AVERAGE CASE TRACTABILITY OF MULTIVARIATE APPROXIMATION WITH GAUSSIAN KERNELS

JIA CHEN, HEPING WANG

ABSTRACT. We study the problem of approximating functions of \( d \) variables in the average case setting for the \( L_2 \) space \( L_{2,d} \) with the standard Gaussian weight equipped with a zero-mean Gaussian measure. The covariance kernel of this Gaussian measure takes the form of a Gaussian kernel with non-increasing positive shape parameters \( \gamma^2_j \) for \( j = 1, 2, \ldots, d \). The error of approximation is defined in the norm of \( L_{2,d} \). We study the average case error of algorithms that use at most \( n \) arbitrary continuous linear functionals. The information complexity \( n(\varepsilon, d) \) is defined as the minimal number of linear functionals which are needed to find an algorithm whose average case error is at most \( \varepsilon \). We study different notions of tractability or exponentially-convergent tractability (EC-tractability) which the information complexity \( n(\varepsilon, d) \) describe how behaves as a function of \( d \) and \( \varepsilon^{-1} \) or as one of \( d \) and \( (1 + \ln \varepsilon^{-1}) \). We find necessary and sufficient conditions on various notions of tractability and EC-tractability in terms of shape parameters. In particular, for any positive \( s > 0 \) and \( t \in (0, 1) \) we obtain that the sufficient and necessary condition on \( \gamma^2_j \) for which

\[
\lim_{d \to \infty, \varepsilon^{-1} \to \infty} \frac{n(\varepsilon, d)}{\varepsilon^{-s} + d^t} = 0
\]

holds is

\[
\lim_{j \to \infty} j^{1-t} \gamma^2_j \ln^+ \gamma^{-2}_j = 0,
\]

where \( \ln^+ x = \max(1, \ln x) \).

1. INTRODUCTION AND MAIN RESULTS

Recently, there has been an increasing interest in \( d \)-variate computational problems with large or even huge \( d \). Examples include problems in computational finance, statistics and physics. Such problems are usually solved by algorithms that use finitely many information operations. The information complexity \( n(\varepsilon, d) \) is defined as the minimal number of information operations which are needed to find an approximating solution to within an error threshold \( \varepsilon \). A central issue is the study of how the information complexity depends on \( \varepsilon \) and \( d \). Such problem is called the tractable problem. There are two kinds of tractability based on polynomial-convergence and exponential-convergence. The (classical) tractability describes how the information complexity \( n(\varepsilon, d) \) behaves as a function of \( d \) and \( \varepsilon^{-1} \), while the exponentially-convergent tractability (EC-tractability) does as one of \( d \) and \( (1 + \ln \varepsilon^{-1}) \). Nowadays study of tractability and EC-tractability has

2010 Mathematics Subject Classification. 41A25, 41A63, 65D15, 65Y20.

Key words and phrases. Tractability; Exponential convergence; EC-tractability; Gaussian covariance kernels; Average case setting.

Supported by the National Natural Science Foundation of China (Project no. 11671271) and the Beijing Natural Science Foundation (1172004) .
become one of the busiest areas of research in information-based complexity (see \[10, 11, 12, 15, 13, 19\] and the references therein).

In this paper, we consider tractability of a multivariate approximation problem defined over the space \(L_{2,d}\) in the average case setting, where

\[
L_{2,d} = \left\{ f \mid \|f\|_{L_{2,d}} = \left( \int_{\mathbb{R}^d} |f(x)|^2 \prod_{j=1}^d \frac{\exp\left(-x_j^2\right)}{\sqrt{\pi}} \, dx \right)^{1/2} < \infty \right\}
\]

is a separable Hilbert space of real-valued functions on \(\mathbb{R}^d\) with inner product

\[
\langle f, g \rangle_{L_{2,d}} = \int_{\mathbb{R}^d} f(x)g(x) \prod_{j=1}^d \frac{\exp\left(-x_j^2\right)}{\sqrt{\pi}} \, dx.
\]

The space \(L_{2,d}\) is equipped with a zero-mean Gaussian measure \(\mu_d\) with Gaussian covariance kernel

\begin{equation}
K_{d,\gamma}(x,y) = \int_{L_{2,d}} f(x)f(y)\mu_d(df) = \prod_{j=1}^d K_{\gamma_j}(x_j, y_j), \, x, y \in \mathbb{R}^d,
\end{equation}

where

\[
K_{\gamma_j}(x, y) = \exp(-\gamma_j^2(x-y)^2), \, x, y \in \mathbb{R},
\]

and \(\gamma = \{\gamma_j^2\}_{j \in \mathbb{N}}\) is a given sequence of shape parameters not depending on \(d\) and satisfying

\begin{equation}
\gamma_1^2 \geq \gamma_2^2 \geq \cdots > 0.
\end{equation}

We consider multivariate approximation which is defined via the embedding operator

\[
\text{App}_d : L_{2,d} \to L_{2,d}, \text{ with } \text{App}_d f = f.
\]

We approximate \(\text{App}_d f\) by algorithms that use only finitely many continuous linear functionals. A function \(f \in L_{2,d}\) is approximated by an algorithm of the form

\begin{equation}
A_{n,d}(f) = \Phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)),
\end{equation}

where \(L_1, L_2, \ldots, L_n\) belong to continuous linear functionals on \(L_{2,d}\), and \(\Phi_{n,d} : \mathbb{R}^n \to L_{2,d}\) is an arbitrary measurable mapping. It is well known (see \([10]\)) that we can restrict ourselves to linear algorithms \(A_{n,d}\) of the form

\begin{equation}
A_{n,d}f = \sum_{k=1}^n L_k(f)\psi_k,
\end{equation}

where \(\psi_k \in L_{2,d}, \, k = 1, 2, \ldots, n\). The average case error for \(A_{n,d}\) is defined by

\[
e(A_{n,d}) = \left( \int_{L_{2,d}} \|\text{App}_d f - A_{n,d}f\|_{L_{2,d}}^2 \mu_d(df) \right)^{1/2}.
\]

The \(n\)th minimal average case error, for \(n \geq 1\), is defined by

\[
e(n, d) = \inf_{A_{n,d}} e(A_{n,d}),
\]

where the infimum is taken over all algorithms of the form \((1.3)\) or \((1.4)\). For \(n = 0\), we use \(A_{0,d} = 0\). We remark that the so-called initial error \(e(0, d)\), defined by

\[
e(0, d) = \left( \int_{L_{2,d}} \|\text{App}_d f\|_{L_{2,d}}^2 \mu_d(df) \right)^{1/2},
\]
is equal to 1. In other words, the normalized error criterion and the absolute error criterion coincide.

The information complexity $n(\varepsilon, d)$ is defined by

$$n(\varepsilon, d) = \inf \{ n \mid e(n, d) \leq \varepsilon \}.$$

Let $\text{App} = \{ \text{App}_d \}_{d \in \mathbb{N}}$. First we consider the classical tractability of $\text{App}$.

Various notions of (the classical) tractability have been studied recently for many multivariate problems. We briefly recall some of the basic tractability notions (see [10, 12, 13, 17]):

We say $\text{App}$ is

- **strongly polynomially tractable (SPT)** iff there exist non-negative numbers $C$ and $p$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,
  $$n(\varepsilon, d) \leq C(\varepsilon^{-1})^p;$$

  The exponent of SPT the exponent is defined to be the infimum of all $p$ for which the above inequality holds;

- **polynomially tractable (PT)** iff there exist non-negative numbers $C$, $p$, and $q$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,
  $$n(\varepsilon, d) \leq Cd^{q(\varepsilon^{-1})^p};$$

- **quasi-polynomially tractable (QPT)** iff there exist two constants $C$, $t > 0$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,
  $$n(\varepsilon, d) \leq C \exp(t(1 + \ln \varepsilon^{-1})(1 + \ln d));$$

- **uniformly weakly tractable (UWT)** iff for all $s, t > 0$,
  $$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, d)}{(\varepsilon^{-1})^s + d^t} = 0;$$

- **weakly tractable (WT)** iff
  $$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0;$$

- **(s, t)-weakly tractable ((s, t)-WT)** for positive $s$ and $t$ iff
  $$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, d)}{(\varepsilon^{-1})^s + d^t} = 0.$$

Clearly, $(1, 1)$-WT is the same as WT. If $\text{App}$ is not WT, then $\text{App}$ is called intractable. We say that $\text{App}$ suffers from the curse of dimensionality if there exist positive numbers $C$, $\varepsilon_0$, $\alpha$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and infinitely many $d \in \mathbb{N}$,

$$n(\varepsilon, d) \geq C(1 + \alpha)^d.$$

SPT and QPT of the above approximation problem $\text{App}$ have been studied in [3] and [6], respectively. The following conditions have been obtained therein:

- SPT holds iff there exists a positive number $\delta > 1$ such that
  $$\sum_{j=1}^{\infty} \gamma_j^{2/\delta} < \infty$$
  if
  $$r(\gamma) > 1,$$

  where

  $$(1.5) \quad r(\gamma) = \sup \{ \delta > 0 \mid \sum_{j=1}^{\infty} \gamma_j^{2/\delta} < \infty \} = \sup \{ \beta \geq 0 \mid \lim_{j \to \infty} j^{\beta} \gamma_j^2 = 0 \}.$$ 

  In this case, the exponent of SPT is $\frac{2}{r(\gamma) - 1}$. 

• QPT holds iff

\[
\sup_{d \in \mathbb{N}} \frac{1}{\ln^+ d} \sum_{j=1}^{d} \gamma_j^2 (1 + \ln(1 + \gamma_j^{-2})) < \infty,
\]

where \( \ln^+ x = \max(1, \ln x) \).

In this paper we obtain complete results about the tractability of App. Specially, we give the necessary and sufficient condition for \((s,t)\)-WT for \( t \in (0, 1) \) and \( s > 0 \). Similar conditions are first given in our paper. We use the new method. We remark that in similar approximation problems with covariance kernels corresponding to Euler and Wiener integrated processes under the normalized error criterion, the necessary and sufficient conditions for \((s,t)\)-WT for \( t \in (0, 1) \) and \( s > 0 \) do not completely match (see [16]).

**Theorem 1.1.** Consider the above approximation problem App with shape parameters \( \gamma = \{\gamma_j^2\} \) satisfying (1.2).

(i) PT holds iff SPT holds iff

\[
(1.6) \quad r(\gamma) = \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j} > 1.
\]

(ii) For \( t > 1 \) and \( s > 0 \), \((s,t)\)-WT holds for all shape parameters.

(iii) For \( t = 1 \) and \( s > 0 \), \((s,1)\)-WT holds iff WT holds iff

\[
(1.7) \quad \lim_{j \to \infty} \gamma_j^2 = 0.
\]

(iv) For \( t \in (0, 1) \) and \( s > 0 \), \((s,t)\)-WT holds iff

\[
(1.8) \quad \lim_{j \to \infty} j^{1-t} \gamma_j^2 \ln^+ \gamma_j^{-2} = 0.
\]

(v) UWT holds iff

\[
(1.9) \quad \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j} \geq 1.
\]

(vi) App suffers from the curse of dimensionality if \( \lim_{j \to \infty} \gamma_j^2 > 0 \).

It is of interest to compare the tractability results of Theorem 1.1 with the ones in the worst case setting from [2], where the behavior of the information complexity in the worst case setting is studied using either the absolute error criterion (ABS) or the normalized error criterion (NOR) (see Subsection 2.3 for related notions in the worst case setting).

For ABS, we have

• SPT holds for all shape parameters with the exponent \( \min(2, \frac{2}{r(\gamma)}) \), where \( r(\gamma) \) is given by (1.5).

• Obviously, SPT implies all PT, QPT, WT, \((s,t)\)-WT for any positive \( s \) and \( t \), as well as UWT, for all shape parameters.

For NOR, we have
• SPT holds iff PT holds iff \( r(\gamma) > 0 \), with the exponent \( \frac{2}{r(\gamma)} \).
• QPT holds for all shape parameters.
• Obviously, QPT implies \((s,t)\)-WT for any positive \( s \) and \( t \), as well as UWT for all shape parameters.

We stress that there is now a difference between ABS and NOR in the worst case setting. For all shape parameters, we always have SPT for ABS and QPT for NOR in the worst case setting, whereas in the average case setting we only have \((s,t)\)-WT for \( s > 0 \) and \( t > 1 \). Also the sufficient and necessary condition for SPT (or PT) in the average case setting is stronger than the one for NOR in the worst case setting.

Next we consider exponential convergence tractability of the approximation problem App. Because the covariance kernel function of the Gaussian measure \( \mu_d \) is an analytic function, the \( n \)th minimal error \( e(n,d) \) can be expected to decay faster than any polynomial. Indeed, we expect exponential convergence. If there exists a number \( q \in (0,1) \) such that for all \( d = 1, 2, \ldots \), there are positive numbers \( C_{1,d}, C_{2,d} \) and \( p_d \) for which

\[
e(n,d) \leq C_{1,d} q^{(n/C_{2,d})^p_d}, \quad \text{for all } n \in \mathbb{N},
\]

then we say that App is \textit{exponential convergence} (EXP). The supremum of positive \( p_d \) in (1.10) is called the exponent of EXP. If \( p_d \) can be chosen as positive and independent of \( d \) we have \textit{uniform exponential convergence} (UEXP).

If App is EXP, then we can discuss the \textit{tractability with exponential convergence} \((EC\text{-tractability})\). Recently, there are many papers where EC-tractability is considered (see [1, 5, 13, 19]).

In the definitions of SPT, PT, QPT, UWT, WT, and \((s,t)\)-WT, if we replace \( 1/\epsilon \) by \( (1 + \ln \frac{1}{\epsilon}) \), we get the definitions of \textit{exponential convergence-strong polynomial tractability} (EC-SPT), \textit{exponential convergence-polynomial tractability} (EC-PT), \textit{exponential convergence-quasi-polynomial tractability} (EC-QPT), \textit{exponential convergence-uniform weak tractability} (EC-UWT), \textit{exponential convergence-weak tractability} (EC-WT), and \textit{exponential convergence-\((s,t)\)-weak tractability} (EC-(\( s,t \))-WT), respectively.

In [18], Sloan and Woźniakowski obtained the following complete results about the EC-tractability in the worst case setting using ABS and NOR.

For ABS or NOR, we have

• EXP holds with the exponent \( p_d^* = 1/d \) and UEXP does not hold for all shape parameters \( \gamma \) satisfying (1.2).
• EC-SPT and EC-PT and EC-QPT do not hold for all shape parameters.
• If \( \max(s,t) > 1 \) then EC-(\( s,t \))-WT holds for all shape parameters.
• EC-WT holds iff \( \lim_{j \to \infty} \gamma_j^2 = 0 \).
• EC-(1,1)-WT with \( t < 1 \) holds iff \( \lim_{j \to \infty} \frac{\ln \gamma_j}{\ln \gamma_j} = 0 \).
• EC-(s,1)-WT with \( s < 1 \) and \( t \leq 1 \) holds iff \( \lim_{j \to \infty} \frac{\ln \gamma_j}{\ln \gamma_j} = 0 \).
• EC-UWT holds iff \( \lim_{j \to \infty} \frac{\ln(\ln \gamma^{-2}_j)}{\ln j} = \infty. \)

EC-tractability in the worst and average case settings has the intimate connection. Specially, according to [19, Theorems 3.2 and 4.2] and [9, Theorem 3.2], we have the same results in the worst and average case settings using ABS concerning EC-WT, EC-UWT, and EC-(s,t)-WT for \( 0 < s \leq 1 \) and \( t > 0 \).

Based on the results of [18], we get the EC-tractability of App in the average case setting.

**Theorem 1.2.** Consider the above approximation problem App with shape parameters \( \gamma = \{\gamma_j^2\} \) satisfying (1.2).

(i) \( \text{EXP} \) holds with the exponent \( p^*_d = 1/d \) and \( \text{UEXP} \) does not hold for all shape parameters \( \gamma \) satisfying (1.2).

(ii) \( \text{EC-SPT} \) and \( \text{EC-PT} \) and \( \text{EC-QPT} \) do not hold for all shape parameters.

(iii) If \( s > 0 \) and \( t > 1 \) then EC-(s,t)-WT holds for all shape parameters.

(iv) \( \text{EC-(s,1)-WT with } s \geq 1 \) holds iff \( \text{EC-WT holds iff } \lim_{j \to \infty} \gamma^{-2}_j = 0. \)

(v) \( \text{EC-(s,t)-WT with } s < 1 \) and \( t \leq 1 \) holds iff

\[
\lim_{j \to \infty} \frac{j^{(1-s)/s}}{\ln \gamma^{-2}_j} = 0.
\]

(vi) \( \text{EC-(1,t)-WT with } t < 1 \) holds iff

\[
\lim_{j \to \infty} \frac{\ln j}{\ln \gamma^{-2}_j} = 0.
\]

(vii) \( \text{EC-(s,t)-WT with } s > 1 \) and \( t < 1 \) holds iff (1.11)

\[
\lim_{j \to \infty} j^{1-t} \gamma^{-2}_j \ln^+ \gamma^{-2}_j = 0.
\]

(viii) \( \text{EC-UWT holds iff } \)

\[
\lim_{j \to \infty} \frac{\ln(\ln \gamma^{-2}_j)}{\ln j} = \infty.
\]

Let us compare the results about EC-tractability in the worst and average case settings. There are the same conclusion for EC-SPT, EC-PT, EC-QPT, EC-UWT, WT, and EC-(s,t)-WT with \( s \leq 1, t > 0 \) or \( s > 0, t > 1 \) in the worst and average case settings. We always have EC-(s,t)-WT with \( s > 1, 0 < t \leq 1 \) for all shape parameters in the worst case setting, whereas in the average case setting, EC-(s,t)-WT with \( s > 1, 0 < t \leq 1 \) holds iff \( (s,t)\)-WT with \( s > 1, 0 < t \leq 1 \) holds iff

\[
\lim_{j \to \infty} j^{1-t} \gamma^{-2}_j \ln^+ \gamma^{-2}_j = 0.
\]

We also compare the results about tractability and EC-tractability in the average case setting. We never have EC-SPT, EC-PT, EC-QPT, whereas SPT, PT, QPT hold for shape parameters decaying fast enough. For all shape parameters, we always have \( \text{EC-(s,t)-WT and } (s,t)\)-WT for \( s > 0 \) and \( t > 1 \). There are the same sufficient and necessary conditions for which \( \text{EC-(s,t)-WT and } (s,t)\)-WT hold with \( s > 1, 0 \leq t \leq 1 \) or \( s = t = 1 \). In the other cases, we need to assume more
conditions about shape parameters to get EC-UWT or EC-\((s,t)\)-WT than ones to get UWT or \((s,t)\)-WT with \(0 < s < 1\), \(0 < t < \frac{1}{2}\) or \(s = 1\), \(0 < t < 1\).

The paper is organized as follows. In Subsection 2.1 we give concept of non-homogeneous tensor product problems in the average case setting. Subsections 2.2 and 2.3 are devoted to introducing the average and worst case approximation problems with Gaussian kernels. In Section 3, we give the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

2.1. Average case non-homogeneous tensor product problems.

We recall the concept of non-homogeneous tensor product problems, see [7]. Let \(F_d, H_d\) are given by tensor products. That is,

\[
F_d = F_1^{(1)} \otimes F_2^{(1)} \otimes \cdots \otimes F_d^{(1)} \quad \text{and} \quad H_d = H_1^{(1)} \otimes H_2^{(1)} \otimes \cdots \otimes H_d^{(1)},
\]

where Banach spaces \(F_k^{(1)}\) are of univariate real functions equipped with a zero-mean Gaussian measure \(\mu_k^{(1)}\), and \(H_k^{(1)}\) are Hilbert spaces, \(k = 1, 2, \ldots, d\). We set

\[
S_d = S_1^{(1)} \otimes S_2^{(1)} \otimes \cdots \otimes S_d^{(1)}, \quad \mu_d = \mu_1^{(1)} \otimes \mu_2^{(1)} \otimes \cdots \otimes \mu_d^{(1)},
\]

where

\[
S_k^{(1)} = F_k^{(1)} \rightarrow H_k^{(1)}, \quad k = 1, 2, \ldots, d
\]

are continuous linear operators. Then \(\mu_d\) is a zero-mean Gaussian measure on \(F_d\) with covariance operator \(C_{\mu_d} : F_d \rightarrow F_d\).

Let \(\nu_d = \mu_d(S_d)^{-1}\) be the induced measure. Then \(\nu_d\) is a zero-mean Gaussian measure on \(H_d\) with covariance operator \(C_{\nu_d} : H_d \rightarrow H_d\) given by

\[
C_{\nu_d} = S_d C_{\mu_d} S_d^*,
\]

where \(S_d^* : H_d \rightarrow F_d^*\) is the operator dual to \(S_d\). Let \(\nu_k^{(1)} = \mu_k^{(1)}(S_k^{(1)})^{-1}\) be the induced zero-mean Gaussian measure on \(H_k^{(1)}\), and let \(C_{\nu_k^{(1)}} : H_k^{(1)} \rightarrow H_k^{(1)}\) be the covariance operator of the measure \(\nu_k^{(1)}\). Then

\[
\nu_d = \nu_1^{(1)} \otimes \nu_2^{(1)} \otimes \cdots \otimes \nu_d^{(1)}, \quad \text{and} \quad C_{\nu_d} = C_{\nu_1^{(1)}} \otimes C_{\nu_2^{(1)}} \otimes \cdots C_{\nu_d^{(1)}}.
\]

The eigenpairs of \(C_{\nu_k^{(1)}}\) are denoted by \(\{(\lambda(k, j), \eta(k, j))\}_{j \in \mathbb{N}^d}\), and satisfy

\[
C_{\nu_k^{(1)}}(\eta(k, j)) = \lambda(k, j) \eta(k, j), \quad \text{with} \quad \lambda(k, 1) \geq \lambda(k, 2) \geq \cdots \geq 0.
\]

Then

\[
\text{trace}(C_{\nu_k^{(1)}}) = \int_{H_k^{(1)}} \|f\|^2_{H_k^{(1)}} \nu_k^{(1)}(df) = \sum_{j=1}^{\infty} \lambda(k, j) < \infty.
\]

The eigenpairs of \(C_{\nu_d}\) are given by

\[
\{(\lambda_{d,j}, \eta_{d,j})\}_{j \in \mathbb{N}^d},
\]

where

\[
\lambda_{d,j} = \prod_{k=1}^{d} \lambda(k, j_k) \quad \text{and} \quad \eta_{d,j} = \prod_{k=1}^{d} \eta(k, j_k).
\]
Let the sequence \( \{\lambda_{d,j}\}_{j \in \mathbb{N}} \) be the non-increasing rearrangement of \( \{\lambda_{d,j}\}_{j \in \mathbb{N}^d} \). Then we obtain
\[
\sum_{j \in \mathbb{N}} \lambda_{d,j}^\tau = \prod_{k=1}^{d} \sum_{j=1}^{\infty} \lambda(k,j)^\tau, \quad \text{for any} \quad \tau > 0.
\]

We approximate \( S_d f \) by algorithms \( A_{n,d} \) of the form (1.3) that use only finitely many continuous linear functionals on \( F_d \). Then the \( n \)th minimal average case error is given by
\[
e(n, S_d) := \inf_{A_{n,d}} \left( \int_{F_d} \| S_d f - A_{n,d} f \|_{H_d}^2 \mu_d(df) \right)^{\frac{1}{2}} = \left( \sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{\frac{1}{2}},
\]
and is achieved by the \( n \)th optimal algorithm
\[
A_{n,d}^*(f) = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{H_d} \eta_{d,j}.
\]

The initial error for \( S_d \) is
\[
e(0, S_d) = \left( \int_{F_d} \| S_d f \|_{H_d}^2 \mu_d(df) \right)^{\frac{1}{2}} = \left( \sum_{j=1}^{\infty} \lambda_{d,j} \right)^{\frac{1}{2}}.
\]

The information complexity for \( S_d \) can be studied using either the absolute error criterion (ABS), or the normalized error criterion (NOR). Then we define the information complexity \( n^X(\varepsilon, S_d) \) for \( X \in \{\text{ABS, NOR}\} \) as
\[
n^X(\varepsilon, S_d) = \min\{n : e(n, S_d) \leq \varepsilon CRI_d\},
\]
where
\[
CRI_d = \left\{ \begin{array}{ll}
1, & \text{for } X=\text{ABS}, \\
\varepsilon(0, S_d), & \text{for } X=\text{NOR}.
\end{array} \right.
\]

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. (See [1] Theorem 6.) Let \( S = \{S_d\} \) be a non-homogeneous tensor product problem. Then for NOR, \( S \) is PT if and only if there exists \( \tau \in (0, 1) \) such that
\[
Q_* := \sup_{d \in \mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} \ln \left( 1 + \sum_{j=2}^{\infty} \frac{\lambda(k,j)}{\lambda(k,1)} \right)^\tau < \infty.
\]

Lemma 2.2. (See [2] Theorem 8 or [3] Lemma 2.1.) Let \( S = \{S_d\} \) be a non-homogeneous tensor product problem. If for \( t > 0 \) there exists a number \( \tau \in (0, 1) \) such that
\[
\lim_{d \to \infty} \frac{1}{d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left( \frac{\lambda(k,j)}{\lambda(k,1)} \right)^\tau = 0,
\]
then \( S \) is \((s,t)\)-WT for this \( t \) and every \( s > 0 \).
2.2. Average case approximation problems with Gaussian kernels.

Let \( C_{\mu_d} \) be the covariance operator of \( \mu_d \) with Gaussian covariance kernel \( K_{d,\gamma} \) given by (1.1), where \( \mu_d \) is a zero-mean Gaussian measure on \( L_{2,d} \). Then for \( f \in L_{2,d} \),

\[
C_{\mu_d}(f)(x) = \int_{\mathbb{R}^d} K_{d,\gamma}(x,y) f(y) \prod_{k=1}^d \exp\left( -\frac{y_k^2}{\gamma} \right) dy, \quad x \in \mathbb{R}^d.
\]

It follows from (2.1) that \( \omega \)

\[
\lambda \Rightarrow (2.4)
\]

and

\[
\mathcal{K} \text{covariance kernel}
\]

Clearly, \( 0 < \omega \) and the corresponding eigenfunctions of \( C_{\mu_1} \), i.e.,

\[
C_{\mu_1} \eta_{\gamma,j} = \lambda_{\gamma,j} \eta_{\gamma,j}, \quad \gamma = 1, 2, \ldots,
\]

and

\[
\lambda_{\gamma,1} \geq \lambda_{\gamma,2} \geq \cdots \geq 0.
\]

Specifically we have, see e.g., [14, Section 4.3.1] and [3, 18],

\[
(2.1) \quad \lambda_{\gamma,j} = (1 - \omega_{\gamma}) \omega_{\gamma}^{j-1}, \quad \text{with} \quad \omega_{\gamma} = \frac{2\gamma^2}{1 + 2\gamma^2 + \sqrt{1 + 4\gamma^2}},
\]

and

\[
\eta_{\gamma,j}(x) = \sqrt{(1 + 4\gamma^2)^{j-1}/2^{j-1}(j-1)!} \exp\left( -\frac{2\gamma^2 x^2}{1 + \sqrt{1 + 4\gamma^2}} \right) H_{j-1}((1 + 4\gamma^2)^{1/4} x),
\]

where \( H_{j-1} \) is the standard Hermite polynomial of degree \( j - 1 \), i.e.,

\[
H_{j-1}(x) = (-1)^{j-1} e^{x^2} \frac{d^{j-1}}{dx^{j-1}} e^{-x^2} \quad \text{for all} \quad x \in \mathbb{R}.
\]

Clearly, \( 0 < \omega_{\gamma} < 1 \),

\[
1 - \omega_{\gamma} = \lambda_{\gamma,1} > \lambda_{\gamma,2} > \cdots > 0, \quad \text{and} \quad \sum_{j=1}^\infty \lambda_{\gamma,j} = 1.
\]

It follows from (2.1) that \( \omega_{\gamma} \) is an increasing function of \( \gamma \) and \( \omega_{\gamma} \) tends to 0 iff \( \gamma \) tends to 0. We also have

\[
(2.2) \quad \lim_{\gamma \rightarrow 0} \frac{\omega_{\gamma}}{\gamma^2} = 1, \quad \text{and} \quad \lim_{\gamma \rightarrow 0} \frac{\ln \omega_{\gamma}}{\ln \gamma^2} = 1.
\]

Due to the tensor product structure of the covariance operator \( C_{\mu_d} \), the eigenvalues and the corresponding eigenfunctions of \( C_{\mu_d} \) have the form

\[
(2.3) \quad \lambda_{\gamma,j} = \prod_{k=1}^d \lambda_{\gamma_k,j_k} = \prod_{k=1}^d \left( 1 - \omega_{\gamma_k} \right) \omega_{\gamma_k}^{j_k-1} \quad \text{and} \quad \eta_{\gamma,j} = \prod_{k=1}^d \eta_{\gamma_k,j_k}(x_k),
\]

for \( j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d \). Let \( \{\lambda_{d,j}\}_{j \in \mathbb{N}^d} \) be the non-increasing rearrangement of \( \{\lambda_{\gamma,j}\}_{j \in \mathbb{N}^d} \). Then we have

\[
(2.4) \quad \lambda_{d,1} = \prod_{k=1}^d (1 - \omega_{\gamma_k}),
\]

and

\[
e(0,d) = \sum_{k=1}^\infty \lambda_{d,k} = \sum_{j \in \mathbb{N}^d} \lambda_{\gamma,j} = \prod_{k=1}^d \sum_{j \in \mathbb{N}^d} \lambda_{\gamma_k,j} = 1.
\]
where \( e(0,d) \) is the initial error. This means that the normalized error criterion and the absolute error criterion are the same.

The \( n \)th minimal average case error is
\[
e(n,d) = \left( \sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{1/2}.
\]

The information complexity \( n(\varepsilon,d) \) of the approximation problem \( \text{App} \) is defined by
\[
n(\varepsilon,d) = \inf \{ n \mid e(n,d) \leq \varepsilon \}.
\]

We emphasize that \( \text{App} \) is a non-homogeneous tensor product problem with
\[
\lambda(k,j) = (1 - \omega \gamma_k)\omega_j - 1 \quad \text{and} \quad \lambda(k,1) = 1 - \omega \gamma_k, \quad k = 1, \ldots, d.
\]

2.3. Worst case approximation problems with Gaussian kernels.

Let \( H(K_d,\gamma) \) be the reproducing kernel Hilbert space with the kernel \( K_d,\gamma \) given by (1.1). The function space \( H(K_d,\gamma) \) has been used widely in numerical computation, statistical learning, and engineering (see e.g., [2, 3, 14, 4]). We consider multivariate approximation problem \( I = \{I_d\}_{d \in \mathbb{N}} \) which is defined via the embedding operator
\[
I_d : H(K_d,\gamma) \to L_{2,d} \quad \text{with} \quad I_d f = f.
\]

We approximate \( I_d \) by algorithms that use only finitely many continuous linear functionals on \( H(K_d,\gamma) \). The worst case error of approximation by an algorithm \( A_{n,d} \) of the form (1.3) or (1.4) is defined as
\[
e_{\text{wor}}(A_{n,d}) = \sup_{\|f\|_{H(K_d,\gamma)} \leq 1} \|I_d f - A_{n,d}(f)\|_{L_{2,d}}.
\]

The \( n \)th minimal worst case error, for \( n \geq 1 \), is defined by
\[
e_{\text{wor}}(n,d) = \inf_{A_{n,d}} e_{\text{wor}}(A_{n,d}),
\]
where the infimum is taken over all algorithms of the form (1.3) or (1.4) using \( n \) information operators \( L_1, L_2, \ldots, L_n \in H(K_d,\gamma)^* \). The error of \( A_{0,d} \) is called the initial error and is given by
\[
e_{\text{wor}}(0,d) = \sup_{\|f\|_{H(K_d,\gamma)} \leq 1} \|I_d f\|_{L_{2,d}} = \|I_d\|.
\]

Let \( \lambda_{d,j}, j \in \mathbb{N} \) be the eigenvalues of the covariance operator \( C_{\mu_d} \) of the Gaussian measure \( \mu_d \) satisfying
\[
\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq \lambda_{d,k} \geq \cdots > 0.
\]

Then the \( n \)th minimal worst case error \( e_{\text{wor}}(n,d) \) and the \( n \)th minimal average case error \( e(n,d) \) are of forms (see [10])
\[
e_{\text{wor}}(n,d) = \lambda_{d,n+1}^{1/2}, \quad \text{and} \quad e(n,d) = \left( \sum_{k=n+1}^{\infty} \lambda_{d,k} \right)^{1/2} \geq e_{\text{wor}}(n,d).
\]

The worst case information complexity can be studied using either ABS or NOR. Then we define the worst case information complexity \( n_{\text{wor},X}(\varepsilon,d) \) for \( X \in \{ \text{ABS}, \text{NOR} \} \) as
\[
n_{\text{wor},X}(\varepsilon,d) = \min \{ n : e_{\text{wor}}(n,d) \leq \varepsilon CRI_d \},
\]
where

\[ CRI_d = \begin{cases} 
1, & \text{for } X=\text{ABS}, \\
\epsilon^{\text{wor}}(0,d), & \text{for } X=\text{NOR} = \begin{cases} 
1, & \text{for } X=\text{ABS}, \\
\lambda_{d,1}^{1/2}, & \text{for } X=\text{NOR}.
\end{cases}
\end{cases} \]

Obviously, we have

\[ n^{\text{wor,ABS}}(\varepsilon, d) \leq n(\varepsilon, d). \tag{2.6} \]

### 3. Proofs of Theorems 1.1 and 1.2

First we give two auxiliary lemmas.

**Lemma 3.1.** Let \( \gamma = \{\gamma_j^2\}_{j \in \mathbb{N}} \) satisfy (1.2) and \( r(\gamma) > 0 \). Then

\[ r(\gamma) = \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j}. \tag{3.1} \]

where \( r(\gamma) \) is given by (1.5).

**Proof.** Since \( r(\gamma) > 0 \), there exists a positive \( \delta \) such that

\[ M_\delta = \sum_{j=1}^{\infty} \frac{\gamma_j^2}{\gamma_j} < \infty. \tag{3.2} \]

It follows that

\[ j \gamma_j^2 \leq \sum_{k=1}^{j} \gamma_k^2 \leq M_\delta. \]

We have

\[ \ln j - \frac{1}{\delta} \ln \gamma_j^{-2} \leq \ln M_\delta, \]

which yields

\[ \frac{\ln \gamma_j^{-2}}{\ln j} \geq \delta \left( \frac{\ln j - \ln M_\delta}{\ln j} \right). \]

Letting \( j \to \infty \) in the above inequality, we conclude that

\[ \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j} \geq \delta. \]

Taking the supremum over all \( \delta \) for which (3.2) holds, we get

\[ r(\gamma) \leq \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j}. \tag{3.3} \]

On the other hand, by (3.3) we know that

\[ \alpha := \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j} > 0. \]

Then for an arbitrary \( \varepsilon \in (0, \alpha/2) \), there exists an integer \( N > 0 \) such that for all \( j \geq N \) we have

\[ \frac{\ln \gamma_j^{-2}}{\ln j} \geq \alpha - \varepsilon. \]

This implies that

\[ \gamma_j^2 \leq j^{-(\alpha - \varepsilon)}. \]
Choosing $\delta = \alpha - 2 \varepsilon > 0$ and noting that $\frac{\alpha - \varepsilon}{\alpha - 2\varepsilon} > 1$, we obtain that

$$\sum_{j=N}^{\infty} \gamma_j^2 \leq \sum_{j=N}^{\infty} j^{-\frac{\alpha(\gamma_j - \varepsilon)}{2\varepsilon}} < \infty,$$

and so $\sum_{j=1}^{\infty} \gamma_j^2 < \infty$. It follows from the definition of $r(\gamma)$ that

$$\delta = \alpha - 2 \varepsilon \leq r(\gamma).$$

Letting $\varepsilon \to 0$ in the above inequality, we conclude that

$$r(\gamma) \geq \alpha = \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j},$$

which combining with (3.3), gives (3.1). Lemma 3.1 is proved.

**Lemma 3.2.** Let $\gamma = \{\gamma_j^2\}_{j \in \mathbb{N}}$ satisfy (1.2). Then

(3.4) \[ \lim_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j} \geq 1, \]

iff for any $t \in (0, 1)$,

(3.5) \[ \lim_{j \to \infty} j^{1-t} \gamma_j^2 \ln^+ \gamma_j^{-2} = 0, \]

iff for any $t \in (0, 1)$,

(3.6) \[ \lim_{j \to \infty} j^{1-t} \gamma_j^2 = 0. \]

**Proof.** Suppose that (3.4) holds. Then for any $t \in (0, 1)$, we have for sufficiently large $j$,

$$\frac{\ln \gamma_j^{-2}}{\ln j} > 1 - t/2,$$

which yields that

$$\gamma_j^2 < j^{1/2-1} \quad \text{and} \quad \gamma_j^2 \ln^+ \gamma_j^{-2} < (1 - t/2) j^{1/2-1} \ln j,$$

where in the last inequality, we used the monotonicity of the function $h(x) = x \ln 1/x$, $x \in (0, 1/e)$. Then (3.5) and (3.6) follow from the above inequalities immediately.

On the other hand, we suppose that for any $t \in (0, 1)$, (3.5) or (3.6) holds. Noting that we can deduce (3.6) from (3.5). So (3.5) holds. For any $t \in (0, 1)$, we have for sufficiently large $j$,

$$j^{1-t} \gamma_j^2 \leq 1,$$

which implies that

$$\frac{\ln \gamma_j^{-2}}{\ln j} \geq 1 - t.$$

Letting $j \to \infty$ and then $t \to 0^+$, we get (3.4). Lemma 3.2 is proved.

**Proof of Theorem 1.1**

(i) It was proved in [3] that SPT holds for App if $r(\gamma) > 1$. Clearly, if SPT holds, then PT holds. So in order to prove (i), by Lemma 3.1 it suffices to show (1.6) whenever PT holds.
Assume that PT holds. According to Lemma 2.1 and (2.5), there exists a \( \tau \in (0, 1) \) such that

\[
Q_\tau := \sup_{d \in \mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} \ln \left( 1 + \sum_{j=2}^{\infty} \omega(j-1)^{\tau} \right) = \sup_{d \in \mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} \ln \left( \frac{1}{1 - \omega(k)^{\tau}} \right) < \infty.
\]

Noting that the function \( \varphi(x) = \ln(1 - x) + x \) is decreasing in \((0, 1)\) due to the fact that \( \varphi'(x) = \frac{x}{1-x} < 0 \) and \( \varphi(0) = 0 \), we get

\[
\ln \frac{1}{1 - x} > x.
\]

This implies that

\[
\ln \left( \frac{1}{1 - \omega(k)^{\tau}} \right) > \omega(k)^{\tau}, \quad \tau \in (0, 1).
\]

It follows from (3.7) and (3.8) that

\[
d\omega_{\gamma_d} \leq \sum_{k=1}^{d} \omega(k)^{\tau} \leq \sum_{k=1}^{d} \ln \left( \frac{1}{1 - \omega(k)^{\tau}} \right) \leq Q_\tau \ln^+ d.
\]

By (3.9) we obtain further

\[
\frac{\ln \omega_{\gamma_d}}{\ln d} \geq \frac{\ln d - \ln(\ln^+ d) - \ln Q_\tau}{\tau \ln d}.
\]

Letting \( d \to \infty \), we get

\[
\lim_{d \to \infty} \frac{\ln \omega_{\gamma_d}}{\ln d} \geq \frac{1}{\tau} > 1.
\]

By (3.9) we have \( \lim_{d \to \infty} \omega_{\gamma_d} = 0 \). It follows from (2.2) and (3.10) that

\[
\lim_{d \to \infty} \frac{\ln \gamma_d}{\ln d} = \lim_{d \to \infty} \frac{\ln \omega_{\gamma_d}}{\ln d} > 1,
\]

which completes the proof of (i).

(ii) Let \( t > 1 \) and \( s > 0 \). By (2.5) and the Stolz theorem we have for any \( \tau \in (0, 1) \),

\[
0 \leq \lim_{d \to \infty} d^{-t} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left( \frac{\lambda(k, j)}{\lambda(k, 1)} \right)^{\tau} = \lim_{d \to \infty} \frac{\sum_{k=1}^{d} \frac{\omega(k)^{\tau}}{1 - \omega(k)^{\tau}}}{d^t} = \lim_{d \to \infty} \frac{d^{1-t} \omega(k)^{\tau}}{1 - \omega(k)^{\tau}} = 0,
\]

where in the last inequality we used the monotonicity of the function \( h(x) = \frac{x}{1-x}, \quad x \in (0, 1) \). By Lemma 2.2, we get that \((s, t)\)-WT holds for \( t > 1 \) and \( s > 0 \). (ii) is proved.

(iii) Let \( t = 1 \) and \( s > 0 \). If \( \lim_{j \to \infty} \gamma_j^2 = 0 \), then by (2.2) we have for any \( \tau \in (0, 1) \),

\[
\lim_{d \to \infty} \frac{\omega(j)^{\tau}}{1 - \omega(j)^{\tau}} = 0.
\]
Similar to (3.11), we get
\[
\lim_{d \to \infty} \left( \sum_{k=1}^{d} \sum_{j=2}^{\infty} \frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} = \lim_{d \to \infty} \frac{\sum_{k=1}^{d} \omega_{k}}{\omega_{1}} = \lim_{d \to \infty} \frac{\omega_{d}}{1 - \omega_{\tau_{d}}} = 0.
\]

By Lemma 2.2, we know that \((s, 1)\)-WT holds for \(s > 0\).

On the other hand, we suppose that \((s, 1)\)-WT holds for some \(s > 0\). We want to show that \(\lim_{j \to \infty} \gamma_{j}^{2} = 0\). It follows from the definition of \(n(\varepsilon, d)\) that
\[
1 - \sum_{k=1}^{n(\varepsilon, d)} \lambda_{d,k} = \sum_{k=n(\varepsilon, d)+1}^{\infty} \lambda_{d,k} \leq \varepsilon^{2}.
\]

We have
\[
(3.12) \quad 1 - \varepsilon^{2} \leq \sum_{k=1}^{n(\varepsilon, d)} \lambda_{d,k} \leq n(\varepsilon, d) \lambda_{d,1}.
\]

This implies that
\[
(3.13) \quad \ln n(\varepsilon, d) \geq \ln(1 - \varepsilon^{2}) + \ln \lambda_{d,1} \geq \ln(1 - \varepsilon^{2}) + \sum_{k=1}^{d} \ln \left( \frac{1}{1 - \omega_{k}} \right) \\
\geq \ln(1 - \varepsilon^{2}) + d \ln \left( \frac{1}{1 - \omega_{d}} \right) \geq \ln(1 - \varepsilon^{2}) + d \omega_{d},
\]

where in the last step, we used the inequality \(\ln \left( \frac{1}{1 - x} \right) \geq x\) for \(x \in [0, 1)\). Since \((s, 1)\)-WT holds for some \(s > 0\), by (3.13) we get
\[
0 = \lim_{d \to \infty} \frac{\ln(n(\varepsilon, d))}{d} \geq \lim_{d \to \infty} \frac{\frac{3}{4} + d \omega_{d}}{d} = \lim_{d \to \infty} \omega_{d} \geq 0,
\]

which implies \(\lim_{j \to \infty} \gamma_{j}^{2} = 0\). This completes the proof of (iii).

(iv) Suppose that \((s, t)\)-WT holds for \(s > 0\) and \(t \in (0, 1)\). We want to show that (1.8) holds. First we show \(\lim_{d \to \infty} d^{1-t} \omega_{d} = 0\). Since \((s, t)\)-WT holds for \(s > 0\) and \(t \in (0, 1)\), by (3.13) we get
\[
0 = \lim_{d \to \infty} \frac{\ln(n(\varepsilon, d))}{\left( \frac{3}{4} \right)^{t} + d^{t}} \geq \lim_{d \to \infty} \frac{\frac{3}{4} + d \omega_{d}}{d^{t}} = \lim_{d \to \infty} d^{1-t} \omega_{d} \geq 0.
\]

Hence
\[
(3.14) \quad \lim_{d \to \infty} d^{1-t} \omega_{d} = 0.
\]

Next we show (1.8) holds. We set
\[
(3.15) \quad u_{k} := \max(\omega_{k}, \frac{1}{2k}), \quad \text{and} \quad s_{k} := \frac{1}{2} \left( \ln^{+} \frac{1}{u_{k}} \right)^{-1}, \quad k \in \mathbb{N}.
\]

By (3.12) we have
\[
1 - \varepsilon^{2} \leq \sum_{k=1}^{n(\varepsilon, d)} \lambda_{d,k} \leq \left( \sum_{k=1}^{n(\varepsilon, d)} \lambda_{d,k}^{1+s_{d}} \right)^{\frac{1}{1+s_{d}}} n(\varepsilon, d)^{\frac{s_{d}}{1+s_{d}}} \leq \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1+s_{d}} \right)^{\frac{1}{1+s_{d}}} n(\varepsilon, d)^{\frac{s_{d}}{1+s_{d}}}.
\]
It follows that
\[
   n(\varepsilon, d) \geq (1 - \varepsilon^2) \frac{1 + s_d}{1 - \varepsilon^2} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1+s_d} \right)^{\frac{1}{s_d}} = (1 - \varepsilon^2) \frac{1 + s_d}{1 - \varepsilon^2} \prod_{k=1}^{d} \frac{1 - \omega_k^{1+s_d}}{(1 - \omega_k)^{1+s_d}}.
\]

We note that the function \( f(x) = \ln \left( \frac{1 - x^{1+s_d}}{(1-x)^{1+s_d}} \right) \) is monotonically increasing in \( x \in (0, 1) \) due to the fact that \( f'(x) = \frac{1 - (1+x)^{1+s_d}}{(1-x)^{1+s_d}(1+x)^{1+s_d}} > 0 \) for \( x \in (0, 1) \). We have
\[
   \ln \left( n\left(\frac{1}{2}, d\right) \right) \geq \frac{1}{s_d} \ln \frac{3}{4} + \frac{1}{s_d} \sum_{k=1}^{d} \ln \left( \frac{1 - \omega_k^{1+s_d}}{(1 - \omega_k)^{1+s_d}} \right)
   \geq \frac{1}{s_d} \ln \frac{3}{4} + \frac{1}{s_d} \sum_{k=1}^{d} \ln \left( 1 - \frac{\omega_k^{1+s_d}}{(1 - \omega_k)^{1+s_d}} \right)
   \geq \frac{1}{s_d} \ln \frac{3}{4} + \frac{d(\omega_d - \omega_d^{1+s_d})}{s_d(1 - \omega_d)} \ln 2,
\]
and
\[
   \lim_{d \to \infty} \frac{1}{d^t s_d} = \lim_{d \to \infty} \frac{2 \ln^+ (2d)}{d^t} = 0.
\]

Since \( (s, t) \)-WT holds for \( s > 0 \) and \( t \in (0, 1) \), we conclude by (3.16) and (3.14) that
\[
   0 = \lim_{d \to \infty} \frac{\ln \left( n\left(\frac{1}{2}, d\right) \right)}{d^t s_d} \geq \lim_{d \to \infty} \frac{\ln \frac{3}{4}}{d^t s_d} + \lim_{d \to \infty} \frac{d(\omega_d - \omega_d^{1+s_d})}{d^t s_d(1 - \omega_d)}
   \geq \lim_{d \to \infty} \frac{d^{1-t}}{s_d} (\omega_d - \omega_d^{1+s_d}) \geq 0,
\]
which yields that
\[
   \lim_{d \to \infty} \frac{d^{1-t}}{s_d} (\omega_d - \omega_d^{1+s_d}) = 0.
\]

Applying the mean value theorem to the function \( g(x) = a^{1+x}(a \in (0, 1)) \), we get for some \( \theta \in (0, 1) \),
\[
   a - a^{1+x} = xa^{1+\theta x} \ln \frac{1}{a} \leq xa \ln \frac{1}{a}.
\]

It follows that
\[
   0 \leq \lim_{d \to \infty} \frac{d^{1-t} \left( \frac{1}{2d} - \left( \frac{1}{2d} \right)^{1+s_d} \right)}{s_d} \leq \lim_{d \to \infty} \frac{d^{1-t} \left( \frac{1}{2d} \right) \ln(2d)}{s_d} = \lim_{d \to \infty} \frac{1}{2d} d^{-t} \ln(2d) = 0,
\]
which gives
\[
   \lim_{d \to \infty} \frac{d^{1-t} \left( \frac{1}{2d} - \left( \frac{1}{2d} \right)^{1+s_d} \right)}{s_d} = 0.
\]
We remark that the function \( u(x) = x - x^{1+s_d} \) is monotonically increasing in 
\([0, (\frac{1}{1+s_d})^{s_d}] \supset (0, \frac{1}{e}) \) and \( \lim_{d \to \infty} u_d = 0 \). By (3.18) and (3.19), we have

\[
(3.20) \quad \lim_{d \to \infty} \frac{d^{1-t}(u_d - u_d^{1+s_d})}{s_d} = 0.
\]

Using the mean value theorem, we conclude for some \( \theta \in (0, 1) \) that,

\[
u_d - u_d^{1+s_d} = u_d u_d^{s_d}(\ln(\frac{1}{u_d})) \geq u_d s_d(\ln(\frac{1}{u_d})) u_d^{s_d} = u_d s_d(\ln(\frac{1}{u_d})) e^{\frac{-\ln \frac{1}{u_d}}{s_d}}.
\]

It follows from (3.20) that

\[
0 = \lim_{d \to \infty} d^{1-t}(u_d - u_d^{1+s_d}) \geq \lim_{d \to \infty} d^{1-t} u_d(\ln(\frac{1}{u_d})) \lim_{d \to \infty} e^{\frac{-\ln \frac{1}{u_d}}{s_d}} = e^{-1/2} \lim_{d \to \infty} d^{1-t} u_d(\ln(\frac{1}{u_d})) \geq 0,
\]

which implies that

\[
\lim_{d \to \infty} d^{1-t} u_d(\ln(\frac{1}{u_d})) = 0.
\]

By the monotonically of the function \( h(x) = x \ln \frac{1}{x} \) in \( x \in (0, \frac{1}{e}) \), we get

\[
0 \leq \lim_{d \to \infty} d^{1-t} \omega_d \ln^+ \left( \frac{1}{\omega_d} \right) \leq \lim_{d \to \infty} d^{1-t} u_d(\ln(\frac{1}{u_d})) = 0,
\]

which combining with (2.2), gives (1.8).

On the other hand, we suppose that (1.8) holds. We want to show that \((s, t)\)-WT holds. We have for any \( k \in \mathbb{N} \),

\[
k \lambda_{d,k}^{1-s_d} \leq \sum_{j=1}^{k} \lambda_{d,j}^{1-s_d} \leq \sum_{j=1}^{\infty} \lambda_{d,j}^{1-s_d},
\]

so that

\[
(3.21) \quad \lambda_{d,k} \leq \left( \sum_{j=1}^{\infty} \lambda_{d,j}^{1-s_d} \right)^{\frac{k^{1-s_d}}{1-s_d}},
\]

where \( s_d \) is given by (3.15). Clearly,

\[
(3.22) \quad \sum_{k=n+1}^{\infty} \frac{1}{k^{1-s_d}} \leq \int_{n}^{\infty} \frac{1}{x^{1-s_d}} dx = \frac{1-s_d}{s_d n^{1-s_d}}.
\]

Combining (3.21) with (3.22) we conclude that

\[
(3.23) \quad \sum_{k=n+1}^{\infty} \lambda_{d,k} \leq \sum_{k=n+1}^{\infty} \frac{1}{k^{1-s_d}} \left( \sum_{j=1}^{\infty} \lambda_{d,j}^{1-s_d} \right)^{\frac{1-s_d}{1-s_d}} \leq \left( \frac{1-s_d}{s_d} \right) n^{1-s_d} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)^{\frac{1-s_d}{1-s_d}}.
\]

Setting

\[
n = \left\lfloor \frac{-2(1-s_d)}{s_d} \right\rfloor \left( \frac{1-s_d}{s_d} \right)^{-\frac{1-s_d}{s_d}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)^{\frac{1-s_d}{s_d}} + 1
\]
in (3.23), we have
\[
\sum_{k=n+1}^{\infty} \lambda_{d,k} \leq \varepsilon^2.
\]

Therefore from the definition of \(n(\varepsilon, d)\), and the inequality \(|x| + 1 \leq 2x\) for \(x > 1\), we get
\[
n(\varepsilon, d) \leq \left[ \frac{\varepsilon}{s_d} \right] + 1
\leq 2\varepsilon \frac{1-s_d}{s_d} \left( \frac{1}{s_d} \right)^{-\frac{1-s_d}{s_d}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)^{\frac{1}{s_d}}.
\]

It follows from (3.17) that
\[
\ln n(\varepsilon, d) \leq \ln 2 + \frac{2(1-s_d)}{s_d} \ln(\varepsilon^{-1}) + \frac{1-s_d}{s_d} \ln \left( \frac{1-s_d}{s_d} \right) + \frac{1-s_d}{s_d} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)
\leq \ln 2 + \frac{2}{s_d} \ln(\varepsilon^{-1}) + \frac{1}{s_d} \ln \left( \frac{1}{s_d} \right) + \frac{1}{s_d} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)
\leq \ln 2 + 4 \ln^+(2d) \ln(\varepsilon^{-1}) + 2 \ln^+(2d) \ln \left( \ln(\varepsilon^{-1}) \right) + \frac{1}{s_d} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)
\leq \ln 2 + (2 \ln^+(2d))^2 + \left( \ln(\varepsilon^{-1}) \right)^2
\leq \ln 2 + \ln^+(2d) \ln(\varepsilon^{-1}) + \frac{1}{s_d} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right).
\]

Note that
\[
\lim_{\frac{1}{s_d} + d \to \infty} \frac{\ln 2 + (2 \ln^+(2d))^2 + \left( \ln(\varepsilon^{-1}) \right)^2 + 2 \ln^+(2d) \ln(2 \ln^+(2d))}{(\frac{1}{s_d})^s + d^t} = 0.
\]

In order to show that \((s, t)\)-WT holds, it suffices to prove that
\[
\lim_{d \to \infty} \frac{\ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right)}{d^t s_d} = 0.
\]

We recall that
\[
\ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right) = \sum_{k=1}^{d} \ln \left( \frac{(1-\omega_k)^{1-s_d}}{1-\omega_k^{1-s_d}} \right).
\]
Note that \( v(x) = \ln \left( \frac{1-x^a}{1-x} \right) (\alpha \in (0, 1)) \) is increasing in \((0, 1)\) due to the fact that 
\[
v'(x) = \frac{(1-x^a)(1-x)}{1-x} > 0.
\]
We get
\[
\ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right) \leq \frac{\sum_{k=1}^{d} \ln \left( \frac{(1-u_k)^{1-s_d}}{1-u_k} \right)}{d^s d_s d}
\]
\[
= \frac{\sum_{k=1}^{d} \ln \left( \frac{1}{1-u_k} \right)}{d^s d_s d} + \frac{\sum_{k=1}^{d} \ln \left( 1 + \frac{u_k^{1-s_d} - u_k}{1-u_k} \right)}{d^s d_s d}
\]
\[
\leq \frac{\sum_{k=1}^{d} \ln \left( \frac{1}{1-u_k} \right)}{d^s d_s d} + \frac{\sum_{k=1}^{d} \frac{u_k^{1-s_d} - u_k}{1-u_k^{1-s_d}}}{d^s d_s d}
\]
\[
=: I_{1,d} + I_{2,d},
\]
where
\[
(3.25) \quad I_{1,d} = \frac{\sum_{k=1}^{d} \ln \left( \frac{1}{1-u_k} \right)}{d^s d_s d} \quad \text{and} \quad I_{2,d} = \frac{\sum_{k=1}^{d} \frac{u_k^{1-s_d} - u_k}{1-u_k^{1-s_d}}}{d^s d_s d}.
\]
By (1.8) and (2.2), we have
\[
\lim_{d \to \infty} d^{1-t} \omega_d \ln^+ \left( \frac{1}{\omega_d} \right) = 0,
\]
which combining the equality
\[
\lim_{d \to \infty} d^{1-t} \left( \frac{1}{2d} \right) \ln(2d) = 0,
\]
yields
\[
\lim_{d \to \infty} d^{1-t} u_d \ln^+ \left( \frac{1}{u_d} \right) = 0.
\]
We have by the Stolz theorem
\[
\lim_{d \to \infty} I_{1,d} = \lim_{d \to \infty} \frac{\ln \left( \frac{1}{u_k} \right)}{d^s d_s d} - \left( \frac{d}{d+1} \right)^t = \lim_{d \to \infty} \frac{u_d}{d^s d_s d-1} = 0.
\]
Applying the mean value theorem, we obtain for some \( \theta \in (0, 1) \),
\[
u_k^{1-s_d} - u_k = s_d u_k^{1-\theta s_d} \ln \left( \frac{1}{u_k} \right) \leq s_d u_k \ln \left( \frac{1}{u_k} \right) u_k^{1-s_d}
\]
\[
\leq s_d u_k \ln \left( \frac{1}{u_k} \right) e^{\frac{1}{\ln(1-u_k)}} \leq e^{1/2} s_d u_k \ln \left( \frac{1}{u_k} \right).
\]
(3.26)
It follows from (3.25), (3.26) and the inequality
\[
1 - u_k^{1-s_d} \geq 1 - u_1^{1-s_d} \geq 1 - u_1^{1-s_1} > 0
\]
that
\[
I_{2,d} \leq \frac{\sum_{k=1}^{d} e^{1/2} u_k \left( \ln \left( \frac{1}{u_k} \right) \right) s_d}{\left( 1 - u_1^{1-s_1} \right) d^s d_s d} = \frac{C \sum_{k=1}^{d} u_k \ln \left( \frac{1}{u_k} \right)}{d^s d_s d},
\]
where \( C = \frac{e^{1/2}}{1 - u_1^2} \). By the Stolz theorem we get

\[
0 \leq \lim_{d \to \infty} I_{2,d} \leq C \lim_{d \to \infty} \frac{\sum_{k=1}^{d} u_k \ln \left( \frac{1}{u_k} \right)}{d^2} = C \lim_{d \to \infty} \frac{u_d \ln \left( \frac{1}{u_d} \right)}{d^2 - (d - 1)^2}.
\]

We obtain further

\[
\lim_{d \to \infty} d^{1-t} u_d \ln \left( \frac{1}{u_d} \right) = 0.
\]

We conclude that if (1.8) holds, then

\[
\lim_{d \to \infty} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d} \right) d^{s_d} = 0.
\]

(v) The proof of (v) follows from (iv) and Lemma 3.2 immediately.

(vi) Suppose that \( \lim_{j \to \infty} \gamma_j^2 > 0 \). Then \( \lim_{j \to \infty} \omega_{\gamma_j} = 2A > 0 \). There exists an \( N \in \mathbb{N} \) such that

\[
\ln(3/4) + d\omega_{\gamma_d} \geq dA
\]

for any \( d > N \). By (3.13) we have for \( \varepsilon \in (0, 1/2) \),

\[
\ln n(\varepsilon, d) \geq \ln n(1/2, d) \geq \ln(3/4) + d\omega_{\gamma_d} \geq dA.
\]

It follows that

\[
n(\varepsilon, d) \geq (e^A)^d, \quad \varepsilon \in (0, 1/2], \quad d > N.
\]

This means that \( \text{App} \) suffers from the curse of dimensionality. (vi) is proved.

The proof of Theorem 1.1 is completed. \( \square \)

**Proof of Theorem 1.2**

(1) We remark that we have the same results about EXP and UEXP in the worst and average case settings. Indeed, using (2.6) and the method in the proof of \( \mathbb{N} \) Theorem 4.1, we obtain that \( \text{App} = \{\text{App}_d\} \) is EXP iff \( I = \{I_d\} \) is EXP with the same exponent. This completes the proof of (i).

(2) Based on the results of [18] and (2.6), we get that EC-SPT, EC-PT, and EC-QPT do not hold for all shape parameters. (ii) is proved.

(3) According to [19] Theorems 3.2 and 4.2 and [9] Theorem 3.2, we know that we have the same results in the worst and average case settings using ABS concerning EC-WT, EC-UWT, and EC-\((s,t)\)-WT for \( 0 < s \leq 1 \) and \( t > 0 \). This implies that (v), (vi) and (viii) hold. We always have EC-\((s,t)\)-WT for \( 0 < s \leq 1 \) and \( t > 1 \). This yields EC-\((s,t)\)-WT for \( s > 1 \) and \( t > 1 \). Hence (iii) holds. This completes the proofs of (iii), (v), (vi), and (viii).

(4) If EC-\((s,1)\)-WT with \( s \geq 1 \) holds, then \((s,1)\)-WT with \( s \geq 1 \) holds. By Theorem 1.1 (iii), we have \( \lim_{j \to \infty} \gamma_j^2 = 0 \).

On the other hand, if \( \lim_{j \to \infty} \gamma_j^2 = 0 \), then EC-WT holds and hence, EC-\((s,1)\)-WT with \( s \geq 1 \) holds. This completes the proof of (iv).
(5) If EC-(s, t)-WT with s > 1 and t < 1 holds, then (s, t)-WT with s > 1 and t < 1 holds. By Theorem 1.1 (iv), we have (1.11).

On the other hand, if (1.11) holds, then by (3.24) we have

\[
\ln n(\varepsilon, d) \leq 2 + 4 \ln^+ (2d) \ln(\varepsilon^{-1}) + 2 \ln^+ (2d) \ln \left(2 \ln^+ (2d)\right) + \frac{1}{s_d} \ln \left(\sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d}\right),
\]

where \(s_d\) is given by (3.15). By (3.27), we obtain

\[
(3.28) \lim_{d \to \infty} \frac{\ln 2 + 2 \ln^+ (2d) \ln \left(2 \ln^+ (2d)\right) + \frac{1}{s_d} \ln \left(\sum_{k=1}^{\infty} \lambda_{d,k}^{1-s_d}\right)}{d^p} = 0.
\]

For \(s > 1\), by the Young inequality \(ab \leq a^p \frac{b}{p} + \frac{b^{p'}}{p'}\), \(a, b \geq 0\), \(1/p + 1/p' = 1\) with \(p = \frac{1+s}{2}\), \(p' = \frac{s+1}{s-1}\) we have

\[
\lim_{\varepsilon^{-1}+d \to \infty} \frac{\ln^+ (2d) \ln(\varepsilon^{-1})}{(1 + \ln \varepsilon^{-1})^s + d^p} = \lim_{\varepsilon^{-1}+d \to \infty} \frac{(\ln \varepsilon^{-1})^{1/p}}{p} + \frac{(\ln^+ (2d))^{p'}}{p'} = 0,
\]

which combining (3.24) and (3.28), leads to

\[
\lim_{\varepsilon^{-1}+d \to \infty} \frac{\ln n(\varepsilon, d)}{(1 + \ln \varepsilon^{-1})^s + d^p} = 0.
\]

This finishes the proof of (vii).

The proof of Theorem 1.2 is completed. \(\square\)

ACKNOWLEDGMENTS

The authors were Supported by the National Natural Science Foundation of China (Project no. 11671271) and the Beijing Natural Science Foundation (1172004).

REFERENCES

[1] J. Dick, G. Larcher, F. Pillichshammer, H. Woźniakowski, Exponential convergence and tractability of multivariate integration for Korobov spaces, Math. Comp. 80 (2011) 905-930.
[2] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, On dimension-independent rates of convergence for function approximation with Gaussian kernels, SIAM J. Numer. Anal., 50 (2012) 247-271.
[3] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, Average case approximation: convergence and tractability of Gaussian kernels, Monte Carlo and Quasi-Monte Carlo 2010, eds. L. Plaskota and H. Woźniakowski, Springer Verlag, 2012, 329-345.
[4] A. I. J. Forrester, A. Sóester, and A. J. Keane, Engineering Design via Surrogate Modelling: A Practical Guide, Wiley, Chichester, 2008.
[5] C. Irrgeher, P. Kritzer, F. Pillichshammer, H. Woźniakowski, Tractability of multivariate approximation defined over Hilbert spaces with exponential weights, J. Approx. Theory 207 (2016) 301-338.
[6] A.A. Khartov, A simplified criterion for quasi-polynomial tractability of approximation of random elements and its applications, J. Complexity, 34 (2016) 30-41.
[7] M. A. Lifshits, A. Papageorgiou, H. Woźniakowski, Average case tractability of non-homogeneous tensor product problems, J. Complexity 28 (2012) 539-561.
[8] Y. Liu, G. Xu, Average case tractability of a multivariate approximation problem, J. Complexity, 43 (2017) 76-102.
[9] Y. Liu, G. Xu, Y. Dong, EC-(s, t)-weak tractability of multivariate linear problems in the average case setting, preprint.
[10] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, EMS, Zürich, 2008.
[11] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume II: Standard Information for Functionals, EMS, Zürich, 2010.
[12] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume III: Standard Information for Operators, EMS, Zürich, 2012.
[13] A. Papageorgiou, I. Petras, A new criterion for tractability of multivariate problems, J. Complexity 30 (2014) 604-619.
[14] C. E. Rasmussen and C. Williams, Gaussian Processes for Machine Learning, MIT Press, 2006 (online version at [http://www.gaussianprocess.org/gpml/](http://www.gaussianprocess.org/gpml/)).
[15] P. Siedlecki, Uniform weak tractability, J. Complexity 29(6) (2013) 438-453.
[16] P. Siedlecki, (s, t)-weak tractability of Euler and Wiener integrated processes, J. Complexity (online)https://doi.org/10.1016/j.jco.2017.10.001.
[17] P. Siedlecki, M. Weimar, Notes on (s, t)-weak tractability: a refined classification of problems with (sub)exponential information complexity, J. Approx. Theory 200 (2015) 227-258.
[18] I. H. Sloan, H. Woźniakowski, Multivariate approximation for analytic functions with Gaussian kernels, in press.
[19] G. Xu, Exponential convergence-tractability of general linear problems in the average case setting, J. Complexity 31 (2015) 617-636.

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.
E-mail address: jiaxenhcd@163.com; wanghp@cnu.edu.cn.