On Bloch seminorm of finite Blaschke products in the unit disk

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Abstract

We prove that, for any finite Blaschke product \( w = B(z) \) in the unit disk, the corresponding Riemann surface over the \( w \)-plane contains a one-sheeted disk of the radius 0.5. Moreover, it contains a unit one-sheeted disk with a radial slit. We apply this result to obtain a universal sharp lower estimate of the Bloch seminorm for finite Blaschke products.

1 Introduction

Let \( D \) be the unit disk \( \{ |z| < 1 \} \) in the complex plane \( \mathbb{C} \). We will consider finite Blaschke products which can be represented in the form

\[
B(z) = \lambda \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j}z}, \quad |z_j| < 1,
\] 

(1)

where \( |\lambda| = 1 \); further, for simplicity of presentation of the results, we will assume that \( \lambda = 1 \). Finite Blaschke products provide very basic examples of bounded analytic functions which have many important and nontrivial properties [15], [14], [21], [17], [8]. Also, they have numerous applications in complex dynamics [22].

Recall that for a function \( f \) analytic in the unit disc \( D \) its Bloch seminorm is given by

\[
\| f \|_B := \sup_{z \in D} |f'(z)|(1 - |z|^2).
\]

The class \( B \) of analytic functions with bounded Bloch seminorm is called the Bloch space. It is a Banach space if endowed with the norm \( |f(0)| + \| f \|_B \).

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The Bloch seminorm is closely related to the inner (conformal) radius of a domain $\Omega$ at a point $w_0 \in \Omega$ [25]:

$$r(w_0) := \frac{1 - |g(w_0)|^2}{|g'(w_0)|}$$

where $g$ conformally maps $\Omega$ onto $\mathbb{D}$; if additionally we consider the mapping $g$ with $g(w_0) = 0$, then $r(w_0) := \frac{1}{|g'(w_0)|}$. One can easily check that $r(f(z_0)) = (1 - |z_0|^2)|f'(z_0)|$ where $f = g^{-1}$. The value $\sup_{w_0 \in \Omega} r(w_0)$ is called the maximal conformal radius of $\Omega$.

We also recall that if $h$ maps conformally $\Omega$ onto the upper half-plane $\mathbb{H}$, then

$$r(w_0) = \frac{2\Im h(w_0)}{|h'(w_0)|}. \quad (2)$$

By the Schwarz–Pick inequality [3], for any $f \in H^\infty$ (the space of all functions bounded and analytic in $\mathbb{D}$) we have $\|f\|_\mathbb{B} \leq \|f\|_\infty$ and, therefore, $\|B\|_\mathbb{B} \leq 1$ for any (even infinite) Blaschke product. On the other hand, if we consider a Möbius transformation, i.e. a Blaschke product of degree one

$$B_a(z) = \frac{z + a}{1 + \overline{a}z}$$

where $|a| < 1$, then $\|B_a\|_\mathbb{B} = 1$. For each $n$ the inequality $\|B\|_\mathbb{B} \leq 1$ cannot be improved because the Blaschke product of degree one can be approximated locally uniformly by Blaschke products of degree $n$. We want to investigate the following

**Problem.** Is there a universal constant $c > 0$ such that for any finite Blaschke product $B$ the inequality $\|B\|_\mathbb{B} \geq c$ holds?

From the results of Aleksandrov, Anderson and Nicolau [2] it follows that, in the case of infinite Blaschke products, the answer is negative, i.e. $c = 0$. Hence, one may expect that the value

$$\inf\{\|B\|_\mathbb{B} : B \text{ is a Blaschke product of degree } n\}$$

goes to zero as $n \to \infty$. Surprisingly, it turned out that our problem has a positive answer. In addition it should be noted that finite Blaschke products belong to the little Bloch space

$$B_0 = \{f \in \mathbb{B} : \lim_{|z| \to 1} |f'(z)|(1 - |z|) = 0\}.$$

**Theorem 1** Suppose that $B$ is a finite Blaschke product. Then

$$\|B\|_\mathbb{B} \geq r_0 = 0.695356\ldots \quad (3)$$

Conversely, for any $k > 2/e = 0.73575\ldots$ there exists a finite Blaschke product for which $\|B\|_\mathbb{B} \leq k$. 

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Theorem 1 is closely related to classical covering theorems in the geometric function theory. The classical Koebe $1/4$–theorem (see, e.g. [16, ch. II, § 4]) implies that, for every univalent analytic function $f : \mathbb{D} \to \mathbb{C}$, the image $f(\mathbb{D})$ contains the disk with center $f(0)$ and radius $|f'(0)|/4$. Now assume that $f$ is an arbitrary analytic function $f : \mathbb{D} \to \mathbb{C}$. We define $B_f$ to be the radius of the largest disk that is the biholomorphic image of a subset of a unit disk. Bloch [7] proved that $B_f \geq |f'(0)|/72$. Ahlfors [1], as an application of his method of "ultrahyperbolic metrics", established that $B_f \geq |f'(0)|\sqrt{3}/4$. Bonk [9] slightly improved this estimate (see also [10]).

The main goal of our paper is to establish an analog of Bloch’s theorem for $f \circ B$ where $B$ is a Blaschke product. More precisely, we are going to investigate the following question: Let $f$ be an analytic in the disk $\mathbb{D}$ function and $f'(0) = 1$. Is it true that there exists a universal constant $c$ such that for all Blaschke products $B$ there is a one-sheeted disk of the radius $c$ that lies in the Riemann surface of the function inverse to $f \circ B$.

As mentioned above, the answer is negative for infinite Blaschke products. It turns out that such bound exists for finite Blaschke products. To prove this, we need a statement on Riemann surfaces generated by Blaschke products.

In this paper, by Riemann surface $R(f)$ we will mean a covering $f : D \to G$, where $D$ and $G$ are some abstract Riemann surfaces or even domains in the complex plane and $f$ is a holomorphic function mapping $D$ onto $G$ (projection). In general, $R(f)$ is either ramified or unramified and need not be unlimited, i.e. we do not require that it has the curve lifting property [13, p. 25, Definition 4.13].

**Definition 1.** Two Riemann surfaces $f_k : D_k \to G$, $k = 1, 2$, are called equivalent if there exists a biholomorphic mapping $h$ of $D_1$ onto $D_2$ such that $f_2 = h \circ f_1$.

As a rule, equivalent Riemann surfaces are not distinguished.

**Definition 2.** We will say that a Riemann surface $f_2 : D_2 \to G_2$ contains a Riemann surface $f_1 : D_1 \to G_1$ if there exists an injective holomorphic function $h : D_1 \to D_2$ such that $f_2 = h \circ f_1$.

If $D$ is a planar domain and $id_D : D \to D$ is the identity mapping in $D$, then we will identify the corresponding Riemann surface with the domain $D$.

Every finite Blaschke product $B(z)$ generates the Riemann surface $B : \mathbb{D} \to \mathbb{D}$ which is an $n$-sheeted covering; we will denote it by $R(B)$. Theorem 2 below describes some subdomains contained in $R(B)$. To formulate the theorem, we need to introduce some notation.
Let $0 \leq a < 1$. Denote by $D_a$ the unit disk $D$ with the slit along the segment $[a, 1]$ of the real axis. Consider also the domain $G_a$ which is the union of the half-strip

$$S := \{ \Re w < 0, \ 0 < \Im w < 2\pi \}$$

and the rectangle

$$\Pi_a := \{ \log a < \Re w < 0, \ 0 < \Im w < 3\pi \},$$

Let $G^*_a$ be the domain which is obtained by the reflection of $G_a$ with respect to the real axis. Denote by $g : \mathbb{C} \to \mathbb{C}$ the exponential mapping $g(w) = e^w$.

Now consider the equation

$$s \frac{1 - a(s)^2}{(s - a(s))^2} \frac{s - 1}{s + 1} = 0.175,$$

where $a(s) := s \frac{2s - (s^2 - 1)}{2s + (s^2 - 1)}$;

on the segment $[1, 1 + \sqrt{2}]$. From Lemma 1 below it follows that it has a unique solution $s_0 = 2.379796 \ldots$, and we define $a = a(s_0) = 0.024286 \ldots$.

**Theorem 2** Let $R(B)$ be the Riemann surface of a finite Blaschke product $B$ and $a = 0.024286 \ldots$. Then $R(B)$ contains, up to a rotation, either the slit disk $D_a$, or $L_a$, or $L^*_a$.

From Theorem 2 we obtain

**Corollary 1** Let $B$ be a finite Blaschke product defined in the unit disk. Then $R(B)$ contains a disk of radius $1/2$.

Theorem 2 allows us to investigate some properties of Blaschke products in the Bloch space. In fact, Theorem 1 is a consequence of Theorem 2.

In the connection with Theorem 1 we should note a recent result by Dubinin [11, Theorem 1.1], giving a sharp upper bound of $|B'(z)|(1 - |z|^2)$ for finite Blaschke products provided that the critical values lie in a given disk.

Theorem 1 allows us to obtain similar results for functions generalizing Blaschke products.

**Theorem 3** Let $f$ be a holomorphic in $D$ function with $f'(0) = 1$ and $B$ be a finite Blaschke product. Then the Bloch seminorm of $g = f \circ B$ satisfies the inequality

$$\|g\| \geq \sqrt{3} r_0/4 = 0.301098 \ldots$$
where $r_0$ is given in (3). Moreover, if additionally $f$ is a convex univalent function, then
\[ \|g\|_B \geq \pi r_0 / 4 = 0.546131 \ldots \]

The proof of the lower bound in Theorem 1 is not constructive. We complement it with a simple observation how to find a point $z$ where the quantity $|B'(z)|(1 - |z|^2)$ admits a universal (but smaller than in Theorem 1) lower bound. This result also applies to some classes of infinite Blaschke products.

Theorem 4 Let $B$ be a Blaschke product with zeros $\{z_j\}_{j \geq 1}$ and assume that there exists a point $\zeta \in \partial D$ such that $\text{dist}(\zeta, \{z_j\}_{j \geq 1}) > 0$ and, moreover,
\[ |B'(\zeta)| \text{dist}(\zeta, \{z_j\}_{j \geq 1}) \geq \delta > 0 \tag{6} \]
for some $0 < \delta \leq 1$. Then for $z_0 = (1 - \frac{\delta}{8|B'(\zeta)|})\zeta$ we have
\[ |B'(z_0)|(1 - |z_0|^2) \geq 0.07\delta. \]

If $B$ is a finite Blaschke product we can take $\zeta$ to be the point where $|B'|$ attains its maximum in the closed disk $\overline{D}$. Then it is clear (see formula (10) below) that $|B'(\zeta)| \geq \frac{1 + |z_i|}{1 - |z_i|}$ and so (6) is satisfied with constant $\delta = 1$.

Condition (6) appears, e.g., in the study of the so-called “one-component” Blaschke products for which the level set $\{z \in D : |B(z)| < \varepsilon\}$ is connected for some $\varepsilon \in (0, 1)$ (see [5, 6]).

2 Proofs of the main results

To prove Theorem 2 we need to establish two lemmas.

Lemma 1 Let $0 \leq a < 1$ and $\mathbb{D}_a$ be the unit disk with the slit along the segment $[a, 1]$ of the real axis. Then the maximal value of the conformal radius of $\mathbb{D}_a$ is equal to the maximum of the function
\[ g(x) = \frac{4(1 - a^2)x(\sqrt{x^2 + 1} - x)^2}{\sqrt{x^2 + 1}(1 - a(\sqrt{x^2 + 1} - x)^2)^2}, \quad x > 0. \]

The maximum is attained at the point $x_0 = (s_0 - 1)/(2\sqrt{s_0})$ where $s_0$ is the unique positive root of the equation
\[ s^3 - (2 - a)s^2 + (2a - 1)s - a = 0, \quad 1 < s \leq 1 + \sqrt{2}. \tag{7} \]

Moreover,
\[ g(x_0) = 4s_0 \frac{1 - a^2}{(s_0 - a)^2} \frac{s_0 - 1}{s_0 + 1} \quad \text{and} \quad a = s_0 \frac{2s_0 - (s_0^2 - 1)}{2s_0 + (s_0^2 - 1)}. \]
Proof. The conformal mapping of the lower half-plane onto $\mathbb{D}_a$ has the form

$$z = h(w) = \frac{(w - \sqrt{w^2 - 1})^2 + a}{1 + a(w - \sqrt{w^2 - 1})^2},$$

consequently, the conformal radius of $\mathbb{D}_a$ at the point $h(-ix)$, $x > 0$, equals

$$2x|h'(-ix)| = g(x).$$

The function $g$ is strictly positive for $x > 0$ and

$$g'(x) = \frac{1}{x(x^2 + 1)} - \frac{2}{\sqrt{x^2 + 1}} \frac{1 + a(\sqrt{x^2 + 1} - x)^2}{1 - a(\sqrt{x^2 + 1} - x)^2}.$$

therefore, $g'(x) = 0$ if and only if

$$\frac{1 - a(\sqrt{x^2 + 1} - x)^2}{1 + a(\sqrt{x^2 + 1} - x)^2} = 2x\sqrt{x^2 + 1}. \quad (8)$$

If $t = \log(\sqrt{x^2 + 1} + x)$, then $x = \sinh t$ and the equation $(8)$ has the form

$$\frac{e^{2t} - a}{e^{2t} + a} = \sinh 2t.$$

It is equivalent to the cubic equation with respect to $s = e^{2t}$:

$$\psi(s) = 0, \quad \text{where} \quad \psi(s) = s^3 - (2 - a)s^2 + (2a - 1)s - a.$$ 

Simple analysis shows that for real $s$ this equation has a unique root $s_0$ satisfying the inequality $1 < s_0 \leq 1 + \sqrt{2}$. Actually, $\psi(1) = 2(a - 1) < 0$ and for $s = 1 + \sqrt{2}$ we have

$$\psi(s) = s^3 - 2s^2 - s + a(s^2 + 2s - 1) = a(s^2 + 2s - 1) \geq 0,$$

therefore, there is at least one real root of the cubic equation on the interval $(1, 1 + \sqrt{2})$. Assume that there is another real root $s_1$ on $(1, 1 + \sqrt{2})$. By Vieta’s formulas, the product of the roots of the equation $\psi(s) = 0$ is equal $-a < 0$, consequently, we have a third root $s_2$ of the cubic equation which is negative. Then $\psi(s) = (s - s_0)(s - s_1)(s - s_2)$ and, therefore, $\psi(1) > 0$ but this contradicts to the fact that $\psi(1) = 2(a - 1) < 0$.

Thus, we have a unique root $s_0$ of $(7)$. Then $x_0 = \sinh \frac{\log s_0}{2} = \frac{s_0 - 1}{2\sqrt{s_0}}$ is the point of maximum of the function $g$. Lemma 1 is proved.

Remark 1 From the proof of Lemma 1 it follows that $g(x_0)$ can be expressed via $s_0$, i.e. $g(x_0) = F(s_0)$ for some increasing function $F$. Calculations give that if $F(s_0) = 0.7$, then $s_0 = F^{-1}(0.7) = 2.379796\ldots$,

$$a = 0.024286\ldots \quad \text{and} \quad \log a = -3.7178547\ldots \quad (9)$$

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**Lemma 2** Let \( \alpha \) be given by (7). The maximum of the conformal radius of the surface \( L_\alpha \) is not less than \( r_0 := 0.695356 \ldots \)

**Proof.** The conformal mapping of the upper half-plane \( \mathbb{H} \) onto \( L_\alpha \) has the form

\[
f(z) = -C \int_{-1}^{z} \frac{\sqrt{1-t} - d}{\sqrt{(t-c)(t^2-1)}} dt,
\]

where \( 1 < c < d < +\infty \), maps \( \mathbb{H} \) onto \( G_\alpha \) which is the union of the strip \( S \) and the rectangle \( \Pi_\alpha \), defined by (4) and (5), supplemented with their common boundary arc. The branches of the square roots are fixed such that the integrand takes positive values for real \( t > d \), and the constant \( C > 0 \).

Investigation of the behavior of \( f(z) \) near \( z = \infty \) gives

\[
f(z) = -C \log z + O(1)
\]

where \( \log z \) takes positive values for real \( z > 0 \). The width of \( S \) is equal to \( 2\pi \), therefore, we conclude that \( C = 2 \). Since the height and the width of the rectangle \( \Pi_\alpha \) equal \( 3\pi \) and \(-\log a = 3.7178547 \ldots \) (see (9)), we obtain

\[
\int_{-1}^{1} |f'(t)| dt = 3\pi, \quad \int_{1}^{c} |f'(t)| dt = -\log a,
\]

therefore,

\[
\int_{-1}^{1} \frac{\sqrt{d-t} dt}{\sqrt{(c-t)(1-t^2)}} = 3\pi/2,
\]

\[
\int_{1}^{c} \frac{\sqrt{d-t} dt}{\sqrt{(c-t)(t^2-1)}} = -\log a/2 = 1.858927 \ldots
\]

Solving the system of equations with respect to \( c \) and \( d \) we find

\[
c = 1.098259 \ldots, \quad d = 1.766556 \ldots
\]

The conformal radius of \( L_\alpha \) at the point \( e^{f(z)} \) is equal to

\[
2 \text{Re} \left[ \left( e^{f(z)} \right)' \right] = 43 \text{Re} \left[ e^{f(z)} \frac{\sqrt{|z-d|}}{\sqrt{|(z-c)(z^2-1)|}} \right].
\]

At the point \( z = -0.0205 + 0.3659i \) the conformal radius equals \( r_0 = 0.695356 \ldots \). Therefore, Lemma 2 is proved.

**Proof of Theorem 2** Consider the surface \( R(B) \) for a finite Blaschke product. It is an \( n \)-sheeted unlimited ramified covering of the unit disk \( \mathbb{D} \). In the case \( n \leq 2 \) the statement of the theorem is evident, therefore, we can assume that \( n > 2 \).
I) First we consider the case where all branch points of \( R(B) \) are simple and their projections \( r_k e^{i\phi_k} \) \( (\phi_k \in [0, 2\pi)) \) on \( \mathbb{D} \) are such that no two of the points lie on the same radius of \( \mathbb{D} \). Then \( R \) can be glued from \( n \) disks slit along the segments \( T_k \) of the form \( r e^{i\phi_k}, 0 < r_k < r < 1 \). We will call them sheets of the Riemann surface \( R(B) \) and denote them by \( S_1, \ldots, S_n \). For every segment \( T_k \) there are exactly two sheets slit along it. From the Riemann–Hurwitz formula for bordered surfaces (see, e.g. [23]) it follows that the number of segments (and branch points of \( R(B) \)) equals \((n - 1)\). Since every segment corresponds to a couple of slits, the number of slits on all the sheets equals \( 2(n - 1) \).

Now we will prove that there exists a sheet \( S_k \) that contains more than one slit and every sheet that is glued to it, with possibly one exception, contains a unique slit. To prove this, we associate with the surface \( R(B) \), glued from the sheets \( S_j \), a connected graph \( \Gamma \), the vertices of which are sheets. A vertex \( S_j \) is connected with \( S_l \), if the sheets \( S_j \) and \( S_l \) are glued with each other, with the help of the slit along some segment \( T_m \). It is easy to see that, because of simply-connectedness of \( R(B) \), the graph is a tree. Now we consider any (oriented) edge path of \( \Gamma \) with the maximal possible number of edges. Then the second vertex \( S_k \) in this path is the required sheet (and, if it exists, the exceptional sheet with more than one slit is the third vertex in this path).

By renumbering the sheets and the points \( r_j e^{i\phi_j} \), we can achieve that \( S_1 \) is the sheet such that the sheets \( S_2, \ldots, S_m, m \leq n - 1 \), are attached to it and each of them has a unique slit along the segment connecting the points \( r_j e^{i\phi_j} \) and \( e^{i\phi_j} \). We can assume that \( 0 < \phi_1 < \phi_2 < \ldots < \phi_m < 2\pi \). We can glue from the sheets \( S_1, \ldots, S_m \) a Riemann surface \( R_1 \subset R(B) \) which has \( m \) sheets, \( m - 1 \) branch points and, possibly, one more slit on some sheet \( S_j, 2 \leq j \leq m \). We will consider the case where there is such slit; in case of its absence, we can always cut the surface along some radial segment. Then we extend the slit so that it coincide with some radius. Thus, the boundary of \( R_1 \) consists of the \( m \) times traversed unit circle and the slit.

If there is \( k, 1 \leq k \leq m \), such that \( r_k > a \), where \( a \) is given by (9), then the sheet \( S_k \) contains a domain which is a rotation of \( \mathbb{D}_a \) by angle \( \phi_k \) and the theorem is proved. Therefore, we can assume that the projections of all branch points of \( R(B) \) are located at a distance from zero less than \( a \). Then \( R_1 \) contains a subsurface \( R_2 \) which is an \( m \)-sheeted non-ramified covering of the annulus \( a < |z| < 1 \), cut along a radial segment. Without loss of generality we can assume that the projection of the segment is on the positive part of the real axis. Then, under an appropriate choice of the branch of the logarithm, the function \( w = \log z \) maps \( R_2 \) onto the rectangle \( \Pi = \{ \log a < \Re w < 0, 0 < \Im w < 2\pi m \} \). The union of \( R_2 \) and every \( S_k, 1 \leq k \leq m \), is a part of \( R(B) \), containing either \( \mathbb{D}_a \) or \( \mathbb{D}_a^* \), turned at an
II) Now let $B$ be an arbitrary Blaschke product. It can be approximated by a sequence of Blaschke products $B_n$ satisfying the requirements considered in I) and the convergence is uniform in the closed unit disk. Because of I), every $R(B_n)$ contains either the slit disk $D_a$, or $L_a$, or $L^*_a$ rotated by some angle $\theta_n \in [0, 2\pi]$. Without loss of generality we can assume that $\theta_n \to \theta_0$ as $n \to \infty$ and every $R(B_n)$ contains a rotation of one of the indicated domains, $D_a$, $L_a$, or $L^*_a$. The sequence of $R(B_n)$ converges to $R(B)$ as to a kernel in the sense of Carathéodory (the kernel convergence of planar domain is described, e.g. in [12, §3.1, p. 77], about the kernel convergence of multi-sheeted Riemann surfaces see, e.g. [24] and the bibliography therein). Let for definiteness, $R(B_n)$ contains $L_a$ rotated by the angle $\theta_n$. Then $R(B)$ contains $L_a$ rotated by the angle $\theta_0$.

Theorem 2 is proved.

Proof of Theorem 1. The norm $\|B\|_B$ is the maximum of the conformal radius of the surface $R(B)$. By Theorem 2, the surface $R(B)$ contains a rotation of either $D_a$, or $L_a$, or $L^*_a$. We note that by the Lindelöf principle (see [16], p. 339), the conformal radius increases under enlargement of domain (Riemann surface). Since the maximum of the conformal radius of $D_a$ equals $0.7$ (see [10]), and, by Lemma 2, the maximum of the conformal radius of $L_a$, and of the symmetric to it surface $L^*_a$, is greater than $r_0 = 0.695356 \ldots$, we conclude that $\|B\|_B \geq r_0$.

To prove the last part of Theorem 1 we consider $B(z) = z^n$. Then it is easy to check that $\|z^n\|_B \to 2/e$ as $n \to \infty$. This fact concludes the proof of Theorem 1.

Proof of Theorem 3. Let $f$ be holomorphic in $D$ function, $f'(0) = 1$ and $B$ be a Blaschke product of order $n$; we can assume that $n > 1$. Then the covering $f : D \to f(D)$ defines a Riemann surface $R$. By a well-known result (see [11] p.364), $R$ contains a one-sheeted disk $K$ of radius $r = \sqrt{3}/4$ centered at some point $a \in C$. Therefore, there is a simply-connected domain $G \subset D$ such that $f$ maps $G$ onto $K$. Denote by $A$ the set of all critical values of $B$. Now we consider a sequence of disks $K_m$ centered at $a$ of radii $r_m$ such that $r_m < r$ and $r_m \to r$, $m \to \infty$. We can choose $r_m$ such that the preimages $\gamma_m$ of the boundary circles $\partial K_m$ under the mapping $f|G$ are disjoint with $A$. It is evident that every $\gamma_m$ is a closed Jordan curve lying in $G$. Now we fix $m$ and consider the set $B^{-1}(\gamma_m)$. Since $\gamma_m$ is disjoint with $A$, the set $B^{-1}(\gamma_m)$ consists of a finite set of disjoint closed Jordan curves. We claim that the interiors of these curves are also disjoint. Indeed, if one of the curves, say, $\alpha$, is in the interior of another one, $\beta$, then it can not be connected with
the boundary of \( D \) without intersecting \( \beta \). On the other hand, every point of \( \gamma_m \) can be connected with \( \partial D \) by a curve \( \omega \) which does not intersect \( \gamma_m \) at other points and does not pass through points of the set \( A \). There is a unique lift of \( \omega \) from some appropriate point of \( \alpha \) on \( D \), with respect to the covering map \( B : D \rightarrow D \), and the lift does not intersect \( \beta \), since \( \omega \) has no common points with \( \gamma \), except for the initial one. This proves that the interiors are disjoint. Now consider one of such curves, \( \alpha \). Denote by \( H \) the interior of \( \alpha \). Then \( g : H \rightarrow K_m \), where \( g = f \circ B \), defines a Riemann surface which is a finite-sheeted ramified covering of \( K_m \). It is easy to see that \( h(z) = (g(z) - a)/r_m \) is a Blaschke product. Applying Theorem 1 to the mapping \( h \), we obtain

\[
\|g\|_B = r_m\|h\|_B \geq r_m \cdot r_0, \quad m \geq 1.
\]

Taking \( m \rightarrow \infty \) we obtain the desired inequality.

If \( f \) is a convex univalent function, then according to [26] the value \( r = \sqrt{3}/4 \) can be replaced by \( \pi/4 \). Theorem 3 is proved.

**Proof of Theorem 4** Let

\[
B(z) = \prod_{j \geq 1} \frac{|z_j|}{\bar{z}_j} \frac{z - z_j}{1 - \bar{z}_j z}.
\]

Then it is easy to see that

\[
B'(z) = B(z) \sum_{j \geq 1} \frac{1 - |z_j|^2}{(z - z_j)(1 - \bar{z}_j z)}, \quad z \in \mathbb{D},
\]

and, in particular,

\[
|B'(\zeta)| = \sum_{j \geq 1} \frac{1 - |z_j|^2}{|\zeta - z_j|^2} . \tag{10}
\]

For some \( d \in (0, 1) \) to be chosen later, we set

\[
z_0 = \left(1 - \frac{d\delta}{|B'(\zeta)|}\right)\zeta.
\]

By (10), \(|\zeta - z_j| = |1 - \bar{z}_j \zeta| \geq \delta |B'(\zeta)|^{-1}\) whence \( z_0 \in \mathbb{D} \) and the following inequalities are valid:

\[
|z - z_j| \geq |\zeta - z_j| - |\zeta - z_0| \geq (1 - d)|\zeta - z_j| \tag{11}
\]

and

\[
|1 - \bar{z}_j z| \geq |1 - \bar{z}_j \zeta| - |\zeta - z_0| \geq (1 - d)|\zeta - z_j| \tag{12}
\]
for any \( z \in [z_0, \zeta] \). Therefore, \(|B'(z)| \leq (1 - d)^{-2}|B'(\zeta)|, \ z \in [z_0, \zeta] \). Hence,

\[
|B(z_0) - B(\zeta)| \leq \frac{|B'(\zeta)|}{(1 - d)^2} \cdot |z_0 - \zeta| = \frac{d\delta}{(1 - d)^2}.
\]

It follows that

\[
|B(z_0)| \geq 1 - \frac{d\delta}{(1 - d)^2}.
\]

Now we can obtain a lower estimate for \(|B'(z_0)|\). We have

\[
\left| \frac{B'(z_0)}{B(z_0)} - \frac{B'(\zeta)}{B(\zeta)} \right| = |z_0 - \zeta| \left( \sum_{j \geq 1} \left( \frac{1 - |z_j|^2}{(z_0 - z_j)(\zeta - z_j)} \right) \right).
\]

Using (6), (11) and (12), we get

\[
\left| \sum_{j \geq 1} \left( \frac{1 - |z_j|^2}{(z_0 - z_j)(\zeta - z_j)} \right) \right| \leq \frac{2|B'(\zeta)|^2}{\delta(1 - d)^2} \cdot |z_0 - \zeta| = \frac{2d|B'(\zeta)|}{(1 - d)^2}.
\]

Estimating analogously the second term and summing up, we get

\[
\left| \frac{B'(z_0)}{B(z_0)} \right| \leq \frac{2d|B'(\zeta)|}{\delta(1 - d)^2} \cdot |z_0 - \zeta| = \frac{2d|B'(\zeta)|}{(1 - d)^2}.
\]

Therefore,

\[
\left| \frac{B'(z_0)}{B(z_0)} \right| \geq \left( 1 - \frac{2d}{(1 - d)^2} \right) |B'(\zeta)|.
\]

This estimate together with inequality (13) yield

\[
(1 - |z_0|^2)|B'(z_0)| = (1 - |z_0|^2)|B(z_0)| \cdot \left| \frac{B'(z_0)}{B(z_0)} \right| \geq\]

\[
\geq d\delta \left( 1 - \frac{d\delta}{(1 - d)^2} \right) \left( 1 - \frac{2d}{(1 - d)^2} \right).
\]

Recall that \( \delta \leq 1 \). It remains to take \( d = 1/7 \) to obtain the required numerical estimate. Theorem 4 is proved.

**Remark 2** The constants in Theorem 4 are by no means optimal. In the case when \( B \) is a finite Blaschke product they can be substantially improved. First, we can choose \( \zeta \) to be a point where \(|B'|\) attains its maximum in \( \mathbb{D} \). Second, using the fact that the Bloch norm is invariant under Möbius transforms we can assume that the zeros are arbitrarily close to the boundary and so \(|B'(\zeta)|\) is arbitrarily large. These improvements make it possible to obtain by this method the lower bound about 0.361..., which is still much smaller than the one obtained in Theorem 4 by geometric methods.
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