Correlations of Quantum Fields on Robertson-Walker Spacetimes

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Abstract

It is a well known fact that quantum fields on Minkowski spacetime are correlated for each pair of spacetime regions. In Robertson-Walker spacetimes there are spacelike separated regions with disjoint past horizons but the absence of correlations in that case was never proved. We derive in this paper formulae for correlations of quantum fields on Robertson-Walker spacetimes. Such correlations could have reasonably influenced the formation of structure in the early universe. We use methods of algebraic and constructive quantum field theory.

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1 Introduction

In this paper we derive formulae for correlations of quantum fields on Robertson-Walker spacetimes. We follow an idea of R. M. Wald [1] who argued that such correlations could be of importance for formation of structure in the early universe.

Considering local properties of quantum fields on Minkowski spacetime, it is a well known fact that correlations are always present, even for spacelike separated regions. On Robertson-Walker spacetimes there are in contrast to Minkowski spacetime spatially separated regions whose past horizons do not intersect. It is usually taken for granted that events occurring in such regions are statistically independent. However, up to our knowledge the absence of correlations was never explicitly proved.

The question whether such correlations do exist may have important relevance in cosmology. A typical example for the application of the argument that events in regions with

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disjoint past horizons are totally independent, is the formation of topological defects in the early universe, such as magnetic monopoles, cosmic strings and domain walls. Such defects are believed to occur if the universe undergoes a phase transition.

We work in the framework of algebraic quantum field theory and use methods of constructive quantum field theory for the derivation of the formulae. To begin with, we consider a neutral free boson field on a Robertson-Walker spacetime and hope to extend this method to more complicated quantum field theories.

The point of the paper is the following. Usually the presence of field correlations in two spatially separated regions is expressed by the formula

\[ \langle \phi(f)\phi(g) \rangle \neq \langle \phi(f) \rangle \langle \phi(g) \rangle, \]

where \( \phi \) is the field operator and \( f, g \) are two test functions with spacelike separated supports. The inequality just tells us that correlations are present.

Complete knowledge about the correlations is obtained if the joint probability of two events to occur in the two spacelike separated regions, respectively, is compared with the product of the probabilities for their respective occurrences. Let \( \mathcal{B}_1, \mathcal{B}_2 \) be any pair of Borel sets on the real line and \( \mathcal{E}_f, \mathcal{E}_g \) be the spectral measures belonging to the operators \( \phi(f), \phi(g) \), respectively. Then the measures \( \langle \mathcal{E}_f\mathcal{E}_g \rangle \) and \( \langle \mathcal{E}_f \rangle \langle \mathcal{E}_g \rangle \) have to be compared. The events \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are correlated if

\[ \langle \mathcal{E}_f(\mathcal{B}_1)\mathcal{E}_g(\mathcal{B}_2) \rangle \neq \langle \mathcal{E}_f(\mathcal{B}_1) \rangle \langle \mathcal{E}_g(\mathcal{B}_2) \rangle, \]

otherwise they are uncorrelated. This means we are interested in a spectral decomposition of the field operator.

A drawback is the fact that the formulae can not be solved analytically for fields located in spacelike separated regions. They can only be solved numerically and related to this we can not analyse the influence of the past horizon.

At present we are not able to compute such correlations for an interacting theory but take this work as a first step. Of course it would be more interesting to derive formulae for a \( \phi^4 \)-theory since the formation of topological defects is expected by such theories. However, we wish to point out that correlations in a noninteracting theory are most unexpected because they can not be considered as a result of a nontrivial interaction. Correlations in a noninteracting theory would strongly suggest the occurrence of stronger correlations in an interacting theory.

The paper is organized as follows. In section 2 we discuss the classical solutions of the Klein-Gordon equation on a Robertson-Walker spacetime. In the next section we construct the Weyl algebra corresponding to the spaces of classical solutions obtained in section 2. In section 4 a distinguished class of states, the adiabatic vacuum states, on these Weyl algebras are discussed. The computation of correlations will be done in these states which are known to be good states for these spacetimes. Since we are interested in a spectral decomposition of the field operator we introduce Gaussian measures on a functional Hilbert space in section 5 and decompose the field operator in section 6 with the help of these spaces. Also in section 6 we summarize the formulae for the correlations.
2 The spaces of classical solutions

We discuss the classical solutions of the Klein-Gordon equation

\[(\Box g - m^2) \varphi = 0,\]

on a Lorentz manifold \((\mathcal{M}, g)\), where \(g\) is a Robertson-Walker metric

\[g = -dt^2 + R(t)^2 [d\theta_1^2 + \sum_2 \left( d\theta_2^2 + \sin^2 \theta_2 d\phi^2 \right) ], \quad R(t) > 0.\]

These are homogeneous and isotropic spaces, topologically of the form \(\mathcal{M} = \mathbb{R} \times S_\varepsilon\), where \(\varepsilon = 1, 0, -1\) corresponds to the spherical, the flat and the hyperbolic case, respectively. We think of the spatial parts as embedded in \(\mathbb{R}^4\) by

\[S_1 = \{ x \in \mathbb{R}^4 \mid \sum_0^3 x_i^2 = 0 \}, \quad \Sigma_1 = \sin \theta_1,\]
\[S_0 = \{ x \in \mathbb{R}^4 \mid x_0 = 0 \}, \quad \Sigma_0 = \theta_1,\]
\[S_{-1} = \{ x \in \mathbb{R}^4 \mid x_0^2 - \sum_1^3 x_i^2 = 1, \ x_0 > 0 \}, \quad \Sigma_{-1} = \sinh \theta_1.\]

The Klein-Gordon operator on these spaces takes the form

\[\Box g - m^2 = -\frac{\partial^2}{\partial t^2} - 3 \frac{\dot{R}(t)}{R(t)} \frac{\partial}{\partial t} + \frac{1}{R^2(t)} \Delta_\varepsilon - m^2,\]

where \(\Delta_\varepsilon\) means the Laplace operator on the respective spatial parts.

2.1 The spherical case

The eigenfunctions of the Laplace operator on the 3-sphere \(S_1\) are functions

\[Y_{\vec{k}}(\vec{x}) = e^{\pm ik_2 \phi} (\sin \theta_1)^{k_1} C_{k_0-k_1}^{k_1+1} (\cos \theta_1)(\sin \theta_2)^{k_2} C_{k_1-k_2}^{k_2+1/2} (\cos \theta_2),\]

\(\vec{x} = (\theta_1, \theta_2, \phi), \vec{k} = (k_0, k_1, k_2), k_0 \in \mathbb{N}_0, 0 < k_1 < k_0, -k_1 < k_2 < k_1\) and \(C_n^l\) are Gegenbauer polynomials (see e.g. [2, Ch. 11.2]). These eigenvectors fulfill the equation

\[\Delta_1 Y_{\vec{k}}(\vec{x}) = -k_0(k_0 + 2)Y_{\vec{k}}(\vec{x}),\]

and every \(k_0\) spans an invariant eigenspace \(\mathcal{H}_{k_0}\) of dimension \((k_0 + 1)^2\). The direct sum of these spaces leads to the quasiregular representation of the isometry group \(I_1 = SO(4)\) of \(S_1\), i.e. by the theorem of Peter-Weyl the Hilbert space of square integrable functions on \(S_1\) is given by

\[L^2(S_1) = \bigoplus_{k_0=0}^{\infty} \mathcal{H}_{k_0}.\]
The Fourier transform
\[ \tilde{h}(\vec{k}) := (Y_{\vec{k}}, h) \]
gives a unitary transformation of \( L^2(S_1) \) to \( L^2(\tilde{S}_1) \), where \( \tilde{S}_1 \) denotes the momentum space associated with \( S_1 \), i.e. the range of values of \( \vec{k} \) equipped with the counting measure.

### 2.2 The flat case
The isometry group of the space \( S_0 \) is the Euclidean group \( I_0 = E(3) \). The generalized eigenvectors of the Laplace operator are
\[ Y_{\vec{k}}(\vec{x}) = (2\pi)^{-3/2} e^{i\vec{k} \cdot \vec{x}} , \]
regarded as distributions over \( S_0 \). They obey the equation
\[ \Delta_0 Y_{\vec{k}}(\vec{x}) = -k^2 Y_{\vec{k}}(\vec{x}) , \quad k := |\vec{k}| . \]
We have the Fourier transform as a unitary mapping of \( L^2(S_0) \) to \( L^2(\tilde{S}_0) \) as usually given by
\[ \tilde{h}(\vec{k}) := (Y_{\vec{k}}, h) , \quad h \in L^2(S_0) . \]
With each \( k \in \mathbb{R}_+ \) we associate a function \( h_k \in C^\infty_0(S_0) \), by \( k \to \tilde{h}_k \), taking values in \( L^2(S^2, d\Omega) \) (\( S^2 \) the two sphere and \( d\Omega \) the invariant measure on it):
\[ \tilde{h}_k(\vec{x}) = \int d\mu(\vec{x}) \overline{Y_{\vec{k}^*}(\vec{x})} h(\vec{x}) . \]
The map \( h \to \tilde{h} \) extends to express the Hilbert space \( L^2(S_0) \) as a direct integral over \( \mathbb{R}_+ \):
\[ L^2(S_0) = \int^\oplus L^2(S^2, d\Omega) k^2 dk . \]

### 2.3 The hyperbolic case
The isometry group \( I_{-1} \) of the hyperbolic space \( S_{-1} \) is the Lorentz group \( L^+ \uparrow_3(4) \). The generalized eigenvectors of the Laplace operator \( \Delta_{-1} \) on this space are distributions of the form
\[ Y_{\vec{k}}(\vec{x}) = (2\pi)^{-3/2}(x \cdot \xi)^{-1+ik} , \quad \vec{k} := k\vec{\xi} \in \mathbb{R}^3 , \]
where \( x \cdot \xi \) means the scalar product in Minkowski spacetime, \( \xi = (1, \vec{\xi}) \in \mathbb{R}^4 \), with eigenvalues given by
\[ \Delta_{-1} Y_{\vec{k}}(\vec{x}) = -(k^2 + 1) Y_{\vec{k}}(\vec{x}) . \]
Similar to the flat case we have a Fourier transformation given by equation (1) and a direct integral decomposition of \( L^2(S_{-1}) \) (cf. [3, Ch.VI 3.3] and [4, Ch.X §4]).
2.4 The propagator

As a consequence of the global hyperbolicity of the manifold \((M, g)\) we have the existence of global fundamental solutions of the Klein-Gordon equation (see [3]). There exist unique operators \(E^\pm: C_0^\infty(M) \to C^\infty(M)\), satisfying

\[
(\Box_g - m^2)E^\pm = E^\pm(\Box_g - m^2) = I,
\]

\[\text{supp}(E^\pm \phi) \subset J^\pm(\text{supp}\phi), \quad \phi \in C_0^\infty(M),\]

where \(J^+(p)\) is the causal future and \(J^-(p)\) is the causal past of \(p \in M\), i.e. all \(q \in M\) that can be joined with \(p\) by a future resp. past directed causal curve and \(I\) the identity. The operator \(E := E^+ - E^-\) is the causal propagator. This operator can be extended continuously to a mapping from the space of distributions with compact support to the space of distributions (on \(M\) resp.): \(E^\pm : D'(M) \to D'(M)\).

Since we know the eigenfunctions and eigenvalues of the operator \(\Delta_\varepsilon\), the propagator is determined by the ordinary differential operator

\[
D = \frac{d^2}{dt^2} + 3 \frac{\dot{R}(t)}{R(t)} \frac{d}{dt} + m^2 + \frac{E(k)}{R^2(t)},
\]

where \(E(k)\) is a unified notation of the eigenvalues of the Laplace operators \(\Delta_\varepsilon\), i.e. \(E(k) = k_0(k_0 + 2)\) for \(\varepsilon = 1\), \(E(k) = k^2\) for \(\varepsilon = 0\) and \(E(k) = k^2 + 1\) for \(\varepsilon = -1\).

Unfortunately this differential equation can be solved analytically only in very limited cases explicitly, namely if the scaling factor \(R(t)\) is proportional to \(t\). There are two fundamental solutions in that case [3, C.2.162]: \(J_\nu(mt)/t\) and \(Y_\nu(mt)/t\), \(\nu := \sqrt{1 - E(k)}\), where \(J_\nu\) and \(Y_\nu\) are Bessel functions of the first and second kind respectively [2].

3 The Weyl algebra

In this section we construct the Weyl algebra \(CCR(D, \sigma)\) associated with the space of classical solutions of the Klein-Gordon equation on Robertson-Walker spacetimes. To this end we need a real symplectic vector space. There are two equivalent methods for constructing an algebra on globally hyperbolic spacetimes [7]. One method is based on the solutions on a Cauchy surface while the other one starts with the vector space \(C^\infty_{0,\mathbb{R}}(M)/\ker E\), where \(\ker E\) is the kernel of the propagator \(E\). We use the first method, because it is appropriate for our problem.

As the real symplectic vector space we take the space of real Cauchy data on a Cauchy surface \(D(S_\varepsilon) := C^\infty_{0,\mathbb{R}}(S_\varepsilon) \oplus C^\infty_{0,\mathbb{R}}(S_\varepsilon)\). For a function \(f \in C^\infty_{0,\mathbb{R}}(M)\) the restriction to \(D(S_\varepsilon)\) is given by \(\rho_0Ef \oplus \rho_1Ef\), where \(\rho_0 : C^\infty_{0,\mathbb{R}}(M) \to C^\infty_{0,\mathbb{R}}(S_\varepsilon)\) is the restriction operator to the Cauchy surface and \(\rho_1 : C^\infty_{0,\mathbb{R}}(M) \to C^\infty_{0,\mathbb{R}}(S_\varepsilon)\) is the forward normal derivative on the Cauchy surface. The Weyl algebra \(CCR(D, \sigma)\) is the algebra generated by the elements \(W(F) \neq 0, F \in D(S_\varepsilon)\), obeying the Weyl form of the canonical commutation relations

\[
W(F)W(G) = e^{-i\sigma(F;G)/2}W(F + G), \quad F, G \in D(S_\varepsilon),
\]
and with the property \( W(F)^* = W(-F) \), see e.g. [8, Ch. 8.2]. The symplectic form is given by

\[
\sigma(F, G) = R^3 \int_{S_\varepsilon} (f_1 g_2 - f_2 g_1) d\mu(S_\varepsilon), \quad F = f_1 \oplus f_2, \quad G = g_1 \oplus g_2,
\]

where the invariant measure on the respective spaces is meant. The net of local observables is defined as

\[
\mathcal{A}(\mathcal{O}) = C^*(W(\rho_0 E f \oplus \rho_1 E f)), \quad \text{supp} (\rho_i E f) \subset \mathcal{O} \subset \mathcal{M}, \quad i = 0, 1, \quad f \in C^\infty_{0, R}(\mathcal{M}),
\]

where we mean the \( C^* \)-algebra generated by these elements. We define the quasilocal algebra by

\[
\mathcal{A} = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{M}),
\]

where \( \mathcal{O} \) runs through all open subsets of \( \mathcal{M} \).

This net fulfills a set of axioms formulated by Dimock [7], generalizing the Haag-Kastler axioms (see [9]) to curved spacetimes. If we have a (regular) representation of \( CCR(D, \sigma) \), we can define a selfadjoint field operator \( \varphi \) with the help of Stone’s theorem by

\[
\exp[it\varphi(F)] = W(tF), \quad F \in D, \quad t \in \mathbb{R}.
\]

4 Adiabatic vacuum states

Adiabatic vacuum states were introduced by Parker [10] with the intention to determine the state in which the creation rate of particles forced by expansion of the universe is minimal. Lüders and Roberts analysed these states in the framework of algebraic quantum field theory [11] and proved that the family of adiabatic vacuum states is consistent with the principle of local definiteness. This principle was introduced by Haag, Narnhofer and Stein [12] in discussing quantum field theory on curved spacetimes (cf. also [13, III.3.1]) for to distinguish the physically realizable states among all positive normalized functionals which have no physical significance. Furthermore it is now known that adiabatic vacuum states are Hadamard states [14].

4.1 Definition and construction of adiabatic vacua

We recall the definition of quasifree and Fock states (cf. e.g. [8, 8.2.3]). Let \( CCR(D, \sigma) \) be the Weyl algebra over the real vector space \( D \) with symplectic form \( \sigma \) and let \( S(\cdot, \cdot) \) be a real scalar product on \( D \), satisfying

\[
|\sigma(F, G)| \leq \sqrt{S(F, F)} \sqrt{S(G, G)}, \quad F, G \in D.
\]
A quasifree state (with vanishing one-point function) is defined by the generating functional
\[ \omega_S(W(F)) = \exp(-S(F,F)/4), \quad F \in D, \]
where \( W(F) \) denotes the Weyl operator. Such a quasifree state \( \omega_S \) determines the two-point function
\[ \langle \Omega|\varphi(F)\varphi(G)\Omega \rangle_S = \left[ S(F,G) + i\sigma(F,G) \right]/2, \quad F,G \in D, \]
where \( \varphi(F) \) is the generator of the Weyl operator in the GNS-representation constructed of \( \omega_S \) and \( \Omega \) the corresponding GNS-vacuum.

If the scalar product \( S \) can be linked with the symplectic form \( \sigma \) by an internal complexification \( J \), i.e. \( J^2 = -1, \sigma(JF,G) = -\sigma(F,JG), \sigma(F,JG) = S(F,G) \), the state is called a Fock state.

We now recall the construction, definition and some properties of adiabatic vacuum states from Lüders and Roberts [11]. In their analysis of the structure of states on a Robertson-Walker spacetime they defined a quasifree state \( \omega \) to be homogeneous and isotropic, if \( \omega \circ \alpha_g = \omega \), where \( g \) is an element of the isometry group \( I_\varepsilon \) of the spaces \( S_\varepsilon \) and \( \alpha_g \) the corresponding automorphism. The analysis of these conditions leads to the function spaces described in section 2. For to find conditions on the scalar product \( \langle \cdot | \cdot \rangle_S \) to define a Fock state they imposed the following continuity condition. There is a \( \nu \in \mathbb{N}_0 \) and a constant \( C > 0 \), such that
\[ \langle F|G \rangle_S \leq C\|F\|_\nu\|G\|_\nu, \quad F,G \in D, \]
where
\[ \|F\|_\nu := (F,(m^2 - \Delta_\varepsilon)^{2\nu}F), \quad \nu \in \mathbb{N}_0, \]
\[ (F,G) := R^3 \int_{S_\varepsilon} (f_1g_1 + f_2g_2) d\mu(S_\varepsilon), \quad F = f_1 \oplus f_2, \quad G = g_1 \oplus g_2. \]

We denote the classical solutions with the uniform notation
\[ F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \int d\vec{k} \begin{pmatrix} c(\vec{k}) \\ \hat{c}(\vec{k}) \end{pmatrix} Y_{\vec{k}}(\vec{x}), \quad (4) \]
\[ G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \int d\vec{k} \begin{pmatrix} b(\vec{k}) \\ \hat{b}(\vec{k}) \end{pmatrix} Y_{\vec{k}}(\vec{x}), \quad F,G \in D. \]
where the integral reduces to a sum in the spherical case. The main result concerning the structure of Fock states is summarized in the following

**Theorem 1** The homogeneous and isotropic Fock states for the Klein-Gordon field on a Robertson-Walker spacetime fulfilling the continuity condition [3] are given by a two-point function of the form
\[ \langle F|G \rangle_S = \int d\vec{k} \left[ c(\vec{k})b(\vec{k})S_{00}(k) + c(\vec{k})\hat{b}(\vec{k})S_{01}(k) + b(\vec{k})\hat{c}(\vec{k})S_{10}(k) + \hat{c}(\vec{k})\hat{b}(\vec{k})S_{11}(k) \right]. \]
The entries of the matrix \( S \) can be expressed in the form
\[
S_{00}(k) = |q(k)|^2, \quad S_{11}(k) = |p(k)|^2, \\
S_{01}(k) = q(k)p(k), \quad S_{10} = \overline{S_{01}},
\]
where \( p \) and \( q \) are essentially polynomially bounded measurable functions satisfying
\[
\overline{q(k)p(k)} - p(k)q(k) = i.
\]
Conversely every pair of polynomially bounded measurable functions satisfying equation (7) yields via (6) and (5) the two-point function of a homogeneous, isotropic Fock state which satisfies the continuity condition (3).

For the proof see [11, Thm. 2.3].

The dynamical evolution of the states is expressed by the coefficients \( c(\vec{k}) \) and \( \hat{c}(\vec{k}) \) in equation (4). An adiabatic vacuum state is given by a specific initial condition on a Cauchy surface which fixes the matrix \( S \). This means that the adiabatic vacuum state depends on the Cauchy surface.

The dynamical equation is
\[
\left( \frac{d^2}{dt^2} + 3 \frac{\dot{R}(t)}{R(t)} \frac{d}{dt} + m^2 + \frac{E(k)}{R^2(t)} \right) c_k(t) = 0, \quad \forall k.
\]
This equation can be solved explicitly only in exceptional cases. In the general case, one tries to solve it by an iteration procedure. For to find the iteration, we consider
\[
T_k(t) = [2R^3(t)\Omega_k(t)]^{-1/2} \exp \left(i \int_{t_0}^{t} \Omega_k(t') dt' \right), \quad \forall k,
\]
where the functions \( \Omega_k \) have to be determined. Inserting this ansatz in equation (8) we find that the functions \( \Omega_k \) have to satisfy
\[
\Omega_k^2 = \omega_k^2 - 3 \frac{\dot{R}(t)}{R(t)} \left( \frac{\dot{R}}{R} \right)^2 - 3 \frac{\dot{R}}{R} + 3 \left( \frac{\dot{\Omega}_k}{\Omega_k} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k}{\Omega_k},
\]
where \( \omega_k^2 = E(k)/R^2 + m^2 \). With
\[
(\Omega_k^{(0)})^2 := \omega_k^2 = E(k)/R^2 + m^2
\]
the iteration is given by
\[
(\Omega_k^{(n+1)})^2 = \omega_k^2 - 3 \frac{\dot{R}(t)}{R(t)} \left( \frac{\dot{R}}{R} \right)^2 - 3 \frac{\dot{R}}{R} + 3 \left( \frac{\dot{\Omega}_k^{(n)}}{\Omega_k^{(n)}} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k^{(n)}}{\Omega_k^{(n)}}.
\]
The functions \( T_k(t) \) and \( \dot{T}_k(t) \) are related to the functions \( q(k) \) and \( p(k) \), which constitute the matrix \( S \) by equation (6). On a Cauchy surface at time \( t \) these relations are
\[
T_k(t) = q(k), \quad \dot{T}_k(t) = R^{-3}(t)p(k).
\]
An adiabatic vacuum state will now be defined by initial values at time \( t \):
Definition 1 For \( t_0, t \in \mathbb{R} \), let
\[
W_k^{(n)}(t) := [2R^3(t)\Omega_k^{(n)}(t)]^{-1/2} \exp \left( i \int_{t_0}^t \Omega_k^{(n)}(t') dt' \right).
\] (10)

An adiabatic vacuum state of order \( n \) is a Fock state, obtained via equations (6) and (9), where the initial values at time \( t \) for equation (8) can be expressed by
\[
T_k(t) = W_k^{(n)}(t), \quad \dot{T}_k(t) = \dot{W}_k^{(n)}(t).
\]

These states depend on the initial time \( t \) in equation (10), the order of iteration \( n \) and the extrapolation of \( \Omega_k^{(n)} \) to small values of \( k \). The main result on the family of adiabatic vacuum states is summarized (see L"uders and Roberts [11, Thm. 3.3]) in

Theorem 2 In a closed Robertson-Walker spacetime any two adiabatic vacuum states for the free Klein-Gordon field are unitarily equivalent.

In a flat or hyperbolic Robertson-Walker spacetime any two adiabatic vacuum states of order \( n \geq 1 \) for the free Klein-Gordon field are locally quasiequivalent.

A generalization to \( n \geq 0 \) also in the flat and hyperbolic case was given by Junker [14, Cor. 3.23], so that all adiabatic vacuum states are locally quasiequivalent and we can perform our computations in an adiabatic vacuum state of order zero.

Local quasiequivalence of two states \( \omega \) and \( \omega' \) means that the density matrices in the GNS Hilbert spaces \( \mathcal{H}_\omega \) and \( \mathcal{H}_{\omega'} \) define the same set of states of \( \mathcal{A}(\mathcal{O}) \) for each \( \mathcal{O} \) in \( \mathcal{M} \). The principle of local definiteness mentioned in section 3 requires local quasiequivalence of the physical realizable states.

The entries of the matrix \( S \) for a zeroth order adiabatic vacuum state are given in terms of \( \Omega \equiv \Omega_k^{(0)} \) by
\[
S_{00}(k) = |q(k)|^2 = \frac{1}{2\Omega R^3},
S_{01}(k) = \overline{q(k)}p(k) = \frac{i}{2} - \frac{3\dot{R}}{4\Omega R} - \frac{\dot{\Omega}}{4\Omega^2},
S_{11}(k) = |p(k)|^2 = \frac{\Omega R^3}{2} + \frac{9R\dot{R}^2}{8\Omega} + \frac{3\dot{\Omega}^2\dot{R}}{8\Omega^2} + \frac{R^3\dot{\Omega}^2}{8\Omega^3}.
\]

The internal complexification \( J \) on \( D \) which leads to adiabatic vacuum states is given by
\[
J = \begin{pmatrix}
\frac{3}{2} \frac{\dot{R}}{R} & \frac{\dot{\Omega}}{2\Omega^2} \\
\frac{1}{R^3\Omega} & -\Omega R^3 - \frac{9R\dot{R}^2}{4\Omega} - \frac{3\dot{\Omega}^2\dot{R}}{8\Omega^2} - \frac{R^3\dot{\Omega}^2}{4\Omega^3}
\end{pmatrix}.
\]

4.2 The field operator

We can define a field operator on the symmetrized Fock space \( \mathcal{F}(L^2(S_\varepsilon)) \) in the usual way as an operator valued distribution following Dimock [7] and L"uders and Roberts [14]. In our case the field operator is
\[
\varphi(f) = \theta(\rho_1 Ef) - \pi(\rho_0 Ef), \quad f \in C_{0,\delta}(\mathcal{M}).
\]
with time-zero fields

\[ \theta(g) = a(qR^3 \tilde{g}) + a^*(qR^3 \tilde{g}), \]
\[ \pi(g) = a(p \tilde{g}) + a^*(p \tilde{g}), \quad g \in C_0^{\infty}(S), \]

where \( a^* \) and \( a \) are the usual creation and annihilation operators. Hence

\[ \varphi(f) = a(qR^3 (\rho_1 Ef)) + a^*(qR^3 (\rho_1 Ef)) - a(p \rho_0 Ef)) - a^*(p \rho_0 Ef)). \]

5 Gaussian measures and the space \( L^2(\mathcal{D}'(N)) \)

The spectral decomposition of the field operator can be achieved on a functional Hilbert space over the distributions \( \mathcal{D}'(N), N \) being a manifold. The inner product on this space is determined by choice of a Gaussian measure \( \mu_C \) with covariance \( C \). We will outline the construction of the measure (cf. e.g. Gelfand and Wilenkin [15, Ch. IV] for details).

Let \( X \) be a locally convex topological vector space and \( C \) a scalar product on \( X \), the covariance. For every finite-dimensional subspace \( U \) of \( X \) with \( \dim U = n \) the Gaussian measure \( \sigma_U \) is defined by

\[ \sigma_U(A) := (2\pi)^{-n/2} \int_A \exp[-C(x,x)/2]dx, \quad A \subset U, \]

where \( dx \) is the Lebesgue measure on \( U \) with respect to the scalar product \( C \). With help of the natural isomorphism \( P : U \to X'/U^\perp \), where \( U^\perp = \{ F \in X'| F(x_i) = 0 \ (i = 1, \ldots, n), x_i \ \text{span} \ U \} \) and \( X' \) the topological dual of \( X \), we define a family of measures \( \nu_U \) on \( X'/U^\perp \) by

\[ \nu_U(A) = \sigma_U(P^{-1}(A)), \quad A \subset X'/U^\perp. \]

This family of measures \( \nu_U \) induces a measure \( \mu \) on the cylinder sets \( Z \) of \( X' \) by

\[ \mu(Z(U, A)) = \nu_U(A), \]

where \( Z(U, A) \in Z \) is the cylinder set with base \( A \) and generating space \( U^\perp \), i.e. \( Z(U, A) = \{ F \in X'| F(x_1), \ldots, F(x_n) \} \in A, \ x_i \ \text{span} \ U \ (i = 1, \ldots, n) \}. \) The measure can be extended to a measure on the \( \sigma \)-algebra generated by the cylinder sets if it is \( \sigma \)-additive. The following theorem states that \( X \) has to be nuclear.

**Theorem 3** Sufficient and necessary for a measure on the dual space \( X' \) of a locally convex vector space \( X \) to be \( \sigma \)-additive is the nuclearity of the space \( X \).

For the proof see Gelfand and Wilenkin [13, Ch. IV, §3].

Hence, the construction presupposes that \( \mathcal{D}(N) \) is nuclear. A sufficient condition is given by the next theorem (see Maurin [16, I.9]).

**Theorem 4** The space \( \mathcal{D}(N) \) is nuclear, if \( N \) is a \( \sigma \)-compact manifold.

This means \( N \) is the countable union of compact subspaces. This condition is not too restrictive. For example every separable and locally compact manifold is \( \sigma \)-compact and especially the spaces \( \mathcal{D}(S) \) are nuclear.
6 Spectral resolution and correlations

For the spectral decomposition of the field operator we choose the functional Hilbert space $L^2(\mathcal{D}'(S_e), \Sigma, \mu_{1/2})$, where $\mathcal{D}'(S_e)$ is the space of distributions over $S_e$, $\mu_{1/2}$ the Gaussian measure on the $\sigma$-algebra $\Sigma$ generated by the cylinder sets of $\mathcal{D}'(S_e)$ with covariance $C(f, g) = \langle f | g \rangle / 2$, $\langle f | g \rangle$ is the scalar product of $L^2(S_e)$. A complete orthonormal system on $L^2(\mathcal{D}'(S_e))$ is given in terms of the Hermite polynomials $H_n$ by $2^{-n/2} n! H_n(\Phi(f))$, $n \in \mathbb{N}_0$, $f \in \mathcal{D}(S_e), \|f\| = 1$ (see [17, Ch. 6.3]). The time-zero field $\theta$ (see equation (11)) acting on the Fock space $\mathcal{F}(L^2(S_e))$ can be represented as an operator of multiplication by $\sqrt{2} \Phi(f)$ on the Hilbert space $L^2(\mathcal{D}'(S_e))$. The unitary transformation $U : \mathcal{F}(L^2(S_e)) \rightarrow L^2(\mathcal{D}'(S_e))$, is given by

$$U : f \otimes \ldots \otimes f \mapsto (n!)^{-1/2} H_n(2^{1/2} \Phi(f)), \quad \|f\| = 1.$$ 

Having represented the field operator $\theta$ as a multiplication operator on $L^2(\mathcal{D}'(S_e), \mu_{1/2})$, we are able to compute the probability distribution for a field operator $\phi(f)$ or correlations for a pair of field operators $\phi(f), \phi(g)$ as a consequence of a certain field configuration.

We compute these correlations in the time-zero field $\theta$ because it is not possible to represent $\theta$ and $\pi$ on $L^2(\mathcal{D}'(S_e), \mu_{1/2})$ simultaneously as multiplication operators. But if we find correlations in the time-zero field $\theta$, they will also be present in the field operator $\varphi = \theta - \pi$.

The spectral projections $P_\Delta(\varphi)$ of an operator $\varphi$ ($\Delta \subset \mathbb{R}$ a Borel set) are given by $P_\Delta(\varphi) \equiv \chi_\Delta(\varphi)$, where $\chi_\Delta$ is the characteristic function of the set $\Delta$. Denoting by $M_{\chi_\Delta(\Phi)}$ the operator on $L^2(\mathcal{D}'(S_e), \mu_{1/2})$ which is defined by multiplication with the function $\chi_\Delta(\Phi)$, we have $\chi_\Delta(\theta(f)) = U^{-1} M_{\chi_\Delta(\sqrt{2} \Phi(f))} U$ and with $U \Omega = 1$

$$\langle \Omega | \chi_\Delta(\theta(f)) \Omega \rangle = \langle \Omega | U^{-1} M_{\chi_\Delta(\sqrt{2} \Phi(f))} U \Omega \rangle = \sqrt{2} \int_\Delta \Phi(f) d\mu_{1/2}.$$ 

These Gaussian integrals can be solved explicitly. The probability measures $\mu_C$ are given by (see Gelfand and Wilenkin [15, Ch. 3, §3])

$$\mu_C\{\Phi(f_1) = x_1, \ldots, \Phi(f_n) = x_n\} =$$

$$= (2\pi)^{-n/2} (\det C)^{-1/2} \int \exp\left(-\frac{1}{2} \sum_{i,j=1}^n (C^{-1})_{ij} x_i x_j \right) \prod_{i=1}^n dx_i,$$

where $C = ((C_{ij})), (i, j = 1, \ldots, n)$ is the covariance matrix. In our case the entries are

$$C_{ij} := \langle f_i | f_j \rangle / 2, \quad f_i \in \mathcal{D}(S_e), \quad i, j = 1, \ldots, n,$$

such that

$$\int d\mu_{1/2} = (2\pi c)^{-1/2} \int \exp(-c^2 x^2 / 2) dx, \quad c^2 := 2/\|f\|^2,$$
and for an interval $\Delta = [a, b] \in \mathbb{R}$, $f \in \mathcal{D}(\mathcal{S}_c)$, we have
\[
\int_a^b \Phi(f) \, d\mu_{1/2} = c(2\pi)^{-1/2} \int_a^b x \exp(-c^2 x^2/2) \, dx
= \frac{\|f\|}{2\sqrt{\pi}} \left[ \exp(-a^2\|f\|^{-2}) - \exp(-b^2\|f\|^{-2}) \right].
\] (11)

For the product $\Phi(f)^2$, $f \in \mathcal{D}(\mathcal{S}_c)$, we find
\[
\int_a^b \Phi(f)^2 \, d\mu_{1/2} = \frac{\|f\|}{2\sqrt{\pi}} \left[ a \exp(-a^2\|f\|^{-2}) - b \exp(-b^2\|f\|^{-2}) \right]
+ \frac{\|f\|^2}{4} \left[ \text{Erf}(b\|f\|^{-1}) - \text{Erf}(a\|f\|^{-1}) \right].
\]

where we have defined the integral
\[
\text{Erf}(d) := \frac{2}{\sqrt{\pi}} \int_0^d \exp(-x^2) \, dx, \quad d \in \mathbb{R}.
\]

With the help of Wicks theorem we find that $c = 2\langle f | g \rangle^{-1/2}$ and for the product of two field operators the formulae are just
\[
\int_a^b \Phi(f)\Phi(g) \, d\mu_{1/2} = \frac{\langle f | g \rangle^{1/2}}{2\sqrt{\pi}} \left[ a \exp(-a^2\langle f | g \rangle^{-1}) - b \exp(-b^2\langle f | g \rangle^{-1}) \right]
+ \frac{\langle f | g \rangle}{4} \left[ \text{Erf}(b\langle f | g \rangle^{-1/2}) - \text{Erf}(a\langle f | g \rangle^{-1/2}) \right].
\] (12)

We are now ready to compute the expectation value
\[
< \chi_\Delta(\theta(f)\theta(g)) > = 2 \int_\Delta \Phi(qR^3(\rho_1 E f)\Phi(qR^3(\rho_1 E g)) \, d\mu_{1/2},
\]
where we can use formula (12) now. On the other hand we have
\[
< \chi_\Delta(\theta(f)) > < \chi_\Delta(\theta(g)) > = 2 \int_\Delta \Phi(qR^3(\rho_1 E f)\Phi(qR^3(\rho_1 E g)),
\]
where we can use formula (11) now.

We summarize some formulae: The expectation value of the product of two field operators to have a value in the interval $\Delta = [a, b]$ is given by
\[
< \chi_\Delta(\theta(f)\theta(g)) > = 2 \int_\Delta \Phi(qR^3(\rho_1 E f)\Phi(qR^3(\rho_1 E g)) \, d\mu_{1/2}(\Phi)
\]

\[
= \langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle^{1/2} \pi^{-1/2} \left[ a \exp(-a^2/\langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle)
- b \exp(-b^2/\langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle) \right]
+ 2^{-1} \langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle \left[ \text{Erf}(b \langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle^{-1/2})
- \text{Erf}(a \langle qR^3(\rho_1 E f) | qR^3(\rho_1 E g) \rangle^{-1/2}) \right].
\]
On the other hand we have
\[
< \chi_\Delta(\theta(f)) > < \chi_\Delta(\theta(g)) > = \\
= 2 \int_\Delta \Phi(q R^3 (\rho_1 E f)) \int_\Delta \Phi(q R^3 (\rho_1 E g)) = \\
= 2 \pi^{-1} \| q R^3 (\rho_1 E f) \|^2 \| q R^3 (\rho_1 E g) \|^2 [\exp(-a^2 \| q R^3 (\rho_1 E f) \|^2) - \exp(-b^2 \| q R^3 (\rho_1 E f) \|^2)] \\
\times [\exp(-a^2 \| q R^3 (\rho_1 E g) \|^2) - \exp(-b^2 \| q R^3 (\rho_1 E g) \|^2)].
\]

Easier formulae are obtained by taking the interval \( \Delta = [0, \infty) \)
\[
< \chi_\Delta(\theta(f)) > < \chi_\Delta(\theta(g)) > = \frac{1}{2\pi} \| q R^3 (\rho_1 E f) \|^2 \| q R^3 (\rho_1 E g) \|^2 \\
< \chi_\Delta(\theta(f)|\theta(g)) > = \frac{1}{2} \langle q R^3 (\rho_1 E f) | q R^3 (\rho_1 E g) \rangle. \tag{13}
\]

Remark: The Hilbert space is built over \( D'(\mathcal{S}_\varepsilon) \), hence one would expect that \( C^\infty_{0,2}(\mathcal{S}_\varepsilon) \)-functions are needed in explicit calculations. But since we are integrating only over cylindrical sets and the propagator can be extended to the spaces of distributions, it is possible to use \( L^2(\mathcal{S}_\varepsilon) \)-functions as well. Nevertheless, by the complexity of the formulae, it is impossible to evaluate them analytically, even in the case when the propagator \( E \) is explicitly known. We will analyse the formulae (13) in a forthcoming paper.

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