Impurity-induced vector spin chirality and anomalous Hall effect in ferromagnetic metals

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Abstract

Scattering by multiple scatterers sometimes gives rise to nontrivial consequences such as anomalous Hall effect. We here study a mechanism for anomalous Hall effect which originates from the correlation of non-magnetic impurities and localized moments; a Hall effect induced by vector spin chirality. Using a scattering theory approach, we study the skew scattering induced by the scattering processes that involve two magnetic moments and a non-magnetic impurity, which is proportional to the vector spin chirality of the spins in the vicinity of the non-magnetic impurity. Furthermore, we show that a finite vector spin chirality naturally exists around an impurity in the usual ferromagnetic metals at finite temperature due to the local inversion-symmetry breaking by the impurity. The result is potentially relevant to magnetic oxides which the anomalous Hall effect is enhanced at finite temperatures.

1. Introduction

Anomalous Hall effect (AHE) in ferromagnetic materials is one of the representative examples of transport phenomena that reflects the quantum nature of electrons in solids [1–3]. Intensive studies over the last half-century have revealed that the mechanism of AHE reflects rich physics such as the Berry phase of electronic bands [4] and scattering of electrons by impurities; the former is called intrinsic mechanism while the later are extrinsic mechanisms. The extrinsic AHE is a consequence of asymmetric scattering induced by impurities, and many different mechanisms are known, such as non-magnetic impurities [5–7], localized magnetic moment [8], and the asymmetric scattering in the Anderson impurity models [9–11]. On the other hand, extrinsic AHE by electrons with Rashba spin–orbit interaction (SOI) is also studied, showing that the bulk SOI also contributes to the skew scattering mechanism [12–14]. Similarly, the Berry curvature of electronic bands leads to extrinsic AHE regardless of its microscopic origin [15, 16]. In addition, related physics is also studied in relation to extrinsic spin Hall effect [17–22]. While a variety of different mechanisms are proposed, most of them are related to a scattering by single impurity. In these mechanisms, the effect of SOI in the scattering process is a crucial ingredient for the anomalous Hall effect, except for that by the Berry curvature.

On the other hand, later studies found that the quantum interference effect induced by multiple scatterers (often by localized magnetic moments) lead to AHE. Unlike the single impurity scattering mechanisms, these mechanisms do not involve SOI in the scattering process. One of such mechanism was theoretically proposed in the double-exchange limit of the Kondo lattice model. In this limit, an effective Peierls phase similar to that by the local inversion-symmetry breaking by the impurity. The result is potentially relevant to magnetic oxides which the anomalous Hall effect is enhanced at finite temperatures.

When the Kondo coupling is perturbative, three-scattering process gives rise to AHE proportional to scalar spin chirality $S_1 \cdot S_2 \times S_3$ [37] (Scalar spin chirality in table 1). It was later discussed that this mechanism
is a distinct mechanism from the effective Peierls phase and is rather similar to the skew scattering but by multiple spins instead of non-magnetic impurities \[38\]. Experimentally, this multi-spin skew scattering was studied in relation to disordered spin systems such as chiral spin glass \[39\], chiral magnets \[36\], and for the skyrmions living on the ferromagnet/semiconductor interfaces \[40\]. In general, this mechanism is also expected to appear in magnets with non-coplanar magnetic orders/correlation.

Different microscopic mechanisms for the extrinsic AHE is summarized in table 1; they are classified by the nature of the conduction electrons and impurities. The skew scattering and side-jump mechanisms belong to the left-bottom quadrant as well as the skew scattering by Rashba SOI and the impurity scattering by Berry curvature; the mechanisms listed in green require SOI while that in yellow and blue don’t. The skew scattering by magnetic impurities (both that by single impurity or by scalar spin chirality) belongs to the right-top quadrant.

In the weak coupling limit, the possibility of the AHE by two spin scattering is also studied, which is proportional to vector spin chirality $S_1 \times S_2$. It was initially discussed to vanish in a clean system \[41\], but was later discussed to remain finite in the presence of non-magnetic impurities \[42\]. In this work, we extend the phenomenological theory in \[42\], and systematically studies the skew scattering mechanism induced by the scattering process involving both spins and non-magnetic impurities. Using second Born approximation, we discuss that the skew scattering appears from the correlation between the non-magnetic impurity and vector spin chirality of spins surrounding the impurity. This mechanism is in contrast to the mechanism studied in \[41\] there, an AHE related to the uniform vector spin chirality, while the mechanism studied here is related to the vector spin chirality defined clockwise surrounding the non-magnetic impurity. We also discuss that the vector spin chirality naturally appears in ferromagnets at a finite temperature when charged non-magnetic impurities exist; the Dzyaloshinskii–Moriya (DM) interaction induced by the impurity induces the vector spin chirality and the correlation between the impurity and the spins. We show the correlation between the vector spin chirality and the impurity is essential for the AHE. Due to these features, this mechanism is possible in ferromagnetic metals on centrosymmetric metals, unlike the scalar spin chirality mechanism \[37, 38, 40\]. These results suggest that the vector-chirality-induced AHE may exists universally in the ferromagnetic metals.

This paper is organized as follows. In section 2, we introduce the Kondo lattice model with non-magnetic impurities which we use to study the scattering process. Using the Kondo lattice model, the scattering process that involves two spins and an impurity is studied in section 3. We show that it gives rise to several different antisymmetric scattering terms; one of them is analogous to the skew scattering which is proportional to the

| Impurities | Conduction electrons |
|------------|----------------------|
| Non-magnetic | Spin-polarized | Unpolarized |
| SOI | No SOI | SOI |
| No SOI | Vector spin chirality | Scalar spin chirality$^{38,39}$ |
| SOI | Frustration-induced non-collinear magnetism$^{35,36}$ | Rashba SOI$^{12-14}$ |
| Rashba SOI$^{12}$ | |
| Side-jump$^{7}$ | | |

Table 1. Classification of the mechanisms of extrinsic anomalous Hall effect based on the nature of scattering process; whether it requires spin-orbit interaction (SOI) and/or magnetism [Spin-polarization in the electron bands is necessary or not, and whether the impurities are magnetic or not]. In this table, we did not consider the origin of the magnetic texture, i.e. whether Dzyaloshinskii–Moriya interaction is necessary for non-coplanar magnetic correlation as it is not related to the scattering mechanism. Each column is for the different nature of conduction electrons and the rows are for the nature of impurities. The non-magnetic, paramagnetic blocks are shaded in gray as the time-reversal symmetry prohibits the Hall effect. Each color shows the different nature of the scattering process: mechanisms that SOI is required (green) and multiple scatterers (blue). Neither SOI nor multiple scatterers are necessary for the orange one. This work is listed as ‘vector spin chirality’ in the left-top quadrant.
vector spin chirality defined anticlockwise around the impurity. In addition, an explicit formula for transverse conductivity is given. In section 4, we discuss that the vector spin chirality generally appears in ferromagnets whenever there is a charged impurity; this is related to the DM interaction induced by the impurity. By a perturbation calculation, we calculate the temperature dependence of the vector spin chirality in the low temperature much below the magnetic transition temperature; we show that a finite vector spin chirality appears at least in the finite temperature, even when the DM interaction is perturbatively small. Section 5 is devoted to summary and discussions.

2. Model

To study how the scattering by impurities and magnetic moments affects transport phenomena, we consider a classical-spin Kondo lattice model in the ferromagnetic phase with non-magnetic impurities,

\[ H = H_0 + H_K + H_i + H_S, \]

where

\[ H_0 = \sum_{k,\sigma} \varepsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}, \]

is the free fermion part of Hamiltonian,

\[ H_K = J_K \sum_j S_i \cdot \{ c_j^\dagger(R_j) \sigma_{\alpha\beta}(R_j) \}, \]

is the Kondo coupling to the localized moments, and

\[ H_i = \sum_l V_l c_{R_l\alpha}^\dagger c_{R_l\beta}, \]

is the coupling to impurities. \( H_K = H_K(\{ S_i \}) \) is the Hamiltonian for the localized moments, which we will introduce in section 4. Here, \( c_{k\sigma} (c_{k\sigma}^\dagger) \) is annihilator (creator) for fermions with momentum \( k \) and spin \( \sigma = \uparrow, \downarrow \) and \( S_i \) is the localized moment at \( R_i \). The coefficient \( \varepsilon_{k\sigma} = \varepsilon_{\sigma}(k) = k^2/2m - \Delta \sigma \) is the eigenenergy of the free electrons with mass \( m \), momentum \( k \) and spin \( \sigma \) \((k \equiv |k|)\); \( V \) is the strength of the impurity potential, and \( J_K \) is the Kondo coupling between the localized moments and itinerant electrons. As we are interested in the ferromagnetic phase, we introduced the Zeeman shift \( \Delta \) to introduce the population difference of up and down spin electrons.

3. Skew scattering by vector spin chirality

In this section, we study how the scattering by both impurities and spins contribute to anomalous Hall effect in ferromagnets. In section 3.1, we study the scattering process that involves both impurities and localized spins. We show that the terms that appear from the second-order Born approximation generally induces asymmetric scattering of spins. In section 3.2, we discuss basic aspects of the scattering term by considering a simple model. We show that, in this limit, the scattering term shows a similar feature to the skew scattering by impurities with SOI. The contribution of this term to the anomalous Hall effect is studied in section 3.3.

3.1. Second Born approximation

We first investigate the scattering probability to study how the multiple spins and impurities affect the scattering in ferromagnetic metals. We consider \( H_i \) and \( H_K \) in equation (3) as the perturbation, and calculate the scattering probability using the second Born approximation. The scattering probability from the state with momentum \( k \) and spin \( \alpha \) \((k\alpha)\) to that with \( k' \) and spin \( \beta \) \((k'\beta)\) reads

\[ W_{k\alpha \rightarrow k'\beta} \sim 2\pi |F^{(1)}(k', k)|^2 \delta(\varepsilon_{k\alpha} - \varepsilon_{k'\beta}) + 2\pi |F^{(1)}(k', k)F^{(2)\dagger}(k', k) + h.c.\delta(\varepsilon_{k\alpha} - \varepsilon_{k'\beta}). \]

Here,

\[ F^{(1)}(k'\beta, k\alpha) = \frac{J_K}{(2\pi)^2} \sum_i S_i \cdot \sigma_{\alpha\beta} e^{-iR_i \cdot (k' - k)} + \frac{1}{(2\pi)^2} \sum_l V_l e^{-iR_l \cdot (k' - k)}, \]
Figure 1. Schematic picture of the scattering processes we consider in this paper. The asymmetric scattering in $O(f^2 V^2)$ appears from the two processes shown in (a) and (b). The blue arrows show localized moments and the crosses are the non-magnetic impurities.

and

$$F^{(2)}(k', k\alpha) =$$

$$= \frac{J^2 m}{(2\pi)^4} \sum_{i, j} \epsilon^{i k R_i - i k' R_j} \frac{1}{2 \delta_{ij}} [S_j^x S_j^z - \sigma_{ij}^z S_j^z] (e^{i k_R} - e^{i k'_R})$$

$$= \frac{J^2 m}{(2\pi)^4} \sum_{i, j} \epsilon^{i k R_i - i k' R_j} \frac{1}{2 \delta_{ij}} \sigma_{ij}^z \cdot S_i (e^{i k_R} + e^{i k'_R})$$

$$= \frac{J^2 m}{(2\pi)^4} \sum_{i, j} V_i S_i \cdot \sigma_{ij} (e^{i k R_i - i k'_R + i \delta_{ij}} - e^{i k R_i - i k'_R + i \delta_{ij}}),$$

(7)

are, respectively, the first and second-order terms in the perturbation expansion; $\delta_{ij} \equiv R_i - R_j$ and $\delta_{ij} \equiv |\delta_{ij}|$ is the distance between the two scatterers. In $F^{(1)}(k', k\alpha)$, the first term is the contribution from the spin scattering and the second term is that by the impurities. Similarly, the first two terms in $F^{(2)}(k', k\alpha)$ come from the scattering by two spins (See the $F^{(2)}(k', k\alpha)$ path in figure 1(a),) and the last term is the scattering by a spin and an impurity ($F^{(2)}(k', k\alpha)$ path in figure 1(b)).

The first term in equation (3) is the contribution from the first Born approximation, which only gives a symmetric scattering. On the other hand, the second term is the additional contribution that appears in the second Born approximation. This contribution includes terms proportional to $f^2 m$, $Vf^2 m$, and $V^3$. The first term $F^1$ is already studied in preceding theoretical works [37, 38]; the anomalous Hall effect appears when the thermal average of scalar spin chirality $S_i \cdot S_j \times S_i$ is finite, such as in chiral spin glass [37, 39] and in chiral magnets [38].

In this work, we focus on the contribution from the impurities by focusing on the leading order term, $O(Vf^2 m)$. As we are interested in the Hall effect, we further split the scattering term into symmetric $[W^{(a)}_{k_0 \rightarrow k'\beta} = (W_{k_0 \rightarrow k'\beta} + W_{k_0 \rightarrow k'\beta})/2]$ and asymmetric parts $[W^{(a)}_{k_0 \rightarrow k'\beta} = (W_{k_0 \rightarrow k'\beta} - W_{k_0 \rightarrow k'\beta})/2]$. The asymmetric part of the scattering appears from the three different contributions

$$W^{(a)}_{k_0 \rightarrow k'\beta} = W^{(a)}_{k_0 \rightarrow k'\beta} + W^{(a)}_{k_0 \rightarrow k'\beta} + W^{(b)}_{k_0 \rightarrow k'\beta}.$$

(8)

These contributions are schematically shown in figure 1. The first two terms $W^{(a)}_{k_0 \rightarrow k'\beta}$ and $W^{(a)}_{k_0 \rightarrow k'\beta}$ come from the interference of the second-order process involving two spins and first-order process for impurities as shown in figure 1(a). These terms read

$$W^{(a)}_{k_0 \rightarrow k'\beta} = \frac{J^2 m}{(2\pi)^4} \sum_{i, j, \ell, \gamma} V_i \sin(k \cdot \delta_{ij})$$

$$\times [\sigma_{ij}^x \cdot S_i \times S_j \{\cos(k \cdot \delta_{ij} - k' \cdot \delta_{ij}) - \cos(k \cdot \delta_{ij} - k' \cdot \delta_{ij})\}$$

$$+ \sin(\gamma) \sigma_{i\ell}^z \{S_i^x S_j^y + S_j^x S_i^y\}$$

$$\times \sin(k \cdot \delta_{ij} - k' \cdot \delta_{ij} + k' \cdot \delta_{ij}),$$

(9)

$$W^{(a)}_{k_0 \rightarrow k'\beta} = \frac{J^2 m}{(2\pi)^4} \sum_{i, j, \ell, \gamma} V_i \sin(k \cdot \delta_{ij})$$

$$\times [\cos(k \cdot \delta_{ij} - k' \cdot \delta_{ij} + k' \cdot \delta_{ij}) + \cos(k \cdot \delta_{ij} - k' \cdot \delta_{ij} + k' \cdot \delta_{ij})$$

$$- \cos(k' \cdot \delta_{ij} - k \cdot \delta_{ij} + k' \cdot \delta_{ij}) - \cos(k' \cdot \delta_{ij} - k \cdot \delta_{ij} + k' \cdot \delta_{ij}).$$

(10)
Here, sgn(γ) is a binary function that is sgn(↑) = 1 and sgn(↓) = −1 and κ, ∈ ℝ (γ = ↑, ↓) is the magnitude of the wave number such that ε₁(κ) = ε₁(κ). The third term \( W^{(b)}_{k_{0} \rightarrow k' \beta} \) comes from the second-order process involving the scattering by a spin and an impurity, and the first-order scattering term by the spins. (For example, the process shown in figure 1(b)). This term reads

\[
W^{(b)}_{k_{0} \rightarrow k' \beta} = -i \frac{\hbar^{2} m}{(2\pi)^{3}} \sum_{i,j,l} V_{i} \delta_{i,j}^{z} (S_{i} \cdot \sigma_{\alpha \beta}) (S_{i} \cdot \sigma_{\alpha \beta})
\]

\[
\times \{ e^{ik_{e} \cdot (k' \cdot \delta_{y} - k \cdot \delta_{y})} + e^{i\kappa_{e} \cdot (k' \cdot \delta_{y} - k \cdot \delta_{y})} \}
\]

+ \text{h.c.}

In general, all these terms contribute to asymmetric scattering and to the anomalous Hall effect.

In the next section, however, we find that only the terms with \( S_{i} \times S_{j} \) in \( W^{(a)}_{k_{0} \rightarrow k' \beta} \) gives rise to skew scattering while the other terms vanish in a most basic setup. In particular, assuming \( k_{\delta_{l}}, k_{\delta_{l}} \ll 1 \) and the net magnetization along the \( z \) axis, we find

\[
W^{(a)}_{k_{0} \rightarrow k' \beta} = \frac{\hbar^{2} m}{(2\pi)^{3}} \sum_{i,j,l} V_{i} \sin(k_{\delta_{l}})(\delta_{i,j}^{z})(\delta_{i,j}^{z})
\]

\[
\times \{ \cos(k \cdot \delta_{y} - k' \cdot \delta_{y}) - \cos(k \cdot \delta_{y} - k' \cdot \delta_{y}) \}
\]

\[
\sim \frac{\hbar^{2} m}{(2\pi)^{3}} \sum_{i,j,l} V_{i} \delta_{i,j}^{z} \sigma_{\alpha \beta}(\delta_{i,j})[(k \times k')(\delta_{i,j} \times \delta_{i,j})]
\]

\[
+ \frac{\hbar^{2} m}{(2\pi)^{3}} \sum_{i,j,l} V_{i} \sigma_{\alpha \beta}(\delta_{i,j}^{z})(\delta_{i,j}^{z}) \cos(k \cdot \delta_{y} - k' \cdot \delta_{y})
\]

\[
\times \{ [(k \cdot \delta_{y})(k \cdot \delta_{y}) + (k' \cdot \delta_{y})(k' \cdot \delta_{y})] \},
\]

where \( \hat{z} \) is the unit vector along the \( z \) axis. Here, we assumed that \( S_{i}^{x}, S_{j}^{y} \ll S_{i}^{z} \sim 1 \) and only considered the leading order terms in \( S_{i}^{x} \) and \( S_{j}^{x} \). The first term in the second line gives a scattering term proportional to \( k \times k' \), which is a typical wave number dependence for the skew scattering [43, 44]. Here, \( \delta_{i,j} \equiv (\delta_{y} + \delta_{y})/2 \) is a vector from the non-magnetic impurity to the bonds connecting \( S_{i} \) and \( S_{j} \) (figure 2(a)). The result indicates that the correlation between the spins and non-magnetic impurity is crucial for the skew scattering. We also note that the skew scattering is proportional to

\[
W^{(a)}_{k_{0} \rightarrow k' \beta} \propto \langle V_{i}(\hat{z} \cdot S_{i} \times S_{j})(\delta_{i,j} \times \delta_{i,j}) \rangle,
\]

where the brackets are for thermal and impurity averages; the skew scattering is proportional to the vector spin chirality defined anticlockwise around the impurity, and not to the uniform vector spin chirality which is finite in helical magnetic states. Equations (13) and (14) show that the skew scattering is related to the correlation between vector spin chirality and the impurity potential; a finite correlation gives a finite net skew scattering even if the average vector spin chirality is zero.

### 3.2. Scattering by an impurity and surrounding moments

To see how the above-mentioned mechanism contributes to the transport phenomena, we here consider a basic model with an impurity and a plaquette of four moments surrounding the impurity. A schematic picture of the model considered is shown in figure 2(b). For simplicity, we assume the impurity is at \( R_{0} = 0 \) and the four surrounding moments are at

[Figure 2 (a) Schematic figure of the electric polarization induced by the canting of two spins \( P_{i} \), and (b) the model we consider as an example.]
\[ R_1 = \left( \frac{a}{2}, -\frac{a}{2}, 0 \right), \quad R_2 = \left( \frac{a}{2}, \frac{a}{2}, 0 \right) , \]
\[ R_3 = \left( -\frac{a}{2}, \frac{a}{2}, 0 \right), \quad R_4 = \left( -\frac{a}{2}, -\frac{a}{2}, 0 \right). \]

We further assume that the magnetic moment is nearly polarized along \( z \) axis with small fluctuation for \( S_i^z \) and \( S_j^z \), and that \( \langle S_i^z \rangle = \xi_i \) for all \( i \) and \( \langle (S_i \times S_j) \| \rangle = 0 \); these assumptions give \( \langle V_i (\hat{\varepsilon} \cdot S_i \times S_j) (\delta_{ij} \times \delta_{ji}) \rangle = -V_0 \xi_i a^2 / 2 \). As we will discuss in the next section, this is a natural assumption for the current model. Using these assumptions and expanding the equations (9)–(11) by \( k \) in the limit of \( ka \ll 1 \), we find
\[
W_{k_\alpha}^{(a)}(k') = \frac{2 J_k V_0 ma^2}{(2\pi)^2} (k_i + k_i)(k \times k')_\sigma \sigma_{ij},
\]
\[
W_{k_\alpha}^{(b)}(k') = 0,
\]
\[
W_{k_\alpha}^{(c)}(k') = 0.
\]

Here, \( W_{k_\alpha}^{(a)}(k') \) vanish as they only contribute to higher-order terms when \( \langle S_i^z \rangle = \langle S_j^z \rangle = 0 \). The results indicate that the scattering process we considered gives rise to an asymmetric scattering term analogous to the skew scattering, i.e. its wave number dependence is proportional to \( k \times k' \). This indicates that only the \( S_i \times S_j \) terms in equation (13) contributes to the skew scattering.

### 3.3. Anomalous Hall effect

We next evaluate the conductivity by using a Boltzmann theory considering the asymmetric scattering term in equation (15). We here assume that the above considered impurity-spin cluster is randomly distributed throughout the system with a density of \( n^{(0)} \). Assuming the system is spatially uniform, the Boltzmann equation reads
\[
q v_{k} \cdot \mathbf{E}^{\prime} = \frac{-g_{k\alpha}}{\tau} + \int d\phi d\theta d\phi' \sin \theta' \rho(k) \frac{4\pi}{k^2} V_{\alpha} \cdot \frac{k \times k'}{k^2} g_{k'\alpha},
\]
where
\[
V_{\alpha} = -2 \text{sign}(\sigma) n^{(0)} J_k^2 m (k_i + k_i) k_i a^2 \langle V_i \hat{\varepsilon}_i \rangle_{\text{imp}} \hat{\varepsilon}.
\]

Here, \( \rho(k) = mk / (2\pi^2) \) is the density of state for the wave number \( k \), \( n^{(0)} \) is the density of impurity, and \( \hat{\varepsilon}_i \) is the thermal average of the vector chirality around the \( i \text{th} \) impurity; \( \langle \cdots \rangle_{\text{imp}} \) is the average over the impurities. In the right-hand side of the equation, the first term is the symmetric scattering term which we approximated by relaxation time approximation, and the second term is the asymmetric scattering term in equation (15).

The Boltzmann equation in equation (18) can be solved without further approximation [16, 38]. The result reads
\[
j = j_\perp + j_\parallel,
\]
\[
j_\parallel = \frac{m q^{2} \tau}{m} (E - 2\pi \tau V_{\alpha} \times E).
\]

Here, \( k_{Fz} \) is the Fermi wave number for the electrons with spin \( \sigma \). Hence, we obtain
\[
\sigma_{xx} = \frac{q^{2} \tau}{m} (n_\uparrow + n_\downarrow),
\]
\[
\sigma_{xy} = \frac{q^{2} \tau}{2m} [\rho(k_{Fz}) V_{\uparrow} n_\downarrow + \rho(k_{Fz}) V_{\downarrow} n_\uparrow],
\]
for the longitudinal and transverse conductivities, respectively (\( V_{\alpha} \equiv V_{\alpha} \)).

The Hall conductivity is linearly proportional to \( V_{\sigma} \propto \langle V_i (\hat{\varepsilon}_i \cdot S_i \times S_j) (\delta_{ij} \times \delta_{ji}) \rangle \), i.e. the vector spin chirality of the localized moments defined anticlockwise around the impurity. The result is also proportional to \( V_{\alpha} \). Therefore, no AHE occurs if there are no correlation between the non-magnetic impurity and the spin-spin correlation, i.e. if \( \langle \langle V_i (\hat{\varepsilon}_i \cdot S_i \times S_j) (\delta_{ij} \times \delta_{ji}) \rangle \rangle = \langle V_i \rangle \langle \langle \hat{\varepsilon}_i \cdot S_i \times S_j \rangle (\delta_{ij} \times \delta_{ji}) \rangle \rangle \), as \( \langle V_i \rangle = 0 \). We also note that the contribution from the up spin and down spin electrons have opposite sign as shown in equation (19).

Therefore, the Hall effect is absent when there is no Zeeman splitting \( \Delta = 0 \). This implies that the leading order in the anomalous Hall conductivity is proportional to the magnetization.
4. Temperature dependence of Hall conductivity

We next study how the AHE studied in the above sections behave at finite temperature. In section 4.1, we introduce the model we consider in this section. In this model, the collinear ferromagnetic state is perturbatively stable against the DM interaction induced by the impurity. Therefore, no AHE appears in the $T \to 0$ limit. Nevertheless, we show that the expectation for vector spin chirality become finite in the finite temperature due to the spin fluctuation. In section 4.2, we introduce the formalism we use, which is based on the spin-wave theory. Using the formalism, we discuss the temperature dependence of the vector spin chirality.

4.1. Impurity-induced Dzyaloshinskii–Moriya interaction

In this section, we discuss how the non-magnetic impurities affect the magnetic ground state. As the spin model, we consider a ferromagnetic Heisenberg model on the cubic lattice with an impurity. The Hamiltonian reads

$$H_S = H_S^{(0)} + H_S^{(1)},$$

where

$$H_S^{(0)} = -J \sum_{\langle R,R' \rangle} S_R \cdot S_{R'} + J h \sum_R S_R^z,$$

is the Heisenberg Hamiltonian. The first term is Heisenberg exchange interaction between the localized moments where $S_R$ is the Heisenberg spin on $R = (n_x + 1/2, n_y + 1/2, n_z)$ site $[n_i \in \mathbb{Z} (\alpha = x, y, z)]$, and $J > 0$ is the exchange coupling between the nearest-neighbor spins; the sum is over nearest-neighbor sites. The second term is the coupling to the external magnetic field which we assume to be along $z$ axis.

In the presence of an impurity, the surrounding spins are affected by the electric potential by the impurity. The spins couples to the electric potential via electric dipole moments induced by spin canting [45], which is proportional to $P_{R,R'} \propto \vec{d}_{R,R} \times (S_R \times S_{R'})$, where $\vec{d}_{R,R} = \vec{R}' - \vec{R}$ is the vector connecting $S_R$ and $S_{R'}$. Therefore, the impurity term of Hamiltonian reads

$$H_S^{(imp)} = -\frac{1}{2} \sum_{R,R'} (\hat{D}_{R,R'} \times \delta_{R,R}) \cdot (S_R \times S_{R'}).$$

Here, $\hat{D}_{R,R'}$ is the coupling vector proportional to the direction of the electric field (or the gradient of the electric potential) at the bond center of $S_R$ and $S_{R'}$. For simplicity, we here assume the impurity is at $R = 0$ and consider only the coupling to the four spins surrounding the impurity (See figure 2(b)):

$$R_i = \left( \frac{1}{2}, -\frac{1}{2}, 0 \right), \quad R_2 = \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \quad R_3 = \left( -\frac{1}{2}, 1, 0 \right), \quad R_4 = \left( -\frac{1}{2}, -\frac{1}{2}, 0 \right).$$

Under this approximation, $H_S^{(imp)}$ simplifies to

$$H_S^{(1)} = -D \sum_{i=1}^{4} [(R_i + R_{i+1}) \times \delta_{R_{i+1},R_i}] \cdot (S_{R_i} \times S_{R_{i+1}}).$$

Here, $D$ is the strength of coupling and $R_5 = R_1$; we assume the electric potential by the impurity is rotationally symmetric. The above argument shows that the electric potential due to the impurity induces DM interaction between the surrounding spins. However, unlike the DM interaction in chiral magnets, the ferromagnetic ground state is stable against infinitesimally small DM interaction. In appendix, we discuss that the collinear ferromagnetic order remains as the classical ground state when $|D| < J + h/2$, unlike the DM interaction in chiral magnets. Therefore, we expect no spin canting in the ground state.

4.2. Temperature dependence of vector spin chirality

In contrast to the ground state, at finite temperature, the DM interaction discussed in the previous section gives rise to finite spin chirality due to the lifting of degeneracy between the excitations with positive and negative vector spin chirality. This may give rise to the finite spin chirality and resulting anomalous Hall effect that appears only at finite temperature. To study whether the weak DM interaction contributes to finite chirality at finite temperature, we focus on the low-temperature region well below the magnetic transition temperature. Assuming the spin fluctuation to be small, we study the temperature dependence of spin chirality using a spin-wave theory.

By using Holsten-Primakov representation

$$S^z_i = S - a_i^+ a_i, \quad S^+ = \sqrt{2S} a_i^+ \left( 1 - \frac{a_i^+ a_i}{2S} \right)^{1/2}, \quad S^- = \sqrt{2S} \left( 1 - \frac{a_i^+ a_i}{2S} \right)^{1/2} a_i,$$

we introduce the model we consider in this section.
and leaving the terms up to the order of $O(S)$, the Hamiltonian in equation (23) become
\[ H_S = \sum_k \varepsilon_k a_k^\dagger a_k - i\sqrt{2} DS \sum_{i=1}^4 a_i^\dagger (R_{i+1}) a(R_i) - a_i^\dagger (R_i) a(R_{i+1}), \]
\[ = \sum_k \varepsilon_k a_k^\dagger a_k + \sum_{kk'} V_{kk'} a_k^\dagger a_{k'}, \tag{29} \]
where $\varepsilon_k = 2JS[3 - \sum_{s=x,y,z} \cos(k_s)] + h$ is the eigenenergy of the spin-wave mode and
\[ V_{kk'} = \sqrt{2} DS \frac{\chi_{kk'}}{N} \]
with
\[ \chi_{kk'} = -i(\epsilon(0,k-k')R_k(\epsilon(k,k') - \epsilon(-k,k') + \epsilon(0,k-k')R_k(\epsilon(-k,k') - \epsilon(k,k'))) \]
\[ + \epsilon(0,k-k')R_k(\epsilon(-k,k') - \epsilon(k,k')) + \epsilon(0,k-k')R_k(\epsilon(k,k') - \epsilon(-k,k')). \tag{31} \]

Using the perturbation expansion with respect to $H_S^{(1)}$, the vector spin chirality reads
\[ \langle (S_i \times S_{i+1}) \rangle = -\frac{DS}{\sqrt{2} J N^2} \sum_{kk'} \frac{|\chi_{kk'}|^2}{\varepsilon_k - \varepsilon_{k'}} n_k. \tag{32} \]
Here, $n_k \equiv 1/(\epsilon^{ik} - 1)$ is the Bose distribution function and $\varepsilon_k \equiv \varepsilon_k/(2JS)$ is the dimensionless energy. In figure 3, we show the temperature dependence of $\langle (S_i \times S_{i+1}) \rangle$. The different curves are for the different strength of magnetic field. At a low temperature, typically below $T \lesssim h$, the result show suppression of the chiral spin fluctuation due to the spin gap induced by the external magnetic field. Above it, we see a linear growth of the chirality, which is expected in the classical approximation up to $T/(2JS) \lesssim 1$ [42].

In addition to the chirality, the temperature dependence of magnetization also affects the temperature dependence of AHE. However, the magnetization only contributes to the subleading order in the temperature dependence because the anomalous Hall conductance is roughly proportional to the product of the vector chirality and the magnetization. Moreover, the reduction of the magnetization is suppressed in the low-temperature region, especially under the magnetic field. Therefore, the temperature dependence of the chirality in figure 3 reflects the temperature dependence of anomalous Hall conductance in the low-temperature region.

5. Discussion and Summary

To summarize, in this work, we discussed the asymmetric scattering and the anomalous Hall effect induced by the correlation of non-magnetic impurities and spins, which brings about a new mechanism for anomalous Hall effect proportional to the vector spin chirality defined anticlockwise around the impurity. By using a scattering theory approach, we show that the Hall conductivity is proportional to the correlation between the non-magnetic impurity and vector spin chirality, $V_{e}(k) \propto \langle V_i(\hat{Z} \times S_i) \rangle$, where $\hat{Z}$ is the direction of the magnetization; this mechanism is interpreted as a skew scattering induced by the quantum phase.
interference due to the multiple scatterers. We further show that the correlation between the non-magnetic impurity and the vector spin chirality of the spins surrounding the impurity naturally appears due to the multiferroic nature of the spin canting.

An interesting aspect of this mechanism is that the correlation between the non-magnetic impurity and the surrounding localized moments generally appears at a finite temperature, even if it is zero at $T = 0$. Therefore, the mechanism contributes to the enhancement of Hall effect at a finite temperature; the temperature dependence of the vector spin chirality is given in figure 3. A previous estimate finds the vector chirality $\bar{\gamma}_l \sim 10^{-2}$ at around $T \sim 10$ K, which results in a Hall angle $\theta_H \sim 10^{-2} \sim 10^{-2}$ for SrCoO$_3$ [42]. In the calculation in section 4.2, the vector chirality is suppressed if the magnetic field of order $h/J \sim 10^{-1}$ is applied. In this case, $T \sim J/k_B \sim 100$ K is required for a similar $\bar{\gamma}$ ($k_B$ is the Boltzmann constant). However, these values are still experimentally reasonable. Therefore, the AHE by the impurity-induced vector chirality is observable in the experiments.

Regarding the materials, in principle, this mechanism is expected to have a non-zero contribution in ferromagnetic metals. In fact, the enhancement of the anomalous Hall effect in a finite temperature is often observed in experiments. For example, the trends of our results resemble that of the finite temperature component of the anomalous Hall effect in SrCoO$_3$ [42], in which we discussed the vector chirality mechanism in a phenomenological way. Another material known for a large anomalous Hall effect at a finite temperature is manganese oxides [31–33]. It is widely discussed that a major contribution to the anomalous Hall effect is expected from the other mechanisms [23, 46]. However, the current mechanism is also expected to take part at a finite temperature, perhaps well below the critical temperature. A similar scenario may also apply to the anomalous Hall effect in a non-collinear antiferromagnet PdCrO$_2$ [47, 48]. This material shows a large anomalous Hall effect in a magnetic field, in which a finite magnetic moment develops due to the external field.

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Appendix. Stability of the ferromagnetic ground state

To study the stability of the ferromagnetic order in the Hamiltonian in equation (23), we first divide the spins in two groups: the four spins surrounding the impurity (A) and other spins (B). Using this grouping, the Hamiltonian is divided into three parts: interactions between the four spins surrounding the impurity ($H_A$), interactions between the other spins ($H_B$), and the interactions between the spins in A and B ($H_{AB}$). The ground state of $H_A$ is obviously the collinear ferromagnetic order and $H_{AB}$ is the ferromagnetic Heisenberg interaction between the spins in A and B. Therefore, if the ground state of $H_A$ is the ferromagnetic order, the ground state of the entire system is the collinear ferromagnetic order.

For the Hamiltonian in equation (23), the term $H_A$ reads

$$H_A = -J \sum_{i=1}^{4} R_i \cdot R_{i+1} - D \sum_{i=1}^{4} (S_R^x S_{R_{i+1}}^x - S_R^z S_{R_{i+1}}^z) - h \sum_i S_i^z. \quad (A.1)$$

In the second term, we used the fact $(R_i + R_{i+1}) \times \delta_{R_{i+1}R_i} = \hat{z}$, where $\hat{z}$ is the unit vector along z axis. To study the local stability of the ferromagnetic state, we consider small fluctuation of the spins round the ferromagnetic order along z axis. Namely, we expand $S_R^z \sim 1 - [(S_R^x)^2 + (S_R^y)^2] \pm 1/2$. The approximated Hamiltonian reads

$$H_A \sim \sum_{i=0,\gamma = \pm}^{3} S_{ki}^x \left( \frac{h}{2} + \gamma D \sin(k_i) \right), \quad (A.2)$$

where $k_i = \pi i/2$ and

$$S_{ki}^x = \frac{1}{2 \sqrt{2}} \sum_{j=1}^{4} (S_R^x - \gamma i S_R^y) e^{-ik_i j}.$$

As the $S_{ki}^z \neq 0$ solutions are also a collinear ferromagnetic state, the ground state remains the ferromagnetic order for $J + h/2 > |D|$.
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