Averaging of dispersion managed nonlinear Schrödinger equations

Mi-Ran Choi\textsuperscript{1} and Young-Ran Lee\textsuperscript{2,∗}

\textsuperscript{1} Research Institute for Basic Science, Sogang University, 35 Baekbeom-ro (Sinsu-dong), Mapo-gu, Seoul 04107, Republic of Korea
\textsuperscript{2} Department of Mathematics, Sogang University, 35 Baekbeom-ro (Sinsu-dong), Mapo-gu, Seoul 04107, Republic of Korea

E-mail: younglee@sogang.ac.kr

Received 15 August 2021, revised 24 January 2022
Accepted for publication 11 February 2022
Published 2 March 2022

Abstract

We consider the dispersion managed power-law nonlinear Schrödinger (DM NLS) equations with a small parameter $\varepsilon > 0$ and the averaged equation, which are used in optical fiber communications. We prove that the solutions of DM NLS equations converge to the solution of the averaged equation in $H^1(\mathbb{R})$ as $\varepsilon$ goes to zero. Meanwhile, in the positive averagedispersion, we obtain the global existence of the solution to DM NLS equation in $H^1(\mathbb{R})$ for sufficiently small $\varepsilon > 0$, even when the exponent of the nonlinearity is beyond the mass-critical power.

Keywords: dispersion managed nonlinear Schrödinger equation, averaging, global existence

Mathematics Subject Classification numbers: 35A01, 35B27, 35Q55, 35Q60.

1. Introduction

The nonlinear Schrödinger equation with varying coefficients arises in modelling many physical phenomena such as the propagation of electromagnetic waves in optical fibers and the mean field dynamics of Bose–Einstein condensates, see, e.g. [18–20, 22]. Let us focus on its application in fiber optics. The envelope equation describing the evolution of a pulse in a frame that moves at the group velocity of the signal along an optical fiber is of the form

$$i\partial_t u + d(t)\partial_x^2 u + |u|^\alpha u = 0,$$

where $u = u(x, t)$, $x, t \in \mathbb{R}$, is a complex-valued function and $\alpha$ a positive constant. Here, $t$ corresponds to the distance along the fiber, $x$ the retarded time, and the varying coefficient...
$d(t)$ the fiber dispersion. A balance between the dispersion and the nonlinearity generates the so-called solitons which are of great importance in optical fiber communications. Since the solitons strongly interact with each other by the nonlinear effect, one way to reduce this negative effect is to try to stay in the linear regime, where the dispersion is dominant, while also controlling the dispersion. Such an idea generated the so-called dispersion management in 1980, see [15], which uses fibers with relatively short alternating sections of positive and negative strong dispersion of almost mean zero. This introduces a periodically rapid and large variation of dispersion, $d(t)$, with almost zero average, which leads to stable soliton-like pulses (breather type solutions) along the fiber. This technique was successful in transferring data at ultra-high speeds over intercontinental distances, see, e.g. [1, 10, 11, 16]. For more detailed information on this technique, see [21] for instance.

This dispersion, in the so-called strong dispersion management regime, can be described by

$$d(t) = d_{av} + \frac{1}{\varepsilon} d_0 \left( \frac{t}{\varepsilon} \right),$$

(1.1)

where $d_0 = O(1)$ is a periodic function of mean zero and $d_{av} \in \mathbb{R}$ the average dispersion over one period, otherwise called the residual dispersion, whose magnitude is very small. Here, the positive parameter $\varepsilon$ has effects on the period and the amplitude of the dispersion $d(t)$. Since the period of the dispersion is relatively small compared with the dispersion length, $\varepsilon \ll 1$, see, e.g. [1]. Therefore, we note that the amplitude of the dispersion is very large since $d = O(\varepsilon^{-1})$.

In this strong dispersion regime, we consider the Cauchy problem

$$\begin{cases}
i \partial_t u + d_{av} \partial_x^2 u + \left| u \right|^{\alpha} u = 0, \\
u(x, 0) = \varphi(x),
\end{cases}$$

(1.2)

where $d_0$ is assumed to be a two-periodic function with $d_0 = \chi_{(0,1)} - \chi_{(1,2)}$ on $[0,2)$. Given $\varepsilon > 0$, by the standard argument, it can be shown that the Cauchy problem (1.2) is globally well-posed in $H^1(\mathbb{R})$ for appropriate values of $\alpha$. Indeed, for any $d_{av} \in \mathbb{R}$, if $\alpha > 0$ and $\varphi \in H^1(\mathbb{R})$, then a local solution exists in $H^1(\mathbb{R})$. Moreover, if $d_{av} \neq 0$, then by the mass and energy conservations it is easy to see that the solution is global under the additional condition $\alpha < 4$ only when $d_{av} > 0$. For the vanishing average dispersion, we use the mass conservation and the regularity of the $L^2$ solution to get the global existence provided $0 < \alpha < 4$. To obtain such results, one should consider the properties of the linear equation associated with (1.2), see [4, 8] for more details.

Now we change the variables $u = T_{D(t)/\varepsilon} v$ in (1.2) to obtain

$$\begin{cases}
i \partial_t v + d_{av} \partial_x^2 v + T^{-1}_{D(t)/\varepsilon} \left( \left| T_{D(t)/\varepsilon} v \right|^{\alpha} T_{D(t)/\varepsilon} v \right) = 0, \\
v(x, 0) = \varphi(x),
\end{cases}$$

(1.3)

where $D(t) = \int_0^t d_0(t')dt'$ and $T_t$ is the solution operator for the free Schrödinger equation in spatial dimension one. We consider the two-scale asymptotic expansion for the solution $v$ of (1.3), that is,

$$v(t) = \sum_{j=0}^{\infty} \varepsilon^j v_j \left( t, \frac{t}{\varepsilon} \right),$$

(1.4)
where all \( v_j = v_j(t, \tau) \) are two-periodic in \( \tau \). Then we see that \( v_0 \) is constant in \( \tau \) at order \( \varepsilon^{-1} \) and
\[
i \partial_t v_0 + i \partial_\tau v_1 + d_\alpha \partial_\tau^2 v_0 + T_{D(\tau)}^{-1} (T_{D(\tau)} v_0)\alpha T_{D(\tau)} v_0 = 0 \tag{1.4}
\]
at order \( \varepsilon^0 \). Averaging (1.4) with respect to \( \tau \) over one period, we have
\[
i \partial_t v_0 + d_\alpha \partial_\tau^2 v_0 + \frac{1}{2} \int_0^1 T_{D(\tau)}^{-1} (T_{D(\tau)} v_0)\alpha T_{D(\tau)} v_0 \, d\tau = 0.
\]
Furthermore, if we use the change of variables \( D(\tau) = r \), we have the following averaged equation of (1.3)
\[
i \partial_t v + d_\alpha \partial_\tau^2 v + \int_0^1 T_r^{-1} (T_r v)\alpha T_r v \, dr = 0, \tag{1.5}
\]
where \( v_0 \) is replaced by \( v \). For more information on the averaging process regarding dispersion management, see, e.g. [1, 10, 11].

The Cauchy problem of the averaged equation (1.5) is globally well-posed in \( H^1(\mathbb{R}) \) when \( 0 < \alpha < 8 \) for \( d_\alpha > 0 \); \( 0 < \alpha < 4 \) for \( d_\alpha = 0 \); \( \alpha > 0 \) for \( d_\alpha < 0 \), see [3, 6]. It is remarkable that this averaged equation has the \( H^1 \) global solution even for \( 4 \leq \alpha < 8 \) when \( d_\alpha > 0 \) in contrast to the classical focusing NLS. In [3], the \( H^1 \) theory for \( s \geq 0 \) in the case of the Kerr nonlinearity was established, while it was shown in [6] that the problem is globally well-posed in \( H^1(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) when \( d_\alpha \neq 0 \) and \( d_\alpha = 0 \), respectively, for more general nonlinearities including even saturated nonlinearities. Furthermore, since a local solution in \( H^1(\mathbb{R}) \) exists for every \( \alpha > 0 \) regardless of the sign of \( d_\alpha \), see [6], in the case \( d_\alpha = 0 \), one can obtain the \( H^1 \) global solution using \( H^1 \) regularity argument from the \( L^2 \) solution, see, e.g. [5].

The averaged equation (1.5) is autonomous and solitary solutions can be found using separation of variables and variational methods, see remark 1.3 (e) below. These solitary solutions for (1.5) yield breather type solutions of the non-autonomous equation (1.2). To make this argument rigorous, it is important to analyze the asymptotic behavior of the solutions, \( v_\varepsilon \), for (1.3) on the maximal life time of the solution \( v \) for (1.5) as \( \varepsilon \to 0 \). When \( \alpha = 2 \), the averaging procedure is first rigorously justified in [23]. More precisely, it is shown that for \( \varepsilon > 0 \), the solutions of (1.3) and (1.5) with the initial datum in \( H^s(\mathbb{R}) \), \( s \) sufficiently large, stay \( \varepsilon \)-close in \( H^{s-3} \) for a long time in \( O(\varepsilon^{-1}) \). Recently, the authors with Kang, in [8], improved this result by verifying the averaging procedure in \( H^1(\mathbb{R}) \) where the solutions exist for the initial datum in \( H^1(\mathbb{R}) \), for \( \alpha = 2 \). The operator associated with this Kerr nonlinearity is multi-linear, which is crucial to get the averaging theorems in [8, 23]. In this paper, we extend the result in [8] to the power-law nonlinearities by overcoming the difficulty caused by the fact that the operator for the general power \( \alpha \) is not multi-linear. Furthermore, it follows from the global well-posedness of the averaged equation in [6] and our main theorem, theorem 1.1, that if \( d_\alpha > 0 \), then, even for \( 4 \leq \alpha < 8 \), the Cauchy problem (1.3) has a global \( H^1(\mathbb{R}) \) solution for sufficiently small \( \varepsilon > 0 \). The main theorem is

**Theorem 1.1.** Let \( d_\alpha \in \mathbb{R}, \alpha \geq 2 \), and the initial datum \( \varphi \in H^1(\mathbb{R}) \). For each \( \varepsilon > 0 \), denote by \( v_\varepsilon \) the maximal solution of (1.3) and by \( v \) the solution of the averaged equation (1.5) defined on the maximal interval \( (-T_-, T_+) \). Then, given \( 0 < M < \min \{ T_-, T_+ \} \), the solution \( v_\varepsilon \) exists on \([ -M, M ] \) for sufficiently small \( \varepsilon > 0 \). Moreover,
Let us start by introducing some notations. The spaces $L^p([-M,M],H^1(\mathbb{R}))$ for $1 \leq p \leq \infty$ and $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ are the usual Lebesgue and Sobolev spaces with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^s}$, respectively.

We use $L^p_t(J,L^q_x(I))$, for $1 \leq p,q < \infty$ and intervals $I,J \subset \mathbb{R}$, to denote the Banach space of functions $u$ with the mixed norm

$$
\| u \|_{L^p_t(J,L^q_x(I))} := \left( \int_J \left( \int_I |u(x,t)|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}}.
$$

If $p = \infty$ or $q = \infty$, use the essential supremum instead. For simple notations, $L^p(J,L^q) is used for $L^p_t(J,L^q_x(I))$.

**Corollary 1.2.** Under the assumptions of theorem 1.1, let $u_\varepsilon$ be the maximal solution of (1.2). Then given $0 < M < \min\{T_-,T_+\}$, the solution $u_\varepsilon$ exists on $[-M,M]$ for sufficiently small $\varepsilon > 0$. Moreover,

$$
\lim_{\varepsilon \to 0} \| u_\varepsilon - v \|_{L^\infty([-M,M],H^1(\mathbb{R}))} = 0.
$$

**Remark 1.3.**

(a) Let $0 < \alpha < 4$ when $d_{av} \geq 0$; $\alpha > 0$ when $d_{av} < 0$, which are naturally assumed. Then, we know that the solution $v_\varepsilon$ of (1.3) for every $\varepsilon > 0$ and the solution $v$ of (1.5) are globally defined in $H^1(\mathbb{R})$. If, in addition, $\alpha \geq 2$, then theorem 1.1 yields that, for each $0 < M < \infty$, $v_\varepsilon$ converge to $v$ in $L^\infty([-M,M],H^1(\mathbb{R}))$ as $\varepsilon \to 0$.

(b) In the case $d_{av} > 0$, the range of $\alpha$ for the global existence increases. Even for $4 \leq \alpha < 8$, by [6], it is known that the solution $v$ of (1.5) globally exists in $H^1(\mathbb{R})$, i.e., $T_+ = T_- = \infty$. Thus, it follows from theorem 1.1 that $v_\varepsilon$ is also globally defined for sufficiently small $\varepsilon$ and that for each $0 < M < \infty$, $v_\varepsilon \to v$ in $L^\infty([-M,M],H^1(\mathbb{R}))$ as $\varepsilon \to 0$.

(c) The restriction on $\alpha$, $\alpha \geq 2$, comes from lemma 2.2. Such a restriction can be removed if we use the initial datum in $H^3(\mathbb{R})$.

(d) Theorem 1.1 can be proved similarly even when the coefficient of the nonlinear term in (1.2) is a bounded periodic function of $t$ with the same period of $d$, see [8] for a special case with the Kerr nonlinearity. Such nonlinearities arise in the presence of fiber loss and amplification, see, e.g., [2].

(e) To investigate the existence of solitons, we use the ansatz, a solution of the form $v(x,t) = e^{i\omega t}f(x)$, $\omega \in \mathbb{R}$, for the averaged equation (1.5). Such an $f$, the so-called dispersion managed soliton, can be found as a minimiser of the corresponding variational problem. The minimisers exist when $0 < \alpha < 4$ for $d_{av} \geq 0$, as well as even when $4 \leq \alpha < 8$ for $d_{av} > 0$ provided $\| f \|_{L^2}$ is large enough, see [7, 13, 14, 23]. Moreover, it is shown in [6, 12] that the set of minimisers is orbitally stable, which together with our averaging result implies that equation (1.2) has breather type stable solutions.

The paper is organised as follows. In section 2, we introduce some notations and gather the bounds of the nonlinearities in two main equations (1.3) and (1.5). In section 3, we prove the local existence of a solution for (1.3) and the main theorem, theorem 1.1.
For a Banach space $X$ with norm $\| \cdot \|_X$ and an interval $J$, we use $C(J, X)$ and $C^1(J, X)$ to denote the space of all continuous functions $u: J \to X$ and the space of all continuously differentiable functions, respectively. For a compact $J$, $C(J, X)$ is the Banach space with norm

$$
\|u\|_{C(J, X)} = \sup_{t \in J} \|u(t)\|_X.
$$

The solution operator $T_t$ for the free Schrödinger equation in spatial dimension one is unitary on $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ and, therefore,

$$
\|T_t f\|_{H^s} = \|f\|_{H^s}
$$

for all $f \in H^s(\mathbb{R})$ and $t \in \mathbb{R}$. The notation $f \lesssim g$ is used when there is a positive constant $C$ such that $f \leq Cg$.

Next, we gather some estimates for the nonlinear terms of two main equations (1.3) and (1.5),

$$
Q(s, f) := T_{D(0)}^{-1} \left( (T_{D(0)} f)^\alpha T_{D(0)} f \right)
$$

and

$$
\langle Q \rangle(f) := \int_0^1 T_r^{-1} \left( (T_r f)^\alpha T_r f \right) dr
$$
defined for $s \in \mathbb{R}$ and $f \in H^1(\mathbb{R})$.

**Lemma 2.1.** Let $\alpha > 0$. Then

$$
\sup_{s \in \mathbb{R}} \|Q(s, f)\|_{H^1} \lesssim \|f\|_{H^1}^{\alpha + 1}
$$

and

$$
\|\langle Q \rangle(f)\|_{H^1} \lesssim \|f\|_{H^1}^{\alpha + 1}
$$

for all $f \in H^1(\mathbb{R})$.

**Proof.** Note that

$$
\|h\|^\alpha h \lesssim \|h\|_{H^1}^{\alpha + 1}
$$

since $\|h\|^\alpha h \leq \|h\|^\alpha h \lesssim \|h\|_{L^2}^{\alpha + 1}$ and $\|(h\|^\alpha h)'\|_{L^2} \lesssim \|h\|_{L^2}^{\alpha + 1} \|h\|_{L^2}^{\alpha + 1} \leq \|h\|_{H^1}^{\alpha + 1}$ for all $h \in H^1(\mathbb{R})$, by the embedding $L^\infty(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$. Using the fact that $T_{D(0)}$ and $\partial_x$ commute and that $T_{D(0)}$ is unitary in $H^1(\mathbb{R})$, we have (2.1). Similarly, we prove bound (2.2) using Minkowski’s inequality.

**Lemma 2.2.** Let $\alpha \geq 2$. Then

$$
\sup_{s \in \mathbb{R}} \|\partial_x^2 Q(s, f)\|_{H^1} \lesssim \|f\|_{H^1}^{\alpha + 1}
$$

and

$$
\|\partial_x^2 \langle Q \rangle(f)\|_{H^1} \lesssim \|f\|_{H^1}^{\alpha + 1}
$$

for all $f \in H^3(\mathbb{R})$.  

2125
Proof. Note that if \( h \in H^3(\mathbb{R}) \), then \( \| (h^{(n)^{m}}) \|_{H^3} \lesssim \| h \|_{H^3}^{n+1} \) since

\[
| (h^{(n)^{m}}) | \lesssim (|h|^{n-1}|h|^2 + |h|^n|h^m|), \\
| (h^{(n)^{m}}) | \lesssim (|h|^{n-2}|h|^3 + |h|^{n-1}|h|^2 + |h|^n|h^m|),
\]

and \( \alpha \geq 2 \). Thus, an argument similar to that used in proving lemma 2.1 completes the proof since \( T_{D(n)} \) is unitary in \( H^3(\mathbb{R}) \), also.

3. Averaging theorem

In this section, we prove the main theorem, theorem 1.1. First, we establish the following two lemmas.

Lemma 3.1. Let \( \alpha \equiv 1 \) and \( M > 0 \). Then

\[
\sup_{\varepsilon > 0} \int_0^M \left\| \mathcal{Q}\left( \frac{t}{\varepsilon}, v_1(t) \right) - \mathcal{Q}\left( \frac{t}{\varepsilon}, v_2(t) \right) \right\|_{L^2} dt \]

\[
\lesssim \left( \| v_1 \|_{L^\infty([0,M],H^1)}^\alpha + \| v_2 \|_{L^\infty([0,M],H^1)}^\alpha \right) \| v_1 - v_2 \|_{L^\infty([0,M],H^1)}
\]

and

\[
\int_0^M \left\| \mathcal{Q}(v_1(t)) - \mathcal{Q}(v_2(t)) \right\|_{H^1} dt \]

\[
\lesssim \left( \| v_1 \|_{L^\infty([0,M],H^1)}^\alpha + \| v_2 \|_{L^\infty([0,M],H^1)}^\alpha \right) \| v_1 - v_2 \|_{L^\infty([0,M],H^1)}
\]

for all \( v_1, v_2 \in C([0, M], H^1(\mathbb{R})) \).

Proof. We prove (3.1) only since (3.2) can be proved analogously. First, we prove

\[
\sup_{\varepsilon > 0} \int_0^M \left\| \mathcal{Q}\left( \frac{t}{\varepsilon}, v_1(t) \right) - \mathcal{Q}\left( \frac{t}{\varepsilon}, v_2(t) \right) \right\|_{L^2} dt \]

\[
\lesssim \left( \| v_1 \|_{L^\infty([0,M],H^1)}^\alpha + \| v_2 \|_{L^\infty([0,M],H^1)}^\alpha \right) \| v_1 - v_2 \|_{L^\infty([0,M],L^2)}.
\]

Note that

\[
\| z_1 \|^\alpha z_1 - |z_2|^\alpha z_2 | \lesssim (|z_1|^\alpha + |z_2|^\alpha) |z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C}.
\]

It follows from the embedding \( L^\infty(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \) that

\[
\sup_{\varepsilon > 0} \int_0^M \left\| \mathcal{Q}\left( \frac{t}{\varepsilon}, v_1(t) \right) - \mathcal{Q}\left( \frac{t}{\varepsilon}, v_2(t) \right) \right\|_{L^2} dt \]

\[
= \sup_{\varepsilon > 0} \int_0^M \left\| T_{D(\varepsilon)} v_1(t) - T_{D(\varepsilon)} v_2(t) \right\|_{L^2} dt \]

\[
\lesssim \int_0^M \left( \| v_1(t) \|_{H^1}^\alpha + \| v_2(t) \|_{H^1}^\alpha \right) \| v_1(t) - v_2(t) \|_{L^2} dt \]

\[
\lesssim \left( \| v_1 \|_{L^\infty([0,M],H^1)}^\alpha + \| v_2 \|_{L^\infty([0,M],H^1)}^\alpha \right) \| v_1 - v_2 \|_{L^\infty([0,M],L^2)}.
\]
Next, to complete the proof of (3.1), we first observe that
\[
\left| |f|^\alpha f - |g|^\alpha g \right| \lesssim |f|^\alpha |f' - g'| + |g|^\alpha |g'| + \|f|^{\alpha - 1} f^2 - |g|^{\alpha - 1} g^2 |g'|
\]
for continuously differentiable functions \(f, g\) on \(\mathbb{R}\). Thus,
\[
\int_0^M \left\| \partial_t \left( Q \left( \frac{t}{\varepsilon}, v_1(t) \right) - Q \left( \frac{t}{\varepsilon}, v_2(t) \right) \right) \right\|_{L^2} \, dt \\
= \int_0^M \left\| \partial_t \left( |T_{D(\varepsilon)} v_1(t)|^\alpha |T_{D(\varepsilon)} v_2(t)| - |T_{D(\varepsilon)} v_2(t)|^\alpha |T_{D(\varepsilon)} v_1(t)\right) \right\|_{L^2} \, dt \\
\lesssim \int_0^M \left\| |T_{D(\varepsilon)} v_1(t)|^\alpha \partial_t \left( T_{D(\varepsilon)} v_1(t) - T_{D(\varepsilon)} v_2(t) \right) \right\|_{L^2} \, dt \\
+ \int_0^M \left\| \left( |T_{D(\varepsilon)} v_1(t)|^\alpha - |T_{D(\varepsilon)} v_2(t)|^\alpha \right) \partial_t T_{D(\varepsilon)} v_2(t) \right\|_{L^2} \, dt \\
+ \int_0^M \left\| |T_{D(\varepsilon)} v_1(t)|^{\alpha - 1} (T_{D(\varepsilon)} v_1(t))^2 - |T_{D(\varepsilon)} v_2(t)|^{\alpha - 1} (T_{D(\varepsilon)} v_2(t))^2 \right\| \partial_t T_{D(\varepsilon)} v_2(t) \right\|_{L^2} \, dt.
\]
(3.3)

It is easy to see that the first term of (3.3) is bounded as
\[
\sup_{\varepsilon > 0} \int_0^M \left\| |T_{D(\varepsilon)} v_1(t)|^\alpha \partial_t \left( T_{D(\varepsilon)} v_1(t) - T_{D(\varepsilon)} v_2(t) \right) \right\|_{L^2} \, dt \\
\lesssim \|v_1\|_{L^\infty([0,M],H^1)} \|\partial_t (v_1 - v_2)\|_{L^\infty([0,M],H^1)}.
\]
Since the third term can be bounded similarly to the second term, we bound the second term only. First, we note that, for \(z_1, z_2 \in \mathbb{C}\),
\[
|z_1|^\alpha - |z_2|^\alpha \lesssim |z_1 - z_2||z_1|^{\alpha - 1} + |z_2|^{\alpha - 1}
\]
since \(\alpha \geq 1\). Thus, similarly as before, we have
\[
\sup_{\varepsilon > 0} \int_0^M \left\| \left( |T_{D(\varepsilon)} v_1(t)|^\alpha - |T_{D(\varepsilon)} v_2(t)|^\alpha \right) \partial_t T_{D(\varepsilon)} v_2(t) \right\|_{L^2} \, dt \\
\lesssim \int_0^M \|v_1(t) - v_2(t)\|_{H^1} \left( \|v_1(t)\|^{\alpha - 1} + \|v_2(t)\|^{\alpha - 1} \right) \|\partial_t v_2(t)\|_{L^2} \, dt \\
\lesssim \left( \|v_1\|_{L^\infty([0,M],H^1)} + \|v_2\|_{L^\infty([0,M],H^1)} \right) \|v_1 - v_2\|_{L^\infty([0,M],H^1)},
\]
which completes the proof.

Using lemma 3.1, we have the following lemma which is the key ingredient in our work. This is inspired by [9, 17] where the nonlinear Schrödinger equation with strong confinement was analyzed.
Lemma 3.2. Let $\alpha \geq 2$ and $M > 0$. If $v \in C([-M,M], H^1(\mathbb{R}))$, then
\[
\sup_{t \in [-M,M]} \left\| \int_0^t \frac{e^{i\alpha s^2}}{s^2} \left[ Q \left( \frac{s}{\varepsilon}, v(s) \right) - \langle Q \rangle (v(s)) \right] \right\|_{H^1} \to 0 \quad (3.4)
\]
as $\varepsilon \to 0$.

Proof. We consider positive times only. It follows from lemma 3.1 that if $v_1, v_2$ belong to $C([-M,M], H^1(\mathbb{R}))$, then
\[
\sup_{t \in [0,M]} \left\| \int_0^t e^{i\alpha s^2} \left( Q \left( \frac{s}{\varepsilon}, v_1(s) \right) - Q \left( \frac{s}{\varepsilon}, v_2(s) \right) \right) \right\|_{H^1} \lesssim \left( \|v_1\|_{L^\infty(0,M), H^1} + \|v_2\|_{L^\infty(0,M), H^1} \right) \|v_1 - v_2\|_{L^\infty(0,M), H^1}
\]
and
\[
\sup_{t \in [0,M]} \left\| \int_0^t e^{i\alpha s^2} \left( Q(v_1(s)) - Q(v_2(s)) \right) \right\|_{H^1} \lesssim \left( \|v_1\|_{L^\infty(0,M), H^1} + \|v_2\|_{L^\infty(0,M), H^1} \right) \|v_1 - v_2\|_{L^\infty(0,M), H^1}.
\]
Therefore, by a density argument, it is enough to prove (3.4) for $v \in C^1([0,M], S(\mathbb{R}))$ only, where the Schwartz space $S(\mathbb{R})$ consists of infinitely differentiable, rapidly decreasing functions.

We define
\[ Q(\theta, f) := \int_0^\theta \left[ Q(s, f) - \langle Q(f) \rangle \right] \, ds \]
on $[0, \infty) \times H^1(\mathbb{R})$. Note that for each $f \in H^1(\mathbb{R})$, $Q(\cdot, f)$ is two-periodic since $Q(\cdot, f)$ is a two-periodic function whose average is $\langle Q(f) \rangle$. Thus, we have
\[
\sup_{\theta \in [0,\infty]} \|Q(\theta, f)\|_{H^1} = \sup_{\theta \in [0,2]} \|Q(\theta, f)\|_{H^1} \lesssim \sup_{\theta \in [0,2]} \int_0^\theta \left( \|Q(s, f)\|_{H^1} + \|Q(f)\|_{H^1} \right) \, ds \quad (3.5)
\]
for all $f \in H^1(\mathbb{R})$, where we use lemma 2.1 in the last inequality. Similarly, it follows from lemma 2.2 that
\[
\sup_{\theta \in \mathbb{R}} \|\partial_\theta^2 Q(\theta, f)\|_{H^1} \lesssim \|f\|_{H^1}^{\alpha+1} \quad (3.6)
\]
for all $f \in H^1(\mathbb{R})$.

Now let $v \in C^1([0,M], S(\mathbb{R}))$. By a simple calculation, we have
Thus, by the same argument as above, we have
\[
\frac{d}{ds} \left( e^{id_{\alpha}(t-s)\partial_x^2} Q \left( \frac{s}{\varepsilon}, v(s) \right) \right) = -id_{\alpha} e^{id_{\alpha}(t-s)\partial_x^2} \partial_x^2 Q \left( \frac{s}{\varepsilon}, v(s) \right)
\]
\[
+ \frac{1}{\varepsilon} e^{id_{\alpha}(t-s)\partial_x^2} \left( Q \left( \frac{s}{\varepsilon}, v(s) \right) - \langle Q \rangle (v(s)) \right) + e^{id_{\alpha}(t-s)\partial_x^2} \int_0^s \frac{d}{ds} \left[ T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) - \int_0^s T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds \right] ds'.
\]
(3.7)

Note that
\[
\frac{d}{ds} \int_0^1 T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds = \int_0^1 T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds
\]
which is two-periodic in $s'$. Moreover, its average over one period is
\[
\frac{d}{ds} \int_0^1 T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds.
\]
Thus, by the same argument as above, we have
\[
\sup_{\theta \in [0,2]} \left\| \int_0^\theta \frac{d}{ds} \left[ T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) - \int_0^s T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds \right] ds' \right\|_{H^1}
\]
\[
= \sup_{\theta \in [0,2]} \left\| \int_0^\theta \frac{d}{ds} \left[ T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) - \int_0^s T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds \right] ds' \right\|_{H^1}
\]
\[
\lesssim \| v(s) \|_{H^1} \| \partial_x v(s) \|_{H^1}.
\]
Therefore, using the bounds (3.5), (3.6) and (3.8) together with (3.7), we obtain
\[
\left\| \int_0 T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)a v(s)) ds \right\|_{L^\infty([0,M],H^1)}
\]
\[
\leq \varepsilon \sup_{r \in [0,M]} \left\| \left[ \frac{s}{\varepsilon}, v(t) \right] \right\|_{H^1} + \varepsilon \| d_{\alpha} \| \left\| \partial_x^2 Q \left( \frac{s}{\varepsilon}, v(s) \right) \right\|_{H^1} ds
\]
\[
+ \varepsilon \left\| \int_0^M \left\| \frac{d}{ds} \left( T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) - \int_0^s T^{-1}_{\alpha}(\frac{s}{\varepsilon}) (T_{\alpha}(s)\partial_x v(s)) ds \right) ds' \right\|_{H^1} ds\right.
\]
\[
\lesssim \varepsilon \left[ \| v \|_{L^\infty([0,M],H^1)}^{\alpha+1} + \| v \|_{L^\infty([0,M],H^1)} \| \partial_x v \|_{L^\infty([0,M],H^1)} \right]
\]
which completes the proof. \qed
In preparation for the proof of the main theorem, theorem 1.1, we provide the local existence of a solution for (1.3). Given \( \varphi \in H^1(\mathbb{R}) \), we consider the Duhamel formula for (1.3)

\[
v_{\varepsilon}(t) = e^{i\varepsilon d \alpha \frac{\partial^2}{\partial t^2}} \varphi + i \int_0^t e^{i(t-s)d \alpha \frac{\partial^2}{\partial x^2}} Q \left( \frac{s}{\varepsilon}, v_{\varepsilon}(s) \right) ds.
\] (3.9)

**Proposition 3.3.** Let \( d \alpha \in \mathbb{R} \) and \( \alpha > 0 \). For every \( K > 0 \), there exist \( M_{\pm} > 0 \) such that if \( \varepsilon > 0 \) and the initial datum \( \varphi \in H^1(\mathbb{R}) \) satisfies \( \| \varphi \|_{H^1} \leq K \), then there is a unique solution \( v_{\varepsilon} \in C([-M_{-}, M_{+}], H^1(\mathbb{R})) \) of (3.9). Moreover,

\[
\| v_{\varepsilon}(t) \|_{H^1} \leq 2K \quad \text{for all } \varepsilon > 0 \quad \text{and } t \in [-M_{-}, M_{+}].
\]

**Proof.** The result is quite standard and its proof is very analogous to that of proposition 3.5 in [6]. Instead of lemma 2.6 in [6], we use lemma 2.1 and

\[
\sup_{s \in \mathbb{R}} \| Q(s, f) - Q(s, g) \|_{L^2} = \sup_{s \in \mathbb{R}} \| T_{D_{\alpha}}(s) f - T_{D_{\alpha}}(s) g \|_{L^2} \leq \left( \| f \|_{H^1} + \| g \|_{H^1} \right) \| f - g \|_{L^2},
\]

for all \( f, g \in H^1(\mathbb{R}) \).

**Proof of Theorem 1.1.** Let \( M > 0 \) be fixed and consider positive times only. Let

\[
K = 2 \sup_{t \in [0, M]} \| v(t) \|_{H^1}.
\] (3.10)

Thus, since \( \| v \|_{H^1} = \| v(0) \|_{H^1} \leq K/2 \), it follows from proposition 3.3 that there exists a positive \( M_1 \) corresponding to \( K/2 \), independent of \( \varepsilon \), such that \( v_{\varepsilon} \in C([0, M_1], H^1(\mathbb{R})) \) and

\[
\sup_{\varepsilon > 0} \sup_{t \in [0, M_1]} \| v_{\varepsilon}(t) \|_{H^1} \leq K.
\] (3.11)

We assume that \( M_1 < M \). Using Duhamel’s formula, for \( t \in [0, M_1] \), we write

\[
v_{\varepsilon}(t) - v(t) = i \mathcal{I}_1(t) + i \mathcal{I}_2(t),
\]

where

\[
\mathcal{I}_1(t) = \int_0^t e^{i\varepsilon d \alpha (t-s)^2} \left[ Q \left( \frac{s}{\varepsilon}, v_{\varepsilon}(s) \right) - Q \left( \frac{s}{\varepsilon}, v(s) \right) \right] ds
\]

and

\[
\mathcal{I}_2(t) = \int_0^t e^{i\varepsilon d \alpha (t-s)^2} \left[ Q \left( \frac{s}{\varepsilon}, v(s) \right) - \langle Q \rangle(v(s)) \right] ds.
\]

Then, it follows from lemma 3.2 that

\[
\| \mathcal{I}_2 \|_{L^\infty([0, M_1], H^1)} \geq \eta_{\varepsilon} \rightarrow 0
\] (3.12)

as \( \varepsilon \rightarrow 0 \). On the other hand, in order to estimate \( \mathcal{I}_1 \), use Minkowski’s inequality and the same
argument in the proof of lemma 3.1, then we obtain
\[
\|I_1(t)\|_{H^1} \leq \int_0^t \left\| Q \left( \frac{s}{\varepsilon}, v_\varepsilon(s) \right) - \frac{s}{\varepsilon} v(s) \right\|_{H^1} ds \\
\lesssim \int_0^t \left( \|v_\varepsilon(s)\|_{H^1} + \|v(s)\|_{H^1} \right) \|v_\varepsilon(s) - v(s)\|_{H^1} ds \\
\lesssim K^\alpha \int_0^t \|v_\varepsilon(s) - v(s)\|_{H^1} ds
\] (3.13)
for all \(0 \leq t \leq M_1\), where we use (3.10) and (3.11) in the last bound. It follows from (3.12) and (3.13) that there exists a positive constant \(C\), independent of \(\varepsilon\), such that
\[
\|v_\varepsilon(t) - v(t)\|_{H^1} \leq \eta_\varepsilon + C \int_0^t \|v_\varepsilon(s) - v(s)\|_{H^1} ds
\]
for all \(0 \leq t \leq M_1\). Thus, by Gronwall’s inequality, we obtain
\[
\sup_{t \in [0, M_1]} \|v_\varepsilon(t) - v(t)\|_{H^1} \leq \eta_\varepsilon e^{Ct} \leq \eta_\varepsilon e^{CM_1} \to 0 \quad \text{as } \varepsilon \to 0.
\]
Note that assuming \(M_1 < M\) is acceptable since, if \(M_1 \geq M\), replacing \(M_1\) on the above by \(M\) completes the proof.

Next, since
\[
\sup_{\varepsilon > 0} \|v_\varepsilon(M_1)\|_{H^1} \leq K
\]
by (3.11), it follows from proposition 3.3 with the initial datum \(v_\varepsilon(M_1)\) that the solution \(v_\varepsilon\) exists on \([M_1, M_1 + M_2]\) for some positive \(M_2\) corresponding to \(K\), independent of \(\varepsilon\), and
\[
\sup_{\varepsilon > 0} \sup_{\varepsilon > 0} \sup_{t \in [0, M_1 + M_2]} \|v_\varepsilon(t)\|_{H^1} \leq 2K.
\]
The last inequality together with (3.11) yields
\[
\sup_{\varepsilon > 0} \sup_{\varepsilon > 0} \sup_{t \in [0, M_1 + M_2]} \|v_\varepsilon(t)\|_{H^1} \leq 2K
\]
and, therefore, by the same argument as above, we obtain
\[
\|v_\varepsilon(t) - v(t)\|_{L^\infty((0, \min(M_1 + M_2), H^1)} \to 0
\] (3.14)
as \(\varepsilon \to 0\). Thus, if \(M_1 + M_2 \geq M\), then the proof is complete.

Now assume that \(M_1 + M_2 < M\). Then, by (3.10) and (3.14), there exists \(\varepsilon_1 > 0\) such that
\[
\sup_{0 < \varepsilon \leq \varepsilon_1} \sup_{\varepsilon > 0} \sup_{t \in [0, M_1 + M_2]} \|v_\varepsilon(t)\|_{H^1} \leq K.
\]
Moreover, since \(v_\varepsilon \in C((0, M_1 + M_2), H^1(\mathbb{R}))\),
\[
\sup_{0 < \varepsilon \leq \varepsilon_1} \|v_\varepsilon(M_1 + M_2)\|_{H^1} \leq K.
\]
Replacing \(v_\varepsilon(M_1)\) and \(\varepsilon > 0\) by \(v_\varepsilon(M_1 + M_2)\) and \(0 < \varepsilon < \varepsilon_1\), respectively, in the previous
argument, we obtain
\[ \|v_\varepsilon - v\|_{L^\infty([0, \min\{M_1, M_1 + 2M_2\}], H^1)} \to 0 \]
as \( \varepsilon \to 0 \). Iterating this procedure finitely many times, until the sum of \( M_1 \) and the integer multiple of \( M_2 \) gets greater than or equal to \( M \), completes the proof. \( \Box \)

Acknowledgments

Young-Ran Lee and Mi-Ran Choi are supported by the National Research Foundation of Korea (NRF) grants funded by the Korean government (MSIT) NRF-2020R1A2C1A01010735 and (MOE) NRF-2021R1I1A1A01045900. The authors are grateful for discussions with Dirk Hundertmark.

References

[1] Ablowitz M J and Biondini G 1998 Multiscale pulse dynamics in communication systems with strong dispersion management Opt. Lett. 23 1668–70
[2] Agrawal G 2012 Nonlinear Fiber Optics 5th edn (New York: Academic)
[3] Albert J and Kahlil E 2017 On the well-posedness of the Cauchy problem for some nonlocal nonlinear Schrödinger equations Nonlinearity 30 2308–33
[4] Antonelli P, Saut J-C and Sparber C 2013 Well-posedness and averaging of NLS with time-periodic dispersion management Adv. Differ. Equ. 18 491–68
[5] Cazanave T 2003 Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics vol 10) (Providence, RI: American Mathematical Society)
[6] Choi M-R, Hundertmark D and Lee Y-R 2020 Well-posedness of dispersion managed nonlinear Schrödinger equations (arXiv:2003.09076)
[7] Choi M-R, Hundertmark D and Lee Y-R 2017 Thresholds for existence of dispersion management solitons for general nonlinearities SIAM J. Math. Anal. 49 1519–69
[8] Choi M-R, Kang Y and Lee Y-R 2021 On dispersion managed nonlinear Schrödinger equations with lumped amplification J. Math. Phys. 62 071506
[9] Frank R L, Méhats F and Sparber C 2017 Averaging of nonlinear Schrödinger equations with strong magnetic confinement Commun. Math. Sci. 15 1933–45
[10] Gabitov I and Turitsyn S K 1996 Breathing solitons in optical fiber links JETP Lett. 63 861–6
[11] Gabitov I R and Turitsyn S K 1996 Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation Opt. Lett. 21 327–9
[12] Hundertmark D, Kunstmann P and Schnaubelt R 2015 Stability of dispersion managed solitons for vanishing average dispersion Arch. Math. 104 283–8
[13] Hundertmark D and Lee Y-R 2012 On non-local variational problems with lack of compactness related to non-linear optics J. Nonlinear Sci. 22 1–38
[14] Kunze M 2004 On a variational problem with lack of compactness related to the Strichartz inequality Calc. Var. Partial Differ. Equ. 19 307–36
[15] Lin C, Cohen L G and Kogelnik H 1980 Optical-pulse equalization of low-dispersion transmission in single-mode fibers in the 1.3–1.7-μm spectral region Opt. Lett. 5 476–8
[16] Mamyshev P V and Mamysheva N A 1999 Pulse-overlapped dispersion-managed data transmission and intrachannel four-wave mixing Opt. Lett. 24 1454–6
[17] Méhats F and Sparber C 2016 Dimension reduction for rotating Bose–Einstein condensates with anisotropic confinement Discrete Contin. Dyn. Syst. 36 5097–118
[18] Pérez-Garcia V M, Torres P J and Konotop V V 2006 Similarity transformations for nonlinear Schrödinger equations with time-dependent coefficients Physica D 221 31–6
[19] Serkin V N and Hasegawa A 2001 Novel soliton solutions of the nonlinear Schrödinger equation model Phys. Rev. Lett. 85 4502–5
[20] Sulem C and Sulem P-L 1999 The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse Applied Mathematical Sciences vol 139 (New York: Springer)

[21] Turitsyn S K, Bale B G and Fedoruk M P 2012 Dispersion-managed solitons in fibre systems and lasers Phys. Rep. 521 135–203

[22] Turitsyn S K, Shapiro E G, Medvedev S B, Fedoruk M P and Mezentsev V K 2003 Physics and mathematics of dispersion-managed optical solitons C. R. Phys. 4 145–61

[23] Zharnitsky V, Grenier E, Jones C K R T and Turitsyn S K 2001 Stabilizing effects of dispersion management Physica D 152–153 794–817