A note on finitely generated quotients of locally compact groups

Linus Kramer*

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Throughout, all topological groups are assumed to be Hausdorff. The identity component of such a group $G$ is denoted $G^\circ$. We call a topological group topologically finitely generated if it contains a finitely generated dense subgroup.

The aim of this note is to prove the following results. The finitely generated group $\Gamma$ that appears in the theorems below is always endowed with the discrete topology.

**Theorem A** Let $G$ be a topologically finitely generated locally compact abelian group, let $\Gamma$ be a finitely generated group and let $\varphi : G \rightarrow \Gamma$ be an abstract epimorphism. Then $\varphi$ is continuous.

**Theorem B** Let $G$ be a Lie group whose identity component $G^\circ$ has finite index in $G$, let $\Gamma$ be a finitely generated group and let $\varphi : G \rightarrow \Gamma$ be an abstract epimorphism. Then $\Gamma$ is finite and $\varphi$ is continuous.

The next result depends on Theorem B and on deep results by Iwasawa, Yamabe and Nikolov–Segal.

**Theorem C** Let $G$ be topologically finitely generated locally compact group, with $G/G^\circ$ compact. Let $\Gamma$ be a finitely generated group and let $\varphi : G \rightarrow \Gamma$ be an abstract epimorphism. Then $\Gamma$ is finite and $\varphi$ is continuous.

**Proof of Theorem A.** We proceed in three steps. The first two steps follow loosely the arguments in [10] Thm. 5.1.

1) The claim of Theorem A is true if $G$ is compact and if $\Gamma$ is finite.

Let $e$ denote the exponent of $\Gamma$. Then $G^e = \{g^e \mid g \in G\} \subseteq G$ is a compact subgroup and $G/G^e$ has finite exponent. Moreover, $\varphi$ factors as

$$
\begin{array}{c}
G \\
\downarrow \pi \\
G/G^e
\end{array}
\xrightarrow{\varphi}
\begin{array}{c}
\Gamma \\
\downarrow \varphi
\end{array}
$$

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Let $\Xi \subseteq G$ be a finitely generated dense subgroup. Then $\pi(\Xi)$ is a finitely generated abelian group of finite exponent and therefore finite. On the other hand, $\pi(\Xi)$ is dense in $G/G^e$. Hence $G/G^e$ is finite and thus $G^e \subseteq \ker(\varphi)$ is open. The claim follows.

2) The claim of Theorem A is true if $G$ is compact.

For $\ell \in \mathbb{N}_{\geq 1}$ consider the composite

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi} & \Gamma \\
& \downarrow{\varphi_\ell} & \\
\Gamma / \Gamma^\ell & \\
\end{array}
$$

By the previous result, $\varphi_\ell$ is continuous. Hence $\ker(\varphi) = \bigcap_{\ell \geq 1} \ker(\varphi_\ell)$ is closed. Therefore $G/\ker(\varphi)$ is a countable locally compact group. A countable locally compact group is, by the Open Mapping Theorem [12] 6.19, discrete. Therefore $\ker(\varphi)$ is open and the claim follows.

3) The claim of Theorem A is true.

We may decompose the group $G$ topologically as $G = \mathbb{R}^n \times H$, where $H$ has a compact open subgroup $K$, see [5] Thm. 7.57 (i). Since $\mathbb{R}^n$ is divisible and since a finitely generated abelian group contains no divisible elements besides the identity, $\varphi$ factors as

$$
\begin{array}{ccc}
\mathbb{R}^n \times H & \xrightarrow{\varphi_0} & \Gamma \\
& \downarrow{pr_2} & \\
H & \xrightarrow{\tilde{\varphi}} & \\
\end{array}
$$

The restriction $\tilde{\varphi} : K \xrightarrow{} \Gamma_0 = \tilde{\varphi}(K)$ is continuous by 2), since $\Gamma_0 \subseteq \Gamma$ is also finitely generated. It follows that the restriction of $\varphi$ to the open subgroup $\mathbb{R}^n \times K \subseteq G$ is continuous and open. Therefore $\varphi$ is continuous.

In the proof of Theorem B we rely on the following fact which was observed in different degrees of generality by Goto [3], Ragozin [11] and George Michael [1]. For the sake of completeness, we include a proof.

**Proposition** Let $G$ be a Lie group whose Lie algebra is semisimple. Suppose that $N \triangleleft G$ is an abstract normal subgroup. Then $N$ is closed.

**Proof.** Since $G^0 \subseteq G$ is closed and open, it suffices to show that $N \cap G^0$ is closed in $G^0$. Hence we may assume that $G$ is connected, and we proceed by induction on the dimension of $G$. The claim is clear if $\dim(G) = 0$. Let $N_1 \subseteq N$ denote the path component of the identity. If $N_1 = \{1\}$, then $N \subseteq \text{Cen}(G)$ and therefore $N$ is closed. If $N_1$ is nontrivial, then $N_1 \triangleleft G$ is a nontrivial virtual normal Lie subgroup, see [3]. Its Lie algebra is then a nontrivial ideal in $\text{Lie}(G)$. Since $\text{Lie}(G)$ is semisimple, the virtual subgroup corresponding to any ideal in $\text{Lie}(G)$
is closed: it is the connected centralizer of the complementary ideal. Therefore \( N_1 \subseteq G \) is closed. Now we may apply the induction hypothesis to \( N/N_1 \subseteq G/N_1 \). It follows that \( N \subseteq G \) is closed.

\[ \therefore \]

**Proof of Theorem B.** The subgroup \( \varphi(G^\circ) = \Gamma_0 \subseteq \Gamma \) has finite index and is therefore also finitely generated. Therefore we may as well assume that \( G \) is connected. Let \( R \trianglelefteq G \) denote the solvable radical, i.e. the unique maximal connected solvable closed normal subgroup of \( G \). Then the Lie algebra \( \text{Lie}(G/R) \) is semisimple. We put \( N = \ker(\varphi) \). By the proposition above, \( NR \subseteq G \) is closed. Since \( G/N \) is countable, \( G/NR \) is countable and hence discrete. It follows that \( NR \subseteq G \) is open, hence \( G = NR \). Now we have \( \Gamma \cong NR/N \cong R/R \cap N \). The group \( R \) is divisibly generated, that is, it has a generating set consisting of divisible elements, see e.g. [7] 9.52. It follows that the abelianization \( \Gamma_{ab} \) of \( \Gamma \) is both divisible and finitely generated. Therefore \( \Gamma_{ab} = 1 \). On the other hand, \( \Gamma \) is solvable because \( R \) is solvable as an abstract group. It follows that \( \Gamma = \{1\} \).

\[ \therefore \]

In the proof of Theorem C, we make use of the following two deep results.

**Theorem (Nikolov–Segal)** Let \( G \) be a compact group and let \( \Gamma \) be a finitely generated group. Suppose that \( \varphi : G \longrightarrow \Gamma \) is an abstract epimorphism. Then \( \Gamma \) is finite. If \( G \) is in addition topologically finitely generated, then \( \varphi \) is continuous and open.

**Proof.** This is Thm. 5.25 and Thm. 5.7 in [10].

**Theorem (Iwasawa)** Let \( G \) be a connected locally compact group. Then there exists a compact connected subgroup \( K \subseteq G \), a simply connected Lie group \( L \) and a continuous open epimorphism \( \varphi : L \times K \longrightarrow G \) with discrete kernel, such that \( \varphi(1, k) = k \) for all \( k \in K \).

**Proof.** Iwasawa proves a local version of this result in [5], p. 547, Theorem 11, assuming that \( G \) is a projective limit of Lie groups. Yamabe showed however that every locally compact group has an open subgroup which is a projective limit of Lie groups, see [9], p. 175. For the global formulation of Iwasawas’s Theorem that we use here, see also [2], [6] and [4] Sec. 4.

**Proof of Theorem C.** Let \( \Gamma_0 = \varphi(G^\circ) \). Since \( G/G^\circ \) is compact, the Theorem by Nikolov–Segal implies that \( \Gamma_0 \subseteq \Gamma \) has finite index. In particular, \( \Gamma_0 \) is also finitely generated. Hence we may assume that \( G \) is connected, and we have to show that then \( \Gamma \) is trivial. We put \( L \times K \longrightarrow G \) as in Iwasawa’s Theorem and we consider the composite \( \bar{\varphi} : L \times K \longrightarrow G \longrightarrow \Gamma \). Then \( \bar{\varphi} \) maps \( L \times \{1\} \) onto the finitely generated group \( \Gamma/\bar{\varphi}(\{1\} \times K) \). By Theorem B, \( \bar{\varphi}(\{1\} \times K) = \Gamma \). By Nikolov–Segal, \( \bar{\varphi}(\{1\} \times K) \) is finite. Since \( K \) is divisible by [6] 9.35, \( \bar{\varphi}(\{1\} \times K) \) is divisible and therefore trivial. Hence \( \Gamma \) is trivial.

\[ \therefore \]

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Linus Kramer
Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
linus.kramer@uni-muenster.de