Predictions of Neutrino Mixing Angles in a $T'$ Model

David A. Eby*, Paul H. Frampton† and Shinya Matsuzaki‡

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599-3255.

Abstract

Flavor symmetry ($T' \times Z_2$) where $T'$ is the binary tetrahedral group predicts for neutrino mixing angles $\theta_{13} = \sqrt{2}(\frac{\pi}{4} - \theta_{23})$ and, with one phenomenological input, provides upper and lower bounds on both $\theta_{13}$ and $\theta_{23}$. The predictions arise from the deviation of the Cabibbo angle $\Theta_{12}$ from its lowest-order value $\tan 2\Theta_{12} = (\sqrt{2})/3$ and from the $T'$ mechanism which relates mixing of $(\nu_\tau, \nu_\mu, \nu_e)$ neutrinos to mixing of $(s, d)$ quarks.

*daeby@physics.unc.edu
†frampton@physics.unc.edu
‡synya@physics.unc.edu
As an attractive alternate to the grand unification of the strong and electroweak interactions, a global flavor symmetry acting in tandem with the standard model gauge group can address the issue of relating the quarks and leptons. Other than the equality of numbers of quarks and leptons no direct connection between them has been discovered.

Evidence for grand unification such as proton decay remains elusive so it seems worth finding testable predictions of family symmetry.

In the present article we shall invoke a family symmetry involving the binary tetrahedral group $T'$ which has sufficient structure to relate quarks and leptons. In a previous study [1], an exact formula for the Cabibbo angle $\Theta_{12}$ was derived in a ($T' \times Z_2$) model where the neutrino mixing angles are of the tribimaximal (TBM) values [2]. In the same model, a striking prediction was made [3] for two-body leptonic decays of the Higgs boson.

The three neutrino mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ are empirically consistent with the TBM values. However, as the experimental accuracy improves, this situation may change. Thus, it is of considerable interest to predict quantitatively what departures from the TBM values

$$\theta_{12} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right), \quad \theta_{23} = (\pi/4), \quad \theta_{13} = 0$$

are to be expected? We are delighted to report that the ($T' \times Z_2$) model allows one to address this question by relating the perturbations around TBM

$$\theta_{ij} = (\theta_{ij})_{TBM} + \epsilon_k,$$

where we use $\epsilon_3$ for $\theta_{12}$, and so on, and the TBM values are in Eq.(1), to the perturbation around the Cabibbo angle value

$$\tan 2(\Theta_{12})_{T'} = \left(\frac{1}{3}\right)(\sqrt{2}).$$

We are less delighted with progress on the quark and lepton masses. Although we understand why $m_t \gg m_b > m_{c,s,d,u}$ for quarks and why $m_3 \gg m_{1,2}$ for neutrinos, when we look more closely at the details we find that masses are not quantitatively explained. It is not clear to us whether this will be corrected in the ($T' \times Z_2$) model by higher order corrections and/or adding $T'$ doublet VEVs. We hope to return to the masses in a further work.

In the present work, we take the view that the model can make reliable predictions about mixing angles even when details of the mass spectra are incomplete.
To analyze the relationship between the perturbations in Eq. (2) and the Cabibbo angle will require, as we shall see, very interesting $T'$ algebra sometimes arriving at astonishingly simple expressions.

Let us begin the analysis.

First we recall a few salient points about the model in [4] based on $A_4$ symmetry [5–7]. The only important scalar for the present analysis is the triplet $H_3(3, +1)$ whose vacuum expectation value in [4] was taken as

$$< H_3 >= (V_1, V_2, V_3) = V(1, -2, 1)$$

(4)

which led to the TBM neutrino mixing in Eq.(1). We consider the perturbation

$$< H_3 >= (V'_1, V'_2, V'_3) = V'(1, -2 + b, 1 + a)$$

(5)

where $|a|, |b| \ll 1$.

We first consider the calculation which makes the perturbation around the $\Theta_{12}$ calculation based on $T'$ symmetry [8–11] in [1] by using Eq.(5) in place of Eq.(4). The down-quark ($2 \times 2$) mass matrix for the first two families ($s, d$) is perturbed to

$$D \equiv \left( \begin{array}{c} \frac{1}{V'Y_S} \\ \sqrt{\frac{2}{3}}(1 + a) \\ \sqrt{\frac{3}{2}}\omega \end{array} \right)$$

$$D' = \left( \begin{array}{ccc} \frac{1}{\sqrt{3}} & (2 + b)\sqrt{\frac{3}{2}}\omega \\ \sqrt{\frac{2}{3}}(1 + a) & \frac{1}{\sqrt{3}}\omega \end{array} \right)$$

(6)

The hermitian square $D \equiv DD^\dagger$ is

$$D \equiv DD^\dagger \simeq \left( \begin{array}{c} 1 \\ \sqrt{2}(-1 + a + b) \\ 3 + 4a \end{array} \right)$$

(7)

The eigenvalues satisfy the quadratic equation

$$(9 - 8b - \lambda)(3 + 4a - \lambda) - 2(1 - a - b)^2 = 0$$

(8)

with solutions

$$\lambda_\pm = (6 \pm \sqrt{11}) + 2a \left(1 \mp \frac{4}{\sqrt{11}}\right) - 2b \left(2 \pm \frac{7}{\sqrt{11}}\right)$$

(9)

An eigenvector $(\alpha, \beta)$ has components satisfying

$$\left( \begin{array}{c} \beta \\ \alpha \end{array} \right) = \left( \begin{array}{c} 3 - \frac{\sqrt{11}}{\sqrt{2}} \\ \sqrt{2}(-1 + a + b) \end{array} \right) \left[ 1 - \frac{a}{\sqrt{11}} + \frac{b}{\sqrt{11}} \right]$$

(10)
whose normalization $N(\alpha, \beta)$ satisfies

$$N^{-2} = 1 + \frac{\beta^2}{\alpha^2}$$  \hspace{1cm} (11)

from which the Cabibbo angle $\sin \Theta_{12} = \frac{N \beta}{\alpha}$ is

$$\sin \Theta_{12} = \sqrt{\left(\frac{1}{2} - \frac{3}{2\sqrt{11}}\right) \left(1 - \frac{3 + \sqrt{11}}{22}(a - b)\right)}$$  \hspace{1cm} (12)

From this one finds at leading order

$$\cos 2\Theta_{12} \simeq \left(\frac{3}{\sqrt{11}}\right) \left(1 + \frac{2}{33}(a - b)\right)$$  \hspace{1cm} (13)

and

$$\sin 2\Theta_{12} \simeq \left(\frac{\sqrt{2}}{\sqrt{11}}\right) \left(1 - \frac{3}{11}(a - b)\right),$$  \hspace{1cm} (14)

whence

$$\tan 2\Theta_{12} \simeq \left(\frac{\sqrt{2}}{3}\right) \left(1 - \frac{1}{3}(a - b)\right)$$  \hspace{1cm} (15)

which is a surprisingly simple generalization of the $a = b = 0$ case [1]!

Our next step is to relate the perturbations $\epsilon_i$ in the neutrino mixing angles, Eq.(2), to the perturbations $a$ and $b$ in the vacuum alignment, Eq.(5).

We use a neutrino mixing matrix relating flavor eigenstates to mass eigenstates $(\nu_1, \nu_2, \nu_3)^T \equiv U(\nu_\tau, \nu_\mu, \nu_e)^T$ with, assuming no CP violation

$$U = \begin{pmatrix} -s_{12}s_{23} - c_{12}c_{23}s_{13} & -s_{12}c_{23} + c_{12}s_{23}s_{13} + c_{12}c_{13} \\ +c_{12}s_{23} - s_{12}c_{23}s_{13} + c_{12}c_{23} + s_{12}s_{23}s_{13} + s_{12}c_{13} \\ +c_{23}c_{13} + s_{23}c_{13} + s_{13} \end{pmatrix}.$$  \hspace{1cm} (16)

This takes the TBM form

$$U_{TBM} = \begin{pmatrix} -\sqrt{\frac{1}{6}} - \sqrt{\frac{1}{6}} + \sqrt{\frac{2}{3}} \\ +\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}} \\ +\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} \end{pmatrix}$$  \hspace{1cm} (17)

for the values of the neutrino mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ given in Eq.(1). Making the perturbations defined in Eq.(2), one has, at first order,
\[ s_{12} \simeq \sqrt{\frac{1}{3}}(1 + \sqrt{2} \epsilon_3); \quad c_{12} \simeq \sqrt{\frac{2}{3}}(1 - \epsilon_3/\sqrt{2}); \]

\[ s_{23} \simeq \sqrt{\frac{1}{2}}(1 + \epsilon_1); \quad c_{23} \simeq \sqrt{\frac{1}{2}}(1 - \epsilon_1). \]

\[ s_{13} \simeq \epsilon_2; \quad c_{13} \simeq 1; \]

Consequently one may write

\[ U \simeq U_{TBM} + \delta U = U_{TBM} + \delta U_1 \epsilon_1 + \delta U_2 \epsilon_2 + \delta U_3 \epsilon_3 \quad (18) \]

in which

\[ \delta U_1 = \begin{pmatrix} -\sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}}&0 \\ +\sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}}&0 \\ -\sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}}&0 \end{pmatrix} \quad (19) \]

\[ \delta U_2 = \begin{pmatrix} -\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}}&0 \\ -\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}}&0 \\ 0&0&0 \end{pmatrix} \quad (20) \]

\[ \delta U_3 = \begin{pmatrix} -\sqrt{\frac{3}{4}} - \sqrt{\frac{3}{4}} - \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} + \sqrt{\frac{3}{4}} - \sqrt{\frac{3}{4}} \\ 0 & 0 & 0 \end{pmatrix} \quad (21) \]

For TBM mixing, one has

\[ (M_\nu)_{TBM} = U^T_{TBM}(M_\nu)_{\text{diag}} U_{TBM} \quad \text{with} \quad (M_\nu)_{\text{diag}} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \quad (22) \]

This gives

\[ (M_\nu)_{TBM} = \left(\frac{1}{6}\right) \begin{pmatrix} m_1 + 2m_2 + 3m_3 & m_1 + 2m_2 - 3m_3 & -2m_{12} \\ m_1 + 2m_2 - 3m_3 & m_1 + 2m_2 + 3m_3 & -2m_{12} \\ 4m_1 + 2m_2 & 4m_1 + 2m_2 & 4m_1 + 2m_2 \end{pmatrix} \quad (23) \]

where \((M_\nu)_{TBM}\) is symmetric and where \(m_{12} \equiv (m_1 - m_2)\).
From Eq. (22) the perturbation in \((M_\nu)_{\text{diag}}\) satisfies
\[
\delta (M_\nu)_{\text{diag}} = \begin{pmatrix}
\delta m_1 & 0 & 0 \\
0 & \delta m_2 & 0 \\
0 & 0 & \delta m_3
\end{pmatrix}
= \delta U (M_\nu)_{\text{TBM}} U^T
+ U_{\text{TBM}} \delta M_\nu U^T
+ U_{\text{TMB}} (M_\nu)_{\text{TBM}} \delta U^T,
\] (24)
in which \(U_{\text{TBM}}\) is known from Eq. (17) and \(\delta U\) from Eqs. (18, 19, 20, 21).

To compute \(\delta M_\nu\) in Eq. (24) we use \((M_\nu)_{\text{TBM}}\) from reference [4]
\[
(M_\nu)_{\text{TBM}} = \begin{pmatrix}
x_1 V_1^2 + 2x_{23} V_2 V_3 & x_1 V_1 V_3 + x_{23} (V_2^2 + V_1 V_3) & x_1 V_1 V_2 + x_{23} (V_2^2 + V_1 V_3) \\
x_1 V_2 V_3 + 2x_{23} V_1 V_2 & x_1 V_2 V_3 + x_{23} (V_1^2 + V_2 V_3) & x_1 V_2^2 + 2x_{23} V_1 V_3
\end{pmatrix},
\] (25)
in which \(< H_3 > = (V_1, V_2, V_3),\ x_1 = Y_1^2/M_1\) and \(x_{23} = Y_2 Y_3/M_{23}\). These variables involve Yukawa couplings and right-handed neutrino masses all of which are empirically unknown. Only the combination \(y = x_{23}/x_1\) survives and our predictions will be obtained by eliminating this unknown.

To find \(\delta M_\nu\) in Eq. (24) we use the perturbation of the vacuum alignment, Eq. (3), in Eq. (25) to find
\[
\delta M_\nu = V_1^2 x_1 \begin{pmatrix}
2(-2a + b)y a + (a - 4b)y & b + (2a + b)y \\
2(a + by) & (-2a + b)(1 + y) \\
-4b + 2ay
\end{pmatrix}.
\] (26)

By inserting this \(\delta M_\nu\) into Eq. (24) we obtain six equations from the \((3 \times 3)\) symmetric matrix to combine with Eq. (15) above. In the \(\delta m_1\) of (I) - (III) a common (unpredicted) normalization factor has been omitted.

- (I) \(\delta m_1 = (2 + y)(a - 2b)\)
- (II) \(\delta m_2 = 0\)
- (III) \(\delta m_3 = -3y(a - 2b)\)
- (IV) \(\epsilon_2 = -\sqrt{2}\epsilon_1\)
- (V) \(a = 6\epsilon_1 = -3\sqrt{2}\epsilon_2\)
- (VI) \((a + b) = \left(\frac{3}{\sqrt{2}}\right) \frac{2 + y}{1 - y} \epsilon_3\)
The result (IV) provides a prediction from $T'$ that
\[
\theta_{13} = \sqrt{2} \left( \frac{\pi}{4} - \theta_{23} \right)
\] (27)
which interestingly links any non-zero value for $\theta_{13}$ to the departure of the atmospheric neutrino mixing angle $\theta_{23}$ from maximal mixing with $\theta_{23} = \pi/4$. This is our most definite prediction from $T'$, independent of phenomenological input \#4.

To arrive at further $T'$ predictions for the neutrino mixings $\theta_{13}$ and $\theta_{23}$ we shall require phenomenological input.

The equation (I) through (III) must be combined with the zeroth order values
\[
m_0^1 = 3(y + 2),
\]
\[
m_0^2 = 0,
\]
\[
m_0^3 = -9y.
\]

It is noted that $m_2 = 0$ remains even at first order. This arises from the zero structures \[13–15\] in the terms of Eq.(24). They are
\[
\delta U(M_\nu)_{TBM}U_{TBM}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
(31)
\[
U_{TBM}\delta M_\nu U_{TBM}^T \begin{pmatrix} 0 & 0 \\ 0 \end{pmatrix},
\]
(32)
\[
U_{TBM}(M_\nu)_{TBM}\delta U^T \begin{pmatrix} 0 & 0 & 0 \\ 0 \end{pmatrix}.
\]
(33)

The necessary phenomenological input, exactly as in [4], is to set
\[
y = -2
\]
(34)
from which equation (VI) gives $(a + b) = 0$ and Eq.(15) becomes simply
\[
\tan 2\Theta_{12} = \left( \frac{\sqrt{2}}{3} \right) (1 - 4\epsilon_1)
\]
(35)

\#4A similar prediction from a different starting point appeared in [12].
Eq. (35) allows us, from the experimental value [16], \((\Theta_{12})_{\text{experiment}} = 13.05 \pm 0.07^\circ\), to identify the limits
\[
-0.0114 < \epsilon_1 < -0.0082 \quad (36)
\]
and
\[
0.011 < \epsilon_2 < 0.016 \quad (37)
\]
The values in Eqs. (36,37) of \(\epsilon_{1,2}\) lead directly to predictions for the neutrino mixing angles. Substitution of Eqs. (36,37) into Eq. (2) gives
\[
0.5 \times 10^{-3} \leq \sin^2 2\theta_{13} \leq 1.0 \times 10^{-3} \quad (38)
\]
and
\[
0.99947 \leq \sin^2 2\theta_{23} \leq 0.99973 \quad (39)
\]
The situation with respect to \(T'\) flavor symmetry is very exciting. The predictions Eq. (27), Eq. (38) and Eq. (39) have different status. The prediction relating \(\theta_{13}\) and \(\theta_{23}\) in Eq. (27) is the sharpest. With the one phenomenological input, Eq. (34), necessary to obtain a sensible neutrino mass spectrum one arrives at the predictions in Eq. (38) and Eq. (39) which also provide targets of opportunity for experiments.

With regard to quark and lepton masses, the flavor symmetry leaves them as enigmatic as before, basically as free parameters just as for the standard model. The mixing angles are, however, significantly constrained by the \(T'\) geometrical structure. In particular, we have shown how the neutrino mixing angles are predicted by the empirical departure from the \(T'\) prediction for the largest quark mixing of the Cabibbo angle.
Acknowledgements

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-06ER41418.
References

[1] P.H. Frampton, T.W. Kephart and S. Matsuzaki. Phys. Rev. D78, 073004 (2008). arXiv:0807.4713 [hep-ph].

[2] P.F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002) hep-ph/0202074

[3] P.H. Frampton and S. Matsuzaki, *Prediction for $\Gamma(H \rightarrow \tau^+\tau^-)/\Gamma(H \rightarrow \mu^+\mu^-)$ from Non-Abelian Flavor Symmetry.* arXiv:0810.1029 [hep-ph]

[4] P.H. Frampton and S. Matsuzaki, *Renormalizable $A_4$ Model for Leptons* arXiv:0806.4592 [hep-ph]

[5] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001). hep-ph/0106291; K.S. Babu, E. Ma and J.W.F. Valle, Phys. Lett. B552, 207 (2003). hep-ph/0206292.

[6] E. Ma, Mod. Phys. Lett. A20, 2601 (2005). Phys. Lett. B632, 352 (2006). hep-ph/0508099. B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M.K. Parida, Phys. Lett. B638, 345 (2006). hep-ph/0603059. E. Ma, Phys. Rev. D73, 057304 (2006). E. Ma, H. Sawanaka, and M. Tanimoto, Phys. Lett. B641, 301 (2006). hep-ph/0606103. E. Ma, Mod. Phys. Lett. A21, 1917 (2006). hep-ph/0607056

[7] G. Altarelli and F. Feruglio, Nucl. Phys. B720, 64 (2005). hep-ph/0504165; Nucl. Phys. B742, 215 (2006). hep-ph/0512103. G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B775, 31 (2007). hep-ph/0610165.

[8] P.H. Frampton and T.W. Kephart, Int. J. Mod. Phys. 10A, 4689 (1995).

[9] P. D. Carr and P. H. Frampton. hep-ph/0701034.

[10] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B775, 120 (2007). hep-ph/0702194.

[11] M.-C. Chen and K.T. Mahanthappa. Phys. Lett. B652, 34 (2007). arXiv:0705.0714

[12] P.F. Harrison and W.G. Scott, Phys. Lett. B628, 93 (2005). hep-ph/0508012.
[13] P.H. Frampton and S.L. Glashow, Phys. Lett. B461, 95 (1999). [hep-ph/9906375]

[14] P.H. Frampton, S.L. Glashow and D. Marfatia, Phys. Lett. 536B, 79 (2002). [hep-ph/0201008]

[15] P.H. Frampton, S.L. Glashow and T. Yanagida, Phys. Lett. B548, 119 (2002). [hep-ph/0208157]

[16] Particle Data Group, Phys. Lett. 667, 1 (2008).