Dynamics of the magnetic flux penetration into type II superconductors

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Abstract

The magnetic flux penetration dynamics of type-II superconductors in the flux flow regime is studied by analytically solving the nonlinear diffusion equation for the magnetic flux induction, assuming that an applied field parallel to the surface of the sample and using a power-law dependence of the differential resistivity on the magnetic field induction. An exact solution of nonlinear diffusion equation for the magnetic induction is obtained using a well known self-similar technique.

Key words: superconductors, nonlinear diffusion, flux flow, flux creep.

§1. Introduction

Theoretical investigations of the magnetic flux penetration dynamics into superconductors in various regimes with a various current-voltage characteristics is one of key problems of electrodynamics of superconductors. Mathematical problem of theoretical study the dynamics of evolution and penetration of magnetic flux into the sample in the viscous flux flow regime can be formulated on the basis of a nonlinear diffusion-like equation [1-3] for the magnetic field induction in a superconductor [4-14]. The dynamics of space-time evolution of the magnetic flux penetration into type-II superconductors, where the flux lines are parallel to the surface of the sample for the viscous flux flow regime with a nonlinear relationship between the field and current density in type II superconductors has been studied by many authors [4-7]. The magnetic flux penetration problem was theoretically studied for the particular case, where the flux flow resistivity independent of the magnetic field by authors [5]. Similar problem has been considered in [6] for the semi-infinite sample in parallel geometry. The situation, where flux flow resistivity depends linearly on the magnetic field induction was considered analytically in [4]. Analogical problem for the creep regime with a nonlinear relationship between the current and field has been considered in [8-14]. The magnetic flux penetration into the superconductor sample, where the flux lines are perpendicular to the surface of the sample is described by a non-local nonlinear diffusion equation [7]. This problem has been exactly solved by Briad and Dorogovstev [7] for the case thin film geometry in the flux flow regime of a type-II superconductors.

§2. Objectives

In the present paper the magnetic flux penetration dynamics of type-II superconductors in the flux flow regime is studied by analytically solving the nonlinear diffusion equation for the magnetic flux induction, assuming that an applied field parallel to the surface of the sample and using a power-law dependence of the differential resistivity on the magnetic field induction. An exact solution of nonlinear diffusion equation for the magnetic induction \( \vec{B}(r, t) \) is obtained by using a well known self-similar technique. We study the problem in the framework of a macroscopic approach, in which all lengths scales are larger than the flux-line spacing; thus, the superconductor is considered as an uniform medium.

§3. Formulation of the problem

Bean [15] has proposed the critical state model which is successfully used to describe magnetic properties of type II superconductors. According to this model, the distribution of the magnetic flux density \( \vec{B} \) and the transport current density \( \vec{j} \) inside a superconductor is given by a solution of the equation

\[ \text{rot} \vec{B} = \mu_0 \vec{j}. \]  

(1)

When the penetrated magnetic flux changes with time, an electric field \( \vec{E}(r, t) \) is generated inside the sample according to Faraday’s law

\[ \text{rot} \vec{E} = \frac{d \vec{B}}{dt}. \]  

(2)

In the flux flow regime the electric field \( \vec{E}(r, t) \) induced by the moving vortices is related with the local current density \( \vec{j}(r, t) \) by the nonlinear Ohm’s law

\[ \vec{E} = \rho \vec{j}. \]  

(3)

In combining the relation (3) with Maxwell’s equation (2), we obtain a nonlinear diffusion equation for the magnetic flux induction \( \vec{B}(r, t) \) in the following form

\[ \frac{d \vec{B}}{dt} = \frac{1}{\mu_0} \nabla \left( \rho(B) \nabla \vec{B} \right). \]  

(4)

Formally, this differential equation is simply a nonlinear diffusion equation with a diffusion coefficient depending on magnetic induction \( B \). The parabolic type diffusion equation (4) allows to obtain a time and space distribution of the magnetic induction profile in a superconductor sample. It is evident that the space-time structure of the solution of the diffusion equation (4) is determined by the character of dependence of the differential resistivity coefficient \( \rho \) on the magnetic field induction \( B \). Usually, in real experimental situation [16], the differential resistivity \( \rho \) grows with an increase of magnetic field induction

\[ \rho = \frac{\vec{B} \phi_0}{\eta c^2} = \rho_n \frac{\vec{B}}{H_{c2}}, \]  

(5)

where \( \rho_n \) is the differential resistivity in the normal state; \( \eta \) is the viscous coefficient, \( \phi_0 = \pi hc/2e \) is the magnetic flux quantum, \( H_{c2} \) is the upper critical field of superconductor [16]. In the case, when the differential resistivity \( \rho \) is a linear function of the magnetic field induction \( B \) an exact solution of the diffusion equation (4) can be easily obtained by using the well-known scaling methods [1, 2]. For the complex dependence of \( \rho(B) \) it can be use by empirical power-law dependence \( \rho(B) = B^n \), where \( n \) is the positive constant parameter.

§4. Basic equations

We formulate the general equation governing the dynamics of the magnetic field induction in a superconductor sample. We study the evolution of the magnetic penetration process in a simple geometry - superconducting semi-infinite sample \( x \geq 0 \). We assume that the external magnetic field induction \( B_e \) is parallel to the z-axis. When the magnetic field with the flux density \( B_e(t) \) is applied in the direction of the z-axis, the transport current \( \vec{j}(r, t) \) and the electric field \( \vec{E}(r, t) \) are induced inside the slab along the y-axis. For this geometry, the spatial and temporal evolution of magnetic field induction \( \vec{B}(r, t) \) is described by the following nonlinear diffusion equation in the generalized dimensionless form [7]

\[ \frac{db}{dt} = \frac{d}{dx} \left[ \rho_n \frac{d}{dx} \left( \rho_n \frac{d}{dx} \right)^n \right], \]  

(6)
where we have introduced the dimensionless parameters \( b = B/B_c \), \( j\sqrt{x'} = x/x_0 \), \( t' = t/\tau \) and variables \( x_0 = B_c/\mu_0 J_0 \) is the magnetic field penetration depth in a Bean model; \( \tau = \mu_0 j_B^2/\mu_0 B^2 \) is the relaxation diffusion time; \( q \) is the constant positive parameter.

The diffusion equation (6) can be integrated analytically subject to appropriate initial and boundary conditions in the center of the sample and on the sample’s edges. We consider the case, when the magnetic field applied to sample increases with time according to a power law with the exponent of \( \alpha \geq 0 \)

\[
b(0, t) = b_0 t^\alpha \tag{7}
\]

Boundary condition (7) is equivalent to a linear increase in the magnetic field with time, which corresponds to a real experimental situation. As can be easily seen that the case \( \alpha = 0 \) describes a constant applied magnetic field at the surface of the sample, while the case \( \alpha = 1 \) corresponds to linearly increasing applied field, respectively.

The other boundary condition follows from the continuity of the flux at the free boundary \( x = x_p \)

\[
b(x_p, t) = 0, \tag{8}
\]

where \( x_p \) is the dimensionless position of the front of the magnetic field. The flux conservation condition for the magnetic field induction can be formulated in the following integral form

\[
\int b(x, 0) \, dx = 1. \tag{9}
\]

It should be noted that the nonlinear diffusion equation (6), completed by the boundary conditions for magnetic induction, totally determines the problem of the space-time distribution of the magnetic flux penetration into superconductor in the flux flow regime with a power-law dependence of differential resistivity on the magnetic field induction. Solution of this equation gives a complete description of the time and space evolution of the magnetic flux in a sample.

\section{5. Scaling solution}

In the following analysis we derive an evolution equation for the magnetic induction profile and formulate a similarity solution for the \( b(x, t) \). As can be shown that the nonlinear diffusion equation (6) can be solved exactly using well known scaling methods [1, 2]. At long times we present a solution of the nonlinear diffusion equation for the magnetic induction (6) in the following scaling form

\[
b(x, t) = t^\alpha f \left( \frac{x}{t^\beta} \right). \tag{10}
\]

The similarity exponents \( \alpha \) and \( \beta \) are of primary physical importance since the parameter \( \alpha \) represents the rate of decay of the magnetic induction \( b(x, t) \), while the parameter \( \beta \) is the rate of spread of the space distribution as time goes on. Inserting this scaling form into differential equation (6) and comparing powers of \( t \) in all terms, we get the following relationship for the exponents \( \alpha \) and \( \beta \)

\[
\alpha + 1 = \alpha(n + q) + \beta(1 + q). \tag{11}
\]

Using the condition of the flux conservation (9) we obtain

\[
\alpha = \beta = \frac{1}{n + 2q}, \tag{12}
\]

which suggests the existence of self-similar solutions in the form

\[
b(z) = t^{-1/(n+2q)} f(z), \quad z = x/t^{1/(n+2q)} . \tag{13}
\]

Substituting this scaling solution (11) into the governing equation (6) yields an ordinary differential equation for the scaling function \( f(z) \) in the form

\[
(n + 2q) \frac{d}{dz} \left[ f^n \left( \frac{df}{dz} \right)^{q} \right] + \frac{df}{dz} + f = 0. \tag{14}
\]

The boundary conditions for the function \( f(z) \) now become

\[
f(0) = 1, \quad f(z_0) = 0. \tag{15}
\]

The above equation (12), depending on the initial and the boundary conditions describes a scaling—like behavior magnetic flux front with a time—dependent velocity in the sample. After a further integration and applying the boundary conditions (13) we get the following solution of the problem

\[
f(z) = f(z_0) \left[ 1 - \left( \frac{z}{z_0} \right)^{1+q} \right]^{1/(n+q-1)}. \tag{16}
\]

The position of the front \( z_0 \) can now be found by substituting the solution (14) into the integral condition (9) and it is given by

\[
z_0^{\frac{1}{n+2q}} = \frac{1}{q+1} \frac{\Gamma \left( \frac{n+q}{n+q-1} \right) + \frac{1}{2}}{\Gamma \left( \frac{n+q}{n+q-1} \right) + \frac{1}{q+1}}. \tag{17}
\]

It is convenient to write the self-similar solution (14) in terms of a primitive variables, as

\[
b(x, t) = b_0(t) \left[ 1 - \left( \frac{x}{x_p(t)} \right)^{1+q} \right]^{1/(n+q-1)}. \tag{18}
\]

\[
b_0(t) = t^{-1/(n+2q)} \left[ \frac{n+q-1}{1+q} \left( \frac{q+1}{n+2q} \right)^{1/(n+q-1)} \right]. \tag{19}
\]

This solution describes the propagation of the magnetic field into the sample, the magnetic induction being localized in the domain between the surface \( x=0 \) and the flux front \( x_p \). This solution is positive in the plane \( x_p^2 > x^2 \) and is zero outside of it. Note, that only the \( x > 0 \) and \( t > 0 \) quarter of the plane is presented, because of it has physical relevance. The penetrating flux front velocity \( x = x_p(t) \) as a function of time can be described by the relation

\[
v \sim \frac{dx_p}{dt} \sim t^{-(2q+n-1)/(n+2q)}. \tag{20}
\]

The velocity of the magnetic flux front decreases rapidly as the magnetic flux propagates (Fig1).
6. Particular case

The spatial and temporal profiles of magnetic flux penetration in the sample depends on the set of three independent parameters, n, q and α. It is of interest to consider the nonlinear diffusion equation for the magnetic induction for different values of the exponents n, q and α. For a given parameter set n, q and α the form of the solution function f(z) can be obtained by solving the nonlinear diffusion equation (6) analytically by a self-similar technique. We solve the nonlinear diffusion equation analytically to provide expressions for the time-space evolution of the magnetic induction for different values of exponents n, q and α. Next, we systematically analyze the effect of different values of exponents on the shape of the magnetic flux front in the sample. Varying the parameters of the equation, we may observe a variety of shapes of the magnetic flux front in the sample. A similar approach has been presented in Ref. [7] within the framework of non-linear flux diffusion in transverse geometry. As can be shown below that different values of exponent n and q generate different space-time magnetic flux fronts. Below we consider a few more practically relevant examples for which the magnetic flux front has a different shape depending on the different values of exponents n and q.

6. 1. Case q = 1

Let us first consider the most interesting case q=1. In this particular case the spatial and temporal evolution of the magnetic flux induction is totally determined by the parameters n and α. In the following analysis we derive an evolution equation for the magnetic induction profile and apply the scalings of the previous section to formulate a similarity solution for the b(x, t). For this particular case nonlinear diffusion equation (6) can be solved exactly using the scaling method. Thus, based on the scalings described in the previous section, we get the following relation for the exponents

$$\alpha = \beta = -\frac{1}{n+2}.$$  

The last relation suggests the existence of solution to equation (6) of the form

$$b(x, t) = t^{-1/(n+2)} f(z), \quad z = x/(1/(n+2)).$$  

(17)

Substituting the similarity solution (17) into the governing equation (6) yields an ordinary differential equation for the scaling function f(z)

$$(2+n) \frac{d}{dz} \left[ f^n \frac{df}{dz} \right] + z \frac{df}{dz} + f = 0.$$  

(18)

Integrating the equation (18) by parts and applying the boundary conditions (13) give

$$f(z) = \left[ \frac{n}{2(n+2)} z_0^2 \right]^{1/n} \left[ 1 - \frac{z^2}{2} \right]^{1/n},$$  

(19)

which is the explicit form of the similarity solution, which we have been seeking. The position of the front z0 can now be found by substituting the last solution into the integral condition (9), so we have

$$\left[ \frac{n}{2(n+2)} z_0^2 \right]^{1/n} \int_0^{z_0} \left[ 1 - \frac{z^2}{2} \right]^{1/n} dz = 1.$$  

By using the following transformation

$$z = z_0 \sin \omega,$$

and after integrating we obtain

$$z_0^{(n+2)/n} \left[ \frac{n}{2(n+2)} \right]^{1/n} = 2 \sqrt{\frac{3}{\pi}} \frac{\Gamma \left( \frac{3}{2} + \frac{1}{n} \right)}{\Gamma \left( 1 + \frac{1}{n} \right)}.$$  

It is convenient to write the self-similar solution (19) in terms of a primitive variables, as

$$b(x, t) = b_0 \left[ 1 - \frac{x^2}{x_p^2} \right]^{1/n},$$  

(20)

where

$$b_0 = \left[ \frac{n}{2(n+2)} \right]^{1/n} \left[ 2 \right]^{-1/(n+2)}.$$  

Equation (20) constitutes an exact solution of the nonlinear flux-diffusion equation for the situation, when q=1. As can be seen the solution (20) describes the propagation of the flux profile inside the sample. The profile of the the normalized flux density b(x, t) for this case is shown schematically in figure 2.

Fig.2. The distribution of the normalized flux density b(x, t) at different times t=0.1, 0.2, 0.3 for n=1, q=1.

The penetrating flux front position x = x_p(t) as a function of time can be described by the relation

$$x_p = z_0 t^{1/(n+2)}.$$  

The velocity of penetration of a magnetic flux into a superconductor can be naturally determined from the last relation

$$v \sim t^{(n+1)/(n+2)}.$$  

Interestingly, that the normalized current density j(x, t) in the region, 0 < x < x_p can be presented using the equations

$$j(x, t) = -\frac{1}{c} \frac{db}{dx}.$$  

After a simple analytical calculation, we can easily obtain the space and time profiles of the normalized current density j(x, t) in the following form

$$j(x, t) = \frac{2b_0}{c n x_p} \left[ 1 - \left( \frac{x}{x_p} \right)^2 \right]^{1/n-1} x.$$  

6. 2. Case n = 0

Let us consider the case n = 0. For this particular case the long-time asymptotic behavior of the magnetic induction has the scaling form

$$b(x, t) = f(z), \quad z = x t^{-1/(q+1)}.$$  

(21)

Substituting the scaling solution into the governing equation (6) yields an ordinary differential equation for for the distribution f(z) in the form

$$(q + 1) \frac{d}{dz} \left[ f^q \right] + z \frac{df}{dz} = 0.$$  

Integrating the last equation and applying the boundary conditions (13) we get the following solution of the problem
The velocity of penetration of a current density into a superconductor can be approximately given as

\[
j(x, t) = \frac{1}{t^{1/(q+1)}} \left[ \frac{q - 1}{2q(q + 1)} \right]^{1/(q+1)} z^2 \left[ 1 - \frac{z^2}{q} \right]^{1/(q+1)}.
\]

The scaling solution (22) can be written in terms of the primitive variables as

\[
j(x, t) = \frac{1}{t^{1/(q+1)}} \left[ \frac{q - 1}{2q(q + 1)} \right]^{1/(q+1)} \left[ x \left( \frac{q}{q + 1} \right) \right]^{2/(q-1)} \left[ 1 - \frac{x^2}{q} \right]^{1/(q-1)}.
\]

The velocity of penetration of a current density into a superconductor is determined from the relation

\[
v = \frac{dx_p}{dt} \sim t^{-q/(q+1)}.
\]

We note that, an analogous problem has also been studied in [10, 11] in connection with the magnetic relaxation of a superconducting slab in the flux creep regime in the framework of an approximate power-law dependence of the electric field E on the current density \( I \). The authors [11] showed that in the case logarithmic barriers the relaxation process causes the system self-organize into critical state. Supposing that the homogeneous magnetic induction \( B_0 \) is induced by a constant magnetic field, they found the expression for the magnetization moment in the limit of \( n \rightarrow 1 \). It has been shown by Koziol [12] that a pinning potential depending logarithmically on current density leads to a similar nonlinear diffusion equation for the spatiotemporal evolution of the flux density with a power-law current-voltage characteristic. An approximate and exact solutions of the diffusion problem have been derived assuming that an external magnetic field directed to parallel the surface of a sample. The scaling relation between the characteristic relaxation time, magnetic field and a sample size has been found. Similar problem has been considered by Gilbert [8, 9]. The flux creep problem has been solved by Wang et al., [13] and for an exponential model by authors [14], numerically.

**§6. 3. Case \( n = 1 \)**

Let us now consider the case \( n = 1 \). In this particular case the spatial and temporal evolution of the magnetic flux induction is determined by the parameters \( q \) and \( \alpha \). In the following analysis we derive an evolution equation for the magnetic induction profile for the case \( n = 1 \) and apply the scalings of the previous section to formulate a similarity solution for the \( b(x, t) \). For this particular case nonlinear diffusion equation (6) can be solved exactly using the scaling method. Thus, based on the scalings described in the previous section, we get the following relation for the exponents

\[
\alpha = \beta = - \frac{1}{2q + 1}.
\]

The last relation suggests the existence of solutions of the form

\[
b(x, t) = t^{-1/(2q+1)} f(z), \quad z = x/t^{1/(2q+1)}.
\]

Substituting the similarity solution (23) into the governing equation (6) yields an ordinary differential equation for the function \( f(z) \)

\[
(2q + 1) \frac{d}{dz} \left( f \left( \frac{df}{dz} \right)^q \right) + \frac{df}{dz} + f = 0.
\]

Integrating the equation (24) by parts and applying the boundary conditions give

\[
f(z) = \left( \frac{q}{q+1} \right)^{\frac{q}{q+1}} \left[ 1 - \left( \frac{z^2}{z_0^2} \right)^{(q+1)/q} \right],
\]

which is the explicit form of the similarity solution we have been seeking.  The position of the front can now be found by substituting the last solution into the integral condition (9), and we have

\[
z_0 = \left[ q^{1/(q+1)}(2q+1) \right]^{(q+1)/(2q+1)}.
\]

It is convenient to write the self-similar solution (25) in terms of a primitive variables, as

\[
b(x, t) = b_0 \left[ 1 - \frac{x^2}{x_p^2} \right]^{1/q},
\]

where

\[
b(x, t) = t^{-1/(2q+1)} \left( \frac{q}{q+1} \right)^{q q^{q+1} + 1/(2q+1)}.
\]

Equation (6) describes an exact solution of the nonlinear flux-diffusion equation for the situation, when \( n = 1 \). As can be seen the solution (26) describes the propagation of the flux profile into the sample. The flux front can be approximately given as \( x_p = t^{1/(2q+1)} \).  The velocity of penetration of a magnetic flux induction front into a superconductor is determined from relation

\[
v = \frac{dx_p}{dt} \sim t^{-2q/(2q+1)}.
\]

The velocity of the magnetic flux front decreases rapidly as the magnetic flux propagates.

**§ 6. 4. Case \( n = 1, \quad q = 1 \)**

Let us now consider the case, where \( n = 1, \quad q = 1 \). For this particular case the nonlinear diffusion equation for the distribution magnetic flux admits an exact self-similar solution with a sharp front moving with a constant velocity. The scaling analysis show that the diffusion equation can be solved for the following values of parameters 1. \( \alpha = 0, \beta = 1/2 \); 2. \( \alpha = 1/3, \beta = 2/3 \); 3. \( \alpha = 1, \beta = 1/3 \); 4. \( \alpha = 1/3, \beta = 1/3 \). Let us consider a solution of the nonlinear diffusion problem for the exponents \( \alpha = 1/3, \beta = 1/3 \). In this case the nonlinear diffusion equation (6) admits an exact solution [4] with the similarity variable

\[
b(x, t) = t^{-1/3} f(x t^{-1/3}).
\]

By substituting the solution (27) into diffusion equation (6) we obtain an ordinary differential equation for the scaling function \( f(z) \) in the form

\[
3 \frac{d}{dz} \left( f \left( \frac{df}{dz} \right)^q \right) + z \frac{df}{dz} + f = 0.
\]

Integrating the equation (28) and using the boundary conditions for the magnetic induction \( f(z) \) we get a simple equation

\[
3 \frac{df}{dz} + z = 0.
\]

Separating variables and integrating equation (29) gives an implicit solution in the form

\[
f = \frac{1}{6} \left( z_0^2 - z^2 \right),
\]

where the integration constant is determined by integral relation (9) and has the form

\[
z_0 = \left( \frac{9}{2} \right)^{1/3}.
\]

Thus, we have solution for the magnetic induction in a primitive variables
The expression (31) describes an exact solution of the nonlinear flux-diffusion equation for the situation, when \( n = 1, \ q = 1 \). As can be seen the solution (31) describes the propagation of the flux profile into the sample. The profiles of the the normalized flux density \( b(x, t) \) and electric field density \( e(x, t) \) for this case is shown schematically in the figures 3a-3b.

![Graph showing normalized flux density and electric field density profiles](image)

Figs.3a and 3b. The distributions of the normalized flux density \( b(x, t) \) and electric field density \( e(x, t) \) at different times \( t = 0.1, 0.2, 0.3 \) for \( n = 1, q = 1 \).

It is evident from the solution (31) that the shock wave like magnetic flux propagates with wave front

\[
x_p = z_0 t^{1/3}.
\]

It can be shown that the sharp flux front moves with the time dependent velocity

\[
v = \frac{dx_p}{dt} \sim z_0 t^{-2/3}.
\]

The case \( \alpha = 1/3, \beta = 1/3 \) implies that the flux front asymptotically expands according to power-law \( x_p \propto t^{1/3} \) and magnetic flux penetration profile decreases as \( b \propto t^{-1/3} \). Similar problem for the case penetration of magnetic field into superconductors has been studied by Bass [4]. An analogous theoretical result in the viscous flow mode for vortices was obtained in [5] using the model of the critical state for oxide high-\( T_c \) superconductors.

Finally, in the case \( \alpha = 0, \beta = 1/2 \) the flux front position has the form of \( x_p \propto t^{1/2} \) and an approximate analytical solution for the case can be easily found solving the diffusion equation with the help of similarity variable \( b(x, t) = f(z) \), where \( z = f(x/t^{1/2}) \). In the vicinity of the front \( x \propto x_p \) the magnetic flux profile is given by the following simple relation \( b \propto \sqrt{2(x - x_p)} \). In this case, which corresponds to a constant external magnetic field the flux front position \( x_p \) grows with time as \( x_p \propto t^{1/2} \) for long times.

\[\text{§6. 5. Case } n = 2, \ q = 1\]

The scaling analysis show that the diffusion equation can be solved for the following values of parameters 1) \( \alpha = 1/4, \beta = 1/4 \); 2) \( \alpha = 1/4, \beta = 1/6 \). The solution to diffusion equation for this case can be presented in the following form

\[
b(x, t) = t^{-1/4} f(x/t^{1/4}).
\]

Substituting the solution (34) into the partial differential equation (6) and integrating with the boundary conditions we can see that magnetic flux profile has an exact solution in the form

\[
f = \frac{z_0}{2} \left[ 1 - \frac{x^2}{z_0^2} \right]^{1/2},
\]

where the integration constant has the form

\[
z_0 = \frac{2}{\sqrt{\pi}}.
\]

The solution for the magnetic induction in a primitive variables has the form

\[
b(x, t) = \frac{1}{\sqrt{\pi t^{1/4}}} \left[ 1 - \frac{\pi x^2}{4 t^{1/2}} \right]^{1/2}.
\]

The expression (35) describes an exact solution of the nonlinear flux-diffusion equation for the situation, when \( n = 2, \ q = 1 \). As can be seen the solution (35) describes the propagation of the flux profile into the sample. The evolution of the self-simulating process of magnetic field penetration into a superconductor is shown schematically in the figures 4a-4b.
Figs. 4a and 4b. The distributions of the normalized flux density \( b(x, t) \) and electric field density \( e(x, t) \) at different times \( t=0.1, 0.2, 0.3 \) for \( n = 2, q = 1 \).

In this case the front coordinate is \( x_p = t^{1/4} \). A main result is that the magnetic flux will move as a shock front with velocity \( v = t^{3/4} \). Similar problem for the case penetration of magnetic field into superconductors has been studied in [5].

§6. Case \( n = 0, q = 1 \)

In this case the diffusion equation can be solved in a different values of \( \alpha \) and \( \beta \), in particular 1). \( \alpha = 0, \beta = 1/2; 2). \( \alpha = -1/2, \beta = -1/2; 3). \alpha = 1, \beta = 1 \). Let us consider the case \( \alpha = 0, \beta = -1/2 \). In this case the differential equation is linear and admits an exact analytical solution. At long times we present a solution of the nonlinear diffusion equation for the magnetic induction (6) in the following scaling form

\[
 b(x, t) = t^{\alpha} f \left( \frac{x}{t^{1/2}} \right). \tag{36}
\]

Inserting the scaling form (36) into differential equation (6) we have

\[
 t^{\alpha+2\beta} \frac{d^2 f}{dz^2} = t^{\alpha-1} \left( \alpha f + \beta^2 \frac{df}{dz} \right). \tag{37}
\]

Comparing powers of \( t \) in all terms in (37), we get the following relationship for the exponents \( \alpha \) and \( \beta \)

\[
 \alpha = \beta = -\frac{1}{2},
\]

Then the problem admits a similarity solution of the form

\[
 b = t^{-1/2} f(xt^{-1/2}). \tag{38}
\]

Substituting the expression (38) into the nonlinear diffusion equation (6) gives an ordinary differential equation for the function similarity function \( f(z) \), namely

\[
 2 \frac{d^2 f}{dz^2} + \beta^2 \frac{df}{dz} + f = 0. \tag{39}
\]

Integrating (39) twice and applying the boundary conditions we have a self-similar solution in the form

\[
 b(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right), \tag{40}
\]

where \( x_p = \sqrt{4t} \).

The last formulae describes an exact solution of the nonlinear flux-diffusion equation for the situation, when \( n = 2, q = 1 \). As can be seen this solution describes the propagation of the flux profile into the sample. The profile of the self-simulating process of magnetic field penetration into a superconductor for this case is shown schematically in the figures 5a-5b.

Figs. 5a and 5b. The distributions of the normalized flux density \( b(x, t) \) and electric field density \( e(x, t) \) at different times \( t=0.1, 0.2, 0.3 \) for \( n = 0, q = 1 \).

Similar problem has been solved in [6] using an extended critical state model for type-II superconductors. Assuming that the critical current density and resistivity are constants a one-dimensional diffusion equation for the magnetic field were solved analytically for the case when an external magnetic field is increased with time according to a power law with the exponent of one- half.

§7. Flux creep

Let us consider the magnetic flux penetration process for the flux creep regime. According to Kim-Anderson theory [17, 18] the thermally activated flux motion in superconductor sample is described by an Arrhenius-type expression

\[
 v = v_0 \exp[-U/kT], \tag{41}
\]

where \( v_0 \) is the resistivity at \( T=0 \), \( U \) is an activation energy for thermally activated flux jumps, and determines the vortex pinning; \( T \) is the temperature and \( k \) is the Boltzmann constant. The activation energy \( U = U(\vec{j}, \vec{B}, T) \) depends on temperature \( T \), magnetic field induction \( \vec{B} \) and current density \( \vec{j} \). For the simple case it can be presented by Kim-Anderson [17] formulae

\[
 U(j) = U_0 \left( 1 - \frac{j}{j_c} \right), \tag{42}
\]

here \( U_0 \) is the characteristic scale of the activation energy and \( j_c = j_c(B) \) is the critical current density. In the flux creep state the effective
activation energy $U$ grows logarithmically [19] with decreasing current density as

$$U(j) = U_0 \ln \left( \frac{j}{j_c} \right)^n, \quad (43)$$

where the exponent $n$ depends upon the flux creep regime. The expression (43) gives a quite realistic description for activation barriers in a wide range of temperatures and magnetic fields. The power law characteristic dependence for $U(j)$ has been observed in numerous experiments, and it has been used extensively in recent theoretical studies for the field and current distributions in superconductors [19, 20]. Equation (43) is equivalent to a power law for the flux-flow resistivity

$$v = v_0 \left( \frac{j}{j_c} \right)^n. \quad (44)$$

In this case the phenomenological relation $\tilde{E}(\tilde{j})$ may be chosen in the power-law form

$$\tilde{E} = v_0|B| \left( \frac{j}{j_c} \right)^n \tilde{j}. \quad (45)$$

If $n = 1$ the last equation reduces to Ohm’s law, describing the normal or flux-flow state. For infinitely large $n$, the equation describes the Bean critical state model $j = j_c$ [15]. When $1 < n < \infty$, the last equation describes nonlinear flux creep.

In such case the analytic solution of the nonlinear creep equation can be constructed by choosing the critical current dependence on the magnetic field. Many models have been for the functional form of $j_c(B)$. For the critical current we adopt the power-law model [21], which can be applied over a relatively wide magnetic field range except in the high field region near the upper critical field

$$j_c(B) = j_0 \left( \frac{B_0}{B} \right)^\gamma, \quad (46)$$

where $j_0$ and $B_0$ are the characteristic values of the current density and magnetic field induction; $\gamma$ is the dimensionless pinning parameter, usually $0 < \gamma < 1$. If we assume $\gamma=0$, the above model reduces to the Bean-London model [15]. This model is applicable to the case where $j_c$ can be regarded approximately field independent. This power-law model for the critical current has also been used by other groups [14]. Another possible decay law would be exponential, which has often been used to take into account its decrease with the magnetic field [22]

$$j = j_0 \exp \left( -\frac{B}{B_0} \right),$$

where $B_0$ is a phenomenological parameter related to the pinning ability: the smaller it is, the more drastic is the decrease of the critical current with field. The numerical methods have been applied to resolve the flux diffusion equation, employing the exponential critical state model [14].

Next, based on the power law and exponential models, we shall study the distribution of the magnetic induction, current density and magnetization of superconductors.

### §7.1. Power law model

For the one-dimensional geometry the spatial and temporal evolution of magnetic field induction $\tilde{B}(r, t)$ is described by the following nonlinear diffusion equation [10] in the dimensionless form

$$\frac{db}{dt} = \frac{dx}{dx} \left[ b^{n+1} \frac{db}{dx} \right]^{n-1} \frac{db}{dx}. \quad (47)$$

where we have introduced the dimensionless variables and parameters

$$b = \frac{B}{B_0}, \quad x_p = \frac{\mu_0 j_0}{B_0} x, \quad t = \frac{\tau}{\tau}, \quad j = \frac{j}{j_0},$$

$$b = \epsilon = \frac{E}{v_0 B_0}, \quad B_0 = \mu_0 j_0 v_0. \quad (48)$$

The solution of above parabolic type diffusion equation allows to obtain the time and space distribution of the magnetic induction profile in the considered sample. Inserting the scaling form (10) into differential equation (47) and comparing powers of $t$ in all terms, we get the following relationship for the exponents $\alpha$ and $\beta$

$$\alpha + 1 = \beta + \alpha(n + 1) + \alpha \beta. \quad (49)$$

Using the condition of the flux conservation we obtain

$$\alpha = \beta = \frac{1}{2(n + \gamma + 1)} \quad (50)$$

which suggests the existence of self-similar solutions in the form

$$b(x, t) = t^{-\gamma/(2n + \gamma + 1)} f(z), \quad z = x^{(1/2n + \gamma + 1)} t^{(2n + \gamma - 1)}. \quad (49)$$

Substituting this scaling solution (49) into the governing equation (47) yields an ordinary differential equation for the scaling function $f(z)$ in the form

$$\frac{d}{dz} \left[ f(n+1) \frac{df}{dz} \right] + \frac{1}{2(n + \gamma + 1)} \frac{d}{dz} \left( \frac{df}{dz} \right) = 0. \quad (50)$$

The solution of the equation (50) must satisfy the following boundary conditions for the scaling function $f(z)$

$$f(0) = 1, \quad f(z_0) = 0. \quad (51)$$

After further integration of equation (50) and applying the boundary conditions we get the following solution of the problem

$$f(z) = f(z_0) \left[ 1 - \left( \frac{z}{z_0} \right)^{n+1} \right]^{1/(n+1)} \quad (52)$$

$$f(z_0) = \left[ \frac{(\gamma + 1)}{(n + 1)} \left( \frac{z_0}{2n + \gamma + 1} \right)^{1/n} \right]^{1/(n+1)}. \quad (53)$$

The position of the front $z_0$ can now be found by substituting the solution (52) into the integral condition (9) and it is given by

$$z_0^{(2n + \gamma + 1)/(n+1)} = \left[ \frac{n}{n + 1} \Gamma(\frac{\gamma + 2}{\gamma + 1}) \Gamma\left( \frac{n}{n + 1} \right) \right]^{1/(n+1)}. \quad (54)$$

The last solution can be presented as

$$b(x, t) = b_0 \left[ 1 - \left( \frac{x}{x_p} \right)^{n+1} \right]^{1/(n+1)}, \quad (53)$$

where

$$b_0(0, t) = t^{-\gamma/(2n + \gamma + 1)} \left[ \frac{(\gamma + 1)}{(n + 1)} \left( \frac{z_0^{n+1}}{2n + \gamma + 1} \right)^{1/n} \right]^{1/(n+1)}. \quad (55)$$

This solution describes the propagation of the magnetic field into the sample, the magnetic induction being localized in the domain between the surface $x=0$ and the flux front $x_p$. The flux front can be approximately given as $x_p = x_0 t^{1/(2n + \gamma + 1)}$. The velocity of penetration of the magnetic flux induction front into the superconductor sample is determined from the relation

$$v \sim t^{-n(2+\gamma)/(2n + \gamma + 1)}. \quad (54)$$

The velocity of the magnetic flux front decreases rapidly as the magnetic flux propagates.

Let us consider the most interesting case $n=1$. In this particular case the spatial and temporal evolution of the magnetic flux induction is totally determined by the parameters $\gamma$, $\alpha$ and $\beta$. In the following analysis we derive an evolution equation for the magnetic induction profile for the case $n=1$ and apply the scalings of the previous section to formulate a similarity solution for the $b(x, t)$.
described in the previous section, we get the following relation for the exponents

$$\alpha = \beta = \frac{1}{\gamma + 3}. \quad (55)$$

The last relation suggests the existence of solutions of the form

$$b(x, t) = t^{-1/(\gamma+3)} f(z), \quad z = x/t^{1/(\gamma+3)}.$$

Substituting the similarity function (56) into the governing equation (47) yields the following solution of the problem

$$f(z) = \left[ \frac{z_0^2 (\gamma + 1)}{2 (\gamma + 3)} \right]^{1/(\gamma+1)} \left[ 1 - \left( \frac{z}{z_0^{1/(\gamma+3)}} \right)^2 \right]^{1/(\gamma+1)}, \quad (57)$$

which is the explicit form of the similarity solution we have been seeking. The position of the front has the form

$$z_0^{(\gamma+3)/(\gamma+1)} = \left[ 2 (\gamma + 3) \right]^{1/(\gamma+1)} \left[ \frac{1}{1 + \frac{3}{2}} \right] \frac{G \left( \frac{1}{1 + \frac{1}{2}} \right)}{G \left( \frac{1}{1 + 1} \right)}.$$

or

$$b(x, t) = b_0 \left[ 1 - \frac{x^2}{x_p^2} \right]^{1/(\gamma+1)}, \quad (59)$$

where

$$b_0 = \left[ \frac{z_0^2 (\gamma + 1)}{2 (\gamma + 3)} \right]^{1/(\gamma+1)} t^{-1/(\gamma+3)}.$$

Equation (59) constitutes an exact solution of the nonlinear flux-diffusion equation for the situation, when n=1. The evolution of the self-simulating process of magnetic field penetration into a superconductor is shown schematically in figure 6a and 6b.

§7.2. Exponential model n=1

Let us consider a solution to the nonlinear diffusion equation for the exponential model, assuming that n=1. The nonlinear differential equation (47), describing evolution of the magnetic flux penetrated into the superconductor sample for the exponential model can be transformed to the following form

$$\frac{db}{dt} = \frac{d}{dx} \left[ e^{\beta x} \frac{db}{dx} \right]. \quad (60)$$

It is remarkable that one of explicit solution of the nonlinear diffusion equation (60) can be obtained by using the method of separation of variables. According to the method of separation of variables, we look for the solution of (60) in the form

$$b(x, t) = a(x) g(t). \quad (61)$$

Substituting this expression for b(x, t) into the partial differential equation (60) and separating variables we obtain the following equations for the distributions of the functions g(t) and a(x)

$$e^{-\gamma g} \frac{dg}{dt} = \lambda, \quad (62)$$

$$\frac{d}{dx} \left( e^{\gamma a} \frac{da}{dx} \right) = \lambda. \quad (63)$$

Integrating the differential equations (62) and (63) with respect to t and x, respectively and using the boundary conditions we obtain

$$e^{-\gamma b} = \lambda (t - t_p), \quad (64)$$

$$e^{\gamma a} = \frac{\lambda}{2} \left[ (x^2 - x_p^2) \right], \quad (65)$$

where $t_p$ is a constant, the parameter $x_p$ is defined by using the integral relation (9). Combining the equations (64) and (65) according to relation (61) we get

$$\phi = \frac{1}{\gamma} \ln \left[ \frac{(x^2 - x_p^2)}{2(t_p - t)} \right]. \quad (66)$$

§8. Magnetization

Let us consider the magnetization profile of the sample which is given by the following relation

$$-\mu_0 M(t) = b_0 - \frac{1}{d} \int_0^d b(x)dx.$$
Substituting here the scaling law, Eq. (15) one obtains

$$-\mu_0 M(t) = b_0 \left[ 1 - \frac{x_p(t)}{d} \psi_n \right],$$

where $d$ is the distance of the flux front penetrating into the sample; $\psi_n$ is a numerical factor which is determined by the average value of the scaling function $f(z)$ in the region $0 < x_p < d$, so

$$\psi_n = \frac{1}{2 \alpha} \int_0^\infty f(z)dz.$$  

Expressing the time dependence explicitly, the magnetization can be written as

$$-\mu_0 M(t) = b_0 \left[ 1 - \psi_n \left( \frac{t}{t^*} \right)^\beta \right],$$

for $0 < t < t^*$, where $t^* = (d/\psi_0)^\beta$. Differentiating the last equation and using an explicit solution for $x_p(t)$ for the case $\gamma = -1/n$ one obtains the expression to the magnetic relaxation rate

$$\frac{dM}{d\ln t} \approx \frac{1}{n+1} \left( \frac{t}{t^*} \right)^{1/(n+1)}.$$  

As the parameter $n$ increases the profile of the relaxation rate is seen to become more linear. At low temperature limit when $n \gg 1$ the relaxation rate varies linearly with time. The time dependence of the magnetic relaxation rate is shown in figure 7.

![Graph](image.png)

Fig.7. The time dependence of the magnetic relaxation rate for $n=3$, $5$, and $11$.

It has been shown by Vinokur [11] that at low temperatures the relationship between the time and relaxation rate is a linear over a very long time period in the early stage of the flux penetration. A such perfect linear relationship characterizes that the system behaves a self-organized criticality.

**Conclusion**

Thus, the problem of magnetic flux penetration into the half-space superconductor sample is studied in the flux flow regime in parallel geometry assuming that an external magnetic field increasing with time in accordance with the power law (7). Assuming that the flux flow resistivity as a power-law function of the magnetic field induction we found an exact analytical solution for the nonlinear local magnetic flux diffusion equation. The obtained solution describes space-time distribution of the magnetic induction in the sample. It was shown that the magnetic flux density profiles at different times follow the different scaling law. Similar scaling profiles can be found for the electric field and current density profiles, too. It was also shown that the spatial and temporal profiles of magnetic flux penetration in the sample depends on the set of three independent parameters, $n$, $q$ and $\alpha$. For a given parameter set $n$, $q$ and $\alpha$ the form of the scaling function $b(x, t)$ was obtained by solving the nonlinear diffusion equation for the magnetic field induction analytically by a self-similar technique. We analyzed the effect of different values of exponents on the propagation of the flux front in the sample. Varying the parameters of the equation, we have observed a various shapes of the magnetic flux front. Finally, we have discussed the flux creep problem, briefly.

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