INFRARED FREEZING OF EUCLIDEAN QCD OBSERVABLES IN THE ONE-CHAIN APPROXIMATION

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We consider the one-chain term in a skeleton expansion for Euclidean QCD observables. Focusing on the particular example of the Adler $D$ function, we show that although there is a Landau pole in the coupling at $Q^2 = \Lambda^2$ which renders fixed-order perturbative results infinite, the Landau pole is absent in the all-orders one-chain result. In this approximation one has finiteness and continuity at $Q^2 = \Lambda^2$, and a smooth freezing as $Q^2 \to 0$.

In this talk I want to describe some recent work with Paul Brooks in which we consider the low-energy behaviour of Euclidean QCD observables. We investigate the $Q^2$-dependence of all-orders perturbative resum- mations obtained using the so-called leading-$b$ approximation, where $b = (33 - 2N_f)/6$ is the leading beta-function coefficient in SU(3) QCD with $N_f$ active quark flavours. This is closely related to the one-chain term in a skeleton expansion in which a single chain of fermion bubbles is inserted into a basic skeleton diagram. We shall demonstrate that in this approximation the Landau pole in the coupling at $Q^2 = \Lambda^2$ is absent, and one has finiteness and continuity at this energy, with a smooth freezing behaviour at lower energies. We focus in this talk on the Adler $D$-function, but DIS sum rules are also studied in Ref.

The Adler $D$ function is directly related to the vacuum polarization function $\Pi^{\mu\nu}(Q^2)$, $(Q^2 = -q^2 > 0)$,

\[
\Pi^{\mu\nu}(Q^2) = \frac{1}{16\pi^2} \int d^4xe^{iqa.x} \langle 0|T(J_\mu(x)J_\nu(0))|0\rangle.
\]

Current conservation dictates that this has the tensor structure

\[
\Pi^{\mu\nu}(Q^2) = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(Q^2). \tag{2}
\]

Only $\Pi(Q^2) - \Pi(0)$ is observable, so it is useful to eliminate the constant and define the Adler Function $D(Q^2)$

\[
D(Q^2) = \frac{3}{4}Q^2 \frac{d}{dQ^2}\Pi(Q^2). \tag{3}
\]

This may be written as a sum of the parton model result and QCD corrections $D(Q^2)$

\[
D(Q^2) = 3 \sum_f Q_f^2[1 + D(Q^2)]. \tag{4}
\]

These QCD corrections are split into a perturbative and non-perturbative part

\[
D(Q^2) = D_{PT}(Q^2) + D_{NP}(Q^2). \tag{5}
\]

The PT component has the form

\[
D_{PT}(Q^2) = a(Q^2) + \sum_{n>0} d_n a^{n+1}(Q^2). \tag{6}
\]

Throughout the talk we will take $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ to be the one-loop form of the QCD coupling

\[
a(Q^2) = \frac{2}{b \ln(Q^2/\Lambda^2)}. \tag{7}
\]

The NP component in Eq.(5) will have the form

\[
D_{NP}(Q^2) = \sum_n c_n \left(\frac{\Lambda^2}{Q^2}\right)^n. \tag{8}
\]

The leading OPE contribution for the Adler function is the dimension 4 gluon condensate

\[
G_0(a(Q^2)) = \frac{1}{q^4}(0|GG|0)C_{GG}(a(Q^2)). \tag{9}
\]

We are interested in the behaviour of $D(Q^2) = D_{PT}(Q^2) + D_{NP}(Q^2)$ as $Q^2 \to 0$. 

\[
D(Q^2) \to \frac{3}{4}Q^2 \frac{d}{dQ^2}\Pi(Q^2) \to \frac{3}{4}Q^2 \frac{d}{dQ^2}\Pi(Q^2).
\]
Clearly at any fixed order perturbation theory breaks down at $Q^2 = \Lambda^2$, the Landau pole in the coupling, and $a(Q^2) \to \infty$. Evidently we need a resummation of perturbation theory to all-orders to address the freezing question, and we need to combine the resummation with the OPE condensates. The large-$N_f$ limit provides a way of formulating this resummation. The coefficient $d_n$ may be expanded in powers of $N_f$ the number of quark flavours

$$d_n = d^{[n]}_n N_f^n + d^{[n-1]}_n N_f^{n-1} + \ldots + d^{[0]}_n \quad (10)$$

The leading large-$N_f$ coefficient $d^{[n]}_n$ may be evaluated to all-orders since it derives from a restricted set of diagrams obtained by inserting a chain of fermion bubbles inside the quark loop. The crucial ingredient in the calculation is the chain of $n$ renormalised bubbles $B^{\mu\nu}(n)$,

$$B^{\mu\nu}(n) = \frac{(k^2 g^{\mu\nu} - k_\mu k_\nu)}{(k^2)^2} \left[ - \frac{N_f}{3} \left( \ln \frac{k^2}{\mu^2} + C \right) \right]^n \quad (11)$$

Here $k$ is the momentum flowing through the chain of fermion bubbles. The factor in the square bracket is the one-loop vacuum polarization contribution $\Pi_0(k^2)$. The constant $C$ depends on the subtraction procedure used to renormalise the bubble. With $\overline{MS}$ subtraction $C = -\frac{5}{2}$. We shall choose to work in the “V-scheme” which corresponds to $\overline{MS}$ with the scale choice $\mu^2 = e^{-5/3}Q^2$, in which case $C = 0$. The result for $d^{(n)}_n(V)$ is

$$d^{(n)}_n(V) = \frac{-2}{3} (n + 1) \left( \frac{1}{6} \right)^n \left[ -2n - \frac{n+6}{2n+2} \right] + \frac{16}{n+1} \sum_{n+1 > m > 0} m(1 - 2^{-2m}) \left[ 1 - 2^{-2m} \zeta_{2m+1} \right] n! \quad (12)$$

This large-$N_f$ result can describe QED vacuum polarization, but for QCD the corrections to the gluon propagator involve gluon and ghost loops, and are gauge-dependent. The result for $\Pi_0(k^2)$ is proportional to $-N_f/3$ which is the first QED beta-function coefficient, $b$. In QCD one expects large-order behaviour of the form $d_n \sim K n^\gamma(b/2)^n n!$ involving the QCD beta-function coefficient $b = (33 - 2N_f)/6$, it is then natural to replace $N_f$ by $(33/2 - 3b)$ to obtain an expansion in powers of $b$.

$$d_n = d^{(n)}_n b^n + d^{(n-1)}_n b^{n-1} + \ldots + d^{(0)}_n \quad (13)$$

The leading-$b$ term $d^{(L)}_n \equiv d^{(n)}_n b^n = (-3)^n d^{[n]} b^n$ can then be used to approximate $d_n$ to all-orders, and an all-orders resummation of these terms performed to obtain $D^{(L)}_{PT}(Q^2)$. If we use the Borel method to define the all-orders perturbative result we obtain

$$D^{(L)}_{PT}(Q^2) = \int_0^\infty dq e^{-q/a(Q^2)} B[D^{(L)}_{PT}](z) \quad (14)$$

The Borel transform $B[D^{(L)}_{PT}](z)$ can be found in Ref. It has the form

$$B[D^{(L)}_{PT}](z) = \sum_{n=1}^\infty A_0(n) - A_1(n) z_n + A_1(n) z_n$$

$$+ \sum_{n=1}^\infty B_0(n) + B_1(n) z_n - B_1(n) z_n$$

The expressions for the $A_0(n)$ and $B_0(n)$ are given in Ref. For the Adler function in leading-$b$ approximation we see that there are single and double poles in $B[D^{(L)}_{PT}](z)$ at positions $z = z_n$ and $z = -z_n$ with $z_n = 2n/b$ $n = 1, 2, 3, \ldots$. The singularities on the positive real semi-axis are the infrared renormalons, $IR_n$ and those on the negative real semi-axis are ultraviolet renormalons, $UV_n$. We shall see that they correspond to integration over the bubble-chain momentum $k^2$ in the regions $k^2 < Q^2$ and $k^2 > Q^2$, respectively. The IR renormalon singularities lie on the Borel integration path along the positive real axis. This leads to an ambiguous imaginary part which is structurally the same as a term in the OPE expansion in Eq.(8). The $IR_n$ renormalon ambiguity is in one-to-one correspondence with non-logarithmic UV divergences present in the $(\Lambda^2/Q^2)^n$ term of the OPE. OPE ambiguities and perturbative
ambiguities can cancel once a definite regulation of the Borel integral, for instance principal value (PV), has been chosen, and the PT and NP components are then separately well-defined. The (PV regulated) Borel integral may be evaluated in terms of exponential integral $Ei$ functions \cite{2} but notice that the Borel integral diverges for $Q^2 < \Lambda^2$, and potentially for $Q^2 = \Lambda^2$!

$$D^{(L)}_{PT}(Q^2) = \sum_{n=1}^{\infty} \left[ z_n e^{-z_n/a(Q^2)} Ei \left( \frac{z_n}{a(Q^2)} \right) \right]$$

$$\times \left[ \frac{z_n}{a(Q^2)} (A_0(n) - z_n A_1(n)) - z_n A_1(n) \right]$$

$$+ (A_0(n) - z_n A_1(n))$$

$$+ \sum_{n=1}^{\infty} \left[ z_n e^{-z_n/a(Q^2)} Ei \left( \frac{z_n}{a(Q^2)} \right) \right]$$

$$\times \left[ \frac{z_n}{a(Q^2)} (B_0(n) + z_n B_1(n)) - z_n B_1(n) \right]$$

$$- (B_0(n) + z_n B_1(n))$$ \hspace{1cm} (16)

This expression has the property that it is finite and continuous at $Q^2 = \Lambda^2$ and freezes smoothly to a freezing limit of $D^{(L)}_{PT}(0) = 0$. Similar behaviour is found for GLS/polarized Bjorken and unpolarized Bjorken DIS sum rules, $K^{(L)}_{PT}(Q^2)$ and $U^{(L)}_{PT}(Q^2)$. See Ref.\cite{1} for their definitions. Interesting connections between these sum rules have been explored in Ref.\cite{3}

The finiteness and continuity are delicate. The $Ei$ functions potentially have a divergence proportional to $\ln a(Q^2)$ as $Q^2 \rightarrow \Lambda^2$, but the coefficient of this divergent term is

$$- \sum_{n=1}^{\infty} \frac{z_n^2}{a} [A_1(n) + B_1(n)].$$ \hspace{1cm} (17)

For $K^{(L)}_{PT}(Q^2)$ and $U^{(L)}_{PT}(Q^2)$ the equivalent coefficients are $(-8 + 2 + 16 - 10 = 0)$ and $(8 - 6 - 2) = 0$, respectively. There is a relation between IR and UV renormalon residues which ensures the divergent term vanishes

$$z_{n+3}^2 B_1(n + 3) = - z_n^2 A_1(n).$$ \hspace{1cm} (18)

This ensures that

$$\sum_{n=1}^{\infty} \frac{z_n^2}{a} [A_1(n) + B_1(n)] = 0. \hspace{1cm} (19)$$

Another similar relation is \cite{2}

$$A_0(n) = - B_0(n + 2). \hspace{1cm} (20)$$

We shall show that these relations are underwritten by continuity of the characteristic function in the skeleton expansion.

The one chain term in the QCD skeleton expansion can be written in the form \cite{2}

$$D^{(L)}_{PT}(Q^2) = \int_0^{\infty} dt \omega_D(t) a(tQ^2) . \hspace{1cm} (21)$$

Here $\omega_D(t)$ is the characteristic function. It satisfies the normalization condition

$$\int_0^{\infty} dt \omega_D(t) = 1 . \hspace{1cm} (22)$$

$t \equiv k^2/Q^2$, and so one is integrating over the momentum flowing through the chain of bubbles. $\omega_D(t)$ and its first three derivatives are continuous \cite{8} at $t = 1$, and the integral divides into an IR and UV part, corresponding to $k^2 < Q^2$ and $k^2 > Q^2$, respectively.

$$D^{(L)}_{PT} = \int_0^1 dt \omega^{IR}_D(t) a(tQ^2) + \int_1^{\infty} dt \omega^{UV}_D a(tQ^2). \hspace{1cm} (23)$$

The first term involving $\omega^{IR}_D$ reproduces the IR renormalon contributions, and the second term involving $\omega^{UV}_D$ the UV renormalon contributions. For $Q^2 > \Lambda^2$ the first integral encounters the Landau pole in the coupling in the region of integration and requires regulation. If one uses a PV definition one can show by a change of variable that the one-chain skeleton expansion result is exactly equivalent to the PV regulated Borel integral of Eq.(14), and yields $D^{(L)}_{PT}(Q^2)$ as in Eq.(16). For $Q^2 < \Lambda^2$ the first integral no longer requires regulation, but the second UV term does. If one uses PV regulation one can show that the one-chain result is exactly equivalent
to a PV-regulated modified Borel representation

\[ \mathcal{D}_{\nu T}^{(L)}(Q^2) = \int_0^{-\infty} dz \, e^{-z/\alpha(Q^2)} \kappa B[\mathcal{D}_{\nu T}^{(L)}](z) \]

Note that the contour of integration now runs along the negative real axis in the Borel plane, and therefore there is now an ambiguous imaginary part due to the UV renormalons. The ambiguity contributed by $UV_n$ is of the form $(Q^2/\Lambda^2)^n$. These ambiguities are associated with IR divergences of dimension-six four-fermion operators associated with UV renormalons, and this suggests that the NP OPE component for $Q^2 < \Lambda^2$ should be replaced by an expansion analogous to Eq. (8), but in powers of $Q^2/\Lambda^2$. Continuity of $\omega_D(t)$ and its first three derivatives at $t = 1$, and equivalently finiteness of $\mathcal{D}_{\nu T}^{(L)}(Q^2)$ and its first three derivatives $d/d\ln Q$ at $Q^2 = \Lambda^2$ is underwritten by the relation of Eq. (19), and by three additional more complicated relations involving the $A_{0,1}$ and $B_{0,1}$ residues. It is easy to show that the ambiguous imaginary part in $\mathcal{D}_{\nu T}^{(L)}$ arising from IR renormalons for $Q^2 > \Lambda^2$, and UV renormalons for $Q^2 < \Lambda^2$, can be written directly in terms of $\omega_D^{IR}$ and $\omega_D^{UV}$. For $Q^2 > \Lambda^2$ one has

\[ Im[\mathcal{D}_{\nu T}^{(L)}(Q^2)] = \pm \frac{2\pi \Lambda^2}{b} \frac{Q^2}{\Lambda^2} \omega_D^{IR} \left( \frac{\Lambda^2}{Q^2} \right) \]

and for $Q^2 < \Lambda^2$,

\[ Im[\mathcal{D}_{\nu T}^{(L)}(Q^2)] = \pm \frac{2\pi \Lambda^2}{b} \frac{Q^2}{\Lambda^2} \omega_D^{UV} \left( \frac{\Lambda^2}{Q^2} \right) \]

Continuity at $Q^2 = \Lambda^2$ then follows from continuity of $\omega_D(t)$ at $t = 1$. In principle the real part of the OPE condensates are independent of the imaginary, but continuity and finiteness involve the set of relations between $A_{0,1}$ and $B_{0,1}$ that we have just noted. Continuity naturally follows if we write $\mathcal{D}_{\nu T}^{(L)}(Q^2)$ in the form

\[ \left( \kappa + \frac{2\pi i}{b} \right) \int_0^{\Lambda^2/Q^2} dt \left( \omega_D(t) + i \frac{d\omega_D(t)}{dt} \right) \]

Here $\kappa$ is an overall real non-perturbative constant. If the PT component is PV regulated then one averages over the $\pm$ possibilities for contour routing, combining with $\mathcal{D}_{\nu T}^{(L)}$ one can then write down the overall result for $\mathcal{D}(Q^2)$. The $Q^2$ evolution is fixed by the non-perturbative constant $\kappa$ and by $\Lambda$. Both the PT and NP components freeze smoothly to zero.

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