Departure Time Choice Models in Urban Transportation Systems Based on Mean Field Games

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Departure time choice models play a crucial role in determining the traffic load in transportation systems. This paper introduces a new framework to model and analyze the departure time user equilibrium (DTUE) problem based on the so-called Mean Field Games (MFGs) theory. The proposed framework is the combination of two main components including (i) the reaction of travelers to the traffic congestion by choosing their departure times to optimize their travel cost; and (ii) the aggregation of the actions of the travelers, which determines the system level of service. The first component corresponds to a classic game theory model while the second one captures the travelers’ interactions at the macroscopic level and describes the system dynamics. In this paper, we first present a continuous departure time choice model and investigate the equilibria of the system. Specifically, we demonstrate the existence of the equilibrium and characterize the DTUE. Then, a discrete approximation of the system is provided based on deterministic differential game models to numerically obtain the equilibrium of the system. To examine the efficiency of the proposed model, we compare it with the departure time choice models in the literature. We apply our framework to a standard test case and observe that the solutions obtained based on our model are 5.6% better in terms of relative cost compared to the solutions determined based on models in the literature. Moreover, our proposed model converges with less number of iterations than the reference solution method in the literature. Finally, the model is scaled-up to the real test case corresponding to the whole Lyon Metropolis with real demand pattern. The results show that the proposed framework is able to tackle much larger test case than usual to includes multiple preferred travel times and heterogeneous trip lengths more accurately than existing models in the literature.

Key words: Departure time choice models, Departure time user equilibrium, Deterministic differential games, Mean Field Games, Macroscopic model, Bathtub model
1. Introduction

In urban transportation systems, representing trip-making behavior requires a deep understanding of the interactions and interrelation between travelers’ decisions and the systems’ performance (Mahmassani, Chang et al. 1985). In this context, studying the dynamics of travelers’ departure time choice behavior, particularly in congested systems, is of fundamental importance (Ben-Akiva and Bierlaire 2003). Departure time choice models represent how travelers choose their departure time considering their own desired arrival time (Hendrickson and Kocur 1981). The goal of each traveler in the system is to optimize its own travel cost (Dafermos 1968, Sheffi 1985). This means that all travelers are assumed to be fully rational decision-makers. They anticipate other travelers’ behavior in order to make an optimal decision, i.e., minimizing the additional travel cost due to the traffic congestion (Perakis and Roels 2006) and narrowing the final arrival time to the desired one (Smith 1984, Guo, Yang, and Huang 2018). Thus, the system behaves as a game in which the winners experience minimum travel cost (Cominetti, Correa, and Larré 2015) considering not only the travel cost but also the departure time. Based on the first principle of Wardrop (Wardrop 1952), the game may have an equilibrium state called Departure Time User Equilibrium (DTUE) where no traveler can improve his individual travel cost by changing his departure time (Mahmassani and Herman 1984, Ran, Boyce, and LeBlanc 1993).

Most of the previous studies on the DTUE problem in the literature are based on solving a classic Nash equilibrium problem coupled with a single point-queue (bottleneck) model based on the pioneering paper of Vickrey (1969). The idea behind the point-queue models is to assume that the travel cost on the transportation system consists of a free-flow travel cost plus a congestion cost represented by a queueing cost (Daganzo 1985). Therefore, DTUE arises because the queue capacity is limited, and travelers should consider the trade-off between the travel time losses and the costs corresponding to arriving later or earlier than the preferred arrival time (Ata and Peng 2018). A comprehensive literature review on the bottleneck models has been recently conducted by Li, Huang, and Yang (2020), which highlights the developments and the applications of bottleneck models to transportation systems in the past half-century.

Representing the urban transportation network by an origin, a destination, and a single bottleneck is not realistic (Lamotte and Geroliminis 2018, Nagel, Wagner, and Woesler 2003). Indeed congestion depends on the detailed topology of the transportation network (e.g., the spatial distribution of origins, destinations, routes, and roads). In addition, congestion is impacted by the
distribution of trips and vehicle densities (Jin 2020a, Ji, Luo, and Geroliminis 2014). To integrate such features while keeping the macroscopic scale, a common is to represent the network dynamics by Macroscopic Fundamental Diagram (MFD) models. This kind of model is also referred to bathtub models in the economic literature (Arnott, Kokoza, and Naji 2016, Mariotte, Leclercq, and Laval 2017, Lamotte and Geroliminis 2016).

1.1. MFD/Bathtub models

The first bathtub model was introduced by Vickrey (1991, 2020). The travel demand is described by (i) the total number of trips, (ii) the distribution of trips’ departure time, and (iii) the distribution of trips’ length. MFD and Bathtub are two names for a set of equations corresponding to macroscopic traffic models. The single bathtub model considers an undifferentiated movement area to represent a dense network of congested links. The motion of travellers is assumed to take place at a speed which is considered to be uniform over the network but varies over time depending on the overall network loading (Bao, Verhoef, and Koster 2020). Therefore, the model does not need the location information of the origin and destination of travelers. A trip is defined by its length and departure time in the dynamic setting. When a trip starts, its remaining distance to travel decreases following the evolution of the network mean speed. Note that the network speed depends on the network characteristics (e.g., network size and road capacities) as well as the load on the network (network density) (Fosgerau 2015). In order to capture the dynamics of the system, Vickrey (1991) defined an ordinary differential equation to describe the evolution of the number of active trips (users in the network). Such a model resort to a strong assumption that the average remaining distance of active trips is constant. Another option is to assume that the remaining trip distance of active trips follows a time-independent negative exponential distribution (Vickrey 1994, 2019). However, based on the empirical studies of Liu et al. (2012), Thomas and Tutert (2013), Tsekeris and Geroliminis (2013), travelers’ trip length distribution is neither time-independent nor exponential.

Recently, Jin (2020a) reviewed and analyzed several studies that relaxed the assumption on the trip length distribution (see, e.g., Leclercq, Sénécat, and Mariotte 2017, Mariotte, Leclercq, and Laval 2017, Lamotte and Geroliminis 2018) and then proposed the generalized bathtub model which captures any distribution of the trip length. From a mathematical point of view, the total number of active trips is the primary variable for most bathtub models. The generalized bathtub model focuses on the number of active trips with remaining distances greater than or equal to a threshold. By this definition, Jin (2020a) derived a set of partial differential equations to track the distribution of the remaining trip lengths. Further properties of the model are discussed in Jin (2020a). In this work, we use the generalized bathtub model to capture the state of the urban transportation network for the departure time equilibrium problem.
1.2. Departure time choice problem

The departure time choice (also known as “morning commute”) problem at the network level is well-reviewed by Lamotte (2018). One of the main questions that have not been well studied in the literature is how a departure time choice model can take into account the heterogeneity of the trip lengths with multiple preferred arrival time (Lamotte 2018). The most complex equilibrium problem that has been addressed in the literature is modeling and numerically solving the DTUE problem for a group of travelers with a single distributed preferred arrival time and heterogeneous trip lengths (Lamotte and Geroliminis 2018). This model is supported by empirical data and simulation. Note that even when a simpler bottleneck formulation is used for solving the DTUE problem (reviewed by Jin (2020b)), few studies in the literature consider multiple preferred arrival time for commuters (Akamatsu et al. 2020, Lindsey 2004, Doan, Ukkusuri, and Han 2011, Ramadurai et al. 2010, Takayama and Kuwahara 2017, Akamatsu et al. 2018, Lindsey, De Palma, and Silva 2019). Recall that the heterogeneity of the travelers’ trip distance is not considered by single bottleneck models because they consider a single origin-destination (Akamatsu et al. 2020). The goal of this paper is to develop a more general mathematical framework to address departure time choice equilibrium with heterogeneous trip lengths and many desired arrival times in an urban transportation network.

The concept of DTUE, originally, comes from game theory and Nash-equilibrium principles (Sun et al. 2017). In general, with rational travelers, the user equilibrium problems represent fixed points (Wang et al. 2018, Bortolomiol, Lurkin, and Bierlaire 2019). The equilibration process of DTUE models is always addressed by population game theory at the network level (Arnott 2013, Yang 2005, Arnott and Buli 2018). Population games have one strong assumption, which is called Myopia. Myopia in our problem means that travelers only take into account the current utilities of each alternative when choosing the departure time, without predicting other users’ reactions (i.e., the departure time adjustment) (Sandholm 2015). User interactions create new system states, which have new perceived utilities as a result of the evolution. This evolution process pushes the system at each iteration or day-to-day process toward equilibria. Iryo (2019) proved that when an evolution dynamics plays the role of the replicator dynamics, no stable equilibrium solution can be determined in the DTUE problem even when the demand profile is homogeneous, i.e., all users have the same travel distance. This study aims to overcome this limitation by employing a mean-field approximation and deriving a macroscopic framework. Note that this approximation does not mean that the users have perfect knowledge about the other users and the system. Because if travelers used perfect information, there is no need for an iterative or day-to-day process. Therefore, we deploy Mean field games to represent a prediction model of users regarding the other users decisions and evolution of the network.
1.3. Mean field games
To propose a new perspective on the DTUE, we resort to the Mean Field Games (MFGs) framework. The mathematical foundations of this theory were introduced in the seminal papers of Lasry and Lions (2006, 2007). The theory and methodology of MFGs have rapidly developed in different engineering fields (Djehiche, Tcheukam, and Tembine 2016). The theory of MFGs studies decision-making problems with an infinite number of interacting players (Adlakha and Johari 2013). The MFGs theory restates the classical game theory model as a micro-macro model (Cardaliaguet 2013). It allows defining players at the microscopic level similar to classical game theory models while translating the effect of players’ decision to macroscopic models (Caines, Huang, and Malhamé 2015). Therefore, instead of solving a large set of highly coupled equations that represent the interactions among players on a microscopic level, the core idea of MFGs is to exploit the “smoothing” effect of large numbers of interacting players. The MFGs’ main assumption (called mean field approximation) states that each player only reacts to a “mass”, which is defined by aggregating the effect of all the players. This approach simplifies the complex multi-agent dynamic systems at a macroscopic level (Degond, Liu, and Ringhofer 2014).

There are a few studies in the literature that apply MFGs to analyze transportation systems and most of them apply MFGs theory in the context of control theory (Chevalier, Le Ny, and Malhamé 2015, Huang et al. 2019), vehicle routing problem (Tanaka et al. 2020, Salhab, Le Ny, and Malhamé 2018) or pedestrian moving models (Aurell and Djehiche 2019). This paper, for the first time, develops a MFGs-based framework for the departure time equilibrium problem. In our framework, each traveler looks for the optimal departure time by predicting the other travelers’ departure time choices, given the current information of the traffic network congestion (mean-field), which is extracted from the generalized bathtub model. Then, the mean field is updated based on the optimal departure time choice of the travelers. The Nash equilibrium state occurs when the initial mean field approximation of the system is equal to the final mean field derived from the travelers’ optimal departure time distribution. This process is equivalent to solving a fixed-point problem (Friesz et al. 1993).

To numerically solve the DTUE model and determine the equilibrium of the system, many studies in the literature limit the feasible space of the DTUE problem by making strong assumptions on the trip-length distribution of the demand profile. For instance, recent studies on the morning commute problem assume that the optimal solution fulfills some sorting property relating to the trip length and departure time, e.g., First-In, First-Out (FIFO) by (Daganzo and Lehe 2015), partial FIFO by Lamotte and Geroliminis (2018), and Last-In, First-Out (LIFO) by Fosgerau (2015). Such assumptions restrict the exploration of the solution space. Moreover, most approaches in the literature have a common drawback: they do not guarantee the solution’s optimality while
they are costly computationally at large-scale (Huang et al. 2020). In this paper, we relax all the assumptions concerning sorting properties in the solution method (i.e., departure time rescheduling process) to better explore the solution space. We also propose a new heuristic method to speed-up the calculation process while converging to a solution which is closer to the DTUE equilibrium compare to the existing methods in the literature.

In this study, we first express the dynamic departure time choice problem at the network level based on the mean field games theory and generalized bathtub model (Section 2). Then, we discuss the properties of the DTUE in the continuous and discrete settings and prove that the model can represent the morning commute problem without any strong assumptions of homogeneity on the demand profile (Sections 3 and 4). Finally, in Section 5, we evaluate the performance of the model against one of the recently proposed approaches in the literature to solve the DTUE (Section 5.1) and apply the proposed model to the real test case of Lyon Metropolis network (Section 5.2). We numerically demonstrate that the model can not only consider heterogeneous demand profile for the morning commute problem but also a large transportation system with a high number of travellers. In Section 6, we provide concluding remarks.

2. Problem Definition

| Table 1 | List of notations |
|---------|------------------|
| $T$     | Time horizon.    |
| $n$     | The total number of trips. |
| $i$     | Index of trips, $i \in N$. |
| $X_{\text{max}}$ | Maximum trip length. |
| $X_{\text{min}}$ | Minimum trip length. |
| $x^i$   | Trip length of trip $i$. |
| $t^i_d$ | Departure time of trip $i$. |
| $T(t_d^i, x^i)$ | Travel time of a trip started at $t_d^i$ with trip length $x^i$. |
| $t^i_a$ | Desired arrival time of trip $i$. |
| $\bar{t}^i_a$ | Actual arrival time of trip $i$. |
| $v_t$   | Velocity of the system at time $t$. |
| $c_t$   | Fraction of the total demand that traveling in the system at time $t$. |
| $z(t)$  | Characteristic travel distance. |
| $o_t$   | Outflow fraction of the system at time $t$. |
| $\varphi(t, \cdot)$ | Probability density function of the active trips’ remaining distances at $t$. |
| $\Phi(t, x)$ | Fractions of active trips with trip lengths more than $x$ at time $t$. |
| $F$     | In-flow measure, the empirical distribution of the departures. |
| $V$     | Speed function, which maps the fraction of active travellers to the velocity. |
| $\Delta t$ | Small time interval. |
| $\Delta x$ | Small space interval. |

Table 1 presents the list of notations used in this paper. Consider a system with $n$ independent trips indexed by $i \in [n] := \{1, 2, \ldots, n\}$ in a time horizon $T := [0, T_{\text{max}}]$. The trip length of the $i$-th
trip is denoted by \( x^i \in \mathcal{X} := [X_{\text{min}}, X_{\text{max}}] \). The goal of the player \( i \) is to choose his departure time \( t^i_d \in T_d \) to arrive at the desired arrival time \( t^i_a \in T_a \), where \( T_d \) and \( T_a \) are two compact subsets of \( \mathcal{T} \). Assume that the joint distribution of the desired arrival times and trip lengths given as the demand profile \( m \). If we define

\[
  m_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{(x^i, t^i_a)},
\]

where \( \delta \) denotes the Dirac delta function, then \( m_n \Rightarrow m \) as \( n \to \infty \). Here \( \Rightarrow \) shows the weak convergence of the measures. That means, \( \int_{T_d \times \mathcal{X}} \phi dm_n \to \int_{T_d \times \mathcal{X}} \phi dm \) for all \( \phi \in C_b(T_d \times \mathcal{X}) \), the set of all bounded continuous functions on \( T_d \times \mathcal{X} \) (see Billingsley (2013), Carmona, Delarue et al. (2018)). Note that \( m \) is a probability measure\(^1\) in the space of all probability measures defined on \( \mathcal{X} \times T_a \), i.e. \( m \in \mathcal{P}(\mathcal{X} \times T_a) \). Therefore, it fully describes the demand characteristics. Hereafter, we assume that \( \{t^i_a, x^i\}_{i=1}^{n} \) are i.i.d random variables with the distribution \( m \). We also make the following regularity assumption on \( m \).

**Assumption 1.** There exists a constant \( M_m \) such that

\[
  m(B, T_a) \leq M_m \lambda(B), \quad \forall B \in \mathcal{B}(\mathcal{X})
\]

where \( \mathcal{B}(\mathcal{X}) \) and \( \lambda \) denote the \( \sigma \)-algebra of Borel sets and the Lebesgue measure on \( \mathcal{X} \), respectively. Broadly speaking, Assumption 1 means that the demand is spread smoothly over \( \mathcal{X} \) and particularly that the demand cannot be concentrated in Dirac distributions.

The congestion in the system at time \( t \) is defined by the fraction of the total demand that is active at time \( t \), which is captured by,

\[
  c_t = \frac{1}{n} \sum_{i=1}^{n} 1_{[t^i_d, \bar{t}^i_a)}(t),
\]

where \( \bar{t}^i_a \) denotes the actual arrival time of the \( i \)-th player and \( 1_{[t^i_d, \bar{t}^i_a)} \) is an indicator function which returns 1 if \( t \in [t^i_d, \bar{t}^i_a) \) and 0 otherwise. We assume that the velocity of the system at time \( t \) depends on the fraction of travelling users in the system \( c_t \) which is defined by a strictly decreasing speed function \( V : \mathbb{R}^+ \to \mathbb{R}^+ \). Therefore, \( V \) represents the mean network speed and is the key collective behavioral characteristic of the generalized bathtub model (Jin 2020a). Recall that the generalized bathtub model considers the network characteristics and the network load to calculate the network speed. We simply denote the velocity at time \( t \) by \( v_t := V(c_t) \) and assume that the velocity is the same for all players who are travelling at the same time.

\(^1\) Any function with values in \([0, 1]\), returning 1 for the entire space and 0 for the empty set that satisfies countable additivity property. A function \( F \) is countable additive if for all countable family \( \{B_i\} \) of pairwise disjoint sets, it holds true that \( F(\bigcup B_i) = \sum F(B_i) \), see Billingsley (2012).
To determine the travel time of a player, we first define a virtual user who starts his trip at time 0. Then, the characteristic travel distance $z(t)$, travelled by this virtual user up to time $t$, is

$$z(t) := \int_0^t v(s) ds = \int_0^t V(c_s) ds.$$  

(4)

Since $v_t > 0 \ \forall t \in T$, $z$ is an invertible function. Let $z^{-1}$ denote the inverse function of $z$. Then, we have $z^{-1}(z(t)) = t$ and $z^{-1}(x)$ represents the time at which the virtual user has reached $x$.

Now, let $T(t_{d}^i,x^i)$ denote the travel time of a player departing at time $t_{d}^i$ with trip length $x^i$. Considering (4), $T(t_{d}^i,x^i)$ can be determined by,

$$T(t_{d}^i,x^i) = z^{-1}(x^i + z(t_{d}^i)) - t_{d}^i.$$  

(5)

To determine the optimal departure time, we assume that each player aims to minimize his travel cost. In the DTUE problem, the travel cost is usually defined based on $\alpha$-$\beta$-$\gamma$ scheduling preferences (Fosgerau 2015). That means, the cost function is defined as the sum of the travel time and a penalty cost for arriving at $t_{d}^i + T(t_{d}^i,x^i)$ instead of the desired arrival time. Specifically, we assume that each player’s cost function is given by,

$$J_i(t_{d}^i,t_{a}^i,t_{d}^{-i},x^{-i}) = \alpha T(t_{d}^i,x^i) + \beta (t_{a}^i - t_{d}^i - T(t_{d}^i,x^i))_+ + \gamma (t_{d}^i + T(t_{d}^i,x^i) - t_{a}^i)_+,$$  

(6)

where $\alpha$ denotes the cost of travelling per unit of time, $\beta$ and $\gamma$ denote, respectively, the cost of earliness and lateness for the traveler arrival. Note that $(y)_+ = \max\{y,0\}$ as well $t_{d}^{-i}$ and $x^{-i}$ respectively express the dependency of $J$ on the departure times and trip lengths of the other users ($\neq i$) via their travel times.

The cost function defined in (6) captures the fact that travelers prefer not to deviate from their desired arrival time (i.e., arrive as close as possible to their desired arrival time) while they do not spend too much time on the traffic. Note that the dependency of the cost function on the trip lengths is not emphasized in the notation, while it holds implicitly. Below, we provide the definition of the optimal strategy that each player adopts to determine his departure time.

**Definition 1.** The departure time vector $\hat{t}_d := (\hat{t}_{d}^1,\ldots,\hat{t}_{d}^n) \in T_d^n$ is a Nash equilibrium (NE) for the cost function given in (6), if for all $i \in [n]$ we have

$$J_i(\hat{t}_{d}^i,t_{a}^i,\hat{t}_{d}^{-i},x^{-i}) \leq J_i(t_{d}^i,t_{a}^i,\hat{t}_{d}^{-i},x^{-i}), \ \forall t \in T_d.$$  

(7)

The above definition indicates that at a NE point $\hat{t}_d$, no player can decrease his travel cost by deviating from his departure time. Based on Definition 1, we define DTUE as a NE of the following Departure Time Choice Problem (DTCP):

$$\min_{t_{d}^i \in T_d} J_i(t_{d}^i,t_{a}^i,t_{d}^{-i},x^{-i}) = \alpha T(t_{d}^i,x^i) + \beta (t_{a}^i - t_{d}^i - T(t_{d}^i,x^i))_+ + \gamma (t_{d}^i + T(t_{d}^i,x^i) - t_{a}^i)_+ \ \forall i \text{ (DTCP)}$$
s.t.
\[
\begin{cases}
    c_t = \frac{1}{n} \sum_{j=1}^{n} 1_{[t^i_d, t^i_d + T(t^i_d, x^i)]}(t), \\
    z(t) = \int_{0}^{t} V(c_s) ds, \\
    T(t^i_d, x^i) = z^{-1}(x^i + z(t^i_d)) - t^i_d.
\end{cases}
\]

(8)

Similar to (6), DTCP provides the cost function of the \(i\)-th player. Note that, given the departure times and trip lengths of others \((t^{j\neq i}_d, x^{j\neq i})\), the player \(i\) is able to find his travel time. Specifically, according to the set of equations given in (8), one can derive the characteristic travel distance \(z(t)\) based on the fraction of active trips \(c_t\). Then, the travel time function \(T\) can be obtained.

Since analyzing the DTCP for a large \(n\) is arduous due to “curse of dimensionality”, in the next section, we examine the behaviour of players in a system where the number of players goes to infinity, i.e., \(n \to \infty\). This means that we adopt the MFGs approach to determine the DTUE.

3. Mean Field Games Framework

In this section, we discuss the DTCP in the framework of the MFGs. First, recall that the idea behind the MFGs is to consider a proxy that represents the macroscopic behavior of all the players at once, instead of taking into account their departure times individually. Therefore, to capture the information of entering trips from the viewpoint of the \(i\)-th player when there are \(n\) players (including player \(i\)) in the game, we define the following empirical measures,

\[
F^i_n := \frac{1}{n-1} \sum_{j \in [n] \setminus i} \delta_{t^j_d, x^j},
\]

(9)

\[
E^i_n := \frac{1}{n-1} \sum_{j \in [n] \setminus i} \delta_{t^j_d, t^j_a},
\]

(10)

where \(\delta\) is the Dirac delta function. Note that the cost function of each player, defined in (6), is a symmetric function\(^2\). That means, the objective function of a player does not change if other players change their labels with each other. Further, as \(n \to \infty\) the impact of each player on the system vanishes. That means by changing either the departure time, desired arrival time, or trip length of a player, the velocity of the system would be left unchanged. Thus, we define the in-flow measure \(F\) and the dis-aggregated in-flow measure \(E\) as the limits of the sequences \(\{F^i_n\}_{n \in \mathbb{N}}\) and \(\{E^i_n\}_{n \in \mathbb{N}}\), respectively. That is,

\[
F := \lim_{n \to \infty} F^i_n,
\]

(11)

\[
E := \lim_{n \to \infty} E^i_n.
\]

(12)

\(^2\) For any \(i \in [n]\), the cost function of the player \(i\) satisfies,

\[
J_i(t^i_d, t^i_a; t^{i\neq i}_d) = J_{\zeta(i)}(t^i_d, t^i_a; t^{i\neq i}_d),
\]

for all permutation \(\zeta\) on \([1, \ldots, n]\), see p. 49 of Lacker (2018)
Note that the limits are independent of \(i\) as the impact of a single player vanishes as \(n\) gets large. Further, \(F\) is a probability measure on the product space \(T_d \times X\), i.e., \(F \in \mathcal{P}(T_d \times X)\), the set of all probability measures define on \(T_d \times X\). In fact, for all \(n \in \mathbb{N}\) and \(i \in [n]\) the function \(F_i^n\) is a probability measure and \(F\) is the limit in the weak convergence sense. Similarly, one can show that \(E \in \mathcal{P}(T_d \times X \times T_a)\). According to (9) and (10), the in-flow measure \(F\) depends on \(E\) in the following sense,

\[
\int_{T_d \times X} \phi dF = \int_{T_d \times X \times T_a} \phi \otimes 1_{T_d} dE, \quad \forall \phi \in \mathcal{C}_b(T_d \times X),
\]

where \(\phi \otimes 1_{T_d}\) is the tensor product of \(\phi\) and \(1_{T_d}\). In fact, (13) assures that the in-flow measure \(F\) is the marginal probability measure of the departure times and trip lengths wrt to the dis-aggregated in-flow measure \(E\), almost surely (a.s.). This linear dependency is continuous by (13) and is denoted by \(F : \mathcal{P}(T_d \times X \times T_a) \rightarrow \mathcal{P}(T_d \times X)\) such that,

\[
F = F(E). \tag{14}
\]

Further, \(E\) is constrained by the demand profile \(m\), see (1), such that,

\[
\int_{T_d \times X \times T_a} 1_{T_d} \otimes \phi dE = \int_{X \times T_a} \phi dm, \quad \forall \phi \in \mathcal{C}_b(X \times T_a). \tag{15}
\]

Constraint (15) restricts the dis-aggregated in-flow measure \(E\) to a subset of \(\mathcal{P}(T_d \times X \times T_a)\) with marginal probability measures which is a.s. equal to the demand profile \(m\). Constraints (13) and (15) together yield to the following demand constraint,

\[
F(T_d, B) = m(B, T_a), \quad \text{(a.s.)} \tag{16}
\]

for all Borel measurable subsets of \(X\) such as \(B\). Roughly speaking, from the demand viewpoint, the fraction of players having a trip length in \(B\), \(m(B, T_a)\), matches the fraction of the departures with a trip length in \(B\), \(F(T_d, B)\). Thus all the demand is served over the overall time period.

Now, given the in-flow measure \(F\), an arbitrary player with the departure time \(t_d\), trip length \(x\) and desired arrival time \(t_a\) can reconsider his cost criteria as a function \(J : T_d \times X \times T_a \times \mathcal{P}(T_d \times X) \rightarrow \mathbb{R}^+\) defined as follows,

\[
J(t_d; x, t_a; F) = \alpha T(t_d, x) + \beta (t_a - t_d - T(t_d, x))_+ + \gamma (t_d + T(t_d, x) - t_a)_+. \tag{17}
\]

To be more specific, assuming \(F\) is known, each player tries to choose his departure time \(t_d\) by minimizing (17)$^3$. Towards this end, we should first state the travel time of a player with departure time \(t_d\) and trip length \(x\), i.e., \(T(t_d, x)\), in terms of the in-flow measure \(F\). That means, we want to find the relation between the travel time of a player and others’ departures and trip lengths.

$^3$ Obviously, each player knows his trip length \(x\) and desired arrival time \(t_a\).
3.1. Dynamics of the System

We assume that the in-flow measure $F$ with the probability density function $f$ is given. The goal of this section is to derive the dynamics of the characteristic travel distance $z$, defined in (4). Then, using (5), we can clarify the relation between the travel time function $T$ and the characteristic travel distance $z$ which completes the definition of (17).

Assuming the number of players in the system goes to infinity ($n \to \infty$), we define the dynamics of the system according to the fraction of active trips instead of the number of them. Therefore, we denote by $\phi(t, \cdot)$ the probability density function of the remaining trip lengths of active trips at time $t$. Then, the out flow of the system at time $t$, $o_t$, can be stated as follows,

$$o_t = c_t v_t \phi(t, 0),$$

where, in the sense of (3), the system congestion in the limit model can be defined as

$$c_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{v_i r_i}^{v_i r_i + \Delta t} (t)$$

(19)

To derive the dynamics of $\phi(t, x)$, we use the idea of the generalized bathtub model, see Jin (2020a). Note that in a system with $n$ trips, for a small time interval $\Delta t$, the number of active trips at $t + \Delta t$ with a remaining trip length in $[x, x + \Delta x]$ is $n c_{t+\Delta t} \phi(t + \Delta t, x) \Delta x$. On the other hand, it is equal to the sum of new departures and trips with remaining trip length in $[x, x + v_t \Delta t, x + v_t \Delta t + \Delta x]$ at time $t$. Thus,

$$n c_{t+\Delta t} \phi(t + \Delta t, x) \Delta x \approx n f(t, x) \Delta x \Delta t + n c_t \phi(t, x + v_t \Delta t) \Delta x,$$

which is equivalent to,

$$c_{t+\Delta t} \phi(t + \Delta t, x) \approx f(t, x) \Delta t + c_t \phi(t, x + v_t \Delta t).$$

To simplify the system dynamics, we approximate $c_{t+\Delta t}$ and $\phi(t, x + v_t \Delta t)$ with $c_t + c'_t \Delta t$ and $\phi(t, x) + v_t \frac{\partial \phi(t, x)}{\Delta t} t$, respectively. Then, dividing both sides by $\Delta t$ and letting $\Delta t$ goes to zero, we get,

$$c_t \partial_t \phi(t, x) + c'_t \phi(t, x) - c_t v_t \partial_x \phi(t, x) = f(t, x).$$

(20)

We use $\partial_t$ and $\partial_x$ to denote, respectively, the partial derivative with respect to time $t$ and space $x$. Integrating both sides of (20) with respect to $x$ from $x$ to $X_{max}$, we get,

$$c_t \partial_t \Phi(t, x) + c'_t \Phi(t, x) - c_t v_t \partial_x \Phi(t, x) = \int_x^{X_{max}} f(t, \xi) d\xi.$$
Note that $\Phi(t,x)$ denotes the fraction of active trips with the remaining trip lengths more than $x$ at time $t$,

$$\Phi(t,x) = \int_x^{X_{\text{max}}} \varphi(t,\xi) d\xi.$$  

Then, using relation $z'(t) = v_t$ as a direct result of (4), we have,

$$\frac{d}{dt} (c_t \Phi(t, x - z(t))) = c_t \partial_x \Phi(t, x - z(t)) + c'_t \Phi(t, x - z(t)) - c_t v_t \partial_x \Phi(t, x - z(t)).$$

Therefore, the equality given in (21) can be written as,

$$\frac{d}{dt} (c_t \Phi(t, x - z(t))) = \int_{x-z(t)}^{X_{\text{max}}} f(t, \xi) d\xi.$$  

Thus, by integrating both sides with respect to time from 0 to $t$ and setting $y = x - z(t)$, the dynamics of the system can be presented as,

$$c_t \Phi(t, y) = \int_0^t \int_{y+z(s)}^{u} f(s, \xi) d\xi ds.$$  

(22)

Note that we assume the system is empty at time 0. Moreover, taking partial derivative with respect to $x$ from both sides, considering that $c_t$ is independent of $x$, and applying Leibniz’s integral rule, we obtain,

$$\partial_x (c_t \Phi(t, x)) = c_t \partial_x \Phi(t, x) = -\int_0^t f(s, x + z(t) - z(s)) ds.$$  

(23)

Finally, using (18) and the definition of in-flow measure $F_t$ and its probability density function $f$, the dynamics of the fraction of the active trips at time $t$, $c_t$ satisfies

$$c'_t = \int_0^{X_{\text{max}}} f(t, x) dx - o_t = \int_0^{X_{\text{max}}} f(t, x) dx + c_t v_t \partial_x \Phi(t, x) |_{x=0}.$$  

Here, $\partial_x \Phi(t, x) |_{x=0}$ demonstrates the right derivative at 0 as the left derivative is not defined. Substituting (23), gives,

$$c'_t = \int_0^{X_{\text{max}}} f(t, x) dx - v_t \int_0^t f(s, z(t) - z(s)) ds.$$  

(24)

In the light of the equality given in (4) and considering that $z'(t) = v_t$, the result of the above discussion about the characteristic travel distance is summarized in Proposition 1. To state the proposition rigorously, we make the following assumption.
Assumption 2. Let $G > 0$ be a constant. Then, for all Borel measurable subset of $\mathcal{T}_d \times \mathcal{X}$ such as $B$, we assume that the in-flow measure $F \in \mathcal{P}(\mathcal{T}_d \times \mathcal{X})$ satisfies

$$F(B) \leq G\lambda_2(B),$$

where $\lambda_2$ is Lebesgue measure on $\mathbb{R}^2$.

Assumption 2 is a regularity condition. It means that the in-flow measure $F$ is spread smoothly both with respect to departure time and trip length. Assumption 2 is a technical condition which essential for most proofs of this section.

Given the demand profile $m$, defined in (1), and the constant $G > 0$, let $\mathcal{P}_{m,G}$ denote the set of all in-flow measures $F \in \mathcal{P}(\mathcal{T}_d \times \mathcal{X})$ that satisfies Assumption 2 and demand constraint given in (16). Then, by Radon–Nikodym theorem (see e.g. Theorem 32.2 of Billingsley (2012)) any $F \in \mathcal{P}_{m,G}$ admits a probability density function denoted by $f$. Further, let $\mathcal{M}_{m,G}$ denotes the set of all positive dis-aggregated in-flow measures $E \in \mathcal{P}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a)$.\par

Considering the definition of $\mathcal{P}_{m,G}$, the system avoids having a mass of departures at the same time (the in-flow cannot be Dirac). For small values of $G$, an in-flow measure $F \in \mathcal{P}_{m,G}$ is very smooth (without any drastic change in a short interval of time). However, when $G$ gets larger, the feasible in-flow measures may have larger fluctuations. Note that Assumption 2 is consistent with the regularity assumption made on the demand profile $m$, i.e., Assumption 1. We have the next proposition about the dynamics of the characteristic travel distance.

**Proposition 1.** Consider a traffic system with speed function $V$ and in-flow measure $F \in \mathcal{P}_{m,G}$. Then, the characteristic travel distance of the system $z_F$ is the solution of the following set of equations,

$$
\begin{cases}
z_F(t) = \int_0^t V(F(S_t(z_F))) \, ds, \\
S_t(z_F) := \{(\tau, \xi) \mid \tau \in [0, t] \cap \mathcal{T}_d, \xi \in (z_F(t) - z_F(\tau), \infty) \cap \mathcal{X}\}.
\end{cases}
$$

(26)

For the proof refer to appendix A. Note that in the set of equations defined in (26), we use subscript $F$ to emphasize the dependency of the variables on the in-flow measure.

Proposition 1 provides the relation between the characteristic travel distance $z_F$ and the in-flow measure $F$, i.e. the distribution of the departures. In fact, $S_t(z_F)$ contains the pairs of the departure times and trip lengths of the users that are travelling at time $t$, in a traffic system with the characteristic travel distance $z_F$. In other words, for all $(t_d, x) \in S_t$, an agent with departure time $t_d$ and trip length $x$ is in the system at time $t$. On the other hand, if $(t_d, x) \notin S_t$, the agent has either not departed or finished her travel before $t$, as illustrated in Figure 1.

\footnote{The Borel measurability of $S_t$ is obvious.}
Note that the set of equations given in (26) should be solved simultaneously. Therefore, we should investigate the existence and uniqueness of the characteristic travel distance derived in Proposition 1. To address this problem, we need to introduce some notations. For any compact subset of $\mathbb{R}^n$ such as $C$, $C(C)$ represents the space of all real valued continuous functions defined on $C$. We assume that $C(C)$ is equipped with the uniform norm, i.e.,

$$\forall u \in C(C): \|u\| := \sup_{t \in C} |u(t)|.$$ 

Also, for a constant $M > 0$, we define the following norm on $C(C)$ which is equivalent to the uniform norm on the compact space $C$,

$$\forall u \in C(C): \|u\|_M := \sup_{t \in C} |e^{-tM} u(t)|.$$ 

We denote by $d(\cdot, \cdot)$ and $d_M(\cdot, \cdot)$ the distances associated to $\|\cdot\|$ and $\|\cdot\|_M$, respectively. Also, we define the function $U : C(T) \times \mathcal{P}_{m,G} \mapsto C(T)$ such that

$$\begin{cases} U(z, F) = \tilde{z}, \\ \tilde{z}(t) = \int_0^t V\left(F(S_s(z))\right) ds. \end{cases} \quad (27)$$

As demonstrated in the next proposition, systems with smooth in-flow measures, in the sense of Assumption 2, and Lipschitz continuous speed functions, admit a unique characteristic travel distance.

**Proposition 2.** For all in-flow measures $F \in \mathcal{P}_{m,G}$ and Lipschitz continuous speed functions $V$, there exists a unique function $z_F \in C(T)$ which satisfies the set of the equations given in (26).
The proof is given in appendix B.

Proposition 2 enables us to obtain $z^*$, the solution of the set of equations given in (26) which is the characteristic travel distance of a system with in-flow measure $F$. The next corollary provides a procedure to obtain $z^*$, based on the successive application of the mapping $U$ defined in (27).

**Corollary 1.** Fix $F \in \mathcal{P}_{m,G}$. Then, starting with an arbitrary element $z_0 \in C(T)$, the sequence $z_l$ defined as

$$z_l := U(z_{l-1}, F), \quad l \geq 1,$$

converges to $z^*$ which is the solution of (26).

See appendix C for the proof.

Consider an arbitrary weakly convergent sequence of probability measures $\{F_k\}_{k \in \mathbb{N}}$ in $\mathcal{P}_{m,G}$ such that $F_k \Rightarrow F$. Then, Proposition 3 clarifies that the limit probability measures $F$ lies in $\mathcal{P}_{m,G}$, too.

**Proposition 3.** For any $G \in \mathbb{R}^+$, $\mathcal{P}_{m,G}$ is a closed subset of $\mathcal{P}(T_d \times X)$ in the weak convergence topology.

For the proof visit appendix D.

In the following proposition we demonstrate that the characteristic travel distance is continuous with respect to the in-flow measure. Suppose that $z^*_k$ is the solution of the set of equations given in (26) for $F_k$. That means $z^*_k$ is the characteristic travel distance of a system having departures with distribution $F_k$. Similarly, consider $z^*$ as the corresponding solution to the probability measure $F$. Further, suppose that the probability space $\mathcal{P}_{m,G}$ and the set of continuous functions $C(T)$ are equipped, respectively, with the weak and uniform convergence topology.

**Proposition 4.** Suppose that the speed function $V$ is Lipschitz continuous. Then, the characteristic travel distance is continuous wrt to the in-flow measure. In other words, if $F_k \Rightarrow F$ then $z^*_k \to z^*$.

In the proof of Proposition 4, see appendix E, we provide a convergence bound for the limit of the characteristic travel distances. Thus, the solution of the equations given in (26) is also continuous wrt the dis-aggregated in-flow measure $E$, and the next corollary can be considered as a consequence of the continuity of $\mathcal{F}$ (recall that $F = \mathcal{F}(E)$).

**Corollary 2.** Suppose that the speed function $V$ is Lipschitz continuous. Then, the characteristic travel distance is continuous wrt to the dis-aggregated in-flow measure $E$.

The proof is given in appendix F.

Note that the characteristic travel distance of the system depends on $E$ be means of $F$, see (26).
3.2. Mean Field Games Departure Time Choice Problem (MFGs-DTCP)

In this section, using the results derived in the previous sections, we provide a DTCP formulation based on the MFGs approach by assuming that the number of travelers goes to infinity in DTCP. Recall that the characteristic travel distance is provided in Proposition 1 and its existence as well as its uniqueness is demonstrated in Proposition 2. Therefore, considering the objective function of an arbitrary player given in (17) and the relation between the travel time and the characteristic travel distance provided in (5), we can define the MFGs-DTCP as follows,

$$\min_{t_d \in T_d} J(t_d; x, t_a; F) = \alpha_T(t_d, x) + \beta(t_a - t_d - T(t_d, x)) + \gamma(t_d + T(t_d, x) - t_a)$$ (MFGs-DTCP)

s.t.

$$z(t) = \int_0^t V\left(F(S_s(z))\right) ds,$$

$$S_t(z) := \{(\tau, \xi) \mid \tau \in [0, t] \cap T_d, \xi \in (z(t) - z(\tau), \infty) \cap \mathcal{X}\},$$

$$T(t_d, x) = z^{-1}(x + z(t_d)) - t_d. \quad (29)$$

Note that in the DTCP model defined in (8), all the three relations should be considered simultaneously, since the choice of an arbitrary player affects the system significantly. However, in the MFGs-DTCP model, the set of equations given in (29) can be investigated independently of the travel time identity provided in (30). This is due to the fact that, as the number of players $n \to \infty$, the impact of a player on the system vanishes. Moreover, note that the MFGs-DTCP model considers the system at a macroscopic level. That is, we do not need to follow the states and decisions of finitely many players\(^5\). Specifically, if finitely many players change their departure times, the relation 29 remains unchanged.

Solving the MFG for the MFGs-DTCP problem is difficult. Indeed it is to be expected that the inflow solution to this problem does not satisfy the regularity Assumption 2 and can exhibit dirac-like concentrations. Therefore we introduce in definition 2 a relaxed problem with relaxation factor $\varepsilon$. From a physical point of view, $\varepsilon$ can be understood as follows: an agent considers his optimum satisfied if his cost lies within $\varepsilon$ of the minimum cost.

**Definition 2.** Given a constant $\varepsilon \geq 0$, $F(E^*) \in P_{m,G}$ (with $E^* \in M_{m,G}$) is an $\varepsilon$-Mean Field Equilibrium ($\varepsilon$-MFE) for the MFGs-DTCP, if the following relation holds,

$$E^* \{ \{(t_d, x, t_a) \in T_d \times \mathcal{X} \times T_a \mid J(t_d; x, t_a; F(E^*)) \leq J(t_d^0; x, t_a; F(E^*) + \varepsilon, \forall t_d^0 \in T_d \} \} = 1.$$ 

Note that, Mean Field Equilibrium (MFE) is $\varepsilon$-MFE with $\varepsilon = 0$. Also, $E^*$ can be expressed as the fixed-point of a map $H : M_{m,G} \mapsto M_{m,G}$ defined as,

$$H(\hat{E}) := \{ E \in M_{m,G} \mid E(\{ (t_d, x, t_a) \in T_d \times \mathcal{X} \times T_a \mid J(t_d; x, t_a; F(\hat{E})) \leq J(t_d^0; x, t_a; F(\hat{E}) + \varepsilon, \forall t_d^0 \in T_d \}) = 1 \}, \quad (31)$$

\(^5\) More precisely, any measure zero subset of agents’ indices set is negligible.
The equivalence of the two definitions provided above holds obviously. In the next section, the existence of an equilibrium for the MFGs-DTCP will be examined.

### 3.3. Existence of the Equilibrium

In this section, we show that there exists an equilibrium solution for the MFGs-DTCP. To prove the existence, we first need to examine whether the cost function given in MFGs-DTCP is jointly continuous.

**Proposition 5.** Suppose that the speed function is bounded from above and below by $V_{\max}$ and $V_{\min}$, respectively, such that $V_{\max} > V_{\min} > 0$. Then, the cost function of the MFGs-DTCP, $J$, is jointly continuous on $(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a \times \mathcal{P}_{m,G})$. Further, the continuity of the cost function on $\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a$ is Lipschitz.

For the proof refer to G.

Considering that the velocity $V$ is a function of the congestion $c_t$, the condition $V \geq V_{\min} > 0$ ensures that the network is not saturated even when $c_t = 1$, i.e., the total demand is less than the capacity of the network.

Note that Proposition 5 demonstrates the joint continuity of the cost function only on $(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a \times \mathcal{P}_{m,G})$. If we assume that the jointly continuity condition is extendable to $(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a \times \mathcal{P}(\mathcal{T}_d \times \mathcal{X}))$, the problem admits a MFE, by Theorem 4.9 in Lacker (2018). Otherwise, we have the next proposition on the existence of the equilibrium.

**Proposition 6.** For an arbitrary $\varepsilon > 0$, there exists a constant $G \in \mathbb{R}^+$ such that MFGs-DTCP admits an $\varepsilon$-MFE in the probability space $\mathcal{P}_{m,G}$. This means that there exists an dis-aggregated in-flow measure $E^* \in \mathcal{M}_{m,G}$ which is the fixed-point of mapping $H$, given in (31).

The proof is outlined through appendix H

**Comment:** The data $m$ (the demand) satisfies the regularity condition expressed by Assumption 1. The regularity constant $M_m$ of the data $m$ and the regularity coefficient $G$ of the $\varepsilon$-MFE should be connected. Indeed the proof of the proposition 6 in Appendix H, formula (62), yields the estimate $G \geq \frac{M_m \text{Lip}(J)}{2\varepsilon}$. Thus $G$ can be chosen proportional to $M_m$. Further, as $\varepsilon \to 0$, $G \to \infty$, which suggests that MFG solutions of the MFGs-DTCP problem cannot be found in any $\mathcal{M}_{m,G}$ space, and that they could exhibit concentrations of departure times on specific instants (Dirac measures).

### 4. MFGs Model for the MFGs-DTCP

In this section, we aim to characterize the departure time user equilibrium (DTUE) for the Mean Field Games model discussed in the previous section. Recall that Proposition 6 guarantees the existence of the departure time equilibrium.
Consider the optimal behavior of an arbitrary player, assuming that the decisions of the other players are known. Specifically, fix a player with the desired arrival time \( t_a \) and the trip length \( x \) as well as an in-flow measure \( F \) as the proxy for the departure times and trip lengths of the other players. By Proposition 2, this system has a unique characteristic travel distance \( z \). Then, based on Proposition 1, we have \( v_t := V(F(S_t)) \), which is the velocity of the system at time \( t \). Also, note that (5) can be written as,

\[
\int_{t_d}^{t_d+T(t_d,x)} v_t dt = x. \tag{32}
\]

Then, taking derivative with respect to \( t_d \) from both sides of the above equality implies\(^6\),

\[
(1 + \partial_t T(t_d,x)) v_{t_d+T(t_d,x)} - v_{t_d} = 0,
\]

which yields to,

\[
\partial_t T(t_d,x) = \frac{v_{t_d}}{v_{t_d+T(t_d,x)}} - 1. \tag{33}
\]

Suppose that \( \alpha > \beta \). We apply the first order condition of optimality to determine the equilibrium departure time considering the cost function \( J \) defined in (17). In the case that \( t_a > t_d + T(t_d,x) \), the third term in the cost function is equal to zero, and we get,

\[
\alpha \partial_t T(t_d,x) - \beta(\partial_t T(t_d,x) + 1) = 0.
\]

Substituting (33) in the above equation, we get,

\[
\frac{v_{t_d}^*}{v_{t_d}^*+T(t_d^*,x)} = \frac{\alpha}{\alpha - \beta}. \tag{34}
\]

Similarly, if \( t_a < t_d + T(t_d,x) \), the second term in the cost function is equal to zero and the following equality can be derived by applying the first order condition,

\[
\frac{v_{t_d}^*}{v_{t_d}^*+T(t_d^*,x)} = \frac{\alpha}{\alpha + \gamma}. \tag{35}
\]

Based on (34) and (35), it is optimal for an agent who arrives before (after) his desired arrival time to choose the departure time such that the ratio of the system velocity at departure and arrival time be equal to \( \frac{\alpha}{\alpha - \beta} \) (\( \frac{\alpha}{\alpha + \gamma} \)). For on-time agents, based on left and right derivatives of the cost function we can get,

\[
\frac{\alpha}{\alpha + \gamma} \leq \frac{v_{t_d}^*}{v_{t_d}^*+T(t_d^*,x)} \leq \frac{\alpha}{\alpha - \beta}. \tag{36}
\]

Summarizing relations provided in (34), (35), and (36), we can conclude the following proposition about the optimal choice of an arbitrary agent given the distribution of the others’ departures. Note that Proposition 7 is in accordance with Proposition 2 in Lamotte and Geroliminis (2018).

\(^6\)Inverse Function Theorem ensures the differentiability of \( T \) at any point \( t_d \) within \((0, T_{max})\), see for example Theorem 7.4 of Protter, Charles Jr et al. (2012).
Proposition 7. The optimal departure time $t_d^*$ of a player having desired arrival time $t_a$ and trip length $x$ with cost function $J$, given in (17), satisfies the following conditions,

$$\frac{\alpha}{\alpha + \gamma} \leq \frac{v_{t_d^*}}{v_{t_d^*} + T(t_d^*, x)} \leq \frac{\alpha}{\alpha - \beta}. \tag{37}$$

Further, for an early and late player we have,

$$\frac{v_{t_d^*}}{v_{t_d^*} + T(t_d^*, x)} = \begin{cases} \frac{\alpha}{\alpha - \beta}, & t_a > t_d + T(t_d, x), \\ \frac{\alpha}{\alpha + \gamma}, & t_a < t_d + T(t_d, x). \end{cases} \tag{38}$$

Note that the cost function $J$ given in (17) is continuous with respect to departure time on a compact set $T_d$, based on Proposition 5. Therefore, there exists a point at which this function is minimized $^7$.

Suppose $D: T_a \times X \mapsto T_d$ defines a solution to Proposition 7. That is, $D$ maps the desired arrival time and trip length to the departure time $t_d^*$ which satisfies (37) and (38), specifically,

$$t_d^* = D(t_a, x). \tag{39}$$

Although Proposition 7 characterizes the function $D$, but it is not possible to calculate $D$ explicitly. Proposition 8 also provides a more detailed characterization of $D$.

Our next goal is to clarify the relation between the demand profile $m$ and the in-flow measure $F$ in terms of $D$. Consider a population of size $n$, and let $t_i^a$ and $x_i^i$ denote the $i$-th player’s desired arrival time and trip length, respectively. Recall that $\{t_i^a, x_i^i\}_{i=1}^n$ are i.i.d random variables with the distribution $m$. By (39), $D$ determines the optimal departure times of the players, i.e., $D(t_i^a, x_i^i) = t_d^i$. Then, based on Glivenko-Cantelli Law of Large Numbers (see e.g. Section 3.2.2 of Cardaliaguet (2018)), almost surely and in $L^1$, $F^n := \frac{1}{n} \sum_{i=1}^n \delta_{t_d^i, x_i^i}$ converges weakly to $F$, which is the distribution of $(t_d^i, x_i^i)^8$. This result shows that the limit of (11), $F$, exists and can be derived based on the demand profile $m$. Using a similar discussion we can show that the limit of (12), $E$, exists and represents the the distribution of $(t_d^i, x_i^i, t_a^i)$.

To clarify the relation between in-flow measure $F$ and demand profile $m$, regarding the function $D$, consider disaggregated in-flow measure $E$. Note that $E(\Delta t_d, \Delta x, \Delta t_a)$ indicates the fraction of the trips having departure time in $\Delta t_d$, trip length in $\Delta x$, and desired arrival time $\Delta t_a$. Then, we can state the following proposition.

$^7$ The minimum could be achieved on the boundary of $T_d$. But, the cost function includes a term aiming to minimize the difference between desired and effective arrival time. Thus, we assume that the minimum of $J$ satisfies (37) and (38).

$^8$ We assume $D$ is measurable; thus, $\{t_d^i, x_i^i\}_{i=1}^n$ are i.i.d RVs, too.
Proposition 8. Suppose $D$, which is defined in (39), is differentiable with respect to $t_a$ and $\partial_t D(t_a, x) > 0, \forall t_a \in T_a$. Also, assume that the demand profile $m$ satisfies Assumption 1 and dis-aggregated in-flow measure $E$ admits a probability density function $e$. Then, we have,

$$
e(D(t_a, x), x, t_a) = \frac{m(dx, dt_a)}{\partial_t D(t_a, x)}.
$$

For the proof see appendix I.

Note that (40) provides the relation between the dis-aggregated in-flow measure and the demand profile that is consistent with the constraint given in (15).

4.1. MFGs system of equations.

In this section, we discuss the MFGs model for the MFGs-DTCP which characterizes the equilibrium of the system. Note that the goal of the MFGs analysis is to examine the equilibrium behavior of travelers (i.e., DTUE) not the individual’s optimal departure time. On the other hand, based on Corollary 1, a generic player would be able to obtain the characteristic travel distance and determine, using Proposition 7, his strategy given the in-flow measure $F$. Therefore, $F$ is the mean field of the MFGs-DTCP, i.e., $F$ captures the required information for a generic agent to describe and analyze the system. Denoting the actual arrival time by $\bar{t}_a = t_d + T(t_d, x)$, we can summarize the discussions and results provided in the previous sections to derive the mean field games model:

\[
\begin{align*}
\alpha + 1_{t_a > t_d} \left( \frac{\alpha - \beta}{\alpha + \gamma} \right) &\le \frac{\nu a}{\nu a} \le 1_{t_a < t_d} \left( \frac{\alpha - \beta}{\alpha + \gamma} \right) \quad \text{with solution } t_d = D(t_a, x), \quad (41) \\
E(D(t_a, x), x, t_a) &= \frac{m(dx, dt_a)}{\partial_t D(t_a, x)}, \quad f(D(t_a, x), x) = \int_{t_a} t_{d} e(D(t_a, x), x, t_a) dt_a, \quad F = \int f(t_d, x) dt_d dx, \quad (42) \\
z(t) &= \int_0^t V \left( F(S_s(z)) \right) ds, \quad (43) \\
S_t(z) &= \{ (\tau, \xi) \mid \tau \in [0, t] \cap T, \xi \in (z(t) - z(\tau), \infty) \cap X \}, \quad (44) \\
T(t_d, x) &= z^{-1}(x + z(t_d)) - t_d, \quad (45) \\
F &\in \mathcal{P}_{m, G}. \quad (46)
\end{align*}
\]

The MFGs-DTCP model given in (41-46) can be explained as follows. Suppose that the decision of the players are captured by the in-flow measure $F$. Using equations (43) and (44), and in the light of Corollary 1, the associated characteristic travel distance $z$ can be obtained. Then, a generic agent can employ the relation given in (45) to determine his travel time. Subsequently, the player is able to obtain the optimal departure time based on (41) along with the function $D$. Finally, the demand profile $m$, which is known, will be transferred according to (42) that specifies the relation between the in-flow measure, dis-aggregated in-flow measure and demand profile. Thus, the optimal distribution of the departure times will be derived as a function $\bar{F}$. Now, based on

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9 Consider two agents with desired arrival times in $t_a^1 < t_a^2$ and the same trip length $x$. Let $t_d^1 = D(t_a^1, x)$ be the departure time of the first player. Then, the virtual user travels a distance of $x$ in the time interval $[t_d^1, t_d^2]$. Therefore, since the velocity is positive, it is rational to assume $D(t_d^1, x) < D(t_d^2, x)$.
Definition 2, the in-flow measure $F$ would be DTUE of the MFGs-DTCP if $\hat{F}$ obtained based on the above procedure is equal to the initial in-flow measure $F$.

Note that (41) and (42) are the main components of the model. While the former gives the optimal condition for the decision of a generic player, the latter captures the distribution of the decisions. The rest are required to make a bridge between relations given in (41) and (42).

**Remark 1.** The MFGs-DTCP model provided in (41-46) can be extended to capture user specific coefficients $\alpha$, $\beta$, and $\gamma$ in the cost function where their distributions are given through demand profile $m$. This paves the way to consider heterogeneous user preferences when solving the DTCP problem.

### 4.2. Discrete approximation of the problem

In this section, we derive the discrete version of the system of equations given in (41-46) to solve the MFGs model numerically. Let $\Delta t$ and $\Delta x$ denote small intervals in the time and space, respectively, such that $\Delta x \geq V_{max} \Delta t$, where $V_{max}$ indicates the maximum of the network free-flow speed. This means that a trip cannot travel more than $\Delta x$ in a time interval $\Delta t$. We denote the time and space discretization as follows,

- **The time discretization:**
  
  $$ (\tau) = [\tau \Delta t, (\tau + 1)\Delta t), $$
  
  (47)

- **The space discretization**

  $$ (\kappa) = [\kappa \Delta x, (\kappa + 1)\Delta x). $$

  (48)

All time intervals $[\tau \Delta t, (\tau + 1)\Delta t)$ will be denoted hereafter by $(\tau)$. A similar interpretation holds for $(\kappa)$. Note that these definitions are matched with time horizon $T$ and space set $X$ such that the union of all the defined intervals is equal to the corresponding set, that is $\cup(\tau) = T$ and $\cup(\kappa) = X$.

Similarly, let $(\tau_d)$ and $(\tau_a)$, respectively, denote the departure and arrival time intervals where the union of $(\tau_d)$ and $(\tau_a)$ is equal to $T_d$ and $T_a$, respectively.

We define an equivalent discrete version of the demand profile $m$ as follows,

$$ \pi(\tau_a, \kappa) := m((\tau_a), (\kappa)) = \int_{(\tau_a)} \int_{(\kappa)} m(dt_a, dx). $$

(49)

We assume that the velocity of the system is constant in each time interval $\tau$ and it is captured by $v_\tau$. Then, the discrete analogous of the optimal condition, given in (7), can be presented as,

$$ \frac{\alpha}{\alpha + \gamma} + 1_{\tau_a \geq \bar{\tau}_a} (\frac{\alpha}{\alpha - \beta} - \frac{\alpha}{\alpha + \gamma}) \leq \frac{v_{\text{ud}}}{v_{\text{ud}}} \leq \frac{\alpha}{\alpha - \beta} + 1_{\tau_a < \bar{\tau}_a} (\frac{\alpha}{\alpha + \gamma} - \frac{\alpha}{\alpha - \beta}), $$

(50)

$^{10}$Indeed, relations in (42) consider the dependency of the in-flow measure on the dis-aggregated in-flow measure in Definition 2.
where \( \bar{\tau}_a := \tau_d + T(\tau_d, \kappa) \) is the actual arrival time interval. Here, with an abuse of notation, \( T(\tau_d, \kappa) \) is the travel time of an agent having departure time in \( (\tau_d) \) and trip length in \( (\kappa) \). Suppose that the function \( D \) is a solution of (50). That means, \( \tau_d^* = D(\tau_a, \kappa) \) is the optimal departure time interval for a traveler having desired arrival time in \( (\tau_a) \) and trip length in \( (\kappa) \). Additionally, let \( \mu(\tau_d, \kappa, \tau_a) \) indicate the fraction of departures in time interval \( (\tau_d) \) with trip length in \( (\kappa) \) having desired arrival time in \( (\tau_a) \). Then, similar to (40), we can capture the relation between the demand profile \( \pi \) and \( \mu \) by,

\[
\mu(\tau_d, \kappa, \tau_a) = \frac{\pi(\tau_a, \kappa) \Delta t}{D(\tau_a + 1, \kappa) - D(\tau_a, \kappa)}.
\]

We also define the discrete characteristic travel distance by,

\[
\zeta(\theta) := \Delta t \sum_{\tau=0}^{\theta-1} v_{\tau}.
\]  

(51)

Then, if we denote by \( \Gamma_\theta(\zeta) \) the indices corresponding to the agents that are travelling in the interval \( \theta \), we can get,

\[
\Gamma_\theta(\zeta) := \{ (\tau_d, \kappa) | \kappa > \zeta(\theta) - \zeta(\tau) \}.
\]

Moreover, the velocity in a system with the discrete characteristic travel distance \( \zeta \) in the time interval \( (\tau) \), \( v_\tau \), would satisfy,

\[
v_\theta = V \left( \sum_{(\tau_d, \kappa) \in \Gamma_\theta(\zeta)} p(\tau_d, \kappa) \right),
\]

where \( p(\tau_d, \kappa) = \sum_{\tau_a} \mu(\tau_d, \kappa, \tau_a) \) is the fraction of departures in \( (\tau_d) \) having trip length in \( (\kappa) \) independent of the desired arrival time. Similarly, we define the the travel time of an agent having departure time in \( (\tau_d) \) and trip length in \( (\kappa) \), \( T(\tau_d, \kappa) \). That is,

\[
T(\tau_d, \kappa) := \zeta^{-1}(\kappa + \zeta(\tau_d)) - \tau_d.
\]

Here, \( \zeta^{-1} \) shows the inverse of the function \( \zeta \), defined in (51).

Therefore, the discrete analogous of the MFGs system defined in (41-46) can be represented as,

\[
\begin{align*}
\alpha &+ 1 \tau_a > \tau_a \left( \frac{\alpha}{\alpha - \beta} \right) \leq \frac{\tau_d}{\tau_a} \leq \frac{\alpha}{\alpha - \beta} + 1 \tau_a < \tau_a \left( \frac{\alpha}{\alpha + \gamma} - \frac{\alpha}{\alpha - \beta} \right) \quad \text{with solution } \tau_d = D(\tau_a, \kappa), \\
\mu(D(\tau_a, \kappa, \tau_a), \kappa) &\lambda = \frac{\pi(\tau_a, \kappa) \Delta t}{D(\tau_a + 1, \kappa) - D(\tau_a, \kappa)}, \quad p(\tau_d, \kappa) = \sum_{\tau_a} \mu(\tau_d, \kappa, \tau_a), \\
\zeta(\theta) &\lambda = \Delta t \sum_{\tau=0}^{\theta-1} V \left( \sum_{(\tau_d, \kappa) \in \Gamma_\theta(\zeta)} p(\tau_d, \kappa) \right), \\
\Gamma_\theta(\zeta) &\lambda = \{ (\tau_d, \kappa) | \kappa > \zeta(\theta) - \zeta(\tau) \}, \\
T(\tau_d, \kappa) &\lambda = \zeta^{-1}(\kappa + \zeta(\tau_d)) - \tau_d, \\
\sum_{\tau_d} p(\tau_d, \kappa) &\lambda = \sum_{\tau_a} \pi(\tau_a, \kappa), \quad \forall \kappa.
\end{align*}
\]  

(52-57)
The set of equations given in (52-57) can be explained similar to the ones provided in (41-46). That means the distribution of the players’ decisions, \( p \), can be treated as the mean-field of the discrete system. Suppose that the decision of the players are given by \( p \) for all \( \tau_d \) and \( \kappa \). Using (54) and (55), an arbitrary player can derive the discrete characteristic travel distance \( \zeta \), and determine his travel time based on (56). Then, the player could find his optimal departure time using (52) and obtain the function \( D \). Finally, using (53), \( \hat{p} \) can be obtained as the distribution of revised departure times wrt the function \( D \), which is derived based on (52). Finally, the DTUE is the fixed-point of this procedure, i.e., \( p = \hat{p} \).

4.3. Equilibrium calculation of the MFGs-DTCP

The equilibrium solution for the DTCP cannot be derived directly from the user optimal control conditions but through an iterative solution method (Zhong et al. 2011, Ameli, Lebacque, and Leclercq 2021). In this section, we present an algorithm that can be utilized to numerically solve the discrete MFGs framework.

Recall that the \( \varepsilon \)-MFE of the DTCP is the fixed-point of (31). Therefore, we can apply fixed-point algorithms with a similar optimality conditions to calculate the equilibrium point of the problem. In the discrete MFGs framework, determining the equilibrium requires obtaining an approximation for \( v_{\bar{\tau}_a} \) given in (52). From the travelers point of view, this approximation enables the players to predict the travel costs required to choose the optimal departure times. The prediction model has to take into account the parameters and evolution of the network, which are captured by the generalized bathtub model. Here, to calculate the equilibrium approximation based on the users decisions, we propose a heuristic algorithm. Our heuristic algorithm is based on the variational inequality theory (Noor 1988). The core idea is to use the delay value \( (\bar{t}_a - t_a) \) to update the departure times in each iteration. In this case, we also consider the travelers mean speed function with the same desired arrival time as the variable to predict the arrival time of the next simulation with respect to their trip length. Indeed, we reschedule the departure time for each traveler based on equations (52-57).

The original algorithm is detailed in Friesz and Han (2019). It is proposed for a continuous dynamic assignment model while we use a discrete version of it based on Ameli, Lebacque, and Leclercq (2020a). Note that in each iteration of the algorithm, a proportion of travelers are selected for rescheduling. This proportion is equal to the product of the total demand and a step size. The step size is a coefficient between zero and one that is decreasing during the optimization process (Ameli, Lebacque, and Leclercq 2020b). In this study, the step size is fixed to one over the iteration index. We also add a smart selection process (inspired from Sbayti, Lu, and Mahmassani (2007)) to the algorithm in order to speed up the convergence. The process sorts all the trips based on
their travel cost (17) and then selects the trips with the higher travel costs for the rescheduling process. Note that in all numerical examples, the length of the time interval in the discrete model is considered as one second.

5. Numerical experiments

In order to examine the efficiency of our MFGs model, we first compare its performance with one of the recently proposed models in the literature. We then apply the MFGs framework to a large-scale test case in order to evaluate its performance and examine how the optimization procedure to determine the DTUE affects the congestion level of the network’s real state. This application is the first one in the literature that addresses the departure time equilibrium on a real large-scale network with a large number of users.

5.1. Validation of the MFGs framework

Lamotte and Geroliminis (2018) used a quadratic function for the network mean speed function (detailed in Lamotte (2018)). They use a trip-based macroscopic fundamental diagram (MFD) model (Lamotte and Geroliminis 2018, Leclercq, Sénécat, and Mariotte 2017) and not the generalized bathtub model. However, both approaches share a common ground and produce similar results in terms of the traffic dynamics. Here, we apply our proposed framework with the exact same demand profile, the same parameters for the cost function, including a smooth approximation of the $\alpha$-$\beta$-$\gamma$ preference, modeled by the marginal utility of the time spent at home $h(t) = \alpha$, the marginal utility of time spent at work $w(t) = \frac{2+\gamma-\beta}{2} + \arctan(4(\bar{t}_i - t_{ia})\frac{2+\beta}{\pi})$ and the same parameters for the mean speed function, i.e., the network capacity and free flow speed. The description of all simulation parameters are presented in Section 5.1 of Lamotte and Geroliminis (2018). Note that trip-lengths are uniformly distributed between 0 and 3. The solution method in Lamotte and Geroliminis (2018) is conducted on a day-to-day basis using a selection method inspired by Method of Successive Average (MSA) and an optimization method based on the grid search (detailed in Lamotte and Geroliminis (2016)). Table 2 compares the optimization results of this model with the one proposed in this paper based on MFGs. The results show that, considering the relative cost, the proposed MFGs method outperforms the previous approach by 5.6%. Interestingly, the improvements are much more significant considering the computational effort (the number of iterations) required to converge. Specifically, our numerical results show that the proposed MGFs approach requires 87% less number of iterations to converge. The computation time of each iteration for the proposed method is approximately 3.5 times higher than the reference algorithm because of the MFG approximation. Therefore, the algorithm works 54% faster than the reference algorithm. Note that the similarity of the solution quality in terms of the average cost and the total travel
TABLE 2 The quality of the equilibrium approximation.

| Solution method | Total number of iteration | Convergence indicator [relative cost] | Average cost | Total travel time [sec] |
|-----------------|--------------------------|--------------------------------------|--------------|------------------------|
| MFGs method     | 259                      | 3.37E-03                             | 12.01        | 26984                  |
| Lamotte and Geroliminis (2018) | 2000                  | 3.57E-03                             | 12.66        | 27362                  |
| Improvement [%] | 87.05                    | 5.61                                 | 5.13         | 1.38                   |

(a) Time series of accumulation
(b) Time series of speed
(c) Cumulative departure and arrival curves.

FIGURE 2 Simulation results with heterogeneous trip-length: Speed MFD based framework with grid search (Lamotte and Geroliminis 2018) versus MFGs framework

time shows that the MFGs framework is consistent with the existing method in the literature for the morning commute problem.

To further compare the properties of the final solution for both algorithms, we consider the cumulative departure and arrival curves that provide the characteristics of all trips, and the time-evolution of accumulation and mean speed in the network, see Figure 2. The MFGs framework
provides a solution with a lower maximum accumulation (Figure 2(a)) and a higher speed (Figure 2(b)) than the grid search algorithm. It means that the system is closer to the system optimum, defined as the solution where the total travel time of all vehicles is minimum. While it is not the objective function we aim to minimize, it is interesting to notice that reducing further the total individual costs has a positive impact on the overall system. Figure 2(c) illustrates how trips are started considering trip lengths and departure time. This is a crucial feature as the existing solution methods require prior assumptions on such a sorting to reduce the exploration of the solution space. For example, in Lamotte and Geroliminis (2018), partial FIFO sorting conditions are mandatory to derive the optimal solutions. Our MFGs framework relaxes such conditions and can provide a full exploration of the solution space. Figure 2 exhibits five time periods (the dash line boxes in each figure) where FIFO patterns are observed in the optimal solution of the grid search solution but no sorting pattern in the MFGs solution. In Figure 2(c), the inflow rate of the MFGs solution is higher than the grid search while the slope of the outflow rate is less than the grid search. This test case shows how important it is to relax the sorting assumptions based on the trip lengths to get the optimal solution, which can only be achieved by the proposed MFGs framework.

5.2. Application of MFGs framework at large scale

The application of the proposed MFGs framework is easily scalable to much larger instances, which is the main advantage of MFGs over the classic game theory approaches. In this section, we consider a real test case corresponding to the northern part of a metropolis in France (Lyon) and all trips during the morning peak hours, i.e., more than 60,000 trips in total. Note that this section presents the application of our DTUE model to the largest real network with real demand pattern compared to the literature of departure time choice models.

5.2.1. Description of the test case and demand profile. We implement and apply the proposed model to the northern part of Lyon Metropolis (Lyon North). Lyon North network covers 25 km² and includes 1,883 nodes and 3,383 links. The map is shown in Figure 3(a). The original demand setting includes all trips during the morning peak hours from 6:30 AM to 10:40 AM (62,450 trips). The data set is published in Ameli, Alisoltani, and Leclercq (2021)11. It has been calibrated to represent realistic traffic conditions (Krug, Burianne, and Leclercq 2019). All trips have an origin and destination on the real network and a departure times. At the link level of the network (Figure 3(b)), the origins set contains 94 points and the destinations set includes 227 points. In this study, we only keep the original trip lengths as the generalized bathtub model does not account for the local traffic dynamics. Some trips have origins or destinations outside the covered area (51,215 trips) and will not be considered in the departure time optimization. Note that 11235 trips are fully

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11 https://research-data.ifsttar.fr/dataset.xhtml?persistentId=doi:10.25578/HWN8KE
interior. For those, the original departure time is disregarded and a desired arrival time is assigned. We divide them into seven classes with different desired arrival times. The desired arrival time of each user is deduced from the real arrival time of the user based on real data (Krug, Burianne, and Leclercq 2019, Alisoltani et al. 2019). The percentage of the trips per class and their desired arrival time are presented in Table 3.

The network speed function has been calculated in Mariotte et al. (2020). The cost function parameters, i.e., the \( \alpha-\beta-\gamma \) scheduling preferences are defined based on the study of Lamotte and Geroliminis (2018): \( \alpha = 1, \beta = 0.4 + \frac{0.2k}{9} \) and \( \gamma = 1.5 + \frac{k}{5} \). In order to consider only the heterogeneity of trip length and desired arrival time distributions, \( k \) is fixed to 5 for all trips in this experiment.
5.2.2. Numerical results. The optimization process is started with an initial solution where the targeted travelers with a higher trip length in all classes start their trip sooner than others based on the network free-flow speed \( v_{\text{max}} = 13.28 \text{m/s} \). The heuristic algorithm converges after 56 iterations to an equilibrium approximation. The results for the convergence pattern is presented in Figure 4(a). The final average cost per traveler is 326.92, and the figure shows that the final solution is stable. As in the previous test case, the MFGs algorithm converges very fast; however, the performance of heuristic/search algorithms depends on the initial solution.

Figure 4(b) presents the evolution of the network’s total travel time during the optimization process. Similar to the convergence pattern, the total travel time decreases and converges to a stable value. Therefore, the final solution can be an equilibrium approximation for the problem.

To provide more insights, we also assess the network and equilibrium features overtime during the optimization process.

![Convergence pattern](image1.png)

![Evolution of the total travel time](image2.png)

**FIGURE 4** Results of the optimization process: The travel cost is calculated using (17) and the travel time is the value of \( T(t_d, x) \) for each traveler.

Figure 5 presents the results for the accumulation of the network at each time step \( \Delta t = 1 \text{ sec} \) for the convergence process and the final iteration of the optimization process. In Figure 5(a), every blue extrema indicates the evolution of the accumulation in 4.17 hours simulation at one iteration. The curve for the next iteration is started right after the previous one. The results show that the accumulation is also decreasing during the optimization process, and as it is expected, the equilibrium approximation has a low value for the maximum accumulation (red line) of the final solution. Remind that the accumulation evolution in Figure 5(a) is drawn for interior trips. The equilibrium accumulation for the full demand is shown in Figure 5(b). The accumulation time
(a) Evolution of accumulation for the interior trips: The red line denotes the maximum accumulation of each iteration.

(b) Accumulation of the real state of the network versus equilibrium approximation for all trips.

FIGURE 5 Results of the network’s performance overtime ($\Delta t = 1s$) in the optimization process. Each iteration contains 4.17 hours simulation [6:30 AM - 10:40 AM].

The convergence results regarding the different classes of trips are presented in Figure 6(a). The algorithm’s convergence pattern improves in the first three iterations continuously. However, after the third iteration, there are small variations for different classes. This is because of our algorithm’s heuristic nature that needs to search (explore) the solution space and then exploit it to find a local
or the global optimum solution. Note that the exploration rate of heuristic methods depends on the complexity of the solution space and the step size. Figure 6(b) illustrates another aspect of the equilibrium approximation where each green diamond represents the departure time, and each red circle represents the arrival time of a trip. The duration of a trip is represented by a horizontal blue line between departure and arrival time. The trips of each class are sorted based on their trip length. In Figure 6(b), the deformations of the distributions for all classes show that non regular sorting pattern matches with the optimal solutions. So, again we show how important it is to not resort to any prerequisite about the sorting when designing the solution method and defining the optimal conditions. For instance, in Figure 6(b), the departure and arrival time distributions for class 4 (with desired arrival time 8:30 and the highest demand level) has a deformation on the trip lengths interval [600-850], which illustrates that the partial sorting pattern like FIFO and LIFO does not stand.

![Figure 6](image)

(a) Absolute value of the average delay for each class of users. (b) Departure and arrival time distributions of the equilibrium approximation.

**FIGURE 6** Optimization results regarding the different classes of trips. Note that there are 11235 interior users in the optimization process.

### 6. Conclusion

This is the first paper that demonstrates the value of the MFGs approach in departure time equilibrium models. Specifically, this work focuses on modeling and characterizing the departure time choice equilibrium, which is mathematically challenging for large-scale networks, using the MFGs approach. We propose a new optimization framework based on the recent findings in transportation systems and game theory. The framework is designed based on mean field game theory coupled with the generalized bathtub model. The MFGs theory allows us to consider a large number of players...
with different desired arrival times. The idea is that each player in the system optimizes its strategy with respect to the mean-field of the strategies of the other players. Besides, the generalized bathtub model can represent more complex interactions of supply and demand in a transportation system.

Departure time choices of a group of rational travelers on a traffic network are intrinsically related to how they predict travel time. In this study, we develop a mathematical model through which a generic player predicts the other players’ macroscopic behavior. Then, based on this prediction, he derives the dynamics of the system by obtaining the velocity. Having velocity, this player optimizes his departure time strategy. Since the setting is a game with rational players, they look for a Nash equilibrium that can be obtained by a fixed point argument in the procedure of decision making.

Moreover, we implemented the proposed model for the known setting of the morning commute problem in the literature, and the morning peak hour of the real traffic network of the Lyon North. The numerical results for the first test case demonstrate the value of the MFGs framework compared to existing models in the literature. The large-scale test case shows that the proposed framework is able to represent the equilibrium problem with multiple desired arrival times for a large number of trips that was little addressed before. For the equilibrium calculation, we adapt a heuristic fixed-point algorithm that converges very fast. The proposed model also provides a good approximation for the equilibrium. The optimization results on both experiments show that optimization based on the mean-field of the users’ strategies performs significantly better than the solution methods with Myopia assumption. The gain is much higher when the number of users and the scale of the problem are increased. The equilibrium approximation obtained by the simulation-based optimization contains partial sorting patterns and provide interesting insights on the prevailing sorting assumptions (FIFO or LIFO). These results are supported by Lamotte and Geroliminis (2018) and underline the importance of the empirical measurements compared to the analytical studies (e.g., Fosgerau (2015), Daganzo and Lehe (2015)).

Future researches can be carried out in the following two aspects: (i) analyzing the characteristics of the equilibrium such as uniqueness and stability, which is not easy with heterogeneity of users (Lamotte and Geroliminis 2021); (ii) extend the model by addressing more complex settings and relaxing some of the assumptions. For instance, considering more complex travel costs or utility functions can be an interesting research direction to build a more realistic mathematical model. Further, developing optimization algorithms based on the MFGs numerical solution methods looks promising for future works.

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Appendix A: Proof of Proposition 1

Since $F \in \mathcal{P}_{m,G}$, by Radon–Nikodym theorem (see e.g. Theorem 32.2 of Billingsley (2012)), $F$ admits a probability density function denoted by $f$. Thus, we can employ relation 24 to describe the system congestion. Then, in the light of the equality given in (4) and considering that $z'(t) = v_t$, we can rewrite (24) as follows,

$$
c_t = \int_0^t \int_0^{X_{\text{max}}} f(r,x)dxdr - \int_0^t \int_0^r v_s f(s,z(s) - z(s))dsdr
$$

$$
= \int_0^t \int_0^{X_{\text{max}}} f(r,x)dxdr - \int_0^t \int_s^r v_s f(s,z(r) - z(s))drds.
$$

$$
= \int_0^t F(dr, X) - \int_0^t F(ds, [0, z(t) - z(ds)])
$$

$$
= \int_0^t F(ds, (z(t) - z(ds), X_{\text{max}}))
$$

$$
= F(S_s(z_e)).
$$

Reusing relation 4 finishes the proof. \hfill \Box

Appendix B: Proof of Proposition 2

Consider the mapping $\mathcal{U}$ defined in (27). Note that the space $\mathcal{C}(\mathcal{T})$ is a Banach space wrt the uniform norm. Therefore, by Banach Fixed-Point Theorem (see Theorem 3.48 of Aliprantis and Border (2006)), it is sufficient to show that the function $\mathcal{U}$ is a contracting mapping, namely there is $\Lambda \in (0, 1)$ such that for all $z_1, z_2 \in \mathcal{C}(\mathcal{T})$, $d(\mathcal{U}(z_1, F), \mathcal{U}(z_2, F)) \leq \Lambda d(z_1, z_2)$. Hence, consider $z_1, z_2 \in \mathcal{C}(\mathcal{T})$ arbitrarily. Using triangle inequality we get:

$$
|\tilde{z}_1(t) - \tilde{z}_2(t)| \leq \int_0^t |V(F(S_s(z_1))) - V(F(S_s(z_2)))|dr.
$$

Then, since $V$ is Lipschitz, we have:

$$
|\tilde{z}_1(t) - \tilde{z}_2(t)| \leq \text{Lip}(V) \int_0^t |F(S_s(z_1)) - F(S_s(z_2))|ds
$$

$$
\leq \text{Lip}(V) \int_0^t F(S_s(z_1) \Delta S_s(z_2))ds.
$$

Using Assumption 2, we get:

$$
|\tilde{z}_1(t) - \tilde{z}_2(t)| \leq G \text{Lip}(V) \int_0^t \int_0^s |z_1(s) - z_2(s)| + |z_1(r) - z_2(r)|drds.
$$

Multiplying both sides of the above inequality by $e^{-Mt}$, we obtain:

$$
|\tilde{z}_1(t) - \tilde{z}_2(t)| e^{-Mt} \leq G \text{Lip}(V) \int_0^t \int_0^s |z_1(s) - z_2(s)| e^{-Ms} e^{-M(t-s)}drds
$$

$$
+ G \text{Lip}(V) \int_0^t \int_0^s |z_1(r) - z_2(r)| e^{-Mr} e^{-M(t-r)}drds
$$

$$
\leq 2G \text{Lip}(V) d_M(z_1, z_2) \frac{Mt - 1 + e^{-Mt} M^2}{M^2},
$$

where the second inequality is based on the following relations:

- $\forall s \in \mathcal{T}$, $|z_1(s) - z_2(s)| e^{-Ms} \leq d_M(z_1, z_2)$,
- $\int_0^t e^{-M(t-s)}ds = \frac{1 - e^{-Ms}}{M}$,
- $\int_0^t re^{-M(t-r)}dr = \frac{Mt - 1 + e^{-Mt} M^2}{M^2}$.
\[ \int_0^t \frac{1-e^{-Mt}}{M} \, dt = \frac{Mt-1+e^{-Mt}}{M^2}. \]

Since \( \frac{Mt-1+e^{-Mt}}{M^2} \) is increasing wrt \( t > 0 \), taking supremum over \( t \in T \) yields to:

\[ d_M(\tilde{z}_1, \tilde{z}_2) \leq 2G \text{Lip}(V) \frac{M_{\text{max}} - 1 + e^{-M_{\text{max}}}}{M^2} d_M(z_1, z_2). \]

Now, choose \( M \) such that \( 2G \text{Lip}(V) \frac{M_{\text{max}} - 1 + e^{-M_{\text{max}}}}{M^2} < 1 \). Considering the equivalency between \( \| \cdot \| \) and \( \| \cdot \|_M \), the proof is complete. \( \square \)

**Appendix C: Proof of Corollary 1**

Based on Banach Fixed-Point Theorem (see Theorem 3.48 of Aliprantis and Border (2006)) and as a result of the proof of Proposition 2, \( z_l \) defined in (28) converges to the fixed point of the function \( U \) given in (27).

**Appendix D: Proof of Proposition 3**

Let \( \{F_k\}_{k \in \mathbb{N}} \subset \mathcal{P}_{m,G} \) be such that \( F_k \Rightarrow F \). Using the definition of \( \mathcal{P}_{m,G} \) and Portmanteau Theorem (see e.g., Theorem 2.1. of Billingsley (2013)), for all open sets \( O \subset T_d \times \mathcal{X} \), we have:

\[ F(O) \leq \liminf_k F_k(O) \leq G\lambda_2(O). \]

Now, note that the Lebesgue measure is outer regular in the sense that a measurable set can be approximated by an open set from outside. That means for all \( \varepsilon > 0 \) and all measurable sets \( B \subset T_d \times \mathcal{X} \), there exists an open set \( O \) such that \( B \subset O \) and \( \lambda_2(O) < \lambda_2(B) + \varepsilon \). Since \( B \subset O \) implies that \( F(B) \leq F(O) \), we can write:

\[ F(B) \leq F(O) \leq G\lambda_2(O) < G\lambda_2(B) + G\varepsilon. \]

As \( \varepsilon > 0 \) is arbitrary, it yields to:

\[ F(B) \leq G\lambda_2(B). \] (58)

Additionally, let \( Q \) be an arbitrary measurable subset of \( \mathcal{X} \). For all \( k \in \mathbb{N} \), since \( F_k \in \mathcal{P}_{m,G} \), by (16), we have,

\[ F_k(T_d, Q) = m(Q, T_a), \quad (\text{a.s.}) \] (59)

where \( m \) is the demand profile. On the other hand, by (58), \( F(\partial Q) \leq G\lambda_2(\partial Q) = 0 \), and \( Q \) is a \( F \)-continuity set\(^{12} \). Then, by the weak convergence of \( F_k \) to \( F \) and relation (59), Portmanteau Theorem implies that:

\[ m(Q, T_a) = \lim_k F_k(T_d, Q) = F(T_d, Q), \quad (\text{a.s.}) \]

Therefore, \( F \in \mathcal{P}_{m,G} \) and \( \mathcal{P}_{m,G} \) is closed under weak convergence. \( \square \)

\(^{12} \partial A \), when \( A \) is a set, refers to its boundary.
Appendix E: Proof of Proposition 4

Let

\[ \Lambda = 2G \text{Lip}(V) \frac{MT_{\text{max}} - 1 + e^{-MT_{\text{max}}}}{M^2}. \]

Following the proof of Proposition 2, we know that,

\[ d_M(\mathcal{U}(z_1, F), \mathcal{U}(z_2, F)) \leq \Lambda d_M(z_1, z_2), \quad \forall F \in \mathcal{P}_{m,G}, \forall z_1, z_2 \in \mathcal{C}(T), \]

where \( \mathcal{U}(z, F) \) denotes the characteristic travel distance of a system having in-flow measure \( F \) and primary characteristic travel distance \( z \), see (27). Then, we have,

\[
\begin{align*}
    d_M(z_k^*, z^*) &= d_M(\mathcal{U}(z^*, F), \mathcal{U}(z_k^*, F_k)) \\
    &\leq d_M(\mathcal{U}(z^*, F_k), \mathcal{U}(z_k^*, F_k)) + d_M(\mathcal{U}(z^*, F), \mathcal{U}(z^*, F_k)) \\
    &\leq \Lambda d_M(z^*, z_k^*) + \sup_{0 \leq t \leq T_{\text{max}}} \exp^{-Mt} \int_0^t |V(F(S_t(z^*))) - V(F_k(S_t(z^*)))| ds.
\end{align*}
\]

Considering that \( M \) is chosen such that \( \Lambda < 1 \), we obtain the following bound:

\[
    d_M(z_k^*, z^*) \leq \frac{\text{Lip}(V)}{1 - \Lambda} \int_0^{T_{\text{max}}} |F(S_t(z^*)) - F_k(S_t(z^*))| ds.
\]

The rhs of (60) is bounded by \( \frac{2\text{Lip}(V)GT_{\text{max}}}{1 - \Lambda} \) and converges to 0 as \( F_k \Rightarrow F \) by Portmanteau Theorem. Then, using Lebesgue dominated convergence, the rhs of (60) converges to 0 as \( k \to \infty \). Finally, considering equivalency between \( \| \cdot \| \) and \( \| \cdot \|_M \), the proof of the proposition is complete. \( \square \)

Appendix F: Proof of Corollary 2

Due to (13), the bound provided in (60) can then be expressed as

\[
    d_M(z_k^*, z^*) \leq \frac{\text{Lip}(V)}{1 - \Lambda} \int_0^{T_{\text{max}}} |E(S_t(z^*) \times T_a) - E_k(S_t(z^*) \times T_a)| ds,
\]

which shows that if \( F_k \Rightarrow E \) as \( k \to \infty \), then \( z_k^* \to z^* \) in the uniform norm. \( \square \)

Appendix G: Proof of Proposition 5

The cost function \( J \) is defined as follows:

\[
J(t_d; x, t_a; F) = \begin{cases} 
\alpha T(t_d, x) + \beta(t_a - t_d + T(t_d, x)), & t_a \geq t_d + T(t_d, x) \\
\alpha T(t_d, x) + \gamma(t_a + T(t_d, x) - t_a), & t_d + T(t_d, x) > t_a.
\end{cases}
\]

In order to prove the proposition, it is sufficient to establish the continuity of \( T \) as a function of \( (t_d, x, t_a) \) and \( F \). Considering that \( T(t_d, x) = z_F^{-1}(x + z_F(t_d)) \), it suffices to establish the continuity for \( z_F \) and \( z_F^{-1} \). Since \( V \) is bounded from above by \( V_{\text{max}} \), \( z_F \) is Lipschitz with \( \text{Lip}(z_F) \leq V_{\text{max}} \). From (61), it follows that for any \( t, t' \in \mathcal{T} \) and any \( F, F' \in \mathcal{P}_{m,G} \):

\[
|z_F(t) - z_F(t')| \leq |z_F(t) - z_F(t)| + |z_F(t) - z_F(t')| \\
\leq |z_F(t) - z_F(t)| + V_{\text{max}}|t - t'| \\
\leq e^{MT_{\text{max}}} \frac{\text{Lip}(V)}{1 - \Lambda} \int_0^{T_{\text{max}}} ds |F(S_t(z_F)) - F'(S_t(z_F))| + V_{\text{max}}|t - t'|.
\]
The continuity of \((t, F) \rightarrow z_F(t)\) is thus established and is obviously Lipschitz wrt \(t\). Now, as \(V\) is bounded from below by \(V_{min}\), it holds that for any \(F \in \mathcal{P}_{m,G}: |z_F(t) - z_F(t')| \geq V_{min}|t - t'|\). Therefore, \(z_F\) is invertible and its inverse is Lipschitz with Lip\((z_F^{-1}) \leq 1/V_{min}\). To conclude, consider \(F, F' \in \mathcal{P}_{m,G}, x, x' \in \mathcal{X}, \) and \(t = z_F^{-1}(x), t' = z_{F'}^{-1}(x')\). We have:

\[
z_F(t) - z_{F'}(t') = z_F(t) - z_F(t) + x - x'.
\]

Since

\[
|z_F(t) - z_{F'}(t')| \geq V_{min}|t - t'| = V_{min}|z_F^{-1}(x) - z_{F'}^{-1}(x')|
\]

and

\[
|z_F(t) - z_F(t)| \leq e^{MT_{max}} \frac{\text{Lip}(V)}{1 - \Lambda} \int_0^{T_{max}} ds \ |F(S_s(z_F)) - F'(S_s(z_F))|,
\]

we get:

\[
|z_F^{-1}(x) - z_{F'}^{-1}(x')| \leq \frac{|x - x'|}{V_{min}} + e^{MT_{max}} \frac{\text{Lip}(V)}{1 - \Lambda} V_{min} \int_0^{T_{max}} ds \ |F(S_s(z_F)) - F'(S_s(z_F))|.
\]

The continuity of \((x, F) \rightarrow z_F^{-1}(x)\) is proved and is Lipschitz wrt \(x\). Since \(J\) is a piecewise linear function of \(t_d, t_a\) and travel time function \(T\), it follows that \(J\) is Lipschitz continuous wrt \(t_d, x\) and \(t_a\). Note that the Lipschitz constant of \(J\) depends only on \(\text{Lip}(V), V_{min}, V_{max}, \) and \(\alpha, \beta, \gamma.\) The continuity coefficient for the dependency of \(J\) on \(F\) is also dependent on \(M,\) and \(\Lambda,\) where \(\Lambda\) itself depends on the constant \(G.\)

**Appendix H: Proof of Proposition 6**

First, note that \(\mathcal{P}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a)\) is a convex compact subset of \(\mathbf{M}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a)\), the set of signed measures with bounded variation on \((\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a),\) see proof of Theorem 4.10 in Lackner (2018). Further, one can show easily that, for all \(G \in \mathbb{R}^+, \mathcal{M}_{m,G}\) is a closed subset of \(\mathcal{P}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a),\) with an argument similar to the one given in the the proof of Proposition 3. Since a closed subset of a compact set is compact, \(\mathcal{M}_{m,G}\) is a compact subset of \(\mathbf{M}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a)\), too. The convexity of \(\mathcal{M}_{m,G}\) is trivial by its definition.

Now, as \(\mathbf{M}(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_a)\) is a locally convex topological vector space, we can apply the fixed-point theorem of Kakutani (see Theorem 8.6 of Granas and Dugundji (2003)) to prove \(H\) admits a fixed point. The convexity and compactness of \(H\) is clear. It remains to show that for all \(E, H(E)\) is non-empty and \(H\) is usc (upper semi-continuous).

i) Consider \(\tilde{E} \in \mathcal{M}_{m,g}.\) We aim to show that \(H(\tilde{E})\) is non-empty. Denote \(\tilde{F} = F(\tilde{E})\) and \(\tilde{J}(t_d; x, t_a; \tilde{F}) = J(t_d; x, t_a; \tilde{F}) - \min_{t \in \mathcal{T}_d} J(t; x, t_a; \tilde{F}).\) The function \(\tilde{J}\) is continuous wrt \(F\) and Lipschitz continuous wrt \(t_d, x, t_a.\) Further, we have \(\text{Lip}(\tilde{J}) \leq 2\text{Lip}(J).\) Indeed, the function \((x, t_a) \rightarrow \min_{t \in \mathcal{T}_d} J(t; x, t_a; F)\) admits the same Lipschitz constant as \(J.\) Notice that:

\[
|\tilde{J}(t; x, t_a; \tilde{F}) - \tilde{J}(t'; x, t_a; \tilde{F})| \leq \text{Lip}(J)|t - t'|, \quad \forall t, t' \in \mathcal{T}.
\]

Denote \(U = \{(t_d, x, t_a), |\tilde{J}(t_d; x, t_a; \tilde{F}) < \varepsilon\} \) and \(U_{x,t_a} = \{t|\tilde{J}(t; x, t_a; \tilde{F}) < \varepsilon\}.\) \(U\) is an open set. More precisely, if \((t_d, x, t_a) \in U,\) then all \((t'_d, x', t'_a)\) such that

\[
|t_d - t'_d| + |x - x'| + |t_a - t'_a| < \frac{\varepsilon - \tilde{J}(t_d, x, t_a; \tilde{F})}{2\text{Lip}(J)}
\]
also belong to $U$. If $t_d$ is in $U_{x,t_a}$, i.e., $\tilde{J}(t,x,t_a;\tilde{F}) < \varepsilon$, then for all $t_d'$ such that

$$|t_d - t_d'| \leq \frac{\varepsilon - \tilde{J}(t,x,t_a;\tilde{F})}{\text{Lip}(J)}.$$

Thus, for all $(x,t_a) \in (X,T_a), U_{x,t_a}$ has Lebesgue measure greater than $\frac{2\varepsilon}{\text{Lip}(J)}$. It follows that $(x,t_a) \to \lambda(U_{x,t_a})$ is lsc (lower semi-continuous) and bounded from below by $\frac{2\varepsilon}{\text{Lip}(J)}$ where $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.

Now define the function $\nu$ on $(T_d \times X \times T_a)$ by:

$$\nu(t_d,x,t_a) = \frac{1_{U}(t_d,x,t_a)}{\lambda(U_{x,t_a})}.$$

This function is lsc and bounded from above by $\frac{\text{Lip}(J)}{2\varepsilon}$.

Finally we define $E$ such that

$$E(dt_d, dx, dt_a) = \nu(t_d,x,t_a)\lambda(dt_d)\mu(dx,dt_a).$$

By construction, $E$ is positive, has total mass 1 and satisfies the constraint (15). Also, the support of $E$ lies in $U$ by construction; hence, $E$ satisfies $E \in H(\tilde{E})$. It remains to be checked that $E \in \mathcal{M}_{m,G}$. It suffices to check that $\mathcal{F}(E)$ satisfies Assumption 2. Consider $B \in \mathcal{B}(T_d \times X)$. Using Fubini Theorem (see, e.g., Theorem 18.3 of Billingsley (2012)) and Assumption 1, regularity condition of $m$, we have,

$$\mathcal{F}(E)(B) = E(B \times T_a) = \int_{B \times T_a} dt_d dm(x,t_a) \nu(t_d,x,t_a) \leq \frac{\text{Lip}(J)}{2\varepsilon} \int_{B \times T_a} dt_d dm(x,t_a) \leq \frac{M_m \text{Lip}(J)}{2\varepsilon} \lambda(B),$$

where $\lambda$ is Lebesgue measure on $\mathbb{R}^2$. The above calculation yields to an estimate from below of $G$:

$$G \geq \frac{M_m \text{Lip}(J)}{2\varepsilon}.$$  \hspace{1cm} (62)

It should be noted that the constant $\text{Lip}(J)$ in (62) does not depend on $\mathcal{F}(E)(B)$ but only on the cost function and on the data $V_{min}, V_{max}$ (refer to the proof in Appendix G). Thus choosing $G \geq \frac{M_m \text{Lip}(J)}{2\varepsilon}$ ensures that $E \in \mathcal{M}_{m,G}$. This completes the proof that $H(\tilde{E})$ is non-empty.

ii) We next show that $H$ is usc (upper semi-continuous). Note that $J$ depends on $E$ via $\mathcal{F}$. In order to simplify the notations, in this paragraph we will write $J(E)$ for $J(\mathcal{F}(E))$. We rewrite the definition of $H$ as follows,

$$H(E) = \left\{ e \in \mathcal{M}_{m,G} \mid e \left( \tilde{J}(E) \leq \varepsilon \right) = 1 \right\}.$$

In order to show that $H$ is usc it is required to prove that for any open set $W \in \mathcal{M}_{m,G}$, the set $H^{-1}W = \{ E \mid H(E) \subseteq W \}$ is open, see Page 166 of Granas and Dugundji (2003). Conversely, denoting $W^c$ and $(H^{-1}W)^c$ the respective complements of $W$ and $H^{-1}W$, it suffices to show that if $W^c$ is closed, $(H^{-1}W)^c$ is closed.

Consider a convergent sequence $\{ E_n \}_{n \in \mathbb{N}}$ of elements of $(H^{-1}W)^c$, and let $E$ be the limit of this sequence. We now show that $E \in (H^{-1}W)^c$. For all $n \in \mathbb{N}$, $H(E_n) \not\subseteq W$ there exists $e_n \in H(E_n) \cap W^c$. By compactness, we can assume, after extracting a sub-sequence, that the sequence $\{ e_n \}_{n \in \mathbb{N}}$ converges weakly towards some $e \in W^c \subseteq \mathcal{M}_{m,G}$. It remains to show that $e \in H(E)$.
Now for any $\eta > 0$ there exists $N(\eta)$ such that for $n \geq N(\eta)$, $|\hat{J}(E_n) - \hat{J}(E)| < \eta$ (uniformly in $C(\mathcal{T}_d \times \mathcal{X} \times \mathcal{T}_d)$). It implies that

$$\{\hat{J}(E_n) \leq \varepsilon\} \subset \{\hat{J}(E) \leq \varepsilon + \eta\}, \quad n \geq N(\eta),$$

and consequently for all $n \geq N(\eta)$,

$$e_n(\hat{J}(e) < \varepsilon + \eta) = 1.$$

Since $e_n \xrightarrow{n \to \infty} e$, it follows by Portmanteau Theorem that

$$e(\hat{J}(e) \leq \varepsilon + \eta) \geq \limsup_{n \to \infty} e_n(\hat{J}(e) \leq \varepsilon + \eta),$$

and thus:

$$e(\hat{J}(e) \leq \varepsilon + \eta) = 1, \quad \forall \eta > 0.$$

Finally, $\eta \mapsto 1_{\{\hat{J}(e) \leq \varepsilon + \eta\}}$ is monotone decreasing; thus, by Monotone Convergence Theorem, Theorem 4.3.2 of Dudley (2018), we have:

$$\lim_{\eta \to 0} e(\hat{J}(e) \leq \varepsilon + \eta) = e(\hat{J}(e) \leq \varepsilon) = 1.$$

Therefore, we proved that $E \in (H^{-1}W)^c$. \hfill \square

**Appendix I: Proof of Proposition 8**

Fix $t_a \in \mathcal{T}_a$ and $x \in \mathcal{X}$. Consider $h$ and $l$ as the small changes in time and space, respectively. The demand with desired arrival time in $\Delta t_a := (t_a - \frac{h}{2}, t_a + \frac{h}{2})$ and trip length in $\Delta x := (x - \frac{l}{2}, x + \frac{l}{2})$ is equal to $m(\Delta x, \Delta t_a)$ which can be approximated by $m(dx, dt_a)hl$.

On the other hand, since $D$ is increasing wrt the desired arrival time, the departure time of the agents with desired arrival times in $\Delta t_a$ and the trip length $x$ is in the interval $(D(t_a - \frac{h}{2}, x), D(t_a + \frac{h}{2}, x))$. Then, we approximate the fraction of agents having departure time in $(D(t_a - \frac{h}{2}, x), D(t_a + \frac{h}{2}, x))$, trip length in $\Delta x$, and desired arrival time in $\Delta t_a$ by

$$e(D(t_a, x), x, t_a)(D(t_a + \frac{h}{2}, x) - D(t_a - \frac{h}{2}, x))l.$$

But, we have:

$$e(D(t_a, x), x, t_a)(D(t_a + \frac{h}{2}, x) - D(t_a - \frac{h}{2}, x))l \approx m(dx, dt_a)hl.$$

Letting $h \to 0$ yields to the desired result. \hfill \square