SPECIALIZING ARONSZAJN TREES BY COUNTABLE APPROXIMATIONS

HEIKE MILDENBERGER AND SAHARON SHELAH

ABSTRACT. We show that there are proper forcings based upon countable trees of creatures that specialize a given Aronszajn tree.

0. Introduction

The main point of this work is finding forcing notions specializing an Aronszajn tree, which are creature forcings, tree-like with halving, but being based on $\omega_1$ (the tree) rather than $\omega$.

For creature forcing in general there is “the book on creature forcing” [4] and for the uncountable case the work is extended in [3] and [5]. Since some of the main premises made in the mentioned work are not fulfilled in our setting, it serves mainly as a guideline, whereas numerous technical details here are different and new.

The norm of creatures (see Definition 1.7) we shall use is natural for specializing Aronszajn trees. It is convenient if there is some $\alpha < \omega_1$ such that the union of the domains of the partial specialization functions that are attached to any branch of the tree-like forcing condition is the initial segment of the Aronszajn tree $(T_A)_{<\alpha}$, i.e. the union of the levels less than $\alpha$. However, allowing for every branch of a given condition finitely many possibilities $(T_A)_{<\alpha_i}$ with finite sets $u_i$ sticking out of $(T_A)_{<\alpha_i}$ is used for density arguments that show that the generic filter leads to a total specialization function.

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1. Tree creatures

In this section we define the tree creatures which will be used later to describe the branching of the countable trees that will serve as forcing conditions. We prove three important technical properties about gluing together (Claim 1.9), about filling up (Claim 1.10) and about changing the base together with thinning out (Claim 1.11) of creatures. We shall define the forcing conditions only in the next section. They will be countable trees with finite branching, such that each node and its immediate successors in the tree are described by a creature in the sense of Definition 1.5. Roughly spoken, in our context, a creature will be an arrangement of partial specialization functions with some side conditions.

We reserve the symbol \((T, \s_T)\) for the trees in the forcing conditions, which are trees of partial specialization functions of some given Aronszajn tree \((T_A, <_{T_A})\). A specialization function is a function \(f: T_A \to \omega\) such that for all \(s, t \in T_A\), if \(s <_{T_A} t\), then \(f(s) \neq f(t)\), see [2, p. 244].

\(\chi\) stands for some sufficiently high regular cardinal, and \(\mathcal{H}(\chi)\) denotes the set of all sets of hereditary cardinality less than \(\chi\). For our purpose, \(\chi = (2^\omega)^+\) is enough.

Throughout this work we make the following assumption:

Hypothesis 1.1. \(T_A\) is an Aronszajn tree ordered by \(<_{T_A}\), and for \(\alpha < \omega_1\) the level \(\alpha\) of \(T_A\) satisfies:

\[(T_A)_\alpha \subseteq [\omega\alpha, \omega\alpha + \omega).\]

Throughout this work, \(T_A\) will be fixed. We define the following finite approximations of specialization maps:

Definition 1.2. For \(u \subseteq T_A\) and \(n < \omega\) we let

\[\text{spec}_n(u) = \{\eta \mid \eta: u \to [0, n) \land (\eta(x) = \eta(y) \rightarrow \neg(x <_{T_A} y))\}.\]

We let \(\text{spec}(u) = \bigcup_{n<\omega} \text{spec}_n(u)\) and \(\text{spec} = \text{spec}^{T_A} = \bigcup\{\text{spec}(u) : u \subset T_A, u \text{ finite}\}\).
Choice 1.3. We choose three sequences of natural numbers \( \langle n_{k,i} : i < \omega \rangle \), \( k = 1, 2, 3 \), such that the following growth conditions are fulfilled:

\[
\begin{align*}
(1.1) & \quad i \cdot n_{1,i} < n_{3,i}, \\
(1.2) & \quad n_{2,i} < n_{1,i+1}, \\
(1.3) & \quad n_{1,i} \cdot n_{1,i} \leq n_{1,i+1}, \\
(1.4) & \quad n_{1,i} \leq n_{2,i}.
\end{align*}
\]

We fix them for the rest of this work.

We compare with the book [4] in order to justify the use of the name creature. However, we cannot just cite that work, because the framework developed there is not suitable for the approximation of uncountable domains \( T_A \).

Definition 1.4. (1.) [4, 1.1.1] A triple \( t = (\text{nor}[t], \text{val}[t], \text{dis}[t]) \) is a weak creature for \( H \) if

\[
\begin{align*}
(a) & \quad \text{nor}[t] \in \mathbb{R}^{\geq 0}, \\
(b) & \quad \text{Let } H = \bigcup_{i \in \omega} H(i) \text{ and let } H(i) \text{ be sets. Let } \prec \text{ be the strict initial segment relation.} \\
& \quad \text{val}[t] \text{ is a non-empty subset of} \\
& \quad \left\{ (x, y) \in \bigcup_{m_0 < m_1 < \omega} \prod_{i < m_0} H(i) \times \prod_{i < m_1} H(i) : x \prec y \right\}. \\
(c) & \quad \text{dis}[t] \in H(\chi).
\end{align*}
\]

(2.) nor stands for norm, val stands for value, and dis stands for distinguish.

In our case, we drop the component dis (in the case of simple creatures in the sense of Definition 1.5) or it will be called \( k \) (in the case of creatures), an additional coordinate, which is a natural number. In order to stress some parts of the weak creatures \( t \) more than others, we shall write \( \text{val}[t] \) in a slightly different form and call it a simple creature, \( c \).

As we will see in the next definition, in this work (b) of 1.4 is not fulfilled: For us \( \text{val} \) is a non-empty subset of \( \{ (x, y) \in \text{spec} \times \text{spec} : x \prec_T y \} \) for some partial order \( \prec_T \) as in Definition 2.1. Though the members of spec are finite partial functions, they cannot be written with some \( n \in \omega \) as a domain, since spec is uncountable and we want to allow arbitrary finite parts. Often properness of a tree creature forcing follows from the countability of \( H \). Note that our analogue
to \( H \) is not countable. In Section 3 we shall prove that the notions of forcing we introduce are proper for other reasons.

Nevertheless the simple creature in the next definition is a specific case for the value of a weak creature in the sense of 1.4 without item (1)(b), and the creature from the next definition can be seen as a case of a value and a distinction part of a weak creature.

**Definition 1.5.** (1) A simple creature is a tuple \( c = (i(c), \eta(c), \text{rge}(\text{val}(c))) \) with the following properties:

(a) The first component, \( i(c) \), is called the kind of \( c \) and is just a natural number.

(b) The second component, \( \eta(c) \), is called the base of \( c \). We require \((\eta(c) = \emptyset \text{ and } i(c) = 0) \text{ or } (i(c) \text{ is the smallest } i \text{ such that } |\text{dom}(\eta(c))| \leq n_{2,i-1})\), and \( \eta(c) \in \text{spec}_{n_{3,i-1}} \).

(c) The range of the value of \( c \), \( \text{rge}(\text{val}(c)) \), is a non-empty subset of \( \{ \eta \in \text{spec}_{n_{3,i}} : \eta \subseteq \eta \land |\text{dom}(\eta)| < n_{2,i} \} \), such that \( |\text{rge}(\text{val}(c))| < n_{1,i} \).

So we have \( \text{val}(c) = \{ \eta(c) \} \times \text{rge}(\text{val}(c)) \). That the domain is a singleton, is typical for tree-creating creatures.

(d) If \( \eta_1 \in \text{rge}(\text{val}(c)) \) and \( x \in \text{dom}(\eta_1) \setminus \text{dom}(\eta(c)) \) then there is some \( \eta_2 \in \text{rge}(\text{val}(c)) \) such that \( x \in \text{dom}(\eta_2) \rightarrow \eta_1(x) \neq \eta_2(x) \).

(2) A creature \( c^+ \) is a tuple \((i(c^+), \eta(c^+), \text{rge}(\text{val}(c^+)), k(c^+))\) where \((i(c^+), \eta(c^+), \text{rge}(\text{val}(c^+)))\) is a simple creature, and \( k(c^+) \in \omega \) is an additional coordinate.

(3) An (simple) \( i \)-creature is a (simple) creature with \( i(c^+) = i \) \((i(c) = i)\).

(4) If \( c^+ \) is a creature we mean by \( c \) the simple creature such that \( c^+ = (c, k(c^+)) \).

(5) The set of creatures is denoted by \( K^+ \), and the set of simple creatures is denoted by \( K \).

**Remark 1.6.** By \( [L, A] \) we have that \( \eta(c) = \bigcap \{ \eta : \eta \in \text{rge}(\text{val}(c)) \} \), and also \( i(c) \) is determined by \( \eta(c) \) and hence from \( \text{rge}(\text{val}(c)) \). Thus, in our specific case, every simple creature is determined by the range of its value.
For a real number \( r \) we let \( m = \lceil r \rceil \) be the smallest natural number such that \( m \geq r \). So, for negative numbers \( r \), \( \lceil r \rceil = 0 \). We let \( \lg \) denote the logarithm function to the base 2. Let \( \log_2(x) = \lceil \lg(x) \rceil \) for \( x > 0 \), and we set \( \log_2 0 = 0 \).

**Definition 1.7.**

1. For a simple \( i \)-creature \( c \) we define \( \text{nor}^0(c) \) as the maximal natural number \( k \) such that if \( a \subseteq n_{3,i} \) and \( |a| \leq k \) and \( B_0, \ldots, B_{k-1} \) are branches of \( T \), then there is \( \eta \in \text{val}(c) \) such that

   \[
   (\forall x \in (\bigcup_{\ell<k} B_\ell \cap \text{dom}(\eta))) \setminus \text{dom}(\eta(c)) (\eta(x) \notin a),
   \]

2. We let \( \text{nor}^*(c) = \log_2(\frac{n_{i}(\text{val}(c))}{n_{2,i}}) \), and \( \text{nor}^\frac{1}{2}(c) = \min(\text{nor}^0(c), \text{nor}^*(c)) \).

3. We define \( \text{nor}^1(c) = \log_2(\text{nor}^0(c)) \), and \( \text{nor}^2(c) = \log_2(\text{nor}^\frac{1}{2}(c)) \).

4. In order not to fall into specific computations, we use functions \( f \) that exhibit the following properties, in order to define norms on (non-simple) creatures:

   \[ (\ast)_1 \quad f \text{ is a two-place function}. \]

   \[ (\ast)_2 \quad f \text{ fulfils the following monotonicity properties: If } n_1 \geq n_2 \geq k_2 \geq k_1 \text{ then } f(n_1, k_1) \geq f(n_2, k_2). \]

   \[ (\ast)_3 \quad (\text{For the 2-bigness, see Definition 1.12}) \quad f(\frac{n}{2}, k) \geq f(n, k) - 1. \]

   \[ (\ast)_4 \quad n \leq k \rightarrow f(n, k) = 0. \]

   \[ (\ast)_5 \quad (\text{For the halving property, see Definition 1.3}) \quad \text{For all } n, k, i: \text{ If } f(n, k) \geq i + 1, \text{ then there is some } k'(n, k) = k' \text{ such that } k < k' < n \text{ and for all } n', \text{ if } k' < n' < n, \text{ then } \]

   \[ f(n', k) = f(n', k') + f(k', k) \geq f(n, k) - 1. \]

For example, \( f(n, k) = \lg(\frac{n}{k}) \) and \( k'(n, k) = \sqrt{nk} \), fulfill these conditions. For a creature \( c^+ \) we define its norm

\[ \text{nor}(c^+) = f(\text{nor}^\frac{1}{2}(c), k(c^+)). \]

**Remark 1.8.**

1. Note that property (1)(d) of simple creatures (Definition 1.4) follows from \( \text{nor}^0(c) > 0 \). So we will not check this property any more, but restrict ourselves to creatures with strictly positive \( \text{nor}^0 \).

2. Definition 1.7 speaks about infinitely many requirements, by ranging over all \( k \)-tuples of branches of \( T \). However, at a crucial point in the proof of Claim 1.10 this boils down to counting the possibilities for \( a \).
The next claim shows that we can extend the functions in the value of a creature and at the same time decrease the norm of the creature only by a small amount.

**Claim 1.9.** Assume that

(a) \( \eta^* \in \text{spec} \),
(b) \( c \) is a simple \( i \)-creature with base \( \eta^* \), \( \text{nor}^0(c) > 0 \),
(c) \( k^* > 1, |\text{rge}(\text{val}(c))| \cdot k^* \leq n_{1,i} \),
(d) for each \( \eta \in \text{rge}(\text{val}(c)) \) and \( k < k^* \) we are given \( \eta \subseteq \rho_{\eta,k} \in \text{spec}_{n_{3,i}} \) with \( |\text{dom}(\rho_{\eta,k})| < n_{2,i} \),
(e) for each \( \eta \in \text{rge}(\text{val}(c)) \), if \( k_1 < k_2 < k^* \) and \( x_1 \in \text{dom}(\rho_{\eta,k_1}) \setminus \text{dom}(\eta) \) and \( x_2 \in \text{dom}(\rho_{\eta,k_2}) \setminus \text{dom}(\eta) \), then \( x_1, x_2 \) are \( < \text{T}_A \)-incomparable,
(f) \( \ell^* = \max\{|\text{dom}(\rho_{\eta,k})| + 1 : \eta \in \text{rge}(\text{val}(c)) \wedge k < k^*\} \).

Then there is a simple \( i \)-creature \( d \) given by

\[ \text{rge}(\text{val}(d)) = \{\rho_{\eta,k} : k < k^*, \eta \in \text{rge}(\text{val}(c))\}. \]

We have \( \eta(d) = \eta^* \), and \( \text{nor}^0(d) \geq m_0 \overset{\text{def}}{=} \min\{\text{nor}^0(c), \log_2(\frac{n_{2,i}}{\ell^*}), k^* - 1\} \).

**Proof.** First of all we are to check Definition 1.3(1). Clauses (a),(b), and (c) follow immediately from the premises of the claim. \( d \) satisfies clause (d): This follows from the proof of the inequality for \( \text{nor}^0(d) \) below. From premise (e) and from the properties of \( c \) it follows that \( \eta(d) = \eta^* \).

Now for the norm: We check clause (a) of Definition 1.7. Let branches \( B_0, \ldots, B_{m_0-1} \) of \( T_A \) and a set \( a \subseteq n_{3,i} \) be given, \( |a| \leq m_0 \). Since \( m_0 \leq \text{nor}^0(c) \), there is some \( \eta \in \text{rge}(\text{val}(c)) \) such that \( (\forall x \in (\bigcup_{\ell<m_0} B_\ell) \cap \text{dom}(\eta)) \setminus \text{dom}(\eta(c)))(\eta(x) \notin a) \). We fix such an \( \eta \). Now for each \( \ell < m_0 \), we let

\[ w_{\eta,\ell} = \{j < k^* : \exists x \in B_\ell \cap \text{dom}(\rho_{\eta,j}) \setminus \text{dom}(\eta)\}. \]

Now we have that \( |w_{\eta,\ell}| \leq 1 \) because otherwise we would have \( k_1 < k_2 < k^* \) in \( w_{\eta,\ell} \) and \( x_i \in B_\ell \cap \text{dom}(\rho_{\eta,k_i}) \setminus \text{dom}(\eta) \). As \( x_1 \) and \( x_2 \) are \( < \text{T}_A \)-comparable, this is contradicting the requirement (d) of 1.9.

Since \( m_0 < k^* \), there is some \( j \in k^* \setminus \bigcup_{\ell<m_0} w_{\eta,\ell} \). For such a \( j \), \( \rho_{\eta,j} \) is as required.
We check clause (β) of Definition 1.7. We take the \( \rho_{\eta,j} \) as chosen above. Then we have

\[
|\text{dom}(\rho_{\eta,j})| \leq \frac{\ell^*}{n_{2,i}} = \frac{1}{2^{\log_2(n_{2,i})}} \leq \frac{1}{2^{m_0}},
\]

as \( m_0 \leq \log_2 \left( \frac{n_{2,i}}{\ell^*} \right) \).

Whereas the previous claim will be used only in Section 3 in the proof on properness (see Claim 3.8), the following two claims will be used in the next section for density arguments in the forcings built from creatures.

**Claim 1.10.** Assume

(a) \( c \) is a simple \( i \)-creature.

(b) \( k = \text{nor}^0(c) \geq 1 \) and \( k \leq n_{1,i} \).

(c) \( x_0, \ldots, x_{m-1} \in T, 1 \leq m \leq \min(k, \frac{n_{2,i}}{2^k}) \).

(d) \( |\text{rge}(\text{val}(c))| \cdot \binom{k}{m} \leq n_{1,i} \).

(e) If \( \eta \in \text{rge}(\text{val}(c)) \), then \( |\{y \in \text{dom}(\eta) : (\exists m' < m)(x_{m'} < T y)\}| < i \).

Then there is \( d \) such that \( \eta(d) = \eta(c) \) and

\[
\text{rge}(\text{val}(d)) \subseteq \{\nu \in \text{spec}_{3,i} : |\text{dom}(\nu)| \leq n_{2,i},
\]

\[
(\exists \eta \in \text{rge}(\text{val}(c)))(\eta \subseteq \nu \land \text{dom}(\nu) = \text{dom}(\eta) \cup \{x_0, \ldots, x_{m-1}\})
\]

such that \( |\text{rge}(\text{val}(d))| \leq n_{1,i} \) and such that for each \( \eta \) there are sufficiently many (\( \binom{k}{m} \) suitable ones suffice) extensions \( \nu \in \text{rge}(\text{val}(d)) \) as described in the first paragraph of the proof.

Moreover we can choose \( d \) such that

(a) \( d \) is a simple \( i \)-creature.

(b) \( \text{nor}^0(d) \geq k - m \).

**Proof.** We take for \( m' < m \), \( z_{m'} \in n_{3,i} \setminus (\bigcup_{\eta \in \text{rge}(\text{val}(c))} \{\eta(y) : x < T_A y\} \cup \{z_{m''} : m'' < m'\}) \). Since \( |\text{rge}(\text{val}(c))| < n_{1,i} \) and by (d) and since by (1.1) \( (i-1) \cdot n_{1,i} + k < n_{3,i} \) there is such a \( z_{m'} \), and indeed, which is important for \( d \) being a creature and for its norm, there are at least \( k - m' \) such \( z_{m'}'s \) for every \( \eta \in \text{rge}(\text{val}(c)) \). We take all these choices \( \nu_{\eta,z} = \eta \cup \{(x_{m'}, z_{m'}) : m' < m\} \) into \( \text{rge}(\text{val}(d)) \). Hence we can choose all \( \nu_{\eta,z} \) so that we avoid any given \( a \) of size \( k - m \) with all the \( z_{m'}'s \).

Now we check the norm: Let \( B_0, \ldots, B_{k-m-1} \) be branches of \( T_A \) and let \( a \subseteq n_{3,i(e)}, |a| \leq k - m \). We have to find \( \nu \in \text{rge}(\text{val}(d)) \) such that (\( \forall \ell < k - \)
For \( m' < m \) we choose \( B_{k - m - 1 + m'} \), a branch containing \( x_{m'} \). We take for \( m' < m \), \( z_{m'} \in n_{3,i} \setminus (\bigcup_{\eta \in \rg(\val(c))} \{ \eta(y) : x < T_a y \}) \cup a \cup \{ z_{m''} : m'' < m' \} \). We set \( a' = a \cup \{ z_0, \ldots, z_{m-1} \} \).

By premise (b), we find \( \eta \in \rg(\val(c)) \) for \( a' \) and \( B_0, \ldots, B_{k-1} \) such that

\[
(1) \quad (\forall \ell < k - 1)(\forall x \in \dom(\eta) \cap B_k \setminus \dom(\eta(c)))(\eta(x) \notin a') \quad \text{and} \\
(2) \quad |\dom(\eta)| \leq \frac{n_{2,i}}{2^k}.
\]

Now \( \nu_{n,2} \) is a witness for the norm. We have \( \frac{n_{2,i}}{2^k} + m \leq \frac{n_{2,i}}{2^m} \), which follows from the premises on \( m \). The only thing to show is that \( \nu \) is really a specialization function. So let \( y \in \dom(\eta) \) and \( y < T_a x_{m'} \). Then \( \nu(y) = \eta(y) \neq \nu(x_{m'}) = z_{m'} \), because \( y \) is on the branch leading to \( x_{m'} \) and because of (1). If \( y > T_a x_{m'} \), then we have taken care of \( y \) simultaneously for all \( \eta \)'s by our choice of the \( z_{m'} \)'s.

An analogous version of Claim 1.10 with \( \nor^{\frac{1}{2}} \) instead of \( \nor^0 \) holds as well. The analogous requirements to premises (c) and (d) are even easier: If we work with \( \nor^{\frac{1}{2}} \) and use \( n_{1,i} \leq n_{2,i} \) from equation (1.4) in the Choice 1.3, then \( 1 \leq m \leq k \) is enough in premise (c). Premise (d) is included in \( \nor^{\frac{1}{2}}(c) = k \).

Suppose we have filled up the range of the value of a creature according to one of the previous claims. Then we want that these extended functions can serve as bases for suitable creatures as well. This is provided by the next claim.

**Claim 1.11.** Assume that

\( (a) \quad c \) is a simple \( i \)-creature.

\( (b) \quad k = \nor^0(c) \geq 1. \)

\( (c) \quad \eta^* \supseteq \eta(c), \eta^* \in \spec_{n_{3,i}} \) (note that we do not suppose that \( \eta^* \in \rg(\val(c)) \)). Furthermore we assume \( |\dom(\eta^*)| \leq n_{2,i(c)-1} \).

\( (d) \quad \) We set

\[ \ell^*_2 = |\dom(\eta^*) \setminus \dom(\eta(c))|, \]

and

\[ \ell^*_1 = |\{ y : (\exists \nu \in \rg(\val(c)))(y \in \dom(\nu) \setminus \dom(\eta(c))) \wedge (\exists x \in \dom(\eta^*) \setminus \dom(\eta(c)))(x < T_a y) \}|. \]

and we assume that \( \ell^*_1 + \ell^*_2 < \nor^0(c) \).
We define $d$ by $\eta(d) = \eta^*$ and
\[
\text{rge}(\text{val}(d)) = \{\nu \cup \eta^* : \nu \in \text{rge}(\text{val}(c)) \land \nu \cup \eta^* \in \text{spec}_{n_3,i} \mid \text{dom}(\nu \cup \eta^*) \leq n_{2,i}\}.
\]
Then
1. $d$ is a simple $i$-creature.
2. $\text{nор}^0(d) \geq \text{nор}^0(c) - \ell^*_2 - \ell_1^*$.

Proof. Item (a) follows from the requirements on $\eta^*$ and from the estimates on the norm, see below. For item (b), we set $k = \text{nор}^0(c) - \ell_1^* - \ell_2^*$. We let $B_0, \ldots, B_{k-1}$ be branches of $T_A$ and $a \subseteq n_{3,i(c)} \mid |a| \leq k$. We set $\ell^* = \ell_1^* + \ell_2^*$. We let $(y_\ell : \ell < \ell_1^*)$ list $Y = \{y : \exists \nu(\nu \in \text{rge}(\text{val}(c)) \land y \in \text{dom}(\nu)) \land \exists x(\exists x \in \text{dom}(\eta^*) \land \text{dom}(\eta(c)) \land x \leq T_A y)\}$ without repetition. Let $B_k, \ldots, B_{k+\ell^*-1}$ be branches of $T_A$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let $(x_\ell : \ell < \ell_2^*)$ list $\text{dom}(\eta^*) \setminus \text{dom}(\eta(c))$. Take for $\ell < \ell_2^*$, $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell) : \ell < \ell_2^*\}$. Since $\text{nор}^0(c) \geq k + \ell^*$ there is some $\nu \in \text{rge}(\text{val}(c))$ such that $\forall x \in ((\text{dom}(\nu) \setminus \text{dom}(\eta(c))) \cap \bigcup_{\ell < \ell_1^*} B_\ell)(\nu(x) \notin a')$. Then, if $x \notin \text{dom}(\eta^*)$, $(\nu \cup \eta^*)(x) \notin a$. Moreover $|\text{dom}(\nu \cup \eta^*)| \leq \frac{n_{2,i}}{2^k} + \ell_2^* \leq \frac{n_{2,i} + 1}{2^k}$, if $\frac{n_{2,i}}{2^k}$ is large enough. (This premise will always be fulfilled in our applications, because $n_{1,i} \leq n_{2,i}$.) We just perform all our operations on forcing conditions only at high levels $i$, compared to the size of the given $\eta^*$. This will be done in the next section.

We have to show that if $\nu \cup \eta^*$ is a partial specialization: Since $\eta^*$ and $\nu$ are specialization maps, we have to consider only the case $x \in \text{dom}(\eta^*) \setminus \text{dom}(\eta(c))$ and $(y \in Y$ or $(y \in \text{dom}(\nu) \setminus \text{dom}(\eta^*))$ and $y <_{T_A} x)$. If $y \in Y$, then we have $\nu(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$. If $y \in \text{dom}(\nu) \setminus \text{dom}(\eta^*)$ and $y <_{T_A} x$, then $y$ is in a branch leading to some $x_\ell, \ell < \ell_2^*$, and hence again $\nu(y) \neq \eta^*(x_\ell)$, $\ell < \ell_2^*$.

In the applications, the proofs of the density properties, $\ell_2^*$ will be small compared to the norm (we add $\ell_2^*$ points to the domain of the functions in the range of the value of a creature with sufficiently high norm) and $\ell_1^* \leq |u|$, were $u$ is the set that sticks out of $(T_A)_{<\alpha(p)}$ (see Definition 2.2 and Remark 2.3). We will suppose that these two are small in comparison to $\text{nор}^0(c)$, so that the premises for Claim 1.11 are fulfilled.

We also need Claims 1.9, 1.10 and 1.11 for $\text{nор}^0$ instead of $\text{nор}^0$. This is proved by easy but a bit tedious accounting of $\text{nор}^0(c) = \log_2(n_{1,i(c)}/\text{val}(c))$. Just see that $|\text{val}(c)|$ increases only in a controllable way in Claim 1.3 and in Claim 1.10 and
does not increase at all in Claim 1.11. Hence also if nor\(^*\) is the part determining the minimum in nor\(^\frac{1}{2}\), the latter falls at most by \(k^*\) in 1.9 from \(c\) to \(d\), and at most by \(\log_2\left(\binom{k}{m}\right)\) in 1.10 and does not decrease in 1.1.

The next claim will help to find large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions.

**Claim 1.12.** (1) The 2-bigness property [4, Definition 2.3.2]. If \(c\) is a simple \(i\)-creature with \(\text{nor}^1(c) \geq k + 1\), and \(c_1, c_2\) are simple \(i\)-creatures such that \(\text{val}(c) = \text{val}(c_1) \cup \text{val}(c_2)\), then \(\text{nor}^1(c_1) \geq k\) or \(\text{nor}^1(c_2) \geq k\). The same holds for \(\text{nor}^2\).

(2) If \(c^+\) is a \(i\)-creature with \(\text{nor}(c) \geq k + 1\), and \(c^+_1, c^+_2\) are \(i\)-creatures such that \(\text{val}(c) = \text{val}(c^+_1) \cup \text{val}(c^+_2)\), and \(k(c^+_1) = k(c^+_2) = k(c^+)\), then \(\text{nor}(c^+_1) \geq k\) or \(\text{nor}(c^+_2) \geq k\).

**Proof.** (1) We first consider \(\text{nor}^0\). Let \(j = \left\lceil \frac{k}{2} \right\rceil - 1\). We suppose that \(\text{nor}^0(c_1) \leq j\) and \(\text{nor}^0(c_2) \leq j\) and derive a contradiction: For \(\ell = 1, 2\) let branches \(B^\ell_0, \ldots, B^\ell_{j-1}\) and sets \(a^\ell \subseteq n_{3,i}\) exemplify this.

Let \(a = a^1 \cup a^2\) and let \(\eta \in \text{rge}(\text{val}(c))\) be such that for all \(x \in (\text{dom}(\eta) \cap \bigcup_{\ell=1,2} B^\ell_j) \setminus \text{dom}(\eta(c))\) we have \(x \notin a\). But then for that \(\ell \in \{1, 2\}\) for which \(\eta \in \text{rge}(\text{val}(c_\ell))\) we get a contradiction. Hence (1) follows for \(\text{nor}^1\) increases or stays when taking subsets of \(\text{val}(c)\), and hence we have the analogous result for \(\text{nor}^\frac{1}{2}\).

Since the \(k\)-components of the creatures coincide, Part (2) follows from the behaviour of \(\text{nor}^\frac{1}{2}\) that was shown in part (1) and from the requirements on \(f\) in Definition 1.7(4): \(f(n, k) \geq f(n, k) - 1\).

## 2. Forcing with tree-creatures

Now we define a notion of forcing with \(\omega\)-trees \(\langle c_t : t \in (T, \triangleleft_T) \rangle\) as conditions. The nodes \(t\) of these trees \((T, \triangleleft_T) = (\text{dom}(p), \triangleleft_p)\) and their immediate successors are described by certain creatures \(c_t\) from Definition 1.3.

First we collect some general notation about trees. The trees here are not the Aronszajn trees of the first section, but trees \(T\) of finite partial specialization functions, ordered by \(\triangleleft_T\) which is a subrelation of \(\subseteq\). Some of these trees will serve as forcing conditions.
Definition 2.1.  
1. A tree \((T, \preceq_T)\) is a set \(T \subseteq \text{spec}\), such that for any \(\eta \in T\), 
   \(\{\nu : \nu \preceq_T \eta\}, \preceq_T\) is a finite linear order and such that in \(T\) there is 
   one least element, called the root, \(\text{rt}(T)\). If \(\eta \preceq_T \nu\) then \(\eta \subset \nu\). Every 
   \(\eta \in T \setminus \text{rt}(T)\) can has just one immediate \(\preceq_T\)-predecessor in \(T\). We shall 
   only work with finitely branching trees.

2. We define the successors of \(\eta\) in \(T\), the restriction of \(T\) to \(\eta\), the splitting 
   points of \(T\) and the maximal points of \(T\) by 
   \[ \text{suc}_T(\eta) = \{\nu \in T : \eta \preceq_T \nu \land \neg(\exists \rho \in T)(\eta \preceq_T \rho \preceq_T \nu)\}, \]
   \[ T^{(\eta)} = \{\nu \in T : \eta \preceq_T \nu\}. \]
   split\((T) = \{\eta \in T : |\text{suc}_T(\eta)| \geq 2\}, \]
   \[ \text{max}(T) = \{\nu \in T : \neg(\exists \rho \in T)(\nu \preceq_T \rho)\}. \]

3. The \(n\)-th level of \(T\) is 
   \[ T^{[n]} = \{\eta \in T : \eta \text{ has } n \preceq_T\text{-predecessors}\}. \]

The set of all branches through \(T\) is 
   \[ \text{lim}(T) = \{(\eta_k : k < \ell) : \ell \leq \omega \land (\forall k < \ell)(\eta_k \in T^{[k]}) \]
   \[ \land (\forall k < \ell - 1)(\eta_k \preceq_T \eta_{k+1}) \]
   \[ \land \neg(\exists \eta_\ell \in T)(\forall k < \ell)(\eta_k \preceq_T \eta_\ell)\}. \]

A tree is well-founded if there are no infinite branches through it.

4. A subset \(F\) of \(T\) is called a front of \(T\) if every branch of \(T\) passes through 
   this set, and the set consists of \(\prec_T\)-incomparable elements.

Definition 2.2. We define a notion of forcing \(Q = Q_{T_A}\). \(p \in Q\) iff 

1. \(p\) is a function from a subset of \(\text{spec} = \text{spec}^{T_A}\) (see Definition 1.12) to \(\omega\).

2. \(p^{[\ell]} = (\text{dom}(p), \prec_p)\) is a tree with \(\omega\) levels, the \(\ell\)-th level of which is denoted 
   by \(p^{[\ell]}\).

3. \(p^{[\ell]}\) has a root, the unique element of level 0, called \(\text{rt}(p)\).

4. We let 
   \[ i(p) \overset{\text{def}}{=} \min\{i : |\text{dom}(\text{rt}(p))| \leq n_{2,i-1}\}. \]
Then for any \( \ell < \omega \) and \( \eta \in p[\ell] \) the set
\[
\text{suc}_p(\eta) = \{ \nu \in p[\ell+1] : \eta \subseteq \nu \}
\]
is \( \text{rge}(\text{val}(c)) \) for a simple \((i(p) + \ell)\)-creature \( c \) with base \( \eta \). We denote this simple creature by \( c_{p,\eta} \) and let \( c_{p,\eta}^+ = (c_{p,\eta}, p(\eta)) \). Furthermore, we require \( p(\eta) \leq \text{nor}_{\frac{1}{2}}(c_{p,\eta}) \).

\( (v) \) If \( \eta \in \text{dom}(p) \) and \( \nu \in \text{dom}(p) \) and if \( \eta \cup \nu \in \text{spec} \), then \( \eta \cup \nu \in \text{dom}(p) \). It is a superset of both \( \tau \) and of \( \nu \), but in \( \prec_p \) it has only one predecessor. Every \( \eta \in \text{spec} \) appears at most once in \( T \).

\( (vi) \) For some \( k < \omega \) for every \( \eta \in p[k] \) there is \( \alpha < \omega_1 \) and a finite \( u \in T_A \setminus (T_A)_{<\alpha} \) such that for every \( \omega \)-branch \( \langle \eta_k : \ell < \omega \rangle \) of \( p[\ell] \) satisfying \( \eta_k = \eta \) we have \( \bigcup_{\ell \in \omega} \text{dom}(\eta_k) \setminus u = (T_A)_{<\alpha} \).

\( (vii) \) For every \( \omega \)-branch \( \langle \eta_k : \ell \in \omega \rangle \) of \( p[\ell] \) we have \( \lim_{\ell \rightarrow \omega} \text{nor}(c_{p,\eta_k}^+) = \omega \).

The order \( \leq \leq_Q \) is given by letting \( p \leq q \) (\( q \) is stronger than \( p \), we follow the Jerusalem convention) iff \( i(p) \leq i(q) \) and there is a projection \( \text{pr}_{p,q} \) which satisfies
\[
(a) \quad \text{pr}_{q,p} \text{ is a function from } \text{dom}(q) \text{ to } \text{dom}(p).
\]
\[
(b) \quad \eta \in q[\ell] \Rightarrow \text{pr}_{q,p}(\eta) \in p[\ell+i(q)-i(p)].
\]
\[
(c) \quad \text{If } \eta_1, \eta_2 \text{ are both in } q[\ell] \text{ and if } \eta_1 \preceq_q \eta_2, \text{ then } \text{pr}_{q,p}(\eta_1) \preceq_p \text{pr}_{p,q}(\eta_2).
\]
\[
(d) \quad q(\eta) \geq p(\text{pr}_{q,p}(\eta)).
\]
\[
(e) \quad \text{If } \eta \in q[\ell] \text{ then } \eta \supseteq \text{pr}_{q,p}(\eta).
\]
\[
(f) \quad \text{If } \nu \in q[\ell] \text{ and } \rho \in q[\ell+1] \text{ and } \nu \subseteq \rho, \text{ pr}_{q,p}(\nu) = \eta, \text{ pr}_{q,p}(\rho) = \tau, \text{ then } \text{dom}(\tau) \cap \text{dom}(\nu) = \text{dom}(\eta).
\]

**Definition 2.3.** For \( p \in Q \) and \( \eta \in \text{dom}(p) \) we let
\[
p(\eta) = p \upharpoonright \{ \rho \in \text{dom}(p) : \eta \subseteq \rho \}.
\]

Let us give some informal description of the \( \leq \)-relation in \( Q \): The stronger conditions’ domain is via \( \text{pr}_{q,p} \) mapped homomorphically w.r.t. the tree orders into \( p(\text{pr}_{q,p}(\text{rt}(q))) \). The stem can grow as well. According to \((b)\), the projection preserves the levels in the trees but for one jump in heights (the \( \ell \)'s in \( p[\ell] \)), due to a possible lengthening of the stem. The partial specialization functions sitting on the nodes of the tree are extended (possibly by more than one extension to one function) in \( q \) as to compared with the ones attached to the image under \( \text{pr}, \)
but by (b) the extensions are so small and so few that it preserves the kind \( i \) of
the creature given by the node and its successors, and according to (f) the new
part of the domain of the extension is disjoint from the domains of the old partial
specification functions living higher up in the new tree.

Let us compare our setting with the forcings given in the book [4]: There the
\( \leq \)-relation is based on a sub-composition function (whose definition is not used
here, because we just deal with one particular forcing notion) whose inputs are
well-founded subtrees of the weaker condition. This well-foundedness condition
[4, 1.1.3] is not fulfilled: if we look at (e) and (f) in the definition of \( \leq \) we see
that we have to look at all the branches of \( p \) that are in the range of \( \text{pr}_{q,p} \) in order
to see whether some \( \nu \in q[\ell] \) fulfills (f) of the definition of \( p \leq q \). On the other
hand, the projections shift all the levels by the same amount \( i(q) - i(p) \), and are
not arbitrary finite contractions as in most of the forcings in the book [4].

**Definition 2.4.** (1) \( p \in Q \) is called normal iff for every \( \omega \)-branch \( \langle \eta_\ell : \ell \in \omega \rangle \)
of \( p[\ell] \) the sequence \( \langle \text{nor}(c_{p,\eta_\ell}) : \ell \in \omega \rangle \) is non-decreasing.

(2) \( p \in Q \) is called smooth iff in clause (vi) of Definition 2.2 the number \( k \) is
0 and \( u \) is empty.

(3) \( p \in Q \) is called weakly smooth iff in clause (vi) of Definition 2.2 the number
\( k \) is 0.

**Remark 2.5.** If \( p \in Q \) is smooth then there is some \( \alpha < \omega_1 \) such that for every
\( \omega \)-branch \( \langle \eta_\ell : \ell \in \omega \rangle \) of \( p[\ell] \) we have \( \bigcup_{\ell < \omega} \text{dom}(\eta_\ell) = T_{< \alpha} \). This \( \alpha \) is denoted by
\( \alpha(p) \).

**Fact 2.6.** (1) Suppose that we have strengthened 2.2(f) to: If \( \nu \in q[\ell] \) and
\( \rho \in q[\ell+1] \) and \( \nu \subseteq \rho \), \( \text{pr}_{q,p}(\nu) = \eta \), \( \tau \supseteq \eta \), \( \tau \in p \), then \( \text{dom}(\tau) \cap \text{dom}(\nu) = \text{dom}(\eta) \). So, we replaced \( \text{pr}_{q,p}(\rho) = \tau \) by \( \tau \supseteq \eta \), \( \tau \in p \) and thus have
information on the \( \tau \in \text{dom}(p) \setminus \text{rge}(\text{pr}_{q,p}) \). Then: If \( p \leq q \) and \( \eta \in \text{dom}(p) \), \( \nu \in \text{dom}(q) \), \( \eta = \text{pr}_{q,p}(\nu) \) and \( \eta \prec \tau \in \text{dom}(p) \), then \( \text{dom}(\nu) \cap \text{dom}(\tau) = \text{dom}(\eta) \).

(2) If \( p \leq q \) and \( p \) is weakly smooth then
\( \nu \in \text{dom}(q) \rightarrow \text{dom}(\nu) \cap (T_{< \alpha(p)} \cup u) = \text{dom}(\text{pr}_{q,p}(\nu)) \).

**Proof.** (1) follows from clauses (e) and the stronger form of clause (f) of the definition
of \( p \leq_q q \). (2): If \( p \) is weakly smooth, 2.2(f) and its stronger form from
the premise of (1) coincide, and hence (2) follows from (1), because each branch of \( p \) has the same union of domains.

**Definition 2.7.** For \( 0 \leq n < \omega \) we define the partial order \( \leq_n \) on \( Q \) by letting \( p \leq_n q \) iff

(i) \( p \leq q \),
(ii) \( i(p) = i(q) \),
(iii) \( p^{[\ell]} = q^{[\ell]} \) for \( \ell \leq n \), and \( p \upharpoonright \bigcup_{\ell \leq n} p^{[\ell]} = q \upharpoonright \bigcup_{\ell < n} q^{[\ell]} \), in particular \( \text{rt}(p) = \text{rt}(q) \),
(iv) if \( \text{pr}_{q,p}(\eta) = \nu \), then
   - \( \eta = \nu \) and \( c^+_{q,\eta} = c^+_{p,\nu} \)
   - or \( \text{nor}(c^+_{q,\eta}) \geq n \).

We state and prove some basic properties of the notions defined above.

**Claim 2.8.** (1) If \( p \leq q \), then \( \text{pr}_{q,p} \) is unique.

(2) If \( p \in Q \) and \( \ell \in \omega \) then \( |p^{[\ell]}| < n_{1,i(p)+\ell} \).

(3) \( (Q, \leq_Q) \) is a partial order.

(4) If \( p \leq q \) and \( \text{pr}_{q,p}(\eta) = \nu \), then \( i(c_{q,\eta}) = i(c_{p,\nu}) \).

(5) If \( p \leq q \) and \( \text{pr}_{q,p}(\eta) = \nu \), then \( \text{nor}^0(c_{q,\eta}) \leq \text{nor}^0(c_{p,\nu}) \).

(6) \( (Q, \leq_n) \) is a partial order.

(7) \( p \leq_{n+1} q \rightarrow p \leq_n q \rightarrow p \leq q \).

(8) If \( c \) is a simple \( i \)-creature with \( k \leq \text{nor}^0(c) \), then there is a simple \( i \)-creature \( c' \) with \( k = \text{nor}^0(c') \) and \( \text{val}(c') \subseteq \text{val}(c) \).

(9) For every \( p \in Q \) there is a \( q \geq p \) such that for all \( \eta \) and \( \nu \)

\[
\text{pr}_{q,p}(\eta) = \nu \rightarrow \text{nor}^0(c_{q,\eta}) = \min\{\text{nor}^0(c_{p,\rho}) : \nu \leq \rho \in p\}.
\]

(10) For every (not necessarily normal) \( p \) we have that \( \lim_{n \to \omega} \min\{\text{nor}(c^+_{p,\eta}) : \eta \in p^{[n]}\} = \infty \).

(11) If \( p \in Q \) and \( \eta \in p^{[\ell]} \) then \( |\text{dom}(\eta)| < n_{2,i(p)+\ell-1} \) or \( \ell = 0 \) and \( i(p) = 0 \) and \( \eta = \emptyset \).
Proof. (1) By induction on \( \ell \) we show that \( pr_{q,p} \upharpoonright \bigcup_{\ell \leq \ell} p[\ell] \) is unique: It is easy to see that \( pr_{q,p}(rt(q)) \) is the \( \subseteq \)-maximal element of \( p \) that is a subfunction of \( rt(q) \). By Definition 2.2(5) such a maximum exists. Then we proceed level by level in \( q[\ell] \), and again Definition 2.2(5) yields uniqueness of \( pr_{q,p} \).

(2) This is also proved by induction on \( \ell \). Note that for \( \eta \in p[\ell] \) we have that \( |rge(val((\eta)))| \leq n_{1,i(p)+\ell-1} \). We have \( |p[\ell]| = 1 \) and by Definition 1.5(c), \( |p[\ell+1]| \leq |p[\ell]| \cdot n_{1,i(p)+\ell} \leq n_{1,i(p)+\ell} \cdot n_{1,i(p)+\ell} \leq n_{1,i(p)+\ell+1} \), by equation (1.3).

(3) Given \( p \leq q \) and \( q \leq r \) we define \( pr_{r,p} = pr_{q,p} \circ pr_{r,q} \). It is easily seen that this function is as required.

(4) Let \( \ell \) be such that \( \eta \in q[\ell] \). Then \( i(c_{q,\eta}) = i(q) + \ell \) and \( \nu \in p^{[\ell+i(q)-i(p)]} \). Hence \( i(c_{p,\nu}) = i(p) + \ell + i(q) - i(p) = i(q) + \ell \).

(5) Suppose \( nor^0(c_{q,\eta}) > nor^0(c_{p,\nu}) \). Let \( k = nor^0(c_{q,\eta}) \) and let \( i = i(c_{q,\eta}) = i(c_{p,\nu}) \). Suppose that \( a \subseteq n_{3,i} \) and the branches \( B_0, \ldots, B_{k-1} \) of \( T \) exemplify that \( nor^0(c_{p,\nu}) < k \). Hence for all \( \tau \in suc_p(\nu) \)

\[(\alpha) \quad \text{there is } x \in (\dom(\tau) \cap \bigcup_{\ell=0}^{k-1} B_\ell) \setminus \dom(\nu) \text{ such that } \tau(x) \in a, \text{ or}
\]

\[(\beta) \quad |\dom(\tau)| \geq \frac{n_{q,i}}{2^k}.
\]

Let \( \tau \in suc_q(\nu) \), and \( pr_{q,p}(\tau) = \tau' \). Suppose (\( \alpha \)) is the case for \( \tau' \). Then the same \( a \) and \( B_0, \ldots, B_{k-1} \) exemplify (\( \alpha \)) for \( \tau \) and \( c_{q,\eta} \), because we have \( \eta \supseteq \nu = pr_{q,p}(\eta) \) and \( pr_{q,p}(suc_\eta(\eta)) \subset suc_p(\nu) \). The same \( x \) will show that \( \exists x \in (\dom(\tau) \cap \bigcup_{\ell=0}^{k-1} B_\ell) \setminus \dom(\eta) \) such that \( \tau(x) \in a \), if we verify that \( x \notin \dom(\eta) \).

But we have for all \( \tau \in suc_q(\eta) \) that \( \dom(\eta) \cap \dom(\tau) = \dom(\nu) \) by 2.2(f), and hence \( x \notin \dom(\eta) \).

Suppose (\( \beta \)) is the case for \( \tau' \). Then \( \tau' \in suc_p(\nu) \) and \( \tau \supseteq \tau' \), and hence \( |\dom(\tau)| \geq \frac{n_{q,i}}{2^k} \).

(6) Suppose that \( p \leq_n q \leq_n r \) and \( pr_{r,q}(\sigma) = \eta \) and \( pr_{q,p}(\eta) = \nu \). By (1) and (3) we have that \( pr_{r,p}(\sigma) = \nu \), and now it is easy to check the requirements for \( p \leq_n r \).

(7) Obvious.

(8) We may assume that \( nor^0(c) > k \), because otherwise \( c \) itself is as required. Look at

\[
Y = \{ d : d \text{ is a simple } i\text{-creature and } val(d) \neq \emptyset \text{ and } nor^0(d) \geq k \text{ and } val(d) \subseteq val(c) \}.
\]
Since $c \in Y$, it is non-empty, and it has a member $d$ with a minimal number of elements. We assume towards a contradiction that $\text{nor}^0(d) > k$. We choose $\eta^* \in \text{rge} (\text{val}(d))$. We let $\text{rge}(\text{val}(d^*)) = \text{rge}(\text{val}(d)) \setminus \{\eta^*\}$.

Claim: $d^* \neq \emptyset$. Otherwise we choose $x \in \text{dom}(\eta^*) \setminus \text{dom}(\eta(c))$, and such an $x$ exists by clause (d) of $(\text{KD})$. Now we let $B_0$ be a branch of $T$ to which $x$ belongs and set $a = \{\eta^*(x)\}$. They witness that $\text{nor}^0(c) \not\geq 1$, so $\text{nor}^0(c) = 0$, which contradicts the assumption that $\text{nor}^0(c) = k > 0$.

Claim: $\text{nor}^0(d^*) \geq k$. Otherwise there are branches $B_0, \ldots, B_{k-1}$ and a set $a \subseteq n_{3,i}$ witnessing $\text{nor}^0(d^*) \not\geq k$. Let $x \in \text{dom}(\eta^*) \setminus \text{dom}(\eta(c))$ and let $B_k$ be a branch such that $x \in B_k$ and set $a' = a \cup \{\eta^*(x)\}$. The $B_0, \ldots, B_k$ and $a'$ witness that $\text{nor}^0(c) \not\geq k + 1$. Hence $d^*$ is a member of $Y$ with fewer elements than $d$, contradiction.

(9) Follows from (8). We can even take $q \subseteq p$. First see: For no $m$ such the set \{\eta \in p such that for densely (in p[\eta]) many \eta' \geq_p \eta we have that \text{nor}^0(c_{p,\eta'}) < m\}. is anywhere dense. Otherwise we choose a branch \langle \eta_\ell : \ell \in \omega \rangle such that there is some $m \in \omega$ such that for all $\ell < \omega$, $\text{nor}^0(c_{p,\eta_\ell}) < m$.

Now we choose by induction of $\ell$, $\text{dom}(q_\ell) \subseteq \text{dom}(p)$, such that $q_\ell$ is has no infinite branch and hence is finite, though we do not have a bound on its height.

First step: Say $\min \{\text{nor}^0(c_{p,\eta}) : \eta \in \text{dom}(p)\} = k$ and it is reached in $\eta \in \text{dom}(p)$. We take $q^{[0]} = \{\eta\}$.

\begin{itemize}
  \item [(1)] Then we take for any $\eta' \in \text{rge}(\text{val}(c_{p,\eta}))$ some $\eta'' \supseteq \eta'$ such that $\eta'' \in p$ and such that for all $\tilde{\eta} \supseteq \eta''$, if $\tilde{\eta} \in p$ then $\text{nor}^0(c_{p,\tilde{\eta}}) \geq k + 1$. By the mentioned nowhere-density result, this is possible. We put such an $\eta''$ in $q^{[\ell]}$, if it is in $p^{[\ell]}$.
  \item [(2)] Then we look at the $\nu$ in the branch between $\eta$ and $\eta''$ in $\text{dom}(p)$. If $\text{nor}^0(c_{p,\nu}) > k$ we take according to (8) a subset of $\text{rge}(\text{val}(c_{p,\nu}))$ with norm $k$ and put this into $q$. We have to put successors to all $\nu' \in \text{rge}(\text{val}(c_{p,\nu}))$ for all $\nu$ in question into $q_\ell$. This is done as in (1), applied to $\nu$ instead of $\eta$. With all the $\nu$ in this subset we do the procedure in (1), and repeat and repeat it. In finitely many (intermediate) steps we reach a subtree $\text{dom}(q_\ell)$ of $\text{dom}(p)$ without any $\omega$-branches such that all its leaves fulfil $\eta'' \in p$ and such that for all $\tilde{\eta} \supseteq \eta''$, if $\tilde{\eta} \in p$ then $\text{nor}^0(c_{p,\tilde{\eta}}) \geq k + 1$, and all its nodes $\eta$ fulfil $\text{nor}^0(c_{p,\eta}) \geq k$.
  \end{itemize}

By König’s lemma, this tree $q_\ell$ is finite.

\begin{itemize}
  \item [(3)] With the leaves of $q_\ell$ and $k + 2$ instead of $k + 1$, we repeat the choice procedure in (1) and (2). We do it successively for all $k \in \omega$. The union of the $q_\ell$, $\ell \in \omega$, is a $q$ as desired in (9).
(10) This follows from König’s lemma: Since $p^\parallel$ is finitely branching, there is a branch through every infinite subset.

(11) Follows from Definitions 1.3 and 2.2. □

The next lemma states that $Q$ fulfils some fusion property:

**Lemma 2.9.** Let $\langle n_i : i \in \omega \rangle$ be a strictly increasing sequence of natural numbers. We assume that for every $i$, $q_i \leq_{n_i} q_{i+1}$, and we set $n_{-1} = 0$. Then $q = \bigcup_{i<\omega} \bigcup_{n_i \leq n < n_i} q_i^{[n]} \in Q$ and for all $i$, $q \geq_{n_i} q_i$.

**Proof.** Clear by the definitions.

The fusion lemma is usually applied in the following setting:

**Conclusion 2.10.** Suppose $p \in Q$ is given and we are to find $q \geq p$ such that $q$ fulfils countably many tasks. For this it is enough to find for any single task and any $p_0$ and $k^* \in \omega$ some $q \geq_{k^*} p_0$ that fulfils the task.

Now we want to fill up the domains of the partial specialization functions and to show that smooth conditions are dense:

**Lemma 2.11.** If $p \in Q$ and $m < \omega$ then for some smooth $q \in Q$ we have $p \leq_m q$. Moreover, if $\bigcup \{ \text{dom}(\eta) : \eta \in p^\parallel \} \subseteq T_{<\alpha}$ then we can demand that $\bigcup \{ \text{dom}(\eta) : \eta \in q^\parallel \} = T_{<\alpha}$. Moreover, $\eta \in q^\parallel$ implies $\text{nor}^1(c_{q,\eta}) \geq \text{nor}^1(c_{p,pr_{q,p}(\eta)}) - 1$, and $q(\eta) = p(pr_{q,p}(\eta))$ implies that $\text{nor}(c_{q,\eta}^+) \geq \text{nor}(c_{p,pr_{q,p}(\eta)}^+) - 1$.

**Proof.** We first use the definition of $p \in Q$: By item (v) there is some $k < \omega$ for every $\eta \in p^{[k]}$ there is $\alpha(\eta) < \omega_1$ and a finite $u_\eta \in T_A \setminus (T_A)_{<\alpha(\eta)}$ such that for every $\omega$-branch $\langle \eta_\ell : \ell < \omega \rangle$ of $p^\parallel$ satisfying $\eta_k = \eta$ we have $\bigcup_{\ell \in \omega} \text{dom}(\eta_\ell) \setminus u_\eta = (T_A)_{<\alpha(\eta)}$. We fix such a $k$ and such $u_\eta$’s.

We can find $n$ such that

\begin{align*}
(\ast)_1 \quad & m \leq n < \omega, \quad k \leq n, \\
(\ast)_2 \quad & |\bigcup_{\eta \in p^{[k]}} u_\eta| < n, \\
(\ast)_3 \quad & \text{for every } \nu \in p^{[n]}, \text{ we have } \text{nor}^0(c_{p,\nu}) > m, \\
(\ast)_4 \quad & \text{if } \eta \in p^{[n]}, \eta \subseteq \nu \in p \text{ then } \text{dom}(\nu) \setminus \text{dom}(\eta) \text{ is disjoint from } u.
\end{align*}

For each $\eta \in p^{[n]}$ let $w_\eta^+ = \{ \nu : \eta <_T \nu \in \text{dom}(p) \land \text{nor}^1(c_{p,\nu}) > \text{nor}^1(c_{p,\eta}) \}$ and $w_\eta = \{ \nu \in w_\eta^+ : (\exists \rho)(\eta \leq_T \rho \land \nu \land \rho \in w_\eta^+ \}$. So $w := \bigcup \{ w_\eta : \eta \in p^{[n]} \}$ is a front of $p^\parallel$. For each $\nu \in w$ let $\nu \in p^{[\ell(\nu)]}$ (so $\ell(\nu) \geq n$) and let $\alpha(\nu) = \alpha(\eta)$.
and \( u_\nu = u_\eta \) when \( \nu \in w_\eta \). Let \( \{ x_\ell^\nu : \ell \in (\ell(\nu), \omega) \} \) enumerate \( T_{<\alpha} \setminus (T_{<\alpha(\nu)} \cup u_\nu) \) without repetition. For each \( \ell, \nu \),

\[
\# \{ \rho \in u_\nu : x_\ell^\nu \triangleleft T \rho \} + \# \{ x_k^\nu : \ell(k) < \ell \} < \ell.
\]

We let

\[
q_\downarrow = \{ \rho : (a) \text{ for some } \nu \in w(\rho \subseteq \nu \vee \nu \subseteq \rho \in p^\downarrow, \alpha = \alpha(\nu) \\
\text{ or } (b) \nu \subseteq \rho, \alpha(\nu) < \alpha \text{ and for some } \ell > \ell(\nu) \\
\text{ and } \exists \tau \in p^{\ell}(\nu \subseteq \tau) \\
\text{ dom}(\rho) = \text{dom}(\tau) \cup \{ x_k : \ell(k) < \ell \text{ and } \rho(x_k^\nu) < n_{3,i(\rho) + k} \} \}
\]

and choose \( q_\downarrow \subseteq q_\uparrow \) by successively climbing upwards in the levels of \( p^\downarrow \), using first Claim 1.10 for the immediate successors of an already chosen node, and then using Claim 1.11 to make these new successors the bases of the creatures attached to them. We choose \( q \) rich enough as in Claim 1.10 but also small enough as to have sufficiently high \( \text{nor}^*(c_{q,\rho}) \). We set \( q(\rho) = p(\tau) \) if \( \rho, \tau \) are as above.

For checking the conditions for \( p \leq q \) and on the norms note that \( \Box \) above gives clause of the premises of Claim 1.10 on a given level and of Claim 1.11 on its successor level. By the choice of \( q \), it is smooth.

**Conclusion 2.12.** Forcing with \( Q \) specializes \( T_A \).

### 3. Decisions taken by the tree creature forcing

In this section we prove that \( Q \) is proper. Indeed we prove that \( Q \) has “continuous reading of names” (this is the property stated in 3.8), which implies Axiom A (see \([\Box]\)) and properness.

**Claim 3.1.**

1. If \( p \in Q \) and \( \{ \eta_1, \ldots, \eta_n \} \) is a front of \( p \), then \( \{ p^{(\eta_1)}, \ldots, p^{(\eta_n)} \} \)
   is predense above \( p \).

2. If \( \{ \eta_1, \ldots, \eta_n \} \) is a front of \( p \) and \( p^{(\eta_n)} \leq q_\ell \in Q \) for each \( \ell \), then there is \( q \geq p \) with \( \{ \eta_1, \ldots, \eta_n \} \subseteq q_\downarrow \) such that for all \( \ell \) we have that \( q^{(\eta_n)} = q_\ell \).
   Hence \( \{ q^{(\eta_n)} : 1 \leq \ell \leq n \} \) is predense above \( q \).

**Claim 3.2.** If \( p \in Q \) and \( X \subseteq \text{dom}(p) \) is upwards closed in \( \triangleleft_p \), and \( \forall \eta \in \text{dom}(p) \text{nor}^0(c_{p,\eta}) > 0 \), then there is some \( q \) such that

1. \( p \leq_0 q \), and either \( (\exists \ell)q^{\geq \ell} \subseteq X \) or \( \text{dom}(q) \cap X = \emptyset \),
2. \( \text{dom}(q) \subseteq \text{dom}(p) \) and \( q = p \restriction \text{dom}(q) \),
(c) For every \( \nu \in \text{dom}(q) \), if \( c_{q,\nu} \neq c_{p,\nu} \), then \( \text{nor}^2(c_{q,\nu}) \geq \text{nor}^2(c_{p,\nu}) - 1 \) and \( \text{nor}(c_{q,\nu}^+) \geq \text{nor}(c_{p,\nu}^+) - 1 \).

**Proof.** We will choose \( \text{dom}(q) \subseteq \text{dom}(p) \) and then let \( q = p \upharpoonright \text{dom}(q) \). For each \( \ell \) we first choose by downward induction on \( j \leq \ell \) subsets \( X_{\ell,j} \subseteq p^{[\leq \ell]} \) and a colouring \( f_{\ell,j} \) of \( X_{\ell,j} \cap p^{[j]} \) with two colours, 0 and 1. The choice is performed in such a way that \( X_{\ell,j} - 1 \subseteq X_{\ell,j} \) and such that \( p_{[i]} \subseteq X_{\ell,j} \) for \( i \leq j \).

We choose \( X_{\ell,\ell} = p^{[\leq \ell]} \) and for \( \nu \in p^{[\ell]} \) we set \( f_{\ell,\ell}(\nu) = 0 \) iff \( (\exists \ell')(p^{[\nu]})_{[\geq \ell']} \subseteq X \) and \( f_{\ell,\ell}(\nu) = 1 \) otherwise.

Suppose that \( X_{\ell,j} \) and \( f_{\ell,j} \) are chosen. For \( \eta \in p^{[j-1]} \cap X_{\ell,j} \) we have

\[
\text{rge}(\text{val}(c_{\eta,p})) = \{ \nu \in \text{rge}(\text{val}(c_{\eta,p})) : f_{\ell,j} = 0 \} \cup \\
\{ \nu \in \text{rge}(\text{val}(c_{\eta,p})) : f_{\ell,j} = 1 \}
\]

Note that the sets would be all the same if we intersect with \( X_{\ell,j} \), because \( p^{[j]} \subseteq X_{\ell,j} \). By Claim 1.12 at least one of the two sets gives a creature \( c \) with \( \text{nor}^2(c) > \text{nor}^2(c_{\eta,p}) - 1 \). So we keep in \( X_{\ell,j-1} \cap p^{[j]} \) only those of the majority colour and close this set downwards in \( p^j \). This is \( X_{\ell,j-1} \). We colour the points on \( p^{[j-1]} \cap X_{\ell,j-1} \) with \( f_{\ell,j-1} \) according to these majority colors, i.e., \( f_{\ell,j-1}(\eta) = i \) iff \( \{ \nu \in \text{rge}(\text{val}(c_{\eta,p})) : f_{\ell,j}(\nu) = i \} \subseteq X_{\ell,j-1} \). We work downwards until we come to the root of \( p \) and keep \( f_{\ell,0}(\text{rt}(p)) \) in our memory.

We repeat the procedure of the downwards induction on \( j \) for larger and larger \( \ell \).

If there is one \( \ell \) where the root got colour 0, we are, because \( X \) is upwards closed, in the first case of the alternative in the conclusion (a). If for all \( \ell \) the root got colour 0, we have for all \( \ell \) finite subtrees \( t \) such that for all \( \nu \in t \), \( p^{[\nu]} \cap t \) has sufficiently high norm at its root. By König’s Lemma (initial segments of trees are taking from finitely many possibilities) we build a condition \( q \) that all of its nodes are not in \( X \), and thus (a) is proved. The item (b) is clear. Item (c) follows from our choice of \( q \) and from 1.12. \( \square \)

The next claim is very similar to 3.2. We want to find \( q \geq_m p \), and therefore we have to weaken the homogeneity property in item (a) of 3.2.

**Claim 3.3.** If \( p \in Q \ k^* \in \omega \), and \( X \subseteq \text{dom}(p) \) is upwards closed, and \( \forall \eta \in \text{dom}(p) \text{nor}^0(c_{p,\eta}) > 0 \), then there is some \( q \) such that
(a) \( p \leq_{k^*} q \), and there is a front \( \{\nu_0, \ldots, \nu_s\} \) of \( p \) which is contained in \( q \) and whose being contained in \( q \) ensures \( p \leq_{k^*} q \), and such that for all \( \nu_i \) we have: either \( (\exists \ell)(q^{(\nu_i)})[\geq \ell] \subseteq X \) or \( \operatorname{dom}(q^{(\nu_i)}) \cap X = \emptyset \),

(b) \( \operatorname{dom}(q) \subseteq \operatorname{dom}(p) \) and \( q = p \upharpoonright \operatorname{dom}(q) \),

(c) for every \( \nu \in \operatorname{dom}(q) \), if \( c^{+}_{q,\nu} \neq c^{+}_{p,\nu} \), then \( \operatorname{nor}(c^{+}_{q,\nu}, k^{+}(c^{+})) \geq \operatorname{nor}(c^{+}_{p,\nu}) - 1 \) and \( \operatorname{nor}(c^{+}_{q,\nu}) \geq \operatorname{nor}(c^{+}_{p,\nu}) - 1 \).

Proof. We repeat the proof of (3.2) for each \( p^{(\nu_i)} \). \( \square \)

Now for the first time we make use of the coordinate \( k(c^{+}) \) of our creatures.

The next lemma states that the creatures have the halving property (compare to \( [4, 2.2.7] \)).

**Definition 3.4.** \( Q \) has the halving property, iff there is a function \( \text{half} : K^+ \to K^+ \) with the following properties:

(1) \( \text{half}(c^{+}) = (c, k(\text{half}(c^{+}))) \),

(2) \( \operatorname{nor}(\text{half}(c^{+})) \geq \operatorname{nor}(c^{+}) - 1 \),

(3) if \( c' \) is a simple creature and \( k \geq k(\text{half}(c^{+})) \) and \( \operatorname{nor}(c', k) > 0 \), then \( \operatorname{nor}(c', k(c^{+})) \geq \operatorname{nor}(c^{+}) \).

**Lemma 3.5.** \( K \) has the halving property.

**Proof.** We set \( k(\text{half}(c^{+})) = k'(\text{nor}^{\frac{1}{2}}(c^{+}), k(c^{+})) \geq k(c^{+}) \) as in \( [4, 7](4) \). Then we have that \( \operatorname{nor}(\text{half}(c^{+})) = f(\text{nor}^{\frac{1}{2}}(c^{+}), k(\text{half}(c^{+}))) \geq \operatorname{nor}(c^{+}) - 1 \), by Definition \( [4, 7](4) \).

If \( c' \) is a simple creature and \( \operatorname{nor}(c', k(c^{+})) > 0 \) and \( \text{nor}^{\frac{1}{2}}(c') \leq \text{nor}^{\frac{1}{2}}(c) \), then

\[
\operatorname{nor}(c', k(c^{+})) = f(\text{nor}^{\frac{1}{2}}(c'), k(c^{+})) \\
\geq f(\text{nor}^{\frac{1}{2}}(c'), k(\text{half}(c^{+}))) + f(\text{nor}^{\frac{1}{2}}(c), k(\text{half}(c^{+}))) \\
\geq 1 + \operatorname{nor}(c^{+}) - 1 \geq \operatorname{nor}(c^{+}).
\]

If \( \text{nor}^{\frac{1}{2}}(c') > \text{nor}^{\frac{1}{2}}(c) \), then the inequality follows from the monotonicity properties in Definition \( [4, 7](4) \). \( \square \)

**Claim 3.6.** Assume that \( \tau \) is a \( Q \)-name for an ordinal, and let \( a \) be a set of ordinals. Let \( m \in \omega \). Let \( p, p^{+} \) be conditions such that
Then for any \( q \in Q \): \( p^+ \leq q \) and \( q \models \tau \in a \) and \( \nu = \text{rt}(q) \) and \( \eta = \text{pr}_{q,p}(\nu) \) imply that there is some \( q' \) such that

(a) \( p^{(\eta)} \leq q' \), \( \nu = \text{rt}(q') \)

(b) \( q' \models \tau \in a \),

(c) for every \( \rho \in \text{dom}(q') \), \( \text{nor}(c_{q',\rho}, q'(\rho)) \geq m \).

Proof. So let \( q \geq p^+ \) and \( q \models \tau \in a \) and \( \eta = \text{pr}_{q,p}(\text{rt}(q)) \). We take some \( n(*) \in \omega \) such that

\[(\forall \rho \in \text{dom}(q))(\rho \in \bigcup_{n' \geq n(*)} q[n'] \rightarrow \text{nor}(c_{q,\rho}^+) > m)\]

We define \( q' \) by \( \text{dom}(q') = \text{dom}(q) \) and \( \rho \in \bigcup_{n' \geq n(*)} q[n'] \rightarrow c_{q',\rho}^+ = c_{q,\rho}^+ \), \( \rho \in \bigcup_{n' < n(*)} q[n'] \rightarrow c_{q',\rho}^+ = (c_{q,\rho}, k(c_{q,\rho}^+)) \).

\( q \) and \( q' \) force the same things, because we weakened \( q \) to \( q' \) only in an atomic part, because there are only finitely many \( k \) such that \( (c_{q,\rho}, k) \) is a creature with \( 0 \leq \text{nor}^0(c_{q,\rho}) \).

From Lemma 3.5 we get \( \rho \in \text{dom}(q) \rightarrow \text{nor}(c_{q',\rho}^+) \geq m \). \( \square \)

As a preparation for the following proof, we define isomorphism types of partial specialization functions over conditions \( p \):

**Definition 3.7.** Let \( \eta_0, \eta_1 \in \text{spec} \) and let \( p \in Q \). We say \( \eta_0 \) is isomorphic to \( \eta_1 \) over \( p \) if there is some injective partial function \( f: T_A \rightarrow T_A \) such that \( x <_{T_A} y \) iff \( f(x) <_{T_A} f(y) \) and \( \text{dom}(\eta_0) \cup \bigcup\{\text{dom}(\eta) : \eta \in \text{dom}(p)\} \subseteq \text{dom}(f) \) and \( f \upharpoonright \bigcup\{\text{dom}(\eta) : \eta \in \text{dom}(p)\} = \text{id} \) and \( f[\text{dom}(\eta_0)] = \text{dom}(\eta_1) \) and \( \eta_0(x) = \eta_1(f(x)) \) for all \( x \in \text{dom}(\eta_0) \).

Facts: For each fixed \( p \), there are only countably many isomorphism types for \( \eta \) over \( p \). If the elements of \( \text{dom}(\eta_0) \) and of \( \text{dom}(\eta_1) \) are pairwise incomparable in \( T_A \) and if they are isomorphic over \( p \) with \( \text{dom}(p) = (T_A)_{<\alpha} \) for some countable \( \alpha \), and if there is some \( r \geq p \) such that \( \eta_0 \in r^{[]} \), then there is some \( r' \geq p \) such that \( \eta_1 \in (r')^{[]} \).
Claim 3.8. Suppose that $p_0 \in Q$ and that $m < \omega$ and that $\tau \bar{}$ is a $Q$-name of an ordinal. Then there is some $q \in Q$ such that

(a) $p_0 \leq m q$,

(b) for some $\ell \in \omega$ we have that for every $\eta \in q^{[\ell]}$ the condition $q^{(n)}$ forces a value to $\tau \bar{}$.

Proof. Choose $n(*)$ such that $\rho \in \bigcup_{n \geq n(*)} p_0^{[n]} \to \text{nor}(c_{p_0,\rho}^+) \geq m + 1$. Then we define $p_1$ by $\text{dom}(p_1) = \text{dom}(p_0)$ and $\rho \in \bigcup_{n' \geq n(*)} p_0^{[n']} \to c_{p_1,\rho}^+ = c_{p_0,\rho}^+$, $\rho \in \bigcup_{n' < n(*)} p_0^{[n']} \to c_{p_1,\rho}^+ = \text{half}(c_{p_0,\rho}^+)$. Then we define

$$X = \left\{ \rho : \rho \in \bigcup_{n \geq n(*)} p_1^{[n]} \land (\exists q) \left( p_1^{(\rho)} \leq q \land q \text{ forces a value to } \tau \right) \land (\forall \nu \in q^{[\ell]})(\text{nor}(c_{q,\nu}^+) \geq 1) \right\}.$$

Let $p_2$ be chosen as in 3.3 for $(p_1, X, n(*))$. By a density argument, there is a front $\{\nu_0, \ldots, \nu_r\}$ of $p_1$ such that for all $\nu_i$ the first clause of the alternative in 3.3(a) holds.

For $\tilde{m} < \omega$, $r, s \in Q$, $\eta \in \text{dom}(r)$, we denote the following property by $(*)_{\tilde{m}, \eta}^{r, s}$:

$$r^{(n)} \leq_0 s \land \forall \nu (\eta \subseteq \nu \in s \to \text{nor}(c_{s,\nu}^+) \geq \tilde{m} + 1) \land (\exists \ell \in \omega)(\forall \rho \in s^{[\ell]})(s^{(\rho)} \text{ forces a value to } \tau).$$

We choose by induction on $t < \omega$ countable $N_t \prec (\mathcal{H}(\chi), \in)$ and an ordinal $\alpha_t$ and pairs $(k_t, q_t)$ such that

1. $p_2, T_A, \tau \in N_0$,
2. $N_t \in N_{t+1}$,
3. $N_t \cap \omega_1 = \alpha_t$,
4. $\delta = \lim_{t \to \omega} \alpha_t$,
5. $k_t$ is increasing with $t$, $k_t \geq n(*)$,
6. $q_t \in Q$ is smooth,
7. $\alpha(q_t) = \alpha_t$. 
(8) $k_t$ is the first $k$ strictly larger than all the $k_i$ for $t_1 < t$ and such that
\[ \rho \in q_t^{[k]} \rightarrow \text{nor}(c^+_{q_t, \rho}) > m + t + 1, \]

(9) $q_t \leq m + t + 1 q_{t+1}$.

(10) if $\eta \in q_t^{[k]}$ and there is $q \in V$ satisfying $(*)_{q_t,q}^{m+t+1,\eta}$, then $q = q_{t+1}^{(\eta)}$ satisfies it,

(11) $q_t \in N_{t+1}$,

(12) if $\eta \in q_t^{[k]}$ and no $q$ satisfies $(*)_{q_t,q}^{m+t+1,\eta}$, then $(q_{t+1}^{(\eta)})[\eta] = (q_t^{(\eta)})[\eta]$ and $\eta \subseteq \rho \in q_t$
implies that $c^+_{q_t+1, \rho} = \text{half}(c^+_{q_t, \rho})$.

It is clear that the definition can be carried out as required. If we are given $q_t$ we can easily find $k_t$. For each $\eta \in q_t^{k}$ we choose $q_t, \eta \in N_t$ such that $(*)_{q_t,q_t}^{m+t+1,\eta}$ is possible and in fact w.l.o.g. $q_t, \eta = q_t^{(\eta)}$, otherwise we follow (12) and apply the halving function.

Having carried out the induction, we let $r = \bigcup_{t \in \omega} (q_t \upharpoonright q_t^{(k_{t-1,k})})$. So, by (7), $r \in Q$ is smooth with $\alpha(r) = \delta$ and for every $t$ we have $q_t \leq m + t + 1 r$, and in particular $p_2 \leq m + 1 r$.

Assume for a contradiction that we are in the bad case

\[ (\forall \ell \in \omega)(\exists \rho^* \in r[\ell])(r^{(\rho^*)} \text{ does not decide the value of } \tau). \]

Choose a minimal $\ell$ as in $\otimes$ and a $\rho^*$ as there. Choose $q \geq r^{(\rho^*)}$ in $Q$ such that $q$ forces a value to $\tau$.

Let $\nu_0 \in q[\eta]$ be such that

\[ \nu_0 \subseteq \nu \in q \rightarrow \text{nor}(c^+_{q, \nu}) \geq m + 1. \]

W.l.o.g. (otherwise we strengthen $q$) we assume that $\nu_0 = rt(q)$.

As $r$ is smooth, by the definition of $q \geq r$ (22(2)) we have that the additional information on partial specialization functions that are in $q$ but not is $r$ does not have the domain in $(T_A)_{< \delta}$.

Let $t(\ast)$ be such that $\text{dom}(rt(q)) \cap (T_A)_{< \delta} \subseteq T_{< \alpha_t(\ast)}$, and w.l.o.g. $\nu_0 \in q^{[k_{t}(\ast) + i(p_0) - i(q)]}$, so $pr_{q,r}(\nu_0) \in r^{[k_{t}]}$, $i(p_0) = i(q_t) = i(r)$. Now easily

if $t(\ast) \leq t$, $\nu_0 \leq \nu \in \text{dom}(q)$, $pr_{q,r}(\nu) \in r^{[k_t]}$,

\[ \text{then } pr_{q,r}(\nu) = pr_{q,q_t}(\nu), \text{dom}(\nu) \cap (T_A)_{\alpha(r)} = \text{dom}(pr_{q,r}(\nu)) = \text{dom}(pr_{q,q_t}(\nu)) \subseteq (T_A)_{< \alpha(q_t)}. \]
Now \( \eta_0 \in r^{[k_{(i)}]} \) and even \( \eta_0 \in q^{[k_{(i)}]}_{t(\cdot)} \). So by the choice of \( \langle q_t : t \in \omega \rangle \) we know that there is no \( q \) with \( \langle * \rangle^{m+t(\cdot)+1}_{q_{t(\cdot)+1}} \), as otherwise \( q^{(n)}_{t(\cdot)+1} \) would be like this and this property would be inherited by \( r \). So clause (12) applies, which means

\[
\text{there is no } \rho \text{ such that } \eta_0 \subseteq \rho \in \text{dom}(q_{t(\cdot)}) \text{ and there is no } s \text{ such that } q^{(n)}_{t(\cdot)} \leq m+t(\cdot)+1/s \text{ and } (\exists r)(\forall \nu \in s[\ell])(s[\nu]) \text{ forces a value to } \tau).
\]

Choose \( \nu^* \) such that \( \nu_0 \subseteq \nu^* \in \text{dom}(q) \) and \( \text{pr}_{q,r}(\nu^*) \in r^{[k_{(i)}]} \). Let \( \eta^* = \text{pr}_{q,q}(\nu^*) \in q^{[k_{(i)}]}_{t(\cdot)} \).

Fix for some time \( \nu \in \text{succ}(\nu^*) \) and let \( \eta = \text{pr}_{q,q}(\nu) \in q^{[k_{(i)}]}_{t(\cdot)} \), so \( \eta \in \text{succ}(\eta^*) \).

So let \( \text{dom}(\nu) \setminus \text{dom}(\eta) = \{x_0, \ldots, x_{\tilde{s}-1}\} \). We just saw that \( x_0, \ldots, x_{\tilde{s}-1} \notin (T_A)_{<\delta} \). Let us define \( \tilde{y} = \langle y_\ell : \ell < \tilde{s} \rangle \) is a candidate for an extended domain

iff:

(a) \( y_\ell \) are without repetitions,

(b) there is some \( r_{\tilde{y}} \) such that

1. \( r_{\tilde{y}} \geq q^{(n)}_{t} \);
2. \( (\forall \rho)(\eta \subseteq \rho \in \text{rt}(r_{\tilde{y}}) \rightarrow \text{nor}(\mathbf{c}_p^{+}, r_{\tilde{y}}) > m + t(\cdot) + 1) \),
3. \( r_{\tilde{y}} \) forces a value to \( \tau \)
4. \( (\langle y_\ell, \nu(x_\ell) \rangle : \ell < \tilde{s}) \in \text{spec}^{T_A} \) is isomorphic over \( (T_A)_{<\delta} \) to \( (\langle x_\ell, \nu(x_\ell) \rangle : \ell < \tilde{s}) \).

We set

\[
Y = Y_\eta = \{ \tilde{y} : \tilde{y} \text{ is a candidate for an extension} \}.
\]

Now we have that \( \langle x_\ell : \ell < \tilde{s} \rangle \in Y \). This is exemplified by \( q^{(n)}_{t} \).

Now we have that \( q_t \in N \) and for all \( \ell, x_\ell \in (T_A)_{<\delta} \), because the \( \alpha_t \) are cofinal in \( \delta \) and since \( \alpha_t = \alpha(q_t) \).

Since \( x_\ell \geq \delta \), counting isomorphism types over \( (T_A)_{<\delta} \) yields \( |Y_\eta| = \aleph_1 \).

By a fact on Aronszajn trees (Jech, or [3 III, 5.4]) we find \( \langle y^n_{j,\ell} : \ell < s, j \in \omega_1 \rangle \) and a root \( \Delta_\eta \) such that

(a) \( y^n_{j,\ell} \in Y_\eta \) are without repetition,

(b) for \( j \neq j' \), \( \{ y^n_{j,\ell} : \ell < \tilde{s} \} \cap \{ y^n_{j',\ell} : \ell < \tilde{s} \} = \Delta_\eta \),
(c) if \( j_1 \neq j_2 \) and if \( y_{j_1, t_1}^\eta \notin \Delta_\eta \) and \( y_{j_2, t_2}^\eta \notin \Delta_\eta \) then they are incompatible in \( \langle T_\lambda \rangle \).

Let \( r_j^{\eta^*}(y_{j_1}, \ell < \bar{s}) \) witness that \( \langle y_{j, \ell}^\eta : \ell < \bar{s} \rangle \in Y_\eta \).

Let \( c = \{ \text{pr}_{q, q}(\nu) : \nu \in \text{rge}(\nu) \} \). This is a simple \((i(q_t) + k_t(\nu^*))\)-creature with \( \text{nor}(c, q_t(\eta^*)) \geq m + t(\nu^* + 2) \) by property (10) of \((k_t(\nu^*), q_t(\nu^*))\). For each \( \eta \in \text{rge}(\nu) \) let \( \langle y_{j, \ell}^\eta : \ell < \bar{s}, j < \omega_1 \rangle \) be as above and let \( r_j^\eta \) be a witness for \( \langle y_{j, \ell}^\eta : \ell < \bar{s} \rangle \in Y_\eta \).

Let \( j^* = \text{nor}^0(c_{q_t(\eta^*)}, \eta^*) \).

For each \( \eta \in \text{rge}(\nu) \) choose a witness \( \nu_\eta \in \text{suc}_q(\nu^*) \) such that \( \text{pr}_{q, q_t(\nu)}(\nu_\eta) = \eta \). Now we define a simple \( i(c_{q_t(\eta^*)}) \)-creature \( d \) by

\[
\eta(d) = \eta(c_{q_t(\eta^*)})
\]

\[
\text{rge}(d) = \{ \eta \cup \{ (x_{j, \ell}, \nu_\eta(x_\ell)) : \ell < \bar{s} \} : \eta \in \text{rge}(c), j < j^* \}.
\]

Then we have by Claim \([3,4]\) that \( d \) is a \( i(c_{q_t(\eta^*)}) \)-creature and \( \text{nor}^0(d) = \text{nor}^0(c_{q_t(\eta^*)}) = m + t(\nu^* + 1) \). \( \text{nor}^* \) drops at most by 1. So we have \( \text{nor}(d, q_t(\nu^*)) > 0 \) and hence by Claim \([3,6]\) \( \text{nor}(d, q_t(\nu^*)) > m \). Now we define \( s \in Q \) as follows:

\[ (\alpha) \quad \text{rt}(s) = \eta^*, s(\eta^*) = q_t(\eta^*), \]

\[ (\beta) \quad c_{s, \eta^*} = d, \]

\[ (\gamma) \quad \text{if } \rho \in \text{rge}(d) \text{ and if } \rho = \eta \cup \{ (x_{j, \ell}, \nu_\eta(x_\ell)) : \ell < \bar{s} \} \text{ then } s(\rho) = r_j^\eta. \]

Clearly \( s \in Q \) and \( q_t(\nu^*) \leq m + t(\nu^* + 1) \) and for every \( \eta \in q[\ell] \) the condition \( q_t(\eta^*) \) forces a value to \( \tau \), in fact \( \ell = k_t(\nu^*) \) is o.k., by the way the \( r_j^\eta \) were chosen. So we get a contradiction to \( \boxtimes \) and to the choice of \( \eta^* \). \( \square \)

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Heike Mildenberger, Institut für formale Logik, Universität Wien, Währinger Str. 25, A-1090 Wien, Austria

Saharon Shelah, Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel

*E-mail address*: heike@logic.univie.ac.at

*E-mail address*: shelah@math.huji.ac.il