Rotation of Polarization vector, in case of Non-Inertial Rotational Frame

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ABSTRACT

In the present work, rotation of the polarization vector of an electromagnetic wave is extensively calculated for a rotating observer, who can also be considered as a non-inertial accelerated observer. The rotation of the polarization vector calculated here is only due to the effect of the non-inertial frame, considering that the light source is at a distance and it is emitting a plane polarized light. Earth has been taken as an example of rotating frame and the amount of rotation of polarization vector is calculated where it is found that the amount of rotation of polarization vector depends on the azimuthal as well as a polar co-ordinate of the rotating frame. Present work also discusses the redshift caused by any rotating frame and the value of the redshift is calculated taking earth as an example of a rotating frame. The possibilities of experimentally verifying the predictions of present work are also discussed.

Keywords: methods: analytical—polarization—reference systems—relativity

1. INTRODUCTION

Light is an electromagnetic wave, and it moves along the curvature of space-time structure. From this very concept, it is a well-known fact that light shows a unique property, known as the gravitational deflection. Following the idea, it is obvious that the light ray will be deflected when it passes through a gravitational field, commonly known as gravitational deflection. The gravitational deflection of the light ray gives an undeniable proof of the general theory of relativity. Since the establishment of the general theory of relativity, many authors Eddington (1919); Dyson (1921); Von Kluber (1960); Vilenkin (1984); Sereno (2004); Sen (2010); Roy & Sen (2015) have carried out investigations for the proper understanding of the gravitational deflection. As gravitational deflection is a well-established phenomena, it gives an opportunity to study the polarization state of the light ray which passes through curved space-time. It has been found by some authors (Balazs (1958); Su & Mallett (1980)) that when a light ray passes through gravitational field having axial symmetry, then its polarization vector will be rotated by the gravitational field, quite analogous to the phenomenon which is being called Faraday Rotation by the magnetic field. Ghosh Sen (2016) recently investigated the rotation of polarization vector in case of both Schwarzschild and Kerr field. They showed that although the Schwarzschild field has no effect on the state of polarization of light, Kerr field would produce a rotation of polarization vector of the electromagnetic wave or light ray. Though there are reported works on the rotation of polarization vector by the space-time geometry introduced by gravitational mass, but there is not much investigation conducted in case of accelerated systems. Thus a natural question lies here, whether the curved space-time generated by accelerated observer has any effect on the polarization state of light ray or not? Here in the present work, the curved space-time generated by non-inertial accelerated frame is the subject of investigation.

In a discussion on the accelerated system, Mashhoon (1990) raised the question about the law which specifies the measurement by an accelerated observer and focussed on the hypothesis of locality which is described as “The presumed equivalence of an accelerated observer with a momentarily co-moving inertial observer underlies the standard relativistic formalism by relating the measurements of an accelerated observer to those of an inertial observer”. Mashhoon (1990) discussed briefly, the significance and the limitations of the hypothesis of locality in his work. To understand the motion under the influence of accelerated frame one must consider the hyperbolic motion in curved space-time, which had been elaborately discussed by Rindler (1960). He was the first person who studied relativistic motion under accelerated system and later Born coined it as hyperbolic motion. Rindler (1960) had obtained the differential equation of motion for a particle in uniform accelerated frame generalising the geometric characteristics.
of a rectangular hyperbola in Minkowskian space-time. It is termed as Hyperbolic motion from the fact that, as seen by the inertial frame observer if we plot the distance against the time on a Minkowski diagram, it describes a hyperbola.

To find out the transformation relation for the linearly accelerated frame let us consider $a$, is the proper linear acceleration of the observer, and it is accelerated parallel to the $+x^1$ axis. In general relativity, proper acceleration is an acceleration measurable by an accelerometer rigidly tied to a frame and does not occur by gravitation. We consider that an object is accelerated and is the coordinate system $\mathbf{K}(x^0, x^1, x^2, x^3)$. Then $\mathbf{K}(x^0, x^1, x^2, x^3)$ defines the co-ordinate frame where the inertial observer is situated.

Now for the constant acceleration, ($a$) the transformation relation parametrised by proper time ($\tau_0$) is given by (Rindler (1960); Castillo & Sanchez (2006); Rindler (2012));

$$ x^0 = c^2 \frac{\alpha}{a} \sinh \left( \frac{a \alpha}{c} \right), $$

$$ x^1 = c \frac{\alpha}{a} \cosh \left( \frac{a \alpha}{c} \right). $$

This is the known form of Rindler co-ordinate. The Rindler co-ordinate for the accelerated observer in instantaneous inertial frame $\tilde{K}$ (as defined above) and inertial frame $K$ can be also written as (Carroll (2004); Stephani (2004); Castillo & Sanchez (2006); Rindler (2012); Misner et al. (2017));

$$ x^0 = \left( \frac{c^2}{a} + \tilde{x}^1 \right) \sinh \left( \frac{a \tilde{x}^1}{c} \right), $$

$$ x^1 = \left( \frac{c^2}{a} + \tilde{x}^1 \right) \cosh \left( \frac{a \tilde{x}^1}{c} \right). $$

This is the most general form of Rindler coordinates for a uniform accelerated observer. Nelson (1987) also derived the transformation equation for a non-rotational uniform linear accelerated frame. He also derived the equation for the motion by the time-dependent, non-gravitational acceleration. Later Nelson (1987) extensively worked on the transformation relation between the rest frame and the accelerated frame considering both linear acceleration and rotational acceleration. Later his work had shown that the relation satisfies the familiar Rindler metric also. A similar transformation relation can be obtained from the work by Alsing & Milonni (2004). In their work Alsing & Milonni (2004), briefly discussed Hawking-Unruh temperature radiation for an accelerated frame. Also from their work, the transformation relation for the uniform accelerated frame can be derived as in Eqn.1. Recently the authors Scarr & Friedman (2016) had obtained the velocity as well as coordinate transformation relation for acceleration of a uniformly accelerated frame. In one of their previous work, Friedman & Scarr (2013) had discussed the four-dimensional covariant relativistic equation and discussed about the concept of a maximal acceleration. Castillo & Sanchez (2006) studied the uniformly accelerated motion, and estimated the red-shift of an electromagnetic wave.

Mashhoon (1990) showed that a uniformly rotating observer with radius $r$ has translational acceleration $r\Omega^2\gamma^2$, where $\gamma$ is the Lorentz factor, and $\Omega$ is the rate of rotation of frame per unit time (in other words angular velocity). Thus in continuation of the work by him, the rotating observer could be considered, as an accelerated system and the metric for the system has been given by (Mashhoon (1990));

$$ g_{00} = \gamma^2 \left[ 1 - \frac{\Omega^2}{c^2} (r + x^1)^2 - \frac{\Omega^2 r^2}{c^2} (x^2)^2 \right] $$

$$ g_{0\alpha} = -\frac{\gamma^2 \Omega}{c} x^\alpha $$

As per Mashhoon (1990), the rotating observer can be characterised by proper acceleration length $\frac{\gamma^2 \Omega}{c}$. In the above $X = (x^1, x^2, x^3)$ vector.

Adams (1925) first confirmed gravitational red-shift from the measurement of the apparent radial velocity of Sirius $B$, since then many observations were made to measure the gravitational red-shift (Freden & White (1959); Snider (1972)). Krisher et al. (1993) measured the gravitational redshift of the sun in the year 1993. Recently Dubey & Sen (2015) gave the analytic result for the gravitational redshift of rotating body and they also calculated the numerical value of redshift of some rotating heavenly bodies. In section 3, of the present work the redshift, $Z$ have been calculated analytically along with their numerical values for an observer on the surface of the earth on equatorial plane.

2. DISCUSSION ON ROTATING OBSERVER

Now consider the observer is sitting on a rotating frame, which has the rate of rotation $\Omega$. According to Mashhoon (1990) the distinction between accelerated observer in Minkowskian space-time and co-moving inertial observer is the presence of acceleration scales associated with non-inertial observer. In this paper, the rotation is considered along with the $x^3$ or $z$-axis. The light source is situated in a space where frame-dragging and other gravitational effect of rotating frame (body) is negligible, and the light ray is coming along $x^1$ or $x$-axis of the Cartesian system. From Eqn.3 the metric elements for the rotating frames can be derived, and these have been given by Mashhoon (1990);

$$ g_{00} = \gamma^2 \left[ 1 - \frac{\Omega^2}{c^2} (r + x^1)^2 - \frac{\Omega^2 r^2}{c^2} (x^2)^2 \right] $$

$$ g_{01} = \frac{\gamma^2 \Omega}{c} x^2 $$

$$ g_{02} = -\frac{\gamma^2 \Omega}{c} x^1 $$

$$ g_{03} = 0 $$

$$ g_{\alpha\beta} = -\delta_{\alpha\beta}. $$

As the only component of $\Omega$ lies along $x^3$ or $z$-axis exists, the other components of $\Omega$ don’t exist (i.e. $\Omega_x = \Omega_y = 0$ and $\Omega_z = \Omega$). Again, if the rate of rotation $\Omega$ becomes
zero from the Eqn.4, the metric elements are given as \( g_{00} = 1, g_{0\alpha} = 0, g_{\alpha\beta} = -\delta_{\alpha\beta} \), which are the metric for flat space-time.

2.1. Rotation of polarization vector

In order to calculate the amount by which the polarization vector of light ray will be rotated, we consider the light ray from the source to be plane polarized (as was done in our earlier work (Ghosh & Sen (2016))). To determine the position angle of the polarization vector of light ray received by the observer from the distant source, one has to consider the initial components of the polarization vector of the light ray \( E_y \) and \( E_z \) along the direction of \( x^2 \) and \( x^3 \)-axis respectively, where one should recall the fact that light ray has been travelling along \( x^1 \)-axis. Let us assume that the magnitude ratio between \( E_y \) and \( E_z \) is \( \xi \), in other words \( E_y = \xi E_z \) (Ghosh & Sen (2016)). So the initial position angle of the polarization vector \( \chi_{\text{source}} \) is given by:

\[
\chi_{\text{source}} = \arctan \left( \frac{E_y}{E_z} \right) = \arctan \xi \tag{5}
\]

Now the position angle of the polarization vector as will be measured by the observer in the rotating frame will be (Ghosh & Sen (2016)).

\[
\chi_{\text{observer}} = \arctan \left( \frac{D_y}{D_z} \right) \tag{6}
\]

Where \( D_y \) and \( D_z \) are the electric displacement component along the \( x^2 \) and \( x^3 \)-axis respectively. To find out the value of \( \chi_{\text{observer}} \) one must find out the relationship between electric displacement vector \( \vec{D} \) and electric vector \( \vec{E} \). The relation between electric displacement vector and the electric vector has been given by (Landau & Lifshitz (1971));

\[
\vec{D} = \frac{\vec{E}}{\sqrt{g_{00}}} + \vec{H} \times \hat{g} \tag{7}
\]

Where \( \vec{H} \) is the magnetic induction of the light ray received. Now from the Eqn.7 the components \( D_y \) and \( D_z \) are deduced as ((Landau & Lifshitz (1971); Ghosh & Sen (2016)):

\[
D_y = \frac{E_y}{\sqrt{g_{00}}} + H_z g_{01} \\
D_z = \frac{E_z}{\sqrt{g_{00}}} - H_y g_{01} \tag{8}
\]

The different components of the magnetic field of the incoming light ray can be written in terms of the electric field using the relation \( H_y = \hat{n} \times E_y \) (Landau & Lifshitz 1971, p112), where \( \hat{n} \) is the unit propagation vector parallel to the propagation of light ray (e.g. parallel to \( x \)-axis). Now taking the note that \( E_y = \xi E_z \) and considering only the magnitude of \( \vec{H} \) we can write from Eqn.8:

\[
\begin{align*}
D_y &= E_z \frac{\xi}{\sqrt{g_{00}}} + g_{01} \\
D_z &= E_z \frac{1}{\sqrt{g_{00}}} - \xi g_{01}
\end{align*} \tag{9}
\]

So from Eqn. 6 with the help from Eqn. 9 it can be written as (Ghosh & Sen (2016)):

\[
\chi_{\text{observer}} = \arctan \left( \frac{\xi + \sqrt{g_{00}g_{01}}}{1 - \xi \sqrt{g_{00}g_{01}}} \right) \tag{10}
\]

So the total rotation of polarization vector of received light ray (say \( \chi \)) can be calculated from Eqn. 10 and Eqn. 5, as:

\[
\begin{align*}
\chi &= \chi_{\text{observer}} - \chi_{\text{source}} \\
&= \arctan \left( \frac{\xi + \sqrt{g_{00}g_{01}}}{1 - \xi \sqrt{g_{00}g_{01}}} \right) - \arctan \xi \\
&= \arctan \left( \sqrt{g_{00}g_{01}} \right) \tag{11}
\end{align*}
\]

It is clear from Eqn.11 that the total rotation of polarization vector of light ray does not depend on the initial position angle of the polarization vector of the light ray emitted from a source. Thus Eqn. 11 gives the change in rotation of polarization vector of light ray received by a rotating observer (i.e non-inertial frame).

2.2. Rotation of polarization vector as seen from rotating frame

To determine the total rotation of polarization vector by a rotating observer, one should recall the metric given in Eqn.4. From subsection 2.1, Eqn. 11, it is clear that only the metric components \( g_{00} \) and \( g_{01} \) given in Eqn.4, are necessary to determine the total rotation of polarization vector by a rotating observer, which can be considered as an accelerated non-inertial frame. Now in Eqn. 4 the Lorentz factor \( \gamma \) has been introduced by ((Mashhoon 1990)):

\[
\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{12}
\]

Let us assume that the observer is situated at the azimuthal position \( \phi \) and polar position \( \theta \) on the surface of a rotating frame of radius \( r \) so the Cartesian \( y \) co-coordinate of the observer is given by \( y^1 = r \sin(\theta) \sin(\phi) \) and \( x \) co-coordinate is given by \( x^1 = r \sin(\theta) \cos(\phi) \). Now with the help of Eqn.4, Eqn. 11 and Eqn. 12 the total rotation of polarization vector for a rotating observer can be written as;
\[ \chi = \arctan(\sqrt{g_{00}}g_{01}) \]

\[ = \arctan \left[ \sqrt{\left( \frac{\gamma^2}{c^2} \left( r + x^2 \right)^2 \right) \left( \frac{\gamma^2\Omega}{c} x^2 \right)} \right] \]

\[ = \arctan \left[ \frac{1}{1 - \left( \frac{\Omega}{c} \right)^2} \left( 1 - \frac{\Omega^2}{c^2} (r + r \sin \theta \cos \phi)^2 - \frac{\Omega^2}{c^2} (r \sin \theta \sin \phi)^2 \right) \right] \]

\[ = \arctan \left[ \frac{c^2}{c^2 - (\Omega r)^2} \left( 1 - \frac{\Omega^2}{c^2} (1 + \cos \phi)^2 - \frac{(c\Omega r)^2}{c^2 - (\Omega r)^2} \sin^2 \phi \right) \right] \] (13)

Eqn. 13 gives the total rotation of polarization vector of light ray received by a rotating observer situated on the surface of a frame of radius \( r \), with azimuthal position \( \phi \) and polar position \( \theta \) having rate of rotation \( \Omega \). These co-ordinate \( \phi \) and \( \theta \) can be identified as longitude and (90° - latitude).

2.2.1. case 1: When \( \theta = 0 \), polar position of observer

Now, consider a case when the polar position of the observer is given by \( \theta = 0 \) (observer is sitting on the north pole of rotating frame), and with the help of Eqn. 13 it is clear that there is no rotation of polarization vector of light ray received by a rotating observer. Putting \( \theta = 0 \) the \( g_{01} \) component in Eqn. 13 becomes zero which in turn makes \( \chi = 0 \) (i.e. there is no rotation of polarization vector). In case of earth the pole of earth is denoted by 90° latitude, where \( \theta = 0 \) and the relation between earth latitude (say \( \Theta \)) is \( \Theta = (90° - \theta) \). So from above discussion, it is clear that there would be no rotation of polarization vector if the light ray has received from any of the two poles of the earth.

2.2.2. case 2: When \( \theta = \frac{\pi}{2} \) equatorial position of observer

In this case, let us consider the polar position of the rotating observer has given by \( \theta = \frac{\pi}{2} \), now with the help of Eqn. 13 the total rotation of the polarization vector is given by;

\[ \chi_{\theta=\frac{\pi}{2}} = \arctan \left[ \sqrt{\left( \frac{c^2}{c^2 - (\Omega r)^2} \left( 1 - \frac{\Omega^2}{c^2} (1 + \cos \phi)^2 - \frac{(c\Omega r)^2}{c^2 - (\Omega r)^2} \sin^2 \phi \right) \right)} \right] \] (14)

From Eqn. 13 it is clear that the total rotation of polarization doesn’t depend on the initial position of polarization vector, as it is previously mentioned that \( \chi \) is independent of variable \( \xi \). Now from Eqn. 14 one can observe that, \( \chi_{\theta=\frac{\pi}{2}} \) (observer on the equator) depends on the azimuthal position \( \phi \) (equivalent to earth’s longitude) of the observer. In case \( \phi = 0 \) there will be no change in the position of polarization vector (as \( \sin 0 = 0 \)) and the rotation is maximum when \( \phi = \frac{\pi}{2} \) or \(-\frac{\pi}{2}\). So the maximum rotation of polarization vector could be observed when the rotating observer is at azimuthal position \( \phi = \frac{\pi}{2} \) or \(-\frac{\pi}{2}\) and polar position \( \theta = \frac{\pi}{2} \). Now for the position \( \phi = \frac{\pi}{2} \) the value of \( \chi \) is +ve and for \( \phi = -\frac{\pi}{2} \) \( \chi \) is -ve. These two positions represent the prograde and retrograde cases respectively. The numerical value of \( \chi \) in both cases are same but they are in opposite sign (i.e. \( \chi_{\theta=0,\phi=\frac{\pi}{2}} = -\chi_{\theta=0,\phi=-\frac{\pi}{2}} \)). So the maximum rotation of polarization vector is given by;

\[ \chi_{\theta=0,\phi=\frac{\pi}{2}} = \arctan \left[ \sqrt{\left( \frac{c^2}{c^2 - (\Omega r)^2} \left( 1 - \frac{\Omega^2}{c^2} (r^2 - \frac{(c\Omega r)^2}{c^2 - (\Omega r)^2} \sin^2 \phi \right) \right)} \right] \] (15)

The rate of rotation of earth, \( \Omega_{\text{earth}} \) is \( 7.27 \times 10^{-5} \) rad/sec and the average radius of earth is \( r_{\text{earth}} \) is \( 63.78 \times 10^5 \) m. On the equatorial plane the latitude of the earth is 0 and, for the azimuthal position, \( \phi = \frac{\pi}{2} \) the longitude is given by 90°. From Eqn. 15 the total rotation of polarization vector of light ray, received by an observer on the equatorial position from a source is \( \chi_{\text{earth},\theta=0,\phi=\frac{\pi}{2}} = 0.33 \) arcsec. Similarly the total rotation of polarization vector at longitude 90° west is, \( \chi_{\text{earth},\theta=0,\phi=-\frac{\pi}{2}} = -0.33 \) arcsec. Now, let us consider an extreme limiting case where \( \Omega r \approx c \), the velocity of the light ray. In that case, Lorentz factor \( \gamma \) becomes...
Table 1. Rotation of polarization vector $\chi$, in arcsec

| $\theta$ | 0    | 15   | 30   | 45   | 60   | 75   | 90   |
|----------|------|------|------|------|------|------|------|
| 0        | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 15       | 0.0000 | 0.0222 | 0.0428 | 0.0606 | 0.0742 | 0.0827 | 0.0856 |
| 30       | 0.0000 | 0.0428 | 0.0827 | 0.1170 | 0.1433 | 0.1598 | 0.1655 |
| 45       | 0.0000 | 0.0606 | 0.1170 | 0.1655 | 0.2027 | 0.2260 | 0.2340 |
| 60       | 0.0000 | 0.0742 | 0.1433 | 0.2027 | 0.2482 | 0.2768 | 0.2866 |
| 75       | 0.0000 | 0.0827 | 0.1598 | 0.2260 | 0.2768 | 0.3088 | 0.3197 |
| 90       | 0.0000 | 0.0856 | 0.1655 | 0.2340 | 0.2866 | 0.3197 | 0.3309 |

Undefined, so the metric elements $g_{00}$, $g_{01}$ and $g_{02}$ also become undefined, as can be seen from Eqn.4. Thus the value of $\chi$ cannot be calculated. The total rotation of polarization vector, $\chi$ of earth received by rotating observer at equatorial plane versus the azimuthal position ($\phi$) of the observer is shown in Fig.1. Again, in the case when $\phi = \frac{\pi}{2}$ and the polar position of the observer is being varied, the total rotation of polarization vector, $\chi$ is been shown in the Fig. 2. From both figures Fig.1 and Fig.2 one can observe that $\chi$ varies in similar fashion for both azimuthal and polar position and attains maximum value when $\theta = \phi = \frac{\pi}{2}$, as mentioned before. This value of rotation which is about 0.33 arc sec, can be measured with a good polarimeter. In principle, by sitting on earth and by making successive measurements at dawn, mid-noon and dusk, we can verify our calculations. The variation of total rotation of polarization vector, against both azimuthal position ($\phi$) and polar position ($\theta$) is shown, taking earth as an example of rotating frame in Table 1

2.3. A different case: When rotation axis of observer is along the direction of propagation of light

Now, consider a case when the light is coming parallel to the rotation axis of observer’s the rotational frame. In
this case the only component of rotational vector (of the frame) is along \( x^1 \)-axis exists. To calculate the rotation of polarization vector \( \chi \), the values of \( g_{00} \) and \( g_{01} \) have to be calculated, from Eqn.11. The value of \( g_{01} \) can be calculated from Eqn.4 which is,

\[
g_{01} = \frac{\gamma^2 \Omega^2}{c^2} \cdot (0) = 0 \quad (16)
\]

So, from Eqn.11 it is clear that there will be no rotation of polarization vector if the light ray is coming parallel to the rotation axis of the frame and we will get, \( \chi = 0 \).

3. REDSHIFT CAUSED BY THE ROTATING FRAME

To calculate the redshift when light is received from a distance source by a rotational observer, the four-velocity of the observer has to be determined and it is obtained as ((Landau & Lifshitz 1971, 23)):

\[
u^i = \frac{dx^i}{ds}
\]

From Eqn. 4 the line element for the rotational observer can be written as:

\[
ds = \sqrt{g_{00}(dx^0)^2 + g_{01}dx^0dx^1 + g_{02}dx^0dx^2 + \delta_{\alpha\beta}g_{\alpha\beta}dx^\alpha dx^\beta}
\]

The values of \( g_{00}, g_{0x}, \) or \( g_{01} \) and \( g_{xx} \) or \( g_{11} \) etc. have been calculated earlier (4), for the co-ordinate system \((ct, x, y, z)\). Now for our convenience in the present geometry, we can rewrite the metric tensors values in terms \((ct, r, \theta, \phi)\), which are as follows:

\[
g_{00} = \left(\frac{1}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) \left[1 - \frac{\Omega^2}{c^2}(r + r \sin \theta \cos \phi)^2 - \frac{\Omega^2\gamma^2}{c^2}(r \sin \theta \sin \phi)^2\right]
\]
\[
g_{01} = \left(\frac{1}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) \Omega r \sin \theta \sin \phi
\]
\[
g_{02} = -\left(\frac{1}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) \Omega r \sin \theta \cos \phi
\]
\[
g_{03} = 0
\]
\[
g_{\alpha\beta} = -\delta_{\alpha\beta}.
\]

Thus we may write:

\[
ds^2 = \left(\frac{1}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) \left[1 - \frac{\Omega^2}{c^2}(r + r \sin \theta \cos \phi)^2 - \frac{\Omega^2\gamma^2}{c^2}(r \sin \theta \sin \phi)^2\right] (cdt)^2
\]
\[
- \left(\frac{\Omega r \sin \theta \sin \phi}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) r \sin \theta \sin \phi d\phi dt + \left(\frac{\Omega r \sin \theta \cos \phi}{1 - \left(\frac{\Omega r \sin \theta \sin \phi}{c}\right)^2}\right) r \sin \theta \cos \phi d\phi dt
\]

(20)

For any calculation of redshift introduced by curved space-time, we must begin with the invariant.

\[
[k_iu^i]_{\text{source}} = [k_iu^i]_{\text{observer}} \quad (21)
\]

Now, the components of four velocity can be written as:

\[
u^0 = \frac{dx^0}{ds}
\]
\[
u^1 = \frac{dx^1}{ds}
\]
\[
u^2 = \frac{dx^2}{ds}
\]
\[
u^3 = \frac{dx^3}{ds}
\]

(22)

Now \( F \) be any arbitrary quantity which describes the wave field. For a plane monochromatic wave \( F \) can be written as ((Landau & Lifshitz 1971, 140)):

\[
F = ae^{\text{i}(k_i x^i) + \alpha}
\]

(23)

Where \( \text{i} = \sqrt{-1} \), \( k_i \) is wave four vector. One can write the expression for the field as, \( F = ae^{\Psi} \). \( \Psi \) is the eikonal. Over small region of space and time intervals, the eikonal, \( \Psi \) can be expanded in series to terms of the first order and it is written as:

\[
\Psi = \Psi_0 + \frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial t}
\]

(24)

From this one can have ((Landau & Lifshitz 1971, 141)):

\[
k_i = -\frac{\partial \Psi}{\partial x^i}
\]

(25)

Again it can be shown that (Landau & Lifshitz (1971)),

\[
k^i k_i = 0
\]

(26)

Eqn.26 is the fundamental equation of geometrical optics and called the eikonal equation. Using Eqn. 25 the
components of the wave four-vector, \( k_i \) are given as:

\[
\begin{align*}
  k_0 &= -\frac{\partial \Phi}{\partial \tau} = -\frac{\partial \Phi}{\partial \tau_0} = \frac{\nu'}{c} \\
  k_1 &= \frac{\partial \Phi}{\partial x} \\
  k_2 &= -\frac{\partial \Phi}{\partial y} \\
  k_3 &= \frac{\partial \Phi}{\partial z}
\end{align*}
\]  

(27)

\( \nu \) is the frequency measured in terms of proper time, \( \tau_0 \) and is defined as (Landau & Lifshitz 1971, 268) \( \nu = -\frac{\partial \Phi}{\partial \tau_0} \). Now, one must note that frequency \( \nu \) expressed in terms of proper time \( \tau_0 \), is different at different point of space. \( \nu' \) is the frequency as measured by the rotating observer. In the present case of interest where the light ray is coming from distance and the observer is rotating and is situated at the surface of radius of rotating frame, we can write from the Eqn. 26 (please also see Dubey & Sen (2015)):

\[
[k_i u^i]_{\text{source}} = [k_0 u_0 + \frac{k_\phi u^\phi}{k_0 u^0}]_{\text{source}}
\]  

(28)

Now for the distance source \( u^0_{\text{source}} = 1 \) and \( u_\phi^{\phi_{\text{source}}} = 0 \), therefore

\[
[k_0 u^0]_{\text{source}} = [k_0 u_0 (1 + \frac{k_\phi u_\phi}{k_0 u^0})]_{\text{source}}
\]  

(29)

Now, for the observer \( \frac{d\phi}{dt} = \Omega \), ( rotational velocity of observer’s frame) \( [u^0]_{\text{observer}} \) is given by, (from Eqn.22)

\[
[u^0]_{\text{observer}} = \frac{1}{\sqrt{1 - \frac{\Omega^2 r^2}{c^2} (1 + \sin \theta \cos \phi) - \frac{\Omega^2 r^2}{c^2} \sin \theta \sin \phi + 2 \frac{\Omega^2 r^2}{c^2} \sin^2 \theta}}
\]  

(31)

Noting that \( \frac{k_\phi}{k_0} \frac{\Omega}{c} \) from Eqn.30 we can have:

\[
\nu' \left[ 1 + \frac{k_\phi}{k_0} \frac{\Omega}{c} \right] = \nu
\]  

(32)

At the location of the observer the observed frequency is defined by \( \nu' \). To calculate the redshift, one must calculate the value of \( \frac{k_\phi}{k_0} \frac{\Omega}{c} \). For a rotating frame like earth, the relativistic action, \( S \) for a particle can be expressed as (Landau & Lifshitz 1971; Dubey & Sen (2015)), \( S = -Et + L\phi + S_r + S_\theta \), where \( E \) is the conserved energy and \( L \) is the components of angular momentum around the axis of symmetry. Now the four momentum \( p_t = -\frac{\partial \Phi}{\partial \tau} \) and for the propagation of photon we can replace action \( S \) by the eikonal, \( \Psi \). Finally, we can write

\[
k_0 = -\frac{\partial \Phi}{\partial \tau} = \frac{E}{c}
\]

(33)

Again, in case of photon the energy \( E \) and linear momentum \( p \) are expressed as, \( E = pc \) considering the rest mass of photon is zero. The angular momentum about the rotation axis of observer can be expressed as (Dubey & Sen (2015)), \( L = pr \sin \phi \sin \theta \). So the value of \( \frac{k_\phi}{k_0} \) is \( r \sin \phi \sin \theta \). So from Eqn. 32 it can be written:

\[
\nu' \left[ 1 + \frac{k_\phi}{k_0} \frac{\Omega}{c} \right] = \nu
\]  

(34)

The redshift, \( Z \) is generally defined by the relation. \( Z = \frac{\nu'}{\nu} - 1 \). So the redshift as estimated by a rotating observer is given by:

\[
Z = \frac{\nu'}{\nu} - 1 = \sqrt{1 - \frac{\Omega^2 r^2}{c^2} (1 + \sin \theta \cos \phi) - \frac{\Omega^2 r^2}{c^2} \sin \theta \sin \phi + 2 \frac{\Omega^2 r^2}{c^2} \sin^2 \theta} - 1
\]  

(35)

Now when \( \frac{\nu'}{\nu} < 1 \) then we call it redshift and when \( \frac{\nu'}{\nu} > 1 \) we call it blueshift. From Eqn.35 one can observe that for equatorial plane there will be no shift of wavelength when the azimuthal position (\( \phi \)) is zero,
but the shift will be extremum at $\phi = \frac{\pi}{2}$ (dusk side) and $\phi = -\frac{\pi}{2}$ (dawn side). We know for earth $\Omega_{\text{earth}}$ is $7.27 \times 10^{-5}$ rad/sec and the average radius of earth is $r_{\text{earth}}$ is 63.78 $\times$ 10^5 m. Therefore, the redshift values under different conditions will be: $Z_{\text{earth}, \phi=\frac{\pi}{2}, \theta=\frac{\pi}{2}} \approx -1.604429 \times 10^{-6}$, i.e. redshift occurs here at dusk. Again for the azimuthal position $\phi = -\frac{\pi}{2}$ (i.e. at dawn) on the equatorial plane of earth, $Z$ will be $Z_{\text{earth}, \phi=-\frac{\pi}{2}, \theta=\frac{\pi}{2}} \approx 1.604436 \times 10^{-6}$ which shows a blueshift. Now if we vary the polar position of the observer then at the both poles there will be no shift of the wavelength of the received light. Let us consider a light ray consisting of Lyman – $\alpha$ line which has frequency, $\nu_{\text{Ly-}\alpha} = 2.47 \times 10^{15} Hz$ (see the difference in the third place after decimal when $\nu_{\text{Ly-}\alpha}$ is on the rotating frame who receives a light ray with wavelength $\lambda_{\text{Dop}}$ from a distant source which initially emits a light ray with wavelength $\lambda_{\text{Dop}}$. The two wavelengths are related by the relation in case of relativistic Doppler effect is (Born & Wolf (2013)): 

$$\frac{\lambda_{\text{Dop}}}{\lambda_{\text{Dop}}} = \gamma (1 - \beta)$$

where $\beta = \frac{v}{c}$, $v$ is the translational velocity of the observer parallel to $x^1$-axis. Here as the observer is sitting on the rotating frame of radius $r$ and the polar and azimuthal positions are given by $\theta$ and $\phi$ respectively, the velocity will be $\gamma \Omega r \sin \theta \cos \phi$. $\gamma$ is the Lorentz factor and $\Omega$ is the angular velocity of the rotating frame. So the $\beta$ is given by the relation $\beta = \frac{v}{c}$ and polar position, $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$. Both $\chi$ and $Z$ become zero when either one of the coordinates between $\phi$ and $\theta$ becomes zero or 180°. Again one must notice that the value of $\chi$ and $Z$ in the azimuthal positions $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ have same modular value but opposite sign.

The Eqn.34 was obtained through a rigorous procedure involving metric for accelerated (rotational) frame. These result for the relativistic redshift is same as the redshift derived considering the rotational frame. Thus the redshift effect for the rotational frame is equivalent to the relativistic redshift caused by translational motion of the observer.

4. CONCLUSION

In this paper, we studied the rotation of polarization vector of light, coming from a distant source and as viewed by an observer sitting on a rotating frame. The amount of rotation depends on the four variables, i.e. $(r, \theta, \phi)$, co-ordinate of the rotating observer, and the rate of rotation $\Omega$, of the co-ordinate frame. The maximum change of polarization vector can be observed, if the position of the observer is on equatorial plane and the azimuthal position is $\pm \frac{\pi}{2}$ of the rotating frame. As an extension of the present work, the earth has been taken as an example of a rotating frame where the observer is on the surface of the earth and the maximum rotation of polarization vector, occurs at latitude $0^\circ$ ($\theta = 90^\circ$), and longitude $\pm 90^\circ$. This maximum value is $\chi_{\text{earth}, \theta=\phi=\frac{\pi}{2}} = 0.33$ arcsec. Again if we consider that the rotating axis of the frame is along along the direction of propagation of light then there will be no rotation of polarization vector.

In the present paper we also report the amount of the redshift caused by the rotational frame. It has been shown that, there will be redshift in dusk (prograde) case and will be blueshift for the dawn (retrograde) case (i.e. maximum redshift at azimuthal position $\phi = \frac{\pi}{2}$ on the equatorial plane and maximum bluhsift at azimuthal position $\phi = -\frac{\pi}{2}$). We calculate the red shifted and blue shifted values of wavelength taking Lyman alpha line as a test case. It is also shown that, results from our calculations considering metric tensors for an accelerated (rotational) frame, come out to be same as the one which can be obtained considering the expression for relativistic Doppler shift. From above discussion, it is clear that both rotation of polarization vector ($\chi$) and redshift ($Z$) is maximum at the azimuthal position, $\phi = \frac{\pi}{2}$ and polar position, $\theta = \frac{\pi}{2}$. Both $\chi$ and $Z$ become zero when either one of the coordinates between $\phi$ and $\theta$ becomes zero or 180°. Again one must notice that the value of $\chi$ and $Z$ in the azimuthal positions $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ have same modular value but opposite sign.

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