THE BORN APPROXIMATION AND CALDERÓN’S METHOD
FOR RECONSTRUCTION OF CONDUCTIVITIES IN 3-D

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Abstract. Two algorithms for the direct reconstruction of conductivities in a bounded domain in \( \mathbb{R}^3 \) from surface measurements of the solutions to the conductivity equation are presented. The algorithms are based on complex geometrical optics solutions and a nonlinear scattering transform. We test the algorithms on three numerically simulated examples, including an example with a complex coefficient. The spatial resolution and amplitude of the examples are well-reconstructed.

1. Introduction. Let \( \Omega \subset \mathbb{R}^3 \) be a smooth and bounded domain and denote by \( \gamma \) a (possibly complex) conductivity coefficient in \( \Omega \). Assume \( \gamma \in L^\infty(\Omega) \), \( C^{-1} \leq \Re(\gamma) \leq C \), for some \( C > 0 \). A voltage potential \( u \in H^1(\Omega) \) generated by the voltage potential \( f \in H^{1/2}(\partial \Omega) \) on the boundary surface is described as the solution to

\[
\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega,
\]

\[
f = u|_{\partial \Omega}.
\]

The voltage \( u \) gives rise to a current flux through the boundary given by

\[
g = \gamma \partial_n u|_{\partial \Omega},
\]

where \( \partial_n \) denotes the outward normal derivative. This definition allows us to define the Dirichlet to Neumann map (or voltage to current density map) \( \Lambda_\gamma \) by

\[
\Lambda_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)
\]

\[
f \mapsto g.
\]

In [7] Calderón formulated an inverse problem consisting of the following two questions:

- Uniqueness: is the mapping \( \gamma \mapsto \Lambda_\gamma \) injective?
- Reconstruction: how can \( \gamma \) be computed from \( \Lambda_\gamma \)?

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The Calderón problem has been one of the most studied inverse problems for the last 30 years with numerous applications, including the emerging technology for medical imaging called Electrical Impedance Tomography or EIT. The reader is referred to the review articles [5, 8] and the monograph [10] for more information about EIT. In Calderón’s seminal paper [7] uniqueness for the linearized problem was established and an approximate reconstruction algorithm for conductivities close to constant was provided. This work stimulated a large body of research on uniqueness results for conductivities with various regularity properties and many reconstruction algorithms. While the research addressing uniqueness relies heavily on the use of complex geometrical optics (CGO) solutions (inspired by Calderón’s proof), most reconstruction algorithms did not, but instead looked most commonly to an iterative approach. The uniqueness problem is surveyed in [21], and algorithms are surveyed in [10].

For the 3-D problem uniqueness for smooth conductivities was proved in [20], and implicitly a reconstruction algorithm was derived. This algorithm was made explicit by several authors [15, 17, 16] for conductivities having two derivatives. The equations in [15, 17] lead to a direct (noniterative) reconstruction algorithm based on CGO solutions. This approach was studied in [2], where a numerical implementation of the direct reconstruction algorithm was presented and tested on radially symmetric examples. The relationship to Calderón’s method was established and the two approaches were compared. In [9] a low frequency direct reconstruction algorithm was outlined that makes use of a D-bar equation to determine the conductivity. In [6, 18] the uniqueness question was relaxed to conductivities having 3/2 derivatives, which is the most general uniqueness result at the time of this publication. Since medically relevant conductivities can be described by piecewise continuous \( L^\infty \) conductivities, it would be desirable to have a uniqueness result for \( L^\infty \) conductivities such as [1] in dimension two or that of Calderón for the linearized problem.

The history of the 2D problem is very extensive, and the reader is referred to [1] for an exposition of the history of the uniqueness question and, for example, to the introduction of [14] for a discussion of reconstruction algorithms. An implementation of Calderón’s algorithm in 2-D for experimental data is given in [4], and a fully nonlinear regularized D-bar algorithm for direct reconstruction of conductivities can be found in [14]. Previous implementations of the 2-D D-bar algorithm included a linearized approximation to the scattering transform, denoted \( t^{\text{exp}} \), which was first introduced in [19] and found to be quite effective on experimental data (see, for example, [11, 12]). These results inspired the use of the Born approximation for the scattering transform used in this paper with the same notation \( t^{\text{exp}} \).

The main purpose of this paper is to illustrate the suitability of the algorithms to reconstruct both non radial and complex conductivities in three dimensions. We stress that conductivities in applications like EIT are often known to be complex (and of course non radial).

The outline of the paper is as follows. In section 2 we will first outline Calderón’s linearized reconstruction algorithm and second a direct non-linear reconstruction algorithm. Details of the implementation are given in section 3. Results of the numerical experiments are found in section 4.

2. Algorithms. In this section we will first describe the two reconstruction algorithms and then compare them. We describe Calderón’s method in the context of
the D-bar methods of inverse scattering so that they can be easily compared, and we use the notation $t^\exp$ for a linearized scattering transform. This will be made clear below.

2.1. Calderón’s method. Let $\Lambda_1$ denote the Dirichlet-to-Neumann map for a constant conductivity distribution of 1, and let $\langle (\Lambda_\gamma - \Lambda_1)f, h \rangle$ be the dual pairing between the action of the difference of the maps $\Lambda_\gamma$ and $\Lambda_1$ on $f$ and $h$. Integration by parts then gives

$$
\langle (\Lambda_\gamma - \Lambda_1)f, h \rangle = \int_\Omega (\gamma - 1) \nabla u \cdot \nabla v \, dx,
$$

where

$$
\nabla \cdot \gamma \nabla u = 0, \quad u|_{\partial \Omega} = f,
$$

$$
\Delta v = 0, \quad v|_{\partial \Omega} = h.
$$

Calderón [7] considered the linearized inverse problem around conductivities equal to one. He took the functions $f, h$ to be restrictions of harmonic functions

$$
f = e^{ix \cdot \zeta}, \quad h = e^{-ix \cdot (\xi + \zeta)}, \quad x \in \partial \Omega,
$$

with $\xi \in \mathbb{R}^3$ and $\zeta = \zeta(\xi) \in \mathbb{C}^3$ such that

$$
(\xi + \zeta)^2 = \zeta^2 = 0.
$$

Plugging in these fields in (1) defines

$$
t^\exp(\xi, \zeta) = \left\langle (\Lambda_\gamma - \Lambda_1)e^{ix \cdot \zeta}, e^{-ix \cdot (\xi + \zeta)} \right\rangle
$$

$$
= \int_\Omega (\gamma - 1) \nabla u^\exp(x, \zeta) \cdot \nabla e^{-ix \cdot (\xi + \zeta)} \, dx,
$$

where

$$
\nabla \cdot \gamma \nabla u^\exp = 0 \text{ in } \Omega, \quad u^\exp|_{\partial \Omega} = e^{ix \cdot \zeta}.
$$

Writing $u^\exp = e^{ix \cdot \zeta} + \delta u$ yields

$$
t^\exp(\xi, \zeta) = -\frac{||\xi||^2}{2} \int_\Omega (\gamma - 1)e^{-ix \cdot \zeta} \, dx + R(\xi, \zeta),
$$

where $R$ denotes the remainder term

$$
R(\xi, \zeta) = \int_\Omega (\gamma - 1) \nabla \delta u \cdot \nabla e^{-ix \cdot (\xi + \zeta)} \, dx.
$$

Thus $-2t^\exp/||\xi||^2$ is the Fourier transform of $\gamma - 1$ up to the remainder term. It is possible to estimate (see for instance [13, formula (53)])

$$
|R(\xi, \zeta)| \leq C||\gamma - 1||^2_{L^\infty(\Omega)} (1 + ||\zeta||^2) e^{2R||\zeta||}, \quad \Omega \subset B_R.
$$

This estimate is useful only in case $\gamma - 1$ is small. The reconstruction now proceeds by taking the $\zeta_M \in \mathbb{C}^3$ with smallest norm satisfying (2). Then multiply by the characteristic function $\chi_K(\xi)$ supported on $B(0, K)$, and apply the inverse Fourier transform to get Calderón’s approximation formula

$$
\gamma^{app}(x) = 1 - \frac{1}{2(2\pi)^n} \int \frac{t^\exp(\xi, \zeta_M)}{||\xi||^2} e^{ix \cdot \zeta} \chi_K(\xi) \, d\xi.
$$
2.2. The non-linear method and the Born approximation. The non-linear reconstruction algorithm is most easily explained by transforming the conductivity equation to a Schrödinger equation. This transformation requires $\gamma$ to have essentially two derivatives. Suppose $u$ solves

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \quad u|_{\partial \Omega} = f.$$  

Then $v = \gamma^{1/2}u$ solves

$$(\Delta + q)v = 0 \text{ in } \Omega, \quad v|_{\partial \Omega} = \gamma^{-1/2}f,$$

with $q = -\Delta \gamma^{1/2}/\gamma^{1/2} \iff (\Delta + q) \gamma^{1/2} = 0$. Associated with the Schrödinger equation is the Dirichlet to Neumann map $\Lambda_q f = \partial_\nu v$. If $\gamma = 1$ in a neighborhood of $\partial \Omega$, then $\Lambda_q = \Lambda_\gamma$.

In the sequel we will therefore assume that $\gamma \in C^2(\mathbb{R}^3)$ is extended beyond $\Omega$ such that $\operatorname{supp}(\gamma - 1) \subset \Omega$.

Let $\zeta \in \mathbb{C}^3$ be such that $\zeta \cdot \zeta = 0$.

For sufficiently large $\zeta$ there is a unique complex geometrical optics solution to the problem

$$(\Delta + q)\psi(x, \zeta) = 0 \text{ in } \mathbb{R}^3,$$  

$$(\psi(x, \zeta)e^{ix \cdot \zeta} - 1) \to 0 \text{ for } |x| \text{ or } |\zeta| \to \infty.$$  

Convolving with the Faddeev Green’s function for the Laplacian defined by

$$G_\zeta(x) = \frac{e^{ix \cdot \zeta}}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\xi \cdot \zeta}}{|\xi|^2 + 2\xi \cdot \zeta} d\xi,$$

in the equation yields the Lippmann-Schwinger-Faddeev (LSF) equation

$$\psi(x, \zeta) = e^{ix \cdot \zeta} + \int_{\Omega} G_\zeta(x - y)q(y)\psi(y, \zeta)dy.$$  

(6)

On the boundary $\partial \Omega$, $\psi|_{\partial \Omega}$ satisfies the boundary integral equation

$$\psi(x, \zeta) + \int_{\partial \Omega} G_\zeta(x - y)(\Lambda_\gamma - \Lambda_1)\psi(y, \zeta)d\sigma(y) = e^{ix \cdot \zeta}, \quad x \in \partial \Omega.$$  

(7)

This is a Fredholm equation of the second kind where the homogenous problem has only the trivial solution.

The key intermediate object in the reconstruction is the non-physical scattering transform defined by

$$t(\xi, \zeta) = \int_{\Omega} e^{-ix \cdot (\xi + \zeta)}q(x)\psi(x, \zeta)dx.$$  

(8)

$$= \int_{\partial \Omega} e^{-ix \cdot (\xi + \zeta)}(\Lambda_\gamma - \Lambda_1)\psi(x, \zeta)|_{\partial \Omega}d\sigma(x), \quad (\xi + \zeta)^2 = 0.$$  

(9)

Since $e^{-ix \cdot \zeta}\psi(x, \zeta)$ is asymptotically 1, it is possible to estimate $t$ by the Fourier transform of $q$. In fact, $t$ satisfies the estimate

$$|\hat{q}(\xi) - t(\xi, \zeta)| = O(1/|\zeta|).$$

The algorithm is now decomposed into the steps

$$\Lambda_\gamma \xrightarrow{1} t(\xi, \zeta) \xrightarrow{2} q(x) \xrightarrow{3} \gamma(x).$$  

(10)

The concrete steps are implemented by

1. Compute $\psi|_{\partial \Omega}$ from boundary measurements by solving (7), and compute $t$ from boundary data and $\psi|_{\partial \Omega}$. 
2. Compute $q$ from $t$ using
\[
\lim_{|\zeta| \to \infty} t(\xi, \zeta) = \hat{q}(\xi).
\]

3. Compute $\gamma$ from $q$ by solving
\[
(\Delta + q)^{\sqrt{\gamma}} = 0 \quad \text{in} \quad \Omega,
\]
\[
\sqrt{\gamma}|_{\partial \Omega} = 1.
\]

If in the definition of (9) we replace the complex geometrical optics solution $\psi$ by its asymptotic value $\psi \sim e^{ix} \zeta$ we obtain a type of Born approximation of the scattering data which is exactly given by (3). By replacing $t$ in (10) by $t^{\exp}$, we then arrive at the following reconstruction procedure based on the Born approximation:
\[
\Lambda_\gamma \xrightarrow{1} t^{\exp}(\xi, \zeta) \xrightarrow{2} q^{\exp}(x) \xrightarrow{3} \gamma^{\exp}(x).
\]

with the concrete steps
1. Compute $t^{\exp}$ by (3).
2. Compute an approximation of $q$ defined by
\[
\hat{q}^{\exp}(\xi) = t^{\exp}(\xi, \zeta_M).
\]
3. Compute an approximation $\gamma^{\exp}$ of $\gamma$ by solving the boundary value problem
\[
(\Delta + q^{\exp})^{\sqrt{\gamma^{\exp}}} = 0 \quad \text{in} \quad \Omega,
\]
\[
\sqrt{\gamma^{\exp}}|_{\partial \Omega} = 1.
\]

The choice of the parameter $\zeta = \zeta_M$ in step 2 is rather delicate. The theory suggests that we let $\zeta \to \infty$ in the scattering transform $t$, but it turns out that $t^{\exp}(\xi, \cdot)$ diverges from $t(\xi, \cdot)$ as $|\zeta| \to \infty$ (see [13, formula (64)] for an explicit estimate in the 2-D case). In addition one can obtain an estimate (cf. (5))
\[
|t^{\exp}(\xi, \zeta) - \hat{q}(\xi)| \leq C||q||_{L^\infty(\Omega)} e^{R(|\zeta|+|\xi+\zeta|)},
\]
which for fixed $q$ is optimal for the minimal $\zeta$. This motivates choosing $\zeta_M$ to be the minimal value of $\zeta$ satisfying (2) rather than a large choice of $\zeta$. Also from a computational point of view it is attractive to work with the minimal exponent in order to reduce numerical errors.

3. Implementation details. For comparison we have computed the non-physical scattering transform $t$. This is done by discretizing the LSF equation (6) using periodic extension and FFT (see [2] for details) and solving it for $\psi$, and then integrating numerically in (8). The approximation $t^{\exp}$ is computed by the volume integral (4). The solution $u^{\exp}$ appearing in the formula is computed numerically using the finite element method implemented in the commercial software COMSOL. We compute on a $\xi$-grid in $\mathbb{R}^3$ inside the cube $[-10, 10]^3$ of size $16^3$. The Fourier transform appearing in both Calderóns method and the Born approximation method is evaluated by numerical integration. No attempt has been done to optimize our computations using e.g. FFT. Finally the boundary value problem appearing in step 3 of the Born approximation is solved using the explicit Dirichlet Green’s function for the sphere.

4. Results. We consider three example conductivities. The first is a radially symmetric conductivity in $C^\infty(\Omega)$, the second is a real and non radially-symmetric conductivity in $C^\infty(\Omega)$, and the third is a complex conductivity given by a superposition of the previous two examples.
4.1. Example 1: radial conductivity. Take $\Omega = B(0,1)$ and define for $\alpha = .3$ and $d = .9$

$$\gamma_1(x) = \begin{cases} 
1 + \alpha e^{-\frac{|x|^2}{(|x|^2-d^2)^2}}, & |x| \leq d \\
1, & d < |x| \leq 1.
\end{cases}$$

In figure 1 the corresponding scattering transform $t$ is displayed in the first row (real and imaginary part). We see that the scattering transform is spherically symmetric and real, which is consistent with the theoretical observations in [2]. The approximation $t^{\exp}$ is displayed in the bottom row. We see that $t^{\exp}$ has several features in common with $t$.

![Figure 1](image1.png)

In figure 2 the true conductivity is displayed in the middle. To the left is the reconstruction $\gamma^{\exp}$ from the Born approximation, to the right is the reconstruction $\gamma^{\app}$ from Calderón’s method. Both reconstructions are fair, but Calderón’s method seem to recover the support of the conductivity slightly better.

4.2. Example 2: non radial conductivity. Take $\Omega = B(0,1)$. The conductivity $\gamma_2$ has uniform background 1 and contains an inclusion centered at $x_0 = (0, 1, 3)$ with radius $d = .6$. The conductivity inside the inclusion is given by the formula

$$\gamma_2(x) = (0.5 \exp(-r^2/(r^2 - d^2)^2) + 1)^2, \quad r = |x - x_0|.$$ 

In figure 4 $t$ and $t^{\exp}$ corresponding to $\gamma_2$ can be seen. We note that the scattering transforms are no longer radially symmetric and real. Also in this example $t^{\exp}$ provides a reasonable approximation of $t$. 
Figure 2. Reconstruction of Example 1. Left reconstruction $\gamma^{\text{exp}}$, middle true conductivity, right reconstruction $\gamma^{\text{app}}$.

Figure 3. Cross-section through plane $\xi_3 = 0$. Upper row real and imaginary part of $t$. Lower row $t^{\text{exp}}$.

The reconstructions of $\gamma_2$ are seen in figure 4. For this example both reconstructions do well, however Calderón’s method again gets the better reconstruction of the support while the Born approximation gets the contrast slightly better.

4.3. Example 3: complex conductivity. Take $\Omega = B(0, 1)$. The conductivity $\gamma_3$ is a complex superposition of the previous two given by

$$\gamma_3(x) = \gamma_2(x) + i(\gamma_1(x) - 1).$$
The scattering transforms $t$ and $t^{\text{exp}}$ are displayed in figure 5. We see that $t^{\text{exp}}$ provides a fair approximation of $t$. In addition the figure illustrates the non-linear nature of the inverse problem: the function displayed in figure 5 is not just a complex linear combination of the functions in figures 1 and 3. The reconstructions of the real and complex part of $\gamma_3$ are seen in figure 6. We see that both methods are able
to separate and recover both the real and complex part even though the contrast in the reconstructions of the real part are just merely acceptable.

![Figure 6](image-url)

Figure 6. Top: Real part of reconstructions of Example 3. Bottom: Imaginary parts. Left reconstruction $\gamma^{\exp}$, middle true conductivity, right reconstruction $\gamma^{\text{app}}$.

5. Discussion and conclusions. We have outlined two different direct (noniterative) algorithms for the reconstruction of conductivities in three dimensions: Calderón’s method and a method based on Born approximation. Both methods use the same scattering transform $t^{\exp}$ as an intermediate object. The algorithms were implemented numerically and their performance was tested on three different examples. Both methods did well in reconstructing the radial, the non-radial, and the complex conductivity with respect to amplitude and spatial resolution.

In order to improve the reconstructions one must face the full non-linearity of the inverse problem. This is done by solving (7) and computing $t$ from boundary data. This is not in the scope of this paper. We note that the ill-posedness of the problem is closely connected with (7), but it is outside the scope of the present paper to discuss the stability of the implementations against noise in boundary measurements.

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RECONSTRUCTION OF CONDUCTIVITIES IN 3-D 853

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