) such that \(h(x) = 0\), where
\[
h(x) := \int_{-1}^{1} (xt + 1)^d (x(xs - 1)(1 - t^2) + t(1 - x^2)) dz.
\]
Since \(h(0) = 0, h'(0) = 2(d - 2)/3\) and the sign of \(h(1)\) is equal to the sign of \(s - 1\), we certainly have a solution \(0 < x < 1\) to \(h(x) = 0\) if \(d > 2\) and \(s < 1\).

For the case \(d = 2\) we calculate directly that
\[
h(x) = \frac{4x^2}{15} (s(x^2 + 5) - 6x)
\]
and there is a solution \(0 < x < 1\) to \(h(x) = 0\) if and only if \(0 < s < 1\).

**Theorem 7.** There are WBF Kähler metrics of the form \((51)\) on:
- \(P(O \oplus L) \to S\), where \(S\) is a compact Ricci-flat Kähler manifold of complex dimension \(\geq 3\) whose Kähler form \(\omega_S\) is integral, and \(L\) is a holomorphic line bundle with \(c_1(L) = [\omega_S/2\pi]\).
Theorem 8. \( P(O \oplus K^{-q/p} \otimes L_0) \rightarrow S \), where \( S \) is a compact negative Kähler–Einstein manifold of complex dimension \( \geq 3 \), \( q \in \mathbb{Z} \) with \( q < 0 \), \( K \) is the canonical bundle on \( S \), and \( L_0 \) is a flat line bundle on \( S \);

\( P(O \oplus K^{-q/p} \otimes L_0) \rightarrow S \), where \( S \) is a compact positive Kähler–Einstein manifold of complex dimension \( \geq 2 \), \( q \in \mathbb{Z} \) with \( q > p \), \( K \) is the canonical bundle on \( S \), and \( L_0 \) is a flat line bundle on \( S \).

For the case \( d = 1 \), we compute that

\begin{equation}
    h(x) = -\frac{2x}{3}(x^2 + 1 - 2sx)
\end{equation}

and there is a solution \( 0 < x < 1 \) to \( h(x) = 0 \) if and only if \( s > 1 \). Since \( S \) in this case is \( \mathbb{C}P^1 \), \( K = O(2) \) and the only possibility is \( s = 2 \), \( L = O(1) \), in accordance with the classification of [3].

4.3.3. WBF Kähler metrics over products of two Kähler–Einstein manifolds. In this section, we give a taste of the case \( N = 2 \), but we postpone a more thorough analysis to a subsequent paper. In this case we are looking for common zeros of the functions

\[ h_s(x_1, x_2) := \int_{-1}^{1} (x_1 t + 1)^{d_1} (x_2 t + 1)^{d_2} (x_a (x_a s_a - 1)(1 - t^2) + t(1 - x_a^2)) dt \]

\( (a = 1, 2) \) with \( 0 < |x_a| < 1 \). Analysing this problem in general involves some delicate calculus arguments, but there are some special cases which are straightforward. One of the simplest is the case that \( d_1 = d_2 \), and \( s_1 = -s_2 \), when symmetry solves the problem for us, and we recover some of the Kähler–Einstein metrics of Koiso and Sakane.

**Theorem 8.** [25, 26] On the total space of \( P(O \oplus O(k, -k)) \rightarrow \mathbb{C}P^d \times \mathbb{C}P^d \), with \( 1 \leq k \leq d \), there is a Kähler–Einstein metric, given on a dense open set by

\[ g = \left( \frac{d + 1}{k} + z \right) g_1 + \left( \frac{d + 1}{k} - z \right) g_2 + \frac{z^2 - \frac{(d+1)^2}{k^2}}{F(z)} dz^2 + \frac{F(z)}{z^2 - \frac{(d+1)^2}{k^2}} \theta^2, \]

where \((g_1, \omega_1)\) and \((g_2, \omega_2)\) are Fubini–Study metrics on the \( \mathbb{C}P^d \) factors with holomorphic sectional curvature \( 2/k \), \( d\theta = \omega_1 - \omega_2 \) and \( F(z) = \int_{-1}^{1} 2 t \left( \frac{(d+1)^2}{k^2} - t^2 \right) dt = -\frac{(d+1)^2}{k^2} (1 - z^2) + \frac{k}{2} (1 - z^k) \).

**Proof.** Let \( s_1 = -s_2 = \frac{d+1}{k} \) and \( x_1 = -x_2 = \frac{k}{d+1} \). Then clearly \( 0 < |x_a| < 1 \) and \( h_s(x_1, x_2) = 0 \) for \( a = 1, 2 \). Further, \( x_a = 1/s_a \) so the WBF metric is Kähler–Einstein. \( \square \)

In a subsequent paper, we generalize these metrics by proving the following.

**Theorem 9.** There is a WBF Kähler metric on the total space \( P(O \oplus O(k_1, k_2)) \rightarrow \mathbb{C}P^{d_1} \times \mathbb{C}P^{d_2} \) in the following cases:

- \( k_1 > d_1 + 1 \) and \( k_2 > d_2 + 1 \);
- \( 1 \leq k_1 \leq d_1 \) and \( 1 \leq -k_2 \leq d_2 \).

We illustrate this with the case \( P(O \oplus O(1, -2)) \rightarrow \mathbb{C}P^2 \times \mathbb{C}P^3 \), where \( d_1 = 2 \), \( d_2 = 3 \), \( s_1 = 3 \), \( s_2 = 4/(-2) = -2 \). The graphs of \( h_1 = 0 \) (solid) and \( h_2 = 0 \) (dashed) for \( 0 < x_1 < 1 \) and \(-1 < x_2 < 0 \) are plotted below. Proving that the graphs do cross as shown is a tedious calculus exercise.

We end this section by giving an example with a blow-down. Consider again \( P(O \oplus O(1, -1)) \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \). This carries a Koiso–Sakane Kähler–Einstein
metric by setting $x_1 = -x_2 = 1/2$, but it also admits two blow-downs in which a $\mathbb{C}P^1$ factor collapses at an endpoint of the momentum interval $[-1, 1]$. Such a collapse corresponds to setting $x_1 = 1$ and/or $x_2 = -1$. If we carry out both blow-downs, the resulting manifold is $\mathbb{C}P^3$, which admits a WBF metric, namely the Fubini–Study metric, so let us consider the case of a single blow-down. The two complex manifolds we obtain are both isomorphic to $P(\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathbb{C}^2) \to \mathbb{C}P^1$, so without loss, we suppose $x_1 = 1, -1 < x_2 < 0$.

This changes the boundary condition for $F'(z)$ at $z = -1$, since $p_c(z) = (z + 1)(z + 1/x_2)$ vanishes at one of the endpoints. By a straightforward application of L'Hôpital's rule we obtain

$$\left(\frac{F'}{p_c}\right)(-1) = 4$$

Thus (55) is replaced by:

$$F'(z) = (z + 1)(z + 1/x_2)(B(z^2 - 1) - 2z + z(z - 1)).$$

Setting $x_1 = 1$ automatically solves one of the integrality constraint with $s_1 = 2$, so it remains to show that we can find $x_2$ to satisfy the second constraint, with $s_2 = -2$. Proceeding as in the case of no blow-downs, this reduces to showing there is $-1 < x < 0$ with $f(x) = 0$ where

$$f(x) = \int_{-1}^{1} (t + 1)(xt + 1)(-x(2x + 1)(1 - t^2) + t(1 - x^2) + \frac{1}{2}(x - 1)(t - 1)(xt + 1))dt.$$  

This holds because $f(-1)$ is negative, while $f(0) = 0$ and $f'(0)$ is negative. Further $f(-1/2)$ is nonzero so the solution $x = x_2$ does not equal $1/s_2$, and the metric is not Kähler–Einstein.

**Theorem 10.** There is a WBF Kähler metric on $P(\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathbb{C}^2) \to \mathbb{C}P^1$ whose normalized Ricci form is a hamiltonian 2-form of order one. In particular, this is an extremal Kähler metric with non-constant scalar curvature.

### 4.3.4. Further extremal Kähler metrics.

Since any WBF metric is extremal, the results presented so far provide new examples of extremal Kähler metrics on projective line bundles and their blow-downs. Furthermore, to obtain an extremal Kähler metric, it suffices that the base manifolds $S_a$ are Hodge Kähler manifolds of constant scalar curvature, giving examples which are not WBF in general.

On the other hand, such an approach is not very satisfactory, since it produces only one extremal Kähler metric in each case, whereas the parameter count of Remark 13 suggests that these metrics should come in $N$ dimensional families (parameterized by admissible Kähler classes on $M$). When the base manifolds have non-negative scalar curvatures, we can obtain such $N$ dimensional families.
Theorem 11. For $a = 1, \ldots, N$, let $(S_a, \pm \omega_a)$ be Hodge Kähler manifolds of constant nonnegative scalar curvature, let $\mathcal{L}_a$ be a holomorphic line bundles on each $S_a$ with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$ and let $\mathcal{L} = \bigotimes_{a=1}^N \mathcal{L}_a$. Then $M = P(O \oplus \mathcal{L})$ admits an $N$ parameter family of extremal Kähler metrics. Furthermore, if the Kähler forms $\pm \omega_a$ do not all have the same sign (i.e., if $c_1(\mathcal{L})$ is strictly indefinite) there is an $N - 1$ dimensional subfamily of constant scalar curvature Kähler metrics on $M$.

This Theorem generalizes results of Hwang [21] and Hwang–Singer [22], and the proof is not materially different. The first of these two papers considers the case that the base manifold has constant eigenvalues of the Ricci tensor (e.g., a product of Kähler–Einstein manifolds) and the idea to weaken this condition is explored in the second paper. However, it is the notion of a hamiltonian 2-form that has selected for us a more general hypothesis for the base. We shall discuss this, and further results, in more detail in a subsequent paper.

Finally, we remark that the parameter count of Remark 13 suggests that one should be able to construct examples of extremal Kähler metrics on projective plane bundles (and their blow downs) over products of constant scalar curvature manifolds. Unfortunately, the existence problem here is considerably less tractible than in the case of WBF metrics on projective line bundles. Nevertheless, we hope to be able obtain examples in subsequent work.

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